A New Time Contour for Equilibrium Real-Time Thermal Field Theories

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Abstract

A new time contour is used to derive real-time thermal field theory. Unlike previous path integral approaches, no contributions to the generating functional are dropped.

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In the path integral approach to equilibrium thermal field theories, one is calculating thermal expectation values, typically

\[ \Gamma_C(\tau_1, \tau_2, ..., \tau_N) = \frac{\text{Tr}\{e^{\Phi - \beta H}T_C\phi_1(\tau_1)\phi_2(\tau_2)...\phi_N(\tau_N)\}}{\text{Tr}\{e^{\Phi - \beta H}\}}. \]  

Here \( T_c \) indicates that the fields are path-ordered with respect to the order along a path, \( C \), of their complex time arguments \( \{\tau\} \) with appropriate sign changes when fermionic fields are involved. Throughout we shall suppress spinor and other indices as well as dependence on spatial variables where necessary. The results are not altered by their inclusion. The \( e^{\Phi - \beta H} \) factor is used to shift the bra states in the trace by \(-i\beta\) in time so that all times involved lie on a path, \( C \), which must run in the complex time plane so that it ends \(-i\beta\) below its starting point.

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Another limitation on $C$ that is often imposed comes from studying the behaviour of the Green functions. Inserting complete sets of energy eigenstates into the thermal Wightman function, say $\Gamma_{a_1...a_N}$, where the $N$ fields are in a fixed order, we have

$$
\Gamma_{a_1...a_N}(\tau_1, ..., \tau_N)
= Tr\{e^{\Phi - \beta H} \phi_{a_1}(\tau_{a_1})\phi_{a_2}(\tau_{a_2})...\phi_{a_N}(\tau_{a_N})\}/Tr\{e^{\Phi - \beta H}\}.
$$

$$
= (-1)^Np \sum_{m_1, m_2, ..., m_N} [exp\{-iE_{m_1}(b - \tau_{a_1} + \tau_{a_N})\}]

\left(\prod_{j=2}^{N} \Phi_{N_j=2} exp\{-iE_{m_j}(\tau_0 - \tau_{a_j} + \tau_{a_{j-1}})\}\right)
\left(\prod_{j=1}^{N} \Phi_{N_j=1} \langle m_j | \phi_{a_j}(0) | m_{j+1} \rangle\right)/Tr\{e^{\Phi - \beta H}\}.
$$

(2)

The $\{|m_j\}$ are a complete set of energy eigenstates for temperature $\beta\Phi - 1 (m_{1+N} = m_1)$. We assume that the energy sums in (2) are uniformly convergent when the exponentials in (2) are of the form $exp\{z_jE_j\}$ with $Re\{z_j\} < 0$. The thermal Wightman functions, $\Gamma_{a_1a_2...a_3}(\{\tau\})$, are therefore bounded when the complex time arguments $\tau_j$ satisfy $\{\tau\} \in A_{a_1a_2...a_N}$ where

$$
A_{a_1a_2...a_N} := \{\tau_{a_1}, \tau_{a_2}, ..., \tau_{a_N} | Im(\tau_{a_N}) \geq Im(\tau_{a_{N-1}}) \geq ... \geq Im(\tau_{a_1}) \geq Im(\tau_{a_N} - i\beta)\}.
$$

(3)

This then implies that the thermal Wightman function is also analytic in this region $[4]$. The path-ordered Green functions (2) are merely a combination of products of these thermal Wightman functions (2) with path ordering theta functions. In order to ensure that the path ordered Green function $\Gamma_c(\{\tau\})$ is also analytic away from hypersurfaces of equal time arguments, one finds that one must ensure that the time curve $C$ always has a decreasing imaginary part as one moves along $C [4]$. This is a further condition that one imposes on the curve $C$ (though a constant real part may be acceptable if one is prepared to accept generalised rather than analytic functions, see section 2.1 in [2]).

Given these two limitations on $C$ one is then free to choose the time curve to suit the problem at hand. The standard choice is to choose one running parallel to the imaginary axis, $C = C_I$ of fig.(4). This gives the ITF (Imaginary Time-Formalism) or Matsubara method. Because this has no real times present, one must make an analytic continuation to real times of the Green functions calculated if one wishes to obtain dynamical information. This is a highly non-trivial because the analytic structure at finite temperature is much more complicated than at zero temperature.

To avoid this problem RTF (Real-Time Formalisms) were developed. The path integral approach was highlighted by Niemi and Semenoff [4] though RTF had also been considered earlier (see [4] for a review). In these approaches one chooses the curve in the complex time plane to be $C = C_R$ where $C_R = C_1 \oplus C_2 \oplus C_3 \oplus C_4$ runs from $-T + \tau_0$ to $+T + \tau_0 (C_1)$, $+T + \tau_0$ to $+T + \tau_0 - i\alpha \beta$ (C2), $+T + \tau_0 - i\alpha \beta$ to $-T + \tau_0 - i\alpha \beta$ (C3), and $-T + \tau_0 - i\alpha \beta$ to $-T + \tau_0 - i\beta$ (C4) [4] [4] [4] [4]. This is shown in Figure 4. The parameters $\alpha = 1 - \bar{\alpha}$ and $\tau_0$ is arbitrary and reflect some of the freedom of choice in the
path $C$. Physical results do not therefore depend on them. The limit of $T \to \infty$ is taken to produce the standard RTF.

The drawback of this approach is that one has to show that the vertical sections $C_3, C_4$ are irrelevant \[1, 2, 5\]. Otherwise one would have an unwieldy combination with few similarities to zero temperature Minkowskii space field theory, and one would not see the close relationship between this path integral method and the Thermo Field Dynamics and $C\Phi$*-algebra approaches to RTF (see \[2\] for a review). To see that $C_3, C_4$ can be ignored, one has to take the $T \to \infty$ limit and then demand that the sources, $j(\tau)$, are zero at $\Re \{\tau\} \to \infty$ in manner reminiscent of the usual asymptotic conditions used in zero temperature field theory.

Dropping these contributions is disquieting to many. The use of an asymptotic condition at infinite times in the past and future must be some sort of approximation in thermal field theories. Any real-time Green function is telling us about dynamics and therefore some sort of deviation from equilibrium. To use equilibrium theories in such problems, we are assuming that deviation from equilibrium is ‘small’ and can be ignored on ‘short’ time scales. After a finite time one must face the fact that the system is returning to equilibrium. This can not be done within equilibrium thermal theories. Thus one should not really be relying on what the theory is doing on infinite time scales.

In practice, calculations of real-time Green functions in RTF and ITF seem to be in complete agreement when we compare the same type of Green functions (the same diagram usually corresponds to different Green functions in different formalisms \[5, 6, 7\]). The one area that still appears to be a problem is when one is interested in static (zero energy) Green functions such as appear in free energy calculations. In RTF, some of the relevant diagrams are infinite due to the appearance of singularities of the form $[\delta(k \Phi^2 - m \Phi^2)] \Phi \geq 2 \ [8]$ (pace \[2, 9\]). Extra rules were shown to be needed for RTF calculations in these situations \[10, 11\]. In the path integral derivation of these rules \[11\] the $C_3, C_4$ sections did contribute when static sources were involved. Only by using a simple trick leading to an extra Feynman rule could one use the standard RTF method.

It would therefore be advantageous to find a path integral approach to RTF that did not involve dropping any parts yet gave the standard RTF Feynman rules. The starting point is to note that it is common to think of the $C_1, C_2$ of the standard RTF curve $C_R$ of fig.\((1)\) as having infinitesimal slopes downwards so that the path ordered Green functions defined on these curves are analytic. The difference in the imaginary part of the ends of the $C_1, C_2$ curves is also infinitesimal. This latter property is unnecessary for the Green functions to be well behaved and it is this condition which we drop. Thus we might try the new curve $C_N = C_{N_1} \oplus C_{N_2}$ shown in fig.\((2)\). This curve satisfies all the requirements discussed above. It is convenient to parameterise the curve using a real variable, $t$, and a thermal label, $a = 1, 2$. We define a complex valued doublet, $\tau \Phi a$, so that

$$\tau \Phi a(t) = \begin{cases} g\Phi 1 t + \tau_0 & \in C_{N_1} \\ g\Phi 2 t - i\bar{\alpha} \beta + \tau_0 & \in C_{N_2} \end{cases} \quad \text{if } a = 1$$

$$\text{if } a = 2 \tag{4}$$
The constants \( g\Phi_1, g\Phi_2 \) are

\[
g\Phi_1 = 1 - \frac{i\gamma\beta}{\alpha T}; \quad g\Phi_2 = -1 - \frac{i(\gamma - \bar{\alpha})\beta}{\alpha T}.
\]  

(5)

Thus \( C_N \) runs from \( \tau\Phi_1(-\alpha T) = -\alpha T + i\gamma\beta + \tau_0 \) to \( \tau\Phi_1(\bar{\alpha}T) = \bar{\alpha}T + i\gamma\bar{\alpha}/\alpha \) \((C_{N1})\) and then from \( \tau\Phi_2(\bar{\alpha}T) = \bar{\alpha}T + i\gamma\bar{\alpha}/\alpha \) to \( \tau\Phi_2(-\alpha T) = -\alpha T + i\gamma\beta + \tau_0 \) \((C_{N2})\). The \( \tau_0 \) is an arbitrary complex time and it is in the region of \( \tau_0 \) that we wish to investigate the Green functions. The ends of the curve must be separated by \(-i\beta\) so that we have \( \gamma + \bar{\gamma} = 1 \), though the Feynman rules will turn out to be completely independent of this \( \gamma \) parameter. We have also defined \( \alpha + \bar{\alpha} = 1 \) which will turn out to play the same role for \( C_N \) as they did for \( C_R \).

The constant \( T \) is a time and one takes the limit \( T \to \infty \) so that the slope of the two sections \( C_N^1, C_N^2 \) becomes infinitesimal as required. Also in this limit, \( C_N \) and \( C_R \) are then essentially identical in the region \( |\tau - \tau_0| \ll T \). The limit of \( T \to \infty \) will therefore generate the usual RTF method and yet no part of the contour has been dropped. There can be no suspicion that RTF contains anything less than any other formalism such as ITF where a different curve is employed.

It is enlightening to check this new approach to RTF by studying the simple example of a real relativistic self-interacting field theory with Lagrangian

\[
\mathcal{L} = \frac{1}{2} \left( \partial \Phi \mu \phi \right) \Phi^2 + \frac{1}{2} m\Phi^2 \phi \Phi^2 - V[\phi].
\]

The generating functional can be written as

\[
Z[j] = \exp\{V[-i\partial_j]\} \cdot Z_0[j]
\]

\[
Z_0[j] = \exp\{-\frac{i}{2} \int_C d\tau d\tau' \Phi j(\tau)\Delta_c(\tau - \tau')j(\tau')\}
\]

where

\[
- (\Box_c + m\Phi^2)\Delta_c(\tau - \tau') = \delta_c(\tau - \tau').
\]  

(6)

The \( \Box_c \) contains time derivatives taken along \( C \). Also \( \theta_c(\tau - \tau') = 1 \) if \( \tau \) is further along \( C \) than \( \tau\Phi \) otherwise \( \theta_c(\tau - \tau') = 0 \). \( \frac{d}{d\tau}\theta_c(\tau - \tau') = \delta_c(\tau - \tau') \) and \( \int_C \delta_c(\tau - \tau') = 1 \) defines the delta function (c.f. [1, 2, 3]).

Using the KMS condition and the equal time commutation relation \([\phi(\tau, \vec{x}), \phi(\tau, \vec{y})] = 0\) as boundary conditions the solution of (6) is

\[
\Delta_c(\tau, \omega) = -\frac{i}{2\omega} \frac{1}{1 - e^{\Phi - \beta}\omega} \{[e\Phi - i\omega\tau + e\Phi - \beta\omega\Phi + i\omega\tau]\theta_c(\tau) + [e\Phi + i\omega\tau + e\Phi - \beta\omega\Phi - i\omega\tau]\theta_c(\tau)^-\}
\]

(7)

where \( \omega = |(k\Phi^2 + m\Phi^2)\Phi^2| \) and we have taken the Fourier transform with respect to the spatial variables.
Now consider $C = C_N$ and define doublet fields and sources with real time arguments \( \{ t \} \) in terms of the original fields and sources, whose time arguments lie on $C_N$, through
\[
\phi \Phi_a(t) = \phi(\tau \Phi_a(t)); \quad j \Phi_a(t) = g \Phi_a j(\tau \Phi_a(t)).
\] (8)
The $a = 1, 2$ is the thermal label, and the $\tau \Phi_a$ is the parameterisation of the new curve given in (4). The generating functional can then be written in the usual final RTF form
\[
Z[j] = \exp \left\{ \sum_{a=1,2} \int_{-\alpha T}^{\alpha T} \Phi \bar{\Phi} \Phi_a V[\phi(\tau \Phi_a)] \right\} Z_0[j]
\]
\[
Z_0[j] = \exp \left\{ -\frac{i}{2} \sum_{a,b=1,2} \int_{-\alpha T}^{\alpha T} \phi \bar{\Phi} \Phi_a \Phi_b \Delta \Phi_a(t-t\Phi_j) \phi \bar{\Phi} \Phi_b(t) \right\}
\]
but where in this case no term has been dropped or absorbed into the normalisation. The two-by-two matrix propagator is defined to be
\[
\Delta \Phi_{ab}(t-t\Phi_j) = \Delta \phi(\tau \Phi_a(t)-\tau \Phi_b(t))
\]
The Feynman rules using these definitions lead to a factor of $g \Phi_a$ (as defined in (5)) associated with each vertex. In the $T \to \infty$ limit this leads to the usual additional factor of $-1$ appearing in type two vertices as compared with the type one vertices. The definition of the doublet fields and sources in (8) is fairly arbitrary and other useful definitions can be used which lead to variations in the Feynman rules (12).

When the limit $T \to \infty$ is taken the doublet fields are then
\[
\lim_{T \to \infty} \phi \Phi_1(t) = \phi(t)
\]
\[
\lim_{T \to \infty} \phi \Phi_2(t) = \phi(t - \bar{\alpha} \beta)
\]
where without loss of generalisation we have chosen $\tau_0=0$. Thus we are indeed calculating the same Green functions as in standard RTF, for example
\[
\frac{\partial \Phi N Z[j]}{\partial j \Phi_1(t_1) ... j \Phi_1(t_N)} = Tr \{ e^{\Phi - \beta HT} \phi_1(\tau_1) \phi_2(\tau_2) ... \phi_N(\tau_N) \} / Tr \{ e^{\Phi - \beta H} \}
\]
which is the connected time-ordered Green function of physical real-time fields.

Calculating the Fourier transform of the propagator in this limit requires care with the various infinitesimal quantities. This is because
\[
\lim_{T \to \infty} \int_{-T}^{T} \phi \Phi_1(t) \quad \phi \Phi_1(t) \quad = \quad \frac{1}{p-\omega+\bar{\epsilon}+\bar{\eta}} [1 - \lim_{T \to \infty} (e^{\Phi 1(p-\omega) T e^{\Phi} - (\epsilon + \omega \eta) T})].
\] (9)
where $p$ and $\omega$ are real and $\epsilon$ and $\eta$ are real and infinitesimal. Consider the calculation of $\Delta \Phi 1$. Then $\eta = (1 - g \Phi 1)/\bar{\nu}$ is positive and comes from the infinitesimal gradient given
to the otherwise horizontal $C_{N1}$. As usual if only particle solutions are propagating at large times, this infinitesimal slope to the time path plays the same role as the usual $\epsilon$ shift in the energy used to regulate the integral. However at non-zero temperatures we have both particle and anti particle solutions, and this means we also have terms where $\omega \to -\omega$ in (9). The slope of the time curve is of course the same so now we require that $\epsilon > \eta$ for a finite result. The same behaviour is found when looking at the particle contributions to the $\theta(-t)$ part of the propagator and so on for all parts of the matrix RTF propagator. In general we find that the infinitesimal slope given to time curves at non-zero temperature is not sufficient to guarantee convergence of the Fourier transform of the free propagator when $T \to \infty$.

The problem is easily avoided. We are still free to give the time curves infinitesimal slopes but we must also have the usual $\epsilon$ shift in the energies. We must then take the slopes to zero before we take the $\epsilon \to 0$ limit. In terms of the RTF curves (both for this new approach and for the old style curves) we must take the $T \to \infty$ limit before the $\epsilon \to 0$ limit.

With this proviso we find that the usual RTF propagators in momentum space are reproduced. In this example where the definitions have been chosen to give a factor of $-1$ appearing in the type two vertices over the type one, we find that

\[
\begin{align*}
\imath\Delta \Phi_{11}(k_0, \omega) &= \frac{\imath}{k_0\Phi^2 - \omega\Phi^2 + \imath\epsilon} + \frac{1}{e\Phi\beta\omega - 1}2\pi\delta(k_0\Phi^2 - \omega\Phi^2), \\
\imath\Delta \Phi_{12}(k_0, \omega) &= \frac{e\Phi\beta\omega/2}{e\Phi\beta\omega - 1}e\Phi(\bar{\alpha} - \frac{1}{2})\beta k_0/2 2\pi\delta(k_0\Phi^2 - \omega\Phi^2), \\
\imath\Delta \Phi_{21}(k_0, \omega) &= \frac{e\Phi\beta\omega/2}{e\Phi\beta\omega - 1}e\Phi - (\bar{\alpha} - \frac{1}{2})\beta k_0/2 2\pi\delta(k_0\Phi^2 - \omega\Phi^2), \\
\imath\Delta \Phi_{22}(k_0, \omega) &= \frac{-\imath}{k_0\Phi^2 - \omega\Phi^2 - \imath\epsilon} + \frac{1}{e\Phi\beta\omega - 1}2\pi\delta(k_0\Phi^2 - \omega\Phi^2).
\end{align*}
\]

The physical origin of these problems is that $T$ represents the time scale over which the system starts to come into equilibrium. In looking at dynamical information in the form of real-time Green functions, we have the inherent contradiction that we must be pushing the system out of equilibrium yet we assume the system is in equilibrium for all time. This is a good approximation on time scales less than $T$ but is bad otherwise. Mathematical consistency therefore demands that this characteristic time scale is taken to infinity in such equilibrium schemes. It is not surprising that the $T \to \infty$ limit must be taken first.

This also explains why we should not be surprised if when using RTF to look at zero energy Green functions we have to work a little harder as these involve averages over all times including those of order $T$. This could be seen in the extra rule found to be necessary [10, 11]. When using the standard RTF time curve the vertical sections were shown to be giving all the contribution to the generating functional [11] but after some manipulations, it was shown the standard RTF approach could be used if one extra
Feynman rule was added. For the new time curve the formal manipulations of [11] work exactly as before. The ends of the curves are significant and so the infinitesimal slope can not be ignored. The same trick as before allows one to take account of this when using the new curve by introducing the same extra rule. Once again the new time curve completely reproduces the standard RTF method without dropping any contributions. This gives us further confidence that ITF and RTF contain the same physical information.

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Figure 1: The time paths used for RTF and ITF.
Figure 2: The new time path for RTF.