An action of $N=8$ self-dual supergravity
in ultra-hyperbolic harmonic superspace

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Abstract

The $N$-extended self-dual supergravity in the ultra-hyperbolic four-dimensional space-time of kleinian signature $(2+2)$ is given in the $N$-extended harmonic superspace. We reformulate the on-shell $N$-extended self-dual supergravity constraints of Siegel to a ‘zero-curvature’ representation, and solve all of them but one in terms of a single superfield prepotential, by using a covariant Frobenius gauge in the Devchand-Ogievetsky approach. An off-shell superspace action, whose equation of motion yields the remaining constraint, is found. Our manifestly Lorentz-covariant action in harmonic superspace is very similar to the non-covariant Chern-Simons-type action, which was proposed earlier by Siegel in the light-cone $N=8$ superspace. Our action is also invariant under the residual superdiffeomorphisms and the residual local $OSp(8|2)$ super-Lorentz rotations, which are left after imposing the Frobenius gauge. The infinitesimal superfield parameters of the residual symmetries are expressed in terms of independent analytic superfields.

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1 Introduction

The self-dual Yang-Mills (SDYM) and self-dual gravity (SDG) in four euclidean spacetime dimensions are usually associated with the instanton solutions to the corresponding non-self-dual classical field theories, which result in a finite euclidean action. The SDYM field strength and the SDG curvature tensor,

\[ F = *F, \quad \text{and} \quad R = *R, \]

(1.1)

imply the equations of motion for a gauge field and metric, respectively, just because of the Bianchi identities. Hence, the notion of self-duality is stronger than that of the equations of motion, and one may wonder whether an off-shell action exists which would yield the self-duality equations, with all the linear symmetries to be manifestly realised in the action. The reason why we would like to find covariant actions for self-dual field theories is closely related to the problem of an off-shell covariant description of BPS-like field configurations, or branes generalizing the instantons in various dimensions.

An educated answer to the problem of constructing the actions for self-dual field theories is crucially dependent upon answering the following two related questions first, namely,

- should the action in question have the same symmetries as a given self-duality condition?
- should the number of physical degrees of freedom in an off-shell action be kept, or it is allowed to increase them?

In other words, on the one hand, there may exist many off-shell actions, which give in particular the desired self-duality relation in some gauge-fixed form, as it is usually the case. On the other hand, naive approaches, e.g. when using Lagrange multipliers, easily do the job but they may simultaneously lead to some additional propagating degrees of freedom whose decoupling may require extra constraints, or result in a trivial (free) theory in the case of self-duality. Therefore, if one insists on maintaining all the linearly realised symmetries and the number of degrees of freedom, the problem of formulating a Lorentz-covariant action for a given self-duality condition becomes non-trivial, and it is not obvious whether a solution exists at all.

Supersymmetry adds new interesting aspects to this problem [1, 2]. The four-dimensional self-duality turns out to be closely related to the existence of Majorana-Weyl (i.e. real chiral) spinors, which only exist in the ultra-hyperbolic spacetime of
kleinian signature (2+2). The SDYM and SDG have essentially one state of ‘helicity’ 
(−1) and (−2), respectively. When being treated off-shell, the Lorentz invariance already implies a need for another state with the opposite ‘helicity’ to compensate the otherwise negative ‘helicity’ of an off-shell action. Of course, it looks like adding a Lagrange multiplier, while such field is already present in the case of the maximally extended $N = 4$ self-dual super-Yang-Mills (SDSYM) and the $N = 8$ self-dual supergravity (SDSG). These are just the only two self-conjugate cases where all particles come in pairs, so that each field in the action can naturally serve as a Lagrange multiplier for another field with the opposite ‘helicity’.

The manifestly Lorentz-covariant action for the $N = 4$ SDSYM theory in components was given by Siegel [1]. He also found the manifestly $N = 4$ supersymmetric (but non-covariant) superfield action for the same theory in light-cone superspace [1]. An off-shell $N = 4$ SDSYM action, which would be both manifestly Lorentz-covariant and $N = 4$ supersymmetric, seems to exist in the harmonic superspace only, i.e. with an infinite number of auxiliary fields. Such action was found by Sokatchev in ref. [3], where it takes the form of the Chern-Simons-type action in terms of properly chosen gauge connections in $N = 4$ harmonic superspace. It is the main purpose of this paper to generalise the result of ref. [3] to the case of the $N = 8$ SDSG.

The $(N = 0)$ SDG in the ultra-hyperbolic spacetime of kleinian signature (2+2) is also known as the effective field theory of interacting (2,2) spinning strings, i.e. the critical closed strings with the (2,2) world-sheet supersymmetry [4]. Because of a ‘topological’ nature of the spinning strings, their non-vanishing amplitudes seem to be only 3-point functions [3]. This very basic observation already implies the cubic (i.e. Chern-Simons) -type of the effective self-interaction (see subsect. 2.1 for more). The effective action of the (2,2) spinning strings is, however, not fully Lorentz-invariant with respect to the $SO(2,2)$ rotations, just like the spinning string amplitudes themselves. It is a particular, gauge-fixed and non-covariant (Plebański) version of SDG that actually appears to be the effective field theory of the (2,2) closed spinning strings [4]. If the effective theory were fully $SO(2,2)$ covariant, this would imply an equivalence between the (4,4) and (2,2) spinning strings, as well as a maximally supersymmetric spectrum of particles in the target four-dimensional ultra-hyperbolic ‘spacetime’. However, since the (4,4) and (2,2) spinning strings have different critical dimensions, and there are no extra fermionic particles interacting with a single physical state of ‘helicity’ (−2) in the spectrum of (2,2) spinning strings [3], no full Lorentz invariance and no any supersymmetry are allowed in the four-dimensional

\[^3\]In the SDYM case, this state is Lie algebra valued.
target (‘spacetime’) of the critical closed (2, 2) strings. Even if one takes into account a possible inequivalence of different pictures or superconformal ghost vacua which, in principle, may lead to an infinite number of massless physical states in the spinning string theory \[7\], all these states cannot be spacetime fermionic because of the spectral flow associated with two-dimensional N=2 superconformal algebras \[8\]. Nevertheless, the effective full Lorentz invariance, as well as the effective (4, 4) world-sheet (twisted) supersymmetry, do appear in the topological reformulation of the interacting closed (2, 2) spinning string theory in the twistor space, not in the ‘spacetime’ \[5\].

Because of the existence of $N$-extended supergravitites \[1, 2\], one may ask about a possible existence of yet another non-trivial ‘spinning’ or ‘heterotic’ closed string theory, which would lead to an $N$-extended SDSG as its effective field theory in the string target space.\[4\] To our knowledge, no string theory is known, which would have the $N = 8$ SDSG as its effective field theory. The covariant component action similar to that of the $N = 8$ SDSG may arise as a consistent finite truncation of the most general effective action of the critical closed spinning strings with the extra massless physical states corresponding to the inequivalent pictures as in ref. \[7\], but with the bosonic states only and no ‘spacetime’ supersymmetry in the conventional (Lie superalgebra) sense. The alternative may be a possible relevance of the maximally extended $N = 8$ SDSG to the F-theory branes \[10\].

It should also be mentioned that the four-dimensional SDYM and SDG equations of motion may also be considered as the master integrable systems because of the fact that they provide a natural classification scheme for many integrable systems in lower dimensions \[11\]. Along these lines, the SDSYM and SDSG can be understood as yet another extensions of integrable systems, whose possible significance for the non-perturbative string theories (= M- and F-theories) is yet to be understood.

Our paper is organized as follows: in sect. 2 we briefly discuss SDG and its relation to the (2, 2) critical closed strings, formulate the standard on-shell constraints defining the $N = 8$ gauged SDSG in superspace, and reformulate the constraints to harmonic superspace. Sect. 3 is devoted to a comparison of the $N = 8$ SDSG in harmonic superspace with the $N = 4$ SDSYM in the formulation of Sokatchev \[3\]. In sects. 4 and 5 we collect some technical details about the Devchand-Ogievetsky approach to self-duality \[12, 13\], and define a Frobenius gauge. The prepotentials in the Frobenius gauge are also introduced in sect. 5, while the action is formulated in sect. 6. We summarize our conclusions in Sect. 7. For the sake of completeness, the known component results about the $N = 8$ SDSG are collected in Appendix.

\[4\]See e.g., ref. \[3\] for some recent efforts in this direction, as regards the $N = 1$ SDSG.
2 From SDG to its supersymmetric $N$-extended (SDSG) generalization

Since the ($N = 0$) SDG in the four-dimensional ultra-hyperbolic spacetime, whose signature is $(2+2)$ and the natural Lorentz group is $SO(2,2)$, is going to be our starting point, we would like to remind the reader some basic facts about SDG and its relation to the spinning strings in subsect. 2.1. The on-shell formulation of the $N = 8$ SDSG [1] is given in subsect. 2.2. A reformulation of the on-shell $N = 8$ SDSG in harmonic superspace is discussed in subsect. 2.3 along the lines of ref. [13]. The whole sect. 2 serves as a technical introduction for the next sections.

2.1 SDG and closed spinning strings

Because of the isomorphism

$$SO(2,2) \cong SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R})' ,$$  \hspace{1cm} (2.1)

it is natural to represent four ultra-hyperbolic spacetime coordinates as $x^{\alpha\alpha'}$, where the spinor indices $\alpha = (1, 2)$ and $\alpha' = (1', 2')$ refer to $SL(2)$ and $SL(2)'$, respectively.

A commutator of the (curved) spacetime covariant derivatives defines the curvature tensor [13],

$$[\nabla_{\beta\beta'}, \nabla_{\alpha\alpha'}] = \varepsilon_{\alpha'\beta'} R_{\alpha\beta} + \varepsilon_{\alpha\beta} R_{\alpha'\beta'} ,$$ \hspace{1cm} (2.2)

where $R_{\alpha\beta}$ can be decomposed with respect to the generators ($M^{\gamma\delta}, M^{\gamma'\delta'}$) of $SL(2) \otimes SL(2)'$ as follows:

$$R_{\alpha\beta} = C_{(\alpha\beta\gamma\delta)} M^{\gamma\delta} + R_{\alpha\beta(\alpha'\beta')} M^{\alpha'\beta'} + \frac{1}{6} R M_{\alpha\beta} ,$$ \hspace{1cm} (2.3)

and similarly for $R_{\alpha'\beta'}$,

$$R_{\alpha'\beta'} = C_{(\alpha'\beta'\gamma'\delta')} M^{\gamma'\delta'} + R_{\gamma'\delta'(\alpha'\beta')} M^{\gamma\delta} + \frac{1}{6} R M_{\alpha'\beta'} .$$ \hspace{1cm} (2.4)

Here $C_{(\alpha'\beta'\gamma'\delta')}$ ($C_{(\alpha\beta\gamma\delta)}$) are the totally symmetric components of the (anti)-self-dual Weyl tensor, $R_{(\alpha\beta)(\alpha'\beta')}$ are the components of the traceless Ricci tensor, and $R$ is the scalar curvature. In this notation, the self-duality of the Riemann curvature in eq. (1.1) just means

$$R_{\alpha'\beta'} = 0 ,$$ \hspace{1cm} (2.5)

\footnote{Any symmetrization of indices (in brackets) is defined with unit weight.}
or, equivalently, because of eq. (2.4),

\[ C_{(\alpha'\beta'\gamma'\delta')} = 0 , \]  

(2.6)

and

\[ R_{\gamma\delta(\alpha'\beta')} = 0 , \quad \text{and} \quad R = 0 . \]  

(2.7)

Eq. (2.7) is equivalent to the Einstein equations without matter, whereas the vanishing of the self-dual Weyl tensor in eq. (2.6) represents the only additional condition needed for self-duality.

In the case of SDG, eqs. (2.2) and (2.3) simplify to

\[ [\nabla_{\beta'}, \nabla_{\alpha}] = \varepsilon_{\alpha'\beta'} R_{\alpha\beta} \]  

(2.8)

and

\[ R_{\alpha\beta} = C_{(\alpha\beta\gamma\delta)} M^{\gamma\delta} , \]  

(2.9)

respectively. It is now obvious that the self-dual curvature takes its values in the \( sl(2) \) algebra only. In other words, it is the \( SL(2) \) factor of the full Lorentz group \( SO(2,2) \) that should be promoted to the local symmetry, whereas the \( SL(2)' \) symmetry should be kept global or rigid because of eq. (2.5). Therefore, we consider the first spinor index (\( \mu \)) of the ultra-hyperbolic spacetime coordinate \( x^{\mu\alpha'} \) as a curved (world) index, and keep another spinor index \( \alpha' \) as a flat (tangential) index associated with the flat \( (sl(2)' \text{-valued}) \) part of the curvature. It leads to the following form of the covariant derivative

\[ \nabla_{\alpha'} = E^{\mu\beta'}_{\alpha'\alpha} \partial_{\mu\beta'} + \omega_{\alpha'\beta'} , \]  

(2.10)

where a ‘vierbein’ \( E^{\mu\beta'}_{\alpha'\alpha} \) and a connection

\[ \omega_{\alpha'\beta'} = \omega_{\alpha'\beta'\gamma} M^{\beta\gamma} \]  

(2.11)

have been introduced. Since the self-duality implies no restrictions on the \( sl(2) \)-valued curvature \( R_{\alpha\beta} \), the solutions to SDG amount to the vierbeins which have no torsion. In other words, it is the vanishing torsion constraint that is the SDG equation of motion in this approach.

As is well-known, the self-duality in four dimensions is equivalent to a Ricci-flat Kähler geometry [11]. A Kähler geometry has a complex structure, while keeping the complex structure intact is only compatible with a part (subgroup) \( U(1,1) \) of the full Lorentz group transformations \( SO(2,2) \cong SU(1,1) \otimes SU(1,1) \). Since the NSR formulation of the spinning string theory with the \( (2,2) \) world-sheet supersymmetry is intrinsically complex, i.e. it actually requires choosing a complex structure for its
definition, it is not very surprising that the spinning string theory is only invaraint under the subgroup $U(1, 1)$ of $SO(2, 2)$, even in the case of a flat background space-time [4, 5]. The missing symmetry $SO(2, 2)/U(1, 1) \sim SU(1, 1)/GL(1, \mathbb{R})$ can be understood as the twistor transformations rotating the complex structure. It is now clear why the full Lorentz symmetry can be formally restored in the extended twistor space [5] but not in the ultra-hyperbolic spacetime which is the target space of a spinning string.

Given a complex structure $x^{a\alpha'} \to (z^i, \bar{z}^\bar{i})$, $i = 1, 2$, and a Kähler potential of the form

$$K = K_0 + 4\kappa^3 \phi, \quad \text{where} \quad K_0 = \eta_{ij} z^i \bar{z}^j, \quad \eta_{ij} = \eta^{ij} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.12)$$

and $\kappa$ is the gravitational coupling constant of the inverse mass dimension, the Ricci-flatness condition takes the form

$$\det(g_{ij}) = -1, \quad (2.13)$$

in terms of the Kähler metric

$$g_{ij} = \partial_i \partial_j K = \eta_{ij} + 4\kappa^3 \partial_i \partial_j \phi, \quad (2.14)$$

where we have used the identity for the Ricci tensor in a Kähler geometry, $R_{ij} = \partial_i \partial_j \log \det(g_{kl})$ and the boundary condition $g_{ij} \to \eta_{ij}$ at infinity. In terms of the Kähler deformation $\phi$, eq. (2.13) can be obtained as an equation of motion from the following (Plebański) action

$$S_{\text{Plebanski}} = \int d^{2+2}z \left( \frac{1}{2} \eta^{ij} \partial_i \phi \bar{\partial}_j \phi + \frac{2\kappa^3}{3} \phi \bar{\partial}_i \phi \wedge \partial_i \phi \right). \quad (2.15)$$

As was shown by Ooguri and Vafa [4], it is exactly the action (2.15) that also appears to be the effective field theory action of the closed spinning strings. The Plebański action (2.15) is only invariant under the $U(1, 1)$ part of the $SO(2, 2)$ Lorentz transformations. Moreover, this action has a dimensionful coupling constant $\kappa$ which is absent in the Lorentz-covaraint equation (1.1) for the SDG.

### 2.2 On-shell SDSG in superspace

The $N$-extended supersymmetrization of SDG essentially amounts to extending the $SL(2)$ local symmetry to the $OSp(N|2)$ local symmetry, while keeping the manifest

$^6$The superconformal symmetry would imply $SO(2, 2) \overset{\text{conf}}{\to} SO(3, 3) \cong SL(4) \overset{\text{susy}}{\to} SL(N|4)$. 

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7
global symmetry $SL(2)'$ intact. Since the supergroup $OSp(N|2)$ contains the $SO(N)$ Lie group in addition, this leads to the *gauged* version of the $N$-extended SDSG, in which the internal symmetry $SO(N)$ rotating $N$ supercharges is gauged. The *on-shell* formulations of the $N$-extended SDSG in superspace are similar for all $N$.

Let $g$ be the corresponding (dimensionless) gauge coupling constant (the gravitational coupling constant $\kappa$ of the inverse mass dimension was already introduced in the previous subsect. 2.1). Let $\eta^{ab}, a, b = 1, 2, \ldots, N,$ be the $so(N)$ Cartan-Weyl metric multiplied by the factor $g/\kappa$, i.e. of mass dimension, and $C^{\alpha\beta} \sim \varepsilon^{\alpha\beta}$ the self-dual part of the charge conjugation matrix for spacetime spinors. The $OSp(N|2)$ metric can now be introduced as $\eta^{AB} = (\eta^{ab}, C^\alpha\beta)$ where $A = (a, \alpha)$. The Grassmann grading is defined by treating the $SL(2)$ and $SL(2)'$ spinor indices $\alpha$ and $\alpha'$ as bosonic indices, and the $SO(N)$ vector indices $a, b, \ldots$ as fermionic indices. The $OSp(N|2)$ generators $M^{AB}$ act on vectors $V^C$ as follows:

$$[M^{AB}, C^C] = V^{[A} \eta^{B]}C,$$  \hspace{1cm} (2.16)

so that one has the identity

$$\frac{1}{2}K_{BC}[M^{CB}, V_A] = K_{AB}\eta^{CB}V_C$$  \hspace{1cm} (2.17)

for any matrix $K_{AB}$.

The on-shell constraints defining the $N$-extended SDSG in the $N$-extended chiral superspace $z^{M\alpha'} = (x^{\mu\alpha'}, \theta^{m\alpha'})$ can be naturally divided into two groups. The first group of (anti)commutators of the superspace covariant derivatives $\nabla^{aa}$ and $\nabla_{b\beta'}$ reads

$$\{\nabla^{aa}, \nabla^{b\beta}\} = C^{\alpha\beta} M^{ab} + \eta^{ab} \eta^{\alpha\beta},$$ \hspace{1cm} (2.18)

$$\{\nabla^{aa}, \nabla_{b\beta}\} = \delta^{a}_{b} C^{\alpha\beta} \nabla_{\beta\beta'}, \quad [\nabla^{aa}, \nabla_{\beta\beta'}] = \delta^{a}_{\beta} \eta^{ab} \nabla_{b\beta'},$$

where $M^{ab}$ are the $SO(N)$ generators of dimension of mass. Eqs. (2.18) naturally define the gauged $N$-extended local supersymmetry, while $M^{ab}, M^{\alpha\beta}$ and $\nabla^{aa}$ can be recognized as the generators $M^{AB}$ of $OSp(N|2)$. In particular, it implies that the spacetime covariant derivatives $\nabla^{aa}$ should be interpreted as some of the $OSp(N|2)$ generators [ ]. The second group of the one-shell superspace constraints defining the $N$-extended SDSG is a rather straightforward generalization of the SDG equation (2.8) to the supersymmetric case, namely,

$$[\nabla_{A\alpha'}, \nabla_{B\beta'}] = C_{\beta'\alpha'} R_{AB},$$ \hspace{1cm} (2.19)

in terms of the supercovariant superspace derivatives (*cf.* eq. (2.10))

$$\nabla_{A\alpha'} = E_{A\alpha'}^{M\beta'} \partial_{M\beta'} + \frac{1}{2} \Omega_{A\alpha'BC} M^{CB},$$ \hspace{1cm} (2.20)
where a supervielbein $E^{M\beta}_{\alpha\alpha'}$ and a superconnection $\Omega_{\alpha\alpha'BC}$ have been introduced. More explicitly, eq. (2.19) can be written down as follows:

$$\{\nabla_a, \nabla_b\} = C^\alpha_{\beta\alpha'} \phi_{ab},$$
$$[\nabla_a, \nabla_{\beta\beta'}] = C^\alpha_{\beta\alpha'} \chi_{ab},$$
$$[\nabla_a, \nabla_{\beta\beta'}] = C^\alpha_{\beta\alpha'} R_{\alpha\beta},$$

where $(\phi_{ab}, \chi_{a\beta}, R_{\alpha\beta}) = R_{AB}$ is the $OSp(N|2)$-valued supercurvature tensor.

The superspace formulation of an $N$-extended SDSG given above is manifestly $N$-supersymmetric, and it is invariant under the local $OSp(N|2)$ symmetry by construction. It is, however, an on-shell formulation since eq. (2.19) has no torsion on its right-hand-side which implies the equations of motion, as we already know from the previous subsection. One of the ways to go off-shell is to turn to a light-cone superspace formulation of the same theory, where only physical (i.e. propagating) field components are kept. As was shown by Siegel [1], all the on-shell SDSG superspace constraints but one can be solved in the light-cone formalism, in terms of a single $N$-extended superfield prepotential $V_{\alpha'\alpha}$ of ‘helicity’ $(-2)$. The ($N = 0$) SDG prepotential originates from the equation

$$C_{\alpha\beta\gamma\delta}(x) \sim \partial_{\alpha'} \partial_{\beta'} \partial_{\gamma'} \partial_{\delta'} V_{\alpha'\alpha}(x),$$

and it can be identified (up to a constant dimensionful factor) with the ‘scalar’ field $\phi$ introduced in the previous subsect. 2.1. It is obvious how to generalize eq. (2.22) in the self-dual $N$-extended superspace to

$$C_{ABCD}(z) \sim \partial_{A'} \partial_{B'} \partial_{C'} \partial_{D'} V_{\alpha'\alpha}(z).$$

The free field equation for the SDSG prepotential $V_{\alpha'\alpha}(z)$ [2],

$$\partial_A \partial_B V_{\alpha'\alpha} = 0,$$

can actually be solved for all the $\theta^{\alpha'\alpha}$-dependent components. It implies that $V_{\alpha'\alpha}(z)$ can be reduced in the light-cone formalism to a self-dual superfield $V_{\alpha'\alpha}(x^{\alpha'\alpha}, \theta^{\alpha'\alpha})$ that merely depends upon a half (Majorana-Weyl) of the anticommuting superspace coordinates. As a result, all the constraints in the light-cone formalism can be reduced to a single non-covariant equation for the self-dual superfield prepotential $V_{\alpha'\alpha}$, which can be obtained from the $N$-extended Plebański action [1]

$$S_{L-c.} = \int d^{2+2}x d^N \theta \left[ \frac{1}{2} V_{\alpha'\alpha} \Box V_{\alpha'\alpha} + \frac{i}{6} V_{\alpha'\alpha} (\partial^{\alpha'} \partial_{\alpha'} \partial_{\alpha'} V_{\alpha'\alpha}) \eta^{BA} (\partial_{B'} \partial_{\alpha'} \partial_{\alpha'} V_{\alpha'\alpha}) \right].$$

(2.25)
This light-cone $N$-extended superspace action is manifestly supersymmetric with respect to a half of the original on-shell supersymmetry since it is written down in terms of a half (Majorana-Weyl) of the anticommuting superspace coordinates. Though the $N=8$ action (2.25) is neutral with respect to the parabolic subgroup $GL(1,\mathbb{R})'$ of the $SL(2,\mathbb{R})'$ Lorentz symmetry, it is still not covariant with respect to the full $SL(2,\mathbb{R})'$ symmetry which is explicitly broken in the light-cone approach. Though a covariant off-shell description of the $N=8$ SDSG exists in components (see ref. [1] and our Appendix), it is not manifestly supersymmetric. Our main purpose in this paper is to ‘covariantize’ the light-cone $N=8$ SDSG action (2.25) in harmonic superspace.

### 2.3 SDSG in harmonic superspace

Since the $SL(2)'$ Lorentz symmetry remains a global symmetry in the on-shell superspace formulation of the $N$-extended SDSG, while it is broken to a parabolic subgroup $GL(1)'$ in the off-shell light-cone formulation, it is natural to ‘covariantize’ the light-cone theory by introducing extra (twistor) harmonic variables $u^{\alpha'}\pm$ valued in the coset $SL(2)'/GL(1)'$, i.e.

\[ u^{\alpha'} \in SL(2,\mathbb{R})' \quad \text{and} \quad u^{\alpha'}u_{\alpha'} = 1 , \quad (2.26) \]

and then apply the formal rules of the harmonic superspace approach along the lines of ref. [15] for this non-compact case. This procedure was successfully applied for a construction of a covariant action of the $N=4$ SDSYM in harmonic superspace in ref. [3], and we are now going to proceed in the similar way, in the case of the $N=8$ SDSG.

Harmonic functions with definite $GL(1)'$ charge $q \geq 0$ are formally defined by their expansion in terms of the harmonic variables, i.e.

\[ F^{(q)}(u) = \sum_{n=0}^{\infty} f^{(\alpha'_1\cdots\alpha'_n+q\beta'_1\cdots\beta'_n)}u^{\alpha'_1+\cdots+u^{\alpha'_{n+q}}u_{\beta'_1+\cdots+u_{\beta'_n}} , \quad (2.27) \]

where $f^{\alpha'_1\cdots\alpha'_n}$ are $SL(2)'$ tensors of ‘spin’ $n + \frac{1}{2}q$. The harmonic covariant derivatives take the form

\[ \partial^{++} = u^{\alpha'}+ \frac{\partial}{\partial u^{\alpha'}} , \quad \partial^{--} = u^{\alpha'}- \frac{\partial}{\partial u^{\alpha'}} , \quad \partial^{0} = u^{\alpha'}+ \frac{\partial}{\partial u^{\alpha'}} - u^{\alpha'}- \frac{\partial}{\partial u^{\alpha'}} , \quad (2.28) \]

and satisfy an $sl(2)$ algebra. It is not difficult to check that

\[ \partial^{++}F^{(q)}(u) = 0 \quad \text{implies} \quad \left\{ \begin{array}{ll}
F^{(q)}(u) = 0 , & \text{when } q < 0 , \\
F^{(q)}(u) = \text{constant} , & \text{when } q = 0 , \\
F^{(q)}(u) = f^{(\alpha'_1\cdots\alpha'_q)}u^{\alpha'_1+\cdots+u_{\alpha'_q}} , & \text{when } q > 0 .
\end{array} \right. \quad (2.29) \]
The integration rules are defined as follows \[15\]:

\[
\int du = 1 , \quad \int du u^+_\alpha \cdots u^+_\alpha \bar{u}^-_{\beta_1} \cdots u^-_{\beta_n} = 0 ,
\]

so that they project out the singlet part of an integrand with vanishing $GL(1)'$ charge. An integration by parts is allowed since

\[
\int du \partial^{+-} F^{--}(u) = 0 .
\]

The availability of harmonic variables allows one to define the Lorentz-covariant $GL(1)'$ projections like $x^\mu = u^\pm_{\alpha'} x^\mu_{\alpha'}$ and $\theta^m = u^\pm_{\alpha'} \theta^m_{\alpha'}$, and similarly for the superspace covariant derivatives.

The on-shell $N$-extended SDSG was defined in subsect. 2.2 by the constraints (2.19) in terms of the curved superspace covariant derivatives (2.20) and the structure group $OSp(N|2)$. By using the harmonic coordinates $u^{\alpha'}$, the covariant derivatives can be rewritten to the form

\[
\nabla_\pm = u^{\alpha'} \pm \nabla \nabla_{\alpha'} = e_\pm + \omega_\pm ,
\]

where the (super)vielbeine

\[
e_\pm = E_M^A \partial_M^+ + E_A^{+M} \partial_M^- ,
\]

have been introduced. The superconnections $\omega_\pm$ are defined by eq. (2.32).

In the original (central) basis of the harmonic superspace, in terms of the central coordinates $z^M = z^{M_{\alpha'}} u^\pm_{\alpha'}$ and $u^\pm_{\alpha'}$, the superdiffeomorphisms are realised via the transformations

\[
\delta z^M = \lambda^M (z^{p \pm}, u) , \quad \delta u^\pm_{\alpha'} = 0 ,
\]

Hence, the harmonic covariant derivatives $(\nabla^{++}, \nabla^{--}, \nabla^0)$ in the central basis are still of the form (2.28), as in the flat harmonic superspace.

It is now straightforward to verify that the $N$-extended SDSG constraints can be put into the form \[13\]:

\[
[\nabla_A^+, \nabla_B^+] = 0 ,
\]

\[
[\nabla^{++}, \nabla_A^+] = 0 ,
\]

\[
[\nabla_A^+, \nabla_{AB}^+ = R_{AB} ,
\]

\[
[\nabla^{++}, \nabla_A] = \nabla_A^+ .
\]
One gets eq. (2.35) after contracting eq. (2.19) with harmonics. In order to get eq. (2.19) back from eq. (2.35), one needs eq. (2.36) which implies that the derivative $\nabla^+_A$ in the central coordinates is linear in the harmonics $u^+$. The most general solution to eq. (2.35) might, however, have torsion terms on the right-hand-side of eq. (2.19). It is eq. (2.37) that takes care of it, since the torsion terms would then also appear on the right-hand-side of eq. (2.37) too. The last eq. (2.38) is just needed to make sure that the covariant derivative $\nabla^-_A$ in the central coordinates is linear in $u^-$. 

To solve the extended superspace constraints, it is usually useful to make a transform to the so-called analytic basis in harmonic superspace, which allows one to realise the relevant symmetries in a smaller analytic subspace [15]. It simultaneously implies introducing more gauge fields, since the harmonic covariant derivatives in the analytic coordinates will no longer be of the form (2.28).

The transform from the central coordinates to the analytic ones is usually described in terms of a ‘bridge’ function $b(z, u)$ [15],

$$z^a_M^\pm = z^M^\pm + b^M^\pm (z, u). \quad (2.39)$$

The analytic subspace $(z^a_M^+, u)$ is supposed to be invariant under the analytic superdiffeomorphisms, i.e.

$$\delta z^M^+ = \lambda^M^+(z^+_a, u), \quad (2.40)$$

whereas

$$\delta z^M^- = \lambda^M^-(z^+_a, z^-_a, u), \quad (2.41)$$

in general. Eq. (2.39) also implies

$$\delta b^M^\pm = \delta z^M^\pm - \delta z^M^\pm \equiv \lambda^M^\pm - \tau^M^\pm. \quad (2.42)$$

All the SDSG constraints (2.35)–(2.38) keep their form after the transform (2.39), since they were written down in terms of the most general covariant derivatives in a curved superspace. However, in the analytic coordinates, the harmonic covariant derivatives receive some extra terms, $\partial^{++} \rightarrow D^{++}$, i.e.

$$D^{++} = \partial^{++}(z^\pm_a, u) \frac{\partial}{\partial(z^\pm_a, u)}$$

$$= \partial^{++} + H^M a \frac{\partial}{\partial z^M_a} + H^M a^+] \frac{\partial}{\partial z^M_a^+} \quad (2.43)$$

$$= \partial^{++} + H^M a M^a + H^M a^M a^+. \quad (2.43)$$
where we have introduced the harmonic vielbeine

$$H^{+M} = \partial^{++}(z^{-M} + b^{-M})$$

$$= z^{+M} + \partial^{++}b^{-M}$$

$$= z^{a+M} - b^{+M} + \partial^{++}b^{-M}$$  (2.44)

and

$$H^{+3M} = \partial^{++}(z^{+M} + b^{+M})$$

$$= \partial^{++}b^{+M}$$,  (2.45)

as well as the notation $\partial^{+}_{aM} = \partial/\partial z^{a-M}$ and $\partial^{-}_{aM} = \partial/\partial z^{a+M}$. The harmonics themselves remain inert under superdiffeomorphisms. Similarly, the vielbein (2.33) gets transformed as

$$e^{+}_{A} \rightarrow (e^{+}_{A}\varphi^{M-})\partial^{+}_{aM} + (e^{+}_{A}\varphi^{M+})\partial^{-}_{aM}$$

$$\rightarrow (e^{+}_{A}\varphi^{M-})\partial^{+}_{aM} \rightarrow E^{M}_{A}\partial^{+}_{aM},$$

where we have used the fundamental property

$$e^{+}_{A}z^{+M} = 0$$  (2.47)

of the analytic coordinates. One finds similarly that

$$e^{-}_{A} \rightarrow (e^{-}_{A}\varphi^{M+})\partial^{+}_{aM} + (e^{-}_{A}\varphi^{M+})\partial^{+}_{aM}$$

$$\rightarrow F^{M}_{A}\partial^{-}_{aM} + F^{-}_{aA}\partial^{+}_{aM}.$$  (2.48)

It follows from the constraint (2.35) that there exists an $OSp(N|2)$ valued superfield $\varphi$ which satisfies

$$\nabla^{+}_{A}\varphi = (e^{+}_{A} + \omega^{+}_{A})\varphi = 0.$$  (2.49)

Hence, a solution to eq. (2.35) takes the form

$$\nabla^{+}_{A} = e^{+}_{A} - (e^{+}_{A}\varphi)\varphi^{-1},$$  (2.50)

whose connection is trivial. Therefore, we can get

$$\nabla^{+}_{A} \rightarrow \varphi^{-1}\nabla^{+}_{A}\varphi = e^{+}_{A}$$  (2.51)

via an $OSp(N|2)$ rotation $e^{+}_{A} \rightarrow e^{+}_{A} = (\phi e^{+})_{A}$. The matrix $\phi$ thus plays the role of a ‘bridge’ between the $OSp(N|2)$ transformations in the central coordinates and that in the analytic ones. In particular, one finds for the harmonic covariant derivative that

$$\nabla^{++} \rightarrow \varphi^{-1}\nabla^{++}\varphi = D^{++} + \omega^{++},$$  (2.52)
where
\[ \omega^{++} = \varphi^{-1} D^{++} \varphi . \] (2.53)

Since the general \( OSp(N|2) \) transformation law of the connection \( \omega^+_A \) is given by
\[ \delta \omega^+_A = D^+_A \Lambda - [\Lambda, \omega^+_A] , \] (2.54)
fixing the gauge \( \omega^+_A = 0 \) implies
\[ D^+_A \Lambda = 0 , \] (2.55)
i.e. the \( OSp(N|2) \)-valued transformation parameter \( \Lambda \) should an analytic superfield.
The transformation laws of all the vielbein superfields can be written down as follows:
\[ \delta E^M_A = E^N_A \partial^+_N \lambda^{M^-} + \lambda^B_A E^M_B , \]
\[ \delta F^M_A = F^N_A \partial^-_N \lambda^{M^+} + \lambda^B_A F^M_B , \]
\[ \delta F^M_{-A} = F^N_A \partial^-_N \lambda^{M^-} + F_{-A}^{--} \partial^+_N \lambda^{M^+} + \lambda^B_A F^M_{-B} , \] (2.56)
\[ \delta H^{+3M} = D^{++} \lambda^{+M} , \]
\[ \delta H^{+M} = D^{++} \lambda^{-M} , \]
where the infinitesimal parameters \( \lambda^{M\pm} \) have been introduced in eqs. (2.40) and (2.41), and \( \lambda^A_B \) are the infinitesimal parameters of local \( OSp(N|2) \) rotations.

We emphasize that an analyticity condition like that in eq. (2.55) is only covariant with respect to the local analytic transformations. In the next sect. 3 the SDSG constraints will be analyzed in the analytic representation. Since in the rest of our paper we are going to deal with the analytic superspace coordinates only, we omit the subscript ‘a’ in what follows.

### 3 \( N = 8 \) SDSG versus \( N = 4 \) SDSYM

The maximally extended \( N = 8 \) SDSG and \( N = 4 \) SDSYM are both self-conjugate in the sense that all their physical states with opposite ‘helicities’ can be naturally paired to form scalars. This implies the existence of a manifestly covariant and supersymmetric action for these theories. Such action for the \( N = 4 \) SDSYM was constructed by Sokatchev [3], by using partial gauges to solve some of the SDSYM constraints in harmonic superspace and then find an action for the rest of the constraints to be interpreted as the equations of motion. In subsect. 3.1 we recapitulate some of the results of ref. [3] since our \( N = 8 \) SDSG construction, in fact, follows

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\[ \text{See also ref. [13].} \]
the Sokatchev pattern for the \(N = 4\) SDSYM up to a point where the differences
between the SDSYM and the SDSG become important.

### 3.1 \(N = 4\) SDSYM action in harmonic superspace

The standard (on-shell) constraints defining the \(N = 4\) SYM theory in the flat \(N = 4\)
superspace \((x^{\alpha \alpha'}, \theta^a_{\alpha}, \theta^{a'}_{\alpha'})\), where \(a, b, \ldots = 1, 2, 3, 4\)
are the indices of the \(SL(4)\) automorphism group of \(N = 4\) supersymmetry, read in terms of the
gauge-covariant and (flat) supercovariant derivatives as follows \([3]\):

\[
\begin{align*}
[\nabla^a_{\alpha}, \nabla^b_{\beta}] &= \varepsilon_{\alpha \beta} \tilde{\phi}^{ab} , \quad [\nabla_{\alpha \alpha'}, \nabla_{b\beta'}] = \varepsilon_{\alpha' \beta'} \phi_{ab} , \\
\{\nabla^a_{\alpha}, \nabla_{b\beta'}\} &= \varepsilon_{\alpha \beta} \tilde{\chi}^a_{\beta'} , \quad \{\nabla_{\alpha \alpha'}, \nabla_{b\beta'}\} = \varepsilon_{\alpha' \beta'} \chi_{ab} , \\
[\nabla^a_{\alpha}, \nabla_{b\beta'}] &= \delta^a_b \nabla_{a\beta'} , \quad \{\nabla_{\alpha \alpha'}, \nabla_{b\beta'}\} = \varepsilon_{\alpha' \beta'} F_{a\beta} + \varepsilon_{a\beta} F_{\alpha' \beta'} ,
\end{align*}
\]

where the real scalars \(\phi_{ab}\) and \(\tilde{\phi}^{ab}\) are related as

\[
\tilde{\phi}^{ab} = \frac{1}{2} \varepsilon^{abcd} \phi_{cd} .
\]

In the case of the \(N = 4\) SDSYM, a half of the superfield strengths on the right-
hand-side of eq. (3.1) vanishes, while there is no constraint (3.2). The on-shell \(N = 4\)
SDSYM constraints can be divided into two groups, namely,

\[
\begin{align*}
[\nabla^a_{\alpha}, \nabla^b_{\beta}] &= 0 , \quad [\nabla^a_{\alpha}, \nabla_{b\beta'}] = \delta^a_b \nabla_{a\beta'} , \quad \{\nabla^a_{\alpha}, \nabla_{b\beta'}\} = 0 ,
\end{align*}
\]

and

\[
\begin{align*}
[\nabla_{\alpha \alpha'}, \nabla_{b\beta'}] &= \varepsilon_{\alpha' \beta'} \phi_{ab} , \quad \{\nabla_{\alpha \alpha'}, \nabla_{b\beta'}\} = \varepsilon_{\alpha' \beta'} \chi_{ab} , \quad \{\nabla_{\alpha \alpha'}, \nabla_{b\beta'}\} = \varepsilon_{\alpha' \beta'} F_{a\beta} ,
\end{align*}
\]

which are similar to the SDSG eqs. (2.18) and (2.21), respectively.

Harmonic superspace is useful in rewriting the \(N = 4\) SDSYM constraints above
into a ‘zero-curvature’ form which is easier to deal with. In particular, the SDSYM
analogues to the SDSG harmonic projections in eq. (2.32) are

\[
\nabla^+_a = u^{a'} \nabla_{a\alpha'} , \quad \text{and} \quad \nabla^+_\alpha = u^{a'} \nabla_{a\alpha'} .
\]

It is quite natural to assume that both the gauge superfield parameters and the
superconnections of the SDSYM are dependent upon the harmonic variables, while
their actual linear dependence in accordance to eq. (3.5) can be recovered as a solution
due to some additional constraints like that in the SDSG eqs. (2.36) and (2.38). One therefore needs a trivial connection to the harmonic covariant derivative $\partial^{++}$, i.e.

$$\nabla^{++} = \partial^{++} + A^{++} ,$$  \hspace{1cm} (3.6)

and the extra constraints

$$\{ \nabla^{++}, \nabla^{a}_a \} = 0 , \quad \{ \nabla^{++}, \nabla^{+}_a \} = 0 , \quad \{ \nabla^{++}, \nabla^{+}_\alpha \} = 0 .$$  \hspace{1cm} (3.7)

The initial $N = 4$ SDSYM constraints (3.3) and (3.4) can now be put into the following equivalent ‘zero-curvature’ form [3]:

\[
\begin{align*}
[\nabla^a_a, \nabla^b_b] &= 0 , \hspace{1cm} (3.8) \\
[\nabla^a_\alpha, \nabla^b_\beta] &= \delta^a_b \nabla^b_\alpha , \hspace{1cm} (3.9) \\
\{ \nabla^a_a, \nabla^{+}_b \} &= 0 , \hspace{1cm} (3.10) \\
[\nabla^+_a, \nabla^+_b] &= 0 , \hspace{1cm} (3.11) \\
[\nabla^+_a, \nabla^{+}_\beta] &= 0 , \hspace{1cm} (3.12) \\
[\nabla^{+}_a, \nabla^{+}_\beta] &= 0 . \hspace{1cm} (3.13)
\end{align*}
\]

The central idea of ref. [3] was the use of a certain supersymmetric gauge, which is a combination of a chiral and a semi-analytic gauges. The chiral gauge

$$\nabla^a_\alpha = \partial^a_\alpha , \quad \text{or} \quad A^a_\alpha = 0 ,$$  \hspace{1cm} (3.14)

is allowed due to eq. (3.8). Together with eq. (3.10), it implies the chirality of the gauge superfields $A^+_a$ and $A^{++}$, i.e. their independence upon $\theta^a_\alpha$. Eq. (3.9) can now be solved as

$$A^+_a = a^+_a (x, \theta^{a\pm}, u) + \theta^a_\alpha A^+_\alpha(x, \theta^{a\pm}, u) \hspace{1cm} (3.15)$$

in terms of two chiral superfields $a^+_a$ and $A^+_\alpha$. After substituting eq. (3.15) into eq. (3.11) one finds

\[
\begin{align*}
\partial^+_a a^+_b + \partial^+_b a^+_a + \{ a^+_a, a^+_b \} &= 0 , \\
\partial^+_a A^+_\beta - \partial^+_\beta a^+_\alpha + [a^+_a, A^+_\beta] &= 0 , \hspace{1cm} (3.16) \\
\partial^+_a A^+_\beta - \partial^+_\beta A^+_\alpha + [A^+_\alpha, A^+_\beta] &= 0 .
\end{align*}
\]

The first line of eq. (3.16) allows one to impose a supersymmetric semi-analytic gauge

$$a^+_a = 0 ,$$  \hspace{1cm} (3.17)

in addition to the chiral gauge (3.14). The rest of equations (3.16), in fact, allows one to fully gauge away $A^+_a$, i.e. take $A^+_\alpha = 0$ also. This twistor transform is useful
in discussing the solutions to the $N = 4$ SDSYM equations \cite{gs}, but it turns out to be too restrictive for writing down an off-shell action \cite{gs}.

In the chiral semi-analytic gauge, the second eqs. (3.7) and (3.16) imply that the remaining harmonic gauge superfields $A^+_\alpha$ and $A^{++}$ are chiral and analytic simultaneously, i.e. they are only dependent upon $(x, \theta^{a+}, u)$, whereas the remaining constraints

\begin{align}
\partial^{\alpha}_+ A^{++} - D^{++} A^+_\alpha + [A^+_\alpha, A^{++}] &= 0 , \\
\partial^{\beta}_+ A^+_\beta - \partial^{\beta}_- A^+_\beta + [A^+_\alpha, A^+_\beta] &= 0 ,
\end{align}

(3.19)

appear as the equations of motion for them. Eqs. (3.19) are obviously invariant under the gauge transformations

\begin{align}
\delta A^+_\alpha &= \partial^{\alpha}_+ \Lambda + [A^+_\alpha, \Lambda] , \\
\delta A^{++} &= D^{++} \Lambda + [A^{++}, \Lambda] ,
\end{align}

(3.20)

whose gauge parameters $\Lambda(x, \theta^{a+}, u)$ are chiral analytic superfields too.

The action, whose variational equations with respect to the independent superfields $A^+_\alpha$ and $A^{++}$ give the $N = 4$ SDSYM equations (3.20), is given by \cite{gs}

\begin{equation}
S_{N=4\ SDSYM} = \int d^2 x d^4 \theta^+ d u \, \text{tr} \left( A^{++} \partial^{a+} A^+_\alpha - \frac{1}{2} A^{a+} D^{++} A^+_\alpha + A^{++} A^{a+} A^+_\alpha \right) .
\end{equation}

(3.21)

This Chern-Simons-type action of the $N = 4$ SDSYM theory is fully covariant and $N = 4$ supersymmetric, and it has no dimensionful coupling constant. In the rest of our paper, our main goal will be to formulate a similar action for the $N = 8$ SDSG.

### 3.2 SDSG in a chirally analytic harmonic superspace

Some of the vanishing SDSG (anti)commutators are quite similar to that of the SDSYM. For instance, eq. (2.35) is equivalent to

\begin{align}
\{ \nabla^+_a, \nabla^+_b \} = 0 , \\
[\nabla^+_a, \nabla^+_\beta] = 0 , \\
[\nabla^+_\alpha, \nabla^+_\beta] = 0 ,
\end{align}

(3.22)

whereas eq. (2.36) amounts to

\begin{align}
[\nabla^{++}, \nabla^+_a] = 0 , \\
[\nabla^{++}, \nabla^+_\alpha] = 0 .
\end{align}

(3.23)

The first eq. (3.22) can be solved as in the previous subsect. 3.1, i.e. by imposing a gauge $\nabla^+_a = \partial^+_a$, where the full covariant derivative $\nabla^+_a$ is given by

\begin{equation}
\nabla^+_a = e^+_a + \omega^+_a ,
\end{equation}

(3.24)
in terms of the vielbein $e^+_a$ and the connection $\omega^+_a$. Under the local $OSp(N|2)$ rotations this covariant derivative transforms as
\[
\delta \nabla^+_a = \left[ e^+_a + \omega^+_a, \Lambda \right] = \left[ e^+_a, \lambda_{CD} M^{DC} \right] + \left[ \omega^+_a, \Lambda \right] = e^+_a \lambda_{CD} M^{DC} - \lambda_{CD} (M^{DC} e^+_a) - [\Lambda, \omega^+_a] = (e^+_a \lambda_{CD}) M^{DC} + (-)^{a(C+D)} \lambda_{CD} (e^+_a M^{DC}) - \lambda_{CD} (M^{DC} e^+_a) - [\Lambda, \omega^+_a] = D^+_a \Lambda - \lambda_{CD} [M^{DC}, e^+_a] - [\Lambda, \omega^+_a] = D^+_a \Lambda - 2 \lambda_a \eta^{DC} e^+_D + [\omega^+_a, \Lambda],
\]
where we have introduced the ‘short’ derivative $D^+_a$ (without a connection) for our convenience. The connection $\omega^+_a$ transforms as usual,
\[
\delta \omega^+_a = D^+_a \Lambda + [\omega^+_a, \Lambda]. \tag{3.25}
\]
However, it follows from this equation that the gauge fixing $\omega^+_a = 0$ implies a restriction $D^+_a \Lambda = 0$. This means that the infinitesimal parameter $\Lambda$ of the super-Lorentz rotations should be a chirally analytic superfield in the gauge $\nabla^+_a = \partial^+_a$, similarly to the supergauge parameter of the SDSYM (subsect. 3.1).

It should be noticed that our gauge for the vierbein, $e^+_a = \delta^m_a \partial^+_m$ or $E^m_a = \delta^m_a$, does not imply the vanishing curvature in SDSG. However, since the vierbein is not inert under the $OSp(N|2)$ gauge transformations and the general coordinate transformations,
\[
\delta E^m_a = \delta \delta^m_a = E^m_a (\partial^+_a \lambda^m) + \lambda_a C^\beta \eta^{DC} E^m_D = \partial^+_a \lambda^m + \lambda_{ab} \eta^{DC} \delta^m_a + \lambda_a C^\beta \lambda^m, \tag{3.26}
\]
our gauge fixing also implies that the infinitesimal parameters $\lambda^m$ of the general coordinate transformations and $\lambda_{ab}$ of the super-Lorentz transformations are to be related, whereas the parameters $\lambda_a$ of the local supersymmetry vanish, $\lambda_a = 0$.

It is now obvious that the edition of a chirally analytic harmonic superspace, in the form used for the SDSYM, has to be $OSp(N|2)$ covariantized in the case of SDSG. The main reason is the supersymmetric nature of the super-Lorentz structure ‘group’ $OSp(N|2)$ which is, in fact, a supergroup that mixes bosonic and fermionic tangent space indices. This simultaneously tells us what should be done in the case of the SDSG, namely, the bosonic and fermionic tangent space indices are to be democratically treated within a single superindex. In other words, we should impose a super-Lorentz covariant gauge $e^+_A = \partial_A^+ \Lambda$ instead of the non-covariant one used above, and relax our analyticity conditions. As will be shown in the next sections, this proposal allows us to formulate an action for the $N = 8$ SDSG.
4 SDSG constraints and Frobenius gauge

In this section we calculate the (anti)commutators of the \( N \)-extended SDSG in some detail. This is needed in order to reduce the number of superfields, as well as the number of symmetries acting on the superfields. We also introduce a Frobenius gauge \([13]\), which will allow us to introduce SDSG superfield prepotentials in the next sect. 5. We follow the method developed by Devchand and Ogievetsky in refs. \([12, 13]\).

4.1 SDSG vielbeine and connections

Our starting point here is the SDSG constraints (2.35)–(2.38) in a supersymmetric gauge \( \omega_A^+ = 0 \). The covariant derivatives in terms of the corresponding vielbein and superconnections read in the chiral superspace of the \( N \)-extended SDSG as follows (see sect. 2.2):

\[
\nabla_A^+ = e_A^+ = E_A^M \partial_M^+ ,
\]

(4.1)

\[
\nabla_A^- = e_A^- + \omega_A^- = F_A^M \partial_M^- + F_{A-M}^+ \partial_M^+ + \omega_A^- ,
\]

(4.2)

\[
\nabla^{++} = D^{++} + \omega^{++} = H^{+M} \partial_M^+ + H^{+3M} \partial_M^- + \omega^{++} .
\]

(4.3)

Substituting eq. (4.1) into the left-hand-side of eq. (2.35) yields

\[
\{ \nabla_A^+, \nabla_B^+ \} = \{ E_A^M \partial_M^+, E_B^N \partial_N^+ \}
\]

\[
= E_A^N (\partial_N^+ E_B^M) \partial_M^+ - (-)^{AB} E_B^N (\partial_N^+ E_A^M) \partial_M^+ ,
\]

where the sign factor \((-)^{AB}\) means the standard grading. Thus we obtain

\[
E_A^N (\partial_N^+ E_B^M) - (-)^{AB} E_B^N (\partial_N^+ E_A^M) = 0 ,
\]

(4.4)

which is nothing but the holonomy condition for the vielbein \( e_A^+ \). Hence, there exists a gauge in which this vielbein is holonomic (see subsect. 4.2).

Substituting eqs. (4.1) and (4.3) into the left-hand-side of eq. (2.36) yields

\[
\{ \nabla^{++}, \nabla_A^+ \} = \{ D^{++} + \omega^{++}, e_A^+ \}
\]

\[
= \{ D^{++}, e_A^+ \} + \{ \omega^{++}, e_A^+ \} ,
\]

(4.5)

where

\[
\{ D^{++}, e_A^+ \} = D^{++} E_A^M \partial_M^+ - D_A^+ H^{+M} \partial_M^+ + D_A^+ H^{+3M} \partial_M^- .
\]

(4.6)
and
\[
[\omega^{++}, e^+_A] = [\omega_{CD}^+ M^D C, e^+_A] \\
= \omega_{CD}^+ (M^D C e^+_A) - e^+_A (\omega_{CD}^+ M^D C) \\
= \omega_{CD}^+ (M^D C e^+_A) - D^+_A \omega^{++} - (-)^{(A+C+D)} \omega_{CD}^+ (e^+_A M^D C) \\
= \omega_{CD}^+ [M^D C, e^+_A] - D^+_A \omega^{++} .
\]

(4.7)

Eqs. (2.16) and (4.7) now imply
\[
[\omega^{++}, e^+_A] = 2 \omega_{AC}^+ \eta^{DC} e^+_D - D^+_A \omega^{++} \\
= 2 \omega_{AC}^+ \eta^{DC} E^M_D \partial^+_M - D^+_A \omega^{++} .
\]

(4.8)

Putting our results together, we find for the left-hand-side of eq. (2.36)
\[
[\nabla^{++}, \nabla^+_A] = D^{++} E^M_A \partial^+_M - D^+_A H^{+M} \partial^+_M - D^+_A H^{+3M} \partial^+_M \\
+ 2 \omega_{AC}^+ \eta^{DC} E^M_D \partial^+_M - D^+_A \omega^{++} .
\]

(4.9)

Hence, all the coefficients in front of $\partial^+_M$ and $\partial_M$, as well as the connection term in eq. (4.9) should vanish. The first vanishing coefficient
\[
D^{++} E^M_A - D^+_A H^{+M} + 2 \omega_{AC}^+ \eta^{DC} E^M_D = 0
\]

(4.10)
determines the harmonic connection $\omega^{++}$ in terms of the other fields. The second vanishing coefficient
\[
D^+_A H^{+3M} = 0
\]

(4.11)
means that the harmonic vielbein $H^{+3M}$ is analytic,
\[
H^{+3M} = H^{+3M}(z^+, u) .
\]

(4.12)
The rest of eq. (4.9) gives the equation
\[
D^+_A \omega^{++} = 0 ,
\]

(4.13)
whose solution means that the harmonic connection $\omega^{++}$ is an analytic superfield too,
\[
\omega^{++} = \omega^{++}(z^+, u) .
\]

(4.14)

Sunstituting eqs. (4.1) and (4.2) into eq. (2.37) yields
\[
[\nabla^+_A, \nabla^+_B] = [e^+_A, e^+_B + \omega^+_B] \\
= [e^+_A, e^+_B] + [e^+_A, \omega^+_B] ,
\]

(4.15)
where
\[ [\epsilon^+_A, \epsilon^-_B] = D^+_A F^M_B \partial^-_M + D^+_A F^{-M}_B \partial^+_M - (-)^{AB} D^-_B E^M_A \partial^+_M , \]  

and
\[ [\epsilon^+_A, \omega^-_B] = e^+_A (\omega^-_{BCD} M^{DC}) - (-)^{AB} \omega^-_{BCD} (M^{DC} e^+_A) \]
\[ = D^+_A \omega^-_B + (-)^{(A(B+C+D)} \omega^-_{BCD} (e^+_A M^{DC}) - (-)^{AB} \omega^-_{BCD} (M^{DC} e^+_A) \]
\[ = D^+_A \omega^-_B - (-)^{AB} \omega^-_{BCD} [M^{DC}, e^+_A] \]
\[ = D^+_A \omega^-_B - (-)^{AB} 2 \omega^-_{BCD} \eta^{DC} e^+_D \]
\[ = D^+_A \omega^-_B - (-)^{AB} 2 \omega^-_{BCD} \eta^{DC} E^M_D \partial^+_M . \]  

Putting this together, one finds
\[ [\nabla^+_A, \nabla^-_B] = D^+_A F^M_B \partial^-_M + D^+_A F^{-M}_B \partial^+_M - (-)^{AB} D^-_B E^M_A \partial^+_M \]
\[ + D^+_A \omega^-_B - (-)^{AB} 2 \omega^-_{BCD} \eta^{DC} E^M_D \partial^+_M . \]  

Comparing the terms having \( \partial^+_M \) with the right-hand-side of eq. (2.37) yields
\[ D^+_A F^{-M}_B - (-)^{AB} D^-_B E^M_A - (-)^{AB} 2 \omega^-_{BCD} \eta^{DC} E^M_D = 0 , \]  

which is an algebraic equation for the connection \( \omega^-_A \) in terms of the vielbeine. Similarly, when comparing the terms having \( \partial^-_M \), one finds
\[ D^+_A F^M_B = 0 . \]  

Hence, the vielbein \( F^M_A \) is also analytic,
\[ F^M_A = F^M_A (\epsilon^+, u) . \]  

Comparing the rest of eq. (4.18) with the right-hand-side of eq. (2.37) yields
\[ D^+_A \omega^-_B = R_{AB} , \]  

which simply defines the super-Riemann tensor \( R_{AB} \).

Finally, after substituting eqs. (4.2) and (4.3) into eq. (2.38), one gets
\[ [\nabla^{++}, \nabla^-_A] = [D^{++} + \omega^{++}, e^-_A + \omega^-_A] \]
\[ = [D^{++}, e^-_A] + [D^{++}, \omega^-_A] + [\omega^{++}, e^-_A] + [\omega^{++}, \omega^-_A] , \]  

where
\[ [D^{++}, e^-_A] = D^{++} F^M_A \partial^-_M + D^{++} F^{-M}_A \partial^+_M - D^-_A H^+M \partial^+_M - D^-_A H^+3M \partial^+_M , \]
\[ [D^{++}, \omega_A] = D^{++} \omega_A^- , \]  
\[ (4.25) \]

and
\[ \begin{align*}
[\omega^{++}, e_A] &= \omega^{++}(M^{DC} e_A) - e_A(\omega^{++}M^{DC}) \\
&= \omega^{++}(M^{DC} e_A) - D_A \omega^{++} - (-)^{A(C+D)} \omega^{++}(e_A M^{DC}) \\
&= \omega^{++}(M^{DC}, e_A) - D_A \omega^{++} \\
&= 2 \omega^{++}_A \eta^{DC} e_D - D_A \omega^{++} \\
&= 2 \omega^{++}_A \eta^{DC} F^M_D \partial_M - 2 \omega^{++}_A \eta^{DC} F^{-M}_D \partial_M - D_A \omega^{++} .
\end{align*} \]

Putting all the terms together on the right-hand-side of eq. (2.38) yields
\[ \begin{align*}
D^{++} F^M_A \partial_M &+ D^{++} F^{-M}_A \partial_M - D_A H^{+M} \partial_M - D_A H^{+3M} \partial_M - D_A \omega^{++} \\
+ 2 \omega^{++}_A \eta^{DC} F^M_D \partial_M &+ 2 \omega^{++}_A \eta^{DC} F^{-M}_D \partial_M + D^{++} \omega_A^- + [\omega^{++}, \omega_A^-] = E^M_A \partial_M .
\end{align*} \]

Hence, when comparing the terms with \( \partial_M \), one finds that
\[ D^{++} F^M_A - 2 \omega^{++}_A \eta^{DC} F^{-M}_D - D_A H^{+M} = E^M_A . \]  
\[ (4.27) \]

This equation gives the vielbein \( E^M_A \) in terms of the other superfields. As we already mentioned at the beginning of this subsection, this vielbein can be put into a holonomic form via a proper coordinate transformation. Eq. (4.27) simultaneously gives an algebraic relation between \( E^M_A \) and \( \omega^{++} \). Since the connection \( \omega^{++} \) was already expressed in terms of the vielbeine in eq. (4.10), eqs. (4.10) and (4.27) together imply a first-order differential equation of motion for the vielbeine.

Comparing the coefficients in front of \( \partial_M \) on the right-hand-side of eq. (2.38) yields the relation
\[ D^{++} F^M_A + 2 \omega^{++}_A \eta^{DC} F^{-M}_D - D_A H^{+3M} = 0 . \]  
\[ (4.28) \]

Similarly, since the vielbein \( F^M_A \) can also be put into a holonomic form (see the next subsect. 4.2), whereas the connection \( \omega^{++} \) is now a certain function of the vielbeine and their first-order derivatives, eq. (4.28) is actually a first-order differential equation of motion for the harmonic vielbeine \( H^{+M} \) and \( H^{+3M} \). Its solution will be given in the next section.

Finally, the rest of the last SDSG constraint (2.38) gives the equation
\[ D^{++} \omega_A^- - D_A^+ \omega^{++} + [\omega^{++}, \omega_A^-] = 0 , \]  
\[ (4.29) \]

which is analogous to the first SDSYM equation of motion in eq. (3.19).
In this subsection we reduced the number of the relevant SDSG constraints and that of the relevant superfields in the on-shell superspace description of SDSG. However, despite of the Ansatz of harmonic superspace in the analytic coordinates, the obtained system of SDSG equations is still rather complicated, and it has some redundant gauge symmetries acting on superfields. An important next step will be the introduction of a Frobenius gauge [13], which will result in a great simplification of the superfield SDSG.

4.2 Frobenius gauge

The vielbeine \((E^M_A, F^M_A, F^{-M}_A, H^{+3M}, H^{+M})\) are subject to the superspace diffeomorphisms with the infinitesimal analytic superfield parameters \((\lambda^{+M}, \lambda^{-M})\), and the local \(OSp(N|2)\) super-Lorentz rotations with the infinitesimal analytic superfield parameters \(\lambda^A_B\), whose form is given by eq. (2.56). These gauge symmetries can be used to impose a supersymmetric gauge which would simplify the SDSG constraints.

Since the covariant derivatives \((\nabla^+_A)\) (anti)commute,

\[
[\nabla^+_A, \nabla^+_B] = 0 ,
\]  

it follows from the Frobenius theorem that there exists a coordinate system in which \(\nabla^+_A = \partial^+_A\). In addition, because of the analyticity of \(F^M_A\) (see eq. (4.20)), there exists an \(OSp(N|2)\) rotation which brings the supermatrix \(F^M_A\) to a unit supermatrix. This leads to the Frobenius gauge [13]

\[
E^M_A = \delta^M_A , \quad F^M_A = \delta^M_A .
\]  

It is obvious that in this gauge there is no difference between the world and tangent superspace indices. The SDSG dynamics is described in terms of the constrained superfields \(H^{+3M}, H^{+M}, F^{-M}_A, \omega^{++}\) and \(\omega^{--}_A\).

Let us now investigate which residual symmetries survive in the gauge (4.31). It follows from \(\delta F^M_A = 0\) that

\[
0 = \delta^N_A \partial^+_N \lambda^{M+} + \lambda^B_A \delta^M_B \\
= \partial^+_A \lambda^{B+} \delta^M_B + \lambda^B_A \delta^M_B \\
= \partial^+_A \lambda^{B+} + \lambda^B_A .
\]  

After taking the supertrace of eq. (4.32) and using the supertracelessness of the \(OSp(N|2)\) parameters \(\lambda^B_A\), one easily finds that the analytic diffeomorphism parameters \(\lambda^{\pm B}(z^+, u)\) satisfy

\[
\partial^-_A \lambda^{A+} = 0 .
\]
A solution to this equation is given by
\[
\lambda^B(z^+, u) = \partial_C \lambda^{++}(z^+, u) \eta^{BC} .
\] (4.34)

Because of eqs. (4.32) and (4.34), the super-Lorentz parameters \( \lambda_A^B \) are now dependent upon the just introduced parameter \( \lambda^{++}(z^+, u) \) as follows:
\[
\lambda_A^B = - \partial_A \lambda^B(z^+, u) = - \partial_A \partial_C \lambda^{++} \eta^{BC} .
\] (4.35)

Similarly, it follows from \( \delta E_A^M = 0 \) that
\[
0 = \partial_A \lambda^B = \partial_A \lambda^B(z^+, u) = \partial_A \partial_C \lambda^{++} \eta^{BC} .
\] (4.36)

A solution to this equation reads
\[
\lambda^{B-} = z^D \partial_D \partial_C \lambda^{++} \eta^{BC} + \bar{\lambda}^{B-}(z^+, u) ,
\] (4.37)
where \( \bar{\lambda}^{B-}(z^+, u) \) is an arbitrary analytic function. In order to check eq. (4.37), it is enough to notice that
\[
\partial_A^+ \lambda^{B-} = \partial_A^+ (z^D \partial_D \partial_C \lambda^{++}) \eta^{BC} + \partial_A^+ \bar{\lambda}^{B-} = \partial_A^+ \partial_C \lambda^{++} \eta^{BC} .
\] (4.38)

Therefore, the residual symmetries of the Frobenius gauge are described in terms of the unconstrained superfield parameters \( \lambda^{++}(z^+, u) \) and \( \bar{\lambda}^{B-}(z^+, u) \) introduced above. We are now in a position to define superfield prepotentials of SDSG in the Frobenius gauge.

5 SDSG prepotentials in the Frobenius gauge

Let us now investigate the SDSG equations of subsect. (4.1) in the Frobenius gauge (4.31).

Eq. (4.10) reduces in the Frobenius gauge to
\[
\partial_A^+ H^{+M} = 2 \omega_A^{++} \eta^{DC} \delta_M^D ,
\] (5.1)
which determines the connection \( \omega^{++} \) in terms of the vielbein \( H^{M+} \). Since \( \omega^{++} \) is analytic (see eq. (4.13)), i.e. \( \partial^+ \omega^{++} = 0 \), the superfield \( \partial^+ H^{+M} \) should therefore be analytic also.

Taking into account eq. (4.11) in the Frobenius gauge, \( \partial^+ H^{+3M} = 0 \), eq. (4.28) goes over to

\[
\partial^- A H^{+M} + 3 = 0,
\]

(5.2)

where we have used the relations

\[
D^- A H^{+3M} = \partial^- A H^{+3M} + F^- - ^N \partial^+ H^{+3M} = \partial^- H^{+3M}.
\]

(5.3)

Comparing eqs. (5.1) and (5.2) obviously yields

\[
\partial^+ H^{+M} = \partial^- A H^{+3M},
\]

(5.4)

whose solution is

\[
H^{+M} = z^{-N} \partial^- H^{+3M}.
\]

(5.5)

The constraint (5.2) can be solved in terms of an arbitrary superfield \( H^{+4}(z^+, u) \) as follows:

\[
2\omega^{++}_{AC} \eta^{DC} \delta^M_D = \partial^- A H^{+3M}
\]

\[
= \partial^- A H^{+3M} \eta^{DC} \delta^M_D
\]

\[
= \partial^- A \partial^- C H^{+4} \eta^{DC} \delta^M_D,
\]

(5.6)

or, equivalently,

\[
\omega^{++}_{AC} = \frac{1}{2} \partial^- A \partial^- C H^{+4},
\]

(5.7)

and

\[
H^{+3M} = \partial^- C H^{+4} \eta^{DC} \delta^M_D.
\]

(5.8)

The vielbeine and connections, which are associated with the harmonic derivative \( \nabla^{++} \), can therefore be entirely expressed in terms of the single analytic prepotential \( H^{+4}(z^+, u) \) of charge (+4) \([\text{3}].\)

Eq. (4.19) in the Frobenius gauge reduces to

\[
2\omega^{++}_{BAC} \eta^{DC} \delta^M_D = (-)^{AB} \partial^+_A F^-_-^M,
\]

(5.9)

which is similar to eq. (5.1). Our Ansatz for its solution, in terms of an independent superfield \( V^{-4} \) of charge (-4), reads

\[
F^-_-^M = \partial^- B \partial^- C V^{-4} \eta^{DC} \delta^M_D.
\]

(5.10)
Substituting eq. (5.10) into eq. (5.9) yields

\[ 2\omega_{B\bar{A}C}\eta^{DC}\delta^M_D = (-)^{AB}\partial^+_A\partial^+_B\partial^+_C V^{-4}\eta^{DC}\delta^M_D, \]

and, hence,

\[ \omega_{B\bar{A}C} = \frac{1}{2}\partial^+_B\partial^+_A\partial^+_C V^{-4}. \]

Unlike the analytic prepotential \( H^{+4} \), the superfield \( V^{-4} \) is, however, not analytic.

The only remaining equations (4.27) and (4.29) in the Frobenius gauge take the form

\[ D^{++}F^{-M}_A + 2\omega^{++}_A\eta^{DC}F^{-M}_D - D_AH^{+M} = \delta^M_A, \]

and (unchanged)

\[ D^{++}\omega^{-}_A - D^{-}_A\omega^{++} + [\omega^{++},\omega^{-}_A] = 0, \]

respectively. When considering eq. (5.13) as an equation on \( V^{-4} \), one might solve it (at least, in principle) in terms of the analytic prepotential \( H^{+4} \). The last equation (5.14) might then be considered as a single equation of motion. However, these two equations (5.13) and (5.14) are actually not independent. Indeed, given eq. (5.13), eq. (5.14) is automatically satisfied, which becomes obvious after differentiating eq. (5.13) with respect to \( \partial^+_B \). This means that, via a solution of eq. (5.13) in terms of an arbitrary prepotential \( H^{+4} \), the SDSG theory is automatically on-shell. Therefore, one arrives at the formal general solution to the SDSG equations via the twistor transform, in terms of an arbitrary analytic superfield \( H^{+4}(z^+,u) \), as in ref. [13]. Of course, this solution is still formal, since its relation to the initial central basis is given via the set of linear differential equations on the ‘bridge’ functions, viz.

\[ H^{+3M} = \partial^{++}b^{M+}, \quad H^{+M} = z^{+M} + \partial^{++}b^{-M}, \quad \omega^{++} = \varphi^{-1}D^{++}\varphi, \]

whose manifest solutions are required in order to express the component vielbeine in terms of the component prepotentials.

In order to write down an off-shell action, we are not allowed to use the analytic prepotential \( H^{+4} \) since it exists only on-shell. Therefore, we should relax our constraints first, which will be described in the next sect. 6.

6 An action of \( N = 8 \) SDSG

It follows from eq. (2.38) that the covariant derivative \( \nabla_{\bar{A}} \) in the central coordinates is linear in harmonics \( u^- \). This implies (see subsects. 2.2 and 2.3) that

\[ [\nabla_{\bar{A}},\nabla_{\bar{B}}] = 0. \]
The left-hand-side of this equation,

\[
\{\nabla_A, \nabla_B\} = \{e^-_A + \omega^-_A, e^-_B + \omega^-_B\} \\
= \{e^-_A, e^-_B\} + \{\omega^-_A, e^-_B\} + \{\omega^-_A, \omega^-_B\},
\]

(6.2)
can be written down in a more explicit form by using the equations

\[
[e^-_A, e^-_B] = D^-_A e^-_B - M \partial^+_M - (-)^{AB} D^-_B e^-_A - M \partial^+_M ,
\]

(6.3)

\[
[e^-_A, \omega^-_B] = e^-_A (\omega^-_{BCD} M^{DC}) - (-)^{AB} \omega^-_{BCD} (M^{DC} e^-_A)
= D^-_A \omega^-_B + (-)^{A(B+C+D)} \omega^-_{BCD} (e^-_A M^{DC}) - (-)^{AB} \omega^-_{BCD} (M^{DC} e^-_A)
= D^-_A \omega^-_B - (-)^{AB} \omega^-_{BCD} [M^{DC} e^-_A - (-)^{A(C+D)} e^-_A M^{DC}]
= D^-_A \omega^-_B - (-)^{AB} \omega^-_{BCD} [M^{DC}, e^-_A]
= D^-_A \omega^-_B - (-)^{AB} 2 \omega^-_{BAC} \eta^{DC} e^-_D
= D^-_A \omega^-_B - (-)^{AB} 2 \omega^-_{BAC} \eta^{DC} D^+_M e^-_D - (-)^{AB} 2 \omega^-_{BAC} \eta^{DC} F^- D^+-M \partial^+_M ,
\]

(6.4)

and

\[
[\omega^-_A, e^-_B] = -(-)^{AB} D^-_B \omega^-_A + 2 \omega^-_{BAC} \eta^{DC} e^-_D
= -(-)^{AB} D^-_B \omega^-_A + 2 \omega^-_{BAC} \eta^{DC} D^+_M e^-_D + 2 \omega^-_{BAC} \eta^{DC} F^- D^+-M \partial^+_M .
\]

(6.5)

Putting all the equations together into eq. (6.1) and comparing the coefficients in
front of the derivative \(\partial^+_M\) yields

\[
D^-_A e^-_B - M \partial^+_M - (-)^{AB} D^-_B e^-_A - M \partial^+_M - (-)^{AB} 2 \omega^-_{BAC} \eta^{DC} F^- D^+-M
+ 2 \omega^-_{BAC} \eta^{DC} F^- D^+-M = 0 .
\]

(6.6)

Similarly, the terms with \(\partial^-_M\) imply the relation

\[
\omega^-_{ABC} = (-)^{AB} \omega^-_{BAC} .
\]

(6.7)

Eq. (6.7) allows us to simplify eq. (6.6) to the form

\[
D^-_A e^-_B - (-)^{AB} D^-_B e^-_A = 0 .
\]

(6.8)

Comparing the terms without derivatives in eq. (6.1) adds the equation

\[
D^-_A \omega^-_B - (-)^{AB} D^-_B \omega^-_A + [\omega^-_A, \omega^-_B] = 0 .
\]

(6.9)

It is easy to verify that differentiating eq. (6.8) with respect to \(\partial^+_A\) yields eq. (6.9). This means that eq. (6.9) is not independent and can be ignored, while eq. (6.8)
represents the true equation of motion. In terms of the prepotential $V^{-4}$ defined by eq. (5.10), eq. (6.8) takes the form

$$ (\partial_A^+ \partial_B^+ - \partial_A^+ \partial_B^-) V^{-4} + \partial_A^+ \partial_E^+ V^{-4} \eta^{FE} \partial_F^+ \partial_G^+ V^{-4} - (-)^{AB} \partial_B^+ \partial_F^+ V^{-4} \eta^{FE} \partial_F^+ \partial_G^+ V^{-4} = 0. \tag{6.10} $$

When using the graded Leibniz-rule as well as the graded antisymmetry of the metric tensor $\eta^{AB}$, one easily finds from eq. (6.10) that

$$ \partial_C^+ \left[ (\partial_A^+ \partial_B^- - \partial_A^+ \partial_B^-) V^{-4} + (\partial_A^+ \partial_D^- V^{-4}) \eta^{ED} (\partial_E^+ \partial_F^+ V^{-4}) \right] = 0. \tag{6.11} $$

This simply means that the function in the rectangular brackets is analytic and, therefore, it can be gauged away to zero by using the invariance of the defining eq. (5.10) with respect to the pre-gauge transformations of the prepotential,

$$ \delta V^{-4} = z^{-A} \Lambda_A^3 (z^+, u) + \Lambda^{-4} (z^+, u), \tag{6.12} $$

with the infinitesimal analytic superfield parameters $\Lambda_A^{-3}$ and $\Lambda^{-4}$. Therefore, the equation of motion in terms of the prepotential $V^{-4}$ takes the form

$$ (\partial_A^+ \partial_B^- - \partial_A^+ \partial_B^-) V^{-4} + (\partial_A^+ \partial_D^- V^{-4}) \eta^{ED} (\partial_E^+ \partial_F^+ V^{-4}) = 0. \tag{6.13} $$

In this form, it is very similar to the Siegel equation of motion, which follows from the light-cone action (2.25). Our equation of motion is, however, Lorentz-covariant unlike that of Siegel.

Eq. (6.13) amounts to the following equations

$$ (\partial_a^+ \partial_b^- - \partial_a^+ \partial_b^-) V^{-4} + (\partial_a^+ \partial_D^- V^{-4}) \eta^{ED} (\partial_E^+ \partial_F^+ V^{-4}) = 0, \tag{6.14a} $$

$$ (\partial_a^- \partial_b^+ - \partial_a^- \partial_b^+) V^{-4} + (\partial_a^+ \partial_D^+ V^{-4}) \eta^{ED} (\partial_E^+ \partial_F^+ V^{-4}) = 0, \tag{6.14b} $$

$$ (\partial_a^- \partial_b^+ - \partial_a^- \partial_b^+) V^{-4} + (\partial_a^+ \partial_D^+ V^{-4}) \eta^{ED} (\partial_E^+ \partial_F^+ V^{-4}) = 0, \tag{6.14c} $$

where we have simply used the fact that $A = (a, \alpha)$ and $B = (b, \beta)$. The Grassmann derivatives $\partial_a^-$ enter the second and third lines of eq. (6.14) only linearly. Hence, as in ref. [4], we can use these two equations in order to solve all the ‘non-analytic’ (i.e. $\theta^+$-dependent) terms in the prepotential $V^{-4}$ as the functions of the remaining ‘anti-analytic’ components. This leads to the effectively anti-analytic prepotential $V^{-4}|_{\theta^+ = 0}$. This statement is, in fact, basis-independent since the expansion rules with respect to the anicommuting superspace coordinates in any basis are all isomorphic.

---

8 A similar prepotential was introduced in ref. [12], in the case of the purely bosonic SDG.
Expanding the $N = 8$ prepotential in terms of the anticommuting anti-analytic coordinates yields

$$V^{-4}(x, \theta^{-}, u) = e^{-4} + \psi_{-}^{-3} \theta^{-} + A_{[ab]}^{-2} \theta^{-2ab} + \chi_{[abc]}^{-3} \theta^{-3abc} + \phi_{[abcd]}^{-4} \theta^{-4abcd}$$

$$+ \chi^{+ [abc]} \theta^{-5} + A^{+ [2ab]} \theta^{-6} + \psi^{+3a} \theta^{-7} + e^{+4} \theta^{-8} ,$$

(6.15)

where we have introduced the notation

$$\theta^{-a} = \theta^{-1} ,$$

$$\theta^{-2ab} = \frac{1}{2!} \theta^{-a} \theta^{-b} ,$$

$$\theta^{-3abc} = \frac{1}{3!} \theta^{-a} \theta^{-b} \theta^{-c} ,$$

$$\theta^{-4abcd} = \frac{1}{4!} \theta^{-a} \theta^{-b} \theta^{-c} \theta^{-d} ,$$

(6.16)

The charges of $2^N$ component fields appearing in the expansion (6.15) can be identified with the component 'helicities' to be multiplied by a factor 2. The prepotential $V^{-4}$ is obviously self-adjoint in the case of $N = 8$ only.

Multiplying the last remaining equation of motion in the first line of eq. (6.14) with $C^\alpha\beta$ yields

$$\Box V^{-4} + \frac{1}{2}(\partial^{+\alpha} \partial^{+}_{D} V^{-4}) \eta^{ED} (\partial^{+}_{E} \partial^{+}_{\alpha} V^{-4}) = 0 ,$$

(6.17)

where we have introduced the d’Alembertian $\Box = (\partial^{+}_{\alpha} \partial^{+}_{\beta} - \partial^{+}_{\alpha} \partial^{-}_{\beta}) C^{\alpha\beta}$.

The action, whose equations of motion are given by eq. (6.17) in the case of the maximally extended $N = 8$ SDSG, reads

$$S = \int d^{4}x d^{+8} \theta du \left\{ \frac{1}{2} V^{-4} \Box V^{-4} + \frac{1}{6} V^{-4} (\partial^{+\alpha} \partial^{+}_{D} V^{-4}) \eta^{ED} (\partial^{+}_{E} \partial^{+}_{\alpha} V^{-4}) \right\} .$$

(6.18)

The action (6.18) is our main result in this paper. Though a similar action can be written down for any $N$, it is Lorentz-invariant only in the case of $N = 8$ SDSG, whose measure is dimensionless and of charge (+8). The action is also invariant under the residual gauge transformations in the Frobenius gauge. In the analytic coordinates, they are given by the analytic diffeomorphisms whose infinitesimal superfield parameters are

$$\lambda^{+A}(z^{+}, u) = \partial_{B} \lambda^{++}(z^{+}, u) \eta^{AB} ,$$

(6.19)
and

\[ \lambda^{-A}(z^+, u) = z^{-C} \partial_C \partial_B \lambda^{++}(z^+, u) \eta^{AB} + \tilde{\lambda}^{-A}(z^+, u), \]  

(6.20)

and the analytic \(OSp(8|2)\) rotations whose infinitesimal superfield parameters are

\[ \lambda^B_A = -\partial^- \partial^- \lambda^{++}(z^+, u) \eta^{BC}, \]  

(6.21)

in terms of the independent analytic parameters \(\lambda^{++}(z^+, u)\) and \(\tilde{\lambda}^{-A}(z^+, u)\) to be evaluated at \(\theta^+ = 0\).

### 7 Conclusion

To conclude, let us briefly summarize what we did in the preceeding sections. The on-shell \(N\)-extended SDSG superspace constraints were reformulated in terms of the analytic coordinates in harmonic superspace. Then the Frobenius gauge was introduced, in which the difference between the world- and tangent- superspace indices disappeared, as in ref. [13]. The harmonic superspace constraints were partially solved in terms of two prepotentials \(H^{+4}(z^+, u)\) and \(V^{-4}(z^+, z^-, u)\). The analytic Devchand-Ogievetsky prepotential \(H^{+4}\) exists only on-shell, where it formally solves the SDSG equations via the twistor transform. In order to describe the theory off-shell, we relaxed the SDSG constraints and solved the vanishing graded commutator of the \(\nabla^-_A\) covariant derivatives in the Frobenius gauge. Along these lines, no \(H^{+4}\) prepotential appears, while the whole theory can be described in terms of \(V^{-4}\) only (cf. ref. [12]). The equations of motion for the prepotential \(V^{-4}\) were derived and divided into two groups. The first group of equations was used to solve all the \(\theta^+\) dependence of the prepotential \(V^{-4}\) in terms of the remaining anti-analytic components. The only remaining superfield equation of motion was then obtained from a Chern-Simons-like harmonic superspace action. This \(N = 8\) SDSG action is very similar to the light-cone \(N = 8\) SDSG action found by Siegel. It also naturally generalizes the \(N = 4\) SDSYM covariant action found by Sokatchev to the case of the \(N = 8\) SDSG.

Our action (6.18) is manifestly Lorentz-invariant, and it is also manifestly supersymmetric with respect to a half of the original supersymmetries of the on-shell \(N = 8\) SDSG by construction. Thus it may be useful e.g., for a study of quantum properties of the \(N = 8\) SDSG. We would like to investigate further the meaning of the residual gauge symmetries of this action, as well as its hidden symmetries. As is well-known, the equations of motion of the non-self-dual (ungauged) \(N = 8\) supergravity have the hidden non-compact global symmetry \(E_{7(7)}\) [10], which is broken in
the $SO(8)$-gauged version of the $N = 8$ theory \cite{17}. Nevertheless, even in the gauged $N = 8$ supergravity its $70 = 133 - 63$ massless physical scalars can still be considered as the (gauged) non-linear sigma-model fields taking their values in the target space $E_7/SU(8)$ \cite{17}. Moreover, a discrete subgroup $E_{7(+7)}(\mathbb{Z})$ survives as the full $U$-duality group in the four-dimensional (toroidally compactified) type-II superstring theory which generalizes the non-self-dual four-dimensional $N = 8$ supergravity \cite{18}. In particular, the $SO(6, 6; \mathbb{Z})$ subgroup of $E_{7(+7)}(\mathbb{Z})$ cab be identified with the type-II superstring T-duality group. Though the $N = 8$ self-dual supergravity is much more simpler than the non-self-dual one, the equations of motion of the ungauged $N = 8$ SDSG seem to have no true duality symmetries beyond the manifest $SL(8; \mathbb{R})$ global rotations. On the other hand, one should expect a very rich spectrum of hidden affine-like symmetries in the $N = 8$ SDSG equations of motion, which generalize the known infinite-dimensional symmetries of the usual $(N = 0)$ SDG \cite{19}. The gauged $N = 8$ SDSG may be relevant for the toroidally compactified four-dimensional heterotic strings with a restricted $U$-duality group to be a subgroup of the full $U$-duality group $O(6, 22; \mathbb{Z}) \otimes SL(2; \mathbb{Z})$ expected in four dimensions \cite{18}. The compactified versions of the gauged $N = 8$ SDSG down to two and three spacetime dimensions may also be relevant for the heterotic spinning strings with the $(2, 1)$ gauged world-sheet supersymmetry and the (compactified) M-theory \cite{20}.

Our analysis does not exclude, in principle, an existence of covariant actions for the non-selfconjugate $N < 4$ SDSYM and $N < 8$ SDSG. \footnote{See e.g., ref. \cite{22} for some recent proposals.} However, in order to compensate the mismatch of ‘helicities’ or $GL(1)\,'$-charges, any covariant action should inevitably have an infinite number of Lagrange multipliers which have to compensate each other altogether. These compensating fields are apparently different from the auxiliary fields present in the harmonic superfield expansion, and they may need extra conditions (i.e. beyond the equations of motion which arise from the action alone) for their actual decoupling on-shell.

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Appendix: $N = 8$ gauged SDSG in components

The covariant component action of the $N = 8$ SDSG was given by Siegel [1]. It this Appendix we merely reproduce his action for the sake of completeness. A direct proof of the on-shell equivalence between the Siegel action (A.1) and our action (6.18) would require eliminating infinitely many auxiliary fields present in the action (6.18) and fixing a gauge in the action (A.1). Though we didn’t do this off-shell, the on-shell equivalence is nevertheless clear from our construction since we started from the same equations of motion in superspace.

The component action was found in ref. [1] by extracting the component equations of motion out of the on-shell $N = 8$ SDSG superspace constraints (subsect. 2.2) in a covariant (Wess-Zumino) gauge. The independent components of the $N = 8$ SDSG and their ‘helicities’ according to ref. [1] are

| $m_{\alpha\alpha'}$ | $\psi_{\mu}^{\alpha\alpha'}$ | $\psi_{m\alpha}$ | $A_{m\alpha\beta}$ | $\chi_{ab}^{\alpha}$ | $\phi_{abcd}$ | $\chi_{abde}^{\alpha}$ | $G_{ab}^{\alpha\beta'}$ | $\tilde{\psi}^{\alpha\alpha'}_{m}$ | $\tilde{R}_{mn\alpha\beta'}^{\alpha\alpha'}$ | $\tilde{\omega}_{mn\alpha'}^{\alpha\beta'}$ |
|------------------|------------------|----------------|-----------------|-----------------|----------------|-----------------|----------------|----------------|-----------------|----------------|
| 2               | 3/2              | 3/2            | 1               | 1/2             | 0              | -1/2            | -1             | -3/2           | -3/2           | -2             |

where we have introduced the notation $m = (\mu\mu')$.

The component covariant Lagrangian is given by [1]

$$
\mathcal{L} = \tilde{\omega}_{m\alpha\beta'}^{\alpha\beta'} T_{m\alpha\beta'}^{\alpha\beta'} + \tilde{\psi}_{m\alpha\beta'}^{\alpha\alpha'} \left[ e_{m\alpha}^{\alpha}\psi_{\mu}^{\alpha\alpha'} - \psi_{m\alpha}^{\alpha} A_{\mu\alpha\beta}^{\alpha} - \frac{1}{2} R_{mn\alpha\beta'}^{\alpha\alpha'} P_{m\alpha\beta'}^{\alpha\alpha'} \right] \\
\quad + e^{-1} \left[ \frac{1}{2} G_{ab}^{\alpha\beta'} F_{ab\alpha\beta'} + \frac{1}{2} \chi_{ab}^{\alpha} \chi_{ab\alpha} - \frac{1}{2} m_{\alpha\beta'}^{\alpha\alpha'} \left( \nabla_{\alpha\alpha'} \phi_{abcd} \right) \left( \nabla_{\alpha\alpha'} \phi_{efgh} \right) \\
\quad - \frac{1}{2} m_{\alpha\beta'}^{\alpha\alpha'} \left( \nabla_{\alpha\alpha'} \phi_{abcd} F_{ab\alpha\beta'} - \frac{1}{2} m_{\alpha\beta'}^{\alpha\alpha'} \chi_{ab\alpha} \chi_{de\alpha} F_{gh\alpha\beta} + \frac{1}{2} \eta_{mn}^{\alpha\beta'} \phi_{abcd} \chi_{efm\alpha} \chi_{gh\alpha} \right) \right]
$$

(A.1)

where the following notation has been introduced for the component torsion:

$$
- T_{mn}^{\alpha\alpha'} = \partial_{\alpha} e_{m\alpha}^{\alpha\alpha'} + e_{m\alpha}^{\beta\alpha'} \omega_{n\mu}^{\beta\alpha} + \psi_{m\alpha}^{\beta} \psi_{n\alpha}^{\mu} + e_{m\alpha}^{\beta\alpha'} \chi_{abcd}^{\beta\alpha'} \\
$$

(A.2)

the supercovariant derivatives and field strengths:

$$
\nabla_{\alpha\alpha'} \phi_{abcd} = D_{\alpha\alpha'} \phi_{abcd} - \frac{1}{2} \psi_{\alpha\alpha'} [\chi_{abcd}]_{\beta} + \psi_{\alpha\alpha'} e_{\beta\alpha'} \chi_{abcd}^{\beta\alpha'} , \\
\nabla_{\alpha\alpha'} \chi_{ab\beta} = D_{\alpha\alpha'} \chi_{abcd} + \psi_{\alpha\alpha'} \phi_{ab\gamma} \left( \frac{1}{2} C_{\alpha\beta'}^{\delta\gamma} \phi_{c\delta}^{\gamma} + \frac{1}{2} C_{\alpha\beta'}^{\delta\gamma} \delta_{[\alpha}^{\delta} F_{bc\delta]} \right) + \psi_{\alpha\alpha'} e_{\beta\alpha'} \chi_{abcd}^{\beta\alpha'} , \\
F_{mnab} = f_{mnab} - \psi_{mn}^{\alpha\beta} \chi_{ab\alpha} + \psi_{mn}^{\alpha\beta} \left( e_{m}^{\alpha} \chi_{ab\alpha} + \frac{1}{2} \psi_{n}^{\beta} \phi_{ab\alpha} \right) , \\
F_{ab\alpha\beta} = e_{\alpha}^{\alpha\alpha'} e_{\beta\alpha'} F_{mnab} , \\
F_{ab\alpha\beta'} = e_{\alpha}^{\alpha\alpha'} e_{\beta\alpha'} F_{mnab} 
$$

(A.3)
in terms of the standard spacetime covariant derivatives $D_{\alpha\alpha'} = e^m_{\alpha\alpha'} D_m$ with the self-dual part of the gravitational (spin) connection $\omega_m^{\alpha\beta}$ as in eq. (2.10), and the standard $SO(8)$ Yang-Mills field strength $f_{mnab}(A)$.

As is clear from eq. (A.1), all the fields of negative ‘helicity’ appear in the Lagrangian as Lagrange multipliers. The scalars $\phi_{abcd}$ are just the Lagrange multipliers for themselves. The vierbein $e_m^{\alpha\alpha'}$ describes on-shell a self-dual graviton, whose anti-self-dual counterpart is given by the abelian gauge field $\tilde{\omega}_{m\alpha'\beta'}$. The self-dual gravitini are described on-shell in terms of the abelian gauge fermionic fields $\tilde{\psi}_{m\alpha'\beta'}$ and $\tilde{R}_{mnab\alpha'}$ since the other fermionic fields $\psi_{m\alpha'\beta'}$ and $\tilde{\psi}_{m\alpha'\beta'}$ of helicity $\pm 3/2$ merely represent the gauge auxiliary degrees of freedom. The equations of motion for the self-dual graviton and gravitini are just the vanishing torsion constraints in eq. (A.2), as they should. The vector field $A_{mnab}$, whose field strength is self-dual on-shell, is paired with its anti-self-dual tensor counterpart $G_{ab\alpha'\beta'}$, and similarly for the spinor fields $\chi_{abc\alpha}$ and $\chi_{abcde\alpha'}$.

The Lagrangian (A.1) can be considered as the $N = 8$ supersymmetric covariant generalization of the self-dual gravity in terms of Ashtekar variables [21]. The corresponding $N = 8$ action is invariant under the following local symmetries:

- general coordinate diffeomorphisms in the ultra-hyperbolic spacetime,
- the $SL(2)$ local Lorentz transformations,
- the $SO(8)$ gauge transformations,
- $N = 8$ on-shell supersymmetry,
- the extra abelian gauge symmetries whose gauge fields transform as
  \[ \delta \tilde{\omega}_{m\alpha'\beta'} = \partial_m \lambda_{\alpha'\beta'}, \ \delta \tilde{\psi}_{m\alpha'\beta'} = \partial_m \lambda^{\alpha'\beta'} \text{ and } \delta \tilde{R}_{mnab\alpha'} = \partial_{(m} \lambda_{n)ab\alpha'}. \]

The $N = 8$ SDSG component action also has the $global \text{ } SL(2, \mathbb{R})'$ Lorentz symmetry, and it is invariant under constant shifts of scalars $\phi_{abcd}$ (the Peccei-Quinn-type symmetry) to be accompanied by corresponding transformations of some other fields too [1].
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