Metric operators for quasi-Hermitian Hamiltonians and symmetries of equivalent Hermitian Hamiltonians

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Abstract
We give a simple proof of the fact that every diagonalizable operator that has a real spectrum is quasi-Hermitian and show how the metric operators associated with a quasi-Hermitian Hamiltonian are related to the symmetry generators of an equivalent Hermitian Hamiltonian.

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1. Introduction

Given a separable Hilbert space \( \mathcal{H} \) and a linear operator \( H : \mathcal{H} \to \mathcal{H} \) that has a real spectrum and a complete set of eigenvectors, one can construct a new (physical) Hilbert space \( \mathcal{H}_{\text{phys}} \) in which \( H \) acts as a self-adjoint operator. This allows for the formulation of a consistent quantum theory where the observables and in particular Hamiltonian need not be self-adjoint with respect to the standard \( (L^2-) \) inner product on \( \mathcal{H} \) [1]. The physical Hilbert space \( \mathcal{H}_{\text{phys}} \) and the observables are determined in terms of a (bounded, everywhere-defined, invertible) positive-definite metric operator \( \eta_+ : \mathcal{H} \to \mathcal{H} \) that renders \( H \) pseudo-Hermitian [2], i.e., \( H \) satisfies

\[
H^\dagger = \eta_+ H \eta_+^{-1}.
\]

This marks the basic significance of the metric operator \( \eta_+ \). The positivity of \( \eta_+ \) implies that \( H \) belongs to a special class of pseudo-Hermitian operators called quasi-Hermitian operators [3].

The fact that for a given linear operator \( H \) with a real (discrete) spectrum and a complete set of eigenvectors, one can always find a (positive-definite) metric operator \( \eta_+ \) fulfilling (1) has been established in [4], and the role of antilinear symmetries such as \( \mathcal{P}\mathcal{T} \) has been elucidated

1 Here and throughout this article, we use \( A^\dagger \) to denote the adjoint of a linear operator \( A \) that is defined using the inner product \( \langle \cdot | \cdot \rangle \) of \( \mathcal{H} \) according to: \( \langle \psi | A \phi \rangle = \langle A \psi | \phi \rangle \) for all \( \psi, \phi \in \mathcal{H} \).
in [5]. Further investigation into the properties of $\eta_+^-$ has revealed its non-uniqueness [3, 7, 9] and the unitary equivalence of $H$ and the Hermitian Hamiltonian
\[ h := \rho H \rho^{-1}, \tag{2} \]
where $\rho := \sqrt{\eta_+^-.}$ The latter observation has been instrumental in providing an objective assessment of the ‘complex ($\mathcal{PT}$-symmetric) extension of quantum mechanics’ [11, 12]. It has also played a central role in clarifying the mysteries associated with the wrong-sign quartic potential [13]. In short, the pseudo-Hermitian quantum theory that is defined by the Hilbert space $\mathcal{H}_{\text{phys}}$ and the Hamiltonian $H$ admits an equivalent Hermitian description in terms of the (standard) Hilbert space $\mathcal{H}$ and the Hermitian Hamiltonian $h$. However, the specific form of $h$ depends on the choice of $\eta_+^-$. This has motivated the search for alternative methods of computing the most general metric operator for a given $H$, [14–19].

In this paper we first give a simple proof of the existence of metric operators $\eta_+^-$ and then relate $\eta_+$ to the symmetries of an equivalent Hermitian Hamiltonian.

2. The existence of metric operators

Let $H : \mathcal{H} \to \mathcal{H}$ be a (closed) operator with a real spectrum, and suppose that it is diagonalizable, i.e., there are operators $T, H_d : \mathcal{H} \to \mathcal{H}$ such that $T$ is invertible (bounded and hence closed),
\[ H = T^{-1} H_d T, \tag{3} \]
and $H_d$ is diagonal in some orthonormal basis of $\mathcal{H}$. The latter property implies that $H_d$ is a normal operator. Furthermore, because $H$ and $H_d$ are isospectral, the spectrum of $H_d$ is also real. This together with the fact that $H_d$ is normal imply that it is Hermitian (self-adjoint).

Next, recall that because $T$ is a closed, invertible operator it admits a polar decomposition [20]:
\[ T = U \rho, \tag{4} \]
where $U$ is a unitary operator and $\rho = |T| := \sqrt{T^\dagger T}$ is invertible and positive (definite). Inserting (4) into (3) and introducing
\[ h := U^\dagger H_d U, \tag{5} \]
we find
\[ H = \rho^{-1} h \rho. \tag{6} \]
Because $\rho$ is positive definite, so is $\eta_+^- := \rho^2$. Because $H_d$ is Hermitian and $U$ is unitary, $h$ is Hermitian. In view of this and the fact that $\rho$ is also Hermitian, (6) implies $H^\dagger = \eta_+^+ H \eta_+^-$. This proves the existence of a metric operator $\eta_+$ that makes $H$, $\eta_+$-pseudo-Hermitian.

The above proof is shorter than the one given in [4]. But it has the disadvantage that it does not offer a method of computing $\eta_+$.

3. Metric operators and symmetry generators

Let $\eta_+$ and $\eta_+$ be a pair of metric operators rendering $H$ pseudo-Hermitian, $\rho := \sqrt{\eta_+}$, and $\rho' := \sqrt{\eta_+^+}$. Then the Hermitian Hamiltonian operators
\[ h := \rho H \rho^{-1}, \quad h' := \rho' H \rho'^{-1}. \tag{7} \]
\[ ^2 \text{The alternative approach using the so-called } \mathcal{CPT}-\text{inner product [6] is equivalent to a specific choice of the metric operator [7, 8].} \]
\[ ^3 \text{Given a positive operator } X : \mathcal{H} \to \mathcal{H}, \sqrt{X} \text{ denotes the unique positive square root of } X. \]
are unitary equivalent to $H$, [10]. It is easy to see that $h$ and $h'$ are related by the similarity transformation

$$h' = AhA^{-1},$$

where

$$A := \rho' \rho^{-1}.$$  \hspace{1cm} (9)

Now, taking the adjoint of both sides of (8) and using the fact that $h$ and $h'$ are Hermitian, we find

$$[A^\dagger A, h] = 0.$$ \hspace{1cm} (10)

This means that $A^\dagger A$ is a (positive-definite) symmetry generator for the Hamiltonian $h$.

Furthermore, (9) and $\eta'_+ = \rho^2$ lead to the curious relation

$$\eta'_+ = \rho A^\dagger A \rho.$$ \hspace{1cm} (11)

Another immediate consequence of (9) is

$$A^\dagger = \rho^{-1} A \rho,$$ \hspace{1cm} (12)

i.e., $A$ is $\rho^{-1}$-pseudo-Hermitian [2].

It is easy to show that the converse relationship also holds, i.e., given an invertible linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ that satisfies (9) and (12), the operator $\eta'$ defined by

$$\eta'_+ := \rho A^\dagger A \rho.$$ \hspace{1cm} (13)

renders $H, \eta'$-pseudo-Hermitian.

The above analysis shows that given a metric operator $\eta_+ = \rho^2$ for the Hamiltonian $H$, we can express any other metric operator for $H$ in the form

$$\eta'_+ = \rho S \rho,$$ \hspace{1cm} (14)

where $S$ is a positive-definite symmetry generator of $h$ such that there is a $\rho^{-1}$-pseudo-Hermitian operator $A$ satisfying

$$S = A^\dagger A.$$ \hspace{1cm} (15)

In practice, the construction of the symmetry generators $S$ of the Hermitian operator $h$ is easier than that of the $\rho^{-1}$-pseudo-Hermitian operators $A$. This calls for a closer look at the structure of $A$.

In view of (15), we can express $A$ in the form

$$A = U \sigma,$$ \hspace{1cm} (16)

where $U : \mathcal{H} \rightarrow \mathcal{H}$ is a unitary operator and $\sigma := \sqrt{S}$. This reduces the characterization of $A$ to that of appropriate unitary operators $U$ that ensure $\rho^{-1}$-pseudo-Hermiticity of $A$.

Inserting (16) in (12) and introducing

$$B := \rho U,$$ \hspace{1cm} (17)

we find

$$B^\dagger = \sigma B \sigma^{-1}.$$ \hspace{1cm} (18)

That is, $B$ is $\sigma$-pseudo-Hermitian. Moreover, (17) implies

$$\eta_+ = BB^\dagger.$$ \hspace{1cm} (19)

Conversely, given a positive-definite symmetry generator $S$ and a $\sqrt{S}$-pseudo-Hermitian operator $B$ satisfying (19), we can easily show that the operator

$$U := \rho^{-1} B$$ \hspace{1cm} (20)
is unitary and $A$ given by (16) is $\rho^{-1}$-pseudo-Hermitian. As a result, the most general metric operator $\eta_+$ is given by (14), alternatively

$$\eta_+ = (\sqrt{S}\rho)^{\dagger}(\sqrt{S}\rho),$$

(21)

where $S$ is a positive-definite symmetry generator of $h$ such that there is a $\sqrt{S}$-pseudo-Hermitian operator $B$ satisfying $\eta_+ = BB^\dagger$.

4. Concluding remarks

The existence of a positive-definite metric operator $\eta_+$ that renders a diagonalizable Hamiltonian operator $H$ with a real spectrum $\eta_+$-pseudo-Hermitian can be directly established using the well-known polar decomposition of closed operators. Previously, we have pointed out that one can describe the most general $\eta_+$ in terms of a given metric operator and certain symmetry generators $A$ of $H$, [7]. Here we offer another description of the most general $\eta_+$ in terms of certain positive-definite symmetry generators $S$ of a given equivalent Hamiltonian $h$. Unlike the symmetry generators $A$ of $H$ that are non-Hermitian, the operators $S$ are standard Hermitian symmetry generators. This makes the results of this paper more appealing.

For the cases that $h$ is an element of a dynamical Lie algebra $\mathcal{G}$ with $\mathcal{H}$ furnishing a unitary irreducible representation of $\mathcal{G}$, one can identify the positive-definite symmetry generators $S$ with certain functions of a set of mutually commuting elements of $\mathcal{G}$ that includes $h$. For example, one can construct $S$ for the two-level system, where $\mathcal{G} = u(2)$, or the generalized (and simple) Harmonic oscillator where $\mathcal{G} = su(1, 1)$, [21]. These respectively correspond to the general two-level quasi-Hermitian Hamiltonians [18] and the class of quasi-Hermitian Hamiltonians that are linear combinations of $x^2$, $p^2$ and $\{x, p\}$ such as the one considered in [22]. For these systems one can also construct a metric operator $\eta_+$ and its positive square root $\rho$. Nevertheless, the implementation of formula (14) for obtaining the most general metric operator proves impractical. This is because it is not easy to characterize the general form of $\sqrt{S}$-pseudo-Hermitian operators $B$ that would fulfil $\eta_+ = BB^\dagger$.

Although formula (14) seems to be of limited practical value, it is conceptually appealing because it traces the non-uniqueness of the metric operator to the symmetries of the equivalent Hermitian Hamiltonians.

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