Graphs of Edge-Intersecting and Non-Splitting One Bend Paths in a Grid

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The families EPT (resp. EPG) Edge Intersection Graphs of Paths in a tree (resp. in a grid) are well studied graph classes. Recently we introduced the graph classes Edge-Intersecting and Non-Splitting Paths in a Tree (ENPT), and in a Grid (ENPG). It was shown that ENPG contains an infinite hierarchy of subclasses that are obtained by restricting the number of bends in the paths. Motivated by this result, in this work we focus on one bend ENPG graphs. We show that one bend ENPG graphs are properly included in two bend ENPG graphs. We also show that trees and cycles are one bend ENPG graphs, and characterize the split graphs and co-bipartite graphs that are one bend ENPG. We prove that the recognition problem of one bend ENPG split graphs is NP-complete even in a very restricted subfamily of split graphs. Last we provide a linear time recognition algorithm for one bend ENPG co-bipartite graphs.

Keywords: Intersection Graphs, Path Graphs, EPT Graphs, EPG Graphs

1 Introduction

1.1 Background

Given a host graph $H$ and a set $\mathcal{P}$ of paths in $H$, the Edge Intersection Graph of Paths (EP graph) of $\mathcal{P}$ is denoted by $\text{Ep}(\mathcal{P})$. The graph $\text{Ep}(\mathcal{P})$ has a vertex for each path in $\mathcal{P}$, and two vertices of $\text{Ep}(\mathcal{P})$ are adjacent if the corresponding two paths intersect in at least one edge. A graph $G$ is EP if there exist a graph $H$ and a set $\mathcal{P}$ of paths in $H$ such that $G = \text{Ep}(\mathcal{P})$. In this case, we say that $\langle H, \mathcal{P} \rangle$ is an EP representation of $G$. We also denote by EP the family of all graphs $G$ that are EP.

The main application area of EP graphs is communication networks. Messages to be delivered are sent through routes of a communication network. Whenever two paths use the same link on the communication network, we say that they conflict. Noting that this conflict model is equivalent to an EP graph, several optimization problems in communication networks (such as message scheduling) can be seen as graph problems (such as vertex coloring) in the corresponding EP graph.

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In many applications it turns out that the host graphs are restricted to certain families such as paths, cycles, trees, grids, etc. Several known graph classes are obtained with such restrictions: when the host graph is restricted to paths, cycles, trees, and grids, we obtain interval graphs, circular-arc graphs, Edge Intersection Graph of Paths in a Tree (EPT) (see Golumbic and Jamison (1985a)), and Edge Intersection Graph of Paths in a Grid (EPG) (see Golumbic et al. (2009)), respectively.

Given a representation \( \langle T, P \rangle \) where \( T \) is a tree and \( P \) is a set of paths of \( T \), the graph of edge intersecting and non-splitting paths of \( \langle T, P \rangle \) (denoted by \( \text{ENPT}(P) \)) is defined as follows in Boyacı et al. (2015a): \( \text{ENPT}(P) \) has a vertex \( v \) for each path \( P_v \) of \( P \) and two vertices \( u, v \) of this graph are adjacent if the paths \( P_u \) and \( P_v \) edge-intersect and do not split (that is, their union is a path). We note that \( \text{ENPT}(P) \) is a subgraph of \( \text{EPT}(P) \). The motivation to study these graphs arises from all-optical Wavelength Division Multiplexing (WDM) networks in which two streams of signals can be transmitted using the same wavelength only if the paths corresponding to these streams do not split from each other (see Boyacı et al. (2015a) for a more detailed discussion). A graph \( G \) is an ENPT graph if there is a tree \( T \) and a set of paths \( P \) of \( T \) such that \( G = \text{ENPT}(P) \). Clearly, when \( T \) is a path, \( \text{EPT}(P) = \text{ENPT}(P) \) and this graph is an interval graph. Therefore, interval graphs are included in the class ENPT. In Boyacı et al. (2015b) we obtain the so-called ENP graphs by extending this definition to the case where the host graph is not necessarily a tree. In the same work, it has been shown that \( \text{ENP} = \text{ENPG} \) where \( \text{ENPG} \) is the family of ENP graphs where the host graphs are restricted to grids. Whenever the host graph is a grid, it is common to use the following notion: a bend of a path on a grid is an internal point in which the path changes direction. An ENPG graph is \( B_k\text{-ENPG} \) if it has a representation in which every path has at most \( k \) bends.

1.2 Related Work

While ENPT and ENPG graphs have been recently introduced, EPT and EPG graphs are well studied in the literature. The recognition of EPT graphs is NP-complete (Golumbic and Jamison (1985b)), whereas one can solve in polynomial time the maximum clique (Golumbic and Jamison (1985b)) and the maximum stable set (Tarjan (1985)) problems in this class.

Several recent papers consider the edge intersection graphs of paths on a grid. Since all graphs are EPG (see Golumbic et al. (2009)), most of the studies focus on the sub-classes of EPG obtained by limiting the number of bends in each path. An EPG graph is \( B_k\text{-EPG} \) if it admits a representation in which every path has at most \( k \) bends. The work of Biedl and Stern (2010) investigates the minimum number \( k \) such that \( G \) has a \( B_k\text{-EPG} \) representation for some special graph classes. The work of Golumbic et al. (2009) studies the \( B_1\text{-EPG} \) graphs. In particular it is shown that every tree is \( B_1\text{-EPG} \), and a characterization of \( C_4 \) representations is given. In Biedl and Stern (2010) the existence of an outer-planar graph which is not \( B_1\text{-EPG} \) is shown. The recognition problem of \( B_1\text{-EPG} \) graphs is shown to be NP-complete in Heldt et al. (2014). Similarly, in the class of \( B_1\text{-EPG} \), the minimum coloring and the maximum stable set problems are NP-complete (Epstein et al. (2013)), however one can solve in polynomial time the maximum clique problem (Epstein et al. (2013)). Asinowski and Ries (2012) give a characterization of graphs that are both \( B_1\text{-EPG} \) and belong to some subclasses of chordal graphs. Recently, Cameron et al. (2016) consider subclasses of \( B_1\text{-EPG} \) obtained by restricting the representations to contain only certain subsets of the four possible single bend rectilinear paths. It is shown that for each possible non-empty subset of these four shapes, the recognition of the corresponding subclass of \( B_1\text{-EPG} \) is an NP-complete problem.

In Boyacı et al. (2015a) we defined the family of ENPT graphs and investigated the representations
of induced cycles. These representations turn out to be much more complex than their counterpart in the EPT graphs (discussed in Golumbic and Jamison [1985a]). In Boyacı et al. [2015b] we extended this definition to the general case in which the host graph is not necessarily a tree. We showed that the family of ENP graphs coincides with the family of ENPG graphs, and that unlike EPG graphs, not every graph is ENPG. We also showed that, in a way similar to the family of EPG graphs, the sub families $B_k$-ENPG of ENPG contains an infinite subset totally ordered by proper inclusion.

1.3 Our Contribution

In this work, we consider $B_1$-ENPG graphs. In Section 2 we present definitions and preliminary results among which we show that cycles and trees are $B_1$-ENPG graphs. In Section 3 we show that the $B_1$-ENPG recognition problem is NP-complete even for a very restricted subfamily of split graphs, i.e. graphs whose vertex sets can be partitioned into a clique and an independent set. In Section 4 we show results among which we show that cycles and trees are $B_1$-ENPG graphs can be recognized in polynomial time within the family of co-bipartite graphs. As a byproduct, we also show that, unlike $B_k$-EPG graphs, $B_k$-ENPG graphs do not necessarily admit a representation where every path has exactly $k$ bends. We summarize and point to further research directions in Section 5.

2 Preliminaries

Given a simple graph (no loops or parallel edges) $G = (V(G), E(G))$ and a vertex $v$ of $G$, we denote by $N_G(v)$ the set of neighbors of $v$ in $G$, and by $d_G(v) = |N_G(v)|$ the degree of $v$ in $G$. A graph is called $d$-regular if every vertex $v$ has $d(v) = d$. Whenever there is no ambiguity we omit the subscript $G$ and write $d(v)$ and $N(v)$. Given a graph $G$ and $U \subseteq V(G)$, $N_U(v) \overset{def}{=} N_G(v) \cap U$. Two adjacent (resp. non-adjacent) vertices $u, v$ of $G$ are twins (resp. false twins) if $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$. For a graph $G$ and $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of $G$ induced by $U$.

A vertex set $U \subseteq V(G)$ is a clique (resp. stable set) (of $G$) if every pair of vertices in $U$ is adjacent (resp. non-adjacent). A graph $G$ is a split graph if $V(G)$ can be partitioned into a clique and a stable set. A graph $G$ is co-bipartite if $V(G)$ can be partitioned into two cliques. Note that these partitions are not necessarily unique. We denote bipartite, co-bipartite and split graphs as $X(V_1, V_2, E)$ where

a) $X = B$ (resp. $C,S$) whenever $G$ is bipartite (resp. co-bipartite, split),

b) $V_1 \cap V_2 = \emptyset, V_1 \cup V_2 = V(G),$

c) for bipartite graphs $V_1, V_2$ are stable sets,

d) for co-bipartite graphs $V_1$ and $V_2$ are cliques,

e) for split graphs $V_1$ is a clique and $V_2$ is a stable set, and

f) $E \subseteq V_1 \times V_2$ (in other words $E$ does not contain the cliques’ edges).

Unless otherwise stated we assume that $G$ is connected and both $V_1$ and $V_2$ are non-empty.

In this work every single path is simple, i.e. without duplicate vertices. However, if a union of paths is a path, the resulting path is not necessarily simple. For example, consider a graph on 5 vertices $v_1, v_2, v_3, v_4, v_5$ and 5 edges $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4, e_4 = v_4v_2, e_5 = v_2v_5$. Each of the
paths $P_1 = \{e_1, e_2, e_3\}$ and $P_2 = \{e_3, e_4, e_5\}$ is simple. On the other hand, though $P_1 \cup P_2$ is a path, it is not simple. Whenever $v$ is an internal vertex of a path $P$, we sometimes say that $P$ crosses $v$. Given two paths $P, P'$, a split of $P, P'$ is a vertex with degree at least 3 in $P \cup P'$. We denote by $\text{split}(P, P')$ the set of all splits of $P$ and $P'$. When $\text{split}(P, P') \neq \emptyset$ we say that $P$ and $P'$ are splitting. Whenever $P$ and $P'$ edge intersect and $\text{split}(P, P') = \emptyset$ we say that $P$ and $P'$ are non-splitting and denote this by $P \sim P'$. Clearly, for any two paths $P$ and $P'$ exactly one of the following holds:

i) $P$ and $P'$ are edge disjoint,

ii) $P$ and $P'$ are splitting,

iii) $P \sim P'$.

A two-dimensional grid graph, also known as a square grid graph, is an $m \times n$ lattice graph that is the Cartesian product graph of two paths $P$ and $P'$ of respectively length $m$ and $n$. Such a grid has vertex set $V = [n] \times [m]$. A bend of a path $P$ in a grid $H$ is an internal vertex of $P$ whose incident edges (in the path) have different directions, i.e. one vertical and one horizontal.

Let $\mathcal{P}$ be a set of paths in a graph $H$. The graphs $\text{E}(\mathcal{P})$ and $\text{EN}(\mathcal{P})$ are such that $V(\text{EN}(\mathcal{P})) = V(\text{E}(\mathcal{P})) = V$, and there is a one-to-one correspondence between $\mathcal{P}$ and $V$, i.e. $\mathcal{P} = \{P_v : v \in V\}$. Given two paths $P_u, P_v \in \mathcal{P}, \{u, v\}$ is an edge of $\text{E}(\mathcal{P})$ if and only if $P_u$ and $P_v$ have a common edge (cases (ii) and (iii)), whereas $\{u, v\}$ is an edge of $\text{EN}(\mathcal{P})$ if and only if $P_u \sim P_v$ (case (iii)). Clearly, $E(\text{EN}(\mathcal{P})) \subseteq E(\text{E}(\mathcal{P}))$. A graph $G$ is EN if there is a graph $H$ and a set of paths $\mathcal{P}$ of $H$ such that $G = \text{EN}(\mathcal{P})$. In this case $\langle H, \mathcal{P} \rangle$ is an EN representation of $G$. When $H$ is a tree (resp. grid) $\text{E}(\mathcal{P})$ is an EPT (resp. EPG) graph, and $\text{EN}(\mathcal{P})$ is an ENPT (resp. ENPG) graph; these graphs are denoted also as $\text{E}(\mathcal{P}), \text{E}(\mathcal{P}), \text{EN}(\mathcal{P})$ and $\text{EN}(\mathcal{P})$, respectively. We say that two representations are equivalent if they are representations of the same graph.

Let $\langle H, \mathcal{P} \rangle$ be a representation of an EN graph $G$. For each edge $e$ of $H$, $\mathcal{P}_e$ denotes the set of the paths of $\mathcal{P}$ containing the edge $e$, i.e. $\mathcal{P}_e \overset{\text{def}}{=} \{P \in \mathcal{P} : e \in P\}$. For a subset $U \subseteq V(G)$ we define $\mathcal{P}_U \overset{\text{def}}{=} \{P_v \in \mathcal{P} : v \in U\}$. Following standard notations, $\cup \mathcal{P}_U \overset{\text{def}}{=} \cup_{P \in \mathcal{P}_U} P$.

Given two paths $P$ and $P'$ of a graph, a segment of $P \cap P'$ is a maximal path that constitutes a sub-path of both $P$ and $P'$. Clearly, $P \cap P'$ is the union of edge disjoint segments. We denote the set of these segments by $S(P, P')$.

Throughout the paper, whenever a representation $\langle H, \mathcal{P} \rangle$ of an ENPG graph is given, we assume the host graph $H$ is a grid on sufficiently many vertices each of which is denoted by an ordered pair of integers.

The following Proposition that is proven in [Boyaci et al. (2015b)] is the starting point of many of our results.

**Proposition 2.1** [Boyaci et al. (2015b)] Let $K$ be a clique of a $B_1$-ENPG graph $G$ with a representation $\langle H, \mathcal{P} \rangle$. Then $\cup \mathcal{P}_K$ is a path with at most 2 bends. Moreover, there is an edge $e_K \in E(H)$ such that every path of $\mathcal{P}_K$ contains $e_K$.

Note that whenever $\cup \mathcal{P}_K$ has two bends, $e_K$ lies between these two bends. Based on the above proposition, given two cliques $K, K'$ of a $B_1$-ENPG graph we denote $S(K, K') \overset{\text{def}}{=} S(\cup \mathcal{P}_K, \cup \mathcal{P}_{K'})$.

By the following two observations, in the sequel we focus on connected twin-free graphs.

**Observation 2.1** Let $G$ be a graph and $G'$ obtained from $G$ by removing a twin vertex until no twins remain. Then, $G$ is $B_1$-ENPG if and only if $G'$ is $B_1$-ENPG.
Observation 2.2 A graph $G$ is $B_k$-ENPG if and only if every connected component of $G$ is $B_k$-ENPG.

We first observe that some well-known graph classes are included in $B_1$-ENPG.

Proposition 2.2
i) Every cycle is $B_1$-ENPG.
ii) Every tree is $B_1$-ENPG, and it has a representation $(H, P)$ where $\cup P$ is a tree.

Proof:
i) For $k = 3$ three identical paths consisting of one edge constitutes a $B_1$-ENPG representation of $C_3$.

For $k = 4$ Figure 1(a) depicts a $B_1$-ENPG representation of $C_4$. Finally for any $k > 4$, we can construct a $C_k$ as shown in Figure 1(b) for the case $k = 11$.

ii) Given a representation $(H, P)$ of a $B_1$-ENPG graph $G$ and $U \subseteq V(G)$, we denote by $R_U$ the bounding rectangle of $P_U$. Let $T$ be a tree with a root $r$. We prove the following claim by induction on the structure of $T$ (see Figure 2). The tree $T$ has a $B_1$-ENPG representation $(H, P)$ in which the corners of the bounding rectangle $R_T$ can be renamed as $a_T, b_T, c_T, d_T$ in counterclockwise order such that i) every path of $P$ has exactly one bend, ii) $b_T$ is a bend of $P_r$, iii) $a_T$ is an endpoint of $P_r$, iv) the line between $a_T$ and $d_T$ is used exclusively by $P_r$, v) $\cup P$ is a tree.

If $T$ is an isolated vertex, any path with one bend is a representation of $T$. Moreover, it is easy to verify that it satisfies conditions i) through v).

Otherwise let $T_1, \ldots, T_k$ be the subtrees of $T$ obtained by the removal of $r$, with roots $r_1, \ldots, r_k$ respectively. By the inductive hypothesis every such subtree $T_i$ has a representation with bounding box $a_{T_i}, b_{T_i}, c_{T_i}, d_{T_i}$ satisfying conditions i) through iv). We now build a representation of $T$ satisfying the same conditions. We shift and rotate the representations of $T_1, \ldots, T_k$ so that the bounding rectangles do not intersect and the vertices $a_{T_1}, b_{T_1}, a_{T_2}, b_{T_2}, \ldots, a_{T_k}, b_{T_k}$ are on the same horizontal line and in this order (See Figure 2). We extend the paths $P_{r_2}, \ldots, P_{r_k}$ representing the roots of the trees $T_2, \ldots, T_k$ such that the endpoint $a_{T_i}$ of $P_r$ is moved to $a_{T_1}$.

Since $a_{T_1}$ is used exclusively by $P_r$, this modification does not cause $P_r$ to split from a path of $P_{V(T_1)}$. Therefore, the individual trees $T_1, \ldots, T_K$ are properly represented. Clearly, if two paths from different subtrees $T_i, T_j$ ($i < j$) intersect, then one of the intersecting paths must be $P_{r_j}$. The path $P_{r_j}$ intersects the bounding rectangle of $T_i$ only at the path between $a_i$ and $b_i$. As every path of $P_{V(T_i)}$, in particular one intersecting $P_{r_j}$ has one bend, such a path splits from $P_{r_j}$. Therefore,
for any pair of vertices \((v_i, v_j) \in T_i \times T_j\) we have that \(v_i\) and \(v_j\) are non-adjacent in ENPG(\(P\)), as required.

We rename the corners of the bounding rectangle \(R_T\) such that \(b_T = a_{T_1}\). We now add the path \(P_r\) from \(b_{T_1}\) to \(a_T\) with a bend at \(b_T\). The conditions i), ii), iii) are satisfied. We extend \(P_r\) by one edge at \(a_T\) to make sure that the line between \(a_T\) and \(d_T\) is exclusively used by \(P_r\), thus satisfying condition iv). The extension of the paths \(P_{r_2}, \ldots, P_{r_k}\) does not add new edges to regions bounded by \(a_{T_1}, b_{T_1}, c_{T_1}, d_{T_1}\) and they don’t introduce cycles between this regions. Moreover, since the line between \(a_{T_1}\) and \(d_{T_1}\) is used only at \(a_{T_1}\) the path \(P_r\) does not introduce any cycles either. Therefore, \(\cup P\) is a tree, i.e. condition v) is satisfied.

The path \(P_r\) intersects only \(R_{T_1}\). This intersection is the path between \(b_{T_1}\) and \(d_{T_1}\), bending at \(a_{T_1}\). Every path that intersects \(P_r\) and does not split from it must bend at \(a_{T_1}\). As \(a_{T_1}\) is used exclusively by \(P_{r_1}\), \(P_{r_1}\) is the only path that possibly satisfies \(P_{r_1} \sim P_r\). We now observe that \(P_{r_i} \sim P_r\) for every \(i \in [k]\). Therefore \(r\) is adjacent to the root of \(T_j\) in ENPG(\(P\)), as required.

\[\square\]

3 Split Graphs

In this section, we present a characterization theorem (Theorem 3.1) for \(B_1\)-ENPG split graphs. In Sections 3.1 and 3.2 we proceed with some properties of these graphs implied by this theorem. An interesting implication of one of these properties is that the family of \(B_1\)-ENPG is properly included in the family of \(B_2\)-ENPG graphs. Finally, using Theorem 3.1 we prove in Section 3.3 that the recognition problem of \(B_1\)-ENPG graphs is NP-complete even in a very restricted subfamily of split graphs. Throughout this section, \(G\) is a split graph \(S(K, S, E)\) unless indicated otherwise. We assume without loss of generality that \(K\) is maximal, i.e. that no vertex in \(S\) is adjacent to all vertices of \(K\), and \(G\) is connected (in particular \(S\) does not contain isolated vertices).

3.1 Characterization of \(B_1\)-ENPG Split Graphs

Consult Figure 3 for the following discussion.
Fig. 3: Regions of $\cup P_K$ and possible representations of a vertex in $S$ for a $B_1$-ENPG split graph $S(K, S, E)$. 

Let $G$ be a $B_1$-ENPG split graph $S(K, S, E)$ with a representation $\langle H, P \rangle$. By Proposition 2.1, we know that $\cup P_K$ is a path with at most two bends, such that there is an edge $e_K$ contained in every path of $P_K$. Moreover, if $\cup P_K$ contains two bends then $e_K$ is between the two bends. Assume without loss of generality that $e_K$ is a horizontal edge. Therefore, $\cup P_K$ consists of a horizontal segment $L_H$ between two vertices $b_L, b_R$ and two vertical segments (each of which is possibly empty). The subgrid $L_H \setminus e_K$ consists of two horizontal subsegments $L_{L,H}, L_{R,H}$. Finally, $\cup P_K \setminus L_H$ consists of two vertical segments $L_{L,V}$ and $L_{R,V}$. The segment $L_{L,Y}$ (resp. $L_{R,Y}$) is on the left (resp. right) of $e_K$ for every $Y \in \{H, V\}$.

For $(X, Y) \in \{L, R\} \times \{H, V\}$, let $K_{X,Y}$ be the set of vertices $v$ of $K$ such that $P_v$ has an endpoint in $L_{X,Y}$. Every path of $P_K$ has its left (resp. right) endpoint on $L_{L,H} \cup L_{L,V}$ (resp. $L_{R,H} \cup L_{R,V}$) since it contains $e_K$. Therefore, $\{K_{X,H}, K_{X,V}\}$ is a partition of $K$ for every $X \in \{L, R\}$. For every $(X, Y) \in \{L, R\} \times \{H, V\}$, let $\sigma_{X,Y}$ be the permutation of $K_{X,Y}$ obtained by ordering the endpoints of $P_K$ in $L_{X,Y}$ in increasing distance from $e_K$. Moreover, $K_{L,V} \cap K_{R,V} = \emptyset$ since otherwise this implies a path containing both $b_L$ and $b_R$ as bends.

The following theorem characterizes the $B_1$-ENPG split graphs. If further provides a canonical representation for them using the above mentioned partitions and by partitioning the vertices of $S$ according their neighborhoods.

**Theorem 3.1** A connected split graph $G = S(K, S, E)$ is $B_1$-ENPG if and only if there are two partitions $\{K_{L,H}, K_{L,V}\}, \{K_{R,H}, K_{R,V}\}$ of $K$ such that $K_{L,V} \cap K_{R,V} = \emptyset$, there is a permutation $\sigma_{X,Y}$ of $K_{X,Y}$ for every $(X, Y) \in \{L, R\} \times \{H, V\}$, and a partition $S = \{S_{X,H}, S_{X,V}, S_{X,H,V}, S_{X,H,V} \mid X \in \{L, R\}\}$ of $S$ such that the following hold.

i) If $s \in S_{X,Y}$ then $N(s)$ is an interval $\sigma_s$ of $\sigma_{X,Y}$.

ii) If $s \in S_{X,H}$ then $N(s)$ consists of the intersection of a prefix $\sigma_s$ of $\sigma_{X,H}$ with $K_{X,H}$ where $X = \{L, R\} \setminus X$.

iii) If $s \in S_{X,V}$ then $N(s)$ is the union of a suffix $\sigma_s$ of $\sigma_{X,H}$ with $K_{X,V}$.
iv) If \( s \in S_{X,H} \cup S_{X,H} \) then there is at most one \( s' \in S_{X,H} \) such that the interval \( \sigma_s \) of \( \sigma_{X,H} \) and the suffix \( \sigma_{s'} \) of \( \sigma_{X,H} \) overlap.

v) If \( S_{X,H} \neq \emptyset \) then \(|S_{X,H}| \leq 1\) where \( X = \{L,R\} \setminus S_{X,H} \).

vi) \( K_{X,V} \subseteq K_{X,H} \) where \( X = \{L,R\} \setminus S_{X,H} \).

**Proof:** (\( \Rightarrow \)) We fix a B1-ENPG representation of \( G \) and consider the sets \( K_{X,Y} \) and their permutations \( \sigma_{X,Y} \) defined by this representation.

For each vertex \( s \in S \) we will determine its membership to one of the sets of the partition \( \mathcal{S} \) depending on its representation \( P_s \). Suppose that there exists a vertex \( s \in S \) such that \(|\mathcal{S}(P_s, \cup \mathcal{P}_K)| > 1\). Then \( P_s \cup \cup \mathcal{P}_K \) contains a cycle, therefore at least 4 bends. But \( P_s \) has at most one bend and \( \cup \mathcal{P}_K \) has at most two bends, a contradiction. Therefore, \( \mathcal{S}(P_s, \cup \mathcal{P}_K) \) consists of one segment \( Q_s \).

Given a vertex \( s \in S \), we consider two disjoint and complementary cases for \( P_s \).

1. \( e_K \notin Q_s \): Let \( c_s \) (resp. \( f_s \)) be the vertex of \( Q_s \) closer to (resp. farther from) \( e_K \). Assume without loss of generality that \( Q_s \subseteq L_L \) where \( L_L = L_{L,H} \cup L_{L,V} \) (we define similarly \( L_R = L_{R,H} \cup L_{R,V} \)). We observe that \( P_s \) does not split from \( L_L \) at \( c_s \), since otherwise \( P_s \) splits from every path of \( \mathcal{P}_K \) that it intersects, implying that \( s \) is an isolated vertex. We further consider three subcases:

   a. \( Q_s \subseteq L_{L,V} \): If \( P_s \) splits from \( L_L \) then \( N(s) \) consists of the vertices \( v \) of \( K \) such that the left endpoint of \( P_v \) is between \( c_s \) and \( f_s \). Therefore, \( N(s) \) is an interval of \( \sigma_{L,V} \). If \( P_s \) does not split from \( L_L \) then \( N(s) \) consists of the vertices \( v \) of \( K \) such that the left endpoint of \( P_v \) is farther than \( c_s \) on \( L_{L,V} \) (with respect to \( b_L \)). Therefore, \( N(s) \) is an interval of \( \sigma_{L,V} \). In both cases we set \( s \in S_{L,V} \).

   b. \( b_l \) is an internal vertex of \( Q_s \): In this case \( P_s \) does not split from \( L_L \). Then \( N(s) \) consists of the vertices \( v \) of \( K \) such that the left endpoint of \( P_v \) is on the left of \( c_s \) on \( L_L \). Therefore, \( N(s) \) consists of the union of a suffix of \( \sigma_{L,H} \) with \( K_{L,V} \), and we set \( s \in S_{L,H} \).

   c. \( Q_s \subseteq L_{L,H} \):

   i. \( P_s \) splits from \( L_L \). In this case \( N(s) \) consists of the vertices \( v \) of \( K \) such that the left endpoint of \( P_v \) is between \( c_s \) and \( f_s \). Therefore, \( N(s) \) is an interval of \( \sigma_{L,H} \) and we set \( s \in S_{L,H} \).

   ii. \( P_s \) does not split from \( L_L \). In this case \( N(s) \) consists of the vertices \( v \) of \( K \) such that the left endpoint of \( P_v \) is on the left of \( c_s \) on \( L_L \). Therefore, \( N(s) \) consists of the union of a suffix of \( \sigma_{L,H} \) with \( K_{L,V} \) and we set \( s \in S_{L,H} \).

2. \( e_K \in Q_s \): In this case, let \( l_s, r_s \) be the endpoints of \( Q_s \) on \( L_L \) and \( L_R \) respectively. The path \( P_s \) splits from \( \cup \mathcal{P}_K \), since otherwise \( s \) is adjacent to every vertex of the clique, contradicting the maximality of \( K \). Since \( P_s \) intersects every path of \( \mathcal{P}_K \), \( N(s) \) consists of the vertices \( v \) of \( K \) such that \( P_v \) does not split from \( P_s \). We consider two subcases:

   a. \( P_s \) splits from exactly one of \( L_L \) and \( L_R \): Assume without loss of generality that \( P_s \) splits from \( L_L \) but not from \( L_R \). We observe that \( l_s \in L_{L,H} \). In this case \( N(s) \) is the set of vertices \( v \) of \( K \) such that the left endpoint of \( P_v \) is closer than \( l_s \) to \( e_K \) which corresponds to a prefix of \( \sigma_{L,H} \). In this case we set \( s \in S_{L,H} \).
every path overlap. The other endpoint is chosen as the endpoint closest to the origin that is farther from the origin \( \sigma \) according to the suffix \( \sigma \) paths with distinct bends on the endpoint of \( \{ l_s, r_s \} \). We first determine the representations of the vertices of \( \{ l_s, r_s \} \) in increasing order of their distances from the origin. Every vertex \( b \) at the corresponding to \( s \) is represented by an ordered pair from \([0, 2|S||K|] \) respectively. By the above case analysis we know that \( P_s \) is either of type 2 or of type 1.c.i, and that \( P_s \) and \( P_s' \) are of one of the types 1.b, 1.c.ii. Note that \( \sigma_{s'} \) and \( \sigma_{s''} \) are determined by the right endpoints of the corresponding paths \( P_{s'} \) and \( P_{s''} \). Since \( \sigma_{s'} \) and \( \sigma_{s''} \), overlap, \( Q_s \) contains the right endpoint of \( P_{s'} \). Therefore these two paths intersect. Moreover \( P_{s'} \) contains the left endpoint of \( Q_s \) (which is the bend point of \( P_s \)) otherwise \( P_s \sim P_{s'} \). By the same arguments \( P_{s''} \) also contains the left endpoint of \( Q_s \) and therefore \( P_s \) intersects \( P_{s''} \). Moreover, since none of them splits from \( L_L \) we have \( P_{s'} \sim P_{s''} \), i.e. \( s \) and \( s' \) are adjacent in \( G \), a contradiction.

Assume, for a contradiction that for some \( X \in \{ L, R \} \), say \( L \), the condition does not hold, i.e. there exists \( s \in S \) such that \( s \) overlaps \( s' \) and \( s'' \). Then both \( P_{s'} \) and \( P_{s''} \) are of one of the types 1.b, 1.c.ii. Moreover, \( P_{s'} \) is of type 2.b with \( l_s = b_s \). We proceed as in the previous case to get a contradiction.

This immediately follow from the already observed fact that \( K_{L,V} \cap K_{R,V} = \emptyset \).

(\( \Leftarrow \)) Suppose that the partitions and the permutations stated in the claim exist. We construct a representation \((H, P)\) as follows (see Figure 5). The host graph \( H \) is a \((2 + 4|S|(|K| + 1))/2|S| \times |K| \) vertices grid, where each vertex is represented by an ordered pair from \([-1 - 2|S| \times |K| + 1, 1 + 2|S| \times |K| + 1] \times [0, 2|S||K|] \). For \((X, Y) \in \{ L, R \} \times \{ H, V \} \), let \( k_{X,Y} = |K_{X,Y}| \). The coordinates of \( b_L \) and \( b_R \) are respectively \((-1 - 2|S| \times (k_L + 1), 0) \) and \((1 + 2|S| \times (k_R + 1), 0) \). The horizontal line between \( b_L \) and \( b_R \) is called \( L_H \). For \((X, Y) \in \{ L, R \} \), \( L_{X,Y} \) is a vertical line of length \( 2|S| \) starting at \( b_X \). We choose \( k_{X,Y} \) vertices on each line \( L_{X,Y} \) such that their distances from each other and from each of \( b_L, b_R, (1, 0) \) are at least \( 2|S| \). We label these vertices as \( W_{X,k_{X,Y}(1)}, ..., W_{X,k_{X,Y}(k_{X,Y})} \) in increasing order of their distances from the origin. Every vertex \( \nu \in \nu \) is represented by a path \( \nu \subseteq \cup L_H \cup \cup L_{X,Y} \) between \( w_{X,\nu} \) and \( w_{Y,\nu} \). Since \( K_{X,Y} \subseteq \bar{K}_{X,H} \) every such path has at most one bend. Since \( e_{K} = (-1, 0) \) is contained in every such path, these paths constitute a proper representation of the clique \( K \).

We proceed with the representation of the vertices of \( S \). Let \( W_X = \{ w_{X,1}, ..., w_{X,|K|} \} \). The endpoints of paths \( Q_s, s \in S \) will be chosen between two vertices of \( W \cup \bar{B}_X \) so that they are all distinct. We first determine the representations of the vertices of \( S \setminus \{ S_{L,H} \cup S_{R,H} \} \) such that all are one bend paths with distinct bends on the endpoint of \( Q_s \) farther from the origin. We determine the endpoints of \( Q_s \) according to the permutation \( \sigma_s \) and as close to the origin as possible. Since all these paths have distinct bends, they represent an independent set. Last, we represent the vertices \( s' \in \{ S_{L,H} \cup S_{R,H} \} \). For each vertex \( \nu \) its representation will be a path \( P_{s'} \subseteq \cup P_{s} \). We choose the endpoint closer to the origin according to the suffix \( \sigma_{s'} \). Let \( O_{s'} \) be the set of vertices such that for all vertices \( s \in O_{s'}, \sigma_s \) and \( \sigma_{s'} \) overlap. The other endpoint is chosen as the endpoint closest to the origin that is farther from the origin than all the endpoints of the paths \( P_s \) where \( s \in O_{s'} \). Conditions \( iv \) and \( v \) guarantee that after this is done, every path \( P_{s'} \) that intersects with \( P_s \) splits from it.

From the proof of Theorem 3.1 we obtain the following corollary where a caterpillar is a tree in which
all vertices are within distance 1 of a central path.

Corollary 3.1

i) Every connected $B_1$-ENPG split graph $G = S(K, S, E)$ has a representation $(H, P)$ such that $\cup P$ is a caterpillar with central path $\cup P_K$ and maximum degree 3.

ii) $B_1$-ENPG $\cap$ SPLIT $\subseteq$ ENPT $\cap$ SPLIT.

3.2 Two Consequences of The Characterization of $B_1$-ENPG Split Graphs

Throughout this section, we use the notation introduced in the previous section. $X$ (resp. $Y$) denotes an element of $\{L, R\}$ (resp. $\{H, V\}$), and $\bar{X} = \{L, R\} \setminus \{X\}$. Given a $B_1$-ENPG split graph $G = S(K, S, E)$, we denote by $K_{X,Y}$, $S_{X,Y}$, $S_{X,HV}$, $S_{X,HH}$ and $\sigma_{X,Y}$ the sets and permutations whose existence are guaranteed by Theorem 3.1. Furthermore $\sigma_X = \sigma_{X,H} \cdot \sigma_{X,V}$ is the permutation of $K$ obtained by the concatenation of the two permutations $\sigma_{X,H}$ and $\sigma_{X,V}$. We define $S_{X,Y,d}$ as the set of vertices of $S_{X,Y}$ having degree $d$ in $G$. The notations $S_{X,HV,d}$ and $S_{X,HH,d}$ are defined similarly.

The following inequalities are easy to show using the definitions of the sets and counting the number of prefixes, suffixes, or intervals of a given permutation having a given length.
Proposition 3.1 If \( G = S(K, S, E) \) is a twin-free and (false twin)-free \( B_1\)-ENPG split graph, then
\[
|S_{X,Y,d}| \leq \max\{|K_{X,Y}| + 1 - d, 0\} \quad (1)
\]
\[
|S_{X,HV,d}| \leq \begin{cases} 
1 & \text{if } d > |K_{X,V}| \\
0 & \text{otherwise}
\end{cases} \quad (2)
\]
\[
|S_{X,V,d} \cup S_{X,HV,d}| \leq \begin{cases} 
1 & \text{if } d > |K_{X,V}| + 1 - d \\
|K_{X,V}| + 1 - d & \text{otherwise}
\end{cases} \quad (3)
\]
\[
|S_{X,HH,d}| \leq \begin{cases} 
1 & \text{if } d \leq |K_{L,H} \cap K_{R,H}| \\
0 & \text{otherwise}
\end{cases} \quad (4)
\]
\[
|S_d| \leq 2(|K| + 2 - d). \quad (5)
\]

Proof: 1. If \( s \in S_{X,Y,d} \) then \( N(s) \) is an interval \( \sigma_s \) of \( \sigma_{X,Y} \) of size \( d \). Since \( G \) is (false twin)-free \( N(s) \neq N(s') \) whenever \( s \neq s' \). Therefore, \( |S_{X,Y,d}| \) is at most the number of such intervals which is given by the right hand size of the inequality.
2. If \( s \in S_{X,HV,d} \) then \( N(s) = K_{X,V} \cup \sigma_s \) where \( \sigma_s \) is a suffix of \( \sigma_{X,H} \). Therefore, \( d > |K_{X,V}| \) and \( \sigma_s \) is the unique suffix of \( \sigma_{X,H} \) of size \( d - |K_{X,V}| \).
3. Follows from (1) and (2).
4. If \( s \in S_{X,HH,d} \), then \( N(s) = K_{X,H} \cap \sigma_s \) where \( \sigma_s \) is a prefix of \( \sigma_{X,H} \) of size \( d \). If \( d > |K_{L,H} \cap K_{R,H}| \) no such prefix exists, otherwise there is exactly one such prefix.
5. By summing up (2), (3), and also (1) for \( Y = H \) and finally multiplying by two for the two possible values of \( X \). \( \square \)

Summing up (5) for all the possible values of \( d \in |K| \) we get the following corollary.

Corollary 3.2 If \( G = S(K, S, E) \) is a (false twin)-free \( B_1\)-ENPG split graph, then \( |S| \) is \( \Omega(|K|^2) \).

Using similar arguments one can show that if \( G = S(K, S, E) \) is twin-free then \( |K| \) is \( \Theta(|S|^2) \) implying that \( |S| \) is \( \Omega(\sqrt{|K|}) \). More specifically, one should consider the set of endpoints or bend points of the paths \( P_u \), all of which are in \( \cup P_K \) and observe that no four such points that are pairwise consecutive in \( \cup P_K \) may surround the endpoints of two paths \( P_u, P_v \in P_K \) since otherwise \( u \) and \( v \) are twins.

Theorem 3.2 The following strict inclusions hold:

- \( B_1\)-ENPG \( \cap \) SPLIT \( \subsetneq \) \( B_2\)-ENPG \( \cap \) SPLIT.
- \( B_1\)-ENPG \( \cap \) SPLIT \( \subsetneq \) ENPT \( \cap \) SPLIT.

Proof: By the definition of a \( B_k\)-ENPG graph, we have \( B_1\)-ENPG \( \cap \) SPLIT \( \subsetneq \) \( B_2\)-ENPG \( \cap \) SPLIT and by Corollary 3.1 we have \( B_1\)-ENPG \( \cap \) SPLIT \( \subsetneq \) ENPT \( \cap \) SPLIT. In the following, we provide a split graph with a representation which is both \( B_2\)-ENPG and ENPT. We show that this split graph is not \( B_1\)-ENPG.

Let \( K = [0, 10], \sigma_L \) be the identity permutation \((0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)\) on \( K \) and \( \sigma_R \) be the permutation \((0, 5, 10, 4, 9, 3, 8, 2, 7, 1, 6)\) on \( K \). Let \( G = S(K, S, E) \) where \( S \) contains 23 vertices: one for every pair that is consecutive in one of \( \sigma_L, \sigma_R \) (there are 10 in every permutation and we note that these pairs are distinct) and one for each of the pairs \( \{0, 2\}, \{0, 3\}, \{0, 4\} \) (which are not consecutive in
Fig. 5: The $B_2$-ENPG representation of a non-$B_1$-ENPG split graph used in the proof of Theorem 3.2. The paths representing vertices of $K$ are not drawn. They are implied by the numbers of the vertices: for every $i \in [0, 10]$, $P_i$ is the shortest path between the two vertices labeled $i$. The paths with two bends, intersects every path of $P_K$, but splits from every path having an endpoint after its bend points. Therefore, these path represent vertices with neighborhood $\{0, 2\}, \{0, 3\}, \{0, 4\}$.

3.3 NP-completeness of $B_1$-ENPG split graph recognition

We now proceed with the NP-completeness of $B_1$-ENPG recognition in split graphs. We first present a preliminary result that can be useful per se.

A graph is $d$-regular if all its vertices have degree $d$. A 3-regular graph is also termed cubic. A diamond is the graph $K_4 - e$ obtained by removing an edge from a clique on four vertices. Clearly, if the edge set of a graph $G$ can be partitioned into two Hamiltonian cycles, then $G$ is 4-regular. However, in the opposite direction we have the following:

**Theorem 3.3** The problem of determining whether the edge set of a diamond-free 4-regular graph can be partitioned into two Hamiltonian cycles is NP-complete.

**Proof:** We prove by reduction from the Hamiltonicity problem of cubic bipartite graphs which is known to be NP-complete (Akiyama et al. 1980). Let $G$ be a cubic bipartite graph whose Hamiltonicity has to be decided, and let $H = L(G)$ be its line graph. $H$ is clearly 4-regular. In the sequel we will show that $H$ is also diamond-free. In addition, we know that $G$ is Hamiltonian if and only if the edge set of its line graph $H$ can be partitioned into two Hamiltonian cycles (Kotzig (1957)). This concludes the proof. It remains to show that $H$ is diamond-free.

Suppose for a contradiction that $H$ contains a diamond on vertices $\{e_1, e_2, e_3, e_4\}$ that are pairwise adjacent except for the pair $e_1, e_4$. Then $\{e_1, e_2, e_3\}$ and $\{e_2, e_3, e_4\}$ are two triangles of $H$. Every triangle of $H$ corresponds to either a triangle of $G$, or to three edges of $G$ incident to a common vertex. Since $G$ is bipartite, only the latter case is possible. Then $e_1, e_2, e_3$ (resp. $e_2, e_3, e_4$) are edges of $G$ incident to a vertex $v$ (resp. $v'$). Since $G$ is cubic we have $v \neq v'$. We conclude that $e_2 = e_3 = vv'$, a contradiction. □

We are now ready to prove the main result of this section.
**Theorem 3.4** The recognition problem of $B_1$-ENPG graphs is NP-complete even when restricted to split graphs.

**Proof:** The proof is by reduction from the problem of decomposing a 4-regular, diamond-free graph into two Hamiltonian cycles which is shown to be NP-complete in Theorem 3.3. Given a 4-regular graph $G$ on $n + 1$ vertices, we remove an arbitrary vertex $v$ of $G$ and obtain the graph $G' = G - v$ on $n$ vertices all of which having degree 4, except the four neighbours $\{v_1, v_2, v_3, v_4\}$ of $v$ each of which having degree 3. We construct the split graph $G'' = S(K, S, E)$ where $K = V(G')$, $S = E(G') \cup \{s_1, s_2, s_3, s_4\}$. Furthermore, the neighborhood of a vertex $s \in S$ is determined as follows. If $s = s_i$ for some $i \in [4]$ then $N_{G''}(s) = K - v_i$, otherwise $s$ is an edge $uv$ of $G'$ in which case $N_{G''}(s) = \{u, v\}$. It remains to show that $G''$ is $B_1$-ENPG if and only if $E(G')$ can be partitioned into two Hamiltonian cycles.

Assume that $E(G')$ can be partitioned into two Hamiltonian cycles $C_L, C_R$. This induces a partition of $E(G')$ into two paths $Q_L$ and $Q_R$ which in turn induces a partition of $S - \{s_1, s_2, s_3, s_4\}$ into $S_L, H = E(Q_L)$ and $S_R, H = E(Q_R)$. Note that the endpoints of $Q_L$ and $Q_R$ are the degree 3 vertices of $G'$, i.e. $\{v_1, v_2, v_3, v_4\}$. Let without loss of generality $v_1, v_2$ (resp. $v_3, v_4$) be the endpoints of $Q_L$ (resp. $Q_R$). For $X \in \{L, R\}$ let $K_{X, H} = K = V(G')$ and $K_{X, V} = \emptyset$. We set $\sigma_{X, H}$ as the order of the vertices of $G'$ in $Q_X$ (which is a permutation of the vertices of $K = V(G')$). Then $v_1$ and $v_2$ (resp. $v_3$ and $v_4$) are the first and last vertices of the permutation $\sigma_{X, H}$ (resp. $\sigma_{R, H}$). For $X \in \{L, R\}$ we set $S_{X, V} = S_{X, H} = S_{X, HV} = \emptyset$. We now verify that these settings satisfy the conditions of Theorem 3.1 Conditions [3], [4], [5], [6] and [7] easily follow since the sets $S_{X, Y}, S_{X, HV}, S_{X, H} H$ and $K_{X, V}$ are empty. As for Condition [8] we consider two cases. If $s = s_i$ for some $i \in [4]$ then $N_{G''}(s) = K - v_i$ is an interval of $\sigma_{X, H}$ for some $X \in \{L, R\}$ since $v_i$ is either the first or the last vertex of one of these permutations. If $s$ is an edge $uv \in E(Q_X)$ of $G'$ then $u$ and $v$ are consecutive in the permutation $\sigma_{X, H}$. Since all the conditions are satisfied, we conclude that $G''$ is $B_1$-ENPG.

Now assume that $G''$ is $B_1$-ENPG. For $X \in \{L, R\}, Y \in \{H, V\}$, let $K_{X, Y}, \sigma_{X, Y}, S_{X, Y}, S_{X, HV}$ and $S_{X, H}$ be sets and permutations whose existence are guaranteed by Theorem 3.1. We first show that $|K_{X, V}| \leq 1$. Assume for a contradiction that $|K_{X, V}| > 1$ for some $X \in \{L, R\}$, say $X = L$. Then we have $|K_{R, H}| > 1$, and $|K_{L, H} |, K_{R, V} | < n - 1$. By Proposition 3.1, these imply $S_{L, H, n} = S_{L, H, n}, S_{R, H, n} = S_{R, H, n} = \emptyset$. Moreover, $S_{R, H, n} \leq 2$ and this may hold with equality only when $K_{R, H} = K$. Finally, we have $|S_{X, Y, n} = 1 |, S_{X, HV, n} = 1 | \leq 1$. Summing up all inequalities we obtain $|S_{n} | \leq 4$. We recall that all the vertices of $S$ have degree 2 except the four special vertices with degree $n - 1$. Therefore, $S_{n} = \{s_1, s_2, s_3, s_4\}$, and we conclude that all the inequalities hold with equality. In particular $|S_{R, H, n} | = 2$, implying $K_{R, H} = K$ and $K_{R, V} = \emptyset$. Then we have $S_{R, Y, n} = S_{R, HV, n} = \emptyset$, i.e. one of the inequalities is strict, a contradiction. Therefore, $|K_{X, V}| \leq 1$, implying

$$S_{X, Y, 2} = \emptyset.$$  

(6)

Recall that $\sigma_X = \sigma_{X, H} \cdot \sigma_{X, V}$. We now show that the set of the first and last vertices of $\sigma_{L, H}$ and $\sigma_{R, H}$ is $\{v_1, v_2, v_3, v_4\}$. Let $i \in [4]$ and consider each one of the cases $s_i \in S_{X, H}, s_i \in S_{X, HV}$ and $s_i \in S_{X, V}$ (the case $s_i \in S_{X, V}$ is impossible since $|K_{X, V}| \leq 1$). It is easy to verify for every case that $v_i$ is either the first or the last vertex of $\sigma_X$. By the pigeonhole principle we conclude that the set of first and last vertices of $\sigma_{L, H}$ and $\sigma_{R, H}$ is $\{v_1, v_2, v_3, v_4\}$. We assume without loss of generality that $v_1$ (resp. $v_2$) is the first (resp. last) vertex of $\sigma_{L, H}$ and that $v_3$ (resp. $v_4$) is the first (resp. last) vertex of $\sigma_{R, H}$.

Our next step is to show that $S_{X, H, 2} = \emptyset$. Assume for a contradiction that this does not hold, and let $s$ be a vertex (without loss of generality) of $S_{L, H, 2}$. Then $\sigma_s$ is a prefix with two vertices of
\[ \sigma_{L,H} \cap K_{R,H} = \sigma_{L,H} \setminus K_{R,V} = \sigma_{L,H} - v_4. \] Clearly, the first vertex of \( \sigma_s \) is \( v_1 \). If the second vertex \( w \) of \( \sigma_{L,H} \) is not \( v_4 \), then \( \sigma_s = v_1w \) is an interval of \( \sigma_{X,H} \) implying that \( s \in S_{L,H} \), a contradiction. Therefore, \( w = v_4 \), and \( \sigma_s = v_1x \) where \( x \) is the third vertex of \( \sigma_{L,H} \). We conclude that \( S_{L,H,2} = \{ v_1x \} \).

Recall that \( v_4 \) has three incident edges in \( G' \). Since \( v_4 \) is the leftmost vertex of \( \sigma_R \), none of these edges is in \( S_{R,H} \cup S_{L,V} \). Moreover, at most one of them is in \( S_{R,H} \cup S_{R,V} \). Therefore, at least two of them are in \( S_{L,H} \). Then these edges are necessarily \( v_4v_1 \) and \( v_4x \). We conclude that \( \{ v_1, v_4, x \} \) induces a triangle in \( G' \). In other words \( \{ v_1, v_4, x, v \} \) induces a diamond on \( G' \), a contradiction. Therefore

\[ S_{X,H,2} = \emptyset. \]

Finally, if \( K_{X,V} = \emptyset \) we have \( S_{X,H,V} = \emptyset \) and \( |S_{X,H,2}| \leq n - 1 \). Otherwise, \( |K_{X,V}| = 1, |K_{X,H}| = n - 1 \) and we have \( |S_{X,H,V}| \leq 1 \) and \( |S_{X,H,2}| \leq n - 2 \). In both cases we have

\[ |S_{X,H,2} \cup S_{X,H,V,2}| \leq n - 1. \]

Combining (6), (7) and (8) we obtain \( |S_2| \leq 2(n - 1) \). Since \( |S_2| = |E(G')| = |E(G)| - 4 = 2(n - 1) \), all the inequalities must hold with equality, in particular \( |S_{X,H,2} \cup S_{X,H,V,2}| = n - 1 \). Therefore, every two consecutive vertices in \( \sigma_X \) are adjacent in \( G' \). In other words, the permutation \( \sigma_X \) corresponds to a path \( Q_X \) of \( G' \). The endpoints of \( Q_L \) (resp. \( Q_R \)) are \( v_1 \) and \( v_2 \) (resp. \( v_3 \) and \( v_4 \)). Adding two edges incident to \( v \) to each \( Q_X \), we get two edge-disjoint Hamiltonian cycles of \( G \).

\[ \square \]

### 4 Co-bipartite Graphs

In Section 4.1 we characterize \( B_1 \)-ENPG co-bipartite graphs. We show that there are two types of representations for \( B_1 \)-ENPG co-bipartite graphs. For each type of representation, we characterize their corresponding graphs. These characterizations imply a polynomial-time recognition algorithm. In Section 4.2 we present an efficient (linear-time) implementation of the algorithm.

#### 4.1 Characterization of \( B_1 \)-ENPG Co-bipartite Graphs

We proceed with definitions and two related lemmas (Lemma 4.1, Lemma 4.2) that will be used in each of the above mentioned characterizations.

Let \( S \) be a path of a graph \( H \) with endpoints \( u, v \). Two sets \( P_u, P_v \) of paths meet at \( S \) if for \( x \in \{ u, v \} \)

(a) every path of \( P_x \) contains \( x \) (b) every path of \( P_x \) has an endpoint that is a vertex of \( S \) different than \( x \), and (c) a pair of paths \( P_u \in P_u, P_v \in P_v \) may intersect only in \( S \) (see Figure 5).

A graph \( G = (V, E) \) is a difference graph (equivalently bipartite chain graph) if every \( v_i \in V \) can be assigned a real number \( a_i \) and there exists a positive real number \( T \) such that (a) \( |a_i| < T \) for all \( i \) and
Let $\delta_1 < \delta_2 < \ldots \delta_s$ be distinct nonzero degrees in $X$. Let $\sigma_1 < \sigma_2 < \ldots \sigma_t$ be distinct nonzero degrees in $Y$. Let $X = X_0 \cup X_1 \cup \ldots \cup X_s$, $Y = Y_0 \cup Y_1 \cup \ldots \cup Y_t$, where $X_i = \{x \in X | d(x) = \delta_i\}$, $Y_j = \{y \in Y | d(y) = \delta_j\}$. Then $s = t$ and for $x \in X_i$, $y \in Y_j$, $\{x, y\} \in E$ if and only if $i + j > t$.

Theorem 4.2 [Hammer et al. (1990)] A graph is a difference graph if and only if it is bipartite and $2K_2$-free.

Lemma 4.1 Let $G_B = B(K, K', E)$ a difference graph, and $t$ be the number of distinct nonzero degrees of vertices of $K$ in $G_B$. Let $H$ be a grid and $S$ be a path of $H$ with length at least $t + 2$ and no bends. Then there is a $B_1$-ENPG representation $\langle H, P \rangle$ of $G_C = C(K, K', E)$ such that $P_K$ and $P_{K'}$ meet at $S$.

Proof: Let $\delta_1 < \delta_2 < \ldots \delta_s$ (resp. $\sigma_1 < \sigma_2 < \ldots \sigma_t$) be the distinct nonzero degrees in $K$ (resp. in $K'$) in $G_B$. By Theorem 4.1 we have $s = t$. Let $-1, 0, 1, \ldots, t + 1$ be $t + 3$ vertices of $S$ such that 0 and $t + 2$ are the endpoints of $S$ and they appear in this order on $S$. Let $x$ (resp. $x'$) be a vertex of $K$ (resp. $K'$), and let $i$ be such that $d_{G_B}(x) = \delta_i$ (resp. $d_{G_B}(x') = \sigma_i$). The path $P_x$ (resp. $P_{x'}$) is constructed between vertices $-1$ and $i$ (resp. $t - j$ and $t + 1$).

With this construction $P_K, P_{K'}$ represent the cliques $K$ and $K'$, moreover they meet at $S$. By the construction two paths $P_x \in P_K, P_{x'} \in P_{K'}$ intersect if and only if $i + j > t$. By Theorem 4.1, $x$ and $x'$ are adjacent if and only if $i + j > t$. Therefore, $P$ is a representation of $G = C(K, K', E)$.

Lemma 4.2

i) If two sets $P_K, P_{K'}$ of one-bend paths meet at a path $S$ then $G_B = B(K, K', E)$ is a difference graph.

ii) If a cobipartite graph $G = C(K, K', E)$ is an interval graph, then $G_B = B(K, K', E)$ is a difference graph.

Proof:

i) Let $u, v$ be the endpoints of $S$. Let $T = |E(S)| + 1$ and $r_i$ (resp. $l_j$) be the endpoint of the path $P_i \in P_K$ (resp. $P_j \in P_{K'}$) among the internal vertices of $S$. Let $a_i = |E(p_S(u, r_i))|$ (resp. $a_j = -|E(p_S(l_j, v))|$) where $p_T(x, y)$ is the unique path between vertices $x$ and $y$ of a tree $T$. By definition, $|a_i| \leq |E(S)| + T < |E(S)| + 1 = T$. Therefore, $G_B$ is a difference graph.

ii) Fix an interval representation of $G$. For $X \in \{K, K'\}$ let $e_X$ be the edge of the representation that is common to all the paths $P_X$ representing the clique $X$. We can assume without loss of generality that $e_K$ and $e_{K'}$ are the leftmost and rightmost edges of the representation. We now subdivide $e_K$ and $e_{K'}$ by adding new vertices $v_K$ and $v_{K'}$ respectively. Finally, if a path contains both $e_K$ and $e_{K'}$,

(b) $\{v_i, v_j\} \in E$ if and only if $|a_i - a_j| \geq T$. Every difference graph is bipartite where the bipartition is according to the sign of $a_i$. 

Theorem 4.1 [Hammer et al. (1990)] If $G = (V, E)$ is a bipartite graph with bipartition $V = X \cup Y$ then the following statements are equivalent:

i) $G$ is a difference graph.

ii) Let $\delta_1 < \delta_2 < \ldots \delta_s$ be distinct nonzero degrees in $X$, and $\sigma_1 < \sigma_2 < \ldots \sigma_t$ be distinct nonzero degrees in $Y$, and $\sigma_0 = 0$. Let $X = X_0 \cup X_1 \cup \ldots \cup X_s$, $Y = Y_0 \cup Y_1 \cup \ldots \cup Y_t$, where $X_i = \{x \in X | d(x) = \delta_i\}$, $Y_j = \{y \in Y | d(y) = \delta_j\}$. Then $s = t$ and for $x \in X_i$, $y \in Y_j$, $\{x, y\} \in E$ if and only if $i + j > t$. 

Theorem 4.2 [Hammer et al. (1990)] A graph is a difference graph if and only if it is bipartite and $2K_2$-free.
we truncate one edge from its end so that it contains \( v_K \) but not \( v_{K'} \). In the new representation, \( P_K \) and \( P_{K'} \) meet at the segment between \( v_K \) and \( v_{K'} \).

\[ \square \]

Two representations \( \langle H, P \rangle \) and \( \langle H', P' \rangle \) are bend-equivalent if they are representations of the same graph \( G \) and the paths \( P_v \in P \) and \( P'_v \in P' \) representing the same vertex \( v \) of \( G \) have the same number of bends. We proceed with the following lemma that classifies all the \( B_1 \)-ENPG representations of a co-bipartite graph into two types.

**Lemma 4.3** Let \( G = C(K, K', E) \) be a connected \( B_1 \)-ENPG co-bipartite graph with a representation \( \langle H, P \rangle \). Then \( G \) has a bend-equivalent representation \( \langle H, P' \rangle \) that satisfies exactly one of the following:

1. \(|S(\cup P'_K, \cup P'_K)| = 1 \) and \( \cup P' \) is a tree with maximum degree at most 3 with at most two vertices of degree 3 as depicted in Figure 7 (a).
2. \(|S(\cup P'_K, \cup P'_K)| = 2 \) and the paths \( \cup P_K \) and \( \cup P_{K'} \) intersect as depicted in Figure 7 (b).

**Proof:** By Proposition 2.1, \( \cup P_K \) and \( \cup P_{K'} \) are two paths with at most 2 bends each. Let \( e_K \) (resp. \( e_{K'} \)) be an arbitrary edge of \( \cup P_K \) (resp. \( \cup P_{K'} \)). The paths \( \cup P_K \) and \( \cup P_{K'} \) intersect in at least one edge, because otherwise \( G \) is not connected. Therefore, \(|S(\cup P_K, \cup P_{K'})| \geq 1 \). We consider two disjoint cases:

- \(|S(\cup P_K, \cup P_{K'})| = 1 \). Let \( T = \cup P \) and \( S \) be the unique segment of \( S(\cup P_K, \cup P_{K'}) \). The collection \( T \) is clearly a tree, since any cycle in \( T \) is a cycle in one of \( \cup P_K, \cup P_{K'} \). Any vertex of degree at least 3 in \( T \) is an endpoint of \( S \), therefore there are at most 2 such vertices. On the other hand an endpoint of \( S \) has degree at most 3. Therefore \( \Delta(T) \leq 3 \) and there are at most 2 vertices of degree 3 in \( T \).

Let \( u \) and \( v \) be the two endpoints of \( S \). Let also \( e_u, e_v \) (respectively \( e'_u, e'_v \)) be the (at most four) edges not in \( S \) but belonging to \( \cup P_K \) (respectively \( \cup P_{K'} \)) and incident respectively to \( u \) and \( v \). Now, shrink all the paths on the branches of \( T \) starting with \( e_u, e_v, e'_u, e'_v \) and not containing \( S \) to respectively the edges \( e_u, e_v, e'_u, e'_v \). Clearly, this transformation does not create or remove any split between two paths and does not remove any intersections since all paths intersecting on one of these branches now intersect on the shrunken edge. To maintain bend-equivalence, we add one more edge to every path that loses a bend during the shrinkage (the same edge for all paths of the same branch). This new representation \( \langle H, P' \rangle \) is bend-equivalent to \( \langle H, P \rangle \) and \( T' = \cup P' \) is a tree with the claimed properties.

- \(|S(\cup P_K, \cup P_{K'})| \geq 2 \). We claim that \( \cup P_K \cap \cup P_{K'} \) contains only horizontal edges, or only vertical edges. Indeed, assume that there is a vertical edge \( e_V \) and a horizontal edge \( e_H \) in \( \cup P_K \cap \cup P_{K'} \). We observe that there is a unique one bend path with \( e_V \) and \( e_H \) its end edges, and that any other connecting these edges contains at least three bends. Therefore, both \( \cup P_K \) and \( \cup P_{K'} \) contain this path. We conclude that \( e_V \) and \( e_H \) are in the same segment. As any other edge is either horizontal or vertical, we can proceed similarly for all the edges of \( \cup P_K \cap \cup P_{K'} \) and prove that they all belong to the same segment, contradicting the fact that we have at least 2 segments. Assume without loss of generality that all the edges of \( \cup P_K \cap \cup P_{K'} \) are vertical. Then every segment is a vertical path. No two segments can be on the same vertical line, because this will require at least one of \( \cup P_K \),
Two types of $B_1$-ENPG representation of connected co-bipartite graphs: (a) Type I: $|S(K, K')| = 1$, $\cup P$ is isomorphic to a tree $T$ with $\Delta(T) \leq 3$ and at most two vertices $u, v$ having degree 3, (b) Type II: $|S(K, K')| = 2$, $P_K$ (resp. $P_{K'}$) has exactly two bend points $u, v$ (resp. $u', v'$)

$\cup P_{K'}$ to contain four bends. Moreover, three vertical segments in distinct vertical lines imply that $P_K$ and $P_{K'}$ contain at least four bends each. Therefore, there are exactly 2 vertical segments and $P_K$ (also $P_{K'}$) has exactly two bends.

Let $u, v$ (resp. $u', v'$) be the bends of $\cup P_K$ (resp. $\cup P_{K'}$). Then $S(\cup P_K, \cup P_{K'})$ where $S_u$ (resp. $S_v$) is on the same vertical line as $u$ and $u'$ (resp. $v$ and $v'$). Moreover $e_K$ (resp. $e_{K'}$) is between $u$ and $v$ (resp. $u'$ and $v'$) since otherwise we would have paths crossing both $u$ and $v$ (resp. $u'$ and $v'$) and thus 2 bends. If both the pairs $u, u'$ and $v, v'$ are on different sides of respectively $S_u$ and $S_v$ (as in Figure 7(b)) then let $H' = H$ and $P' = P$ be the desired representation. Now consider the situation where $u$ and $u'$ are on the same side of $S_u$. Every path intersecting $S_u$ crosses the same endpoint of $S_u$ say without loss of generality $u$, implying that if a pair of paths from distinct cliques intersect at $S_u$, they split at this endpoint. Then remove $S_u$ from every path of $P_{K'}$ to obtain a bend-equivalent representation that contains one segment. The resulting representation can be transformed into a bend-equivalent representation $\langle H, P' \rangle$ as described the previous bullet.

Based on Lemma 4.3, a $B_1$-ENPG representation of a connected co-bipartite graph $G = C(K, K', E)$ is Type I (resp. Type II) if $|S(K, K')| = 1$ (resp. $|S(K, K')| = 2$).

We proceed with the characterization of $B_1$-ENPG graphs having a Type II representation that turns out to be simpler than the characterization of the others. In the following lemma, a trivial connected component is an isolated vertex.

**Lemma 4.4** A connected twin-free co-bipartite graph $G = C(K, K', E)$ has a Type II $B_1$-ENPG representation if and only if the bipartite graph $G_B = B(K, K', E)$ contains at most two non-trivial connected components each of which is a difference graph.

**Proof:** ($\Rightarrow$) Let $\langle H, P \rangle$ be a Type II $B_1$-ENPG representation of $G$ and $u, v$ (resp. $u', v'$) be the bends of $\cup P$ (resp. $\cup P'$) as depicted in Figure 7(b). For $x \in \{u, v\}$, let $S_x$ be the segment contained in the path between $x$ and $x'$. The paths of $P$ not intersecting with any of $S_u, S_v$ correspond to isolated vertices of $G_B$. Since $G$ is twin-free, there is at most one such path in $P_K$ (resp. $P_{K'}$).
Each one of the remaining paths intersects exactly one of \( S_u, S_v \), as otherwise such a path would contain two bends. For \( X \in \{ K, K' \} \) and \( x \in \{ u, v \} \) let \( P_{X,x} \) be the paths of \( P_X \) intersecting \( S_x \). Then \( P_{K_x} \) and \( P_{K'_x} \) meet at \( S_x \). By Lemma \( 4.2 \), \( G_B[\cup P_{K_x}] \) is a difference graph.

\( (\Leftarrow) \) It is sufficient to construct a Type II representation for the maximal case, i.e. \( G_B \) contains exactly two trivial connected components and two non-trivial connected components. Let \( w \in K \) and \( w' \in K' \) be the trivial connected components and \( B(K_u, K'_u, E_u), B(K_v, K'_v, E_v) \) be the non-trivial connected components of \( G_B \). We construct a rectangle as depicted in Figure 7 b) having vertical lines with \( \max(\min(|K_u|, |K'_u|), \min(|K_v|, |K'_v|)) + 2 \) edges, and horizontal lines with one edge \( e_K = \{ u, v \} \) and \( e_{K'} = \{ u', v' \} \). For \( X \in \{ K, K' \} \), and \( x \in \{ u, v \} \) the paths \( P_{X,x} \) start with \( x \) and enter segment \( S_x \). The other endpoints of the paths will be in the segment \( S_x \). Then, for \( x \in \{ u, v \}, P_{K_x} \) and \( P_{K'_x} \) meet at \( S_x \). Since \( B(K_x, K'_x, E_x) \) is a difference graph, by Lemma \( 4.1 \), the endpoints can be determined such that \( P_{K_x} \cup P_{K'_x} \) is a representation of \( B(K_x, K'_x, E_x) \). The path \( P_w \) (resp. \( P_{w'} \)) consists of the edge \( e_K \) (resp. \( e_{K'} \)). It is easy to verify that this is a representation of \( G \). \( \square \)

We proceed with the characterization of the B₂-ENPG graphs with a Type I representation. For this purpose we resort to the following definitions from Fouquet et al. (2004).

Let \( G = (V, V', E) \) be a bipartite graph and \( M \subseteq V \cup V' \). A vertex \( v \in V \setminus M \) (resp. \( v \in V' \setminus M \)) distinguishes \( M \) if it has a neighbour in \( M \cap V' \) (resp. \( M \cap V \)) and a non-neighbour in \( M \cap V' \) (resp. \( M \cap V \)). A non-empty subset \( M \) of \( V \cup V' \) is a bimodule of \( G \) if no vertex distinguishes \( M \). It follows from the definition that \( V \cup V' \) is a bimodule of \( G \), and so are all the singletons and all the pairs of vertices with exactly one from \( V \). These bimodules are the trivial bimodules of \( G \).

A zed is a graph isomorphic to a \( P_4 \) or any induced subgraph of it. We note that a trivial bimodule different from \( V \cup V' \) is a zed.

**Lemma 4.5** A connected twin-free co-bipartite graph \( G = C(K, K', E) \) has a Type I B₂-ENPG representation if and only if there is a set of vertices \( Z \) of \( G \) such that

i) \( Z \) is a zed of \( G \),

ii) \( Z \) is a bimodule of \( G_B = (V, V', E) \), and

iii) \( G_B \setminus Z \) is a difference graph.

Moreover, if \( Z \) is a set of vertices of minimum size that satisfies \( \text{iii} \) and \( Z \) is a set of two non-adjacent vertices of \( G \), then for the unique segment \( S \) of \( S(\cup K, \cup K') \) the following hold in every representation \( \langle H, \mathcal{P} \rangle \):

a) \( S \) is contained in at least one of the paths of \( \mathcal{P}_Z \).

b) the endpoints of \( S \) have degree 3 in \( \cup \mathcal{P} \) and these endpoints constitute split(\( \cup \mathcal{P}_K, \cup \mathcal{P}_{K'} \)).

**Proof:** \( (\Rightarrow) \) Let \( \langle H, \mathcal{P} \rangle \) be a Type I B₂-ENPG representation of \( G \). By Lemma \( 4.3 \), \( |S(K, K')| = 1 \) and \( \cup \mathcal{P} \) is a tree. Let \( u, v \) be the endpoints of the unique segment \( S \) of \( S(K, K') \). We consider the following disjoint cases:

- \( \{ e_K, e_{K'} \} \not\subseteq E(S) \): Without loss of generality, suppose that \( e_K \not\subseteq E(S) \) and \( u \) is closer to \( e_K \) than \( v \). Consider two paths \( P_{v'}, P_{v''} \in \mathcal{P}_{K'} \) that cross \( u \). We observe that these paths are indistinguishable by the paths of \( \mathcal{P}_K \). Namely, every path of \( \mathcal{P}_K \) either does not intersect any one of
Fig. 8: (a) Four special paths corresponding to a zed (b) The type of vertices and edge relations of a $B_1$-ENPG co-bipartite graph having a Type I representation. $K^u$ (resp. $K^{u'}$) is the set of vertices corresponding to the paths of $P_K$ (resp. $P_{K'}$) crossing neither $u$ nor $v$.

$P_{x'}, P_{y'}$, or intersects both and splits from both at $u$. Therefore the corresponding vertices $x', y'$ are twins. As $G$ is twin-free we conclude that there is at most one path of $P_{K'}$ that crosses $u$. Similarly, consider two paths $P_x, P_y \in P_K$ that cross $v$. These paths cross also $u$ since $e_K$ is an edge of both paths. Therefore, every path of $P_{K'}$ either does not intersect any one of $P_x, P_y$, or intersects both and splits from both at either $u$ of $v$, or intersects both and does not split from any of them. We conclude that there is at most one path of $P_{K}$ that crosses $v$. Let $P_{Z'}$ be a set of these at most two paths. Namely, $P_{Z'}$ consists of all the paths of $P_{K'}$ crossing $u$ and all the paths of $P_{K}$ that cross $v$. We now observe that $\cup (P \setminus P_{Z'})$ is either a path or the union of two edge-disjoint paths. In both cases no two paths split from each other, and their adjacency is determined only by the intersections. Therefore, the resulting graph $G \setminus Z'$ is an interval graph implying that $G_B \setminus Z'$ is a difference graph. We note that the path $P_{x'} \in P_{K'}$ that crosses $u$ is an isolated vertex of $G_B$, therefore for $Z = Z' \setminus \{x'\}$ we have that $G_B \setminus Z$ is a difference graph too, i.e. $Z$ satisfies (iii). Since $|Z| \leq 1$, $Z$ satisfies (ii) and (iii) trivially. This completes the proof of the first part of the claim. As for the second part, since $|Z| \leq 1$, any set of minimum size satisfying the conditions has at most one vertex. Therefore, the second part of the claim holds vacuously.

- $\{e_K, e_{K'}\} \subseteq E(S)$: We first note that we can assume $e_K \neq e_{K'}$ since otherwise we can subdivide this edge into two and rename the new edges as $e_K$ and $e_{K'}$. Assume without loss of generality that $e_K$ is closer to $u$ than $e_{K'}$. (see Figure 8). Consider two paths $P_{x'}, P_{y'} \in P_{K'}$, that cross $u$ but not $v$. We observe that these paths are indistinguishable by the paths of $P_{K}$. Therefore, the corresponding vertices are twins. As $G$ is twin-free we conclude that there is at most one path $P_{K'}$ of $P_{K'}$ that
crosses \( u \) and does not cross \( v \). Similarly there is at most one path \( P^u_0 \) of \( \mathcal{P}_K \) that crosses \( v \) but does not cross \( u \), at most one path \( P^u_0 \) of \( \mathcal{P}_K \) that crosses both \( u \) and \( v \), and at most one path \( P^u_0 \) of \( \mathcal{P}_K \) that crosses both \( u \) and \( v \). Let \( \mathcal{P}_Z \) be the set of these at most four paths. As in the previous case, \( \cup(\mathcal{P} \setminus \mathcal{P}_Z) \) is a path, thus \( G_B \setminus Z \) is a difference graph, i.e. \( Z \) satisfies (iii). Assuming that all the four paths exist, it is easy to verify that their corresponding vertices \( K^u, K^u, K^u, K^u, v, K^u, v, K^u, v \) constitute a \( P_4 \) with endpoints \( K^u, v, K^u, v \). Therefore, \( Z \) is a zed, i.e. \( Z \) satisfies (i). Finally, we observe that \( P^u_0 \) and \( P^u_0 \) are distinguishable only by \( P^u_0 \in \mathcal{P}_Z \). In other words, they are indistinguishable by paths from \( \mathcal{P}_K \setminus \mathcal{P}_Z \). By symmetry, we conclude that \( Z \) is a bimodule of \( G_B \), i.e. it satisfies (ii). This concludes the proof of the first part of the claim. To prove the second part, assume by contradiction that there is a minimal set \( Z \) satisfying (i)-(iii) consisting of two vertices and none of the corresponding paths contains the segment \( S \). Then these paths are \( P^v_0 \) and \( P^v_0 \). We now observe that \( P^v_0 \sim P^v_0 \), i.e. \( K^v \) and \( K^v \) are adjacent in \( G \), contradicting the assumption that the vertices of \( Z \) are non-adjacent in \( G \). This concludes the proof of (ii). If both paths contain \( S \), then these paths are \( P^v_0 \) and \( P^v_0 \) and we have \( \text{split}(\cup \mathcal{P}_K, \cup \mathcal{P}_K) \supseteq \text{split}(P^v_0, P^v_0) = \{u, v\} \), proving (ii) for this case. Otherwise, one of the paths does not contain \( S \). Let, without loss of generality this path be \( P^v_0 \). Then no path of \( \mathcal{P}_K \) crosses \( v \). We conclude that \( \cup(\mathcal{P} \setminus \{P^v_0\}) \) is a path, implying that the corresponding vertices induce a difference graph on \( G_B \), contradicting the assumption that \( Z \) is a minimal set satisfying (i)-(iii).

\((\Rightarrow)\) Given a zed \( Z \) of \( G \) satisfying the conditions of the lemma, we construct a Type I representation \( \langle H, \mathcal{P} \rangle \) as follows. Without loss of generality we assume that \( Z \) is a \( P_4 \) with endpoints \( y \in K, y' \in K' \) and internal vertices \( x \in K, x' \in K' \). Let \( \ell = \min(|K|, |K'|) + 2 \). The graph \( H \) is a 3 by \( \ell + 3 \) vertices grid where each vertex is represented by an ordered pair from \([-1, \ell + 1] \times [-1, 1]\). The path \( P_y \) (resp. \( P_y' \)) is between \((0, 0)\) (resp. \((-1, 0)\)) and \((\ell, 1)\) with a bend at \((\ell, 0)\). The path \( P_x \) (resp. \( P_x' \)) is between \((\ell, 0)\) (resp. \((\ell + 1, 0)\)) and \((0, -1)\) with a bend at \((0, 0)\). It is easy to verify that this correctly represents \( Z \). The representation of the difference graph \( G_B \setminus Z \) is two sets of paths that meet at the line segment between \((0, 0)\) and \((\ell, 0)\). By Lemma 4.1 the endpoints of the paths within this segment can be determined in accordance with the difference graph \( G_B \setminus Z \). The other endpoints of these paths are determined so as to satisfy the adjacencies of vertices of \( Z \) with other vertices, as follows: The other endpoint of every path of \( \mathcal{P}_K \cap \mathcal{N}_G(y) \) (resp. \( \mathcal{P}_K \cap \mathcal{N}_G(y') \)) is \((\ell, 0)\) (resp. \((\ell + 1, 0)\)). The other endpoint of every path of \( \mathcal{P}_K \cap \mathcal{N}_G(y) \) (resp. \( \mathcal{P}_K \cap \mathcal{N}_G(y') \)) is \((0, 0)\) (resp. \((-1, 0)\)).

By Lemmas 4.4 and 4.5 we have the following Theorem.

**Theorem 4.3** Let \( G = C(K, K', E) \) be a connected, twin-free co-bipartite graph, and \( G_B = B(K, K', E) \). Then, \( G \in \mathcal{B}_1\text{-ENPG} \) if and only if at least one of the following holds:

i) \( G_B \) contains at most two non-trivial connected components each of which is a difference graph.

ii) \( G \) contains a zed \( Z \) that is a bimodule of \( G_B \) such that \( G_B \setminus Z \) is a difference graph.

Since all the properties mentioned in Theorem 4.3 can be tested in polynomial time we have the following corollary.

**Corollary 4.1** \( \mathcal{B}_1\text{-ENPG} \) co-bipartite graphs can be recognized in polynomial time.
4.2 Efficient Recognition Algorithm

In this section we describe an efficient algorithm, namely Algorithm 1, to recognize whether a co-bipartite graph is $B_1$-ENPG using the characterization of Theorem 4.3. In Algorithm 1, $\text{isTypeI}$ is a function taking as input a connected twin-free cobipartite graph and a subset $Z$ of vertices to decide if there is $Z' \supseteq Z$ for the graph being $B_1$-ENPG of Type I. Similarly, $\text{isTypeII}$ takes a connected twin-free cobipartite graph $G$ and returns "YES" if $G$ is $B_1$-ENPG of Type II, and "NO" otherwise. As for function $\text{FindBimoduleZed}$, it takes a twin-free cobipartite graph $G$ and a $Z$ of $G$ to return the minimum superset of $Z$ that is a zed of $G$ and a bimodule of $G_B$, if any. Lastly, the function $\text{isDifference}$ in Algorithm 1 takes a bipartite graph $G$ and either indicates that $G$ is a difference graph or provides a $2K_2$ certifying that $G$ is not a difference graph.

Theorem 4.4 Given a co-bipartite graph $G = C(K_1, K_2, E)$, Algorithm 1 decides in time $O(|K| + |K'| + |E|)$ whether $G$ is $B_1$-ENPG.

Proof: Let $n = |K| + |K'|$, $m = |E|$. Let $T_{\text{diff}}(n, m)$ be the running time of $\text{isDifference}$ on a graph with $n$ vertices and $m$ edges, and let $T_{\text{bm}}(n, m)$ be the running time of $\text{FindBimoduleZed}$ that finds a minimum zed of $G$ that is a bimodule of $G_B$ and contains a given zed $Z$. Finally let $\alpha(n, m) \overset{\text{def}}{=} T_{\text{diff}}(n, m) + T_{\text{bm}}(n, m)$. The correctness of the algorithm follows from Observations 2.1, 2.2, Lemma 4.3 and from the correctness of the functions $\text{isTypeI}$ and $\text{isTypeII}$ that we prove in the sequel.

The correctness of $\text{isTypeI}$ is based on Lemma 4.5. A subset $Z$ of vertices of $G$ satisfying (ii)–(iii) of Lemma 4.5 is termed as a certificate through this proof. We now show that a given twin-free co-bipartite graph $G$ and $Z \subseteq V(G)$, $\text{isTypeI}$ returns "YES" if and only if there exists a certificate $Z' \supseteq Z$. Moreover, we show that its running time is at most $5^{|Z|} \alpha(n, m)$ when $|Z| \leq 4$ and constant otherwise.

We first observe that if $Z$ is not a zed, then no superset of $Z$ is a zed, and the algorithm returns correctly "NO" in constant time at line 8. Therefore, our claim is correct whenever $Z$ is not a zed. We proceed by induction on $5 - |Z|$. If $5 - |Z| = 0$, then $Z$ is not a zed and the algorithm returns "NO" in constant time. In the sequel we assume that $Z$ is a zed. In this case, $Z$ is verified to be a zed by $\text{isTypeI}$ in constant time and $\text{isTypeI}$ proceeds to line 2 to find (in time $T_{\text{bm}}(n, m)$) the minimal bimodule $Z'$ of $G_B$ that contains $Z$ and is a zed of $G$. We consider three cases according to the branching of $\text{isTypeI}$.

- $Z' = Z$ (i.e. $Z$ is a bimodule of $G_B$), and $G_B \setminus Z$ is a difference graph: $G_B \setminus Z$ is verified to be a difference graph by $\text{isTypeI}$ at line 11. It returns "YES" which is correct by Lemma 4.5 since $Z$ is a certificate. The running time is $\alpha(n, m)$, and the result follows since $1 \leq 5^{|Z|}$.

- $Z' = Z$ (i.e. $Z$ is a bimodule of $G_B$), but $G_B \setminus Z$ is not a difference graph: As $G_B \setminus Z$ is not a difference graph, there is a set $U \subseteq K \cup K' \setminus Z$ such that $G_B[U]$ is a $2K_2$. Every certificate $Z' \supseteq Z$ must contain at least one vertex of $U$ because otherwise $G_B \setminus Z'$ contains $G_B[U]$ which is a $2K_2$. Therefore, $\text{isTypeI}$ proceeds recursively calling $\text{isTypeI}$ on $(G, Z \cup \{u\})$ for each $u \in U$. The algorithm returns "YES" if and only if one of the guesses succeeds. Then, the total running time is at most $\alpha(n, m) + 4 \cdot 5^{|Z| + 1} \alpha(n, m) < (1 + 4 \cdot 5^{|Z|}) \alpha(n, m)$. Since $1 \leq 5^{|Z|}$ we conclude that the running time is at most $5^{|Z|} \alpha(n, m)$.

- $Z' \neq Z$ (i.e. $Z$ is not a bimodule of $G_B$): If $Z'$ exists, the definition of a bimodule implies that any certificate that contains $Z$ has to contain $Z'$. Therefore, $\text{isTypeI}(G, Z')$ is invoked and its result is
Algorithm 1 is O graphs. A graph is B edges of the host grid (and consequently all the paths containing the related edges) as needed to introduce that every path has exactly k bends; indeed, if there is a path with less than k bends, one can subdivide the edges of the host grid (and consequently all the paths containing the related edges) as needed to introduce bends, one can subdivide the edges of the host grid (and consequently all the paths containing the related edges) as needed to introduce bends, one can subdivide the edges of the host grid (and consequently all the paths containing the related edges) as needed to introduce bends, one can subdivide the edges of the host grid (and consequently all the paths containing the related edges) as needed to introduce bends, one can subdivide the edges of the host grid (and consequently all the paths containing the related edges) as needed to introduce bends, one can subdivide the edges of the host grid (and consequently all the paths containing the related edges) 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new bends until it has exactly $k$ bends, without creating any new intersection or split [Golumbic (2015)]. The following proposition states that this does not hold for $B_k$-ENPG graphs.

**Proposition 4.1** Every $B_1$-ENPG representation of a graph $G = C(K, K', E)$ such that $G_B = B(K, K', E)$ is isomorphic to $3K_2$ contains at least one path with zero bend.

**Proof:** Consider a set $Z$ consisting of two non-adjacent vertices of $G$. Then $Z$ is a trivial bimodule of $G_B$ and a zed of $G$. Moreover, by Theorem 4.2 $G_B \setminus Z$ is a difference graph since it does not contain a $2K_2$. Therefore, $Z$ satisfies conditions [i]-[iii] of Lemma 4.5. Then $G$ is $B_1$-ENPG.

Let $(H, P)$ be a $B_1$-ENPG representation of $G$. Since $G_B$ has three non-trivial connected components, by Lemma 4.4 $(H, P)$ is a Type I representation. For any single vertex $v$ of $G$, the graph $G_B \setminus \{v\}$ contains a $2K_2$ therefore fails to satisfy condition [iii]. We conclude that $Z$ is a set of minimum size satisfying the conditions [i]-[iii] of Lemma 4.5. Moreover, $Z$ consists of two non-adjacent vertices of $G$. Therefore, the unique segment $S$ of $S(K, K')$ has the properties [i] and [ii] mentioned in the same Lemma.

Let $Z = \{x, y\}$ where $x \in K$ and $y' \in K'$, and let $y$ and $x'$ be the unique neighbors in $G_B$ of $x$ and $y'$ respectively. Let also $u, v$ be the endpoints of $S$. By property [a] without loss of generality $P_x$ contains $S$. Therefore, $P_{x'}$ is contained in $S$ as otherwise it would split from $P_x$ in at least one of $u, v$, contradicting the fact that $x$ and $x'$ are adjacent. By property [b] of the lemma, $u$ and $v$ are split points. To conclude the claim, we now show that $P_{x'}$ has no bends. Assume by contradiction that $P_{x'}$ has a bend $w$. Then $w$ is a bend of $S$ and also of $P_x$. Therefore, $P_x$ does not bend neither at $u$ nor in $v$ as otherwise it would contain 2 bends. We conclude that both $u$ and $v$ are bends of $\cup P_{K'}$. Clearly, $w$ is also a bend of $\cup P_{K'}$. Then $\cup P_{K'}$ has 3 bends, contradicting Proposition 2.1. $\square$

5 Summary and Future Work

In [Boyacı et al. (2015b)] we showed that ENPG contains an infinite hierarchy of subclasses that are obtained by restricting the number of bends. In this work we showed that $B_1$-ENPG graphs are properly included in $B_2$-ENPG graphs. The question whether $B_2$-ENPG $\subseteq B_3$-ENPG $\subseteq \ldots$ remains open.

In this work, we studied the intersection of $B_1$-ENPG with some special chordal graphs. We showed that the recognition problem of $B_1$-ENPG graphs in NP-complete even for a very restricted sub family of split graphs. On the other hand we showed that this recognition problem is polynomial-time solvable within the family of co-bipartite graphs. A forbidden subgraph characterization of $B_1$-ENPG co-bipartite graphs is also work in progress.

We also showed that unlike $B_k$-EPG graphs that always have a representation in which every path has exactly $k$ bends, some $B_1$-ENPG graphs can not be represented using only paths having (exactly) one bend. One can define and study the graphs of edge intersecting non-splitting paths with exactly $k$ bends. Another possible direction is to follow the approach of [Cameron et al. (2016)] and consider $B_1$-ENPG representations restricted to subsets of the four possible rectilinear paths with one bend.

We showed that trees and cycles are $B_1$-ENPG. The characterization of their representations is work in progress. A natural extension of such a characterization is to investigate the relationship of $B_1$-ENPG graphs and cactus graphs. Another possible extension is to use the characterization of the special case of $C_4$ to characterize induced sub-grids. A non-trivial characterization would imply that not every bipartite graph is $B_1$-ENPG. Therefore, it would be natural to consider the recognition problem of $B_1$-ENPG bipartite graphs. The following interpretation of our results suggests that the latter problem is NP-hard:
Algorithm 1 $B_1$-ENPG $\cap$ Co-bipartite Recognition

**Require:** A co-bipartite graph $G = C(K, K', E)$
1. if $G$ is not connected then return "YES" \hspace{1cm} \triangleright G has a trivial $B_1$-ENPG representation.
2. Make $G$ twin-free using modular decomposition.
3. if $\text{ISTypeI}(G, \emptyset)$ then return "YES".
4. if $\text{ISTypeII}(G)$ then return "YES".
5. return "NO".

6. function $\text{ISTypeI}(G = C(K, K', E), Z)$

**Require:** $G$ is connected, twin-free, $Z \subseteq V(G)$

**Ensure:** returns whether there is a certificate $Z' \supseteq Z$ for $G$ being Type I
7. $G_B \leftarrow B(K, K', E)$.
8. if $G[Z]$ is not a zed then return "NO".
9. $Z' \leftarrow \text{FINDBIMODULEZED}(G, Z)$.
10. if $Z' = Z$ then \hspace{1cm} \triangleright Z is a zed of $G$ and also a bimodule of $G_B$
11. if $\text{ISDIFFERENCE}(G_B \setminus Z)$ then return "YES".
12. Let $U \subseteq (K \cup K') \setminus Z$ such that $G_B[U]$ is a $2K_2$.
13. for $u \in U$ do
14. if $\text{ISTypeI}(G, Z \cup \{u\})$ then return "YES".
15. return "NO".
16. else
17. if $Z' \neq \text{NULL}$ then return $\text{ISTypeI}(G, Z')$.
18. else return "NO".

19. function $\text{ISTypeII}(G = C(K, K', E))$

**Require:** $G$ is connected, twin-free

**Ensure:** returns whether $G$ has a Type II representation
20. $G_B \leftarrow B(K, K', E)$.
21. Remove all isolated vertices from $G_B$. \hspace{1cm} \triangleright There are at most two of them
22. Calculate the connected components $G_1, \ldots, G_k$ of $G_B$.
23. if $k > 2$ then return "NO".
24. if not $\text{ISDIFFERENCE}(G_1)$ then return "NO".
25. if not $\text{ISDIFFERENCE}(G_2)$ then return "NO".
26. return "YES".

27. function $\text{FINDBIMODULEZED}(G = C(K, K', E), Z)$

**Require:** $G$ is twin-free, $Z$ is a zed of $G$

**Ensure:** Returns the minimum superset of $Z$ that is a zed of $G$ and a bimodule of $G_B$
28. if $|Z \cap K| \leq 1$ and $|Z \cap K'| \leq 1$ then return $Z$.
29. Let without loss of generality $Z \cap K = \{u_1, u_2\}$.
30. $Z' \leftarrow (N_{G_B}(u_1) \cup N_{G_B}(u_2)) \cup Z$.
31. if $Z'$ is not a zed then return NULL.
32. $U' \leftarrow Z' \cap K'$.
33. if $|U'| \leq 1$ then return $Z'$.
34. Let without loss of generality $U' = \{u'_1, u'_2\}$.
35. $Z'' \leftarrow (N_{G_B}(u'_1) \cup N_{G_B}(u'_2)) \cup Z'$.
36. if $Z'' = Z'$ then return $Z'$
37. else return NULL.

38. function $\text{ISDIFFERENCE}(G)$

**Require:** $G$ is bipartite

**Ensure:** Returns "YES" if $G$ is a difference graph and a $2K_2$ of $G$ otherwise.
A clique provides substantial information on the representation, and when the graph is partitioned into two cliques we are able to recognize $B_1$-ENPG graphs. However, the absence of one such clique (in case of split graphs) already makes the problem NP-hard. In case of bipartite graphs both of the cliques are absent.

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