THE IDEALS OF
FREE DIFFERENTIAL ALGEBRAS

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ABSTRACT. We consider the free $\mathbb{C}$-algebra $\mathcal{B}_q$ with $N$ generators $\{\xi_i\}_{i=1,\ldots,N}$, together with a set of $N$ differential operators $\{\partial_i\}_{i=1,\ldots,N}$ that act as twisted derivations on $\mathcal{B}_q$ according to the rule $\partial_i \xi_j = \delta_{ij} + q_{ij} \xi_j \partial_i$; that is, $\forall x \in \mathcal{B}_q$, $\partial_i (\xi_j x) = \delta_{ij} x + q_{ij} \xi_j \partial_i x$, and $\partial_i C = 0$. The suffix $q$ on $\mathcal{B}_q$ stands for $\{q_{ij}\}_{i,j=1,\ldots,N}$ and is interpreted as a point in parameter space, $q = \{q_{ij}\} \in \mathbb{C}^N$. A constant $C \in \mathcal{B}_q$ is a nontrivial element with the property $\partial_i C = 0$, $i = 1, \ldots, N$. To each point in parameter space there corresponds a unique set of constants and a differential complex. There are no constants when the parameters $q_{ij}$ are in general position. We obtain some precise results concerning the algebraic surfaces in parameter space on which constants exist. Let $\mathcal{I}_q$ denote the ideal generated by the constants. We relate the quotient algebras $\mathcal{B}'_q = \mathcal{B}_q / \mathcal{I}_q$ to Yang-Baxter algebras and, in particular, to quantized Kac-Moody algebras. The differential complex is a generalization of that of a quantized Kac-Moody algebra described in terms of Serre generators. Integrability conditions for $q$-differential equations are related to Hochschild cohomology. It is shown that $H^p(\mathcal{B}'_q, \mathcal{B}'_q) = 0$ for $p \geq 1$. The intimate relationship to generalized, quantized Kac-Moody algebras suggests an approach to the problem of classification of these algebras.
Introduction.

A recent study of the universal R-matrix of quantum groups led to the study of a type of free differential algebras. The positive Serre generators generate an algebra with certain relations, Drinfeld’s quantized Serre relations, and the action of the other generators can be expressed in terms of $q$-differentiation operators. But the Serre relations can be replaced by others, and wide generalizations are possible. This leads to the study of free differential algebras and their ideals. In this paper we obtain results that bear on the classification of ideals in quantum groups and Kac-Moody algebras, and on the classification of a new series of quantum Hopf algebras (generalized quantum groups).

To illustrate the type of generalization that is encompassed here, consider the generalized Cartan matrices

$$
\begin{pmatrix}
2 & -1 & n \\
-1 & 2 & -1 \\
 n & -1 & 2
\end{pmatrix}, \quad n = -1, 0, 2.
$$

For $n = 0$ it corresponds to a simple Lie algebra, and to a finite quantum group. For $n = -1$ it is the Cartan matrix of a Kac-Moody algebra of affine type, also quantizable. The quantized Serre ideals include generators of the type

$$
n = 0 : \ [e_1, e_3]_q := e_1 e_3 - q e_3 e_1,
$$

$$
n = -1 : \ [e_1, [e_1, e_3]_q]_{q'}.
$$

For $n = 2$, there is no generator of this type, constructed from $e_1$ and $e_3$, but instead there is a generator of a new type, namely

$$
n = 2 : \ [[e_2, e_1]_q, e_3]_{q'} + [[e_2, e_3]_{q'}, e_1].
$$

The parameters $q, q'$ tend to 1 in the “classical” limit. As this example shows, generalized quantized Kac-Moody algebras of indefinite type are characterized by unexpected Serre-type ideals.

Section 1 begins with the abstract definition of a family of free algebras with $q$-differential structure. The connection with quantum groups and universal R-matrices is reviewed in Section 2. We demonstrate the advantage of our methods by evaluating the highest root vectors for $U_q(A_l)$ (already found by Jimbo [J]) and for $U_q(C_l)$.

Let $B$ be the $C$-algebra freely generated by $\xi_1, \ldots, \xi_N$. A $q$-differential structure is a set of operators $\partial_1, \ldots, \partial_N$ that act on $B$ by the rule

$$
\partial_i(\xi_j x) = \partial_{ij} x + q_{ij} \xi_j \partial_i x, \quad \partial_i C = 0.
$$

This action involves a set $\{q_{ij}\}$ of complex parameters. We denote by $B_q$ the algebra $B$ endowed with the $q$-differential structure. An interesting property of $B_q$ is that, when the parameters are in general position, $q$-differential equations of the form $\partial_i x = y_i, i = 1, \ldots, N$, can be solved for $x$, for any choice of the “one-form” $\{y_i\}$; all one-forms are exact. Let us call “exceptional” those points in parameter space for which this is not true. These are precisely the same as those for which there exist homogeneous elements in $B_q$, of degree higher than zero, satisfying $\partial_i C = 0, i = 1, \ldots, N$; such elements are called constants. The constants in $B_q$ generate an ideal $I_q$ in $B_q$ and allows to define the quotient algebra $B'_q = B_q/I_q$. Quantized Kac-Moody algebras are a particular case that is the subject of Section 2. A main goal is the complete classification of all special points in parameter space; in the following sense. Two points $q$ and $q'$ in the space of
parameters are said to be equivalent if the ideals \( I_q \) and \( I_q' \) of \( B_q \) and \( B_q' \) coincide (as subalgebras of \( B \)). An alternative and probably more fundamental classification, where equivalence is instead based on isomorphism of the respective quotient algebras, is not contemplated in this paper. The complete classification of \( B_q' \)-algebras would provide, in particular, a partial classification of quantized Kac-Moody algebras of the most general type.

The existence of constants in \( B_q \) is revealed by the reduction in rank, at exceptional points in parameter space, of the matrix \( S \) defined by

\[
S_{i_1 \ldots i_p}^{j_1 \ldots j_p} := \partial_{i_p} \ldots \partial_{i_1} (\xi_{j_1} \ldots \xi_{j_p}).
\]

The projection of this matrix on \( B_q' \) is invertible and the components of the inverse matrix appear as coefficients of the expansion of the universal R-matrix in terms of Serre generators. Here we make contact with the work of Varchenko [V] on quantum groups.

An action of \( B_q' \) on \( B_q' \) is defined via the homomorphism that sends \( \xi_i \) to \( \partial_i \), \( i = 1, \ldots, N \).

It is shown that \( H^p(B_q', B_q') \), defined via this action, is zero for \( p \geq 1 \). The meaning of this result in the case of quantized Kac-Moody algebras is as follows. Let \( \hat{A}_+ \) be the algebra freely generated by \( \{e_i\}_{i=1,\ldots,N} \), identified with \( B_q \). Serre type ideals in \( \hat{A}_+ \) are generated by elements annihilated by the \( f_i \)'s; they are precisely the constants in \( B_q \). One can therefore draw the conclusion that the generalized, quantized Kac-Moody algebra associated to \( B_q' \) is rigid with respect to deformations that respect the Cartan decomposition.

Triviality of cohomology in this algebraic setting is not surprising. Interesting, nontrivial cohomology depends on completion. A particular type of completion is implied by the interpretation of the differential structure in terms of finite difference operators, possibly related to difference equations of the type studied by Smirnov [S][FR].

The advantages of the Serre formulation of (quantized) Kac-Moody algebras are obvious. It is natural to ask whether some aspects of the cohomology of these algebras (as generalized) can be formulated in terms of linear forms on the span of the Serre generators. We show, in Section 3.5, that the natural definition of “Serre cohomology” is in terms of multilinear \( p \)-cochains restricted to closed chains. For cochains valued in \( B_q' \) the differential is the projection on closed chains of the map that sends a \( p \)-cochain \( z \) to the \((p+1)\)-cochain \( dz \) according to the formula

\[
dz(\xi_{i_0} \otimes \xi_{i_1} \otimes \ldots \otimes \xi_{i_p}) = \partial_{i_0} z(\xi_{i_1} \otimes \ldots \otimes \xi_{i_p}).
\]

The cohomology is not trivial, but \( z \) is exact iff it is “strongly closed”; that is, iff

\[
\sum_{\hat{i}} C^\hat{i} \partial_{i_1} \ldots \partial_{i_k} = 0 \quad \Rightarrow \quad \sum_{\hat{i}} C^\hat{i} \partial_{i_1} \ldots \partial_{i_{k-1}} z(\xi_{i_k} \otimes \xi_{j_1} \otimes \ldots \otimes \xi_{j_p}) = 0.
\]

We must explain this statement. If there are coefficients \( C^\hat{i} \) such that

\[
\sum_{\hat{i}} C^\hat{i} \xi_{i_1} \ldots \xi_{i_k} := \sum_{i_1,\ldots,i_k} C^{i_1 \ldots i_k} \xi_{i_1} \ldots \xi_{i_k} \in \mathcal{B}_q
\]

is a constant, then

\[
\sum_{\hat{i}} C^\hat{i} \xi_{i_1} \ldots \xi_{i_k} = 0
\]

* Here, and in the formulas that follow, the summation is over all repeated indices, running independently over \( \{1,\ldots,N\} \). We use a multi-index notation where \( \hat{i} \) stands for \( i_1,\ldots,i_k \).
is a relation in $\mathcal{B}_q'$, and (as is shown in subsection 1.3.3) the operator

$$\sum_i C^i \partial_{i_1} \ldots \partial_{i_k}$$

is identically zero. The cohomology is nontrivial if there are irreducible constants of polynomial order higher than 2. For more details please turn to Section 3.5.

Section 4 contains results pertaining to the classification of the algebras $\mathcal{B}_q'$; more precisely, we try to determine the exceptional points in parameter space at which constants occur in $\mathcal{B}_q$. Let $Q = (1, \ldots, n)$ and $Q_i = (1 \ldots \hat{i} \ldots n)$. Let $\mathcal{B}_{1 \ldots n} = \mathcal{B}_Q$ be the space of polynomials linear in $\xi_1, \ldots, \xi_n$ separately. It is shown that, if the parameters associated with $Q_i$, $i = 1, \ldots, n$, namely $\{q_{jk}, \{j, k\} \subset Q_i\}$, are in general position; more precisely, if there are no constants in $\mathcal{B}_{Q_i}, i = 1, \ldots, n$, then constants exist in $\mathcal{B}_{1 \ldots n}$ if and only if

$$1 - \prod_{i \neq j \in Q} q_{ij} = 0 \ ,$$

and that the dimension of the subspace of constants in $\mathcal{B}_{1 \ldots n}$ is then $(n - 2)!$. Analogous results, for arbitrary sets $Q$ (with repetitions), will be reported elsewhere.

Physical applications of hyperbolic Kac-Moody algebras appear in connection with dimensional reduction of general relativity [N]. In other contexts it is interesting to look for finite-dimensional representations. It seems likely that finite representations of quantized Kac-Moody algebras of hyperbolic or more general type exist only for parameters at roots of unity. It is interesting to notice that all the constraints turn out to imply a factorization of unity. In this connection it may be productive to take another point of view. Instead of regarding the $q_{ij}$ as complex parameters, one may regard them as generators of a commutative algebra, and replace the field $\mathbb{C}$ by the ring of polynomials in $q_{ij}, i, j = 1, \ldots, n$. In this interpretation there is a unique algebra generated by $q_{ij}, \xi_i$, and the left hand side of Eq.(1) generates an ideal. The problem is then one of classification of certain ideals of a commutative algebra. Compare [V].

Completion of the work contained in Section 4 would go some way towards the classification of the algebras $\mathcal{B}_q'$. For more complete results we expect that geometrical methods, such as those of Varchenko [V], may be the most powerful. We suggest, in particular, that a study of the holonomy of the arrangement of surfaces defined by the exceptional points in parameter space may be useful and interesting.

Section 5 gives a complete account of constants in the subspace $\mathcal{B}_{123}$ of polynomials separately linear in three generators.

1. q-differential algebras.

In this section we present the principal players: the freely generated algebra $\mathcal{B}$ with its $q$-differential structure and the symmetric form $S$, with their most basic properties.

1.1. Free algebra and $q$-differential structure.

1.1.1. Let $\mathcal{B}$ denote the unital $\mathbb{C}$-algebra freely generated by $\xi_1, \ldots, \xi_N$, with its natural grading, $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}(n)$, where $\mathcal{B}(n)$ contains homogeneous polynomials of degree $n$. Suppose given a map $q : [N] \times [N] \to \mathbb{C}, \ (i, j) \mapsto q_{ij}, \ [N] := \{1, \ldots, N\}.$
The \( q_{ij} \)'s are the parameters; a choice of parameters will be interpreted as selecting a point in the space \( V = \mathbb{C}^{N^2} \).

1.1.2. For a fixed choice of parameters, let \( \partial_1, \ldots, \partial_N \) be the set of linear, \( q \)-differential operators

\[
\partial_i : \mathcal{B}(n) \to \mathcal{B}(n-1), \ n \geq 1, \ \mathcal{B}(0) \to 0, \ i = 1, \ldots, N,
\]

defined by

\[
\partial_i(\xi_j x) = \delta_{ij} x + q_{ij} \xi_j x, \ \forall x \in \mathcal{B}.
\]

In particular,

\[
\partial_i(\xi_i^r) = [r]_q, \ \xi_i^{-1} = [r]_q := 1 + q + \ldots + q^{r-1}.
\]

Let \( \mathcal{B}_q \) denote the algebra \( \mathcal{B} \) endowed with this \( q \)-differential structure. Thus \( \mathcal{B}_q \), as an algebra, is identified with \( \mathcal{B} \).

1.1.3. Let \( \hat{\mathcal{B}} \) denote the unital \( \mathbb{C} \)-algebra freely generated by \( \hat{\partial}_1, \ldots, \hat{\partial}_N \), with the natural grading. There is a unique homomorphism

\[
D_q : \hat{\mathcal{B}} \to \text{End} \mathcal{B},
\]

such that \( \hat{\partial}_i \mapsto \partial_i, i = 1, \ldots, N. \) Let \( \mathcal{B}_q^* \) denote the image of \( \hat{\mathcal{B}} \) in \( \text{End} \mathcal{B} \); it is the algebra of linear differential operators generated by \( \partial_1, \ldots, \partial_N \), but unlike \( \hat{\mathcal{B}} \) it is not always freely generated.

1.1.4. Example. If \( q_{12}q_{21} = 1 \), then \( (\partial_1 \partial_2 - q_{21} \partial_2 \partial_1)x = 0 \), for all \( x \in \mathcal{B}_q \), and hence \( \partial_1 \partial_2 - q_{21} \partial_2 \partial_1 = 0 \) is a relation of \( \mathcal{B}_q^* \). Proof. For all \( y \in \mathcal{B}_q \), if \( q_{12}q_{21} = 1 \), we have

\[
(\partial_1 \partial_2 - q_{21} \partial_2 \partial_1)(\xi_i y) = q_{i1}q_{21} \xi_i(\partial_1 \partial_2 - q_{21} \partial_2 \partial_1)y, \ i = 1, 2, \ldots, N.
\]

Hence for all homogeneous \( x \in \mathcal{B}_q \) there is \( f_x \in \mathbb{C} \) such that

\[
(\partial_1 \partial_2 - q_{21} \partial_2 \partial_1)x = f_x(x)(\partial_1 \partial_2 - q_{21} \partial_2 \partial_1)1 = 0.
\]

1.1.5. Example. If \( q_{11} \neq 1 \) is an \( n \)'th root of unity, then \( \partial_1^r x = 0 \) for all \( x \in \mathcal{B}_q \).

1.1.6. Theorem. For generic \( q \), \( D_q \) is an isomorphism. The exceptional values of \( q \), for which \( \ker D_q \neq \{0\} \), are determined by polynomial equations in \( q_{ii}, i = 1, \ldots, N \) and in \( \sigma_{ij} = q_{ij}q_{ji}, i \leq j = 1, \ldots, N. \) The proof is in 1.3.

1.2. Constants.

To prepare for the proof of the theorem we introduce the constants of \( \mathcal{B}_q \).

1.2.1. Definition. For a fixed set of parameters, let \( \mathcal{B}_q = \bigoplus_{n \geq 0} \mathcal{B}_q(n) \) denote the algebra \( \mathcal{B} \) endowed with the differential structure 1.1.2. A constant in \( \mathcal{B}_q \) is an element \( C \neq 0 \) in \( \bigoplus_{n \geq 1} \mathcal{B}_q(n) \) that satisfies \( \partial_i C = 0, i = 1, \ldots, N \). A homogeneous constant is a constant that is homogeneous in each generator.

Note the exclusion of \( \mathcal{B}(0) \). Every constant is a sum of homogeneous constants. The simplest examples are (1) if \( q_{11} = -1 \), then \( \xi_1^2 \) is a constant, (2) if \( q_{12}q_{21} = 1 \), then \( \xi_1 \xi_2 - q_{21} \xi_2 \xi_1 \) is a constant.
1.2.2. Lemma. For \( q \) in general position there are no constants in \( \mathcal{B}_q \). The set of exceptional points in \( V \) is determined by polynomial equations in the \( q_{ij}, i, j = 1, \ldots, N \).

Proof. It is enough to consider homogeneous constants. A homogeneous constant is an element

\[
C = \sum_{\mathcal{I}} X_{i_1}^{\mathcal{I}} \xi_{i_1} \cdots \xi_{i_n} \in \mathcal{B}_q(n),
\]

with complex coefficients \( X_i \), where the sum is over effective permutations of the indices. The condition \( \partial_i C = 0, i = 1, \ldots, N \) is equivalent to

\[
MC = 0, \quad M := \sum_{i=1}^N \xi_i \partial_i.
\]

On the monomial basis for \( \mathcal{B}_q(n) \), the operator \( M \) is a matrix with coefficients that are monomials in the \( q_{ij} \)'s. Nonzero solutions exist if and only if the determinant of this matrix is equal to zero. This condition is a polynomial equation in the \( q_{ij} \)'s.

1.2.3. Lemma. The exceptional set in 1.2.2 is determined by polynomial equations in \( q_{ii}, i = 1, \ldots, N \), and in \( \sigma_{ij} = q_{ij}q_{ji}, i \leq j = 1, \ldots, N \).

Proof. A change of monomial basis leads to

\[
C = \sum_{\mathcal{I}} Y_{i_1}^{\mathcal{I}} \prod_{i \leq j} (q_{ij}^{i_1}) \xi_{i_1} \cdots \xi_{i_n},
\]

where the product includes a factor \( q_{ij} \) for each occurrence of the pair \( (i, j), i < j \), in the order \( j \) before \( i \), in the index set \( \mathcal{I} = i_1, \ldots, i_n \). The conditions \( \partial_i C = 0 \) reduce to linear equations for the coefficients \( Y_{i_1}^{\mathcal{I}} \). After multiplying each equation by a monomial in the \( q_{ij} \)'s, one obtains a set of equations that involve only the \( q_{ii} \) and the \( \sigma_{ij} \)'s.

1.3. Proof of the theorem 1.1.6.

1.3.1. Fix the parameters \( q = \{q_{ij}\} \) and let \( \mathcal{B}_q \) denote the algebra \( \mathcal{B} \) endowed with the differential structure 1.1.2. Let \( \partial^t_i, i = 1, \ldots, N \), be another set of differential operators, defined in the same way but with \( q \) replaced by \( q^t \), where \( q^t_{ij} := q_{ji} \) for \( i, j = 1, \ldots, N \). Let \( \mathcal{B}_q^t \) denote the algebra \( \mathcal{B} \) endowed with this new differential structure.

1.3.2. Let \( \hat{\mathcal{B}}_q \) be the universal, unital algebra generated by \( \xi_1, \ldots, \xi_N, \hat{\partial}_1, \ldots, \hat{\partial}_N \), with relations*.

\[
\hat{\partial}_i \xi_j = \delta_{ij} + q_{ij} \xi_j \hat{\partial}_i,
\]

Let \( \hat{\mathcal{B}}_q^t \) be the same, but with \( q_{ij} \) replaced by \( q_{ji} \). There is a unique anti-isomorphism

\[
\Phi : \hat{\mathcal{B}}_q \to \hat{\mathcal{B}}_q^t,
\]

* This algebra appears in work of Lusztig [L] and Kashiwara [K], in the context of quantized Kac-Moody algebras.
such that $\xi_i \mapsto \hat{\partial}_i, \hat{\partial}_i \mapsto \xi_i, i = 1, ..., N$.

1.3.3. Now suppose that there is a homogeneous constant in $B_q$. Then $q = \{q_{ij}\}$ is exceptional, and so is $q' = \{q'_{ij}\}$, $q'_{ii} = q_{ii}$, by 1.2.3. Hence there is a homogeneous constant $C \in B_{q'}$. This implies that there are $f_i \in C, i = 1, ..., N$, such that

$$\hat{\partial}_i C = f_i C \hat{\partial}_i \in \tilde{B}B_{q'}.$$ 

Applying $\Phi^{-1}$ one gets

$$\Phi^{-1} (C) \xi_i = f_i \xi_i \Phi^{-1} (C) \in \tilde{B}B_q,$$

This implies that, $\forall x \in B_q$, homogeneous in each variable, there is $f_x \in C$ such that

$$\Phi^{-1} (C) x = f_x x \Phi^{-1} (C) \in \tilde{B}B_q;$$ 

hence $\Phi^{-1} (C) \neq 0$ belongs to $\text{Ker} D_q$. Conversely, any element of $\text{Ker} D_q$ is a sum of homogeneous elements. Let $C'$ be a homogeneous element of $\text{Ker} D_q$, of total degree $p$. There are complex coefficients $f'_i, i = 1, ..., N$, such that $C' \xi_i = \hat{C}'_i + f'_i \xi_i C'' \in \tilde{B}B_q$, with $\hat{C}'_i \in \text{Ker} D_q$ of total order $p - 1$. Iterating this by evaluating $\hat{C}'_i \xi_i$ and so on, one eventually obtains an element $\hat{C}' \in \text{Ker} D_q$, of order $m \geq 2$, (since there are no elements of order 1 in $\text{Ker} D_q$), such that $\hat{C}' \xi_i = f'_i \xi_i \hat{C}' \in \tilde{B}B_q$.

Applying $\Phi$ one obtains $\hat{\partial}^k \Phi (C') = f'_i \Phi (\hat{C}') \hat{\partial}^k_i$ and thus $\hat{\partial}^k \Phi (C') = 0$. By 1.2.3 it follows that there is $C \neq 0$ in $B_q(m)$ such that $\partial_i C = 0, i = 1, ..., N$. Hence $q$ is exceptional, in the sense of both lemmas, and the theorem is proved.

1.3.4. Comments. The proof of the direct part of the theorem demonstrates the existence of an monomorphism from the space of constants in $B_q$ into $\text{Ker} D_q$. In the proof of the converse, however, the reduction to elements of minimal degree is essential; there is no degree preserving vector space isomorphism between the two spaces.

1.3.5. Example. If $q_{12}q_{21} = 1$, then $A := \xi_1 \xi_2 - q_{21} \xi_2 \xi_1$ is a constant of $B_q$, and so is $A' := A \xi_2 = q_{31} q_{32} \xi_3 A$. Generically, the space of constants of degree 1 in each generator $\xi_1, \xi_2, \xi_3$ is $\mathbb{C}A'$. But $A := \hat{\partial}_1 \hat{\partial}_2 - q_{21} \hat{\partial}_2 \hat{\partial}_1$ belongs to $\text{Ker} D_q$ and therefore so do $\hat{A} \hat{\partial}_3$ and $\hat{\partial}_3 \hat{A}$. So $\text{Ker} D_q$ is bigger than the space of constants. We shall show that $\text{Ker} D_q$ is isomorphic to the ideal in $B_q$ generated by the constants.

1.4. More about constants.

The significance of the constants lies in their relation to problems of integrability.

1.4.1. Proposition. Fix $q$ and a positive integer $n$. The following statements are equivalent:

(a) There are no constants in $B_q(n)$.

(b) The equations

$$\partial_i x = y_i, \quad i = 1, ..., N,$$

have a solution $x \in B_q(n)$ for arbitrary $y_1, ..., y_N \in B_q(n - 1)$. In this case the solution is unique.

Proof. A straightforward extension of the proof of Lemma 1.2.2.
1.4.2. Proposition. Let $C$ be a homogeneous constant, and $x \in B_q$ any monomial of total degree $k$. There exists a constant of the form

$$C' = \sum_{m=0}^{k} \sum_{\mathcal{I}} \xi_{i_1} \cdots \xi_{i_m} a(\mathcal{I}) = Cx + \sum_{m=1}^{k} \sum_{\mathcal{I}} \xi_{i_1} \cdots \xi_{i_m} Ca(\mathcal{I}),$$

where $a(\mathcal{I})$ is a monomial of degree $k - m$.

Proof. There is a simple construction, using induction on the degree $k$ of $x$, that leads to a (generally unique) constant of this form.

1.4.3. Definition. Let $I_q$ denote the two-sided ideal of $B_q$ generated by the constants in $B_q$. Let $B'_q$ be the quotient algebra $B_q/I_q$, and $\pi$ the projection of $B_q$ on $B'_q$. Since $I_q$ is invariant under differentiation, there is a natural action of $\partial_i$ on $B'_q$, namely $\partial_i \pi x = \pi \partial_i x$. Thus $B'_q$ inherits the differential structure of $B_q$.

1.4.4. Let $\hat{J}$ be the isomorphism $B \to \hat{B}$ such that $\xi_i \mapsto \hat{\partial}_i$, and $J_q = D_q \circ \hat{J} : B_q \to B'_q$ the unique homomorphism that maps $\xi_i$ to $\partial_i$, $i = 1, \ldots, N$.

1.4.5. Theorem. The mapping $\hat{J} : B \to \hat{B}$ induces an isomorphism $I_q \to \ker D_q$, and $J_q$ induces an isomorphism $B'_q \to B'_q$. Hence $B'_q$ is the topological dual of $B'_q$.

The proof is in 1.6.

1.5. The symmetric form $S$.

This will prepare the way for a proof of theorem 1.4.5.

1.5.1. Definition. Denote by $S_q$ the 2-form on $B_q$ defined by *

$$S_q(x,y) = \left((J_q x)y\right)_0, \quad x, y \in B_q.$$

Here $(\cdot)_0$ is the projection $B_q \to B_q(0)$.

1.5.2. Proposition. The form $S_q$ is symmetric.

Proof. It is enough to consider the case when $x$ and $y$ are monomials of the same degree. Pairing the operator $\partial_\alpha$ with each $\xi_\gamma$ in turn we have

$$(... \partial_\gamma \partial_\beta \partial_\alpha \xi_\alpha \xi_\beta \xi_\gamma ...)_0 = \delta_{\alpha \alpha} (... \partial_\gamma \partial_\beta \xi_\alpha \xi_\beta \xi_\gamma ...)_0 + \delta_{\alpha \beta} q_{\alpha \alpha} (... \partial_\gamma \partial_\beta \xi_\alpha \xi_\beta \xi_\gamma ...)_0 + \delta_{\alpha \gamma} q_{\alpha \beta} \xi_\alpha \xi_\beta \xi_\gamma ...)_0 + \cdots \cdots .$$

A similar pairing of $\xi_\alpha$ with each $\partial_\gamma$ gives

$$(\partial_\gamma \partial_\beta \partial_\alpha \cdots \xi_\gamma \xi_\beta \xi_\alpha)_0 = \delta_{\alpha \alpha} (\partial_\gamma \partial_\beta \partial_\alpha \cdots \xi_\gamma \xi_\beta)_0 + \delta_{\alpha \beta} q_{\alpha \alpha} (\partial_\gamma \partial_\beta \partial_\alpha \cdots \xi_\gamma \xi_\beta)_0 + \delta_{\alpha \gamma} q_{\alpha \beta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\beta \partial_\alpha \cdots \xi_\gamma \xi_\beta)_0 + \cdots \cdots .$$

* This form was studied by Kashiwara [K], and by Varchenko; both in the context of Kac-Moody algebras. It appears in the present, wider context in a study of the standard universal R-matrix [F].
The result follows by induction on the degree.

1.6. Proof of the theorem 1.4.5.

1.6.1. Let \( x \in \mathcal{I}_q \), a sum of homogeneous polynomials each of which contains a constant factor. It was shown that \( \tilde{J} \) maps constants into \( \text{Ker}D_q \); it defines a monomorphism from \( \mathcal{I}_q \) into \( \text{Ker}D_q \).

1.6.2. Conversely, let \( x' \in \tilde{\mathcal{B}} \), then there is \( x \in \mathcal{B} \) such that \( x' = \tilde{J}x \). In particular, let \( \tilde{J}x \in \text{Ker}D_q \), homogeneous of degree \( k \). Then for all \( y \in \mathcal{B}_q \) of the same degree, \( (J_qy)x = 0 \), and by the symmetry of \( S \),

\[
(J_qy)x = 0, \quad \forall \tilde{J}x \in \text{Ker}D_q, \quad \deg(y) = \deg(x).
\]

It remains to be shown that this result implies that \( x \) belongs to \( \mathcal{I}_q \).

1.6.3. Lemma. (a) There are no constants in \( \mathcal{B}'_q \).
(b) If \( x \in \mathcal{B}_q \) is homogeneous of degree \( k \), and if \( x \) is annihilated by all differential operators of degree \( k \), then \( x \in \mathcal{I}_q \).
(c) Every \( x \in \mathcal{B} \) can be expressed as a “Taylor series”,

\[
x = c(x) + \sum_{n \geq 1} \sum_{\mathbf{i}} A_{\mathbf{i}} \partial_{i_1} \cdots \partial_{i_n} x, \quad A_{\mathbf{i}} = \sum_\sigma A^\sigma \xi_{i_{\sigma 1}} \cdots \xi_{i_{\sigma n}},
\]

where \( \partial_i c(x) = 0, i = 1, \ldots, N \); the second sum is over the permutations of \( 1, \ldots, n \), and \( A^\sigma \in \mathbb{C} \). (The coefficients are universal, independent of \( x \); they are calculated in 1.6.5 and examples are given in 1.6-7.)

Proof. (a) means that, if \( \partial_i \pi x = 0 \), then \( \pi x = 0 \); that is, if \( \pi \) annihilates the derivatives of \( x \), then \( \pi \) annihilates \( x \). Suppose that \( x \) is homogeneous of degree \( k \), and that all derivatives of order \( k \) are zero, \( \partial_{i_1} \cdots \partial_{i_n} x = 0 \). Then \( \pi \partial_{i_1} \cdots \partial_{i_n} x = \partial_{i_1} \cdots \partial_{i_n} \pi x = 0 \). Assume (a), then it follows that \( \pi x = 0 \). Hence (a) implies (b). On the other hand, (c) implies (a) (as is seen by applying \( \pi \) to both sides of the formula) so it is enough to prove (c).

To prove (c), it is enough to prove that the coefficients \( A_{\mathbf{i}} \) can be chosen so that both sides of the equation have the same first derivatives, which means that

\[
\sum_\mathbf{i} \left( \partial_{k} A_{i_1} \cdots i_n + \delta_{k,i_1} q_{i_1 i_2} \cdots q_{i_1 i_n} A_{i_2} \cdots i_n \right) \partial_{i_1} \cdots \partial_{i_n} = 0, \quad n = 1, 2, \ldots . \tag{1.1}
\]

Here \( A_{i_1} \cdots i_n \) is defined to be equal to \(-1\) for \( n = 0 \). This looks like a sequence of equations of the form \( \partial_i x = y_i \), to the solutions of which the presence of constants represents an obstruction. But the obstructions are, in fact, circumvented. Consider the operator that sends \( A \in \mathcal{B}(n) \) to the \( \mathcal{B}(1) \) one-form \( (\partial_k A)_{k=1, \ldots, N} \) valued in \( \mathcal{B}(n-1) \). On suitable bases, denote by \( M \) the associated square matrix. The obstructions to solving

\[
\partial_{k} A_{i_1} \cdots i_n + \delta_{k,i_1} q_{i_1 i_2} \cdots q_{i_1 i_n} A_{i_2} \cdots i_n = 0, \tag{1.2}
\]

are the constants in \( \mathcal{B}(n) \), the null vectors of \( M \). To the null space of \( M \) there corresponds the null space of the transposed matrix, of the same dimension. Indeed, the map \( J_q \) defined in 1.4.4 induces a natural bijection from one to the other. Let

\[
C = \sum_\mathbf{j} C^{j_1 \cdots j_n} \xi_{j_1} \cdots \xi_{j_n} = \sum_\mathbf{j} C_{\mathbf{j}}^{i} \xi_{j_1} \cdots \xi_{j_n}
\]
be a constant. Then

$$J_q C = \sum \frac{C^2}{j} \partial_{j_1} \ldots \partial_{j_n}$$

vanishes identically and this represents a null vector of the transposed of $M$:

$$\sum \left( C^2 \partial_{j_1} \ldots \partial_{j_{n-1}} \right) \partial_{j_n} A = 0. \quad (1.3)$$

Now assume that Eq.(1.1) can be solved for $n = 1, \ldots, m-1$, then (1.2) is valid for $n = 1, \ldots, m-1$; not identically, but as a substitution under $\sum \partial_{i_1} \ldots \partial_{i_n}$. Eq.(1.3) says that the one-form $(\partial_k A)$ is closed; the obstruction to solving Eq.(1.2) consists of the fact that the second term is not closed. However, using (1.2) for $n = 1, \ldots, m-1$, in the sense just explained, we find that

$$\sum \frac{C^2}{j} \partial_{j_1} \ldots \partial_{j_{n-1}} \partial_{j_n} \delta_{j_n, i_1} q_{i_1 i_2} \ldots q_{i_1 i_n} A^{i_2 \ldots i_n} \propto C^2 \approx 0.$$ 

Therefore, the obstructions to solving Eq.(1.2) do not affect Eq.(1.1), and we conclude that (1.1) is always solvable, giving a unique solution for $\sum \partial_{i_1} \ldots \partial_{i_n}$. The lemma is established by induction in $n$, and with that, Theorem 1.4.5 is proved.

1.6.4. Corollary. The radical of the form $S_q$ is the ideal $I_q$. By the projection $\pi : B_q \to B'_q$, we get a nondegenerate two-form on $B'_q$ that will also be denoted $S_q$; it can be interpreted as an invertible map $S_q : B'_q \to B^*_q$.

1.6.5. We can actually determine the coefficients $A^i$ explicitly. To this end let $q$ be in general position and set

$$A^{i_1 \ldots i_n} = (-)^{n+1} \prod_{k < l} q_{i_k i_l} T^{i_1 \ldots i_n},$$

Then the zero-rank tensor $T = 1$, and the recursion relation (2) reduces to

$$\partial_k T^{i_1 \ldots i_n} = \delta_{k, i_1} T^{i_2 \ldots i_n}.$$ 

Iteration gives

$$\partial_{k_n} \ldots \partial_{k_1} T^{i} = \delta_{k_1, i_1} \ldots \delta_{k_n, i_n}.$$ 

Setting

$$T^i = \sum \frac{T^i}{j} \delta_{j_1 \ldots j_n}$$ 

we obtain

$$\sum \frac{T^i}{j} S^i_k = \delta_{k_1, i_1} \ldots \delta_{k_n, i_n},$$

where the coefficients of the form $S$ are defined by

$$S^{i_1 \ldots i_n}_{j_1 \ldots j_n} = \partial_{j_1} \ldots \partial_{j_n} (\xi_{i_1} \ldots \xi_{i_n}).$$

Hence $T$ can be interpreted as the contragredient two-form, inverse to $S$. This interpretation survives at exceptional points in the space of parameters where $S$ is a non-degenerate two-form on $B'_q$. The coefficients $T^i_j$ are not unique, but $\sum_2 T^i_j \partial_{j_1} \ldots \partial_{j_n}$ is unique and so is $\pi \sum_2 T^i_j \delta_{j_1 \ldots j_n}$. 

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1.6.6. **Example.** Suppose $N = 1$. If $q := q_{11}$ is not a root of unity, then, for all $x \in \mathcal{B}_q$,

$$x = x_0 + \xi_1 \partial_1 x + \sum_{n \geq 2} (-1)^{n-1} q_{11}^{(n)} [n]_q^{1/2} \xi_1^n \partial_1^n x,$$

where $x_0$ is the projection of $x$ on $\mathcal{B}(0)$. When $q^m = 1$, $q \neq 1$, the expansion truncates at $n = m - 1$, and $x_0$ is replaced by $c(x)$, with $\partial_1 c(x) = 0$.

1.6.7. **Example.** Suppose $N$ arbitrary. If $q_{ij}$ are in general position, then, for all $x \in \mathcal{B}$,

$$x = x_0 + \sum_i \xi_i \partial_i x - \sum_i \frac{q_{ii}}{2 q_{i1}} \xi_i \partial_i^2 x - \sum_{i \neq j} \frac{q_{ij}}{1 - \sigma_{ij}} (\xi_i \xi_j - q_{ji} \xi_j \xi_i) \partial_i \partial_j x + \ldots,$$

where $x_0$ is the projection of $x$ on $\mathcal{B}(0)$. When there is the constraint $\sigma_{12} = 1$, the above expansion holds provided that $x_0$ is replaced by some $c(x)$, with $\partial_i c(x) = 0$, $i = 1, \ldots, N$. Terms that blow up due to the factor $(1 - \sigma_{12})$ in the denominators are replaced by expressions that can be obtained either solving (1) or from the generic expansion by a limiting procedure; thus

$$- \sum_{i \neq j = 1,2} \frac{q_{ij}}{1 - \sigma_{ij}} (\xi_i \xi_j - q_{ji} \xi_j \xi_i) \partial_i \partial_j x \mapsto (\alpha q_{12} \xi_1 \xi_2 + \beta \xi_2 \xi_1) \partial_1 \partial_2 x + (\gamma \xi_1 \xi_2 + \delta q_{21} \xi_2 \xi_1) \partial_1 \partial_2 x$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are arbitrary complex numbers satisfying $\alpha + \beta + \gamma + \delta + 1 = 0$.

Ambiguities in these parameters are absorbed into the constant term.

2. **Yang-Baxter algebras.**

Our interest now focuses (in this section only) on the exceptional values of the parameters and on the quotient algebras $\mathcal{B}'_q$. Here we shall connect all these algebras to *Yang-Baxter* algebras and some of them to quantized Kac-Moody algebras.

To every $\mathcal{B}'_q$ corresponds a Yang-Baxter algebra. This family of algebras includes the quantized Kac-Moody algebras and complete them in a natural way. They all admit a coboundary Hopf structure with a universal $R$-matrix of a standard form. (The Hopf structure is reviewed in [F].)

2.1. **Generators.**

The generators $(\xi_i, \partial_i)$ of $\mathcal{B}_q$ are related to the Chevalley-Serre generators $(e_i, f_i)$ of quantized Kac-Moody algebras. We first introduce the “Cartan subalgebra”.

2.1.1. Let $\hat{K}^i$ and $\hat{K}_i$ be the unique automorphisms of $\mathcal{B}_q$ such that

$$\hat{K}^i(\xi_j) = q_{ij} \xi_j, \quad \hat{K}_i(\xi_j) = \frac{1}{q_{ji}} \xi_j.$$

We expand the algebra $\mathcal{B}_q$ by including new generators $K^i, K_i, i = 1, \ldots, N$ that implement these automorphisms, thus

$$K^i \xi_j = q_{ij} \xi_j K^i, \quad K_i \xi_j = \frac{1}{q_{ji}} \xi_j K_i, \quad i, j = 1, \ldots, N.$$
2.1.2. Let \((\tilde{\partial}_i)_{i=1,...,N}\) be differential operators that act on \(x \in B_q\) from the right, such that
\[
x\xi_i \tilde{\partial}_j = x\delta_{ij} + xq_{ij} \tilde{\partial}_j \xi_i.
\]

2.1.3. Proposition. [F] The ideal in \(B_q\) that is generated by the constants with respect to the operators \(\partial_i\) coincides with the ideal generated by the constants with respect to the operators \(\tilde{\partial}_i\).

2.1.4. For any \(q = \{q_{ij}\}\), set
\[e_i = \xi_i, \quad f_i = \tilde{\partial}_i K_i - K^i \partial_i,
\]
then the following relations hold for \(i,j = 1,...,N\):
\[
[K_i, K_j] = [K_i, K^j] = [K^i, K^j] = 0,
K^i e_j = q_{ij} e_j K^i, \quad K_i e_j = (q_{ji})^{-1} e_j K_i,
K^i f_j = (q_{ij})^{-1} f_j K^i, \quad K_i f_j = q_{ji} f_j K_i,
[e_i, f_j] = \delta_{ij} (K^i - K_i), \quad i,j = 1,...,N.
\]

These are the relations of (the multiparameter version of) Drinfel’d’s quantization of Kac-Moody algebras, except (i) for the omission of Serre relations and (ii) certain conditions on \(q\) that we discuss next. Of course, (i) and (ii) are very closely connected.

2.2. Serre relations.

The ideals \(I_q\) of \(B_q\) that appear for exceptional values of the parameters have not yet been classified, but it is not difficult to find examples. We specialize, temporarily, to quantized Kac-Moody algebras.

Suppose that for each ordered pair \((i,j)\) \(i,j = 1,...,N\) there is a positive integer \(k_{ij}\) such that
\[
\sigma_{ij} q_{ii}^{k_{ij}-1} = 1, \quad \sigma_{ij} := q_{ij} q_{ji}.
\]

In this case the following elements of \(B_q\) are constants,
\[
\sum_{m=0}^{k_{ij}} Q_{km}^{ij} (\xi_i)^m (\xi_j)^{k_{ij}-m},
Q_{km}^{ij} := (-q_{ij})^m (q_{ii})^{m(m-1)/2} \binom{k_{ij}}{m} q_{ii}^{\binom{k_{ij}}{m}},
(\binom{m}{k})_q := \frac{[k]_q!}{[m]_q! [k-m]_q!}.
\]

and the image of each one by \(J_q\) is identically zero. The correspondence in 2.1.4, and passage to the quotient, now yields the quantized Kac-Moody algebra with Cartan matrix
\[
A_{ij} = 1 - k_{ij}.
\]
2.3. An alternative presentation.

2.3.1. Definition. Let $\mathcal{M}, \mathcal{N}$ be two finite sets, $\varphi, \psi$ two maps,

\[
\begin{align*}
\varphi &: \mathcal{M} \otimes \mathcal{M} \to \mathbb{C}, \quad a, b \mapsto \varphi_{ab}, \\
\psi &: \mathcal{M} \otimes \mathcal{N} \to \mathbb{C}, \quad a, i \mapsto H_a(i).
\end{align*}
\]

Let $\mathcal{A}$ or $\mathcal{A}(\varphi, \psi)$ be the universal, associative, unital algebra over $\mathbb{C}$ with generators $\{H_a\}_{a \in \mathcal{M}}, \{e_i, f_i\}_{i \in \mathcal{N}}$, and relations

\[
\begin{align*}
[H_a, H_b] &= 0, \\
[H_a, e_i] &= H_a(i) e_i, \quad [H_a, f_i] = -H_a(i) f_i, \\
[e_i, f_j] &= \delta_{ij} \left( e^\varphi(i, \cdot) - e^{-\varphi(\cdot, i)} \right),
\end{align*}
\]

with $\varphi(i, \cdot) = \sum_{a,b} \varphi_{ab} H_a(i) H_b$, $\varphi(\cdot, i) = \sum_{a,b} \varphi_{ab} H_a H_b(i)$, $\varphi(i, \cdot) + \varphi(\cdot, i) \neq 0$, $i \in \mathcal{N}$. (The last condition on the parameters is included in order to avoid having to make some rather trivial exceptions.) For $H \in \mathcal{A}$, $e^H$ is the formal series $e^H := \sum_n \frac{1}{n!} H^n$. We take $\mathcal{N} = \{1, \ldots, N\}$.

These relations imply those in 2.1.4 if we set

\[
K^i = e^{\varphi(i, \cdot)}, \quad K_i = e^{-\varphi(\cdot, i)},
\]

and choose $\varphi$ so that

\[
e^\varphi(i, j) = q_{ij}, \quad \varphi(i, j) := \sum_{a,b} \varphi_{ab} H_a(i) H_b(j).
\]

This alternative presentation is more cumbersome, but it seems to be necessary for the introduction of a Universal $R$-matrix.

2.3.2. The free subalgebra generated by $\{e_i\}_{i=1,\ldots,N}$ (resp. $\{f_i\}_{i=1,\ldots,N}$) will be denoted $\mathcal{A}^+$ (resp. $\mathcal{A}^-$). The Cartan subalgebra generated by $\{H_a\}_{a \in \mathcal{M}}$ is denoted $\mathcal{A}^0$. The passage to the quotient $\mathcal{B}'_q$ gives a quotient $\mathcal{A}'$ with subalgebras $\mathcal{A}'^\pm, \mathcal{A}'^0$.

2.3.3. Yang-Baxter element. (Universal $R$-matrix.)

The Universal $R$-matrix for $\mathcal{A}$ exists for parameters in general position and is given by $[F]$

\[
R = e^\varphi \sum_{n=0}^\infty t_n, \quad t_n = \sum_{\frac{1}{2}, \ldots} T_{ij...} f_{i_1}...f_{i_n} \otimes e_{j_1}...e_{j_n}.
\]

The “Cartan potential”,

\[
e^\varphi := e^{\sum_{a,b} \varphi_{ab} H_a \otimes H_b},
\]

satisfies the following relations

\[
e^\varphi (e_i \otimes 1) = (e_i \otimes K^i) e^\varphi, \quad (1 \otimes e_i) e^\varphi = e^\varphi (K_i \otimes e_i).
\]

For exceptional values of the parameters the expression for $R$ makes sense in $\mathcal{A}' \otimes \mathcal{A}'$, and gives the standard Universal $R$-matrix for the quotient algebra $\mathcal{A}'$. 

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2.3.4. Remark. For every set of parameters \( q_{ij}, q_{ii} \neq 1, 1 = 1, \ldots, N \), there are \( A_{ij} \in \mathbb{C} \) such that \( \sigma_{ij} = q_{ij}^{A_{ij}}, i \neq j \), \( A_{ii} = 2 \), and such that the matrix \( (A_{ij}) \) is symmetrizable. Any such matrix may be called the Cartan matrix of \( B_q \).

2.4. Highest root vectors and quantized loop algebras.

2.4.1. We ask whether the quantized enveloping algebra of a finite dimensional Lie algebra, of standard (multidimensional) type, admits a highest root generator; that is, an element \( E \in U_q(\mathfrak{g}) \), with the same weight relatively to the Cartan subalgebra as the highest root in \( \mathfrak{g} \), and a set of complex coefficients \( a_1, \ldots, a_l \) \((l \text{ is the rank of } \mathfrak{g})\) such that

\[
\sigma_{ij} = q_{A_{ij} ii}, i \neq j, \quad A_{ii} = 2, \quad \text{and such that the matrix } (A_{ij}) \text{ is symmetrizable. Any such matrix may be called the Cartan matrix of } B_q.
\]

An affirmative answer is given for \( g \) in the series \( A_l \) and \( C_l \); and a negative one for \( B_3, G_2 \).

Proofs are included for the case of \( C_l \) only. When \( E \) exists, let \( F \) be the corresponding negative root generator. Let \( U_z q(\mathfrak{g}) = U_q(\mathfrak{g}) \otimes \mathbb{C}[z, 1/z] \); this algebra, generated by the set of generators of \( U_q(\mathfrak{g}) \) augmented by \( e_0 := zF \) and \( f_0 := z^{-1} E \), satisfies all the relations of a quantized Kac-Moody algebra of affine type except, possibly, for the relation \([e_0, f_0] = e^{H(0)} - e^{-H(0)}\).

2.4.2. The case of \( A_l \).

The constraints on the parameters are

\[
\sigma_{ij} = 1, \quad |i - j| > 1, \quad \sigma_{ij} := q_{ij} q_{ji}, \quad i \neq j,
\]

\[
\sigma_{ij} q_{jj} = 1, \quad |i - j| = 1, \quad i, j = 1, \ldots, \ell,
\]

and the ideal \( I_q \) is generated by

\[
\xi_i \xi_j - q_{ij} \xi_j \xi_i, \quad |i - j| > 1, \quad i, j = 1, \ldots, \ell,
\]

\[
C_{ij} := \frac{1}{q_{ij}} \xi_i^2 \xi_j - [2]_{q_i} \xi_i \xi_j \xi_i + \frac{1}{q_{ij}} \xi_j \xi_i^2, \quad |i - j| = 1.
\]

Following Jimbo [J], we introduce a sequence of “root vectors”

\[
X^1 := \xi_1, \quad X^n = [X^{n-1}, \xi_n]_{a_n}, \quad n = 2, \ldots, \ell,
\]

\[
a_n = q_{n} \ldots q_{n-n+2}/q_{n-1,n}.
\]

Proposition 2.4.3. Let \( X^0 = 1 \), then

\[
\partial_i X^n \propto \delta_{in} X^{n-1}, \quad i, n = 1, \ldots, \ell,
\]

\[
[X^n, \xi_i]_{q_{i1} \ldots q_{in}} = 0, \quad i = n + 2, \ldots, \ell.
\]

The constants of proportionality are \( \neq 0 \).

Proposition 2.4.4. With

\[
a(i, n) = q_{i1} \ldots q_{in}, \quad i \neq n, n + 1,
\]

\[
a(n, n) = q_{n1} \ldots q_{n,n-1}, \quad n \neq 1, \quad a(1, 1) = 1,
\]

\[
a(n, n - 1) = a_n = q_{n1} \ldots q_{n,n-2}/q_{n-1,n},
\]

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one has for \( n, i = 1, \ldots, \ell \),
\[
[X^n, \xi_i]_{a(i, n)} = \delta_{i, n+1} X^{n+1}.
\]  \hfill (2.7)

**Corollary 2.4.5.** The element \( X^\ell \) is a highest root vector for \( A_\ell \).

**2.4.6. The case of \( B_2, B_3 \).**

Here, we had intended to present our study of \( B_\ell \), but when this case turned out to be more
difficult we scaled back our ambition and attacked \( B_2 \) and \( B_3 \) instead. In the first case there
is a unique (up to normalization) highest weight generator. The unexpected negative result for
\( B_3 \) is stated as Theorem 2.4.8, below.

We begin with \( B_2 \). The data encoded in the Cartan matrix is \( q_{11} = q_{22}^2 = 1/\sigma_{12} \). The long
simple root corresponds to \( \xi_1 \) and \( E \in B_{122} \); that is, \( E \) is of first order in \( \xi_1 \) and of second order
in \( \xi_2 \). The ideal \( \mathcal{I}_q \) is generated by \( C_{112} \), Eq. (2.3), and
\[
C_{2221} := \frac{1}{q_{12}} \xi_2^3 \xi_1 - [3]_{q_{22}} \xi_2^2 \xi_1 \xi_2 + [3]_{q_{22}} q_{12} q_{22} \xi_2 \xi_1 \xi_2^2 - q_{12}^2 q_{22} \xi_1 \xi_2^3.
\]  \hfill (2.8)

As the space of constants in \( B_{1222} \) is 1-dimensional, we may set \( w_2 := [E, \xi_2]_{a_2} \propto C_{2221} \); that is,
we look for an element \( E \) such that
\[
C_{2221} \propto E \xi_2 - a_2 \xi_2 E.
\]

There are exactly 3 (linearly independent) solutions of this equation. They are
\[
E = \xi_1 \xi_2^2 - q_{21} q_{22} (1 + q_{22}) \xi_2 \xi_1 \xi_2 + q_{21}^2 q_{22}^2 \xi_1^2, \quad a_2 = q_{21},
\]
\[
E = \xi_1 \xi_2^2 - q_{21} (1 + q_{11}) \xi_2 \xi_1 \xi_2 + q_{21}^2 q_{22} \xi_1^2, \quad a_2 = q_{21} q_{22},
\]
\[
E = \xi_1 \xi_2^2 - q_{21} (1 + q_{22}) \xi_2 \xi_1 \xi_2 + q_{21}^2 q_{22} \xi_1^2, \quad a_2 = q_{21} q_{22}.
\]

The task is finished if there is an element in \( \mathcal{I}_q \) of the form \( w_1 := E \xi_1 - a_1 \xi_1 E \). It is easy
to see that \( \partial_1 w_1 = 0 \) requires that \( \partial_1 E = 0 \). The first two solutions satisfy this requirement.
There remains to satisfy \( \partial_2 w_1 = 0 \), and this makes the solution unique:
\[
E = C_{221} = [\xi_1, \xi_2]_{q_{21}, \xi_2}^{q_{21}, \xi_1}, \quad a_1 = q_{12}^2 q_{22}, \quad a_2 = q_{21} q_{22}.
\]

Note that \( C_{221} \) is not a constant in this case.

On to \( B_3 \).

**Remark 2.4.7.** A convenient basis for \( B_3 \sim so(7) \) may be found in [J]. If \( E_{ij} \) is the \( 7 \times 7 \) matrix
that has 1 in position \((i, j)\) and 0's elsewhere, then a system of simple root generators is
\[
e_1 = E_{12} - E_{67}, \quad e_2 = E_{23} - E_{56}, \quad e_3 = E_{34} - E_{45}.
\]

The highest root is \( r_1 + 2r_2 + 2r_3 \), and \( r_3 \) is the short root. The highest root generator can be
expressed in several different ways, but if it is written as \([A, e_i], A \in B_3\), then always \( i = 2 \),
\[
E = [A, e_2].
\]  \hfill (2.9)
In the quantized case, we would need an element \( E \in \mathcal{B}'_{12233} \); that is, a sum of permutations of \( \xi_1 \xi_2 \xi_3 \xi_4 \), and constants \( a_1, a_2, a_3 \), such that
\[
w_i := [E, \xi_i]_{a_i} = 0, \quad i = 1, 2, 3.
\] (We do not assume that \( E \) has the form \([A, \xi_2]_a\) suggested by (2.9).)

**Theorem 2.4.8.** There is no element \( E \not= 0 \) in \( \mathcal{B}'_{12233} \) that satisfies (2.10), and that tends to the highest root generator of \( B_3 \) in the Lie limit.

**2.4.9. The case of \( C_\ell \).**

The highest root of \( C_\ell \), in terms of the simple roots, is \( 2r_1 + \ldots + 2r_{\ell-1} + r_\ell; \) where \( r_\ell \) is the long root. To the sequence
\[
1, 2, \ldots, (l-1), l, (l-1), \ldots, 2, 1
\] we associate a sequence \( X^1, \ldots, X^l, X^{l+1}, \ldots, X^{2l-1} \) of root vectors defined as follows,
\[
X^1 = \xi_1, \quad X^n = [X^{n-1}, \xi_n]_{a_n}, \quad n = 2, \ldots, \ell, \quad X^{\ell+n} = [X^{\ell+n-1}, \xi_{\ell-n}]_{b_n}, \quad n = 1, \ldots, \ell - 1,
\]
with coefficients \( a_n, b_n \) to be chosen.

The constraints are
\[
\sigma_{ij} = 1, \quad |i-j| > 1,
\]
\[
\sigma_{ij} q_{ii} = 1, \quad |i-j| = 1, \quad i, j = 1, \ldots, \ell - 1,
\]
\[
\sigma_{\ell, \ell-1} q_{\ell \ell} = 1, \quad q_{11} = \ldots = q_{\ell-1, \ell-1} =: q, \quad q_{\ell \ell} = q^2.
\]

The ideal is generated by (2.2), except that \( C_{\ell-1, \ell-1, \ell} \) is replaced by \( C_{\ell-1, \ell-1, \ell-1, \ell} \) defined in (2.8).

We take over the definition (2.4) of \( X^1, \ldots, X^\ell \), and the relations (2.5) still hold. Of the relations (2.7), all but one remains valid; \([X^\ell, \xi_{\ell-1}]_{a(\ell-1, \ell)} \) is no longer zero because the constraint \( \sigma_{\ell, \ell-1} q_{\ell-1, \ell-1} = 1 \) no longer applies, since \( r_{\ell-1} \) is short. Thus
\[
[X^n, \xi_i]_{a(i, n)} = \delta_{i, n+1} X^{n+1}, \quad n, i = 1, \ldots, l,
\]
except for \( n = l, \quad i = l - 1 \).

This invites us to construct the sequence (2.13), the properties of which we shall now explore.

**Proposition 2.4.10.** For \( n = 1, \ldots, \ell - 1 \), define
\[
X^{\ell+n} = [X^{\ell+n-1}, \xi_{\ell-n}]_{b_n},
\]
\[
b_n = q_{\ell-1, n-1} \cdots q_{\ell-1, n} q_{\ell-1, \ell-1} \cdots q_{\ell-1, l} =: q_{\ell-1, n} \cdots q_{\ell-1, \ell-n}.
\]

Then
\[
\partial_i X^{\ell+n} \propto \delta_{i, \ell-n} X^{\ell+n-1}, \quad n = 1, \ldots, \ell - 1, \quad i = 1, \ldots, l.
\]

**Proof.** For \( n = 1 \) we have
\[
\partial_i X^{\ell+1} = \partial_i [X^\ell, \xi_{\ell-1}]_{b_1} = [\partial_i X^\ell, \xi_{\ell-1}]_{b_1 q_{i, \ell-1}} + \delta_i^{\ell-1} (q_{i1} \cdots q_{i \ell} - b_1) X^\ell
\]
This is zero for \( i = 1, \ldots, \ell - 2 \). With the help of (2.5) one gets
\[
\partial_i X^{\ell+1} \propto [X^{\ell-1}, \xi_{\ell-1}] b_{1q_{\ell-1}}.
\]
The constraint \( q_{\ell-1}^2 q_{\ell-1} q_{\ell-1} \xi_{\ell-1} = 1 \) makes
\[
b_{1q_{\ell-1}} = q_{\ell-1,1} \cdots q_{\ell-1,\ell-2} = a(\ell - 1, \ell - 1),
\]
and
\[
\partial_i X^{\ell+1} \propto [X^{\ell-1}, \xi_{\ell-1}] a(\ell - 1, \ell - 1) = 0.
\]
Finally, \( \partial_{\ell-1} X^{\ell+1} \propto X^{\ell} \), as required. This establishes a base for induction in \( n \). Suppose that the statement of the proposition is true for \( n = 1, \ldots, m \); we have for \( n = 1, \ldots, \ell - 1, \ i = 1, \ldots, \ell \),
\[
\partial_i X^{\ell+m+1} = \partial_i [X^{\ell+m}, \xi_{\ell-m-1}] b_{m+1}
\]
\[
= [\partial_i X^{\ell+m}, \xi_{\ell-m-1}] b_{m+1} + \delta_{i}^{\ell-m-1} (q_{1,1}, \ldots, q_{i,\ell-m} - b_{m+1}) X^{\ell+m}.
\]
By hypothesis this is zero for \( i \neq \ell - m - 1, \ell - m \), while
\[
\partial_{\ell-m} X^{\ell+m+1} = [\partial_{\ell-m} X^{\ell+m}, \xi_{\ell-m-1}] b_{m+1} q_{\ell-m, \ell-m-1}
\]
\[
\propto [X^{\ell+m-1}, \xi_{\ell-m-1}] b_{m+1} q_{\ell-m, \ell-m-1} \tag{2.18}
\]
For \( m = 0 \) this was already seen to vanish. For \( m = 1 \) it is zero because of (2.15) and
\[
b_{2q_{\ell-1}, \ell-2} = q_{\ell-2,1} \cdots q_{\ell-2,\ell} = a(\ell - 2, \ell).
\]
For \( m > 1 \) we again invoke the principle according to which (2.18) vanishes if and only all its derivatives vanish. In fact,
\[
\partial_i [X^{\ell+m-1}, \xi_{\ell-m-1}] b_{m+1} q_{\ell-m, \ell-m-1} \tag{2.19}
\]
vanishes for \( i \neq \ell - m \pm 1 \) by the induction hypothesis, so we have to prove that it vanishes for \( i = \ell - m \pm 1 \) as well. First, taking \( i = \ell - m - 1 \),
\[
\partial_{\ell-m-1} [X^{\ell+m-1}, \xi_{\ell-m-1}] b_{m+1} q_{\ell-m, \ell-m-1}
\]
\[
= (q_{\ell-m-1,1} \cdots q_{\ell-m-1,\ell-m+1} - b_{m+1} q_{\ell-m, \ell-m-1}) X^{\ell+m-1} = 0,
\]
by virtue of the constraint \( \sigma_{\ell-m-1, \ell-m-1} q_{\ell-m-1, \ell-m-1} = 1 \), \( m > 0 \). For \( i = \ell - m + 1 \), (2.19) becomes
\[
\partial_{\ell-m+1} [X^{\ell+m-1}, \xi_{\ell-m-1}] b_{m+1} q_{\ell-m, \ell-m-1}
\]
\[
\propto [X^{\ell+m-2}, \xi_{\ell-m-1}] b_{m+1} q_{\ell-m, \ell-m-1} q_{\ell-m+1, \ell-m-1} \tag{2.20}
\]
For \( m = 2 \) it is
\[
[X^\ell, \xi_{\ell-3}] q_{\ell-3,1} \cdots q_{\ell-3,\ell} q_{\ell-3, \ell-1} q_{\ell-3, \ell-2} q_{\ell-3, \ell-3} q_{\ell-3, \ell-4} q_{\ell-3, \ell-5} q_{\ell-3, \ell-6} = [X^\ell, \xi_{\ell-3}] a(\ell-3, \ell) = 0.
\]
Now (2.20) looks like a shifted form of (2.18) and this suggests to repeat the steps that led from one to the other. Thus, to show that (2.20) is zero, it is enough to verify that the expression is
annihilated by $\partial_{\ell-m+2}$ and by $\partial_{\ell-m-1}$. That $\partial_{\ell-m-1}$ gives zero is obvious since this operator commutes with $\partial_{\ell-m+1}$. (Two operators $A, B$ commute if there is $\alpha$ in the field such that $[A, B]_\alpha := AB - \alpha BA = 0$.) So it is enough to consider
\[
\partial_{\ell-m+2}[X^{\ell+m-2}, \xi_{\ell-m-1}]_{b_{m+1}q_{\ell-m,\ell-m-1}q_{\ell-m+1,\ell-m-1}} \propto [X^{\ell+m-3}, \xi_{\ell-m-1}]_{b_{m+1}q_{\ell-m,\ell-m-1}q_{\ell-m+1,\ell-m-1}q_{\ell-m+2,\ell-m-1}}.
\]
This vanishes for $m = 3$ since the coefficient is then
\[
q_{\ell-4,1} \cdots q_{\ell-4,\ell} q_{\ell-4,\ell-1} \cdots q_{\ell-4,\ell-4}/q_{\ell-4,\ell-4} q_{\ell-4,\ell-3} q_{\ell-4,\ell-2} q_{\ell-4,\ell}, \text{ etc.}
\]
\[
= a(\ell - 4, \ell).
\]
The pattern is clear, after $m - 1$ iterations we end up with $[X^{\ell}, \xi_{\ell-m-1}]_{a(\ell, \ell-m-1)} = 0$. The theorem is proved.

We collect all the relations obtained so far, and some new ones.

2.4.11. Properties of root vectors. (a) From (2.15) and (2.16), now valid for $i, n = 1, \ldots, \ell$ except for $n = \ell, i = \ell - 1$:
\[
[X^n, \xi_{i}]_{a(i,n)} = \delta_{i}^{\ell+1} X^{n+1},
\]
where $a(i, n)$ is as in (2.6).

(b) From (2.13),
\[
[X^{\ell+n-1}, \xi_{\ell-n}]_{b_n} = X^{\ell+n}, \quad n = 1, \ldots, \ell - 1,
\]
where $b_n$ is as in (2.16).

(c) For $m = 0, \ldots, \ell - 1, n = 2, 3, \ldots, 2m$,
\[
[X^{\ell+m-n}, \xi_{\ell-m}]_{b(m,n)} = 0,
\]
with $b(m, n) = q_{\ell-m,1} \cdots q_{\ell-m,\ell-m+n}$. See the proof of Proposition 2.4.10.

(d) For $m = 0, \ldots, \ell - 1, n = 0, \ldots, \ell - m - 1$,
\[
[X^{\ell+m+n}, \xi_{\ell-m}]_{c(m,n)} = 0,
\]
with
\[
c(m, n) = q_{\ell-m,1} \cdots q_{\ell-m,\ell-m-n}, \quad n \neq 0
\]
\[
c(m, 0) = q_{\ell-m,1} \cdots q_{\ell-m,\ell-m+1}.
\]

Proof of (d). The special case $m = n = 0$ is included in Part (a), and we check that $c(0, 0) = a(\ell, \ell)$. We shall verify that all the derivatives of (2.24) vanish. First, when $n = 0$, there is only one nontrivial case, namely
\[
\partial_{\ell-m}[X^{\ell+m}, \xi_{\ell-m}]_{c(m,0)} = \partial_{\ell-m} X^{\ell+m}, \xi_{\ell-m}]_{c(m,0)} q_{\ell-m,\ell-m} \\
+ (q_{\ell-m,1} \cdots q_{\ell-m,\ell-m} - c(m,0)) X^{\ell+m}.
\]
To show that this vanishes we calculate
\[
\partial_{\ell-m} \partial_{\ell-m}[X^{\ell+m}, \xi_{\ell-m}]_{c(m,0)} \\
= (1 + q_{\ell-m,\ell-m})(q_{\ell-1,1} \cdots q_{\ell-m,\ell-m+1} - c(m,0)) X^{\ell+m} = 0,
\]
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\[
\partial_{t^{-m+1}} \partial_{t^{-m}} [X^t, \xi_{t^{-m}}]_{c(m,0)} \\
\times \partial_{t^{-m+1}} [X^{t^{-m+1}}, \xi_{t^{-m}}]_{c(m,0)q_{t^{-m+1}}^{-1}} \propto [X^{t^{-m+2}}, \xi_{t^{-m}}]_{b(m,2)} = 0.
\]

The statement is thus true for \( n = 0 \). For \( n \neq 0 \) there are two items,
\[
\partial_{t^{-m}} [X^{t^{-m+n}}, \xi_{t^{-m}}]_{c(m,n)} = ((Q_{t^{-m}}, ..., Q_{t^{-m+n}} - c(m,n)) X^{t^{-m+n}} = 0,
\]
\[
\partial_{t^{-m+n}} [X^{t^{-m+n}}, \xi_{t^{-m}}]_{c(m,n)} \propto [X^{t^{-m+n-1}}, \xi_{t^{-m}}]_{c(m,n)q_{t^{-m+n}}, t^{-m}}.
\]

We note that \( c(m,n)q_{t^{-m+n}}, t^{-m} = c(m,n - 1) \). When \( n \neq 1 \) this depends on the constraint \( \sigma_{t^{-m}, t^{-m-n}} = 1 \). When \( n = 1 \), it makes use of \( \sigma_{t^{-m}, t^{-m-1}} q_{t^{-m}, t^{-m}} = 1 \), always valid. The validity of (2.24) follows by induction on \( n \).

Corollary 2.4.12. The element \( X^{2t-1} \) is a highest root vector for \( C_t \).

3. Cohomology.

To each point \( q \) in parameter space there corresponds a free differential algebra \( B_q \), an ideal \( \mathcal{I}_q \) generated by the irreducible constants in \( B_q \), and a quotient algebra \( B'_q \leq B_q/\mathcal{I}_q \). In this section we shall introduce a differential complex generated by the constants. The first three subsections are tentative, exploratory and motivational. The reader may prefer to skip them.

3.1. The \( q \)-differential complex of a quantum plane.

Until further notice, a one-form is a map from \( B_q(1) \) to \( B_q \) or to \( B'_q \).

3.1.1. Recall the notation \( \sigma_{t} = q_{ij}q_{ji}, i, j = 1, ..., N \). Suppose first that \( \sigma_{t} = 1, i \neq j = 1, ..., N \), and that these are the only constraints. Then the polynomials

\[
C_{ij} = \sum_{k,l} C_{i,j}^{k,l} \xi_k \xi_l = \xi_i \xi_j - q_{ij} \xi_j \xi_i, \quad i \neq j = 1, ..., N,
\]

are constants and the operators \( D_q \hat{C}_{ij} \) are identically zero,

\[
\hat{C}_{ij} := \partial_i \partial_j - q_{ij} \partial_j \partial_i, \quad D_q \hat{C}_{ij} = \partial_i \partial_j - q_{ij} \partial_j \partial_i = 0, \quad i \neq j = 1, ..., N.
\]

A one-form \( y = (y_1, ..., y_N) \) on \( B_q(1) \) is called exact if there is \( x \in B_q \) such that \( y_i = \partial_i x, i = 1, ..., N \), and it is said to be closed if

\[
C_{ij}(y) := \sum_{k,l} C_{i,j}^{k,l} \partial_k y_l = \partial_i y_j - q_{ij} \partial_j y_i = 0, \quad i \neq j = 1, ..., N.
\]

It is usual to look upon the collection \( \{C_{ij}(y)\} \) as the components of an exact 2-form, but that is a point of view that does not have a natural generalization. Instead, we shall say that a two-form \( z = (z_{ij}), z_{ij} = z(\xi_i, \xi_j), \) is exact if there is a 1-form \( y \) such that \( \sum_{k,l} C_{ij}^{k,l} \xi_k z_{kl} = C_{ij}(y), i \neq j = 1, ..., N \), and that it is closed if for all complex coefficients \( C_{ij}^{k,j} \), such that \( \sum_{i,j} C_{ij}^{k,j} \xi_i \xi_j = 0 \), \( \sum_{i,j,k} C_{ij}^{k,j} \partial \xi_k z_{ik} = 0 \). It is fairly clear how this development can be completed to a \( q \)-differential complex on quantum planes.

3.1.2. We consider the case when there is just one condition on the parameters, \( \sigma_{ij} = 1 \), and just one constant \( C_{12} = \sum_{k,l} C_{12}^{k,l} \xi_k \xi_l = \xi_1 \xi_2 - q_{21} \xi_2 \xi_1 \). Then it makes sense to say that \( y \) is
closed if $C_{12}(y) = \partial_1 y_2 - q_{21} \partial_2 y_1 = 0$, with no conditions on the other components, and that $z$ is exact if $\sum_{k,l} C_{12}^{kl} z_{kl} = C_{12}(y)$. One feels that, in this case as well, an associated differential complex is lurking.

3.2. The q-differential complex of a quantized Kac-Moody algebra.

3.2.1 The Serre constraints that define the quantization of $A_2$ are

$$\sigma_{12} q_{11} = \sigma_{12} q_{22} = 1.$$  

The ideal $I_q$ is generated by two constants in $B_q(3)$,

$$C_{112} = \frac{1}{q_{12}} \xi_2 \xi_1^2 - (1 + q_{11}) \xi_1 \xi_2 \xi_1 + \frac{1}{q_{21}} \xi_1^2 \xi_2,$$

and $C_{221}$ defined analogously. Define

$$\hat{C}_{112} := \frac{1}{q_{12}} \hat{\partial}_2 \hat{\partial}_1^2 - (1 + q_{11}) \hat{\partial}_1 \hat{\partial}_2 \hat{\partial}_1 + \frac{1}{q_{21}} \hat{\partial}_1^2 \hat{\partial}_2.$$

then the differential operator $D_q \hat{C}_{112}$ vanishes identically. A one-form $y = (y_1, ..., y_N)$ may be called exact if $y_i = \partial_i x, i = 1, ..., N$ and it is natural to say that it is strongly closed if it satisfies

$$C_{112}(y) := \sum_{i,j,k} C_{112}^{ijk} \partial_i \partial_j y_k = \frac{1}{q_{12}} \partial_2 \partial_1 y_1 - (1 + q_{11}) \partial_1 \partial_2 y_1 + \frac{1}{q_{21}} \partial_1^2 y_2 = 0,$$

as well as $C_{221}(y) = 0$.

3.3. q-differential complex in general.

3.3.1. If $C \in B_q$ is a constant, then $\hat{C} := J^C \in \text{Ker} D_q$ and $C(y)$ is obtained from the latter by replacing the right-most $\hat{\partial}_i$ by $y_i$ and the other $\hat{\partial}_i$’s by $\partial_i$’s operating on $y_i$.

3.3.2. Definition. Proposition. A one-form $y$ is said to be exact if there is $x \in B_q$ such that $y_i = \partial_i x, i = 1, ..., N$, and it is called closed (strongly closed) if, $\forall C \in B_q(2)$ ($\forall C \in B_q$), constant, $C(y) = 0$. Every exact one-form is strongly closed, and therefore closed, and every strongly closed one-form is exact.

Proof. Only the last statement needs justification. It is a corollary of Lemma 1.6.3, part (c). For consider the expression

$$x := \sum_{n \geq 1} \sum_{i} A^i_1 \partial_i \cdots \partial_{i_{n-1}} y_{i_n}.$$  

Taking the derivatives of both sides of this equation we get

$$\partial_k x - y_k = \sum_{i} (\partial_k A^{i_1 \cdots i_n} + \delta_k^{i_1} q_{i_1 i_2} \cdots q_{i_{n-1} i_n} A^{i_2 \cdots i_n}) \partial_{i_1} \cdots \partial_{i_{n-1}} y_{i_n}.$$  

This is equal to zero if $y$ is strongly closed, as can be checked by consulting subsection 1.6.3.
3.3.3. Definition. A homogeneous constant (a constant that is a sum of reorderings of a monomial, Definition 1.2.1) is irreducible if it does not belong to the ideal generated by constants of lower order. A reducible constant is a sum of polynomials of total degree \( k \) each of which contains a constant factor of lower degree.

It is clear that the ideal \( I_q \) generated by all the constants is generated by the irreducible constants. It is easy to check that a one-form \( y \) is strongly closed iff \( \forall C \in B_q, C \) an irreducible constant, \( C(y) = 0 \).

3.3.4. Before proceeding to complete the construction of a differential complex in the general case we present an example of a different kind, beyond the context of Kac-Moody algebras. Suppose at first that the parameters satisfy the condition \( \sigma_{123} := \sigma_{12}\sigma_{13}\sigma_{23} = 1 \), but are otherwise in general position. Then there is just one irreducible constant, and it can be written as follows

\[
C_{123} = \frac{1}{q_{12}} \left( \xi_2 (\xi_3 \xi_1 + \sigma_{12} q_{13} \xi_1 \xi_3) - q_{32} q_{12} (\xi_3 \xi_1 + \sigma_{12} q_{13} \xi_1 \xi_3) \xi_2 \right) + \text{cyclic.}
\]

Now suppose that

\[
\sigma_{12}\sigma_{23}\sigma_{13} =: \sigma_{123} = \sigma_{124} = \sigma_{134} = \sigma_{234} = 1.
\]

This implies that either \( \sigma_{ij} = \sigma_{kl} \) or else \( \sigma_{ij} = -\sigma_{kl} \), for \( i,j,k,l \) all different. Now there are 4 irreducible constants. One can verify that, if \( \sigma_{ij} = \sigma_{kl} \), then there is an identity

\[
\hat{\partial}_4 \hat{C}_{123} + q_{14} q_{13} q_{34} \hat{\partial}_1 \hat{C}_{234} + \frac{q_{21} q_{13}}{q_{42} q_{43}} \hat{\partial}_2 \hat{C}_{134} + q_{32} q_{34} q_{24} \hat{\partial}_3 \hat{C}_{124} - \frac{q_{24}}{q_{31}} \hat{C}_{412} \hat{\partial}_3 - \frac{q_{34}}{q_{12}} \hat{C}_{423} \hat{\partial}_1 - q_{13} q_{32} q_{34} \hat{C}_{413} \hat{\partial}_2 - q_{14} q_{24} q_{34} \hat{C}_{123} \hat{\partial}_4 = 0.
\]

Consequently, for any one-form \( y \),

\[
\partial_4 C_{123}(y) + q_{14} q_{13} q_{34} \partial_1 C_{234}(y) + \frac{q_{21} q_{13}}{q_{42} q_{43}} \partial_2 C_{134}(y) + q_{32} q_{34} q_{24} \partial_3 C_{124}(y) = 0,
\]

or, if \( (dy)_i = C_{jkl}(y) \) for each cyclic permutation \( i,j,k,l \) of \( \{1,2,3,4\} \),

\[
\partial_4(dy)_4 + q_{14} q_{13} q_{34} \partial_1(dy)_1 + \frac{q_{21} q_{13}}{q_{42} q_{43}} \partial_2(dy)_2 + q_{32} q_{34} q_{24} \partial_3(dy)_3 = 0,
\]

The equations \( dy_i = z_i, i = 1,2,3,4 \) are not integrable unless the two-form \( z \) satisfies

\[
\partial_4 z_4 + q_{14} q_{13} q_{34} \partial_1 z_1 + \frac{q_{21} q_{13}}{q_{42} q_{43}} \partial_2 z_2 + q_{32} q_{34} q_{24} \partial_3 z_3 = 0;
\]

we may ask whether this condition is sufficient to guarantee that \( z \) can be expressed as \( dy \). Complete answers to all these problems of integrability will now be found within a study of the Hochschild cohomology of \( B'_q \).

3.4. Hochschild complex.

Here we shall see that the idea of constructing a differential complex based on \( B'_q \) can be realized within the setting of the ordinary Hochschild complex of \( B'_q \).
3.4.1. Let $\mathcal{E}$ be an associative $\mathbf{C}$-algebra. On $p$-chains $a_1 \otimes \ldots \otimes a_p \in C_p(\mathcal{E}), a_i \in \mathcal{E}, p \geq 1$, define a linear boundary operator $\partial : C_p \to C_{p-1}, p \geq 2$, and $C_1 \to 0$, by

$$\partial(a_1 \otimes \ldots \otimes a_p) = \sum_{i=1}^{p-1} (-)^i (a_1 \otimes a_2 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes \ldots \otimes a_p).$$

One has $\partial \circ \partial = 0$.

3.4.2. Examples.

$$\partial(a_1) = 0,$$
$$\partial(a_1 \otimes a_2) = a_1 a_2,$$
$$\partial(a_1 \otimes a_2 \otimes a_3) = (a_1 a_2 \otimes a_3) - (a_1 \otimes a_2 a_3).$$

For $a \in C_p(\mathcal{E})$ we use the notation

$$a = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \otimes \ldots \otimes a_{(p)} = \sum a_{(1)} \otimes a_{(2, \ldots, p)}.$$

The formula in 3.4.1 applies with indices in parentheses and summation, for example $\partial(\sum a_{(1)} \otimes a_{(2)}) = \sum a_{(1)} a_{(2)}$. The summation sign will usually be omitted.

3.4.3. Proposition. The $\mathbf{C}$-algebra $\mathcal{B}(+) = \sum_{n \geq 1} \mathcal{B}(n)$, freely generated by $\xi_1, \ldots, \xi_N$, has homology quotients $H_1(\mathcal{B}(+)) \sim \mathcal{B}(1)$ and $H_p(\mathcal{B}(+)) = 0, p \geq 2$.

Proof. All 1-chains are closed, and all homogeneous 1-chains of degree higher than 1 are of the form $a = \sum_i \xi_i a^i = \partial(\sum_i \xi_i \otimes a^i)$, so $H_1(\mathcal{B}(+)) \sim \mathcal{B}(1)$. Every $a \in C_{p+1}(\mathcal{B}(+)), p \geq 1$,

$$a = a_{(0)} \otimes a_{(1)} \otimes \ldots \otimes a_{(p)}, \quad a_{(i)} \in \mathcal{B}(+), \quad i = 0, 1, \ldots, p,$$

is homologous to a chain of the form

$$\hat{a} = \sum_{i=1}^N \xi_i \otimes b^i \in C_{p+1}(\mathcal{B}(+)), \quad b^i = b_{(1)}^i \otimes \ldots \otimes b_{(p)}^i \in C_p(\mathcal{B}(+)).$$

Hence $\partial a = \partial \hat{a} = \partial(\sum_i \xi_i \otimes b^i) = \sum_i (\xi_i b_{(1)}^i) \otimes b_{(2)}^i \otimes \ldots \otimes b_{(p)}^i - \sum_i \xi_i \otimes \partial b^i$. The degree of $\xi_i b_{(1)}^i$ is greater than 1; therefore, if $\partial a = 0$, then $\xi_i b_{(1)}^i = 0$. Since there are no relations in $\mathcal{B}$, $b^i = 0$ and $a$ is homologous to zero.

As a corollary of the proof we have

3.4.4. Proposition. One has $H_1(\mathcal{B}_q'(+) \sim \mathcal{B}(1)$ and $H_p(\mathcal{B}_q'(+) \sim \{\sum_i \xi_i \otimes b^i \in Z_p(\mathcal{B}_q')\}, p \geq 2$.

3.4.5. We introduce the spaces $C^p$ of linear maps from $C_p$ to a left $\mathcal{E}$-module $M$ and set $C^0 = M$. The action of $a \in \mathcal{E}$ in $M$ will be denoted $\pi(a)$. A linear coboundary operator $d : C^{p-1} \to C^p$ is defined by

$$d \tau(a) = \pi(a_1) \tau(a_2 \otimes \ldots \otimes a_p) - \tau(\partial a), \quad a = a_1 \otimes \ldots \otimes a_p.$$ 

One has $d \circ d = 0$. 

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3.4.6. **Examples.**

\[ d\tau(a_1) = \pi(a_1)\tau, \]
\[ d\tau(a_1 \otimes a_2) = \pi(a_1)\tau(a_2) - \tau(a_1a_2), \]
\[ d\tau(a_1 \otimes a_2 \otimes a_3) = \pi(a_1)\tau(a_2 \otimes a_3) - \tau(a_1a_2 \otimes a_3) + \tau(a_1 \otimes a_2a_3). \]

The linear extension takes the form

\[ a \in C^1, \quad d\tau(a) = \pi(a)\tau, \]
\[ a \in C^2, \quad d\tau(a) = \sum \pi(a_{(1)})\tau(a_{(2)}) - \tau(\partial a), \]
\[ a \in C^3, \quad d\tau(a) = \sum \pi(a_{(1)})\tau(a_{(23)}) - \tau(\partial a). \]

The homomorphism \( J_q : B_q \to B_q^\prime \), generated by \( \xi_i \mapsto \pi(\xi_i) = \partial_i \), provides a new action of \( B_q \) on \( B_q \) and on \( B_q^\prime \). From now on we take \( M = B_q \) or \( M = B_q^\prime \) and write \( \pi(a) \) for this action.

In the next proposition \( M = B_q \).

3.4.7. **Proposition.** The Hochschild cohomology of \( B_q^\prime(+):= \bigoplus_{n \geq 1} B_q(n) \), with values in \( B_q \), vanishes except that \( H^0(B_q^\prime(+), B_q) \) is the linear span of \( B(0) \) with the space of constants in \( B_q \).

**Proof.** For \( p \geq 1 \), let \( z \) be a closed \( p \)-cochain; we try to find a \((p - 1)\)-cochain \( y \) such that

\[ z(a) = dy(a) = \pi(a_{(1)})y(a_{(2 \ldots p)}) - y(\partial a). \]

When \( p = 1 \) the existence of \( y \) is assured by Proposition 3.3.2, so from now on suppose that \( p > 1 \). We interpret the equation, recursively in the degree, as the definition of the last term, where the argument \( \partial a \) has the highest degree. The obstruction is \( \partial a = 0, a \neq 0 \). If all \( a_{(i)}, i = 1, \ldots, p \) are of degree 1 then, since there are no relations, \( \partial a = 0 \) implies that \( a = 0 \); this establishes the basis for the recursion. In general, when \( \partial a = 0 \), by Proposition 3.4.3, there is a \((p + 1)\)-chain \( b \) such that \( a = \partial b \) and we need to satisfy

\[ z(\partial b) = \pi(b_{(0)}b_{(1)})y(b_{(2 \ldots p)}) - \pi(b_{(0)})y(\partial b_{(1 \ldots p)}), \]

Since \( z \) is closed, \( z(\partial b) = \pi(b_{(0)})z(b_{(1 \ldots p)}) \), this holds if

\[ z(b_{(1 \ldots p)}) = \pi(b_{(1)})y(b_{(2 \ldots p)}) - y(\partial b_{(1 \ldots p)}). \]

This throws the solution of the equation \( z(a) = dy(a) \) back on the solution of the same equation with \( a \) replaced by \( p \)-chains of lower degree. The existence of a base for the recursion was established, and this completes the proof of the theorem.

3.4.8. **Theorem.** The Hochschild cohomology of \( B_q^\prime(+):= \bigoplus_{n \geq 1} B_q^\prime(n) \), with values in \( B_q \) or in \( B_q^\prime \), vanishes except that \( H^0(B_q^\prime(+), B_q) \) is the union of \( B(0) \) and the space of constants and \( H^0(B_q^\prime(+), B_q^\prime) = B(0) = C \).

**Proof.** We begin as in the proof of 3.4.7, but \( \partial a = 0 \) no longer implies that \( a \) is exact. By Proposition 3.4.4 every closed \( p \)-chain is homologous to one of the form \( a = \xi_i \otimes b^\prime \), so we have to show that, when \( z \) is closed, there is \( y \) such that

\[ z(\xi_i \otimes b) = \partial_i y(b) - y(\partial(\xi_i \otimes b)). \]
We need a lemma.

3.4.9. Lemma. Let \( \{ x_\alpha \} \) be any finite collection of homogeneous elements of \( B'_q \), all of the same degree in each variable, satisfying a linear functional relation \( \sum_\alpha A^\alpha(x_\alpha) = 0 \). Then there is a family \( \{ \Pi^\alpha \} \) of differential operators, such that \( \sum_\alpha A^\alpha(x_\alpha) = \sum_\alpha \Pi^\alpha x_\alpha \).

Proof. This is just the statement that the algebra \( B^*_q \) of differential operators is the algebraic dual of \( B'_q \), see Theorem 1.4.5.

3.4.10. Proof of the Theorem. To settle the integrability of Eq.(3.1), note that it is a linear relation in a finite dimensional vector space. Given the left side, (3.1) has a solution \( y \) if and only if the left side satisfies all linear functional relations that hold for the right hand side identically in \( y \). By the lemma, such relations take the form

\[
\sum_\alpha \pi(c^{i,\alpha}) \left[ \partial_i y(b_\alpha) - y(\partial(\xi_i \otimes b_\alpha)) \right] = 0,
\]

where \( \{ b_\alpha \} \) is a family of \((p - 1)\)-chains and \( \{ c^{i,\alpha} \} \) is a family of elements of \( B'_q \). Since this is required to hold identically for all \( y \), both terms must vanish separately,

\[
\sum_\alpha \pi(c^{i,\alpha}) y(b_\alpha) = 0, \quad \sum_\alpha \pi(c^{i,\alpha}) y(\partial(\xi_i \otimes b_\alpha)) = 0.
\]

The second relation is equivalent to

\[
\sum_\alpha c^{i,\alpha} \otimes \partial(\xi_i \otimes b_\alpha) = 0;
\]

the first relation is satisfied for all \( y \) if and only if the differential operators are 0; hence (3.1) is integrable if and only if

\[
\sum_\alpha \pi(c^{i,\alpha}) z(\xi_i \otimes b_\alpha) = 0,
\]

for all families \( \{ b_\alpha \} \) and all \( \{ c^{i,\alpha} \} \), such that \( \sum_\alpha c^{i,\alpha} \otimes \partial(\xi_i \otimes b_\alpha) = 0 \) and \( \sum_i c^{i,\alpha} \xi_i = 0 \); in other words for all \( \{ b_\alpha, c^{i,\alpha} \} \) such that \( \sum_\alpha c^{i,\alpha} \otimes \xi_i \otimes b_\alpha \) is closed. But that is true if \( z \) is closed. The theorem is proved.

3.5 Serre cohomology.

3.5.1. A zero-cochain \( x \in C^0(B'_q, B'_q) \) is an element \( x \) of \( B'_q \), it is exact only if \( x = 0 \), and it is closed if \( \pi(a)x = 0, \forall a \in B'_q \), which (because there are no constants in \( B'_q \)) is true iff \( x \in B(0) = C \).

3.5.2. A one-cochain \( y \) on \( B'_q \) is exact if there is \( x \in B'_q \) such that \( \forall a \in B'_q \),

\[
y(a) = \pi(a)x.
\]

Let \( y_1 \) denote the restriction of \( y \) to \( B_q(1) \), then if \( y \) is exact we have

\[
y_1(\xi_i) = \partial_i x,
\]

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in which case we say that \( y_1 \) is exact. Conversely, if \( y_1 \) is an exact one-form on \( B_q(1) \), then there is a unique, exact one-cochain \( y \) on \( B'_q \) that restricts to \( y_1 \) on \( B_q(1) \). *

### 3.5.3

A one-cochain \( y \) on \( B'_q \) is closed if \( \forall a, b \in B'_q \):

\[
dy(a \otimes b) = \pi(a) y(b) - y(ab) = 0 .
\]

This implies that, if \( c = \sum_i a_i \xi_i \), then

\[
\sum_i \pi(a_i) y(\xi_i) = y(c) ,
\]

and in particular that

\[
c = \sum_i a_i \xi_i = 0 \Rightarrow c(y_1) := \sum_i \pi(a_i) y_1(\xi_i) = 0 .
\]

Compare 3.1.1. Conversely, if \( y_1 \) satisfies (3.6), then a unique, closed one-cochain \( y \) on \( B'_q \) is determined by solving (3.4), recursively in the degree of \( ab \).

**Definition 3.5.4.** A \( p \)-form \( z_1 \) on \( B_q(1) \), with values in \( B'_q \), is exact if there is a \( (p - 1) \)-form \( y_1 \) on \( B_q(1) \) such that

\[
z_1(a) = \pi(a(1)) y_1(a(2) \otimes \ldots \otimes a(p)) , \quad \forall a = a(1) \otimes \ldots \otimes a(p) , \quad \partial a = 0 .
\]

It is closed (strongly closed) if

\[
\pi(a(0)) z_1(a(1) \otimes \ldots \otimes a(p)) = 0 , \quad \forall a = a(0) \otimes a(1) \otimes \ldots \otimes a(p) , \quad \partial a = 0 ,
\]

where \( a(1), \ldots, a(p) \in B_q(1) \) in both cases, and \( a(0) \in B_q(1) \) (\( a(0) \in B'_q \)).

#### 3.5.5

It is obvious that, if \( z \) is an exact \( p \)-cochain on \( B'_q \), then its restriction \( z_1 \) is an exact \( p \)-form on \( B_q(1) \). Conversely, if \( z_1 \) is an exact \( p \)-form on \( B_q(1) \), expressed in terms of a \( (p - 1) \)-form \( y_1 \) as in (3.7), let \( y \) be any \( (p - 1) \)-cochain on \( B'_q \) that extends \( y_1 \); then the formula \( z = dy \) extends \( z_1 \) to an exact \( p \)-cochain on \( B'_q \).

#### 3.5.6

If \( z \) is a closed \( p \)-cochain on \( B'_q \) then its restriction \( z_1 \) is a strongly closed \( p \)-form on \( B_q(1) \). Conversely, if \( z_1 \) is a strongly closed \( p \)-form on \( B_q(1) \), then it extends to a closed \( p \)-cochain on \( B'_q \). To show this consider the condition

\[
dz(a) := \pi(a(0)) z(a(1) \otimes \ldots \otimes a(p)) - z(\partial a) = 0
\]

for \( a = a(0) \otimes a(1) \otimes \ldots \otimes a(p) \), with \( \deg(a(i)) \geq 1, i = 1, \ldots, p \) and of total polynomial degree

\[
n(a) = \sum_{i=0}^p \deg(a(i)) \geq p + 1 .
\]

* There are no constants in \( B_q(1) \); hence \( B'_q(1) = B_q(1) \).
When $n(a) = p + 1$, this condition determines $z(\partial a)$ in terms of $z_1$; this amounts to a partial determination of $z(b)$ for $n(b) = p + 2$. Proceed recursively in the total degree $n(a)$. Eq. (3.9) is solvable for $n(a) = p + 1$. Suppose it can be solved for $n(a) = p + 1, p + 2, \ldots, p + k$, and let $n(a) = m = p + k + 1$. Then $\partial a$ is of order $m + 1$, and (3.9) can be solved for $z(\partial a)$ provided

$$\pi(a(0))z(a(1) \otimes \ldots \otimes a(p)) = 0, \quad \forall a = a(0) \otimes \ldots \otimes a(p), \quad \partial a = 0. \quad (3.10)$$

The apparent obstruction to (3.10) comes from the fact that $z(...)$ on the left hand side is already known for the case that the argument is exact. Setting this argument equal to $\partial b$, and using (3.9) for lower degrees, we reduce the left hand side of (3.10) to $\pi(a(0)b(0))z(b(1 \ldots p))$. But this is the same as $z(\partial a)$ and hence zero by induction in the degree of $a(1 \ldots p)$.

We conclude as follows.

**Theorem 3.5.7.** Let $D$ denote the differential complex of which the cochains are $B_q(1)$-forms restricted to closed chains, and the differential is given, for cochains valued in $B'_q$ by the formula

$$dz(a(0) \otimes a(1) \otimes \ldots \otimes a(p)) = \pi(a(0))z(a(1) \otimes \ldots \otimes a(p)) . \quad (3.11)$$

The cohomology is nontrivial if $B_q$ admits irreducible constants* of order higher than 2. If

$$C = \sum_I C^I_\xi \xi_{i_1} \ldots \xi_{i_k}$$

is a constant, then

$$\sum_I C^I_\xi \partial_{i_1} \ldots \partial_{i_k} = 0,$$

and $z$ is exact iff $z$ is strongly closed; that is, iff for all constants $C$,

$$\sum_I C^I_\xi \partial_{i_1} \ldots \partial_{i_{k-1}} z(\xi_{i_k} \otimes \xi_{j_1} \otimes \ldots \otimes \xi_{j_p}) = 0,$$

(If $x$ is a closed 0-form, then $\partial_i x = 0$, so $x \in B_0 = C$, and no 0-cochain is exact, so $H^0 = C$.)

The restriction to closed chains is very natural; in the ($q$-) commutative case it restricts the cochains to be ($q$-) alternating. But if there are no constants (in $B_q$) of order two, then the restriction is moot.

**4. Classification of Constraints.**

**4.1. The Constants in $B_{1 \ldots n}$.**

4.1.1. From now on we drop the suffix $q$ on $B$ and denote by $B_n$ the space of polynomials of order $n$. For any subset $s = \{i_1, \ldots, i_{|s|}\} \subset \{1, \ldots, n\}$, with cardinality $|s|$, denote by $B_s(\xi)$ the subspace of $B_{|s|}$ consisting of all polynomials of degree $|s|$ that are separately linear in $\xi_{i_1}, \ldots, \xi_{i_{|s|}}$, and let

$$\sigma(s) = \prod_{i \neq j \in s} q_{ij} . \quad (4.1)$$

* See Definition 3.3.3.
4.1.2. **Theorem.** Assume that there are no constants in $B_s$, for any proper subset $s \subset \{1, \ldots, m\}$. Then the dimension of the space of constants in $B_{1\ldots m}$ is

\[(m - 2)! \quad \text{if } \sigma_{1\ldots m} = 1,
\]
\[0 \quad \text{otherwise.} \quad (4.2)\]

**Proof.** By induction. The statement is true for $m = 2$. We assume that it is true for $m = 2, \ldots, n - 1$ and prove that it is true for $m = n$. Since there are no constants in $B_s$ and the statement is true for $m = 2, \ldots, n - 1$ it follows that $\sigma(s) \neq 1$, for any proper subset $s$ of $1\ldots n$.

4.1.3. Let $X = \sum X(1\ldots n) \xi_1 \ldots \xi_n$ (sum over all permutations of $12\ldots n$) be a constant in $B_{1\ldots n}$. The equations $\partial_i X = 0, i = 1, \ldots, n$ are

\[
X(n12\ldots) + q_{n1}X(1n2\ldots) + q_{n1}q_{n2}X(12n3\ldots) + \ldots + q_{n1}\ldots q_{mn''}X(12\ldots n'' n'n') + q_{n1}\ldots q_{mn'}X(12\ldots n'n) = 0,
\]
\[
X(1n2\ldots) + q_{n1}X(n12\ldots n') = 0,
\]
\[
X(12n3\ldots) + q_{12}q_{1n}X(2n13\ldots n') = 0
\]
\[\ldots\]
\[
X(12\ldots n'n) + q_{12}\ldots q_{1n}X(23\ldots n'n1) = 0.
\]

and those obtained from this set by permuting the indices $1, \ldots, n'$. We write $12\ldots n'$ instead of $i_1\ldots i_{n'}$ for any permutation of $1, \ldots, n'$.

4.1.4. **Conventions.** We have used $n' = n - 1, n'' = n - 2$, and

\[
X(\ldots 1\ldots k\ldots) := S(1\ldots k)X(\ldots 1\ldots k\ldots)
\]
\[
:= X(\ldots 1\ldots k\ldots) + q_{12}X(\ldots 21\ldots k\ldots) + \ldots + q_{12}\ldots q_{1k}X(\ldots 23\ldots k1\ldots),
\]

In addition, set $k' = k + 1, k'' = k + 2$ and

\[
X(1\ldots p\ldots mn\ldots) = S(p\ldots m)S(2\ldots m)S(1\ldots m)X(1\ldots p\ldots mn\ldots).
\]

Here $m$ is by definition the index that stands to the left of $n$. Note that the permutations act on the symbols, not on the spaces, and that they affect the parameters, for example $S(12)q_{13} = q_{23}S(12)$. Finally, a lot of typing is saved by introducing

\[
Z(1\ldots mn\ldots) := q_{n1}\ldots q_{nm}X(1\ldots mn\ldots).
\]

4.1.5. **Remark.** The operator that sends $X(\ldots i_1\ldots i_k\ldots)$ to $X(\ldots i_1\ldots i_k\ldots)$ for each permutation $i_1\ldots i_k$ of $1\ldots k$ corresponds to differentiation; $X(\ldots 1\ldots k\ldots)$ is the coefficient of $\xi_2\ldots \xi_k$ in $\partial_1 \sum X(\ldots i_1\ldots i_k\ldots)\xi_{i_1}\ldots \xi_{i_k}$. It is invertible if and only if $\partial_i X = 0, i = 1, \ldots, k$ implies that $X = 0$.

4.1.6. With this notation equations (4.3) take the form

\[
Z(n12\ldots) + Z(1n2\ldots) + Z(12n\ldots) + \ldots + Z(12\ldots n'n) = 0,
\]
\[
Z(1\ldots kn\ldots n') + q_{12}\ldots q_{1n}(1\ldots kn1'\ldots n') = 0, \quad k = 1, \ldots, n'.
\]

(4.4 – 5)
Applying $S(1...n')$ to the long equations and using the short ones to push the index $n$ towards the right we obtain

$$(1 - \sigma_{1n})(Z(1n2...n') + Z(12n3...n') + ... + Z(12...n'n)) = 0.$$ 

This completes Step 1. By stipulation $\sigma_{1n} \neq 1$, so that the first factor can be dropped; hence

$$\sum_{p=1}^{n'} Z(1...pnp'...n') = 0.$$ 

Next apply $S(2...n')$ to this equation and use the short equations (4.5) in the same way. We claim that the result after Step $k$ is

$$(1 - \sigma_{1...kn}^{-1}) \sum_{p=k'}^{n'} Z(1...k'k''...pnp'...n') = 0,$$ 

where $P_{k...1}$ is the cyclic permutation that takes $m \mapsto m - 1$ ($m = 2,...,k$), 1 $\mapsto k$. To verify this we carry out the next step. We need a simple Lemma.

4.1.7. Lemma. The first factor in (4.6) is invertible if and only if $\sigma_{1...kn} \neq 1$.

Proof of the Lemma. The permutation $P_{k...1}$ is of order $k$. Iteration of (4.6) leads to 

$$(1 - A) \sum Z = 0,$$ 

with $A := \left(\frac{1}{q_{k1}...q_{kk-1}\sigma_{kn}} P_{k...1}\right)^k = \sigma_{1...kn}$. 

Hence, if $A \neq 1$, then (4.6) implies that $\sum Z = 0$.

4.1.8. The first factor in (4.6) can therefore be dropped. Next, apply $S(k'...n')$ to get

$$\sum_{p=k'}^{n'} Z(1...k'k''...pnp'...n') + \sum_{p=k}^{n'} q_{k'p} Z(1...k''...p'nkp''...n') = 0.$$ 

(4.7)

The second term is

$$\sum_{p=k'}^{n'} QZ(1...k''...pnp'...n') = \sum_{p=k'}^{n'} \bar{Q} P_{k'...1} Z(1...k''...p'1p...n), \quad \bar{Q} = q_{k'k''}...q_{k'p}.$$ 

Using the short equations we transform the summand to

$$-\bar{Q} P_{k'...1} [q_{12}...q_{1p}\sigma_{1n}]^{-1} Z(1...k''...pnp'...n')$$

$$= -[q_{k'1}...q_{k'p}\sigma_{k'n}]^{-1} P_{k'...1} Z(1...k''...pnp'...n').$$

Thus (4.7) becomes

$$\left(1 - \frac{1}{q_{k'1}...q_{k'p}\sigma_{k'n}} P_{k'...1}\right) \sum_{p=k'}^{n'} Z(1...k''...pnp'...n') = 0.$$ 

(4.8)
This proves our claim (4.6). The process ends with \( k = n' \), when (4.6) reduces to
\[
\left( 1 - \frac{1}{q_{n'1} \cdots q_{n'n'\sigma_{n'n}}} P_{n' \cdots 1} \right) Z(1, n, n') = 0. \tag{4.9}
\]
By Lemma 4.1.3 the first factor is invertible unless \( \sigma_1 \cdots n = 1 \).

4.1.9. We have shown that, if \( \sigma_1 \cdots n \neq 1 \), then \( Z(1, n, n') \) must vanish, along with all the coefficients related to it by permutations of the indices \( 1 \cdots n' \). By Remark 4.1.5 it follows that \( Z(i_1 \cdots i_{n'1} \cdots i_{n'n}) = 0 \) for any permutation \( i_1 \cdots i_{n'} \) of \( 1 \cdots n' \). In the same way we use the short equations (4.5) to show that in that case all the coefficients are zero. Thus, when \( \sigma_1 \cdots n \neq 1 \) there are no constants in \( B_{1 \cdots n} \). Conversely, if \( \sigma_1 \cdots n = 1 \), then Eq.s (4.9) has nontrivial solutions. Now Eq.(4.9) fixes the ratios of \( n' \) coefficients related by \( P_{n' \cdots 1} \). Among all the coefficients \( Z(i_n \cdots i_{n'n}) \), \((n-2)! \) remain arbitrary. Choosing any solution of (4.9) we use the short equations to construct a unique constant in \( B_{1 \cdots n} \). The space of all these constants has dimension \((n-2)! \) and Theorem 4.1.2 is proved.

4.1.10. As to determining the constants in \( B_{(s)} \) when the set \( s \) contains repetitions, we do not yet have complete results.

4.1.11. We have nothing to say about the space of constants in \( B_{(s)} \) if \( \sigma_{(r)} = 1 \) for some proper subset \( r \subset s = \{1, \ldots, n\} \). That seems to present a much more difficult problem.

4.2. Constants of order 3.

4.2.1. We denote by \( B_{123} \) the subspace of polynomials linear in \( \xi_1, \xi_2, \xi_3 \), separately.

Proposition. There are constants in \( B_{123} \) iff
\[
(\sigma_{12} - 1)(\sigma_{23} - 1)(\sigma_{13} - 1)(\sigma_{123} - 1) = 0.
\]

Corollary. Generically, there are no constants and \( \dim B'_{(123)} = 6 \).

4.2.2. Of special cases there are 5 essentially different kinds.

(1) The constraint \( \sigma_{123} = 1 \). The space of constants is 1-dimensional with basis
\[
C_{123} = \left( \frac{1}{q_{31}} - q_{13} \right) (\xi_1 \xi_2 \xi_3 + q_{31} q_{32} q_{21} \xi_3 \xi_2 \xi_1) + \text{cycl. perm.}.
\]
The intersection \( I_q \cap B_{123} \) is generated by \( C_{123} \) and the subspace \( B'_{123} \) of the quotient is 5-dimensional.

(2) The constraint \( \sigma_{12} = 1 \). Then
\[
C_{12} = \xi_1 \xi_2 - q_{21} \xi_2 \xi_1
\]
is a constant. The space of constants in $B_{123}$ is 1-dimensional with basis
\[ C_{12}\xi_3 - q_{31}q_{32}\xi_3 C_{12}. \]
The ideal $I_q$ is generated by $C_{12}$, $I_q \cap B_{123}$ is two-dimensional and $B'_{123}$ is 4-dimensional.

(3) Two constraints: $\sigma_{12} = 1 = \sigma_{13}$. The space of constants in $B_{123}$ is two-dimensional and $B'_{123}$ is two-dimensional with basis $\{\xi_1\xi_2\xi_3, \xi_1\xi_3\xi_2\}$.

(4) Three constraints: $\sigma_{12} = \sigma_{13} = \sigma_{23} = 1$. The space of constants in $B_{123}$ is the same as in the previous case, but $B'_{123}$ is 1-dimensional.

(5) The other case of two constraints, $\sigma_{12} = 1, \sigma_{123} = 1$ yields a surprise. When $\sigma_{12} = 1$, and $\sigma_{13} \neq 1 \neq \sigma_{23}$, then there are no special cases: no further constant appears as $\sigma_{123}$ takes the value 1.

4.2.3. Summary. We have the following complete list of generators of ideals in $B_{123}$,

| Constraints | Generators | $\#$ const. | $\in B_{123}$ | dim $I \cap B_{123}$ |
|-------------|------------|-------------|----------------|-----------------------|
| $\sigma_{12}\sigma_{23}\sigma_{13} = 1$ | $C_{123}$ | 1 | 1 | |
| $\sigma_{12} = 1$ | $\xi_1\xi_2 - q_{21}\xi_2\xi_1$ | 1 | 2 | |
| $\sigma_{12} = 1, \sigma_{23} = 1$ | $\xi_1\xi_2 - q_{21}\xi_2\xi_1, \xi_2\xi_3 - q_{32}\xi_3\xi_2$ | 2 | 4 | |
| $\sigma_{ij} = 1, \ i \neq j$ | $\xi_\xi_j - q_{ji}\xi_\xi_i, \ i \neq j$ | 2 | 5 | |

4.2.4. Constants of order 3, in two generators, are of the Serre type. Constants in one generator, $C = \xi_1^3$ require $q_{11}^3 = 1$. This completes the survey of constants of order 3.

Appendix. Alternative expressions for some constants.

The constants $C_{ij}, i \neq j$, given in 3.1.1 can be written, up to a factor, as
\[ C_{ij} = \text{antisym}(\sqrt{q_{ij}}\xi_i\xi_j), \]
where the antisymmetrizer is just $1 - P_{ij}$, $P_{ij}$ being the permutation that exchanges the indices $i, j$.

The constants $C_{ijk}, i \neq j \neq k$, given in 3.3.4 for the case 123, can be written, up to a factor, as
\[ C_{ijk} = \text{sym}(\sqrt{q_{ij}q_{jk}q_{ki}}(q_{ki}^{-1} - q_{ik})\xi_i\xi_j\xi_k), \]
where the symmetrizer is $1 + P_{ij} + P_{jk} + P_{ki} + P_{ij}P_{jk} + P_{ij}P_{ki}$.

These constants satisfy the identity given in 3.3.4, and which we rewrite up to a factor as
\[ (\sqrt{q_{12}q_{13}q_{14}} \hat{\partial}_1 \hat{C}_{234} - \sqrt{q_{21}q_{31}q_{41}} \hat{C}_{234}\hat{\partial}_1) + \text{cycl.} = 0. \]

Acknowledgements. We thank Moshe Flato for criticism. A.G. thanks the Fundación Del Amo for financial support and the Department of Physics of UCLA for hospitality.
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