FREE KLEENE ALGEBRAS WITH DOMAIN

BRETT MCLEAN

Abstract
First we identify the free algebras of the class of algebras of binary relations equipped with the composition and domain operations. Elements of the free algebras are pointed labelled finite rooted trees. Then we extend to the analogous case when the signature includes all the Kleene algebra with domain operations; that is, we add union and reflexive transitive closure to the signature. In this second case, elements of the free algebras are ‘regular’ sets of the trees of the first case. As a corollary, the axioms of domain semirings provide a finite quasiequational axiomatisation of the equational theory of algebras of binary relations for the intermediate signature of composition, union, and domain. Next we note that our regular sets of trees are not closed under complement, but prove that they are closed under intersection. Finally, we prove that under relational semantics the equational validities of Kleene algebras with domain form a decidable set.

2010 Mathematics subject classification: primary 08B15; secondary 20M20.

Keywords and phrases: Kleene algebra, domain, binary relation, equational theory, decidable.

1. Introduction

Reasoning about binary relations, and ways of combining them, has an extensive literature and a multitude of applications. Classically, in algebraic logic, binary relations model logical formulas with two free variables [30]. In computer science, we can find binary relations modelling the actions of programs [20, 24], and elsewhere representing relationships between items of data that compose a tree [1], or a graph [21].

When an algebraic logician thinks of binary relations, the first signature to come to mind will always be that of Tarski’s relation algebras. In computer science, the Kleene algebra signature has the greatest prominence. In the latter case, the operations are relational composition, union, and reflexive transitive closure, as well as constants for the empty relation and the identity relation.

Any set of binary relations closed under the five Kleene algebra operations/constants can be viewed as an algebra in the sense of universal algebra/model theory, that is, a structure over a signature of function symbols (but no predicate symbols). It is well known that this class of algebras contains its free algebras, and that the free algebra

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant agreement No. 670624).
generated by a given finite set Σ is precisely the set of all regular languages over the alphabet Σ. The importance of regular languages in theoretical computer science goes almost without saying [29]. The algebraic perspective on sets of regular languages was employed to great effect by Eilenberg in his celebrated variety theorem [7], and continues to yield valuable new insights to this day [12, 13].

In this paper, we identify the analogous free algebras in the case where the signature is expanded with one extra operation on binary relations: the unary domain operation

\[ D(R) = \{(x, x) \mid \exists y : (x, y) \in R\}, \]

which provides a record of all points having at least one image under the given relation. This expanded signature is that of Kleene algebra with domain, a certain finite set of algebraic laws extending Kozen’s theory of Kleene algebras with a domain operation and a small number of associated equations [6]. One intended model for this theory is indeed algebras of binary relations, and there is a hope that the theory will prove useful for reasoning about the actions of nondeterministic computer programs [4, 5].

In addition to identifying the free algebras, we also show that it is decidable whether an equation in this signature is valid over all algebras of binary relations.

**Structure of the paper** In Section 2 we give the necessary definitions and some context regarding algebras of binary relations, and their free algebras.

In Section 3 we introduce the trees that we use for describing our free algebras, and certain relations and operations on those trees.

In Section 4 we prove an intermediate result: we identify the free algebras for the reduced signature that omits the ‘nondeterministic’ union and reflexive transitive closure operators, and also the empty relation constant, that is, the signature with the composition and domain operations and the identity constant. For this signature the elements of the free algebras are ‘reduced’ pointed labelled finite rooted trees (Theorem 4.6).

In Section 5 we extend the result of the previous section to identify the free algebras for the full Kleene algebra with domain signature. In this case, elements of the free algebras are certain sets of the trees of the previous case (Theorem 5.6). We term these sets ‘regular’ sets of trees, by analogy with the regular languages of the Kleene algebra signature. By combining with an existing result, it follows as a corollary that the axioms of domain semirings provide a finite quasiequational axiomatisation of the equational theory of algebras of binary relations for the signature of composition, union, domain, and the two constants (but not reflexive transitive closure).

Section 6 is devoted to closure properties of regular sets of trees. We use automata to show the regular sets of trees are closed under intersection. We also note the regular sets of trees are not closed under complement and pose the question of whether they are closed under residuation.

In Section 7 we again use automata to prove the decidability of validity for equations in the signature of Kleene algebra with domain under relational semantics (Theorem 7.2).
2. Algebras of binary relations

We begin by making precise what is meant by an algebra of binary relations.

**Definition 2.1.** An **algebra of binary relations** of the signature \( \{ ;, +, *, 0, 1 \} \) is a universal algebra \( \mathfrak{A} = (A, ;, +, *, 0, 1) \) where the elements of the universe \( A \) are all binary relations on some (common) set \( X \), the **base**, and the interpretations of the symbols are given as follows:

- the binary operation \( ; \) is interpreted as **composition** of relations:
  \[ R ; S := \{ (x, y) \in X^2 \mid \exists z \in X : (x, z) \in R \land (z, y) \in S \} \],

- the binary operation \( + \) is interpreted as set-theoretic **union**:
  \[ R + S := \{ (x, y) \in X^2 \mid (x, y) \in R \lor (x, y) \in S \} \],

- the unary operation \( * \) is interpreted as **reflexive transitive closure**:
  \[ R^* := \{ (x, y) \in X^2 \mid \exists n \in \mathbb{N} \exists x_0 \ldots x_n : (x_0 = x) \land (x_n = y) \land (x_0, x_1) \in R \land \ldots \land (x_{n-1}, x_n) \in R \} \],

- the constant 0 is interpreted as the **empty** relation:
  \[ 0 := \emptyset, \]

- the constant 1 is interpreted as the **identity** relation on \( X \):
  \[ 1 := \{ (x, x) \in X^2 \}. \]

We let \( \text{Rel}(; , +, *, 0, 1) \) denote the isomorphic closure of the class of all algebras of binary relations of the signature \( \{ ;, +, *, 0, 1 \} \).

**Remark 2.2.**

(i) It is easy to see that \( \text{Rel}(; , +, *, 0, 1) \) is not a first-order axiomatisable class, not even closed under elementary equivalence, by a simple argument showing that \( \text{Rel}(; , +, *, 0, 1) \) is not closed under ultrapowers.

(ii) Despite \( \text{Rel}(; , +, *, 0, 1) \) being far from a variety, it is easily seen to be closed under subalgebras and products. (An element of a product of algebras of binary relations is the disjoint union of all its component binary relations.) Hence, by a basic theorem of universal algebra (see, for example, [3, Theorem 10.12]), the class \( \text{Rel}(; , +, *, 0, 1) \) contains its free algebras.

(iii) It is a folk theorem that the free \( \text{Rel}(; , +, *, 0, 1) \)-algebra generated by a finite set \( \Sigma \) is the set of all regular languages over the alphabet \( \Sigma \) (with the operations of language concatenation, union, and so on).
(iv) It is well known that the variety $HS \mathcal{P} \mathcal{R}(;+,\ast,0,1)$ generated by $\mathcal{R}(;+,\ast,0,1)$ has no finite equational axiomatisation \[27\].

(v) We do however have Kozen’s quasivariety of Kleene algebras \[19\], defined by a finite number of equations/quasiequations,\(^1\) intermediate to $\mathcal{R}(;+,\ast,0,1)$ and $HS \mathcal{P} \mathcal{R}(;+,\ast,0,1)$. That is,

$$\mathcal{R}(;+,\ast,0,1) \subseteq \text{Kleene algebras} \subseteq HS \mathcal{P} \mathcal{R}(;+,\ast,0,1),$$

and so

$$HS \mathcal{P}(\text{Kleene algebras}) = HS \mathcal{P} \mathcal{R}(;+,\ast,0,1).$$

Of course, the operations of Definition 2.1 are not the only operations that can be defined on binary relations. In particular, various unary ‘test’ operations can be defined; here is a selection.

**Definition 2.3.**

- The unary operation $D$ is the operation of taking the diagonal of the domain of a relation:
  $$D(R) = \{(x, x) \in X^2 | \exists y \in X : (x, y) \in R\}.$$

- The unary operation $R$ is the operation of taking the diagonal of the range of a relation:
  $$R(R) = \{(y, y) \in X^2 | \exists x \in X : (x, y) \in R\}.$$

- The unary operation $A$ is the operation of taking the diagonal of the antidomain of a relation—those points of $X$ at which the image of the relation is empty:
  $$A(R) = \{(x, x) \in X^2 | \forall y \in X : (x, y) \in R\}.$$

One can vary the operations from those of Definition 2.1 and/or restrict the binary relations to some particular form. The resulting class will again contain its free algebras. If the class is of interest, then it is useful to establish a description of these free algebras.

Restricting the binary relations to be some type of function (total functions, partial functions, or injective partial functions, for example) tends to yield free algebras whose elements are a ‘single object’, rather than a ‘set of objects’. The class of semigroups, for example, is the variety generated by $\text{Tot}(;)$—algebras of total functions with composition—and an element of a free semigroup is a single string, rather than a set of strings as we have in the case $\mathcal{R}(;+,\ast,0,1)$. Similarly, elements of free groups are also strings, with groups forming isomorphs of algebras of bijective functions, with the familiar operations.

There is also an observable pattern when test operations are added to the signature: strings are replaced by (labelled) trees. The following results are known.

\(^1\) A **quasiequation** is a conditional equation where the condition is a finite conjunction of equations. That is, a quasiequation is a formula of the form $s_1 = t_1 \land \cdots \land s_n = t_n \rightarrow u = v$. 

1. The class $\text{Inj}(; , ^{-1})$, of isomorphs of algebras of injective partial functions with composition and inverse, is the class of inverse semigroups [26, 32]. Elements of free inverse semigroups are certain trees, so-called Munn trees [23].

2. The class $\text{Par}(; , D)$—partial functions with composition and domain—is a variety [31], most commonly known as the restriction semigroups. A description of the free algebras has been given, and again, elements can be viewed as trees [9].

3. The class $\text{Par}(; , D, R)$—partial functions with composition, domain, and range—is a proper quasivariety; a finite quasiequational axiomatisation was given by Schein [28]. Once more, a description of the free algebras has been given, and elements can be viewed as trees [10].

We should also mention at this stage Hollenberg’s finite equational axiomatisation of the equational theory of the quasivariety $\text{Rel}(; , A)$ (in which $D$, $0$, and $1$ are easily expressible) [17]. Of course, this result amounts to an implicit description of the corresponding free algebras, as quotients of term algebras by this theory. Another result involving binary relations (but not tests), is Bloom, Ésik, and Stefanescu’s explicit description of the free algebras for the case $\text{Rel}(; , +, *, 0, 1, \sim)$, where $\sim$ is the converse operation $R \sim := \{(x, y) \in X^2 \mid (y, x) \in R\}$ [2]. There, the elements of the algebras are sets of strings.

Having noted that binary relations $\sim$ sets, functions $\sim$ singletons, and tests $\sim$ trees, one can anticipate that when tests are added to the case $\text{Rel}(; , +, *, 0, 1)$, elements of free algebras will be sets of labelled trees. We will prove that this is indeed the case (Theorem 5.6). On the way to doing this, we also identify the free algebras for the case without the ‘nondeterministic’ operators $;$. and $*$, more precisely, for the case $\text{Rel}(; , 1, D)$ (Theorem 4.6).

**Remark 2.4.** The term ‘Kleene algebra with domain’ was originally used for a certain quasiequational theory extending the two-sorted Kleene algebra with tests with a domain operation [4]. It was subsequently redefined as a (strictly less expressive) one-sorted quasiequational theory extending Kleene algebra with a domain operation [6].

### 3. Trees

The central objects we will be working with throughout will be labelled rooted trees. We will give two definitions of these. The first, Terminology 3.1, is the usual graph-theoretic definition, and we give it in order to make use of basic graph-theoretic terminology: vertex, edge, and so on. The second, Definition 3.2, is cleaner, in the sense that isomorphic trees are identical, and will serve as the ‘official’ definition in this paper.

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2 In this signature, in which $^{-1}$ is available, the inverse semigroups form a quasivariety, since the existence of unique inverses can be expressed as a quasiequation.
**Terminology 3.1.**

- A **tree** is a connected acyclic undirected graph (reflexive edges are prohibited). All trees will be assumed to be finite unless otherwise stated.

- A **rooted tree** is a tree with a distinguished vertex called the **root**.

- By a labelled tree, we will mean an **edge-labelled** tree. That is, given a set $\Sigma$ of labels, a **labelled** tree is a tree $T$ together with a function from the edges of $T$ to $\Sigma$.

**Definition 3.2.** Given a set $\Sigma$ of labels, a **labelled rooted tree** is defined recursively as a set of pairs $(a, T)$, where $a \in \Sigma$ and $T$ is a labelled rooted tree.

Some explanation may be helpful. According to Definition 3.2, the empty set is a labelled rooted tree (this is the base case of the definition). This empty set should be thought of as encoding what is, in the graph-theoretic view, the tree with a single vertex. Figure 1 illustrates how some simple examples of labelled rooted trees should be viewed.

![Figure 1. The labelled rooted trees encoded as $\emptyset$, $\{(a, \emptyset)\}$, and $\{(a, \emptyset), (a, \{(b, \emptyset)\})\}$, respectively (with roots at the top)](image)

The reader may note that Definition 3.2 is more restrictive than the definition of labelled rooted trees obtained from Terminology 3.1—it cannot describe any tree having a vertex with two distinct but isomorphic child subtrees. However, we will have no need of such trees in this paper.

**Definition 3.3.** A **pointed tree** is a tree with a distinguished vertex called the **point**.

We will primarily work with pointed labelled rooted trees. The notion of a **homomorphism** of (possibly pointed) labelled rooted trees is the evident one: the root maps to the root, the point, if there is one, maps to a point, and a label-preserving map on edges must be induced.

We want to be able to reduce trees to forms without any redundant branches. In order to do that, we first define a preorder on trees.
**Definition 3.4.** The preorder $\leq$ on (possibly pointed) labelled rooted trees is defined recursively as follows. For trees $T_1$ and $T_2$ with roots $r_1$ and $r_2$ respectively, $T_1 \leq T_2$ if and only if

(a) $r_2$ is not the point vertex,

(b) for each child $v_2$ of $r_2$, there is a child $v_1$ of $r_1$ such that

(i) the labels of the edges $r_1v_1$ and $r_2v_2$ are equal,

(ii) $T_{v_1} \leq T_{v_2}$, where $T_{v_1}$ and $T_{v_2}$ are the $v_1$-rooted and $v_2$-rooted subtrees respectively.

That $\leq$ is indeed a preorder is clear. In fact, by induction on the height of the trees, it is easy to see that

$$T_1 \leq T_2 \iff \text{there exists a homomorphism } \theta : T_2 \to T_1. \quad (3.1)$$

**Definition 3.5.** Let $T$ be a labelled tree with root $r$. The **reduced form** of $T$ is the tree formed recursively as follows.

(a) For each child $v$ of $r$, replace the $v$-rooted subtree with its reduced form.

(b) Remove all but the $\leq$-minimal child subtrees of the tree obtained after the first step.

**Example 3.6.** The pointed tree on the left of Figure 2 reduces as shown. The tree on the right is already reduced.

![Figure 2. Reduction of pointed labelled rooted trees](image)

**Proposition 3.7.**

1. The preorder $\leq$ is a partial order on reduced labelled rooted trees.

2. If a labelled rooted tree $T$ reduces to $T'$, then $T \leq T'$ and $T' \leq T$. 
Proof.

1. An easy induction on the maximum height of the two trees being compared. Indeed for the base case, note that for trees $T_1 \neq T_2$ of height 0, without loss of generality, $T_2$ is pointed and $T_1$ is not. Then $T_1 \not\leq T_2$, so antisymmetry holds.

2. Another easy induction. \[\square\]

Thus reduction selects a unique member of every $\leq$-equivalence class. Note it is also easy to see that the partial order $\leq$ on reduced trees is Noetherian (converse well-founded). Informally, we can think of $\leq$ on reduced trees as corresponding to the inclusion relation on binary relations—if $T_1 \leq T_2$ then $T_1$ is a more specific description than $T_2$.

Definition 3.8. Let $\Sigma$ be a set and let $T$ and $S$ be reduced pointed $\Sigma$-labelled rooted trees.

- The pointed tree concatenation $T ; S$ of $T$ and $S$ is the tree formed by
  1. identifying the point of $T$ and the root of $S$ (the root is now the root of $T$ and the point is the point of $S$),
  2. reducing the resulting tree to its reduced form.

- The domain $D(T)$ of $T$ is the tree formed by
  1. reassigning the point of $T$ to the current root of $T$,
  2. reducing the resulting tree to its reduced form.

Notation 3.9. For a symbol $a$ from an alphabet, we write $a$ for the pointed labelled rooted tree with two vertices, whose point is the child vertex and whose single edge is labelled by $a$. We write $\epsilon$ for the pointed labelled rooted tree with a single vertex.

Remark 3.10. A very similar setup to that presented in this section has already been used for investigating Kleene algebras with domain $[22]$. In that thesis, the graph-theoretic definition of trees is used, and pointed labelled finite rooted trees are called ‘trees with a top’. There, the relation $\leq$ is termed ‘simulates’, and trees are only considered up to simulation equivalence. Thus there is no notion of a reduced form; the operations of Definition 3.8 are defined without their reduction steps.

We will return to say more about the thesis $[22]$ at the end of Section 5.

4. Composition, identity, and domain

In this section we will identify the free algebras of the class $\text{Rel}(\cdot, 1, D)$. From there, it is straightforward to accommodate the addition of $+$ and $*$ (and 0).

Definition 4.1. We define the single-tree interpretation $[\cdot]$ of $\{\cdot, 1, D\}$-terms as follows.
1. \([a] := a\), for any variable \(a\),

2. \([1] := \varepsilon\),

3. \([s ; t] := [s] ; [t]\),

4. \([D(s)] := D[s]\).

Here the operations and constants on the right-hand side are those defined for trees in Section 3.

**Lemma 4.2.** The map \([ \cdot ]\) is a surjection from the \([; 1, D]\)-terms in variables \(\Sigma\) onto the reduced pointed \(\Sigma\)-labelled rooted trees. Hence \([ \cdot ]\) is a surjective homomorphism from the term algebra onto the reduced pointed \(\Sigma\)-labelled rooted trees, viewed as an algebra of the signature \([; 1, D]\) with the operations we have defined.

**Proof.** We show that every reduced tree is the interpretation of some term by induction on the size of the tree.

A tree whose root has no children is just \(\varepsilon\)—the interpretation of 1—so assume we have a tree \(T\) given by a nonempty set \((a_1, T_1), \ldots, (a_n, T_n)\), and a distinguished point \(p\). First suppose \(p\) is the root of \(T\). For each \(1 \leq i \leq n\), let \(t_i\) be a term whose interpretation is the pointed tree whose tree is \(T_i\) and whose point is its root. Then we can realise \(T\) by the term \(D(a_1 ; t_1) ; \cdots ; D(a_n ; t_n)\). (We write iterated ; without brackets because the positioning of the brackets is immaterial.)

Alternatively, suppose \(p\) is not the root of \(T\); so without loss of generality \(p\) is a vertex in \(T_n\). Let \(t_1, \ldots, t_{n-1}\) be as before and now let \(t_n\) be a term whose interpretation is the pointed tree whose tree is \(T_n\) and whose point is \(p\). Then we can realise \(T\) by the term \(D(a_1 ; t_1) ; \cdots ; D(a_{n-1} ; t_{n-1}) ; a_n ; t_n\). \(\square\)

The following lemma says that the single-tree interpretation of a term records for us a tree that, in a relational model, ‘connects’ \(x\) and \(y\) if and only if the term holds on the pair \((x, y)\).

**Lemma 4.3.** Let \(\mathcal{A}\) be a \([; 1, D]\)-algebra of binary relations, with base \(X\). Let \(t\) be a \([; 1, D]\)-term, and let \(f\) be an assignment of elements of \(\mathcal{A}\)—that is, binary relations—to the variables in \(t\). Then for any \(x, y \in X\), the following are equivalent.

1. The pair \((x, y)\) belongs to the interpretation of \(t\) under the assignment \(f\).

2. There is a homomorphism of \([t]\) into \(X\) such that the root of \([t]\) is mapped to \(x\) and the point of \([t]\) is mapped to \(y\).\(^3\)

\(^3\) By a relational model we mean an algebra of relations together with an assignment of relations to (some suitable set of) variables.

\(^4\) Here, homomorphism means homomorphism of labelled graphs, with \([t]\) and \((X, f)\) viewed as labelled graphs in the obvious way.
Structural induction on terms (making use of the fact that there is a homomorphism of a tree into \( X \) if and only if there is a homomorphism of its reduced form into \( X \)). □

Corollary 4.4 (soundness with respect to relations). For any pair \( s \) and \( t \) of \([; , 1, D]\)-terms

\[
[s] = [t] \implies \text{Rel}(; , 1, D) \models s = t.
\]

Lemma 4.5 (completeness with respect to relations). For any pair \( s \) and \( t \) of \([; , 1, D]\)-terms

\[
\text{Rel}(; , 1, D) \models s = t \implies [s] = [t].
\]

Proof. Suppose \( \text{Rel}(; , 1, D) \models s = t \). We will show that \([s] \leq [t]\). Then by symmetry, also \([t] \leq [s]\). So \([s] = [t]\) and we are done.

Let \( r \) be the root and \( p \) the point of \([s]\). We can view \([s]\) as (generating) a relational model (with its vertex set as base). On this relational model, evidently the pair \((r, p)\) belongs to the interpretation of \( s \). Hence, by the assumption \( \text{Rel}(; , 1, D) \models s = t \), we have that \((r, p)\) belongs to the interpretation of \( t \). Then by Lemma 4.3 there is a homomorphism of labelled trees from \([t]\) into \([s]\) mapping the root of \([t]\) to \( r \) and the point of \([t]\) to \( p \). That is, there is a homomorphism of pointed labelled rooted trees from \([t]\) to \([s]\). We know this is equivalent to the conclusion \([s] \leq [t]\) we seek, so we are done. □

With Corollary 4.4 and Lemma 4.5 available, we can now complete the proof that we have identified the free algebras of the class by \( \text{Rel}(; , 1, D) \).

Theorem 4.6. Let \( \Sigma \) be an alphabet, and let \( R_\Sigma \) be the set of reduced pointed \( \Sigma \)-labelled finite rooted trees. Then the free \( \text{Rel}(; , 1, D) \)-algebra over \( \Sigma \) is \( R_\Sigma \) equipped with the operations of pointed tree concatenation and of domain from Definition 3.8 (and the constant \( \varepsilon \)).

Proof. Since \( R_\Sigma \) is generated by \( \Sigma \), it follows by the first isomorphism theorem of universal algebra that \( R_\Sigma \) is isomorphic to a quotient \( \Xi \) of the term algebra (for signature \([; , 1, D]\)) over variables \( \Sigma \). The congruence relation \( \sim \) defining the quotient is given by \( s \sim t \iff [s] = [t] \), for terms \( s \) and \( t \) over \( \Sigma \). So by Corollary 4.4 and Lemma 4.5, the congruence \( \sim \) is given by equational validity in \( \text{Rel}(; , 1, D) \). It is a basic result of universal algebra that \( \Xi \) is then precisely the free algebra over \( \Sigma \) of the class \( \text{Rel}(; , 1, D) \). □

The inclusion of 1 and exclusion of 0 from the signature was in fact not essential. We can easily add and remove them according to our wishes.

Corollary 4.7. Let \( \Sigma \) and \( R_\Sigma \) (viewed as an algebra) be as in Theorem 4.6.

- The free \( \text{Rel}(; , D) \)-algebra over \( \Sigma \) is given by removing the tree \( \varepsilon \) from \( R_\Sigma \).
• The free Rel(;, 0, D)-algebra over $\Sigma$ is given by the addition of a zero element—an element validating $0; T = T; 0 = 0$ and $D(0) = 0$—to the free Rel(;, D)-algebra over $\Sigma$.

• The free Rel(;, 1, 0, D)-algebra over $\Sigma$ is given by the addition of a zero element to $R_\Sigma$.

Proof. For Rel(;, D), first note that every nontrivial tree in $R_\Sigma$ is the interpretation of a $\{; D\}$-term, and conversely, if a tree is the interpretation of a $\{; D\}$-term then it cannot be trivial—by induction, all such interpretations have at least one edge. Hence the nontrivial trees indeed form a $\{; D\}$-algebra generated by (the interpretations of elements of) $\Sigma$. Since every $\{; D\}$-algebra of relations embeds in a $\{; 1, D\}$-algebra of relations, it follows from Corollary 4.4 that every equation validated by the nontrivial trees is validated by all $\{; D\}$-algebras of relations. The converse follows immediately from Lemma 4.5.

For Rel(;, 0, D) and Rel(;, 1, 0, D), note that by the definition of a zero element, a term is interpreted as the zero of the algebra if and only if the symbol 0 appears in the term, and similarly a term is interpreted as $\emptyset$ in every algebra of relations if and only if 0 appears in the term. These observations are sufficient to extend Corollary 4.4 and Lemma 4.5 to terms that may contain 0. □

We remark that the class Rel(;, 1, D) is observed to form a quasivariety. It has been shown that this quasivariety is not finitely axiomatisable in first-order logic [16]. Naturally, the same statements hold when we add/remove 0 and 1.

5. Expansion by union and reflexive transitive closure

In this section, we extend the result of the previous section to provide a description of the free algebras of the class of relational Kleene algebras with domain, that is, the free algebras of Rel(;, +, *, 0, 1, D).

Definition 5.1. For a set $K$ of reduced trees, let $\text{maximal}(K)$ denote the set of $\leq$-maximal elements of $K$. We lift the notation $;$ and $D$ of Definition 3.8 to sets of trees by using elementwise application. We define the standard tree interpretation $\llbracket \cdot \rrbracket$ of $\{; +, *, 0, 1, D\}$-terms as follows.

1. For $a \in \Sigma$, $\llbracket a \rrbracket := \{a\},$
2. $\llbracket 0 \rrbracket := \emptyset,$
3. $\llbracket 1 \rrbracket := \{\epsilon\},$
4. $\llbracket s + t \rrbracket := \text{maximal}(\llbracket s \rrbracket \cup \llbracket t \rrbracket),$
5. $\llbracket s ; t \rrbracket := \text{maximal}(\llbracket s \rrbracket ; \llbracket t \rrbracket),$

The short proof of this using general model-theoretic results consists of noting that the class is both closed under direct products and—almost by definition—has a pseudouniversal axiomatisation.
6. \([s]^* := \text{maximal}( \bigcup_{i=1}^{\infty} [s]^i )\), where \([s]^0 := 1\) and \([s]^{i+1} := [s]^i + [s]\).

7. \([D(s)] := \text{maximal}(D[s])\).

Note that \([a]\), for \(a \in \Sigma\), \([0]\), and \([1]\), contain only reduced trees, and \(\cup\) preserves this property on sets of trees (as do the lifted \(\cdot\) and \(D\), by definition). Hence the maximal operation is applicable whenever it is used in Definition 5.1, and standard interpretations contain only reduced trees.

**Definition 5.2.** Let \(\Sigma\) be an alphabet. A set of pointed \(\Sigma\)-labelled rooted trees is **regular** if it is the standard tree interpretation of some \(\{\cdot, +, *, 0, 1, D\}\)-term.\(^6\)

We can think of a regular set \(L\) of trees as a concise record of all the reduced trees in the downwards-closed set \(\downarrow \ L\) (with respect to the \(\leq\) ordering). In this view (thinking of \(L\) as \(\downarrow \ L\)), the operation \(+\) corresponds to the real set union operation, and \(\cdot\) and \(D\) correspond to pointwise application of the operations of Definition 3.8. The advantage of using the arrangement of Definition 5.1 is that regular sets remain **finite** until such time that Kleene star is used.

The following lemma is the analogue of Lemma 4.3.

**Lemma 5.3.** Let \(\mathcal{A}\) be a \(\{\cdot, +, *, 0, 1, D\}\)-algebra of binary relations, with base \(X\). Let \(t\) be a \(\{\cdot, +, *, 0, 1, D\}\)-term, and let \(f\) be an assignment of elements of \(\mathcal{A}\) to the variables in \(t\). Then for any \(x, y \in X\), the following are equivalent.

1. The pair \((x, y)\) belongs to the interpretation of \(t\) under the assignment \(f\).

2. There is a tree \(T\) in \([t]\) and a homomorphism of \(T\) into \(X\) such that the root of \(T\) is mapped to \(x\) and the point of \(T\) is mapped to \(y\).

**Proof.** Structural induction on terms. \(\square\)

**Proposition 5.4** (soundness with respect to relations). For any pair \(s\) and \(t\) of \(\{\cdot, +, *, 0, 1, D\}\)-terms

\(\hspace{1cm} [s] = [t] \implies \text{Rel}(\cdot, +, *, 0, 1, D) \models s = t.\)

**Proof.** Suppose \([s] = [t]\). Let \(\mathcal{A}\) be a \(\{\cdot, +, *, 0, 1, D\}\)-algebra of binary relations, with base \(X\). Let \(f\) be an assignment of elements of \(\mathcal{A}\) to the variables appearing in \(s = t\). Write \(\cdot \mathcal{A}_f^\mathcal{A}\) for the interpretations in \(\mathcal{A}\) under \(f\). Then by Lemma 5.3, for any \(x, y \in X\), we have that \((x, y) \in [s]^{\mathcal{A}_f^\mathcal{A}}\) if and only if there is a \(T \in [s]\) with \(T\) connecting \(x\) and \(y\). As \([s] = [t]\), this is equivalent to there being a \(T \in [t]\) with \(T\) connecting \(x\) and \(y\), which in turn is equivalent, by Lemma 5.3 again, to having \((x, y) \in [t]^{\mathcal{A}_f^\mathcal{A}}\). As \(x\) and \(y\) were arbitrary, we have \([s]^{\mathcal{A}_f^\mathcal{A}} = [t]^{\mathcal{A}_f^\mathcal{A}}\). As \(\mathcal{A}\) and \(f\) were arbitrary, we conclude that \(\text{Rel}(\cdot, +, *, 0, 1, D) \models s = t.\) \(\square\)

\(^6\) It is not claimed that this notion of regular sets of pointed trees is the same as the notion of a regular tree language coming from the theory of tree automata \([11]\).
Proposition 5.5 (completeness with respect to relations). For any pair \( s \) and \( t \) of \( \{;+,\ast,0,1,D\}\)-terms

\[
\text{Rel}(;+,\ast,0,1,D) \models s = t \implies \llbracket s \rrbracket = \llbracket t \rrbracket.
\]

Proof. Just like the proof of Lemma 4.5. Given \( S \in \llbracket s \rrbracket \), we obtain the existence of a \( T \in \llbracket t \rrbracket \) with \( S \leq T \). By symmetry, there is an \( S' \in \llbracket s \rrbracket \) with \( T \leq S' \). Since \( \llbracket s \rrbracket \) is a \( \leq \)-antichain, \( S = T \). Hence \( \llbracket s \rrbracket \subseteq \llbracket t \rrbracket \). By symmetry, also \( \llbracket t \rrbracket \subseteq \llbracket s \rrbracket \). \( \square \)

With Proposition 5.4 and Proposition 5.5, we can now complete the proof of Theorem 5.6 in the familiar way.

Theorem 5.6. Let \( \Sigma \) be an alphabet, and let \( \mathcal{R}_\Sigma \) be the set of reduced pointed \( \Sigma \)-labelled rooted trees. Then the free \( \text{Rel}(;+,\ast,0,1,D) \)-algebra over \( \Sigma \) has as its universe all the regular subsets of \( \mathcal{R}_\Sigma \). The operations are the following, where \( L, L_1, \) and \( L_2 \), are regular sets of reduced trees.

1. \( 0 := \emptyset \),
2. \( 1 := \{ \varepsilon \} \),
3. \( L_1 + L_2 := \text{maximal}(L_1 \cup L_2) \),
4. \( L_1 ; L_2 := \text{maximal}(L_1 ; L_2) \),
5. \( L^* := \text{maximal}(\bigcup_{i=1}^{\infty} L^i) \), where \( L^0 := \{ \varepsilon \} \) and \( L^{i+1} := L^{i+1} ; L \),
6. \( D(L) := \text{maximal}(D(L)) \).

Proof. Similar to the proof of Theorem 4.6. \( \square \)

Corollary 5.7. Let \( \Sigma \) and \( \mathcal{R}_\Sigma \) be as in Theorem 5.6.

- The free \( \text{Rel}(;+,\ast,0,D) \)-algebra over \( \Sigma \) consists of the regular subsets of \( \mathcal{R}_\Sigma \) that do not contain the trivial tree \( \varepsilon \).
- The free \( \text{Rel}(;+,\ast,1,D) \)-algebra over \( \Sigma \) consists of the nonempty regular subsets of \( \mathcal{R}_\Sigma \).
- The free \( \text{Rel}(;+,\ast,D) \)-algebra over \( \Sigma \) consists of the nonempty regular subsets of \( \mathcal{R}_\Sigma \) that do not contain \( \varepsilon \).

Proof. Similar to the proof of Corollary 4.7. \( \square \)

Corollary 5.8. Let \( \Sigma \) and \( \mathcal{R}_\Sigma \) be as in Theorem 5.6. Then the free \( \text{Rel}(;+,0,1,D) \)-algebra over \( \Sigma \) consists of all finite regular subsets of \( \mathcal{R}_\Sigma \).

Proof. First note that the regular subsets interpreting \( \{;+,0,1,D\} \)-terms are precisely the finite regular subsets. Then soundness and completeness follow from Proposition 5.4 and Proposition 5.5 respectively. \( \square \)
In [22], Mbacke proves (Theorem 5.3.3 there) that a certain finite equational theory over the signature \{; +, 0, 1, D\}—the theory of domain semirings—is complete for the equational validities of what amounts to the algebras of trees identified in Corollary 5.8. Hence, we obtain the following corollary.

**Corollary 5.9.** The axioms of domain semirings provide a finite equational axiomatisation of the equational theory of Rel(\{; +, 0, 1, D\}).

In other words, the axioms of domain semirings are (sound and) **equationally complete** for algebras of binary relations. In [18], Jipsen and Struth study the singly-generated free domain semiring. Corollary 5.8 now subsumes the description of that paper, though that is not to say that these free algebras are uncomplicated objects.

The other main result of [22] (Theorem 5.3.12 there) is an axiomatisation of the equational validities of our algebras of regular sets of trees. The axiomatisation used consists of the second-order theory of star-continuous Kleene algebras, augmented with one additional second-order axiom:

\[
    a \cdot (\sum_{b \in B} b) ; c = \sum_{b \in B} (a \cdot b ; c) \rightarrow a \cdot (\sum_{b \in B} D(b)) ; c = \sum_{b \in B} (a ; D(b) ; c),
\]

where \(\sum\) indicates supremum. Unfortunately, this axiom is not sound for algebras of binary relations. (And so, in particular, is not a consequence of the axioms of star-continuous Kleene algebras, which are sound for relations.) Hence, we do not obtain an analogue of Corollary 5.9 for the signature \{; +, *, 0, 1, D\}.

### 6. Automata, and closure under intersection

It is well known that the set of regular languages over a finite alphabet \(\Sigma\) is closed under complement with respect to \(\Sigma^*\). It is clear that, even over a (nonempty) finite alphabet, the regular sets of trees are not closed under the complement operation (with respect to the set of reduced trees). And, more meaningfully, this is true even in the view that a regular set \(L\) represents the downward-closed set \(\downarrow L\) (since complement does not preserve downward closure). However, as we will show, the regular sets of trees are closed under the following ‘intersection’ operation.

\[
    L_1 \cdot L_2 := \text{maximal}(\downarrow L_1 \cap \downarrow L_2)
\]

In [15], and its extended journal version [14], **condition automata** are defined. They are an extension of finite-state automata designed specifically for working with relational queries that may contain \(D\) (among other tests such as range and antiderandom). In this section, we will use a slightly simplified definition of condition automata.

**Definition 6.1.** A **domain condition automata** is a 6-tuple \((\Sigma, S, I, T, \delta, c)\), where \((\Sigma, S, I, T, \delta)\) is a finite-state automata (nondeterministic, with \(\epsilon\)-transitions permitted), and \(c\) is a function that assigns a \{; +, *, 0, 1, D\}-term (over \(\Sigma\)) to each state \(S\).
A domain condition automata accepts a path \( x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} x_n \) precisely when it is accepted by the finite-state automata with a trace such that at each step the condition \( D(c(s)) \), where \( s \) is the current state, is satisfied at the corresponding vertex in the model.

**Lemma 6.2.** Let \( t \) be a \([;+,\ast,0,1,D]\)-term, and let \((T,p)\) be a pointed labelled rooted tree, with root \( r \) and point \( p \). Then the pair \((r,p)\) in the relational model \( T \) satisfies \( t \) if and only if there is a \((T',p')\) in \([t]\) with \((T,p) \leq (T',p')\).

**Proof.** This is just a specialisation of Lemma 5.3 (and the equivalence of the condition \((T,p) \leq (T',p')\) with the existence of a homomorphism from \((T',p')\) into \((T,p)\)). \(\square\)

**Proposition 6.3.** Let \( \Sigma \) be an alphabet and let \( L_1 \) and \( L_2 \) be two sets of reduced pointed \( \Sigma \)-labelled finite rooted trees. If \( L_1 \) and \( L_2 \) are regular, then \( L_1 \cdot L_2 \) is regular.

**Proof.** Let \( L_1 = [t_1] \) and \( L_2 = [t_2] \). Then by [15, Proposition 5], there is a domain condition automata \( \mathcal{A}_1 \) such that for any pointed labelled rooted tree \((T,p)\), with root \( r \) and point \( p \), the pair \((r,p)\) satisfies \( t_1 \) if and only if the path from \( r \) to \( p \) is accepted by \( \mathcal{A}_1 \). Similarly, there is such a domain condition automata \( \mathcal{A}_2 \) for \( t_2 \). By [15, Proposition 6], there is a domain condition automata \( \mathcal{A} \) that accepts a path if and only if that path is accepted by both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). Then using [15, Proposition 5] in the other direction, applied to \( \mathcal{A} \), there is a \([;+,\ast,0,1,D]\)-term \( t \) such that for any pointed labelled rooted tree \((T,p)\), with root \( r \) and point \( p \), the pair \((r,p)\) satisfies \( t \) if and only if it satisfies both \( t_1 \) and \( t_2 \).

Applying Lemma 6.2, we have (for reduced \((T,p)\)) that \((T,p) \in [t]\) if and only if \((r,p)\) satisfies \( t \) if and only if \((r,p)\) satisfies both \( t_1 \) and \( t_2 \) if and only if \((T,p) \in [t_1]\) and \((T,p) \in [t_2]\) if and only if \((T,p) \in [t_1] \cap [t_2]\). Hence \([t] = [t_1] \cap [t_2] \). So

\[
[t] = \text{maximal}([t_1]) = \text{maximal}([t_2] = L_1 \cdot L_2. \]  \(\square\)

**Problem 6.4.** Are the regular sets of reduced trees closed under the following reduction operations?

- \( L_1 \setminus L_2 := \text{maximal}\{T \in \mathcal{R}_\Sigma \mid \forall S \in \downarrow L_1, \ S ; T \in \downarrow L_2\}\)
- \( L_1 / L_2 := \text{maximal}\{T \in \mathcal{R}_\Sigma \mid \forall S \in \downarrow L_2, \ T ; S \in \downarrow L_1\}\)

7. Decidability of equational theory

In this section we describe how to decide the validity of a \([;+,\ast,0,1,D]\)-equation with respect to relational semantics.

First, we give a definition of condition automata closer to that found in [15] and [14]. Recall (Definition 2.3) that \( A \) is a unary function symbol whose relational interpretation is the antdomain operation.

**Definition 7.1.** A **condition automata** is a 6-tuple \((\Sigma,S,I,T,\delta,c)\), where \((\Sigma,S,I,T,\delta)\) is a finite-state automata (nondeterministic, with \(e\)-transitions permitted), and \(c\) is a function that assigns a \([;+,\ast,0,1,A]\)-term (over \( \Sigma \)) to each state \( S \).
Acceptance for condition automata is defined just like acceptance for domain condition automata, where the symbol $D$ is now shorthand for two applications of $\lambda$.

**Theorem 7.2.** The equational theory of the class of algebras of binary relations of the signature $\{;+, *, 0, 1, D\}$ is decidable.

**Proof.** Let $s$ and $t$ be $\{;+, *, 0, 1, D\}$-terms, and let $\Sigma$ be the set of variables appearing in either $s$ or $t$. Now $s = t$ is valid in $\text{Rel}(;+, *, 0, 1, D)$ if and only if the pointed $\Sigma$-labelled rooted trees satisfying $s$ are precisely those satisfying $t$. (We proved the stronger statement involving reduced trees.) In the equivalence just stated, we can temporarily use the usual graph-theoretic definition of a pointed $\Sigma$-labelled rooted tree; in particular we do not limit to finite trees. According to [15, Proposition 5], there are (domain) condition automata $A_s$ and $A_t$ that accept precisely the finite pointed trees satisfying $s$ and $t$ respectively. But in fact the finiteness condition plays no role, and hence can be dropped. The same remark can be made for the following statements and we will make no further mention of it. By [15, Corollary 3], there are condition automata $A_{s-t}$ and $A_{t-s}$ such that a pointed tree is accepted by $A_{s-t}$ precisely if it is accepted by $A_s$, but not by $A_t$, and a tree is accepted by $A_{t-s}$ precisely if it is accepted by $A_t$ but not $A_s$. By [15, Proposition 5], there exist $\{;+, *, 0, 1, A\}$-terms $\tau_{s-t}$ and $\tau_{t-s}$ such that the pointed trees satisfying $\tau_{s-t}$ and $\tau_{t-s}$ are precisely those accepted by $A_{s-t}$ and $A_{t-s}$ respectively. Hence $s = t$ is valid in $\text{Rel}(;+, *, 0, 1, D)$ if and only if the sets of pointed $\Sigma$-labelled rooted trees satisfying $\tau_{s-t}$ and $\tau_{t-s}$ are both empty. Finally, we reduce the problem of deciding if a $\{;+, *, 0, 1, A\}$-term $\tau$ is satisfiable by a pointed labelled rooted tree to the problem of deciding the satisfiability of a formula of propositional dynamic logic. This latter problem is known to be decidable (in EXPTIME [25]), so then we are done.

We define a translation from $\{;+, *, 0, 1, A\}$-terms to (propositional variable-free) formulas of propositional dynamic logic as follows. First we define the translation $P$ from $\{;+, *, 0, 1, A\}$-terms to program terms by structural induction as follows.

$$
P(a) := a$$

$$P(s ; t) := P(s) ; P(t)$$

$$P(s + t) := P(s) + P(t)$$

$$P(t^*) := P(t)^*$$

$$P(0) := \bot?$$

$$P(1) := \top?$$

$$P(A(t)) := (\neg P(t))?$$
Free Kleene algebras with domain

Then we simply define the translation \( \varphi(t) \) of a \( \{; +, *, 0, 1, A\} \)-term \( t \) to be \( \langle P(t) \rangle^T \). It is clear that for any given regular frame, satisfiability of \( \varphi(t) \) is equivalent to satisfiability of \( t \) on the corresponding relational model. Since propositional dynamic logic has the\nunwinding property (any frame can be ‘unwound’ to an equivalent tree), we obtain the equivalence of satisfiability of \( \varphi(t) \) on regular frames and satisfiability of \( t \) on tree-based relational models. But if \( t \) holds on a pair \( (x, y) \) of vertices in a tree-based model, then clearly \( y \) is a descendent of \( x \), and \( t \) is satisfied by the tree rooted at \( x \) and having point \( y \). (And conversely, satisfaction by a pointed rooted tree implies satisfaction by a tree-based model.) Hence we have the required equivalence between satisfaction of \( t \) with respect to pointed labelled rooted trees and satisfaction of \( \varphi(t) \) with respect to regular frames. \( \square \)

The procedure described in the proof of Theorem 7.2 hardly seems efficient. The best upper bound that can be obtained from it is a \( 3\text{EXPTIME} \) bound. The constructions of \( \mathcal{A}_{s^{-1}} \) and \( \mathcal{A}_{t^{-5}} \) rely on determinisations involving a subset construction, so can result in an exponential increase in problem size. Likewise, construction of the terms \( \tau_{s^{-1}} \) and \( \tau_{t^{-5}} \) from automata can also add an exponent in general. Lastly, we already mentioned that the final step, deciding satisfiability of propositional dynamic logic formulas, is in \( \text{EXPTIME} \), and in fact this problem also has an exponential time lower bound (no \( O(2^n^\varepsilon) \) algorithm for any \( \varepsilon < 1 [8] \))

We finish with the obvious problem.

**Problem 7.3.** *Determine the precise complexity of deciding validity of \( \{; +, *, 0, 1, D\} \)-equations with respect to relational semantics.*

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Brett McLean, Laboratoire J. A. Dieudonné, Université Nice Sophia Antipolis, 06108 Nice Cedex 02
e-mail: brett.mclean@unice.fr