“Wormhole” geometry for entrapping topologically-protected qubits in non-Abelian quantum Hall states and probing them with voltage and noise measurements

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We study a tunneling geometry defined by a single point-contact constriction that brings to close vicinity two points sitting at the same edge of a quantum Hall liquid, shortening the trip between the otherwise spatially separated points along the normal chiral edge path. This “wormhole”-like geometry allows for entrapping bulk quasiparticles between the edge path and the tunnel junction, possibly realizing a topologically protected qubit if the quasiparticles have non-Abelian statistics. We show how either noise or simpler voltage measurements along the edge can probe the non-Abelian nature of the trapped quasiparticles.

Perhaps the largest challenge for constructing a functioning quantum computer is how to control the loss of coherence in quantum mechanical systems. There are several different fronts being pursued for attacking this problem, one of these being the idea of topological quantum computation [1, 2]. Decoherence originates from the contact of the system with its external environment, and the resilience of topological quantum computation to decoherence is rooted at degeneracies that cannot be lifted by any local perturbation or disturbance by the environment. These degeneracies are associated with topology and not to symmetries, and were originally proposed by Wen [3] in the context of the fractional quantum Hall (FQH) effect.

An example of a system with topological degeneracies in its excited states is the fractional quantum Hall state at $\nu = 5/2$, which is believed to contain quasiparticle excitations with non-Abelian statistics, and be described by Pfaffian wavefunctions (paired states) [4, 5] as constructed by Read and Moore. Numerical calculations (exact diagonalizations) lend strong support that the Pfaffian states do describe the $\nu = 5/2$ FQH state. [4, 5]. The excited state with $2N$ far-apart quasiholes is $2^{N-1}$-fold degenerate [3], and the degeneracy should be accurate to exponential order in the characteristic separation between pairs of quasiparticles. Hence, by keeping the quasiparticles sufficiently apart, the environment is ineffective in damaging a given quantum superposition in the $2^{N-1}$-dim Hilbert space. Computation with non-Abelian anyons is achieved by braiding anyons around each other, which leads to unitary operations in the degenerate $2^{N-1}$-dim Hilbert space $[3]$. In other words, unitary operators in the degenerate Hilbert space are associated to the elements of the braid group acting on the $2N$ particles.

Certainly, the first step in order to actually realize such quantum computing scheme is to find a candidate physical system that realizes non-Abelian statistics. Recently, there have been a few concrete proposals for experimental probes to investigate the possible manifestations of non-Abelian statistics in the $\nu = 5/2$ FQH state. One of them explores noise correlations [6], and four of them [7] contain information on the quasiparticle statistics, which are manifest both in the voltage drops and their noise correlations.

In this paper we propose a simple geometry for trapping quasiparticles and probing non-Abelian statistics through voltage measurements in quantum Hall systems. The assembly requires manipulating a single chiral edge of the Hall bar with a single point-contact, a simplification with respect to the geometries of Refs. [11, 12, 13, 14, 15]. Non-Abelian statistics can be probed either by measuring voltage drops at different locations along the edge of the Hall bar, or by measuring noise correlations between the voltages at these different points.
While the former is certainly an easier measurement than the latter, noise measurements of similar complexity have been carried out almost a decade ago and successfully provided evidence for fractional charge in quantum Hall systems \[15\,16\].

The geometry we propose is depicted in Fig. 2 and requires a single point contact that brings into close vicinity two points labeled by \(A_-\) and \(A_+\), separated by a distance \(2a\) measured along the chiral edge path. The tunneling path shortens the travel between points \(A_-\) and \(A_+\), as compared to the length measured along the edge, acting as a “wormhole” passage that allows edge quasiparticles to jump over a length \(2a\) along the edge. The edge segment of length \(2a\) between \(A_-\) and \(A_+\), together with the tunneling path between these two points, encircles a droplet of bulk FQH liquid. Within this region, quasiparticles can be trapped. If the statistics of the quasiparticles are non-Abelian, and their number inside the bounded region is odd, “half” of a topologically protected qubit is entrapped in the island. The qubit corresponds, if formulated as in Refs. \[20, \,21\], to the complex fermion (occupation states \(|0\rangle\) and \(|1\rangle\)) on the string connecting one quasiparticle inside the region and its long-distance partner somewhere outside the island.

To detect whether a topologically protected qubit resides inside the trap, one can measure voltages along the edge. As shown in Fig. 2, one electrode should be placed before the island (lead 1), one touching the perimeter of the island (lead 2), and one past the island (lead 3). We focus on the voltage differences \(V_{12} = V_1 - V_2\), \(V_{13} = V_1 - V_3\), and \(V_{23} = V_2 - V_3\). (One can simply use leads 1 and 2 instead, but as we discuss below, the third lead allows for redundancy and checks.) The voltage drop \(\langle V_{12}\rangle\), we will show, should be sensitive to the parity of quasiparticles inside the island, and be a signature of their non-Abelian statistics. As opposed to the geometries proposed in Refs. \[11\,12\,13\,14\,16\,17\], there is an absolute baseline for the discrimination of non-Abelions: there should generically be a voltage drop when tunneling is allowed, unless the statistics is non-Abelian and there is an odd number of quasiparticles in the island, in which case \(\langle V_{12}\rangle = 0\). Notice that \(\langle V_{13}\rangle = 0\) should always be the case, which can be seen as a consequence of zero longitudinal resistance in the FQH effect, or conservation of charge of the chiral quasiparticles scattered at the junction. Noise fluctuations of \(V_{12}(t)\), we show, can also tell about the statistics of the quasiparticles inside.

It is instructive to consider the following argument before going to the voltage and noise calculation using the edge theory. The leading processes that interfere to give rise to a voltage differential between leads 1 and 2 are shown in Fig. 2. For an incoming many-body scattering state \(|\Psi\rangle\),

\[
\langle V_{12}\rangle \propto \left|\langle 12 | (\mathbb{1} - i\Gamma^* U^1) | \Psi\rangle \right|^2 - \left|\langle \Psi | U | \Psi\rangle \right|^2 = -2 \text{ Im}(\Gamma \langle \Psi | U | \Psi\rangle),
\]

and the interference term \(\langle \Psi | U | \Psi\rangle = 0\) when an odd number of quasiparticles are in the island because the crossing of the tunneling quasiparticle with the string (a unitary operation \(U\) flips the qubit state with respect to the initial state, so that \(U|\Psi\rangle\) is orthogonal to \(|\Psi\rangle\).

The edge theory of \(c = 5/2\) has a charge sector, corresponding to a free boson theory, and a neutral sector which is one of the chiral components of the critical theory of the 2D Ising model with \(c = 1/2\) \[11\,22\]. The former part is characterized by the chiral Luttinger liquid with Lagrangian density,

\[
L_0 = -\frac{1}{4\pi} \partial_x \phi(\partial_t + v \partial_y) \phi, \tag{2}
\]

where \(v\) is the velocity of the (right moving) edge excitations. The boson fields satisfy the commutation relation, \([\phi(t, x), \phi(t, y)] = i\pi \text{ sgn}(x - y)\). Ignoring the filled Landau levels and focusing on the half filled one, we will work with \(\tilde{v} = 1/2\) hereafter. Working in units such that \(v = 1\), the density and current operators are both given by \(\rho(t, x) = \frac{\tilde{v}}{2\pi} \partial_x \phi(t, x)\) and \(J(t, x) = \partial_t \phi(t, x)\). For the geometry shown in Fig. 2, the tunneling operators across the point contact correspond to the annihilation of a quasiparticle at \(x = -a\) and creation at \(a\), or vice versa. Thus, the Lagrangian for the tunneling perturbation is

\[
L_t = -\Gamma \sigma(t, a) \sigma(t, -a) e^{-i \sqrt{\pi} \phi(t, a)} e^{i \sqrt{\pi} \phi(t, -a)} + \text{h.c.} \tag{3}
\]

Hopping of quasiparticles with higher charge is less relevant. The vertex operators constructed from the charge mode field \(\phi\) have opposite vertex charges, and will correspond to the insertion of dipoles in the perturbative Coulomb gas expansion.

The quantities that we will focus on are the densities at the three probe locations at leads \(i = 1, 2, 3:\) \(\rho(t, x_i)\). A few comments are in order. First, the current and density (which are essentially the same up to a velocity factor, which we set to unit) are proportional to the voltage measured at the leads, \(v [\rho(t, x_i) - \rho(t, x_j)] = \tilde{v} \frac{e^2}{h} V_{ij}(t)\). For example, a drop or step up in the voltage \(V_{12}\) corresponds to a current that passes through the “wormhole”.

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**FIG. 2:** Processes up to order \(\mathcal{O}(\Gamma)\) that interfere and lead to a detectable voltage difference \(V_{12}\).
If it were not for the point contact and the particular overhang geometry, no difference in voltage should occur, according to a vanishing longitudinal Hall conductance. Second, seen as a many body scattering problem, the incoming and outgoing chiral states do not simply differ by a phase, and we thus use the Schwinger-Keldysh nonequilibrium formalism. Some of the results to first order are not dependent on using this approach, but other higher order results that we discuss below do require it.

Within the interaction picture, one defines the scattering operator as $S(\infty, -\infty) \equiv \exp[i \oint ds L_n]$, where the integral is done over a closed-time path $2\pi$. The expectation value of arbitrary operators, ordered along the Keldysh contour, are expressed as

$$\langle T_c [O_1 \cdots O_n] \rangle = \sum_{n=0}^{\infty} \langle \Psi | T_c \left( \frac{\oint ds L_c}{n!} O_1 \cdots O_n \right) | \Psi \rangle$$

(4)

where $| \Psi \rangle$ is the initial state of the system.

Instead of calculating the density expectation value directly, we can insert a test charge and compute the expectation of the vertex operator $\langle e^{i 2 \pi / a} \phi(t, x) \rangle$ (constructed using only the charge mode $\phi$) in perturbation to the desired order and use the following identity

$$\langle \rho(t, x) \rangle = -i \sqrt{\frac{\sigma}{2\pi}} \frac{\partial}{\partial \lambda} \langle e^{i 2 \pi / a} \phi(t, x) \rangle \bigg|_{\lambda=0}.$$  

(5)

Each order of the perturbation expansion of the test field will contain the integral over correlation function and has the general form up to some factors as

$$\oint ds \prod_{j=1}^{m} dt_j \prod_{i=1}^{n} \langle O_i(s_i) A_i(t_j) \rangle = e^{-\sum_{i>j} \chi_{ij} \langle T \phi(t_i, x_i) \phi(t_j, x_j) \rangle},$$

(6)

and the more involved multi-point contour-ordered correlations of the twist fields $\sigma$.

The lowest non-vanishing contribution to the density/current appears at the first order:

$$\langle \rho(t, x) \rangle = \frac{\gamma^* - \gamma}{2i} \sqrt{\nu} \left[ \text{sgn}(x + a) - \text{sgn}(x - a) \right],$$

(8)

where $\gamma$ and $\gamma^*$ include the self-interaction of the $\phi$-vertex dipole insertion, and the correlation of the twist fields $\sigma$ that encodes the information on the non-Abelian statistics of the tunneling quasiparticles:

$$\gamma \equiv \langle \Psi | \sigma(s, a) \sigma(s, -a) | \Psi \rangle e^{g \langle T \phi(t, x) \phi(t, x) \rangle}.$$  

(9)

The equal-time correlation function of boson field gives a constant that can be simply absorbed into $\Gamma$. The matrix element $M = \langle \Psi | \sigma(s, a) \sigma(s, -a) | \Psi \rangle$ of the two twist fields at points $A_-$ and $A_+$ with respect to the quantum state $| \Psi \rangle$ does capture information on the bulk FQH state encircled by the edge perimeter path and the tunneling bridged path between points $A_-$ and $A_+$. The matrix element can be calculated using the operator product expansion fusion rules of the $\sigma$ fields, and has been carried out in this way in Ref. [14] for the $\nu = 5/2$ state (and extended in similar ways to the $\nu = 12/5$ state in Refs. [17]). $M$ vanishes if the number of quasiparticles inside the island is odd, while it is non-zero if the number is even.

It follows from Eq. (8) using $x_1 < -a$ and $x_3 > a$ that

$$\langle \rho(t, x_1) \rangle = \langle \rho(t, x_3) \rangle = 0$$

(we just compute the fluctuations, and subtract a constant term, the zeroth order term, which in any case cancels out when we calculate density – and voltage – differences). Moreover, this result can be shown to hold order by order in $\Gamma, \Gamma^*$, and it is simple to understand why physically. The incoming chiral edge current, at lead 1, cannot be affected by tunneling because of causality, and the outgoing current, at lead 3, must equal the incoming one by charge conservation. Notice that this result imply that $\langle V_{13} \rangle = 0$, which is consistent with a vanishing longitudinal Hall resistance.

However, if one probe is located along the island perimeter at $-a < x < a$, one measures

$$\langle V_{12}(t) \rangle \propto \langle \rho(t, x_2) - \rho(t, x_1) \rangle = -2 \frac{\text{Im}(\gamma)}{\sqrt{\nu}}$$

(10)

because of tunneling currents through the junction that are responsible for local deviation from the vanishing longitudinal Hall resistance. This is the simplest measurement that can probe whether the bulk quasiparticles trapped within the inside of the island have non-Abelian statistics. Basically, there should be a detectable voltage difference $\langle V_{12} \rangle$ whenever the number of quasiparticles trapped inside the island is even, while there should be none if the number is odd $\text{Im}(\gamma) = 0$ as $M = 0$.

Let us now turn to the behavior of noise correlations, and define

$$S_{ij}(\omega) = S_{ji}(\omega) = \int_{-\infty}^{\infty} e^{i \omega t} \langle \{ \rho(t, x_1), \rho(0, x_j) \} \rangle,$$

(11)

which can be computed using contour ordered perturbation theory once we express the expectation value of the anti-commutator as

$$\langle \{ \rho(t, x_1), \rho(0, x_j) \} \rangle = \sum_{\mu} \langle T \rho_\mu (t, x_1) \rho_{-\mu} (0, x_j) \rangle,$$

(12)

where $\mu = \pm$ indicate insertions at the top or bottom branch of the contour, respectively. From the $S_{ij}(\omega)$, we can obtain the noise auto-correlations of the quantities $\delta \rho_{ij}(t) \equiv \rho(t, x_1) - \rho(t, x_j)$ via

$$S_{ij}(\omega) = S_{ii}(\omega) + S_{jj}(\omega) - S_{ij}(\omega) - S_{ji}(\omega),$$

(12)
which are the quantities that are detected if the noise auto-correlation of the voltage differences $V_{ij}(t)$ are measured.

Carrying out this program, we find to zeroth and first order in the tunneling amplitude the following:

$$S_{12}^{(0)}(\omega) = \frac{\tilde{V}}{\pi |\omega|} \sin^2 \left[ \frac{[\omega(x_2 - x_1)]}{2} \right]$$

$$S_{12}^{(1)}(\omega) = 8g\tilde{\nu} \gamma + \gamma^* \left\{ \sin [|\omega|(x_2 - x_1)] \sin^2(\omega a) + \sin^2 \left[ \frac{[\omega(x_2 - x_1)]}{2} \right] \sin(2|\omega|a) \right\}. \quad (13)$$

A simplification occurs if the position of probe 1 is dithered, or else the edge path length is modulated, or the distance $|x_1 - x_2|$ is large enough for dephasing to occur (here we actually profit from loss of coherence). In this case, as long as the observation frequencies are large compared to the inverse of the time of flight through the length $|x_1 - x_2|$, one can average over $x_1 - x_2$ and obtain

$$\overline{S}_{12}^{(0)}(\omega) = \frac{\tilde{V}}{\pi |\omega|}$$

$$\overline{S}_{12}^{(1)}(\omega) = 4g\tilde{\nu} \operatorname{Re}(\gamma) \sin(2|\omega|a). \quad (15)$$

It is noteworthy that $\overline{S}_{12}^{(1)}(\omega) = S_{12}^{(1)}(\omega)$, i.e., the excess noise comes all from the auto-correlation measured at lead 2.

Notice that the excess noise $\overline{S}_{12}^{(1)}(\omega)$ in the voltage difference $V_{12}$ is proportional to $\operatorname{Re}(\gamma)$, and hence depends on whether the number of quasiparticles within the island is even or odd. Moreover, in this geometry, the voltage difference measurement and the noise are not proportional to each other; instead they are in quadrature, one proportional to $\operatorname{Re}(\gamma)$ and the other to $\operatorname{Im}(\gamma)$. If both are measured, (un)fortuitous cancellations of one due to destructive interference will be accompanied by a maximum in the other. Hence, no signal can be detected for both measurements only if $\gamma = 0$. This scheme provides a redundant check and a strong constraint on the even/odd detection recipe.

Let us turn the discussion to what should happen in the regime of strong coupling. When tunneling is strong, the incoming edge current should pass mostly through the “wormhole”, leaving an isolated island aside. Within this isolated FQH puddle an integer number of electrons must reside, and thus only an even number of quasiparticles is permitted (actually, a multiple of 4, since $e^* = e/4$). Therefore, if a topological qubit was realized for an odd number of quasiparticles in the island at weak coupling, it should be “screened” at strong coupling. We speculate that this screening takes place as in the Kondo effect at strong coupling, where the magnetic impurity is screened by the conduction electrons. The two-state system corresponding to the qubit can be represented as a complex fermion shared by a quasiparticle in the island and another outside, and as the tunneling amplitude gets larger, the string connecting this pair gets crossed more and more frequently, and the qubit is rapidly flipped, and should “hybridize” with the edge quasiparticles and be screened. The connection between the trapped topological qubit and the Kondo effect seems appealing, but in order to substantiate this scenario, one must learn how to deal with the non-Abelian phase factors due to the products of a number of twist operators $\sigma$, which is certainly a non-obvious problem, and one that deserves further investigation.

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