Holographic dark energy with cosmological constant

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Received March 2, 2015
Revised May 5, 2015
Accepted July 11, 2015
Published August 10, 2015

Abstract. Inspired by the multiverse scenario, we study a heterotic dark energy model in which there are two parts, the first being the cosmological constant and the second being the holographic dark energy, thus this model is named the ΛHDE model. By studying the ΛHDE model theoretically, we find that the parameters $d$ and $\Omega_{hde}$ are divided into a few domains in which the fate of the universe is quite different. We investigate dynamical behaviors of this model, and especially the future evolution of the universe. We perform fitting analysis on the cosmological parameters in the ΛHDE model by using the recent observational data. We find the model yields $\chi^2_{\text{min}} = 426.27$ when constrained by Planck + SNLS3 + BAO + HST, comparable to the results of the HDE model (428.20) and the concordant $\Lambda$CDM model (431.35). At 68.3% CL, we obtain $-0.07 < \Omega_0 < 0.68$ and correspondingly $0.04 < \Omega_{hde0} < 0.79$, implying at present there is considerable degeneracy between the holographic dark energy and cosmological constant components in the ΛHDE model.

Keywords: dark energy theory, dark energy experiments

ArXiv ePrint: 1502.01156

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1 Introduction

The holographic dark energy model (HDE) [1, 2] was motivated by the holographic principle [3–10], as one of promising models to solve the nature of dark energy [11–21]. The basic idea behind the HDE is that our universe is a sense finite and can be described by a two dimensional spherical holographic screen, thus there must be finite size effects, and one of these effects is the contribution to the zero point energy, depending on the size of the screen. Parametrically, this contribution assumes the form

\[ \rho_{hde} = 3d^2 M_{pl}^2 L^{-2}, \]  

(1.1)

where \( d \) is a dimensionless parameter to be determined by experiments, \( M_{pl} \) is the reduced Planck mass, and \( L \) denotes the size of holographic screen. For convenience, we work in the natural units, where \( \hbar = c = 1 \). In [2], one of the present authors suggested to choose the future event horizon of the universe as the size of the holographic screen, given by

\[ R_h = a \int_t^{+\infty} \frac{dt}{a}. \]  

(1.2)

This choice not only gives a reasonable value for the dark energy density, but also leads to an accelerated expansion.

The holographic dark energy (HDE) model based on eq. (1.1) and eq. (1.2) has proven to be a promising dark energy candidate. In the original paper [2], Li showed that the HDE can explain the coincidence problem. In [22], it is proven that the model is perturbatively stable. Other studies show that the model is in good agreement with the current cosmological observations [23–25]. Thus, the HDE model becomes one of the most competitive and popular dark energy candidates, and attracts a lot of interests [26–44].
It remains quite a mystery that to date all the papers on the HDE assume that dark energy is dominated by the HDE given by eq. (1.1). In retrospect, this can be explained only by the reason that all the authors believed that the universe dominated by the HDE is the unique universe thus the form of dark energy is also unique. In the multiverse scenario, however, our observable universe is only one out of numerous universes, and the cosmological constant is one of the physics constants varying from one universe to another. Thus, it is not reasonable to simply assume it be vanishing.

If for whatever reason that the multiverse scenario is true, then it is natural that the cosmological constant indeed is a constant to be determined by observations. On the other hand, the HDE on a general ground must be present too, according to the holographic principle. Thus, dark energy must consist of two parts, the first is a constant, the second is of the finite size effect. Indeed, in a calculation of the photon contribution to the zero point energy [40, 45], it is found that in addition to the usual UV divergent part, there is a second divergent part proportional to $L^{-2}$ where $L$ is the radius of the de Sitter space. The usual quartically divergent part can be absorbed into the cosmological constant, the second part is the same form of the HDE. Thus, in general

$$\rho_{IR} = \Lambda + b M_{pl}^2 L^{-2} + b_1 L^{-4} + \ldots .$$  

(1.3)

It just happens that in our universe, the first term and the second term are comparable, and the third term is much smaller and can be neglected completely.

In this paper, we shall study this heterotic model of dark energy, in which there are the cosmological constant and the HDE. Thus we have

$$\rho_{de} = \rho_\Lambda + \rho_{hde}.$$  

(1.4)

We shall name this model ΛHDE model. This model raises interesting theoretically questions: is the CC positive or negative? If it is positive, whether it is greater or smaller than the HDE? Is the future of our universe dominated by the CC or by HDE? Will the big rip happen or not? In the rest of this paper, we shall try to answer these questions.

This paper is organized as follows. We study the ΛHDE model theoretically in section 2, and find that the parameters $d$ and $\Omega_{hde}$ are divided into a few domains in which the fate of the universe is quite different. In section 3, we fit the model to the combined Planck + SNLS3+BAO+HST and Planck+SNLS3+BAO+HST+SDSS−Lyα datasets, and present the fittings results in section 4. Many interesting issues, including the ratio of HDE and the equation of state (EoS) of the ΛHDE are discussed. Some concluding remarks are given in section 5. In this work, we assume today’s scale factor $a_0 = 1$, so the redshift $z$ satisfies $z = 1/a - 1$. The subscript “0” indicates the present value of the corresponding quantity unless otherwise specified.

2 ΛHDE model: theoretical analysis

In this section, we will write down the basic equations for the ΛHDE model in a non-flat universe and study the fate of the universe in this model.

2.1 HDE with cosmological constant

In a spatially non-flat Friedmann-Robertson-Walker universe, the Friedmann equation can be written as

$$3M_{pl}^2 H^2 = \rho_{dm} + \rho_b + \rho_r + \rho_k + \rho_{de},$$  

(2.1)
where \( \rho_k = -3M^2_{\text{pl}} \frac{k}{a^2} \) is the effective energy density of the curvature component. In the \( \Lambda \)HDE model, the dark energy density is

\[
\rho_{\text{de}} = \rho_{\Lambda} + \rho_{hde} = M^2_{\text{pl}} \Lambda + 3M^2_{\text{pl}}d^2 R_h^{-2}.
\]

(2.2)

For convenience, we define the fractional energy densities of the various components, i.e.,

\[
\Omega_k = -\frac{k}{H^2 a^2}, \quad \Omega_{\text{de}} = \frac{\rho_{\text{de}}}{\rho_c}, \quad \Omega_{hde} = \frac{\rho_{hde}}{\rho_c}, \quad \Omega_{\Lambda} = \frac{\rho_{\Lambda}}{\rho_c},
\]

\[
\Omega_{\text{dm}} = \frac{\rho_{\text{dm}}}{\rho_c}, \quad \Omega_{b} = \frac{\rho_{b}}{\rho_c}, \quad \Omega_{r} = \frac{\rho_{r}}{\rho_c},
\]

(2.3)

where \( \rho_c = 3M^2_{\text{pl}}H^2 \) is the critical density of the universe. The subscripts, “k”, “de”, “hde”, “\( \Lambda \)”, “dm”, “b” and “r” represent curvature, total dark energy, holographic dark energy, cosmological constant, dark matter, baryon and radiation, respectively. By definition, we have

\[
\Omega_{hde} + \Omega_{\Lambda} + \Omega_{\text{dm}} + \Omega_{b} + \Omega_{r} + \Omega_k = 1.
\]

(2.4)

The energy conservation equations for the components in the universe take the forms

\[
\dot{\rho}_{hde} + 3H(\rho_{hde} + p_{hde}) = 0, \quad (2.5)
\]

\[
\dot{\rho}_{\Lambda} = 0, \quad \dot{\rho}_{dm} + 3H\rho_{dm} = 0, \quad \dot{\rho}_{b} + 3H\rho_{b} = 0, \quad \dot{\rho}_{r} + 4H\rho_{r} = 0, \quad \dot{\rho}_{k} + 2H\rho_{k} = 0.
\]

(2.6)

Combining eq. (2.5) and eq. (2.6) together, we can obtain the form of \( p_{hde} \),

\[
p_{hde} = -2 \frac{\dot{H}}{3H^2 \rho_c} - \rho_c - \frac{1}{3} \rho_r + 1 \frac{1}{3} \rho_k + \rho_{\Lambda}.
\]

(2.7)

Substituting \( p_{hde} \) into eq. (2.5), follow the similar procedure in ref. [46], we get a differential equation of \( \dot{H} \) and \( \dot{\Omega}_{hde} \):

\[
2(\Omega_{hde} - 1) \dot{H} + \dot{\Omega}_{hde} + H(3\Omega_{hde} - 3 + 3\Omega_{\Lambda} + \Omega_k - \Omega_r) = 0.
\]

(2.8)

From the energy density of the HDE in eq. (1.1), we have

\[
L = \frac{d}{H \sqrt{\Omega_{hde}}},
\]

(2.9)

Following ref. [47], in a non-flat universe, the IR cut-off length scale \( L \) takes the form

\[
L = ar(t),
\]

(2.10)

and \( r(t) \) satisfies

\[
\int_{0}^{r(t)} \frac{dr}{\sqrt{1 - kr^2}} = \int_{t}^{+\infty} \frac{dt}{a(t)}.
\]

(2.11)

By carrying out the integration, we find

\[
r(t) = \frac{1}{\sqrt{|k|}} \sin \left( \sqrt{|k|} \int_{t}^{+\infty} \frac{dt}{a} \right) = \frac{1}{\sqrt{|k|}} \sin \left( \sqrt{|k|} \int_{a(t)}^{+\infty} \frac{da}{Ha^2} \right),
\]

(2.12)
where the function \( \sinh(x) \) is defined as

\[
\sinh(x) = \begin{cases} 
\sin(x) & k > 0; \\
0 & k = 0; \\
\sinh(x) & k < 0. 
\end{cases}
\]

Equation (2.10) leads to another equation about \( r(t) \), namely,

\[
r(t) = \frac{L}{a} = \frac{d}{\sqrt{\Omega_{\text{hde}} H a}}.
\]  

(2.13)

Combining eqs. (2.12) and (2.13) yields

\[
\sqrt{|k|} \int_{t}^{+\infty} \frac{dt}{a} = \text{arsin}\left(\frac{\sqrt{|k|} d}{\sqrt{\Omega_{\text{hde}} a H}}\right). 
\]  

(2.14)

Taking derivative of eq. (2.14) with respect to \( t \), one gets

\[
\frac{\dot{\Omega}_{\text{hde}}}{2\Omega_{\text{hde}}} + \frac{H}{H} = \frac{\Omega_{\text{hde}} H^2}{d^2} - \frac{k}{a^2}. 
\]  

(2.15)

### 2.2 Evolution equations of \( E(z) \) and \( \Omega_{\text{hde}}(z) \)

Combining eq. (2.8) with eq. (2.15), we eventually obtain the following two equations governing the dynamical evolution of the AHDE model in a non-flat universe,

\[
\frac{1}{E(z)} \frac{dE(z)}{dz} = -\Omega_{\text{hde}} \left( \frac{3\Omega_{\Lambda} + \Omega_{k} - \Omega_{r} - 3}{2\Omega_{\text{hde}}} + \frac{1}{2} + \sqrt{\frac{\Omega_{\text{hde}}}{d^2} + \Omega_{k}} \right),
\]  

(2.16)

\[
\frac{d\Omega_{\text{hde}}}{dz} = -2\Omega_{\text{hde}} (1 - \Omega_{\text{hde}}) \left( \frac{\Omega_{\text{hde}}}{d^2} + \Omega_{k} + \frac{1}{2} - \frac{3\Omega_{\Lambda} + \Omega_{k} - \Omega_{r}}{2(1 - \Omega_{\text{hde}})} \right),
\]  

(2.17)

where \( E(z) \equiv H(z)/H_0 \) is the dimensionless Hubble expansion rate. Notice that we have

\[
\Omega_{k}(z) = \frac{\Omega_k(1 + z)^2}{E(z)^2}, \quad \Omega_{r}(z) = \frac{\Omega_r(1 + z)^4}{E(z)^2}, \quad \Omega_{\Lambda}(z) = \frac{\Omega_{\Lambda 0}}{E(z)^2}, \quad \Omega_{b}(z) = \frac{\Omega_{b 0}(1 + z)^3}{E(z)^2},
\]  

(2.18)

and the fractional density of dark matter is given by \( \Omega_{\text{dm}}(z) = 1 - \Omega_{k}(z) - \Omega_{\text{hde}}(z) - \Omega_{\Lambda}(z) - \Omega_{r}(z) - \Omega_{b}(z) \). Equations (2.16) and (2.17) can be solved numerically and will be used in the data analysis procedure.

### 2.3 Dark energy equation of state

The EoS of the HDE takes the form [48]

\[
w_{\text{hde}} = \frac{p_{\text{hde}}}{\rho_{\text{hde}}} = -\frac{1}{3} - \frac{2}{3} \sqrt{\frac{\Omega_{\text{hde}}}{d^2} + \Omega_{k}}.
\]  

(2.19)

So according to the partial pressure law, the EoS of the total dark energy is

\[
w_{\text{de}} = \frac{p_{\text{de}}}{\rho_{\text{de}}} = \frac{p_{\text{hde}} + p_{\Lambda}}{\rho_{\text{hde}} + \rho_{\Lambda}} = \frac{w_{\text{hde}} \Omega_{\text{hde}} - \Omega_{\Lambda}}{\Omega_{\text{hde}} + \Omega_{\Lambda}}.
\]  

(2.20)

Obviously the property of \( w_{\text{de}} \) is closely related to values of \( d \) and \( \Omega_{\Lambda} \).
2.4 The fate of the Universe in ΛHDE model

For convenience, we transform eq. (2.1) into the following form

\[(1 - \Omega_{hde})H^2 = \Omega_{m0}H_0^2a^{-3} + \Omega_{r0}H_0^2a^{-4} + \Omega_{k0}H_0^2a^{-2} + \Omega_{\Lambda0}H_0^2,\] (2.21)

and define

\[f(a) \equiv \Omega_{m0}H_0^2a^{-1} + \Omega_{r0}H_0^2a^{-2} + \Omega_{k0}H_0^2 + \Omega_{\Lambda0}H_0^2a^2.\] (2.22)

Let \(x \equiv \log a\), then we obtain

\[H = \sqrt{\frac{f(a)}{a^2(1 - \Omega_{hde})}},\] (2.23)

\[\frac{d}{dx} \ln \left| \frac{\Omega_{hde}}{1 - \Omega_{hde}} \right| + \frac{d}{dx} \ln |f(a)| = \frac{2}{d} \sqrt{\Omega_{hde} - (1 - \Omega_{hde}) \frac{kd^2}{f(a)}}.\] (2.24)

Eq. (2.24) tells us how the HDE evolves with \(a\). For simplicity, in this section we only study the \(k = 0\) case. Then eq. (2.24) becomes

\[\frac{d}{dx} \ln \left| \frac{\Omega_{hde}}{1 - \Omega_{hde}} \right| + \frac{d}{dx} \ln |f(a)| = \frac{2}{d} \sqrt{\Omega_{hde}}.\] (2.25)

This equation can not be solved exactly. But for the purpose of studying the fate of the universe, we can introduce a good approximation.

During a period in which \(f(a)\) is dominated by a single term on the right hand side of eq. (2.22), we can use a constant \(k_r\) to approximate \(\frac{d}{dx} \ln |f(a)|\). Then the eq. (2.25) becomes

\[\frac{d}{dx} \ln \left| \frac{\Omega_{hde}}{1 - \Omega_{hde}} \right| = \frac{2}{d} \sqrt{\Omega_{hde}} - k_r.\] (2.26)

It is easy to see that the cosmological constant term on the right hand side of eq. (2.22) will dominate when scale factor \(a\) evolves to infinity. Thus we get \(k_r = 2\) in this limit. Fortunately, it can be proved that \(a\) will always evolve to infinity. The proof is given in the appendix.

With \(k_r = 2\), we can rewrite the eq. (2.26) and get its solution as follows:

\[\frac{d\Omega_{hde}}{dx} = \frac{2}{d}(\sqrt{\Omega_{hde}} - d)(1 - \Omega_{hde})\Omega_{hde},\] (2.27)

\[x + x_1 = \frac{d}{2d - 2} \ln \left| 1 - \sqrt{\Omega_{hde}} \right| - \frac{1}{(d^2 - 1)} \ln \sqrt{\Omega_{hde} - d} - \frac{1}{2} \ln \Omega_{hde} + \frac{d}{2d + 2} \ln (1 + \sqrt{\Omega_{hde}}),\] (2.28)

where \(x_1\) is the constant of integration. Then let us exhibit how the HDE evolves: the first three pictures in figure 1 show how \(\sqrt{\Omega_{hde}}\) evolves with the initial condition \(\sqrt{\Omega_{hde0}} < 1\). In both picture (1a) and picture (1b), \(\sqrt{\Omega_{hde0}} < d\), the right hand of eq. (2.27) is negative, so \(\sqrt{\Omega_{hde0}}\) will decrease with \(x\). Thus the right hand of eq. (2.27) will always be negative. So \(\sqrt{\Omega_{hde0}}\) will keep decreasing to 0. According to eq. (2.28) when \(\sqrt{\Omega_{hde0}} \to 0^+, x \to +\infty\). Thus once \(\sqrt{\Omega_{hde0}} < 1\), either \(d < 1\) or \(d > 1\), according to eq. (2.23), \(H\) will always tend to a constant, which means space-time will be de Sitter in the future. In picture (1c),
Figure 1. The evolution of $\sqrt{\Omega_{\text{hde}}}$ with different initial conditions.

$\sqrt{\Omega_{\text{hde}}}$ will always be positive. $\sqrt{\Omega_{\text{hde0}}}$ will keep increasing to 1, according to eq. (2.28) when $\sqrt{\Omega_{\text{hde0}}} \to 1^+, x \to +\infty$. So this solution describes a HDE dominated universe with big rip.

The last three pictures in figure 1 show how $\sqrt{\Omega_{\text{hde}}}$ evolves with the initial condition $\sqrt{\Omega_{\text{hde0}}} > 1$. In picture (1d), $\sqrt{\Omega_{\text{hde0}}} > d$, the right hand of eq. (2.27) is negative, so when $d < 1$, the $\sqrt{\Omega_{\text{hde0}}}$ will keep decreasing with x all the way up to 1. According to eq. (2.28) when $\sqrt{\Omega_{\text{hde0}}} \to 1^+, x \to +\infty$. So picture (1d) describes a HDE dominated universe with big rip. In picture (1e) (or in picture (1f)), $d < \sqrt{\Omega_{\text{hde0}}}$ (or $d > \sqrt{\Omega_{\text{hde0}}}$) the right hand of eq. (2.27) will be negative (positive), so the $\sqrt{\Omega_{\text{hde0}}}$ will keep decreasing (or increasing) to d. According to eq. (2.28), when $\sqrt{\Omega_{\text{hde0}}} \to d, x \to +\infty$. So they both describe an universe whose space-time will be de Sitter in the future.

3 The observational data and methodology

We explore cosmological constraints on the $\Lambda$HDE model with the most recent observational data. For comparison, we will also present the fitting results of the original HDE and ΛCDM models. Data used in our analysis include:

- The SNLS3 combined sample [49, 50], consisting of 472 SNIa, combining the results of two light-curve fitting codes SiFTO [51] and SALT2 [52].

It should be mentioned that, previous studies on the SNLS3 data sets [53] found strong evidence for the redshift-dependence of color-luminosity parameter $\beta$, and this conclusion has significant effects on parameter estimation of various cosmological models [54–57]. In addition, different light-curve fitters of SNIa can also affect the results of cosmology-fits [58–60]. But in this work, for simplicity, we just adopt the most mainstream recipe of processing SNLS3 data and do not consider the factors of time-varying $\beta$ and different light-curve fitters.
where $C$ is a $472 \times 472$ covariance matrix capturing the statistic and systematic uncertainties, and $\Delta \vec{m} = \vec{m}_B - \vec{m}_{\text{mod}}$ is a vector of model residuals of the SNIa sample, with $m_B$ the rest-frame peak $B$ band magnitude of the SNIa and $m_{\text{mod}}$ the predicted magnitude of the SNIa, given by

$$m_{\text{mod}} = 5 \log_{10} D_L - \alpha (s - 1) + \beta C + M, \quad (3.2)$$

where $D_L$ is the Hubble-constant free luminosity distance, the stretch $s$ is a measure of the shape of SN light-curve, $C$ is the color measure for the SN, and $\alpha, \beta$ are two nuisance parameters characterizing the stretch-luminosity and color-luminosity relationships, respectively. Following [49], we treat $\alpha$ and $\beta$ as free parameters of $\chi^2$ function.

- The Planck “distance priors” provided in [62], which are extracted from Planck first year [63, 64] observations. The data include the baryon component $\omega_b \equiv \Omega_b h^2$, the “acoustic scale” $l_a \equiv \pi r(z_\ast)/r_s(z_\ast)$, and the “shift parameter” $R \equiv \sqrt{\Omega_m H_0^2} r(z_\ast)$, where $z_\ast$ is the redshift to the photon-decoupling surface [65], $r(z_\ast)$ is our comoving distance to $z_\ast$, and $r_s(z_\ast)$ is the comoving sound horizon at $z_\ast$. The distance priors provide an efficient summary of the CMB data in regards to dark energy constraints [66].

- The BAO data including the measurement of $r_s/D_V$ at $z = 0.106$ from 6dFGS [67], the isotropic measurement of $D_V/r_d$ at $z = 0.32$ from the BOSS DR11 LOWZ sample [68], the anisotropic measurement of $D_A/r_d$ and $H r_d$ at $z = 0.57$ from the BOSS DR11 CMASS sample [68], and the improved measurements of $D_V/r_s$ at $z = 0.44, 0.60, 0.73$ from the WiggleZ Dark Energy Survey [69]. Here $r_d$ is the comoving sound horizon at the “drag” epoch when the baryons are “released” from the drag of the photons [70], and $D_V$ is a volume averaged distance indicator similar to the angular diameter distance $D_A$ [71].

- The Hubble constant measurement $H_0 = 73.8 \pm 2.4$km/s/Mpc from the WFC3 on the HST (Hubble Space Telescope) [72].

- The high-redshift BAO measurement from the Quasar-Ly$\alpha$-forest cross-correlation of the BOSS DR11 of SDSS-III [73], namely $\alpha_\parallel^{0.7} \alpha_\perp^{0.3} = 1.025 \pm 0.0211$; $\alpha_\parallel$ and $\alpha_\perp$ are defined as $\alpha_\parallel = \frac{[D_m(z)/r_d]_{\text{fid}}}{[D_H(z)/r_d]_{\text{fid}}}$, $\alpha_\perp = \frac{[D_A(z)/r_d]_{\text{fid}}}{[D_A(z)/r_d]_{\text{fid}}}$, where $\bar{z} = 2.34$, the fiducial values $[D_H(z)/r_d]_{\text{fid}}$ and $[D_A(z)/r_d]_{\text{fid}}$ are 8.708 and 11.59 respectively.

In the following context, we will use “SNLS3”, “Planck”, “BAO”, “HST” and “SDSS – Ly$\alpha$” to represent these five datasets.

We combine the above data sets to perform $\chi^2$ analyses. Since SNLS3, Planck, BAO, HST and SDSS-Ly$\alpha$ are effectively independent measurements, the total $\chi^2$ function is just the sum of all individual $\chi^2$ functions:

$$\chi^2_{\text{total}} = \chi^2_{\text{SNLS3}} + \chi^2_{\text{Planck}} + \chi^2_{\text{BAO}} + \chi^2_{\text{HST}} + \chi^2_{\text{SDSS–Ly$\alpha$}}. \quad (3.3)$$

In our work, for a detailed investigation, we do fittings with two datasets: Planck + SNLS3 + BAO + HST and Planck + SNLS3 + BAO + HST + SDSS – Ly$\alpha$, respectively.

The LHDE model has two dark energy parameters $d$, and $\Omega_\Lambda$. Including four other cosmological parameters $\Omega_m h^2$, $\omega_b$, $\Omega_k$ and $h$, and two nuisance parameters $\alpha$, $\beta$ characterizing the systematic errors of the SNLS3 dataset [49], the full set of free parameters in our analysis is

$$P = \{\Omega_m h^2, \omega_b, h, c, \Omega_\Lambda, \Omega_k, \alpha, \beta\}. \quad (3.4)$$
### Table 1. Fitting results for the ΛHDE model

| Parameter | Planck + SNLS3 + BAO + HST | Planck + SNLS3 + BAO + HST + SDSS − Lyα |
|-----------|-----------------------------|------------------------------------------|
| $\Omega_m h^2$ | 0.1405, 0.1413$^{+0.0025}_{-0.0025}$ | 0.1375, 0.1400$^{+0.0025}_{-0.0025}$ |
| $H_0$ | 73.3, 70.5$^{+1.4}_{-1.8}$ | 71.6, 70.5$^{+1.3}_{-1.4}$ |
| $d$ | 0.003, 0.570$^{+0.320}_{-0.180}$ | 0.001, 0.308$^{+0.075}_{-0.308}$ |
| $\Omega_{\Lambda 0}$ | 0.61, 0.07$^{+0.14}_{-0.07}$ | 0.63, 0.42$^{+0.25}_{-0.10}$ |
| $\Omega_{k 0}$ | $-0.0012$, 0.0037$^{+0.0037}_{-0.0059}$ | $-0.0063$, $-0.0013$ $^{+0.0028}_{-0.0032}$ |

Where $\gamma$ represents photons, and $N_{\text{eff}}$ is the effective number of neutrino species.

We modify the public available CosmoMC package [75] to explore the parameter space using the Markov Chain Monte Carlo (MCMC) algorithm. All the parameters listed in eq. (3.4) are fitted simultaneously.

### 4 Dynamical behaviors and the cosmic expansion history

#### 4.1 Fitting results

In table 1, we give best-fit parameters as well as 68.3% confidence limits for constrained parameters. The results show that the spatial curvatures are close to zero in both cases (the 68.3% confidence limits are $|\Omega_{k 0}| < 0.007$ and $|\Omega_{k 0}| < 0.004$, respectively). Thus the results are impressively consistent with a spatially flat universe. Table 1 also gives the constraint on the parameter $\Omega_{\Lambda 0}$:

- Planck + SNLS3 + BAO + HST: $0.07 < \Omega_{\Lambda 0} < 0.68$ (68.3% CL);
- Planck + SNLS3 + BAO + HST + SDSS − Lyα: $0.32 < \Omega_{\Lambda 0} < 0.67$ (68.3% CL).

Combine with the constraint results of $\Omega_m h^2$ and $H_0$ listed in table 1 together, we get the corresponding constraint on $\Omega_{hde 0}$:

- Planck + SNLS3 + BAO + HST: $0.04 < \Omega_{hde 0} < 0.79$ (68.3% CL);
- Planck + SNLS3 + BAO + HST + SDSS − Lyα: $0.06 < \Omega_{hde 0} < 0.41$ (68.3% CL).

Figure 2 shows the marginalized likelihood distributions of parameter $d$, $\Omega_m h^2$ and $\Omega_{\Lambda 0}$ constrained by Planck+SNLS3+BAO+HST and Planck+SNLS3+BAO+HST+SDSS−Lyα datasets, respectively. The 68.3% and 95.4% contours of $\Omega_{\Lambda 0}$-$\Omega_m h^2$, $d$-$\Omega_m h^2$ and $\Omega_{\Lambda 0}$-$d$ planes are plotted in figure 3.

The results show that, compared with the Planck + SNLS3 + BAO + HST dataset, the Planck + SNLS3 + BAO + HST + SDSS − Lyα dataset makes a more tighter constraints on
Figure 2. Marginalized likelihood distribution of $\Omega_m h^2$, $\Omega_{\Lambda 0}$, $d$ constrained by Planck + SNLS3 + BAO + HST (solid lines) and Planck + SNLS3 + BAO + HST + SDSS – Ly$\alpha$ (dashed lines) datasets.

Figure 3. Marginalized 68.3% and 95.4% CL contours of $\Omega_{\Lambda 0}$, $\Omega_m h^2$, $d$, $\Omega_m h^2$ and $\Omega_{\Lambda 0}$–$d$ planes constrained by Planck + SNLS3 + BAO + HST (solid lines) and Planck + SNLS3 + BAO + HST + SDSS – Ly$\alpha$ (dotted lines) datasets.

$d$ and $\Omega_{\Lambda 0}$ parameters. We also find that a smaller value of $\Omega_m h^2$ and a bigger value of $\Omega_{\Lambda 0}$ is favored by the Planck + SNLS3 + BAO + HST + SDSS – Ly$\alpha$ dataset.

There is a degeneracy between $d$ and $\Omega_{\Lambda 0}$, which can be seen in figure 3. The reason is that both the cosmological constant and HDE are good candidate for explaining the feature of cosmic acceleration revealed by current observational data. Therefore, when we combine the HDE and cosmological constant components together (thus $\Lambda$HDE model), we may probably get the degeneracy.

In addition to the cosmological consequence of the $\Lambda$HDE model, we are also interested in its comparison with the $\Lambda$CDM and the original HDE models. Therefore we also perform the $\chi^2$ analysis of the $\Lambda$CDM and the HDE models by using the same datasets. To assess different models, here we adopt the Akaike information criteria (AIC) [76] and Bayesian information criteria (BIC) [77], defined as

$$ AIC = \chi^2_{\text{min}} + 2k, \quad BIC = \chi^2_{\text{min}} + k \ln N, $$

(4.1)

where $k$ is the number of free parameters, and $N$ is the number of data points used in the fits. A model with smaller AIC (BIC) is more favored.

Table 2 shows the $\chi^2_{\text{min}}$s, AICs and BICs of the $\Lambda$CDM, HDE and $\Lambda$HDE models. Notice that the values of the AIC and BIC themselves are not interesting, thus we only list the difference between the $\Lambda$HDE (HDE) and $\Lambda$CDM models, i.e.,

$$ \Delta \text{AIC} \equiv \text{AIC}_{\text{model}} - \text{AIC}_{\Lambda \text{CDM}}, \quad \Delta \text{BIC} \equiv \text{BIC}_{\text{model}} - \text{BIC}_{\Lambda \text{CDM}} $$

(4.2)
Planck + SNLS3 + BAO + HST Planck + SNLS3 + BAO + HST + SDSS − Lyα

| Model   | $\chi^2_{\text{min}}$ | $\Delta\text{AIC}$ | $\Delta\text{BIC}$ | $\chi^2_{\text{min}}$ | $\Delta\text{AIC}$ | $\Delta\text{BIC}$ |
|---------|------------------------|---------------------|---------------------|------------------------|---------------------|---------------------|
| $\Lambda$CDM | 431.35                | 0                   | 0                   | 438.22                 | 0                   | 0                   |
| HDE     | 428.20                | −1.15               | 3.03                | 438.19                 | 1.97                | 6.15                |
| $\Lambda$HDE | 426.27               | −1.06               | 7.27                | 431.79                 | −2.43               | 5.92                |

Table 2. The $\chi^2_{\text{min}}$ s, $\Delta\text{AIC}s$ and $\Delta\text{BIC}s$ of the $\Lambda$CDM, HDE and $\Lambda$HDE models, obtained by using the Planck + SNLS3 + BAO + HST and Planck + SNLS3 + BAO + HST + SDSS − Lyα datasets, respectively.

Figure 4. Reconstructed evolution history of $H(z)$ (95.4% CL) in $\Lambda$HDE model, constrained by the Planck + SNLS3 + BAO + HST (red solid lines) and Planck + SNLS3 + BAO + HST + SDSS − Lyα (black dashed lines) datasets, respectively.

Compared with the $\Lambda$CDM and HDE models, the $\Lambda$HDE model provides a better fit to the data. For Planck + SNLS3 + BAO + HST dataset, the $\Lambda$HDE model reduces the $\chi^2_{\text{min}}$ s by amount of 5.08 (1.93) compared with the $\Lambda$CDM (HDE) model. While for Planck + SNLS3 + BAO + HST + SDSS − Lyα dataset, the $\Lambda$HDE model reduces the $\chi^2_{\text{min}}$ s by amount of about 6.4 compared with the other two models. By adopting the AIC and BIC, table 2 also shows that, compared with $\Lambda$CDM model, the $\Lambda$HDE model is slightly favored by AIC. But both the $\Lambda$HDE and HDE models are not favored by BIC, though these models have slightly smaller $\chi^2_{\text{min}}$ s than the $\Lambda$CDM model.

4.2 The expansion history

It is worth investigating the cosmic expansion history of the $\Lambda$HDE model by the fitting results.

In figure 4, we plot the reconstructed evolution history of $H(z)$ (95.4% CL) in $\Lambda$HDE model, constrained by the Planck+SNLS3+BAO+HST and Planck+SNLS3+BAO+HST+SDSS − Lyα datasets, respectively. We find that, in low redshift region, the reconstructed evolution history $H(z)$ of the Planck + SNLS3 + BAO + HST and Planck + SNLS3 + BAO + HST + SDSS − Lyα datasets are almost the same. However, in the high redshift region, the
constraint of the Planck + SNLS3 + BAO + HST + SDSS – Lyα dataset is much more tighter than that of the Planck + SNLS3 + BAO + HST dataset. It is clear that this feature is due mainly to the SDSS-Lyα data at redshift $z = 2.34$. As revealed by [78, 79], the high redshift datasets play a big part in constraining the cosmic expansion history.

For a comparison, it is also of value to compare the $\Lambda$HDE model with the original HDE model. Figure 5 shows the reconstructed evolution history of $H(z)$ (95.4% CL) in the $\Lambda$HDE and the original HDE model constrained by Planck + SNLS3 + BAO + HST + SDSS – Lyα dataset. We find that the reconstructed $H(z)$ of $\Lambda$HDE and HDE models have negligible difference in the low redshift region. However, in the high redshift region, the $H(z)$ in $\Lambda$HDE model has slightly lower value, which should be due mainly to the existence of a cosmological constant component in the model.

4.3 Equation of state

In this subsection we discuss the EoS $w$, which is believed to be the most important marker of the properties of dark energy.

Figure 6 shows the reconstructed evolution history of $w(z)$ at $0 \leq z \leq 2.5$ (68.3% and 95.4% CL) constrained by Planck + SNLS3 + BAO + HST + SDSS – Lyα dataset. It shows that $w$ slightly cross -1 from above roughly at the current epoch. However, in the past we have $w$ slightly bigger than -1, which can be viewed as a feature of diluted holographic dark energy. This behavior is consistent with the results shown by figure 5.

As mentioned above, the dynamical evolution of dark energy have not be confirmed by the current observational data. Our results is consistent with this statement.

From figure 4, we can conclude that, if we want to break the degeneracy between the HDE and cosmological constant components, one way is to get more observational data at high redshifts.
5 Conclusion

In this work, we study the ΛHDE model in which there are two parts, the first being the cosmological constant and the second being the holographic dark energy. This model has similar dynamical equations with the original HDE model, except a cosmological term. By studying the ΛHDE model theoretically, we find that the parameters \( d \) and \( \Omega_{hde} \) are divided into a few domains in which the fate of the universe is quite different.

Using the Planck + BAO + SNLS3 + HST and Planck + BAO + SNLS3 + SDSS − Ly\( \alpha \) datasets, we investigate the dynamical properties and cosmic expansion history of the ΛHDE model. The results shows that the goodness-of-fit of the ΛHDE model are \( \chi^2_{\text{min}}=426.27 \) (Planck+SNLS3+BAO+HST) and \( \chi^2_{\text{min}}=431.79 \) (Planck+SNLS3+BAO+SDSS−Ly\( \alpha \)) which is smaller then the results of the original HDE model (Planck + SNLS3 + BAO + HST:428.20; Planck + SNLS3 + BAO + HST + SDSS − Ly\( \alpha \):438.19) and the concordant ΛCDM model (Planck + BAO + SNLS3 + HST:431.35; Planck + SNLS3 + BAO + HST + SDSS − Ly\( \alpha \):438.22) obtained using the same datasets. Especially when constrained by the Planck+SNLS3+BAO+HST+SDSS−Ly\( \alpha \) dataset, The \( \chi^2_{\text{min}} \) of ΛHDE model shrinks more than 6, compared with both the HDE and ΛHDE model. Thus, the ΛHDE model provides a nice fit to the cosmological data.

For parameter \( \Omega_{A0} \), the 68.3% confidence level constrained by the Planck + SNLS3 + BAO + HST and the Planck + SNLS3 + BAO + HST + SDSS − Ly\( \alpha \) dataset is \(-0.07 < \Omega_{A0} < 0.68 \) and \( 0.32 < \Omega_{A0} < 0.67 \), respectively. This gives the corresponding components of the holographical dark energy, namely

\[
\text{Planck + BAO + SNLS3 + HST : } 0.04 < \Omega_{hde0} < 0.79; \\
\text{Planck + SNLS3 + BAO + HST + SDSS − Ly\( \alpha \) : } 0.06 < \Omega_{hde0} < 0.41.
\]

We also find that there is degeneracy between the cosmological constant and the holographic dark energy component when constrained by current cosmological observations. By reconstructing the evolution of the EoS of dark energy, we find the ΛHDE mainly differs from the original HDE model at high redshift (as shown in figure 5).
From the constraint results by Planck + SNLS3 + BAO + HST + SDSS − Lya dataset, it shows that if we want to break the degeneracy between the HDE and cosmological constant components, one way is to get more observational data at high redshifts.

Acknowledgments

We are grateful to the Referee for the valuable suggestions. We also thank Xiao-Dong Li and Shuang Wang for their helps on fitting issues. ML is supported by the National Natural Science Foundation of China (Grant No. 11275247, and Grant No. 11335012) and 985 grant at Sun Yat-Sen University.

A Proof for log a not having a maximum in ΛHDE model

In this appendix, we prove that \( x \) does not have a maximum as a function of time. Since \( x \) does not have a maximum in either ΛCDM or HDE model, here we only consider ΛHDE model, which includes the coexistence case of HDE and cosmological constant.

In a flat universe, the Friedmann equation is

\[
3M_p^2 H^2 = \rho_{dm} + \rho_b + \rho_r + \rho_\Lambda + \rho_{hde},
\]

Let us note that

\[
g(x) = \frac{d}{dx} \ln |f(a)|, \tag{A.2}
\]

then

\[
g(x) = \frac{2\Omega_\Lambda e^{2x} - \Omega_m e^{-x} - 2\Omega_r e^{-2x}}{\Omega_\Lambda e^{2x} + \Omega_m e^{-x} + \Omega_r e^{-2x}}. \tag{A.3}
\]

According to eq. (2.25), we get

\[
\frac{d\Omega_{hde}}{dx} = \left[ \frac{2}{d} \sqrt{\Omega_{hde} - g(x)} \right] \Omega_{hde} (1 - \Omega_{hde}). \tag{A.4}
\]

We first consider the cases that \( \Omega_\Lambda > 0 \) or \( \Omega_\Lambda + \Omega_m + \Omega_r < 0 \). In these cases, \( g(x) \) is a bounded function in the region \( (x_0, +\infty) \). If \( x \) has a maximum \( x_m \), \( H \) would approach zero when \( x \) goes to \( x_m \). According to eq. (A.1), \( \Omega_{hde} \) would approach infinity. However, eq. (A.4) shows that, once \( \sqrt{\Omega_{hde}} > \frac{d}{dx} g(x) \) and \( \Omega_{hde} > 1, \frac{d}{dx} \sqrt{\Omega_{hde}} < 0 \). Thus \( \Omega_{hde} \) would never approach infinity. So \( x \) has no maximum in these cases.

Another case is \( \Omega_\Lambda < 0 \) and \( \Omega_\Lambda + \Omega_m + \Omega_r > 0 \). In this case

\[
\exists x_c > 0 : \quad \Omega_\Lambda e^{2x_c} + \Omega_m e^{-x_c} + \Omega_r e^{-2x_c} = 0. \tag{A.5}
\]

So eq. (A.4) has a singularity at \( x = x_c \). Similar to the previous analysis, \( x \) does not have a maximum in the region \( (x_0, x_c) \). Now we prove that \( x_c \) is not a maximum of \( x \). If it is, \( g(x) \) would approach negative infinity and \( \Omega_{hde} \) would approach infinity when \( x \) goes to \( x_c \). However, once \( \Omega_{hde} > 1, \frac{d}{dx} \sqrt{\Omega_{hde}} < 0 \). Thus \( \Omega_{hde} \) would never approach infinity. So \( x_c \) is not a maximum of \( x \). When \( x > x_c \), we can see that \( \Omega_\Lambda + \Omega_m + \Omega_r < 0 \). This is just the case we have discussed. So \( x \) also has not a maximum when \( x > x_c \).

In summary, we have proved that \( x \) does not have a maximum. So we can use a constant \( k_r \) to approximate \( \frac{d}{dx} \ln |f(a)| \) when we study the fate of the universe.
References

[1] A.G. Cohen, D.B. Kaplan and A.E. Nelson, Effective field theory, black holes and the cosmological constant, *Phys. Rev. Lett.* 82 (1999) 4971 [hep-th/9803132] [inSPIRE].

[2] M. Li, A Model of holographic dark energy, *Phys. Lett.* B 603 (2004) 1 [hep-th/0403127] [inSPIRE].

[3] G. ’t Hooft, Dimensional Reduction in Quantum Gravity, *gr-qc/9310026*.

[4] L. Susskind, The World as a hologram, *J. Math. Phys.* 36 (1995) 6377 [hep-th/9409089] [inSPIRE].

[5] J.D. Bekenstein, Black holes and entropy, *Phys. Rev.* D 7 (1973) 2333 [inSPIRE].

[6] J.D. Bekenstein, Generalized second law of thermodynamics in black hole physics, *Phys. Rev.* D 9 (1974) 3292 [inSPIRE].

[7] J.D. Bekenstein, A Universal Upper Bound on the Entropy to Energy Ratio for Bounded Systems, *Phys. Rev.* D 23 (1981) 287 [inSPIRE].

[8] J.D. Bekenstein, Entropy bounds and black hole remnants, *Phys. Rev.* D 49 (1994) 1912 [gr-qc/9307035] [inSPIRE].

[9] S.W. Hawking, Particle Creation by Black Holes, *Commun. Math. Phys.* 43 (1975) 199 [Erratum ibid. 46 (1976) 206] [inSPIRE].

[10] S.W. Hawking, Black Holes and Thermodynamics, *Phys. Rev.* D 13 (1976) 191 [inSPIRE].

[11] SUPERNova Search Team collaboration, A.G. Riess et al., Observational evidence from supernovae for an accelerating universe and a cosmological constant, *Astron. J.* 116 (1998) 1009 [astro-ph/9805201] [inSPIRE].

[12] SUPERNova Cosmology Project collaboration, S. Perlmutter et al., Measurements of Omega and Lambda from 42 high redshift supernovae, *Astrophys. J.* 517 (1999) 565 [astro-ph/9812133] [inSPIRE].

[13] V. Sahni and A.A. Starobinsky, The case for a positive cosmological Lambda term, *Int. J. Mod. Phys.* D 9 (2000) 373 [astro-ph/9912133] [inSPIRE].

[14] P.J.E. Peebles and B. Ratra, The cosmological constant and dark energy, *Rev. Mod. Phys.* 75 (2003) 559 [astro-ph/0207347] [inSPIRE].

[15] T. Padmanabhan, Cosmological constant: The weight of the vacuum, *Phys. Rept.* 380 (2003) 235 [hep-th/0212290] [inSPIRE].

[16] E.J. Copeland, M. Sami and S. Tsujikawa, Dynamics of dark energy, *Int. J. Mod. Phys.* D 15 (2006) 1753 [hep-th/0603057] [inSPIRE].

[17] V. Sahni and A. Starobinsky, Reconstructing Dark Energy, *Int. J. Mod. Phys.* D 15 (2006) 2105 [astro-ph/0610026] [inSPIRE].

[18] J. Frieman, M. Turner and D. Huterer, Dark Energy and the Accelerating Universe, *Ann. Rev. Astron. Astrophys.* 46 (2008) 385 [arXiv:0803.0982] [inSPIRE].

[19] S. Tsujikawa, Dark energy: investigation and modeling, arXiv:1004.1493 [inSPIRE].

[20] M. Li, X.-D. Li, S. Wang and Y. Wang, Dark Energy, *Commun. Theor. Phys.* 56 (2011) 525 [arXiv:1103.5870] [inSPIRE].

[21] M. Li, X.-D. Li, S. Wang and Y. Wang, Dark Energy: A Brief Review, *Front. Phys. China* 8 (2013) 828 [arXiv:1209.0922] [inSPIRE].

[22] M. Li, C. Lin and Y. Wang, Some Issues Concerning Holographic Dark Energy, *JCAP* 05 (2008) 023 [arXiv:0801.1407] [inSPIRE].
[23] Q.-G. Huang and Y.-G. Gong, Supernova constraints on a holographic dark energy model, JCAP 08 (2004) 006 [astro-ph/0403590] [SPIRE].

[24] X. Zhang and F.-Q. Wu, Constraints on Holographic Dark Energy from Latest Supernovae, Galaxy Clustering and Cosmic Microwave Background Anisotropy Observations, Phys. Rev. D 76 (2007) 023502 [astro-ph/070405] [SPIRE].

[25] M. Li, X.-D. Li, S. Wang and X. Zhang, Holographic dark energy models: A comparison from the latest observational data, JCAP 06 (2009) 036 [arXiv:0904.0928] [SPIRE].

[26] C.J. Hogan, Quantum Gravitational Uncertainty of Transverse Position, astro-ph/0703775 [SPIRE].

[27] C.J. Hogan, Spacetime Indeterminacy and Holographic Noise, arXiv:0706.1999 [SPIRE].

[28] Q.-G. Huang and M. Li, Anthropic principle favors the holographic dark energy, JCAP 03 (2005) 001 [hep-th/0410095] [SPIRE].

[29] X. Zhang, Statefinder diagnostic for holographic dark energy model, Int. J. Mod. Phys. D 14 (2005) 1597 [astro-ph/0504586] [SPIRE].

[30] X. Zhang, Reconstructing holographic quintessence, Phys. Lett. B 648 (2007) 1 [astro-ph/0604484] [SPIRE].

[31] X. Zhang, Dynamical vacuum energy, holographic quintom and the reconstruction of scalar-field dark energy, Phys. Rev. D 74 (2006) 103505 [astro-ph/0609699] [SPIRE].

[32] B. Chen, M. Li and Y. Wang, Inflation with Holographic Dark Energy, Nucl. Phys. B 774 (2007) 256 [astro-ph/0611623] [SPIRE].

[33] J. Zhang, X. Zhang and H. Liu, Holographic tachyon model, Phys. Lett. B 651 (2007) 84 [arXiv:0706.1185] [SPIRE].

[34] Y.-Z. Ma and X. Zhang, Possible Theoretical limits on holographic quintessence from weak gravity conjecture, Phys. Lett. B 661 (2008) 239 [arXiv:0709.1517] [SPIRE].

[35] M. Li, X.-D. Li, C. Lin and Y. Wang, Holographic Gas as Dark Energy, Commun. Theor. Phys. 51 (2009) 181 [arXiv:0811.3332] [SPIRE].

[36] M. Li, R.-X. Miao and Y. Pang, Casimir Energy, Holographic Dark Energy and Electromagnetic Metamaterial Mimicking de Sitter, Phys. Lett. B 689 (2010) 55 [arXiv:0910.3375] [SPIRE].

[37] M. Li, R.-X. Miao and Y. Pang, More studies on Metamaterials Mimicking de Sitter space, Opt. Express 18 (2010) 9026 [arXiv:0912.4837] [SPIRE].

[38] M. Li and Y. Wang, Quantum UV/IR Relations and Holographic Dark Energy from Entropic Force, Phys. Lett. B 687 (2010) 243 [arXiv:1001.4466] [SPIRE].

[39] Y. Gong and T. Li, A Modified Holographic Dark Energy Model with Infrared Infinite Extra Dimension(s), Phys. Lett. B 683 (2010) 241 [arXiv:0907.0860] [SPIRE].

[40] L.N. Granda, A. Oliveros and W. Cardona, Age problem in holographic dark energy, Mod. Phys. Lett. A 25 (2010) 1625 [arXiv:0905.1976] [SPIRE].

[41] Z.-P. Huang and Y.-L. Wu, Holographic Dark Energy Characterized by the Total Comoving Horizon and Insights to Cosmological Constant and Coincidence Problem, Phys. Rev. D 85 (2012) 103007 [arXiv:1202.4228] [SPIRE].
[45] M. Li, R.-X. Miao and Y. Pang, \textit{Casimir Energy, Holographic Dark Energy and Electromagnetic Metamaterial Mimicking de Sitter}, \textit{Phys. Lett. B} 689 (2010) 55 \[arXiv:0910.3375]\ [nSPIRE].

[46] Z. Zhang, S. Li, X.-D. Li, X. Zhang and M. Li, \textit{Revisit of the Interaction between Holographic Dark Energy and Dark Matter}, \textit{JCAP} 06 (2012) 009 \[arXiv:1204.6135]\ [nSPIRE].

[47] Q.-G. Huang and M. Li, \textit{The holographic dark energy in a non-flat universe}, \textit{JCAP} 08 (2004) 013 \[astro-ph/0404229]\ [nSPIRE].

[48] M. Li and R.-X. Miao, \textit{A New Model of Holographic Dark Energy with Action Principle}, \textit{arXiv:1210.0966}\ [nSPIRE].

[49] SNLS collaboration, J. Guy et al., \textit{The Supernova Legacy Survey 3-year sample: Type Ia Supernovae photometric distances and cosmological constraints}, \textit{Astron. Astrophys.} 523 (2010) A7 \[arXiv:1010.4743]\ [nSPIRE].

[50] SNLS collaboration, M. Sullivan et al., \textit{SNLS3: Constraints on Dark Energy Combining the Supernova Legacy Survey Three Year Data with Other Probes}, \textit{Astrophys. J.} 737 (2011) 102 \[arXiv:1104.1444]\ [nSPIRE].

[51] SNLS collaboration, A.J. Conley et al., \textit{SiFTO: An Empirical Method for Fitting SNe Ia Light Curves}, \textit{Astrophys. J.} 681 (2008) 482 \[astro-ph/0701828]\ [nSPIRE].

[52] SNLS collaboration, J. Guy et al., \textit{SALT2: Using distant supernovae to improve the use of Type Ia supernovae as distance indicators}, \textit{Astron. Astrophys.} 466 (2007) 11 \[astro-ph/0701828]\ [nSPIRE].

[53] S. Wang and Y. Wang, \textit{Exploring the Systematic Uncertainties of Type Ia Supernovae as Cosmological Probes}, \textit{Phys. Rev. D} 88 (2013) 043511 \[arXiv:1306.6423]\ [nSPIRE].

[54] S. Wang, Y.-H. Li and X. Zhang, \textit{Exploring the evolution of color-luminosity parameter $\beta$ and its effects on parameter estimation}, \textit{Phys. Rev. D} 89 (2014) 063524 \[arXiv:1310.6109]\ [nSPIRE].

[55] S. Wang, Y.-Z. Wang, J.-J. Geng and X. Zhang, \textit{Effects of time-varying $\beta$ in SNLS3 on constraining interacting dark energy models}, \textit{Eur. Phys. J. C} 74 (2014) 3148 \[arXiv:1406.0072]\ [nSPIRE].

[56] S. Wang, Y.-Z. Wang and X. Zhang, \textit{Effects of a Time-Varying Color-Luminosity Parameter $\beta$ on the Cosmological Constraints of Modified Gravity Models}, \textit{Commun. Theor. Phys.} 62 (2014) 927 \[arXiv:1407.7322]\ [nSPIRE].

[57] S. Wang, J.-J. Geng, Y.-L. Hu and X. Zhang, \textit{Revisit of constraints on holographic dark energy: SNLS3 dataset with the effects of time-varying $\beta$ and different light-curve fitters}, \textit{Sci. China Phys. Mech. Astron.} 58 (2015) 019801 \[arXiv:1312.0184]\ [nSPIRE].

[58] G.R. Bengochea, \textit{Supernova light-curve fitters and Dark Energy}, \textit{Phys. Lett. B} 696 (2011) 5 \[arXiv:1010.4014]\ [nSPIRE].

[59] G.R. Bengochea and M.E. De Rossi, \textit{Dependence on supernovae light-curve processing in void models}, \textit{Phys. Lett. B} 733 (2014) 258 \[arXiv:1402.3167]\ [nSPIRE].

[60] Y. Hu, M. Li, N. Li and S. Wang, \textit{Impacts of different SNLS3 light-curve fitters on cosmological consequences of interacting dark energy models}, \textit{arXiv:1501.06962}\ [nSPIRE].

[61] Z. Zhang, M. Li, X.-D. Li, S. Wang and W.-S. Zhang, \textit{Generalized Holographic Dark Energy and its Observational Constraints}, \textit{Mod. Phys. Lett. A} 27 (2012) 1250115 \[arXiv:1202.5163]\ [nSPIRE].

[62] Y. Wang and S. Wang, \textit{Distance Priors from Planck and Dark Energy Constraints from Current Data}, \textit{Phys. Rev. D} 88 (2013) 043522 \[arXiv:1304.4514]\ [nSPIRE].

[63] PLANCK collaboration, P.A.R. Ade et al., \textit{Planck 2013 results. I. Overview of products and scientific results}, \textit{Astron. Astrophys.} 571 (2014) A1 \[arXiv:1303.5062]\ [nSPIRE].
[64] PLANCK collaboration, P.A.R. Ade et al., Planck 2013 results. XVI. Cosmological parameters, Astron. Astrophys. 571 (2014) A16 [arXiv:1303.5076] [inSPIRE].

[65] W. Hu and N. Sugiyama, Small scale cosmological perturbations: An analytic approach, Astrophys. J. 471 (1996) 542 [astro-ph/9510117] [inSPIRE].

[66] M. Li, X.-D. Li, S. Wang and Y. Wang, Dark Energy, Commun. Theor. Phys. 56 (2011) 525 [arXiv:1103.5870] [inSPIRE].

[67] F. Beutler et al., The 6dF Galaxy Survey: Baryon Acoustic Oscillations and the Local Hubble Constant, Mon. Not. Roy. Astron. Soc. 416 (2011) 3017 [arXiv:1106.3366] [inSPIRE].

[68] BOSS collaboration, L. Anderson et al., The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: baryon acoustic oscillations in the Data Releases 10 and 11 Galaxy samples, Mon. Not. Roy. Astron. Soc. 441 (2014) 24 [arXiv:1312.4877] [inSPIRE].

[69] E.A. Kazin et al., The WiggleZ Dark Energy Survey: improved distance measurements to $z = 1$ with reconstruction of the baryonic acoustic feature, Mon. Not. Roy. Astron. Soc. 441 (2014) 3524 [arXiv:1401.0358] [inSPIRE].

[70] D.J. Eisenstein and W. Hu, Baryonic features in the matter transfer function, Astrophys. J. 496 (1998) 605 [astro-ph/9709112] [inSPIRE].

[71] SDSS collaboration, D.J. Eisenstein et al., Detection of the baryon acoustic peak in the large-scale correlation function of SDSS luminous red galaxies, Astrophys. J. 633 (2005) 560 [astro-ph/0501171] [inSPIRE].

[72] A.G. Riess et al., A 3% Solution: Determination of the Hubble Constant with the Hubble Space Telescope and Wide Field Camera 3, Astrophys. J. 730 (2011) 119 [Erratum ibid. 732 (2011) 129] [arXiv:1103.2976] [inSPIRE].

[73] BOSS collaboration, T. Delubac et al., Baryon acoustic oscillations in the Lyα forest of BOSS DR11 quasars, Astron. Astrophys. 574 (2015) A59 [arXiv:1404.1801] [inSPIRE].

[74] WMAP collaboration, E. Komatsu et al., Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation, Astrophys. J. Suppl. 192 (2011) 18 [arXiv:1001.4538] [inSPIRE].

[75] A. Lewis and S. Bridle, Cosmological parameters from CMB and other data: A Monte Carlo approach, Phys. Rev. D 66 (2002) 103511 [astro-ph/0205436] [inSPIRE].

[76] H. Akaike, A new look at the statistical model identification, IEEE Trans. Automatic Control 19 (1974) 716.

[77] G. Schwarz, Estimating the dimension of a model, Ann. Stat. 6 (1978) 461.

[78] V. Sahni, A. Shafieloo and A.A. Starobinsky, Model independent evidence for dark energy evolution from Baryon Acoustic Oscillations, Astrophys. J. 793 (2014) L40 [arXiv:1406.2209] [inSPIRE].

[79] Y. Hu, M. Li and Z. Zhang, Test $\Lambda$CDM model with High Redshift data from Baryon Acoustic Oscillations, arXiv:1406.7695 [inSPIRE].