Polynomial Invariants for Torus Knots
and Topological Strings

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We make a precision test of a recently proposed conjecture relating Chern-Simons gauge theory to topological string theory on the resolution of the conifold. First, we develop a systematic procedure to extract string amplitudes from vacuum expectation values (vevs) of Wilson loops in Chern-Simons gauge theory, and then we evaluate these vevs in arbitrary irreducible representations of $SU(N)$ for torus knots. We find complete agreement with the predictions derived from the target space interpretation of the string amplitudes. We also show that the structure of the free energy of topological open string theory gives further constraints on the Chern-Simons vevs. Our work provides strong evidence towards an interpretation of knot polynomial invariants as generating functions associated to enumerative problems.

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1. Introduction

Ever since the Jones polynomial and its generalizations were discovered [1], knot theorists have been searching for an interpretation of the integers entering these polynomials. Though it seems rather natural to regard these polynomials as generating functions associated to enumerative problems, no much progress has been achieved in this direction. One of the main goals of this paper is to point out that the situation has changed dramatically after the recent work by Ooguri and Vafa in [2]. Based on their results, we will provide strong evidence to affirm that from the ordinary polynomial invariants associated to arbitrary irreducible representations of the group SU(N) one can construct new ones whose integer coefficients can be interpreted as the solutions to specific enumerative problems in the context of string theory. Thus, in what regards to a picture of polynomial invariants as generating functions, these new polynomials are more fundamental than the ordinary ones.

At the heart of this development is Chern-Simons gauge theory [3] and the relationship between large $N$ gauge theories and gravity in the light of the AdS/CFT correspondence (see [4] for a review).

The proposal of [5][6][7][2], which can be regarded as a simpler version of the AdS/CFT correspondence, relates Chern-Simons gauge theory on $S^3$ to topological string theory
whose target is the resolution of the conifold. This proposal is very interesting from the point of view of knot theory and three-manifold topology, since it reformulates the invariants obtained in the context of Chern-Simons gauge theory in terms of invariants associated to topological strings and related to the counting of BPS states. In particular, in [2] a generating function of vacuum expectation values (vevs) of Wilson loops was expressed in terms of certain integers counting the number of D2 branes ending on D4 branes. This reformulation makes some predictions about the structure of Chern-Simons vevs and provides an interpretation for the integer coefficients of some related polynomial invariants. It was verified in [2] that these predictions were true in the simple case of the unknot.

One purpose of this paper is to make a precision test of the proposal of [2] for a wide class of nontrivial knots. As a preliminary step, we present a systematic procedure to extract from the vevs of Wilson loops a series of polynomials arising naturally in the context of topological strings. This is the content of equation (2.19) below. These polynomials, that we denote by \( f_R \), are labeled by the irreducible representations, \( R \), of \( SU(N) \), and according to the conjecture in [2] they have a very precise structure dictated by the BPS content of the “dual” theory. We then test this conjecture with actual computations in Chern-Simons gauge theory.

The technical challenge associated to the conjecture in [2], on the Chern-Simons side, is that it involves vevs of Wilson loops in arbitrary irreducible representations of \( SU(N) \), with \( N \) generic. For the fundamental representation, the vevs are related (up to a normalization) to the HOMFLY polynomial [12]. Not much is known about these vevs for other irreducible representations, except in the case of \( SU(2) \), where they are related to the Akutsu-Wadati polynomials (for a review, see [13]). There are also some sample computations for a few knots in [14], for representations of \( SU(N) \) with only one row in their Young diagram. However, in the case of torus knots, one can compute these vevs using the formalism of knot operators introduced in [16]. Knot operators were used in [14] [17] [18] [19] to compute the vevs of Wilson loops for torus knots and links in the fundamental representations of \( SU(N) \)

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1 The relation between Chern-Simons gauge theory and string theory has been addressed also in [8] [9] [10]. The connection between Chern-Simons and topological open string theory was discovered by Witten in [11].

2 Though we will refer to the \( f_R \) as polynomials, they are not. They are polynomials up to a common factor as stated in (2.8) and (4.22).

3 There are also a few computations in [14] for the gauge group \( SU(3) \).
and \( SO(N) \), and in arbitrary irreducible representations of \( SU(2) \). The computation of
vevs for torus knots in arbitrary irreducible representations of \( SU(N) \), as needed to test the
conjecture of [4], is technologically difficult, but fortunately many of the intermediate results
were already obtained in [18]. This leads to a general formula for these vevs, which can
be found in (3.29) below. Despite its intimidating aspect, it is not difficult to implement
it in a computer routine to obtain the vevs for any torus knot. The equation (3.29) is
of course a result interesting in its own, and we hope that it will be helpful in exploring
the generalizations of the HOMFLY polynomial to arbitrary irreducible representations of
\( SU(N) \).

Using the general formula (3.29), we will test the conjecture presented in [2] for some
nontrivial knots. We will find that, in all the examples that we have checked, the polyno-
mials \( f_R \) have in fact the structure predicted by [2]. This is a highly nontrivial fact from
the point of view of Chern-Simons gauge theory, and we regard it as a strong evidence for
the duality advocated in [5][6][7][2].

There are in fact two different predictions in [2], which are in a sense complementary.
The first one predicts the structure of the polynomials \( f_R \); it is based on a target space
interpretation, and it is nonperturbative. The second one is perturbative and it is based
on the worldsheet interpretation of the Chern-Simons vevs presented in [11]. These two
predictions are related in a very interesting way. More precisely, it turns out that the
perturbative structure of the free energy of the open string gives some “sum rules” on the
integers that count BPS configurations. We have also found complete agreement with the
perturbative prediction in all the examples that we have checked.

The paper is organized as follows: in section 2, we describe the conjecture presented
in ref. [2], which expresses a generating functional of Chern-Simons gauge theory in terms
of certain polynomials \( f_R \). We extract from the conjecture a “master equation” which
allows us to obtain these functions from usual vevs in Chern-Simons gauge theory through
a recursive procedure. In section 3, we obtain a general formula for the vevs of torus knots
in arbitrary irreducible representations of \( SU(N) \). This section contains the arguments
leading to formula (3.29), which are independent of the rest of the paper. It could be
skipped in a first reading. In section 4, we use formula (3.29) to obtain some of the
polynomials \( f_R \), taking as an example the right-handed trefoil knot. The results are in
full agreement with the conjecture of [2]. In section 5 we show that the perturbative
point of view gives some nontrivial constraints among the integer invariants that appear
in the polynomials \( f_R \), and we also show that the connected vevs of Chern-Simons have
the structure dictated by these constraints. Finally, in section 6, we conclude with some comments and open problems. An appendix collects the expressions of $f_R$ for the right-handed trefoil knot for all irreducible representations of $SU(N)$ whose associated Young Tableaux contains four boxes.

2. Extracting string amplitudes from Chern-Simons gauge theory

We first recall some basic aspects of Chern-Simons gauge theory, mainly to fix our notation. Chern-Simons gauge theory is a topological gauge theory whose action is,

$$S = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

(2.1)

where $A$ is a gauge connection on some vector bundle over a three-manifold $M$, and $k$ is the coupling constant. From the holonomy of the gauge field around a closed loop $\gamma$ in $M$,

$$U = \text{P exp} \oint_{\gamma} A,$$

(2.2)

one can construct a natural class of topological observables, the gauge-invariant Wilson loop operators, which are given by

$$W^\gamma_R(A) = \text{Tr}_R U,$$

(2.3)

where $R$ denotes an irreducible representation of $SU(N)$. Some of the standard topological invariants that have been considered in the context of Chern-Simons gauge theory are vevs of products of these operators:

$$\langle W_{R_1}^{\gamma_1} \cdots W_{R_n}^{\gamma_n} \rangle = \frac{1}{Z(M)} \int [DA] \left( \prod_{i=1}^{n} W_{R_i}^{\gamma_i} \right) e^{iS},$$

(2.4)

where $Z(M)$ is the partition function of the theory. In this paper we will consider an enlarged set of operators which, to our knowledge, has not been studied from a Chern-Simons gauge theory point of view for non-trivial knots. These operators involve, besides the standard Wilson loops and their products, additional products with traces of powers of the holonomy (2.2). We will compute their vevs for the case of torus knots. In the process
we will derive a formula for the vevs of Wilson loops in arbitrary irreducible representations of the gauge group $SU(N)$. The resulting vevs will be expressed in terms of the variables
\[
 t = \exp\left(\frac{2\pi i}{k + N}\right), \quad \lambda = t^N. \tag{2.5}
\]

In order to make a precise test of the conjecture presented in ref. [2], we will consider the vev of the operator,
\[
 Z(U, V) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} U^n \text{Tr} V^n\right], \tag{2.6}
\]
where $U$ is the holonomy of the Chern-Simons $SU(N)$ gauge field (2.2), and $V$ is an $SU(M)$ matrix that can be regarded as a source term. In this operator the trace is taken over the fundamental representation. In what follows, when no representation is indicated in a trace, it should be understood that it must be taken in the fundamental representation.

The main conjecture of [2] has two parts. First, it states that the vev of (2.6) can be written as,
\[
 \langle Z(U, V) \rangle = \exp\left(\sum_{n=1}^{\infty} \sum_{R} f_R(t^n, \lambda^n) \text{Tr}_R V^n\right), \tag{2.7}
\]
where the sum over $R$ is a sum over irreducible representations of $SU(M)$. Second, it predicts the following structure for the functions $f_R(t, \lambda)$:
\[
 f_R(t, \lambda) = \sum_{s, Q} N_{R,Q,s} \lambda^Q t^s, \tag{2.8}
\]
where $N_{R,Q,s}$ are integer numbers, and the $Q$ and $s$ are, in general, half-integers (however, for a given $f_R$, the $Q$ differ by integer numbers). In writing (2.7), and to be able to compare to the results in Chern-Simons gauge theory, we have performed an analytic continuation, as suggested in [2]. The prediction (2.8) is based on the duality between Chern-Simons theory and topological string theory. As explained in [2], given a knot $K$ in $S^3$ one constructs a Lagrangian submanifold $C_K$ in the noncompact Calabi-Yau $\mathcal{O}(-1) + \mathcal{O}(-1) \rightarrow S^2$ (the resolution of the conifold). The integers $N_{R,Q,s}$ count, very roughly, holomorphic maps from Riemann surfaces with boundaries to the Calabi-Yau, in such a way that the boundaries are mapped to $C_K$. A more precise understanding of the integers $N_{R,Q,s}$

\[\text{A word of caution about notation: in [18], the variable $\lambda$ is denoted $t^{N-1}$. Also, in order to compare to [2], notice that our $t$ is their exp$(i\lambda)$, and our $\lambda$ is their exp$t$.} \]
is given by the target space interpretation of the string amplitudes. In this interpretation, one reformulates the counting problem in terms of D-branes. One considers configurations of D2 branes ending on $C_K$, in the presence of $M$ D4 branes wrapping $C_K$ and filling an $\mathbb{R}^2$ in the uncompactified spacetime. The D2 branes are BPS particles from the two-dimensional point of view. These particles are characterized by their magnetic charge, their bulk D2 brane charge, and their spin, which correspond, respectively, to $R$, $Q$, and $s$ in (2.8). The integer $N_{R,Q,s}$ counts the number of BPS states with these quantum numbers. We then see that the conjecture of [2] makes a remarkable connection between knot invariants and an enumerative problem in the context of symplectic and algebraic geometry, and that the polynomials $f_R$ can be regarded as counting functions for this enumerative problem.

In this section we will prove the first part of the conjecture. It follows from simple group theoretical arguments. Thus, it will be established that the vevs of Wilson loops in arbitrary irreducible representations of the gauge group can be encoded in the functions $f_R(t, \lambda)$. This also gives a concrete procedure to compute these functions from Chern-Simons vevs, and using this procedure we will present a highly nontrivial evidence for (2.8) in the case of torus knots.

Our starting point is the construction of a set of linear equations for the functions $f_R(t, \lambda)$ in terms of vevs of standard Wilson loops in arbitrary irreducible representations. To carry this out, it is convenient to use the following basis of class functions (see, for example, [20][21][22]). Take a vector $\vec{k}$ with an infinite number of entries, almost all zero, and whose nonzero entries are positive integers. Given such a vector, we define:

\[
\ell = \sum j k_j, \quad |\vec{k}| = \sum k_j. \tag{2.9}
\]

We can associate to any vector $\vec{k}$ a conjugacy class $C(\vec{k})$ of the permutation group $S_\ell$. This class has $k_1$ cycles of length 1, $k_2$ cycles of length 2, and so on. The number of elements of the permutation group in such a class is given by [23]

\[
|C(\vec{k})| = \frac{\ell!}{\prod k_j! \prod j^{k_j}}. \tag{2.10}
\]

Equivalently, the vectors $\vec{k}$ with $\sum_j jk_j = \ell$ are in one-to-one correspondence with the partitions of $\ell$. Given an ordered $h$-uple of positive integers $(n_1, \cdots, n_h)$, we can map it to a vector $\vec{k}$ by putting $k_i$ equal to the number of $i$'s in the $h$-uple. Notice that $h = |\vec{k}|$, and that there are $h!/\prod k_j!$ different $h$-uples giving the same vector $\vec{k}$. 

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We now introduce the following basis in the space of class functions, labeled by the vectors \( \vec{k} \):

\[
\Upsilon_{\vec{k}}(U) = \prod_{j=1}^{\infty} \left( \text{Tr} U^j \right)^{k_j}. \tag{2.11}
\]

It is easy to see that:

\[
Z(U, V) = 1 + \sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} \Upsilon_{\vec{k}}(U) \Upsilon_{\vec{k}}(V), \tag{2.12}
\]

since we are assuming \( \ell > 0 \).

Let’s now consider the expansion of the exponent in (2.7) in terms of the basis (2.11). We first recall Frobenius formula to express traces in an arbitrary irreducible representation of \( SU(M) \) in terms of the elements of the basis (2.11) referred to this group:

\[
\text{Tr}_R(V) = \sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} \chi_R(C(\vec{k})) \Upsilon_{\vec{k}}(V). \tag{2.13}
\]

In this formula, the irreducible representation \( R \) can be associated to a Young diagram in the standard way. The sum is then over conjugacy classes with \( \ell \) equal to the number of boxes in the diagram.

To analyze the expansion in (2.7), we have to write \( \text{Tr}_R V^n \) in terms of the basis (2.11). To do this it is convenient to define the following vector \( \vec{k}_{1/n} \). Fix a vector \( \vec{k} \), and consider all the positive integers that satisfy the following condition: \( n|j \) for every \( j \) with \( k_j \neq 0 \). Notice that \( n = 1 \) always satisfies this condition. When this happens, we will say that “\( n \) divides \( \vec{k} \),” and we will denote this as \( n|\vec{k} \). We can then define the vector \( \vec{k}_{1/n} \) whose components are:

\[
(\vec{k}_{1/n})_i = k_{ni}. \tag{2.14}
\]

The vectors which satisfy the above condition and are “divisible by \( n \)” have the structure \((0, \ldots, k_n, 0, \ldots, 0, k_{2n}, \ldots)\), and the vector \( \vec{k}_{1/n} \) is then given by \((k_n, k_{2n}, \ldots)\). It is a simple combinatorial exercise to prove that the exponent in (2.7) is given by:

\[
\sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} \sum_{n|\vec{k}} n^{\vec{k}} \sum_{R} \chi_R(C(\vec{k}_{1/n})) f_R(t^n, \lambda^n) \Upsilon_{\vec{k}}(V). \tag{2.15}
\]

In this equation, the third sum is over representations of \( S_\ell \). We now define a generalization of the cumulant expansion for the vevs we are considering. First, associate to any \( \vec{k} \) the
polynomial $p_{\vec{k}}(x) = \prod_j x_j^{k_j}$ in the variables $x_1, x_2, \ldots$. We then define the “connected” coefficients $a_{\vec{k}}^{(c)}$ as follows:

$$\log \left( 1 + \sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} a_{\vec{k}} p_{\vec{k}}(x) \right) = \sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} a_{\vec{k}}^{(c)} p_{\vec{k}}(x).$$

(2.16)

One has, for example:

$$a_{(2,0,...)}^{(c)} = a_{(2,0,...)} - a_{(1,0,...)}^2,$$
$$a_{(1,1,0,...)}^{(c)} = a_{(1,1,0,...)} - a_{(1,0,...)} a_{(0,1,0,...)},$$
$$a_{(0,...,0,1,0,...)}^{(c)} = a_{(0,...,0,1,0,...)},$$

and so on. For vectors of the form $(n, 0, \ldots)$, this is just the cumulant expansion.

Define now the vevs:

$$G_{\vec{k}}(U) = \langle \Upsilon_{\vec{k}}(U) \rangle.$$  

(2.18)

Using (2.3) (2.12) (2.16) and (2.18), we find:

$$\log \langle Z(U, V) \rangle = \sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} G_{\vec{k}}^{(c)}(U) \Upsilon_{\vec{k}}(V).$$

(2.19)

Since the $\Upsilon_{\vec{k}}(V)$ are a basis in the space of class functions, we find that the equation (2.7) can be written as

$$G_{\vec{k}}^{(c)}(U) = \sum_{n|\vec{k}} n^{[\vec{k}]-1} \sum_R \chi_R(C(\vec{k}_1/n)) f_R(t^n, \lambda^n).$$

(2.20)

This is our “master equation”. It allows us to obtain the functions $f_R(t, \lambda)$ once we compute the vevs that appear on the left hand side. The way to do that is to consider all vectors $\vec{k}$ with a fixed $\ell$, where $\ell$ will be considered as the “order” of the expansion. The number of these vectors is the number of partitions of $\ell$, $p(\ell)$. At every order there are then $p(\ell)$ vevs $G_{\vec{k}}^{(c)}(U)$ and also $p(\ell)$ representations $R$ of $S_\ell$. The relations (2.20) provide $p(\ell)$ equations with $p(\ell)$ unknowns, the functions $f_R(t, \lambda)$. The data to solve the equations are the vevs (2.18) and the $f_{R'}(t, \lambda)$ for representations with $\ell' < \ell$ boxes. The procedure to find the polynomials is then recursive, and the structure one finds is very similar, in fact, to the recursive procedure which determines the integer invariants introduced in [24], as it is explained in [25].
The above system of linear equations has a unique solution. This follows from the fact that the associated matrix, \(|\mathbf{C}(\mathbf{k})|\chi_R(\mathbf{C}(\mathbf{k}))\), is invertible due to orthonormality of the characters. Thus, the first part of the conjecture, eq. (2.7), is proved.

There are two cases of the above expression which are particularly interesting. The first one is for \(\mathbf{k} = (\ell, 0, \ldots)\). In this case, the corresponding conjugacy class in \(S_\ell\) is the identity, and one finds

\[
\langle (\text{Tr} \, U)^\ell \rangle^{(c)} = \sum_R (\dim R) f_R(t, \lambda),
\]

where the sum is over the representations \(R\) of \(S_\ell\). The left hand side is the usual connected vev. The second example corresponds to the vector \(\mathbf{k} = (0, \ldots, 0, 1, 0, \ldots)\), where the nonzero entry is in the \(\ell\)-th position. In this case, we have to sum in (2.20) over all the divisors of \(\ell\), that we will denote by \(n\). The vector \(\mathbf{k}_{1/n}\) is then \((0, \ldots, 0, 1, 0, \ldots)\), where the nonzero entry is in the \(\ell/n\)-th position. The characters \(\chi(\mathbf{C}(\mathbf{k}_{1/n}))\) are different from zero only for the hook representations, i.e., those corresponding to Young diagrams with \((\ell/n) - s\) boxes in the first row, and one box in the remaining ones, for example,

\[
\begin{array}{cccccccc}
    & & & & & & & \\
    & & & & & & & \\
    & & & & & & & \\
    & & & & & & & \\
\end{array}
\]

The character is then \((-1)^s\) (see [23], 4.16). The formula (2.20) reads in this case:

\[
\langle \text{Tr} \, U^\ell \rangle = \sum_{n|\ell} \sum_{s=0}^{\ell/n-1} (-1)^s f_{\text{hook},s}(t^n, \lambda^n).
\]

3. Polynomial invariants for torus knots in arbitrary irreducible representations of \(SU(N)\)

In order to obtain the functions \(f_R\) from the master equation (2.20) one needs to compute the connected functions \(G_{\mathbf{k}}^{(c)}(U)\). After using (2.18) and the inverse of Frobenius formula (2.13), it turns out that these involve the computation of vevs of Wilson loops in arbitrary irreducible representations of \(SU(N)\). As stated in the introduction these vevs are known only for some particular cases. In order to have a good testing ground of the conjecture (2.7) it would be desirable to have a formula for these vevs valid for any representation, at least for some particular class of knots. The goal of this section is to derive such a formula for torus knots. The result is contained in eq. (3.29) below. The arguments leading to it are independent of the rest of the paper and thus this section could be skipped in a first reading. The techniques used to obtain the formula (3.29) are based on the application of the operator formalism to Chern-Simons gauge theory [26], that in the case of torus knots leads to the useful concept of knot operators [16].
3.1. Knot operators

Knot operators for torus knots were introduced in [13]. They allow the computation of vevs of Wilson loops corresponding to this type of knots for arbitrary irreducible representations of the gauge group. The first piece of data we need to introduce these operators is the Hilbert space of Chern-Simons gauge theory on a torus [26]. This space has an orthonormal basis $|p\rangle$ labeled by weights $p$ in the fundamental chamber of the weight lattice of $SU(N)$, $F_l$, where $l = k + N$. We take as representatives of $p$ the ones of the form $p = \sum_i p_i \lambda_i$, where $\lambda_i$, $i = 1, \cdots, N - 1$, are the fundamental weights, $p_i > 0$ and $\sum_i p_i < l$. The vacuum is the state $|\rho\rangle$, where $\rho$ is the Weyl vector (i.e., the sum of all the fundamental weights).

Torus knots are labeled by two coprime integers $(n, m)$. They correspond to winding numbers around the two non-contractible classes of cycles, $A$ and $B$ on the torus. Let $m$ be the number of times that the torus knot winds around the axis of the torus, and let $\Lambda$ be the highest weight of an irreducible representation. Then, the Wilson loop corresponding to that torus knot is represented by the following operator:

$$W_{\Lambda}^{(n,m)}|p\rangle = \sum_{\mu \in M_{\Lambda}} \exp\left[-i\pi \mu^2 \frac{nm}{k + N} - 2\pi i \frac{m}{k + N} p \cdot \mu\right] |p + n\mu\rangle. \quad (3.1)$$

In this equation, $M_{\Lambda}$ is the set of weights corresponding to the irreducible representation $\Lambda$.

To compute the vev of the Wilson loop around a torus knot in $S^3$, one proceeds as follows: first of all, one makes a Heegard splitting of $S^3$ into two solid tori. Then, one puts the torus knot on the surface of one of the solid tori by acting with the knot operator (3.1) on the vacuum. Finally, one glues together the tori by performing an $S$-transformation. There is an extra subtlety related to the framing dependence in Chern-Simons gauge theory, since the vev computed in this way has to be corrected with a phase. In the standard framing the vev of the Wilson loop is given by:

$$\langle W_{\Lambda}^{(n,m)} \rangle = e^{2\pi i nm h_{\rho + \Lambda}} \frac{\langle \rho | SW_{\Lambda}^{(n,m)} | \rho \rangle}{\langle \rho | S | \rho \rangle}, \quad (3.2)$$

where,

$$h_{\rho} = \frac{p^2 - \rho^2}{2(k + N)}, \quad (3.3)$$

is the conformal weight of the primary fields in the associated WZW model at level $k$. 

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Our next task is to provide a more precise expression for the vev \((3.2)\). When acting with the knot operator \((3.1)\) on the vacuum, we get the set of weights \(\rho + n\mu\), where \(\mu \in M_\Lambda\). These weights will have representatives in the fundamental chamber, which can be obtained by a series of Weyl reflections. If the representative has a vanishing component, then the corresponding state in the Hilbert space is zero due to antisymmetry of the wave function under Weyl reflections. The set of weights that have a nonzero representative in \(F_l\) will be denoted by \(M(n, \Lambda)\), and it depends on the irreducible representation with highest weight \(\Lambda\), and on the integer number \(n\). The representative of \(\rho + n\mu\) in \(M(n, \Lambda)\) will be denoted by \(\rho + \mu_n\). The matrix elements of \(S\) have the explicit expression,

\[
S_{p,p'} = c(N, k) \sum_{w \in W} \epsilon(w) \exp \left[ -\frac{2\pi i p \cdot w(p')}{k + N} \right],
\]

where \(c(N, k)\) is a constant depending only on \(N\) and \(k\), and the sum is over the Weyl group of \(SU(N)\), \(W\). Using this, the vev \((3.2)\) can be written as:

\[
e^{2\pi i n m h_{\rho + \Lambda}} \sum_{\mu \in M(n, \Lambda)} \exp \left[ -i\pi \mu^2 \frac{nm}{k + N} - 2\pi i \frac{m}{k + N} p \cdot \mu \right] \text{ch}_{\mu_n} \left[ -\frac{2\pi i}{k + N} \rho \right].
\]

In this expression, we have used the Weyl formula for the character:

\[
\text{ch}_\Lambda(a) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\Lambda + \rho) \cdot a}}{\sum_{w \in W} \epsilon(w) e^{w(\rho) \cdot a}}.
\]

Notice that, since the representatives \(\mu_n\) live in \(F_l\), they can be considered as highest weights for a representation, hence the above expression \((3.5)\) makes sense.

In practice, the main problem to compute this vevs explicitly is to find the nonzero representatives of the weights that appear in \((3.1)\), and to find an expression for the characters in \((3.3)\). Fortunately, this has been done in [18] in a slightly different context. In that paper, these problems were solved for all the weights in the product representation \(V^{\otimes s}\), where \(V\) is the fundamental representation of \(SU(N)\) and \(s\) is any integer. Since all the representations of \(SU(N)\) that correspond to Young diagrams with \(s\) boxes are in fact contained in the reducible tensor product \(V^{\otimes s}\), we only have to combine the results of [18] with some simple group theory. This will give an explicit expression for the vev value of Wilson loops for torus knots in arbitrary representations of \(SU(N)\).
3.3. Group theory

To obtain the expression for the vev of the Wilson loop, we need the weight space associated to arbitrary representations of $SU(N)$. It is very convenient to regard this space as a subspace of the weight space associated to the reducible representation $V^\otimes s$. Let’s denote by $\mu_i$, $i = 1, \ldots, N$ the weights of the fundamental representation of $SU(N)$. Any weight in $V^\otimes s$ will have the form

$$k_1 \mu_{i_1} + \cdots + k_r \mu_{i_r}, \ 1 \leq i_1 < \ldots < i_r \leq N,$$

(3.7)

where $(k_\lambda) = (k_1, \ldots, k_r)$ is an ordered partition of $s$, i.e. an $r$-tuple that sums up to $s$. The $k_\lambda$ will be taken as strictly positive integers, therefore $1 \leq r \leq s$. The corresponding unordered partition will be simply denoted by $k$. Unordered partitions for $SU(N)$ will be written as $N$-tuples with nonincreasing components, as in [23]. The set of weights (3.7), for a fixed $(k_\lambda)$, will be denoted by $M_{k_\lambda}$.

Consider now a irreducible representation $R$ of $SU(N)$, associated to the highest weight

$$\Lambda = \sum_{i=1}^{N-1} a_i \lambda_i.$$

(3.8)

This representation can be labeled by a Young diagram with $s = \sum_i ia_i$ boxes in the usual way. Equivalently, we can assign to the highest weight (3.8) an unordered partition of $s$:

$$a = (a_1 + \cdots + a_{N-1}, a_2 + \cdots + a_{N-1}, \ldots, a_{N-1}, 0).$$

(3.9)

The weight space of this representation can always be written as follows

$$M_\Lambda = \bigcup_{k_\lambda} m^\Lambda_{(k_\lambda)} M_{k_\lambda},$$

(3.10)

where the $m^\Lambda_{(k_\lambda)}$ are nonnegative integers giving the multiplicities of the weights (3.7) in $M_\Lambda$. This can be proved as follows (see [23] for more details). The irreducible representation associated to $\Lambda$ is given by $S_a(V)$, where $S_a$ is the Schur-Weyl functor. Any endomorphism of $V$ will extend to $S_a(V)$, and its character will be given by the Schur polynomial $S_a(x_1, \cdots, x_N)$, where $x_1, \cdots, x_N$ are the eigenvalues of $g$. The Schur polynomials can be expanded in terms of the symmetric polynomials $F_k$, which are also labeled by unordered partitions of $s$, $k = (k_1, \cdots, k_N)$ (with $k_1 \geq \cdots \geq k_N$). $F_k$ is the sum of the elementary monomial $X^k = x_1^{k_1} \cdots x_N^{k_N}$ and all the monomials obtained from it by
permuting the variables. The set of $X^k$ and its permutations is then labeled by ordered partitions. The expansion of the Schur polynomials is given by:

$$S_a = \sum_k N_{ak} F_k,$$

(3.11)

where the $N_{ak}$ are called the Kostka numbers. These numbers are nonnegative integers and can be also computed as the number of ways one can fill the diagram $a$ with $k_1$ 1’s, $k_2$ 2’s, ..., $k_r$ r’s in such a way that the entries in each row are nondecreasing and those in each column are strictly increasing. Since the $x_i$, $i = 1, \cdots, N$, correspond to the weights $\mu_i$ of the fundamental representation, each of the monomials in $F_k$ corresponds to a one-dimensional weight space with a weight of the form (3.7). The different monomials in $F_k$ are in one-to-one correspondence with the different ordered partitions associated to the unordered partition $k$. We have then proved the equality (3.10).

From the proof above follows that in the decomposition (3.10) all the ordered partitions corresponding to the same unordered partition appear with the same multiplicity, and moreover that

$$m^\Lambda_{(k, \lambda)} = N_{ak},$$

(3.12)

where $a$ is the partition associated to $\Lambda$. We can then compute the multiplicities in (3.10) very easily. For example, for $R = \text{Sym}^s(V)$, we have $\Lambda = s\lambda_1$, and $m^s_{(k, \lambda)} = 1$ for every ordered partition of $s$. For $R = \wedge^s V$ one has $\Lambda = \lambda_s$, and $m^\lambda_{(k, \lambda)} = 0$ for every $(k, \lambda)$ except for $(k, \lambda) = (1, 1, \cdots, 1)$, where the multiplicity is one. For the diagram $\mathfrak{P}$, we can represent (3.10) as:

$$\mathfrak{P} = 2(1,1,1) + (2,1) + (1,2),$$

(3.13)

where the vectors in the r.h.s. represent ordered partitions, and the coefficients are the multiplicities. For representations with four boxes one has:

$$\begin{align*}
\mathfrak{P} &= 3(1,1,1,1) + (1,3) + (3,1) + (2,2) + 2\{(2,1,1) + (1,2,1) + (1,1,2)\}, \\
\mathfrak{Q} &= 3(1,1,1,1) + (2,1,1) + (1,2,1) + (1,1,2), \\
\mathfrak{R} &= 2(1,1,1,1) + (2,2) + (2,1,1) + (1,2,1) + (1,1,2).
\end{align*}$$

(3.14)

3.4. General formula

We are now in a position to be more explicit about the expression (3.3). Using the decomposition (3.10), we can write all the weights for the irreducible representation $R$ in
the form (3.7). We have to find now which vectors of the form $\rho + n\mu$ have a representative in $\mathcal{F}_l$, and the explicit structure of such a weight. This has been completely solved in Theorem 4.1 of [18]. The main output of this theorem is that the weights

$$\rho + n(k_1\mu_1 + \cdots + k_r\mu_r)$$

(3.15)

associated to a partition of cardinal $r$ give a representative only for certain values of the indices $i_1, \ldots, i_r$. The procedure to get these indices, as well as the corresponding representative, is rather involved, but we will give it here for completeness. For further details, we refer the reader to [18].

The arrangement of indices $i_\lambda, \lambda = 1, \ldots, r$, producing a weight in $\mathcal{F}_l$ is contained in the set specified by the following conditions:

1. $i_\lambda \leq k_\lambda n$,
2. $i_\lambda = i_\mu + k_\lambda n, \mu < \lambda$,

(3.16)

in such a way that, in (II), no previous index $i_\nu, \nu < \lambda$, has the form $i_\nu = i_\mu + k_\nu n, \mu < \nu$. Given an arrangement of indices like this, with $r - k$ indices verifying condition (I) (which will be called of type I) and $k$ indices verifying condition (II) (which will be called of type II), a weight belonging to $\mathcal{F}_l$ is obtained if and only if:

$$i_\mu - i_\nu + (k_\nu - k_\mu)n \neq 0,$$

(3.17)

for every pair of indices $i_\mu, i_\nu$, verifying (I). The set of arrangements of indices selected in this way will be denoted by $\mathcal{I}_{(k,\lambda)}(n)$, and the corresponding set of weights will be denoted by $M_{\mathcal{I}_{(k,\lambda)}(n)}$.

To each arrangement of indices in $\mathcal{I}_{(k,\lambda)}(n)$ we will associate a canonical representative in $\mathcal{F}_l$ accompanied by a sign. This association is carried out by the following procedure:

1) For indices of type I, which will be denoted by $i_{\lambda_1}, \ldots, i_{\lambda_{r-k}}$, one defines a total order relation according to:

$$i_{\lambda_p} > i_{\lambda_q} \text{ iff } i_{\lambda_p} - i_{\lambda_q} + (k_{\lambda_q} - k_{\lambda_p})n > 0.$$

(3.18)

This relation defines a permutation $\tau$ of the set of indices of type I under consideration with respect to their natural ordering:

$$\tau = \begin{pmatrix} i_{\lambda_1} & i_{\lambda_2} & \cdots & i_{\lambda_{r-k}} \\ i_{\tau(\lambda_1)} & i_{\tau(\lambda_2)} & \cdots & i_{\tau(\lambda_{r-k})} \end{pmatrix}.$$
2) For the \( k \) indices of type II, \( i_{\nu_1}, \ldots, i_{\nu_k}, \ i_{\nu_1} < \cdots < i_{\nu_k} \), one takes the set of indices \( \hat{i}_{\nu_1}, \ldots, \hat{i}_{\nu_k} \), verifying \( i_{\nu_p} = \hat{i}_{\nu_p} + k_{\nu_p} n \), and defines on it the order relation inherited from the natural ordering of the indices \( i_{\nu_p} \):

\[
i_{\nu_p} \succ i_{\nu_q} \iff i_{\nu_p} > i_{\nu_q}.
\] (3.20)

This gives again a permutation \( \sigma \) with respect to the natural ordering of the set \( \hat{i}_{\nu_p} \):

\[
\sigma = \begin{pmatrix}
i_{\sigma^{-1}(\hat{i}_1)} & i_{\sigma^{-1}(\hat{i}_2)} & \cdots & i_{\sigma^{-1}(\hat{i}_k)} \\
i_{\hat{i}_1} & i_{\hat{i}_2} & \cdots & i_{\hat{i}_k}
\end{pmatrix},
\] (3.21)

with \( i_{\sigma^{-1}(\hat{i}_1)} < i_{\sigma^{-1}(\hat{i}_2)} < \cdots < i_{\sigma^{-1}(\hat{i}_k)} \).

3) Define \( r - k \) numbers \( \xi(\lambda_p), \ p = 1, \ldots, r - k \), associated to type I indices as follows: \( \xi(\lambda_p) \) is the number of type II indices preceding the type I index \( i_{\lambda_p} \) in the original arrangement of indices \( i_1, \ldots, i_r \), in (3.13).

The canonical representative in \( F_1 \) of the weight (3.13) is the weight:

\[
\rho + p_1 \lambda_1 + p_2 \lambda_2 + \cdots + p_{r-k} \lambda_{r-k} + \lambda_{i_{\mu_1}+r-k-1} + \lambda_{i_{\mu_2}+r-k-2} + \cdots + \lambda_{i_{\mu_{r-k}}},
\] (3.22)

where \( p_i, i = 1, \ldots, r - k \), are given by:

\[
p_1 = i_{\tau(\lambda_2)} - i_{\tau(\lambda_1)} + (k_{\tau(\lambda_1)} - k_{\tau(\lambda_2)}) n - 1,
p_2 = i_{\tau(\lambda_3)} - i_{\tau(\lambda_2)} + (k_{\tau(\lambda_2)} - k_{\tau(\lambda_3)}) n - 1,
\]

\[
\vdots
\]

\[
p_{r-k} = k_{\tau(\lambda_{r-k})} n - i_{\tau(\lambda_{r-k})},
\] (3.23)

and the indices \( i_{\mu_p} \) in (3.22) are the complementary ones to the indices \( \{\hat{i}_{\nu_p}\}_{p=1,\ldots,k} \), i.e., those indices \( i_{\mu_p} \) such that no index \( i_{\nu} > i_{\mu_p} \) has the form \( i_{\nu} = i_{\mu_p} + k_{\nu} n \). They are ordered according to their natural ordering: \( i_{\mu_1} < \cdots < i_{\mu_{r-k}} \).

Finally, the sign associated to this weight because of the Weyl reflections needed to obtain it is:

\[
\epsilon(\tau)\epsilon(\sigma)(-1)^{\sum_{p=1}^{r-k} i_{\mu_p} - i_{\mu_p} + \xi(\lambda_p)}.
\] (3.24)

This result gives then an explicit description of the set of weights \( M(n, \Lambda) \) in (3.5):

\[
M(n, \Lambda) = \bigcup_{(k_\lambda)} m^\Lambda_{(k_\lambda)} M_{\I(X_\lambda)}(n),
\] (3.25)
and the representatives of these weights have the form $|\lambda\rangle$.

The last ingredient in (3.3) is the character, which has also been computed in [18] for weights with the structure of (3.22). Before doing this, it is useful to introduce $q$-numbers and $q$-combinatorial numbers as follows:

$$[x] = t^x - t^{-x}, \quad (x) = t^x - 1,$$

$$\frac{x}{y} = \frac{[x]!}{[x-y]![y]!}. \quad (3.26)$$

One can then easily prove that

$$\begin{bmatrix} N + p \\ p \end{bmatrix} = \lambda^{-\frac{1}{2}} p! \frac{\prod_{i=1}^{p-1}(\lambda - i^2)}{(p)!},
$$

$$\begin{bmatrix} N \\ i \end{bmatrix} = \lambda^{-\frac{1}{2}} \sum_{i=0}^{\frac{i-1}{2}}(\lambda - i^2) \frac{(i)!}{(i)!}. \quad (3.27)$$

The character for the weight (3.22) is given by

$$\text{ch}_\lambda \left[ -\frac{2\pi i}{k+N\rho} \right] = \prod_{k=1}^{r-1} [p_k + 1] \prod_{\lambda=1}^{r-1} p_{\lambda + r - 1} \prod_{1 \leq j < k \leq r} [i_k - i_j]
\times \prod_{k=1}^{r} \prod_{i=1}^{\lambda=1} \left[ i_k \right] \left[ N + \sum_{\lambda=1}^{r} p_{\lambda + r - k} \right] \left[ N + \sum_{\lambda=1}^{r} p_{\lambda + r - k} \right] \left[ N \right]. \quad (3.28)$$

and it is in principle a rational function of $t^\pm$ and $\lambda^\pm$.

Taking all this into account, and evaluating the phase factor in (3.3), we can finally write the expression for the vev of a Wilson loop corresponding to an $(n, m)$ torus knot in an arbitrary irreducible representation $R$:

$$\langle W^{(n,m)} \rangle = \lambda^{\frac{mn(n-1)}{2}} t^m \sum \epsilon(\sigma) \epsilon(\tau) (-1)^{\sum_{\lambda=1}^{r} i_{\mu_{\lambda}} - i_{\mu_{\lambda}}}
\sum_{(k_\lambda)} m_{(k_\lambda)} t^{-\frac{mn}{2}} s_{\lambda=1} k_\lambda^2 \sum_{I_{(k_\lambda)}} \sum_{(n)} \tau_{\lambda=1}^\tau \epsilon(\sigma) \epsilon(\tau) (-1)^{\sum_{\lambda=1}^{r-1} i_{\mu_{\lambda}} - \mu_{\lambda} + \xi_\lambda} t^m \sum_{\lambda=1}^{r} k_\lambda i_\lambda
\times \prod_{\tau=1}^{r-1} [p_\tau + 1] \prod_{\lambda=1}^{r-1} p_{\lambda + r - k - 1} \prod_{1 \leq \sigma < \tau \leq r-k} [i_{\mu_{\tau}} - i_{\mu_{\sigma}}]
\times \prod_{\tau=1}^{r-k} \prod_{\mu_{\tau}} [i_{\mu_{\tau}}] \left[ N + \sum_{\lambda=1}^{r-k} p_{\lambda + r - k - \tau} \right] \left[ N \right], \quad (3.29)$$

where the highest weight associated to $R$ has been written as in (3.8), and the multiplicities are given by Kostka numbers. This expression is fairly complicated. It can be evaluated using a simple computer routine, and in some simple cases it can be explicitly computed, as we will see in the next section.
3.5. Some particular cases. Akutsu-Wadati polynomials

In this subsection, we will study some particular cases of the general formula (3.29). In particular, we will see that for gauge group $SU(2)$ it reduces to the Akutsu-Wadati polynomials of torus knots first obtained in [17].

As a first check of (3.29), and for completeness, let us consider the fundamental representation of $SU(N)$, with Young tableau $\young(\ )$ and $s = 1$. The vev in this case was obtained in the context of Chern-Simons gauge theory in [18]. There is only one index $1 \leq i \leq n$ involved in the procedure described above. The representatives of the weights that appear in the computation are, according to (3.22) and (3.23),

$$\rho + (n - i)\lambda_1 + \lambda_i,$$

and the character in this case is simply:

$$\text{ch}_{p\lambda_1 + \lambda_i} = \frac{[\bar{p}]}{[p + i]} \left[ N + p \right] \left[ N \right] / \left[ i \right].$$

(3.31)

Using the explicit expressions (3.27), we finally obtain

$$\langle W^{(n,m)} \rangle = t^{\frac{j}{2}} \lambda^{-\frac{j}{2}} \left( \frac{(m-1)(n-1)}{2} \right) \sum_{p+i+1 = n} (-1)^i t^{-m+i+\frac{j}{2}p(p+1)} \prod_{j=1}^{p} \frac{(\lambda - t^j)}{(i)!p!(p)!},$$

(3.32)

where we have redefined the index $i$. This is in fact the unnormalized HOMFLY polynomial of an $(n, m)$ torus knot. If we divide by the vev of the unknot, we find the expression for the HOMFLY polynomial first obtained in [1].

An interesting check of (3.29) is that, for $SU(2)$ and representations with isospin $j/2$, one obtains in fact the Akutsu-Wadati polynomials for torus knots. The discussion is very similar to the reduction of the HOMFLY polynomial for torus links to the Jones polynomial studied in [18], section 5.

To obtain the Akutsu-Wadati polynomial for isospin $j/2$, we consider the representation of $SU(N)$ associated to $\text{Sym}^j(V)$, and given by the Young tableau

\[
\begin{array}{cccccccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ }\\
\end{array}
\]

with $j$ boxes. We then put $N = 2$. Notice that there are other representations of $SU(N)$ that reduce to the $j/2$ for $SU(2)$, but the above is the simpler one. From the expression
for the character (3.28) we see that it vanishes for \( i_k > 2 \). This implies, in particular, that the only partitions contributing in this limit have at most cardinal two. From the partition \((j)\), we obtain the following weights in \( F_1 \): \( j n \lambda_1 \) (with sign \(-1\)) and \((jn - 2)\lambda_1 + \lambda_2 \) (with sign \(+1\)). For the partitions \((l, j - l)\), with \( 1 \leq l \leq j - 1 \), one finds weights of the form \( p\lambda_1 + q\lambda_2 \), where \( p = n(2l - j) \), \( q = n(j - l) \) if \( 1 + n(2l - j) > 0 \), and \( p = (j - 2l)n - 2 \), \( q = nl + 1 \) if \( 1 + n(2l - j) < 0 \). In the first case, the sign is \(+1\), and it is \(-1\) in the second case. The character of this weight for \( \lambda = t^2 \) is simply

\[
\text{ch}_{p\lambda_1 + q\lambda_2} = t^{-\frac{p}{2}} \frac{t^{p+1} - 1}{t - 1}.
\]

(3.33)

Taking into account that \( m_j^{\lambda_1} = 1 \) for all the ordered partitions, one finds, after a short calculation:

\[
\langle W_j^{(n,m)} \rangle = t^{-\frac{j}{2}} \frac{t^{(n-1)(m-1)}}{t - 1} \sum_{l=0}^{j} t^{m(1+nl)(j-l)} (t^{1+nl} - t^{n(j-l)}) ,
\]

(3.34)

where the summands with \( l = 0, j \) come from the partition \((s)\), and the summands with \( 1 \leq l \leq j - 1 \) come from the partition \((l, j - l)\). The expression in (3.34) is in fact the unnormalized Akutsu-Wadati polynomial for the \((n, m)\) torus knot, in the representation of isospin \( j/2 \) [17].

4. Explicit results for \( f_R \)

The results of the previous section will allow us to compute the quantities on the left hand side of the master equation (2.20). These quantities are products of traces of powers of the holonomy associated to a given knot. Computations of this type are delicate from a field theory point of view because they involve products of operators evaluated for the same loop. The corresponding calculations are plagued with singularities which must be regularized. One way to do this, advocated in [15] and also suggested in [2], involves the use of Frobenius formula. In particular, what is needed is the inverse of (2.13):

\[
\Upsilon_{\vec{k}}(U) = \sum_R \chi_R(C(\vec{k})) \text{Tr}_R(U).
\]

(4.1)

All problems arising from products of operators evaluated for the same loop are avoided using this formula since one ends computing vevs of standard Wilson loops. Actually, it
is rather simple to prove that the choice (4.1) leads to the following general form for the functions \( f_R(t, \lambda) \):

\[
 f_R(t, \lambda) = \langle \text{Tr}_R(U) \rangle + \text{lower order terms},
\]

(4.2)

where “lower order terms” involves \( f_{R'}(t, \lambda) \) for representations \( R' \) with \( \ell' < \ell \). Thus the set of functions \( f_R(t, \lambda) \) is equivalent to the set of vevs in arbitrary irreducible representations.

This relation implies that the new polynomial invariants are basically the ordinary ones plus correction terms. As it follows from the master equation (2.20) these corrections terms are linear combinations of products of lower-order \( f_R \) evaluated at different arguments. The remarkable consequence that follows from the validity of the conjecture (2.8) is that the corrected polynomials possess integer coefficients which can be interpreted as the solutions to counting problems in the context of string theory.

Using (4.1) and the result for \( \text{Tr}_R(U) \) in (3.29) we will be able to obtain the functions \( f_R(t, \lambda) \) for torus knots after solving the master equation (2.20). We will present in this section the computations up to third order, where the order is set by \( \ell \), as we explained in section 2.

4.1. \( \ell = 1 \)

In this case, \( \vec{k} = (1, 0, \cdots) \) and (2.21) just says that

\[
 \langle \text{Tr}\Box U \rangle = f_{\Box}(t, \lambda).
\]

(4.3)

The left hand side of this equation is the unnormalized HOMFLY polynomial. To normalize it we have to divide it by the vev of the unknot:

\[
\langle \text{Tr}\Box U \rangle_u = \frac{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}.
\]

(4.4)

Due to the skein relations, the normalized HOMFLY polynomial always has the structure [27]:

\[
\frac{\langle \text{Tr}\Box U \rangle}{\langle \text{Tr}\Box U \rangle_u} = \sum_s p_s(\lambda)t^s.
\]

(4.5)

In this equation, the \( s \) take integer values, and \( p_s(\lambda) = \sum_j a_{s,j}\lambda^j \) are Laurent polynomials in \( \lambda \). The \( a_{s,j} \) are integer numbers. Therefore, \( f_{\Box}(t, \lambda) \) has indeed the structure predicted in (2.8). The integers \( N_{\Box,Q,s} \) are given by:

\[
 N_{\Box,j+1/2,s} = a_{s,j} - a_{s,j+1}.
\]

(4.6)
We then see that, for the fundamental representation, the integers introduced in (2) are simple linear combinations of the coefficients in the normalized HOMFLY polynomial. Notice that (4.4) is valid for any knot, since we have only used the general structure of the HOMFLY polynomial. As an example, let us consider the right-handed trefoil, which is the $(2, 3)$ torus knot. One obtains that
\[
f(t, \lambda) = \frac{1}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \left( -2\lambda^{\frac{1}{2}} + 3\lambda^{\frac{3}{2}} - \lambda^2 \right) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(-\lambda^{\frac{3}{2}} + \lambda^{\frac{5}{2}}),
\]
and from here one can easily extract the values of $N_{Q,s}$.

4.2. $\ell = 2$

In this case, there are two possible vectors corresponding to conjugacy classes: $(2, 0, \cdots)$, and $(0, 1, 0, \cdots)$. From (2.20), (2.21) and (2.23) we find two equations:
\[
\langle (\text{Tr} U)^2 \rangle - \langle \text{Tr} U \rangle^2 = f_{\Box}(t, \lambda) + f_{\Box}(t, \lambda),
\]
\[
\langle \text{Tr} U^2 \rangle = f_{\Box}(t, \lambda) - f_{\Box}(t, \lambda) + f_{\Box}(t^2, \lambda^2).
\]
A432s argued above, Frobenius formula (1.1) allows to express the new quantities appearing on the left of this equation in terms vevs of Wilson loops. For the case under consideration it leads to:
\[
\langle (\text{Tr} U)^2 \rangle = \langle \text{Tr}_{\Box} U \rangle + \langle \text{Tr}_{\Box} U \rangle, \\
\langle \text{Tr} U^2 \rangle = \langle \text{Tr}_{\Box} U \rangle - \langle \text{Tr}_{\Box} U \rangle.
\]
From these relations and eq. (4.8) we obtain, after taking into account (4.3):
\[
f_{\Box}(t, \lambda) = \langle \text{Tr}_{\Box} U \rangle - \frac{1}{2} \left( f_{\Box}(t, \lambda)^2 + f_{\Box}(t^2, \lambda^2) \right),
\]
\[
f_{\Box}(t, \lambda) = \langle \text{Tr}_{\Box} U \rangle - \frac{1}{2} \left( f_{\Box}(t, \lambda)^2 - f_{\Box}(t^2, \lambda^2) \right).
\]

We will now present explicit formulae for these functions in the simplest nontrivial case, namely the right-handed trefoil knot. The vev in the symmetric representation is given by:
\[
\langle \text{Tr}_{\Box} U \rangle = \frac{(\lambda - 1)(\lambda t - 1)}{\lambda(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2(1 + t)} \left( (\lambda t^{-1})^2(1 - \lambda t^2 + t^3 - \lambda t^3 + t^4 - \lambda t^5 + \lambda^2 t^5 + t^6 - \lambda t^6) \right).
\]
In this equation, we have explicitly factored out the vev for the unknot in the symmetric representation. The polynomial multiplying the fraction in the right hand side is then
the normalized polynomial invariant in the symmetric representation. One can see that this expression agrees with the result presented in [14]. It can also be easily checked that, when we substitute $\lambda \to t^2$, we obtain the Akutsu-Wadati polynomial for the right-handed trefoil in the $j = 2$ representation, as it should be.

For the antisymmetric representation we find:

$$\langle \text{Tr} U \rangle = \frac{(\lambda - 1)(\lambda - t)}{\lambda(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 (1 + t)} \left((\lambda t^{-2})^2(1 - \lambda - \lambda t + \lambda^2 t^2 + t^3 - \lambda t^3 - \lambda^2 t^4 + t^6)\right). \quad (4.12)$$

Notice that, when $N = 2$ (i.e., when $\lambda = t^2$), $\Box$ becomes the trivial representation and indeed (4.12) becomes 1.

Using (4.10) one finds:

$$f_{\Box}(t, \lambda) = \frac{t^{-\frac{1}{2}} \lambda(\lambda - 1)^2 (1 + t^2)(t + \lambda^2 t - \lambda (1 + t^2))}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}$$

$$f_\Box(t, \lambda) = -\frac{1}{t^3} f_{\Box}(t, \lambda). \quad (4.13)$$

The structure of these functions is in perfect agreement with (2.8). This computation makes clear that the prediction (2.8) is far from being trivial from the Chern-Simons side. The vevs (4.11) and (4.12) have complicated denominators that have to cancel out except for a single factor of $t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ when one subtracts the lower order contributions as in (4.10). Also notice that the coefficients of the functions in (4.13) are in fact integers, and again this is not obvious from (4.10) (which involves dividing by 2). These features become more and more remarkable as we increase the number of boxes of the representations, as we will see.

4.3. $\ell = 3$

At this order there are three vectors that contribute, $\vec{k} = (3, 0, \cdots)$, $\vec{k} = (1, 1, 0, \cdots)$, and $\vec{k} = (0, 0, 1, 0, \cdots)$. From (2.21) we obtain:

$$\langle (\text{Tr} U)^3 \rangle - 3 \langle \text{Tr} U \rangle (\langle \text{Tr} U \rangle) + 2 \langle \text{Tr} U \rangle^3 = f_{\Box \Box}(t, \lambda) + f_{\Box}(t, \lambda) + 2f_{\Box}(t, \lambda), \quad (4.14)$$

while from (2.23) one has:

$$\langle \text{Tr} U^3 \rangle = f_{\Box \Box}(t, \lambda) + f_{\Box}(t, \lambda) - f_{\Box}(t, \lambda) + f_{\Box}(t^3, \lambda^3). \quad (4.15)$$
Finally, the vector \((1, 1, 0, \ldots)\) gives us:
\[
\langle \text{Tr} U \text{Tr} U^2 \rangle - \langle \text{Tr} U \rangle \langle \text{Tr} U^2 \rangle = f_{\begin{array}{c}3 \\ \end{array}}(t, \lambda) - f_{\begin{array}{c}1 \\ \end{array}}(t, \lambda).
\]  
(4.16)

Using again Frobenius formula, we find:
\[
f_{\begin{array}{c}3 \\ \end{array}}(t, \lambda) = \langle \text{Tr} U \rangle - f_{\begin{array}{c}1 \\ \end{array}}(t, \lambda) f_{\begin{array}{c}3 \\ \end{array}}(t, \lambda) - \frac{1}{6} f_{\begin{array}{c}3 \\ \end{array}}(t, \lambda)^3
- \frac{1}{2} f_{\begin{array}{c}1 \\ \end{array}}(t, \lambda) f_{\begin{array}{c}2 \\ \end{array}}(t^2, \lambda^2) - \frac{1}{3} f_{\begin{array}{c}3 \\ \end{array}}(t^3, \lambda^3),
\]
\[
f_{\begin{array}{c}2 \\ \end{array}}(t, \lambda) = \langle \text{Tr} U \rangle - f_{\begin{array}{c}1 \\ \end{array}}(t, \lambda) f_{\begin{array}{c}2 \\ \end{array}}(t, \lambda) - \frac{1}{3} f_{\begin{array}{c}2 \\ \end{array}}(t, \lambda)^3 + \frac{1}{3} f_{\begin{array}{c}3 \\ \end{array}}(t^3, \lambda^3),
\]
\[
f_{\begin{array}{c}1 \\ \end{array}}(t, \lambda) = \langle \text{Tr} U \rangle - f_{\begin{array}{c}1 \\ \end{array}}(t, \lambda) f_{\begin{array}{c}1 \\ \end{array}}(t, \lambda) - \frac{1}{6} f_{\begin{array}{c}1 \\ \end{array}}(t, \lambda)^3 + \frac{1}{2} f_{\begin{array}{c}1 \\ \end{array}}(t, \lambda) f_{\begin{array}{c}2 \\ \end{array}}(t^2, \lambda^2) - \frac{1}{3} f_{\begin{array}{c}3 \\ \end{array}}(t^3, \lambda^3).
\]  
(4.17)

Let’s now present some results for the right-handed trefoil knot. For the representation \(\begin{array}{c}3 \\ \end{array}\), we find:
\[
\langle \text{Tr} \begin{array}{c}3 \\ \end{array} U \rangle = \frac{\lambda t - 1)(\lambda t^2 - 1)}{\lambda^3/2(t^{1/2} - t^{-1/2})^3(1 + t)(1 + t + t^2)} \left( (\lambda t^5 - \lambda t^6 + \lambda t^9 + \lambda^2 t^8 + t^{10} - \lambda t^{10} \\
- \lambda t^{11} + \lambda t^{12} + \lambda t^{12} - \lambda t^{13} + \lambda^2 t^{13} \right),
\]  
(4.18)

which also agrees with the computation in [14]. For the representation \(\begin{array}{c}2 \\ \end{array}\), we find:
\[
\langle \text{Tr} \begin{array}{c}2 \\ \end{array} U \rangle = \frac{(\lambda - 1)(\lambda - t)(\lambda t - 1)}{\lambda^3/2(t^{1/2} - t^{-1/2})^3(1 + t + t^2)} \left( \lambda^3 t^{-10}(-\lambda + \lambda^2 + t - \lambda t + \lambda^2 t \\
- \lambda^3 t - \lambda t^2 + \lambda^2 t^2 + \lambda t^3 + t^4 - 2 \lambda t^4 + \lambda^2 t^4 + t^5 - 2 \lambda t^5 \\
+ \lambda^2 t^5 - \lambda t^6 + \lambda^2 t^6 + t^7 + t^8 - \lambda t^8 + t^9 - \lambda t^9 - \lambda t^{10} + t^{13}) \right).
\]  
(4.19)

Notice that, when \(\lambda \rightarrow t^2\), the normalized polynomial (which is the polynomial inside the parentheses in (4.19)) goes to the Jones polynomial of the right-handed trefoil, since the representation \(\begin{array}{c}2 \\ \end{array}\) reduces to the fundamental representation \(j = 1\) when \(N = 2\).

Finally, for the representation \(\wedge^3 V\), with Young diagram \(\begin{array}{c}1 \\ \end{array}\), one has:
\[
\langle \text{Tr} \begin{array}{c}1 \\ \end{array} U \rangle = \frac{(\lambda - 1)(\lambda - t)(\lambda - t^2)}{\lambda^3/2(t^{1/2} - t^{-1/2})^3(1 + t)(1 + t + t^2)} \left( \lambda^3 t^{-10}(-\lambda + \lambda^2 + t - \lambda t + \lambda^2 t \\
- \lambda^3 t - \lambda t^2 + \lambda^2 t^2 + \lambda t^3 + t^4 - 2 \lambda t^4 + \lambda^2 t^4 + t^5 - 2 \lambda t^5 \\
+ \lambda^2 t^5 - \lambda t^6 + \lambda^2 t^6 + t^7 + t^8 - \lambda t^8 + t^9 - \lambda t^9 - \lambda t^{10} + t^{13}) \right).
\]  
(4.20)
Using these vevs, one finds:

\[
\begin{align*}
\textstyle f_{\infty}(\lambda, t) &= - \frac{\lambda^{\frac{3}{2}} t^{-1} (\lambda - 1)^2}{t^{\frac{3}{2}} - t^{-\frac{3}{2}}} \left( t^3 (1 + t + t^3) + \lambda^4 t^4 (1 + t + t^2 + t^3 + t^4 + t^6) \right. \\
\textstyle & \quad \quad - \lambda t \left( 1 + 3 t + 3 t^2 + 4 t^3 + 5 t^4 + 2 t^5 + 2 t^6 + t^7 \right) \right. \\
\textstyle & \quad \quad - \lambda^3 t \left( 1 + 3 t + 3 t^2 + 5 t^3 + 5 t^4 + 4 t^5 + 2 t^6 + 3 t^7 + t^9 \right) \\
\textstyle & \quad \quad + \lambda^2 \left( 1 + 2 t + 3 t^2 + 7 t^3 + 7 t^4 + 6 t^5 + 6 t^6 + 4 t^7 + t^8 + 2 t^9 \right), \\
\textstyle f_{\cal P}(t, \lambda) &= \frac{\lambda^{\frac{3}{2}} t^{-4} (\lambda - 1)^2 (1 + t + t^2)}{t^{\frac{3}{2}} - t^{-\frac{5}{2}}} \left( t^3 + \lambda^4 (t^2 + t^4) - \lambda t \left( 1 + 2 t + t^2 + 2 t^3 + t^4 \right) \right. \\
\textstyle & \quad \quad - \lambda^3 t \left( 2 + t + 3 t^2 + t^3 + 2 t^4 \right) + \lambda^2 \left( 1 + t + 3 t^2 + 3 t^3 + 3 t^4 + t^5 + t^6 \right), \\
\textstyle f_{\cal B}(t, \lambda) &= - \frac{(\lambda^{\frac{3}{2}} t^{-3})^3 (\lambda - 1)^2}{t^{\frac{3}{2}} - t^{-\frac{5}{2}}} \left( t^4 + t^6 + t^7 + \lambda^4 \left( t^3 + t^4 + t^5 + t^6 + t^7 \right) \right. \\
\textstyle & \quad \quad - \lambda^2 \left( 1 + 2 t + 2 t^2 + 5 t^3 + 4 t^4 + 3 t^5 + 3 t^6 + t^7 \right) \\
\textstyle & \quad \quad + \lambda^2 t \left( 2 + t + 4 t^2 + 6 t^3 + 6 t^4 + 7 t^5 + 7 t^6 + 3 t^7 + 2 t^8 + t^9 \right) \\
\textstyle & \quad \quad - \lambda^3 \left( 1 + 3 t^2 + 2 t^3 + 4 t^4 + 5 t^5 + 5 t^6 + 3 t^7 + 3 t^8 + t^9 \right). \\
\end{align*}
\]

(4.21)

Again, this is in perfect agreement with (2.8). In the appendix we list the \( f_R(t, \lambda) \) for the right-handed trefoil knot, for representations with four boxes.

4.4. Structure of \( f_R \)

The functions \( f_R(t, \lambda) \) that we have listed in this section, as well as many other examples that we have explicitly computed, have the structure predicted by (2.8). In all cases they can be written as

\[
\begin{align*}
\textstyle f_R(t, \lambda) &= \frac{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} P_R(t^{\frac{1}{2}}, \lambda^{\frac{1}{2}}),
\end{align*}
\]

(4.22)

where \( P_R(t^{\frac{1}{2}}, \lambda^{\frac{1}{2}}) \) is a Laurent polynomial in \( t^{\frac{1}{2}}, \lambda^{\frac{1}{2}} \) with integer coefficients. Notice that we have factored out the vev of the unknot in the fundamental representation. The above structure is far from being obvious from its definition, or from the explicit expressions given above: to get \( f_R(t, \lambda) \) we have to add up functions with rather complicated denominators, however the result has the simpler structure given in (4.22). Also, to obtain \( f_R(t, \lambda) \) we have to divide by \( \ell! \), however the coefficients of the resulting polynomial in (4.22) have integer coefficients. Notice that (4.22) has an extra piece of information when compared to
(2.8): namely, that one can extract a common factor $\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}$ from the functions $f_R(t, \lambda)$. It would be interesting to see if this is a general fact, and if it can be also predicted from the string side.

It also follows from our computations that, for a given irreducible representation $R$, the integers $N_{R,Q,s}$ are only different from zero for a finite number of values of $Q$ and $s$. However, the functions $f_R$ become more and more complicated as we increase the number of boxes, even for the trefoil knot (which is the simplest nontrivial knot). This seems to indicate that, given an irreducible representation $R$, there are always values of $Q$ and $s$ for which $N_{R,Q,s} \neq 0$. Therefore, for every nontrivial knot there seems to be an infinite number of nonzero integers $N_{R,Q,s}$.

4.5. The functions $f_R$ for the unknot

In [2] it was explicitly shown that, for the unknot, the functions $f_R(t, \lambda)$ vanish for all $R$ but the fundamental representation. This property can be easily checked using the fact that for the unknot,

$$\langle \text{Tr}_R U \rangle_u = \text{dim}_t R,$$

where $\text{dim}_t R$ is the quantum dimension of the representation $R$. Recall that this dimension is easily computed for $SU(N)$ using the hook rule to calculate the ordinary dimension of $R$. This rule assigns a quotient to $\text{dim} R$ obtained in the following way: for the numerator, products of $N \pm i$, $i = 0, 1, 2, \ldots$, each coming from a box located on the parallel to the diagonal placed $\pm i$ times away from the diagonal, taking the plus sign for the upper part and the minus sign for the lower part; a denominator provided by the hook lengths. For example, for the Young tableau:

$$R = \ydiagram{1},$$

the dimension is,

$$\text{dim} R = \frac{N(N + 1)(N - 1)}{3 \cdot 1 \cdot 1}.$$

The corresponding quantum dimension is obtained after replacing each of the integers $n$ appearing in the quotient by its corresponding quantum number,

$$\{x\} = \frac{t^\frac{n}{2} - t^{-\frac{n}{2}}}{t^\frac{1}{2} - t^{-\frac{1}{2}}}.$$

so that,

$$\text{dim}_t R = \frac{\{N\}\{N + 1\}\{N - 1\}}{\{3\}\{1\}\{1\}}.$$
Using (4.3) we obtain for the unknot,

\[
f_{\square}(t, \lambda)_u = \langle \text{Tr} U \rangle_u = \dim \square = \{N\},
\]

which is consistent with (4.4). This relation is also consistent with the results in [2]. Let us now test the rest of the expressions for the functions \(f_R\) which we have obtained. Taking (4.10) one finds that, indeed,

\[
f_{\square}(t, \lambda)_u = \dim \square - \frac{1}{2} \left( \{N\}^2 + \{N\} \right)_{t \to t^2} = 0, \\
f_{\Box}(t, \lambda)_u = \dim \Box - \frac{1}{2} \left( \{N\}^2 - \{N\} \right)_{t \to t^2} = 0.
\]

Similarly, using these results and (4.17) one confirms that for the functions of third order:

\[
f_{\mathcal{M}}(t, \lambda)_u = f_{\mathcal{P}}(t, \lambda)_u = f_{\mathcal{E}}(t, \lambda)_u = 0.
\]

Eqs. (4.29) and (4.30) constitute an important check of our calculations.

4.6. Perturbative expansions and Vassiliev invariants

Using the same arguments as in [28] to prove that the coefficients of the perturbative series expansion are Vassiliev invariants [29] one can easily show that the vevs (2.18) lead to a perturbative series expansion whose coefficients are also Vassiliev invariants. This implies that the functions \(f_R(t, \lambda)\) share the same properties. In other words, if one considers the power series expansion,

\[
f_R(e^x, e^{Nx}) = \sum_{i=0}^{\infty} \alpha_i x^i,
\]

the coefficients \(\alpha_i, i = 0, 1, 2, \ldots\), are Vassiliev invariants of order \(i\). The explicit form of the Vassiliev invariants for torus knots \((n, m)\) are known [30][31] up to order six. They turn out to be polynomials in \(n\) and \(m\). At lowest orders, the form of these invariants imply the following structure for the polynomials \(P_R\) in (4.22):

\[
P_R(e^x, e^{Nx}) = g_0 + g_2(R) \beta_{2,1} x^2 + g_3(R) \beta_{3,1} x^3 + \mathcal{O}(x^4),
\]

where,

\[
\beta_{2,1} = \frac{1}{24} (n^2 - 1)(m^2 - 1), \\
\beta_{3,1} = \frac{1}{144} nm(n^2 - 1)(m^2 - 1).
\]
and \(g_2(R)\) and \(g_3(R)\) are constants (independent of \(n\) and \(m\)) which depend on the representation \(R\). After computing \(P_R\) for a variety of torus knots we find,

\[
\begin{align*}
P_{\square}(x) &= 2N(N^2 - 1)\beta_{3,1}x^3 + O(x^4), \\
P_{\square}(x) &= -2N(N^2 - 1)\beta_{3,1}x^3 + O(x^4), \\
P_{\square\square}(x) &= 6N(N^2 - 1)\beta_{3,1}x^3 + O(x^4), \\
P_{\square}(x) &= -6N(N^2 - 1)\beta_{3,1}x^3 + O(x^4), \\
P_{\square}(x) &= 6N(N^2 - 1)\beta_{3,1}x^3 + O(x^4),
\end{align*}
\]

in full agreement with (4.32).

These results constitute a test of the fact that the coefficients of the perturbative series expansion associated to the polynomials \(P_R\) must be Vassiliev invariants. But the test indicates the existence of more structure. As argued above, the functions \(f_R\) have a very simple structure, many cancellations occur in such a way that these functions have a simple denominator and the vev for the unknot factorizes. The results (4.34) indicate that they might satisfy more striking properties. Though the fact that \(g_0 = 0\) is a simple consequence of (4.29) and (4.30), there is no reason to expect that \(g_2(R) = 0\) for the representations under consideration. This property implies that the second derivative respect to \(t\) of \(P_R\) (after replacing \(\lambda \rightarrow t^N\)) vanishes at \(t = 1\). This might be a first indication of the existence of some important properties shared by the polynomials \(P_R\). Further work is needed to study their general features and applications. In particular, it would be very interesting to understand in more detail the relation between the coefficients \(N_{R,Q,s}\) of (2.8) and Vassiliev invariants.

5. A conjecture for the connected vevs

In the previous sections we have shown how to extract string amplitudes from Chern-Simons vevs, and that the amplitudes computed in that way have in fact the structure predicted in (2.8) by using the target space interpretation. The worldsheet interpretation of the amplitudes is in principle more complicated, since it involves open string instantons. However, the structure of the free energy of topological string theory dictated by worldsheet perturbative considerations gives a remarkable set of constraints on the integers \(N_{R,Q,s}\) or, equivalently, on the connected vevs of Chern-Simons gauge theory. As explained in [3],
the arguments in [11] imply that the free energy $F(V) = -\log\langle Z(U,V) \rangle$ is given by the expression

$$F(V) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \sum_{n_1, \ldots, n_h} x^{2g-2+h} F_{g,n_1,\ldots,n_h}(\lambda) \Tr V^{n_1} \cdots \Tr V^{n_h},$$

(5.1)

where $x$ is $2\pi i/(k + N)$, and $g$ and $h$ denote the genus and number of boundaries of the string worldsheet. Let us now compare this expression with the equations (2.7) and (2.8), based on the target space interpretation. To do this we assume that there is some analytical continuation which turns the series (5.1) into a series involving only positive integers $n_i$. If we then choose the basis (2.11) for the class functions, we find:

$$F(V) = \sum_{\vec{k}} |\vec{k}|! \prod_{j} j^{k_j} \sum_{g=0}^{\infty} F_{g,\vec{k}}(\lambda) x^{2g-2+|\vec{k}|} \Tr_{\vec{k}}(V).$$

(5.2)

Comparing to (2.13) we immediately obtain:

$$G_{\vec{k}}^{(c)}(U) = -|\vec{k}|! \prod_{j} j^{k_j} \sum_{g=0}^{\infty} F_{g,\vec{k}}(\lambda) x^{2g-2+|\vec{k}|}.$$  

(5.3)

This makes a highly nontrivial prediction about the structure of the connected vevs: if we put $t = e^x$, keeping $\lambda$ as an independent variable, then the expansion in $x$ should start with a power greater or equal than $|\vec{k}| - 2$. Moreover, the expansion should contain powers of the same parity $i.e.$ the powers should be all even or all odd, depending on the parity of $|\vec{k}|$). This implies that the functions $G_{\vec{k}}^{(c)}(U)$ are even (odd) under $t \leftrightarrow t^{-1}$ when $|\vec{k}|$ is even (odd).

Let us now analyze this prediction. For $\ell = 1$ (and therefore $|\vec{k}| = 1$), the left hand side of (5.3) is the unnormalized HOMFLY polynomial. The fact that the expansion in $x$ starts with $x^{-1}$ is a consequence of (1.3). Since the normalized HOMFLY polynomial is even under the exchange of $t$ and $t^{-1}$ [27], $G_{(1,0,\ldots)}^{(c)}(U)$ is odd, in agreement with the prediction. These two facts were already noted in [7] in this context (indeed, the equation (5.3) generalizes equation (2.3) of [7] to more complicated vevs). For $\ell > 1$, the prediction (5.3) is far from being obvious: the vevs of Wilson loops in the representation $R$ start typically with the power $x^{-\ell}$ when we do not expand $\lambda$, and they do not have any a priori symmetry under $t \leftrightarrow t^{-1}$. However, we have found that the prediction (5.3) is in fact true.
in all the cases that we have checked. For example, in the case of the right-handed trefoil knot, and for the connected vevs at order four, we have obtained:

\[
G^{(c)}_{(0,0,0,1,\ldots)}(U) = \frac{1}{4} \lambda^2(\lambda - 1)(-134 + 1498\lambda - 6278\lambda^2 + 13146\lambda^3 - 15129\lambda^4 + 9735\lambda^5 - 3289\lambda^6 + 455\lambda^7)x + O(x),
\]

\[
G^{(c)}_{(0,0,0,0,\ldots)}(U) = 12\lambda^2(\lambda - 1)^4(9 - 72\lambda + 198\lambda^2 - 176\lambda^3 + 49\lambda^4) + O(x^2),
\]

\[
G^{(c)}_{(1,0,0,0,\ldots)}(U) = 9\lambda^2(\lambda - 1)^4(10 - 92\lambda + 233\lambda^2 - 200\lambda^3 + 55\lambda^4) + O(x^2),
\]

\[
G^{(c)}_{(2,0,0,0,\ldots)}(U) = 72(\lambda - 1)^5\lambda^2(-3 + 27\lambda - 58\lambda^2 + 28\lambda^3)x + O(x^3),
\]

\[
G^{(c)}_{(4,0,0,0,\ldots)}(U) = 432(\lambda - 1)^6\lambda^2(1 - 9\lambda + 16\lambda^2)x^2 + O(x^4).
\]

In addition, one finds that the expansion only contains powers of \( x \) of the same parity, in agreement with (5.3). We think that this result gives another important check of the open string interpretation of Chern-Simons gauge theory.

The prediction (5.3) can be stated in terms of the integer invariants \( N_{R,Q,s} \) by using our master equation (2.20). Notice that, from (2.20), the most we can say about the expansion of the connected vevs is that they start with \( x^{-1} \). However, more is true, as we have just seen. This means that there must be some constraints on the integer invariants \( N_{R,Q,s} \). Let us obtain these constraints. Using the definition of the Bernoulli polynomials,

\[
\frac{e^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^{m-1}}{m!},
\]

we find the following equation for \( F_{g,\vec{k}} \):

\[
F_{g,\vec{k}}(\lambda) = -\frac{1}{|\vec{k}|!} \prod j^k \sum \sum \chi_R(C(\vec{k}_1/n)) N_{R,Q,s} B_{2g-1+|\vec{k}|}(s + 1/2)/(2g - 1 + |\vec{k}|)! \lambda^{nQ}.
\]

This expression can be interpreted as a multicovering formula for open string instantons, in the spirit of [24]. Notice that the sum over representations in this equation is finite, as in (2.20). The structure of the expansion in (5.2) also implies the following sum rules. Fix a vector \( \vec{k} \) and a half-integer \( j \). Then, one has:

\[
\sum \sum \chi_R(C(\vec{k}_1/n)) N_{R,j/n,s} B_{m}(s + 1/2) = 0,
\]

when \( m = 1 \) mod 2, and also when \( m = 0, 1, \ldots, |\vec{k}| - 2 \) (for \( |\vec{k}| > 2 \)). \( N_{R,j/n,s} \) is taken to be zero if \( j/n \) is not a half-integer. Notice that the sum in (5.7) involves only a finite number of terms. The sum rules (5.7) encode the properties about the perturbative expansion of the connected vevs that we discussed above, in terms of the integers \( N_{R,Q,s} \).
6. Conclusions and open problems

In this paper we have presented strong evidence for the existence of new polynomial invariants, \( f_R \), whose integer coefficients \( N_{R,Q,s} \) can be regarded as the solutions of certain enumerative problem in the context of string theory. These polynomials are labeled by irreducible representations of \( SU(N) \), and for the fundamental representation they correspond to the unnormalized HOMFLY polynomials. For other irreducible representations they have the form of the corresponding unnormalized ordinary polynomial invariants, plus a series of correction terms which involve representations whose associated Young tableaux have a lower number of boxes. Their existence would answer a basic question in knot theory which has remained open for many years: polynomial invariants, appropriately corrected, can indeed be regarded as generating functions.

The evidence for the existence of the new polynomials is a consequence of the precision test of the correspondence between Chern-Simons gauge theory and topological strings carried out in this paper. We have proved that one can in fact extract the string amplitudes from the Chern-Simons vevs following a recursive procedure. This makes possible to compute the integer invariants \( N_{R,Q,s} \) starting from the Chern-Simons side. Using explicit results for torus knots, we have been able to give remarkable evidence for the predictions of [2], and we have also exploited the interplay between worldsheet and target results to give further checks of the string theory interpretation of Chern-Simons gauge theory.

There are clearly two different avenues for future research. On the Chern-Simons side, it would be extremely interesting to extend these results to more general knots, \( n \)-component links, and/or other gauge groups. It would be also very interesting to explore in more detail the relations between the integers \( N_{R,Q,s} \) and the two other sets of integer invariants of knots: the coefficients of the normalized polynomials, and the Vassiliev invariants. On the string side, the duality with Chern-Simons gauge theory opens the possibility of extracting information about open string instantons in the resolved conifold geometry. The procedure we have developed in this paper gives a very concrete strategy to compute the string amplitudes and obtain the relevant spectrum of BPS states associated to D2 branes ending on D4 branes. As a preliminary step, one should make more precise the geometry of the Lagrangian submanifold in the resolved geometry. We hope to report on these and other related issues in the near future.
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Appendix A. The functions $f_R(t, \lambda)$ for $\ell = 4$

In this appendix, we list the functions $f_R(t, \lambda)$ for representations with four boxes, in the case of the right-handed trefoil knot. The results are:

\[
\begin{align*}
\mathbf{f}(t, \lambda) &= \frac{t^{-3/2}}{t^{3/2} - t^{-3/2}} (-1 + \lambda)^2 \lambda^2 (\lambda - t) (-1 + \lambda t) \\
&\quad \left( t^3 (1 + t + 2 t^2 + t^3 + 2 t^4 + t^6) \\
&\quad - \lambda t (1 + 3 t + 6 t^2 + 9 t^3 + 11 t^4 + 11 t^5 \\
&\quad + 10 t^6 + 8 t^7 + 5 t^8 + 3 t^9 + 2 t^{10} + t^{11}) \\
&\quad + \lambda^4 t^3 (1 + t + 3 t^2 + 2 t^3 + 4 t^4 + 2 t^5 + 4 t^6 \\
&\quad + 2 t^7 + 3 t^8 + t^9 + 2 t^{10} + t^{11} + t^{12} + t^{14}) \\
&\quad + \lambda^2 (1 + 2 t + 7 t^2 + 10 t^3 + 18 t^4 + 19 t^5 \\
&\quad + 24 t^6 + 19 t^7 + 20 t^8 + 13 t^9 + 12 t^{10} + 6 t^{11} + 5 t^{12} + 2 t^{13} + 2 t^{14}) \\
&\quad - \lambda^3 t (1 + 3 t + 6 t^2 + 10 t^3 + 13 t^4 + 15 t^5 + 15 t^6 \\
&\quad + 14 t^7 + 12 t^8 + 10 t^9 + 8 t^{10} + 6 t^{11} + 4 t^{12} + 2 t^{13} + 2 t^{14} + t^{15}) \right), \\
\end{align*}
\]

\[
\begin{align*}
\mathbf{f}(t, \lambda) &= -\frac{t^{-9/2}}{t^{3/2} - t^{-3/2}} (-1 + \lambda)^2 \lambda^2 (1 + t) (\lambda - t) (-1 + \lambda t) \\
&\quad \left( t^3 (1 + t + t^2 + t^3) + \lambda^2 (1 + t + t^2 + t^3)^2 (1 + t + 2 t^2 + t^4) \\
&\quad - \lambda t (1 + 3 t + 5 t^2 + 8 t^3 + 7 t^4 + 6 t^5 + 3 t^6 + 2 t^7) \\
&\quad + \lambda^4 t^2 (1 + t + 3 t^2 + 2 t^3 + 3 t^4 + t^5 + 2 t^6 + t^8) \\
&\quad - \lambda^3 t (2 + 4 t + 8 t^2 + 10 t^3 + 11 t^4 + 10 t^5 + 7 t^6 + 4 t^7 \\
&\quad + 3 t^8 + t^9 + t^{10}) \right), \\
\end{align*}
\]

\[
\begin{align*}
\mathbf{f}(t, \lambda) &= \frac{t^{-19/2}}{t^{5/2} - t^{-5/2}} (-1 + \lambda)^2 \lambda^2 (1 + t) (\lambda - t) (-1 + \lambda t) \\
&\quad \left( t^5 (1 + t + t^2 + t^3) + \lambda^2 t (1 + t + t^2 + t^3)^2 (1 + 2 t^2 + t^3 + t^4) \\
&\quad - \lambda t^3 (2 + 3 t + 6 t^2 + 7 t^3 + 8 t^4 + 5 t^5 + 3 t^6 + t^7) \\
&\quad + \lambda^4 (t + 2 t^3 + t^4 + 3 t^5 + 2 t^6 + 3 t^7 + t^8 + t^9) \\
&\quad - \lambda^3 t (1 + t + 3 t^2 + 7 t^4 + 10 t^5 + 11 t^6 + 10 t^7 \\
&\quad + 8 t^8 + 4 t^9 + 2 t^{10}) \right), \\
\end{align*}
\]
\[ f(t, \lambda) = -t^{-6} (-1 + \lambda)^2 \lambda^2 (\lambda - t) (-1 + \lambda t) (1 + t + t^2) \]
\[ \left( t^3 + t^5 - \lambda t (1 + t + t^2 + t^3)^2 \right. \]
\[ - \lambda^3 t (1 + t)^2 (1 + t + t^2 + t^3 + t^4) + \lambda^4 (t + 2 t^3 + t^4 + 2 t^5 + t^7) \]
\[ + \lambda^2 (2 + 2 t + 6 t^2 + 6 t^3 + 9 t^4 + 6 t^5 + 6 t^6 + 2 t^7 + 2 t^8) \right) , \]
(A.4)

\[ f(t, \lambda) = - \frac{t^{-35/2}}{t^{17} - t^{-17}} (-1 + \lambda)^2 \lambda^2 (\lambda - t) (-1 + \lambda t) \]
\[ \left( t^8 (1 + 2 t^2 + t^3 + 2 t^4 + t^5 + t^6) \right. \]
\[ - \lambda t^5 (1 + 2 t + 3 t^2 + 5 t^3 + 8 t^4 + 10 t^5 + 11 t^6 + 11 t^7 + 9 t^8) \]
\[ + 6 t^9 + 3 t^{10} + t^{11}) + \lambda^4 (1 + t^2 + t^3 + 2 t^4 + t^5 \]
\[ + 3 t^6 + 2 t^7 + 4 t^8 + 2 t^9 + 4 t^{10} + 2 t^{11} + 3 t^{12} + t^{13} + t^{14}) \]
\[ + \lambda^2 t^3 (2 + 2 t + 5 t^2 + 6 t^3 + 12 t^4 + 13 t^5 + 20 t^6 + 19 t^7 + 24 t^8 \]
\[ + 19 t^9 + 18 t^{10} + 10 t^{11} + 7 t^{12} + 2 t^{13} + t^{14}) - \lambda^3 t (1 + 2 t + 2 t^2 + 4 t^3 \]
\[ + 6 t^4 + 8 t^5 + 10 t^6 + 12 t^7 + 14 t^8 + 15 t^9 + 15 t^{10} + 13 t^{11} \]
\[ + 10 t^{12} + 6 t^{13} + 3 t^{14} + t^{15}) \right) . \]

(A.5)
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