Study of boundary conditions in the Iterative Filtering method for the decomposition of nonstationary signals

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Abstract

Nonstationary and nonlinear signals are ubiquitous in real life. Their decomposition and analysis is an important topic of research in signal processing. Recently a new technique, called Iterative Filtering, has been developed with the goal of decomposing such signals into simple oscillatory components. Several papers have been published regarding the analysis of this technique from a mathematical point of view. All these work start with the assumption that each compactly supported signal is extended outside the boundaries periodically.

In this work we tackle the problem of studying the influence of different boundary conditions on the decompositions produce by the Iterative Filtering method.

1 Introduction

Given a real life nonstationary signal \( s(x), \ x \in \mathbb{R} \), we may be interested in decomposing it into simple components in order to identify features and quasi periodicities hidden in it. We can think, for instance, to an economic index, like the GDP of a nation, or a geophysical signal, like the sea level during a tsunami, as well as to an engineering measure, like the vibrations of a structure or machinery. Standard techniques, like Fourier or wavelet transform, cannot help, in general, to decompose meaningfully nonstationary signals. Whereas, following the idea proposed by Huang et al. in [1], we can iteratively decompose such signals into a finite sequence of simple components, defined Intrinsic Mode Functions (IMFs), which fulfill two properties: i) the number of extrema and the number of zero crossings must either equal or differ at most by one; ii) considering upper and lower envelopes connecting respectively all the local maxima and minima of the function, their mean has to be zero at any point.

The method originally proposed by Huang et al. in [1], called Empirical Mode Decomposition (EMD), proved to be unstable to small perturbations. For this reason an alternative algorithm, called Ensemble Empirical Mode Decomposition (EEMD), was proposed in [10] which is based on the idea of applying EMD to an ensemble of signals produced perturbing hundreds of times the given one with noise. The decomposition is then derived as the average decomposition of the ensemble. In [2] the authors proposed an alternative technique to the EMD, called Iterative Filtering (IF), which has the very same structure of EMD, but it is stable and convergent both in the continuous setting [6,3] and in the discrete one [5,7]. We point out that IF has been also generalized producing the so called Adaptive Local Iterative Filtering (ALIF) algorithm whose convergence and stability are under investigation [5,4]. For further details on why standard techniques may
fail in decomposing a nonstationary signal and the advantages of using EEMD or IF we refer the interested reader to [9].

In this work we consider the case of compactly supported and discrete signals. For simplicity we assume that each signal $s = [s(x_j)]_{j=0}^{n-1}$ (with $n \in \mathbb{N}$) is supported on $[0,1]$ and it is sampled at $n$ points $x_j = \frac{j}{n-1}$, with $j = 0, \ldots, n-1$. Without loosing generality we can further assume that $\|s\|_2 = 1$.

Any signal decomposition method which deals with a compactly supported signal requires assumptions on how the signal extends outside the boundaries, the so called Boundary Conditions (BCs). For Iterative Filtering in [8] the authors addressed its convergence when it is assumed that the signals extend periodically outside the boundaries. The questions which are still open are: does IF converge also for other kinds of BCs? Given a signal extended artificially outside the boundaries in a certain way, how do the errors introduced outside the boundaries effect the decomposition in the iterations? Given a compactly supported signal what is the best choice in terms of BCs?

The rest of the paper is organized as follows. In Section 2 we review the IF algorithm applied to discrete and compactly supported signals. Section 3 is devoted to a summary of boundary conditions and their properties. In Section 4 we study the IF convergence when nonperiodic BCs are chosen for the signal.

2 The iterative Filtering algorithm

In this section we review the Iterative Filtering method focusing on the case of discrete and compactly supported signals $s$. We start from the definition of filter

**Definition 1.** A nonnegative vector $w = (0, \ldots, 0, w_{-l}, \ldots, w_0, w_1, \ldots, w_l, 0, \ldots, 0) \in \mathbb{R}^n$ such that $w_j > 0$ for $j = -l, \ldots, l$ and $\sum_{j=-l}^{l} w_j = 1$ is called a filter of length $l$, with $0 < l \leq \left\lfloor \frac{n}{2} \right\rfloor$. If it also holds that $w_{-j} = w_j$ for $j = 1, \ldots, l$, then $w$ is called a symmetric filter.

In this work we consider only symmetric filters. When we have a symmetric filter associated with some step $m$, we employ the following notation

$$w_m = (0, \ldots, 0, w_{-l_m}^m, \ldots, w_1^m, w_0^m, w_1^m, \ldots, w_{l_m}^m, 0, \ldots, 0),$$

(1)

with $w_0^m + 2 \sum_{j=1}^{l_m} w_j^m = 1$.

If we assume that some filter shape $h : [-1,1] \rightarrow \mathbb{R}$ (symmetric with respect to $y$-axis) has been selected a priori, like one of the Fokker-Planck filters described in [5], then the elements $w_j^m$ can be computed, for $j = 0, 1, \ldots, l_m$, by the linear scaling formula

$$w_j^m = h \left( \frac{j}{l_m} \right) \frac{1}{l_m},$$

(2)

where $l_m$ is the length that characterizes the filter. Assuming $s_1^m = s$, where the two indices will become
clear in the next paragraph, the main step of the IF method, for \( i = 0, \ldots, n - 1 \), is

\[
\mathbf{s}_m^{k+1}(x_i) = \mathbf{s}_m^k(x_i) - \int_{x_i - \frac{l_m}{m}}^{x_i + \frac{l_m}{m}} \mathbf{s}_m^k(y) h \left( \frac{(x_i - y)(n - 1)}{l_m} \right) \frac{n - 1}{l_m} dy
\]

\[
\approx \mathbf{s}_m^k(x_i) - \sum_{x_j = x_i - \frac{l_m}{m}}^{x_j = x_i + \frac{l_m}{m}} \mathbf{s}_m^k(x_j) h \left( \frac{(x_i - x_j)(n - 1)}{l_m} \right) \frac{1}{l_m}
\]

\[
= \mathbf{s}_m^k(x_i) - \sum_{j = i - l_m}^{i + l_m} \mathbf{s}_m^k(x_j) h \left( \frac{i - j}{l_m} \right) \frac{1}{l_m}
\]

\[
= \mathbf{s}_m^k(x_i) - \sum_{j = i - l_m}^{i + l_m} \mathbf{s}_m^k(x_j) w_m^{m_{ij}}
\]

Algorithm 1 provides the pseudocode of the Discrete version of the Iterative Filtering Algorithm. We observe that the first while loop is called Outer Loop, whereas the second one Inner Loop. In the notation \( s_m^k, m \) denotes the step relative to the Outer Loop, while \( k \) denotes the step relative to the Inner Loop. The

\[
\text{Algorithm 1 Discrete Iterative Filtering } \text{IMFs} = \text{IF}(s, h)
\]

\[
m = 1
\]

\[
\mathbf{s}_1^m = s
\]

\[
\text{while the number of extrema of } \mathbf{s}_1^m \geq 2 \text{ do}
\]

\[
\begin{align*}
&k = 1 \\
&\text{compute the filter length } l_m \text{ for the signal } \mathbf{s}_1^m \\
&\text{compute } w_m \text{ (having } h \text{ and } l_m) \\
&\text{while the stopping criterion is not satisfied do}
&\begin{align*}
&(\mathbf{s}_1^m)^{BC}(x_i) = \mathbf{s}_1^m(x_i), \ i = 0, \ldots, n - 1 \\
&\text{apply BCs for computing } (\mathbf{s}_1^m)^{BC}(x_i), \ i = -l_m, \ldots, -1 \text{ and } i = n, \ldots, n - 1 + l_m \\
&\mathbf{s}_1^{m+1}(x_i) = \mathbf{s}_1^m(x_i) - \sum_{j = i - l_m}^{i + l_m} (\mathbf{s}_1^m)^{BC}(x_j) w_m^{m_{ij}}, \ i = 0, \ldots, n - 1 \\
&k = k + 1
&\end{align*}
\end{align*}
\]

\[
\text{end while}
\]

\[
\mathbf{f}_m = \mathbf{s}_1^m
\]

\[
\mathbf{s}_1^{m+1} = \mathbf{s}_1^m - \mathbf{f}_m
\]

\[
m = m + 1
\]

\[
\text{end while}
\]

\[
\mathbf{f}_m = \mathbf{s}_1^m
\]

\[
\text{IMFs} = \{\mathbf{f}_1, \ldots, \mathbf{f}_m\}
\]

idea is to suitably choose the filter length \( l_m \) in order to capture the desired frequencies in the IMF \( \mathbf{f}_m \), and this is done by subtracting iteratively from the signal its moving average computed as convolution of the signal itself with the selected filter. In matrix form we have

\[
\mathbf{s}_m^{k+1} = (I - W_m)\mathbf{s}_m^k,
\]

where \( W_m \) is a structured matrix constructed from the filter \( w_m \) and the boundary conditions imposed. In particular \( W_m \) can be written as the sum of two matrices

\[
W_m^{BC} = T_m + K_m^{BC},
\]

where the first one is a Toeplitz matrix, while the second one is a correction matrix which depends on BCs (for details, see Section 3).

3
From (3), it follows immediately that
\[ s_{k+1}^m = (I - W_m)^k s_1^m, \]  
so ideally the first IMF is given by
\[ f_1 = \lim_{k \to \infty} (I - W_1)^k s. \]  
However, in the implemented algorithm we do not let \( k \) to go to infinity, instead we use a stopping criterion. We can define, for instance, the following quantity
\[ \Delta_k^m := \frac{\|s_{k+1}^m - s_k^m\|_2}{\|s_k^m\|_2}, \]  
so we can either stop the process when the value \( \Delta_k^m \) reaches a certain threshold or we can introduce a limit on the maximal number of iterations for all the Inner Loops. It is also possible to adopt different stopping criteria for different Inner Loops.

3 Boundary conditions

Boundary conditions deal with the problem of extending the signal outside the field of view in which the detection is made. So let \( s \) be the signal inside the boundaries and let \( p \) be the parameter relative to the space outside the boundaries, we have that, for \( j = 1, \ldots, p \), Zero BCs are defined as
\[ s(x-j) = 0, \quad s(x_{n-1+j}) = 0, \]  
Periodic BCs are defined as
\[ s(x-j) = s(x_{n-j}), \quad s(x_{n-1+j}) = s(x_{j-1}), \]  
Reflective BCs are defined as
\[ s(x-j) = s(x_{j-1}), \quad s(x_{n-1+j}) = s_{n-j}, \]  
Anti-Reflective BCs are defined as
\[ s(x-j) = 2s(x_0) - s(x_j), \quad s(x_{n-1+j}) = 2s(x_{n-1}) - s(x_{n-1-j}). \]

According to the BCs imposed, we have a different kind of structured matrix as \( W^{BC} \), whose elements are defined from values of the filter \( w \) of length \( l \). As said, here we take into account symmetric filters, since this choice allows to have useful theoretical properties. In particular, symmetry is not necessary to get the algebra of Circulant matrices, associated with Periodic BCs, while it is necessary to get the algebra of Reflective matrices and the algebra of Anti-Reflective matrices. The symmetry property also allows in all these three cases to have a fast transform that can be employed for computing matrix-vectors products in an efficient way. In particular, we have Fast Fourier Transform (FFT) for Circulant matrices, Discrete Cosine Transform of type III (DCT-III) for Reflective matrices and Discrete Sine Transform of type I (DST-I) for Anti-Reflective matrices.
Unfortunately, this does not hold for Toeplitz matrices, associated with Zero BCs, i.e. \( W^Z = T \), where

\[
T = \begin{pmatrix}
  w_0 & w_1 & w_2 & \ldots & w_l \\
  w_1 & w_0 & w_1 & \ddots & w_l \\
  w_2 & w_1 & w_0 & \ddots & \vdots \\
  \vdots & w_2 & w_1 & \ddots & \ddots \\
  w_l & \ddots & w_2 & \ddots & w_1 \\
  w_l & \ddots & w_2 & \ddots & w_1 \\
  w_l & \ddots & w_2 & \ddots & w_1 \\
  w_l & \ddots & w_2 & \ddots & w_1 \\
  w_l & \ddots & w_2 & \ddots & w_1 \\
  w_l & \ddots & w_2 & \ddots & w_1 \\
\end{pmatrix}_{n \times n}.
\] (12)

In case of Periodic BCs, we have Circulant matrices that have this form

\[
W^P = \begin{pmatrix}
  w_0 & w_1 & \ldots & w_l & w_l & \ldots & w_1 \\
  w_1 & w_0 & w_1 & \ddots & w_l & \ddots & \vdots \\
  \vdots & w_1 & w_0 & w_1 & \ddots & \ddots & \ddots \\
  w_l & \ddots & w_1 & w_0 & w_1 & \ddots & w_l \\
  w_l & \ddots & w_1 & w_0 & w_1 & \ddots & \ddots \\
  \vdots & w_l & \ddots & w_0 & w_1 & \ddots & \ddots \\
  w_l & \ddots & \ddots & w_l & w_1 & w_0 & w_1 \\
  \vdots & \ddots & \ddots & \ddots & w_l & w_1 & \ldots & w_1 \\
  w_l & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & w_l \\
\end{pmatrix}_{n \times n}.
\] (13)

The eigenvalues of \( W^P \) can be computed by this formula, for \( i = 1, \ldots, n \),

\[
\lambda_i^P = w_0 + 2 \sum_{j=1}^{l} w_j \cos \left( \frac{2j(i-1)\pi}{n} \right).
\] (14)
In the Reflective case, we have $W^R = T + H^R$, where $H^R$ is a Hankel matrix

$$H^R = \begin{pmatrix}
w_1 & w_2 & w_3 & \ldots & w_l \\
w_2 & w_3 & \ddots & \ddots & \vdots \\
w_3 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
w_l & \ldots & w_3 & w_2 & w_1
\end{pmatrix}_{n \times n}.
$$

$W^R$ can be diagonalized by $Q^R$, that is the $n$-dimensional Discrete Cosine Transform of type III (DCT-III), having entries

$$[Q^R]_{i,j} = \sqrt{\frac{2 - \delta_{i1}}{n}} \cos \left( \frac{(i - 1)(2j - 1)\pi}{2n} \right), \quad i, j = 1, \ldots, n,$$

where $\delta_{ij}$ denotes the Kronecker delta. Eigenvalues of $W^R$ can be obtained by the following formula

$$\lambda^R_i = \frac{[Q^R (W^R e_1)]_i}{[Q e_1]_i},$$

where $e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$. Therefore, using the variable $t = \frac{(i - 1)\pi}{2n}$, and exploiting

$$\cos(2jt - t) = \cos(2jt) \cos(t) + \sin(2jt) \sin(t)$$

and

$$\cos(2jt + t) = \cos(2jt) \cos(t) - \sin(2jt) \sin(t),$$

we get that in the Reflective case eigenvalues can be computed by this formula, for $i = 1, \ldots, n,$

$$\lambda^R_i = \sum_{j=1}^{n} [W^R]_{i,j} \cos \left( \frac{(2j - 1)(i - 1)\pi}{2n} \right) \cos \left( \frac{(i - 1)\pi}{2n} \right)
= w_0 + \sum_{j=1}^{l} w_j \cos((2j - 1)t) + \cos((2j + 1)t)
= w_0 + 2 \sum_{j=1}^{l} w_j \cos(2jt)
= w_0 + 2 \sum_{j=1}^{l} w_j \cos \left( \frac{j(i - 1)\pi}{n} \right).$$
In the Anti-Reflective case, the structure of the matrix is more involved, namely

\[
W^{\text{AR}} = \begin{pmatrix} 
  z_1 + w_0 & 0 & \ldots & \ldots & 0 & 0 \\
  z_2 + w_1 & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  z_l + w_{l-1} & \ddots & \ddots & \ddots & \ddots & \ddots \\
  w_l & W^{\text{AR}} & \ddots & \ddots & \ddots & \ddots \\
  0 & w_l & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & \ldots & \ldots & 0 & z_1 + w_0 \\
\end{pmatrix}_{n \times n},
\]

where \( z_j = 2 \sum_{k=j}^{l} w_k \) and \( \tilde{W}^{\text{AR}} = PW^{\text{AR}}P^T \), with

\[
P = \begin{pmatrix} 
  0 & 1 & 0 \\
  0 & 1 & 0 \\
  \vdots & \ddots & \vdots \\
  0 & 1 & 0 \\
  0 & 1 & 0 \\
\end{pmatrix}_{(n-2) \times n}.
\]

Moreover, \( \tilde{W}^{\text{AR}} = \tilde{T} - \tilde{H}^{\text{AR}} \), where \( \tilde{T} \) is a Toeplitz matrix

\[
\tilde{T} = \begin{pmatrix} 
  w_0 & w_1 & \ldots & w_l \\
  w_1 & w_0 & \ldots & w_l \\
  \vdots & \ddots & \ddots & \ddots \\
  w_l & \ddots & \ddots & \ddots \\
  w_l & \ddots & \ddots & \ddots \\
  w_l & \ddots & \ddots & \ddots \\
  w_l & \ddots & \ddots & \ddots \\
  w_l & \ldots & w_1 & w_0 \\
\end{pmatrix}_{(n-2) \times (n-2)},
\]

while \( \tilde{H}^{\text{AR}} \) is a Hankel matrix

\[
\tilde{H}^{\text{AR}} = \begin{pmatrix} 
  w_2 & w_3 & \ldots & w_l \\
  w_3 & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots \\
  w_l & \ddots & \ddots & \ddots \\
  w_l & \ddots & \ddots & \ddots \\
  w_l & \ddots & \ddots & \ddots \\
  w_l & \ldots & w_3 & w_2 \\
\end{pmatrix}_{(n-2) \times (n-2)}.
\]

\( \tilde{W}^{\text{AR}} \) can be diagonalized by \( \tilde{Q}^{\text{AR}} \), that is the \((n-2)\)-dimensional Discrete Sine Transform of type I (DST-I),
having entries

\[ [\hat{Q}^{AR}]_{i,j} = \sqrt{\frac{2}{n-1}} \sin \left( \frac{ij \pi}{n-1} \right), \quad i, j = 1, \ldots, n-2. \]  

(23)

Eigenvalues of \( \hat{W}^{AR} \) can be obtained by the following formula

\[ \hat{\lambda}^{AR}_i = \frac{[\hat{Q}^{AR} (\hat{W}^{AR} \hat{e}_1)]_i}{[\hat{Q}^{AR} \hat{e}_1]_i} \]

(24)

where \( \hat{e}_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^{n-2} \). Therefore, using the variable \( t = \frac{\pi n}{n-1} \), and exploiting

\[ \sin(2t) = 2 \sin(t) \cos(t) \]

and

\[ \cos(2t) = 1 - 2 \sin^2(t) \]

and

\[ \sin((j+2)t) - \sin jt = \sin(jt) \cos(2t) + \sin(2t) \cos(jt) - \sin(jt) = \sin(jt)(1 - 2 \sin^2(t)) + 2 \sin(t) \cos(jt) - \sin(jt) = -2 \sin(jt) \sin^2(t) + 2 \sin(t) \cos(jt) \cos(jt) \]

and

\[ 2 \sin(jt) \sin(t) = \cos(jt - t) - \cos(jt + t) \]

and

\[ 2 \cos(t) \cos(jt) = \cos(jt - t) - \cos(jt + t) \]

we get that eigenvalues of \( \hat{W}^{AR} \) can be computed by this formula, for \( i = 1, \ldots, n-2 \),

\[ \hat{\lambda}^{AR}_i = \sum_{j=1}^{n-2} [\hat{W}^{AR}]_{1,j} \frac{\sin \left( \frac{j \pi}{n-1} \right)}{\sin \left( \frac{i \pi}{n-1} \right)} = w_0 + w_1 \cos(t) + \sum_{j=1}^{l-1} w_{j+1} \frac{\sin((j+2)t)}{\sin(t)} - w_{j+1} \frac{\sin(jt)}{\sin(t)} = w_0 + 2w_1 \cos(t) + \sum_{j=1}^{l-1} w_{j+1} (2 \cos(t) \cos(jt) - 2 \sin(jt) \sin(t)) = w_0 + 2w_1 \cos(t) + 2 \sum_{j=1}^{l-1} w_{j+1} \cos((j+1)t) = w_0 + 2w_1 \cos(t) + 2 \sum_{j=2}^{l} w_j \cos(jt) = w_0 + 2 \sum_{j=1}^{l} w_j \cos \left( \frac{j \pi}{n-1} \right) \]

(25)

Finally, by Lemma 3.1 in [11], we know that the eigenvalues of \( W^{AR} \) are given by 1 with multiplicity two and by the eigenvalues \( \{ \hat{\lambda}^{AR}_i \}_{i=1,\ldots,n-2} \) of \( \hat{W}^{AR} \).
4 Spectral and convergence properties

By exploiting the structures presented in Section 3, we are able to prove spectral properties of $W^{BC}$ and convergence properties of Discrete Iterative Filtering algorithm.

**Lemma 1.** Let $W^{BC} \in \mathbb{R}^{n\times n}$ be the structured matrix constructed from a symmetric filter $w$ of length $0 < l \leq \lfloor \frac{n+1}{2} \rfloor$, imposing some BCs (Zero or Periodic or Reflective or Anti-Reflective). Then $\sigma(W^{BC}) \subseteq [-1, 1]$.

**Proof.** By definition $W^{BC}$ is a symmetric matrix, so it has a real spectrum. By considering the following estimate relative to the spectral radius

$$\rho(W^{BC}) \leq \|W^{BC}\|_1 = \max_j \sum_{i=1}^{n} |(W^{BC})_{i,j}| = w_0 + 2 \sum_{k=1}^{l} w_k = 1,$$

we can conclude that all eigenvalues lies in the interval $[-1, 1]$. \hfill \Box

**Theorem 1.** Let $s \in \mathbb{R}^n$ the signal that has to be decomposed. Let $v$ be a symmetric filter of length $0 < l' \leq \lfloor \frac{n+1}{4} \rfloor$ and $w = v * v$ be another symmetric filter of length $0 < l \leq \lfloor \frac{n+1}{2} \rfloor$ defined by making the convolution of $v$ with itself. Then for the matrix $W^{BC} \in \mathbb{R}^{n\times n}$ constructed from $w$ and some BCs (Periodic or Reflective or Anti-Reflective), it holds that $\sigma(W^{BC}) \subseteq [0, 1]$.

**Proof.** As already said, in case of Periodic or Reflective or Anti-Reflective BCs we are in a matrix algebra; this means that the matrix product gives rise to a matrix which can still be interpreted as a matrix constructed from a filter and the same BCs. In particular, thanks to the fact that $l' \leq \lfloor \frac{n+1}{4} \rfloor$, such filter can be computed by means of convolution. Thus, if we denote by $W^{BC}$ the structured matrix associated with filter $v$, we have that $W^{BC} = (V^{BC})^2$. By Lemma 2 $\sigma(V^{BC}) \subseteq [-1, 1]$, therefore $\sigma(W^{BC}) \subseteq [0, 1]$. \hfill \Box

**Theorem 2.** Let $W^{BC} \in \mathbb{R}^{n\times n}$ be the matrix constructed from $w = v * v$ and some BCs (Periodic or Reflective or Anti-Reflective). Then $\lambda_1^P = 1$ and $\{\lambda_i^P\}_{i=2,\ldots,n} \subseteq [0,1)$, $\lambda_1^R = 1$ and $\{\lambda_i^R\}_{i=2,\ldots,n} \subseteq [0,1)$, $\lambda_1^{AR} = \lambda_2^{AR} = 1$ and $\{\lambda_i^{AR}\}_{i=3,\ldots,n} \subseteq [0,1)$.

**Proof.** By Theorem 1 the spectrum of $W^{BC}$ is in $[0,1]$.

In case of Periodic BCs, if we consider $\lambda_i^P$ with $i = 1$, we get $\lambda_1^P = 1$, while for $1 < i \leq n$ we have a convex combination of cosine values (each of them strictly less than 1), except for the first term multiplied by $w_0$ which is equal to 1. Clearly this sum cannot equal to 1, so it has a value $0 \leq x < 1$.

In case of Reflective BCs, if we consider $\lambda_i^R$ with $i = 1$, we get $\lambda_1^R = 1$, while for $1 < i \leq n$ we have a convex combination of cosine values (each of them strictly less than 1), except for the first term multiplied by $w_0$ which is equal to 1. Clearly this sum cannot equal to 1, so it has a value $0 \leq x < 1$.

In case of Anti-Reflective BCs, if we consider $\lambda_i^{AR}$ for $i = 1, \ldots, n-2$ we have a convex combination of cosine values (each of them strictly less than 1), except for the first term multiplied by $w_0$ which is equal to 1. Clearly this sum cannot equal to 1, so it has a value $0 \leq x < 1$. Therefore $n-2$ eigenvalues of $W^{BC}$ belong to the interval $[0,1)$, while the remaining two are equal to 1, as we already know. \hfill \Box

Looking at the structure of $W^{BC}$, it can be easily verified that the following Corollary holds.

**Corollary 1.** The eigenvector $u_i^P$ associated with $\lambda_i^P = 1$ and $\lambda_i^R = 1$ is $(1, \ldots, 1)^T$; the eigenvectors associated with $\lambda_i^{AR} = \lambda_i^{AR} = 1$ are of the form $u_i^{AR} = (0,1,2,\ldots,n-2,n-1)^T$ and $u_i^{AR} = (n-1,n-2,\ldots,2,1,0)^T$.

In the case of zero, periodical, reflective and antireflective boundary conditions, the matrix $W^{BC}$ can be diagonalized by means of a unitary matrix $U$ containing as columns the orthonormal eigenvectors of $W^{BC}$.

We are ready now to discuss the convergence of the DIF method.
Proposition 1. Given the DIF algorithm applied to a signal \( s \in \mathbb{R}^n \), and considering periodical, reflective or antireflective boundary conditions. Assuming that we are considering a doubly convolved filter, and the half filter support length is constant throughout all the steps of an inner loop, given the unitary matrix \( U \) which diagonalizes the matrix \( W^{BC} \), and assuming that \( W^{BC} \) has \( k \) zero eigenvalues, where \( k \) is a number in the set \( \{0, 1, \ldots, n-1\} \).

Then the first outer loop step of the DIF method converges to
\[
I_1 = \lim_{m \to \infty} (I - W)^m s = UZU^T s
\]
where \( Z \) is a diagonal matrix with entries all zero except \( k \) elements in the diagonal which are equal to one.

The proof follows trivially from the previous Corollary.

So the DIF method in the limit produces IMFs that are projections of the given signal \( s \) onto the eigenspace of \( W^{BC} \) corresponding to the zero eigenvalue which has algebraic and geometric multiplicity \( k \) \( \in \{0, 1, \ldots, n-1\} \).

Clearly, if \( W^{BC} \) has only a trivial kernel then the method converges to the zero vector.

If we enforce a stopping criterion like (7) we discontinue the calculations of the DIF algorithm after finitely many steps and we produce approximated IMFs.

Theorem 3. Given \( s \in \mathbb{R}^n \), we consider the convolution matrix \( W^{BC} \) defined before, associated with a filter vector \( w \) given as a symmetric filter \( \tilde{w} \) convolved with itself. Assuming that \( W^{BC} \) has \( k \) zero eigenvalues, where \( k \) is a number in the set \( \{0, 1, \ldots, n-1\} \), and fixed \( \delta > 0 \).

Then, calling \( \tilde{s} = U^T s \), for the minimum \( N_0 \in \mathbb{N} \) such that it holds true the inequality
\[
\frac{N_0^N_0}{(N_0 + 1)^{N_0 + 1}} < \frac{\delta}{\|s\|_\infty \sqrt{n - 1 - k}}
\]
we have that \( \|s_{m+1} - s_m\|_2 < \delta \) \( \forall m \geq N_0 \) and the first IMF is given by
\[
I_1 = U(I - D)^{N_0}U^T s = U P \begin{bmatrix}
0 & (1 - \lambda_1)^{N_0} & \cdots & (1 - \lambda_{n-1-k})^{N_0} & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & 0
\end{bmatrix} \cdot P^T U^T s
\]
where \( P \) is a permutation matrix which allows to reorder the columns of \( U \), which correspond to eigenvectors of \( W^{BC} \), so that the corresponding eigenvalues \( \{\lambda_p\}_{p=1,...,n-1} \) are in decreasing order.

The proof follows directly from -one of Theorem 5 [8].

5 Extended approach

Now, once defined the matrix
\[
R = \begin{pmatrix}
0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & \cdots & 0
\end{pmatrix}_{n \times (n+2p)}
\]
in Algorithm 1 we present the pseudocode of Extended Iterative Filtering algorithm, which is not based (as in Discrete Iterative Filtering algorithm) on a structured matrix $W_{m}^{BC} \in \mathbb{R}^{n \times n}$ (whose structure depends on BCs), associated with the original signal $s \in \mathbb{R}$, but is based on a Circulant matrix $\hat{W}_{m}^{P} \in \mathbb{R}^{(n+2p) \times (n+2p)}$ (whose structure does not depend on BCs), associated with a signal $s^{BC} \in \mathbb{R}^{(n+2p)}$ extended by means of any BCs. This means that, since in this framework BCs does not affect the matrix structure, we have more freedom in the choice of BCs. For instance, we can think to address this issue by employing a method that, given the original signal, extends it in a suitable way. Moreover, we notice that in Algorithm 2 there is only one step involving the use of BCs, while in Algorithm 1 this happens at every step of the Inner Loop. Considering that every time we make use of BCs we are adding errors to the decomposition method, we think that the proposed approach improves the standard one.

**Algorithm 2 Extended Iterative Filtering**

$$IMFs = IF(s, h)$$

$$s^{BC}(x_i) = s(x_i), \quad i = 0, \ldots, n - 1$$

apply BCs for computing $s^{BC}(x_i), \quad i = -p, \ldots, -1$ and $i = n, \ldots, n - 1 + p$

$m = 1$

$s_{1}^{m} = s^{BC}$

while the number of extrema of $s_{1}^{m} \geq 2$

$k = 1$

compute the filter length $l_{m}$ for the signal $s_{k}^{m}$

compute $w_{m}$ (having $h$ and $l_{m}$)

while the stopping criterion is not satisfied

$s_{k+1}^{m} = (I - \hat{W}_{m}^{P})s_{k}^{m}$

$k = k + 1$

end while

$f_{m} = s_{k}^{m}$

$s_{1}^{m+1} = s_{m}^{m} - f_{m}$

$m = m + 1$

end while

$f_{m} = s_{1}^{m}$

IMFs = \{ $Rf_{1}, \ldots, Rf_{m}$ \}

We denote by $f_{1}$ the first approximated IMF computed by Extended Iterative Filtering algorithm, by $\bar{f}_{1}$ the first exact IMF, by $s^{BC}$ the signal extended according to boundary conditions and by $\hat{s}$ the real extended signal. Then, we can split both of them considering the portion of signal inside and outside the field of view, getting

$$s^{BC} = s^{BC}_{\text{ins}} + s^{BC}_{\text{out}}$$  \hspace{1cm} (30)

and

$$\hat{s} = \tilde{s}_{\text{ins}} + \tilde{s}_{\text{out}}.$$  \hspace{1cm} (31)

Clearly $s^{BC}_{\text{ins}} = \tilde{s}_{\text{ins}}$.

We have that

$$f_{1} = R(I - \hat{W}_{1}^{P})^{k}s^{BC}$$

$$= R(I - \hat{W}_{1}^{P})^{k}(s^{BC}_{\text{ins}} + s^{BC}_{\text{out}} + \tilde{s}_{\text{out}} - \hat{s}_{\text{out}})$$

$$= R(I - \hat{W}_{1}^{P})^{k}\tilde{s} + R(I - \hat{W}_{1}^{P})^{k}(s^{BC}_{\text{out}} - \hat{s}_{\text{out}})$$

where the first term tends to $\bar{f}_{1}$, while the second term tends to zero in norm, but, when $k$ grows, it propagates itself from the boundary inside the field of view.
6 Conclusion and Outlook

Based on the results shown in this work we can say that the influence of the errors from the boundaries propagates inside the signal for roughly one period (half mask length) each side, if we consider Fokker-Plank filters, or two periods (one mask length) each side, if we consider a triangular filter.

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