Leibniz Superalgebras and Central Extensions

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ABSTRACT

Dialgebras are generalizations of associative algebras which give rise to Leibniz algebras instead of Lie algebras. In this paper we study super dialgebras and Leibniz superalgebras, which are \( \mathbb{Z}_2 \)-graded dialgebras and Leibniz algebras. We also study universal central extensions of Leibniz superalgebras and obtain some results as in the case of Lie superalgebras. We determine universal central extensions of basic classical Lie superalgebras in the category of Leibniz superalgebras. They play a key role in studying Leibniz superalgebras graded by finite root systems.

Key Words: Leibniz superalgebra; super dialgebra; universal central extension.

1 INTRODUCTION

Central extensions play an important role in the theory of Lie algebras, and it is therefore not surprising that there are many results on central extensions of various classes of Lie algebras. Recently several authors have considered central extensions of Leibniz algebras ([Lo2], [Gao1], [Gao2], [L], [LH1], etc.) and Lie superalgebras ([IK], [MP], etc.).

Universal central extensions of simple Lie algebras over commutative rings were studied in [Gar] and [Kas]. To be precise, let \( \mathfrak{g} \) be a simple finite dimensional Lie algebra over the field of complex numbers \( \mathbb{C} \). It is known that the kernel of the universal central extension of the Lie algebra \( \mathfrak{g} \otimes A \), where \( A = \mathbb{C}[t_1^\pm, \ldots, t_\nu^\pm] (\nu \geq 1) \), is \( \Omega_A^1/dA \), the \( A \)-module of Kähler differentials over \( \mathbb{C} \) ([Gar], [Kas], [KL], [MRY]). Some authors ([Gao1], [Gao2], [LL], etc.) studied the universal central extension of \( \mathfrak{g} \otimes A \) in the category of Leibniz algebras. The kernel of this universal central extension of \( \mathfrak{g} \otimes A \) is \( \Omega_A^1 \). The universal central extensions of basic classical Lie superalgebras over an associative algebra have also been studied ([IK], [MP], etc.). Let \( \mathfrak{g} \) be a basic classical simple Lie superalgebra ([Kac]) over \( \mathbb{C} \), then the universal central extension \( \tilde{U} \) of \( \mathfrak{g} \otimes A \) is given by [IK]

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as follows:

\[ \mathfrak{g}(A) := \tilde{\mathfrak{g}} \otimes A \oplus \Omega^1_A/dA, \]

where

\[ \tilde{\mathfrak{g}} = \begin{cases} \mathfrak{g}, & \text{if } \mathfrak{g} \text{ is not of type } A(n,n)(\forall n) \\ \mathfrak{sl}(n+1,n+1), & \text{if } \mathfrak{g} \text{ is of type } A(n,n)(n > 1) \\ \mathfrak{d}, & \text{if } \mathfrak{g} \text{ is of type } A(1,1) \end{cases} \]

and \( \mathfrak{d} \) is defined as in [IK].

The concepts of Leibniz superalgebra and its cohomology were first introduced by Dzhumadil'daev in [D]. Moreover pre-simplicial structure on the cochain complex of colour Leibniz algebras is constructed in [D]. The aim of this paper is to study super dialgebras and Leibniz superalgebras. In Section 2 we recall some notions of dialgebras and Leibniz algebras and their (co)homology and Lie superalgebras. In Section 3 and Section 4 we define super dialgebras and Leibniz superalgebras and their universal central extensions. In Section 5 we mainly study the universal central extensions of \( \mathfrak{g} \otimes D \) for a basic classical Lie superalgebra \( \mathfrak{g} \) in the category of Leibniz superalgebras over a unital dialgebra \( D \). Especially we obtain that the universal central extension \( \tilde{\mathfrak{g}} \) of \( \mathfrak{g} \otimes A \) is

\[ \tilde{\mathfrak{g}} := \tilde{\mathfrak{g}} \otimes A \oplus \Omega^1_A. \]

Throughout this paper \( \mathbb{C}, \mathbb{Z}, K \) denote the field of complex numbers, the ring of integers and a field respectively, \( |x| \) denotes the degree of a homogeneous element \( x \) in a \( \mathbb{Z}_2 \)-graded vector space, \( \mathfrak{g} \) denotes a basic classical simple Lie superalgebra.

## 2 BASICS

We recall some notions of dialgebras and Leibniz algebras and their (co)homology as defined in [Lo1]—[Lo3] and some notions of Lie superalgebras introduced in [Kac] and [IK].

### 2.1 Dialgebras and Leibniz algebras

**Definition 2.1.** [Lo2] A (associative) dialgebra \( D \) over \( K \) is a \( K \)-vector space equipped with two operations \( \lhd, \vdash \): \( D \otimes D \to D \), called left and right products, satisfying the following five axioms:

\[ \begin{align*}
(a \lhd (b \vdash c)) &= (a \lhd b) \lhd c = a \lhd (b \vdash c), \\
(a \vdash b) \lhd c &= a \vdash (b \lhd c), \\
(a \vdash b) \vdash c &= a \vdash (b \vdash c) = (a \lhd b) \vdash c. 
\end{align*} \]  

(D)

A dialgebra is called unital if it is given a specified bar-unit: an element \( 1 \in D \) which is a unit for the left and right products only on the bar-side, that is \( 1 \vdash a = a = 1 \lhd a \), for any \( a \in D \). Denote by \( \text{Dias} \) and \( \text{Ass} \) the categories of dialgebras and associative algebras over \( K \) respectively. Then the category \( \text{Ass} \) is a full subcategory of \( \text{Dias} \).

Obviously a dialgebra is an associative algebra if and only if \( a \vdash b = a \lhd b = ab \).

The (co)homology of dialgebras can be seen in [F1] and [F2]. Now we introduce the Kähler differential modules over a dialgebra \( D \) defined in [L].
For a commutative associative dialgebra $D$ over $K$ (i.e. $a \vdash b = b \dashv a$), the module of differential $(\Omega^1_D, d)$ of $D$ is defined in the following way. Let $\{a_i\}$ be any basis for $D$ over $K$ and let $F$ be the free left $D$-module on a basis $\{\tilde{d}a_i\}$, where $\{\tilde{d}a_i\}$ is some set equipotent with $\{a_i\}$. We treat $F$ as a 2-sided $D$-module by setting $b \vdash (\tilde{d}a_i) = (\tilde{d}a_i) \dashv b$ and $b \dashv (\tilde{d}a_i) = (\tilde{d}a_i) \vdash b$ for all $a, b \in D$. Let $\tilde{d} : D \to F$ be the $K$-linear map $\sum c_i \vdash a_i \mapsto \sum b_i \dashv (\tilde{d}a_i)$ and let $N$ be the $D$-submodule generated by the relations $\tilde{d}(a \star b) - (\tilde{d}a) \star b + a \star (\tilde{d}a), a, b \in D$, $\star = \dashv, \vdash$. Then $\Omega^1_D := F/N$ and the canonical quotient map $a \mapsto \tilde{d}a + K$ is the differential map $d : D \to \Omega^1_D$.

Up to evident isomorphism, $(\Omega^1_D, d)$ is characterized by the property that for every $D$-module $M$ and every derivation $f : D \to M$ there is a unique $D$-module map $g : \Omega^1_D \to M$ such that $f = g \circ d$. In this way $\text{Der}_K(D, M) \cong \text{Hom}_D(\Omega^1_D, M)$.

A Leibniz algebra [Lo2] $L$ is a vector space over a field $K$ equipped with a $K$-bilinear map $\lbrack \cdot, \cdot \rbrack : L \times L \to L$ satisfying the Leibniz identity
\[
\lbrack \lbrack x, y \rbrack, z \rbrack = \lbrack x, [y, z] \rbrack - [y, [x, z]], \quad \forall x, y, z \in L. \tag{2.1}
\]

Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if $\lbrack x, x \rbrack = 0$ for all $x \in L$.

### 2.2 Chevalley basis of basic classical simple Lie superalgebras

Let $g$ be a basic classical simple Lie superalgebra (see [Kac] in details), $A = (a_{ij})_{i,j \in I}$ for some fixed index set $I$ be the Cartan matrix of $g$. Let $D = \text{diag}(\varepsilon_i)_{i \in I}$ and $B = (b_{ij})$ be diagonal and symmetric matrices such that $A = DB$. Let $\Delta$ be its root system. We fix a Dynkin diagram. Hence an odd simple root is unique, and is denoted by $\alpha_i$.

For $\alpha = \sum_{i \in I} k_i \alpha_i \in \Delta$, we define
\[\varepsilon_\alpha := \begin{cases} 2, & (\alpha, \alpha) \neq 0 \\ \varepsilon_i, & (\alpha, \alpha) = 0 \end{cases}\]
and
\[H_\alpha := \varepsilon_\alpha \sum_{i \in I} k_i \varepsilon_i^{-1} h_i,\]
where $h_i$ is the $i$-th simple coroot (see [IK] in detail).

By Lemma 2.23 in [IK] we have
\[\alpha(H_\alpha) = \begin{cases} 2, & (\alpha, \alpha) \neq 0 \\ 0, & (\alpha, \alpha) = 0 \end{cases}.\]

We also set
\[\sigma_\alpha := \begin{cases} -1, & \alpha \in -\Delta_i^+ \\ 1, & \text{otherwise} \end{cases},\]
and by definition we have
\[\sigma_{-\alpha} = (-1)^{\lvert \alpha \rvert} \sigma_\alpha.\]
For each $\alpha \in \Delta$, choose a root vector $X_\alpha \in \mathfrak{g}_\alpha$ such that
\[ [x_\alpha, x_{-\alpha}] = \sigma_\alpha H_\alpha, \]
then we have for these root vectors, we define $N_{\alpha, \beta} \in \mathbb{C}(\alpha, \beta \in \Delta)$ by
\[ N_{\alpha, \beta} := \begin{cases} 
\text{the coefficient of } X_{\alpha+\beta} \text{ in } [X_\alpha, X_\beta], & \alpha + \beta \in \Delta \\
0, & \alpha + \beta \notin \Delta
\end{cases} \]

The Chevalley basis of a basic classical Lie superalgebra was determined in [IK].

Theorem 2.2. [IK]

1. Let $\mathfrak{g}$ be a basic classical Lie superalgebra over $\mathbb{C}$. For each $\alpha \in \Delta$, we define $\sigma_\alpha$ and $H_\alpha$ as above. Then there exist vectors $\{X_\alpha \in \mathfrak{g}_\alpha | \alpha \in \Delta\}$ such that
   
   \begin{enumerate}
   \item $[X_\alpha, X_{-\alpha}] = \sigma_\alpha H_\alpha,$
   \item $[X_\alpha, X_\beta] = N_{\alpha, \beta} X_{\alpha+\beta},$ where the structure constants $\{N_{\alpha, \beta}\}$ satisfy the following conditions:
   \begin{enumerate}
   \item If $\alpha \in \Delta_0$ or $\beta \in \Delta_0$ (we assume $\alpha \in \Delta_0$) and $\alpha + \beta \in \Delta$, then
     \[ N_{\alpha, \beta}^2 = (p + 1)^2. \]
   \item If $\alpha, \beta \in \Delta_1$ and $(\alpha, \alpha) \neq 0$ or $(\beta, \beta) \neq 0$ (we assume $(\alpha, \alpha) \neq 0$) and $\alpha + \beta \in \Delta$, then
     \[ N_{\alpha, \beta}^2 = (p + 1)^2. \]
   \item If $\alpha, \beta \in \Delta_1$ and $(\alpha, \alpha) = (\beta, \beta) = 0$ and $\alpha + \beta \in \Delta$, then
     \[ N_{\alpha, \beta}^2 = (H_\alpha)^2. \]
   \end{enumerate}
\end{enumerate}

Where $p := \max\{i | \beta - i\alpha \in \Delta\}$.

2. Let $\{X'_\alpha | \alpha \in \Delta\}$ be another set of root vectors satisfying the above conditions and $\{N_{\alpha, \beta} | \alpha, \beta \in \Delta\}'$ be corresponding structure constants. Then there exists $\{u_\alpha | \alpha \in \Delta\} \subset \{\pm 1\}$ such that $u_\alpha u_{-\alpha} = 1$ and
\[ N'_{\alpha, \beta} = u_\alpha u_\beta u_{\alpha+\beta}^{-1} N_{\alpha, \beta}, \]
for any $\alpha, \beta \in \Delta$.

In the sequel, we call the set $\{H_\alpha, X_\alpha | \alpha \in \Delta\}$ a Chevalley basis of $\mathfrak{g}$ if it satisfies the condition in Theorem 2.2.

We also need to use the following Lemma, which is given in [IK].

Lemma 2.3. [IK] Let $\Delta$ be the root system of a basic classical Lie superalgebra $\mathfrak{g}$. If $\alpha \in \Delta^+_1$ such that $(\alpha, \alpha) = 0$, then there exists $\gamma \in \Delta$ such that $\alpha(H_\gamma) = 0$.

3 SUPER DIALGEBRAS AND LEIBNIZ SUPERALGEBRAS

3.1 Super dialgebras.

To study Leibniz superalgebras, we introduce super dialgebras.
**Definition 3.1.** Let $K$ be a field. A super dialgebra over $K$ is a $\mathbb{Z}_2$-graded $K$-vector space $D$ with two operations $\vdash, \dashv : D \otimes D \to D$, called left and right products, satisfying the axiom (D) (section 2) and 
$$D_\sigma \vdash D_{\sigma'}, D_\sigma \dashv D_{\sigma'} \subset D_{\sigma + \sigma'}, \forall \sigma, \sigma' \in \mathbb{Z}_2.$$ 

The definitions of bar-unit and (homogeneous) morphisms of super dialgebras are similar. Denote by $\text{SDias}$, $\text{SAss}$ the categories of super dialgebras and associative superalgebras over $K$ respectively. Then the category $\text{SAss}$ is a full subcategory of $\text{SDias}$.

**Examples.** 1. Obviously an associative superalgebra is a super dialgebra if $a \vdash b = a \dashv b = ab$.

2. *Super differential dialgebra.* Let $(A = A_0 \oplus A_1, d)$ be a differential associative super algebra($|d| = 0$). So by hypothesis, $d(ab) = (da)b + adb$ and $d^2 = 0$. Define left and right products on $A$ by the formulas 
$$x \vdash y = xdy, \quad x \dashv y = (dx)y.$$ 
Then $A$ equipped with these two products is a super dialgebra.

3. *Tensor product.* If $D$ and $D'$ are two super dialgebras, then the tensor product $D \otimes D'$ is also a super dialgebra with 
$$(a \otimes a') \star (b \otimes b') = (-1)^{|a'||b|}(a \star b) \otimes (a' \star b') \quad (3.1)$$
for $\star = \vdash, \dashv$.

For instance $M_{m+n}(D) := M_{m+n}(K) \otimes D$ is a super dialgebra if $D$ is a super dialgebra and $M_{m+n}(K)$ is the superalgebra of all $(m+n) \times (m+n)$-matrices over $K$.

Similar to the result in [Lo3], we also have:

**Theorem 3.2.** The free super dialgebra on a $\mathbb{Z}_2$-graded vector space $V$ is the dialgebra $\text{Dias}(V) = T(V) \otimes V \otimes T(V)$ equipped with the induced $\mathbb{Z}_2$-grading.

A bimodule over a super dialgebra $D$, also called a representation, is a $\mathbb{Z}_2$-graded $K$-module $M$ equipped with two linear maps 
$$\dashv, \vdash : D \otimes M \to M$$
$$\dashv, \vdash : M \otimes D \to M$$
satisfying the axioms (D) whenever they make sense and preserving $\mathbb{Z}_2$-gradation.

For a super dialgebra $S$, let $S_{\text{SAss}}$ be the quotient of $S$ by the ideal generated by elements 
$$x \vdash y - x \dashv y \text{ for all } x, y \in S.$$ It is clear that $S_{\text{SAss}}$ is an associative superalgebra. The canonical surmorphism $S \to S_{\text{SAss}}$ is universal among the maps from $S$ to associative superalgebras. In other words the associativization functor $(-)_{\text{SAss}} : \text{SDias} \to \text{SAss}$ is left adjoint to $\text{inc} : \text{SAss} \to \text{SDias}$.

### 3.2 Leibniz superalgebra

**Definition 3.3.** [D] A Leibniz superalgebra is a $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ over a field $K$ equipped with a $K$-bilinear map $[-,-] : L \times L \to L$ satisfying 
$$[L_\sigma, L_{\sigma'}] \subset L_{\sigma + \sigma'}, \forall \sigma, \sigma' \in \mathbb{Z}_2$$
and the Leibniz identity

\[ [a, b, c] = [a, [b, c]] - (-1)^{|a||b|}[b, [a, c]], \quad \forall a, b, c \in L. \quad (3.2) \]

Obviously, \( L_0 \) is a Leibniz algebra. Moreover any Lie superalgebra is a Leibniz superalgebra and any Leibniz algebra is a trivial Leibniz superalgebra. A Leibniz superalgebra is a Lie superalgebra if and only if

\[ [a, b] + (-1)^{|a||b|}[b, a] = 0, \quad \forall a, b \in L. \]

**Examples.** 1. Let \( g \) be a Lie superalgebra, \( D \) be a unital commutative dialgebra, then \( g \otimes D \) with Leibniz bracket \([x \otimes a, y \otimes b] = [x, y] \otimes (a \triangleright b)\) is a Leibniz superalgebra. Let \( g \) be a basic classical simple Lie superalgebra, then \( \tilde{g} = g \otimes D \oplus \Omega^1_D \) with the bracket

\[ [x \otimes a, y \otimes b] = [x, y] \otimes (a \triangleright b) + (x, y)b \leftarrow da, \quad \forall a, b \in D, x, y \in \tilde{g}, \quad (3.4) \]

\[ [\Omega^1_D, \tilde{g}] = 0 \quad (3.5) \]

is also a Leibniz superalgebra, where \( \tilde{g} \) defined in (1.1) and \((-,-)\) is an even invariant bilinear form of \( g \). In fact we shall prove that it is the universal central extension of \( g \otimes D \) in Section 5.

2. Tensor product. Let \( g \) be a Lie superalgebra, then the bracket

\[ [x \otimes y, a \otimes b] = [[x, y], a] \otimes b + (-1)^{|a||b|}a \otimes [[x, y], b] \]

defines a Leibniz superalgebra structure on the vector space \( g \otimes g \) (see [KP] for that in Leibniz algebras case).

3. The general linear Leibniz superalgebra \( gl(m, n, D) \) is generated by all \( n \times n \) matrices with coefficients from a dialgebra \( D \), and \( m, n \geq 0, n + m \geq 2 \) with the bracket

\[ [E_{ij}(a), E_{kl}(b)] = \delta_{jk}E_{il}(a \triangleright b) - (-1)^{|a||b|}\delta_{il}E_{kj}(b \leftarrow a), \quad (3.7) \]

for all \( a, b \in D \).

Clearly, \( gl(m, n, D) \) is a Leibniz superalgebra. If \( D \) is an associative superalgebra, then \( gl(m, n, D) \) becomes a Lie superalgebra.

By definition, the special linear Leibniz superalgebra with coefficients in \( D \) is

\[ sl(m, n, D) := [gl(m, n, D), gl(m, n, D)]. \]

Notice that if \( n \neq m \) the Leibniz superalgebra \( sl(m, n, D) \) is simple.

The special linear Leibniz superalgebra \( sl(m, n, D) \) has generators \( E_{ij}(a), 1 \leq i \neq j \leq m + n, a \in D \), which satisfy the following relations:

\[ [E_{ij}(a), E_{kl}(b)] = 0, \quad \text{if} \; i \neq l, \; \text{and} \; j \neq k; \]

\[ [E_{ij}(a), E_{kl}(b)] = E_{il}(a \triangleright b), \quad \text{if} \; i \neq l, \; \text{and} \; j = k; \]

\[ [E_{ij}(a), E_{kl}(b)] = -(-1)^{|a||b|}E_{kj}(b \leftarrow a), \quad \text{if} \; i = l, \; \text{and} \; j \neq k, \]

4. The Steinberg Leibniz superalgebra \( stl(m, n, D) \) ([L]) is a Leibniz superalgebra generated by symbols \( u_{ij}(a), 1 \leq i \neq j \leq n, a \in D \), subject to the relations

\[ v_{ij}(k_1a + k_2b) = k_1v_{ij}(a) + k_2v_{ij}(b), \quad \text{for} \; a, b \in D, \; k_1, k_2 \in K; \]
\[
[v_{ij}(a), v_{kl}(b)] = 0, \text{ if } i \neq l, \text{ and } j \neq k;
\]
\[
[v_{ij}(a), v_{kl}(b)] = v_{il}(a \leftarrow b), \text{ if } i \neq l, \text{ and } j = k;
\]
\[
[v_{ij}(a), v_{kl}(b)] = -(−1)^{τ_{i,j}τ_{k,l}} v_{kj}(b \leftarrow a), \text{ if } i = l, \text{ and } j \neq k,
\]
where \(1 \leq i \neq j \leq m+n, a \in D\). It is clear that the last two relations make sense only if \(m+n \geq 3\). See [L] and [HL] for more details about the Steinberg Leibniz superalgebra.

We also denote by \(\text{SLeib}\) and \(\text{SLie}\) the categories of Leibniz superalgebras and Lie superalgebras over \(K\) respectively.

For any super dialgebra \(D\), define
\[
[x, y] = x \leftarrow y - (-1)^{|x||y|} y \leftarrow x,
\]
then \(D\) equipped with this bracket is a Leibniz superalgebra. We denoted it by \(D_L\). The canonical morphism \(D \to D_L\) induces a functor \((-) : \text{SDias} \to \text{SLeib}\).

Remark. For a super dialgebra \(D\), if we define
\[
[x, y] = x \leftarrow y - (-1)^{|x||y|} y \rightarrow x,
\]
then \((D, [\cdot, \cdot])\) is a right Leibniz superalgebra.

For a Leibniz superalgebra \(L\), let \(L_{LS}\) be the quotient of \(L\) by the ideal generated by elements
\[
[x, y] + (-1)^{|x||y|} [y, x], \text{ for all } x, y \in L.
\]
It is clear that \(L_{LS}\) is a Lie superalgebra. The canonical epimorphism : \(L \to L_{LS}\) is universal among the maps from \(L\) to Lie superalgebras. In other words the functor \((-)_{LS} : \text{SLeib} \to \text{SLie}\) is left adjoint to \(\text{inc} : \text{SLie} \to \text{SLeib}\).

Moreover we have the following commutative diagram of categories and functors
\[
\begin{align*}
\text{SDias} & \xrightarrow{\sim} \text{SLeib} \\
\downarrow & \downarrow \\
\text{SAss} & \xrightarrow{\sim} \text{SLie}
\end{align*}
\]

As in the Leibniz algebra case, the universal enveloping super dialgebra of a Leibniz superalgebra \(L\) is
\[
\text{Ud}(L) := (T(L) \otimes L \otimes T(L))/\{[x, y] - x \leftarrow y + (-1)^{|x||y|} y \leftarrow x, x, y \in L\}.
\]

Proposition 3.4. The functor \(\text{Ud} : \text{SLeib} \to \text{SDias}\) is left adjoint to the functor \(- : \text{SDias} \to \text{SLeib}\).

Let \(V\) be a \(\mathbb{Z}_2\)-graded \(K\)-vector space. The free Leibniz superalgebra \(\mathcal{L}(V)\) is the universal Leibniz superalgebra for maps from \(V\) to Leibniz superalgebras.

Let \(\bar{T}(V) := \bigoplus_{n \geq 1} V^\otimes n\) be the reduced tensor module. Then we have

Lemma 3.5. \(\bar{T}(V)\) is the free Leibniz superalgebra over \(V\) with the bracket defined inductively by
\[
\begin{align*}
(1) & \quad [v, x] = v \otimes x, \quad \forall x \in \bar{T}(V), v \in V, \\
(2) & \quad [y \otimes v, x] = [y, v \otimes x] - (-1)^{|y||v|} v \otimes [y, x], \text{ if } x, y \in \bar{T}(V) \text{ and } v \in V.
\end{align*}
\]
**Proof.** By direct calculation we can check that \( \overline{T}(V) \) is a Leibniz superalgebra.

For any Leibniz superalgebra \( g \) and \( \phi : V \to g \), define \( f : \overline{T}(V) \to g \) inductively by
\[
f(v) = \phi(v) \quad \text{and} \quad f(v_1 \otimes \cdots \otimes v_n) = [f(v_1 \otimes \cdots \otimes v_{n-1}), f(v_n)],
\]
where the latter is the bracket in \( g \). Note this definition is forced by relation (1) . Since \( g \) is a Leibniz superalgebra, \( f \) satisfies (2).

Let \( L \) be a Leibniz superalgebra. We call a \( \mathbb{Z}_2 \)-graded space \( M = M_0 \oplus M_1 \) a module over \( L \) if there are two bilinear maps:
\[
[-, -] : L \times M \to M \quad \text{and} \quad [-, -] : M \times L \to M
\]
satisfying the following three axioms

- **SLLM**
  \[
  [[x, y], m] = [x, [y, m]] - (-1)^{|x||y|}[y, [x, m]],
  \]

- **SLML**
  \[
  [x, [m, y]] = [x, m, y] - (-1)^{|x||m|}[m, [x, y]],
  \]

- **SMLL**
  \[
  [m, x, y] = [m, [x, y]] - (-1)^{|m||x|}[x, [m, y]],
  \]

for any \( m \in M \) and \( x, y \in L \).

Given a Leibniz superalgebra \( L \), let \( C^n(L, M) \) be the space of all super skew-symmetric \( K \)-linear homogeneous mapping \( L^\otimes n \to M \), \( n \geq 0 \) and \( C^0(L, M) = M \). Let
\[
d^n : C^n(L, M) \to C^{n+1}(L, M)
\]
be an \( F \)-homomorphism defined by
\[
(d^n f)(x_1, \ldots, x_{n+1}) := \sum_{i=1}^n (-1)^{(|f|+|x_1|+\cdots+|x_{i-1}|)|x_i|}(-1)^{i-1}[x_i, f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})]
- (-1)^{(|f|+|x_1|+\cdots+|x_{n}|)|x_{n+1}|}(-1)^n f(x_1, \ldots, x_n, x_{n+1})
+ \sum_{1\leq i<j\leq n+1} (-1)^{|x_i|(|x_{i+1}|+\cdots+|x_{j-1}|)}(-1)^{j-1} f(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}),
\]
where \( f \in C^n(L, M) \).

From [D] we see that
\[
d^{n+1}d^n = 0, \quad n \geq 0.
\]
Therefore, \( (C^*(L, M), d) \) is a cochain complex, whose cohomology is called the cohomology of the Leibniz algebra \( L \) with coefficients in the representation \( M \):
\[
HL^*(L, M) = H^*(C^*(L, M), d).
\]

Suppose that \( L \) is a Leibniz superalgebra over \( K \). For any \( z \in L \), we define \( \text{ad} z \in \text{End}_K L \) by
\[
\text{ad} z(x) = [z, x], \quad \forall x \in L.
\]
It follows (2.1) that
\[
\text{ad } z([x, y]) = [\text{ad } z(x), y] + (-1)^{|z||x|}[x, \text{ad } z(y)]
\]
(3.11)
for all \(x, y \in L\). This says that \text{ad } z is a super derivation of degree \(|z|\) of \(L\). We also call it the inner derivation of \(L\).

**Remark.** In right Leibniz superalgebras case, (3.11) also holds if we define
\[
\text{ad } z(x) = -(−1)^{|z||x|}[x, z], \forall x \in L.
\]
(3.12)

Similarly we also have the definition of general super derivation of a Leibniz superalgebra \(L\).

By definition a super derivation of degree \(s, s \in \mathbb{Z}_2\) of \(L\) is an endomorphism \(\mu \in \text{End}_s L\) with the property
\[
\mu([a, b]) = [\mu(a), b] + (-1)^{s|a|}[a, \mu(b)].
\]
We denote by \(\text{Inn}(L), \text{Der}(L)\) the sets of all inner derivations, super derivations of \(L\) respectively. They are also Leibniz superalgebras.

For a Lie superalgebra \(L\), \(HL^1(L, M) = H^1(L, M) = \text{Der}(L, M)/\text{Inn}(L, M)\).

### 4 UNIVERSAL CENTRAL EXTENSIONS OF LEIBNIZ SUPERALGEBRAS

As in the cases of Lie algebras and Lie superalgebras and Leibniz algebras, the universal central extensions of Leibniz superalgebras also play an important role in the theory of Leibniz superalgebras.

#### 4.1 Central extensions

**Definition 4.1.** A central extension of a Leibniz superalgebra \(L\) is a short exact sequence in the category \(\text{SLeib}\):
\[
0 \to Z \to \hat{L} \xrightarrow{\alpha} L \to 0,
\]
(4.1)
where \(Z\) is in the center of \(\hat{L}\) and \(Z\) is called the kernel of the central extension (4.1).

We also call a Leibniz superalgebra \(L\) **perfect** if \([L, L] = L\) and sometimes denote the above central extension by \(\alpha : \hat{L} \to L\).

Two central extensions
\[
0 \to C \to \hat{L} \to L \to 0
\]
\[
0 \to C \to \hat{L}' \to L \to 0
\]
of \(L\) are said to be **equivalent** if there exists an isomorphism of Leibniz superalgebras from \(\hat{L} \to \hat{L}'\) such that the following diagram
\[
\begin{array}{c}
0 \to C \to \hat{L} \to L \to 0 \\
\| \quad \downarrow j \quad \|
\end{array}
\]
\[
\begin{array}{c}
0 \to C \to \hat{L}' \to L \to 0
\end{array}
\]
follows. Set the second cohomology group \( H^2(L, C) \). To be precise, we first introduce \( Z^2(L, C) \) and \( B^2(L, C) \) as follows. Set

\[
Z^2(L, C) = \{ \psi : L \times L \to C \mid \psi([a, b], c) = \psi(a, [b, c]) - (-1)^{|a||b|} \psi(b, [a, c]), \quad \forall a, b, c \in L \} \tag{4.2}
\]

and

\[
B^2(L, C) = \{ f : L \times L \to C \mid f(a, b) = g([a, b]) \text{ for some } K\text{-linear } g : L \to C \}. \tag{4.3}
\]

An element in \( Z^2(L, C) \) is called a Leibniz super 2-cocycle.

**Lemma 4.2.** Let \( C \) be a \( K \)-module. The second cohomology group \( H^2(L, C) = Z^2(L, C)/B^2(L, C) \) is in one-to-one correspondence with the set of the equivalence classes of central extensions of \( L \) by \( C \).

**Proof.** The proof is classical and essentially the same as in the case of Lie superalgebras.

### 4.2 Universal central extensions

Now we define the universal central extension of a Leibniz superalgebra \( L \).

**Definition 4.3.** The central extension (4.1) of \( L \) is called the universal central extension (UCE) if the following conditions hold.

1. \( \hat{L} \) is perfect.
2. For any central extension \( \beta : \hat{L}' \to L \), there exists \( \gamma : \hat{L} \to \hat{L}' \) such that \( \alpha = \beta \cdot \gamma \).

**Remark.** A UCE is unique up to isomorphism of Leibniz superalgebras if it exists.

We notice that the mapping \( \gamma \) in the Definition 4.3 is unique. In fact we have:

**Lemma 4.4.** Let \( (X, f) \) and \( (Y, g) \) be central extensions of a Leibniz superalgebra \( L \). If \( Y \) is perfect, then there exists only one homomorphism \( h \) from \( Y \) to \( X \) such that \( f \circ h = g \).

**Proof.** Suppose that there are two homomorphisms \( h_i : Y \to X \) such that \( f \circ h_i = g, i = 1, 2 \).

For \( a, b \in Y \), let \( h_i(a) = a_i \in X, h_i(b) = b_i \in X, i = 1, 2 \). Then \( f(a_1) = f(h_1(a)) = g(a) = f(h_2(a)) = f(a_2) \). Similarly, \( f(b_1) = f(b_2) \). Therefore \( a_1 = a_2 + r_a, b_1 = b_2 + r_b \), where \( r_a, r_b \in \ker f \). Hence

\[
h_1([a, b]) = [h_1(a), h_1(b)] = [a_1, b_1] = [a_2 + r_a, b_2 + r_b] = [a_2, b_2] = [h_2(a), h_2(b)] = h_2([a, b]).
\]

Since \( Y \) is perfect, \( h_1 = h_2 \).

**Proposition 4.5.** A Leibniz superalgebra \( L \) admits a universal central extension \( \hat{L} \) if and only if \( L \) is perfect.

**Proof.** Suppose that \( \alpha : \hat{L} \to L \) is the UCE. By definition, \( \hat{L} \) is perfect, and hence

\[
L = \alpha(\hat{L}) = \alpha([\hat{L}, \hat{L}]) = [\alpha(\hat{L}), \alpha(\hat{L})] = [L, L].
\]
Next let us suppose that $L$ is perfect. We set
\[ W = (L \otimes L)/I, \]
where $I$ is the ideal generated by $[a, b] \otimes c - a \otimes [b, c] + (-1)^{|a||b|} b \otimes [a, c]$ for all $a, b, c \in L$. Let $w : L \otimes L \to W$ be the canonical projection. It is clear that $w \in Z^2(L, W)$. We consider the central extension
\[ 0 \to W \to L_w \to L \to 0 \]
defined by $w$. Using this central extension, we construct the universal central extension of $L$. Let $V$ be an arbitrary $K$-module and $f \in Z^2(L, V)$. Since
\[ f([x, y], z) - f(x, [y, z]) + (-1)^{|x||y|} f(y, [x, z]) = 0, \]
we have a $K$-linear map $\psi' : W \to V$ such that $w(x, y) \to f(x, y)$.

Let us define $\phi' : L_w \to L_f$ by
\[ \phi'((x, u)) = (x, \psi'(u)). \]
It is clear that
\[ \alpha = \beta \cdot \phi'. \]
Now we set $\hat{L} = [L_w, L_w]$. Since $L$ is perfect, it follows that $\hat{L} + W = L_w$. This implies that $\hat{L}$ is perfect since
\[ \hat{L} = [\hat{L} + W, \hat{L} + W] = [\hat{L}, \hat{L}]. \]
Furthermore if we set
\[ C = W \cap \hat{L}, \]
then we have a central extension
\[ 0 \to C \to \hat{L} \to L \to 0 \]
such that $\hat{L}$ is perfect.

Now if we define $\phi$ as the restriction of $\phi'$ to the subalgebra $\hat{L}$, then we have
\[ \alpha|_{\hat{L}} = \beta \cdot \phi. \]
Therefore, $\hat{L} \to L$ is the universal central extension of $L$.

The following two propositions are clear (see [BM] and [P] for the Lie algebra case, see [LL] for the Leibniz algebra case).

**Proposition 4.6.** If $L$ is a perfect Leibniz superalgebra and $\hat{L}$ is a universal central extension of $L$, then every derivation of $L$ lifts to a derivation of $\hat{L}$. If $L$ is centerless, the lift is unique and $\text{Der}(\hat{L}) = \text{Der}(L)$.

**Proof.** We use the notation as the proof Proposition 4.5 and may assume that $\hat{L} = [L \oplus W, L \oplus W]$. Let $\mu \in \text{Der}(L)$. Then $\mu$ induces a linear mapping on $L \otimes L$ in the usual way:
\[ \mu(x \otimes y) = \mu(x) \otimes y + (-1)^{|x||\mu|} x \otimes \mu(y). \]
This mapping stabilizes $I$, so it induces a map, also denoted by $\mu$, on $W$. 


Thus \( x + u \mapsto \mu(x) + \mu(u) \) defines a derivation on \( L_w \) whose restriction to \( L \) is the required lifting.

Suppose that the kernel of the central extension is \( C \). Since \( L \) is centerless, the center of \( \hat{L} \) is \( C \), and every derivation of \( L \) induces a derivation of \( L \cong \hat{L}/C \). If \( \mu_1 \) and \( \mu_2 \) are derivations of \( \hat{L} \), both lifts of \( \mu \), then for all \( x \in \hat{L} \), \( \mu_2(x) = \mu_1(x) + a(x) \) where \( a(x) \in C \). Then calculation

\[
\mu_2([x, y]) = [\mu_2(x), y] + [x, \mu_2(y)] = [\mu_1(x), y] + [x, \mu_1(y)] = \mu_1([x, y])
\]

shows that \( \mu_2 = \mu_1 \). It follows that

\[
\text{Der}(\hat{L}) = \text{Der}L.
\]

**Proposition 4.7.** Let \( L \) be a perfect Leibniz algebra and \( \hat{L} \) be its universal central extension. Every automorphism \( \theta \) of \( L \) admits a unique extension \( \hat{\theta} \) to an automorphism \( \hat{L} \). Furthermore, the map \( \theta \rightarrow \hat{\theta} \) is a group monomorphism. If \( L \) is centerless, then \( \text{Aut} L \cong \text{Aut} \hat{L} \).

**Proof.** We use again the notation as in the proof of Proposition 4.5 and may assume that \( \hat{L} = [L \oplus W, L \oplus W] \). Every automorphism \( \theta \) induces an automorphism \( \theta_W \) of \( W \) via \( \theta_W(x \otimes y) = \theta(x) \otimes \theta(y) \).

It is clear that \( \theta \) extends to an automorphism \( \theta_{L_w} \) of \( L_w \) satisfying \( \theta_{L_w} : x + u \rightarrow \theta(x) + \theta_W(u) \).

By restriction, \( \theta_{L_w} \) induces an automorphism \( \hat{\theta} \) of \( \hat{L} \).

It is clear from the definition that \( \xi : \theta \mapsto \hat{\theta} \) is a group homomorphism. Suppose \( \hat{\theta} = 1 \). Then for all \( x, y \in L \subset L_w \) we have

\[
[x, y] + \psi(x, y) = [x, y]_{L_w} = \hat{\theta}([x, y]_{L_w}) = \theta_{L_w}([x, y]_{L_w}) = [\theta_{L_w}(x), \theta_{L_w}(y)]_{L_w} = [\theta(x), \theta(y)]_{L_w}
\]

\[
= [\theta(x), \theta(y)] + \psi(\theta(x), \theta(y)) = \theta([x, y]) + \psi(\theta(x), \theta(y)).
\]

Thus \( \theta \) is the identity on \([L, L] = L \). It follows that \( \xi \) is injective. Next we show that the lifting of \( \theta \) to \( \hat{L} \) is unique (and hence equal to \( \hat{\theta} \)). Let \( \theta_1 \) and \( \theta_2 \) be two lifts of \( \theta \) to \( \hat{L} \). Then for \( x \in \hat{L} \) we have \( \theta_1(x) = \theta_2(x) + x_C \) for some \( x_C \in C \), where \( C \) is the kernel of the universal central extension. Thus for all \( x, y \in \hat{L} \) we have

\[
\theta_1([x, y]_{\hat{L}}) = [\theta_2(x) + x_C, \theta_2(y) + y_C]_L = [\theta_2(x), \theta_2(y)]_L = \theta_2([x, y]_L).
\]

Thus \( \theta_1 = \theta_2 \) on \([\hat{L}, \hat{L}] = \hat{L} \).

For a subspace \( E \) of \( C \), we set

\[
\text{Aut}(L, E) := \{ \theta \in \text{Aut}(L) \mid \theta_W(E) = E \}.
\]

(4.5)

If \( \theta \in \text{Aut}(L, E) \), then \( \hat{\theta} \) induces an automorphism of \( \hat{L}/E \) that we will denote by \( \hat{\theta}_E \). With similar arguments we can show that \( \theta \rightarrow \hat{\theta}_E \) is a group monomorphism from \( \text{Aut}(L, E) \) into \( \text{Aut}(\hat{L}/E) \).

Now we show that this map is surjective if \( L \) is centreless.

Let \( \sigma \in \text{Aut}(\hat{L}/E) \), we know that \( \sigma \) admits a unique extension \( \hat{\sigma} \) to \( \hat{L} \) since \( \hat{L} \) is the UCE of \( \hat{L}/E \). If \( L \) is centreless, the center of \( \hat{L}/E \) is \( C/E \). In this case \( \sigma \in \text{Aut}(\hat{L}/E) \) induces an automorphisms of \( (\hat{L}/E)/(C/E) \cong L \) that will be denoted by \( \theta \). Then both \( \theta \) and \( \hat{\sigma} \) are extensions of \( \theta \) to \( \hat{L} \), and therefore equal. Now \( \hat{\sigma} \) stabilizes \( E \) (since \( E \) is the kernel of \( \hat{L} \rightarrow \hat{L}/E \)) and therefore so does \( \hat{\theta} \). But the restriction of \( \hat{\theta} \) to \( C \subset W \) coincides with \( \theta_W \). This show that \( \theta \) is in fact an element of \( \text{Aut}(L, E) \). By taking \( E = \{0\} \), we obtain

\[
\text{Aut}(\hat{L}) = \text{Aut}(L).
\]
5 UNIVERSAL CENTRAL EXTENSIONS OF THE BASIC CLASSICAL LIE SUPERALGEBRAS

Throughout this section \( \mathfrak{g} \) always denotes a basic classical simple Lie superalgebra and \( D \) denotes a unital commutative dialgebra over \( \mathbb{C} \) and \( A = \mathbb{C}[t^\pm_1, \ldots, t^\pm_\nu] \) \((\nu \geq 1)\). For convenience, we always assume that \( \mathfrak{g} \) is not of type \( A(n,n) \).

In this section we shall determine the universal central extension of the Lie superalgebra \( \mathfrak{g} \otimes D \).

Now we describe the main theorem of this section.

Theorem 5.1. The universal central extension of \( \mathfrak{g} \otimes D \) is

\[ \widehat{\mathfrak{g}} = \mathfrak{g} \otimes D \oplus \Omega_D \]

with the bracket

\[ [X \otimes a, Y \otimes b] = [X,Y] \otimes (a \vdash b) + (X,Y)b \dashv da, \quad X,Y \in \widehat{\mathfrak{g}}, a,b \in D, \]

where \((,\) is an even supersymmetric invariant bilinear form on \( \mathfrak{g} \) describe in section 3.

In what follows, we mainly prove Theorem 5.1.

First we introduce some notation. Let

\[ 0 \to Z \to u \xrightarrow{\pi} a \to 0 \]

be a central extension. Notice that, for \( x,y \in a \) and \( x',y' \in u \) such that \( \pi(x') = x, \pi(y') = y \), the commutator \([x',y']\) does not depend on the choice of the inverse images \( x' \) and \( y' \). Hence we denote \([x',y']\) by \([x,y]'\).

Proposition 5.2. Suppose \( \mathfrak{g} \) is a basic classical Lie superalgebra which is not of type \( A(n,n) \). Let \( Z \) be a free \( \mathbb{C} \)-module and

\[ 0 \to Z \to \mathfrak{g}'(D) \xrightarrow{\pi} \mathfrak{g} \otimes D \to 0 \]

a central extension of \( \mathfrak{g} \otimes D \). Then the bracket of \( \mathfrak{g}'(D) \) can be described as

\[ [(X \otimes a)', (Y \otimes b)'] = ([X,Y] \otimes (a \vdash b))' + (X,Y)\{a,b\}, \quad \{,\} : D \times D \to Z \]

where \((,\) is a non-degenerate even supersymmetric invariant bilinear form on \( \mathfrak{g} \), \( \{,\} : D \times D \to Z \) satisfies

\[ \{a \vdash b, c\} = \{a, b \vdash c\} + \{b, a \vdash c\}. \quad (5.2) \]

Proof. (1) We first consider the case of \( \mathfrak{sl}_2 \), which is regard as a trivial Lie superalgebra, so its proof is same as that in Leibniz central extension case given in [LL] and [Gao1]. It is also similar to Lemma 4.12 in [IK] (just delete \( \{a,b\} = -\{b,a\} \) and replace \( ab \) by \( a \vdash b \)).

(2) Now we consider the case of \( \mathfrak{osp}(1,2) \):

\[ \mathfrak{osp}(1,2) = \mathbb{C}X_+ \oplus \mathbb{C}x_+ \oplus \mathbb{C}H \oplus \mathbb{C}x_- \oplus \mathbb{C}X_- \]

with

\[ [H, X_\pm] = \pm 4X_\pm, \quad [X_+, X_-] = \frac{1}{2}H, \quad X_\pm = \pm \frac{1}{4}[x_\pm, x_\pm]. \]
$$[H, x_\pm] = \pm 2x_\pm, \quad [x_+, x_-] = H, \quad [X_\pm, x_\pm] = -x_\pm.$$ 

With this Chevalley basis ($|x_\pm| = \bar{1}$, $|X_\pm| = |H| = 0$), we can give a non-dengenerate even supersymmetric bilinear form on \textit{osp}(1, 2) as follows:

$$(H, H) = 2, \quad (x_+, x_-) = 1, \quad (X_+, X_-) = \frac{1}{4}.$$ 

We set

$$(X_\pm \otimes a)' : = \pm \frac{1}{4}[H \otimes 1, X_\pm \otimes a]$$

$$(x_\pm \otimes a)' : = \pm \frac{1}{2}[H \otimes 1, x_\pm \otimes a]$$

$$(H \otimes a)' : = [x_+ \otimes 1, x_- \otimes a]'$$

$$\{a, b\} : = \frac{1}{2}[(H \otimes a)', (H \otimes b')]$$.

We shall show these elements satisfy (5.2). It is suffice to show the following formulas:

F1) $[(H \otimes a)', (Y \otimes b)'] = \beta(H)(Y \otimes (a \mp b))'$ and $[(Y \otimes a)', (H \otimes b)'] = -\beta(H)(Y \otimes (a \mp b))'$,

where $Y \in g_{\beta}$.

F2) $[(x_\pm \otimes a)', (x_{\mp} \otimes b)'] = \pm (H \otimes (a \mp b))' + \{a, b\}$ and $[(H \otimes a)', (H \otimes b)'] = 2\{a, b\}$.

F3) $[(x_\pm \otimes a)', (x_\mp \otimes b)'] = \pm 4(X_\pm \otimes (a \mp b))'$.

F4) $[(X_\pm \otimes a)', (x_{\mp} \otimes b)'] = -(x_\pm \otimes (a \mp b))'$ and $[(x_\pm \otimes a)', (X_{\mp} \otimes b)'] = (x_\pm \otimes (a \mp b))'$.

F5) $[(X_\pm \otimes a)', (X_{\mp} \otimes b)'] = \pm \frac{1}{2}(H \otimes (a \mp b))' + \frac{1}{4}\{a, b\}$.

F6) $[(X_\pm \otimes a)', (x_\mp \otimes b)'] = 0 = [(x_\pm \otimes a)', (X_{\mp} \otimes b)'], [(X_\pm \otimes a)', (X_{\mp} \otimes b)'] = 0$.

We prove F1). By definition and Jacobi identity

$$[(H \otimes a)', (X_{\pm} \otimes b)']$$

$$= [(H \otimes a)', \pm \frac{1}{4}(H \otimes 1)', (X_{\pm} \otimes b)']$$

$$= \pm \frac{1}{4}([(H \otimes a)', (H \otimes 1)'], (X_{\pm} \otimes b)'] + [(H \otimes 1)', [(H \otimes a)', (X_{\pm} \otimes b)']]).$$

Since

$$[(H \otimes a)', (H \otimes 1)'] \equiv 0(\text{mod} Z)$$

$$[(H \otimes a)', (X_{\pm} \otimes b)'] = [H \otimes a, X_{\pm} \otimes b]' \equiv \pm 4(X_{\pm} \otimes (a \mp b))'(\text{mod} Z).$$

So for $Y = X_{\pm}$, F1) is true. Similar we can prove F1) for $Y = x_{\pm}$.

The formulas F2)–F6) are just follows from direct calculation similar that in Lemma 4.13 in [IK].

(3) Suppose $g$ is a basic classical Lie superalgebra which is not of type $A(n, n)$.

Let $\{X_\alpha, H_\alpha | \alpha \in \Delta\}$ be a Chevalley basis as in Section 3.

For $\alpha \in \Delta_0$ such that $\frac{1}{2} \alpha \notin \Delta$, we set

$$(X_\beta \otimes a)' := \frac{1}{\beta(H_\alpha)}[H_\alpha \otimes 1, X_\beta \otimes a]'$$
\((H_\beta \otimes a)\)' := \sigma(\beta)[X_\beta \otimes 1, X_{-\beta} \otimes a]',\)

for \(\beta = \pm \alpha\) and
\[
\tilde{g}(\alpha) := \{\mathbb{C}X_\alpha \oplus \mathbb{C}H_\alpha \oplus \mathbb{C}X_{-\alpha}\} \otimes D,
\]
\[
\tilde{g}'(\alpha) := \pi^{-1}(\tilde{g}(\alpha)).
\]

For \(\alpha \in \Delta_1\) such that \(2\alpha \in \Delta\), we set
\[
(X_\beta \otimes a)' := \frac{1}{\beta(H_\alpha)}[H_\alpha \otimes 1, X_\beta \otimes a]'
\]
\[
(H_\beta \otimes a)' := \sigma(\beta)[X_\beta \otimes 1, X_{-\beta} \otimes a]',
\]
for \(\beta = \pm \alpha, \pm 2\alpha\) and
\[
\tilde{g}(\alpha) := \{\mathbb{C}X_{2\alpha} \oplus \mathbb{C}X_\alpha \oplus \mathbb{C}H_\alpha \oplus \mathbb{C}X_{-2\alpha} \oplus \mathbb{C}X_{-\alpha}\} \otimes D,
\]
\[
\tilde{g}'(\alpha) := \pi^{-1}(\tilde{g}(\alpha)).
\]

Then from Case (2) and Case (3), we see that for \(\alpha \in \Delta\) such that \((\alpha, \alpha) \neq 0\), there exists \(\{,\} : D \times D \to \mathbb{Z}\) such that
\[
([X \otimes a]' , (Y \otimes b)'] = (\sigma(X,Y) \otimes (a \vdash b))' + (X,Y)\{a,b\}
\]
and
\[
\{a \vdash b, c\} = \{a, b \vdash c\} + \{b, a \vdash c\}.
\]

For \(\alpha \in \Delta_1\) such that \(2\alpha \notin \Delta\) i.e. \((\alpha, \alpha) = 0\), we fix \(\gamma \in \Delta\) as in Lemma 3.5, and set
\[
(X_\beta \otimes a)' := \frac{1}{\beta(H_\gamma)}[H_\gamma \otimes 1, X_\beta \otimes a]'
\]
\[
(H_\beta \otimes a)' := \sigma(\beta)[X_\beta \otimes 1, X_{-\beta} \otimes a]',
\]
for \(\beta = \pm \alpha\).

For above \(\gamma\), we also introduce the following notation:
\[
\tilde{\alpha} := \begin{cases} 
\alpha, & (\alpha, \alpha) \neq 0 \\
\gamma, & (\alpha, \alpha) = 0
\end{cases}.
\]

Now it is sufficient to prove the following formulas, whose proofs are similar to Lemma 4.16—4.18, 4.20—4.22 in [IK].

1) For \(\alpha, \beta \in \Delta\), we have
\[
[(H_\alpha \otimes a)' , (X_\beta \otimes b)'] = \beta(H_\alpha)(X_\beta \otimes (a \vdash b))' + (X_\alpha, X_{-\alpha})(a,b).
\]

2) For each \(\alpha \in \Delta\), we set \(\{a,b\} = [(H_\alpha \otimes a)' , (H_\alpha \otimes b)']\) and \(\{a,b\} = \frac{\sigma_{\alpha}}{\{X_\alpha, X_{-\alpha}\}}\{a,b\}_\alpha\), where \(\sigma_{\alpha}\) is defined as before. Then \(\{,\}\) is independent of the choice of \(\alpha\) and
\[
[(X_\alpha \otimes a)', (X_{-\alpha} \otimes b)'] = \sigma_{\alpha}(H_\alpha \otimes (a \vdash b))' + (X_\alpha, X_{-\alpha})(a,b),
\]
\[
[(H_\alpha \otimes a)', (H_\beta \otimes b)'] = (H_\alpha, H_\beta)(a,b).
\]
3) For $\alpha, \beta \in \Delta$ such that $\alpha + \beta \neq 0$,

$$[(x_\alpha \otimes a)', (x_\beta \otimes b)'] = N_{\alpha, \beta}(X_{\alpha + \beta} \otimes (a \dashv b))',$$

where $N_{\alpha, \beta}$ is defined as in section 3.

4) For $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$, we have

$$(H_{\alpha + \beta} \otimes a)' = \frac{\varepsilon_{\alpha + \beta}}{\varepsilon_\alpha}(H_\alpha \otimes a)' + \frac{\varepsilon_{\alpha + \beta}}{\varepsilon_\beta}(H_\beta \otimes a)'.$$

5) $\{,\} : D \times D \rightarrow \mathbb{Z}$ satisfies

$$\{a \dashv b, c\} = \{a, b \dashv c\} + \{b, a \dashv c\}.$$

Let us complete the proof of Theorem 5.1 by using Proposition 5.2. Let $\phi : D \times D \rightarrow \Omega_D^1$ defined by $(a, b) \rightarrow b \dashv da$. Since the Leibniz superalgebra $g \otimes D$ is perfect, it has a universal central extension by Proposition 4.1. By definition of $\Omega_D^1$ and Proposition 5.2, for any central extension

$$0 \rightarrow \mathbb{Z} \rightarrow g'(D) \rightarrow g \otimes D \rightarrow 0$$

there exists $\psi : \Omega_D^1 \rightarrow \mathbb{Z}$ such that

$$\{,\} = \psi \cdot \phi.$$

Hence the central extension

$$0 \rightarrow \Omega_D^1 \rightarrow \tilde{g} \rightarrow g \otimes D \rightarrow 0$$

is the universal central extension.

**Corollary 5.3.** Under the assumption of Theorem 5.1, we have

$$HL_2(g \otimes D) \cong \Omega_D^1.$$

Especially, if $D$ degenerate a unital commutative algebra $A = \mathbb{C}[t_1^\pm, \cdots, t_{\nu}^\pm] (\nu \geq 1)$, we have

**Corollary 5.4.** The UCE of $g \otimes A$ in the category of Leibniz superalgebras is

$$\tilde{g} = g \otimes A \oplus \Omega_A^1$$

with the bracket

$$[X \otimes a, Y \otimes b] = [X, Y] \otimes (ab) + (X, Y) bda, \quad X, Y \in \tilde{g}, \ a, b \in A,$$

where $(,)$ is an even supersymmetric invariant bilinear form on $g$ described in section 3.

With Theorem 4.7 in [IK], the methods of (4.6) in [LP] and Corollary 5.5 we also have

**Corollary 5.5.** Let $g(A)$ be the UCE of $g \otimes A$ in the category of Lie algebras (see section 1) and $\tilde{g}$ as in Corollary 5.4. Then the UCE of $g(A)$ in the category of Leibniz superalgebras is $\tilde{g}$ with kernel $\text{Im} B$, where $B$ is the modified Connes operator defined in [HL] (also see [Lo3]).
Corollary 5.6. The derivation algebra of the Leibniz superalgebra \( \tilde{g} \) is

\[
\text{Der}(\tilde{g}) = \text{Der}(A) \ltimes \text{Ion}(\tilde{g})
\]

and the automorphism group of \( \tilde{g} \) is

\[
\text{Aut}(\tilde{g}) = \text{Aut}(g \otimes A),
\]

where \( \text{Der}(A) = \sum_{i=1}^{\nu} A \frac{\partial}{\partial t_i} \) is the derivation algebra of \( A \) as an associative algebra and \( \text{Aut}(g \otimes A) \) is the automorphism group of Lie superalgebras \( g \otimes A \).

Remarks. 1. If we consider \( g \) is the type \( A(n, n) \), we also show that \( \tilde{g} := \tilde{g} \otimes D \oplus \Omega^1_D \), where \( \tilde{g} \) defined as in section 1, is also the universal central extension of \( g \), i.e.

Proposition 5.7. If \( g \) is a basic classical Lie superalgebra and \( D \) is a unital commutative dialgebra, then we have

\[
\text{HL}_2(g \otimes D) \cong \begin{cases} 
\Omega^1_D, & \text{if } g \text{ is not of type } A(n, n), \\
\Omega^1_D \oplus D^{\oplus 3}, & \text{if } g \text{ is of type } A(1,1), \\
\Omega^1_D \oplus D & \text{if } g \text{ is of type } A(n, n), n > 1.
\end{cases}
\]

(see [IK]).

2. For a general associative algebra \( A \), the universal central extension of the Lie superalgebra \( \mathfrak{sl}(m,n) \otimes A \) in the Leibniz superalgebras category is obtained in [HL].

Theorem 5.8. [HL] Let \( k \) be a field (char \( k \neq 2,3 \)) and \( A \) an associative and unital \( k \)-algebra. For \( m+n \geq 3, m,n \geq 0 \), the universal extension of \( \mathfrak{sl}(m,n,A) := \mathfrak{sl}(m,n) \otimes A \) in the category of Leibniz superalgebras is

\[
0 \to \text{HH}_1(A) \to \text{stl}(m,n,A) \to \mathfrak{sl}(m,n) \otimes A \to 0.
\]

In particular, if \( A \) is commutative, then

\[
\text{HL}^2(\mathfrak{sl}(m,n) \otimes A) \cong \Omega^1_{A|k},
\]

where \( \text{stl}(m,n,A) \) is the special Steinberg Leibniz superalgebra \( \text{stl}(m,n,D) \) (section 4) for \( D \) degenerated by an associative algebra \( A \).

3. More generally, for a unital dialgebra \( D \), the universal central extension of the Leibniz superalgebra \( \mathfrak{sl}(m,n,D) \) is obtained in [L].

Theorem 5.9. [L] Let \( k \) be a field (char \( k \neq 2,3 \)) and \( D \) a unital dialgebra. For \( m+n \geq 3, m,n \geq 0 \), the universal extension of \( \mathfrak{sl}(m,n,D) \) is

\[
0 \to \text{HHL}_1(D) \to \text{stl}(m,n,D) \to \mathfrak{sl}(m,n,D) \to 0.
\]

In particular, if \( D \) is commutative (i.e. \( a \vdash b = b \dashv a \) for all \( a, b \in D \)), then

\[
\text{HL}^2(\mathfrak{sl}(m,n,D)) \cong \Omega^1_D,
\]

where \( \text{HHL}_1(D) \) is first modified Frabetti’s Homology group for dialgebras defined in [L] (also see [F1] and [F2], \( \text{HHL}_1(D) \cong \Omega^1_D \) if \( D \) is commutative).
4. Let $D$ be a unital super dialgebra, then $\mathfrak{sl}(m, n, D)$ and $g \otimes D$ ($D$ commutative) are also Leibniz superalgebras according to (4.1) and (4.7) and (4.8). They play key roles in studying the Leibniz superalgebras graded by finite root systems as in the Lie algebra, Leibniz algebra and Lie superalgebra case (see [BM], [BE], [BZ] and [LH2]).

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