A General Framework for Nonlinear Model Predictive Control with Abstract Updates

J. Pannek ∗ J. Michael ∗ M. Gerdts ∗

∗ Faculty of Aerospace Engineering, University of the Federal Armed Forces, 85577 Munich, Germany (johannes.michael@unibw.de, matthias.gerdts@unibw.de, juergen.pannek@unibw.de).

Abstract: Considering nonlinear processes which are subject to unknown but measurable disturbances, we provide both stability and feasibility proofs for nonlinear model predictive controllers with abstract updates without the use of stabilizing terminal constraints or cost. The result utilizes a relaxed Lyapunov inequality for the nominal system and reasonable affine Lipschitz conditions on the open loop change of the optimal value function and the stage costs. For this methodology, we provide proofs that the known MPC updating techniques based on sensitivities, realtime iterations and hierarchically structured MPC updates satisfy our assumptions revealing a common stability framework. To illustrate our approach we present a quartercar example and show performance improvements of the updated solution with respect to comfort and handling properties.

Keywords: predictive control; nonlinear systems; robust stability; sensitivity analysis; realtime iterations; control applications

1. INTRODUCTION

Within the last decades, model predictive control (MPC) has grown mature for both linear and nonlinear systems, see, e.g., Bitmead et al. (1990); Camacho and Bordons (2004) and Gröne and Pannek (2011); Rawlings and Mayne (2009). Although analytically and numerically challenging, this method of approximating an infinite horizon optimal control is attractive due to its simplicity: in each sampling interval, based on a measurement of the current state, a truncated finite horizon optimal control is computed and the first element (or sometimes also more) of the resulting optimal control sequence is applied to the process as input for the next sampling interval(s). Then, the entire problem is shifted forward in time rendering the scheme to be iteratively applicable.

As a consequence of the truncation of the infinite horizon, stability and optimality of the closed loop may be lost. Yet, stability of the resulting closed loop can be guaranteed, either by imposing terminal point constraints as shown in Alamir (2006); Keerthi and Gilbert (1988) or Lyapunov type terminal costs and terminal regions, see Chen and Allgöwer (1998); Mayne et al. (2000). Apart from these modifications, an alternative approach shown in Gröne and Rantzer (2008) utilizes a relaxed Lyapunov condition which can be shown to hold if the system is controllable in terms of the stage costs, cf. Gröne (2009). Additionally, this method allows for computing an estimate on the degree of suboptimality with respect to the infinite horizon controller, see also Nevistić and Príms (1997); Shamma and Xiong (1997) for earlier works on this topic and Gröne and Pannek (2009) for a method to compute this degree.

In this work, we extend the third approach to the case of parametric control systems subject to measured disturbances including subsequent disturbance rejection updates. In order to be as general as possible, an abstract update law is considered to handle the disturbances and a respective algorithm is presented. While feasibility of the resulting closed loop can be shown via self–concordant set conditions, stability requires affine Lipschitz conditions on the open loop change of the optimal value function as well as the stage cost function. Combining these conditions with the relaxed Lyapunov condition for the nominal open loop solution allows us to show a generalization of the stability proof given in Pannek and Gerdts (2012) for the closed loop solution despite the presence of disturbances. Bringing the general framework to life, we show that the known MPC update laws utilizing realtime iterations Diehl et al. (2005) as well as hierarchical approaches Bock et al. (2007) and sensitivity information Zavala and Biegler (2009) are special cases of our abstract update law. Despite the fact that these methods where designed for MPC problems with Lyapunov type terminal costs and terminal regions to ensure stability of the closed loop, these modifications are not required to prove that the stated affine Lipschitz conditions hold and the closed loop to be stable.

The paper is organized as follows: The problem formulation and the concept of practical stability are defined in Section 2. Additionally, the basic MPC algorithm with abstract updates is introduced. In the subsequent Section 3, conditions for feasibility and practical stability of the closed loop are given and respective proofs are shown. Thereafter, the stability conditions are verified for the known sensitivity based update law, realtime iteration and the hierarchical MPC approach in Section 4. Moreover, we present simulation results for a quartercar subject to...
disturbed measurements of both the state and the sensor inputs in Section 5. Concluding our paper, we shortly summarize the results and point out areas of future research.

2. SETUP AND PRELIMINARIES

The set \( \mathbb{N} \) denotes the natural numbers, \( \mathbb{N}_0 \) the natural numbers including zero, \( \mathbb{R} \) denotes the real numbers and \( \mathbb{R}_0^+ \) the nonnegative reals. We call a continuous function \( p : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) a class \( K_\infty \)-function if it satisfies \( p(0) = 0 \), is strictly increasing and unbounded. A continuous function \( \beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) is said to be of class \( K \) if for each \( r > 0 \) the limit \( \lim_{t \rightarrow \infty} \beta(r, t) = 0 \) holds and for each \( t \geq 0 \) the condition \( \beta(\cdot, t) \in K_\infty \) is satisfied.

Throughout this paper we consider plant models driven by the dynamics

\[
\dot{x}(n+1) = f(x(n), u(n), p(n))
\]

where \( x \) denotes the state of the system, \( u \) the external control and \( p \) a parameter which can be measured. These variables are elements of the respective state, control and parameter spaces \( (X, \mathcal{X}), (U, \mathcal{U}) \) and \( (P, \mathcal{P}) \). Here, we allow \( X, U \) and \( P \) to be arbitrary metric spaces which renders our results applicable to discrete time dynamics induced by a sampled finite or infinite dimensional system. In this setting, constraints on both state and control are introduced by considering suitable subsets \( X \subseteq X \) and \( P \subseteq U \).

In most cases, simulated and true trajectories do not coincide. Here, we denote the simulated or nominal trajectory of \( x(\cdot) \) corresponding to an initial state \( x_0 \in X \), a control sequence \( u = (u(k))_{k \in \mathbb{N}_0} \) and a nominal parameter sequence \( p = (p(k))_{k \in \mathbb{N}_0} \) by \( x(\cdot) = x_{up}(\cdot ; x_0) \). The true system dynamics, however, are given by

\[
\dot{x}(n+1) = \overline{f}(x(n), \overline{u}(n), \overline{p}(n)) \tag{2}
\]

where \( \overline{f}, \overline{u}, \overline{p} \) correspond to measurement errors and \( \delta_f \) represents the plant–model mismatch. To distinguish between solutions resulting from (1) and (2), \( \overline{x}(\cdot) = \overline{x}_{up}(\cdot ; \overline{x}_0) \) denotes the true or distorted solution of (2) which is subject to a disturbed initial value \( \overline{x}_0 \), a possibly modified control sequence \( \overline{u} = (\overline{u}(k))_{k \in \mathbb{N}_0} \) and the disturbed parameter sequence \( \overline{p} = (\overline{p}(k))_{k \in \mathbb{N}_0} \). We call a point \( x_0 \in X \) a controlled equilibrium of (2) if there exist a control \( u_* \in U \) and a nominal parameter \( p_* \in P \) satisfying \( f(x_0, u_*, p_*) = x_0 \).

To shorten notation, we define the abbreviation \( \|x\| = \delta_f(x, y) \) for \( \delta_f \in \{\delta_x, \delta_u, \delta_p\} \). This allows us to formulate the following assumption which we suppose to hold throughout the paper:

**Assumption 1.** There exist a control forward invariant set \( F \subseteq X \) and constants \( \Delta_x, \Delta_u, \Delta_p > 0 \) such that for all \( x \in F \) we have \( \|\delta_x\| = \|\delta_u\| \leq \Delta_x \) and additionally the bounds \( \|\delta_p\| \leq \Delta_p \) and \( \|p(0) + 1\| \leq \Delta_p \) hold for all \( n \in \mathbb{N}_0 \).

For this setting, we wish to design a feedback controller based on the model (1) which despite the measurement and modelling errors semiglobally practically asymptotically stabilizes the plant (2) at a controlled equilibrium, i.e. which satisfies the following ISS property:

**Definition 2.** Suppose \( x_0 \in X \) is a controlled equilibrium of (2). Then a control sequence \( u = (u(n))_{n \in \mathbb{N}_0} \) semiglobally practically asymptotically stabilizes system (2) if there exist \( r > 0, \beta \in K_\infty \) and \( \gamma \in K_\infty \) such that the inequality

\[
\|x_{up}(n, p(n))\| \leq \beta(\|x_0\|, n) + \gamma(\|\delta_x, \delta_u, \delta_p\|)
\]

(3) holds for all \( x_0 \in B_r(x_0) \), all \( n \in \mathbb{N}_0 \) and all \( \delta_x, \delta_u, \delta_p \) satisfying Assumption 1.

To accomplish this task, we propose a two stage feedback. In the first stage, we apply model predictive control (MPC) to generate an approximation of a minimizer of the infinite horizon cost functional \( J_\infty(x, u, p) = \sum_{n=0}^\infty \ell(x(n), u(n), p(n)) \). As indicated by the notation, the basis of this computation is the model (1) together with a nominal parameter sequence \((p(n))_{n \in \mathbb{N}_0}\). This step is applied in an advanced step setting, cf. Findeisen and Allgöwer (2004); Zavala and Biegler (2009), resulting in a control that is to be implemented at some future time instant. The second stage utilizes the precomputed control to locally update it based on newly obtained measurements of \( x \) and \( p \). Examples of updating techniques include sensitivities which have been analysed in the nonlinear optimization context for quite some time, see, e.g., Grötschel et al. (2001), but have also been applied in the MPC context, cf. Zavala and Biegler (2009), as well as realtime iterations Diehl et al. (2005), and hierarchical MPC Bock et al. (2007).

Within the first stage, we assume the stage cost \( \ell : X \times U \times P \rightarrow \mathbb{R}_0^+ \) to be continuous and to satisfy \( \ell(x_0, u_0, p_0) = 0 \) and \( \ell(x, u, p) > 0 \) for all \( u \in U \) for each \( x \neq x_0 \) and each \( p \neq p_0 \). Due to the possible existence of constraints, we only consider the set of admissible controls within the minimization. To formally define this set, we introduce \( U^\downarrow \) and \( P^\downarrow \) denoting the set of all control and parameter sequences where \( I := \{0, 1, \ldots, N - 1\} \) and \( N \in \mathbb{N} \cup \{\infty\} \) specifies the length of these sequences. Then, the set of admissible controls for a fixed parameter sequence \( p \in P^\downarrow \) is given by

\[
U^\downarrow(x_0, p) := \{u \in U^\downarrow \mid f(x(n), u(n), p(n)) \in X \text{ and } u(n) \in U \text{ for all } n \in I\}.
\]

Since solving the infinite horizon optimal control problem is in most cases computationally intractable, the idea of MPC is to approximate such a solution via a series of finite horizon problems. To this end, the truncated cost functional

\[
J_N(x, u, p) := \sum_{k=0}^{N-1} \ell(x(k), u(k), p(k))
\]

with horizon length \( N \in \mathbb{N}_{\geq 2} \) is minimized revealing a finite optimal control sequence \( u^*(\cdot, x, p) \). Then, only the first element is implemented defining the feedback law \( \mu_N(x, p) := u^*(0, x, p) \) and the optimization horizon is shifted forward in time which allows the method to be iteratively applicable, see also Alamir (2006); Camacho and Bordons (2004); Grune and Pannek (2011); Rawlings and Mayne (2009) for further details. Incorporating the second stage of the proposed feedback design, the only modification of the MPC methodology is to compute an updated control \( U_N(x, \overline{r}) \) to replace \( \mu_N(x, p) \) upon implementation. Here, we assume such an update to be instantly computable.

As a result we obtain the following Algorithm 1 where we used the short notation \( p_n := (p(n + k))_{k \in \mathbb{Z}} \) abbreviating the parameter sequence used for computing or updating the open loop control at time instant \( n \).
Algorithm 1 MPC Algorithm

1. Obtain measurement of $\mathcal{X}(n)$ and $\mathcal{P}_n$.
2. Update control $u^*(x(n), p_n)$ to $\mathcal{X}(n), \mathcal{P}_n$, and apply control $\mathcal{P}_N(x(n), p_n) = \mathcal{X}(n), \mathcal{P}_n$.
3. Set $x(n) := \mathcal{X}(n)$ and $p_{n+1} := \mathcal{P}_n$, predict $x(n+1)$ using dynamic (1) together with $\mathcal{X}(n), \mathcal{P}_N(x(n), p_n)$ and $\mathcal{P}_N$.
4. Compute $u^*(x(n+1), p_{n+1})$, set $n := n+1$ and goto step (1).

Remark 3. Within Algorithm 1 a sequence of nominal future parameters $p$ is necessary to evaluate both the dynamic and the cost function. A respective sequence can be computed by extrapolation methods or measured via forward sensors such as road scanning laser sensors in a car or thermometers at intake pipes of a chemical plant.

Here, given $N \in \mathbb{N}$ we suppose that for each $x \in \mathcal{X}$ and $p \in \mathcal{P}^2$ there exists a minimizer $u^*(x, p) \in \mathcal{U}^2$ of (4). Note that this assumption guarantees the existence of a feasible solution for each $x \in \mathcal{X}$ and each $p \in \mathcal{P}^2$, see also Kerrigan and Maciejowski (2001); Primbs and Nevišć (2000) for relaxed conditions. Hence, we can define the optimal value function on a finite horizon via

$$V_N(x, p) := \min_{u \in \mathcal{U}^2(x,p)} J_N(x, u, p).$$

If additionally there exists a feasible updated control feedback $\mathcal{P}_N(\mathcal{X}, \mathcal{P})$ then we define the associated closed loop costs associated via

$$V_{\mathcal{P}_N}(x, p) := \sum_{n=0}^{\infty} \ell(\mathcal{X}(n), \mathcal{P}_N(\mathcal{X}(n), \mathcal{P}_n), \mathcal{P}(n)).$$

Instead of showing semiglobal practical asymptotic stability of system (2) via ISS Lyapunov functions, cf. Jiang and Wang (2001); Magni and Scattolini (2007), we utilize the concept of $\mathcal{P}$-practical asymptotic stability and “truncated” Lyapunov functions. As outlined in (Grüne and Pannek, 2011, Chapter 8.5), ISS of a system (2) can be shown via a suitable choice of $\gamma \in K_{X}$ in (3) if the following holds:

Definition 4. Let $A \subset \mathcal{X}$ be a forward invariant set with respect to all possible disturbances satisfying Assumption 1 and let $\mathcal{P} \subset A$. Then $x^* \in \mathcal{P}$ is $\mathcal{P}$-practically asymptotically stable on $A$ if there exists $\beta \in K_{L}$ such that

$$||\mathcal{X}(n)||_{x^*} \leq \beta(||\mathcal{X}(n)||_{x^*}, n)$$

holds for all $\mathcal{X}_0 \in A$ and all $n \in \mathbb{N}_0$ with $\mathcal{X}(n) \notin \mathcal{P}$.

If we limit the Lyapunov property to hold outside the practical region $\mathcal{P}$ and define $\mathcal{P}_N^\text{max} := \max_{n \in \mathbb{N}_0} ||\mathcal{X}(n)||_{\mathcal{P}(n)}$, then $\mathcal{P}$-practical asymptotic stability can be guaranteed, cf. (Grüne and Pannek, 2011, Theorem 2.20) for a corresponding proof.

Theorem 5. Suppose $A \subset \mathcal{X}$ is a forward invariant set with respect to disturbances satisfying Assumption 1, $\mathcal{P} \subset A$ and $x^* \in \mathcal{P}$. If there exist $\alpha_1, \alpha_2, \alpha_3 \in K_{\infty}$ and a Lyapunov function $V$ on $D \setminus \mathcal{P}$ satisfying

$$\alpha_1(||\mathcal{X}||_{x^*} + ||\mathcal{P}_N^\text{max}||_{p^*}) \leq V(\mathcal{X}(p), \mathcal{P}) \leq \alpha_2(||\mathcal{X}||_{x^*} + ||\mathcal{P}_N^\text{max}||_{p^*}),$$

then $x^*$ is $\mathcal{P}$-practically asymptotically stable on $A$.

3. STABILITY AND FEASIBILITY

Before approaching the stability problem of the proposed Algorithm 1, we first need to guarantee existence of a feasible updated control $\mathcal{P}_N(\mathcal{X}, \mathcal{P})$.

Theorem 6. Suppose Assumption 1 holds. If there exist a set $A \subset \mathcal{X}$ and a feedback $\mathcal{P}_N(x, p)$ such that

(i) $\inf_{x \in \mathcal{F}, y \in \mathcal{X}} \alpha_N(x, y) \geq \Delta_\nu$ holds and
(ii) for each $x \in \mathcal{F}$ we have $f(x, \mu_N(x, p), p) \in \mathcal{F}$,

then the closed loop solution resulting from Algorithm 1 is feasible for each initial value $x_0 \in \mathcal{F}$, i.e. $\mathcal{X}(n) \in \mathcal{X}$ for all $n \in \mathbb{N}_0$.

Proof: To prove the assertion by induction, first we consider estimates $x(0) = x_0 \in \mathcal{F}$ of the initial value and $p_0 \in \mathcal{P}$ of the parameter sequence to be given. Then, by Assumption 1 we have $||\mathcal{X}(0)||_{x(0)} \leq \Delta_\nu$, and by (i) it follows that $\mathcal{X}(0) \in A \subset \mathcal{X}$. In the induction step we consider estimates $x(n) \in \mathcal{F}$ and $p_n \in \mathcal{P}$. Now, we can use (ii) to obtain $x(n+1) = f(x(n), \mu_N(x(n), p_n), p_n) \in \mathcal{F}$. Hence, Assumption 1 reveals we have $||\mathcal{X}(n+1)||_{x(n+1)} \leq \Delta_\nu$ and again (i) allows us to conclude $\mathcal{X}(n) \in A \subset \mathcal{X}$.

Remark 7. We like to note that Assumption 1 is restrictive in terms of one step boundedness of the dynamics. While this condition may be weakened easily, it does not make sense from an updating technique point of view for the control which requires a certain closeness of nominal and disturbed solution.

Now that we have certified existence of a solution resulting from Algorithm 1 we show that the obtained feedback semiglobally practically asymptotically stabilizes system (2) in the sense of Definition 2. In contrast to approaches based on terminal constraints or Lyapunov type terminal costs, cf., e.g., Diehl et al. (2005); Bock et al. (2007); Zavala and Biegler (2009), we consider a relaxed Lyapunov condition to hold for the nominal case, cf. Grüne and Rantzer (2008); Lincoln and Rantzer (2006). This property is not artificial but can be shown to hold if $N$ is chosen sufficiently large, see Alami and Bornard (1995); Grimm et al. (2005); Jadamba and Hauser (2005); Pannek and Worthmann (2011).

One way to prove a similar result in the disturbed case of system (2) is to modify the stage cost $\ell$ to be positive definite with respect to a robustly stabilizable forward invariant neighbourhood of $x^*$. The computation of such a neighbourhood, however, may be impossible. Here, we consider the stage cost $\ell$ to be positive definite with respect to $x^*$ only, i.e. we ignore the effects of disturbances. Since the stage cost typically decrease towards the desired equilibrium $x^*$, convergence of the closed loop to a neighbourhood of $x^*$ may still be expected, that is $\mathcal{P}$-practical stability of the closed loop.

In order to show such a performance results, we assume the following: (2)

Assumption 8. For sets $\mathcal{F} \subset A \subset \mathcal{X}$ containing $x^*$ there exist constants $L_\ell, L_V, S_\ell$ and $S_V$ such that the bounds

$$\left| \ell(\mathcal{X}, \mathcal{P}_N(\mathcal{X}, \mathcal{P}), \mathcal{P}) - \ell(x, \mu_N(x, p), p) \right| \leq L_\ell (||\mathcal{X}||_{x^*} + ||\mathcal{P}_N^\text{max}||_{p^*}) + S_\ell$$

$$||V_N(\mathcal{X}, \mathcal{P}) - V_N(x, p)|| \leq V_N(\mathcal{X}, \mathcal{P}) - V_N(x, p)$$
\[
\ell(x, u, p) := \begin{cases} \max \{\ell(x, u, p) - \varepsilon, 0\} & x \in \mathcal{F} \setminus \mathcal{L} \\ 0 & x \in \mathcal{L} \end{cases}
\]

and \(\sigma := \inf\{V(f(x, \mu_N(x), p), p) | x \in \mathcal{F} \setminus \mathcal{L}, p \in P\}\) we have
\[
\alpha\bar{\ell}_\infty(\pi, p) \leq V(\pi, \bar{p}) - \sigma \leq V_\infty(\pi, p) - \sigma
\] for all \(\pi \in A\).

**Proof:** Choose an arbitrary initial value \(x_0 \in \mathcal{F}\) and let \(n_0 \in \mathbb{N}\) be minimal with \(x(n_0 + 1) \in \mathcal{L}\) where we set \(n_0 := \infty\) if this case does not occur.

Now we reformulate (8) to obtain
\[
\alpha\ell(x, \mu_N(x), p) \leq V_N(x, p) - V_N(f(x, \mu_N(x), p), p).\]

which allows us to incorporate the effects of disturbances and updates of the control using Assumption 8:
\[
\alpha\ell(\pi(n), \pi_N(\pi(n)), \bar{p}(n), p(n))
\]
\[
\leq \alpha\ell(x(n), \mu_N(x(n)), p(n), p(n))
\]
\[
+ \alpha L_\ell \left( \|\pi(n)\|_{x(n)} + \|\bar{p}(n)\|^{\max} \right) + \alpha S_\ell
\]
\[
\leq V_N(x(n), p(n)) - V_N(f(x(n), \mu_N(x(n)), p(n)), p(n))
\]
\[
+ \alpha L_\ell \left( \|\pi(n)\|_{x(n)} + \|\bar{p}(n)\|^{\max} \right) + \alpha S_\ell
\]
\[
= V_N(x(n), p(n)) - V_N(x(n + 1), p(n))
\]
\[
+ \alpha L_\ell \left( \|\pi(n)\|_{x(n)} + \|\bar{p}(n)\|^{\max} \right) + \alpha S_\ell
\]
\[
\leq V_N(x(n), p(n)) - V_N(x(n + 1), p(n + 1))
\]
\[
+ \alpha L_\ell \left( \|\pi(n)\|_{x(n)} + \|\bar{p}(n)\|^{\max} \right) + \alpha S_\ell
\]
\[
+ L_V \|\bar{p}(n + 1)\|^{\max} + S_V
\]

Now we can use the feasibility result from Theorem 6 to obtain
\[
\alpha\ell(\pi(n), \pi_N(\pi(n)), \bar{p}(n), p(n))
\]
\[
\leq V_N(\pi(n), \bar{p}_n) - V_N(\pi(n + 1), \bar{p}_{n + 1})
\]
\[
+ (\alpha L_\ell + L_V) \left( \|\pi(n)\|_{x(n)} + \|\bar{p}(n)\|^{\max} \right)
\]
\[
+ L_V \left( \|\pi(n + 1)\|_{x(n+1)} + \|\bar{p}_{n+1}\|^{\max} \right)
\]
\[
+ L_V \|\bar{p}_{n+1}\|^{\max} + \alpha S t + 2S_V.
\]

Hence, using boundness from Assumption 1 reveals
\[
\alpha\ell(\pi(n), \pi_N(\pi(n)), \bar{p}(n), p(n))
\]
\[
\leq V_N(\pi(n), \bar{p}_n) - V_N(\pi(n + 1), \bar{p}_{n + 1})
\]
\[
+ \alpha (L_\ell(\Delta x + \Delta p) + S_\ell) + L_V(2\Delta x + 3\Delta p) + 2S_V.
\]

Combining the last inequality with the condition on \(\varepsilon\) in (ii), we can use condition (iii) and (i) to obtain \(V_N(\pi, \bar{p}) \geq V_N(f(\pi, \pi_N(\pi, \bar{p}), \bar{p}) \forall x \in \mathcal{F} \setminus \mathcal{L} \land \bar{p} \in P)\) satisfying Assumption 1. Hence, again using the bound on \(\varepsilon\) and the definition of \(\bar{\ell}\) in (9) we have
\[
\alpha\bar{\ell}(\pi(n), \pi_N(\pi(n)), \bar{p}(n), p(n))
\]
\[
= \max \{\alpha\ell(\pi(n), \pi_N(\pi(n), \bar{p}(n))), p(n) - \alpha\varepsilon, 0\}
\]
\[
\leq V_N(\pi(n), \bar{p}_n) - V_N(\pi(n + 1), \bar{p}_{n + 1}).
\]

for \(n \geq n_0 + 1\) the invariance of \(\mathcal{L}\) gives us \(\pi(\pi) \in \mathcal{L}\) and hence \(\bar{\ell}(\pi(n), \pi_N(\pi(n), \bar{p}(n))) = 0\). Additionally, since \(\sigma\) is the minimal cost after entry in \(\mathcal{L}\), we have \(V_N(\pi(n), \bar{p}_n) \geq \sigma\) for all \(n \leq n_0 + 1\). Now the feasibility result from Theorem 6 allows us to sum the stage costs over \(n\) and obtain
\[
\alpha \sum_{n=0}^{K} \bar{\ell}(\pi(n), \pi_N(\pi(n), \bar{p}_n), p(n))
\]
\[
= \alpha \sum_{n=0}^{K_0} \bar{\ell}(\pi(n), \pi_N(\pi(n), \bar{p}_n), p(n))
\]
\[
\leq V_N(\pi(0), \bar{p}_0) - V_N(\pi(K_0 + 1), \bar{p}_{K_0 + 1})
\]
\[
\leq V_N(\pi(0), \bar{p}_0) - \sigma.
\]

where \(K_0 := \min\{K, n_0\}\). Using that \(K \in \mathbb{N}\) was arbitrary we can conclude that \(V_N(\pi(0), \bar{p}_0) - \sigma) / \alpha\) is an upper bound for \(\bar{\ell}_\infty(\pi(0), \bar{p})\). Last, using arbitrariness of the initial value \(x_0 \in \mathcal{F}\), assertion (10) follows. □

**Remark 10.** Within Theorem 9 \(\mathcal{L}\) is implicitly defined. For approximation techniques of this section we refer to Grimm et al. (2005); Grune and Rantzer (2008).

Last, we can use the previous performance result to show \(P\)-practical asymptotic stability of system (2).

**Theorem 11.** Suppose the conditions of Theorem 9 hold and additionally there exist \(\kappa\)-functions \(\alpha_1, \alpha_2\) such that
\[
\alpha_1(\|\pi\|_{x^*} + \|\bar{p}\|^{\max}) \leq V(\pi, p) \leq \alpha_2(\|\pi\|_{x^*} + \|\bar{p}\|^{\max})
\]
holds for all \(\pi \in A \setminus \mathcal{P}\) with \(\mathcal{P} = \mathcal{L}\). Then \(x^*\) is \(P\)-practically asymptotically stable on \(A\).

**Proof:** The definition of \(\bar{\ell}\) in (9) and the property \(V(\pi, \bar{p}) \geq V_N(f(\pi, \pi_N(\pi, \bar{p}), \bar{p}), \bar{p})\) shown in the proof of Theorem 9 guarantee the conditions of Theorem 5 to hold showing the assertion. □

**Remark 12.** (i) To cover the case of \(P\)-practically stable nominal systems (1) condition (8) can be relaxed to
\[
V_N(x, p) - V_N(f(x, \mu_N(x), p), p) \leq \min(\alpha\ell(x, \mu_N(x), p), p - \varepsilon)\ell(x, \mu_N(x), p) - \varepsilon
\]
as shown in Grune and Rantzer (2008). Then the performance and stability results from Theorems 9 and 11 hold if we consider the modified lower bound \(\varepsilon \geq L_\ell(\Delta x + \Delta p) + S_\ell\).
\[ S_t + (L_V(2\Delta_x + 3\Delta_p) + 2S_V + \tau)/\alpha. \]

(ii) Using identical arguments as in the proof of Theorem 11, \( m \)-step MPC control laws can be handled. To this end, condition (8) can be replaced by
\[
V_N(x, p) - V_N(x_{u,p}(m; x), x)
\geq \alpha \sum_{k=0}^{m-1} \ell(x_{u,p}(k; x), \mu_N(x_{u,p}(k; x), x), p)
\]
and both \( \bar{\ell}(x, u, p) \) and \( \sigma \) need to be redefined accordingly.

Here, we like to stress that the stability result of Theorem 11 holds for any updated feedback law \( \pi_X \) satisfying Assumption 8. In the next section, we verify these requirements for particular updating strategies known from the MPC literature.

4. UPDATE TECHNIQUES

Currently, there are two main lines of updating techniques used in MPC which are heavily influenced by nonlinear optimization theory: For one, as shown in Zavala and Biegler (2009), sensitivity information Fiacco (1983) can be applied in a similar manner as in offline optimal control, cf., e.g. Grötschel et al. (2001), to update the control law based on newly arrived information. Different from that, interim updates on optimality or feasibility can be employed to cope with measurable disturbances. The key idea here is that the solutions of two consecutive MPC iterations are close to each other. Hence, for the realtime iteration approach Diehl et al. (2005), one can show that typically one iteration of the underlying nonlinear optimization routine is sufficient to guarantee stability. Extending this idea, hierarchical methods Bock et al. (2007) may operate on different levels and different time scales of the problem to robustify the MPC method.

While the ideas of these updating methods are simple, corresponding proofs are quite involved and require a lot of notation used in nonlinear optimization theory. Since we make use of respective properties of these methods only, we condense the additional notation to a minimum and refer to the original publications Bock et al. (2007); Diehl et al. (2005); Fiacco (1983) for details.

4.1 Sensitivity Analysis

Given the advanced step setting of Algorithm 1, it is possible to solve a nominal optimal control problem ahead of time and to additionally compute sensitivity information \( \frac{\partial u^*}{\partial x} \) and \( \frac{\partial u^*}{\partial p} \). Hence, upon arrival of new measurements of \( \pi \) and \( p \), the optimal control can be updated via
\[
\pi(\cdot, \pi, p) := u^*(\cdot, x, p) + \left( \frac{\partial u^*}{\partial x}, (x, p) \right) (\pi - x) + \left( \frac{\partial u^*}{\partial p}, (x, p) \right) (p - p)
\]
which allows us to set \( \pi_X(\pi, p) = \pi(0, \pi, p) \). A proof under which conditions such an update reveals an optimal control is given in the following theorem from Fiacco (1983) where we adapted the notation to match the considered MPC case:

Theorem 13. Consider \( f \) and \( \ell \) to be twice continuously differentiable in a neighbourhood \( A \subset \mathbb{X} \) of the nominal solution \( u^*(\cdot, x, p) \) of Problem (5). If the linear independence constraint qualification (LICQ), the second order sufficient optimality conditions (SSOC) and the strict complementarity condition (SCC) hold in this neighbourhood, then
\begin{itemize}
  \item \( u^*(\cdot, x, p) \) is an isolated local minimizer and the respective Lagrange multipliers are unique,
  \item there exists a unique local minimizer \( u^*(\cdot, x, p) \) for \( (\pi, p) \) in a neighbourhood of \( (x, p) \) which satisfies LICQ, SSOC and SCC and is differentiable with respect to \( x \) and \( p \),
  \item there exist a Lipschitz constant \( L_V \) such that
  \[ |V_N(\pi, p) - V_N(x, p)| \leq L_V \left( \|\pi\|_x + \|p\|_p^\text{max} \right) \]
  \end{itemize}
holds and
\[ \pi(\cdot, \pi, p) \text{ from (12) the following estimate holds:} \]
\[ \|\pi(\cdot, \pi, p)\|_{u^*(\cdot, x, p)} \leq L_u \left( \|\pi\|_x + \|p\|_p^\text{max} \right) \]

Now, the following result allows us to conclude that Assumption 8 holds if the conditions of Assumption 1 and Theorem 13 apply.

Proposition 14. If there exist sets \( F \subset A \subset \mathbb{X} \) containing \( x^* \) such that \( F \) is control forward invariant and the conditions of Theorem 13 hold for all tuples \( (\pi, x, p, p) \) with \( \pi \in A, x \in F \) and \( p, p' \in P^2 \) satisfying Assumption 1, then Assumption 8 holds.

Proof: The conclusion follows directly from (13) and the fact that differentiability of \( f \) implies the existence of a local Lipschitz constant \( L_f \).

Hence, we can conclude the following:

Conclusion 15. Suppose Assumption 1 and the conditions of Theorem 6 and Proposition 14 to hold. If there exist \( K_\infty \) functions \( \alpha_1, \alpha_2 \) such that (11) is true for all \( \pi \in A \setminus \mathcal{P} \) with \( \mathcal{P} = \mathcal{L} \), then the control computed by Algorithm 1 \( \mathcal{L} \)–practically asymptotically stabilizes \( x^* \) on \( A \).

Proof: Assumption 1 and Proposition 14 guarantee the conditions of Theorem 11 to hold showing the assertion.

4.2 Realtime Iterations and Hierarchical MPC

In contrast to sensitivity based updates, realtime iterations can be performed without precomputing sensitivity matrices. Instead, only a single Newton step of the optimization routine is executed and the updated control is applied right away. It has been shown in (Diehl et al., 2005, Theorem 4.1), that such a procedure leads to a contraction of the disturbance around the nominal solution and the updated solution converges towards the stationary point of the disturbed problem. Considering the notation introduced in this paper, this results reads as follows:

Theorem 16. Consider \( f \) and \( \ell \) to be twice continuously differentiable and the approximation of the second order derivative of the Lagrangian to be continuous with bounded inverse in a neighbourhood \( A \subset \mathbb{X} \) of the nominal solution \( u^*(\cdot, x, p) \) of Problem (5). Additionally suppose that both the approximation error of the Lagrangian for a linear interpolation of the disturbance and the disturbance impact on the approximation of the Lagrangian are bounded. If the disturbance is sufficiently small and
the disturbed state trajectory $\tau_{u,p}(\tau; \tau)$ is feasible, then applying one Newton step we obtain

(i) feasibility of the updated state trajectory $\tau_{u,p}(\tau; \tau)$ and

(ii) existence of constants $c_1 < 1$ and $c_2 < \infty$ such that the following error bound is guaranteed:

$$
||\tau_{u,p}(\tau; \tau)||_{x_{u,p}(\tau; \tau)} \leq c_1 ||\tau_{u,p}(\tau; \tau)||_{x_{u,p}(\tau; \tau)}^{\max} + c_2 \left(||\tau_{u,p}(\tau; \tau)||_{x_{u,p}(\tau; \tau)}^{\max}\right)^2.
$$

Additionally, the Newton sequence converges to the exact stationary point of the disturbed problem.

Based on this result and Assumption 1, we can show that our fundamental Assumption 8 is satisfied:

**Proposition 17.** If there exist sets $F \subset A \subset X$ containing $x^*$ such that $F$ is control forward invariant and the conditions of Theorem 16 hold for all tuples $(\tau, x, \tau, p)$ with $\tau \in A$, $x \in F$ and $\tau, p \in P^T$ satisfying Assumption 1, then Assumption 8 holds.

**Proof:** Since $f$ is Lipschitz, boundedness of the state and parameter disturbances given by Assumption 1 allows us to conclude that there exists a constant $L_x$ such that the quadratic bound (14) can be overbounded linearly by $||\tau_{u,p}(\tau; \tau)||_{x_{u,p}(\tau; \tau)}^{\max} \leq L_x (||x||_x + ||p||_{p_{\max}})$. Moreover, since the Newton iterations are contracting and therefore bounded, we obtain that there exists a constant $L_u$ such that $||\tau_{u,p}(\tau; \tau)||_{x_{u,p}(\tau; \tau)}^{\max} \leq L_u (||x||_x + ||p||_{p_{\max}})$ holds. Now, due to continuity of $f$ and $\ell$ boundedness of the disturbed state trajectory $\tau_{u,p}(\tau; \tau)$ and the updated control $\tau(\tau, \tau, \tau, \tau, \tau)$ allows us to conclude that there exists a constant $L_\ell$ such that (13) holds. Similar to the proof of Proposition 14, differentiability of $\ell$ implies the existence of a local Lipschitz constant $L_\ell$ showing the assertion. □

Hence, $P$-practical asymptotic stability of the closed loop can be concluded similar to the sensitivity based scenario:

**Conclusion 18.** Consider Assumption 1 and the conditions of Theorem 6 and Proposition 17 to apply. If there exist $K_{\infty}$ functions $\alpha_1$, $\alpha_2$ such that (11) holds for all $\tau \in P$ with $P = L$, then the control computed by Algorithm 1 $P$-practically asymptotically stabilizes $x^*$ on $A$.

**Proof:** Similar to the proof of Conclusion 15. □

Here, we like to stress that the convergence result for hierarchically structured MPC updates show the properties (i) and (ii) of Theorem 16, cf. (Bock et al., 2007, Theorem 6). Hence, the results of Proposition 17 and Conclusion 18 also apply in the context of hierarchically MPC updates.

**Remark 19.** (i) Note that the feasibility and update sets $F$ and $A$ within Theorems 9, 13, 16 and Propositions 14, 17 are not necessarily large. This become clear by the requirement of identical open loop control structures in Theorem 13 which is typically only satisfied in a small neighbourhood. Hence, both the maximal disturbances $\Delta_x$, $\Delta_p$ and the Lipschitz constants $L_f$, $L_\ell$ are comparably small. Since the properties are required only locally, i.e. for one MPC step, the closed loop control structure may still vary. The closed loop feasibility and update sets and constants can then be obtained by the union of the local sets and maximum of local values respectively.

(ii) Within the proofs in the section the jump condition constants $S_f$ and $S_\ell$ in Assumption 8 can be set to zero. Consequently, the results from Section 3 apply to a wider range of updates than the ones we considered here.

5. **NUMERICAL EXAMPLE**

The previous analysis was motivated by considering a quarter car application with forward sensors for measuring road data and state sensors for the position of both wheel and chassis, cf. Figure 1.

![Figure 1. Test bench and schematic drawing of the quartercar](image)

The dynamics of the quarter car model are given by the set of second order ordinary differential equations

$$
m_w \ddot{x}_w(t) = c_w(x_c(t) - x_w(t)) + u(t)(\dot{x}_c(t) - \dot{x}_w(t)) - c_w(x_w(t) - p(t)) - d_w(\dot{x}_w(t) - \dot{p}(t))
$$

where $x_w(t)$ and $x_c(t)$ denote the state of the wheel and the chassis and $m_w$ and $m_c$ the respective wheel and chassis masses. The parameters $d_w$ and $c_w$ represent the spring and the damper coefficients modelling the wheel to ground interaction whereas $c_d$ denotes the spring constant of the chassis, see Table 1 for respective values. Moreover, the functions $u(t)$ and $p(t)$ represent the controllable damper constant of the chassis and the road profile. Note that within system (15), the states $x_w$ and $x_c$ do not correspond to the actual heights above ground of the center of masses of both wheel and body. Instead, the equilibrium of the system has been shifted to the origin. In the following, we use our standard notation and abbreviate the system state via $x(t) := [x_w, x_c]^T$.

Within the MPC setup we considered the criteria comfort and safety in accordance with ISO/IEC 2631 ISO (1997). In particular, the jerk of the chassis mass was employed in

$$
J_{\text{comfort}}(x(t), u(t), p(t)) = \int_0^T (\dot{x}_c)^2 dt,
$$

| Name               | Symbol | Quantity | Unit  |
|--------------------|--------|----------|-------|
| mass wheel         | $m_w$  | 35       | kg    |
| mass chassis       | $m_c$  | 325      | kg    |
| spring constant wheel | $c_w$ | 0.2 kN/m |       |
| damper constant wheel | $d_w$ | 150 kN/s/m |     |
| spring constant chassis | $c_d$ | 20 kN/m  |       |

Table 1. Parameters for the quarter car example from Retting and von Stryk (2005)
to measure the comfort of the driver. To measure safety, the functional

\[ J_{\text{safety}}(x(t), u(t), p(t)) = \int_0^T (c_w(x_w(t) - p(t))) \]

is used which represent the deviation from the equilibrium tire–surface force. In order to adapt to different driving situations and driver adjustments, these criteria are weighted by parameters \( \mu_{\text{comfort}} \) and \( \mu_{\text{safety}} \)

\[
J_N(x, u, p) = \sum_{k=0}^{N-1} \mu_{\text{comfort}} \int_{kT}^{(k+1)T} J_{\text{comfort}}(x(t), u(t), p(t))dt + \mu_{\text{safety}} \int_{kT}^{(k+1)T} J_{\text{safety}}(x(t), u(t), p(t))dt.
\]

For our computations, we considered the control value set \( U = [0.5, 3.0] \) and a sampling period \( T = T_u = 0.1 \text{s} \) during which the control is held constant. Measurements of the road data, however, are taken at a sampling period of \( T_p = 0.002s \) from an artificial test track, cf. the dashed line in Figure 2. Note that since we utilized a point model of the tire, we focused on large road excitations assuming that a more realistic tire model such as CDTire Baecker et al. (2011) reduces the effect of small excitations.

In order to solve the resulting optimal control problem, we set \( N = 5 \), \( \mu_{\text{comfort}} = 10 \), \( \mu_{\text{safety}} = 1 \) and applied a direct approach. To this end, we discretized the problem using the sampling period of the control \( T_u \) and set the optimization tolerance of the used SQP method to \( 10^{-6} \).

Within each sampling period, the dynamics and the cost functional are evaluated via the DoPri5 method with error tolerance \( 10^{-6} \). Last, the required twice continuously differentiable road profile \( p(t) \) is obtained by means of a Fast Fourier Transformation (FFT) for the nominal measurements over the optimization horizon. The FFT itself is based on the \( N \cdot T_u/T_p + 1 = 251 \) road measurements contained within the open loop optimization horizon and recomputed for each sampling point of the control.

Here, we like to mention that although the coefficients need to be calculated only once for each MPC step, the evaluation of the FFT interpolation polynomial remains the computationally most expensive part since it is required by the differential equation solver. To reduce this additional effort, the integration method can be restricted to operate on the sampling instances of the road measurements only. Alternatively, since a more realistic tire model compensates small excitations with high frequencies, a low pass filter for the FFT may be used to reduce the number of coefficients.

As one observes from Figure 2, the \( x_w \) and \( x_c \) trajectories follow the road excitation profile nicely. Additionally, the expected overshoot of the chassis fades out within a few sampling instances due to the computed control strategy shown in Figure 3.

To incorporate disturbances, we first introduced measurement errors of the road profile due to sensor scattering. Here, we considered the error to be uniformly distributed in the interval \([-0.005, 0.005]\) which corresponds to the typical accuracy of a laser scanner of \( 5\text{mm} \) in our simulated application. Regarding Assumption 1 we therefore obtain

\[
\|x\| \leq \Delta_x = 0.005 \text{ in Assumption 1, on the closed loop performance regarding different updating techniques. As we suppose that no more precise information is available to our system we utilized the scattered road data as input parameters and calculated the updates due to the state measurements. Since the sensitivity based approach is theoretically unsuitable even without these additional errors, we focused on the realtime iteration and the hierarchical MPC approach. As expected, the updated control laws show an improved performance in terms of the closed loop performance.}
\]
costs shown in Figure 4: For the chosen road profile the closed loop costs using realtime iterations with sampling time $T_s = 0.1$ decrease by approximately 10.5%. While this may appear to be a small improvement given the additional computing costs, we added full reoptimization results which show that at most a reduction of approximately 14.6% is to be expected for our chosen setting.

Figure 4. Closed loop costs for different updating techniques

Considering hierarchical MPC, however, allows us to increase the sampling rate which is still covered by our theoretical results. To be comparable, we performed one „D-level“ step each 0.1s and one „A-level“ step each 0.002s, i.e. $T_a = T_p = 0.002$s and we obtain 50 times as many updates as in the other approaches, cf. Bock et al. (2007) for details on the updating steps. Even without the intermediate „B-level“ and „C-level“ steps, we observed an improvement of approximately 35.4%.

Figure 5. Stage costs for different updating techniques

Taking a closer look at the occurring stage costs depicted in Figure 5, it becomes clear that the faster switching ability allows not only to reduce the costs during jumps of the road but also to recover faster from these jumps, i.e. both the overshoot and the decay rate of excitations are improved.

6. CONCLUSIONS AND OUTLOOK

We presented a feasibility and stability proof for nonlinear model predictive control with abstract updates using relaxed Lyapunov arguments. In particular, we have shown that the known updating techniques via sensitivity information, realtime iterations and hierarchical MPC are covered by our result.

Utilizing the jump constants within our assumptions, future research will focus on incorporating model reduction effects on the control into the stability analysis as well as extensions towards hybrid systems by covering integer jumps in variables.

ACKNOWLEDGEMENTS

This work was partially funded by the German Federal Ministry of Education and Research (BMBF), grant no. 05M10WNA.

REFERENCES

Alamir, M. (1995). Stability of a truncated infinite constrained receding horizon scheme: the general discrete nonlinear case. *Automatica*, 31(9), 1353–1356.

Baecker, M., Gallrein, A., Hack, M., and Toso, A. (2011). A Method to Combine a Tire Model with a Flexible Rim Model in a Hybrid MBS/FEM Simulation Setup. *SAE Technical Paper*, 1–186. doi:10.4271/2011-01-0186.

Bitmead, R., Gevers, M., and Wertz, V. (1990). *Adaptive optimal control: The thinking man’s GPC*. International Series in Systems and Control Engineering. Prentice-Hall, New York.

Bock, H., Diehl, M., Kostina, E., and Schlöder, J. (2007). Constrained optimal feedback control of systems governed by large differential algebraic equations. In L. Biegler, O. Ghattas, M. Heinkenschloss, D. Keyes, and B. Bloemen Waanders (eds.), *Real-Time PDE-Constrained Optimization*, 3–22. SIAM.

Camacho, E. and Bordons, C. (2004). *Model Predictive Control*. Springer.

Chen, H. and Allgöwer, F. (1998). A quasii-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34(10), 1205–1218.

Diehl, M., Bock, H., and Schlöder, J. (2005). A Real–Time Iteration Scheme for Nonlinear Optimization in Optimal Feedback Control. *SIAM Journal on Control and Optimization*, 43(5), 1714–1736.

Fiacco, A. (1983). *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*. Academic Press Inc.

Findeisen, R. and Allgöwer, F. (2004). Computational Delay in Nonlinear Model Predictive Control. In *Proceedings of the International Symposium on Advanced Control of Chemical Processes*.

Grimm, G., Messina, M.J., Tuna, S.E., and Teel, A.R. (2005). Model predictive control: For want of a local control Lyapunov function, all is not lost. *IEEE Transactions on Automatic Control*, 50(5), 546–558.
Grötschel, M., Krumke, S., and Rambau, J. (2001). *Online Optimization of Large Scale Systems*. Springer.

Grüne, L. (2009). Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems. *SIAM Journal on Control and Optimization*, 48, 1206–1228.

Grüne, L. and Pannek, J. (2009). Practical NMPC sub-optimality estimates along trajectories. *Sys. & Contr. Lett.*, 58(3), 161–168.

Grüne, L. and Pannek, J. (2011). *Nonlinear Model Predictive Control: Theory and Algorithms*. Springer.

Grüne, L. and Rantzer, A. (2008). On the Infinite Horizon Performance of Receding Horizon Controllers. *IEEE Transactions on Automatic Control*, 53(9), 2100–2111.

Jadbabaie, A. and Hauser, J. (2005). On the stability of receding horizon control with a general terminal cost. *IEEE Transactions on Automatic Control*, 50(5), 674–678.

Jiang, Z.P. and Wang, Y. (2001). Input-to-state Stability for Discrete-time Nonlinear Systems. *Automatica*, 37(6), 857 – 869.

Keerthi, S. and Gilbert, E. (1988). Optimal infinite horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving horizon approximations. *Journal of Optimization Theory and Applications*, 57, 265–293.

Kerrigan, E. and Maciejowski, J. (2001). Robust feasibility in model predictive control: necessary and sufficient conditions. In *Proceedings of the 40th IEEE Conference on Decision and Control, 2001*, 728–733. doi:10.1109/41.980192.

Lincoln, B. and Rantzer, A. (2006). Relaxing Dynamic Programming. *IEEE Transactions on Automatic Control*, 51(8), 1249–1260.

Magni, L. and Scattolini, R. (2007). Robustness and robust design of MPC for nonlinear discrete-time systems. In R. Findeisen, F. Allgöwer, and L.T. Biegler (eds.), *Assessment and future directions of nonlinear model predictive control*, volume 358 of Lecture Notes in Control and Information Sciences, 239–254. Springer, Berlin.

Mayne, D., Rawlings, J., Rao, C., and Scokaert, P. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36(6), 789–814.

Nevistić, V. and Primbs, J.A. (1997). Receding Horizon Quadratic Optimal Control: Performance Bounds for a Finite Horizon Strategy. In *Proceedings of the European Control Conference*.

Pannek, J. and Gerdts, M. (2012). Robust Stability and Performance Bounds for MPC with Abstract Updates. In *Proceedings of the 4th IFAC Nonlinear Model Predictive Control Conference*, 311–316.

Pannek, J. and Worthmann, K. (2011). Reducing the Prediction Horizon in NMPC: An Algorithm Based Approach. In *Proceedings of the 18th IFAC World Congress*, 7969–7974.

Primbs, J. and Nevistić, V. (2000). Feasibility and stability of constrained finite receding horizon control. *Automatica*, 36, 965–971.

Rawlings, J.B. and Mayne, D.Q. (2009). *Model Predictive Control: Theory and Design*. Nob Hill Publishing.

Rettig, U. and von Stryk, O. (2005). Optimal and Robust Damping Control for semi-active Vehicle Suspension. In *Proceedings of the 5th EUROMECH Nonlinear Dynamics Conference (ENOC)*, [URL](http://tubiblio.ulb.tu-darmstadt.de/24542/). Paper no. 20-316.

Shamma, J. and Xiong, D. (1997). Linear Nonquadratic Optimal Control. *IEEE Transactions on Automatic Control*, 42(6), 875–879.

Zavala, V.M. and Biegler, L.T. (2009). The advanced-step NMPC controller: Optimality, stability and robustness. *Automatica*, 45(1), 86–93.