ANALYTIC TWISTS OF $\text{GL}_2 \times \text{GL}_2$ AUTOMORPHIC FORMS

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Abstract. Let $f$ and $g$ be holomorphic or Maass cusp forms for $\text{SL}_2(\mathbb{Z})$ with normalized Fourier coefficients $\lambda_f(n)$ and $\lambda_g(n)$, respectively. In this paper, we prove nontrivial estimates for the sum
$$\sum_{n=1}^{\infty} \lambda_f(n)\lambda_g(n)e\left(t\varphi\left(\frac{n}{X}\right)\right)V\left(\frac{n}{X}\right),$$
where $e(x) = e^{2\pi ix}$, $V(x) \in C^\infty_c(1, 2)$, $t \geq 1$ is a large parameter and $\varphi(x)$ is some nonlinear real valued smooth function. Applications of these estimates include a subconvex bound for the Rankin-Selberg $L$-function $L(s, f \otimes g)$ in the $t$-aspect, an improved estimate for a nonlinear exponential twisted sum and the following asymptotic formula for the sum of the Fourier coefficients of certain $\text{GL}_5$ Eisenstein series
$$\sum_{n \leq X} \lambda_{11\mathfrak{E}(f \times g)}(n) = L(1, f \times g)X + O(X^{\frac{3}{4} + \varepsilon}),$$
for any $\varepsilon > 0$.

1. Introduction

When studying number theory problems, one often runs into nonlinear exponential sums of the form
$$\sum_{n=1}^{\infty} a_n e\left(t\varphi\left(\frac{n}{X}\right)\right)V\left(\frac{n}{X}\right),$$
where $a_n$ is some arithmetic function, here and throughout the paper, $e(x) = e^{2\pi ix}$, $V(x) \in C^\infty_c(1, 2)$ is a smooth function with support contained in $(1, 2)$, $t, X \geq 1$ are large parameters and $\varphi(x)$ is some nonlinear real valued smooth function. For example, for an automorphic $L$-function $L(s, F)$, the subconvexity problem of $L(s, F)$ in the $t$-aspect boils down to a nontrivial estimate for this sum with $a_n = \lambda_F(n)$ being the Fourier coefficients of the automorphic form $F$ and $\varphi(x) = -(\log x)/2\pi$. Here we remind that for $a_n$ ($n \sim X$) satisfying $\|a_n\|^2 = \sum_n |a_n|^2 \ll X$, the trivial bound of this nonlinear exponential sum is $O(X)$. On the other hand, it is worth noting that the square-root cancellation phenomenon should not hold in general, as first found...
by Iwaniec, Luo and Sarnak \cite{ILS00} (see Appendix C, (C.17) and (C.18)) that
\[
\sum_{n=1}^{\infty} \lambda_F(n)e(-2\sqrt{qn})V\left(\frac{n}{X}\right) = \frac{\lambda_F(q)}{q^{1/4}} \hat{V}(0)X^{3/4} + O((qX)^{1/4+\varepsilon}),
\]
for any positive integer \(q\) and any \(\varepsilon > 0\), where \(\lambda_F(n)\) are the normalized Fourier coefficients of a \(SL_2(\mathbb{Z})\) holomorphic cusp form \(F\) of weight \(\kappa\) and \(\hat{V}(0) = 2^{-1}i\pi(1-i)\int_0^{\infty} V(x)x^{-1/4}dx\). Moreover, Kaczorowski and Perelli \cite{KP05} improved and extended this result for Selberg class and this was later revisited by Ren and Ye \cite{RY15b} for \(GL_m\) Maass cusp forms.

For \(a_n = \lambda_F(n)\) being the Fourier coefficients of an automorphic form \(F\), a natural way to study the associated nonlinear exponential twisted sum is to directly use the functional equation of the automorphic \(L\)-function \(L(s, F)\) or equivalently, the Voronoi formula for \(\lambda_F(n)\), as shown in \cite{KP05} and \cite{RY15b}. However, if the nonlinear exponential function \(e(t\varphi(n/X))\) oscillates strong enough, there is a chance to get more savings by separating the oscillations of \(\lambda_F(n)\) and \(e(t\varphi(n/X))\) using the \(\delta\)-method. Kumar, Mallesham and Singh \cite{KMS19} first implemented this idea for \(GL_3\) Maass cusp forms by using the Duke-Friedlander-Iwaniec \(\delta\)-method given in \cite{IK04} together with the conductor-lowering trick due to Munshi \cite{Mun15}, and proved that for \(t = X^\beta\) and \(\varphi(x) = \alpha x^\beta\) \((\alpha \in \mathbb{R}\setminus\{0\}, 0 < \beta < 1)\)
\[
\sum_{n=1}^{\infty} \lambda_\pi(1,n)e\left(t\varphi\left(\frac{n}{X}\right)\right) V\left(\frac{n}{X}\right) \ll_{\pi,\alpha,\beta} t^{3/10}X^{3/4+\varepsilon},
\]
which improved the estimate \(O(X^{3/2}\log X)\) by Ren and Ye \cite{RY15a} for \(\beta > 5/8\). Here \(\lambda_\pi(1,n)\) are the normalized Fourier coefficients of a Hecke-Maass cusp form \(\pi\) for \(GL_3(\mathbb{Z})\). See also the first author \cite{Hua21}. For cusp forms on \(GL_2\), the associated nonlinear exponential twisted sums were studied in Aggarwal, Holowinsky, Lin and Qi \cite{AHLQ20} by a Bessel \(\delta\)-method. Recently, Lin and the second author \cite{LS21} studied the \(GL_3 \times GL_2\) case by using the Duke-Friedlander-Iwaniec \(\delta\)-method in \cite{IK04}, but unlike \cite{KMS19} without the conductor-lowering trick (as in Aggarwal \cite{Agg21}).

The goal of this paper is to study nonlinear exponential twists of \(GL_2 \times GL_2\) automorphic forms. More precisely, let \(f\) and \(g\) be either holomorphic or Maass cusp forms for \(SL_2(\mathbb{Z})\) with normalized Fourier coefficients \(\lambda_f(n)\) and \(\lambda_g(n)\), respectively. Define
\[
S(X, t) = \sum_{n=1}^{\infty} \lambda_f(n)\lambda_g(n)e\left(t\varphi\left(\frac{n}{X}\right)\right) V\left(\frac{n}{X}\right).
\]

Our main result states as follows.

**Theorem 1.1.** Let \(\varphi(x) = \alpha \log x\) or \(\alpha x^\beta\) \((\beta \in (0, 1) \setminus \{1/2, 3/4\}, \alpha \in \mathbb{R}\setminus\{0\})\). Let \(V(x) \in C^\infty_c(1, 2)\) with total variation \(\text{Var}(V) \ll 1\) and satisfying the condition
\[
V^{(j)}(x) \ll_j \Delta^j
\]
for any integer \(j \geq 0\) with \(\Delta \ll t^{1/2-\varepsilon}\) for any \(\varepsilon > 0\). Then we have
\[
S(X, t) \ll_{f, g, \varphi, \varepsilon} t^{2/5}X^{3/4+\varepsilon}
\]
for \(t^{8/5} < X < t^{12/5}\).

**Remark 1.** The assumption \(\Delta \ll t^{1/2-\varepsilon}\) arises when we use stationary phase analysis for certain oscillatory integral in the proof (see (7.6)). For the sake of simplicity, we have restricted \(f\) and
$g$ to be on the full modular group. In fact, Theorem 1 can be similarly extended to modular
forms of arbitrary level and nebentypus without taking much effort.

Since the test function $V$ in Theorem 1.1 allows oscillations, we can remove it from the sum.

**Corollary 1.2.** Same notation and assumptions as in Theorem 1.1. We have
\[ \sum_{X<n \leq 2X} \lambda_f(n) \lambda_g(n) e \left( t \varphi \left( \frac{n}{X} \right) \right) \ll_{f,g,\varphi,\varepsilon} t^{2/5} X^{3/4+\varepsilon} \]
for $t^{8/5} < X < t^{12/5}$.

A special case of Theorem 1.1 is that $t = X^\beta$ and $\varphi(x) = \alpha x^\beta$ ($\alpha \in \mathbb{R}, 0 < \beta < 1, \beta \neq 1/2, 3/4$). Then Corollary 1.2 implies

**Corollary 1.3.** For any $\alpha \in \mathbb{R} \setminus \{0\}$, we have
\[ \sum_{n \leq X} \lambda_f(n) \lambda_g(n) e(\alpha n^\beta) \ll_{f,g,\alpha,\beta,\varepsilon} X^{3/4+2\beta/5+\varepsilon} \]
for $5/12 < \beta < 5/8, \beta \neq 1/2$.

For $15/32 < \beta < 5/8, \beta \neq 1/2$, Corollary 1.3 improves the estimate $O_{f,g,\alpha,\beta,\varepsilon}(X^{2\beta})$ by Czarnecki [Cza16].

Theorem 1.1 also admits an application in bounding Rankin-Selberg $L$-functions on the critical line. We recall
\[ L(s, f \otimes g) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n) \lambda_g(n)}{n^s} \]
for $\Re s > 1$. The convexity bound in the $t$-aspect is $L(1/2 + it, f \otimes g) \ll t^{1+\varepsilon}$ and recently Acharya, Sharma and Singh [ASS20] proved the subconvexity bound $O_{f,g,\varepsilon}(t^{1-1/16+\varepsilon})$ by using the Duke-Friedlander-Iwaniec $\delta$-method given in [IK04] together with the conductor-lowering trick due to Munshi [Mun15]. An application of the approximate functional equation implies
\[ L \left( \frac{1}{2} + it, f \otimes g \right) \ll \sup_{N \ll t^{1+\varepsilon}} \left| \frac{1}{N} \sum_{n=1}^{\infty} \lambda_f(n) \lambda_g(n) n^{-it} V \left( \frac{n}{N} \right) \right| + t^{-100}. \]

We demonstrate that the conductor-lowering trick in Acharya, Sharma and Singh’s proof can be removed and applying Theorem 1.1 with $\varphi(x) = -(\log x)/2\pi$, we improve the result of Acharya, Sharma and Singh.

**Corollary 1.4.** We have
\[ L(1/2 + it, f \otimes g) \ll_{f,g,\varepsilon} (1 + |t|)^{9/10+\varepsilon}. \]

The best record bound for $L(1/2 + it, f \otimes g)$ is the Weyl type bound $L(1/2 + it, f \otimes g) \ll (1 + |t|)^{2/3+\varepsilon}$ due to Blomer, Jana and Nelson [BJN21] by combining in a substantial way representation theory, local harmonic analysis, and analytic number theory. Bernstein and Reznikov showed the bound $(1 + |t|)^{5/6+\varepsilon}$ in [BR10] (see Remarks 7.2.2.2).

Now we consider another application of Theorem 1.1. Let $L(s, F)$ be an $L$-function of degree $d$ with coefficients $A_F(1) = 1$, $A_F(n) \in \mathbb{C}$. It is a fundamental problem to prove an asymptotic formula for the sum
\[ A(X, F) = \sum_{n \leq X} \lambda_F(n). \]
Let $(\mu_1, \ldots, \mu_d, F)$ be the Satake parameter of $F$ at $\infty$. Assume $L_\infty(s, F) = \prod_{1 \leq j < d} \Gamma_{\mathbb{R}}(s - \mu_j, F)$ does not have poles for $\Re(s) > 1/2 + 1/d$, where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$. Under the Ramanujan-Petersson conjecture $\lambda_F(n) \ll n^\varepsilon$, Friedlander and Iwaniec \cite{FI05} established the following identity which relates $A(X, F)$ to its dual sum $B(X, N)$

$$A(X, F) = \text{Res}_{s=1} \frac{L(s, F)}{s} X + c_F X \frac{d+1}{2s} B(X, N) + O \left( N^{-\frac{1}{2}} X^{\frac{d+1}{2s} + \varepsilon} \right),$$

(1.4)

where $c_F$ is some constant depending on the form $F$ only and Let

$$B(X, N) = \sum_{n \leq N} \lambda_F(n) n^{-\frac{d+1}{2s}} \cos(2\pi d (nX)^{1/d}).$$

In particular, by estimating the sum $B(X, N)$ trivially and choosing $N = X^{(d-1)/(d+1)}$, Friedlander and Iwaniec showed that

$$A(X, F) = \text{Res}_{s=1} \frac{L(s, F)}{s} X + O \left( X^\frac{d+1}{2s} \right)$$

(1.5)

for any $\varepsilon > 0$. For $\lambda_F(n) = \sum_{\chi \bmod n} \chi(1) \chi_2(n)(n) \chi_3(n_3)$, $\chi_j$ being primitive Dirichlet characters, Friedlander and Iwaniec \cite{FI05} proved an asymptotic formula with the error term $O(X^{1/2-1/150+\varepsilon})$. Recently, for $F = 1 \bmod f$ and $\lambda_F(n) = \sum_{\ell \bmod n} \lambda_f(m)$, where $f$ is a holomorphic cusp form for $SL_2(\mathbb{Z})$, Huang, Lin and Wang \cite{HLW21} proved an asymptotic formula with the error term $O(X^{1/2-4/739+\varepsilon})$. For $F = 1 \bmod \text{sym}^2 F$ and $\lambda_F(n) = \sum_{\ell \bmod n} \lambda_f(m)^2$, where $f$ is a Hecke-holomorphic or Hecke-Maass cusp form for $SL_2(\mathbb{Z})$, Huang \cite{Hua21} proved an asymptotic formula with the error term $O(X^{3/5-1/560+\varepsilon})$. Under the Ramanujan-Petersson conjecture Lin and the second author \cite{LS21} considered the $GL_3 \times GL_2$ case and proved the bound $O(X^{5/7-1/364+\varepsilon})$ for $F = \pi \otimes f$, where $\pi$ is a Hecke–Maass cusp form for $SL_3(\mathbb{Z})$ and $f$ is a holomorphic or Maass cusp form for $SL_2(\mathbb{Z})$.

As an application of Theorem \ref{thm:main} we improve (1.5) for $F$ being certain $GL_5$ Eisenstein series, namely when $F = 1 \bmod (f \times g)$ and $L(s, F) = \zeta(s) L(s, f \times g)$. For simplification, we consider the holomorphic case. In fact, our argument holds also for Maass cusp forms under the Ramanujan-Petersson conjecture. Now let $f$ and $g$ be holomorphic Hecke cusp forms for $SL_2(\mathbb{Z})$ of weight $k$, $\kappa$, with $k \geq \kappa \geq 12$, with normalized Fourier coefficients $\lambda_f(n)$ and $\lambda_g(n)$, respectively. For $\Re(s) > 1$, we define

$$L(s, 1 \bmod (f \times g)) = \zeta(s) L(s, f \times g) = \zeta(s) \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n) \lambda_g(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}(f \times g)}(n)}{n^s} X^{\frac{d+1}{2s}},$$

where $\lambda_{\text{sym}(f \times g)}(n) := \sum_{\ell \bmod n} \lambda_f(r) \lambda_g(r)$. Note that (1.5) reads

$$\sum_{n \leq X} \lambda_{\text{sym}(f \times g)}(n) = L(1, f \times g) X + O(X^{\frac{k+1}{2s}}).$$

We shall prove the following result.

**Corollary 1.5.** Let $f$ and $g$ be holomorphic Hecke cusp forms for $SL_2(\mathbb{Z})$ of weight $k$ and $\kappa$ with $12 \leq \kappa \leq k$, with normalized Fourier coefficients $\lambda_f(n)$ and $\lambda_g(n)$, respectively. Assume $f \perp g$. Then we have

$$\sum_{n \leq X} \lambda_{\text{sym}(f \times g)}(n) = L(1, f \times g) X + O(X^{\frac{k}{2s}} + \frac{1}{350}X^{\varepsilon}).$$
for any $\varepsilon > 0$.

**Remark 2.** At the end of the proof of Corollary 1.5 we will use the exponent pair $(\frac{13}{194} + \varepsilon, \frac{76}{97} + \varepsilon)$ which is a consequence of Bourgain’s exponential pair in [Bou17] and the A-process in the theory of exponential pairs. This is the best known exponent pair we find for our problem. We essentially need to choose an exponent pair $(p, q)$ to minimize $\frac{38+33p−28q}{58+48p−43q}$.

The paper is organized as follows. In Section 2, we provide a quick sketch and key steps of the proof. In Section 3, we review some basic materials of automorphic forms on $\text{GL}_2$ and estimates on exponential integrals. Sections 4 and 7 give details of the proof for Theorem 1.1 and in Sections 5 and 6 we complete the proofs for Corollaries 1.2 and 1.5, respectively.

**Notation.** Throughout the paper, the letters $q$, $m$, and $n$, with or without subscript, denote integers. The letters $\varepsilon$ and $A$ denote arbitrarily small and large positive constants, respectively, not necessarily the same at different occurrences. We use $A \asymp B$ to mean that $c_1 B \leq |A| \leq c_2 B$ for some positive constants $c_1$ and $c_2$. The symbol $\ll_{a,b,c}$ denotes that the implied constant depends at most on $a$, $b$ and $c$, and $q \sim C$ means $C < q \leq 2C$.

2. **OUTLINE OF THE PROOF**

In this section, we provide a quick sketch of the proof for Theorem 1.1. Suppose we are working with the following sum

\[ S = \sum_{n \sim X} \lambda_f(n)\lambda_g(n) e \left( t\varphi \left( \frac{n}{X} \right) \right). \]

The first step is writing

\[ S = \sum_{n \sim X} \lambda_f(n) \sum_{m \sim X} \lambda_g(m) e \left( t\varphi \left( \frac{m}{X} \right) \right) \delta(m - n, 0), \]

and using the $\delta$-method to detect the Kronecker delta symbol $\delta(m - n, 0)$. As in [LS21], we use the Duke-Friedlander-Iwaniec’s $\delta$-method [4,3] to write

\[
S = \frac{1}{Q} \int_{-X^\varepsilon}^{X^\varepsilon} \sum_{q \sim Q} \frac{1}{q} \sum_{a \mod q} \sum_{n \sim X} \lambda_f(n) \left( -\frac{na}{q} \right) e \left( -\frac{n\zeta}{qQ} \right) \delta(m - n, 0),
\]

\[
\sum_{m \sim X} \lambda_g(m) \left( \frac{ma}{q} \right) e \left( t\varphi \left( \frac{m}{X} \right) + \frac{m\zeta}{qQ} \right) d\zeta,
\]

(2.1)

where the $\ast$ in the sum over $a$ means that the sum is restricted to $(a, q) = 1$.

Next, we use the $\text{GL}_2$ Voronoi summation formulas to dualize the $m$- and $n$-sums. The $m$-sum can be transformed into the following

\[
\sum_{m \sim X} \lambda_g(m) \left( \frac{ma}{q} \right) e \left( t\varphi \left( \frac{m}{X} \right) + \frac{m\zeta}{qQ} \right) V \left( \frac{m}{X} \right)
\]

\[
\leftrightarrow \frac{X}{Qt^{1/2}} \sum_{m \sim Q^2t^2/X} \sum_{\pm} \lambda_g(m) e \left( -\frac{ma}{q} \right) \Phi \left( m, q, \zeta \right),
\]

(2.2)
where

\[ \Phi^\pm(m, q, \zeta) = \int_0^\infty V(y)y^{-1/4}e\left(t \varphi(y) + \frac{\zeta Xy}{qQ} \pm \frac{2\sqrt{mXy}}{q}\right) dy. \]

If we assume for example \( \varphi'(x) > 0 \), then by integration by parts, \( \Phi^+ (m, q, \zeta) \ll X^{-A} \), and we only need to consider the minus sign contribution. Similarly, for the \( n \)-sum, we have

\[ \sum_{n \sim X} \lambda_f(n)e\left(-\frac{n\alpha}{q}, -\frac{n\zeta}{qQ}\right) U\left(\frac{n}{X}\right) \]

\[ \leftrightarrow X^{1/2} \sum_{n \sim X/Q^2} \lambda_f(n)e\left(\frac{n\alpha}{q}\right) \Psi^+ (n, q, \zeta) + O_A(X^{-A}), \tag{2.3} \]

where

\[ \Psi^+ (n, q, \zeta) = \int_0^\infty U(y)y^{-1/4}e\left(-\frac{\zeta Xy}{qQ} + \frac{2\sqrt{mXy}}{q}\right) dy. \]

We perform a stationary phase argument to get (note that \( n \sim X/Q^2 \))

\[ \Psi^+ (n, q, \zeta) \asymp \frac{q^{1/2}}{(nX)^{1/4}} \left(\frac{nQ}{Xq}\right) U^2\left(\frac{nQ^2}{X^2}\right) \asymp \frac{Q}{X^{1/2}} \left(\frac{nQ}{q}\right) U^2\left(\frac{nQ^2}{X^2}\right) \]

for some smooth compactly supported function \( U^2(y) \). Then by plugging the dual sums (2.2) and (2.3) back into (2.1) and switching the orders of integration over \( \zeta \) and \( y \), we roughly get

\[ S \approx \frac{X}{Q^{1/2}} \sum_{q \sim Q} \sum_{m \sim Q^{2/3}X} \lambda_g(m) \sum_{n \sim X/Q^2} \lambda_f(n) S(m - n, 0; q) \]

\[ \times \int_0^\infty V(y)y^{-1/4}e\left(t \varphi(y) - \frac{2\sqrt{mXy}}{q}\right) K(y; n, q) dy \tag{2.4} \]

where

\[ K(y; n, q) = \int_{-X^e}^{X^e} U^2\left(\frac{nQ^2}{X^2}\right) e\left(\frac{\zeta Xy}{qQ} + \frac{nQ}{q}\right) d\zeta. \]

We evaluate the integral \( K(y; n, q) \) using the stationary phase method (note that \( n \sim X/Q^2 \))

\[ K(y; n, q) \asymp \frac{n^{1/2}q^{1/2}Q}{X^{3/4}} e\left(\frac{2\sqrt{nXy}}{q}\right) F(y) \asymp \frac{Q}{X^{1/2}} e\left(\frac{2\sqrt{nXy}}{q}\right) F(y) \]

for some smooth compactly supported function \( F(y) \). Hence putting things together and writing the Ramanujan sum \( S(m - n, 0; q) \) as \( \sum_{d|m-n,q} d\mu(q/d) \), \( S \) in (2.4) is roughly equal to

\[ \frac{X^{1/2}}{Q^{1/2}} \sum_{q \sim Q} \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{m \sim Q^{2/3}X} \lambda_g(m) \sum_{n \sim X/Q^2} \lambda_f(n) \mathcal{I}(m, n, q), \]

where

\[ \mathcal{I}(m, n, q) = \int_0^\infty \tilde{V}(y)e\left(t \varphi(y) + \frac{2\sqrt{nXy}}{q} - \frac{2\sqrt{mXy}}{q}\right) dy \tag{2.5} \]
for some smooth compactly supported function $\tilde{V}(y)$. Assume $\varphi(x) = c\log x$ or $cx^\beta$ with $\beta \in (0,1), \beta \neq 1/2$. We apply the stationary phase analysis to the integral $I(m,n,q)$ to get

$$I(m,n,q) \sim e(t\varphi(y_0^2) - D_{y_0}) I^*(m,n,q)$$

where $y_0 = (ct/D)^{1/\beta} \asymp 1$ with $D = 2q^{-1}(mX)^{1/2}$ and

$$I^*(m,n,q) \asymp t^{-1/2}e\left(\frac{2y_0n^{1/2}X^{1/2}}{q}\right).$$

To prepare for an application of the Poisson summation in the $m$-variable, we now apply the Cauchy-Schwarz inequality to smooth the $m$-sum and put the $n$-sum inside the absolute value squared to get

$$S \ll \frac{X^{1/2}}{Qt^{1/2}} \sum_{q \sim Q} \sum_{d \mid q} \frac{1}{d} \left(\sum_{m \sim Q^2t^2/X} |\lambda_g(m)|^2\right)^{1/2} \left(\sum_{m \sim Q^2t^2/X} \sum_{n \sim X/Q^2} \lambda_f(n)I^*(m,n,q)\right)^2 \frac{1}{X^{1/2}}$$

$$\ll t^{1/2} \sum_{q \sim Q} \sum_{d \mid q} \frac{1}{d} \left(\sum_{m \sim Q^2t^2/X} \sum_{n \sim X/Q^2} \lambda_f(n)I^*(m,n,q)\right)^2 \frac{1}{X^{1/2}}.$$

**Remark 3.** If we open the absolute value squared, by the Rankin-Selberg estimate for $\lambda_f(n)$ and the trivial estimate $I^*(m,n,q) \ll t^{-1/2}$, the contribution from the diagonal term $n = n'$ is given by

$$S_{\text{diag}} \ll t^{1/2} \sum_{q \sim Q} \sum_{d \mid q} \frac{1}{d} \left(\sum_{m \sim Q^2t^2/X} \sum_{n \sim X/Q^2} |\lambda_f(n)|^2 |I^*(m,n,q)|^2\right)^{1/2}$$

$$\ll Q^{3/2} t,$$

which will be fine for our purpose (i.e., $S_{\text{diag}} = o(X)$) as long as $Q \ll (X/t)^{3/2}$.

Note that the oscillation in the $m$-variable of $I^*(m,n,q)$ in (2.5) is of size $2y_0n^{1/2}X^{1/2}/q \approx X/Q^2$. So opening the absolute value squared and applying the Poisson summation formula in the $m$-variable, we have

$$\sum_{m \sim Q^2t^2/X \atop m \equiv n \mod d} I^*(m,n,q)I^*(m,n',q) \leftrightarrow \frac{Q^2t^2}{dX} \sum_{\tilde{m} \equiv \frac{dX/Q^2}{Q^2t^2/X}} \mathcal{H}\left(\tilde{m}Q^2t^2\right),$$

where

$$\mathcal{H}(x) = \int_{\mathbb{R}} I^*(Q^2t^2\xi/X,n,q) \overline{I^*(Q^2t^2\xi/X,n',q)} e(-x\xi) \, d\xi. \quad (2.7)$$

The contribution to $S$ from the zero-frequency $\tilde{m} = 0$ will roughly correspond to the diagonal contribution $S_{\text{diag}}$ in (2.6). For the non-zero frequencies from the terms with $\tilde{m} \neq 0$, we note that by performing stationary phase analysis, when $|x|$ is “large”, the expected estimate for the triple integral $\mathcal{H}(x)$ in (2.7) is

$$\mathcal{H}(x) \ll t^{-1/2} \cdot t^{-1/2} \cdot |x|^{-1/2}, \quad (2.8)$$
which comes from the square-root cancellation of the two inner integrals and the square-root cancellation in the \( \xi \)-variable. Note that this estimate does not hold for “small” \( |x| \). In fact, for these exceptional cases the “trivial” bound \( \mathcal{H}(x) \ll t^{-1/2} \cdot t^{-1/2} \) will suffice for our purpose. (These are the content of Lemma 4.2). We ignore these exceptions and plug the expected estimate (2.8) for \( \mathcal{H}(x) \) into \( S \). It turns out that the non-zero frequencies contribution \( S_{\text{off-diag}} \) from \( \tilde{m} \neq 0 \) to \( S \) is given by

\[
S_{\text{off-diag}} \ll t^{1/2} \sum_{q \sim Q} \sum_{d | q} d \left( \sum_{n \sim X/q^2} |\lambda_f(n)|^2 \sum_{n' \sim X/q^2 \atop n' \equiv n \mod d} \frac{Q^2 t^2}{dX} \sum_{0 \neq \tilde{m} \ll dX^2/(Q t^2)} \frac{d^{1/2} X^{1/2}}{|\tilde{m}|^{1/2}} \right)^{1/2}
\]

\[
\ll \frac{X^{5/4}}{Q} + X^{3/4} Q^{1/2}
\]

\[
\ll \frac{X^{5/4}}{Q}
\]

provided that \( Q < X^{1/3} \). Hence combining this with the diagonal contribution \( S_{\text{diag}} \) in (2.6), we get

\[
S \ll Q^{3/2} t + \frac{X^{5/4}}{Q}.
\]

By choosing \( Q = X^{1/2} / t^{2/5} \) we obtain \( S \ll t^{2/5} X^{3/4} \) provided that \( X < t^{12/5} \), which improves over the trivial bound \( S \ll X \) as long as \( t^{8/5} \ll X \).

3. Preliminaries

First we recall some basic results on automorphic forms for \( GL_2 \).

3.1. Holomorphic cusp forms for \( GL_2 \). Let \( f \) be a holomorphic cusp form of weight \( \kappa \) for \( SL_2(\mathbb{Z}) \) with Fourier expansion

\[
f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(\kappa - 1)/2} e(nz)
\]

for \( \text{Im } z > 0 \), normalized such that \( \lambda_f(1) = 1 \). By the Ramanujan-Petersson conjecture proved by Deligne [Del74], we have \( \lambda_f(n) \ll \tau(n) \ll n^{\varepsilon} \) with \( \tau(n) \) being the divisor function.

For \( h(x) \in C_c(0, \infty) \), we set

\[
\Phi_h(x) = 2\pi i^\kappa \int_0^\infty h(y) J_{\kappa-1}(4\pi\sqrt{xy}) dy,
\]

where \( J_{\kappa-1} \) is the usual \( J \)-Bessel function of order \( \kappa - 1 \). We have the following Voronoi summation formula (see [KMV02, Theorem A.4]).

**Lemma 3.1.** Let \( q \in \mathbb{N} \) and \( a \in \mathbb{Z} \) be such that \( (a, q) = 1 \). For \( X > 0 \), we have

\[
\sum_{n=1}^{\infty} \lambda_f(n) e \left( \frac{an}{q} \right) h \left( \frac{n}{X} \right) = \frac{X}{q} \sum_{n=1}^{\infty} \lambda_f(n) e \left( -\frac{\overline{a} n}{q} \right) \Phi_h \left( \frac{nX}{q^2} \right),
\]

where \( \overline{a} \) denotes the multiplicative inverse of \( a \) modulo \( q \).

The function \( \Phi_h(x) \) has the following asymptotic expansion when \( x \gg 1 \) (see [LS21, Lemma 3.2]).
Lemma 3.2. For any fixed integer \( J \geq 1 \) and \( x \gg 1 \), we have
\[
\Phi_h(x) = x^{-1/4} \int_0^\infty h(y) y^{-1/4} \sum_{j=0}^J c_je^{(2\sqrt{xy})} + d_j e^{-2\sqrt{xy}} dy + O_{\kappa,J} \left( x^{-J/2-3/4} \right),
\]
where \( c_j \) and \( d_j \) are constants depending on \( \kappa \).

3.2. Maass cusp forms for \( \text{GL}_2 \). Let \( f \) be a Hecke-Maass cusp form for \( \text{SL}_2(\mathbb{Z}) \) with Laplace eigenvalue \( 1/4 + \mu^2 \). Then \( f \) has a Fourier expansion
\[
f(z) = \sqrt{y} \sum_{n \neq 0} \lambda_f(n) K_{i\mu}(2\pi |n| y) e(nz),
\]
where \( K_{i\mu} \) is the modified Bessel function of the third kind. The Fourier coefficients satisfy
\[
\lambda_f(n) \ll n^\vartheta,
\]
where, here and throughout the paper, \( \vartheta \) denotes the exponent towards the Ramanujan conjecture for \( \text{GL}_2 \) Maass forms. The Ramanujan conjecture states that \( \vartheta = 0 \) and the current record due to Kim and Sarnak \([KS03]\) is \( \vartheta = 7/64 \). We also need the following average bound (see for instance \([Mur85\), Lemma 1])
\[
\sum_{n \leq X} |\lambda_f(n)|^2 = c_f X + O(X^{3/5}).
\]

For \( h(x) \in C_c^\infty(0, \infty) \), we define the integral transforms
\[
\Phi_h^+(x) = \frac{-\pi}{\sin(\pi i \mu)} \int_0^\infty h(y) (J_{2i\mu}(4\pi \sqrt{xy}) - J_{-2i\mu}(4\pi \sqrt{xy})) dy,
\]
\[
\Phi_h^-(x) = 4\varepsilon_f \cosh(\pi \mu) \int_0^\infty h(y) K_{2i\mu}(4\pi \sqrt{xy}) dy,
\]
where \( \varepsilon_f \) is an eigenvalue under the reflection operator. We have the following Voronoi summation formula (see \([KMV02\), Theorem A.4]).

Lemma 3.3. Let \( q \in \mathbb{N} \) and \( a \in \mathbb{Z} \) be such that \( (a,q) = 1 \). For \( X > 0 \), we have
\[
\sum_{n=1}^{\infty} \lambda_f(n) e \left( \frac{an}{q} \right) h \left( \frac{n}{X} \right) = \frac{X}{q} \sum_{n=1}^{\infty} \sum_{\pm} \lambda_f(n) e \left( \mp \frac{an}{q} \right) \Phi_h^\pm \left( \frac{nX}{q^2} \right),
\]
where \( \overline{a} \) denotes the multiplicative inverse of \( a \) modulo \( q \).

For \( x \gg 1 \), we have (see (3.8) in \([LS21]\))
\[
\Phi_h^-(x) \ll_{\mu,A} x^{-A}.
\]

For \( \Phi_h^+(x) \) and \( x \gg 1 \), we have a similar asymptotic formula as for \( \Phi_h(x) \) in the holomorphic case (see \([LS21\), Lemma 3.4]).

Lemma 3.4. For any fixed integer \( J \geq 1 \) and \( x \gg 1 \), we have
\[
\Phi_h^+(x) = x^{-1/4} \int_0^\infty h(y) y^{-1/4} \sum_{j=0}^J c_je^{(2\sqrt{xy})} + d_j e^{-2\sqrt{xy}} dy + O_{\mu,J} \left( x^{-J/2-3/4} \right),
\]
where \( c_j \) and \( d_j \) are some constants depending on \( \mu \).
Remark 4. For $x \gg X^\varepsilon$, we can choose $J$ sufficiently large so that the contribution from the $O$-terms in Lemmas 3.2 and 3.4 is negligible. For the main terms we only need to analyze the leading term $j = 1$, as the analysis of the remaining lower order terms is the same and their contribution is smaller compared to that of the leading term.

3.3. Estimates for exponential integrals. Let

$$I = \int_{\mathbb{R}} w(y) e^{i\varphi(y)} dy.$$ 

Firstly, we have the following estimates for exponential integrals (see [BKY13, Lemma 8.1] and [AHLQ20, Lemma A.1]).

Lemma 3.5. Let $w(x)$ be a smooth function supported on $[a, b]$ and $\varphi(x)$ be a real smooth function on $[a, b]$. Suppose that there are parameters $Q, U, Z, R > 0$ such that

$$\varphi^{(i)}(x) \ll_i Y/Q^i, \quad w^{(j)}(x) \ll_j Z/U^j,$$

for $i \geq 2$ and $j \geq 0$, and

$$|\varphi'(x)| \geq R.$$ 

Then for any $A \geq 0$ we have

$$I \ll_A (b-a) Z \left( \frac{Y}{R^2 Q^2} + \frac{1}{R Q} + \frac{1}{R U} \right)^A.$$ 

Next, we need the following evaluation for exponential integrals which are Lemma 8.1 and Proposition 8.2 of [BKY13] in the language of inert functions (see [KPY19, Lemma 3.1]).

Let $\mathcal{F}$ be an index set, $Y : \mathcal{F} \to \mathbb{R}_{\geq 1}$ and under this map $T \mapsto Y_T$ be a function of $T \in \mathcal{F}$. A family $\{w_T\}_{T \in \mathcal{F}}$ of smooth functions supported on a product of dyadic intervals in $\mathbb{R}_{d_{\geq 0}}^d$ is called $Y$-inert if for each $j = (j_1, \ldots, j_d) \in \mathbb{Z}_{d_{\geq 0}}^d$ we have

$$C(j_1, \ldots, j_d) = \sup_{T \in \mathcal{F}} \sup_{(y_1, \ldots, y_d) \in \mathbb{R}_{d_{\geq 0}}^d} |Y_T^{j_1} \cdots y_d^{j_d} w_T^{(j_1, \ldots, j_d)}(y_1, \ldots, y_d)| < \infty.$$ 

Lemma 3.6. Suppose that $w = w_T(y)$ is a family of $Y$-inert functions, with compact support on $[Z, 2Z]$, so that $w^{(j)}(y) \ll (Z/Y)^{-j}$. Also suppose that $\varphi$ is smooth and satisfies $\varphi^{(j)}(y) \ll H/Z^j$ for some $H/X^2 \geq R \geq 1$ and all $y$ in the support of $w$.

1. If $|\varphi'(y)| \gg H/Z$ for all $y$ in the support of $w$, then $I \ll A Z R^{-A}$ for $A$ arbitrarily large.
2. If $\varphi''(y) \gg H/Z^2$ for all $y$ in the support of $w$, and there exists $y_0 \in \mathbb{R}$ such that $\varphi'(y_0) = 0$ (note $y_0$ is necessarily unique), then

$$I = \frac{e^{i\varphi(y_0)}}{\sqrt{\varphi''(y_0)}} F(y_0) + O_A(Z R^{-A}), \quad \text{(3.6)}$$

where $F(y_0)$ is an $Y$-inert function (depending on $A$) supported on $y_0 \approx Z$.

We also need the second derivative test (see [Hux96, Lemma 5.1.3]).

Lemma 3.7. Let $\varphi(x)$ be real and twice differentiable on the open interval $[a, b]$ with $\varphi''(x) \gg \lambda_0 > 0$ on $[a, b]$. Let $w(x)$ be real on $[a, b]$ and let $V_0$ be its total variation on $[a, b]$ plus the maximum modulus of $w(x)$ on $[a, b]$. Then

$$I \ll \frac{V_0}{\sqrt{\lambda_0}}.$$
4. Proof of the main theorem

In this section, we provide the details of the proof for Theorem 1.1. Recall

\[ S(X, t) = \sum_{n=1}^{\infty} \lambda_f(n) \lambda_g(n) e \left( t\varphi \left( \frac{n}{X} \right) \right) V \left( \frac{n}{X} \right), \]  

where \( V(x) \in C^{\infty}_c(1, 2) \) with total variation \( \text{Var}(V) \ll 1 \) and satisfying (1.3) that \( V^{(j)}(x) \ll j \Delta^j \) for any integer \( j \geq 0 \) with \( \Delta \ll t^{1/2-\varepsilon} \). Without loss of generality, we assume that the function \( \varphi \) satisfies

\[ \varphi'(x) > 0, \quad \varphi''(x) \gg 1. \]  

(The case \( \varphi'(x) < 0 \) can be analyzed analogously.)

4.1. Applying DFI’s \( \delta \)-method. Define \( \delta : \mathbb{Z} \to \{0, 1\} \) with \( \delta(0) = 1 \) and \( \delta(n) = 0 \) for \( n \neq 0 \). As in [LS21], we will use a version of the circle method of Duke, Friedlander and Iwaniec (see [IK04, Chapter 20]) which states that for any \( n \in \mathbb{Z} \) and \( Q \in \mathbb{R}^+ \), we have

\[ \delta(n) = \frac{1}{Q} \sum_{q \sim Q} \sum_{a \mod q}^* e \left( \frac{na}{q} \right) \int_{\mathbb{R}} g(q, \zeta) e \left( \frac{n\zeta}{qQ} \right) d\zeta \]  

where the \( * \) on the sum indicates that the sum over \( a \) is restricted to \( (a, q) = 1 \). The function \( g \) has the following properties (see (20.158) and (20.159) of [IK04] and Lemma 15 of [Hua21])

\[ g(q, \zeta) \ll |\zeta|^{-A}, \quad g(q, \zeta) = 1 + h(q, \zeta) \quad \text{with} \quad h(q, \zeta) = O \left( \frac{Q}{q} \left( \frac{q}{Q} + |\zeta| \right)^A \right) \]  

for any \( A > 1 \) and

\[ \zeta^j \frac{\partial^j}{\partial \zeta^j} g(q, \zeta) \ll (\log Q) \min \left\{ \frac{Q}{q}, \frac{1}{|\zeta|} \right\}, \quad j \geq 1. \]  

In particular the first property in (4.4) implies that the effective range of the integration in (4.3) is \([ -X^{\varepsilon}, X^{\varepsilon} ]\).

We write (4.1) as

\[ S(X, t) = \sum_{n=1}^{\infty} \lambda_f(n) U \left( \frac{n}{X} \right) \sum_{m=1}^{\infty} \lambda_g(m) e \left( t\varphi \left( \frac{m}{X} \right) \right) V \left( \frac{m}{X} \right) \delta(m - n), \]

where \( U(x) \in C^{\infty}_c(1/2, 5/2) \) satisfying \( U(x) = 1 \) for \( x \in [1, 2] \) and \( U^{(j)}(x) \ll_j 1 \) for any integer \( j \geq 0 \). Plugging the identity (4.3) for \( \delta(m - n) \) in and exchanging the order of integration and summations, we get

\[ S(X, t) = \frac{1}{Q} \int_{\mathbb{R}} \sum_{q \sim Q} \frac{g(q, \zeta)}{q} \sum_{a \mod q}^* \left\{ \sum_{n=1}^{\infty} \lambda_f(n) e \left( -\frac{na}{q} \right) U \left( \frac{n}{X} \right) e \left( -\frac{n\zeta}{qQ} \right) \right\} \]  

\[ \left\{ \sum_{m=1}^{\infty} \lambda_g(m) e \left( \frac{ma}{q} \right) V \left( \frac{m}{X} \right) e \left( t\varphi \left( \frac{m}{X} \right) + \frac{m\zeta}{qQ} \right) \right\} d\zeta. \]
Note that the contribution from $|\zeta| \leq X^{-B}$ is negligible for $B > 0$ sufficiently large. Moreover, by the first property in (4.4), we can restrict $\zeta$ in the range $|\zeta| \leq X^{\varepsilon}$ up to an negligible error. So we can insert a smooth partition of unity for the $\zeta$-integral and get

$$S(X, t) = \sum_{X^{-B} \ll \Xi \ll X^{\varepsilon}} \frac{1}{Q} \int_{\mathbb{R}} W \left( \frac{\zeta}{X} \right) \sum_{q \sim Q} \frac{g(q, \zeta)}{q} \sum_{m \text{ mod } q}^{*} \left\{ \sum_{n=1}^{\infty} \lambda_f(n) e \left( -\frac{n a}{q} \right) U \left( \frac{n}{X} \right) e \left( -\frac{n \zeta}{qQ} \right) \right\}$$

$$\left\{ \sum_{m=1}^{\infty} \lambda_g(m) e \left( \frac{ma}{q} \right) V \left( \frac{m}{X} \right) e \left( t \varphi \left( \frac{m}{X} \right) + \frac{m \zeta}{qQ} \right) \right\} \frac{d\zeta}{\zeta} + O(X^{-A}),$$

where $\tilde{W}(x) \in \mathcal{C}^{\infty}_{\varepsilon}(1, 2)$, satisfying $\tilde{W}(j)(x) \ll 1$ for any integer $j \geq 0$.

Next we break the $q$-sum $\sum_{q \sim Q}$ into dyadic segments $q \sim C$ with $1 \ll C \ll Q$ and write

$$S(X, t) = \sum_{X^{-B} \ll \Xi \ll X^{\varepsilon}} \sum_{1 \ll C \ll Q} \mathcal{S}(C, \Xi) + O(X^{-A}), \quad (4.6)$$

where $\mathcal{S}(C, \Xi) = \mathcal{S}(X, t, C, \Xi)$ is

$$\mathcal{S}(C, \Xi) = \frac{1}{Q} \int_{\mathbb{R}} W \left( \frac{\zeta}{X} \right) \sum_{q \sim C} \frac{g(q, \zeta)}{q} \sum_{m \text{ mod } q}^{*} \left\{ \sum_{n=1}^{\infty} \lambda_f(n) e \left( -\frac{n a}{q} \right) U \left( \frac{n}{X} \right) e \left( -\frac{n \zeta}{qQ} \right) \right\}$$

$$\left\{ \sum_{m=1}^{\infty} \lambda_g(m) e \left( \frac{ma}{q} \right) V \left( \frac{m}{X} \right) e \left( t \varphi \left( \frac{m}{X} \right) + \frac{m \zeta}{qQ} \right) \right\} \frac{d\zeta}{\zeta}. \quad (4.7)$$

Without loss of generality, for the $\zeta$-integral, we only consider the contribution from $\zeta \geq 0$ (the contribution from $\zeta \leq 0$ can be estimated similarly). By abuse of notation, we still write the contribution from $\zeta \geq 0$ as $\mathcal{S}(C, \Xi)$. We now proceed to estimate $\mathcal{S}(C, \Xi)$.

### 4.2. Voronoi summations.

In what follows, we dualize the $n$-and $m$-sums in (4.7) using Voronoi summation formulas.

We first consider the sum over $m$. Depending on whether $f$ is holomorphic or Maass, we apply Lemma 3.1 or Lemma 3.3 respectively with $h_1(y) = V(y)e \left( t \varphi(y) + \zeta Xy/qQ \right)$, to transform the $m$-sum in (4.7) into

$$\frac{X}{q} \sum_{m=1}^{\infty} \lambda_g(m) e \left( \pm \frac{ma}{q} \right) \Phi_{h_1}^{\pm} \left( \frac{mX}{q^2} \right), \quad (4.8)$$

where if $g$ is holomorphic, $\Phi_{h_1}^{+}(x) = \Phi_{h_1}(x)$ with $\Phi_{h_1}(x)$ given by (3.1) and $\Phi_{h_1}^{-}(x) = 0$, while for $g$ a Hecke–Maass cusp form, $\Phi_{h_1}^{\pm}(x)$ are given by (3.4).

Similarly, we apply Lemma 3.1 or Lemma 3.3 with $h_2(y) = U(y)e \left( -\zeta Xy/qQ \right)$ to transform the $n$-sum in (4.7) into

$$\frac{X}{q} \sum_{n=1}^{\infty} \lambda_f(n) e \left( \pm \frac{nn^2}{q} \right) \Phi_{h_2}^{\pm} \left( \frac{nX}{q^2} \right), \quad (4.9)$$

where if $f$ is holomorphic, $\Phi_{h_2}^{+}(x) = \Phi_{h_2}(x)$ with $\Phi_{h_2}(x)$ given by (3.1) and $\Phi_{h_2}^{-}(x) = 0$, while for $f$ a Hecke–Maass cusp form, $\Phi_{h_2}^{\pm}(x)$ are given by (3.4).
As is typical in applying the $\delta$-method, we assume that
\[ Q < X^{1/2-\varepsilon}. \]  
Then we have $mX/q^2 \gg X^\varepsilon$ and $nX/q^2 \gg X^\varepsilon$. In particular, by (3.5), the contributions from $\Phi_{b_1}(mX/q^2)$ and $\Phi_{b_2}(nX/q^2)$ are negligible. For $\Phi_{b_1}(mX/q^2)$ and $\Phi_{b_2}(nX/q^2)$, we apply Lemma 3.2, Lemma 3.4 and Remark 4 and find that the sum (4.8) is asymptotically equal to
\[ \frac{X^{3/4}}{q^{1/2}} \sum_{m=1}^{\infty} \frac{\lambda_g(m)}{m^{1/4}} e \left( -\frac{ma}{q} \right) \int_0^\infty V(y)y^{-1/4}e \left( t\varphi(y) + \frac{\zeta_Xy}{qQ} \pm \frac{2\sqrt{mXy}}{q} \right) dy, \]  
and the sum (4.9) is asymptotically equal to
\[ \frac{X^{3/4}}{q^{1/2}} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1/4}} e \left( \frac{n\bar{a}}{q} \right) \int_0^\infty U(y)y^{-1/4}e \left( -\frac{\zeta_Xy}{qQ} \pm \frac{2\sqrt{nXy}}{q} \right) dy. \]
Note that by the assumption (4.2), the first derivative of the phase function of the exponential function in (4.11) in the plus case is
\[ t\varphi'(y) + \frac{\zeta_X}{qQ} + \frac{\sqrt{mX/y}}{q} \gg X^\varepsilon. \]
By applying integration by parts repeatedly, one finds that the contribution from the plus case is negligible. Similarly, the contribution from the minus case in (4.12) is negligible.

Assembling the above results, $\mathcal{S}(C, \Xi)$ in (4.7) is asymptotically equal to
\[ \frac{X^{3/2}}{Q} \sum_{q\sim C} q^{-2} \sum_{m=1}^{\infty} \frac{\lambda_g(m)}{m^{1/4}} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1/4}} S(m-n, 0; q) \mathcal{I}(m, n, q, \Xi), \]  
where
\[ \mathcal{I}(m, n, q, \Xi) = \int_0^\infty W \left( \frac{\zeta_X}{q} \right) g(q, \zeta) \Phi(m, q, \zeta) \Psi(n, q, \zeta) d\zeta \]
with
\[ \Phi(m, q, \zeta) = \int_0^\infty V(y)y^{-1/4}e \left( t\varphi(y) + \frac{\zeta_Xy}{qQ} - \frac{2\sqrt{mXy}}{q} \right) dy \]
and
\[ \Psi(n, q, \zeta) = \int_0^\infty U(y)y^{-1/4}e \left( -\frac{\zeta_Xy}{qQ} + \frac{2\sqrt{nXy}}{q} \right) dy. \]
Note that for $\triangle < t^{1/2-\varepsilon}$, defined in (1.3), by Lemma 3.5, the integral $\Phi(m, q, \zeta)$ is negligibly small unless $\sqrt{mX/q} \ll X^\varepsilon \max \{ t, X\Xi/qQ \}$. Thus we only need to consider those $m$ in the range $m \ll X^\varepsilon \max \{ C^2t^2/X, X\Xi^2/Q^2 \}$. Similarly, up to a negligible error, we only need to consider those $n$ in the range $n \asymp X\Xi^2/Q^2$. Making smooth partitions of unity into dyadic segments to the sums over $m$ and $n$ in (4.13), we arrive at
\[ \mathcal{S}(C, \Xi) \ll \sum_{M \asymp X^\varepsilon \max \{ C^2t^2/X, X\Xi^2/Q^2 \} \text{ dyadic}} \sum_{N \sim X\Xi^2/Q^2 \text{ dyadic}} |T|, \]  
where
\[ T = \left( \sum_{m=1}^{\infty} \frac{\lambda_g(m)}{m^{1/4}} e \left( -\frac{ma}{q} \right) \int_0^\infty V(y)y^{-1/4}e \left( t\varphi(y) + \frac{\zeta_Xy}{qQ} \pm \frac{2\sqrt{mXy}}{q} \right) dy \right) \left( \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1/4}} e \left( \frac{n\bar{a}}{q} \right) \int_0^\infty U(y)y^{-1/4}e \left( -\frac{\zeta_Xy}{qQ} \pm \frac{2\sqrt{nXy}}{q} \right) dy \right). \]
where temporarily, $T := T(X, C, M, N_1, Ξ)$ is given by

$$T = X^{3/2} Q \sum_{q \sim C} q^{-2} \sum_{m \sim M} \lambda_g(m) m^{1/4} \sum_{n \sim N_1} \lambda_f(n) n^{1/4} S(m - n, 0; q) I(m, n, q, Ξ).$$  \hfill (4.18)

Now we consider the integral $I(m, n, q, Ξ)$ in \hfill (4.14). First we apply the stationary phase method to the integral $Ψ (n, q, ζ)$ in \hfill (4.16). The stationary point $y_0$ is given by $y_0 = nQ^2/(Xζ^2)$. Applying Lemma \hfill 3.6 (2) with $Y = Z = 1$ and $H = R = \sqrt{nX}/q \gg X^ε$, we obtain

$$Ψ (n, q, ζ) = \frac{q^{1/2}}{(nX)^{1/4}} e \left( \frac{nQ}{q} \right) U^z \left( \frac{nQ^2}{Xζ^2} \right) + O_A \left( X^{-A} \right),$$

where $U^z$ is an 1-inert function (depending on $A$) supported on $y_0 \asymp 1$. Plugging this asymptotic formula for $Ψ (n, q, ζ)$ and \hfill (4.15) into \hfill (4.14) and switching the order of integrations, one has

$$I(m, n, q, Ξ) = \frac{q^{1/2}}{(nX)^{1/4}} \int_0^∞ K(y; n, q, Ξ)V(y) y^{-1/4} e \left( tφ(y) - \frac{2\sqrt{mXy}}{q} \right) dy + O_A \left( X^{-A} \right) \hfill (4.19)$$

with

$$K(y; n, q, Ξ) = \int_0^∞ g(q, ζ) W \left( \frac{ζ}{X} \right) \widehat{W} \left( \frac{ζ}{Ξ} \right) U^z \left( \frac{nQ^2}{Xζ^2} \right) e \left( \frac{ζXy}{qQ} + \frac{nQ}{qζ} \right) dζ.$$

Next, we derive an asymptotic expansion for $K(y; n, q, Ξ)$. By changing variable $nQ^2/(Xζ^2) \rightarrow ζ$,

$$K(y; n, q, Ξ) = \frac{n^{1/2}Q}{X^{1/2}} \int_0^∞ \phi(ζ) \exp \left( iζ(ζ) \right) dζ,$$

where

$$\phi(ζ) := -\frac{1}{2} ζ^{-3/2} U^2(ζ) g \left( q, \frac{n^{1/2}Q}{ζ^{1/2}X^{1/2}} \right) W \left( \frac{n^{1/2}Q}{ζ^{1/2}X^{1/2+ε}} \right) \widehat{W} \left( \frac{n^{1/2}Q}{ζ^{1/2}X^{1/2+ε}} \right)$$

and the phase function $ζ(ζ)$ is given by

$$ζ(ζ) = \frac{2πn^{1/2}X^{1/2}}{q} \left( yζ^{-1/2} + ζ^{1/2} \right).$$

Note that

$$ζ'(ζ) = \frac{πn^{1/2}X^{1/2}}{q} \left( -yζ^{-3/2} + ζ^{-1/2} \right),$$

and for $j \geq 2$,

$$ζ^{(j)}(ζ) = \left( \frac{3}{2} \right) \cdots \left( \frac{1}{2} - j \right) \frac{πn^{1/2}X^{1/2}}{q} \left( -yζ^{-1/2-j} + \frac{1}{2j-1} ζ^{1/2-j} \right).$$

Thus the stationary point is $ζ_0 = y$ and $ζ^{(j)}(ζ) \ll_j n^{1/2}X^{1/2}/q$ for $j \geq 2$. By \hfill (4.13), we have $φ^{(j)}(ζ) \ll_j X^ε$. Applying Lemma \hfill 3.6 (2) with $Y = Z = 1$ and $H = R = n^{1/2}X^{1/2}/q \gg X^ε$, we obtain

$$K(y; n, q, Ξ) = \frac{n^{1/4}q^{1/2}Q}{X^{3/4}} e \left( \frac{2\sqrt{nXy}}{q} \right) F(y) + O_A \left( X^{-A} \right) \hfill (4.20)$$
where \( F(y) = F(y; \Xi) \) is an inert function (depending on \( A \) and \( \Xi \)) supported on \( \zeta_0 \cong 1 \). Substituting (4.20) into (4.19), we get

\[
I(m,n,q,\Xi) = \frac{qQ}{X} \int_0^\infty V(y)F(y)y^{-1/4}e\left( t\varphi(y) + \frac{2\sqrt{nXYy}}{q} - \frac{2\sqrt{mXYy}}{q} \right)dy + O_A(X^{-\Lambda}). \tag{4.21}
\]

Further substituting (4.21) into (4.18) and writing the Ramanujan sum \( S(m-n,0;q) \) as \( \sum_{d|\gcd(m-n,q)}d\mu(q/d) \), one has

\[
T = X^{1/2} \sum_{q \sim C} q^{-1} \sum_{d|q} \mu\left( \frac{q}{d} \right) \sum_{m \sim M} \frac{\lambda_g(m)}{m^{1/4}} \sum_{n \sim N_1, \ n \equiv n \mod d} \frac{\lambda_f(n)}{n^{1/4}} I(m,n,q) + O_A \left( X^{-\Lambda} \right), \tag{4.22}
\]

where \( I(m,n,q) = I(m,n,q;\Xi) \) is given by

\[
I(m,n,q) = \int_0^\infty \tilde{V}(y)e\left( t\varphi(y) + \frac{2\sqrt{nXYy}}{q} - \frac{2\sqrt{mXYy}}{q} \right)dy. \tag{4.23}
\]

Here \( \tilde{V}(y) = V(y)F(y)y^{-1/4} \) satisfying \( \tilde{V}^{(j)}(y) \ll_j \Delta^j \) and \( \text{Var}(\tilde{V}) \ll 1 \). Recall \( \Delta \) denotes the quantity such that \( V^{(j)}(x) \ll \Delta^j \) (see (3.3)).

Making a change of variable \( y \to y^2 \) in (4.23), one has

\[
I(m,n,q) = 2 \int_0^\infty y \tilde{V}(y^2)e\left( t\varphi(y^2) + \frac{2X^{1/2}y}{q} \left( n^{1/2} - m^{1/2} \right) y \right)dy. \tag{4.24}
\]

Since the properties of the integral \( I(m,n,q) \) depend on the size of \( C \), we split the modulus \( C \) according to \( C \leq X^{1+\varepsilon}\Xi/(Qt) \) or \( X^{1+\varepsilon}\Xi/(Qt) \leq C \ll Q \).

**4.3. The case of small modulus.** We first deal with the case \( 1 \ll C \leq X^{1+\varepsilon}\Xi/(Qt) \). If we assume \( (\varphi(y^2))'' \gg 1 \), equivalently \( \varphi(y) \neq cy^{1/2} + c_0 \) for any constant \( c_0 \), then the second derivative of the phase function satisfies

\[
t(\varphi(y^2))'' \gg t
\]

and by Lemma 3.7, we have

\[
I(m,n,q) \ll t^{-1/2}.
\]

By this estimate, (3.2) and (3.3), \( T \) in (4.22) can be bounded by

\[
T \ll \frac{X^{1/2}N_0}{t^{1/2}(MN_1)^{1/4}} \sum_{q \sim C} q^{-1} \sum_{d|q} d \sum_{m \sim M} |\lambda_g(m)| \sum_{n \sim N_1, n \equiv n \mod d} 1
\]

\[
\ll \frac{X^{1/2}M^{3/4}N_0}{t^{1/2}N_1^{1/4}} \sum_{q \sim C} q^{-1} \sum_{d|q} d \left( 1 + \frac{N_1}{d} \right)
\]

\[
\ll t^{-1/2}X^{1/2}M^{3/4}N_1^{-1/4+\varepsilon}(C + N_1)
\]

\[
\ll t^{-1/2}X^{1+\varepsilon}X^{1/2+\varepsilon} \frac{X^{1+\varepsilon}X^{1+\varepsilon}}{Q^{1+2\varepsilon}} \left( \frac{X\Xi}{Qt} + \frac{X\Xi^2}{Q^2} \right)
\]

\[
\ll \frac{X^{2+\varepsilon}}{Q^{2+2\varepsilon}t^{1/2}} \left( \frac{1}{t} + \frac{1}{Q} \right)
\]
recalling \( \Xi \ll X^{\varepsilon} \), \( M \ll X^{\varepsilon} \max\{C^2t^2/X, X\Xi^2/Q^2\} \ll X^{1+\varepsilon}\Xi^2/Q^2 \) and \( N_1 \ll X\Xi^2/Q^2 \) in (4.17). Assuming

\[
Q < t
\]

Then the contribution from \( 1 \ll C \leq X^{1+\varepsilon}\Xi/(Qt) \) to \( J(C, \Xi) \) in (4.17) is

\[
X^{2+\varepsilon}\Xi \ll \frac{Q^{3+2\varepsilon}t^{1/2}}{4t^{1/2}}.
\]

4.4. The case of large modulus. In the subsequent sections, we deal with the case \( X^{1+\varepsilon}\Xi/(Qt) \leq C \ll Q \). In this case, we will evaluate the integral \( J(m, n, q) \) more precisely. The integral \( J(m, n, q) \) has the following properties which will be proved in Section 7.

**Lemma 4.1.** Assume \( V^{(j)}(x) \ll \Delta^j \) as defined in (1.3) with \( \Delta < t^{1/2-\varepsilon} \) and \( C \) satisfies \( C \geq X^{1+\varepsilon}\Xi/(Qt) \). Further assume \( \varphi(x) = c \log x \) or \( cx^\beta \) with \( \beta \in (0, 1), \beta \neq 1/2. \) Then we have

\[
J^*(m, n, q) = e\left(t\varphi(y_0^2) - Dy_0\right) J^*(m, n, q) + O_A(t^{-A}),
\]

where \( y_0 = (ct/D)^{1/\beta} \) with \( D = 2q^{-1}(mX)^{1/2} \) and

\[
J^*(m, n, q) = \frac{1}{\sqrt{t}} G_2(y_0) e\left(By_0 + \frac{y_0^2}{2c\beta^2} B^2 + B \sum_{j=2}^{K_2} g_{c,\beta,j}(y_0) \left(\frac{B}{t}\right)^j\right) + O_A(t^{-A}).
\]

Here \( B = 2q^{-1}(nX)^{1/2} \), \( y_\ast \) is defined in (7.5), \( G_2(x) \) is some inert function supported on \( x \ll 1 \) and \( g_{c,\beta,j}(x) \) is some polynomial function depending only on \( c, \beta, j \).

4.4.1. Cauchy-Schwarz and Poisson summation. Applying the Cauchy-Schwarz inequality to (??) and using the Rankin-Selberg estimate \( (3.3) \), one sees that

\[
T \ll \frac{X^{1/2}}{M^{1/4}} \sum_{q \sim C} q^{-1} \sum_{d|q} d \left( \sum_{m \sim M} |\lambda_g(m)|^2 \right)^{1/2} \left( \sum_{n \sim N_1} \frac{\lambda_f(n)n^{-1/4}J^*(m, n, q)}{m \equiv m \mod d} \right)^{1/2}
\]

\[
\ll X^{1/2} M^{1/4} \sum_{q \sim C} q^{-1} \sum_{d|q} d \sqrt{\Omega(q, d)},
\]

where

\[
\Omega(q, d) = \sum_{m \in \mathbb{Z}} \omega\left(\frac{m}{M}\right) \sum_{n \sim N_1 \atop n \equiv m \mod d} \lambda_f(n)n^{-1/4}J^*(m, n, q)^2.
\]

Here \( \omega \) is a nonnegative smooth function on \( (0, +\infty) \), supported on \( [2/3, 3] \), and such that \( \omega(x) = 1 \) for \( x \in [1, 2] \).

Opening the absolute square, we break the \( m \)-sum into congruence classes modulo \( d \) and apply the Poisson summation formula to the sum over \( m \) to get

\[
\Omega(q, d) = \sum_{n_1 \sim N_1} \lambda_f(n_1)n_1^{-1/4} \sum_{n_2 \sim N_1 \atop n_2 \equiv n_1 \mod d} \lambda_f(n_2)n_2^{-1/4} \sum_{m \equiv n_1 \mod d} \omega\left(\frac{m}{M}\right) J^*(m, n_1, q) J^*(m, n_2, q)
\]

\[
= \frac{M}{d} \sum_{n_1 \sim N_1} \lambda_f(n_1)n_1^{-1/4} \sum_{n_2 \sim N_1 \atop n_2 \equiv n_1 \mod d} \lambda_f(n_2)n_2^{-1/4} \sum_{\tilde{m} \in \mathbb{Z}} e\left(-\frac{\tilde{m}n_1}{d}\right) H\left(\frac{\tilde{m}M}{d}\right),
\]

(4.31)
where the integral $\mathcal{H}(x) = \mathcal{H}(x; n_1, n_2, q)$ is given by

$$
\mathcal{H}(x) = \int_{\mathbb{R}} \omega(x) \sum_{M \geq 1} e(-xM) \log M \mathcal{J}(M, n_1, n_2, q) e(-xM) dM.
$$

(4.32)

We have the following estimates for $\mathcal{H}(x)$, whose proofs we postpone to Section 4.2.

Lemma 4.2. Assume $\varphi(x) = c \log x$ or $cx^\beta$ with $\beta \in (0, 1) \setminus \{1/2, 3/4\}$. Further assume $V^{(j)}(x) \ll x^{1/2}$ as defined in (1.23) with $\Delta < x^{1/2-\varepsilon}$ and $C$ satisfies $C \geq X^{1+\varepsilon}N/(Qt)$.

1) We have $\mathcal{H}(x) \ll t^{-1}$ for any $x \in \mathbb{R}$.

2) For $x \gg X^{1+\varepsilon}N/(CQ)$, we have $\mathcal{H}(x) \ll X^{-A}$.

3) For $x \neq 0$, we have $\mathcal{H}(x) \ll t^{-1}|x|^{-1/2}$.

4) $\mathcal{H}(0)$ is negligibly small unless $|n_1-n_2| \ll X^{\varepsilon}$.

With estimates for $\mathcal{H}(x)$ ready, we now continue with the treatment of $\Omega(q,d)$ in (4.31). By Lemma 4.2 (2), the contribution from the terms with $\bar{m}$ to $\Omega(q,d)$ is negligible. So we only need to consider the range $0 \leq |ar{m}| \leq N_2$.

We treat the cases where $\bar{m} = 0$ and $\bar{m} \neq 0$ separately and denote their contributions to $\Omega(q,d)$ by $\Omega_0$ and $\Omega \neq 0$, respectively.

4.4.2. The zero frequency. Let $\Sigma_0$ denote the contribution of $\Omega_0$ to (4.31). Correspondingly, we denote its contribution to (4.29) by $\Sigma_0$.

Lemma 4.3. Assume

$$
Q > (X/t)^{1/2}.
$$

(4.34)

We have

$$
\Sigma_0 \ll X^{\varepsilon}N^{1/2} t.
$$

Proof. Splitting the sum over $n_1$ and $n_2$ according as $n_1 = n_2$ or not, and applying Lemma 4.2 (4), the Rankin-Selberg estimate (3.3) and using the inequality $|\lambda_f(n_1)\lambda_f(n_2)| \leq |\lambda_f(n_1)|^2 + |\lambda_f(n_2)|^2$, we have

$$
\Omega_0 \ll \frac{M}{dtN_1^{1/2}} \sum_{n_1 \sim N_1} \sum_{n_2 \sim N_1} |\lambda_f(n_1)||\lambda_f(n_2)|
$$

$$
\ll \frac{M}{dtN_1^{1/2}} \sum_{n_1 \sim N_1} \sum_{n_2 \sim N_1} |\lambda_f(n_1)|^2 \sum_{n_2 \sim N_1} \frac{1}{N_1} \frac{1}{N_2}
$$

$$
\ll \frac{X^{\varepsilon}N^{1/2}}{dt}.
$$

This bound when substituted in place of $\Omega(q,d)$ into (4.29) yields that

$$
\Sigma_0 \ll X^{1/2+\varepsilon}M^{1/4} \sum_{q \sim C} q^{-1} \sum_{d \sim q} \frac{M^{1/2}N_1^{1/4}}{dt^{1/2}q^{1/2}} \ll \frac{X^{1/2+\varepsilon}M^{3/4}N_1^{1/4}C^{1/2}}{t^{1/2}}.
$$

(4.35)
Recall $C \ll Q$ and from (4.17) that $1 \ll M \ll X^\varepsilon \max \{C^2 t^2 / X, X \Xi^2 / Q^2\}$, $N_1 \ll X \Xi^2 / Q^2$. In particular, if we further assume $Q$ satisfies $Q > (X / t)^{1 / 2}$, then $1 \ll M \ll X^\varepsilon Q^2 t / X$. Thus

$$\Sigma_0 \ll X^3 \varepsilon^2 t.$$ 

This proves the lemma.

4.4.3. The non-zero frequencies. Recall $\Omega_{\neq 0}$ denotes the contribution from the terms with $\tilde{m} \neq 0$ to $\Omega(q, d)$ in (4.31). Correspondingly, we denote its contribution to (4.29) by $\Sigma_{\neq 0}$. Using the inequality $|\lambda_f(n_1) \lambda_f(n_2)| \leq |\lambda_f(n_1)|^2 + |\lambda_f(n_2)|^2$, we have

$$\Omega_{\neq 0} \ll \frac{M}{dN_1^{1 / 2}} \sum_{n_1 \sim N_1} |\lambda_f(n_1)|^2 \sum_{n_2 \sim N_1 \mod d} \sum_{n_2 \equiv \tilde{m} \ll N_2} |\mathcal{H} \left( \frac{\tilde{m} M}{d} \right)|,$$

(4.36)

where $N_2 = dC^{-1}Q^{-1}M^{-1}X^{1 + \varepsilon} \Xi$ is defined in (4.33).

Lemma 4.4. Assume

$$Q < \min \{t, X^{1 / 3}\}.$$ 

(4.37)

We have

$$\Sigma_{\neq 0} \ll X^{5 / 4 + \varepsilon} / Q.$$ 

Proof. For $x = \tilde{m} M / d$, in order to apply the estimates for $\mathcal{H}(x)$ in Lemma 4.2 consider the cases where $x \ll X^\varepsilon$ and $x \gg X^\varepsilon$, separately, and split the sum over $\tilde{m}$ accordingly. Set

$$N_3 := dX^\varepsilon / M.$$ 

(4.38)

Then for $0 \neq \tilde{m} \ll N_3$ we will use the bound $\mathcal{H}(x) \ll t^{-1}$ in Lemma 4.2 (1), and for the remaining part we apply the bound $\mathcal{H}(x) \ll t^{-1} |x|^{-1 / 2}$ in Lemma 4.2 (3). By (4.36), we have

$$\Omega_{\neq 0} \ll \frac{M}{dN_1^{1 / 2}} \sum_{n_1 \sim N_1} |\lambda_f(n_1)|^2 \sum_{n_2 \sim N_1 \mod d} \sum_{n_2 \equiv \tilde{m} \ll N_3} t^{-1}
$$

$$+ \frac{M}{dN_1^{1 / 2}} \sum_{n_1 \sim N_1} |\lambda_f(n_1)|^2 \sum_{n_2 \sim N_1 \mod d} \sum_{N_3 \ll \tilde{m} \ll N_2} t^{-1} \left( |\tilde{m}| M / d \right)^{-1 / 2}
$$

$$\ll \frac{MN_1^{1 / 2} N_3}{d t} \left( 1 + \frac{N_1}{d} \right) + \frac{M^{1 / 2} N_1^{1 / 2} N_2^{1 / 2}}{d^{1 / 2} t} \left( 1 + \frac{N_1}{d} \right)
$$

$$\ll \frac{M^{1 / 2} N_1^{1 / 2}}{d^{1 / 2} t} \left( 1 + \frac{N_1}{d} \right) \left( \frac{M^{1 / 2} N_3}{d^{1 / 2} + N_2^{1 / 2}} \right).$$

Here we have applied the Rankin–Selberg estimate (3.3). Recall (4.33) $N_2 = dC^{-1}Q^{-1}M^{-1}X^{1 + \varepsilon} \Xi$ and (4.38) $N_3 = dX^\varepsilon / M$. Thus

$$\Omega_{\neq 0} \ll \frac{X^\varepsilon M^{1 / 2} N_1^{1 / 2}}{t} \left( 1 + \frac{N_1}{d} \right) \left( \frac{X^\varepsilon}{M^{1 / 2}} + \frac{X^{1 / 2} \Xi^{1 / 2}}{C^{1 / 2} Q^{1 / 2} M^{1 / 2}} \right)
$$

$$\ll \frac{X^{1 / 2 + \varepsilon} N_1^{1 / 2}}{C^{1 / 2} Q^{1 / 2} t} \left( 1 + \frac{N_1}{d} \right).$$
since $\Xi \ll X^{\varepsilon}$ and $Q < X^{1/2-\varepsilon}$ by (4.10).

Since $1 \ll M \ll X^{\varepsilon}$ max $\left\{ C^2 t^2/X, X \Xi^2/Q^2 \right\} = X^{\varepsilon} C^2 t^2/X$ in (4.17) as $X^{1+\varepsilon} \Xi/(Qt) \leq C \ll Q$, this bound when substituted in place of $\Omega(q, d)$ in (4.29) gives that

$$\Sigma_{\neq 0} \ll X^{1/2+\varepsilon} M^{1/4} \sum_{q \sim C} q^{-1} \sum_{d \mid q} \frac{dX^{1/4+\varepsilon} N_{1/4}^{1/4}}{C^{1/4} Q^{1/4} t^{1/2}} \left( 1 + \frac{N_{1/2}^{1/2}}{d^{1/2}} \right)$$

$$\ll \frac{M^{1/4} N_{1/4}^{1/4} X^{3/4+\varepsilon} C^{1/4}}{Q^{1/4} t^{1/2}} \left( C^{1/2} + N_{1/2}^{1/2} \right)$$

$$\ll N_{1/4}^{1/4} X^{1/2+\varepsilon} C^{3/4} Q^{-1/4} \left( C^{1/2} + N_{1/2}^{1/2} \right)$$

$$\ll N_{1/4}^{1/4} X^{1/2+\varepsilon} Q^{1/2} \left( Q^{1/2} + N_{1/2}^{1/2} \right)$$

Recall $1 \ll N_{1} \times X \Xi^2/Q^2 \ll X^{1+\varepsilon}/Q^2$ in (4.17) and $C \ll Q$, we further imply

$$\Sigma_{\neq 0} \ll X^{3/4+\varepsilon} \left( Q^{1/2} + \frac{X^{1/2}}{Q} \right)$$

$$\ll X^{5/4+\varepsilon}/Q$$

provided that $Q < X^{1/3}$.

4.5. **Conclusion.** By inserting the upper bounds in Lemmas 4.3 and 4.4 into (4.29), we have shown the following

$$T \ll X^{\varepsilon} \left( Q^{3/2} t + \frac{X^{5/4}}{Q} \right),$$

under the assumption $X^{1+\varepsilon} \Xi/(Qt) \leq C \ll Q$ and

$$(X/t)^{1/2} < Q < \min \{ t, X^{1/3} \}$$

(4.39)

which is a combination of (4.10), (4.34) and (4.37). We set $Q = X^{1/2}/t^{2/5}$ to balance the contribution. Then for $X^{1+\varepsilon} \Xi/(Qt) \leq C \ll Q$,

$$T \ll t^{2/5} X^{3/4+\varepsilon}$$

(4.40)

provided $X < t^{12/5}$. Moreover, for this choice of $Q$, when $C \leq X^{1+\varepsilon} \Xi/(Qt)$, by (4.20), $T$ is bounded by

$$\frac{X^{2+\varepsilon} t^{1/2}}{Q^{6+2\varepsilon} t^{1/2}} = X^{1/2+\varepsilon} t^{7/10+4\varepsilon/5}.$$  

Note that the estimate $t^{2/5} X^{3/4+\varepsilon}$ is superior to the trivial bound $X^{1+\varepsilon}$ for $X > t^{8/5}$. So we assume $X > t^{8/5}$ and in this case the term $X^{1/2+\varepsilon} t^{7/10+4\varepsilon/5}$ is dominated by the estimate in (4.40), since we can take $\vartheta = 7/64$ by [KS03].

Substituting the estimate in (4.40) for $T$ into (4.17) and using (4.6), we obtain

$$S(X, t) \ll t^{2/5} X^{3/4+\varepsilon}$$

provided $t^{8/5} < X < t^{12/5}$.
Notice that by Lemma 4.2 (3), $\varphi(x)$ further satisfies $\varphi(x) = c \log x$ or $cx^\beta$ ($\beta \in (0, 1) \setminus \{1/2, 3/4\}$, $c \in \mathbb{R} \setminus \{0\}$); see (7.3). The assumption $\Delta < t^{1/2-\varepsilon}$ arises also in applying Lemma 4.2 (3); see (7.6). This completes the proof of Theorem 1.1.

5. Proof of Corollary 1.2

In this section, we prove Corollary 1.2 in Section 1. Without loss of generality, we assume that $f$ and $g$ are both Maass cusp forms. If one of $f$ and $g$ is holomorphic, the proof is similar and simpler. Note that from (3.3), we have

$$\sum_{X<n \leq X+X/\Delta} |\lambda_f(n)|^2 \ll X/\Delta + X^{3/5}.$$ 

In particular, if $\Delta \leq X^{2/5}$, one has

$$\sum_{X<n \leq X+X/\Delta} |\lambda_f(n)|^2 \ll X/\Delta.$$ 

Similarly, under the same assumption $\Delta \leq X^{2/5}$, we have

$$\sum_{X<n \leq X+X/\Delta} |\lambda_g(n)|^2 \ll X/\Delta.$$ 

We choose the smooth function $V$ in (1.2) to be supported on $[1, 2]$ and $V(x) = 1$ on $[1 + 1/\Delta, 2 - 1/\Delta]$. Then, Theorem 1.1 yields

$$\sum_{X<n \leq 2X} \lambda_f(n)\lambda_g(n)e\left(t\varphi\left(\frac{n}{X}\right)\right) \ll t^{7/16}X^{3/4+\varepsilon} + \sum_{X<n \leq X+X/\Delta} |\lambda_f(n)\lambda_g(n)| + \sum_{2X-X/\Delta<n \leq 2X} |\lambda_f(n)\lambda_g(n)|$$

$$\ll t^{2/5}X^{3/4+\varepsilon} + \left(\sum_{X<n \leq X+X/\Delta} |\lambda_f(n)|^2\right)^{1/2} \left(\sum_{X<n \leq X+X/\Delta} |\lambda_g(n)|^2\right)^{1/2}$$

$$\ll t^{2/5}X^{3/4+\varepsilon} + X/\Delta,$$

as long as $\Delta \leq X^{2/5}$ and $t^{8/5} < X < t^{12/5}$. Corollary 1.2 then follows by choosing $\Delta = t^{1/2-\varepsilon}$ and by noting that $t^{1/2-\varepsilon} \leq X^{2/5}$ if and only if $t^{5/4-\varepsilon} \leq X$.

6. Proof of Corollary 1.5

In this section, we prove Corollary 1.5. We introduce a lemma about "Functional equation" of $L(s, 1 \boxplus (f \times g))$ before proving Corollary 1.5.

**Lemma 6.1.** For $\text{Re}(s) > 1$, we have

$$L(1-s, 1 \boxplus (f \times g)) = \frac{1}{\varepsilon(f \times g)}\gamma(s)L(s, 1 \boxplus (f \times g)),$$

where $\varepsilon(f \times g)$ is the root number of $L(f \times g)$ with $|\varepsilon(f \times g)| = 1$,

$$\gamma(s) = (\pi^{-5}s^4\frac{1}{2})^{\frac{5}{2}}\prod_{j=1}^{5} \Gamma\left(\frac{s + \kappa_j}{2}\right)\Gamma\left(\frac{1-s + \kappa_j}{2}\right)^{-1},$$
with $\kappa_1 = 0$, $\kappa_2 = \frac{k-\kappa}{2}$, $\kappa_3 = \frac{k-\kappa}{2} + 1$, $\kappa_4 = \frac{k+\kappa}{2} - 1$, $\kappa_5 = \frac{k+\kappa}{2}$, and

$$\gamma(\sigma - it) = \omega_k \left( \frac{t}{2\pi} \right)^{5(\sigma - 1/2)} \left( \frac{2\pi e}{t} \right)^{5it} \left\{ 1 + O \left( \frac{1}{t} \right) \right\},$$

for $\sigma > 1/2$, $t > 1$, $\omega_k = e^{\left( \frac{ik}{8} \right)}$.

**Proof.** First by the functional equation

$$\Lambda(s, 1 \boxtimes (f \times g)) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) \cdot (2\pi)^{-2s} \Gamma \left( s + \frac{k-\kappa}{2} \right) \Gamma \left( s + \frac{1+k\kappa}{2} - 1 \right) L(s, f \times g)$$

$$= \varepsilon(f \times g) \Lambda(1 - s, 1 \boxtimes (f \times g)) = \varepsilon(f \times g) \Lambda(1 - s, 1 \boxtimes (f \times g)),$$

we can write the functional equation as follows,

$$L(1 - s, 1 \boxtimes f \times g) = \frac{1}{\varepsilon(f \times g)} \frac{\pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \Gamma \left( s + \frac{k-\kappa}{2} \right) \Gamma \left( s + \frac{1+k\kappa}{2} - 1 \right)}{(2\pi)^{-2s} \Gamma \left( 1 - s + \frac{k-\kappa}{2} \right) \Gamma \left( 1 - s + \frac{1+k\kappa}{2} - 1 \right)} L(s, f \times g)$$

$$= \frac{1}{\varepsilon(f \times g)} \gamma(s) L(s, 1 \boxtimes (f \times g)),$$

where

$$\gamma(s) = \pi^{\frac{s}{2} - s} (2\pi)^{2-4s} \frac{\Gamma \left( \frac{s}{2} \right)}{\Gamma \left( 1 - s + \frac{k-\kappa}{2} \right) \Gamma \left( 1 - s + \frac{k+\kappa}{2} - 1 \right)}.$$

Note that $\Gamma(z) \Gamma \left( \frac{1}{2} \right) = 2^{1-2z}\pi^{\frac{1}{2}}\Gamma(2z)$. Taking $z = \frac{s+k-\kappa}{2}$, $z = \frac{s+k+\kappa}{2} - 1$ respectively, we obtain

$$\Gamma \left( s + \frac{k-\kappa}{2} - 1 \right) = 2^{s+k-\kappa - 1} \pi^{-\frac{1}{2}} \Gamma \left( s + \frac{k-\kappa}{2} \right) \Gamma \left( s + \frac{k+\kappa}{2} - 1 \right) \Gamma \left( s + \frac{k+\kappa}{2} + 1 \right),$$

$$\Gamma \left( s + \frac{k+\kappa}{2} - 1 \right) = 2^{s+k+\kappa - 2} \pi^{-1/2} \Gamma \left( s + \frac{k+\kappa}{2} - 1 \right) \Gamma \left( s + \frac{k+\kappa}{2} + 1 \right).$$

With the notation of $\gamma(s)$, we derive

$$\gamma(s) = \left( \pi^{-5} \right)^{s/2} \prod_{j=1}^{5} \Gamma \left( \frac{s + \kappa j}{2} \right) \Gamma \left( \frac{1 - s + \kappa j}{2} \right)^{-1} \Gamma \left( \frac{1 - s + \kappa j}{2} \right)^{-1},$$

with $\kappa_1 = 0$, $\kappa_2 = \frac{k-\kappa}{2}$, $\kappa_3 = \frac{k-\kappa}{2} + 1$, $\kappa_4 = \frac{k+\kappa}{2} - 1$, $\kappa_5 = \frac{k+\kappa}{2}$. Hence, by the argument of Friedlander-Iwaniec [PI05] Section 1, we complete the proof of Lemma. \hfill \Box

Take $s = 1 + \varepsilon - it$. Lemma 6.1 yields

$$L(-\varepsilon + it, 1 \boxtimes (f \times g)) = \frac{\omega_k}{\varepsilon(f \times g)} \left( \frac{t}{2\pi} \right)^{\frac{5}{2} + 5\varepsilon} \left( \frac{t}{2\pi e} \right)^{-5it} L(1 + \varepsilon - it, 1 \boxtimes (f \times g)) \left\{ 1 + O \left( \frac{1}{t} \right) \right\} \ .$$

(6.1)

**Proof of Corollary 1.5.** We first approximate $\sum_{n \leq X} \lambda_1 \boxtimes (f \times g)(n)$ by a smooth sum. Let

$$Y = X^{2/3 - \delta}$$

for some $\delta \in (0, 2/39)$. 

\hfill \Box
Let $W$ be a smooth function supported on $[1/2 − Y/X, 1 + Y/X]$ such that $W(u) = 1$, $u ∈ [1/2, 1]$ and $W(u) ∈ [0, 1]$, $u ∈ [1/2 − Y/X, 1/2] ∪ [1, 1 + Y/X]$, and $W^{(m)}(u) ≪ m (X/Y)^m$ for all $m ≥ 1$. Then
\[
\sum_{X/2 < n ≤ X} \lambda_{1,[(f, g)]}(n) = \sum_{X/2 − Y < n < X + Y} \lambda_{1,[(f, g)]}(n)W\left(\frac{n}{X}\right)
+ O\left(\sum_{X/2 − Y < n < X/2} |\lambda_{1,[(f, g)]}(n)| + \sum_{X < n < X + Y} |\lambda_{1,[(f, g)]}(n)|\right) \tag{6.2}
= \sum_{n ≥ 1} \lambda_{1,[(f, g)]}(n)W\left(\frac{n}{X}\right) + O\left(X^{2/3−δ+ε}\right),
\]
where we have used Deligne’s bound $\lambda_{1,[(f, g)]}(n) = \sum_{tm^2r=n} λ_f(r)λ_g(r) ≪ n^ε$. Thus we only need to show
\[
\sum_{n ≥ 1} \lambda_{1,[(f, g)]}(n)W\left(\frac{n}{X}\right) = L(1, f × g)\tilde{W}(1)X + O\left(X^{2/3−δ+ε}\right) \tag{6.3}
\]
where $\tilde{W}(s) = \int_0^∞ W(x)x^{s−1}dx$ is the Mellin transform of $W(x)$ and $\tilde{W}(1) = 1/2 + O(Y/X)$. By breaking the sum into dyadic intervals and plugging (6.3) into (6.2), we get
\[
\sum_{n ≤ X} \lambda_{1,[(f, g)]}(n) = 2L(1, f × g)\tilde{W}(1)X + O\left(X^{2/3−δ+ε}\right)
= L(1, f × g)X + O\left(X^{2/3−δ+ε}\right).
\]
Now we consider the sum $\sum_{n ≥ 1} \lambda_{1,[(f, g)]}(n)W\left(\frac{n}{X}\right)$ in (6.3). By the Mellin inversion formula
\[
W(u) = \frac{1}{2πi} ∫_{(2)} \tilde{W}(s)u^{−s}ds,
\]
we get
\[
\sum_{n ≥ 1} \lambda_{1,[(f, g)]}(n)W\left(\frac{n}{X}\right) = \frac{1}{2πi} ∫_{(2)} \tilde{W}(s)L(s, 1 ⊕ (f × g))X^sds.
\]
Next we move the integration to the parallel segment with $\text{Re}(s) = −ε$. Note that inside the contour the integrand has only a simple pole at $s = 1$ with residue $L(1, f × g)\tilde{W}(1)X$, since $L(s, 1 ⊕ (f × g)) = ζ(s)L(s, f × g)$. Hence,
\[
\sum_{n ≥ 1} \lambda_{1,[(f, g)]}(n)W\left(\frac{n}{X}\right) = L(1, f × g)\tilde{W}(1)X + \frac{1}{2πi} ∫_{−ε} \tilde{W}(s)L(s, 1 ⊕ (f × g))X^sds. \tag{6.4}
\]
Let
\[
I(X) := \frac{1}{2πi} ∫_{−ε} \tilde{W}(s)L(s, 1 ⊕ (f × g))X^sds.
\]
Inserting a dyadic smooth partition of unity to the $t$-integral, we get
\[
I(X) = \sum_{T \text{ dyadic}} I(X, T), \tag{6.5}
\]
where
\[
I(X, T) := \frac{X^{−ε}}{2π} ∫_R X^{it}\tilde{W}(−ε + it)L(−ε + it, 1 ⊕ (f × g))V\left(\frac{t}{T}\right)dt
\]
and
\[
X^{it}\tilde{W}(−ε + it)L(−ε + it, 1 ⊕ (f × g))V\left(\frac{t}{T}\right)
\]
for some fixed compactly supported function $V$. For $\tilde{W}(s)$, by applying integration by parts, we have, for any $m \geq 1$

$$\tilde{W}(s) = \frac{(-1)^m}{s(s+1)\cdots(s+m-1)} \int_0^\infty W^{(m)}(u)u^{s+m-1}du \ll_m \frac{1}{|s|^m} \left(\frac{X}{Y}\right)^{m-1},$$

(6.6)

since $\text{supp} \ W^{(m)} \subset [1/2 - Y/X, 1/2] \cup [1, 1 + Y/X]$. By (6.6), one finds that the contribution from the $t$-integral of $I(X, T)$ is negligible if $t \gg X^{1+\varepsilon}/Y$. In addition, by the upper bound $L(-\varepsilon + it, 1 \boxplus (f \times g)) \ll (1 + t)^{5/2 + \varepsilon}$ which follows from Lemma 6.1 and the Phragmén–Lindelöf principle and by (6.6) with $m = 1$, we deduce that

$$I(X, T) \ll X^{\varepsilon}T^{5/2 + \varepsilon} \ll Y$$

if $T \ll Y^{2/5 - \varepsilon}$. Thus, up to a negligible error, we only need to consider those $T$ in (6.5) in the range $Y^{2/5 - \varepsilon} \ll T \ll X^{1+\varepsilon}/Y$. And we only consider positive $T$’s, since negative $T$’s can be handled similarly. Next, for $I(X, T)$, by the first equality in (6.6) with $m = 1$, we get

$$I(X, T) = \frac{-X^{-\varepsilon}}{2\pi} \int_{1/3}^3 W'(u)u^{-\varepsilon} \int_{\mathbb{R}} \frac{(Xu)^it}{-\varepsilon + it} L(-\varepsilon + it, 1 \boxplus (f \times g))V\left(\frac{t}{T}\right)dt \, du$$

$$\ll \frac{X^{-\varepsilon}}{T} \sup_{u \in [1/3, 3]} \left|\int_{\mathbb{R}} (Xu)^it L(-\varepsilon + it, 1 \boxplus (f \times g))V_1\left(\frac{t}{T}\right)dt\right|. \tag{6.7}$$

Hence, in the following, we only need to estimate

$$J(X, T) := \int_{\mathbb{R}} X^it L(-\varepsilon + it, 1 \boxplus (f \times g))V_1\left(\frac{t}{T}\right)dt. \tag{6.8}$$

We shall apply functional equation for $L(-\varepsilon + it, 1 \boxplus (f \times g))$ to change the variable $s = -\varepsilon + it$ into $1 - s = 1 + \varepsilon - it$. By inserting the functional equation (6.1) into (6.8), we have

$$J(X, T) = \int_{\mathbb{R}} X^it \frac{1}{\varepsilon (f \times g)} \left(\frac{t}{2\pi}\right)^{\frac{5(1+\varepsilon)}{2}} \left(\frac{t}{2\pi\varepsilon}\right)^{-5it} L(1 + \varepsilon - it, 1 \boxplus (f \times g))V_1\left(\frac{t}{T}\right)dt$$

$$+ O\left(\frac{1}{T} \cdot T^{5/2 + \varepsilon} \cdot T\right)$$

$$\ll T^{5/2 + \varepsilon} \left|\int_{\mathbb{R}} \sum_{n \geq 1} \frac{\lambda_{\text{HE}}(f \times g)(n)}{n^{1+\varepsilon - it}} X^it \left(\frac{t}{2\pi\varepsilon}\right)^{-5it} V_2\left(\frac{t}{T}\right)dt\right| + T^{5/2 + \varepsilon}$$

for some smooth compactly supported function $V_2$.

Exchanging the order of the integration and summation above, and making a change of variable $\frac{t}{T} \rightarrow \xi$, we get

$$J(X, T) \ll T^{5/2 + \varepsilon} \left|\sum_{n \geq 1} \frac{\lambda_{\text{HE}}(f \times g)(n)}{n^{1+\varepsilon}} \int_{\mathbb{R}} (nX)^it \left(\frac{t}{2\pi\varepsilon}\right)^{-5it} \left(\frac{t}{T}\right)dt\right| + T^{5/2 + \varepsilon}$$

$$\ll T^{7/2 + \varepsilon} \left|\sum_{n \geq 1} \frac{\lambda_{\text{HE}}(f \times g)(n)}{n^{1+\varepsilon}} \int_{\mathbb{R}} e^{i\xi T \log(nX(\frac{2\pi\varepsilon}{iT})^{-\varepsilon})} V_2(\xi)d\xi\right| + T^{5/2 + \varepsilon}.$$
Let $h(\xi) := T\xi \log(nX(\frac{2\pi}{T})^{-5})$, then $h'(\xi) = 5T \log \frac{2(nX)\frac{1}{5}}{\xi} T$, $h^{(j)}(\xi) = (-1)^{j-1}(j-2)!\frac{5T}{\xi^{j-2}}$ for $j \geq 2$. If $2\pi(nX)^{1/5}/T \not\in \text{supp } V_2$, it is not difficult to see that $h''(\xi) \gg T^{\varepsilon}$. Applying Lemma 3.6 (1), we have the integral over $\xi$ is $O(T^{-2021})$. Now for the above integral over $\xi$, we consider the case $2\pi(nX)^{1/5}/T \in \text{supp } V_2$. Note that the stationary point is $\xi_0 = \frac{2\pi(nX)^{1/5}}{T}$, $h(\xi_0) = 5T\xi_0$, $h''(\xi_0) = -\frac{5\pi}{\xi_0} \times T$ and $V^{(j)}(\xi) \ll_j 1$ for $j \geq 0$, $h^{(j)}(\xi_0) \asymp T$ for $j \geq 2$. Applying Lemma 3.6 (2) with $Y = Z = 1$ and $H = R = T$, we obtain

$$
\int_{R} V_2(\xi)e^{it\xi\log(nX(\frac{T}{\xi})^{-5})} d\xi = \frac{e^{ih(\xi_0)}}{T^{1/2}} W_1(\xi_0) + O\left(\frac{1}{T^{2021}}\right)
$$

$$
= e^{(5(nX)^{1/5})} W_2\left(\frac{n}{T^{3/5}/X}\right) + O\left(\frac{1}{T^{2021}}\right),
$$

for some inert functions $W_1, W_2$. Consequently,

$$
J(X, T) \ll T^{3+\varepsilon} \left| \sum_{n \geq 1} \frac{\lambda_1(\lambda x g)(n)}{n^{1+\varepsilon}} e\left(5(nX)^{1/5}\right) W_2\left(\frac{n}{T^{3/5}/X}\right) \right| + T^{5/2+\varepsilon}
$$

$$
\ll \frac{X^{1+\varepsilon}}{T^2} \left| \sum_{n \geq 1} \lambda_1(\lambda x g)(n)e\left(5(nX)^{1/5}\right) W_3\left(\frac{n}{T^{3/5}/X}\right) \right| + T^{5/2+\varepsilon}
$$

for some inert function $W_3$. Note that

$$
X^{1/5+\varepsilon} \ll Y^{2/5+\varepsilon} \ll T \ll \frac{X^{1+\varepsilon}}{Y}.
$$

(6.10)

So the above sum over $n$ is non-empty. Combining (6.5), (6.7), (6.8) and (6.9), we have

$$
I(X) \ll \sum_{T \text{ dyadic}} \left(\frac{X^{1+\varepsilon}}{T^3} \left| \sum_{n \geq 1} \frac{\lambda_1(\lambda x g)(n)}{n^{1+\varepsilon}} e\left(5(nX)^{1/5}\right) W\left(\frac{n}{T^{3/5}/X}\right) \right| + T^{3/2+\varepsilon}\right).
$$

(6.11)

Here $X$ on the right-hand side of (6.11) should be understood as the original $X u$ in (6.7) with $u \in [1/3, 3]$, and $W$ is a smooth compactly supported function with $\text{supp } W \in [1/4, 4]$. So we only need to consider the case $n \asymp T^{3/5}/X$. Now we make use of the fact that $\lambda_1(\lambda x g)(n) = \sum_{lm^2r=n} \lambda_f(r)\lambda_g(r)$. Inserting dyadic partitions to the $l$-sum and $m$-sum and making a smooth partition of unity into dyadic segments to the $r$-sum, we arrive at

$$
I(X) \ll \sum_{T \text{ dyadic}} \left(\frac{X^{1+\varepsilon}}{T^3} \sup_{L^2M^2R \geq T^{3/5}/X} \left| B(L, M, R) \right| + T^{3/2+\varepsilon}\right),
$$

where

$$
B(L, M, R) := \sum_{l \sim L} \sum_{m \sim M} \sum_{r \geq 1} \lambda_f(r)\lambda_g(r) e\left(5(lm^2rX)^{1/5}\right) V\left(\frac{T}{R}\right).
$$

We distinguish two cases.
Combining (6.11), (6.12) and (6.13), we have

\[
B(L, M, R) = \sum_{m=M} \sum_{r \geq 1} \lambda_f(r) \lambda_g(r) V \left( \frac{r}{R} \right) \left( \sum_{l \sim L} e \left( 5lm^2rX^{1/5} \right) \right).
\]

For the inner sum over \( l \), we apply the method of exponent pairs with A-process (see for example GK91 Chapter 3), by taking the exponent pair \((p, q)\) as

\[
(p, q) = \left( \frac{k}{2k+2}, \frac{k+h+1}{2k+2} \right) = \left( \frac{13}{194} + \varepsilon, \frac{76}{97} + \varepsilon \right),
\]

where \((k, h) = \left( \frac{12}{59} + \varepsilon, \frac{25}{64} + \varepsilon \right)\) is an exponent pair according to Bourgain [Bou17 Theorem 6]. Hence,

\[
B(L, M, R) \ll \sum_{m=M} \sum_{r \geq 1} \mid \lambda_f(r) \lambda_g(r) \mid V \left( \frac{r}{R} \right) \left( \sum_{l \sim L} e \left( 5lm^2rX^{1/5} \right) \right)
\]

\[
\ll T^\varepsilon MR \cdot (T/L)^p L^q
\]

\[
\ll T^{193/194 + \varepsilon} X^{-1 + \varepsilon} M^{-1 + \varepsilon} L^{-55/194 + \varepsilon}
\]

\[
\ll T^{316/69 + \varepsilon} M^{-152/207 + \varepsilon} X^{-359/414 + \varepsilon}
\]

\[
\ll T^{316/69 + \varepsilon} X^{-359/414 + \varepsilon}.
\]

In the last inequality we have used the fact \( M \gg 1 \).

Case 1. \( L \gg T^{593/345} M^{-149/207} X^{-97/207} \). We rewrite \( B(L, M, R) \) as

\[
B(L, M, R) = \sum_{l \sim L} \sum_{m=M} \left( \sum_{r \geq 1} \lambda_f(r) \lambda_g(r) e \left( 5lm^2rX^{1/5} \right) \right) V \left( \frac{r}{R} \right)
\]

\[
\ll \sum_{l \sim L} \sum_{m=M} \left( \sum_{r \geq 1} \lambda_f(r) \lambda_g(r) e \left( 5T \left( \frac{r}{R} \right)^{1/5} \right) \right) V \left( \frac{r}{R} \right).
\]

In order to apply Theorem 1.1, we need to verify that \( R \) satisfies the condition \( R \ll T^{12/5} \). Note that \( R \ll LM^2R \times T^5/X \) and \( Y^{2/5 + \varepsilon} \ll T \ll X^{1/3 + \varepsilon} \). Since we assume \( \delta < 2/39 \), we have \( T \ll X^{5/13} \) and hence \( R \ll T^5/X \ll T^{12/5} \). Therefore, by Theorem 1.1, we have

\[
B(L, M, R) \ll LM^{7/2} R^{3 + \varepsilon} X^{-1/4 + \varepsilon} T^{83/20 + \varepsilon} X^{-3/4 + \varepsilon}
\]

\[
\ll T^{316/69 + \varepsilon} M^{-152/207 + \varepsilon} X^{-359/414 + \varepsilon}
\]

\[
\ll T^{316/69 + \varepsilon} X^{-359/414 + \varepsilon}.
\]

Combining (6.11), (6.12) and (6.13), we have

\[
I(X) \ll \sum_{T \text{dyadic}} \left( \frac{X^{1 + \varepsilon}}{T^3} \cdot T^{316/69 + \varepsilon} X^{-359/414 + \varepsilon} + T^{3/2 + \varepsilon} \right)
\]

\[
\ll \sum_{T \text{dyadic}} \left( X^{55/414 + \varepsilon} T^{109/69 + \varepsilon} + T^{3/2 + \varepsilon} \right)
\]

\[
\ll X^{405/69 + \varepsilon} + T^{3/2 + \varepsilon}.
\]
Finally, putting together the above estimates (6.2), (6.4) and (6.14), we conclude that
\[
\sum_{X/2 < n \leq X} \lambda_{1\text{int}}(f \times g)(n) = L(1, f \times g)\tilde{W}(1)X + O\left(X^{m/69 + \frac{91}{138} + \epsilon}\right) + O\left(X^{2/3 - \delta + \epsilon}\right).
\]
So we complete the proof of Corollary 1.5 by taking \(\delta \leq 1/356\). \(\square\)

7. Estimation of integrals

We first prove Lemma 4.1.

Proof of Lemma 4.1. By (4.24), we write
\[
I(m, n, q) = 2 \int_0^\infty y\tilde{V}(y^2)e^{t\varphi(y^2) + By - Dy} dy,
\]
where
\[
B = 2q^{-1}(nX)^{1/2} \times \sqrt{XN_1/C}, \quad D = 2q^{-1}(mX)^{1/2} \times (MX)^{1/2}/C. \tag{7.1}
\]
Recall the range of \(N_1\) in (4.17) that \(N_1 \asymp X\Xi^2/Q^2\). Thus for \(X^{1+\epsilon}\Xi/(Qt) \leq C \ll Q\), we have
\[
B \ll \frac{X^{1+\epsilon}\Xi}{CQ} \ll X^{-\epsilon}t. \tag{7.2}
\]
Therefore, the integral \(I(m, n, q)\) is negligibly small unless \(D \asymp t\).

Assume
\[
(\varphi(y^2))' = cy^{-\beta} \quad \text{with} \quad \beta \neq 0, \tag{7.3}
\]
where \(c > 0\) is an absolute constant, i.e.,
\[
\varphi(y) = \frac{c}{2} \log y + c_1 \quad \text{or} \quad \varphi(y) = \frac{c}{1 - \beta}y^{(1-\beta)/2} + c_2 \quad \text{with} \quad \beta \neq 0, 1, \tag{7.4}
\]
where \(c_i \in \mathbb{R}, i = 1, 2\), are absolute constants. Without loss of generality, we further assume \(c_i = 0, i = 1, 2\). Let \(\rho(y) = t\varphi(y^2) + By - Dy\). Then
\[
\rho'(y) = cty^{-\beta} + B - D, \quad \rho^{(j)}(y) = t(\varphi(y^2))^{(j)}, \quad j = 2, 3, \ldots.
\]
The stationary point \(y_*\), which is the solution to the equation \(\rho'(y) = cty^{-\beta} + B - D\) is \(y_* = \left(\frac{ct}{D-B}\right)^{1/\beta}\). Denote
\[
C^j_\alpha = \frac{\alpha(\alpha - 1) \cdots (\alpha - j + 1)}{j!}.
\]
Then by the Taylor series approximation, \(y_*\) can be written as
\[
y_* = \left(\frac{ct}{D}\right)^{1/\beta} \left(1 + \sum_{j=1}^{K_1} C^j_{-1/\beta} \left(\frac{-B}{D}\right)^j + O_{\beta, K_1} \left(\frac{B^{K_1+1}}{t^{K_1+1}}\right)\right) + O_{c, \beta, K_1} \left(\frac{B^{K_1+1}}{t^{K_1+1}}\right), \tag{7.5}
\]
where
\[
y_0 = \left(1 + \sum_{j=1}^{K_1} y_j + O_{c, \beta, K_1} \left(\frac{B^{K_1+1}}{t^{K_1+1}}\right)\right).
\]
where here and after, $K_j \geq 1$, $j = 1, 2, 3, \ldots$, denote integers, and
\[
y_0 = \left(\frac{ct}{D}\right)^{1/\beta} \times 1,
y_j = C_{-1/\beta}^j \left(-\frac{B}{t}\right)^j \times \left(\frac{B}{t}\right)^j.
\]
By (7.2), the $O$-term in (7.5) is $O(N^{-\varepsilon K_1})$, which can be arbitrarily small by taking $K_1$ sufficiently large.

Note that $\rho(y) \approx t$ for any integer $j \geq 1$. Recall $\tilde{V}(y) \ll \Delta$, where $\Delta < t^{1-\varepsilon}$ (see (1.3)). To make sure that the stationary phase analysis is applicable to the integral $\mathcal{I}(M\xi, n, q)$, we assume $\Delta$ satisfies
\[
\Delta < t^{1/2-\varepsilon}
\]
(7.6)
Now applying Lemma 3.6 with $Z = 1$, $Y = \Delta$, $H = t$ and $R = H/X^2 \gg t^\varepsilon$, we have
\[
\mathcal{I}(m, n, q) = e^{(\rho(y))} \frac{e^{(\rho(y))}}{\sqrt{2\pi \rho''(y)}} G(y) + O_A(t^{-A}),
\]
for any $A > 0$, where $G(y)$ is some inert function supported on $y \approx 1$. From (7.4), (7.5) and using Taylor series approximation, we have
\[
\rho(y) = t\varphi(y_0^2) + By - Dy_0 + B\frac{y_0^2}{2c\beta^2} \frac{B^2}{t} + B \sum_{j=2}^{K_2} g_{c,\beta,j}(y_0) \left(\frac{B}{t}\right)^j + O_{c,\beta,K_2} \left(\frac{B^{K_2+2}}{t^{K_2+1}}\right)
\]
and
\[
\rho''(y) = -c\beta ty_0^{-\beta-1} - c\beta ty_0^{-\beta-1} + B(\beta+1)y_0^{-1} + B \sum_{j=1}^{K_3} h_{c,\beta,j}(y_0) \left(\frac{B}{t}\right)^j + O_{c,\beta,K_3} \left(\frac{B^{K_3+2}}{t^{K_3+1}}\right)
\]
for some functions $g_{c,\beta,j}(x)$, $h_{c,\beta,j}(x)$ of polynomially growth, depending only on $c, \beta, j$, and supported on $x \approx 1$. Note that $\rho''(y) \approx t$. Hence,
\[
\mathcal{I}(m, n, q) = \frac{1}{\sqrt{t}} G_z(y) e \left(t\varphi(y_0^2) - Dy_0 + B\frac{y_0^2}{2c\beta^2} \frac{B^2}{t}\right) \times e \left(B \sum_{j=2}^{K_2} g_{c,\beta,j}(y_0) \left(\frac{B}{t}\right)^j\right) + O_A(t^{-A}),
\]
where $G_z(y) = (t/(2\pi \rho''(y)))^{1/2} G(y)$ satisfies $G_z^{(j)}(y) \ll_j 1$. This finishes the proof of the lemma.

Next we prove Lemma 4.2.

Proof of Lemma 4.2. The proof is similar to [LS21, Lemma 4.3]. Recall (1.32) which we relabel as
\[
\mathcal{H}(x) = \int_{\mathbb{R}} \omega(\xi) \mathcal{I}^*(M\xi, n_1, q) \mathcal{I}^*(M\xi, n_2, q) e(-x\xi) d\xi,
\]
(7.7)
where by (4.28),

$$\mathcal{J}^\ast(M\xi, n, q) = \frac{1}{\sqrt{t}} G_2(y_\ast) e \left( B y_0 + \frac{y_0^2}{2c\beta^2} B^2 + B \sum_{j=2}^{K_2} g_{c,\beta,j}(y_0) \left( \frac{B}{t} \right)^j \right) + O_A(t^{-A}).$$

(7.8)

Here $y_0, y_\ast$ are as in (7.3), $G_2(x)$ is some inert function supported on $x \asymp 1$, $B = 2q^{-1}(nX)^{1/2}$ is defined in (7.1) and $g_{c,\beta,j}(x)$ some polynomial function depending only on $c, \beta, j$. Trivially, one has

$$\mathcal{H}(x) \ll t^{-1}.$$

This proves the first statement of Lemma 4.2.

Plugging (7.8) into (7.7), we obtain

$$\mathcal{H}(x) = \frac{1}{t} \int_{\mathbb{R}} \omega(\xi) G_2(y_\ast) \frac{\mathcal{J}^\ast}{\mathcal{J}^\ast} e \left( -x \xi + (B - B')y_0 \xi^{-1/(2\beta)} + (B^2 - B'^2)\frac{y_0^2}{2c\beta^2t} \xi^{-1/\beta} \right)$$

$$\times e \left( \sum_{j=2}^{K_2} g_{c,\beta,j}(y_0\xi^{-1/(2\beta)}) \left( B \left( \frac{B}{t} \right)^j - B' \left( \frac{B'}{t} \right)^j \right) \right) d\xi + O_A(t^{-A}),$$

where $\bar{y}_0 = y_0\xi^{1/(2\beta)} = (ct/\bar{D})^{1/\beta} \asymp 1$ with $\bar{D} = D\xi^{-1/2} = 2q^{-1}(MX)^{1/2}$ is defined in (7.1), $y_0, y_\ast$ are as in (7.3), and $B$ is defined in (7.1) and $B'$ is defined in the same way but with $n_1$ replaced by $n_2$. Note that the first derivative of the phase function in the above integral equals

$$-x - \frac{1}{2\beta} (B - B')\bar{y}_0 \xi^{-1/(2\beta)} - \frac{1}{\beta} (B^2 - B'^2)\frac{\bar{y}_0^2}{2c\beta^2t} \xi^{-1/\beta - 1}$$

$$-\frac{1}{2\beta} \bar{y}_0 \xi^{-1/(2\beta) - 1} \sum_{j=2}^{K_2} g'_{c,\beta,j}(\bar{y}_0\xi^{-1/(2\beta)}) \left( B \left( \frac{B}{t} \right)^j - B' \left( \frac{B'}{t} \right)^j \right)$$

(7.9)

which is $\gg |x| \gg X^\varepsilon$ if $|x| \gg X^{1+\varepsilon}N_1/C \asymp X^{1+\varepsilon}\Xi/(CQ)$ since $B, B' \asymp \sqrt{XN_1/C}$ and $N_1 \asymp X\Xi^2/Q^2$ in (4.17). Then repeated integration by parts shows that the contribution from $x \gg X^{1+\varepsilon}\Xi/(CQ)$ is negligible. Thus the second statement of Lemma 4.2 is clear.

Moreover, if $-1/(2\beta) - 1 \neq 0$, i.e., $\beta \neq -1/2$ or equivalently, $\varphi(x) \neq cx^{3/4}$, the second term in (7.9) is of size

$$|B - B'| = \frac{2N^{1/2}}{q} |n_1^{1/2} - n_2^{1/2}| \asymp \frac{X^{1/2}}{C^{1/2}} |n_1 - n_2| \asymp \frac{Q}{C\Xi} |n_1 - n_2|$$

since $N_1 \asymp X\Xi^2/Q^2$. Thus repeated integration by parts shows that $\mathcal{H}(x)$ is negligibly small unless $|x| \asymp \frac{Q}{C\Xi} |n_1 - n_2|$. Now by applying the second derivative test in Lemma 5.7 we infer that for $x \neq 0$ and $\varphi(x) \neq cx^{3/4}$,

$$\mathcal{H}(x) \ll t^{-1}|x|^{-1/2}.$$

This proves (3).
Finally, for $x = 0$, using the identity $a^{j+1} - b^{j+1} = (a - b)(a^j + a^{j-1}b + \cdots + ab^{j-1} + b^j)$ and (7.2), one sees that, for $j \geq 1$,
\[
B\left(\frac{B}{t}\right)^j - B'\left(\frac{B'}{t}\right)^j = (B - B') \left( \left(\frac{B}{t}\right)^j + \left(\frac{B'}{t}\right)^{j-1} \cdot \frac{B'}{t} + \cdots + \frac{B}{t} \left(\frac{B'}{t}\right)^{j-1} + \left(\frac{B'}{t}\right)^j \right)
\]
\[
\ll |B - B'| X^{-\varepsilon}.
\]
Thus the first derivative of the phase function in (7.3) is
\[
\gg |B - B'| \times \frac{Q}{C^\Xi |n_1 - n_2|}.
\]
By repeated integration by parts, $\mathcal{H}(0)$ is negligible small unless $|n_1 - n_2| \ll C^\Xi N^\varepsilon / Q$. Since $\Xi \ll N^\varepsilon$ and $C \ll Q$, we have that $\mathcal{H}(0)$ is negligibly small unless $|n_1 - n_2| \ll N^\varepsilon$. This completes the proof of Lemma 1.2.

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