A Discussion on the Forms of Planar Mixed-Mode Fracture Criteria

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Abstract: For the mixed-mode fracture on an original crack plane, the fracture criteria are usually based on the coupling of $K_1/K_{IC}$ and $K_{II}/K_{IC}$. Extended to general potential fracture planes, they become $K_1(\theta)/K_{IC}$ and $K_{II}(\theta)/K_{IC}$. Specifically, three forms have been widely used: linear, elliptical, and quadratic forms. The question of how these criteria perform is addressed by this research. This study does not intend to provide a general assessment but focused on only one case in which an analytical solution can readily be obtained. Specifically, we considered the performance of these criteria under Mode I loading, namely, $K_1 \neq 0$, $K_{II} = 0$. The analytical solution for fracture initiation angle under each of the three mixed-mode criteria were presented, where the results demonstrated that only the elliptical form yielded accurate results. That is, under Mode I loading, the initiation of pure Mode I fracture takes precedence that agrees well with standard results. What’s more, the mathematical requirements are obtained for the applicable mixed-mode fracture criteria.

Keywords: mixed-mode, fracture criteria, mathematical requirement, initiation angle

1. Introduction

Classical fracture mechanics defines three basic fracture modes: Mode I, Mode II, and Mode III, and are expected to propagate along the plane of a pre-existing defect according to the stress analysis of Paris and Sih [1]. But these kinds of coplanar fractures are not always observed in experiments. For example, under the geometry of Mode II, a tensile crack at a large angle to the pre-existing crack plane has been widely observed in experiments instead of a planar shear fracture [2-5]. Thus, it is worth noting that loading condition and fracture mode are different concepts [6]. Nevertheless, loading conditions correspond to the classical definitions, and fracture modes are classified according to the failure mechanism of the fracture. The Mode I fracture is caused by only tensile stress, and thus can also be named as a pure tensile fracture. The Mode II fracture is caused by only shear stress, and thus can be referred to as a shear fracture. Both tensile and shear stresses cause the mixed-mode fracture, and thus can be referred to as the tensile shear fracture. This definition of fracture mode was already used in some literature [7-10], especially the clarification made by Rao et al. [10].
Because the concepts of loading condition and fracture mode are different, the mixed-mode fracture criteria in the literature can be classified into two categories. For the first category, the mixed-mode fracture criterion is constructed using the stress intensity factors on the initial crack plane, which are $K_I$, $K_{II}$. For example, Aliha [11] employed a minimum strain energy density criterion to predict Mode II fracture toughness of asphalt concrete. Demir et al. [12] also developed an improved empirical in-plane I/II mixed-mode fracture criterion with consistent terms. This category of fracture criterion can predict the initiation direction of a fracture and the fracture toughness. However, most of these criteria are based on the assumption of tensile fractures that are not along the original crack plane. The fracture toughness obtained corresponds to the stress intensity factors on the initial crack plane. Thus, this kind of fracture toughness only represents the loading conditions when the fracture initiates regardless of the direction and varies with loading conditions, meaning that it cannot be used as a material property. Besides, these criteria yield a Mode II fracture toughness less than Mode I, contrary to the fracture property of most rocks with much higher shear strength than tensile strength.

Another category of mixed-mode fracture criteria is constructed by the stress intensity factors on the potential fracture plane, $K_I(\theta)$, $K_{II}(\theta)$, where $\theta$ indicates the direction of fracture propagation. For example, Hou et al. [13] proposed a generalized maximum energy release rate criterion for mixed-mode fracture analysis. Rao et al. [10] and Liu et al. [14] employed the Mohr-Coulomb criterion to study compressive shear fracture. This category of criteria is based on the stress state around the crack tip, and the resulted fracture toughness is consistent with the stress intensity factor. Thus, this category of criteria is more appropriate for revealing fracture properties of brittle materials and, therefore, is this study's focus.

Most of the current research are based on are experimental studies of mixed-mode fractures. Some fracture criteria are fitted according to the test results of fracture toughness [15-17] and then are used as the basic assumptions for the theoretical analysis of mixed-mode fractures [18-20]. However, none of these researches considered the mathematical modeling of the mixed-mode fracture criterion. Specifically, the mathematical form of the second category of mixed-mode fracture criteria is not arbitrary. An inaccurate form could yield unrealistic analysis results, as presented in the following sections. Hence, the mathematical modeling is developed and discussed to avoid such mistakes in future researches.

2. Stress intensity factor on slant crack plane
A straight crack with sharp tips in a planar plate is considered in this study. The Mode I and II stress intensity factors are expressed as $K_I$ and $K_{II}$ under any planar loading. Figure 1 shows the stress in terms of the polar coordinate system. For the mixed-mode loading, the stress equations are expressed as [21]

$$
\begin{bmatrix}
\sigma_r \\
\sigma_\theta \\
\tau_{r\theta}
\end{bmatrix}
=
\begin{bmatrix}
\frac{K_I}{\sqrt{2\pi r}}
\cos\frac{\theta}{2}
&
\frac{\sin\frac{\theta}{2}}{2}
&
\frac{\cos^2\frac{\theta}{2}}{2}
\end{bmatrix}
\begin{bmatrix}
1
\frac{\sin^2\frac{\theta}{2}}{2}
\end{bmatrix}
+\begin{bmatrix}
\frac{K_{II}}{\sqrt{2\pi r}}
\sin\frac{\theta}{2}(1-3\sin^2\frac{\theta}{2})
\cos\frac{\theta}{2}
&
-3\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2}
\end{bmatrix},
$$

(1)

where $\theta$ is the angle away from the original crack plane, as shown in Figure 1, and ranges from $-180^\circ$ to $+180^\circ$. Mode I’s stress intensity factor is negative if the crack is under compression in its normal direction. Eq. (1) is still applicable when the crack is open under compression; hence, an open crack is assumed through all the analyses.
Figure 1. Definitions of the coordinate system and positive direction. (a) Stress components at a point near a crack tip in the polar coordinate system. (b) Positive direction of $\theta$ when $K_{II} > 0$, (c) Positive direction of $\theta$ when $K_{II} < 0$.

As shown in Figure 1(c), a negative Mode II stress intensity factor exists, but its absolute value could always be used in Eq. (1) because the shear direction is meaningless. For example, if Figure 1(c) is flipped along the original crack plane, the solution should be the same as that shown in Figure 1(b). Therefore, the positive direction of $\theta$ is defined differently for the loadings shown in Figures 1(b) and 1(c) to obtain the same result and for derivation convenience. Here a simple demonstration is presented for this equivalence. Substituting $-K_{II}$ and $-\theta$ which represent the condition of Figure 1(c) in the coordinate of Figure 1(a) into Eq. (1) yields:

$$
\begin{bmatrix}
\sigma_r \\
\sigma_\theta \\
-\tau_{r\theta}
\end{bmatrix} = \frac{K_{II}}{\sqrt{2\pi r}} \begin{bmatrix}
1 + \sin^2 \frac{\theta}{2} \\
\cos \frac{\theta}{2} \\
\sin \frac{\theta}{2} \cos \frac{\theta}{2}
\end{bmatrix} + \frac{K_{I}}{\sqrt{2\pi r}} \begin{bmatrix}
\sin \frac{\theta}{2} \left(1 - 3\sin^2 \frac{\theta}{2}\right) \\
-3\sin \frac{\theta}{2} \cos \frac{\theta}{2} \\
\cos \frac{\theta}{2} \left(1 - 3\sin^2 \frac{\theta}{2}\right)
\end{bmatrix}.
$$

Compared with Eq. (1), Eq. (2) has a negative sign for $\tau_{r\theta}$. The shear direction does not affect fracture initiation, and with its absolute value being the essential parameter. As a result, the solution from Eq. (1) with a positive value of $K_{II}$ and positive directions of $\theta$ shown in Figures 1(b) and 1(c) is sufficient enough to represent all loading conditions.

From the perspective of the original definition, the stress intensity factor is defined based on an original crack plane. The pure Mode I loading signifies only tensile stress is applied on the original crack plane, whereas the pure Mode II loading signifies only shear stress is applied on same. Furthermore, the mixed-mode loading signifies applying both tensile and shear stresses on the original crack plane. But tensile and shear stresses also exist on the other planes under any external loading. The principal stress in a tension may exist on a specific plane, where a pure tensile fracture may be caused under any loading other than Mode I loading. So stress intensity factor should also be defined on a slant plane as presented by Rao et al. [10]:

$$
\begin{align*}
K_I(\theta) &= \lim_{r \to 0} \sigma_\theta \sqrt{2\pi r} \\
K_{II}(\theta) &= \lim_{r \to 0} \tau_{r\theta} \sqrt{2\pi r}.
\end{align*}
$$

Substituting Eq. (1) into Eq. (3) yields
\[
K_1(\theta) = K_i \cos^2 \theta - 3K_{II} \sin \frac{\theta}{2} \cos \frac{\theta}{2}, \\
K_{II}(\theta) = K_i \sin \frac{\theta}{2} \cos^2 \theta + K_{II} \cos \frac{\theta}{2} \left(1-3 \sin \frac{\theta}{2}\right).
\] (4)

Note that stress intensity factors \(K_N\) and \(K_N(\theta)\) (\(N\) indicates the mode of stress state) are different. Here \(K_N\) is a specific value of \(K_N(\theta)\) under pure Mode \(N\) loading along the original crack plane.

This kind of definition facilitates studies on mixed-mode fracture problems and gives a new perspective on the fracture mode of crack. In the traditional perspective, Mode I fracture means that tensile crack initiates and propagates along the original crack plane. Mode II fracture means that shear crack initiates and propagates along the original crack plane. Still, such kind of pure shear crack has never been observed in the laboratory for brittle materials. A mixed-mode fracture means that apart from the original crack plane, a slanting crack initiates and propagates. The fracture criterion for Mode I is given by \(K_1 = K_{IC}\), since Mode I fracture is on the original crack plane. The fracture criterion for Mode II is given by \(K_{II} = K_{IIIC}\), but \(K_{II}\) is the stress intensity factor of the original crack plane, and will not be equal to \(K_{IIIC}\) after the crack initiates obliquely, meaning this criterion for Mode II fracture does not work. The fracture criterion for mixed-mode does not have a uniform equation and can be classified as maximum stress criterion [22], maximum energy release rate criterion [23], and minimum strain energy density criterion [24] in practice. All these criteria give the same result \(K_{IIIC} < K_{IC}\) [21], which is not true for brittle materials since the shear strength of brittle materials is known to be larger than its tensile strength [10]. The reason for misinterpreting \(K_{IC}\) is not the inaccuracy of these criteria; besides, they are correct but incorrectly treat a slant tensile fracture as a fracture crack of Mode II. So the traditional definition of stress intensity factor seems insufficient in solving Mode II and mixed-mode fracture problems.

With this new perspective, no matter which direction the crack initiates and propagates, fracture mode is defined as pure Mode I if the material is split by pure tensile stress, and shear mode (including pure Mode II and compressive shear mode) if a material is split by shear stress with zero or nonzero compressive stress, and mixed-mode if a material is split by both tensile and shear stresses. It is easier to inspect the criteria and angles of fracture initiation for different fracture modes in the whole material domain instead of on the original crack plane.

3. Performance of some fracture criteria in Mode I fracture problem

The mixed-mode fracture criterion is a basic assumption for fracture analysis. Results may vary if the adopted mixed-mode fracture criterion is different, so the criterion cannot be an arbitrary expression.

As two special cases of mixed-mode fracture, the criteria for pure Mode I and pure Mode II must be considered in the mixed-mode fracture criterion, and are expressed as

\[
\begin{align*}
K_1(\theta)/K_{IC} &= 1 \text{ when } K_{II}(\theta) = 0, \\
K_{II}(\theta)/K_{IC} &= 1 \text{ when } K_1(\theta) = 0.
\end{align*}
\] (5)

But these two conditions are still not enough. A criterion that satisfies both requirements may lead to an unreasonable fracture initiation when the mixed-mode fracture criterion is used in fracture analysis. Here three criteria obtained from reference [21] are taken as examples, and their equations are transformed from the expression of \(K_1\) and \(K_{II}\) to the expression of \(K_1(\theta)\) and \(K_{II}(\theta)\).
where $C$ is a constant value.

All these three criteria satisfy the conditions of Eq. (5). Multiplying $K_{IC}$ on both sides of the straight line criterion yields $K_{IC}$ on the right side, and multiplying $K_{IC}^2$ on both sides of the ellipse and the quadratic criterion yields $K_{IC}^2$ on the right side. Thus the equivalent stress intensity factors $K_{\text{mix}}^e(\theta)$ are defined as

$$
K_{\text{mix}}^e(\theta) = K_1(\theta) + K_0(\theta)
$$

$$
K_{\text{mix}}^e(\theta) = \sqrt{K_1^2(\theta) + k_c^2 K_0^2(\theta)}
$$

$$
K_{\text{mix}}^e(\theta) = \sqrt{K_1^2(\theta) + C k_c K_1(\theta) K_0(\theta) + C_k K_0^2(\theta)}
$$

where $k_c = K_{IC}/K_{IC}$.

Although the angle of maximum mixed-mode stress intensity factor can be solved by

$$
\frac{\partial K_{\text{mix}}^e(\theta)}{\partial \theta} = 0,
\frac{\partial^2 K_{\text{mix}}^e(\theta)}{\partial \theta^2} < 0, -180^\circ < \theta \leq \theta_{\text{II pure}},
$$

where $\theta_{\text{II pure}}$ is determined according to $K_0(\theta) \geq 0$ to obtain the solution in the tensile stress domain, and can be solved as [14]:

$$
\theta_{\text{II pure}} = 2 \arctan \left( \frac{K_1}{K_0} \right),
$$

According to Eq. (8), the procedure is so complicated such that it would be difficult to solve the third equation. Thus, only Mode I loading ($K_1 \neq 0, K_0 = 0$), which is enough to explain this problem, is considered here. Substituting Eq. (4) and $K_0 = 0$ into Eq. (7) yields:

$$
K_{\text{mix}}^e(\theta) = K_1 \cos \theta \left( \frac{\theta}{2} + k_c \frac{\sin \theta}{2} \right)
$$

$$
K_{\text{mix}}^e(\theta) = K_1 \cos \theta \left( \frac{\cos \theta}{2} + k_c \frac{\sin \theta}{2} \right)
$$

$$
K_{\text{mix}}^e(\theta) = K_1 \cos \theta \left( \frac{\cos \theta}{2} + C k_c \frac{\sin \theta}{2} \cos \theta \right) + k_c \frac{\sin \theta}{2}
$$

Then, the procedure of solution becomes much easier, and the angles $\theta_{\text{mix max}}$ of maximum stress intensity factor are respectively solved as:
\[
\begin{align*}
\theta_{\text{mix max}} &= 2 \arctan \left( \frac{1}{4} \sqrt{\frac{9}{k_c^2} + 8 - \frac{3}{4k_c}} \right) \neq 0 \\
\theta_{\text{mix max}} &= 0 \quad (k_c < \sqrt{3}) \\
\theta_{\text{mix max}} &= 2 \arctan \left( \frac{3}{2} p + \sqrt{p^2 + q^3} + \frac{3}{2} p - \sqrt{p^2 + q^3} - \frac{5C}{12k_c} \right) \neq 0
\end{align*}
\]

where \( p = \frac{5C(6 - 2k_c^2)}{96k_c^3} - \frac{5C}{8k_c} \), \( q = \frac{3 - k_c^2}{6k_c^2} - \frac{5C}{12k_c} \). Note that here the solution of the quadratic criterion is solved for the case of \( k_c = 1.0 \) and \( C = 1.0 \), which make \( p^2 + q^3 \) larger than 0, otherwise it should be solved according to another equation presented in the reference [14].

For most brittle materials whose fracture toughness ratio \( k_c \) is less than 1, when a notched structure subjects to pure Mode I loading, the fracture should be in Mode I and the initiation angle should be \( \theta_{\text{pure}} = 0 \). To clarify this, a Mode I fracture test of sandstone was carried out by employing a three-point bending test, and the fracture trajectory is shown in Figure 2. The fracture initiates and propagates along the plane of the original crack plane, where \( \theta = 0 \), thus, it is clear that the initiation angle should be zero under Mode I loading. Also, note that the fracture deflects to another direction after a short distance of propagation, this is mainly because of the heterogeneity of sandstone materials and the complexity of stress state after the fracture initiation, which are out of the research scope of the current theoretical study.

Among these three criteria, the ellipse criterion is the only one that yields \( \theta_{\text{mix max}} = \theta_{\text{pure}} = 0 \), while the others yield the initiation of a slant fracture in mixed-mode. Furthermore, the curves of \( K^e_{\text{mix}}(\theta) \) for different criteria are plotted in Figure 3. For two other criteria, it is shown that the maximum value is obtained at a plane away from the original crack plane where \( K^e_{\text{mix}}(\theta_{\text{mix max}}) > K^e_{\text{mix}}(\theta_{\text{pure}}) = K_1 \), which means that mixed-mode fracture could initiate before the initiation of pure Mode I fracture. Thus, the straight line criterion and the quadratic criterion fail to explain such a simple mechanism of pure Mode I loading, and the ellipse criterion is a reasonable one that allows the existence of pure Mode I fracture.
4. Discuss on basic mathematical requirements for fracture criterion

According to the above analysis, except for these two conditions in Eq. (5), another important condition must be considered, that is Mode I fracture must be ensured to initiate before two other modes of fracture, especially mixed-mode fracture, for some loading conditions by which Mode I fracture has been widely confirmed, for example, Mode I fracture has been widely observed for a beam constructed by a common material under pure Mode I loading. This additional condition seems unnecessary, but mixed-mode fracture may take place first under pure Mode I loading for common materials when the mixed-mode fracture criterion is actually taken into fracture analysis. The expression of this condition is derived in the following analysis.

To ensure the initiation of pure Mode I fracture for the materials which have relatively small tensile strength compared with shear strength, the mixed-mode fracture criteria must satisfy the following condition:

\[ \frac{\partial K_{\text{mix}}^e (\theta)}{\partial \theta} = 0 \text{ when } \theta = \theta_{\text{pure}}. \]  

(12)

It is a necessary condition. If \( \frac{\partial K_{\text{mix}}^e (\theta)}{\partial \theta} \neq 0 \) at \( \theta_{\text{pure}} \), one of \( K_{\text{mix}}^e (\theta_{\text{pure}} + \delta\theta) \) and \( K_{\text{mix}}^e (\theta_{\text{pure}} - \delta\theta) \) must be larger than \( K_{\text{mix}}^e (\theta_{\text{pure}}) \), which yields one of other modes of fracture must be initiate first.

The general form of the mixed-mode fracture criterion can be expressed a function of \( K_i (\theta) \) and \( K_{II} (\theta) \).

\[ F_M (K_i (\theta), K_{II} (\theta)) = 1. \]  

(13)

Some transformations make it possible to express the fracture criterion as another form of the equivalent stress intensity factor.

\[ K_{\text{mix}}^e (\theta) = F (F_M, K_{IC}) = K_{IC}. \]  

(14)

According to the chain rule, the derivative of \( K_{\text{mix}}^e (\theta) \) is expressed as:
\[
\frac{\partial K_{\text{mix}}^e (\theta)}{\partial \theta} = \frac{\partial F}{\partial F_M} \frac{\partial F_M}{\partial \theta} + \frac{\partial F}{\partial F_M} \left( \frac{\partial K_1 (\theta)}{\partial \theta} + \frac{\partial K_{II} (\theta)}{\partial \theta} \right). \tag{15}
\]

As presented in Eq. (12), \(\partial K_{\text{mix}}^e (\theta)/\partial \theta\) needs to be zero when \(\theta = \theta_{\text{pure}}\). Obviously, \(\partial F/\partial F_M\) can’t be zero, so
\[
\left( \frac{\partial F_M}{\partial K_1 (\theta)} \frac{\partial K_1 (\theta)}{\partial \theta} + \frac{\partial F_M}{\partial K_{II} (\theta)} \frac{\partial K_{II} (\theta)}{\partial \theta} \right)_{\theta = \theta_{\text{pure}}} = 0. \tag{16}
\]

It can be easily demonstrated that
\[
\frac{\partial K_1 (\theta)}{\partial \theta} = -\frac{3}{2} \cos \theta \left[ K_1 \cos \theta \sin \theta + K_{II} \left( 1 - 3 \sin^2 \theta \right) \right] = -\frac{3}{2} K_{II} (\theta). \tag{17}
\]

When \(\theta = \theta_{\text{pure}}\), \(K_{II} (\theta) = 0\) which leads to \(\partial K_1 (\theta)/\partial \theta = 0\), so the first term of Eq. (16) equals zero and the second term must be zero too. Furthermore, \(\partial K_{II} (\theta)/\partial \theta = 0\) only when the maximum value of \(K_{II} (\theta)\) is obtained. However \(\partial K_{II} (\theta)/\partial \theta \neq 0\) when \(\theta = \theta_{\text{pure}}\) since the maximum values of \(K_1 (\theta)\) and \(K_{II} (\theta)\) could not be obtained at the same plane. Thus, Eq. (16) stands only when
\[
\frac{\partial F_M}{\partial K_{II} (\theta)} = 0 \tag{18}
\]

This condition ensures \(\partial K_{\text{mix}}^e (\theta)/\partial \theta = 0\) at \(\theta_{\text{pure}}\) and the existence of a pure Mode I fracture. Actually, this condition is a restriction on the shape of the fracture envelope. If the fracture envelope \(F_M (K_1 (\theta), K_{II} (\theta)) = 1\) is plotted in a chart the abscissa and the ordinate are \(K_1 (\theta)\) and \(K_{II} (\theta)\) respectively, the envelope must cross the abscissa axis vertically. Three criteria in Eq. (6) are presented in Figure 4. It is clearly shown that the ellipse criterion is the only one that satisfies the proposed condition among these three criteria. Of course, other criteria may be constructed according to the conditions of Eq. (5) and Eq. (18).
5. Conclusion

For the mixed-mode fracture on the original crack plane, the fracture criteria used are often based on the coupling of $K_I/K_{IC}$ and $K_{II}/K_{IC}$. However, recent studies showed that the forms of the fracture criteria could be extended to a more general case, on any potential fracture plane, and then become a function of $K_I(\theta)/K_{IC}$ and $K_{II}(\theta)/K_{IC}$. In particular, three forms have been widely used: linear, elliptical, and quadratic form. The performances of these criteria are the central question addressed here.

This study focuses on the cases for which an analytical solution can readily be obtained. Specifically, the performance of these criteria under Mode I loading, namely, $K_I \neq 0, K_{II} = 0$ is studied. Notice that in this case, $K_I(\theta) = K_I \cos^2(\theta/2)$ and $K_{II}(\theta) = K_I \sin(\theta/2) \cos(\theta/2)$. An analytical solution for fracture initiation angle under each of the three mixed-mode criteria is presented, and the results demonstrate that only the elliptical form yields the correct results. Under mode I loading, pure Mode I fracture initiation takes precedence, conforming to the well-known consequences. The results showed that the fracture paths of two other fracture criteria deflect obviously from the well-known paths except for the elliptical form. What’s more, the mathematical requirements are obtained for the applicable mixed-mode fracture criterion. The envelope of fracture criterion must cross the abscissa axis vertically in a chart where the abscissa and the ordinate are $K_I(\theta)$ and $K_{II}(\theta)$ respectively. Based on the mathematical requirements, more fracture criteria could be constructed to yield a right initiation angle of Mode I, Mode II, and mixed-mode fracture.

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