Pressures for Asymptotically Sub-additive Potentials
Under a Mistake Function

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Abstract. This paper defines the pressure for asymptotically sub-additive potentials under a mistake function, including the measure-theoretical and the topological versions. Using the advanced techniques of ergodic theory and topological dynamics, we reveals a variational principle for the new defined topological pressure without any additional conditions on the potentials and the compact metric space.

Key words and phrases Variational principle; Topological pressure; Asymptotically sub-additive

1 Introduction

Throughout this paper, \((X, T)\) denotes a topological dynamical systems(TDS for short) in the sense that \(T : X \to X\) is a continuous transformation on the compact metric space.
space $X$ with metric $d$. The term $C(X)$ denotes the space of continuous functions from $X$ to $\mathbb{R}$. Invariant Borel probability measures are are associated with $(X,T)$. The terms $\mathcal{M}(X,T)$ and $\mathcal{E}(X,T)$ represent the space of $T$--invariant Borel probability measures and the set of $T$--invariant ergodic Borel probability measures, respectively.

In classical ergodic theory, measure-theoretic entropy and topological entropy are important determinants of complexity in dynamical systems. The important relationship between these two quantities is the well-known variational principle. Topological pressure is an important generalization of topological entropy. Ruelle first introduced the concept of topological pressure for additive potentials for expansive dynamical systems in [15], in which he formulated a variational principle for topological pressure. Later, Walters [19] generalized these results to continuous maps on compact metric spaces. For an arbitrary set, we emphasize that it need not be invariant or compact, as it generalizes the notion of topological pressure proposed by Pesin and Pitskel’ in [12], and these notions of lower and upper capacity topological pressures introduced by Pesin in [13]. The theories of topological pressure, variational principle and equilibrium states play a fundamental role in statistical mechanics, ergodic theory and dynamical systems, see [3, 16, 18].

Since Bowen [4], topological pressure has become a basic tool for studying dimension theory in conformal dynamical systems [14]. To study dimension theory in non-conformal cases, experts in dimension theory and dynamical systems introduced thermodynamic formalism for non-additive potentials [1, 2, 5, 6, 7, 11, 20]. Cao, Feng and Huang introduced the sub-additive topological pressure via separated sets in [5] on general compact metric spaces, and obtained the variational principle for sub-additive potentials without any additional assumptions on the sub-additive potentials or the TDS $(X,T)$.

This paper defines the pressure for asymptotically sub-additive potentials under a mistake function, including the measure-theoretical and the topological versions. This paper also obtains a variational principle for this newly defined topological pressure. As a physical process evolves, it is natural for the evolving process to change or produce some errors in the evaluation of orbits. However, a self-adaptable system should decrease errors over time. This is the motivation for this study to investigate the dynamical systems under a mistake function. The following paragraphs provide some notations and definitions.

For $x, y \in X$ and $n \in \mathbb{N}$, $d_n(x,y) := \max\{d(T^i(x), T^i(y)) : i = 0, 1, ..., n-1\}$ gives a new metric on $X$. The term $B_n(x, \epsilon) := \{y \in X : d_n(x,y) < \epsilon\}$ denotes a ball centered at $x$ with radius $\epsilon$ under the metric $d_n$. Let $Z \subseteq X$, $n \in \mathbb{N}$ and $\epsilon > 0$. A set $F \subseteq Z$ is an $(n, \epsilon)$--spanning set for $Z$ if for every $z \in Z$, there exists $x \in F$ with $d_n(x,z) \leq \epsilon$. A set $E \subseteq Z$ is an $(n, \epsilon)$--separated set for $Z$ if for every $x, y \in E$ implies $d_n(x, y) > \epsilon$. Given $\delta > 0$ and $\mu \in \mathcal{M}(X,T)$, a set $S$ is a $(n, \epsilon, \delta)$--spanning set if $\mu(\bigcup_{x \in S} B_n(x, \epsilon)) > 1 - \delta$.  

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A sequence $F = \{f_n\}_{n=1}^{\infty} \subseteq C(X)$ is an asymptotically sub-additive potentials (ASP for short) on $X$, if for each $k > 0$, there exists a sub-additive potentials $\Phi_k = \{\varphi^k_n\}_{n \geq 1}$, i.e. $\varphi^k_{n+m}(x) \leq \varphi^k_n(x) + \varphi^k_m(T^n x)$, $\forall x \in X, n, m \in \mathbb{N}$, such that

$$\limsup_{n \to \infty} \frac{1}{n}||f_n - \varphi^k_n|| \leq \frac{1}{k}$$

where $||f_n - \varphi^k_n|| := \max_{x \in X} |f_n(x) - \varphi^k_n(x)|$. This kind of potential appears naturally in the study of the dimension theory in dynamical systems, see [7, 22] for related examples. Along with Cao, Feng and Huang’s paper [6], Feng and Huang defined asymptotically sub-additive topological pressure in [7] as follows:

$$P(T, F, n, \epsilon) = \sup \{ \sum_{y \in E} f_n(y) : E \text{ is an } (n, \epsilon) \text{ - separated subset of } X \}$$

$$P(T, F) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P(T, F, n, \epsilon)$$

the term $P(T, F)$ is the asymptotically sub-additive topological pressure of $T$ with respect to (w.r.t.) $F$.

Let $F = \{f_n\}_{n=1}^{\infty}$ be an ASP. For a $T$-invariant Borel probability measure $\mu$, let $h_\mu(T)$ denote the measure-theoretic entropy, and denote

$$F_\ast(\mu) = \lim_{n \to \infty} \frac{1}{n} \int f_n \, d\mu.$$ 

When $\mu \in \mathcal{E}(X, T)$, the above limit exists $\mu$–almost everywhere without integrating against $\mu$. See the appendix in [7] for a proof of the above results. However, it is easy to show that $F_\ast(\mu) = \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \int \varphi^k_n \, d\mu$.

With a minor modification of the proof in [5], Feng and Huang obtained the relationships among $P(T, F)$, $h_\mu(T)$ and $F_\ast(\mu)$ in [7].

**Theorem 1.1.** Let $(X, T)$ be a TDS, and $F = \{f_n\}_{n \geq 1}$ an ASP. Then

$$P(T, F) = \begin{cases} 
- \infty, & \text{if } F_\ast(\mu) = -\infty \text{ for all } \mu \in \mathcal{M}(X, T), \\
\sup \{ h_\mu(T) + F_\ast(\mu) : \mu \in \mathcal{M}(X, T), F_\ast(\mu) \neq -\infty \}, & \text{otherwise.}
\end{cases}$$

**Remark 1.** For each $\mu \in \mathcal{M}(X, T)$, let $\mu = \int_{\mathcal{E}(X, T)} m \, d\tau(m)$ be its ergodic decomposition. Thus, $h_\mu(T) = \int_{\mathcal{E}(X, T)} h_\tau(T) \, d\tau(m)$ and $F_\ast(\mu) = \int_{\mathcal{E}(X, T)} F_\ast(\mu) \, d\tau(m)$, see [18] and [7] for details. It is then possible to prove that

$$\sup \{ h_\mu(T) + F_\ast(\mu) : \mu \in \mathcal{M}(X, T), F_\ast(\mu) \neq -\infty \}$$

$$= \sup \{ h_\mu(T) + F_\ast(\mu) : \mu \in \mathcal{E}(X, T), F_\ast(\mu) \neq -\infty \}.$$

Thus, we can replace $\mathcal{M}(X, T)$ with $\mathcal{E}(X, T)$ in the supremum of theorem 1.1.
The thermodynamic formalism for a single function and a sequence of functions arose from various considerations in physics and mathematics. This study extends thermodynamic formalism to asymptotically sub-additive potentials under a mistake function without any condition on the potentials and the dynamics.

The remainder of this paper is organized as follows. Section 2 defines the pressure for ASP under a mistake function, including the measure-theoretical and the topological versions. And we state our main result and give some preliminary results. Section 3 provides the proof of the results. The analysis in this study relies on the techniques of ergodic theory and topological dynamics.

2 Preliminaries

This section first defines pressure for ASP under a mistake function, and then presents the main results. The following section presents the proof.

First, recall the definitions of the mistake function and mistake dynamical balls presented by Thompson [17].

**Definition 2.1.** Given \( \epsilon_0 > 0 \) the function \( g : \mathbb{N} \times (0, \epsilon_0] \to \mathbb{N} \) is called a mistake function if for all \( \epsilon \in (0, \epsilon_0] \) and all \( n \in \mathbb{N} \), \( g(n, \epsilon) \leq g(n+1, \epsilon) \) and

\[
\lim_{n \to \infty} \frac{g(n, \epsilon)}{n} = 0.
\]

Given a mistake function \( g \), if \( \epsilon > \epsilon_0 \) set \( g(n, \epsilon) = g(n, \epsilon_0) \).

For any subset of integers \( \Lambda \subset [0, N] \) we will use the family of distances in the metric space \( X \) given by \( d_{\Lambda}(x, y) = \max\{d(f^i x, f^i y) : i \in \Lambda\} \) and consider the balls \( B_{\Lambda}(x, \epsilon) = \{y \in X : d_{\Lambda}(x, y) < \epsilon\} \).

**Definition 2.2.** Let \( g \) be a mistake function and let \( \epsilon > 0 \) and \( n \geq 1 \). The mistake dynamical ball \( B_n(g; x, \epsilon) \) of radius \( \epsilon \) and length \( n \) associated to \( g \) is defined as follows:

\[
B_n(g; x, \epsilon) = \{y \in X \mid y \in B_{\Lambda}(x, \epsilon) \text{ for some } \Lambda \in I(g; n, \epsilon)\}
\]

where \( I(g; n, \epsilon) = \{\Lambda \subset [0, n - 1] \cap \mathbb{N} \mid \#\Lambda \geq n - g(n, \epsilon)\} \) and \( \#\Lambda \) denotes the cardinality of the set \( \Lambda \). A set \( F \subset Z \) is \( (g; n, \epsilon) \)-separated for \( Z \) if for every \( x, y \in F \) implies \( d_{\Lambda}(x, y) > \epsilon, \forall \Lambda \in I(g; n, \epsilon) \). The dual definition is as follows. A set \( E \subset Z \) is \( (g; n, \epsilon) \)-spanning for \( Z \) if for all \( z \in Z \), there exists \( x \in E \) and \( \Lambda \in I(g; n, \epsilon) \) such that \( d_{\Lambda}(x, y) \leq \epsilon \). Given \( \delta > 0 \) and \( \mu \in \mathcal{M}(X, T) \), a set \( S \) is \( (g; n, \epsilon, \delta) \)-spanning set if \( \mu(\bigcup_{x \in S} B_n(g; x, \epsilon)) > 1 - \delta \).
Let $F = \{f_n\}_{n=1}^\infty$ be an ASP and let $g$ be a mistake function. For $\mu \in \mathcal{E}(X, T)$, the definition of asymptotically sub-additive measure-theoretic pressure is as follows:

$$P_\mu(g; T, F, n, \epsilon, \delta) = \inf \left\{ \sum_{x \in S} \exp \left[ \sup_{y \in B_n(g; x, \epsilon)} f_n(y) \right] | S \text{ is a } (g; n, \epsilon, \delta) - \text{spanning set} \right\}$$

$$P_\mu(g; T, F) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P_\mu(g; T, F, n, \epsilon, \delta)$$

The term $P_\mu(g; T, F)$ is an asymptotically sub-additive measure-theoretic pressure of $T$ w.r.t. $F$ under a mistake function $g$. The following theorem presents the main findings of this paper, which imply that the small errors cannot affect the important factors of dynamical systems. The following section presents the proof.

**Theorem A.** Let $(X, T)$ be a TDS, let $g$ be a mistake function, and let $F = \{f_n\}_{n=1}^\infty$ be an ASP. For each $\mu \in \mathcal{E}(X, T)$ with $F^\ast(\mu) \neq -\infty$, we have

$$P_\mu(g; T, F) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P_\mu(g; T, F, n, \epsilon, \delta) = h_\mu(T) + F^\ast(\mu).$$

The formula remains true if we replace the $\liminf$ by $\limsup$, and the value taken by the $\liminf$ (or $\limsup$) is independent of $\delta$ and the mistake function $g$.

This result generalizes Katok’s entropy formula [9], and the results in [8] and [21]. The main virtue of this approach is that we do not require any condition on the ASP and the TDS. The proof of the above theorem requires the following lemma.

**Lemma 2.1.** Let $(X, T)$ be a TDS, let $g$ be a mistake function, and let $F = \{f_n\}_{n=1}^\infty$ be an ASP. Given some $k > 0$, there exist sub-additive potentials $\Phi_k = \{\varphi_k^i\}_{i=0}^{n-1}$ such that for any positive integer $l$ and small number $\eta > 0$, there exists $\epsilon_0 > 0$ so that for any $0 < \epsilon < \epsilon_0$, the following inequalities hold for sufficiently large $n$

$$\sup_{y \in B_n(g; x, \epsilon)} f_n(y) \leq \sum_{i=0}^{n-1} \frac{1}{l} \varphi_i^k(T^i x) + C(g(n, \epsilon) + 1) + n\left(\frac{1}{k} + \eta\right)$$

where $C$ is a constant.

**Proof.** Given some $k > 0$, since $F = \{f_n\}_{n=1}^\infty$ is an ASP, there exist sub-additive potentials $\Phi_k = \{\varphi_k^i\}_{i=0}^{n-1}$, such that $\limsup_{n \to \infty} \frac{1}{n} \|f_n - \varphi_k^i\| \leq \frac{1}{k}$. This implies that

$$f_n(x) \leq \varphi_k^n(x) + \frac{n}{k}, \quad \forall x \in X$$

(2.1)

for sufficiently large $n$.

Let us fix any positive integer $l$. Since $\frac{1}{l} \varphi_i^k(x)$ is continuous, for each $\eta > 0$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, we have

$$d(x, y) < \epsilon \Rightarrow d\left(\frac{1}{l} \varphi_i^k(x), \frac{1}{l} \varphi_i^k(y)\right) < \eta.$$
Note that for each \( y \in B_n(g; x, \epsilon) \), there exists \( \Lambda \subset I(g; n, \epsilon) \) so that \( y \in B_\Lambda(x, \epsilon) \), therefore
\[
\sum_{i=0}^{n-1} \frac{1}{l} \varphi_i^k(T^i y) \leq \sum_{i \in \Lambda} \left( \frac{1}{l} \varphi_i^k(T^i x) + \eta \right) + \sum_{i \not\in \Lambda} \left( \frac{1}{l} \varphi_i^k(T^i x) \right) \\
\leq \sum_{i=0}^{n-1} \frac{1}{l} \varphi_i^k(T^i x) + \epsilon + C_1 g(n, \epsilon)
\]
(2.2)
where \( C_1 = 2(\|\frac{1}{l} \varphi_i^k\| + \epsilon) \).

For each sufficiently large \( n \), it is possible to rewrite \( n = sl + r \), where \( 0 \leq s, 0 \leq r < l \). Then, for any \( 0 \leq j < l \), we have
\[
\varphi_n^k(x) \leq \varphi_j^k(x) + \varphi_i^k(T^j x) + \cdots + \varphi_i^k(T^{(s-2)l} T^j x) + \varphi_i^k(T^{(s-1)l} T^j x)
\]
where \( \varphi_0^k(x) \equiv 0 \). Summing \( j \) from 0 to \( l - 1 \) leads to
\[
\frac{1}{l} \varphi_n^k(x) \leq 2IC_2 + \sum_{i=0}^{(s-1)l-1} \varphi_i^k(T^i x)
\]
where \( C_2 = \max_{j=1, \ldots, 2l} \max_{x \in X} |\varphi_j^k(x)| \). Hence,
\[
\varphi_n^k(x) \leq 2C_2 + \sum_{i=0}^{(s-1)l-1} \frac{1}{l} \varphi_i^k(T^i x) \leq 4C_2 + \sum_{i=0}^{n-1} \frac{1}{l} \varphi_i^k(T^i x).
\]
(2.3)
Let \( C = \max\{C_1, 4C_2\} \), we have that
\[
\sup_{y \in B_n(g; x, \epsilon)} f_n(y) \leq \sup_{y \in B_n(g; x, \epsilon)} \left( C + \sum_{i=0}^{n-1} \frac{1}{l} \varphi_i^k(T^i y) + \frac{n}{k} \right)
\]
\[
\leq \sum_{i=0}^{n-1} \left( \frac{1}{l} \varphi_i^k(T^i x) + \eta \right) + C(g(n, \epsilon) + 1) + \frac{n}{k}.
\]
where the first inequality follows from (2.1) and (2.3), and the second inequality follows from (2.2). This completes the proof of the lemma.

Let \( \mathcal{F} = \{f_n\}_{n=1}^\infty \) be an ASP. The following discussion defines the topological version of asymptotically sub-additive pressure under a mistake function. This study first gives an equivalent definition of asymptotically sub-additive topological pressure via spanning set, and then gives a new definition of asymptotically sub-additive topological pressure under a mistake function.

For each positive integer \( n \) and \( \epsilon > 0 \), put
\[
P^*(T, \mathcal{F}, n, \epsilon) = \inf \{ \sum_{x \in \mathcal{F}} \exp \left[ \sup_{y \in B_n(g; x, \epsilon)} f_n(y) \right] : F \text{ is an } (n, \epsilon) - \text{spanning subset of } X \}
\]
\[
P^*(T, \mathcal{F}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P^*(T, \mathcal{F}, n, \epsilon).
\]
The following lemma says that this newly defined quantity equals the asymptotically sub-additive topological pressure defined by separated sets.

**Proposition 2.1.** $P^*(T, \mathcal{F}) = P(T, \mathcal{F})$.

**Proof.** Let $F$ be an $(n, \epsilon/2)$—spanning subset of $X$, and let $E$ be an $(n, \epsilon)$—separated subset of $X$. Define a map $\phi : E \to F$ by choosing for each $x \in E$ some $\phi(x) \in F$ such that $d_n(x, \phi(x)) \leq \epsilon/2$. Then, it is easy to see that $\phi$ is injective. Therefore,

$$P^*(T, \mathcal{F}, n, \epsilon/2) \geq \sup \{ \sum_{y \in E} e^{f_n(y)} : E \text{ is an } (n, \epsilon) - \text{ separated subset of } X \}.$$

This immediately yields $P^*(T, \mathcal{F}) \geq P(T, \mathcal{F})$.

Next, we prove that $P^*(T, \mathcal{F}) \leq P(T, \mathcal{F})$. Given $n \in \mathbb{N}$ and $\epsilon > 0$. Choose $x_1 \in X$ with $f_n(x_1) = \sup_{x \in X} f_n(x)$, and then choose $x_2 \in X \setminus B_n(x_1, \epsilon)$ with $f_n(x_2) = \sup_{x \in X \setminus B_n(x_1, \epsilon)} f_n(x)$. We continue this process. More precise, in step $m$ we choose $x_m \in X \setminus \bigcup_{j=1}^{m-1} B_n(x_j, \epsilon)$ with $f_n(x_m) = \sup_{x \in X \setminus \bigcup_{j=1}^{m-1} B_n(x_j, \epsilon)} f_n(x)$. This process stops at some step $l$, and produces a maximal $(n, \epsilon)$—separated set $E = \{x_1, x_2, \cdots, x_l\}$ (meaning that $E$ is also an $(n, \epsilon)$—spanning set of $X$). Therefore,

$$P^*(T, \mathcal{F}, n, \epsilon) \leq \sum_{x \in E} \exp \left( \sup_{y \in B_n(x, \epsilon)} f_n(y) \right) = \sum_{x \in E} e^{f_n(x)} \leq \sup \{ \sum_{y \in E} f_n(y) : E \text{ is an } (n, \epsilon) - \text{ separated subset of } X \}$$

This immediately implies that $P^*(T, \mathcal{F}) \leq P(T, \mathcal{F})$, and completes the proof. \hfill \Box

Next, this study modifies the definition of $P(T, \mathcal{F})$ to define asymptotically sub-additive topological pressure under a mistake function.

Let $\mathcal{F} = \{ f_n \}_{n=1}^\infty$ be an ASP and let $g$ be a mistake function. For each $n \in \mathbb{N}$ and $\epsilon > 0$, put

$$P(g; T, \mathcal{F}, n, \epsilon) = \sup \{ \sum_{x \in F} e^{f_n(x)} : F \text{ is an } (g; n, \epsilon) - \text{ separated subset of } X \}$$

and

$$P(g; T, \mathcal{F}) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log P(g; T, \mathcal{F}, n, \epsilon).$$

The term $P(g; T, \mathcal{F})$ is the asymptotically sub-additive topological pressure of $T$ w.r.t. $\mathcal{F}$ under a mistake function $g$. The asymptotically sub-additive topological pressure under mistake function $P(g; T, \mathcal{F})$ equals $P(T, \mathcal{F})$, which means that the dynamical system is self adaptable if the amount of errors decrease as time goes by.

**Theorem B.** Let $(X, T)$ be a TDS, let $g$ be a mistake function, and let $\mathcal{F} = \{ f_n \}_{n=1}^\infty$ be an ASP. Then $P(g; T, \mathcal{F}) = P(T, \mathcal{F})$. 

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Theorem B and theorem 2.1 immediately imply the following corollary, i.e., the variational principle for the asymptotically sub-additive topological pressure under a mistake function.

Corollary 1. Let \((X, T)\) be a TDS, let \(g\) be a mistake function, and let \(\mathcal{F} = \{f_n\}_{n=1}^\infty\) be an ASP. Then

\[
P(g; T, \mathcal{F}) = \begin{cases} 
-\infty, & \text{if } \mathcal{F}_*(\mu) = -\infty \text{ for all } \mu \in \mathcal{M}(X, T), \\
\sup\{h_\mu(T) + \mathcal{F}_*(\mu) : \mu \in \mathcal{M}(X, T), \mathcal{F}_*(\mu) \neq -\infty\}, & \text{otherwise.}
\end{cases}
\]

To prove theorem B, we need an analogue of proposition 2.1. Thus, we define

\[
P^*(g; T, \mathcal{F}, n, \epsilon) = \inf\{\sum_{x \in \mathcal{F}} \exp[\sup_{y \in B_n(g; x, \epsilon)} f_n(y)] : F \text{ is a } (g; n, \epsilon) - \text{spanning subset of } X\}
\]

\[
P^*(g; T, \mathcal{F}) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log P(g; T, \mathcal{F}, n, \epsilon).
\]

And the following lemma holds.

Proposition 2.2. \(P(2g; T, \mathcal{F}) \leq P^*(g; T, \mathcal{F}) \leq P(g; T, \mathcal{F})\).

Proof. Let \(F\) be an \((g; n, \epsilon/2)\)--spanning subset of \(X\), and let \(E\) be an \((2g; n, \epsilon)\)--separated subset of \(X\). Define a map \(\phi : E \to F\) by choosing for each \(x \in E\) some \(\phi(x) \in F\) and some \(\Lambda_x \in I(g; n, \epsilon/2)\) such that \(d_{\Lambda_x}(x, \phi(x)) \leq \epsilon/2\). Suppose that \(x, y \in E\) with \(x \neq y\), let \(\Lambda = \Lambda_x \cap \Lambda_y\). Since \(\Lambda \in I(2g; n, \epsilon/2)\), \(d_\Lambda(\phi(x), \phi(y)) > 0\) and thus \(\phi(x) \neq \phi(y)\). Hence, \(\phi\) is injective. Therefore,

\[
P^*(g; T, \mathcal{F}, n, \epsilon/2) \geq \sup\{\sum_{y \in E} e^{f_n(y)} : E \text{ is an } (2g; n, \epsilon) - \text{separated subset of } X\}.
\]

This immediately shows that \(P^*(g; T, \mathcal{F}) \geq P(2g; T, \mathcal{F})\).

Next, we prove that \(P^*(g; T, \mathcal{F}) \leq P(g; T, \mathcal{F})\). Given \(n \in \mathbb{N}\) and \(\epsilon > 0\), choose \(x_1 \in X\) with \(f_n(x_1) = \sup_{x \in X} f_n(x)\), and then choose \(x_2 \in X \setminus B_n(g; x_1, \epsilon)\) with \(f_n(x_2) = \sup_{x \in X \setminus B_n(g; x_1, \epsilon)} f_n(x)\). We continue this process. More precise, in step \(m\) choose \(x_m \in X \setminus \bigcup_{j=1}^{m-1} B_n(g; x_j, \epsilon)\) with \(f_n(x_m) = \sup_{x \in X \setminus \bigcup_{j=1}^{m-1} B_n(g; x_j, \epsilon)} f_n(x)\). This process stops at some step \(l\), producing a maximal \((g; n, \epsilon)\)--separated set \(E = \{x_1, x_2, \ldots, x_l\}\) (meaning that \(E\) is also an \((g; n, \epsilon)\)--spanning set of \(X\), see lemma 3.3 in [17] for a proof). Therefore,

\[
P^*(g; T, \mathcal{F}, n, \epsilon) \leq \sum_{x \in E} \exp[\sup_{y \in B_n(g; x, \epsilon)} f_n(y)] = \sum_{x \in E} e^{f_n(x)} \leq \sup\{\sum_{y \in E} f_n(y) : E \text{ is an } (g; n, \epsilon) - \text{separated subset of } X\}
\]

This immediately implies that \(P^*(g; T, \mathcal{F}) \leq P(g; T, \mathcal{F})\), and completes the proof of the lemma.
3 Proof of main results

This section proves theorems A and B presented in the former section.

3.1 Proof of Theorem A

This subsection gives the proof of theorem A by following the arguments in [14] and [17], but the proof here is more complicated. This means that the asymptotically sub-additive measure-theoretic pressure is stable under a mistake function.

Proof. Assume that \( \mu \in \mathcal{E}(X, T) \) with \( \mathcal{F}_s(\mu) \neq -\infty \). Note that \( B_n(x, \epsilon) \subset B_n(g; x, \epsilon) \) implies that an \((n, \epsilon, \delta)\)-spanning set must be a \((g; n, \epsilon, \delta)\)-spanning set, and this leads to the following inequality

\[
P_\mu(g; T, \mathcal{F}, n, \epsilon, \delta) \leq \inf \left\{ \sum_{x \in S} \exp \left[ \sup_{y \in B_n(g; x, \epsilon)} f_n(y) \right] \mid S \text{ is a } (n, \epsilon, \delta) \text{-spanning set} \right\}
\]

\[
\leq e^{C(g(n, \epsilon)+1)+n(\frac{1}{k}+\eta)} \inf \left\{ \sum_{x \in S} \exp \sum_{i=0}^{n-1} \frac{1}{l} \varphi^k_i(T^i x) \mid S \text{ is a } (n, \epsilon, \delta) \text{-spanning set} \right\}
\]

where the second inequality follows from lemma 2.1. The terms \( l, C, \eta, k \) and \( \Phi_k = \{ \varphi^k_n \}_{n \geq 1} \) are all the same as lemma 2.1. Previous authors [8] proved that

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \inf \left\{ \sum_{x \in S} \exp \sum_{i=0}^{n-1} \frac{1}{l} \varphi^k_i(T^i x) \mid S \text{ is a } (n, \epsilon, \delta) \text{-spanning set} \right\} = h_\mu(T) + \int \frac{1}{l} \varphi^k_i(x) d\mu.
\]

Therefore, based on the fact that \( g \) is a mistake function,

\[
P_\mu(g; T, \mathcal{F}) \leq h_\mu(T) + \int \frac{1}{l} \varphi^k_i(x) d\mu + \frac{1}{k} + \eta.
\]

Let \( l \to \infty \) and \( k \to \infty \), and the arbitrariness of \( \eta \) implies that \( P_\mu(g; T, \mathcal{F}) \leq h_\mu(T) + \mathcal{F}_s(\mu) \).

Now, we turn to prove the reverse inequality that \( P_\mu(g; T, \mathcal{F}) \geq h_\mu(T) + \mathcal{F}_s(\mu) \). This method is similar to the proof of theorem A2.1 in [14]. For each \( \eta > 0 \), there exists \( 0 < \gamma \leq \eta \), a finite partition \( \xi = \{C_1, C_2, \cdots, C_m\} \) and a finite open cover \( \mathcal{U} = \{U_1, U_2, \cdots, U_k\} \) of \( X \), where \( k \geq m \). The following properties holds(assuming the regularity of the measure \( \mu \)):

1) \( |U_i| \leq \eta \) and \( |C_j| \leq \eta, 1 \leq i \leq k, 1 \leq j \leq m \), here \( | \cdot | \) denote the diameter of set;
2) \( \overline{U_i} \subset C_i, 1 \leq i \leq m \), where \( \overline{A} \) denotes the closure of the set \( A \);
3) \( \mu(C_i \setminus U_i) \leq \gamma, 1 \leq i \leq m \) and \( \mu(\bigcup_{i=m+1}^{k} U_i) \leq \gamma \);
4) \( 2\gamma \log m \leq \eta \).
Next, fix $\eta$ so $1 - \delta > \eta > 0$ and take the corresponding $\gamma$, partition $\xi$ and covering $\mathcal{U}$. Fix $Z \subset X$ with $\mu(Z) > 1 - \delta$ and put $t_n(x) := \#\{0 \leq l < n : T^lx \in \bigcup_{i=m+1}^k U_i\}$. Let $\xi_n = \bigcup_{i=0}^{n-1} T^{-i}\xi$ and $\xi_n(x)$ denote the element of $\xi_n$ contains $x$.

We claim that: there exists $A \subset Z$ and $N > 0$ with $\mu(A) \geq \mu(Z) - \gamma$ such that for every $x \in A$ and $n \geq N$, we have (i) $t_n(x) \leq 2\eta n$; (ii) $\mu(\xi_n(x)) \leq \exp(-(h_\mu(T, \xi) - \gamma)n)$; (iii) $\mathcal{F}_s(\mu) - \gamma \leq \frac{1}{n}f_n(x) \leq \mathcal{F}_s(\mu) + \gamma$.

Proof of the claim: Let $g = \chi_{\bigcup_{i=m+1}^k U_i}$, then $t_n(x) = \sum_{j=0}^{n-1} g(T^jx)$. According to the Birkhoff ergodic theorem and Egorov theorem, we can find a set $A_1 \subset Z$ with $\mu(A_1) \geq \mu(Z) - \frac{2}{3}$ such that

$$\lim_{n \to \infty} \frac{1}{n}t_n(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^jx) = \int g d\mu = \mu\left( \bigcup_{i=m+1}^k U_i \right) \leq \gamma$$

holds uniformly on $A_1$. Therefore, we can choose $N_1$ such that if $n \geq N_1$ and $x \in A_1$, then $t_n(x) \leq 2\eta n$. Using the Shannon-McMillan-Brieman theorem and Egorov theorem, it is possible to find a set $A_2 \subset Z$ with $\mu(A_2) \geq \mu(Z) - \frac{2}{3}$. By the same argument, it is possible to choose $N_2$ such that if $n \geq N_2$ and $x \in A_2$, then $\mu(\xi_n(x)) \leq \exp(-(h_\mu(T, \xi) - \gamma)n)$. Then, using Egorov theorem and the fact that

$$\lim_{n \to \infty} \frac{1}{n}f_n(x) = \mathcal{F}_s(\mu)(\neq -\infty), \quad \mu - a.e. \ x \in X.$$

we can find a set $A_3 \subset Z$ with $\mu(A_3) \geq \mu(Z) - \frac{2}{3}$. By the same argument, it is possible to choose $N_3$ such that if $n \geq N_3$ and $x \in A_3$, then $\mathcal{F}_s(\mu) - \gamma \leq \frac{1}{n}f_n(x) \leq \mathcal{F}_s(\mu) + \gamma$. Set $A = A_1 \cap A_2 \cap A_3$ and $n = \max\{N_1, N_2, N_3\}$ to prove the claim.

Set $\xi_n^* := \{\xi_n(x) \in \xi_n | \xi_n(x) \cap A \neq \emptyset\}$. Using (ii) of the claim shows that

$$\#\xi_n^* \geq \sum_{\xi_n(x) \in \xi_n^*} \mu(\xi_n(x)) \exp[(h_\mu(T, \xi) - \gamma)n] \geq \mu(A) \exp[(h_\mu(T, \xi) - \gamma)n], \ \forall n \geq N (3.4)$$

Let $2\epsilon$ be the Lebesgue number of the open cover $\mathcal{U}$ and let $S$ be a $(g; n, \epsilon)$-spanning set for $Z$. Picking a suitable $\Lambda \in I(g; n, \epsilon)$ leads to $Z \subset \bigcup_{x \in S} \overline{B}_{\Lambda}(x, \epsilon)$. Let $S' \subset S$ such that $\overline{B}_{\Lambda}(x, \epsilon) \cap A \neq \emptyset$ for each $x \in S'$. Fix $x \in S'$ and $B = \overline{B}_{\Lambda}(x, \epsilon)$, let $\xi_{\Lambda} := \bigvee_{j \in \Lambda} T^{-j}\xi$, $p(B, \xi_{\Lambda}) := \#\{C \in \xi_{\Lambda} \mid C \cap A \neq \emptyset\}$ and $p(B, \xi_n) := \#\{C \in \xi_n \mid C \cap A \neq \emptyset\}$.

We now estimate the number $p(B, \xi_{\Lambda})$. Note that $\overline{B}(T^jx, \epsilon) \subset U_{i_j}$ for some $U_{i_j} \in \mathcal{U}$, since $2\epsilon$ is the Lebesgue number of the open cover $\mathcal{U}$. If $i_j \in \{1, 2, \ldots, m\}$ then $T^{-1}U_{i_j} \subset T^{-1}C_{i_j}$. If $i_j \in \{m + 1, \ldots, k\}$, then there are at most $m$ sets of the form $T^{-l}C_{i_j}$ may have non-empty intersection with $T^{-l}U_{i_j}$. Using (i) of the claim shows that

$$p(B, \xi_{\Lambda}) \leq m^{2^{m}} = \exp(2\gamma n \log m).$$

Therefore,

$$p(B, \xi_n) \leq p(B, \xi_{\Lambda}) m^{g(n, \epsilon)} \leq \exp[(2\gamma n + g(n, \epsilon)) \log m].$$

Therefore,
It follows that
\[
\#\xi^*_n \leq \sum_{x \in S'} p(B_{\Lambda^*_n}(x, \epsilon), \xi_n) \leq \#S' \exp[[2\gamma n + g(n, \epsilon)] \log m].
\]

Therefore,
\[
\sum_{x \in S} \exp[\sup_{y \in B_n(y: x, \epsilon)} f_n(y)] \geq \sum_{x \in S'} \exp[\sup_{y \in B_n(y: x, \epsilon)} f_n(y)] \geq \#S' \exp[n(F_*(\mu) - \gamma)]
\]
where the second inequality follows from the fact that \(B_n(g; x, \epsilon) \cap A \neq \emptyset\) for each \(x \in S'\) and (iii) of the claim, and the third inequality follows from (3.4) and (3.5). This leads to
\[
\frac{1}{n} \log P_*(g; T, F, n, \epsilon, \delta) \geq \frac{1}{n} \log \mu(A) + h_\mu(T, \xi) + F_*(\mu) - 2\gamma - \frac{(g(n, \epsilon) + 2\gamma \log m)}{n}
\]
Since \(\gamma < \eta\), \(2\gamma \log m < \eta\), \(\frac{g(n, \epsilon)}{n} \to 0\) as \(n \to \infty\), \(|\xi| := \max_{1 \leq i \leq m} |C_i| < \eta\), and \(\eta\) is arbitrary,
\[
P_\mu(g; T, F) \geq h_\mu(T) + F_*(\mu).
\]
This completes the proof of the theorem.

### 3.2 Proof of Theorem B

This subsection combines the results in theorem A and proposition 2.2 to give the proof of theorem B. This proof says that the asymptotically sub-additive topological pressure is stable under a mistake function.

**Proof.** If \(E\) is a \((g; n, \epsilon)\)–separated set, then \(E\) must be an \((n, \epsilon)\)–separated set. Therefore,
\[
P(g; T, F, n, \epsilon) \leq \sup \left\{ \sum_{y \in E} f_n(y) : E \text{ is an } (n, \epsilon) - \text{separated subset of } X \right\}
\]
Hence, \(P(g; T, F) \leq P(T, F)\).

Now it is enough to prove that \(P(g; T, F) \geq \sup \{h_\mu(T) + F_*(\mu) : \mu \in \mathcal{E}(X, T), F_*(\mu) \neq -\infty\}\) by remark [1]. To illustrate this statement, for each \(\mu \in \mathcal{E}(X, T)\) with \(F_*(\mu) \neq -\infty\), a \((g; n, \epsilon)\)–spanning set must a \((g; n, \epsilon, \delta)\)–spanning set. Therefore,
\[
P^*(g; T, F, n, \epsilon) \geq P_\mu(g; T, F, n, \epsilon, \delta).
\]
According to theorem A and proposition 2.2,
\[
P(g; T, F) \geq P^*(g; T, F) \geq h_\mu(T) + F_*(\mu), \forall \mu \in \mathcal{E}(X, T) \text{ with } F_*(\mu) \neq -\infty.
\]
Combining the above arguments, theorem B immediately follows. \qed
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