Computationally Efficient Bayesian Estimation of High Dimensional Copulas with Discrete and Mixed Margins

D. Gunawan∗ M.-N. Tran† K. Suzuki‡ J. Dick‡ R. Kohn∗§

November 1, 2016

Abstract

Estimating copulas with discrete marginal distributions is challenging, especially in high dimensions, because computing the likelihood contribution of each observation requires evaluating $2^J$ terms, with $J$ the number of discrete variables. Currently, data augmentation methods are used to carry out inference for discrete copula and, in practice, the computation becomes infeasible when $J$ is large. Our article proposes two new fast Bayesian approaches for estimating high dimensional copulas with discrete margins, or a combination of discrete and continuous margins. Both methods are based on recent advances in Bayesian methodology that work with an unbiased estimate of the likelihood rather than the likelihood itself, and our key observation is that we can estimate the likelihood of a discrete copula unbiasedly with much less computation than evaluating the likelihood exactly or with current simulation

∗UNSW Business School, University of New South Wales
†The University of Sydney Business School
‡School of Mathematics and Statistics, University of New South Wales
§The research of D. Gunawan and R. Kohn was partially supported by Australian Research Discovery Grant DP150104630 and Australian Center of Excellence Grant CE140100049. The research of J. Dick and K. Suzuki was partially supported by Australian Research Discovery Grant DP150101770.
methods that are based on augmenting the model with latent variables. The first approach builds on the pseudo marginal method that allows Markov chain Monte Carlo simulation from the posterior distribution using only an unbiased estimate of the likelihood. The second approach is based on a Variational Bayes approximation to the posterior and also uses an unbiased estimate of the likelihood. We show that Monte Carlo and randomised quasi-Monte Carlo methods can be used with both approaches to reduce the variability of the estimate of the likelihood, and hence enable us to carry out Bayesian inference for high values of \( J \) for some classes of copulas where the computation was previously too expensive. Our article also introduces a correlated quasi random number pseudo marginal approach into the literature. The methodology is illustrated through several real and simulated data examples.

Key words: Markov chain Monte Carlo; Correlated pseudo marginal Metropolis-Hastings; Randomized quasi random numbers; Variational Bayes

1 Introduction

Copula models provide a flexible approach for modeling multivariate distributions by capturing the joint dependence structure by a copula and modeling the marginal distributions of the variables separately and flexibly (see, for example, Trivedi and Zimmer, 2005; Smith and Khaled, 2012). There are now a number of copula models that allow for a wide range of dependence.

In many applications in the literature, multivariate data are modeled as parametric copulas and unknown parameters are often estimated by maximum likelihood. However, for high dimensional data with discrete variables, maximum likelihood estimation (MLE) is expensive as it requires \( 2^J \) evaluations of a \( J \) dimensional cumulative distribution function to compute the probability mass function (pmf) at a single data point. Recently, Bayesian methods have been developed which offer, to some extent, solutions to this problem. Pitt et al. (2006) propose an efficient
Bayesian method to estimate the parameters of a Gaussian copula model with all its margins discrete. They augment latent variables to the model and generate these latent variables within an MCMC scheme. Smith and Khaled (2012) extend this data augmentation approach to discrete D-vine copula, which is constructed from a sequence of bivariate 'pair-copulas'. They consider Archimedean and elliptical copulas as the building block of D-vine copula. They also extend the method to combinations of discrete and continuous marginals and give two dimensional example. These Bayesian methods typically use Gibbs, or Metropolis within Gibbs, sampling. As the number of latent variables is the same size as the data, i.e. a matrix of size \(n \times J\), with \(n\) the number of observations and \(J\) the number of dimensions, these methods still suffer from computational issues when either \(n\) or \(J\) is large because they require sampling \(nJ\) latent variables. In particular, generating these latent variables for Archimedean copulas is very expensive since the inverses of the conditional distributions for the copula model are usually unavailable in closed form and need to be computed numerically. Furthermore, for Archimedean copula, the conditional copula distribution functions and their densities are also expensive to compute for \(J\) large; see section S8. A second problem with data augmentation approaches is that for large \(J\) they are likely to induce high correlations in the MCMC iterates; see section 6.

Our article introduces several methodological innovations for Bayesian inference in discrete and mixed marginal copulas to overcome the problems experienced when using the latent variable data augmentation approach. Our key observation is that the likelihood of a copula model is a product of terms each of which is expensive to evaluate, but it is relatively cheap to estimate each term, and hence the likelihood, unbiasedly. Based on this insight, we adapt to discrete and mixed margin copulas two recent approaches to Bayesian inference which work with unbiased estimates of the likelihood. The first approach is based on the pseudo marginal (PM) method (Andrieu and Roberts, 2009) and the second (approximate) approach is based on
the Variational Bayes with intractable likelihood (VBIL) method (Tran et al., 2015). Section 3 discusses both approaches.

In particular, our first contribution is to introduce into the copula literature PM approaches based on Monte Carlo (MC) and randomised quasi-Monte Carlo (RQMC) methods. These approaches include the standard PM and the correlated PM approaches discussed in section 3.2 and the block sampling method discussed in section 3.4. We use RQMC for both the PM and the VBIL approaches to help reduce the variance of the log of the estimator of the likelihood based on MC. Second, we introduce into the (general) pseudo marginal literature the correlated RQMC approach in section 3.3, which can be used as a more efficient alternative to the correlated PM approach of Deligiannidis et al. (2016); i.e., it can also be used in a number of applications such as random effects models, time series models, etc. Third, we introduce into the copula literature a variational Bayes approach that works with an unbiased estimate of the likelihood and is much faster than all the PM approaches. Although this approach is approximate, we show in our applications that the approximations are very accurate. The attraction of both approaches is that: (i) they can be used for high dimensional problems (large $J$) and for large data sets (large $n$), where computation based on latent variable methods is prohibitively expensive or infeasible. See section 6. Our article considers 50 dimensional real discrete data examples and a 100 dimensional simulated example. To the best of our knowledge, the highest dimension discrete data examples handled in the literature are less than 30; see, for example, 16-dimensions in Smith and Khaled (2012)), 20-dimensions in Panagiotelis et al. (2012), and 6-dimensions in Panagiotelis et al. (2015). (ii) Both approaches are single-step procedures that can simultaneously estimate both the parameters of the copula and of the marginal densities. In a two-step procedure, the marginal parameters are first estimated and then the copula parameters are estimated based on the transformed marginals, (see, e.g., Joe, 2015). (iii) Both approaches apply to general parametric copulas and can easily incorporate
covariates and random effects.

An online supplement to our article gives further technical and empirical results. All equations, lemmas, tables, etc in the article are referred to as equation (1), lemma 1, table 1, etc, and in the supplement they are referred to as equation (S1), lemma S1 and table S1, etc.

2 The Copula Model

2.1 Definition

A copula \( C(u) \) of dimension \( J \) is a multivariate distribution function defined on \([0,1]^J\), that has each of its margins uniformly distributed on \([0,1]\). Sklar (1959) showed that for \( \mathbf{x} := (x_1, \ldots, x_J) \), a joint cumulative distribution function \( F(\mathbf{x}) \), with marginal distribution functions \( F_j(x_j), j = 1, \ldots, J \), can be written as

\[
F(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \ldots, F_J(x_J)).
\] (1)

If \( C(\cdot) \) has density \( c(\cdot) \), then

\[
\Pr(\mathbf{X} \in \prod_{j=1}^J [a_j^X, b_j^X]) = \int_{a_1}^{b_1} \cdots \int_{a_J}^{b_J} c(u) du
\] (2)

where \( a_j := F_j(a_j^X) \) and \( b_j := F_j(b_j^X) \), for \( j = 1, \ldots, m \).

2.2 Discrete \( \mathbf{X} \)

If \( \mathbf{X} = (X_1, \ldots, X_J) \) is a vector of discrete random variables, then

\[
\Pr(\mathbf{X} \in d\mathbf{X}) = \int_{a_1}^{b_1} \cdots \int_{a_J}^{b_J} c(u) du
\]

\[
= \left( \prod_{j=1}^J (b_j - a_j) \right) \int_0^1 \cdots \int_0^1 c((b_1 - a_1)v_1 + a_1, \ldots, (b_J - a_J)v_J + a_J) dv
\] (3)
with \( u_j = (b_j - a_j)v_j + a_j \), and \( \mathbf{v} := (v_1, \ldots, v_J) \). We can now apply Monte Carlo (MC) or randomized quasi-Monte Carlo (RQMC) to estimate the integral unbiasedly.

We can simplify the integrals (2) and (3) when some of the \( a_j \) are 0. Without loss of generality, suppose that \( a_1, \ldots, a_K \neq 0 \) and \( a_{K+1}, \ldots, a_J = 0 \). Then,

\[
\hat{b}_1 a_1 \cdots \hat{b}_K a_K \int_{u_1:K} D(u_{1:K}, b_{K+1:J}) \, du_{1:K},
\]

where \( u_{1:K} := (u_1, \ldots, u_K) \), \( b_{K+1:J} := (b_{K+1}, \ldots, b_J) \) and

\[
D(u_{1:K}, b_{K+1:J}) := \partial_{u_1} \cdots \partial_{u_K} C(u_{1:K}, b_{K+1:J}) := \frac{\partial^K C(u_{1:K}, b_{K+1:J})}{\partial u_1 \cdots \partial u_K}.
\]

We can rewrite the integral (4) as

\[
\int_{a_1}^{b_1} \cdots \int_{a_K}^{b_K} D(u_{1:K}, b_{K+1:J}) \, du_{1:K} = \left( \prod_{j=1}^{K} (b_j - a_j) \right) \\
\times \int_{0}^{1} \cdots \int_{0}^{1} D((b_1 - a_1)v_1 + a_1, \ldots, (b_K - a_K)v_K + a_K, b_{K+1:J}) \, dv_{1:K}
\]

with \( u_j = (b_j - a_j)v_j + a_j \), \( j = 1, \ldots, K \), and we can now estimate it unbiasedly using MC or RQMC. This also leads to faster and more stable computation as long as we can evaluate \( D(u_K, b_{K+1:J}) \). See sections 2.3 and S3.1 for the Clayton and Gumbel copula cases, respectively.

### 2.3 Example: The Clayton copula

The Clayton \( J \) dimensional copula is an example of an Archimedean copula. Its cdf is

\[
C(\mathbf{u}) := \left( \sum_{j=1}^{J} u_j^{-\theta} - J + 1 \right)^{-\frac{1}{\theta}}, \quad \theta > 0,
\]

(7)
and its density is
\[ c(u) = \partial_{u_1} \cdots \partial_{u_J} C(u) = \prod_{k=0}^{J-1} (\theta k + 1) \left( \prod_{j=1}^{J} u_j \right)^{-(1+\theta)} \left( \sum_{j=1}^{J} u_j^{-\theta} - J + 1 \right)^{-(J+\frac{1}{\theta})}. \]

Equation (8)

We use (4) to evaluate the integral (3) if \( a_j = 0 \), for \( j = K+1, \ldots, J \) and \( a_j > 0 \), \( j = 1, \ldots, K \), in (3). It is readily checked that
\[ D(u_1:K, b_{K+1}:J) = \prod_{k=0}^{K-1} (\theta k+1) \left( \prod_{j=1}^{K} u_j \right)^{-(1+\theta)} \left( \sum_{j=1}^{K} u_j^{-\theta} + \sum_{j=K+1}^{J} b_j^{-\theta} - J + 1 \right)^{-(K+\frac{1}{\theta})}. \]

This integration (4) is preferable since \( D(u_1:K, b_{K+1}:J) \) is bounded on the domain of integration and the dimension of the integration is reduced.

2.4 The Gaussian copula

Thus far, we have taken the copula \( C \) and its density \( c \) as given. Often, however, \( C \) is derived from a parametric multivariate distribution model. We discuss, in particular, the case of the Gaussian copula, but it is straightforward to generalize the discussion to other multivariate distributions such as the multivariate t distribution.

Let \( Z \sim \mathcal{N}(0, \Sigma) \) with \( \Sigma \) the covariance matrix. We say that \( C \) is a Gaussian copula if \( C \) is the joint CDF of the random vector \( (U_1, \ldots, U_J) \), \( U_j = \Phi(Z_j/\sqrt{\sigma_{jj}}), j = 1, \ldots, J \), where \( \Phi(\cdot) \) is the standard normal CDF. In this case, (3) can be rewritten as
\[ \text{Pr}(X \in dX) = \int_{\sqrt{\sigma_{11}}\Phi^{-1}(a_1)}^{\sqrt{\sigma_{11}}\Phi^{-1}(b_1)} \cdots \int_{\sqrt{\sigma_{JJ}}\Phi^{-1}(a_J)}^{\sqrt{\sigma_{JJ}}\Phi^{-1}(b_J)} n(z; 0, \Sigma) \, dz \quad (9) \]

where \( n(z; \mu, \Sigma) \) is the multivariate normal density function with mean \( \mu \) and covariance matrix \( \Sigma \). We estimate the multidimensional integral in (9) unbiasedly using the approach by Genz (1992), which is described in section S4.1.
2.5 Mixed continuous and discrete marginals

We now extend copula framework to accommodate the case where $X$ has both discrete and continuous marginals, with the distribution of $X$ generated by the copula $C(\cdot)$ with density $c(\cdot)$. Without loss of generality, suppose that $X_1, \ldots, X_r$ are the discrete marginals and $X_{r+1}, \ldots, X_J$ are the continuous marginals with CDF $F_j(x_j)$ and PDF $f_j(x_j)$. Then, similarly to (2)

$$
\Pr(X_{1:r} \in dX_{1:r} | x_{r+1:j}) p(x_{r+1:j}) = \int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} c(u_1, \ldots, u_r, u_{r+1}, \ldots, u_J) du_1 \cdots \prod_{j=r+1}^{J} f_j(x_j)
$$

(10)

where $u_j = F_j(x_j)$ for $j = r+1, \cdots, J$.

The Gaussian copula case combines (10) and section 2.4 and is discussed in section S4.2.

3 Bayesian inference

This section discusses estimation and inference using the PM and VBIL methods. In the statistical literature, Beaumont (2003) was the first to propose the PM approach, and Andrieu and Roberts (2009) studied some of its theoretical properties. The PM method carries out Markov chain Monte Carlo (MCMC) on an expanded state space and uses an unbiased estimate of the likelihood, instead of the likelihood. Doucet et al. (2015) show that the variance of the log of the estimated likelihood should be around 1 for the optimal performance of the standard PM method (defined more precisely in section 3.2), and that the performance of the standard PM deteriorates exponentially as the variance of the log of the estimated likelihood increases beyond 1. Thus, a serious drawback of the standard PM method is that it is highly sensitive to the variability of the log of the estimated likelihood (see, e.g., Flury and Shephard, 2011). It may therefore be very computationally demanding to
ensure that the variance of the log of the estimated likelihood is around 1 for high dimensional discrete copulas. To reduce the variation in the Metropolis-Hastings acceptance probability, Deligiannidis et al. (2016) modified the standard PM method by correlating the pseudo-random numbers used in constructing the estimators of the likelihood at the current and proposed values of the Markov chain. This correlated PM approach helps the chain to mix well even if highly variable estimates of the likelihood are used. Thus, the correlated PM requires far fewer computations at every iteration than the standard PM. Tran et al. (2016) propose a modification of the correlated PM approach which samples the latent variables in blocks and shows that for some problems it can much more efficient than the correlated PM approach of Deligiannidis et al. (2016).

The VBIL method, described in section 3.5, developed by Tran et al. (2015) provides a fast variational approximation of the posterior distribution when the likelihood is intractable, but can be estimated unbiasedly. Tran et al. (2015) show both theoretically and empirically that the VBIL method still works well when only highly variable estimates of likelihood are available.

3.1 Estimating the likelihood unbiasedly

Suppose that we have \( n \) observations on the discrete copula at the point \( X_i, i = 1, \ldots, n \). Define \( L_t(\theta) := \Pr(dx_t|\theta) \), where \( \theta \) is the vector of parameters in the copula model, which includes the parameters in the copula itself and the parameters in the marginal distributions. Define the likelihood as \( L(\theta) := \prod_{t=1}^{n} L_t(\theta) \). The term \( \Pr(dx_t|\theta) \) is defined as in (2), or (9) for the Gaussian copula case, and is the density of \( X_t \) with respect to a discrete measure as defined in Smith and Khaled (2012). We can estimate each \( L_t(\theta) \) unbiasedly by MC or RQMC as

\[
\hat{L}_t(\theta) = \left( \prod_{j=1}^{J} (b_j - a_j) \right) \times \frac{1}{M} \sum_{i=1}^{M} c((b_1 - a_1)u_{1}^{(t,i)} + a_1, \ldots, (b_J - a_J)u_{J}^{(t,i)} + a_J),
\]

(11)
where the $\mathbf{u}^{(t,i)} := (u_1^{(t,i)}, \ldots, u_j^{(t,i)})$ are uniformly distributed pseudo or quasi random numbers. A similar estimator can be obtained for the integral in (6). We define the likelihood estimate as

$$
\hat{L}(\theta) := \prod_{t=1}^n \hat{L}_t(\theta).
$$

To indicate that $\hat{L}(\theta)$ also depends on the random variates $\mathbf{u} := \{\mathbf{u}^{(t)}, t = 1, \ldots, n\}$, with $\mathbf{u}^{(t)} := \{u^{(t,i)}, i = 1, \ldots, M\}$, we will sometimes write $\hat{L}(\theta, \mathbf{u})$.

Section S4.1 describes how we estimate the likelihood unbiasedly for the Gaussian copula model.

### 3.2 The Pseudo Marginal (PM) method

This section discusses the PM approach. Let $p_U(\mathbf{u})$ be the density function of $\mathbf{u}$ and $p_\Theta(\theta)$ the prior for $\theta$. We define the joint density of $\theta$ and $\mathbf{u}$ as

$$
\pi(\theta, \mathbf{u}) := \hat{L}(\theta, \mathbf{u}) p_\Theta(\theta) p_U(\mathbf{u}) / \overline{L},
$$

where $\overline{L} := \int L(\theta) p_\Theta(\theta) d\theta$ is the marginal likelihood. Clearly,

$$
\pi(\theta) = \int \pi(\theta, \mathbf{u}) d\mathbf{u} = L(\theta) p_\Theta(\theta) / \overline{L} = \pi(\theta)
$$

is the posterior of $\theta$, because $\int \hat{L}(\theta, \mathbf{u}) p_U(\mathbf{u}) d\mathbf{u} = L(\theta)$ by unbiasedness. Hence, we can obtain samples from the posterior density $\pi(\theta)$ by sampling $\theta$ and $\mathbf{u}$ from $\pi(\theta, \mathbf{u})$.

Let $q_\Theta(\theta'; \theta)$ be a proposal density for $\theta'$ with current state $\theta$ and $q_U(\mathbf{u'}; \mathbf{u})$ the proposal density for $\mathbf{u'}$ given $\mathbf{u}$. We assume that $q_U(\mathbf{u'}; \mathbf{u})$ satisfies the reversibility condition

$$
q_U(\mathbf{u'}; \mathbf{u}) p_U(\mathbf{u}) = q_U(\mathbf{u}; \mathbf{u'}) p_U(\mathbf{u'}).
$$

Then, we generate a proposal $\theta'$ from $q_\Theta(\theta'; \theta)$ and $\mathbf{u'}$ from $q_U(\mathbf{u'}; \mathbf{u})$, and accept
these proposals with the acceptance probability

$$\alpha(\theta, u; \theta', u') := \min \left\{ 1, \frac{\hat{L}(\theta', u') p_\theta(\theta') p_U(u') q_\theta(\theta'; \theta) q_U(u'; u)}{\hat{L}(\theta, u) p_\theta(\theta) p_U(u) q_\theta(\theta; \theta') q_U(u; u')} \right\}$$

using (13).

In our applications, we take $q_\theta(\theta'; \theta)$ as a version of the Adaptive Metropolis-Hastings algorithm of Haario et al. (2001) and Roberts and Rosenthal (2009) using the proposal density given at iteration $j$ by

$$q_\theta(\theta; \theta[j]) := n(\theta; \theta[j], (0.1)^2 I_J/J)$$

for $j < 100$ and

$$q_\theta(\theta; \theta[j]) := 0.05n(\theta; \theta[j], (0.1)^2 I_J/J) + 0.95n(\theta; \theta[j], (2.38)^2 \Sigma[j]/J),$$

for $j \geq 100$, where $\Sigma[j]$ is the current empirical estimate of the covariance structure of the posterior distribution of the parameter $\theta$.

**Definition 1** (The standard pseudo marginal). *In the standard PM method, $q_U(u'; u) = p_U(u')$ so that a new independent set of pseudo-random numbers $u'$ is generated each time we estimate the likelihood.*

The performance of the PM approach depends on the number of samples $M$ used to estimate the likelihood. Pitt et al. (2012) suggest selecting $M$ such that the variance of the log of the estimated likelihood to be around 1 to obtain an optimal tradeoff between computing time and statistical efficiency. However, in many real applications such as the high dimensional copula models that we are interested in, it may be computationally very expensive to ensure that the variance of the log-likelihood is around 1.

The correlated PM proposed by Deligiannidis et al. (2016) correlates the random numbers/auxiliary variables, $u$, used in constructing the estimators of the likelihood at the current and proposed values of the parameters to reduce the variance of
the difference \( \log \hat{L}(\theta', u') - \log \hat{L}(\theta, u) \) appearing in the MH acceptance ratio (14). This method tolerates a much larger variance of the likelihood estimator without getting stuck. The correlated PM approach is given in Algorithm 1. The reversibility condition (13) is satisfied under this scheme.

We now present the approach we use in the paper to select \( M \) in the correlated PM approach. We set the correlation parameter \( \phi = 0.99 \) and then run a short MCMC scheme with a reasonably large \( M \) to determine an approximate estimate \( \tilde{\theta} \) for the posterior mean of \( \theta \). We then obtain \( R \), e.g. \( R = 50 \), independent estimates \( \log \hat{L}^{(i)}(\tilde{\theta}, u) \) and \( \log \hat{L}^{(i)}(\tilde{\theta}, u') \), and compute their sample correlation \( \hat{\rho} \). The optimal variance of the log of the likelihood estimate \( \sigma^2_{opt} \) is approximately \( 2.16^2/(1-\hat{\rho}^2) \) (Tran et al., 2016). We then select the number of samples \( M \) to obtain this optimal variance.

3.3 Correlated PM based on Scrambled \((t, m, s)\)-nets

We refer the reader to section S1 for some background on quasi-Monte Carlo (QMC) and randomized QMC.

Generating correlated Scrambled \((t, m, s)\)-nets

Assume we are given a \((t, m, s)\)-nets \( P_N = \{w_0, ..., w_{N-1}\} \). We apply the scrambling algorithm as described in section S1 using the permutations \( \sigma_j, \sigma_{j,a_{1,j}}, \sigma_{j,a_{1,j},a_{2,j}}, \ldots \)
for \(a_{j,k} \in \{0, \ldots, b-1\}\) and \(j \in \{1, \ldots, J\}\) to get a randomly scrambled \((t, m, s)\)-nets \(\tilde{P}_N = \{y_0, \ldots, y_{N-1}\}\).

The goal is now to generate another scrambled \((t, m, s)\)-nets \(\tilde{Q}_N = \{z_0, \ldots, z_{N-1}\}\) which is highly correlated with \(\tilde{P}_N\). We introduce correlation index \(L \in \{0, 1, 2, \ldots\} \cup \{\infty\}\), where \(L = 0\) implies that \(\tilde{P}_N\) is scrambled independently of \(\tilde{Q}_N\) and \(L = \infty\) means that \(\tilde{P}_N = \tilde{Q}_N\).

To obtain the scrambled point set \(\tilde{Q}_N\) we proceed as follows. Let \(L\) be given. For \(L \in \{2, 3, 4, \ldots\}\), we reuse the permutations \(\sigma_j, \sigma_{j,a_j,1}, \sigma_{j,a_j,1,a_j,2}, \ldots, \sigma_{j,a_j,1,\ldots,a_j,L-1}\) for \(a_{j,k} \in \{0, \ldots, b-1\}\) and \(j \in \{1, \ldots, J\}\), which have been used to obtain the scrambled point set \(\tilde{P}_N\). For \(L = 1\) we only reuse the permutations \(\sigma_j\), whereas for \(L = 0\) we do not reuse any of the permutations and for \(L = \infty\) we reuse all of the permutations. For \(L \in \{0, 1, 2, \ldots\}\) we replace the remaining permutations (which were used to generate \(\tilde{P}_N\)) by a set of new, independently chosen, random set of permutations

\[
\tilde{\sigma}_{j,a_j,1,\ldots,a_j,L-1,a_j,L}, \tilde{\sigma}_{j,a_j,1,\ldots,a_j,L-1,a_j,L+1}, \ldots
\]

for \(a_{j,k} \in \{0, \ldots, b-1\}\) and \(j \in \{1, \ldots, J\}\). We now use these permutations to generate \(\tilde{Q}_N\). Hence, by construction

\[
|y_n - z_n| \in [0, b^{-L}]^J, \quad n = 0, \ldots, N - 1.
\]

The parameter \(L\) therefore determines the level of correlation between the scrambled nets. For \(L = 0\), the point sets \(\tilde{P}_N\) and \(\tilde{Q}_N\) are obtained by independent scramblings, whereas for \(L = \infty\) we obtain that \(y_n = z_n\) for \(n = 0, \ldots, N - 1\) and so \(\tilde{P}_N = \tilde{Q}_N\).

Randomly shifted lattice rules give an alternative method for the correlated PM algorithm, however, the random shift does not preserve the correlation between the points. Let \(\varepsilon > 0\) be small (in some sense). We illustrate this for a real number \(w \in [1 - \varepsilon, 1)\) which lies close to boundary 1. Then, by adding a shift \(\varepsilon < \Delta < 1\) we obtain \(\{w + \Delta\} = w + \Delta - 1\) and so \(|w - (w + \Delta - 1)| = 1 - \Delta\). So even if
we choose the shift $\Delta$ is small, it does not guarantee that $w$ and the shifted point \{\(w + \Delta\)\} are close to each other.

### 3.3.1 The correlated PM based on RQMC

The correlated PM algorithm based on RQMC proceeds similarly to the correlated PM based on pseudo random numbers except that with probability 0.05 we take $L = 0$ to ensure the ergodicity of the sampling scheme.

### 3.4 The block PM algorithm

The block PM algorithm of Tran et al. (2016) is an alternative to the correlated PM of Deligiannidis et al. (2016) by updating $u$ in blocks. Suppose that $u$ is partitioned into $G$ blocks $u_{(1)}, \ldots, u_{(G)}$. We write the target density in $\theta$ and $u$ as

$$\pi(\theta, u) := \tilde{L}(\theta, u_{(1)}, \ldots, u_{(G)})p_\theta(\theta)p_U(u_{(1)}, u_{(2)}, \ldots, u_{(G)})/L$$

Instead of updating the full set of $(\theta, u)$ at each iteration of the PM algorithm, the block PM algorithm updates $\theta$ and a block $u_{(k)}$ at a time. Block PM always takes a less CPU time in each MCMC iteration than the standard and correlated PM approaches as it does not generate the entire set of random numbers $u$. The block index $k$ is selected at random from $1, \ldots, G$ with $\Pr(\mbox{\(K = k\)}) > 0$ for every $k = 1, \ldots, G$. Our article uses $\Pr(\mbox{\(K = k\)}) = 1/G$. Using this scheme, the acceptance probability becomes

$$\min \left\{ 1, \frac{\tilde{L}(\theta', u_{(1)}, \ldots, u_{(k-1)}, u'_{(k)}, u_{(k+1)}, \ldots, u_{(G)})p_\theta(\theta)}{\tilde{L}(\theta, u_{(1)}, \ldots, u_{(k-1)}, u_{(k)}, u_{(k+1)}, \ldots, u_{(G)})p_\theta(\theta)} \times \frac{q_\theta(\theta'; \theta)}{q_\theta(\theta'; \theta')} \right\}.$$

Tran et al. (2016) show that in the block PM, the correlation between $\log \tilde{L}(\theta', u')$ and $\log \tilde{L}(\theta, u)$ is approximately $\tilde{\rho} = 1 - 1/G$, where $u' = (u_{(1)}, \ldots, u_{(k-1)}, u'_{(k)}, u_{(k+1)}, \ldots, u_{(G)})$. We can then compute the optimal variance as $2.16^2/(1 - \tilde{\rho}^2)$ and choose the number of samples as in section 3.2.
3.5 Variational Bayes with an Intractable Likelihood (VBIL)

Variational Bayes (VB) is a fast method to approximate the posterior distribution \( \pi(\theta) \) by a distribution \( q_\lambda(\theta) \) within some tractable class, such as an exponential family, where \( \lambda \) is a variational parameter which is chosen to minimise the Kullback-Leibler divergence between \( q_\lambda(\theta) \) and \( \pi(\theta) \) (Ormerod and Wand, 2010). We can write the Kullback-Leibler divergence of \( q_\lambda(\theta) \) from \( \pi(\theta) \) as a function of \( \lambda \) as

\[
KL(\lambda) = KL(q_\lambda(\theta) || \pi(\theta)) := \int \log \frac{q_\lambda(\theta)}{\pi(\theta)} q_\lambda(\theta) d\theta.
\]

Most current VB algorithms require that the likelihood \( L(\theta) \) is computed analytically for any \( \theta \). Tran et al. (2015) proposed the VBIL algorithm that works with an unbiased estimate of the likelihood. Define \( z := \log \hat{L}(\theta, u) - \log L(\theta) \) so that \( \hat{L}(\theta, u) = L(\theta) \exp(z) \), and denote by \( g(z|\theta) \) the density of \( z \) given \( \theta \). The reason for introducing \( z \) is that it is easier to work with a scalar \( z \) rather than the high dimensional random numbers \( u \). In this section we also write \( \hat{L}(\theta, u) \) as \( \hat{L}(\theta, z) \). Due to the unbiasedness of the estimator \( \hat{L}(\theta, u) \), we have \( \int \exp(z) g(z|\theta) dz = 1 \).

We now define the corresponding target joint density \( \pi(\theta, z) \) of \( \theta \) and \( z \) as

\[
\pi(\theta, z) := L(\theta) p_N(\theta) \exp(z) g(z|\theta) / \hat{L} = \pi(\theta) \exp(z) g(z|\theta)
\]

which admits the posterior density \( \pi(\theta) \) as its marginal. Tran et al. (2015) approximate \( \pi(\theta, z) \) by \( q_\lambda(\theta, z) := q_\lambda(\theta) g(z|\theta) \), where \( \lambda \) is the vector of variational parameters that are estimated by minimising the Kullback-Leibler divergence between \( q_\lambda(\theta, z) \) and \( \pi(\theta, z) \) in the augmented space, i.e.,

\[
KL(\lambda) = KL(q_\lambda(\theta, z) || \pi(\theta, z)) := \int q_\lambda(\theta) g_N(z|\theta) \log \frac{q_\lambda(\theta) g(z|\theta)}{\pi(\theta, z)} dz d\theta.
\]
The gradient of $KL(\lambda)$ is

$$\nabla_{\lambda} KL(\lambda) = E_{q_{\lambda}} \left\{ \nabla_{\lambda} \left[ \log q_{\lambda}(\theta) \right] (\log q_{\lambda}(\theta) - \log (p_{\Theta}(\theta) \hat{L}(\theta, z))) \right\},$$  \hspace{1cm} (16)$$

where the expectation is with respect to $q_{\lambda}(\theta, z)$. See Tran et al. (2015) for details.

We obtain an unbiased estimator $\nabla_{\lambda} KL(\lambda)$ of the gradient $\nabla_{\lambda} KL(\lambda)$ by generating $\theta \sim q_{\lambda}(\theta)$ and $z \sim g(z|\theta)$ and computing the likelihood estimate $\hat{L}(\theta, z)$. MC or RQMC can be used to estimate the gradient unbiasedly and stochastic optimization is then used to find the optimal $\lambda$.

Algorithm 2 gives general pseudo code for the VBIL method. We note that each iteration of the algorithm can be parallelized because the gradient is estimated by importance sampling. The performance of VBIL depends mainly on the variance of the noisy gradient estimator. Following Tran et al. (2015), we employ a range of methods, such as control variates and factorisation, to reduce this variance.

The VB approximation density $q_{\lambda}(\theta)$ for the Archimedean copulas in our article is the inverse gamma discussed in section S3.2 and for the Gaussian copula it is discussed in section S4.4.

4 Discrete and mixed marginal examples using Archimedean copulas

This section purposely keeps the dimension of the parameter $\theta$ low in order to study the effect of estimating the likelihood without confounding this effect with the quality of the proposal for $\theta$.

4.1 Description of the well-being and life events dataset

The data used in the examples is obtained from the Household, Income, and Labour Dynamics in Australia (HILDA) survey for the year 2013. We use 50 categorical
Algorithm 2 The VBIL algorithm

Initialise \( \lambda^{(0)} \) and let \( S \) be the number of samples used to estimate the gradient (16).

1. Initialisation: Set \( t = 0 \)
   - (a) Generate \( \theta_s^{(t)} \sim q_\lambda(\theta) \) and \( z_s^{(t)} \sim g(z|\theta) \), for \( s = 1, \ldots, S \)
   - (b) Set
     \[
     c^{(t)} = \frac{\hat{\text{Cov}}(\log(\hat{L}(\theta, z)p_\theta(\theta)\nabla_\lambda \log q_\lambda(\theta), \nabla_\lambda \log q_\lambda(\theta)))}{\hat{\text{V}}(\nabla_\lambda \log q_\lambda(\theta))},
     \]
     where \( \hat{\text{Cov}}(\cdot) \) and \( \hat{\text{V}}(\cdot) \) are sample estimates of covariance and variance based on the samples \( (\theta_s^{(t)}, z_s^{(t)}) \), for \( s = 1, \ldots, S \). The control variate \( c^{(t)} \) is employed to reduce the variance in the gradient estimation.

2. Cycle: Repeat the following until the stopping criterion in section S2 is satisfied.
   - (a) Set \( t = t + 1 \) and generate \( \theta_s^{(t)} \sim q_\lambda(\theta) \) and \( z_s^{(t)} \sim g(z|\theta) \), for \( s = 1, \ldots, S \)
   - (b) Estimate the gradient
     \[
     \hat{H}^{(t)} = \frac{1}{S} \sum_{s=1}^{S} \left( h(\theta_s^{(t)}, z_s^{(t)}) - \log q_\lambda(\theta_s^{(t)}) - c^{(t-1)} \right) \nabla_\lambda \log q_\lambda(\theta_s^{(t)})
     \]
   - (c) Estimate the control variate \( c^{(t)} \) as in step 1(b).
   - (d) Update the variational parameter \( \lambda \) by
     \[
     \lambda^{(t+1)} = \lambda^{(t)} - a_t I_F(\lambda^{(t)})^{-1} \hat{H}^{(t)}
     \]

The learning rate sequence \( \{a_t, t \geq 1, a_t > 0\} \) satisfies the Robbins-Monro conditions \( \sum_t a_t = \infty \) and \( \sum_t a_t^2 < \infty \) (Robbins and Monro, 1951), and \( I_F(\lambda^{(t)}) = \text{Cov}(\nabla_\lambda \log q_\lambda(\theta)) \). Ong et al. (2016) give a method to tune the sequence \( \{a_t\} \) adaptively.
variables which include a range of well-being attributes, such as health (items 1-36), income (item 50), education (item 49), and life satisfaction (item 37), and a range of major life-shock events, such as getting married (item 38), separated from spouse (item 39), getting back together with the spouse (item 40), serious personal injury (item 41), death of spouse or child (item 42), death of a close friend (item 43), being a victim of a property crime (item 44), promotion at work (item 45), major improvement (item 46) and major worsening in personal finances (item 47), and change of residence (item 48). We transformed the response of each person to each item into a 0 or 1. Thus, for questions on health we classified a person as healthy (0) and unhealthy (1). Similarly, for income (item 50), education (item 49), and life satisfaction variables (item 37), we classified people into rich (0) and poor (1), high education (0) and low education (1), and high life satisfaction (0) and low life satisfaction (1). In this example, the unit of analysis was a male aged above 45, who have non-missing information on all the variables being considered, and who are not in the labour force and married, resulting in \( n = 1210 \) individuals. Section S5.1 gives further details on this dataset.

### 4.1.1 Discrete Clayton and Gumbel copulas

We estimated the joint binary distribution of the well-being attributes and life shock events by fitting Clayton and Gumbel copula models using the two correlated and blocked PM methods and the VBIL method. Each MCMC chain consisted of 15,000 iterates with the first 5000 iterates used as burnin. For this example, the parameters of the marginal distributions were set at their sample estimates. For VBIL, the variational distribution \( q_{\lambda} (\theta) \) was the inverse gamma distribution \( IG (a, b) \), with \( S = 256 \) points used to estimate the gradient unbiasedly. We used the adaptive random walk method with automatic scaling of Garthwaite et al. (2015) to ensure that the overall acceptance rate was around 0.44 (Roberts et al., 1997).

To define our measure of the inefficiency of a PM scheme that takes computing
time into account, we first define the Integrated Autocorrelation Time (IACT\(\theta\)) of an MCMC scheme with respect to a parameter \(\theta\), where \(M\) iterates are used to estimate the posterior mean \(\mathbb{E}(\theta|D)\) by \(\hat{\theta}\).

\[
\text{IACT}_\theta := \lim_{M \to \infty} \frac{M \text{V}(\hat{\theta})}{\text{V}(\theta|D)},
\]

where \(D\) is the data, \(\text{V}(\theta|D)\) is the posterior variance of \(\theta\) and \(\text{V}(\hat{\theta})\) is the variance of \(\hat{\theta}\). We note that in a PM scheme, IACT\(\theta\) also depends on the number of samples \(N\) used to estimate the likelihood. Let \(\overline{\text{IACT}}\) be the average of the IACT’s over all the parameters. Our measure of the inefficiency of a PM scheme is the time normalized variance

\[
\text{TNV} := \overline{\text{IACT}} \times CT,
\]

where CT is the computing time.

Table 1 shows the variance of the log of the estimated likelihood for different numbers of points or samples for the 50 dimensional Clayton and Gumbel copulas using MC or RQMC. In particular, the table shows that even 16384 points or samples standard PM based on MC or RQMC would get stuck. We therefore do not report results for standard PM methods in this section and the next as their TNV would be much higher than the correlated or block PM methods.

Tables 2 and 3 summarize the estimation results and show that: (i) The block PM using RQMC is the best PM method, and is over twice as good as the correlated PM using MC. (ii) The VBIL method is at least twice as fast as the best PM method. (iii) All the estimates are close to each other. Section S5.2 gives further results for this example, which confirm that all the PM samplers, as well as the VBIL estimator, converged adequately.
Table 1: The variance of log of the estimated likelihood of the Clayton (columns 2 and 3), the Gumbel copula (columns 4 and 5) with discrete marginals and the Gumbel copula with mixed marginals (columns 7 and 8) as a function of the number of samples (MC) or points (RQMC) and evaluated at the posterior mean estimates for the well-being example.

| Points | Discrete Clayton | Discrete Gumbel | Mixed Gumbel |
|--------|------------------|-----------------|-------------|
|        | RQMC MC          | RQMC MC         | Points RQMC MC |
| 256    | 57.99 88.75      | 169.82 250.71   | 64 51.40 47.90 |
| 512    | 37.46 51.40      | 149.16 151.92   | 128 31.78 37.70 |
| 1024   | 24.66 41.56      | 60.52 131.22    | 256 18.41 17.03 |
| 2048   | 19.17 20.95      | 38.48 106.31    | 512 12.57 12.36 |
| 8192   | 13.95 11.49      | 38.93 52.86     | 1024 5.43 11.03 |
| 16384  | 5.72 8.51        | 19.65 33.77     | 16384 2.36 2.55 |

Table 2: Posterior mean estimates (with posterior standard deviations in brackets) for a 50 dimensions Clayton copula model with \( n = 1210 \) for the well-being example. The TNV is defined above and the rel. TNV := \( \frac{\text{TNV}_{\text{method}}}{\text{TNV}_{\text{corr,MC}}} \).

| Points | Corr. RQMC | Block RQMC | Corr. MC | Block MC | VBIL |
|--------|------------|------------|----------|----------|------|
|        |            |            |          |          |      |
| 0.392  | 0.391      | 0.389      | 0.395    | 0.386    |      |
| Points | (0.01)     | (0.01)     | (0.01)   | (0.01)   |      |
| 512    | 37.46      | 37.46      | 41.56    | 41.56    | 37.46 |
| #iterations | 15000 | 15000 | 15000 | 15000 | 7     |
| CPU time (mins) | 2017.5 | 745 | 1817.5 | 1027.5 | 171.78 |
| IACT   | 4.30       | 4.53       | 4.31     | 5.24     |      |
| TNV    | 8675.25    | 3374.85    | 7833.43  | 5384.10  |      |
| rel. TNV | 1.11 | 0.43 | 1 | 0.69 |      |

Table 3: Posterior mean estimates (with posterior standard deviations in brackets) for a 50 dimensional Gumbel copula model with \( n = 1210 \) for the well-being data example. The rel. TNV := \( \frac{\text{TNV}_{\text{method}}}{\text{TNV}_{\text{corr,MC}}} \).

| Points | Corr. RQMC | Block RQMC | Corr. MC | Block MC | VBIL |
|--------|------------|------------|----------|----------|------|
|        |            |            |          |          |      |
| 1.201  | 1.199      | 1.208      | 1.204    | 1.212    |      |
| Points | (0.01)     | (0.01)     | (0.01)   | (0.01)   |      |
| 1024   | 60.52      | 60.52      | 106.31   | 106.31   | 60.52 |
| #iterations | 15000 | 15000 | 15000 | 15000 | 12     |
| CPU time (mins) | 3427.5 | 1730 | 4377.50 | 2865 | 628.9 |
| IACT   | 10.11      | 7.37       | 20.53    | 4.93     |      |
| TNV    | 34652.03   | 12750.10   | 89870.08 | 14124.5  |      |
| rel. TNV | 0.39 | 0.14 | 1 | 0.16 |      |
4.2 Mixed Marginals Example

The data used in this example is also obtained from the Household, Income, and Labour Dynamics in Australia (HILDA) survey for the year 2014. We use 50 variables consisting of 30 categorical variables and 20 continuous variables. The continuous variables include income, SF36 continuous health score, weight in kg, height in cm, and hours/mins per week for the following activities: paid employment, traveling to/from paid employment, household errands, housework, outdoor tasks, playing with the children, playing with other people children, volunteer work, and caring for disabled relatives. The categorical variables include community participation activities (11 variables), personal satisfaction variables (8 variables), including satisfaction with financial situation, personal safety, employment opportunities, questions about the current job situation (10 variables), and a question about the availability of internet at home. In this example, we fit a Gumbel copula model for a subset of \( n = 1000 \) individuals.

Table 4 summarises the estimation results and shows that: (i) The four PM samplers perform equally well for this example. (ii) The VBIL estimator is at least three times faster than the PM estimators. (iii) All estimates are close to each other.

Section S5.2 gives further results for this example, which confirm that all the PM samplers, as well as the VBIL estimator, converge adequately.

Table 4: Posterior mean estimates (with posterior standard deviation in brackets) for a 50 dimensional Gumbel copula model with \( n = 1000 \) for the well-being example with mixed marginals. The rel. TNV := TNV_{method}/TNV_{corr,MC}.

| Estimate | Corr. RQMC | Block RQMC | Corr. MC | Block MC | VBIL |
|----------|------------|------------|----------|----------|------|
|          | 1.013      | 1.013      | 1.013    | 1.013    | 1.013|
| post. std. dev. | (0.0008) | (0.0008) | (0.0008) | (0.0008) | (0.0008) |
| Points   | 128        | 128        | 128      | 128      | 128  |
| Est. var. $\log \hat{L}(\theta, u)$ | 31.78      | 31.78      | 37.70    | 37.70    | 31.78 |
| #iteration | 15000    | 15000      | 15000    | 15000    | 6    |
| CPU time (mins) | 1280     | 995        | 1027.50  | 975      | 300  |
| IACT     | 4.38       | 4.47       | 4.57     | 4.16     |      |
| TNV      | 5606.4     | 4447.65    | 4695.68  | 4056     |      |
| rel. TNV | 1.19       | 0.94       | 1        | 0.86     |      |
5 Gaussian copula models

5.1 Gaussian copula model for the Melbourne bicycle path data

5.1.1 Data description

We illustrate our methodology using a longitudinal time series of hourly counts of bicycles on an inner city off-road bicycle path in the city of Melbourne. Data was collected on working days between 12 December 2005 and 19 June 2008, resulting in \( n = 565 \) daily observations on hourly counts between 05:01 and 22:00. Smith and Khaled (2012) also fitted a Gaussian copula to this data.

5.1.2 Gaussian copula fitting

To capture the multivariate dependence, the counts during each of the \( J = 16 \) hourly periods are modelled using their empirical distributions and with the Gaussian copula factor model described in section S4.3. We estimated the parameters using the standard PM with MC or RQMC and the VBIL method. We did not use the correlated PM or the block PM methods in this example, as table S1 shows that even with a with a small number of samples or points the variances of the logs of the estimated likelihoods were already less than 0.3. We chose the number of factors using two measures: the lower bound (LB) (which approximates the log of the marginal likelihood) when using the VBIL method, and the five-fold cross-validated log predictive density score (LPDS) (Good, 1952; Geisser, 1980), when using PM with MC. Suppose that the dataset \( \mathcal{D} \) is split into roughly \( B = 5 \) equal parts \( \mathcal{D}_1, ..., \mathcal{D}_5 \). Then the 5−fold LPDS is defined as

\[
LPDS (\hat{p}) := \sum_{j=1}^{B} \sum_{x_j \in \mathcal{D}_j} \log \hat{p} (x_j | \mathcal{D} \setminus \mathcal{D}_j).
\]
For this example, we use the variational distribution $q_{\lambda}(\theta) = q(\beta)$ ($\beta = \text{vec}(B)$), where $q(\beta)$ is a $d$-variate normal $\mathcal{N}(\mu, \Sigma)$ and we used RQMC with $S = 256$ points to estimate the gradient unbiasedly. Each MCMC chain consists of 60,000 iterates with the first 10,000 used as burnin. After convergence of the draws, the $M = 50,000$ iterates $\{\beta^{(k)} : k = 1, \ldots, M\}$ are assumed to come from the posterior $\pi(\beta)$ and are used to estimate the posterior means of the parameters and the dependence measures, together with the corresponding posterior credible intervals. The variance of the log of the estimated likelihood is set as approximately 1 for the optimal independent PM.

Table 5 shows that the LB measure chooses the 3 factor model and the LPDS score chooses the 2 factor model. For simplicity, we choose the 2 factor model for further analysis. Table S1 summarizes the estimation results and shows that: (i) The PM with RQMC is a little better than that with MC. (ii) The VBIL method is at least 30 times faster than the PM methods. Section S6 in the supplementary material further summarizes the estimation results in this example and shows that all estimators converged adequately.

Table 5: Model Selection for the bicycle path data. The lower bound is obtained using VBIL and the LPDS is obtained using the standard PM with MC.

| Model    | 1 factor | 2 factor | 3 factor |
|----------|----------|----------|----------|
| LPDS     | $-3.1530 \times 10^4$ | $-3.1020 \times 10^4$ | $-3.1094 \times 10^4$ |
| LB       | $-3.8684$ | $-3.8042$ | $-3.7995$ |

Table 6: Bicycle path data: Summary of the estimation using a 2 factor Gaussian copula model. The $\text{rel. TNV} := \text{TNV}_{\text{RQMC}}/\text{TNV}_{\text{MC}}$. $\text{IACT}$ is the average IACT over all the parameters.

|                  | RQMC | MC  | VBIL |
|------------------|------|-----|------|
| Points           | 8    | 8   | 8    |
| Est. var. log $\hat{L}(\theta, u)$ | 0.25 | 0.21 | 0.25 |
| #iterations      | 50000 | 50000 | 23   |
| CPU time (mins)  | 891.67 | 1233.33 | 23   |
| $\text{IACT}$    | 258.76 | 274.23 |      |
| rel. TNV         | 0.68  | 1    |      |
5.2 Gaussian copula model for the well-being example

The following 25 variables used in this example are a subset of the variables used in Section 4.1: health variables (item 1-15), a range of major life events (item 41-48), education (item 49), and income (item 50). We estimated their joint distribution using a Gaussian copula factor model using the standard PM methods and the VBIL method. Each MCMC chain consisted of 105000 iterates, with the first 5000 iterates used as burnin, and we only used every 10th iterate of the subsequent 100000. We used the variational distribution \( q_{\lambda}(\theta) = q(\beta) (\beta = \text{vec}(B)) \), where \( q(\beta) \) is a \( d \)-variate normal \( N(\mu, \Sigma) \) and we use RQMC with \( S = 512 \) points to estimate the gradient unbiasedly.

Table 7 summarizes the estimation results and shows that: (i) The block PM using RQMC performs the best and is over twice as good as the correlated PM using MC. (ii) VBIL is over 16 times faster than the PM methods. (iii) All estimates are close to each other. Section S7 gives further results for this example and confirms that all the methods converged adequately.

Table 7: Well being example: Summary of the estimation using a 1 factor Gaussian copula model. The rel. TNV := \( \frac{TNV_{\text{method}}}{TNV_{\text{corr MC}}} \)

| Points | Corr. RQMC | Block RQMC | Corr. MC | Block MC | VBIL |
|--------|-------------|-------------|----------|----------|------|
| Est. var. log \( \hat{p}_N(x|\theta) \) | 18.48 | 18.48 | 22.01 | 22.01 | 18.48 |
| iteration | 105000 | 105000 | 105000 | 105000 | 10 |
| CPU time (mins) | 2800 | 1645 | 3377.50 | 1697.5 | 97.3 |
| IACT | 8.9424 | 8.5059 | 8.6447 | 8.7738 | |
| TNV | 25038.72 | 13992.21 | 29197.47 | 14893.53 | |
| rel. TNV | 0.85 | 0.47 | 1 | 0.51 |
6 Comparing the pseudo marginal and data augmentation approaches

Pitt et al. (2006) proposed algorithm 3 as an efficient Bayesian data augmentation (DA) method to estimate the parameters of a Gaussian copula with discrete marginals and Smith and Khaled (2012) generalized it to arbitrary copulas. Section S8 gives more details of the algorithm. We now compare the PM method to the

\begin{algorithm}
\textbf{Algorithm 3} Data augmentation algorithm
\begin{itemize}
  \item Generate the $j$ marginal $u_{(j)}$, $j = 1, \ldots, J$, from $p\left(u_{(j)}|\theta, u_{(k \neq j)}, x\right)$ for $j = 1, \ldots, J$.
  \item Generate $\theta$ from $p\left(\theta|u\right)$
\end{itemize}
\end{algorithm}

data augmentation method by generating 500 observations from the Gumbel copula with $J = 10$ and from the Clayton copula with $J = 15$, with the copulas having Bernoulli marginals. The value of $\theta$ for the Gumbel was 1.5 and for the Clayton it was 1.0. We used a uniform prior from 1.0 to 1000 for $\theta$ for the Gumbel and a uniform prior from 0.0 to 1000 for the Clayton. We used the block RQMC method for the PM and the data augmentation algorithm 3. Each MCMC chain consisted of 15,000 iterates with the first 5000 iterates used as burnin. For this example, the parameters of the marginal distributions were set at their true values.

Table 8 summarises the simulation results and shows that: (i) The estimates from the two methods are close to each other; (ii) the PM method is much faster than the data augmentation method, especially for the Gumbel copula; (iii) the PM method has a much smaller IACT since it does not condition on the latent variables because it integrates them out; (iv) hence the time normalized variance of the PM method is much smaller than for the data augmentation method.
Table 8: Posterior mean estimates for a 10 dimensional Gumbel copula model with \( n = 500 \) and a 15 dimensional Clayton copula model with \( n = 1000 \). The posterior standard deviations are in brackets. The rel. TNV := \( \frac{\text{TNV}_{\text{block,RQMC}}}{\text{TNV}_{\text{DA}}} \).

|                   | 10 dimensional Gumbel | 15 dimensional Clayton |
|-------------------|------------------------|-------------------------|
|                   | RQMC                   | Data Aug.               |
| Estimate post. std. dev. | 1.52 (0.031)         | 1.56 (0.034)         |
| Points            | 32                     | 32                     |
| Est. var. \( \log \hat{p}_N(\mathbf{x}|\theta) \) | 19.75                  | 14.78                  |
| iteration         | 15000                  | 15000                  |
| CPU time (mins)   | 65                     | 6960                   |
| IACT              | 5.76                   | 33.08                  |
| TNV               | 374.40                 | 230236.80              |
| rel. TNV          | 0.001                  | 1                      |
|                   | 15 dimensional Clayton | Data Aug.               |
| Estimate post. std. dev. | 1.05 (0.051)         | 1.12 (0.056)         |
| Points            | 32                     | 32                     |
| Est. var. \( \log \hat{p}_N(\mathbf{x}|\theta) \) | 19.75                  | 14.78                  |
| iteration         | 15000                  | 15000                  |
| CPU time (mins)   | 65                     | 6960                   |
| IACT              | 5.76                   | 33.08                  |
| TNV               | 374.40                 | 230236.80              |
| rel. TNV          | 0.005                  | 1                      |

7 Online supplementary material

The online supplementary material gives further technical details and empirical results on the real and simulated data.

8 Summary and conclusions

Our article proposes computationally efficient methods for estimating high-dimensional copulas with discrete or mixed marginals and large sample sizes. The proposed methods work with an unbiased estimate of the likelihood, based on pseudo- or quasi-MC numbers, using correlated and block PM sampling. We also propose the VBIL method, which approximates the posterior distribution. The empirical results in both the paper and the supplementary material suggest that for high \( nJ \): (a) The PM samplers and the VBIL method are appreciably more efficient than data augmentation approaches, which can become computationally infeasible for large \( J \) and \( n \); (b) The correlated and block PM samplers are appreciably more efficient than the standard PM sampler; (c) The block PM samplers perform at least as well as the correlated PM samplers, and sometimes appreciably better; (d) RQMC is usually better than MC. (d) The VBIL method is the fastest method, and usually produces
good approximations of the posterior. However, it is only an approximate method. For example, for the well-being dataset, figures S3 and S7 show that the VBIL posterior density estimates are very close to the PM estimates for the 50 dimensional Clayton copula and the 50 dimensional mixed marginal copula, while figure S5 shows that for the 50 dimensional Gumbel copula the VBIL estimate is somewhat different than the PM estimates.

References

Andrieu, C. and Roberts, G. (2009). The pseudo-marginal approach for efficient Monte Carlo computations. The Annals of Statistics, 37:697–725.

Beaumont, M. A. (2003). Estimation of population growth or decline in genetically monitored populations. Genetics, 164:1139–1160.

Deligiannidis, G., Doucet, A., and Pitt, M. (2016). The correlated pseudomarginal method. http://arxiv.org/abs/1511.04992.

Dick, J., Kuo, F., and (2013), I. S. (2013). High-dimensional integration: the quasi-Monte Carlo way. Acta Numerica, 22:133–288.

Dick, J. and Pillichshammer, F. (2010). Digital nets and sequence. Discrepancy theory and quasi-Monte Carlo integration. Cambridge University Press, Cambridge.

Doucet, A., Pitt, M., Deligiannidis, G., and Kohn, R. (2015). Efficient implementation of Markov chain Monte Carlo when using an unbiased likelihood estimator. Biometrika, pages 1–19.

Flury, T. and Shephard, N. (2011). Bayesian inference based only on simulated likelihood: Particle filter analysis of dynamic economic models. Econometric Theory, 1:1–24.
Garthwaite, P. H., Fan, Y., and Sisson, S. A. (2015). Adaptive optimal scaling of Metropolis-Hastings algorithms using the Robbins-Monro process. *Communications in Statistics - Theory and Methods.*

Gassmann, H., Deak, I., and Szantai, T. (2002). Computing multivariate normal probabilities: a new look. *Journal of Computational and Graphical Statistics*, 11(4):920–949.

Geisser, S. (1980). Discussion of “Sampling and ‘Bayes’ inference in scientific modelling and robustness” by G.E.P. Box. *Journal of the Royal Statistical Society A*, 143:416–417.

Genz, A. (1992). Numerical computation of multivariate normal probabilities. *Journal of Computational and Graphical Statistics*, 1:141–149.

Genz, A. and Bretz, F. (1999). Numerical computation of multivariate t-probabilities with application to power calculation of multiple contrasts. *Journal of Statistical Computation and Simulation*, 63:361–378.

Genz, A. and Bretz, F. (2002). Comparison of methods for the computation of multivariate t-probabilities. *Journal of Computational and Graphical Statistics*, 11(4):950–971.

Genz, A. and Bretz, F. (2009). *Computation of multivariate normal and t probabilities*, volume 195. Springer.

Geweke, J. F. and Zhou, G. (1996). Measuring the pricing error of the arbitrage pricing theory. *Review of Financial Studies*, 9:557–587.

Ghosh, J. and Dunson, D. B. (2009). Default prior distributions and efficient posterior computation in Bayesian factor analysis. *Journal of Computational and Graphical Statistics*, 18:2:306–320.
Good, I. J. (1952). Rational decisions. *Journal of the Royal Statistical Society B*, 14:107–114.

Haario, H., Saksman, E., and Tamminen, J. (2001). An adaptive Metropolis algorithm. *Bernoulli*, 7:223–242.

Hickernell, F. J., Lemieux, C., and Owen, A. B. (2005). Control variates for quasi-Monte Carlo. *Statistical Science*, 20:1–31.

Joe, H. (2015). *Dependence Modeling with Copulas*. CRC Press, Taylor and Francis Group, Boca Raton, FL.

Kuo, F. Y., Dunsmuir, W. T. M., Sloan, I., Wand, M., and Womersley, R. S. (2008). Quasi-Monte Carlo for highly structured generalised response models. *Methodol. Comput. Appl. Probab.*, 10:239–275.

L’Ecuyer, P., Lecot, C., and Tuffin, B. (2006). A randomised quasi-Monte Carlo simulation method for Markov chains. In Monte Carlo and quasi-Monte Carlo methods 2004, Springer Berlin Heidelberg.

Lopes, H. F. and West, M. (2004). Bayesian model assessment in factor analysis. *Statistica Sinica*, pages 41–67.

Matoušek, J. (1998). On the $l_2$-discrepancy for anchored boxes. *J. Complexity*, 14:527–556.

Murray, J. S., Dunson, D. B., Carin, L., and Lucas, J. (2013). Bayesian Gaussian copula factor models for mixed data. *Journal of the American Statistical Association*, 108:502:656–665.

Ong, V., Nott, D. J., Tran, M. N., Sisson, S. A., and Drovandi, C. (2016). Synthetic likelihood variational Bayes. http://arxiv.org/abs/1608.03069.

Ormerod, J. T. and Wand, M. P. (2010). Explaining variational approximations. *American Statistician*, 64:140–153.
Owen, A. (1997a). Monte Carlo variance of scrambled net quadrature. *SIAM J. Numer. Anal.*, 34:1884–1910.

Owen, A. (1997b). Scrambled net variance for integrals of smooth functions. *Annals of Statistics*, 25(4):1541–1562.

Owen, A. B. (1998). Scrambling Sobol’ and Niederreiter-Xing points. *Journal of Complexity*, 14(4):466–489.

Panagiotelis, A., Czado, C., and Joe, H. (2012). Pair copula constructions for multivariate discrete data. *Journal of American Statistical Association*, 107:1063–1072.

Panagiotelis, A., Czado, C., Joe, H., and Stober, J. (2015). Model selection for discrete regular vine copulas. *Working paper*.

Pitt, M., Chan, D., and Kohn, R. (2006). Efficient bayesian inference for gaussian copula regression models. *Biometrika*.

Pitt, M. K., Silva, R. S., Giordani, P., and Kohn, R. (2012). On some properties of Markov chain Monte Carlo simulation methods based on the particle filter. *Journal of Econometrics*, 171(2):134–151.

Robbins, H. and Monro, S. (1951). A stochastic approximation method. *The Annals of Mathematical Statistics*, 22(3):400–407.

Roberts, G. O., Gelman, A., and Gilks, W. R. (1997). Weak convergence and optimal scaling of random walk Metropolis-Hastings. *Annals of Applied Probability*, 7:110–120.

Roberts, G. O. and Rosenthal, J. S. (2009). Examples of adaptive MCMC. *Journal of Computational and Graphical Statistics*, 18(2):349–367.

Sklar, A. (1959). Fonctions de rpartition n dimensions et leurs marges. *Publications de l’Institut de Statistique de L’Universit de Paris*. 
Smith, M. and Khaled, M. A. (2012). Estimation of copula models with discrete margins via Bayesian data augmentation. *Journal of the American Statistical Association*.

Tran, M. N., Kohn, R., Quiroz, M., and Villani, M. (2016). Block-wise pseudo marginal Metropolis-Hastings. *preprint arXiv:1603.02485v2*.

Tran, M.-N., Nott, D., and Kohn, R. (2015). Variational Bayes with intractable likelihood. https://arxiv.org/abs/1503.08621.

Trivedi, P. and Zimmer, D. (2005). Copula modeling: An introduction to practitioners. *Foundation and Trends in Econometrics*.

Wand, M. P. (2014). Fully simplified multivariate normal updates in non-conjugate variational message passing. *Journal of Machine Learning Research*, 15:1351–1369.

Ware, J. E., Snow, K. K., Kolinski, M., and Gandeck, B. (1993). *SF-36 Health Survey Manual and Interpretation Guide*. The Health Institute New England Medical Centre, Boston, MA.
S1 Some Background on quasi-Monte Carlo

This section provides some background on quasi-Monte Carlo (QMC) rules. Further information, aimed at statisticians, can be found in Hickernell et al. (2005) and Kuo et al. (2008). The monograph Dick and Pillichshammer (2010) and the survey article Dick et al. (2013) provide extensive background on QMC rules.

QMC rules are designed to approximate integrals over the unit cube $\int_{[0,1]^J} f(u) \, du$ by the average over some function values $\frac{1}{N} \sum_{n=0}^{N-1} f(u_n)$. Plain Monte Carlo (MC) algorithms (using pseudo random numbers) choose the quadrature points i.i.d. uniformly distributed, $u_n \sim U[0,1]^J$. In QMC rules the aim is to choose the points more uniformly than random in the sense that it minimizes the so-called star discrepancy of the point set. For point sets in one dimension (i.e. in $[0,1]$), the star-discrepancy is the Kolmogorov-Smirnov statistic of the empirical distribution of a point set and the uniform distribution. This can be generalised to arbitrary dimension by setting

$$D^*(P_N) = \sup_{a \in [0,1]^J} \left| \frac{1}{N} \sum_{n=0}^{N-1} 1_{[0,a]}(u_n) - \operatorname{Vol}(0,a) \right|$$

where $P_N = \{u_0, \ldots, u_{N-1}\}$, $a = (a_1, \ldots, a_J)$, $[0,a) = [0,a_1) \times \ldots \times [0,a_J)$, and $1_{[0,a]}(u_n)$ is the indicator function, i.e., it is 1 if $u_n \in [0,a)$ and 0 otherwise. The Koksma-Hlawka inequality then asserts that

$$\left| \int_{[0,1]^J} f(u) \, du - \frac{1}{N} \sum_{n=0}^{N-1} f(u_n) \right| \leq D^*(P_N) V(f)$$

where $V(f)$ is the variation of the function $f$ in the sense of Hardy and Krause,
see for instance Dick and Pillichshammer (2010, Chapter 2.4) for more information. There are explicit constructions of deterministic point sets $P_N$ for which $D^*(P_N)$ is of order $N^{-1} (\log N)^{J-1}$. One such construction are so-called $(t,m,s)$-nets in base $b$ (where $b \geq 2$ is an integer). A point set $P_N$ with $N = b^m$ points is called a $(t,m,s)$-net in base $b$ if for all $d_1, \ldots, d_J \geq 0$ with $d_1 + \ldots + d_J = m-t$, every interval

$$
\prod_{j=1}^J \left[ \frac{a_j}{b^{d_j}}, \frac{a_j+1}{b^{d_j}} \right], \quad 0 \leq a_j < b^{d_j} \text{ for } j = 1, \ldots, J
$$

contains exactly $b^t$ points of $P_N$.

Although QMC yields a convergence rate of almost $N^{-1}$ for integrands of finite variation, MC still has some advantages. Using plain MC, one can get a statistical estimate of the variance of the error and one gets a bound on the variance of the integration error of the form $\sigma(f) N^{-1/2}$ (where $\sigma^2(f)$ is the variance of the integrand $f$). As long as $\sigma(f)$ is independent of the dimension $J$, so is the error bound.

To get the best of both MC and QMC, a randomised version of QMC was invented. We shall focus on scrambled $(t,m,s)$-nets in base $b$ here, which were first studied in Owen (1997a,b). It is easiest to describe how the scrambling algorithm applies to a single point $u = (x_1, \ldots, x_J) \in [0,1)^J$. Let

$$
w_j = \xi_{j,1} \frac{1}{b} + \xi_{j,2} \frac{1}{b^2} + \ldots, \quad \text{with } \xi_{j,k} \in \{0,1,\ldots,b-1\},
$$

denote the base $b$ expansion of $x_j$, unique in the sense that, for each $j = 1, \ldots, J$, infinitely many of the digits $\xi_{j,k}$ are different from $b-1$ (thereby excluding a situation where $x_j = \xi_{j,1} b^{-1} + \ldots + \xi_{j,k_0} b^{-k_0} + (b-1) b^{-k_0-1} + (b-1) b^{-k_0-2} + \ldots$).

The scrambling algorithm is applied to each coordinate independently. Hence, it is sufficient to consider only one coordinate $w_j$. We randomly choose a set of permutations of the set $\{0,1,\ldots,b-1\}$ of the following form: We choose permutations $\sigma_j$, $\sigma_{j,a_{j,1}}$, $\sigma_{j,a_{j,1},a_{j,2}}$, $\ldots$ uniformly from the set of permutations for each value of $a_{j,k} \in \{0,\ldots,b-1\}$ for $k = 1,2,\ldots$. Permutations with different indices are indepen-
dent of each other. The scrambled point \( y = (y_1, ..., y_J) \) is obtained by setting

\[
y_j = \frac{\sigma_j(\xi_{j,1})}{b} + \frac{\sigma_{j,1}(\xi_{j,2})}{b^2} + \frac{\sigma_{j,1,1}(\xi_{j,3})}{b^3} + ..., \quad j = 1, ..., J.
\]

To obtain a scrambled \((t, m, s)\)-net \( \{y_0, ..., y_{N-1}\} \), we apply the scrambling to each point in the \((t, m, s)\)-net (note that the permutations do not depend on the points of the \((t, m, s)\)-net). An illustration of this algorithm can for instance be found in Dick and Pillichshammer (2010, Chapter 13.1).

It can be shown that each point of the scrambled \((t, m, s)\)-net is uniformly distributed in \([0, 1)^J\) and that, with probability one, the scrambled \((t, m, s)\)-net is again a \((t, m, s)\)-net. In particular, the fact that each point \(y_n\) of the scrambled \((t, m, s)\)-net is uniformly distributed, implies that the estimator \( \frac{1}{N} \sum_{n=0}^{N-1} f(y_n) \) is unbiased, i.e.

\[
E \left( \frac{1}{N} \sum_{n=0}^{N-1} f(y_n) \right) = \int_{[0,1)^J} f(u) \, du.
\]

Note that Matlab uses a simplified scrambling method due to Matoušek (1998), which has the properties mentioned above but runs faster on a computer.

Owen (1998) shows that the variance of the integration error for scrambled \((0, m, s)\)-nets for a finite number of points is never more than \( e = 2.71828... \) times as large as the Monte Carlo variance, and that the variance of scrambled \((t, m, s)\)-nets is never more than \( b^j ((b+1)/b - 1)^J \) times the Monte Carlo variance for a least-favourable integrand and a finite number of sample points. However, it is also known that asymptotically, the variance of the integration using a scrambled \((t, m, s)\)-nets converges to 0 faster than \( 1/N \), which is better than the plain Monte Carlo rate of \( 1/N \).

Shifted lattice rules are another popular QMC method of the form

\[
\frac{1}{N} \sum_{n=0}^{N-1} f(\{ng/N + \triangle\}) ,
\]

S3
where $g \in \{1, 2, \ldots, N - 1\}^J$ is called the generating vector of the lattice rule, $\{u\} = u - \lfloor u \rfloor$ is the fractional part of a nonnegative real number $w$ (taken componentwise for vector $w$), and $\Delta \in [0, 1)^J$ is called shift. For randomly shifted lattice rules, the shift $\Delta$ is chosen uniformly distributed in $[0, 1)^J$. The variance of the integration error using lattice rules for integrands of bounded variance decays with order $O(N^{-1+\delta})$ for any $\delta > 0$, for integrands in a Sobolev space with partial mixed derivatives up to order 1 in $L_2$, whereas for scrambled digital nets the variance decays with order $O(N^{-3/2+\delta})$ for any $\delta > 0$.

Now consider the problem of estimating the high-dimensional integral
\[ \int_{[0,1]^J} f(u) \, du \] for $n$ observations. A simple approach to this problem would be to generate randomised QMC point set $y$ in $[0, 1)^{Jn}$ and use it to estimate the high-dimensional integrals for all $t$. The problem with this simple approach is that the dimension $Jn$ of $y$ can be very large, and the equidistribution properties of QMC point sets (as measured by the star discrepancy) deteriorate as a function of dimensions. An alternative to this approach is the array-RQMC algorithm proposed by L’Ecuyer et al. (2006). The idea of this approach is to replace the RQMC point set in $[0, 1)^{Jn}$ by $n$ RQMC point sets in $[0, 1)^J$.

Figure S1: QMC versus Plain Monte Carlo: $N = 256$ points sampled independently and uniformly in $[0, 1)^2$ (left); QMC scrambled Sobol sequence in $[0, 1)^2$ of the same length (right)
S2 Stopping criterion for the VBIL algorithm

This section describes the stopping criterion for Algorithm 2 (the VBIL algorithm) in section 3.5. Let $\mathcal{L}$ be the marginal likelihood. Then,

$$\log \mathcal{L} = LB(\lambda) + KL(\lambda),$$  \hspace{1cm} (S1)

where the lower bound $LB(\lambda)$ at iteration $t$ is

$$LB(\lambda^{(t)}) := \frac{1}{n} \left\{ \frac{1}{S} \sum_{s=1}^{S} \hat{h}(\theta^{(t)}_s, z^{(t)}_s) \right\}.$$  \hspace{1cm} (S2)

Clearly, minimising $KL(\lambda)$ with respect to $\lambda$ is equivalent to maximising the lower bound $LB(\lambda)$. The updating step in algorithm 2 is stopped if the change in an average value of the lower bounds over a window of $K$ iterations $\overline{LB}(\lambda^{(t)}) = (1/K) \sum_{k=1}^{K} \hat{LB}(\lambda^{(t-k+1)})$, is less than some threshold $\epsilon$. The lower bound $LB(\lambda)$ is also a good approximation of $\log \mathcal{L}$, which is an important quantity for model selection.
S3  Some further technical results for the Gumbel and Clayton copulas

S3.1 The Gumbel copula

The $J$-dimensional Gumbel copula is another popular example of an Archimedean copula. Its cdf $C(u)$ and density $c(u)$ are

$$C(u) := \exp \left\{ - \left[ \sum_{j=1}^{J} \left( - \log (u_j) \right)^{\theta} \right]^{1/\theta} \right\}$$

$$c(u) := \theta^J \exp \left\{ - \left[ \sum_{j=1}^{J} \left( - \log (u_j) \right)^{\theta} \right]^{\frac{1}{\theta}} \right\} \times \frac{\prod_{j=1}^{J} (- \log (u_j))^{\theta - 1}}{\left( \sum_{j=1}^{J} (- \log (u_j))^{\theta} \right)^{\frac{J}{\theta}} \prod_{j=1}^{J} u_j} \times P_{j,\theta}^G \left( \left[ \sum_{j=1}^{J} (- \log (u_j))^{\frac{1}{\theta}} \right]^{\frac{1}{\theta}} \right)$$

where

$$P_{j,\theta}^G (x) = \sum_{k=1}^{J} a_{mk}^G (\theta) \ x^k,$$

and

$$a_{mk}^G (\theta) = \frac{J!}{k!} \sum_{j=1}^{k} \binom{k}{j} \binom{j/\theta}{J} (-1)^{J-j}.$$

The dependence parameter $\theta$ is defined on $[1, \infty)$, where a value of 1 represents the independence case. The Gumbel copula is an appropriate choice if the data exhibit weak correlation at lower values and strong correlation at the higher values.

If some of the $a_j$ are zero, then directly estimating the integral (3) is computationally inefficient for the same reasons as given in section 2.3 for the Clayton copula.
It can be readily checked that
\[
D(u_{1:K}, b_{K+1:J}) := \theta^K \exp \left\{ - \left[ \sum_{j=1}^{K} (-\log (u_j))^\theta + \sum_{j=K+1}^{J} (-\log (b_j))^\theta \right] \right\}^{\frac{1}{\theta}} \times \frac{\prod_{j=1}^{K} (-\log (u_j))^\theta - 1}{\left( \sum_{j=1}^{K} (-\log (u_j))^\theta + \sum_{j=K+1}^{J} (-\log (b_j))^\theta \right) \prod_{j=1}^{K} u_j} \times P^G_{K,\theta} \left( \left[ \sum_{j=1}^{K} (-\log (u_j))^\theta + \sum_{j=K+1}^{J} (-\log (b_j))^\theta \right]^{\frac{1}{\theta}} \right)
\]

Then, we can rewrite the integral as
\[
\int_{a_1}^{b_1} \cdots \int_{a_K}^{b_K} D(u_{1:K}, b_{K+1:J}) du_{1:K} = \prod_{j=1}^{K} (b_j - a_j) \times \int_{0}^{1} \cdots \int_{0}^{1} D((b_1 - a_1)v_1 + a_1, \ldots, (b_K - a_K)v_K + a_K, b_{K+1:J}) dv_{1:K}
\]

### S3.2 The VBIL approximation distribution

For the Clayton copula, the VB approximation to the posterior of \(\theta\) is the inverse gamma density
\[
q_\lambda(\theta) = \frac{a^b}{\Gamma(a)} (\theta)^{-1-a} \exp \left( -\frac{b}{\theta} \right), \ \theta > 0,
\]
and for the Gumbel copula
\[
q_\lambda(\theta) = \frac{a^b}{\Gamma(a)} (\theta - 1)^{-1-a} \exp \left( -\frac{b}{(\theta - 1)} \right), \ \theta > 1,
\]
with the natural parameters \(a\) and \(b\). The Fisher information matrix for the inverse gamma is
\[
I_F(a, b) = \begin{pmatrix}
\nabla_{aa} \left[ \log \Gamma(a) \right] & -1/b \\
-1/b & a/b^2
\end{pmatrix}
\]
with gradient

$$\nabla_a [\log q_\lambda (\theta)] = -\log (\theta) + \log (b) - \nabla_a [\log \Gamma (a)] \quad \text{and} \quad \nabla_b [\log q_\lambda (\theta)] = -\frac{1}{\theta} + \frac{a}{b}.$$ 



### S4 Computation for the Gaussian copula model

Section S4.1 discusses the Genz and Bretz numerical integration approach, section S4.2 discusses the Gaussian copula representation for mixed marginals, section S4.3 presents a factor representation for the Gaussian copula, and section S4.4 discusses the VBIL approximation for the Gaussian factor copula model.

#### S4.1 The Genz and Bretz numerical integration method

In the Gaussian copula with discrete marginals it is necessary to evaluate the integral

$$I = \int_{a_1}^{b_1} \cdots \int_{a_J}^{b_J} n(z; 0, \Sigma) \, dz,$$

where $z = (z_1, ..., z_J)'$ and $\Sigma$ is an $J \times J$ symmetric positive definite covariance matrix. Genz (1992) and Genz and Bretz (1999) propose a method to estimate this integral using either pseudo random or quasi random numbers. Extensive comparisons amongst the numerous proposals in the literature indicate that the Genz (1992) method is the most accurate across a wide range of medium and large dimensional problems (Genz and Bretz, 2002, 2009; Gassmann et al., 2002). Genz’ method uses a sequence of transformations to express $I$ as an integral over a unit hypercube. First, we use the Cholesky decomposition $\Sigma = CC'$, where $C$ is a lower triangular matrix with positive diagonal elements $c_{ii}$. Define $y := C^{-1} z$ so that $z' \Sigma^{-1} z = y'y$ and $dz = |\Sigma|^{1/2} dy = |C| dy$. Then,

$$I = \frac{1}{(2\pi)^{J/2}} \int_{a_1'}^{b_1'} e^{-\frac{y_1^2}{2}} \int_{a_2'(y_1)}^{b_2'(y_1)} e^{-\frac{y_2^2}{2}} \cdots \int_{a_J'(y_1, ..., y_{J-1})}^{b_J'(y_1, ..., y_{J-1})} e^{-\frac{y_J^2}{2}} dy.$$
where $a'_i (y_1, ..., y_{i-1}) = \left( a_i - \sum_{j=1}^{i-1} c_{ij} y_j \right) / c_{ii}$ and $b'_i (y_1, ..., y_{i-1}) = \left( b_i - \sum_{j=1}^{i-1} c_{ij} y_j \right) / c_{ii}$.

Next, define $r_i := \Phi (y_i)$, so that

$$I = \int_{d_1}^{e_1} \int_{d_2(r_1)}^{e_2(r_1)} ... \int_{d_J(r_1,...,r_{J-1})}^{e_J(r_1,...,r_{J-1})} \mathbf{d} \mathbf{r},$$

where

$$d_i (r_1, ..., r_{i-1}) = \Phi \left( \left( a_i - \sum_{j=1}^{i-1} c_{ij} \Phi^{-1} (r_j) \right) / c_{ii} \right).$$

We analogously define

$$e_i (r_1, ..., r_{i-1}) := \Phi \left( \left( b_i - \sum_{j=1}^{i-1} c_{ij} \Phi^{-1} (r_j) \right) / c_{ii} \right).$$

Finally, we use the transformation $r_i = d_i + w_i (e_i - d_i)$ to express $I$ as an integral over the unit cube,

$$I = (e_1 - d_1) \int_0^1 (e_2 - d_2) ... \int_0^1 (e_J - d_J) \int_0^1 \mathbf{d} \mathbf{w},$$

where now

$$d_i = \Phi \left( \left( a_i - \sum_{j=1}^{i-1} c_{ij} \Phi^{-1} (d_j + w_j (e_j - d_j)) \right) / c_{ii} \right)$$

and

$$e_i = \Phi \left( \left( b_i - \sum_{j=1}^{i-1} c_{ij} \Phi^{-1} (d_j + w_j (e_j - d_j)) \right) / c_{ii} \right).$$

The Genz and Bretz algorithm is summarized as follows.
Algorithm S1 The Genz and Bretz algorithm

1. Input $\Sigma$, $a$, $b$, rep

2. Compute lower triangular Cholesky factor $C$ for $\Sigma$.

3. Initialise $\text{Intval} = 0$, $\text{Varsum} = 0$, $d_1 = \Phi\left(\frac{a_1}{c_{1,1}}\right)$, $e_1 = \Phi\left(\frac{b_1}{c_{1,1}}\right)$, and $f_1 = e_1 - d_1$

4. For $s = 1, \ldots, \text{rep}$,
   
   (a) Generate pseudo uniform or randomised quasi random numbers $w_1, \ldots, w_{J-1} \in [0, 1]$
   
   (b) For $i = 2, \ldots, J$, set $y_{i-1} = \Phi^{-1}\left(d_{i-1} + w_{i-1} (e_{i-1} - d_{i-1})\right)$
   
   (c) Set $d_i = \Phi\left(\left(a_i - \sum_{j=1}^{i-1} c_{i,j} y_j\right) / c_{i,i}\right)$ and $e_i = \Phi\left(\left(b_i - \sum_{j=1}^{i-1} c_{i,j} y_j\right) / c_{i,i}\right)$, and $f_i = (e_i - d_i) f_{i-1}$

   (d) Compute $\delta = \left((f_m - \text{Intsum}) / N\right)$, $\text{Intsum} = \text{Intsum} + \delta$, $\text{Varsum} = (s - 2) \frac{\text{Varsum}}{s} + \delta$

   (e) Compute $\text{Error} = \alpha \sqrt{\text{Varsum}}$

5. Output $\text{Varsum}$, $\text{Intsum}$, $\text{Error}$

S4.2 Mixed marginals for the Gaussian copula

This section specializes the discussion in section 2.5 to the Gaussian copula case. Suppose that $X_{1:r}$ have discrete marginals and $X_{r+1:J}$ have continuous marginals and partition $Z$ conformally as $Z_D := Z_{1:r}$ and $Z_C := Z_{r+1:J}$. Define $\mu_{D|C} := \mathbb{E}(Z_D|Z_C)$, $\Sigma_{D|C} := \mathbb{V}(Z_D|Z_C)$, and $\Sigma_C := \mathbb{V}(Z_C)$. It then follows from (9) that

$$
\Pr(X_D \in dX_d|X_C) = \int_{\sqrt{\sigma_{11}^{-1}(a_1)}}^{\sqrt{\sigma_{11}^{-1}(b_1)}} \cdots \int_{\sqrt{\sigma_{rr}^{-1}(a_r)}}^{\sqrt{\sigma_{rr}^{-1}(b_r)}} n(z_D; \mu_{D|C}, \Sigma_{D|C}) dz_D
$$

and

$$
p(x_C) = n(z_C; 0, \Sigma_C) \times \prod_{j=r+1}^{J} \frac{\sqrt{\sigma_{jj}} f_j(x_j)}{n(\Phi^{-1}(F_j(x_j)); 0, 1)}.
$$

with $z_j := \sqrt{\sigma_{jj}} \Phi^{-1}(F_j(x_j))$ for $j = r + 1, \ldots, J$ and where $\Phi(\cdot)$ is the standard normal cdf. The density of observation $x$ is then $p(dx_D|x_C)p(x_C)$ with respect to a
mixed measure. Analogously to section 3, we can then define the likelihood of the observations \( x_1, \ldots, x_n \) as the product of the densities of these observations.

**S4.3 Factor model representation for the Gaussian copula**

To reduce the number of parameters in the covariance matrix \( \Sigma \) of the copula in high dimensions we follow Murray et al. (2013) and use a factor copula model by writing \( \Sigma \) in the factor form as

\[
\Sigma := BB' + I_J,
\]

(S4)

with \( B = (b_{ij}), i = 1, \ldots, J, j = 1, \ldots, k, \) the \( J \times k \) loading matrix. Then, \( \sigma_{ii} = \sum_{j=1}^k b_{ij}^2 + 1 \). To identify the loading matrix \( B \) it is necessary to constrain it. We follow Geweke and Zhou (1996), Lopes and West (2004), Ghosh and Dunson (2009), and Murray et al. (2013) and assume that \( B \) is a full-rank lower-triangular matrix with strictly positive diagonal elements.

We now specify the prior distribution of \( B \) based on these constraints. For \( i > j \), we use independent normal priors \( B_{ij} \sim N(\mu_0, \sigma_0^2) \); for \( i = j \), with \( j = 1, \ldots, k \), we use \( B_{jj} \propto N(\mu_0, \sigma_0^2) I(B_{jj} > 0) \), where \( \mu_0 = 0 \) and \( \sigma_0^2 = 10 \).

**S4.4 VB approximation for the Gaussian factor copula**

The VBIL approximation to the posterior of \( \theta \) for the Gaussian factor copula is the multivariate normal density with natural parameters \( (\mu, \Sigma) \). For a \( d \times d \) matrix \( A \), we define \( \text{vec}(A) \) as the \( d^2 \)-vector is obtained by stacking the columns of \( A \), and \( \text{vech}(A) \) as the \( \frac{1}{2}d(d+1) \)-vector is obtained by stacking the columns of the lower triangular part of \( A \). The duplication matrix \( D_d \) of order \( d \) is the \( d^2 \times \frac{1}{2}d(d+1) \) matrix of zeros and ones such that for any symmetric matrix \( A \)

\[
D_d \text{vech}(A) = \text{vec}(A).
\]
Define $D_d^+ := (D_d'D_d)^{-1} D_d'$ as the Moore-Penrose inverse of $D_d$ and let $\text{vec}^{-1}$ be the inverse of the operator $\text{vec}$. Then, the exponential family for of the multivariate normal density is $q(\theta) = \exp \left( T(\theta)' \lambda - Z(\lambda) \right)$ with

$$T(\theta) = \begin{bmatrix} \theta \\ \text{vech} (\theta \theta') \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \Sigma^{-1} \mu \\ -0.5 D_d' \text{vec} (\Sigma^{-1}) \end{bmatrix}.$$

The usual mean and variance parameterization is

$$\mu = -\frac{1}{2} \left\{ \text{vec}^{-1} \left( D_d^+ \lambda_2 \right) \right\}^{-1} \lambda_1 \quad \text{and} \quad \Sigma = -\frac{1}{2} \left\{ \text{vec}^{-1} \left( D_d^+ \lambda_2 \right) \right\}^{-1}.$$

Wand (2014) derives the following formula for the inverse of the Fisher information matrix

$$I_F(\lambda)^{-1} = \begin{bmatrix} \Sigma^{-1} + M'S^{-1}M & -M'S^{-1} \\ -S^{-1}M & S^{-1} \end{bmatrix},$$

with

$$M = 2D_d^+ (\mu \otimes I_d), \quad \text{and} \quad S = 2D_d^+ (\Sigma \otimes \Sigma) D_d^+.$$  

where $\otimes$ is the Kronecker product and $I_d$ the identity matrix of order $d$. Then, the gradient $\nabla_{\lambda} \left[ \log q(\theta) \right]$ is

$$\nabla_{\lambda} \left[ \log q(\theta) \right] = \begin{bmatrix} \theta - \mu \\ \text{vech} (\theta \theta' - \Sigma - \mu \mu') \end{bmatrix}.$$

**S5 Further description and analysis of the well-being and life-shock events dataset**

This section gives further details of the well-being and life-shock events dataset (abbreviated to ‘well-being dataset’) described in section 4.1, as well as analyses based on it in the main paper.
S5.1 Description of the dataset

The health data used in this paper is obtained from the SF-36 data collected by the HILDA survey. The SF-36 (Medical Outcome Trust, Boston, MA) is a multipurpose and short form health survey with 36 items. Each item provides multiple choice answers for respondents to select from in regard to different aspects of their health. SF-36 is one of the most widely used generic measures of health-related quality of life (HRQoL) in clinical research and general population health. It is a standardised questionnaire used to assess patient health across eight attributes (Ware et al., 1993). These are physical functioning (PF, items 3 to 12), role-physical (RP, items 13 to 16), bodily pain (BP, items 21 and 22), general health (GH, items 1, 2, 33-36), vitality (VT, items 28-31), social functioning (SF, items 20 and 32), role-emotional (RE, items 17 to 19), and mental health (MH, items 23-27). The details of the survey questions can be found in Ware et al. (1993).

S5.2 Discrete and mixed marginal Clayton and Gumbel copulas applied to the well-being dataset

This section presents further results on the analysis of the well-being dataset with discrete margins reported in section 4.1.1 and for mixed margins reported in section 4.2. Figures S2, S4 and S6 show that all PM sampling schemes mix well. Figures S3, S5 and S7 show that all the PM estimates of the posterior density of $\theta$ are very similar. These plots also show that the VBIL estimator is very close to the PM estimates for the discrete Clayton copula and the mixed marginal Gumbel copula, but that there is some discrepancy between the VBIL density estimator and the PM estimators for the discrete Gumbel copula.
Figure S2: Well-being dataset: Sample paths of the 50 dimensional Clayton copula parameter estimated using correlated and block PM methods with MC (pseudo) or RQMC (quasi).

Figure S3: Well-being dataset: Kernel smoothing density estimates of the marginal posterior distributions of 50 dimensional Clayton copula parameter estimated using correlated and block PM with MC (pseudo), RQMC (quasi) and VBIL.

Figure S4: Well-being dataset: Sample paths of the 50 dimensional Gumbel copula parameter estimated using correlated and block PM methods with MC (pseudo) or RQMC (quasi)
Figure S5: Well-being dataset: Kernel smoothing density estimates of the marginal posterior distributions of 50 dimensional Gumbel copula parameter estimated using correlated and block PM with MC (pseudo), RQMC (quasi) and VBIL.

Figure S6: Well-being dataset with mixed marginals: Sample paths of the 50 dimensional Gumbel copula parameter estimated using the correlated and block PM methods with MC (pseudo) or RQMC (quasi).

Figure S7: Well-being dataset with mixed marginals: Kernel smoothing density estimates of the marginal posterior distributions of 50 dimensional Gumbel copula parameter estimated using correlated and block PM with MC (pseudo), RQMC (quasi) and VBIL.
S6  Gaussian copula fitted to the bicycle example

This section further summarizes the Gaussian copula 2 factor model for the bicycle data discussed in section 5.1. Table S1 and figure S8 show that the PM estimators perform well and give very similar results. The VBIL estimator also performs well, but gives slightly different results for some of the parameters.

Table S1: Bicycle example: Posterior mean estimates (with posterior standard deviations in brackets) for the two factor Gaussian copula model. The rel.TNV := $\frac{TNV_{\text{quasi}}}{TNV_{\text{pseudo}}}$

| Param. | Standard RQMC | Standard MC | VBIL | Param. | Standard RQMC | Standard MC | VBIL |
|--------|---------------|-------------|------|--------|---------------|-------------|------|
| $\hat{\beta}_1$ | 1.1145 (0.0567) | 1.1175 (0.0535) | 1.142 (0.057) | $\hat{\beta}_{17}$ | 0.1654 (0.0857) | 0.1760 (0.0864) | 0.199 (0.057) |
| $\hat{\beta}_2$ | 1.8559 (0.0838) | 1.8591 (0.0827) | 1.823 (0.082) | $\hat{\beta}_{18}$ | 0.2209 (0.0828) | 0.2229 (0.0804) | 0.253 (0.054) |
| $\hat{\beta}_3$ | 1.5954 (0.0768) | 1.5846 (0.0738) | 1.564 (0.071) | $\hat{\beta}_{19}$ | 0.5637 (0.0639) | 0.5662 (0.0625) | 0.568 (0.053) |
| $\hat{\beta}_4$ | 0.8381 (0.0591) | 0.8295 (0.0540) | 0.796 (0.051) | $\hat{\beta}_{20}$ | 0.7412 (0.0601) | 0.7376 (0.0588) | 0.763 (0.053) |
| $\hat{\beta}_5$ | 0.5484 (0.0589) | 0.5457 (0.0576) | 0.504 (0.050) | $\hat{\beta}_{21}$ | 1.0659 (0.0769) | 1.0742 (0.0735) | 1.075 (0.068) |
| $\hat{\beta}_6$ | 0.4611 (0.0718) | 0.4553 (0.0705) | 0.388 (0.055) | $\hat{\beta}_{22}$ | 1.1279 (0.0703) | 1.1220 (0.0717) | 1.142 (0.069) |
| $\hat{\beta}_7$ | 0.3063 (0.0762) | 0.3074 (0.0743) | 0.249 (0.053) | $\hat{\beta}_{23}$ | 1.0988 (0.0691) | 1.0930 (0.0729) | 1.145 (0.070) |
| $\hat{\beta}_8$ | 0.3889 (0.0761) | 0.3835 (0.0709) | 0.314 (0.053) | $\hat{\beta}_{24}$ | 1.0774 (0.0757) | 1.0850 (0.0750) | 1.099 (0.071) |
| $\hat{\beta}_9$ | 0.4432 (0.0745) | 0.4436 (0.0728) | 0.376 (0.052) | $\hat{\beta}_{25}$ | 1.0476 (0.0754) | 1.0524 (0.0690) | 1.089 (0.066) |
| $\hat{\beta}_{10}$ | 0.7633 (0.0721) | 0.7628 (0.0713) | 0.704 (0.056) | $\hat{\beta}_{26}$ | 0.2588 (0.0689) | 0.2601 (0.0645) | 0.279 (0.049) |
| $\hat{\beta}_{11}$ | 1.2076 (0.0608) | 1.2089 (0.0619) | 1.206 (0.058) | $\hat{\beta}_{27}$ | 0.1866 (0.1222) | 0.1940 (0.1163) | 0.220 (0.058) |
| $\hat{\beta}_{12}$ | 2.6468 (0.1209) | 2.6621 (0.1313) | 2.667 (0.123) | $\hat{\beta}_{28}$ | 0.1965 (0.1154) | 0.2024 (0.1155) | 0.219 (0.0537) |
| $\hat{\beta}_{13}$ | 2.3967 (0.1218) | 2.3953 (0.1079) | 2.378 (0.102) | $\hat{\beta}_{29}$ | 0.2613 (0.0811) | 0.2695 (0.0817) | 0.287 (0.0502) |
| $\hat{\beta}_{14}$ | 1.5991 (0.0637) | 1.5936 (0.0684) | 1.590 (0.067) | $\hat{\beta}_{30}$ | 0.3169 (0.0685) | 0.3237 (0.0652) | 0.338 (0.0514) |
| $\hat{\beta}_{15}$ | 1.0654 (0.0605) | 1.0572 (0.0533) | 1.059 (0.054) | $\hat{\beta}_{31}$ | 0.2506 (0.0569) | 0.2547 (0.0533) | 0.260 (0.0468) |
| $\hat{\beta}_{16}$ | 0.5875 (0.0471) | 0.5859 (0.0481) | 0.571 (0.046) |
Figure S8: Bicycle example: Kernel density estimates of the marginal posterior densities for the Gaussian copula 2 factor model estimated using VBIL and standard pseudo marginal MC and RQMC.
S7 Gaussian Copula fitted to the well-being example

This section further summarizes the Gaussian copula one factor model for the well-being data discussed in section 5.2. Figures S9 to S12 show that all PM schemes mix well. Figure S13 shows that all estimators produce very similar posterior densities.

Figure S9: Sample paths of the 25 dimensional Gaussian copula parameter estimated using the block PM method with MC

![Sample paths of the 25 dimensional Gaussian copula parameter estimated using the block PM method with MC]
Figure S10: Sample paths of the 25 dimensional Gaussian copula parameter estimated using correlated PM method with RQMC

Figure S11: Sample paths of the 25 dimensional Gaussian copula parameter estimated using the correlated PM method with RQMC
Figure S12: Sample paths of the 25 dimensional Gaussian copula parameter estimated using block PM method with RQMC
Figure S13: Kernel smoothing density estimates of the marginal posterior distributions of the 25 dimensional Gaussian copula parameter estimated using correlated and block PM with MC (pseudo) or RQMC (quasi) and VBIL method.
S8  Details of the data augmentation approach

This section gives further details of Algorithm 3. The conditional distribution of
\( p \left( u_{(j)} | \theta, u_{(k \neq j)}, x \right) \) is given by

\[
p \left( u_{(j)} | \theta, u_{(k \neq j)}, x \right) \propto p \left( x | \theta, u \right) p \left( u_{(j)} | \theta, u_{(k \neq j)} \right)
\]

\[
\propto \prod_{i=1}^{n} I \left( a_{i,j} \leq u_{i,j} < b_{i,j} \right) c \left( u_i; \theta \right)
\]

\[
\propto \prod_{i=1}^{n} I \left( a_{i,j} \leq u_{i,j} < b_{i,j} \right) c_{j\mid k \neq j} \left( u_{i,j} \mid u_{i,k \neq j}; \theta \right)
\]

The latents \( u_{i,j} \) are generated from the conditional densities \( c_{j\mid k \neq j} \) constrained
to \([a_{i,j}, b_{i,j})\) and an iterate of \( u_{(j)} \) obtained. In this sampling scheme, the copula parameter \( \theta \) is generated conditional on \( u \) from

\[
p \left( \theta \mid u, x \right) = p \left( \theta \mid u \right) \propto \prod_{i=1}^{n} c \left( u_i; \theta \right) p \left( \theta \right)
\]

The following algorithm is used to generate the latent variables one margin at a
time.

For \( j = 1, \ldots, J \) and for \( i = 1, \ldots, n \)

- Compute

\[
A_{ij} = C_{j\{1,\ldots,J\}\setminus j} \left( a_{i,j} \mid \{u_{i1}, \ldots, u_{ij} \} \setminus u_{ij}, \theta \right)
\]

and

\[
B_{ij} = C_{j\{1,\ldots,J\}\setminus j} \left( b_{i,j} \mid \{u_{i1}, \ldots, u_{ij} \} \setminus u_{ij}, \theta \right)
\]

- Generate \( w_{i,j} \sim Uniform \left( A_{i,j}, B_{i,j} \right) \)

- Compute \( u_{i,j} = C_{j\{1,\ldots,J\}\setminus j}^{-1} \left( w_{i,j} \mid \{u_{i1}, \ldots, u_{ij} \} \setminus u_{ij}, \theta \right) \)
S9  Simulation study for discrete marginals

This simulation study considers a Clayton copula model with \( J \in \{10, 25, 50\} \) and \( n = 1000 \) with the true value of \( \theta = 1 \) and a \( J = 100 \) with \( n = 500 \) and with \( \theta = 0.2 \). All the margins are assumed Bernoulli. The prior distribution for \( \theta \) is inverse gamma with \( \alpha = 2.2 \) and \( \beta = 1.1 \). The posterior distribution of \( \theta \) is estimated using the two correlated and block PM methods and the VBIL method. Each MCMC chain consisted of 15,000 iterates with the first 5000 iterates used as burnin. For VBIL, we used the inverse gamma distribution for \( q_\lambda(\theta) \) as in section S3.2. In this example, the parameters of the marginal distributions are set at their true values.

Tables S2, S3, S4, and S5 summarize the simulation results for \( J = 10, 25, 50, \) and 100 dimensions respectively. They show that all the PM estimates of the posterior mean of \( \theta \) are close to the true values, as are the VBIL estimates, with the VBIL estimate for \( J = 100 \) a bit further away. The tables also show that the IACT’s for \( \theta \) are quite small. Figures S14, S17, S20, and S23 show the sampled paths of the Clayton copula parameters for the 10, 25, 50, and 100 dimensional problems, respectively. They suggest that the chains mix well. Figures S15, S18, S21, and S24 plot the estimates of the posterior marginal densities \( \pi(\theta) \) of \( \theta \) for the PM methods and VBIL. The posterior estimates for all the PM methods are very similar. The VBIL estimates for \( J = 10 \) and 25 are close to the PM estimates, while there is more of a discrepancy between the VBIL estimates and the PM estimates for \( J = 50 \) and 100. estimated using different PM methods are very similar to each other.
Table S2: Summary of simulation for a Clayton copula with $J = 10$ and $n = 1000$.
Posterior mean estimates (with posterior standard deviations in brackets) The rel.
TNV := TNV_{method}/TNV_{corr\_MC}

| true $\theta = 1$ | Corr. RQMC | Block RQMC | Corr. MC | Block MC | VBIL |
|-------------------|------------|------------|----------|----------|------|
| Estimate post. std. dev. | 1.0675 (0.0551) | 1.0682 (0.0546) | 1.0607 (0.0562) | 1.0735 (0.0557) | 1.0668 (0.0691) |
| Points | 16 | 16 | 32 | 32 | 16 |
| Est. var. log $\hat{L}(\theta, u)$ | 12.73 | 12.73 | 13.91 | 13.91 | 12.73 |
| #iteration | 15000 | 15000 | 15000 | 15000 | 10 |
| CPU time (mins) | 127.5 | 85 | 57.5 | 55 | 6.26 |
| IACT | 4.97 | 4.40 | 4.64 | 4.88 |
| TNV | 633.67 | 374 | 266.8 | 268.4 |
| rel. TNV | 2.38 | 1.40 | 1 |

Table S3: Summary of simulation for a Clayton copula with $J = 25$ and $n = 1000$.
Posterior mean estimates (with posterior standard deviations in brackets) The rel.
TNV := TNV_{method}/TNV_{corr\_MC}

| true $\theta = 1$ | Corr. RQMC | Block RQMC | Corr. MC | Block MC | VBIL |
|-------------------|------------|------------|----------|----------|------|
| Estimate post. std. dev. | 1.0100 (0.0432) | 1.0116 (0.0436) | 1.0113 (0.0445) | 1.0122 (0.0426) | 1.007 (0.051) |
| Points | 128 | 128 | 256 | 256 | 128 |
| Est. var. log $\hat{L}(\theta, u)$ | 15.58 | 15.58 | 20.16 | 20.16 | 15.58 |
| #iteration | 15000 | 15000 | 15000 | 15000 | 9 |
| CPU time (mins) | 277.5 | 155 | 242.5 | 185 | 23.85 |
| IACT | 4.49 | 4.45 | 4.53 | 4.50 |
| TNV | 1245.98 | 689.75 | 1098.53 | 832.50 |
| rel. TNV | 1.13 | 0.63 | 1 | 0.75 |

Table S4: Summary of simulation for a Clayton copula with $J = 50$ and $n = 1000$.
Posterior mean estimates (with posterior standard deviations in brackets) The rel.
TNV := TNV_{method}/TNV_{corr\_MC}

| true $\theta = 1$ | Corr. RQMC | Block RQMC | Corr. MC | Block MC | VBIL |
|-------------------|------------|------------|----------|----------|------|
| Estimate post. std. dev. | 1.029 (0.042) | 1.013 (0.042) | 1.028 (0.042) | 1.015 (0.041) | 1.027 (0.059) |
| Points | 256 | 256 | 512 | 512 | 256 |
| Est. var. log $\hat{L}(\theta, u)$ | 30.06 | 30.06 | 36.68 | 36.68 | 30.06 |
| #iteration | 15000 | 15000 | 15000 | 15000 | 9 |
| CPU time (mins) | 832.5 | 330 | 740 | 437.50 | 30.71 |
| IACT | 4.395 | 7.250 | 4.908 | 5.289 |
| TNV | 3658.84 | 2392.50 | 3631.92 | 2313.94 |
| rel. TNV | 1.01 | 0.66 | 1 | 0.63 |
Table S5: Summary of simulation for a Clayton copula with $J = 100$ and $n = 500$. Posterior mean estimates (with posterior standard deviations in brackets) The rel. TNV := $\text{TNV}_{\text{method}}/\text{TNV}_{\text{corr,MC}}$

| true $\theta = 0.2$ | Corr. RQMC | Block RQMC | Corr. MC | Block MC | VBIL |
|---------------------|------------|------------|----------|----------|------|
| Estimate            | 0.212      | 0.216      | 0.211    | 0.212    | 0.242|
| post. std. dev.     | (0.014)    | (0.014)    | (0.014)  | (0.013)  | (0.016)|
| Points              | 512        | 512        | 1024     | 1024     | 512  |
| Est. var. log $\hat{L}(\theta, u)$ | 43.22    | 43.22      | 35.91    | 35.91    | 43.22|
| #iteration          | 15000      | 15000      | 15000    | 15000    | 13   |
| CPU time (mins)     | 3097.5     | 1011       | 1767.50  | 1087.50  | 200.98|
| IACT                | 4.782      | 6.970      | 4.142    | 6.795    |      |
| TNV                 | 14812.25   | 7046.67    | 7320.99  | 7389.56  |      |
| rel. TNV            | 2.02       | 0.96       | 1        | 1        | 1.00 |

Figure S14: Plots of the iterates of $\theta$ for the 10 dimensional Clayton copula using correlated and block PM methods with MC (pseudo) and RQMC (quasi) methods
Figure S15: Kernel smoothing density estimates of the posterior density of $\theta$ for the 10 dimensional Clayton copula using correlated and block PM with MC (pseudo) and RQMC (quasi).

Figure S16: Autocorrelation estimates of the iterates of $\theta$ for the 10 dimensional Clayton copula using correlated and block PM with MC (pseudo) and RQMC (quasi).
Figure S17: Plots of the iterates of $\theta$ for the 25 dimensional Clayton copula using correlated and block PM methods with MC (pseudo) and RQMC (quasi) methods.

Figure S18: Kernel smoothing density estimates of the posterior density of $\theta$ for the 25 dimensional Clayton copula using correlated and block PM with MC (pseudo) and RQMC (quasi)
Figure S19: Autocorrelation estimates of the iterates of $\theta$ for the 25 dimensional Clayton copula using correlated and block PM with MC (pseudo) and RQMC (quasi) methods.

Figure S20: Plots of the iterates of $\theta$ for the 50 dimensional Clayton copula using correlated and block PM methods with MC (pseudo) and RQMC (quasi) methods.
Figure S21: Kernel smoothing density estimates of the posterior density of $\theta$ for the 50 dimensional Clayton copula using correlated and block PM with MC (pseudo) and RQMC (quasi)
Figure S22: Autocorrelation estimates of the iterates of $\theta$ for the 50 dimensional Clayton copula using correlated and block PM with MC (pseudo) and RQMC (quasi) methods.

Figure S23: Plots of the iterates of $\theta$ for the 100 dimensional Clayton copula using correlated and block PM methods with MC (pseudo) and RQMC (quasi) methods.
Figure S24: Kernel smoothing density estimates of the posterior density of $\theta$ for the 100 dimensional Clayton copula using correlated and block PM with MC (pseudo) and RQMC (quasi)
Figure S25: Autocorrelation estimates of the iterates of \( \theta \) for the 100 dimensional Clayton copula using correlated and block PM with MC (pseudo) and RQMC (quasi)