Local Differential Geometry
(as a Representation of the SUSY Oscillator)

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Abstract

The choice of a coordinate chart on an analytical $\mathbb{R}^n$ ($\mathbb{R}^n_a$) provides a representation of the $n$-dimensional SUSY oscillator. The 1-parameter group of dilations provides a Euclidean evolution moving the system through a sequence of charts, that at each instant supplies a Hilbert space by Cartan’s exterior algebra endowed with a suitable scalar product. Stationary states and coherent states are eigenstates of the Lie derivatives generating the dilations and the translations respectively.
I. Introduction

In this paper the presence of differential geometric structures in a special but important quantum system will be worked out. The $n$-dimensional SUSY oscillator [1-6] turns out to be a perfect manifestation of the differential geometry of the most simple manifold: analytical $R^n (R^n_a)$, where a coordinate system exists globally. The quantum structures are due to a selected chart on this manifold. Hence, it is important to observe that coordinates are not auxiliary quantities. In this respect this representation is similar to the (anti-)holomorphic representations of Fock [7], Bargmann [8] and Berezin [9]. However, as a major difference, the present work involves exclusively real quantities.

The paper is divided into two parts. We start by restating basic differential geometrical notions [10] and show that the choice of a coordinate system on $R^n_a$ amounts to finding familiar geometrical objects that are reinterpreted as operators and state vectors from a Fock space point of view. In particular, forms consisting of commuting coordinates and anticommuting differentials are suited for a representation of a bosonic/fermionic Fock space. In contrast to [11], where square integrable differential forms are employed, and the metric provides the scalar product, the treatment of bosonic and fermionic degrees of freedom is perfectly analogous in our approach. This requires, as the only additional ingredient from a geometrical point of view, the introduction of a scalar product for the forms. A metric is not needed on $R^n_a$. From the discussion of the exterior derivative the algebra of the SUSY oscillator is derived. The corresponding Hamiltonian is, on the one hand, the Lie derivative generating the dilations on $R^n_a$, on the other hand, it is a total derivative giving rise to a euclidean Schrödinger equation. The euclidean evolution enables the formalism to be maintained strictly real. Stationary states are defined to be the eigenstates of the Hamiltonian. Analogously, coherent states are defined as eigenstates of the Lie derivatives generating the translations on $R^n_a$.

For an illustration, in the second part, we concentrate on $R^1_a$, i.e. the 1-dimensional SUSY oscillator, which is among a number of physical SUSY quantum mechanical systems [11], [12], [13], [14]. We begin with a discussion of stationary states. Finally, coherent states are found to exhibit classical motion within the inverted oscillator potential propagating in euclidean time.
II. Geometry of $R^n_a$

and the $n$-dimensional SUSY oscillator

Consider $R^n$ as a differentiable manifold [10]. We pick an arbitrary point $\in R^n$ and take it as the origin of coordinates $\{x^i\}$ that parametrize $R^n$ entirely. We restrict ourselves to functions that are analytical around the origin

$$f(\{x^i\}) = f_0 + f_ix^i + f_{ij}x^i x^j + \ldots, \quad f_{ij...} = \text{const.} \in R$$

(1)

defining $R^n_a$. Immediately, we have two kinds of operators acting on these functions (from the left). The coordinate function $x^i$ is an operator that acts on $f(\{x^i\})$ by multiplication. $\partial_{x^i} := \partial/\partial x^i$, generating displacements on $R^n_a$, acts on $f(\{x^i\})$ by performing the partial derivative with respect to $x^i$. They yield the algebra

$$[x^i, x^j]_\ominus = [\partial_{x^i}, \partial_{x^j}]_\ominus = 0, \quad [\partial_{x^i}, x^j]_\ominus = \delta^j_i,$$

de$[\partial_{x^i}, x^j]_\ominus = \delta^j_i$. These and all following commutators are graded by the form degrees of the entries, i.e. $[A, B] = [A, B]_+$, if both $A$ and $B$ have odd form degree and $[A, B] = [A, B]_- \text{ otherwise.}$ Eq.(1) is regarded as a bosonic algebra of creators $x^i$ and annihilators $\partial_{x^i}$. From this point of view the function in eq.(1) represents an element of a bosonic Fock space, if we interpret the Taylor series as a power expansion in the creators $x^i$ applied to a constant number. Having defined analytical 0-forms in eq.(1), we consider now analytical $p$-forms on $R^n_a$. In our chart they can be expressed as

$$F^{(p)} = F_{i_1 \ldots i_p}(\{x^i\}) dx^{i_1} \ldots dx^{i_p} \in \Lambda^p R^n_a,$$

(3)

with the factors $F_{i_1 \ldots i_p}(\{x^i\})$ being analytical functions as in eq.(1). The product of the $dx^i$ (to be read $(dx^i)$ in the present terminology) is anti-commutative, often symbolized by a wedge that is omitted here. Just as a function is a series in $x^i$, a form is defined as a series in the Grassmann numbers $dx^i$, which is finite in $dx^i$ for finite dimension $n$ in contrast to eq.(1),

$$\Psi(\{x^i\}, \{dx^i\}) = F_0(\{x^i\}) + F_1(\{x^i\}) dx^j + F_{jk}(\{x^i\}) dx^j dx^k + \ldots \in \Lambda R^n_a = \bigoplus_{p=0}^n \Lambda^p R^n_a,$$

(4)
spanning Cartan’s exterior algebra, which is a combined Fock space with \( n \) bosonic \( x^i \) and \( n \) fermionic \( dx^i \). The contraction of a \( p \)-form, with a vector \( v = v^i \partial_x^i \in TR^n \) yields a \( (p - 1) \)-form. This is the interior product operation usually indicated by \( i_v \). We write alternatively
\[
v^i \partial_{dx^i} := v^i \frac{\partial}{\partial dx^i} \equiv i_v, \tag{5}\]
where \( \partial_{dx^i} \) is the left derivative with respect to \( dx^i \) carrying form degree \(-1\). The duality of \( \partial_x^i \in TR^n \) and \( dx^j \in T^*R^n \) is expressed by
\[
[dx^i, dx^j]_+ = [\partial_{dx^i}, \partial_{dx^j}]_+ = 0, \quad [\partial_{dx^i}, dx^j]_+ = \delta^j_i, \tag{6}\]
Obviously, we obtained a fermionic algebra analogous to eq.(2) consisting of creators \( dx^i \) and annihilators \( \partial_{dx^i} \). Any form can be created by applying 1-forms \( dx^i \) (from the left) and bosonic factors \( x^i \) to a constant number. We complete eqns. (2) and (6) by the the mixed commutators
\[
[x^i, dx^j]_- = [x^i, \partial_{dx^j}]_- = [\partial_x^i, dx^j]_- = [\partial_x^i, \partial_{dx^j}]_- = 0. \tag{7}\]
We now define the adjoints of the fundamental operators
\[
(x^i)^+ := \partial_x^i, \quad (dx^i)^+ := \partial_{dx^i}, \tag{8}\]
while real numbers are self-adjoint. The adjoint of an arbitrary operator is obtained by decomposing it into the fundamental operators above and applying the usual rules \((A + B)^+ = A^+ + B^+\), \((AB)^+ = B^+ A^+\) and \((A^+)^+ = A\). In particular, the adjoint operation applied to the Fock space vector eq.(4) supplies the dual vector \( \Psi^+ \{\partial_{dx^i}, \{\partial_x^i\}\} \) from the dual of \( \Lambda R^n \). We denote the scalar product of two forms \( \Psi, \Xi \in \Lambda R^n \) by
\[
\langle \Psi, \Xi \rangle := (\Psi^+ \Xi) |_{x^i = dx^i = 0}. \tag{9}\]
It is calculated by performing all the derivations of \( \Psi^+ \) on \( \Xi \) and putting the remaining factors \( x^i \) and \( dx^i \) to zero. This scalar product is symmetric and induces a positive definite norm. Therefore, endowed with this scalar product the Fock space \( \Lambda R^n \) becomes a Hilbert space. In [15] a similar scalar product was introduced for the space of functionals in quantum field theory.
Until now, we implicitly used the exterior derivative $d$ that transforms a bosonic $x^i$ into a fermionic $dx^i$. It can be written as $d = dx^i \partial_{x^i}$ and is obviously nilpotent $[d, d]_+ = 0$. We can construct its adjoint which turns a fermionic $dx^i$ into a bosonic $x^i$, hence $d^+ = x^i \partial_{dx^i}$ being also nilpotent $[d^+, d^+]_+ = 0$. This operator is also defined in [16] and slightly different in [10] for the proof of Poincaré’s lemma. We calculate the commutator of $d$ and $d^+$, defining the self-adjoint Hamiltonian

$$H := [d, d^+]_+ = x^i \partial_{x^i} + dx^i \partial_{dx^i}$$

The first term counts the powers in the coordinates of an expression that it is applied on. So we define the boson number operator $N := x^i \partial_{x^i}$, which is bosonic. The second term counts the form degree, if it is applied to a $p$-form. Accordingly, we define the fermion number operator $P := dx^i \partial_{dx^i}$, which is also bosonic. $d$ and $d^+$ are conserved

$$[d, H]_- = [d^+, H]_- = 0.$$  

Equations (10) and (11) represent the algebra of an $n$-dimensional SUSY oscillator with the supersymmetry charges $d$ and $d^+$ [1, 4]. The Hamiltonian reads $H = (N + n/2) + (P - n/2)$, the zero-point energies reinserted.

Generally, a Lie derivative with respect to the vector field $v = v^i \partial_{x^i}$ can be written as $L_v = [d, v^i \partial_{dx^i}]_+$. Therefore, the Hamiltonian $H$ is a Lie derivative with respect to the vector field $x^i \partial_{x^i}$ with integral lines which are rays emanating from the origin. In fact, $H \equiv L_{x^i \partial_{x^i}}$ is a total derivative with respect to a parameter $\tau$ [10]

$$- \frac{d}{d\tau} \Psi(\tau) = H \Psi(\tau)$$

for any

$$\Psi(\{x^i(\tau)\}, \{dx^j(\tau)\}) = \Psi(\{x^i e^{-\tau}\}, \{dx^j e^{-\tau}\}) = e^{-\tau H} \Psi(\{x^i\}, \{dx^j\}) \in \Lambda R^n_a,$$

where $x^i \equiv x^i(0)$. $\tau$ parametrizes a sequence of charts on $R^n_a$ each representing the quantum system at an instant $\tau$. The rule for the adjoint operation is to be applied within each chart, hence

$$(x^i(\tau))^+ = \partial_{x^i(\tau)}, \quad (dx^i(\tau))^+ = \partial_{dx^i(\tau)}.$$
As a consequence, the norm is independent of $\tau < \infty$. The compatibility of eqns. (8) and (14) implies $\tau^+ = -\tau$, therefore, while in the geometric framework $\tau$ is to be regarded a real number, in the sense of the scalar product, relating the Fock space and its dual, it is not. Thus, unlike in quantum mechanics, the sequence of Hilbert spaces, representing the evolution of the system, is not a Hilbert space itself.

Observe, that for $\tau = \infty$ all states but the constant functions, representing the only non-vanishing cohomology-class on $R^n$, vanish.

Obviously, we have found a euclidean Schrödinger type equation determining the $\tau$-propagation of any $\Psi$ similar to quantum mechanics. (In fact, by performing an analytical continuation $\tau \to it$, using complex coordinates and constants, where the adjoint of a constant is its complex conjugate, one ends up with the real-time quantum mechanics governing the $n$-dimensional SUSY oscillator in creator/annihilator language.)

We define stationary states to have the evolution $\phi(\tau) = \phi e^{-\epsilon \tau}$ and therefore to be eigenstates of $H$

$$H \phi = \epsilon \phi.$$ (15)

The definition for coherent states here is inspired by that of the stationary states. Consider the Lie derivative with respect to a constant vector field

$$\mathcal{L}_{c^i} \partial x^i = [d, c^i \partial x^i]_+ = c^i \partial x^i,$$ (16)

which turns out to be a simple directional derivative on any element of Cartan's exterior algebra. The coherent states are defined to be solutions of the corresponding eigenvalue problem

$$c^i \partial x^i \kappa = \gamma \kappa, \quad \gamma \in R.$$ (17)

Since $[\mathcal{L}_v, d]_- = 0$, from a given eigenstate of $\mathcal{L}_v$ another one is immediately found by application of $d$, such that eigenstates of $\mathcal{L}_v$ will always come in SUSY pairs.

We conclude this section by remarking, that this representation is the one where the creation operator $x^i$ is diagonal. Its continuous spectrum is given by the points on $R^n$. The corresponding eigenstates are not normalizable as is well-known [17] and easily shown. This is analogous to quantum mechanics where the position and momentum operators provide representations without having square-integrable eigenstates themselves.
III. Stationary states and coherent states on $R^1_a$

The eigenvalue problem of the Hamiltonian on $R^1_a$ is

$$H \phi_\epsilon = \epsilon \phi_\epsilon, \quad H = x \partial_x + dx \partial_{dx}$$

(18)

Due to \([H, P]_\epsilon = 0\), there are eigenstates of definite form degree, indicated by superscripts in parentheses. We begin by choosing form degree zero, such that

$$x \partial_x \phi_\epsilon^{(0)} = \epsilon \phi_\epsilon^{(0)}.$$  

(19)

Obviously, $\phi_\epsilon^{(0)} \sim x^\epsilon$, where a priori $\epsilon$ is any real number. However, the condition that all states have to be analytic, immediately rules out all solutions of (19) with the exception of $\phi_\epsilon^{(0)} = (1/\sqrt{(\epsilon)!})x^\epsilon$, ($\epsilon = 0, 1, 2, ...$). There is however a degeneracy of $H$. Application of $d$ to eq.(18) yields

$$H d \phi_\epsilon^{(0)} = \epsilon d \phi_\epsilon^{(0)},$$

(20)

such that $d \phi_\epsilon^{(0)}$ is as well an eigenfunction of $H$ corresponding to the same eigenvalue as $\phi_\epsilon^{(0)}$. Therefore, by application of $d$ to the set of eigenstates $\phi_\epsilon^{(0)}$, we get a second series of eigenstates $\phi_\epsilon^{(1)} = (1/\sqrt{(\epsilon - 1)!})x^{\epsilon - 1}dx$, ($\epsilon = 1, 2, ...$). The only state without degeneracy is the vacuum $\phi_0^{(0)} = 1$ satisfying $d1 = d^+1 = 0$, implying that supersymmetry is not spontaneously broken. The complete set of eigenstates consists of both \{\phi_\epsilon^{(0)}\} and \{\phi_\epsilon^{(1)}\}, obeying the orthonormality relation

$$\langle \phi_\epsilon^{(p)} , \phi_\epsilon^{(p')} \rangle = \delta_{\epsilon \epsilon'} \delta_{pp'}, \quad p = 0, 1.$$  

(21)

The eigenvalue problem for coherent states on $R^1_a$ reads

$$\partial_x \kappa_\gamma = \gamma \kappa_\gamma, \quad \gamma \in R.$$  

(22)

Note that $[\partial_x, P]_\gamma = 0$, such that again we can find solutions of definite form degree $p$, coming in pairs of supersymmetric partners due to $[\partial_x, d]_\gamma = 0$

$$\kappa_\gamma^{(p)} = e^{-\frac{1}{2} \gamma^2} e^{\gamma x} (dx)^p$$  

(23)
after normalization. Coherent states of different form degree are clearly orthogonal. But the scalar product of two arbitrary $p$-form coherent states is

$$\langle \kappa^{(p)}_\gamma, \kappa^{(p')}_{\gamma'} \rangle = e^{-\frac{1}{2} (\gamma - \gamma')^2} \delta_{pp'}.$$  \hspace{1cm} (24)

For general coherent states there exist three major definitions in the literature, which coincide in the case of the harmonic oscillator: (i) the displacement operator definition, (ii) the annihilation operator eigenstate definition, (iii) the minimum uncertainty definition \cite{18}. By employing supergroups, also coherent states in SUSY quantum mechanical systems can meet all of these definitions \cite{19}. The coherent states in the present scenario deviate from these "super-coherent states", since Grassmann eigenvalues are not involved. The definition of coherent states eq.(17) corresponds to (ii), but in the following, we will show that they are also minimum-uncertainty states (iii).

Consider the operators

$$q = (1/\sqrt{2})(x + \partial_x) = q^+$$

and

$$\pi = (1/\sqrt{2})(x - \partial_x) = -\pi^+$$

with $[q, \pi]_- = -1$. The $\tau$ evolution of a coherent state of definite form degree yields

$$\kappa^{(p)}_\gamma(\tau) = e^{-\frac{1}{2} \gamma^2} e^{\gamma e^{-\tau} x} (dxe^{-\tau})^p.$$  \hspace{1cm} (25)

The expectation values

$$\langle \kappa^{(p)}_\gamma(\tau), q \kappa^{(p)}_\gamma(\tau) \rangle = \sqrt{2} \gamma \cosh \tau,$$

$$\langle \kappa^{(p)}_\gamma(\tau), \pi \kappa^{(p)}_\gamma(\tau) \rangle = \sqrt{2} \gamma \sinh \tau$$  \hspace{1cm} (26)

satisfy the classical equations of motion in $\tau$ within an inverted oscillator potential with energy $> 0$, if $q$ corresponds to the position and $\pi$ to the momentum in the classical phase space. By analytical continuation of $\tau \to it$ and $\pi \to -i$ times momentum, we recover the classical motion of the harmonic oscillator.

Finally, $\Delta q = \Delta \pi = 1/\sqrt{2}$ for coherent states, with $(\Delta q)^2 = \langle \Psi, (q^2 - \langle \Psi, q\Psi \rangle^2)\Psi \rangle$ and $(\Delta \pi)^2 = \langle \Psi, (\langle \Psi, \pi\Psi \rangle^2 - \pi^2)\Psi \rangle$, implying $\Delta q \Delta \pi = 1/2$ for any $\tau$.

**IV. Conclusions**

Since the geometrically most simple situation of an $R^n_a$ already supplies a fundamental physical application by the above formalism, it is tempting to consider extensions of the geometry in view of extending the range of physical
applicability. In particular, the promotion of the presented concepts to non-trivial manifolds appears to be attractive having in mind a correspondence between cohomology classes and SUSY vacua.

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References

[1] H. Nicolai, J. Phys. A9 (1976), 1467.
[2] Berezin F.A., Marinov M.S., JETP Lett. 21 (1975), 320.
[3] Brink L., Deser S., Zumino B., Di Vecchia P., Howe P.S., Phys. Lett. 64B, 435 (1976).
[4] E. Witten, Nucl. Phys. B185, 513 (1981).
[5] P. Salomonson, J.W. van Holten, Nucl. Phys. B196, 509 (1982).
[6] Gendenstein L.E., Krivé I.V., Soviet Physics Uspekhi 28:8, 645-660 (1985).
[7] V.A. Fock, Z. Phys. 49 (1928), 339.
[8] V. Bargmann, Commun. Pure and Appl. Math. 14 (1961), 187.
[9] F.A. Berezin: The Method of Second Quantization (Academic Press, New York 1966).
[10] Y. Choquet-Bruhat, C. DeWitt-Morette, M. Dillard-Bleick: Analysis, Manifolds and Physics, revised edition (North-Holland, Amsterdam 1982).
[11] E. Witten, J. Diff. Geom., 17, (1982), 661-692.
[12] L. D. Landau, E.M. Lifshitz: Quantum Mechanics. Non-Relativistic Theory, 3rd ed., Pergamon Press, 1977.
[13] V. A. Kostelecky, V. I. Man’ko, M. M. Nieto, D. R. Truax, Phys. Rev. A 48, 951 (1993).
[14] R. J. Hughes, V. A. Kostelecky, M. M. Nieto, Phys. Rev. D 34, 1100 (1986).
[15] M. Dubois-Violette, Nuovo Cimento, 62 B, 235 (1969).
[16] F. Brandt, N. Dragon, M. Kreuzer, Phys. Lett. 231B, 263 (1989).
[17] M. M. Miller, E. A. Mishkin: Phys.Rev. 152 (1966) 1110.
[18] M. M. Nieto, L.M. Simmons, Jr., Phys. Rev. D 20, 1321 (1979).
[19] B. W. Fatyga, V. A. Kostelecky, M. M. Nieto, D. R. Truax, Phys. Rev. D 43, 1403 (1991).