WEIGHTED VECTOR-VALUED BOUNDS FOR THE SINGULAR INTEGRAL OPERATORS WITH NONSMOOTH KERNELS

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ABSTRACT. Let $T$ be a singular integral operator with non-smooth kernel which was introduced by Duong and McIntosh. In this paper, we prove that this operator and its corresponding grand maximal operator satisfy certain weak type endpoint vector-valued estimate of $L \log L$ type. As an application we established a refined weighted vector-valued bound for this operator.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 1$. Let $A_p(\mathbb{R}^n)$ ($p \in (1, \infty)$) be the weight functions class of Muckenhoupt, that is, $w \in A_p(\mathbb{R}^n)$ if $w$ is nonnegative and locally integrable, and

$$[w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)\,dx \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}(x)\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$, $[w]_{A_p}$ is called the $A_p$ constant of $w$, see [6] for properties of $A_p(\mathbb{R}^n)$. In the last several years, there has been significant progress in the study of sharp weighted bounds with $A_p$ weights for the classical operators in Harmonic Analysis. The study was begin by Buckley [1], who proved that if $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator $M$ satisfies

$$(1.1) \quad \|Mf\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n, p} [w]_{A_p}^{\frac{1}{p'\gamma}} \|f\|_{L^p(\mathbb{R}^n, w)},$$

Moreover, the estimate (1.1) is sharp since the exponent $1/(p-1)$ can not be replaced by a smaller one. Hytönen and Pérez [9] improved the estimate (1.1), and showed that

$$(1.2) \quad \|Mf\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n, p} ([w]_{A_p}[w^{-\frac{1}{p-1}}]_{A_{\infty}})^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}^n, w)},$$

where and in the following, for a weight $u$, $[u]_{A_{\infty}}$ is defined by

$$[u]_{A_{\infty}} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{u(Q)} \int_Q M(u\chi_Q)(x)\,dx.$$

It is well known that for $w \in A_p(\mathbb{R}^n)$, $[w^{-\frac{1}{p-1}}]_{A_{\infty}} \lesssim [w]_{A_p}^{\frac{1}{p'}}$. Thus, (1.2) is more subtle than (1.1).

The sharp dependence of the weighted estimates of singular integral operators in terms of the $A_p(\mathbb{R}^n)$ constant was first considered by Petermichl [14, 15], who...
solved this question for Hilbert transform and Riesz transform. Hytönen [7] proved that for a Calderón-Zygmund operator \( T \) and \( w \in A_2(\mathbb{R}^n) \),
\[
\|Tf\|_{L^2(\mathbb{R}^n, w)} \lesssim_n \|w\|_{A_2} \|f\|_{L^2(\mathbb{R}^n, w)}.
\]
This solved the so-called \( A_2 \) conjecture. Combining the estimate (1.3) and the extrapolation theorem in [3], we know that for a Calderón-Zygmund operator \( T \),
\[
(1.4) \quad \|Tf\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n, p} \|w\|_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(\mathbb{R}^n, w)}.
\]
In [12], Lerner gave a much simple proof of (1.4) by controlling the Calderón-Zygmund operator using sparse operators. Lerner [12] proved that

**Theorem 1.1.** Let \( T \) be a sublinear operator and \( \mathcal{M}_T \) be the corresponding grand maximal operator defined by

\[
\mathcal{M}_T f(x) = \sup_{Q \ni x} \sup_{\xi \in Q} |T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|.
\]

Suppose that both \( T \) and \( \mathcal{M}_T \) are bounded from \( L^1(\mathbb{R}^n) \) to \( L^{1, \infty}(\mathbb{R}^n) \). Then for \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}^n) \), \( T \) satisfies (1.4).

Let \( p, r \in (0, \infty) \) and \( w \) be a weight. As usual, for a sequence of numbers \( \{a_k\}_{k=1}^\infty \), we denote \( \|\{a_k\}\|_r = (\sum_k |a_k|^r)^{1/r} \). The space \( L^p(l^r; \mathbb{R}^n, w) \) is defined as
\[
L^p(l^r; \mathbb{R}^n, w) = \{\{f_k\}_{k=1}^\infty : \|\{f_k\}\|_{L^p(l^r; \mathbb{R}^n, w)} < \infty\}
\]
where
\[
\|\{f_k\}\|_{L^p(l^r; \mathbb{R}^n, w)} = \left( \int_{\mathbb{R}^n} \|f_k(x)\|_r^p w(x) \, dx \right)^{1/p}.
\]
When \( w \equiv 1 \), we denote \( \|\{f_k\}\|_{L^p(l^r; \mathbb{R}^n, w)} \) by \( \|\{f_k\}\|_{L^p(l^r; \mathbb{R}^n)} \) for simplicity. Hu [10] extended Lerner’s result to the vector-valued case, proved that

**Theorem 1.2.** Let \( T \) be a sublinear operator and \( \mathcal{M}_T \) be the corresponding grand maximal operator. Suppose that for some \( q \in (1, \infty) \),
\[
\left| \left\{ y \in \mathbb{R}^n : \|Tf_k(y)\|_r + \|\mathcal{M}_T f_k(y)\|_r > \lambda \right\} \right| \lesssim \int_{\mathbb{R}^n} \|\{f_k(y)\}\|_r^q \log \left( 1 + \frac{\|\{f_k(y)\}\|_r^q}{\lambda} \right) \, dy.
\]

Then for all \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}^n) \),
\[
\|\{Tf_k\}\|_{L^p(l^r; \mathbb{R}^n, w)} \lesssim_{n, p} \|w\|_{A_p}^{\frac{1}{p}} \left( \|w^{-\frac{1}{p-1}}\|_{A_\infty}^{\frac{1}{p}} + \|w\|_{A_\infty} \right) \|\{f_k\}\|_{L^p(l^r; \mathbb{R}^n, w)}.
\]

Let \( T \) be a \( L^2(\mathbb{R}^n) \) bounded linear operator with kernel \( K \) in the sense that for all \( f \in L^2(\mathbb{R}^n) \) with compact support and a.e. \( x \in \mathbb{R}^n \setminus \text{supp } f \),
\[
(1.5) \quad T f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy.
\]

where \( K \) is a measurable function on \( \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) : x = y\} \). To obtain a weak \((1, 1)\) estimate for certain Riesz transforms, and \( L^p \) boundedness with \( p \in (1, \infty) \) of holomorphic functional calculi of linear elliptic operators on irregular domains, Duong and McIntosh [11] introduced singular integral operators with nonsmooth kernels on spaces of homogeneous type via the following generalized approximation to the identity.
Definition 1.3. A family of operators \( \{A_t\}_{t>0} \) is said to be an approximation to the identity, if for every \( t > 0 \), \( A_t \) can be represented by the kernel at in the following sense: for every function \( u \in L^p(\mathbb{R}^n) \) with \( p \in [1, \infty] \) and almost everywhere \( x \in \mathbb{R}^n \),

\[
A_t u(x) = \int_{\mathbb{R}^n} a_t(x, y) u(y) dy,
\]

and the kernel \( a_t \) satisfies that for all \( x, y \in \mathbb{R}^n \) and \( t > 0 \),

\[
|a_t(x, y)| \leq h_t(x, y) = t^{-n/s} h \left( \frac{|x - y|}{t^{1/s}} \right),
\]

where \( s > 0 \) is a constant and \( h \) is a positive, bounded and decreasing function such that for some constant \( \eta > 0 \),

\[
\lim_{r \to \infty} r^{n+\eta} h(r) = 0.
\]

Assumption 1.4. There exists an approximation to the identity \( \{A_t\}_{t>0} \) such that the composite operator \( T A_t \) has an associated kernel \( K_t \) in the sense of (1.6), and there exists a positive constant \( c_1 \) such that for all \( y \in \mathbb{R}^n \) and \( t > 0 \),

\[
\int_{|x-y| \geq c_1 t^s} K(x, y) - K_t(x, y) dx \lesssim 1.
\]

An \( L^2(\mathbb{R}^n) \) bounded linear operator with kernel \( K \) satisfying Assumption 1.4 is called a singular integral operator with nonsmooth kernel, since \( K \) does not enjoy smoothness in space variables. Duong and McIntosh [4] proved that if \( T \) is an \( L^2(\mathbb{R}^n) \) bounded linear operator with kernel \( K \), and satisfies Assumption 1.4 then \( T \) is bounded from \( L^1(\mathbb{R}^n) \) to \( L^{1, \infty}(\mathbb{R}^n) \). To consider the weighted estimates with \( A_p(\mathbb{R}^n) \) boundedness of singular integral operators with non-smooth kernel, Martell [13] introduced the following assumptions.

Assumption 1.5. There exists an approximation to the identity \( \{D_t\}_{t>0} \) such that the composite operator \( D_t T \) has an associated kernel \( K^t \) in the sense of (1.6), and there exist positive constants \( c_2 \) and \( \alpha \in (0, 1] \), such that for all \( t > 0 \) and \( x, y \in \mathbb{R}^n \) with \( |x - y| \geq c_2 t^s \),

\[
|K(x, y) - K^t(x, y)| \lesssim \frac{t^\alpha/s}{|x-y|^{n+\alpha}}.
\]

Martell [13] proved that if \( T \) is an \( L^2(\mathbb{R}^n) \) bounded linear operator, satisfies Assumption 1.4 and Assumption 1.5 then for any \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}^n) \), \( T \) is bounded on \( L^p(\mathbb{R}^n, w) \). The first purpose of this paper is to establish the endpoint vector-valued estimates for the corresponding grand maximal operator of singular integral operators with nonsmooth kernels. Our main result can be stated as follows.

Theorem 1.6. Let \( T \) be an \( L^2(\mathbb{R}^n) \) bounded linear operator with kernel \( K \) as in (1.5). Suppose that \( T \) satisfies Assumption 1.4 and Assumption 1.5. Then for each \( \lambda > 0 \),

\[
\left| \left\{ x \in \mathbb{R}^n : \|\{T f_k(x)\}\|_{l^q} + \|\{M T f_k(x)\}\|_{l^q} > \lambda \right\} \right| \\
\lesssim \int_{\mathbb{R}^n} \|\{f_k(x)\}\|_{l^q} \frac{1}{\lambda} \log \left( 1 + \|\{f_k(x)\}\|_{l^q} \right) dx.
\]
If we further assume that the kernels \( \{K^t\}_{t>0} \) in Assumption [1.5] also satisfy that for all \( t>0 \) and \( x,y \in \mathbb{R}^n \) with \( |x-y| \leq ct^{\frac{1}{2}}, \)
\[(1.9) \quad |K^t(x,y)| \lesssim t^{-\frac{n}{2}}, \]
then (1.8) is also true for \( T^* \), here and in the following, \( T^* \) is the maximal singular integral operator defined by
\[
T^* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|,
\]
with
\[
T_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x,y) f(y) dy.
\]
As a consequence of Theorem [1.6] and Theorem [1.1] we obtain the following weighted vector-valued bounds for \( T \) and \( T^* \).

**Corollary 1.7.** Let \( T \) be an \( L^2(\mathbb{R}^n) \) bounded linear operator with kernel \( K \) in the sense of (1.9). Suppose that \( T \) satisfies Assumption [1.1] and Assumption [1.2] Then for \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}^n), \)
\[(1.10) \quad \|\{Tf_k\}\|_{L^p(q;\mathbb{R}^n,w)} \lesssim_n, p \left( [w]_{A_p}^{\frac{1}{q}} + [w]_{A_\infty}^{\frac{1}{q}} \right) \|\{f_k\}\|_{L^p(q;\mathbb{R}^n,w)}, \]
with \( \sigma = w^{-\frac{1}{p-1}}. \) Moreover, if the kernels \( \{K^t\}_{t>0} \) in Assumption [1.5] satisfy (1.9), then the weighted estimate (1.10) also holds for \( T^* \).

**Remark 1.8.** We do not know if the weighted bound in (1.10) is sharp.

In what follows, \( C \) always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol \( A \lesssim B \) to denote that there exists a positive constant \( C \) such that \( A \leq CB. \)
Constant with subscript such as \( C_1, \) does not change in different occurrences. For any set \( E \subset \mathbb{R}^n, \chi_E \) denotes its characteristic function. For a cube \( Q \subset \mathbb{R}^n \) and \( \lambda \in (0, \infty), \) we use \( \ell(Q) \) (diam\(Q)) to denote the side length (diam\(eter\)) of \( Q, \) and \( \lambda Q \) to denote the cube with the same center as \( Q \) and whose side length is \( \lambda \) times that of \( Q. \) For \( x \in \mathbb{R}^n \) and \( r > 0, \) \( B(x, r) \) denotes the ball centered at \( x \) and having radius \( r. \)

2. **Proof of Theorem 1.6**

We begin with some preliminary lemmas.

**Lemma 2.1.** Let \( q, p_0 \in (1, \infty), q \in [0, \infty) \) and \( S \) be a sublinear operator. Suppose that
\[
\|\{Sf_k\}\|_{L^{p_0}(q;\mathbb{R}^n)} \lesssim \|\{f_k\}\|_{L^{p_0}(q;\mathbb{R}^n)},
\]
and for all \( \lambda > 0, \)
\[
\left| \{x \in \mathbb{R}^n : \|\{Sf_k(x)\}\|_{L^q} \geq \lambda \} \right| \lesssim \int_{\mathbb{R}^n} \frac{\|\{f_k\}\|_{L^q}}{\lambda} \log^\theta \left( 1 + \frac{\|\{f_k\}\|_{L^q}}{\lambda} \right) dx.
\]
Then for cubes \( Q_2 \subset Q_1 \subset \mathbb{R}^n, \)
\[
\frac{1}{|Q_1|} \int_{Q_1} \|\{S(f_k\chi_{Q_2})(x)\}\|_{L^{q}} dx \lesssim \|\{f_k\}\|_{L^{q}} \|L_{(\log L)^{p_0+1}, Q_2} \|_{L^{(\log L)^{p_0+1}, Q_2}},
\]
here and in the following, for $\beta \in [0, \infty)$,

$$\|f\|_{L^0, \infty} = \inf \{\lambda > 0 : \frac{1}{|Q|} \int_Q \frac{|f(y)|}{\lambda} \log^\beta \left(1 + \frac{|f(y)|}{\lambda}\right) dy \leq 1\}.$$

**Proof.** Lemma 2.1 is a generalization of Lemma 3.1 in [11]. Their proofs are very similar. By homogeneity, we may assume that $\|\{f_k\}\|_{L^0, \infty} = 1$, which implies that

$$\int_{Q_2} \|\{f_k(x)\}\|_{\infty} \log^{\beta+1} (1 + \|\{f_k(x)\}\|_{\infty}) dx \leq |Q_2|.$$

For each fixed $\lambda > 0$, set $\Omega_\lambda = \{x \in \mathbb{R}^n : \|\{f_k(x)\}\|_{\infty} > \lambda^{\frac{p_0-1}{2p_0}}\}$. Decompose $f_k$ as

$$f_k(x) = f_k(x)\chi_{\Omega_\lambda}(x) + f_k(x)\chi_{\mathbb{R}^n\setminus\Omega_\lambda}(x) = f_k^1(x) + f_k^2(x).$$

It is obvious that $\|\{f_k^1\}\|_{L^\infty(\Omega_\lambda; \mathbb{R}^n)} \leq \lambda^{\frac{p_0-1}{2p_0}}$. A trivial computation leads to that

$$\int_1^\infty \left|\left\{x \in \mathbb{R}^n : \|\{S(f_k^1(x))\}\|_{\infty} > \lambda/2\right\}\right| d\lambda \lesssim \int_1^\infty \int_{Q_2} \|\{f_k^1(x)\}\|_{\infty}^p d\lambda \lesssim \int_{Q_2} \|\{f_k^1(x)\}\|_{\infty} d\lambda \int_1^\infty \lambda^{-p_0} \frac{(\log \lambda)^2}{\lambda} d\lambda \lesssim |Q_2|.$$ 

On the other hand,

$$\int_1^\infty \left|\left\{x \in \mathbb{R}^n : \|\{S(f_k^1(x))\}\|_{\infty} > \lambda/2\right\}\right| d\lambda \lesssim \int_1^\infty \int_{Q_2} \|\{f_k^1(x)\}\|_{\infty} \log^\beta (1 + \|\{f_k^1(x)\}\|_{\infty}) d\lambda \lesssim \int_{Q_2} \|\{f_k^1(x)\}\|_{\infty} \log^\beta (1 + \|\{f_k^1(x)\}\|_{\infty}) d\lambda \int_1^{\|\{f_k(x)\}\|_{\infty}^{2p_0}} \frac{1}{\lambda} d\lambda dx \lesssim \int_{Q_2} \|\{f_k^1(x)\}\|_{\infty} \log^{\beta+1} (1 + \|\{f_k^1(x)\}\|_{\infty}) dx.$$ 

Combining the estimates above then yields

$$\int_0^\infty \left|\left\{x \in Q_1 : \|\{S(f_k^1(x))\}\|_{\infty} > \lambda\right\}\right| d\lambda \lesssim \int_0^1 \left|\left\{x \in Q_1 : \|\{S(f_k^1(x))\}\|_{\infty} > \lambda\right\}\right| d\lambda + \int_1^\infty \left|\left\{x \in \mathbb{R}^n : \|\{S(f_k^1(x))\}\|_{\infty} > \lambda/2\right\}\right| d\lambda + \int_1^\infty \left|\left\{x \in \mathbb{R}^n : \|\{S(f_k^1(x))\}\|_{\infty} > \lambda/2\right\}\right| d\lambda \lesssim |Q_1|.$$

This completes the proof of Lemma 2.1.
Recall that the standard dyadic grid in \( \mathbb{R}^n \) consists of all cubes of the form 
\[ 2^{-k}([0, 1)^n + j), \quad k \in \mathbb{Z}, \quad j \in \mathbb{Z}^n. \]

Denote the standard grid by \( \mathcal{D} \). For a fixed cube \( Q \), denote by \( \mathcal{D}(Q) \) the set of dyadic cubes with respect to \( Q \), that is, the cubes from \( \mathcal{D}(Q) \) are formed by repeating subdivision of \( Q \) and each of descendants into \( 2^n \) congruent subcubes.

As usual, by a general dyadic grid \( \mathcal{D} \), we mean a collection of cube with the following properties: (i) for any cube \( Q \in \mathcal{D} \), it side length \( \ell(Q) \) is of the form \( 2^k \) for some \( k \in \mathbb{Z} \); (ii) for any cubes \( Q_1, Q_2 \in \mathcal{D} \), \( Q_1 \cap Q_2 \in \{Q_1, Q_2, 0\} \); (iii) for each \( k \in \mathbb{Z} \), the cubes of side length \( 2^k \) form a partition of \( \mathbb{R}^n \). By the one-third trick, (see \cite{S} Lemma 2.5), there exist dyadic grids \( \mathcal{D}_1, \ldots, \mathcal{D}_{3^n} \), such that for each cube \( Q \subset \mathbb{R}^n \), there exists a cube \( I \in \mathcal{D}_j \) for some \( j \), \( Q \subset I \) and \( \ell(I) \approx \ell(Q) \).

Let \( \{D_t\}_{t \geq 0} \) be an approximation to the identity. Associated with \( \{D_t\}_{t \geq 0} \), define the sharp maximal operator \( M_{f, D}^l \) by

\[
M_{D}^l f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - D_t f(y)| \, dy, \quad f \in L^p(\mathbb{R}^n), \quad p \in [1, \infty)
\]

with \( t_B = r_B^s \) and \( s \) the constant appeared in (1.6), the supremum is taken over all balls in \( \mathbb{R}^n \). This operator was introduced by Martell \cite{M} and plays an important role in the weighted estimates for singular integral operators with non-smooth kernels. Let \( q \in (1, \infty) \), \( \{f_k\} \subset L^p(\mathbb{R}^n) \) for some \( p \in (1, \infty) \), set

\[
M_{f, D}^q (\{f_k\})(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B \|\|f_k(y) - D_{t_B} f_k(y)\|_q\| \, dy.
\]

**Lemma 2.2.** Let \( \lambda > 0 \), \( \{f_k\} \subset L^1(\mathbb{R}^n) \) with compact support, \( B \subset \mathbb{R}^n \) be a cube such that there exists \( x_0 \in B \) with \( M(\|\{f_k\}\|_{L^q}) (x_0) < \lambda \). Then, for every \( \zeta \in (0, 1) \), we can find \( \gamma > 0 \) (independent of \( \lambda, B, f, x_0 \)), such that

\[
|\{x \in B : M(\|\{f_k\}\|_{L^q}) (x) > A \lambda, M_{f, D}^q (\{f_k\})(x) \leq \gamma \lambda\}| \leq \zeta |B|,
\]

where \( A > 1 \) is a fixed constant which only depends on the approximation of the identity \( \{D_t\}_{t \geq 0} \).

**Proof.** Let \( A \in (1, \infty) \) be a constant which will be chosen later. For \( \lambda > 0 \), set

\[
E_{\lambda} = \{x \in B : M(\|\{f_k\}\|_{L^q}) (x) > A \lambda, M_{f, D}^q (\{f_k\})(x) \leq \gamma \lambda\}.
\]

We assume that there exists \( x_E \in E_{\lambda} \), for otherwise there is nothing to prove. As in the proof of Proposition 4.1 in \cite{M} (see also the proof of Lemma 2.6 of \cite{M}), we can verify that for each \( x \in E_{\lambda} \) and \( A = 2^{-2n} A \),

\[
M(\|\{f_k\}\|_{L^q} \chi_{A B})(x) > A \lambda.
\]

Now let \( t = r_{16B} \). For \( y \in 4B \), write

\[
|D_t (f_k \chi_{16B})(y)| \leq \int_{16B} |h_t(y, z)f_k(z)| \, dz.
\]

By Minkowski’s inequality, we deduce that

\[
\|\{D_t (f_k \chi_{16B})(y)\}\|_{L^p} \leq \int_{16B} |h_t(y, z)||\{f_k(z)\}|_{L^q} \, dz \lesssim M(\|\{f_k\}\|_{L^q})(x_0),
\]
since $h$ is bounded on $[0, \infty)$. Also, we have that for $y \in 4B$,

$$\|\{D_t(f_k \chi_{R^n \setminus 16B})(y)\}\|_{\ell^v} \leq \sum_{l=1}^{\infty} \frac{1}{|B|} \int_{2^{l+1}B \setminus 2^{l}B} |h_t(y, z)| \|\{f_k(z)\}\|_{\ell^v} dz$$

$$\lesssim \sum_{l=1}^{\infty} \frac{1}{|B|} \int_{2^{l+1}B \setminus 2^{l}B} h(2^{l+1}) \|\{f_k(z)\}\|_{\ell^v} dz$$

$$\lesssim M(\|\{f_k\}\|_{\ell^v})(x_0).$$

This, in turn implies that for all $y \in \mathbb{R}^n$,

$$M(\|\{(D_t f_k \chi_{4B})\|_{\ell^v}\})(x) \lesssim M(\|\{f_k\}\|_{\ell^v})(x_0) \leq C_1 \lambda,$$

with $C_1 > 0$ a constant. Therefore, for each $x \in E_\lambda$,

$$M(\|\{f_k \chi_{4B}\}\|_{\ell^v})(x) \leq M(\|\{(f_k - D_t f_k \chi_{4B})\|_{\ell^v}\})(x) + M(\|\{(D_t f_k \chi_{4B})\|_{\ell^v}\})(x)$$

$$\leq M(\|\{(f_k - D_t f_k \chi_{4B})\|_{\ell^v}\})(x) + C_1 \lambda.$$

We choose $A > 1$ such that $\tilde{A} = C_1 + 1$. It then follows that

$$E_\lambda \subset \{x \in B : M(\|\{(f_k - D_t f_k \chi_{4B})\|_{\ell^v}\})(x) > \lambda\}.$$

This, via the weak type $(1, 1)$ estimate of $M$, tells us that

$$|E_\lambda| \leq C_2 \lambda^{-1} \int_{4B} \|h_k(y) - D_t h_k(y)\|_{\ell^v} dy \leq C_2 16^n \lambda^{-1} |B|M_2^v(\{f_k\})(x_E) \leq C_2 16^n \gamma |B|.$$

For each $\zeta \in (0, 1)$, let $\gamma = \zeta (2C_2 16^n)^{-1}$. The inequality (2.1) holds for this $\gamma$. □

As in the proof of the Fefferman-Stein inequality (see [6], pp 150-151), or the proof of Theorem 2.2 in [11]), we can deduce from Lemma 2.2 that

**Corollary 2.3.** Let $\Phi$ be an increasing function on $[0, \infty)$ satisfying

$$\Phi(2t) \leq C \Phi(t), \quad t \in [0, \infty).$$

$\{D_t\}_{t > 0}$ be an approximation to the identity as in Definition 1.4. Let $\{f_k\}$ be a sequence of functions such that for any $R > 0$,

$$\sup_{0 < \lambda < R} \Phi(\lambda)|\{x \in \mathbb{R}^n : M(\|\{f_k\}\|_{\ell^v})(x) > \lambda\}| < \infty.$$

Then

$$\sup_{\lambda > 0} \Phi(\lambda)|\{x \in \mathbb{R}^n : M(\|\{f_k\}\|_{\ell^v})(x) > \lambda\}| \leq \sup_{\lambda > 0} \Phi(\lambda)|\{x \in \mathbb{R}^n : M_2^v(\{f_k\})(x) > \lambda\}|.$$

**Lemma 2.4.** Let $T$ be an $L^2(\mathbb{R}^n)$ bounded linear operator with kernel $K$ as in (1.5). Suppose that $T$ satisfies Assumption [14]. Then for any $q \in (1, \infty)$, $T$ is bounded from $L^1(\ell^q; \mathbb{R}^n)$ to $L^{1, \infty}(\ell^q; \mathbb{R}^n)$

**Proof.** We only consider the case $c_1 = 2$. The other cases can be treated in the same way. For $\lambda > 0$, by applying the Calderón-Zygmund decomposition to $\|\{f_k\}\|_{\ell^v}$ at level $\lambda$, we obtain a sequence of cubes $\{Q_l\}$ with disjoint interiors, such that

$$\lambda < \frac{1}{|Q_l|} \int_{Q_l} \|\{f_k(x)\}\|_{\ell^v} dx \lesssim \lambda,$$

and $\|\{f_k(x)\}\|_{\ell^v} \lesssim \lambda$ for a.e. $x \in \mathbb{R}^n \setminus \cup_l Q_l$. For each fixed $k$, set

$$f^l_k(x) = f_k(x) \chi_{\mathbb{R}^n \setminus \cup_l Q_l}(x),$$

\text{for } x \in Q_l.$$
with $b_{k,l}(y) = f(x)\chi_{Q_l}(y)$, $t_{Q_l} = \{\ell(Q_l)\}^\ast$. By the fact that $\|\{f_k\}\|_{L^\infty(\nu;\mathbb{R}^n)} \lesssim \lambda$, we deduce that

$$\|\{f_k\}\|_{L^q(\nu;\mathbb{R}^n)}^q \lesssim \lambda^{q-1} \|\{f_k\}\|_{L^1(\nu;\mathbb{R}^n)}.$$ 

Recalling that $T$ is bounded on $L^q(\mathbb{R}^n)$, we have that

$$\left|\left\{Tf_k(x)\right\}\right|_{\nu} \geq \lambda / 3 \right| \lesssim \lambda^{-q} \|\{f_k\}\|_{L^q(\nu;\mathbb{R}^n)} \lesssim \lambda^{-1} \|\{f_k\}\|_{L^1(\nu;\mathbb{R}^n)}.$$ 

On the other hand, we get from (1.5) and (1.6) that

$$\int_{\mathbb{R}^n} |v_k(y)A_{t_{Q_l}}b_{k,l}(y)|dy \leq \int_{Q_l} |b_{k,l}(z)| \left|\int_{\mathbb{R}^n} h_{t_{Q_l}}(z, y)|v_k(z)|dz\right|dy \lesssim \int_{Q_l} |b_{k,l}(z)|dz \inf_{y \in Q_l} Mv_k(y).$$

A straightforward computation involving Minkowski’s inequality gives us that

$$\left(\sum_k \|b_{k,l}\|_{L^1(\nu;\mathbb{R}^n)}^q\right)^{1/q} \leq \int_{Q_l} \left(\sum_k |f_k(y)|^q\right)^{1/q} dy \lesssim \lambda |Q_l|.$$ 

Therefore, by Minkowski’s inequality and the vector-valued inequality of the Hardy-Littlewood maximal operator $M$ (see [7]),

$$\left\|\left(\sum_k \left|\sum_l A_{t_{Q_l}}b_{k,l}\right|^q\right)^{1/q}\right\|_{L^q(\mathbb{R}^n)} \leq \sup_{\|v_k\|_{L^{q'}(\nu';\mathbb{R}^n)} \leq 1} \sum_k \sum_l \int_{\mathbb{R}^n} |v_k(y)A_{t_{Q_l}}b_{k,l}(y)|dy \lesssim \sup_{\|v_k\|_{L^{q'}(\nu';\mathbb{R}^n)} \leq 1} \sum_k \sum_l \int_{Q_l} |b_{k,l}(z)|dz \inf_{y \in Q_l} Mv_k(y) \lesssim \sup_{\|v_k\|_{L^{q'}(\nu';\mathbb{R}^n)} \leq 1} \sum_k \left(\sum_l \left(\int_{Q_l} |b_{k,l}(z)|dz\right)^q\right)^{1/q} \inf_{y \in Q_l} \|Mv_k(y)\|_{L^{q'}} \lesssim \sup_{\|v_k\|_{L^{q'}(\nu';\mathbb{R}^n)} \leq 1} \int_{Q_l} \left\|Mv_k(y)\right\|_{L^{q'}} dy \lesssim \lambda \sup_{\|v_k\|_{L^{q'}(\nu';\mathbb{R}^n)} \leq 1} \int_{Q_l} \left\|Mv_k(y)\right\|_{L^{q'}} dy \lesssim \lambda^{1 - q} \|\{f_k\}\|_{L^1(\nu;\mathbb{R}^n)}.$$ 

This, along with the fact that $T$ is bounded from $L^q(\mathbb{R}^n)$, leads to that

$$\|T\|_{L^q(\mathbb{R}^n)} \lesssim \lambda / 3.$$ 

We turn our attention to $Tf_k$. Let $\Omega = \cup_{l} 4nQ_l$. It is obvious that $|\Omega| \lesssim \lambda^{-1} \|\{f_k\}\|_{L^1(\nu;\mathbb{R}^n)}$. For each $x \in \mathbb{R}^n \setminus \Omega$, write

$$|Tf_k(x)| \leq \sum_l \int_{\mathbb{R}^n} |K(x; y) - K_{A_{t_{Q_l}}}(x; y)|b_{k,l}(y)|dy$$

for the Littlewood maximal operator $\lambda$.
Applying Minkowski’s inequality twice, we obtain
\[ \|\{Tf_k^2(x)\}\|_{L^p} \leq \sum_l \int_{\mathbb{R}^n} |K(x; y) - K_{A_lQ_l}(x; y)| \|\{b_{k,l}(y)\}\|_{L^q} dy \]

Therefore,
\[ (2.4) \quad \{x \in \mathbb{R}^n \setminus \Omega : \|\{Tf_k^2(x)\}\|_{L^p} > \lambda/3\} \]
\[ \lesssim \lambda^{-1} \sum_l \int_{\mathbb{R}^n \setminus 4nQ_l} |K(x; y) - K_{A_lQ_l}(x; y)| dx \|\{b_{k,l}(y)\}\|_{L^q} dy \]
\[ \lesssim \lambda^{-1} \|\{f_k\}\|_{L^q(r^3; \mathbb{R}^n)}. \]
Combining the inequalities (2.2)-(2.4) leads to our conclusion. \( \square \)

**Lemma 2.5.** Let \( T \) be the singular integral operator in Theorem 1.2, then for each \( N \in \mathbb{R}^n \) and functions \( \{f_k\}_{k=1}^N \subset L^p(\mathbb{R}^n) \) for some \( p_0 \in [1, \infty) \),
\[ M^2_p(Tf_k)(x) \lesssim M_{L \log L}(\|\{f_k\}\|_{L^q}(x) + \|Mf_k(x)\|_{L^q}). \]

**Proof.** Let \( x \in \mathbb{R}^n \), \( B \) be a ball containing \( x \) and \( t_B = r_B^2 \). Write
\[ \frac{1}{|B|} \int_B \|\{(Tf_k(y) - D_{t_B}Tf_k(y))\}\|_{L^p} dy \leq E_1 + E_2 + E_3, \]
with
\[ E_1 = \frac{1}{|B|} \int_B \|\{T(f_k\chi_{4B}(y))\}\|_{L^q} dy, \]
\[ E_2 = \frac{1}{|B|} \int_B \|\{D_{t_B}T(f_k\chi_{4B}(y))\}\|_{L^q} dy, \]
and
\[ E_3 = \frac{1}{|B|} \int_B \|\{(Tf_k\chi_{R^n\setminus4B}) - D_{t_B}T(f_k\chi_{R^n\setminus4B})(y)\}\|_{L^q} dy. \]
Recall that \( T \) is bounded on \( L^2(\mathbb{R}^n) \) (and so is bounded on \( L^q(\mathbb{R}^n) \)). Thus by Lemma 2.1 and Lemma 2.4
\[ E_1 \lesssim \|\|\{f_k\}\|_{L^q}\|_{L^p} log L, 4B \lesssim M_{L \log L}(\|\{f_k\}\|_{L^q}(x)). \]
On the other hand, it follows from Minkowski’s inequality that
\[ \|\{D_{t_B}T(f_k\chi_{4B})(y)\}\|_{L^q} \lesssim \int_{\mathbb{R}^n} |h_{t_B}(y, z)| \|\{T(f_k\chi_{4B})(z)\}\|_{L^q} dz \]
Let
\[ F_0 = \int_{16B} |h_{t_B}(y, z)| \|\{T(f_k\chi_{4B})(z)\}\|_{L^q} dz \]
and for \( j \in \mathbb{N} \),
\[ F_j = \int_{2^{j+5}B \setminus 2^{j+4}B} |h_{t_B}(y, z)| \|\{T(f_k\chi_{4B})(z)\}\|_{L^q} dz. \]
By the estimate (1.7) and Lemma 2.1 we know that
\[ F_0 \leq \|\|\{f_k\}\|_{L^q}\|_{L^p} log L, 4B, \]
and
\[ F_j \leq \frac{1}{|B|} h(2^j) \int_{2^{j+5}B} \|\{T(f_k\chi_{4B})(z)\}\|_{L^q} dz \lesssim 2^{-\delta j} \|\{f_k\}\|_{L^q} \|\{f_k\}\|_{L^p} log L, 4B. \]
This, in turn gives us that

\[ E_2 \lesssim \|\{f_k\}\|_{L^q} \log L, AB. \]

Finally, Assumption 1.3 tells us that for each \(k\) and \(y \in B\),

\[ |T(f_k\chi_{\mathbb{R}^n \setminus AB})(y) - D_{tb}T(f_k\chi_{\mathbb{R}^n \setminus AB}(y)| \lesssim Mf_k(x), \]

which implies that

\[ E_3 \lesssim \|\{Mf_k(x)\}\|_t. \]

Combining the estimates for \(E_1, E_2\) and \(E_3\) then leads to our desired conclusion. \(\Box\)

Let \(\mathcal{D}\) be a dyadic grid. Associated with \(\mathcal{D}\), define the sharp maximal function \(M^\sharp_{\mathcal{D}}\) as

\[ M^\sharp_{\mathcal{D}}f(x) = \sup_{Q \subset \mathcal{D}} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy. \]

For \(\delta \in (0, 1)\), let \(M^\sharp_{\mathcal{D}, \delta}f(x) = [M^\sharp_{\mathcal{D}}(|f|^\delta)(x)]^{1/\delta}\). Repeating the argument in [16, p. 153], we can verify that if \(\Phi\) is a increasing function on \([0, \infty)\) which satisfies that

\[ \Phi(2t) \leq C\Phi(t), \quad t \in [0, \infty), \]

then

\[ (2.5) \quad \sup_{\lambda > 0} \Phi(\lambda)\{x \in \mathbb{R}^n : |h(x)| > \lambda\} \lesssim \sup_{\lambda > 0} \Phi(\lambda)\{x \in \mathbb{R}^n : M^\sharp_{\mathcal{D}, \lambda}h(x) > \lambda\}, \]

provided that \(\sup_{\lambda > 0} \Phi(\lambda)\{x \in \mathbb{R}^n : M_{\mathcal{D}, \lambda}h(x) > \lambda\} < \infty\).

**Lemma 2.6.** Under the assumption of Theorem 1.2, for each \(\lambda > 0\),

\[ \{x \in \mathbb{R}^n : \{\{MTf_k(x)\}\}\|_t > \lambda\} \lesssim \int_{\mathbb{R}^n} \frac{\|\{f_k\}\|_t}{\lambda} \log \left(1 + \frac{\|\{f_k\}\|_t}{\lambda}\right) dx. \]

**Proof.** By the well known one-third trick (see [5, Lemma 2.5]), we only need to prove that, for each dyadic grid \(\mathcal{D}\), the inequality

\[ (2.6) \quad \{\{x \in \mathbb{R}^n : \{M_{\mathcal{D}}(Tf_k(x))\}\|_t > 1\} \}
\]

\[ \lesssim \int_{\mathbb{R}^n} \|\{f_k(x)\}\|_t \log \left(1 + \|\{f_k(x)\}\|_t\right) dx. \]

holds when \(\{f_k\}\) is finite. As in the proof of Lemma 8.1 in [2], we can very that for each cube \(Q \in \mathcal{D}\), \(\delta \in (0, 1)\),

\[ \inf_{c \in \mathbb{C}} \left(\frac{1}{|Q|} \int_Q \|\{M_{\mathcal{D}}f_k(y)\}\|_t - c\|_t^\delta \right)^{\frac{1}{\delta}} \lesssim \left(\frac{1}{|Q|} \int_Q \|\{M_{\mathcal{D}}(f_k\chi_Q)\}\|_t^\delta \right)^{\frac{1}{\delta}} \lesssim \|\{f_k\chi_Q\}\|_t, \]

where in the last inequality, we invoked the fact that \(M_{\mathcal{D}}\) is bounded from \(L^1(\mathbb{R}^n)\) to \(L^{1, \infty}(l^q, \mathbb{R}^n)\). This, in turn, implies that

\[ (2.7) \quad M^\sharp_{\mathcal{D}, \delta}(\|\{M_{\mathcal{D}}f_k\}\|_t)(x) \lesssim M_{\mathcal{D}}(\|\{f_k\}\|_t)(x). \]
Let $\Phi(t) = t \log^{-1}(1 + t^{-1})$. It follows from (2.5), (2.7), (2.1) and Lemma 2.6 that

$$\left\{ x \in \mathbb{R}^n : \| \{ M_{T_f} f_k(x) \} \|_\nu > 1 \right\}$$

$$\lesssim \sup_{t > 0} \Phi(t) \left\{ x \in \mathbb{R}^n : M_{T_f}^2 \| \{ M_{T_f} f_k \} \|_\nu(x) > t \right\}$$

$$\lesssim \sup_{t > 0} \Phi(t) \left\{ x \in \mathbb{R}^n : M \| \{ T f_k \} \|_\nu(x) > \lambda \right\}$$

$$\lesssim \sup_{t > 0} \Phi(t) \left\{ x \in \mathbb{R}^n : M_{\log L}^2 \| \{ T f_k \} \|_\nu(x) > t \right\}$$

$$\lesssim \sup_{t > 0} \Phi(t) \left\{ x \in \mathbb{R}^n : M_{L \log L} \| \{ f_k \} \|_\nu(x) + \| M f_k(x) \|_\nu > t \right\}$$

$$\lesssim \int_{\mathbb{R}^n} \| \{ f_k(x) \} \|_\nu \log\left(1 + \| \{ f_k(x) \} \|_\nu\right) dx,$$

where in the last inequality, we have invoked the fact that $M$ is bounded from $L^1(I_\nu; \mathbb{R}^n)$ to $L^{1, \infty}(I_\nu; \mathbb{R}^n)$ (see [5]). This establish (2.6) and completes the proof of Lemma 2.6.

Proof of Theorem 1.6 Let $q \in (1, \infty)$. We know by Lemma 2.4 that $T$ is bounded from $L^1(I_\nu; \mathbb{R}^n)$ to $L^{1, \infty}(I_\nu; \mathbb{R}^n)$. On the other hand, it was proved in [4] (see also [13]) that under the assumption of Theorem 1.6

$$T^* f(x) \leq MT f(x) + M f(x).$$

Thus by Lemma 2.6 for each $\lambda > 0$,

$$\left\{ x \in \mathbb{R}^n : \| \{ T^* f_k(x) \} \|_\nu > \lambda \right\} \lesssim \int_{\mathbb{R}^n} \| \{ f_k(x) \} \|_\nu \log\left(1 + \| \{ f_k(x) \} \|_\nu\right) dx.$$

Therefore, it suffices to consider $M_T$ and $M_{T^*}$. On the other hand, it was proved in that maximal operator $M_{L \log L}$ satisfies that

$$\left\{ x \in \mathbb{R}^n : \| \{ M_{L \log L} f_k(x) \} \|_\nu > \lambda \right\}$$

$$\lesssim \int_{\mathbb{R}^n} \| \{ f_k(x) \} \|_\nu \log\left(1 + \| \{ f_k(x) \} \|_\nu\right) dx.$$

Thus, by Lemma 2.6 our proof is now reduced to proving that the inequalities

(2.8) $\quad \mathcal{M}_T f(x) \lesssim MT f(x) + M_{L \log L} f(x).$

and

(2.9) $\quad \mathcal{M}_{T^*} f(x) \lesssim MT f(x) + M_{L \log L} f(x).$

hold. Without loss of generality, we assume that $c_2 > 1$.

Let $Q \subset \mathbb{R}^n$ be a cube and $x, \xi \in Q$. Set $t_Q = \frac{1}{c_2 \sqrt{n} \Omega(Q)}$ and write

$$T(f \chi_{\mathbb{R}^n \setminus \delta_3 Q})(\xi) = D_{t_Q} T f(\xi) - D_{t_Q} T(f \chi_{\delta_3 Q})(\xi)$$

$$+ \left( T(f \chi_{\mathbb{R}^n \setminus \delta_3 Q})(\xi) - D_{t_Q} T(f \chi_{\mathbb{R}^n \setminus \delta_3 Q})(\xi) \right).$$
A trivial computation involving (1.8) leads to that

\[ |D_{tQ} T f(\xi)| \lesssim |Q|^{-1} \sum_{j=1}^{\infty} \int_{2^{j-1}n t_Q^\frac{1}{2} < |\xi - y| \leq 2^j n t_Q^\frac{1}{2}} h \left( \frac{|\xi - y|}{t_Q^\frac{1}{2}} \right) |T f(y)| dy \]

+ |Q|^{-1} \int_{|\xi - y| \leq 2n t_Q^\frac{1}{2}} |T f(y)| dy

\[ \lesssim |Q|^{-1} \sum_{j=1}^{\infty} \int_{2^{j-1}n t_Q^\frac{1}{2} < |x - y| \leq 2^j n t_Q^\frac{1}{2}} h \left( \frac{|x - y|}{2 t_Q^\frac{1}{2}} \right) |T f(y)| dy \]

+ |Q|^{-1} \int_{|x - y| \leq 3n t_Q^\frac{1}{2}} |T f(y)| dy

\[ \lesssim M T f(x). \]

On the other hand, it follows from Lemma 2.11 that

\[ |D_{tQ} T (f \chi_{3Q})(\xi)| \lesssim \frac{1}{|Q|} \sum_{j=1}^{\infty} \int_{2^{j-1}n t_Q^\frac{1}{2} < |x - y| \leq 2^j n t_Q^\frac{1}{2}} h \left( \frac{|x - y|}{2 t_Q^\frac{1}{2}} \right) |T (f \chi_{3Q})(y)| dy \]

+ |Q|^{-1} \int_{|x - y| \leq 3n t_Q^\frac{1}{2}} |T (f \chi_{3Q})(y)| dy

\[ \lesssim M_L \log L f(x). \]

Finally, Assumption 1.3 tells us that

\[ |T (f \chi_{R^n \setminus 3Q})(\xi) - D_{tQ} T (f \chi_{R^n \setminus 3Q})(\xi)| \lesssim \int_{R^n \setminus 3Q} |K(\xi, y) - K^{tQ}(\xi, y)||f(y)| dy \]

\[ \lesssim t_Q^\alpha \int_{R^n \setminus 3Q} \frac{1}{|\xi - y|^{n+\alpha}} |f(y)| dy \]

\[ \lesssim M f(x). \]

Combining the estimates above leads to (2.8).

It remains to prove (2.9). Let \( x, \xi \in Q \). Observe that \( \operatorname{supp} \chi_{R^n \setminus 3Q}(y) \subset \{ y : |y - x| \geq \ell(Q) \} \).

(2.10) \[ T^*(f \chi_{R^n \setminus 3Q})(\xi) \leq |T (f \chi_{R^n \setminus 3Q})(\xi)| + \sup_{\epsilon \geq \ell(Q)} |T_\epsilon f(\xi)|. \]

Now let \( \epsilon \geq \ell(Q) \). Write

\[ T_\epsilon (f \chi_{R^n \setminus 3Q})(\xi) = D_{(\epsilon/c^2)^*} T f(\xi) - D_{(\epsilon/c^2)^*} T (f \chi_{3Q})(\xi) + (T_\epsilon (f \chi_{R^n \setminus 3Q})(\xi) - D_{\epsilon^*} T (f \chi_{R^n \setminus 3Q})(\xi)). \]

As in the argument for \( M_T \), we can verify that

\[ |D_{(\epsilon/c^2)^*} T f(\xi)| \lesssim M(T f)(x) \]

and

\[ |D_{(\epsilon/c^2)^*} T (f \chi_{3Q})(\xi)| \lesssim M_L \log L f(x). \]
As in [4], write
\[
T_\epsilon(f \chi_{R^n \setminus 3Q})(\xi) - D_\epsilon^s T(f \chi_{R^n \setminus 3Q})(\xi) = \int_{|\xi - y| \leq \epsilon} K^{(\epsilon / c_2)^s}(\xi, y) f(y) \chi_{R^n \setminus 3Q}(y) dy \\
+ \int_{|\xi - y| > \epsilon} (K(\xi, y) - K^{(\epsilon / c_2)^s}(\xi, y)) f(y) \chi_{R^n \setminus 3Q}(y) dy.
\]

The fact that \(K^{(\epsilon / c_2)^s}\) satisfies the size condition (1.9), implies that
\[
\left| \int_{|\xi - y| \leq \epsilon} K^{(\epsilon / c_2)^s}(\xi, y) f(y) dy \right| \lesssim \epsilon^{-n} \int_{|\xi - y| < \epsilon} |f(y)| dy \lesssim M f(x).
\]

On the other hand, by the Assumption 1.5, we obtain that
\[
\left| \int_{|\xi - y| > \epsilon} (K(\xi, y) - K^{(\epsilon / c_2)^s}(\xi, y)) f(y) \chi_{R^n \setminus 3Q}(y) dy \right| \lesssim M f(x).
\]

Therefore,
\[
\sup_{\epsilon \geq \ell(Q)} |T_\epsilon(f \chi_{R^n \setminus 3Q})(\xi)| \lesssim M T f(x) + M L \log L f(x),
\]

which, via the estimates (2.8) and (2.10), shows that
\[
M_{T^*} f(x) \lesssim M T f(x) + M L \log L f(x).
\]

This completes the proof of Theorem 1.6. \(\square\)

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