On no-go results for the nonlinear Klein–Gordon–Maxwell equations

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Abstract
In this paper we propose a new proof of some non-existence results for the nonlinear Klein–Gordon–Maxwell system of equations. The proof is based on the scaling arguments, i.e. special variations of the fields, only. We also apply the obtained results to the case of the simplest Q-balls and present some restrictions on their existence.

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1. Introduction

The nonlinear Klein–Gordon–Maxwell system of equations has been extensively studied during the past few years. A lot of existence and non-existence results were obtained, see, for example, [1–6] and references therein. A more general case of the Yang–Mills field coupled to a scalar field was considered in [7] for a non-negative scalar filed potential (see also [8]).

In this paper, we will present a method for obtaining non-existence results for the Klein–Gordon–Maxwell system of equations. This method is based on the use of the scaling arguments proposed in [9] and known as the Derrick theorem. Some of the restrictions which can be obtained by our method coincide with those obtained in [10], and some of them are in agreement with the restrictions presented in [7, 8] for a more general Yang–Mills–Klein–Gordon system. The difference between our proof and the proofs of [7, 10] is that the proofs of [7, 10] are based on some explicit properties of possible solutions to the system of equations of motion, as well as on the use of these equations itself, whereas our proof is based on the arguments of [9], which were applied to the system of electromagnetic and scalar fields. It should also be mentioned that analogous scaling arguments were discussed in [10] as an alternative method which can be used at some step of that proof. Our method is based on these scaling arguments, although we had to use (only one) equation of motion in a special limiting case.

We will also apply the obtained results to the case of the simplest Q-balls (a solitons in a system of a single complex scalar field) and find that these results do not contradict the known existence [11] and non-existence [12] conditions and the existence of the solitons [13, 14].
2. The setup

Let us consider the following form of the four-dimensional action:

\[ S = \int d^4x \left[ \eta^{\mu\nu} (D_\mu \varphi)^* D_\nu \varphi - m^2 \varphi^* \varphi - V(\varphi^* \varphi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right], \]  

(1)

where \( \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the flat Minkowski metric, \( D_\mu \varphi = \partial_\mu \varphi - ieA_\mu \varphi \). Of course, the term \( m^2 \varphi^* \varphi \) can be incorporated into \( V(\varphi^* \varphi) \), but we will retain it in order to correspond to the notations used in [10]. Note that in [7, 8] the mass term is incorporated into \( V(\varphi^* \varphi) \).

In what follows we will focus on the standing wave solutions:

\[ \varphi(t, \vec{x}) = e^{i\omega t} \phi(\vec{x}), \]  

(2)

with the real function \( \phi(\vec{x}) \). We also suppose that

\[ A_i \equiv 0, \quad A_0(t, \vec{x}) = A_0(\vec{x}) \]  

(3)

for \( i = 1, 2, 3 \). In this case we can use the following three-dimensional effective action with \( V(\varphi^* \varphi) = V(\phi) \) instead of (1):

\[ S = \int d^3x \left[ -\partial_\nu \phi \partial_\nu \phi + \frac{1}{2} \partial_\nu \partial_\nu A_0 + (\omega^2 - m^2) \phi^2 - 2e\omega A_0 \phi^2 + e^2 A_0^2 \phi^2 - V(\phi) \right], \]  

(4)

where \( \partial_\nu \phi \partial_\nu \phi = \sum_{k=1}^3 \partial_k \phi \partial_k \phi \). We will be looking for non-topological solitons such that

\[ \int d^3x \phi^2 < \infty, \quad \int d^3x \partial_\nu \phi \partial_\nu \phi < \infty, \]

\[ \int d^3x V(\phi) < \infty, \quad \int d^3x \frac{dV(\phi)}{d\phi} \phi < \infty, \]

\[ \int d^3x \partial_\nu A_0 \partial_\nu A_0 < \infty, \quad \lim_{x^i \to \pm\infty} A_0 = 0, \quad \lim_{x^i \to \pm\infty} \phi = 0. \]  

(5)

It is straightforward to get from (4) the corresponding equations of motion for the fields \( \phi, A_0 \); they can be found, for example, in [10]. We do not present these equations here to stress that we will not use them in their explicit form to obtain the no-go results.

Let \( \phi(\vec{x}), A_0(\vec{x}) \) be a localized solution to the equations of motion following from action (4). It means that

\[ \delta S(\phi, A_0) = 0. \]  

(6)

Let us denote

\[ \int d^3x \phi^2 = V_1 \geq 0, \quad \int d^3x \partial_\nu \phi \partial_\nu \phi = \Pi_1 \geq 0, \]

\[ \frac{1}{2} \int d^3x \partial_\nu A_0 \partial_\nu A_0 = \Pi_2 \geq 0, \quad e^2 \int d^3x A_0^2 \phi^2 = I_2 \geq 0, \]

\[ 2e\omega \int d^3x A_0 \phi^2 = I_1, \quad \int d^3x V(\phi) = V_2. \]  

(7)

Now we can proceed to specific examples.
3. No-go results

3.1. The Klein–Gordon–Maxwell system

First, let us consider the potential $V(\phi)$ to have the form

$$V(\phi) = \gamma(\phi^2)^{p/2} = \gamma|\phi|^p,$$

with $p > 1$, which is often used for examining the Klein–Gordon–Maxwell system of equations. In this case

$$V_2 = \int d^3x \, V(\phi) = \gamma \int d^3x \, |\phi|^p = \gamma \bar{V}_2,$$

where $\bar{V}_2 \geq 0$. The potential of the form (8) allows one to see how the generalized scale change method works in a simple case.

Let us consider the following modification of the solution $\phi, A_0$:

$$\phi(\vec{x}) \rightarrow \lambda^\alpha \phi(\lambda \vec{x}),$$

$$A_0(\vec{x}) \rightarrow \lambda^\beta A_0(\lambda \vec{x}).$$

Since $\lambda = 1$ corresponds to the solution to (6), the following identity holds:

$$\frac{dS(\lambda)}{d\lambda} \bigg|_{\lambda=1} = 0.$$  

Substituting (10) and (11) into (4) and using (7), (12) we get

$$-(2\alpha - 1) \Pi_1 + (2\beta - 1) \Pi_2 + (2\alpha - 3)(\omega^2 - m^2)V_1$$

$$- (2\alpha + \beta - 3)I_1 + (2\alpha + 2\beta - 3)I_2 - (p\alpha - 3)\gamma \bar{V}_2 = 0.$$  

(13)

The term $I_1$ is not of a definite sign, so below we will consider the case $\beta = 3 - 2\alpha$. Thus we obtain from (13)

$$-(2\alpha - 1) \Pi_1 + (5 - 4\alpha) \Pi_2 + (2\alpha - 3)(\omega^2 - m^2)V_1 + (3 - 2\alpha)I_2 - (p\alpha - 3)\gamma \bar{V}_2 = 0.$$  

(14)

Now we are ready to discuss restrictions on the existence of solitons coming from (14) for different values of $\alpha$, $\gamma$ and $p$.

**Proposition 1.** For the potential of the form (8) non-topological solitons of the form (2), (3), (5) are absent if

(i) $\gamma = 0$.

(ii) $\gamma > 0$

- $p \geq 2$,
- $1 < p < 2$ and $m^2 \geq \omega^2$.

(iii) $\gamma < 0$

- $1 < p \leq 2$,
- $p \geq 6$ and $m^2 \geq \omega^2 > 0$.

**Remark 1.** The absence of solitons in the case $\gamma = 0$ was shown in [10] (the absence of spherically symmetric solitons in this case was shown in [15]); restrictions for the case $\gamma > 0$ are in agreement with those presented in [7] for a more general case of the Yang–Mills–Klein–Gordon system (they are $p \geq 4$ for $m = 0$ and $p > 2$ for $m \neq 0$ in our notations); restrictions for the case $\gamma < 0$ coincide with those obtained in paper [10].
Proof.

(i) $\gamma = 0$.
In this case it is convenient to take $\alpha = \frac{3}{2}$. Equation (14) takes the form

$$-2\Pi_1 - \Pi_2 = 0.$$ 

Since by definition $\Pi_1 \geq 0$, $\Pi_2 \geq 0$, the latter identity implies $\Pi_1 = \Pi_2 = 0$ and thus $\phi = A_0 \equiv 0$. There are no solitons of the form (2), (3), (5) for $\gamma = 0$. Analogous considerations will be used below for the other values of $\gamma$.

(ii) $\gamma > 0$.
In this case it is also convenient to take $\alpha = \frac{3}{2}$. Equation (14) takes the form

$$-2\Pi_1 - \Pi_2 - 3\left(\frac{p^2}{2} - 1\right)\gamma \tilde{V}_2 = 0.$$ 

Since $\tilde{V}_2 \geq 0$, it implies that for $p \geq 2$ $\phi = A_0 \equiv 0$ and there are no solitons of the form (2), (3), (5).

Now let us take $\alpha > \frac{3}{2}$. Identity (14) takes the form

$$(2\alpha - 1)\Pi_1 + (4\alpha - 5)\Pi_2 + (2\alpha - 3)(m^2 - \omega^2)V_1 + (2\alpha - 3)I_2 + (p\alpha - 3)\gamma \tilde{V}_2 = 0.$$ 

In this case if $m^2 \geq \omega^2$ and $p\alpha \geq 3$ then $\phi = A_0 \equiv 0$. Considering $\alpha > \frac{3}{2}$ and $p\alpha \geq 3$ together one can conclude that for any $p > 1$ there exist $\alpha : \alpha > \frac{3}{2}$ and $p\alpha \geq 3$ hold simultaneously. Thus, for $m^2 \geq \omega^2$ there are no solitons of the form (2), (3), (5).

Considering other values of the parameter $\alpha$ does not provide any additional restrictions on the existence of solitons.

(iii) $\gamma < 0$ (this case was considered in [10]).
First let us again take $\alpha = \frac{3}{2}$. Equation (14) again takes the form

$$-2\Pi_1 - \Pi_2 - 3\left(\frac{p^2}{2} - 1\right)\gamma \tilde{V}_2 = 0.$$ 

But since $\gamma < 0$, it implies that now $\phi = A_0 \equiv 0$ and there are no solitons of the form (2), (3), (5) if $p \leq 2$.

It is also useful to consider the case $\alpha = \frac{1}{2}$. Identity (14) takes the form

$$3\Pi_2 - 2(\omega^2 - m^2)V_1 + 2I_2 - \left(\frac{p^2}{2} - 3\right)\gamma \tilde{V}_2 = 0. \quad (15)$$ 

Now $\phi = A_0 \equiv 0$ and there are no solitons of the form (2), (3), (5) if $p \geq 6$ and $m^2 > \omega^2$ or $p > 6$ and $m^2 \geq \omega^2$. The case $p = 6$ and $m^2 = \omega^2 \neq 0$ should be considered separately. Equation (15) gives us only the condition $A_0 \equiv 0$ for $p = 6$ and $m^2 = \omega^2$. Nevertheless, it is easy to see that $\phi \equiv 0$ also holds. Indeed, equation of motion for the field $A_0$ coming from (4) takes the form $\phi^2 \equiv 0$ for $A_0 \equiv 0$, $\omega \neq 0$ and, consequently, $\phi \equiv 0$ whenever $A_0 \equiv 0$ and $\omega \neq 0$. The latter statement is quite obvious from the physical point of view, because only the solution with the zero charge density can provide the absence of the electric field in the whole space.

Considering other values of the parameter $\alpha$ also does not provide any additional restrictions on the existence of solitons.

It should be noted that an analogous generalized rescaling of the field $\phi$ (but with $\alpha = 1$) was discussed in [10] as a possible alternative method which can be used at some step of the proof presented in [10].
The method of generalized rescaling presented above can be used in more general cases. Let us consider quite a general form of the potential $V(\phi)$ such that

$$V(\phi)|_{\phi=0} = 0, \quad \frac{dV(\phi)}{d\phi}|_{\phi=0} = 0. \tag{16}$$

The latter condition ensures that the trivial solution is $\phi \equiv 0$, $A_0 \equiv 0$. Using (10) one can get

$$V_2(\lambda) = \int d^3 x \ V(\lambda^a \phi(x)) = \frac{1}{\lambda^3} \int d^3 \bar{x} \ V(\lambda^a \phi(\bar{x}))$$

$$= \frac{1}{\lambda^3} \int d^3 x \ V(\lambda^a \phi(x)), \tag{17}$$

where $V_2(1) = V_2$, and

$$\frac{dV_2(\lambda)}{d\lambda}|_{\lambda=1} = \alpha \int d^3 x \ (\frac{dV(\phi)}{d\phi} \phi(x) - 3V_2). \tag{18}$$

Equation (14) transforms into

$$-2(\alpha - 1)\Pi_1 + (5 - 4\alpha)\Pi_2 + (2\alpha - 3)(\omega^2 - m^2)V_1 + (3 - 2\alpha)I_2$$

$$= \alpha \int d^3 x \ (\frac{dV(\phi)}{d\phi} \phi(x) - 3V_2) = 0. \tag{19}$$

Now we are ready to obtain more general non-existence results.

**Theorem 1.** For the potential of the form (16) non-topological solitons of the form (2), (3), (5) are absent at least if one of the following inequalities fulfills:

- $\frac{dV(\phi)}{d\phi} \phi - 2V(\phi) \geq 0$,
- $4(m^2 - \omega^2)\phi^2 \geq \frac{dV(\phi)}{d\phi} \phi - 6V(\phi), \omega \neq 0$,
- $V(\phi) - (\omega^2 - m^2)\phi^2 \geq 0$

for any $\phi$ (or at least for that range of values of the field $\phi$ which is supposed for a solution).

**Remark 2.** The first two relations coincide with those obtained in [10] (up to the notations). The first relation also coincides with the restriction presented in [7] for a more general case of the Yang–Mills–Klein–Gordon system with $m \neq 0$ and $m^2 \phi^2 + V(\phi) \geq 0$ (in our notations). The third relation for the static case $\omega = 0$ and for $m = 0$ simply gives the Derrick theorem [9].

**Proof.** First we take $\alpha = \frac{3}{2}$. Equation (19) takes the form

$$-2\Pi_1 - \Pi_2 - \int d^3 x \ \frac{3}{2} \ (\frac{dV(\phi)}{d\phi} \phi(x) - 2V(\phi)) = 0. \tag{20}$$

So if $\frac{dV(\phi)}{d\phi} \phi - 2V(\phi) \geq 0$ for any $\phi$ (or at least for that range of values of the field $\phi$ which is supposed for a solution), then solitons of the form (2), (3), (5) are absent in the theory (again because (20) implies that $\Pi_1 = \Pi_2 = \int d^3 x \ (\frac{dV(\phi)}{d\phi} \phi(x) - 2V(\phi)) = 0$). The latter inequality can be rewritten as

$$\phi^3 \frac{d(V(\phi))}{d\phi} \geq 0. \tag{21}$$

Second we take $\alpha = \frac{1}{7}$. Identity (19) takes the form

$$3\Pi_2 - 2(\omega^2 - m^2)V_1 + 2I_2 - \left(\frac{1}{2} \int d^3 x \ (\frac{dV(\phi)}{d\phi} \phi(x) d^3 x - 3V_2) \right) = 0. \tag{22}$$
Using \( \int d^3x \phi^2 = V_1 \) one easily realizes that \( A_0 \equiv 0 \) if
\[
4(m^2 - \omega^2)\phi^2 \geq \frac{dV(\phi)}{d\phi} \phi - 6V(\phi),
\]
for any \( \phi \). Using the equation of motion for the field \( A_0 \) we can also get \( \phi \equiv 0 \) provided \( \omega \neq 0 \) (see the paragraph just after equation (15)).

Now let us take \( \alpha = 0 \). Equation (19) takes the form
\[
\Pi_1 + 5\Pi_2 - 3(\omega^2 - m^2)V_1 + 3I_2 + 3V_2 = 0,
\]
from which we get that solitons of the form (2), (3), (5) are absent in the theory if
\[
V(\phi) - (\omega^2 - m^2)\phi^2 \geq 0.
\]

Now let us consider transformations of form
\[
\phi(\vec{x}) \rightarrow \lambda^\alpha \phi(\vec{x}),
\]
\[
A_0(\vec{x}) \rightarrow \lambda^\beta A_0(\vec{x}),
\]
for the system with action (4), i.e. transformations without rescaling of the coordinates. With \( \beta = -2\alpha \) (this choice vanishes the term \( \sim I_1 \) we get from (12) for any \( \alpha \neq 0 \)
\[
-2\Pi_1 - 4\Pi_2 + 2(\omega^2 - m^2)V_1 - 2I_2 - \int d^3x \frac{dV(\phi)}{d\phi}\phi(x) = 0,
\]
which leads to a new additional restriction: solitons are absent if
\[
\frac{dV(\phi)}{d\phi}\phi - 2(\omega^2 - m^2)\phi^2 \geq 0.
\]
This relation for the static case \( \omega = 0 \) is in agreement with that presented in [8] for the static Yang–Mills–Klein–Gordon system. But since \( V(\phi) - (\omega^2 - m^2)\phi^2|_{\phi=0} = 0 \) (see (16)), from inequality (29) automatically follows inequality (25).

It should be noted that all the restrictions presented in proposition 1 follow from theorem 1. But the restrictions presented in theorem 1 are more general, because they deal not only with the parameters of the potential \((\gamma, m, p)\) and frequency \( \omega \), but with the values of the scalar field potential and values of its first derivative for all values of the field \( \phi \) (or at least for that range of values of the scalar field which is supposed for a solution).

### 3.2. Q-balls

The arguments presented above can be easily applied to the case of the simplest Q-balls [11, 16], which are solitary wave solutions in the case of the absence of the electromagnetic field (we also set \( m = 0 \) in this case).

**Proposition 2.** For the potential of the form (16) Q-ball solutions of the form (2) and such that \( \lim_{x \to \pm \infty} \phi = 0 \) are absent at least if one of the following inequalities fulfills:

- \( \frac{dV(\phi)}{d\phi}\phi - 2V(\phi) \geq 0 \),
- \( 4\omega^2\phi^2 - 6V(\phi) < 0 \) or \( 4\omega^2\phi^2 + \frac{dV(\phi)}{d\phi}\phi - 6V(\phi) > 0 \),
- \( V(\phi) - \omega^2\phi^2 \geq 0 \)
for any $\phi \neq 0$ (or at least for that range of values of the field $\phi$ which is supposed for a solution).

**Remark 3.** The second and third relations coincide with those obtained in [12] (see theorem 1 in [12] in a three-dimensional case).

**Proof.** The proof directly follows from equations (20), (22) and (24) if one simply takes $\Pi_2 = I_1 = I_2 \equiv 0$ and $m = 0$. The difference is only in the case of item 2. The situation differs from that of the Klein–Gordon–Maxwell system because we have no equation for the field $A_0$ which can provide additional constraints in the limiting case $4\omega^2\phi^2 + \frac{dV(\phi)}{d\phi} \phi - 6V(\phi) = 0$ (see equation (22)). The absence of terms $\Pi_2$ and $I_2$ also leads to the existence of two inequalities instead of one (equation (22) with $\Pi_2 = I_1 = I_2 \equiv 0$ and $m = 0$ does not hold if $4\omega^2\phi^2 + \frac{dV(\phi)}{d\phi} \phi - 6V(\phi) > 0$ or $4\omega^2\phi^2 + \frac{dV(\phi)}{d\phi} \phi - 6V(\phi) < 0$). □

It is illustrative to apply these restrictions to the Q-ball solution proposed in [11]. It was shown in [11] that a Q-ball can exist if $V(\phi)$ has a minimum at $\phi_0 \neq 0$. For simplicity suppose that $\phi_0 > 0$. The existence of a minimum implies that $\frac{dV(\phi)}{d\phi}$ changes its sign at $\phi = \phi_0$ and condition (21) is not fulfilled. In this case in principle, $\int d^3x \left( \frac{dV(\phi)}{d\phi} \phi(x) - 2V(\phi) \right)$ can be negative and there are no restrictions on the existence of solitons. Thus, the results described above do not contradict those obtained in [11].

Other examples of solutions, for which we can check our results, can be found in [13, 14]. Solution in [13] corresponds to $V(\phi) = \mu^2 \phi^2 - g^2 \phi^4$ and was found numerically, whereas solution in [14] corresponds to $V(\phi) = \mu^2 \phi^2 - g\phi^2 \ln(\phi^2)$ and was found analytically. Although the first solution was found numerically and appeared to be unstable, it can be easily shown by using its explicit form (see [13]) that all the non-existence conditions for the Q-balls presented above (four conditions) appear to be violated by this solution. The same analysis can be made for the ‘at rest’ solution of [14] with $g > 0$, in this case all the non-existence conditions also appear to be violated by the solution. Thus, the presented results again do not contradict the existence of the solitons.

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