Generalized Pair Weights of Linear Codes and MacWilliams Extension Theorem*

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Abstract
In this paper, we introduce the notion of generalized pair weights of an \([n, k]\)-linear code over a finite field and the notion of pair \(r\)-equiweight codes, where \(1 \leq r \leq k - 1\). We give some properties of generalized pair weights of linear codes over finite fields. We obtain a necessary and sufficient condition for an \([n, k]\)-linear code to be a pair equiweight code and characterize the pair \(r\)-equiweight codes for any \(1 \leq r \leq k - 1\). In addition, a relationship between the pair equiweight code and the pair \(r\)-equiweight code is also given. Finally, we prove the MacWilliams extension theorem for the pair weight case.

Keywords: generalized pair weight, pair equiweight code, pair \(r\)-equiweight code, MacWilliams extension theorem.

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1 Introduction
Let \(\mathbb{F}_q\) be the finite field of order \(q\), where \(q = p^e\) and \(p\) is a prime. An \([n, k]\)-linear code \(C\) of length \(n\) over \(\mathbb{F}_q\) is an \(\mathbb{F}_q\)-subspace of dimension \(k\) of \(\mathbb{F}_q^n\).

Symbol-pair read channels, in which the outputs of the read process are pairs of consecutive symbols, were studied by Cassuto and Blaum [2] in 2011. In the same paper, the pair weight and the pair distance of linear codes (see Definition 2.3 below) were defined, which can be used to check and correct pair errors from read channels. This new paradigm was motivated by the limitations of the reading process in high density data storage systems. Chen, Lin and Liu [3] presented three lower bounds for the minimum pair distance of constacyclic codes and obtained new MDS symbol-pair codes with minimum pair distance seven and eight through repeated-root cyclic codes. In [9], Liu, Xing and Yuan presented the list decodability of symbol-pair codes and a list decoding algorithm.

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of Reed-Solomon codes beyond the Johnson-type bound in the pair weight. In [5] and [4], the authors calculated the symbol-pair distances of repeated-root constacyclic codes of lengths $p^s$ and $2p^s$, respectively. In [18], Yaakobi, Bruck and Siegel generalized the notion of symbol-pair weight to $b$-symbol weight, and they considered the case where the read channel output is the number $b$ of consecutive symbols with $b \geq 2$. The authors provided some extensions of several concepts, results, and code constructions to this setting.

Motivated by cryptographical applications, the algebraic structure of linear codes from a new perspective was studied. By viewing the minimum Hamming weight as a certain minimum property of subspaces of dimension one, the notion of generalized Hamming weights was introduced in coding theory by Wei [13]. These weights were described in a geometric setting in [11]. The generalized Hamming weights were introduced to codes over finite chain rings and principal ideal rings, and bounds on the minimum generalized Hamming weights were given in [6]. In [19], two general formulas on $d_r(C)$ for irreducible $[n, k]$-cyclic codes were presented using Gauss sums, and the weight hierarchy \{$d_1(C), d_2(C), \ldots, d_k(C)$\} was completely determined for several cases, where $d_r(C)$ is the $r$-minimal Hamming weight of the code. In [8], The authors investigated the generalized Hamming weights of three classes of linear codes constructed through defining sets.

The Hamming equiweight code is a linear code of the constant weight for all nonzero codewords. This class of codes was firstly studied by Weiss [14] in 1966. In the same paper, Weiss gave a necessary and sufficient condition for a linear code to be a Hamming equiweight code. In [16], Wood determined completely the structure of linear equiweight codes over $\mathbb{Z}_n$, where the weights include: Hamming weight, Lee weight and pre-homogeneous weights.

MacWilliams [10] and later Bogart, Goldberg, and Gordon [1] proved that, every linear isomorphism preserving the Hamming weight between two linear codes over finite fields can be extended to a monomial transformation. This classical result was called MacWilliams extension theorem. In [15], Wood proved MacWilliams extension theorem for all linear codes over finite Frobenius rings equipped with the Hamming weight. In the commutative case, he showed that the Frobenius property was not only sufficient but also necessary. In the non-commutative case, the necessity of the Frobenius property was proved in [17].

Compare with the Hamming weight of a code, the pair weight is a new weight and it can be used widely to check and correct pair errors in the communication system of symbol-pair read channel. This gives that the study of some special classes of linear codes with pair weights is interesting and valuable. Furthermore, MacWilliams extension theorem for the Hamming weight case is a fundamental result in classical coding theory. However, this result is not true in general for the pair weight case. The study of this fundamental result for the pair weight case is a challenge problem. In this paper, we first introduce the notion of generalized pair weights of an $[n, k]$-linear code over a finite field and the notion of pair $r$-equiweight codes, where $1 \leq r \leq k - 1$. Then we characterize the generalized pair weights of linear codes over finite fields. We study the relationship between the pair equiweight codes and the pair $r$-equiweight codes for any $1 \leq r \leq k - 1$. Finally, we provide the MacWilliams extension theorem for the pair weight case.
This paper is organized as follows. Section 2 gives some preliminaries, the definition of the generalized pair weights of linear codes and a characterization of the pair weight of any codeword of linear codes. In Section 3, we give some basic results and bounds about the generalized pair weights of linear codes. In Section 4, we first obtain a necessary and sufficient condition for a linear code to be a pair equiweight code. Then we provide a relationship between the pair equiweight code and the pair $r$-equiweight code for $1 \leq r \leq k - 1$. Necessary and sufficient conditions for an $[n, k]$-linear code to be a pair $r$-equiweight code are given when $r = k - 2$ and $k - 1$. A sufficient condition for an $[n, k]$-linear code to be a pair $r$-equiweight code is also obtained when $2 \leq r \leq k - 3$. In Section 5, we give and prove the MacWilliams extension theorem for the pair weight case.

2 Preliminaries

Throughout this paper, let $\mathbb{F}_q$ be the finite field of order $q$, where $q = p^e$ and $p$ is a prime. Let $n$ be a positive integer, and let $\mathbb{F}_q^n$ be the $n$-dimensional vector space over $\mathbb{F}_q$. An $\mathbb{F}_q$-subspace $C$ of dimension $k$ of $\mathbb{F}_q^n$ is called an $[n, k]$-linear code. If $k = 0$, then $C = \{0\}$ is the zero code. We assume all codes in this paper are nonzero linear codes.

Let $(G, +)$ be a finite abelian group. A character of $G$ is a group homomorphism $\zeta : (G, +) \rightarrow (\mathbb{C}^\times, \cdot)$, where $\mathbb{C}^\times$ is the multiplication group of all nonzero complex numbers. The character of $G$ which maps all elements in $G$ to 1 is called the trivial character, denoted by $1_G$. Let $\hat{G}$ be the finite group of all characters of $G$, then $G \cong \hat{G}$ as an abelian group.

The following result is well-known.

Lemma 2.1. ([12]) Assume the notation is given above. Let $\chi \in \hat{G}$. Then

$$\sum_{g \in G} \chi(g) = \begin{cases} |G|, & \text{if } \chi = 1_G; \\ 0, & \text{if } \chi \neq 1_G. \end{cases}$$

Let $G = (\mathbb{F}_q, +)$, and let $\xi$ be a primitive $p^e$th root of unity in $\mathbb{C}^\times$. Then $\exp(x) = \xi^{tr(x)}$ for any $x \in \mathbb{F}_q$ is a nontrivial character of $\mathbb{F}_q$, where $tr : \mathbb{F}_q \rightarrow \mathbb{F}_p$ is the trace function from $\mathbb{F}_q$ to its prime subfield $\mathbb{F}_p$. It turns out that any other character of $\mathbb{F}_q$ has the form $\exp(ax) = \xi^{tr(ax)}$, where $a \in \mathbb{F}_q$. Define a scalar operation on $\hat{\mathbb{F}}_q$ as follows:

$$\mathbb{F}_q \times \hat{\mathbb{F}}_q \rightarrow \hat{\mathbb{F}}_q, (a, \chi) \mapsto a \cdot \chi,$$

where $a \cdot \chi(g) = \chi(ag)$. Then $\hat{\mathbb{F}}_q$ is a vector space of dimension one over $\mathbb{F}_q$.

If $G = V$, an arbitrary vector space of dimension $k$ over $\mathbb{F}_q$, we denote by $V^* = Hom_{\mathbb{F}_q}(V, \mathbb{F}_q)$ the $\mathbb{F}_q$-dual space of $V$. Then we have the following $\mathbb{F}_q$-linear isomorphism, denoted by $\exp$ again:

$$\exp : V^* = Hom_{\mathbb{F}_q}(V, \mathbb{F}_q) \rightarrow \hat{V}, \lambda \mapsto \exp(\lambda) = \exp \circ \lambda,$$

where $\exp \circ \lambda(v) = \exp(\lambda(v))$ for any $v \in V([12])$.

The following corollary is straightforward from Lemma 2.1.
Corollary 2.2. ([12]) Assume the notation is given above. Let \( \lambda \in V^* \) and \( v \in V \). Then
\[
\sum_{a \in \mathbb{F}_q} \zeta^{tr(a\lambda(v))} = \begin{cases} q, & \text{if } \lambda(v) = 0; \\ 0, & \text{if } \lambda(v) \neq 0. \end{cases}
\]

Let \( \pi : \mathbb{F}_q^n \to (\mathbb{F}_q \times \mathbb{F}_q)^n \) be the map which is defined as
\[
\pi(x) = ((x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_0))
\]
for any \( x = (x_0, x_1, \ldots, x_{n-1}) \in \mathbb{F}_q^n \). The definition of the pair distance and the pair weight in \( \mathbb{F}_q^n \) was given by Cassuto and Blaum [2] in 2011 as follows.

Definition 2.3. ([2]) For any \( x, y \in \mathbb{F}_q^n \), the pair distance between \( x \) and \( y \) is defined as
\[
d_p(x, y) = |\{0 \leq i \leq n-1 | (x_i, x_{i+1}) \neq (y_i, y_{i+1})\}|
\]
where the indices are taken modulo \( n \). The pair weight of \( x \) is defined as \( w_p(x) = d_p(x, 0) \).

Let \( C \) be a code over \( \mathbb{F}_q \). The minimal pair distance of \( C \) is defined as
\[
d_p(C) = \min_{c \neq c' \in C} d_p(c, c').
\]
The pair weight of \( C \) is defined as \( w_p(C) = \{w_p(c) | 0 \neq c \in C\} \). Note that if \( C \) is an \([n, k] \)-linear code, then \( d_p(C) = w_p(C) \).

The following generalized Hamming weights of any \( \mathbb{F}_q \)-subspace of \( \mathbb{F}_q^n \) and the \( r \)-minimal Hamming weight of \([n, k] \)-linear codes over \( \mathbb{F}_q \) for \( 1 \leq r \leq k \) were defined by Wei [13].

Definition 2.4. ([13]) Let \( D \) be an \( \mathbb{F}_q \)-subspace of \( \mathbb{F}_q^n \). The Hamming support of \( D \), denoted by \( \chi_H(D) \), is the set of all non-always-zero bit positions of \( D \), i.e.,
\[
\chi_H(D) = \{0 \leq i \leq n-1 | \exists x = (x_0, x_1, \ldots, x_{n-1}) \in D, x_i \neq 0\},
\]
and the generalized Hamming weight of \( D \) is defined as \( w_H(D) = |\chi_H(D)| \).

Definition 2.5. ([13]) Let \( C \) be an \([n, k] \)-linear code over \( \mathbb{F}_q \). For \( 1 \leq r \leq k \), the \( r \)-minimal Hamming weight of \( C \) is defined as \( d^*_H(C) = \min\{w_H(D) | D \leq C, \dim(D) = r\} \).

Note that if \( r = 1 \), the 1-minimal Hamming weight of \( C \) is just the minimal Hamming weight of \( C \). In [13], the following result was proved.

Lemma 2.6. ([13, Theorem 1]) Let \( C \) be an \([n, k] \)-linear code over \( \mathbb{F}_q \). Then we have
\[
1 \leq d^1_H(C) < d^2_H(C) < \cdots < d^{k-1}_H(C) < d^k_H(C) \leq n.
\]
The set \( \{d^1_H(C), d^2_H(C), \ldots, d^k_H(C)\} \) is called the generalized Hamming weight hierarchies.

The following definition of the Hamming \( r \)-equiweight code over \( \mathbb{F}_q \) for \( 1 \leq r \leq k - 1 \) was defined in [7] in 2003.
Definition 2.7. Let $C$ be an $[n,k]$-linear code over $\mathbb{F}_q$ and $1 \leq r \leq k-1$. We say that $C$ is a Hamming $r$-equiweight code if $d_p^r(C) = w_H(D)$ for any subspace $D$ of dimension $r$ of $C$.

Let $M_n(\mathbb{F}_q)$ be the set of all $n \times n$ matrices over $\mathbb{F}_q$. For $A \in M_n(\mathbb{F}_q)$, let $A^T$ denote the transpose of $A$. Let $GL_n(\mathbb{F}_q)$ be the set of all $n \times n$ invertible matrixes over $\mathbb{F}_q$. A monomial matrix over $\mathbb{F}_q$ is a square matrix such that in every row and in every column there is exactly one nonzero element. Let $MO_n(\mathbb{F}_q)$ denote the set of all the $n \times n$ monomial matrices over $\mathbb{F}_q$.

The following theorem is called classical MacWilliams extension theorem, which was first proved by (MacWilliams [10]). Bogart, Goldberg, and Gordon provided an alternative proof in 1978 ([1]).

Proposition 2.8 (MacWilliams Extension Theorem ([10],[1])). Let $C$ and $\tilde{C}$ be two $[n,k]$-linear codes over $\mathbb{F}_q$. Then there exists an $\mathbb{F}_q$-linear isomorphism $f : C \rightarrow \tilde{C}$ which preserves Hamming weights if and only if there exists a monomial matrix $M \in MO_n(\mathbb{F}_q)$ such that $f(c) = cM$ for all $c \in C$.

In this paper, we introduce the notion of generalized pair weights of any $\mathbb{F}_q$-subspace of $\mathbb{F}_q^n$ and $r$-minimal generalized pair weight of $[n,k]$-linear codes over $\mathbb{F}_q$, where $1 \leq r \leq k$.

Definition 2.9. Let $D$ be an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^n$. The pair support of $D$, denoted by $\chi_p(D)$, is the set of non-always-zero pair-positions of $\pi(D)$, i.e.,

$$\chi_p(D) = \{0 \leq i \leq n-1 \mid \exists x = (x_0, \cdots, x_{n-1}) \in D, (x_i, x_{i+1}) \neq (0,0)\},$$

where $\pi(D) = \{\pi(x) \mid x \in D\}$ and the indices are taken modulo $n$. The generalized pair weight of $D$ is defined as $w_p(D) = |\chi_p(D)|$.

Definition 2.10. Let $C$ be an $[n,k]$-linear code over $\mathbb{F}_q$. For $1 \leq r \leq k$, the $r$-minimal pair weight of $C$ is defined as $d_p^r(C) = \min\{w_p(D) \mid D \leq C, \dim(D) = r\}$.

Remark 2.11. If $r = 1$, the 1-minimal pair weight of the code $C$ is just the minimal pair weight of $C$.

Let $U$ be an $\mathbb{F}_q$-vector space of dimension $k$. We denote by $\langle V, W \rangle$ the subspace generated by the subspaces $U, W$ of $U$, and let $U/W$ denote the quotient space modulo $W$. For any $1 \leq r \leq k$, let

$$PG^r(U) = \{V \leq U \mid \dim(V) = r\}$$

be the set of all subspaces of dimension $r$ of $U$. Let $n_{r,k}$ denote the number of all $r$-dimensional subspaces of $k$-dimensional vector space $U$. It is easy to see that $n_{r,k} = |PG^r(U)| = \prod_{i=0}^{r-1} \frac{q^k-q^i}{q^r-q^i}$. In particular, $n_{1,k} = |PG^1(U)| = |PG^{k-1}(U)| = n_{k-1,k} = \frac{q^{k-1}}{q-1}$.
Let $C$ be an $[n, k]$-linear code with a generator matrix $G = (G_0, \ldots, G_{n-1})$, where $G_i$ is the column vector of $G$. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all natural numbers. For any $L \in PG^i(\mathbb{F}_q^k)$, the function $\hat{m}_G : PG^i(\mathbb{F}_q^k) \rightarrow \mathbb{N}$ is defined as follows.

$$
\hat{m}_G(L) = |\{0 \leq i \leq n - 1 \mid \langle G_i, G_{i+1} \rangle = L\}|
$$

Using the function $\hat{m}_G$, for $1 \leq r \leq k$, we define the function $\hat{m}_G^r : PG^r(\mathbb{F}_q^k) \rightarrow \mathbb{N}$ to be

$$
\hat{m}_G^r(U) = \sum_{L \in PG^i(U)} \hat{m}_G(L)
$$

for any $U \in PG^r(\mathbb{F}_q^k)$. When $r = 0$, we let $\hat{m}_G^0 = 0$ be the zero function.

The function $\hat{m}_G : PG^2(\mathbb{F}_q^k) \rightarrow \mathbb{N}$ is defined as follows. For any $W \in PG^2(\mathbb{F}_q^k)$,

$$
\hat{m}_G(W) = |\{0 \leq i \leq n - 1 \mid \langle G_i, G_{i+1} \rangle = W\}|
$$

And the function $\hat{m}_G^r : PG^r(\mathbb{F}_q^k) \rightarrow \mathbb{N}$ induced by $\hat{m}_G$ is defined to be

$$
\hat{m}_G^r(U) = \sum_{W \in PG^2(U)} \hat{m}_G(V)
$$

for any $U \in PG^r(\mathbb{F}_q^k)$ and $2 \leq r \leq k$. When $0 \leq r \leq 1$, let $\hat{m}_G^r = 0$ be the zero functions. Let $m_r = \hat{m}_G^r + \hat{m}_G^{k-r}$ be the function from $PG^r(\mathbb{F}_q^k)$ to $\mathbb{N}$ for $0 \leq r \leq k$.

For an $[n, k]$-linear code $C$ over $\mathbb{F}_q$ with a generator matrix $G$, we know that for any nonzero codeword $c \in C$, there exists a unique nonzero vector $y \in \mathbb{F}_q^k$ such that $c = yG = (yG_0, yG_1, \ldots, yG_{n-1})$, where $G = (G_0, G_1, \ldots, G_{n-1})$. We have

**Lemma 2.12.** Assume the notations are given above. Let $n_0(G) = |\{0 \leq i \leq n - 1 \mid \langle G_i, G_{i+1} \rangle = 0\}|$. Then for any $0 \neq c \in C$, $w_p(c) = n - n_0(G) - m_G^{k-1}(\langle y \rangle^\perp)$.

**Proof.** By the definitions of $w_p$ and $m_G^{k-1}$, we have

$$
w_p(c) = |\{0 \leq i \leq n - 1 \mid (c_i, c_{i+1}) \neq (0, 0)\}|
$$

$$
= n - |\{0 \leq i \leq n - 1 \mid (c_i, c_{i+1}) = (0, 0)\}|
$$

$$
= n - |\{0 \leq i \leq n - 1 \mid yG_i = yG_{i+1} = 0\}|
$$

$$
= n - |\{0 \leq i \leq n - 1 \mid \langle G_i, G_{i+1} \rangle \subseteq \langle y \rangle^\perp\}|
$$

$$
= n - |\{0 \leq i \leq n - 1 \mid \langle G_i, G_{i+1} \rangle \subseteq \langle y \rangle^\perp, \langle G_i, G_{i+1} \rangle = 0\}|
$$

$$
- |\{0 \leq i \leq n - 1 \mid \langle G_i, G_{i+1} \rangle \subseteq \langle y \rangle^\perp, \dim(\langle G_i, G_{i+1} \rangle) = 1\}|
$$

$$
- |\{0 \leq i \leq n - 1 \mid \langle G_i, G_{i+1} \rangle \subseteq \langle y \rangle^\perp, \dim(\langle G_i, G_{i+1} \rangle) = 2\}|
$$

$$
= n - n_0(G) - \sum_{L \in PG^1(\langle y \rangle^\perp)} |\{0 \leq i \leq n - 1 \mid \langle G_i, G_{i+1} \rangle = L\}|
$$

$$
- \sum_{W \in PG^2(\langle y \rangle^\perp)} |\{0 \leq i \leq n - 1 \mid \langle G_i, G_{i+1} \rangle = W\}|
$$

$$
= n - n_0(G) - m_G^{k-1}(\langle y \rangle^\perp) - \hat{m}_G^{k-1}(\langle y \rangle^\perp)
$$

$$
= n - n_0(G) - m_G^{k-1}(\langle y \rangle^\perp).
$$

\[\square\]
Remark 2.13. The dimension of $\langle G_i, G_{i+1} \rangle$ could be 0 for a generator matrix $G = (G_0, \cdots, G_{n-1})$ of an $[n, k]$-linear code $C$. If $\langle G_i, G_{i+1} \rangle = 0$, we can construct a new linear code $\tilde{C}$ with a generator matrix $\tilde{G} = (G_0, \cdots, G_i, G_{i+2}, \cdots, G_{n-1})$ and a linear isomorphism from $C$ to $\tilde{C}$ keeping the pair weight invariant. Without loss of generality, we will assume that $n_0(G) = 0$ for a generator matrix $G = (G_0, \cdots, G_{n-1})$ of a linear code $C$ in the rest of the paper.

Definition 2.14. Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$ and $1 \leq r \leq k - 1$, we say that $C$ is a pair $r$-equiweight code if $d^*_p(C) = w_p(D)$ for any subspace $D$ of dimension $r$ of $C$.

Remark 2.15. If $r = 1$, the Hamming $r$-equiweight code is the full equiweight code, and a pair $r$-equiweight code is a pair equiweight code. In general, a Hamming equiweight code is not a pair equiweight code.

Example 2.16. Let $C_1$ be the linear code with a generator matrix \[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}
\] over $\mathbb{F}_2$. Then $C_1$ is a pair equiweight code but not a Hamming equiweight code. Let $C_2$ be the linear code with a generator matrix \[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix}
\] over $\mathbb{F}_2$. Then $C_2$ is a Hamming equiweight code but not a pair equiweight code.

The following proposition gives a method to construct a pair equiweight code from a Hamming equiweight code.

Proposition 2.17. Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$ with a generator matrix $G = (G_0, \cdots, G_{n-1})$, and $\hat{C}$ be a $[2n, k]$-linear code over $\mathbb{F}_q$ with a generator matrix $\hat{G} = (G_0, 0, \cdots, G_{n-1}, 0)$, where $0$ is the column zero vector. Then for any $1 \leq r \leq k - 1$, $C$ is a Hamming $r$-equiweight code if and only if $\hat{C}$ is a pair $r$-equiweight code.

Proof. Let $\varphi$ be a map from $C$ to $\hat{C}$ such that $\varphi(c) = (c_0, 0, \cdots, c_{n-1}, 0) \in \hat{C}$ for any $c = (c_0, \cdots, c_{n-1}) \in C$. Then $\varphi$ is an $\mathbb{F}_q$-linear isomorphism and $w_p(\varphi(c)) = 2w_H(c)$. The rest part of the proposition is trivial. 

3 Generalized pair weights of linear codes

In this section, we give some general properties of the generalized pair weights of linear codes. We obtain some bounds about the generalized pair weights of linear codes.

We first give a description on the relationship between the Hamming weight $w_H(D)$ and the pair weight $w_p(D)$ for any $\mathbb{F}_q$-subspace $D$ of $\mathbb{F}_q^n$. If $w_H(D) = n$, then $w_p(D) = n$. We have the following lemma.

Lemma 3.1. Let $D$ be an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^n$, and suppose $w_H(D) < n$. Let $\bigcup_{l=1}^L B_l$ be a minimal partition of the set $\chi_H(D)$ to subsets of consecutive indices (Indices may wrap modulo $n$) such that each subset $B_l = \{s_l, s_l + 1, \cdots, s_l + e_l\}$ and $L$ being the smallest integer that achieves such partition. Then $w_p(D) = w_H(D) + L$. 

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Proof. If \( i \in \chi_H(D) \), there exists \( x = (x_0, \cdots, x_{n-1}) \in D \) such that \( x_i \neq 0 \). Then the two pairs \((x_{i-1}, x_i)\) and \((x_i, x_{i+1})\) are not \((0,0)\) and \(\{i-1, i\} \subseteq \chi_p(D)\). Hence \( \chi_p(D) = \bigcup_{i=1}^d B_i = \bigcup_{i=1}^d \{s_l - 1, s_l, s_l + 1, \cdots, s_l + e_l\} \). Since \( s_l \geq s_{l-1} + e_{l-1} + 2 \), we have \( s_l - 1 > s_{l-1} + e_{l-1} \) and

\[
\{s_{l-1} - 1, s_{l-1}, s_{l-1} + 1, \cdots, s_{l-1} + e_{l-1}\} \cap \{s_l - 1, s_l, s_l + 1, \cdots, s_l + e_l\} = \emptyset.
\]

Hence \( w_p(D) = |\chi_p(D)| = |\chi_H(D)| + L = w_H(D) + L \).

\[\square\]

**Theorem 3.2.** Let \( C \) be an \([n, k]\)-linear code over \( \mathbb{F}_q \). Then we have

(a) \( d^r_H(C) + 1 \leq d^r_p(C) \leq 2d^r_H(C) \) for any \( 1 \leq r \leq k - 1 \).

(b) If \( d^k_H(C) < n \), then \( d^k_H(C) + 1 \leq d^k_p(C) \leq 2d^k_H(C) \).

(c) If \( d^k_H(C) = n \), then \( d^k_p(C) = n \).

Proof. (a) For \( 1 \leq r \leq k - 1 \), let \( D \) be an \( \mathbb{F}_q \)-subspace of \( C \) such that \( \dim(D) = r \) and \( d^r_p(C) = w_p(D) \). If \( w_H(D) = |\chi_H(D)| = n \), then

\[
d^r_p(C) = w_p(D) = |\chi_p(D)| = n.
\]

There exists an \( \mathbb{F}_q \)-subspace \( \tilde{D} \) of \( C \) such that \( \dim(\tilde{D}) = r \) and \( w_H(\tilde{D}) = d^r_H(C) < n \) by Lemma 2.6. Then \( n = w_p(D) = d^r_p(C) \leq w_p(\tilde{D}) \) and hence \( d^r_p(C) = w_p(D) \). Without loss of generality, we can assume \( w_H(D) < n \). Then by Lemma 3.1 we have \( w_p(D) = w_H(D) + L \). Hence

\[
d^r_p(C) = w_p(D) = w_H(D) + L \geq w_H(D) + 1 \geq d^r_H(C) + 1.
\]

Let \( E \) be an \( \mathbb{F}_q \)-subspace of \( C \) such that \( \dim(E) = r \) and \( d^r_H(C) = w_H(E) \). Since \( d^r_H(C) = w_H(E) < n \), by Lemma 3.1 we have \( w_p(E) = w_H(E) + L_1 \). Hence \( d^r_p(C) \leq w_p(E) = w_H(E) + L_1 \leq 2w_H(E) = 2d^r_H(C) \).

(b) We have \( w_p(C) = w_H(C) + L_2 \) by Lemma 3.1 since \( |\chi_H(C)| = d^k_H(C) < n \). Hence

\[
d^r_H(C) + 1 = w_H(C) + 1 \leq d^k_H(C) = w_p(C) = w_H(C) + L_2 \leq 2w_H(C) = 2d^r_H(C).
\]

(c) If \( d^k_H(C) = |\chi_H(C)| = n \), then \( d^k_p(C) = |\chi_p(C)| = n \).

\[\square\]

**Theorem 3.3.** Let \( C \) be an \([n, k]\)-linear code over \( \mathbb{F}_q \) with \( n \geq 2 \). Then we have

\[
2 \leq d^1_p(C) < d^2_p(C) < \cdots < d^{k-1}_p(C) \leq d^k_p(C) \leq n.
\]

Proof. The inequality \( d^r_p(C) \leq d^{r+1}_p(C) \) is trivial for \( 1 \leq r \leq k - 1 \). For any subspace \( D \) of \( C \) over \( \mathbb{F}_q \) with one dimension, there exists \( 0 \neq x = (x_0, \cdots, x_{n-1}) \in D \) such that \( x_i \neq 0 \). Hence \( w_p(D) \geq 2 \) and \( d^1_p(C) \geq 2 \).
For any $2 \leq r \leq k - 1$, by Lemma 2.6 we have $d^r_H(C) < n$. Note that there exists a subspace $D$ of $C$ such that $\dim(D) = r$ and $w_p(D) = d^r_p(C)$ by the definition of the $r$-minimal pair weight of $C$. If $w_H(D) = |\chi_H(D)| = n$, then

$$d^r_p(C) = w_p(D) = |\chi_p(D)| = n.$$  

There exists an $F_q$-subspace $\tilde{D}$ of $C$ such that $\dim(\tilde{D}) = r$ and $w_H(\tilde{D}) = d^r_H(C) < n$ by Lemma 2.6. Then $n = w_p(D) = d^r_p(C) \leq w_p(\tilde{D})$ and hence $d^r_p(C) = w_p(\tilde{D})$. Without loss of generality, we can assume $w_H(\tilde{D}) < n$. Then there exists an index $i \in \chi_H(D)$ such that $i + 1 \not\in \chi_H(D)$, where the indices are taken modulo $n$. Let $\hat{D} = \{x \in D | x = (x_0, \ldots, x_{n-1}), x_i = 0\}$. We know that $\dim(\hat{D}) = r - 1$, $i \not\in \chi_H(\hat{D})$ and $\chi_H(\hat{D}) \cup \{i\} = \chi_H(D)$. Hence $i \not\in \chi_p(\hat{D})$ and $i \in \chi_p(D)$. Therefore, $d^{r-1}_p(C) \leq |\chi_p(D)| < |\chi_p(D)| = d^r_p(C)$ for $2 \leq r \leq k - 1$. \hfill \Box

**Remark 3.4.** There exists a linear code $C$ of length $n$ such that $d^{k-1}_p(C) = d^k_p(C) = n$. For example, let $C$ be the linear code over $F_2$ with generator matrix \( \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \). Then we have $d^2_p(C) = d^3_p(C) = 3 = n$.

**Corollary 3.5.** Let $C$ be an $[n, k]$-linear code over $F_q$ with $k \geq 2$. Then

(a) if $d^{k-1}_p(C) = d^k_p(C)$ then $d^k_H(C) = n$.

(b) $d^{k-1}_p(C) = d^k_p(C)$ if and only if $d^{k-1}_p(C) = d^k_p(C) = n$.

**Proof.** (a) We first prove $d^{k-1}_p(C) = d^k_p(C)$ implies $d^k_H(C) = n$. Suppose otherwise that $d^k_H(C) < n$. Then there exists an index $i \in \chi_H(C)$ such that $i + 1 \not\in \chi_H(C)$, where the indices are taken modulo $n$. Let

$$\hat{C} = \{x \in C | x = (x_0, \ldots, x_{n-1}), x_i = 0\}.$$ 

We know $\dim(\hat{C}) = k - 1$, $i \not\in \chi_H(\hat{C})$ and $\chi_H(\hat{C}) \cup \{i\} = \chi_H(C)$. Hence $i \not\in \chi_p(\hat{C})$ and $i \in \chi_p(C)$. Therefore, we have

$$d^{k-1}_p(C) \leq |\chi_p(\hat{C})| < |\chi_p(C)| = d^k_p(C)$$

which is a contradiction.

(b) By (a), $d^k_H(C) = n = |\chi_H(C)|$, hence $d^k_p(C) = |\chi_p(C)| = n$. Therefore, $d^{k-1}_p(C) = d^k_p(C) = n$. \Box

The claim of "$d^k_H(C) = n$ implies $d^{k-1}_p(C) = d^k_p(C)$" is not true in general. For example, let $C$ be a $[4, 2]$-linear code over $F_2$ with the generator matrix $G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then $d^2_H(C) = 4$, $d^1_p(C) = 3$ and $d^2_p(C) = 4$.

**Corollary 3.6.** Let $C$ be an $[n, k]$-linear code over $F_q$. Then
(a) for all $1 \leq r \leq k - 1$, $r + 1 \leq d_p^r(C) \leq n - k + r + 1$.

(b) $k \leq d_p^k(C) \leq n$.

**Proof.** (a) By Theorem 3.3, for all $1 \leq r \leq k - 1$, we get

$$d_p^r(C) \leq d_p^{r+1}(C) - 1 \leq \cdots \leq d_p^{k-1}(C) + r - k + 1 \leq n + r - k + 1$$

and

$$d_p^r(C) \geq d_p^{r-1}(C) + 1 \geq \cdots \geq d_p^1(C) + r - 1 \geq r + 1.$$

The proof of (b) is also straightforward from Theorem 3.3.

For an $[n, k]$-linear code $C$ over $\mathbb{F}_q$, the upper bound $d_p^r(C) \leq n - k + r + 1$ for all $1 \leq r \leq k - 1$ and $d_p^k(C) \leq n$ in the last corollary is called the Singleton Bound respect to the generalized pair distance of the linear code $C$. If $C$ satisfies $d_p^r(C) = n - k + r + 1$ for all $1 \leq r \leq k - 1$ and $d_p^r(C) = n$, then $C$ is called a maximum generalized pair distance separable code (MGPDS). In particular, when $r = 1$, we get $d_p^1(C) \leq n - k + 2$ which is the Singleton Bound of the pair distance of $C$. And if $C$ satisfies $d_p^1(C) = n - k + 2$, then $C$ is called a maximum pair distance separable code (MPDS).

**Corollary 3.7.** Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$. Then $C$ is an MPDS code if and only if $C$ is an MGPDS code.

**Proof.** It is trivial by using the inequality in the proof of Corollary 3.6.

### 4 Symbol pair $r$-equiweight codes

In this section, we study symbol pair $r$-equiweight codes. Before we provide our main theorems in this section, we need some notions and a lemma.

Recall that $n_{r, k} = \prod_{i=0}^{r-1} \frac{q^{k-i} - q^i}{q^r - q}$, where $r, k \in \mathbb{N}$ with $1 \leq r \leq k$. For convenience, we assume $n_{r, k} = 0$ if $r > k$. For $r, s, k \in \mathbb{N}$ and $1 \leq r \leq s \leq k$, let $PG^r(\mathbb{F}_q^k) = \{V_1, V_2, \ldots, V_{n_{r, k}}\}$, and let $PG^s(\mathbb{F}_q^k) = \{U_1, U_2, \ldots, U_{n_{s, k}}\}$. In particular, let $PG^1(\mathbb{F}_q^k) = \{L_1, L_2, \ldots, L_{n_{1, k}}\}$, $PG^2(\mathbb{F}_q^k) = \{W_1, W_2, \ldots, W_{n_{2, k}}\}$. Let $PG^{k-1}(\mathbb{F}_q^k) = \{M_1 = L_1^1, M_2 = L_2^1, \ldots, M_{n_{k,k}} = L_{n_{1,k}}^1\}$, where $M_j$ is the orthogonal subspace of $L_j$ for $1 \leq j \leq n_{1, k}$.

Let $T_{r, s}$ be a matrix in $M_{n_{r, k} \times n_{s, k}}(\mathbb{Q})$ such that

$$T_{r, s} = (t_{ij})_{n_{r, k} \times n_{s, k}}, \quad t_{ij} = \begin{cases} 1, & \text{if } V_i \subseteq U_j; \\ 0, & \text{if } V_i \not\subseteq U_j, \end{cases}$$

where $\mathbb{Q}$ is the set of all rational numbers. Let $J_{m \times n}$ be the $m \times n$ matrix with all entries being 1. i.e., $J_{m \times n} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$. In particular, $J_{1 \times n} = 1 = (1, \ldots, 1)$. **
Lemma 4.1. Assume the notations are given above, and let \( k \geq 2 \). Then

(a) The sum of all rows of \( T_{1,k-1} \) is a row vector \( t = n_{1,k-1}1 \). The matrix \( T_{1,k-1}^{-1} \) is an invertible matrix and

\[
T_{1,k-1}^{-1} = \frac{1}{q^{k-2} - 1} (T_{1,k-1} - \frac{q^{k-2} - 1}{q^{k-1} - 1} J_{n_{1,k} \times n_{1,k}}).
\]

The sum of all rows of \( T_{1,k-1}^{-1} \) is a constant row vector.

(b) The sum of all rows of \( T_{r,k-1} \), the sum of all rows of \( T_{1,r} \) and the sum of all rows of \( T_{2,r} \) are all constant row vectors for \( 2 \leq r \leq k-1 \).

(c)\( T_{2,k-1} T_{1,k-1} = \frac{q^{k-3} T_{1,r} T_{1,2} + \frac{q^{k-3} - 1}{q-1} J_{n_{2,k} \times n_{1,k}}}{q} \) and \( T_{2,k-1} T_{1,k-1}^{-1} = \frac{1}{q} (T_{1,2} - \frac{q-1}{q^{k-1} - 1} J_{n_{2,k} \times n_{1,k}}) \) for \( k \geq 3 \).

(d) \( T_{r,s} T_{s,z} = n_{s-r,z-r} T_{r,z} \) for \( 1 \leq r \leq s \leq z \leq k \).

Proof. (a) Since the number of all subspaces of dimension 1 of \( M_i \) is \( n_{1,k-1} \) for any \( 1 \leq i \leq n_{1,k} \), we know that the sum of the rows of \( T_{1,k-1} \) is a constant row vector,

\[
t = (t_1, t_2, \ldots, t_{n_{1,k}}),
\]

with \( t_i = n_{1,k-1} \) for all \( 1 \leq i \leq n_{1,k} \). And \( J_{n_{1,k} \times n_{1,k}} T_{1,k-1} = n_{1,k-1} J_{n_{1,k} \times n_{1,k}} \) by the same argument.

Note that \( M_i = L_i^+ \) for all \( i \), we have \( T_{1,k-1} = T_{1,k-1}^T \). Let

\[
T_{1,k-1} T_{1,k-1} = T_{1,k-1}^T = (b_{ij})_{n_{1,k} \times n_{1,k}}.
\]

Since \( b_{ij} \) is the number of all subspaces of dimension 1 of \( M_i \cap M_j \) for \( 1 \leq i, j \leq n_{1,k} \), we have \( b_{ij} = \begin{cases} n_{1,k-1} & \text{if } i = j; \\ n_{1,k-2} & \text{if } i \neq j. \end{cases} \)

Then

\[
\frac{1}{n_{1,k-1} - n_{1,k-2}} (T_{1,k-1} - \frac{n_{1,k-2}}{n_{1,k-1}} J_{n_{1,k} \times n_{1,k}}) T_{1,k-1}
\]

is an identity matrix. This implies that \( T_{1,k-1} \) is an invertible matrix and

\[
T_{1,k-1}^{-1} = \frac{1}{n_{1,k-1} - n_{1,k-2}} (T_{1,k-1} - \frac{n_{1,k-2}}{n_{1,k-1}} J_{n_{1,k} \times n_{1,k}}) = \frac{1}{q^{k-2} - 1} (T_{1,k-1} - \frac{q^{k-2} - 1}{q^{k-1} - 1} J_{n_{1,k} \times n_{1,k}}).
\]

Since the sum of all rows of \( T_{1,k-1} \) and the sum of all rows of \( J_{n_{1,k} \times n_{1,k}} \) are constant row vectors, the sum of all rows of \( T_{1,k-1}^{-1} \) is a constant row vector.

(b) It is easy to prove by the technique used in (a).

(c) By the definitions of \( T_{1,k-1} \) and \( T_{2,k-1} \), we have

\[
T_{2,k-1} T_{1,k-1} = T_{2,k-1} T_{1,k-1}^r = (c_{ij})_{n_{2,k} \times n_{1,k}},
\]

where

\[
c_{ij} = |\{ M_s | 1 \leq s \leq n_{1,k}, W_i \leq M_s, L_j \leq M_s \}|.
\]
for $1 \leq i \leq n_{2,k}$ and $1 \leq j \leq n_{1,k}$. If $L_j \leq W_i$, then
\[ c_{ij} = |\{M_s | 1 \leq s \leq n_{1,k}, W_i \leq M_s\}| = |\{M | M \leq \mathbb{F}_q^k/W_i, \dim(M) = k-3\}| = n_{1,k-2}. \]

If $L_j \not\subseteq W_i$, then
\[ c_{ij} = |\{M_s | 1 \leq s \leq n_{1,k}, (L_j, W_i) \subseteq M_s\}| = |\{M | M \leq \mathbb{F}_q^k/(L_j, W_i), \dim(M) = k-4\}| = n_{1,k-3}. \]

Hence
\[ c_{ij} = \begin{cases} n_{1,k-2}, & \text{if } L_j \subseteq W_i; \\ n_{1,k-3}, & \text{if } L_j \not\subseteq W_i, \end{cases} \]

and
\[ T_{2,k-1}T_{1,k-1} = (n_{1,k-2} - n_{1,k-3})T_{1,2}^T + n_{1,k-3}J_{n_{2,k} \times n_{1,k}} = q^{k-3}T_{1,2}^T + \frac{q^{k-3} - 1}{q - 1}J_{n_{2,k} \times n_{1,k}}. \]

Since $T_{2,k-1}J_{n_{1,k} \times n_{1,k}} = n_{1,k-2}J_{n_{2,k} \times n_{1,k}}$, we have
\[ T_{2,k-1}T_{1,k-1}^{-1} = \frac{1}{n_{1,k-1} - n_{1,k-2}}T_{2,k-1}(T_{1,k-1} - \frac{n_{1,k-2}}{n_{1,k-1}}J_{n_{1,k} \times n_{1,k}}) \]
\[ = \frac{1}{q}T_{1,2}^T - \frac{q - 1}{q(q^{k-1} - 1)}J_{n_{2,k} \times n_{1,k}}. \]

(d) Let $PG^z(\mathbb{F}_q^k) = \{K_1, K_2, \ldots, K_{n_{z,k}}\}$. By the definitions of $T_{r,s}$ and $T_{s,z}$, we have $T_{r,s}T_{s,z} = (d_{ij})_{n_{r,k} \times n_{z,k}}$, where
\[ d_{ij} = |\{U_i | 1 \leq i \leq n_{s,k}, V_i \leq U_i \leq K_j\}| \]

for $1 \leq i \leq n_{r,k}$ and $1 \leq j \leq n_{z,k}$. If $V_i \subseteq K_j$, then
\[ d_{ij} = |\{U | V_i \leq U \leq K_j\}, \dim(U) = s\}| = |\{U | U \leq K_j/V_i, \dim(U) = s-r\}| = n_{s-r,z-r}. \]

If $V_i \not\subseteq K_j$, then $d_{ij} = 0$. Hence $T_{r,s}T_{s,z} = (d_{ij})_{n_{r,k} \times n_{z,k}} = n_{s-r,z-r}T_{r,z}$. \hfill \Box

It is easy to see that when $k = 1$, any $[n,1]$-linear code is a pair equiweight code. In the following we assume $k \geq 2$, and study pair equiweight linear codes.

**Theorem 4.2.** Assume the notations are given above. Let $C$ be an $[n,k]$-linear code over $\mathbb{F}_q$ with a generator matrix $G = (G_0, \ldots, G_{n-1})$. Suppose
\[ \{W_1, W_2, \ldots, W_t\} = \{W \in PG^2(\mathbb{F}_q^k) | \hat{m}_G(W) > 0\}. \]

(a) If $k = 2$, then $C$ is a pair equiweight code if and only if the function $\hat{m}_G$ for $G$ is a constant function.
(b) If $k \geq 3$, then $C$ is a pair equiweight code if and only if $\hat{m}_G(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_G(W_s)$ is constant for all $1 \leq i \leq n_{1,k}$.

Proof. (a) There is $0 \neq y \in \mathbb{F}_q^2$ such that $c = yG$ for any $0 \neq c \in C$, since $G$ is a generator matrix of $C$. By Lemma 2.12, we have

$$w_p(c) = n - |\{0 \leq i \leq n - 1 | \langle G_i, G_{i+1} \rangle \subseteq \langle y \rangle \}|$$

$$= n - |\{0 \leq i \leq n - 1 | \langle G_i, G_{i+1} \rangle \subseteq \langle y \rangle, dim(\langle G_i, G_{i+1} \rangle) = 1 \}|$$

$$= n - \hat{m}_G(\langle y \rangle).$$

The rest part of statement (1) is trivial.

(b) Let $PG^1(\mathbb{F}_q^k) = \{L_1, \cdots, L_{n_{1,k}}\}$, $PG^{k-1}(\mathbb{F}_q^k) = \{M_1 = L_1^\perp, \cdots, M_{n_{1,k}} = L_{n_{1,k}}^\perp\}$ and let $PG^2(\mathbb{F}_q^k) = \{W_1, \cdots, W_{n_{2,k}}\}$. Let

$$\alpha_L = (\hat{m}_G(L_1), \cdots, \hat{m}_G(L_{n_{1,k}})), \quad \alpha_M = (m_{k-1}^{-1}(M_1), \cdots, m_{k-1}^{-1}(M_{n_{1,k}})),$$

and let $\alpha_W = (\hat{m}_G(W_1), \cdots, \hat{m}_G(W_{n_{2,k}}))$. By the definition of $m_{k-1}^{-1}$ and Lemma 4.1 we get

$$\alpha_M = \alpha_LT_{1,k-1} + \alpha_WT_{2,k-1} = (\alpha_L + \alpha_WT_{2,k-1}T_{1,k-1}^{-1})T_{1,k-1}. \tag{4.1}$$

Now suppose $\hat{m}_G(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_G(W_s)$ is constant for all $1 \leq i \leq n_{1,k}$. Since the element in the ith position of the vector $\alpha_L + \alpha_WT_{2,k-1}T_{1,k-1}^{-1}$ is

$$\hat{m}_G(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_G(W_s) - \frac{q-1}{q(q^k-1)} \sum_{s=1}^{t} \hat{m}_G(W_s), \tag{4.2}$$

by Lemma 4.1 the Equation (4.2) is constant for all $1 \leq i \leq n_{1,k}$. We have $\alpha_L + \alpha_WT_{2,k-1}T_{1,k-1}^{-1}$ is a constant vector. By Lemma 4.1 again, the sum of all rows of $T_{1,k-1}$ is a constant row vector. Therefore,

$$\alpha_M = \alpha_LT_{1,k-1} + \alpha_WT_{2,k-1}$$

is a constant vector by Equation (4.1). Hence $m_{k-1}^{-1}$ is a constant function and $C$ is a pair equiweight code by Lemma 2.12.

On the contrary, suppose $C$ is a pair equiweight code. Then $\alpha_M$ and $\alpha_MT_{1,k-1}^{-1} = \alpha_L + \alpha_WT_{2,k-1}T_{1,k-1}^{-1}$ are both constant vectors by Lemma 2.12, Lemma 4.1 and Equation (4.1). Then

$$\hat{m}_G(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_G(W_s) - \frac{q-1}{q(q^k-1)} \sum_{s=1}^{t} \hat{m}_G(W_s)$$

is constant for all $1 \leq i \leq n_{1,k}$, since the element in the ith position of the vector $\alpha_L + \alpha_WT_{2,k-1}T_{1,k-1}^{-1}$ is

$$\hat{m}_G(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_G(W_s) - \frac{q-1}{q(q^k-1)} \sum_{s=1}^{t} \hat{m}_G(W_s).$$

Hence $\hat{m}_G(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_G(W_s)$ is constant for all $1 \leq i \leq n_{1,k}$. \qed
In particular, we have the following corollary.

**Corollary 4.3.** Assume the notations are given above. Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$ with a generator matrix $G = (G_0, \cdots, G_{n-1})$ and $k \geq 3$. If the function $\hat{m}_G$ is constant function, then $C$ is a pair equiweight code if and only if the function $\hat{m}_G$ for $G$ is a constant function.

**Proof.** Suppose the function $\hat{m}_G$ is constant function with value $a \in \mathbb{N}$, then

$$\hat{m}_G(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_G(W_s) = \hat{m}_G(L_i) + \frac{n_{2, k-1}}{q} a.$$ 

Hence the function $\hat{m}_G$ for $G$ is a constant function if and only if

$$\hat{m}_G(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_G(W_s)$$

is constant for all $1 \leq i \leq n_{1,k}$, if and only if $C$ is a pair equiweight code by statement (b) in Theorem 4.2.

**Example 4.4.** Let $\alpha_1 = (0110000010100100101011100010100101001)$, $\alpha_2 = (001011000001011001010)$ and $\alpha_3 = (000001011001001011001) \in \mathbb{F}_2^{21}$. Let $C$ be the linear code with a generator matrix

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

over $\mathbb{F}_2$. Then $C$ is a pair equiweight code and the value of the pair weight of $C$ is 14 by Corollary 4.3. Also we can directly calculate to get the following table such that the first and the second column are non-zero vectors in $C$ and the third column is the pair weight of the vector which is at the same row.

| $\alpha$          | Value                        | Pair Weight |
|-------------------|------------------------------|-------------|
| $\alpha_1$        | (0110000010100100101011100010100101001) | 14          |
| $\alpha_2$        | (001011000001011001010)           | 14          |
| $\alpha_3$        | (000001011001001011001)           | 14          |
| $\alpha_1 + \alpha_2$ | (010011001011010000001)         | 14          |
| $\alpha_1 + \alpha_3$ | (01100101001100010001001010)      | 14          |
| $\alpha_2 + \alpha_3$ | (00101010011000010010010010010011) | 14          |
| $\alpha_1 + \alpha_2 + \alpha_3$ | (010010010011001011000110110001) | 14          |

In order to give the relationship between the pair $r$-equiweight code for $2 \leq r \leq k - 1$ and the pair equiweight code in the following, we first introduce Lemma 4.5 which was proved in (7). Here we provide an alternative proof.

**Lemma 4.5.** (7, Lemma 1) Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$ with a generator matrix $G = (G_0, \cdots, G_{n-1})$. Then $\hat{m}_G^r$ is a constant function from $PG^n(\mathbb{F}_q)$ to $\mathbb{N}$ for some $1 \leq r_0 \leq k - 1$ if and only if $\hat{m}_G$ is a constant function for any $1 \leq r \leq k - 1$. 

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Proof. Suppose there exists $1 \leq r_0 \leq k - 1$ such that $\hat{m}_G^{r_0}$ is a constant function. If $r_0 = 1$, it is obvious that $\hat{m}_G = \hat{m}_G^{r_0}$ is a constant function.

Now suppose $2 \leq r_0 \leq k - 1$. We construct two vectors over $\mathbb{Q}$ as follows:

$$\alpha_L = (\hat{m}_G(L_1), \hat{m}_G(L_2), \ldots, \hat{m}_G(L_{n,k}))$$

for $PG^1(\mathbb{F}_q) = \{L_1, L_2, \ldots, L_{n,k}\}$ and

$$\alpha_V = (\hat{m}_G(V_1), \hat{m}_G(V_2), \ldots, \hat{m}_G(V_{n_{r_0,k}}))$$

for $PG^{r_0}(\mathbb{F}_q) = \{V_1, V_2, \ldots, V_{n_{r_0,k}}\}$. By the definition of $\hat{m}_G^{r_0}$ in Section 2, we have $\alpha_V = \alpha_L T_{1,r_0}$. Then

$$\alpha_V T_{r_0,k-1} = \alpha_L T_{1,r_0} T_{r_0,k-1} = \alpha_L n_{r_0-1,k-2} T_{1,k-1}$$

and

$$\frac{1}{n_{r_0-1,k-2}} \alpha_V T_{r_0,k-1} T_{1,k-1}^{-1} = \alpha_L$$

(4.3)

by Lemma 4.1. Since $\alpha_V$ is a constant vector, $\alpha_L$ is a constant vector and $\hat{m}_G$ is a constant function by Lemma 4.1.

Therefore, $\hat{m}_G$ is a constant function for all $1 \leq r \leq k - 1$. \vspace{1em}

**Theorem 4.6.** Assume the notations are given above. Let $C$ be an $[n,k]$-linear code over $\mathbb{F}_q$ with a generator matrix $G = (G_0, \ldots, G_{n-1})$ and $k \geq 2$. If the function $\hat{m}_G$ is a constant function, then $C$ is a pair $r_0$-equiweight code for some $1 \leq r_0 \leq k - 1$ if and only if $C$ is a pair $r$-equiweight code for any $1 \leq r \leq k - 1$.

Proof. Suppose the function $\hat{m}_G$ is a constant function with value $a \in \mathbb{N}$ and $C$ is a pair $r_0$-equiweight code for some $1 \leq r_0 \leq k - 1$. Note that the $\mathbb{F}_q$-linear map $\phi : \mathbb{F}_q^k \rightarrow C$ such that $\phi(y) = yG$ for any $y \in \mathbb{F}_q^k$ is a linear isomorphism. There is a unique $\mathbb{F}_q$-subspace $\bar{D}$ of $\mathbb{F}_q^k$ such that $D = \phi(\bar{D})$ for any $\mathbb{F}_q$-subspace $D$ of dimension $r_0$ of $C$. Then

$$w_p(D) = |\{0 \leq i \leq n - 1 | \exists x = (x_0, \ldots, x_{n-1}) \in D, (x_i, x_{i+1}) \neq (0,0)\}|$$

$$= n - |\{0 \leq i \leq n - 1 | \forall x = (x_0, \ldots, x_{n-1}) \in D, (x_i, x_{i+1}) = (0,0)\}|$$

$$= n - |\{0 \leq i \leq n - 1 | \forall y \in \bar{D}, yG_i = 0 = yG_{i+1}\}|$$

$$= n - |\{0 \leq i \leq n - 1 | \langle G_i, G_{i+1} \rangle \subseteq \bar{D}^\perp\}|$$

$$= n - |\{0 \leq i \leq n - 1 | \langle G_i, G_{i+1} \rangle \subseteq \bar{D}^\perp, \dim(\langle G_i, G_{i+1} \rangle) = 1\}|$$

$$- |\{0 \leq i \leq n - 1 | \langle G_i, G_{i+1} \rangle \subseteq \bar{D}^\perp, \dim(\langle G_i, G_{i+1} \rangle) = 2\}|. \quad (4.4)$$

If $r_0 = k - 1$, we have

$$w_p(D) = n - |\{0 \leq i \leq n - 1 | \langle G_i, G_{i+1} \rangle \subseteq \bar{D}^\perp, \dim(\langle G_i, G_{i+1} \rangle) = 1\}| = n - \hat{m}_G(\bar{D}^\perp).$$

Then the function $\hat{m}_G$ is a constant function.
If $1 \leq r_0 \leq k - 2$, we have
\[
w_p(D) = n - \{0 \leq i \leq n - 1\} \langle G_i, G_{i+1} \rangle \subseteq \tilde{D}^\perp, \dim(\langle G_i, G_{i+1} \rangle) = 1\} - \{0 \leq i \leq n - 1\} \langle G_i, G_{i+1} \rangle \subseteq \tilde{D}^\perp, \dim(\langle G_i, G_{i+1} \rangle) = 2\} = n - \hat{m}_{G}^{k-r_0}(\tilde{D}^\perp) - \hat{m}_{G}^{k-r_0}(\tilde{D}^\perp).
\]

(4.5)

Since the functions $\hat{m}_G$ is a constant function with value $a \in \mathbb{N}$, we have
\[
\hat{m}_{G}^{k-r_0}(\tilde{D}^\perp) = \sum_{V \in PG_2(\tilde{D}^\perp)} \hat{m}_G(V) = \frac{(q^{k-r_0} - 1)(q^{k-r_0} - q)}{(q^2 - 1)(q^2 - q)} a.
\]

Then we have
\[
w_p(D) = n - \hat{m}_{G}^{k-r_0}(\tilde{D}^\perp) - \frac{(q^{k-r_0} - 1)(q^{k-r_0} - q)}{(q^2 - 1)(q^2 - q)} a
\]

and the function $\hat{m}_{G}^{k-r_0}$ and $\hat{m}_G$ are constant functions by Lemma 4.5 since $C$ is a pair $r_0$-equiweight code.

For any $1 \leq r \leq k - 1$. There is a unique $\mathbb{F}_q$-subspace $\tilde{E}$ of $\mathbb{F}_q^k$ such that $E = \phi(\tilde{E})$ for any $\mathbb{F}_q$-subspace $E$ of dimension $r$ of $C$. If $r = k - 1$, by Equation (4.4), we have
\[
w_p(E) = n - \{0 \leq i \leq n - 1\} \langle G_i, G_{i+1} \rangle \subseteq \tilde{E}^\perp, \dim(\langle G_i, G_{i+1} \rangle) = 1\} = n - \hat{m}_G(\tilde{E}^\perp).
\]

Then $C$ is a pair $(k - 1)$-equiweight code since $\hat{m}_G$ is a constant function.

If $1 \leq r \leq k - 2$, by Equation (4.5), we have
\[
w_p(E) = n - \hat{m}_{G}^{k-r}(\tilde{E}^\perp) - \frac{(q^{k-r} - 1)(q^{k-r} - q)}{(q^2 - 1)(q^2 - q)} a
\]

Then $C$ is a pair $r$-equiweight code since $\hat{m}_G$ and $\hat{m}_{G}^{k-r}$ are constant functions.

In the next theorem, we give a necessary and sufficient condition for an $[n, k]$-linear code to be a pair $r$-equiweight code when $r = k - 1$.

**Theorem 4.7.** Assume the notations are given above. Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$ with a generator matrix $G = (G_0, \cdots, G_{n-1})$, and $k \geq 3$. Then $C$ is a pair $(k - 1)$-equiweight code if and only if the function $\hat{m}_G$ is a constant function.

**Proof.** Since the $\mathbb{F}_q$-linear map $\phi : \mathbb{F}_q^k \rightarrow C$ such that $\phi(y) = yG$ for any $y \in \mathbb{F}_q^k$ is a linear isomorphism, there is a unique $\mathbb{F}_q$-subspace $\tilde{D}$ of $\mathbb{F}_q^k$ such that $D = \phi(\tilde{D})$ for any $\mathbb{F}_q$-subspace $D$ with $\dim(D) = k - 1$ of $C$. By $\dim(\tilde{D}^\perp) = 1$ and Equation (4.4), we have
\[
w_p(D) = n - \{0 \leq i \leq n - 1\} \langle G_i, G_{i+1} \rangle \subseteq \tilde{D}^\perp, \dim(\langle G_i, G_{i+1} \rangle) = 1\} = n - \hat{m}_G(\tilde{D}^\perp).
\]

Hence the function $\hat{m}_G$ is a constant function if and only if $w_p(D)$ is invariant for any $D \in PG(C)^{k-1}$ if and only if $C$ is a pair $(k - 1)$-equiweight code.
In the next theorem, we obtain a sufficient condition for an \([n,k]-\)linear code to be a pair \(r\)-equiweight code when \(2 \leq r \leq k - 2\).

**Theorem 4.8.** Assume the notations are given above. Let \(C\) be an \([n,k]-\)linear code over \(\mathbb{F}_q\) with a generator matrix \(G = (G_0, \cdots, G_{n-1})\), and let \(2 \leq r \leq k - 2\). If \(\hat{m}_G(W_i) + \frac{1}{n_1,k-r-1} \sum_{L \in P G^2(W_i)} \hat{m}_G(L)\) is constant for all \(1 \leq i \leq n_{2,k}\), then \(C\) is a pair \(r\)-equiweight code.

**Proof.** Since the \(\mathbb{F}_q\)-linear map \(\phi : \mathbb{F}_q^k \rightarrow C\) such that \(\phi(y) = yG\) for any \(y \in \mathbb{F}_q^k\) is a linear isomorphism, there is a unique \(\mathbb{F}_q\)-subspace \(\tilde{D}\) of \(\mathbb{F}_q^k\) such that \(D = \phi(\tilde{D})\) for any \(\mathbb{F}_q\)-subspace \(D\) of \(C\) with \(\dim(D) = r\) for any \(2 \leq r \leq k - 2\). Then by Equation (4.5),

\[
w_k(D) = n - \hat{m}_G(\tilde{D}^\perp) - \hat{m}_G^k(\tilde{D}^\perp).
\]

We construct three vectors over \(\mathbb{Q}\) as follows:

\[
\alpha_V = (m_G^{k-r}(V_1), m_G^{k-r}(V_2), \cdots, m_G^{k-r}(V_{n_{k-r,k}}))
\]

for \(PG^{k-r}(\mathbb{F}_q^k) = \{V_1, V_2, \cdots, V_{n_{k-r,k}}\}\),

\[
\alpha_L = (\hat{m}_G(L_1), \hat{m}_G(L_2), \cdots, \hat{m}_G(L_{n_{1,k}}))
\]

for \(PG^1(\mathbb{F}_q^k) = \{L_1, L_2, \cdots, L_{n_{1,k}}\}\) and

\[
\alpha_W = (\hat{m}_G(W_1), \hat{m}_G(W_2), \cdots, \hat{m}_G(W_{n_{2,k}}))
\]

for \(PG^2(\mathbb{F}_q^k) = \{W_1, W_2, \cdots, W_{n_{2,k}}\}\).

By the definition of \(m_G^{k-1}\) and Lemma 4.1, we have

\[
\alpha_V = \alpha_L T_{1,k-r} + \alpha_W T_{2,k-r} = \frac{1}{n_{1,k-r-1}} \alpha_L T_{1,2} T_{2,k-r} + \alpha_W T_{2,k-r}
\]

\[
= \left(\frac{1}{n_{1,k-r-1}} \alpha_L T_{1,2} + \alpha_W\right) T_{2,k-r}.
\]

(4.6)

Since the element in \(i\) position of vector \(\frac{1}{n_{1,k-r-1}} \alpha_L T_{1,2} + \alpha_W\) is

\[
\hat{m}_G(W_i) + \frac{1}{n_{1,k-r-1}} \sum_{L \in PG^1(W_i)} \hat{m}_G(L)
\]

which is constant for all \(1 \leq i \leq n_{2,k}\) by assumption, we have that \(C\) is a pair \(r\)-equiweight code by Equation (4.6) and Lemma 4.1. \(\Box\)

In particular, when \(r = k - 2\), we have the following a necessary and sufficient condition for an \([n,k]-\)linear code to be a pair \((k - 2)\)-equiweight code.

**Corollary 4.9.** Let the notations be the same as above. Let \(C\) be an \([n,k]-\)linear code over \(\mathbb{F}_q\) with a generator matrix \(G = (G_0, \cdots, G_{n-1})\) and \(r = k - 2\). Then \(C\) is a pair \((k - 2)\)-equiweight code if and only if \(\hat{m}_G(W_i) + \sum_{L \in PG^1(W_i)} \hat{m}_G(L)\) is constant for all \(1 \leq i \leq n_{2,k}\).

**Proof.** When \(r = k - 2\), \(T_{2,k-r}\) in Equation (4.6) is the identity matrix. Hence \(C\) is a pair \((k - 2)\)-equiweight code if and only if \(\alpha_L T_{1,2} + \alpha_W\) is a constant vector. \(\Box\)
5 MacWilliams extension theorem for pair weights

In this section, we prove the MacWilliams extension theorem for the pair weight case.

The following example shows that an $\mathbb{F}_q$-linear isomorphism between two linear codes which preserves the pair weight maybe not induced by a monomial matrix.

Example 5.1. Let $C$ and $\tilde{C}$ be two linear codes with the generator matrices
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1
\end{pmatrix}
\]
over $\mathbb{F}_2$ respectively, let $f$ be an $\mathbb{F}_q$-linear isomorphism from $C$ to $\tilde{C}$ such that $f(x) = xM$, where $M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$. It is obvious that $f$ preserves the pair weight of any codeword in $C$, but $f$ cannot be induced by a monomial matrix.

In order to give our main result in this section, we have the following notations.

Let $C$ be an $[n, k]$-linear code with a generator matrix $G = (G_0, \cdots, G_{n-1})$. Suppose $|\{W \in PG^2(\mathbb{F}_q^k) | \tilde{m}_G(W) > 0\}| = t$, and assume $\{W \in PG^2(\mathbb{F}_q^k) | \tilde{m}_G(W) > 0\} = \{W_1, W_2, \cdots, W_t\}$.

Let $M \in M_n(\mathbb{F}_q)$ and $\tilde{G} = GM = (\tilde{G}_0, \tilde{G}_1, \cdots, \tilde{G}_{n-1})$. Note that $n_0(\tilde{G}) = n_0(GM) = |\{0 \leq i \leq n-1 | \langle G_i, \tilde{G}_{i+1} \rangle = 0\}|$. Let $\hat{m}_{G, \tilde{G}} = \tilde{m}_G - \hat{m}_G$, and let $\hat{m}_{G, \tilde{G}} = \tilde{m}_G - \hat{m}_G$.

Let $\Omega_p^G$ be a subset of $M_n(\mathbb{F}_q)$ such that for any $M \in \Omega_p^G$, $M$ satisfies the following two conditions.

**Condition A:** for any $1 \leq i \leq n_{1,k}$,
\[
\hat{m}_{G, \tilde{G}}(L_i) + \frac{1}{q} \left( \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_{G, \tilde{G}}(W_s) - \frac{q-1}{q^k-1} \sum_{s=1}^{t} \hat{m}_{G, \tilde{G}}(W_s) \right) = \frac{n_0(\tilde{G})}{n_{1,k-1}}.
\]

**Condition B:** Let $M = (M_0, \cdots, M_{n-1})$. For any $0 \leq i \leq n-1$, $w_p(M_i^T) = 2$ or $3$.

Then we have the following theorem.

**Theorem 5.2.** Let the notations be the same as above. Let $C$ and $\tilde{C}$ be two $[n, k]$-linear codes over $\mathbb{F}_q$ and $G = (G_0, \cdots, G_{n-1})$ be a generator matrix of $C$ for $k \geq 2$. Then there exists an $\mathbb{F}_q$-linear isomorphism $f : C \rightarrow \tilde{C}$ which preserves the pair weight if and only if there exists a matrix $M \in \Omega_p^G$ such that $f(c) = cM$ for all $c \in C$.

**Proof.** Suppose there exists an $M \in \Omega_p^G$ such that $f(c) = cM$ for all $c \in C$. Then for any $0 \neq c \in C$, there is a nonzero vector $y \in \mathbb{F}_q^k$ such that $c = yG$ and
\[
f(c) = cM = (yG)M = y(GM) = y\tilde{G}.
\]
By Lemma 2.12

\[ w_p(c) = n - m_G^{k-1}(\langle y \rangle^\perp), \quad w_p(f(c)) = n - n_0(\tilde{G}) - m_G^{k-1}(\langle y \rangle^\perp), \]  

(5.1)

where \( \tilde{G} = GM = (\tilde{G}_0, \tilde{G}_1, \ldots, \tilde{G}_{n-1}) \) and \( n_0(\tilde{G}) = |\{0 \leq i \leq n-1 | \langle \tilde{G}_i, \tilde{G}_{i+1} \rangle = 0 \}| \) by Lemma 2.12. Recall that

\[ PG^1(\mathbb{F}_q^k) = \{L_1, L_2, \ldots, L_{n_1,k}\}, \quad PG^2(\mathbb{F}_q^k) = \{W_1, W_2, \ldots, W_{n_2,k}\}, \]

\[ PG^{k-1}(\mathbb{F}_q^k) = \{M_1 = L_1^\perp, M_2 = L_2^\perp, \ldots, M_{n_1,k} = L_{n_1,k}^\perp\}, \]

\[ \{W_1, W_2, \ldots, W_t\} = \{W \in PG^2(\mathbb{F}_q^k) | \tilde{m}_G(W) > 0\}. \]

We construct six vectors over \( \mathbb{Q} \) as follows:

\[ \alpha_M = (m_G^{k-1}(M_1), \ldots, m_G^{k-1}(M_{n_1,k})), \quad \beta_M = (m_G^{k-1}(M_1), \ldots, m_G^{k-1}(M_{n_1,k})), \]

\[ \alpha_L = (\tilde{m}_G(L_1), \ldots, \tilde{m}_G(L_{n_1,k})), \quad \beta_L = (\tilde{m}_G(L_1), \ldots, \tilde{m}_G(L_{n_1,k})), \]

\[ \alpha_W = (\tilde{m}_G(W_1), \ldots, \tilde{m}_G(W_{n_2,k})), \quad \beta_W = (\tilde{m}_G(W_1), \ldots, \tilde{m}_G(W_{n_2,k})). \]

By the definition of \( m_G^{k-1} \), it is easy to check that \( \alpha_M = \alpha_L T_{1,k-1} + \alpha_W T_{2,k-1} \) and \( \beta_M = \beta_L T_{1,k-1} + \beta_W T_{2,k-1} \). By Lemma 4.1 we have

\[ \alpha_M - \beta_M = (\alpha_L - \beta_L) T_{1,k-1} + (\alpha_W - \beta_W) T_{2,k-1} \]

\[ = ((\alpha_L - \beta_L) + (\alpha_W - \beta_W) T_{2,k-1} T_{1,k-1}^{-1}) T_{1,k-1}. \]  

(5.2)

Note that \( M \in \Omega_{p}^G \). By Lemma 4.1 for all \( 1 \leq i \leq n_1,k \), the \( i \)th entry of \( (\alpha_L - \beta_L) + (\alpha_W - \beta_W) T_{2,k-1} T_{1,k-1}^{-1} \) is

\[ \tilde{m}_{G,\tilde{G}}(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} \tilde{m}_{G,\tilde{G}}(W_s) - \frac{q-1}{q(q^k-1)} \sum_{s=1}^{t} \tilde{m}_{G,\tilde{G}}(W_s) = \frac{n_0(\tilde{G})}{n_{1,k-1}}. \]

We have

\[ \alpha_M - \beta_M = \left(\frac{n_0(\tilde{G})}{n_{1,k-1}}, \frac{n_0(\tilde{G})}{n_{1,k-1}}, \ldots, \frac{n_0(\tilde{G})}{n_{1,k-1}}\right) T_{1,k-1} = (n_0(\tilde{G}), n_0(\tilde{G}), \ldots, n_0(\tilde{G})). \]

For any \( c \in C \), by Equation (5.1) and \( \langle y \rangle^\perp \in PG^{k-1}(\mathbb{F}_q^k) \), we have

\[ w_p(c) - w_p(f(c)) = n_0(\tilde{G}) + m_G^{k-1}(\langle y \rangle^\perp) - m_G^{k-1}(\langle y \rangle^\perp) = 0. \]

Hence \( \mathbb{F}_q \)-linear map \( f : C \rightarrow \tilde{C} \) which preserves the pair weight and \( ker(f) = 0 \), since the vector \( (0, 0, \ldots, 0) \) is only vector in \( C \) which is pair weight 0. Therefore, \( f : C \rightarrow \tilde{C} \) is an \( \mathbb{F}_q \)-linear isomorphism which preserves the pair weight of any codeword in \( C \).
Now we assume there exists an $\mathbb{F}_q$-linear isomorphism $f: C \to \tilde{C}$ such that $w_p(c) = w_p(f(c))$ for any $c \in C$. Let $G = \left( \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \right)$ for some $g_i \in \mathbb{F}^n_q$. Then $\tilde{G} = \left( \begin{array}{c} f(g_1) \\ \vdots \\ f(g_k) \end{array} \right)$ is a generator matrix of $\tilde{C}$.

Let $\lambda$ be an $\mathbb{F}_q$-linear isomorphism from $\mathbb{F}_q^k$ to $C$ such that $\lambda(y) = yG$ for any $y \in \mathbb{F}_q^k$, and let $\mu$ be an $\mathbb{F}_q$-linear isomorphism from $\mathbb{F}_q^k$ to $\tilde{C}$ such that $\mu(y) = y\tilde{G}$ for any $y \in \mathbb{F}_q^k$. Hence $\mu = f \circ \lambda$. Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ and $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$ such that $\lambda_i$ and $\mu_i$ are $\mathbb{F}_q$-linear maps from $\mathbb{F}_q^k$ to $\mathbb{F}_q$. Then $w_p(c) = w_p(f(c))$ for any $c \in C$ if and only if

$$w_p((\lambda_0(y), \lambda_1(y), \ldots, \lambda_{n-1}(y))) = w_p((\mu_0(y), \mu_1(y), \ldots, \mu_{n-1}(y)))$$

for any $y \in \mathbb{F}_q^k$. By the definition of pair weights, the above equality gives that

$$|\{0 \leq i \leq n-1 | (\lambda_i(y), \lambda_{i+1}(y)) \neq (0, 0)\}| = |\{0 \leq i \leq n-1 | (\mu_i(y), \mu_{i+1}(y)) \neq (0, 0)\}|$$

for any $y \in \mathbb{F}_q^k$. This implies that

$$|\{0 \leq i \leq n-1 | (\lambda_i(y), \lambda_{i+1}(y)) = (0, 0)\}| = |\{0 \leq i \leq n-1 | (\mu_i(y), \mu_{i+1}(y)) = (0, 0)\}|$$

for any $y \in \mathbb{F}_q^k$. By Corollary 2.2, for any $y \in \mathbb{F}_q^k$, we have

$$\sum_{i=0}^{n-1} (\sum_{a \in \mathbb{F}_q} \zeta^{\text{tr} a \lambda_i(y)})(\sum_{b \in \mathbb{F}_q} \zeta^{\text{tr} b \lambda_{i+1}(y)}) = \sum_{i=0}^{n-1} (\sum_{a \in \mathbb{F}_q} \zeta^{\text{tr} a \mu_i(y)})(\sum_{b \in \mathbb{F}_q} \zeta^{\text{tr} b \mu_{i+1}(y)}).$$

The above equality is

$$\sum_{i=0}^{n-1} \sum_{a \in \mathbb{F}_q} \sum_{b \in \mathbb{F}_q} \zeta^{\text{tr}(a \lambda_i(y) + b \lambda_{i+1}(y))} = \sum_{i=0}^{n-1} \sum_{a \in \mathbb{F}_q} \sum_{b \in \mathbb{F}_q} \zeta^{\text{tr}(a \mu_i(y) + b \mu_{i+1}(y))}$$

for any $y \in \mathbb{F}_q^k$. Therefore, we have

$$\sum_{i=0}^{n-1} \sum_{a \in \mathbb{F}_q} \sum_{b \in \mathbb{F}_q} \exp(a \lambda_i + b \lambda_{i+1}) = \sum_{i=0}^{n-1} \sum_{a \in \mathbb{F}_q} \sum_{b \in \mathbb{F}_q} \exp(a \mu_i + b \mu_{i+1}).$$

Hence,

$$\sum_{i=0}^{n-1} \sum_{a,b \in \mathbb{F}_q, (a,b) \neq (0,0)} \exp(a \lambda_i + b \lambda_{i+1}) = \sum_{i=0}^{n-1} \sum_{a,b \in \mathbb{F}_q, (a,b) \neq (0,0)} \exp(a \mu_i + b \mu_{i+1})$$

which is an equation of characters of $\mathbb{F}_q^k$.

Since the characters of $\mathbb{F}_q^k$ are linearly independent over $\mathbb{C}$, the characters on the left side of the last equation must match up with those on the right side. Then there must be an index $0 \leq s(i) \leq n-1$ such that $a_{s(i)}, b_{s(i)} \in \mathbb{F}_q$, $(a_{s(i)}, b_{s(i)}) \neq (0, 0)$ and

$$\exp(\mu_i) = \exp(a_{s(i)} \lambda_{s(i)} + b_{s(i)} \lambda_{s(i)+1})$$

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for $0 \leq i \leq n - 1$. Since $\exp : V = \text{Hom}_{\mathbb{F}_q}(V, \mathbb{F}_q) \to \hat{V}$ is an $\mathbb{F}_q$-linear isomorphism, we have $\mu_i = a_{s(i)}\lambda_{s(i)} + b_{s(i)}\lambda_{s(i)+1}$. Then we can construct a matrix $M$ in $M_n(\mathbb{F}_q)$ such that

$$M = (a_{ij})_{n \times n}, \text{ where } a_{ij} = \begin{cases} a_{s(j)}, & \text{if } i = s(j); \\ b_{s(j)}, & \text{if } i = s(j) + 1; \\ 0, & \text{other.} \end{cases}$$

It is easy to check that $M$ satisfies Condition B. Hence for any $y \in \mathbb{F}_q^k$, we have

$$(\mu_0(y), \mu_1(y), \ldots, \mu_{n-1}(y)) = (\lambda_0(y), \lambda_1(y), \ldots, \lambda_{n-1}(y)) M$$

and $f(c) = c M$.

Recall that $w_p(c) = n - n^{k-1}_G(\langle y \rangle^\perp)$, $w_p(f(c)) = n - n_0(\tilde{G}) - n^{k-1}_G(\langle y \rangle^\perp)$, and $w_p(f(c)) = w_p(f(c))$ for any $c \in C$, we have

$$(n_0(\tilde{G}), n_0(\tilde{G}), \ldots, n_0(\tilde{G})) = \alpha_M - \beta_M = (\alpha_L - \beta_L)T_1_{k-1} + (\alpha_W - \beta_W)T_{2,k-1}$$

by Equation (5.1) and $\langle y \rangle^\perp \in PG^{k-1}(\mathbb{F}_q^k)$. By Lemma 4.1 again, for all $1 \leq i \leq n_{1,k}$, the $i$th element of the vector $(\alpha_L - \beta_L) + (\alpha_W - \beta_W)T_{2,k-1}T_{1,k-1}^{-1}$ is

$$m_{G,\tilde{G}}(L_i) + \frac{1}{q} \sum_{1 \leq s \leq t, L_i \subseteq W_s} m_{G,\tilde{G}}(W_s) - \frac{q - 1}{q(q^k - 1)} \sum_{s=1}^t m_{G,\tilde{G}}(W_s).$$

Therefore, we have

$$m_{G,\tilde{G}}(L_i) + \frac{1}{q} \left( \sum_{1 \leq s \leq t, L_i \subseteq W_s} m_{G,\tilde{G}}(W_s) - \frac{q - 1}{q^k - 1} \sum_{s=1}^t m_{G,\tilde{G}}(W_s) \right) = \frac{n_0(\tilde{G})}{n_{1,k-1}},$$

which is Condition A. Hence $f(x) = x M$, where $M \in \Omega_p^G$. \hfill \Box

**Remark 5.3.** It is easy to check that the matrix $M$ in Example 5.1 satisfies $M \in \Omega_p^G$.

We denote the set of all $n \times n$ permutation matrices over $\mathbb{F}_q$ and the set of all $n \times n$ invertible diagonal matrices over $\mathbb{F}_q$ by $P_n(\mathbb{F}_q)$ and $D_n(\mathbb{F}_q)$, respectively. It is easy to see that $MO_n(\mathbb{F}_q) \cong D_n(\mathbb{F}_q) \rtimes P_n(\mathbb{F}_q)$ and $P_n(\mathbb{F}_q) \cong S_n$ which is the symmetric group on the set $X = \{0, 1, \ldots, n - 1\}$.

Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$ with a generator matrix $G = \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}$. For any $\gamma \in S_n$, we define $\gamma(x) = (x_{\gamma(0)}, x_{\gamma(1)}, \ldots, x_{\gamma(n-1)})$ for $x = (x_0, x_1, \ldots, x_{n-1})$, $\gamma(G) = \begin{pmatrix} \gamma(g_1) \\ \vdots \\ \gamma(g_k) \end{pmatrix}$ and $\gamma(C) = \{\gamma(c) | c \in C\}$. It is easy to see that $\gamma(G)$ is a generator matrix of $\gamma(C)$.

By using Theorem 5.2, we can see whether a permutation preserves the pair weight or not in Corollary 5.3 below.
Corollary 5.4. Let the notations be the same as above. Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$ with a generator matrix $G = (G_0, \cdots, G_{n-1})$, and let $n \geq 4$. Then a permutation $\gamma \in S_n$ is an $\mathbb{F}_q$-linear isomorphism preserving the pair weight from $C$ to $\gamma(C)$ if and only if for all $1 \leq i \leq n_{1,k}$, 

$$\hat{m}_{G, \gamma(G)}(L_i) + \frac{1}{q} \left( \sum_{1 \leq s \leq t, L_i \subseteq W_s} \hat{m}_{G, \gamma(G)}(W_s) - \frac{q-1}{q^k-1} \sum_{s=1}^{t} \hat{m}_{G, \gamma(G)}(W_s) \right) = \frac{n_0(\gamma(G))}{n_{1,k-1}}.$$ 

Proof. For any $\gamma \in S_n$, there exists $M \in P_n(\mathbb{F}_q)$ such that $\gamma(c) = cM$ for any $c \in C$ since $P_n(\mathbb{F}_q) \cong S_n$. The rest part of proof is trivial by using Theorem 5.2. \qed

Remark 5.5. In the corollary above, we only consider the condition of $n \geq 4$. When $n = 1, 2$ or $3$, any permutation $\gamma \in S_n$ is an $\mathbb{F}_q$-linear isomorphism preserving the pair weight from $C$ to $\gamma(C)$.

Example 5.6. Let $C$ be the linear code with the generator matrix

$$\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}$$

over $\mathbb{F}_2$, then the set of all permutations preserving the pair weight of any element in $C$ is

$$\{(0), (01), (13), (23), (01)(23), (0123), (0321)\} \cong D_8.$$ 

So if we want to know a linear isomorphism induced by $M \in \text{MO}_n(\mathbb{F}_q)$ which preserves the pair weight of any element in $C$ or not, we only need to find out that the permutation part $\gamma_M$ of $M$ by $\text{MO}_n(\mathbb{F}_q) \cong D_n(\mathbb{F}_q) \rtimes P_n(\mathbb{F}_q)$ is a permutation preserving the pair weight of any element in $C$ or not, since any linear isomorphism induced by an element in $D_n(\mathbb{F}_q)$ preserves the pair weight of any element in $C$.

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References

[1] K. Bogart, D. Goldberg, and J. Gordon, ”An elementary proof of the MacWilliams theorem on equivalence of codes,” Information and Control, vol. 37, no. 1, pp. 19-22, 1978.

[2] Y. Cassuto and M. Blaum, ”Codes for symbol-pair read channels,” IEEE Transactions on Information Theory, vol. 57, no. 12, pp. 8011-8020, 2011.

[3] B. Chen, L. Lin, and H. Liu, ”Constacyclic symbol-pair codes: lower bounds and optimal constructions,” IEEE Transactions on Information Theory, vol. 63, no. 12, pp. 7661-7666, 2017.

[4] H. Q. Dinh, B. T. Nguyen, A. K. Singh, and S. Sriboonchitta, ”On the symbol-pair distance of repeated-root constacyclic codes of prime power lengths,” IEEE Transactions on Information Theory, vol. 64, no. 4, pp. 2417-2430, 2017.
[5] H. Q. Dinh, X. Wang, H. Liu, and S. Sriboonchitta, "On the symbol-pair distances of repeated-root constacyclic codes of length $2p^s$," Discrete Mathematics, vol. 342, no. 11, pp. 3062-3078, 2019.

[6] S. T. Dougherty, S. Han, and H. Liu, "Higher weights for codes over rings," Applicable Algebra in Engineering, Communication and Computing, vol. 22, no. 2, pp. 113-135, 2011.

[7] Y. Fan and H. Liu, "Generalized Hamming EquiWeight Linear Codes," Acta Electronica Sinica, vol. 31, no. 10, pp. 1591-1593, 2003.

[8] G. Jian, R. Feng, and H. Wu, "Generalized Hamming weights of three classes of linear codes," Finite Fields and Their Applications, vol. 45, no. 5, pp. 341-354, 2017.

[9] S. Liu, C. Xing, and C. Yuan, "List Decodability of Symbol-Pair Codes," IEEE Transactions on Information Theory, vol. 65, no. 8, pp. 4815-4821, 2019.

[10] J. MacWilliams, "A theorem on the distribution of weights in a systematic code," Bell System Technical Journal, vol. 42, no. 1, pp. 79-94, 1963.

[11] M. A. Tsfasman and S. G. Vladut, "Geometric approach to higher weights," IEEE Transactions on Information Theory, vol. 41, no. 6, pp. 1564-1588, 1995.

[12] H. N. Ward and J. A. Wood, "Characters and the equivalence of codes," Journal of Combinatorial Theory, Series A, vol. 73, no. 2, pp. 348-352, 1996.

[13] V. K. Wei, "Generalized Hamming weights for linear codes," IEEE Transactions on information theory, vol. 37, no. 5, pp. 1412-1418, 1991.

[14] E. Weiss, "Linear codes of constant weight," SIAM Journal on Applied Mathematics, vol. 14, no. 1, pp. 106-111, 1966.

[15] J. A. Wood, "Duality for modules over finite rings and applications to coding theory," American Journal of Mathematics, vol. 121, no. 3, pp. 555-575, 1999.

[16] J. A. Wood, "The structure of linear codes of constant weight," Transactions of the American Mathematical Society, vol. 354, no. 3, pp. 1007-1026, 2002.

[17] J. A. Wood, "Code equivalence characterizes finite Frobenius rings," Proceedings of the American Mathematical Society, vol. 136, no. 2, pp. 699-706, 2008.

[18] E. Yaakobi, J. Bruck, and P. H. Siegel, "Constructions and decoding of cyclic codes over $b$-symbol read channels," IEEE Transactions on Information Theory, vol. 62, no. 4, pp. 1541-1551, 2016.

[19] M. Yang, J. Li, K. Feng, and D. Lin, "Generalized Hamming weights of irreducible cyclic codes," IEEE Transactions on Information Theory, vol. 61, no. 9, pp. 4905-4913, 2015.