All Multiparty Quantum States Can Be Made Monogamous

Salini K.1, R. Prabhu2, Aditi Sen(De)2, and Ujjwal Sen2
1School of Physics, IISER TVM, Thiruvananthapuram, Kerala, India
2Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211 019, India

We show that arbitrary multiparty quantum states can be made to satisfy monogamy by considering increasing functions of any bipartite quantum correlation that may itself lead to a non-monogamous feature. This is true for states of an arbitrary number of parties in arbitrary dimensions, and irrespective of whether the state is pure or mixed. The increasing function of the quantum correlation satisfies all the expected quantum correlation properties as the original one. We illustrate this by considering a thermodynamic quantum correlation measure, known as quantum work-deficit. We find that although quantum work-deficit is non-monogamous for certain three-qubit states, there exist polynomials of the measure that satisfy monogamy for those states.

I. INTRODUCTION

Sharing of quantum correlations among many parties is known to play an important role in quantum phenomena, ranging from quantum communication protocols [1–4] to cooperative events in quantum many-body systems [5, 6]. It is therefore important to conceptualize and quantify quantum correlations, for which investigations are usually pursued in two directions, viz. the entanglement-separability [7] and the information-theoretic [8] ones. Any such measure of quantum correlation is expected to satisfy a monotonicity (precisely, non-increasing) under an intuitively satisfactory set of local quantum operations.

In case the quantum state is shared between more than two parties, one also expects that all measures of quantum correlation would additionally follow a monogamy property [9–11], which restricts the sharability of quantum correlations among many parties. In the case of three parties, say, Alice, Bob and Charu, monogamy of a measure says that the sum of quantum correlations of the two-party local states between the Alice-Bob and the Alice-Charu pairs, should not exceed the quantum correlation of Alice with Bob and Charu taken together. Alice is therefore allotted a special status, and is called the “nodal observer”. The concept has also been carried over to more than two extra-nodal observers. Classical correlations certainly do not satisfy such a monogamy constraint. The monogamous nature of quantum correlations plays a key role in the security of quantum cryptography [12]. Surprisingly however, there are important and useful entanglement measures that do not satisfy monogamy for certain multiparty quantum states, an example being the entanglement of formation [13], which quantifies the amount of entanglement required for preparation of a given bipartite quantum state. Nevertheless, it was found that for three-qubit systems, the concurrence squared [14], a monotonically increasing function of the entanglement of formation is monogamous [9–11]. Recently, it was shown that the information-theoretic quantum correlation measure, quantum discord [15–17], can violate monogamy [18–20] (cf. [21, 22]), and again a monotonically increasing function of the quantum discord satisfies monogamy for three-qubit pure states [23].

In this paper, we show that if any bipartite quantum correlation measure, of an arbitrary number of parties in arbitrary dimensions, is non-increasing under loss of a part of a local subsystem, any multiparty quantum state is either already monogamous with respect to that measure or an increasing function of the bipartite measure can make it so. Note that the result holds for both pure and mixed states. It is interesting to note that the increasing function also satisfies all the properties for being a measure of quantum correlation, which include monotonicity under local operations and vanishing for “classically correlated” states (which is the set of separable states for measures of entanglement). To illustrate the result, we show that although the quantum work-deficit [24], an information-theoretic quantum correlation measure, violates monogamy even for three-qubit pure states, the states become monogamous when one considers integer powers of the measure. In stark contrast to what happens for concurrence and quantum discord, we show that for the three-qubit generalized W states [25, 26], the fourth power of quantum work-deficit is required to obtain monogamy for these states. In case of arbitrary three-qubit W-class states [25, 26] and the GHZ-class states [26, 27], to obtain monogamy of quantum work-deficit, one requires higher polynomials. We also find that three-qubit pure states that are monogamous with respect to quantum discord are also so with respect to quantum work-deficit.

In Sect. II, we prove the result about the transformability of all non-monogamous multiparty states into monogamous ones. We illustrate this result in the next section (Sect. III) by using the concept of quantum work-deficit, where we prove certain general results about monogamy of quantum work-deficit for arbitrary three-party quantum states. We present a conclusion in Sect. IV. A brief introduction to quantum work-deficit is given in Appendix A.
II. TURNING NON-MONOGAMOUS MULTISITE QUANTUM STATES INTO MONOGAMOUS ONES

In proving the results, we will work with three-party quantum states. However, they can easily be generalized to an arbitrary number of parties. Let \( Q \) be a quantum correlation measure that is defined for arbitrary bipartite states (pure or mixed) in arbitrary dimensions. Consider a three-party quantum state (pure or mixed), \( \varrho_{ABC} \), in arbitrary dimensions, shared between three observers, Alice (A), Bob (B), and Charu (C). Let \( Q_{AB} \) denote the quantum correlation \( Q \) for the two-party reduced state \( \varrho_{AB} = \text{tr}_C \varrho_{ABC} \). \( Q_{AC} \) is similarly defined. Let \( Q_{A:BC} \) denote the quantum correlation for the state \( \varrho_{ABC} \) in the \( A:BC \) partition.

The measure \( Q \) is said to satisfy monogamy for the state \( \varrho_{ABC} \) if \( Q_{A:BC} \geq Q_{AB} + Q_{AC} \). The idea is that a measure will be called monogamous for a certain shared quantum state if the amount of quantum correlations that Alice has with Bob and Charu separately would be smaller than what she has with her partners taken together. The measure will be called strictly monogamous for \( \varrho_{ABC} \) if \( Q_{A:BC} > Q_{AB} + Q_{AC} \). On the other hand, \( Q_{A:BC} < Q_{AB} + Q_{AC} \), will imply that the measure is non-monogamous for the corresponding state.

The following theorem demonstrates that the non-monogamous nature of any measure for any state can be transformed into a monogamous one (in fact, strictly so), by considering an increasing function of the measure. Let \( R \) be the set of all real numbers.

**Theorem 1:** If \( Q \) violates monogamy for an arbitrary three-party quantum state \( \varrho_{ABC} \) in arbitrary dimensions, there always exists an increasing function \( f: R \to R \) such that

\[
f(Q_{A:BC}) > f(Q_{AB}) + f(Q_{AC}),
\]

provided that \( Q \) is monotonically decreasing under discarding systems and invariance under discarding systems occurs only for monogamy-satisfying states.

**Proof:** Let us first rename

\[
Q_{A:BC} = x, \quad Q_{AB} = y, \quad Q_{AC} = z,
\]

for notational simplicity. Then the constraints in the premise of the theorem (non-monogamy and monotonicity of \( Q \)) can be rewritten as

\[
x < y + z, \quad x > y > 0, \quad x > z > 0.
\]

Hence it follows that \( 0 < \frac{y}{x} < 1 \) and \( 0 < \frac{z}{x} < 1 \). This implies that

\[
\lim_{n \to \infty} \left( \frac{y}{x} \right)^n = 0, \quad \lim_{n \to \infty} \left( \frac{z}{x} \right)^n = 0
\]

Hence \( \forall \epsilon > 0 \), there exists positive integers \( n_1(\epsilon), n_2(\epsilon) \) such that

\[
\left( \frac{y}{x} \right)^m < \epsilon \quad \forall \text{ positive integers } m \geq n_1(\epsilon),
\]

\[
\left( \frac{z}{x} \right)^m < \epsilon \quad \forall \text{ positive integers } m \geq n_2(\epsilon).
\]

Let us now choose \( \epsilon = \epsilon_1 < \frac{1}{2} \). Therefore, \( \left( \frac{y}{x} \right)^m < \epsilon_1 \) and \( \left( \frac{z}{x} \right)^m < \epsilon_1 \), \( \forall \) positive integers \( m \geq n(\epsilon_1) \), where \( n(\epsilon_1) = \max\{n_1(\epsilon_1), n_2(\epsilon_1)\} \). Adding the inequalities, we have \( \left( \frac{y}{x} \right)^m + \left( \frac{z}{x} \right)^m < 2\epsilon_1 < 1 \), \( \forall \) positive integers \( m \geq n(\epsilon_1) \). Hence the proof.

Note here that invariance under discarding part of a subsystem implying monogamy, holds for many quantum correlation measures, including entanglement of formation and concurrence for three-qubit systems and quantum discord in arbitrary-dimensional three-party states. Note also that the power of a measure vanishes for the same class of states for which the original measure vanishes, so that the set of states that is indicated to be “classical” by the original measure, is invariant after the transformation of the original measure into the new one. Let us also mention here that if a measure is monotonically non-increasing for a certain class of local operations (possibly assisted by classical communication between the parties), a positive integer power of the measure also has the same property. Note that while the cases of vanishing \( x, y, z \) have been ignored in the proof, they can be handled easily.

We now show that the class of monogamous states is closed under the operation of taking positive integral powers of the corresponding measure.

**Theorem 2:** If a quantum correlation measure is monogamous for a three-party quantum state, any positive integer power of the measure is also monogamous for the same state.

**Proof:** The premise implies that \( x \geq y + z \). Then for any positive integer \( m \), we have

\[
x^m \geq (y + z)^m = \sum_{k=0}^{m} \binom{m}{k} y^k z^{m-k},
\]

which in turn is \( \geq y^m + z^m \), as \( y, z \) are non-negative. Hence the proof.

III. ON MONOGAMY OF QUANTUM WORK-DEFICIT

We will now consider the monogamy properties of the information-theoretic quantum correlation measure, called quantum work-deficit (WD) \([24]\), for arbitrary three-qubit pure states. In particular, this will help to illustrate that powers of a measure can lead to monogamous nature for a state, when the measure itself is not so.

We begin by relating the monogamy properties of quantum discord, quantum work deficit, and entanglement of formation. Consider an arbitrary pure
three-party state $|\psi\rangle_{ABC}$. Let us denote the quantum discord for the state $\sigma_{AB} = \text{tr}_C |\psi\rangle\langle\psi| \text{ by } D_{AB}$, where the measurement is performed by the observer $B$. $D_{AC}$ is similarly defined, with the measurement being performed by the observer $C$. The entanglements of formation of $\sigma_{AB}$ and $\sigma_{AC}$ are denoted by $E_{AB}^f$ and $E_{AC}^f$ respectively. Similar notations are used for the different varieties of the quantum work-deficits, $\Delta^t$, $\Delta^t_→$, and $\Delta^→$. See Appendix A for the definition of WD.

**Proposition 1**: For an arbitrary three-party pure state, $D_{AB} + D_{AC} + H(\{p_i^B\}) + H(\{p_j^C\}) = E_{AB}^f + E_{AC}^f + H(\{p_i^B\}) + H(\{p_j^C\}) \geq \Delta^→_{AB} + \Delta^→_{AC} \geq \Delta_{AB} + \Delta_{AC}$, where $H(\{p_i^B\})$ is the entropy produced by the measurement in $B$, and similarly for $H(\{p_j^C\})$.

**Proof**: It can be obtained from Ref. [10] that for an arbitrary pure state $|\psi\rangle_{ABC}$,

$$E_{AB}^f - \sum_i p_i^C S(I \otimes M_i \rho_{AC} I \otimes M_i^f / p_i^C) = 0, \quad (5)$$

where $\{M_i\}$ forms the optimal measurement by the observer $C$ and $p_i^C$ are the corresponding probabilities. Here $S(\cdot)$ denotes the von Neumann entropy of its argument. Therefore, $E_{AB}^f + H(\{p_i^C\}) - S(\sum_i I \otimes M_i \rho_{AC} I \otimes M_i^f) = 0$, where $H(\cdot)$ denotes the Shannon entropy of the probability distribution in its argument. Here we assume that projective measurements attain optimality, which is indeed the case for rank-2 states [23]. Consequently, $E_{AB}^f + H(\{p_i^C\}) \geq \Delta^→_{AB} + S(\sigma_{AB}) \geq \Delta^→_{AB} \geq \Delta_{AB}$. Hence the result.

Performing measurements on the first parties will lead to $2E_{BC}^f + H(\{p_i^A\}) + H(\{q_i^A\}) \geq \Delta^→_{AB} + \Delta^→_{AC} \geq \Delta_{AB} + \Delta_{AC}$, where $H(\{p_i^A\}) (H(\{q_i^A\}))$ is the entropy produced in the measurement at $A$ on $\sigma_{AB}$ ($\sigma_{AC}$).

**Theorem 3**: For an arbitrary pure three-party quantum state $|\psi\rangle_{ABC}$, quantum discord is monogamous whenever the quantum work-deficit, $\Delta^→_→$, is so.

**Proof**: From the definitions of quantum discord and WD, we obtain

$$D_{AB} = S_B + \Delta_{AB} - H(\{p_i^B\}), \quad (6)$$

where $S_B$ is the von Neumann entropy of $\sigma_B = \text{tr}_AC |\psi\rangle\langle\psi|$. Since $S_B \geq 0$, $D_{AB} \leq \Delta_{AB}$. For states for which WD is monogamous, we have

$$D_{AB} + D_{AC} \leq \Delta^→_{AB} + \Delta^→_{AC} \leq \Delta^→_{ABC} = S_A = D_{ABC}. \quad (7)$$

Here we assume that the minimum of work-deficit and quantum discord are attained by the same measurement. It is easy to see that the theorem holds even if the first parties perform the measurements.

**A. Monogamy of Work-deficit for W-class**

We now consider the monogamy properties of quantum work-deficit for an important class of three-qubit pure states, viz. the generalized W states [23, 24], given by

$$|\phi_{GW}\rangle = \sin \theta \cos \phi |011\rangle + \sin \theta \sin \phi |101\rangle + \cos \theta |110\rangle, \quad (8)$$

where $\theta \in (0, \frac{\pi}{2}]$ and $\phi \in (0, 2\pi]$. We find that quantum work-deficit is non-monogamous for almost all members of this class (see Fig. 1 (left)). In other words, setting

$$\delta_Q(\sigma_{ABC}) \equiv Q_{ABC} - Q_{AB} - Q_{AC} \quad (9)$$

for an arbitrary bipartite quantum correlation measure $Q$ and an arbitrary three-party state $\sigma_{ABC}$, we find that

$$\delta_{\Delta^→_→} < 0 \quad (10)$$

for about 98.97% of randomly chosen generalized W states. Note here that another information-theoretic quantum correlation measure, the quantum discord, can also be non-monogamous for these states [17, 20]. However, recently it has been shown that the square of (one variety of) quantum discord is a monogamous quantity for all three-qubit pure states [23]. This however is no longer valid for WD. As stated in Theorem 1, suitably chosen integral powers of WD will be monogamous for any given state. And we find that for WD, monogamy for almost all generalized W states is obtained for the fifth power (see Fig. 1 (right)), i.e.

$$\delta_{\Delta^→_→^5} > 0 \quad (11)$$

for about 99.72% of the generalized W states.

We have also considered the monogamy properties of general three-qubit pure states with respect to quantum work-deficit, $\Delta^→$. A histogram showing the relative
Quantum correlation measures can be monogamous or non-monogamous for multisite quantum states. This can happen for measures within the entanglement-separability paradigm, as well as those in the information-theoretic one. We demonstrated that any quantum correlation measure that is non-monogamous for a multiparty quantum state can be made monogamous for the same by considering an increasing function of the measure. The transformed measure retains the important properties, like monotonicity under local operations and vanishing for “classical” states, of the original measure. We illustrate the results by using the concept of quantum work-deficit, an information-theoretic quantum correlation measure. We show that while the generalized W states are non-monogamous with respect to quantum work-deficit, its integral powers, for arbitrary three-qubit pure states.

IV. CONCLUSION

Quantum correlation measures can be monogamous or non-monogamous for multisite quantum states. This can happen for measures within the entanglement-separability paradigm, as well as those in the information-theoretic one. We demonstrated that any quantum correlation measure that is non-monogamous for a multiparty quantum state can be made monogamous for the same by considering an increasing function of the measure. The transformed measure retains the important properties, like monotonicity under local operations and vanishing for “classical” states, of the original measure. We illustrate the results by using the concept of quantum work-deficit, an information-theoretic quantum correlation measure. We show that while the generalized W states are non-monogamous with respect to quantum work-deficit, the fourth power of the measure makes the states monogamous. We also discuss the monogamy properties of quantum work-deficit, and its powers, for arbitrary three-qubit pure states.

ACKNOWLEDGMENTS

RP acknowledges an INSPIRE-faculty position at the Harish-Chandra Research Institute (HRI) from the Department of Science and Technology, Government of India, and SK thanks HRI for hospitality and support.
[1] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[2] C.H. Bennett, G. Brassard, C. Crépeau, R. Josza, A. Peres, and W.K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[3] J.-W. Pan, Z.-B. Chen, M. Żukowski, H. Weinfurter, and A. Zeilinger, arXiv:0805.2853 [quant-ph]. H. Häffner, C.F. Roos, and R. Blatt, Phys. Rep. 469, 155 (2008); L.-M. Duan and I. L. Chuang, Rev. Mod. Phys. 82, 155 (2010); D. Jaksch and P. Zoller, Ann. Phys. 315, 52 (2005); L.M.K. Vandersypen and I. L. Chuang, Rev. Mod. Phys. 75, 565 (2001).
[4] For a recent review, see e.g. A. Sen(De) and U. Sen, Physics News 40, 17 (2010) (arXiv:1105.2412).
[5] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen(De), and U. Sen, Adv. Phys. 56, 243 (2007).
[6] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).
[7] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[8] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, arXiv:1112.0238.
[9] V. Coffman, J. Kundu, and W.K. Wootters, Phys. Rev. A 61, 052306 (2000).
[10] M. Koashi and A. Winter, Phys. Rev. A 69, 022309 (2004).
[11] See also T.J. Osborne and F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006); G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A 73, 032345 (2006); T. Hiroshima, G. Adesso, and F. Illuminati, Phys. Rev. Lett. 98, 050503 (2007); M. Seevinck, Phys. Rev. A 76, 012106 (2007); S. Lee and J. Park, ibid. 79, 054309 (2009); A. Kay, D. Kaszlikowski, and R. Ramanathan, Phys. Rev. Lett. 103, 050501 (2009); F.F. Fanchini, M.C. de Oliveira, and A.O. Caldeira, Phys. Rev. A 84, 012313 (2011); M. Hayashi and L. Chen, ibid. 84, 012325 (2011), and references therein.
[12] A.K. Ekert, Phys. Rev. Lett. 67, 661 (1991); N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).
[13] C.H. Bennett, H.J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996).
[14] S. Hill and W.K. Wootters, Phys. Rev. Lett. 78, 5022 (1997); W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[15] L. Henderson and V. Vedral, J. Phys. A 34, 6899 (2001).
[16] H. Ollivier and W.H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).
[17] R. Prabhu, A.K. Pati, A. Sen(De), and U. Sen, Phys. Rev. A 85, 052102(R) (2012).
[18] G.L. Giorgi, Phys. Rev. A 84, 054301 (2011).
[19] R. Prabhu, A.K. Pati, A. Sen(De), and U. Sen, arXiv:1109.4318.
[20] X.-J. Ren and H. Fan, arXiv:1111.5163.
[21] A. Sen(De) and U. Sen, Phys. Rev. A 85, 052103 (2012).
[22] A. Streltsov, G. Adesso, M. Piani, and D. Bruß, arXiv:1112.3967.
[23] Y.-K. Bai, N. Zhang, M.-Y. Ye, and Z.D. Wang, arXiv:1206.2096.
[24] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 89, 180402 (2002); M. Horodecki, K. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), and U. Sen, ibid. 90, 100402 (2003); M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak-Radtke, Phys. Rev. A 71, 062307 (2005).
[25] A. Zeilinger, M.A. Horne, and D.M. Greenberger, in Proc. Squeezed States & Quantum Uncertainty, eds. D. Han, Y.S. Kim, and W.W. Zachary, NASA Conf. Publ. 3135 (1992).
[26] W. Dür, G. Vidal, and J.I. Cirac, Phys. Rev. A 62, 062314 (2000).
[27] D.M. Greenberger, M.A. Horne, and A. Zeilinger, in Bell’s Theorem, Quantum Theory, and Conceptions of the Universe, ed. M. Kafatos (Kluwer Academic, Dordrecht, 1989).
[28] S. Hamieh, R. Kobes, and H. Zaraket, Phys. Rev. A 70, 052325 (2004).