Popescu, S., Sainz, A. B., Short, A. J., & Andreas, W. (2020). Reference frames which separately store noncommuting conserved quantities. Physical Review Letters, 125, [090601]. https://doi.org/10.1103/PhysRevLett.125.090601

Peer reviewed version

Link to published version (if available):
10.1103/PhysRevLett.125.090601

Link to publication record in Explore Bristol Research

PDF-document

This is the author accepted manuscript (AAM). The final published version (version of record) is available online via American Physical Society at https://doi.org/10.1103/PhysRevLett.125.090601 . Please refer to any applicable terms of use of the publisher.

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Reference frames which separately store non-commuting conserved quantities

Supplemental Material

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I. EXPLICIT BOUNDS FOR $O\left(\frac{1}{N^2}\right)$ AND $O\left(\frac{1}{N}\right)$ TERMS FOR SPIN-$\frac{1}{2}$ SYSTEMS

Here we provide more detailed technical proofs giving explicit bounds for the $O\left(\frac{1}{N^2}\right)$ and $O\left(\frac{1}{N}\right)$ terms in our bounds for spin-$\frac{1}{2}$ systems, obtained using similar techniques to those introduced in Ref. [9].

A. Proof of Eq. (6)

First notice that

$$\text{tr}_R \left\{ V_\alpha \rho_S \otimes \tau'_y \otimes \tau''_z V_\alpha^\dagger \right\} = \rho_S - i \frac{4\alpha}{N} \text{tr}_R \left\{ [T, \rho_S \otimes \tau'_y \otimes \tau''_z] \right\} + O\left(\frac{1}{N^2}\right),$$

$$= \rho_S - i \frac{4\alpha}{N} \sum_{j,k,l \in \{x,y,z\}} \epsilon_{jkl} [s_j, \rho_S] \text{tr} \{ s'_k \tau'_y \} \text{tr} \{ s''_l \tau''_z \} + O\left(\frac{1}{N^2}\right),$$

$$= \rho_S - i \frac{\alpha}{N} [s_x, \rho_S] + O\left(\frac{1}{N^2}\right),$$

$$= U_{\alpha,x} \rho_S U_{\alpha,x}^\dagger + O\left(\frac{1}{N^2}\right). \quad (16)$$

What we need to do now is to provide a rigorous computation for the $O\left(\frac{1}{N^2}\right)$ term. To do this, we must first show that

$$\xi = \left\| \text{tr}_R \left\{ V_\alpha \rho_S \otimes \tau'_y \otimes \tau''_z V_\alpha^\dagger \right\} - (\rho_S - i \frac{4\alpha}{N} \text{tr}_R \left\{ [T, \rho_S \otimes \tau'_y \otimes \tau''_z] \right\}) \right\|_1 \leq O\left(\frac{1}{N^2}\right) \quad (17)$$
where \( V_\alpha = \exp\{-i \frac{4\alpha}{N} T\} \). Expanding the exponentials we obtain

\[
\xi = \left\| \sum_{n=2}^{\infty} \sum_{k=0}^{n} \text{tr}_R \left( \left( -i \frac{4\alpha}{N} T \right)^k \rho_S \otimes \tau_y' \otimes \tau_z' \left( i \frac{4\alpha}{N} T \right)^{n-k} \right) \right\|_1 \leq \sum_{n=2}^{\infty} \left( \frac{4\alpha}{N} \right)^n \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \left\| T^k \rho_S \otimes \tau_y' \otimes \tau_z' T^{n-k} \right\|_1 \leq \sum_{n=2}^{\infty} \left( \frac{4\alpha}{N} \right)^n \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \left\| T \right\|^{n-k} \leq \sum_{n=2}^{\infty} \frac{1}{n!} \left( \frac{3\alpha}{N} \right)^n \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \leq \sum_{n=2}^{\infty} \frac{1}{n!} \left( \frac{6\alpha}{N} \right)^n \leq 36 \left( \frac{\alpha}{N} \right)^2 \left( e - 2 \right) \left( \frac{\alpha}{N} \right)^2 . \tag{18} \]

where in the fifth line we have used the fact that \( T = \sum_{j,k,l \in \{x,y,z\}} \epsilon_{jkl} s_j \otimes s_k' \otimes s'' \) has 6 non-zero terms, each of which has operator norm \( \frac{1}{3} \) (as \( \|s_j\| = \frac{1}{2} \)), giving \( \|T\| \leq \frac{3}{2} \). Hence for \( N \geq 6\alpha \),

\[
\xi \leq 36 \left( e - 2 \right) \left( \frac{\alpha}{N} \right)^2 . \tag{19} \]

On the other hand, applying Eq. (D5) of Ref. [9] to our setup yields

\[
\left\| U_{\alpha,x} \rho_S U_{\alpha,x}^\dagger - \left( \rho_S - i \frac{\alpha}{N} [s_x, \rho_S] \right) \right\|_1 \leq 4(e - 2) \left( \frac{\alpha}{N} \right)^2 , \tag{20} \]

where \( U_{\alpha,x} = \exp\{-i \frac{\alpha}{N} s_x\} \).

Combining Eqs. (19) and (20) through the triangle inequality, we obtain

\[
\left\| \text{tr}_R \left\{ V_\alpha \rho_S \otimes \tau_y' \otimes \tau_z' V_\alpha^\dagger \right\} - U_{\alpha,x} \rho_S U_{\alpha,x}^\dagger \right\|_1 \leq 40(e - 2) \left( \frac{\alpha}{N} \right)^2 , \tag{21} \]

where we have used the fact that \( \text{tr}_R \{[T, \rho_S \otimes \tau_y' \otimes \tau_z']\} = \frac{1}{4} [s_x, \rho_S] \).

**B. Proof of Eqs. (7) and (8)**

Here we will discuss the case of Eq. (7), since Eq. (8) follows similarly. First notice that

\[
\Delta s_i' = \text{tr} \left\{ \left( \mathbb{1} \otimes s_i' \otimes \mathbb{1} \right) V_\alpha \left( \rho_S \otimes \tau_y' \otimes \tau_z' \right) V_\alpha^\dagger \right\} - \text{tr} \left\{ s_i' \tau_y' \right\} ,
\]

\[
= -i \frac{4\alpha}{N} \sum_{j,k,l \in \{x,y,z\}} \epsilon_{jkl} \text{tr} \left\{ s_j \rho_S \right\} \text{tr} \left\{ s_i' [s_k', \tau_y'] \right\} \text{tr} \left\{ s_j' \tau_z'' \right\} + \mathcal{O} \left( \frac{1}{N^2} \right) ,
\]

\[
= -\frac{\alpha}{N} \delta_{i,z} \text{tr} \{ s_y \rho_S \} + \mathcal{O} \left( \frac{1}{N^2} \right) , \tag{22} \]

where in the last step we have noted that the expression is only non-zero when \( \ell = z, k = x \) and \( j = y \), and used the spin commutation relation \([s_a, s_b] = i \epsilon_{abc} s_c\).
Now, applying similar arguments to those in Eqs. (17)–(19) to the changes in spin given by Eq. (22) we obtain
\[
\left| \Delta s'_i + \frac{\alpha}{N} \delta_{i,z} \text{tr} \{ s_y \rho_S \} \right| \leq 18 (e - 2) \left( \frac{\alpha}{N} \right)^2 , \tag{23}
\]
\[
\left| \Delta s''_i - \frac{\alpha}{N} \delta_{i,y} \text{tr} \{ s_z \rho_S \} \right| \leq 18 (e - 2) \left( \frac{\alpha}{N} \right)^2 . \tag{24}
\]

C. Proofs of Eqs. (12), (13) and (14)

Let us start by proving Eq. (12). When considering a general small rotation, we must obtain a bound on
\[
\zeta \equiv \left\| \text{tr}_R \left\{ V_{\alpha_x} V_{\alpha_y} V_{\alpha_z} (\rho_S \otimes \tau_R) V_{\alpha_x}^\dagger V_{\alpha_y}^\dagger V_{\alpha_z}^\dagger \right\} - (\rho_S - i \frac{1}{N} [H, \rho_S]) \right\|_1 . \tag{25}
\]
Following a similar argument to that below Eq. (D7) of Ref. [9], expanding the unitaries and collecting terms in \( \left( \frac{1}{N} \right)^n \) for \( n \geq 2 \) we obtain at most 6\(^n\) contributions (as each of the 6 unitaries could provide each power of \( \frac{1}{N} \)), with a constant coefficient upper bounded by \( (4\alpha_{\text{max}})^n \) where \( \alpha_{\text{max}} \leq \pi \). Each term also contains the trace norm of an operator with \( n \) copies of \( T \), acting on different parties, distributed either before or after the initial state. Following the argument above Eq. (19), such a term is upper bounded by \( \left( \frac{3}{4} \right)^n \). Combining all of these observations we obtain
\[
\zeta \leq \sum_{n=2}^{\infty} \left( \frac{18\pi}{N} \right)^n . \tag{26}
\]
For \( N \geq 36\pi \) we therefore obtain \( \zeta \leq 648\pi^2 \left( \frac{1}{N} \right)^2 \). Combining this with Eq. (D11) of Ref. [9], for \( D = 3 \), we obtain
\[
\left\| V_{\alpha_x} V_{\alpha_y} V_{\alpha_z} (\rho_S \otimes \tau_R) V_{\alpha_x}^\dagger V_{\alpha_y}^\dagger V_{\alpha_z}^\dagger R - (\rho_S - i \frac{1}{N} [H, \rho_S]) \right\|_1 \leq (648 + 16(e - 2)) \frac{\pi^2}{N^2} \tag{27}
\]
for sufficiently large \( N \). Iterating this transformation \( N \) times and using the inductive argument in Appendix C of Ref. [9], gives
\[
\left\| \text{tr}_R \left\{ V \rho_S \otimes \rho_R V^\dagger \right\} - U_S \rho_S U_S^\dagger \right\|_1 \leq (648 + 16(e - 2)) \frac{\pi^2}{N} . \tag{28}
\]
This completes the proof of Eq. (12).

To now prove Eqs. (13) and (14), we can apply a similar argument as above. This leads to
\[
\left| \Delta s_j + \Delta s'_j \right| \leq 648 \pi^2 \left( \frac{1}{N} \right), \tag{29}
\]
\[
\left| \Delta s''_j \right| \leq 324 \pi^2 \left( \frac{1}{N} \right) , \tag{30}
\]
from which the argument follows.

II. SEPARATING CONSERVED QUANTITIES FOR HIGHER DIMENSIONAL SYSTEMS

We will now consider whether we can separate changes in conserved quantities for higher dimensional systems, with dimension \( d > 2 \).
A. Introductory thoughts on systems of dimension \( d > 2 \) and spin conservation laws

Let us first consider angular momentum conservation of spin \( s \) systems, with dimension \( d = 2s + 1 \). In this case, we can use essentially the same procedure as in the main text (now taking \( \tau_j \) as the eigenstate of \( s_j \) with maximum spin in the \( j \)-direction, and \( V_a = \exp\{-i\frac{a}{2}S_a\} \)) to approximately implement any single system rotation of the form \( e^{-i\theta_n s^n} \) on a spin-\( s \) system, whilst conserving the three components of total angular momentum \( S_x, S_y \) and \( S_z \). Any changes in angular momentum will be separated into three different ‘batteries’ as before. Furthermore, we could again implement interacting unitaries between different spins such as \( e^{-i\theta_s s^{(1)}s^{(2)}} \), as this is rotationally invariant.

However, unlike in the case of spin-\( \frac{1}{2} \) particles, spatial rotations of the form \( e^{-i\theta_n n^a} \) no longer represent the complete set of local unitary transformations for spin-\( s \) particles. It would be interesting to explore what additional transformations are required in order to give a universal gate set, and whether these can be implemented in such a way as to localise any changes to angular momentum.

Moving from angular momentum conservation to more general conservation laws, an interesting question is whether we can construct a complete basis of conserved quantities, such that arbitrary unitary transformations of a system can be performed whilst separating the changes in all of these conserved quantities into different ‘batteries’. Later in this section we generalise the proofs to show that this is possible when the dimension of the system is \( 2^n \).

B. Generalisation to arbitrary dimension: what we can and may not do (yet)

Here we present a generalisation of the reference frame defined in the main text for spin-\( \frac{1}{2} \) systems, and of the operator \( T \) of Eq. (4), to higher dimensional systems. In particular, we give sufficient conditions to be able to construct a complete basis of extensive conserved quantities for such systems, and a reference frame that allows us to perform arbitrary unitary transformations of the system whilst storing any changes in these conserved quantities in different ‘batteries’ (up to arbitrarily small corrections).

Consider a system \( S \), of dimension \( d > 2 \). In this case there exist \( K = d^2 - 1 \) possible linearly independent conserved quantities, where without loss of generality we do not include the scalar. In what follows, we will consider the case in which there exists a particular choice for these conserved quantities given by \( K \) Hermitian operators \( \mathcal{M} = \{O_k\}_{k=0,\ldots,K-1} \) with the following properties

(i) Traceless: \( \text{tr} \{O_k\} = 0 \) for all \( k \)

(ii) Orthogonal: \( \text{tr} \{O_kO_\ell\} = 0 \) if \( k \neq \ell \)

(iii) Closed under commutation: For each \( k, \ell \) either \( \{O_k, O_\ell\} = 0 \), or there exists an \( m \) such that \( \{O_k, O_\ell\} \propto O_m \).

Together with the identity these operators form an orthogonal operator basis for the \( d \)-dimensional system. In (iii), note that the constant of proportionality could depend on \( k \) and \( \ell \), and that \( m \neq \ell \) as \( \text{tr} \{[O_k, O_\ell]O_\ell\} = \text{tr} \{O_k[O_\ell, O_\ell]\} = 0 \) (and similarly \( m \neq k \)).

First let us consider how to perform the small unitary transformation \( U_{a,0} = \exp(-i\frac{\tau_0^a}{2}O_0) \). To do this, we first prepare a reference frame consisting of \( D = K - 1 \) particles of the same type as the system, initialised in the state

\[ \rho_R = \rho^{(1)} \otimes \ldots \otimes \rho^{(D)}, \tag{31} \]

where \( \rho^{(k)} = \frac{1}{d} \left( I + \frac{1}{\|O_k\|} O_k \right) \). Given that the operators \( O_k \) are traceless and orthogonal, it follows that \( \text{tr} \{O_k\rho^{(r)}\} = n_k \delta_{k,r} \) where \( n_k = \text{tr} \{O_k^2\}/(d\|O_k\|) \).

A generalisation of the interaction \( T \) of Eq. (4) to systems of arbitrary dimensions is

\[ T = \sum_{a,a_1,\ldots,a_D} f_{a,a_1,\ldots,a_D} O_a \otimes O_{a_1} \otimes \ldots \otimes O_{a_D}, \tag{32} \]
where \( f_{a_1 \ldots a_D} = 0 \) unless all the sub-indices are different, in which case \( f_{a_1 \ldots a_D} = \pm 1 \), where the sign indicates if the sub-indices are an even or odd permutation of \( 0, 1, \ldots, D \).

Notice that for \( d = 2 \), \( f \) is just the antisymmetric symbol \( \epsilon_{ijk} \). In addition, when \( d = 2 \) and \( M = \{ s_k \}_{k=x,y,z} \), we recover the interaction defined in Eq. (4) and the batteries defined in the main text for spin-\( \frac{1}{2} \) systems.

The main requirement that \( T \) should satisfy is \( [T, C] = 0 \) for any extensive conserved quantity \( C \). We will show this in the next subsection, but for now we focus on using it to perform transformations of the system. The operator \( T \) allows us to define a global unitary interaction among \( K \) systems given by

\[
V = \exp \left( -i \frac{1}{\eta_1 \ldots \eta_D} \alpha \frac{1}{N} T \right),
\]

similarly to Eq. (5).

The effective action of this global unitary \( V \) on our system \( A \) is given by \( \text{tr}_R \{ V \rho_S \otimes \rho_R V^\dagger \} \). Computing this trace, we find that only the term in \( T \) containing \( f_{012 \ldots D} \) gives a non-zero contribution to first order in \( \frac{1}{N} \), leading to

\[
\text{tr}_R \{ V \rho_S \otimes \rho_R V^\dagger \} = \rho_S - i \frac{\alpha}{N} [O_0, \rho_S] + \mathcal{O} \left( \frac{1}{N^2} \right)
= U_{\alpha,0} \rho_S U_{\alpha,0}^\dagger + \mathcal{O} \left( \frac{1}{N^2} \right),
\]

\( \sum_{f_{r1} \ldots f_{rD}} \) gives a non-zero contribution to first order in \( \frac{1}{N} \). By iterating this process \( N \) times, we can then implement an arbitrary transformation \( \exp(-iH) \) on the system. As the error per step is \( \mathcal{O} \left( \frac{1}{N^2} \right) \) and there are \( \mathcal{O} (N) \) steps, the overall error can be made as small as desired by choosing \( N \) sufficiently large.

Next, let us see how the the conserved quantities are stored in the batteries (i.e., the reference frame). For simplicity, we again consider implementing the small transformation \( U_{\alpha,0} \) generated by \( O_0 \) on the system. The change in the conserved quantity \( O_k \) for particle \( r \) in the reference frame is given by

\[
\Delta O_k^r = \text{tr} \{ O_k^r V \rho_S \otimes \rho_R V^\dagger \} - \text{tr} \{ O_k^r \rho^{(r)} \},
\]

where \( O_k^r := I \otimes \ldots \otimes \underbrace{O_k \otimes \ldots \otimes I}_r \) is the operator that is \( I \) everywhere except for the \( r \)-th frame particle where it is \( O_k \). Expanding \( V \) to first order in \( \frac{1}{N} \) we obtain

\[
\Delta O_k^r = i \frac{1}{\eta_1 \ldots \eta_D} \frac{\alpha}{N} \text{tr} \{ O_k^r [T, \rho_S \otimes \rho_R] \} + \mathcal{O} \left( \frac{1}{N^2} \right)
= i \frac{1}{\eta_1 \ldots \eta_D} \frac{\alpha}{N} \sum_{a,a_1,\ldots,a_D} f_{a_1 \ldots a_D} \text{tr} \{ O_k^r [O_{a_1}, \rho^{(r)}] \} \text{tr} \{ O_a \rho_S \} \prod_{j \neq r} \text{tr} \{ O_a, \rho^{(j)} \} + \mathcal{O} \left( \frac{1}{N^2} \right)
= -i \frac{1}{\eta_r} \frac{\alpha}{N} \sum_{a,a_r} f_{a_1,\ldots,r-1,a_r,r+1,D} \text{tr} \{ O_a \rho_S \} \text{tr} \{ O_k^r [O_{a_r}, \rho^{(r)}] \} + \mathcal{O} \left( \frac{1}{N^2} \right). \tag{35}
\]

Now notice that the sum in Eq. (35) consists of two terms: \( \{ a = 0, a_r = r \} \) and \( \{ a = r, a_r = 0 \} \), since any other assignment of values to \( a \) or \( a_r \) will render \( f_{a_1,\ldots,r-1,a_r,r+1,D} = 0 \). Hence,

\[
\Delta O_k^r = -i \frac{1}{\eta_r} \frac{\alpha}{N} \left( f_{0,1,\ldots,r-1,r+1,D} \text{tr} \{ O_0 \rho_S \} \text{tr} \{ O_k^r [O_r, \rho^{(r)}] \} \right)
+ f_{r,1,\ldots,r-1,0,r+1,D} \text{tr} \{ O_r \rho_S \} \text{tr} \{ O_k^r [O_0, \rho^{(r)}] \} + \mathcal{O} \left( \frac{1}{N^2} \right). \tag{36}
\]

\[
\Delta O_k^r = -i \frac{1}{\eta_r} \frac{\alpha}{N} \left( f_{0,1,\ldots,r-1,r+1,D} \text{tr} \{ O_0 \rho_S \} \text{tr} \{ O_k^r [O_r, \rho^{(r)}] \} \right)
+ f_{r,1,\ldots,r-1,0,r+1,D} \text{tr} \{ O_r \rho_S \} \text{tr} \{ O_k^r [O_0, \rho^{(r)}] \} + \mathcal{O} \left( \frac{1}{N^2} \right). \tag{37}
\]
Using \( \rho^{(r)} = \frac{1}{d} \left( I + \frac{1}{\|O_k\|} O_k \right) \) and \( f_{r,1,\ldots,r-1,0,r+1,D} = -1 \), we obtain

\[
\Delta O^r_k = i \frac{\alpha}{N} \text{tr} \left\{ O_k \rho S \right\} \text{tr} \left\{ O_k [O_0, O_r] \right\} + \mathcal{O} \left( \frac{1}{N^2} \right). \tag{38}
\]

So far, we have only used the first two properties of the operators \( O_k \). Given the third property that the operators are closed under commutation, we find that either \( [O_0, O_r] = 0 \), in which case the \( r^{th} \) subsystem in the reference frame accumulates no changes in any conserved quantity (i.e., \( \Delta O^r_k = 0 \) for all \( k \)), or \( [O_0, O_r] \propto O_m \). In the latter, the \( r^{th} \) subsystem will only accumulate changes in the conserved quantity \( O_m \) (i.e., \( \Delta O^r_k = 0 \) unless \( k = m \)). As each subsystem accumulates changes in at most one conserved quantity, it is possible to separate all of the conserved quantities into different batteries. A similar result will apply for small rotations generated by any \( O_k \) and thus for the overall transformation.

C. \( T \) preserves extended conserved quantities

In this subsection, we show that \( T \) commutes with all extensive conserved quantities. Let us begin by revisiting the case with \( d = 2 \), where our conserved quantities are \( s_x \), \( s_y \) and \( s_z \), as presented in the main text for spin-\( \frac{1}{2} \) systems. Here, \( T \) acts on three particles as

\[
T = \sum_{a,a_1,a_2} f_{aa_1a_2} s_a \otimes s_{a_1} \otimes s_{a_2}.
\]

Now consider the conserved quantity corresponding to the total angular momentum in the \( x \)-direction, \( S_x = s_x \otimes I \otimes I + I \otimes s_x \otimes I + I \otimes I \otimes s_x \), which is the sum of the angular momentum of the three individual particles in the \( x \)-direction. The commutator \([S_x, T]\) may be expressed as

\[
[S_x, T] = \sum_{a,a_1,a_2} f_{aa_1a_2} [s_x, s_a] \otimes s_{a_1} \otimes s_{a_2} + \sum_{a,a_1,a_2} f_{aa_1a_2} s_a \otimes [s_x, s_{a_1}] \otimes s_{a_2} + \sum_{a,a_1,a_2} f_{aa_1a_2} s_a \otimes s_{a_1} \otimes [s_x, s_{a_2}]. \tag{39}
\]

We will now see how each term in each sum is either zero or cancelled out by a similar term appearing in a different sum but with the opposite sign. First, note that all terms in the final answer must contain the same spin operator on exactly two particles and a different spin operator on the third particle. This is because \( a, a_1 \), and \( a_2 \) must all be distinct in order for \( f_{aa_1a_2} \) to be non-zero, and the spin operator generated by the commutator is different from the spin operators appearing within it. As an example, a term proportional to \( s_y \otimes s_y \otimes s_x \) can be generated by \([s_x, s_z] \otimes s_y \otimes s_x \) and \( s_y \otimes [s_x, s_z] \otimes s_x \). The former of these will contribute to \([S_x, T]\) with a coefficient \( f_{zyx} \) via the first sum, whereas the latter will contribute with a coefficient \( f_{yxx} \) via the second sum. As \( f_{zyx} = -f_{yxx} \) these two terms will cancel out, and the same applies to all other terms, giving a total commutator of zero.

Now let us see how a similar reasoning applies to the higher dimensional case. We want to show that the total conserved quantity \( O_k^{\text{tot}} = O_k + \tilde{O}_k^R \) is preserved by \( T \), i.e., that \([O_k^{\text{tot}}, T] = 0\), where

\[
[O_k^{\text{tot}}, T] = \sum_{r=0}^{n} [O_k^r, T]
\]

with

\[
[O_k^r, T] = \sum_{a,a_1,\ldots,a_D} f_{aa_1\ldots a_D} O_a \otimes \ldots \otimes [O_k, O_{a_D}] \otimes \ldots \otimes O_{a_D}.
\]
We will see how each term in \([O_k^r, T]\), for every \(r\), is either 0 or cancelled out by a term in \([O_k^s, T]\), for some other \(u\). To begin, take a fixed but arbitrary \(r\), and a non-zero term in \([O_k^r, T]\). This term is identified by its indices \(aa_1 \ldots a_D\). Given the ‘closed under commutation’ property of the operators in \(\mathcal{M}\), \([O_k, O_{ar}] = \varepsilon O_b\), where \(b \in \{a, a_1, \ldots, a_D\} \setminus \{a_r\}\), and \(\varepsilon\) is a constant that may depend on \(k\) and \(a_r\). Let \(u\) be such that \(a_u = b\), and for simplicity, and without loss of generality, assume \(u > r\). Hence,

\[
O_a \otimes \ldots \otimes [O_k, O_{ar}] \otimes \ldots \otimes O_{au} \otimes \ldots \otimes O_{ad} = \varepsilon O_a \otimes \ldots \otimes O_b \otimes \ldots \otimes O_k \otimes \ldots \otimes O_{ad} \\
= O_a \otimes \ldots \otimes O_{au} \otimes \ldots \otimes [O_k, O_{ar}] \otimes \ldots \otimes O_{ad}.
\]

Now, the term with indices \((aa_1 \ldots a_r \ldots a_u \ldots a_D)\) will contribute to \([O_k^u, T]\) via the sum in the term \([O_k^r, T]\) with coefficient \(f_{a a_1 \ldots a_r \ldots a_u \ldots a_D}\). The same term will however appear in the sum in the term \([O_k^s, T]\), with indices \((aa_1 \ldots a_u \ldots a_r \ldots a_D)\), and will contribute to \([O_k, T]\) with coefficient \(f_{a a_1 \ldots a_r \ldots a_u \ldots a_D}\). By the properties of \(f\), \(f_{a a_1 \ldots a_r \ldots a_u \ldots a_D} = -f_{a a_1 \ldots a_u \ldots a_r \ldots a_D}\), and the two terms cancel out. As every term in the final answer can be generated in exactly two ways with opposite coefficients, the result that \([O_k^u, T] = 0\) then follows. Although we will think of the operators \(O_k^u\) as the conserved quantities of interest, note that any other extensive conserved quantity can be expressed as a linear combination of the operators \(O_k^u\) and the identity, this means that \(T\) commutes with all extensive conserved quantities.

Interestingly, one can also show that \([O_k^u, T] = 0\) without the ‘closed under commutation’ property, by using the fact that \([O_k, O_\ell] = \sum_{m \neq k, \ell} w_{km} O_m\) for some coefficients \(w_{km}\). As above, every non-zero term in \([O_k^u, T] = 0\) can be generated in exactly two ways with opposite coefficients. Hence for any complete set of traceless orthonormal operators \(O_k\), the operator \(T\) constructed from those operators commutes with all extensive conserved quantities.

D. Dimension \(2^n\)

Let us consider the case where we have a system \(S\) of dimension \(d = 2^n\). Here, we can indeed find an operator basis with the properties specified above, and can thus separate any changes to a complete basis of conserved quantities. In particular, we take the set \(\mathcal{M}\) of \(K = 2^n - 1\) operators \(\{O_k\}\) to consist of products of spin-\(\frac{5}{2}\) operators. That is, each \(O \in \mathcal{M}\) has the form \(O = s_1 \otimes \ldots \otimes s_n\), where \(s_k \in \{\frac{1}{2}, \frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z\}\), but where not all \(s_k\) may equal the identity.

It is easy to verify that these operators are traceless and orthogonal. Furthermore, the product of any two distinct operators in \(\mathcal{M}\) is proportional to another operator in the set, i.e., there exists an \(m\) such that \(O_k O_\ell \propto O_m\). By taking the Hermitian conjugate, we also have \(O_\ell O_k \propto O_m\). Hence either \([O_k, O_\ell] = 0\), or \([O_k, O_\ell] \propto O_m\) and these operators are hence closed under commutation as defined above.

E. Explicit bounds

As in the previous case, we can also calculate explicit bounds on the \(O(\frac{1}{N^2})\) terms in the bounds above, using similar techniques to those in Appendix I and in Ref. [9].

In particular, we find

\[
\| \text{tr}_R \{ V \rho_S \otimes \rho_S V^\dagger \} - (\rho_S - i \frac{\alpha}{N} [O_0, \rho_S]) \|_1 \leq \left( 2K \| O_0 \| \| O_1 \| \ldots \| O_D \| \right)^2 \| \frac{\| O_0 \| \| O_1 \| \ldots \| O_D \|}{\eta_1 \eta_2 \ldots \eta_D} \| \alpha \right)^2 \leq (\frac{N}{\alpha})^2,
\]

(40)

where we have used an approach similar to that in the derivation of Eq. (18), assuming that \(N > \left( 2K \| O_0 \| \| O_1 \| \ldots \| O_D \| \right) \alpha \).

Applying now Eq. (D5) of Ref. [9] to this setup yields

\[
\| U_{a,0} \rho_S U_{a,0}^\dagger - (\rho_S - i \frac{\alpha}{N} [O_0, \rho_S]) \|_1 \leq 4(e-2) \left( \frac{\alpha}{N} \right)^2,
\]

(41)

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Combining Eqs. (40) and (41) through the triangle inequality, we find

\[ \left\| \text{tr} \left\{ V \rho S \otimes \rho R V^\dagger \right\} - U_{\alpha,0} \rho S U_{\alpha,0}^\dagger \right\|_1 \leq 4(e - 2) \left( 1 + \left( K! \frac{\|O_0\| \|O_1\| \ldots \|O_D\|}{\eta_1 \eta_2 \ldots \eta_D} \right)^2 \right) \left( \frac{\alpha}{N} \right)^2. \]  

(42)

Finally, following a similar approach we obtain

\[ \left| \Delta O_k^r - \frac{\alpha}{N} \frac{\text{tr} \{ O_r \rho S \} \text{tr} \{ O_k [O_0, O_r] \} \text{tr} \{ O_2^r \}}{\text{tr} \{ O_2^r \}} \right| \leq \|O_k\| \left( 2K! \frac{\|O_0\| \|O_1\| \ldots \|O_D\|}{\eta_1 \eta_2 \ldots \eta_D} \right)^2 (e - 2) \left( \frac{\alpha}{N} \right)^2. \]  

(43)

In the particular case considered above of dimension \( d = 2^n \), in which the operators \( O_k \) are products of spin-\( \frac{1}{2} \) operators and \( I/2 \), note that \( \eta_k = \|O_k\| = \frac{1}{d} \) and hence these expressions simplify considerably.