Abstract

The objective of this paper is to develop a duality between a novel Entropy Martingale Optimal Transport problem \((A)\) and an associated optimization problem \((B)\). In \((A)\) we follow the approach taken in the Entropy Optimal Transport (EOT) primal problem by Liero et al. “Optimal entropy-transport problems and a new Hellinger-Kantorovic distance between positive measures”, Invent. math. 2018, but we add the constraint, typical of Martingale Optimal Transport (MOT) theory, that the infimum of the cost functional is taken over martingale probability measures, instead of finite positive measures, as in Liero et al. The Problem \((A)\) differs from the corresponding problem in Liero et al. not only by the martingale constraint, but also because we admit less restrictive relaxation terms \(D\), which may not have a divergence formulation. In Problem \((B)\) the objective functional, associated via Fenchel conjugacy to the terms \(D\), is not any more linear, as in OT or in MOT. This leads to a novel optimization problem which also has a clear financial interpretation as a non linear subhedging value. Our results allow us to establish a novel nonlinear robust pricing-hedging duality in financial mathematics, which covers a wide range of known robust results in its generality.

Keywords: Martingale Optimal Transport problem, Entropy Optimal Transport problem, Pricing-hedging duality, Robust finance, Pathwise finance.

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JEL Classification: C61, G13

1 Introduction

In this research we exploit optimal transport theory to develop the duality

\[
A := \inf_{Q \in \text{Mart}(\Omega)} (E_Q [c] + D_U (Q)) = \sup_{\Delta \in \mathcal{H}} \sup_{\varphi \in \Phi_{\Delta(c)}} S^U (\varphi) := B. \tag{1}
\]

In \((A)\) we recognize the approach taken in the Entropy Optimal Transport primal problem (Liero et al. [42]) with the additional constraints, typical of Martingale Optimal Transport (MOT),
that the infimum of the cost functional $c$ is taken over *martingale probability* measures, instead of finite positive *measures*, as in [42]. This is a consequence of the additional supremum over the integrands $\Delta \in H$ in problem (B), and of the cash additivity of the functional $S^U$. The functional $S^U$ is associated to a, typically non linear, utility functional $U$ and represents the pricing rule over continuous bounded functions $\varphi$ defined on $\Omega$. We observe that the marginal constraints, typical of Optimal Transport (OT) problems, in (A) are relaxed by introducing the functional $D_U$, also associated to the map $U$, which may have a divergence formulation. The counterpart of this in Problem (B) is that the functional $S^U$, associated via Fenchel coniugacy to the penalization functional $D_U$ is not necessarily linear, as in OT or in MOT. Both $S^U$ and $D_U$ may also depend on some martingale measure $\hat{Q}$ which in the classical OT and MOT theories, plays the role of fixing the marginals. The duality (1) generalizes the well known robust pricing hedging duality in financial mathematics.

We provide a clear financial interpretation of both problems and observe that the novel concept of a non linear subhedging value expressed by (B) was not previously considered in the literature.

### 1.1 Pricing-hedging Duality in Financial Mathematics

The notion of subhedging price is one of the most analyzed concepts in financial mathematics. In this introduction we will take the point of view of the subhedging price, but obviously an analogous theory for the superhedging price can be developed as well. We are assuming a discrete time market model with zero interest rate. It may be convenient for the reader to have at hand the summary described in Table 1 on page 11.

**The classical setup** In the classical setup of stochastic securities market models, one considers an adapted stochastic process $X = (X_t)_{t=0,...,T}$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, representing the price of some underlying asset. Let $\mathcal{P}(P)$ be the set of all probability measures on $\Omega$ that are absolutely continuous with respect to $P$, $\text{Mart}(\Omega)$ be the set of all probability measures on $\Omega$ under which $X$ is a martingale and $\mathcal{M}(P) = \mathcal{P}(P) \cap \text{Mart}(\Omega)$. We also let $\mathcal{H}$ be the class of admissible integrands and $I^\Delta := I^\Delta(X)$ be the stochastic integral of $X$ with respect to $\Delta \in \mathcal{H}$. Under reasonable assumptions on $\mathcal{H}$, the equality

$$E_Q \left[I^\Delta(X)\right] = 0$$

holds for all $Q \in \mathcal{M}(P)$ and, as well known, all linear pricing functionals compatible with no arbitrage are expectations $E_Q[\cdot]$ under some probability $Q \in \mathcal{M}(P)$ such that $Q \sim P$.

We denote with $p$ the **subhedging price** of a contingent claim $c : \mathbb{R} \to \mathbb{R}$ written on the payoff $X_T$ of the underlying asset. If we let $\mathcal{L}(P) \subseteq L^0((\Omega, \mathcal{F}_T, P))$ be the space of random payoff and let $Z := c(X_T) \in \mathcal{L}(P)$, then $p : \mathcal{L}(P) \to \mathbb{R}$ is defined by

$$p(Z) := \sup \left\{ m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H} \text{ s.t. } m + I^\Delta(X) \leq Z, \ P \text{ - a.s.} \right\}.$$  

(3)

The subhedging price is independent from the preferences of the agents, but it depends on the reference probability measure via the class of $P$-null events. It satisfies the following two key properties:
(CA) Cash Additivity on $\mathcal{L}(P)$: $p(Z + k) = p(Z) + k$, for all $k \in \mathbb{R}$, $Z \in \mathcal{L}(P)$.

(IA) Integral Additivity on $\mathcal{L}(P)$: $p(Z + I^\Delta) = p(Z)$, for all $\Delta \in \mathcal{H}$, $Z \in \mathcal{L}(P)$.

When a functional $p$ satisfies (CA), then $Z, k$ and $p(Z)$ must be expressed in the same monetary unit and this allows for the monetary interpretation of $p$, as the price of the contingent claim. This will be one of the key features that we will require also in the novel definition of the nonlinear subhedging value. The (IA) property and $p(0) = 0$ imply that the $p$ price of any stochastic integral $I^\Delta(X)$ is equal to zero, as in (2).

Since the seminal works of El Karoui and Quenez [28], Karatzas [41], Delbaen and Schachermayer [24], it was discovered that, under the no arbitrage assumption, the dual representation of the subhedging price $p$ is

$$p(Z) = \inf_{Q \in \mathcal{M}(P)} \mathbb{E}_Q[Z].$$

More or less in the same period, the concept of coherent risk measure was introduced in the pioneering work by Artzner et al. [3]. A Coherent Risk Measure $\rho : \mathcal{L}(P) \to \mathbb{R}$ determines the minimal capital required to make acceptable a financial position and its dual formulation is assigned by

$$-\rho(Y) = \inf_{Q \in \mathcal{Q} \subseteq \mathcal{P}(P)} \mathbb{E}_Q[Y],$$

where $Y$ is a random variable representing future profit-and-loss and $\mathcal{Q} \subseteq \mathcal{P}(P)$. Coherent Risk Measures $\rho$ are convex, cash additive, monotone and positively homogeneous. We take the liberty to label both the representations in (4) and in (5) as the “sublinear case”.

In the study of incomplete markets the concept of the (buyer) indifference price $p^b$, originally introduced by Hodges and Neuberger [39], received, in the early 2000, increasing consideration (see Frittelli [30], Rouge and El Karoui [45], Delbaen et al. [23], Bellini and Frittelli [7]) as a tool to assess, consistently with the no arbitrage principle, the value of non replicable contingent claims, and not just to determine an upper bound (the superhedging price) or a lower bound (the subhedging price) for the price of the claim. Differently from the notion of subhedging, $p^b$ is based on some concave increasing utility function $u : \mathbb{R} \to [-\infty, +\infty)$ of the agent. By defining the indirect utility function

$$U(w_0) := \sup_{\Delta \in \mathcal{H}} \mathbb{E}_P[u(w_0 + I^\Delta(X))],$$

where $w_0 \in \mathbb{R}$ is the initial wealth, the indifference price $p^b$ is defined as

$$p^b(Z) := \sup\{m \in \mathbb{R} \mid U(Z - m) \geq U(0)\}.$$

Under suitable assumptions, the dual formulation of $p^b$ is

$$p^b(Z) = \inf_{Q \in \mathcal{M}(P)} \{\mathbb{E}_Q[Z] + \alpha_u(Q)\},$$

and the penalty term $\alpha_u : \mathcal{M}(P) \to [0, +\infty]$ is associated to the particular utility function $u$ appearing in the definition of $p^b$ via the Fenchel conjugate of $u$. We observe that in case of the exponential utility function $u(x) = 1 - \exp(-x)$, the penalty is $\alpha_{\exp}(Q) := H(Q, P) - \min_{Q \in \mathcal{M}(P)} H(Q, P)$, where

$$H(Q, P) := \int F\left(\frac{dQ}{dP}\right) dP, \text{ if } Q \ll P \text{ and } F(y) = y \ln(y),$$

and

$$\alpha_{\exp}(Q) := \int F\left(\frac{dQ}{dP}\right) dP, \text{ if } Q \ll P \text{ and } F(y) = y \ln(y),$$

for $Q \ll P$. This completes the discussion on the sublinear case.
is the relative entropy. In this case, the penalty \( \alpha_{\exp} \) is a divergence functional, as those that will be considered below in Section 3.4. Observe that the functional \( p^b \) is concave, monotone increasing and satisfies both properties (CA) and (IA), but it is not necessarily linear on the space of all contingent claims. As recalled in the conclusion of Frittelli [30], “there is no reason why a price functional defined on the whole space of bundles and consistent with no arbitrage should be linear also outside the space of marketed bundles”.

It was exactly the particular form (6) of the indifference price that suggested to Frittelli and Rosazza Gianin [31] to introduce the concept of Convex Risk Measure (also independently introduced by Follmer and Schied [29]), as a map \( \rho : \mathcal{L}(P) \to \mathbb{R} \) that is convex, cash additive and monotone decreasing. Under good continuity properties, the Fenchel Moreau Theorem shows that any convex risk measure admits the following representation

\[
-\rho(Y) = \inf_{Q \in \mathcal{P}(P)} \{ E_Q[Y] + \alpha(Q) \}
\]

for some penalty \( \alpha : \mathcal{P}(P) \to [0, +\infty] \). We will then label functional in the form (6) or (7) as the “convex case”. As a consequence of the cash additivity property, in the dual representations (6) or (7) the infimum is taken with respect to probability measures, namely with respect to normalized non negative elements in the dual space, which in this case can be taken as \( L^1(P) \). Differently from the indifference price \( p^b \), convex risk measures do not necessarily take into account the presence of the stochastic security market, as reflected by the absence of any reference to martingale measures in the dual formulation (7) and (5), in contrast to (6) and (4).

**Pathwise finance** As a consequence of the financial crises in 2008, the uncertainty in the selection of a reference probability \( P \) gained increasing attention and led to the investigation of the notions of arbitrage and the pricing hedging duality in different settings. On the one hand, the single reference probability \( P \) was replaced with a family of - a priori non dominated - probability measures, leading to the theory of Quasi-Sure Stochastic Analysis (see Bayraktar and Zhang [4], Bayraktar and Zhou [5], Bouchard and Nutz [12], Cohen [18], Denis and Martini [25], Peng [43], Soner et al. [48]). On the other hand, taking a even more radical approach, a probability free, pathwise, theory of financial markets was developed, as in Acciaio et al. [1], Burzoni et al. [16], Burzoni et al. [17], Burzoni et al. [15], Riedel [44]. In such framework, Optimal Transport theory became a very powerful tool to prove pathwise pricing-hedging duality results with very relevant contributions by many authors (Beiglböck et al. [6], Davis et al. [22], Dolinski and Soner [26], Dolinsky and Soner [27]; Galichon et al. [33], Henry-Labordère [34], Henry-Labordère et al. [35]; Hou and Obłój [40], Tan and Touzi [49]). These contributions mainly deal with what we labeled above as the sublinear case, while our main interest in this paper is to develop the convex case theory, as explained below.

From now on, we will abandon the classical setup described above and work without a reference probability measure. We consider \( T \in \mathbb{N}, T \geq 1 \), and

\[
\Omega := K_0 \times \cdots \times K_T
\]

for \( K_0, \ldots, K_T \) subsets of \( \mathbb{R} \) and denote with \( X_0, \ldots, X_T \) the canonical projections \( X_t : \Omega \to K_t \),
for \( t = 0, 1, \ldots, T \). We denote

\[
\text{Mart}(\Omega) := \{\text{Martingale probability measures for the canonical process of } \Omega\},
\]

and when \( \mu \) is a measure defined on the Borel \( \sigma \)-algebra of \((K_0 \times \cdots \times K_T)\), its marginals will be denoted with \( \mu_0, \ldots, \mu_T \). We consider a contingent claim \( c : \Omega \to (-\infty, +\infty] \) which is now allowed to depend on the whole path and we will adopt semistatic trading strategies for hedging. This means that in addition to dynamic trading in \( X \) via the admissible integrands \( \Delta \in \mathcal{H} \), we may invest in “vanilla” options \( \varphi_t : K_t \to \mathbb{R} \). For modeling purposes we take vector subspaces \( \mathcal{E}_t \subseteq C_b(K_t) \) for \( t = 0, \ldots, T \), where \( C_b(K_t) \) is the space of real-valued, continuous, bounded functions on \( K_t \). For each \( t \), \( \mathcal{E}_t \) is the set of static options that can be used for hedging, say affine combinations of options with different strikes and same maturity \( t \). The key assumption in the robust, Optimal Transport based, formulation is that the marginals \( (\hat{Q}_0, \hat{Q}_1, \ldots, \hat{Q}_T) \) of the underlying price process \( X \) are known. This assumption can be justified (see the seminal papers by Breeden and Litzenberger [13] and Hobson [38], as well as the many contributions by Hobson [36], Cox and Obłój [19], [20], Cox and Wang [21], Labordère et al. [35], Brown et al. [14], Hobson and Klimmerk [37]) by assuming the knowledge of a sufficiently large number of plain vanilla options maturing at each intermediate date. Thus the class of arbitrage-free pricing measures that are compatible with the observed prices of the options is given by

\[
\mathcal{M}(\hat{Q}_0, \hat{Q}_1, \ldots, \hat{Q}_T) := \left\{ Q \in \text{Mart}(\Omega) \mid X_t \sim \hat{Q}_t \text{ for each } t = 0, \ldots, T \right\}.
\]

In this framework,

\[
\mathcal{H} := \{ \Delta = [\Delta_0, \ldots, \Delta_{T-1}] \mid \Delta_t \in C_b(K_0 \times \cdots \times K_t; \mathbb{R}) \}\]

(8)

and the sub-hedging duality, obtained in [6] Th. 1.1, takes the form:

\[
\inf_{Q \in \mathcal{M}(\hat{Q}_0, \hat{Q}_1, \ldots, \hat{Q}_T)} E_Q[c] = \sup \left\{ \sum_{t=0}^{T} E_{\hat{Q}_t}[\varphi_t] \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi_t(x_t) + I^\Delta(x) \leq c(x) \forall x \in \Omega \right\},
\]

(10)

where the RHS of (10) is known as the robust subhedging price of \( c \). Comparing (10) with the duality between (3) and (4), we observe that: (i) the \( P\)–a.s. inequality in (3) has been replaced by an inequality that holds for all \( x \in \Omega \); (ii) in (10) the infimum of the price of the contingent claim \( c \) is taken under all martingale measure compatible with the option prices, with no reference to the probability \( P \); (iii) static hedging with options is allowed. As can be seen from the LHS of (10), this case falls into the category labeled above as the sublinear case, and the purpose of this paper is to investigate the convex case, in the robust setting, using the tools from Entropy Optimal Transport (EOT) recently developed in Liero et al. [42]. Let us first describe the financial interpretation of the problems that we are going to study.

**The dual problem** The LHS of (10), namely \( \inf_{Q \in \mathcal{M}(\hat{Q}_0, \hat{Q}_1, \ldots, \hat{Q}_T)} E_Q[c] \), represents the dual problem in the financial application, but is typically the primal problem in Martingale Optimal Transport (MOT)
We label this case as the sublinear case of MOT. In [42], the primal Entropy Optimal Transport (EOT) problem takes the form

\[ \inf_{\mu \in \text{Meas}(\Omega)} \left( E_Q[c] + \sum_{t=0}^{T} D_{F_t, \hat{Q}_t}(\mu_t) \right), \] (11)

where Meas(\Omega) is the set of all positive finite measures \( \mu \) on \( \Omega \), and \( D_{F_t, \hat{Q}_t}(\mu_t) \) is a divergence in the form:

\[ D_{F_t, \hat{Q}_t}(\mu_t) := \begin{cases} \int K_t F_t \left( \frac{d\mu_t}{d\hat{Q}_t} \right) d\hat{Q}_t, & \text{if } \mu_t \ll \hat{Q}_t, \\ +\infty, & \text{otherwise} \end{cases} \] (12)

We label with \( F := (F_t)_{t=0,...,T} \) the family of divergence functions appearing in (12). Problem (11) represents the convex case of OT theory.

Notice that in the EOT primal problem (11) the typical constraint that \( \mu \) has prescribed marginals \( (\hat{Q}_0, \hat{Q}_1, ..., \hat{Q}_T) \) is relaxed (as the infimum is taken with respect to all positive finite measures) by introducing the divergence functional \( D_{F_t, \hat{Q}_t}(\mu_t) \), which penalizes those measures \( \mu \) that are “far” from some reference marginals \( (\hat{Q}_0, \hat{Q}_1, ..., \hat{Q}_T) \). We are then naturally let to the study of the convex case of MOT, i.e. to the Entropy Martingale Optimal Transport (EMOT) problem

\[ D_{F, \hat{Q}}(c) := \inf_{Q \in \text{Mart}(\Omega)} \left( E_Q[c] + \sum_{t=0}^{T} D_{F_t, \hat{Q}_t}(Q_t) \right) \] (13)

having also a clear financial interpretation. The marginals are not any more fixed a priori, as in (10), because we may not have sufficient information to detect them with enough accuracy. So the infimum is taken over all martingale probability measures, but those that are far from some estimate \( (\hat{Q}_0, \hat{Q}_1, ..., \hat{Q}_T) \) are appropriately penalized through \( D_{F_t, \hat{Q}_t}(\mu_t) \). Of course, when \( D_{F_t, \hat{Q}_t}(\cdot) = \delta_{\hat{Q}_t}(\cdot) \), we recover the sublinear MOT problem, where only martingale probability measures with fixed marginals are allowed. Observe that in addition to the martingale property, the elements \( Q \in \text{Mart}(\Omega) \) in (13) are required to be probability measures, while in the EOT (11) theory all positive finite measure are allowed. As it was recalled after equation (7), this normalization feature of the dual elements \( (\mu(\Omega) = 1) \) is not surprising when one deals with dual problems of primal problems with a cash additive objective functional as, for example, in the theory of coherent and convex risk measures.

Potentially, we could push our smoothing argument above even further: in place of the functionals \( D_{F_t, \hat{Q}_t}(\mu_t), t = 0, ..., T \), we might as well consider more general marginal penalizations, not necessarily in the divergence form (12), yielding the problem

\[ \mathcal{D}(c) := \inf_{Q \in \text{Mart}(\Omega)} \left( E_Q[c] + \sum_{t=0}^{T} D_t(Q_t) \right). \] (14)

These penalizations \( \mathcal{D}_0, ..., \mathcal{D}_T \) will be better specified later.

**The primal problem: the Nonlinear Subhedging Value** We provide the financial interpretation of the primal problem which will yield the EMOT problem \( \mathcal{D}_{F, \hat{Q}} \) as its dual. It is convenient to reformulate the robust subhedging price in the RHS of (10) in a more general setting.
Definition 1.1. Consider a measurable function \( c : \Omega \to \mathbb{R} \) representing a (possibly path dependent) option, the set \( V \) of hedging instruments and a suitable pricing functional \( \pi : V \to \mathbb{R} \). Then the robust Subhedging Value of \( c \) is defined by

\[
\Pi_{\pi,V}(c) = \sup \{ \pi(v) \mid v \in V \text{ s.t. } v \leq c \}.
\]

In the classical setting, functionals of this form (and even with a more general formulation) are known as general capital requirement, see for example Frittelli and Scandolo [32]. We stress however that in Definition 1.1 the inequality \( v \leq c \) holds for all elements in \( \Omega \) with no reference to a probability measure whatsoever. The novelty in this definition is that a priori \( \pi \) may not be linear and it is crucial to understand which evaluating functional \( \pi \) we may use. For our discussion, we assume that the vector subspaces \( \mathcal{E}_t \subseteq C_b(K_t) \) satisfies \( \mathcal{E}_t + \mathbb{R} = \mathcal{E}_t \), for \( t = 0, \ldots, T \). We let \( \mathcal{E} := \mathcal{E}_0 \times \cdots \times \mathcal{E}_T \), and \( V := \mathcal{E}_0 + \cdots + \mathcal{E}_T + \mathcal{I} \). Suppose we took a linear pricing rule \( \pi : V \to \mathbb{R} \) defined via a \( \bar{Q} \in \text{Mart}(\Omega) \) by

\[
\pi(v) := E_{\bar{Q}} \left[ \sum_{t=0}^{T} \varphi_t + I^\Delta \right] \overset{(i)}{=} E_{\bar{Q}} \left[ \sum_{t=0}^{T} \varphi_t \right] \overset{(ii)}{=} \sum_{t=0}^{T} E_{\bar{Q}}[\varphi_t],
\]

where we used (2) and the fact that \( \bar{Q}_t \) is the marginal of \( \bar{Q} \). In this case, we would trivially obtain for the robust subhedging value of \( c \)

\[
\Pi_{\pi,V}(c) = \sup \{ \pi(v) \mid v \in V \text{ s.t. } v \leq c \}
\]

\[
= \sup \left\{ \sum_{t=0}^{T} E_{\bar{Q}_t}[\varphi_t] \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi_t(x_t) + I^\Delta(x) \leq c(x) \forall x \in \Omega \right\}
\]

\[
= \sup \left\{ m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H}, \varphi \in \mathcal{E}, \text{ s.t. } m - \sum_{t=0}^{T} E_{\bar{Q}_t}[\varphi_t] + \sum_{t=0}^{T} \varphi_t + I^\Delta \leq c \right\}
\]

\[
= \sup \left\{ m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H}, \varphi \in \mathcal{E}, \text{ with } E_{\bar{Q}_t}[\varphi_t] = 0 \text{ s.t. } m + \sum_{t=0}^{T} \varphi_t + I^\Delta \leq c \right\},
\]

where in the last equality we replaced \( \varphi_t \) with \( (E_{\bar{Q}_t}[\varphi_t] - \varphi_t) \in \mathcal{E}_t \), which satisfies:

\[
E_{\bar{Q}_t} \left[ E_{\bar{Q}_t}[\varphi_t] - \varphi_t \right] = 0.
\]

Interpretation: \( \Pi_{\pi,V}(c) \) is the supremum amount \( m \in \mathbb{R} \) for which we may buy options \( \varphi_t \) and dynamic strategies \( \Delta \in \mathcal{H} \) such that \( m + \sum_{t=0}^{T} \varphi_t + I^\Delta \leq c \), where the value of both the options and the stochastic integrals are computed as the expectation under the same martingale measure (\( \bar{Q} \) for the integral \( I^\Delta \); its marginals \( \bar{Q}_t \) for each option \( \varphi_t \)).

However, as mentioned above when presenting the indifferent price \( p^b \), there is a priori no reason why one has to allow only linear functional in the evaluation of \( v \in V \).

We thus generalize the expression for \( \Pi_{\pi,V}(c) \) by considering valuation functionals \( S : V \to \mathbb{R} \) and \( S_t : \mathcal{E}_t \to \mathbb{R} \) more general than \( E_{\bar{Q}}[\cdot] \) and \( E_{\bar{Q}_t}[\cdot] \).

Nonetheless, in order to be able to repeat the same key steps we used in (16)-(17) and therefore to keep the same interpretation, we shall impose that such functionals \( S \) and \( S_t \) satisfy the property in (18) and the two properties (i) and (ii) in equation (15), that is:
(a) $S_t[\varphi_t + k] = S_t[\varphi_t] + k$ and $S_t[0] = 0$, for all $\varphi_t \in C_b(K_t)$, $k \in \mathbb{R}$, $t = 0, \ldots, T$.

(b) $S \left[ \left( \sum_{t=0}^{T} \varphi_t \right) + I^\Delta(x) \right] = S \left[ \sum_{t=0}^{T} \varphi_t \right]$ for all $\Delta \in \mathcal{H}$ and $\varphi \in \mathcal{E}$.

(c) $S \left[ \sum_{t=0}^{T} \varphi_t \right] = \sum_{t=0}^{T} S_t[\varphi_t]$ for all $\varphi \in \mathcal{E}$.

We immediately recognize that (a) is the Cash Additivity (CA) property on $C_b(K_t)$ of the functional $S_t$ and (b) implies the Integral Additivity (IA) property on $\mathcal{V}$. As a consequence, repeating the same steps in (16)-(17), we will obtain as primal problem the nonlinear subhedging value of $c$:

$$\mathcal{P}(c) = \sup \left\{ S(v) \mid v \in \mathcal{V} : v \leq c \right\}$$

$$= \sup \left\{ \sum_{t=0}^{T} S_t(\varphi_t) \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi_t(x_t) + I^\Delta(x) \leq c(x) \forall x \in \Omega \right\}$$

$$= \sup \left\{ m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H}, \varphi \in \mathcal{E}, \text{ with } S_t(\varphi_t) = 0 \text{ s.t. } m + \sum_{t=0}^{T} \varphi_t + I^\Delta \leq c \right\},$$

(19)

(20)

to be compared with (17).

Interpretation: $\mathcal{P}(c)$ is the supremum amount $m \in \mathbb{R}$ for which we may buy zero value options $\varphi_t$ and dynamic strategies $\Delta \in \mathcal{H}$ such that $m + \sum_{t=0}^{T} \varphi_t + I^\Delta \leq c$, where the value of both the options and the stochastic integrals are computed with the same functional $S$.

Before further elaborating on these issues, let us introduce the concept of Stock Additivity, which is the natural counterpart of properties (IA) and (CA) when we are evaluating hedging instruments depending solely on the value of the underlying stock $X$ at some fixed date $t \in \{0, \ldots, T\}$. Let $\text{Id}_t$ be the identity function $x_t \mapsto x_t$ on $K_t$. As before, the set of such instruments is denoted by $\mathcal{E}_t \subseteq C_b(K_t)$ and we will suppose that $\text{Id}_t \in \mathcal{E}_t$ (that is, we can use units of stock at time $t$ for hedging) and that $\mathcal{E}_t + \mathbb{R} = \mathcal{E}_t$ (that is, deterministic amounts of cash can be used for hedging as well).

**Definition 1.2.** A functional $p_t : \mathcal{E}_t \rightarrow \mathbb{R}$ is stock additive on $\mathcal{E}_t$ if $p_t(0) = 0$ and

$$p_t(\varphi_t + \alpha_t\text{Id}_t + \lambda_t) = p_t(\varphi_t) + \alpha_t x_0 + \lambda_t \quad \forall \varphi_t \in \mathcal{E}_t, \lambda_t \in \mathbb{R}, \alpha_t \in \mathbb{R},$$

We now clarify the role of stock additive functionals in our setup. Suppose that $S_t : \mathcal{E}_t \rightarrow \mathbb{R}$ are stock additive on $\mathcal{E}_t$, $t = 0, \ldots, T$. It can be shown (see Lemma A.7) that if there exist $\varphi, \psi \in \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$ and $\Delta \in \mathcal{H}$ such that $\sum_{t=0}^{T} \varphi_t = \sum_{t=0}^{T} \psi_t + I^\Delta$ then

$$\sum_{t=0}^{T} S_t(\varphi_t) = \sum_{t=0}^{T} S_t(\psi_t).$$

This allows us to define a functional $S : \mathcal{V} = \mathcal{E}_0 + \cdots + \mathcal{E}_T + \mathbb{R} \rightarrow \mathbb{R}$ by

$$S(v) := \sum_{t=0}^{T} S_t(\varphi_t), \quad \text{for } v = \sum_{t=0}^{T} \varphi_t + I^\Delta.$$  

(21)

Then $S$ is a well defined, integral additive functional on $\mathcal{V}$, and $S, S_0, \ldots, S_T$ satisfy the properties (a), (b), (c). There is a natural way to produce a variety of Stock Additive functionals, as explained in Example 1.3 below.
Example 1.3. Consider a Martinagle measure $\hat{Q} \in \text{Mart}(\Omega)$ and a concave non decreasing utility function $u_t : \mathbb{R} \to [-\infty, +\infty]$, satisfying $u(0) = 0$ and $u(x_t) \leq x_t \forall x_t \in \mathbb{R}$. We can then take

$$S_t(\varphi_t) = U_{\hat{Q}_t}(\varphi_t) := \sup_{\alpha \in \mathbb{R}, \lambda \in \mathbb{R}} \left( \int_{\Omega} u_t (\varphi_t(x_t) + \alpha x_t + \lambda) \ d\hat{Q}_t(x_t) - (\alpha x_0 + \lambda) \right).$$

As shown in Lemma 4.2 the stock additivity property is then satisfied for these functionals.

When we consider stock additive functionals $S_0, \ldots, S_T$ that induce the functional $S$ as explained in (21), we can focus our attention to the optimization problem (19) or (20), that will be referred to as our primal problem.

The Duality  As a consequence of our main results we prove the following duality (see Theorem 3.3). If

$$D_t(Q_t) := \sup_{\varphi_t \in \mathcal{E}_t} \left( S_t(\varphi_t) - \int_{K_t} \varphi_t \ dQ_t \right)$$

for $Q_t \in \text{Prob}(K_t)$, $t = 0, \ldots, T$, and $D(c)$ and $\mathcal{P}(c)$ are defined respectively in (14) and (19), then

$$D(c) = \mathcal{P}(c).$$

In the particular case of $S_0, \ldots, S_T$ induced by utility functions, as explained in Example 1.3, the problem corresponding to (19) will be denoted by $\mathcal{P}_{U,\hat{Q}}(c)$, that is

$$\mathcal{P}_{U,\hat{Q}}(c) := \sup \left\{ \sum_{t=0}^{T} U_{\hat{Q}_t}(\varphi_t) \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi_t(x_t) + I^{\Delta}(x) \leq c(x) \forall x \in \Omega \right\}. \quad (22)$$

We also show the duality between (13) and (22), namely we prove in Section 4.1

$$D_{F_i,\hat{Q}}(c) := \inf_{Q \in \text{Mart}(\Omega)} \left( E_Q[c] + \sum_{t=0}^{T} D_{F_i,\hat{Q}_t}(Q_t) \right) = \mathcal{P}_{U,\hat{Q}}(c). \quad (23)$$

The divergence functions $F_i$ appearing in $D_{F_i,\hat{Q}}$ (via $D_{F_i,\hat{Q}_t}$) are associated to the utility functions $u_t$ appearing in $U_{\hat{Q}_t}$ and in $\mathcal{P}_{U,\hat{Q}}$ via the coniugacy relation:

$$F_i(y) := v^*_i(y) = \sup_{x_t \in \mathbb{R}} \left\{ x_t y - v(y) \right\} = \sup_{x_t \in \mathbb{R}} \left\{ u_t(x_t) - x_t y \right\},$$

where $v(y) := -u(-y)$. Thus, depending on which utility function $u$ is selected in the primal problem $\mathcal{P}_{U,\hat{Q}}(c)$ to evaluate the options through $U_{\hat{Q}_t}$, the penalization term $D_{F_i,\hat{Q}_t}$ in the dual formulation $D_{F_i,\hat{Q}}(c)$ has a particular form induced by $F_i = v^*_i$. In the special case of linear utility functions $u_t(x_t) = x_t$, we recover the sublinear MOT theory. Indeed, in this case, $v^*_i(y) = +\infty$, for all $y \neq 1$ and $v^*_i(1) = 0$, so that $D_{F_i,\hat{Q}_t}(\cdot) = \delta_{\hat{Q}_t}(\cdot)$ and thus we obtain the robust pricing-hedging duality (10) of the classical MOT.

1.2 Entropy Martingale Optimal Transport Duality

In the previous section we described and provided the financial interpretation of the new duality (23). This will be a particular case of a more general duality established in Theorem 2.3 and Theorem 2.4 and announced in the first section of this Introduction.
In our main result (Theorem 2.4) we start by introducing two general functionals, $U : \mathcal{E} \to [-\infty, +\infty)$ and $D_U : \text{ca}(\Omega) \to (-\infty, +\infty]$, that are associated through a Fenchel Moreau type relation (see 26). The vector space $\mathcal{E} \subseteq \mathcal{C}_b(\Omega; \mathbb{R}^{T+1})$ consists of continuous and bounded functions defined on some Polish space $\Omega$ and with values in $\mathbb{R}^{T+1}$. The map $U$ is not necessarily cash additive. We then rely on the notion of the Optimized Certainty Equivalent (OCE), that was introduced in Ben Tal and Teboulle [8] and further analyzed in Ben Tal and Teboulle [9]. As it is easily recognized, any OCE is, except for the sign, a particular convex risk measure and so it is cash additive. We introduce the Generalized Optimized Certainty Equivalent associated to $U$ as the functional $S^U : \mathcal{E} \to [-\infty, +\infty]$ defined by

$$S^U(\varphi) := \sup_{\xi \in \mathbb{R}^{T+1}} \left( U(\varphi + \xi) - \sum_{t=0}^{T} \xi_t \right), \quad \varphi \in \mathcal{E}. \quad (24)$$

Thus we obtain a cash additive map $S^U(\varphi + \xi) = S^U(\varphi) + \sum_{t=0}^{T} \xi_t$, which will guarantee that in the problem (11) the elements $\mu \in \text{Meas}(\Omega)$ are normalized, i.e. are probability measures. Then the duality will take the form

$$\inf_{Q \in \text{Mart}(\Omega)} (E_Q [c(X)] + D_U(Q)) = \sup_{\Delta \in \mathcal{H}} \sup_{\varphi \in \Phi_{\Delta}(c)} S^U(\varphi), \quad (25)$$

where

$$\Phi_{\Delta}(c) := \left\{ \varphi \in \text{dom}(U), \sum_{t=0}^{T} \varphi_t(x_t) + I^\Delta(x) \leq c(x) \ \forall \ x \in \Omega \right\},$$

and we also prove the existence of the optimizer for the problem in the LHS of (25), see Proposition 2.6.

The penalization term $D := D_U$ associated to $U$ does not necessarily have an additive structure ($D(Q) = \sum_{t=0}^{T} D_t(Q_t)$) nor needs to have the divergence formulation, as described in (12), and so it does not necessarily depend on a given martingale measure $\hat{Q}$. As explained in Sections 4.1.2 and 4.2 this additional flexibility in choosing $D$ allows several different interpretation and constitutes one generalization of the Entropy Optimal Transport theory of [42]. Of course, the other additional difference with EOT is the presence in (25) of the additional supremum with respect to admissible integrand $\Delta \in \mathcal{H}$. As a consequence, in the LHS of (25) the infimum is now taken with respect to martingale measures. We also point out that in [42], the cost functional $c$ is required to be lower semicontinuous and nonnegative and that the theory is developed only for the bivariate case ($t = 0, 1$), while in this paper we take $c$ lower semicontinuous and with compact level sets, hence bounded from below, and consider the multivariate case ($t = 0, ..., T$).

In [42] the authors work with a Hausdorff topological space $\Omega$, while we request $\Omega$ to be a Polish space and for some of the results even a compact subset of $\mathbb{R}^N$. This stronger assumption however is totally reasonable for the applications we deal with (see Remark 4.4) and, in case of the Polish space assumption, it would be also compatible with a theory for stochastic processes $X$ in continuous time, a topic left for future investigation.

To conclude, our framework allows to establish and comprehend several different duality results, even if under different type of assumptions:

1. The new non linear robust pricing-hedging duality with options described in (23) and proved in Section 4.1, Corollary 4.3.
2. The new non linear robust pricing-hedging duality with options and singular components (see Corollary 4.5).

3. The linear robust pricing-hedging duality with options (see [6] Th. 1.1, or Acciaio et al. [1] Th 1.4) described in (10) and proved in Corollary 4.7.

4. The linear robust pricing-hedging duality without options (see for example Burzoni et al. [17] Th. 1.1) proved in Corollary 4.8.

5. A new robust pricing-hedging duality with penalization function based on market data (see Section 4.1.2).

6. A new dual robust representation for the Optimized Certainty Equivalent functional (see Section 4.2.1).

We will explain in Section 4.2 why the degree of freedom in the choice of $\mathcal{E}$ may be relevant also for the application in the financial context.

We summarize the preceding discussion in the following Table and we point out that in this paper we develop the duality theory sketched in the last line of the Table and provide its financial interpretation. Differently from rows 1, 2, 5, 6, in rows 3, 4, 7, 8, the financial market is present and martingale measures are involved in the dual formulation. In rows 1, 2, 3, 4 we illustrate the classical setting, where the conditions in the functional form hold $P$-a.s., while in the last four rows Optimal Transport is applied to treat the robust versions, where the inequalities holds for all elements of $\Omega$.

Table 1: $\Pi(\Omega)$ is the set of all probabilities on $\Omega$; $\mathcal{P}(P) = \{Q \in \Pi(\Omega) \mid Q \ll P\}$; $\text{Mart}(\Omega)$ is the set of all martingale probabilities on $\Omega$; $\mathcal{M}(P) = \text{Mart}(\Omega) \cap \mathcal{P}(P)$; $\Pi(Q_1, Q_2) = \{Q \in \Pi(\Omega) \text{ with given marginals} \}$; $\text{Mart}(Q_1, Q_2) = \{Q \in \text{Mart}(\Omega) \text{ with given marginals} \}$; $\text{Meas}(\Omega)$ is the set of all positive finite measures on $\Omega$; $\text{Sub}(c)$ is the set of static parts of semistatic subhedging strategies for $c$; $U$ is a concave proper utility functional and $S^U$ is the associated generalized Optimized Certainty Equivalent.

|   | FUNCTIONAL FORM | SUBLINEAR | CONVEX |
|---|----------------|-----------|---------|
| 1 | - Coherent R.M. | $- \inf \{m \mid c + m \in \mathcal{A}\}$, $\mathcal{A}$ cone | $\inf_{Q \in \mathcal{P}(P)} E_Q [c]$ | $\inf_{Q \in \mathcal{P}(P)} (E_Q [c] + \alpha_\mathcal{A}(Q))$ |
| 2 | - Convex R.M. | $- \inf \{m \mid c + m \in \mathcal{A}\}$, $\mathcal{A}$ convex | $\inf_{Q \in \mathcal{P}(P)} E_Q [c]$ | $\inf_{Q \in \mathcal{P}(P)} (E_Q [c] + \alpha_\mathcal{A}(Q))$ |
| 3 | Subreplic. price | $\sup \{m \mid \exists \Delta : m + f_{\Delta}(X) \leq c\}$ | $\inf_{Q \in \mathcal{P}(P)} E_Q [c]$ | $\inf_{Q \in \mathcal{P}(P)} (E_Q [c] + \alpha_\mathcal{A}(Q))$ |
| 4 | Indiff. price | $\sup \{m \mid U(c - m) \geq U(0)\}$ | $\inf_{Q \in \mathcal{P}(P)} E_Q [c]$ | $\inf_{Q \in \mathcal{P}(P)} (E_Q [c] + \alpha_\mathcal{A}(Q))$ |
| 5 | O.T. | $\sup_{\varphi + \psi \leq c} (E_{Q_1}[\varphi] + E_{Q_2}[\psi])$ | $\inf_{Q \in \mathcal{P}(P)} E_Q [c]$ | $\inf_{Q \in \mathcal{P}(P)} (E_Q [c] + \alpha_\mathcal{A}(Q))$ |
| 6 | E.O.T. | $\sup_{\varphi + \psi \leq c} U(\varphi, \psi)$ | $\inf_{Q \in \mathcal{P}(P)} E_Q [c]$ | $\inf_{Q \in \mathcal{P}(P)} (E_Q [c] + \alpha_\mathcal{A}(Q))$ |
| 7 | M.O.T. | $\sup_{[\varphi, \psi] \in \text{Sub}(c)} (E_{Q_1}[\varphi] + E_{Q_2}[\psi])$ | $\inf_{Q \in \mathcal{P}(P)} E_Q [c]$ | $\inf_{Q \in \mathcal{P}(P)} (E_Q [c] + \alpha_\mathcal{A}(Q))$ |
| 8 | E.M.O.T. | $\sup_{[\varphi, \psi] \in \text{Sub}(c)} S^U(\varphi, \psi)$ | $\inf_{Q \in \mathcal{P}(P)} E_Q [c]$ | $\inf_{Q \in \mathcal{P}(P)} (E_Q [c] + \alpha_\mathcal{A}(Q))$ |
2 A Generalized Optimal Transport Duality

For unexplained concepts on Measure Theory we refer to the Appendix A.1. We let $\Omega$ be a Polish Space and define the following sets:

$$
ca(\Omega) := \{ \gamma : \mathcal{B}(\Omega) \to (-\infty, +\infty) \mid \gamma \text{ is finite signed Borel measures on } \Omega \},
$$

$$
\text{Meas}(\Omega) := \{ \mu : \mathcal{B}(\Omega) \to [0, +\infty) \mid \mu \text{ is a non negative finite Borel measures on } \Omega \},
$$

$$
\text{Prob}(\Omega) := \{ Q : \mathcal{B}(\Omega) \to [0, 1] \mid Q \text{ is a probability Borel measures on } \Omega \}.
$$

$$
C_b(\Omega, \mathbb{R}^M) := (C_b(\Omega))^M = \{ \phi : \Omega \to \mathbb{R}^M \mid \phi \text{ is bounded and continuous on } \Omega \}.
$$

We let $E \subseteq C_b(\Omega; \mathbb{R}^{M+1})$ be a vector subspace, $U : E \to (-\infty, +\infty)$ be a proper concave functional and set

$$
V(\phi) := -U(-\phi) = \sup_{\phi \in E} \left( \sum_{m=0}^M \int_{\Omega} \phi_m d\gamma - V(\phi) \right), \quad \gamma \in ca(\Omega).
$$

(26)

$D : ca(\Omega) \to (-\infty, +\infty]$ is a convex functional and is $\sigma(ca(\Omega), E)$-lower semicontinuous, even if we do not require that $U$ is $\sigma(E, ca(\Omega))$-upper semicontinuous.

The following Assumption will hold throughout all the paper without further mention.

**Standing Assumption 2.1.** $D$ is proper, i.e. $\text{dom}(D) = \{ \gamma \in ca(\Omega) \mid D(\gamma) < +\infty \} \neq \emptyset$.

**Remark 2.2.** Another way to introduce our setting, that will be used in Subsection 4.1.2, is to start initially with a proper convex functional $D : ca(\Omega) \to (-\infty, +\infty]$ which is $\sigma(ca(\Omega), E)$-lower semicontinuous for some vector subspace $E \subseteq C_b(\Omega, \mathbb{R}^{M+1})$. By Fenchel-Moreau Theorem we then have the representation

$$
D(\gamma) = \sup_{\phi \in E} \left( \sum_{m=0}^M \int_{\Omega} \phi_m d\gamma - V(\phi) \right),
$$

where now $V$ is the Fenchel-Moreau (convex) conjugate of $D$, namely

$$
V(\phi) := \sup_{\gamma \in ca(\Omega)} \left( \sum_{m=0}^M \int_{\Omega} \phi_m d\gamma - D(\gamma) \right).
$$

(27)

Setting

$$
U(\phi) := -V(-\phi), \quad \phi \in E,
$$

(28)

we get back that $D$ satisfies (26) and additionally that $U$ is $\sigma(E, ca(\Omega))$-upper semicontinuous. In conclusion, a pair $(U, D)$ satisfying (26) might be defined either providing a proper concave $U : E \to [-\infty, +\infty)$, as described at the beginning of this section, or assigning a proper convex and $\sigma(E, ca(\Omega))$-lower semicontinuous functional $D : ca(\Omega) \to (-\infty, +\infty]$ as explained in this Remark.

We set

$$
\text{dom}(U) := \{ \phi \in E \mid U(\phi) > -\infty \}.
$$

(29)
Theorem 2.3. Let \( c : \Omega \to (-\infty, +\infty] \) be proper lower semicontinuous with compact sublevel sets and assume the following holds:

There exists a sequence \((k^n)_n \subseteq \mathbb{R}^{M+1}\) with \( \limsup_n \sum_{m=0}^{M} k^n_m = +\infty \) such that \( U(-k^n) > -\infty \) \( \forall n \).

Then
\[
\inf_{\mu \in \text{Meas}(\Omega)} \left( \int_{\Omega} c \, d\mu + D(\mu) \right) = \sup_{\varphi \in \Phi(c)} U(\varphi),
\]
where
\[
\Phi(c) := \left\{ \varphi \in \text{dom}(U) \mid \sum_{m=0}^{M} \varphi_m(x) \leq c(x) \ \forall x \in \Omega \right\}.
\]

Proof. We start applying (26) to get that
\[
\int_{\Omega} c \, d\mu + D(\mu) = \int_{\Omega} c \, d\mu + \sup_{\varphi \in \mathcal{C}} \left( U(\varphi) - \sum_{m=0}^{M} \int_{\Omega} \varphi_m \, d\mu \right).
\]

We then consider \( \mathcal{L} : \text{Meas}(\Omega) \times \text{dom}(U) \to (-\infty, +\infty] \) defined by
\[
\mathcal{L}(\mu, \varphi) := \int_{\Omega} \left( c - \sum_{m=0}^{M} \varphi_m \right) \, d\mu + U(\varphi)
\]
and we set \( M := \{ \mu \in \text{Meas}(\Omega) \mid \int_{\Omega} c \, d\mu < +\infty \} \). We observe that \( \mathcal{L} \) is real valued on \( M \times \text{dom}(U) \) and for any \( \mu \in \text{Meas}(\Omega) \setminus M \) we have \( \mathcal{L}(\mu, \varphi) = +\infty \) for all \( \varphi \in \text{dom}(U) \) (since \( c \) is bounded from below). We also see that setting \( C := \text{dom}(U) \)
\[
\inf_{\mu \in \text{Meas}(\Omega)} \left( \int_{\Omega} c \, d\mu + D(\mu) \right) = \inf_{\mu \in M} \sup_{\varphi \in \mathcal{C}} \mathcal{L}(\mu, \varphi).
\]

The aim is now to interchange sup and inf in RHS of (32), using Theorem A.8.

To this end, without loss of generality we can assume \( \alpha := \sup_{\varphi \in \mathcal{C}} \inf_{\mu \in \text{Meas}(\Omega)} \mathcal{L}(\mu, \varphi) < +\infty \) and we have to find \( \varphi \in \mathcal{C} \) and \( C > \alpha \) such that the sublevel set \( \mu_C := \{ \mu \in \text{Meas}(\Omega) \mid \mathcal{L}(\mu, \varphi) \leq C \} \) is weakly compact. The functional \( c \) is proper, lower continuous and has compact sublevel sets, hence it attains a minimum on \( \Omega \). Therefore, for any \( \varepsilon > 0 \) we can choose, by Assumption (30), a deterministic vector \( \varphi \in \mathcal{C} \) having all components \( \varphi_m \) equal to some constant \( -k^n_m < 0 \), such that
\[
\inf_{x \in \Omega} \left( c(x) - \sum_{m=0}^{M} \varphi_m(x) \right) > \varepsilon > 0.
\]

For such choice of \( \varphi \) and for a sufficiently big constant \( C > \alpha \) there exists another constant \( D := C - U(\varphi) \geq 0 \), independent of \( \mu \), such that
\[
\mu_C = \left\{ \mu \in \text{Meas}(\Omega) \mid \int_{\Omega} \left( c - \sum_{m=0}^{M} \varphi_m \right) \, d\mu \leq D \right\} = \left\{ \mu \in \text{Meas}(\Omega) \mid \int_{\Omega} \left( c - \sum_{m=0}^{M} \varphi_m \right) \, d\mu + \varepsilon \mu(\Omega) \leq D \right\}.
\]

Consequently, the set \( \mu_C \) is:
1. Nonempty, as the measure $\mu \equiv 0$ belongs to $\mu_C$.

2. Narrowly closed. Indeed, for each $\varphi \in C$ the function $c - \varphi$ is lower semicontinuous on $\Omega$, and so it is the pointwise supremum of bounded continuous functions $(c_n)_n \subseteq C_b(\Omega)$. For each $n$, $\mu \mapsto \int_{\Omega} c_n \, d\mu$ is narrowly lower semicontinuous on $\text{Meas}(\Omega)$, by definition. Hence by Monotone Convergence Theorem the map $\mu \mapsto \int_{\Omega} \left(c - \sum_{m=0}^{M} \varphi_m \right) \, d\mu$ is the pointwise supremum of narrowly lower semicontinuous functions, and is lower semicontinuous with respect to the narrow topology itself. We conclude that for each $\varphi \in C$ the functional $L(\cdot, \varphi)$ is narrowly lower semicontinuous, and has closed sublevel sets. This implies that in particular $\mu_C$ is narrowly closed, using the central expression in (33).

3. Bounded: having a sequence of measures in $\mu_C$ with unbounded total mass would result in a contradiction with the constraint in the last item of (33), taking into account that $c - \sum_{m=0}^{M} \varphi_m - \varepsilon \geq 0$ and $\varepsilon > 0$.

4. Tight: let $0 \leq f := c - \sum_{m=0}^{M} \varphi_m - \varepsilon$. Since $\varepsilon \mu(\Omega) \geq 0$ for all $\mu \in \text{Meas}(\Omega)$, by the central expression of (33) the inclusion $\mu_C \subseteq \{\mu \in \text{Meas} \mid \int_{\Omega} f \, d\mu \leq D\}$ holds. Now it is easy to check that for all $\mu \in \mu_C$ and $\alpha > 0$

$$D \geq \int_{\Omega} f \, d\mu \geq \int_{f > \alpha} f \, d\mu \geq \alpha \mu(\{f > \alpha\}).$$

Observing that the sublevels of $f$ are compact, by lower semicontinuity of $c$ and compactness of its sublevel sets, we see that $\{f > \alpha\}$ are complementaries of compact subsets of $\Omega$ and can be taken with arbitrarily small measure, just by increasing $\alpha$, uniformly in $\mu \in \mu_C$. Thus tightness follows.

5. A subset of $M$.

These properties in turns yield narrow compactness of $\mu_C$ in $\text{Meas}(\Omega)$, by Theorem A.6, and therefore $\sigma(\text{Meas}(\Omega), C_b(\Omega))$ compactness (recalling that weak and narrow topology coincide in our setup). As a consequence, by Item 5, $\mu_C$ is compact in the relative topology $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_M$.

We now may apply Theorem A.8. Indeed, $L$ is real valued on $M \times C$. Items 1 and 2 of Theorem A.8 are fulfilled for: $A = M$ endowed with the topology $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_M$; $B = C$; and $C$ taken as above. We only justify explicitly lower semicontinuity $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_M$ for Item 1, which can be obtained arguing as in Item 4 above and observing that narrow topology and weak topology coincide in our setup (see Proposition A.4). Hence we may interchange sup and inf in RHS of (32), obtaining

$$\inf_{\mu \in M} \sup_{\varphi \in C} L(\mu, \varphi) = \sup_{\varphi \in C} \inf_{\mu \in M} L(\mu, \varphi) = \sup_{\varphi \in C} \inf_{\mu \in \text{Meas}(\Omega)} L(\mu, \varphi)$$

(34)

where the last equality follows from the fact that $L(\mu, \varphi) = +\infty$ on the complement of $M$ in $\text{Meas}(\Omega)$ for every $\varphi \in C$. It is now easy to check that for every $\varphi \in C$

$$\inf_{\mu \in \text{Meas}} L(\mu, \varphi) = \begin{cases} U(\varphi) & \text{if } \sum_{m=0}^{M} \varphi_m(x) \leq c(x) \forall x \in \Omega \\ -\infty & \text{otherwise} \end{cases}$$
thus
\[
\sup_{\varphi \in \mathcal{C}} \inf_{\mu \in \text{Meas}(\Omega)} \mathcal{L}(\mu, \varphi) = \sup_{\varphi \in \Phi(c)} U(\varphi)
\]
which concludes the proof, given (32) and (34).

\[\square\]

### 2.1 The Entropy Martingale Optimal Transport Duality

In order to describe a suitable theory to develop the entropy optimal transport duality in a dynamic setting, in this section we will adopt a particular product structure of the set \(\Omega\).

To this end, in addition to the notations already introduced in Section 2, we consider \(T \in \mathbb{N}\), \(T \geq 1\), and
\[
\Omega := K_0 \times \cdots \times K_T
\]
for \(K_0, \ldots, K_T \subseteq \mathbb{R}\). We denote with \(X_0, \ldots, X_T\) the canonical projections \(X_t : \Omega \to K_t\) and we set \(X = [X_0, \ldots, X_T] : \Omega \to \mathbb{R}^{T+1}\), to be considered as discrete-time stochastic process. We denote with:
\[
\text{Mart}(\Omega) := \{\text{Martingale measures for the canonical process of } \Omega\}
\]

When \(\mu \in \text{Meas}(K_0 \times \cdots \times K_T)\), its marginals will be denoted with: \(\mu_0, \ldots, \mu_T\).

We recall, respectively from (8) and (9), that \(H\) is the set of admissible trading strategies and \(I\) is the set of elementary stochastic integral. We take \(E = E_0 \times \cdots \times E_T\) where \(E_t \subseteq C_b(K_0 \times \cdots \times K_t)\) is a vector subspace, for every \(t = 0, \ldots, T\). Then \(E\) is clearly a vector subspace of \(C_b(\Omega; \mathbb{R}^{T+1})\), and in the stochastic processes interpretation its elements are processes adapted to the natural filtration of the process \((X_t)_t\).

We suppose that \(U : E \to [-\infty, +\infty)\) is proper and concave, \(D : \text{Meas}(\Omega) \to (-\infty, +\infty]\) is defined in (26) and, as in (24),
\[
S^U(\varphi) := \sup_{\xi \in \mathbb{R}^{T+1}} \left( U(\varphi + \xi) - \sum_{t=0}^T \xi_t \right), \quad \varphi \in \mathcal{E}.
\]

**Theorem 2.4.** Assume that \(\Omega := K_0 \times \cdots \times K_T\) for compact sets \(K_0, \ldots, K_T \subseteq \mathbb{R}\), that \(c : \Omega \to (-\infty, +\infty]\) is lower semicontinuous, that \(D : \text{Meas}(\Omega) \to (-\infty, +\infty]\) is lower bounded on \(\text{Meas}(\Omega)\) and proper. Suppose also \(U\) satisfies (30), and that
\[
\mathcal{N} := \left\{ \mu \in \text{Meas}(\Omega) \cap \text{dom}(D) \mid \int_{\Omega} c \, d\mu < +\infty \right\} \neq \emptyset \quad \text{and} \quad \text{dom}(U) + \mathbb{R}^{T+1} \subseteq \text{dom}(U).\]  

Then the following holds:
\[
\inf_{Q \in \text{Mart}(\Omega)} (E_Q[c(X)] + D(Q)) = \sup_{\Delta \in \mathcal{H}} \sup_{\varphi \in \Phi_\Delta(c)} S^U(\varphi) \tag{37}
\]

where for each \(\Delta \in \mathcal{H}\)
\[
\Phi_\Delta(c) := \left\{ \varphi \in \text{dom}(U), \sum_{t=0}^T \varphi_t(x_t) + \sum_{t=0}^{T-1} \Delta_t(x_0, \ldots, x_t)(x_{t+1} - x_t) \leq c(x) \quad \forall x \in \Omega \right\}. \tag{38}
\]

**Proof.** The first part of the proof is inspired by [6] Equations (3.4)-(3.3)-(3.2)-(3.1).

\[
\inf_{Q \in \text{Mart}(\Omega)} (E_Q[c(X)] + D(Q)) \tag{39}
\]
We define now that the inner supremum over \(\lambda\) is a martingale measure on \(\Omega\); (41)=(42) and (42)=(43) are trivial; (43)=(44) follows observing that the inner supremum over \(\lambda \in \mathbb{R}\) explodes to \(+\infty\) unless \(\mu(\Omega) = 1\); (44)=(45) is trivial.

We define now \(K : \text{Meas}(\Omega) \times (\mathcal{H} \times \mathbb{R}^T) \to (-\infty, +\infty]\) as

\[
K(\mu, \Delta, \lambda) := \int_\Omega \left[ c - I^\Delta + \sum_{t=0}^T \lambda_t \right] d\mu - \sum_{t=0}^T \lambda_t + \mathcal{D}(\mu).
\]

From (36), we observe that \(K\) is real valued on \(N \times (\mathcal{H} \times \mathbb{R}^{T+1})\) and that \(K(\mu, \Delta, \lambda) = +\infty\) if \(\mu \in \text{Meas}(\Omega) \setminus N\), for all \((\Delta, \lambda) \in \mathcal{H} \times \mathbb{R}^{T+1}\). This, together with our previous computations, provides

\[
\inf_{Q \in \text{Mart}(\Omega)} \left( E_Q [c(X)] + \mathcal{D}(Q) \right) = \inf_{\mu \in \text{Meas}(\Omega)} \sup_{\Delta \in \mathcal{H} \atop \lambda \in \mathbb{R}^{T+1}} K(\mu, \Delta, \lambda) = \inf_{\mu \in N} \sup_{\Delta \in \mathcal{H} \atop \lambda \in \mathbb{R}^{T+1}} K(\mu, \Delta, \lambda).
\]

As in the proof of Theorem 2.3, we wish to apply the Minimax Theorem A.8 in order to interchange inf and sup in RHS of (46) and without loss of generality we can assume that \(\alpha := \sup_{\Delta \in \mathcal{H} \atop \lambda \in \mathbb{R}^{T+1}} K(\mu, \Delta, \lambda) < +\infty\). The functional \(K\) is real valued on \(N \times (\mathcal{H} \times \mathbb{R}^{T+1})\) and convexity in Item 1, concavity in 2 of Theorem A.8 are clearly satisfied. We have to find \(\Delta \in \mathcal{H}, \lambda \in \mathbb{R}^{T+1}\) and \(C > \alpha\) such that the sublevel set \(M_C := \{\mu \in \text{Meas}(\Omega) \mid K(\mu, \Delta, \lambda) \leq C\}\) is weakly compact. Fix a \(\varepsilon > 0\). As the functional \(c\) is lower semicontinuous on the compact \(\Omega\), it is lower bounded on \(\Omega\) and we can take \(\Delta = 0\) and \(\lambda\) sufficiently big in such a way that \(\inf_{x \in \Omega} \left( c(x) + \sum_{t=0}^T \lambda_t \right) > \varepsilon\). For such a choice of \((\Delta, \lambda)\) we have that \(M_C\) satisfies

\[
M_C \subseteq \left\{ \mu \in \text{Meas}(\Omega) \mid \int_\Omega \left[ c + \sum_{t=0}^T \lambda_t - \varepsilon \right] d\mu(x) + \varepsilon \mu(\Omega) \leq C + \lambda - \inf_{\mu \in \text{Meas}(\Omega)} \mathcal{D}(\mu) \right\} =: D
\]

where \(D \in \mathbb{R}\) since \(\mathcal{D}(\cdot)\) is lower bounded by hypothesis. By (36) and for large enough \(C\), the set \(M_C\) is nonempty, and the same arguments in Items 2, 3 and 4 of the proof of Theorem 2.3 can be
applied to conclude that the set $\mathcal{M}_C$ is narrowly closed, bounded and tight, hence narrowly and $\sigma(\text{Meas}(\Omega), C_b(\Omega))$-compact. Moreover we see that $\mathcal{M}_C \subseteq \mathcal{N}$, hence it is also compact in the topology $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$. We finally verify $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$-lower semicontinuity of $\mathcal{K}(\cdot, \Delta, \lambda)$ on $\mathcal{N}$ for every $(\Delta, \lambda) \in (\mathcal{H} \times \mathbb{R}^{T+1})$. To see this, observe that arguing as in Item 2 of the proof of Theorem 2.3 we get that $\mathcal{M}_C$ is narrowly closed, bounded and tight, hence narrowly and $\sigma(\text{Meas}(\Omega), C_b(\Omega))$-compact. Moreover we see that $\mathcal{M}_C \subseteq \mathcal{N}$, hence it is also compact in the topology $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$. We finally verify $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$-lower semicontinuity of $\mathcal{K}(\cdot, \Delta, \lambda)$ on $\mathcal{N}$ for every $(\Delta, \lambda) \in (\mathcal{H} \times \mathbb{R}^{T+1})$. To see this, observe that arguing as in Item 2 of the proof of Theorem 2.3 we get that $\mathcal{M}_C$ is narrowly closed, bounded and tight, hence narrowly and $\sigma(\text{Meas}(\Omega), C_b(\Omega))$-compact. Moreover we see that $\mathcal{M}_C \subseteq \mathcal{N}$, hence it is also compact in the topology $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$. We finally verify $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$-lower semicontinuity of $\mathcal{K}(\cdot, \Delta, \lambda)$ on $\mathcal{N}$ for every $(\Delta, \lambda) \in (\mathcal{H} \times \mathbb{R}^{T+1})$. To see this, observe that arguing as in Item 2 of the proof of Theorem 2.3 we get that $\mathcal{M}_C$ is narrowly closed, bounded and tight, hence narrowly and $\sigma(\text{Meas}(\Omega), C_b(\Omega))$-compact. Moreover we see that $\mathcal{M}_C \subseteq \mathcal{N}$, hence it is also compact in the topology $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$. We finally verify $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$-lower semicontinuity of $\mathcal{K}(\cdot, \Delta, \lambda)$ on $\mathcal{N}$ for every $(\Delta, \lambda) \in (\mathcal{H} \times \mathbb{R}^{T+1})$. To see this, observe that arguing as in Item 2 of the proof of Theorem 2.3 we get that $\mathcal{M}_C$ is narrowly closed, bounded and tight, hence narrowly and $\sigma(\text{Meas}(\Omega), C_b(\Omega))$-compact. Moreover we see that $\mathcal{M}_C \subseteq \mathcal{N}$, hence it is also compact in the topology $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$. We finally verify $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$-lower semicontinuity of $\mathcal{K}(\cdot, \Delta, \lambda)$ on $\mathcal{N}$ for every $(\Delta, \lambda) \in (\mathcal{H} \times \mathbb{R}^{T+1})$. To see this, observe that arguing as in Item 2 of the proof of Theorem 2.3 we get that $\mathcal{M}_C$ is narrowly closed, bounded and tight, hence narrowly and $\sigma(\text{Meas}(\Omega), C_b(\Omega))$-compact. Moreover we see that $\mathcal{M}_C \subseteq \mathcal{N}$, hence it is also compact in the topology $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$. We finally verify $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$-lower semicontinuity of $\mathcal{K}(\cdot, \Delta, \lambda)$ on $\mathcal{N}$ for every $(\Delta, \lambda) \in (\mathcal{H} \times \mathbb{R}^{T+1})$. To see this, observe that arguing as in Item 2 of the proof of Theorem 2.3 we get that $\mathcal{M}_C$ is narrowly closed, bounded and tight, hence narrowly and $\sigma(\text{Meas}(\Omega), C_b(\Omega))$-compact. Moreover we see that $\mathcal{M}_C \subseteq \mathcal{N}$, hence it is also compact in the topology $\sigma(\text{Meas}(\Omega), C_b(\Omega))|_{\mathcal{N}}$.
Remark 2.5. (i) The assumptions of Theorem 2.4 are reasonably weak and are satisfied, for example, if: \( \text{dom}(U) = \mathcal{E} \), there exists a \( \hat{\mu} \in \text{Meas}(\Omega) \cap \partial U(0) \) such that \( c \in L^1(\hat{\mu}) \), and \( c \) is lower semicontinuous. Indeed, for all \( \mu \in \text{Meas}(\Omega) \), \( D(\mu) \geq U(0) - 0 > -\infty \). Clearly \( \text{dom}(U) + \mathbb{R}^{T+1} = \text{dom}(U) \). Finally, \( \hat{\mu} \in \mathcal{N} \), because \( c \in L^1(\hat{\mu}) \) and \( -\infty < U(0) \leq D(\hat{\mu}) \leq 0 \), by definition of \( D \).

(ii) The step (40)=(41) is the crucial point where compactness of the sets \( K_0, \ldots, K_T \subseteq \mathbb{R} \) is necessary for a smooth argument, since integrability of the underlying stock process is in this case automatically satisfied for all \( Q \in \text{Prob}(\Omega) \), not only for \( Q \in \text{Mart}(\Omega) \). Also, compactness is key in guaranteeing that the cost functional \( c - I^\Delta + \sum_{t=0}^T \lambda_t \) is bounded from below, in order to apply Theorem 2.3.

Proposition 2.6. Suppose that LHS of (37) is finite and that \( D|_{\text{Meas}(\Omega)} \) is \( \sigma(\text{Meas}(\Omega),C_b(\Omega)) \)-lower semicontinuous. Then, under the same the assumptions of Theorem 2.4, the problem in LHS of (37) admits an optimum.

Proof. Similarly to what we argued in Item 2 of the proof of Theorem 2.3, the map \( \mu \mapsto \int_\Omega c \, d\mu \) is \( \sigma(\text{Meas}(\Omega),C_b(\Omega)) \)-lower semicontinuous, and we deduce the lower semicontinuity of
\[
J(Q) := E_Q[c] + \sum_{t=0}^T D(Q), \quad Q \in \text{Mart}(\Omega).
\]
Moreover for \( C \) big enough the sublevel \( \{Q \in \text{Mart}(\Omega) \mid J(Q) \leq C\} \) is nonempty (since we are assuming LHS of (37) is finite), hence \( J \) is proper on \( \text{Mart}(\Omega) \). Since \( K_0, \ldots, K_T \) are compact, \( \text{Prob}(\Omega) \) is \( \sigma(\text{Meas}(\Omega),C_b(\Omega)) \) compact (see [2] Theorem 15.11), and \( \text{Mart}(\Omega) \) is \( \sigma(\text{Meas}(\Omega),C_b(\Omega)) \) closed because, arguing as in [6] Lemma 2.3,
\[
\text{Mart}(\Omega) = \bigcap_{\Delta \in \mathcal{H}} \left\{ Q \in \text{Prob}(\Omega) \mid \int_\Omega \left( \sum_{t=0}^{T-1} \Delta_t(x_0,\ldots,x_t)(x_{t+1} - x_t) \right) \, dQ(x) \leq 0 \right\}.
\]
We conclude that \( \text{Mart}(\Omega) \) is \( \sigma(\text{Meas}(\Omega),C_b(\Omega)) \)-compact, and \( J \) is lower semicontinuous and proper on it, hence it attains a minimum. \hfill \Box

Remark 2.7. The lower semicontinuity assumption in Proposition 2.6 is satisfied in many cases, as it will become clear in Section 3.

2.2 A useful rephrasing of Theorem 2.4

We now rephrase our findings in Theorem 2.4, with minor additions, to get the formulation in Corollary 2.8 which will simplify our discussion of Section 4. In particular, this reformulation will come in handy when dealing with subhedging and superhedging dualities in Corollaries 4.3-4.8 and Proposition 4.9.

For a given proper concave \( U : \mathcal{E} \to \mathbb{R} \), we recall the definition of \( S^U \) in (24) and, for \( V(\cdot) = -U(\cdot) \), we define \( \text{dom}(V) := \{ \varphi \in \mathcal{E} \mid V(\varphi) < +\infty \} = -\text{dom}(U) \) and
\[
S_V(\varphi) := \inf_{\lambda \in \mathbb{R}^{T+1}} \left( V(\varphi + \lambda) - \sum_{t=0}^T \lambda_t \right) = -S^U(\varphi), \quad \varphi \in \text{dom}(V).
\]
Furthermore, given functions $c : \Omega \to (-\infty, +\infty)$, $d : \Omega \to [-\infty, +\infty)$ we introduce the sets
\[
S_{\text{sub}}(c) := \left\{ \varphi \in \text{dom}(U) \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi(x_t) + I^\Delta(x) \leq c(x) \ \forall x \in \Omega \right\} \tag{48}
\]
\[
S_{\text{sup}}(d) := \left\{ \varphi \in \text{dom}(V) \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi(x_t) + I^\Delta(x) \geq d(x) \ \forall x \in \Omega \right\} \tag{49}
\]
and observe that $S_{\text{sup}}(\Psi) = -S_{\text{sub}}(-\Psi)$.

**Corollary 2.8.** Suppose that the assumptions in Theorem 2.4 are satisfied, that $d : \Omega \to [-\infty, +\infty)$ is upper semicontinuous and that $\{\mu \in \text{Meas}(\Omega) \cap \text{dom}(D) \mid \int_{\Omega} d d\mu > -\infty \} \neq \emptyset$. Then the following hold
\[
\inf_{Q \in \text{Mart}(\Omega)} (E_Q [c(X)] + D(Q)) = \sup_{\varphi \in S_{\text{sub}}(c)} S^U(\varphi), \tag{50}
\]
\[
\sup_{Q \in \text{Mart}(\Omega)} (E_Q [d(X)] - D(Q)) = \inf_{\varphi \in S_{\text{sup}}(d)} S^V(\varphi). \tag{51}
\]

**Proof.** Equation (50) is an easy rephrasing of the corresponding (37). As to (51), we observe that for $c := -d$ we get from (50)
\[
\sup_{\varphi \in S_{\text{sub}}(-d)} S^U(\varphi) = \inf_{Q \in \text{Mart}} (E_Q [-d(X)] + D(Q)) = -\sup_{Q \in \text{Mart}} (E_Q [d(X)] - D(Q)).
\]
Observing that $S_{\text{sup}}(d) = -S_{\text{sub}}(-d)$ and that $S^V(\cdot) = -S^U(\cdot)$ on dom(V) we get $\sup_{\varphi \in S_{\text{sub}}(-d)} S^U(\varphi) = -\inf_{\varphi \in S_{\text{sup}}(d)} S^V(\varphi)$. This completes the proof. \qed

## 3 Additive structure

In Section 2, we did not require any particular structural form of the functionals $D, U$. Here instead, we will assume in addition to (35) also an additive structure of $U$ and, complementarily, an additive structure of $D$. In the whole Section 3 we take for each $t = 0, \ldots, T$ a vector subspace $\mathcal{E}_t \subseteq C_b(K_t)$ such that $\mathcal{E}_t + \mathbb{R} = \mathcal{E}_t$ and set $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$. Observe that we automatically have $\mathcal{E} + \mathbb{R}^{T+1} = \mathcal{E}$. It is also clear that $\mathcal{E}$ is a subspace of $C_b(\Omega, \mathbb{R}^{T+1})$, if we interpret $\mathcal{E}_0, \ldots, \mathcal{E}_T$ as subspaces of $C_b(\Omega)$.

### 3.1 Additive Structure of $U$

**Setup 3.1.** For every $t = 0, \ldots, T$ we consider a proper concave functional $U_t : \mathcal{E}_t \to [-\infty, +\infty)$. We define $D_t$ on $\text{ca}(K_t)$ similarly to (26) as
\[
D_t(\gamma_t) := \sup_{\varphi_t \in \mathcal{E}_t} \left( U_t(\varphi_t) - \int_{K_t} \varphi_t d\gamma_t \right) \quad \gamma_t \in \text{ca}(K_t)
\]
and observe that $D_t$ can also be thought to be defined on $\text{ca}(\Omega)$ using for $\gamma \in \text{ca}(\Omega)$ the marginals $\gamma_0, \ldots, \gamma_T$ and setting $D_t(\gamma) := D_t(\gamma_t)$. We may now define, for each $\varphi \in \mathcal{E}$, $U(\varphi) := \sum_{t=0}^{T} U_t(\varphi_t)$ and define $\mathcal{D}$ on $\text{ca}(\Omega)$ using (26) with $M = T$. Recall from (24)
\[
S_t^U(\varphi) := \sup_{\xi \in \mathbb{R}^{T+1}} \left( U(\varphi + \xi) - \sum_{t=0}^{T} \xi_t \right), \quad \varphi \in \mathcal{E}, \quad S_t^{U_t}(\varphi_t) := \sup_{\alpha \in \mathbb{R}} (U(\varphi_t + \alpha) - \alpha), \quad \varphi_t \in \mathcal{E}_t.
Proof. See Appendix A.2.

3.2 Duality for the general Cash Additive setup

**Theorem 3.3.** Suppose for each $t = 0, \ldots, T$ $\mathcal{E}_t \subseteq \mathcal{C}_b(K_t)$ is a vector subspace satisfying $\text{Id}_t \in \mathcal{E}_t$ and $\mathcal{E}_t + \mathbb{R} = \mathcal{E}_t$ and that $S_t : \mathcal{E}_t \to \mathbb{R}$ is a concave, cash additive functional null in $0$. Consider for every $t = 0, \ldots, T$ the penalizations

$$D_t(Q_t) := \sup_{\varphi_t \in \mathcal{E}_t} \left( S_t(\varphi_t) - \int_{K_t} \varphi_t \, dQ_t \right) \text{ for } Q_t \in \text{Prob}(K_t),$$

and set $D(Q) := \sum_{t=0}^T D_t(Q_t)$. Let $c : \Omega \to (-\infty, +\infty]$ be lower semicontinuous and let $\Psi(c)$ be defined respectively in (14) and (19). If $N := \{ \mu \in \text{Meas}(\Omega) \cap \text{dom}(D) \mid \int_{\Omega} c \, d\mu < +\infty \} \neq \emptyset$ then $\Psi(c) = D(c)$.

Proof. Set $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$ and $U(\varphi) := \sum_{t=0}^T S_t(\varphi_t)$, for $\varphi \in \mathcal{E}$, and let $D$ defined as in (26) for $M = T$. For any $\mu \in \text{Meas}(\Omega)$ we have $D(\mu) \geq \sum_{t=0}^T S_t(0) - 0 = 0$ hence $D$ is lower bounded on $\text{Meas}(\Omega)$. Observe that $\text{dom}(U) = \mathcal{E}$, which implies $\text{dom}(U) + \mathbb{R}^{T+1} = \text{dom}(U)$, and that we are in Setup 3.1. Lemma 3.2 tells us that $S^U(\varphi) = \sum_{t=0}^T S^U_t(\varphi_t) = \sum_{t=0}^T S_t(\varphi_t)$, since $S_0, \ldots, S_T$ are Cash Additive, and that $D$ coincides on $\text{Mart}(\Omega)$ with the penalization term $Q \mapsto \sum_{t=0}^T D_t(Q_t)$, as provided in the statement of this Theorem. Since all the assumptions of Theorem 2.4 are fulfilled, we can apply Corollary 2.8, which yields exactly $D(c) = \Psi(c)$.

3.3 Additive Structure of $D$.

The results of this subsection will be applied in Subsection 4.1.2. In the spirit of Remark 2.2, we may now reverse the procedure taken in the previous subsection: we start from some functionals $D_t$ on $\text{ca}(K_t)$, for $t = 0, \ldots, T$, and build an additive functional $D$ on $\text{ca}(\Omega)$. Our aim is to find the counterparts of the results in Section 3.1.

**Setup 3.4.** For every $t = 0, \ldots, T$ we consider a proper, convex, $\sigma(\text{ca}(K_t), \mathcal{E}_t)$-lower semicontinuous functional $D_t : \text{ca}(K_t) \to (-\infty, +\infty]$. We can then extend the functionals $D_t$ to $\text{ca}(\Omega)$ by using, for any $\gamma \in \text{ca}(\Omega)$, the marginals $\gamma_0, \ldots, \gamma_T$. If $\gamma \in \text{ca}(\Omega)$, we set

$$D_t(\gamma) := D_t(\gamma_t) \quad \text{and} \quad D(\gamma) := \sum_{t=0}^T D_t(\gamma) = \sum_{t=0}^T D_t(\gamma_t).$$

We define $V(\varphi)$ for $\varphi \in \mathcal{E}$ and $V_t(\varphi_t)$ for $\varphi_t \in \mathcal{E}_t$, for $t = 0, \ldots, T$ similarly to (27), as

$$V(\varphi) := \sup_{\gamma \in \text{ca}(\Omega)} \left( \int_{\Omega} \left( \sum_{t=0}^T \varphi_t \right) \, d\gamma - D(\gamma) \right) \quad \text{and} \quad V_t(\varphi_t) := \sup_{\gamma \in \text{ca}(K_t)} \left( \int_{K_t} \varphi_t \, d\gamma - D_t(\gamma) \right).$$
We define on $\mathcal{E}$ the functional $U(\cdot) = -V(\cdot)$, as in (28), and similarly $U_t(\cdot) = -V_t(\cdot)$ on $\mathcal{E}_t$, for $t = 0, \ldots, T$. Finally, $S^U(\varphi)$, $S^{U_0}(\varphi_0), \ldots, S^{U_T}(\varphi_T)$ are defined as in Setup 3.1.

**Lemma 3.5.** In Setup 3.4 we have:

1. $D_0, \ldots, D_T$, as well as $D$, are $\sigma(\mathcal{C}(\Omega), \mathcal{E})$-lower semicontinuous.

2. Under the additional assumption that $\text{dom}(D_t) \subseteq \text{Prob}(K_t)$ for every $t = 0, \ldots, T$, for any $\varphi = [\varphi_0, \ldots, \varphi_T] \in \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$

   
   \[
   U(\varphi) = \sum_{t=0}^T U_t(\varphi_t) = \sum_{t=0}^T -V_t(-\varphi_t),
   \]

   \[
   S^U(\varphi) = \sum_{t=0}^T S^{U_t}(\varphi_t).
   \]

**Proof.** See Appendix A.2
\[\square\]

### 3.4 Divergences induced by utility functions

**Assumption 3.6.** We consider concave, upper semicontinuous nondecreasing functions $u_0, \ldots, u_T : \mathbb{R} \to [-\infty, +\infty)$ with $u_0(0) = \cdots = u_T(0) = 0$, $u_t(x) \leq x \forall x \in \mathbb{R}$ (that is $1 \in \partial u_0(0) \cap \cdots \cap \partial u_T(0)$).

For each $t = 0, \ldots, T$ we define $v_t(x) := -u_t(-x)$, $x \in \mathbb{R}$ and

\[
 v_t^*(y) := \sup_{x \in \mathbb{R}} (xy - v_t(x)) = \sup_{x \in \mathbb{R}} (u_t(x) - xy), \quad y \in \mathbb{R}.
\]

We observe that $v_t(y) = v_t^*(y) = \sup_{x \in \mathbb{R}} (xy - v_t^*(y))$ for all $y \in \mathbb{R}$ by Fenchel-Moreau Theorem and that $v_t^*$ is convex, lower semicontinuous and lower bounded on $\mathbb{R}$.

**Example 3.7.** Assumption 3.6 is satisfied by a wide range of functions. Just to mention a few with various peculiar features, we might take $u_t$ of the following forms: $u_t(x) = 1 - \exp(-x)$, whose convex conjugate is given by $v_t^*(y) = -\infty$ for $y < 0$, $v_t^*(0) = 0$, $v_t^*(y) = (y \log(y) - y + 1)$ for $y > 0$; $u_t(x) = \alpha x 1_{(-\infty, 0]}(x)$ for $\alpha \geq 1$, so that $v_t^*(y) = +\infty$ for $y < 0$, $v_t^*(y) = 0$ for $y \in [0, \alpha]$, $v_t^*(y) = +\infty$ for $y > \alpha$; $u_t(x) = \log(x+1)$ for $x > -1$, $u_t(x) = -\infty$ for $x \leq -1$, so that $v_t^*(y) = +\infty$ for $y \leq 0$, $v_t^*(y) = y - \log(y) - 1$ for $y > 0$; $u_t(x) = +\infty$ for $x \leq -1$, $u_t(x) = \frac{1}{x+1}$ for $x > -1$ so that $v_t^*(y) = -\infty$ for $y < 0$, $v_t^*(y) = y - 2\sqrt{y} + 1$ for $y \geq 0$; $u_t(x) = -\infty$ for $x < 0$, $u_t(x) = 1 - \exp(-x)$ for $x \geq 0$, so that $v_t^*(y) = +\infty$ for $y < 0$, $v_t^*(y) = y \log(y) - y + 1$ for $0 \leq y \leq 1$, $v_t^*(y) = 0$ for $y > 1$.

Fix $\tilde{\mu}_t \in \text{Meas}(K_t)$. We pose for $\mu \in \text{Meas}(K_t)$

\[
\mathcal{D}_{v_t^*, \tilde{\mu}_t}(\mu) := \begin{cases} f_K \ v_t^* \left( \frac{d\mu}{d\tilde{\mu}_t} \right) d\tilde{\mu}_t, & \text{if } \mu \ll \tilde{\mu}_t, \\ +\infty, & \text{otherwise} \end{cases}
\]

In the next two propositions, whose proofs are postponed to the Appendix A.2, we provide the dual representation of the divergence terms.

**Proposition 3.8.** Take $u_0, \ldots, u_T$ satisfying Assumption 3.6, and suppose $\text{dom}(u_0) = \cdots = \text{dom}(u_T) = \mathbb{R}$. Let $\tilde{\mu}_t \in \text{Meas}(K_t)$ and $v_t(\cdot) := -u_t(\cdot)$, $t = 0, \ldots, T$. Then
\[ D_{\nu, \hat{\eta}}(\mu) = \sup_{\varphi \in \mathcal{C}_b(K_t)} \left( \int_{K_t} \varphi(t(x)) \, d\mu(x) - \int_{K_t} \nu((\varphi(x))) \, d\hat{\mu}(x) \right). \quad (57) \]

Set:

\[ (v^*_t)'_\infty := \lim_{y \to +\infty} \frac{v^*_t(y)}{y}, \quad t = 0, \ldots, T. \]

As \( \text{dom}(u) \supseteq [0, +\infty), (v^*_t)'_\infty \in [0, +\infty]. \) Let \( \hat{Q}_t \in \text{Prob}(K_t) \) and, for \( \mu \in \text{Meas}(K_t), \) let \( \mu = \mu_a + \mu_s \) be the Lebesgue Decomposition of \( \mu \) with respect to \( \hat{Q}_t, \) where \( \mu_a \ll \hat{Q}_t \) and \( \mu_s \perp \hat{Q}_t. \) Then we can define for \( \mu \in \text{Meas}(K_t) \)

\[ \mathcal{F}_t(\mu \mid \hat{Q}_t) := \int_{K_t} v^*_t \left( \frac{d\mu_a}{d\hat{Q}_t} \right) \, d\hat{Q}_t + (v^*_t)'_\infty \mu_s(K_t) \]

where we use the convention \( \infty \times 0 = 0, \) in case \( (v^*_t)'_\infty = +\infty, \mu_s(K_t) = 0. \) Observe that the restriction of \( \mathcal{F}(\cdot \mid \hat{Q}_t) \) to \( \text{Meas}(K_t) \) coincides with the functional in [42] (2.35) with \( F = v^*_t, \) and that whenever \( \text{dom}(u_t) = \mathbb{R} \) we have \( (v^*_t)'_\infty = \lim_{y \to +\infty} \frac{v^*_t(y)}{y} = +\infty \) and \( \mathcal{F}(\cdot \mid \hat{Q}_t) \) coincides with \( D_{v^*_t, \hat{\eta}}(\cdot) \) (see (56)) on \( \text{Meas}(K_t) \).

**Proposition 3.9.** Suppose that \( u_0, \ldots, u_T : \mathbb{R} \to [-\infty, +\infty) \) satisfy Assumption 3.6 and \( v_t(\cdot) := -u_t(-\cdot). \) If \( \hat{Q}_t \in \text{Prob}(K_t), t \in \{0, \ldots, T\}, \) has full support then

\[ \mathcal{F}_t(\mu \mid \hat{Q}_t) = \sup_{\varphi \in \mathcal{C}_b(K_t)} \left( \int_{K_t} \varphi(x) \, d\mu(x) - \int_{K_t} v_t((\varphi(x))) \, d\hat{Q}_t(x) \right). \quad (58) \]

**Example 3.10.** The requirement that \( \hat{Q}_0, \ldots, \hat{Q}_T \) have full support is crucial for the proof of Proposition 3.9. We provide a simple example to the fact that (58) does not hold in general when such an assumption is not fulfilled. To this end, take \( K = \{-2, 0, 2\}, \hat{Q} = \frac{1}{2} \delta_{-2} + \frac{1}{2} \delta_{+2}, \mu = \delta_{0}, \)
\[ u(x) := \frac{x}{\sqrt{1+x^2}} \text{ for } x \geq -1 \text{ and } u(x) = -\infty \text{ for } x < -1. \] It is easy to see that the associated \( v^* \) via (55) is defined by \( v^*(y) = 1 + y - 2\sqrt{y} \) for \( y \geq 0 \) and \( v^*(y) = -\infty \) for \( y < 0, \) so that \( (v^*_t)'_\infty = 1. \) It is also easy to see that \( \mu \perp \hat{Q} \), hence in the Lebesgue decomposition with respect to \( \hat{Q} \), \( \mu_a = 0 \) and \( \mu_s = \mu. \) Hence \( \mathcal{F}(\mu \mid \hat{Q}) = 1 + 1\mu(K) = 2. \) At the same time we see that taking \( \varphi_N \in \mathcal{C}_b(K) \) defined via \( \varphi_N(-2) = \varphi_N(2) = 0, \varphi_N(0) = -N \) (observe that for \( N \) sufficiently large \( u(\varphi_N) \notin \mathcal{C}_b(K) \)) we have

\[ \sup_{\varphi \in \mathcal{C}_b(K)} \left( \int_K \varphi \, d\mu - \int_K \nu((\varphi(x))) \, d\hat{Q} \right) = \sup_{\varphi \in \mathcal{C}_b(K)} \left( \int_K u(\varphi(x)) \, d\hat{Q} - \int_K \varphi \, d\mu \right) \geq \sup_N \left( \int_K u(\varphi_N) \, d\hat{Q} - \int_K \nu \, d\mu \right) \geq \sup_N \left( 0 \cdot \frac{1}{2} + (0) \cdot \frac{1}{2} - (-N) \right) = +\infty. \]

### 4 Applications of the Main Theorems of Section 2

In this section we suppose the following requirements are fulfilled:

**Standing Assumption 4.1.** \( \Omega := K_0 \times \cdots \times K_T \) for compact sets \( K_0, \ldots, K_T \subseteq \mathbb{R} \) and \( K_0 = \{x_0\}; \) the functional \( c : \Omega \to (-\infty, +\infty) \) is lower semicontinuous and \( d : \Omega \to [-\infty, +\infty) \) is upper semicontinuous; \( \text{Mart}(\Omega) \neq \emptyset; \hat{Q} \in \text{Mart}(\Omega) \) is a given probability measure with marginals \( \hat{Q}_0, \ldots, \hat{Q}_T; c, d \in L^1(\hat{Q}). \)
4.1 Subhedging and Superhedging

As it will become clear from the proofs, in all the results in Section 4.1 the functional $U$ is real valued on the whole $E$, that is $\text{dom}(U) = E$. Thus we will exploit Theorem 2.4 and Corollary 2.8, in particular (48) and (49), in the case $\text{dom}(U) = \text{dom}(V) = E$.

We set for $\varphi_t \in \mathcal{C}_b(K_t)$

$$U^{\hat{Q}_t}_t(\varphi_t) = \sup_{\alpha, \lambda \in \mathbb{R}} \left( \int_{K_t} u_t(\varphi_t(x_t)) + \alpha \text{Id}_t(x_t) + \lambda \right) d\hat{Q}_t(x_t) - (\alpha x_0 + \lambda),$$

(59)

$$V^{\hat{Q}_t}_t(\varphi_t) = -U^{\hat{Q}_t}_t(-\varphi_t) = \inf_{\alpha, \lambda \in \mathbb{R}} \left( \int_{K_t} v_t(\varphi_t(x_t)) + \alpha \text{Id}_t(x_t) + \lambda \right) d\hat{Q}_t(x_t) + (\alpha x_0 + \lambda).$$

We observe that Assumption 3.6 does not impose that the functions $u_t$ are real valued on the whole $\mathbb{R}$. Nevertheless, for the functionals $U^{\hat{Q}_t}_t, V^{\hat{Q}_t}_t$, we have:

**Lemma 4.2.** Under Assumption 3.6, for each $t = 0, \ldots, T$

1. $U^{\hat{Q}_t}_t$ and $V^{\hat{Q}_t}_t$ are real valued on $\mathcal{C}_b(K_t)$ and null in $0$.

2. $U^{\hat{Q}_t}_t$ and $V^{\hat{Q}_t}_t$ are concave and convex respectively, and both nondecreasing.

3. $U^{\hat{Q}_t}_t$ and $V^{\hat{Q}_t}_t$ are stock additive on $\mathcal{C}_b(K_t)$, namely for every $\alpha, \lambda_t \in \mathbb{R}$ and $\varphi_t \in \mathcal{C}_b(K_t)$

$$U^{\hat{Q}_t}_t(\varphi_t + \alpha_1 \text{Id}_t + \lambda_t) = U^{\hat{Q}_t}_t(\varphi_t) + \alpha_1 x_0 + \lambda_t, \quad V^{\hat{Q}_t}_t(\varphi_t + \alpha_1 \text{Id}_t + \lambda_t) = V^{\hat{Q}_t}_t(\varphi_t) + \alpha_1 x_0 + \lambda_t.$$

**Proof.** Since $V^{\hat{Q}_t}_t(\varphi_t) = -U^{\hat{Q}_t}_t(-\varphi_t)$, w.l.o.g. we prove the claims only for $U^{\hat{Q}_t}_t$. Clearly $U^{\hat{Q}_t}_t(\varphi_t) > -\infty$, as we may choose $\lambda_t \in \mathbb{R}$ so that $(\varphi_t + 0 \text{Id}_t + \lambda_t) \in \text{dom}(u) \supseteq [0, +\infty)$. Furthermore,

$$U^{\hat{Q}_t}_t(\varphi_t) \quad 1 \in \partial U_t(0) \quad \sup_{(\alpha, \lambda) \in \mathbb{R}} \left( \int_{K_t} (\varphi_t + \alpha \text{Id}_t + \lambda) d\hat{Q}_t - (\alpha x_0 + \lambda) \right) \quad \hat{Q} \in \text{Mart}(\Omega) \quad \sup_{(\alpha, \lambda) \in \mathbb{R}} \left( \int_{K_t} \varphi_t d\hat{Q}_t + (\alpha x_0 + \lambda - \alpha x_0 - \lambda) \right) \leq \|\varphi_t\|_\infty.$$

Finally, $0 = \int_{K_t} u(0) d\hat{Q}_t \leq U^{\hat{Q}_t}_t(0) \leq \|0\|_\infty$.

**Item 2:** trivial from the definitions. **Item 3:** we see that

$$U^{\hat{Q}_t}_t(\varphi_t + \alpha_1 \text{Id}_t + \lambda_t) = \sup_{(\alpha, \lambda) \in \mathbb{R}} \left( \int_{K_t} u_t(\varphi_t(x_t)) + (\alpha + \alpha_1) x_t + (\lambda + \lambda_t)) d\hat{Q}_t(x_t) - (\alpha x_0 + \lambda) \right)$$

$$= \sup_{(\alpha, \lambda) \in \mathbb{R}} \left( \int_{K_t} u_t(\varphi_t(x_t)) + (\alpha + \alpha_1) x_t + (\lambda + \lambda_t)) d\hat{Q}_t(x_t) - (\alpha x_0 + \lambda_1) \right) + \alpha_1 x_0 + \lambda_t,$$

in which we recognize the definition of $U^{\hat{Q}_t}_t(\varphi_t) + \alpha_1 x_0 + \lambda_t$. □

As in [6], in the next two Corollaries we suppose that the elements in $\mathcal{E}_t$ represent portfolios obtained combining call options with maturity $t$, units of the underlying stock at time $t$ ($x_t$) and deterministic amounts, that is $\mathcal{E}_t$ consists of all the functions in $\mathcal{C}_b(K_t)$ with the following form:

$$\varphi_t(x_t) = a + bx_t + \sum_{n=1}^{N} c_n(x_t - K_n)^+, \text{ for } a, b, c_n, k_n \in \mathbb{R}, x_t \in K_t$$

and take $E = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$. As shown in the proof, one could as well take $E = \mathcal{C}_b(K_0) \times \cdots \times C_b(K_T)$ preserving validity of (60), (61), (62) and (63).
Corollary 4.3. Take $u_0, \ldots, u_T$ satisfying Assumption 3.6, and suppose $\text{dom}(u_0) = \cdots = \text{dom}(u_T) = \mathbb{R}$. Then the following equalities hold:

$$
\inf_{Q \in \text{Mart}(\Omega)} \left( E_Q[c(X)] + \sum_{t=0}^{T} D_{v_t^*, \widehat{Q}_t}(Q_t) \right) = \sup_{Q \in \text{Mart}(\Omega)} \left( E_Q[d(X)] - \sum_{t=0}^{T} D_{v_t^*, \widehat{Q}_t}(Q_t) \right) = \inf_{Q \in \text{Mart}(\Omega)} \left( \sum_{t=0}^{T} U_{\widehat{Q}_t}(\varphi_t) \mid \varphi \in S_{\text{sub}}(c) \right) \quad (60)
$$

$$
\sup_{Q \in \text{Mart}(\Omega)} \left( E_Q[d(X)] - \sum_{t=0}^{T} D_{v_t^*, \widehat{Q}_t}(Q_t) \right) = \inf_{Q \in \text{Mart}(\Omega)} \left( \sum_{t=0}^{T} V_{\widehat{Q}_t}(\varphi_t) \mid \varphi \in S_{\text{sup}}(d) \right) \quad (61)
$$

Proof. We prove (60), since (61) can be obtained in a similar fashion. Set $U(\varphi) = \sum_{t=0}^{T} U_{\widehat{Q}_t}(\varphi_t)$ for $\varphi \in \mathcal{E}$. We observe that $\mathcal{E}_t$ consists of all piecewise linear functions on $K_t$, which are norm dense in $C_0(K_t)$. By Lemma 4.2 for each $t = 0, \ldots, T$ the monotone concave functional $\varphi_t \mapsto U_{\widehat{Q}_t}(\varphi_t)$ is actually well defined, finite valued, concave and nondecreasing on the whole $C_0(K_t)$. Hence, by the Extended Namioka-Klee Theorem (see [10]) it is norm continuous on $C_0(K_t)$ and we can take $\mathcal{E} = C_0(K_t) \times \cdots \times C_0(K_T)$ in place of $\mathcal{E}_0 \times \cdots \times \mathcal{E}_T$ in the RHS of (60) and prove equality to LHS in this more comfortable case (notice that $S_{\text{sub}}(c)$ depends on $\mathcal{E}$). We also observe that in this case we are in Setup 3.1. Define $\mathcal{D}$ as in (26) with $M = T$. Using the facts that if $\varphi_t \in \mathcal{E}_t$, $\alpha, \lambda \in \mathbb{R}$ then $(\varphi_t + \alpha \text{Id}_t + \lambda) \in \mathcal{E}_t$, that $Q \in \text{Mart}(\Omega)$ and that $v_t(\cdot) := -u_t(\cdot)$ one may easily check that

$$
\mathcal{D}(Q) := \sup_{\varphi \in \mathcal{E}} \left( U(\varphi) - \sum_{t=0}^{T} \int_{K_t} \varphi_t \, dq_t \right) = \sup_{\varphi \in \mathcal{E}} \left( \sum_{t=0}^{T} \int_{K_t} u_t(\varphi_t(x_t)) \, dq_t(x_t) - \sum_{t=0}^{T} \int_{K_t} \varphi_t \, dq_t \right)
$$

where the last equality follows from Proposition 3.8 Equation (57). The Standing Assumption 2.1 is satisfied. Indeed, from Assumption 3.6 we have $v_0^*(1), \ldots, v_T^*(1) < +\infty$, hence $D_{v_t^*, \widehat{Q}_t}(\widehat{Q}_t) = \int_{K_t} v_t^* \left( \frac{d\widehat{Q}_t}{dq_t} \right) \, dq_t < +\infty$ and therefore $\widehat{Q} \in \text{dom}(\mathcal{D})$. Recalling that $c \in L^1(\widehat{Q})$, this in turns yields $\widehat{Q} \in \mathcal{N} = \{ \mu \in \text{Meas}(\Omega) \cap \text{dom}(\mathcal{D}) \mid \int_{\Omega} c \, d\mu < +\infty \}$. Moreover, by Lemma 4.2 Item 1, $\text{dom}(U) = \mathcal{E}$, and for every $\mu \in \text{Meas}(\Omega)$ $D(\mu) \geq U(0) - 0 = 0$, hence $\mathcal{D}$ is lower bounded on the whole $\text{Meas}(\Omega)$. We conclude that $U$ and $\mathcal{D}$ satisfy the assumptions of Theorem 2.4.

Using Lemma 3.2 and the fact that $U_{\widehat{Q}_0}, \ldots, U_{\widehat{Q}_T}$ are cash additive we get $S^U(\varphi) = \sum_{t=0}^{T} S^U(\varphi_t) = \sum_{t=0}^{T} U_{\widehat{Q}_t}(\varphi_t) = U(\varphi)$, and by Corollary 2.8 Equation (50) we obtain

$$
\inf_{Q \in \text{Mart}(\Omega)} \left( E_Q[c(X)] + \sum_{t=0}^{T} D_{v_t^*, \widehat{Q}_t}(Q_t) \right) = \sup_{Q \in \text{Mart}(\Omega)} \left( \sum_{t=0}^{T} U_{\widehat{Q}_t}(\varphi_t) \mid \varphi \in S_{\text{sub}}(c) \right).
$$

\[\Box\]

Remark 4.4. At this point we can add some more discussion on the compactness condition of the underlying sets $K_0, \ldots, K_T$ in Assumption 4.1. In the classical non-robust setup the requirement of (essential) boundedness of the underlying stock is quite common. We observe that in our canonical setup for the underlying space, compactness is essentially tantamount to requiring that the stock $(\text{Id}_t)_t$ is bounded (everywhere). But we can actually say more: when $\mathcal{D}$ is taken as in Corollary 4.3, we automatically have that $Q_t \ll \widehat{Q}_t$, whenever $\mathcal{D}(Q) < +\infty$. If the marginals $\widehat{Q}_t$, for every $t = 0, \ldots, T$, satisfy $\text{Id}_t \in L^\infty(\widehat{Q}_t)$ we then get automatically that $\text{Id}_t \in L^\infty(Q_t)$, with $\|\text{Id}_t\|_{L^\infty(Q_t)} \leq \|\text{Id}_t\|_{L^\infty(\widehat{Q}_t)}$, for any $Q \in \text{dom}(\mathcal{D})$. Thus, it is possible to reformulate the hypotheses
in Corollary in 4.3 using $K_0 = \cdots = K_T = \mathbb{R}$ but requesting that the marginals $\hat{Q}_0, \ldots, \hat{Q}_T$ have compact support. A version of Corollary 4.3 should hold even without the compactness requirement in Assumption 4.1, but it is a delicate issue. It would require a modification of the settings, as the set of continuous functions would not work well any more, and a generalization of [42] Theorem 2.7, which is not trivial. We leave these interesting issues for future research.

We stress the fact that in Corollary 4.3 we assume that all the functions $u_0, \ldots, u_T$ are real valued on the whole $\mathbb{R}$. A more general result can be obtained when weakening this assumption, but it requires an additional assumption on the marginals of $\hat{Q}$.

**Corollary 4.5.** Suppose Assumption 3.6 is fulfilled. Assume $\hat{Q}_0, \ldots, \hat{Q}_T$ have full support on $K_0, \ldots, K_T$ respectively. Then Equations (60), (61) hold true replacing $\mathcal{D}_{u_t, \hat{Q}_t}(Q_t)$ with $\mathcal{F}_t(Q_t, \hat{Q}_t)$.

**Proof.** The proof can be carried over almost literally as the proof of Corollary 4.3, with the exception of replacing the reference to Proposition 3.8 with the reference to Proposition 3.9.

**Remark 4.6.** Observe that we are requesting the full support property on $K_0, \ldots, K_T$ with respect to their induced (Euclidean) topology. In particular, this means that whenever $k_t \in K_t$ is an isolated point, $\hat{Q}_t(\{k_t\}) > 0$. This is consistent with our assumption $K_0 = \{x_0\}$, which implies $\text{Prob}(K_0)$ reduces to the Dirac mesure, $\text{Prob}(K_0) = \{\delta_{x_0}\}$.

We now take $u_t(x) = x$ for each $t = 0, \ldots, T$, and get $U_{\hat{Q}_t}(\varphi_t) = V_{\hat{Q}_t}(\varphi_t) = \mathbb{E}_{\hat{Q}_t}[\varphi_t]$. Hence with an easy computation we have

$$\mathcal{D}_{u_t, \hat{Q}_t}(Q_t) = \begin{cases} 0 & \text{if } Q_t \equiv \hat{Q}_t \\ +\infty & \text{otherwise.} \end{cases}$$

Recalling that $\text{Mart}(\hat{Q}_1, \ldots, \hat{Q}_T) = \{Q \in \text{Mart}(\Omega) \mid Q_t \equiv \hat{Q}_t \ \forall \ t = 0, \ldots, T\}$, from Corollary 4.3 we can recover the following result of [6] (under more stringent assumptions on the underlying space).

**Corollary 4.7** ([6] Theorem 1.1 and Corollary 1.2). The following equalities hold:

$$\inf_{Q \in \text{Mart}(\hat{Q}_1, \ldots, \hat{Q}_T)} E_Q[c] = \sup \left\{ \sum_{t=0}^{T} \mathbb{E}_{\hat{Q}_t}[\varphi_t] \mid \varphi \in \mathcal{S}_{\text{sub}}(c) \right\}$$

$$\sup_{Q \in \text{Mart}(\hat{Q}_1, \ldots, \hat{Q}_T)} E_Q[d] = \inf \left\{ \sum_{t=0}^{T} \mathbb{E}_{\hat{Q}_t}[\varphi_t] \mid \varphi \in \mathcal{S}_{\text{sup}}(d) \right\}$$

### 4.1.1 Superhedging and Subhedging without Options

**Corollary 4.8.** The following equalities hold:

$$\inf_{Q \in \text{Mart}(\Omega)} E_Q[c] = \sup \{ m \in \mathbb{R} \mid m \in \mathcal{S}_{\text{sub}}(c) \} := \Pi^{\text{sub}}(c),$$

$$\sup_{Q \in \text{Mart}(\Omega)} E_Q[d] = \inf \{ m \in \mathbb{R} \mid m \in \mathcal{S}_{\text{sup}}(d) \} := \Pi^{\text{sup}}(d).$$

**Proof.** We take $\mathcal{E}_0 = \cdots = \mathcal{E}_T = \mathbb{R}$ and $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T = \mathbb{R}^{T+1}$. We first focus on (64). For each $\varphi \in \mathcal{E}$ with $\varphi = [m_1, \ldots, m_T]$, $m \in \mathbb{R}^{T+1}$ we select $U(\varphi) := \sum_{t=0}^{T} m_t$ (we notice that when
\( \mathcal{E} = \mathbb{R}^{T+1}, \ u_t(x_t) = x_t, \ t = 0, \ldots, T, \) and \( \hat{Q} \in \text{Mart}(\Omega) \), the functional \( U_{\hat{Q}_t} \) defined in (59) is given by \( U_{\hat{Q}_t}(m_t) = m_t \) and so \( U(m) = \sum_{t=0}^{T} U_{\hat{Q}_t}(m_t) = \sum_{t=0}^{T} m_t \) for all \( m \in \mathcal{E} \). Then applying the definition of \( D \) in (26) we get

\[
D(\gamma) = \begin{cases} 0 & \text{for } \gamma \in \text{ca}(\Omega) \text{ s.t. } \gamma(\Omega) = 1 \\ +\infty & \text{otherwise.} \end{cases}
\]

In particular \( D(Q) = 0 \) for every \( Q \in \text{Mart}(\Omega) \). Moreover we observe that \( D(U) = U(\varphi) \) for every \( \varphi \in \mathcal{E} \). Applying Corollary 2.8 (whose assumptions are clearly satisfied here), from Equation (50) we get that

\[
\inf_{Q \in \text{Mart}(\Omega)} E_{Q} [c] = \sup \left\{ \sum_{t=0}^{T} m_t \mid m_0, \ldots, m_T \in \mathbb{R} \text{ s.t. } \exists \Delta \in \mathcal{H} \text{ with } \sum_{t=0}^{T} m_t + I^\Delta \leq c \right\}.
\]

We recognize in the RHS above the RHS of (64). Equation (65) can be obtained in a similar way using Corollary 2.8 Equation (51).

4.1.2 Penalization with market price

In this section we change our perspective. Instead of starting from a given \( U \), we will give a particular form of the penalization term \( D \) and proceed in identifying the corresponding \( U \) in the spirit of Remark 2.2. For each \( t = 0, \ldots, T \) we suppose that finite sequences \((c_{t,n})_{1 \leq n \leq N_t} \subseteq \mathbb{R}\) and \((f_{t,n})_{1 \leq n \leq N_t} \subseteq C_b(K_t)\) are given. The functions \((f_{t,n})_{1 \leq n \leq N_t} \subseteq C_b(K_t)\) represent payoffs of options whose prices \((c_{t,n})_{1 \leq n \leq N_t} \subseteq \mathbb{R}\) are known from the market. Furthermore, we consider penalization functions \( \Psi_{t,n} : \mathbb{R} \rightarrow (-\infty, +\infty] \) which are convex, null in 0, symmetric in 0, proper and lower semicontinuous. For such functions we define the conjugates \( \Psi_{t,n}^* : \mathbb{R} \rightarrow (-\infty, +\infty] \) as \( \Psi_{t,n}^*(y) = \sup_{x \in \mathbb{R}} (xy - \Psi_{t,n}(x)) \).

Define

\[
\text{Mart}_t(K_t) = \{ \gamma_t \in \text{Prob}(K_t) \mid \exists Q \in \text{Mart}(\Omega) \text{ with } \gamma_t = Q_t \} \subseteq \text{ca}(K_t)
\]

and for \( \gamma_t \in \text{ca}(K_t) \)

\[
D_t^\Psi(\gamma_t) := \begin{cases} \sum_{n=1}^{N_t} \Psi_{t,n} \left( \int_{K_t} f_{t,n} \, d\gamma_t - c_{t,n} \right) & \text{for } \gamma_t \in \text{Mart}_t(K_t) \\ +\infty & \text{otherwise} \end{cases}
\]

**Proposition 4.9.** Suppose that the martingale measure \( \hat{Q} \in \text{Mart}(\Omega) \) in Standing Assumption 4.1 also satisfies \( \int_{K_t} f_{t,n} \, d\hat{Q}_t - c_{t,n} \in \text{dom}(\Psi_{t,n}) \) for every \( t = 0, \ldots, T, n = 0, \ldots, N_t \). Then setting for \( n = 1, \ldots, N_t, t = 0, \ldots, T \) \( g_{t,n} := f_{t,n} - c_{t,n} \in C_b(K_t) \) we have

**Subhedging Duality:**

\[
\inf_{Q \in \text{Mart}(\Omega)} \left( E_{Q} [c] + \sum_{t=0}^{T} D_t^\Psi(Q_t) \right) = \sup \left\{ \sum_{t=0}^{T} U_t^\Psi(\varphi_t) \mid \varphi \in \mathcal{S}_{\text{sub}}(c) \right\},
\]

where

\[
U_t^\Psi(\varphi_t) := \sup_{y_t \in \mathbb{R}^{N_t}} \left( \Pi^\text{sub} \left( \varphi_t + \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) - \sum_{n=1}^{N_t} \Psi_{t,n}^* \left( |y_{t,n}| \right) \right)
\]

is a stock additive functional and \( \Pi^\text{sub} \) is given in (64).
Superhedging Duality:

\[
\sup_{Q \in \text{Mart}(\Omega)} \left( E_Q [d] - \sum_{t=0}^{T} D^\Psi_t(Q_t) \right) = \inf \left\{ \sum_{t=0}^{T} V^\Psi_t(\varphi_t) : \varphi \in S_{\sup}(d) \right\},
\]

where

\[
V^\Psi_t(\varphi_t) = -U^\Psi_t(-\varphi_t) = \inf_{y_t \in \mathbb{R}^{N_t}} \left( \Pi_{\sup} \left( \varphi_t - \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) + \sum_{n=1}^{N_t} \Psi^*_t(|y_{t,n}|) \right)
\]
is a stock additive functional and \( \Pi_{\sup} \) is given in (65).

Before providing a proof, we state an auxiliary result.

**Lemma 4.10.** Suppose \( K_0, \ldots, K_t \subseteq \mathbb{R} \) are compact. Then \( \text{Mart}_t(K_t) \) is \( \sigma(\text{ca}(K_t), C_b(K_t)) \)-compact.

**Proof.** We see that \( \text{Mart}(\Omega) \) is \( \sigma(\text{ca}(\Omega), C_b(\Omega)) \)-closed subset of the \( \sigma(\text{ca}(\Omega), C_b(\Omega)) \)-compact set \( \text{Prob}(\Omega) \) (which is compact since \( \Omega \) is a compact Polish Space, see [2] Theorem 15.11), hence it is compact itself. \( \text{Mart}_t(K_t) \) is then the image of a compact set via the marginal map \( \gamma \mapsto \gamma_t \) which is \( \sigma(\text{ca}(\Omega), C_b(\Omega)) \) continuous, hence it is \( \sigma(\text{ca}(K_t), C_b(K_t)) \) compact. \( \square \)

**Proof of Proposition 4.9.** We focus on (66) first.

**STEP 1:** for any \( t \in \{0, \ldots, T\} \) we prove the following: the functional \( D^\Psi_t \) is \( \sigma(\text{ca}(K_t), C_b(K_t)) \)-lower semicontinuous and for every \( \varphi_t \in C_b(K_t) \) its Fenchel-Moreau (convex) conjugate satisfies

\[
V^\Psi_t(\varphi_t) := \sup_{\gamma_t \in \text{ca}(K_t)} \left( \int_{K_t} \varphi_t \, d\gamma_t - D^\Psi_t(\gamma_t) \right) = \inf_{y_t \in \mathbb{R}^{N_t}} \left( \Pi_{\sup} \left( \varphi_t - \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) + \sum_{n=1}^{N_t} \Psi^*_t(|y_{t,n}|) \right),
\]

and thus

\[
U^\Psi_t(\varphi_t) := -V^\Psi_t(-\varphi_t) = \sup_{y_t \in \mathbb{R}^{N_t}} \left( \Pi_{\sup} \left( \varphi_t + \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) - \sum_{n=1}^{N_t} \Psi^*_t(|y_{t,n}|) \right).
\]

We observe that \( D^\Psi_t \) is \( \sigma(\text{ca}(K_t), C_b(K_t)) \)-lower semicontinuous (it is a sum of functions, each being composition of a lower semicontinuous function and a continuous function on \( \text{Mart}_t(K_t) \)) which is \( \sigma(\text{ca}(K_t), C_b(K_t)) \)-compact by Lemma 4.10. We now need to compute

\[
V^\Psi_t(\varphi_t) = \sup_{\gamma_t \in \text{ca}(K_t)} \left( \int_{K_t} \varphi_t \, d\gamma_t - D^\Psi_t(\gamma_t) \right) = \sup_{Q_t \in \text{Mart}_t(K_t)} \left( \int_{K_t} \varphi_t \, dQ_t - D^\Psi_t(Q_t) \right).
\]

Recall now that from Fenchel-Moreau Theorem and symmetry \( \Psi_{t,n}(|x|) = \Psi_{t,n}(x) = \sup_{y \in \mathbb{R}} (xy - \Psi^*_{t,n}(y)) \). Hence, setting \( g_{t,n} = f_{t,n} - c_{t,n} \),

\[
V^\Psi_t(\varphi_t) = \sup_{Q_t \in \text{Mart}_t(K_t)} \left( \int_{K_t} \varphi_t \, dQ_t - \sum_{n=1}^{N_t} \sup_{y_{t,n} \in \mathbb{R}} \left( y_{t,n} \int_{K_t} \varphi_t \, dQ_t - \Psi^*_t(|y_{t,n}|) \right) \right) = \sup_{Q_t \in \text{Mart}_t(K_t)} \left( \int_{K_t} \varphi_t \, dQ_t - \sum_{n=1}^{N_t} \sup_{y_{t,n} \in \text{dom}(\Psi^*_t)} \left( y_{t,n} \int_{K_t} \varphi_t \, dQ_t - \Psi^*_t(|y_{t,n}|) \right) \right) = \inf_{Q_t \in \text{Mart}_t(K_t), y_{t,n} \in \text{dom}(\Psi^*_t)} \left( \int_{K_t} \left( \varphi_t - \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) \, dQ_t + \sum_{n=1}^{N_t} \Psi^*_t(|y_{t,n}|) \right) =: \inf_{Q_t \in \text{Mart}_t(K_t), y_{t,n} \in \text{dom}(\Psi^*_t)} T(y_t, Q_t),
\]
where $\text{dom} = \text{dom}(\Psi^*_{t,1}) \times \cdots \times \text{dom}(\Psi^*_{t,N_t}) \subseteq \mathbb{R}^{N_t}$. We now see that $\mathcal{T}$ is real valued on $\text{dom} \times \text{Mart}_t(K_t)$, is convex in the first variable and concave in the second. Moreover, $\{\mathcal{T}(y_t, \cdot) \geq C\}$ is $\sigma(\text{Mart}_t(K_t), C_b(K_t))$-closed in $\text{Mart}_t(\Omega)$ for every $y_t \in \text{dom}$, and $\text{Mart}_t(K_t)$ is $\sigma(\text{Mart}_t(K_t), C_b(K_t))$-compact (by Lemma 4.10). As a consequence $\mathcal{T}(y_t, \cdot)$ is $\sigma(\text{Mart}_t(K_t), C_b(K_t))$-lower semicontinuous on $\text{Mart}_t(K_t)$. We can apply [47] Theorem 3.1 with $A = \text{dom}$ and $B = \text{Mart}_t(K_t)$ endowed with the topology $\sigma(\text{Mart}_t(K_t), C_b(K_t))$, and by definition $\text{dom}(\Psi^*_{t,1})$ is convex in the first variable and concave in the second. Moreover, $\mathcal{T}(y_t, \cdot)$ is $\sigma(\text{Mart}_t(K_t), C_b(K_t))$-lower semicontinuous on $\text{Mart}_t(K_t)$.

To express this, we might take for $\mathcal{T}_{\ast}$ compact (by Lemma 4.10). As a consequence $\mathcal{T}(y_t, \cdot)$ is $\sigma(\text{Mart}_t(K_t), C_b(K_t))$-lower semicontinuous on $\text{Mart}_t(K_t)$. We can apply [47] Theorem 3.1 with $A = \text{dom}$ and $B = \text{Mart}_t(K_t)$ endowed with the topology $\sigma(\text{Mart}_t(K_t), C_b(K_t))$, and by definition $\text{dom}(\Psi^*_{t,1})$ is convex in the first variable and concave in the second. Moreover, $\mathcal{T}(y_t, \cdot)$ is $\sigma(\text{Mart}_t(K_t), C_b(K_t))$-lower semicontinuous on $\text{Mart}_t(K_t)$.

**STEP 2:** conclusion. We are clearly in the setup of Theorem 2.4 with $\mathcal{D}$ given as in Setup 3.4 from $\mathcal{D}_0^\Phi, \ldots, \mathcal{D}_T^\Phi$, and by definition $\text{dom}(\mathcal{D}_t^\Phi) \subseteq \text{Prob}(K_t)$ for each $t = 0, \ldots, T$. Using Lemma 3.5 Item 2, together with the computations in STEP 1 and the fact that clearly $\mathcal{S} U^\Phi_t \equiv U^\Phi_t$ by cash additivity of $U^\Phi_t$, we get the desired equality from Corollary 2.8 Equation (50): observe that our assumption on the existence of the measure $\widehat{Q} \in \text{Mart}(\Omega)$ guarantees, together with the fact that $\mathcal{D}$ is clearly lower bounded on $\text{Meas}(\Omega)$, that the hypotheses of Theorem 2.4 are satisfied (hence so is Standing Assumption 2.1). Equality (67) can now be obtained similarly to (66). \qed

**Remark 4.11.** Our assumption of existence of a particular $\widehat{Q} \in \text{Mart}(\Omega)$ in Proposition 4.9 expresses the fact that we are assuming our market prices $(c_{t,n})_{t,n}$ are close enough to those given by expectations under some martingale measure.

**Remark 4.12.** Proposition 4.9 covers a wide range of penalizations. For example, we might use power-like penalizations, i.e. $\Psi^*_{t,n}(x) = \frac{|x|^{p_{t,n}}}{p_{t,n}}$, for $p_{t,n} \in (1, +\infty)$. In such a case $\Psi^*_{t,n}(x) = \frac{|x|^{q_{t,n}}}{q_{t,n}}$, for $\frac{1}{p_{t,n}} + \frac{1}{q_{t,n}} = 1$. Alternatively, we might impose a threshold for the fitting, that is take into account only those martingale measure $Q$ such that $\int_{\Omega} f_{t,n} dQ_{t} - c_{t,n} \leq \varepsilon_{t,n}$ for some $\varepsilon_{t,n} > 0$. To express this, we might take for $x, y \in \mathbb{R}$

\[
\Psi^*_{t,n}(x) = \begin{cases} 0 & \text{if } |x| \leq \varepsilon_{t,n} \\ +\infty & \text{otherwise} \end{cases} \implies \Psi^*_{t,n}(y) = \frac{|y|}{\varepsilon_{t,n}}
\]

### 4.2 Beyond uniperiodal semistatic hedging

We now explore the versatility of Corollary 2.8, which can be used beyond the semistatic subhedging and superhedging problems in Section 4.1. Note that in Section 4.1 we chose for static hedging...
portfolios the sets $\mathcal{E}_t$, $t = 0, \ldots, T$ consisting of deterministic amounts, units of underlying stock at
time $t$ and call options with different strike prices and same maturity $t$. This affected the primal
problem in the fact that the penalty $\mathcal{D}$ turned out to depend solely on the (one dimensional)
marginals of $\hat{Q}$. Nonetheless, Theorem 2.4 allows to choose for each $t = 0, \ldots, T$ a subspace
$\mathcal{E}_t \subseteq \mathcal{C}_b(K_0 \times \cdots \times K_t)$, potentially allowing to consider also Asian and path dependent options in
the sets $\mathcal{E}_t$. We expect that this would translate in the penalty $\mathcal{D}$ depending no more only on the
one dimensional marginals of $\hat{Q}$. The study of these less restrictive, yet technically more complex
cases is left for future research.

In the following we will treat a slightly different problem, which however helps understanding how
also the extreme case $\mathcal{E}_t = \mathcal{C}_b(K_0 \times \cdots \times K_t)$, $t = 0, \ldots, T$ is of interest.

### 4.2.1 Dual representation for generalized OCE associated to the indirect utility function

Theorem 2.4 yields the following dual robust representation of the generalized Optimized Certainty
Equivalent associated to the indirect utility function. We stress here the fact that, again, $\hat{Q} \in \text{Mart}(\Omega)$ is a fixed martingale measure, but we will not focus anymore on its marginals only, as
will become clear in the following.

**Theorem 4.13.** Take $u : \mathbb{R} \to \mathbb{R}$ such that $u_0 = \ldots, u_T := u$ satisfy Assumption 3.6 and let $v^*$ be
defined in (55) with $u$ in place of $u_t$. Let $U^H_\hat{Q} : \mathcal{C}_b(\Omega) \to \mathbb{R}$ be the associated indirect utility

$$U^H_\hat{Q}(\varphi) := \sup_{\Delta \in \mathcal{R}} \int_\Omega u(\varphi + I^\Delta) \, d\hat{Q}.$$ 

and $S^{U^H_\hat{Q}}$ be the associated Optimized Certainty Equivalent defined according to (24), namely

$$S^{U^H_\hat{Q}}(\varphi) := \sup_{\xi \in \mathbb{R}} \left( U^H_\hat{Q}(\varphi + \xi) - \xi \right), \quad \varphi \in \mathcal{C}_b(\Omega).$$

Then for every $c \in \mathcal{C}_b(\Omega)$

$$S^{U^H_\hat{Q}}(c) = \inf_{Q \in \text{Mart}(\Omega)} \left( \int_\Omega c \, dQ + \mathcal{D}_\hat{Q}(Q) \right)$$

where for $\mu \in \text{Meas}(\Omega)$

$$\mathcal{D}_\hat{Q}(\mu) := \begin{cases} \int_\Omega v^* \left( \frac{d\mu}{d\hat{Q}} \right) \, d\hat{Q} & \text{if } \mu \ll \hat{Q} \\ +\infty & \text{otherwise} \end{cases}.$$ 

**Proof.** Take $\mathcal{E}_t = \mathcal{C}_b(K_0 \times \cdots \times K_t)$ for $t = 0, \ldots, T$. Define for $\psi \in \mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$
$U(\psi) := U^H_\hat{Q} \left( \sum_{t=0}^T \psi_t \right)$. Clearly $U(\psi) > -\infty$ for any $\psi \in \mathcal{E}$, and since $\hat{Q} \in \text{Mart}(\Omega)$ and $u(x) \leq x$ for all $x \in \mathbb{R}$ we also have $U(\psi) \leq \sum_{t=0}^T \|\psi_t\|_\infty < +\infty$. Moreover it is easy to verify that defining $\mathcal{D}$ as
in (26) for any $Q \in \text{Mart}(\Omega)$ we have

$$\mathcal{D}(Q) := \sup_{\psi \in \mathcal{E}} \left( U(\psi) - \int_\Omega \sum_{t=0}^T \psi_t \, dQ \right) = \sup_{\varphi \in \mathcal{C}_b(\Omega)} \left( \int_\Omega u(\varphi) \, d\hat{Q} - \int_\Omega \varphi \, dQ \right)$$

and arguing as in Proposition 3.8 we get $\mathcal{D}(Q) = \mathcal{D}_\hat{Q}(Q)$. From the fact that $u(x) \leq x$ for every
$x \in \mathbb{R}$ we have $v^*(1) < +\infty$, hence from Assumption 4.1 $\hat{Q} \in \text{dom}(\mathcal{D})$. This and $c \in L^1(\hat{Q})$ in
turns yields \( \hat{Q} \in \mathcal{N} \) (see (36)). Moreover \( \text{dom}(U) = \mathcal{E} \) and by definition of \( \mathcal{D} \) for any \( \mu \in \text{Meas}(\Omega) \) we have \( \mathcal{D}(\mu) \geq U(0) - 0 = 0 \), hence \( \mathcal{D} \) is lower bounded on the whole \( \text{Meas}(\Omega) \). We conclude that \( U \) and \( \mathcal{D} \) satisfy the assumptions of Theorem 2.4. We then get

\[
\inf_{Q \in \text{Mart}(\Omega)} (E_Q[c(X)] + \mathcal{D}(Q)) = \inf_{Q \in \text{Mart}(\Omega)} (E_Q[c(X)] + \mathcal{D}(Q)) = \sup_{\Delta \in \mathcal{H}} \sup_{\psi \in \Phi_\Delta(c)} S^U(\psi) .
\]

Observe now that \( S^U \) satisfies

\[
S^U(\psi) := \sup_{\lambda \in \mathbb{R}^{T+1}} \left( U(\psi + \lambda) - \sum_{t=0}^T \lambda_t \right) = \sup_{\lambda \in \mathbb{R}^{T+1}} \left( U_{\hat{Q}} \left( \sum_{t=0}^T \psi_t + \sum_{t=0}^T \lambda_t \right) - \sum_{t=0}^T \lambda_t \right)
\]

\[
= \sup_{\xi \in \mathbb{R}} \left( U_{\hat{Q}} \left( \sum_{t=0}^T \psi_t + \xi \right) - \xi \right) =: S^{U_{\hat{Q}}}(\sum_{t=0}^T \psi_t).
\]

\( S^{U_{\hat{Q}}} : C_b(\Omega) \to \mathbb{R} \) is (IA) and is nondecreasing, thus

\[
\sup_{\Delta \in \mathcal{H}} \sup_{\psi \in \Phi_\Delta(c)} S^{U_{\hat{Q}}}(\sum_{t=0}^T \psi_t) = \sup_{\Delta \in \mathcal{H}} \sup_{\psi \in \Phi_\Delta(c)} S^{U_{\hat{Q}}}(\sum_{t=0}^T \psi_t + I^\Delta) = S^{U_{\hat{Q}}}(c)
\]

by definition of \( \Phi_\Delta(c) \) and since \( c \in C_b(\Omega) \).

\[\square\]

### Appendix

#### A Setting

##### A.1 Measures

We start fixing our setup and some notation. Let \( \Omega \) be a Polish space and endow it with the Borel sigma algebra \( B(\Omega) \) generated by its open sets. A set function \( \mu : B(\Omega) \to \mathbb{R} \) is a \textbf{finite signed measure} if \( \mu(\emptyset) = 0 \) and \( \mu \) is \( \sigma \)-additive. A \textbf{finite measure} \( \mu \) is a finite signed measure such that \( \mu(B) \geq 0 \) for all \( B \in B(\Omega) \). A finite measure \( \mu \) such that \( \mu(\Omega) = 1 \) will be called a \textbf{probability measure}. Recall from Section 2 the notations for \( \text{ca}(\Omega), \text{Meas}(\Omega), \text{Prob}(\Omega) \). The following result is well known, see e.g. [11] Theorem 1.1 and 1.3.

\[\text{Proposition A.1. Every finite measure } \mu \text{ on } B(\Omega) \text{ is a Radon Measure, that is for every } B \in B(\Omega) \text{ and every } \varepsilon > 0 \text{ there exists a compact } K_\varepsilon \subseteq B \text{ such that } \mu(B \setminus K_\varepsilon) \leq \varepsilon.\]

A measure \( \mu \in \text{Meas}(\Omega) \) has \textbf{full support} if \( \mu(A) > 0 \) for every nonempty open set \( A \subseteq \Omega \). We also introduce for \( M \in \mathbb{N}, M \geq 1 \) the sets

\[
C_b(\Omega) := C_b(\Omega, \mathbb{R}) = \{ \varphi : \Omega \to \mathbb{R} \mid \varphi \text{ is bounded and continuous on } \Omega \},
\]

\[
C_b(\Omega, \mathbb{R}^M) := (C_b(\Omega))^M = \{ \varphi : \Omega \to \mathbb{R}^M \mid \varphi \text{ is bounded and continuous on } \Omega \},
\]

\[
LSC_b(\Omega) := LSC_b(\Omega, \mathbb{R}) = \{ \varphi : \Omega \to \mathbb{R} \mid \varphi \text{ is bounded and lower semicontinuous on } \Omega \}.
\]

Given a vector subspace \( \mathcal{E} \subseteq C_b(\Omega, \mathbb{R}^{M+1}) \) we will consider the dual pair \((\text{ca}(\Omega), C_b(\Omega, \mathbb{R}^{M+1}))\) with pairing given by the bilinear functional \( (\gamma, \varphi) \mapsto \int_{\Omega} \left( \sum_{m=0}^M \varphi_m \right) d\gamma \). We will induce on \( \text{ca}(\Omega) \) the topology \( \sigma(\text{ca}(\Omega), \mathcal{E}) \), which is the coarsest topology on \( \text{ca}(\Omega) \) making the functional \( \gamma \mapsto \int_{\Omega} \left( \sum_{m=0}^M \varphi_m \right) d\gamma \) continuous for each \( \varphi \in \mathcal{E} \). Similarly, we will induce on \( \mathcal{E} \) the topology \( \sigma(\mathcal{E}, \text{ca}(\Omega)) \) which is the coarsest topology on \( \mathcal{E} \) making the functional \( \gamma \mapsto \int_{\Omega} \left( \sum_{m=0}^M \varphi_m \right) d\gamma \) continuous for each \( \gamma \in \text{ca}(\Omega) \).
A.1.2 Weak and Narrow Topology

Definition A.2. The **Weak Topology** on $\text{Meas}(\Omega)$ is the coarsest (Hausdorff) topology for which all maps $\mu \mapsto \int_{\Omega} \varphi \, d\mu$ are continuous, for all $\varphi \in C_b(\Omega)$. The **Narrow Topology** is the coarsest (Hausdorff) topology for which all maps $\mu \mapsto \int_{\Omega} \varphi \, d\mu$ are lower semicontinuous, for all $\varphi \in LSC_b(\Omega)$.

Remark A.3. The weak topology on $\text{Meas}(\Omega)$ is the topology $\sigma(\text{Meas}(\Omega), C_b(\Omega, \mathbb{R}))$, which is the relative topology $\sigma(\text{ca}(\Omega), C_b(\Omega, \mathbb{R}))|_{\text{Meas}(\Omega)}$ induced by $\sigma(\text{ca}(\Omega), C_b(\Omega, \mathbb{R}))$ on $\text{Meas}(\Omega) \subseteq \text{ca}(\Omega)$ (see [2] Lemma 2.53).

**Proposition A.4.** When $\Omega$ is a Polish Space, the weak and narrow topologies coincide.

**Proof.** See [46] page 371.

**Remark A.5.** Even though the two topologies coincide in our setting, because of their different definitions we will find more convenient to exploit the one or the other topology in our proofs.

We now turn our attention to compactness issues in $\text{Meas}(\Omega)$ under the narrow topology. We recall first that a family $\Gamma \subseteq \text{Meas}(\Omega)$ is **bounded** if $\sup_{\mu \in \Gamma} \mu(\Omega) < +\infty$ and **tight** if for every $\varepsilon > 0$ there exists a compact $K_{\varepsilon} \subseteq \Omega$ such that $\sup_{\mu \in \Gamma} \mu(\Omega \setminus K_{\varepsilon}) \leq \varepsilon$. The following generalization of Prokhorov’s Theorem holds:

**Theorem A.6.** If a subset $\Gamma \subseteq \text{Meas}(\Omega)$ is bounded and tight, it is relatively compact in the narrow topology.

**Proof.** See [46] Theorem 3 pg. 379.

A.2 Auxiliary Results and Proofs

**Lemma A.7.** Take compact $K_1, \ldots, K_T \subseteq \mathbb{R}$, and suppose that $K_0 = \{x_0\}$ and $\text{card}(K_{t+1}) \geq \text{card}(K_t)$ for every $t = 0, \ldots, T - 1$. Take $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$ for vector subspaces $\mathcal{E}_t \subseteq C_b(K_t)$ such that $\text{Id}_t \in \mathcal{E}_t$ and $\mathcal{E}_t + \mathbb{R} = \mathcal{E}_t$, for $t = 0, \ldots, T$. Suppose there exist $\varphi, \psi \in \mathcal{E}$ and $\Delta \in \mathcal{H}$, where $\mathcal{H}$ is defined in (8), such that $\sum_{t=0}^{T} \phi_t = \sum_{t=0}^{T} \psi_t + I^\Delta$. Then there exist constants $k_0, \ldots, k_T, h_0, \ldots, h_T \in \mathbb{R}$ such that for each $t = 0, \ldots, T$ $\psi_t(x_t) = \phi_t(x_t) + k_t x_t + h_t, \forall x_t \in K_t$. In particular for $S_t : \mathcal{E}_t \to \mathbb{R}$, $t = 0, \ldots, T$ Stock Additive functionals we have

$$\sum_{t=0}^{T} S_t(\varphi_t) = \sum_{t=0}^{T} S_t(\psi_t).$$

and for $V := \sum_{t=0}^{T} \mathcal{E}_t + \mathcal{I}$ (see (9)) the map

$$v = \sum_{t=0}^{T} \varphi_t + I^\Delta \mapsto S(v) := \sum_{t=0}^{T} S_t(\varphi_t)$$

is well defined on $V$, (CA) and (IA).

**Proof.**

**STEP 1:** we prove that if $\sum_{t=0}^{T} \phi_t = \sum_{t=0}^{T} \psi_t + I^\Delta$ then $\Delta = [\Delta_0, \ldots, \Delta_{T-1}] \in \mathcal{H}$ is a deterministic vector $\Delta \in \mathbb{R}^T$. If $\text{card}(K_T) = 1$ this is trivial. We can then suppose $\text{card}(K_T) \geq 2$ We see that

$$\varphi_T(x_T) - \psi_T(x_T) = \sum_{t=0}^{T-1} (\psi(x_t) - \varphi_t(x_t)) + \sum_{t=0}^{T-2} \Delta_t(x_0, \ldots, x_t)(x_{t+1} - x_t) +$$

31
Additivity is trivial from the definition. As a consequence, given with $\sum$ is clear that there exist constants $\Delta_1$.

An argument similar to the one we used in the previous time step shows that $\Delta_2(x_0, \ldots, x_{T-2}) - \Delta_1(x_0, x_1) = 0$. Hence $\sum_{t=0}^{T-1} \Delta_t(x_0, \ldots, x_t) = 0$ for each $t = 0, \ldots, T$ that $\varphi_t(x_t) - (\psi_t(x_t) + k_t x_t)$ does not depend on $x_t$, hence is constant, call it $-h_t$. Then $k_0, \ldots, k_T, h_0, \ldots, h_T \in \mathbb{R}$ satisfy our requirements. The last claim $\sum_{t=0}^{T} S_t^U(\varphi_t) = \sum_{t=0}^{T} S_t^U(\psi_t)$ is then an easy consequence of stock additivity.

Proof of Lemma 3.2. We will only focus on (52), since the remaining claims are easily checked. We have that

$$D(\gamma) = \sup_{\varphi \in \mathcal{E}} \left( \sum_{t=0}^{T} U_t(\varphi_t) - \sum_{t=0}^{T} \int_{K_t} \varphi_t \, d\gamma \right) = \sum_{t=0}^{T} \int_{K_t} \varphi_t \, d\gamma = \sum_{t=0}^{T} D_t(\gamma_t) = \sum_{t=0}^{T} D_t(\gamma).$$

As to the second claim in (52), we observe that

$$\sup_{\xi \in \mathbb{R}^{T+1}} \left( U(\varphi + \xi) - \sum_{t=0}^{T} \xi_t \right) = \sum_{t=0}^{T} \sup_{\xi \in \mathbb{R}} (U_t(\varphi_t + \xi) - \xi) = \sum_{t=0}^{T} S_t^U(\varphi_t).$$
Proof of Lemma 3.5.

**Item 1.** For each \( t = 0, \ldots, T \) \( D_t(\gamma) = D_t \circ \pi_t(\gamma) \), where \( D_t \) is \( \sigma(ca(K_t), E_t) \)-lower semicontinuous and \( \pi_t \), the projection to the \( t \)-th marginal, is \( \sigma(ca(\Omega), E) - \sigma(ca(K_t), E_t) \) continuous. Hence, for each \( t = 0, \ldots, T \) \( \gamma \mapsto D_t(\gamma) \) is \( \sigma(ca(\Omega), E) \)-lower semicontinuous. Lower semicontinuity of \( D \) is then a consequence of the fact that the sum of lower semicontinuous functions is lower semicontinuous.

**Proof of Proposition 3.9.** We will exploit again [42] Theorem 2.7 and [42] Remark 2.8 (with 

**Item 2, equation (53).** We have that for \( \psi = -\varphi \)

\[
-U(\varphi) = V(\psi) = \sup_{\mu \in ca(\Omega)} \left( \int_\Omega \left( \sum_{t=0}^T \psi_t \right) d\mu - D(\mu_t) \right) = \sup_{\mu \in ca(\Omega)} \sum_{t=0}^T \left( \int_{K_t} \psi_t d\mu - D_t(\mu_t) \right)
\]

\[
\overset{(i)}= \sup \left\{ \sum_{t=0}^T \left( \int_{K_t} \psi_t d\gamma_t - D_t(\gamma_t) \right) \mid \gamma \in ca(\Omega) \text{ with } \gamma_t \in \text{Prob}(K_t) \forall t = 0, \ldots, T \right\}
\]

\[
\overset{(ii)}= \sup \left\{ \sum_{t=0}^T \left( \int_{K_t} \psi_t dQ_t - D_t(Q_t) \right) \mid [Q_0, \ldots, Q_T] \in \text{Prob}(K_0) \times \cdots \times \text{Prob}(K_T) \right\}
\]

\[
= \sum_{t=0}^T \sup_{Q_t \in \text{Prob}(K_t)} \left( \int_{K_t} \psi_t dQ_t - D_t(Q_t) \right) \overset{(iii)}= \sum_{t=0}^T \sup_{\gamma_t \in \text{ca}(K_t)} \left( \int_{K_t} \psi_t d\gamma_t - D_t(\gamma_t) \right)
\]

\[
= \sum_{t=0}^T V_t(\psi_t) = \sum_{t=0}^T -U_t(\varphi_t)
\]

Note that (i) follows from \( \text{dom}(D) \subseteq Z := \{ \gamma \in ca(\Omega) \mid \gamma_t \in \text{Prob}(K_t) \forall t = 0, \ldots, T \} \). In (ii) we used the facts that any vector of probability measures \( (Q_0, \ldots, Q_T) \) with \( Q_t \in \text{Prob}(K_t) \), \( t = 0, \ldots, T \), identifies \( \gamma := Q_0 \otimes \cdots \otimes Q_T \in Z \) with \( D(\gamma) = \sum_{t=0}^T D_t(Q_t) \) (note that this does not hold for a general vector of signed measures, which is why we need the additional assumption on the domains of the penalization functionals for Item 2) and that for every \( \gamma \in Z \), setting \( Q_t := \gamma_t \in \text{Prob}(K_t) \), we have \( D(\gamma) = \sum_{t=0}^T D_t(Q_t) \). Equality (iii) follows from \( \text{dom}(D_t) \subseteq \text{Prob}(K_t) \) for each \( t = 0, \ldots, T \).

**Item 2, equation (54).** The argument is identical to the one in the proof of Lemma 3.2, using the additive structure of \( U \) we obtained in the previous step of the proof. 

Proof of Proposition 3.8. We will use [42] Theorem 2.7 and [42] Remark 2.8. To do so, let us rename \( F := v^*_t \) (see (55) for the definition of \( v^* \)), which implies that \( F^\circ(y) := -F^\ast(-y) \) of [42] Equation (2.45) satisfies \( F^\circ(y) := -F^\ast(-y) = -v^*_t(-y) = v_t(y) \), by Fenchel-Moreau Theorem. All the assumptions of [42] Section 2.3 on \( F \) are satisfied, since for every \( y \geq 0 \)

\[
F(y) \geq u_t(0) - 0y = 0 \quad \text{and} \quad F(1) = \sup_{x \in \mathbb{R}} (u_t(x) - x) \leq 0 \quad \text{(recall } u_t(x) \leq x, \forall x \in \mathbb{R}) \text{.}
\]

Also, since \( \text{dom}(u_t) = \mathbb{R} \), \( \lim_{y \to +\infty} \frac{F(y)}{y} = F^\prime(1) = +\infty \). We can then apply [42] Theorem 2.7 and [42] Remark 2.8, obtaining (57). We stress the fact that since \( u_t \) is finite valued on the whole \( \mathbb{R} \), it is continuous there and for every \( \varphi_t \in C_b(K_t) \), \( F^\circ(\varphi_t) = u_t(\varphi_t) \in C_b(K_t) \), hence the additional constraint \( F^\circ(\varphi_t) \in C_b(K_t) \) (below [42] (2.49)) would be redundant in our setup.

Proof of Proposition 3.9. We will exploit again [42] Theorem 2.7 and [42] Remark 2.8 (with \( u_t \) in place of \( F^\circ \) ), as we explain now. Since \( u_t \) is nondecreasing, either its domain is in the form \( [M, +\infty) \) or \( (M, +\infty) \), with \( M \leq 0 \). Given a \( \varphi_t \in C_b(K_t) \) and a \( \mu \in \text{Meas}(K_t) \)
• Either \( \inf(\varphi_t(\mathbb{R})) > M \), in which case \( u_t(\varphi_t) \in C_b(K_t) \) since \( u_t \) is continuous on the interior of its domain.

• Or \( \inf(\varphi_t(\mathbb{R})) < M \), in which case \( \{\varphi_t < M\} \) is open nonempty and hence has positive \( \hat{Q}_t \) measure, as \( \hat{Q}_t \) has full support. Thus \( \int_{K_t} u_t(\varphi_t) \, d\hat{Q}_t = -\infty \).

• Or \( \inf(\varphi_t(\mathbb{R})) = M \) in which case \( u_t(\varphi_t) = \lim_{\varepsilon \downarrow 0} u_t(\max(\varphi_t, M + \varepsilon)) \) (since \( u_t \) is nondecreasing and upper semicontinuous) \( u_t(\max(\varphi_t, M + \varepsilon)) \in C_b(K_t) \) (see first bullet) and by Monotone Convergence Theorem

\[
\int_{K_t} u_t(\varphi_t) \, d\hat{Q}_t - \int_{K_t} \varphi_t \, d\mu = \lim_{\varepsilon \downarrow 0} \left( \int_{K_t} u_t(\max(\varphi_t, M + \varepsilon)) \, d\hat{Q}_t - \int_{K_t} \max(\varphi_t, M + \varepsilon) \, d\mu \right).
\]

Then we infer that

\[
\sup_{\varphi_t \in C_{\mathbb{R}}(K_t)} \left( \int_{K_t} \varphi_t \, d\mu - \int_{K_t} v_t(\varphi_t) \, d\hat{Q}_t \right) = \sup_{\varphi_t \in C_{\mathbb{R}}(K_t)} \left( \int_{K_t} u_t(\varphi_t) \, d\hat{Q}_t - \int_{K_t} \varphi_t \, d\mu \right)
\]

\[
= \sup \left\{ \int_{K_t} u_t(\varphi_t) \, d\hat{Q}_t - \int_{K_t} \varphi_t \, d\mu : \varphi_t, u_t(\varphi_t) \in C_b(K_t) \right\},
\]

and from [42] Theorem 2.7, [42] Remark 2.8 and from (69) we conclude the thesis.

\[\square\]

### A.3 On Minimax Duality Theorem

The following theorem is stated, without the proof, in [42], Th. 2.4. For the sake of completeness and without claiming any originality, we here provide the short proof.

**Theorem A.8 (Minimax Duality Theorem).** Let \( A, B \) be nonempty convex subsets of some vector spaces and suppose \( A \) is endowed with a Hausdorff topology. Let \( L : A \times B \to \mathbb{R} \) be a function such that

1. \( a \mapsto L(a,b) \) is convex and lower semicontinuous in \( A \) for every \( b \in B \).

2. \( b \mapsto L(a,b) \) is concave in \( B \) for every \( a \in A \).

When \( \alpha := \sup_{b \in B} \inf_{a \in A} L(a,b) < +\infty \), suppose that there exist \( C > \alpha \) and \( b^* \in B \) such that \( \{a \in A \mid L(a,b^*) \leq C\} \) is compact in \( A \). Then

\[
\inf_{a \in A} \sup_{b \in B} L(a,b) = \sup_{b \in B} \inf_{a \in A} L(a,b).
\]  

**Proof.** We start observing that in general \( \inf_{a \in A} \sup_{b \in B} L(a,b) \geq \sup_{b \in B} \inf_{a \in A} L(a,b) \), hence if \( \alpha = +\infty \) then (70) trivially holds. We then assume \( \alpha < +\infty \) and modify the proof of [47] Theorem 3.1. Let \( b_1, \ldots, b_N \in B \) be given and set \( b_0 = b^* \). By [47] Lemma 2.1.(a), using \( f_i(\cdot) := L(\cdot, b_i) \) we get constants \( \lambda_0, \ldots, \lambda_N \geq 0 \) with \( \sum_{i=0}^{N} \lambda_i = 1 \) such that

\[
\inf_{a \in A} \left( \max_{i=0,\ldots,N} L(a,b_i) \right) = \inf_{a \in A} \left( \sum_{i=0}^{N} \lambda_i L(a,b_i) \right) \leq \inf_{a \in A} L \left( a, \sum_{i=0}^{N} \lambda_i b_i \right) \leq \sup_{b \in B} \inf_{a \in A} L(a,b) = \alpha,
\]
where we used the concavity in \( B \) to obtain the first inequality. We now observe that for all \( \varepsilon > 0 \) there exists an \( a \in A \) such that

\[
a \in \left\{ \max_{i=0,\ldots,N} L(a, b_i) \leq \alpha + \varepsilon \right\} = \bigcap_{i=0}^{N} \left\{ L(a, b_i) \leq \alpha + \varepsilon \right\} = \bigcap_{i=1}^{N} \{ L(a, b_i) \leq \alpha + \varepsilon \} \cap \{ L(a, b^*) \leq \alpha + \varepsilon \}.
\]

Hence for \( A^* = \{ L(a, b^*) \leq \alpha + \varepsilon \} \) the family \( A^*_\varepsilon := \{ a \in A^* \mid L(a, b) \leq \alpha + \varepsilon \} \) is a collection of closed subsets of \( A^* \) having the finite intersection property. Now take \( \varepsilon > 0 \) such that \( \alpha + \varepsilon < C \). Then \( A^* \) is Hausdorff and compact, being a closed subset of the compact set \( \{ a \in A \mid L(a, b^*) \leq C \} \).

As a consequence \( \bigcap_{b \in B} A^*_\varepsilon \neq \emptyset \). This yields the existence of an \( a^* \) such that \( a^* \in A^* \) and \( L(a^*, b) \leq \alpha + \varepsilon \ \forall \ b \in B \). Hence

\[
\inf_{a \in A} \sup_{b \in B} L(a, b) \leq \sup_{b \in B} L(a^*, b) \leq \varepsilon + \alpha
\]

and letting \( \varepsilon \downarrow 0 \) we get

\[
\inf_{a \in A} \sup_{b \in B} L(a, b) \leq \sup_{b \in B} \inf_{a \in A} L(a, b) \leq \inf_{a \in A} \sup_{b \in B} L(a, b).
\]

\[\square\]

References

[1] B. Acciaio, M. Beiglböck, F. Penkner, and W. Schachermayer. A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. Math. Finance, 26(2):233–251, 2016.

[2] C. D. Aliprantis and K. C. Border. Infinite dimensional analysis. Springer, Berlin, third edition, 2006. A hitchhiker’s guide.

[3] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. Math. Finance, 9(3):203–228, 1999.

[4] E. Bayraktar and Y. Zhang. Fundamental theorem of asset pricing under transaction costs and model uncertainty. Math. Oper. Res., 41(3):1039–1054, 2016.

[5] E. Bayraktar and Z. Zhou. Super-hedging American options with semi-static trading strategies under model uncertainty. Int. J. Theor. Appl. Finance, 20(6):1750036, 10, 2017.

[6] M. Beiglböck, P. Henry-Labordère, and F. Penkner. Model-independent bounds for option prices—a mass transport approach. Finance and Stochastics, 17(3):477–501, 2013.

[7] F. Bellini and M. Frittelli. On the existence of minimax martingale measures. Math. Finance, 12(1):1–21, 2002.

[8] A. Ben-Tal and M. Teboulle. Expected utility, penalty functions, and duality in stochastic nonlinear programming. Management Sci., 32(11):1445–1466, 1986.

[9] A. Ben-Tal and M. Teboulle. An old-new concept of convex risk measures: the optimized certainty equivalent. Math. Finance, 17(3):449–476, 2007.
[10] S. Biagini and M. Frittelli. On the extension of the Namioka-Klee theorem and on the Fatou property for risk measures. In Optimality and risk—modern trends in mathematical finance, pages 1–28. Springer, Berlin, 2009.

[11] P. Billingsley. Convergence of probability measures. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[12] B. Bouchard and M. Nutz. Arbitrage and duality in nondominated discrete-time models. Ann. Appl. Probab., 25(2):823–859, 2015.

[13] D. T. Breeden and R. H. Litzenberger. Prices of state-contingent claims implicit in option prices. The Journal of Business, 51(4):621–651, 1978.

[14] H. Brown, D. Hobson, and L. C. G. Rogers. Robust hedging of barrier options. Math. Finance, 11(3):285–314, 2001.

[15] M. Burzoni, M. Frittelli, Z. Hou, M. Maggis, and J. Obłój. Pointwise arbitrage pricing theory in discrete time. Math. Oper. Res., 44(3):1034–1057, 2019.

[16] M. Burzoni, M. Frittelli, and M. Maggis. Universal arbitrage aggregator in discrete-time markets under uncertainty. Finance Stoch., 20(1):1–50, 2016.

[17] M. Burzoni, M. Frittelli, and M. Maggis. Model-free superhedging duality. Ann. Appl. Probab., 27(3):1452–1477, 2017.

[18] S. N. Cohen. Quasi-sure analysis, aggregation and dual representations of sublinear expectations in general spaces. Electron. J. Probab., 17:no. 62, 15, 2012.

[19] A. M. G. Cox and J. Obłój. Robust hedging of double touch barrier options. SIAM J. Financial Math., 2(1):141–182, 2011.

[20] A. M. G. Cox and J. Obłój. Robust pricing and hedging of double no-touch options. Finance Stoch., 15(3):573–605, 2011.

[21] A. M. G. Cox and J. Wang. Root’s barrier: construction, optimality and applications to variance options. Ann. Appl. Probab., 23(3):859–894, 2013.

[22] M. Davis, J. Obłój, and V. Raval. Arbitrage bounds for prices of weighted variance swaps. Math. Finance, 24(4):821–854, 2014.

[23] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker. Exponential hedging and entropic penalties. Math. Finance, 12(2):99–123, 2002.

[24] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. Math. Ann., 300(3):463–520, 1994.

[25] L. Denis and C. Martini. A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. Ann. Appl. Probab., 16(2):827–852, 2006.
[26] Y. Dolinsky and H. M. Soner. Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Related Fields*, 160(1-2):391–427, 2014.

[27] Y. Dolinsky and H. M. Soner. Martingale optimal transport in the Skorokhod space. *Stochastic Process. Appl.*, 125(10):3893–3931, 2015.

[28] N. El Karoui and M.-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.*, 33(1):29–66, 1995.

[29] H. Föllmer and A. Schied. Convex measures of risk and trading constraints. *Finance Stoch.*, 6(4):429–447, 2002.

[30] M. Frittelli. Introduction to a theory of value coherent with the no-arbitrage principle. *Finance Stoch.*, 4(3):275–297, 2000.

[31] M. Frittelli and E. Rosazza Gianin. Putting order in risk measures. *Journal of Banking & Finance*, 26(7):1473–1486, 2002.

[32] M. Frittelli and G. Scandolo. Risk measures and capital requirements for processes. *Math. Finance*, 16(4):589–612, 2006.

[33] A. Galichon, P. Henry-Labordère, and N. Touzi. A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Probab.*, 24(1):312–336, 2014.

[34] P. Henry-Labordère. Automated option pricing: numerical methods. *Int. J. Theor. Appl. Finance*, 16(8):1350042, 27, 2013.

[35] P. Henry-Labordère, J. Obłój, P. Spoida, and N. Touzi. The maximum maximum of a martingale with given n marginals. *Ann. Appl. Probab.*, 26(1):1–44, 2016.

[36] D. Hobson. The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, pages 267–318. Springer, Berlin, 2011.

[37] D. Hobson and M. Klimmek. Maximizing functionals of the maximum in the Skorokhod embedding problem and an application to variance swaps. *Ann. Appl. Probab.*, 23(5):2020–2052, 2013.

[38] D. G. Hobson. Robust hedging of the lookback option. *Finance and Stochastics*, 2(4):329–347, 1998.

[39] S. D. Hodges and A. Neuberger. Optimal replication of contingent claims under transaction costs. *Review Futures Market*, 8:222–239, 1989.

[40] Z. Hou and J. Obłój. Robust pricing-hedging dualities in continuous time. *Finance Stoch.*, 22(3):511–567, 2018.

[41] I. Karatzas. *Lectures on the mathematics of finance*, volume 8 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1997.
[42] M. Liero, A. Mielke, and G. Savaré. Optimal entropy-transport problems and a new Hellinger-Kantorovich distance between positive measures. *Inventiones mathematicae*, 211(3):969–1117, 2018.

[43] S. Peng. *Nonlinear expectations and stochastic calculus under uncertainty*, volume 95 of *Probability Theory and Stochastic Modelling*. Springer, Berlin, 2019. With robust CLT and G-Brownian motion.

[44] F. Riedel. Financial economics without probabilistic prior assumptions. *Decis. Econ. Finance*, 38(1):75–91, 2015.

[45] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.

[46] L. Schwartz. *Radon measures on arbitrary topological spaces and cylindrical measures*. Studies in mathematics. Published for the Tata Institute of Fundamental Research [by] Oxford University Press, 1973.

[47] S. Simons. *Minimax and monotonicity*. Lecture Notes in Mathematics. Springer, Berlin, 1998.

[48] H. M. Soner, N. Touzi, and J. Zhang. Quasi-sure stochastic analysis through aggregation. *Electron. J. Probab.*, 16: no. 67, 1844–1879, 2011.

[49] X. Tan and N. Touzi. Optimal transportation under controlled stochastic dynamics. *Ann. Probab.*, 41(5):3201–3240, 2013.