COMPLETE SUM OF PRODUCTS OF
(h, q)-EXTENSION OF EULER POLYNOMIALS AND NUMBERS

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Abstract

By using the fermionic $p$-adic $q$-Volkenborn integral, we construct generating functions of higher-order $(h, q)$-extension of Euler polynomials and numbers. By using these numbers and polynomials, we give new approach to the complete sums of products of $(h, q)$-extension of Euler polynomials and numbers one of which is given by the following form:

\[ E_{n,q}^{(h,v)}(y_1 + y_2 + \ldots + y_v) = \sum_{l_1, l_2, \ldots, l_v \geq 0} \binom{n}{l_1, l_2, \ldots, l_v} \prod_{j=1}^v E_{l_j,q}^{(h)}(y_j), \]

where \( \binom{n}{l_1, l_2, \ldots, l_v} \) are the multinomial coefficients and \( E_{m,q}^{(h)}(y) \) is the $(h, q)$-extension of Euler polynomials. Furthermore, we define some identities involving $(h, q)$-extension of Euler polynomials and numbers.

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1. Introduction, Definitions and Notations

The main aim of this paper is to study higher-order $(h, q)$-extension of Euler numbers and polynomials. Bernoulli and Euler numbers and polynomials were studied by many authors (see for detail [18], [19], [9], [7], [10], [11], [12], [13], [16], [8], [20], [21], [22], [24], [14]) . We introduce some of them here. In [7], [8], [10], Kim constructed $p$-adic $q$-Volkenborn integral identities. By using these identities, he proved $p$-adic $q$-integral representation of $q$-Euler and Bernoulli numbers and polynomials. In [20], [21], we constructed generating functions of $q$-generalized Euler numbers and polynomials and twisted $q$-generalized Euler numbers and polynomials. We also constructed a complex analytic twisted $l$-series which is interpolated twisted $q$-Euler numbers at non-positive integers. In [9], by using $q$-Volkenborn integration,
Kim constructed the new \((h, q)\)-extension of the Bernoulli numbers and polynomials. He defined \((h, q)\)-extension of the zeta functions which are interpolated new \((h, q)\)-extension of the Bernoulli numbers and polynomials. In [22], we defined twisted \((h, q)\)-Bernoulli numbers, zeta functions and \(L\)-function. He also gave relations between these functions and numbers.

By the same motivation of the above studies, in this paper, we construct new approach to the complete sums of products of \((h, q)\)-extension of Euler polynomials and numbers. By the Multinomial Theorem and multinomial relations, we find new identities related to these polynomials and numbers.

Throughout this paper \(\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p\) will be denoted by the ring of rational integers, the ring of \(p\)-adic integers, the field of \(p\)-adic rational numbers and the completion of \(\mathbb{Q}_p\), respectively. Let \(v_p\) be the normalized exponential valuation of \(\mathbb{C}_p\) with \(|p|_p = p^{-v_p(p)} = p^{-1}\). When one talks of \(q\)-extension, \(q\) is variously considered as an indeterminate, a complex number \(q \in \mathbb{C}\), or \(p\)-adic number \(q \in \mathbb{C}_p\). If \(q \in \mathbb{C}_p\), then we normally assume \(|q - 1|_p < p^{-p-1}\), so that

\[q^x = \exp(x \log q) \text{ for } |x|_p \leq 1.\]

If \(q \in \mathbb{C}\), then we normally assume \(|q| < 1\). cf. ([7], [4], [10], [11], [12], [8], [22]). We use the notations as

\[\lfloor x \rfloor_q = \frac{1 - q^x}{1 - q}, \quad \lfloor x \rfloor_{-q} = \frac{1 - (-q)^x}{1 + q}.\]

Let \(UD(\mathbb{Z}_p)\) be the set of uniformly differentiable function on \(\mathbb{Z}_p\). For \(f \in UD(\mathbb{Z}_p)\), Kim [7] defined the \(p\)-adic invariant \(q\)-integral on \(\mathbb{Z}_p\) as follows:

\[I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x,\]

where \(N\) is a fixed natural number. The bosonic integral was considered from a physical point of view to the bosonic limit \(q \to 1\), as follows:

\[I_1(f) = \lim_{q \to 1} I_q(f) , \text{ cf. (7), (10), (11), (17) ).}\]

We consider the fermionic integral in contrast to the conventional bosonic, which is called the \(q\)-deformed fermionic integral on \(\mathbb{Z}_p\). That is

\[I_{-q}(f) = \lim_{q \to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) , \text{ cf. (10), (11), (8), (17) }.\]

Recently, twisted \((h, q)\)-Bernoulli and Euler numbers and polynomials were studied by several authors cf.(see [15], [4], [20], [21], [22], [14], [13], [16]).

By definition of \(\mu_{-q}(x)\), we see that

\[I_{-1}(f_1) + I_{-1}(f) = 2f(0) , \text{ cf. (10), (1.1) }\]

where \(f_1(x) = f(x + 1)\).
In [18], Ozden and Simsek defined new \((h, q)\)-extension of Euler numbers and polynomials. By using derivative operator to these functions, we derive \((h, q)\)-extension of zeta functions and \(l\)-functions, which interpolate \((h, q)\)-extension of Euler numbers at negative integers.

\((h, q)\)-extension of Euler polynomials, \(E_{n,q}^{(h)}(x)\) are defined by

\[
F_h^q(t, x) = F_h^q(t)e^{tx} = \frac{2e^{tx}}{q^h e^t + 1} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!} \text{ cf. [18]},
\]

(1.2)

For \(x = 0\), we have

\[
F_h^q(t) = \frac{2e^{tx}}{q^h e^t + 1} = \sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!} \text{ cf. [18]},
\]

(1.3)

**Theorem 1.** ([18]) **(Witt formula)** For \(h \in \mathbb{Z}, q \in \mathbb{C}_p\) with \(|1 - q|_p < p^{-\frac{1}{p-1}}\).

\[
\int_{\mathbb{Z}_p} q^{hx} x^n d\mu_{-1}(x) = E_{n,q}^{(h)},
\]

(1.4)

\[
\int_{\mathbb{Z}_p} q^{hy}(x + y)^n d\mu_{-1}(y) = E_{n,q}^{(h)}(x).
\]

(1.5)

**Theorem 2.** ([18]) **(Distribution Relation)** For \(d\) is an odd positive integer, \(k \in \mathbb{N}\), we have

\[
E_{k,q}^{(h)}(x, q) = d^k \sum_{a=0}^{d-1} (-1)^a q^{ha} E_{k,q}^{(h)} \left( \frac{x + a}{d} \right).
\]

2. **Higher-order \((h, q)\)-Euler Polynomials and Numbers**

Our main purpose in this section is to give complete sums of products of \((h, q)\)-Euler polynomials and numbers. We define \((h, q)\)-Euler polynomials and numbers of higher-order, \(E_{n,q}^{(h,k)}(z)\) and \(E_{n,q}^{(h,v)} \in \mathbb{C}_p\), respectively, by making use multiple of \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) in the fermionic sense:

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{v} hx_j} \exp \left( \sum_{j=1}^{v} x_j \right) \prod_{j=1}^{v} d\mu_{-1}(x_j)
\]

\[
= \sum_{n=0}^{\infty} E_{n,q}^{(h,v)} \frac{t^n}{n!},
\]

(2.1)

where

\[
\prod_{j=1}^{v} d\mu_{-1}(x_j) = d\mu_{-1}(x_1)d\mu_{-1}(x_2)\ldots d\mu_{-1}(x_v).
\]
By using Taylor series of \( \exp(tx) \) in the above equation, we have
\[
\sum_{n=0}^{\infty} \left( \int_{Z_p \times \cdots \times Z_p} q^{v \sum_{j=1}^{v} hx_j} \left( \sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_1(x_j) \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \frac{E_{n,q}^{(h,v)} t^n}{n!}.
\]
By comparing coefficients \( \frac{t^n}{n!} \) in the above equation, we arrive at the following theorem.

**Theorem 3.** For positive integers \( n, v, \) and \( h \in \mathbb{Z} \), we have
\[
E_{n,q}^{(h,v)} = \int_{Z_p \times \cdots \times Z_p} q^{v \sum_{j=1}^{v} hx_j} \left( \sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_1(x_j).
\]

By (2.1), \((h, q)\)-Euler numbers of higher-order, \( E_{n,q}^{(h,v)} \) are defined by means of the following generating function
\[
\left( \frac{2}{q^h e^t + 1} \right)^v = \sum_{n=0}^{\infty} \frac{E_{n,q}^{(h,v)} t^n}{n!}.
\]

Observe that for \( v = 1 \), the above equation reduces to (1.3).

By using Taylor series of \( \exp(tx) \) in the above equation, we have
\[
\sum_{n=0}^{\infty} \left( \int_{Z_p \times \cdots \times Z_p} q^{v \sum_{j=1}^{v} hx_j} \left( z + \sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_1(x_j) \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \frac{E_{n,q}^{(h,v)}(z) t^n}{n!}.
\]
By comparing coefficients \( \frac{t^n}{n!} \) in the above equation, we arrive at the following theorem.

**Theorem 4.** For \( z \in \mathbb{C}_p \) and positive integers \( n, v, \) and \( h \in \mathbb{Z} \), we have
\[
E_{n,q}^{(h,v)}(z) = \int_{Z_p \times \cdots \times Z_p} q^{v \sum_{j=1}^{v} hx_j} \left( z + \sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_1(x_j)
\]
\[
(2.3)
\]
By (2.1), \((h, q)\)-Euler polynomials of higher-order, \(E_{n,q}^{(h,v)}(z)\) are defined by means of the following generating function

\[
F_{h,v,q,w}^q(z, t) = e^{tz} \left( \frac{2}{q^h e^t + 1} \right)^v = \sum_{n=0}^{\infty} E_{n,q}^{(h,v)}(z) \frac{t^n}{n!} .
\]

Note that when \(v = 1\), then we have (1.2), when \(h = 1, q \to 1\), then we have

\[
F_{1,v}^1(z, t) = e^{tz} \left( \frac{2}{e^t + 1} \right)^v = \sum_{n=0}^{\infty} E_n^{(v)}(z) \frac{t^n}{n!} ,
\]

where \(E_n^{(v)}(z)\) are denoted classical higher-order Euler polynomials cf. [24].

**Theorem 5.** For \(z \in \mathbb{C}_p\) and positive integers \(n, v, h \in \mathbb{Z}\), we have

\[
E_{n,q}^{(h,v)}(z) = \sum_{l=0}^{n} \binom{n}{l} z^{n-l} E_{l,q}^{(h,v)} . \tag{2.4}
\]

**Proof.** By using binomial expansion in (2.3), we have

\[
E_{n,q}^{(h,v)}(z) = \sum_{l=0}^{n} \binom{n}{l} z^{n-l} \int_{z_p} \ldots \int_{z_p} q^{\sum_{j=1}^{v} l x_j} \left( \sum_{j=1}^{v} x_j \right)^l \prod_{j=1}^{v} d\mu_{-1}(x_j).
\]

By (2.2) in the above, we arrive at the desired result. \(\square\)

3. **The complete sums of products of \((h, q)\)-extension of Euler polynomials and numbers**

In this section, we prove main theorems on the complete sums of products of \((h, q)\)-extension of Euler polynomials and numbers. Firstly, we need the Multinomial Theorem, which is given as follows cf. ([1, 2]):

**Theorem 6.** (Multinomial Theorem)

\[
\left( \sum_{j=1}^{v} x_j \right)^n = \sum_{l_1 + l_2 + \ldots + l_v = n} \binom{n}{l_1, l_2, \ldots, l_v} \prod_{a=1}^{v} l_a ! x_a^{l_a} ,
\]

where \(\binom{n}{l_1, l_2, \ldots, l_v}\) are the multinomial coefficients, which are defined by

\[
\binom{n}{l_1, l_2, \ldots, l_v} = \frac{n!}{l_1 ! l_2 ! \ldots l_v !} .
\]

**Theorem 7.** For positive integers \(n, v\), we have

\[
E_{n,q}^{(h,v)} = \sum_{l_1 + l_2 + \ldots + l_v = n} \binom{n}{l_1, l_2, \ldots, l_v} \prod_{j=1}^{v} E_{l_j,q}^{(h)} . \tag{3.1}
\]
where \( \binom{n}{l_1, l_2, \ldots, l_v} \) is the multinomial coefficient.

Proof. By using Theorem 6 in (2.2), we have

\[
E_{n,q}^{(h,v)} = \sum_{l_1, l_2, \ldots, l_v \geq 0} \binom{n}{l_1, l_2, \ldots, l_v} \prod_{j=1}^{v} \int_{Z_p} q^{hx_j} x_j^{l_j} d\mu_{-1}(x_j).
\]

By (1.4) in the above, we obtain the desired result. \( \square \)

Remark 1.

\[
\lim_{q \to 1} \lim_{h \to 1} E_{n,q}^{(h,v)} = E_n^{(v)} = \sum_{l_1, l_2, \ldots, l_v \geq 0} \binom{n}{l_1, l_2, \ldots, l_v} E_{l_1} E_{l_2} \ldots E_{l_v},
\]

where \( \binom{n}{l_1, l_2, \ldots, l_v} \) is the multinomial coefficient, and \( E_{l_j}, 1 \leq j \leq v \), are denoted classical Euler numbers.

By substituting (3.1) into (2.4), after some elementary calculations, we arrive at the following corollary:

Corollary 1. For \( z \in \mathbb{C}_p \) and positive integers \( n, v \), we have

\[
E_{n,q}^{(h,v)}(z) = \sum_{m=0}^{n} \sum_{l_1, l_2, \ldots, l_v \geq 0} \binom{n}{m} \binom{m}{l_1, l_2, \ldots, l_v} z^{n-m} \prod_{j=1}^{v} E_{l_j,q}^{(h)}(y_j),
\]

where \( \binom{n}{l_1, l_2, \ldots, l_v} \) and \( \binom{n}{m} \) are the multinomial coefficient and the binomial coefficient, respectively.

One of the main theorem of this section is to give complete sum of products of \((h,q)\)-Euler polynomials.

Theorem 8. For \( y_1, y_2, \ldots, y_v \in \mathbb{C}_p \) and positive integers \( n, v \), we have

\[
E_{n,q}^{(h,v)}(y_1 + y_2 + \ldots + y_v) = \sum_{l_1, l_2, \ldots, l_v \geq 0} \binom{n}{l_1, l_2, \ldots, l_v} \prod_{j=1}^{v} E_{l_j,q}^{(h)}(y_j), \quad (3.2)
\]

where \( \binom{n}{l_1, l_2, \ldots, l_v} \) is the multinomial coefficient.
Proof. By substituting $z = y_1 + y_2 + ... + y_v$ into (2.3), we have

$$E_{n,q}^{(h,v)}(y_1 + y_2 + ... + y_v) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{v} h x_j} \left( \sum_{j=1}^{v} (y_j + x_j) \right)^n \prod_{j=1}^{v} d\mu_1(x_j).$$

By using Theorem 6 in the above, and after some elementary calculations, we get

$$E_{n,q}^{(h,v)}(y_1 + y_2 + ... + y_v) = \sum_{l_1, l_2, ..., l_v \geq 0 \atop l_1 + l_2 + ... + l_v = n} \left( \begin{array}{c} n \\ l_1, l_2, ..., l_v \end{array} \right) \prod_{j=1}^{v} q^{h x_j} (y_j + x_j)^{l_j} d\mu_1(x_j).$$

By substituting (1.5) into the above, we arrive at the desired result. \qed

Remark 2. If we take $y_1 = y_2 = ... = y_v = 0$ in Theorem 8, then Theorem 8 reduces to Theorem 7. Substituting $h = 1$ and $q \to 1$ into (3.2), we obtain the following relation:

$$E_{n,q}^{(v)}(y_1 + y_2 + ... + y_v) = \sum_{l_1, l_2, ..., l_v \geq 0 \atop l_1 + l_2 + ... + l_v = m} \left( \begin{array}{c} m \\ l_1, l_2, ..., l_v \end{array} \right) \prod_{j=1}^{v} E_{l_j}(y_j).$$

I-C. Huang and S-Y. Huang [3] found complete sums of products of Bernoulli polynomials. Kim [6] defined Carlitz’s $q$-Bernoulli number of higher order using an integral by the $q$-analogue of the ordinary $p$-adic invariant measure. He gave different proof of complete sums of products of higher order $q$-Bernoulli polynomials. In [5], Jang et al gave complete sums of products of Bernoulli polynomials and Frobenious Euler polynomials. In [23], Simsek et al gave complete sums of products of $(h,q)$-Bernoulli polynomials and numbers.

By applying the following multinomial relations cf. ([11] pp. 25, 56), ([2] pp. 168)

$$(x + y + z)^n = \sum_{0 \leq l_1, l_2, l_3 \leq n \atop l_1 + l_2 + l_3 = n} \frac{(l_1 + l_2 + l_3)!}{l_1! l_2! l_3!} x^{l_1} y^{l_2} z^{l_3},$$

and

$$\left( \begin{array}{c} l_1 + l_2 + ... + l_v \\ l_1, l_2, ..., l_v \end{array} \right) = \frac{(l_1 + l_2 + ... + l_v)!}{l_1! l_2! \cdots l_v!} = \left( \begin{array}{c} l_1 + l_2 + l_3 + ... + l_v \\ l_2 + l_3 + ... + l_v \end{array} \right) \cdots \left( \begin{array}{c} l_{v-1} + l_v \\ l_v \end{array} \right),$$

and the above method, one can also evaluate the other sums related to Bernoulli Polynomials or Euler Polynomials (see [3], [6], [23]).
Theorem 9. Let $n \in \mathbb{N}$. Then we have
\[ E^{(h,v)}_{n,q}(z+y) = \sum_{l=0}^{n} \binom{n}{l} E^{(h,v)}_{l,q}(y)z^{n-l}. \]

Proof.
\begin{align*}
E^{(h,v)}_{n,q}(z+y) &= (E^{(h,v)}_{q} + z + y)^n \\
&= \sum_{l=0}^{n} \binom{n}{l} E^{(h,v)}_{l,q}(y + z)^{n-l} \\
&= \sum_{l=0}^{n} \binom{n}{l} E^{(h,v)}_{l,q} \sum_{m=0}^{n-l} \binom{n-l}{m} y^m z^{n-l-m} \\
&= \sum_{0 \leq l \leq m \leq n} \binom{n}{l} \binom{n-l}{m-l} E^{(h,v)}_{l,q} y^{m-l} z^{n-m} \\
&= \sum_{0 \leq l \leq m \leq n} \binom{n}{m} \binom{m}{l} E^{(h,v)}_{l,q} y^{m-l} z^{n-m} \\
&= \sum_{m=0}^{n} \binom{n}{m} \sum_{l=0}^{m} \binom{m}{l} E^{(h,v)}_{l,q} y^{m-l} z^{n-m},
\end{align*}

with usual convention of symbolically replacing $E^{(h,v)}_{q}$ by $E^{(h,v)}_{l,q}$. By using (2.4) in the above, we have
\[ E^{(h,v)}_{n,q}(z+y) = \sum_{m=0}^{n} \binom{n}{m} E^{(h,v)}_{m,q}(y)z^{n-m}. \]

Thus the proof is completed. \qed

From Theorem 8 and Theorem 9 after some elementary calculations, we arrive at the following interesting result:

Corollary 2. Let $n \in \mathbb{N}$. Then we have
\[ \sum_{m=0}^{n} \binom{n}{m} E^{(h,v)}_{m,q}(y_1) y_2^{n-m} = \sum_{l_1, l_2 \geq 0} \binom{n}{l_1, l_2} E^{(h)}_{l_1,q}(y_1) B^{(h)}_{l_2,q}(y_2). \]

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