Abstract—Information compression techniques are majorly employed to address the concern of reducing communication cost over peer-to-peer links. In this paper, we investigate distributed Nash equilibrium (NE) seeking problems in a class of noncooperative games over directed graphs with information compression. To improve communication efficiency, a compressed distributed NE seeking (C-DNES) algorithm is proposed to obtain a NE for games, where the differences between decision vectors and their estimates are compressed. The proposed algorithm is compatible with a general class of compression operators, including both unbiased and biased compressors. Moreover, our approach only requires the adjacency matrix of the directed graph to be row-stochastic, in contrast to past works that relied on balancedness or specific global network parameters. It is shown that C-DNES not only inherits the advantages of conventional distributed NE algorithms, achieving linear convergence rate for games with restricted strongly monotone mappings, but also saves communication costs in terms of transmitted bits. Finally, numerical simulations illustrate the advantages of C-DNES in saving communication cost by an order of magnitude under different compressors.

Index Terms—Nash equilibrium seeking, information compression, distributed networks, noncooperative games.

I. INTRODUCTION

Game theory has been studied extensively on account of its significant role in analyzing the interactions among rational decision-makers. Engineering applications of game-theoretic methods include congestion control in traffic networks [1], charging or discharging of electric vehicles [2] and demand-side management in smart grid [3]. As an important issue, Nash equilibrium (NE) seeking has attracted ever-increasing attention with the emergence of multi-agent networks. Conceptually, NE is a proposed solution in multiplayer noncooperative games, where a number of selfish players aim to minimize their own cost functions by making decisions according to others’ actions. A large body of work on NE seeking has been reported in the recent literature (see e.g. [4]–[6] and references therein). Most of them assumed that each player can access the actions of all other players. This global knowledge assumption requires a central coordinator to broadcast the information to the network, which is sometimes impractical [7], [8]. Hence, distributed NE seeking algorithms have become a subject of great interest recently, where a distributed information sharing protocol is adopted to exchange local messages among players. For example, Ye et al. [9] designed a fixed-step gradient algorithm to seek a NE over directed graphs with the knowledge of Perron-Frobenius (PF) eigenvector of the adjacency matrix.

All the approaches mentioned above assume unlimited communication bandwidth. However, in practical systems such as underwater vehicles and low-cost unmanned aerial vehicles, the communication capacity and bandwidth are often limited by the environments. As the dimension of data increases, the burden of information exchange between players will cause communication bottleneck, which deteriorates the efficiency of the algorithm. Thus, in order to fulfill the requirement of limited bandwidth, a compressor is necessary for each agent to send compressed information that is encoded with fewer number of bits. Before transmitting local messages, each agent firstly compresses the information then transmits the compressed one to their neighbors, as shown Fig. 1. Recently, various information compression methods, such as quantization and sparsification, have been adopted in distributed optimization in centralized networks [15]–[20]. For decentralized optimization, with certain compression error feedback techniques, some works achieved linear convergence rate for optimization algorithms with compressed information [21]–[24].

Though the distributed NE seeking problem may be viewed as an extension of distributed optimization problems, the existing information compression methods cannot be directly applied to distributed NE seeking problems due to the fact that the objective function of each player in noncooperative games relies on the actions of all players. Owing to the more complex information exchange among players, such as estimation of joint action profile, more difficulties are faced when designing the adaptive compression approach. Thus, few works studied distributed NE seeking problem with compressed communication. Nekouei et al. [25] investigated the impact of quantized
communication on the behavior of NE seeking algorithms over fully connected communication graphs, which is usually an impractical assumption in distributed networks. A continuous-time distributed NE seeking algorithm with finite communication bandwidth is proposed by Chen et al. [26], where a specific quantizer is used. Recently, logarithmic and uniform quantizations are adopted by Ye et al. [27] to develop a quantized NE seeking strategy in continuous-time systems. However, the proposed method in [27] only converge to a neighborhood of NE point.

All the above motivates us to further develop a discrete-time distributed NE seeking algorithm with information compressed by general compressors before transmission. To the best of our knowledge, this paper is the first to propose a communication-efficient discrete-time distributed algorithm over directed graphs, which can converge linearly to an NE for noncooperative games under a general class of compressors. The main contributions of this paper are summarized as follows:

1) In designing a compressed distributed NE seeking algorithm (C-DNES), we consider a general class of compressors instead of specific compression schemes in [26], [27]. The assumption in this paper encompasses both biased and unbiased compressors, as well as norm-sign compressors and the composition of quantization and sparsification.

2) In contrast to earlier studies [11]–[14], [26] that relied on balancedness or awareness of specific global network parameters (PF eigenvector), our method only necessitates the row-stochasticity assumption for the adjacency matrix of the communication graph. This mild assumption in our proposed C-DNES framework allows for the accommodation of both undirected and directed topology graphs, which is advantageous for practical implementation.

3) We prove that there exists a unique NE for games with restricted strongly monotone mappings (unlike the strongly monotone mappings in [26], [27]) and C-DNES is guaranteed to linearly converge to the unique NE. This relatively moderate assumption of game mapping expands applicability of C-DNES to a more extensive range of games.

4) Through simulation examples, we illustrate that the proposed C-DNES algorithm is applicable to various compressors and has convergence performance comparable to those of state-of-the-art algorithms with accurate communication. Meanwhile, we show that C-DNES can decrease the transmitted bits by an order of magnitude.

Notations: In this paper, \( 1 \in \mathbb{R}^n \) represents the column vector with each entry given by 1. We denote \( e_i \in \mathbb{R}^n \) as the \( i \)-th unit vector which takes zero except for the \( i \)-th entry that equals to 1. \( \mathbb{R}_{++} \) denotes the set of all positive real numbers. The spectral radius of matrix \( A \) is denoted by \( \rho(A) \). The smallest nonzero eigenvalue of a positive semidefinite matrix \( M \succeq 0 \) is denoted as \( \lambda_{\min}(M) \). Given a vector \( x \), we denote its \( i \)-th element by \( [x]_i \). Given a matrix \( A \), we denote its \( i \)-th row and \( j \)-th column by \( [A]_{ij} \). Let \( \text{sgn}() \) and \( | \cdot | \) be the element-wise sign function and absolute value function, respectively. We denote \( ||x||_2 \) and \( ||X||_F \) respectively as \( l_2 \) norm function of vector \( x \) and the Frobenius norm of matrix \( X \). We denote by \( A \odot B \), the Hadamard product of two matrices, \( A \) and \( B \).

In addition, \( x \preceq y \) is denoted as component-wise inequality between vectors \( x \) and \( y \). For a matrix \( A \in \mathbb{R}^{n \times n} \), denote diag(\( A \)) as its diagonal vector, i.e., \( \text{diag}(A) = \left[ [A]_{11}, \ldots, [A]_{nn} \right] ^T \). For a vector \( a \in \mathbb{R}^n \), denote \( \text{Diag}(a) \) as the diagonal matrix with the vector \( a \) on its diagonal. A matrix \( A \in \mathbb{R}^{n \times n} \) is consensual if it has equal row vectors.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Network Model

We consider a group of agents communicating with each other over a directed graph \( G \equiv (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{1, 2, 3, \ldots, n\} \) denotes the agent set and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) denotes the edge set, respectively. A communication link from agent \( j \) to agent \( i \) is denoted by \( (i, j) \in \mathcal{E} \), indicating that agent \( j \) can send messages to agent \( i \). The set of neighbors of agent \( i \) is denoted as \( N_i = \{ j \in \mathcal{V} | (i, j) \in \mathcal{E} \} \). The adjacency matrix of the graph is denoted as \( W = [w_{ij}]_{n \times n} \) with \( w_{ij} > 0 \) if \( (i, j) \in \mathcal{E} \) or \( i = j \), and \( w_{ij} = 0 \) otherwise. The graph \( G \) is called strongly connected if there exists at least one directed path from any agent \( i \) to any agent \( j \) in the directed graph with \( i \neq j \).

Assumption 1: The directed graph \( G \equiv (\mathcal{V}, \mathcal{E}) \) is strongly connected. Moreover, the adjacency matrix \( W \) associated with \( G \) is row-stochastic, i.e., \( \sum_{i=1}^{n} w_{ij} = 1, \forall i \in \mathcal{V} \).

B. Compressors

A stochastic compressor \( C(x, \xi) : \mathbb{R}^d \times \mathcal{Z} \to \mathbb{R}^d \) is a mapping that convert a \( \mathbb{R}^d \)-valued vector \( x \) to a compressed one, where \( \xi \) is a random variable with range \( \mathcal{Z} \). Note that the realizations \( \xi \) of the compressor \( C(\cdot) \) are independent among different agents and time steps. In other words, given an sequence \( \{\xi^k_i\} \) for each agent \( i \in \mathcal{V} \), and any time iteration \( k \geq 0 \), the randomness \( \xi^k_i \) with each usage of compressor \( C(\xi^k_i, \xi^k_j) \) are i.i.d., where \( C(\cdot, \cdot) \) denotes the compressor used by agent \( i \) at iteration \( k \). For deterministic compressors, which can be treated as special cases of the random ones, the random variable \( \xi \) is not included, thus the notation reduces to \( C(x) \). For stochastic compressors, the notation \( \mathbb{E}_\xi \) is used to denote the expectation over the inherent randomness in the stochastic compressor \([28]–[31]\). For deterministic compressors, the expected value operator will return the deterministic argument. Hereafter, we drop \( \xi \) and write \( C(x) \) for notational simplicity.

Next, we introduce a general assumption on the compression operators which is considered in our paper.

Assumption 2: For any agent \( i \in \mathcal{V} \) and any iteration \( k \geq 0 \), the compression operator \( C^k_i \) satisfies
\[
\mathbb{E}_\xi(||C^k_i(x) - x||^2) \leq C||x||^2, \forall x \in \mathbb{R}^d, \tag{1}
\]
where \( C \geq 0 \) and the \( r \)-scaling of \( C^k_i \) satisfies
\[
\mathbb{E}_\xi(||C^k_i(x)/r - x||^2) \leq (1 - \delta)||x||^2, \forall x \in \mathbb{R}^d, \tag{2}
\]
where \( \delta \in (0, 1] \) and \( r > 0 \).

Remark 1: Assumption 2 requires the mean square of the relative compression error to be bounded. Note that if \( C \) is unbiased, i.e., \( \mathbb{E}_\xi(C(x,x)) = x \), Assumption 2 degenerates to the condition of unbiased compressors in [16], [19], [20], [23], [32]. Moreover, if \( 0 \leq C < 1 \), compressors satisfying Assumption 2 is equivalent to the class of biased but contractive compressors which are widely adopted in practice [33]–[35]. In a word, Assumption 2 is a general assumption of unbiased and biased compressors.

Some commonly used compressors satisfying Assumption 2 are given below.

1) Unbiased compressor: Stochastic quantization:

The \( b \)-bit \( q \)-norm quantization compressor is defined as follows:
\[
C_q(x) = \frac{||x||_q}{2^{b-1}} \text{sgn}(x) \odot \left[ 2^{b-1} \frac{||x||_q}{||x||_q} + u \right], \tag{3}
\]
where \( u \) is a random vector uniformly sampled from \([0, 1]^d\), which is independent of \( x \). For \( q = \infty \), the compressor satisfies Assumption 2 with \( C = d/4^b \). Owing to the fact that only the norm \( ||x||_q \), \( \text{sgn}(x) \) and integers in the bracket need to be transmitted, this compressor...
is widely adopted in compressed distributed learning [19], [20], [32] with $l_2$-norm quantization, while $l_2$ norm quantization is adopted in [16], [22], [33].

2) Biased but contractive compressor: Top-$k$ sparsification:

The largest $k$ coordinates in magnitude is selected among the vector $x$, i.e.,

$$C_{top}(x) = x \odot e,$$

(4)

where the elements of $x$ is reordered as $[x]_1 \geq [x]_2 \geq \cdots [x]_d$ and $e$ satisfies $[e]_l = 1$ for $l \leq k$ and $[e]_l = 0$ for $l > k$. This compressor meets Assumption 2 with $C = 1 - k/d$.

3) General compressor: Norm-sign compressor:

$$C(x) = ||x||_q \text{sgn}(x), \quad q \geq 2.$$  

(5)

**Claim 1:** For the compressor defined in (5), we have the following the compression constants satisfying Assumption 2.

$$C = d - 1, \quad r = d, \quad \delta = \frac{1}{2}.$$  

**Proof:** See Appendix VII-A.

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C. Problem Statement

Consider a noncooperative game in a multi-agent system of $n$ agents, where each agent has an unconstrained action set $\Omega_i = \mathbb{R}$. Without loss of generality, we assume that each agent’s decision variable is a scalar. Let $J_i$ denote the cost function of the agent $i$. Then, the game is denoted as $\Gamma(n, \{J_i\}, \{\Omega_i = \mathbb{R}\})$. The goal of each agent $i \in \mathcal{V}$ is to minimize its objective function $J_i(x_i, x_{-i})$, which depends on both the local variable $x_i$ and the decision variables of the other agents $x_{-i}$. We then make the following assumptions with respect to game $\Gamma$.

**Assumption 3:** For all $i \in \mathcal{V}$, the cost function $J_i(x_i, x_{-i})$ is strongly convex and continuously differentiable in $x_i$ for each fixed $x_{-i}$.

**Definition 1:** The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, referred to the game mapping of $\Gamma(n, \{J_i\}, \{\Omega_i = \mathbb{R}\})$ is denoted as

$$F(x) \triangleq [\nabla_1 J_1(x_1, x_{-1}), \ldots, \nabla_n J_n(x_n, x_{-n})]^\top,$$  

(6)

where $\nabla_i J_i(x_i, x_{-i}) = \frac{\partial J_i(x_i, x_{-i})}{\partial x_i}$, $\forall i \in \mathcal{V}$.

**Assumption 4:** The game mapping $F(x)$ is restricted strongly monotone to any NE $x^*$ with constant $\mu > 0$, i.e.,

$$(F(x) - F(x^*), x - x^*) \geq \mu \|x - x^*\|^2, \forall x \in \mathbb{R}.$$  

**Remark 2:** In this paper, we relax the strongly monotone mapping condition in [4], [9], [13] to a restricted one as Assumption 4. Thus, a wider variety of games are taken into consideration.

**Assumption 5:** Each function $\nabla_i J_i(x_i, x_{-i})$ is Lipschitz continuous in $x_i$ for every fixed $x_{-i}$, i.e., $\exists L_i \geq 0$, we have $\forall x_i, y_i$,

$$|\nabla_i J_i(x_i, x_{-i}) - \nabla_i J_i(y_i, x_{-i})| \leq L_i |x_i - y_i|.$$  

Moreover, each function $\nabla_i J_i(x_i, x_{-i})$ is Lipschitz continuous in $x_{-i}$ for every fixed $x_i$, i.e., $\exists L_{-i} \geq 0$, we have $\forall x_i, y_{-i}$,

$$|\nabla_i J_i(x_i, x_{-i}) - \nabla_i J_i(y_i, x_{-i})| \leq L_{-i} |x_{-i} - y_{-i}|.$$  

The concept of NE is given below.

**Definition 2:** A vector $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^\top$ is a NE if for any $i \in \mathcal{V}$ and $x_i \in \mathbb{R}$,

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*).$$  

(7)

In this paper, we are interested in distributed seeking of a NE in a game $\Gamma(n, \{J_i\}, \{\Omega_i = \mathbb{R}\})$ where Assumption 4 and 5 hold. Note that a NE can be alternatively characterized by using the first-order optimality conditions [36]. To be specific, $x^* \in \mathbb{R}^n$ is a NE if and only if for all $x_i$, we have

$$|\nabla_i J_i(x_i^*, x_{-i}^*)| \geq 0.$$  

(8)

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III. COMPRESSED DISTRIBUTED NASH EQUILIBRIUM SEEKING

A. Nash Equilibrium Seeking in Distributed Settings

To cope with incomplete information, we assume that each agent maintains a local variable $x(i) = [\tilde{x}(i)_1, \ldots, \tilde{x}(i)_{i-1}, x_i, \tilde{x}(i)_{i+1}, \ldots, \tilde{x}(i)_n]^\top \in \mathbb{R}^n$, which is its estimation of the joint action profile $x = [x_1, x_2, \ldots, x_n]^\top$, where $\tilde{x}(i)_j \in \mathbb{R}$ denotes agent $i$’s estimate of $x_j$.

Denote estimation matrix, the compact form of action-profile estimates from all agents, as

$$X = [x(1), x(2), \ldots, x(n)]^\top \in \mathbb{R}^{n \times n},$$

where the $i$th row is the estimation vector $x(i), i \in \mathcal{V}$. At $k$-th iteration, action-profile estimates $X$ are denoted by $X^k$.

Moreover, for any given action-profile estimates, we define a diagonal matrix

$$\tilde{F}(X) \triangleq \text{Diag}(\nabla_1 J_1(x(1), \ldots, \nabla_n J_n(x(n))) \in \mathbb{R}^{n \times n}.$$

Next, we present the following proposition showing a equivalent condition for NE in the game $\Gamma(n, \{J_i\}, \{\Omega_i\})$.

**Proposition 1:** ([37]) Consider the game $\Gamma(n, \{J_i\}, \{\Omega_i\})$ satisfies Assumption 4 and 5. Then the following statements are equivalent:

1) The vector $x^* = [x_1^*, \ldots, x_n^*]^\top$ is a NE in $\Gamma(n, \{J_i\}, \{\Omega_i\})$.

2) There exists an estimation matrix $X^*$ with the diagonal vector $x^*$ and the corresponding diagonal matrix $\tilde{F}(X^*)$ such that for any $X$ the following holds

$$(I - W)X^* + \eta \tilde{F}(X^*) (X^* - X) \geq 0,$$

(10)

where $\eta > 0$ is an arbitrary constant.

**Proposition 2** shows that any solution matrix $X^*$ of the variational inequality (10) is consensual and its diagonal vector is a NE in $\Gamma(n, \{J_i\}, \{\Omega_i\})$. Furthermore, from the variational inequality (10), we can define the matrix mapping $F_a : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ as

$$F_a(X) = (I - W)X + \eta \tilde{F}(X),$$

(11)

which is referred to as the augmented mapping [37] of game $\Gamma(n, \{J_i\}, \{\Omega_i\})$.

B. Distributed Nash Equilibrium Seeking with Information Compression

With the above analysis for NE seeking in distributed networks, we are now ready to introduce our proposed compressed algorithm, where compressors $C(\cdot)$ satisfying Assumption 2 are adopted to propose a communication-efficient algorithm to find a NE of the game $\Gamma$ in a fully distributed manner. The agents aim to asymptotically reconstruct the true values of the actions of the other agents, based on the compressed data received from their neighbors.

The detailed procedures are presented in Algorithm 1 and the notations used throughout Algorithm 1 are summarized in Table 1.

**Algorithm 1** A Compressed Distributed Nash Equilibrium Seeking (C-DNES) Algorithm

**Input:** stopping time $K$, step-size $\eta$, consensus step-size $\gamma$, scaling parameters $\alpha > 0$, and initial values $x(0)$

**Output:** $x^K$

1: for each agent $i \in \mathcal{V}$ do
2: \hspace{1em} $h_{i,w}^0 = \sum_{j \in N_i} w_{ij} h_j^0$
3: end for
for $k = 0, 1, 2, \ldots, K - 1$ do locally at each agent $i \in \mathcal{V}$
5. $\mathbf{q}_i^k = C^k_i (\mathbf{x}_i^k - \mathbf{h}_i^k)$.
6. $\hat{\mathbf{x}}_i^k = \mathbf{h}_i^k + C^k_i$.
7. $h_i^{k+1} = (1 - \alpha) h_i^k + \alpha \hat{\mathbf{x}}_i^k$.
8. Send $\mathbf{q}_i^k$ to agent $l \in \mathcal{N}_i$ and receive $\mathbf{q}_j^k$ from agent $j \in \mathcal{N}_i$.
9. $\hat{\mathbf{x}}_{i,w}^k = h_i^{k+1} + \sum_{j=1}^{n} \mathbf{w}_{ij} \mathbf{q}_j^k$.
10. $h_i^{k+1} = (1 - \alpha) h_i^k + \alpha h_i^{k+1}$.
11. $x_i^{k+1} = x_i^k - \gamma (\hat{\mathbf{x}}_i^k - \hat{x}_i^{k+1}) - \gamma \eta \nabla J_i (x_i^{k+1}) e_i$
12. end for

Remark 3: In C-DNES, instead of compressing the local variable $x_i^k$, we maintain an auxiliary variable $h_i^k$, acting as a reference point of $x_i^k$, and compress the difference $\hat{x}_i^k - h_i^k$. As $h_i^k$ approaches $x_i^k$, by Assumption 2 the variance of compression error will tends to 0. After receiving the compressed value $q_i^k = C (x_i^k - h_i^k)$, each agent $i$ obtains an estimator of $x_i^k$ by assembling from $h_i^k$ and the received value. Then $h_i^{k+1}$ is obtained as the weighted average of its previous value $h_i^k$ and $h_i^{k+1}$ with mixing weight $\alpha$, indicating that $h_i^k$ is tracking the motions of $x_i^k$. The update procedure of $h_i^k$ is motivated from DIANA [19] and LEAD [23], which controls the compression error, particularly for a relatively large constant $C$ in Assumption 2. Moreover, the variable $h_{i,w}$ is a weighted averaged version of $h_i^k$, which can be regarded as a backup copy for the neighboring information. The introduction of this auxiliary variable eliminates the need to store all the neighbors’ variable $h$. Denote the compact form of stochastic approximation of action estimates as $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n]^T \in \mathbb{R}^{n \times n}$, where the $i$th row is the approximation vector $\hat{x}_i$, $i \in \mathcal{V}$. Auxiliary variables in compact form at $k$-th iteration are denoted as $\hat{\mathbf{x}}^k, H^k, H_{i,w}^k, Q^k$ and $\tilde{X}_{i,w}^k$, respectively. Algorithm 1 can be written in compact form as follows:

\begin{align}
Q^k &= C^k (X^k - H^k), \\
\hat{X}^k &= H^k + Q^k, \\
\hat{X}_{i,w}^k &= H_{i,w}^k + W^k Q^k, \\
H^{k+1} &= (1 - \alpha) H^k + \alpha \hat{X}^k, \\
H_{i,w}^{k+1} &= (1 - \alpha) H_{i,w}^k + \alpha \hat{X}_{i,w}^k, \\
x^{k+1} &= x^k - \gamma (\hat{X}^k - \hat{X}_{i,w}^k) - \gamma \eta \nabla J_i (x_i^{k+1}) e_i
\end{align}

where $X^0$ and $H^0$ are arbitrary chosen.

Note that after initialization $H_0^w = \mathbf{W}^0$, we have

\begin{align}
H_1^w &= (1 - \alpha) H_0^w + \alpha (\mathbf{W}^0 + W^0 Q^0) \\
= W^0 \{(1 - \alpha) H_0^w + \alpha \hat{X}_i^0\} = \mathbf{W}^1.
\end{align}

Hence, by induction, we obtain $H^w = \mathbf{W}^k$ and $\hat{X}_{i,w}^k = W \hat{X}$

for all $k$. Then, the state variable update in [12] becomes

\begin{align}
X^{k+1} &= X^k - \gamma (\hat{X}^k - W \hat{X}) - \gamma \eta \nabla J_i (x_i^{k+1}) e_i \\
&= (1 - \gamma) X^k + \gamma W \hat{X} - \gamma \eta \nabla J_i (x_i^{k+1}) e_i \\
&= X^k - \gamma (I - W) (X^k - E^k) - \gamma \eta \nabla J_i (x_i^{k+1}) e_i \\
&= X^k - \gamma F_0 (x_i^{k+1}) + \gamma (I - W) E^k,
\end{align}

where $E^k = X^k - \hat{X}$ denotes the compression error for the decision variable. The consensus step-size $\gamma$ ensures the algorithmic convergence, which is proved theoretically in Section IV.

Remark 4: It is worth noting that the above equation implies that C-DNES performs an implicit error compensation mechanism that alleviates the impact of compression error. The $(I - W) E^k$ term in [14] shows that each agent $i$ transmits compression error to its neighbors and compensates this error locally by adding $e_i^k$, where $e_i^k \in \mathbb{R}^n$ is the $i$-th row of $E^k$.

IV. CONVERGENCE ANALYSIS

In this section, we analyze the convergence performance of C-DNES. The main idea of our strategy is to bound $E_\xi (\|X^{k+1} - X^*\|^2_2)$ and $E_\xi (\|H^{k+1} - H^*\|^2_2)$ on the basis of the linear combinations of their previous values. By establishing a linear system of inequalities, we can derive the convergence result.

Let $\mathcal{F}^k$ be the $\sigma$-algebra generated by $\{E^0, E^1, \ldots, E^{k-1}\}$, and denote $E_\xi [\cdot | \mathcal{F}^k]$ as the conditional expectation given $\mathcal{F}^k$. The following flow graph illustrates the relation between iterative variables in Algorithm 1 where the solid arrows show the dynamics of the algorithm.

\begin{align}
&\text{Fig. 2. Flow chart of variables during iterations.}
\end{align}

Since the inherent randomness of the compressor is not correlated across the iteration steps, we can obtain

\begin{align}
E_\xi (\|X^{k+1} - H^{k+1} - C^k (X^k - H^k)\|^2_2 | \mathcal{F}^k) \\
= E_\xi (\|X^{k+1} - H^{k+1} - C^k (X^k - H^k)\|^2_2 | \{X^k, H^k\}^k_{j=0}) \\
= E_\xi (\|X^{k+1} - H^{k+1} - C^k (X^k - H^k)\|^2_2 | \mathcal{F}^{k-1}) \\
\leq C \|X^k - H^k\|^2_2.
\end{align}

A. Technical lemmas

We first prepare a few supporting lemmas for further convergence analysis.

Lemma 1: The variables $X^k, H^k$ are measurable with respect to $\mathcal{F}^k$. Furthermore, we have

\begin{align}
E_\xi (\|E^k\|^2_2 | \mathcal{F}^{k-1}) \leq C \|X^k - H^k\|^2_2.
\end{align}

Proof: By expanding [12a] and [12b] recursively, $X^k$ and $H^k$ can be represented by linear combinations of $X^0, H^0$ and random variables $\{E^j_{1:j=0}\}$, i.e., $X^k, H^k$ are measurable respect to $\mathcal{F}^k$. Moreover, from Equation [12a] and [15], we have

\begin{align}
E_\xi (\|E^k\|^2_2 | \mathcal{F}^{k-1}) \\
&= E_\xi (\|X^k - H^k - C^k (X^k - H^k)\|^2_2 | \mathcal{F}^{k-1}) \\
&\geq C \|X^k - H^k\|^2_2.
\end{align}
Lemma 2: (Lemma 7 in [21]) For any $U, V \in \mathbb{R}^{n \times n}$, the following inequality is satisfied:
\[
||U + V||^2 \leq \tau'||U||^2 + \tau'||V||^2,
\]
where $\tau' > 1$. Moreover, for any $U_1, U_2, U_3 \in \mathbb{R}^{n \times n}$, we have:
\[
||U_1 + U_2 + U_3||^2 \leq \tau' ||U_1||^2 + \frac{27\tau'}{16} ||U_2||^2 + ||U_3||^2 \text{ and }
||U_1 + U_2 + U_3|| \leq 3||U_1|| + 3||U_2|| + 3||U_3||^2.
\]

Lemma 3: (Lemma 1 in [37]) Given Assumption 5, the augmented mapping $F_a$ of game $\Gamma(n, \{J_i\}, \{\Omega_i\})$ is Lipschitz continuous with $L_F = \eta L_m + \|I - W\|_F$, where $L_m = \max_i \{\sqrt{L^2_i + L^2_{-i}}\}$.

Lemma 4: (Lemma 3 in [37]) Given Assumption 5, the augmented mapping $F_a$ of game $\Gamma(n, \{J_i\}, \{\Omega_i\})$ is restricted strongly monotone to any NE matrix $X^* = (x^*)^T$ with the constant $\mu_F = \min\{b_1, b_2\} > 0$, where $b_1 = \eta \mu_r/2n, b_2 = (\beta^2 \lambda_{\text{min}}(I - W))/((\beta^2 + 1)) - \eta^2 L_m$ and $\beta$ is a positive constant such that $\beta^2 + 2\beta = \frac{\mu_r}{\eta^2 L_m}$.

B. Main results

The following lemmas are crucial for establishing a linear system of inequalities that bound $\mathbb{E}_\xi[||X^{k+1} - X^*||^2_2]$ and $\mathbb{E}_\xi[||X^{k+1} - H^{k+1}||^2_2]$ with respect to time $k$.

Lemma 5: Given Assumptions 3, 4, and 5 when $\alpha \in (0, \frac{1}{\tau})$, the following linear system of component-wise inequalities holds:
\[
\mathbb{E}_\xi[||X^{k+1} - X^*||^2_2 | F^k] \leq A \mathbb{E}_\xi[||X^k - X^*||^2_2 | F^k],
\]
where the elements of matrix $A = [a_{ij}]$ are given by
\[
A = \begin{bmatrix}
c_1(1 + L_F^2 \gamma^2 - 2 \mu_F \gamma) & c_2 \gamma^2 \\
c_3 \gamma^2 L_F & c_4 \gamma^2
\end{bmatrix},
\]
with positive constants $c_1$'s and $c_2$ given in Appendix VII-B.

Proof: See Appendix VII-B.

Remark 5: In terms of the linear inequalities in Lemma 5 the optimization error $\mathbb{E}_\xi[||X^{k+1} - X^*||^2_2]$ and the compression error $\mathbb{E}_\xi[||X^{k+1} - H^{k+1}||^2_2]$ all converge exponentially to 0 if the spectral radius $\rho(A) < 1$. The following lemma states a sufficient condition for ensuring $\rho(A) < 1$.

Lemma 6: (Corollary 8.1.29 in [38]) Let $M \in \mathbb{R}^{n \times n}$ be a matrix with nonnegative elements and $v \in \mathbb{R}^n$ be a vector with positive elements. If $Mv \leq \theta v$, there is $\rho(M) \leq \theta$.

Before analyzing the convergence performance of C-DNES, we first give a theorem that proves the existence and uniqueness of NE in game $\Gamma(n, \{J_i\}, \{\Omega_i\})$.

Theorem 1: Under Assumption 3 a Nash equilibrium $x^*$ exists in the game $\Gamma(n, \{J_i\}, \{\Omega_i\})$. Furthermore, under Assumption 3 this NE $x^*$ is unique.

Proof: We first prove the existence of NE in $\Gamma(n, J_i, \Omega_i)$. Consider a function $\rho(x, y)$ defined by
\[
\rho(x, y) = \sum_{i=1}^n J_i(x_1, \ldots, y_i, \ldots, x_n),
\]
where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Under Assumption 3 it can be seen that $\rho(x, y)$ is continuous in $x$ and $y$ and is strongly convex in $y$ for every fixed $x$. Hence, $\rho(x, y)$ has a global minimum $x^0 \in \mathbb{R}^n$ in $y$, i.e.,
\[
\rho(x^0, x) = \min_{x \in \mathbb{R}^n} \rho(x^0, z).
\]

To show $x^0$ is a NE $x^*$ in game $\Gamma(n, \{J_i\}, \{\Omega_i\})$, we will prove by contradiction. Assume for $i = 1$ there exists a point $x_i' \neq x_i^0$ such that $x' = (x_0^0, \ldots, x_i', \ldots, x_n^0) \in \mathbb{R}^n$ and $J_i(x') < J_i(x^0)$. Then we have $\rho(x^0, x') < \rho(x^0, x^0)$, which contradicts (21). Hence, the minimum point $x^0$ is a NE $x^*$ in game $\Gamma(n, \{J_i\}, \{\Omega_i\})$ satisfying (7).

Next, we show the uniqueness of NE under Assumption 3. If there exists a NE $x^* \in \mathbb{R}^n$ in $\Gamma(n, \{J_i\}, \{\Omega_i\})$, we can obtain $\rho(\mathbb{E}(x^*), x - x^*) \geq 0, \forall x \in \mathbb{R}^n$ based on inequality (9). Assume there exists another NE $y^*$, we have $\rho(F(x^*), x^* - y^*) \geq 0$ and $\rho(F(y^*), x^* - y^*) \geq 0$. Thus, $\rho(F(x^*) - F(y^*), x^* - y^*) \leq 0$. Based on Assumption 3 that $\rho(F(y^*), x^* - y^*) \geq \mu_r |x^* - y^*|^2$, we can derive that $x^* = y^*$.

In summary, there exists a unique NE in game $\Gamma(n, \{J_i\}, \{\Omega_i\})$ under Assumption 3 and 4.

The following theorem shows the convergence properties for the C-DNES algorithm in Algorithm 1.

Theorem 2: Under Assumption 1, 2, 3, 4, and 5 hold, we take $\alpha \in (0, \frac{1}{2})$, the consensus step-size $\gamma = \mu_L/L_F$, and gradient step-size $\eta$ satisfying
\[
\eta \leq \min\left\{\frac{2\mu_r}{\gamma L_F^2} \left(1 - \frac{c_1}{m_1} \min(J - W) m_2^2 + \frac{1}{m_1} \right)\right\},
\]
where $m_1, m_2$ are defined in (26) and (29), respectively. Then the optimization error $\mathbb{E}_\xi[||X^{k+1} - X^*||_2^2]$ and the compression error $\mathbb{E}_\xi[||X^{k+1} - H^{k+1}||_2^2]$ both converge to 0 at the linear rate $O(\rho(A)^k)$ with $\rho(A) < 1$, where matrix $A$ is defined in (22).

Proof: Since $\gamma = \mu_L/L_F$, we have $|A|_{11} = c_1(1 + L_F^2 \gamma^2 - 2 \mu_F \gamma) = (1 - \mu_F^2/2L_F^2) < 1$. Recalling from Lemma 6 to ensure $\rho_A < 1$, we can derive the range of $\eta$ and a positive vector $\epsilon := [c_1, c_2]^T \in \mathbb{R}^2_+$, such that
\[
A \epsilon \leq (1 - \frac{\mu_F^2}{4L_F^2}) \epsilon
\]
It suffices to have
\[ \eta \leq \min \left\{ \frac{2n}{\mu_k} \sqrt{1 - c_k x}, \frac{\lambda_{\min}(I - W)m_2^2}{L_m(m_2^2 + 1)} \right\}, \tag{29} \]
where \( m_2 = -1 + \sqrt{1 + \frac{n^2}{4n^2}L_m \sqrt{1 - c_k x}} \).

To wrap up, if the positive constants \( \epsilon_1, \epsilon_2 \) and the step-size \( \eta \) satisfy the following conditions,
\[ \epsilon_1 \geq \frac{4c_2}{L_F^2} \epsilon_2, \quad \epsilon_2 > 0, \]
\[ \eta \leq \min \left\{ \frac{2n}{\mu_k} \sqrt{1 - c_k x}, \frac{\lambda_{\min}(I - W)m_2^2}{L_m(m_2^2 + 1)} \right\}, \tag{30} \]
the linear system of element-wise inequalities in (22) can be shown and we can conclude that the optimization error \( E_{\star} \) and the compression error \( E_{c} \) both converge to 0 at the linear rate \( O(\rho(A)^k) \), where \( \rho(A) < 1 \).

**Remark 6:** It is worth noting that C-DNES can be equipped with different types of compressors in different time iteration while existing quantized distributed NE seeking methods ([26], [27]) use a specific compressor for each time iteration. Furthermore, based on the mild assumption for compressors (Assumption 2), we can even adopt multi-step compressions, such as the composition of quantization and scarification \( \rho(Q(x)) \), to further reduce communication bits. In summary, C-DNES enjoys more flexibility in choosing compression methods.

**Remark 7:** All the above results can be adopted for games with different dimensions of the action sets. The scalar case is considered for the sake of notational simplicity.

**V. SIMULATIONS**

Consider a random generated directed communication network with \( n = 50 \) agents. The weight matrix \( W \) is defined as follows,
\[ [W]_{ij} = \begin{cases} 1/\max_{i \in V} |N_i|, & \text{if } j \in N_i; \\ 1 - \sum_{j \in N_i} w_{ij}, & \text{if } j = i; \\ 0, & \text{Otherwise.} \end{cases} \tag{31} \]

The connectivity control game defined in [39] is considered, where the sensors in the network try to find a tradeoff between the local objective, e.g., source seeking and positioning and the global objective, e.g., maintaining connectivity with other sensors. The cost function of sensor \( i \) is defined as follows,
\[ J_i(x) = l_i^c(x_i) + l_i^p(x), \tag{32} \]
where \( l_i^c(x_i) = x_i^T r_{ii} x_i + x_i^T r_i x_i + b_i, \]
\[ l_i^p(\mathbf{x}) = \sum_{j \in S_i} c_{ij} \sqrt{x_i^T x_j}, \]
\( S_i \subseteq V \) and \( x_i = [x_{i1}, x_{i2}]^T \in \mathbb{R}^2 \) denotes the position of sensor \( i \), \( r_{ii}, r_i, b_i, c_{ij} > 0 \) are constants.

In this simulation, the parameters are set as \( r_{ii} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \), \( r_i = [i \ i]^T \), \( b_i = i \) and \( c_{ij} = 1, \forall i, j \in V \). Furthermore, there is \( S_i = \{i + 1\} \) for \( i \in \{2, \ldots, 49\} \) and \( S_{50} = \{1\} \). The unique NE of the game is \( x_{ij} = -0.5 \) for \( i \in \{1, 2, \ldots, 50\} \), \( j \in \{1, 2\} \). Meanwhile, \( x_{0j} \) are randomly generated in \([0, 1]^{100} \), \( h_0^i = 0 \), the scaling parameter \( \alpha \) and the consensus step-size \( \gamma \) are set to 1.

We tested three different compressors, the stochastic quantization compressor with \( b = 2 \) and \( q = \infty \) in (Q), the Top-k compressor with \( k = 2 \) in (TOP-2) and the Norm-sign compressor with \( q = \infty \) in (N-S). By setting the step-size as \( \eta = 0.01 \), Fig. 3 shows that C-DNES with different compressors all converge linearly to the NE of the game and achieve the same performance as the conventional distributed Nash equilibrium seeking algorithm without information compression [13]. Moreover, the effectiveness of different compressors is illustrated in Fig. 4 and Fig. 5, which shows that Norm-sign compressor achieves the lowest communication cost among all algorithms. Furthermore, all compressed schemes outperform the conventional distributed NE seeking algorithm in terms of the communication burden.

**VI. CONCLUSION AND FUTURE WORK**

In this paper, we study the problem of distributed Nash equilibrium seeking over a directed graph with communication information compression. Specifically, we propose a novel compressed distributed NE seeking approach (C-DNES) and prove its linear convergence. C-DNES not only inherits the advantages of the conventional distributed...
NE algorithm for games with strongly monotone mappings, but also works with a general class of compression operators, e.g., unbiased and biased compressors. Future works may focus on the extensions to networks over time-varying directed graphs. Distributed NE seeking with constrained action space will also be considered. In addition, it is of interest to investigate the combination of acceleration techniques with C-DNES to further speed up the algorithm.

VII. APPENDIX

A. Proof of Claim 1

From the property of vector norm, if $q \geq 2$, we have $\|x\|_q^2 \leq \|x\|^2$ and $\|x\|_q \geq \max_i |x_i|$ for any vector $x$.

$$\|C(x) - x\|^2 = \sum_{i=1}^d (\|x\|_q \text{sgn}(x_i) - x_i)^2$$

$$\leq (d - 2)\|x\|^2 + \|x\|^2$$

$$\leq (d - 1)\|x\|^2.$$  (33)

Thus, we can obtain

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^* + \gamma F_a(x^k)\|^2$$

$$\leq \tau_1 \|x^k - x^* + \gamma F_a(x^k)\|^2$$

$$+ \frac{\tau_1}{\tau_2 - 1} \gamma \|x^k - x^*\|^2,$$  (35)

where the second inequality comes from Lemma 2 with $\tau_1 > 1$.

Next, we bound

$$\|x^{k+1} - x^*\|^2 \leq c_1 (1 + \gamma^2 L_F^2 - 2\gamma \mu_F)|x^k - x^*|^2$$

$$+ \frac{c_1}{c_1 - 1} \gamma^2 \|I - W\|^2 \|E[x^k]\|^2,$$  (37)

where $c_1 = \frac{2L_F^2 - 2\mu_F}{2L_F^2 - 3\mu_F}$.

Then, from Lemma 1 we can obtain

$$\mathbb{E}_x[\|x^{k+1} - x^*\|^2 \mid x]\leq c_1 (1 + \gamma^2 L_F^2 - 2\gamma \mu_F)\mathbb{E}_x[\|x^k - x^*\|^2 \mid x]$$

$$+ c_2 \gamma^2 \mathbb{E}_x[\|x^k - x^*\|^2 \mid x],$$  (38)

where $c_2 = \frac{c_1}{c_1 - 1}$.

Compression error of the decision variable:

Denote $c^k(x^k) = c^k(x^k)/r$, according to (120), for $0 < \alpha \leq \frac{1}{r}$, we have

$$\|x^{k+1} - H^{k+1}\|^2$$

$$= \|x^{k+1} - x^k + \alpha Q_k\|^2$$

$$= \|x^{k+1} - x^k + \alpha R_k(x^k - H^k - c^k(x^k - H^k)) + (1 - \alpha)(x^k - H^k)\|^2$$

$$\leq \tau_2 \|\alpha R_k(x^k - H^k - c^k(x^k - H^k)) + (1 - \alpha)(x^k - H^k)\|^2$$

$$+ \frac{\tau_2}{\tau_2 - 1} \|x^{k+1} - x^k\|^2$$

$$\leq \tau_2 [\alpha R_k(x^k - H^k - c^k(x^k - H^k)) + (1 - \alpha)(x^k - H^k)]^2$$

$$+ \frac{\tau_2}{\tau_2 - 1} \|x^{k+1} - x^k\|^2,$$  (39)

where in the first inequality we use the result of Lemma 2 with $\tau_2 > 1$.

Taking conditional expectation on both sides of (39), we obtain

$$\mathbb{E}_x[\|x^{k+1} - H^{k+1}\|^2 \mid x^k]$$

$$\leq \tau_2 [\alpha R_k(1 - \delta) + (1 - \alpha)\mathbb{E}_x[\|x^k - H^k\|^2 \mid x^k]]$$

$$+ \frac{\tau_2}{\tau_2 - 1} \mathbb{E}_x[\|x^{k+1} - x^k\|^2 \mid x^k],$$  (40)

where the inequality holds based on Assumption 2.
Moreover, we have
\[
E_ξ[∥X^{k+1} - X^k∥^2 | F^k] = E_ξ[∥(I - W)(X^k - X^k) - γF_a(X^k) + γF_a(X^*)∥^2 | F^k] ≤ 2γ^2∥(I - W)∥^2E_ξ[∥X^k - X^k∥^2 | F^k] + 2γ^2L_F^2E_ξ[∥X^k - X^*∥^2 | F^k],
\]
(41)

Bringing (41) into (40) and denoting \(c_3 = \frac{2γ^2}{3γ^2 - 1} > 1, c_5 = τ_2(1 - α^2) + (1 - αr)\), we have
\[
E_ξ[∥X^{k+1} - H^{k+1}∥^2 | F^k] ≤ c_3γ^2L_F^2E_ξ[∥X^k - X^k∥^2 | F^k] + (c_4 + c_5γ^2)E_ξ[∥X^k - H^k∥^2 | F^k],
\]
(42)

where \(c_4 = c_3C_1(∥I - W∥)^2\).

Combining the above equalities (38) and (42) yields the desired result.

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