Natural configurations and normal frequencies of a vertically suspended, spinning, loaded cable with both extremities pinned

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Abstract
The resonant configurations and normal frequencies of a heavy elastic cable that is hanging in the field of gravity and is rotating uniformly about the vertical are examined from a theoretical perspective. The cable is assumed pinned at both extremities, with an extra load added to the lower one for stability. The equation of motion for this system is obtained and solved exactly. It is shown that the various resonant configurations of the cable are described by Bessel functions of order zero. The normal configurations and frequencies are obtained and the configuration of the cable in its lowest mode is presented as a photograph of demonstrations effected on three different samples. The results presented in this note would be useful for students in intermediate or advanced courses in classical mechanics, computation physics or waves and vibrations.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The dynamical behaviour of a massive, vertically hanging elastic cable with a free lower extremity has been studied extensively over the years. For example, the early texts of Routh [1] and Lamb [2] discuss the transverse modes of such a system with respect to oscillations that take place in a fixed vertical plane. More recently, Bailey [3] discussed the time required by a short transverse pulse to propagate up and down the cable. In other independent studies,
Satterly [4], Levinson [5], Morse [6], Coomer et al [7] and McCreech et al [8] also considered various aspects of the transverse modes, normal frequencies and swaying of a chain of discrete links or of a heavy cable. Young [9] considered the longitudinal standing waves on a vertical slinky with a free lower extremity. Other workers studied the effects of rotation around a vertical axis on the various stationary configurations that a heavy cable can take, again when the lower extremity is free. Western [10] considered a cable with free lower extremity; he determined its normal frequencies, compared the results to a few experimentally measured modes and also produced time-exposure photographs of the resonant configurations of the system. It has recently been shown, by Allen and Schmidt [11], for example, that the modes of vibration of a classical, non-relativistic string with a heavy mass at one end are of relevance in the study of quantum strings. Generally speaking, the problem of a heavy loaded cable is of pedagogical interest because it provides an opportunity for students to gain a deeper understanding of the methods of mathematical physics, by working in a concrete manner with Bessel functions.

The present note considers the motion of a heavy cable that has its upper extremity attached to a uniformly rotating vertical axis while its lower extremity is pinned yet allowed to rotate without friction about the same axis. The lower extremity of the specimen is further subjected to a non-vanishing external load, for stability. The equation of motion for this system is presented and solved, and the resonant configurations and normal modes are obtained and analysed. We have been unable to find reference to this particular problem, be it in textbooks or in the pedagogical literature, and we believe that it can provide an effective demonstration relating to the normal modes of a complex system that may exhibit some interesting features for an undergraduate physics student.

Section 2 solves the equation of motion for the normal configurations and modes of the system. Section 3 is devoted to a numerical evaluation of the lowest modes and to presenting photographs of demonstrations that we set up to illustrate the configuration of the lowest mode for students.

2. The equation of motion, the resonant configurations and normal frequencies

Consider a heavy, uniform elastic cable of total mass \( M \), relaxed length \( L \) and linear mass density \( \mu \equiv M/L \) whose position of static equilibrium in the gravitational field is the vertical \( z \)-axis. The upper extremity of the cable is located at \( z = L \) and forced by a motor to rotate around the vertical axis at a known uniform angular velocity \( \Omega \). The lower extremity of the cable, which is located at \( z = 0 \), is assumed attached to a frictionless bearing and put under an additional external tension, \( T_0 \). The cable moves away from the vertical axis, by a small time-independent horizontal amount, \( r = r(z) \), which is to be determined.

The general form of the differential equation describing the rotational motion of the cable is

\[
\frac{\partial}{\partial z} \left[ T(z) \frac{\partial}{\partial z} R(z, t) \right] = \frac{\partial^2}{\partial t^2} [\mu R(z, t)].
\]

(1)

In this expression, \( T(z) \) is the net vertical tension on the cable, including its own weight:

\[
T(z) = T_0 + \mu g z.
\]

(2)

In this expression, \( g \) is the acceleration due to gravity, \( \mu \) is the linear mass density and \( T_0 \) is the additional tension that is applied at the lower extremity of the cable. We are looking for stationary solutions of equation (1) so we impose that the motion be oscillatory in nature; thus,

\[
R(z, t) = r(z) e^{i \Omega t}.
\]

(3)
In this expression, the parameter $\Omega$ is to be determined and will give the natural angular frequencies of rotation of the system.

Insertion of equation (3) into equation (1) gives the stationary form of the equation that must be solved for the normal configurations and modes of the cable:

$$\frac{d}{dz} \left[ T(z) \frac{d}{dz} r(z) \right] + \Omega^2 \mu r(z) = 0. \quad (4)$$

Once a resonant configuration has been established, the boundary conditions are taken to be

$$r(z = 0) = r(z = L) = 0. \quad (5)$$

An equation similar to equation (4) was discussed by Western [10] for the motion of a rotating vertical chain with a free lower extremity, when $r(z = 0) \neq 0$. In the present situation, however, the tension conditions described by equation (2) as well as the boundary conditions on the lower extremity are different.

In order to solve equation (4) for the allowed angular frequencies, $\Omega$, and the corresponding configuration functions, $r(z)$, we first set

$$x = \frac{z}{L} + \frac{T_0}{Mg}. \quad (6)$$

Then we change variables from $z$ to $x$ in equation (4) and define $\sigma^2 \equiv \Omega^2 L/g$. Inserting this new variable in equation (2) then gives that $T = (\mu g L) x$. Also, we may replace $\frac{d}{dz}$ by $\frac{1}{L} \frac{d}{dx}$ and transform equation (4) to the following form:

$$\frac{d^2 r}{dx^2} + \frac{1}{x} \frac{dr}{dx} + \sigma^2 r = 0. \quad (7)$$

Finally, we introduce a second change of variable; thus,

$$w^2 = 4\sigma^2 x. \quad (8)$$

In terms of this latter variable, we have that $w = 2\sigma \sqrt{x}$, $\frac{d}{dx} = \frac{2\sigma^2}{w} \frac{d}{dw}$, and $\frac{d^2}{dx^2} = \frac{4\sigma^4}{w^5} \left( \frac{d^2}{dw^2} - \frac{1}{w} \frac{d}{dw} \frac{d}{dw} \right)$. The differential equation then takes the following familiar form:

$$\frac{d^2 r}{dw^2} + \frac{1}{w} \frac{dr}{dw} + r = 0. \quad (9)$$

This is a Bessel equation of the first kind and of order zero in the unknown configuration function, $r(z)$. The general solution of this equation is [12]

$$r(z) = C_1 J_0(w) + C_2 Y_0(w)$$

$$= C_1 J_0(2\sigma \sqrt{x}) + C_2 Y_0(2\sigma \sqrt{x})$$

$$= C_1 J_0 \left( \frac{2\Omega}{g} \sqrt{g L x} \right) + C_2 Y_0 \left( \frac{2\Omega}{g} \sqrt{g L x} \right)$$

$$= C_1 J_0 \left( \frac{2\Omega}{g} \sqrt{\frac{1}{\mu} (\mu g z + T_0)} \right) + C_2 Y_0 \left( \frac{2\Omega}{g} \sqrt{\frac{1}{\mu} (\mu g z + T_0)} \right). \quad (10)$$

In this expression, $J_0$ and $Y_0$ are zeroth-order Bessel functions of the first and second kinds, respectively. By performing dimensional analysis, one can readily verify that the argument of the Bessel functions is a dimensionless quantity, as it should. Also, $C_1$ and $C_2$ are constants that are determined next.
From the boundary conditions given in equation (5), we get the following system of two homogeneous equations:

\[ C_1 J_0 \left( \frac{2\Omega}{g} \sqrt{\frac{1}{\mu} T_0} \right) + C_2 Y_0 \left( \frac{2\Omega}{g} \sqrt{\frac{1}{\mu} T_0} \right) = 0; \quad (11) \]

\[ C_1 J_0 \left( \frac{2\Omega}{g} \sqrt{\frac{1}{\mu} (Mg + T_0)} \right) + C_2 Y_0 \left( \frac{2\Omega}{g} \sqrt{\frac{1}{\mu} (Mg + T_0)} \right) = 0. \quad (12) \]

This homogeneous system of equations has a nontrivial solution only if

\[ J_0 \left( \gamma /\Omega_1 \right) Y_0 \left( \gamma /\Omega_1 \sqrt{Mg T_0} + 1 \right) - Y_0 \left( \gamma /\Omega_1 \right) J_0 \left( \gamma /\Omega_1 \sqrt{Mg T_0} + 1 \right) = 0. \quad (13) \]

The following variable was introduced to simplify the appearance of the above constraints:

\[ \gamma = \frac{2}{g} \sqrt{\frac{T_0}{\mu}}. \quad (14) \]

The discrete normal frequencies of the motion, \( \Omega = \Omega_m \), with \( m = 0, 1, 2, \ldots \), are obtained by solving equation (13) numerically. The corresponding configurations, \( r_m(z) \), are then to be found by expressing \( C_2 \) in terms of \( C_1 \) with the help of either equation (11) or equation (12) and inserting the result in the last line of equation (10). After simplifying we get the following result:

\[ r_m(z) = C_m \left[ J_0 \left( \gamma \Omega_m \sqrt{\frac{\mu g z}{T_0} + 1} \right) - Y_0 \left( \gamma \Omega_m \right) Y_0 \left( \gamma \Omega_m \sqrt{\frac{\mu g z}{T_0} + 1} \right) \right]. \quad (15) \]

In this expression, \( C_m \) is an arbitrary constant that has units of length. The normal frequencies \( \Omega_m \) and their corresponding normalized configurations \( r_m(z)/C_m \) are evaluated numerically in the following section.

3. The eigenfrequencies and the associated configurations

In figure 1, we present a graph of the numerical solutions for the first five roots of equation (13). The fundamental mode has the lowest frequency and is shown as the lowest curve, labelled \( m = 0 \). The first overtone is the next lowest frequency and corresponds to the curve labelled \( m = 1 \). Finally, the second overtone is labelled \( m = 2 \). The vertical axis corresponds to \( \gamma \Omega \), where \( \Omega \) is the angular velocity of the cable and the horizontal axis corresponds to the ratio of the total weight of the cable and the total tension that was applied at the bottom of the cable.

The theoretical curves indicate that as the tension increases (moving from right to left) the resonant angular frequencies also increase; this is in line with what would be expected. On this figure, we have superimposed observed frequencies and tensions for three different specimens that were used in an experimental apparatus to compare the theoretical result with those found experimentally. The samples used were two chains and one cable, as follows: stainless steel chain (asterisks); brass chain (triangles); steel cable (open circles). Note that the observed experimental data fall very close to the theoretical curve for the mode corresponding to \( m = 0 \). The results for the second overtone exhibit somewhat larger deviations from the predicted behaviour. The mode \( m = 1 \) is absent because we could not excite it long enough to measure the tension at the base of the apparatus or determine the value of the rotational
Figure 1. The numerical solutions of equation (13) for the first five modes (solid lines). The observed frequencies are plotted for the stainless steel chain (asterisks), the brass chain for $m = 0$ (inverted triangles), the brass chain for $m = 2$ (triangles) and steel cable (open circles).

frequency before the mode decayed into the $m = 0$ configuration. We also noted that for the $m = 0$ mode, the chains give slightly smaller eigenfrequencies than do the cables. We attribute this effect to the physical differences between a chain and a cable. Chains are composed of links that must initially rotate through some angle before engaging the next link to move along, a process that is continued until all links are engaged in the motion. This causes a twist or corkscrew effect down the chain, thereby lowering the angular frequency of the assembly with respect to that of a cable. This particular aspect of the motion is not built into the mathematical model that we used.

In figure 2 we present the configuration of the chain for the three lowest eigenfrequencies that were found using equation (13) for three different ratios of the weight of the cable to the applied tension, $Mg/T_0$. The left panel shows the relative displacement for $m = 0$, $m = 1$ and $m = 2$ modes for $Mg/T_0 = 1/8$, the middle panel shows the relative displacement for the same $m$ values for $Mg/T_0 = 1$ while the right panel shows the displacement for $Mg/T_0 = 8$. We note from figure 2 that, as the tension becomes less important, i.e. as we move to larger values of $Mg/T_0$ (from left to right), the relative displacement of the cable with respect to its mid-point becomes increasingly asymmetric. Conversely, as the tension becomes more important than the weight of the chain, i.e. small $Mg/T_0$ ratio, the relative displacement of the chain from its equilibrium position becomes increasingly symmetric about its mid-point. We conclude from this that when $T_0$ is large compared to the weight of the chain, we can effectively neglect the weight of the chain when we want to evaluate the modes. On the other hand, when the applied tension is small compared to the weight of the chain or when we are dealing with a massive chain, we cannot neglect its mass and we should use equation (13) to find the permitted modes of rotation.

We also proceeded to compare the theoretical shape of the chain to the actual physical shape of the chain during our demonstration. Figure 3 shows an example of a frame taken from a movie clip that shows the physical shape of two different chains while they were rotating in
**Figure 2.** Plots of the relative displacement of the cable \((r(z)/C_m)\) from its equilibrium position for the three lowest eigenfrequencies that were found using equation (13) for three given ratios of weight to applied tension: the left panel corresponds to \(Mg/T_0 = 1/8\), the middle panel corresponds to \(Mg/T_0 = 1\) while the third panel corresponds to \(Mg/T_0 = 8\).

**Figure 3.** The shape of the rotating chain for \(Mg/T_0 = 0.03\). The theoretical result is shown in the left panel; the stainless steel and brass chains are shown in the middle and in the right panels, respectively.
their fundamental modes. The left-hand panel shows the numerical results that were obtained using equation (15). The middle and right-hand panels show frames for the stainless steel and the brass chains, respectively. One can see from these picture frames that the general features of the predicted shapes are in good agreement with the actual shapes of the rotating specimens.

Larger values of the parameter $\frac{Mg}{T_0}$ would have provided for more impressive demonstration photos than shown in figure 3, but we were limited by safety concerns. Indeed, the cables and chains that were used in our demonstrations ranged in mass values from 0.277 to 0.699 kg and it was already hazardous to rotate these at 300 rpm with our apparatus. Using heavier cables or chains would have required a complete re-engineering of our demonstration apparatus and we estimated that the gains would be greatly outweighed by the risks of harm to students so we did not pursue this possibility.

4. Concluding remarks

The motion of a vertical elastic cable that is pinned at both ends and made to rotate around the vertical in the gravitational field was considered. A simple model for the rotational dynamics of the system was used and the resulting equation of motion solved for the stationary configurations and normal frequencies. The predictions of the model were compared to data collected from three different specimens, two chains and one cable. The results for the lower modes agree reasonably well with the predictions of the model for a small $\frac{Mg}{T_0}$ ratio.

The study of this system may have pedagogical value by allowing students to develop a more concrete understanding and appreciation of Bessel functions and of the complex behaviour of real systems.

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