Density-metric unimodular gravity: vacuum spherical symmetry

Amir H Abbassi¹ and Amir M Abbassi²

¹ Department of Physics, School of Sciences, Tarbiat Modares University, PO Box 14155-4838, Tehran, Iran
² Department of Physics, University of Tehran, PO Box 14155-6455, Tehran, Iran

E-mail: ahabbasi@modares.ac.ir and amabasi@khayam.ut.ac.ir

Received 22 February 2008, in final form 17 June 2008
Published 19 August 2008
Online at stacks.iop.org/CQG/25/175018

Abstract
We analyze an alternative theory of gravity characterized by metrics that are tensor density of rank (0, 2) and weight $-\frac{1}{2}$. The metric compatibility condition is supposed to hold. The simplest expression for the action of gravitational field is used. Taking the metric and trace of connections as dynamical variables, the field equations in the absence of matter and other kinds of sources are derived. The solutions of these equations are obtained for the case of vacuum static spherical symmetric spacetime. The null geodesics and advance of perihelion of ellipses are discussed. We confirm a subclass of solutions is regular for $r > 0$ and there is no event horizon while it is singular at $r = 0$.

PACS numbers: 04.20.Cv, 04.20.Fy, 04.90.+e

1. Introduction
Unimodular theory of relativity was an alternative theory of gravity considered by Einstein in 1919. It turned out that it is equivalent to general relativity with the cosmological constant appearing as an integration constant. In unimodular theory and its modified versions the determinant of the metric does not serve as a dynamical variable and this imposes a specific constraint on the variations of the metric [1–6]. The reason for this very feature of the theory is obscure. It is being considered as the substantial essence of the system and is often not invoked by any decisive evidence or physical interpretation. It will be more appropriate if this feature be the emergent of the theory and attributes a natural property of it. This paper is an attempt to fulfil this task. What should be noted in advance is that in special relativity all the Jacobians of the propounded transformations are identically equal to one. Therefore all tensors including the metrics may be considered as tensor densities with arbitrary weight. This raises the question that in passing toward general relativity which one should be kept to be dealt with, metric or density metric? Certainly in general relativity where there is not such a
constraint on the Jacobians, consideration of a tensor density as a metric will bring significant change in the obtained equations. As the starting point let us propose that the metric should be considered as a symmetric tensor density of rank \((0, 2)\) and weight \(-\frac{1}{2}\), denoting it by \(g_{\mu\nu}\). This is chosen and accepted so that the determinant of \(g_{\mu\nu}\) becomes a scalar. Without loss of generality this scalar may be normalized to one. We should distinctly emphasize that this proposal for the role of the metric is novel in our approach and has no acquaintance in the literature. The metric compatibility condition is supposed to hold for the new metric too. In short we may summarize

\[
\begin{align*}
g_{\mu\nu}, \text{ symmetric tensor density of weight } & -\frac{1}{2} \quad (1.1) \\
|g| = 1 \quad (1.2) \\
\nabla_{\lambda} g_{\mu\nu} = 0. \quad (1.3) 
\end{align*}
\]

The inverse of the metric \(g^{\mu\nu}\) is a density tensor of rank \((2, 0)\) with weight \(+\frac{1}{2}\) which satisfies the following:

\[
\begin{align*}
g^{\mu\nu} g_{\nu\lambda} &= \delta^{\mu}_{\lambda}, \quad \nabla_{\lambda} g^{\mu\nu} = 0. \quad (1.4) 
\end{align*}
\]

The metric compatibility condition equation \((1.3)\) gives

\[
\partial_{\lambda} g_{\mu\nu} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\mu\rho} + \frac{1}{2} \Gamma^{\rho}_{\mu\rho} g_{\lambda\nu} = 0. \quad (1.5) 
\]

Rewriting equation \((1.5)\) by a cyclic permutation of the indices makes

\[
\begin{align*}
\partial_{\nu} g_{\lambda\mu} - \Gamma^{\rho}_{\nu\lambda} g_{\rho\mu} - \Gamma^{\rho}_{\nu\mu} g_{\lambda\rho} + \frac{1}{2} \Gamma^{\rho}_{\nu\rho} g_{\lambda\mu} &= 0 \quad (1.6) \\
\partial_{\mu} g_{\nu\lambda} - \Gamma^{\rho}_{\mu\nu} g_{\rho\lambda} - \Gamma^{\rho}_{\mu\lambda} g_{\nu\rho} + \frac{1}{2} \Gamma^{\rho}_{\mu\rho} g_{\nu\lambda} &= 0. \quad (1.7) 
\end{align*}
\]

Adding equations \((1.5)\) and \((1.6)\) and subtracting equation \((1.7)\) gives

\[
\partial_{\lambda} g_{\mu\nu} + \partial_{\nu} g_{\lambda\mu} - \partial_{\mu} g_{\lambda\nu} - 2 \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu} + \frac{1}{2} \left( \Gamma^{\rho}_{\nu\rho} g_{\lambda\mu} - \Gamma^{\rho}_{\nu\mu} g_{\lambda\rho} \right) = 0. \quad (1.8) 
\]

Multiplying equation \((1.8)\) by \(g^{\alpha\nu}\) leads to

\[
\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\nu} \left( \partial_{\nu} g_{\mu\nu} + \partial_{\nu} g_{\mu\lambda} - \partial_{\lambda} g_{\mu\nu} + \frac{1}{2} \left( \Gamma^{\rho}_{\nu\rho} g_{\mu\lambda} - \Gamma^{\rho}_{\nu\lambda} g_{\mu\rho} \right) \right). \quad (1.9) 
\]

Equation \((1.9)\) does not fix all the components of the connection in terms of the metric and its derivatives; the components of the trace of connection \(\Gamma^{\rho}_{\mu\lambda}\) remain undetermined. Since the determinant of \(g_{\mu\nu}\) is taken to be one this automatically yields the following constraint on the variations of \(g_{\mu\nu}\):

\[
\begin{align*}
g^{\mu\nu} \delta g_{\mu\nu} = 0, \quad (1.10) 
\end{align*}
\]

which is a required condition for the unimodular relativity. Using the connection \((1.9)\), we may define the Riemann curvature and Ricci tensors respectively by

\[
\begin{align*}
\mathcal{R}^{\rho}_{\lambda\mu\nu} &= \partial_{\lambda} \Gamma^{\rho}_{\nu\mu} - \partial_{\nu} \Gamma^{\rho}_{\lambda\mu} - \Gamma^{\alpha}_{\nu\lambda} \Gamma^{\rho}_{\mu\alpha} - \Gamma^{\alpha}_{\lambda\mu} \Gamma^{\rho}_{\nu\alpha} \quad (1.11) \\
\mathcal{R}_{\rho\lambda} &= \partial_{\lambda} \Gamma^{\rho}_{\rho\nu} - \partial_{\rho} \Gamma^{\rho}_{\lambda\nu} - \Gamma^{\alpha}_{\nu\lambda} \Gamma^{\rho}_{\rho\alpha} - \Gamma^{\alpha}_{\rho\lambda} \Gamma^{\rho}_{\rho\alpha}. \quad (1.12) 
\end{align*}
\]

\(\mathcal{R}^{\rho}_{\lambda\mu\nu}\) and \(\mathcal{R}_{\rho\lambda}\) both are regular tensors, i.e. are tensor densities of weight zero.

Now a scalar density of weight \(+\frac{1}{2}\) may be constructed by using the Ricci tensor \((1.12)\) and inverse metric \(g^{\mu\nu}\),

\[
\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu}. \quad (1.13) 
\]

Since \(d^{4}x\) is a scalar density of weight \(-1\), it is no longer possible to use the determinant of \(g_{\mu\nu}\) to construct a scalar.
2. Action

Let us consider gravitational fields in the absence of matter and other kinds of sources. $\mathcal{R}$ and $d^4x$ both are scalar densities of weights $\frac{1}{2}$ and $-1$ respectively. Therefore $\mathcal{R}^2 d^4x$ is a scalar. We may define the action as

$$I = \int \kappa \mathcal{R}^2 d^4x$$  \hspace{1cm} (2.1)

where $\kappa$ is a proper constant. It is a functional of the metric, its first and second derivatives, and the unspecified trace of the connections $\Gamma_{\rho\lambda}^\mu$ and its first derivatives. It is somehow similar to the Palatini approach where both the metric and the connection are considered as independent dynamical variables. Here our dynamical variables are $g^{\mu\nu}$ and $\Gamma_{\rho\lambda}^\mu$. The variations of the action with respect to the first lead to

$$\delta I = \int 2\kappa \mathcal{R}(\mathcal{R}_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta \mathcal{R}_{\mu\nu}) d^4x = 0$$ \hspace{1cm} (2.2)

while its variations with respect to the second give

$$\delta I = \int 2\kappa g^{\mu\nu}\delta \mathcal{R}_{\mu\nu} d^4x = 0.$$ \hspace{1cm} (2.3)

Generally, we have

$$\delta \mathcal{R}_{\mu\nu} = \nabla_\rho (\delta \Gamma_{\rho\mu\nu}^\sigma) - \nabla_\delta (\delta \Gamma_{\rho\mu\nu}^\delta).$$ \hspace{1cm} (2.4)

The variation of $\Gamma_{\mu\nu}^\rho$ with respect to $\Gamma_{\rho\lambda}^\mu$ is

$$\delta \Gamma_{\mu\nu}^\rho = \frac{1}{4} \left( \delta \Gamma_{\rho\nu}^{\mu} + \delta \Gamma_{\rho\mu}^{\nu} - \delta g_{\mu\nu} \right) \delta \Gamma_{\rho\lambda}^\mu. $$ \hspace{1cm} (2.5)

Inserting equation (2.5) into equation (2.4) and then equation (2.4) into equation (2.3) leads to the following equation:

$$\nabla_\rho \mathcal{R} = 0.$$ \hspace{1cm} (2.6)

The variation of the action with respect to the metric and applying the unimodular condition equation (1.10) by the method of Lagrange undetermined multipliers and inserting field equation (2.6) leads to

$$\mathcal{R}_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \mathcal{R} = 0.$$ \hspace{1cm} (2.7)

It is worth noting that equations (2.6) and (2.7) are consistent with the Bianchi identity. The field equation (2.7) is traceless and since $\mathcal{R}$ is a scalar density so equation (2.6) is not a trivial relation.

In the next we are going to find the analog of Schwarzschild spacetime in this alternative theory, that is find the solution of equations (2.6) and (2.7) for a spherically symmetric spacetime.

3. Spherical symmetry

In the Schwarzschild solution the starting ansatz is that there exists a spherical coordinate system $x^\mu = (t, r, \theta, \phi)$ in which the line element has the form

$$ds^2 = B(r) \, dt^2 - A(r) \, dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).$$  \hspace{1cm} (3.1)
If we define $g_{\mu\nu} = \frac{g_{\mu\nu}}{\sqrt{\gamma}}$ then it will have the desired property. So the components of the $g_{\mu\nu}$ are

\[
\begin{align*}
    g_{rr} &= -\frac{B^2}{A^2 \sin^2 \theta}, &
    g_{\theta\theta} &= \frac{A^2}{B^2 \sin^2 \theta}, &
    g_{\phi\phi} &= \frac{A^2}{B^2 \sin^2 \theta}, &
    g_{\theta\phi} &= \frac{B^2}{A^2 \sin^2 \theta}, \\
    g_{\theta r} &= \frac{r}{(AB)^2 \sin^2 \theta}, &
    g_{\phi r} &= \frac{r \sin^2 \theta}{(AB)^2},
\end{align*}
\]

and the components of the inverse metric $g^{\mu\nu}$ are

\[
\begin{align*}
    g^{rr} &= -\frac{B^2}{A^2 \sin^2 \theta}, &
    g^{\theta\theta} &= \frac{A^2}{B^2 \sin^2 \theta}, &
    g^{\phi\phi} &= \frac{A^2}{B^2 \sin^2 \theta}, &
    g^{\theta\phi} &= \frac{B^2}{A^2 \sin^2 \theta}, \\
    g^{\theta r} &= \frac{1}{r \sin^2 \theta}, &
    g^{\phi r} &= \frac{1}{r \sin^2 \theta},
\end{align*}
\]

The nonzero components of the connection by using equations (1.9), (3.2) and (3.3) are as follows:

\[
\begin{align*}
    \Gamma^\theta_{rt} &= \frac{1}{4} \Gamma^r_{\theta t}, &
    \Gamma^\theta_{sr} &= \frac{3 B^4}{8 A^2} - \frac{1}{4} A^4 - \frac{1}{2} + \frac{1}{8} \Gamma^r_{\theta r}, &
    \Gamma^\theta_{\theta r} &= -\frac{1}{4} \cot \theta + \frac{1}{8} \Gamma^r_{\theta \theta},
\end{align*}
\]

\[
\begin{align*}
    \Gamma^\phi_{rt} &= \frac{1}{4} \Gamma^r_{\phi t}, &
    \Gamma^\phi_{sr} &= \frac{3 B^4}{8 A^2} - \frac{1}{4} A^4 - \frac{1}{2} + \frac{1}{8} \Gamma^r_{\phi r}, &
    \Gamma^\phi_{\phi r} &= -\frac{1}{4} \cot \theta + \frac{1}{8} \Gamma^r_{\phi \phi},
\end{align*}
\]

\[
\begin{align*}
    \Gamma^\phi_{\theta r} &= \frac{1}{4} \Gamma^r_{\phi \theta}, &
    \Gamma^\phi_{\phi \theta} &= \frac{3 B^4}{8 A^2} - \frac{1}{4} A^4 - \frac{1}{2} + \frac{1}{8} \Gamma^r_{\phi \phi}, &
    \Gamma^\phi_{\phi \phi} &= -\frac{1}{4} \cot \theta + \frac{1}{8} \Gamma^r_{\phi \phi},
\end{align*}
\]

where $(\cdot)$ denotes derivative with respect to $r$.

By inserting equation (3.4) into equation (1.12) we obtain the components of the Ricci tensor as follows:

\[
\begin{align*}
    R_{rt} &= -\left( \frac{3 B^4}{8 A^2} - \frac{B A'}{8 A^2} - \frac{B}{2 A} + \frac{B}{4 A} \Gamma^r_{\mu r} \right) - \frac{B}{4 r^2} \left( 1 + \cot^2 \theta + \frac{\partial}{\partial \theta} \Gamma^r_{\phi \phi} \right) \\
    &\quad + 2 \left( \frac{3 B^4}{8 A^2} - \frac{B A'}{8 A^2} - \frac{B}{2 A} + \frac{1}{8} \Gamma^r_{\mu r} \right) \left( \frac{3 B^4}{8 A^2} - \frac{B A'}{8 A^2} - \frac{B}{2 A} + \frac{B}{4 A} \Gamma^r_{\mu r} \right) \\
    &\quad + \frac{B}{8 r^2} \left( -\cot \theta + \Gamma^r_{\phi \phi} \right)^2 - \frac{B}{8 r^2} \sin \theta \Gamma^r_{\phi \phi} \\
    &\quad - \left( \frac{3 B^4}{8 A^2} - \frac{B A'}{8 A^2} - \frac{B}{2 A} + \frac{B}{4 A} \Gamma^r_{\mu r} \right) \Gamma^r_{\mu r} - \frac{B}{4 r^2} \left( -\cot \theta + \Gamma^r_{\phi \phi} \right) \Gamma^r_{\phi \phi}.
\end{align*}
\]
\[
\mathcal{R}_{rr} = -\left(\frac{3A'}{8A} - \frac{B'}{8B} - \frac{1}{2r} - \frac{3\Gamma_{\rho\phi}^\rho}{4\Gamma_{\rho\phi}^\rho}\right) + \frac{A}{4r^2} \left(1 + \cot^2\theta + \frac{\partial}{\partial \theta} \Gamma_{\rho\phi}^\rho\right) + \left(\frac{3B'}{8B} - \frac{A'}{8A} - \frac{1}{2r} + \frac{1\Gamma_{\rho\phi}^\rho}{4\Gamma_{\rho\phi}^\rho}\right) \right.
\]
\[
\left. + \frac{A}{8r^2} \left(\cot \theta - \Gamma_{\rho\phi}^\rho\right)^2 + \frac{A}{8r^2 \sin^2 \theta} \Gamma_{\rho\phi}^\rho \right) + 2 \left(\frac{3A'}{8A} - \frac{B'}{8B} - \frac{1}{2r} + \frac{1\Gamma_{\rho\phi}^\rho}{4\Gamma_{\rho\phi}^\rho}\right)^2 \right) \right) \]
\[ \mathcal{R}_{rr} = -\left( \frac{3A'}{8A} - \frac{B'}{8B} - \frac{1}{2r} - \frac{3}{4} \Gamma_{\rho \sigma}^\rho \right)^2 + \left( \frac{3B'}{8B} - \frac{A'}{8A} - \frac{1}{2r} + \frac{1}{4} \Gamma_{\rho \sigma}^\rho \right)^2 \]

\[
+ \left( \frac{3A'}{8A} - \frac{B'}{8B} - \frac{1}{2r} + \frac{1}{4} \Gamma_{\rho \sigma}^\rho \right)^2 + 2\left( \frac{1}{2r} - \frac{A'}{8A} - \frac{B'}{8B} + \frac{1}{4} \Gamma_{\rho \sigma}^\rho \right)^2 \]

\[
- \left( \frac{3A'}{8A} - \frac{B'}{8B} - \frac{1}{2r} + \frac{1}{4} \Gamma_{\rho \sigma}^\rho \right) \Gamma_{\rho \sigma}^\rho \]

(3.10)

\[ \mathcal{R}_{\theta \theta} = -1 - \left( -\frac{r}{2A} + \frac{r^2 A'}{8A^2} + \frac{r^2 B'}{4AB} - \frac{r^2}{4A} \Gamma_{\rho \sigma}^\rho \right)^2 \]

\[
+ 2\left( \frac{r}{2A} + \frac{r^2 A'}{8A^2} + \frac{r^2 B'}{4AB} - \frac{r^2}{4A} \Gamma_{\rho \sigma}^\rho \right)^2 \left( \frac{1}{2r} - \frac{A'}{8A} - \frac{B'}{8B} + \frac{1}{4} \Gamma_{\rho \sigma}^\rho \right) \]

\[
- \left( \frac{r}{2A} + \frac{r^2 A'}{8A^2} + \frac{r^2 B'}{4AB} - \frac{r^2}{4A} \Gamma_{\rho \sigma}^\rho \right) \]

(3.11)

\[ \mathcal{R}_{\phi \phi} = \sin^2 \theta \mathcal{R}_{\theta \theta}. \] (3.12)

Using equations (3.10)–(3.12), (1.13) and (3.3) with some manipulation \( \mathcal{R} \) takes the form:

\[ \mathcal{R} = B \Gamma_{\rho \sigma}^\rho \sin^2 \theta \left[ \frac{3}{2} \Gamma_{\rho \sigma}^\rho + \frac{B'}{4B} - \frac{A'}{4A} + \left( \frac{1}{2} - 2A \right) \frac{1}{r} + \frac{11}{32} \left( \frac{A'}{A} \right)^2 + \frac{35}{32} \left( \frac{B'}{B} \right)^2 + \frac{15}{16} \frac{A'B'}{AB} \right] \]

\[
- \frac{9}{8} \frac{A'}{A} + \frac{5}{16} \frac{B'}{B} - \frac{3}{2r} \Gamma_{\rho \sigma}^\rho - \frac{A'}{8A} \Gamma_{\rho \sigma}^\rho - \frac{5B'}{8B} \Gamma_{\rho \sigma}^\rho - \frac{1}{8} \Gamma_{\rho \sigma}^\rho. \] (3.13)

Multiplying the \( tt \) component of the field equation (2.7) by \( 4Af \) and substituting equations (3.9)–(3.11) into it leads to

\[ \frac{1}{2} \Gamma_{\rho \sigma}^\rho = \frac{5B'}{4B} - \frac{A'}{4A} + \left( \frac{1}{2} - 2A \right) \frac{1}{r} + \frac{11}{32} \left( \frac{A'}{A} \right)^2 + \frac{35}{32} \left( \frac{B'}{B} \right)^2 + \frac{15}{16} \frac{A'B'}{AB} \]

\[
- \frac{9}{8} \frac{A'}{A} + \frac{5}{16} \frac{B'}{B} - \frac{3}{2r} \Gamma_{\rho \sigma}^\rho - \frac{A'}{8A} \Gamma_{\rho \sigma}^\rho - \frac{5B'}{8B} \Gamma_{\rho \sigma}^\rho - \frac{1}{8} \Gamma_{\rho \sigma}^\rho = 0. \] (3.14)

Multiplying the \( rr \) component of equation (2.7) by \( \frac{4}{A} \) and substituting equations (3.9)–(3.11) into it gives

\[ \frac{3}{2} \Gamma_{\rho \sigma}^\rho + \frac{B'}{4B} - \frac{3A'}{4A} + \frac{3}{2r} + \frac{33}{32} \left( \frac{A'}{A} \right)^2 + \frac{9}{32} \left( \frac{B'}{B} \right)^2 = - \frac{3A'B'}{16AB} \]

\[
- \frac{3A'}{4rA} - \frac{7B'}{4rB} + \frac{1}{2r} \Gamma_{\rho \sigma}^\rho + \frac{B'}{8B} \Gamma_{\rho \sigma}^\rho - \frac{3A'}{8A} \Gamma_{\rho \sigma}^\rho - \frac{2A}{r^2} - \frac{3}{8} \Gamma_{\rho \sigma}^\rho = 0. \] (3.15)

Multiplying of the \( \theta \theta \) component of equation (2.7) by \( \frac{4}{f} \) and insertion of equations (3.9)–(3.11) into it gives the final relation

\[ -\frac{1}{2} \Gamma_{\rho \sigma}^\rho - \frac{3B'}{4B} + \frac{A'}{4A} - \left( \frac{1}{2} + 2A \right) \frac{1}{r} - \frac{11}{32} \left( \frac{A'}{A} \right)^2 + \frac{13}{32} \left( \frac{B'}{B} \right)^2 + \frac{9A'B'}{16AB} \]

\[
- \frac{3A'}{4rA} + \frac{B'}{4rB} + \frac{1}{2r} \Gamma_{\rho \sigma}^\rho + \frac{A'}{8A} \Gamma_{\rho \sigma}^\rho - \frac{3B'}{8B} \Gamma_{\rho \sigma}^\rho + \frac{1}{8} \Gamma_{\rho \sigma}^\rho = 0. \] (3.16)

The \( \phi \phi \) component of equation (2.7) does not yield a new equation and equation (3.16) is repeated. It is interesting to note that equations (3.14), (3.15) and (3.16) are not independent

6
and we have indeed two independent relations. If we add equations (3.14) and (3.16) will result in

\[ \frac{B''}{B} = -\frac{2A}{r^2} + \frac{3}{4} \left( \frac{B'}{B} \right)^2 + \frac{3A'B'}{4AB} - \frac{3A'}{2rA} - \frac{B'}{2rB} \Gamma_{\rho\sigma}^\rho + \frac{1}{r} \Gamma_{\rho\sigma}^\rho. \]  

(3.17)

Also, addition of equations (3.14) and (3.15) and substitution of the value of \( \frac{B''}{B} \) from equation (3.17) gives

\[ \frac{A''}{A} = 2\Gamma_{\rho\sigma}^\rho + \frac{2(1 + A)}{r^2} + \frac{11}{8} \left( \frac{A'}{A} \right)^2 + \frac{5}{8} \left( \frac{B'}{B} \right)^2 - \frac{3A'}{2rA} \]

\[ -\frac{5B'}{2rB} + \frac{1}{r} \Gamma_{\rho\sigma}^\rho - \frac{A'}{2A} \Gamma_{\rho\sigma}^\rho - \frac{1}{2} \frac{\Gamma_{\rho\sigma}^\rho}{r^2}. \]  

(3.18)

Using equation (3.15), equation (3.13) for \( \Re \) can be written as

\[ \Re = \frac{B^{1/2} r \sin^2 \theta}{A^{1/2}} \left[ -(1 + 4A) + \frac{3}{16} \left( \frac{A'}{A} \right)^2 - \frac{5}{16} \left( \frac{B'}{B} \right)^2 - \frac{A'B'}{8AB} - \frac{A'}{2rA} \right. \]

\[ + \left. \frac{3B'}{2rB} + \frac{1}{r} \Gamma_{\rho\sigma}^\rho + \frac{3A'}{4A} \Gamma_{\rho\sigma}^\rho + \frac{B'}{4B} \Gamma_{\rho\sigma}^\rho + \frac{3}{4} \frac{\Gamma_{\rho\sigma}^\rho}{r^2} \right]. \]  

(3.19)

Now the field equation (2.6) will be satisfied automatically if we insert equation (3.19) into equation (2.6) and replace \( \frac{B''}{B} \) and \( \frac{A''}{A} \) from equations (3.17) and (3.18) in it. This means that the field equation (2.6) does not lead to an independent relation.

An empty space is spherically symmetric at each point so it is homogeneous and isotropic. For an empty space \( A \) and \( B \) are independent of radial coordinate and consequently are constant. Then equations (3.17) and (3.19) with constants \( A \) and \( B \) imply \( \Gamma_{\rho\sigma}^\rho = \frac{2}{r} \) for this case. The action for a vacuum spherical symmetry space is the same as the action of an empty space plus an additional matter term corresponding to a single point source, its existence at the origin being the cause of spherical symmetry. Actually this additional term does not depend on \( \Gamma_{\rho\sigma}^\rho \). So the field equations corresponding to the variation with respect to \( \Gamma_{\rho\sigma}^\rho \) are the same for both cases. Meanwhile both these cases satisfy the same boundary conditions. So this necessitates the same result for \( \Gamma_{\rho\sigma}^\rho \) in both cases. A rigorous proof for \( \Gamma_{\rho\sigma}^\rho = \frac{2}{r} \) will be given in the following section in the context of a gauge invariance discussion.

4. Gauge fixing

First we demonstrate that at any point \( P \) there exists a locally inertial coordinate \( x' \) at which \( g_{\mu'\nu'} \) takes its canonical form \( \eta_{\mu'\nu'} \) and the first partial derivatives of the metric \( \partial_{\mu'} g_{\mu'\nu'} \), and the components of the trace of the connection \( \Gamma_{\rho'\lambda'}^\rho \), all vanish. Meanwhile some of the components of the second partial derivatives of the metric \( \partial_{\mu'} \partial_{\nu'} g_{\mu'\nu'} \) and the first partial derivative of the trace of the connection cannot be made to vanish. Let us consider the transformation law for the metric and the trace of the connection;

\[ g_{\mu'\nu'} = \left( \frac{\partial x}{\partial x'} \right)^2 \frac{\partial x'^\mu}{\partial x'} \frac{\partial x'^\nu}{\partial x'} g_{\mu\nu}, \]  

(4.1)

\[ \Gamma_{\rho'\lambda'}^{\rho'} = \frac{\partial x'^\lambda}{\partial x^i} \Gamma_{\rho\lambda}^\rho + \frac{\partial x'^\nu}{\partial x^i} \frac{\partial^2 x'^\lambda}{\partial x'^\nu \partial x^i} \]  

(4.2)

and expand both sides in Taylor series in the sought-after coordinate \( x'^\mu \). The expansion of
the old coordinate looks like
\[ x^\mu = \left( \frac{\partial x^\mu}{\partial x'^\mu} \right)_p x'^\mu + \frac{1}{2} \left( \frac{\partial^2 x^\mu}{\partial x'^\mu \partial x'^\nu} \right)_p x'^\mu x'^\nu + \frac{1}{6} \left( \frac{\partial^3 x^\mu}{\partial x'^\mu \partial x'^\nu \partial x'^\rho} \right)_p x'^\mu x'^\nu x'^\rho + \ldots \] (4.3)

(For simplicity we have \( x^\mu(P) = x'^\mu(P) = 0 \)). Then using some schematic notation, the expansion of equations (4.1) and (4.2) to second order is
\[
\begin{align*}
(g')_p + (\partial' g')_p x' + (\partial' \partial' g')_p x' x' + \cdots &= \left( \frac{\partial x}{\partial x'} \right)^{-1}_p \frac{\partial x}{\partial x'} \left( \frac{\partial^2 x}{\partial x' \partial x'} \right)_p + \left[ \left( \frac{\partial x}{\partial x'} \right)^{-1}_p \frac{\partial^2 x}{\partial x' \partial x'} \right] \frac{\partial x}{\partial x'} \left( \frac{\partial^2 x}{\partial x' \partial x'} \right)_p x' + \cdots \\
(\Gamma')_p + (\partial' \Gamma')_p x' + \cdots &= \left[ \frac{\partial x}{\partial x'} \frac{\partial^2 x}{\partial x' \partial x'} \right]_p + \left[ \frac{\partial^2 x}{\partial x' \partial x'} \right]_p \frac{\partial x}{\partial x'} + \left[ \frac{\partial^2 x}{\partial x' \partial x'} \right]_p \frac{\partial x}{\partial x'} \frac{\partial^2 x}{\partial x' \partial x'} x' + \cdots.
\end{align*}
\] (4.4)

We can set terms of the same order in \( x' \) on each side to be equal. Therefore according to equation (4.4), the components \( g_{\mu'\nu'} \), ten numbers in all, are determined by the matrix \( \left( \frac{\partial x^{\mu'}}{\partial x'^\mu} \right)_p \). This is a \( 4 \times 4 \) matrix with no constraint. Thus we are free to choose 16 numbers. This is enough freedom to put the ten numbers of \( g_{\mu'\nu'}(P) \) into the canonical form \( \eta_{\mu'\nu'} \).

The six remaining degrees of freedom can be interpreted as exactly the six parameters of the Lorentz group which leave the canonical form unchanged. At first order we have the derivative \( \partial' \eta_{\mu'\nu'}(P) \), four derivatives of ten components for a total of 40 numbers. Since \( g^{\mu\nu} \eta_{\mu'\nu'} = 0 \) we have merely 36 independent numbers for them. Now looking at the right-hand side of equation (4.4), we have the additional freedom to choose \( \frac{\partial^2 x^{\mu'}}{\partial x'^\mu \partial x'^\nu} \). In this set of numbers there are a total number of 40 degrees of freedom. Precisely the number of choices we need to determine all the first derivatives of the metric is 36, which we can set to zero, while four numbers of the 40 degrees of freedom of \( \frac{\partial^2 x^{\mu'}}{\partial x'^\mu \partial x'^\nu} \) remain unused. According to equation (4.5) the four components of \( \Gamma_{\rho'\lambda}(P) \) may be determined by the four remaining components of the matrix \( \left( \frac{\partial^2 x^{\mu'}}{\partial x'^\mu \partial x'^\nu} \right)_p \) which we can also set equal to zero. We should emphasize that \( \Gamma_{\rho'\lambda} = 0 \) in polar coordinates takes the form \( (0, 2, \cot \theta, 0) \). So it is always possible to find a locally inertial coordinate system in which the components of the Riemann curvature tensor take the form
\[
\mathcal{R}_{\mu'\nu'\rho'\lambda'} = \eta_{\mu'\lambda'}(\partial' \Gamma_{\nu'\rho'} - \partial_\sigma \Gamma_{\nu'\rho'}^\sigma) \]
\[
= \left( \frac{1}{2} \eta_{\mu'\lambda'} \eta_{\nu'\rho'} \eta_{\sigma'\nu'} \eta_{\lambda'\rho'} \eta_{\nu'\sigma'} \eta_{\lambda'\mu'} \right) \left( \eta_{\mu'\lambda'} \eta_{\nu'\rho'} \eta_{\sigma'\nu'} \eta_{\lambda'\rho'} \eta_{\nu'\sigma'} \eta_{\lambda'\mu'} \right)^{-1} \delta_{\mu'\nu'\rho'\lambda'} - \eta_{\mu'\lambda'} \partial_\nu' \Gamma_{\rho'\lambda'}^\nu - \eta_{\nu'\rho'} \partial_\lambda' \Gamma_{\mu'\lambda'}^\lambda - \eta_{\lambda'\mu'} \partial_\nu' \Gamma_{\rho'\lambda'}^\nu - \eta_{\rho'\lambda'} \partial_\nu' \Gamma_{\mu'\lambda'}^\lambda.
\] (4.6)

The Ricci tensor comes from the contracting over \( \mu \) and \( \rho \),
\[
\mathcal{R}_{\nu'\sigma'} = \eta^{\mu'\nu'} \mathcal{R}_{\mu'\nu'\rho'\lambda'}. \]
(4.7)

The change of the metric which is a tensor density of rank (0, 2) and weight \(-\frac{1}{2}\) under a diffeomorphism along the vector field \( \xi^\mu \) is
\[
\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \xi^\lambda.
\] (4.8)
Equation (4.8) in the local inertial frame takes the form
\[ \delta g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2} \eta_{\mu\nu} \partial_\lambda \xi^\lambda. \] (4.9)

The change of the trace of the connection in the local inertial frame under this diffeomorphism is
\[ \delta \Gamma^\rho_{\rho\mu} = \partial_\mu \partial_\lambda \xi^\lambda. \] (4.10)

We may call equations (4.9) and (4.10) a gauge transformation. The Ricci tensor equation (4.7) under the gauge transformation (4.9) and (4.10) is invariant. This gauge degree of freedom has to be fixed before going further. This may be achieved by taking
\[ \Gamma^\rho_{\rho\mu} = 0. \] (4.11)

As has been mentioned, the gauge (4.11) in polar coordinates leads to
\[ \Gamma^\rho_{\rho\mu} = \left( 0, \frac{2}{r}, \cot \theta, 0 \right). \] (4.12)

So we take \( \Gamma^\rho_{\rho r} = \frac{2}{r} \) without any loss of generality in our calculations. This spacetime has four Killing vectors just the same as the Schwarzschild spacetime, one timelike and three spacelike that are the same as those on \( S^2 \).

5. \( \Gamma^\rho_{\rho r} = \frac{2}{r} \)

Now let us investigate the solutions of equations (3.17) and (3.18) for the given form of \( \Gamma^\rho_{\rho r} = \frac{2}{r} \). This leads to the following two relations for \( A \) and \( B \):
\[ \frac{B''}{B} = \frac{2(1 - A)}{r^2} + \frac{3}{4} \left( \frac{B'}{B} \right)^2 + \frac{3}{4} \frac{A'B'}{AB} - \frac{3}{2r} A' - \frac{3}{2r} B' \] (5.1)
\[ \frac{A''}{A} = \frac{2(-1 + A)}{r^2} + \frac{11}{8} \left( \frac{A'}{A} \right)^2 + \frac{5}{8} \left( \frac{B'}{B} \right)^2 - \frac{5}{2r} A' - \frac{5}{2r} B'. \] (5.2)

Combining equations (5.1) and (5.2) with some manipulation gives
\[ \left( \frac{A'}{A} + \frac{B'}{B} \right)' = \frac{3}{8} \left( \frac{A'}{A} + \frac{B'}{B} \right)^2 - \frac{4}{r} \left( \frac{A'}{A} + \frac{B'}{B} \right) \] (5.3)

Equation (5.3) can easily be integrated with respect to \( r \), then it results
\[ \left( \frac{A'}{A} + \frac{B'}{B} \right) = \frac{C(AB)^{\frac{3}{4}}}{r^2} \] (5.4)
where \( C \) is the constant of integration. Now if we introduce the new parameter \( y = AB \), then integration of equation (5.4) with respect to \( r \) gives the solution
\[ y^{-\frac{1}{2}} = \frac{C}{8r^3} + \tilde{C} \] (5.5)
where \( \tilde{C} \) is another integration constant. To be consistent with the asymptotic form of \( A = B = 1 \) at infinity we should have \( \tilde{C} = 1 \)
\[ AB = \left( 1 + \frac{C}{8r^3} \right)^{-\frac{1}{2}}. \] (5.6)
Then by inserting equation (5.6) into equation (5.1) we find an equation for $B$,
\[ B'' = \frac{2\left[ B - r^8 \left( r^3 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} \right]}{r^2} + \frac{C \left( \frac{1}{2} r B' - \frac{3}{2} B \right)}{r^2 \left( r^3 + \frac{C'}{r^7} \right)}. \] (5.7)

Equation (5.7) has a general solution as follows:
\[ B(r) = \frac{1}{\left( 1 + \frac{C'}{r^7} \right)^{\frac{3}{2}}} \left[ \frac{\alpha r^2}{1 + \frac{C'}{r^7}} + \frac{\beta}{1 + \frac{C'}{r^7}} r \right] \] (5.8)

where $C' = \frac{C}{r^7}$, $\alpha$ and $\beta$ are constants of integration.

The Newtonian limit requires that ($c = 1$)
\[ B \rightarrow 1 - \frac{2GM}{r} \] as $r \rightarrow \infty$. (5.9)

Applying condition (5.9) to the equation (5.8) implies that
\[ \beta - \alpha C' = -2GM. \] (5.10)

Here $\alpha$ has the dimension $[L]^{-2}$ and may be called the cosmological constant. So we may take
\[ \beta = -2GM + \Lambda C' \] (5.11)
and will have
\[ B(r) = \left( 1 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} \left[ 1 - \frac{2GM - \Lambda C'}{r} \left( 1 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} + \Lambda r^2 \left( 1 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} \right] \] (5.12)

\[ A(r) = \left( 1 + \frac{C'}{r^7} \right)^{-2} \left[ 1 - \frac{2GM - \Lambda C'}{r} \left( 1 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} + \Lambda r^2 \left( 1 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} \right]^{-1}. \] (5.13)

A special case of this general solution is when $\Lambda = 0$,
\[ B(r) = \left( 1 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} \left[ 1 - \frac{2GM}{r} \left( 1 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} \right] \] (5.14)

\[ A(r) = \left( 1 + \frac{C'}{r^7} \right)^{-2} \left[ 1 - \frac{2GM}{r} \left( 1 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} \right]^{-1}. \] (5.15)

Equations (5.14) and (5.15) show that the case $C' = 0$ gives the familiar Schwarzschild solution.

The term $\left( 1 + \frac{C'}{r^7} \right)$ is always nonnegative for $C' \geq 0$, so $B(r)$ and $A(r)$ are always nonnegative if the condition
\[ 1 - \frac{2GM}{r} \left( 1 + \frac{C'}{r^7} \right)^{-\frac{3}{2}} \geq 0 \] (5.16)
holds for the whole range of $r$. Equation (5.16) can be rewritten as
\[ r^3 \geq (2GM)^3 - C'. \] (5.17)

Equation (5.17) implies that equation (5.16) will be true for the whole range of $r$ if we take
\[ C' \geq (2GM)^3. \] (5.18)

If we assume that the condition (5.18) holds then $A$ and $B$ will be analytic on the whole range of $r$. A remarkable point about $\Lambda$ which has the same role of cosmological constant is that
choosing $\Lambda = 0$ is merely to simplify the calculations. Keeping $\Lambda \neq 0$ does not change the character of $C'$ significantly. We omit the detail of calculations to avoid mathematical complications. The solution asymptotically becomes de Sitter–Schwarzschild.

Considering a null radial geodesic, its line element gives

$$\pm dr = \sqrt{\frac{A}{B}} \, dr.$$  \hspace{1cm} (5.19)

Putting equations (5.14) and (5.15) into equation (5.19) gives

$$\pm dt = \frac{r^2 \, dr}{(r^3 + C')^{\frac{1}{3}} ((r^3 + C')^{\frac{1}{3}} - 2GM)}.$$ \hspace{1cm} (5.20)

Let us define the new parameter $R = (r^3 + C')^{\frac{1}{3}}$. The range of $R$ will be from $(C')^{\frac{1}{3}}$ to infinity.
We have

$$R^2 \, dR = r^2 \, dr.$$ \hspace{1cm} (5.21)

Then equation (5.20) takes the form

$$\pm dt = \frac{R \, dR}{R - 2GM}.$$ \hspace{1cm} (5.22)

Integrating equation (5.22) gives

$$\pm (t - t_0) = (r^3 + C')^{\frac{1}{3}} + 2GM \ln[(r^3 + C')^{\frac{1}{3}} - 2GM]$$

$$- (r_0^3 + C')^{\frac{1}{3}} + 2GM \ln[(r_0^3 + C')^{\frac{1}{3}} - 2GM].$$ \hspace{1cm} (5.23)

Since we have assumed that $C' > (2GM)^{\frac{1}{3}}$, then equation (5.23) shows no sign of singularity over the whole range of $r$.

6. Upper bound on $C'$

It is natural to anticipate a classical test like the advance of the perihelion of Mercury to put some upper bound on the probable values of $C'$. Fortunately the high degree of symmetry greatly simplifies our task. There are four Killing vectors, each of these will lead to a constant of the motion for a free particle. In addition we always have another constant of the motion for the geodesics; the geodesic equation implies that the quantity

$$\epsilon = -g_{\mu\nu} \frac{dx^\mu}{dx^\tau} \frac{dx^\nu}{dx^\lambda}$$ \hspace{1cm} (6.1)

is constant along the path. For massive particles we typically choose $\lambda$ so that $\epsilon = 1$. For massless particles we always have $\epsilon = 0$. Invariance under time translations leads to conservation of energy while invariance under spatial rotations leads to conservation of the three components of angular momentum. Conservation of direction of angular momentum means that a particle moves on a plane. So we may choose $\theta = \frac{\pi}{2}$. The two remaining Killing vectors correspond to the energy and the magnitude of angular momentum. The energy arises from the timelike Killing vector

$$(\partial_t)^\mu = (1, 0, 0, 0),$$ \hspace{1cm} (6.2)

while the Killing vector whose conserved quantity is the magnitude of angular momentum is

$$(\partial_\theta)^\mu = (0, 0, 0, 1).$$ \hspace{1cm} (6.3)

The two conserved quantities are

$$E = - \left(1 + \frac{C'}{r^3}\right)^{-\frac{1}{3}} \left[1 - \frac{2GM}{r} \left(1 + \frac{C'}{r^3}\right)^{-\frac{1}{3}}\right] \frac{dr}{d\lambda}.$$ \hspace{1cm} (6.4)
and

$$L = r^2 \frac{d\phi}{d\lambda}. \quad (6.5)$$

Expanding expression (6.1), multiplying it by equation (5.14) and using equations (6.4) and (6.5) gives

$$-E^2 + \left(1 + \frac{C'}{r^3}\right)^{-\frac{3}{2}} \left(1 + \sqrt{1 + \frac{2GM}{r}} \left(1 + \frac{C'}{r^3}\right)^{-\frac{1}{2}} \left(L^2 + \epsilon\right) = 0. \quad (6.6)$$

Expanding in powers of $r$, we shall content ourselves here with only the lowest order of approximation in equation (6.6). Then we have

$$\left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right) \left(\frac{L^2}{r^2} + \epsilon\right) - E^2 \left(1 + \frac{8C'}{3r^3}\right) + \frac{2\epsilon C'}{r^3} = 0. \quad (6.7)$$

Now by taking $\epsilon = 1$, multiplying equation (6.7) by $\left(\frac{d\phi}{d\lambda}\right)^{-2} = \frac{L^2}{GM}$ and defining $x = \frac{L^2}{GM}$ we have

$$\left(\frac{dx}{d\phi}\right)^2 - 2x + x^2 - \frac{2G^2M^2}{L^2}x^3 + \frac{GM}{L^4} \left(2C' - \frac{8E^2C'}{3}\right) x^3 = \frac{E^2L^2}{G^2M^2} - \frac{L^2}{G^2M^2}. \quad (6.8)$$

Differentiating equation (6.8) with respect to $\phi$ gives

$$\frac{d^2x}{d\phi^2} - 1 + x = \left(\frac{3G^2M^2}{L^2} - \frac{3GM}{L^4} C' + \frac{4E^2C'GM}{L^4}\right) x^2. \quad (6.9)$$

After some manipulation we obtain that the perihelion advances by an angle

$$\Delta \phi = \frac{6\pi G^2M^2}{L^2} \left(1 - \frac{C'}{GML^2} + \frac{4E^2C'GM}{3GML^2}\right). \quad (6.10)$$

On the other hand we have

$$E^2 = 1 + \frac{G^2M^2}{L^2}(e^2 - 1) \quad (6.11)$$

and

$$L^2 \cong GM(1 - e^2)a, \quad (6.12)$$

where $e$ is the eccentricity and $a$ is the semi-major axis of the ellipse. Putting equations (6.11) and (6.12) into equation (6.10) yields

$$\Delta \phi = \frac{6\pi GM}{(1 - e^2)a} \left(1 + \frac{C'}{3G^2M^2(1 - e^2)a} + \frac{4C'}{3GM(1 - e^2)a^2}\right). \quad (6.13)$$

The third term in the parenthesis is smaller than the second term by a factor $\frac{GM}{a}$ and may be neglected:

$$\Delta \phi = \frac{6\pi GM}{(1 - e^2)a} \left(1 + \frac{C'}{3G^2M^2(1 - e^2)a}\right). \quad (6.14)$$

This shows a severe dependence on $C'$ and should not conflict with observational data, i.e. we should have

$$C' \ll G^2M^2a \quad (6.15)$$

by an order of magnitude.
7. Conclusions and remarks

A new field equation for gravity is proposed by introducing a new character for the metric as a tensor density in the absence of matter. We obtained the vacuum spherical symmetry solutions of the field equations. A considerable large part of these solutions is regular for the whole range of \( r \) except at \( r = 0 \). Since \( \partial_r \) remains timelike everywhere, there is no event horizon in the spacetime. The solutions for the metric show a singularity at \( r = 0 \). To find the nature of the singularity we need to check some scalar densities made of Riemann tensor. First let us check the behavior of Riemann scalar density of weight \( +\frac{1}{2} \), \( \mathcal{R} \). Substituting \( r^{\mu \nu} = \frac{1}{r} \) into equation (3.19) results in

\[
\mathcal{R} = \frac{B^{\mu \nu} r \sin^2 \theta}{A^{\mu \nu}} \left\{ \frac{4(1 - A)}{r^2} + \left( \frac{B' - A'}{B} - A'\right) \right\} - \frac{2}{r} - \frac{1}{4} \left( \frac{A'}{A} + \frac{B'}{B} \right)\right] - \frac{1}{16} \left( \frac{A'}{A} + \frac{B'}{B} \right)^2.
\]

(7.1)

In the range of Newtonian limit when \( r \gg 2GM \), \( A \) and \( B \) in equations (5.14), (5.15) reduce to the Schwarzschild form, so equation (7.1) gives \( \mathcal{R} = 0 \). For the range of \( r \ll 2GM \) we have two different asymptotic forms for \( A \) and \( B \) depending on the value of \( C' \). If \( C' > (2GM)^3 \) then we have

\[
B \approx (C')^{-\frac{3}{2}} \left( 1 - \frac{2GM}{(C')^{\frac{3}{2}}} \right)^2 r^2
\]

(7.2)

\[
A \approx (C')^{-2} \left( 1 - \frac{2GM}{(C')^{\frac{3}{2}}} \right)^2 r^6.
\]

(7.3)

For \( C' = (2GM)^3 \), \( A \) and \( B \) reduce to

\[
B = \frac{1}{3} \left( \frac{r}{2GM} \right)^3
\]

(7.4)

\[
A = 3 \left( \frac{r}{2GM} \right)^5.
\]

(7.5)

For both cases by inserting equations (7.2)–(7.5) into equation (7.1), \( \mathcal{R} \) becomes

\[
\mathcal{R} = -4(C')^{-\frac{3}{2}} r \sin^2 \theta.
\]

(7.6)

Immediately we may calculate \( \mathcal{R}^{\mu \nu} \mathcal{R}_{\mu \nu} \). According to the field equation (2.7) we have \( \mathcal{R}_{\mu \nu} = \frac{1}{2} r^{\mu \nu} \mathcal{R} \). This leads to

\[
\mathcal{R}^{\mu \nu} \mathcal{R}_{\mu \nu} = \frac{1}{2} \mathcal{R}^2 = 4(C')^{-\frac{3}{2}} r^2 \sin \theta.
\]

(7.7)

Equations (7.6) and (7.7) are quite different from that of the Schwarzschild metric where \( \mathcal{R} \) and \( \mathcal{R}^{\mu \nu} \mathcal{R}_{\mu \nu} \) are zero everywhere including the neighborhood of the origin. The Jacobian of the transformation from polar coordinates \((t, r, \theta, \phi)\) to the coordinate system \((t, x, y, z)\) is \( r^2 \sin \theta \). Since \( \mathcal{R} \) is a scalar density of weight \( +\frac{1}{2} \) and \( \mathcal{R}^{\mu \nu} \mathcal{R}_{\mu \nu} \) is a scalar density of weight \(+1\), in the \((t, x, y, z)\) coordinate system we have \( \mathcal{R} = -(C')^{-\frac{3}{2}} \) and \( R^{\mu \nu} R_{\mu \nu} = \frac{1}{4}(C')^{-\frac{3}{2}} \). That is, \( \mathcal{R} \) is constant and negative while \( \mathcal{R}^{\mu \nu} \mathcal{R}_{\mu \nu} \) is constant and positive. To specify the nature of the singularity it is necessary that the scalar density \( \mathcal{R}^{\mu \nu} \mathcal{R}_{\mu \nu} \mathcal{R}_{\nu \mu \rho \delta} \) be calculated too. The nonzero components of the Riemann tensor are

\[
\mathcal{R}^{\mu \nu \rho \delta} = \left( \frac{3}{8} \frac{B''}{B} - \frac{A''}{8A} + \frac{3}{16} \left( \frac{A'}{A} \right)^2 - \frac{3}{16} \left( \frac{B'}{B} \right)^2 - \frac{1}{4} \frac{A'B'}{AB} \right)
\]

(7.8)
small distances is a novel feature of this work. It is not hard to imagine that taking distances of the order which is the counterpart of the de Sitter spacetime will show an event horizon at cosmological

\[ r_{\text{hor}} = \frac{r^2}{A} \left( \frac{1}{64} \left( \frac{A'}{A} \right)^2 - \frac{3}{64} \left( \frac{B'}{B} \right)^2 - \frac{1}{32} \frac{A'B'}{AB} + \frac{3}{8} \frac{B'}{rB} - \frac{A'}{8rA} \right) \]  \quad (7.9)

\[ \mathcal{R}_{\phi\phi} = \sin^2 \theta \mathcal{R}_{\theta\theta} \]  \quad (7.10)

\[ \mathcal{R}_{\phi\phi} = \frac{r^2}{A} \left( \frac{A''}{8A} - \frac{B''}{8B} + \frac{3}{16} \left( \frac{A'}{A} \right)^2 + \frac{1}{8} \left( \frac{B'}{B} \right)^2 + \frac{1}{16} \frac{A'B'}{AB} - \frac{5}{8r} \frac{A'}{A} - \frac{1}{8} \frac{B'}{B} \right) \]  \quad (7.11)

\[ \mathcal{R}_{\phi\phi} = \sin^2 \theta \mathcal{R}_{\phi\phi} \]  \quad (7.12)

\[ \mathcal{R}_{\phi\phi} = \frac{r^2}{A} \sin^2 \theta \left( -1 + \frac{1}{r^2} + \frac{1}{64} \left( \frac{A'}{A} \right)^2 + \frac{1}{64} \left( \frac{B'}{B} \right)^2 + \frac{1}{32} \frac{A'B'}{AB} - \frac{A'}{4rA} - \frac{1}{4rB} \right) \]  \quad (7.13)

Using equations (7.8)–(7.13) we have

\[ \mathcal{R}^{\mu\nu\rho\lambda} \mathcal{R}_{\mu\nu\rho\lambda} = \frac{2B'r^2 \sin \theta}{A^2} \left( \left[ \frac{3}{8A} \frac{B''}{8B} - \frac{A''}{A} + \frac{3}{16} \left( \frac{A'}{A} \right)^2 - \frac{3}{16} \left( \frac{B'}{B} \right)^2 - \frac{1}{4} \frac{A'B'}{AB} \right]^2 \\
+ 2 \left[ \frac{1}{64} \left( \frac{A'}{A} \right)^2 - \frac{3}{64} \left( \frac{B'}{B} \right)^2 - \frac{1}{32} \frac{A'B'}{AB} + \frac{3}{8} \frac{B'}{rB} - \frac{A'}{8rA} \right]^2 \\
+ 2 \left[ -\frac{A''}{8A} - \frac{B''}{8B} + \frac{3}{16} \left( \frac{A'}{A} \right)^2 + \frac{1}{8} \left( \frac{B'}{B} \right)^2 + \frac{1}{16} \frac{A'B'}{AB} - \frac{5}{8r} \frac{A'}{A} - \frac{1}{8} \frac{B'}{B} \right]^2 \\
+ \left[ -1 + \frac{1}{r^2} + \frac{1}{64} \left( \frac{A'}{A} \right)^2 + \frac{1}{64} \left( \frac{B'}{B} \right)^2 + \frac{1}{32} \frac{A'B'}{AB} - \frac{A'}{4rA} - \frac{B'}{4rB} \right]^2 \right). \]  \quad (7.14)

Now inserting the asymptotic forms (7.2)–(7.5) for \( A \) and \( B \) into equation (7.14) yields

\[ \mathcal{R}^{\mu\nu\rho\lambda} \mathcal{R}_{\mu\nu\rho\lambda} \propto \frac{\sin \theta}{r^6} \text{ for } C' > (2GM)^3 \]  \quad (7.15)

and

\[ \mathcal{R}^{\mu\nu\rho\lambda} \mathcal{R}_{\mu\nu\rho\lambda} \propto \frac{\sin \theta}{r^4} \text{ for } C' = (2GM)^3 \]  \quad (7.16)

where \( \mathcal{R}^{\mu\nu\rho\lambda} \mathcal{R}_{\mu\nu\rho\lambda} \) is a scalar density of weight +1, so in a \((t,x,y,z)\) coordinate system equations (7.15) and (7.16) are proportional to \( \frac{1}{r^6} \) and \( \frac{1}{r^4} \) respectively. This is enough to convince us that \( r = 0 \) represents an actual singularity. Nonexistence of event horizon in small distances is a novel feature of this work. It is not hard to imagine that taking \( \Lambda < 0 \) which is the counterpart of the de Sitter spacetime will show an event horizon at cosmological distances of the order \( |\Lambda|^{-\frac{1}{2}} \). It is worth investigating the alternative version in the presence of matter.

**Acknowledgments**

AMA appreciates the support of the research council at the University of Tehran.
References

[1] Einstein A 1919 *Sitzungshber. Preuss Akad. Wissen* 1 433
   Lorentz H A et al 1952 *The Principle of Relativity* (New York: Dover) (English transl.)
[2] Anderson J and Finkelstein D 1971 *Am. J. Phys.* 39 901
[3] Finkelstein D et al 2001 *J. Math. Phys.* 42 340–6
[4] Van der Bij J J et al 1982 *Physica* A 116 367
[5] Weinberg S 1989 *Rev. Mod. Phys.* 61 1–23
[6] Alvarez E 2005 *J. High Energy Phys.* JHEP03(2005)002 JHEP03(2005)002