Enhanced approximate cloaking by SH and FSH lining

Jingzhi Li\(^1\), Hongyu Liu\(^2\) and Hongpeng Sun\(^3\)

\(^1\) South University of Science and Technology of China, Shenzhen, 518055, People’s Republic of China
\(^2\) Department of Mathematics and Statistics, University of North Carolina, Charlotte, NC 28223, USA
\(^3\) Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People’s Republic of China

E-mail: li.jz@sustc.edu.cn, hliu28@uncc.edu and hpsun@amss.ac.cn

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Abstract

We consider approximate cloaking from a regularization viewpoint introduced in Kohn \textit{et al} (2008 \textit{Inverse Problems} \textbf{24} 015016) for EIT and further investigated in Kohn \textit{et al} (2010 \textit{Commun. Pure Appl. Math.} \textbf{63} 0973–1016) and Liu (2009 \textit{Inverse Problems} \textbf{25} 045006) for the Helmholtz equation. The cloaking schemes given by Kohn \textit{et al} and Liu are shown to be (optimally) within \(|\ln \rho|^{-1}\) in 2D and \(\rho\) in 3D of perfect cloaking, where \(\rho\) denotes the regularization parameter. In this paper, we show that by employing a sound-hard layer right outside the cloaked region, one could (optimally) achieve \(\rho^N\) in \(\mathbb{R}^N\), \(N \geq 2\), which significantly enhances the near-cloak. We then develop a cloaking scheme by making use of a lossy layer with well-chosen parameters. The lossy-layer cloaking scheme is shown to possess the same cloaking performance as the one with a sound-hard layer. Moreover, it is shown that the lossy layer could be taken as a finite realization of the sound-hard layer. Numerical experiments are also presented to assess the cloaking performances of all the cloaking schemes for comparisons.

(Some figures may appear in colour only in the online journal)

1. Introduction

A region is said to be \textit{cloaked} if its contents together with the cloak are invisible to certain measurements. From a practical viewpoint, these measurements are made in the exterior of the cloak. Blueprints for making objects invisible to electromagnetic waves were proposed by Pendry \textit{et al} [30] and Leonhardt [22] in 2006. In the case of electrostatics, the same idea was discussed by Greenleaf \textit{et al} [18] in 2003. The key ingredient is that optical parameters have transformation properties and could be \textit{pushed forward} to form new material parameters. The obtained materials/media are called \textit{transformation media}. We refer to [10, 15, 16, 29, 33, 35].
for state-of-the-art surveys on the rapidly growing literature and many striking applications of the so-called transformation optics.

In this work, we shall be mainly concerned with the acoustic cloaking via ‘transformation acoustics’. The transformation media proposed in [18, 30] are rather singular. This poses much challenge to both theoretical analysis and practical fabrication. In order to avoid the singular structures, several regularized approximate cloaking schemes are proposed in [14, 20, 21, 23, 31]. The basic idea is to introduce regularization into the singular transformation underlying the ideal cloaking, and instead of the perfect invisibility, one would consider the ‘near-invisibility’ depending on a regularization parameter. Our study is closely related to the one introduced in [21] for approximate cloaking in EIT, where the ‘blow-up-a-point’ transformation in [18, 30] is regularized to be the ‘blow-up-a-small-region’ transformation. The idea was further explored in [20] and [23] for the Helmholtz equation. In [23], the author imposed a homogeneous Dirichlet boundary condition at the inner edge of the cloak and showed that the ‘blow-up-a-small-region’ construction gives successful near-cloak. In [20], the authors introduced a special lossy layer between the cloaked region and the cloaking region, and also showed that the ‘blow-up-a-small-region’ construction gives successful near-cloak. For both cloaking constructions, it was shown that the near-cloaks come, respectively, within $\frac{1}{|\ln \rho|}$ in 2D and $\rho$ in 3D of the perfect cloaking, where $\rho$ is the relative size of the small region blown up for the construction and plays the role of a regularization parameter. These estimates are also shown to be optimal for their constructions.

It is worth noting that in [20] a large loss parameter is required, and if one lets it go to infinity, this limit corresponds to the imposition of a homogeneous Dirichlet boundary condition at the inner edge of the cloak. On the other hand, the imposition of a homogeneous Dirichlet boundary condition at the inner edge of the cloak is equivalent to employing a sound-soft (SS) layer right outside the cloaked region. In this sense, the lossy-layer lining in [20] is a finite realization of the SS lining in [23]. However, we would also like to note that a high loss layer would produce significant radiating in the infrared regime, and hence deteriorates the near-cloaks. On the other hand, it is emphasized that employing some special lining is necessary for the near-cloak construction, since otherwise it is shown in [20] that there exists certain resonant inclusion which defies any attempt to achieve near-cloak.

In this work, we shall impose a homogeneous Neumann boundary condition on the inner edge of the cloak, which amounts to employing a sound-hard (SH) layer right outside the cloaked region. The cloaking scheme is referred to as an SH construction. For the SH construction, we show that one could achieve significantly enhanced cloaking performance. Actually, it is shown that one could achieve, respectively, $\rho^2$ in 2D and $\rho^3$ in 3D within the perfect cloaking for such construction. We then develop a cloaking scheme by making use of a high-density lossy-layer lining. The properly designed lossy layer could be taken as a finite realization of the SH lining. The cloaking scheme is referred to as a finite sound-hard (FSH) construction. The FSH construction is shown to possess the same cloaking performance as the SH construction.

The analysis of cloaking must specify the type of exterior measurements. In [14, 20, 21], the near-cloaks are assessed in terms of boundary measurement encoded into the boundary Dirichlet-to-Neumann (DtN) map. The scattering measurement is considered for the near-cloak in [23]. In this work, we shall assess our near-cloak construction with respect to scattering measurement encoded into the scattering amplitude.

In this paper, we focus entirely on the transformation-optics approach in constructing cloaking devices. But we would like to mention in passing other promising cloaking schemes including one based on anomalous localized resonance [28] and another based on special (object-dependent) coatings [1]. It is also interesting to note a recent work in [3], where the
authors implement multi-coatings to enhance the near-cloak in EIT. The same idea has also been extended to acoustic cloaking for achieving enhancement in [4, 5].

The rest of the paper is organized as follows. In section 2, we develop the cloaking scheme by employing the SH lining. In section 3, we present the cloaking scheme with a properly designed lossy layer. Section 4 is devoted to discussions on different cloaking schemes. In section 5, we present the numerical examples.

2. Transformation acoustics and cloaking construction

Let \( q \in L^\infty(\mathbb{R}^N) \) be a scalar function and \( \sigma = (\sigma^{ij})_{i,j=1}^N \in \text{Sym}(N) \) be a symmetric-matrix-valued function on \( \mathbb{R}^N \), which is bounded in the sense that, for some constants \( 0 < c_0 < C_0 < \infty \),

\[
    c_0 k^T \xi \leq \xi^T \sigma(x) \xi \leq C_0 k^T \xi
\]

for all \( x \in \mathbb{R}^N \) and \( \xi \in \mathbb{R}^N \). In acoustics, \( \sigma^{-1} \) and \( q \), respectively, represent the mass density tensor and the bulk modulus of a regular acoustic medium. We shall denote \( \{\mathbb{R}^N; \sigma, q\} \) an acoustic medium as described above. It is assumed that the inhomogeneity of the acoustic medium is compactly supported, namely \( \sigma = I \) and \( q = 1 \) in \( \mathbb{R}^N \setminus \bar{\Omega} \), with \( \Omega \) being a bounded Lipschitz domain in \( \mathbb{R}^N \). In \( \mathbb{R}^N \), the time-harmonic acoustic wave propagation is governed by the heterogeneous Helmholtz equation:

\[
    \text{div}(\sigma \nabla u) + k^2 q u = 0,
\]

where \( k > 0 \) represents the wave number and \( u \) denotes the wave pressure. Stationary scattering theory is to seek a solution to (2) admitting the following asymptotic development:

\[
    u(x) = e^{ik|x|} + \frac{e^{ik|x|}}{|x|^{(N-1)/2}} \left\{ A\left(\frac{x}{|x|}, \frac{\xi}{|\xi|}\right) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty, \tag{3}
\]

where \( \xi = kd \), with \( d \in S^{N-1} \). \( A(\hat{x}, d) \), with \( \hat{x} := x/|x| \) being the so-called scattering amplitude. An important problem arising in practical application is to recover \( \{\Omega; \sigma, q\} \) from the measurement of the corresponding scattering amplitude. In the following, for clarity and also for the convenience of our subsequent study, we give a more detailed description of the scattering problem. We shall let \( u'(x) := e^{ikd \cdot x} \) denote a time-harmonic plane wave, where \( d \in S^{N-1} \) denotes the incident direction. Let \( u^{\text{int}} \) and \( u^{\text{ext}} \) denote the total wave fields inside and outside the inhomogeneous medium, respectively, which satisfy the following PDE system:

\[
    \begin{align*}
        \nabla \cdot (\sigma \nabla u^{\text{int}}) + k^2 q u^{\text{int}} &= 0 & &\text{in } \Omega, \\
        \Delta u^{\text{ext}} + k^2 u^{\text{ext}} &= 0 & &\text{in } \mathbb{R}^N \setminus \bar{\Omega}, \\
        u^{\text{int}}|_{\partial \Omega} = u^{\text{ext}}|_{\partial \Omega}, & & \sum_{i,j=1}^n v_i \sigma^{ij} \partial_j u^{\text{int}}|_{\partial \Omega} = \partial u^{\text{ext}}/\partial n|_{\partial \Omega}, \\
        u^{\text{ext}}(x) &= u'(x) + u'(x), & &x \in \mathbb{R}^N \setminus \bar{\Omega}, \\
        \lim_{r \to \infty} r^{(N-1)/2} \left[ \frac{\partial u'}{\partial r} -iku' \right] &= 0,
    \end{align*}
\]

where \( r = |x| \) for \( x \in \mathbb{R}^N \). We know \( u' \in H^1_{\text{loc}}(\mathbb{R}^N) \) (see, e.g., [19, 27]) and clearly, \( A(\hat{x}, d) \) can be read off from the large \( |x| \) asymptotics of \( u' \).

In this paper, we shall be concerned with the construction of a layer of cloaking medium which makes the inside scatterer invisible to scattering amplitude. To that end, we present a quick discussion on transformation acoustics. Let \( \hat{x} = F(x) : \Omega \to \bar{\Omega} \) be a bi-Lipschitz and
orientation-preserving mapping. For an acoustic medium $\{\Omega; \sigma, q\}$, we let the pushed-forward medium be defined as

$$\{\tilde{\Omega}; \tilde{\sigma}, \tilde{q}\} = F_*\{\Omega; \sigma, q\} := \{\Omega; F_*\sigma, F_*q\},$$

(5)

where

$$\tilde{\sigma}(\tilde{x}) = F_*\sigma(x) := \frac{1}{J}M\sigma(x)M^T|_{x=F^{-1}(\tilde{x})}$$

$$\tilde{q}(\tilde{x}) = F_*q(x) := q(x)/J|_{x=F^{-1}(\tilde{x})}$$

(6)

and $M = (\partial x_i/\partial \tilde{x}_j)_{i,j=1}^n, J = \det(M)$. Then, $u \in H^1(\Omega)$ solves the Helmholtz equation

$$\nabla \cdot (\sigma \nabla u) + k^2qu = 0 \quad \text{on} \ \Omega,$$

if and only if the pull-back field $\tilde{u} = (F^{-1})^*u := u \circ F^{-1} \in H^1(\tilde{\Omega})$ solves

$$\nabla \cdot (\tilde{\sigma} \nabla \tilde{u}) + k^2\tilde{q}\tilde{u} = 0.$$

We have made use of $\nabla$ and $\nabla$ to distinguish the differentiations respectively in $x$- and $\tilde{x}$-coordinates. We refer to [20, 23] for a proof of this invariance.

We are in a position to construct the cloaking device. In the following, let $D \Subset \Omega$ be a Lipschitz domain such that $\Omega \setminus D$ is connected. Without loss of generality, we assume that $D$ contains the origin. Let $\rho > 0$ be sufficiently small and $D_\rho := \{\rho x; x \in D\}$. Suppose

$$F_{\rho} : \tilde{\Omega} \setminus D_\rho \rightarrow \tilde{\Omega} \setminus D,$$

(7)

which is a bi-Lipschitz and orientation-preserving mapping and $F_{\rho}|_{\partial \Omega} = \text{identity}$. A celebrated example of such blow-up mapping is given by

$$y = F_{\rho}(x) := \left(\frac{R_1 - \rho}{R_2 - \rho}, \frac{R_2 - R_1}{R_2 - \rho}\right) \frac{x}{|x|}, \quad \rho < R_1 < R_2$$

(8)

which blows up the central ball $B_\rho$ to $B_{R_2}$ within $B_{R_1}$. Now, we set

$$\{\Omega; \tilde{D}; \sigma^\rho, q^\rho\} = (F_{\rho})_*\{\Omega \setminus \tilde{D}_\rho; I, 1\}.$$  

(9)

Let $M \Subset D$ represent the cloaked region. Then, we claim that the following construction gives a near-cloaking device:

$$\{\mathbb{R}^N; \sigma, q\} = \begin{cases} 
[I, 1] & \text{in } \mathbb{R}^N \setminus \tilde{\Omega}; \\
[\sigma^\rho, q^\rho] & \text{in } \Omega \setminus \tilde{D}; \\
a \text{SH layer} & \text{in } D \setminus M; \\
\text{arbitrary target object} & \text{in } M.
\end{cases}$$

(10)

In (10), by a SH layer we mean a layer of material which prevents acoustic waves from penetrating inside and the normal velocity of the underlying wave field vanishes from the exterior boundary of the layer. The wave equation governing the wave scattering corresponding to the cloaking device constructed in (10) is

$$\begin{cases}
\nabla \cdot (\sigma \nabla u) + k^2qu = 0 & \text{in } \mathbb{R}^N \setminus \tilde{D}, \\
\sum_{i,j=1}^N (\sigma^\rho)^{ij}v_i\partial_j u = 0 & \text{on } \partial D,
\end{cases}$$

(11)

where $v = (v_i)_{i=1}^N$ is the exterior unit normal vector to $\partial D$. In the following, we shall let $A(\tilde{x}, d)$ denote the scattering amplitude to the PDE system (11) corresponding to the cloaking device. Let

$$F = F_{\rho} \quad \text{on } \Omega \setminus \tilde{D}_\rho; \quad \text{identity on } \mathbb{R}^N \setminus \Omega.$$
Set \( v = F^* u \in H^1_{\text{loc}}(\mathbb{R}^N \setminus \bar{D}_\rho) \) and \( v^r(x) := v(x) - e^{ikx \cdot d} \). By transformation acoustics, together with straightforward calculations, one can show that
\[
\begin{align*}
(\Delta + k^2)v &= 0 \quad \text{in} \quad \mathbb{R}^N \setminus \bar{D}_\rho, \\
\frac{\partial v}{\partial \nu}|_{\partial D_\rho} &= 0, \\
v(x) &= v^r(x) + e^{ix \cdot \xi} \quad x \in \mathbb{R}^N \setminus \bar{D}_\rho, \\
lim_{r \to \infty} r^{(N-1)/2} \left\{ \frac{\partial v^r}{\partial r} - i k v^r \right\} &= 0. 
\end{align*}
\]
(12)

In terms of the terminologies in [23], (11) describes the scattering in the physical space and (12) describes the scattering in the virtual space. Since \( v = u \) in \( \mathbb{R}^N \setminus \bar{O} \), we see that the scattering in the physical space is the same as that in the virtual space. That is, \( A(\hat{x}, d) \) could also be read off from the large \(|x|\) asymptotics of \( v^l \). Next, we give one of the main results of this paper, which justifies the near-invisibility of the above construction.

**Theorem 2.1.** There exists \( \rho_0 > 0 \) such that when \( \rho < \rho_0 \) the scattering amplitude \( A(\hat{x}, d) \) to (12) satisfies
\[
|A(\hat{x}, d)| \leq C \rho^N, \quad \hat{x}, d \in S^{N-1},
\]
(13)

where \( C \) is a constant dependent only on \( k, \rho_0 \) and \( D \), but completely independent of \( \rho \).

**Remark 2.2.** According to our earlier discussion, the scattering amplitude \( A(\hat{x}, d) \) in (13) could be read off from the large \(|x|\) asymptotics of \( v^l \) in (12). Hence, in order to assess the near-cloaking performance, it suffices to evaluate the scattering in (12) corresponding to a small SH obstacle. It is noted that the study on scattering due to small SH obstacles could be found, e.g., in [12, 26]. Nonetheless, we shall present a proof based on a layer potential technique which works for a more general setting. We would also like to mention the closely related study in the literature on scattering estimates due to small inhomogeneities in [6–9].

**Proof of theorem 2.1.** In order to ease the exposition, we shall only prove the theorem for \( N = 2, 3 \). But we would like to emphasize that for \( N > 3 \), the proof follows by completely similar arguments.

Let
\[
G(x) = \frac{i}{4} \left( \frac{k}{2\pi |x|} \right)^{(N-2)/2} H^{(1)}_{(N-2)/2}(k|x|)
\]
(14)

be the outgoing Green’s function. By Green’s representation, we know for \( x \in \mathbb{R}^N \setminus \bar{D}_\rho \),
\[
\begin{align*}
v^l(x) &= \int_{D_\rho} \left\{ \frac{\partial G(x - y)}{\partial v(y)} v^r(y) - G(x - y) \frac{\partial v^r(y)}{\partial v(y)} \right\} \, ds(y) \\
&= (Kv^r)(x) + g(x),
\end{align*}
\]
(15)

where we have set
\[
(Kv^r)(x) := \int_{\partial D_\rho} \frac{\partial G(x - y)}{\partial v(y)} v^r(y) \, ds(y),
\]
(16)

\[
g(x) := -\int_{\partial D_\rho} G(x - y) \frac{\partial v^r(y)}{\partial v(y)} \, ds(y).
\]
(17)

Clearly, \( v^l(x)|_{\partial D_\rho} \in H^{1/2}(\partial D_\rho) \). By the jump properties of the double-layer potential operator \( \mathcal{K} \) (cf [27]), we have from (15)
\[
\frac{1}{2} v^l(x) = (Kv^r)(x) + g(x), \quad x \in \partial D_\rho.
\]
(18)

5
Let \( x' = x/\rho \), then (18) is read as
\[
\frac{1}{2} u'(\rho x') = (Ku')(\rho x') + g(\rho x'), \quad x' \in \partial D. \tag{19}
\]

Next, we claim
\[
\|g(\rho \cdot)\|_{L^2(\partial D)} \leq C \rho, \tag{20}
\]
where \( C \) remains uniform as \( \rho \to 0^+ \). In the following, we shall make use of the following asymptotic developments of the 2D \( G(x) \) (cf [11]):
\[
G(x) = -\frac{1}{2\pi} \ln |x| + \frac{i}{4} - \frac{1}{2\pi} \ln \frac{k}{2} - \frac{E}{2\pi} + O(|x|^2 \ln |x|) \tag{21}
\]
for \( |x| \to 0 \), where \( E \) is Euler’s constant. Moreover, we know that
\[
G(x) = \frac{e^{ik|x|}}{4\pi |x|} \quad \text{when} \ N = 3. \tag{22}
\]

In order to prove (20), we first assume \( x \in \partial D_\rho \) with \( 1 < t \leq 2 \). Then by Green’s formula, we have
\[
\int_{\partial D_\rho} G(x - y) \frac{\partial e^{iy \cdot y}}{\partial y} dy = \int_{D_\rho} \Delta_y e^{iy \cdot y} G(x - y) dy + \int_{D_\rho} \nabla_y G(x - y) \cdot \nabla_y e^{iy \cdot y} dy
\]
\[
= -k^2 \int_{D_\rho} e^{iy \cdot y} G(x - y) dy + \int_{D_\rho} \nabla_y G(x - y) \cdot \nabla_y e^{iy \cdot y} dy. \tag{23}
\]

Using (21) and (22), we have for \( x' \in D_t \),
\[
|g_t(\rho x')| = k^2 \int_{D_\rho} e^{iy \cdot y} G(\rho x' - y) dy
\]
\[
\leq k^2 \int_D |G(\rho (x' - y'))| \rho^s dy'. \tag{24}
\]

By the mapping properties of the volume potential operator, one has
\[
|g_t(x)|_{C(\partial D_\rho)} \leq C \rho, \tag{25}
\]
where \( C \) is independent of \( \rho \) and \( t \). In like manner, one can show that
\[
|g_2(x)|_{C(\partial D_\rho)} = \left| \int_{D_\rho} \nabla_y G(x - y) \cdot \nabla_y e^{iy \cdot y} dy \right|_{C(\partial D_\rho)} \leq C \rho. \tag{25}
\]

By (24) and (25), we see
\[
|g(x)|_{C(\partial D_\rho)} \leq C \rho. \tag{26}
\]

Next, by the mapping property of the single-layer potential operator, we know
\[
g(x)|_{\partial D_\rho} = \lim_{\tau \to 1, x' \to x} (g(x)|_{\partial D_\rho}), \tag{27}
\]

which together with (26) implies (20).

We proceed to the integral equation (18). First, by using change of variables in integrals, it is straightforward to show that
\[
(Ku')(\rho x') = (K_0 u'(\rho \cdot))(\rho x') + (Ru')(\rho x'), \quad x' \in \partial D, \tag{28}
\]
where \( K_0 \) is an integral operator with the kernel given by
\[
G_0(x - y) = \begin{cases} 
-\frac{1}{2\pi} \ln |x - y| & N = 2, \\
\frac{1}{4\pi} \frac{1}{|x - y|} & N = 3,
\end{cases}
\]
which is the fundamental solution to \(-\Delta;\) and \(R\) satisfies
\[
\|R\|_{L^2(D) \to L^2(D')} \lesssim \begin{cases} \rho \ln \rho & \text{when } N = 2; \\
\rho & \text{when } N = 3. \end{cases} \quad (29)
\]
Hence, the integral equation (19) can be reformulated as
\[
\left( I - \frac{1}{\tilde{\kappa}_0} R \right) v' (\cdot) = \tilde{g} (\cdot) \quad \forall x' \in D.
\]
By the well-known result in [34], \(I - \frac{1}{\tilde{\kappa}_0} R\) is invertible from \(L^2 (\partial D) \to L^2 (\partial D)\). Then, by using (20), we have from (30) that
\[
\|v' (\cdot)\|_{L^2 (\partial D)} \leq C \|\tilde{g} (\cdot)\|_{L^2 (\partial D)} \leq C \rho.
\]
Noting \(\|v' (\cdot)\|_{L^2 (\partial D)} = \rho^{(N-1)/2} \|v (\cdot)\|_{L^2 (\partial D)}\), we further have from (31) that
\[
\|v' (\cdot)\|_{L^2 (\partial D)} \leq \begin{cases} C \rho^{3/2} & \text{when } N = 2, \\
C \rho^2 & \text{when } N = 3. \end{cases} \quad (32)
\]
Finally, by letting \(|x| \to \infty\) in (15), we have
\[
\mathcal{A}(\hat{x}, d) = \gamma \int_{D_D} \left[ \frac{\partial}{\partial y (y)} v' (y) + e^{-ik\hat{x} \cdot y} \frac{\partial}{\partial y (y)} \right] \text{d}s(y),
\]
where \(\gamma = e^{i\pi} / \sqrt{8\pi k}\) when \(N = 2\), and \(\gamma = 1/4\pi\) when \(N = 3\). By using (32) and the Schwarz inequality in (33), we have
\[
\left| \int_{D_D} \frac{\partial}{\partial y (y)} v' (y) \text{d}s(y) \right| \leq C \rho^N. \quad (34)
\]
By Green’s formula, we have
\[
\left| \int_{D_D} e^{-ik\hat{x} \cdot y} \frac{\partial}{\partial y (y)} e^{ik\hat{x} \cdot y} \text{d}s(y) \right| = \left| \int_{D_D} (\Delta e^{ik\hat{x} \cdot y}) e^{-ik\hat{x} \cdot y} \text{d}y + \int_{D_D} \nabla e^{ik\hat{x} \cdot y} \cdot \nabla e^{-ik\hat{x} \cdot y} \text{d}y \right|
\]
\[
= |k^2 (d \cdot \hat{x} - 1) \int_{D_D} e^{ik(d - \hat{x}) \cdot y} \text{d}y| \leq C \rho^N. \quad (35)
\]
By (33), (34) and (35), we have (13).

The proof is completed. \(\square\)

By theorem 2.1, we know that construction (10) gives a near-invisibility cloaking within \(\rho^N\) of the perfect cloaking. For a special case by taking \(D_D = B_{\rho}\), namely the central ball of radius \(\rho > 0\), using wavefunctions expansion, one can show (cf [25, 32])
\[
\mathcal{A}(\hat{x}, d) = -e^{i\pi^2} \sqrt{\frac{2}{\pi k}} \left[ \frac{\mathcal{F}_0 (k\rho)}{H_0^{(1)} (k\rho)} + 2 \sum_{n=1}^{\infty} \mathcal{F}_n (k\rho) \frac{H_n^{(1)} (k\rho) \cos n \theta}{H_0^{(1)} (k\rho)} \right] \quad (36)
\]
in \(\mathbb{R}^2\), where \(\theta = \angle (\hat{x}, d)\); and
\[
\mathcal{A}(\hat{x}, d) = \frac{i}{k} \sum_{n=0}^{\infty} \frac{(2n + 1)}{(2n + 1)} \frac{\mathcal{F}_n (k\rho)}{H_n^{(1)} (k\rho)} P_n (\cos \theta) \quad (37)
\]
in \(\mathbb{R}^3\), where \(P_n\) is the Legendre polynomial of degree \(n\). By the asymptotic developments of spherical Bessel functions (cf [25, 32]), one has from (37)
\[
\mathcal{A}(\hat{x}, d) = \frac{i}{k} \left( \frac{\cos \theta}{2} - \frac{1}{2} \right) (k\rho)^3 + \mathcal{O}((k\rho)^5). \quad (38)
\]
Similarly, from (36) one can show that in $\mathbb{R}^2$,
\[
A(\hat{x}, d) = -e^{i\pi/4} \sqrt{2\pi k} \left( \cos \theta - \frac{1}{4} \right) (k\rho)^2 + O((k\rho)^4).
\]
(39)

From (38) and (39), it is readily seen that the estimates in theorem 2.1 are optimal for full scattering measurements, namely $\hat{x} \in S^{N-1}$ and $d \in S^{N-1}$. Nevertheless, it is interesting to note from (38) and (39) that for some specific scattering measurements, e.g., $A(\hat{x}, d)$ with $\angle(\hat{x}, d) = \pm \arccos \frac{2}{3}$ in 3D, and with $\angle(\hat{x}, d) = \pm \frac{\pi}{3}$ in 2D, one would have even more enhanced invisibility cloaking effects.

3. Near-cloak construction with a lossy layer

In this section, we shall develop a lossy approximate cloaking scheme. To that end, we first introduce the following transformation $T : \mathbb{R}^N \rightarrow \mathbb{R}^N$:
\[
y = T(x) := \begin{cases} 
x & \text{for } x \in \mathbb{R}^N \setminus \bar{\Omega}, 
\end{cases}
\]
(40)
where $F_\rho$ is given in (7). Let
\[
[\mathbb{R}^N; \sigma, q] = \begin{cases} 
I, 1 & \text{in } \mathbb{R}^N \setminus \Omega, 
T \sigma, T q & \text{in } \Omega \setminus D, 
T \sigma l, T q l & \text{in } D \setminus D_{1/2}, 
\sigma' a, q' a & \text{in } D_{1/2}, 
\end{cases}
\]
(41)
be the cloaking device in the physical space. Here, $[D_{1/2}; \sigma', q']$ is an arbitrary but regular medium, which represents the target object being cloaked, and
\[
[D_{\rho'} \setminus D_{\rho''}; \sigma l, q l]
\]
(42)
is a lossy layer whose parameters shall be specified in the following. Similar to our earlier argument in section 3, using transformation acoustics we see that the scattering amplitude in the physical space corresponding to the cloaking device is the same as the one in the virtual space. In the virtual space, the wave scattering is governed by the following PDE system:
\[
\begin{cases} 
(\Delta + k^2) u = 0 & \text{in } \mathbb{R}^N \setminus \bar{D}_{\rho'}, 
\nabla \cdot (\sigma l \nabla u) + k^2 q l u = 0 & \text{in } D_{\rho'} \setminus D_{\rho'/2}, 
\nabla \cdot (\sigma a \nabla u) + k^2 q a u = 0 & \text{in } D_{\rho'/2},
\end{cases}
\]
(43)
where
\[
[D_{\rho'/2}; \sigma a, q a] = (T^{-1})_* [D_{1/2}; \sigma', q']
\]
(44)
is arbitrary but regular, and $u \in H^1_{\text{loc}}(\mathbb{R}^N)$ satisfies
\[
\lim_{r \to \infty} r^{(N-1)/2} \left( \frac{\partial u}{\partial r} - i k u \right) = 0.
\]
We shall choose
\[
\sigma l = C \rho^{2+\delta} I \quad \text{and} \quad q l = a + ib,
\]
(45)
with $C, \delta, a$ and $b$ being some fixed positive constants in our construction. We shall show that it will yield an enhanced approximate cloaking scheme. We would only consider this for a
special case with spherical geometry and uniform cloaked contents. In the following, we let \( D_\rho = B_\rho \) and, \( \sigma'_a \) and \( q'_a \) be the arbitrary positive constants but independent of \( \rho \). With some abuse of notation, we shall also write

\[
\sigma_l = C \rho^{2+2\delta}.
\]

By (44), one can show by direct calculations that in the virtual space

\[
\sigma_a, q_a = \begin{cases} 
\sigma'_a, q'_a \rho^2 & \text{in } D_{\rho/2} \text{ when } N = 2 \\
\sigma'_a, q'_a \rho^3 & \text{in } D_{\rho/2} \text{ when } N = 3.
\end{cases}
\]

For the PDE system (43), we let \( u = u_0 \) in \( \mathbb{R}^N \setminus \tilde{D}_\rho \), \( u = u_2 \) in \( D_\rho \setminus \tilde{D}_{\rho/2} \) and \( u = u_{\text{int}} \) in \( D_{\rho/2} \).

Under transmission conditions, we have

\[
u_2 = u_0 \quad \text{and} \quad \frac{\partial u_2}{\partial \nu} = \frac{\partial u_0}{\partial \nu} \quad \text{on } \partial D_\rho, \tag{46}
\]

and

\[
u_2 = u_{\text{int}} \quad \text{and} \quad \frac{\partial u_2}{\partial \nu} = \sigma_l \frac{\partial u_{\text{int}}}{\partial \nu} \quad \text{on } \partial D_{\rho/2}. \tag{47}
\]

Set \( \tilde{k} = k \sqrt{\frac{\sigma'}{\rho}} \) and \( k_2 = k \sqrt{\frac{\sigma}{\rho}} \). We choose the complex branch of \( \tilde{k} \) such that \( \Im(\tilde{k}) > 0 \).

We first consider the 2D case. By [11], one has the following series expansions:

\[
u_0(x) = e^{ikx} + u^t = \sum_{n=-\infty}^{\infty} \hat{J}_n(k|x|) e^{in\theta} + \sum_{n=-\infty}^{\infty} d_n H_n^{(1)}(k|x|) e^{in\theta},
\]

\[
u_2(x) = \sum_{n=-\infty}^{\infty} a_n J_n(\tilde{k}|x|) e^{in\theta} + \sum_{n=-\infty}^{\infty} b_n H_n^{(1)}(\tilde{k}|x|) e^{in\theta}, \tag{48}
\]

\[u_{\text{int}} = \sum_{n=-\infty}^{\infty} c_n J_n(k_2|x|) e^{in\theta}.
\]

By (46) and (48), we have

\[
\begin{bmatrix}
a_n J_n(\tilde{k}|\rho) + b_n H_n^{(1)}(\tilde{k}|\rho) = \hat{r} J_n(k|\rho) + d_n H_n^{(1)}(k|\rho) \\
\sigma_l \left[ a_n J_n'(\tilde{k}|\rho) + b_n H_n^{(1)}(\tilde{k}|\rho) \right] = \hat{r} k J_n'(k|\rho) + d_n k H_n^{(1)}(k|\rho).
\end{bmatrix} \tag{49}
\]

In a similar manner, by (47) and (48), we have

\[
\begin{bmatrix}
a_n J_n(\tilde{k}|\rho/2) + b_n H_n^{(1)}(\tilde{k}|\rho/2) = c_n J_n(k_2|\rho/2) \\
\sigma_l \left[ a_n J_n'(\tilde{k}|\rho/2) + b_n H_n^{(1)}(\tilde{k}|\rho/2) \right] = c_n k_2 J_n'(k_2|\rho/2).
\end{bmatrix} \tag{50}
\]

Here, \( J_n'(k_2|\rho) \) and \( J_n'(k_2|\rho/2) \) are understood in the same sense. By letting \( C_0 = 1/\sqrt{\sigma_l q'_a} \) and \( A = \sqrt{q'_a/\sigma_l} \), \( q'_a/\sigma_l \), \( k = \sqrt{\frac{\sigma}{\rho}} \), and (49) and (50) are read as

\[
\begin{bmatrix}
a_n J_n(\tilde{k}|\rho) + b_n H_n^{(1)}(\tilde{k}|\rho) = \hat{r} J_n(k|\rho) + d_n H_n^{(1)}(k|\rho) \\
\left[ a_n J_n'(\tilde{k}|\rho) + b_n H_n^{(1)}(\tilde{k}|\rho) \right] = C_0 \left[ \hat{r} J_n(k|\rho) + d_n H_n^{(1)}(k|\rho) \right].
\end{bmatrix} \tag{51}
\]

and

\[
\begin{bmatrix}
a_n J_n(\tilde{k}|\rho/2) + b_n H_n^{(1)}(\tilde{k}|\rho/2) = c_n J_n(k_2|\rho/2) \\
\left[ a_n J_n'(\tilde{k}|\rho/2) + b_n H_n^{(1)}(\tilde{k}|\rho/2) \right] = C_0 c_n A J_n'(k_2|\rho/2).
\end{bmatrix} \tag{52}
\]
We next investigate the asymptotic development of $B_n(k\rho/2)$, if $J_n(k\rho/2) \neq 0$, then by direct calculations, we have from (52)

$$c_n = \frac{a_n J_n(k\rho/2) + b_n H_n^{(1)}(k\rho/2)}{J_n(k\rho/2)}$$

and

$$b_n = -\frac{J_n(k\rho/2) - C_0 \frac{\ell_k J_n(\rho k/2)}{J_n(k\rho/2)} J_n(k\rho/2)}{H_n^{(1)}(k\rho/2) - C_0 \frac{\ell_k J_n(\rho k/2)}{J_n(k\rho/2)} H_n^{(1)}(k\rho/2)} a_n. \quad (53)$$

In the case $J_n(k\rho/2) = 0$, we have from the first equation in (52) that

$$b_n = \frac{c_n J_n(\rho k/2)}{H_n^{(1)}(\rho k/2)} a_n. \quad (54)$$

Let $\gamma_0$ denote the fraction in (53) or (54) and hence $b_n = \gamma_0 a_n$. Plugging $b_n$ into (51), we have by straightforward calculations

$$a_n = \frac{\tau_0^2 J_n(\rho k) + \gamma_0 H_n^{(1)}(\rho k)}{J_n(\rho k) + \gamma_0 H_n^{(1)}(\rho k)},$$

and

$$d_n = \frac{\tau_0^2 J_n(\rho k) - \frac{1}{\tau_0^{2}} \frac{\gamma_0 H_n^{(1)}(\rho k)}{J_n(\rho k)} + \tau_0 J_n(\rho k)}{H_n^{(1)}(\rho k) - \frac{1}{\tau_0^{2}} \frac{\gamma_0 H_n^{(1)}(\rho k)}{J_n(\rho k)} H_n^{(1)}(\rho k)}.$$

We next investigate the asymptotic development of

$$H(\rho, \gamma_0, \rho) := \frac{1}{c_0} J_n(\rho) + \gamma_0 H_n^{(1)}(\rho) = \frac{1}{\tau_0} \frac{J_n(\rho)}{J_n(\rho)} + \frac{\gamma_0}{\tau_0} \frac{H_n^{(1)}(\rho)}{J_n(\rho)} \left(1 + \frac{\gamma_0 H_n^{(1)}(\rho)}{J_n(\rho)}\right). \quad (56)$$

Clearly, we only need to study the asymptotic behavior of $\frac{1}{\tau_0} \frac{J_n(\rho)}{J_n(\rho)}$, $\frac{1}{\tau_0} \frac{H_n^{(1)}(\rho)}{J_n(\rho)}$ and $\gamma_0 \frac{H_n^{(1)}(\rho)}{J_n(\rho)}$. We note that due to (45)

$$\Re(\hat{\rho}^*) \rightarrow +\infty \quad \text{and} \quad \Im(\hat{\rho}^*) \rightarrow +\infty \quad \text{as} \quad \rho \rightarrow 0^+,$$

which implies the following asymptotic developments for $z = \hat{\rho}^*$ (see formulæ 9.2.1 and 9.2.3 in [2]):

$$\begin{cases}
J_n(z) \sim \sqrt{\frac{2}{\pi z}} \cos \left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) + e^{i\arg(z)}\mathcal{O}(|z|^{-1}), \quad |\arg z| < \pi \\
H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)}, \quad -\pi < \arg z < 2\pi.
\end{cases} \quad (57)$$

We also note the following recurrence relation (see formulæ 9.1.27 in [2]) for subsequent use:

$$B_n(z) = \frac{n}{z} B_n(z) - B_{n+1}(z), \quad (58)$$

where $B_n(z) = J_n(z)$ or $H_n^{(1)}(z)$. Since

$$J_n(z) = (-1)^n J_{-n}(z) \quad \text{and} \quad H_n^{(1)}(z) = (-1)^n H_{-n}(z),$$

(see formulæ 9.1.5 in [2]), it suffices for us to consider the case with $n \geq 0$ in the following.
Noting $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$, by (57) we further have, as $\Re(z), \Im(z) \to \infty$,

$$
\begin{align*}
J_0(z) &\sim \sqrt{\frac{2}{\pi z}} e^{iz} \left(1 - \frac{z^2}{4} + \frac{z^4}{64} + \frac{z^6}{256} + \frac{z^8}{16384} + \cdots\right), \quad |\arg z| < \pi \\
H_{n}^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{-\Im(z)} \left(1 + \frac{z^2}{4} + \frac{z^4}{64} + \frac{z^6}{256} + \frac{z^8}{16384} + \cdots\right), \quad -\pi < \arg z < 2\pi \\
|H_{n}^{(1)'}(z)| &\sim \sqrt{\frac{2}{\pi |z|}} e^{-\Im(z)}, \quad -\pi < \arg z < 2\pi.
\end{align*}
$$

(59)

It is remarked that the third equation in (59) is obtained by using the recurrence relation (58).

Hence, we see for $z = \tilde{k}\rho$, $J_n(z)$ blows up exponentially, while $H_{n}^{(1)}(z)$ decreases exponentially as $\rho \to 0^+$. In the following, we let $\sqrt{\alpha + i\beta} = \alpha + i\beta$, with $\alpha, \beta > 0$. For $\frac{1}{C_0}J_n(k\rho)$, as $\rho \to 0^+$, by (59) we have

$$
\frac{1}{C_0}J_n(k\rho) = 1 = -e^{iz/2} \frac{1}{C_0} = -e^{iz/2} \sqrt{\alpha + i\beta} \to +0.
$$

(60)

For $\mathcal{Y}_0 \frac{H_n^{(1)}(k\rho)}{J_n(k\rho)}$, if $\mathcal{Y}_0$ is the fraction in (53), we have

$$
\begin{align*}
\mathcal{Y}_0 \frac{H_n^{(1)}(k\rho)}{J_n(k\rho)} &= -\frac{J_n(k\rho/2) - C_0 A_{J_n(k\rho/2)} J_n(k\rho/2)}{H_n^{(1)}(k\rho/2) - C_0 A_{H_n^{(1)}(k\rho/2)} H_n^{(1)}(k\rho/2)} \\
&= -\frac{J_n(k\rho/2) - C_0 A_{J_n(k\rho/2)} J_n(k\rho/2)}{H_n^{(1)}(k\rho/2) - C_0 A_{H_n^{(1)}(k\rho/2)} H_n^{(1)}(k\rho/2)} \\
&= \mathcal{Y}_1 \times \mathcal{Y}_2.
\end{align*}
$$

(61)

By straightforward asymptotic analysis, one can show as $\rho \to 0^+$,

$$
|\mathcal{Y}_1| = \left|\frac{J_n(k\rho/2) - C_0 A_{J_n(k\rho/2)} J_n(k\rho/2)}{J_n(k\rho)} \right| \\
\sim \left|\frac{\left(e^{iz/2} - C_0 A_{J_n(k\rho/2)} J_n(k\rho/2)\right)}{J_n(k\rho)} \right| \\
\leq \tilde{C}(1 + (n + 1)|\alpha + i\beta|\rho^{-2}) |e^{-\Im(z/\rho)}|,
$$

(62)

where $\tilde{C}$ is a constant independent of $\rho$. That is, $|\mathcal{Y}_1|$ decreases to 0 more quickly than $\rho^r$ for any $r > 0$. Similarly, one can show that $|\mathcal{Y}_2|$ decreases to 0 more quickly than $\rho^r$ for any $r > 0$. Furthermore, if $\mathcal{Y}_0$ is the fraction in (54), by completely similar arguments one can show that $\mathcal{Y}_0 \frac{H_n^{(1)}(k\rho)}{J_n(k\rho)}$ also decays more quickly than $\rho^r$ for any $r > 0$. Hence, by (56)–(62) and $\sigma_i = \rho^{r+2}$, we have

$$
\mathcal{H}(\sigma_i, \rho) \sim \frac{1}{C_0} \frac{J_n(k\rho)}{J_n(k\rho)} \sim -e^{iz/2} \frac{1}{C_0} = -e^{iz/2} (\alpha + i\beta) \rho^{1+\delta} \quad \text{as} \quad \rho \to 0^+.
$$

(63)

Now, by (63) and (55) we have

$$
d_n = \frac{-\tilde{p} J_n(k\rho) + e^{iz/2} (\alpha + \beta i) \rho^{1+\delta} J_n(k\rho)}{H_n^{(1)}(k\rho) + e^{iz/2} (\alpha + \beta i) \rho^{1+\delta} H_n^{(1)}(k\rho)}.
$$

(64)
Using the asymptotic developments of Bessel functions and their derivatives (cf [25]), we further have

\[
\begin{align*}
    d_0 & \sim -\frac{k \rho}{2} + e^{i \pi/2} (\alpha + i \beta) \rho^{1+\delta} + \frac{1}{2} \frac{\alpha}{\rho} + e^{i \pi/2} (\alpha + i \beta) \rho^{1+\delta} \frac{\rho}{2} 
    n & = 0, \\
    d_n & \sim \frac{\rho n (k \rho)^{1-1}}{2 \Gamma(n+1)} + e^{i \pi/2} (\alpha + i \beta) \rho^{1+\delta} p^n \frac{(k \rho)^n}{2 \Gamma(n+1)} - e^{i \pi/2} (\alpha + i \beta) \rho^{1+\delta} \frac{2^n (n-1 \ldots 2)}{\pi (k \rho)^{1+n}}, 
\end{align*}
\]

(65)

which imply

\[
\begin{align*}
    d_0 & \sim \rho^2, & n & = 0, \\
    d_n & \sim \rho^{2n}, & n & \in \mathbb{N}, 
\end{align*}
\]

(66)

and

\[
\begin{align*}
    |d_0| & \leq \pi k |\alpha + i \beta| \rho^{2+\delta}, \\
    |d_n| & \leq \frac{4 \pi |\alpha + i \beta|}{k^{1+\delta}} \frac{1}{(2^n!)} (k \rho)^{2n+2+\delta}.
\end{align*}
\]

(67)

Since

\[
\begin{align*}
    u^s(x) & = \sum_{n=-\infty}^{\infty} d_n H_n^{(1)}(k|x|) e^{in\theta},
\end{align*}
\]

(68)

by taking |x| → +∞, together with (66), one has by direct calculations that the corresponding scattering amplitude satisfies

\[
|A(\hat{\chi}, d)| \leq C \rho^2,
\]

(69)

where C is a positive constant that remains uniform as ρ → 0+. That is, construction (41) gives an anti-cloaking device within ρ^2 of the ideal cloaking.

Now, we look into the series representation of the scattered wave field (68). If Dρ is a SH obstacle, the scattered wave corresponding to e^{ikd} is given by (see equation (3.19) in [25])

\[
\tilde{u}^s(x) = -\sum_{n=-\infty}^{\infty} \frac{\tilde{F}_n(k \rho)}{H_n^{(1)}(k \rho)} H_n^{(1)}(k x) e^{in\theta},
\]

(70)

By (67), (68) and (70), one has by direct verification that for ρ sufficiently small

\[
|u^s(x) - \tilde{u}^s(x)| \leq C \rho^{2+\delta}
\]

(71)

for any x ∈ R^2 \ B_ρ, with ε > ρ being a fixed constant, where C depends only on k but is independent of ρ. We remark that the estimate in (71) would reduce to C ρ^{1+\delta} for x ∈ R^2 \ D_ρ.

Hence, construction (41) is actually a finite realization of the SH lining construction (10).

The 3D case can be proved following similar arguments, which we shall sketch in the following. We shall make use of the same notation, u₀, u₂, uₙ, ℓ and k₂, etc. We have the following series representations of the wave fields:

\[
\begin{align*}
    u₀(x) & = e^{ikd} + u^s(x) \\
    & = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n 4\pi Y_m^s(d) j_n(k|x|) Y_n^m(\hat{x}) + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} d_n^m h_n^{(1)}(k|x|) Y_n^m(\hat{x}), \\
    u₂(x) & = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} d_n^m j_n(\tilde{k}|x|) Y_n^m(\hat{x}) + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m h_n^{(1)}(\tilde{k}|x|) Y_n^m(\hat{x}), \\
    uₙ(x) & = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_n^m j_n(k₂|x|) Y_n^m(\hat{x}).
\end{align*}
\]

(72)
By the transmission conditions, respectively, on $\partial D_\rho$ and $\partial D_{\rho/2}$, we have
\begin{equation}
\begin{aligned}
&\left[ d_n^m j_n(k\rho) + b_n^m h_n^{(1)}(k\rho) \right] = i^44\pi \tilde{Y}_n^m(d)\tilde{j}_n(k\rho) + d_n^m h_n^{(1)}(k\rho), \\
&\left[ \alpha \left[ k\tilde{e}_n^m j_n(k\rho) + b_n^m h_n^{(1)}(k\rho) \right] \right] = i^44\pi k\tilde{Y}_n^m(d)\tilde{j}_n(k\rho) + kd_n^m h_n^{(1)}(k\rho),
\end{aligned}
\tag{73}
\end{equation}
and
\begin{equation}
\begin{aligned}
&\left[ d_n^m j_n(k\rho/2) + b_n^m h_n^{(1)}(k\rho/2) \right] = c_n^m j_n(k_2\rho/2), \\
&\left[ \alpha \left[ k\tilde{e}_n^m j_n(k\rho/2) + b_n^m h_n^{(1)}(k\rho/2) \right] \right] = \sigma_A c_n^m k_2 j_n(k_2\rho/2).
\end{aligned}
\tag{74}
\end{equation}

For this 3D case, we have $A = \sqrt{\sigma/j_0^2}/\rho^2$ and $k_2 = \frac{\rho}{\sqrt{\sigma/j_0}}$. Similar to the 2D case (see (52)–(54)), we would solve the linear systems (73) and (74) and we need to distinguish two cases $j_n(k_2\rho/2) \neq 0$ and $j_n(k_2\rho/2) = 0$. In the following, we only present the more complicated case with $j_n(k_2\rho/2) \neq 0$ and the other case could be handled in a completely similar manner. By solving (73) and (74), we have
\begin{equation}
d_n^m = \frac{i^44\pi \tilde{Y}_n^m(d)\tilde{j}_n(k\rho) - 1}{i^44\pi C_0 j_n(k\rho)/\tilde{j}_n^{(1)}(k\rho)} + i^44\pi \tilde{Y}_n^m(d)\tilde{j}_n(k\rho),
\end{equation}
where
\begin{equation}
\begin{aligned}
&\tilde{Y}_0 := -\frac{\tilde{j}_n(\tilde{k}\rho/2) - C_0 A\tilde{j}_n^{(1)}(k\rho/2)}{\tilde{j}_n^{(1)}(\tilde{k}\rho/2) - C_0 A\tilde{j}_n(\tilde{k}\rho/2)}.
\end{aligned}
\end{equation}

By similar asymptotic analyses as those for the 2D case, one can show that as $\rho \to 0^+$,
\begin{equation}
\begin{aligned}
&\frac{1}{C_0} \frac{\tilde{j}_n(\tilde{k}\rho) + \tilde{Y}_0 h_n^{(1)}(\tilde{k}\rho)}{\tilde{j}_n^{(1)}(\tilde{k}\rho)} \to -\frac{1}{C_0} e^{i\pi/2} = -e^{i\pi/2}(\alpha + i\beta)\sqrt{\sigma_1},
\end{aligned}
\end{equation}
Then, by (75) we have
\begin{equation}
\begin{aligned}
&\left| d_0^0 \right| \sim \frac{4\pi \tilde{Y}_0^0(d)k\rho/3 + 4\pi \tilde{Y}_0^0(d)e^{i\pi/2}(\alpha + i\beta)\sqrt{\sigma_1}}{2\pi\sqrt{\beta}} - e^{i\pi/2}(\alpha + i\beta)\sqrt{\sigma_1}\frac{1}{2\pi\sqrt{\beta}}, \\
&\left| d_0^m \right| \sim \frac{i^44\pi \tilde{Y}_0^m(d)^2\alpha n(k\rho)^{\gamma-1}}{2\pi^2\sigma_1^{(2m+1)}(2m+1)} - e^{i\pi/2}(\alpha + i\beta)\sqrt{\sigma_1}\frac{1}{2\pi^2\sigma_1^{(2m+1)}(2m+1)},
\end{aligned}
\tag{76}
\end{equation}
which in turn implies
\begin{equation}
\begin{aligned}
&\left| d_0^0 \right| \sim O(\rho^3), \\
&\left| d_0^m \right| \sim O(\rho^{2m+1}),
\end{aligned}
\tag{77}
\end{equation}
and
\begin{equation}
\begin{aligned}
&\left| d_0^0 \right| \leq 2|\alpha + i\beta|k^24\pi|Y_0^0(d)|\rho^{3+\delta}, \\
&\left| d_0^m \right| \leq 2|\alpha + i\beta|k^{2m+1}4\pi|Y_0^m(d)||\alpha + i\beta|k^{-1-\delta}(\rho^2)^{2\delta+3+\delta}.
\end{aligned}
\tag{78}
\end{equation}

With the above preparations, one can show that the corresponding scattering amplitude corresponding to the cloaking device (41) in $\mathbb{R}^3$ satisfies
\begin{equation}
|A(\hat{x}, \hat{d})| \leq C\rho^3,
\end{equation}
where $C$ remains uniform as $\rho \to 0^+$. That is, (41) gives a near-cloaking device within $\rho^3$ of the ideal cloaking. Furthermore, by comparing the scattered wave field $\sigma^\prime$ in (72) to that
of a 3D SH ball \( B_{\rho} \) (cf [25]), together with (78), one can show that the deviation between them is within \( \rho^{3+\delta} \) in \( \mathbb{R}^3 \setminus B_{\rho_0} \) for any fixed \( \epsilon_0 > \rho \), and within \( \rho^{2+\delta} \) in \( \mathbb{R}^3 \setminus B_{\rho} \). Hence again, we come to the conclusion that construction (41) is a finite realization of the SH lining construction (10).

In summary, we have shown in this section that

**Theorem 3.1.** Let \( D_{\rho} = B_{\rho} \) and \( \{ D_{\rho} \setminus D_{\rho/2}; \sigma_l, q_l \} \) be given by (45), and \( \{ D_{\rho/2}; \sigma_a, q_a \} \) be arbitrary but uniform. Then, construction (41) produces a near-cloaking device within \( \rho N \) of ideal cloaking. Furthermore, (41) is a finite realization of (10) in the sense that the scattered wave fields corresponding to (41) and (10), respectively, deviate within \( \rho^N \) outside the cloaking device.

We remark that theorem 3.1 equally holds for the general case with general geometry and arbitrary cloaked contents (see [24]). Actually, in the subsequent section, our numerical examples are presented for variable cloaked contents.

4. Some discussion on different cloaking schemes

As we discussed earlier in section 1, two different cloaking schemes were developed in [20] and [23]. One of the key ingredients is to introduce a special layer between the cloaked region and the cloaking region, which are respectively \( D_{1/2} \) and \( \Omega \setminus \hat{D} \) in the physical space described in (10). They correspond to, respectively, \( D_{\rho/2} \) and \( \Omega \setminus \hat{D}_{\rho} \) in the virtual space. In [20], the authors introduced a lossy layer occupying \( D \setminus D_{1/2} \) and showed that the lossy layer is indispensable to achieving successful near-cloak. In the virtual space, the lossy layer in [20] is given as

\[
\{ D_{\rho} \setminus D_{\rho/2}; I, 1 + i\beta \}, \quad \beta \sim \rho^{-2}.
\]

If one lets the loss parameter \( \beta \) go to infinity, the limit corresponds to the imposition of a homogeneous Dirichlet boundary condition, which is the one considered in [23]. It is interesting to mention that the improvement of cloaking performance by imposing the specific boundary condition on the cloaking interface is also considered in [17]. In this context, imposition of a homogeneous Dirichlet boundary condition amounts to the lining of a layer of SS material in \( D \setminus D_{\rho/2} \). In this sense, the lossy layer (80) is a finite realization of the SS layer lining. We shall refer the construction in [23] with a SS layer as **SS construction**, and the one in [20] with a finite realization of SS layer as **FSS construction**.

In our cloaking construction (10), the inclusion of a SH layer will significantly enhance the cloaking performance, and as specified in the introduction we call this scheme an SH construction. In order to achieve a finite realization of the SH layer, we utilize the following lossy layer

\[
\{ D_{\rho} \setminus D_{\rho/2}; \sigma_l, a + ib \}, \quad \sigma_l \sim \rho^{2+\delta} I, \quad a \sim 1, \quad b \sim 1.
\]

Clearly, the lossy layer (81) is of different physical and mathematical nature from (80), and it could produce significantly improved near-cloaking performances. This losses scheme is called an FSH construction.

We would like to note that in [13], the singular ideal cloaking problem was treated directly by considering the so-called **finite energy solution**. In this ideal limiting case, it is shown that the wave field satisfies the SH condition on the interface between the cloaking and cloaked regions. Hence, the SH layer produces a compatible interface condition with the perfect cloaking problem in the limiting singular case. This is a main reason that the SH layer produces enhanced cloaking performance. On the other hand, from a practical standpoint, the
FSH and FSS layers would be easier to implement than the actual SH and SS layers since one could achieve them with finite material parameters.

5. Numerical examples

In this section, we present some numerical experiments to demonstrate the theoretical results established in the previous sections. There are totally four near-cloak schemes investigated, namely SS lining, FSS lining, SH lining and FSH lining.

First, all the numerical experiments are conducted in $\mathbb{R}^2$ and we choose $D$ and $D_\rho$ as $B_{R_1}$ and $B_{\rho}$, respectively. Some key parameters are set as follows: $R_1 = 2$, $R_2 = 3$ and $R_3 = 4$, thus $R_0 = R_1/2 = 1$. Experimental settings are shown in figure 1. For the setting with respect to the scattering measurement, a Cartesian PML layer of width 1 is attached to the square $(-5, 5)^2$ to truncate the whole space into a finite domain with scattering boundary conditions enforced on the outer boundary. $(-5, 5)^2 \setminus B_{R_1}$ and $B_{R_1} \setminus B_{R_3}$ are homogeneous surrounding media. $B_{R_1} \setminus B_{R_0}$ is the cloaking medium. $B_{R_1} \setminus B_{R_0}$, $B_{R_0}$ are the lossy layer and the cloaked region, respectively. The regularization parameter $\rho$ ranges from 0.5 to $10^{-5}$. We fix the wave number $k = 2$ and the incident direction $d = (1, 0)^T$.

In addition, for scheme FSS, the lossy parameters are chosen according to [20], i.e. $\sigma_l = I$, $q_l = \rho^2 (1 + 2.5 \rho^{-2} i)$ in $B_{R_1} \setminus B_{R_0}$. For scheme FSH, the lossy parameters are chosen by (41), (45) and the definition of the map (8), namely $\sigma_l = \rho^{2.5} I$, $q_l = \rho^2 (3 + 2i)$ in the cloaking medium of the physical space, namely $C = 1$, $\delta = 0.5$, $a = 3$ and $b = 2$.

For schemes FSS and FSH, we choose the medium coefficients in the cloaked region to be variable functions $\sigma_a = 1 + 0.3 x^2 y^2$, $q_a = 5 + x$. For scheme SH, the incident wave is shifted to the right by a tenth of the wavelength such that the origin is neither in the valley nor at the peak and the incident wave does not vanish at the origin.

From the transformation map $F$ defined in (8), we can determine the cloaking medium coefficients by (41) for any $\rho < R_1$. It is emphasized that as $\rho$ tends to zero, the singularity of $\sigma$ increases very fast. For example, it is observed that $\sigma_{11}$ grows asymptotically as $1/\rho$ and exhibits a thin layer of large values except some constant background as $\rho$ decreases, which requires many local refinements for better resolution and accounts for the boundary layer in the finite element solutions.
The scattering amplitude, or the far field data, can be computed in the following way (cf [11]). We can represent the scattered wave in the boundary integral form

\[ u'(x) = \int_{\partial B_3} u'(y) \frac{\partial \Phi(x, y)}{\partial v(y)} - \frac{\partial u'(y)}{\partial v(y)} \partial \nu(y) \, ds(y), \quad x \in \mathbb{R}^2 \setminus B_3. \]  

(82)

Using the asymptotic expansion of the fundamental solution \( \Phi(x, y) = iH_0^{(1)}(k|x - y|)/4 \), we then have

\[ A(\hat{x}) := \exp\left(-i\pi/4\right) \frac{\sqrt{8k\pi}}{\hat{x}} \int_{\partial B_3} u'(y) \frac{\partial e^{-ik\hat{x}y}}{\partial v(y)} - e^{-ik\hat{x}y} \frac{\partial u'(y)}{\partial v(y)} \, ds(y), \]  

(83)

where \( \hat{x} = x/|x| \) is the observation direction and \( v(y) \) is the outward unit normal vector pointing to the infinity. That is, the far field data can be approximated by the numerical quadrature of (82) using the Cauchy data of the scattered wave \( u' \) on the boundary of \( B_3 \).

Two groups of tests are carried out in the following.

5.1 Near-cloaks of schemes SS and FSS

It is pointed out that as \( \rho \) tends to zero, the medium coefficients exhibit more and more singularity close to \( \partial B_{R_1} \). Thus, finite element solutions require more degrees of freedom to resolve the boundary layer close to \( \partial B_{R_1} \). Either the SS boundary condition or the FSS lossy layer enforces a strong influence to the local behavior of the transmitted wave close to \( \partial B_{R_1} \), which amounts to introducing a point source in the virtual space. Such a virtual point source makes the boundary data show a very slow convergence as we decrease \( \rho \). To tackle the difficulty of the boundary layer amounts to more than two million DOFs to achieve the desired relative error tolerance when \( \rho = 10^{-5} \). The same observation holds for the other schemes. To avoid the influence of the finite element discretization error, all the following tests are carried out on the finest mesh to show the sharpness of the theoretical error bounds in terms of \( \rho \).

The scattered wave and the transmitted wave are plotted for schemes SS and FSS, respectively, for different \( \rho \)s in figures 2.

The scattering measurement data are generated according to (83) by solving the Helmholtz system on the truncated domain with a PML layer. In the discrete sense, all the norms are equivalent. So the discrete maximum norm is used to measure the decay rate of the far-field data \( A(\hat{x}) \), namely the maximum of the modulus of \( A(\hat{x}) \) is taken over 100 equidistant observation directions on \( S^1 \). We investigate the convergence rate by testing \( \rho = 1/2^j, j = 1, 2, \ldots, 7 \) for schemes SS and FSS. By plotting in figure 3 the convergence history of the discrete maximum norm of \( A(\hat{x}) \) over \( S^1 \) with respect to the upper bound \( 1/|\log_{10} \rho| \), we see clearly that schemes SS and FSS indeed achieve near-cloak, but we see that the convergence curve demonstrates linearity asymptotically except the first few outliers to the left of the plots in figure 3, which verifies the sharpness of the upper bound established in [20]. But the convergence slows down significantly as \( \rho \) decreases, which is reflected by the clustering of the blue star points in the curve. More importantly, it can be seen from figure 3 that the difference between the discrete maximum norms for schemes SS and FSS gets smaller as \( \rho \) decreases because the FSS lossy layer tends effectively to the SS boundary condition for small \( \rho \) and thus scheme FSS approaches scheme SS in the limit sense as \( \rho \to 0 \).

5.2 Near-cloaks of schemes SH and FSH

Contrary to schemes SS and FSS, our new schemes by SH and FSH lining can achieve significantly better near-cloak performance.
Figure 2. The scattered wave and the transmitted wave (real part) with respect to $\rho = 10^{-1}$ and $10^{-5}$ from top to bottom, respectively. Left: scheme SS; right: scheme FSS.

Figure 3. Convergence history of scattering measurement data versus $1/\log_{10}(\rho)$ for scheme SS (left) and scheme FSS (right).

Figure 4 shows the scattered wave and the transmitted wave for scheme SH for different $\rho$s. Compared with scheme SS, the scattered wave of scheme SH decays significantly as $\rho$ decreases. Moreover, the transmitted wave approaches more and more a deformed plane incident wave, the interior value of the transmitted wave tends to a constant as $\rho \to 0$. 
Figure 4. The scattered wave $u'$ (real part) and the transmitted wave $u'$ with respect to $\rho = 10^{-1}$ and $10^{-5}$ from top to bottom, respectively, for scheme SH.

We plot the scattered wave and the transmitted wave for scheme FSH for different $\rho$ in figure 5. We can observe a thin red layer near the outer boundary of the lossy layer, on which both the acoustic potential and the normal flux match with those from the cloaking medium. Compared with scheme FSS, the scattered wave of scheme FSH decays significantly as $\rho$ decreases. The transmitted wave approaches more and more a deformed plane incident wave, and the interior acoustic potential of the transmitted wave within the cloaked region vanishes as $\rho \to 0$. In other words, the medium scatterer in the cloaked region is approximately isolated from the exterior world and has negligible affect on the scattering measurement, which means that we achieve significant near invisibility by SH and FSH constructions.

Finally, we study the convergence history of the discrete maximum norm of $A(x, \hat{x})$ with respect to the regularization parameter $\rho$. From figure 6, we see clearly the second-order decay rate of the discrete maximum norm of $A(x, \hat{x})$ in terms of $\rho$ for the near-cloak construction schemes SH and FSH, compared with the red reference line of second-order decay in figure 6, which confirms the sharpness of our theoretical upper bounds for schemes SH and FSH. Similar to their SS counterparts, it can be seen from figure 6 that the discrete maximum norms for schemes SH and FSH have nearly the same values as $\rho$ decreases because the FSH lossy layer tends effectively to the SH boundary condition for small $\rho$ and thus scheme FSH approaches scheme SH in the limit sense as $\rho \to 0$. The significantly improved cloaking performance for schemes SH and FSH makes it easier for engineers to design practical near-cloak devices in a variety of industrial applications.
Figure 5. The scattered wave $u_s$ (real part) and the transmitted wave $u_t$ with respect to $\rho = 10^{-1}$ and $10^{-5}$ from top to bottom, respectively, for scheme FSH.

Figure 6. Convergence history of scattering measurement data versus $\rho$ for scheme SH (left) and scheme FSH (right).
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