Self-similar orbit-averaged Fokker-Planck equation for isotropic spherical dense clusters (iii) Application to Galactic globular clusters

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Abstract Fitting parametric models to globular clusters’ structural profiles has been essential for the study of stellar dynamics. It provides their important structural parameters, such as the concentrations and core radii of the clusters. However, existing parametric models can apply only to non-collapsing-core clusters in the early relaxation-evolution stage. Hence, a single parametric model cannot provide globular clusters’ structural parameters in both the early and late evolution stages. We have recently found an accurate spectral solution for the self-similar orbit-averaged Fokker-Planck (OAFP) equation to model collapsing-core clusters at the late evolution stage. The present work establishes a new parametric model by combining the self-similar OAFP- and polytropic- models. Although it is a single-mass and isotropic model, the new model applies to at least fifty-five Galactic globular clusters with resolved cores in all the evolution stages. As a main result, we show the characteristics of the relaxation times against the concentrations of the clusters. We also affirm that the structures of low-concentration clusters are polytropic in the Milky Way.

Key words: (Galaxy:) globular clusters: general — Galaxy: kinematics and dynamics — methods: numerical

1 INTRODUCTION

In the late relaxation-evolution stage, standard stellar dynamics expect globular clusters to experience a self-similar evolution and core-collapse (e.g., Spitzer 1988; Heggie & Hut 2003; Binney & Tremaine 2011). The corresponding mathematical model is called the self-similar orbit-averaged Fokker-Planck (OAFP) equation (Heggie & Stevenson 1988). We recently found an accurate Gauss-Chebyshev spectral solution of the equation for isotropic, spherical clusters in the pre-collapse phase (Ito 2021). Based on the pre-collapse solution, the present paper proposes an energy-truncated self-similar OAFP (ss-OAFP) model. It reasonably fits the projected structural profiles of Galactic globular clusters, including collapsing-core and collapsed-core clusters, with resolved cores. In the rest of the present section, we review the applicability of fitting models to Galactic globular clusters. We primarily focus on the most fundamental parametric fitting model (the King model, King 1966) (Sect. 1.1) and time-dependent OAFP models (Sect. 1.2). We also explain why we may apply the pre-collapse solution of the ss-OAFP equation to even the post-collapse clusters (Sect. 1.3).

1.1 The Applicability of the King Model to Galactic Globular Clusters

The King model (King 1966) is the most fundamental parametric fitting model for globular clusters’ structures. It is a single-component, isotropic, spherical cluster model. The model can reasonably apply to clusters in the early relaxation-evolution stage. The fitting of the King model depends on three numerical parameters; the central projected density $\Sigma_c$, core radius $r_c$ and dimensionless central potential $K(=\varphi(r=0)/\sigma_c)$, where $\varphi(r=0)$ is the central cluster potential and $\sigma_c$ the central velocity dispersion. With only the three degrees of freedom, the King model adequately fits the structural profiles of approximately 80% of globular clusters in the Milky Way. The rest of the clusters are considered to be undergoing or have undergone a core-collapse at least once (Djorgovski & King 1986). If the King model fits a globular cluster’s structural profile well, then one
conventionally calls the cluster a ‘normal’ or ‘King-model’ (KM) cluster, otherwise a ‘post-collapsed-core’ or ‘post-core-collapse’ (PCC) cluster. There are three differences between the KM- and PCC- clusters. (i) The projected structural profile of a typical KM cluster flattens in the core while a typical PCC cluster has a cusp (a power-law projected density profile) of approximately $r^{-1}$, where $r$ is the distance from the center of the cluster. (ii) The concentrations $c$ of PCC clusters are high ($c \gtrsim 2.0$) while those of KM clusters are low ($0.7 \lesssim c \lesssim 1.8$) (see, e.g., Meylan & Heggie 1997). The concentration $c$ of a cluster is a possible measure to describe the dynamical state of the cluster and is defined by $\log \left( r_{\text{tid}}/r_c \right)$, where $r_{\text{tid}}$ is the tidal radius of the cluster. (iii) The outer halos of PCC clusters are more anisotropic than those of KM clusters because of the cores’ relaxation processes. Realistic outer halo structures depend not only on the effect of the relaxation process but also tidal effects from the Galaxy and other factors.

1 On one hand, numerical simulations (Giersz & Spurzem 1994; Takahashi 1995; Drukier et al. 1999) demonstrated that the anisotropy could be less significant in the core and inner halo due to the shorter relaxation time. Hence, one may expect that even isotropic models, like the ss-OAFP model, may reasonably apply to the central parts of globular clusters.

1.2 The Applicabilities of OAFP- and Other Models to PCC Clusters

Time-dependent spherical OAFP models may fit PCC clusters’ structural profiles. However, it is not an easy task to self-consistently solve a time-dependent OAFP equation coupled with Poisson’s equation. Generally, one applies a time-dependent OAFP model to a certain globular cluster as a case study to examine the detailed structure. Some examples of such clusters are NGC 6838 (Drukier et al. 1992), NGC 6397 (Drukier 1995) and NGC 7088 (Murphy et al. 2011). On the one hand, the King model builds only on the Poisson equation and is easy to solve for the potential of a spherical cluster. It has been preferably applied in homogeneous surveys to capture the common properties (the characteristic sizes of clusters and dynamical states) of as many globular clusters as possible. The surveys typically neglect the detailed information on each cluster. The concentration and core- and tidal radii obtained from the King model have been the fundamental structural parameters in the compilation works for globular-cluster studies (e.g., Peterson & King 1975; Trager et al. 1995; Noyola & Gebhardt 2006; Miocchi et al. 2013; Merafina 2017) and in the Harris catalog (Harris 1996, 2010 edition). To the best of our knowledge, there does not exist a single mass isotropic model only based on the Poisson equation that applies to both PCC- and KM- clusters due to their different core structures. For PCC clusters, one should employ a modified power-law profile (e.g., Lugger et al. 1995; Ferraro et al. 2003) or non-parametric model (e.g., Noyola & Gebhardt 2006). For KM clusters, one can utilize the single-component King model, its variants (e.g., Woolley 1961; Wilson 1975) and generalized models (e.g., Gomez-Leyton & Velazquez 2014; Gieles & Zocchi 2015). Although the present work focuses on a single mass model for simplicity, a multi-mass King model is known to fit some PCC clusters (King et al. 1995). There is no strict argument that rules out the multi-mass King model from a proper PCC cluster model (Meylan & Heggie 1997).

1.3 The Relationship of the Self-similar OAFP Model with PCC Clusters

We expect that the ss-OAFP model in the pre-collapse phase can fit PCC clusters with resolved cores. In principle, the ss-OAFP model applies only to globular clusters at the moment of complete core-collapse and approximately collapsing-core clusters in the late relaxation-evolution stage (Heggie & Stevenson 1988; Ito 2021). Also, the ss-OAFP model itself is unrealistic. In actual clusters, the cores’ densities can reach high enough to form binaries from single stars. The binaries release gravitational energy and halt the collapse before the cores develop an infinite central density. After the core-collapse holds, time-dependent- and self-similar-conductive gaseous models predict that the clusters successively repeat a core expansion (due to the energy released from binaries) and core-collapse (due to the relaxation process and self-gravity) (Sugimoto & Bettwieser 1983; Bettwieser & Sugimoto 1984; Goodman 1984, 1987). This process is called ‘gravothermal oscillation’ in the PCC phase since it manifests a nonlinear oscillation of the core density with time. Time-dependent OAFP models (Cohn et al. 1989; Murphy et al. 1990; Takahashi 1996) and $N$-body simulations (Makino 1996; Breen & Heggie 2012) also predict the same oscillation.

PCC clusters may form different structures in the post-collapse phase depending on binary stars and total stellar numbers $N$. A time-dependent conductive gaseous model (Sugimoto & Bettwieser 1983; Bettwieser & Sugimoto 1984) implied that the core structure is similar to the non-singular isothermal sphere (except at the moment of core-collapse) in the post-collapse phase. This feature contradicts the result of a self-similar gaseous model (Inagaki & Lynden-Bell 1983) that revealed a central-
cusp structure after a core-collapse. The latter result appears proper to model PCC clusters. However, the formation of a cusp in the core is conditional. Another self-similar gaseous model (Goodman 1984), including the mass-ejection effect from the core, examined the evolution on longer time scales compared to Inagaki & Lynden-Bell (1983). It also showed a cusp in the core. Moreover, the model clarified that the core radius gets smaller with an increasing stellar number $N$. On the one hand, the same model as Goodman (1984) confirmed that gravothermal oscillation could occur if the cluster had enough stars in it ($N \gtrsim 7 \times 10^3$), and more efficient binary heating was employed (Goodman 1987). In other words, efficient binary heating is necessary to produce a core like a non-singular isothermal sphere. To avoid unrealistically small cores, efficient binary heating with primordial binaries must also occur (Goodman & Hut 1989). These discussions infer that PCC clusters may have various core structures, such as a non-singular isothermal core, resolved core with a cusp or unresolved core. Especially, if binary heating is efficient, the structural profiles of both the gaseous model (Goodman 1987) and the OAFP model (Takahashi 1996) are similar in between the post-collapse and pre-collapse phases. There is no way to differentiate the structural profiles in the two phases only from observational data (Meylan & Heggie 1997) unless one acquires accurate kinematic data to see the temperature inversion. Hence, the ss-OAFP model can model some PCC clusters with resolved cores. This idea motivates us to apply the (pre-collapse) ss-OAFP model to PCC clusters.

The present paper’s purpose is to establish a parameteric model comparable with the King model. We propose an energy-truncated ss-OAFP model. It can apply to Galactic KM- and PCC- clusters with resolved cores reported in Kron et al. (1984), Dzorgovski & King (1986), Trager et al. (1995), Luger et al. (1995), Drukier et al. (1993), Ferraro et al. (2003) and Miocchi et al. (2013). Unfortunately, we did not have access to the data of Dzorgovski & King (1986); Luger et al. (1995); Miocchi et al. (2013). Hence, we employed WebPlotDigitizer (Rohatgi, Ankit 2019) to extract the data points and uncertainties of the projected structural profiles from their figures. The present paper is organized as follows. Section 2 introduces the energy-truncated ss-OAFP model that we applied to Galactic globular clusters’ projected structural profiles. Section 3 explains the result of fitting the new model to PCC clusters. Section 4 demonstrates the relationship between the completion rate of the core-collapse and concentration based on fitting our model to KM- and PCC- clusters. It also suggests that a polytropic model could fit low-concentration globular clusters in the Milky Way. Hence, Section 5 discusses whether the low-concentration clusters can have structures described by polytropic spheres of index $m$. Section 6 concludes this paper. For the sake of brevity, in Appendixes B and C, we show the majority of the projected structural profiles fitted by the energy-truncated ss-OAFP model.

2 ENERGY-TRUNCATED SS-OAFP MODEL

The present section introduces a new model, i.e., an energy-truncated ss-OAFP model. Section 2.1 highlights the relationship between the ss-OAFP model and the isothermal sphere first, and then explains the motivation for truncating the energy domain of the ss-OAFP model. Section 2.2 details the new model. The new model does not depend on dimensionless central potential $K$, unlike the King model. Hence, Section 2.3 explains how to regularize the new model’s concentration and core radius so that the structural parameters are comparable with those of the King model. Also, our model is composed of a polytrope of $m$ and the ss-OAFP model. Section 2.4 describes how we found an optimal value of $m$.

2.1 The Relationship between the ss-OAFP Model and the Isothermal Sphere

The ss-OAFP model can model KM clusters since it has a flat core like the isothermal sphere. Figure 1 depicts that the ss-OAFP model’s core has almost the same density profile as that of the isothermal sphere. For the figure, we rescaled the ss-OAFP model’s radius by multiplying by 3.739. This scaling makes the two models’ cores approximately the same in terms of size. This core structure infers that one may obtain a model similar to the King model at small radii by adequately truncating the energy domain of the ss-OAFP model. We discuss how to truncate the energy in Section 2.2.

2.2 Energy-truncated ss-OAFP Model

We energy-truncate the ss-OAFP model so that the new model’s outer halo behaves like a polytrope of $m$. Hence, the model is phenomenological, unlike the King model (King 1966). The King model counts the effects of the escaping stars or imitates the tidal effect from the Galaxy’s potential by truncating energies available.
A prescribed DF for the isothermal sphere and ss-OAFP model at dimensionless radius \( R \). The profiles were obtained by utilizing the codes from Ito et al. (2018) and Ito (2020, 2021).

Fig. 1 Dimensionless densities \( D(R) \) of the isothermal sphere and ss-OAFP model at dimensionless radius \( R \). The profiles were obtained by utilizing the codes from Ito et al. (2018) and Ito (2020, 2021).

...to stars. Simple arguments have explained the model’s physical origin based on the ‘test particle’ method of kinetic theories. The method assumes that particles (stars) follow a Maxwellian-Boltzmann distribution function (DF) (hereafter Maxwellian DF), but the test particle (star) does not have to. It explains that the King model is compatible with the isothermal sphere enclosed in a square well. Their relationship was studied for a stationary Fokker-Planck model (Spitzer & Harm 1958; Michie 1962; King 1965) and OAFP model (Spitzer & Shapiro 1972).

Also, the stellar DF for the King model is proportional to \( -E \rightarrow 0 \). This feature may be related to the asymptotic stationary solution of the OAFP model with a constant stellar flux at the fringe (Spitzer & Shapiro 1972). To obtain the King model (or lowered-Maxwellian DF), one must subtract the DF for polytrope of \( m = 2.5 \) from the Maxwellian DF

\[
F(E) = \begin{cases} 
\frac{\rho_c}{(2\pi\sigma_c^2)^{3/2}}I(K)(\exp[-E/\sigma_c^2] - 1), & (E < 0) \\
0, & (E > 0)
\end{cases}
\]

\[
I(K) = \exp(K)\text{erf}\left(\sqrt{K}\right) - \left(1 + \frac{2K}{3}\right)\sqrt{\frac{4K}{\pi}},
\]

where the parameter \( K \) and the error function \( \text{erf}(x) \) read

\[
K = -\frac{\varphi(r = 0)}{\sigma_c^2},
\]

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}}\exp\left(x^2\right)\int_0^x\exp\left(-y^2\right)dy.
\]

A prescribed DF \( F(E) \) provides the mean-field (MF) potential \( \varphi(r) \) through Poisson’s equation

\[
\frac{d^2\varphi}{dr^2} + \frac{2}{r} \frac{d\varphi}{dr} = 4\pi GD(\varphi)
\]

\[
\equiv 16\pi^2G \int_{\varphi(r)}^{0} F(E)\sqrt{2[E - \varphi(r)]}dE',
\]

where \( G \) is the gravitational constant.

Our new model incorporates the escaping stars’ effect in the same way as the King model by truncating high energies available to stars. However, the mathematical operation for combining DFs differs from that of the King model. Our model adds the DFs up for a polytrope of \( m \) and the ss-OAFP model as follows

\[
F(E) = \begin{cases} 
\frac{\rho_c}{4\sqrt{2\pi}\sigma_c^3}D_o(\varphi = -1) + \delta B(m - 1/2, 3/2), & (E > 0) \\
0, & (E < 0)
\end{cases}
\]

\[
(D_o(\varphi = -1) + \delta B(m - 1/2, 3/2))^{-1}
\]

where \( \delta \) and \( m \) are positive real numbers, \( D_o(\varphi) \) is the density of the ss-OAFP model, and \( B(a, b) \) is the beta function defined as \( B(a, b) = 2\int_0^1 t^{a-1}(1 - t)^{b-1}dt \) with \( a > 1/2 \) and \( b > 1 \). The factor \( (D_o(\varphi = -1) + \delta B(m - 1/2, 3/2))^{-1} \) is inserted in the DF so that the density profile for the DF \( F(E) \) has a certain central density \( \rho_c \) as \( R \rightarrow 0 \). In Equation (2.4), \( F_o(E) \) is the DF for the ss-OAFP model and is proportional to \((-E)^{b-1/2}\) as \( E \rightarrow 0 \) (Ito 2021). On the one hand, we limit \( m \) to be less than 5, which results in \( F(E) \propto (-E)^{m-3/2} \) as \( E \rightarrow 0 \).

Our model can be finite in size by this limit, following the discussion of Chandrasekhar (1939). Hence, a polytrope of \( m \) controls the model’s outer halo. Also, the new DF (Eq. (2.4)) behaves like the ss-OAFP model’s DF beyond the order of \( \delta \), but it is approximately like a polytropic sphere’s DF below \( \delta \).

We employ a polytrope of \( m \) to truncate the energy in Equation (2.4) since a simple argument cannot generally provide an explicit form of energy truncation. The tidal effect on globular clusters significantly depends on where they orbit in the Milky Way. Also, more realistic effects (mass spectrum, collisionless relaxation and dark matter) could have affected the current outer halos’ structures. Hence, one cannot universally determine their structures. For example, the King model does not reasonably fit all the KM clusters’ outer halos. The Wilson model (Wilson 1975) and Woolley model (Woolley 1961) can fit some clusters’ structural profiles better than the King model. The Wilson model’s outer halo behaves like a polytrope of \( m = 3.5 \), while that of the Woolley model is approximately the polytrope of \( m = 1.5 \) in the limit of \( K \rightarrow 0 \). Using more different values of \( m \) provides a more flexible fitting to outer halos. This generalization was carried out in the truncated \( \gamma \) exponential (fractional-power) model proposed in Gomez-Leyton & Velazquez (2014). Our model has the same parameter-dependence as the model. Of course, our purpose is to build a parametric model comparable with the King model. Hence, we must first specify the value of \( m \) for our model based on physical arguments (with numerical experiments) and observational
The numerical values of the Chebyshev coefficients are given in Appendix A. The dimensionless form of the truncated model reads

$$D(\varphi) = \rho_c D_0(\varphi) + \delta B(m - 1/2, 3/2) (-\varphi)^m \overline{D}_o(\varphi = -1) + \delta B(m - 1/2, 3/2).$$  

The density of the ss-OAFP model in a Chebyshev-expansion form was obtained from Ito (2021) as follows

$$D_o(\varphi) = (-\varphi)^{\beta+3/2} \sum_{n=1}^{63} D_n T_{n-1}(\varphi),$$  

$$\beta = 8.1783711596581004,$$

where $$T_n$$ is the Chebyshev polynomials of the first kind. The numerical values of the Chebyshev coefficients $$\{D_n\}$$ are given in Appendix A. The dimensionless form of the Poisson’s Equation (2.3) reads

$$\frac{d^2 \varphi}{dr^2} + 2 \frac{d \varphi}{r \, dr} = \bar{D}(\varphi)$$

$$= \frac{D_o(\varphi) + \delta B(m - 1/2, 3/2) (-\varphi)^m}{D_o(\varphi = -1) + \delta B(m - 1/2, 3/2)},$$

where the potential $$\varphi(r)$$, radius $$r$$ and density $$\rho(r)$$ are defined in dimensionless form using

$$\varphi(r) = -\frac{\varphi(r)}{\sigma^2_c},$$  

$$\bar{r} = r \left( \frac{4\pi G \rho_c}{\sigma^2_c} \right),$$  

$$\bar{D}(r) = \frac{D(r)}{\rho_c},$$

where the variables with subscript $$c$$ correspond to the time-dependent variables in the self-similar analysis (Heggie & Stevenson 1988; Ito 2021). The boundary conditions for Poisson’s Equation (2.7) are

$$\varphi(\bar{r} = 0) = 1, \quad \frac{d \varphi}{d \bar{r}}(\bar{r} = 0) = 0.$$  

The dimensionless potential $$\varphi$$ is an independent variable in the ss-OAFP model (Heggie & Stevenson 1988; Ito 2021). Hence, following the method of inverse mapping (Ito et al. 2018; Ito 2021), we must solve the following Poisson’s equation in its inverse form for $$R(\varphi)$$

$$R \frac{d^2 R}{d \varphi^2} - 2 \left( \frac{d R}{d \varphi} \right)^2 = \left( \frac{d R}{d \varphi} \right)^3 \times \frac{D_o(\varphi) + \delta B(m - 1/2, 3/2) (-\varphi)^m}{D_o(\varphi = -1) + \delta B(m - 1/2, 3/2)}.$$  

The numerical integration of the Poisson’s equation provided the density profile (Fig. 2) and MF potential (Fig. 3) for an optimal index $$m = 3.9$$. (We explain the reason why $$m = 3.9$$ is an optimal choice in Sect. 2.4.) In the figures, the value of $$\delta$$ spans $$10^{-5}$$ through $$10^5$$. For large $$\delta$$ ($$> 1$$), the profiles appear to be the same regardless of the value of $$\delta$$. They behave like the polytrope of $$m = 3.9$$. On the one hand, the profile is similar to the ss-OAFP model for small $$\delta$$ ($$ \lesssim 10^{-2}$$).

In applying the density $$\bar{D}(\varphi)$$ to globular clusters, one needs to convert $$\bar{D}(\varphi)$$ to the projected density profile

$$\Sigma(r) = 2 \int_0^\infty \frac{D(\varphi)}{\sqrt{1 - (r/r')^2}} dr'.$$

The corresponding inverse form with dimensionless variables is

$$\Sigma(\varphi) = -2 \int_0^{\varphi} \frac{1 - \mu_R(\varphi, \varphi')}{1 + \mu_R(\varphi, \varphi')} \left[ -2 \frac{d D}{d \varphi} \frac{R(\varphi')}{S(\varphi')} + D(\varphi') S(\varphi') \frac{1 + 2 \mu_R(\varphi, \varphi')}{{1 + \mu_R(\varphi, \varphi')}} \right] d \varphi',$$

where $$\mu_R(\varphi, \varphi') \equiv R(\varphi)/R(\varphi')$$ and $$S \equiv -dR/d\varphi$$. Figure 4 depicts the projected density profiles for different $$\delta$$. As $$\delta$$ decreases, the power-law profile $$R^{-1.23}$$ develops.
more clearly in the inner halo (as expected from the asymptotic density profile of the ss-OAFP model, i.e., $D \propto R^{-2.23}$ as $R \to \infty$). This power-law profile appears at radii between $R \sim 10$ and $R \sim 100$ for $\delta = 10^{-4}$. In addition, one can find a similar power-law profile for larger $\delta$. For $\delta = 10^{-2}$ and $10^{-3}$, $\Sigma$ shows power-law-like structures $R^{-1.0} \sim R^{-1.1}$ at radii between $R \sim 1$ and $R \sim 10$. This property is desirable to fit our model to PCC clusters’ structural profiles. PCC clusters with resolved cores have similar power-law profiles near the core (e.g., Djorgovski & King 1986; Lugger et al. 1995).

2.3 Regularization for the Concentration and King Radius

The energy-truncated ss-OAFP model differs from the King model in the sense of how concentrations and core radii depend on the dimensionless central potential $K$. We must properly regularize the structural parameters for the sake of comparison. By plugging the lowered-Maxwellian DF (Eq. (2.1)) into the Poisson’s Equation (2.3) and employing the dimensionless variables (Equations (2.8a) – (2.8c)) with new variable $\bar{\varphi} \equiv \varphi / K$, one can reduce the King model to

$$\frac{d^2 \bar{\varphi}}{d\bar{r}^2} + \frac{2}{\bar{r}} \frac{d\bar{\varphi}}{d\bar{r}} = \frac{1}{K} \frac{I(\bar{\varphi}^m)}{I(1)} = 0,$$  

(2.13)

with the boundary conditions $\bar{\varphi}(\bar{r} = 0) = 1$ and $\frac{d\bar{\varphi}}{d\bar{r}}(\bar{r} = 0) = 0$. Due to the $K$-dependence of the equation, $c \to 0$ as $K \to 0$. Of course, one can make Equation (2.13) independent of $K$ by further regularizing the radius $\bar{r}$ as

$$\bar{r} = r / \sqrt{K}.$$  

(2.14)

Then, the King model’s tidal radius is equal to that of polytrope of $m = 2.5$, that is, 5.355275459 (e.g., Boyd 2011) regardless of the value of $K$. On the one hand, the energy-truncated ss-OAFP model does not depend on $K$ since the dimensionless central potential $\bar{\varphi}(\bar{r} = 0)$ is unity (Eq. (2.9)), unlike the King model. One may like to find the same order of the structural parameters as the King model. To do so, one can regularize the structural parameters$^4$ as follows

$$\bar{r}_{\text{Kin}} \equiv \frac{r_{\text{Kin}}}{\sqrt{K^{(m)}}},$$  

(2.15a)

$$\bar{c} \equiv \log \left( \frac{r_{\text{tid}}}{r_{\text{Kin}}^{(m)}} \sqrt{K^{(m)}} \right),$$  

(2.15b)

where $K^{(m)}$ is the central potential, $r_{\text{Kin}}$ the King (core) radius and $r_{\text{tid}}$ the tidal radius of the energy-truncated ss-OAFP model. Equation (2.15) reduces Equation (2.7) to

$$\frac{d^2 \bar{\varphi}}{d\bar{r}^2} + \frac{2}{\bar{r}} \frac{d\bar{\varphi}}{d\bar{r}} - \bar{D}(\bar{\varphi})^{m} = 0,$$  

(2.16)

which has the same $K$-dependence as Equation (2.13).

Using Equation (2.15), one can obtain the concentration of the energy-truncated ss-OAFP model. Strictly speaking, the value of $K^{(m)}$ depends on both $m$ and $\delta$ since we would like to determine the value of $K^{(m)}$ so that the ss-OAFP model’s structural parameters are the same order as those of the King model. To reduce the complexity of determining $K^{(m)}$, we resort to the property that our model’s outer halo builds on a polytrope. We consider our model’s size to be approximately equal to that of the polytrope of $m$. For a specific value of $m$, we can calculate the tidal radius $r_{\text{tid}}^{(m)}$ of the polytrope through the Lane-Emden equation of the first kind (Chandrasekhar 1939)

$$\frac{d^2 \bar{\varphi}}{d\bar{r}^2} + \frac{2}{\bar{r}} \frac{d\bar{\varphi}}{d\bar{r}} = (-\bar{\varphi})^m = 0,$$  

(2.17)

with the boundary conditions (Eq. (2.9)). Then, we must solve the King model (Eq. (2.13)) and find the value of

$^4$ One should not regularize the tidal radius of the energy-truncated ss-OAFP model since it is still the radius at which the projected density reaches zero numerically or on a graph. After adequately fitting the model to a globular cluster’s projected structural profile, one can find the tidal radius.
Table 1 displays the results obtained by fitting our model. It provided the same order of the structural parameters obtained from our model are reasonably close to their statistical analyses. However, the structural parameters depend on different instruments, photometry methods and compilation works are inhomogeneous surveys in the sense that they chose to compare this time. Many of the compilation works based on the King model (Peterson (1984). Only the six clusters were reported in all the compilation works following Spitzer & Shapiro (1972), though it was not the case. The values in 3.5 ≤ m ≤ 4.4 were useful. Among the values of m, we chose 3.9 as the optimal value in the present work. It provided the same order of the structural parameters as that of the existing works based on the King model. Table 1 displays the results obtained by fitting our model with m = 3.9 to the projected surface densities of six Milky-Way globular clusters reported in Kron et al. (1984). Only the six clusters were reported in all the compilation works based on the King model (Peterson & King 1975; Kron et al. 1984; Chernoff & Djorgovski 1989; Trager et al. 1993; Micocchi et al. 2013) that we chose to compare this time. Many of the compilation works are inhomogeneous surveys in the sense that they depend on different instruments, photometry methods and statistical analyses. However, the structural parameters obtained from our model are reasonably close to their results. On the one hand, when we chose m ≤ 3.8 or 4.3 ≤ m for our model’s fitting, the structural parameters’ orders are approximately ten times less or greater than those of the compilation works. Interestingly, our tidal radii are close to those obtained from the King model rather than the Wilson model (Table 1). The Wilson model’s index m is 3.5 in the limit of K → 0 and greater than that of the King model. A higher index m provides a larger tidal radius of the polytropic sphere (e.g., Chandrasekhar 1939). The reason why our model with the high m (= 3.9) does not overestimate the tidal radii would be that the density profile of the ss-OAFP model more rapidly decays compared to the isothermal sphere in the inner halo (Fig. 1). Refer to Appendix B to find the energy-truncated ss-OAFP model with m = 3.9 reasonably fitted to the projected structural profiles of 39 KM clusters reported in Kron et al. (1984); Micocchi et al. (2013).

Another reason why we chose m = 3.9 is that the energy-truncated ss-OAFP model with m = 3.9 agreeably fits the relatively new data for NGC 6752 reported in Ferraro et al. (2003). The result of Ferraro et al. (2003) provided data and errors of the projected surface density for NGC 6752. Their data were convenient for us to test our model since we did not need to artificially extract data from their graph. The King model does not fit the central part of the projected density profile well in their work since the cluster is one of the (possible) PCC clusters with a power-law profile near the core. Hence, following Lugger et al. (1995), they employed a modified power-law profile Σ ∝ (1 + (R/3.1)2)−0.525, where R is measured in log [arcsec]. This profile fits the central part well, as shown in Figure 6 (top left). On the one hand, our model with m = 3.0 does not match the cluster’s profile at all on the figure. However, our model more reasonably fits the same data with a greater m. The model with m = 4.2 best fits the data except in the tail of the cluster. Even with m = 3.9, one can find a good fit to the data. Hence, the present work chose m = 3.9 as the optimal value to consistently accumulate the data for both KM clusters and PCC clusters.

2.4 An Optimal Value of the Polytropic Index: m = 3.9

We determined the index m to be 3.9 in the energy-truncated ss-OAFP model after having preliminarily applied the model to the projected surface densities of six KM clusters and a PCC cluster that we chose. Initially, we expected that m = 2.5 could be an optimal choice for the energy-truncated ss-OAFP model following Spitzer & Shapiro (1972), though it was not the case. The values in 3.5 ≤ m ≤ 4.4 were useful. Among the values of m, we chose 3.9 as the optimal value in the present work. It provided the same order of the structural parameters as that of the existing works based on the King model. Table 1 displays the results obtained by fitting our model with m = 3.9 to the projected surface densities of six Milky-Way globular clusters reported in Kron et al. (1984). Only the six clusters were reported in all the compilation works based on the King model (Peterson & King 1975; Kron et al. 1984; Chernoff & Djorgovski 1989; Trager et al. 1993; Micocchi et al. 2013) that we chose to compare this time. Many of the compilation works are inhomogeneous surveys in the sense that they depend on different instruments, photometry methods and statistical analyses. However, the structural parameters obtained from our model are reasonably close to their results. On the one hand, when we chose m ≤ 3.8 or 4.3 ≤ m for our model’s fitting, the structural parameters’ orders are approximately ten times less or greater than those of the compilation works. Interestingly, our tidal radii are close to those obtained from the King model rather than the Wilson model (Table 1). The Wilson model’s index m is 3.5 in the limit of K → 0 and greater than that of the King model. A higher index m provides a larger tidal radius of the polytropic sphere (e.g., Chandrasekhar 1939). The reason why our model with the high m (= 3.9) does not overestimate the tidal radii would be that the density profile of the ss-OAFP model more rapidly decays compared to the isothermal sphere in the inner halo (Fig. 1). Refer to Appendix B to find the energy-truncated ss-OAFP model with m = 3.9 reasonably fitted to the projected structural profiles of 39 KM clusters reported in Kron et al. (1984); Micocchi et al. (2013).

Another reason why we chose m = 3.9 is that the energy-truncated ss-OAFP model with m = 3.9 agreeably fits the relatively new data for NGC 6752 reported in Ferraro et al. (2003). The result of Ferraro et al. (2003) provided data and errors of the projected surface density for NGC 6752. Their data were convenient for us to test our model since we did not need to artificially extract data from their graph. The King model does not fit the central part of the projected density profile well in their work since the cluster is one of the (possible) PCC clusters with a power-law profile near the core. Hence, following Lugger et al. (1995), they employed a modified power-law profile Σ ∝ (1 + (R/3.1)2)−0.525, where R is measured in log [arcsec]. This profile fits the central part well, as shown in Figure 6 (top left). On the one hand, our model with m = 3.0 does not match the cluster’s profile at all on the figure. However, our model more reasonably fits the same data with a greater m. The model with m = 4.2 best fits the data except in the tail of the cluster. Even with m = 3.9, one can find a good fit to the data. Hence, the present work chose m = 3.9 as the optimal value to consistently accumulate the data for both KM clusters and PCC clusters.

3 FITTING OF THE SS-OAFP MODEL TO PCC CLUSTERS

The energy-truncated ss-OAFP model with m = 3.9 reasonably fits the projected structural profiles of PCC clusters with resolved cores at R ≲ 1 arcmin. Other than NGC 6752, we also had access to the numerical data of the projected density profile for NGC 6397 from Drukier (1995). Our model reasonably fits the density profile of NGC 6397 with χ2 = 1.52 (Fig. 7) where the reduced chi-square is defined as follows

\[
\chi^2 = \sum \frac{\bar{\rho}^2}{\sigma^2},
\]

where \(\bar{\rho}\) is the projected density, and \(\sigma\) is the error of the projected density.
where \( \chi^2 \) is the chi-square value between the observed data and our model, and \( n_{\text{d.f.}} \) is the number of degrees of freedom. We chose \( n_{\text{d.f.}} = 3 \) in the same way as the King model since we fixed index \( m \) of our model to \( 3.90 \). The only fitting parameters of our model are \( \Sigma_c \), \( r_c \) and \( \delta \). Unfortunately, we did not have access to numeric values for the rest of the projected structural profiles of PCC clusters reported in Djorgovski & King (1986) and Lugger et al. (1995). Hence, our error analysis becomes less trustworthy hereafter. However, it appears enough to capture the applicability of our model for the PCC clusters. For example, a review work (Meylan & Heggie 1997) introduced NGC 6388 and Terzan 2 as examples of a KM cluster and PCC cluster by citing the clusters’ surface brightness profiles from Djorgovski & King (1986). The energy-truncated ss-OAFP model reasonably fits both the density profiles at radii \( R \lesssim 1 \) arcmin (Fig. 8). In addition to NGC 6752 and NGC 6397, we applied our model to thirteen PCC clusters with resolved and unresolved cores reported in Djorgovski & King (1986) and Lugger et al. (1995) (see Appendix C for the results).

Table 2 lists the values of \( \chi^2_{\nu} \) for both the KM- and PCC- clusters for which we could obtain uncertainties in the observed densities from the numeric values or graphs. The result affirms that the ss-OAFP model adequately fits the KM clusters’ profiles at all the data points given. On the one hand, the model fits only the PCC clusters with resolved cores reported in Lugger et al. (1995). For example, our model fits PCC clusters with partially-resolved cores (NGC 6453, NGC 6522 and NGC 7099) and resolved cores (NGC 6397 and NGC 6752) at \( R \lesssim 1 \) arcmin with \( \chi^2_{\nu} \lesssim 2 \). It even reasonably applies similarly to a PCC cluster with an unresolved core (NGC 6342), though the present work does not count the ‘seeing-effect’ that comes from the seeing-disk’s finiteness. However, the

### Table 1

Concentrations and core- and tidal- radii obtained by fitting the energy-truncated ss-OAFP model to six KM clusters. The structural parameters are compared to the previous compilation works based on the King- and Wilson-models.

| Cluster     | KM         | Wilson         | Wilson         |
|-------------|------------|----------------|----------------|
| NGC 1904    | 1.86 0.191 | 1.24 0.410     | 1.54 0.779     |
| NGC 2419    | 1.65 0.13  | 1.45 0.178     | 1.83 0.116     |
| NGC 6205    | 1.72 0.159 | 1.50 0.125     | 1.75 0.118     |
| NGC 6229    | 2.14 0.18  | 1.45 0.178     | 1.75 0.178     |
| NGC 6341    | 1.45 0.178 | 1.65 0.13  | 1.83 0.116     |
| NGC 6397    | 1.74 0.243 | 1.65 0.13  | 1.83 0.116     |
| NGC 6752    | 1.74 0.243 | 1.65 0.13  | 1.83 0.116     |
| NGC 6864    | 2.14 0.18  | 1.45 0.178     | 1.75 0.118     |

For the fitting of our model, we first determined the value of \( \delta \). We then numerically integrated Equation (2.16) with the fixed \( \delta \). We fitted the computed projected-density profile to the observed data by adjusting the values of \( \Sigma_c \) and \( r_c \) so that \( \chi^2_{\nu} \)-value reached its minimum for the fixed \( \delta \). We repeated these steps for different \( \delta \) until we found the minimum of \( \chi^2_{\nu} \) for any \( \delta \).
model does not fit the rest of PCC clusters’ structural profiles with unresolved cores (e.g., NGC 5946 and NGC 6624). It does not even only fit the observed cores since the cores have steeper power-law profiles than our model. This failure was expected since our model does not correctly count the binaries’ effect, as explained in Section 1.3. Also, we do not assess that our (single-component) model is the best model for PCC clusters. PCC clusters are supposed to have experienced mass segregation. Hence, we need to incorporate the mass spectrum’s effect in our model and would be able to improve the fitting, as done for the King model (Costa & Freeman 1976).

4 MAIN RESULT: RELAXATION TIME AND COMPLETION RATE OF CORE COLLAPSE AGAINST CONCENTRATION

Concentration \( \bar{c} \) is a possible measure to characterize the globular clusters’ states in the relaxation evolution, especially for the cores. The present section compares \( \bar{c} \) to the core relaxation time and completion rate of core-collapse. The energy-truncated ss-OAFP model can reasonably apply not only to KM clusters (Appendix B) but also to PCC clusters (Appendix C). One may systematically discuss their relationship. We first discuss how the core relaxation time depends on the concentration. Figure 9(a) depicts the characteristics of the core relaxation
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Fig. 7 Fitting the energy-truncated ss-OAFP model \((m = 3.9)\) to the projected surface density of NGC 6397 reported in Drukier et al. (1993). The unit of \(\Sigma\) is stellar number per arcminute squared, and \(\Sigma\) is normalized so that the density is unity at the smallest radius for data. In the legends, (c) means a PCC cluster as judged in Djorgovski & King (1986). \(\Delta \log[\Sigma]\) is the corresponding deviation of \(\Sigma\) from the model on log scale.

Fig. 8 Fitting of the energy-truncated ss-OAFP model \((m = 3.9)\) to the surface brightness profiles of Terzan 2 and NGC 6388 reported in Djorgovski & King (1986). The unit of surface brightness (SB) is B magnitude per arcsecond squared. The brightness is normalized by the magnitude SB\(_{o}\) observed at the smallest radius point. In the legends, (n) means a normal or KM cluster and (c) means a PCC cluster as judged in Djorgovski & King (1986). \(\Delta(SB\_o - SB)\) is the corresponding deviation of \((SB\_o - SB)\) from the model.

time \(t_{c.r}\) against the concentration \(\bar{c}\). Figure 9(b) displays the corresponding characteristics based on the King model reported in the Harris catalog (Harris 1996, 2010 edition). All the relaxation times on both the figures are the values reported for PCC- and KM- clusters in the catalog (Harris 1996, 2010 edition). The catalog features some concentrations as ‘2.50c’ for clusters whose projected structural profiles are not well fitted by the King model. Hence, we assumed that the concentrations of the clusters are 2.50 in Figure 9(b). In the figure, the relaxation time decreases with increasing concentration \(c\) for KM clusters. However, it is not clear if the PCC clusters have the same tendency. On the one hand, Figure 9(a) shows not only that the relaxation time decreases with concentration \(\bar{c}\) for KM clusters but also that the time drops down almost vertically for PCC clusters when the relaxation time is long. This tendency captures the feature of PCC clusters well. Their projected profiles can be close to the ss-OAFP model near the moment of a core-collapse. They are still similar to the King model (KM clusters) in the expansion phase after a core-collapse.

We compare our and the King model using the completion rate \(\eta_c\) of core-collapse. To find the value of \(\eta_c\), we rely on the formula employed in Lightman (1982)

\[
\eta_c \equiv \frac{t_{\text{age}}}{t_{\text{c.r.o}}} = -\left(1 + Aq_o\right) + \sqrt{\left(1 + Aq_o\right)^2 + 4ABq_o^2} \cdot \frac{2B}{\bar{c}}.
\]  

(4.1)
Table 2  Values of $\chi^2_\nu$ between the energy-truncated ss-OAFP model and observed structural data for Galactic globular clusters. The data we considered for the KM clusters are from Miocchi et al. (2013), for NGC 6397 from Drukier et al. (1993), for Terzan 2 from Djorgovski & King (1986), for NGC 6752 from Ferraro et al. (2003) and for the rest of the PCC clusters from Lugger et al. (1995). $N_b$ is the number of data points at large radii excluded from the calculation.

| KM cluster | $\chi^2_\nu$ | $N_b$ |
|------------|--------------|------|
| NGC 288    | 0.45         | 0    |
| NGC 1851   | 0.56         | 0    |
| NGC 5466   | 2.07         | 0    |
| NGC 6121   | 0.72         | 0    |
| NGC 6205   | 1.05         | 0    |
| NGC 6254   | 0.57         | 0    |
| NGC 6626   | 0.47         | 0    |
| NGC 6809   | 0.44         | 0    |
| Pal 3      | 0.06         | 0    |
| Pal 4      | 0.34         | 0    |
| Pal 14     | 0.31         | 0    |
| Trz 5      | 2.23         | 0    |

| PCC cluster | $\chi^2_\nu$ | $N_b$ |
|-------------|--------------|------|
| NGC 6342    | 1.73         | 3    |
| NGC 6397    | 1.52         | 0    |
| NGC 6453    | 1.89         | 5    |
| NGC 6522    | 2.52         | 5    |
| NGC 6558    | 2.17         | 5    |
| NGC 6752    | 2.00         | 6    |
| NGC 7099    | 2.12         | 2    |
| Trz 1       | 2.41         | 5    |
| Trz 2       | 1.94         | 0    |

### Fig. 9
Core relaxation time against (a) concentration $\bar{c}$ obtained from the energy-truncated ss-OAFP model ($m = 3.9$) and (b) concentration $c$ based on the King model reported in Harris (1996, 2010 edition).

where $A = 35$, $B = 4.8$ and $\eta_0 = t_{o,age}/\xi_c$. The time $t_{o,age}$ is the order of globular clusters’ ages, $\sim 10^{10}$ years. The time $t_{c,xo}$ is the estimated relaxation time at the beginning of each cluster’s relaxation evolution based on an $N$-body simulation. Figure 10(a) plots the completion rate against concentration $\bar{c}$ obtained from the energy-truncated ss-OAFP model. The majority of data plots lie within the region between two lines $\eta_c = 0.75(\bar{c} - 2.0) + 1.05$ and $\eta_c = 0.75(\bar{c} - 2.0) + 0.40$ that are empirical lines of the equations, not based on physical arguments. Figure 10(b) shows that the corresponding characteristics of $\eta_c$ against concentration $c$ based on the King model, and the same two lines reasonably include most data plots between them. The horizontal difference between the two lines is approximately unity in both Figures 10(a) and (b). Recalling the logarithmic dependence of the concentration, we assess that our model can provide the same order of structural parameters as the King model.

From Figure 10(a), we can find several results. (i) Clusters with $c > 2.0$ are PCC clusters, as explained...
5 DISCUSSION: ARE LOW-CONCENTRATION GLOBULAR CLUSTERS LIKE SPHERICAL POLYTROPES?

The present section discusses the relationship between the polytropic sphere and low-concentration ($\bar{c} = 1$) globular clusters. The result of Section 4 shows that some clusters (e.g., Palomar 3 and Palomar 4) have concentrations $\bar{c}$ close to one. This result indicates that the polytropic spheres of $m \approx 3.9$ can model the low-concentration clusters’ structural profiles. Our interest in the present section is to see whether the projected structural profiles of low-concentration globular clusters are polytropic. We computed the density profile of the polytropic sphere of $m$ using Equation (2.17). Appendix D features the fitting of polytropic-sphere models to the globular clusters’ structural profiles reported in Kron et al. (1984), Trager et al. (1995) and Miocchi et al. (2013). We found that the model could fit the projected structural profiles of 18 low-concentration globular clusters well. In the present section, we showcase the results for NGC 288 and NGC 6254 as examples (Fig. 11). Their concentrations are 1.30 and 1.64 based on the energy-truncated ss-OAFP model, while those of the King model are 1.0 and 1.41. NGC 288 is a good example of a polytropic globular cluster, while NGC 6254 is an example of a non-polytropic cluster. Figure 12 also depicts another example of a polytropic globular cluster (NGC 5139) whose surface brightness profile was reported in Meylan (1987). The central part of the cluster data deviates from the polytrope model due to the weak cusp, though the polytropic sphere fits the inner- and outer- halos well. In the rest, we first discuss that many-body relaxation may bring the low-concentration globular clusters into structures like polytropic spheres (Sect. 5.1). We then demonstrate that the many-body relaxation effect could characterize polytropic globular clusters well (Sect. 5.2). Lastly, we criticize the concept of polytropic globular clusters (Sect. 5.3).

5.1 Discussion of Why Low-concentration Cluster Structures Are Like Polytropes

The low-concentration cluster cores are in non-equilibrium states, possibly modeled by polytropic spheres rather than a state of (local) thermodynamic equilibrium, if mass loss from the clusters is less significant. The results of Section 4 and Appendix B indicate that both the King- and energy-truncated-ss-OAFP- models reasonably fit the projected structural profiles of low-concentration globular clusters in the Milky Way. These results imply that the low-concentration cluster cores are well relaxed. However, it is not evident whether their DFs have already reached a local Maxwellian DF since their core-relaxation times $t_{c.r.}$ are long ($\gtrsim 1$ Gyr; see, e.g., Harris 1996, 2010 edition). Also, even the cores of star clusters undergoing a complete core-collapse cannot reach a local Maxwellian DF in principle (Ito 2020).

In the initial relaxation-evolution stage, the cores of globular clusters can be described by a non-Maxwellian DF because of the non-dominant (many-body-relaxation) effect. According to standard stellar dynamics (e.g., Spitzer 1988; Heggie & Hut 2003; Binney & Tremaine 2011), two-body relaxation drives the evolution of globular clusters. The two-body relaxation was originally introduced as the ‘dominant effect’ of evolution in Chandrasekhar (1943). During the two-body relaxation evolution, a star can ‘encounter’ another star when their distance is as large as the mean stellar distance ($n^{-1/3} \sim$...)

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9 We obtained the concentrations in Fig. 11 by applying the same regularization explained in Sect. 2.3.

10 We consider the concentration of NGC 288 to be 1.0 based on our fitting of the King model. We confirmed the concentrations $c$ and values of $\chi_r$ reported in Table 2 of Miocchi et al. (2013) based on our calculation, but not for NGC 288. We found the same result ($\chi_r = 1.7$ with $W_o = 5.8$) for NGC 288 as their result. However, we found that the concentration for NGC 288 was $c = 1.0$ for $W_o = 5.0$ that provided $\chi_r = 0.48$. This value is smaller than their value $\chi_r = 1.7$ with $W_o = 5.8$ and close to unity.

11 The encounter causes the deflection of a star’s path from its original path determined by the MF potential. The deflection is due to pair-wise Newtonian interaction between the two stars.
Fig. 10 Completion rate of core-collapse against (a) concentration $\bar{c}$ based on the energy-truncated ss-OAFP model with $m = 3.9$ and (b) concentration $c$ based on the King model reported in Harris (1996, 2010 edition).

Fig. 11 Fitting of the polytropic sphere of index $m$ to the projected densities $\Sigma$ of NGC 288 and NGC 6254 reported in Miocchi et al. (2013). The unit of $\Sigma$ is number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius data point. In the legends, (n) means a ‘normal’ or KM cluster as judged in Djorgovski & King (1986). \( \Delta \log[\Sigma] \) is the corresponding deviation of $\log[\Sigma]$ from the model.

$N^{-1/3}$, where $n$ is the mean stellar density) and as small as the order of $r_{\text{tid}}/N (\sim 1/N)^{12}$. Hence, classical stellar-dynamics theory expects the two-body relaxation to not occur between two distant stars. However, direct $N$-body simulations (e.g., Aarseth & Heggie 1998) have assessed that two-body relaxation could be not enough to quantify the evolution. One must include the non-dominant (many-body-relaxation) effect. However, the mathematical
Fig. 12 Fitting of the polytropic sphere of index $m$ to the surface brightness profile of NGC 5139 reported in Meylan (1987). The unit of the surface brightness (SB) is V magnitude per arcminute squared. The brightness is normalized by the magnitude SB$_o$ observed at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster as judged in Djorgovski & King (1986). ∆(SB$_o$ − SB) is the corresponding deviation of (SB$_o$ − SB) from the model.

expression of the relaxation time, including the many-body effect, is complicated. It participates in both the multiple collision integrals and the summation of Fourier-series expansion in an orbit-averaged kinetic equation (Polyachenko & Shukhman 1982; Ito 2018). One has conventionally assumed that the two-body relaxation is dominant even at the apocenter $r_{\text{tid}} \sim 1$ (Cohen et al. 1950). Standard stellar dynamics assumes the half-mass relaxation time to be $t_i/t_{\text{cross}} \sim N/\ln[0.1N]$, where $t_{\text{cross}}$ is the crossing time (e.g., Heggie & Hut 2003). The time $t_i$ is proper to characterize the two-body relaxation process in homogeneous clusters. On the one hand, actual clusters are inhomogenous systems. We also need to consider the many-body relaxation effect on secular-evolution time scales $t_{\text{sec}} \sim Nt_{\text{cross}}$. Loosely speaking, during the late relaxation-evolution stage, the two-body relaxation is approximately $\ln[0.1N]$ times more significant than the many-body one. However, it may not be significant in the early stage, and the non-dominant effect can be more effective. The pericenter of stars may be much larger on average. Ideally, many stars are separated from other stars by at least the order of $1/n^{1/3}$. Extreme cases were discussed and mathematically formulated in Kandrup (1981) and Kandrup (1988). Kandrup (1985) discussed some simple examples of this matter by neglecting the effect of escaping stars. The work confined a self-gravitating system in a box and examined its secular evolution. This analysis concluded that the many-body relaxation could cause a deviation of the stellar DF from the Maxwellian DF on secular-evolution time scales at the beginning of the evolution, given that evaporation did not occur.

Our concern is whether little evaporation could bring globular clusters’ DFs into those of polytropes. Taruya & Sakagami (2003) discussed the realization of polytropic clusters based on an N-body simulation. The study examined a self-gravitating system of equal-masses enclosed in an adiabatic container. It found that DFs for polytropes can approximate the simulated DFs well even on time scales much longer than the half-mass relaxation time. This result was also confirmed relying on a time-dependent Fokker-Planck model (Taruya & Sakagami 2004). Taruya & Sakagami (2003) also tested the system without an adiabatic wall. Of course, due to evaporation, the simulated stellar DF largely deviates from the stellar polytrope as time elapses. On the one hand, in the early relaxation-evolution stage, the simulated DF seems reasonably fitted by the DF for a polytrope (see $m = 5.7$ at $T = 50$ in their work). Also, the inner parts of their model clusters and stellar DF at low energies are modeled well by the DFs for polytropes, regardless of the effect of escaping stars. Their results imply that the stellar DF and structural profile of a star cluster can be like a polytrope unless the effect of evaporation is dominant.

5.2 Secular Evolution and Polytropic Globular Clusters in the Milky Way

We found that the secular-evolution time $t_{\text{sec}}$ could characterize the physical states of polytrope globular clusters in the Milky Way well. Table 3 shows the time scales and parameters for the globular clusters, such as current and estimated-initial relaxation times $t_i$ and $t_{\text{cross}}$, cluster’s age $t_{\text{age}}$ and the total stellar mass $M$. We estimated the values of $t_{\text{cross}}$ using the analysis of Lightman (1982) that we employed in Section 4. Fortunately, all the polytropic clusters’ ages are available in the literature. Hence, we employed the actual ages rather than the order of the ages in the present section, unlike Section 4. The present section examines how many initial secular-evolution times have already passed during the cluster ages. We measure this property by defining the ‘secular-
isolated (shock). In an segregation with stellar evolution and tidal effects globular clusters are subject to mass spectrum Section 5.1 is oversimplified in the sense that actual The discussion of polytropic globular clusters in 5.3 Criticism of the Concept of Polytropic Globular Clusters (e.g., Fukushige & Heggie 1996; Portegies Zwart et al. 1990; Takahashi & Lee 2000) and multi-mass OAFP models (e.g., Chernoff & Weinberg which leads to a core-collapse, as discussed for both mass segregation and tidal effect accelerate the process (Baumgardt et al. 2002). However, more realistically, stellar mass in the first five initial-relaxation time scales (Baumgardt et al. 2008) explained one possible interpretation for this issue. It showed that the low-concentration clusters had already undergone primordial mass segregation in the early relaxation-evolution stage due to stellar evolution. This idea was extended to a sophisticated case study for one of the low-concentration clusters, Palomar 4 (Zonoozi et al. 2017). The study reported that the total mass of Palomar 4 rapidly decreased only in the first 0.1 Gyr, and the mass of the cluster calmly kept decreasing with time. The decrease in the mass depends on the orbit of Palomar 4 in the Milky Way, though the total stellar number decreases by approximately 60% in 10 Gyr. Based on the results from the literature, it appears that the reason why the low-concentration clusters have polytropic structures is not directly because of little mass loss from the clusters.

However, we cannot jump to conclusions without more careful consideration. The direct relationship is currently unknown between the DFs for polytropes and globular clusters that have experienced mass segregation.

5.3 Criticism of the Concept of Polytropic Globular Clusters

The discussion of polytropic globular clusters in Section 5.1 is oversimplified in the sense that actual globular clusters are subject to mass spectrum (segregation) with stellar evolution and tidal effects (shock). In an isolated N-body system with equal masses, the cluster loses a small fraction (≈ 0.1%) of the total stellar mass in the first five initial-relaxation time scales (Baumgardt et al. 2002). However, more realistically, mass segregation and tidal effect accelerate the process which leads to a core-collapse, as discussed for both multi-mass OAFP models (e.g., Chernoff & Weinberg 1990; Takahashi & Lee 2000) and N-body simulations (e.g., Fukushige & Heggie 1996; Portegies Zwart et al. 1998; Baumgardt & Makino 2003) in tidal fields. Also, a relatively new observation (Marchi et al. 2007) showed an unexpected feature of low-concentration clusters. Low-mass stars are more depleted in low-concentration clusters’ mass functions than high-concentration ones. This result implies that the lower-concentration clusters have lost more stars due to evaporation or tidal stripping. However, the excessive loss of low-mass stars from low-concentration clusters contradicts standard stellar dynamics. Conventionally, higher-concentration clusters are supposed to have lost more low-mass stars due to more frequent two-body relaxation processes and mass stratification (segregation) (Spitzer 1988). An N-body simulation (Baumgardt et al. 2008) explained one possible interpretation for this issue. It showed that the low-concentration clusters had already undergone primordial mass segregation in the early relaxation-evolution stage due to stellar evolution. This idea was extended to a sophisticated case study for one of the low-concentration clusters, Palomar 4 (Zonoozi et al. 2017). The study reported that the total mass of Palomar 4 rapidly decreased only in the first 0.1 Gyr, and the mass of the cluster calmly kept decreasing with time. The decrease in the mass depends on the orbit of Palomar 4 in the Milky Way, though the total stellar number decreases by approximately 60% in 10 Gyr. Based on the results from the literature, it appears that the reason why the low-concentration clusters have polytropic structures is not directly because of little mass loss from the clusters.
The total mass $M_\text{cluster}$ ages in Forbes & Bridges (2010). clusters from Forbes & Bridges (2010) and resorted to other sources when we located more recent data or could not find polytropic- clusters. The current relaxation time $t_\text{c.r.}$ and concentration $c$ are values reported in Harris (1996, 2010 edition).

### Table 3

Secular-evolution parameters calculated from the core relaxation times and ages of polytropic- and non-polytropic- clusters. The current relaxation time $t_\text{c.r.}$ and concentration $c$ are values reported in Harris (1996, 2010 edition).

| Polytropic cluster | $c$ | $t_\text{c.r.}$ (Gyr) | $t_\text{c.r.o.}$ (Gyr) | $\log \frac{M_\odot}{M}$ | $\eta_\text{m}$ | $t_\text{age}$ (Gyr) | Reference for $t_\text{age}$ |
|--------------------|-----|------------------------|-------------------------|--------------------------|--------------|----------------------|------------------|
| NGC 288            | 0.99| 0.98                   | 2.0                     | 4.64                     | 0.63         | 10.62                | (Forbes & Bridges 2010) |
| NGC 1261           | 1.16| 0.39                   | 1.15                    | 5.17                     | 0.913        | 10.24                | (Forbes & Bridges 2010) |
| NGC 5053           | 0.74| 6.5                    | 8.2                     | 4.41                     | 0.19         | 12.29                | (Forbes & Bridges 2010) |
| NGC 5139           | 1.31| 4.0                    | 5.5                     | 6.38                     | 0.17         | 11.52                | (Forbes & Bridges 2010) |
| NGC 5466           | 1.04| 2.2                    | 3.6                     | 4.85                     | 0.41         | 13.57                | (Forbes & Bridges 2010) |
| NGC 5897           | 0.86| 2.1                    | 3.5                     | 4.83                     | 0.40         | 12.3                 | (Forbes & Bridges 2010) |
| NGC 5986           | 1.23| 0.38                   | 1.24                    | 5.48                     | 0.94         | 12.16                | (Forbes & Bridges 2010) |
| NGC 6101           | 0.80| 1.6                    | 2.9                     | 4.83                     | 0.48         | 12.54                | (Forbes & Bridges 2010) |
| NGC 6205           | 1.53| 0.32                   | 1.12                    | 5.59                     | 0.98         | 11.65                | (Forbes & Bridges 2010) |
| NGC 6402           | 0.99| 1.14                   | 2.35                    | 5.89                     | 0.47         | 12.6                 | (Santos & Piatti 2004)  |
| NGC 6496           | 0.70| 0.87                   | 2.0                     | 4.29                     | 0.82         | 12.42                | (Forbes & Bridges 2010) |
| NGC 6712           | 1.05| 0.40                   | 1.2                     | 4.98                     | 0.95         | 10.4                 | (Forbes & Bridges 2010) |
| NGC 6723           | 1.11| 0.62                   | 1.7                     | 5.15                     | 0.81         | 13.06                | (Forbes & Bridges 2010) |
| NGC 6809           | 0.93| 0.72                   | 1.8                     | 5.03                     | 0.77         | 13.0                 | (Wang et al. 2016)      |
| NGC 6981           | 1.21| 0.52                   | 1.4                     | 4.80                     | 0.89         | 10.88                | (Forbes & Bridges 2010) |
| Pal 3              | 0.99| 4.5                    | 5.8                     | 4.36                     | 0.21         | 9.7                  | (Forbes & Bridges 2010) |
| Pal 4              | 0.93| 5.2                    | 6.5                     | 4.21                     | 0.19         | 9.5                  | (Forbes & Bridges 2010) |
| Pal 14             | 0.80| 7.1                    | 8.6                     | 3.83                     | 0.22         | 13.2                 | (Sollima et al. 2010)   |
| NGC 1851           | 1.86| 0.027                  | 0.38                    | 5.42                     | 2.36         | 9.2                  | (Salaris & Weiss 2002)  |
| NGC 5634           | 2.07| 0.047                  | 0.53                    | 5.18                     | 2.30         | 11.8                 | (Forbes & Bridges 2010) |
| NGC 6121           | 1.65| 0.079                  | 0.61                    | 4.83                     | 2.11         | 11.5                 | (Wang et al. 2016)      |
| NGC 6144           | 1.55| 0.60                  | 1.7                     | 4.76                     | 0.94         | 13.82                | (Forbes & Bridges 2010) |
| NGC 6254           | 1.38| 0.16                  | 0.81                    | 5.06                     | 1.51         | 11.39                | (Forbes & Bridges 2010) |
| NGC 6273           | 1.53| 0.33                  | 1.1                     | 6.03                     | 1.47         | 11.90                | (Forbes & Bridges 2010) |
| NGC 6352           | 1.10| 0.29                  | 1.1                     | 4.57                     | 1.37         | 12.67                | (Forbes & Bridges 2010) |
| NGC 6388           | 1.75| 0.052                 | 0.553                   | 6.16                     | 1.82         | 12.03                | (Forbes & Bridges 2010) |
| NGC 6626           | 1.67| 0.042                 | 0.52                    | 5.36                     | 2.29         | 12.1                 | (Kerber et al. 2018)     |
| NGC 6656           | 1.38| 0.34                  | 1.2                     | 5.53                     | 1.00         | 12.67                | (Forbes & Bridges 2010) |
| NGC 7099(c)        | 2.50| 0.0023                | 0.35                    | 4.91                     | 2.97         | 12.93                | (Forbes & Bridges 2010) |
| NGC 3201           | 1.29| 0.41                  | 1.2                     | 5.05                     | 0.92         | 10.24                | (Forbes & Bridges 2010) |
| NGC 4590           | 1.41| 0.28                  | 1.1                     | 4.95                     | 1.29         | 13.0                 | (Dotter et al. 2009)     |

Table 3: Secular-evolution parameters calculated from the core relaxation times and ages of polytropic- and non-polytropic- clusters. The current relaxation time $t_\text{c.r.}$ and concentration $c$ are values reported in Harris (1996, 2010 edition).

The total mass $M_\text{cluster}$ ages in Forbes & Bridges (2010). clusters from Forbes & Bridges (2010) and resorted to other sources when we located more recent data or could not find polytropic- clusters. The current relaxation time $t_\text{c.r.}$ and concentration $c$ are values reported in Harris (1996, 2010 edition).

Also, the present work does not discuss the projected line-of-sight velocity dispersion profiles of the energy-truncated ss-OAFP model. Many of the polytropic clusters are low-concentration clusters, which imply that accurate observational data are hard to obtain compared to high-concentration clusters (e.g., Meylan & Heggie 1997). Perhaps, more recent data from Gaia Data Release 2 (e.g., Baumgardt et al., 2018), the ESO Multi-Instrument Kinematic Survey (MIKiS) (Ferraro et al. 2018) and more accurate kinematic data may differentiate the polytropic model from other models, including the King- and energy-truncated ss-OAFP models.

### 6 CONCLUSIONS

The present work introduced a phenomenological model, i.e., the energy-truncated ss-OAFP model. It can apply to at least fifty-five Galactic globular clusters, including PCC clusters with resolved cores. Our new model is a linear algebraic combination of the DFs for the ss-OAFP model and a polytropic sphere of $m$. We weighed the latter by a factor $\delta$. We determined the optimal value of $m$ to be 3.9 by comparing the structural parameters obtained from the King- and our models. After this procedure, the new model has only three degrees of freedom that are the same as those of the King model.

Our new model can reasonably fit the projected structural profiles of both the PCC- and KM- clusters with resolved cores. The fitting provided the completion rates of core-collapse against the concentrations of the clusters. The characteristics of the completion rates are consistent with standard stellar dynamics. Also, our model provided the same order of structural parameters as those of the King model. This feature infers that our model can apply to globular clusters as adequately as the King model. On the other hand, our model is more useful compared to the King model to single out KM clusters whose structure is similar to the complete core-collapse cluster, i.e., high concentration ($\bar{c} \geq 2.0$) clusters. The examples are NGC 1851, NGC 6626 and NGC 6517. We also found that low-concentration ($\bar{c} \approx 1$) clusters may have structures close to the polytropes of $m = 3.9$. This result motivated
us to discuss the relationship between low-concentration clusters and polytropic-sphere models.

We found that eighteen low-concentration globular clusters could be polytropic in the Milky Way. Polytrope models fitted their projected structural profiles well. We also confirmed that the secular-evolution time $t_{\text{sec}}$ could characterize well the low-concentration clusters’ dynamical states in the cores. This feature implies that many-body relaxation processes are more significant than two-body ones in the early relaxation-evolution stage. However, one has not carried out detailed physical arguments and numerical and observational studies regarding the polytropic globular clusters, including the effects of mass spectrum and segregation. Hence, we consider that the polytropic clusters are a tentative idea, which intrigues us to work on two topics in the future. We will examine (i) the relationship between the mass spectrum (segregation) in star clusters and the stellar DF for polytropes and (ii) the applicability of the King-, polytrope- and ss-OAFP-models to kinematic profiles of Galactic globular clusters.

We will determine whether $m = 3.9$ is the best value in our model utilizing more recent data, such as the Gaia DR2 survey, in the future. The present paper primarily focused on whether our model can fit globular clusters’ structural profiles as reasonably as the King model. Strictly speaking, we do not need to choose the optimal value by comparing our and the King models, unlike Section 2.4. We also do not need to employ the regularization made in Section 2.3. We introduced those tedious processes in the present work only for the sake of comparison with the King model. We will examine our model’s parametric-space dependence more carefully using data based on a recent homogeneous survey. Part of the work has been done in Ito (2020).

We will also extend our model by incorporating extra effects. As explained in Section 1, the King model has been extended to anisotropic and multi-component models including the effects of black holes, dark matter and others. We will follow the same path as the King model to “fine-tune” our model for specific clusters. We especially consider the multi-mass effect to be important in our model. As we examined in Section 3, our model does not fit PCC clusters as adequately as KM clusters near the cores. We believe that the multi-mass effect resolves the issue. Hence, we are planning to resort to the method of Costa & Freeman (1976) with a realistic mass spectrum.

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Appendix A: CHEBYSHEV COEFFICIENTS FOR THE DENSITY PROFILE OF THE SS-OAFP MODEL

Table A.1 provides the Chebyshev coefficients for the density profile obtained from the Gauss-Chebyshev spectral solution of the ss-OAFP equation (Ito 2021).

Appendix B: FITTING THE ENERGY-TRUNCATED SS-OAFP MODEL TO KM CLUSTERS

The present appendix describes the fitting of the energy-truncated ss-OAFP model to Galactic KM clusters’ projected structural profiles reported in Kron et al. (1984); Miocchi et al. (2013). For fitting the model to data of Miocchi et al. (2013), we determined the fitting parameters’ values when $\chi^2$ reached its minimum. To fit the data of Kron et al. (1984), we minimized the model’s absolute deviation from the observed data. Sections B.1 and B.2 show that our model fits the KM stars reported in Miocchi et al. (2013) and Kron et al. (1984).

B.1. KM Cluster (Miocchi 2013)

Figures B.1, B.2 and B.3 depict the energy-truncated ss-OAFP model fitted to KM clusters’ projected density profiles reported in Miocchi et al. (2013). Table B.1 compares the structural parameters obtained from our, the King- and the Wilson-models. The majority of parameters obtained from our model are slightly greater than those of the King model. However, they are less than those of the Wilson model. Our model does not fit the structures of NGC 5466 and Terzan 5 well, but the King- and Wilson-models fit them. This result implies that the clusters are less close to the core-collapse cluster and the polytropic sphere of $m = 3.9$.

B.2. KM Clusters (Kron 1984)

Figures B.4–B.7 plot the projected density profiles reported in Kron et al. (1984), fitted by the energy-truncated ss-OAFP model. In Kron et al. (1984), the King model’s fitting does not include several data points near the clusters’ centers because of uncertainty in the data originating from too high brightness in the cores. However, the present work included them since our model adequately fits almost all the data plots (except for NGC 5053 and NGC 5897). Also, the data near the center did not affect the fitting parameters significantly.

Appendix C: FITTING THE ENERGY-TRUNCATED SS-OAFP MODEL TO PCC CLUSTERS

The present appendix shows the results of applying the energy-truncated ss-OAFP model to PCC clusters reported
Table A.1 Chebyshev Coefficients for $D(\varphi)$ Obtained from the Solution of the ss-OAFP Equation in Ito (2021)

| Index | Coefficient $D_n$ | Index | Coefficient $D_n$ | Index | Coefficient $D_n$ |
|-------|-------------------|-------|-------------------|-------|-------------------|
| 1     | 0.0102046967906   | 26    | -0.00000000116399 | 51    | 0.00000000000015 |
| 2     | 0.0022456865877   | 27    | -0.00000000468238 | 52    | 0.00000000000003 |
| 3     | 0.0025245701257   | 28    | 0.000000000229369 | 53    | 0.00000000000006 |
| 4     | 0.0016537221704   | 29    | 0.00000000101426 | 54    | -0.00000000000001 |
| 5     | 0.00043226567655  | 30    | -0.00000000134350 | 55    | -0.00000000000006 |
| 6     | -0.000004376748035 | 31    | 0.0000000012193  | 56    | -0.00000000000005 |
| 7     | 0.000003421960193  | 32    | 0.00000000050729 | 57    | -0.00000000000003 |
| 8     | 0.00005947570967   | 33    | -0.00000000025939 | 58    | 0.00000000000001 |
| 9     | -0.00000928241976  | 34    | -0.00000000101419 | 59    | 0.00000000000002 |
| 10    | -0.00001498128930  | 35    | 0.0000000015245  | 60    | 0.00000000000002 |
| 11    | 0.00000781023967   | 36    | -0.0000000002059 | 61    | 0.00000000000002 |
| 12    | 0.00000346620655   | 37    | -0.0000000005568 | 62    | 0.00000000000002 |
| 13    | -0.00000374902390  | 38    | 0.0000000003277 | 63    | 0.00000000000001 |
| 14    | -0.0000016966157   | 39    | 0.0000000000392 | 64    | -0.00000000000002 |
| 15    | 0.0000013122735    | 40    | -0.0000000001808 | 65    | 0.00000000000002 |
| 16    | -0.00000037843594  | 41    | 0.0000000000392 | 66    | -0.00000000000002 |
| 17    | -0.000000041661024  | 42    | -0.0000000000605 | 67    | 0.00000000000002 |
| 18    | 0.0000000327834197  | 43    | -0.0000000000441 | 68    | -0.00000000000002 |
| 19    | 0.00000006739235   | 44    | -0.0000000000062 | 69    | 0.00000000000002 |
| 20    | -0.0000000012986644 | 45    | 0.00000000000219 | 70    | -0.00000000000002 |
| 21    | 0.000000001901015  | 46    | -0.0000000000073 | 71    | 0.00000000000002 |
| 22    | 0.000000001414535  | 47    | -0.0000000000061 | 72    | -0.00000000000002 |
| 23    | -0.000000002355931 | 48    | 0.0000000000059 | 73    | 0.00000000000002 |
| 24    | -0.000000000894979 | 49    | 0.00000000000000 | 74    | 0.00000000000002 |
| 25    | 0.000000001267818  | 50    | -0.0000000000028 | 75    | 0.00000000000002 |

Fig. B.1 Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density $\Sigma$ of NGC 288, NGC 1851, NGC 5466 and NGC 6121 reported in Miocchi et al. (2013). The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986). $\Delta \log[\Sigma]$ is the corresponding deviation of $\Sigma$ from the model on the log scale.
Fig. B.2 Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density $\Sigma$ of NGC 6254, NGC 6626, Palomar 3 and Palomar 4 reported in Miocchi et al. (2013). The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986). $\Delta \log[\Sigma]$ is the corresponding deviation of $\Sigma$ from the model on the log scale.

Fig. B.3 Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density $\Sigma$ of Palomar 14, Terzan 5 and NGC 6809 reported in Miocchi et al. (2013). The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986). $\Delta \log[\Sigma]$ is the corresponding deviation of $\Sigma$ from the model on the log scale.
Fig. B.4 Fitting of the energy-truncated ss-OAFP model \( (m = 3.9) \) to the projected density profiles of NGC 2419, NGC 4590, NGC 5272, NGC 5634, NGC 5694 and NGC 5824 reported in Kron et al. (1984). The unit of the projected density \( \Sigma \) is stellar number per arcminute squared. In the legends, (n) means a ‘normal’ cluster or KM, as judged in Djorgovski & King (1986).

Fig. B.5 Fitting of the energy-truncated ss-OAFP model \( (m = 3.9) \) to the projected density profiles of NGC 6093, NGC 6205, NGC 5229, NGC 6273, NGC 6304 and NGC 6333 reported in Kron et al. (1984). The unit of the projected density \( \Sigma \) is stellar number per arcminute squared. In the legends, (n) means a ‘normal’ or KM cluster and (n?) a ‘probable normal’ cluster, as judged in Djorgovski & King (1986).
Table B.1 Core- and tidal- radii obtained by fitting the energy-truncated ss-OAFP model to projected density profiles of KM clusters reported in Miocchi et al. (2013).

| Cluster   | ss-OAFP model | King model (Miocchi et al. 2013) | Wilson model (Miocchi et al. 2013) |
|-----------|---------------|-----------------------------------|------------------------------------|
|           | c  r_c  r_rad | c  r_c  r_rad                       | c  r_c  r_rad                      |
| NGC 288   | 1.30 1.43 28.9| 1.20 1.17 21                        | 1.10 1.53 25.8                     |
| NGC 1851  | 2.04 0.10 11.1| 1.95 0.09 8.3                       | 3.33 0.09 204                      |
| NGC 5466  | 1.41 1.26 32.4| 1.31 1.20 26.3                      | 1.42 1.33 40                       |

| Cluster   | ss-OAFP model | King model (Miocchi et al. 2013) | Wilson model (Miocchi et al. 2013) |
|-----------|---------------|-----------------------------------|------------------------------------|
|           | c  r_c  r_rad | c  r_c  r_rad                       | c  r_c  r_rad                      |
| NGC 6121  | 1.81 1.13 73.6| 1.68 1.07 53                        | 2.08 1.08 1200                     |
| NGC 6254  | 1.64 0.68 30.1| 1.41 0.68 19.0                      | 1.80 0.73 52                       |
| NGC 6626  | 2.12 0.19 25.4| 1.79 0.26 16                        | 3.1 0.26 380                      |

| Cluster   | ss-OAFP model | King model (Miocchi et al. 2013) | Wilson model (Miocchi et al. 2013) |
|-----------|---------------|-----------------------------------|------------------------------------|
|           | c  r_c  r_rad | c  r_c  r_rad                       | c  r_c  r_rad                      |
| Pal 3     | 1.03 0.55 5.91| 0.8 0.47 3.6                       | 0.81 0.49 5.33                     |
| Pal 4     | 1.16 0.46 6.79| 1.1 0.37 4.9                       | 1.3 0.38 9                        |
| Pal 14    | 1.04 0.85 9.23| 0.9 0.68 6.4                       | 1.0 0.70 10                       |
| Trz 5     | 1.69 0.15 7.25| 1.59 0.13 5.2                      | 2.4 0.14 39                      |

Fig. B.6 Fitting of the energy-truncated ss-OAFP model (m = 3.9) to the projected density profiles NGC 6341, NGC 6356, NGC 6401, NGC 6440, NGC 6517 and NGC 6553 reported in Kron et al. (1984). The unit of the projected density \( \Sigma \) is stellar number per arcminute squared. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986).
Fig. B.7 Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density profiles of NGC 6569, NGC 6638, NGC 6715, NGC 6864, NGC 6934 and NGC 7006 reported in Kron et al. (1984). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986). Following Kron et al. (1984), data at small radii are ignored due to depletion of the projected density profiles.

Fig. B.8 Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density profiles of NGC 5053 and NGC 5897 reported in Kron et al. (1984). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986). Following Kron et al. (1984), data at small radii are ignored due to depletion of the projected density profiles.
Fig. C.1 Fitting of the energy-truncated ss-OAFP model to the projected densities of NGC 1904, NGC 4147, NGC 6544 and NGC 6652 reported in Kron et al. (1984). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (c) means a ‘core-collapse’ or PCC cluster, (c?) a probable/possible PCC cluster, and (n?c?) weak indications of a PCC cluster as judged in Djorgovski & King (1986).

Fig. C.2 Fitting of the energy-truncated ss-OAFP model to the surface brightness profiles of NGC 5946, NGC 6342, NGC 6624 and NGC 6453 reported in Lugger et al. (1995). The unit of surface brightness (SB) is U magnitude per arcsecond squared. The brightness is normalized by the magnitude $SB_o$ at the smallest radius point. In the legend, (c) means a PCC cluster as judged in Djorgovski & King (1986). $\Delta(SB_o - SB)$ is the corresponding deviation of $(SB_o - SB)$ from the model.
Fig. C.3  Fitting of the energy-truncated ss-OAFP model to the surface brightness profiles of NGC 6522, NGC 6558 and NGC 7099 reported in Lugger et al. (1995) and Terzan 1 reported in Djorgovski & King (1986). The unit of surface brightness (SB) is U magnitude per arcsecond squared, except for Terzan 1 for which B band is used. The brightness is normalized by the magnitude $SB_0$ at the smallest radius point. In the legend, (c) means a PCC cluster as judged in Djorgovski & King (1986). $\Delta(SB_0 - SB)$ is the corresponding deviation of $(SB_0 - SB)$ from the model.

C.1. PCC? Clusters (Kron 1984)

Figure C.1 depicts the energy-truncated ss-OAFP model with $m = 3.9$ fitted to the projected density profiles of possible PCC clusters reported in Kron et al. (1984). The work of Kron et al. (1984) considered NGC 1904, NGC 4147, NGC 6544 and NGC 6652 as possible PCC clusters.

C.2. PCC Clusters (Lugger 1995) and (Djorgovski 1986)

Figures C.2 and C.3 show the results of fitting the energy-truncated ss-OAFP model with $m = 3.9$ to the surface brightness profiles of PCC clusters reported in Lugger et al. (1995) and Terzan 1 in Djorgovski & King (1986). For clusters with resolved cores, our model fits the core and halo up to 1 arcminute. As expected, the fitting to the clusters with unresolved cores (NGC 5946 and NGC 6624) is not satisfactory. On the one hand, NGC 6342 is one of them, but our model reasonably fits the profile.

Appendix D: FITTING POLYTROPIC SPHERE MODEL TO LOW-CONCENTRATION CLUSTERS

The present appendix features the result of fitting polytropic spheres of index $m$ to the projected structural profiles of low-concentration globular clusters reported in Miocchi et al. (2013), Trager et al. (1995) and Kron et al. (1984). When we fitted the polytropes to the data of Miocchi et al. (2013), we minimized $\chi^2$ and determined the structural parameters. We minimized the infinite norm of the differences between the fitted curve and the data of Kron et al. (1984); Trager et al. (1995).

D.1. Polytropic Cluster (Miocchi 2013) and (Kron 1984)

Figures D.1 and D.2 present successful applications of the polytrope models to the projected density profiles of NGC 5466, NGC 6809, Palomar 3, Palomar 4 and Palomar 14 reported in Miocchi et al. (2013). In the figures, the polytropes of $3.0 < m < 5$ reasonably fit the projected surface densities of the globular clusters whose
Fig. D.1 Fitting the polytropic spheres of index $m$ to the projected density profiles $\Sigma$ of NGC 5466, NGC 6809, Palomar 3 and Palomar 4 reported in Miocchi et al. (2013). The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986). $\Delta \log[\Sigma]$ is the corresponding deviation of $\Sigma$ from the model on the log scale.

Fig. D.2 Fitting of the polytropic sphere of index $m$ to the projected density of Palomar 14 reported in Miocchi et al. (2013). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986).

concentrations range $1 < \bar{c} < 1.4$. Figure D.3 displays the projected density of NGC 4590 fitted with a polytrope. NGC 4590 is one of the examples that could fall in either the polytropic or non-polytropic clusters.

D.2. Polytropic Cluster (Trager 1995)

Figures D.4 and D.5 depict the fitting of the polytropic sphere model to the Chebyshev approximation to the surface brightness profiles reported in Trager et al. (1995). The polytropic indices $m = 3.3 \sim 4.99$ are useful to fit the polytrope models to the low-concentration clusters whose core-relaxation times are the order of 1 Gyr (from Harris 1996, 2010 edition’s catalog). The polytropes themselves do not rapidly decay near their tidal radii. Hence, their concentrations are relatively high, such as $\bar{c} = 3.34$ for $m = 4.99$. On the one hand, Figure D.6 shows the clusters whose surface brightness profiles are not close to the polytropes. Such non-polytropic clusters have shorter core-relaxation times ($< 0.5$ Gyr) and relatively high
Fig. D.3  Partial success of fitting the polytropic sphere of index $m$ to the projected density of NGC 4590 reported in Kron et al. (1984). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986). Following Kron et al. (1984), data at small radii are ignored due to the depletion of projected density profile in the core.

Fig. D.4  Fitting of the polytropic spheres of index $m$ to the surface brightness profiles of NGC 1261, NGC 5053, NGC 5897, NGC 5986, NGC 6101 and NGC 6205 reported in Trager et al. (1995). The unit of surface brightness (SB) is $V$ magnitude per arcsecond squared. The brightness is normalized by the magnitude $SB_0$ at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986).

concentrations ($c > 1.5$ based on the King model), as displayed in Table 3.

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Fig. D.5  Fitting of the polytropic sphere of index \( m \) to the surface brightness profiles of NGC 6402, NGC 6496, NGC 6712, NGC 6723 and NGC 6981 reported in Trager et al. (1995). The unit of the surface brightness (SB) is V magnitude per arcsecond squared. The brightness is normalized by the magnitude SB\(_o\) at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986).

Fig. D.6  Failure of fitting the polytrope models of index \( m \) to the surface brightness profiles of NGC 3201, NGC 6144, NGC 6273, NGC 6352, NGC 6388 and NGC 6656 reported in Trager et al. (1995). The unit of the surface brightness (SB) is V magnitude per arcsecond squared. The brightness is normalized by the magnitude SB\(_o\) at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged in Djorgovski & King (1986).
