The Chen groups of a group $G$ are the lower central series quotients of the maximal metabelian quotient of $G$. Under certain conditions, we relate the ranks of the Chen groups to the first resonance variety of $G$, a jump locus for the cohomology of $G$. In the case where $G$ is the fundamental group of the complement of a complex hyperplane arrangement, our results positively resolve Suciu’s Chen ranks conjecture. We obtain explicit formulas for the Chen ranks of a number of groups of broad interest, including pure Artin groups associated to Coxeter groups, and the group of basis-conjugating automorphisms of a finitely generated free group.

## 1. Introduction

Let $G$ be a group, with commutator subgroup $G' = [G, G]$, and second commutator subgroup $G'' = [G', G']$. The Chen groups of $G$ are the lower central series quotients $gr_k(G/G'')$ of $G/G''$. These groups were introduced by K.T. Chen in \cite{5}, so as to provide accessible approximations of the lower central series quotients of a link group. For example, if $G = F_n$ is the free group of rank $n$ (the fundamental group of the $n$-component unlink), the Chen groups are free abelian, and their ranks, $\theta_k(G) = \text{rank } gr_k(G/G'')$, are given by

$$\theta_k(F_n) = (k-1) \cdot \binom{k+n-2}{k}, \quad \text{for } k \geq 2.$$  

While apparently weaker invariants than the lower central series quotients of $G$ itself, the Chen groups sometimes yield more subtle information. For instance, if $G = P_n$ is the Artin pure braid group, the ranks of the Chen groups distinguish $G$ from a direct product of free groups, while the ranks of the lower central series quotients fail to do so, see \cite{9}. In this paper, we study the Chen ranks $\theta_k(G)$ for a class of groups which includes all arrangement groups (fundamental groups of complements of complex hyperplane arrangements), and potentially fundamental groups of more general smooth quasi-projective varieties. We relate these Chen ranks to the first resonance variety of the cohomology ring of $G$. For arrangement groups, our results positively resolve Suciu’s Chen ranks conjecture, stated in \cite{39}.

Let $A = \bigoplus_{k=0}^{\infty} A^k$ be a finite-dimensional, graded, graded-commutative, connected algebra over an algebraically closed field $k$ of characteristic 0. For each $a \in A^1$, we have $a^2 = 0$, so right-multiplication by $a$ defines a cochain complex

\begin{equation} \label{eq:complex} (A, a): \quad 0 \longrightarrow A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots \xrightarrow{a} A^k \xrightarrow{a} \cdots. \end{equation}
In the context of arrangements, with $A$ the cohomology ring of the complement, the complex $(A,a)$ was introduced by Aomoto \[1\], and subsequently used by Esnault-Schechtman-Viehweg \[16\] and Schechtman-Terao-Varchenko \[36\] in the study of local system cohomology. In this context, if $a \in A^1$ is generic, the cohomology of $(A,a)$ vanishes, except possibly in the top dimension, see Yuzvinsky \[41\].

In general, the resonance varieties of $A$, or of $G$ in the case where $A = H^*(G;\mathbb{k})$, are the cohomology jump loci of the complex $(A,a)$,

$$R^k_d(A) = \{ a \in A^1 \mid \dim H^k(A,a) \geq d \},$$

homogeneous algebraic subvarieties of $A^1$. These varieties, introduced by Falk \[17\] in the context of arrangements, are isomorphism-type invariants of the algebra $A$. They have been the subject of considerable recent interest in a variety of areas, see, for instance, Dimca-Papadima-Suciu \[14\], Yuzvinsky \[42\], and references therein.

We will focus on the first resonance variety of the group $G$,

$$R^1(A) = R^1(H^*(G;\mathbb{k})) = \{ a \in A^1 \mid H^1(A,a) \neq 0 \}.$$ 

Assume that $G$ is finitely presented. The group $G$ is 1-formal (over the field $\mathbb{k}$) if and only if the Malcev Lie algebra of $G$ is isomorphic, as a filtered Lie algebra, to the completion with respect to degree of a quadratic Lie algebra (see \[33\] for details), and is said to be a commutator-relators group if it admits a presentation $G = F/R$, where $F$ is a finitely generated free group and $R$ is the normal closure of a finite subset of $[F,F]$. For any finitely generated 1-formal group, Dimca-Papadima-Suciu \[14\] show that all irreducible components of the resonance variety $R^1_d(H^*(G;\mathbb{k}))$ are linear subspaces of $H^1(G;\mathbb{k})$. For a finitely presented, commutator-relators group, the resonance variety $R^1(G)$ may be realized as the variety defined by the annihilator of the linearized Alexander invariant $\mathcal{B}$ of $G$, a module over the polynomial ring $S = \text{Sym}(H_1(G;\mathbb{k}))$, $R^1(G) = V(\text{ann}(\mathcal{B}))$, see Section \[2\] below. We can thus view $R^1(G)$ as a scheme.

Let $A = H^*(G;\mathbb{k})$, and let $\mu: A^1 \times A^1 \to A^2$ be the cup product map, $\mu(a \cup b) = a \cup b$. A non-zero subspace $U \subseteq A^1$ is said to be $p$-isotropic with respect to the cup product map if the restriction of $\mu$ to $U \times U$ has rank $p$. For instance, $A^1$ is $0$-isotropic if $G$ is a free group, while $A^1$ is 1-isotropic if $G$ is the fundamental group of a closed, orientable surface. Call subspaces $U$ and $V$ of $A^1$ projectively disjoint if they meet only at the origin, $U \cap V = \{0\}$. In the formulas below, we use the convention $\binom{m}{n} = 0$ if $n < m$. Our main result is as follows.

**Theorem A.** Let $G$ be a finitely presented, 1-formal, commutator-relators group. Assume that the components of $R^1(G)$ are (i) 0-isotropic, (ii) projectively disjoint, and (iii) reduced (viewing $R^1(G)$ as a scheme). Then, for $k \gg 0$,

$$\theta_k(G) = (k - 1) \sum_{m \geq 2} h_m \binom{m+k-2}{k},$$

where $h_m$ denotes the number of $m$-dimensional components of $R^1(G)$.

**Example 1.1.** Examples illustrating the necessity of the hypotheses in the theorem include the following. Let $[u,v] = uvu^{-1}v^{-1}$ denote the commutator of $u,v \in G$.

(a) The Heisenberg group $G = \langle g_1, g_2 \mid [g_1,g_2], [g_2,[g_1,g_2]] \rangle$ is not 1-formal. Here, $R^1(G) = H^1(G;\mathbb{k})$ is 0-isotropic since the cup product is trivial. But $\theta_k(G) \neq \theta_k(F_2)$. Since $G$ is nilpotent, the Chen groups of $G$ are trivial. See \[13\].
(b) The fundamental group $G$ of a closed, orientable surface of genus $g \geq 2$ is 1-formal. But $R^1(G) = H^1(G; \mathbb{k})$ is not 0-isotropic, and $\theta_0(G) \neq \theta_0(F_{2g})$. See [31].

(c) Let $G = G_\Gamma$ be the right-angled Artin group corresponding to the graph $\Gamma$ with vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{12, 13, 24, 34, 45\}$. The resonance variety $R^1(G)$ is the union of two 3-dimensional subspaces in $H^1(G; \mathbb{k})$ which are not projectively disjoint, and $\theta_k(G) \neq 2\theta_k(F_3)$. See [32].

(d) Let $G = \langle g_1, g_2, g_3, g_4 \mid [g_2, g_3], [g_1, g_4], [g_3, g_4], [g_1, g_3] \rangle$. As a variety, $R^1(G)$ is a 2-dimensional subspace of $H^1(G; \mathbb{k})$. But $R^1(G)$ is not reduced, and $\theta_k(G) \neq \theta_k(F_2)$. We do not know if this group is 1-formal. See Example 3.3.

Groups which satisfy the hypotheses of Theorem A include all arrangement groups. More generally, let $X$ be a smooth quasi-projective variety, and assume that $G = \pi_1(X)$ is a commutator-relators group. If $X$ admits a smooth compactification with trivial first Betti number, work of Deligne [13] and Morgan [28] implies that $G$ is 1-formal. In this instance, Dimca-Papadima-Suciu [14] show that the irreducible components of $R^1(G)$ are all projectively disjoint and 0-isotropic. Thus, Theorem A applies when the components of $R^1(G)$ are reduced. Another interesting example is provided by the “group of loops.”

Let $F_n$ be the free group of rank $n$. The basis-conjugating automorphism group, or pure symmetric automorphism group, is the group $P\Sigma_n$ of all automorphisms of $F_n$ which send each generator $x_i$ to a conjugate of itself. Results of Dahm [12] and Goldsmith [21] imply that this group may also be realized as the “group of loops,” the group of motions of a collection of $n$ unknotted, unlinked oriented circles in 3-space, where each circle returns to its original position.

**Theorem B.** For $k \gg 0$, the ranks of the Chen groups of the basis-conjugating automorphism group $P\Sigma_n$ are

$$\theta_k(P\Sigma_n) = (k - 1) \binom{n}{2} + (k^2 - 1) \binom{n}{3}.$$

Our interest in the relationship between Chen ranks and resonance stems from the theory of hyperplane arrangements. Let $A = \{H_1, \ldots, H_n\}$ be an arrangement in $\mathbb{C}^d$, with complement $M = \mathbb{C}^d \setminus \bigcup_{i=1}^n H_i$. It is well known that the fundamental group $G = \pi_1(M)$ is a commutator-relators group, and is 1-formal. Furthermore, in low dimensions, the cohomology of $G$ is isomorphic to that of $M$, $H^{\leq 2}(G; \mathbb{k}) \cong H^{\leq 2}(M; \mathbb{k})$, see Matei-Suciu [20]. Consequently, the first resonance varieties of $G$ and of the Orlik-Solomon algebra $A = H^*(M; \mathbb{k})$ coincide. Falk and Yuzvinsky observed that the irreducible components of $R^1(G)$ are 0-isotropic (see [17] and [19]), and Libgober-Yuzvinsky [24] showed that these components are projectively disjoint. In Section 5 below, we show that the components of $R^1(G)$ are reduced. Thus, Theorem A yields a combinatorial formula for the Chen ranks of the arrangement group $G$ in terms of the Orlik-Solomon algebra of $A$. In [39], Suciu conjectured that this formula, the Chen ranks conjecture, holds.

Theorem A facilitates the explicit calculation of the Chen ranks of a number of arrangement groups of broad interest. For example, let $W$ be a finite reflection group, and $A_W$ the arrangement of reflecting hyperplanes. The fundamental group $PW$ of the complement of $A_W$ is the pure braid group associated to $W$. If $W = A_n$ is the type A Coxeter group, then $PW = PA_n = P_{n+1}$ is the classical Artin pure...
braid group on \(n+1\) strings, whose Chen ranks were determined in [9]. We complete the picture for the remaining infinite families.

**Theorem C.** Let \(PA_n, PB_n\), and \(PD_n\) be the pure braid groups associated to the Coxeter groups \(A_n, B_n\), and \(D_n\). For \(k \gg 0\), the ranks of the Chen groups of these pure braid groups are

\[
\theta_k(PA_n) = (k - 1) \left(\frac{n + 2}{4}\right),
\theta_k(PB_n) = (k - 1) \left[16 \left(\frac{n}{3}\right) + 9 \left(\frac{n}{4}\right)\right] + (k^2 - 1) \left(\frac{n}{2}\right),
\theta_k(PD_n) = (k - 1) \left[5 \left(\frac{n}{3}\right) + 9 \left(\frac{n}{4}\right)\right].
\]

2. Preliminaries

2.1. Alexander invariant. Let \(G\) be a finitely presented group, with abelianization \(G/G'\) and \(a: G \to G/G'\) the natural projection. Let \(\mathbb{Z}G\) be the integral group ring of \(G\), and let \(J_G = \ker(\epsilon)\) be the kernel of the augmentation map \(\epsilon: \mathbb{Z}G \to \mathbb{Z}\), given by \(\epsilon(\sum m_g g) = \sum m_g\). Classically associated to the group \(G\) are the \(\mathbb{Z}(G/G')\)-modules \(A_G = \mathbb{Z}(G/G') \otimes_{\mathbb{Z}G} J_G\) and \(B_G = G'/G''\). The Alexander module \(A_G\) is induced from \(J_G\) by the extension of the abelianization map \(a\) to group rings. The action of \(G/G'\) on the Alexander invariant \(B_G\) is given on cosets of \(G''\) by \(gG'' \cdot hG'' = ghg^{-1}G''\) for \(g \in G\) and \(h \in G'\). These modules, and the augmentation ideal of \(\mathbb{Z}(G/G')\), comprise the Crowell exact sequence \(0 \to B_G \to A_G \to J_{G/G'} \to 0\).

Let \(J = J_{G/G'}\), and consider the \(J\)-adic filtration \(\{J^kB_G\}_{k \geq 0}\) of the Alexander invariant. Let \(\text{gr}(B_G) = \bigoplus_{k \geq 0} J^kB_G/J^{k+1}B_G\) be the associated graded module over the ring \(\text{gr}(\mathbb{Z}(G/G')) = \bigoplus_{k \geq 0} J^k/J^{k+1}\). A basic observation of Massey [23] shows that

\[
\text{gr}_k(G'/G'') = \text{gr}_{k-2}(B_G)
\]

for \(k \geq 2\), where the associated graded on the left is taken with respect to the lower central series filtration. Consequently, the Chen ranks of \(G\) are given by

\[
\sum_{k \geq 0} \theta_{k+2}(G) t^k = \text{Hilb}(B_G \otimes \mathbb{k}, t),
\]

for any field \(\mathbb{k}\) of characteristic zero.

Now assume that \(G = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_m \rangle\) admits a commutator-relators presentation with \(n\) generators, and let \(p: F_n \to G\) be the natural projection, where \(F_n = \langle g_1, \ldots, g_n \rangle\). Set \(t_i = a \circ p(g_i)\). The choice of basis \(t_1, \ldots, t_n\) for \(G/G' \cong \mathbb{Z}^n\) identifies the group ring \(\mathbb{Z}(G/G')\) with the Laurent polynomial ring \(\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]\). With this identification, the augmentation ideal is given by \(J = (t_1 - 1, \ldots, t_n - 1)\).

Presentations for the Alexander module \(A_G\) and Alexander invariant \(B_G\) may be obtained using the Fox calculus [20]. For \(1 \leq i \leq n\), let \(\partial_i = \frac{\partial}{\partial r_i}: F_n \to \mathbb{Z}F_n\) be the Fox free derivatives. The Alexander module has presentation

\[
\Lambda^m \xrightarrow{D} \Lambda^n \to A_G \to 0,
\]

where \(D = (a \circ p \circ \partial_i(r_j))\) is the Alexander matrix of (abelianized) Fox derivatives.
Let \((C_*,d_*)\) be the standard Koszul resolution of \(\mathbb{Z}\) over \(\Lambda\), where \(C_0 = \Lambda\), \(C_1 = \Lambda^n\) with basis \(e_1, \ldots, e_n\), and \(C_k = \Lambda^{(k)}\). The differentials are given by

\[
d_k(e_I) = \sum_{r=1}^k (-1)^{r+k}(t_i - 1)e_{I\setminus\{i\}},\]

where \(e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}\) if \(I = \{i_1, \ldots, i_k\}\). By the fundamental formula of Fox calculus [20], we have \(d_1 \circ D = 0\). This yields a chain map

\[
\begin{array}{cccc}
\Lambda^m & \xrightarrow{D} & \Lambda^n & \xrightarrow{d_1} & \Lambda \\
\downarrow \alpha & & \downarrow \text{id} & & \downarrow \text{id} \\
\cdots & \xrightarrow{d_4} & \Lambda^{(3)} & \xrightarrow{d_3} & \Lambda^{(2)} & \xrightarrow{d_2} & \Lambda^n & \xrightarrow{d_1} & \Lambda
\end{array}
\]

Basic homological algebra insures the existence of the map \(\alpha\) satisfying \(d_2 \circ \alpha = D\). Analysis of the mapping cone of this chain map as in [25] then yields a presentation for the Alexander invariant \(B_G = \ker(d_1)/\text{im}(D)\):

\[
\Lambda^{(3)+m} \xrightarrow{\Delta \circ d_3 + \alpha} \Lambda^{(2)} \rightarrow B_G \rightarrow 0.
\]

2.2. Linearized Alexander invariant. The ring \(\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]\) may be viewed as a subring of the formal power series ring \(P = \mathbb{Z}[x_1, \ldots, x_n]\) via the Magnus embedding, defined by \(\psi(t_i) = 1 + x_i\). Note that the augmentation ideal \(J = (t_1 - 1, \ldots, t_n - 1)\) is sent to the ideal \(m = (x_1, \ldots, x_n)\). Passing to associated graded rings, with respect to the filtrations by powers of \(J\) and \(m\) respectively, the homomorphism \(\text{gr}(\psi)\) identifies \(\text{gr}(\Lambda)\) with the polynomial ring \(S = \text{gr}(P) = \mathbb{Z}[x_1, \ldots, x_n]\).

For \(q \geq 0\), let \(\psi^{(q)}: \Lambda \to P/m^{q+1}\) denote the \(q\)-th truncation of \(\psi\). Since \(G\) is a commutator-relators group, all entries of the Alexander matrix \(D\) are in the augmentation ideal \(J\). It follows that \(\psi^{(0)}(D)\) is the zero matrix. Consequently, all entries of the linearized Alexander matrix \(\psi^{(1)}(D)\) are in \(m/m^2 = \text{gr}_1(P)\), so may be viewed as linear forms in the variables of \(S\).

For the \(n\)-generator, commutator-relators group \(G\), the polynomial ring \(S = \mathbb{Z}[x_1, \ldots, x_n]\) may be identified with the symmetric algebra on \(H_1(G) = G/G' = \mathbb{Z}^n\). Let \(E = \bigwedge H_1(G)\) be the exterior algebra (over \(\mathbb{Z}\)), and let \((E \otimes S, \partial_*\) be the Koszul complex of \(S\). Observe that \(\partial_k = \psi^{(1)}(d_k)\), in particular, \(\partial_1\) is the matrix of variables of \(S\), with image the ideal \(m = (x_1, \ldots, x_n)\). Note also that \(\partial_1 \circ \psi^{(1)}(D) = 0\). The linearized Alexander invariant of \(G\) is the \(S\)-module \(\mathfrak{B}_S = \ker(\partial_1)/\text{im}(\psi^{(1)}(D))\).

**Theorem 2.1** (Papadima-Suciu [31] Cor. 9.7). Let \(G\) be a 1-formal, commutator-relators group with associated linearized Alexander invariant \(\mathfrak{B}_S\), and let \(k\) be a field of characteristic zero. Then

\[
\sum_{k \geq 0} \theta_{k+2}(G)t^k = \text{Hilb}(\mathfrak{B}_S \otimes k, t).
\]

A presentation for the linearized Alexander invariant may be obtained by a procedure analogous to that used to obtain one for the Alexander invariant itself. Linearizing the equality \(d_2 \circ \alpha = D\), we obtain \(\partial_k \circ \alpha_2 = \psi^{(1)}(D)\), where \(\alpha_2 = \psi^{(0)}(\alpha)\), the entries of which are in \(S/m\). Thus, \(\alpha_2\) is induced by a map \(\mathbb{Z}^m \to \mathbb{Z}^{(2)}\),
which we denote by the same symbol. These considerations yield a chain map
\begin{equation}
\begin{aligned}
S^m & \xrightarrow{D^{(1)}} S^n \xrightarrow{\partial_1} S \\
\alpha_2 & \downarrow \quad \text{id} \downarrow \quad \text{id} \downarrow \\
\cdots & \xrightarrow{\partial_1} S^{(n)} \xrightarrow{\partial_2} S^{(2)} \xrightarrow{\partial_2} S^n \xrightarrow{\partial_1} S
\end{aligned}
\end{equation}
where $D^{(1)} = \psi^{(1)}(D)$. Analysis of the mapping cone of this chain map then yields a presentation for the linearized Alexander invariant:
\begin{equation}
S^{(n)+m} \xrightarrow{\Delta_{\text{lin}} = \partial_1 + \alpha_2} S^{(2)} \xrightarrow{} \mathcal{B}_Z \xrightarrow{} 0.
\end{equation}

2.3. Resonance. In the case where $G$ is an arrangement group, it is known [11] that the annihilator of the linearized Alexander invariant defines the first resonance variety. We show that this holds for an arbitrary commutator-relators group. Let $k$ be a field of characteristic zero, and consider $\mathcal{B} := \mathcal{B}_Z \otimes k$, the linearized Alexander invariant with coefficients in $k$. Abusing notation, let $S = k[x_1, \ldots, x_n]$.

**Theorem 2.2.** Let $G$ be an $n$ generator, commutator-relators group. The resonance variety $R^1(G)$ is the variety defined by the annihilator of the linearized Alexander invariant $\mathcal{B}$, $R^1(G) = V(\text{ann } \mathcal{B})$.

**Proof.** Let $X$ be the 2-dimensional CW-complex corresponding to a commutator-relators presentation of $G$ with $n$ generators, and let $R^1(X) \subset k^n$ be the first resonance variety of the cohomology ring $H^*(X; k)$. We first show that $R^1(G) = R^1(X)$.

A $K(G, 1)$ space may be obtained from $X$ by attaching cells of dimension at least 3. The resulting inclusion map $X \hookrightarrow K(G, 1)$ induces an isomorphism between $H^1(G; k)$ and $H^1(X; k)$. Identify these cohomology groups. Dualizing the Hopf exact sequences reveals that $H^2(G; k) \hookrightarrow H^2(X; k)$. Consequently, if $a, b \in H^1(G; k) = H^1(X; k)$, then $a \cup b = 0$ in $H^2(G; k)$ if and only if $a \cup b = 0$ in $H^2(X; k)$. It follows that $R^1(G) = R^1(X)$.

Consider the chain map (2.2), now with $S = k[x_1, \ldots, x_n]$. For $f$ a map of free $S$-modules and $a \in k^n$, denote the evaluation of $f$ at $a$ by $f(a)$. The resonance variety $R^1(G) = R^1(X)$ may be realized as the variety in $H^1(G; k) = k^n$ defined by the vanishing of the $(n-1) \times (n-1)$ minors of the linearized Alexander matrix $D^{(1)}$, see [26, 40]. In other words,
\[
R^1(G) = \{ a \in k^n | \text{rank } D^{(1)}(a) < n - 1 \}.
\]
An exercise with the mapping cone of (2.2) reveals that rank $D^{(1)}(a) < n - 1$ if and only if rank $\Delta_{\text{lin}}(a) < \left(\begin{array}{c} n \\ 2 \end{array}\right)$. This implies that $R^1(G) = V(\text{ann } \mathcal{B})$. \hfill \square

Over the field $k$, the presentation (2.3) of the linearized Alexander invariant may be simplified. Recall that $X$ is the presentation 2-complex corresponding to an $n$-generator, commutator-relators presentation of $G$. Let $H = H_1(X; \mathbb{Z}) = H_1(G; \mathbb{Z}) = \mathbb{Z}^n$. Specializing (2.2) at $x_i = 0$ (resp., (2.1) at $t_i = 1$) yields $\alpha_2: H_2(X; k) \rightarrow H_2(H; k)$, which is dual to the cup product map $\mu: H \wedge H \rightarrow H^2(X; k)$, see [26]. The cohomology ring $E = H^*(H; k)$ is an exterior algebra. Let $I \subset E$ be the ideal generated by $\ker(\mu): E^1 \wedge E^1 \rightarrow H^2(X; k))$. Writing $A = E/I$, for $a \in A^1 = E^1$, we obtain a short exact sequence of cochain complexes
0 \rightarrow (I, a) \rightarrow (E, a) \rightarrow (A, a) \rightarrow 0. \text{ If } a \neq 0, (E, a) \text{ is acyclic, and } H^1(A, a) \neq 0 \text{ if and only if ker}(I^2 \xrightarrow{a} I^3) \neq 0. \text{ Thus, }
\mathcal{R}^1(G) = \{a \in A^1 \mid \text{ker}(I^2 \xrightarrow{a} I^3) \neq 0\}.

If e_1, \ldots, e_n \in E^1 \text{ generate the exterior algebra } E, \text{ let } \delta^k : E^{k-1} \otimes S \rightarrow E^k \otimes S \text{ denote multiplication by } \sum_{i=1}^n x_ie_i, \text{ dual to the Koszul differential } \partial_k : E^k \otimes S \rightarrow E^{k-1} \otimes S. \text{ Tensoring the exact sequence } 0 \rightarrow I \rightarrow E \rightarrow A \rightarrow 0 \text{ with } S, \text{ we obtain a commutative diagram of cochain complexes, with exact columns}

\begin{align*}
\cdots & \xleftarrow{\delta^3} A^3 \otimes S & \xleftarrow{\delta^3} A^2 \otimes S & \xleftarrow{\delta^3} A^1 \otimes S & \xleftarrow{\delta^1} A^0 \otimes S \\
& \quad \uparrow & \mu \otimes \text{id} & \quad \text{id} & \quad \text{id} \\
\cdots & \xleftarrow{\delta^3} E^3 \otimes S & \xleftarrow{\delta^3} E^2 \otimes S & \xleftarrow{\delta^3} E^1 \otimes S & \xleftarrow{\delta^3} E^0 \otimes S \\
& \quad \uparrow & \text{id} & \quad \text{id} & \quad \text{id} \\
\cdots & \xleftarrow{\delta^1} I^3 \otimes S & \xleftarrow{\delta^1} I^2 \otimes S
\end{align*}

where } \delta^1_k \text{ denotes the restriction, and } \delta^3_k \text{ denotes the induced map on the quotient. Referring to the diagram (2.4), we have } A_2 \otimes S \cong S^m/\text{ker}(\alpha_2 : S^m \rightarrow S^{(2)}). \text{ The map } D^{(1)} = \psi_1(D) \text{ is trivial on this kernel, let } \bar{D}^{(1)} \text{ denote the induced map on the quotient. Then the linearized Alexander invariant may be realized as } \mathfrak{B} = \text{ker}(\partial_1)/\text{im}(\bar{D}^{(1)}), \text{ and the map } \delta^3_k \text{ is dual to } D^{(1)}. \text{ Dualizing the diagram (2.4), we obtain an exact sequence of chain complexes}

\begin{align*}
0 \rightarrow (A \otimes S, \partial^A) \rightarrow (E \otimes S, \partial^E) \rightarrow (I \otimes S, \partial^I) \rightarrow 0,
\end{align*}

where } \partial^* \text{ is dual to } \delta^*. \text{ Since } (E \otimes S, \partial^E) \text{ is acyclic, we have } \mathfrak{B} = H_1(A \otimes S, \partial^A) = H_2(I \otimes S, \partial^I), \text{ yielding the following presentation for } \mathfrak{B}:

\begin{align*}
I^3 \otimes S \xrightarrow{\delta^1_k} I^2 \otimes S \rightarrow \mathfrak{B} \rightarrow 0.
\end{align*}

3. Chen ranks from resonance

In this section, we prove Theorem A. For a group } G \text{ satisfying the hypotheses of Theorem A, we determine the ranks of the Chen groups of } G \text{ from the resonance variety } \mathcal{R}^1(G).

Let } G \text{ be a finitely presented, 1-formal, commutator-relators group. These assumptions on } G \text{ insure (i) that the ranks of the Chen groups are given by the Hilbert series of the linearized Alexander invariant } \mathfrak{B} \text{ of } G \text{ (with coefficients in } k): \n\begin{align*}
\sum_{k \geq 2} \theta_k(G)t^k = \text{Hilb}(\mathfrak{B}, t),
\end{align*}

see Papadima-Suciu [31, Cor. 9.7]; and (ii) that all irreducible components of } \mathcal{R}^1(G) \text{ are linear subspaces of } H^1(G; k), \text{ see Dimca-Papadima-Suciu [14, Thm. B]. Assume that these components of } \mathcal{R}^1(G) \text{ are 0-isotropic, projectively disjoint, and reduced. Let } L \text{ be an irreducible component of } \mathcal{R}^1(G), \text{ and let } \{l_1, \ldots, l_m\} \text{ be a basis for } L, \text{ where the } l_i \text{ are linearly independent elements of } A^1 = E^1 = H^1(G; k). \text{ Since } L \text{ is 0-isotropic, we have } l_i \wedge l_j = 0 \text{ in } A^2 \text{ for } 1 \leq i < j \leq m. \text{ Consequently, the ideal}

\begin{align*}
I_L = \langle l_i \wedge l_j \mid 1 \leq i < j \leq m \rangle
\end{align*}
Proposition 3.1. The ideals $I_L$ and $J_L$ are equal if and only if $(\mathfrak{B}_L)_q \simeq (\mathfrak{B}_L)_q$.
Proof. Choose a basis for \( E^1 \) so that \( L = \text{span}\{e_1, \ldots, e_m\} \) and \( I_L \) is consequently generated by \( e_ie_j := e_i \land e_j, \ 1 \leq i < j \leq m. \)

First, suppose \( I_L = J_L. \) Since \( \{e_1, \ldots, e_m\} \cap J = I_L \), we may choose (independent) elements \( \{f_1, \ldots, f_r\} \) in \( E^2 \) whose images form a basis for the quotient \( I^2/I_L^2, \) and whose initial terms in \( (i_f) = g_i \) are distinct elements of the ideal generated by \( e_ie_j, \ m < i < j \leq n. \) Note that \( \{e_1f_1, \ldots, e_if_r\} \) are independent in \( I^3, \) for if not, then

\[
\sum_{i=1}^r c_if_i = e_1 \sum_{i=1}^r c_if_i = 0,
\]
so \( \sum_{i=1}^r c_if_i \in \langle e_1 \rangle. \) Since the initial terms in \( (i_f) = g_i \) are distinct, \( \sum_{i=1}^r c_if_i = 0 \) is impossible, and \( \sum_{i=1}^r c_if_i \) cannot be a nontrivial multiple of \( e_1 \) for the same reason.

Recall from (3.3) that \( \psi^3: I^2/I_L^2 \otimes S \to I^3/I_L^3 \otimes S \) denotes the map induced by \( \delta^3: I^2 \otimes S \to I^3 \otimes S \) given by multiplication by \( \sum_{i=1}^n x_ie_i. \) Let \( M \) be the submatrix of (the matrix of) \( \psi^3 \) with rows corresponding to \( e_1f_1, \ldots, e_if_r \in I^3. \)

\[
M = x_1 \cdot \text{id} + M', \quad \text{with } M' \in k[x_2, \ldots, x_n]^{r \times r},
\]
where \( \text{id} \) is the identity matrix of size \( r = \dim I^2/I_L^2. \) To see this, note that

\[
\delta^3_i(f_j) = x_1e_if_j + \sum_{i=2}^n x_ie_if_j.
\]

While there can be syzygies with \( e_if_j = e_1f_k + \cdots, \) for such a relation we have \( i \geq 2, \) hence only \( x_i \) with \( i \geq 2 \) will appear. Thus, \( \det(M) = x_1^r + \text{higher degree terms}, \) with \( \deg x_1(h) < r. \)

In the localization \( S_q, \) since \( x_1 \notin q = (x_{m+1}, \ldots, x_n), \) \( \det(M) \) is a unit. This implies that the maximal Fitting ideal \( \text{Fitt}_q((\psi^3)_q) \) contains a unit. Consequently, after localizing, the cokernel of \( \psi^3 \) vanishes. From the long exact sequence arising from the dual of (3.3), this cokernel is the kernel of the map \( (\mathfrak{B}_q) \to (\mathfrak{B}_L)_q \to 0, \) so the map \( \pi_L \) of (3.4) induces an isomorphism \( (\mathfrak{B}_q) \simeq (\mathfrak{B}_L)_q. \)

Now suppose the irreducible component \( L \) of \( \mathcal{R}^1(G) \) is such that \( I_L \neq J_L. \) Let \( g \in (J_L^2 \setminus I_L^2), \) assume as above that \( L \) has basis \( \{e_1, \ldots, e_m\}, \) and write \( L^* = \text{span}\{e_{m+1}, \ldots, e_n\}. \) Note that \( g \) is indecomposable, for if \( g = l_1 \land l_2 \) with \( l_i \in L, \) \( l_2 \in L^* \), then the plane \( \text{span}\{l_1, l_2\} \subseteq R^1(G) \) intersects \( L \) in a line, contradicting the assumption that the components of \( \mathcal{R}^1(G) \) are projectively disjoint.

Denote the ideal \( I_L + g \) in the exterior algebra \( E \) by \( K_L, \) and let \( \delta^3_L: K_L^{-1} \otimes S \to K_L^* \otimes S \) be multiplication by \( \omega = \sum_{i=1}^n x_ie_i. \) Let \( \mathfrak{B}_{K_L} \) be the \( S \)-module presented by the dual \( \partial^3_{K_L} \) of \( \delta^3_L, \)

\[
(3.7)
I^3_{K_L} \otimes S \xrightarrow{\delta^3_{K_L}} I^2_{K_L} \otimes S \rightarrow \mathfrak{B}_{K_L}.
\]

We will construct a free resolution \( K_* \) of \( \mathfrak{B}_{K_L}. \)

First, consider the cochain complex \( C^* = (I^*_L \otimes S, \delta^*_L) \) corresponding to the ideal \( I_L \) in \( E. \) This complex is acyclic. To see this, let \( C^*(0) \) denote the Koszul (cochain) complex \( \wedge^* L \otimes S, \) with differential given by multiplication by \( \omega' = \sum_{i=1}^n x_ie_i. \) Multiplication by \( x_{m+1} \) induces a chain map \( C^*(0) \to C^*(0). \) Let \( C^*(1) \) be the corresponding mapping cone. Since \( C^*(0) \) is acyclic, so is \( C^*(1). \) Continuing in this manner, multiplication by \( x_{m+j+1} \) induces a chain map \( C^*(j) \to C^*(j), \) and the resulting mapping cone \( C^*(j+1) \) is acyclic. Thus, \( C^* = C^*(n-m) \) is acyclic. Let \( C_* = (I^*_L \otimes S, \partial^*_L) \) be the chain complex dual to \( C^*. \) Since \( C^* \) is acyclic and
free, the dual complex $C_\bullet$ gives a free resolution of the module $\mathfrak{B}_L$,
\begin{equation}
\cdots \to I^k \otimes S \xrightarrow{\partial^k I} \cdots \to I^3 \otimes S \xrightarrow{\partial^3 I} I^2 \otimes S \to \mathfrak{B}_L.
\end{equation}

Let $\hat{C}^\bullet$ be the Koszul complex $\bigwedge^\bullet L^* \otimes S$, with differential given by multiplication by $-\omega'' = -\sum_{i=m+1}^n x_i e_i$. Multiplication by $\omega' g = \sum_{i=1}^m x_i e_i g$ induces a chain map $\hat{C}^\bullet \to C^\bullet$. Let $K^\bullet$ be the mapping cone of this chain map. Then we have a short exact sequence of cochain complexes $0 \to C^\bullet \to K^\bullet \to \hat{C}^\bullet \to 0$. Since $C^\bullet$ and $\hat{C}^\bullet$ are acyclic and free, $K^\bullet$ is acyclic and free. It is readily checked that the map $\delta^3_{K^3}: I^2_{K^3} \otimes S \to I^3_{K^3} \otimes S, f \mapsto \omega f = \omega' f + \omega'' f$, coincides with the first differential of $K^\bullet$. It follows that the chain complex $K^\bullet$, dual to $K^\bullet$, is a free resolution of $\mathfrak{B}_{K_L}$.

Recall that $q = (x_{m+1}, \ldots, x_n)$. The above considerations yield a short exact sequence of chain complexes $0 \to \hat{C}_\bullet \to K_\bullet \to C_\bullet \to 0$, where $\hat{C}_\bullet$ is dual to $\hat{C}^\bullet$, and is a free resolution of $S/q$. We thus have a short exact sequence of $S$-modules
\begin{equation}
0 \to S/q \to \mathfrak{B}_{K_L} \to \mathfrak{B}_L \to 0.
\end{equation}
Since the localization $(S/q)_q$ is nontrivial, we have $(\mathfrak{B}_{K_L})_q \neq (\mathfrak{B}_L)_q$. The surjection $\pi_L: \mathfrak{B} \to \mathfrak{B}_L$ of \eqref{eq:3.4} factors through $\mathfrak{B}_{K_L}$, so the fact that $(\mathfrak{B}_{K_L})_q \neq (\mathfrak{B}_L)_q$ implies that $(\mathfrak{B})_q \neq (\mathfrak{B}_L)_q$, which completes the proof. \hfill $\square$

Proposition \ref{prop:3.1} implies that if $I_L = J_L$ for each irreducible component $L$ of $\mathcal{R}^1(G)$, then the modules $\mathfrak{A} = \ker \pi$ and $\mathfrak{C} = \text{coker} \pi$ in the exact sequence \eqref{eq:3.5} are of finite length, and hence the Hilbert polynomials of the modules $\mathfrak{B}$ and $\bigoplus_{L=1}^k \mathfrak{B}_L$, are equal. If $\dim L = m$, a straightforward exercise using the resolution \eqref{eq:3.8} shows that the Hilbert polynomial of $\mathfrak{B}_L$ is $\binom{k}{m} \binom{m+k-2}{m-1}$. Consequently, to complete the proof of Theorem \ref{thm:A} it suffices to prove the following.

**Proposition 3.2.** The ideals $I_L$ and $J_L$ are equal if and only if the irreducible component $L$ of $\mathcal{R}^1(G)$ is reduced.

**Proof.** We continue with the notation established in the proof of Proposition \ref{prop:3.1}. Let $L = \text{span}\{e_1, \ldots, e_m\}$ be an $m$-dimensional irreducible component of $\mathcal{R}^1(G)$, and $q = I(L)$ the prime ideal in $S$ with $V(q) = L$. The component $L$ of $\mathcal{R}^1(G)$ is reduced if in the primary decomposition
\[
\text{ann}(\mathfrak{B}) = \bigcap Q_i \quad \text{with} \quad \sqrt{Q_i} = q_i,
\]
the $q_i$-primary component is equal to $q$.

Assume that $I_L = J_L$. For simplicity, bigrade $E$, viewing elements of $L$ as of bidegree $(1, 0)$ and elements of $L^* = \text{span}\{e_{m+1}, \ldots, e_n\}$ as of bidegree $(0, 1)$. Order monomial bases for $I^2_L$ and $I^3_L$ in a grading where $(1, 0) > (0, 1)$ (e.g., lex order). Then, the map $\partial^2 I: I^2_L \otimes S \to I^3_L \otimes S$ of \eqref{eq:3.2} presenting the module $\mathfrak{B}_L$ has matrix
\[
[\partial^2 I] = [d^2_I(x_1, \ldots, x_m) \quad x_{m+1} \cdot \text{id} \quad x_{m+2} \cdot \text{id} \quad \cdots \quad x_n \cdot \text{id}] = [d^2_I \quad X],
\]
where $d^2_I(x_1, \ldots, x_m)$ is the second Koszul differential on $x_1, \ldots, x_m$, $\text{id}$ is the identity matrix of size $|I^2_L| = \binom{m}{2}$, and $X = [x_{m+1} \cdot \text{id} \quad \cdots \quad x_n \cdot \text{id}]$. Using this, one can check that $\text{ann}(\mathfrak{B}_L) = \langle x_{m+1}, \ldots, x_n \rangle = q$. Since $(\mathfrak{B})_q \cong (\mathfrak{B}_L)_q$ when $I_L = J_L$ by Proposition \ref{prop:3.1} and localization commutes with taking annihilators, it follows that $L$ is reduced in this instance.

For the other direction, we show that $I_L \neq J_L$ implies that $L$ is not reduced. Let $g$ be an indecomposable element in $J^2_L \setminus I^2_L$, and let $K_L = I_L + g$. Changing bases in $L$ and $L^* = \text{span}\{e_{m+1}, \ldots, e_n\}$, we may assume that $g = e_1 e_{m+1} + \cdots + e_k e_{m+k}$
for some \( k, 2 \leq k \leq m \). To see that \( L \) is not reduced, it suffices to exhibit an element \( \beta \) in the module \( \mathcal{B}_L \), which is not annihilated by \( q = (x_{m+1}, \ldots, x_n) \).

Choose ordered bases for \( K_2^L \) and \( K_1^L \) whose initial segments are the bases of \( I_2^L \) and \( I_1^L \) above, so that \( g \) appears last in the basis for \( K_2^L \), and \( g \wedge L^* \) is the final segment of the basis for \( K_1^L \). With respect to these ordered bases (in light of the mapping cone construction in the proof of Proposition [4]), the map \( \partial_3^{K_L} : K_2^L \otimes S \to K_1^L \otimes S \) of (3.7) presenting \( \mathcal{B}_L \) has matrix

\[
[\partial_3^{K_L}] = \begin{bmatrix}
d_1 & X & 0 \\
0 & y & x
\end{bmatrix},
\]

where \( x = [x_{m+1} \cdots x_n] \). As \( \partial_3^{K_L} \) is dual to the multiplication map \( \partial_3^{K_L} : K_2^L \otimes S \to K_1^L \otimes S \), the last row of \( [\partial_3^{K_L}] \) corresponds to \( \partial_3^{K_L}(g) = (\sum_{i=1}^n x_i e_i) \wedge g \). Using this, and \( g = \sum_{i=1}^k e_i e_{m+i}, \) one can check that the entries of \( y \) are in \( \text{k}[x_1, \ldots, x_m] \).

Let \( \beta \) be the class of \( e_1 e_2 \in K_2^L \otimes S \) in \( \mathcal{B}_L \), and assume that \( x_{m+1} \beta = 0 \), i.e., that \( x_{m+1} e_1 e_2 \in \text{im}(\partial_3^{K_L}) \). Then there exists \( u \in K_2^L \otimes S \) so that \( \partial_3^{K_L}(u) = x_{m+1} e_1 e_2 \).

From the form of the matrix of \( \partial_3^{K_L} \) above, we must have \( u = e_1 e_2 e_{m+1} + v \) for some \( v \in K_2^L \otimes S \). Since the coefficient of \( e_1 e_2 e_{m+1} \) in \( \partial_3^{K_L}(g) \) is \( -x_2 \), the transpose of the column of \( [\partial_3^{K_L}] \) corresponding to \( e_1 e_2 e_{m+1} \) is \( [x_{m+1} 0 \cdots 0 -x_2] \). In other words, \( \partial_3^{K_L}(e_1 e_2 e_{m+1}) = x_{m+1} e_1 e_2 - x_2 g \). Thus, \( \partial_3^{K_L}(v) = \partial_3^{K_L}(u - e_1 e_2 e_{m+1}) = x_2 g \), and the assumption that \( x_{m+1} e_1 e_2 \in \text{im}(\partial_3^{K_L}) \) implies that \( x_2 g \in \text{im}(\partial_3^{K_L}) \) as well. This in turn implies that the kernel of the map \( \mathcal{B}_L \to \mathcal{B}_L \) is annihilated by \( x_2 \), a contradiction since this kernel is \( S/q = S/(x_{m+1}, \ldots, x_n) \), see (3.3).

**Example 3.3.** For the group \( G \) in Example [14](d), \( I = \langle e_1 e_2, e_1 e_3 + e_2 e_4 \rangle \), and \( R^1(G) = L = \text{span}\{e_1, e_2\} \), so that \( I_L = \langle e_1 e_2 \rangle \). Since \( J_L = I \neq I_L \), the resonance component \( L \) is not reduced. A calculation reveals that the Chen ranks of \( G \) are given by \( \theta_k(G) = 2(k - 1) \), which of course differ from \( \theta_k(F_2) = k - 1 \) for \( k \geq 2 \).

4. **Basis-conjugating automorphism groups**

As a first application of Theorem [A], we compute the Chen ranks of the basis-conjugating automorphism group \( G = P\Sigma_n \), proving Theorem [B].

Let \( F_n \) be the free group generated by \( x_1, \ldots, x_n \). The basis-conjugating automorphism group \( P\Sigma_n \) is the group of all automorphisms of \( F_n \) which send each generator \( x_i \) to a conjugate of itself. As noted in the introduction, this group may be realized as the group of motions of a collection of \( n \) unknotted, unlinked oriented circles in 3-space, where each circle returns to its original position. McCool [27] found the following presentation for the basis-conjugating automorphism group:

\[(4.1) \quad P\Sigma_n = \langle \beta_{i,j}, 1 \leq i \neq j \leq n \mid [\beta_{i,j}, \beta_{k,l}], [\beta_{i,k}, \beta_{j,k}], [\beta_{i,j}, (\beta_{i,k} \cdot \beta_{j,k})] \rangle,\]

where \([u, v] = uvu^{-1}v^{-1}\), the indices in the relations are distinct, and the generators \( \beta_{i,j} \) are the automorphisms of \( F_n \) defined by

\[
\beta_{i,j}(x_k) = \begin{cases} x_k & \text{if } k \neq j, \\
x_j^{-1} x_i x_j & \text{if } k = i. \end{cases}
\]

The integral cohomology of \( P\Sigma_n \) was determined by Jensen-McCammond-Meier [23], resolving a conjecture of Brownstein-Lee [4]. We rephrase their result for a field \( k \) of characteristic zero. Let \( E \) be the exterior algebra over \( k \) generated by
Then the cohomology algebra of the basis-conjugating automorphism group $R(4.2)$
generated by

Proof of Theorem B. Checking that the components $C_{k,i,j}$ and $C_{i,j,k}$ is 1-formal. And it is clear from (4.1) that

It follows that

We will show that if

Let $L = C_{1,2,3} = \text{span}\{e_{2,1} - e_{3,1}, e_{1,2} - e_{3,2}, e_{1,3} - e_{2,3}\}$, and let $g \in I^2$, where

Write

where $a_{i,j}, b_{i,j}^k \in k$. We will show that if $l \wedge g \in I_L$ for all $l \in I$, then $g \in I_L$. This implies that $I_L = J_L$, and hence that $L$ is reduced by Proposition 3.1.

Note that $s_{2,3}^1 = b_{2,3}^1(e_{2,1} - e_{3,1})(e_{1,2} - e_{3,2})(e_{1,3} - e_{2,3})$ is in $I_L$. So $(e_{2,1} - e_{3,1})g \in I_L$ implies that $h = (e_{2,1} - e_{3,1})g - s_{2,3}^1$ is also in $I_L$. Write

(4.3)

where $c_{1,3}^2 = \sum x_{i,j} e_{i,j}, c_{2,3}^3 = \sum y_{i,j} e_{i,j}$, and $c_{1,2}^2 = \sum z_{i,j} e_{i,j}$ with $x_{i,j}, y_{i,j}, z_{i,j} \in k$. Comparing coefficients of $e_{2,1} e_{i,j} e_{i,j}$ in the left- and right-hand sides of (4.3) reveals that $a_{i,j} = 0$ for $\{i,j\} \neq \{1,2,3\}$. Similarly, if $k \neq \{1,2,3\}$ and $i < j$, by considering the coefficients of $e_{2,1} e_{k,j} e_{i,j}$, $e_{2,1} e_{k,i} e_{j,k}$, and $e_{2,1} e_{k,j} e_{i,k}$ in (4.3), we see that $b_{i,j}^k = 0, b_{j,k}^i = 0 (if i < k)$ or $b_{k,i}^j = 0 (if k < i)$, and $b_{i,j}^3 = 0$.

Thus, we have $b_{i,j}^k = 0$ if $\{i,j,k\} \neq \{1,2,3\}$.

These considerations imply that $h = \sum a_{i,j}(e_{2,1} - e_{3,1})e_{i,j} e_{i,j}$. Using (4.3) again, comparing coefficients of $e_{i,j} e_{1,2} e_{1,3}, e_{i,j} e_{2,1} e_{2,3}$ and $e_{i,j} e_{1,3} e_{1,2}$ reveals that $x_{i,j} = 0, y_{i,j} = 0$, and $z_{i,j} = 0$ for $\{i,j\} \neq \{1,2,3\}$. So, for instance, $c_{2,3}^3 = \sum x_{i,j} e_{i,j}$ and similarly for $c_{2,3}^2$ and $c_{1,2}$. Then, comparing coefficients in (4.3) as indicated below yields the following:

It follows that $x_{2,1} = x_{3,1} = 0$, and $c_{1,2}^3 = x(e_{2,1} - e_{3,1})$, where $x = x_{2,1}$. Similar considerations yield $c_{1,3}^2 = y(e_{1,2} - e_{3,2})$ and $c_{1,2}^3 = z(e_{1,3} - e_{2,3})$. 

degree one elements $e_{p,q}, 1 \leq p \neq q \leq n$, and let $I$ be the two-sided ideal in $E$
generated by

e_{i,j} e_{j,i}, 1 \leq i < j \leq n, \quad (e_{k,j} - e_{i,j})(e_{k,j} - e_{i,j}),

Then the cohomology algebra of the basis-conjugating automorphism group $PS_n$ is isomorphic to the quotient of $E$ by $I$, $H^*(PS_n; k) \cong E/I$. Using this description of the cohomology, Cohen [3] computed the first resonance variety of $PS_n$:

(4.2) \[ R^1(PS_n) = \bigcup_{1 \leq i < j \leq n} C_{i,j} \cup \bigcup_{1 \leq i < j < k \leq n} C_{i,j,k} \subset H^1(PS_n; k) = k^{n(n-1)}, \]

where $C_{i,j} = \text{span}\{e_{i,j}, e_{j,i}\}$ and $C_{i,j,k} = \text{span}\{e_{i,j} - e_{k,i}, e_{i,j} - e_{k,j}, e_{i,k} - e_{j,k}\}$.
Summarizing, we have
\[ h = \sum_{1 \leq i < j \leq 3} a_{i,j}(e_{2,1} - e_{3,1})e_{i,j}e_{j,i} = \lambda(e_{2,1} - e_{3,1})(e_{1,2} - e_{3,2})(e_{1,3} - e_{2,3}), \]
where \( \lambda = x + y + z \). Comparing coefficients of, for instance, \( e_{1,3}e_{1,2}e_{2,1} \) here yields \( \lambda = 0 \), which implies that \( a_{i,j} = 0 \) for all \( i, j \). Hence, we have
\[ g = b_{1,2,3}^1(e_{1,2} - e_{3,2})(e_{1,3} - e_{2,3}) + b_{1,2,3}^2(e_{2,1} - e_{3,1})(e_{1,3} - e_{1,2}) + b_{1,2,3}^3(e_{3,1} - e_{2,1})(e_{3,2} - e_{1,2}) \]
and \( g \in I_L \). Thus, \( I_L = J_L \), and the component \( L = C_{1,2,3} \) of \( R^1(P\Sigma_n) \) is reduced.

A similar (easier) argument, which we leave to the reader, shows that the component \( C_{1,2} \) of \( R^1(P\Sigma_n) \) is also reduced. Thus, Theorem A may be used to compute the Chen ranks of \( P\Sigma_n \). Noting that \( R^1(P\Sigma_n) \) has \( \binom{n}{3} \) two-dimensional components and \( \binom{n}{4} \) three-dimensional components completes the proof of Theorem B. \( \square \)

**Remark 4.1.** The upper triangular McCool groups illustrate the necessity of the hypotheses of Theorem A. For each \( n \geq 2 \), the upper triangular McCool group is the subgroup \( P\Sigma_n^+ \) of \( P\Sigma_n \) generated by the elements \( \beta_{i,j} \) with \( 1 \leq i < j \leq n \), subject to the relevant relations \( \{4\} \). Thus, \( P\Sigma_n^+ \) is a commutator-relators group. Moreover, in [3], Berceanu-Papadima remark that \( P\Sigma_n^+ \) is 1-formal.

In [8], Cohen-Pakianathan-Vershchin-Wu determine the integral cohomology of \( P\Sigma_n^+ \), see also [2]. We rephrase their result for a field \( k \) of characteristic zero. Let \( E^+ \) be the exterior algebra over \( k \) generated by elements \( e_{p,q} \), \( 1 \leq p < q \leq n \), of degree one, and let \( I^+ \) be the two-sided ideal in \( E^+ \) generated by \( e_{i,j}(e_{i,k} - e_{i,j}) \), \( 1 \leq i < j < k \leq n \). Then, \( H^*(P\Sigma_n^+; k) \cong E^*/I^+ \).

Using the above description of the cohomology ring, one can check that the resonance variety \( R^1(P\Sigma_n^+) \) has components
\[ \text{span}\{e_{1,3} - e_{1,2}, e_{2,3}\}, \text{span}\{e_{1,4} - e_{1,2}, e_{2,4}\}, \text{and} \]
\[ \text{span}\{e_{1,3} - e_{1,4}, e_{2,3} - e_{2,4}, e_{3,4}\}. \]
These components are projectively disjoint. However, the 3-dimensional component \( L = \text{span}\{e_{1,3} - e_{1,4}, e_{2,3} - e_{2,4}, e_{3,4}\} \) is neither 0-isotropic nor reduced. For the former, note that \( (e_{1,3} - e_{1,4})(e_{2,3} - e_{2,4}) \) is nonzero in \( H^*(P\Sigma_n^+; k) \). For the latter, \( L \) has an embedded component \( \text{span}\{e_{3,4}\} \). Accordingly (see Proposition 3.2), the ideals \( I_L \) and \( J_L \) are not equal. For instance, \( e_{2,4}(e_{1,4} - e_{1,2}) - e_{2,3}(e_{1,3} - e_{1,2}) \) is in \( J_L \), but not in \( I_L \).

In light of the above observations, it is not surprising that the Chen ranks formula of Theorem A does not hold for the upper triangular McCool groups. For example, a computation reveals that, for \( k \gg 0 \), the Chen ranks of \( P\Sigma_n^+ \) are given by \( \theta_k(P\Sigma_n^+) = 1 + 2\theta_k(F_2) + \frac{1}{2}\theta_k(F_3) \), which differs from the value \( 2\theta_k(F_2) + \theta_k(F_3) \) naively predicted from the resonance variety \( R^1(P\Sigma_n^+) \).

## 5. Hyperplane Arrangements

Let \( A = \{H_1, \ldots, H_n\} \) be a hyperplane arrangement in \( \mathbb{C}^n \), with complement \( M(A) = \mathbb{C}^n \setminus \bigcup_{i=1}^n H_i \). We assume that \( A \) is a central arrangement, i.e., that each hyperplane of \( A \) passes through the origin. Let \( L(A) = (\bigcap_{H \in B} H \mid B \subseteq A) \) be the intersection lattice of \( A \), with rank function given by codimension. We refer to elements of \( L(A) \) as flats. Recall that \( k \) is a field of characteristic zero. A well known theorem of Orlik-Solomon [29] yields a presentation for the cohomology ring \( A = H^*(M(A); k) \), the Orlik-Solomon algebra, in terms of the lattice \( L(A) \). Let
A multiarrangement \((\mathcal{A}, \pi)\) be the fundamental group of the complement. As noted in the introduction, the arrangement group \(G = \pi_1(M(\mathcal{A}))\) is a 1-formal, commutator-relators group, and the resonance varieties \(\mathcal{R}^1(G)\) and \(\mathcal{R}^1(\mathcal{A})\) coincide. In this context, we denote this variety by \(\mathcal{R}^1(\mathcal{A})\), the first resonance variety of the arrangement \(\mathcal{A}\).

Falk [17] initiated the study of resonance varieties in the context of arrangements. Among his main innovations was the concept of a neighborly partition. A partition \(\Pi\) of \(\mathcal{A}\) is neighborly if, for any rank two flat \(Y \in \mathcal{L}_2(\mathcal{A})\) and any block \(\pi \in \Pi\),

\[|Y| - 1 \leq |Y \cap \pi| \implies Y \subseteq \pi,\]

Partitions with a single block will be called trivial, others nontrivial. Flats contained in a single block of \(\Pi\) will be referred to as monochrome, others polychrome. Flats of multiplicity two are necessarily monochrome.

Falk showed that all components of \(\mathcal{R}^1(\mathcal{A})\) arise from nontrivial neighborly partitions of subarrangements of \(\mathcal{A}\), and conjectured that \(\mathcal{R}^1(\mathcal{A})\) was a subspace arrangement. This was proved simultaneously by Cohen-Suciu in [11] and Libgober-Yuzvinsky in [24], and the latter also showed that the irreducible components of \(\mathcal{R}^1(\mathcal{A})\) are projectively disjoint. As noted by Falk-Yuzvinsky [19] (see also [17]), these components are 0-isotropic. Arrangements which admit nontrivial neighborly partitions, and corresponding resonance components, include all central arrangements in \(\mathbb{C}^2\), the rank 3 braid arrangement, the Pappus, Hessian, and type B Coxeter arrangements in \(\mathbb{C}^3\), etc., see [11, 17, 35, 39] among others.

Let \(\Pi\) be a neighborly partition of a subarrangement \(\mathcal{A}'\) of \(\mathcal{A}\). Following [17, 18, 24], we explicitly describe the corresponding component \(L_{\Pi}\) of \(\mathcal{R}^1(\mathcal{A})\). Let \(E\) be the exterior algebra over \(k\), with generators \(e_1, \ldots, e_n\) corresponding to the hyperplanes of \(\mathcal{A}\). The Orlik-Solomon algebra is given by \(H^*(M(\mathcal{A}); k) = A = E/I\), where \(I\) is the Orlik-Solomon ideal of \(\mathcal{A}\). This ideal is generated by boundaries of circuits, \(\partial e_1 \cdots e_k\), \(\{H_1, \ldots, H_{ik}\}\) a minimally dependent set of hyperplanes in \(\mathcal{A}\) and \(\partial : E \to E\) is defined by \(\partial 1 = 0\), \(\partial e_i = 1\), and \(\partial(uv) = (\partial u)v + (-1)^{|u|}u(\partial v),|u|\) denoting the degree of \(u\), see [30] Ch. 3. For \(u = \sum u_i e_i \in E^1\), write \(\partial u = u_1\) and \(\partial_X u = \sum_{X \subset H} u_i\) for a rank 2 flat \(X\). Let \(\text{poly}(\Pi)\) denote the set of rank two flats \(X\) which are polychrome with respect to \(\Pi\). Then the resonance component corresponding to the neighborly partition \(\Pi\) of \(\mathcal{A}' \subset \mathcal{A}\) is given by

\[(5.1) \quad L_{\Pi} = \{u \in E^1 | \partial u = 0, \partial_X u = 0 \forall X \in \text{poly}(\Pi), \partial_i u = 0 \forall H_i \notin \mathcal{A}'\}.
\]

Since the irreducible components of \(\mathcal{R}^1(\mathcal{A})\) are 0-isotropic and projectively disjoint, to bring Theorem [A] to bear in the context of arrangement groups, it suffices to show that these components are also reduced. To establish this, in addition to Falk’s description of components of \(\mathcal{R}^1(\mathcal{A})\) in terms of neighborly partitions described above, we will also use the more recent Falk-Yuzvinsky [19] description of resonance components in terms of multitnets, which utilizes the Libgober-Yuzvinsky [24] treatment in terms of the Vinberg classification of generalized Cartan matrices.

A multiarrangement is an arrangement \(\mathcal{A}\) together with a multiplicity function \(\nu : \mathcal{A} \to \mathbb{N}\) which assigns a positive integer \(\nu(H)\) to each hyperplane in \(\mathcal{A}\). A \((k,d)\)-multinet on a multiarrangement \((\mathcal{A}, \nu)\) (of lines in \(\mathbb{C}^n\)) is a pair \((\Pi, \mathcal{X})\), where \(\Pi = \{\beta_1 | \beta_2 | \cdots | \beta_k\}\) is a partition of \(\mathcal{A}\) with \(k \geq 3\) blocks, and \(\mathcal{X}\) is a set of rank 2 flats of \(\mathcal{A}\), satisfying

(i) For each block \(\beta_i\) of the partition \(\Pi\), the sum \(\sum_{H \in \beta_i} \nu(H) = d\) is constant, independent of \(i\);

(ii) For each \(H \in \beta_i\) and \(H' \in \beta_j\) with \(i \neq j\), the flat \(H \cap H'\) is in \(\mathcal{X}\);
For each flat \( X \in \mathcal{X} \), \( \sum_{H \in \beta_i, X \subset H} \nu(H) \) is constant, independent of \( i \);

(iv) For each \( i \) and \( H, H' \in \beta_i \), there is a sequence \( H = H_0, H_1, \ldots, H_r = H' \) of hyperplanes of \( A \) such that \( H_{j-1} \cap H_j \notin \mathcal{X} \) for \( 1 \leq j \leq r \).

As shown by Falk-Yuzvinsky \[19\], an \( \ell \)-dimensional component of \( \mathcal{R}^1(A) \) corresponds to a multiplicity function and an \( (\ell + 1, d) \)-multinet on a subarrangement of \( A \), for some \( d \). Let \( e_1, \ldots, e_n \) be the standard generators of the exterior algebra \( E \), in correspondence with the hyperplanes of \( A \). If \( (\Pi, X) \) is a multinet on \( (A', \nu) \), where \( A' \subset A \) and \( \Pi = (\beta_1 | \beta_2 | \cdots | \beta_{\ell+1}) \), the corresponding resonance component \( L = L_{\Pi} \) has basis

\[
(5.2) \quad \{ \nu_1 - \nu_{\ell+1}, \nu_2 - \nu_{\ell+1}, \ldots, \nu_{\ell} - \nu_{\ell+1} \},
\]

where \( \nu_i = \sum_{H_j \in \beta_i} \nu(H_j)e_j = \sum_{H_j \in \beta_i} \nu_{i,j}e_j \), see \[19\] Thms. 2.4, 2.5. Note that \( \dim L = \ell \), the partition \( \Pi \) is neighborly, the elements of \( \mathcal{X} \) are the polychrome flats of \( \Pi \), the coefficients \( \nu_{i,j} = \nu(H_j) \) are positive integers, and \( \partial \nu_i = \partial \nu_j \) for \( 1 \leq i, j \leq \ell + 1 \) by (i) above.

**Theorem 5.1.** For any hyperplane arrangement \( A \), the irreducible components of the resonance variety \( \mathcal{R}^1(A) \) are reduced.

**Remark.** The conclusion reached at the end of the second paragraph of the proof of this result in the published version of the paper [Adv. Math. 285 (2015), 1–27] is not valid. We thank Claudiu Raicu for bringing this to our attention, and provide a alternate proof of the theorem here.

**Proof of Theorem 5.1.** We first note that it suffices to consider components of \( \mathcal{R}^1(A) \) which are supported on the entire arrangement. If \( L \subset \mathcal{R}^1(A) \) is a component supported on a proper subarrangement \( A' \) of \( A \), and \( L \) is nonreduced in \( H^1(M(A); \mathbb{k}) \), then \( L \subset H^1(M(A'); \mathbb{k}) \subset H^1(M(A); \mathbb{k}) \) is necessarily nonreduced in \( H^1(M(A'); \mathbb{k}) \) as well.

So let \( L \subset \mathcal{R}^1(A) \) be an irreducible component which is supported on the entire arrangement \( A \). By Propositions 3.1 and 3.2, it suffices to show that the ideals \( I_L = \Lambda^2 L \) and \( J_L = \{ g \in I^2 \mid xg \in I_L \forall x \in L \} \) in the exterior algebra \( E \) are equal.

The component \( L \) corresponds to multinet structure on \( A \), with associated multiplicity function \( \nu : A \to \mathbb{N} \), neighborly partition \( \Pi \), and \( \mathcal{X} = \text{poly}(\Pi) \subset L_2(A) \) the polychrome flats of \( \Pi \). If the partition \( \Pi \) of \( A \) is given by \( \Pi = (\beta_1 | \beta_2 | \cdots | \beta_{\ell+1}) \), the resonance component \( L \) has basis given by (5.2) above and \( \dim L = \ell \).

Recall from (3.6) that \( J_L = \{ \nu_1 - \nu_{\ell+1}, \ldots, \nu_{\ell} - \nu_{\ell+1} \} \cap I \). Consequently, if \( g \in J_L \) is a generator (not necessarily in \( I_L \)), we can write

\[
g = (\nu_1 - \nu_{\ell+1})u_1 + \cdots + (\nu_{\ell} - \nu_{\ell+1})u_{\ell}.
\]

We will show that \( g \in I_L \) by showing that \( u_i \in L = L_{\Pi} \) for each \( i, 1 \leq i \leq \ell \), that is, \( u_i \) satisfies the conditions of (5.1). Note that the last of these conditions is vacuous.

Since \( \nu_i - \nu_{\ell+1} \in L \), we have \( \partial(\nu_i - \nu_{\ell+1}) = 0 \). Similarly, since \( g \) is in the Orlik-Solomon ideal \( I \), we have \( \partial g = 0 \). Computing

\[
\partial g = -(\nu_1 - \nu_{\ell+1})\partial u_1 - \cdots - (\nu_{\ell} - \nu_{\ell+1})\partial u_{\ell} = 0,
\]

the fact that \( \{ \nu_1 - \nu_{\ell+1}, \ldots, \nu_{\ell} - \nu_{\ell+1} \} \) is linearly independent implies that \( \partial u_1 = \cdots = \partial u_{\ell} = 0 \). It remains to show that \( \partial_X(u_i) = 0 \) for \( X \in \text{poly}(\Pi) \) a rank two flat of \( A \) which is polychrome with respect to \( \Pi \).
Let $X \in L_2(A)$ be a polychrome flat, that is, $X \in \mathcal{X}$. Then $X$ meets each block of $\Pi$, see [24, Rem. 3.10]. We can assume that $X$ is contained in the hyperplanes $H_1, \ldots, H_{\ell+1}$ of $A$ (and possibly others), and that $H_j \in \beta_j$ for $1 \leq j \leq \ell+1$. Write the basis elements of $L$ as $\nu_i - \nu_{i+1} = y_i + z_i$, where $y_i \in E^1_X = \text{span}\{e_r \mid X \subset H_r\}$ and $z_i = \nu_i - \nu_{\ell+1} - y_i$ for each $i$, $1 \leq i \leq \ell$. From our assumptions concerning the hyperplanes containing $X$ and $[24]$, we have

$$y_i = \nu_i, i; e_i - \nu_{i+1}, \ell+1 e_{\ell+1} + \sum_{k>\ell+1; X \subset H_k} (\nu_i, k - \nu_{\ell+1}, k) e_k.$$

Since the multiplicities $\nu_{i,j} = \nu(H_j)$ are positive integers, it is readily checked that \(\{y_1, \ldots, y_\ell\}\) is linearly independent.

To analyze $\partial X(u_i)$, first note that, since $\nu_i - \nu_{i+1} = y_i + z_i \in L$, we have $\partial_X(\nu_i - \nu_{i+1}) = 0$, which implies that $\partial_X y_i = \partial y_i = 0$. This, together with $\partial(\nu_i - \nu_{i+1}) = 0$, yields $\partial z_i = 0$ for each $i$. Writing

$$g = \sum_{i=1}^\ell (\nu_i - \nu_{i+1}) u_i = \sum_{i=1}^\ell (y_i + z_i) u_i = \sum_{i=1}^\ell y_i u_i + \sum_{i=1}^\ell z_i u_i = g_1 + g_2$$

we have $g_1 = \sum_{i=1}^\ell y_i u_i \in I$ since $g \in J_L$ is in $I$ and $\partial z_i = \partial u_i = 0$ implies that $g_2 = \sum_{i=1}^\ell z_i u_i$ is also in $I$.

Now write $u_i = v_i + w_i$, where $v_i \in E^1_X$ and $w_i = u_i - v_i$, so that $\partial_X(u_i) = \partial v_i$. Then

$$g_1 = \sum_{i=1}^\ell y_i u_i = \sum_{i=1}^\ell y_i (v_i + w_i) = \sum_{i=1}^\ell y_i v_i + \sum_{i=1}^\ell y_i w_i = h + k.$$ 

It is known that the degree two part of the Orlik-Solomon ideal $I$ decomposes as $I^2 = \bigoplus_{L \in L_2(A)} I^2_X$, see [30] Prop. 3.24. We assert that $h = \sum_{i=1}^\ell y_i v_i$ is in $I^2_X$.

For a contradiction, assume otherwise. Then $h$ represents a nontrivial element in $A^2(A_X)$, the Orlik-Solomon algebra of the subarrangement $A_X \subset A$. Then, by the Brieskorn lemma [30] Cor. 3.27, $h$ represents a nontrivial element in $A^2 = A^2(A) = \bigoplus_{Y \in L_2(A)} A^2(Y)$. Since $k = \sum_{i=1}^\ell y_i w_i \in E^1_X \wedge (E^1 \wedge E^1_X)$, $g_1 = h + k$ also represents a nontrivial element in $A^2$, contradicting the fact that $g_1 \in I$.

Using $h \in I^2_X \implies \partial h = 0$ and the fact that $\partial y_i = 0$ noted above, we have

$$\partial h = \partial y_1 v_1 + \cdots + \partial y_\ell v_\ell - y_1 \partial v_1 - \cdots - y_\ell \partial v_\ell = -y_1 \partial v_1 - \cdots - y_\ell \partial v_\ell = 0.$$ 

The linear independence of the set $\{y_1, \ldots, y_\ell\}$ then yields $\partial v_1 = \cdots = \partial v_\ell = 0$, that is, $\partial_X u_i = 0$ for each $i$. But the polychrome flat $X$ was arbitrary, so $u_1, \ldots, u_\ell$ satisfy $\partial_X u_i = 0$ for each $i$ and every flat $X$ of $A$ which is polychrome with respect to $\Pi$. Thus, $u_i \in L$ for each $i$, and $g = \sum_{i=1}^\ell (\nu_i - \nu_{i+1}) u_i$ is in the ideal $I_L$. \hfill \Box

Remark. One can alternatively reduce to the case of resonance components supported on the entire arrangement $A$ (as in the first paragraph of the above proof) using the ideals $I_L$ and $J_L$. Let $A' \subset A$ be a proper subarrangement, and write $I^2(A) = I^2(A') \oplus I^2(A, A')$, where

$$I^2(A') = \langle \partial e_i e_j e_k \mid H_i \cap H_j \cap H_k \in L_2(A') \rangle,$$

$I^2(A, A') = \langle \partial e_i e_j e_k \mid H_i \cap H_j \cap H_k \in L_2(A) \rangle$ and $\{|H_i, H_j, H_k| \cap (A \setminus A')| \geq 1\}$. If $L$ is a resonance component supported on $A'$ and $g \in J_L$ is a generator, write $g = g_1 + g_2$, where $g_1 \in I^2(A')$ and $g_2 \in I^2(A, A')$. One can then use the condition $xg \in I_L$ for any $x \in L$ to show that $g_2 = 0$. 


This chore is facilitated by results on the structure of multinets, namely there are no multinets with more than four blocks, see [2] and the references therein.

In light of Theorem 5.1, Theorem A provides a formula for the ranks of the Chen groups of $G$ in terms of the resonance variety $\mathcal{R}^1(\mathcal{A})$. This formula, $\theta_k(G) = \sum_{m \geq 2} h_m \theta_k(F_m)$ for $k \gg 0$, where $h_m$ is the number of irreducible components of $\mathcal{R}^1(\mathcal{A})$ of dimension $m$, for the Chen ranks was conjectured by Suciu [39]. In [38], Schenck-Suciu proved that $\theta_k(G) \geq \sum_{m \geq 2} h_m \theta_k(F_m)$ for $k \gg 0$. Suciu’s original conjecture predicted equality in this Chen ranks formula for all $k \geq 4$, but in [38] it is shown the value for which $\theta_k(G)$ is given by a fixed polynomial in $k$ depends on the Castelnuovo-Mumford regularity of the linearized Alexander invariant of $G$.

**Example 5.2.** Let $\mathcal{A}$ be the Hessian arrangement in $\mathbb{C}^3$, defined by the polynomial $Q = xyz \prod_{1 \leq i,j,k \leq 3}(x + \omega^i x + \omega^j z)$, where $\omega = \exp(2\pi i/3)$. The projectivization of $\mathcal{A}$ consists of the twelve lines in $\mathbb{CP}^2$ passing through the nine inflection points of a smooth plane cubic curve. Four lines meet at each of the nine inflection points, yielding nine rank 2 flats in $L(\mathcal{A})$ of cardinality 4, and associated 3-dimensional components of $\mathcal{R}^1(\mathcal{A})$. The arrangement $\mathcal{A}$ has 54 subarrangements lattice-isomorphic to the rank 3 braid arrangement. Each of these contributes a 2-dimensional component to $\mathcal{R}^1(\mathcal{A})$. The arrangement $\mathcal{A}$ itself admits a nontrivial neighborly partition, and has a corresponding 3-dimensional component of $\mathcal{R}^1(\mathcal{A})$. A calculation reveals that these (10 3-dimensional and 54 2-dimensional) components constitute all irreducible components of $\mathcal{R}^1(\mathcal{A})$. Consequently, if $G$ is the fundamental group of the complement of $\mathcal{A}$, by Theorem A we have $\theta_k(G) = 10(k^2 - 1) + 54(k - 1)$ for $k \gg 0$.

6. Coxeter arrangements

In this section, we use Theorem A to determine the Chen ranks of the pure braid groups associated to the Coxeter groups of types A, B, and D, proving Theorem C.

**Example 6.1.** Let $\mathcal{A}_n$ be the braid arrangement, the type A Coxeter arrangement in $\mathbb{C}^n$ with hyperplanes $\ker(x_i - x_j)$, $1 \leq i < j \leq n$. The complement of $\mathcal{A}_n$ is the configuration space of $n$ ordered points in $\mathbb{C}$, with fundamental group the Artin pure braid group $G = P_n$. The resonance variety $\mathcal{R}^1(\mathcal{A}_n)$ has $\binom{n+1}{4}$ two-dimensional irreducible components, see [11, 34]. Theorem A yields $\theta_k(P_n) = (k-1)\binom{n+1}{4}$ for $k \gg 0$, as first calculated in [9].

More generally, let $\Gamma$ be a simple graph on vertex set $\{1, \ldots, n\}$, and let $\mathcal{A}_\Gamma$ be the corresponding graphic arrangement in $\mathbb{C}^n$, consisting of the hyperplanes $\ker(x_i - x_j)$ for which $\{i, j\}$ is an edge of $\Gamma$. The resonance variety $\mathcal{R}^1(\mathcal{A}_\Gamma)$ has $\kappa_3 + \kappa_4$ two-dimensional irreducible components, where $\kappa_m$ denotes the number of complete subgraphs on $m$ vertices in $\Gamma$, see [35]. If $G$ is the fundamental group of the complement of $\mathcal{A}_\Gamma$, Theorem A yields $\theta_k(G) = (k-1)(\kappa_3 + \kappa_4)$ for $k \gg 0$, as first calculated in [38].

6.1. Resonance of the Coxeter arrangement of type D. Let $\mathcal{D}_n$ be the type D Coxeter arrangement in $\mathbb{C}^n$, with $n(n-1)$ hyperplanes $H_{i,j}^{\pm} = \ker(x_i \pm x_j)$, $1 \leq i < j \leq n$. The rank 2 flats of $\mathcal{D}_n$ are

$$H_{i,j}^{-} \cap H_{i,k}^{-} \cap H_{j,k}^{-}, \quad H_{i,j}^{-} \cap H_{i,k}^{+} \cap H_{j,k}^{+}, \quad H_{p,q}^{-} \cap H_{p,q}^{+},$$

$$H_{i,j}^{+} \cap H_{i,k}^{-} \cap H_{j,k}^{+}, \quad H_{i,j}^{+} \cap H_{i,k}^{+} \cap H_{j,k}^{-}, \quad H_{p,q}^{+} \cap H_{r,s}^{\pm},$$

where $i,j,k \in \{1, \ldots, n\}$, $p,q \in \{1, \ldots, n\}$, and $r,s \in \{1, \ldots, n\}$.
where \(1 \leq i < j < k \leq n\), \(1 \leq p < q \leq n\), \(1 \leq r < s \leq n\), and \(\{p, q\} \cap \{r, s\} = \emptyset\). Note that \(D_n\) has \(\binom{n}{3}\) rank two flats of multiplicity 3, and \(\binom{n}{2} + 12\binom{n}{4}\) rank two flats of multiplicity 2. We determine the structure of the variety \(R^1(D_n) \subset \mathbb{k}^{n(n-1)}\).

Recall that \(A_n\) is the type A Coxeter arrangement in \(\mathbb{C}^n\). Denote the hyperplanes of \(A_n\) by \(H_{i,j} = \ker(x_i - x_j)\), \(1 \leq i < j \leq n\). Note that \(A_3\) is lattice-isomorphic to \(D_3\). It is well known that \(A_3\) and \(A_4\) subarrangements give rise to two-dimensional components of the first resonance variety, see Falk [17]. It is also known that \(D_4\) subarrangements also yield two-dimensional components of the first resonance variety, see Pereira-Yuzvinsky [35]. These facts yield a number of components of \(R^1(D_n)\), which we record explicitly.

Each triple \(1 \leq i < j < k \leq n\) yields four \(A_3\) subarrangements of \(D_n\):

\[
1 \{H_{i,j}, H_{i,k}, H_{j,k}\}, \quad 2 \{H^+_{i,j}, H^-_{i,k}, H^-_{j,k}\}, \quad 3 \{H^-_{i,j}, H^+_{i,k}, H^+_{j,k}\}, \quad 4 \{H^+_{i,j}, H^+_{i,k}, H^-_{j,k}\}.
\]

Denote the corresponding components of \(R^1(D_n)\) by \(U_{i,j,k}^q\), \(1 \leq q \leq 4\). Each such triple also yields the \(A_4\) subarrangement \(\{H^\pm_{i,j}, H^\pm_{i,k}, H^\pm_{j,k}\}\), with corresponding resonance component \(V_{i,j,k}\).

Each 4-tuple \(1 \leq i < j < k < l \leq n\) yields eight \(A_4\) subarrangements of \(D_n\):

\[
1 \{H_{i,j}, H^-_{i,k}, H^-_{i,l}, H^-_{j,k}, H^-_{j,l}\}, \quad 2 \{H_{i,j}, H^+_{i,k}, H^+_{i,l}, H^-_{j,k}, H^-_{j,l}\}, \quad 3 \{H_{i,j}, H^-_{i,k}, H^+_{i,l}, H^-_{j,k}, H^-_{j,l}\}, \quad 4 \{H_{i,j}, H^+_{i,k}, H^-_{i,l}, H^-_{j,k}, H^-_{j,l}\},
\]

\[
5 \{H^+_{i,j}, H^-_{i,k}, H^+_{i,l}, H^-_{j,k}, H^-_{j,l}\}, \quad 6 \{H^+_{i,j}, H^-_{i,k}, H^-_{i,l}, H^-_{j,k}, H^-_{j,l}\}, \quad 7 \{H^+_{i,j}, H^-_{i,k}, H^-_{i,l}, H^+_{j,k}, H^-_{j,l}\}, \quad 8 \{H^+_{i,j}, H^-_{i,k}, H^-_{i,l}, H^-_{j,k}, H^+_{j,l}\}.
\]

Denote the corresponding components of \(R^1(D_n)\) by \(V_{i,j,k,l}^q\), \(1 \leq q \leq 8\). Each such 4-tuple also yields the subarrangement \(\{H^\pm_{i,j}, H^\pm_{i,k}, H^\pm_{i,l}, H^\pm_{j,k}, H^\pm_{j,l}\}\) isomorphic to \(D_4\), with corresponding resonance component \(W_{i,j,k,l}\).

**Theorem 6.2.** The first resonance variety of the arrangement \(D_n\) is given by

\[
R^1(D_n) = \bigcup_{i<j<k} \left( V_{i,j,k} \cup \bigcup_{q=1}^4 U_{i,j,k}^q \right) \cup \bigcup_{i<j<k<l} \left( W_{i,j,k,l} \cup \bigcup_{q=1}^8 V_{i,j,k,l}^q \right).
\]

Hence, \(R^1(D_n) \subset \mathbb{k}^{n(n-1)}\) is a union of \(\binom{n}{3} + 9\binom{n}{4}\) two-dimensional components.

**Proof.** The inclusion of the union in \(R^1(D_n)\) follows from the preceding discussion, so it suffices to establish the opposite inclusion. For this, it is enough to show that a subarrangement of \(D_n\) not isomorphic to \(A_3\), \(A_4\), or \(D_4\) does not contribute a component to \(R^1(D_n)\). Let \(B\) be such a subarrangement. We show that \(B\) does not admit a nontrivial neighborly partition.

If \(B \subset D_n\) is a subarrangement of cardinality at most 3 that is not isomorphic to \(A_3\), then \(B\) is in general position, and admits no nontrivial neighborly partition. So we may assume that \(|B| \geq 4\).

Let \(\Pi\) be a neighborly partition of \(B\). If \(B\) contains 3 hyperplanes \(H^a_{i,j}, H^b_{k,l}, H^c_{r,s}\), where \(a, b, c \in \{+, -, \}\), with \(|\{i, j, k, l, r, s\}| \geq 5\), we assert that \(\Pi\) must be trivial. If \(|\{i, j, k, l, r, s\}| = 6\), then the hyperplanes \(H^a_{i,j}, H^b_{k,l}, H^c_{r,s}\) are in general position, so must lie in the same block, say \(\Pi_0\), of \(\Pi\). Let \(H^d_{p,q}\) be any other hyperplane in \(B\), where \(d \in \{+, -, \}\). Then \(H^d_{p,q}\) must be in general position with at least one of \(H^a_{i,j}, H^b_{k,l}, H^c_{r,s}\), which implies that \(H^d_{p,q} \subset \Pi_0\). Thus, \(\Pi = \Pi_0\) is trivial.
If $H$ and $H'$ are hyperplanes of an arrangement $B$ such that the rank two flat $H \cap H'$ is not contained in any other hyperplane $H''$ of $B$, i.e., codim $H \cap H' \cap H'' > 2$, we will write $H \cap H'$ in $B$.

If $|\{i,j,k,l,r,s\}| = 5$, permuting indices if necessary, we can assume that $\{i,j\} = \{1,2\}$, $\{k,l\} = \{3,4\}$, and $\{r,s\} = \{4,5\}$. Let $\Pi_0$ be the block of $\Pi$ containing $H_{1,2}$. Then, $H_{1,2} \cap H_{3,4} \cap H_{4,5} \cap H_{1,5} \in B$, which implies that $H_{3,4} \neq H_{1,5} \in \Pi_0$.

Let $H_{p,q}^d$ be any other hyperplane in $B$. We must show that $H_{p,q}^d \cap \Pi_0$. If $\{p,q\} = \{1,2\}$ or $3 \leq p$, then $H_{p,3}^e \cap H_{p,4}^d \cap H_{p,5}^d \in \Pi_2$, and hence $\Pi_2$ is trivial. As before, let $\Pi = \Pi_0$.

If $q = 3$, then $H_{4,5}^e \cap H_{p,q}^d \cap H_{p,3}^d \in \Pi_0$. It remains to consider the instance $p \in \{1,2\}$ and $q = 4$.

For such $B$, there are pairs of indices $i < j$ and $k < l$ so that $|\{H_{i,j}^k\} \cap B| = 1$ and $|\{H_{i,j}^k\} \cap B| = 2$. If $k < l$ is the only pair of indices with $|\{H_{i,j}^k\} \cap B| = 2$, then $B = \{A_4 \cap \{H_{k,l}^a\} \subseteq D_4$ is an arrangement of 7 hyperplanes containing a copy of $A_4$. Checking that the hyperplane $H_{k,l}^a \in B \cap \{A_4 \subseteq D_4$ is transverse to $A_4$, we see that $B$ admits no nontrivial neighborly partition. Consequently, we may assume that there is more than one pair of indices $i < j$ and $k < l$ with $|\{H_{i,j}^k\} \cap B| = 2$. Write $H_{i,j}^a \in B$ and $H_{i,j}^b \notin B$, and assume that $\{H_{i,j}^a, H_{i,j}^b\} \subseteq B$, where $\{a, b\} = \{+,-\}$.

6.2. Resonance of the Coxeter arrangement of type $B$. Let $B_n$ be the type $B_n$ Coxeter arrangement in $\mathbb{C}^n$, consisting of the $n^2$ hyperplanes $H_i = \ker(x_i)$,
1 ≤ i ≤ n, and \( H^+_{i,j} = \ker(x_i \pm x_j) \), 1 ≤ i < j ≤ n. The rank 2 flats of \( B_n \) are

\[
H_{i,j}^-, H_{i,j,k}^-, H_{i,j,k}^+, H_{i,j,k}^+, H_{i,j}^+, H_{i,j}^-, \ H_i \cap H_j \cap H^{-,j}_{i,j}, H^+_{i,j, \ k}, H^+_{p,q} \cap H^-_{p,q},
\]

where 1 ≤ i < j < k ≤ n, 1 ≤ p < q ≤ n, 1 ≤ r < s ≤ n, and \( \{p, q\} \cap \{r, s\} = \emptyset \). Note that \( B_n \) has \( \binom{n}{2} \) rank two flats of multiplicity 4, \( \binom{n}{4} \) rank two flats of multiplicity 3, and \( \binom{n}{5} + 12 \binom{n}{6} \) rank two flats of multiplicity 2. We determine the structure of the variety \( R^1(B_n) \subset k^n^2 \).

Since \( D_n \subset B_n \), there is an inclusion \( R^1(D_n) \subset R^1(B_n) \). As noted previously, \( A_3 \) and \( A_4 \) subarrangements give rise to two-dimensional components of the first resonance variety. It is also known that \( B_2 \) and \( B_3 \) subarrangements yield resonance components, of dimensions 3 and 2 respectively, see [17]. These facts yield a number of components of \( R^1(B_n) \), which we now specify.

Each 2-tuple 1 ≤ i < j ≤ n yields a subarrangement \( B_2(i, j) \) of \( B_n \), isomorphic to \( B_2 \), and a corresponding rank two flat \( H_i \cap H_j \cap H^{-,j}_{i,j} \cap H^+_{i,j, \ k} \). Let \( L_{i,j} \) be the three-component of \( R^1(B_n) \) corresponding to this flat (resp., to \( B_2(i, j) \)).

Each 3-tuple 1 ≤ i < j < k ≤ n determines a subarrangement \( B_3(i, j, k) \) of \( B_n \), defined by \( x_i x_j x_k (x_i^2 - x_j^2)(x_j^2 - x_k^2)(x_k^2 - x_i^2) \), which is isomorphic to \( B_3 \). This subarrangement yields 12 two-dimensional components of \( R^1(B_n) \), 11 corresponding to \( A_4 \) subarrangements of \( B_3(i, j, k) \), and 1 component corresponding to \( B_3(i, j, k) \) itself. The \( A_4 \) subarrangements of \( B_3(i, j, k) \) are

\[
1 \ \{H_i, H_j, H_k, H^-_{i,j}, H^-_{i,k}, H^-_{j,k}, H^+_{i,j,k} \}, \quad 2 \ \{H_i, H_j, H_k, H^-_{i,j}, H^-{i,k}, H^+_{j,k} \},
\]

\[
3 \ \{H_i, H_j, H_k, H^-_{i,j}, H^-{i,k}, H^+_{j,k} \}, 4 \ \{H_i, H_j, H_k, H^-_{i,j}, H^+_{i,k}, H^-_{j,k} \},
\]

\[
5 \ \{H_i, H^-_{i,j}, H^+_{i,j}, H^-{i,k}, H^-{j,k}, H^+_{j,k} \}, 6 \ \{H_i, H^-_{i,j}, H^+_{i,j}, H^+_{i,k}, H^-{j,k} \},
\]

\[
7 \ \{H^-_{i,j}, H^+_{i,j}, H^-{i,k}, H^-{j,k}, H^+_{j,k} \}, 8 \ \{H^-_{i,j}, H^+_{i,j}, H^+{i,k}, H^-{j,k} \},
\]

\[
9 \ \{H^-_{i,j}, H^-{i,k}, H^-{j,k}, H^+_{j,k} \}, 10 \ \{H^+_{i,j}, H^-{i,k}, H^-{j,k}, H^+_{j,k} \},
\]

\[
11 \ \{H^-_{i,j}, H^-{i,k}, H^-{j,k}, H^+_{j,k} \}.
\]

Note that only the last of these is contained in \( D_n \). For the 10 other \( A_4 \) subarrangements of \( B_3(i, j, k) \), let \( Y^q_{i,j,k} \), 1 ≤ q ≤ 10, be the corresponding components of \( R^1(B_n) \). Let \( Z_{i,j,k} \) be the component of \( R^1(B_n) \) corresponding to \( B_3(i, j, k) \) itself.

**Theorem 6.3.** The first resonance variety of the arrangement \( B_n \) is given by

\[
R^1(B_n) = R^1(D_n) \cup \bigcup_{i<j} L_{i,j} \cup \bigcup_{i<j<k} \left( Z_{i,j,k} \cup \bigcup_{q=1}^{10} Y^q_{i,j,k} \right).
\]

Hence, \( R^1(B_n) \subset k^n^2 \) is a union of \( 16 \binom{n}{3} + 9 \binom{n}{4} \) two-dimensional components and \( \binom{n}{5} \) three-dimensional components.

**Proof.** The inclusion of the union in \( R^1(B_n) \) follows from the preceding discussion, so it suffices to establish the opposite inclusion. For this, it is enough to show that a subarrangement of \( B_n \) not isomorphic to \( A_3, A_4, B_2, B_3, \) or \( D_4 \) does not contribute a component to \( R^1(B_n) \). Let \( A \) be such a subarrangement. We show that \( A \) does not admit a nontrivial neighborly partition.

In light of Theorem **6.2**, we may assume that \( A \) is not contained in any \( D_k \) subarrangement of \( B_n \) for \( k \leq n \). Thus, \( A = A' \cup A'' \), where \( A' \subset \{H_i\} \) is nonempty, and \( A'' \subset \{H^\pm_{i,j} \} \). If \( |A''| \leq 2 \), it is readily checked that \( A \not\cong A_3, B_2 \) admits no nontrivial neighborly partition.
Suppose $A'' \supset \{H_{i,j}^a, H_{k,l}^b, H_{r,s}^c \}$. If $|\{i,j,k,l,r,s\}| \geq 5$, these three hyperplanes must lie in the same block $\Pi_0$ of a neighborly partition $\Pi$ of $A$. For each hyperplane $H_p \in A' \subseteq A$, one of these three and $H_p$ forms a rank two flat of $A$ of multiplicity two, which implies that $H_p \in \Pi_0$ as well. Then, arguing as in the proof of Theorem 6.2 reveals that $\Pi$ is trivial.

We are left with the case where $A = A' \cup A''$, $|A'| \geq 1$, $|A''| \geq 3$, and all hyperplanes of $A'' \subseteq D_n$ involve at most 4 indices, say $\{i,j,k,l\}$. Assume first that $H_m \in A'$ for some $m \notin \{i,j,k,l\}$. Then $L_2(A)$ contains the flats $H_m \cap H_{r,s}^a$ for each $\{r,s\} \subset \{i,j,k,l\}$ for which $H_{r,s}^a \in A''$. If $\Pi$ is a neighborly partition of $A$, it follows that $H_m$ and $H_{r,s}^a$ lie in the same block $\Pi_0$ of $\Pi$ for all such $\{r,s\}$. If $H_t \in A'$ for $t \in \{i,j,k,l\}$, then since $m \notin \{i,j,k,l\}$, $H_m \cap H_t$ is a multiplicity two flat of $A$, which implies that $H_t \in \Pi_0$ as well and $\Pi$ is trivial.

Consequently, we can assume that all hyperplanes of $A = A' \cup A''$ involve only the indices $\{i,j,k,l\}$, so $A$ is a subarrangement of the $B_4$ arrangement involving these indices. We consider the various possibilities for $|A'|$.

If $|A'| = 4$, then $H_i, H_j, H_k, H_l \in A$. Suppose, without loss, that $H_{k,l}^a \in A$. Then $H_j \cap H_k^a$ and $H_j \cap H_l^a$ are rank two flats of $A$, so $H_j, H_k, H_l^a$ must lie in the same block $\Pi_0$ of a neighborly partition $\Pi$ of $A$. If $H_{i,q}^b$ or $H_{i,q}^c$ are in $A$, for $q \in \{k,l\}$, then $H_j \cap H_{i,q}^b \cap H_{i,q}^c \in L_2(A)$, which implies $H_{i,q}^b, H_{i,q}^c \in \Pi_0$. The rank two flat $H_j \cap H_{i,q}^a \cap H_{i,q}^c \in L_2(B_n)$ yields a rank two flat in $A$. Since $H_i, H_{i,q}^a, H_{i,q}^c \in \Pi_0$ (if either of the latter two hyperplanes are in $A$), this flat in $A$ must be monochrome. Hence, $H_k, H_l \in \Pi_0$, and $\Pi$ is trivial.

If $|A'| = 3$, we can assume that $H_i, H_j, H_k, H_{k,l}^a \notin A$ and $H_l \notin A$. As above, $H_i, H_j, H_k \in \Pi_0$ must lie in the same block of a neighborly partition $\Pi$ of $A$. Using rank two flats of $A$ of multiplicity two, as in the previous case, any hyperplane $H_{i,q}^b \in A$ must also lie in $\Pi_0$. Since $H_i \notin A$, the flat $H_k \cap H_i \cap H_{i,j}^a \cap H_{k,l}^a \in L_2(B_n)$ yields a flat of multiplicity two or three in $A$, which must be monochrome. Hence, $H_k \in \Pi_0$, and $\Pi$ is trivial.

If $|A'| = 2$, assume that $H_i, H_j \in A$ and $H_k, H_l \notin A$. If $H_{k,l}^a \in A$, then $H_i, H_j, H_{k,l}^a \in \Pi_0$ must lie in the same block of a neighborly partition $\Pi$ of $A$, and one can show that $\Pi$ must be trivial by considering multiplicity two rank two flats of $A$ as above. So assume that $\{H_{k,l}^a \} \cap A = \emptyset$. Since the hyperplanes of $A$ involve all four indices $i, j, k, l$, there are hyperplanes $H_{p,k}^a, H_{q,l}^b \in A$ with $\{p,q\} = \{i,j\}$. There is a corresponding multiplicity two rank two flat $H_{p,k}^a \cap H_{q,l}^b \in L_2(A)$ (as $H_{i,j}^a \notin A$). So $H_{p,k}^a, H_{q,l}^b \in \Pi_0$ must lie in the same block of a neighborly partition $\Pi$ of $A$. Since $H_q \cap H_{p,k}^a, H_p \cap H_{q,l}^b \in L_2(A)$, we have $H_i, H_j \in \Pi_0$, and it follows that $\Pi$ must be trivial.

If $|A'| = 1$ and $A$ has a hyperplane which does not involve the index $i$, we can assume that $H_{j,k}^a, H_{p,l}^b \in A$, where $p \in \{i,j,k\}$. Then $H_i \cap H_{j,k}^a \in L_2(A)$, so $H_i, H_{j,k}^a \in \Pi_0$ lie in the same block of a neighborly partition $\Pi$ of $A$. If $p \neq i$,
then $H_i \cap H^b_{p,j} \in L_2(A)$, while if $p = i$, then $H^a_{r,s} \cap H^b_{p,j} \in L_2(A)$. So $H^b_{p,j} \in \Pi_0$ as well. If $H^a_{r,s} \in A$ and $i \neq r$, then $H_i \cap H^a_{r,s} \in L_2(A)$, and $H^a_{r,s} \in \Pi_0$. If $H^a_{r,s} \in A$, then $H^a_{r,s} \cap H^b_{p,j} \in L_2(A)$, and $H^a_{r,s} \in \Pi_0$. Suppose $H^a_{r,s} \in A$ for $s \in \{j, k\}$. If \{i, s\} ∪ \{p, l\} = \{i, j, k, l\}, then $H^a_{i,s} \cap H^b_{p,l} \in L_2(A)$, and $H^a_{i,s} \in \Pi_0$. Otherwise, we have either $p = i$ or $p = s$. If $p = i$, there is a flat $H^a_{i,s} \cap H^b_{p,l} \cap H^d_{r,s} \in L_2(B_n)$. If $H^d_{r,s} \in A$, then $H^d_{r,s} \in \Pi_0$ since $i \neq s$. Consequently, this flat yields a flat of multiplicity two or three in $A$, which must be monochrome, so $H^a_{i,s} \in \Pi_0$. If $p = s$, there is a flat $H^a_{i,s} \cap H^b_{p,l} \cap H^d_{r,s} \in L_2(B_n)$. If $H^d_{r,s} \in A$, then $H^d_{r,s} \in \Pi_0$ since $H^a_{r,s} \cap H^d_{r,s} \in L_2(A)$. Consequently, this flat yields a flat of multiplicity two or three in $A$, which must be monochrome, so $H^a_{i,s} \in \Pi_0$. Thus, II is trivial. \qed

Proof of Theorem C. See Example 6.1 for the type A pure braid group. Let $PB_n = \pi_1(M(B_n))$ and $PD_n = \pi_1(M(D_n))$ be the type B and D pure braid groups. The resolution of the Chen ranks conjecture and the determination of the resonance varieties of the arrangements $B_n$ and $D_n$ yield the Chen ranks of these groups. The remaining portions of Theorem C are immediate consequences of Theorem A

Theorem C.1, Theorem C.2, and Theorem C.3. \qed

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