A note on helicity, chirality and spin of optical fields

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Abstract. Helicity $H$, chirality $C$ and spin angular momentum $S$ are three physical observables that play an important role in the study of optical fields. These quantities are closely related, but their connection is hidden by the use of four different vector fields for their representation, namely the electric and magnetic fields $E$ and $B$, and the two transverse potential vectors $C^\perp$ and $A^\perp$. Helmholtz’s decomposition theorem restricted to solenoidal vector fields, entails the introduction of a bona fide inverse curl operator, which permits to express the above three quantities in terms of the electric and magnetic fields only. This gives clear expressions for $H, C$ and $S$, which are automatically gauge-invariant and show electric-magnetic democracy.

Keywords: Optical fields, helicity, chirality, spin angular momentum of light

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1. Introduction

In both classical and quantum mechanics, physical systems can be characterised by conserved quantities that do not change with time [1]. A conserved quantity $F = F[f]$ is a linear functional of a space- and time-dependent density $f = f(r, t)$ of the form

$$F[f] = \int d^3r \, f(r, t),$$

where $f(r, t)$ can be either a scalar or a tensor [2]. The free electromagnetic field possesses an infinite set of conserved quantities associated with densities which are bilinear functions of the field variables $E, B, C, A$. Not all of these quantities have a clear physical meaning. Among the meaningful ones, the helicity $H = H[h]$, the chirality $C = C[\chi]$ and the spin angular momentum $S = S[s]$ are particularly significant for the study of optical fields [7, 8, 9, 10, 11, 12, 13]. The corresponding densities $h = h(r, t), \chi = \chi(r, t)$ and $s = s(r, t)$, are given by

$$h = \frac{1}{2} \left( \sqrt{\frac{\varepsilon_0}{\mu_0}} \mathbf{A} \cdot (\nabla \times \mathbf{A}) + \sqrt{\frac{\mu_0}{\varepsilon_0}} \mathbf{C} \cdot (\nabla \times \mathbf{C}) \right),$$

(2a)
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\[ \chi = \frac{\epsilon_0}{2} \left[ \mathbf{E} \cdot (\nabla \times \mathbf{E}) + c^2 \mathbf{B} \cdot (\nabla \times \mathbf{B}) \right], \]  
\[ s = \frac{1}{2} \left[ \epsilon_0 \mathbf{E} \times \mathbf{A} + \mathbf{B} \times \mathbf{C} \right], \]

where \( \mathbf{E} = \mathbf{E}(r, t) \) and \( \mathbf{B} = \mathbf{B}(r, t) \) are the electric and magnetic fields, respectively, and \( \mathbf{C} = \mathbf{C}^\perp(r, t) \) and \( \mathbf{A} = \mathbf{A}^\perp(r, t) \) are the transverse vector potentials implicitly defined in the Coulomb gauge by

\[ \mathbf{E} = -\frac{1}{\epsilon_0} \nabla \times \mathbf{C}, \]  
\[ \mathbf{B} = \nabla \times \mathbf{A}. \]

Substituting \((3a)\) and \((3b)\) into \((2a)-(2c)\), one could express \( h, \chi \) and \( s \) in terms of the gauge fields \( \mathbf{C} \) and \( \mathbf{A} \) only. However, as \( \mathbf{C} \) and \( \mathbf{A} \) are not directly observables, it would be more appealing instead to write \( h, \chi \) and \( s \) as functions of the physical fields \( \mathbf{E} \) and \( \mathbf{B} \) solely. For this we would have to invert \((3a)\) and \((3b)\) to obtain,

\[ \mathbf{C} = -\epsilon_0 (\nabla \times)^{-1} \mathbf{E}, \]  
\[ \mathbf{A} = (\nabla \times)^{-1} \mathbf{B}, \]

where \((\nabla \times)^{-1}\) would denote the formal inverse curl operator.

The aim of this work is to determine such representation of helicity, chirality and spin densities in terms of physical fields only, in order to clarify their connection. To achieve this result, we invert \((3a)\) and \((3b)\) using Helmholtz’s decomposition theorem, to obtain \( \mathbf{C} = \mathbf{C}[\mathbf{E}] \) and \( \mathbf{A} = \mathbf{A}[\mathbf{B}] \). Inserting these functionals into \((2a)-(2c)\), we obtain the sought physical-field representation in real space. Then, we write \( h, \chi \) and \( s \) in reciprocal space by means of the Fourier transforms of the physical fields. This provides further information on their connection and shows that, unlike the corresponding conserved quantities, these densities do not separate into the sum of right-handed and left-handed terms. Finally, the work is completed by three appendices of a mainly didactic nature, in which all the details of the calculations omitted in the main text are presented.

2. Helicity, chirality and spin in real space

In this section we calculate \( H[h], C[\chi] \) and \( S[s] \) in real space via \((1)\). We evaluate explicitly the densities \( h = h(r, t), \chi = \chi(r, t) \) and \( s = s(r, t) \), in terms of the electric and magnetic fields only.

2.1. Helicity

From the definition \((2a)\), it follows that

\[ H = \int d^3 r \frac{1}{2} \left[ \frac{\epsilon_0}{\mu_0} \mathbf{A} \cdot (\nabla \times \mathbf{A}) + \frac{\mu_0}{\epsilon_0} \mathbf{C} \cdot (\nabla \times \mathbf{C}) \right], \]
\[
H = \frac{\varepsilon_0}{2c} \int d^3r \left\{ E \cdot [(\nabla \times)^{-1}E] + c^2 B \cdot [(\nabla \times)^{-1}B] \right\},
\]

where (4a) and (11a), have been used. Next, we use (2.17) with \( G = E \) and \( G = B \), respectively, to write \((\nabla \times)^{-1}E\) and \((\nabla \times)^{-1}B\) explicitly in (5), thus obtaining

\[
H = \int d^3r \left\{ \frac{\varepsilon_0}{8\pi c} \int d^3r' \frac{E(r, t) \cdot (\nabla' \times E(r', t)) + c^2 B(r, t) \cdot (\nabla' \times B(r', t))}{|r - r'|} \right\},
\]

where here and hereafter \( \nabla' \) denotes the gradient with respect to the primed coordinates \( r' = (x', y', z') \). An equivalent expression for \( H \) was already given in (14).

### 2.2. Chirality

From (2b) it follows that \( C \) is automatically fulfilling electric-magnetic democracy. However, to highlight its connection with the helicity and the spin, we can recast the expression (2b) in a form similar to (6), as follows. We start from the defining equation

\[
C = \int d^3r \left\{ \frac{\varepsilon_0}{2} \left[ E \cdot (\nabla \times E) + c^2 B \cdot (\nabla \times B) \right] \right\},
\]

and we rewrite it as a double spatial integral with the help of the Dirac delta function:

\[
C = \frac{\varepsilon_0}{2} \int d^3r \int d^3r' \delta(r - r') E(r, t) \cdot (\nabla' \times E(r', t))
+ \frac{\varepsilon_0 c^2}{2} \int d^3r \int d^3r' \delta(r - r') B(r, t) \cdot (\nabla' \times B(r', t))
\equiv C_E + C_B.
\]

Let us consider first the electric-field contribution \( C_E \) in (5). Using

\[
\delta(r - r') = -\frac{1}{4\pi} \nabla^2 \frac{1}{|r - r'|},
\]

and swapping the order of integration, we can rewrite \( C_E \) as

\[
C_E = -\frac{1}{4\pi} \int d^3r' \left[ (\nabla' \times E(r', t)) \right] \cdot \int d^3r E(r, t) \left( \frac{\nabla^2}{R} \frac{1}{|r - r'|} \right).
\]

Next, we notice that

\[
E \left( \frac{\nabla^2}{R} \right) = \frac{1}{R} (\nabla^2 E) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[ \left( \frac{\partial}{\partial x_i} \frac{1}{R} \right) E - \frac{1}{R} \left( \frac{\partial}{\partial x_i} E \right) \right],
\]

yields a surface term that goes \( \to 0 \), when integrated

where \( R \equiv |r - r'|. \) Inserting (11) into (10), we obtain

\[
C_E = -\frac{1}{4\pi} \int d^3r \int d^3r' \frac{\nabla^2 E(r, t) \cdot (\nabla' \times E(r', t))}{|r - r'|}
= -\frac{1}{4\pi c^2} \int d^3r \int d^3r' \frac{\partial^2 E(r, t) \cdot (\nabla' \times E(r', t))}{|r - r'|},
\]

(12)
where the wave equation
\[ \nabla^2 E(r, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E(r, t) = 0, \] (13)
has been used. The same procedure can be followed to calculate \( C_B \), so that eventually we can write
\[ C = \int d^3 r \left\{ -\frac{\epsilon_0}{8\pi c^2} \int d^3 r' \frac{\partial^2 E(r, t)}{\partial t^2} \cdot [\nabla' \times E(r', t)] + c^2 \frac{\partial^2 B(r, t)}{\partial t^2} \cdot \frac{[\nabla' \times B(r', t)]}{|r - r'|} \right\}. \] (14)

2.3. Spin

From (2c) it follows that
\[ S = \frac{\epsilon_0}{2} \int d^3 r \left\{ E \times [(\nabla \times)^{-1} B] - B \times [(\nabla \times)^{-1} E] \right\}, \] (15)
where (4a) and (17) have been used. To begin with, let us consider the first term in the equation above. Our goal is to evaluate the functional
\[ \int d^3 r E \times [(\nabla \times)^{-1} B] = \int d^3 r E \times \left[ \nabla \times \int \frac{d^3 r'}{4\pi} \frac{B(r', t)}{|r - r'|} \right] \]
\[ \equiv \int d^3 r E \times (\nabla \times F), \] (16)
where (2.11) has been used, and we have defined \( F = F(r, t) \), as
\[ F(r, t) = \int \frac{d^3 r'}{4\pi} \frac{B(r', t)}{|r - r'|}. \] (17)
Note that (2.20) in Appendix B implies
\[ \nabla \cdot F = 0. \] (18)

Next, we show that
\[ \int d^3 r E \times (\nabla \times F) = \int d^3 r (\nabla \times E) \times F. \] (19)

For this, first we notice that
\[ \nabla (E \cdot F) = E \times (\nabla \times F) + F \times (\nabla \times E) + (F \cdot \nabla) E + (E \cdot \nabla) F. \] (20)
The integral of \( \nabla (E \cdot F) \) with respect to \( d^3 r \) is a surface term that goes to zero when the surface of integration goes to infinity. The integral of \( (F \cdot \nabla) E + (E \cdot \nabla) F \) is also zero because
\[ \int d^3 r (F \cdot \nabla) E = \int d^3 r \frac{\partial F_i}{\partial x_i} \]
\[ = \int d^3 r \frac{\partial F_i}{\partial x_i} - \int \text{surface term} = 0 \text{ from } (18) \]
\[ = 0, \] (21)
where $\partial_i = \partial/\partial x_i$, with $i = 1, 2, 3$, and summation over repeated indices is understood. In the same way we can show that

$$\int d^3r \left( \mathbf{E} \cdot \nabla \right) \mathbf{F} = 0.$$  \hspace{1cm} (22)

Thus, (19) is demonstrated.

Now, using (19) and Faraday’s law

$$\nabla \times \mathbf{E}(r, t) = -\frac{\partial \mathbf{B}(r, t)}{\partial t},$$  \hspace{1cm} (23)

we can rewrite (16) as,

$$\int d^3r (\nabla \times \mathbf{E}) \times \mathbf{F} = \int d^3r \left[ -\frac{\partial \mathbf{B}(r, t)}{\partial t} \right] \times \int \frac{d^3r'}{4\pi |r - r'|} \mathbf{B}(r', t)$$

$$= \frac{1}{4\pi} \int d^3r \int d^3r' \frac{\mathbf{B}(r, t) \times \frac{\partial \mathbf{B}(r', t)}{\partial t}}{|r - r'|},$$  \hspace{1cm} (24)

where in the last line we have swapped the dummy integration variables $r$ and $r'$.

Following the same procedure as above, and using Ampère’s law

$$\nabla \times \mathbf{B}(r, t) = \frac{1}{c^2} \frac{\partial \mathbf{E}(r, t)}{\partial t},$$  \hspace{1cm} (25)

we can directly prove that

$$\int d^3r \mathbf{B} \times \left[ (\nabla \times)^{-1} \mathbf{E} \right] = -\frac{1}{4\pi c^2} \int d^3r \int d^3r' \frac{\mathbf{E}(r, t) \times \frac{\partial \mathbf{E}(r', t)}{\partial t}}{|r - r'|},$$  \hspace{1cm} (26)

Finally, gathering (15), (25) and (26), we can write

$$S = \frac{\epsilon_0}{8\pi c^2} \int d^3r \int d^3r' \frac{\mathbf{E}(r, t) \times \frac{\partial \mathbf{E}(r', t)}{\partial t} + c^2 \mathbf{B}(r, t) \times \frac{\partial \mathbf{B}(r', t)}{\partial t}}{|r - r'|},$$  \hspace{1cm} (27)

or, equivalently,

$$S = \int d^3r \left\{ -\frac{\epsilon_0}{8\pi c^2} \int d^3r' \frac{\frac{\partial \mathbf{E}(r, t)}{\partial t} \times \mathbf{E}(r', t) + c^2 \frac{\partial \mathbf{B}(r, t)}{\partial t} \times \mathbf{B}(r', t)}{|r - r'|} \right\}.$$  \hspace{1cm} (28)

As expected, this expression displays electric-magnetic democracy.

2.4. Discussion

Equations (6), (14) and (28), show that $h(r, t)$, $\chi(r, t)$ and $s(r, t)$ have the same form:

$$f(r, t) = \frac{\epsilon_0}{8\pi c} \int d^3r' f_E(r, r', t) + f_B(r, r', t) \frac{\mathbf{E}(r', t) + c^2 \mathbf{B}(r', t)}{|r - r'|},$$  \hspace{1cm} (29)
where \( f \in \{ h, \chi, s \} \), and

\[
\begin{align*}
  h_E(r, r', t) &= E(r, t) \cdot [\nabla' \times E(r', t)], \\
  \chi_E(r, r', t) &= -\frac{1}{c} \frac{\partial^2 E(r, t)}{\partial t^2} \cdot [\nabla' \times E(r', t)], \\
  s_E(r, r', t) &= -\frac{1}{c} \frac{\partial E(r, t)}{\partial t} \times E(r', t).
\end{align*}
\]

The corresponding magnetic densities \( h_B, \chi_B \) and \( s_B \), can be obtained from (30a)-(30c) by replacing \( E \) with \( cB \) everywhere. Equations (29) and (30a)-(30c) are the main result of this work.

Next, we will make some remarks on the advantages of expressing \( h, \chi \) and \( s \) in terms of physically observable electric and magnetic fields, compared to the traditional formulas (2a) -(2c).

(i) In (29), electric-magnetic democracy is clearly displayed.

(ii) From (29), the nonlocal nature of the three densities \( h(r, t), \chi(r, t) \) and \( s(r, t) \) is evident. The importance of this point has been thoroughly discussed by Bialynicki-Birula [14].

(iii) By definition, equations (29) and (30a)-(30c), are manifestly gauge invariant.

(iv) Comparing (30a) and (30b), we can see directly why for monochromatic fields of frequency \( \omega_0 \), helicity and chirality are proportional to each other. In this case \( \partial^2 E(r, t)/\partial t^2 = -\omega_0^2 E(r, t) \), so that \( \chi_E(r, r', t) = (\omega_0^2/c) h_E(r, r', t) \).

(v) The densities (30a)-(30c) are not uniquely defined. To obtain these equations, we repeatedly used integration by parts with respect to \( d^3r \), which caused the elimination of some surface terms. This is particularly evident for the chirality, where we have two distinct expressions for \( \chi(r, t) \), given by (7) and (14).

3. Helicity, chirality and spin of optical fields in reciprocal space

Further insights into \( H[h], C[\chi] \) and \( S[s] \), come from their expressions in reciprocal space, obtained by substituting the Fourier transforms of \( E \) and \( B \) into equations (29) and (30a)-(30c). As shown in Appendix C, equations (3.9), (3.22), (3.26) and (3.36), demonstrate that it is possible to write

\[
  f(r, t) = -\frac{\epsilon_0}{c} \sum_{\sigma, \sigma' = \pm 1} \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}} \frac{e^{i\mathbf{r} \cdot (\mathbf{k} - \mathbf{k}')}}{|\mathbf{k}'|} f_{\sigma\sigma'}(\mathbf{k}, \mathbf{k}') A_{\sigma\sigma'}(\mathbf{k}, \mathbf{k}', t) + c.c.,
\]

where, as before, \( f \in \{ h, \chi, s \} \), and the common term \( A_{\sigma\sigma'}(\mathbf{k}, \mathbf{k}', t) \) is defined by

\[
  A_{\sigma\sigma'}(\mathbf{k}, \mathbf{k}', t) = a_\sigma(\mathbf{k}) a_{\sigma'}(-\mathbf{k}') \left( \frac{\sigma' - \sigma}{2} \right) \exp[-i(\omega + \omega')t] + a_\sigma(\mathbf{k}) a_{\sigma'}^*(\mathbf{k}') \left( \frac{\sigma' + \sigma}{2} \right) \exp[-i(\omega - \omega')t],
\]

(32)
with $\omega = c |\mathbf{k}|$, $\omega' = c |\mathbf{k}'|$, and

$$h_{\sigma\sigma'}(\mathbf{k}, \mathbf{k'}) = \mathbf{\hat{e}}_{\sigma}(\mathbf{k}) \cdot \mathbf{\hat{e}}^{*}_{\sigma'}(\mathbf{k}')$$, \hspace{1cm} \text{from (3.9)} \hspace{1cm} (33a)$$

$$\chi_{\sigma\sigma'}(\mathbf{k}, \mathbf{k'}) = c |\mathbf{k}|^2 \mathbf{\hat{e}}_{\sigma}(\mathbf{k}) \cdot \mathbf{\hat{e}}^{*}_{\sigma'}(\mathbf{k}')$$, \hspace{1cm} \text{from (3.22)} \hspace{1cm} (33b)$$

$$\chi_{\sigma\sigma'}(\mathbf{k}, \mathbf{k'}) = c |\mathbf{k}'|^2 \mathbf{\hat{e}}_{\sigma}(\mathbf{k}) \cdot \mathbf{\hat{e}}^{*}_{\sigma'}(\mathbf{k}')$$, \hspace{1cm} \text{from (3.26)} \hspace{1cm} (33c)$$

$$\mathbf{s}_{\sigma\sigma'}(\mathbf{k}, \mathbf{k'}) = -i \sigma \mathbf{\hat{e}}_{\sigma}(\mathbf{k}) \times \mathbf{\hat{e}}^{*}_{\sigma'}(\mathbf{k}')$$, \hspace{1cm} \text{from (3.36)} \hspace{1cm} (33d)$$

There are several features of (31)-(32) and (33a)-(33d), which are worth highlighting.

(i) It is well known that the helicity $H$, the chirality $C$ and the spin angular momentum $S$, are diagonal with respect to the helicity polarisation basis. In fact, from (3.11), (3.23) and (3.38) it follows that

$$H = \frac{2 \epsilon_0}{c} \int d^3 \mathbf{k} \frac{1}{|\mathbf{k}|} \left[ |a_-(\mathbf{k})|^2 - |a_+(\mathbf{k})|^2 \right]$$, \hspace{1cm} (34a)$$

$$C = 2 \epsilon_0 \int d^3 \mathbf{k} |\mathbf{k}| \left[ |a_-(\mathbf{k})|^2 - |a_+(\mathbf{k})|^2 \right]$$, \hspace{1cm} (34b)$$

$$S = \frac{2 \epsilon_0}{c} \int d^3 \mathbf{k} \frac{\mathbf{\hat{k}}}{|\mathbf{k}|} \left[ |a_-(\mathbf{k})|^2 - |a_+(\mathbf{k})|^2 \right]$$, \hspace{1cm} (34c)$$

Conversely, the corresponding densities are not diagonal and cross-helicity terms do not vanish. However, from (3.22) it follows that in the monochromatic limits the rapidly oscillating factors $\exp \left[ \pm i (\omega + \omega') t \right]$ go to zero after averaging over a period of oscillation $|15|$. The remaining terms proportional to $\exp \left[ \pm i (\omega - \omega') t \right]$ average to 1, so that

$$A_{\sigma\sigma'}(\mathbf{k}, \mathbf{k'}, t) \to a_{\sigma}(\mathbf{k}) a^{*}_{\sigma'}(\mathbf{k'}) \left( \frac{\sigma' + \sigma}{2} \right) = \sigma \delta_{\sigma\sigma'} a_{\sigma}(\mathbf{k}) a^{*}_{\sigma}(\mathbf{k'})$$, \hspace{1cm} (35)$$

and the cross-helicity terms disappear.

(ii) Equations (33b) and (33c) give another view on the non-uniqueness of the densities. Note, however, that the two expressions coincide for monochromatic light where $|\mathbf{k}| = |\mathbf{k'}|$. 

(iii) Equation (33d) shows that the spin density possesses both a longitudinal and a transverse part, because from (1.66) it follows that,

$$\mathbf{s}_{\sigma\sigma'}(\mathbf{k}, \mathbf{k'}) = -i \sigma \mathbf{\hat{e}}_{\sigma}(\mathbf{k}) \times \mathbf{\hat{e}}^{*}_{\sigma'}(\mathbf{k'})$$

$$= \mathbf{\hat{k}} \left[ \mathbf{\hat{e}}_{\sigma}(\mathbf{k}) \cdot \mathbf{\hat{e}}^{*}_{\sigma'}(\mathbf{k'}) \right] - \mathbf{\hat{e}}_{\sigma}(\mathbf{k}) \left[ \mathbf{\hat{k}} \cdot \mathbf{\hat{e}}^{*}_{\sigma'}(\mathbf{k'}) \right]$$, \hspace{1cm} (36)$$

However, the spin $\mathbf{S}$ is purely longitudinal because spatial integration yields a delta function $\delta(\mathbf{k} - \mathbf{k'})$, so that the transverse part disappears, due to the transverse character of the electromagnetic field.
4. Concluding remarks

In this note I have presented some alternative expressions for the well-known helicity $H$, chirality $C$ and spin angular momentum $S$ of an optical field, based solely on observable quantities of the electromagnetic field. The two main results of this work are summarised as follows.

(i) Equations (29) and (30a)-(30c) give manifestly gauge-invariant expressions for the helicity, chirality and spin densities of the electromagnetic field, without using the auxiliary transverse vector potentials $A$ and $C$. The simple form of these equations makes clear the connection between $H$, $C$ and $S$, and the electromagnetic democracy is fully displayed.

(ii) In the reciprocal space the helicity, chirality and spin densities reveals their connection at a deeper level, as shown by (31)-(32) and (33a)-(33d). These equations also highlight the special role played by monochromatic fields.

It would be desirable if the results presented here could stimulate further research into the physical interpretation of the (virtually infinite) conserved quantities of the electromagnetic field.

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Appendix A. Notation

In this appendix we quickly review the notation used throughout this work, following [15]. The electric and magnetic fields $E$ and $B$, respectively, are given by

$$E(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ a(k)e^{i(k\cdot r - \omega t)} + a^*(k)e^{-i(k\cdot r - \omega t)} \right], \quad (1.1a)$$

$$cB(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ b(k)e^{i(k\cdot r - \omega t)} + b^*(k)e^{-i(k\cdot r - \omega t)} \right], \quad (1.1b)$$

where $\omega = c|k|$, and

$$b(k) = \hat{k} \times a(k). \quad (1.2)$$

The time-independent vector amplitude $a(k)$ can be calculated as

$$a(k) = \frac{1}{2} \int \frac{d^3r}{(2\pi)^{3/2}} \left[ E(\mathbf{r}, t) + i \frac{\partial}{\partial t} E(\mathbf{r}, t) \right] \exp(-i\mathbf{k} \cdot \mathbf{r} + i\omega t). \quad (1.3)$$

We use a Cartesian coordinate system with the three perpendicular axes parallel to the unit vectors $\hat{e}_1(\hat{k})$, $\hat{e}_2(\hat{k})$ and $\hat{e}_3(\hat{k}) = \hat{k}$, such that

$$\hat{e}_a(\hat{k}) \cdot \hat{e}_b(\hat{k}) = \delta_{ab}, \quad \text{and} \quad \hat{e}_a(\hat{k}) \times \hat{e}_b(\hat{k}) = \varepsilon_{abc} \hat{e}_c(\hat{k}), \quad (1.4)$$
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where \( \varepsilon_{abc} \) denotes the Levi-Civita symbol with \( a, b, c \in \{1, 2, 3\} \), and summation over repeated indices (Einstein’s summation convention), is understood. Using the transverse basis \( \{ \hat{e}_1(\mathbf{k}), \hat{e}_2(\mathbf{k}) \} \), we can build the so-called helicity (or, circular) polarisation basis \([16], \{ \hat{e}_+(\mathbf{k}), \hat{e}_-(\mathbf{k}) \} \), defined by

\[
\hat{e}_\sigma(\mathbf{k}) = \frac{\hat{e}_1(\mathbf{k}) - i \sigma \hat{e}_2(\mathbf{k})}{\sqrt{2}}, \quad (\sigma = \pm 1).
\]  

(1.5)

Note that according to this definition, \( \hat{e}_+(\mathbf{k}) \) represents right-hand circular polarisation, and \( \hat{e}_-(\mathbf{k}) \) represents left-hand circular polarisation. These two orthogonal unit complex vectors have the following properties:

\[
\hat{\mathbf{k}} \cdot \hat{e}_\sigma(\mathbf{k}) = 0,
\]  

(1.6a)

\[
\hat{\mathbf{k}} \times \hat{e}_\sigma(\mathbf{k}) = i \sigma \hat{e}_\sigma(\mathbf{k}),
\]  

(1.6b)

\[
\hat{e}_\sigma^*(\mathbf{k}) \cdot \hat{e}_{\sigma'}(\mathbf{k}) = \delta_{\sigma \sigma'},
\]  

(1.6c)

\[
\hat{e}_\sigma(-\mathbf{k}) = \hat{e}_\sigma(\mathbf{k}),
\]  

(1.6d)

\[
\hat{e}_\sigma(\mathbf{k}) \times \hat{e}_{\sigma'}^*(\mathbf{k}) = i \mathbf{k} \delta_{\sigma \sigma'},
\]  

(1.6e)

where \( \sigma, \sigma' = \pm 1 \). We can write \( a(\mathbf{k}) \) and \( b(\mathbf{k}) \), in the helicity basis as

\[
a(\mathbf{k}) = \sum_{\sigma = \pm 1} a_\sigma(\mathbf{k}) \hat{e}_\sigma(\mathbf{k}),
\]  

(1.7a)

\[
b(\mathbf{k}) = \sum_{\sigma = \pm 1} b_\sigma(\mathbf{k}) \hat{e}_\sigma(\mathbf{k}),
\]  

(1.7b)

where the components

\[
a_\sigma(\mathbf{k}) = \hat{e}_\sigma^*(\mathbf{k}) \cdot a(\mathbf{k}),
\]  

(1.8a)

\[
b_\sigma(\mathbf{k}) = \hat{e}_\sigma^*(\mathbf{k}) \cdot b(\mathbf{k}),
\]  

(1.8b)

satisfy the equation

\[
b_\sigma(\mathbf{k}) = i \sigma a_\sigma(\mathbf{k}), \quad (\sigma = \pm 1).
\]  

(1.9)

Appendix B. Helmholtz’s decomposition theorem for solenoidal vector fields

In this appendix we briefly illustrate the Helmholtz decomposition theorem for solenoidal vector fields, following closely § 6-3 in [17] and appendix F in [18]. For the sake of definiteness, we consider the magnetic field \( \mathbf{B} \) and the potential vector \( \mathbf{A} \) as prototypical solenoidal fields connected by

\[
\mathbf{B} = \nabla \times \mathbf{A}.
\]  

(2.1)

We assume that both \( \mathbf{A} \) and \( \mathbf{B} \) go to zero faster than \( 1/r \) as \( r \to \infty \) [19]. With \( \hat{\mathbf{A}} = \hat{\mathbf{A}}(\mathbf{k}, t) \) and \( \hat{\mathbf{B}} = \hat{\mathbf{B}}(\mathbf{k}, t) \) we denote their spatial Fourier transform, defined by

\[
\mathbf{A}(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{\mathbf{A}}(\mathbf{k}, t) \exp(ik \cdot \mathbf{r}),
\]  

(2.2a)
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\[ B(r, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \tilde{B}(k, t) \exp(i k \cdot r), \]          \hspace{1cm} (2.2b)

From \( \nabla \cdot A = 0 = \nabla \cdot B \), it follows that

\[ k \cdot \tilde{A} = 0 = k \cdot \tilde{B}. \]  \hspace{1cm} (2.3)

Our goal is to find \( A \) given \( B \), that is to give a meaning to the symbolic equation

\[ A = (\nabla \times)^{-1} B, \]  \hspace{1cm} (2.4)

where \((\nabla \times)^{-1}\) denotes the formal inverse curl operator. Substituting \((2.2a)\) and \((2.2b)\) into \((2.1)\), and using

\[ \nabla \times \left[ a \exp(i k \cdot r) \right] = i (k \times a) \exp(i k \cdot r), \]  \hspace{1cm} (2.5)

where \( a \) is an arbitrary constant (\( r \)-independent) vector, we obtain

\[ \tilde{B} = i k \times \tilde{A} \]

\[ = i \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \end{pmatrix}. \]  \hspace{1cm} (2.6)

The antisymmetric matrix above has zero determinant, therefore we cannot obtain \( \tilde{A} \) directly from \((2.6)\) by matrix inversion. However, multiplying both sides of \((2.6)\) by \( i k \times \), and using \( a \times (b \times c) = (a \cdot c) b - (a \cdot b) c \), we find

\[ i k \times \tilde{B} = -k \times \left( k \times \tilde{A} \right) \]

\[ = -\left[ (k \cdot \tilde{A}) k - (k \cdot k) \tilde{A} \right] \]

\[ = k^2 \tilde{A}, \]  \hspace{1cm} (2.7)

where \( k^2 = k \cdot k = |k|^2 \), and \((2.3)\) has been used. From \((2.7)\) it immediately follows that

\[ \tilde{A} = i \frac{k \times \tilde{B}}{k^2}. \]  \hspace{1cm} (2.8)

Substituting this equation into \((2.2a)\), we obtain

\[ A(r, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ i \frac{k \times \tilde{B}(k, t)}{k^2} \right] \exp(i k \cdot r) \]

\[ = \nabla \times \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{k^2} \tilde{B}(k, t) \exp(i k \cdot r), \]  \hspace{1cm} (2.9)

where \((2.5)\) has been used. Writing \( 1/k^2 \) as

\[ \frac{1}{k^2} = \frac{1}{4\pi} \int \frac{d^3r'}{4\pi} \exp \left[-i k \cdot (r - r') \right] |r - r'|, \]  \hspace{1cm} (2.10)
and using the definition (2.2a), we can rewrite (2.9) as

\[ \mathbf{A}(r, t) = \nabla \times \int \frac{d^3k}{(2\pi)^3/2} \int \frac{d^3r'}{4\pi} \exp (i \mathbf{k} \cdot \mathbf{r}') \mathbf{B}(k, t) \]

\[ = \nabla \times \int \frac{d^3r'}{4\pi} \frac{\mathbf{B}(r', t)}{|r - r'|}. \tag{2.11} \]

This equation can be cast in a different but equivalent form using the vector identity

\[ \nabla \times (\phi \mathbf{a}) = \phi (\nabla \times \mathbf{a}) - \mathbf{a} \times (\nabla \phi), \tag{2.12} \]

and

\[ \nabla f(r - r') = -\nabla' f(r - r'), \tag{2.13} \]

where here and hereafter \( \nabla' \) denotes the gradient with respect to the primed coordinates \( r' = (x', y', z') \), with \( f \) an arbitrary function. Applying (2.12) and (2.13) to the integrand in (2.11), we find

\[ \nabla \times \left[ \frac{\mathbf{B}(r', t)}{|r - r'|} \right] = \nabla' \times \frac{\mathbf{B}(r', t)}{|r - r'|} - \nabla' \times \left[ \frac{\mathbf{B}(r', t)}{|r - r'|} \right]. \tag{2.14} \]

Inserting (2.14) in (2.11) yields two terms:

\[ \mathbf{A}(r, t) = \int \frac{d^3r'}{4\pi} \nabla' \times \frac{\mathbf{B}(r', t)}{|r - r'|} - \int \frac{d^3r'}{4\pi} \nabla' \times \left[ \frac{\mathbf{B}(r', t)}{|r - r'|} \right]. \tag{2.15} \]

The last volume integral in (2.15) can be written as a surface integral with the surface of integration lying at infinity, and it vanishes if \( \mathbf{B}(r, t) \) goes to zero faster than \( 1/r \) as \( r \to \infty \).

Thus, from (2.11) and (2.15), it follows that for any solenoidal field \( \mathbf{G} = \mathbf{G}(r, t) \), we can write

\[ \left[ (\nabla \times)^{-1} \mathbf{G} \right](r, t) = \nabla \times \int \frac{d^3r'}{4\pi} \frac{\mathbf{G}(r', t)}{|r - r'|} \]

\[ = \int \frac{d^3r'}{4\pi} \frac{\nabla' \times \mathbf{G}(r', t)}{|r - r'|}. \tag{2.16} \]

This is a faithful representation of the inverse curl operator for solenoidal fields. Indeed, it is straightforward to prove that \( \nabla \times \left[ (\nabla \times)^{-1} \mathbf{G} \right](r, t) = \mathbf{G}(r, t) \) using either (2.16) or (2.17). The simplest way to prove this is by using the first equation:

\[ \nabla \times \left[ (\nabla \times)^{-1} \mathbf{G} \right](r, t) = \nabla \times \left[ \nabla \times \int \frac{d^3r'}{4\pi} \frac{\mathbf{G}(r', t)}{|r - r'|} \right] \]

\[ = \nabla \left( \nabla \cdot \int \frac{d^3r'}{4\pi} \frac{\mathbf{G}(r', t)}{|r - r'|} \right) - \nabla^2 \int \frac{d^3r'}{4\pi} \frac{\mathbf{G}(r', t)}{|r - r'|}. \tag{2.18} \]

The first integral in (2.18) is zero because from (2.13) and

\[ \nabla \cdot (\phi \mathbf{a}) = \mathbf{a} \cdot (\nabla \phi) + \phi (\nabla \cdot \mathbf{a}), \tag{2.19} \]
it follows that
\[ \nabla \cdot \int \frac{d^3r'}{4\pi |r - r'|} G(r', t) \equiv \int \frac{d^3r'}{4\pi} G(r', t) \cdot \nabla \frac{1}{|r - r'|} \]
\[ = - \int \frac{d^3r'}{4\pi} G(r', t) \cdot \nabla' \frac{1}{|r - r'|} \]
\[ = - \left( \frac{d^3r'}{4\pi} \nabla' \cdot \frac{G(r', t)}{|r - r'|} \right) + \int \frac{d^3r'}{4\pi} \frac{1}{|r - r'|} \nabla' \cdot G(r', t) = 0 \]
\[ \text{surface term} = 0 \]
\[ = 0 . \] (2.20)

Finally, using the relation
\[ \nabla^2 \frac{1}{|r - r'|} = -4\pi \delta (r - r') , \] (2.21)
we can rewrite the second integral in (2.18) as
\[ -\nabla^2 \int \frac{d^3r'}{4\pi} \frac{G(r', t)}{|r - r'|} = \int \frac{d^3r'}{4\pi} G(r', t) \delta (r - r') \]
\[ = G(r, t) . \] (2.22)

This completes the proof.

Appendix C. Helicity, chirality and spin in reciprocal space

In this appendix we calculate the expressions of \( H, C \) and \( S \) in reciprocal space, in terms of the Fourier transform of the electric and magnetic fields. Note that here and hereafter we will use the asterisk symbol “∗” to denote ordinary multiplication, as in \( 2 \ast 3 = 6 \), when we split equations over two (or more) lines, instead of the traditional “times” symbol \( \times \). This particular notation is necessary to avoid confusion between ordinary multiplication and the scalar and vector products, denoted by•” and “×”, respectively.

Appendix C.1. Helicity in reciprocal space

It is convenient to rewrite (3) as \( H = H_E + H_B \), where
\[ H_E = \frac{\varepsilon_0}{8\pi c} \int d^3r \int d^3r' \frac{E(r, t) \cdot \nabla' \times E(r', t)}{|r - r'|} , \] (3.1)
and \( H_B \) is obtained from (3.1) replacing \( E \) with \( cB \), everywhere. Next, substituting (1.1) into (3.1), we obtain
\[ H_E = \frac{\varepsilon_0}{8\pi c} \int d^3r \int d^3r' \frac{1}{|r - r'|} \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}} \]
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where we have made the change of variables

\[ a(k) e^{i(k \cdot r - \omega t)} + a^*(k) e^{-i(k \cdot r - \omega t)} \]

\[ \cdot \left\{ i k' \times \left[ a(k') e^{i(k' \cdot r' - \omega' t)} - a^*(k') e^{-i(k' \cdot r' - \omega' t)} \right] \right\} , \quad (3.2) \]

where (2.5) has been used. Calculating the scalar product in the second and third line above, we obtain after a little calculation

\[ H_E = \frac{\epsilon_0}{8\pi c} \int \frac{d^3r}{(2\pi)^3} \int \frac{d^3r'}{|r - r'|} \int \frac{d^3k}{|k'|} \int d^3k' \left| k' \right| e^{i r \cdot k} \]

\[ \ast \left\{ \alpha(k, k') \exp \left[ i (r \cdot k + r' \cdot k') - i (\omega + \omega') t \right] \right\} \]

\[ - \beta(k, k') \exp \left[ i (r \cdot k - r' \cdot k') - i (\omega - \omega') t \right] \right\} + \text{c.c.} , \quad (3.3) \]

where we have defined

\[ \alpha(k, k') = i a(k) \cdot \left[ k' \times a(k') \right] , \quad (3.4a) \]

\[ \beta(k, k') = i a(k) \cdot \left[ k' \times a^*(k') \right] , \quad (3.4b) \]

with \( \omega' = c |k'| \), and c.c. stands for complex conjugate. It is not difficult to show that we can rewrite \( H_E \) as

\[ H_E = \frac{\epsilon_0}{2c} \int \frac{d^3r}{(2\pi)^3} \int \frac{d^3k}{|k|} \int d^3k' \frac{e^{i r \cdot (k - k')}}{|k'|} \]

\[ \ast \left\{ \alpha(k, -k') e^{-(\omega + \omega') t} - \beta(k, k') e^{-i (\omega - \omega') t} \right\} + \text{c.c.} , \quad (3.5) \]

where we have made the change of variables \( k' \to -k' \) in the part of the integrand proportional to \( \alpha(k, k') \). Using the helicity basis (1.5), we can rewrite

\[ \alpha(k, -k') = - \sum_{\sigma, \sigma' = \pm 1} \sigma \sigma' a_{\sigma}(k) a_{\sigma'}(-k') \left[ \hat{e}_{\sigma}(k) \cdot \hat{e}^*_{\sigma'}(k') \right] , \quad (3.6a) \]

\[ \beta(k, k') = \sum_{\sigma, \sigma' = \pm 1} \sigma \sigma' a_{\sigma}(k) a_{\sigma'}(k') \left[ \hat{e}_{\sigma}(k) \cdot \hat{e}^*_{\sigma'}(k') \right] , \quad (3.6b) \]

where (1.6c)-(1.6e) and (1.7d), have been used.
Substituting (3.6a) and (3.6b) into (3.5), we eventually obtain
\[ H_E = -\frac{\epsilon_0}{2c} \sum_{\sigma,\sigma' = \pm 1} \int \frac{d^3r}{(2\pi)^3} \int d^3k \int d^3k' \frac{e^{i\mathbf{r} \cdot (\mathbf{k} - \mathbf{k}')}}{|\mathbf{k}'|} \hat{\mathbf{e}}_\sigma(\mathbf{k}) \cdot \hat{\mathbf{e}}^*_\sigma(\mathbf{k}') \]

\[ \ast \sigma' a_\sigma(\mathbf{k}) \left[ a_{\sigma'}(-\mathbf{k}')e^{-i(\omega' + \omega)t} + a^*_{\sigma'}(\mathbf{k}')e^{-i(\omega - \omega')t} \right] + \text{c.c.} \]  
(3.7)

To calculate \( H_B \) we simply take the expression above for \( H_E \) and we replace \( a_\sigma(\mathbf{k}) \) with \( b_\sigma(\mathbf{k}) = i\sigma a_\sigma(\mathbf{k}) \), according to (1.9). This implies that
\[ \sigma' a_\sigma(\mathbf{k}) a_{\sigma'}(-\mathbf{k}') \rightarrow \sigma' b_\sigma(\mathbf{k}) b_{\sigma'}(-\mathbf{k}') = -\sigma a_\sigma(\mathbf{k}) a_{\sigma'}(-\mathbf{k}') , \]  
(3.8a)
\[ \sigma' a_\sigma(\mathbf{k}) a^*_\sigma(\mathbf{k}') \rightarrow \sigma' b_\sigma(\mathbf{k}) b^*_\sigma(\mathbf{k}') = \sigma a_\sigma(\mathbf{k}) a^*_\sigma(\mathbf{k}') . \]  
(3.8b)

From this result and (3.7), it follows that
\[ H = H_E + H_B \]

\[ = -\frac{\epsilon_0}{c} \sum_{\sigma,\sigma' = \pm 1} \int \frac{d^3r}{(2\pi)^3} \int d^3k \int d^3k' \frac{e^{i\mathbf{r} \cdot (\mathbf{k} - \mathbf{k}')}}{|\mathbf{k}'|} \hat{\mathbf{e}}_\sigma(\mathbf{k}) \cdot \hat{\mathbf{e}}^*_\sigma(\mathbf{k}') \]

\[ \ast a_\sigma(\mathbf{k}) \left\{ a_{\sigma'}(-\mathbf{k}') \left( \frac{\sigma' - \sigma}{2} \right) \exp[-i(\omega + \omega')t] \right. \]

\[ \left. + a^*_\sigma(\mathbf{k}') \left( \frac{\sigma' + \sigma}{2} \right) \exp[-i(\omega - \omega')t] \right\} + \text{c.c.} \]  
(3.9)

Integration of this expression with respect to \( d^3r \) yields \((2\pi)^3\) times the delta function \( \delta(\mathbf{k}' - \mathbf{k}) \), so that
\[ \delta(\mathbf{k}' - \mathbf{k}) \hat{\mathbf{e}}_\sigma(\mathbf{k}) \cdot \hat{\mathbf{e}}^*_\sigma(\mathbf{k}') = \delta(\mathbf{k}' - \mathbf{k}) \delta_{\sigma\sigma'} . \]  
(3.10)

Using this result, we directly obtain
\[ H = -\frac{\epsilon_0}{c} \sum_{\sigma = \pm 1} \sigma \int \frac{d^3k}{|\mathbf{k}|} |a_\sigma(\mathbf{k})|^2 + \text{c.c.} \]

\[ = \frac{2\epsilon_0}{c} \int \frac{d^3k}{|\mathbf{k}|} \left[ |a_-(\mathbf{k})|^2 - |a_+(\mathbf{k})|^2 \right] . \]  
(3.11)

We show now that this result is in agreement with the expressions for \( H \) given in the literature in terms of photon-number operators (see, e.g., Eq. (2.8) in [10]). First, we use our notation to rewrite Eq. (10.4-39) in [20], which gives the electric field quantum operator in a finite quantization volume \( L^3 \), as
\[ \hat{\mathbf{E}}(\mathbf{r}, t) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \sum_{\sigma} \left( \frac{\hbar \omega}{2\epsilon_0} \right)^{1/2} \left[ i \hat{a}_{k\sigma} \epsilon_{k\sigma} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \text{h.c.} \right] , \]  
(3.12)
where \text{h.c.} stands for the Hermitian conjugate of the preceding term. To compare this expression with \( \mathbf{E}(\mathbf{r}, t) \) given by (1.1a), we must convert the integral in (1.1d), to a sum according to the general rule (see, e.g., Eq. (10.8-4) in [20]),
\[ \sum_k \rightarrow \left( \frac{L}{2\pi} \right)^3 \int d^3k , \]  
(3.13)
so that (1.1a) becomes
\[ E(r, t) = \frac{1}{L^{3/2}} \sum_k \sum_{\sigma} \left( \frac{2\pi}{L} \right)^{3/2} a_{\sigma}(k) \hat{e}_{\sigma}(k) e^{i(k \cdot r - \omega t)} + \text{c.c.} \],
(3.14)
where (1.7a) has been used. From the comparison between (3.12) and (3.14), it follows that
\[ E(r, t) \rightarrow \hat{E}(r, t), \quad \text{if} \quad a_{\sigma}(k) \rightarrow \left[ \frac{\hbar \omega L^3}{2(2\pi)^3 \epsilon_0} \right]^{1/2} \hat{a}_{\sigma} k \sigma. \]
(3.15)
This implies that we can define the classical quantities \( n_k \sigma \) corresponding to the quantum photon-number operators \( \hat{n}_k \sigma \), as
\[ n_k \sigma \equiv \frac{2(2\pi)^3 \epsilon_0}{\hbar \omega L^3} |a_{\sigma}(k)|^2 \rightarrow \hat{a}_{\sigma}^\dagger \hat{a}_{\sigma} \equiv \hat{n}_k \sigma, \]
(3.16)
So, if we rename \( n_k^- \) and \( n_k^+ \), as \( n_{kL} \) and \( n_{kR} \), respectively, where the subscripts \( L \) and \( R \) label left- and right-handed circular polarisation, we can rewrite (3.11) with the help of (3.13), as
\[ H = \sum_k \hbar (n_{kL} - n_{kR}), \]
(3.17)
in agreement with [10].

Appendix C.2. Chirality in reciprocal space

We have two distinct expressions for \( C \), given by (7) and (14). Here we rewrite both equations in the reciprocal space.

Appendix C.2.1. Chirality in reciprocal space from (7). The expression of
\[ C_E = \frac{\epsilon_0}{2} \int \frac{d^3 r}{4\pi} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} \]
\[ \times \left( \left[ a(k)e^{i(k \cdot r - \omega t)} + a^*(k)e^{-i(k \cdot r - \omega t)} \right] \right. \]
\[ \cdot \left. \left\{ i k' \times \left[ a(k')e^{i(k' \cdot r - \omega' t)} - a^*(k')e^{-i(k' \cdot r - \omega' t)} \right] \right\} \right), \]
(3.19)
where (2.5) has been used. The evaluation of the scalar product in (3.19) above, gives
\[ C_E = \frac{\epsilon_0}{2} \int \frac{d^3 r}{(2\pi)^3} \int \frac{d^3 k}{4\pi} \int d^3 k' |k'| \exp[i r \cdot (k - k')] \]
\[ \times \left\{ \alpha (k, -k') \exp[-i(\omega + \omega') t] \right. \]
\[ - \beta (k, k') \exp[-i(\omega - \omega') t] \} + \text{c.c.}, \]
(3.20)
where (3.4\text{a}) and (3.4\text{b}) have been used. Next, substituting (3.6\text{a}) and (3.6\text{b}) into (3.20), we obtain

$$C_E = -\frac{\epsilon_0}{2} \sum_{\sigma, \sigma' = \pm 1} \int \frac{d^3r}{(2\pi)^3} \int d^3k \int d^3k' |k'| e^{i\mathbf{r} \cdot (k-k')} \hat{\epsilon}_\sigma(\mathbf{k}) \cdot \hat{\epsilon}_{\sigma'}^*(\mathbf{k'})$$

$$\times \sigma' a_\sigma(\mathbf{k}) \left[ a_{\sigma'}(-\mathbf{k'}) e^{-i(\omega+\omega')t} + a_{\sigma'}^*(\mathbf{k'}) e^{-i(\omega-\omega')t} \right] + \text{c.c.} \quad (3.21)$$

To calculate $C_B$ we replace $a_\sigma(\mathbf{k})$ with $b_\sigma(\mathbf{k}) = i\sigma a_\sigma(\mathbf{k})$, in (3.21). Using (3.8\text{a}) and (3.8\text{b}) we can eventually write

$$C = C_E + C_B$$

$$= -\epsilon_0 \sum_{\sigma, \sigma' = \pm 1} \sigma \int \frac{d^3k}{(2\pi)^3} \int d^3k' \frac{e^{i\mathbf{r} \cdot (k-k')}}{|k'|} |k'|^2 \hat{\epsilon}_\sigma(\mathbf{k}) \cdot \hat{\epsilon}_{\sigma'}^*(\mathbf{k'})$$

$$\times a_\sigma(\mathbf{k}) \left\{ a_{\sigma'}(-\mathbf{k'}) \left( \frac{\sigma' - \sigma}{2} \right) \exp \left[ -i (\omega + \omega') t \right] \right. \right.$$  

$$\left. + a_{\sigma'}^*(\mathbf{k'}) \left( \frac{\sigma' + \sigma}{2} \right) \exp \left[ -i (\omega - \omega') t \right] \right\} + \text{c.c.} \quad (3.22)$$

Performing the integration in real space with respect to $d^3\mathbf{r}$, we obtain

$$C = -\epsilon_0 \sum_{\sigma = \pm 1} \sigma \int d^3k \left| \mathbf{k} \right| |a_\sigma(\mathbf{k})|^2 + \text{c.c.}$$

$$= 2 \epsilon_0 \int d^3k \left| \mathbf{k} \right| \left[ |a_-(\mathbf{k})|^2 - |a_+(\mathbf{k})|^2 \right] \quad (3.23)$$

Finally, using (3.13)-(3.16) it is not difficult to show that we can write the chirality in a quantum-like language as

$$C = \sum_\mathbf{k} \hbar c |\mathbf{k}|^2 (n_{kL} - n_{kR}) \quad (3.24)$$

\textbf{Appendix C.2.2.  Chirality in reciprocal space from (L/4).}  In this case it is not necessary to make new calculations because by comparing (14) with (3.1), we can see that

$$C_E = -\frac{1}{c} H_E \bigg|_{E(r,t) \rightarrow \frac{\partial E(r,t)}{\partial t}} \quad (3.25)$$

Then, we can use this relation and (4.9) to write directly

$$C = -\epsilon_0 \sum_{\sigma, \sigma' = \pm 1} \int \frac{d^3r}{(2\pi)^3} \int d^3k \int d^3k' \frac{e^{i\mathbf{r} \cdot (k-k')}}{|k'|} |k'|^2 \hat{\epsilon}_\sigma(\mathbf{k}) \cdot \hat{\epsilon}_{\sigma'}^*(\mathbf{k'})$$

$$\times a_\sigma(\mathbf{k}) \left\{ a_{\sigma'}(-\mathbf{k'}) \left( \frac{\sigma' - \sigma}{2} \right) \exp \left[ -i (\omega + \omega') t \right] \right. \right.$$  

$$\left. + a_{\sigma'}^*(\mathbf{k'}) \left( \frac{\sigma' + \sigma}{2} \right) \exp \left[ -i (\omega - \omega') t \right] \right\} + \text{c.c.} \quad (3.26)$$
Appendix C.3. Spin in reciprocal space

We calculate

\[
S = \frac{\epsilon_0}{8\pi c^2} \int d^3r \int d^3r' \frac{E(r, t) \times \partial E(r', t)}{|r - r'|} \nonumber
\]

\[
+ \frac{\epsilon_0}{8\pi c^2} \int d^3r \int d^3r' \frac{c^2 B(r, t) \times \partial B(r', t)}{|r - r'|} \nonumber
\]

\[\equiv S_E + S_B, \quad (3.27)\]

in reciprocal space. Substituting (1.1a) into (3.27), we obtain

\[
S_E = \frac{\epsilon_0}{8\pi c^2} \int d^3r \int d^3r' \frac{1}{|r - r'|} \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}} (-ic|k'|) \nonumber
\]

\[\times \left\{ \left[ a(k)e^{i(k \cdot r - \omega t)} + a^*(k)e^{-i(k \cdot r - \omega t)} \right] \nonumber \right. \]

\[\times \left[ a(k')e^{i(k' \cdot r' - \omega' t)} - a^*(k')e^{-i(k' \cdot r' - \omega' t)} \right] \right\}. \quad (3.28)\]

Performing the vector products, we obtain

\[
S_E = -\frac{\epsilon_0}{8\pi c^2} \int d^3r \int d^3r' \frac{1}{|r - r'|} \int d^3k \int d^3k' |k'| \nonumber
\]

\[\times \left\{ \alpha (k, k') \exp [i (r \cdot k + r' \cdot k') - i (\omega + \omega') t] \nonumber \right. \]

\[- \beta (k, k') \exp [i (r \cdot k - r' \cdot k') - i (\omega - \omega') t] \right\} + \text{c.c.}, \quad (3.29)\]

where we have defined

\[
\alpha (k, k') = i a(k) \times a(k'), \quad (3.30a)\]

\[
\beta (k, k') = i a(k) \times a^*(k'). \quad (3.30b)\]

A few more calculations give

\[
S_E = -\frac{\epsilon_0}{8\pi c} \int \frac{d^3r}{(2\pi)^3} \int d^3k \int d^3k' |k'| e^{i k \cdot r} \nonumber
\]

\[\times \left\{ \alpha (k, -k') \exp [-i (\omega + \omega') t] - \beta (k, k') \exp [-i (\omega - \omega') t] \right\} \nonumber \right. \]

\[\times \left. \int d^3r' e^{-i k' \cdot r'} \right) + \text{c.c.} \nonumber \]

\[= e^{-i k' \cdot r} 4\pi/|k'|^3 \]
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where \( (1.6^e) \) , we obtain

\[
\beta (k, k') = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{|k'|} e^{i r \cdot (k-k')} \frac{e^{i (\omega+\omega') t}}{|k'|} [i \hat{\epsilon}_\sigma (\hat{k}) \times \hat{\epsilon}^*_\sigma (\hat{k'})] * [\alpha (k, -k') e^{-i (\omega+\omega') t} - \beta (k, k') e^{-i (\omega-\omega') t}] + c.c., \tag{3.31}
\]

where we made the change of variables \( k' \rightarrow -k' \) in the part of the integrand proportional to \( \alpha (k, k') \).

In the helicity basis \([1.5]\), we can write

\[
\alpha (k, -k') = \sum_{\sigma, \sigma' = \pm 1} a_\sigma (k) a_{\sigma'} (-k') \left[ \hat{\epsilon}_\sigma (\hat{k}) \times \hat{\epsilon}^*_{\sigma'} (\hat{k'}) \right], \tag{3.32a}
\]

\[
\beta (k, k') = \sum_{\sigma, \sigma' = \pm 1} a_\sigma (k) a^*_{\sigma'} (k') \left[ \hat{\epsilon}_\sigma (\hat{k}) \times \hat{\epsilon}^*_{\sigma'} (\hat{k'}) \right], \tag{3.32b}
\]

where \((1.6^e)\) and \((1.7^a)\), have been used. Next, substituting \((3.32a)\) and \((3.32b)\) into \((3.31)\), we obtain

\[
S_E = -\frac{\epsilon_0}{2c} \sum_{\sigma, \sigma' = \pm 1} \int \frac{d^3 r}{(2\pi)^3} \int d^3 k \int d^3 k' \frac{e^{i r \cdot (k-k')}}{|k'|} \left[ i \hat{\epsilon}_\sigma (\hat{k}) \times \hat{\epsilon}^*_{\sigma'} (\hat{k'}) \right] * a_\sigma (k) \left[ a_{\sigma'} (-k') e^{-i (\omega+\omega') t} - a^*_{\sigma'} (k') e^{-i (\omega-\omega') t} \right] + c.c., \tag{3.33}
\]

\(S_B\) is obtained from \(S_E\) by replacing \(a_\sigma (k)\) with \(b_\sigma (k) = i \sigma a_\sigma (k)\), in \((3.33)\). Eventually, we obtain

\[
S = S_E + S_B
\]

\[
= -\frac{\epsilon_0}{2c} \sum_{\sigma, \sigma' = \pm 1} \int \frac{d^3 r}{(2\pi)^3} \int d^3 k \int d^3 k' \frac{e^{i r \cdot (k-k')}}{|k'|} \left[ i \hat{\epsilon}_\sigma (\hat{k}) \times \hat{\epsilon}^*_{\sigma'} (\hat{k'}) \right] * a_\sigma (k) \left\{ a_{\sigma'} (-k') \left( \frac{1 - \sigma \sigma'}{2} \right) \exp [-i (\omega + \omega') t] - a^*_{\sigma'} (k') \left( \frac{1 + \sigma \sigma'}{2} \right) \exp [-i (\omega - \omega') t] \right\} + c.c. \tag{3.34}
\]

To put this expression in a form useful for the comparison with \((3.9)\), \((3.22)\) and \((3.26)\), we note that

\[
-\sigma \left( \frac{\sigma' - \sigma}{2} \right) = \left( \frac{1 - \sigma \sigma'}{2} \right) \quad \text{and} \quad \sigma \left( \frac{\sigma' + \sigma}{2} \right) = \left( \frac{1 + \sigma \sigma'}{2} \right). \tag{3.35}
\]

Substituting \((3.35)\) into \((3.34)\), we obtain

\[
S = S_E + S_B
\]

\[
= \frac{\epsilon_0}{c} \sum_{\sigma, \sigma' = \pm 1} \int \frac{d^3 r}{(2\pi)^3} \int d^3 k \int d^3 k' \frac{e^{i r \cdot (k-k')}}{|k'|} \left[ i \sigma \hat{\epsilon}_\sigma (\hat{k}) \times \hat{\epsilon}^*_{\sigma'} (\hat{k'}) \right] * a_\sigma (k) \left\{ a_{\sigma'} (-k') \left( \frac{\sigma' - \sigma}{2} \right) \exp [-i (\omega + \omega') t] \right\}
\]
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\[ + a^\ast_{\sigma'}(k') \left( \frac{\sigma' + \sigma}{2} \right) \exp \left\{ -i \left( \omega - \omega' \right) t \right\} \right\} + \text{c.c.}, \]  

(3.36)

where, according to (1.6b),

\[ i \sigma \hat{\epsilon}_{\sigma}(\hat{k}) \times \hat{e}^\ast_{\sigma'}(\hat{k}') = \hat{\epsilon}_{\sigma}(\hat{k}) \left[ \hat{k} \cdot \hat{e}^\ast_{\sigma'}(\hat{k}') \right] - \hat{k} \left[ \hat{\epsilon}_{\sigma}(\hat{k}) \cdot \hat{e}^\ast_{\sigma'}(\hat{k}') \right]. \]  

(3.37)

As a last step, we perform the integration with respect to \( d^3r \) in (3.36), to obtain, after a little calculation,

\[ S = -\frac{\epsilon_0}{c} \sum_{\sigma = \pm} \sigma \int d^3k \frac{k}{|k|^2} |a_{\sigma}(k)|^2 + \text{c.c.} \]

\[ = \frac{2\epsilon_0}{c} \int d^3k \frac{\hat{k}}{|k|} \left[ |a_-(k)|^2 - |a_+(k)|^2 \right]. \]  

(3.38)

Finally, using (3.13)-(3.16) it is not difficult to show that we can write the spin in a quantum-like language as

\[ S = \sum_k \hbar \hat{k} (n_{kL} - n_{kR}) \]  

(3.39)

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