Performance of Complex Langevin Simulation in 0+1 dim. massive Thirring model at finite density

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Abstract: Statistical sampling with the complex Langevin (CL) equation is applied to (0+1)-dimensional Thirring model, and its uniform-field variant, at finite fermion chemical potential $\mu$. The CL simulation reproduces a crossover behavior which is similar to but actually deviating from the exact solution in the transition region, where we confirm that the CL simulation becomes susceptible to the drift singularities, \textit{i.e.}, zeros of the fermion determinant. In order to simulate the transition region with the CL method correctly, we examine two approaches, a reweighting method and a model deformation, in both of which a single thimble with an attractive fixed point practically covers the integration domain and the CL sampling avoids the determinant zeros. It turns out that these methods can reproduce the correct crossover behavior of the original model with using reference ensembles in the complexified space. However, they need evaluation of the reweighting factor, which scales with the system size exponentially. We discuss feasibility of applying these methods to the Thirring model and to more realistic theories.
1 Introduction

Simulating a fermion system at finite chemical potentials often leads to the notorious sign problem with a complex action in the path-integral formalism. Study of QCD phase diagram at finite densities is one of the most challenging subjects of this category. Recently, among other extensive attempts[1], two approaches which involve field complexification have been studied as a promising direction to the solution of the problem. Application of the Langevin equation to a complex action problem is a classic idea [20–22], and has been attracting renewed attention with recent progress [23–34] made in the last decade including exploratory applications to QCD at finite density in [35–39], also in [40] (see [41–43] for review). Generalization of the steepest-descent-path idea for oscillatory integrals to the multi-dimensional cases is called now the Lefschetz-thimble approach [2–4] and has been applied to simple bosonic theories [3–9]
as well as fermionic theories [11–19]. Furthermore several remarks are found in [44–47] in conjunction with the two approaches.

Long-time evolution of the Langevin equation of real action $S(x)$

$$\frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta(\tau)$$ (1.1)

with a stochastic noise $\eta(\tau)$ is a standard method to generate the equilibrium ensemble $P(x) \propto e^{-S(x)}$ of configurations $x$, which is shown by the spectral analysis of the associated Fokker-Planck operator. Apparently, the Langevin evolution can be applied to a complex action system $S \in \mathbb{C}$, once the configuration space is complexified: $x \in \mathbb{R}^n \rightarrow z = x + iy \in \mathbb{C}^n$, which results in the complex Langevin (CL) equation. One can solve CLE in practice by applying adaptive stepsize in the Langevin time $\tau$ to suppress numerical instability of the evolution trajectories [25].

It has been recognized, however, that the resultant distribution $P(x, y)$ in the complexified space $z = x + iy$ converges to an incorrect solution in some cases, including fermionic systems at finite density [48–53]. Correctness of the CL method is argued relying on the holomorphic property of the associated Fokker-Planck equation with repeated use of integration by parts, with requiring the distribution $P(x, y)$ localized in imaginary directions [26, 30, 33]. But the fermion determinant gives rise to a meromorphic drift force, and the condition for correctness of the method is revisited in Refs. [54–57], where subtleties with the meromorphic drift term is carefully studied. It is pointed out that the singularities of the drift force give additional surface-term contribution in the integration by parts, and also that they prevent us from taking the continuum limit of the discretized CL equation, depending of the support of the CL sample distribution. It is claimed in Ref. [56] that CL method gives the correct results if the distribution $P(x, y)$ is suppressed at large drift magnitude $|K| \equiv |\partial S/\partial z|$ exponentially, $P(x, y) \sim e^{-\text{const.}|K(z)|}$ for large $|K|$, or stronger.

In this work we examine statistical simulations with the CL equation applied to the (0+1) dimensional Thirring model, and its uniform-field variant, at finite chemical potential $\mu$ [15, 49, 50, 66]. This model is analytically solvable on a finite lattice, showing a smooth crossover with increasing $\mu$. It has been used as a good benchmark for testing new methods for the sign problem in a fermionic system [15–17, 49, 66]. The CL simulations are performed in (0+1) dimensional Thirring model and the results are compared with the analytic results [15, 49], and the consistency condition are examined [49]. It has been known that CL simulation is successful at small and large $\mu$ outside the crossover, but converges to an incorrect solution in between.

Here we confirm numerically that for $\mu$ in the crossover region the CL simulations of the model becomes susceptible to the singularities of the drift term and the ensemble distribution is only power-suppressed in the vicinities of the determinant zeros, which makes the method unjustifiable there [56]. From the structure of the drift flow field of the model, it seems inevitable that the CL sampling hits the vicinity of the drift singularities (fermion determinant
zeros) for \( \mu \) in the crossover region. At small or large \( \mu \), where CL method is successful, the determinant zeros locate outside the support of the CL ensemble.

In view of the thimble (steepest descent path) integration, the abrupt crossover behavior is resulted from the contributions of the multi thimbles connected via the determinant zeros\(^{14–16}\). The destructive interference between multi-thimble contributions poses the global sign problem in numerical evaluation. In contrast, at zero or large \( \mu \), a single thimble \( J_{\mu_0} \) practically covers the integration domain. In this latter case, the CL sampling \( P(x,y) \) around the thimble does not hit the determinant zeros.

Several attempts are made so far in order to extend the applicability of CL simulations to the crossover region. One is the reweighting method: Observables in the crossover region are computed by reweighting with the CL ensembles generated at a reference point within the (multi-) parameter space where the CL method is valid \(^{58–60}\), at the cost of the so-called reweighting factor. Deformation of the models is proposed in \(^{61, 62}\), where the fermion determinant is modified so that the determinant zeros become irrelevant in the CL sampling, and observables of the original model are re-expressed exactly in terms of those of the deformed models. Another model-deformation method is studied in \(^{63}\) with introduction of an auxiliary parameter into the fermion determinant to study the spontaneous symmetry breaking. The CL method is applied to a fermionic model within the applicable parameter region, and then the zero limit of the parameter is taken by extrapolation. The gauge cooling method is extended in \(^{64}\) for avoiding the artificial singular drift problem in a random matrix model.

In this work, we examine the reweighting method\(^ {59}\) and model-deformation method\(^ {61, 62}\). The latter may be regarded as a kind of reweighting method, for it needs evaluation of the reweighting factor, too, as seen in Eq. (4.1) below. In reweighting we use the CL ensembles generated with zero or large \( \mu \), for which the ensemble supports have no overlap with the vicinity of the zeros and also the corresponding thimble structure becomes simple. The reweighting method can reproduce the correct crossover behavior (for a small system size), which enforces the correctness of the CL ensembles generated outside the crossover region. Then we study the efficiency of the reweighting method by changing the system size. For deformation method, we find a subtlety in dealing with the pole of the fermion observables such as number and scalar densities. We will comment on this point, in addition to the efficiency of the method.

This paper is organized as follows: In the next section we present the definition of the model and its uniform-field variant, and perform CL simulations for them. We then analyze the success and failure of CL simulations by inspecting the ensemble distributions on the complex plane in relation to the zeros, critical points, and thimble structure in some detail, and make histograms of the drift force to test the criterion for the validity of CL method. We end Sec. 2 with a brief summary. In Sec. 3 we apply the re-weighting method to simulate the model in the crossover region using zero- and large-\( \mu \) reference ensembles. Main focus is the efficiency of the re-weighting w.r.t. the system size toward larger systems. In Sec. 4 we examine the deformation method, where we encounter a difficulty due to the poles in the observable and/or numerical efficiency in this approach. Sec. 5 is devoted to Summary. Appendix A
present analytic expressions for the determinant zeros in the deformed model.

2 Model

2.1 Thirring model in (0+1) dimensions

We deal with the lattice Thirring model with a single staggered fermion in (0+1) dimensions at finite chemical potential \( \mu \). After unfolding the vector-type four-point interaction, we can write the partition function \( Z \) as a path integral over a compact auxiliary field \( A_n \) [15, 49, 50, 66]:

\[
Z = \int_{-\pi}^{\pi} \prod_{n=1}^{L} \frac{dA_n}{2\pi} e^{-\beta S_b[A]} \det D[A; \mu]
\]

(2.1)

with the bosonic action \( S_b[A] = \sum_{n=1}^{L} (1 - \cos A_n) \) and the fermion determinant \(^1\)

\[
\det D[A; \mu] = \cosh(L\mu + i\sum_{n=1}^{L} A_n) + \cosh L\hat{m},
\]

(2.2)

where \( L \) is the temporal lattice size, \( \beta = 1/(2g^2) \) the inverse coupling constant, \( \hat{m} = \sinh^{-1} m \) the fermion mass term. We have set the lattice unit \( a = 1 \) and all the dimensionful quantities are measured in the lattice unit hereafter. We notice here that \( \det D[A; \mu] \) depends on the field configuration only through the sum \( s = \sum_{n=1}^{L} A_n \). The sign problem arises from the complex determinant \( \det D \) at nonzero \( \mu \) in this model.

The analytic expression for \( Z \) is known as

\[
Z = e^{-\beta L} \left[ I_1(\beta)^L \cosh L\mu + I_0(\beta)^L \cosh L\hat{m} \right],
\]

(2.3)

where \( I_{0,1}(x) \) are the modified Bessel function of the first kind. The fermion number density \( \langle n \rangle \) and scalar density \( \langle \sigma \rangle \) are obtained, respectively, as

\[
\langle n \rangle = \frac{1}{L} \frac{\partial \log Z}{\partial \mu} = \frac{1}{Z} \int_{-\pi}^{\pi} \prod_{n=1}^{L} \frac{dA_n}{2\pi} e^{-S_b(A)} \det D[A; \mu] n[A; \mu]
\]

\[
= \frac{I_1(\beta)^L}{I_1(\beta)^L \cosh L\mu + I_0(\beta)^L \cosh L\hat{m}} \sinh L\mu
\]

(2.4)

\[
\langle \sigma \rangle = \frac{\sinh(L\mu + is)}{\det D[A; \mu]},
\]

(2.5)

\(^1\) Note that we have omitted an irrelevant factor of \( 2^{L-1} \) in \( \det D \) for notational simplicity.
and
\[
\langle \sigma \rangle = \frac{1}{L} \frac{\partial \log Z}{\partial m}
\]
\[
= \frac{1}{Z} \int_{-\pi}^{\pi} \prod_{n=1}^{L} \frac{dA_n}{2\pi} e^{-S_b[A]} \det D[A; \mu] \sigma[A; \mu]
\]
\[
= \frac{I_0(\beta L)}{I_L(\beta L) \cosh L \mu + I_0(\beta L) \cosh L \hat{m}} \frac{1}{\cosh \hat{m}}
\]
(2.6)

with
\[
\sigma[A; \mu] = \frac{\sinh L \hat{m}}{\det D_0[A; \mu]} \frac{1}{\cosh \hat{m}}.
\]
(2.7)

At zero chemical potential \( \mu = 0 \), the fermion number density vanishes \( \langle n \rangle = 0 \) and the scalar density is nonzero \( \langle \sigma \rangle \neq 0 \). These densities show crossover-transition behavior as the chemical potential \( \mu \) is increased at fixed \( L \), or temperature \( T = 1/L \). In the zero temperature limit \( L \to \infty \), the crossover behavior changes to a first-order transition at a critical value of \( \mu_c \).

### 2.2 Uniform-field model

For studying analytic properties of the CL sampling in the complexified configuration space, it is instructive to introduce a single-variable model which can be deduced from the \( (0+1) \) dimensional Thirring model by setting uniform-field condition, \( x = A_n \) for all \( n = 1, \cdots, L \). Then the bosonic action reduces to \( S_b(x) = \beta L (1 - \cos x) \) and the “determinant” term becomes

\[
\det D_0(x; \mu) = \cosh L (\mu + ix) + \cosh L \hat{m}.
\]
(2.8)

The partition function of this model is given as

\[
Z_0 = \int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{-S_b(x)} \det D_0(x; \mu)
\]
\[
= e^{-\beta L} [I_L(\beta L) \cosh L \mu + I_0(\beta L) \cosh L \hat{m}].
\]
(2.9)

The fermion number density and scalar density are defined in the same way as before:

\[
\langle n \rangle = \frac{1}{Z_0} \int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{-S_b(x)} \det D_0(x; \mu) n(x; \mu)
\]
\[
= \frac{I_L(\beta L) \sinh L \mu}{I_L(\beta L) \cosh L \mu + I_0(\beta L) \cosh L \hat{m}}
\]
(2.10)

with
\[
n(x; \mu) = \frac{\sinh L (\mu + ix)}{\det D_0(x; \mu)},
\]
(2.11)

\[
-5-
\]
and

\[
\langle \sigma \rangle = \frac{1}{Z_0} \int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{-S_b(x)} \det D_0(x; \mu) \sigma(x; \mu) = \frac{I_0(\beta L) \sinh L\hat{m}}{I_L(\beta L) \cosh L\mu + I_0(\beta L) \cosh L\hat{m} \cosh \hat{m}} \frac{1}{\det D_0(x; \mu) \cosh \hat{m}}.
\]

(2.12)

with

\[
\sigma(x; \mu) = \frac{\sinh L\hat{m}}{\det D_0(x; \mu) \cosh \hat{m}} \frac{1}{\det D_0(x; \mu) \cosh \hat{m}}.
\]

(2.13)

The parameter \( L \) is a remnant of the lattice size, with which we can study the system size dependence in this uniform-field model. We use this model for examining the analytic properties of the CL sampling and the severity of the sign problem.

2.3 CL simulation

For the Thirring model in (0+1) dimensions, the CL evolution in the Langevin time \( \tau \) with discrete step \( \varepsilon \), is written collectively for the complexified variables \( z_\ell (\ell = 1, \ldots, L) \) as

\[
z(\tau + \varepsilon) = z(\tau) + \varepsilon K(z(\tau)) + \sqrt{\varepsilon} \eta(\tau)
\]

(2.14)

with the drift force

\[
K_\ell(z) = -\frac{\partial S(z)}{\partial z_\ell} = -\beta \sin z_\ell + i \frac{\sinh(L\mu + is)}{\det D(s; \mu)}
\]

(2.15)

and with the real stochastic force \( \eta_\ell(\tau) \) whose average satisfies \( \langle \eta_\ell(\tau) \rangle_\eta = 0 \) and \( \langle \eta_\ell(\tau) \eta_{\ell'}(\tau') \rangle_\eta = 2\delta_{\ell\ell'}\delta_{\tau\tau'} \). The sum \( s \equiv \sum_{n=1}^{L} z_n \) becomes complex-valued, too. For the uniform-field model we have a single equation with the same structure as above, where the drift force reads

\[
K(z) = -\frac{dS(z)}{dz} = -L\beta \sin z + iL \frac{\sinh L(\mu + iz)}{\det D_0(z; \mu)}.
\]

(2.16)

Notice that the zeros of the determinant give rise to singularities of the drift force in both cases.

Expectation values of observables \( O(x) \) are computed with the ensemble distribution \( P(x, y) \) sampled along with a CL trajectory as

\[
\langle O \rangle_{CL} = \int dx dy P(x, y) O(x + iy) \sim \frac{1}{N_{\text{sample}}} \sum_{\tau} O(z(\tau)).
\]

(2.17)

We have performed the CL simulation for these models at finite \( \mu \). We set the model parameters as \( \beta = 1, m = 1, L = 8 \), and we choose \( \varepsilon = 10^{-5} \) for the Langevin time step. After \( 10/\varepsilon \) steps for thermalization, we evolve the equation by \( 10^9 \) steps with sampling the configurations in every \( 10^3 \) steps. When the drift magnitude \( |K| \) becomes large such that \( \varepsilon|K|/L > \varepsilon_0 = 10^{-3} \), we evolve the equation with a smaller step size \( \varepsilon_v \) to keep \( \varepsilon_v|K|/L < \varepsilon_0 \)
Figure 1. (a): Fermion number \( \langle n \rangle \) (filled circle) and scalar \( \langle \sigma \rangle \) (open circle) densities as a function of the chemical potential \( \mu \) in \((0+1)\) dimensional Thirring model for \( L = 8, \beta = 1 \) and \( m = 1 \). (b): the same plot for the uniform-field model.

till \( \varepsilon \leq \sum \varepsilon_{v} \) is achieved. Since the model is periodic in \( \text{Re} z_{\ell} \), we restricted \( \text{Re} z_{\ell} \in [-\pi, \pi) \) in the simulations.

The simulation results for the fermion number density \( \langle n \rangle \) and scalar condensate \( \langle \sigma \rangle \) in the Thirring model are plotted as a function of \( \mu \) in Fig. 1 (a). The exact results (solid curves) clearly show smooth crossover behavior in the region of \( 1 \lesssim \mu \lesssim 2.3 \). In contrast, the CL results for \( \langle n \rangle \) and \( \langle \sigma \rangle \) deviate from the exact ones, showing milder dependence on \( \mu \) around the crossover region. They agree with the exact results for small and large values of \( \mu \). A similar trend is also seen in the CL result for the uniform-field model as shown in Fig. 1 (b); the exact solution shows crossover transition in the region of \( 0.8 \lesssim \mu \lesssim 2.0 \), while the CL result behaves almost linearly with increasing \( \mu \) in the corresponding region.

2.4 CL ensembles, determinant zeros and thimbles

It has been noticed for some time that the CL method fails in crossover regions for simple fermionic models with singular drift terms, such as the chiral random matrix model, the models with modified Rayleigh distributions, Hubbard model, etc., in addition to the \((0+1)\) dimensional Thirring model [48–53]. In order to understand the situation clearly, we present in Fig. 2 the scatter plots of the CL samples in the uniform field model for \( \mu = 0.5, 0.6, 1.5, 2.0 \) and 2.1 with \( L = 8, \beta = 1, m = 1 \). We collected \( 10^{5} \) samples and show them with yellow dots in the complex \( z \) plane. In the same figure, we also display the drift flow field (2.16) with gray arrows, and the zeros of the determinant in red diamonds, where the drift become singular. Green circles denote the fixed points \( z_{c} \) of the drift force field.

Blue curves are the steepest descent paths \( J_{z_{c}} \) for \( e^{-S} \), or the thimbles in Lefschetz theory.
The thimbles are defined by the union of integral curves of the anti-holomorphic flow

\[ \frac{dz}{d\tau} = \frac{\partial S[z]}{\partial z} \]  \hspace{1cm} (2.18)

with the boundary condition, \( \lim_{\tau \to -\infty} z(\tau) \to z_c \). It is known that a set of thimbles \( \{J_{zc}\} \) becomes equivalent to the original integration contour:

\[ Z = \int_{-\pi}^{\pi} dx e^{-S} = \int_{\{J\}} dz e^{-S}. \] \hspace{1cm} (2.19)

One important property of the anti-holomorphic flow is that \( \text{Re} S \) is monotonically increasing with \( \text{Im} S \) kept constant along the flow, especially on a thimble. This property motivated us to perform Monte Carlo simulations on thimbles[2–4].

This anti-holomorphic flow field is obtained from the drift force field by flipping the sign of the real component. Under this transformation, zero points \( z_{\text{zero}} \) and critical points \( z_c \), which are saddle points and attractive (or repelling) points in CL method, respectively become attractive points and saddle points in Lefschetz theory. Since the total set of thimbles form a skeleton graph without loops, there must be an associated critical point \( z_c \) for each zero point \( z_{\text{zero}} \), which provides an endpoint for a thimble \( J_{zc} \).

In Fig. 2, we first notice that the critical point \( z_0 \) on the imaginary axis is an attractive point of the drift force field for all the values of \( \mu \) studied here, and it is always included in the CL ensemble. Indeed, thimble \( J_{z_0} \) is always the most dominating one in the thimble integration in this model. At \( \mu = 0.5 \) (Fig. 2 (a)), the distribution of the CL samples is elongated along the original integration contour, the real axis \( [-\pi, \pi] \), having small extent in the imaginary direction. It also has a good overlap with the thimble \( J_{z_0} \) emerging from the critical point \( z_0 \). We note that the distribution apparently has no overlap with the vicinity of any \( z_{\text{zero}} \).

At a slightly larger chemical potential \( \mu = 0.6 \) (Fig. 2 (b)), however, the distribution gets expanded in the imaginary direction to have an overlap with the vicinity of some zeros \( z_{\text{zero}} \) of the determinant. This situation is understood as follows: With increasing \( \mu \), the zeros \( \{z_{\text{zero}}\} \) move upward together with their associated critical points \( \text{3} \), and a reconnection of the thimbles (the Stokes phenomenon) occurs between \( \mu = 0.5 \) and \( 0.6 \). At \( \mu = 0.6 \) the thimble \( J_{z_0} \) terminates at the two zeros closest to the origin, and the integration contour now consists of multi thimbles. A change of the thimble structure must accompany a change of the drift flow pattern, because the anti-holomorphic flow defining the thimbles is obtained just by flipping the sign of the real component of the drift flow 2.16 of CL method.

We show a representative configuration of a CL ensemble, drift field and thimbles at \( \mu = 1.5 \) in Fig. 2 (c), where the thimbles are connected with each other at \( z_{\text{zero}} \)'s, and the CL samples are distributed closely to the thimbles \( \text{4} \). Until \( \mu \sim 2.0 \) (Fig. 2 (d)), the CL sampling

2 Since we assume that \( S \) is bounded below, we don’t have repelling points here.
3 See Appendix A for the explicit expressions for \( z_{\text{zero}} \)'s.
4 It is known that there is other models where a critical point of a relevant thimble becomes repulsive and not sampled in the CL sampling.
Figure 2. Scattering plot of the CLE sampling for $\mu = 0.5$ (a), 0.6 (b) 1.5 (c), 2.0 (d) and 2.1 (e) with $L = 8$, $m = 1$, $\beta = 1$. Arrows depict the field of the drift force and blue curves are thimbles. Red diamonds and green circles are singular and zero points of the field, respectively.
Figure 3. Scatter plot of the CL samples of 0+1 dim model with $L = 8$, $\beta = 1$ in complex $s/L = \sum_{n=1}^{L} z_n/L$ space for $\mu = 0.5$, 1, 1.5, 2, and 2.5 from bottom to top. Singular points of the drift force are shown with circles, triangles, squares, diamonds, and crosses for respective values of $\mu$. For $\mu = 1.0, 1.5$ and 2.0, these singular points locate at the edges of scatter distributions. Note that the distribution for $\mu = 2.5$ is almost oval-shaped, but part of it is hidden below that for $\mu = 2.0$.

includes the vicinities of the zeros, and exactly in this region of $\mu$, the CL results converge to the incorrect solution. Above this value of the chemical potential, as is seen in Fig. 2 (e), the CL samples are distributed away from any zeros and localized around the single thimble $\mathcal{J}_z$ covering practically the whole integration contour. In this case, the CL simulation gives the correct result (see Fig. 1 (b)).

Importance of multi-thimble contributions in the transition region is emphasized in Refs. [14, 15]; each thimble $\mathcal{J}_z$ has a complex phase of $\text{Im}S(z_c)$, and destructive interference among multi-thimble contributions is crucial for realizing the Silver-Blaze phenomenon[67], i.e., no response of the system to $\mu$ below a certain critical value at zero temperature.

Because the thimbles in the set which is equivalent to the original integration contour are connected at the determinant zeros and because the CL samples are well localized around the thimbles in this model, it seems almost inevitable that the CL samples the neighborhoods of the drift singularities (determinant zeros). In Fig. 2 (c), the critical points and zero points are aligned almost in a straight-line parallel to the real axis, and then these critical points become attractive nodes and these zeros are saddle points in the drift flow field. We also notice that in the thimble integration one needs to include correctly the complex phases from $\text{Im}S(z_c)$ and the Jacobian in the change of the variables, while in the CL method observables are computed as a simple average of the samples $P(x, y)$ distributed around the thimbles in
the complex $z = x + iy$ plane without any additional phase factors.

Outside the crossover region, the integration contour is dominated solely by a single thimble, and relative phases among the thimbles become irrelevant. In the CL approach, this implies that the vicinities of the determinant zeros are not sampled so that the simulations give the correct results.

In Fig. 3, we show a scatter plot of the (0+1) dimensional model in complex $s/L = \sum_{n=1}^{L} z_n/L$ space, for $\mu = 0.5, 1, 1.5, 2$ and 2.5 with $L = 8$, $\beta = 1$, and $m = 1$ in the complex $s/L$ plane. The zero manifold, where $K = 0$, projected on this plane becomes a set of points, which we denote with circles, triangles, squares, diamonds, and crosses for respective values of $\mu$ in Fig. 3.

We confirm that the CL ensembles overlap with the neighborhoods of the determinant zeros for $\mu$ in the crossover region ($\mu = 1, 1.5, 2$). We also note that the samples are scattered wider in imaginary direction than in the uniform-field model.

2.5 Histogram of drift force

Recently authors of Ref. [56] claimed that an exponential fall-off of the probability of the drift term at large magnitude is the necessary and sufficient condition for the correctness of the CL results. Indeed, the histogram of $|K|$ is very helpful to quantify the contributions from the vicinities of the zeros in the CL simulations.

We make histograms of the drift force $|K|$ of the uniform-field model with $L = 8$, $\beta = 1$ and $m = 1$ for $\mu = 0.5, 1, 1.5, 2, 2.5$ and 3 in Fig. 4 (a) by binning the drift magnitude as $a^{n-1} \leq |K|/L < a^n$ (we chose here $a = 2^{1/4}$ and $n$ integer). Outside the crossover region, $\mu = 0.5, 2.5$ and 3, we find that the histograms fall off very sharply and become vacant at

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{(a): Histogram of magnitude of the drift term $|K(z)|$ in the uniform-field model for $\mu = 0.5, 1, 1.5, 2, 2.5$ and 3 with $10^5$ samples. (b): The same plot for in (0+1)-dim. model for typical values of $\mu = 0.5, 1.0, 1.5, 2.0, 2.5$ and 3.0}
\end{figure}
large $|K|$. But it develops a power-law tail for $\mu = 1, 1.5, 2$ in the crossover region, which clearly show the fact that the CL sampling hits the vicinities of the determinant zeros. This result is consistent with the argument of Ref. [56].

In Fig. 4 (b), we make histograms of $|K|$ of the $(0+1)$ dimensional Thirring model with $L=8$, $\beta = 1$ and $m = 1$ for $\mu = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$. At small $\mu = 0.5$, for which the CL simulation is successful, the histogram is well localized around $|K|/L \sim 0.2$. But we find that a power-law-like distribution of $|K|$ appears for $\mu \gtrsim 1.0$, which may be caused by the singularities of the force at the determinant zeros. From Fig. 1 (a), however, the value $\mu = 3$ is well outside the crossover region and the CL simulation successfully reproduce the correct result. Moreover, from Fig. 3 the vicinities of the zeros are little sampled at this value of $\mu$, in spite of the power-low tail appearing in the histogram.

To measure the contributions from the vicinities of the drift singularities, we make histograms of $|\frac{d}{ds} \log \det D(s)|$ in Fig. 5 (a). We find that the histogram is empty at large $|\frac{d}{ds} \log \det D(s)|$ for $\mu = 0.5$ and 3.0, i.e., for $\mu$ outside the crossover region, while the power-law tail appears for $\mu$ in the crossover. It is concluded, therefore, that the power-law-tail contribution from the neighborhoods of the drift singularities exists only in the crossover region to make the CL method dubious there.

Then the power-law tail at large $|K|$ in the histogram Fig. 4 (a) for large $\mu$ must come from the first term $-\beta \sin(z_{\ell})$ in the drift force, (2.15). This is indeed the case as can be confirmed by looking at the histogram of $|\sin(z_{1})|$, for example, in Fig. 5 (b). We see that the probability of large $|\sin(z_{1})|$ develops as $\mu$ increases.

This power-law tail of the $|K|$-distribution for large $\mu$ (-3) appears by fluctuations of $z_{\ell}$ ($\ell = 1, 2, 3, \cdots, L$) in imaginary direction. In the $(0+1)$ dimensional model, with $s = \sum_{n=1}^{L} z_{n}$.
localized around the critical point $z_0$ (Fig. 3), a component $z_\ell$ can first fluctuate in real direction to the region $z_\ell \sim \pm \pi + iy_\ell$ with $y_\ell > 0$. Then a trajectory rides on a drift flow to imaginary direction, which is caused by $-\beta \sin z_\ell$ term in the flow, (2.15), and the trajectory makes an excursion in the imaginary direction which then comes back to the region near $z_0$ (See Fig. 6). We have confirmed that the histogram of $y_\ell$ integrated over other variables is consistent with an exponentially decaying distribution $\propto e^{-cy_\ell}$ with $c \sim 2.6$ for $y_\ell \gtrsim 2$. When plotted with respect to $|\sin z_\ell| \sim \frac{1}{2}e^{y_\ell}$, the histogram decays with power law $|\sin(z_\ell)|^{-(c+1)}$ in large $y_\ell$ region, which is seen in Fig. 5 (b).

2.6 Short summary

We have seen that the CL simulation converges to an incorrect solution within the crossover region of the model, where CL samples the neighborhoods of the determinant zeros, $z_{\text{zero}}$, as seen in the histograms of the drift force and also in the scatter plots. This is the very situation where CL method becomes dubious as is argued in Refs. [56, 57]. We have also seen that it is almost inevitable for CL simulations to sample the vicinities of these zeros $z_{\text{zero}}$’s from the observation of the drift flow pattern in the crossover region, where contributions from multi-thimble connected with each other via these zeros become important in the thimble integration.

Outside the crossover region, on the other hand, the CL simulation successfully reproduces the correct solution. In this region, the thimble structure in the complexified configuration space is so simple that a single thimble becomes practically equivalent to the original integration domain, and the CL ensemble is well localized around this single thimble without sampling the vicinity of any determinant zeros $z_{\text{zero}}$’s.

In order to apply the CL simulation in the crossover region, therefore, we need methods which avoid sampling the vicinity of $z_{\text{zero}}$’s. One possible approach to overcome this situation is the so-called re-weighting method, in which one evaluates the observables in the crossover.
region by using the ensemble generated in other parameter region where the CL method gives the correct results[59]. Another approach is to express the observables with a deformed (or modified) model for which the CL simulation is successful without sampling the vicinity of its zeros[61].

We examine these two approaches one by one in the next two sections.

3 Reweighting method at finite $\mu$

In order to compute the observables correctly in the crossover region, one can try the reweighting method with CL ensembles in the parameter space $({\beta, m, \mu})$ where CL method works well [59]. We have observed that the CL simulation is successful for $\mu$ outside the crossover region, where a single thimble practically covers the original integration contour and the CL ensembles are distributed along this thimble. Based on this observation, we study here the reweighting method with the CL ensembles generated at a large reference chemical potential, which we call $\nu$.

In this approach, we recast the expectation value of an observable $O$ as

$$\langle O \rangle_\mu \equiv \frac{\int_{-\pi}^{\pi} dx e^{-S_b(x)} \det D(x; \mu) O(x)}{\int_{-\pi}^{\pi} dx e^{-S_b(x)} \det D(x; \mu)} = \left( \frac{\det D(\mu)}{\det D(\nu)} \right) \langle O \rangle_\nu,$$

(3.1)

where the subscript $\langle \rangle_\mu$ indicates that the expectation value is evaluated with the weight $\exp(-S_b(x)) \det D(x; \mu)$ and similarly for $\langle \rangle_\nu$. In usual reweighting, one introduces a real weight function so that one can perform Monte Carlo sampling. For our model we can choose $\exp(-S_b(x)) \det D(x; 0)$ or $\exp(-S_b(x))|\det D(x; \nu)|$ as the weight function. But in the context of CL simulations, one may take the system with large chemical potential $\nu$ outside the crossover region as the reference system, and perform the CL simulation to evaluate the observables using Eq. (3.1). The reweighting method with the CL method is expected to work as long as $\nu$ is outside the crossover region but the reference ensembles still have sufficient overlap with the physical one, as discussed below.

3.1 Reweighting factor

Severity of the sign problem in this approach appears in the evaluation of the reweighting factor with the reference ensemble

$$\left( \frac{\det D(\mu)}{\det D(\nu)} \right)_\nu = \frac{\int_{-\pi}^{\pi} dx e^{-S_b(x)} \det D(x; \mu) \det D(x; \nu)}{\int_{-\pi}^{\pi} dx e^{-S_b(x)} \det D(x; \nu)} = \left( \frac{Z(\mu)}{Z(\nu)} \right)$$

$$= \frac{\int \sum J dz e^{-S_b(z)} \det D(z; \mu) \det D(z; \nu)}{\int \sum J dz e^{-S_b(z)} \det D(z; \nu)}.$$

(3.2)

In the second line, we have changed the integration contour from the real axis to a set of the thimbles for the function $e^{-S_b(z)} \det D(z; \nu)$, because the integrands are holomorphic. In the
Figure 7. Average of the phase factors $\langle e^{i\varphi} \rangle_{\mu / m}$ and $\langle e^{i\varphi} \rangle_{\nu / m}$ as a function of $\mu / m$ for $L = 8, 16, 32$ with $\beta = 1, m = 1$, and for $L = 16$ with $\beta = 2, m = 1/2$. The reference chemical potential $\nu / m$ is set to 3 (solid) and 2 (dashed).

CL simulation with large $\nu$, the ensembles are localized along a single thimble which covers the dominant part of the integration contour in the complex plane. This suggests that the integration with the CL method is closely related to the integration along the thimble $J$, i.e., the second expression of eq. (3.2).

It is noteworthy that the reweighting factor is expressed with the average phase fluctuation factor Eq. (3.4) as

$$\frac{\langle \det D(\mu) \rangle}{\langle \det D(\nu) \rangle} = \langle e^{i\varphi} \rangle_{\nu / m} \cdot \left| \frac{\langle \det D(\mu) \rangle}{\langle \det D(\nu) \rangle} \right|_{\nu},$$

where the average phase fluctuation is defined as the integral along the thimbles:

$$\langle e^{i\varphi} \rangle_{\nu / m} \equiv \int_{\sum J} dz \ e^{-S_b(z)} \det D(\nu) / \int_{\sum J} dz \ e^{-S_b(x)} \det D(\nu) \left| \frac{\det D(\mu)}{\det D(\nu)} \right|.$$  (3.4)

This clearly shows that the average phase factor (3.4) is a good measure of the severity of the sign problem in evaluation of the reweighting factor. Note that the denominator of Eq. (3.4) is now an integral of a non-holomorphic function, and that the average phase factor (3.4) depends on the choice of the integration path. By setting $\nu = 0$, the integration path reduces to the real axis $[-\pi, \pi)$ and the expression (3.4) coincides with the usual phase factor

$$\langle e^{i\varphi} \rangle_{\nu_0} \equiv \frac{\int_{-\pi}^{\pi} dx \ e^{-S_b(x)} \det D(x; \mu)}{\int_{-\pi}^{\pi} dx \ e^{-S_b(x)} \left| \det D(x; \mu) \right|}.$$  (3.5)

because $\det D(x; \nu)$ is real positive there.

We plot in Fig. 7 the averaged phase factors, $\langle e^{i\varphi} \rangle_{\mu / m}$ and $\langle e^{i\varphi} \rangle_{\nu / m}$, as a function $\mu / m$ with fixed $\nu = 2$ (dashed) and 3 (solid) for $L = 8, 16$ and 32 in the uniform-field model. The
Figure 8. Fermion number $n$ and scalar $\sigma$ densities as a function of $\mu/m$, evaluated with the reweighting method at $\nu = 0$ (left) and $\nu/m = 2.5$ (3) for $L = 8, \beta = 1, m = 1 (16, 2, 1/2)$ in uniform-field model ($10^8/10^2$ samples collected with $\varepsilon = 10^{-3}, \varepsilon_0 = 10^{-1}$).

phase average $\langle e^{i\phi} \rangle_{p,q}$ with $\mu = 0$ is unity at small $\mu$ and starts to decrease around $\mu \sim 0.8$, which is close to the threshold value for the failure of the simple CL simulation. It continues to decrease until around $\mu \sim 1.5$ and becomes constant for larger $\mu$. But the value of the phase average $\langle e^{i\phi} \rangle_{p,q}$ at large $\mu$ is suppressed exponentially in $L$; it becomes as small as $10^{-2}$ for $L = 8$ at large $\mu$, and in order to get signal at large $\mu$, one needs at least $O(10^4)$ samples in statistical sampling. For $L = 16$ the number of required samples is likely to become more than $O(10^6)$. For $L = 32$ it is more than $O(10^{13})$.

In contrast, the phase average $\langle e^{i\phi} \rangle_{\nu,p,q}$ evaluated along the thimbles is almost unity on the large $\mu$ side. This can be understood by noting the fact that the location of the dominant thimble $J_{z_0}$ becomes less sensitive to the value of $\mu$ when $\mu$ is set to a larger value above the crossover region. One can see that for large $\mu$ the critical point $z_0$ appears on the imaginary axis, independently of $\mu$. Then the thimbles $J_{z_0}$ for large $\mu$ and $\nu$ locate very close to each other, on which the integrands are real positive up to the Jacobian, and therefore the phase factor in eq. (3.4) becomes close to unity.\footnote{In CL simulations the integration measure is replaced with the ensemble average, and the ensembles generated by the CL evolution with these $\mu$ and $\nu$ are closely overlapped with each other.} At small $\mu \lesssim \hat{m}$, on the other hand, the phase average $\langle e^{i\phi} \rangle_{\nu,p,q}$ becomes small because the integrand of $Z(\mu)$ is now complex-valued and very oscillatory along the dominant thimble $J_{z_0}$ of $Z(\nu)$ in the complex $z$ plane. We notice here that the situation here is quite opposite to the reweighting at $\nu = 0$: one needs large ensembles exponentially in the size $L$, to get signals at small $\mu$, while the simulations at large $\mu$ becomes easier.

We also show in Fig. 7 the reweighting factor for $L = 16$ but with $\beta = 2$, intending one step toward the continuum limit from $L = 8$ and $\beta = 1$. In this change of parameters, the reweighting factors take the similar values to those in the case with $L = 8$ and $\beta = 1$.\footnote{In CL simulations the integration measure is replaced with the ensemble average, and the ensembles generated by the CL evolution with these $\mu$ and $\nu$ are closely overlapped with each other.}
3.2 Uniform-field model

Now we apply the reweighting method to the uniform-field model with the parameters $(L, \beta, m) = (8, 1, 1)$ with choosing the reference chemical potential to $\nu/m = 0$ and 2.5. As seen in Fig. 1, the value $\mu/m = 2.5$ is well above the crossover region, where CL method gives the correct results. For this parameter set, the phase average $\langle e^{i\phi}\rangle_{p,q}$ or $\langle e^{i\varphi}\rangle_{\nu,p,q}$ becomes as small as of order $O(10^{-2})$, which implies that we will lose numerical accuracy by this factor in the reweighting. Thus, in order to detect significant signals, we collected $10^6$ samples in the CL simulations of $10^8$ steps with the step size $\varepsilon = 10^{-3}$ by making measurements every 100 interval steps.

In Fig. 8, results of reweighting at $\nu = 0$ (left panel) and 2.5 (right panel) for $L = 8$ are presented with red circles. The bars indicate the statistical uncertainties assuming that the samples with interval $\Delta \tau = 100\varepsilon = 0.1$ are independent. We find that the reweighting method reproduces the correct results reasonably for the fermion number $\langle n \rangle$ and scalar $\langle \sigma \rangle$ densities. If we look closer in the crossover region $\mu/m = 1 \sim 2$, we find that the reweighting at $\nu = 2.5$ works slightly better than the reweighting at $\nu = 0$. This can be understood from the smallness of the phase factors, $\langle e^{i\phi}\rangle_{p,q}$ and $\langle e^{i\varphi}\rangle_{\nu,p,q}$, presented in Fig. 7.

The results of the same model on the finer lattice ($L = 16, \beta = 2, m = 1/2$) at the one step toward the continuum limit, are also shown with squares in Fig. 8. We recognize here that the reweighting method works with similar quality between $L = 8$ and 16.

We perform the same simulations for lower-temperature case, $L = 16, \beta = 1, m = 1$, and show the result in Fig. 9. The statistical uncertainties become substantially larger in this case than in the previous case. This is again consistent with the smallness of the phase average shown in Fig. 7. The ensemble size $10^6$ is still insufficient to assure a good accuracy of the numerical results. The size required for obtaining the significant numerical accuracy will
Figure 10. Expectation values of $\langle e^{i\varphi}\rangle_{\nu,p,q}$ as a function $\mu/m$ in 0+1 dimensional Thirring model, evaluated in Langevin simulations. Left: $(L,\beta,m) = (8,1,1)$ and $(16,2,1/2)$. Right: $(16,1,1)$.

increase exponentially with $L$. We notice here that in the crossover region of $\mu$ the reweighting results at $\nu = 2.5$ have smaller uncertainty bars than those at $\nu = 0$ in Fig. 9.

3.3 Thirring model in (0+1) dimensions

We collected $10^7$ samples with 200 interval steps ($\varepsilon = 5 \cdot 10^{-4}$, $\varepsilon_0 = 5 \cdot 10^{-2}$) to draw the figures in this subsection. First we show in Fig. 10 the expectation value of the phase average, the CL counterpart of Eq. (3.4). The left panel shows the results for $L = 8$ and $\beta = 1, m = 1$. We see that the phase average obtained in the phase quench simulation at $\nu = 0$ (filled circles) decreases to about $10^{-3}$ at large $\mu$. In contrast, the phase average evaluated with the reference parameter $\nu = 3.0$ (open circles) becomes a few times $10^{-2}$ in the crossover region but even turns to be negative with substantial errors for $\mu \lesssim 1$. On the finer lattice, $(L,\beta,m) = (16,2,1/2)$, the result for the phase average evaluated with $\nu = 0$ ensemble (filled squares) becomes a bit larger in large $\mu$ region, while the phase factor evaluated with $\nu/m = 3.0$ ensemble (open squares) takes smaller values in the crossover region, as compared to $L = 8$ case. This results suggest that the sample points more than $10^6$ will be required to study these system with the CL simulations, if the statistical uncertainty scales with square root of the sample size.

However, when we decrease the temperature by changing $L = 16$ from $L = 8$ with $\beta = 1, m = 1$ fixed, the phase average $\langle e^{i\varphi}\rangle$ becomes smaller by one order of magnitude with substantially larger statistical errors, as is shown in the right panel of Fig. 10. We performed CL simulations with $\nu = 0$ and 3 to generate the reference ensembles of size $10^7$, but after reweighting we did not obtain any significant results in the crossover region. This result suggests that we need to search more efficient reference parameters $\beta$ and $m$ in addition to $\mu$, in order to apply the reweighting method at lower temperature, which we leave for future study.
Figure 11. Number $\langle n \rangle$ and scalar $\langle \sigma \rangle$ densities obtained by reweighting at $\nu/m = 0$ (left) and 3 (right) for 0+1 dim Thirring model with $(L, \beta, m) = (8, 1, 1)$ (red circles) and $(16, 2, 1/2)$ (blue squares).

Figure 12. Reweighting result of 0+1 dim Thirring model with $\beta = 1$, $m = 1$, $L = 16$. (data points to be added)

4 Deformation method

Tsutsui and Doi[61] proposed a model deformation for a simple fermionic model to avoid the problem of the determinant zeros, by noting an empirical relation between the multi-thimble structure of the model and the failure of the CL simulation. In this section, we examine the deformation method by applying it to the uniform field model.

4.1 Exact expressions

In Ref. [61] an extra term is added to the polynomial term of the original model so that the deformed model has a single thimble covering the integration contour and the CL method becomes applicable to the model. Then the observables of the original model are expressed...
Figure 13. Thimbles (blue lines) and gradient flow fields in the $z = (x,y)$ plane for the modified model $Z_{f+g}$ for $\mu_f = 1$ (left) and $2$ (right) ($\mu_g = 3; \beta = 1; m = 1; L = 16$). The zeros and critical points are shown in red and green, respectively.

exactly in terms of those of the deformed models, by utilizing the following identity $^6$:

\begin{equation}
\langle O \rangle_f = \langle O \rangle_{f+g} + \left( \langle O \rangle_{f+g} - \langle O \rangle_g \right) \frac{\langle g \rangle_0}{\langle f \rangle_0} \\
= \langle O \rangle_{f+g} + \left( \langle O \rangle_{f+g} - \langle O \rangle_g \right) \left( \frac{f}{g} \right)_{-1}^{1} ,
\end{equation}

where $\langle O \rangle_{f,f+g}$ denote respectively the expectation values taken with the partition functions,

\begin{equation}
Z_f \equiv \int \frac{dx}{2\pi} e^{-S_b(x)} f(x) , \quad Z_{f+g} \equiv \int \frac{dx}{2\pi} e^{-S_b(x)} [f(x) + g(x)]
\end{equation}

with certain functions, $f(x)$ and $g(x)$. In the present model, we identify $f(x) = \det D_0(x;\mu)$, and among many possibilities for $g(x)$ we choose here $g(x) = \det D_0(x;\mu_g)$ with $\mu_g$ a reference chemical potential. (Hereafter, we denote the physical chemical potential $\mu = \mu_f$. The expectation value $\langle O \rangle_0$ is taken with the fermion quenched model without $f$ and $g$ factor. This identity will be beneficial if one can find $g(x)$ such that CL simulation produces the correct results for $Z_{g,f+g}$ even when it does not work for $Z_f$. Vanishing $\langle g \rangle_0 = 0$ is the most beneficial case. But there is no general proof for the existence of such a special choice of $g(x)$ for a given model.

In the uniform-field model, we have seen that CL method gives the correct results for a large $\mu_f$, where the thimble structure becomes simple and the original integration contour is practically covered with a single thimble. By choosing the reference chemical potential $\mu_g$ large, we can make the thimble structure of the deformed model $Z_{f+g}$ (as well as $Z_g$) simple for $\mu_f$ within the range considered here. For $\mu_g = 3$ we draw the thimble structure of the

$^6$The second expression is due to S. Shimazaki.
deformed model $Z_{g+f}$ with $L = 16, \beta = 1, m = 1$ and $\mu_f = 1$ (left) and 2 (right) in Fig. 13. All the zero points locate well above the real axis in the imaginary direction and the single dominant thimble practically covers the integration domain for $\mu$ in the range considered here (see Appendix A for analytic expressions for the zero points).

The sign problem in this expression remains in the multiplier $\langle g \rangle_0 / \langle f \rangle_0$ appearing in Eq. (4.1): Both the integrands in the denominator and numerator become very oscillatory for large $\mu_{f,g}$. This multiplier is actually the inverse of the reweighting factor as is indicated in the second line of Eq. (4.1).

We evaluate the exact expression (4.1) with CL method for the observables $n(x; \mu)$ (2.11) and $\sigma(x; \mu)$ (2.13). We notice here, however, that CL simulations may hit the poles of these observables depending on the value of $\mu_f$, although the poles of the drift term of the deformed model are avoided by choosing $\mu_g$ large. The CL simulation will not be justified for such values of $\mu_f$ in this approach.

Concerning this point, we like to remark that if we adopt also for the number density observable the deformed expressions,

$$\tilde{n}_{f+g}(x) \equiv \frac{\sinh L(\mu_f + i x) + \sinh L(\mu_g + i x)}{\det D_0(x; \mu_f) + \det D_0(x; \mu_g)}, \quad (4.3)$$

$$\tilde{n}_g(x) \equiv \frac{\sinh L(\mu_g + i x)}{\det D_0(x; \mu_g)}, \quad (4.4)$$

then we find another, but similar identity:

(II): $\langle \tilde{n}_f \rangle_f = \langle \tilde{n}_{f+g} \rangle_{f+g} + \left( \langle \tilde{n}_{f+g} \rangle_{f+g} - \langle \tilde{n}_g \rangle_g \right) \left( \frac{\det D_0(\mu_f)}{\det D_0(\mu_g)} \right)^{-1}. \quad (4.5)$

We also have a corresponding identity for $\langle \sigma \rangle$ with deformed expressions $\tilde{\sigma}_{g,f+g}(z; \mu)$. The expression (II) has two good properties: (i) the pole locations of $\tilde{n}_{g,f+g}$ are the same as those of the drift term, which are avoided in the CL evolution by taking $\mu_g$ large enough, and (ii) we have analytic expressions for $\langle \tilde{n}_{g,f+g} \rangle$ to be compared with the numerical results.

### 4.2 Numerical results

We present numerical results of the deformation method applied to the uniform-field model for $L = 8$ and 16 with $\beta = 1, m = 1$. In the simulations we chose here $\varepsilon = 10^{-4}$ and $\varepsilon_0 = 10^{-2}$, and collected the samples $N = 10^6$ with $10^2$ interval steps.

#### 4.2.1 Case (I)

We display the numerical results of the deformation method (I) applied to the uniform field model for $L = 8$ in the upper row of Fig. 14, and those for $L = 16$ in the lower row. We choose here $\mu_g = 2.2$. For this value of $\mu_g$ the integration contours practically get covered by the dominant thimbles $J_{z_0}$ of $Z_{g,f+g}$, respectively. The left panels show the density obtained $\langle n \rangle_f$ by formula (I), and the middle and right panels show the intermediate quantities appearing on RHS of formula (I).
Figure 14. Fermion number density obtained in CL simulation using the formula (I) [left], its decomposition $\langle n \rangle_{f+g}$ and $\langle n \rangle_{f+g} - \langle n \rangle_{g}$ (offset by $-0.5$ for visibility) [middle], and reweighting factor [right] for $L = 8$ (upper) and 16 (lower) with $\beta = 1$, $m = 1$, and $\mu_g = 2.2$.

We show $\langle n \rangle_{f+g}$ in middle panels of Fig. 14 as a function of $\mu_f$. The CL results for both $\langle n \rangle_{f+g}$ and $\langle n \rangle_{g}$ actually show abrupt changes around $\mu_f \sim 1.8$ (and $\sim 0$). We have confirmed that they are almost consistent with the values obtained by integration along the thimbles of $Z_{g,f+g}$. In the thimble integration, these densities make jumps as the poles in the integrand (2.11) cross the integration contours, but they show abrupt but smooth change here because the CL ensembles extend over the complex plane, not on a line. We also show with an offset $-0.5$ the difference $\langle n \rangle_{f+g} - \langle n \rangle_{g}$ there, which has a spike because the abrupt change of each term occurs at different $\mu_f$ because the CL ensembles for $Z_{g,g+f}$ are not identical.

As shown in left panel of Fig. 14, the numerical result of (I) for $L = 8$ reproduces the crossover behavior fairly well except in the regions $\mu \sim 1.8$ and $\sim 0$, where we see the spike structure. In the CL simulations, a complete cancellation of the abrupt changes in the subtraction term on RHS is impossible to reproduce the exact result. Moreover the poles in the expression of the observable make the correctness of CL simulation very dubious.

In formula (I), the gradual $\mu_f$ dependence of the exact curve should be reproduced by a delicate combination of $\langle n \rangle_{f+g} - \langle n \rangle_{g}$ and $(Z_f/Z_g)^{-1} = (D_0(\mu_f)/D_0(\mu_g))^{-1}$, because $\langle n \rangle_{f+g}$ is almost flat except for the abrupt changing point of $\mu_f$. It is statistically challenging to evaluate accurately the subtraction $\langle n \rangle_{f+g} - \langle n \rangle_{g}$ and the reweighting factor (right panel) for larger size systems. Note the magnification factors $10$ and $10^3$ in the plot of the subtracted

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7 If we evaluate $\langle n \rangle_{f+g}$ and $\langle n \rangle_{g}$ by integration along the thimble, the poles of their integrands cross the thimbles at different values of $\mu_f$ because the thimbles of $Z_{g,f+g}$ are slightly different. As a result, the subtracted density $\langle n \rangle_{f+g} - \langle n \rangle_{g}$ shows a spike. In order for (I) to hold, one must choose the same integration contour in each term on RHS so that the pole contributions must cancel out on LHS of Eq. (4.1).
density for $L = 8, 16$, respectively. The numerical severity is seen clearly in the result for $L = 16$ as shown in the lower panels of Fig. 14, where we have much larger statistical errors.

4.2.2 Case (II)

In case (II), expressions for the observables $\tilde{n}_{g,f+g}$ have the poles at the same locations as the drift term, which are avoided in the CL simulation by taking $\mu_g$ large. Thus the fermion number densities, $\langle \tilde{n} \rangle_{g,f+g}$, will not show any abrupt dependence on $\mu_f$, unlike to case (I). We show in Fig. 15 the fermion number density $\langle \tilde{n} \rangle_f$ obtained for $L = 8$ (upper) and 16 (lower) with $\beta = 1, m = 1$ and $\mu_g = 2.2$.

Figure 15. Fermion number density obtained in CL simulation using formula (II) [left], its decomposition $\langle n_{f+g} \rangle - 1$ and $\langle n_g \rangle - 1$ [right] for $L = 8$ (upper) and 16 (lower) with $\beta = 1, m = 1$ and $\mu_g = 2.2$.

However, for the reference value $\mu_g$ large above the crossover value, both $\langle \tilde{n} \rangle_{g,f+g}$ become nearly unity regardless of the value of $\mu_f$ within the range of our interest (see Eqs. (4.3) and (4.4)). After taking subtraction, $\langle \tilde{n} \rangle_{f+g} - \langle \tilde{n} \rangle_g$, we are left with a tiny number as seen in the right panel of Fig. 15. In order to get the fermion number density $\langle n_f \rangle$, we multiply to this density difference the reweighting factor $Z_f/Z_g = \langle D_0(\mu_f)/D_0(\mu_g) \rangle_g^{-1}$, whose evaluation becomes more difficult at the smaller $\mu$ and the larger $L$. Although formula (II) gives a fair result in the crossover region for $L = 8$, the problem becomes already challenging for $L = 16$, as is seen in Fig. 15.
5 Summary

We have studied applicability of the LC simulation method to the Thirring model in (0+1) dimensions at finite $\mu$, and its uniform-field variant.

The CL simulations reproduce the exact solution at small and large $\mu$. However, they deviate from the exact one in the crossover region, where the CL ensemble distribution has an overlap with the neighborhoods of the determinant zeros, where the drift force becomes singular. This observation is consistent with the argument for correctness of the CL simulation, given in Refs. [56] (see also [57]).

We have found that the ensemble distribution localizes around the thimbles in the CL simulations for the Thirring model, in which the critical points of the relevant thimbles behave as attractive points of the drift flow field and the zeros of the determinant become the saddle points of the flow. It seems inevitable in this model that the CL ensemble has an overlap with the determinant zeros whenever the corresponding thimble integration has contributions from multi thimbles connected via these zeros.

In order to extend the applicability of the CL simulation to the crossover region, we have attempted two methods, both of which make the thimble structure of the model simple to cover the integration path with a single thimble in view of the thimble integration. But we have noticed that both methods need the evaluation of the reweighting factor. With the direct reweighting method at large reference chemical potential $\nu$, we can reproduce the correct crossover behavior of the observables as a function of physical chemical potential $\mu$. In this sense, one can think of the ensemble generated by CL method physical when CL simulation is successful and can be used for the reweighting. In the deformation method, we also reproduce the correct result but there appears an unphysical structure when the observables have the poles coming from the original determinant zeros. But one can remove this unphysical structure by employing the deformed expressions for the observables in accord with the model.

For a large system size, however, both methods encounter a difficulty in evaluating the reweighting factor; the phase factor average amounts to a small value which scales exponentially with the system size. In the deformation method, we need to evaluate the subtraction of the two expectation values of the similar magnitudes, which gives an additional limitation on the method.

Here we only choose the chemical potential $\mu$ as a reference parameter, but one can consider other reweighting by taking other model parameters, such as the mass $m$ or temperature $T$, etc., as reference parameters for the reweighting. We leave the examination of the multi-parameter reweighting method for future work.

The eigenvalue distribution of the fermion operator of the models here in the configuration space complexified by CL method is also intriguing quantity to be studied, as was discussed in Ref. [49, 52, 65].

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A Determinant zeros in uniform-field model

We show the zero analytic of the uniform-field model and those for the deformed model.

First, the analytic structure of the reference system $Z_g$ is the same as $Z_f$ with a simple replacement of $\mu_f \to \mu_g$; the zeros in the uniform-field model, $D(z_{\text{zero}}, \mu) = 0$ are given as

$$Lz_{\text{zero}} = (2n + 1)\pi + iL(\mu \pm \tilde{m}),$$  \hspace{1cm} (A.1)

and there appear critical points associated to these zeros as seen in Fig. 2.

In order to understand the thimble structure of the modified model, $Z_{f+g}$, let us first find the zeros of

$$D(z_{\text{zero}}, \mu_f) + D(z_{\text{zero}}, \mu_g) = 0.$$  \hspace{1cm} (A.2)

Denoting the ratio $r = \cosh L\tilde{m}/\cosh L\delta$, we easily find the zeros as

$$Lz_{\text{zero}} = iL\frac{\mu_f + \mu_g}{2} + L\tilde{z}_{\text{zero}}$$

$$= iL\frac{\mu_f + \mu_g}{2} + (2\ell + 1)\pi + \begin{cases} 
\pm \cos^{-1} r & \text{for } r \leq 1, \\
\pm i\cosh^{-1} r & \text{for } r > 1.
\end{cases}$$  \hspace{1cm} (A.3)

For $r \leq 1$ the $z_{\text{zero}}$’s line up in the $z$ plane along a straight line parallel to the real axis offset by $(\mu_f + \mu_g)/2$ to the imaginary direction. For $r > 1$ they appear with an equal separation $2\pi/L$ along the two straight lines separated by $2\cosh^{-1} r$. In both cases the modified model $Z_{f+g}$ has $2L$ zeros in total, just as many as the original model. Setting $\mu_g$ large enough, we can make these zeros (and the associated critical points, too) appear with large imaginary axis in the complex $z$ plane. We note, however, that one critical point on the imaginary part remains sitting at $\sinh z_c \sim i/\beta$ (and another critical point on the $x = \pi$ axis) for large $\mu_g$ with fixed $\mu_f$.

We show the thimble structure of $Z_{f+g}$ for $\mu_f = 1$ and 2 with $\mu_g = 3$ in Fig. 13, for example. One recognizes that the original integration contour on the real axis $[-\pi, \pi]$ can be replaced with a set of thimbles which includes the thimble starting at the critical point on the imaginary axis. This thimble gives the most important contribution to the integral. By increasing the $\mu_g$ value, we can move the critical points associated to the zeros further upward so that the single thimble becomes equivalent to the original integration contour.

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