Article

An Analytical Elastic Solution for Right-Angle Trapezoidal Opening in Steeply Inclined Coal Seam

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Abstract: In the process of underground mining, steeply inclined rocks or coal seams are often encountered, forming the openings of right-angle trapezoid. According to the geological conditions of a mining project in China, an analytical elastic solution of stress and displacement around right-angle trapezoidal opening in a homogeneous, isotropic, and linear elastic geomaterial is presented, which is based on the evaluation of the conformal mapping representation by an appropriate numerical calculation and the complex potential functions. The different results from other shaped openings are shown as follows. In a right-angle trapezoidal opening, the maximum displacements of roof falling occur on the low side, while the most horizontal displacements on the low side are around the roof and the most horizontal displacements on the high side are around the middle of the high side in this opening. These results are also compared with the numerical calculations in FLAC software, illustrating that the solution may be easily applied to rock mechanics or rock engineering for understanding the deformation of floor heave and roof falling down. The solution is also suitable for optimum design of bolt supporting in a right-angle trapezoidal opening, which is different from the traditional concept of symmetrical bolt supporting. Finally, a methodology is proposed for the estimation of conformal mapping coefficients for a given cross-sectional shape of an opening without symmetrical axis.

Keywords: right-angle trapezoidal opening; conformal mapping; plane elasticity; steeply inclined coal seam; deformation analysis

1. Introduction

Underground openings in rocks and coal seams are excavated in a wide range of geometries, such as circle [1–4], ellipse [5], ovaloid [6], rectangle [7], etc. The common feature of all these openings is that the confining stress on the boundary of the opening drops to zero and the rock or coal seam deforms elastically at the very least. An understanding of the elastic deformation around the opening is quite important for underground engineering problems, especially for mining engineering problems. The majority of elastic solutions are applied to underground openings with one or two axes of symmetry [8,9]. Although the visualizing solutions for openings without symmetrical axes can be obtained by the use of numerical modeling, the analytical solution for openings is helpful to understand the solution. On the other hand, the engineer can analyze the correctness of numerical calculations because of the simplification of numerical modeling; however, the analytical solution provides a basis for grasping the correctness.

During underground mining, steeply inclined rocks and coal seams are often encountered and may result in instability problems, which has become a significant research topic in recent years [10–14]. Because of the presence of steeply inclined dips, the openings in rocks or coal seams do not have symmetrical axes, such as right-angle trapezoidal shape, for decreasing the exposed area of the opening roof and withstanding great lateral pressure.
on both sides of the opening. It should be noticed that the right-angle trapezoidal shape is not the only option for opening without symmetrical axes, but most openings are designed as right-angle trapezoidal shapes in steeply inclined rocks or coal seams in China, and some numerical calculations are carried out as shown in references from [15,16]. However, there are few analytical results of stresses or displacements in the right-angle trapezoidal openings. The excavation-induced asymmetrical elastic deformation in surrounding rock or coal seam is a difficult problem for the stability of supporting the openings. Therefore, considering the excavation of openings in steeply inclined rocks or coal seams, the analytical solution for right-angle trapezoidal openings should be given to obtain the stress or displacement on the boundary of the opening.

Based on the geological conditions in the Chen-man-zhuang mining project in China, a theoretical solution of right-angle trapezoidal opening in elastic, isotropic, and homogeneous coal seam is proposed based on the powerful conformal mapping method. Subsequently, the deformation on the boundary of opening is obtained and analyzed, which may bring about a better insight to the stability of opening in underground rock (coal seam).

2. Chen-Man-Zhuang Mining Project

The Chen-man-zhuang underground mining project is situated in the East of Shanxian Coalfields in Heze City of Shandong Province, China. The center of the Chen-man-zhuang project is 16 km away from Shanxian and 90 km from Heze Economic Development Zone, which is shown in Figure 1. It is also interconnected by an excellent highway system with quick access to Jiangsu Province, Henan Province, and Anhui Province.

![Figure 1. Schematic diagram of geographical location of Chen-man-zhuang project.](image)

It is reported that the recoverable reserves of coal seam are 50.766 million tons, and the average thickness of coal seam is more than 3.50 m in the Chen-man-zhuang project. The total mining area of the Chen-man-zhuang project is 25.69 km$^2$. There is a coal seam named 3-top coal seam, which is 3.50 m thick and at a depth of 900 m. It belongs to steeply inclined seam with an average inclined dip angle of 40°. The immediate roof of 3-top
coal seam is mudstone with 0.9–1.3 m thick and low strength, and the main roof is full medium sandstone with high strength. The floor of 3-top coal seam is mudstone with low strength, and contains small amounts of sandstone. The opening often needs to be excavated before mining the coal resource. In this project, the opening in 3-top coal seam is designed for mining the coal resource in 3402 working face. According to the characteristics of steeply inclined seam and the common design method in this mining project, the opening is designed to be a right-angle trapezoidal shape, which is shown in Figure 2. Based on the previous mining experience in this project, there is always great deformation and invalid bolt supporting because of the steeply inclined rock (seam) and low strength of floor and roof for the opening. Therefore, it is very important to predict the deformation and provide a bolt supporting scheme for the opening before mining, which can provide the basis for safety mining.

![Figure 2](Image)

**Figure 2.** Schematic diagram of the opening with a right-angle trapezoidal shape in 3-top coal seam.

### 3. The Conformal Mapping Representation of Right-Angle Trapezoidal Opening

In the theory of plane elasticity [17], elastic stress/displacement around an opening can be approximated by that of a hole, having the same shape, in an infinitely large and elastic plate subjected to edge loads. As a matter of fact, classical solutions for the stresses/displacements around a circular, elliptic, oval, or rectangular hole have interesting applications in rock engineering. For the solution of plane elasticity problems with complicated shapes, the conformal mapping methodology with complex variables is a feasible technology, in which the problem geometry involving an awkwardly shaped region is transformed into one of a simple shape [18–20].

In this section, a plane strain elastic model is considered for the opening with a right-angle trapezoidal shape on a homogeneous infinitely large and elastic plate subjected to principal stresses \( \sigma_{xy} \) and \( \sigma_{yz} \) referred to a Cartesian coordinate system \( Oxyz \). That is, the opening axis is assumed to be aligned with the direction of the third out-of-plane principal stress \( \sigma_{zxy} \). The direction of \( \sigma_{zxy} \) is parallel to the \( O\chi \)-axis. The methodology starts with the conformal mapping of the boundary of the right-angle trapezoidal opening and its exterior region \( S \) (Figure 3a) into the interior region \( \Sigma \) of the circle with unit radius (Figure 3b). The
position of every point in the physical z-plane with \( z = x + iy = re^{i\alpha} \), where \( r, \alpha \) denote polar coordinates and \( i = \sqrt{-1} \) is the imaginary unit, is mapped into the unit circle in the \( \zeta \)-plane with \( \zeta = \xi + i\eta = pe^{i\theta} \) with complex function:

\[
z = x + iy = \omega(\zeta) = c_{-1} \zeta + \sum_{k=0}^{n} c_k \zeta^k, \ |\zeta| \leq 1
\]

(1)

where, the constant coefficient \( c_{-1} \) is a real number and the constant coefficients \( c_k \) are in general complex numbers with \( c_k = a_k + ib_k \) (\( k = 0, 1, 2, 3, \ldots, n \)). These coefficients should be chosen in such a way that the curve is fully close to the right-angle trapezoidal shape in Figure 3a. Because of the exterior region of opening in z-plane being mapped into the interior region in \( \zeta \)-plan, the point moves along the outline \( C \) in a counterclockwise direction in z-plane; however, this point moves along the clockwise direction in \( \zeta \)-plane (Figure 3b). Meanwhile, the outline \( C \) of a right-angle trapezoidal shape is mapped into the outline \( \gamma \) of unit circle (for \( \zeta = e^{i\theta} \) along \( \gamma \)).

![Diagram](image)

**Figure 3.** (a) Schematic diagram of the opening and system of coordinates; (b) unit circle and system of coordinates.

The parametric representation of the curves in the \( Oxy \)-plane transformed by Equation (1) are expressed as follows:

\[
\begin{align*}
x &= c_{-1} \frac{\cos \theta}{p} + \sum_{k=0}^{n} \rho^k (a_k \cos k\theta - b_k \sin k\theta) \\
y &= -c_{-1} \frac{\sin \theta}{p} + \sum_{k=0}^{n} \rho^k (a_k \sin k\theta + b_k \cos k\theta)
\end{align*}
\]

(2)

By setting \( \rho = 1 \) in Equation (2), the parametric equations defining the boundary of the opening are obtained. Then, it is possible to define some tunnels or cavern shapes widely used in mining and civil engineering if a certain number of terms and their values are assigned to the coefficients of Equation (2).

The computation of constant coefficients \( c_{-1}, a_k, b_k \) (\( k = 0, 1, 2, 3, \ldots, n \)) are shown as follows:

(i) First, the outline \( C \) of the opening is divided into \( m + 1 \) points (\( m \geq 1 \)). It should be noticed that the first point and the last point must be the same point. For approximate calculations, it is supposed that these discrete points match with \( \theta = \frac{2\pi}{m} \) equidistant points along the outline \( \gamma \) of unit circle;
(ii) Next, the constant coefficients of Equation (2) are obtained by solving the 2m linear equations with 2n + 3 unknown numbers with least square method:

\[ A_{pq}X_q = R_p \Leftrightarrow X_q = \left(A_{pq}^T A_{pq}\right)^{-1} A_{pq}^T R_p \]  \hspace{1cm} (3)

where \( m \) is the number of points and the coefficient matrix is shown as follows:

\[
A_{pq} = \begin{bmatrix}
\cos \theta_1 & 1 & 0 & \cos \theta_1 & -\sin \theta_1 & \cos(2\theta_1) & -\sin(2\theta_1) & \cdots & \cos(n\theta_1) & -\sin(n\theta_1) \\
\cos \theta_2 & 1 & 0 & \cos \theta_2 & -\sin \theta_2 & \cos(2\theta_2) & -\sin(2\theta_2) & \cdots & \cos(n\theta_2) & -\sin(n\theta_2) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\cos \theta_m & 1 & 0 & \cos \theta_m & -\sin \theta_m & \cos(2\theta_m) & -\sin(2\theta_m) & \cdots & \cos(n\theta_m) & -\sin(n\theta_m) \\
-\sin \theta_1 & 0 & 1 & \sin \theta_1 & \cos \theta_1 & \sin(2\theta_1) & \cos(2\theta_1) & \cdots & \sin(n\theta_1) & \cos(n\theta_1) \\
-\sin \theta_2 & 0 & 1 & \sin \theta_2 & \cos \theta_2 & \sin(2\theta_2) & \cos(2\theta_2) & \cdots & \sin(n\theta_2) & \cos(n\theta_2) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
-\sin \theta_m & 0 & 1 & \sin \theta_m & \cos \theta_m & \sin(2\theta_m) & \cos(2\theta_m) & \cdots & \sin(n\theta_m) & \cos(n\theta_m)
\end{bmatrix}
\]

\[
X_q = \begin{bmatrix}
c_{-1} & a_0 & b_0 & a_1 & b_1 & a_2 & b_2 & \cdots & a_n & b_n
\end{bmatrix}^T
\]

\[
R_p = \begin{bmatrix}
x_1 & x_2 & \cdots & x_m & y_1 & y_2 & \cdots & y_m
\end{bmatrix}^T
\]

(iii) After the first results of the coefficients \( c_{-1}, a_k, b_k \) (\( k = 0, 1, 2, 3, \ldots, n \)), the new points on the outline \( C \) of opening in the \( z \)-plane are obtained by the mapping function Equation (1). Subsequently, the \( x \) coordinate or \( y \) coordinate of every point obtained by Equation (1) is substituted into \( R_p \) and Equation (3). The new values are solved again for modified values of \( c_{-1}, a_k, b_k \) (\( k = 0, 1, 2, 3, \ldots, n \));

(iv) This overlay process is carried out until an allowable error is achieved.

Figure 4 shows the predictions of the opening outline through the above calculation. It is shown that the prediction of opening outline is completely close to the real boundary when \( n = 10 \).

![Figure 4. Approximation of the opening boundary of \( n = 10 \).](image)

**4. Elastic Solution for Right-Angle Trapezoidal Opening**

Based on elastic mechanics, the boundary condition for unit circle can be taken as follows:

\[
\varphi_0(\xi) + \frac{1}{2\pi i} \int_\sigma \frac{\omega'(\sigma)}{\omega(\sigma)} \frac{\varphi_0'(-\xi)}{\sigma - \xi} d\sigma = \frac{1}{2\pi i} \int_\sigma \sigma \frac{f_0 d\sigma}{\sigma - \xi} \quad (4)
\]
in which $\sigma = e^{i\theta}$ represents an arbitrary point along the outline $\gamma$, and:

$$f_0 = i \int \left( \frac{X + iY}{2\pi} + \ln r - \frac{1 + \mu}{8\pi} (X - iY) \frac{\omega'(r)}{\omega'(0)} \right) \cdot (B' - iC') \omega'(r)$$

(5)

where $X$ and $Y$ is the sum of surface traction on the contour $C$ along Ox-axes and Oy-axes, respectively.

Additionally:

$$B = \frac{1}{4} (\sigma_{x\circ{o}} + \sigma_{y\circ{o}}), \quad B' = -\frac{1}{2} (\sigma_{x\circ{o}} - \sigma_{y\circ{o}})$$

(6)

where $\sigma_{x\circ{o}}, \sigma_{y\circ{o}}$ denotes to the primary stress at infinity along x-axis and y-axis, respectively (e.g., Figure 3a).

An expansion of $\varphi_0(\zeta)$ should be employed for solving the Cauchy integral Equation (4), which has been presented elegantly in the solution for the elliptical opening, rectangular opening, and notched circular opening with one or two axis of symmetry:

$$\varphi_0(\zeta) = \sum_{k=0}^{n} a_k \zeta^k$$

(7)

where $a_k (k = 0, 1, 2, \ldots, n)$ are unknown complex numbers and are obtained by the boundary conditions.

Proceeding formally, according to Equations (1) and (7), the left integral item of Equation (4) may be derived that we may write:

$$\frac{1}{2\pi i} \int_\gamma \frac{\varphi_0(\zeta)}{\omega'(\zeta)} \frac{d\zeta}{\sigma - \zeta} = -\frac{1}{2\pi} \sum_{k=1}^{9} \left[ \pi_2 (a_k + ib_k + 1) \right]$$

$$+ \frac{1}{2\pi} \sum_{k=1}^{9} \left[ \pi_2 \left( \frac{\zeta^k (a_{k+2} + ib_{k+2})}{2\pi \sigma k} \right) + \pi_2 \sum_{k=1}^{7} \left[ \frac{\zeta^k (a_{k+3} + ib_{k+3})}{2\pi \sigma k} \right] \right]$$

$$+ \frac{1}{2\pi} \sum_{k=1}^{9} \left[ \pi_2 \left( \frac{\zeta^k (a_{k+4} + ib_{k+4})}{2\pi \sigma k} \right) + 4\pi_4 \sum_{k=1}^{5} \left[ \frac{\zeta^k (a_{k+5} + ib_{k+5})}{2\pi \sigma k} \right] + 5\pi_5 \sum_{k=1}^{4} \left[ \frac{\zeta^k (a_{k+6} + ib_{k+6})}{2\pi \sigma k} \right] \right]$$

$$+ \frac{1}{2\pi} \sum_{k=1}^{9} \left[ \pi_2 \left( \frac{\zeta^k (a_{k+7} + ib_{k+7})}{2\pi \sigma k} \right) + 7\pi_7 \sum_{k=1}^{2} \left[ \frac{\zeta^k (a_{k+8} + ib_{k+8})}{2\pi \sigma k} \right] + 8\pi_8 \zeta (a_{10} + ib_{10}) \right]$$

(8)

The bars $\overline{\cdot}$ in Equation (8) denote complex conjugates and the primes $'\cdot'$ denote differentiation. Simplifying the right items of Equation (8) into polynomials it can be given as the following relation:

$$\frac{1}{2\pi i} \int_\gamma \frac{\varphi_0(\zeta)}{\omega'(\zeta)} \frac{d\zeta}{\sigma - \zeta} = -\frac{1}{2\pi} \sum_{k=1}^{9} \left[ \pi_2 (a_k + ib_k + 1) \right]$$

$$- \frac{1}{2\pi} \sum_{k=1}^{9} \left[ \pi_3 \left( \frac{\zeta^k (a_{k+1} + ib_{k+1})}{2\pi \sigma k} \right) - \pi_2 \sum_{k=1}^{8} \frac{\zeta^k (a_{k+2} + ib_{k+2})}{2\pi \sigma k} - \pi_2 \sum_{k=1}^{7} \frac{\zeta^k (a_{k+3} + ib_{k+3})}{2\pi \sigma k} - \pi_2 \sum_{k=1}^{6} \frac{\zeta^k (a_{k+4} + ib_{k+4})}{2\pi \sigma k} + \frac{1}{\pi} \right]$$

(9)

in which the constant complex coefficients $d_{ik} (k = 0, 1, 2, \ldots, 8), d_{2k} (k = 0, 1, 2, \ldots, 7), d_{3k} (k = 0, 1, 2, \ldots, 6), d_{4k} (k = 0, 1, 2, \ldots, 5), d_{5k} (k = 0, 1, 2, \ldots, 4), d_{6k} (k = 0, 1, 2, 3), d_{7k} (k = 0, 1, 2), d_{8k} (k = 0, 1), 1$ are represented in closed form in terms of the constant conformal mapping coefficients $c_1, a_k, b_k (k = 0, 1, 2, 3, 4, \ldots, 10)$ and $O(\cdot)$ represents high order of magnitude.

For solving the complex coefficients $a_k$, the right integral item of Equation (4) should be given according to the expression $f_0$ in Equation (5). In this paper, we mainly research the case that there is no surface traction on the contour $C$ in Figure 3a. Therefore, the expression $f_0$ is given as the following relation:

$$f_0 = -\frac{1}{2} (\sigma_{x\circ{o}} + \sigma_{y\circ{o}})[c_{-1} (1 - \zeta^k) + \sum_{k=0}^{10} (a_k + ib_k) \zeta^k] + \frac{1}{2} (\sigma_{x\circ{o}} - \sigma_{y\circ{o}})[c_{-1} (1 + \zeta^k) + \sum_{k=0}^{10} (a_k - ib_k) \zeta^k]$$

(10)
The right integral item of Equation (4) is written as follows.

\[
\frac{1}{2\pi i} \int_{\sigma} \frac{f_{0d}d\sigma}{\sigma - \zeta} = -\frac{1}{2} \left( \sigma_{xvy} + \sigma_{yyo} \right) \left| \sum_{k=0}^{10} (a_k + ib_k) \zeta^k \right| + \frac{1}{2} \left( \sigma_{xvy} - \sigma_{yyo} \right) c_{-1} \zeta + \frac{1}{2} \left( \sigma_{xvy} - \sigma_{yyo} \right) \left( a_0 - ib_0 \right)
\]  

\( (11) \)

By substituting Equations (7), (9), and (11) into Equation (4), we obtain the relation for evaluating the unknown complex coefficients \( a_k \) in Equation (7):

\[
\sum_{k=0}^{10} \sigma_k \zeta^k - \frac{1}{2} \sum_{k=1}^{9} \left[ k\sigma_k (a_{k+1} + ib_{k+1}) \right] - \overline{\alpha} \sum_{k=0}^{8} d_{1k} \zeta^k - \overline{\alpha_2} \sum_{k=0}^{7} d_{2k} \zeta^k - \overline{\alpha_3} \sum_{k=0}^{6} d_{3k} \zeta^k - \overline{\alpha_4} \sum_{k=0}^{5} d_{4k} \zeta^k
\]

\[\overline{\alpha_5} \sum_{k=0}^{4} d_{5k} \zeta^k - \overline{\alpha_6} \sum_{k=0}^{3} d_{6k} \zeta^k - \overline{\alpha_7} \sum_{k=0}^{2} d_{7k} \zeta^k - \overline{\alpha_8} \sum_{k=0}^{1} d_{8k} \zeta^k \]

\(- \frac{1}{2} \left( \sigma_{xvy} + \sigma_{yyo} \right) \left| \sum_{k=0}^{10} (a_k + ib_k) \zeta^k \right| + \frac{1}{2} \left( \sigma_{xvy} - \sigma_{yyo} \right) c_{-1} \zeta + \frac{1}{2} \left( \sigma_{xvy} - \sigma_{yyo} \right) \left( a_0 - ib_0 \right) \]

\( (12) \)

By comparing the same powers of the variable \( \zeta \) and separating the real values and imaginary values in Equation (12), the linear equations are gained for solving the coefficients \( a_k \):

\[
B_{aq} X_a \equiv R_{aq} \Leftrightarrow X_a = (B_{aq})^{-1} R_{aq}
\]

(13)

In the above relations, \( \text{Re}() \) and \( \text{Im}() \) denote the real and imaginary value of what it encloses, respectively. By setting the value of \( \sigma_{xvy} \) and \( \sigma_{yyo} \) is equal to \(-33.75 \text{ MPa} \) and \(-22.5 \text{ MPa} \) according to the in situ stress measurement in the Chen-man-zhuang mining project, respectively, and using the values of coefficients in Tables 1 and 2, the real values and imaginary values of the complex coefficients \( a_k \) are listed in Table 3.

| Table 1. The values of \( c_{-1}, a_k, b_k \) \( (k = 0, 1, 2, 3, \ldots, 10) \). |
|-----------------|-----------------|-----------------|-----------------|
| Coefficient     | Value           | Coefficient |
| \( c_{-1} \)    | 1.943007098870969 | \( a_0 \)  | 0.123864538423751 |
| \( a_1 \)       | -0.76277576573139 | \( b_1 \)  | 0.229952727807971 |
| \( a_2 \)       | -0.183253154614749 | \( b_2 \)  | -0.108592194784094 |
| \( a_3 \)       | -0.210508431449733 | \( b_3 \)  | -0.11912990218219 |
| \( a_4 \)       | 0.059058447475317 | \( b_4 \)  | 0.114679930116000 |
| \( a_5 \)       | 0.041584641384929 | \( b_5 \)  | 0.012230502784915 |
| \( a_6 \)       | 0.00902374775452 | \( b_6 \)  | -0.041315476242453 |
| \( a_7 \)       | -0.017357739598054 | \( b_7 \)  | 0.01363905406708 |
| \( a_8 \)       | -0.00390225809779 | \( b_8 \)  | -0.01382808675274 |
| \( a_9 \)       | -0.002407424676143 | \( b_9 \)  | 0.004481421558265 |
| \( a_{10} \)    | -0.00993907030865 | \( b_{10} \) | 0.017387660145686 |
Table 2. The values of the complex coefficients $d_{1k}, d_{2k}, d_{3k}, d_{4k}, d_{5k}, d_{6k}, d_{7k}, d_{8k}$.

| $d_{1k}$ | Value | $d_{1k}$ | Value |
|----------|-------|----------|-------|
| $d_{10}$ | $-0.00613127527563420 - 0.0212816433307378i$ | $d_{11}$ | $-0.117842059038110 - 0.0680388091648117i$ |
| $d_{12}$ | $0.0293443760564685 + 0.0707224081150705i$ | $d_{13}$ | $0.0274885674993676 + 0.0090292634928232i$ |
| $d_{14}$ | $0.00303458256474199 - 0.0218746391688664i$ | $d_{15}$ | $-0.00820945608106655 + 0.0040098771607415i$ |
| $d_{16}$ | $0.00105898412971797 - 0.0100245445605539i$ | $d_{17}$ | $-0.00123902001055050 + 0.00230643601913176i$ |
| $d_{18}$ | $-0.00511530313329306 + 0.00894880425683173i$ | $d_{19}$ | | |
| $d_{20}$ | $0.190009864787558 - 0.0134533527904577i$ | $d_{21}$ | $0.058688752129370 + 0.14144816230141i$ |
| $d_{22}$ | $0.05497771349987352 + 0.018045852695646i$ | $d_{23}$ | $0.00606916512488398 - 0.0437492783377329i$ |
| $d_{24}$ | $-0.0164189121261331 + 0.00800197543214830i$ | $d_{25}$ | $0.00217796825943594 - 0.0200490891211079i$ |
| $d_{26}$ | $-0.00247804002110100 + 0.00461287203826352i$ | $d_{27}$ | $-0.01023036062665861 + 0.01789786805136635i$ |
| $d_{28}$ | $-0.00313502474218401 + 0.0351015869869336i$ | $d_{29}$ | $0.0824675024981029 + 0.027067790478470i$ |
| $d_{30}$ | $0.00910337469422596 - 0.0656239175065993i$ | $d_{31}$ | $-0.0246283682431996 + 0.0120029631482225i$ |
| $d_{32}$ | $0.003176695238915391i - 0.030073636816618i$ | $d_{33}$ | $-0.00371706003165149 + 0.0069193080573927i$ |
| $d_{34}$ | $-0.0153459093998792 + 0.0268465207704952i$ | | | |
| $d_{36}$ | $0.0243454394191507 + 0.010913201845532i$ | $d_{41}$ | $0.0121383302589680 - 0.0874985566756858i$ |
| $d_{38}$ | $-0.0328387243242662 + 0.0160399508642966i$ | $d_{42}$ | $0.00423959651887188 - 0.0400918782242158i$ |
| $d_{44}$ | $-0.00495608004220199 + 0.00922574407652703i$ | $d_{45}$ | $-0.020461212531722 + 0.0357953610273269i$ |
| $d_{50}$ | $-0.00804817520189555 - 0.00306491820535773i$ | $d_{51}$ | $-0.0410782040533227 + 0.0200049385803708i$ |
| $d_{52}$ | $0.02592492064858985 - 0.050122728027697i$ | $d_{53}$ | $-0.0061951005257249 + 0.0115321800956588i$ |
| $d_{54}$ | $-0.0255765155664653 + 0.0447442012841587i$ | | | |
| $d_{60}$ | $0.00434391049648474 - 0.018115587543296i$ | $d_{61}$ | $0.00635390477830782 - 0.060147267633236i$ |
| $d_{62}$ | $-0.00743412006330299 + 0.0138386161147905i$ | $d_{63}$ | $-0.0306918187997584 + 0.0536930415409904i$ |
| $d_{70}$ | $0.0214705733520627 - 0.0203538957756759i$ | $d_{71}$ | $-0.00867314007385349 + 0.0161450521339223i$ |
| $d_{72}$ | $-0.035807121930514 + 0.0626418817978211i$ | | | |
| $d_{80}$ | $0$ | $d_{81}$ | $-0.0409224250663445 + 0.0715907220546539i$ |

Table 3. The real and imaginary values of the complex coefficients $a_k$.

| $a_k$ | Real Value | Imaginary Value |
|-------|------------|-----------------|
| $a_0$ | $6,048,774.18703198$ | $-10,298,069.9676466$ |
| $a_1$ | $-31,352,104.1442811$ | $9,158,635.9361744$ |
| $a_2$ | $-5,619,295.03514821$ | $-4,860,490.56892524$ |
| $a_3$ | $-6,300,981.33006717$ | $-3,869,348.92054447$ |
| $a_4$ | $1,524,019.59546642$ | $4,091,707.21872871$ |
| $a_5$ | $1,662,297.65786878$ | $494,696.756351372$ |
| $a_6$ | $118,164.270916096$ | $-1,132,658.59704894$ |
| $a_7$ | $-457,719.446320244$ | $172,337.55670975$ |
| $a_8$ | $132,590.210627846$ | $-622,630.541593787$ |
| $a_9$ | $-67,208,819,0165163$ | $126,039.98132619$ |
| $a_{10}$ | $-279,536.352211837$ | $489,027.941597429$ |
Finally, the second complex potential function is also supposed in a Laurent series expansion, which is written as follows:

$$
\psi_{0}(\zeta) = \sum_{k=0}^{n} \beta_{k} \zeta^{k}
$$  \hspace{1cm} (14)

where $\beta_{k}$ ($k = 0, 1, 2, \ldots, n$) are unknown coefficients and can be obtained using the next relation:

$$
\psi_{0}(\zeta) + \frac{1}{2\pi i} \int_{c} \frac{\omega(\sigma)}{\omega'_{0}(\sigma)} \frac{\psi'_{0}(\sigma)}{\sigma - \zeta} d\sigma = \frac{1}{2\pi i} \int_{c} \frac{f_{0} d\sigma}{\sigma - \zeta}
$$  \hspace{1cm} (15)

Proceeding similarly, according to Equations (1) and (7), the left integral item of Equation (15) may be derived, which we may write as follows:

$$
\frac{1}{2\pi i} \int_{c} \frac{\omega(\sigma)}{\omega'_{0}(\sigma)} \frac{\psi'_{0}(\sigma)}{\sigma - \zeta} d\sigma = \frac{9c_{-1}a_{0}}{10(a_{0} + ib_{0})} + \frac{10\alpha_{10}c_{-1}c_{10}}{c_{-1}^{10} + \sum_{k=0}^{10} k(a_{k} + ib_{k})\zeta^{k-10}} + \frac{\alpha_{10}(a_{0} - ib_{0})}{(a_{0} + ib_{0})}
$$  \hspace{1cm} (16)

Additionally, by simplifying the right items of Equation (16) into polynomials it can be given as the following relation:

$$
\frac{1}{2\pi i} \int_{c} \frac{\omega(\sigma)}{\omega'_{0}(\sigma)} \frac{\psi'_{0}(\sigma)}{\sigma - \zeta} d\sigma = \frac{\alpha_{10}(a_{0} - ib_{0})}{(a_{0} + ib_{0})} + \gamma_{11} \zeta + O\left(\frac{1}{\zeta}\right)
$$  \hspace{1cm} (17)

in which, $\gamma_{11} = \frac{\alpha_{10}c_{-1}}{(a_{0} + ib_{0})}$. According to Equations (5) and (6), the right integral item of Equation (15) is written as follows:

$$
\frac{1}{2\pi i} \int_{c} \frac{f_{0} d\sigma}{\sigma - \zeta} = -\frac{1}{2} (\sigma_{\phi\phi} + \sigma_{\theta\phi} c_{1-1} \zeta - \frac{1}{2} (\sigma_{\phi\phi} + \sigma_{\theta\phi}) (a_{0} - ib_{0}) + \frac{1}{2} (\sigma_{\phi\phi} - \sigma_{\phi\phi}) \sum_{k=0}^{10} (a_{k} + ib_{k}) \zeta^{k}
$$  \hspace{1cm} (18)

By substituting Equations (14), (17), and (18) into Equation (15), the value of the unknown complex coefficients $\beta_{k}$ in Equation (7) can be obtained:

$$
\sum_{k=0}^{n} \beta_{k} \zeta^{k} + \frac{\alpha_{10}(a_{0} - ib_{0})}{(a_{0} + ib_{0})} + \gamma_{11} \zeta = -\frac{1}{2} (\sigma_{\phi\phi} + \sigma_{\theta\phi}) c_{1-1} \zeta - \frac{1}{2} (\sigma_{\phi\phi} + \sigma_{\theta\phi}) (a_{0} - ib_{0}) + \frac{1}{2} (\sigma_{\phi\phi} - \sigma_{\phi\phi}) \sum_{k=0}^{10} (a_{k} + ib_{k}) \zeta^{k}
$$  \hspace{1cm} (19)

Next, by comparing the same powers of the variable $\zeta$, the linear equations are also gained for solving the coefficients $\beta_{k}$. By using the values of $\sigma_{\phi\phi}$ and $\sigma_{\theta\phi}$ and the values of the constant conformal mapping coefficients $a_{k}, b_{k}$ ($k = 0, 1, 2, 3, \ldots, 10$) in Table 1, the real values and imaginary values of the complex coefficients $\beta_{k}$ are listed in Table 4.

Then, in polar coordinates $(\rho, \theta)$ the radial, tangential, and shear stresses are denoted as $\sigma_{\rho}, \sigma_{\theta},$ and $\tau_{\rho\theta}$, respectively; the radial and tangential incremental displacements $u_{\rho}, u_{\theta}$ due to stress relief at the breakout boundary, may be computed by virtue of the following formulae:

\[
\begin{align*}
\sigma_{\rho} - \sigma_{\rho} + 2i \tau_{\rho\theta} &= \frac{2\zeta}{\rho^{2} \omega'(\zeta)} \left\{ \omega(\zeta) \frac{\phi'_{0}(\zeta) \omega'(\zeta) - \phi_{0}(\zeta) \omega''(\zeta)}{[\omega'(\zeta)]^{2}} + \omega'(\zeta) [B' + iC'] + \frac{\psi_{0}(\zeta)}{\omega'(\zeta)} \right\} \\
\frac{E}{1+v} (u_{\rho} + iu_{\theta}) &= \frac{\zeta}{\rho \omega'_{0}(\zeta)} \left[ k [B \omega(\zeta) + \phi_{0}(\zeta)] - \frac{\omega'(\zeta)}{\omega'_{0}(\zeta)} (B' + iC') \omega'(\zeta) + \phi'_{0}(\zeta) - [(B' + iC') \omega(\zeta) + \phi_{0}(\zeta)] \right]
\end{align*}
\]  \hspace{1cm} (20)
where primes denote differentiation, $| \cdot |$ represents the absolute value for real number or the modulus for complex number, $\kappa$ equals to $3 - 4\mu$ for plane strain problem or $\kappa$ equals to $(3 - \mu)/(1 + \mu)$ for plain stress problem, $E$ and $\mu$ is the Young’s modulus and Poisson’s ratio of rock (coal seam), respectively. Finally, the displacements are given in the Cartesian coordinate system as follows:

$$\frac{E}{1 + \mu} (u_x + iu_y) = \kappa [B\omega(\zeta) + \psi_0(\zeta)] - \frac{\omega'(\zeta)}{\omega(\zeta)} B\omega'(\zeta) + \psi_0'(\zeta) - [(B' + iC')\omega'(\zeta) + \psi_0'(\zeta)]$$ (21)

Table 4. The real values and imaginary values of the complex coefficients $\beta_k$.

| $\beta_k$ | Real Value | Imaginary Value |
|-----------|------------|-----------------|
| $\beta_0$ | -696,738.028633599 | 1,930,106.51793591 |
| $\beta_1$ | 4,290,613.68134891 | -1,293,484.09391984 |
| $\beta_2$ | 1,030,798.99470796 | 610,831.095660530 |
| $\beta_3$ | 1,184,109.92690475 | 670,105.739352481 |
| $\beta_4$ | -332,203.767048660 | -645,074.606902502 |
| $\beta_5$ | -233,913.607790227 | -68,796.5781651473 |
| $\beta_6$ | -50,758.5812369165 | 232,377.455386432 |
| $\beta_7$ | 97,637.2852390529 | -76,719.9116627314 |
| $\beta_8$ | 21,948.8951800059 | 77,782.9537984135 |
| $\beta_9$ | 13,541.7638033033 | -25,207.9962652390 |
| $\beta_{10}$ | 55,907.270423675 | -97,805.5883194858 |

5. Displacements Analysis for Right-Angle Trapezoidal Opening

By setting the Young’s modulus $E = 1.5$ GPa and the Poisson’s ratio $\mu = 0.16$ according to laboratory testing with the rock specimen from the Chen-man-zhuang mining project, the displacements on opening boundary can be obtained with the computational software (e.g., Matlab, Mathcad, Maple, etc.) according to the implementation of Equation (20) or (21). Geotechnical engineers have begun to take advantage of the results of elasticity theory in rock mechanics or rock engineering applications. The displacements on the boundary of right-angle trapezoidal openings are illustrated below for the applications in rock engineering or mining engineering. Meanwhile, the numerical model in FLAC software was developed for comparing the results of displacement according to the analytical elastic solution. The length of numerical model in horizontal and vertical direction is 100 m and 100 m, respectively; the overall mesh of numerical model is $500 \times 500$. The constraints on the left and right boundary of the numerical model are both that the horizontal displacement is equal to zero, while the constraint on the bottom of the numerical model is that the vertical displacement is equal to zero. Uniform loading is applied on the top of the numerical model to simulate the load of overlying strata. The value of the Young’s modulus $E$ and Poisson’s ratio $\mu$ in numerical model is the same with the value in analytical computation, respectively.

Figures 5 and 6 show the vertical displacement ($y$-displacement for short) on the floor and roof for right-angle trapezoidal openings as shown in Figure 3a, respectively. The positive value of $y$-displacement denotes that the vertical displacement is along $Oy$-axis; on the contrary, the negative value of $y$-displacement indicates that the vertical displacement is the opposite direction to $Oy$-axis. It can be seen that the most values of $y$-displacement on the floor are positive numbers, which is consistent with the phenomenon of floor heave for opening in mining engineering. The $y$-displacement on the floor around corner $A$ or corner $B$, as shown on Figure 3a, are negative numbers because of stress concentration applied by principal stresses $\sigma_{x0}$ and $\sigma_{y0}$. The most values of $y$-displacement on the roof are negative numbers, which indicates that the roof is almost falling down for opening.
in mining engineering. The most displacements of falling down on the roof are around corner $E$, which is close to the left side of the opening (also called “low side” in right-angle trapezoidal openings). It can be concluded that the roof near the low side in right-angle trapezoidal openings need more bolt supporting to resist downward deformation, which is an important difference from the deformation of opening with other shapes.

**Figure 5.** The $y$-displacement on floor for right-angle trapezoidal opening.

**Figure 6.** The $y$-displacement on roof for right-angle trapezoidal opening.

Figures 7 and 8 show the horizontal displacement ($x$-displacement for short) on the left side (also called “low side”) and right side (also called “high side”) for right-angle trapezoidal openings as shown in Figure 3a, respectively. In the same way, the positive value of $x$-displacement denotes that the horizontal displacement is along the $Ox$-axis, and the negative value indicates the opposite direction to the $Ox$-axis. It can be clearly obtained that the values of $x$-displacement on the left side are all positive numbers and these are almost negative numbers on the right side, which indicate that the left and right sides of opening are close to each other. The $x$-displacements on the left side around corner $E$ are bigger than that around corner $A$ and the $x$-displacements on the middle of the right side are bigger than that around corner $B$ and corner $F$. This indicates that more bolt supporting is needed around corner $A$ on the low side and the middle of the high side for right-angle
trapezoidal openings, which is the other difference from the deformation of opening with other shapes.

In conclusion, the geotechnical engineers should change the traditional concept of symmetrical bolt supporting for right-angle trapezoidal openings because there is not a symmetrical axis in right-angle trapezoidal shapes and no symmetrical deformation on the boundary of right-angle trapezoidal openings.

![Analytical FLAC](image)

**Figure 7.** The $x$-displacement on the left side for right-angle trapezoidal opening.

![FLAC Analytical](image)

**Figure 8.** The $x$-displacement on the right side for right-angle trapezoidal opening.

### 6. Conclusions

An analytical elastic solution for stresses and displacements around right-angle trapezoidal openings is presented, which is usually applied to steeply inclined coal seam or rock. It can be shown that the conformal mapping and the complex potentials can be used to analyze the displacements or stresses in plane elasticity problems for the opening without a symmetrical axis. The different results from other shaped openings are shown as follows. The most displacements of falling down on the roof are around the low side in right-angle trapezoidal openings, but the most horizontal displacements on the low side are around the roof and the most horizontal displacements on the high side are around the middle of the high side in this opening. Meanwhile, the deformation results are consistent with those of the numerical calculation in FLAC software. The formulation employed here is a good method to gain insight into deformation and the optimum design of bolt supporting in
right-angle trapezoidal openings. Finally, a methodology is proposed for the estimation of conformal mapping coefficients for a given cross-sectional shape of an opening without a symmetrical axis.

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References

1. Jaeger, J.C.; Cook, N.G.W. *Fundamentals of Rock Mechanics*, 2nd ed.; Chapman & Hall: London, UK, 1976.
2. Hoek, E.; Brown, E.T. *Underground Excavations in Rock*; Institute of Mining and Metallurgy: London, UK, 1980.
3. Brady, B.H.G.; Brown, E.T. *Rock Mechanics for Underground Mining*; Allen & Unwin: London, UK, 1985.
4. Sulem, J.; Panet, M.; Guenot, A. An analytical solution for time-dependent displacements in a circular tunnel. *Int. J. Rock Mech. Min. Geomech. Abstr.* 1987, 24, 155–164. [CrossRef]
5. Inglis, C.E. Stresses in a plate due to the presence of cracks and sharp corners. *Trans. Inst. Naval Arch.* 1913, 55, 219–230.
6. Greenspan, M. Effect of a small hole on the stresses in a uniformly loaded plate. *Q. J. Appl. Math.* 1944, 2, 60–71. [CrossRef]
7. England, A.H. *Complex Variable Methods in Elasticity*; Wiley: London, UK, 1971.
8. Gerczek, H. Stresses around tunnels with arched roof. In *Proceedings of the Seventh International Congress on Rock Mechanics*, Balkema, Rotterdam, The Netherlands, 16–20 September 1991; Volume 2, pp. 1297–1299.
9. Gerczek, H. An elastic solution for stresses around tunnels with conventional shapes. *Int. J. Rock Mech. Min. Sci.* 1997, 34, 96.e1–96.e14. [CrossRef]
10. He, S.; Song, D.; Li, Z.; He, X.; Chen, J.; Li, D.; Tian, X. Precursor of Spatio-temporal Evolution Law of MS and AE Activities for Rock Burst Warning in Steeply Inclined and Extremely Thick Coal Seams Under Caving Mining Conditions. *Rock Mech. Rock Eng.* 2019, 52, 2415–2435. [CrossRef]
11. Jawed, M.; Sinha, R.K. Design of rhombus coal pillars and support for Roadway Stability and mechanizing loading of face coal using SDLs in a steeply inclined thin coal seam—A technical feasibility study. *Arab. J. Geosci.* 2018, 11, 415. [CrossRef]
12. Yang, Y.; Lai, X.; Shan, P.; Cui, F. Comprehensive analysis of dynamic instability characteristics of steeply inclined coal-rock mass. *Arab. J. Geosci.* 2020, 13, 241. [CrossRef]
13. Gong, W.; Peng, Y.; He, M.; Wang, J. Thermal image and spectral characterization of roadway failure process in geologically 45 inclined rocks. *Tunn. Undergr. Space Technol.* 2015, 49, 156–173. [CrossRef]
14. Wu, G.; Jia, S.; Chen, W.; Yuan, J.; Yu, H.; Zhao, W. An anchorage experimental study on supporting a roadway in steeply inclined geological formations. *Tunn. Undergr. Space Technol.* 2018, 82, 125–134. [CrossRef]
15. Xiong, X.; Dai, J.; Chen, X. Analysis of stress asymmetric distribution law of surrounding rock of roadway in inclined coal seam: A case study of shitanjing No. 2 coal seam. *Adv. Civ. Eng.* 2020, 2, 1–14. [CrossRef]
16. Yang, H.; Cao, S.; Li, Y.; Sun, C.; Guo, P. Soft roof failure mechanism and supporting method for gob-side entry retaining. *Minerals* 2015, 5, 707–722. [CrossRef]
17. Novozhilov, V.V. *Theory of Elasticity*; Pergamon Press: New York, NY, USA, 1961.
18. Panet, M. Understanding deformations in tunnels. In *Comprehensive Rock Engineering*; Hudson, J.A., Ed.; Fundamentals; Pergamon Press: Oxford, UK, 1993; Volume 1, pp. 663–690.
19. Exadaktylos, G.E.; Stavropoulu, M.C. A closed-form elastic stress-displacement solution for stresses and displacements around tunnels. *Int. J. Rock Mech. Min. Sci.* 2002, 39, 905–916. [CrossRef]
20. Exadaktylos, G.E.; LioiIos, P.A.; Stavropoulu, M.C. A semi-analytical elastic stress-displacement solution for notched circular openings in rocks. *Int. J. Solids Struct.* 2003, 40, 1165–1187. [CrossRef]