ON THE HADRONIC CONTRIBUTION TO LIGHT-BY-LIGHT SCATTERING IN $g_\mu - 2$.

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ABSTRACT

We comment on the theoretical uncertainties involved in estimating the hadronic effects on the light-by-light scattering contribution to the anomalous magnetic moment of the muon, especially based on the analysis and results of Ref. 1. From the point of view of an effective field theory and chiral perturbation theory, we suggest that the charged pion contribution may be better determined than has been appreciated. However, the neutral pion contribution needs greater theoretical insight before its magnitude can be reliably estimated.
The construction of an extremely high-precision experiment to determine the anomalous magnetic moment of the muon, \( a_\mu \equiv (g_\mu - 2)/2 \), is underway at Brookhaven National Laboratory (BNL).\(^\ast\) The anticipated design sensitivity, \( \Delta a_\mu = 4 \times 10^{-10} \), will, if achieved, be about 20 times better than the results from CERN,\(^2\) corresponding to a magnitude 5 times smaller than the Standard Model weak corrections. If the theory of known contributions is sufficiently good, such a measurement would either constrain or reveal physics beyond the Standard Model.\(^{[3−5]}\) To draw such an inference, or even to check the weak radiative correction, requires a highly accurate determination of hadronic contributions. The \( O(\alpha^2) \) correction to \( a_\mu \) coming from the \( O(\alpha) \) contribution to hadronic vacuum polarization to the photon propagator, has received a great deal of attention\(^3\) since it is approximately 35 times larger than the weak correction. The systematic error on this contribution can be reduced to the level of accuracy of the BNL experiment by a more precise low-energy measurement of \( R \equiv \sigma(hadron)/\sigma(\mu^-\mu^+) \), at VEPP-2M,\(^4\) currently operating, or, in the future, at DAΦNE.\(^6\)

There are other hadronic contributions in \( O(\alpha^3) \) coming from the \( O(\alpha^2) \) contribution to the four-photon vacuum polarization tensor \( \Pi_{\nu\rho\lambda\sigma} \), the so-called contribution from light-by-light scattering. At first sight, since hadronic interactions are strong, this would appear to be very difficult to determine accurately, but Kinoshita et.al.\(^1\) have estimated this contribution to be \( 49 \times 10^{-11} \) with an quoted accuracy of about 10\%. If correct, this contribution is a bit larger than the anticipated accuracy of the experiment, but four times smaller than the Standard Model weak correction. The method of calculation employed elementary pion contributions to the hadronic amplitude, together with vector meson dominance (VMD) of the photon couplings to the pion. They noted that their results happened to agree with a one-loop calculation involving elementary quarks with constituent quark masses, but they did not place great store by this second method because it is sensitive to the choice of the quark mass and would not appear to be a correct physical picture for momenta on the order of \( m_\mu \) or so.

Barbieri and Remiddi\(^7\) have recently raised doubts about the degree of certainty to be associated with the magnitude of the hadronic contribution in light-by-light scattering. As this is important for the interpretation of the experiment and crucial for drawing inferences about potential new physics, it is the purpose of this note to stimulate further discussion.

\(^\ast\) For a summary and review of earlier measurements, see, e.g., Ref. 2.
of these important issues.

The criticism in Ref. 7 is two-fold: (1) It is argued that the estimation in Ref. 1 by a quark loop is untrustworthy since it is so sensitive to the value chosen for the (constituent) quark mass. In this respect, they are actually in agreement with Ref. 1, who did not base their errors on the agreement with the estimation via a quark loop. So this should not be a point of dispute. (2) They doubt the accuracy claimed in Ref. 1, because it results “from the cancellation of different (gauge invariant) contributions, each of which is larger than the electroweak correction itself, ....”[7] In this regard, we believe that they are mistaken in suspecting that the approximation is unstable. In fact, the separate contributions, labelled A, B, and C in Ref. 1, while gauge-invariant with respect to the external photon, are not gauge-invariant contributions to Π_{νρλσ}; in particular, Bose symmetry among the four photons, while true of the sum, is not respected by the three separate sets of diagrams. Although each set, A, B, and C, yields a finite contribution to $a_\mu$, it commonly happens in the calculation of radiative corrections that there are large cancellations among classes of diagrams that are not separately gauge invariant, and such an occurrence in and of itself is not necessarily cause to distrust the final result.

Let us consider, ab initio how one may approximate the hadronic contributions to Π_{νρλσ}. The relevant momentum scale for the external photon momenta attached to Π_{νρλσ} is set by the muon mass, an energy scale small compared to typical hadronic scales, with the exception of the pion. Therefore, it should be a good approximation to describe this by an effective field theory involving pions and photons only. At low energies, the pions may be regarded as the Goldstone bosons associated with chiral symmetry breaking, whose self-interactions are described by the $O(3)$ non-linear sigma model, gauged with respect to the $U_1$ of electromagnetism. As a result, the interactions among pions at low-energy are actually weak, the scale of the chiral perturbation expansion being set by the mass $m_\rho$ of the $\rho$ meson or thereabouts, well above the mass of the muon or pion. As a first approximation, then, one may neglect the self-interactions of pions entirely. For the charged pion, this is precisely what was done in Ref. 1, but one must understand that this is not a model for Π_{νρλσ}, but the first term of a systematic expansion. The result for this contribution in Ref. 1 was given in Eq. (3.20). To illustrate that there should be no mystery here, a back-of-the-envelope estimate for this contribution to $a_\mu$ is as follows: all such contributions are proportional to $m_\mu^2$: one power comes from the definition of $a_\mu$; the other comes from
the observation that the photon couplings are all chiral conserving whereas the anomalous moment requires a chiral-flip, so a mass insertion is required. This contribution will then be proportional to \((m_\mu/m_\pi)^{-2}\). There are 6 factors of the coupling \(e\) and a factor of \(1/16\pi^2\) for each loop. While each graph contributes a logarithmic subdivergence to \(\Pi_{\nu\rho\lambda\sigma}\), we know from gauge invariance that this divergence is cancelled in the sum. So we estimate

\[
(m_\mu/m_\pi)^2(\alpha/4\pi)^3 \approx .01(\alpha/\pi)^3. \tag{01}
\]

The actual result of \(-.04(\alpha/\pi)^3\), given in Eq. (3.20) of Ref. 1, supports the view that, even though there are 21 graphs to be added, the cancellation among divergent pieces also holds for the finite remainder. Thus, the final result is no bigger than our estimate, reinforcing our belief that the cancellation between subsets of graphs is simply a gauge cancellation. We do not have an argument for the sign of the contribution.

Leaving aside the matter of the chiral anomaly for a moment, what are the corrections from interactions among the pions? The pion interactions do not become strong until one reaches to vicinity of the \(\rho\)-meson, so one would expect such contributions to \(g_\mu - 2\) to be suppressed compared to the free pion contribution by a factor of approximately \(m_\pi^2/m_\rho^2 \approx 4\%\). In Ref. 1, some of these corrections are introduced by VMD for the photon propagators, and it was found that they make corrections of more than 30\% to various groups of diagrams contributing to \(a_\mu\) and reduce the final answer for the charged pion contribution by a factor of more than 3. This casts a seemingly reliable approximation scheme in doubt. The VMD approximation used in Ref. 1 simply multiplies \(\Pi_{\nu\rho\lambda\sigma}\) by a common factor. Therefore, \(\Pi_{\nu\rho\lambda\sigma}\) remains gauge invariant, even though this VMD prescription does not respect the Ward identities for the couplings of photons to charged pions when off-mass-shell. Since \(\Pi_{\nu\rho\lambda\sigma}\) produces a convergent integral for the free pion loop, it is hard to understand how the VMD approximation, modifying the integrand on scales large compared to the pion mass, can change the final answer by a factor of three. This is a puzzle which suggests that the numerical integration be confirmed. It would add great confidence to the error estimates if the VMD corrections could be recalculated and

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Actually there are only 8 different integrals; see Fig. 5 of Ref. 1.

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I thank T. Kinoshita for emphasizing this point.

\*\*In fact, chiral perturbation theory has been extended to include the \(\rho\), but it is not clear how that formalism would be helpful here.
found to be of order $m^2 / m^2$ as expected. Of course, the integrand is not positive definite, so if the low-momentum region contributed little to the final answer, then modification of the high-momentum regime conceivably could produce a large percentage change in the final result. However, in that case, it may be important to have a gauge-invariant VMD model. For the time being, we tend to trust the free pion result more.

Another approach to estimating corrections would be to determine the effects of the higher order terms in the chiral perturbation expansion. That would be a complicated calculation that we have not attempted. However, we may anticipate one feature of these corrections, viz., because they involve higher-derivative interactions, they will lead to divergent contributions to $a_\mu$. That simply signifies that, when the muon is included in the effective field theory, one must include a “direct” contribution to $g_\mu - 2$ for which such a divergence provides a renormalization. That is to say, the effective field theory contains a term of the form

$$\frac{\alpha_d}{\Lambda^2} (\bar{\psi} \sigma^{\mu\nu} \psi) F_{\mu\nu},$$

(02)

where $\alpha_d$ represents some effective coupling constant and $\Lambda$ represents the scale at which this effective field theory breaks down. In the present context, we would expect $\Lambda$ to be on the order of the masses of the vector mesons $m_\rho$ and $m_\omega$. It has been tacitly assumed in Ref. 1 that the direct contribution is negligibly small. Strictly speaking, one requires a renormalizable description of strong interactions in order to be able to assume that the bare $\alpha_d$ is zero. QCD is, of course, such a theory but, unfortunately, we cannot reliably calculate its behavior in the low-energy regime relevant here. Even if we could, one must expect the physics omitted from any approximate description, renormalizable or not, to produce a direct interaction of this sort. For example, the weak correction itself may be regarded as a direct contribution at scales below $M_W$. One might explore various renormalizable models for chiral-symmetry-breaking that are somewhat simpler than QCD; for example, one might choose the linear sigma model, gauged with respect to electromagnetism. This effectively provides a cutoff at the mass of the $\sigma$, and the sensitivity of the result to $\sigma$ mass (or the strength of the pion self-coupling) may provide an indication of the size of the

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* This sort of contribution to $a_\mu$ and the corresponding effective field theory is described in detail in Ref. 5. It occurs also for the $\pi^0$ contribution; see below.

† We shall revisit this assumption below in connection with the neutral pion contribution.
uncertainty in the corrections; however, extracting the contribution to $a_{\mu}$ from a two-loop contribution to $\Pi_{\nu\rho\lambda\sigma}$ is a bit daunting.

In sum, we suggest that the result given for the non-interacting pion loop, Eq. (3.20) of Ref. 1, may be a reliable estimate of the charged pion contribution to an accuracy of about 4%. The VMD model is not gauge-invariant, although it does leave $\Pi_{\nu\rho\lambda\sigma}$ gauge invariant. As a result, we would be inclined to trust the non-interacting result more than the somewhat smaller one favored in Ref. 1 resulting from their implementation of VMD.

We now turn to consider the other contribution at the pion mass scale, the contribution of the $\pi^0$.

The effective field theory must be amended to include the effect of the chiral anomaly by adding $^{[10]}$

$$G \frac{\alpha}{8\pi f_{\pi}} \pi^0 \tilde{F}_{\mu
u} F^{\mu\nu} \quad (03)$$

in the linear representation or the corresponding expression for the nonlinear realization in chiral perturbation theory. $^{[11]}$ (Here, the pion decay constant $f_{\pi} \approx 93$ MeV.) Eq. (03) describes the coupling of the $\pi^0$ to two photons at low energy, and this local vertex was used as the first approximation in Ref. 1 to the contribution of the $\pi^0$ to $\Pi_{\nu\rho\lambda\sigma}$. The coupling constant $G$ is predicted to be 1 in QCD in the “chiral limit” in which the pion is massless, and the experimental value agrees extremely well with this, to about 0.25%. Power counting shows that this local approximation to the interaction leads to a (logarithmically) divergent result for $a_{\mu}$. This point-like interaction will be damped on scales associated with pion substructure, so, at scales on the order of $m_{\rho}$, this local approximation to the coupling $G$ must be modified. This is qualitatively equivalent to introducing damping factors via VMD as in Ref. 1, with a result that is given in their Eq. (3.27). In this case, their approximation to the vertex is gauge-invariant, and, therefore, this can be expected to provide a believable estimate of the size of this correction. The result should not be very different from simply introducing a cutoff on the divergence at the scale $m_{\rho}$, and, since the divergence is only logarithmic, the result will not be very sensitive to the precise value chosen. Again, a back-of-the-envelope calculation gives for any one diagram

$$a_{\mu} \sim m_{\mu}^2 \left( \frac{1}{16\pi^2} \right)^2 e^2 \left( \frac{\alpha}{\pi f_{\pi}} \right)^2 \ln(m_{\rho}^2/m_{\pi}^2) = \frac{1}{4} \left( \frac{m_{\mu}}{4\pi f_{\pi}} \right)^2 \left( \frac{\alpha}{\pi} \right)^3 \ln(m_{\rho}^2/m_{\pi}^2) \approx 0.006 (\alpha/\pi)^3.$$
There are two equal pairs of two diagrams each, so we ought to multiply this by at least a factor of two; that is still about a factor of 4 less than the result of $0.05(\alpha/\pi)^3$ given in Eq. (3.27) of Ref. 1, using VMD to cut off the divergence.

It is actually quite surprising that the magnitude of the $\pi^0$ contribution is as large as the charged pion contribution. Even though it is obviously the same order in $\alpha$, it is in effect one-loop order higher. That is, the anomalous coupling $\alpha/\pi f_\pi$ itself is a one-loop contribution, and it enters squared in a two-loop graph. Thus, this is in effect a four-loop contribution, to be compared with the three-loop contributions involving the charged pion. So, $a priori$, one would anticipate this contribution being suppressed by $1/16\pi^2$. Even though there are many more diagrams in the charged pion case, the cancellations among diagrams as required by gauge invariance arranges for the final result to be no bigger than our estimate. Thus, whereas the charged pion contribution is somewhat smaller than one might have guessed given the number of diagrams, the $\pi^0$ contribution is somewhat larger. This enhancement may be attributed to a host of small factors: the anomaly is twice as large as we would have guessed and it enters squared; the log enhancement is about a factor of 3; a factor of 2 from $(m_\pi/f_\pi)^2$; finally the various diagrams may add constructively in this case and destructively in the charged pion case. Thus, remarkably, the additional factor of $1/16\pi^2$ may be largely compensated, but it remains rather surprising and would be worth double-checking.

Combining the neutral pion result with the charged pion loop contribution (Eq. (3.20) rather than Eq. (3.24) of Ref. 1,) one obtains the result

$$a_\mu(\text{had2}) = 0.014 \left[ \frac{\alpha}{\pi} \right]^3,$$

(04)

coincidentally about 3 times smaller than that given there. This depends upon the significant cancellation between the charged and neutral pion contributions.

If this were the whole story, the situation would be quite satisfactory. However, the fact that the local coupling Eq. (03) results in a ultraviolet divergence implies that there must also be a “direct” contribution to $a_\mu$ coming from physics above the cutoff ($m_\rho$) whose natural size cannot be much smaller than the radiative correction due to the $\pi^0$.

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* The two diagrams of Fig. 5 of Ref. 1 must be added to two having the muon line reversed.

* According to Ref. 1, the free pion result is about equal but of opposite sign.
There are several ways to see this: If we regard $\alpha_d$ as a bare coupling, then the divergence of the $\pi^0$ contribution renormalizes it. Alternatively but equivalently, we may infer from the logarithmic divergence a contribution to the $\beta$ function for the renormalized coupling $\alpha_d(M)$, where $M$ is the renormalization scale. It would be unnatural, in the technical sense of the word, \cite{footnote:12} for it to be small compared to the size implied by varying $M$ a bit over the range of interest. Since $\ln(m_\rho/m_\pi) \approx 1.6$, we would expect the direct term to be not much smaller than the $\pi^0$ contribution calculated in Ref. 1. This suggests that important physics has been omitted from this method of calculation whose magnitude may well be comparable to the contribution calculated.

Of course, we expect that the $\pi^0\gamma\gamma$ vertex, even for the simple triangle diagram, will be damped whenever one of the external momenta becomes large compared to the typical scale of strong interactions. On the other hand, the Adler-Bardeen theorem\cite{footnote:10} guarantees that, for zero external momenta, the strong interactions do not in fact modify the anomaly associated with the underlying fundamental quark-loop diagram. The theoretical challenge is to figure out how to move away from the local approximation in a manner that is sufficiently well-controlled to provide an estimate of the error on the calculated contribution to $g_\mu - 2$.

The implications of this uncertainty associated with the $\pi^0$ contribution are severe: If the data turn out not to agree with the theoretically predicted value and the experimental errors are sufficiently small, then, rather than necessarily a signature of new physics, the deviations could just as well be ascribed to an erroneous assumption about the magnitude of the hadronic contribution from light-by-light scattering. In this respect, we agree with Ref. 7 that more work needs to be done.

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