General Null Lagrangians and Their Novel Role in Classical Dynamics

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Abstract. A method for constructing general null Lagrangians and their higher harmonics is presented for dynamical systems with one degree of freedom. It is shown that these Lagrangians can be used to obtain non-standard Lagrangians, which give equations of motion for the law of inertia and some dissipative dynamical systems. The necessary condition for deriving equations of motion by using null Lagrangians is presented, and it is demonstrated that this condition plays the same role for null Lagrangians as the Euler-Lagrange equation plays for standard and non-standard Lagrangians. The obtained results and their applications establish a novel role of null Lagrangians in classical dynamics.
1. Introduction

There are three families of Lagrangians that can be classified as standard, non-standard and null Lagrangians. The main characteristic of standard Lagrangians is the presence of the kinetic and potential energy-like terms, and these Lagrangians are commonly used in Classical Mechanics (CM) to derive equations of motion for different dynamical systems (e.g., [1-5]). On the other hand, in non-standard Lagrangians, neither the kinetic nor the potential energy-like terms can be identified. Different methods have been developed to derive the standard and non-standard Lagrangians for given ordinary differential equations (ODEs) [6-12].

The third family of null Lagrangians (NLs) has the following two main equivalent characteristics: (i) they must satisfy identically the Euler-Lagrange (E-L) equation: the null condition, and (ii) they must be expressed as the total derivative of any scalar function known as gauge function: the gauge condition. The properties and applications of these null Lagrangians have been extensively explored in different fields of mathematics (e.g., 13-20]) as well as in some physical applications (e.g., [21,22]) that include restoring Galilean invariance of Lagrangians in Newtonian dynamics [23,24], and introducing forces to CM [25-27].

Several methods of constructing the NLs have been proposed and most of them rely on specifying a gauge function and using it to obtain the resulting null Lagrangian [13,14,26,27]. However, in this paper, we develop a new method that is based on a generating function that differs from the gauge function. It is shown that using this generating function a general null Lagrangian can be obtained. Depending on the form of the generating function, either standard or non-standard null Lagrangians can be constructed [28]. Moreover, the method also allows us to derive higher harmonics of these null Lagrangians, and the harmonics are new mathematical objects in calculus of variations; therefore, their role and possible applications are discussed.

Since the NLs satisfy identically the E-L equation, they can be added to any standard or non-standard Lagrangian, and this addition does not affect the resulting equation of motion [13-18]. Despite this well-known property of the NLs, we demonstrate that they may be used to construct non-standard Lagrangians, which then give equations of motion when substituted into the E-L equations. Using this approach, the NLs for some equations of motion can be found and compared to other Lagrangians for these equations, which may give new insights into symmetry of the Lagrangians and the resulting equations of motion [29-33]. The fact that equations of motion can be derived by using the NLs demonstrates a novel role of these Lagrangians in classical dynamics.

Applications of the obtained results show that the law of inertia can be derived from its null Lagrangian, and that other constructed NLs give equations of motion for some dissipative systems with linear and quadratic dissipative terms. There have been many attempts to establish Lagrangian formalism for dissipative systems [7-9,27,34-47]. However, none of the previous work on this problem has involved null Lagrangians and shown that they can be used to obtain equations of motion. The results of this paper
demonstrate the validity of this approach and presents the necessary condition that null Lagrangians must obey in order to give equations of motion. The condition plays the same role for null Lagrangians as the Euler-Lagrange equation plays for standard and non-standard Lagrangians.

The paper is organized as follows. In Section 2, Lagrangian formalism and its Lagrangians are briefly described; construction of null Lagrangians and non-standard Lagrangians resulting from them is presented in Sections 3 and 4, respectively; applications of the results to dynamical systems are considered and discussed in Section 5; and conclusions are given in Section 6.

2. Lagrangian formalism and its Lagrangians

2.1. Standard and non-standard Lagrangians

Dynamical systems with holonomic constraints can be analyzed using the Lagrangian formalism. The foundation of this formalism is the smooth configuration manifold $\mathcal{Q}$ constructed from the generalized coordinates of the system of interest with holonomic constraints and its tangent bundle $T\mathcal{Q}$. A Lagrangian $\mathcal{L}$ for this system is commonly defined as a $C^\infty$ map from $T\mathcal{Q}$ to the real line $\mathbb{R}$, i.e., $\mathcal{L} : T\mathcal{Q} \to \mathbb{R}$ [3].

In the case of one-dimensional dynamical systems, a Lagrangian can be symbolized as $\mathcal{L} = \mathcal{L}(x, \dot{x})$ in an open neighborhood $\mathcal{U} \subset T\mathcal{Q}$ containing a point $p \in T\mathcal{Q}$ in a local coordinate system defined by the coordinate function $x : \mathcal{Q} \to \mathbb{R}$ and its associated tangent vector $\dot{x}$ at the point $p$ using a diffeomorphism $\phi : p \in \mathcal{U} \subset T\mathcal{Q} \to \mathcal{Q} \times \mathbb{R}$ known as a local trivialization. Similarly, the explicit dependence of a Lagrangian on time can be defined as $\mathcal{L} : T\mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ and thus expressing $\mathcal{L} = \mathcal{L}(x, \dot{x}, t)$. After a suitable Lagrangian for an interested system is constructed or discovered, the time evolution of the system is analyzed by applying the Euler-Lagrange operator on the chosen Lagrangian, i.e., $\hat{\mathcal{E}}L[\mathcal{L} (\dot{x}, x, t)] = 0$, with $\hat{\mathcal{E}}L$ being the Euler-Lagrange (E-L) operator [2-4].

The Euler-Lagrange (EL) operator in each degree of freedom for a system is a linear operator acting on the vector space of $C^\infty$ functions defined on $T\mathcal{Q}$ and thus $\mathcal{L} \in \mathcal{F}(T\mathcal{Q})$ with $\mathcal{F}$ denoting the space of $C^\infty$ functions. Most commonly, a standard strategy for the construction of the most common Lagrangians (thus known as standard Lagrangians) are carried out from the difference of the kinetic and potential energy of the respective system under consideration, i.e., $\hat{\mathcal{E}}L[\mathcal{L}(\dot{x}, x, t)] = \frac{1}{2}\dot{x}^2 - V(x)$, with the potential energy $V(x)$ for a system with only one degree of freedom [1-3].

On the other hand, non-standard Lagrangians are dependent on functions of time in addition to the usual dynamical variables such as $x$ and $\dot{x}$ for systems with only one degree of freedom [7-11]. In general, the construction of these non-standard Lagrangians for the set of desired equations of motion requires solving nonlinear (Riccati-type) equations [9,12]. However, the problem can be simplified by replacing functions to be determined by constants, for instance, we may consider the following form of the
non-standard Lagrangian

\[ \mathcal{L}_{ns}(\dot{x}, x, t) = \frac{1}{a_1 \dot{x} + a_2 t + a_3}, \]  

with constants \( a_1, a_2, \) and \( a_3 \) describes a system undergoing a constant acceleration. It should be emphasized that the above method for constructing non-standard Lagrangians \( C^\infty \) functions demands an extra constraint that the denominators do not vanish, i.e., \( a_1 \dot{x} + a_2 t + a_3 \neq 0 \) in the domain \( U \subset TQ \). Therefore, non-standard Lagrangians in general represent dynamical systems with more constraints than the usual accompanying holonomic constraints. The construction of these Lagrangians can be carried out with the assumption that the corresponding Lagrangians are well-behaved on the domain \( U \subset TQ \).

### 2.2. Null Lagrangians

Since the null space of the Euler-Lagrange (EL) operator is itself a vector subspace, the members of this null space, rightly known as the null Lagrangians, are nullified by the E-L operator. More explicitly, an action functional defined in terms of null Lagrangians as

\[ A[x; t_e, t_o] = \int_{t_o}^{t_e} L_{\mathrm{null}}(t, \dot{x}) dt, \]  

satisfies the property \( A[x; t_e, t_o] = A[x + \eta; t_e, t_o] \) for continuous functions \( x(t) \) and \( \eta(t) \) of time on the same domain. This implies through approximations \([13,14]\) that \( A[x; t_e, t_o] = A[\eta; t_e, t_o] \) for all permissible continuous functions \( x(t) \) and \( \eta(t) \) on the domain such that \( x(t_e) = \eta(t_e) \) and \( x(t_0) = \eta(t_0) \). In other words, the above action integral produces the same real number for all admissible trajectories \( x(t) \)’s between \( t_e \) and \( t_o \) in the same domain, which in turn implies that the variation of the action integral is rendered identically to zero. This also implies that no preferred trajectory that extremizes the action integral exists for the relevant null Lagrangians. Therefore, the E-L operator renders all the null Lagrangians defined on the same domain identically to zero, i.e., no equation of motion can be obtained directly from the application of the E-L operator on null Lagrangians.

However, the study of null space of the E-L operator has been of particular interest due to its applicability to the exploration of symmetries in field theories. An effort to construct these null Lagrangians has been an active practice for a while \([13,14]\). The most prevalent constructions centered around the observation that the E-L operator nullifies the total time derivative of an arbitrary \( C^1 \) (continuous first derivatives) function \( \Phi(x, t) \), known as the gauge function, defined on the domain of interest \( U \subset TQ \). More explicitly, a Lagrangian is null if, and only if, it is the total time derivative of a scalar function \( \Phi(x, t) \) \([13]\) that can be expressed as

\[ L_{\mathrm{null}}(\dot{x}, x, t) = \frac{d\Phi(x, t)}{dt} = \frac{\partial \Phi}{\partial x} \dot{x} + \frac{\partial \Phi}{\partial t}. \]
Again, it should be emphasized that the above form of a null Lagrangian is rendered identically to zero by the E-L operator, i.e., \( \hat{E}L[\mathcal{L}_{null}] := \frac{d\Phi(x,t)}{dt} \) = 0. Then a natural strategy to construct a standard null Lagrangian is to cast it in the following form:

\[
\mathcal{L}_{null}(\dot{x}, x, t) = B(x, t)\dot{x} + C(x, t)
\]

such that \( B(x, t) := \frac{\partial\Phi}{\partial x} \) and \( C(x, t) := \frac{\partial\Phi}{\partial t} \). A similar construction has been presented in [14]. A simple example of a standard null Lagrangian can be obtained from a given gauge function \( \Phi(x, t) := f_1(t)x^2 + f_2(t) \) as

\[
\mathcal{L}_{null} = 2f_1(t)x\dot{x} + \dot{f}_2(t)
\]

with the obvious identifications \( B(x, t) = 2f_1(t)x \) and \( C(x, t) = \dot{f}_1(t)x^2 + \dot{f}_2(t) \). It is also possible to construct non-standard null Lagrangians following the same method as above. For example, the following non-standard null Lagrangian

\[
\mathcal{L}_{ns, test1}(\dot{x}, x, t) = \frac{a_1}{a_2^2} \left( a_2 x + a_4 \right)
\]

can be obtained from \( \Phi(x, t) = \frac{a_1}{a_2} \ln |a_2 x + a_4| \) [28]. Therefore, it is always possible to construct a standard as well as a non-standard null Lagrangian in the above form in Eq. (4) if a corresponding gauge function \( \Phi(x, t) \) is known.

On the contrary, it is also possible to construct some standard null Lagrangians in the above form in Eq. (4) without any reference to the corresponding gauge functions. In the following section, an earnest humble effort is made to construct a set of both standard and non-standard null Lagrangians and their higher harmonics in the similar form as in Eq. (4) without explicitly resorting to any gauge function.

3. Method to construct null Lagrangians

3.1. Standard null Lagrangians

The construction of null Lagrangians is motivated by the recognition of the null-condition \( \hat{E}L[\mathcal{L}_{null}(\dot{x}, x, t)] = 0 \), i.e., the action of the E-L operator on these smooth functions nullifies them identically to zero. In other words, these null Lagrangians must satisfy the following relation:

\[
\frac{dp_{null}}{dt} = \frac{\partial\mathcal{L}_{null}}{\partial x},
\]

with the generalized momentum \( p_{null} = \frac{\partial\mathcal{L}_{null}}{\partial \dot{x}} \).

**Proposition 1:** Let \( B(x, t), C(x, t) \) and \( f(t) \) be locally \( C^1 \) functions on \( U \subset TQ \) such that they satisfy the following condition:

\[
\frac{dB(x, t)}{dt} = \left[ \frac{\partial B(x, t)}{\partial x} \right] \dot{x} + \frac{\partial[xC(x, t)]}{\partial x}.
\]

Then \( B(x, t) \) generates a family of null Lagrangians \( \mathcal{L}_{null}(\dot{x}, x, t) \) of the form

\[
\mathcal{L}_{null}(\dot{x}, x, t) = B(x, t)\dot{x} + C(x, t)x + f(t).
\]

**Proof:** The proof follows from the null-condition \( \hat{E}L[\mathcal{L}_{null}(\dot{x}, x, t)] = 0 \) and Eq. (4).
It is interesting to note that the only restriction on $B(x, t)$ and $C(x, t)$ is that they be continuously differentiable functions on $\mathcal{U} \subset TQ$ and they satisfy the null condition, i.e., Eq. (5). A similar construction is presented in [14] albeit with a different constraint.

However, the application of the null condition, Eq. (7), on a Lagrangian of the form in Eq. (8) implies that there exists a gauge function $\Phi(x, t)$ such that the gauge condition, $L_{null}(\dot{x}, x, t) = d\Phi(x, t)/dt$, is automatically satisfied. Therefore, the null condition on the function $B(x, t)$ is equivalent to the gauge condition on the gauge function $\Phi(x, t)$ related to the same null Lagrangian. In this paper, no direct reference to a gauge function $\Phi(x, t)$, i.e., no a priori assumption of the gauge condition on the gauge function is made to construct the corresponding null Lagrangian. Instead, a general function $B(x, t)$, called a generating function, is chosen along with its null condition to construct the appropriate null Lagrangian. Now, some choices of $B(x, t)$ generating different null Lagrangians would clarify the applications of the above proposition.

(i) For the simplest choice of a constant function $B(x, t)$, Eq. (7) forces both $xC(x, t)$ and $L_{null}(\dot{x}, x, t)$ to be functions of $t$ only, implying that $EL[L_{null}(\dot{x}, x, t)] = 0$.

(ii) If $B(x, t)$ is a linear function of $x$, then the null-Lagrangian presented in [27] is reproduced. Let $B(x, t) = f_1(t)x + f_2(t)t + f_3(t)$ such that $f_1(t)$, $f_2(t)$, $f_3(t)$ and $f_4(t)$ are assumed to be arbitrary but at least twice differentiable functions of the independent variable, then the following null Lagrangian [26] is obtained

$$L_{null1}(\dot{x}, x, t) = [f_1(t)x + f_2(t)t + f_3(t)] \dot{x}$$

$$+ \left[ \frac{1}{2} \dot{f}_1(t)x + \dot{f}_2(t)t + f_2(t) + \dot{f}_3(t) \right] x + f_4(t) .$$

(iii) Now, let $B(x, t) = f_1(t)x^2 + f_2(t)t + f_3(t)$. Then the corresponding null Lagrangian turns out to be

$$L_{null2}(\dot{x}, x, t) = [f_1(t)x^2 + f_2(t)t + f_3(t)] \dot{x}$$

$$+ \left[ \frac{1}{2} \dot{f}_1(t)x^2 + \dot{f}_2(t)t + f_2(t) + \dot{f}_3(t) \right] x + f_4(t) .$$

(iv) It follows from the above special choices of $B(x, t)$ that it can be any polynomial of $x$. However, in general, $B(x, t)$ can be any continuously differentiable function of any order such as a $C^k$ function to obtain a null Lagrangian. To show that, let $B(x, t) = f_1(t)\sin(x) + f_2(t)e^{\pi t} + f_3(t)$. Then, the corresponding null Lagrangian turns out to be

$$L_{null3}(\dot{x}, x, t) = [f_1(t)\sin(x) + f_2(t)e^{\pi t} + f_3(t)] \dot{x}$$

$$- \dot{f}_1(t)\cos(x) + \dot{f}_2(t)e^{\pi t} + f_2(t)e^{\pi} + \dot{f}_3(t)x + f_4(t) .$$

Furthermore, since the addition of a total time derivative of a function whose variation vanishes at the both ends of the trajectory does not affect the dynamics of the system, the total time derivative of $B(x, t)$ can be added to the null-Lagrangian in Eq. (8) to generate its higher harmonics, which is presented below in Proposition 2.
Proposition 2: Let $B(x,t)$, and $C(x,t)$ be locally $C^\infty$ functions on $\mathcal{U} \subset TQ$ such that they are differentiable up to order $n$ to satisfy the following condition:

$$\frac{dB_n(x,t)}{dt} = \left[ \frac{\partial B_n(x,t)}{\partial x} \right] \dot{x} + \frac{\partial [xC(x,t)]_n}{\partial x},$$

(12)

with

$$B_n(x,t) := \sum_{i=0}^{n} C_{n-i} B^{(i)}(x,t)$$

(13)

and

$$[xC(x,t)]_n := \sum_{i=0}^{n} C_{n-i} [xC(x,t)]^{(i)}.$$  

(14)

Here, $(i)$ represents the $i$-th spatial derivative and $C_{n-i} = n! / [(n-i)!]$. Then $B(x,t)$ generates a family of higher harmonics of null Lagrangians $L_{null}(\dot{x}, x, t)$ of the form

$$L_{null}^{(n)}(\dot{x}, x, t) = B_n(x,t)\dot{x} + [xC(x,t)]_n + f(t).$$

(15)

Proof: Starting from the base case, i.e., the first harmonic, $L_{null}^{(1)} = L_{null} + \frac{dB(x,t)}{dt},$ it follows by induction that $L_{null}^{(n)} = L_{null}^{(n-1)} + \frac{dB_{n-1}(x,t)}{dt}$. Then the application of the null condition for the higher harmonics, i.e., Eq. (12), proves equation Eq. (15).

Again, if $B(x,t)$ is a linear function of $x$, then the null-Lagrangian presented in [26,27] is reproduced. Let $B(x,t) = f_1(t)x + f_2(t)t + f_3(t)$ and the following harmonics of the null-Lagrangian in Eq. (9) are obtained

$$L_{null}^{(1)}(\dot{x}, x, t) = [f_1(t)(x + 1) + f_2(t)t + f_3(t)] \dot{x} \\
+ \dot{f}_1(t)(\frac{x}{2} + 1)x + [\dot{f}_2(t)t + f_2(t) + \dot{f}_3(t)] (x + 1) + f_4(t),$$

(16)

and the highest harmonics

$$L_{null}^{(2)}(\dot{x}, x, t) = [f_1(t)(x + 2) + f_2(t)t + f_3(t)] \dot{x} \\
+ \dot{f}_1(t)(\frac{x^2}{2} + 2x + 1) + [\dot{f}_2(t)t + f_2(t) + \dot{f}_3(t)] (x + 2) + f_4(t).$$

(17)

Let $B(x,t) = f_1(t)\sin(x) + f_2(t)e^x t + f_3(t)$. Then, the corresponding null Lagrangian turns out to be

$$L_{null}^{(1)}(\dot{x}, x, t) = [f_1(t)[\sin(x) + \cos(x)] + 2f_2(t)e^x t + f_3(t)] \dot{x} \\
+ \dot{f}_1(t)[\sin(x) - \cos(x)] + 2\dot{f}_2(t)e^x t + 2f_2(t)e^x + \dot{f}_3(t)(x + 1) + f_4(t),$$

(18)

and the higher harmonics can be computed according to Eq. (15). It turns out that the above strategy can also be employed to generate a family of non-standard null Lagrangians as presented below.
3.2. Non-standard null Lagrangians

The key idea to generate a non-standard null-Lagrangian is to make a judicious choice of $B(x, t)$ as a fraction defined as $B(x, t) = f(t)/h(x, t)$ with $h(t)$ being a non-vanishing differentiable and integrable function. Once a choice on $B(x, t)$ is made, the corresponding non-standard null Lagrangian can be obtained by following the procedure described above in Proposition 1. Let a generating function $B(x, t)$ be expressed as

$$B(x, t) = \frac{f_1(t)}{f_2(t)x + f_3(t)t + f_4(t)},$$

where $f(t)$’s are differential functions of time and $f_2(t)x + f_3(t)t + f_4(t) \neq 0$. Then, the corresponding non-standard null Lagrangian can be obtained by using Eqs. (7) and (8) as

$$L_{\text{nsn}}(\dot{x}, x, t) = \frac{f_1(t)\dot{x}}{f_2(t)x + f_3(t)t + f_4(t)} + \frac{h_2(t)}{f_2^2} \left[ \ln|f_2x + f_3t + f_4| + \frac{f_3t + f_4}{f_3x + f_3t + f_4(t)} \right] - \frac{h_3(t)t + h_4(t)}{f_2[f_2(t)x + f_3(t)t + f_4(t)]} + f(t),$$

where $h_2(t) := [\dot{f}_1(t)f_2(t) - f_1(t)\dot{f}_2(t)]$, $h_3(t) := [\dot{f}_1(t)f_3(t) - f_1(t)\dot{f}_3(t)] - f_1(t)f_3(t)$, and $h_4(t) := [\dot{f}_1(t)f_4(t) - f_1(t)\dot{f}_4(t)]$. It is worth mentioning that any other suitably chosen differentiable and integrable function $B(x, t)$ of the form in Eq. (19) can be used as a generating function to construct its corresponding non-standard null Lagrangian.

Again, the harmonics of any non-standard null-Lagrangian can be obtained using the procedure presented in Proposition 2. For instance, the first harmonics of the above non-standard null Lagrangian in Eq. (20) can be calculated as

$$L_{\text{nsn}}^{(1)}(\dot{x}, x, t) = \left[ \frac{f_1(t)}{f_2(t)x + f_3(t)t + f_4(t)} - \frac{f_1(t)f_2(t)}{[f_2(t)x + f_3(t)t + f_4(t)]^2} \right] \dot{x} + \frac{h_2(t)}{f_2^2} \left[ \ln|f_2x + f_3t + f_4| + \frac{f_3t + f_4}{f_3x + f_3t + f_4(t)} \right] - \frac{h_3(t)t + h_4(t)}{f_2[f_2(t)x + f_3(t)t + f_4(t)]} + \frac{h_2(t)x + h_3(t)t + h_4(t)}{[f_2(t)x + f_3(t)t + f_4(t)]^2} + f(t).$$

After the exploration of the above strategy for constructing both standard and non-standard null Lagrangians, it is natural to seek for similar strategy for constructing non-standard Lagrangians. In the following section, it is shown that indeed a simple strategy for constructing non-standard Lagrangians from the above null-Lagrangians can be adopted.
4. From null to non-standard Lagrangians

The construction of non-standard Lagrangians follows from the availability of the smooth and invertible Lagrangians defined as $C^\infty$ functions on the domain $\mathcal{U} \subset TQ$. After a desirable Lagrangian, $L(\dot{x}, x, t)$, is identified, a non-standard Lagrangian, $L(\dot{x}, x, t)$ may be constructed by function composition as $L(\dot{x}, x, t) = (F \circ L)(\dot{x}, x, t)$ over the same domain $\mathcal{U} \subset TQ$. The following proposition encodes this construction of a non-standard Lagrangian and the derivation of its corresponding evolution equation.

**Proposition 3:** Let $L(\dot{x}, x, t) : \mathcal{U} \subset TQ \rightarrow \mathbb{R}$ be an invertible and $C^\infty$ function. Then a non-standard Lagrangian $L : \mathcal{U} \subset TQ \rightarrow \mathbb{R}$ can be defined by using at least a $C^2$ function $F : \mathbb{R} \rightarrow \mathbb{R}$ as $L(\dot{x}, x, t) = (F \circ L)(\dot{x}, x, t) = F(L(\dot{x}, x, t))$ and the corresponding equation of motion is given by

$$ p_L \left[ \frac{d^2 F}{dL^2} \right] \frac{dL}{dt} + (\dot{p}_L - f_L) \frac{dF}{dL} = 0, $$

(22)

where $p_L := \partial L / \partial \dot{x}$, $\dot{p}_L = dp_L / dt$ and $f_L := \partial L / \partial x$. 

**Proof:** First of all, the composition is well defined since $L(\dot{x}, x, t) : \mathcal{U} \subset TQ \rightarrow \mathbb{R}$ is invertible and $F : \mathbb{R} \rightarrow \mathbb{R}$ is at least a $C^2$ function and thereby implying $L := (F \circ L) : \mathcal{U} \subset TQ \rightarrow \mathbb{R}$ is at least a $C^2$ function, which can therefore be used to define the action functional such as Eq. (2) for extremization. Therefore, the derivation of Eq. (22) follows directly from $\dot{E}L[\mathcal{L} = F(L(\dot{x}, x, t))] = 0$. 

It should be noted that no assumption on Lagrangians $L(\dot{x}, x, t) \in \mathcal{F}(\mathcal{U} \subset TQ)$ being null is made in the above proposition. As a simple check, the equation of motion for $\mathcal{L} = F(L(\dot{x}, x, t)) = L(\dot{x}, x, t)$ reproduces the expected $\dot{p}_L - f_L = 0$. Some interesting forms of $\mathcal{L} = F(L(\dot{x}, x, t))$ are $F(L(\dot{x}, x, t)) = \exp(L)$ [11], $F(L(\dot{x}, x, t)) = \ln(L)$ and $F(L(\dot{x}, x, t)) = L^{-1}$. Upon inserting these forms in Eq. (22), the respective equations of motion are obtained as $dL / dt + (\dot{p}_L - f_L)/p_L = 0$, $dL / dt - (\dot{p}_L - f_L)/p_L = 0$ and $dL / dt - [(\dot{p}_L - f_L)/2p_L]L = 0$.

Next, it is interesting to note that Proposition 3 simplifies considerably for functions of null Lagrangians. The use of null Lagrangians in Proposition 3 eliminates the second term in Eq. (22), for $\dot{p}_{null} = f_{null}$ for all null Lagrangians. In particular, inserting $\mathcal{L} = F(L_{null}(\dot{x}, x, t)) = L_{null}(\dot{x}, x, t)$ in Eq. (22) gives the expected $\dot{E}L[\mathcal{L} = L_{null}] = 0$ validating that no equation of motion is obtained from $\mathcal{L} = L_{null}$. However, most importantly, the final equation of motion derived from a function of a null Lagrangian turns out to be independent of the form of the function provided that $p_L \left[ \frac{d^2 F}{dL_{null}^2} \right] \neq 0$ for a given function $L_{null}$, which is encoded in the following Corollary.

**Corollary 1:** Let $\mathcal{L} := (F \circ L_{null}) : \mathcal{U} \subset TQ \rightarrow \mathbb{R}$ be a set of $C^\infty$ functions composed from $L_{null} : \mathcal{U} \subset TQ \rightarrow \mathbb{R}$ and $F : \mathbb{R} \rightarrow \mathbb{R}$ then, for any permissible form of function $\mathcal{L} = (F \circ L_{null})(\dot{x}, x, t)$ satisfying $p_{L_{null}} \left[ \frac{d^2 F}{dL_{null}^2} \right] \neq 0$, the corresponding equation of motion is given by

$$ \frac{d}{dt} [L_{null}] = 0. $$

(23)
The simplicity of the derivation of an equation of motion from Eq. (23) is rather remarkable, for it generates an equation of motion describing a dynamical system simply from the total time derivative of a permissible null Lagrangian defined over a suitable domain. In other words, there exists a dynamical system corresponding to every permissible null Lagrangian over a domain of interest. Furthermore, the form of the corresponding equation of motion is independent of the functional form defined over different domain for each functional. For example, the simplified equation of motion for \( F[\mathcal{L}_{\text{null}}(\dot{x}, x, t)] = \exp(\mathcal{L}_{\text{null}}) \) hinges upon the assumption that \( p_{\mathcal{L}_{\text{null}}} \neq 0 \) and \( \mathcal{L}_{\text{null}} \) is finite (but can be zero at some points) over the chosen domain. On the contrary, both \( p_{\mathcal{L}_{\text{null}}} \neq 0 \) and \( \mathcal{L}_{\text{null}} \neq 0 \) must be satisfied over the same domain for \( F[\mathcal{L}_{\text{null}}(\dot{x}, x, t)] = \ln(\mathcal{L}_{\text{null}}) \).

At this point, it is desirable to explicitly express the equation of motion for a null Lagrangian in the form presented in the previous sections. Therefore, an invertible null Lagrangian, \( \mathcal{L}(\dot{x}, x, t) = B(x, t)\dot{x} + C(x, t)x + f(t) \) over a domain \( U \subset TQ \), generates the following equation of motion:

\[
B(x, t)\ddot{x} + \left[ B'(x, t)\dot{x} + 2\dot{B}(x, t) \right] \dot{x} + C(x, t)x + \dot{f}(t) = 0, \tag{24}
\]

where primes and dots are defined respectively as \( B' := \partial B(x, t)/\partial x \) and \( \dot{B} := \partial B(x, t)/\partial t \).

Similarly, the equation of motion for \( n \)-th harmonics of an invertible null Lagrangian, \( \mathcal{L}_{\text{null}}(\dot{x}, x, t) = B(x, t)\dot{x} + C(x, t)x + f(t) \) over some domain \( U \subset TQ \), can be computed from

\[
\frac{d\mathcal{L}^{(n)}_{\text{null}}}{dt} = \frac{d\mathcal{L}^{(n-1)}_{\text{null}}}{dt} + \frac{d^2B_{n-1}(x, t)}{dt^2} = 0, \tag{25}
\]

where again \( \mathcal{L}^{(n-1)}_{\text{null}} \) and \( B_{n-1}(x, t) \) are defined in Eqs. (15) and (13) respectively.

Now, after constructing the non-standard Lagrangians from null Lagrangians and the simplified method for obtaining the corresponding equation of motion, it is imperative that the above formulation be applied to explore the range of allowed physical systems. Therefore, a few specific examples are presented below to apply these discoveries for a range of allowed dynamical systems.

5. Applications to dynamical systems

5.1. Newton’s law of inertia

The general form of the equation of motion (see Eq. 24) resulting from Eq. (23) for the null Lagrangian \( \mathcal{L}_{\text{null}}(\dot{x}, x, t) = B(x, t)\dot{x} + C(x, t)x + f(t) \) may imply that the equation represents a broad variety of dynamical systems. However, it must be kept in mind that there is also the null condition that can be written as

\[
\left( \frac{\partial B(x, t)}{\partial t} \right) = \left[ \frac{\partial [xC(x, t)]}{\partial x} \right], \tag{26}
\]
and that this condition imposes stringent constraints on admissible functions $B(x, t)$ and $C(x, t)$. As a result, only certain dynamical systems may have null Lagrangians that allow deriving their equations of motion directly from Eq. (23), which is a necessary condition for obtaining the equation of motion directly from a given null Lagrangian. The existence of this condition is a new phenomenon in the calculus of variations.

In the simplest case when $B(x, t) = c_1 = \text{const}$, $C(x, t) = c_2 = \text{const}$, and $f(t) = c_3 = \text{const}$, Eq. (24) reduces to $\ddot{x} = 0$, which is the Newton law of inertia. Then, $L_{null}(\dot{x}) = c_1 \dot{x}$, which is the simplest null Lagrangian. It is easy to see that substitution of this Lagrangian into Eq. (23) gives the required equation of motion. Thus, the law of inertia can be derived by using the null Lagrangian, which is a new result of this paper.

5.2. Dissipative systems with constant coefficients

To identify dynamical systems, whose equations of motion can be obtained from the null Lagrangians by using Eq. (23), we consider the following equation

$$\ddot{x} + \alpha_o \dot{x}^2 + \beta_o \dot{x} + \gamma_o x = 0,$$

where $\alpha_o = \text{const}$, $\beta_o = \text{const}$ and $\gamma_o = \text{const}$. Comparison of this equation to Eq. (24) allows finding the functions $B(x, t)$ and $C(x, t)$ and obtaining the corresponding null Lagrangian.

However, if the coefficients in Eq. (27) are constant, we obtain

$$B(x, t) = B_o e^{\alpha_o x + \beta_o t/2},$$

and

$$C(x, t) = 2B_o \left( \frac{\gamma_o}{\beta_o} \right) e^{\alpha_o x + \beta_o t/2}.$$

Moreover, the null condition gives $(1+\alpha_o x)\gamma_o = \beta_o^2/4$. Since the coefficients are constant, the dependence on $x$ must be eliminated by either $\alpha_o = 0$ or $\alpha_o \neq 0$ but $\beta_o = \gamma_o = 0$. This shows that there is no null Lagrangian for the equation of motion $\ddot{x} + \gamma_o x = 0$, which for $\gamma_o$ being a spring constant represents a harmonic oscillator. Our method of deriving equations of motion from null Lagrangians seems to be restricted to dissipative systems, with the law of inertia being an exception.

We now present the resulting ODEs for the cases when null Lagrangians exist and can be used used to derived them. With $\alpha_o = 0$ and $\gamma_o = \beta_o^2/4$, Eq. (27) reduces to

$$\ddot{x} + \beta_o \dot{x} + \frac{1}{4} \beta_o^2 x = 0,$$

and its null Lagrangian is

$$L_{null}(\dot{x}, x, t) = \left( \dot{x} + \frac{1}{2} x \right) B_o e^{\beta_o t/2}.$$

It is easy to verify that $L_{null}(\dot{x}, x, t)$ is the null Lagrangian and that this Lagrangian gives the equation of motion (see Eq. (29)) after it is substituted into Eq. (23). The obtained equation of motion describes a damped harmonic oscillator in which the damping term and the natural frequency of this system are related to each other.
In case $\alpha_o \neq 0$ and $\beta_o = \gamma_o = 0$, the resulting equation of motion is
\[
\ddot{x} + \alpha_o \dot{x}^2 = 0 ,
\] (32)
and the null Lagrangian that gives this equation is
\[
L_{\text{null}}(\dot{x}, x, t) = B_o \dot{x} e^{\alpha_o x} .
\] (33)
In this case the equation of motion describes a dynamical system with the quadratic damping and its coefficient $\alpha_o$ being arbitrary.

5.3. Dissipative systems with time-dependent coefficients

Let us generalize Eq. (27) and consider the following equation
\[
\ddot{x} + \alpha_1(t) \dot{x}^2 + \beta_1(t) \dot{x} + \gamma_1(t) x = 0 ,
\] (34)
where $\alpha_1(t)$, $\beta_1(t)$ and $\gamma_1(t)$ are given functions of time. By comparing this equation to Eq. (24), the functions $B(x, t)$ and $C(x, t)$ are obtained
\[
B(x, t) = B_o e^{\alpha_1(t)x + I_\beta(t)} ,
\] (35)
where
\[
I_\beta(t) = \frac{1}{2} \int^t \beta_1(\tilde{t}) d\tilde{t} ,
\] (36)
and
\[
C(x, t) = B_o \int^t \gamma_1(\tilde{t}) e^{\alpha_1(\tilde{t})x + I_\beta(\tilde{t})} d\tilde{t} .
\] (37)

Then, the null condition is
\[
\left[ \dot{\alpha}_1(t) x + \frac{1}{2} \beta_1(t) \right] e^{\alpha_1(t)x + I_\beta(t)} = \int^t \gamma_1(\tilde{t}) e^{\alpha_1(\tilde{t})x + I_\beta(\tilde{t})} d\tilde{t} + x \int^t \alpha_1(\tilde{t}) \gamma_1(\tilde{t}) e^{\alpha_1(\tilde{t})x + I_\beta(\tilde{t})} d\tilde{t} .
\] (38)

Since the coefficients are only functions of $t$, the dependence on $x$ must be eliminated. This can be achieved by taking either $\alpha_1(t) = 0$ or $\alpha_1(t) \neq 0$ but $\beta_1(t) = \gamma_1(t) = 0$. Both cases are similar to those considered in Section 5.1 but the resulting equations of motion and the null Lagrangians are more general, thus, we present them now.

Having obtained $B(x, t)$ and $C(x, t)$, we take $\alpha_1(t) = 0$ and find the following null Lagrangian
\[
L_{\text{null}}(\dot{x}, x, t) = B_o \left[ \dot{x} e^{I_\beta(t)} + x \int^t \gamma_1(\tilde{t}) e^{I_\beta(\tilde{t})} d\tilde{t} \right] .
\] (39)
Substituting this Lagrangian into Eq. (23), we obtain
\[
\ddot{x} + \beta_1(t) \dot{x} + \gamma_1(t) x = 0 ,
\] (40)
which is the correct equation of motion. However, the functions $\beta_1(t)$ and $\gamma_1(t)$ are related to each other through the null condition (see Eq. 38) given by
\[
\frac{1}{2} \beta_1(t) e^{I_\beta(t)} = \int^t \gamma_1(\tilde{t}) e^{I_\beta(\tilde{t})} d\tilde{t} .
\] (41)
This equation generalizes that given by Eq. (30) and describes a damped harmonic oscillator, whose coefficients are functions of \( t \) and they are dependent.

Now, let us consider the second case of \( \beta_1(t) = \gamma_1(t) = 0 \), which reduces the null condition given by Eq. (38) to \( \dot{\alpha}_1(t) = 0 \) or \( \alpha_1(t) = \alpha_o = \text{const} \). This is the same case as considered in Section 5.1 and the resulting equation of motion and its null Lagrangian are given by Eqs (32) and (33), respectively. The obtained result shows that the null Lagrangian exists only for the equation of motion with the quadratic damping when the damping coefficient is constant.

5.4. Dissipative systems with displacement-dependent coefficients

We may also generalize Eq. (27) and consider the following equation

\[
\ddot{x} + \alpha_2(x)\dot{x}^2 + \beta_2(x)\dot{x} + \gamma_2(x)x = 0,
\]

where \( \alpha_2(x), \beta_2(x) \) and \( \gamma_2(x) \) are given functions of displacement. To find \( B(x, t) \) and \( C(x, t) \), we compare the above equation to Eq. (24), and obtain

\[
B(x, t) = B_o e^{I_{\alpha}(x) + \beta_2(x)t/2},
\]

where

\[
I_{\alpha}(x) = \int^t \alpha_2(\tilde{x})d\tilde{x},
\]

and

\[
C(x, t) = 2B_o \frac{\gamma_2(x)}{\beta_2(x)} e^{I_{\alpha}(x) + \beta_2(x)t/2}.
\]

Then, the null condition can be written as

\[
\frac{1}{4} \beta_o^2(x) = 1 + \left[ \frac{\gamma_2'(x)}{\gamma_2(x)} - \frac{\beta_2'(x)}{\beta_2(x)} + \alpha_2(x) + \frac{1}{2} \beta_2'(x)t \right].
\]

Since the coefficients depend only on \( x \), the dependence on \( t \) must be eliminated by taking \( \beta_2'(x) = 0 \), which gives \( \beta_2(x) = \beta_o = \text{const} \). As a result, the null condition reduces to

\[
\frac{1}{4} \beta_o^2 = x\gamma_2'(x) + \gamma_2(x) \left[ 1 + \alpha_2(x)x \right].
\]

The most general case when the above null condition is satisfied requires \( \alpha_2(x), \beta_2(x) = \beta_o \) and \( \gamma_2(x) \). Then, the resulting null Lagrangian is

\[
L_{\text{null}}(\dot{x}, x, t) = B_o \left[ \dot{x} + 2x \frac{\gamma_2(x)}{\beta_o} \right] e^{I_{\alpha}(x) + \beta_o t/2},
\]

which gives the following equation of motion

\[
\ddot{x} + \alpha_2(x)\dot{x}^2 + \beta_o \dot{x} + \gamma_2(x)x = 0.
\]

In this equation of motion, the functions \( \alpha_2(x) \) and \( \gamma_2(x) \) are related to each other by the null condition (see Eq. 47), which also depends on the damping coefficient \( \beta_o \).
There are also three special cases: $\alpha_2(x) = \alpha_o/x$, $\alpha_2(x) = 0$ and $\beta_o = 0$. In the first case, we find

$$\gamma_2(x) = \frac{1}{4} \frac{\beta_o^2}{1 + \alpha_o} + \tilde{c}_1 x^{-(1+\alpha_o)} ,$$  \hspace{1cm} (50)

where $\tilde{c}_1$ is an integration constant. The null Lagrangian is the same as that given by Eq. (48) but with the replacement $\alpha_2(x) = \alpha_o/x$. The same replacement in Eq. (49) gives the required equation of motion. In the second case, $\alpha_2(x) = 0$, and the solution to the null condition can be found and written as

$$\gamma_2(x) = \frac{\tilde{c}_2}{x} + \frac{1}{4} \beta_o^2 .$$  \hspace{1cm} (51)

where $\tilde{c}_2$ is an integration constant. It is seen that $\gamma_2(x)$ must have a special dependence on $x$ and on the coefficient $\beta_o$. The null Lagrangian and the resulting equation of motion are given by Eqs (48) and (49), respectively, by taking $\alpha_2(x) = 0$.

The third case requires that $\beta_o = 0$, which gives

$$B(x, t) = B_o e^{I_o(x)} ,$$  \hspace{1cm} (52)

and

$$C(x, t) = B_o t \gamma_2(x) e^{I_o(x)} .$$  \hspace{1cm} (53)

Then, the null Lagrangian becomes

$$L_{null}(\dot{x}, x, t) = B_o [\dot{x} + x t \gamma_2(x)] e^{I_o(x)} ,$$  \hspace{1cm} (54)

and the equation of motion is

$$\ddot{x} + \alpha_2(x) \dot{x}^2 + \gamma_2(x) x = 0 .$$  \hspace{1cm} (55)

Again, the functions $\alpha_2(x)$ and $\gamma_2(x)$ are not arbitrary but instead they must satisfy the following condition

$$\gamma_2(x) = \frac{\tilde{c}_3}{x} e^{I_o(x)} ,$$  \hspace{1cm} (56)

where $\tilde{c}_3$ is an integration constant.

5.5. Comparison to other Lagrangians

The presented results show that null Lagrangians can be obtained for the law of inertia and some dynamical systems. An interesting result is that, except the first Newton law, these systems must be dissipative, which suggests a novel role of null Lagrangians in classical dynamics of dissipative systems. For some systems considered in this paper, standard and nonstandard Lagrangians were already obtained, so now we compare those Lagrangians to the null Lagrangians derived in this paper.

For the law of inertia, $\ddot{x} = 0$, the standard [1-4] and non-standard [28] Lagrangians are given by

$$L_{sd}(\dot{x}) = \frac{1}{2} \dot{x}^2 ,$$  \hspace{1cm} (57)
and
\[ L_{nlsd}[\dot{x}, x, t] = \frac{1}{C_1(a_o t + v_o)^2 (a_o t + v_o) \dot{x} - a_o x + C_2}, \]  
(58)
where \( C_1 \) and \( C_2 \) being constants of integration, and \( v_o \) and \( a_o \) being specified by the initial conditions for solving the auxiliary differential equation [28]. Both Lagrangians, when substituted into the E-L equation, give the law of inertia.

A new result of this paper is that the null Lagrangian
\[ L_{null}(\dot{x}) = c_1 \dot{x}, \]  
(59)
where \( c_1 \) is a constant, gives also the law of inertia after it is substituted into Eq. (23). Thus, the law of inertia can be derived by using either the standard or non-standard or null Lagrangian. Among these three Lagrangians, the simplest one is the null Lagrangian and the most complicated one is the non-standard Lagrangian, yet all of them give the same law of inertia. The advantage of the Lagrangian formulation presented in this paper is that the law of inertia is derived from the null Lagrangian by simply taking its time derivative.

The equation of motion with the quadratic damping term, \( \ddot{x} + \alpha_o \dot{x}^2 = 0 \), was also previously studied [7,8], and the following standard and non-standard Lagrangians were obtained
\[ L_{sd}(\dot{x}) = \frac{1}{2} \dot{x}^2 e^{2\alpha_o x}, \]  
(60)
and
\[ L_{nlsd}[\dot{x}, x, t] = \frac{1}{\dot{x} e^{\alpha_o x} + 1}. \]  
(61)
Both Lagrangians, when substituted into the E-L equation, give the equation of motion. The same equation of motion is obtained when the null Lagrangian
\[ L_{null}(\dot{x}) = c_2 \dot{x} e^{\alpha_o x}, \]  
(62)
where \( c_2 \) is a constant, is substituted into Eq. (23). Comparison of the above Lagrangians shows that each one of them contains \( \dot{x} \) and \( e^{\alpha_o x} \), and that their forms are similar.

Finally, we consider the equation of motion for a damped oscillator, \( \ddot{x} + \beta_o \dot{x} + \frac{1}{4} \beta_o^2 x = 0 \), whose standard and non-standard Lagrangians [7,8] are
\[ L_{sd}(\dot{x}, t) = \frac{1}{2} \left( \dot{x}^2 - \frac{1}{4} \beta_o^2 x^2 \right) e^{\beta_o t}, \]  
(63)
and
\[ L_{nlsd}[\dot{x}, x, t] = \frac{e^{-\beta_o x/2}}{\dot{x} + \frac{1}{2} \beta_o x}. \]  
(64)
The null Lagrangian for the equation of motion is
\[ L_{null}(\dot{x}, x, t) = c_3 \left( \dot{x} + \frac{1}{2} \beta_o x \right) e^{\beta_o t/2}. \]  
(65)
where \( c_2 \) is a constant. Comparison of \( L_{nlsd}(\dot{x}, x, t) \) and \( L_{null}(\dot{x}, x, t) \) shows that the non-standard Lagrangian is exactly the inverse of the null Lagrangian, if \( c_3 = 1 \).
6. Conclusions

General null Lagrangians and their higher harmonics are constructed for dynamical systems with one degree of freedom. Since the null Lagrangians satisfy identically the E-L equation, they cannot be directly used to obtain equations of motion. However, as the results of this paper show, the derived null Lagrangians can be used to construct non-standard Lagrangians that do give equations of motion. The main result of this paper is the following condition

$$\frac{d}{dt} [L_{null}] = 0,$$

which gives an equation of motion for any null Lagrangian. Thus, the condition plays the same role for null Lagrangians as the E-L equation plays for standard or non-standard Lagrangians.

The presented results show how to determine a null Lagrangian for a given dynamical system and then how to use this Lagrangian to obtain the resulting equation of motion. Specific applications of the method to general dynamical systems show that the method preferentially works for dissipative systems but it also can be used to some conservative systems such as the law of inertia. The results of this paper demonstrate a novel role of null Lagrangians in classical dynamics.

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