THE NOVIKOV-BOTT INEQUALITIES

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ABSTRACT. We generalize the Novikov inequalities for 1-forms in two different directions: first, we allow non-isolated critical points (assuming that they are non-degenerate in the sense of R. Bott), and, secondly, we strengthen the inequalities by means of twisting by an arbitrary flat bundle. We also obtain an $L^2$ version of these inequalities with finite von Neumann algebras.

The proof of the main theorem uses Bismut’s modification of the Witten deformation of the de Rham complex; it is based on an explicit estimate on the lower part of the spectrum of the corresponding Laplacian.

1. The generalized Novikov numbers. Let $M$ be a closed manifold and let $\mathcal{F}$ be a complex flat vector bundle over $M$. We will denote by $\nabla : \Omega^i(M, \mathcal{F}) \to \Omega^{i+1}(M, \mathcal{F})$ the covariant derivative on $\mathcal{F}$. Given a closed 1-form $\omega \in \Omega^1(M)$ on $M$ with real values, it determines a family of connections on $\mathcal{F}$ (the Novikov deformation) parameterized by the real numbers $t \in \mathbb{R}$

$$\nabla_t : \Omega^i(M, \mathcal{F}) \to \Omega^{i+1}(M, \mathcal{F}), \quad \nabla_t : \theta \mapsto \nabla \theta + t \omega \wedge \theta. \quad (1)$$

All the connections $\nabla_t$ are flat, i.e. $\nabla_t^2 = 0$. Denote by $\mathcal{F}_t$ the flat vector bundle defined by the connection (1). Note that changing $\omega$ by a cohomologious 1-form determines a gauge equivalent connection $\nabla_t$ and so the cohomology $H^i(M, \mathcal{F}_t)$ depends only on the cohomology class $\xi = [\omega] \in H^1(M, \mathbb{R})$ of $\omega$. One can show that there exists a finite subset $S \subset \mathbb{R}$ (the set jump points) such that $\dim H^i(M, \mathcal{F}_t)$ is constant for $t \notin S$ and it jumps up for $t \in S$. The dimension of $\dim H^i(M, \mathcal{F}_t)$ for $t \notin S$ is called the $i$-th (generalized) Novikov number $\beta_i(\xi, \mathcal{F})$.

2. Assumptions on the 1-form. Let $C$ denote the set of critical points of $\omega$ (i.e. the set of points of $M$, where $\omega$ vanishes). We assume that $\omega$ is non-degenerate in the sense of Bott, i.e. $C$ is a submanifold of $M$ and that the Hessian of $\omega$ is a non-degenerate quadratic form on the normal bundle $\nu(C)$ to $C$ in $M$. Here by the Hessian of $\omega$ we understand the Hessian of the unique function $f$ defined in a tubular neighborhood of $C$ and such that $df = \omega$ and $f|_C = 0$.

3. The main result. Let $Z$ be a connected component of the critical point set $C$ and let $\nu(Z)$ denote the normal bundle to $Z$ in $M$. Since the Hessian of $\omega$ is non-degenerate, the bundle $\nu(Z)$ splits into the Whitney sum of two subbundles $\nu(Z) = \nu^+(Z) \oplus \nu^-(Z)$, such that the Hessian is strictly positive on $\nu^+(Z)$ and strictly negative on $\nu^-(Z)$. The dimension of the bundle $\nu^-(Z)$ is called the index of $Z$ (as a critical submanifold of $\omega$) and is denoted by $\text{ind}(Z)$. Let $o(Z)$ denote the

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orientation bundle of $\nu^{-}(Z)$, considered as a flat line bundle. Consider the twisted Poincaré polynomial of $Z$

$$\mathcal{P}_{Z,\mathcal{F}}(\lambda) = \sum \lambda^{i} \dim_{\mathbb{C}} H^{i}(Z, \mathcal{F}_{|Z} \otimes o(Z))$$

(here $H^{i}(Z, \mathcal{F}_{|Z} \otimes o(Z))$ denote the cohomology of $Z$ with coefficients in the flat vector bundle $\mathcal{F}_{|Z} \otimes o(Z)$) and define using it the following Morse counting polynomial

$$\mathcal{M}_{\omega,\mathcal{F}}(\lambda) = \sum_{Z} \lambda^{\text{ind}(Z)} \mathcal{P}_{Z,\mathcal{F}}(\lambda),$$

where the sum is taken over all connected components $Z$ of $C$.

With one-dimensional cohomology class $\xi = [\omega] \in H^{1}(M, \mathbb{R})$ and the flat vector bundle $\mathcal{F}$, one can associate the Novikov polynomial

$$\mathcal{N}_{\xi,\mathcal{F}}(\lambda) = \sum_{i=0}^{n} \lambda^{i} \beta_{i}(\xi, \mathcal{F}), \quad n = \dim M.$$  

**Theorem 4.** There exists a polynomial $Q(\lambda)$ with non-negative integer coefficients, such that

$$\mathcal{M}_{\omega,\mathcal{F}}(\lambda) - \mathcal{N}_{\xi,\mathcal{F}}(\lambda) = (1 + \lambda) Q(\lambda).$$

The main novelty in this theorem is that it is applicable to the case of 1-forms with non-isolated singular points. Thus, we obtain, in particular, a new proof of the degenerate Morse inequalities of R.Bott. Moreover, Theorem 4 yields a generalization of the Morse-Bott inequalities to the case of an arbitrary flat vector bundle $\mathcal{F}$; this generally produces stronger inequalities as shown in Section 7.

**Corollary 5 (Euler-Poincaré theorem).** Under the conditions of Theorem 4, the Euler characteristic of $M$ can be computed as $\chi(M) = \sum (\chi(Z) - \text{ind}(Z)) \chi(Z)$, where the sum is taken over all connected components $Z \subset C$.

**6. The case of isolated critical points.** Consider now the special case when all critical points of $\omega$ are isolated. Theorem 4 gives the inequalities

$$\sum_{i=0}^{p} (-1)^{i} m_{p-i}(\omega) \geq d^{-1} \cdot \sum_{i=0}^{p} (-1)^{i} \beta_{p-i}(\xi, \mathcal{F}), \quad p = 0, 1, 2, \ldots,$$

where $d = \dim \mathcal{F}$ and $m_{p}(\omega)$ denotes the number of critical points of $\omega$ of index $p$. The last inequalities coincide with the Novikov inequalities [N1] when $\mathcal{F} = \mathbb{R}$ with the trivial flat structure. Examples described in Section 7, show that using of flat vector bundles $\mathcal{F}$ gives sharper estimates in general.

On the other hand, (6) also generalizes the Morse type inequalities obtained by S.P.Novikov in [N2], using Bloch homology (which correspond to the case, when $[\omega] = 0$ in (6)).
7. Examples. Here we describe examples, where the Novikov numbers twisted by a flat vector bundle $F$ (as defined above) give greater values (and thus stronger inequalities) than the usual Novikov numbers (where $F = \mathbb{R}$ or $\mathbb{C}$, cf. [N1, Pa]).

Let $k \subset S^3$ be a smooth knot and let the 3-manifold $X$ be the result of $1/0$-surgery on $S^3$ along $k$. Note that the one-dimensional homology group of $X$ is infinite cyclic and thus for any complex number $\eta \in \mathbb{C}$, $\eta \neq 0$, there is a complex flat line bundle $F_\eta$ over $X$ such that the monodromy with respect to the generator of $H_1(X)$ is $\eta$. By a choice of the knot $k$ and the number $\eta \in \mathbb{C}^\ast$, we may make the group $H^1(X, F_\eta)$ arbitrarily large, while $H^1(X, \mathbb{C})$ is always one-dimensional.

Consider now the 3-manifold $M$ which is the connected sum $M = X \# (S^1 \times S^2)$. Thus $M = X_+ \cup X_-$, where $X_+ \cap X_- = S^2$, $X_+ = X - \{\text{disk}\}$ and $X_- = (S^1 \times S^2) - \{\text{disk}\}$. Suppose $F$ is a flat complex line bundle $F$ over $M$ such that its restriction to $X_+$ is isomorphic to $F_\eta|_{X_+}$. Consider the class $\xi \in H^1(X, \mathbb{R})$ such that its restriction onto $X_+$ is trivial and its restriction to $X_-$ is the generator.

By using the Mayer-Vietoris sequence, we show that $\beta_1(\xi, F) = \dim_{\mathbb{C}} H^1(X, F_\eta)$. As we noticed above, this number can be arbitrarily large, while $\dim_{\mathbb{C}} H^1(M, \mathbb{C}) = 2$.

8. Sketch of the proof of Theorem 4. Our proof of Theorem 4 is based on a slight modification of the Witten deformation [Wi] suggested by Bismut [Bi] in his proof of the degenerate Morse inequalities of Bott. However our proof is rather different from [Bi] even in the case $[\omega] = 0$. We entirely avoid the probabilistic analysis of the heat kernels, which is the most difficult part of [Bi]. Instead, we give an explicit estimate on the number of the “small” eigenvalues of the deformed Laplacian. We now will explain briefly the main steps of the proof.

Let $U$ be a small tubular neighborhood of $C$ in $M$. We identify $U$ with a neighborhood of the zero section in the normal bundle $\nu(C)$. Fix an affine connection on $\nu(C)$. This connection defines a bigrading

$$\Omega^\bullet(M, F) = \bigoplus \Omega^{i,j}(M, F),$$

where $\Omega^{i,j}(M, F)$ is the space of forms having degree $i$ in the horizontal direction and degree $j$ in the vertical direction. For $s \in \mathbb{R}$, let $\tau_s$ be the map from $\Omega^\bullet(M, F)$ to itself which sends $\alpha \in \Omega^{i,j}(M, F)$ to $s^i \alpha$.

Following Bismut, we introduce a 2-parameter deformation

$$\nabla_{t, \alpha} : \Omega^\bullet(M, F) \rightarrow \Omega^{\bullet+1}(M, F), \quad t, \alpha \in \mathbb{R}$$

of the covariant derivative $\nabla$, such that, for large values of $t, \alpha$ the Betti numbers of the deformed de Rham complex $\left(\Omega^\bullet(M, F), \nabla_{t, \alpha}\right)$ are equal to the Novikov numbers $\beta_\nu(\xi, F)$. Let $e(\omega) : \Omega^\bullet(M, F) \rightarrow \Omega^{\bullet+1}(M, F)$ denote the external multiplication by $\omega$. Then on $U$ the deformation (7) is given by

$$\nabla_{t, \alpha} = (\tau_{t\sqrt{2}})^{-1} \circ \left(\nabla + t\omega e(\omega)\right) \circ \tau_{\sqrt{2}},$$

while outside of some larger neighborhood $V \supset U$, we have $\nabla_{t, \alpha} = \nabla + t\omega e(\omega)$.

There exists a unique function $f : U \rightarrow \mathbb{R}$ such that $df = \omega$ and $f|_C = 0$. By the parameterized Morse lemma there exist an Euclidean metric $h^{\nu(C)}$ on $\nu(C)$ such that $\nu(C)$ decomposes into an orthogonal direct sum $\nu(C) = \nu^+(C) \oplus \nu^-(C)$ and if $(y^+, y^-) \in U$, then $f(y) = \frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}$. Fix an arbitrary Riemannian metric $g^C$ on
The metrics $h^\nu(C), g^C$ define naturally a Riemannian metric $g^\nu(C)$ on $\nu(C)$ (here we consider $\nu(C)$ as a non-compact manifold).

Let $g^M$ be any Riemannian metric on $M$ whose restriction to $U$ is equal to $g^\nu(C)$. We also choose a Hermitian metric $h^F$ on $F$. Let us denote by $\Delta_{t,\alpha}$ the Laplacian associated with the differential (7) and with the metrics $g^M, h^F$.

Fix $\alpha > 0$ sufficiently large. It turns out that, when $t \to \infty$, the eigenfunctions of $\Delta_{t,\alpha}$ corresponding to “small” eigenvalues localize near the critical points set $C$ of $\omega$. Hence, the number of the “small” eigenvalues of $\Delta_{t,\alpha}$ may be calculated by means of the restriction of $\Delta_{t,\alpha}$ on $U$. We are led, thus, to study of a certain Laplacian on $\nu(C)$. The latter Laplacian may be decomposed as $\bigoplus_Z \Delta^Z_{t,\alpha}$ where the sum ranges over all connected components of $C$ and $\Delta^Z_{t,\alpha}$ is a Laplacian on the normal bundle $\nu(Z) = \nu(C)|_Z$ to $Z$. We denote by $\Delta^Z_{t,\alpha}$ ($p = 0, 1, 2, \ldots$) the restriction of $\Delta^Z_{t,\alpha}$ on the space of $p$-forms.

It follows from (8), that the spectrum of $\Delta^Z_{t,\alpha}$ does not depend on $t$. Moreover, if $\alpha > 0$ is sufficiently large, then

$$\dim \ker \Delta^Z_{t,\alpha} = \dim H^{p-\text{ind}(Z)}(Z, F|_Z \otimes o(Z)).$$

In the case when $F$ is a trivial line bundle, (9) is proven by Bismut [Bi, Theorem 2.13].

Let $E^p_{t,\alpha}$ ($p = 0, 1, \ldots, n$) be the subspace of $\Omega^p(M, F)$ spanned by the eigenvectors of $\Delta_{t,\alpha}$ corresponding to the “small” eigenvalues. The cohomology of the de Rham complex $\left(\Omega^p(M, F), \nabla_{t,\alpha}\right)$ may be calculated as the cohomology of the subcomplex $\left(E^p_{t,\alpha}, \nabla_{t,\alpha}\right)$.

Using the method of [Sh1], we show that, if the parameters $t$ and $\alpha$ are large enough, then

$$\dim E^p_{t,\alpha} = \sum_Z \dim \ker \Delta^Z_{t,\alpha},$$

where the sum ranges over all connected components $Z$ of $C$. The Theorem 4 follows now from (9),(10) by standard arguments (cf. [Bo2]).

Remark. In [HS], Helffer and Sjöstrand gave a very elegant analytic proof of the degenerate Morse inequalities of Bott. Though they also used the ideas of [Wi], their method is completely different from [Bi]. It is not clear if this method may be applied to the case $\xi \neq 0$.

9. $L^2$ generalization. Our Theorem 4, combined with the results of W.Lück [Lü], gives the following $L^2$ version of the Novikov-Bott inequalities (5). Recall that $L^2$ generalization of the usual Morse inequalities for Morse functions were obtained first by S.P.Novikov and M.A.Shubin in [NS]. $L^2$-version of Novikov inequalities for 1-forms (allowing only isolated critical points) is considered in a recent preprint [MS] of V.Mathai and M.Shubin. They use different technique and their assumptions do not require residual finiteness.

Let $\pi$ be a countable residually finite group and let $\mathcal{N}(\pi)$ denote the von Neumann algebra of $\pi$ acting on the Hilbert space $l^2(\pi)$ from the left and commuting with the standard action of $\pi$ on $l^2(\pi)$ from the right. The algebra $\mathcal{N}(\pi)$ is supplied with the
canonical finite trace and all von Neumann dimensions later will be understood with respect to this trace.

Suppose that a flat bundle $L^\pi$ of Hilbert $\mathcal{N}(\pi)$-modules $l^2(\pi)$ over a closed manifold $M$ is given. (Here $\pi$ is not necessarily the fundamental group of $M$). Any such bundle can be constructed by the standard construction from a representation of the fundamental group of $M$ into $\pi$. Let $F$ denote a finite dimensional flat vector bundle over $M$ as above. The tensor product $L^\pi \otimes F$ (the tensor product taken over $C$) is again a bundle of Hilbert $\mathcal{N}(\pi)$-modules over $M$.

Let $\omega$ be a closed real valued 1-form on $M$, which is non-degenerate in the sense of Bott. It determines a family of flat bundles $F_t$ as in Section 1. Then there exists a countable subset $S \in \mathbb{R}$ (the set of jump points) such that the von Neumann dimension $\text{dim}_{\mathcal{N}(\pi)} H^i_2(M, L^\pi \otimes F_t)$ is constant for $t \notin S$ and it jumps up for $t \in S$. This fact follows from Theorem 0.1 of Lück [Lück]. We will define the von Neumann - Novikov numbers $\beta_i(\xi, L^\pi \otimes F)$ as the value of $\text{dim}_{\mathcal{N}(\pi)} H^i_2(M, L^\pi \otimes F_t)$ for $t \notin S$. This value clearly depends only on the cohomology class $\xi \in H^1(M, \mathbb{R})$ of $\omega$. Define the von Neumann - Novikov polynomial $N_{\xi, L^\pi \otimes F}(\lambda) = \sum \lambda^i \beta_i(\xi, L^\pi \otimes F)$.

For any component $Z$ of the set of critical points $C$ of $\omega$ define the following von Neumann - Poincaré polynomial $P_{Z, L^\pi \otimes F}(\lambda) = \sum \lambda^i \text{dim}_{\mathcal{N}(\pi)} H^i_2(Z, L^\pi_{|Z} \otimes F_{|Z} \otimes o(Z))$, and then the von Neumann - Morse counting polynomial $M_{\omega, L^\pi \otimes F}(\lambda) = \sum_{Z} \lambda^{\text{ind}(Z)} P_{Z, L^\pi \otimes F}(\lambda)$, the sum is taken over the set of connected components $Z$ of $C$.

**Theorem 10.** There exists a polynomial $Q(\lambda)$ with real non-negative coefficients, such that

$$M_{\omega, L^\pi \otimes F}(\lambda) - N_{\xi, L^\pi \otimes F}(\lambda) = (1 + \lambda) Q(\lambda).$$

The proof is based on theorem (0.1) of Lück [Lück] and Theorem 4. Let’s briefly indicate the main points.

Let $\pi \supset \Gamma_1 \supset \Gamma_2 \supset \ldots$ be a sequence of subgroups of finite index in $\pi$ having trivial intersection. The flat $\mathcal{N}(\pi)$ bundle $L^\pi$ is constructed by means of a representation $\psi : \pi_1(M) \to \pi$. Let $\psi_m : \pi_1(M) \to \pi/\Gamma_m$ denote the composition of $\psi$ with the reduction modulo $\Gamma_m$. The representation $\psi_m$ determines a flat vector bundle $L^\pi_m$ whose fiber is the group ring $\mathbb{C}[\pi/\Gamma_m]$ for any $m$. Slightly generalizing theorem (0.1) of Lück [Lück], we obtain that

$$\text{dim}_{\mathcal{N}(\pi)} H^i_2(M, L^\pi \otimes F) = \lim_{m \to \infty} |\pi/\Gamma_m|^{-1} \text{dim}_C H^i(M, L^\pi_m \otimes F)$$

This allows to approximate the von Neumann - Novikov polynomial $N_{\xi, L^\pi_m \otimes F}(\lambda)$ by the polynomials $|\pi/\Gamma_m|^{-1}N_{\xi, L^\pi_m \otimes F}(\lambda)$. 
Similarly, the von Neumann - Morse polynomial $\mathcal{M}_{\omega, L^\pi \otimes F}(\lambda)$ is approximated by the polynomials $|\pi/\Gamma_m|^{-1} \mathcal{M}_{\omega, L^\pi m \otimes F}(\lambda)$. Application of Theorem 4 then finishes the proof.

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