ON THE EVALUATION OF DEFLECTIONS
OF THE VERTICAL USING FFT TECHNIQUE

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ABSTRACT There exist three types of convolution formulae for the efficient evaluation of gravity field convolution integrals, i.e., the planar 2D convolution, the spherical 2D convolution and the spherical 1D convolution. The largest drawback of both the planar and the spherical 2D FFT methods is that, due to the approximations in the kernel function, only inexact results can be achieved. Apparently, the reason is the meridian convergence at higher latitudes. As the meridians converge, the $\Delta \phi, \Delta \lambda$ blocks do not form a rectangular grid, as is assumed in 2D FFT methods. It should be pointed out that the meridian convergence not only leads to an approximation error in the kernel function, but also causes an approximation error during the implementation of 2D FFT in computer. In order to meet the increasing need for precise determination of the vertical deflections, this paper derives a more precise planar 2D FFT formula for the computation of the vertical deflections. After having made a detailed comparison between the planar and the spherical 2D FFT formulae, we find out the main source of errors causing the loss in accuracy by applying the conventional spherical 2D FFT method. And then, a modified spherical 2D FFT formula for the computation of the vertical deflections is developed in this paper. A series of numerical tests have been carried out to illustrate the improvement made upon the old spherical 2D FFT. The second part of this paper is to discuss the influences of the spherical harmonic reference field, the limited capsize, and the singular integral on the computation of the vertical deflections. The results of the vertical deflections over China by applying the spherical 1D FFT formula with different integration radii have been compared to the astro-observed vertical deflections in the South China Sea to obtain a set of optimum deflection computation parameters.

1 Introduction

The fast Fourier transform (FFT) technique is a very powerful tool for the efficient evaluation of gravity field convolution integrals. Thanks to the good computation efficiency, the FFT technique, in the mid-1980s, began to find widespread use in geoid determination, when compared to classical numerical integration.

In the earlier time, the planar, two-dimensional (2D) FFT had been used for many years. To overcome the approximation errors of the planar convolution, a 2D spherical convolution expression was developed by Strang van Hees in 1990. However, each of these FFT implementations is still subject to several approximation errors, the most notable of which is the simplification of Stokes’ s kernel. Therefore, Forsberg and Sideris (1993) proposed the spherical multi-band FFT, which reduces the impact of the simplified kernel. Haagmans et al.
(1993) refined this approach to give the spherical one-dimensional (1D) FFT, which requires no simplification of Stokes’ kernel. In fact, the 1D FFT is a combination of the FFT and numerical integration, but is indisputably faster than numerical integration alone. In a review paper, Schwarz et al. (1990) had given the planar 2D FFT formulae for the efficient evaluation of deflections of vertical. Then, Ning et al. (1994) and Liu et al. (1997) had derived the spherical evaluation formula using 2D FFT and 1D FFT in succession. The 1D FFT approach, although, has the advantage of accuracy over the 2D FFT, many people, up to now, are still used to applying the planar and spherical 2D FFT methods, due to the consideration of their gains in computer time, to perform the convolution evaluations in physical geodesy. It means that it is worthy while discussing the question about making any possible improvement on the conventional 2D FFT approaches.

2 The modification of evaluation formula

2.1 Mathematical model

According to Schwarz et al. (1990) and Ning et al. (1994), the planar Vening-Meinesz formula can be expressed as:

\[ \xi(x_p, y_p) = \frac{1}{2 \pi} \int \Delta g(x, y) [x_p - x] \cdot \left[ (x_p - x)^2 + (y_p - y)^2 \right]^{-\frac{3}{2}} \, dx \, dy \]  
\[ \eta(x_p, y_p) = \frac{1}{2 \pi} \int \Delta g(x, y) [y_p - y] \cdot \left[ (x_p - x)^2 + (y_p - y)^2 \right]^{-\frac{3}{2}} \, dx \, dy \]  

the corresponding spectral expressions of above formulae are:

\[ \xi(x, y) = \frac{1}{2 \pi} F^{-1} \left[ \Delta G(u, v) L_\xi(u, v) \right] \]  
\[ \eta(x, y) = \frac{1}{2 \pi} F^{-1} \left[ \Delta G(u, v) L_\eta(u, v) \right] \]  

where \( \Delta G(u, v) \), \( L_\xi(u, v) \) and \( L_\eta(u, v) \) represent the spectral functions of anomalies and two kernel functions as follows, respectively:

\[ l_\xi(x, y) = x (x^2 + y^2)^{-\frac{3}{2}} \]  
\[ l_\eta(x, y) = y (x^2 + y^2)^{-\frac{3}{2}} \]  

The approximations introduced by the planar form of Vening-Meinesz’ integral can be minimized or avoided by using the spherical Vening-Meinesz integral. Strang van Hees (1990) suggested to apply FFT directly to the spherical Stokes formula. With different approximations of Vening-Meinesz’ kernel function on the sphere, the deflections of the vertical can be evaluated efficiently by means of the 2D DFT (Ning et al., 1994) and the 1D FFT (Liu et al., 1997) as follows.

\[ \xi = \frac{1}{4 \pi} F^{-1} \left[ F(\Delta g \cos \varphi) F \left[ V(\varphi_M, \Delta \varphi, \Delta \lambda) SC(\varphi_M, \Delta \varphi, \Delta \lambda) / (4s) \right] \right] \]  
\[ \eta = \frac{1}{4 \pi} F^{-1} \left[ F(\Delta g \cos^2 \varphi) F \left[ V(\varphi_M, \Delta \varphi, \Delta \lambda) (\sin \Delta \lambda / (4s)) \right] \right] \]  

\[ \xi = \frac{1}{4 \pi} \sum_{\varphi_p \in \varphi} F^{-1} \left[ F(\Delta g \cos \varphi) F \left[ V(\varphi_p, \varphi, \Delta \lambda) \cdot \right. \right] \]  
\[ SC(\varphi_p, \varphi, \Delta \lambda) / (4s) \right] \]  
\[ \eta = \frac{1}{4 \pi} \sum_{\varphi_p \in \varphi} F^{-1} \left[ F(\Delta g \cos^2 \varphi) F \left[ V(\varphi_p, \varphi, \Delta \lambda) (\sin \Delta \lambda / (4s)) \right] \right] \]  

where

\[ V(s) = \frac{dS(s)}{ds} = -s^{-2} - 6 + 20s - (3 - 6s^2) \cdot (1 + 2s)(s + s^2)^{-1} + 12s \ln(s + s^2) \]  
\[ SC(\varphi_p, \varphi, \Delta \lambda) = \sin(\varphi_p + \varphi) \sin^2 \frac{1}{2} \Delta \lambda - \sin(\varphi_p - \varphi) \cos^2 \frac{1}{2} \Delta \lambda \]  
\[ s^2 = \sin^2 \frac{\Psi}{2} = \sin^2 \frac{1}{2} \Delta \varphi + \sin^2 \frac{1}{2} \Delta \lambda \cos \varphi \cos \varphi \]  
\[ = \sin^2 \frac{1}{2} \Delta \varphi + \sin^2 \frac{1}{2} \Delta \lambda \left[ \cos^2 \frac{1}{2} (\varphi_p + \varphi) - \sin^2 \frac{1}{2} \Delta \varphi \right] \]  

and \( S(s) \) indicates Stokes’ kernel function. In 2D DFT, \( \varphi_p + \varphi \) is expressed as two times the mean latitude (\( \varphi_M \)) of the computation area.

Many numerical examples have been carried out to show that the results derived from the 2D spherical FFT approach, in most of cases, are superior to those obtained by the 2D planar FFT technique. Apparently, the reason is that the kernel function used in the planar form of Vening-Meinesz’ integral follows only the principal term of Vening-Meinesz’ kernel function used in the spherical form.
of Vening-Meinesz's integral. In order to eliminate or avoid the truncation errors of Stokes' kernel function in the 2D planar FFT approach, Li et al. (1997) proposed to take account of all terms of Stokes' kernel function by transforming the variable \( \sin \left( \frac{\Psi}{2} \right) \) of the function to the \( l \), a straight linear length corresponding to the spherical distance \( \Psi \). In that case, Stokes' kernel function could be strictly expressed in planar coordinates, and there exist no truncation errors any more. In [11] (Li et al., 1997), the variable sin \( \left( \frac{\Psi}{2} \right) \) was expressed as

\[
s = \sin \left( \frac{\Psi}{2} \right) = \frac{l}{2R} \sqrt{(x_p - x)^2 + (y_p - y)^2}
\]

(14)

As a generalization of the above idea, a more accurate evaluation formula of deflections of the vertical in the form of 2D planar FFT is suggested here as

\[
\xi(x_p, y_p) = -\frac{1}{4\pi R^2} \int \Delta g(x, y) V(\Delta x, \Delta y)
\]

\[
(x_p - x)/(4sR^2) \, dx \, dy
\]

(15)

\[
\eta(x_p, y_p) = -\frac{1}{4\pi R^2} \int \Delta g(x, y) V(\Delta x, \Delta y)
\]

\[
(y_p - y)/(4sR^2) \, dx \, dy
\]

(16)

The numerical tests in [11] (Li et al., 1997) showed the improvement of the new method used in the computation of geoid. Furthermore, their numerical tests also showed that the results derived from the new 2D planar FFT approach are even better than those obtained from the 2D spherical FFT method. A similar result can be found in Zhang and Sideris (1996; see Table 1 to Table 6 in the paper). In our opinion, that is an unexpected conclusion, because it is not identified with the conventional view that the spherical methods are always superior from a theoretical point of view. So it is necessary to make a further research on the problem mentioned above. The intention of this paper is to make an effort in that direction.

2.2 Comments on 2D convolution evaluation

First of all, let us make a comparison between the two functions in Eqs. (7) and (8), and Eqs. (15) and (16), respectively. Apparently, the form of the two kernel functions is the same as that of Vening-Meinesz's kernel function. The difference between them is they use different approximate expressions to compute the variable \( s = \sin \left( \frac{\Psi}{2} \right) \). Eq. (13) is used in the 2D spherical FFT, while Eq. (14) is used in the 2D planar FFT. It should be pointed out that here the gravity field is usually given in blocks with constant \( \Delta \varphi \) and \( \Delta \lambda \). And when the planar FFT is applied, the constitution of gravity anomalies as input remains unchanged, i.e., they are still expressed in geographical coordinates. The only change is that the constant \( \Delta \varphi \) and \( \Delta \lambda \) spacing is transformed to the planar equivalent spacing \( \Delta x = R\Delta \varphi \) and \( \Delta y = R\Delta \lambda \cos \varphi_M \). In this case, Eq. (14), due to \( x_p - x = R(\varphi_p - \varphi) \), \( y_p - y = R(\lambda_p - \lambda) \cdot \cos \varphi_M \), can be rewritten as

\[
s = \sin \left( \frac{\Psi}{2} \right) = \frac{1}{2} \sqrt{(\varphi_p - \varphi)^2 + (\lambda_p - \lambda)^2 \cos^2 \varphi_M}
\]

(17)

\[
sin^2 \left( \frac{\Psi}{2} \right) = \frac{(\varphi_p - \varphi)^2}{4} + \frac{(\lambda_p - \lambda)^2 \cos^2 \varphi_M}{4}
\]

(18)

Making a comparison between Eqs. (13) and (18), it can be seen that Eq. (18) is only the approximation of Eq. (13), despite when used in 2D DFT, Eq. (13) itself is also an approximate expression. It is shown that, if viewing only from the kernel functions, we can not come to the conclusion that the new 2D planar FFT approach is superior to the conventional 2D spherical FFT method. Then, what happened in [11] (Li et al., 1997) and Zhang's and Sideris's (1996)? In the following, let us have a deep look at the implementation of 2D FFT in computer.

According to Strang van Hees (1990), the error of geoid computed with 2D spherical FFT is proportional to tan. The reason is that the meridians converge at higher latitudes, which leads to larger approximation errors in Stokes' kernel function. It must be pointed out that the meridian convergence has influence not only on the kernel function but also on the constitution of input data. In fact, as the meridians converge, the \( \Delta \varphi \) and \( \Delta \lambda \) blocks do not form a rectangular grid, that is to say, the gravity anomalies as input are not sampled in equivalent distance at different latitudes, whereas the assumption that the data blocks form a rectangular grid is an essential requirement of 2D FFT method. During the implementation of 2D FFT in computer, the
\( \Delta \varphi \) and \( \Delta \lambda \) blocks corresponding to the sampled points of input data are always simply considered to be rectangular grid, otherwise one must introduce a map projection. If the latter is chosen, the extra work to compute new grid anomalies is usually not worthwhile in comparison to the advantage of using FFT (Strang van Hees, 1990). In view of the above fact, in practice, one always chooses the former, i.e., approximately considers the \( \Delta \varphi \) and \( \Delta \lambda \) blocks to be rectangular grid. Obviously, such approximations will inevitably introduce somewhat error in 2D convolution evaluation. The pity is that in the existing literatures the researchers have only concentrated their attention on the modification of Stokes’ and Vening-Meinesz’ kernel function, and little attention has been paid to the requirement of FFT method for the input data. In fact, it is just because the \( \Delta \varphi \) and \( \Delta \lambda \) blocks in which the gravity anomalies as input distribute do not form a rectangular grid that Stokes’ and Vening-Meinesz’ kernel function can not be rigorously expressed as a function of latitude and longitude differences alone. Therefore, in 2D convolution evaluation, the input data and the kernel function are complements of each other. A good approximation of Stokes’ and Vening-Meinesz’ kernel function only fulfills the requirement of a convolution integral. It does not change the constitution of gravity anomalies as input. That is to say, it does not fulfill the requirement of 2D FFT method.

As we know, in the practical implementation of the Fourier transform formulae, two approximations are employed: 1) the continuous integrations are replaced by discrete summations, i.e., the Continuous Fourier Transform (CFT) will be replaced by the Discrete Fourier Transform (DFT); 2) the infinite limits of summation are replaced by finite ones. However, in most computer software for DFTs, such as the FFT subroutines in the IMSL library, the DFT is always simply defined by using the wave number instead of the ‘wavelength’ and also by omitting the period (record length). This means that the space (or time) interval is taken as unit and all other parameters dependent on it are omitted (Sideris, 1994). Consequently, the final practical formulae of 2D planar and spherical FFT should be rewritten as follows, they are corresponding to Eqs. (15) and (7), but here only the evaluation formula of \( \xi \) is given in order to save space on the assumption that the input data are \( M \times N \) grid-ded point gravity anomalies with spacing \( \Delta x \) and \( \Delta y \).

\[
\xi = -\frac{T_x T_y}{4\pi \gamma R^2} \left[ F[\Delta g(x, y)] \cdot \frac{F[V(x, y) / (4\pi R^2)]]}{SC(\varphi_M, \Delta \varphi, \Delta \lambda) / (4\pi)} \right] \quad (19)
\]

where

\[
T_x = M \Delta x = M \lambda \varphi,
T_y = N \Delta y = N \Delta \lambda \cos \varphi_M, T_\varphi = M \Delta \varphi, T_\lambda = N \Delta \lambda \quad (21)
\]

It should be pointed out that the above relations, i.e., Eqs. (19) and (20) hold on the condition that the input data is strictly distributed in a rectangular grid. Unfortunately, just as mentioned in the preceding section, the fact is not as satisfying as expected. This means that, when Eqs. (19) and (20) are used to implement 2D planar and spherical FFT, their input and output data do not match each other in geometrical location. That is to say, the input data is located with a varying spacing \( (\Delta \lambda \cos \varphi) \) at different latitudes in longitude direction, whereas the output data is defined to distribute in a constant spacing \( (\Delta \lambda \cos \varphi_M \text{ or } \Delta \lambda) \) in the same direction. Before the implementation of 2D planar and spherical FFT, to meet the requirement of FFT method, the input data has to be approximately considered to locate with a constant spacing. Adversely, when having finished the computation of 2D planar and spherical FFT, the output data has to be approximately again considered to distribute to the same varying spacing as the input data does. Obviously, the errors introduced by such repeated approximations are dependent on the latitude of the area in question. And their influences on the computation of geoid are propagated through the record length of data \( (T_x \text{ and } T_y \text{ or } T_\varphi \text{ and } T_\lambda) \).

As mentioned above, because the constitution of the input information remains unchanged, the case that the input and output data do not match each
other in geometrical location will also happen during the practical implementation of 2D planar FFT. As we know, the sampled spacing of input data in longitude direction is $\Delta \lambda \cos \phi$. If we neglect temporarily the difference in unit, according to Eqs. (21) and (22), the intervals of computation points in longitude direction, corresponding to the 2D planar and spherical FFT, are $\Delta \lambda \cos \phi_M$ and $\Delta \lambda$, respectively. It means that the difference between input and output data in 2D planar FFT is $\Delta L_P = \Delta \lambda \cos \phi - \Delta \lambda \cos \phi_M = \Delta \lambda (\cos \phi - \cos \phi_M)$, while in 2D spherical FFT, the corresponding location difference is $\Delta L_S = \Delta \lambda \cos \phi - \Delta \lambda = \Delta \lambda (\cos \phi - 1)$. It can be seen easily that in most of the cases (except the areas around the equator), the following relation holds:

$$\text{abs}(\Delta L_P) < \text{abs}(\Delta L_S) \quad (23)$$

Up to now, it can be understood why the results derived from the new 2D planar FFT approach mentioned in Li et al. (1997) are superior to those obtained by the conventional 2D spherical FFT technique.

To improve the conventional 2D spherical FFT evaluation, we suggest that Eq. (20) be rewritten as follows (a similar formula can be written for $\eta$)

$$\xi = \frac{T_M T_s \cos \phi_M}{4 \pi} F_1^{-1} [F[\Delta g]] \cdot F[V(\phi_M, \Delta \phi, \Delta \lambda) SC(\phi_M, \Delta \phi, \Delta \lambda)/(4\pi)]$$

$$\quad (24)$$

Although, at first appearance, Eq. (24) is only the approximation of Eq. (20), the practical importance is that, as mentioned above, such modification will satisfy more closely the requirement of 2D FFT method. Consequently, it will not lead to the loss but gain in computation accuracy. This can be proved through numerical tests in the next section.

2.3 Numerical tests

In order to demonstrate the improvement of the new 2D planar and spherical FFTs proposed here upon the conventional ones, several consistent sets of gravity and vertical deflections data, generated from a spherical harmonic model, are used to make numerical tests of FFT vertical deflections predictions. In the sequel, an ultra-high degree geopotential model called MOD99c up to degree and order 1440, developed on the basis of altimeter-derived marine anomalies on a global scale (Huang, 1998) and the local data at high resolution in China, and EMG96 as starting model (Huang et al., 1999), is used in different degree ranges, in which the lowest spherical harmonic coefficients have been removed as to simulate the typical prediction case. Here, thanks to the fact that the results with the model are not affected by the error resources mentioned above, the 1D FFT vertical deflections will be, first, used as "standard values" for comparing the results of 2D FFT vertical deflections predictions. Using the MOD99c geopotential model, several sets of gravity anomalies were produced on a $5' \times 5'$ grid, then used as input to the planar and spherical FFT software. The vertical deflections computed by such derived anomalies were compared with the "standard values" from 1D FFT and the RMS discrepancies between them may be used to judge the accuracy of vertical deflections predictions. Three $10' \times 10'$ blocks located at different latitudes are chosen as our test areas in this study, each of which corresponds to a $120 \times 120$ grids (14400 points).

The locations of the three test areas are: (A) $0^\circ$--$10^\circ$ N, $105^\circ$--$115^\circ$ E; (B) $35^\circ$--$45^\circ$ N, $80^\circ$--$90^\circ$ E; (C) $75^\circ$--$85^\circ$ N, $80^\circ$--$90^\circ$ E. A statistical result of gravity anomalies in different degree ranges is summarized in Table 1.

Here the vertical deflections predictions have been carried out by using different 2D planar FFTs, including the conventional and the improved one, and

| Table 1 Statistical result of gravity anomalies in the test areas/mGal |
|------------------------|------------------------|------------------------|------------------------|------------------------|
| Degree ranges | Area (A) | Area (B) | Area (C) |
| max | min | RMS | max | min | RMS | max | min | RMS |
| 2~1440 | 101.6 | 67.4 | 27.0 | 424.3 | 228.1 | 87.2 | 67.5 | 49.5 | 19.3 |
| 37~1440 | 79.5 | 86.5 | 16.3 | 462.7 | 215.4 | 77.4 | 64.4 | 54.5 | 18.5 |
| 181~1440 | 75.3 | 49.2 | 11.4 | 294.8 | 190.8 | 47.4 | 38.9 | 35.1 | 11.6 |
| 361~1440 | 52.4 | 29.8 | 5.6 | 188.2 | 141.4 | 29.0 | 20.3 | 20.9 | 5.2 |
different 2D spherical FFTs, including the conventional and the new one proposed here, respectively. The computation results are then compared with the corresponding "standard value". Table 2 shows the statistical results of the differences for every whole computation area. It should be noted here that in all cases 100% zero-padding (Haagmans et al., 1993) has been used in our computations.

Table 2  The RMS differences between predictions and "standard values" of the $\xi$ deflections from different formulae in different areas

| Test area | Degree ranges | Conventional | | Improved |
|-----------|---------------|--------------|------------------|------------------|
|           |               | planar       | spherical        | planar           | spherical        |
| A         | 2~1 440       | 0.153        | 0.011            | 0.004            | 0.004            |
|           | 37~1 440      | 0.043        | 0.006            | 0.003            | 0.003            |
|           | 181~1 440     | 0.010        | 0.004            | 0.003            | 0.003            |
|           | 361~1 440     | 0.002        | 0.002            | 0.002            | 0.002            |
| B         | 2~1 440       | 0.422        | 0.514            | 0.130            | 0.128            |
|           | 37~1 440      | 0.226        | 0.503            | 0.132            | 0.123            |
|           | 181~1 440     | 0.083        | 0.284            | 0.084            | 0.082            |
|           | 361~1 440     | 0.015        | 0.167            | 0.003            | 0.003            |
| C         | 2~1 440       | 0.208        | 0.541            | 0.207            | 0.204            |
|           | 37~1 440      | 0.194        | 0.514            | 0.208            | 0.202            |
|           | 181~1 440     | 0.116        | 0.375            | 0.115            | 0.114            |
|           | 361~1 440     | 0.046        | 0.168            | 0.047            | 0.046            |

From the results in Table 2, a number of interesting conclusions can be drawn. First of all, the results are significantly improved in most of cases, when the 2D planar FFT is performed with the perfect form of Vening-Meinesz' kernel function. And the results derived through the improved planar FFT approach are superior to those obtained by the conventional 2D spherical FFT technique, indeed. Of course, the most interesting result is the fact that the new 2D spherical FFT proposed in this paper gives as good prediction values as expected. The accuracy improvements gained in the new 2D spherical FFT are very significant, especially in degree ranges (2~1 440) and (37~1 440). It also can be seen from Table 2 that although a very satisfactory agreement between the improved 2D planar and new spherical FFTs is obtained, comparatively speaking, the latter is slightly superior to the former. In this sense, our work is going on to support the conventional view that the spherical methods are always superior to the planar ones.

It is worth mentioning here that the new 2D spherical FFT method proposed in this paper is not restricted to vertical deflections predictions. Indeed, any problem which can be expressed in spherical formula, and in which the input data are given in geographical grid, may be treated correspondingly. Examples include such problems as upward continuation, the Stokes' formulae, the inverse Stokes' formula, many terrain effect integrals, and are especially important from a theoretical point of view, many of the integrals in the modern convolution series expansions of the Molodensky problem. In fact, we have also carried out the numerical tests of the Stokes' formulae and the inverse Stokes' formula by using the new 2D spherical FFT proposed in the last section and obtained as good results as we expected.

3 Practical computation of the vertical deflections in China

With the increasing need for precise geoid determinations, recently, much attention has been paid to the optimization of parameters in geoid computation (Featherstone and Sideris, 1997; Forsberg and Featherstone, 1997; Higgins et al., 1997). Many discussions were concentrated on the influences of a spherical harmonic reference field, a limited cap size and a modified Stokes kernel. According to the previous studies the authors mentioned above, the performance of various geoid models depends strongly
on the cap size, and results can be seriously degraded if a too large cap size is used. In practical computation of the vertical deflections in China, we have carried out a series of numerical tests with different computation parameters. And some practical considerations have been taken into account.

First of all, as we know, it is best to compute the vertical deflections in the center of the given $\Delta g$ blocks. This leads to the singularity problem at $\Psi = 0$ where the computation point $P$ and the integration point $Q$ coincide. Therefore one should exclude this block by defining $S(\Psi) = 0$ if $\Psi = 0$. The influence of this central block can be computed separately. When the central area is a circle with radius $r$, its contributions to $\xi$ and $\eta$ had been given by Heiskanen and Moritz (1967)

$$\begin{align*}
\xi_0 &= -\frac{r}{2\gamma} \left( g_x \right) \\
\eta_0 &= -\frac{r}{2\gamma} \left( g_y \right)
\end{align*}$$

(25)

where $g_x = \frac{\partial g}{\partial x}$ and $g_y = \frac{\partial g}{\partial y}$; thus the innermost zone effects for the vertical deflections depend on the gradients of gravity anomaly, as discrete data $g_x$ and $g_y$ can be obtained by numerically differentiating $\Delta g$ along the $x$ and $y$ directions, respectively. By introducing planar approximation (which is permitted for small distance), and on the assumption that the planar grid intervals are $a$ and $b$, the radius of the innermost zone may be approximated by $r = \sqrt{ab}/\sqrt{\pi}$, and the gradients of gravity anomaly by

$$
g_x(i) = \frac{1}{2a} [\Delta g(i + 1) - \Delta g(i - 1)]$$

(26)

$$
g_y(j) = \frac{1}{2b} [\Delta g(j + 1) - \Delta g(j - 1)]$$

(27)

Inserting $r$, $g_x$ and $g_y$ into Eq. (25), finally, we have

$$\xi_0 = -\frac{\sqrt{b}}{4\gamma \sqrt{a} \sqrt{\pi}} [\Delta g(i + 1) - \Delta g(i - 1)]$$

(28)

$$\eta_0 = -\frac{\sqrt{a}}{4\gamma \sqrt{b} \sqrt{\pi}} [\Delta g(j + 1) - \Delta g(j - 1)]$$

(29)

Using the same gravity anomalies in test area B as used in the last section, we have made a comparison between the 1D FFT deflections and the "ground truth" generated from the corresponding geopotential expansion. Two sets of comparison results have been computed. One is corresponding to the case that the innermost zone effect is omitted, and the other to the case that the effect is considered. The RMS discrepancies between the computed values and the "ground truth" are listed in Table 3. It indicates the inner block contributions are so large that we cannot omit them in practical application.

| Table 3 | The inner block contribution to the $\xi$ deflection/rad·s |
|---------|--------------------------------------------------|
| Degree range | 2°-1 440 | 37°-1 440 | 181°-1 440 | 361°-1 440 |
| inner block contribution | 1.049 | 1.243 | 1.159 | 0.953 |
| RMS to the case one | 3.250 | 3.005 | 1.945 | 1.499 |
| RMS to the case two | 2.814 | 2.310 | 1.030 | 0.648 |

The computation of the vertical deflections in China was done in this study by gridding available free-air data, including marine gravimetry data near the coast, onto a 5' x 5' grid by the well-known weighted average, covering the region (0°-55°N, 70°E-140°E). A set of gravimetric deflections of the vertical was computed by 1D spherical FFT method, using a 100% zero-padded grid of 1 320 x 1 680 points, with subsequent restoration of the EGM96 geoid. Several deflection solutions were computed, corresponding to different integration radii $\Psi_0$.

The results of the vertical deflections over China by applying the spherical 1D FFT formula with different integration radii have been compared with the astro-observed vertical deflections in the South China Sea. The statistical results of differences are given in Table 4.

| Table 4 | Comparison of the deflections between the observed and the computed values/rad·s |
|---------|--------------------------------------|
| $\Psi_0$ | 15' | 30' | 1° | 1.5° | 2° |
| $\xi$ | 3.737 | 3.668 | 3.708 | 3.688 | 3.698 |
| $\eta$ | 2.518 | 2.474 | 2.503 | 2.498 | 2.499 |

From Table 4, it can be seen that the changes of integration caps slightly affect the computed re-
suits. And comparatively speaking, the optimum cap size is somewhere at $\Psi_0 = 30'$. Finally, we choose $\Psi_0 = 30'$ and EGM96 model as the computation parameters and use different FFT methods, including the new 2D FFT approaches mentioned in the second section, to finish the computation of deflections of the vertical in China. The comparison results between different FFT solutions and the observed values are given in Table 5.

Table 5 Comparison of the deflections between the observed and the 2D FFT-computed values/rad·s

|            | $\delta$ | $\eta$ |
|------------|----------|--------|
| new 2D planar | 3.680    | 2.486  |
| new 2D spherical | 3.679    | 2.486  |

Table 5 shows that the results from the new 2D planar and spherical FFT have approached very much those derived from the rigorous 1D FFT method. That means that our work is still valuable to the practical determination of the earth's gravity field, especially over large regions.

It should be pointed out that, as mentioned in Forsberg and Featherstone (1997), depending on the actual area in question, cap sizes may need to be limited, or modified Vening-Meinesz' kernel should be used, in order to obtain optimum computed results, however, different results can be expected, due to probably different systematic errors in the spherical harmonic reference models and/or in the observed gravity data. The lesson is, therefore, that the best kernels and integration cap radii are area-specific, and no universal “optimum” computation parameters exist.

4 Conclusion

The spherical FFT approach has opened the way for rigorous FFT predictions in areas bounded by meridians and parallels. However, in the original form of such method, people have only focused their attentions on the approximation of integration kernel function so that it can be expressed as a function of latitude and longitude differences, as required by the convolution integral. And little attention has been paid to the requirement of 2D FFT method for the input data, i.e., the data blocks must form a rectangular grid. Our work has showed that, if neglecting the latter, it will produce somewhat approximation errors. In order to minimize such errors, we have proposed a new 2D spherical FFT formula. The numerical tests have illustrated the effectiveness of the new formula.

Spectral computation of deflections of the vertical should use a spherical cap of limited extent instead of the whole gravity over China. This causes the FFT approach to more closely mimic numerical integration. Moreover, a geopotential model, indeed, reduces the truncation error associated with performing the convolution over a limited area. Therefore, it is recommended that limited integration caps and a geopotential model up to degree and order 360, at least, be used in future FFT deflection determinations of China.

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