Symbolic Control for Stochastic Systems via Parity Games

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We consider the problem of computing the maximal probability of satisfying an $\omega$-regular specification for stochastic, continuous-state, nonlinear systems evolving in discrete time. The problem reduces, after automata-theoretic constructions, to finding the maximal probability of satisfying a parity condition on a (possibly hybrid) state space. While characterizing the exact satisfaction probability is open, we show that a lower bound on this probability can be obtained by (I) computing an under-approximation of the qualitative winning region, i.e., states from which the parity condition can be enforced almost surely, and (II) computing the maximal probability of reaching this qualitative winning region.

The key insight in our work is that the stochastic nature of the systems evolving in discrete time form a general model for temporal decision making under stochastic uncertainty. In recent years, the problem of finding or approximating optimal policies in CMPs for specifications given in temporal logics or automata has received a lot of attention. While there is a steady progression towards more powerful techniques for reachability policy synthesis in stochastic nonlinear systems, we get an abstraction-based symbolic algorithm for finding a lower bound on the maximal satisfaction probability.

We have implemented our approach and evaluated it on the nonlinear model of the perturbed Dubins vehicle. We show empirically that the lower bound on the winning region computed by our approach is precise, by comparing against an over-approximation of the qualitative winning region.

1 INTRODUCTION

Controlled Markov processes (CMPs) over continuous state spaces and evolving in discrete time form a general model for temporal decision making under stochastic uncertainty. In recent years, the problem of finding or approximating optimal policies in CMPs for specifications given in temporal logics or automata has received a lot of attention. While there is a steady progression towards more expressive models and properties [11, 12, 15, 26, 32, 37], a satisfactory solution that can handle nonlinear models for general $\omega$-regular specifications in a symbolic way is still open. In this paper, we make progress toward a solution to this general problem.

For finite-state Markov decision processes (MDP), one can find optimal policies for $\omega$-regular specifications by decomposing the problem into two parts [2, 3, 8, 9]. (I) Using graph-theoretic techniques that ignore the actual transition probabilities, one can find the set of states that ensures the satisfaction of the specification almost surely. Further, for any state in this almost sure winning region, an optimal policy for almost sure satisfaction of the specification can be derived. (II) One finds an optimal policy to reach the almost sure winning region using linear programming or traditional dynamic programming approaches. Combining both policies returns an optimal policy for the overall synthesis problem.

Unfortunately, this two-step solution approach does not carry over to optimal policy synthesis for all $\omega$-regular specifications given a continuous-state CMP. First, we do not have characterizations of optimal policies for almost sure satisfaction in this case—such as whether randomization or memory can be necessary. Second, in contrast to finite-state MDPs, it is possible that the almost sure winning region of an CMP is empty, even if there is a policy that satisfies the specification with positive probability [26].

However, as we show in this paper, the same decomposition can be used to compute an under-approximation for the optimal policy instead: that is, the resulting policy gives a lower bound on the probability of satisfying a given $\omega$-regular specification from every state. While existing techniques [15, 30, 37] can be used in step (II) to compute the reachability probability with any given precision, we provide a new technique to under-approximate the set of states of a CMP that almost surely satisfies a parity specification in step (I) of the decomposition. A parity specification is a canonical representation for all $\omega$-regular properties [13, 34]; thus, our approach provides a way to under-approximate any $\omega$-regular specification.

The main contribution of our paper is to show that the approximate solution to step (I) can be computed by a symbolic algorithm over a finite state abstraction of the underlying CMP that is using only the support of the probabilistic evolution of the system. This abstraction-based policy synthesis technique is inspired by abstraction-based controller design (ABCD) for non-stochastic systems [27, 28, 33]. In ABCD, a nonlinear dynamical system is abstracted into a discrete two-player game over a finite discrete state space obtained by partitioning the continuous state space into a finite set of cells. The resulting abstract two-player game is then used to synthesize a discrete controller which is then refined into a continuous controller for the original system.

In ABCD, the abstract two-player game models the interplay between the controller (Player 0) and the dynamics (Player 1) s.t. the resulting abstract controller (i.e., the winning strategy of Player 0) can be correctly refined to the original system. This requires a very powerful Player 1; in every instance of the play (corresponding to the system being in one particular abstract cell 5) and for any input $u$ chosen by Player 0, Player 1 can adversarially choose both (a) the actual continuous state $s$ within 5 to which $u$ is applied and (b) any continuous disturbance affecting the system at this state $s$.

The key insight in our work is that the stochastic nature of the underlying CMP does not require a fully adversarial treatment of continuous disturbances by Player 1 in the abstract game to allow

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for controller refinement. Intuitively, disturbances need to be handled in a fair way - in the long run, all transitions with positive probability will eventually occur. We show that the resulting fairness assumption on the behavior of Player 1 can be modeled by an additional random player (also called 1/2 player) resulting in a so-called 2/3-player game [4–6] as the abstraction.

This provides a conceptually very appealing result of our paper. Using 2/3-player games as abstractions of CMPs allows to utilize the machinery of symbolic game solving, analogous to ABCD techniques for non-stochastic systems, while capturing the intuitive differences between the problem instances by the use of a random player in the abstract game. Most interestingly, the stochastic nature of the resulting abstract game eases the abstract synthesis problem compared to standard ABCD where disturbances are non-stochastic. In conclusion, we obtain a symbolic algorithm to compute an under-approximation of the almost sure winning region in a continuous-state CMP for all 0-regular specifications. Moreover, similar to the results for finite-state MDPs, this shows that the approximate solution to step (I) does not need to handle the actual transition probabilities. They are only needed in step (II), where existing techniques can be used.

**Related Work.** Our paper extends the recent results of Majumdar et al. [26] from Büchi specifications to parity specifications. Seem through the lens of 2/3-player games, the algorithm of [26] can be seen as directly solving a Büchi game symbolically on a non-probabilistic abstraction, by implicitly reducing the 2/3-player games to two player games on graphs with extreme fairness assumptions [16]. While it may be possible to present a similar "direct" symbolic algorithm for parity games, the details of handling fairness symbolically get difficult. Our exposition in this paper helps separate out the different combinatorial aspects: the representation of the abstraction and the solution of the game on the abstraction, leading to a clean proof of correctness.

2/3-player games have been used as abstractions of probabilistic systems, both in the finite case [21] and for stochastic linear systems [32]. Our paper subsumes the result of [32] by showing a computational procedure to abstract a general, nonlinear CMP into a finite-state 2/3-player game. Further [32] only consider specifications in the GR(1) fragment of linear temporal logic while we can handle any 0-regular specification. However, restricting attention to linear systems as well as polytopic predicates in [32] enables the use of symbolic algorithms based on polyhedral manipulation. Instead, our abstractions are based on gridding the state space, as in ABCD for non-stochastic nonlinear systems.

Stochastic nonlinear systems were abstracted to finite-state interval Markov decision processes by Dutreix et al. [11, 12]. By using algorithms for model checking finite state Markov chains against deterministic Rabin automata, they provide an alternative approach for approximating the almost sure winning region for CMPs against 0-regular specifications. Both methods are conceptually very different. Dutreix et al. explicitly compute lower and upper bounds of all involved probabilities and construct winning regions by an enumerative algorithm taking these probability bounds into account. On the other hand, our approach shows a clean separation between step (II), which requires to know explicit transition probabilities but can be solved by existing techniques, and step (I), for which this knowledge is not needed. This allows us to provide a conceptually simpler symbolic algorithm approximately solving (I) via abstract 2/3-player games.

## 2 Stochastic Nonlinear Systems

### 2.1 Preliminaries

For any set $A$, a sigma-algebra on $A$ comprises subsets of $A$ as events that includes $A$ itself and is closed under complement and countable unions. We consider a probability space $(\Omega, F, \mathbb{P})$, where $\Omega$ is the sample space, $F$ is a sigma-algebra on $\Omega$, and $\mathbb{P}$ is a probability measure that assigns probabilities to events. An $((S, F_S), \mathbb{P})$-valued random variable $X$ is a measurable function of the form $X: (\Omega, F) \rightarrow (S, F_S)$, where $S$ is the codomain of $X$ and $F_S$ is a sigma-algebra on $S$. Any random variable $X$ induces a probability measure on its space $(S, F_S)$ as $\mathbb{P}(A) = \mathbb{P}(X^{-1}(A))$ for any $A \in F_S$. We often directly discuss the probability measure on $(S, F_S)$ without explicitly mentioning the underlying probability space $(\Omega, F, \mathbb{P})$ and the function $X$ itself.

A topological space $S$ is called a Borel space if it is homeomorphic to a Borel subset of a Polish space (i.e., a separable and completely metrizable space). Examples of a Borel space are the Euclidean spaces $\mathbb{R}^n$, its Borel subsets endowed with a subspace topology, as well as hybrid spaces. Any Borel space $S$ is assumed to be endowed with a Borel sigma-algebra (i.e., the one generated by the open sets in the topology), which is denoted by $\mathcal{B}(S)$. We say that a map $f : S \rightarrow Y$ is measurable whenever it is Borel measurable.

Given an alphabet $A$, we use the notation $A^*$ and $A^+$ to denote respectively the set of all finite words, the set of all infinite words formed using the letters of the alphabet $A$, and use $A^n$ to denote the set $A^n \cup A^n$. Let $X$ be a set and $R \subseteq X \times X$ be a relation. For simplicity, let us assume that $\text{dom } R := \{ x \in X \mid \exists y \in X . (x, y) \in R \} = X$. For any given $x \in X$, we use the notation $R(x)$ to denote the set $\{ y \in X \mid (x, y) \in R \}$. We extend this notation to sets: For any given $Z \subseteq X$, we use the notation $R(Z)$ to denote the set $\cup_{z \in Z} R(z)$. Given a set $A$, we use the notation $\text{Dist}(A)$ to denote the set of all probability distributions over $A$.

We denote the set of nonnegative integers by $\mathbb{N} := \{ 0, 1, 2, \ldots \}$ and the set of integers in an interval by $[a, b] := \{ k \in \mathbb{N}, k \leq b-a \}$. We also use the symbols "even" and "odd" to denote memberships in the set of even and odd integers within a given set of integers: For example, for a given set of natural numbers $M \subseteq \mathbb{N}$, the notation $n \in_{\text{even}} M$ is equivalent to $n \in M \cap \{ 0, 2, 4, \ldots \}$, and the notation $n \in_{\text{odd}} M$ is equivalent to $n \in M \cap \{ 1, 3, 5, \ldots \}$.

### 2.2 Controlled Markov Processes

A controlled Markov process (CMP) is a tuple $\mathcal{S} = (S, \mathcal{U}, T_s)$, where $S$ is a Borel space called the state space, $\mathcal{U}$ is a finite set called the input space, and $T_s$ is a conditional stochastic kernel $T_s : \mathcal{B}(S) \times S \times \mathcal{U} \rightarrow [0, 1]$ with $\mathcal{B}(S)$ being the Borel sigma-algebra on the state space and $(S, \mathcal{B}(S))$ being the corresponding measurable space. The kernel $T_s$ assigns to any $s \in S$ and $u \in \mathcal{U}$ a probability measure $T_s(s, u)$ on the measurable space $(S, \mathcal{B}(S))$ so that for any set $A \in \mathcal{B}(S)$, $P_{s,u}(A) = \int_A T_s(ds, u)$, where $P_{s,u}$ denotes the conditional probability $P(\cdot | s, u)$.
In general, the input space $\mathcal{U}$ can be any Borel space and the set of valid inputs can be state dependent. While our results can be extended to this setting, for ease of exposition, we consider the special case where $\mathcal{U}$ is a finite set and any input can be taken at any state. This choice is motivated by the digital implementation of control policies with a finite number of possible actions.

The evolution of a CMP is as follows. For $k \in \mathbb{N}$, let $X^k$ denote the state at the $k$th time step and $A^k = u \in \mathcal{U}$, then the system moves to the next state $X^{k+1}$, according to the probability distribution $P_{\rho,u}$. Once the transition into the next state has occurred, a new input is chosen, and the process is repeated.

Given a CMP $\Sigma$, a finite path of length $n + 1$ is a sequence
\[ \omega^n = (s^0, s^1, \ldots, s^n), \quad n \in \mathbb{N}, \]
where $s^i \in \mathcal{S}$ are state coordinates of the path. The space of all paths of length $n + 1$ is denoted $\mathcal{S}^{n+1}$. An infinite path of the CMP $\Sigma$ is the sequence $\omega = (s^0, s^1, \ldots)$, where $s^i \in \mathcal{S}$ for all $i \in \mathbb{N}$. The space of all infinite paths is denoted by $\mathcal{S}^{\omega}$. The spaces $\mathcal{S}^{n+1}$ and $\mathcal{S}^{\omega}$ are endowed with their respective product topologies and are Borel spaces.

A stationary control policy is a universally measurable function $\rho : \mathcal{S} \to \mathcal{U}$ such that at any time step $n \in \mathbb{N}$, the input $u^n$ is taken to be $\rho(s^n) \in \mathcal{U}$. As we only deal with stationary control policies in this paper, we simply refer to them as policies for short. We denote the class of all such policies by $\Pi$. The function $\rho$ is also called the state feedback controller in control theory.

For a CMP $\Sigma$, any policy $\rho \in \Pi$ together with an initial probability measure $\alpha : \mathcal{B}(\mathcal{S}) \to [0, 1]$ of the CMP induces a unique probability measure on the canonical sample space of paths [17], denoted by $P_\alpha^\omega$ with the expectation $E_\alpha$. In the case when the initial probability measure is supported on a single state $s \in \mathcal{S}$, i.e., $\alpha(s) = 1$, we write $P_\alpha^\omega$ and $E_\alpha$ in place of $P_\alpha^0$ and $E_\alpha$, respectively. We denote the set of probability measures on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ by $\mathcal{S}$.

Given any $\omega$-regular specification $\varphi$ defined using a set of predicates over the state space $\mathcal{S}$ of $\Sigma$, we use the notation $\Sigma \models \varphi$ to denote the set of all paths of $\Sigma$ which satisfy $\varphi$. Thus, $P_\rho^\omega(\Sigma \models \varphi)$ denotes the probability of satisfaction of $\varphi$ by $\Sigma$ under the effect of the control policy $\rho$, when the initial probability measure is given by $\alpha$. Often we will use Linear Temporal Logic (LTL) notation to express $\omega$-regular properties. The syntax and semantics of LTL can be found in standard literature [2].

A stochastic dynamical system $\Sigma$ is described by a state evolution
\[ s^{k+1} = f(s^k, u^k, s^k), \quad k \in \mathbb{N}, \quad (1) \]
where $s^k \in \mathcal{S}$ and $u^k \in \mathcal{U}$ are states and inputs for each $k \in \mathbb{N}$, and $(s^0, s^1, \ldots)$ is assumed to be a sequence of independent and identically distributed (i.i.d.) random variables representing a stochastic disturbance. The map $f$ gives the next state as a function of current state, current input, and the disturbance. One can construct a CMP over states $\mathcal{S}$ and inputs $\mathcal{U}$ from (1) by noticing that for any given state $s^k$ and input $u^k$ at time $k$, the next state is a random variable defined as a function of $s^k$. Thus, $P_\rho(\cdot | s^k, u^k)$ is exactly the distribution of the random variable $f(s^k, u^k, s^k)$ and can be computed based on the distribution of $s^k$ and the map $f$ itself [19].

2.3 Parity Specifications
Let $\Sigma = (\mathcal{S}, \mathcal{U}, \mathcal{T}_p)$ be a CMP and suppose $\mathcal{P} = \{B_0, B_1, \ldots, B_I\}$ is a partition of $\mathcal{S}$ with measurable sets $B_0, \ldots, B_I$; that is, $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^I B_i = \mathcal{S}$. We allow some $B_i$’s to be empty. For each $B_i$, we call the integer $i$ its priority.

Intuitively, an infinite path $w \in \mathcal{S}^{\omega}$ satisfies the parity specification w.r.t. $\mathcal{P}$ if the highest subset $B_i$ visited infinitely often has even priority. This specification is formalized in Linear Temporal Logic (LTL) notation [2] using the following formula:
\[ \text{Parity}(\mathcal{P}) := \bigwedge_{i \in \text{odd}[1,I]} \left( \bigcirc B_i \rightarrow \bigvee_{j \in \text{even}[1+1,I]} \bigcirc B_j \right), \quad (2) \]
which requires that infinitely many visits to an odd priority subset $(\bigcirc B_i)$ must imply infinitely many visits to a higher even priority subset $(\bigcirc B_j)$. We denote the set of all infinite paths $w \in \mathcal{S}^{\omega}$ of a CMP $\Sigma$ that satisfy the property $\text{Parity}(\mathcal{P})$ by $\Sigma \models \text{Parity}(\mathcal{P})$. The proof of measurability of the event $\Sigma \models \text{Parity}(\mathcal{P})$ goes back to the work by Vardi [38] that provides the proof for probabilistic finite state programs. The proof for a CMP follows similar principles, using the observation that $\Sigma \models \text{Parity}(\mathcal{P})$ can be written as a Boolean combination of events $\bigwedge S \models \bigcirc B_i$, where $b$ is a measurable set, and $\bigwedge A$ is a canonical $G_\delta$ set in the Borel hierarchy.

It is well-known that every $\omega$-regular specification whose propositions range over measurable subsets of the state space of a CMP can be modeled as a deterministic parity automaton [14, Thm. 1.19]. By taking a synchronized product of this parity automaton with the CMP, we can obtain a product CMP and a parity specification over the product state space such that every path satisfying the parity specification also satisfies the original $\omega$-regular specification and vice versa. Moreover, a stationary policy for the parity objective gives a (possibly history-dependent) policy for the original specification. Thus, without loss of generality, we assume that an $\omega$-regular objective is already given as a parity condition using a partition of the state space of the system.

3 PROBLEM DEFINITION
We are interested in the maximal probability that a given parity specification can be satisfied by paths of a CMP $\Sigma$ starting from a particular state $s \in \mathcal{S}$ under stationary policies. Given a control policy $\rho \in \Pi$ and an initial state $s \in \mathcal{S}$, we define the satisfaction probability and the supremum satisfaction probability as
\[ f(s, \rho) := P_\rho^0(\Sigma \models \text{Parity}(\mathcal{P})) \quad (3) \]
\[ f^*(s) := \sup_{\rho \in \Pi} P_\rho^0(\Sigma \models \text{Parity}(\mathcal{P})), \quad (4) \]
respectively. An optimal control policy for the parity condition is a policy $\rho^*$ such that $f^*(s) = f(s, \rho^*)$ for all $s \in \mathcal{S}$. Note that an optimal policy need not exist, since the supremum may not be achieved by any policy. Our goal is to study the following optimal policy synthesis problem.

Problem 1 (Optimal Policy Synthesis). Given $\Sigma$ and a parity specification $\text{Parity}(\mathcal{P})$, find an optimal control policy $\rho^*$, if it exists, together with $f^*(s)$ such that $P_{\rho^*}^0(\Sigma \models \text{Parity}(\mathcal{P})) = f^*(s)$. 

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While the satisfaction probability (3) and the supremum satisfaction probability (4) are both well-defined, we are not aware of any work characterizing necessary or sufficient conditions for existence of optimal control policies on continuous-space CMPs for parity specifications. Additionally, we restrict attention to stationary policies. While it is possible to define more general classes of policies, that depend on the entire history and use randomization over $\mathcal{U}$, we are again unaware of any work that characterizes the class of policies that are sufficient for optimal control of CMPs for parity specifications. For finite-state systems, stationary policies are sufficient and we restrict attention to them.

Since we cannot prove existence or computability of optimal policies, in this paper, we focus on providing an approximation procedure to compute a possibly sub-optimal control policy and guaranteed lower bounds on the optimal satisfaction probability. Our procedure relies on first approximating almost sure winning regions (i.e., where the specification can be satisfied with probability one), and then solving a reachability problem as formalized next.

**Definition 3.1 (Almost sure winning region).** Given a CMP $\mathcal{S}$, a policy $\rho$, and a parity specification $\text{Parity}(\mathcal{P})$, the state $s \in \mathcal{S}$ satisfies the specification almost surely if $f(s, \rho) = 1$. The almost sure winning region of the policy $\rho$ is defined as

$$\text{WinDom}(\mathcal{S}, \rho) := \{s \in \mathcal{S} \mid f(s, \rho) = 1\}. \quad (5)$$

We also define the maximal almost sure winning region as

$$\text{WinDom}^+ (\mathcal{S}) := \{s \in \mathcal{S} \mid f^+(s) = 1\}. \quad (6)$$

Note that $\text{WinDom}(\mathcal{S}, \rho) \subseteq \text{WinDom}^+(\mathcal{S})$ for any control policy $\rho \in \Pi$. It is clear by definition of the winning region that for any control policy, the satisfaction probability $P^\rho_\omega(\mathcal{S} \models \text{Parity}(\mathcal{P}))$ is equal to 1 for any $s$ in the winning region $W := \text{WinDom}(\mathcal{S}, \rho)$. The next theorem states that this satisfaction probability is lower bounded by the probability of reaching the winning region $W$ for any $s \notin W$. We denote such a reachability by $(\mathcal{S} \models \diamondsuit W) := \{w = (s^0, s^1, s^2, \ldots) \mid \exists n \in \mathbb{N}. s^n \in W\}$.

**Theorem 3.2.** For any control policy $\rho \in \Pi$ on CMP $\mathcal{S}$, and $W := \text{WinDom}(\mathcal{S}, \rho)$, we have

$$P^\rho_\omega(\mathcal{S} \models \text{Parity}(\mathcal{P})) = 1 \quad \text{if } s \in W \quad \text{and}$$

$$P^\rho_\omega(\mathcal{S} \models \text{Parity}(\mathcal{P})) \geq P^\rho_\omega(\mathcal{S} \models \diamondsuit W) \quad \text{if } s \notin W. \quad (7)$$

The proof can be found in the appendix. The inequality in the second part of (7) is because the $\text{Parity}(\mathcal{P})$ specification may be satisfied with positive probability even though the path always stays outside of $W$. When the state space is finite (i.e., for finite Markov decision processes), equality holds [2]. However, equality need not hold for general CMPs: [26] shows an example where the maximal almost sure winning region is empty even though a Büchi specification is satisfied with positive probability.

The next theorem establishes that for any policy $\rho$, the winning region is an absorbing set, i.e., paths starting from this set will stay in the set almost surely.

**Theorem 3.3.** For any control policy $\rho$, The set $W = \text{WinDom}(\mathcal{S}, \rho)$ is an absorbing set, i.e., $T_\omega(W|s, \rho(s)) = 1$ for all $s \in W$. This implies $P^\rho_\omega(\mathcal{S} \models \diamondsuit S \setminus W) = 0$ for all $s \in W$.

The proof of this theorem utilizes the fact that $\text{Parity}(\mathcal{P})$ is a long-run property and its satisfaction is independent of the prefix of a path. The proof is provided in the appendix.

Thm. 3.2 and Thm. 3.3 enable us to decompose the maximization of $P^\rho_\omega(\mathcal{S} \models \text{Parity}(\mathcal{P}))$ with respect to policies $\rho$ into two subproblems. First, find a policy that gives the largest winning region $W$ and employ that policy when the state $s \in W$. Then, find a policy that maximizes the reachability probability $P^\rho_\omega(\mathcal{S} \models \diamondsuit W)$ and employ that policy as long as $s \notin W$.

Computation of the reachability probability has been studied extensively in the literature for both infinite horizon [15, 35–37] and finite horizon [18, 20, 22–24, 29–31, 39] using different abstract models and computational methods. Together with an algorithm that underapproximates the region of almost sure satisfaction, these approaches can be used to provide a lower bound on the probability of satisfaction of the parity condition. In the rest of the paper, we focus on the following problem (the first half of (7)).

**Problem 2 (Approximate Maximal Winning Region).** Given $\mathcal{S}$ and a parity specification $\text{Parity}(\mathcal{P})$, find a (sub-optimal) control policy $\rho \in \Pi$, its winning region $\text{WinDom}(\mathcal{S}, \rho) \neq \emptyset$, and a bound on the volume of the set difference $\text{WinDom}^+(\mathcal{S}) \setminus \text{WinDom}(\mathcal{S}, \rho)$.

In Sec. 4-5, we provide a solution for Prob. 2 via the paradigm of abstraction-based controller design. Not surprisingly, we get a tighter (i.e., larger) approximation of $\text{WinDom}^+(\mathcal{S})$ if we use a finer discretization of the state space during the abstraction step. We also provide an over-approximation of $\text{WinDom}^+(\mathcal{S})$, and show closeness of the under- and over-approximation of $\text{WinDom}^+(\mathcal{S})$ in the numerical example provided in Sec. 6.

## 4 ABSTRACTION-BASED POLICY SYNTHESIS

The main result of our paper is a solution to Prob. 2 via a symbolic algorithm over abstract $2/\omega$-player games in the spirit of abstraction-based controller design (ABCD). ABCD is typically used to compute temporal-logic controllers for non-stochastic nonlinear dynamical systems [27, 28, 33] in two steps. First, the system is abstracted into an abstract finite-state two-player game. This game is then used to synthesize a discrete controller which is then refined into a continuous controller for the original system. In standard ABCD techniques, the abstract game has two players: Player 0 simulating the controller and choosing the next control input $u$ based on the currently observed abstract state $s$, and Player 1 simulating the adversarial effect of (a) choosing any continuous state $s$ in $\mathcal{S}$ to which $u$ is applied and (b) choosing any continuous disturbance $d$ that effects the resulting transition.

The key insight in our abstraction step is that the stochastic nature of the underlying CMP allows choosing disturbances in a fair random way instead of purely adversarially. We model this by introducing an additional random player (also called the player) resulting in a so called $2/\omega$-player game [4–6]. In the resulting abstract game, only the effect of the discretization is handled by Player 1 in an adversarial manner. The random player picks the applied disturbance uniformly at random.

After introducing necessary preliminaries on $2/\omega$-player games in Sec. 4.1, we show how a CMP can be abstracted into a $2/\omega$-player game in Sec. 4.2. We then recall in Sec. 4.3 a symbolic procedure to
find winning regions in 2-player games for parity specifications. Finally, we state in Sec. 4.4 how an almost-sure winning strategy in the abstract 2-player game is refined, and that the resulting control policy is almost sure winning for the original CMP and its associated parity specification. This establishes soundness of our ABCD technique to solve Problem 2.

4.1 Preliminaries: 2-player Parity Games

A 2-player game graph is a tuple \( G = (V, E, (V_0, V_1, V_r)) \), where \( V \) is a finite set of vertices, \( E \) is a set of directed edges \( E \subseteq V \times V \), and the sets \( V_0, V_1, V_r \) form a partition of the set \( V \). A 2-player parity game is a tuple \( (G, \mathcal{P}) \), where \( G \) is a 2-player game graph, and \( \mathcal{P} = (b_0, b_1, \ldots, b_k) \) is a tuple of \( t \) disjoint subsets of \( V \), some of which can possibly be empty. The tuple \( \mathcal{P} \) induces the parity specification \( \text{Parity}(\mathcal{P}) \) over the set of vertices \( V \) in the natural way. In order to ensure that every infinite run must have infinitely many occurrences of vertices from at least one of the sets in \( \mathcal{P} \). In other words, we require that every set of vertices \( U \subseteq V \) for which there is no \( i \in [1:t] \) with \( U \cap b_i \neq \emptyset \) must be “transient” vertices.

The players and their strategies. We assume that there are two players Player 0 and Player 1, who are playing a game by moving a token along the edges of the game graph \( G \). In every step, if the token is located on a vertex \( v \in V \), the next step the token moves to a vertex \( v' \) which is chosen uniformly at random from the set \( E(v) \). Strategies of Player 0 and Player 1 are respectively the functions \( \sigma_0 : V^* V_0 \rightarrow \text{Dist}(V) \) and \( \sigma_1 : V^* V_1 \rightarrow \text{Dist}(V) \) such that for all \( w \in V^* \), \( w_0 \in V_0 \) and \( v_1 \in V_1 \), we have \( \text{supp} \ \sigma_0(w) \subseteq E(w_0) \) and \( \text{supp} \ \sigma_1(w) \subseteq E(v_1) \). We use the notation \( \Pi_0 \) and \( \Pi_1 \) to denote the set of all strategies of Player 0 and Player 1 respectively. A strategy \( \sigma_i \) of Player \( i \), for \( i \in \{0, 1\} \), is deterministic memoryless if for every \( w_1, w_2 \in V^* \) and for every \( v \in V_i \), \( \sigma_i(w_1) \equiv \sigma_i(w_2) \) holds; we simply write \( \sigma_i(v) \) in this case. We use the notation \( \Pi_i^{\text{DM}} \) to denote the set of all deterministic memoryless strategies of Player \( i \). Observe that \( \Pi_0^{\text{DM}} \subseteq \Pi_0 \).

Runs and winning conditions. An infinite (finite) run of the game graph \( G \), compatible with the strategies \( \sigma_0 \in \Pi_0 \) and \( \sigma_1 \in \Pi_1 \), is an infinite (finite) sequence of vertices \( r = v^0 \sigma_0 v^1 \sigma_1 \ldots \) for some \( n \in \mathbb{N} \) such that for every \( k \in \mathbb{N} \), (a) \( \sigma_k \in V_0 \) implies \( v^{k+1} \in \text{supp} \ \sigma_0(v^k \ldots v^1) \), (b) \( \sigma_k \in V_1 \) implies \( v^{k+1} \in \text{supp} \ \sigma_1(v^0 \ldots v^k) \), and (c) \( \sigma_k \in V_r \) implies \( v^{k+1} \in E(v^k) \). Given an initial vertex \( v^0 \) and a fixed pair of strategies \( \sigma_0 \in \Pi_0 \) and \( \sigma_1 \in \Pi_1 \), we obtain a probability distribution over the set of infinite runs of the system. For a measurable set of runs \( R \subseteq V^\omega \), we use the notation \( P_{\sigma_0, \sigma_1}(R) \) to denote the probability of obtaining the set of runs \( R \) when the initial vertex is \( v^0 \) and the strategies of Player 0 and Player 1 are fixed to \( \sigma_0 \) and \( \sigma_1 \). For an \( \alpha \)-regular specification \( \varphi \), defined using a predicate over the set of vertices of \( G \), we write \( (G, \mathcal{P}) \models \varphi \) to denote the set of all infinite runs for all possible strategies of Player 0 and Player 1 which satisfy \( \varphi \). For example, \( (G, \mathcal{P}) \models \text{Parity}(\mathcal{P}) \) denotes the set of all infinite runs for all possible strategies of Player 0 and Player 1 which satisfy the parity condition \( \text{Parity}(\mathcal{P}) \). We say that Player 0 wins \( \text{Parity}(\mathcal{P}) \) almost surely from a vertex \( v \in V \) (or \( v \) is almost sure winning for Player 0) if Player 0 has a strategy \( \sigma_0 \in \Pi_0 \) such that for all \( \sigma_1 \in \Pi_1 \), we have \( P_{\sigma_0, \sigma_1}(G, \mathcal{P}) = 1 \). We collect all vertexes for which this is true in the almost-sure winning region \( W(G, \mathcal{P}) \).

4.2 Abstraction: CMPs to 2-player Parity Games

Given a CMP \( \mathcal{G} = (\mathcal{S}, \mathcal{U}, T_0) \) and a parity specification \( \text{Parity}(\mathcal{P}) \) for a partition \( \mathcal{P} \) of the state space \( \mathcal{S} \) we construct an abstract 2-player game.

State-space abstraction. We introduce a finite partition \( \mathcal{S} = \{S_i\}_{i \in I} \) such that for every \( i \neq j \) there is a state \( s \in S_i \) and \( s \in S_j \). Furthermore, we assume that the partition \( \mathcal{S} \) is consistent with the given priorities \( \mathcal{P} \), i.e., for every partition element \( s \in S_i \) and \( s \in S_j \), and for every \( x, y \in S_i \) and \( x, y \) belong to the same partition element in \( \mathcal{P} \) (i.e., \( x \) and \( y \) are assigned the same priority). We call the set \( \mathcal{S} \) the abstract state space and each element \( s \in \mathcal{S} \) an abstract state.

We introduce the abstraction function \( Q : \mathcal{S} \rightarrow \mathcal{S} \) as a mapping from the continuous to the abstract states: For every \( s \in \mathcal{S} \), \( Q : s \mapsto \tilde{s} \) such that \( s \in \tilde{s} \). We define the concretization function as the inverse of the abstraction function: \( Q^{-1} : \tilde{s} \mapsto s \). We generalize the use of \( Q \) and \( Q^{-1} \) to sets of states: For every \( U \subseteq \mathcal{S} \), \( Q(U) = \bigcup_{s \in U} Q(s) \), and for every \( \tilde{U} \subseteq \mathcal{S} \), \( Q^{-1}(\tilde{U}) = \bigcup_{s \in \tilde{U}} Q^{-1}(s) \).

Transition abstraction. We also introduce an over- and under-approximation of the probabilistic transitions of the CMP \( \mathcal{G} \) using the non-deterministic abstract transition functions \( \tilde{F} : \tilde{S} \times \tilde{U} \rightarrow \tilde{S} \) and \( \tilde{E} : \tilde{S} \times \tilde{U} \rightarrow \tilde{S} \) with the following properties:

\[
\tilde{F}(\tilde{s}, u) \subseteq \{\tilde{s}' \in \tilde{S} | \exists s \in s. T_0(s') \& | s, u > 0\}, \\
\tilde{E}(\tilde{s}, u) \subseteq \{\tilde{s}' \in \tilde{S} | \exists s \in s. T_0(s') \& | s, u | > \varepsilon\}.
\]

To understand the need for both \( \tilde{F} \) and \( \tilde{E} \) and the way they are constructed, consider the following examples. Intuitively, given an abstract state \( s \) and an input \( u \), the set \( \tilde{F} \) over-approximates the set of abstract states reachable by probabilistic transitions from \( s \) on input \( u \). On the other hand, \( \tilde{E} \) under-approximates those abstract states which can be reached by every state in \( s \) with probability bounded away from one.

Example 4.1. Consider the two CMPs, \( \mathcal{G}_A \) and \( \mathcal{G}_B \):

\[
\mathcal{G}_A : \quad \mathcal{G}_B : \quad S_1 \quad S_2 \quad S_3
\]

\[
S_1 \quad S_2 \quad S_3
\]

The circles are concrete states \( s_i \), the dashed boxes denote abstract states \( \tilde{s}_i \), and the edges denote transitions with positive probability between concrete states \( s_i \). Consider the left abstract state \( \tilde{s}_1 \). Here, the adversary decides which concrete state (i.e., \( s_1 \) or \( s_2 \)) the game is in. In both \( \mathcal{G}_A \) and \( \mathcal{G}_B \), \( \tilde{F} \) says that both \( \tilde{s}_1 \) and \( \tilde{s}_2 \) are reachable from \( \tilde{s}_1 \). In \( \mathcal{G}_A \), \( \tilde{E} \) contains both \( \tilde{s}_1 \) and \( \tilde{s}_2 \), in \( \mathcal{G}_B \), \( \tilde{E} \) is empty. An adversary that plays according to \( \tilde{F} \) is too strong: it can keep playing the self loop in \( s_2 \), while the stochastic nature of the CMP ensures that
eventually \( s_2 \) will transition to \( s_3 \). In order to follow the probabilistic semantics, we must ensure the adversary picks a distribution whose support contains both abstract states.

In \( \mathcal{S}_A \), the probabilistic behavior of the two concrete states \( s_1 \) and \( s_2 \) are very different: \( s_1 \) stays in \( s_1 \) with probability one and \( s_2 \) stays in \( s_3 \) or moves probabilistically to \( s_2 \). To ensure correct behavior, we look at possible supports of distributions induced by the dynamics; these are the possible subsets of abstract states between \( F \) and \( \bar{F} \).

Here, the game either stays in \( s_1 \) (eventually) moves to \( s_2 \) and, in our reduction, we force the adversary to commit to one of the two options.

\[ \mathcal{S}_A \]

The parameter \( \varepsilon \) states that there is a uniform lower bound on transition probabilities for all states in an abstract state. This ensures that, provided \( \mathcal{S} \) is visited infinitely often and \( u \) is applied infinitely often from \( \mathcal{S} \), then \( \mathcal{S} \) will be reached almost surely from \( \mathcal{S} \). In the absence of a uniform lower bound, this property need not hold for infinite state systems; for example, if the probability goes to zero, the probability of escaping \( \mathcal{S} \) can be strictly less than one.

**Algorithmic computation of \( \bar{F} \) and \( F \).** While it is difficult to compute \( \bar{F} \) and \( F \) in general, they can be approximated for the important subclass of stochastic nonlinear systems with affine disturbances

\[
s^{k+1} = f(s^k, u^k) + \varepsilon^k, \quad k \in \mathbb{N},
\]

where \( \varepsilon^0, \varepsilon^1, \ldots \) are i.i.d. random variables from a distribution with bounded support and we assume we are only interested in a compact region \( \mathcal{S} \) of the state space. In this case, for any abstract state \( \mathcal{S} \) and any \( u \in U \), one can compute an approximation \( \text{ReachSet}(\mathcal{S}, u) \) with \( \text{ReachSet}(\mathcal{S}, u) \supseteq \{ s' \in \mathcal{S} \mid \exists s \in \mathcal{S} : f(s, u) = s' \} \) using standard techniques \([1, 7, 28]\). Define \( S_1, S_2 : 2^\mathcal{S} \times U \to 2^\mathcal{S} \) such that

\[
S_1 : (\mathcal{S}, u) \mapsto D \cup \text{ReachSet}(\mathcal{S}, u), \quad S_2 : (\mathcal{S}, u) \mapsto D \cap (\text{ReachSet}(\mathcal{S}, u)),
\]

where the minus sign \( (\text{ReachSet}(\mathcal{S}, u)) \) is applied to each individual element of \( \text{ReachSet}(\mathcal{S}, u) \) and \( \cup \) and \( \cap \) are Minkowski sum and difference, respectively. Using \( S_1 \) and \( S_2 \), the functions \( \bar{F}(\cdot, \cdot) \) and \( F(\cdot, \cdot) \) can be computed as \([26, \text{Thm. 6.1}]\): (1) \( \mathcal{S} \in \bar{F}(\mathcal{S}, u) \) if either \( \mathcal{S} \subseteq \mathcal{S}' \) and \( \mathcal{S}' \cap \mathcal{S} \neq \emptyset \) or \( \mathcal{S} \) is a special sink state, and (2) \( \mathcal{S} \in F(\mathcal{S}, u) \) if either \( \lambda(\mathcal{S}' \cap \mathcal{S}, u) > 0 \) or \( \lambda(\mathcal{S} \setminus \mathcal{S}', u) > 0 \) and \( \mathcal{S}' \) is a special sink state, and where \( \lambda(\cdot) \) denotes the Lebesgue measure (generalized volume) of a set.

**Abstract \( 2^i \)-player game graph.** Given the abstract state space \( \mathcal{S} \) and the over and under-approximations of the transition functions \( F \) and \( \bar{F} \), we are ready to construct the abstract \( 2^i \)-player game graph induced by a CMP.

**Definition 4.2.** Let \( \mathcal{S} \) be a given CMP. Then its induced abstract \( 2^i \)-player game graph is given by \( G = (V, E, (V_0, V_1, V_r)) \) s.t.

- \( V_0 = \mathcal{S} \) and \( V_1 = \mathcal{S} \times U \);
- \( V_r = \bigcup_{v_0 \in V_0} V_r(v_0) \), where
  \[
  V_r(v_0) = \{ v_r \in \mathcal{S} \mid F(v_0) \subseteq v_r \subseteq F(v_0) + 1, v_r \in U \}
  \]
- and it holds that
  - for all \( v_0 \in V_0, E(v_0) = \{ (v_0, u) \mid u \in U \} \)
  - for all \( v_1 \in V_1, E(v_1) = \{ V_r(v_1) \} \), and
  - for all \( v_r \in V_r, E(v_r) = \{ (v_0, v_r) \mid v_0 \in v_r \} \).

Fig. 1. Illustration of the construction of the abstract \( 2^i \)-player game (right) from a continuous-state CMP (left). The state space of the CMP is discretized into rectangular abstract states \( A_1, \ldots, D_3; F(A_2, u) = \{ C_2, C_3, C_4 \} \) (intersecting the green region), and \( F(A_2, u) = \{ C_1, C_2, C_3, C_4 \} \) (intersecting orange region). \( V_0, V_1 \) and \( V_r \) are indicated by circle, rectangular and diamond-shaped nodes. Random vertices are dashed.

Note that \( V_r(v_1) \) contains non-empty subsets of \( \mathcal{S} \) that includes all the abstract states in \( F(v_0) \) and possibly include only one additional element from \( F(v_1) \). The construction is illustrated in Fig. 1.

In the reduced game, \( \text{Player} \ 0 \) models the controller, \( \text{Player} \ 1 \) models the effect of discretization of the state space of \( \mathcal{S} \), and the random edges from the states in \( V_r \) model the stochastic nature of the transitions of \( \mathcal{S} \). Intuitively, the game graph in Def. 4.2 captures the following interplay which is illustrated in Fig. 1: At every time step, the control policy for \( \mathcal{S} \) has to choose a control input \( u \in U \) based on the current vertex \( s \) of \( G \). Since the control policy is oblivious to the precise continuous state \( s \in \mathcal{S} \) of \( \mathcal{S} \), hence \( u \) required to be an optimal choice for every continuous state \( s \in \mathcal{S} \). This requirement is materialized by introducing a fictitious adversary (i.e. \( \text{Player} \ 1 \)) who, given \( \mathcal{S} \) and \( u \), picks a continuous state \( s \in \mathcal{S} \) from which the control input \( u \) is to be applied. Now, we know that no matter what continuous state \( s \) is chosen by \( \text{Player} \ 1 \), \( V_r(v_1) \subseteq \mathcal{S} \) of \( \mathcal{S} \) contains the set of vertices \( F(s, u) \), for some \( \varepsilon > 0 \). This explains why every successor of \( (s, u) \in V_r \) states contains the set of vertices \( F(s, u) \). Moreover, depending on which exact \( s \in \mathcal{S} \) \( \text{Player} \ 1 \) chooses, with positive probability the system might go to some state in \( F(s, u) \). This is materialized by adding every state in \( F(s, u) \) to \( F(s, u) \) at a time to the successors of the states in \( V_r \) (see Def. 4.2). Finally, we assume that the successor from every state in \( V_r \) is chosen uniformly at random (indicated by dotted edges in Def. 4.2). Later, it will be evident that the exact probability values are never used for obtaining the almost sure winning region, and so we could have chosen any other probability distribution.

**Abstract parity specification.** To conclude the abstraction of a given CMP \( \mathcal{S} \) and its parity specification \( P = \{ B_1, \ldots, B_k \} \), we have to formally translate the priority sets \( B_i \) defined over subsets of states of the CMP into a partition of the vertices of the abstract \( 2^i \)-player game graph \( G \) induced by \( \mathcal{S} \). To this end, recall that we have assumed that the state space abstraction \( \mathcal{S} \) respects the priority set \( P \).

**Definition 4.3.** Let \( \mathcal{S} \) be a CMP with parity specification \( \text{Parity}(P) \) and \( G \) the abstract \( 2^i \)-player game graph induced by \( \mathcal{S} \). Then the induced abstract parity specification \( \bar{P} = \{ \bar{B}_0, \ldots, \bar{B}_r \} \) is defined such that \( \bar{B}_i = \{ v_0 \in V_0 \mid Q^{-1}(v_0) \subseteq B_i \} \) for all \( i \in [0, r] \). We denote the resulting \( 2^i \)-player parity game by the tuple \((G, \bar{P})\).
We note that the choice of the abstract parity set $\widehat{P}$ does not partition the state space. Indeed, we implicitly assign an "undefined" color "-" to all nodes $V_1 \cup V_r$. Thereby, we only interpret the given parity specification over a projection of a run to its player 0 nodes. Formally, a run $r$ over the abstract game graph $\widehat{G}$ starting from a vertex $s^0 \in V_0$ is of the form:

$$r = s^0, (s^0, u^0), (s^1, u^1), (s^2, u^2), \ldots$$

where $s^k \in \{s^0, \ldots, s^k, i \}$ for all $k \in \mathbb{N}$. The projection of the run $r$ to the player 0 states is defined as $\text{Proj}_{v_0}(r) = s^0, s^1, \ldots$. Let $\varphi$ be an $\omega$-regular specification defined using a set of predicates over $V_0$. We use the convention that $(G, \widehat{P})$ will denote the set of every infinite run $r$ of $\widehat{G}$, for any arbitrary pair of strategies of Player 0 and Player 1, such that $\text{Proj}_{v_0}(r)$ satisfies $\varphi$. This convention is well-defined because every infinite run of $\widehat{G}$ will have infinitely many occurrences of vertices from $V_0$ in it: This follows from the strict alternation of the vertices in $V_0, V_1,$ and $V_r$, as per Def. 4.2.

### 4.3 Abstract Controller Synthesis

Once the 2/2-player parity game $(G, \widehat{P})$ is constructed from the CMP $\mathcal{S}$ according to Def. 4.2, one can use existing to compute the almost-sure winning states of Player 0 along with an associated almost-sure Player 0 winning policy $\rho$ over $(G, \widehat{P})$. The best-known algorithm is due to Chatterjee, Jurdzinski and Henzinger [5]. It first converts a 2/2-player parity game $(G, \widehat{P})$ into a two-player parity game and then uses well-known symbolic fixed-point algorithms [13, 25] to solve the latter game. The resulting Player 0 winning strategy $\rho$ for the two-player game is known to be memoryless. Further, the same strategy constitutes a deterministic memoryless almost-sure Player 0 winning policy in the original 2/2-player game [5]. This implies that 2/2-player parity games are determined from 2-player games being determined; that is, from every vertex, either Player 0 has a deterministic memoryless strategy to win almost surely, or Player 1 has a deterministic memoryless strategy to win with positive probability bounded away from zero [5, Thm. 2].

### 4.4 Controller Refinement

Now consider the abstract 2/2-player parity game $(G, \widehat{P})$ constructed from the CMP $\mathcal{S}$ via Def. 4.2 and Def. 4.3 has been solved as discussed in Sec. 4.3. Hence, we know the almost-sure Player 0 winning region $W(G \models \text{Parity}(\widehat{P}))$ and the associated deterministic memoryless almost-sure Player 0 winning policy $\rho_0 \in \Pi^{DM}$. We then refine $\rho_0$ to a control policy $\rho \in \Pi$ for the CMP $\mathcal{S}$ under parity condition $\text{Parity}(\widehat{P})$ as follows.

**Definition 4.4.** Let $\mathcal{S}$ be a CMP with parity specification $\text{Parity}(P)$ and $(G, \widehat{P})$ its induced finite 2/2-player parity game with deterministic memoryless almost-sure Player 0 winning strategy $\rho_0 \in \Pi^{DM}$. Then the control policy $\rho \in \Pi$ is defined as $\rho(s) = \text{refinement of } \rho_0$ if for every $s \in \mathcal{S}$, if $s \in \widehat{S}$ for some $\widehat{s} \in \widehat{S}$, and if $\rho_0(\widehat{s}) = (\widehat{s}, u) \in V_1$ for some $u \in \mathcal{U}$, then $\rho(s) = u$.

With the completion of this last step of our ABCD method for stochastic nonlinear systems to Problem 2, which we prove in Sec. 5.

### Theorem 4.5 (Solution of Problem 2)

Let $\mathcal{S}$ be a CMP and $\text{Parity}(P)$ be a given parity specification. Let $(G, \widehat{P})$ be the abstract 2/2-player game defined in Def. 4.2. Suppose, a vertex $s \in V_0$ is almost sure winning for Player 0 in the game $(G, \widehat{P})$, and $\rho_0 \in \Pi^{DM}$ is the corresponding Player 0 winning strategy. Then the refinement $\rho$ of $\rho_0$ ensures that $s \in \text{WinDom}(\mathcal{S}, \rho)$.

**Remark 1.** An over-approximation of the optimal almost sure winning domain $\text{WinDom}^*(\mathcal{S})$ of $\mathcal{S}$ w.r.t. $\text{Parity}(P)$ can be computed via $(G, \widehat{P})$ as well. To obtain an over-approximation, we solve this abstract game cooperatively. That is, we let for $\rho_0$ to both its own moves and the moves of player $p_i$ to win almost surely w.r.t. $\text{Parity}(\widehat{P})$. We use this over-approximation to check the quality of the under-approximation in Sec. 6.

### 5 PROOF OF THEOREM 4.5

**Proof outline.** To prove Thm. 4.5, we first decompose both the original and the abstract parity specifications $\text{Parity}(P)$ and $\text{Parity}(\widehat{P})$ into a combination of more manageable safety and reachability sub-parts. That is, for every state reachable by a finite run in $\mathcal{S}$, we consider a local safety specification $\psi_S$ and a local reachability specification $\psi_R$ defined by

$$\psi_S := \square B_1$$

and

$$\psi_R := \diamond (\psi_S \lor \bigvee_{j \in \mathbb{N}, i \geq 1} B_j).$$

Intuitively, $\psi_R$ requires that every time an odd priority—say $B_i$—is visited in $\mathcal{S}$, eventually either $B_i$ should never occur or an even priority $B_j$ with $j > i$ should occur, almost surely. Similarly, for the abstract 2/2-player parity game $(G, \widehat{P})$ we consider the local safety specification $\widehat{\psi}_S$ and a local reachability specification $\widehat{\psi}_R$ defined by

$$\widehat{\psi}_S := \square \neg B_i$$

and

$$\widehat{\psi}_R := \diamond (\widehat{\psi}_S \lor \bigvee_{j \in \mathbb{N}, i \geq 1} \widehat{B}_j).$$

While the above decomposition needs to be established both for $G$ and for $\mathcal{S}$, the directions of the respective proof differ. For $\mathcal{S}$ we show that if $\widehat{\psi}_R$ holds for a state reachable by a finite run over $\mathcal{S}$, then the original specification $\text{Parity}(P)$ is satisfied by a continuation of the run using the refined policy $\rho$ (Step 1). For $G$ we show that if $\text{Parity}(\widehat{P})$ is satisfied, then $\widehat{\psi}_R$ holds for every state visited by a run compatible with the almost-sure winning strategy $\rho_0$ in $(G, \widehat{P})$ (Step 2). Further, we show that satisfaction of $\widehat{\psi}_S$ (resp. $\widehat{\psi}_R$) in $\mathcal{S}$ implies satisfaction of $\psi_S$ (resp. $\psi_R$) in $\mathcal{S}$ (Step 3-5). With this, we have all ingredients to prove Thm. 4.5 (Step 6).

**Step 1: Decomposition of $\text{Parity}(P)$**

We prove a sufficient condition for satisfaction of $\text{Parity}(P)$ in $\mathcal{S}$ if $\psi_R$ holds.

**Lemma 5.1.** Let $\mathcal{S}$ be a CMP, $\text{Parity}(P)$ be a parity specification, $s^0 \in \mathcal{S}$ be a given initial state, and $\rho$ be a control policy. Suppose the following holds for every finite path $(s^0, \ldots, s^k) \in S^{<\omega+1}$ of $\mathcal{S}$ and every $i \in \mathbb{N}, i \geq 1$:

$$s^i \in B_i \Rightarrow P_{\rho_i}^{\mathcal{S}}(\mathcal{S} \models \psi_R) = 1.$$  

Then $P_{\psi_R}^{\mathcal{S}}(\mathcal{S} \models \text{Parity}(P)) = 1$.

**Proof of Lemma 5.1.** Define for any arbitrary $i \in \mathbb{N}, i \geq 1$ the event $E_i := (\mathcal{S} \models \psi_i)$ with the specification $\psi_i := \bigvee_{j \in \mathbb{N}, i \geq 1} B_i \land B_j$. We want to show that $P_{\psi_i}^{\mathcal{S}}(\mathcal{S} \models \psi_i)$
Parity($\mathcal{P}$) = $P_0^\rho \left( \bigcap_i E_i \right) = 1$ where $i \in odd \{1; k\}$. We prove this by showing $P_0^\rho (E_i) = 0$ for every $i \in odd \{1; k\}$. Once we show this, the result follows according to the standard inequalities:

$$P_0^\rho \left( \bigcap_i E_i \right) = 1 - P_0^\rho (\bigcup_i \overline{E}_i) \geq 1 - \sum_i P_0^\rho (\overline{E}_i) = 1$$

where $P_0^\rho (\overline{E}_i) = P_0^\rho (\{ (G | \models \square B_i) \cap (G | \models \neg \bigcirc B_i) \})$

with $i \in odd \{1; k\}$ and $j \in even \{1; j; k\}$. Define the random variable $x$ to be the largest time instance when the trajectory visits one of the sets $B_j$. Also define $x' > x$ to be the first time instance after $x$ when the trajectory visits $B_j$ again. Note that for any trajectory satisfying $\square B_j$ and $\bigcirc \neg B_j$, both $x$ and $x'$ are well-defined and bounded. According to the assumption (11), we have

$$P_0^\rho (E_i | (G | \models \square B_j) \cap (G | \models \neg B_j)) = 0.$$ 

By taking the expectation with respect to the condition $(x', s^0, s^1, \ldots, s^p)$, we conclude that $P_0^\rho (E_i | (G | \models \square B_j) \cap (G | \models \neg B_j)) = 0$.

**Step 2: Decomposition of Parity($\mathcal{P}$).** We present a necessary condition for satisfaction of Parity($\mathcal{P}$) in $G$ if $\psi_R$ holds.

**Lemma 5.2.** Let $(G, \mathcal{P})$ be a 2/1-player parity game, and $v^0$ be a given vertex of $G$. Suppose $\pi_0^1 \in \Pi^{DM}$ is a Player 0 strategy such that $\pi_0^1 \in \Pi^{DM}$. Suppose $\pi_0^1 \models \text{Parity}(\mathcal{P}) = 1$. Then given any finite path $v^0 \ldots v^n \in V^+$ such that there exists a Player 1 strategy $\pi_1^1 \in \Pi^0$ with $P_{\pi_0^1, \pi_1^1}^* (G \models v^0 \ldots v^n > 0)$, the following holds for every $i \in odd \{1; k\}$:

$$v^n \in \overline{B}_i \Rightarrow \inf_{\pi_1^1 \in \Pi^0} P_{\pi_0^1, \pi_1^1}^* (G \models v^n | \psi_R) = 1.$$ (12)

The only new factor in Eq. (12) is the presence of the adversarial effect of the Player 1 strategies.

**Proof.** It follows from the definition of the parity specification in (2) that a vertex $v^0 \in V$ is almost sure winning using the strategy $\pi_0^1$ if the condition following is fulfilled:

$$\inf_{\pi_1^1 \in \Pi^0} P_{\pi_0^1, \pi_1^1}^* (G \models v^0 \ldots v^n | \psi_R) = 1.$$ (13)

From the semantics of LTL, (13) implies:

$$\inf_{\pi_1^1 \in \Pi^0} P_{\pi_0^1, \pi_1^1}^* (G \models v^0 \ldots v^n | \psi_R) = 1,$$ (14)

with $i \in odd \{1; k\}$. We show that (14) implies for every finite run $v^0 \ldots v^n \in V^*$ occurring with a positive probability $p_1 > 0$ for some strategy of Player 1, (12) holds. Suppose, for contradiction’s sake, there exists some $i \in odd \{1; k\}$ such that $v^0 \in \overline{B}_i$ and (12) does not hold, i.e., $\inf_{\pi_1^1 \in \Pi^0} P_{\pi_0^1, \pi_1^1}^* (G \models v^n | \psi_R) < 1$, implying existence of some $0 < p_2 \leq 1$ with $\sup_{\pi_1^1 \in \Pi^0} P_{\pi_0^1, \pi_1^1}^* (G \models v^n | \psi_R) = p_2$. This results in satisfaction of the parity specification with a probability of at most $(1 - p_1 \cdot p_2) < 1$, contradicting (14).

**Step 3: Refinement of $\psi_R$ to $\psi_3$.** We show that almost sure safety with respect to a given set $U$ in $G$ implies the same with respect to the set $\text{Q}^{-1}(U)$ in $G$; this will later be used to infer $\psi_R \Rightarrow \psi_3$.

**Proposition 5.3.** Let $\mathcal{S} = \mathcal{S}$ and $G = \mathcal{S}$ be a finite 2/1-player game graph as defined in Def. 4.2. Suppose $U, V_0$ is a given set of vertices of $G$, and assume that there is a Player 0 vertex $v$ in $U$ for which there is a strategy $\pi_0 \in \Pi^{DM}$ of Player 0 such that $\inf_{\pi_1 \in \Pi^0} P_{\pi_0, \pi_1}^*(G \models v) = 1$. Then the refinement $\rho \in \Pi_0$ ensures that for every state $s \in v, P_s^\rho (G | \models Q^{-1}(U)) = 1$.

**Proof.** It is known that for safety properties, almost sure satisfaction coincides with sure satisfaction, i.e., $\inf_{\pi_1 \in \Pi^0} P_{\pi_0, \pi_1}^*(G \models v) = 1$ if and only if for every strategy $\pi_1 \in \Pi^0$, every infinite run of $G$ stays inside $U$ at all time [10]. In other words, there must be a controlled invariant set $W$ inside $U$ for the strategy $\pi_0$ and $v \in W$. This controlled invariant set can be obtained by considering the 2-player game, obtained from $G$ by removing all the random vertices, and redirecting the outgoing transitions of a given Player 1 vertex $v' \in V_1$ to the Player 0 vertices within the set $\overline{F}(v', \pi_0(v')) \subseteq V_0$. Since $\overline{F}(v', \pi_0(v'))$ overapproximates the set of all the continuous states reachable from $v'$ using the input $\pi_0(v')$, hence if Player 0 can fulfill $\overline{F}(v')$ using the strategy $\pi_0(v')$, then $\rho$ can fulfill $\overline{Q}(U)$ from every state $s \in v'$ in $G$. (This follows from the standard arguments in abstraction-based control using over-approximation based abstractions [28].)

**Step 4: Refinement for $\psi_R$ to $\psi_3$.** We show that almost sure reachability with respect to a given set $U$ in $G$ implies the same with respect to the set $\text{Q}^{-1}(U)$ in $G$; this will be used to infer $\psi_R \Rightarrow \psi_3$.

Let $\mathcal{S} = \mathcal{S}$ be a CMP. Let us consider a reachability specification $U$ for a set $U \subseteq V_0$, in the game $G$ defined in Def. 4.2. Suppose $\pi_0 \in \Pi^{DM}$ is some strategy of Player 0.

We introduce a ranking function $r : V_0 \rightarrow N \cup \{\infty\}$ as a certificate for almost sure satisfaction of the specification $\phi U$. The ranking function $r$ is defined inductively as follows:

$$r(v) = \begin{cases} 0 & v \in U, \\ i + 1 \min \{n \in N | \inf_{\pi_1 \in \Pi^0} P_{\pi_0, \pi_1}^*(G \models v | \square^{r^{-1}(n)} > 0) \} & i \in \Pi^0 \text{ and } \inf_{\pi_1 \in \Pi^0} P_{\pi_0, \pi_1}^*(G \models v | \square^{r^{-1}(\infty)}) = 1. \end{cases}$$

Note that every vertex $v \in V$ gets a rank: If $r(v) \neq \infty$, then $\inf_{\pi_1 \in \Pi^0} P_{\pi_0, \pi_1}^*(G \models v | \square^{r}) = 1$ by definition of $r$. In this case, there must exist some path to $U$, i.e., $\inf_{\pi_1 \in \Pi^0} P_{\pi_0, \pi_1}^*(G \models \square^{r^{-1}(\infty)}) = 1$ must be true, and moreover $\inf_{\pi_1 \in \Pi^0} P_{\pi_0, \pi_1}^*(G \models \square^{r^{-1}(n)} > 0) = i$ will be true for some $n$. Thus, $r(v) = n + 1$.

From the ranking function $r(\cdot)$ defined in (15), it is clear that $\inf_{\pi_1 \in \Pi^0} P_{\pi_0, \pi_1}^*(G \models v | \square^{r}) = 1$ implies $r(v) = \infty$. We first identify some local structural properties of the abstract transition functions $\overline{F}$ and $\overline{E}$ evaluated on some abstract states with finite ranking.

**Lemma 5.4.** Suppose $\pi_0 \in \Pi^{DM}$ is some strategy of Player 0. For every $v \in V_0$ with $r(v) = i \neq \infty$, $i > 0$, both $\overline{F}(v, \pi_0(v)) \cap r^{-1}(\infty) = \emptyset$ and either of the following holds:

1. $\overline{F}(v, \pi_0(v)) \cap r^{-1}(i - 1) \neq \emptyset$,

2. $\overline{F}(v, \pi_0(v)) \cap r^{-1}(i) = \emptyset$. 


(2) $F(v, \pi_0(v)) = \emptyset$ and $\overline{F}(v, \pi_0(v)) \subseteq r^{-1}(i - 1)$.

**Proof.** Firstly, $F(v, \pi_0(v)) \cap r^{-1}(\infty) = \emptyset$ should always hold as otherwise Player 1 would have a strategy to reach a state in $r^{-1}(\infty)$ with nonzero probability in the next step.

Suppose (2) does not hold, implying either (a) $F(v, \pi_0(v)) \neq \emptyset$, or (b) the existence of a vertex $v' \in \overline{F}(v, \pi_0(v))$ with $r(v') = i - 1$. Then $F(v, \pi_0(v)) \cap r^{-1}(i - 1) = \emptyset$ must hold, as otherwise, for case (a) and (b) Player 1 would have a strategy $\pi_1$ with $\pi_1(v, \pi_0(v)) = (F(v, \pi_0(v)))$ and $\pi_1(v, \pi_0(v)) = (\overline{F}(v, \pi_0(v)) \cup \{v'\})$, respectively, such that $P^{\pi_0,\pi_1}_v(G) \models \bigcirc r^{-1}(i - 1) = 0$.

On the other hand, suppose (1) does not hold. Then $F(v, \pi_0(v)) = \emptyset$ must be true, as otherwise Player 1 would have a strategy $\pi_1$ with $\pi_1(v, \pi_0(v)) = (F(v, \pi_0(v)))$ such that $P^{\pi_0,\pi_1}_v(G) \models \bigcirc r^{-1}(i - 1) = 0$. Moreover, $\overline{F}(v, \pi_0(v)) \subseteq r^{-1}(i - 1)$ must also be true, as otherwise there would exist a vertex $v' \in \overline{F}(v, \pi_0(v))$ with $r(v') \neq i - 1$, and Player 1 would have a strategy $\pi_1$ with $\pi_1(v, \pi_0(v)) = (\overline{F}(v, \pi_0(v)) \cup \{v'\})$ such that $P^{\pi_0,\pi_1}_v(G) \models \bigcirc r^{-1}(i - 1) = 0$. □

The following lemma establishes soundness of the reduction with respect to reachability specifications.

**Proposition 5.5.** Let $\mathcal{S}$ be a CMP and $G$ be a finite 2|\vdash\vdash|/player game graph as defined in Def. 4.2. Suppose there is a Player 0 vertex $v \in V_0$ in $G$ and a set of vertices $U \subseteq V_0$ for which there is a strategy $\pi_0 \in \Pi^0_{DM}$ of Player 0 such that $\inf_{\pi_1 \in \Pi} P^{\pi_0,\pi_1}_v(G) \models \bigcirc U = 1$. Then the refinement $\rho \in \Pi(\pi_0)$ ensures that for every state $s \in v$, $P^\rho_s(\mathcal{S} \models \boxdot Q^{-1}(U)) = 1$.

**Proof.** It follows from the definition of the ranking function in (15) that the set of almost sure winning vertices for the specification $\boxdot U$ is given by all the vertices with finite rank. We show that for every vertex $v$ with a finite rank, the refinement $\rho \in \Pi(\pi_0)$ ensures that from every state $s \in v$, $P^\rho_s(\mathcal{S} \models \boxdot Q^{-1}(U)) = 1$.

First, trajectories starting from any state $s \in v$ with $r(s) \neq \infty$ never go to the region $Q^{-1}(r^{-1}(\infty))$. This follows from the identity $\overline{F}(v, \pi_0(v)) \cap r^{-1}(\infty) = \emptyset$ in Lem. 5.4 and because $\overline{F}(v, \pi_0(v))$ is an overapproximation of the one step reachable set from the states within vertex $v$. Hence, every infinite trajectory of $\mathcal{S}$ starting from $s$ will visit the states in $S \setminus Q^{-1}(r^{-1}(\infty))$ infinitely often.

The rest of the proof shows that if a trajectory visits the states $S \setminus Q^{-1}(r^{-1}(\infty)) \cup r^{-1}(0)$ infinitely often, then the trajectory will almost surely satisfy $Q^{-1}(r^{-1}(0)) = Q^{-1}(U)$. This is by induction over the largest rank assigned by $r$. For the base case, let the largest rank assigned by $r$ be 2. We show that every state $s \in S$ starting from inside a vertex $v$ with $r(v) = 1$ or $r(v) = 2$ will almost surely reach $Q^{-1}(U)$, i.e., $P^\rho_s(\mathcal{S} \models \boxdot Q^{-1}(r^{-1}(1)) \cup r^{-1}(2)) = 0$. Note that the events $\{\boxdot Q^{-1}(r^{-1}(2))\}$ and $\{\boxdot Q^{-1}(r^{-1}(1))\}$ form a partition of the event of $\{\boxdot Q^{-1}(r^{-1}(1)) \cup r^{-1}(2)\}$. Therefore, $P^\rho_s(\mathcal{S} \models \boxdot Q^{-1}(r^{-1}(1)) \cup r^{-1}(2))$. The first term is upper bounded by $P^\rho_s(\mathcal{S} \models \boxdot Q^{-1}(r^{-1}(2)))$ which is zero, because $P^\rho_s(\mathcal{S} \models \boxdot Q^{-1}(r^{-1}(2))) = \prod_{i=1}^{n}(1 - \rho^a) = 0$. The second term is also zero because the event requires the number of transitions from $Q^{-1}(r^{-1}(1))$ to be infinite. To see this, let $\lambda_n = (i_0, i_1, \ldots, i_n)$ be the first $(n + 1)$ time instances that a trajectory visits $Q^{-1}(r^{-1}(1))$. Then,$P^\rho_s(\mathcal{S} \models \boxdot Q^{-1}(r^{-1}(1)) \cup r^{-1}(2)) = \sum_{\lambda_n} P^\rho_s(\mathcal{S} \models \boxdot Q^{-1}(r^{-1}(1)) \cup r^{-1}(2)) \land \boxdot Q^{-1}(r^{-1}(1))) \propto i_n)P^\rho_s(i_n) \leq \sum_{\lambda_n} (1 - \rho^a)P^\rho_s(i_n) = (1 - \rho)^a$.

The last inequality is due to either Cond. (1) or Cond. (2) of Lem. 5.4 applied to the vertices in $v \in r^{-1}(1)$. Note that this inequality holds for any $n$. By taking the limit when $n$ goes to infinity, we have that this second term is also zero. Hence the base case is established.

For the induction hypothesis, assume that the claim holds when the maximum rank assigned by the function $r(i)$. Then for the induction step, i.e., when the maximum rank is $i + 1$, we can follow the same argument, as we did for the states with rank 2 in the base case, to show that every infinite trajectory inside $S \setminus Q^{-1}(r^{-1}(\infty)) \cup r^{-1}(0))$ will never get trapped inside $Q^{-1}(r^{-1}(i + 1))$, which will mean that the trajectory will visit the states in $S \setminus Q^{-1}(r^{-1}(\infty)) \cup r^{-1}(0)) \cup r^{-1}(i + 1))$ infinitely often. Then it follows from the induction hypothesis that the trajectories will reach $Q^{-1}(U)$ almost surely. □

**Step 5: Refinement of runs.** We show that every finite path in $\mathcal{S}$ can be mapped to a positive probability finite run in $G$ by using the universal quantification over finite paths in $\mathcal{S}$ to the universal quantification over finite runs in $G$.

**Lemma 5.6.** Let $\mathcal{S}$ be a CMP, $G$ be the abstract game graph as defined in Def. 4.2, $\pi_0 \in \Pi^0_{DM}$ be an arbitrary Player strategy in the game $G$, and $s \in S$ be a state of $\mathcal{S}$. Suppose $\rho \in \Pi$ is the refinement of $\pi_0$. Then for every finite trajectory $s_0 \ldots s_n \in S^* \subseteq \mathcal{S}$ in the support of the distribution $P^{\rho,s}_s$, there exists a Player 1 strategy $\pi_1 \in \Pi_1$ such that $P^{\pi_0,\pi_1}_s(G) \models \exists s_0 \ldots s_n > 0$, where $s_0 = Q(s')$ for every $i \in \{0, n\}$.

**Proof.** The initial state $s_0 \in \mathcal{S}$, and for every $0 \leq i < n$, from the definition of $F$ it follows that $s_{i+1} \in F(s_i, \pi_0(s'))$. Thus, from every Player 1 vertex $(s', \pi_0(s'))$, there is a successor vertex in $V_1$ whose successor is $s_{i+1}$. Hence, for every $0 \leq i < n$, there is some move of Player 1 which causes a transition to $s_{i+1}$ with some positive probability $p^s$. Then $P^{\pi_0,\pi_1}_s(G) \models \exists s_0 \ldots s_n = \prod_{i=0}^{n-1} p^s > 0$. □

**Step 6: The final assembly of the proof.** Finally, we finish the proof of Thm. 4.5 by stitching everything together. It is known that memoryless strategies suffice for winning almost surely in $2|\vdash\vdash$-player parity games [40]. Let $\pi_0^w \in \Pi^0_{DM}$ be the witness strategy of Player 0 to almost surely win from the vertex $s_0$ in the game $(G, \rho^s)$, and $P^{\pi_0,\pi_1}_s(G) \models \alpha \in \WinDom(\mathcal{S}, \rho^s)$. We will show that for every finite path of $\mathcal{S}$ starting within $s_0$ and ending in some odd priority state $B_i$, eventually either $B_i$ will not be visited any more, or a state of higher even color will be visited. Then the claim will follow from Lem. 5.1. We know from Lem. 5.6 that this existence of a finite path $s_0 \ldots s_n \in \mathcal{S}$ implies existence of an abstract run $s_0 \ldots s_n$ such that $sup_{\pi_1 \in \Pi} P^{\pi_0,\pi_1}(G) \models \exists s_0 \ldots s_n > 0$. Moreover, due to the priority preserving partitions we have $s_n \in \bar{B}_i$.
where $i$ is odd. Since $\pi^*_n$ is an almost sure winning strategy, hence, by using Lem. 5.2, we know that the following holds:

$$\inf_{\pi_i \in \mathcal{H}_1} P^\pi_i \pi^*_n (\mathcal{G} \equiv 0 (\square \neg B_j \vee \bigcup_{k \in \mathbb{N}} \square (\square [1:i+1] B_j)) = 1. \quad (16)$$

From Prop. 5.3, we know that the set of abstract states from which the specification $\square \neg B_j$ is satisfied (almost) surely using the strategy $\pi^*_n$ are also the set of continuous states from which the specification $\square (\square [1:i+1] B_j)$ is satisfied almost surely using the controller $\rho^*$. Together with Prop. 5.5 and Lem. 5.1, we can infer Thm. 4.5 from (16).

### 6 NUMERICAL EXAMPLE

We consider the controller synthesis problem for a mobile robot, modeled using the sampled-time version of perturbed Dubins vehicle [26]. The system has three state variables, denoted as $s_1$, $s_2$, and $s_3$, and representing respectively the position along the X-coordinate, the position along the Y-coordinate, and the steering angle. The vehicle moves with a constant forward velocity $V$ (maintained by, e.g., a low level cruise control system), which is set to 1 unit in this example. The single control input $u$ is responsible for moving the steering wheel, and thus changing the direction of the movement. The sampled-time dynamics for all $k \in \mathbb{N}$ and for all inputs $u^k \neq 0$ is given as follows:

$$s_1^{k+1} = s_1^k + \frac{V}{u^k} \left[ \sin(s_3^k + u^k \tau) - \sin(s_3^k) \right] + \xi_1^k$$

$$s_2^{k+1} = s_2^k - \frac{V}{u^k} \left[ \cos(s_3^k + u^k \tau) - \cos(s_3^k) \right] + \xi_2^k$$

$$s_3^{k+1} = s_3^k + u^k \tau + \xi_3^k,$$

when $u^k = 0$ then the dynamics can be obtained by taking limit $u^k \to 0$ in the right hand side of the above equations. for all $k \in \mathbb{N}$ with $u^k = 0$. The sampling time is $\tau = 0.1$ sec and $(\xi_1^k, \xi_2^k, \xi_3^k)$ is a collection of stochastic noise samples drawn from a piecewise continuous density function with the support $D = [-0.06, 0.06] \times [-0.06, 0.06] \times [-0.06, 0.06]$. We assume that the states of the vehicle moves inside the domain $[-0.6, 0.96] \times [-1.2, 1.98] \times [-\pi, \pi]$.

Fig. 2 shows the state space of the robot with various annotations for certain sets of states. The specification is provided using the following atomic propositions: (1) $A_0 \equiv \text{Door is open}$, (2) $C_0 \equiv \text{Robot inside office}$, (3) $G_1 \equiv \text{Robot inside kitchen}$, and (4) $\text{Crash} \equiv \text{Robot hits the door when it is closed}$. There is a safety requirement that the robot should never hit the closed door, i.e., $\square \neg \text{Crash}$. The rest of the specification is provided in an implication form. We assume that the following property is satisfied by the environment:

(a) the door opens infinitely often, i.e., $\square \diamond A_0$,  
(b) whenever the door is open, it remains open until the robot reaches the kitchen, i.e., $\square (A_0 \rightarrow (A_0 \diamond G_1))$.

If the environment satisfies the above, then the robot has to fulfill the following:

(a) The robot serves the request infinitely often, i.e., $\square \diamond C_0$, and  
(b) the robot goes to the kitchen infinitely often, i.e., $\square \diamond G_1$.

The overall specification for the robot can be summarized as:

$$\square \neg \text{Crash} \land (\square \diamond A_0 \land \square (A_0 \rightarrow (A_0 \diamond G_1)) \rightarrow \square \diamond C_0 \land \square \diamond G_1). \quad (17)$$

The specification in (17) can be modeled as a 3-color parity automaton, whose description can be found in the Appendix. We computed the synchronous product of the parity automaton and the vehicle’s dynamics model.

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Table 1. Performance evaluation of our method on the Dubins vehicle:

| Size of abstract states | Volume of the gap | Computation time |
|------------------------|-------------------|-----------------|
|                        |                   | Abs. | Over-approx. | Under-approx. |
| $0.1 \times 0.1 \times 0.1$ | 6.6              | < 1m | 9m          | 31m          |
| $0.08 \times 0.08 \times 0.08$ | 4.8              | 2m  | 84m         | 4h           |
| $0.06 \times 0.06 \times 0.06$ | 4.5              | 7m  | 102m        | 9h           |

Fig. 3. The figures in the top and the bottom row respectively show the trajectories of the states $s_1$ and $s_2$ with respect to time. The green regions show when the assumption $A_0$ was satisfied, and the red and blue plot markers show when the guarantees $G_0$ and $G_1$ were satisfied respectively.
We used the infrastructure of Mascot-SDS [26] to compute a 2/3-player game and to synthesize an almost sure winning controller for the product system. We performed the experiments on a computer with 3.3GHz Intel Xeon E5 v2 processor and 256 GB RAM. We used three different levels of discretization for the abstract state space for computing the 2/3-player game. The results are summarized in Tab. 1. We would like to highlight two key facts which came out of the experiments: (a) In all three cases, when we treated the noise in the worst case fashion, the synthesis process failed to provide us any controller, and (b) as we decreased the size of the abstract states (i.e., finer abstraction), the gap between the over and the under-approximation of the controller domain got monotonically smaller, which empirically confirms the intuition that the quality of the controller improves with finer abstraction.

We also visualize a couple of different simulations with the obtained controller in Figs. 3a–3b. We empirically show that whenever the assumption $A_0$ continues to hold recurrently, the $G_0$ and $G_1$ also hold recurrently. In contrary, when $A_0$ does not hold persistently, $G_0$ and $G_1$ also does not hold persistently. This empirically validates our claim that the synthesized controller is sound.

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7 APPENDIX

7.1 Proof of Statements

Proof of Thm. 3.2. By the definition of the winning set, we already know that $P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P})) = 1$ for all $s \in \text{WinDom}(\mathcal{S}, \rho)$. Take any $s \not\in W := \text{WinDom}(\mathcal{S}, \rho)$. Define $\tau$ to be the first time step when the path visits the set $W$. Note that $\tau$ is a random variable taking values in $\mathbb{N} \cup \{\infty\}$. We use the law of total probability by making the event $(\mathcal{S} \models Parity(\mathcal{P}))$ conditional on $\tau$. Then we have

$$P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P})) = \sum_{n=0}^{\infty} P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P}) | \tau = n)P_{\mathcal{S}}^0(\tau = n)$$

$$+ P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P}) | \tau = \infty)P_{\mathcal{S}}^0(\tau = \infty)$$

$$= \mathbb{E}_{\mathcal{S}}\left[ P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P}) | s^0, s^1, \ldots, s^n, \tau = n) \right]$$

$$+ P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P}) \land \mathcal{P} \models \square(s^0 \in W))$$

$$= \mathbb{E}_{\mathcal{S}}\left[ \sum_{n=0}^{\infty} P_{\mathcal{S}}^0(s^1, s^2, \ldots, s^{n-1} \in S \setminus W, s^n \in W) \right]$$

$$+ P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P}) \land \mathcal{P} \models \square(s^0 \in W))$$

$$\geq P_{\mathcal{S}}^0(\mathcal{S} \models \square W).$$

The equality (*) holds due to $s^n \in W$ and $P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P})) = 1$. □

Proof of Thm. 3.3. For any $s \in W$, we have

$$P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P})) = \int_S P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P}))T_s(ds_1[s, \rho(s)])$$

$$= \int_W T_s(ds_1[s, \rho(s)]) + \int_{S \setminus W} P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P}))T_s(ds_1[s, \rho(s)])$$

This means

$$\int_{S \setminus W} (1 - P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P})))T_s(ds_1[s, \rho(s)]) = 0$$

$$\Rightarrow \forall \epsilon > 0, P_{\mathcal{S}}^0(1 - P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P})))1_{S \setminus W}(s_1) \geq \epsilon \leq -\epsilon = 0,$$

where the last inequality is a consequence of Markov’s inequality for non-negative random variables. By taking the union over a monotone positive sequence $(\epsilon_n \to 0)$, we get

$$P_{\mathcal{S}}^0(1 - P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P})))1_S(s_1) > 0,$$

$$P_{\mathcal{S}}^0[s_1 \in S \setminus W \land P_{\mathcal{S}}^0(\mathcal{S} \models Parity(\mathcal{P})) < 0] = 0,$$

$$P_{\mathcal{S}}^0(s_1 \in S \setminus W) = 0.$$  □

7.2 The parity automaton for the specification used for the Dubin’s vehicle example

The states of the automaton can be encoded using a collection of Boolean variables and counters: At any given state, the variables $a_0, g_0, g_1$ denote the truth values of the respective atomic propositions $A_0, G_0, G_1$, the variables $\text{good}$ and $\text{bad}$ represent whether certain sink states have been reached upon violation of the safety components in the specification, and the counter $\text{counterG}$ will cycle through 0–1 upon seeing a $G_0$ followed by (not necessarily immediately) a $G_1$. The colors of the states are given in Fig. 4.

current state variable $a_0, g_0, g_1, \text{good}, \text{bad} \in \{\text{true, false}\}$,

guarantee counter $\text{counterG} \in \{0, 1, 2\}$

atomic proposition $AP = \{A_0, G_0, G_1, \text{Crash}\}$

init $a_0 = g_0 = g_1 = \text{false}$

transition

$\begin{aligned}
& a_0 = \text{false} \xrightarrow{A_0} a_0 = \text{true} \\
& g_0 = \text{false} \xrightarrow{G_0} g_0 = \text{true} \\
& g_1 = \text{false} \xrightarrow{G_1} g_1 = \text{true} \\
& \text{counterG} < 2 \xrightarrow{G_0} \text{counterG} = 1 \\
& \text{counterG} = 1 \xrightarrow{G_1} \text{counterG} = 2 \\
& \text{counterG} = 2 \xrightarrow{\text{true}} \text{counterG} = 0 \\
& a_0 = \text{true} \xrightarrow{\neg A_0 \land G_0} a_0 = \text{false} \\
& g_0 = \text{true} \xrightarrow{\neg G_0} g_0 = \text{false} \\
& g_1 = \text{true} \xrightarrow{\neg G_1} g_1 = \text{false} \\
& a_0 = \text{true} \land \text{good} = \text{false} \land \text{bad} = \text{false} \xrightarrow{\neg A_0 \land \neg G_1} \text{good} = \text{true} \\
& \text{good} = \text{false} \land \text{bad} = \text{false} \xrightarrow{\text{Crash}} \text{bad} = \text{true} \\
& \text{good} = \text{true} \xrightarrow{\text{true}} \text{good} = \text{true} \\
& \text{bad} = \text{true} \xrightarrow{\text{true}} \text{bad} = \text{true}
\end{aligned}$

color $\text{col} = 2$ if $\text{good} = \text{true}$ or $\text{counterG} = 2$

$\text{col} = 1$ if $\text{bad} = \text{true}$ or $a_0 = \text{true}$

$\text{col} = 0$ otherwise.

Fig. 4. The equivalent parity automaton for the specification of the robot in Eq. (17).