Universality for persistence exponents of local times of self-similar processes with stationary increments

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Abstract

We show that $P(\ell_X(0, T] \leq 1) = (c_X + o(1))T^{-(1-H)}$, where $\ell_X$ is the local time measure at 0 of any recurrent $H$-self-similar real-valued process $X$ with stationary increments that admits a sufficiently regular local time and $c_X$ is some constant depending only on $X$. A special case is the Gaussian setting, i.e. when the underlying process is fractional Brownian motion, in which our result settles a conjecture by Molchan [Commun. Math. Phys. 205, 97-111 (1999)] who obtained the upper bound $1 - H$ on the decay exponent of $P(\ell_X(0, T] \leq 1)$. Our approach establishes a new connection between persistence probabilities and Palm theory for self-similar random measures, thereby providing a general framework which extends far beyond the Gaussian case.

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1. Introduction: Persistence probabilities for fractional Brownian motion

We study local times of stochastic processes from the point of view of persistence probabilities, i.e. the probabilities that a stochastic process remains inside a relatively small subset of its state space for a long time. The problem of calculating persistence probabilities is a very active field of mathematical research, see e.g. the recent articles [AB18, AGPP18, LS18, Mol18, PS18, AMZ17], an overview of the developments over the last decades is given by Aurzada and Simon in [AS15]. The main motivation to study the persistence behaviour of stochastic systems is its great significance to certain areas of statistical physics, see e.g. [CD08, CDC+04, Maj99] and the survey by Bray et al. [BMS13].

The starting point of the present investigation are Molchan’s celebrated and by now classical results [Mol99] concerning the persistence of linear fractional Brownian motion, which we briefly summarise now. Let \( B = (B_t)_{t \in \mathbb{R}} \) denote a 1-dimensional fractional Brownian motion (FBM) of Hurst index \( H \in (0, 1) \). \( B \) can be characterised as the unique (up to multiplication by a constant) Gaussian process which is \( H \)-self-similar with stationary increments (H-ssi). In [Mol99] it is shown that the maximum process \( \bar{B}_t = \max_{0 \leq s \leq t} B_s, t \geq 0 \), of \( B \) satisfies

\[
P(\bar{B}_T \leq 1) = T^{-(1-H)+o(1)}. \tag{1}
\]

Subsequently, improved bounds on the error estimate implicit in (1) have been derived by Aurzada [Aur11] and by Aurzada et al. [AGPP18]. Note that, using self-similarity, we may replace the boundary 1 in (1) by any fixed value \( x > 0 \) without changing the order of decay. The probability in (1) is called the persistence probability of \( B \) and the corresponding exponent \( \bar{\kappa} = 1 - H \) the persistence exponent of \( B \). In [Mol99], Molchan also showed that the lower tail probabilities of several other path functionals of FBM are governed by the persistence exponent. In particular, he studied

- \( \ell(0, T) := \lim_{\epsilon \to 0} (2\epsilon)^{-1} \int_0^T \mathbb{1} \{ B_t \in (-\epsilon, \epsilon) \} dt \), the local time at 0,
- \( \tau^+ := \inf \{ t \geq 1 : B_t = 0 \} \), the first zero after time 1,
- \( \sigma_T^+ := \int_0^T \mathbb{1} \{ B_t > 0 \} dt \), the time spent in the positive half-axis,
- \( \tau_T^{\max} := \arg \max \{ B_t, t \in [0, T] \} \), the time at which the maximum is achieved;

and his results imply that

\[
\lim_{T \to \infty} \frac{\mathbb{P}(\tau^+ \geq T)}{\log T} = \lim_{T \to \infty} \frac{\mathbb{P}(\sigma_T^+ \leq 1)}{\log T} = \lim_{T \to \infty} \frac{\mathbb{P}(\tau_T^{\max} \leq 1)}{\log T} = H - 1. \tag{2}
\]

These asymptotics can be viewed as a general (and very weak) form of Lévy’s arcsine-laws for Brownian motion. Intuitively, the agreement of exponents can be explained
by observing that the dominating events contributing to each of the probabilities in (1) and (2) are long (negative) excursions of $B$ from $B_0 = 0$. This type of event also entails a small local time at 0, and one is inclined to believe that the probability of the local time being small is of the same order. However, the result in [Mol99] for the local time is only a lower bound, namely that there is a constant $b \in (0, \infty)$ such that

$$
P(\ell(0, T] \leq 1) \geq T^{-(1-H)} be^{-\sqrt{\log T}},$$

for sufficiently large $T$. Hence, the local time persistence exponent

$$
\kappa = - \lim_{T \to \infty} \frac{P(\ell(0, T] \leq 1)}{\log T}
$$

of $B$ satisfies

$$
\kappa \leq 1 - H,
$$

and this upper bound with the error estimate given in (3) is still the best known lower tail estimate for the local time of FBM with index $H \in (0, 1) \setminus \{1/2\}$. The Markovian case $H = 1/2$, is of course exceptional – the exact distribution of $\ell(0, T]$ for 1/2-FBM (i.e. Brownian motion) had already been determined by Lévy [Lev48] 50 years prior to Molchan’s paper.

Molchan’s proofs rely on the connection of the persistence probability to a certain path integral functional and this relation is in fact also useful outside the FBM context, see e.g. [AGP15]. Based on the bounds for the persistence probability obtained in this manner, he then derives the tail bounds for the distribution of the other functionals by explicitly relating the events in question. However, determining $\kappa$ this way is harder than determining $\bar{\kappa}$ – as a functional of the path of $B$, the local time $\ell$ is in general analytically more involved than the other quantities $\bar{B}_T, \sigma_T^+, \tau_T^{\max}$ and $\tau^+$. Thus, relating distributional properties of $\ell$ to the behaviour of $B$ in a path-wise manner is a challenging task. In addition, Molchan’s argument requires some technical tools, namely Slepian’s Lemma and reproducing kernel Hilbert spaces, which are specific to the Gaussian setting.

The goal of this paper is to show how to circumvent these obstacles and establish the equality

$$
\kappa = 1 - H
$$

for FBM directly by studying the local time. In fact, we prove a significantly stronger result, namely that there is a constant $C \in (0, \infty)$, such that

$$
P(\ell([0, T] \leq 1) \sim CT^{-(1-H)},
$$

where here and in what follows we use the notation $f(T) \sim g(T)$ to indicate that the ratio of the two functions $f, g$ converges to 1 as the argument $T$ approaches $\infty$. Our
approach to show (5) does not use that FBM is a Gaussian process. Consequently, (5) not only holds for FBM, but for any $H$-s.ssi process which admits sufficiently regular local time measures.

A heuristic interpretation of the equality (4) is that it relates the time $B$ spends at 0 to the box-counting dimension $1-H$ of its zero set, which equals $1-H$. Indeed, there is a well known non-rigorous box-counting argument, see e.g. [DY95], which suggests that the probability of observing an excursion from 0 of length greater than $T$ is of order $T^{-(1-H)}$. One way of looking at our result is that it makes this connection rigorous; our method indeed enables us to prove (4) using only the invariance properties of the underlying processes, without recurrence to specific distributional structures such as Gaussianity, the Markov or Martingale property, etc.

It is immediate from (4) that we have $\bar{\kappa} = \kappa$ for $B$ and the author believes that this is also true in a more general context. In particular, one should be able to combine the arguments in this paper with the methods developed by Aurzada et al. in [AGPT18, AM18] to show that both persistence exponents coincide for all $H$-s.ssi processes which are positively associated.

The technique for establishing $\kappa = 1-H$ proposed below is completely novel in the context of persistence probabilities. It combines three principal ingredients: A distributional representation of the local times using Palm theory; a simple bi-variate scaling relation for an associated point process, which is equivalent to the $H$-s.ssi property; and a well-known invariance property of Palm distributions which is the measure theoretic counterpart to cycle-stationary [Tho95] in the point processes setting. Only the first part requires a few abstract results from the theory of random measures which are not based on simple calculations. More precisely, we exploit the fact that the local time of an $H$-s.ssi process can be constructed as the Palm distribution associated to certain stationary non-finite distributions of measures on the real line. This approach was originally developed by Zähle [Zäh88, Zäh90] who applied it to determine the carrying (Hausdorff) dimension of local times and other random measures derived from $H$-s.ssi processes [Zäh91].

The remainder of this text is organised as follows. In the next section we fix our notation and present our main result for the local time persistence probabilities in a general setting, Theorem 2.1, and in two special cases, namely for FBM and the Rosenblatt process. The main arguments to prove Theorem 2.1 are given in Section 3 subject to some auxiliary results which require a more extensive discussion. The subsequent three sections are devoted to this groundwork. The necessary background for the in-

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1Heuristic scaling arguments often use the box-counting dimension to capture the fractality of a set due to its rather intuitive definition, whereas the Hausdorff dimension is generally preferable from a mathematical point of view, see e.g. [Fal04] for a discussion. Both notions of fractal dimension coincide for many random fractals and in particular for the zero set of FBM. In fact, it was recently shown by Mukeru [Muk18] that the level sets of FBM even have Fourier dimension $1-H$. 

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variance results is developed in Sections 4 and 5 and Zähle’s construction of local time as a Palm distribution is discussed in Section 6. The concluding Section 7 contains some historical remarks on related ideas. Also, an appendix with some useful results from the literature and some auxiliary calculations is provided for convenience of the reader.

2. Notation and main results

We assume throughout the remainder of this article that \((X_t)_{t \in \mathbb{R}}\) is a real-valued stochastic process defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), which is continuous in probability, i.e.

\[
\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| > \epsilon) = 0, \quad \text{for all } t \in \mathbb{R}.
\]

More importantly, \(X\) is also taken to be H-self-similar, i.e. satisfy the invariance relations

\[
(X_{t+s} - X_t)_{s \in \mathbb{R}} \overset{d}{=} (X_{u+s} - X_u)_{s \in \mathbb{R}}, \quad \text{for any } t, u \in \mathbb{R}, \quad \text{(stationarity of increments),}
\]

and

\[
(X_{rs})_{s \in \mathbb{R}} \overset{d}{=} (r^H X_s)_{s \in \mathbb{R}}, \quad \text{for every } r \in (0, 1), \quad \text{(H-self-similarity),}
\]

where \(\overset{d}{=}\) denotes equality of finite dimensional distributions. We extend the definition of H-self-similarity to processes indexed by \([0, \infty)\) by restricting the stationarity of increments to positive shifts only. Note that, for stationary increment processes, continuity in probability follows from continuity in probability at time 0. Moreover, self-similarity and continuity in probability at 0 imply that \(\mathbb{P}(X_0 = 0) = 1\) [EM09, Lemma 1.1.1] and thus we may rewrite the stationary increment property as

\[
(X_{t+s} - X_t)_{s \in \mathbb{R}} \overset{d}{=} (X_s)_{s \in \mathbb{R}}, \quad \text{for all } t \in \mathbb{R}.
\]

Let us now turn our attention to the main object of interest, the local time measure of \(X\) at 0. We use the following notational conventions related to measures: \(\mathfrak{B}(\cdot)\) denotes the Borel-\(\sigma\)-field of the space in brackets. If \(\nu\) is a measure on \(\mathfrak{B}(\mathbb{R})\), the Borel-sets of the real line, and \((a, b)\) is an interval, we use the notation \(\nu(a, b)\) instead of \(\nu((a, b))\) and an analogous shorthand for closed and half open intervals. We frequently associate a measure \(\nu\) with its additive functional

\[
\nu_t = \begin{cases} 
\nu(0, t], & \text{if } t > 0, \\
-\nu(-t, 0], & \text{if } t \leq 0,
\end{cases}
\]

and vice versa. If \(\nu\) is a random measure, then \((\nu_t)_{t \in \mathbb{R}}\) is a non-decreasing stochastic process. We define the occupation measure of \(X\) on \(\mathfrak{B}(\mathbb{R} \times \mathbb{R})\) by setting

\[
\psi(A \times B) = \int_A 1\{X_r \in B\}dr, \quad A, B \in \mathfrak{B}(\mathbb{R}), \quad (6)
\]
and recall that \((6)\) yields a well-defined Borel measure as long as the trajectories of \(X\) are Borel-functions. We say that \(X\) has local times, or shorter \(X\) is \(\text{LT}\), if for each \(n = 1, 2, \ldots\), \(\mathbb{P}\text{-a.s.},\)

\[
\psi\left((-n, n) \times \cdot\right) \text{ is absolutely continuous w.r.t. Lebesgue measure.}
\]

Since \(X\) has stationary increments it is in fact sufficient for \(X\) to be \(\text{LT}\), that the Radon-Nikodym density \(d\psi(I, dy)/dy\) exists a.s. for an arbitrary open set \(I\). Disintegration yields, for every \(y\) outside some Lebesgue-negligible set \(R\), a locally finite measure \(\ell_y\) on \(\mathcal{B}(\mathbb{R})\) such that

\[
\psi(A \times B) = \int_B \ell_y(A)dy, \quad A, B \in \mathcal{B}(\mathbb{R}),
\]

and we call \(\ell_y\) the local time of \(X\) at level \(y\). Moreover, it can be shown, see e.g. [GH74, Lemma (3)], that for every \(y \notin \mathcal{R}\), we can choose a version of \(\ell_y(0, t]\) which is right-continuous in the time variable. A similar statement holds for \(\ell_y(-t, 0]\). Recall that we have \(X(0) = 0\) a.s., i.e. \(\ell^0 = \ell^{X(0)\text{, if } 0 \notin \mathcal{R}}\), but a priori the existence of \(\ell^0\) cannot be guaranteed using the above construction of local times. This technical issue is addressed in Section 6; for the time being let us assume that \(0 \notin \mathcal{R}\) and that \(\ell^0\) is well-defined.

We are chiefly interested in \(\ell^0\) and therefore just abbreviate \(\ell = \ell^0\) and call it local time, without reference to the level 0. From the construction of \(\ell\) we can straightforwardly derive a path-wise representation. Let

\[
\ell^0_\varepsilon(A) := \frac{1}{2\varepsilon} \psi(A \times (y - \varepsilon, y + \varepsilon)),
\]

then we have that for all \(y \notin \mathcal{R}\)

\[
\lim_{\varepsilon \to 0} \ell^y_\varepsilon(A) = \ell^y(A), \quad A \in \mathcal{B}(\mathbb{R}),
\]

which shows that our definition agrees with the formula for \(\ell\) given in the introduction.

We now introduce two further structural conditions on \(\ell\) which are necessary for our derivation of the lower tail probabilities of \(\ell(0, T]\). Let

\[
\text{supp}(\nu) = \{t : \nu(t - \varepsilon, t + \varepsilon) > 0 \text{ for all } \varepsilon > 0\}
\]

denote the support of a measure \(\nu\) on \(\mathcal{B}(\mathbb{R})\) and recall that a nowhere dense set of real numbers is a set whose (topological) closure does not contain any interval.

**Assumptions on the local time.** With probability 1,

\(\ell\) has no atoms, \hspace{1cm} \text{AL}

\text{supp}(\ell) is nowhere dense. \hspace{1cm} \text{ND}
Condition AL is equivalent to demanding that \((\ell_t)_{t \in \mathbb{R}}\) be continuous a.s. Both conditions entail a rather erratic behaviour of the trajectories of \(X\), which is not surprising in view of our main example, fractional Brownian motion. The validity of AL and ND are indispensable for the approach to local times taken in this paper. To exclude pathologies, we also restrict ourselves to situations where \(\ell\) is a.s. not the zero measure, we then say \(\ell\) is non-zero. We are now prepared to state our main result.

**Theorem 2.1** (Persistence of local time for \(H\)-sssi processes). Let \(X\) be continuous in probability, \(H\)-sssi and \(LT\) and denote by \(\ell\) its local time at 0. If \(\ell\) is non-zero and satisfies AL and ND, then there exists a constant \(c_X \in (0, \infty)\) such that, as \(T \to \infty\),

\[
P(\ell([0,T]) \leq 1) \sim c_X T^{-(1-H)}.
\]

There are two important observations needed for the proof of Theorem 2.1. The first one is a result stating that the lengths of the excursions of \(X\) from 0 follow, in a certain sense, a hyperbolic distribution. The precise formulation is given in Section 3 as Proposition 3.1. The second one is that, under AL and ND \(\ell\) is entirely encoded in the excursions of \(X\) from 0, which is manifested in the fact that the right-continuous inverse of \((\ell_t)_{t \in \mathbb{R}}\) is a.s. a pure jump process. Before we develop the details we devote the remainder of this section to some of the implications of Theorem 2.1.

To this end we provide two example processes, for which Theorem 2.1 can be applied. The first one, naturally, is fractional Brownian motion. The local times and level sets of FBM have been studied by several authors, the pioneering work was done by Kahane in the late 1960’s, see in particular [Kah85, Chapter 18].

**Theorem 2.2** (Persistence of local time for FBM). Let \(B\) denote fractional Brownian motion with Hurst index \(H \in (0, 1)\) and denote by \(\ell\) its local time at 0. Then there is a constant \(c_B \in (0, \infty)\) such that, as \(T \to \infty\),

\[
P(\ell([0,T]) \leq 1) \sim c_B T^{-(1-H)}.
\]

**Proof.** We only need to verify conditions AL and ND. The continuity in time of fractional Brownian local time is well known, e.g. by applying the criterion proposed by Geman [Gem76] for Gaussian processes. Let \(Z_B\) denote the set of zeroes of the FBM trajectory. Kahane [Kah85] showed that the Hausdorff dimension \(\dim Z_B\) of the zero set equals \(1 - H < 1\) a.s. Since \(B\) is a.s. continuous, it follows that \(Z_B\) is closed. Together with its non-integer dimension this implies that \(Z_B\) is nowhere dense and therefore \(\text{supp}(\ell)\) is nowhere dense, since \(\text{supp}(\ell) \subset Z_B\).

\[\square\]

To illustrate the power of our approach, we now discuss a non-Gaussian example, namely the Rosenblatt process \(R = (R_t)_{t \in \mathbb{R}}\). This process was introduced by Taqqu

\[\text{However, the author strongly believes that the conditions listed are not minimal an that in particular condition ND is a consequence of condition AL for any H-sssi process which is continuous in probability, but is not aware of any proof of this implication.}\]
[Taq75], see also [DM79] and arises as a limiting process in so-called (functional) non-central limit theorems, analogously to FBM appearing in central limit theorems for correlated random walks. We will not give a formal definition of the Rosenblatt process, an ad hoc definition can be given using an iterated Wiener-Itô integral, see [Taq78]. Instead, we restrict ourselves to listing the properties of which are relevant to verify the local time persistence result. A comprehensive source for all stated facts is Taqqu’s survey article [Taq11]. Unlike FBM, the Rosenblatt process can only be defined for \( H \in \left(\frac{1}{2}, 1\right) \). For any such \( H \), \( R \) is uniquely defined (up to multiplication by a constant) and satisfies

- \( R \) has Hölder continuous paths a.s. for any Hölder exponent \( \gamma < H \),
- \( R \) is \( H \)-sssi.

**Theorem 2.3** (Persistence of local time for the Rosenblatt process). Let \( R \) denote the \( H \)-sssi Rosenblatt process, \( H \in \left(\frac{1}{2}, 1\right) \) and denote by \( \ell \) its local time at 0. Then there is a constant \( c_R \in (0, \infty) \) such that, as \( T \to \infty \),

\[
P(\ell(0, T] \leq 1) \sim c_R T^{-(1-H)}
\]

**Proof.** In principle, we can apply the same arguments as for FBM, but the corresponding preliminary results for the Rosenblatt process needed to verify conditions \( \text{AL} \) and \( \text{ND} \) are less well known. We thus give a slightly more explicit version of the argument. Existence of square integrable (in space) local times for \( R \) has been shown in [She11].

Let us show continuity of the cumulative local time process \((\ell_t)_{t \geq 0}\). Geman’s sufficient criterion [Gem76, Therorem B (I)] for the continuity of the local time can be restated as follows for two-sided stationary increment processes:

\[
\int_{-1}^{1} \sup_{\epsilon > 0} \frac{1}{\epsilon} \mathbb{P}(|R_s| < \epsilon) ds < \infty.
\]

(9)

To show that (9) holds, we use results of Veillette and Taqqu [VT13], who studied the distribution of \( R_1 \) extensively. In particular, they show that \( R_1 \) has a smooth density and the same holds, by self-similarity, for \( R_s, s \in \mathbb{R} \setminus \{0\} \). Let \( f_s \) denote the density of \( R_s \). Then (9) is satisfied, if

\[
g(s) := \limsup_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f_s(u) du
\]

is integrable around the origin. But by smoothness of \( f_s \), we have \( g(s) = f_s(0) < \infty \) and thus \( g(s) = s^{-H} g(1) \) for any \( s > 0 \). Consequently, \( g \) is integrable around 0 and (9) is satisfied.

Turning to \( \text{ND} \), we argue using the well known fact, see e.g. [Kah85], about Hölder continuity and Hausdorff dimension of the level sets of a real function \( f \): If \( f \) is \( \gamma \)-Hölder continuous with \( \gamma \in (0, 1) \) then the Hausdorff dimension of its level sets is at most \( 1 - \gamma \). This is sufficient to complete the argument in the same fashion as for FBM.
Remark 2.4. The following general recipe may be used to verify the conditions of Theorem 2.1: If an $H$-sssi LT process has sufficiently high moments, then Hölder-continuity of the paths can always be inferred from the Kolmogorov-Chentsov Continuity Theorem [Che56] and ND is then always satisfied. Additionally, if the transition density at 0 exists, then AL is always satisfied.

3. Proof of Theorem 2.1

Let $X$ be as in Theorem 2.1 and let $\ell$ be its local time at 0. Recall that the corresponding additive functional $(\ell_t)_{t\in\mathbb{R}}$, is given by

$$\ell_t = \begin{cases} \ell(0, [t]) & \text{if } t > 0 \\ -\ell(t, 0] & \text{if } t \leq 0 \end{cases},$$

and denote its right-continuous inverse by $L = (L_x)_{x\in\mathbb{R}}$. It is straightforward from (8) that $(\ell_t)_{t\in\mathbb{R}}$ is $(1 - H)$-self-similar, hence $L$ is $1/(1-H)$ self-similar. By AL, $\ell$ has no atoms and hence $L$ is strictly increasing. By ND, $\text{supp}(\ell)$ is nowhere dense. $L$ is then a monotone pure jump process. Consequently, it induces a purely atomic random measure $\hat{\ell} \in \mathcal{M}_0$. We say a random measure is $\beta$-scale-invariant, if its additive functional is a $\beta$-self-similar process. It thus follows from self-similarity of $L$ that $\hat{\ell}$ is $1/(1-H)$-scale-invariant. Because $\hat{\ell}$ is purely atomic, we may identify it with a point process on $\mathbb{R} \times (0, \infty)$, see Lemma 5.2. This point process is denoted by $\hat{N}$ and its intensity measure by $\hat{\Lambda}$. The key observation of our argument is that $\hat{\Lambda}$ is entirely determined (up to a multiplicative constant) by the invariance properties of $\hat{\ell}$.

Proposition 3.1. Let $\hat{N}$ denote the point process representation of the inverse local time measure $\hat{\ell}$. Then the corresponding intensity measure $\hat{\Lambda}$ is given by

$$\hat{\Lambda}(dx \times dm) = cm^{-1-(1-H)}dx dm,$$

for some finite constant $c > 0$.

We postpone the proof of Proposition 3.1 to the end of Section 6 but note that subject to the validity of Proposition 3.1, all that remains to establish Theorem 2.1 is to relate the tail behaviour of $\hat{\Lambda}$ to the tail behaviour of $\ell$.

Proof of Theorem 2.1. We observe that

$$\mathbb{P}(\ell(0, T] \leq 1) = \mathbb{P}(\ell(0, T] < 1) = \mathbb{P}(L_1 > T), \quad T > 0,$$

i.e. we obtain lower tail bounds for $\ell_t$ from upper tail bounds for $L_1$. Let $\hat{N}$ denote the point process representation of $\hat{\ell}$ and note that $L_1 = \int_0^1 \int_0^\infty m\hat{N}(dx \times dm)$. Fix any $r > 0$. Since $\hat{N}$ is a simple point process and $\hat{\ell}$ is purely atomic, we have by standard results from random measure theory, e.g. [DVJ07, Prop. 9.1.3(v)],

$$\hat{N}([0, 1] \times (r, \infty)) = \lim_{n \to \infty} \sum_{k=1}^n \mathbb{I} \left\{ \hat{\ell} \left( \frac{k - 1}{n}, \frac{k}{n} \right] > r \right\}, \quad \text{a.s.}$$

(11)
Set 
\[ P_{k,n} := \mathbb{P} \left( \ell \left( \frac{k - 1}{n}, \frac{k}{n} \right) > r \right), \quad 1 \leq k \leq n, n = 1, 2, \ldots, \]
taking expectations in (11), we obtain
\[ \limsup_{n \to \infty} \sum_{k=1}^{n} P_{k,n} \leq \mathbb{E} \hat{N}(\{0, 1\} \times (r, \infty)) \leq \liminf_{n \to \infty} \sum_{k=1}^{n} P_{k,n}, \]
having applied Fatou’s Lemma and the inverse Fatou’s Lemma, i.e.
\[ \sum_{k=1}^{n} P_{k,n} \xrightarrow{n \to \infty} \mathbb{E} \hat{N}(\{0, 1\} \times (r, \infty)). \]
For any \( \delta \in (0, 1) \) we may thus fix \( 1 \leq N_\delta < \infty \) such that
\[ (1 - \delta) \int_{r}^{\infty} cm^{-1-(1-H)} dm \leq \sum_{k=1}^{N_\delta} P_{k,N_\delta} \leq (1 + \delta) \int_{r}^{\infty} cm^{-1-(1-H)} dm, \] (12)
where we have used that
\[ \mathbb{E} \hat{N}(\{0, 1\} \times (r, \infty)) = \hat{\Lambda}(\{0, 1\} \times (r, \infty)) = \int_{r}^{\infty} cm^{-1-(1-H)} dm, \]
according to Proposition 3.1. Using that \( P_{k,N_\delta} = \mathbb{P}(L_{1/N_\delta} > r) \) by the stationarity of the increments of \( L \), we can rewrite (12) as
\[ (1 - \delta) \frac{c}{1 - H} r^{-1-(1-H)} \leq N_\delta \mathbb{P}(L_{1/N_\delta} > r) \leq (1 + \delta) \frac{c}{1 - H} r^{-1-(1-H)}, \]
and applying the \( 1/(1-H) \)-self-similarity of \( L \) and rearranging terms yields
\[ (1 - \delta) \frac{c}{1 - H} \left( r N_\delta^{1/(1-H)} \right)^{-(1-H)} \leq \mathbb{P} \left( L_{1} > r N_\delta^{1/(1-H)} \right) \leq (1 + \delta) \frac{c}{1 - H} \left( r N_\delta^{1/(1-H)} \right)^{-(1-H)}, \]
i.e.
\[ \mathbb{P}(L_{1} > T) = \frac{c}{1 - H} (1 + o(1)) T^{-(1-H)}, \quad \text{as } T \to \infty, \]
and Theorem 2.1 is proved, subject to Proposition 3.1.

4. Palm distributions and duality

We now provide the background needed to complete the proof of Theorem 2.1, starting with some basics of random measure theory, namely we introduce Palm distributions and discuss some of their key properties.

Here and in the following two sections, we take a general point of view on \( \ell \) and its distributional properties as a random measure, in particular we can forget about the
The process $X$ from which it is derived. We instead consider some complete measurable space $(\Omega, \mathcal{F})$, equipped with a $\sigma$-finite measure $Q$. We denote by $E_Q(\cdot)$ integration with respect to $Q$. A measurable map $\xi$ from $\Omega$ into the space $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ of locally finite measures on $\mathbb{R}$, equipped with its Borel-$\sigma$-field is called a random measure, even though we stress that $Q$ need not be a probability distribution. We call $Q$ a quasi-distribution, to distinguish it from the measures which are elements of $\mathcal{M}$ and set $Q_\xi = Q \circ \xi^{-1}$, i.e.

\[ Q_\xi(G) = Q(\xi \in G), \quad G \in \mathcal{B}(\mathcal{M}). \]

The intensity measure of $\xi$ under $Q$ (or $Q_\xi$) is given by

\[ \Lambda_\xi(A) := E_Q(\xi(A)) = \int \nu(A)Q_\xi(d\nu), \quad A \in \mathcal{B}(\mathbb{R}). \]

In what follows, we frequently consider $Q$ directly as a quasi-distribution on $\mathcal{B}(\mathcal{M})$ without explicit reference to a (canonical) random measure $\xi$ with distribution $Q$, consequently we denote the associated intensity measure by $\Lambda_Q$. Whenever we discuss a probability measure, then we indicate this by using blackboard-face symbols, e.g. $P, P_\xi, E_\xi$, etc. Furthermore, to formalise our discussion of stationarity properties, we use the shift group $\left(\theta_t\right)_{t \in \mathbb{R}}$ on $\mathbb{R}$. Note that the $\theta_t, t \in \mathbb{R}$, act measurably on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, and in particular we have that

\[ \theta_{-t}\nu(A) = \nu(A + t) = \nu \circ \theta_t(A), \quad A \in \mathcal{B}(\mathbb{R}), \]

where $A + t := \{a + t, a \in A\}$. A quasi-distribution $Q$ on $\mathcal{B}(\mathcal{M})$ is invariant under the shifts $\left(\theta_t\right)_{t \in \mathbb{R}}$, i.e. stationary, if

\[ Q(G) = Q(\{\theta_t\nu, \nu \in G\}), \quad t \in \mathbb{R}. \]

If $Q$ is stationary and satisfies

\[ \lambda_Q := \Lambda_Q((0, 1]) \in (0, \infty) \]

then it follows immediately that $\Lambda_Q(ds) = \lambda_Q ds$, i.e. $\Lambda_Q$ is a constant multiple of Lebesgue measure. We call $\lambda_Q \in [0, \infty]$ the intensity of $Q$. Note that the quasi-distribution $Q$ is a place holder for a stationarised version of the distribution of the local time measure $\ell$. The corresponding construction is given in Section 6. At the moment it is more beneficial to stay in the general setting. However, the following assumptions on the support of $Q$

\[ S_Q := \text{supp}(Q) = \mathcal{M} \setminus \bigcup_{N \in \mathcal{B}(\mathcal{M}): Q(N) = 0} N \]

are justified in view of our applications. Let $\mathcal{O}$ denote the 0-measure, then

\[ \mathcal{O} \notin S_Q, \text{ i.e. } Q \text{ is non-zero}, \]
{\nu_t, t \in \mathbb{R}} = \mathbb{R}, \text{ for all } \nu \in S_Q.

We have \( \lambda_Q > 0 \), if \( Q \) is non-zero and stationary. Note that \( [3] \) is in congruence with condition \( [A.1] \), the details are given in Lemma \( [A.1] \).

We now turn to the subject of Palm distributions of a stationary, non-zero quasi-distribution \( Q \). Fix any \( A \in \mathcal{B}(\mathbb{R}) \) with finite and positive Lebesgue measure, then the quasi-distribution defined by

\[
P_Q(G) = \frac{1}{\lambda_Q} \int_A \int_A \mathbb{1}_{\{\theta - \nu \in G\}} \nu(dt)Q(d\nu), \quad G \in \mathcal{B}(\mathcal{M}),
\]

is independent of the choice of \( A \), see Lemma \( [A.2] \). It is referred to as the Palm measure of \( Q \). If \( Q \) has finite intensity \( \lambda_Q \) then a probability distribution is defined by

\[
P_Q(G) = \frac{P_Q(G)}{\lambda_Q}, \quad G \in \mathcal{B}(\mathcal{M}),
\]

and is called the Palm distribution of \( Q \). Conversely, we say that a probability distribution \( P \) (on \( \mathcal{B}(\mathcal{M}) \)) is Palm-distributed, if it is the Palm distribution of some stationary quasi-distribution \( Q \). Similarly, a random measure is Palm distributed if its distribution is the Palm distribution of some \( Q \). It is well known, see e.g. [Zäh88, Lemma 3.3], that the almost sure properties of \( Q \) and \( P_Q \) agree (up to shifts). We may thus assume that any \( \nu \in S(P_Q) \) has the property indicated in \( [3] \). The latter entails that the right continuous inverse \( (\nu_t)^{-1} \) of the additive functional \( (\nu_t)_{t \in \mathbb{R}} \) of \( \nu \) is strictly increasing and that we have

\[
\nu_{\hat{\theta}_x} = x, \quad \text{for all } x \in \mathbb{R}.
\]

This allows us to define the measurable group of random time shifts

\[
\hat{\theta}_x := \hat{\theta}_x(\nu) := \theta_{\nu_t^{-1}}, \quad x \in \mathbb{R}.
\]

Note that we may pick \( A = [0, 1] \) in the definition of \( P_Q \) and change variables according to the random time change to obtain the alternative representation

\[
P_Q(G) = \frac{1}{\lambda_Q} \int_0^{\nu_1} \int_{\hat{\theta}_x \nu} \mathbb{1}_{\{\theta - \nu \in G\}} dx Q(d\nu), \quad G \in \mathcal{B}(\mathcal{M}).
\]

The following basic lemma is crucial for our argument. It is a generalisation of [MN94, Lemma 2.3], see also [MNS00, Theorem 3.1].

**Lemma 4.1** (Duality lemma). Let \( Q \) be a stationary non-zero measure on \( \mathcal{B}(\mathcal{M}) \) with \( \lambda_Q < \infty \), then its Palm distribution \( P_Q \) is stationary with respect to the random shifts \( (\hat{\theta}_x)_{x \in \mathbb{R}} \).

Before we give the proof, we briefly discuss the intuition behind Palm distributions in general and Lemma \( [4.1] \) in particular. Palm distributions originated in queuing theory [Pal43]. The concept is easiest understood for simple stationary point processes,
which corresponds to \( Q \) being the distribution of a random counting measures in our setting. In this case, the Palm distribution may be interpreted as a description of the distribution of the point process seen from a 'typical point', an intuition which can be made precise using ergodic theory, see e.g. the discussion in [Tho00, Chapter 8]. At the heart of Palm theory lies a duality principle [Tho95], which can be paraphrased as “A point process is stationary, if and only if its Palm version is stationary under point-shifts.”

For the purpose of this paper, the backward implication in this statement is not needed. We only rely on the observation, that a Palm distributed random measure is stationary w.r.t. to intrinsic shifts, i.e. shifts by mass points of its realisation, which is exactly what is expressed in Lemma 4.1.

**Proof of Lemma 4.1**. Let \( G \in \mathcal{B}(\mathcal{M}) \) and fix any \( r > 0 \). Then the stationarity of \( Q \) implies that, for any \( y > 0 \),

\[
\int \int_{\nu(0,r]} \mathbf{1}_G \circ \hat{\theta}_x \mathrm{d}xQ(\mathrm{d}\nu) = \int \int_{\nu(0,r]} \mathbf{1}_G \circ \hat{\theta}_x \mathrm{d}xQ(\mathrm{d}\nu \circ \theta_{-r})
= \int \int_{-\nu(r,0]} \mathbf{1}_G \circ \hat{\theta}_{x-\nu(-r,0]} \mathrm{d}xQ(\mathrm{d}\nu)
= \int \int_{0}^{y} \mathbf{1}_G \circ \hat{\theta}_x \mathrm{d}xQ(\mathrm{d}\nu),
\]

and an analogous calculation can be made for \( r < 0 \). Thus we have that, using (13),

\[
\mathbb{P}_Q(\theta_x^{-1}G) = \frac{1}{\lambda_Q} \int \int_{\nu(0,r]} \mathbf{1}_G \circ \hat{\theta}_{z+x} \mathrm{d}zQ(\mathrm{d}\nu)
= \frac{1}{\lambda_Q} \left( \int \int_{\nu(0,r]} \mathbf{1}_G \circ \hat{\theta}_{z} \mathrm{d}zQ(\mathrm{d}\nu) - \int \int_{0}^{x} \mathbf{1}_G \circ \hat{\theta}_z \mathrm{d}zQ(\mathrm{d}\nu)
+ \int \int_{\nu(0,r]} \mathbf{1}_G \circ \hat{\theta}_z \mathrm{d}zQ(\mathrm{d}\nu) \right)
= \frac{1}{\lambda_Q} \int \int_{\nu(0,r]} \mathbf{1}_G \circ \hat{\theta}_{z} \mathrm{d}zQ(\mathrm{d}\nu) = \mathbb{P}_Q(G).
\]

\[\square\]

5. Bi-scale-invariance

So far, we have focussed our discussion of random measures on invariance with respect to time shifts only. Now we additionally consider scale-invariance of measures, which is the counterpart to self-similarity of processes. The essential observation of
this section is that stationarity combined with scale-invariance of a quasi-distribution or Palm distribution determines the corresponding intensity measures entirely up to a multiplicative constant.

We illustrate this by means of marked point processes on the real line. We consider $(\Omega, \mathcal{F}, P)$, i.e. we work under a probability measure. An extended marked point process with positive marks (EMPP) is a point process on $\mathbb{R} \times (0, \infty)$ which is a.s. finite on all sets of the form $A \times M$ for bounded $A \in \mathcal{B}(\mathbb{R})$ and Borel sets

$$M \subset \left( \epsilon, \frac{1}{\epsilon} \right), \text{ for some } \epsilon > 0.$$  

Fix $\beta > 0$ and $r \in (0, 1)$. We define a rescaled point process by

$$S^\beta_r N(A \times M) := N(rA \times r^\beta M),$$

where $cM := \{cm, m \in M\}$ for any $c \in \mathbb{R} \setminus \{0\}$, hence a contraction by factor $r$ in the time domain is combined with a contraction by $r^\beta$ in the mark space into the operator $S^\beta_r$. An EMPP on $\mathbb{R} \times (0, \infty)$ is called $\beta$-bi-scale-invariant if, for any $r \in (0, 1)$,

$$P \circ (S^\beta_r N)^{-1} = P \circ N^{-1}.$$  

Similarly, the EMPP is stationary, if its distribution is invariant under the shifts $(\theta_t)_{t \in \mathbb{R}}$ applied to the time domain only. We now recall a well known result about point processes: if the intensity measure of the EMPP is finite, then it follows from bi-scale invariance together with stationarity that the intensity measure must be a product of a multiple of Lebesgue measure in time and a hyperbolic law on the marks. That this is the case can be seen by noting that stationarity implies homogeneity in time of the intensity measure, i.e. it must be a multiple of Lebesgue measure. Additionally, bi-scale-invariance is transferred into stationarity on the mark space when mapped to logarithmic coordinates, hence the intensity on the marks is a multiple of Lebesgue measure in logarithmic coordinates which is translated into a hyperbolic law when reversing the coordinate transform.

**Proposition 5.1.** Let $N$ be a bi-scale invariant, stationary, extended marked point process on $\mathbb{R} \times (0, \infty)$ with positive and locally finite intensity measure $\Lambda_N$. Then $\Lambda_N$ necessarily is of the form

$$\Lambda_N(dt \times dm) = cm^{-1-1/\beta}dt dm.$$  

(14)

**Proof.** We only give an outline of the precise argument. A similar derivation in more detail can be found in [DVJ07, Chapter 12], for the case of an extended marked Poisson process. Let $\Lambda_N$ denote the intensity measure of $N$. We assume that

$$\Lambda_N(\{0\} \times (0, \infty)) = 0,$$

Instead of using the notion of an extended point process, one can also consider a classical locally finite point process if one equips the closure of the mark space with a metric which places 0 infinitely far away from any positive mark.
which holds if $N$ has almost surely no points at $0$ and that $\Lambda_N$ is absolutely continuous with respect to 2-dimensional Lebesgue measure. We use a logarithmic change of coordinates, which makes it necessary to decompose $N$ into a marked point process $N_+$ on $(0, \infty)$ and a marked point process $N_-$ on $(-\infty, 0)$ with associated intensity measures $\Lambda_{\pm}$.

Let us first consider $\Lambda_+$ only. By the logarithmic change of coordinates $$(t, m) \mapsto (\log t, \beta \log t - \log m), \quad t \in \mathbb{R}, m \in (0, \infty),$$
bi-scale-invariance is turned into shift-invariance and consequently under the new coordinates, $\Lambda_+$ must be a product of Lebesgue measure and some absolutely continuous measure $\rho_+$ on the (coordinate transformed) mark space, whenever the intensity in time is finite. In particular, reversing the coordinate transform, we obtain

$$\Lambda_+(dt \times dm) = \frac{\phi_+(t^\beta / m)}{tm} dt dm$$

for some locally integrable density $\phi_+$ of $\rho_+$ on $(0, \infty)$, see [DVJ07, p. 258]. A similar representation holds for $\Lambda_-$ with a density $\phi_-$. The additional assumption of stationarity in the time domain now implies that we must have

$$\phi_+(m) = \phi_-(m) = cm^{-1/\beta},$$

for some $c > 0$ and thus (14) must be satisfied.

To apply this representation to random measures we recall that atomic random measures can be bijectively mapped to EMPPs. Let $N$ be the space of locally finite extended marked point processes with positive marks $N$ on $(\mathbb{R} \times (0, \infty))$ satisfying

$$\int (m \wedge 1) \Lambda_N(A \times dm) < \infty \quad (15)$$

for any bounded Borel set $A$, and let $\mathcal{M}_a \subset \mathcal{M}$ denote the locally finite, purely atomic random measures on $\mathcal{B}(\mathbb{R})$.

**Lemma 5.2 ([DVJ07, Lemma 9.1.VII]).** There is a bijection mapping $N$ onto $\mathcal{M}_a$.

In principle, the bijection of Lemma 5.2 just consists of interpreting, for given $\xi \in \mathcal{M}_a$, a point $x \in \text{supp}(\xi)$ with $\xi(\{x\}) = m > 0$ as a pair $(x, m) \in \mathbb{R} \times (0, \infty)$ and the collection of all such points forms a marked point process $N_\xi$. To deal with accumulation points of $\text{supp}(\xi)$, one needs to consider extended MPPs. Note that if the intensity measure $\Lambda_\xi$ of some random measure $\xi$ under this bijection assigns infinite mass to a bounded open

---

4 As noted above, the result there is for a Poisson process, however the statement transfers immediately to the general case. This is also discussed in [DVJ07] on pp. 260-262, however not in as much detail as the Poisson case.
interval $A$, then by local finiteness of $\xi$ this must be the consequence of infinitely many smaller and smaller atoms. Thus $N_\xi$ puts finite mass on any open set $A \times (\delta, \infty)$ for $\delta > 0$ and is thus locally finite on $(0, \infty)$ in the extended sense. Since we wish to apply Lemma 5.2 to measures without fixed atoms, we observe that the bijection is preserved when restricted to the subspace of measures without fixed points and the subspace of without fixed points, respectively.

The version of $\beta$-bi-scale-invariance for quasi-distributions is just called $\beta$-scale-invariance, as defined Section 3 for probability distributions. Let us quickly recall this definition in the notation of this section. Let $P$ be a probability distribution on $\mathfrak{B}(\mathcal{M})$. $P$ is called $\beta$-scale-invariant, if for any $r \in (0,1)$, $B \in \mathfrak{B}(\mathbb{R})$ we have

$$P(G) = P\left(\{R^\beta_r \nu, \nu \in G\}\right), \quad G \in \mathfrak{B}(\mathcal{M})$$

where

$$\left(R^\beta_r \nu\right)(A) := r^{-\beta}\nu(rA),$$

or short $P \circ (R^\beta_r)^{-1} = P$. When considering a non-finite quasi-distribution, one needs to add an additional factor rescaling the total mass: A stationary non-finite quasi-distribution $Q$ with finite intensity $\lambda_Q$ is called $\beta$-scale-invariant if

$$Q \circ (R^\beta_r)^{-1} = r^{\beta-1}Q.$$

That this is the correct notion of scale-invariance for quasi-distributions can be seen by looking at their Palm distributions:

**Lemma 5.3 ([Zäh88, Statement 2.3]).** A stationary, non-zero, non-finite quasi distribution $Q$ is $\beta$-scale-invariant if and only if its Palm distribution $P_Q$ is $\beta$-scale-invariant.

### 6. Local time as a Palm distribution

Zähle observed in [Zäh88] that scale-invariance of random measures can be based on a notion of scaling around a typical point of mass of the measure, i.e. by scale-invariance of Palm distributions. Clearly, the local time of a centered self-similar process is always distributionally scale-invariant in the usual sense, i.e. when scaling is performed w.r.t. to the origin. To fit local time and other fractal measures derived from an $H$-sssi process $X$ into the framework developed in [Zäh88, Zäh90] one has to express them as Palm distributions. This is done in [Zäh91]. We do not require the main results of this work which are concerned with the carrying Hausdorff dimension of the realisations of random measures derived from $X$, but remark in passing, that they can be useful for verifying assumption [ND] cf. the proofs of Theorems 2.2 and 2.3. In view of the previous two sections, we only need to know that the local time of an $H$-sssi process can be viewed as a Palm distribution. The precise result, in the notation introduced in Section 5 is as follows:
Proposition 6.1 ([Zäh91, Proposition 6.9]). If \( X \) is \( H \)-sssi and \( \ell T \), then \( \ell \) is an \((1 - H)\)-scale-invariant, Palm-distributed random measure.

We may rephrase the statement of Proposition 6.1 as a statement about \( P_\ell \), the distribution of the local time as a random measure: There exists some quasi-distribution \( Q_\ell \), such that \( P_\ell \) is the Palm-distribution of \( Q_\ell \).

Remark 6.2. As mentioned in Section 2, it cannot be a priori excluded that \( 0 \in \mathcal{R} \) and thus \( \ell = \ell^0 = \ell^X(0) \) is not defined. Note that in [Zäh91], this is avoided by working with the averaged version

\[ \tilde{P}_\ell(\cdot) = \int_0^1 P_{\ell,t}(\cdot) dt, \]

where \( P_{\ell,t}(\cdot) \) denotes the distribution of \( \theta_{-t} \ell^X_t, t \in \mathbb{R} \). This trick can always be used in the stationary increment case, but it has the drawback that this version of the local time does not have the usual path-wise interpretation. It is, however, easily seen that \( \tilde{P}_\ell \) and \( P_\ell \) have the same distribution as random measures, if the latter is defined, see the discussion in [Zäh91, p. 132]. We remark that the use of \( \tilde{P}_\ell \) is obsolete, if \( X \) a.s. has continuous paths.

Aside from the technical issue of Remark 6.2 we can give a path-wise interpretation to the quasi-distribution \( Q_\ell \). Let us set

\[ Q_\ell(G) = \mathbb{E} \int 1\{\ell^y(\cdot) \in G\} dy, \quad G \in \mathcal{B}(\mathcal{M}), o \notin G, \]

recalling that \( o \) is the trivial measure. We can think of \( Q_\ell(G) \) as the ‘local time at the origin’ of a trajectory in the flow of \( X \), i.e. the quasi-distribution \( Q_X \) on trajectories obtained via

\[ Q_X(H) = \int 1\{X + y \in H\} dy, \quad H \in \mathcal{F}(\mathcal{C}), \]

where \( \mathcal{C} \) is a suitable path space equipped with the \( \sigma \)-field \( \mathcal{F}(\cdot) \) of cylinder sets. \( Q_X \) can be thought of as mixture of the law of \( X \) w.r.t. Lebesgue measure in the ‘origin’ \( X(0) \) to obtain a stationary measure on paths. From \( Q_X \) we can derive a stationary version of the occupation measure and then disintegrate to obtain \( Q_\ell \). Note however, that \( Q_\ell \) and \( P_\ell \) need not be interpreted in this way – it is only necessary that the distribution of local time as a random measure is a Palm distribution.

Finally, we are in the position to prove Proposition 3.1 and thus conclude the proof of Theorem 2.1.

**Proof of Proposition 3.1** The representation (10) follows immediately from Proposition 5.1 upon showing \( \ell / 1 - H \)-bi-scale-invariance of \( \hat{N} \). This is, in turn, equivalent to \( 1 / 1 - H \)-scale-invariance and stationarity of \( \hat{\ell} \). The scale-invariance has already been established in the opening paragraph of Section 3. We now show that \( L \) has stationary increments and thus \( \ell \) is a stationary random measure. By Proposition 6.1 \( \ell \) is Palm-distributed.
For $x \in \mathbb{R}$ set $t(x) := \inf\{ t : \ell(t) > x \} = L_x$. Fix $x_0 \in \mathbb{R}$ and consider a finite family of points $x_1, \ldots, x_n$ with $x_0 < x_1 < \cdots < x_n$ and the corresponding random times $t(x_i), i = 1, \ldots, n$. Almost surely, $\{t(x_i), i = 1, \ldots, n\} \subset \text{supp}(\ell)$ and because $\ell$ is Palm-distributed, we may apply Lemma 4.1 to obtain

$$
(t(x_i) - t(x_0))_{i=1}^n \overset{d}{=} (t(x_i - x_0) - t(0))_{i=1}^n = (L_{x_i-x_0})_{i=1}^n
$$

and thus $L$ has stationary increments. Consequently, $\hat{\ell}$ is $\frac{1}{1-H}$-scale-invariant and stationary and this concludes the proof of the Proposition 3.1.

7. Related work and historical remarks

As already noted, the key part of our approach is Zähle’s construction of the local time as a Palm distribution. His main goal was to show that scale-invariance of random measures already determines their carrying Hausdorff dimension. He did however already notice the relation between index of self-similarity and tails of the ‘gap lengths’ in fractal sets, see [Zäh88, Theorem 5.3] and even mentioned the zero set of Brownian motion as an explicit example. However, he did not pursue this investigation further in the more general set up of [Zäh90, Zäh91]. In fact [Zäh88, Theorem 5.3] prompted the present author’s investigation of Palm distributions as a means to derive persistence exponents from invariance properties and be regarded as a precursor of Proposition 3.1.

One great advantage of Zähle’s theory is that it works in arbitrary space and time dimension, and it would certainly be fruitful to try and extend the discussion presented here to higher dimensions.

The ‘Palm duality’ mentioned in Section 4 is actually also true in the context of random measures. The invariance relation expressed in Lemma 4.1 holds in a more general setting [Mec75]. A converse statement of equal generality was proved in [HL05], where it is also shown that Palm distributions are characterised by the duality principle even in the random measure setting. The random time change approach we have chosen is taken from [GH73]. In fact, its importance for studying local times was already pointed out there. Earlier uses of the same concept can be found in [Tott66, Mar65].

Generalising the EMPP representation from completely random measures to stationary random measures was first suggested by Vere-Jones in [VJ05], see also [DVJ07]. It is also conjectured there, that it might prove fruitful to pursue this generalisation in the setting of quasi-distributions introduced in [Zäh88, Zäh90, Zäh91] and this view was justified as the results of this article illustrate.

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A. Some auxiliary statements used in the text

Lemma A.1. Let $\ell$ be a non-zero local time of an $H$-self-similar process satisfying $AL$ then

$$\{\ell_t, t \in \mathbb{R}\} = \mathbb{R}.$$ 

Proof. From the path-wise representation of $\ell$, it follows immediately that $(\ell_t)_{t \geq 0}$ is an $1-H$-self-similar process. Assume that $P(\ell_t \in [0, K])$ for all $t \geq 0$ for some $K < \infty$, $\delta > 0$. Fix $b > 0$, then we also have $P(\ell_{Tb} \in [0, K]) > \delta$ for any $T$ and by self-similarity, this means $P(\ell_b \in [0, KT^{-(1-H)}]) > \delta$ for any $T > 0$. Thus $\ell(0, b) = 0$ with probability exceeding $\delta$. Since $b$ was arbitrary, this means $\ell_t \equiv 0$ on $[0, \infty)$. A similar argument works for negative $b$ and we obtain a contradiction to $\ell$ being non-zero. Hence the range of $(\ell_t)_{t \geq 0}$ is a.s. not bounded on either side. By $AL$, i.e. continuity of $\ell_t$, the range must cover all of $\mathbb{R}$. ⊓⊔

Lemma A.2. The definition of the Palm measure $P_Q$ of $Q$ does not depend on the choice of the set $A$.

Proof. For probability distributions this is shown in [Kal06, Lemma 11.2]. The proof transfers easily to quasi-distributions. Let $Q$ be stationary and non-zero. Fix $G \in \mathcal{B}(M)$, let $g(\nu) = 1_{\{\nu \in G\}}$ and consider the measures

$$\nu g(B) = \int_B g(\theta_s \nu) \nu(ds), \ B \in \mathcal{B}(\mathbb{R}),$$

then the definition of the Palm measure reads

$$P_Q(G) = \frac{\int \nu g(A)Q(d\nu)}{\int_A \nu g(A)ds}. \quad (16)$$

We claim that $\theta_{-t}\nu g = (\theta_{-t}\nu)_g$, then we have for any measurable function $h$,

$$\int h(\theta_{-t}\nu g) \ dQ = \int h((\theta_{-t}\nu)_g) \ dQ = \int h(\nu g) \ dQ$$

by stationarity of $Q$. This means that stationarity of $Q$ is preserved under the operation $(\cdot)_g$, hence the numerator in (16) does not change under translations of $A$, i.e. is a multiple of Lebesgue measure. To prove the claim we note that, for any Borel set $B$,

$$\theta_{-t}\nu g(B) = \nu g(B + t) = \int_{B+t} g(\theta_s \nu) \nu(ds)$$

$$= \int 1_{\{s - t \in B\}} g(\theta_s \nu) \nu(ds) = \int 1_{\{u \in B\}} g(\theta_{-u} \nu) \nu(du + t)$$

$$= \int_B g(\theta_{-u} \theta_{-t} \nu) \theta_{-t} \nu(du) = (\theta_{-t} \nu)_g(B).$$
References

[AB18] F. Aurzada and M. Buck. Persistence probabilities of two-sided (integrated) sums of correlated stationary Gaussian sequences. *Journal of Statistical Physics*, 170(4):784–799, 2018.

[AGP15] F. Aurzada and N. Guillotin-Plantard. Persistence exponent for discrete-time, time-reversible processes. *arXiv preprint arXiv:1502.06799*, 2015.

[AGPP18] F. Aurzada, N. Guillotin-Plantard, and F. Pene. Persistence probabilities for stationary increment processes. *Stochastic Processes and their Applications*, 128(5):1750–1771, 2018.

[AM18] F. Aurzada and C. Mönch. Persistence probabilities and a decorrelation inequality for the Rosenblatt process and Hermite processes. *Teor. Verojatnost. i Primenen.*, 63(4):817–826, 2018.

[AMZ17] F. Aurzada, S. Mukherjee, and O. Zeitouni. Persistence exponents in Markov chains. *arXiv preprint arXiv:1703.06447*, 2017.

[AS15] F. Aurzada and T. Simon. Persistence probabilities and exponents. In *Lévy matters V*, pages 183–224. Springer, 2015.

[Aur11] F. Aurzada. On the one-sided exit problem for fractional Brownian motion. *Electron. Commun. Probab*, 16:392–404, 2011.

[BMS13] A. J. Bray, S. N. Majumdar, and G. Schehr. Persistence and first-passage properties in nonequilibrium systems. *Advances in Physics*, 62(3):225–361, 2013.

[CD08] F. Caravenna and J.-D. Deuschel. Pinning and wetting transition for (1+1)-dimensional fields with Laplacian interaction. *The Annals of Probability*, 36(6):2388–2433, 2008.

[CDC+04] M. Constantin, C. Dasgupta, P. P. Chatraphorn, S. N. Majumdar, and S. D. Sarma. Persistence in nonequilibrium surface growth. *Physical Review E*, 69(6):061608, 2004.

[Che56] N. N. Chentsov. Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the “heuristic” approach to the Kolmogorov-Smirnov tests. *Theory of Probability & Its Applications*, 1(1):140–144, 1956.
[DM79] R. L. Dobrushin and P. Major. Non-central limit theorems for non-linear functional of Gaussian fields. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 50(1):27–52, 1979.

[DVJ07] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes: Volume II: General theory and structure*. Springer Science & Business Media, 2007.

[DY95] M. Ding and W. Yang. Distribution of the first return time in fractional Brownian motion and its application to the study of on-off intermittency. *Physical Review E*, 52(1):207, 1995.

[EM09] P. Embrechts and M. Maejima. *Selfsimilar processes*, volume 21. Princeton University Press, 2009.

[Fal04] K. Falconer. *Fractal geometry: mathematical foundations and applications*. John Wiley & Sons, 2004.

[Gem76] D. Geman. A note on the continuity of local times. *Proceedings of the American Mathematical Society*, 57(2):321–326, 1976.

[GH73] D. Geman and J. Horowitz. Remarks on Palm measures. *Annales de l’IHP Probabilités et statistiques*, 9(3):215–232, 1973.

[GH74] D. Geman and J. Horowitz. Local times and supermartingales. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 29(4):273–293, 1974.

[HL05] M. Heveling and G. Last. Characterization of Palm measures via bijective point-shifts. *Annals of probability*, pages 1698–1715, 2005.

[Kah85] J.-P. Kahane. Some random series of functions, volume 5 of cambridge studies in advanced mathematics, 1985.

[Kal06] O. Kallenberg. *Foundations of modern probability*. Springer Science & Business Media, 2006.

[Lév48] P. Lévy. *Processus stochastiques et mouvement brownien*. Gauthier-Villars, 1948.

[LS18] H. Lyu and D. Sivakoff. Persistence of sums of correlated increments and clustering in cellular automata. *Stochastic Processes and their Applications*, 2018.

[Maj99] S. N. Majumdar. Persistence in nonequilibrium systems. *Current Science*, pages 370–375, 1999.

[Mar65] G. Maruyama. Theory of stationary processes and ergodic theory, 1965. A lecture at the Symposium held at Kyoto University.
[Mec75] J. Mecke. Invarianzeigenschaften allgemeiner palmscher maße. Mathematische Nachrichten, 65(1):335–344, 1975.

[MN94] M. Miyazawa and G. Nieuwenhuis. Modified Palm and modified time-stationary distributions for random measures and applications. Katholieke Universiteit Brabant, 1994.

[MNS00] M. Miyazawa, G. Nieuwenhuis, and K. Sigman. Palm theory for random time changes. International Journal of Stochastic Analysis, 14(1):55–74, 2000.

[Mol99] G. M. Molchan. Maximum of a fractional Brownian motion: probabilities of small values. Communications in Mathematical Physics, 205(1):97–111, 1999.

[Mol18] G. M. Molchan. Persistence exponents for Gaussian random fields of fractional Brownian motion type. Journal of Statistical Physics, pages 1–11, 2018.

[Muk18] S. Mukeru. The zero set of fractional Brownian motion is a Salem set. Journal of Fourier Analysis and Applications, 24(4):957–999, Aug 2018.

[Pal43] C. Palm. Intensitatsschwankungen im fernsprechverkehr. Ericsson Technics, 1943.

[PS18] M. Poplavskyi and G. Schehr. Exact persistence exponent for the 2d-diffusion equation and related Kac polynomials. arXiv preprint arXiv:1806.11275, 2018.

[She11] G. Shevchenko. Properties of trajectories of a multifractional Rosenblatt process. Theory of Probability and Mathematical Statistics, 83:163–173, 2011.

[Taq75] M. S. Taqqu. Weak convergence to fractional Brownian motion and to the Rosenblatt process. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 31(4):287–302, 1975.

[Taq78] M. S. Taqqu. A representation for self-similar processes. Stochastic Processes and their Applications, 7(1):55–64, 1978.

[Taq11] M. S. Taqqu. The Rosenblatt process. Selected Works of Murray Rosenblatt. Springer Verlag, New York, 2011.

[Tho95] H. Thorisson. On time-and cycle-stationarity. Stochastic processes and their applications, 55(2):183–209, 1995.

[Tho00] H. Thorisson. Coupling, stationarity, and regeneration. Springer, 2000.

[Tot66] H. Totoki. Time changes of flows. Memoirs of the Faculty of Science, Kyushu University. Series A, Mathematics, 20(1):27–55, 1966.
[VJ05] D. Vere-Jones. A class of self-similar random measure. *Advances in Applied Probability*, 37(4):908–914, 2005.

[VT13] M. S. Veillette and M. S. Taqqu. Properties and numerical evaluation of the Rosenblatt distribution. *Bernoulli*, 19(3):982–1005, 2013.

[Zäh88] U. Zähle. Self-similar random measures I. Notion, Carrying Hausdorff Dimension, and Hyperbolic Distribution. *Probability Theory and Related Fields*, 80(1):79–100, 1988.

[Zäh90] U. Zähle. Self-similar random measures II. A generalization to self-affine measures. *Mathematische Nachrichten*, 146(3-6):85–98, 1990.

[Zäh91] U. Zähle. Self-similar random measures III. Self-similar random processes. *Mathematische Nachrichten*, 151(1):121–148, 1991.