RETRACEMENTS OF FREE MV-ALGEBRAS AND UNITAL ℓ-GROUPS
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Abstract. A number of papers deal with the problem of counting the number of retractions of a structure $S$ onto a substructure $T$. In the particular case when $S$ is a free algebra, this number is $\geq 1$ iff $T$ is projective. In this paper we consider the case when $T$ is a projective lattice-ordered abelian group with a distinguished strong order unit, or equivalently, a projective MV-algebra. Let $A$ be a retract of the free $n$-generator MV-algebra $M([0,1]^n)$ of McNaughton functions on $[0,1]^n$. We prove that the number $r(A)$ of retractions of $M([0,1]^n)$ onto $A$ is finite if, and only if, the maximal spectral space $\mu_A$ is homeomorphic to a (Kuratowski) closed domain $M$ of $[0,1]^n$, in the sense that $M = cl(int(M))$. Further, the closed domain condition is decidable and $r(A)$ is computable, once a retraction onto $A$ is explicitly given. Thus every finitely generated projective MV-algebra $B$ comes equipped with a new invariant $\iota(B) = \sup\{r(A) | A \cong B \text{ for } A \text{ a retract of } M([0,1]^n)\}$, where $k$ is the smallest number of generators of $B$. We compute $\iota(B)$ for many projective MV-algebras $B$ considered in the literature. Various problems concerning retractions of free MV-algebras are shown to be decidable. Via the $\Gamma$ functor, our results and computations automatically transfer to finitely generated projective abelian ℓ-groups with a distinguished strong unit.

1. Foreword

Several papers deal with the problem of counting the number $r(T)$ of retractions (= idempotent endomorphisms) of a structure $S$ onto a substructure $T \subseteq S$. See, e.g., [5, 20, 26, 29], [16, p.174], [4, p.122]. In the particular case when $S$ is a free algebra, $r(T) \geq 1$ iff $T$ is projective.

In this paper we will compute $r(T)$ when $T$ is a projective MV-algebra or equivalently, a projective unital ℓ-group, which is short for “lattice-ordered abelian group with a distinguished strong order unit”. As a particular case of the equivalence $\Gamma$ established in [22, Theorem 3.9], finitely presented MV-algebras are categorically equivalent to finitely presented unital ℓ-groups. Further, both categories are dually equivalent to rational polyhedra, i.e., finite unions of simplexes with rational vertices in the same euclidean space $\mathbb{R}^n$, $n = 1, 2, \ldots$, with morphisms given by $\mathbb{Z}$-maps, i.e, piecewise-linear maps $f$ with a finite number of linear pieces, such that each linear piece of $f$ has integer coefficients, [21], [24, §3]. The synergy between these three categories has received increasing attention in the last few years, [7]–[11], [14], [18], [21], [24].

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Differently from finitely presented $\ell$-groups, finitely presented unital $\ell$-groups, as well as finitely presented MV-algebras $A$ and their dual rational polyhedra, are endowed with a wealth of computable invariants, such as: the number of rational points of a given denominator $d = 1, 2, \ldots$ in the maximal spectral space $\mu_A$, [24, Proposition 3.15 and Theorem 4.16]; the rational measure of the $t$-dimensional part of $\mu_A$, $(t = 0, \ldots, \dim(\mu_A))$. [24, §14], [25]. None of these invariants makes sense for finitely presented $\ell$-groups.

A new numerical invariant, the index $i(A)$, will be introduced in this paper, by counting the maximum number of retractions of a free $n$-generator algebra onto $A$, where $n$ is the smallest number of generators of $A$.

Not surprisingly, the isomorphism problem for finitely presented unital $\ell$-groups is still open, although Markov’s celebrated unrecognizability theorem [27] is to the effect that the isomorphism problem for finitely presented $\ell$-groups is undecidable, [19].

Another main point of distinction between $\ell$-groups and unital $\ell$-groups is the characterization of finitely generated projectives. On the one hand, from the Baker-Beynon duality [1, 2, 3] one easily obtains that finitely generated projective $\ell$-groups coincide with the finitely presented ones. On the other hand, finitely generated projective unital $\ell$-groups (resp., finitely generated projective MV-algebras) are a proper subclass of finitely presented unital $\ell$-groups (resp., finitely presented MV-algebras). Their characterization is a tour de force in algebraic topology, [7, 8].

In this paper we focus on $n$-generator projective MV-algebras, $n = 1, 2, \ldots$, using their rich algebraic, geometric, arithmetic and algorithmic structure. It is well known that any such MV-algebra is isomorphic to a retract $A$ of the free MV-algebra $\mathcal{M}([0, 1]^n)$ of McNaughton functions over the unit $n$-cube $[0, 1]^n$. Let $r(A)$ denote the number of retractions of $\mathcal{M}([0, 1]^n)$ onto $A$. In Theorem 7.4 we prove that $r(A)$ is Turing computable.

Following Kuratowski, [15, p.20], we say that a subset $D$ of a topological space $X$ is a closed domain in $X$ if $D$ coincides with the closure of its interior in $X$, in symbols, $\text{cl}(\text{int}(D)) = D$. For any finitely generated projective MV-algebra $B$, letting $k_B$ be the smallest number of its generators, we define the index $i(B)$ as the sup of all $r(A)$ as $A$ ranges over retracts of $\mathcal{M}([0, 1]^{k_B})$ isomorphic to $B$. Then in Corollary 4.3 we prove that $i(B)$ is finite iff the maximal ideal space of $B$ is homeomorphic to a closed domain in $\mathbb{R}^{k_B}$. Depending on $B$, $i(B)$ can be an arbitrarily large finite number, already in the two-dimensional case, (Theorem 5.1). Various estimates and computations of the multiplicity and of the index are carried on (respectively in §3-6 and §7), and various related problems are shown to be Turing decidable.

Via the mentioned $\Gamma$ equivalence, the results of this paper automatically transfer to finitely generated projective unital $\ell$-groups. Anyway, in this paper we will mostly work in the MV-algebraic framework, because all the algebraic machinery concerning finite presentations and projectives, (resp., all the algorithmic machinery needed to compute invariants) naturally arises from MV-algebras (resp., from the underlying Lukasiewicz calculus of MV-algebras). For all necessary background on MV-algebras we refer to the monographs [12] and [24].

### 2. Polyhedra and retracts of free MV-algebras and unital $\ell$-groups

A rational polyhedron $P \subseteq \mathbb{R}^n$ is the union of finitely many simplexes in $\mathbb{R}^n$ with rational vertices. By a $\mathbb{Z}$-map $\zeta: P \to [0, 1]^n$ we mean a piecewise linear map where each linear piece has integer coefficients, and the number of linear pieces is finite. (Throughout this paper the adjective “linear” is understood in the affine sense.) A $\mathbb{Z}$-homeomorphism $\theta$ of a rational polyhedron $P \subseteq [0, 1]^n$ onto a rational
polyhedron $Q \subseteq [0, 1]^n$ is a $Z$-map of $P$ onto $Q$ such that also the inverse $\theta^{-1}$ is a $Z$-map. A $Z$-map $\sigma : [0, 1]^n \rightarrow [0, 1]^n$ is said to be a $Z$-retraction of $[0, 1]^n$ if it satisfies the idempotence condition $\sigma \circ \sigma = \sigma$. The set $R_\sigma = \text{range}(\sigma) \subseteq [0, 1]^n$
is said to be a $Z$-retract of $[0, 1]^n$. $R_\sigma$ is a rational polyhedron, and we have the identity $R_\sigma = \{ x \in [0, 1]^n \mid x = \sigma(x) \}$.

For $n = 1, 2, \ldots$, we let $M([0, 1]^n)$ denote the MV-algebra of $[0, 1]$-valued $Z$-maps defined over $[0, 1]^n$, equipped with the pointwise operations of the standard MV-algebra $[0, 1]$. $M([0, 1]^n)$ is a free MV-algebra, which throughout this paper comes equipped with the free generating set $\{ \pi_1, \ldots, \pi_n \}$, where $\pi_i : [0, 1]^n \rightarrow [0, 1]$ is the $i$th coordinate map. Elements of $M([0, 1]^n)$ are known as McNaughton functions.

For any MV-term $q(X_1, \ldots, X_n)$ we write $\hat{q} : [0, 1]^n \rightarrow [0, 1]$ for the McNaughton function associated to $q$. In the notation of [12, §3.1], $\hat{q}$ is written $q^{M([0, 1]^n)}$. In particular, $X_i$ is the $i$th coordinate function $\pi_i : [0, 1]^n \rightarrow [0, 1]$. More generally, for any $n$-tuple $t = (t_1, \ldots, t_n)$ of MV-terms, where all $t_i$ are in the same variables $X_1, \ldots, X_n$, we let $\hat{t}$ denote the $Z$-map $(\hat{t}_1, \ldots, \hat{t}_n) : [0, 1]^n \rightarrow [0, 1]^n$.

Following [21], let $M$ denote the functor from the category of rational polyhedra with $Z$-maps to finitely presented MV-algebras, [24, §3], [21]. For any rational polyhedron $P \subseteq [0, 1]^n$, the MV-algebra $M(P)$ is defined by restricting to $P$ every element of $M([0, 1]^n)$, in symbols, $M(P) = \{ f \mid P \mid f \in M([0, 1]^n) \}$, where $\mid$ denotes restriction. Further, the action of $M$ on any $Z$-map $\sigma$ is given by

$$M_\sigma = \sigma \circ -.$$  

(1)

If in particular $\sigma : [0, 1]^n \rightarrow [0, 1]^n$ is a $Z$-retraction, $M_\sigma$ is a retraction that maps $M([0, 1]^n)$ onto the MV-subalgebra $\text{gen}(\sigma_1, \ldots, \sigma_n)$ of $M([0, 1]^n)$ generated by $\sigma_1, \ldots, \sigma_n$. Thus by (1), $M_\sigma$ is the uniquely determined homomorphism of $M([0, 1]^n)$ into $M([0, 1]^n)$ extending the map $\pi_i \mapsto \sigma_i$, $(i = 1, \ldots, n)$.

Conversely, for any retraction $\epsilon : M([0, 1]^n) \rightarrow M([0, 1]^n)$, the $n$-tuple $Z_\epsilon = (\epsilon(\pi_1), \ldots, \epsilon(\pi_n)) : [0, 1]^n \rightarrow [0, 1]^n$ is a $Z$-retraction of $[0, 1]^n$. The range $R_{Z_\epsilon}$ of $Z_\epsilon$ is a rational polyhedron and coincides with the set $\{ x \in [0, 1]^n \mid x = Z_\epsilon(x) \}$.

It is easy to see that two maps $M$ and $Z$ are inverses of each other, 

$$M_{Z_\epsilon} = \epsilon \quad \text{and} \quad Z_{M_\sigma} = \sigma.$$  

(2)

Throughout we let $\text{id}_X$ denote the identity map on a set $X$. By a retract we mean the range of a retraction.

**Theorem 2.1.** Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a $Z$-retraction of $[0, 1]^n$ onto the rational polyhedron $R_\sigma$. Let $\text{gen}(\sigma_1, \ldots, \sigma_n)$ be the retract of $M([0, 1]^n)$ associated with $\sigma$.

(a) The map $\tau \mapsto R_\tau$ yields a one-one correspondence between:

- $Z$-retractions $\tau = (\tau_1, \ldots, \tau_n)$ of $[0, 1]^n$ such that $\text{gen}(\tau_1, \ldots, \tau_n) = \text{gen}(\sigma_1, \ldots, \sigma_n)$, and
- rational polyhedra $Q \subseteq [0, 1]^n$ such that $\sigma \upharpoonright Q \cong_Z R_\sigma$.

(b) Thus there exists a one-one correspondence between:

- retractions of $M([0, 1]^n)$ onto the MV-algebra $\text{gen}(\sigma_1, \ldots, \sigma_n)$.
- rational polyhedra $Q \subseteq [0, 1]^n$ such that $\sigma \upharpoonright Q \cong_Z R_\sigma$, and

**Proof.** (a) Let $\tau : [0, 1]^n \rightarrow [0, 1]^n$ be a $Z$-retraction satisfying the condition $\text{gen}(\tau_1, \ldots, \tau_n) = \text{gen}(\sigma_1, \ldots, \sigma_n)$.

Then there are MV-terms $t_1, \ldots, t_n$ and $s_1, \ldots, s_n$ such that $\tau_i = t_i(\sigma_1, \ldots, \sigma_n)$ and $\sigma_i = s_i(\tau_1, \ldots, \tau_n)$. Hence $\hat{t} = (\hat{t}_1, \ldots, \hat{t}_n)$ and $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_n) : [0, 1]^n \rightarrow [0, 1]^n$ are $Z$-maps satisfying

$$\sigma = \hat{s} \circ \tau \quad \text{and} \quad \tau = \hat{t} \circ \sigma.$$  

(3)
Claim. $\sigma |_{R_{n}}$ is a $Z$-homeomorphism onto $R_{n}$ satisfying the identity $$(\sigma |_{R_{n}})^{-1} = \tau |_{R_{n}}$$ (4)

As a matter of fact, let us pick an arbitrary $x \in R_{n}$. The identities $(\sigma \circ \tau)(x) = (\hat{\delta} \circ \tau)(x) = (x \circ \tau)(x) = x$ show that $\sigma |_{R_{n}}$ is onto $R_{n}$. Similarly, for all $y \in R_{n}$ we have $(\tau \circ \sigma)(y) = y$. It follows that $\sigma |_{R_{n}}$ is one-one. The identity (4) is now immediate, and the claim is proved.

To complete the proof of (a), let us assume that, conversely, $Q \subseteq [0, 1]^{n}$ is a rational polyhedron such that $\sigma |_{Q} : Q \cong_{Z} R_{n}$. Let us write $\zeta = (\zeta_{1}, \ldots, \zeta_{n})$ as an abbreviation of the $Z$-homeomorphism $\sigma |_{Q}^{-1}$ of $R_{n}$ onto $Q$.

$$\zeta = (\sigma |_{Q})^{-1} : R_{n} \cong_{Z} Q.$$ (5)

Observe that $\zeta$ is piecewise linear with integer coefficients, and is defined over the rational polyhedron $R_{n}$. For short, $\zeta$ is a $Z$-map on $R_{n} \subseteq [0, 1]^{n}$. So by [[24, Proposition 3.2]], we have a $Z$-map $\zeta : [0, 1]^{n} \rightarrow [0, 1]^{n}$ extending $\zeta$. By McNaughton theorem, [[12, Theorem 9.1.5]], for each $i+1, \ldots, n$, $\zeta_{i}$ is the McNaughton function of some $n$-variable $M$-term. The composite map $\rho = \zeta \circ \sigma$ is a $Z$-retraction of $[0, 1]^{n}$ onto $Q$, because $\zeta \circ \sigma \circ \sigma \circ \sigma = \zeta \circ \id_{R_{n}} \circ \sigma = \zeta \circ \sigma$. Since $R_{n}$ is a $Z$-retract of $[0, 1]^{n}$, then so is the rational polyhedron $Q = \zeta(R_{n})$. From $\rho = \zeta \circ \sigma$, we get $\text{gen}(\rho) \subseteq \text{gen}(\sigma)$. From $\sigma = \zeta^{-1} \circ \rho$, we get $\text{gen}(\sigma) \subseteq \text{gen}(\rho)$. Further, by (3) and (4) we can write $\zeta \circ \sigma = \tau |_{R_{n}} \circ \sigma = \tau \circ \sigma = \iota \circ \sigma \circ \iota = \tau \circ \sigma = \tau$, and $R_{\zeta \circ \sigma} = \zeta \circ \sigma([0, 1]^{n}) = \zeta(R_{n}) = Q$. Thus the maps $\tau \mapsto R_{\tau}$ and $Q \mapsto \zeta \circ \sigma$ are inverse of each other, and (a) is proved.

(b) This immediately follows from (a) and (2).

For the proof of Theorem 2.3 below, we record the following elementary fact:

**Lemma 2.2.** Let $\eta : [0, 1]^{n} \rightarrow [0, 1]^{n}$ be a $Z$-map and $P, Q \subseteq [0, 1]^{n}$ be rational polyhedra satisfying the following conditions:

(i) both $\text{int}(P)$ and $\text{int}(Q)$ are connected;

(ii) $P = \text{cl}(\text{int}(P))$ and $Q = \text{cl}(\text{int}(Q))$;

(iii) $\eta(P) = \eta(Q)$;

(iv) $\eta |_{P} : P \cong_{Z} \eta(P)$ and $\eta |_{Q} : Q \cong_{Z} \eta(Q)$.

Then either $P = Q$ or $\text{int}(P) \cap \text{int}(Q) = \emptyset$.

**Proof.** By way of contradiction, let us assume $P \neq Q$ and there is $x \in \text{int}(P) \cap \text{int}(Q)$. Without loss of generality assume that $y \in P \setminus Q$ for some $y$. By (ii), $P = \text{cl}(\text{int}(P))$, whence we may insist that $y \in \text{int}(P)$. Since by (i) the interior of $P$ is an open connected subset of $\mathbb{R}^{n}$, it is also path connected. (See Figure 1.) Let $\gamma : [0, 1] \rightarrow P$ be a path such that $\gamma([0, 1]) \subseteq \text{int}(P)$, $\gamma(0) = x$, and $\gamma(1) = y$. Since $\gamma$ is continuous and $Q$ is closed, the set $J = \{ \delta \in [0, 1] | \gamma(\delta) \in Q \} \subseteq [0, 1]$ is closed. Let $\lambda$ be the largest element of $J$. From $\gamma(1) = y \not\in Q$ we get $\lambda < 1$. Let $z = \gamma(\lambda)$. Then $z \in \text{int}(P)$ and $z \in Q \setminus \text{int}(Q)$. By (iv), $\eta$ maps $z$ to a point $\eta(z)$ that simultaneously belongs to the interior of $\eta(P)$ and to the boundary of $\eta(Q)$, which contradicts (iii).

Up to isomorphism, any $n$-generator projective $M$-algebra $B$ has the form $\mathcal{M}(P) = \{ f | P \mid f \in \mathcal{M}([0, 1]^{n}) \}$ for some $Z$-retract $P$ of $[0, 1]^{n}$. Specifically, by [[8, Theorem 5.1]] or [[24, Proposition 17.5]], there is a $Z$-retraction $\sigma$ of $[0, 1]^{n}$ such that $B \cong \mathcal{M}(R_{n}) \cong \text{range}(\mathcal{M} \sigma) = \text{range}(\mathcal{- \circ \sigma})$. By [[24, Corollary 4.18]], the $Z$-retract $Q = R_{n}$ is homeomorphic to the maximal spectral space $\mu(B)$. If another $Z$-retract $Q' \subseteq [0, 1]^{n}$ is chosen such that $B \cong \mathcal{M}(Q')$, then $Q$ is $Z$-homeomorphic to $Q'$ ([24, Corollary 3.10]). Thus in particular $Q$ is a closed domain in $[0, 1]^{n}$ iff so
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Figure 1. The path $\gamma: [0, 1] \to \text{int}(P)$ in the proof of Lemma 2.2 joins $x \in \text{int}(P) \cap \text{int}(Q)$ and $y \in P \setminus Q$, and has a nonempty intersection with the boundary of $Q$.

is $Q'$. (See [15, p.20] for this terminology, going back to Kuratowski.) This state of affairs can be unambiguously described by saying that the maximal spectral space $\mu_B$ is a closed domain in $[0,1]^n$.

Theorem 2.3. Suppose $A$ is a retract of $\mathcal{M}([0,1]^n)$ and $\mu_A$ is a closed domain in $[0,1]^n$. Then the number of retractions of $\mathcal{M}([0,1]^n)$ onto $A$ is finite.

Proof. Let us choose a retraction $\epsilon$ of $\mathcal{M}([0,1]^n)$ onto $A$, along with its associated $Z$-retraction $Z_\epsilon = \sigma$ as given by (2). Since $R_\sigma$ is a polyhedron (it is compact and) the connected components of $\text{int}(R_\sigma) \subseteq [0,1]^n$ are finitely many. Let $O_{\sigma,1}, \ldots, O_{\sigma,k} \subseteq \text{int}(R_\sigma) \subseteq [0,1]^n$ be the list of these connected components.

With reference to the notation (1) for the functor $\mathcal{M}$, let $\zeta$ be a $Z$-retraction of $[0,1]^n$ such that $M_\zeta$ is a retraction of $\mathcal{M}([0,1]^n)$ onto $A$. By Theorem 2.1, $\sigma \circ R_\zeta$ is $Z$-homeomorphism onto $R_\sigma$. Therefore, $\text{int}(R_\zeta)$ has $k$ connected components $O_{\zeta,1}, \ldots, O_{\zeta,k}$, and we can write $\sigma(O_{\zeta,j}) = O_{\sigma,j}$. We have $Z$-homeomorphisms

$$\sigma \circ \text{cl}(O_{\zeta,j}) : \text{cl}(O_{\zeta,j}) \to \text{cl}(O_{\sigma,j}), \quad (j = 1, \ldots, k)$$

(5)

Let the family $\mathcal{O}$ of open sets in $[0,1]^n$ be defined by

$$\mathcal{O} = \{O_{\zeta,j} \mid j = 1, \ldots, k\},$$

and the map $- \circ \zeta$ is a retraction of $\mathcal{M}([0,1]^n)$ onto $A$.

Let $O_{\zeta,j}, O_{\zeta,j'} \in \mathcal{O}$. If $O_{\zeta,j} \neq O_{\zeta,j'}$, then either $j \neq j'$ or $\zeta \neq \zeta'$. If $j \neq j'$, then $\sigma(O_{\zeta,j}) \cap \sigma(O_{\zeta,j'}) = \emptyset$, whence $O_{\zeta,j} \cap O_{\zeta,j'} = \emptyset$. If $j = j'$, then $\zeta \neq \zeta'$. From Lemma 2.2, (with $n = \sigma$, $P = \text{cl}(O_{\zeta,j})$, and $Q = \text{cl}(O_{\zeta,j'})$) it follows that $O_{\zeta,j} \cap O_{\zeta,j'} = \emptyset$. Therefore, the elements of $\mathcal{O}$ are pairwise disjoint.

Since $Z$-homeomorphisms preserve the Lebesgue measure of $n$-dimensional polyhedra in $[0,1]^n$ ([24, Lemma 14.3], [23, Theorem 2.1(iii)]), by (5) each $O_{\zeta,j} \in \mathcal{O}$ has the same $(n$-dimensional) Lebesgue measure as $O_{\sigma,j}$, because $O_{\sigma,j}$ has the same Lebesgue measure as $\text{cl}(O_{\zeta,j})$. Let $O_{\sigma,j}$ be chosen among $O_{\sigma,1}, \ldots, O_{\sigma,k}$ as having the smallest $n$-dimensional Lebesgue measure. Say that $\lambda$ is its measure. Since the
elements of \( \mathcal{O} \) are pairwise disjoint, we have

\[
\text{number of elements in } \mathcal{O} \leq \lceil 1/\lambda \rceil = \max\{l \in \mathbb{Z} \mid l \leq 1/\lambda \}.
\]

By Theorem 2.1, the number \( r(A) \) of retractions of \([0,1]^n\) onto \( A \) satisfies the inequality

\[
r(A) \leq \left( \frac{\lceil 1/\lambda \rceil}{k} \right).
\]

This completes the proof. \( \square \)

Throughout we let \( \mathcal{M}_\mathbb{Z}([0,1]^n) \) denote the unital \( \ell \)-group of piecewise linear functions \( f: [0,1]^n \to \mathbb{R} \), where each linear piece of \( f \) has integer coefficients. In view of the categorical equivalence \( \Gamma \) between unital \( \ell \)-groups and MV-algebras, [22, Theorem 3.9], such notions as “free unital \( \ell \)-group” and “finitely presented \( \ell \)-group” make perfect sense, not only as \( \Gamma \)-correspondents of free and finitely presented MV-algebras, but also from the categorical viewpoint, (respectively see [22, Corollary 4.16] and [11, Remark 5.10].)

The maximal spectral space \( \mu_G \) of every unital \( \ell \)-group \((G,u)\) is canonically homeomorphic to the maximal spectral space of its associated MV-algebra \( \Gamma(G,u) \), [12, §7.2]. Precisely as in the case of MV-algebras, it makes perfect mathematical sense to say that \( \mu_G \) is a closed domain in \([0,1]^n\).

By [22, Theorem 4.15], \( \Gamma(\mathcal{M}_\mathbb{Z}([0,1]^n)) = \mathcal{M}([0,1]^n) \). Thus by [12, §7.2], up to unital \( \ell \)-isomorphism every finitely generated projective unital \( \ell \)-group has the form \( \mathcal{M}_\mathbb{Z}(P) = \{ f \mid P \mid f \in \mathcal{M}_\mathbb{Z}([0,1]^n) \} \) for some \( n = 1, 2, \ldots \) and \( \mathbb{Z} \)-retract \( P \) of \([0,1]^n\).

**Corollary 3.2.** Given a retract \((G,u)\) of \( \mathcal{M}_\mathbb{Z}([0,1]^n) \), suppose \( \mu_G \) is a closed domain in \([0,1]^n\). Then the number of retractions of \( \mathcal{M}_\mathbb{Z}([0,1]^n) \) onto \((G,u)\) is finite.

**Proof.** Immediate from Theorem 2.3, using the preservation properties of the \( \Gamma \) equivalence, [12, §7.2]. \( \square \)

## 3. The index of a projective MV-algebra and of a unital \( \ell \)-group

**Definition 3.1.** The *multiplicity* \( r(A) \) of a retract \( A \) of \( \mathcal{M}([0,1]^n) \) is the number of distinct retractions of \( \mathcal{M}([0,1]^n) \) onto \( A \) if this number is finite, and \( \infty \) otherwise. The *index* \( \iota(B) \in \{1, 2, \ldots \} \cup \{ \infty \} \) of a finitely generated projective MV-algebra \( B \) is the supremum of the multiplicities of all retractions \( A \cong B \) of \( \mathcal{M}([0,1]^n) \), with \( k \) the smallest number of generators of \( B \). One similarly defines the index of finitely generated projective unital \( \ell \)-groups.

**Proposition 3.2.** (a) Let \( P \subseteq [0,1]^n \) be a \( \mathbb{Z} \)-retract and a closed domain in \([0,1]^n\). Let \( m \) be the maximum number of \( \mathbb{Z} \)-homeomorphic pairwise disjoint copies of \( P \) in \([0,1]^n\). Then \( \iota(\mathcal{M}(P)) \geq m \).

(b) An upper bound for the index \( \iota(\mathcal{M}(P)) \) is given by (7).

**Proof.** (a) Our assumption \( P = \text{cl}(\text{int}(P)) \) ensures that \( n \) is the smallest number of generators of \( \mathcal{M}(P) \). As a matter of fact, if \( \mathcal{M}(P) \) had \( n - 1 \) generators (absurdum hypothesis) then by [12, Theorem 3.6.7] \( \mathcal{M}(P) \) would be isomorphic to an MV-algebra of the form \( \mathcal{M}(X) \) for some closed subset \( X \) of \([0,1]^{n-1}\). By [24, Corollary 4.18] the maximal spectral space \( \mu_{\mathcal{M}(X)} \) is homeomorphic to \( X \), whence its dimension is \( \leq n - 1 \). On the other hand, from the isomorphism \( \mathcal{M}(P) \cong \mathcal{M}(X) \) we get the homeomorphism \( P \cong X \), so \( \text{dim}(P) \leq n - 1 \), thus contradicting the assumption that \( P \) is a closed domain in \([0,1]^n\).

Let \( Q_1, Q_2, \ldots, Q_m \) be a (maximal) set of pairwise disjoint \( \mathbb{Z} \)-homeomorphic copies of \( P \) in \([0,1]^n\). Since by [24, Corollary 3.10] \( \mathcal{M}(Q_1) \cong \mathcal{M}(P) \) and the index is an isomorphism invariant, we may assume \( Q_1 = P \) without loss of generality. If
Corollary 3.3. Similarly, for every $m = 1$ we have nothing to prove. So assume $m \geq 2$. For each $i = 2, \ldots, m$ there is a $\mathbb{Z}$-homeomorphism $\eta_i$ of $Q_i$ onto $Q_1$. For completeness let us set $\eta_1 = \text{id}_P$. Since the $Q_i$ are pairwise disjoint ($j = 1, \ldots, m$) the set $\bigcup_{j=1}^m \eta_j$ is a $\mathbb{Z}$-map of $\bigcup_{j=1}^m Q_j$ onto $P$. By [24, Proposition 3.2(ii)] there is a $\mathbb{Z}$-map $\eta: [0, 1]^n \to [0, 1]^n$ simultaneously extending each $\eta_j$. Pick a $\mathbb{Z}$-retraction $\sigma$ of $[0, 1]^n$ onto $P$. Then the composite map $\sigma \circ \eta$ is a $\mathbb{Z}$-retraction of $[0, 1]^n$ onto $P$, and for each $j = 1, \ldots, m$ the restriction $\sigma \circ \eta| Q_j = \sigma \circ \eta_j$ is a $\mathbb{Z}$-homeomorphism of $Q_j$ onto $P$. By Theorem 2.1, the multiplicity of the retract $A = \text{gen}(\sigma_1, \ldots, \sigma_m)$ is $\geq m$. By [24, Lemma 3.6] $A \cong \mathcal{M}(P)$, whence the desired conclusion follows by definition of the index, recalling that $n$ is the smallest number of generators of $\mathcal{M}(P)$.

(b) By [24, Lemma 14.3] or [23, Theorem 2.1(iii)], $\mathbb{Z}$-homeomorphisms preserve the Lebesgue measure of $n$-dimensional polyhedra in $[0, 1]^n$. By [24, Corollary 3.10], $\mathcal{M}(P) \cong \mathcal{M}(Q)$ implies $P \cong_{\mathbb{Z}} Q$. $\square$

Corollary 3.3. The index of every finitely generated free MV-algebra is 1. Similarly, for every $n = 1, 2, \ldots$, the index of the free unital $\ell$-group $\mathcal{M}_\mathbb{Z}([0, 1]^n)$ is 1. In particular, the two-element MV-algebra $\{0, 1\}$ is the free MV-algebra over 0 generators. Its index is equal to 1.

The following example explains why in the definition of the index of $B$ we restrict to those isomorphic copies of $B$ which are retracts of $\mathcal{M}([0, 1]^n)$ with $k$ the smallest number of generators of $B$. As in [24, p. 11], or [25], for any rational point $r \in \mathbb{R}^n$, the denominator $\text{den}(r)$ is defined by

$$\text{den}(r) = \text{least common denominator of the coordinates of } r. \quad (8)$$

Example 3.4. For $n \geq 1$ let $\text{cyl}(n, \mathcal{M}([0, 1])) \subseteq \mathcal{M}([0, 1]^n)$ be the isomorphic copy of $\mathcal{M}([0, 1])$ obtained by cylindrifying each $f \in \mathcal{M}([0, 1])$ into the function $c \in \mathcal{M}([0, 1]^n)$ given by $c(x, x_2, \ldots, x_n) = f(x)$ for all $(x, x_2, \ldots, x_n) \in [0, 1]^n$. By Corollary 3.3 the index of the free MV-algebra $\mathcal{M}([0, 1])$ is 1. We claim that the multiplicity of its isomorphic copy $\text{cyl}(2, \mathcal{M}([0, 1]))$ is $\infty$. Let the $\mathbb{Z}$-retraction $\xi = (\xi_1, \xi_2): [0, 1]^2 \to [0, 1]^2$ be given by $\xi_1(x, y) = x$, $\xi_2(x, y) = 0$. $\xi$ projects any point of the unit square onto the x-axis. A direct inspection shows that $\xi$ preserves the denominator of a rational point $(x, y) \in [0, 1]^2$ iff the denominator of $y$ is a divisor of the denominator of $x$. This is the case, in particular, when the point $(x, y)$ belongs to the graph $W$ of a McNaughton function $f$ in $\mathcal{M}([0, 1])$, because every linear piece of $f$ has integer coefficients. By [24, Proposition 3.15], $\xi$ acts $\mathbb{Z}$-homeomorphically over the broken line $W \subseteq [0, 1]^2$. There are countably many such broken lines $W$, one for each $f \in \mathcal{M}([0, 1])$. By Theorem 2.1(c) there are countably many retractions of $\mathcal{M}([0, 1]^2)$ onto $\text{cyl}(2, \mathcal{M}([0, 1]))$. Thus the multiplicity of $\text{cyl}(2, \mathcal{M}([0, 1]))$ is $\infty$, and our claim is proved. One similarly proves that the multiplicity of $\text{cyl}(n, \mathcal{M}([0, 1]))$ is $\infty$ for each $n \geq 2$. As already noted in Corollary 3.3, $\iota(\mathcal{M}([0, 1])) = 1$ whence $\iota(\text{cyl}(n, \mathcal{M}([0, 1]))) = 1$ for each $n$.

The proof of the following result is immediate:

Proposition 3.5. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a $\mathbb{Z}$-retraction of $[0, 1]^n$, and $\alpha$ a $\mathbb{Z}$-homeomorphism of $[0, 1]^n$ onto $[0, 1]^n$. Then the range of the composite map $\sigma \circ \alpha$ is a $\mathbb{Z}$-retract, and so is its $\mathbb{Z}$-homeomorphic copy $R_{\alpha} \subseteq [0, 1]^n$. If $R_{\alpha}$ is $n$-dimensional, $n$ is the smallest number of generators of the retract $\text{gen}(\sigma_1, \ldots, \sigma_n)$ of $\mathcal{M}([0, 1]^n)$. Letting $\tau = \alpha \circ \sigma \circ \alpha^{-1}$, it follows that $\tau$ is a $\mathbb{Z}$-retraction of $[0, 1]^n$, and the two isomorphic retracts $\text{gen}(\sigma_1, \ldots, \sigma_n)$ and $\text{gen}(\tau_1, \ldots, \tau_n)$ have equal multiplicities and equal indexes.

Figure 2 shows the special case of Proposition 3.5 for $n = 2$, where $\alpha$ is Panti’s $\mathbb{Z}$-homeomorphism, [13] and $\sigma = \pi_1 \land \lnot \pi_1: [0, 1]^2 \to [0, 1]^2$. 
The effective computability of the index of a one-generator projective MV-algebra is taken care of by the following easy result:

**Proposition 3.6.** Let \( B \neq \{0, 1\} \) be a one-generator projective MV-algebra.

(a) For a unique rational \( 0 < r \in [0, 1] \) we have the isomorphism \( B \cong \mathcal{M}(0, r) \). Then \( \iota(B) \in \{1, 2\} \). Further, \( \iota(B) = 2 \) iff \( r \leq 1/2 \).

(b) In equivalent algebraic-topological terms, \( \iota(B) = 2 \), unless the maximal spectral space \( \mu_B \) contains an element \( m \) such that \( B/m \cong \{0, 1/2, 1\} \) and \( \mu_B \setminus \{m\} \) is disconnected—in which case \( \iota(B) = 1 \).

**Proof.** (a) The first statement is a particular case of [24, Proposition 17.5], upon noting that every \( Z \)-retract of \( [0, 1] \) is \( Z \)-homeomorphic to \( \mathcal{M}(0, r) \) for some \( r \in \mathbb{Q} \cap [0, 1] \). Further, \( r > 0 \), for otherwise \( B \) would be isomorphic to the two-element MV-algebra. In case \( r > 1/2 \) the measure-theoretic argument in the proof of Theorem 2.3 shows that \( \iota(A) = 1 \). On the other hand, if \( r \leq 1/2 \), the only other rational polyhedron in \( [0, 1] \) which is \( Z \)-homeomorphic to \( [0, r] \) is \( [1 - r, 1] \). By Theorem 2.1(b) and Proposition 3.2, \( \iota(\mathcal{M}(0, r)) = 2 \).

(b) This is just a reformulation of part (a) in the light of the spectral theory of MV-algebras, [24, §4.5], and the duality between finitely presented MV-algebras and rational polyhedra, [24, §3], [21]. \( \square \)

While the index is invariant under isomorphisms, in the following example we present two isomorphic retracts of \( \mathcal{M}(0, 1) \) having different multiplicities.
Example 3.7. Let the $\mathbb{Z}$-retraction of $[0, 1]$ onto $[0, 1/2]$ be given by $\sigma(x) = x \land \neg x$. The retract $A = \text{gen}(\sigma) = \text{gen}(\pi_1 \land \neg \pi_1) \subseteq \mathcal{M}([0, 1])$ is the MV-algebra of all one-variable McNaughton functions $f$ such that $f(1 - x) = f(x)$. Since the restriction of $\sigma$ to $[1/2, 1]$ is a $\mathbb{Z}$-homeomorphism onto $[0, 1/2]$ and $(\sigma | [1/2, 1])^{-1} = \pi_1 \lor \neg \pi_1$, by Theorem 2.1 the map $\mathcal{M}_{\pi_1 \lor \neg \pi_1}$ is a second retraction $\mathcal{M}([0, 1])$ onto $A$. Moreover, $\mathcal{M}_\pi$ and $\mathcal{M}_\sigma$ are the only rejections of $\mathcal{M}([0, 1])$ onto $A$. Thus $r(A) = 2$. Let $\tau: [0, 1] \to [0, 1]$ be given by $\tau(x) = (x \land \neg x) \land ((\neg x \lor \neg x) \lor (\neg x \lor \neg x))$. Then $\tau$ is a $\mathbb{Z}$-retraction of $[0, 1]$ onto $[0, 1/2]$. Let $B = \text{gen}(\tau)$. We have $A \cong B \cong \mathcal{M}([0, 1/2])$. For no other segment $J$ other than $[0, 1/2]$ it is the case that $\tau|J$ is a $\mathbb{Z}$-homeomorphism of $J$ onto $[0, 1/2]$. By Theorem 2.1, $r(B) = 1$.

4. When the maximal spectral space is not a closed domain in $[0, 1]^n$

For any rational $m$-simplex $T = \text{conv}(v_0, \ldots, v_m) \subseteq \mathbb{R}^n$, let us display each vertex $v_j$ of $T$ as $(a_{j1}/b_{j1}, \ldots, a_{jn}/b_{jn})$, for uniquely determined integers $a_{jt}, b_{jt}$ $(t = 1, \ldots, n)$ such that $b_{jt} > 0$. With the notation of (8) we let the homogeneous correspondent $\tilde{v}_j$ of $v_j$ be defined by

$$\tilde{v}_j = \text{den}(v_j)(a_{j1}/b_{j1}, \ldots, a_{jn}/b_{jn}, 1) \in \mathbb{Z}^{n+1}.$$

Conversely, $v_j$ is said to be the affine correspondent of $\tilde{v}_j$.

An $m$-simplex $U = \text{conv}(w_0, \ldots, w_m) \subseteq \mathbb{R}^n$ is said to be regular if it is rational and the set of integer vectors $\{\tilde{w}_0, \ldots, \tilde{w}_m\}$ can be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$.

For every simplicial complex $\Sigma$ the point-set union of the simplexes of $\Sigma$ is called the support of $\Sigma$, and is denoted $|\Sigma|$. We also say that $\Sigma$ is a triangulation of $|\Sigma|$. A simplicial complex is said to be a regular triangulation (of its support) if all its simplexes are regular. Regular triangulations (called “unimodular” in [23]) are the affine counterparts of the regular, or nonsingular, fans of toric algebraic geometry, [17]. Suppose $\Sigma$ and $\Theta$ are two simplicial complexes with the same support, and every simplex of $\Theta$ is contained in some simplex of $\Sigma$. Then $\Theta$ is said to be a subdivision of $\Sigma$.

Let $\Sigma$ be a simplicial complex and $b \in |\Sigma| \subseteq \mathbb{R}^n$. Following [17, III, 2.1], the simplicial complex $\Sigma_{(b)}$ is obtained by the following procedure: replace every simplex $S \subseteq \Sigma$ containing $b$ by the set of all simplexes of the form $\text{conv}(b, F)$, where $F$ is any face of $S$ that does not contain $b$. The subdivision $\Sigma_{(b)}$ of $\Sigma$ is known as the blow-up $\Sigma_{(b)}$ of $\Sigma$ at $b$.

For any $m \geq 1$ and regular $m$-simplex $U = \text{conv}(w_0, \ldots, w_m) \subseteq \mathbb{R}^n$ the Farey mediant of $U$ is the affine correspondent of the vector $\tilde{w}_0 + \cdots + \tilde{w}_m \in \mathbb{Z}^{n+1}$. In the particular case when $\Sigma$ is a regular triangulation and $b$ is the Farey mediant of a simplex of $\Sigma$, the blow-up $\Sigma_{(b)}$ is regular.

The short proof of the following proposition is a template for the main construction in the proof of Theorem 4.2, yielding a converse of Theorem 2.3.

**Proposition 4.1.** There is a retract of $\mathcal{M}([0, 1]^2)$ having an infinite index.

**Proof.** Let $L$ be the union of the two edges $\text{conv}((0, 1), (0, 0))$ and $\text{conv}((1, 0), (0, 0))$. Let $\rho = (\rho_1, \rho_2): [0, 1]^2 \to L$ be the $\mathbb{Z}$-retraction of $[0, 1]^2$ onto $L$ given by

$$\rho(x, y) = (x \lor y, y \lor x) = \begin{cases} (0, y - x) & \text{if } y \geq x; \\ (x - y, 0) & \text{if } x \geq y. \end{cases}$$

A direct inspection shows that $\rho$ sends each point $(x, y) \in [0, 1]^2$ to the point of $L$ whose coordinates are $x - \min(x, y)$ and $y - \min(x, y)$. Geometrically, $\rho$ moves down
by 45 degrees in the south-west direction each point \((x, y) \in [0, 1]^2\), by subtracting the same quantity \(\min(x, y)\) to each coordinate.

Claim. The MV-algebra \(A = \text{gen}(\rho_1, \rho_2) \subseteq \mathcal{M}([0, 1]^2)\) has infinite multiplicity.

As a matter of fact, (see Figure 3) for each integer \(p \geq 3\) let the broken line \(W_p \subseteq [0, 1]^2\) be the union of the segment \(\text{conv}((0, 1), (0, 0))\) with the three segments

\[
\text{conv} \left( (0, 0), \left( \frac{2}{p}, \frac{1}{p} \right) \right), \text{ conv} \left( \left( \frac{2}{p}, \frac{1}{p} \right), \left( \frac{1}{p-1}, 0 \right) \right), \text{ conv} \left( \left( \frac{1}{p-1}, 0 \right), (1, 0) \right).
\]

It is not hard to check that \(\rho\) is a \(\mathbb{Z}\)-homeomorphism of \(W_p\) onto \(L\). To this purpose one notes that the triangle \(\text{conv}((0, 0), (\frac{2}{p}, \frac{1}{p}), (\frac{1}{p-1}, 0))\) is the union of the regular triangles \(\text{conv}((0, 0), (\frac{2}{p}, \frac{1}{p}), (\frac{1}{p}, 0))\) \(\text{conv}((\frac{1}{p-1}, 0), (\frac{1}{p}, 0))\). Further

- \(\rho\) fixes the segment \(\text{conv}((0, 1), (0, 0))\);
- \(\rho\) maps \(\text{conv}((0, 0), (\frac{2}{p}, \frac{1}{p}))\) one-one onto \(\text{conv}((0, 0), (\frac{1}{p}, 0))\);
- \(\rho\) maps \(\text{conv}((\frac{2}{p}, \frac{1}{p}), (\frac{1}{p-1}, 0))\) one one onto \(\text{conv}((\frac{1}{p}, 0), (\frac{1}{p-1}, 0))\);
- \(\rho\) fixes \(\text{conv}((\frac{1}{p-1}, 0), (1, 0))\).

By [24, Lemma 3.7, Proposition 3.15], \(\rho\) is an invertible \(\mathbb{Z}\)-map of \(W_p\) onto \(L\). To see that the multiplicity of \(A\) is infinite, recall Theorem 2.1, and let \(p\) range over all integers \(\geq 3\). Having thus settled our claim, the proof of the proposition is complete.

Following [8, Definition 4.1], a triangulation \(\Delta\) of a rational polyhedron \(P\) is said to be strongly regular if it is regular and for each maximal simplex \(T\) of \(\Delta\) the greatest common divisor of the denominators of the vertices of \(T\) is equal to 1. \(P\) is called strongly regular if it has a strongly regular triangulation. Then every regular triangulation of \(P\) is strongly regular ([8, Remark 5.1]). Every \(\mathbb{Z}\)-retract of \([0, 1]^n\) is strongly regular, [8, Theorem 5.2(iii)].
Theorem 4.2. Fix a $\mathbb{Z}$-retraction $\rho = (\rho_1, \ldots, \rho_n)$ of $[0, 1]^n$. Let $P = \mathbb{R}_\rho$ be the range of $\rho$. If some (equivalently, every) triangulation of $P$ contains a maximal $m$-simplex with $m < n$ then the number of retractions of $\mathcal{M}([0, 1]^n)$ onto $\text{gen}(\rho_1, \ldots, \rho_n)$ is infinite.

Proof. Since $P$ is a $\mathbb{Z}$-retract of $[0, 1]^n$ then $\mathcal{M}(P)$ projective, [8, Theorem 5.1], [24, Proposition 17.5(ii)]. By Theorem 2.1 there is a one-one correspondence between the set of retractions of $\mathcal{M}([0, 1]^n)$ onto the MV-algebra $\text{gen}(\rho_1, \ldots, \rho_n) \subset \mathcal{M}([0, 1]^n)$ and the set of rational polyhedra $R \subset [0, 1]^n$ such that the restriction $\rho | R$ is a $\mathbb{Z}$-homeomorphism domain for $\rho$. So it suffices to show that the number of such domains $R$ is infinite.

Let $\Delta$ be a regular triangulation of $P$. The existence of $\Delta$ follows from [24, Corollary 2.10]. By assumption, $\Delta$ has a maximal $m$-simplex $T$ with $m < n$. It follows that $m \geq 1$, for otherwise by [8, Theorem 5.2(i)-(ii)] the $\mathbb{Z}$-retract $P$ would coincide with a vertex of $[0, 1]^n$, so $\mathcal{M}(P)$ is the free MV-algebra over $0$ generators, and $n = m = 0$, which is impossible.

Since $P$ is a $\mathbb{Z}$-retract of $[0, 1]^n$, $P$ is strongly regular, [8, Theorem 5.2(iii)]. Thus, for some prime number $p$ there exists a rational point $c$ of denominator $p$ with

$$c \in \text{relint} T.$$  

(9)

Actually, such $c$ exists for all sufficiently large primes $p$, because $T$ is a strongly regular $m$-simplex with $m > 0$.

As another consequence of the strong regularity of $P$, the affine hull $\text{aff}(T) \subset \mathbb{R}^n$ of $T$ contains some integer point of $\mathbb{Z} \subset \mathbb{R}^n$, [6, Theorem 4.17]. Then the construction of [10, Lemma 5] yields integer points $j_0, \ldots, j_m \in \mathbb{Z}^n$ such that $\text{aff}(T) = \text{aff}(\text{conv}(j_0, \ldots, j_m))$ and the $m$-simplex $I = \text{conv}(j_0, \ldots, j_m) \subset \mathbb{R}^n$ is regular and contains $c$ in its relative interior.

Let $G_n = \text{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ denote the $n$-dimensional affine group over the integers. By [10, Lemma 1] some function $\gamma \in G_n$ maps $\text{aff}(T)$ one-one onto the $m$-dimensional space

$$F_m = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{m+1} = \cdots = x_n = 0 \}.$$  

Thus the $m$-simplex $\gamma(I)$ lies in $F_m$, and we can write without loss of generality

$$\gamma(I) = \text{conv}(0, (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 1, 0, \ldots, 0)) = (a_1/p, \ldots, a_m/p, 0, \ldots, 0) + (0, \ldots, 0).$$

Since $\gamma$ is a (linear) $\mathbb{Z}$-homeomorphism, it preserves denominators of rational points and maps regular simplexes one-one onto regular simplexes, [24, Proposition 3.15]. Let us display the point $c' = \gamma(c)$ as follows:

$$c' = (c'_1, \ldots, c'_m, 0, \ldots, 0) = (a_1/p, \ldots, a_m/p, 0, \ldots, 0) + (0, \ldots, 0)$$

for suitable relatively prime integers $0 \leq a_1, \ldots, a_m \leq p$. Note that $\text{den}(c') = \text{den}(c) = p$. By (9),

$$c' \in \text{relint}(\gamma(T)).$$  

(10)

We next define the point $l \in \mathbb{R}^n$ by

$$l = (c'_1, \ldots, c'_m, 1/p, 0, \ldots, 0) = (a_1/p, \ldots, a_m/p, 1/p, 0, \ldots, 0).$$

(11)

Permuting, if necessary, the coordinates in $\mathbb{R}^n$, for all sufficiently large primes $p$ we can safely assume

$$\gamma^{-1}(l) \in [0, 1]^n.$$  

(12)
Since \( P \) is a polyhedron and \( T \in \Delta \), then by (9) for all sufficiently small \( \epsilon > 0 \) the closed ball \( B_{\epsilon,c} \) of radius \( \epsilon \) centered at \( c \) satisfies the condition
\[
B_{\epsilon,c} \cap P \subseteq T.
\] (13)

The affine transformation \( \gamma \) sends \( B_{\epsilon,c} \) one-one onto an \( n \)-dimensional ellipsoid \( \gamma(B_{\epsilon,c}) \) containing the point \( \gamma(c) \) in its relative interior. Further, by (13) we can write
\[
\gamma(B_{\epsilon,c}) \cap \gamma(P) \subseteq \gamma(T).
\] (14)

The map \( \rho' = \gamma \circ \rho \circ \gamma^{-1} \) is a \( \mathbb{Z} \)-retraction of the \( n \)-parallelepiped \( \gamma([0,1]^n) \) onto the rational polyhedron \( \gamma(P) \). By (14), all points sufficiently close to \( c' \) are mapped by \( \rho' \) into points lying in the \( m \)-simplex \( \gamma(T) \). For all sufficiently small \( \epsilon > 0 \) the piecewise linear map \( \rho' \) is linear over \( \gamma(B_{\epsilon,c}) \). A continuity argument recalling (10) ensures that the point \( l^* = \rho'(l) \) lies in the relative interior of \( \gamma(T) \), because \( l \) tends to \( c' \) as \( p \) tends to \( \infty \).

The De Concini-Procesi theorem in the version of [24, Theorem 5.3] (or the affine version of the desingularization procedure of [17, p.70]) yields a regular triangulation \( \nabla \) of \( \gamma(T) \) such that \( l^* \) is a vertex of some simplex of \( \nabla \). The set \( S \) of \( m \)-simplexes of \( \nabla \) is now defined by
\[
S = \{ B \mid B \in \nabla \text{ is an } m \text{-simplex having } l^* \text{ among its vertices} \}.
\]

Fix now \( B \in S \) and write \( B = \text{conv}(v_0, v_1, \ldots, v_m) \) for suitable points \( v_i \in \mathbb{R}^n \).

For each \( i = 0, \ldots, m \) let us display the homogeneous correspondent \( \tilde{v}_i \in \mathbb{Z}^{n+1} \) of vertex \( v_i \) as follows:
\[
\tilde{v}_i = (b_{i1}, \ldots, b_{im}, 0, \ldots, 0, d_i).
\]

From the regularity of \( B \in S \subseteq \nabla \) we get
\[
\det \begin{pmatrix} b_{01} & b_{02} & \cdots & b_{0m} & d_0 \\ b_{11} & b_{12} & \cdots & b_{1m} & d_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} & d_m \end{pmatrix} = \pm 1.
\] (15)

Recalling (11) we can similarly write
\[
\bar{l} = (a_1, \ldots, a_m, 1, 0, \ldots, 0, p).
\]

Let \( b_1, \ldots, b_m, d \) be the column vectors of the integer matrix (15). We then have
\[
\det \begin{pmatrix} b_1 & b_2 & \cdots & b_m & 0 & d \\ a_1 & a_2 & \cdots & a_m & 1 & p \end{pmatrix} = \pm 1,
\]
showing the regularity of the \((m+1)\)-simplex \( P_B = \text{conv}(B,l) \) for every \( B \in S \).

The basis of the pyramid \( P_B \) is the \( m \)-simplex \( B \subseteq F_m \). The lateral \( m \)-surface of \( P_B \) is the point set union of all \( m \)-simplexes of \( P_B \) having \( l^* \) as a vertex. Let the \((m+1)\)-dimensional pyramid \( P_S \subseteq \mathbb{R}^n \) be defined by
\[
P_S = \bigcup_{B \in S} P_B.
\]

Its basis \( B_S \) is the point set union of the bases \( B \)'s, for all \( B \in S \). Let \( L_B^* \subseteq L_B \) be obtained by stripping \( L_B \) of all \( m \)-simplexes of \( P_B \) having \( l^* \) as a vertex. Then the lateral surface \( L_S \) of \( P_S \) is given by
\[
L_S = \text{cl} \bigcup_{B \in S} L_B^* = \bigcup_{B \in S} \text{cl}(L_B^*).
\]

Since the \( \mathbb{Z} \)-retraction \( \rho' \) is linear over the ellipsoid \( \gamma(B_{\epsilon,c}) \) then \( \rho' \) maps \( L_S \) one-one onto \( B_S \). Intuitively, \( \rho' \) collapses the lateral surface of \( P_S \) one-one onto its basis \( B_S \).
This can be directly verified for each $B \in S$, noting that $\rho'$ maps $\text{cl}(L_B^f)$ one-one onto $B$. Since $\rho'$ preserves the denominators of the vertices of each $m$-simplex $P_B$, and $P_B$ is regular, then by [24, Proposition 3.15] $\rho'$ maps $\mathbb{Z}$-homeomorphically $\text{cl}(L_B^f)$ onto $B$. Thus $\rho'$ maps $\mathbb{Z}$-homeomorphically $L_S$ onto $B_S$. Further, $\rho'$ is identity over $\gamma(P) \setminus B_S$, whence $\rho'$ is a $\mathbb{Z}$-homeomorphism of $(\gamma(P) \setminus B_S) \cup L_S$ onto $\gamma(P)$.

In conclusion, the set $(\gamma(P) \setminus B_S) \cup L_S$ is a $\mathbb{Z}$-homeomorphism domain of $\rho'$. Going back via $\gamma^{-1}$ we see that the $\mathbb{Z}$-retraction $\rho$ sends $(P \setminus \gamma^{-1}(B_S)) \cup \gamma^{-1}(L_S)$ $\mathbb{Z}$-homeomorphically onto $P$. (Note that (12) ensures that $\gamma^{-1}(L_S)$ lies in the $n$-cube). The choice of $c \in \text{relint}(T)$ and of the large prime $p$ being arbitrary, it follows that there are infinitely many $\mathbb{Z}$-homeomorphism domains of $\rho$. By Theorem 2.1 the number of retractions of $\mathcal{M}([0,1]^n)$ onto $\text{gen}(\rho_1, \ldots, \rho_n)$ is infinite. □

Combining the foregoing theorem with Theorem 2.3 we immediately obtain:

**Corollary 4.3.** Let $k$ be the smallest number of generators of a finitely generated projective MV-algebra $B$. Then the index of $B$ is finite iff the maximal spectral space of $B$ is homeomorphic to a regular domain in $[0,1]^k$.

**Proof.** Identify $B$ with $\mathcal{M}(P)$ for some $\mathbb{Z}$-retract $P$ of $[0,1]^k$.

$(\Rightarrow)$ If the maximal spectral space of $B$ is not homeomorphic to a regular domain in $[0,1]^k$, then the same holds for its homeomorphic copy $P$. As a consequence, every (equivalently, some) triangulation $\Delta$ of $P$ contains some maximal $l$-simplex with $l < k$. By the foregoing theorem, the index of $B$ is infinite.

$(\Leftarrow)$ If the maximal spectral space of $B$ is homeomorphic to a regular domain in $[0,1]^k$, then so is its homeomorphic copy $P$. Now apply Theorem 2.3. □

5. Arbitrarily high finite index

**Theorem 5.1.** For every $j = 1, 2, \ldots$ there is retract $A_j$ of $\mathcal{M}([0,1]^2)$ such that the maximal spectral space of $A_j$ is a closed domain in $[0,1]^2$ and $j < \nu(A_j) \in \mathbb{Z}$.

**Proof.** For each rational point $x = (x_1, x_2) \in [0,1]^2$ let $d = \text{den}(x)$ be the least common multiple of the denominators of $x_1$ and $x_2$. Then for uniquely determined integers $n_1, n_2$ we can write $x_1 = n_1/d$ and $x_2 = n_2/d$. Throughout this proof we will specify $x$ in terms of its homogeneous integer coordinates as in [24, §2.1]. Identifying $x$ with its homogeneous correspondent we will write

$$x = (n_1/d, n_2/d) = [n_1, n_2, d].$$

(16)

We will also write $o$ for the origin $[0,0,1]$ in $\mathbb{R}^2$.

The proof amounts to a construction of $\mathbb{Z}$-retracts $\sigma^{(1)}, \sigma^{(2)}, \ldots$ of $[0,1]^2$ such that the multiplicity of the retract $A_n = \text{range}(\sigma^{(n)})$ is $> 2^n$. We assume familiarity with regular triangles, regular triangulations, and Farey blow-ups [24, §2.2, §5.1].

**Step 0.**

Let the regular triangles $U_1, V_1 \subseteq [0,1]^2$ be defined by

$$V_1 = \text{conv}(o, [1,1,1], [0,1,1]) \text{ and } U_1 = \text{conv}(o, [1,0,1], [1,1,1]).$$

Let $\zeta^{(1)} : V_1 \to U_1$ be the unique linear extension of the map

$$o \mapsto o, [1,1,1] \mapsto [1,1,1], [0,1,1] \mapsto [1,0,1].$$
By [24, Lemma 3.7, Corollary 3.10], $\zeta^{(1)}$ is a $\mathbb{Z}$-homeomorphism of $V_1$ onto $U_1$, in symbols, $\zeta^{(1)} : V_1 \cong \mathbb{Z} \ U_1$. Next let $\rho^{(1)} = \sigma^{(1)} = \zeta^{(1)} \cup \text{id}_{U_1}$. Then $\sigma^{(1)}$ is a $\mathbb{Z}$-retraction of $[0, 1]^2$ onto $U_1$, acting $\mathbb{Z}$-homeomorphically over $V_1$. By Theorem 2.1, the multiplicity of the retract $A_1 = \text{range}(\sigma^{(1)})$ is equal to $2^1$.

The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ be defined by

\[ F_1 = 1, \ F_2 = 1, \ F_{n+1} = F_n + F_{n-1}. \]  

(17)

**Step 1.**

Let $\Sigma_1$ be the regular simplicial complex given by $U_1$ and all its faces. Let $b_1$ be the Farey mediant of the edge of $U_1$ opposite to the origin $o$. Then the blow-up of $\Sigma_1$ at $b_1$ yields a regular simplicial complex, whose maximal triangles $V_2, U_2$ are given by

\[ V_2 = \text{conv}(o, [1, 2, F_3], [1, 1, F_2]) \text{ and } U_2 = \text{conv}(o, [1, 0, F_1], [2, 1, F_3]). \]

Let $\zeta^{(2)} : V_2 \to U_2$ be the unique linear extension of the map

\[ o \mapsto o, \ [2, 1, F_3] \mapsto [2, 1, F_3], \ [1, 1, F_2] \mapsto [1, 0, F_1]. \]

Then $\zeta^{(2)}$ is a $\mathbb{Z}$-homeomorphism of $V_2$ onto $U_2$, in symbols, $\zeta^{(2)} : V_2 \cong \mathbb{Z} \ U_2$. Next let $\rho^{(2)} = \zeta^{(2)} \cup \text{id}_{U_2}$. This is a $\mathbb{Z}$-retraction of $U_1$ onto $U_2$ acting $\mathbb{Z}$-homeomorphically over $V_2$. Let

\[ \sigma^{(2)} = \rho^{(2)} \circ \sigma^{(1)} = \rho^{(2)} \circ \rho^{(1)}. \]

This is a $\mathbb{Z}$-retraction of $[0, 1]^2$ onto $U_2$ acting $\mathbb{Z}$-homeomorphically over the following $2^2$ triangles: $U_2, V_2, (\zeta^{(1)})^{-1}(U_2), (\zeta^{(1)})^{-1}(V_2)$. By Theorem 2.1, the multiplicity of the retract $A_2$ of $\mathcal{M}([0, 1]^2)$ defined by $A_2 = \text{range}(\sigma^{(2)})$ is equal to $2^2$.

**Step 2.**

Let $\Sigma_2$ be the regular simplicial complex given by the triangle

\[ U_2 = \text{conv}(o, [1, 0, F_2], [2, 1, F_3]) \]

and all its faces. In homogeneous coordinates, let $b_2 = [3, 1, F_4]$ be the Farey mediant of the edge $\text{conv}([1, 0, F_2], [2, 1, F_3])$ of $U_2$ opposite to the origin $o$. Then the blow-up of $\Sigma_2$ at $b_2$ yields a regular simplicial complex, whose maximal triangles $V_3, W_3$ are given by

\[ V_3 = \text{conv}(o, [3, 1, F_4], [2, 1, F_3]) \text{ and } W_3 = \text{conv}(o, [1, 0, F_2], [3, 1, F_4]). \]

We now let $[1, 0, F_3]$ be the Farey mediant of $[1, 0, F_2]$ and $o = [0, 0, F_1]$. Let $\mathcal{W}_3$ be the (regular) simplicial complex given by $W_3$ and all its faces. By blowing-up $\mathcal{W}_3$ at $[1, 0, F_3]$ we obtain a regular triangulation of $W_3$ whose maximal triangles $U_3$ and $T_3$ are given by

\[ U_3 = \text{conv}(o, [1, 0, F_3], [3, 1, F_4]) \text{ and } T_3 = \text{conv}([1, 0, F_3], [1, 0, F_2], [3, 1, F_4]). \]

By construction $U_2 = V_3 \cup W_3 = V_3 \cup U_3 \cup T_3$.

Let $\zeta^{(3)} : V_3 \to U_3$ be the unique linear extension of the map

\[ o \mapsto o, \ [3, 1, F_4] \mapsto [3, 1, F_4], \ [2, 1, F_3] \mapsto [1, 0, F_3]. \]

As above, $\zeta^{(3)}$ is a $\mathbb{Z}$-homeomorphism of $V_3$ onto $U_3$, in symbols, $\zeta^{(3)} : V_3 \cong \mathbb{Z} \ U_3$.

Let $\lambda^{(3)} : T_3 \to U_3$ be the unique linear extension of the map

\[ [1, 0, F_3] \mapsto [1, 0, F_3], \ [3, 1, F_4] \mapsto [3, 1, F_4], \ [1, 0, F_2] \mapsto o. \]
Then the map \( \rho^{(3)} = \zeta^{(3)} \cup \lambda^{(3)} \cup \text{id}_{U_3} \) is a \( \mathbb{Z} \)-retraction of \( U_2 \) onto \( U_3 \) acting \( \mathbb{Z} \)-homeomorphically over \( V_3 \) and, trivially, over \( U_3 \). (Actually, \( \rho^{(3)} \) also acts \( \mathbb{Z} \)-homeomorphically over \( T_3 \), but for our purposes it is sufficient to restrict attention to the action of \( \rho^{(3)} \) over the two triangles \( V_3 \) and \( U_3 \).) The map
\[
\sigma^{(3)} = \rho^{(3)} \circ \sigma^{(3)} = \rho^{(3)} \circ \sigma^{(2)} \circ \rho^{(1)}
\]
is a \( \mathbb{Z} \)-retraction of \( [0,1]^2 \) onto \( U_3 \) acting \( \mathbb{Z} \)-homeomorphically over (among others) the following 2\( 4 \) triangles:
\[
U_3, \ V_3, \ (\zeta^{(2)})^{-1}(U_3), \ (\zeta^{(2)})^{-1}(V_3), \ (\zeta^{(1)})^{-1}(\text{the foregoing 4 triangles}). \quad (18)
\]
By Theorem 2.1(c), the multiplicity of the retract \( A_3 \) of \( \mathcal{M}([0,1]^2) \) defined by \( A_3 = \text{range}(- \circ \sigma^{(3)}) \) is \( \geq 2^4 \).

\textbf{Step 3.}

Let the regular simplex \( \Sigma_n \) consist of the triangle
\[
U_3 = \text{conv}(o, [1,0,F_3], [3,1,F_4])
\]
and all its faces. In homogeneous coordinates, let \( b_3 = [4,1,F_5] \) be the Farey mediant of the edge \( \text{conv}([1,0,F_3], [3,1,F_4]) \) of \( U_3 \) opposite to the origin \( o \). Then the blow-up of \( \Sigma_3 \) at \( b_3 \) yields a regular simplicial complex, whose maximal triangles \( V_4, W_4 \) are given by
\[
V_4 = \text{conv}(o, [4,1,F_5], [3,1,F_4]) \quad \text{and} \quad W_4 = \text{conv}(o, [1,0,F_3], [4,1,F_5]).
\]
We now let \([1,0,F_4] \) be the Farey mediant of \([1,0,F_3] \) and \( o = [0,0,F_1] \).

Let \( W_4 \) be the (regular) simplicial complex given by \( W_4 \) and all its faces. By blowing-up \( W_4 \) at \([1,0,F_4] \), we obtain a regular triangulation of \( W_4 \) whose maximal triangles \( U_4, T_4 \) are given by
\[
U_4 = \text{conv}(o, [1,0,F_4], [4,1,F_5], [3,1,F_4]) \quad \text{and} \quad T_4 = \text{conv}([4,0,F_4], [1,0,F_3], [4,1,F_5]).
\]
Observe that \( U_4 = V_4 \cup W_4 = V_4 \cup U_4 \cup T_4 \).

Let \( \zeta^{(4)} : V_4 \to U_4 \) be the unique linear extension of the map
\[
o \mapsto o, \quad [4,1,F_5] \mapsto [4,1,F_5], \quad [3,1,F_4] \mapsto [1,0,F_4].
\]
As above, \( \zeta^{(4)} \) is a \( \mathbb{Z} \)-homeomorphism of \( V_4 \) onto \( U_4 \), in symbols, \( \zeta^{(4)} : V_4 \cong \mathbb{Z} U_4 \).

Let \( \lambda^{(4)} : T_4 \to U_4 \) be the unique linear extension of the map
\[
[1,0,F_4] \mapsto [1,0,F_4], \quad [4,1,F_5] \mapsto [4,1,F_5], \quad [1,0,F_3] \mapsto o.
\]
Then the map
\[
\rho^{(4)} = \zeta^{(4)} \cup \lambda^{(4)} \cup \text{id}_{U_4}
\]
is a \( \mathbb{Z} \)-retraction of \( U_3 \) onto \( U_4 \) acting \( \mathbb{Z} \)-homeomorphically over \( V_4 \). The map
\[
\sigma^{(4)} = \rho^{(4)} \circ \sigma^{(4)} = \rho^{(4)} \circ \rho^{(3)} \circ \rho^{(2)} \circ \rho^{(1)}
\]
is a \( \mathbb{Z} \)-retraction of \( [0,1]^2 \) onto \( U_4 \) acting \( \mathbb{Z} \)-homeomorphically over the following 2\( 4 \) triangles:
\[
U_4, \ V_4, \ (\zeta^{(3)})^{-1}(U_4), \ (\zeta^{(2)})^{-1}(V_4), \ \text{etc. etc. etc. unfolding.}
\]
(As in the previous step, \( \sigma^{(4)} \) acts \( \mathbb{Z} \)-homeomorphically over other triangles, but for our present purposes it is convenient to restrict attention to these 2\( 4 \) only. See Figure 4.) By Theorem 2.1(c), the multiplicity of the retract \( A_4 \) of \( \mathcal{M}([0,1]^2) \) defined by \( A_4 = \text{range}(- \circ \sigma^{(4)}) \) is \( \geq 2^4 \).

\textbf{Step \( n - 1 \), (\( n = 5,6,\ldots \)).}

Inductively let the regular simplex \( \Sigma_{n-1} \) consist of the triangle
\[
U_{n-1} = \text{conv}(o, [1,0,F_{n-1}], [n-1,1,F_n])
\]
and all its faces. In homogeneous coordinates, let \( b_{n-1} = [n, 1, F_{n+1}] \) be the Farey mediant of the edge \( \text{conv}([1, 0, F_{n-1}], [n - 1, 1, F_n]) \) of \( U_{n-1} \) opposite to the origin \( o \). Then the blow-up of \( \Sigma_{n-1} \) at \( b_{n-1} \) yields a regular simplicial complex, whose maximal simplexes \( V_n, W_n \) are given by

\[
V_n = \text{conv}(o, [n, 1, F_{n+1}], [n - 1, 1, F_n]) \quad \text{and} \quad W_n = \text{conv}(o, [1, 0, F_{n-1}], [n, 1, F_{n+1}]).
\]

Let the regular triangle \( U_n \subseteq W_n \) be given by \( U_n = \text{conv}(o, [1, 0, F_n], [n, 1, F_{n+1}]) \). Let \( \zeta^{(n)}: V_n \rightarrow U_n \) be the unique linear extension of the map

\[
o \mapsto o, \quad [n, 1, F_{n+1}] \mapsto [n, 1, F_{n+1}], \quad [n - 1, 1, F_n] \mapsto [1, 0, F_n].
\]

The regularity of \( V_n \) and \( U_n \) ensures that \( \zeta^{(n)} \) is a \( \mathbb{Z} \)-homeomorphism of \( V_n \) onto \( U_n \), in symbols, \( \zeta^{(n)}: V_n \cong \mathbb{Z} U_n \).

For each \( j = 0, \ldots, F_{n-2} - 1 \), let the triangle \( T_{n,j} \) be defined by

\[
T_{n,j} = \text{conv}([n, 1, F_{n+1}], [1, 0, F_{n-1} + j], [1, 0, F_{n-1} + j + 1]).
\]

A direct verification shows that every \( T_{n,j} \) is regular. As a matter of fact, the triangle \( W_n = \text{conv}(o, [1, 0, F_{n-1}], [n, 1, F_{n+1}]) \) is regular; the points \([1, 0, F_{n-1} + 1], [1, 0, F_{n-1} + 2], \ldots, [1, 0, F_{n-1} + F_{n-2} - 1], [1, 0, F_{n-1} + F_{n-2}] = [1, 0, F_n] \), are obtained by taking the (always Farey) mediant \([1, 0, F_{n-1} + 1] \) of \( o \) and \([1, 0, F_{n-1}] \), and then taking the mediant \([1, 0, F_{n-1} + 2] \) of \( o \) and \([1, 0, F_{n-1} + 1], \ldots, \) and finally taking the mediant \([1, 0, F_n] \) of \( o \) and \([1, 0, F_n - 1] = [1, 0, F_{n-1} + F_{n-2} - 1] \). Let \( W_n \) be the regular simplicial complex given by \( W_n \) and all its faces. Then \( U_n \) and the \( T_{n,j} \) are the maximal simplexes of a regular triangulation of \( W_n \), which is obtained from \( W_n \) by consecutive Farey blow-ups as described in Figure 5. Observe that \( U_{n-1} = V_n \cup W_n = V_n \cup U_n \cup \bigcup_j T_{n,j} \).
For each $j = 0, \ldots, F_n - 2$, let

$$
\lambda_j^{(n)}: \text{conv}(1, 0, F_n, [n, 1, F_{n+1}], [1, 0, F_{n-1} + j]) \to U_n
$$

be the unique linear extension of the map

$$
[1, 0, F_n] \mapsto [1, 0, F_n], \quad [n, 1, F_{n+1}] \mapsto [n, 1, F_{n+1}], \quad [1, 0, F_{n-1} + j] \mapsto o.
$$

By [24, Lemma 3.7], each $\lambda_j^{(n)}$ is linear with integer coefficients (i.e., $\lambda_j^{(n)}$ is a linear $\mathbb{Z}$-map) sending the regular triangle $\text{conv}(n, 1, F_{n+1}, 1, 0, F_n - 1)$ onto $U_n$, and mapping all other triangles $T_j$ onto the segment $\text{conv}(o, [n, 1, F_{n+1}]) \subseteq U_n$.

The map

$$
\rho(n) = \zeta^{(n)} \cup \text{id}_{U_n} \cup \bigcup_{j=0}^{F_n-2} \lambda_j^{(n)}
$$

is a $\mathbb{Z}$-retraction of $U_{n-1}$ onto $U_n$ acting $\mathbb{Z}$-homeomorphically over $V_n$. The map

$$
\sigma(n) = \rho(n) \circ \sigma^{(n-1)} = \rho(n) \circ \rho^{(n-1)} \circ \cdots \circ \rho^{(1)}
$$

is a $\mathbb{Z}$-retraction of $[0, 1]^2$ onto $U_n$. Generalizing (18), $\sigma^{(n)}$ is a $\mathbb{Z}$-homeomorphism onto $U_n$ of each of the following $2^n$ triangles: $U_n$, $V_n$, $(\zeta^{(n-1)})^{-1}(U_n)$, $(\zeta^{(n-2)})^{-1}(V_n)$, $(\zeta^{(n-2)})^{-1}(n)$, $\cdots$, $(\zeta^{(2)})^{-1}(n)$, $(\zeta^{(1)})^{-1}(n)$, $(\zeta^{(1)})^{-1}(n)$, $(\zeta^{(1)})^{-1}(n)$, $(\zeta^{(1)})^{-1}(n)$, $(\zeta^{(1)})^{-1}(n)$. Actually, $\sigma^{(n)}$ is a $\mathbb{Z}$-homeomorphism also of other triangles onto $U_n$, but these are irrelevant to our purposes. By Theorem 2.1, the multiplicity of the retract $A_n = \text{range}(\text{range}(\cdots \circ \sigma^{(n)}))$ of $\mathcal{M}([0, 1]^2)$ is $\geq 2^n$. Since the area of $U_n$ is $> 0$, by (6) the multiplicity of $A_n$ is finite.
Iterating this inductive procedure we obtain retracts $A_m$ of $M([0,1]^2)$ whose maximal ideal space is a closed domain in $[0,1]^2$, and whose multiplicity and index are finite and arbitrarily large.

\[\text{Corollary 5.2.} \text{ Adopt the notation of (16)-(17). For each } n = 1, 2, \ldots \text{ let the triangle } U_n \subseteq [0,1]^2 \text{ be defined by } U_n = \text{conv}([0,0,1],[1,0,F_n],[n,1,F_{n+1}]). \text{ Then } 2^n \leq \iota(M(U_n)) = \iota(M_{\mathbb{R}}(U_n)) \in \mathbb{Z}.\]

\[\text{Proof.} \text{ By [24, Lemma 3.6], the retract } A_n = \text{range}(\sigma^{(n)}) \text{ of } M([0,1]^2) \text{ in the foregoing theorem is isomorphic to } M(U_n). \text{ So } \iota(M(U_n)) \geq 2^n. \text{ The preservation properties of the } \Gamma \text{ functor ensure that } \iota(M_{\mathbb{R}}(U_n)) = \iota(M(U_n)).\]

\[\text{Corollary 5.3.} \text{ For every } j = 1, 2, \ldots, \text{ there is retract } R_j \text{ of the free unital } \ell \text{-group } M_{\mathbb{Z}}([0,1]^2) \text{ such that } \iota(R_j) > j, \text{ and the maximal spectral space of } R_j \text{ is a closed domain in } [0,1]^2.\]

While the index of a finitely generated projective MV-algebra arises by taking the \text{sup} of multiplicities, taking the \text{inf} is of little interest:

\[\text{Proposition 5.4.} \text{ Let } A \text{ be a retract of } M([0,1]^n), \text{ say } A = \text{range}(\sigma^{(n)}) \text{ for some } Z \text{-retraction } \sigma \text{ of } [0,1]^n \text{ onto the rational polyhedron } P. \text{ Suppose } P \text{ is a closed domain in } [0,1]^n. \text{ Then } A \text{ has an isomorphic copy } A' = \text{range}(\rho) \text{ where } \rho \text{ is a } Z \text{-retraction of } [0,1]^n \text{ onto } P \text{ and the multiplicity of } A' \text{ is equal to } 1.\]

\[\text{Proof.} \text{ If the multiplicity of } A \text{ is } 1 \text{ we are done. Otherwise, let } m > 1 \text{ be the multiplicity of } A. \text{ The finiteness of } m \text{ follows from Theorem 2.3 since by assumption, } P = \text{cl}(\text{int}(P)). \text{ By Theorem 2.1, there are exactly } m \text{ rational polyhedra } P = Q_1, Q_2, \ldots, Q_m \subseteq [0,1]^n \text{ such that for each } i = 1, \ldots, m, \text{ } \rho | Q_i \text{ is a } Z \text{-homeomorphism of } Q_i \text{ onto } P. \text{ Now consider } Q_2. \text{ By the final part of the proof of Theorem 2.3 some connected component, say } Q, \text{ of the interior of } Q_2 \text{ is disjoint from the interior of } P. \text{ Let } \nabla \text{ be a regular triangulation of } [0,1]^n \text{ having the following properties:}

\begin{enumerate}
  \item \nabla \text{ linearizes } \rho \text{ (i.e., } \rho \text{ is linear over each simplex of } \nabla); 

  \item \nabla \text{ has an } n \text{-simplex } T = \text{conv}(t_0, t_1, \ldots, t_n) \text{ lying in the interior of } Q, \text{ where } d = \text{den}(t_0); 

  \item \nabla \text{ also has a vertex } t^* \in Q \setminus T \text{ of denominator } d.
\end{enumerate}

The existence of \nabla is ensured by [24, Proposition 3.2]. Applying [24, Lemma 3.7] to the regular simplex } T \text{ and to the } n+1 \text{ rational points } t^*, t_1, \ldots, t_n \text{ we obtain a } Z \text{-map } \zeta : [0,1]^n \to [0,1]^n \text{ such that } \zeta \text{ is identity over } [0,1]^n \setminus \text{int}(T), \zeta(t_0) = t^*, \text{ and } \zeta(T) \text{ is contained in } Q. \text{ It follows that } \zeta | U \text{ is not one-one.}

Thus the composite } Z \text{-map } \rho^{(1)} = \rho \circ \zeta, \text{ while being a } Z \text{-retraction of } [0,1]^n \text{ onto } P, \text{ does not act } Z \text{-homeomorphically over } Q_2. \text{ By Theorem 2.1, the multiplicity of the retract } A_1 = \text{range}(\sigma^{(1)}) \text{ is equal to } m - 1. \text{ Proceeding in this way we can find a } Z \text{-retraction } \rho^{(m-1)} \text{ of } [0,1]^n \text{ onto } P \text{ such that the multiplicity of the retract } A_{m-1} = \text{range}(\sigma^{(m-1)}) \text{ is equal to } 1. \text{ Since all } Z \text{-retractions } \rho^{(1)}, \ldots, \rho^{(m-1)} \text{ are onto the same rational polyhedron } P, A_{m-1} \text{ is isomorphic to } A. \text{ Now set } A' = A_{m-1}. \]

6. Comparing retracts of free MV-algebras and unital \ell-groups

Two sets \(A, B\) are said to be comparable if either \(A \subseteq B\) or \(B \subseteq A\).

\[\text{Proposition 6.1.} \text{ Any two } Z \text{-homeomorphic comparable rational polyhedra } P, Q \subseteq [0,1]^n \text{ are equal. However, two isomorphic comparable finitely presented subalgebras of } M([0,1]^n) \text{ need not be equal.}\]
Proof. Suppose \( P \subseteq Q \) and \( P \neq Q \). Then for some suitably large integer \( d \) the number of points of denominator \( d \) in \( P \) is strictly less than in \( Q \). By [24, Proposition 3.15], \( P \) and \( Q \) are not \( \mathbb{Z} \)-homeomorphic. For the second statement, the subalgebra of \( \mathcal{M}([0,1]) \) generated by \( x \oplus x \) is isomorphic to \( \mathcal{M}([0,1]) \) but is not equal to \( \mathcal{M}([0,1]) \).

\begin{align*}
\textbf{Theorem 6.2.} \text{ Retracts } A, B \text{ of } \mathcal{M}([0,1]^n) \text{ are equal if they are comparable and isomorphic.}
\end{align*}

Proof. For the nontrivial direction, assume \( A \cong B \) and \( A \subseteq B \), with the intent of proving \( A = B \). For suitable McNaughton functions \( \sigma_1, \ldots, \sigma_n \) and \( \tau_1, \ldots, \tau_n \) with \( \sigma \circ \sigma = \sigma \) and \( \tau \circ \tau = \tau \) we can write \( A = \gen(\sigma_1, \ldots, \sigma_n) \) and \( B = \gen(\tau_1, \ldots, \tau_n) \). The restriction to \( B \) of the retraction \( - \circ \sigma : \mathcal{M}([0,1]^n) \to A \) is a retraction of \( B \) onto \( A \), and we have a commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{(-\circ \sigma)} & A \\
\downarrow \text{id} & & \downarrow \text{id} \\
B & \xleftarrow{\text{inclusion}} & A
\end{array}
\]

Dually, [21], we get the commutative diagram

\[
\begin{array}{ccc}
R_r & \xleftarrow{\epsilon} & R_\sigma \\
\downarrow \text{id} & & \downarrow \text{id} \\
R_r & \xrightarrow{\delta} & R_\sigma
\end{array}
\]

(19)

Since \( A \subseteq B \), from [6, Theorem 3.2(ii)] it follows that \( \delta \) is onto \( R_\sigma \). By [6, Theorem 3.5], \( \epsilon \) is one-one and preserves denominators. Since \( A \cong B \), by the (Cantor-Bernstein) theorem [6, Theorem 3.7], \( \epsilon \) is a \( \mathbb{Z} \)-homeomorphism of \( R_\sigma \) onto \( R_r \),

\[
\epsilon : R_\sigma \cong_{\mathbb{Z}} R_r.
\]

Now from (19) it follows that \( \delta \) and \( \epsilon \) are inverses of each other, whence

\[
\delta : R_r \cong_{\mathbb{Z}} R_\sigma.
\]

Therefore, the inclusion map of \( A \) into \( B \) (which is the dual of \( \delta \)) is surjective, and \( A = B \).

The following result is a special case of [9, Theorem 4.6]. We have included it here because of its simple proof.

\begin{proposition}
Let \( A \) be a separating retract of \( \mathcal{M}([0,1]^n) \), in the sense that for all distinct \( x, y \in [0,1]^n \) there is \( f \in A \) with \( f(x) \neq f(y) \). Then \( A \) coincides with \( \mathcal{M}([0,1]^n) \).
\end{proposition}

Proof. By hypothesis we have a retraction \( \epsilon \) of \( \mathcal{M}([0,1]^n) \) onto \( A \). Letting \( \sigma_i = \epsilon(\pi_i) \), \( (i = 1, \ldots, n) \), and recalling the notational stipulations in the introductory part of Section 2, the retraction \( \epsilon \) determines the \( \mathbb{Z} \)-retraction \( Z_\epsilon = \sigma = (\sigma_1, \ldots, \sigma_n) : [0,1]^n \to [0,1]^n \), and we can write \( A = \gen(\sigma_1, \ldots, \sigma_n) \). If \( R_\sigma = [0,1]^n \) we are done, and \( \sigma \) is identity on \( [0,1]^n \). If \( R_\sigma \) is strictly contained in \( [0,1]^n \) then some rational point \( r \in [0,1]^n \) of sufficiently high denominator does not belong to \( R_\sigma \). Since \( \sigma(r) \) lies in \( R_\sigma \), necessarily \( \sigma(r) \neq r \). Since \( \sigma(r) = \sigma(\sigma(r)) \) then for each \( f \in A \) we must have \( f(r) = f(\sigma(r)) \), because \( f \) has the form \( g \circ \sigma \) for some \( g \in \mathcal{M}([0,1]^n) \). We conclude that \( A \) is not a separating subalgebra of \( \mathcal{M}([0,1]^n) \).

\[ \square \]
Proposition 6.4. For any two $\mathbb{Z}$-retractions $\sigma \neq \tau$ of $[0,1]^n$ with equal range, the retracts $A_\sigma = \text{gen}(\sigma_1, \ldots, \sigma_n)$ and $A_\tau = \text{gen}(\tau_1, \ldots, \tau_n)$ are (isomorphic and) incomparable.

Proof. Isomorphism immediately follows from [24, Corollary 3.10]. Concerning incomparability, the solutions of the equation

$$\xi \circ \sigma = \sigma$$

in the unknown $\xi = (\xi_1, \ldots, \xi_n) : [0,1]^n \to [0,1]^n$, $\xi_i \in \mathcal{M}([0,1]^n)$, are precisely those elements of $(\mathcal{M}([0,1]^n))^n$ which act identically on the range $R_\sigma$. If $A_\sigma$ is a subalgebra of $A_\tau$ then for some $\chi = (\chi_1, \ldots, \chi_n)$ with $\chi_i \in \mathcal{M}([0,1]^n)$ we have $\chi \circ \sigma = \tau$, because $\{\sigma_1, \ldots, \sigma_n\}$ is a generating set of $A_\sigma$. Over the polyhedron $R_\sigma = R_\tau$, the function $\chi$ must act identically, because so do $\sigma$ and $\tau$. Similarly, for each $y \in [0,1] \setminus R_\sigma$ the point $\sigma(y)$ lies in $R_\tau$. We have proved the identity $(\chi \circ \sigma)(y) = \sigma(y)$ for all $y \in [0,1]^n$. From $\chi \circ \sigma = \sigma$ and $\chi \circ \sigma = \tau$ we get $\sigma = \tau$. □

For any fixed $\mathbb{Z}$-retract $P \subseteq [0,1]^n$ the set $\Omega_P$ of MV-algebras is defined by

$$\Omega_P = \{\text{gen}(\sigma_1, \ldots, \sigma_n) \mid \sigma \text{ any possible } \mathbb{Z}\text{-retraction of } [0,1]^n \text{ onto } P\}.$$ 

By duality, any two algebras in $\Omega_P$ are isomorphic.

Proposition 6.5. In general, the intersection of two MV-algebras in $\Omega_P$ need not be in $\Omega_P$. The smallest MV-algebra containing two MV-algebras in $\Omega_P$ need not be in $\Omega_P$.

Proof. For both statements we have examples already for $n = 1$.

For the first statement, let $\sigma = \pi_1 \land \neg \pi_1$. Then the map $f \mapsto f \circ \sigma$ amounts to taking the mirror image of the first half of $f$. Let now $\tau : [0,1] \to [0,1]$ act identically on the interval $[0,1/2]$, then descend to 0 with slope $\sim -3$, and finally vanish over $[2/3,1]$. All functions $f \in A_\sigma \cap A_\tau$ are symmetric around the axis $y = 1/2$, and are constant over the interval $[2/3,1]$, so they are also constant over the interval $[0,1/3]$. As a consequence, $A_\tau \cap A_\sigma$ does not have a maximal quotient isomorphic to $\Gamma([1,2],1)$. By [24, Lemma 3.6], every MV-algebra $A$ in $\Omega_{[0,1/2]}$ is isomorphic to $\mathcal{M}([0,1/2])$. So, in particular, $A$ has a maximal quotient isomorphic to $\Gamma([1,2],1)$. So $A_\sigma \cap A_\tau \notin \Omega_{[0,1/2]}$.

For the second statement take two different $\mathbb{Z}$-retractions $\sigma, \tau$ of $[0,1]$ onto the the same range $[0,q] \subseteq [0,1]$. The interval $[0,q]$ is a $\mathbb{Z}$-retract of $[0,1]$. Every MV-algebra in $\Omega_{[0,q]}$ is isomorphic to $A_\sigma$ and hence it is projective. By duality we can write $A_\tau \cong A_\tau \in \Omega_{[0,q]}$. Now for definiteness assume both $\sigma$ and $\tau$ to have exactly three linear pieces. We claim that the range $R$ of the map $\langle \sigma, \tau \rangle : [0,1] \to [0,1]^2$ is not simply connected. As a matter of fact, let us proceed along the trajectory $t \in [0,q] \to (\sigma(t), \tau(t)) \in [0,1]^2$ starting from the $(0,0)$ at time $t = 0$. Then we go up along the diagonal $x_1 = x_2$ of $[0,1]^2$ until, at time $t = q$, we reach the point $(q,q)$; we then go down until we reach, say, the $x$-axis, and finally move leftward until we reach the origin, at time $t = 1$. The resulting piecewise linear curve $R = \text{range}(\sigma, \tau)$ is the perimeter of a quadrangle, whence it is not simply connected. Our claim is settled.

Let $\text{gen}(\sigma, \tau)$ denote the subalgebra of $\mathcal{M}([0,1]^n)$ generated by $\sigma$ and $\tau$. This is the smallest MV-algebra containing $A_\sigma \cup A_\tau$. By [24, Lemma 3.6] we have the isomorphism $\text{gen}(\sigma, \tau) \cong \mathcal{M}(R)$. By [24, Corollary 4.18], the maximal spectrum of $\text{gen}(\sigma, \tau)$ is homeomorphic to $R$, so it is not simply connected, and $\text{gen}(\sigma, \tau)$ is not projective. We conclude that $\text{gen}(\sigma, \tau) \notin \Omega_{[0,q]}$. □
7. Decision problems for projective algebras

Unless otherwise specified, all MV-terms in this section are in the same variables $X_1, \ldots, X_n$. We use the adjective “decidable” (resp., “computable”) as an abbreviation of “Turing decidable” (resp., “Turing computable”).

**Proposition 7.1.** The following problem is decidable:

INSTANCE: MV-terms $t_1, \ldots, t_n$.

QUESTION: Is the map $i = (\hat{i}_1, \ldots, \hat{i}_n)$ a $\mathbb{Z}$-retraction of $[0, 1]^n$?

Proof. Checking the idempotency property $i \circ i = \hat{i}$ amounts to deciding whether the MV-term $t_i \mapsto t_i \circ i$ is a tautology in infinite-valued Lukasiewicz logic ($i = 1, \ldots, n$). The latter problem is decidable, [12, Corollary 4.5.3]. □

The foregoing innocent looking result should be contrasted with the following:

**Proposition 7.2.** When a rational polyhedron $R \subseteq [0, 1]^n$ is presented as a union of rational simplexes in $[0, 1]^n$, or even by a rational triangulation, checking whether $R$ is a $\mathbb{Z}$-retract is not a decidable problem.

Proof. As is well known, $R$ is contractible if it is a retract of $[0, 1]^n$, [8, Proposition 5.1]. Using both directions of the characterization theorems of $\mathbb{Z}$-retracts, (respectively in [8] and [7]) it follows that $R$ is a $\mathbb{Z}$-retract iff it is contractible and satisfies the following two conditions:

(i) $R$ has a nonempty intersection with the set of vertices of $[0, 1]^n$;

(ii) $R$ has a strongly regular triangulation i.e., [8, Definition 4.1] a regular triangulation $\Delta$ such that the greatest common divisor of the vertices of each maximal simplex of $\Delta$ is equal to 1.

Property (i) is trivially decidable. Also (ii) is decidable, because it is equivalent to the strong regularity of every regular triangulation of $R$.

By way of contradiction, assume the $\mathbb{Z}$-retract problem is decidable. Then we can decide the contractibility of rational polyhedra in $[0, 1]^n$, whence the contractibility of rational polyhedra in $\mathbb{R}^n$ would be a decidable problem. This contradicts [28, p.242]. □

**Proposition 7.3.** The following problem is decidable:

INSTANCE: MV-terms $t_1, \ldots, t_n$ such that the map $i: [0, 1]^n \to [0, 1]^n$ is idempotent (a decidable hypothesis, by Proposition 7.1). Let $\mu_A$ denote the maximal spectrum of the MV-algebra $A \subseteq \mathcal{M}([0, 1]^n)$ generated by $\hat{t}_1, \ldots, \hat{t}_n$.

QUESTION: Is $\mu_A$ homeomorphic to a closed domain in $[0, 1]^n$?

Proof. By [24, Corollary 4.18] there is a homeomorphism of $\mu_A$ onto the set $E = \{x \in [0, 1]^n \mid \hat{i}(x) = R_i\}$. The rational polyhedron $E$ can be computed from the input MV-terms $t_1, \ldots, t_n$. By [24, Lemma 18.1], a (regular) triangulation $\nabla$ of $E$ can be computed. Then $\mu_A \cong E$ is a closed domain iff all maximal simplexes of $\nabla$ are $n$-dimensional. This property is decidable. □

**Theorem 7.4.** Let $\sigma = (\hat{s}_1, \ldots, \hat{s}_n)$ be the $\mathbb{Z}$-retraction of $[0, 1]^n$ determined by the MV-terms $s_1, \ldots, s_n$. Let $P = R_\sigma$ be the range of $\sigma$, and $A = \text{gen}(\hat{s}_1, \ldots, \hat{s}_n)$ be the retract of $\mathcal{M}([0, 1]^n)$ associated to $\sigma$. If $P$ is a closed domain, the multiplicity $\rho(A)$ is computable from the input terms $s_1, \ldots, s_n$.

Proof. Given the input terms $s_1, \ldots, s_n$ the idempotency of $\sigma$ is decidable by Proposition 7.1, and so is the hypothesis that $P$ is a closed domain, by Proposition 7.3. Let us write $P = R_\sigma$. For any $\mathbb{Z}$-retraction $\tau = (\tau_1, \ldots, \tau_n)$ of $[0, 1]^n$ we have $\text{gen}(\tau_1, \ldots, \tau_n) = \text{gen}(\hat{s}_1, \ldots, \hat{s}_n)$ iff $\sigma \upharpoonright R_\sigma$ is a $\mathbb{Z}$-homeomorphism of $R_\tau$ onto $R_\sigma$. This is proved in Theorem 2.1. Let $\Delta$ be a regular triangulation of $[0, 1]^n$ that
Suppose the rational polyhedron
Claim 1. \( \nabla = \{ \sigma \in \Delta \mid \sigma \restriction S \text{ is a } \mathbb{Z}\text{-homeomorphism of } S \text{ onto } \sigma(S) \} \).

Also \( \nabla \) is computable from the input MV-terms \( s_i \). Let the subcomplex \( \nabla \subseteq \Delta \) of simplexes be defined by
\[
|\nabla| = \bigcup \{ S \mid S \in \nabla \}.
\]

Claim 1. Suppose the rational polyhedron \( Q \subseteq [0,1]^n \) satisfies \( \sigma \restriction Q : Q \cong Z P \). Then \( Q \subseteq |\nabla| \).

As a matter of fact, since \( P \) is a closed domain in \([0,1]^n\) then so is \( Q \). Fix \( x \in \text{int}(Q) \) together with an \( n \)-simplex \( S \in \Delta \) such that \( x \in S \). There is a rational simplex \( T \) satisfying \( T \subseteq Q \cap S \). From \( \sigma \restriction Q : Q \cong Z P \) we get \( \sigma \restriction T : T \cong Z \sigma(T) \). Since \( T \) and \( S \) are \( n \)-simplexes and \( \sigma \) is linear over \( S \) (because \( S \in \Delta \) and \( \Delta \) linearizes \( \sigma \)) then \( \sigma \restriction S : S \cong Z \sigma(S) \). We have thus shown that int(\( Q \)) is contained in \( |\nabla| \). Since \( Q \) is a closed domain and \( |\nabla| \) is closed then \( Q \) is contained in \( |\nabla| \), and our claim is settled.

We now strengthen Claim 1 as follows:

Claim 2. Suppose the rational polyhedron \( Q \subseteq [0,1]^n \) satisfies \( \sigma \restriction Q : Q \cong Z P \). Then \( Q = \bigcup \{ S \in \nabla \mid S \subseteq Q \} \).

To prove this claim, again fix \( x \in \text{int}(Q) \). By Claim 1 there is \( S \in \nabla \) with \( x \in S \). By way of contradiction suppose \( S \) is not contained in \( Q \). Then by Claim 2 in Theorem 2.3 (using the connectedness of \( \text{int}(S) \)) there is \( y \in \text{int}(S) \) satisfying \( y \in Q \cap \text{int}(Q) \). From \( \sigma \restriction Q : Q \cong Z P \) it follows that \( \sigma(y) \in P \cap \text{int}(P) \). From \( \sigma \restriction S : S \cong Z \sigma(S) \) it follows that \( \sigma(y) \in \text{int}(\sigma(S)) \subseteq \text{int}(P) \). This contradiction settles Claim 2.

To conclude the proof, for each subset \( S \) of \( \nabla \) only consisting of \( n \)-dimensional simplexes, it is decidable whether \( \sigma \restriction \bigcup S \) is a \( \mathbb{Z}\text{-homeomorphism of } \bigcup S \text{ onto } P \). Injectivity is equivalent to the following property: For any two distinct \( k \) simplexes \( V, W \in S \), from relint(\( V \)) \cap relint(\( W \)) = \( \emptyset \) it must follow that \( \sigma(\text{relint}(V)) \cap \sigma(\text{relint}(W)) = \emptyset \). This amounts to a routine linear algebra problem involving intersections of rational hyperplanes in \( \mathbb{R}^n \), once \( V \) and \( W \) are presented as intersections of rational hyperplanes—in an effective way as in [24, Lemma 18.1]. Once the injectivity of \( \sigma \restriction \bigcup S \) has been verified, we check surjectivity by computing the \( n \)-dimensional Lebesgue measure \( \lambda \) of the union of all \( n \)-dimensional simplexes in \( S \). This is computable because \( \Delta \) is a rational (actually, a regular) triangulation. We finally check that \( \lambda \) is equal to the measure of \( \bigcup \{ \sigma(T) \mid T \in S \} \). This, too, is computable, once the set \( \bigcup \{ \sigma(T) \mid T \in S \} \) has been equipped with a regular triangulation. In this way, some Turing machine can compute the set \( \Lambda = \Sigma_1, \ldots, \Sigma_w \) of all subsets \( S \) of \( \nabla \) such that \( \sigma \restriction \bigcup S \) is a \( \mathbb{Z}\text{-homeomorphism of } \bigcup S \text{ onto } P \). By Theorem 2.1, the number of elements in \( \Lambda \) coincides with the multiplicity of \( A, w = r(A) \).

\[ \square \]

Proposition 7.5. The following problem is decidable:

\textbf{Instance:} MV-terms \( t_1, \ldots, t_n \) such that the map \( \hat{t} : [0,1]^n \to [0,1]^n \) is idempotent, and the maximal spectral space \( \mu_A \) of \( A = \text{gen}(t_1, \ldots, t_n) \) is homeomorphic to a closed domain (both conditions being decidable, respectively by Proposition 7.1 and 7.3).

\textbf{Question:} Let \( \text{int}(\mu_A) \) denote the interior of \( \mu_A \). Is \( \text{int}(\mu_A) \) connected?
Theorem 7.8. The following problem is decidable:

 INSTANCE : MV-terms \( s_1, \ldots, s_n \) and \( t_1, \ldots, t_n \) providing \( \mathbb{Z} \)-retractions \( \hat{s}, \hat{t} \) of \([0,1]^n\).

 QUESTION : \( \mathbb{Z} \)-retractions have the same range?

 Proof. The ranges of \( \hat{s} \) and \( \hat{t} \) are computable from the input terms \( s_1, \ldots, s_n \) and \( t_1, \ldots, t_n \). It is decidable whether the two rational polyhedra \( R_s \) and \( R_t \) coincide, [24, Corollary 18.4].

 Proposition 7.6. The following problem is decidable:

 INSTANCE : MV-terms \( s_1, \ldots, s_n \) and \( t_1, \ldots, t_n \) providing \( \mathbb{Z} \)-retractions \( \hat{s}, \hat{t} \) of \([0,1]^n\) with the same range, (both assumption being decidable, by Propositions 7.1 and 7.6).

 QUESTION : Does the MV-algebra generated by \( \hat{s}_1, \ldots, \hat{s}_n \) coincide with the MV-algebra generated by \( \hat{t}_1, \ldots, \hat{t}_n \) ?

 Proof. By Proposition 6.4, the answer is positive answer iff \( \hat{s} = \hat{t} \). This in turn is equivalent to checking whether the MV-term \( s_i \leftrightarrow t_i \) is a tautology for all \( i = 1, \ldots, n \), which is a decidable problem, [12, Corollary 4.5.3].

 Dropping the hypothesis that \( \hat{s} \) and \( \hat{t} \) have the same range, the problem remains decidable, yet with a much subtler proof:

 Theorem 7.8. The following problem is decidable:

 INSTANCE : MV-terms \( s_1, \ldots, s_n \) and \( t_1, \ldots, t_n \) determining \( \mathbb{Z} \)-retractions \( \hat{s} \) and \( \hat{t} \) of \([0,1]^n\) with the same range, (a decidable condition, by Proposition 7.1).

 QUESTION : Does the MV-algebra \( A \) generated by \( \hat{s}_1, \ldots, \hat{s}_n \) coincide with the MV-algebra \( B \) generated by \( \hat{t}_1, \ldots, \hat{t}_n \) ?

 Proof. Let \( P = R_s \) be the range of \( \hat{s} \) and \( Q = R_t \) be the range of \( \hat{t} \). If \( P \) coincides with \( Q \) (a decidable condition, by Proposition 7.6) then Proposition 7.7 shows that the problem \( A = B \) is decidable. So it is sufficient to argue in case \( P \neq Q \). We have

 \[ A = B \iff \hat{s} \upharpoonright Q \text{ is a } \mathbb{Z} \text{-homeomorphism of } Q \text{ onto } P, \text{ and } \hat{t} = (\hat{s} \upharpoonright Q)^{-1} \circ \hat{s}. \]  

 (20)

 The \( \Rightarrow \)-direction is proved in Theorem 2.1. For the \( \Leftarrow \)-direction, the hypothesis shows that \( (\hat{s} \upharpoonright Q) \circ \hat{t} = \hat{s} \), whence \( A = \text{gen}(\hat{s}_1, \ldots, \hat{s}_n) = \text{gen}(\hat{t}_1, \ldots, \hat{t}_n) = B \).

 Next, in order to check the right-hand side of (20) we proceed as follows:

 (i) Using the effective procedure of [24, Corollary 2.9], we compute a regular triangulation \( \Lambda \) of \( Q \) such that \( \hat{s} \upharpoonright Q \) is linear over each simplex of \( \Lambda \). In the light of the characterization of \( \mathbb{Z} \)-homeomorphisms, [24, Proposition 3.15], we then check whether

 — each maximal simplex \( M \) of \( \Lambda \) is sent by \( \hat{s} \) onto a regular simplex \( \Lambda(M) \subseteq P \) with preservation of the denominators of the vertices of \( M \);
— the relative interiors of any two distinct simplexes $M', M''$ of $\Lambda$ are sent to disjoint simplexes $\Lambda(M'), \Lambda(M'')$;
— the $i$-dimensional rational measure $[25]$ of $\Lambda(Q)$ coincides with the $i$-dimensional rational measure of $Q$, for each $i = 0, 1, \ldots, n$.

(ii) The three conditions above are necessary and sufficient for $\hat{s} | Q$ to be a $\mathbb{Z}$-homeomorphism of $Q$ onto $P$.

(iii) Using the extension argument, [24, Theorem 5.8(ii)] it is easy to compute MV-terms $r_1, \ldots, r_n$ such that the $\mathbb{Z}$-map $\hat{r} = (\hat{r}_1, \ldots, \hat{r}_n)$ coincides with $(\hat{s} | Q)^{-1}$ over $P$.

(iv) The verification of the identity $\hat{t} = (\hat{s} | Q)^{-1} \circ \hat{s}$ now amounts to checking whether the MV-term $t_i \leftrightarrow r_i \circ (s_1, \ldots, s_n)$ is a tautology in Lukasiewicz logic for all $i = 1, \ldots, n$, which, as we have seen, is decidable.

The proof is complete.  

Replacing identity by isomorphism in the foregoing theorem we have an open problem:

**Problem 7.9.** The following problem is open:

**INSTANCE**: MV-terms $s_1, \ldots, s_n$, and $t_1, \ldots, t_n$ yielding $\mathbb{Z}$-retractions $\hat{s}$ and $\hat{t}$ of $[0, 1]^n$, (a decidable condition, by Proposition 7.1).

**QUESTION**: Is the subalgebra of $M([0, 1]^n)$ generated by $\hat{s}_1, \ldots, \hat{s}_n$ isomorphic to the subalgebra of $M([0, 1]^n)$ generated by $\hat{t}_1, \ldots, \hat{t}_n$?

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