On bi-variate poly-Bernoulli polynomials

Claudio Pita-Ruiz

Abstract. We introduce poly-Bernoulli polynomials in two variables by using a generalization of Stirling numbers of the second kind that we studied in a previous work. We prove the bi-variate poly-Bernoulli polynomial version of some known results on standard Bernoulli polynomials, as the addition formula and the binomial formula. We also prove a result that allows us to obtain poly-Bernoulli polynomial identities from polynomial identities, and we use this result to obtain several identities involving products of poly-Bernoulli and/or standard Bernoulli polynomials. We prove two generalized recurrences for bi-variate poly-Bernoulli polynomials, and obtain some corollaries from them.

1 Introduction

Bernoulli numbers are one of the most important mathematical objects that have been studied by mathematicians since they appeared in the 18-th century (see [13]). A recent important generalization of the Bernoulli numbers $B_n$ and Bernoulli polynomials $B_n(x)$ is about the so-called poly-Bernoulli numbers and poly-Bernoulli polynomials. Poly-Bernoulli numbers $B_n^{(k)}$, where $k$ is a given positive integer, were introduced by M. Kaneko [10] in 1997, by means of the generating function

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

where $\text{Li}_k(z) = \sum_{j=1}^{\infty} z^j / j^k$ is the polylogarithm function. The case $k = 1$ corresponds to the standard Bernoulli numbers $B_n$ (except the sign of $B_1^{(1)}$). Poly-Bernoulli polynomials

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\(B_n^{(k)}(x)\) can be defined by the generating function

\[
\frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-tx} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!},
\]

(see [5]). The case \(k = 1\) corresponds to \((-1)^n B_n(x)\), and the case \(x = 0\) corresponds to the poly-Bernoulli numbers \(B_n^{(k)}\) mentioned before. Some slightly different definitions of poly-Bernoulli polynomials \(B_n^{(k)}(x)\), with \(x\) replaced by \(-x\), and/or with an additional factor \((-1)^n\), can be found in some related papers (see [1], [5], [6], [9]). In this work we use the following explicit formula

\[
B_n^{(k)}(x) = \sum_{l=0}^{n} \frac{1}{(l+1)^k} \sum_{j=0}^{l} (-1)^j \binom{l}{j} (j+x)^n,
\]

(1) as our definition of poly-Bernoulli polynomials (see formula (1.8) in [1]). It is important to mention that the notation \(B_n^{(k)}(x)\) is also used for a different kind of mathematical objects, namely, Bernoulli polynomials of \(k\)-th order (see [3]).

A different generalization of Bernoulli polynomials, studied in the past few years, is about considering Bernoulli polynomials in several variables \(B_{p_1,\ldots,p_t}(x_1,\ldots,x_t)\), that is, polynomials of degree \(p_i\) in the variable \(x_i\), with

\[
B_{0,\ldots,p_1,\ldots,0}(x_1,\ldots,x_t) = B_{p_i}(x_i) \quad \text{for each } i \in \{1,\ldots,t\},
\]

seeking that reasonable generalizations of the known properties in the one-variable case, remain valid. This kind of work is done in [16], with a flavor of multivariable analysis and working with Jack polynomials. A different approach is presented in [17] (see also [2], [7]).

In this work we study poly-Bernoulli polynomials in two variables (bi-variate poly-Bernoulli polynomials). We define the bi-variate poly-Bernoulli polynomials by using a generalization of Stirling numbers of the second kind we studied in [14], and then we use the results in [14] to obtain results for the bi-variate poly-Bernoulli polynomials considered in this work.

We present now the definitions and results in [14] that we will use in the remaining sections.

The generalized Stirling numbers of the second kind (GSN, for short), denoted as \(S_{a_1,b_1}^{(a_2,b_2,p_2)}(p_1,k)\), where \(a_j,b_j \in \mathbb{C}\), \(a_j \neq 0\), \(j = 1,2\), and \(p_1,p_2\) non-negative integers, are defined by means of the expansion

\[
(a_1 n + b_1)^{p_1} (a_2 n + b_2)^{p_2} = \sum_{k=0}^{p_1+p_2} k! S_{a_1,b_1}^{(a_2,b_2,p_2)}(p_1,k) \binom{n}{k},
\]

(2)

\((S_{a_1,b_1}^{(a_2,b_2,p_2)}(p_1,k) = 0 \text{ if } k < 0 \text{ or } k > p_1 + p_2 \)). An explicit formula for these numbers is

\[
S_{a_1,b_1}^{(a_2,b_2,p_2)}(p_1,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (a_1(k-j) + b_1)^{p_1} (a_2(k-j) + b_2)^{p_2}.
\]

(3)
If \( p_2 = 0 \), we write the GSN \( S_{a,b}^{a_2,b_1,0}(p,k) \) as \( S_{a,b}(p,k) \). We have

\[
S_{a,b}(p,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (a(k-j)+b)^p.
\] (4)

In the case \( a = 1, b = 0 \), the corresponding GSN \( S_{1,0}(p,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^p \) are the known Stirling numbers of the second kind. We will refer to them as “standard Stirling numbers”, and in this case we use the known notation \( S(p,k) \).

From (3) it is clear that \( S_{a,b}^{a_2,b_1,2}(p_1,k) = S_{a,b}(p_1 + p_2,k) \). We can see directly from (4) that

\[
S_{1,1}(p,k) = S(p+1,k+1),
\]
\[
S_{1,2}(p,k) = S(p+2,k+2) - S(p+1,k+2).
\] (6)

In this work we will use GSN of the form \( S_{1,x_1}^{1,x_2,p_2}(p_1,k) \). Some important facts about the GSN \( S_{1,x_1}^{1,x_2,p_2}(p_1,k) \) are the following:

- Some values of the GSN \( S_{1,x_1}^{1,x_2,p_2}(p_1,k) \) are

\[
S_{1,x_1}^{1,x_2,p_2}(p_1,0) = x_1^{p_1} x_2^{p_2},
\]
\[
S_{1,x_1}^{1,x_2,p_2}(p_1,1) = (x_1 + 1)^{p_1} (x_2 + 1)^{p_2} - x_1^{p_1} x_2^{p_2},
\]
\[
\vdots
\]
\[
S_{1,x_1}^{1,x_2,p_2}(p_1,p_1 + p_2) = 1.
\] (7)

- The GSN \( S_{1,x_1}^{1,x_2,p_2}(p_1,k) \) can be written in terms of the GSN \( S_{1,y_1}^{1,y_2,p_2}(p_1,k) \) as follows

\[
S_{1,x_1}^{1,x_2,p_2}(p_1,k) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x_1 - y_1)^{p_1-j_1} (x_2 - y_2)^{p_2-j_2} S_{1,y_1}^{1,y_2,p_2}(j_1,k).
\] (8)

- The GSN \( S_{1,x_1}^{1,x_2,p_2}(p_1,k) \) can be written in terms of standard Stirling numbers as follows

\[
k! S_{1,x_1}^{1,x_2,p_2}(p_1,k) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x_1 - m)^{p_1-j_1} (x_2 - m)^{p_2-j_2}
\]
\[
\times \sum_{i=0}^{m} \binom{m}{i} (k+i)! S(j_1 + j_2, k + i),
\] (9)

where \( m \) is an arbitrary non-negative integer.
• The GSN \( S^{1,x_2,p_2}_{1,x_1}(p_1, k) \) can be written in terms of standard Stirling numbers as follows

\[
S^{1,x_2,p_2}_{1,x_1}(p_1, k) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x_1 - n)^{p_1-j_1} (x_2 - n)^{p_2-j_2} \\
\times \sum_{i=0}^{n-1} (-1)^i s(n, n-i) S(j_1 + j_2 + n - i, k + n),
\tag{10}
\]

where \( n \) is an arbitrary positive integer, and \( s(\cdot, \cdot) \) are the unsigned Stirling numbers of the first kind.

• The GSN \( S^{1,x_2,p_2}_{1,x_1}(p_1, k) \) satisfy the identity

\[
S^{1,x_2+1,p_2}_{1,x_1}(p_1, k) = S^{1,x_2,p_2}_{1,x_1}(p_1, k) + (k + 1) S^{1,x_2,p_2}_{1,x_1}(p_1, k + 1).
\tag{11}
\]

• The GSN \( S^{1,x_2,p_2}_{1,x_1}(p_1, k) \) satisfy the recurrence

\[
S^{1,x_2,p_2}_{1,x_1}(p_1, k) = S^{1,x_2,p_2}_{1,x_1}(p_1 - 1, k - 1) + (k + x_1) S^{1,x_2,p_2}_{1,x_1}(p_1 - 1, k).
\tag{12}
\]

## 2 Definitions and preliminary results

The relation of Bernoulli (numbers and polynomials) with Stirling (numbers of the second kind) is an old story, that dates back to Worpitsky [18] (see also [8, p. 560] and [11, p. 5]). We have the following formula for Bernoulli numbers

\[
B_p = \sum_{l=0}^{p} S(p, l) \frac{(-1)^l l!}{l+1},
\tag{13}
\]

and in the case of Bernoulli polynomials we have

\[
B_p(x) = \sum_{l=0}^{p} \sum_{j=0}^{p} \binom{p}{j} x^{p-j} S(j, l) \frac{(-1)^l l!}{l+1}.
\tag{14}
\]

An important observation of formula (1) is that poly-Bernoulli polynomial \( B_p^{(k)}(x) \) can be written in terms of the GSN as

\[
B_p^{(k)}(x) = \sum_{l=0}^{p} S_{1,x}(p, l) \frac{(-1)^l l!}{(l+1)^k}.
\tag{15}
\]

The generalization of (15) to the case of two variables comes through the GSN: we define poly-Bernoulli polynomial in the variables \( x_1, x_2 \), denoted by \( B^{(k)}_{p_1,p_2}(x_1, x_2) \), as

\[
B^{(k)}_{p_1,p_2}(x_1, x_2) = \sum_{l=0}^{p_1+p_2} S^{1,x_2,p_2}_{1,x_1}(p_1, l) \frac{(-1)^l l!}{(l+1)^k}.
\tag{16}
\]
If \( p_2 = 0 \), formula (16) becomes (15). By using (9) we can write \( B_{p_1, p_2}^{(k)}(x_1, x_2) \) in terms of standard Stirling numbers as

\[
B_{p_1, p_2}^{(k)}(x_1, x_2) = \sum_{i=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) (x_1 - m)^{p_1-j_1} (x_2 - m)^{p_2-j_2} \\
\times \sum_{i=0}^{m} \left( \begin{array}{c} m \\ i \end{array} \right) S(j_1 + j_2, l + i) \frac{(-1)^i (l + i)!}{(l + 1)^k},
\]

where \( m \) is an arbitrary non-negative integer. Similarly, by using (10) we have that

\[
B_{p_1, p_2}^{(k)}(x_1, x_2) = \sum_{i=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) (x_1 - n)^{p_1-j_1} (x_2 - n)^{p_2-j_2} \\
\times \sum_{i=0}^{n-1} (-1)^i s(n, n - i) S(j_1 + j_2 + n - i, l + n) \frac{(-1)^i l!}{(l + 1)^k},
\]

where \( n \) is an arbitrary positive integer.

The simplest cases of (17) and (18) are

\[
B_{p_1, p_2}^{(k)}(x_1, x_2) = \sum_{i=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) x_1^{p_1-j_1} x_2^{p_2-j_2} S(j_1 + j_2, l) \frac{(-1)^i l!}{(l + 1)^k},
\]

and

\[
B_{p_1, p_2}^{(k)}(x_1, x_2) = \sum_{i=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) (x_1 - 1)^{p_1-j_1} (x_2 - 1)^{p_2-j_2} S(j_1 + j_2 + 1, l + 1) \frac{(-1)^i l!}{(l + 1)^k},
\]

respectively.

Two examples are the following

\[
B_{1,1}^{(k)}(x_1, x_2) = \frac{2}{3^k} - \frac{1}{2^k} (x_1 + x_2 + 1) + x_1 x_2,
\]

\[
B_{1,2}^{(k)}(x_1, x_2) = \frac{1}{3^k} (2x_1 + 4x_2 + 6) - \frac{1}{2^k} (x_1 (2x_2 + 1) + (x_2 + 1)^2) + x_1 x_2 - \frac{6}{4^k}.
\]

Clearly we have

\[
B_{0,0}^{(k)}(x_1, x_2) = 1.
\]

Observe also that

\[
B_{p_1, p_2}^{(k)}(x, x) = B_{p_1+p_2}^{(k)}(x).
\]

In particular, we have

\[
B_{p_1, p_2}^{(k)}(0, 0) = B_{p_1+p_2}^{(k)}(0).
\]
From (3) and (16) we have

\[ B_{p_1,p_2}^{(k)}(x_1, x_2) = \sum_{l=0}^{p_1+p_2} \sum_{j=0}^{l} (-1)^j \binom{l}{j} (l-j+x_1)^{p_1} (l-j+x_2)^{p_2} \frac{(-1)^l}{(l+1)^k}, \quad (24) \]

from where we see that

\[ \frac{\partial}{\partial x_1} B_{p_1,p_2}^{(k)}(x_1, x_2) = p_1 B_{p_1-1,p_2}^{(k)}(x_1, x_2), \]

\[ \frac{\partial}{\partial x_2} B_{p_1,p_2}^{(k)}(x_1, x_2) = p_2 B_{p_1,p_2-1}^{(k)}(x_1, x_2). \]

We can use (8) to write \( B_{p_1,p_2}^{(k)}(x_1, x_2) \) in terms of \( B_{j_1,j_2}^{(k)}(y_1, y_2), \ 0 \leq j_i \leq p_i, \ i = 1, 2, \) as

\[ B_{p_1,p_2}^{(k)}(x_1, x_2) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x_1-y_1)^{p_1-j_1} (x_2-y_2)^{p_2-j_2} B_{j_1,j_2}^{(k)}(y_1, y_2), \quad (25) \]

that generalizes the known addition formula \( B_p^{(k)}(x) = \sum_{j=0}^{p} \binom{p}{j} (x-y)^{p-j} B_j^{(k)}(y) \) for one-variable poly-Bernoulli polynomials. In fact, we have

\[ B_{p_1,p_2}^{(k)}(x_1, x_2) = \sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2} (p_1, l) \frac{(-1)^l l!}{(l+1)^k} \]

\[ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x_1-y_1)^{p_1-j_1} (x_2-y_2)^{p_2-j_2} \sum_{l=0}^{j_1+j_2} S_{1,y_1}^{1,y_2,j_1} (j_1, l) \frac{(-1)^l l!}{(l+1)^k} \]

\[ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x_1-y_1)^{p_1-j_1} (x_2-y_2)^{p_2-j_2} B_{j_1,j_2}^{(k)}(y_1, y_2), \]

as claimed. In particular, if we set \( y_1 = y_2 = y \) in (25) we obtain an expression for the bi-variate poly-Bernoulli polynomial \( B_{p_1,p_2}^{(k)}(x_1, x_2) \) in terms of one-variable poly-Bernoulli polynomials \( B_j^{(k)}(y), \ 0 \leq j \leq p_1 + p_2, \) namely

\[ B_{p_1,p_2}^{(k)}(x_1, x_2) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x_1-y)^{p_1-j_1} (x_2-y)^{p_2-j_2} \sum_{l=0}^{j_1+j_2} S_{1,y}^{1,y,j_1} (j_1, l) \frac{(-1)^l l!}{(l+1)^k} \]

\[ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x_1-y)^{p_1-j_1} (x_2-y)^{p_2-j_2} B_{j_1,j_2}^{(k)}(y), \quad (26) \]

and then we can write the bi-variate poly-Bernoulli polynomial \( B_{p_1,p_2}^{(k)}(x_1, x_2) \) in terms of poly-Bernoulli numbers \( B_j^{(k)}, \ 0 \leq j \leq p_1 + p_2, \) as

\[ B_{p_1,p_2}^{(k)}(x_1, x_2) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} x_1^{p_1-j_1} x_2^{p_2-j_2} B_{j_1,j_2}^{(k)}. \quad (27) \]
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The cases \( k = 0 \) and \( k = -1 \) of (27) are

\[
B_{p_1,p_2}^{(0)}(x_1, x_2) = (x_1 - 1)^{p_1} (x_2 - 1)^{p_2},
\]

and

\[
B_{p_1,p_2}^{(-1)}(x_1, x_2) = (x_1 - 2)^{p_1} (x_2 - 2)^{p_2},
\]

respectively. In fact, according to (16), formulas (28) and (29) are the particular cases \( r = 0 \) and \( r = 1 \) of the identity

\[
\sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2} (p_1, l) (-1)^l (l + r)! = r! (x_1 - r - 1)^{p_1} (x_2 - r - 1)^{p_2},
\]

where \( r \) is a non-negative integer. We leave the proof of (30) to the reader.

Observe also that (25) implies that

\[
B_{p_1,p_2}^{(k)}(x_1 + 1, x_2 + 1) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B_{j_1,j_2}^{(k)}(x_1, x_2),
\]

which generalizes the known binomial formula for standard Bernoulli polynomials

\[
B_p(x + 1) = \sum_{j=0}^{p} \binom{p}{j} B_j(x).
\]

If we set \( x_1 = x_2 = x \) in (25), we obtain a formula for the standard poly-Bernoulli polynomial \( B_{p_1+p_2}^{(k)}(x) \) in terms of the bi-variate poly-Bernoulli polynomials \( B_{j_1,j_2}^{(k)}(y_1, y_2) \), \( 0 \leq j_i \leq p_i, \ i = 1, 2 \), namely

\[
B_{p_1+p_2}^{(k)}(x) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x - y_1)^{p_1-j_1} (x - y_2)^{p_2-j_2} B_{j_1,j_2}^{(k)}(y_1, y_2).
\]

Some additional simple observations are the following

\[
B_{p_1,0}^{(k)}(x_1, x_2) = B_{p_1}^{(k)}(x_1),
\]

\[
B_{0,p_2}^{(k)}(x_1, x_2) = B_{p_2}^{(k)}(x_2),
\]

and

\[
B_{p_1,0}^{(k)}(0, x_2) = \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} x_2^{p_2-j_2} B_{p_1+j_2}^{(k)},
\]

\[
B_{p_1,0}^{(k)}(x_1, 0) = \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} x_1^{p_1-j_1} B_{j_1+p_2}^{(k)}.
\]
3 Some identities

In this section we obtain some identities involving poly-Bernoulli polynomials, by using the following result:

**Theorem 3.1.** The polynomial identity

\[
\sum_{r=0}^{n} a_{n,r}(x + \alpha)^r = \sum_{r=0}^{n} b_{n,r}(x + \beta)^r. \tag{38}
\]

implies the poly-Bernoulli polynomial identity

\[
\sum_{r=0}^{n} a_{n,r}B_r^{(k)}(x + \alpha) = \sum_{r=0}^{n} b_{n,r}B_r^{(k)}(x + \beta). \tag{39}
\]

**Proof.** Observe that the hypothesis of the polynomial identity (38), comes together with the identity of its derivatives:

\[
\sum_{r=0}^{n} \binom{r}{j} a_{n,r}(x + \alpha)^{r-j} = \sum_{r=0}^{n} \binom{r}{j} b_{n,r}(x + \beta)^{r-j}
\]

where \(j\) is a non-negative integer.

We have

\[
\sum_{r=0}^{n} a_{n,r}B_r^{(k)}(x + \alpha) = \sum_{r=0}^{n} a_{n,r} \sum_{l=0}^{r} \binom{r}{j} S_{1,x+\alpha}(r, l) \frac{(-1)^l l!}{(l + 1)^k}
\]

\[
= \sum_{l=0}^{n} \sum_{j=0}^{n} \left( \sum_{r=0}^{n} \binom{r}{j} a_{n,r}(x + \alpha)^{r-j} \right) S(j, l) \frac{(-1)^l l!}{(l + 1)^k}
\]

\[
= \sum_{l=0}^{n} \sum_{j=0}^{n} \left( \sum_{r=0}^{n} \binom{r}{j} b_{n,r}(x + \beta)^{r-j} \right) S(j, l) \frac{(-1)^l l!}{(l + 1)^k}
\]

\[
= \sum_{r=0}^{n} b_{n,r} \sum_{l=0}^{r} \sum_{j=0}^{r} \binom{r}{j} (x + \beta)^{r-j} S(j, l) \frac{(-1)^l l!}{(l + 1)^k}
\]

\[
= \sum_{r=0}^{n} b_{n,r} S_{1,x+\beta}(r, l) \frac{(-1)^l l!}{(l + 1)^k}
\]

\[
= \sum_{r=0}^{n} b_{n,r} B_r^{(k)}(x + \beta),
\]

as desired. \(\square\)
Remark 3.2. The case \( k = 1 \) of Theorem 3.1 is an old result: based on [12], we obtained Theorem 3.1 in the case \( k = 1 \) and we used it to generate several identities in [15].

For example, by using Theorem 3.1 in the trivial identity \( x^p = \sum_{j=0}^{p} \binom{p}{j} (x - y)^{p-j} y^j \) we obtain the addition formula for poly-Bernoulli polynomials

\[
B_p^{(k)}(x) = \sum_{j=0}^{p} \binom{p}{j} (x - y)^{p-j} B_j^{(k)}(y),
\]

that we can write as

\[
\sum_{j=0}^{p} \binom{p}{j} x^{p-j} B_j^{(k)} = \sum_{j=0}^{p} \binom{p}{j} (x - y)^{p-j} B_j^{(k)}(y).
\] (40)

We can use again Theorem 3.1 to obtain from (41) that

\[
\sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)}(x) B_j^{(k)} = \sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k)}(x - y) B_j^{(k)}(y).
\] (42)

Set \( y = x \) in (42) to get the identity

\[
\sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)}(x) B_j^{(k)} = \sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k)} B_j^{(k)}(x).
\] (43)

If we set \( k = 1 \) in (43), and replace \( x \) by \( x + 1 \), we obtain

\[
\sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)}(x+1) B_j = \sum_{j=0}^{p} \binom{p}{j} B_{p-j}^{(k_1)} B_j(x + 1).
\] (44)

By using that \( B_j(x + 1) = B_j(x) + j x^{j-1} \) together with (43), we obtain from (44), after some elementary algebraic steps, the curious identity

\[
\sum_{j=0}^{p} \binom{p}{j} \left( B_{p-j}^{(k_1)}(x+1) - B_{p-j}^{(k_1)}(x) \right) B_j = p B_{p-1}^{(k_1)}(x).
\] (45)

From (17), (18) and (19) we have that

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_1+j_2}^{p_1+j_2} \binom{p_1}{j_1} \binom{p_2}{j_2} x_1^{p_1-j_1} x_2^{p_2-j_2} S(j_1 + j_2, l) \frac{(-1)^l l!}{(l + 1)^{k_0}} (x_1 - m)^{p_1-j_1} (x_2 - m)^{p_2-j_2}
\] (46)
\[ \times \sum_{i=0}^{m} \binom{m}{i} S(j_1 + j_2, l + i) \frac{(-1)^l (l + i)!}{(l + 1)^{k_0}} \]

\[ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \binom{p_1}{j_1} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) (x_1 - n)^{j_1} (x_2 - n)^{j_2} \]

\[ \times \sum_{i=0}^{n-1} (-1)^i s(n, n - i) S(j_1 + j_2 + n - i, l + n) \frac{(-1)^l!}{(l + 1)^{k_0}}. \]

where \( m, n \) are arbitrary integers, \( m \geq 0, n > 0 \). Now we use Theorem 3.1 in (46) and then set \( x_1 = x_2 = m + n \), to obtain the identities

\[ \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \binom{p_1}{j_1} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) B_{p_1-j_1}^{(k_1)} (m + n) B_{p_2-j_2}^{(k_2)} (m + n) S(j_1 + j_2, l) \frac{(-1)^l!}{(l + 1)^{k_0}} \]

\[ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \binom{p_1}{j_1} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) B_{p_1-j_1}^{(k_1)} (n) B_{p_2-j_2}^{(k_2)} (n) \]

\[ \times \sum_{i=0}^{m} \binom{m}{i} S(j_1 + j_2, l + i) \frac{(-1)^l (l + i)!}{(l + 1)^{k_0}} \]

\[ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \binom{p_1}{j_1} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) B_{p_1-j_1}^{(k_1)} (m) B_{p_2-j_2}^{(k_2)} (m) \]

\[ \times \sum_{i=0}^{n-1} (-1)^i s(n, n - i) S(j_1 + j_2 + n - i, l + n) \frac{(-1)^l!}{(l + 1)^{k_0}}. \]

From (26) we see that

\[ \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) (x_1 - y)^{j_1} (x_2 - y)^{j_2} B_{j_1+j_2}^{(k_0)} (y) \]

\[ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) (x_1 - z)^{j_1} (x_2 - z)^{j_2} B_{j_1+j_2}^{(k_0)} (z). \]

We can use Theorem 3.1 to get from (48), the identity

\[ \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) B_{p_1-j_1}^{(k_1)} (x_1 - y) B_{p_2-j_2}^{(k_2)} (x_2 - y) B_{j_1+j_2}^{(k_0)} (y) \]

\[ = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) B_{p_1-j_1}^{(k_1)} (x_1 - z) B_{p_2-j_2}^{(k_2)} (x_2 - z) B_{j_1+j_2}^{(k_0)} (z). \]
Set \( x_1 = x_2 = y = x \) to obtain from (49) that

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B^{(k_1)}_{p_1-j_1} B^{(k_2)}_{p_2-j_2} B^{(k_0)}_{j_1+j_2}(x) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B^{(k_1)}_{p_1-j_1}(x-z) B^{(k_2)}_{p_2-j_2}(x-z) B^{(k_0)}_{j_1+j_2}(z).
\]

With \( z = 0, z = 1 - (q - 1) x, \) and \( z = qx - 1, \) where \( q \) is an arbitrary parameter, we obtain from (50) the identities

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B^{(k_1)}_{p_1-j_1} B^{(k_2)}_{p_2-j_2} B^{(k_0)}_{j_1+j_2}(x) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B^{(k_1)}_{p_1-j_1}(x) B^{(k_2)}_{p_2-j_2}(x) B^{(k_0)}_{j_1+j_2}
\]

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B^{(k_1)}_{p_1-j_1}(q x - 1) B^{(k_2)}_{p_2-j_2}(q x - 1) B^{(k_0)}_{j_1+j_2}(1 - (q - 1) x)
\]

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B^{(k_1)}_{p_1-j_1}(1 - (q - 1) x) B^{(k_2)}_{p_2-j_2}(1 - (q - 1) x) B^{(k_0)}_{j_1+j_2}(q x - 1).
\]

In the case \( q = 2, \) if some (or all) of the parameters \( k_0, k_1, k_2 \) are equal to 1, we can use the known property \( B_\nu(1 - x) = (-1)^\nu B_\nu(x) \) to simplify the corresponding expression in (51). For example, if \( k_0 = k_1 = k_2 = 1 \) and \( q = 2, \) we have the following identities involving standard Bernoulli numbers and polynomials

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B_{p_1-j_1} B_{p_2-j_2} B_{j_1+j_2}(x) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B_{p_1-j_1}(x) B_{p_2-j_2}(x) B_{j_1+j_2}
\]

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{j_1+j_2} B_{p_1-j_1}(2 x - 1) B_{p_2-j_2}(2 x - 1) B_{j_1+j_2}(x)
\]

\[
= (-1)^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{j_1+j_2} B_{p_1-j_1}(x) B_{p_2-j_2}(x) B_{j_1+j_2}(2 x - 1).
\]

In particular, by setting \( x = 1 \) in (52), we see that if \( p_1 + p_2 \) is odd, then

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{j_1+j_2} B_{p_1-j_1} B_{p_2-j_2} B_{j_1+j_2} = 0.
\]
From (30) together with (9) (with $m = 0$) and (10) (with $n = 1$), we have

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \left( p_1 \right)_{j_1} \left( p_2 \right)_{j_2} x_1^{p_1-j_1} x_2^{p_2-j_2} S(j_1 + j_2, l)(-1)^l(l + r)!
\]

\[= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \left( p_1 \right)_{j_1} \left( p_2 \right)_{j_2} (x_1 - 1)^{p_1-j_1} (x_2 - 1)^{p_2-j_2} S(j_1 + j_2 + 1, l + 1)(-1)^l(l + r)!
\]

\[= r!(x_1 - r - 1)^{p_1} (x_2 - r - 1)^{p_2},
\]

and then, applying Theorem 3.1 in (54) we get

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \left( p_1 \right)_{j_1} \left( p_2 \right)_{j_2} B_{p_1-j_1}(x_1) B_{p_2-j_2}(x_2) S(j_1 + j_2, l)(-1)^l(l + r)!
\]

\[= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \left( p_1 \right)_{j_1} \left( p_2 \right)_{j_2} B_{p_1-j_1}(x_1 - 1) B_{p_2-j_2}(x_2 - 1) S(j_1 + j_2 + 1, l + 1)(-1)^l(l + r)!
\]

\[= r!B_{p_1}(x_1 - r - 1) B_{p_2}(x_2 - r - 1),
\]

where $r$ is an arbitrary non-negative integer.

Now let us consider the difference

\[
B_{p_1,p_2}(x_1 + r, x_2 + r) - B_{p_1,p_2}(x_1, x_2),
\]

where $r$ is an arbitrary positive integer. In the case $k_0 = 1$, we know that (56) is equal to (see [16])

\[
p_1 \sum_{t=0}^{r-1} (x_2 + t)^{p_2}(x_1 + t)^{p_1-1} + p_2 \sum_{t=0}^{r-1} (x_1 + t)^{p_1}(x_2 + t)^{p_2-1}.
\]

We can use (26) to write (56) as

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( p_1 \right)_{j_1} \left( p_2 \right)_{j_2} \left( (x_1 + r)^{p_1-j_1}(x_2 + r)^{p_2-j_2} B_{j_1+j_2}^{(k_0)} - x_1^{p_1-j_1} x_2^{p_2-j_2} B_{j_1+j_2}^{(k_0)} \right)
\]

\[= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( p_1 \right)_{j_1} \left( p_2 \right)_{j_2} \left( (x_1 + r - y)^{p_1-j_1}(x_2 + r - y)^{p_2-j_2} B_{j_1+j_2}^{(k_0)}(y)
\]

\[- (x_1 - z)^{p_1-j_1}(x_2 - z)^{p_2-j_2} B_{j_1+j_2}^{(k_0)}(z) \),
\]

where $y$ and $z$ are arbitrary parameters. If $k_0 = 1$, we have from (57) and (58) that

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( p_1 \right)_{j_1} \left( p_2 \right)_{j_2} \left( (x_1 + r)^{p_1-j_1}(x_2 + r)^{p_2-j_2} - x_1^{p_1-j_1} x_2^{p_2-j_2} B_{j_1+j_2} \right)
\]

\[
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( p_1 \right)_{j_1} \left( p_2 \right)_{j_2} \left( (x_1 + r)^{p_1-j_1}(x_2 + r)^{p_2-j_2} - x_1^{p_1-j_1} x_2^{p_2-j_2} \right) B_{j_1+j_2}
\]
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\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} ((x_1 + r - y)^{p_1-j_1}(x_2 + r - y)^{p_2-j_2} B_{j_1+j_2}(y) \\
- (x_1 - z)^{p_1-j_1}(x_2 - z)^{p_2-j_2} B_{j_1+j_2}(z))
\]

\[
= p_1 \sum_{t=0}^{r-1} (x_2 + t)^{p_2} (x_1 + t)^{p_1-1} + p_2 \sum_{t=0}^{r-1} (x_1 + t)^{p_1} (x_2 + t)^{p_2-1}.
\]

By using Theorem 3.1 in (59) we get

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left( B_{p_1-1-j_1}(x_1 + r) B_{p_2-j_2}(x_2 + r) - B_{p_1-1-j_1}(x_1) B_{p_2-j_2}(x_2) \right) B_{j_1+j_2} (60)
\]

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left( B_{p_1-1-j_1}(x_1 + r - y) B_{p_2-j_2}(x_2 + r - y) B_{j_1+j_2}(y) \\
- B_{p_1-1-j_1}(x_1 - z) B_{p_2-j_2}(x_2 - z) B_{j_1+j_2}(z) \right)
\]

\[
= p_1 \sum_{t=0}^{r-1} B_{p_1-1}^{(k_1)}(x_1 + t) B_{p_2}^{(k_2)}(x_2 + t) + p_2 \sum_{t=0}^{r-1} B_{p_1}^{(k_1)}(x_1 + t) B_{p_2-1}^{(k_2)}(x_2 + t).
\]

Set \((y, z) = (r, r), (r, 0), (0, -r), (2r, r)\) in (60), to obtain the following identities involving poly-Bernoulli polynomials, standard Bernoulli polynomials and Bernoulli numbers

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left( B_{p_1-1-j_1}(x_1 + r) B_{p_2-j_2}(x_2 + r) - B_{p_1-1-j_1}(x_1) B_{p_2-j_2}(x_2) \right) B_{j_1+j_2} (61)
\]

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left( B_{p_1-1-j_1}(x_1) B_{p_2-j_2}(x_2) - B_{p_1-1-j_1}(x_1 - r) B_{p_2-j_2}(x_2 - r) \right) B_{j_1+j_2}(r)
\]

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B_{p_1-1-j_1}(x_1) B_{p_2-j_2}(x_2) (B_{j_1+j_2}(r) - B_{j_1+j_2})
\]

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B_{p_1-1-j_1}(x_1 + r) B_{p_2-j_2}(x_2 + r) (B_{j_1+j_2} - B_{j_1+j_2}(-r))
\]

\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B_{p_1-1-j_1}(x_1 - r) B_{p_2-j_2}(x_2 - r) (B_{j_1+j_2}(2r) - B_{j_1+j_2}(r))
\]

\[
= p_1 \sum_{t=0}^{r-1} B_{p_1-1}^{(k_1)}(x_1 + t) B_{p_2}^{(k_2)}(x_2 + t) + p_2 \sum_{t=0}^{r-1} B_{p_1}^{(k_1)}(x_1 + t) B_{p_2-1}^{(k_2)}(x_2 + t).
\]

4 Generalized recurrences

In this section we show two generalized recurrences for bi-variate poly-Bernoulli polynomials, and obtain some consequences of them.
Proposition 4.1. We have

\[
\sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} B_{p_1+l,p_2}^{(k)}(x_1, x_2) = \sum_{l=0}^{q} \binom{q}{l} (-x_2)^{q-l} B_{p_1,p_2+l}^{(k)}(x_1, x_2) = - \sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1, l) (-1)^l l! R_{q-1,k}(l) \frac{(\prod_{i=1}^{q+1}(l+i))^k}{(l!)^k}.
\]

where \(R_{-1,k}(y) = -1\), and for \(\mu \geq 0\) the functions \(R_{\mu,k}(y)\) are defined recursively by

\[
R_{\mu,k}(y) = y(y + \mu + 2)^k R_{\mu-1,k}(y) - (y + 1)^{k+1} R_{\mu-1,k}(y + 1).
\]

Proof. We prove that

\[
\sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} B_{p_1+l,p_2}^{(k)}(x_1, x_2) = - \sum_{l=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1, k) (-1)^l l! R_{q-1,k}(l) \frac{(\prod_{i=1}^{q+1}(l+i))^k}{(l!)^k}
\]

by induction on \(q\). The case \(q = 0\) of (63) is the definition (16). Let us suppose formula (63) is true for a given \(q \in \mathbb{N}\). Then

\[
\sum_{l=0}^{q+1} \binom{q+1}{l} (-x_1)^{q+1-l} B_{p_1+l,p_2}^{(k)}(x_1, x_2)
\]

\[
= -x_1 \sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} B_{p_1+l,p_2}^{(k)}(x_1, x_2) + \sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} B_{p_1+1+l,p_2}^{(k)}(x_1, x_2)
\]

\[
= x_1 \sum_{l=0}^{p_1} S_{1,x_1}^{1,x_2,p_2}(p_1, l) (-1)^l l! R_{q-1,k}(l) \frac{(\prod_{i=1}^{q+1}(l+i))^k}{(l!)^k} - \sum_{l=0}^{p_1+1} S_{1,x_1}^{1,x_2,p_2}(p_1 + 1, l) (-1)^l l! R_{q-1,k}(l) \frac{(\prod_{i=1}^{q+1}(l+i))^k}{(l!)^k}.
\]

Now we use the recurrence (12) to write

\[
\sum_{l=0}^{q+1} \binom{q+1}{l} (-x_1)^{q+1-l} B_{p_1+l,p_2}^{(k)}(x_1, x_2)
\]

\[
= x_1 \sum_{l=0}^{p_1} S_{1,x_1}^{1,x_2,p_2}(p_1, l) (-1)^l l! R_{q-1,k}(l) \frac{(\prod_{i=1}^{q+1}(l+i))^k}{(l!)^k}
\]

\[
- \sum_{l=0}^{p_1+1} S_{1,x_1}^{1,x_2,p_2}(p_1 + 1, l - 1 + (l + x_1) S_{1,x_1}^{1,x_2,p_2}(p_1, l)) (-1)^l l! R_{q-1,k}(l) \frac{(\prod_{i=1}^{q+1}(l+i))^k}{(l!)^k}.
\]

Some further simplifications give us

\[
\sum_{l=0}^{q+1} \binom{q+1}{l} (-x_1)^{q+1-l} B_{p_1+l,p_2}^{(k)}(x_1, x_2)
\]
as desired. The proof of

\[
\sum_{l=0}^{q} \binom{q}{l} (-x_2)^{q-l} B_{p_1+p_2+l}^{(k)}(x_1, x_2) = - \sum_{l=0}^{p_1+p_2} S_{1, x_1}^{1, x_2, p_2}(p_1, l) \frac{(-1)^l l! \mathcal{R}_{q-1}(l)}{(\prod_{i=1}^{q+1} (l + i))^{k}},
\]

is similar. \hfill \Box

For example, we have

\[
\mathcal{R}_{0,k}(y) = (y + 1)^{k+1} - y(y + 2)^k,
\]
\[
\mathcal{R}_{1,k}(y) = (2y + 1)(y + 1)^{k+1} + (y + 3)^k - y^2(y + 2)^k(y + 3)^k - (y + 1)^{k+1}(y + 2)^{k+1},
\]

and then, formula (62) with \( q = 1 \) is

\[
-x_1 B_{p_1+p_2}^{(k)}(x_1, x_2) + B_{p_1+1, p_2}^{(k)}(x_1, x_2) = -x_2 B_{p_1, p_2}^{(k)}(x_1, x_2) + B_{p_1, p_2+1}^{(k)}(x_1, x_2)
\]

\[
= - \sum_{l=0}^{p_1+p_2} S_{1, x_1}^{1, x_2, p_2}(p_1, l) (-1)^l l! \left( \frac{l + 1}{(l + 2)^k} - \frac{l}{(l + 1)^k} \right),
\]

and with \( q = 2 \) is

\[
x_1^2 B_{p_1+p_2}^{(k)}(x_1, x_2) - 2x_1 B_{p_1+1, p_2}^{(k)}(x_1, x_2) + B_{p_1+2, p_2}^{(k)}(x_1, x_2)
\]

\[
= x_2^2 B_{p_1+p_2}^{(k)}(x_1, x_2) - 2x_2 B_{p_1, p_2+1}^{(k)}(x_1, x_2) + B_{p_1, p_2+2}^{(k)}(x_1, x_2)
\]

\[
= - \sum_{l=0}^{p_1+p_2} S_{1, x_1}^{1, x_2, p_2}(p_1, l) (-1)^l l! \left( \frac{2l + 1}{(l + 1)^k} - \frac{l^2}{(l + 2)^k} - \frac{l + 1}{(l + 3)^k} \right).
\]

We can write (62) by using (26) as

\[
\sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} \sum_{j_1=0}^{p_1+l} \sum_{j_2=0}^{p_2} \left( \binom{p_1 + l - j_1}{j_1} \binom{p_2}{j_2} (x_1 - y)^{p_1+l-j_1} (x_2 - y)^{p_2-j_2} B_{j_1+j_2}^{(k)}(y) \right)
\]
By setting \( y, z = x_1, x_2 \) in (67), we get the identities

\[
\sum_{l=0}^{q} \begin{pmatrix} q \end{pmatrix} (-x_1)^{q-l} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \frac{p_1 + l}{j_1} \left( x_1 - x_2 \right)^{p_1+l-j_1} B_{j_1+p_2}^{(k)} (x_1),
\]

If we set \( x_2 = 0 \) in (68), we get

\[
\sum_{l=0}^{q} \begin{pmatrix} q \end{pmatrix} (-x_1)^{q-l} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \frac{p_1 + l}{j_1} \left( x_1 \right)^{p_2-j_2} B_{j_1+p_2}^{(k)} (x_1).
\]

The case \( q = 0, x_1 = 1, k = 1 \) of (69) reduces to

\[
(-1)^{p_1+p_2} \sum_{j_2=0}^{p_2} \left( \begin{pmatrix} p_2 \end{pmatrix} B_{p_1+j_2} \right) = \sum_{j_1=0}^{p_1} \frac{p_1}{j_1} B_{j_1+p_2}.
\]
Formula (70) is the famous Carlitz identity [4]. In terms of bi-variate Bernoulli polynomials, Carlitz identity is written as

\[ B_{p_1,p_2}(1, 0) = (-1)^{p_1+p_2} B_{p_1,p_2}(0, 1). \]  

(71)

For example, we can use (71) to write the following version of (62) in the case \( k = 1 \), when \( x_1 = 0, x_2 = 1 \)

\[
\sum_{l=0}^{q} \left( \begin{array}{c} q \\ l \end{array} \right) (-1)^{q-l} B_{p_1+l,p_2}(1, 0) = (-1)^{p_1+p_2+q} \sum_{l=0}^{q} \left( \begin{array}{c} q \\ l \end{array} \right) B_{p_1+l,p_2}(0, 1)
\]

\[
= B_{p_1,p_2+q}(1, 0) = (-1)^{p_1+p_2+q} B_{p_1,p_2+q}(0, 1)
\]

\[
= - \sum_{l=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} S_{1,1}^{l,0,p_2} (p_1, l) \frac{(-1)^l R_{q-1,1}(l)}{\prod_{i=1}^{q+1}(l+i)}
\]

or, explicitly

\[
\sum_{l=0}^{q} \left( \begin{array}{c} q \\ l \end{array} \right) (-1)^{q-l} \sum_{j_1=0}^{p_1+l} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) B_{j_1+p_2}
\]

\[
= (-1)^{p_1+p_2+q} \sum_{l=0}^{q} \left( \begin{array}{c} q \\ l \end{array} \right) \sum_{j_2=0}^{p_2} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) B_{p_1+l+j_2}
\]

\[
= \sum_{j_1=0}^{p_1} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) B_{j_1+p_2+q} = (-1)^{p_1+p_2+q} \sum_{j_2=0}^{p_2+q} \left( \begin{array}{c} p_2+q \\ j_2 \end{array} \right) B_{p_1+j_2}
\]

\[
= - \sum_{l=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) S(j_1 + p_2, l) \frac{(-1)^l R_{q-1,1}(l)}{\prod_{i=1}^{q+1}(l+i)}
\]

It is easy to check that in the case \( k = 0 \), the functions \( R_{\mu,k}(y) \) of Proposition 4.1 are \( R_{\mu,0}(y) = (-1)^\mu \). Thus, the case \( k = 0 \) of (62) is the case \( r = 0 \) of (30). Also, in the case \( k = -1 \), we have that \( R_{\mu,-1}(y) = \frac{(-1)^\mu e^{2\mu+1} (y+1)}{\prod_{i=2}^{\mu+2}(y+i)} \), and then the case \( k = -1 \) of (62) is the case \( r = 1 \) of (30).

**Proposition 4.2.** For non-negative integers \( p_1, p_2, q \) we have

\[
\sum_{l=0}^{q} (-1)^l B^{(k)}_{p_1+l,p_2}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_1} \prod_{i=0}^{q-1}(x_1 + i)
\]

\[
= \sum_{l=0}^{q} (-1)^l B^{(k)}_{p_1,p_2+l}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_2} \prod_{i=0}^{q-1}(x_2 + i)
\]

(72)
true for a given $q$ by induction on $q$. The case $q = 0$ of (73) is the definition (16). If we suppose that (73) is true for a given $q \in \mathbb{N}$, then

\[
\begin{align*}
\sum_{l=0}^{q+1} (-1)^l B_{p_1+l,p_2}^{(k)}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i) &= \sum_{l=0}^{q+1} (-1)^l B_{p_1+l,p_2}^{(k)}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_1^l} \left( \prod_{i=0}^{q-1}(x_1 + i) \right) \\
&= \sum_{l=0}^{q+1} (-1)^l B_{p_1+l,p_2}^{(k)}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_1^l} \left( \prod_{i=0}^{q-1}(x_1 + i) \right) + \sum_{l=1}^{q+1} (-1)^l B_{p_1+l,p_2}^{(k)}(x_1, x_2) \frac{1}{(l-1)!} \frac{d^{l-1}}{dx_1^{l-1}} \prod_{i=0}^{q-1}(x_1 + i) \\
&= (x_1 + q) \sum_{l=0}^{q} (-1)^l B_{p_1+l,p_2}^{(k)}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i) \\
&\quad - \sum_{l=0}^{q} (-1)^l B_{p_1+l,p_2}^{(k)}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i) \\
&= (x_1 + q) \sum_{l=0}^{q} (-1)^l B_{p_1+l,p_2}^{(k)}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i) \\
&\quad - \sum_{l=0}^{q} (-1)^l B_{p_1+l,p_2}^{(k)}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i) \\
&= (x_1 + q) \sum_{l=0}^{q} \left( (-1)^l \frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i) \right) \\
&\quad - \sum_{l=0}^{q} \left( (-1)^l \frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i) \right) \\
&= \sum_{l=0}^{q} \left( (-1)^l \frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i) \right).
\end{align*}
\]

Now we use the recurrence (12) and formula (11) to write

\[
\begin{align*}
\frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i) &= \sum_{l=0}^{q+1} (-1)^l B_{p_1+l,p_2}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_1^l} \prod_{i=0}^{q}(x_1 + i).
\end{align*}
\]
On bi-variate poly-Bernoulli polynomials

\[(x_1 + q) \sum_{l=0}^{p_1+p_2} S^{1,x_2+q,p_2}_{1,x_1+q}(p_1, l) \frac{(-1)^l(l + q)!}{(l + q + 1)^k} - \sum_{l=0}^{p_1+p_2+1} (S^{1,x_2+q,p_2}_{1,x_1+q}(p_1, l - 1) + (l + x_1 + q)S^{1,x_2+q,p_2}_{1,x_1+q}(p_1, l)) \frac{(-1)^l(l + q)!}{(l + q + 1)^k}\]

\[= \sum_{l=0}^{p_1+p_2} S^{1,x_2+q+1,p_2}_{1,x_1+q+1}(p_1, l) \frac{(-1)^l(l + q + 1)!}{(l + q + 2)^k},\]

as desired. The proof of

\[\sum_{l=0}^{q} (-1)^l B^{(k)}_{p_1,p_2+l}(x_1, x_2) \frac{1}{l!} \frac{d^l}{dx_2^l} \prod_{i=0}^{q-1} (x_2 + i) = \sum_{l=0}^{p_1+p_2} S^{1,x_2+q,p_2}_{1,x_1+q}(p_1, l) \frac{(-1)^l(l + q)!}{(l + q + 1)^k},\]

is similar. \(\square\)

Formula (72) with \(x_1 = 0, x_2 = 1\) looks as

\[\sum_{l=0}^{q} (-1)^l s(q, l) B^{(k)}_{p_1+l,p_2}(0, 1) = \sum_{l=0}^{q} (-1)^l s(q + 1, l + 1) B^{(k)}_{p_1,p_2+l}(0, 1) \quad (74)\]

or, explicitly

\[\sum_{l=0}^{q} (-1)^l s(q, l) \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} B^{(k)}_{p_1+l+j_2} = \sum_{l=0}^{q} (-1)^l s(q + 1, l + 1) \sum_{j_2=0}^{p_2+l} \binom{p_2 + l}{j_2} B^{(k)}_{p_1,j_2} \quad (75)\]

\[= \sum_{l=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} q^{p_1-j_1} (q + 1)^{p_2-j_2} S(j_1 + j_2, l) \frac{(-1)^l(l + q)!}{(l + q + 1)^k}.\]

If we set \(k = 1\) in formula (74), we can use Carlitz identity (71) to write the following enriched version of (74)

\[\sum_{l=0}^{q} (-1)^l s(q, l) B_{p_1+l,p_2}(0, 1) = (-1)^{p_1+p_2} \sum_{l=0}^{q} s(q, l) B_{p_1+l,p_2}(1, 0) \quad (76)\]
or, explicitly
\[
\sum_{l=0}^{q} (-1)^l s(q, l) \sum_{j_2=0}^{p_2} \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) B_{p_1+l+j_2}^{(1)} (77)
\]
\[
= (-1)^{p_1+p_2} \sum_{l=0}^{q} s(q, l) \sum_{j_1=0}^{p_1+l} \left( \begin{array}{c} p_1 + l \\ j_1 \end{array} \right) B_{j_1+p_2}
\]
\[
= \sum_{l=0}^{q} (-1)^l s(q + 1, l + 1) \sum_{j_2=0}^{p_2+l} \left( \begin{array}{c} p_2 + l \\ j_2 \end{array} \right) B_{p_1+j_2}
\]
\[
= (-1)^{p_1+p_2} \sum_{l=0}^{q} s(q + 1, l + 1) \sum_{j_1=0}^{p_1} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) B_{j_1+p_2+l}
\]
\[
= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{l=0}^{j_1+j_2} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) q^{p_1-j_1} (q + 1)^{p_2-j_2} S(j_1 + j_2, l) \frac{(-1)^l (l + q)!}{l + q + 1}.
\]

To end this section, let us consider the case \( q = 1 \) of the first two lines of (72). That is
\[
B_{p_1,p_2+1}^{(k)}(x_1, x_2) = B_{p_1+1,p_2}^{(k)}(x_1, x_2) = (x_2-x_1)B_{p_1,p_2}^{(k)}(x_1, x_2).
\]
Formula (78) is the first step of two results contained in the following proposition.

**Proposition 4.3.** We have the following identities:

a) \[
\sum_{j=0}^{q} \left( \begin{array}{c} q \\ j \end{array} \right) (-1)^j B_{p_1+j,p_2+q-j}^{(k)}(x_1, x_2) = (x_2-x_1)^q B_{p_1,p_2}^{(k)}(x_1, x_2).
\]  

b) \[
\sum_{j=0}^{q} \left( \begin{array}{c} q \\ j \end{array} \right) (x_2-x_1)^j B_{p_1+q-j,p_2}^{(k)}(x_1, x_2) = B_{p_1,p_2+q}(x_1, x_2).
\]

**Proof.** Let us prove (79) by induction on \( q \). The case \( q = 0 \) is a trivial identity. Let us suppose that (79) is true for a given \( q \in \mathbb{N} \). Then
\[
\sum_{j=0}^{q+1} \left( \begin{array}{c} q + 1 \\ j \end{array} \right) (-1)^j B_{p_1+j,p_2+q+1-j}^{(k)}(x_1, x_2)
\]
identity. Let us suppose that (80) is true for a given \(q\) as desired. In the last step we used (78).

Now let us prove (80). Again we proceed by induction on \(p\). The case \(p = 1\) of (81) is the addition formula for standard poly-Bernoulli polynomials, namely

\[
\sum_{j=0}^{q+1} \binom{q+1}{j} (x_2 - x_1)^j B_{p_1+q+1-j,p_2}^{(k)}(x_1, x_2) = B_{p_1+q+1,p_2}^{(k)}(x_1, x_2) = B_{p_1+1,p_2+q}(x_1, x_2)
\]

as desired. We used (78) (with \(p_2\) replaced by \(p_2 + q\)) in the last step.

The case \(p_1 = 0\) of (80) is

\[
\sum_{j=0}^{q} \binom{q}{j} (x_2 - x_1)^j B_{q-j,p_2}^{(k)}(x_1, x_2) = B_{p_2+q}^{(k)}(x_2).
\]

The case \(p_2 = 0\) of (81) is the addition formula for standard poly-Bernoulli polynomials, namely

\[
\sum_{j=0}^{q} \binom{q}{j} (x_2 - x_1)^{q-j} B_j^{(k)}(x_1) = B_{q}^{(k)}(x_2).
\]

The case \(p_1 = p_2 = 0\) of (79) is

\[
\sum_{j=0}^{q} \binom{q}{j} (-1)^j B_{j,q-j}^{(k)}(x_1, x_2) = (x_2 - x_1)^q.
\]
As a final comment, we mention that by considering the GSN $S_{1,x_1}^{(1,x_2,p_2),\ldots,(1,x_n,p_n)}(p_1,k)$ involved in the expansion

$$(m + x_1)^{p_1} \cdots (m + x_n)^{p_n} = \sum_{l=0}^{p_1 + \cdots + p_n} l! S_{1,x_1}^{(1,x_2,p_2),\ldots,(1,x_n,p_n)}(p_1,l) \binom{m}{l}, \quad (83)$$

where $p_1, p_2, \ldots, p_n$ are non-negative integers given, one can define poly-Bernoulli polynomials in $n$ variables $x_1, \ldots, x_n$, denoted as $B_{p_1,\ldots,p_n}^{(k)}(x_1, \ldots, x_n)$, as

$$B_{p_1,\ldots,p_n}^{(k)}(x_1, \ldots, x_n) = \sum_{l=0}^{p_1 + \cdots + p_n} S_{1,x_1}^{(1,x_2,p_2),\ldots,(1,x_n,p_n)}(p_1,l) \frac{(-1)^l!}{(l+1)^{k}},$$

or explicitly as

$$B_{p_1,\ldots,p_n}^{(k)}(x_1, \ldots, x_n) = \sum_{l=0}^{p_1 + \cdots + p_n} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \cdots \sum_{j_n=0}^{p_n} \binom{p_1}{j_1} \cdots \binom{p_n}{j_n} \frac{(-1)^l!}{(l+1)^{k}} x_1^{p_1-j_1} \cdots x_n^{p_n-j_n} B_{j_1+\cdots+j_n, k}^{(k)}.$$

In this more general setting we have natural generalizations of results (62), (72), (79) and (80). We show the corresponding results in the case of poly-Bernoulli polynomials in 3 variables:

$$B_{p_1,p_2,p_3}^{(k)}(x_1, x_2, x_3) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} \binom{p_1}{j_1} \binom{p_2}{j_2} \binom{p_3}{j_3} x_1^{p_1-j_1} x_2^{p_2-j_2} x_3^{p_3-j_3} B_{j_1+j_2+j_3, k}^{(k)},$$

(a) (See (62)) We have the generalized recurrences

$$\sum_{l=0}^{q} \binom{q}{l} (-x_1)^{q-l} B_{p_1+l,p_2,p_3}^{(k)}(x_1, x_2, x_3)$$

$$= \sum_{l=0}^{q} \binom{q}{l} (-x_2)^{q-l} B_{p_1,p_2+l,p_3}^{(k)}(x_1, x_2, x_3)$$

$$= \sum_{l=0}^{q} \binom{q}{l} (-x_3)^{q-l} B_{p_1,p_2,p_3+l}^{(k)}(x_1, x_2, x_3)$$

$$= - \sum_{l=0}^{p_1+p_2+p_3} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} \binom{p_1}{j_1} \binom{p_2}{j_2} \binom{p_3}{j_3} x_1^{p_1-j_1} x_2^{p_2-j_2} x_3^{p_3-j_3} \times S(j_1+j_2+j_3, l) \frac{(-1)^l l! R_{q-1, k}(l)}{(\prod_{i=1}^{q+1} (l+i))^{k}},$$

where $R_{q-1, k}(l)$ is defined in Proposition 4.1.
(b) (See (72)) We have the generalized recurrences
\[ \sum_{l=0}^{q} (-1)^l B_{p_1 + l, p_2, p_3}^{(k)}(x_1, x_2, x_3) \frac{d^l}{dx_1} \prod_{i=0}^{q-1} (x_1 + i) \]
\[ = \sum_{l=0}^{q} (-1)^l B_{p_1, p_2 + l, p_3}^{(k)}(x_1, x_2, x_3) \frac{d^l}{dx_2} \prod_{i=0}^{q-1} (x_2 + i) \]
\[ = \sum_{l=0}^{q} (-1)^l B_{p_1, p_2, p_3 + l}^{(k)}(x_1, x_2, x_3) \frac{d^l}{dx_3} \prod_{i=0}^{q-1} (x_3 + i) \]
\[ = \sum_{j_1=0}^{p_1-1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} \left( \begin{array}{c} p_1 \\ j_1 \end{array} \right) \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) \left( \begin{array}{c} p_3 \\ j_3 \end{array} \right) (x_1 + q)^{p_1-j_1} (x_2 + q)^{p_2-j_2} (x_3 + q)^{p_3-j_3} \]
\[ \times S(j_1 + j_2 + j_3, l) \frac{(-1)^l(l + q)!}{(l + q + 1)^k} . \]

(c) (See (79)) We have the identities
\[ \sum_{j=0}^{q} \left( \begin{array}{c} q \\ j \end{array} \right) (-1)^j B_{p_1, p_2 + j, p_3 + q - j}^{(k)}(x_1, x_2, x_3) = (x_3 - x_2)^q B_{p_1, p_2, p_3}^{(k)}(x_1, x_2, x_3), \]
\[ \sum_{j=0}^{q} \left( \begin{array}{c} q \\ j \end{array} \right) (-1)^j B_{p_1 + j, p_2, p_3 + q - j}^{(k)}(x_1, x_2, x_3) = (x_3 - x_1)^q B_{p_1, p_2, p_3}^{(k)}(x_1, x_2, x_3), \]
\[ \sum_{j=0}^{q} \left( \begin{array}{c} q \\ j \end{array} \right) (-1)^j B_{p_1 + j, p_2, p_3 + q - j}^{(k)}(x_1, x_2, x_3) = (x_2 - x_1)^q B_{p_1, p_2, p_3}^{(k)}(x_1, x_2, x_3). \]

(d) (See (80)) We have the identities
\[ \sum_{j=0}^{q} \left( \begin{array}{c} q \\ j \end{array} \right) (x_1 - x_2)^j B_{p_1, p_2 + q - j, p_3}^{(k)}(x_1, x_2, x_3) \]
\[ = \sum_{j=0}^{q} \left( \begin{array}{c} q \\ j \end{array} \right) (x_1 - x_3)^j B_{p_1, p_2, p_3 + q - j}^{(k)}(x_1, x_2, x_3) = B_{p_1, p_2, p_3}^{(k)}(x_1, x_2, x_3). \]
\[ \sum_{j=0}^{q} \left( \begin{array}{c} q \\ j \end{array} \right) (x_2 - x_1)^j B_{p_1 + q - j, p_2, p_3}^{(k)}(x_1, x_2, x_3) \]
\[ = \sum_{j=0}^{q} \left( \begin{array}{c} q \\ j \end{array} \right) (x_2 - x_3)^j B_{p_1, p_2, p_3 + q - j}^{(k)}(x_1, x_2, x_3) = B_{p_1, p_2, p_3}^{(k)}(x_1, x_2, x_3). \]
\[\sum_{j=0}^{q} \binom{q}{j} (x_3 - x_1)^j B_{p_1+q-j,p_2,p_3}^{(k)}(x_1,x_2,x_3) = \sum_{j=0}^{q} \binom{q}{j} (x_3 - x_2)^j B_{p_1,p_2+q-j,p_3}^{(k)}(x_1,x_2,x_3) = B_{p_1,p_2,p_3+q}(x_1,x_2,x_3).\]

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