Separation of variables for the $D_n$ type periodic Toda lattice

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Abstract

We prove separation of variables for the most general ($D_n$ type) periodic Toda lattice with $2 \times 2$ Lax matrix. It is achieved by finding proper normalisation for the corresponding Baker-Akhiezer function. Separation of variables for all other periodic Toda lattices associated with infinite series of root systems follows by taking appropriate limits.

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1. Introduction

Bogoyavlensky [3] introduced periodic Toda lattices corresponding to the root systems of affine algebras. In this case the integrable potentials in the Hamiltonian

\[ H = \sum_{j=1}^{n} \frac{p_j^2}{2} + V(q), \quad \{p_j, q_k\} = \delta_{jk}, \]  

(1.1)

for the loop algebras \( A_n^{(1)} \), \( B_n^{(1)} \), \( C_n^{(1)} \), and \( D_n^{(1)} \) have the form

\[
\begin{align*}
V_{A_n^{(1)}} &= V_{A_n} + \exp(q_n - q_1), \\
V_{B_n^{(1)}} &= V_{A_n} + \exp(q_n) + \exp(-q_1 - q_2), \\
V_{C_n^{(1)}} &= V_{A_n} + \exp(2q_n) + \exp(-2q_1), \\
V_{D_n^{(1)}} &= V_{A_n} + \exp(q_{n-1} + q_n) + \exp(-q_1 - q_2),
\end{align*}
\]

where

\[ V_{A_n} = \sum_{j=1}^{n-1} \exp(q_j - q_{j+1}). \]

For the twisted loop algebras the integrable potentials are as follows [21]:

\[
\begin{align*}
V_{A_n^{(2)}} &= V_{A_n} + \exp(q_n) + \exp(-2q_1), \\
V_{A_{2n+1}^{(2)}} &= V_{A_n} + \exp(-q_1 - q_2) + \exp(2q_n), \\
V_{D_{n+1}^{(2)}} &= V_{A_n} + \exp(q_n) + \exp(-q_1).
\end{align*}
\]

Inozemtsev [7] found a generic (\( D_n \) type) periodic Toda lattice with 4 more parameters (\( A, B, C, D \)) in the potential,

\[
V(q) = V_{A_n} + \exp(-q_1 - q_2) + \exp(q_{n-1} + q_n)
\]

\[
+ \frac{A}{\sinh^2 \frac{q_1}{2}} + \frac{B}{\sinh^2 q_1} + \frac{C}{\sinh^2 \frac{q_n}{2}} + \frac{D}{\sinh^2 q_n},
\]

(1.2)

which includes all the above potentials as limiting cases. He gave \( 2n \times 2n \) Lax representation and proved Liouville integrability for this system.

Sklyanin [23] found \( 2 \times 2 \) Lax representations for all cases (including \( BC_n \)) except \( B_n^{(1)}, D_n^{(1)}, A_{2n+1}^{(2)} \), and (1.4), introducing reflection equation which also provided quantisation of those systems. The \( 2 \times 2 \) Lax matrices (\( L \)-operators) for the rest 3 cases and for Inozemtsev’s extension (1.2) were found in [10, 11, 20]. See also [14] where Inozemtsev’s case was interpreted as the \( A_n \) type open Toda lattice interacting with two Lagrange tops (one on each end of the lattice).

Periodic Toda lattice (of the \( A_n^{(1)} \) type) was separated in [6]. In [22] it was treated within the \( R \)-matrix method which allowed separation of its quantum counterpart. Partial results on separation of variables for other Toda lattices were scattered in several places [11, 21, 14], essentially repeating the basic technique of [22] for the
case of reflection equation algebra introduced in [23]. As for a detailed algebro-

dynamical treatment of many of these Toda lattices we refer to [1].

In the present paper we prove separation of variables for the generic potential

\[ L(u) f(u) = v f(u) , \quad (f(u) = (f_1(u), f_2(u))^t) . \]

We recall that (usually) the separation variables are obtained as poles of the BA

function \( f(u) \). The standard normalisation \( f_1(u) = 1 \) (or \( f_2(u) = 1 \))

which was valid, for instance, for the \( A_n^{(1)} \) case [22] does not work here, giving too

many poles which are not in involution with respect to the Poisson bracket. The

reason is extra symmetries of the Lax matrix. To obey the symmetry and reduce

the number of poles to the number of degrees of freedom, one has to find a specific

normalisation \( \alpha(u) = (\alpha_1(u), \alpha_2(u)) \) of the BA vector:

\[ \alpha_1(u) f_1(u) + \alpha_2(u) f_2(u) = 1 . \]

Structure of the paper is following. In Section 2 we give an overview of the

method of separation of variables and apply it then, in the Section 3, to the integrable

system in question. In Section 4 there are some concluding remarks.

2. The method

The method of separation of variables plays an important role in studying Liouville

integrable systems.

**Definition 1.** A Liouville integrable system possesses a Lax matrix if there is a

matrix \( L(u) \) dependent on a “spectral parameter” \( u \in \mathbb{C} \) such that its characteristic

polynomial obeys two conditions

\[ \{ \det(L(u) - v \cdot 1), \det(L(\tilde{u}) - \tilde{v} \cdot 1) \} = 0 , \quad \forall u, \tilde{u}, v, \tilde{v} \in \mathbb{C} ; \]

\[ \{ \det(L(u) - v \cdot 1) \} \text{ generates all integrals of motion } H_i . \]

**Definition 2.** By separation of variables (SoV) in the classical mechanics we call an

existence of a canonical transformation \( M : (x, p) \mapsto (u, v) , \) \( M : H_i(x, p) \mapsto H_i(u, v) \)

such that \( H_i(u, v) \) are in the separated form:

\[ \Phi(u_i, v_i; H_1, \ldots, H_n) \equiv \det(L(u_i) - v_i \cdot 1) = 0 , \quad i = 1, \ldots, n . \]

The above definition corresponds precisely to the standard definition of SoV in the

Hamilton-Jacobi equation [2].
We would like to notice here that we have connected our definition of SoV to Lax representation and to associated spectral curve of the Lax matrix \( L(u) \), so it might be not unique (if exists) in the case when a chosen integrable system has, for instance, two or more inequivalent Lax representations.

One of the main questions in the theory is: How to constructively define those new separation variables \((u_j, v_j)\) sitting on the spectral curve of an \( L \)-matrix for a given integrable system?

For a very long time a great deal of attention has been given to so-called coordinate separation of variables or to separation in the configuration space (see, for instance, \([8, 24, 12, 13, 4, 25]\) and references therein). In this case the separation variables \( u_j \) are functions of \( x_i \)'s only:

\[
    u_j = u_j(x_1, \ldots, x_n). \tag{2.1}
\]

Such kinds of integrable systems admitting a coordinate (spatial) separation of variables were studied in detail, although in the same time it was understood that far not every Liouville integrable system can be separated through a transition (2.1) to new “coordinates” \( u_i \). The class of admissible transformations should be enlarged for a generic integrable system up to a general canonical transformation

\[
    u_j = u_j(x_1, \ldots, x_n, p_1, \ldots, p_n), \quad v_j = v_j(x_1, \ldots, x_n, p_1, \ldots, p_n). \tag{2.2}
\]

The very existence of SoV according to the above definition is still unproved in general, to author’s knowledge; although there are powerful methods which have been applied to many families of integrable systems (see recent review \([25]\)) showing that separability is one of the most important features of integrability, and that hopefully latter always implies former. The method of SoV in its modern formulation can be found in \([25]\). See also the works \([16, 17, 18, 19, 15]\). Here we describe very briefly its main steps.

The first difficulty is: How to find separation variables \( u_j \)? There is a general answer to this question, which has been inspired by the whole experience of the inverse scattering method, and it is a very simple one:

**Answer:** They \( \{u_j\} \) are poles of the Baker-Akhiezer function which is properly normalized.

There is, however, a slight further problem of choosing the right normalisation for the BA function; the problem which was not completely solved by powerful and successful method of inverse scattering. So, a general theory connecting the symmetry of the Lax matrix to proper normalisation vector of the BA function is still incomplete. But, supposing that one somehow knows the right normalisation, then one could proceed further and put the above general recipe into the formulas (cf. \([25]\)).

The linear problem for the BA function \( f(u) \) is of the form

\[
    L(u) f(u) = v(u) f(u), \quad (\det(L(u) - v \cdot 1) = 0). \tag{2.3}
\]
The normalisation $\bar{\alpha}(u)$ of the eigenvectors $f(u)$ has to be fixed

$$\sum_{i=1}^{N} \alpha_i(u) \ f_i(u) = 1, \quad (f(u) \equiv (f_1(u), \ldots, f_N(u))^t). \quad (2.4)$$

Let $L(u)$ be a meromorphic function in $u$ then $f(u)$ is also meromorphic in $u$. Let us look at its $[f(u)'s]$ poles $u_j$:

$$f_i^{(j)} = \text{res}_{u=u_j} f_i(u).$$

Then from (2.3)–(2.4) we have

$$\left\{ \begin{array}{l}
L(u_j) \ f^{(j)} = v_j \ f^{(j)}, \quad v_j \equiv v(u_j), \\
\sum_{i=1}^{N} \alpha_i(u_j) \ f^{(j)}_i = 0.
\end{array} \right. \quad (2.5)$$

Equations (2.3) are $N + 1$ linear homogeneous equations for the separation variables $u = u_j$ and $v = v_j$ which are bounded by definition to the spectral curve (cf. (2.3)). These equations have to be compatible. The system (2.5) is equivalent to the condition:

$$\text{rank} \left( \begin{array}{c}
\bar{\alpha}(u) \\
L(u) - v \cdot 1
\end{array} \right) = N - 1 \quad (2.6)$$

where $\bar{\alpha}$ is thought of as a row-vector. Finally, the condition (2.6) can be rewritten as the following vector equation:

$$\bar{\alpha} \cdot (L(u) - v \cdot 1) = 0, \quad (2.7)$$

where wedge denotes the classical adjoint matrix (matrix of cofactors).

**Proposition 1** Excluding $v$, one can derive from equations (2.7) the equation for $u$ in the form

$$B(u) = \det \begin{pmatrix}
\bar{\alpha} \\
\bar{\alpha} \cdot L(u) \\
\vdots \\
\bar{\alpha} \cdot L^{N-1}(u)
\end{pmatrix} = 0. \quad (2.8)$$

**Proof** When $u = u_j$ we have the equations (cf. (2.3))

$$L(u) \ f = v(u) \ f, \quad \bar{\alpha} \ f = 0. \quad (2.9)$$

Hence

$$\bar{\alpha} \ L^k \ f = 0, \quad k = 0, 1, 2, \ldots.$$ 

Then (2.8) follows because $f$ is a non-zero vector.

Also, from equations (2.7) we can get formulas for $v$ in the form

$$v = A(u)$$

5
with $A(u)$ being some rational functions of the entries of $L(u)$ (cf. [15]).

What is left is just to verify (somehow) the canonical brackets between the whole set of separation variables, namely: between zeros $u_j$ of $B(u)$ and their conjugated variables $v_j \equiv v(u_j) = A(u_j)$. To do this final calculation we need information about Poisson brackets between entries of the Lax matrix $L(u)$ which is usually provided by corresponding $r$-matrix (standard or dynamical).

In order to perform a SoV, say, in a strong sense, one has to try also to get an explicit representation for the corresponding generating function $F(u|x)$ of the separating canonical transform $M$ from the set $(x_j, p_j)$ to the set $(u_j, v_j)$. Actually, to find the generating function $F(u|x)$ one has to solve the system of non-linear equations of the form

$$\tilde{\alpha}(u_j) \left( L(u_j) + \frac{\delta F}{\delta u_j} \cdot 1 \right) \bigg|^{p_k = \frac{\delta F}{\delta x_k}} = 0,$$

In the quantum case the function $F(u|x)$ has a quantum counterpart: the kernel $\mathcal{M}_h(u|x)$ of the separating integral transform $M_h$, so that

$$\mathcal{M}_h(u|x) \sim \exp\left(\frac{i}{\hbar} F(u|x)\right), \quad \hbar \to 0.$$

For some integrable systems such special functions of many variables ($F$ and $\mathcal{M}_h$) can be obtained in very explicit terms (cf. [25, 16, 17, 18]). Remark here that the other generating function has been traditionally associated to constructions of the method of separation of variables in the Hamilton-Jacobi equation, namely: the action function $S(H|u)$ given in terms of separation variables, $u_j$, and integrals of motion, $H_j$. Our choice of arguments of the generating function is justified by the quantum case where $F(y|x)$ has a direct quantum analog, while the action function $S(H|u)$ does not have such a nice quantum counterpart at all.

Very often the above prescription of SoV should be read “in the opposite direction” (because one does not usually know the separating normalisation in advance). Sometimes, regardless of choosing the vector $\tilde{\alpha}$, the Baker-Akhiezer function $f(u)$ has just needed number of poles in involution. Sometimes, and this is very important, $f(u)$ has too many poles and they are not mutually in involution, showing that there are some constraints between them. In the latter case, one should find proper (and quite unique) normalisation vector $\tilde{\alpha}(u)$ so that to fix all extra poles of $f(u)$ being constants. The prescription then makes us to search for a way to resolve possible constraints on poles of the Baker-Akhiezer function by using the freedom of choosing its normalisation. In this paper we show that it is the case in the $D_n$ type periodic Toda lattice and give the right normalisation for corresponding $f(u)$, thereby producing a SoV for this system which was not solved before by this method.

If we make a similarity transformation for the $L$-matrix

$$\tilde{L}(u) = V(u) \ L(u) \ V^{-1}(u)$$

with a non-degenerate matrix $V(u)$ then the linear problem

$$L(u) \ f(u) = v(u) \ f(u), \quad \tilde{\alpha} \cdot f = 1$$

(2.10)
turns into
\[ \tilde{L}(u) \tilde{f}(u) = v(u) \tilde{f}(u), \quad \tilde{\alpha}_0 \cdot \tilde{f} = 1 \]
where
\[ \tilde{f}(u) = V(u) f(u), \quad \tilde{\alpha}(u) = \tilde{\alpha}_0(u) V(u). \]

This shows that the freedom of choosing the normalisation vector \( \tilde{\alpha} \) is equivalent to the freedom of making similarity transformations to the initial Lax representation.

Let us put \( N = 2 \), so that we assume from now on that we have a \( 2 \times 2 \) Lax representation for our integrable system. In this case the equations of SoV (2.6) have the form
\[ \text{rank} \begin{pmatrix} \alpha_1(u) & \alpha_2(u) \\ L_{11}(u) - v & L_{12}(u) \\ L_{21}(u) & L_{22}(u) - v \end{pmatrix} = 1. \]
From which we conclude that
\[
\begin{cases}
\alpha_1 L_{12} = \alpha_2 (L_{11} - v) \\
\alpha_1 (L_{22} - v) = \alpha_2 L_{21}
\end{cases} \iff \begin{cases}
B(u) = \alpha_2^2 L_{12} - \alpha_1 \alpha_2 (L_{11} - L_{22}) - \alpha_2^2 L_{21} = 0 \\
v = A(u) = L_{11} - \frac{\alpha}{\alpha_2} L_{12} = L_{22} - \frac{\alpha}{\alpha_1} L_{21}
\end{cases}.
\]
Suppose we have found a non-degenerate matrix \( V(u) \) such that the Lax matrix \( \tilde{L}(u) = V(u)L(u)V^{-1}(u) \) ends up in SoV with the standard normalisation vector \( \tilde{\alpha}_0 = (1, 0) \). That would imply the separability for the matrix \( L(u) \) with the normalisation vector \( \tilde{\alpha} \) (cf. (2.11))
\[ \tilde{\alpha} = \tilde{\alpha}_0 \cdot V = (V_{11}(u), V_{12}(u)). \]

3. The separation

Let us remind first the construction of the \( 2 \times 2 \) Lax matrix for the \( D_n \) type periodic Toda lattice with four extra parameters (Inozemtsev’s case) [10, 11, 20, 14].

Given the rational classical \( 4 \times 4 \) \( r \)-matrix of the form
\[ r(u) = \frac{\kappa}{u} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]
one considers two algebras: the Sklyanin quadratic algebra
\[ (S) \quad \{ L^{(1)}(u), L^{(2)}(v) \} = [ r(u - v), L^{(1)}(u) L^{(2)}(v) ], \]
and the reflection equation algebra
\[ (RE) \quad \{ L^{(1)}(u), L^{(2)}(v) \} = [ r(u - v), L^{(1)}(u) L^{(2)}(v) ]. \]
where the Lax matrix

\[ L^{(1)}(u) r(u+v) L^{(2)}(v) - L^{(2)}(v) r(u+v) L^{(1)}(u). \]

These two algebras appeared in the quantum inverse scattering method. Their representations play an important role in the classification and studies of classical integrable systems (see, for instance, \[3, 27, 23, 24\] and references in there). Here the sup-indices (1) and (2) mean standard tensoring of the 2 \times 2 matrix \( L(u) \) with the 2 \times 2 unit matrix \( \mathbf{1} \): \( L^{(2)}(u) = \mathbf{1} \otimes L(u) \), \( L^{(1)}(u) = L(u) \otimes \mathbf{1} \).

The following 2 \times 2 \( L \)-operators

\[ L_1(u) = \begin{pmatrix} u^2 x_1 + u [i(x_1^2 - 1) p_1 + c_1 x_1 + c_2] + c_1 c_2 \\ u (u^2 + (x_1^2 - 1) p_1^2 - 2i p_1 (c_1 x_1 + c_2) - c_1^2) \end{pmatrix} + \begin{pmatrix} u (x_1^2 - 1) \\ u^2 x_1 - u [i(x_1^2 - 1) p_1 + c_1 x_1 + c_2] + c_1 c_2 \end{pmatrix} \]

and

\[ L_2(u) = \begin{pmatrix} -u^2 x_2 + u [i(x_2^2 - 1) p_2 + c_3 x_2 + c_4] - c_3 c_4 \\ u (u^2 + (x_2^2 - 1) p_2^2 - 2i p_2 (c_3 x_2 + c_4) - c_3^2) \end{pmatrix} + \begin{pmatrix} -u^2 x_2 - u [i(x_2^2 - 1) p_2 + c_3 x_2 + c_4] - c_3 c_4 \end{pmatrix} \]

satisfy the \( (RE) \) algebra with \( \kappa = i \). Here the \( (x_j, p_j) \) are canonical Darboux variables, i.e., the Poisson brackets are \( \{p_j, x_k\} = \delta_{jk} \). These \( L \)-operators were found in \[10, 11\] (see also \[20, 14\]). They generate the \( D_n \) type periodic Toda lattice having four additional (singular) potential terms with the parameters \( c_1, c_2, c_3, c_4 \). Namely, consider the following Lax matrix

\[ T(u) = L_3(u) \cdots L_n(u) \cdot L_1(u) \cdot L_n^{-1}(-u) \cdots L_3^{-1}(-u) \cdot L_2(u), \]

where the \( L \)-operators \( L_3, \ldots, L_n \) satisfy the \( (S) \) algebra with \( \kappa = i \) and have the form:

\[ L_k(u) = \begin{pmatrix} 0 & -x_k^{-1} \\ x_k & u + ip_k x_k \end{pmatrix}, \quad k = 3, \ldots, n. \]

The constructed Lax matrix describes an integrable system with the following Hamiltonian:

\[ H_1 = \sum_{i=3}^{n} (x_i p_i)^2 + p_1^2 (x_1^2 - 1) + p_2^2 (x_2^2 - 1) - 2 \sum_{i=3}^{n-1} \frac{x_i}{x_{i+1}} + 2 \frac{x_2}{x_3} + 2 x_1 x_N - 2i p_1 (c_1 x_1 + c_2) - 2i p_2 (c_3 x_2 + c_4). \]

This Hamiltonian turns into the one for Inozemtsev’s Toda lattice (cf. \[1, 2\]) under the following change of variables: \( x_1 = \cosh q_1, \ x_2 = \cosh q_2, \ x_j = \exp(q_j), \ j = 3, \ldots, n, \) and obvious gauge-type canonical transformation for two particles (with the variables \( (x_1, p_1) \) and \( (x_2, p_2) \)) to get rid of terms linear in \( p_1, p_2 \) in \(3.2\).

Our problem is to separate variables in this system and restore the Lax matrix \( T(u) \) in terms of (new) separation variables. This is performed in the following three Propositions.
Spectral curve has the following form:

\[
\det (T(u) - v \cdot 1) = v^2 - v \left[ (-1)^n u^{2n+2} + (-1)^n H_1 u^{2n} + H_2 u^{2n-2} + \ldots + H_n u^2 - 2c_1c_2c_3c_4 \right] \\
+ \prod_{i=k}^4 (u^2 - c_k^2) = 0.
\]

(3.3)

**Proposition 2** Let

\[
V(u) = \begin{pmatrix}
1 - x_2 & u + c_3 - ip_2 (1 - x_2) \\
0 & \frac{1}{1 - x_2}
\end{pmatrix}.
\]

(3.4)

Then it is easy to verify that \(V(u)\) obeys the (S) algebra with \(\kappa = i\) and, moreover, it sends the matrix \(L_2(u)\) into the triangular form:

\[
\tilde{L}_2(u) \equiv V(-u) \cdot L_2(u) \cdot V^{-1}(u) = \begin{pmatrix}
(u - c_3)(u + c_4) & 0 \\
-u \frac{1 + x_2}{1 - x_2} & (u + c_3)(u - c_4)
\end{pmatrix}.
\]

**Proof** It is a simple and straightforward algebraic calculation. The second part of the statement is crucial for the following procedure of separation of variables and is absolutely non-trivial since we apply *almost* similarity transformation to the boundary matrix \(L_2(u)\) to put it into the triangular form (notice the changed sign of the spectral parameter \(u\)).

**Proposition 3** Consider the representation of the (RE) algebra of the following form:

\[
\tilde{T}(u) = V(u) \cdot L_3(u) \cdot \ldots \cdot L_n(u) \cdot L_1(u) \cdot L_n^{-1}(-u) \cdot \ldots \cdot L_3^{-1}(-u) \cdot V^{-1}(-u)
\]

\[
= \begin{pmatrix}
\tilde{A}(u) & \tilde{B}(u) \\
\tilde{C}(u) & \tilde{D}(u)
\end{pmatrix}.
\]

(3.5)

Then the matrix \(\tilde{T}(u)\) which is similar to the \(T(u)\) can be represented as follows:

\[
\tilde{T}(u) \equiv V(u) \cdot T(u) \cdot V^{-1}(u) = \tilde{T}(u) \cdot \tilde{L}_2(u).
\]

(3.6)

Hence

\[
\text{tr} T(u) = (u - c_3)(u + c_4) \tilde{A}(u) + (u + c_3)(u - c_4) \tilde{D}(u) - u \frac{1 + x_2}{1 - x_2} \tilde{B}(u),
\]

with

\[
\det \tilde{T}(u) = (u^2 - c_1^2)(u^2 - c_2^2), \quad \det \tilde{L}_2(u) = (u^2 - c_3^2)(u^2 - c_4^2),
\]

\[
\det T(u) = \prod_{k=1}^4 (u^2 - c_k^2).
\]
If we choose \( n \) zeros \( u_k \) of the polynomial \( \tilde{B}(u) \) as \( n \) separation variables:

\[
\tilde{B}(\pm u_k) = 0, \quad \lambda_k^\pm = \tilde{D}(\pm u_k), \quad k = 1, \ldots, n, \tag{3.7}
\]

then they satisfy the relations

\[
\{u_j, u_k\} = 0, \\
\{u_k, \lambda_k^\pm\} = \pm i \lambda_k^\pm, \\
\lambda_k^+ \lambda_k^- = (u_k^2 - c_1^2)(u_k^2 - c_2^2), \\
\{\lambda_j^\pm, \lambda_k^\pm\} = \{\lambda_j^\pm, u_k\} = 0, \quad j \neq k.
\]

Moreover, from their definition it follows that they satisfy the equalities \((k = 1, \ldots, n)\)

\[
\text{tr} T(u_k) = (u_k - c_3)(u_k + c_4) \lambda_k^- + (u_k + c_3)(u_k - c_4) \lambda_k^+ \tag{3.8}
\]

(the separation equations).

**Proof** The matrix \( \tilde{T}(u) \) satisfies the involution

\[
\tilde{T}(-u) = [\det \tilde{T}(u)] \cdot \tilde{T}^{-1}(u) = \sigma_2 \tilde{T}^t(u) \sigma_2
\]

or, in component-wise form,

\[
\tilde{A}(-u) = \tilde{D}(u), \quad \tilde{B}(-u) = -\tilde{B}(u), \quad \tilde{C}(-u) = -\tilde{C}(u).
\]

Moreover, its polynomial in \( u \) entries have the degrees

\[
\deg \tilde{T}(u) = \left( \begin{array}{cc} 2n & 2n + 1 \\ 2n - 1 & 2n \end{array} \right)
\]

The matrix \( \tilde{T}(u) \) obeys the \((RE)\) algebra of Poisson brackets according to the Proposition 2 from [22] because \( L_j(u) , j = 3, \ldots, n, \) and \( V(u) \) obey the \((S)\) algebra brackets. Using our Proposition 2 we establish the decomposition \((3.6)\) for the matrix \( \tilde{T}(u) \) which is similar to the Lax matrix \( T(u) \). The rest of the formulas are obvious. The polynomial \( \tilde{B}(u) \) has exactly \( n \) non-trivial zeros \( u_k, k = 1, \ldots, n \) (doubled by the obvious \( \pm\)-symmetry). The related \( \lambda_k^\pm \) variables are defined according to \((3.7)\).

These new variables \( u_k, \lambda_k^\pm \) are bounded to the equalities \((3.8)\) by their definition. The calculation of all the Poisson brackets between \( u_k, \lambda_k^\pm \) is a standard procedure nowadays which was originally invented in [22]. Let us recall, for instance, how one calculates the brackets between 

\[
-i\{\tilde{B}(u), \tilde{D}(v)\} = \frac{\tilde{D}(u)\tilde{B}(v) - \tilde{D}(v)\tilde{B}(u)}{u - v} + \frac{\tilde{D}(-u)\tilde{B}(v) + \tilde{D}(v)\tilde{B}(u)}{u + v}.
\]

Combining it with the equation

\[
0 = \{\tilde{B}(u_k), \tilde{D}(v)\} = \{\tilde{B}(u), \tilde{D}(v)\} |_{u = u_k} + \tilde{B}'(u_k)\{u_k, \tilde{D}(v)\}
\]
we obtain
\[ \{ u_k, \lambda^+_k \} = \frac{-i}{B'(u_k)} \left( \frac{\lambda^+_k}{u_k-v} + \frac{\lambda^-_k}{u_k+v} \right) \tilde{B}(v) \big|_{v=u_k} = i \lambda^+_k. \]

Proposition 4 The interpolation problem to restore the matrix \( \tilde{T}(u) \) in terms of new (separation) variables \( u_k, \lambda^\pm_k \) has the following solution:

\[
\tilde{B}(u) = (-1)^n u \prod_{k=1}^n (u^2 - u_k^2),
\]

\[
\tilde{D}(u) = (-1)^n c_1 c_2 \prod_{k=1}^n \frac{u^2 - u_k^2}{u_k^2} + \sum_{k=1}^n \left[ \frac{u(u+u_k)}{2u_k^2} \lambda^+_k + \frac{u(u-u_k)}{2u_k^2} \lambda^-_k \right] \prod_{j \neq k} \frac{u^2 - u_j^2}{u_k^2 - u_j^2},
\]

\[
\tilde{A}(u) = \tilde{D}(-u), \quad \tilde{C}(u) = \frac{\tilde{A}(u)\tilde{D}(u) - (u^2 - c_1^2)(u^2 - c_2^2)\tilde{D}(0)}{\tilde{B}(u)}.
\]

Proof The formula for \( \tilde{B}(u) \) is obvious. The polynomial \( \tilde{D}(u) \) of degree \( 2n \) is restored in terms of the separation variables by interpolation with \( 2n + 1 \) data of the form

\[ \tilde{D}(\pm u_k) = \lambda^\pm_k, \quad \tilde{D}(0) = c_1 c_2. \]

Now we can derive, in principle, the formulas connecting old and new variables. For instance, noticing that \( \tilde{D}(u) \) has the asymptotics

\[ \tilde{D}(u) = \frac{(-1)^n}{1 - x^2} u^{2n} + \ldots, \quad u \to \infty, \]

we find that

\[ \frac{1}{1 - x^2} = \frac{c_1 c_2}{\prod_{k=1}^n u_k^2} + (-1)^n \sum_{k=1}^n \frac{\lambda^+_k + \lambda^-_k}{2u_k^2} \prod_{j \neq k} \frac{u_k^2 - u_j^2}{u_k^2 - u_j^2}. \]

We can express some other combinations of initial variables in terms of new (separation) variables, comparing the coefficients of entries of \( \tilde{T}(u) \) in both representations. Considering the tr \( T(u) \), we could as well get the expressions for the integrals of motion \( H_1, \ldots, H_n \) in terms of the separation variables.

Corollary 1 The separating normalisation vector for the \( D_n \) type periodic Toda lattice with the Hamiltonian (3.2) and with the Lax matrix (3.1) has the form

\[ \vec{\alpha} = (1 - x_2, u + c_3 - ip_2 (1 - x_2)). \]

The separation variables \( u_k \) and \( v_k^\pm \equiv (u_k \pm c_3)(u_k \mp c_4) \lambda_k^\pm \), \( k = 1, \ldots, n \), are sitting on the spectral curve (3.3) of the Lax matrix \( T(u) \) (3.1)

\[ (v_k^\pm)^4 - v_k^\pm \text{tr} T(u_k) + \det T(u_k) = 0, \]
i.e.

\[ v_k^+ + v_k^- = \text{tr} \ T(u_k), \quad v_k^+ v_k^- = \det \ T(u_k). \]

They have the following Poisson brackets

\[ \{ u_k, v_k^\pm \} = \pm i \ v_k^\pm. \]

**Remark 1** The (obvious) alternative choice of the separating normalisation vector follows if we put the matrix \( L_1(u) \) instead of the matrix \( L_2(u) \) (cf. Proposition 2) into the triangular form. This would correspond to interchanging two edge particles in the lattice.

It would be interesting to (explicitly) construct the generating function \( F(u|x) \) of this separating canonical transform.

If we introduce the canonically conjugate variables \( \pi_j \)

\[ \{ \pi_j, u_k \} = \delta_{jk} \]
then we can put

\[ v_k^\pm = [\det \ T(u_k)]^{\frac{1}{2}} \exp(\mp i \ \pi_k) \]
and get the separation equations in the form

\[ 2 [\det \ T(u_k)]^{\frac{1}{2}} \cos(\pi_k) = \text{tr} \ T(u_k). \]

Hence, the action variables \( S_k(H_1, \ldots, H_n) \) have the form

\[ S_k(H_1, \ldots, H_n) = \oint_{\alpha_k} \arccos \left( \frac{\text{tr} \ T(u)}{2[\det \ T(u)]^{\frac{1}{2}}} \right) \ du, \quad k = 1, \ldots, n, \]

where \( \alpha_k \) are the \( \alpha \)-cycles on the Riemannian surface of \( \sqrt{\text{tr}^2 T(u) - 4 \det T(u)}. \)

One can get the quasiclassical spectrum \( H_k(N_1, \ldots, N_n) \) of the integrals of motion \( H_1, \ldots, H_n \) (cf. [3]) inverting the integrals (Bohr-Sommerfeld quantisation)

\[ S_k(H_1, \ldots, H_n) = h N_k, \quad k = 1, \ldots, n, \]

where \( N_k \)'s are the quantum numbers, \( N_k = 1, 2, 3, \ldots \). Obtaining of true discrete spectrum of the integrals of quantum \( \mathcal{D}_n \) type periodic Toda lattice is the problem of quantum separation of variables.
4. Concluding remarks

We refer reader to the review [25] (cf. also the work [13]) where it was illustrated that the simplest choice of the normalisation vector $\vec{\alpha}$, when one of the components of the Baker-Akhiezer function $f(u)$ (for instance the first one) is equal to 1, i.e. when

$$\vec{\alpha} = (1, 0, \ldots, 0),$$

provides a SoV for many integrable systems of the $A_n$ type. If a chosen integrable system can not be separated with this simplest normalisation, and this usually means that its $L$-matrix has some extra symmetries/involutions (i.e. is of the $BC_n$ or $D_n$ type or obeys elliptic $r$-matrix), then the main problem is to find proper $\vec{\alpha}$. For the time being there is no theory to give a general prescription for finding right normalisation vector $\vec{\alpha}$ in those cases. Although one practical rule can be suggested. Usually, if one looks at the poles of the Baker-Akhiezer function with the simplest normalisation (4.1), one finds that there are too many poles and they do not respect the symmetry presenting in the problem. Then the rule is the following: take an ansatz for $\vec{\alpha}(u)$ with some dependence on $u$ and with some indeterminates in it, derive equations for those indeterminates demanding that (a) $f(u)$ with such a normalisation has the right number of moving poles respecting involutions of the spectral curve and (b) all extra poles are equal to constants. Then solve the equations ... .

In this paper we applied this approach to the $D_n$ type periodic Toda lattice with four additional singular terms in the potential. This system is not separated with the simplest choice of the normalisation vector $\vec{\alpha}$ (4.1), so we have derived the right normalisation $\vec{\alpha}$ producing the SoV. For some of the root systems the separating normalisation vector is a constant vector (cf. the $BC_n$ case in [11, 20, 14]). For the generic $D_n$ case the separating $\vec{\alpha}(u)$ depends on the spectral parameter $u$ and on the phase variables, so it is dynamical. We think that it is an important feature of this kind of problems (the ones with extra involutions), that the separating choice of $\vec{\alpha}$ is not quite arbitrary, as it was for some of the $A_n$ type of systems, but is quite unique and dynamical.

The specific situation with the $D_n$ type periodic Toda lattice, i.e. that the right $\vec{\alpha}$ is $u$-dependent and dynamical, is surely connected with the fact that we have the dynamical boundary $L_{1,2}$-matrices in construction of the corresponding Lax matrix $T(u)$ (3.1) for this case.

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