Spin splitting spectroscopy of heavy quark antiquark systems

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Abstract
Phenomenological potentials describe the quarkonium systems like Charmonia, Bottomonia and \( B_c \) Meson. They give a good accuracy for the mass spectra. In the present work we extend one of our previous works in the central case by adding spin dependent terms to allow for relativistic corrections. By using non-central terms, we get better accuracy than other previous theoretical calculation. In the present work, the mass spectra of the bound states of heavy quarks are studied within the framework of the nonrelativistic Schrödinger’s equation. First, we solve Schrödinger’s equation by Nikiforov Uvarov (NU) method. The energy eigenvalues are presented using a non-central potential. The results obtained are in good agreement with the experiment and are better than previous theoretical estimates.

I. Introduction
In the twenty’s century, quarkonium systems have been discovered. Theorists have been trying to explain some aspects of those systems like mass spectra and decay modes properties, …etc.. [1][2][3][4][5]. Some of them used lattice quantum chromodynamics view [6] [7] [8] [9][10][11][12], effective filed theory[13], relativistic potential models [14][15], semi-relativistic potential models [16] and non-relativistic potential models[17][18][19] which have shared in common the Coulomb and linear potentials. There are other groups which use confinement power potential \( r^n \) [20][21][22], the Bethe-Salpeter approach[23][24][25]. In the present work we use mixed potential; nonrelativistic potential models (Coulomb + linear) and confinement power potentials plus spin dependent splitting terms as a correction. Schrödinger’s equation is solved by the Nikiforov-Uvarov (NU) method, which gives asymptotic expressions for the eigenfunctions and eigenvalues of the Schrödinger’s equation.
II. Methodology

In the quarkonium system which deals with quark and antiquark interaction in the center of mass frame, the masses of the quark and antiquark are bigger than chromodynamics scaling i.e. \( M_{q,\bar{q}} \gg \Lambda_{QCD} \). So this allows for non-relativistic treatment and is considered as a heavy bound systems. By using Schrödinger equation of two-body system in spherical symmetric potential.

\[
\frac{d^2 Q(r)}{dr^2} + \left[ \frac{2\mu}{\hbar^2} (E - V_{tot}(r)) - \frac{l(l + 1)}{r^2} \right] Q(r) = 0
\]

Where \( \mu \) is reduced mass, \( E \) is energy eigenvalue, \( l \) is orbital quantum number, \( V_{tot}(r) \) is total potential of the system, \( Q(r) = rR(r) \) and \( R(r) \) is a radial wavefunction solution of Schrödinger's equation. Our radial potential is taken as:

\[
V(r) = -\frac{b}{r} + ar + dr^2 + pr^4
\]

Also, we use spin dependent splitting terms: spin-spin interaction, spin-orbital interaction and tensor interaction respectively[26][27][28][29][30][31][32][33][34].

\[
V_{s-l-T}(r) = V_{s-s}(r) + V_{s-l}(r) + V_T(r)
\]

Where

\[
V_{s-s}(r) = \frac{2}{3m_q m_{\bar{q}}} \nabla^2 V_V(r) \left[ \vec{S}_q \cdot \vec{S}_{\bar{q}} \right]
\]

\[
V_{s-l}(r) = \frac{1}{2m_q m_{\bar{q}}} \left[ \frac{3}{r} \frac{dV_V(r)}{dr} - \frac{dV_V(r)}{dr} \right] \left[ \vec{L} \cdot \vec{S} \right]
\]

\[
V_T(r) = \frac{1}{12m_q m_{\bar{q}}} \left[ \frac{1}{r} \frac{dV_V(r)}{dr} - \frac{d^2V_V(r)}{dr^2} \right] \left[ 6 \left( \vec{S}_q \cdot \frac{\vec{r}}{|r|} \right) \left( \vec{S}_{\bar{q}} \cdot \frac{\vec{r}}{|r|} \right) - 2 \vec{S}_q \cdot \vec{S}_{\bar{q}} \right]
\]

\( V_V \) is a vector potential term and \( V_s \) is a scalar potential term.

So, the total potential becomes

\[
V_{tot}(r) = V(r) + V_{s-l-T}(r)
\]

\[
V_{tot}(r) = \frac{4a_v}{3m_q m_{\bar{q}}} v(ss) + \frac{v(ls)}{m_q m_{\bar{q}}} \left[ \frac{3a_v - a_s}{2r} + \frac{3b}{2r^3} - d - 2pr^2 \right] + \frac{v(T)}{12m_q m_{\bar{q}}} \left[ \frac{a_v}{r} + \frac{3b}{r^3} \right]
\]

\[
-\frac{b}{r} + ar + dr^2 + pr^4
\]

Where \( v(ss) = \left[ \vec{S}_q \cdot \vec{S}_{\bar{q}} \right] \), \( v(ls) = \left[ \vec{L} \cdot \vec{S} \right] \), \( v(T) = -2 \left[ \vec{S}_q \cdot \vec{S}_{\bar{q}} - 3 \left( \vec{S}_q \cdot \frac{\vec{r}}{|r|} \right) \left( \vec{S}_{\bar{q}} \cdot \frac{\vec{r}}{|r|} \right) \right] \).
\[ a_s + a_v = a \]

By substituting in equation (1), we get

\[
\frac{d^2 Q}{dr^2} + \left[ \varepsilon - \frac{B}{r^3} - \frac{l(l+1)}{r^2} - \frac{C}{r} - \frac{Ar - Fr^2 - Pr^4}{r^4} \right] Q = 0 \tag{8}
\]

Where

In natural units

\[
\varepsilon = 2\mu E + \frac{2\mu d\nu(\ell s)}{m_q m_q}, \quad B = \frac{\mu b[6\nu(\ell s) + \nu(t)]}{2m_q m_q} \tag{9}
\]

\[
A = 2\mu a, \quad F = \frac{2\mu}{m_q m_q}[-2p\nu(\ell s) + dm_q m_q] \tag{10}
\]

\[
C = \frac{\mu[16a_v v(ss) + 6(3a_v - a_s)\nu(\ell s) + a_v\nu(T) - 12m_q m_q b]}{6m_q m_q}, \quad P = 2\mu p \tag{11}
\]

Let \( x = \frac{1}{r} \) and by substituting in equation (8), we obtain

\[
\frac{d^2 Q}{dx^2} + \frac{2}{x} \frac{dQ}{dx} + \frac{1}{x^2} \left[ \varepsilon - \frac{Bx - C}{x} - \frac{l(l+1)}{x^3} - \frac{A}{x^4} - \frac{F}{x^5} - \frac{P}{x^6} \right] Q = 0 \tag{12}
\]

In equation (12), one can use the Nikiforov-Uvarov method (NU) to get eigenvalue and eigenfunction equations[35][36][37][38][39][40][41][42][43]. Due to the singularity point in equation (12), let \( y + \delta = x \), and using the Taylor’s series to expand to second order terms, one obtains

\[
\frac{d^2 Q}{dx^2} + \frac{2}{x} \frac{dQ}{dx} + \frac{1}{x^2} [-q + wx - zx^2] Q = 0 \tag{13}
\]

Where

\[
- \frac{6\varepsilon}{\delta^2} + \frac{3C}{\delta} + l(l+1) + \frac{10A}{\delta^3} + \frac{15F}{\delta^4} + \frac{28P}{\delta^6} = q \tag{14}
\]

\[
- \frac{8\varepsilon}{\delta^3} - B + \frac{3C}{\delta^2} + \frac{15A}{\delta^4} + \frac{24F}{\delta^5} + \frac{48P}{\delta^7} = w \tag{15}
\]

\[
- \frac{3\varepsilon}{\delta^4} + \frac{C}{\delta^3} + \frac{6A}{\delta^5} + \frac{10F}{\delta^6} + \frac{21P}{\delta^8} = z \tag{16}
\]

We get

\[
z = \left[ \frac{w}{2n + 1 + 2\sqrt{q + \frac{1}{4}}} \right]^2 \tag{17}
\]
By substituting equations (14-16) in equation (17) and arrange it, we get
\[
\epsilon = \frac{C\delta}{3} + \frac{2A}{\delta} + \frac{10F}{3\delta^2} + 7P + \frac{8E}{\delta} - B\delta^2
\]
\[
- \frac{1}{3} \left[ \frac{3C}{\delta^2} + \frac{15A}{\delta^3} + \frac{24F}{\delta^4} + \frac{48P}{\delta^5} - \frac{8\epsilon}{\delta} - B\delta^2 \right]^2
\]
(18)

We substitute equations (9-11) into equation (18), to obtain the energy eigenvalue equation
\[
\epsilon = \frac{[16a_v(\tau) + 6(b_v - a_v)\psi(\tau) + a_v(\tau) - 12m_m\mu)]}{\delta^{3/2}} + 2a_0 + \frac{10}{3\mu\mu_q}[2\psi(\tau) + \delta m_m\mu] - 20\alpha_r + 30\mu\mu_q [2\psi(\tau) + \mu\mu_q m_m\mu] - 56\mu_r^2 - 6\mu_r^2 [2E + 2E(\tau)]
\]
\[
\left[ \frac{[16a_v(\tau) + 6(b_v - a_v)\psi(\tau) + a_v(\tau) - 12m_m\mu)]}{\delta^{3/2}} + 2a_0 + \frac{10}{3\mu\mu_q}[2\psi(\tau) + \delta m_m\mu] - 20\alpha_r + 30\mu\mu_q [2\psi(\tau) + \mu\mu_q m_m\mu] - 56\mu_r^2 - 6\mu_r^2 [2E + 2E(\tau)] \right]^{1/2}
\]
(19)

Where \( \delta = \frac{1}{r_0} \)

Knowing that
\[
M(q\bar{q}) = E + m_q + m_q \quad \rightarrow \quad E = M(q\bar{q}) - (m_q + m_q)
\]
(20)

So, the mass spectra equation becomes,
\[
M(q\bar{q}) = \left[ \frac{[16a_v(\tau) + 6(b_v - a_v)\psi(\tau) + a_v(\tau) - 12m_m\mu)]}{\delta^{3/2}} + 2a_0 + \frac{10}{3\mu\mu_q}[2\psi(\tau) + \delta m_m\mu] - 20\alpha_r + 30\mu\mu_q [2\psi(\tau) + \mu\mu_q m_m\mu] - 56\mu_r^2 - 6\mu_r^2 [2E + 2E(\tau)] \right]^{1/2}
\]
(21)

The eigenfunction equation is
\[
Q(r) = N_{nls} \frac{1}{r^{3/2}} \sqrt{\left(\frac{q+2}{2}\right)} e^{-\frac{\sqrt{2}}{r}} L_n^2 \left(\frac{q+1}{2}\right) \left(\frac{2\sqrt{2}}{r}\right)
\]
(22)

Where \( L_n^2 \left(\frac{q+1}{2}\right) \left(\frac{2\sqrt{2}}{r}\right) \) is the Rodrigues’s formula of the associated Laguerre polynomial and \( N_{nls} \) is a normalization constant.

So, the radial wavefunction solution of Schrödinger’s equation is given by
\[
R(r) = N_{nls} r^{-\frac{1}{2}} \sqrt{\left(\frac{q+1}{2}\right)} e^{-\frac{\sqrt{2}}{r}} L_n^2 \left(\frac{q+1}{2}\right) \left(\frac{2\sqrt{2}}{r}\right)
\]
(23)
The energy eigenvalue equation (19) has spin-orbital-tensor coefficients $v(ss), v(sl), v(T)$ and those can be given from references [26][27][28]. Also, it has potential parameters $(a_s, a_v, b, d, p)$ and $r_0$ due to the expansion, so we have six parameters of the eigenvalue equation which can be obtained from the experimental data[44] by best fitting.

### III. Numerical Results and Discussions

In table (1), potential parameters are shown for each system. It is noticed that the values of these parameters are different for different systems and this is due to the properties of those systems like energy scale, decay mode...etc. We use spectroscopic notation for the levels $(n^{2S+1}L_J)$.

S is total spin of the system, L is the orbital quantum number, n is the principal quantum number, J is the total (orbital + spin) quantum number.

By using equation (21) and table (1), we get the mass spectra of different quantum states as shown in the tables (2-7). Previously, we used the phenomenological potential in equation (2) without spin dependent corrections(central dependent potential) [45]. The results obtained were good in comparison with the experimental data.

#### Table 1. Parameter values of each system

| Systems     | $m_q$ | $m_{\bar{q}}$ | $r_0$ | as  | av  | b    | d    | p     |
|-------------|-------|----------------|-------|-----|-----|------|------|-------|
| cc system   | 1.317 | 1.317          | 12.82 | -0.0796 | -0.7349 | 0.009 | 0.10686 | -0.000184 |
| bb system   | 4.584 | 4.584          | 7.23795 | 0.0505 | 2.5771 | 15.02 | -0.07969 | -0.00102 |
| bc system   | 4.584 | 1.317          | 11.434 | -2.5549 | 1.5369 | 0.039 | 0.11453 | -0.00018438 |

#### Table 2. Charmonia mass spectrum of S and P-wave in MeV

| Level     | Present | [24] | [46] | [28] | [47] | [48] | [15] | [49] | [31] | [50] | [51] | [44] |
|-----------|---------|------|------|------|------|------|------|------|------|------|------|------|
| $1^3S_0$  | 3.0326  | 2.93 | 2.981 | 2.984 | 2.989 | 2.979 | 2.980 | 2.982 | 3.088 | 2.979 | 2.984 |
| $1^3S_1$  | 3.1257  | 3.11 | 3.096 | 3.097 | 3.094 | 3.097 | 3.097 | 3.097 | 3.168 | 3.096 | 3.097 |
| $2^5S_0$  | 3.6657  | 3.68 | 3.635 | 3.637 | 3.602 | 3.623 | 3.597 | 3.633 | 3.630 | 3.669 | 3.600 | 3.639 |
| $2^5S_1$  | 3.7006  | 3.68 | 3.685 | 3.679 | 3.681 | 3.673 | 3.685 | 3.690 | 3.672 | 3.707 | 3.680 | 3.686 |
|      | 3^1S_0 | 4.1580 | -- | 3.989 | 4.004 | 4.058 | 3.991 | 4.014 | 3.992 | 4.043 | 4.067 | 4.011 | -- |
|------|--------|--------|----|--------|--------|--------|--------|--------|--------|--------|--------|--------|----|
| 3^3S_1 | 4.0549 | 3.80 | 4.039 | 4.030 | 4.129 | 4.022 | 4.095 | 4.030 | 4.072 | 4.094 | 4.077 | 4.039 |
| 4^1S_0 | 4.4145 | -- | 4.401 | 4.264 | 4.448 | 4.250 | 4.433 | 4.244 | 4.384 | 4.398 | 4.397 | -- |
| 4^3S_1 | 4.4146 | -- | 4.427 | 4.281 | 4.514 | 4.273 | 4.477 | 4.273 | 4.406 | 4.420 | 4.454 | 4.421 |
| 5^1S_0 | 4.6074 | -- | 4.811 | 4.459 | 4.799 | 4.446 | -- | 4.440 | -- | -- | -- | -- |
| 5^3S_1 | 4.5845 | -- | 4.837 | 4.472 | 4.863 | 4.463 | -- | 4.464 | -- | -- | -- | -- |
| 6^1S_0 | 4.7543 | -- | 5.155 | -- | 5.124 | 4.595 | -- | 4.601 | -- | -- | -- | -- |
| 6^3S_1 | 4.7333 | -- | 5.167 | -- | 5.185 | 4.608 | -- | 4.621 | -- | -- | -- | -- |
| 1^3P_0 | 3.4067 | 3.32 | 3.413 | 3.415 | 3.428 | 3.433 | 3.416 | 3.392 | 3.424 | 3.448 | 3.488 | 3.415 |
| 1^1P_1 | 3.4865 | 3.49 | 3.511 | 3.521 | 3.468 | 3.510 | 3.508 | 3.491 | 3.505 | 3.520 | 3.514 | 3.511 |
| 1^3P_1 | 3.5023 | 3.43 | 3.525 | 3.526 | 3.470 | 3.519 | 3.527 | 3.524 | 3.516 | 3.536 | 3.536 | 3.525 |
| 1^1P_2 | 3.5224 | 3.55 | 3.555 | 3.553 | 3.480 | 3.556 | 3.558 | 3.570 | 3.556 | 3.564 | 3.565 | 3.556 |
| 2^3P_0 | 3.8987 | 3.83 | 3.870 | 3.848 | 3.897 | 3.842 | 3.844 | 3.845 | 3.852 | 3.870 | 3.947 | 3.918 |
| 2^1P_1 | 3.7858 | 3.67 | 3.906 | 3.914 | 3.938 | 3.901 | 3.940 | 3.902 | 3.925 | 3.934 | 3.972 | -- |
| 2^3P_1 | 3.8209 | 3.75 | 3.926 | 3.916 | 3.943 | 3.908 | 3.960 | 3.922 | 3.934 | 3.950 | 3.996 | -- |
| 2^3P_2 | 3.9054 | -- | 3.949 | 3.937 | 3.955 | 3.937 | 3.994 | 3.949 | 3.972 | 3.976 | 4.021 | 3.927 |
| 3^3P_0 | 4.1202 | -- | 4.301 | 4.146 | 4.296 | 4.131 | -- | 4.192 | 4.202 | 4.214 | -- | -- |
| 3^3P_1 | 4.1226 | 3.91 | 4.319 | 4.192 | 4.338 | 4.178 | -- | 4.178 | 4.271 | 4.275 | -- | -- |
| 3^3P_2 | 4.1642 | -- | 4.337 | 4.193 | 4.344 | 4.184 | -- | 4.137 | 4.279 | 4.291 | -- | -- |
| 4^3P_2 | 4.1444 | -- | 4.354 | 4.211 | 4.358 | 4.208 | -- | 4.212 | 4.317 | 4.316 | -- | -- |
| 4^3P_0 | 4.3615 | -- | 4.698 | -- | 4.653 | -- | -- | -- | -- | -- | -- | -- |
| 4^3P_1 | 4.3729 | -- | 4.728 | -- | 4.696 | -- | -- | -- | -- | -- | -- | -- |
| 4^3P_1 | 4.4195 | -- | 4.744 | -- | 4.704 | -- | -- | -- | -- | -- | -- | -- |
| 4^3P_2 | 4.4111 | -- | 4.763 | -- | 4.718 | -- | -- | -- | -- | -- | -- | -- |
| 5^3P_0 | 4.5429 | -- | -- | -- | 4.983 | -- | -- | -- | -- | -- | -- | -- |




Table 3.: Charmonia mass spectrum of D and F-wave in MeV

| Level | Present  | [47] | [46] | [28] | [24] | [48] | [15] | [49] | [31] | [50] | [51] |
|-------|----------|------|------|------|------|------|------|------|------|------|------|
|       |          |      |      |      |      |      |      |      |      |      |      |
| 1^3D_3 | 3.3067   | 3.755 | 3.813 | 3.808 | 3.869 | 3.799 | 3.831 | 3.844 | 3.806 | 3.809 | 3.798 |
| 1^3D_2 | 3.3755   | 3.765 | 3.807 | 3.805 | 3.739 | 3.796 | 3.824 | 3.802 | 3.799 | 3.803 | 3.796 |
| 1^3D_2 | 3.3481   | 3.772 | 3.795 | 3.807 | 3.550 | 3.798 | 3.824 | 3.788 | 3.800 | 3.804 | 3.794 |
| 1^3D_1 | 3.3744   | 3.775 | 3.783 | 3.792 | --    | 3.787 | 3.804 | 3.729 | 3.785 | 3.789 | 3.792 |
| 2^1D_3 | 3.7972   | 4.176 | 4.220 | 4.112 | 3.806 | 4.103 | 4.202 | 4.132 | 4.167 | 4.167 | 4.425 |
| 2^1D_2 | 3.8359   | 4.182 | 4.196 | 4.108 | --    | 4.099 | 4.191 | 4.105 | 4.158 | 4.158 | 4.224 |
| 2^1D_2 | 3.8005   | 4.188 | 4.190 | 4.109 | --    | 4.100 | 4.189 | 4.095 | 4.158 | 4.159 | 4.223 |
| 2^1D_1 | 3.8002   | 4.188 | 4.105 | 4.095 | --    | 4.089 | 4.164 | 4.057 | 4.142 | 4.143 | 4.222 |
| 3^3D_3 | 4.1627   | 4.549 | 4.574 | 4.340 | --    | 4.331 | --    | 4.351 | --    | --    | --    |
| 3^3D_2 | 4.1763   | 4.553 | 3.549 | 4.336 | --    | 4.326 | --    | 4.330 | --    | --    | --    |
| 3^3D_2 | 4.1345   | 4.557 | 4.544 | 4.337 | --    | 4.327 | --    | 4.322 | --    | --    | --    |
| 3^3D_1 | 4.11298  | 4.555 | 4.507 | 4.324 | --    | 4.317 | --    | 4.293 | --    | --    | --    |
| 4^3D_3 | 4.4358   | 4.890 | 4.920 | --    | --    | --    | --    | 4.526 | --    | --    | --    |
| 4^3D_2 | 4.4293   | 4.892 | 4.898 | --    | --    | --    | --    | 4.509 | --    | --    | --    |
| 4^3D_2 | 4.3825   | 4.896 | 4.896 | --    | --    | --    | --    | 4.504 | --    | --    | --    |
| 4^3D_1 | 4.345    | 4.891 | 4.857 | --    | --    | --    | --    | 4.480 | --    | --    | --    |
| 1^1F_2 | 3.4031   | 3.990 | 4.041 | --    | --    | --    | 4.068 | --    | 4.029 | --    | --    |
| 1^1F_3 | 3.3752   | 4.012 | 4.068 | --    | 3.999 | --    | 4.070 | --    | 4.029 | --    | --    |
| 1^1F_3 | 3.403    | 4.017 | 4.071 | --    | 4.037 | --    | 4.066 | --    | 4.026 | --    | --    |
Table 4. Bottomonia mass spectrum of S and P-wave in MeV

| Level | Present | [52] | [46] | [28] | [24] | [53] | [15] | [49] | [54] | [44] |
|-------|---------|------|------|------|------|------|------|------|------|------|
| 1^3F_4 | 3.3152 | 4.036 | 4.093 | --   | --   | --   | 4.062 | --   | 4.021 | --   | --   |
| 2^3F_2 | 3.81158 | 4.378 | 4.361 | --   | --   | --   | --   | --   | 4.351 | --   | --   |
| 2^3F_3 | 3.823 | 4.396 | 4.400 | --   | --   | --   | --   | --   | 3.352 | --   | --   |
| 2^3F_3 | 3.8583 | 4.400 | 4.406 | --   | --   | --   | --   | --   | 4.350 | --   | --   |
| 2^3F_4 | 3.814 | 4.415 | 4.434 | --   | --   | --   | --   | --   | 4.348 | --   | --   |
| 3^3F_2 | 4.111 | 4.730 | --   | --   | --   | --   | --   | --   | --   | --   | --   |
| 3^3F_3 | 4.152 | 4.746 | --   | --   | --   | --   | --   | --   | --   | --   | --   |
| 3^3F_3 | 4.1944 | 4.749 | --   | --   | --   | --   | --   | --   | --   | --   | --   |
| 3^3F_4 | 4.1857 | 4.761 | --   | --   | --   | --   | --   | --   | --   | --   | --   |
|   | 6$^1$S$_1$ | 1$^3$P$_0$ | 1$^3$P$_1$ | 1$^3$P$_2$ | 2$^3$P$_0$ | 2$^3$P$_1$ | 2$^3$P$_2$ | 3$^3$P$_0$ | 3$^3$P$_1$ | 3$^3$P$_2$ | 4$^3$P$_0$ | 4$^3$P$_1$ | 4$^3$P$_2$ | 5$^3$P$_0$ | 5$^3$P$_1$ | 5$^3$P$_2$ |
|---|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
|   | 11.02145  | 9.8395    | 9.87509   | 9.88423   | 10.202    | 10.22867  | 10.23734  | 10.29912  | 10.33934  | 10.40635  | 10.53183  | 10.5711   | 10.5941   | 10.6370   | 10.73045  | 10.76889  | 10.79183  |
|   | 11.102    | 9.847     | 9.876     | 9.882     | 10.226    | 10.255     | 10.260     | 10.552    | 10.538    | 10.550    | 10.775     | 10.788    | 10.790    | 10.798    | 11.004    | 11.014    | 11.016    |
|   | 11.088    | 9.859     | 9.892     | 9.900     | 10.233    | 10.249     | 10.254     | 10.521    | 10.541    | 10.550    | 10.781     | 10.802    | 10.804    | 10.812    | 11.004    | 11.014    | 11.016    |
|   | 10.988    | 9.864     | 9.903     | 9.909     | 10.220    | 10.120     | 10.154     | 10.490    | 10.515    | 10.528    | 10.732     | 10.753    | 10.757    | 10.767    | --        | --        | --        |
|   | --        | 9.815     | 9.842     | 9.816     | 10.254    | 10.251     | 10.256     | --        | 10.303    | --        | --        | --        | --        | --        | --        | --        | --        |
|   | --        | 9.865     | 9.897     | 9.890     | 10.226    | 10.255     | 10.262     | --        | 10.524    | --        | --        | --        | --        | --        | --        | --        | --        |
|   | --        | 9.861     | 9.891     | 9.900     | 10.230    | 10.255     | 10.261     | --        | 10.507    | --        | --        | --        | --        | --        | --        | --        | --        |
|   | 10.988    | 9.862     | 9.888     | 9.896     | 10.241    | 10.255     | 10.261     | --        | 10.513    | --        | --        | --        | --        | --        | --        | --        | --        |
|   | --        | 9.855     | 9.874     | 9.879     | 10.221    | 10.263     | 10.260     | --        | 10.516    | --        | --        | --        | --        | --        | --        | --        | --        |
|   | 10.995    | 9.859     | 9.874     | 9.899     | 10.232    | 10.255     | 10.260     | --        | 10.521    | --        | --        | --        | --        | --        | --        | --        | --        |
|   | 11.019    | 9.893     | 9.893     | 9.912     | 10.269    | 10.263     | 10.260     | --        | 10.500    | --        | --        | --        | --        | --        | --        | --        | --        |

The table displays energy levels for various atomic states, with columns representing different states and rows showing the energy values for each state.
| Level | Present | [52]   | [46]   | [28]   | [24]   | [53]   | [15]   | [49]   | [54]   | [44]   |
|-------|---------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $1^3D_3$ | 9.84905 | 10.115 | 10.166 | 10.157 | 10.232 | 10.156 | 10.163 | 10.177 | 10.127 | --     |
| $1^3D_2$ | 9.76645 | 10.148 | 10.163 | 10.153 | 10.194 | 10.152 | 10.158 | 10.166 | 10.123 | --     |
| $1^3D_2$ | 10.09624 | 10.147 | 10.161 | 10.153 | 10.145 | 10.151 | 10.157 | 10.162 | 10.122 | 10.163 |
| $1^3D_1$ | 9.6664  | 10.138 | 10.154 | 10.146 | --     | 10.145 | 10.149 | 10.147 | 10.117 | --     |
| $2^3D_3$ | 10.1746 | 10.455 | 10.449 | 10.436 | --     | 10.442 | 10.456 | 10.447 | 10.422 | --     |
| $2^3D_2$ | 10.0943 | 10.450 | 10.445 | 10.432 | --     | 10.439 | 10.452 | 10.440 | 10.419 | --     |
| $2^3D_2$ | 10.0712 | 10.449 | 10.443 | 10.432 | --     | 10.438 | 10.450 | 10.437 | 10.418 | --     |
| $2^3D_1$ | 9.99643 | 10.441 | 10.435 | 10.425 | --     | 10.432 | 10.443 | 10.428 | 10.414 | --     |
| $3^3D_3$ | 10.4462 | 10.711 | 10.717 | --     | --     | 10.680 | 10.652 | --     | --     | --     |
| $3^1D_2$ | 10.3679 | 10.706 | 10.713 | --     | --     | 10.677 | 10.646 | --     | --     | --     |
| $3^3D_2$ | 10.3448 | 10.705 | 10.711 | --     | --     | 10.676 | 10.645 | --     | --     | --     |
| $3^3D_1$ | 10.2718 | 10.698 | 10.704 | --     | --     | 10.670 | 10.637 | --     | --     | --     |
| $4^3D_3$ | 10.6756 | 10.939 | 10.963 | --     | --     | 10.886 | 10.817 | --     | --     | --     |
| $4^1D_2$ | 10.5988 | 10.935 | 10.959 | --     | --     | 10.883 | 10.813 | --     | --     | --     |
| $4^3D_2$ | 10.5758 | 10.934 | 10.957 | --     | --     | 10.882 | 10.811 | --     | --     | --     |
| $4^3D_1$ | 10.5043 | 10.928 | 10.949 | --     | --     | 10.877 | 10.811 | --     | --     | --     |
| $1^3F_2$ | 9.64171 | 10.350 | 10.343 | 10.338 | --     | --     | 10.353 | --     | 10.315 | --     |
| $1^3F_3$ | 9.75433 | 10.355 | 10.346 | 10.340 | 10.302 | --     | 10.356 | --     | 10.321 | --     |
| $1^3F_3$ | 9.77753 | 10.355 | 10.347 | 10.339 | 10.319 | --     | 10.356 | --     | 10.322 | --     |
| $1^3F_4$ | 9.89613 | 10.358 | 10.349 | 10.340 | --     | --     | 10.357 | --     | --     | --     |
| $2^3F_2$ | 9.97096 | 10.615 | 10.610 | --     | --     | --     | 10.610 | --     | --     | --     |
| $2^3F_3$ | 10.0806 | 10.619 | 10.614 | --     | --     | --     | 10.613 | --     | --     | --     |
| $2^3F_3$ | 10.1038 | 10.619 | 10.647 | --     | --     | --     | 10.613 | --     | --     | --     |
| $2^3F_4$ | 10.219  | 10.622 | 10.617 | --     | --     | --     | 10.615 | --     | --     | --     |
| Level | Present | [47] | [55] | [46] | [56] | [57] | [44] |
|-------|---------|------|------|------|------|------|------|
| 3^3F_2 | 10.2457 | 10.850 | -- | -- | -- | -- | -- |
| 3^3F_3 | 10.3529 | 10.853 | -- | -- | -- | -- | -- |
| 3^1F_3 | 10.376 | 10.853 | -- | -- | -- | -- | -- |
| 3^3F_4 | 10.48872 | 10.856 | -- | -- | -- | -- | -- |

Table 6. B_c Meson mass spectrum of S and P-wave in MeV
| Level | Present  | [47] | [55] | [46] | [56] | [57] |
|-------|----------|------|------|------|------|------|
| 2^1P | 6.845372 | 7.168| 7.156| 7.094| 7.150| 7.322| --  |
| 2^3P | 6.916870 | 7.173| 7.162| 7.157| 7.164| 7.232| --  |
| 3^1P | 7.2273759| 7.536| 7.463| 7.474|-- |-- |-- |
| 3^3P | 7.278255 | 7.555| 7.479| 7.510|-- |-- |-- |
| 3^P  | 7.287580 | 7.559| 7.479| 7.500|-- |-- |-- |
| 3^3P | 7.3549761| 7.565| 7.485| 7.524|-- |-- |-- |
| 4^1P | 7.5825397| 7.885|-- |7.817|-- |-- |-- |
| 4^3P | 7.630816 | 7.905|-- |7.853|-- |-- |-- |
| 4^P  | 7.6400914| 7.908|-- |7.844|-- |-- |-- |
| 4^3P | 7.703863 | 7.915|-- |7.867|-- |-- |-- |
| 5^1P | 7.8668871| 8.207|-- |--|-- |-- |-- |
| 5^3P | 7.9129099| 8.226|-- |--|-- |-- |-- |
| 5^P  | 7.9221732| 8.230|-- |--|-- |-- |-- |

Table 7. *Bc* Meson mass spectrum of D and F-wave in MeV
### IV. CONCLUSIONS

The above tables show that, spin dependent terms are important factors to give high accuracy and complete quantitative description of the quarkonium systems for the cases where experimental values are available. The theoretical work agrees with experimental data. This shows also that the Nikiforov-Uvarov method is a good method to get the energy eigenvalues for the meson spectra. The results are even better than other previous works.

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