A Calabi’s Type Correspondence

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Abstract

Calabi observed that there is a natural correspondence between the solutions of the minimal surface equation in $\mathbb{R}^3$ with those of the maximal spacelike surface equation in $L^3$. We are going to show how this correspondence can be extended to the family of $\varphi$-minimal graphs in $\mathbb{R}^3$ when the function $\varphi$ is invariant under a two-parametric group of translations. We give also applications in the study and description of new examples.

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1 Introduction

Differential geometry of surfaces and partial differential equations (PDEs) have a strong link by means of which both theories benefit mutually. Actually, many classic PDEs are linked to interesting geometric problems. The geometry allows to integrate these equations, to establish non trivial properties of the solutions and to give some superposition principles which determine new solutions in terms of already known solutions. The classical theory of surfaces shows that geometric transformations may also be used to construct new surfaces from a given one.

Two of the most studied geometric PDEs has been

(1.1) \[ \text{div} \left( \frac{\nabla u}{\sqrt{1 + \epsilon |\nabla u|^2}} \right) = 0, \quad \epsilon \in \{1, -1\}. \]

They are the Euclidean minimal equation ($\epsilon = 1$) and the Lorentzian spacelike maximal equation ($\epsilon = -1$). Calabi proved in [2], there is a natural one to one correspondence between the solutions of (1.1) with $\epsilon = 1$ and the solutions of (1.1) with $\epsilon = -1$. This relationship gives a (local) connection between Euclidean minimal graphs and Lorentzian spacelike maximal graphs which is

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useful for describing examples and for applying similar methods to the study of their geometrical and topological properties. What we offer in this work is an extension of this correspondence to solutions of

\[ \text{div} \left( \frac{\nabla u}{\sqrt{1 + \epsilon |\nabla u|^2}} \right) = \frac{G(u)}{\sqrt{1 + \epsilon |\nabla u|^2}}, \quad \text{in } \Omega_\epsilon \subseteq \mathbb{R}^2, \]

where \( \Omega_\epsilon \) is a simply connected domain, \( G(u) \) is a function of \( u \) and \( \epsilon \in \{1, -1\} \).

For \( \epsilon = 1 \), (1.2) describes the equilibrium condition of a graph in \( \mathbb{R}^3 \) in an external conservative force field whose potential is invariant under a two-parameter group of translations. When \( G(u) \equiv 1 \), the graphs of a solutions of (1.2) are translating solitons, that is, solutions of the mean curvature flow which move by either a vertical translation in \( \mathbb{R}^3 \) if \( \epsilon = 1 \) or in \( \mathbb{L}^3 \) if \( \epsilon = -1 \).

The paper is organized as follows, Sections 2 and 3 deal with some elementary facts of the theory of \( \varphi \)-minimal surfaces in \( \mathbb{R}^3 \) and spacelike \( \varphi \)-maximal surfaces in \( \mathbb{L}^3 \). In Section 4, we give a Calabi’s type correspondence between the solutions of (1.2) with \( \epsilon = 1 \) and the solutions of (1.2) with \( \epsilon = -1 \). Finally, Section 5 is devoted to give some applications in the description and study of new examples.

## 2 The \( \varphi \)-minimal equation

The equilibrium condition for a surface \( \Sigma \) in \( \mathbb{R}^3 \) in an external force field \( \mathcal{F} \) was described by Poisson [11, pp. 173-187]. When the intrinsic forces of the surface are assumed to be equal, \( \mathcal{F} \) must be conservative and it writes as \( \mathcal{F} = \nabla (e^\varphi) \), for some smooth function \( \varphi \) on a domain of \( \mathbb{R}^3 \) which contains \( \Sigma \). In this case, the equilibrium condition is given in terms of the mean curvature vector \( \mathbf{H} \) of \( \Sigma \) as follows:

\[ \mathbf{H} = (\nabla \varphi)^\perp \]

where \( \nabla \) is the gradient operator in \( \mathbb{R}^3 \) and \( \perp \) denotes the projection to the normal bundle of \( \Sigma \).

A surface \( \Sigma \) satisfying (2.1) is called \( \varphi \)-minimal and it can be also viewed either as a critical point of the weighted volume functional

\[ V_\varphi(\Sigma) := \int_{\Sigma} e^\varphi \ dA_\Sigma, \]

where \( dA_\Sigma \) is the volume element of \( \Sigma \), or as a minimal surface in the conformally changed metric

\[ G_\varphi := e^\varphi \langle ., . \rangle. \]

For the particular case that the function \( \varphi \) is invariant under a two-parameter group of translations in \( \mathbb{R}^3 \) we have some interesting examples:
• **Minimal surfaces**, if \( \varphi \) is constant.

• **Translating solitons**, if \( \varphi \) is a linear function.

• **Singular minimal surfaces** (also called cupolas because they arise by considering \( \mathcal{F} \) as the gravitational force field) if \( \varphi(p) = \alpha \log(\langle p, \vec{v} \rangle) \), where \( \alpha \in \mathbb{R} \) and \( \vec{v} \) is a constant vector.

Some references in the study of the above examples are \([7, 8, 9, 10]\).

When \( \varphi \) is invariant under a two-parameter group of translations then, up to a motion in \( \mathbb{R}^3 \), we can consider that the external force field \( \mathcal{F} \) is always a vertical vector field, that is, \( \mathcal{F} \wedge e_3 = e^\varphi \nabla \varphi \wedge e_3 \equiv 0 \), where \( e_3 = (0, 0, 1) \) and \( \varphi \) only depends on the third coordinate in \( \mathbb{R}^3 \). In this case, the condition \((2.1)\) writes as

\[
(2.4) \quad \mathbf{H} = \dot{\varphi} e_3^\perp,
\]

where \( (\cdot) \) denotes derivative respect to the third coordinate.

**Definition 2.1.** A \( \varphi \)-minimal surface \( \Sigma \) whose mean curvature vector satisfies \((2.4)\) will be called \([\varphi, e_3]\)-minimal.

Let \( u : \Omega_1 \to \mathbb{R} \) be a regular function on a simply connected planar domain \( \Omega_1 \) and consider \( \psi(x, y) = (x, y, u) \) its graph. Then, the induced metric, Gauss map and mean curvature vector of \( \psi \) write, respectively, by

\[
(2.5) \quad g := (1 + u_x^2) dx^2 + (1 + u_y^2) dy^2 + 2u_x u_y dxdy,
\]

\[
(2.6) \quad N := \frac{1}{W} (-u_x, -u_y, 1), \quad W = \sqrt{1 + u_x^2 + u_y^2},
\]

\[
(2.7) \quad \mathbf{H} : = -\frac{1}{W^3} \mathcal{L} u \mathcal{N},
\]

where

\[
\mathcal{L} u = (1 + u_x^2) u_{yy} + (1 + u_y^2) u_{xx} - 2u_x u_y u_{xy},
\]

Hence and from \((2.4)\), we get that \( \psi \) is a \([\varphi, e_3]\)-minimal graph if and only if \( u \) is a solution of the following elliptic partial differential equation:

\[
(2.8) \quad (1 + u_x^2) u_{yy} + (1 + u_y^2) u_{xx} - 2u_x u_y u_{xy} = \dot{\varphi}(u) W^2.
\]

Using \((2.8)\), we can see

\[
\left( \frac{1 + u_x^2}{W} e^{\varphi(u)} \right)_x - \left( \frac{u_x u_y}{W} e^{\varphi(u)} \right)_y = - \frac{u_x e^{\varphi(u)} \mathcal{L} u}{W^3} + \frac{u_x e^{\varphi(u)} \dot{\varphi}(u)}{W} = 0,
\]

\[
\left( \frac{1 + u_y^2}{W} e^{\varphi(u)} \right)_y - \left( \frac{u_x u_y}{W} e^{\varphi(u)} \right)_x = - \frac{u_y e^{\varphi(u)} \mathcal{L} u}{W^3} + \frac{u_y e^{\varphi(u)} \dot{\varphi}(u)}{W} = 0.
\]
The equation (2.8), is equivalent to the integrability of the following differential system,

\[
\begin{align*}
\phi_{xx} &= 1 + \frac{u_x^2}{W} e^{\varphi(u)}, \\
\phi_{xy} &= \frac{u_x u_y}{W} e^{\varphi(u)}, \\
\phi_{yy} &= 1 + \frac{u_y^2}{W} e^{\varphi(u)},
\end{align*}
\]

for a convex function \( \phi : \Omega \rightarrow \mathbb{R} \) (unique, module linear polynomials).

From (2.5) and (2.8), the laplacian operator \( \Delta_g \) of \( g \) is given by

\[
W^2 \Delta_g := (1 + u_x^2) \partial_{yy}^2 + (1 + u_y^2) \partial_{xx}^2 - 2u_x u_y \partial_{xy}^2 - \dot{\varphi}(u)(u_x \partial_x + u_y \partial_y),
\]

and a straightforward computation gives,

\[
\begin{align*}
\Delta_g \psi &= (\nabla \varphi) \cdot N = \langle N, \epsilon_3 \rangle \dot{\varphi}(u) W, \\
\Delta_g \phi_x &= \frac{e^{\varphi(u)} \dot{\varphi}(u)}{W} u_x = \langle N, \epsilon_3 \rangle e^{\varphi(u)} \dot{\varphi}(u) u_x, \\
\Delta_g \phi_y &= \frac{e^{\varphi(u)} \dot{\varphi}(u)}{W} u_y = \langle N, \epsilon_3 \rangle e^{\varphi(u)} \dot{\varphi}(u) u_y, \\
\Delta_g (\int e^{\varphi(u)} du) &= e^{\varphi(u)} \dot{\varphi}(u) = \langle N, \epsilon_3 \rangle e^{\varphi(u)} \dot{\varphi}(u) W.
\end{align*}
\]

### 3 The \( \varphi \)-maximal equation

Let \( \mathbb{L}^3 \) be the Minkowski space \( \mathbb{R}^3 \) with the Lorentz metric

\[
\ll \cdot, \cdot \gg = dx^2 + dy^2 - dz^2.
\]

A surface in \( \mathbb{L}^3 \) is called spacelike if the induced metric on the surface is a positive definite Riemannian metric. This kind of surfaces have played a major role in Lorentzian geometry, for a survey of some results we refer to \( \Pi \).

A spacelike surface \( \tilde{\Sigma} \) in \( \mathbb{L}^3 \) is called \( \varphi \)-maximal if its mean curvature vector \( \tilde{H} \) satisfies

\[
\tilde{H} = \left( \nabla^{\mathbb{L}^3} \varphi \right) \cdot N,
\]

where \( \nabla^{\mathbb{L}^3} \) denotes the gradient operator in \( \mathbb{L}^3 \) and \( \varphi \) is a smooth function on a domain in \( \mathbb{L}^3 \) containing \( \tilde{\Sigma} \).

As in the Euclidean case, a \( \varphi \)-maximal spacelike surface can be also viewed either a critical point of the weighted volume functional

\[
\tilde{V}_\varphi(\tilde{\Sigma}) := \int_{\tilde{\Sigma}} e^\varphi \ dA_{\tilde{\Sigma}},
\]

or a maximal (zero mean curvature) spacelike surface in the conformally changed metric

\[
\tilde{G}_\varphi := e^\varphi \ll \cdot, \cdot \gg.
\]
**Definition 3.1.** If $\varphi$ only depend on the third coordinate, any spacelike surface with mean curvature vector satisfying (3.2) will be called $[\varphi, \vec{e}_3]$-maximal.

Well known examples of $[\varphi, \vec{e}_3]$-maximal are the maximal surfaces and the translating solitons, whose study is an exciting and already classical mathematical research field, see [3, 4] for some results. In analogy to the Euclidean case, a $[\varphi, \vec{e}_3]$-maximal spacelike with $\varphi(p) = \alpha \log <p, \vec{e}_3>$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$ will be called singular $\alpha$-maximal surface.

By a straightforward computation, if $\Omega_{-1}$ is a simply connected planar domain, it is not difficult to prove that the vertical graph in $\mathbb{L}^3$ of a function $u: \Omega_{-1} \rightarrow \mathbb{R}$ is a $[\varphi, \vec{e}_3]$-maximal spacelike if and only if $u$ is a solution of the following elliptic partial differential equation:

$$
(3.5) \quad (1 - u_x^2) u_{yy} + (1 - u_y^2) u_{xx} + 2u_xu_yu_{xy} + ˙\varphi(u)\sqrt{1 - u_x^2 - u_y^2} = 0,
$$

where $\sqrt{W} = \sqrt{1 - u_x^2 - u_y^2}$. From (3.1) and (3.5), the laplacian operator $\Delta_{\overline{g}}$ of the induced metric $\overline{g}$ is given by

$$
(3.6) \quad \overline{W}^2 \Delta_{\overline{g}} := (1 - u_x^2) \partial_{yy}^2 + (1 - u_y^2) \partial_{xx}^2 + 2u_xu_y\partial_{xy}^2 - \varphi(u)(\overline{u}_x\partial_x + \overline{u}_y\partial_y),
$$

and the equation (3.5), is equivalent to the integrability of the following differential system,

$$
(3.7) \quad \overline{\phi}_{xx} = \frac{1 - \pi_x^2}{W} e^{\varphi(\overline{u})}, \quad \overline{\phi}_{xy} = -\frac{\overline{u}_x\overline{u}_y}{W} e^{\varphi(\overline{u})}, \quad \overline{\phi}_{yy} = \frac{1 - \pi_y^2}{W} e^{\varphi(\overline{u})}
$$

for a convex function $\overline{\phi}: \Omega_{-1} \rightarrow \mathbb{R}$ (unique, module linear polynomials).

4 The correspondence

Calabi observed, [2], there is a natural correspondence between all solutions of the minimal surface equation in $\mathbb{R}^3$ with those of the maximal spacelike surface equation in $\mathbb{L}^3$. We are going to show how this correspondence can be extended to the family of $[\varphi, \vec{e}_3]$-minimal graphs in $\mathbb{R}^3$.

**Theorem 4.1.** Let $\Omega_1$ be a simply connected planar domain, $\psi: \Omega_1 \rightarrow \mathbb{R}^3$, $\psi(x, y) = (x, y, u)$ be a $[\varphi, \vec{e}_3]$-minimal graph in $\mathbb{R}^3$, $\phi$ be a solution to the system (2.9) and $\vartheta$ be a primitive function of $e^\varphi$ (that is, $\dot{\vartheta} = e^\varphi$). Then $\tilde{\psi}: \Omega_1 \rightarrow \mathbb{L}^3$ given by

$$
(4.1) \quad \tilde{\psi} := (\phi_x, \phi_y, \vartheta(u)),
$$

is a $[-\varphi\circ \vartheta^{-1}, \vec{e}_3]$-maximal spacelike graph in the Lorentz-Minkowski space whose Gauss map $\tilde{N}$ writes

$$
(4.2) \quad \tilde{N} = (u_x, u_y, W),
$$
The induced metrics $g$ and $\tilde{g}$ of $\psi$ and $\tilde{\psi}$, respectively, are conformal and the mean curvature $H$ ($\tilde{H}$) and Gauss curvature $K$ ($\tilde{K}$) of $\psi$ ($\tilde{\psi}$) satisfy

\begin{align}
\label{4.3}
\tilde{H} + W^2 e^{-\phi(u)} H &= 0, \\
\label{4.4}
\tilde{K} + W^4 e^{-2\phi(u)} K &= 0.
\end{align}

**Proof.** From (2.9), (4.2) and (4.1), we have

\begin{align}
\label{4.5}
\ll \tilde{\psi}_x, \tilde{N}\gg = \ll \tilde{\psi}_y, \tilde{N}\gg = 0, \quad \ll \tilde{N}, \tilde{N}\gg = -1,
\end{align}

\begin{align}
\label{4.6}
\tilde{g} = \ll d\tilde{\psi}, d\tilde{\psi}\gg = \left(\frac{d\phi}{du}\right)^2 + \left(\frac{d\phi_d}{du}\right)^2 - \left(\frac{d\vartheta}{du}\right)^2 = e^{2\phi(u)} W^2 g.
\end{align}

So, $\tilde{\psi}$ is spacelike graph with Gauss map $\tilde{N}$. As $\tilde{g}$ and $g$ are conformal metrics, from (2.12), (2.13), (4.2) and (4.6), we get that

\begin{align}
\tilde{H} = \Delta \tilde{\psi} = -W^2 \phi(u) \tilde{N} = \frac{d\phi}{d\vartheta} W \tilde{N} = \left(-\nabla^3 \phi \circ \vartheta^{-1}\right)^{\perp},
\end{align}

and $\tilde{\psi}$ is $[-\phi \circ \vartheta^{-1}, e_3]$-maximal.

Finally, from (2.9) and (4.2), we have

\begin{align}
\ll \tilde{\psi}_x, \tilde{N}_x\gg &= \frac{e^{\phi(u)}}{W} u_{xx} = -e^{\phi(u)} < \psi_x, N_x>, \\
\label{4.8}
\ll \tilde{\psi}_x, \tilde{N}_y\gg &= \frac{e^{\phi(u)}}{W} u_{xy} = -e^{\phi(u)} < \psi_x, N_y>, \\
\ll \tilde{\psi}_y, \tilde{N}_y\gg &= \frac{e^{\phi(u)}}{W} u_{yy} = -e^{\phi(u)} < \psi_y, N_y>.
\end{align}

Thus, the shapes operators $A$ ($\tilde{A}$) of $\psi$ ($\tilde{\psi}$) are related as follows:

\begin{align}
\label{4.10}
\tilde{A} + e^{-\phi(u)} W^2 A = 0,
\end{align}

which, together (4.6) let us to prove that (4.3) and (4.4) hold. \qed

**Remark 4.2.** Observe that $\tilde{\psi}$ is a graph on the Legendre transform domain $\Omega_{-1}$ of $\phi$.

**Remark 4.3.** The correspondence (4.1) can be given globally as follows

\begin{align}
\label{4.11}
\tilde{\psi} = \int e^{\phi(<\psi, e_3>)}(\tilde{e}_3 \wedge (d\psi \wedge N) + < d\psi, \tilde{e}_3 > \tilde{e}_3),
\end{align}

where $\wedge$ denotes the cross product in $\mathbb{R}^3$. Moreover, the singularities of $\tilde{\psi}$ hold where the angle function $<\tilde{e}_3, N>$ vanishes.

Arguing as in Theorem 4.1 we can prove,
Theorem 4.4. Let $\Omega_{-1}$ be a simply connected planar domain, $\tilde{\psi} : \Omega_{-1} \to \mathbb{L}^3$, $\tilde{\psi}(x,y) = (x,y,\bar{u})$ be a $[\varphi,\vec{e}_3]$-maximal graph in $\mathbb{L}^3$, $\varphi$ be a solution to the system (3.7) and $\vartheta$ be a primitive of $e^\varphi$. Then the immersion given by

\begin{equation}
\psi := (\varphi_x, \varphi_y, \vartheta(\bar{u})) ,
\end{equation}

is a $[-\varphi \circ \vartheta^{-1}, \vec{e}_3]$-minimal graph in $\mathbb{R}^3$ on the Legendre transform domain of $\varphi$, whose induce metric, mean curvature $H$ and Gauss curvature $K$ satisfy

\begin{align}
\label{eq:4.13}
g &:= \frac{e^{2\varphi(\bar{u})}}{W^4} \bar{g} , \\
\label{eq:4.14}H + e^{-\varphi(\bar{u})} W^2 \bar{H} &= 0 , \\
\label{eq:4.15}K + e^{-2\varphi(\bar{u})} W^4 \bar{K} &= 0 ,
\end{align}

where $W = \sqrt{1 - \bar{u}_x^2 - \bar{u}_y^2}$ and $\bar{g}$, $\bar{H}$ and $\bar{K}$ are the induced metric, the mean curvature and the Gauss curvature of the spacelike graph of $\bar{u}$.

Remark 4.5. The correspondence (4.12) also writes as follows

\begin{equation}
\psi = \int e^{\varphi(\vartheta)} \left( \vec{e}_3 \wedge_{\mathbb{L}^3} (d\tilde{\psi} \wedge_{\mathbb{L}^3} \bar{N}) - \vartheta d\tilde{\psi}, \vec{e}_3 \right) ,
\end{equation}

where $\wedge_{\mathbb{L}^3}$ denotes the cross product in $\mathbb{L}^3$ and $\bar{N}$ is the Gauss map of $\tilde{\psi}$. The singular points of $\psi$ hold where the angle function $\vartheta \vec{e}_3, \bar{N}$ vanishes.

Definition 4.6. If $\psi$ and $\tilde{\psi}$ are related as in either (4.11) or (4.16) we say that $(\psi, \tilde{\psi})$ is a Calabi-pair.

Corollary 4.7. Let $(\psi, \tilde{\psi})$ be a Calabi-pair.

- If $\psi$ is a translating solitons in $\mathbb{R}^3$ then $\tilde{\psi}$ is a singular $(-1)$-maximal spacelike surface in $\mathbb{L}^3$, see Figure 4.1.
- If $\tilde{\psi}$ is a translating solitons in $\mathbb{L}^3$ then $\psi$ is a singular $(-1)$-minimal surface in $\mathbb{R}^3$, see Figure 4.2.

Remark 4.8. Observe that if one of the surfaces in a Calabi-pair is convex (respectively, of vanishing Gauss curvature), then both are convex (respectively, of vanishing Gauss curvature).

5 Applications

In this section we show some applications to the description and study of new examples.
5.1 Radially symmetric is a preserved condition

The elliptic equations (2.8) and (3.5) have radially symmetric solutions $u(r)$ and $\pi(r)$, $r = \sqrt{x^2 + y^2}$, respectively, if and only if the following ODE equations are satisfied,

\begin{align*}
\frac{u''}{1 + u'^2} &= \dot{\varphi}(u) - \frac{u'}{r}, \\
\frac{\pi''}{1 - \pi'^2} &= -\dot{\varphi}(\pi) - \frac{\pi'}{r}.
\end{align*}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{Figure_4.1}
\caption{Soliton in $\mathbb{R}^3$ and its corresponding singular (-1)-maximal surface in $L^3$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{Figure_4.2}
\caption{Cupola in $\mathbb{R}^3$ and its corresponding translating soliton in $L^3$.}
\end{figure}
From these equations we get,

\[
\left( \frac{e^\varphi(u)}{\sqrt{1 + u'^2}} \right)' = e^\varphi(u) \sqrt{1 + u'^2},
\]

(5.3)

\[
\left( \frac{e^\varphi(\pi)}{\sqrt{1 - \pi'^2}} \right)' = e^\varphi(\pi) \sqrt{1 - \pi'^2}.
\]

(5.4)

Hence, if we consider \( \varphi \) and \( \tilde{\varphi} \) the two radial functions

\[
\varphi = \int e^{\varphi(u)} r \cos(z) dr, \quad \tilde{\varphi} = \int e^{\varphi(\pi)} r \cosh(z) dr
\]

(5.5)

where

\[ u'(r) = \tan z, \quad \pi'(r) = \tanh z, \]

we have,

\[
\varphi'' = \frac{e^{\varphi(u)}}{\cos(z)}, \quad \tilde{\varphi}'' = \frac{e^{\varphi(\pi)}}{\cosh(z)}.
\]

(5.6)

From (5.5), (5.6) and by a straightforward computation, we can prove that \( \varphi \) and \( \tilde{\varphi} \) are, respectively, solutions of the differential systems (2.9) and (3.7).

Thus, from Theorem 4.1 and Theorem 4.4 we have,

**Proposition 5.1.**

- Let \( \psi := (r \cos t, r \sin t, u(r)) \) be a revolution \([\varphi, \hat{e}_3]\)-minimal surface in \( \mathbb{R}^3 \). Then, the corresponding \([-\varphi \circ \vartheta^{-1}, \hat{e}_3]\)-maximal spacelike surface in (4.1) is the rotational surface of elliptic type given by,

\[
\tilde{\psi} := \left( \hat{\vartheta}(u) r \cos(z) \cos t, \hat{\vartheta}(u) r \cos(z) \sin t, \hat{\vartheta}(u) \right),
\]

- Let \( \tilde{\psi} := (r \cos t, r \sin t, \pi(r)) \) be a revolution \([\pi, \hat{e}_3]\)-maximal spacelike surface of elliptic type in \( \mathbb{L}^3 \). Then, the corresponding \([-\varphi \circ \vartheta^{-1}, \hat{e}_3]\)-minimal surface in (4.12) is the rotational surface given by,

\[
\psi := \left( \hat{\vartheta}(\pi) r \cosh(z) \cos t, \hat{\vartheta}(\pi) r \cosh(z) \sin t, \hat{\vartheta}(\pi) \right),
\]

where

\[ u'(r) = \tan z, \quad \pi'(r) = \tanh z, \]

5.2 Translating solitons and singular \( \alpha \)-maximal surfaces with \( \alpha > 1 \).

The behaviour of a rotational singular \( \beta \)-minimal surface \( \psi \) in \( \mathbb{R}^3 \), \( \beta \leq -1 \), was described in [7, Theorem 8]. In fact, if

\[ \gamma(s) = (x(s), 0, u(s)), \quad s \in [0, L], \]


is the arc-length parametrized generating curve of $\psi$, there are only two possibilities:

**CASE I: $\gamma$ intersects the rotation axis**

In this case, although the ordinary differential equation associated to problem has a singularity at $x = 0$, see (5.1), the curve $\gamma$ is the graph of a symmetric and concave function $u(x)$, $u \in C^2(-R, R)$ and meets orthogonally the rotation axis and to the $x$-axis, see Figure 5.1. In particular,

\[ u(0) = u(L) = 0, \quad x(0) = -R, \quad x(L) = R, \quad x'(0) = x'(L) = 0. \]
\[ x(L/2) = 0, \quad u(L/2) > 0, \quad u'(L/2) = 0. \]

**Theorem 5.2** (Existence of spacelike singular $\alpha$-maximal bowls of elliptic type).

(i) There exists a rotationally symmetric, entire, smooth, strictly convex spacelike translating soliton (unique up to translation) and of linear growth, Figure 5.2 left (see [4, 6] for a proof based on ODE theory).

(ii) For any $\alpha > 1$, there is a rotationally symmetric, entire, smooth, strictly convex spacelike singular $\alpha$-maximal graph (unique up to homothety) and of linear growth, Figure 5.2 right.

These examples will be called singular $\alpha$-maximal bowls of elliptic type.
Proof. As $\gamma$ is the generating curve of $\psi$, from (5.1), (5.5) and (5.6), we have
\begin{align}
  x'(s) &= \cos z(s) \quad u'(s) = \sin z(s), \\
  z'(s) &= \beta \cos z(s) \quad u'(s) = \sin z(s) / x(s),
\end{align}
(5.7)
(5.8)
From Proposition 5.1, the curve $\tilde{\gamma} = (\lambda, 0, \vartheta(u))$ with
\begin{align}
  \lambda &= u^\beta x \cos z, \\
  \frac{d\vartheta}{du} &= \dot{\vartheta}(u) = u^\beta,
\end{align}
(5.9)
is the generating curve of either a revolution spacelike singular $\beta + 1$-maximal surface of elliptic type with $\beta < -1$ or a translating soliton with $\beta = -1$ in $\mathbb{L}^3$, $\tilde{\psi}$ given by $\tilde{\psi} := (\lambda \cos(t), \lambda \sin(t), \vartheta(u))$.

From (a), (b), (5.3), (5.7), (5.8) and (5.9), we have,
\begin{align}
  \lim_{s \to 0, L} \frac{d\lambda}{ds} &= \lim_{s \to 0, L} u^\beta = \infty, \\
  \lim_{s \to 0, L} \frac{d\vartheta}{ds} &= \lim_{s \to 0, L} u^\beta \sin z = -\infty
\end{align}
Hence, $\lambda(s)$ increases in $[0, L]$ from $-\infty$ to $\infty$ and
\begin{align}
  \lim_{s \to 0, L} \dot{\vartheta}(s) &= -\infty.
\end{align}
Moreover,
\begin{align}
  \lim_{s \to L/2} \lambda(s) = 0, \\
  \lim_{s \to L/2} \vartheta &= u(L/2)^{\beta + 1} / (\beta + 1), \quad (= \log u(L/2)) \quad \text{if} \quad \beta < -1 \quad (\beta = -1).
\end{align}
Finally, from (5.7), (5.8) and (5.9), we have also,
\begin{align}
  \frac{d\vartheta}{d\lambda} &= \sin z, \\
  \frac{d^2\vartheta}{d\lambda^2} &= \frac{\beta \cos^2 z}{u^{\beta + 1}} - \frac{2z}{xu^\beta} \leq 0.
\end{align}
(5.10)
So, $\vartheta$ is a function of $\lambda$ which satisfies: $\vartheta \in C^2(\mathbb{R})$, is concave, has a maximum at the origin and has a linear growth, see Figure 5.3.

The convex singular $\alpha$-maximal bowl, $\alpha = \beta / (\beta + 1)$, is obtained by the reflection in $z = 0$ of $\tilde{\psi}$.

The uniqueness follows by using the tangency principle for $[\varphi, \vec{c}_3]$-minimal and spacelike $[\varphi, \vec{c}_3]$-maximal vertical graphs and making into account that a vertical translation moves any translating soliton into another translating soliton and a Euclidean homothety with center at the origin moves any spacelike singular $\alpha$-maximal surface into another spacelike singular $\alpha$-maximal surface. $\square$

Consider either $f(\pi) \equiv 1$ or $f(\pi) = \alpha / \pi$, for some $\alpha > 1$. Then the above theorem gives the following existence result:
Corollary 5.3. For each positive real number \( a \), the initial value problem
\[
\frac{\pi''}{1 - \pi'^2} = f(\pi) - \frac{\pi'}{r}, \quad r \in [0, \infty],
\]
(5.11) \( \pi(0) = a > 0, \quad \pi'(0) = 0. \)
(5.12) has an unique convex solution \( \pi \in C^2[0, \infty] \) such that
\[
\lim_{r \to \infty} \pi' = 1.
\]

Case II: \( \gamma \) does not intersects the rotation axis
In this case, \( \gamma \) is embedded, has a winglike-shape and intersects orthogonally the \( x \)-axis in two points, see Figure 5.4, that is,
\[
\begin{align*}
&\text{(c) } u(0) = u(L) = 0, \quad x'(0) = x'(L) = 0. \\
&\text{(d) } \text{There exists } s_1 \in [0, L], \text{ such that } x'(s_1) = 0, \ u(s_1) > 0 \text{ and } u \text{ is concave in } [s_1, L].
\end{align*}
\]

Figure 5.4: Generating curves \( \gamma \) for \( \beta = -1 \) and for \( \beta = -2 \).

Theorem 5.4 (Existence of winglike spacelike singular \( \alpha \)-maximal surfaces).
For any \( \alpha > 1 \), there exist, up to an homothety (translation), two entire spacelike singular \( \alpha \)-maximal graphs (translating solitons) in \( \mathbb{L}^3 \) with linear growth and with an isolated singularity at the origin which is asymptotics to the light cone.
This kind of examples will be called either winglike solitons or winglike singular \( \alpha \)-maximal surfaces, see Figure 5.5.

![Figure 5.5: winglike soliton and singular 3-maximal winglike in \( \mathbb{L}^3 \)](image)

**Proof.** As in the proof of Theorem 5.2, the curve \( \bar{\gamma} = (\lambda, 0, \vartheta(u)) \) with

\[
\lambda = u^\beta x \cos z, \quad \dot{\vartheta}(u) = u^\beta,
\]

is the generating curve of a revolution spacelike \( \beta/\beta+1 \)-maximal surface of elliptic type (respectively, translating soliton) with \( \beta < -1 \) (respectively, with \( \beta = -1 \)) in \( \mathbb{L}^3 \), \( \tilde{\psi} \) given by

\[
\tilde{\psi} := (\lambda \cos(t), \lambda \sin(t), \vartheta(u)),
\]

which verifies,

\[
\frac{d\lambda}{ds} = u^\beta, \quad \frac{d\vartheta}{ds} = u^\beta \sin z.
\]

Hence, \( \lim_{s \to 0, L} \frac{d\lambda}{ds} = \infty \), \( \lim_{s \to 0, L} \frac{d\vartheta}{ds} = -\infty \) and \( \lambda(s) \) increases in \( [0, L] \) from \( -\infty \) to \( \infty \) and

\[
\lim_{s \to 0, L} \vartheta(s) = -\infty.
\]

On the other hand,

\[
\lim_{s \to s_1} \lambda(s) = 0, \quad \lim_{s \to s_1} \vartheta = \frac{u(s_1)^{\beta+1}}{\beta+1}, \quad (= \log u(s_1)) \quad \text{if} \ \beta < -1 \quad (\beta = 1).
\]

and

\[
\frac{d\vartheta}{d\lambda} = \sin z, \quad \frac{d^2\vartheta}{d\lambda^2} = \frac{\beta \cos^2 z}{u^{\beta+1}} - \frac{\sin 2z}{2x u^\beta}.
\]

Finally,

\[
\lim_{\lambda \to -\infty, \infty} \frac{d\vartheta}{d\lambda} = -1, \quad \lim_{\lambda \to 0} \frac{d\vartheta}{d\lambda} = 1,
\]

which says that the isolated singularity is asymptotic to the light cone. Moreover, as a function of \( \lambda \), \( \vartheta \) is of linear growth (see Figure 5.6). Uniqueness
follows by applying uniqueness of solution of (5.7) and (5.8) with the same initial conditions at \( s_1 \) and having in mind that a vertical translation moves any translating soliton into another translating soliton and a Euclidean homothety with center at the origin moves any spacelike singular \( \alpha \)-maximal surface into another spacelike singular \( \alpha \)-maximal surface.

In Figure 5.5 we have pictures of the rotational surfaces with generating curve \((\lambda, 0, -\varrho(u))\) for \( \beta = -1 \) and for \( \beta = -1.5 \).

![Figure 5.6: generating curve \( \tilde{\gamma} \) with \( \beta = -1 \) and \( \beta = -1.5 \).](image)

If \( f(\pi) \equiv 1 \) or \( f(\pi) = \alpha/\pi \), for some \( \alpha > 1 \), the Theorem 5.4 gives

Corollary 5.5. For each positive real number \( a \),

\[
(5.14) \quad \begin{cases} 
\frac{\pi''}{1-\pi^2} = f(\pi) - \frac{\pi'}{r}, & r \in [0, \infty], \\
\pi(0) = a > 0, & \pi'(0) = -1
\end{cases}
\]

has an unique solution \( \pi \in C^2([0, \infty]) \). Moreover, it is strictly convex, has a minimum in \([0, \infty]\) and satisfies,

\[
\lim_{r \to \infty} \frac{\pi'}{r} = 1.
\]

Moreover,

\[
(5.15) \quad \begin{cases} 
\frac{\pi''}{1-\pi^2} = f(\pi) - \frac{\pi'}{r}, & r \in [0, \infty], \\
\pi(0) = a > 0, & \pi'(0) = 1
\end{cases}
\]

has an unique solution \( \pi \in C^2[0, \infty] \) such that

\[
\lim_{r \to \infty} \frac{\pi'}{r} = 1.
\]

5.3 Rotational surfaces of hyperbolic type

In this section, we study rotational \( \alpha \)-maximal singular surfaces of hyperbolic type in \( L^3 \).
These surfaces are invariant by the 1-parameter group of hyperbolic rotations of the Lorentz group which fix the \( e_1 \) spacelike direction. A such surface with generating curve the arc-length parametrized spacelike curve \( \tilde{\gamma} = (x(s), 0, u(s)) \), \( u > 0, s \in I \subseteq \mathbb{R} \) is given by,

\[
(5.16) \quad \tilde{\psi}(s,t) = (x(s), u(s) \sinh(t), u(s) \cosh(t)), \quad (s,t) \in I \times \mathbb{R},
\]

with

\[
(5.17) \quad x'(s) = \cosh(z(s)), \quad u'(s) = \sinh(z(s)).
\]

The Gauss map of \( \tilde{\psi} \) writes as

\[
(5.18) \quad \tilde{N}(s,t) = (\sinh(z(s)), \cosh(z(s)) \sinh(t), \cosh(z(s)) \cosh(t)),
\]

and we have,

\[
\begin{align*}
\ll \tilde{\psi}_s, \tilde{\psi}_s \gg &= 1, & \ll \tilde{\psi}_s, \tilde{N}_s \gg &= z'(s), \\
\ll \tilde{\psi}_t, \tilde{\psi}_t \gg &= u(s)^2, & \ll \tilde{\psi}_t, \tilde{N}_t \gg &= u(s) \cosh(z(s)), \\
\ll \tilde{\psi}_s, \psi_t \gg &= 0, & \ll \tilde{\psi}_s, \tilde{N}_t \gg &= 0.
\end{align*}
\]

From (5.19), the mean curvature field of \( \tilde{\psi} \) is given by

\[
(5.20) \quad \tilde{H} = \left( \frac{u \, z' + \cosh(z)}{u} \right) \tilde{N} = \left( \frac{u \, K + \cosh(z)}{u} \right) \tilde{N},
\]

where \( K \) is the curvature of \( \tilde{\gamma} \).

Consequently, from (3.2), (5.16) and (5.18), the spacelike surface \( \tilde{\psi} \) is a singular \( \alpha \)-maximal spacelike surface if and only if

\[
(5.21) \quad \frac{z'(s)}{\cosh(z(s))} + \frac{1 + \alpha}{u(s)} = 0,
\]

or equivalently,

\[
(5.22) \quad \cosh(z) \, u^{\alpha+1} = k, \quad \text{for some positive constant } k.
\]

Moreover, from (5.21) and (5.22), the curvature of \( \tilde{\gamma} \) satisfies

\[
(5.23) \quad K = -(1 + \alpha)k \, u^{-\alpha-2}.
\]

From the spacelike condition, (5.17), (5.21) and (5.23), if \( 1 + \alpha > 0 \) (respectively, \( 1 + \alpha < 0 \)), the generating curve \( \tilde{\gamma} \) is the graph of a strictly concave (respectively, convex) function \( u(x) \) solution of the following ordinary differential equation,

\[
(5.24) \quad \frac{d^2 u}{dx^2} = -\frac{\alpha + 1}{u}(1 - \left(\frac{du}{dx}\right)^2),
\]
or equivalently,

\begin{align}
\frac{du}{dx} &= \tanh(z), \\
\frac{dz}{dx} &= -\frac{1 + \alpha}{u}.
\end{align}

As a first integrate of this system is given by \((5.22)\), if \(1 + \alpha \neq 0\), there exists a unique \(x_0 \in \mathbb{R}\) such us \(u'(x_0) = 0\) (see Figure 5.7 for a representation of the trajectories of \((5.25)\)). So, up to translation in the \(x\)-axis, we may assume that \(x_0 = 0\) and consider the solutions to \((5.25)\) satisfying

\begin{align}
\tag{5.26}
\text{ } \\
u(0) = u_0 > 0, \quad \frac{du}{dx}(0) = 0.
\end{align}

By taking \(\overline{u}(x) = u(-x)\), we see easily that, if \(1 + \alpha \neq 0\), a solution to \((5.24)-(5.26)\) is even and, from \((5.22)\) and \((5.25)\), it is defined in the interval \([-\Lambda_{u_0}, \Lambda_{u_0}]\), where

\begin{align}
\tag{5.27}
\Lambda_{u_0} &= |\alpha + 1| \int_0^\infty \frac{u_0}{\cosh \tau} d\tau.
\end{align}

Figure 5.7: Phase portrait of \((5.25)\) for \(\alpha = 1\) (left) and for \(\alpha = -3\) (right).

So, from \((5.22)\), \((5.25)\) and \((5.27)\), we have

**Proposition 5.6.** Let \(\overline{\gamma} = (x, 0, u)\) be the generating curve of a rotational space-like \(\alpha\)-maximal surface of hyperbolic type. Then, up to horizontal translation, we have, see figure \(5.8\),

- if \(1 + \alpha > 0\), \(\overline{\gamma}\) is the graph of a symmetric and strictly concave function \(u(x)\) in a bounded interval \([-\Lambda_{u_0}, \Lambda_{u_0}]\) which has a maximum at 0 and meets (asymptotically to the light cone) the \(x\)-axis in \(\pm\Lambda_{u_0}\), that is

\[
\lim_{x \to \pm\Lambda_{u_0}} u(x) = 0, \quad \lim_{x \to \pm\Lambda_{u_0}} \frac{du}{dx} = -1.
\]
• if $1 + \alpha < 0$, $\tilde{\gamma}$ is the graph of a symmetric and strictly convex function $u(x)$ in $]-\infty, \infty[$ which has a minimum at 0 and has linear growth, in fact
  \[
  \lim_{x \to \pm \infty} u(x) = \infty, \quad \lim_{x \to \pm \infty} \frac{du}{dx} = 1.
  \]

• if $1 + \alpha = 0$, then $\tilde{\gamma}$ is a straight line.

Figure 5.8: Generating curve $\tilde{\gamma}$ for $\alpha = 1$ (left) and for $\alpha = -2$ (right).

Remark 5.7. Observe that from (5.16), (5.19) and (5.22), the Gauss curvature of $\tilde{\psi}$ is given by
\[
K = (\alpha + 1) \frac{k^2}{u^{2\alpha+4}},
\]
and so, see Figure 5.9

Theorem 5.8. Let $\tilde{\psi}$ be a spacelike surface $\tilde{\psi}$ a singular $\alpha$-maximal spacelike graph, parametrized as in (5.16) with $\tilde{\gamma} = (x, 0, u)$ as in Proposition 5.6, then

• if $1 + \alpha > 0$, the Gauss curvature of $\tilde{\psi}$ is positive (that is, the second fundamental form is semidefinite and nondegenerate). Moreover, \[
\lim_{x \to \pm \Lambda_\alpha} K = \infty.
\]

• if $1 + \alpha < 0$, $\tilde{\psi}$ is an entire graph, strictly convex with a flat behaviour at infinity, that is, \[
\lim_{x \to \pm \infty} K = 0.
\]

(These examples will be called singular $\alpha$-maximal bowls of hyperbolic type.)

• If $\alpha = -1$, $\tilde{\psi}$ is flat.

Remark 5.9. Observe that, from (5.22), the divergent generating curve $\tilde{\gamma}$ of a $\alpha$-maximal bowl of hyperbolic type, has infinite length if and only if
\[
\int_0^\infty u^{\alpha+1} dx = \infty
\]
which, because $u$ has a linear growth, only holds when $\alpha + 2 \geq 0$. So, a singular $\alpha$-maximal bowl of hyperbolic type is complete if and only if $-1 > \alpha \geq -2$. 
5.4 Rotational singular $\alpha$-maximal spacelike surfaces of hyperbolic type in a Calabi-pair

In this section we study the Calabi-pairs $(\psi, \tilde{\psi})$ with $\tilde{\psi}$ a rotational singular $\alpha$-maximal spacelike surfaces of hyperbolic type.

Let $\tilde{\psi}$ be the rotational singular $\alpha$-maximal spacelike surface of hyperbolic type given by (5.16) with Gauss map (5.18). If $(\psi, \tilde{\psi})$ is a Calabi-pair, then from Theorem 4.4 and (4.16), $\psi$ is either a singular $\alpha$-$\alpha + 1$-minimal surface if $\alpha + 1 \neq 0$ or a translating soliton if $\alpha + 1 = 0$ in $\mathbb{R}^3$ which can be parametrized as

\[ \psi(s, t) = \int u(s)^\alpha \cosh^{\alpha + 1} t \left( \vec{e}_3 \wedge \vec{N} \wedge \vec{L}_3 (d\tilde{\psi} \wedge \vec{L}_3 \vec{N}) - \ll d\tilde{\psi}, \vec{e}_3 \gg \vec{e}_3 \right). \]

**Case I: If $\psi$ is a singular $\frac{\alpha}{\alpha + 1}$-minimal surfaces**

Then, from Theorem 4.4, Proposition 5.6, Theorem 5.8 and Remark 5.7, we have

**Theorem 5.10.** Let $(\psi, \tilde{\psi})$ be a Calabi-pair such that $\tilde{\psi}$ is the rotational singular $\alpha$-maximal, $\alpha + 1 \neq 0$, spacelike surface of hyperbolic type given by (5.16). Then $\psi$ is

- either a singular $\frac{\alpha}{\alpha + 1}$-minimal $\psi$ in $\mathbb{R}^3$ whose Gauss curvature is strictly negative if $1 + \alpha > 0$.
- or a singular strictly convex $\frac{\alpha}{\alpha + 1}$-minimal $\psi$ in $\mathbb{R}^3$ if $1 + \alpha < 0$.

In both cases a parametrization of $\psi$ is given by,

\[ \psi(x, t) = \left( \frac{k}{k + 1} u'(x) \cosh^{\alpha + 1}(t), k \int \cosh^\alpha (t) dt, \frac{u^\alpha(x) \cosh^{\alpha + 1}(t)}{\alpha + 1} \right), \]

where $k = u_0^{\alpha + 1}$ and $u(x)$ is the solution of (5.24)-(5.26) satisfying the properties describe in Proposition 5.6, see Figure 5.10.
Figure 5.10: singular $\frac{\alpha}{\alpha + 1}$-minimal surfaces for $\alpha = 1$ and for $\alpha = -2$

**CASE II: IF $\psi$ IS A TRANSLATING SOLITONS ($\alpha = -1$)**

From the Proposition 5.6, $\tilde{\psi}$ could be parametrize (up to translation) by,

$$\tilde{\psi}(x, t) = (x, (\tanh(\zeta_0) x + u_0) \sinh(t), (\tanh(\zeta_0) x + u_0) \cosh(t)),$$

for some $\zeta_0, x_0 \in \mathbb{R}$, where $t \in \mathbb{R}$ and $\tanh(\zeta_0) x + u_0 > 0$.

Consequently, from (5.28) and (5.29) we have

**Theorem 5.11.** Let $(\psi, \tilde{\psi})$ be a Calabi-pair such that $\tilde{\psi}$ is the revolution singular $(-1)$-maximal spacelike surface given by (5.29). Then, $\psi$ is either the titled Grim-Reaper parametrized as

$$\psi(y, t) = \left(\frac{y}{\lambda} - \lambda \log(\cosh(t)), 2 \sqrt{1 + \lambda^2} \arctan(\tanh(t/2)), y + \cosh(t))\right),$$

if $\lambda = \sinh(\zeta_0) \neq 0$, or to the Grim-Reaper parametrized as,

$$\psi(y, t) = \left(-\frac{y}{u_0}, 2 \arctan(\tanh(t/2)), \log(u_0 \cosh(t))\right),$$

if $\lambda = 0$ (see Figure 5.11).

6 Concluding remarks

- Formulas (4.11) and (4.16) are the source of most the results in this paper and doubtless they will have other applications. For example, by using the classification of flat translating solitons in $\mathbb{R}^3$ given in [5] and (4.11) one can give a complete description of flat singular $(-1)$-maximal spacelike
graphs in $L^3$. In this sense it is interesting to classify complete flat $[\varphi, \vec{e}_3]$-minimal graphs in $\mathbb{R}^3$ at least when $\varphi$ is an strictly increasing (decreasing) diffeomorphism and apply our correspondence to give the corresponding classification result in $L^3$. 

- The existence of $\alpha$-maximal bowls of elliptic and hyperbolic type, motivates the interest for obtaining Bernstein type results for this kind of surfaces.

- More generally, it would be interesting to know if the examples of winglike surfaces described in Theorem 5.4 are the only ones either translating solitons or singular $\alpha$-maximal surfaces which are graphs on the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$.

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