SHEAF QUANTIZATION IN WEINSTEIN SYMPLECTIC MANIFOLDS

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Abstract. Using the microlocal theory of sheaves, we associate a category to each Weinstein manifold. By constructing a microlocal specialization functor, we show that exact Lagrangians give objects in our category, and that the category is invariant under Weinstein homotopy.

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1. Introduction

1.1. **Principles.** For a particle traveling on a manifold $M$, geometric quantization relates semiclassical states — half-densities on Lagrangian submanifolds $L \subset T^*M$ — to quantum states — half-densities on the base $M$. Sheaf quantization is a categorical analogue: a functor from the category of locally constant sheaves on $L$ to the category of sheaves on $M$. In both cases, singularities of the object on $M$ are controlled by the asymptotics of $L$.

The physical setting appropriate to sheaf quantization is the topological A-twist of the 2d supersymmetric sigma model from [55]. In mathematical terms [15], one studies Lagrangian submanifolds $L_1, L_2$ in a symplectic manifold $(X, \omega)$, and considers their transverse intersections $L_1 \cap L_2$. Instantons, i.e. pseudo-holomorphic strips running along $L_1, L_2$, provide a differential on the space of functions on $L_1 \cap L_2$; the resulting Floer cohomology $HF^*(L_1, L_2)$ is invariant under Hamiltonian isotopy [15]. Polygons running along multiple Lagrangians provide the structure constants of Fukaya categories [16, 43].

Sheaf quantization arises from Fukaya categorical considerations as follows. Given an object $\mathcal{L}$ of the Fukaya category of $T^*M$, for instance given by an exact Lagrangian carrying a local system, one obtains a sheaf $\mathcal{F}_\mathcal{L}$ on $M$ roughly by the prescription $(\mathcal{F}_\mathcal{L})_p = \text{Hom}_{\mathcal{Fuk}(T^*M)}(T^*_p M, \mathcal{L})$. Variants on this theme appear in [39, 34, 1, 51, 52, 21].

More remarkably, it is possible to construct sheaf quantization in cotangent bundles entirely in sheaf theoretic terms, in particular with no reference to instantons [22]. Our purpose here is to do the same in a general Weinstein (in particular exact; $\omega = d\lambda$) symplectic manifold. This will provide, for such manifolds, an alternative categorical framework for Lagrangian intersection theory, constructed via topology rather than analysis.

Returning to the cotangent bundle $X = T^*M$, consider a graphical Lagrangian $\Gamma_{df} \subset T^*M$ with primitive a Morse function $f : M \to \mathbb{R}$. Form the family $L_t = \varphi_t(L) = \Gamma_{tdf} \subset T^*M$, for $t \in \mathbb{R}_{>0}$, and note the limit $L_0 = \lim_{t \to 0} L_t \subset T^*M$ is simply the zero-section. The intersection $L_t \cap M$, for $t \neq 0$, is a finite set, while $L_0 \cap M$ is the entire zero-section. These are not as different as they at first appear: for $t \neq 0$, functions on $L_t \cap M$ underlie the Morse complex of $f$, while functions on the derived intersection $L_0 \cap M$ are differential forms...
on $M$. Imposing the Morse differential on the former and the de Rham differential on the latter renders them quasi-isomorphic; in fact, the family provides an interpolation between the two [54]. Nevertheless, while the differential for $L_0 \cap L_t$ involves global “instantons”, the de Rham differential does not.

More generally, by Stokes’ theorem, pseudo-holomorphic strips along exact Lagrangians where $\lambda|_{L_1} = df_1, \lambda|_{L_2} = df_2$ have area given by the change in the “action” $f_1 - f_2$ at intersection points. In particular, if all intersection points have the same action, there can be no instanton corrections. To arrive at such a situation, one can attempt to move exact Lagrangians $\lambda|_L = df$ so they are conic for the Liouville flow $v_\lambda = (d\lambda)^{-1}(\lambda)$ and hence have constant primitive functions ($df = 0$). In particular, writing $\varphi_t = e^{\log(t)\nu_\lambda} : X \to X$, for $t \in \mathbb{R}_{>0}$, for the Liouville flow obtained by integrating $v_\lambda$, one can attempt to take limits

$$L_0 = \lim_{t \to 0} \varphi_t(L) \subset X$$

Making sense of the categorical intersection theory in this limit would come with a substantial payoff: the absence of instantons implies the theory should be local in nature.

In general, and in contrast to the case of graphical Lagrangians in cotangent bundles, $L_0$ will typically be very singular. No existing account of Lagrangian Floer theory is compatible with the singularities which arise. Instead, we construct by different means a category for $L_0$, and show it enjoys the familiar properties of Fukaya categories. Our construction will be local in nature as suggested by the absence of instantons highlighted above.

Our main tool will be the microlocal theory of sheaves, as pioneered by Kashiwara-Schapira [29]. This is originally formulated in the context of cotangent bundles; in order to adapt it to a general exact symplectic manifold $X$ we embed the latter as a symplectic submanifold in some contact cosphere bundle $S^*N$, and consider microlocal sheaves along its core $\mathfrak{c}(X)$.

While we will not use any Floer theory in our constructions, let us give a heuristic Floer theoretic discussion here to motivate certain expectations which we will later establish rigorously with sheaf theoretical methods. We restrict to the case when $X \subset S^*N$ is a hypersurface and $\mathfrak{c}(X) \subset S^*N$ is a singular Legendrian. Were $\mathfrak{c}(X)$ smooth, it would be natural to consider the Legendrian contact homology of $\mathfrak{c}(X)$ in $S^*N$ [12]; most relevant for us would be the positive augmentation category [40]. Homs in this category are generated by Reeb chords from $\mathfrak{c}(X)$ to itself, some of finite length and some infinitesimal. As this is a purely heuristic discussion, let us imagine ourselves in possession of a (currently nonexistent) version of Legendrian contact homology for singular Legendrians.

We are interested only in the phenomena intrinsic to $X$, and wish to decouple these from anything related to the embedding. In our hypothetical setup, it would be natural to proceed by considering the chord-length filtration on Hom spaces and restricting attention to the zero length part. By analogy with typical behavior of quantum systems, it is plausible that this

1In fact the works [19, 20, 21], relying on the “antimicrolocalization” result of the present work, show that the category constructed here is equivalent to the Fukaya category. As presently written in those works this requires a stable normal polarization, but it may be expected that techniques similar to those appearing in Sec. 10 of the present article, combined with the local-to-global principle of [20], will allow the removal of the polarizability assumption.

2For 1d Legendrians with tripod singularities, a combinatorial LCH has been constructed [3] and satisfies the expected comparison with sheaves [4]. In general one can replace the Legendrian contact homology with the endomorphism algebra of the linking disks to $\mathfrak{c}(X)$ as in [21, Sec. 6.4]; this is always sensible but it is no longer evident how to give a filtration whose associated graded is generated by self chords.
“ground state” category remains constant under deformations of the inclusion \( c(X) \subset S^*N \) that are \textit{gapped} in the sense that the length of the shortest chord is bounded below.

The workhorse technical result of this article, proven entirely by sheaf theoretical methods, is that gapped specialization is indeed well behaved on categories of microlocal sheaves.\(^3\)

This is applied to the problem of sheaf quantization as follows. Consider a compact exact Lagrangian \( L \subset X \). Once flowed a large but finite distance under the Liouville flow, it lifts to a Legendrian \( \tilde{L} \) in the contactization \( X \times (-\epsilon, \epsilon) \), which we may regard as embedded in \( S^*N \). A contact lift of the Liouville flow of \( \tilde{L} \) limits to a subset of \( c(X) \); this flow is gapped because a short self-chord of \( \tilde{L} \) would be a self-intersection of \( L \). Our theorem on gapped specialization implies the existence of a fully faithful functor from microsheaves on \( L \) to microsheaves on \( c(X) \). A similar argument establishes invariance of the category associated to \( X \) under Weinstein homotopy. This is a highly nontrivial result: such homotopies will generally be accompanied by significant variation of the geometry of \( c(X) \).

Sect. 1.2 immediately following contains a rough summary of the techniques we will use in our constructions. Afterwards, in Sect. 1.3, we state theorems in a manner intended to be intelligible to a non-specialist in microlocal sheaf theory and calibrated for comparisons to the Fukaya category.

1.2. Methods. After reviewing here some basic notions of the microlocal theory of sheaves, we will sketch in this section how we adapt and extend this theory to study general exact symplectic manifolds.

Let \( M \) be a smooth manifold. Denote by \( sh(M) \) sheaves on \( M \) valued in some symmetric monoidal presentable stable \( \infty \)-category, for instance the dg derived category of (unbounded) complexes of sheaves of \( \mathbb{Z} \)-modules on \( M \). In particular, an object \( F \in sh(M) \) assigns to any open submanifold \( U \subset M \) with smooth boundary \( \partial U \subset M \) an object (e.g. a complex of \( k \)-modules) \( F(U) \).\(^4\)

1.2.1. Microsheaves on cotangent and cosphere bundles. The microlocal study of sheaves might begin with the question: for what isotopies of an open \( U_t \subset M \) does the complex \( F(U_t) \) remain homologically unchanged? Nonobviously, it is profitable to formulate the answer in terms of conic subsets of the cotangent bundle \( T^*M \rightarrow M \).

Indeed, to each sheaf \( F \in sh(M) \), there is a closed conic coisotropic subset \( ss(F) \subset T^*X \) called its microsupport. As long as the intersection \( ss(F) \cap T_{\partial U_t}^*M \subset T^*M \) of the singular support and outward conormal of the boundary lies in the zero-section \( M \subset T^*M \), the sections \( F(U_t) \) remain homologically unchanged (see e.g. [29, Proposition 2.7.2, Cor. 5.4.19]). In particular one finds that \( F \) is locally constant if and only if \( ss(F) \) lies in the zero-section.

It is then possible to study sheaves “along their microsupports”. More precisely, given an open subset \( \Omega \subset T^*M \), consider the full subcategory \( \text{Null}(\Omega) \subset sh(M) \) of objects \( F \) with disjoint singular support \( ss(F) \cap \Omega = \emptyset \). If we measure objects of \( sh(M) \) by the change in their sections along the codirections lying in \( \Omega \), then \( \text{Null}(\Omega) \) consists of objects for which such measurements vanish. Thus from the viewpoint of \( \Omega \) such objects are trivial, and we

\(^3\)We remark that [20, Thm. 1.4] translated through [21] gives a different criterion (constancy of the contact complement); we do not know a comparison.

\(^4\)In this article, \textit{all} functors are derived, in particular \( F(U) \) means what is elsewhere written \( R\Gamma(U, F) \). Our sheaf conventions are detailed in Sect. 1.5, with also some notes on the use of unbounded sheaves.
may as well pass to the quotient category
\[ \mu sh^{pr} (\Omega) = sh(M)/\text{Null}(\Omega) \]

This assignment provides a presheaf \(\mu sh^{pr}\) of dg categories on open subsets \(\Omega \subset T^*M\), but not in general a sheaf: the hom spaces do not form a sheaf [29, Exercise VI.6], and moreover it may happen that for some cover \(\Omega = \bigcup_i \Omega_i\), there are objects \(\mathcal{F}_i \in \mu sh^{pr}(\Omega_i)\) whose restrictions to intersections are coherently isomorphic, but which nevertheless do not glue to any object \(\mathcal{F} \in \mu sh^{pr}(\Omega)\). In general, the results on \(\mu sh^{pr}\) in [29] are formulated in terms of its stalks. We consider its sheafification \(\mu sh\), and use that its properties can often be checked stalkwise and hence reduce to results of [29].

Since singular support is always conic, the sheaf \(\mu sh\) only depends on the saturation \(\mathbb{R}_{>0} \cdot \Omega \subset T^*M\) of open subsets \(\Omega \subset T^*M\). Thus we lose nothing by viewing \(\mu sh\) as a sheaf on the conic topology where opens \(\Omega \subset T^*M\) are assumed to be saturated \(\Omega = \mathbb{R}_{>0} \cdot \Omega\). Said another way, \(\mu sh\) descends along the quotient map \(T^*M \rightarrow T^*M/\mathbb{R}_{>0}\), i.e., is the pullback of a sheaf from the quotient. Away from the zero-section, the latter is just the cosphere bundle \(S^*M = (T^*M \setminus \{0\})/\mathbb{R}_{>0}\). Thus the restriction of \(\mu sh\) away from \(M \subset T^*M\) provides a sheaf (in the usual topology) on \(S^*M\) which we also denote by \(\mu sh\).

Finally, for any subset \(X \subset T^*X\) (resp. \(X \subset S^*M\)), there is a subsheaf \(\mu sh_X \subset \mu sh\) of full dg subcategories of objects whose singular support (resp. projectivized singular support) lies in \(X\). When \(X\) is closed, note that \(\mu sh_X\) is the pushforward of a sheaf on \(X\) which we will also denote by \(\mu sh_X\).

1.2.2. Microsheaves on Liouville manifolds. We define microlocal sheaves on more general spaces by fixing an embedding into a cotangent bundle. Of course, we will need to understand the dependence of our constructions on the choice of embedding.

As a first example, suppose given a smooth manifold \(N\) embedded as a Legendrian in \(S^*M\). A local calculation shows that \(\mu sh_N\) is locally isomorphic to the sheaf of (derived) local systems on \(N\). Globally, \(\mu sh_N\) is twisted by the Maslov class of \(N\) and related topological data (for a precise formulation see Rem. 10.8). In particular, \(\mu sh_N\) is largely ignorant of the subtle contact topology of the embedding; for example, it cannot tell whether \(N\) is loose. By contrast, the quotient category \(sh_N(M)/\text{Loc}(M)\) is sensitive to the contact topology; for instance, it vanishes for \(N\) loose. The sheafification of \(\mu sh^{pr}\) into \(\mu sh\) has destroyed our access to such subtle global information. While this would be undesirable for defining invariants of the embedding, it is well suited for defining invariants of \(N\) itself.

Now consider a Liouville manifold \((X, \lambda)\) with Liouville vector field \(v_\lambda = (d\lambda)^{-1}(\lambda)\) and compact core \(c(X) \subset X\). By an exact embedding of \(X\) into a cosphere bundle \(S^*M\), we will mean the germ along \(c(X)\) of a smooth embedding, such that there is a contact form on \(S^*M\) restricting to the Liouville form \(\lambda\). At present, we can only prove good invariance results when \(c(X) \subset X\) is (possibly singular) Lagrangian, which we assume henceforth.

Suppose we have an exact embedding of \(X\) as a (codimension one) hypersurface in \(M\). Then \(\lambda |_{c(X)} = 0\), hence \(c(X) \subset S^*M\) is Legendrian. Let us consider the sheaf \(\mu sh\) of microlocal sheaves on \(S^*M\), and its subsheaf \(\mu sh_{c(X)} \subset \mu sh\) of microlocal sheaves supported along \(c(X) \subset S^*M\). Ultimately, we are interested in its global sections
\[ \mathfrak{G}h(X) := \mu sh_{c(X)}(c(X)) \]

For example, when \(X = T^*N\) so that \(c(X) = N\), an exact embedding provides a Legendrian embedding \(N \subset S^*M\), and conversely, by a standard neighborhood theorem, a
Legendrian embedding $N \subset S^*M$ extends to an exact embedding. In this case, as discussed above, $\mu sh_{\mathcal{X}}$ is locally isomorphic to the sheaf of derived local systems on $N$.

In complete generality, we do not know whether every Liouville manifold $(X, \lambda)$ admits a codimension one exact embedding into a cosphere bundle $S^*M$. To get around this, one can imagine repeating the above constructions locally on $X$ and then gluing them together. We will instead follow [45] in using higher codimensional exact embeddings, whose existence and uniqueness are completely controlled by Gromov’s h-principle for contact embeddings. Then we will choose a normally polarized thickening of $X$ to a codimension one exact embedding and proceed with the above constructions. This is explained in Sec. 8.1.

Finally, we study the extent to which the resulting $\mu sh_{\epsilon(X)}$ depends on such an embedding and thickening. We will check the dependence factors through a choice of topological “Maslov data” on $X$, which is in fact the same data required to define the Fukaya category on $X$. As a result we can relax the requirement of the existence of a normal polarization. This is explained in Section 10.

1.2.3. **Lagrangian objects via nearby cycles.** Suppose given a Liouville manifold $(X, \lambda)$ with Liouville vector field $v_\lambda = (d\lambda)^{-1}(\lambda)$, compact core $c(X) \subset X$, and equipped with Maslov data. Then we have assigned the sheaf $\mu sh_{\epsilon(X)}$ of microlocal sheaves along $c(X)$, and thus have its global sections $\mathcal{G}h(X) = \mu sh_{\epsilon(X)}(c(X))$. We outline here how compact exact Lagrangian submanifolds $L \subset X$ give rise to objects $\mathcal{F}_L \in \mathcal{G}h(X)$.

Ultimately, this will depend on certain “secondary Maslov data”. To avoid this complication in the introduction, suppose we have constructed $\mu sh_{\epsilon(X)}$ using a codimension one exact embedding of $X$ into a cosphere bundle $S^*M$. By a standard neighborhood theorem, such an exact embedding extends to the germ along $c(X) \times \{0\}$ of a contact embedding of the contactization $X \times \mathbb{R}$.

Fix a compact exact Lagrangian $L \subset X$ with primitive $f : L \to \mathbb{R}$ with $\lambda|_L = df$, and its Legendrian lift $\Lambda = \text{Graph}(f) \subset L \times \mathbb{R} \subset X \times \mathbb{R}$. For $t \in \mathbb{R}$, apply the Liouville flow to obtain the family

$$L_t = e^{tv_\lambda} \cdot L, \quad f_t = e^t f, \quad \Lambda_t = \text{Graph}(f_t)$$

Note for any open neighborhood $U \subset X$ of $c(X)$, and $\epsilon > 0$, there is $t_0 \in \mathbb{R}$ such that for $t < t_0$, we have $\Lambda_t \subset U \times (-\epsilon, \epsilon)$. Thus for $t < t_0$, we have a family of Legendrian embeddings $\Lambda_t \subset S^*M$. As discussed above, since $\Lambda_t \subset S^*M$ is a smooth Legendrian, the subsheaf $\mu sh_{\Lambda_t} \subset \mu sh$ of microlocal sheaves supported along $\Lambda_t$ is a twisted form of the sheaf of (derived) local systems on $\Lambda_t$. Thus objects of the global section category $\mu sh_{\Lambda_t}(\Lambda_t) \subset \mu sh(S^*M)$ can be identified with certain topological “decorations” on $\Lambda_t \simeq L$. Moreover this category itself constant for $0 < t < t_0$.

Fix some object $\mathcal{F}_t \in \mu sh_{\Lambda_t}(\Lambda_t)$. We would like to take its limit as $t \to -\infty$ to obtain a conic global object

$$\lim_{t \to -\infty} \mathcal{F}_t \in \mu sh_{\epsilon(X)}(c(X)) = \mathcal{G}h(X)$$

Such limits can be made sense of in sheaf theory using the formalism of nearby cycles. To apply this notion in our microlocal setting, we construct an “antimicrolocalization” embedding of $\mu sh_{\Lambda_t}(\Lambda_t) \hookrightarrow \text{Sh}(M)$, whose image is characterized as sheaves microsupported on a certain “double” of $\Lambda_t$ (Theorems 6.17 and 6.28). This antimicrolocalization is of independent interest, being in particular a key step of the [19, 20, 21] programme to prove Kontsevich’s localization conjecture [30] (see particularly [21, Sec. 6.6]).
In order that the antimicrolocalization can be performed uniformly in the family (hence compatibly with the limit), it is necessary to impose that the $\Lambda_t$ are uniformly displaceable from themselves by a positive flow. That is, the length of chords for this flow must be bounded below as $t \to -\infty$. We term such a family \textit{gapped}.

In fact, the notion of gappedness also appears naturally in the study of a more fundamental question: when is the nearby cycles functor fully faithful? We give a criterion in Theorem 4.2, in which gappedness is an essential ingredient.

The antimicrolocalization construction and the criterion for full faithfulness of nearby cycles are major new technical results of this article. Combining them, we obtain in Theorem 7.3 a fully faithful functor (“gapped specialization”)

$$\mu sh_{\Lambda}(\Lambda) \to \mu sh_{\xi(X)}(\xi(X)) = \mathcal{Sh}(X)$$

\textbf{Remark 1.1.} A key consequence of full faithfulness is the following. Recall that an object $L \in \mu sh_{\Lambda}(\Lambda)$ can be locally identified with a local system; suppose it is rank 1. Then $\text{End}(L) \cong H^*(L)$. By full faithfulness, the same holds for the image of $L$ in $\mathcal{Sh}(X)$. Thus, exact Lagrangians (with trivial Maslov class) define objects whose endomorphisms are the cohomology of the Lagrangian, just as in the Fukaya category.

\textbf{Example 1.2.} Suppose we consider a smooth Lagrangian immersion $L \to X$ lifting to a Legendrian embedding $\Lambda \subset X \times \mathbb{R}$. Then the double points in the image correspond to self Reeb chords of $\Lambda_t \subset S^*M$ whose length shrinks to zero as $t \to -\infty$. This violates the gapped condition.

\textbf{Example 1.3.} Consider a pair of smooth Lagrangians $L, L' \subset X$ intersecting in a single point. The Legendrian lifts $\Lambda, \Lambda' \subset X \times \mathbb{R}$ can be chosen either to intersect at a single point (in which case the gapped specialization theorem applies) or disjoint. In the latter case, note that objects of $\mu sh_{\Lambda \cup \Lambda'}(\Lambda \cup \Lambda')$ supported on different components will be orthogonal. But given that $L, L'$ intersect in a single point, we would certainly expect such objects to have non-trivial homs between them! Again, we have violated the gapped condition: the intersection point $L \cap L'$ corresponds to self-Reeb chords of $\Lambda_t \cup \Lambda'_t$ whose length shrinks to zero as $t \to -\infty$.

We have been considering smooth $L$ (and hence $\Lambda$), but in fact the gapped specialization theorem $\mu sh_{\Lambda}(\Lambda) \to \mu sh_{\xi(X)}(\xi(X))$ does not require smoothness (though the identification of objects of $\mu sh_{\Lambda}(\Lambda)$ in simple terms as topological brane structures does).

In particular, we can apply gapped specialization to show that $\mathcal{Sh}(X)$ is in fact invariant under varying the exact symplectic primitive $\lambda$, at least as long as the variation is through primitives with isotropic core. Note that $\xi(X)$ may undergo drastic changes during this process. The basic idea is the following. Denote the core for some variant primitive $\lambda + df$ by $\xi'(X)$, then we may apply gapped specialization to get a fully faithful functors $\mu sh_{\xi'(X)}(\xi'(X)) \to \mu sh_{\xi(X)}(\xi(X))$ and $\mu sh_{\xi(X)}(\xi(X)) \to \mu sh_{\xi'(X)}(\xi'(X))$. Showing these functors compose to the identity can be done by studying an interpolation between the cores, and is done in Theorem 8.15.

The gapped specialization can be applied more generally to noncompact but eventually conical Lagrangians, though in this case we must increase the core. That is, there is a fully faithful functor

$$\mu sh_{\Lambda}(\Lambda) \to \mu sh_{\xi(X) \cup RL^\infty}(\xi(X) \cup RL^\infty) =: \mathcal{Sh}(X; L^\infty)$$
We can compose this functor with the (not fully faithful) left adjoint to the inclusion \( \mathcal{Sh}(X) \to \mathcal{Sh}(X; L^\infty) \) to get a functor \( \mu \text{sh}_\Lambda(\Lambda) \to \mathcal{Sh}(X) \). This is not fully faithful, as should be expected from the counterpart of using noncompact Lagrangians to define objects of the wrapped Fukaya category.

**Remark 1.4.** The idea of using the gapped condition in microlocal sheaf theory was first proposed in [37, 35], where a certain kind of deformation of Legendrians (“arborealization”) was constructed and shown to leave invariant the sheaf categories, and it was suggested that this could be demonstrated more abstractly under a gappedness hypothesis plus some sort of topological constancy. In [56] it was shown that Legendrian ribbotopies (deformations of skeleta induced by isotopies of a Weinstein ribbon) induce equivalences on sheaf categories; note that a ribbotopy necessarily leaves invariant the homotopy type of the skeleton. (Strictly speaking, [56] does not in fact imply [35] as it is unknown if or when the arborealization procedure used in that paper is in fact a ribbotopy.) Here we show that *absent any hypotheses of topological constancy or any hypotheses on the existence on Weinstein thickenings*, the gapped condition implies full faithfulness. For ribbotopies this automatically upgrades to an equivalence since one can construct full faithful maps in both directions and interpolate between their composition and the identity. The current version is essential to show that Lagrangians give objects.

### 1.3. Results

In this section, we give a sample formulation of our results intelligible without any familiarity with microlocal sheaf theory. We work over some fixed symmetric monoidal presentable stable \( \infty \)-category \( \mathcal{C} \). For technical foundations we use [32, 33]. The reader unfamiliar with these works may take \( \mathcal{C} \) to be the dg derived category of \( \mathbb{Z} \)-modules and will lose none of the new ideas.

We say a map of sheaves of categories has some property (e.g. is fully faithful) if this holds on all sections. If \( Y \subset X \) is a closed inclusion, we often omit the notation for the pushforward functor identifying sheaves on \( Y \) and sheaves on \( X \) supported in \( Y \).

We put off defining precisely the notion of “Maslov data” (and later “secondary Maslov data”) until Sect. 10; suffice it to say here that it is certain topological data related to trivializing natural maps related to Lagrangian Grassmannians; the data required is the same as is usually used in setting up the Fukaya category.

**Theorem 1.5.** Let \( (X, \lambda) \) be a Liouville manifold, equipped with Maslov data.

1. **Locality.** For conic \( \Lambda \subset X \), there is a sheaf of presentable stable \( \infty \)-categories \( \mu \text{sh}_\Lambda \) on the conic topology on \( \Lambda \). The restriction maps are continuous and co-continuous.

   In addition, for \( \Lambda \subset \Lambda' \) there is a fully faithful, continuous and cocontinuous embedding \( \mu \text{sh}_\Lambda \to \mu \text{sh}_{\Lambda'} \), functorial in composition of inclusions. In particular, there is such a functor on global sections \( \mu \text{sh}_\Lambda(\Lambda) \to \mu \text{sh}_{\Lambda'}(\Lambda') \).

2. **Removal.** Given an inclusion of closed conic \( \Lambda \subset \Lambda' \), the adjoints to the maps above fit into a localization sequence:

   \[
   \mu \text{sh}_{\Lambda'}(\Lambda' \setminus \Lambda) \to \mu \text{sh}_{\Lambda'}(\Lambda') \to \mu \text{sh}_\Lambda(\Lambda) \to 0
   \]

   (The first morphism is not in general fully faithful.)

3. **Involutivity.** The sheaf \( \mu \text{sh} \) vanishes wherever \( \Lambda \) is not co-isotropic.
(4) **Microstalks.** Given a smooth contractible Lagrangian open \( U \subset \Lambda \) and \( p \in U \), the natural restriction is an equivalence

\[
\mu sh(U) \xrightarrow{\sim} \mu sh_p \cong \mathcal{C}
\]

where the isomorphism to the coefficient category is non-canonical.

(5) **Künneth.** For conic \( \Lambda \subset X \) and \( \Xi \subset Y \), there is a fully faithful functor

\[
\mu sh_{\Lambda} \boxtimes \mu sh_{\Xi} \to \mu sh_{\Lambda \times \Xi}
\]

which is an isomorphism at least whenever either \( \Lambda \) or \( \Xi \) is smooth Lagrangian.

When \( X \) is a cotangent bundle, the construction of \( \mu sh \) is essentially due to Kashiwara and Schapira [29], though working technically with sheaves of homotopical categories requires some foundations such as [32, 33]. We review the construction in Sect. 5. For cotangent bundles, (1) and (2) follow formally from the definition of \( \mu sh \) and elementary properties of microsupports, as does the existence and full faithfulness of the Künneth map. (3) follows from the deep theorem [29, 6.5.4] that microsupports are coisotropic. The facts about Lagrangians in (4) and (5) are well known (see e.g. [22, 36]) and follow from a calculation which is easy after contact transformation, as we recall in Lemmas 5.2 and 5.3 and Cor. 5.4.

The new content of the theorem is the generalization to other Liouville \( X \). We do this by embedding \( X \) in a cotangent bundle; once this is done the importation of facts about \( \mu sh \) from the cotangent setting is immediate. This was described in [45]; we give more details here. The construction is in two stages; for \( X \) equipped with a stable normal polarization, the construction is in Sect. 8.1; in fact this already suffices for typical applications. In full generality, we explain in Sect. 10 how to weaken the requirement of a normal polarization to that of Maslov data.

**Remark 1.6.** For comparisons to the Fukaya category and applications in mirror symmetry, it is preferable to regard \( \mu sh \) as a cosheaf by passing to the left adjoints of the restriction maps. The corestriction maps in the cosheaf are then left adjoints of left adjoints, and hence preserve compact objects. The first morphism of (2) above is such a corestriction, and the sequence should be compared to the stop removal of [20].

**Remark 1.7.** When \( \Lambda \) is “Lagrangian enough”, we may give compact generators for \( \mu sh_{\Lambda} \). Indeed, by (4) we obtain from each Lagrangian point \( p \) a (noncanonical) map \( \mu_p : \mathcal{C} \to \mu sh_{\Lambda}(\Lambda) \). Let \( \Lambda^\circ \subset \Lambda \) be the Lagrangian locus, and assume that \( \Lambda \setminus \Lambda^\circ \) contains no co-isotropic subset (e.g. say \( \Lambda \) admits an isotropic Whitney stratification). Then by (2) we have

\[
\mu sh_{\Lambda}(\Lambda^\circ) \to \mu sh_{\Lambda}(\Lambda) \to \mu sh_{\Lambda \setminus \Lambda^\circ}(\Lambda) \to 0
\]

but by the first assertion of (4), \( \mu sh_{\Lambda \setminus \Lambda^\circ}(\Lambda) = 0 \) and hence \( \mu sh_{\Lambda}(\Lambda^\circ) \to \mu sh_{\Lambda}(\Lambda) \) is surjective. Because \( \mu sh \) is a cosheaf, the former category is generated by the images of the \( \mu_p \), hence so is \( \mu sh_{\Lambda}(X) \). A similar argument shows that if both \( \Lambda \setminus \Lambda^\circ \) and \( \Xi \setminus \Xi^\circ \) have no co-isotropic subsets, then the Künneth map is an isomorphism.

The cospecialization maps, hence also \( \mu_p \), carry compact objects of \( \mathcal{C} \) to compact objects of \( \mu sh_{\Lambda}(X) \). Thus if \( \Lambda \setminus \Lambda^\circ \) is not co-isotropic and \( \mathcal{C} \) is compactly generated, then \( \mu sh_{\Lambda}(X) \) is compactly generated by the images of the \( \mu_p \). More generally, in this situation all sections
of the sheaf $\mu_{sh}^L$ are compactly generated. We may take adjoints to view $\mu_{sh}$ as a cosheaf, and then pass to compact objects and obtain a cosheaf$^5$ $\mu_{sh}^c$ valued in small categories.

These compact objects were called “wrapped microlocal sheaves” in [36]. There is no wrapping in their definition: the name indicated an expected comparison to the wrapped Fukaya category, since established in [21]. It is also natural to study the dual sheaf of pseudoperfect modules, $\mu_{sh}^{pp} : U \mapsto \text{Fun}^{ex}(\mu_{sh}^c(U), C^c)$.

**Remark 1.8.** Under appropriate tameness hypotheses on $\Lambda$, e.g. $\Lambda$ subanalytic Lagrangian, $\mu_{sh}^{pp}(U)$ is obtained by microlocalizing the bounded derived category of constructible sheaves (see [36] for a proof by “arborealization” or [21] for a more elementary argument when $U$ is the cotangent bundle). Under similar hypotheses, it is possible to show that $\mu_{sh}^c(\Lambda)$ is categorically smooth, in particular giving a natural inclusion $\mu_{sh}^{pp}(\Lambda) \subset \mu_{sh}^c(\Lambda)$ (e.g. by arborealization, the sections are a finite colimit of representation categories of tree quivers; these being smooth and proper the colimit is smooth).

Microlocal sheaves in the sense of Theorem 1.5 respect the conic topology on $X$ hence depend on the symplectic primitive $\lambda$. In particular, there is no a priori role for non-conic Lagrangians. On the other hand, the Fukaya category does not depend on $\lambda$, and certainly contains non-conic Lagrangians. In order to produce something like the Fukaya category from $\mu_{sh}$, we must resolve this tension; this is the main purpose of the present article.

Recall that the core $c(X)$ of a Liouville manifold $(X, \lambda)$ is the locus which does not escape under the Liouville flow. Given $\Lambda \subset \partial X$ a closed subset of the contact boundary $\partial^\infty X$, let $\text{Cone}(\Lambda) \subset X$ denote its closed Liouville cone, and form the union

$$c(X, \Lambda) = c(X) \cup \text{Cone}(\Lambda) \subset X$$

In particular, we have $c(X, \emptyset) = c(X)$.

Given $L \subset X$ with conic end, we define $L_0$ as the limit of $L$ under the Liouville flow. More precisely, we consider $L_t \subset X \times (0, \infty) \subset X \times [0, \infty)$ the images of $L$ under the Liouville flow, and take $L_0 = \bigcup L_t \cap (X \times 0)$. Note $L_0 \subset c(X, L^\infty)$.

We will use the phrase “sufficiently isotropic” below to mean a certain collection of technical properties which appear in the body of the text, all of which follow from for example the existence of an isotropic Whitney stratification. We say a symplectic primitive is sufficiently Weinstein if it gives rise to a sufficiently isotropic core. (See Def. 8.11.)

**Theorem 1.9.** For $(X, \lambda)$ Liouville, equipped with Maslov data, and $\Lambda \subset X^\infty$ we write

$$\mathfrak{Sh}_\lambda(X; \Lambda) := \mu_{sh}(X, \Lambda \cup L^\infty)$$

This category has the following properties:

1. **Lagrangian objects.** Let $L \subset X$ be a smooth closed (not necessarily compact) exact Lagrangian $L \subset X$ with conic end $L^\infty$. Fix secondary Maslov data on $L$, and let $\text{Loc}(L)$ be the (derived) category of local systems on $L$.

   If $L_0$ is sufficiently isotropic (e.g. if $c(X, L^\infty)$ is sufficiently isotropic), then there is a fully faithful functor

   $$\psi : \text{Loc}(L) \hookrightarrow \mathfrak{Sh}_\lambda(X, \Lambda \cup L^\infty)$$

---

$^5$It is a cosheaf because colimits commute with cocompletion. We could not analogously get a sheaf of compact objects, since limits do not.
(2) Invariance. If $\lambda, \lambda'$ are sufficiently Weinstein for $(X, \Lambda)$, then there is a fully faithful morphism $\mathcal{G}\mathcal{H}_\lambda(X, \Lambda) \hookrightarrow \mathcal{G}\mathcal{H}_{\lambda'}(X, \Lambda)$. If $\lambda$ and $\lambda'$ admit a sufficiently Weinstein homotopy, this morphism is an isomorphism.

The major new technical ingredients in the proof of Theorem 1.9 are the criterion for full faithfulness of nearby cycles (Theorem 4.2) and the antimicrolocalization (Theorem 6.28).

Remark 1.10. Note that full faithfulness means in particular for $E \in \text{Loc}(L)$, we have

$$\text{End}_{\mathcal{G}\mathcal{H}_\lambda(X, \Lambda \cup L^\infty)}(\psi(E)) \simeq C^*(L; \text{End}(E))$$

In other words, endomorphisms of an object on a given Lagrangian are given by the cohomology of that Lagrangian, as one expects for e.g. compact exact Lagrangians, or more generally noncompact Lagrangians in an infinitesimally wrapped Fukaya category.

Remark 1.11. (Wrapped Lagrangian objects.) Given $L$ as in (1) above, we may compose $\psi$ with the (not fully faithful) “removal” functor $\rho: \mathcal{G}\mathcal{H}_\lambda(X, \Lambda \cup L^\infty) \to \mathcal{G}\mathcal{H}_\lambda(X, \Lambda)$ to obtain a (not fully faithful) functor

$$\rho\phi: \text{Loc}(L) \to \mathcal{G}\mathcal{H}_\lambda(X, \Lambda)$$

Let us consider this map in the case where $L$ is conic (so $\psi$ does nothing) and $L$ is a disk transverse to $c(X)$ at a Lagrangian point $p$, e.g. a critical co-core for a Weinstein Morse function. As $L$ is a disk, $\text{Loc}(L) \cong C$ (noncanonically), and a local calculation shows that $\rho\psi: C \to \mathcal{G}\mathcal{H}(X; \emptyset) = \mu\text{sh}_{c(X)}(c(X))$ is isomorphic to the map $\mu_p: C \to \mu\text{sh}_{c(X)}(c(X))$ described in Rem. 1.7. (After contact transformation, the local calculation is just the fact that the stalk of a local system is given by Hom from the skyscraper sheaf.)

When $X = T^*M$ and $L = T^*_pM$, one can check (e.g. because the cosheaf of local systems is the constant cosheaf of stable categories, which, by an $\infty$-version of the van Kampen theorem, is modules over chains on the based loop space) that

$$\text{End}_{\mathcal{G}\mathcal{H}(T^*M, \emptyset)}(\rho\psi(E)) \simeq C_*(\Omega M; \text{End}(E))$$

Thus $\rho\psi$ is very far from being fully faithful, but it does behave as one might expect for Lagrangians in the wrapped Fukaya category.

Remark 1.12. In fact for any exact and sufficiently isotropic (but possibly singular) $L$ with conic end, we construct a fully faithful functor from a category of microsheaves on $L$ to $\mathcal{G}\mathcal{H}_\lambda(X, \Lambda \cup L^\infty)$. This construction is independent of the choice of secondary Maslov data; the role of which when $L$ is smooth is only to identify the microsheaf category on $L$ with local systems.

Taking adjoints and passing to compact objects provides the analogue of the “Viterbo restriction” functor of [2].
Remark 1.13. Under suitable tameness hypotheses on the Liouville flow of $L$ (e.g. if $X$ is subanalytic Weinstein in the sense of [21, Sect. 6.8] and $L$ is subanalytic), our constructions preserve boundedness and constructibility thus $\psi : \text{Loc}(L) \hookrightarrow \mathcal{Gh}_\Lambda(X, \Lambda \cup L^\infty)$ will carry finite rank local systems to microlocalizations of constructible sheaves. As noted in Rem. 1.7, under subanalyticity assumptions these are pseudoperfect modules and moreover pseudoperfect modules are compact. It follows that $\rho\psi$ carries pseudoperfect modules to compact objects.

1.4. Motivations. While this article is wholly foundational, we mention here some motivations for the present work.

1.4.1. Homological mirror symmetry. The ideas of [39, 34] indicated that the ‘A’ side of mirror symmetry, while usually defined using Fukaya categories, may sometimes be calculated using sheaf theory. The calculational strength of this approach was first illustrated by the determination of the category associated to the mirrors of a toric variety [14, 31]. Another interesting calculation concerns cluster algebras associated to surfaces [47].

More recent calculations have run somewhat ahead of the foundational work, insofar as they calculate some well defined category of microlocal sheaves, sometimes with a somewhat ad-hoc definition, which was nevertheless neither known to match the corresponding Fukaya category nor even known to be a priori a symplectic invariant. Examples include calculations of microlocal categories associated to very affine hypersurfaces, [36, 18], to Landau-Ginzburg models [38], Lagrangian mutations [46], and multiplicative quiver varieties [7] and multiplicative hypertoric varieties [17].

One purpose of the present article is to remedy this situation. On the one hand, our main results construct an invariant category of microlocal sheaves which the prior calculations determine in interesting situations. On the other, the antimicrolocalization lemma proven here is a key ingredient in the comparison theorem to Fukaya categories anticipated by [30, 36] and proven in [21].

1.4.2. Geometric representation theory. The use of stratifications to define categories of sheaves has a long history in geometric representation theory. A paradigmatic example is the BGG category $\mathcal{O}_0$ (a certain category of modules of a semisimple complex Lie algebra $\mathfrak{g}$) is identified (by Beilinson-Bernstein localization followed by the Riemann-Hilbert correspondence) with perverse sheaves on the $\mathfrak{g}$ flag variety, constructible with respect to the stratification by Schubert cells. As always, one can impose this constructibility via a microsupport condition, namely the conormals to the Schubert cells. In the present context there is a good reason to do this. There is a braid group action on the derived category of $\mathcal{O}_0$, given explicitly by sheaf kernels [42, Sect. 5]. However, the fact that this is a braid group action is not apparent a priori and must be calculated.

It would be more satisfying to have on abstract grounds the action of some geometrically defined group, which a posteriori is calculated to be the braid group. (See [25] for work in this direction.) Using [23], one can see that loops of contactomorphisms of $S^*\text{Flag}(\mathfrak{g})$ will act, which in fact is enough to construct the braid group action for $\mathfrak{sl}_2$. From our present work, one sees more generally that loops of gapped deformations will act; this may provide a geometric construction in general. We pause to remark that we do not know whether the conormals to the Schubert cells meet $S^*\text{Flag}(\mathfrak{g})$ in the skeleton of some Weinstein manifold; it would be of some interest to demonstrate this. Our methods can also be applied beyond cotangent bundles, e.g. to the analogous study of category $\mathcal{O}$ for symplectic resolutions as in
Relatedly, it may be of interest to produce a sheaf-theoretic account of the symplectic Khovanov homology of $[44]$. The next step beyond symplectic resolutions is the Hitchin moduli space associated to an algebraic curve $C$. Here the relevant category appears on one side of the “Betti geometric Langlands” of $[6]$, and asserts an equivalence between sheaves on $Bun_G(C)$ with prescribed singular support, and a category of coherent sheaves on the character variety of $C$. In particular, the latter being independent of the complex structure on $C$, the same is predicted for the former. This is nonobvious: while the topology of $Bun_G$ does not vary with $C$, the prescribed singular support does. One may hope to approach this question as well using our theorem on gapped specialization.

1.5. Sheaf conventions. For a manifold $M$, we write $\text{sh}(M)$ for the category of sheaves valued in the symmetric monoidal stable $\infty$-category $C$ of the reader’s choice. For instance, the dg category of complexes of sheaves of abelian groups on $M$, localized along the acyclic sheaves. Another possibility for $C$ is the stable $\infty$-category of spectra.

We appeal often to results from the foundational reference on microlocal sheaf theory $[29]$. This work was written in the then-current terminology of bounded derived categories. The hypothesis of boundedness can typically be removed, though one needs in some cases different arguments, such as for proper base change, where to work in the unbounded setting one needs to use $[49]$, and in one lemma pertaining to the noncharacteristic deformation lemma, where to work in the unbounded setting one needs to use $[41]$. There are some instances when boundedness is fundamental, e.g. whenever one wants to apply Verdier duality twice, but we will never do this. Passing to the dg or stable $\infty$ setting is solely a matter of having adequate foundations; in the dg setting one can use $[11, 50]$ and in general $[32, 33]$. The latter also provide foundations for working with sheaves of categories more generally, which we will require as well.

For a sheaf $F \in \text{sh}(M)$, we write $\text{ss}(F) \subset T^*M$ for its microsupport, i.e. the locus of codirections along which the space of local sections is nonconstant. For $X \subset T^*M$, we write $\text{sh}_X(M)$ for sheaves microsupported in $X$.

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2. Propagation and displacement

A fundamental tool in the arsenal of the microlocal sheaf theorist is the noncharacteristic propagation lemma. Essentially it says that given a sheaf $F$ and a family of open sets $U_t$, if $F(U_t)$ changes as $t$ varies, then it must change locally at some point at the “boundary” of some particular $U_t$; a precise statement can be found in $[29$, Lemma 2.7.2]. It is particularly useful in its microlocal formulation:

**Lemma 2.1. (Noncharacteristic propagation $[29$, Cor. 5.4.19])** For a sheaf $F$ on $M$ and a $C^1$ function $\phi : M \to \mathbb{R}$ proper on the support of $F$, if $d\phi \notin \text{ss}(F)$ over $\phi^{-1}[a, b)$ then the natural maps

$$F(\phi^{-1}(-\infty, b)) \to F(\phi^{-1}(-\infty, a)) \to F(\phi^{-1}(-\infty, a])$$
are isomorphisms.

Note that $d\phi$ over the boundary of a regular level set of $\phi$ is just the outward conormal to the boundary. To apply noncharacteristic propagation, we will want to know when such conormals can be displaced from $ss(F)$. We introduce the following:

**Definition 2.2.** For a contact manifold $V$, a closed subset $\Lambda \subset V$ is positively displaceable from legendrians (pdfl) if given any Legendrian submanifold $L$ (compact in a neighborhood of $V$), there is a 1-parameter positive family of Legendrians $L_t$ for $t \in (\epsilon, \epsilon)$ (constant outside a compact set), such that $L_t$ is disjoint from $\Lambda$ except at $t = 0$.

To explain the term “positive” in the above definition, we briefly recall the notion of positive contact vector fields. On a contact manifold $(V, \xi)$, a contact vector field is by definition one whose flow preserves the distribution $\xi$; they are in natural natural bijection with sections of $TV/\xi$. When $\xi$ is co-oriented, it makes sense to discuss positive vector fields; a contact form $\alpha$ identifies sections of $TV/\xi$ with functions and positive vector fields with positive functions. The Reeb vector field is the contact vector field associated to the constant function 1; in other words it is characterized by $\alpha(v_{\text{Reeb}}) = 1$ and $d\alpha(v_{\text{Reeb}}, \cdot) = 0$. In fact any positive vector field is a Reeb vector field, namely the vector field corresponding to $f$ is the Reeb field for the contact form $f^{-1}\alpha$.

Another way of expressing the above is in terms of the symplectization $\hat{V} = T^*V \cong \xi$, the space of covectors $(n, \alpha) \in T^*V$ with $\alpha|_\xi = 0$ and $\alpha = c\lambda$, with $c > 0$, for any cooriented local contact form $\alpha$. For example, if $V = S^*X$, then $\hat{V} \cong T^*X$.

We view $\hat{V} \to V$ as an $\mathbb{R}_{>0}$-bundle with $\mathbb{R}_{>0}$-action given by scaling $\alpha$. A $\mathbb{R}_{>0}$-equivariant trivialization $\hat{N} \cong N \times \mathbb{R}_{>0}$ over $N$ is the same data as an $\mathbb{R}_{>0}$-equivariant function $f: \hat{V} \to \mathbb{R}_{>0}$. Its Hamiltonian vector field $v_f$ is $\mathbb{R}_{>0}$-invariant, preserves the level-sets $f^{-1}(c) \cong N$, for all $c \in \mathbb{R}_{>0}$, and the corresponding Hamiltonian flow is a contact isotopy on each. For example, if $N = S^*X$, so that $\hat{N} = T^cX$, a metric on $X$ provides such a trivialization $T^cX \cong S^*X \times \mathbb{R}_{>0}$ with the function $|\xi|: T^cX \to \mathbb{R}_{>0}$ given by the length of covectors and positive contact isotopy given by normalized geodesic flow.

By a positive contact isotopy, we mean the flow of a (possibly time dependent) positive contact vector field. The typical example is the geodesic flow, which corresponds to the Reeb vector field for the form $pdq$ restricted to the cosphere bundle in a given metric. The image of a given cosphere fiber under this flow is the boundary of some round ball. We recall that any positive contact vector field also locally determines balls.

A family of Legendrians is positive if it is the flow of a positive Legendrian under a positive contact isotopy. (By Gray’s theorem, this can be checked just on the Legendrians without need to explicitly construct an ambient positive isotopy.) An important example is the geodesic flow of a cosphere. For time positive and smaller than the injectivity radius, the image of the cosphere is the outward conormal to a ball; for time negative it will be the inward conormal. Generalizing this we have:

**Lemma 2.3.** Let $\eta_t : S^*M \to S^*M$ be the time $t$ contactomorphism of some positive contact flow. For small time $t < T(x)$, the cosphere $\eta_t(S^*_xM)$ is the outward conormal to the boundary of some open topological ball $B_t(x)$, with closure $\overline{B}_t(x)$. For $t < t'$ one has $B_t(x) \subset B_{t'}(x)$.

For negative time, $\eta_t(S^*_xM)$ will be similarly an inward conormal.

---

6A stickler might reserve the term ‘positively displaceable’ for a variant of the above definition when we consider only $t \in [0, \epsilon)$. 

Proof. Because the projection of the cosphere to $M$ is degenerate, it is convenient to change coordinates before making a transversality argument. We may as well assume $M = \mathbb{R}^n$. There is a contactomorphism $J^1S^{n-1} \to T^*\mathbb{R}^n$ sending the zero section of $J^1S^{n-1}$ to the cosphere over 0, which induces on front projections the “polar coordinates” map $\mathbb{R} \times S^{n-1} \to \mathbb{R}^n$. Embeddedness of the front is a generic condition, so any sufficiently small isotopy of the zero section in $J^1S^{n-1}$ will leave the front in $\mathbb{R} \times S^{n-1}$ embedded. A positive isotopy will carry the front to a subset of $\mathbb{R}^{>0} \times S^{n-1}$, hence its image under the embedding $\mathbb{R}^{>0} \times S^{n-1} \hookrightarrow \mathbb{R}^n$ will remain embedded. □

Definition 2.4. In the situation of Lemma 2.3, it is always possible to choose a function $f_x : M \to \mathbb{R}_{>0}$ with $f_x^{-1}(t)$ the front projection of $\eta_t(S_x^*M)$ for small $t$. We term such a function a neighborhood defining function at $x$ for the flow $\eta_t$.

Note that a similar argument to Lemma 2.3 shows that the positive perturbation of the conormal to a submanifold will be the outward conormal to its boundary. Correspondingly, we may also speak of the neighborhood defining functions for a submanifold.

By the injectivity radius at a point (or submanifold) we mean the first time $T > 0$ at which $\eta_T$ applied to the conormal fails to have an embedded projection. The neighborhood defining function is well defined for $0 < t < T$, and we use it to give sense to the notion of distance to the point or submanifold. (For the Reeb flow for a metric, these are of course the usual notions of injectivity radius and distance.)

The basic purpose of the pdfl condition is to ensure the following:

Lemma 2.5. For $F \in \text{sh}(M)$, if $ss(F) \subset T^*M$ is pdfl, then for any $x \in M$, there exists $r(x) > 0$ so that for any $0 < r < r(x)$, the natural restrictions are isomorphisms

$$\Gamma(\overline{B}_r(x), F) \sim \Gamma(B_r(x), F) \sim F_x$$

Proof. For $\phi$ the distance squared, $\phi^{-1}[0, \epsilon)$ and $\phi^{-1}[0, \epsilon]$ are the open and closed balls of radius $\epsilon^2$; $d\phi$ over the boundary of these is the outward conormal to the boundary. Now apply the noncharacteristic propagation lemma. □

Remark 2.6. Since we formulated the pdfl hypothesis including negative times, a similar formula holds for costalks. It follows from Lemma 2.5 (and its costalk version) that if $ss(F)$ is pdfl then $F$ satisfies the first two conditions of “cohomological constructibility” of [29, Def. 3.4.1].

We will want to formulate displaceability conditions in terms of chord lengths.

Definition 2.7. Fix a co-oriented contact manifold $(\mathcal{V}, \xi)$ and positive contact isotopy $\eta_t$. For any subset $Y \subset \mathcal{V}$ we write $Y[s] := \eta_s(Y)$.

Given $Y, Y' \subset \mathcal{V}$ we define the chord length spectrum of the pair to be the set of lengths of Reeb trajectories from $Y$ to $Y'$:

$$\text{cls}(Y \to Y') = \{ s \in \mathbb{R} \mid Y[s] \cap Y' \neq \emptyset \}$$

We term $\text{cls}(Y) := \text{cls}(Y \to Y)$ the chord length spectrum of $Y$.

Remark 2.8. An entirely similar argument shows that for any (closed) submanifold $N \subset M$, any small positive contact perturbation of $S^*_N M$ gives the outward conormal to a neighborhood of $N$. We also use the notion of neighborhood defining function in this context.
Borrowing terminology from quantum mechanics or functional analysis, we say that the chord length spectrum is \emph{gapped} if its intersection with some interval \((-\epsilon, \epsilon)\) contains only 0.

\textbf{Definition 2.9.} Given a parameterized family of pairs \((Y_b, Y'_b)\) in some \(V\) over \(b \in B\), we say it is gapped if there is some interval \((0, \epsilon)\) uniformly avoided by all \(\text{cls}(Y_b \to Y'_b)\). In case \(Y = Y'\), we simply say \(Y\) is gapped.

\textbf{Example 2.10.} One can use this notion meaningfully even if \(B\) is a point: Def. 2.2 can be reformulated as: \(\Lambda\) is pdfl if, for every smooth Legendrian \(L\), there is some positive contact isotopy for which \((L, \Lambda)\) is gapped.

Note that in the notion of gapped, some positive contact isotopy is assumed given and fixed. We will often specify this explicitly and say that the family is “gapped with respect to \(\eta\)”. In the typical case where \(\eta\) is generated by a time independent positive Hamiltonian and hence is Reeb flow for some fixed contact form \(\alpha\), we may say that the family is “gapped with respect to \(\alpha\)”.

We also say:

1. \(Y\) is \(\epsilon\)-chordless if \((0, \epsilon] \cap \text{cls}(Y) = \emptyset\).
2. \(Y\) is chordless if \(\text{cls}(Y) = \{0\}\).
3. \(Y\) is \emph{locally chordless} if for any point \(y \in Y\), there exists \(\epsilon > 0\) and an open neighborhood \(U\) of \(y\) such that \(Y \cap U\) is \(\epsilon\)-chordless.

Note that a compact, locally chordless subset is \(\epsilon\)-chordless for some \(\epsilon\).

\textbf{Example 2.11.} Smooth Legendrians are locally chordless for any positive isotopy. This may be seen by an argument similar to the proof of Lemma 2.3.

\textbf{Example 2.12.} The curve selection lemma can be used to show that a subanalytic subset which is Legendrian at all smooth points is locally chordless for the Reeb flow for an analytic metric. (Corollary: such a subanalytic subset is pdfl. Proof: its union with any cosphere remains subanalytic, hence locally chordless; now apply Ex. 2.10.)

\textbf{Example 2.13.} An example of a singular Legendrian which is not locally chordless for the geodesic flow: consider the Legendrian in \(S^*\mathbb{R}^2\) whose front projection is the union of the \(x\)-axis and the graph of \(e^{-1/x^2} \sin(x)\).

\textbf{Example 2.14.} A submanifold \(W \subset V\) is said to be a (exact) symplectic hypersurface if, for some choice of contact form \(\lambda\), the restriction \(d\lambda|_W\) provides a symplectic form. Such a hypersurface is locally chordless, since the Reeb flow is along the kernel of \(d\lambda\).

Such a hypersurface is said to be Liouville if \(\lambda|_W\) gives \(W\) the structure of a Liouville domain or manifold. Liouville domains themselves are the subject of much inquiry; in particular the subclass of Weinstein domains, see [9]. The elementary contact geometry of such Liouville hypersurfaces is studied in e.g. [5, 13].

\textbf{Example 2.15.} In particular, if \(\Lambda \subset V\) is a compact subset for which there exists a Liouville hypersurface \(W \subset V\) with \(\Lambda = \text{Core}(W)\) (“\(\Lambda\) admits a ribbon”), then \(\Lambda\) is locally chordless.

We note some properties and examples of positive flows. Generalizing the relationship between jet and cotangent bundles, we have:

\textbf{Lemma 2.16.} Let \(M, N\) be manifolds. Then \(T^*M \times S^*N \subset T^*(M \times N)\) is a contact level, which is identified (by real projectivization) with \(S^*(M \times N) \setminus 0_{T^*(N)}\).
Given any positive contact isotopy \( \phi_t \) on \( S^*N \), the product \( \phi_t \times 1 \) on \( T^*M \times S^*N \) remains positive. In particular, the Reeb vector field on \( T^*M \times S^*N \) is just the pullback of the Reeb vector field on \( S^*N \).

**Proof.** Pull back the contact Hamiltonian by the projection map. \( \square \)

**Remark 2.17.** The most typical appearance of this fact is when \( N = \mathbb{R} \), in which case it provides the contactomorphism embedding \( J^1M \hookrightarrow S^*(M \times \mathbb{R}) \). Note that the Reeb direction in \( J^1M \) happens only along the \( \mathbb{R} \) factor; from this embedding we can see that along a certain subset of \( S^*(M \times \mathbb{R}) \), just moving in the \( \mathbb{R} \) direction is a positive contact isotopy.

We will use the lemma later in the opposite case \( M = \mathbb{R} \), for the purpose of learning similarly that using the “Reeb flow in the \( M \) direction” is positive along a certain subset of \( S^*(M \times \mathbb{R}) \).

Consider now manifolds \( X, Y \) and positive contact isotopies \( \phi_t \) on \( S^*X \), \( \eta_t \) on \( S^*Y \) corresponding in the symplectization to \( \mathbb{R}_{>0} \)-equivariant Hamiltonians \( x : T^oX \to \mathbb{R}_{>0}, y : T^oY \to \mathbb{R} \). Observe we may define an \( \mathbb{R}_{>0} \)-equivariant Hamiltonian by the formula

\[
x \# y := (x^2 + y^2)^{1/2} : T^o(X \times Y) \to \mathbb{R}_{>0}
\]

Although \( x, y \) are only defined on \( T^oX, T^oY \), since they are \( \mathbb{R}_{>0} \)-equivariant, their squares \( x^2, y^2 \) smoothly extend to \( T^*X, T^*Y \). Let us write \( (\phi \# \eta)_t \) for the flow on \( S^*(X \times Y) \) generated by \( x \# y \), and note it is again a positive contact isotopy.

**Example 2.18.** If \( x, y \) are the length of covectors for some metrics on \( X, Y \), so that \( \phi_t, \eta_t \) are normalized geodesic flow, then \( x \# y \) is the length of covectors for the product metric on \( X \times Y \), and \( (\phi \# \eta)_t \) is again normalized geodesic flow.

**Lemma 2.19.** Let \( V, W \subset T^0X \) be conic subsets, and \( \phi_t \) the positive contact isotopy on \( S^*X \) generated by a time-independent \( \mathbb{R}_{>0} \)-equivariant Hamiltonian \( f : T^oX \to \mathbb{R}_{>0} \).

Then the following are in length-preserving bijection:

- Chords for \( \phi_t \) from \( V \) to \( W \)
- Chords for \( (\phi \# \phi)_t \) from the conormal of the diagonal to \( -V \times W \).

Let us write \( \rho \) for the \( \phi_t \) distance, and \( \rho^\# \) for the \( \phi_t \# \phi_t \) distance, and \( \Delta \) for the diagonal. When both are defined, \( \rho(x_1, x_2) = \sqrt{2}\rho^\#((x_1, x_2), \Delta) \). If \( X \) is compact (or the contact Hamiltonian is bounded below), there is some \( \epsilon \in \mathbb{R} \) such that both distances are always defined in the \( \epsilon \) neighborhood of the diagonal, hence \( B_\epsilon(\Delta) \).

Chords of length smaller than \( \epsilon \) are then additionally in bijection with intersections in \( T^*(B_\epsilon(\Delta)) \) between the graph \( df(x_1, x_2) \) and \( -V \times W \), with the chord length matching the value of \( \rho \) at the intersection.

**Proof.** Let \( v_f = \omega\inv(df) \) denote the vector field generating \( \phi_t \). Let \( t \mapsto \gamma(x, \xi)_t \) denote the integral curve of \( v_f \) through \( (x, \xi) \in T^oX \).

Write \( (x_1, \xi_1, x_2, \xi_2) \) for a point of \( T^o(X \times X) \), and set \( h := f \# f = (f^2(x_1, \xi_1) + f^2(x_2, \xi_2))^{1/2} \).

By direct calculation, \( (\phi \# \phi)_t \) is generated by the vector field \( v_{f \# f} = h^{-1}(f(x_1, \xi_1)v_f + f(x_2, \xi_2)v_f) \).

Although \( v_f \) is only defined on \( T^0X \), the scaling \( f v_f \) smoothly extends to \( T^*X \), vanishing on the 0-section. Indeed, it is the Hamiltonian vector field for \( \frac{1}{2} f^2 \) and its integral curve through \( (x, \xi) \in T^oX \) is given by \( t \mapsto \gamma(x, \xi)_{t'} \) where \( t' = f(x, \xi) t \).

Thus the integral curve of \( v_{f \# f} \) through \( (x_1, \xi_1, x_2, \xi_2) \in T^o(X \times X) \) is given by \( t \mapsto \gamma(x_1, \xi_1)_{t_1} \times \gamma(x_1, \xi_2)_{t_2} \) where \( t_1 = h^{-1} f(x_1, \xi_1) t, t_2 = h^{-1} f(x_2, \xi_2) t \). Note that \( h^{-1} f(x_1, \xi_1) \),
$h^{-1}f(x_2, \xi_2)$ are constant along the integral curve, so give a linear reparametrization of time. In particular, if we have $f(x_1, \xi_1) = f(x_2, \xi_2) = 1$, then the reparametrization constant is $h^{-1} = \sqrt{2}/2$.

With the above formulas, all of the assertions are elementary to check.

For example, suppose $\gamma(x_1, \xi_1) = (x_2, \xi_2)$ at $t = \ell$, i.e. provides a chord of $\rho$-length $\ell$ from $(x_1, \xi_1)$ to $(x_2, \xi_2)$. Suppose without loss of generality that $f(x_1, \xi_1) = f(x_2, \xi_2) = 1$ so that $h = \sqrt{2}$. Set $(x', \xi') = (x_1, \xi_1)_{\ell/2}$. Then the integral curve $t \mapsto \gamma(x', -\xi')_{t_1} \times \gamma(x', \xi')_{t_2}$ lies in the conormal to the diagonal at $t = 0$, and equals $(x_1, -\xi_1, x_2, \xi_2)$ at $t = \frac{\ell}{\sqrt{2}}$, i.e. provides a chord of $\rho^\#$-length $\frac{\ell}{\sqrt{2}}$ from the conormal to the diagonal to $(x_1, -\xi_1, x_2, \xi_2)$.

Conversely, given an integral curve $t \mapsto \gamma(x', -\xi')_{t_1} \times \gamma(x', \xi')_{t_2}$, the integral curve $t \mapsto \gamma(x', \xi')_{t-t/2}$ provides the inverse construction.

Finally, note that the level-sets of $\rho$ are the fronts of the flow of the conormal to the diagonal. Thus $d\rho$ lies in a subset precisely when the flow of the conormal intersects the subset. \hfill \Box

**Remark 2.20.** In particular, Lemma 2.19 allows the chord length spectrum and the gapped condition to be reformulated in terms of the graph of the derivative of the distance from the diagonal.

**Remark 2.21.** We will want to use the above lemma (and remark) in connection with the “gapped” condition of Def. 2.9, and so henceforth we will only consider gappedness with respect to a fixed contact form. This is solely due to the above lemma; if a similar result could be shown for time dependent flows then we could use them throughout.

### 3. Relative microsupport

Let $\pi : E \to B$ be a smooth fiber bundle. We write $T^*\pi$ for the relative cotangent bundle, i.e. the bundle over $E$ defined by the short exact sequence

$$0 \to \pi^*T^*B \to T^*E \xrightarrow{\nabla} T^*\pi \to 0$$

The fibers of $T^*\pi$ are the relative cotangent spaces: for $e \in E$,

$$(T^*\pi)_e = T^*_e(E_{\pi(e)})$$

We may also view $T^*\pi$ as a fiber bundle over $B$, with fibers the cotangent bundles to the fibers of $\pi$: for $b \in B$,

$$(T^*\pi)_b = T^*(E_b)$$

In this section, we present some lemmas concerning the microlocal theory of sheaves on $E$, on the fibers $E_b$, and on the base $B$.

Regarding the comparison with the base, we have the following consequence of the standard estimate on microsupport of a pushforward ([29, Proposition 5.4.4]):

**Definition 3.1.** We say $\Lambda \subset T^*E$ is $\pi$-noncharacteristic, or synonymously $B$-noncharacteristic, if $\Lambda \cap \pi^*T^*B \subset 0_{T^*X}$. We say a sheaf $\mathcal{F}$ on $X$ is $\pi$-noncharacteristic if its microsupport $ss(\mathcal{F})$ is $\pi$-noncharacteristic, and $\pi$ is proper on the support of $\mathcal{F}$.

**Lemma 3.2.** If $\mathcal{F}$ is $\pi$-noncharacteristic, then $\pi_*\mathcal{F}$ has microsupport contained in the zero section of $B$ and hence is locally constant.

The interaction with the fiber will require more subtle microsupport estimates, recalled in the next subsection.
3.1. Microsupport estimates. We recall from [29, Chap. 6] the standard estimates on how microsupport interacts with various functors.

These are expressed in terms of certain operations on conical subsets. A special case is the $\hat{+}$ construction. It is an operation on conical subsets of $T^*X$, the result of which is larger than the naive sum. It is defined in [29, Chap. 6.2] in terms of a normal cone construction. In practice, one uses the equivalent pointwise characterization [29, 6.2.8.ii], which we will just take as the definition.

Definition 3.3. Given conical $A, B \subset T^*X$, a point $(x, \xi) \in A \hat{+} B$ if, in local coordinates, there are sequences $(a_n, \zeta_n) \in A$ and $(b_n, \eta_n) \in B$ with $a_n, b_n \to x$ and $\eta_n + \zeta_n \to \xi$, satisfying the estimate $|a_n - b_n||\eta_n| \to 0$ (or equivalently $|a_n - b_n||\zeta_n| \to 0$).

Evidently $a_n, b_n \to x$ implies that $|a_n - b_n| \to 0$. So certainly sequences with bounded $\zeta_n$ satisfy the estimate. Of course, in this case passing to a subsequence gives $(a_n, \zeta_n)$ convergent in $A$, hence the element $(x, \xi)$ would be already in $A + B$. The additional points of $A + B$ arise when $|\zeta_n| \to \infty$, in a manner controlled by $|a_n - b_n||\zeta_n| \to 0$.

Remark 3.4. Omitting the condition $|a_n - b_n||\zeta_n| \to 0$ would result in a larger $\hat{+}$, for which all the below estimates would be true, but weaker – the term with the $\hat{+}$ is always used as an upper bound. However, these weaker statements would still be strong enough for our purposes here: we never actually use the constraint $|a_n - b_n||\zeta_n| \to 0$.

Example 3.5. Consider the loci $A$ and $B$ given by the conormals to the $x$ and $y$ axes in $\mathbb{R}^2_{x,y}$. Then $A \hat{+} B = A + B$ consists of the union of these conormals and the conormal to the origin.

Example 3.6. Consider the loci $A = \{y = 0\}$ and $B = \{y = x^2\}$ in $\mathbb{R}^2_{x,y}$. Then $A + B$ is just the union of these conormals, whereas $A \hat{+} B$ is the union of these with the conormal to the origin.

A related operation arises in the context of a map $f : Y \to X$. Denote the natural maps $T^*Y \leftarrow f^*T^*X \rightarrow f_!T^*X$.

Definition 3.7. Given conic $A \subset T^*X$ and $B \subset T^*Y$, then $(y, \eta) \in f^!(A, B)$ if (in coordinates) there is a sequence $(x_n, \xi_n) \times (y_n, \eta_n) \in A \times B$ with $y_n \to y, x_n \to f(y)$, and $df_{y_n}(\xi_n) - \eta_n \to \eta$, while respecting the estimate $|x_n - f(y_n)||\xi_n| \to 0$.

One writes $f^!(A) := f^!(A, T^*_Y Y)$. The $\hat{+}$ construction is the special case $A \hat{+} B = id^!(A, -B)$.

Just as $A + B \subset A \hat{+} B$, we have $-B + df(f^{-1}(A)) \subset f^!(A, B)$ as the locus where the $\xi_n$ remain bounded.

Example 3.8. Of particularly frequent use is the case when $f$ is (locally) a closed embedding. We take coordinates $(z, y, \zeta, \eta)$ on $T^*X$, with $y$ the coordinates on $Y$, $\zeta$ the directions conormal to $Y$, and $\eta$ the directions cotangent to $Y$. Then a point $(y, \eta)$ is in $f^!(A) \subset T^*Y$ if there is a sequence $(z_n, y_n, \zeta_n, \eta_n) \in A \subset T^*X$ with $(y_n, \eta_n) \to (y, \eta)$ while $z_n \to 0$ and $|z_n||\zeta_n| \to 0$.

These constructions are useful to describe how microsupport is affected by various functors.

Lemma 3.9. Some microsupport estimates:
• [29, Theorem 6.3.1] For $j : U \to X$ an open inclusion, and $\mathcal{F} \in \sh(U)$
\[ ss(j_*\mathcal{F}) \subset ss(\mathcal{F}) + \mathbb{N}^+ U \]
\[ ss(j^\#(\mathcal{F})) \subset ss(\mathcal{F}) - \mathbb{N}^- U \]

In particular, if $U$ is the complement of a closed submanifold $Y \subset X$, then [29, Proposition 6.3.2]
\[ ss(j_*\mathcal{F})|_Y \subset ss(\mathcal{F}) + T_Y X \]
\[ ss(j^\#(\mathcal{F}))|_Y \subset ss(\mathcal{F}) - T_Y X \]

• [29, Cor. 6.4.4] For $f : Y \to X$,
\[ ss(f^*\mathcal{F}) \subset f^\#(ss(\mathcal{F})) \]
\[ ss(f^\#(ss(\mathcal{F}))) \subset ss(\mathcal{F}) - \mathbb{N}^+ ss(\mathcal{G}) \]

• [29, Cor 6.4.5] For sheaves $\mathcal{F}, \mathcal{G}$
\[ ss(\mathcal{F} \otimes \mathcal{G}) \subset ss(\mathcal{F}) + ss(\mathcal{G}) \]
\[ ss(\sh(\mathcal{F}, \mathcal{G})) \subset -ss(\mathcal{F}) + ss(\mathcal{G}) \]

3.2. Relative microsupport. We return to our smooth fiber bundle $\pi : E \to X$. Recall we write $\Pi : T^* E \to T^* \pi$ for the projection to the relative cotangent bundle.

Definition 3.10. For $\mathcal{F} \in \sh(E)$, we define the relative microsupport to be the conical locus in $T^* \pi$ given by
\[ ss_\pi(\mathcal{F}) := \Pi(ss(\mathcal{F})) \]

The definition is motivated by the following connection with the $\hat{+}$ construction.

Lemma 3.11. Let $\pi : E \to B$ be a fiber bundle, and let $\Lambda \subset T^* E$ be conical. Let $\Pi : T^* E \to T^* \pi$ be the natural projection to the relative cotangent bundle. Then
\[ \Lambda \hat{+} \pi^* T^* B = \Pi^{-1}(\Pi(\Lambda)) \]

Proof. The assertion is local on $E$; choose local coordinates $(x, e)$ with $x$ the coordinates along $B$ and $e$ the bundle coordinates. Let $(x, e, \xi, \eta)$ be corresponding coordinates on $T^* E$.

Then $(x, e, \xi, \eta) \in \Lambda \hat{+} f^* T^* B$ iff there is a sequence $(x_n, e_n, \xi_n, \eta_n) \in \Lambda$ and some $(x'_n, e'_n, \xi'_n, 0)$ such that $x_n, x'_n \to x$ and $e_n, e'_n \to e$ and $\xi_n + \xi'_n \to \xi$ and $\eta_n \to \eta$, and some estimate holds. But we may as well take $x_n = x'_n$ and $e_n = e'_n$, so the estimate is vacuous. The condition on $\xi_n$ is vacuous as well, since $\xi'_n$ can be chosen arbitrarily. Thus the condition with content is $\eta_n \to \eta$. I.e., $(x, e, \eta)$ is a limit point of $\Pi(\Lambda)$. \hfill $\square$

Suppose given in addition some submanifold $\alpha : A \subset B$. We write $\pi_A : E_A \to A$ for the restricted bundle, and $\Pi_A : T^* E_A \to T^* \pi_A$ for the relative cotangent bundle. Note the natural identification $T^* \pi_A = T^* \pi|_{E_A}$. In this setting, we may write $\pi_B := \pi$ and $\Pi_B := \Pi$ for clarity.

We have the following estimate on images in the relative cotangent:

Lemma 3.12. For $\Lambda \subset T^* E$ conical, we have $\Pi_A(\alpha^\#(\Lambda)) \subset \Pi_B(\Lambda)|_{E_A}$.

Proof. We take local coordinates $(y, z, e, \gamma, \zeta, \eta)$ on $T^* E$, where $y$ are coordinates on $A$, the $z$ are coordinates on $B$ in the normal directions to $A$, the $e$ are bundle coordinates, and the $\gamma, \zeta, \eta$ are corresponding cotangent coordinates. We use corresponding notations for coordinates on related spaces.

Consider a point $(y, e, \gamma, \eta) \in \alpha^\#(\Lambda) \subset T^* E_A$. By definition, there must be a sequence $(y_n, z_n, e_n, \gamma_n, \zeta_n, \eta_n) \in \Lambda$ with $(y_n, e_n, \gamma_n, \zeta_n, \eta_n) \to (y, e, \gamma, \eta)$ and $z_n \to 0$ (and also $|z_n| \zeta_n \to 0$, though we won’t use it).
The image of \((y, e, \gamma, \eta)\) in \(T^*\pi_A\) is \((y, e, \eta)\). We must show this point is already in \(\Pi_B(ss(F))\). Thus consider the image of the above sequence in \(T^*\pi_B\), i.e. \((y_n, z_n, e_n, \eta_n)\). By the above hypotheses, this converges to \((y, 0, e, \eta)\).

This lemma implies that the relative microsupport interacts well with restriction to submanifolds of the base.

**Lemma 3.13.** For \(F \in sh(E)\), we have \(ss_{\pi}(\alpha^*F) \subset ss_{\pi}(F)|_{E_A}\), and \(ss_{\pi}(\alpha^!F) \subset ss_{\pi}(F)|_{E_A}\).

**Proof.** What is being asserted is that \(\Pi_A(ss(\alpha^*F)) \subset \Pi_B(ss(F))|_{E_A}\) and \(\Pi_A(ss(\alpha^!F)) \subset \Pi_B(ss(F))|_{E_A}\). This follows from the previous lemma and the estimate \(ss(\alpha^*F) \subset \alpha^# ss(F)\).

**Corollary 3.14.** In particular, if \(P \in B\) is a point, \(ss(F|_{E_P}) \subset ss_{\pi}(F)|_{E_P}\).

**Corollary 3.15.** Let \(\pi_1, \pi_2 : E \times_B E \to B\) denote the projections to the factors. For sheaves \(F_1, F_2\) on \(E\), consider the relative external hom

\[
\mathcal{H}om_{E \times_B E}(\pi_1^*F_1, \pi_2^*F_2) \in sh(E \times_B E)
\]

Let \(\bar{\pi} : E \times_B E \to B\) be the structure map, and \(\bar{\Pi} : T^*(E \times_B E) \to T^*\bar{\pi}\) the projection to the relative cotangent. Then

\[
ss_{\bar{\pi}}(\mathcal{H}om_{E \times_B E}(\pi_1^*F_1, \pi_2^*F_2)) \subset -ss_{\pi}(F_1) \boxplus ss_{\pi}(F_2)
\]

**Proof.** This follows by applying Lemma 3.13 to \(\delta : B \to B \times B\). (Here we use the general identity \(f^!\mathcal{H}om(F, G) = \mathcal{H}om(f^*F, f^!G)\). This is Prop [29, 3.1.13], which is there stated under some boundedness hypotheses on \(F, G\), but the proof is a series of adjunctions which hold in general.)

### 3.3. Microsupport of nearby cycles.

The nearby cycles functor in sheaf theory is a useful notion for taking limits of families of sheaves. We recall its construction. Let \(X\) be a topological space and \(\pi : X \to \mathbb{R}_{\geq 0}\) a continuous map. We consider the diagram

\[
\begin{array}{ccc}
X_{>0} & \xrightarrow{j} & X \\
\pi \downarrow & & \pi \downarrow \\
\mathbb{R}_{>0} & \xrightarrow{j} & \mathbb{R}_{\geq 0} \hookrightarrow \{0\}
\end{array}
\]

where \(X_{>0} := \pi^{-1}(\mathbb{R}_{>0})\) and \(X_0 = \pi^{-1}(0)\). The nearby cycles functor is by definition

\[
\psi = i^*j_* : sh(X_{>0}) \longrightarrow sh(X_0)
\]

By precomposing with \(j^*\), we may also consider \(\psi = i^*j_*j^* : sh(X) \to sh(X_0)\). In this setting we also have the vanishing cycles functor, \(\phi := Cone(i^* \to i^*j_*j^*)\).

Consider the case when \(X_0\) is a manifold, and \(X = X_0 \times \mathbb{R}_{\geq 0}\) and \(\pi\) is the projection to the second factor.

\[
\begin{array}{ccc}
X_0 \times \mathbb{R}_{>0} & \xrightarrow{j} & X_0 \times \mathbb{R}_{\geq 0} \\
\pi \downarrow & & \pi \downarrow \\
\mathbb{R}_{>0} & \xrightarrow{j} & \mathbb{R}_{\geq 0} \hookrightarrow \{0\}
\end{array}
\]

In this case it makes sense to ask about the microsupport of the nearby cycles.
Recall our notation for the cotangent sequence
\[ 0 \to \pi^*T_{\geq 0} \to T^*(X_0 \times \mathbb{R}_{\geq 0}) \xrightarrow{\Pi} T^*\pi \to 0 \]
Note the natural identification \( T^*\pi|_0 \simeq T^*X_0 \).

**Lemma 3.16.** For \( \mathcal{F} \in \text{sh}(X_0 \times \mathbb{R}_{\geq 0}) \), we have
\[ \text{ss}(\psi(\mathcal{F})) \subset \Pi(\text{ss}(\mathcal{F})) \cap T^*X_0 \]

**Proof.** Follows from Lemma 3.9 and Lemma 3.11. \( \square \)

**Definition 3.17.** Given a subset \( \Lambda \subset T^*(X_0 \times \mathbb{R}_{\geq 0}) \), we define its nearby subset as
\[ \psi(\Lambda) := \Pi(\Lambda) \cap T^*X_0 \]
so as to write the statement of the previous lemma as
\[ \text{ss}(\psi(\mathcal{F})) \subset \psi(\text{ss}(\mathcal{F})) \]

4. On the full faithfulness of nearby cycles

4.1. **Main statement.** Let \( M \) be a manifold, and consider the diagram:
\[
\begin{array}{ccc}
U = M \times \mathbb{R}_{\neq 0} & \xrightarrow{j} & N = M \times \mathbb{R} \\
\pi & \downarrow & \pi \\
\mathbb{R}_{\neq 0} & \xrightarrow{j} & \mathbb{R} \xleftarrow{i} \{0\}
\end{array}
\]

Consider the functor
\[ \psi = i^*j_* : \text{sh}(U) \longrightarrow \text{sh}(M) \]
(This is nearby cycles if we restrict attention to sheaves supported always over \( \mathbb{R}_{\geq 0} \), which we will do in all applications. It is evident that all results of Sect. 3.3 hold in the present situation.)

In particular, given \( \mathcal{F}, \mathcal{G} \in \text{sh}(U) \), there is an induced map between the following sheaves on \( M \):

(1) \[ \psi : \psi\text{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}(\psi\mathcal{F}, \psi\mathcal{G}) \]

We are interested to know when (1) induces an isomorphism on global sections. (Note we may replace \( \mathbb{R} \) with any neighborhood of 0.)

We factor (1) as:

(2) \[ i^*j_*\text{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{j^*} i^*\text{Hom}(j_*\mathcal{F}, j_*\mathcal{G}) \xrightarrow{i^*} \text{Hom}(i^*j_*\mathcal{F}, i^*j_*\mathcal{G}) \]

Since \( j \) is an open embedding, the counit of adjunction is an equivalence \( j^*j_* \simeq \text{id} \), and hence the arrow above labelled \( j_* \) is always an isomorphism.

Thus we may as well begin with \( \mathcal{F}, \mathcal{G} \in \text{sh}(N) \), with the properties \( \mathcal{F} = j_*\mathcal{F}|_U \) and \( \mathcal{G} = j_*\mathcal{G}|_U \), and inquire when the functorial map

(3) \[ i^* : i^*\text{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}(i^*\mathcal{F}, i^*\mathcal{G}) \]

induces an isomorphism on global sections.
Example 4.1. The following example shows the global sections of (3) can easily fail to be an isomorphism for sheaves not of the form $\mathcal{F} = j_* \mathcal{F}_U, \mathcal{G} = j_* \mathcal{G}_U$.

Set $M = pt$, so $N = \mathbb{R}$. Take $\mathcal{F} = k_0, \mathcal{G} = k_0$. Then we have

$$i^* \text{Hom}_N(\mathcal{F}, \mathcal{G}) \simeq k_0[-1] \quad \text{Hom}_M(i^* \mathcal{F}, i^* \mathcal{G}) \simeq k_0$$

so (3) does not induce an isomorphism on global sections.

Theorem 4.2. Let $\mathcal{F}_U, \mathcal{G}_U$ be sheaves on $U = M \times \mathbb{R}_{\neq 0}$. Assume

- $ss(\mathcal{F}_U)$ and $ss(\mathcal{G}_U)$ are $\mathbb{R}_{\neq 0}$-noncharacteristic;
- $\psi(ss(\mathcal{F}_U))$ and $\psi(ss(\mathcal{G}_U))$ are pdfl;
- The family of pairs in $S^*M$ determined by $(ss_*(\mathcal{F}_U), ss_*(\mathcal{G}_U))$ is gapped for some fixed contact form on $S^*M$.

Set $\mathcal{F} = j_* \mathcal{F}_U, \mathcal{G} = j_* \mathcal{G}_U$. Then the natural map

$$i^* : \Gamma(M, i^* \text{Hom}_N(\mathcal{F}, \mathcal{G})) \to \text{Hom}_M(i^* \mathcal{F}, i^* \mathcal{G})$$

is an isomorphism.

Proof. Here we will reduce the proof to three technical results whose proofs occupy the remainder of the section.

Let $\delta_M : M \to M \times M$ be the diagonal embedding. We write $i$ for a base-change of $i : \{0\} \to \mathbb{R}$. Consider the diagram:

\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
M & \xrightarrow{i} & M \times \mathbb{R} \\
\downarrow{\delta_M} & & \downarrow{\delta_M} \\
M \times M & \xrightarrow{i \times id} & (M \times M) \times \mathbb{R} \\
\downarrow{id \times i} & & \downarrow{\delta_R} \\
(M \times \mathbb{R}) \times M & \xrightarrow{id \times i} & (M \times \mathbb{R}) \times (M \times \mathbb{R})
\end{array}
\end{array}
\end{align*}

We write

$$\mathcal{H} := \text{Hom}_{(M \times \mathbb{R}) \times (M \times \mathbb{R})}(p_1^* \mathcal{F}, p_2^* \mathcal{G}) \in \text{sh}(M \times \mathbb{R} \times M \times \mathbb{R})$$

Note $\text{Hom}_{M \times \mathbb{R}}(\mathcal{F}, \mathcal{G}) = \delta_M^! \delta_R^! \mathcal{H}$. A diagram chase shows we may factor (3) as:

\begin{align*}
\Gamma(M, i^* \text{Hom}_N(\mathcal{F}, \mathcal{G})) &= \Gamma(M, i^* \delta_M^! \delta_R^! \mathcal{H}) \\
&\to \Gamma(M, \delta_M^! i^* \delta_R^! \mathcal{H}) \\
&\to \Gamma(M, \delta_M^! (i \times id)^! (id \times i)^* \mathcal{H}) \\
&\to \text{Hom}_M(i^* \mathcal{F}, i^* \mathcal{G})
\end{align*}

We must check that the arrows are in fact isomorphisms. Each is an example of a base change formula which does not hold in general. They will however be valid under the microsupport hypotheses we have imposed. The first arrow, we check in Lemma 4.14 below. For the second, we will check below that the map $i^* \delta_R^! \mathcal{H} \to (i \times id)^! (id \times i)^* \mathcal{H}$ is an isomorphism, in Lemma 4.13. The third arrow is an isomorphism by Cor. 4.11. \qed
4.2. Microlocal criterion for comparing hyperbolic restrictions. Let $X$ be a manifold, and $Y_1, Y_2 \subset X$ transverse submanifolds. Let $Y = Y_1 \cap Y_2$ so we have a Cartesian diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{i_1} & Y_1 \\
\downarrow i_2 & & \downarrow y_1 \\
Y_2 & \xrightarrow{y_2} & X
\end{array}
\]

We will write $p : Y \to pt$ for the map to a point.

We will be interested in “hyperbolic restriction” from sheaves on $X$ to sheaves on $Y$. There is a natural map of functors

\[
i^*_1 y_1^! \xrightarrow{\sim} i^*_2 y_2^!
\]

More specifically, we will be interested in (6) after taking global sections

\[
p_* i^*_1 y_1^! \xrightarrow{\sim} p_* i^*_2 y_2^!
\]

Example 4.3. In general, (6) and (7) are not equivalences. Take $X = \mathbb{R}^2_{>0}$, $Y_1 = \mathbb{R}_{>0} \times \{0\}$, $Y_2 = \{0\} \times \mathbb{R}_{>0}$, $Y = \{(0,0)\}$. Take $F = k_{\Delta_{>0}}$ to be the standard extension of the constant sheaf on the diagonal $\Delta_{>0} = \{(t,t)\mid t \in \mathbb{R}_{>0}\}$. Then $i^*_1 y_1^! F \simeq 0$, while $i^*_2 y_2^* F \simeq k_Y$.

The following is a natural “correction” of the issue arising in the prior example.

Lemma 4.4. Take $X = \mathbb{R}^2_{>0}$, $Y_1 = \mathbb{R}_{>0} \times \{0\}$, $Y_2 = \{0\} \times \mathbb{R}_{>0}$, $Y = \{(0,0)\}$.

Suppose $\mathcal{H} \in \text{sh}(X)$ is locally constant on the open submanifold $\tilde{X} = \mathbb{R}^2_{>0}$ near to $Y$. Then the base-change map is an equivalence

\[
i^*_1 y_1^! \mathcal{H} \xrightarrow{\sim} i^*_2 y_2^* \mathcal{H}
\]

Proof. It is elementary to check the base-change map is an equivalence for $\mathcal{H}$ supported on $Y_1$ or $Y_2$. Thus we may assume $\mathcal{H}$ is the standard extension of a sheaf on $\tilde{X} = \mathbb{R}^2_{>0}$. With the given assumption, it is elementary to verify this case as well. □

Let us reduce the general situation of (7) to the model situation of the lemma.

Proposition 4.5. Let $X$ be a manifold, and $Y_1, Y_2 \subset X$ transverse submanifolds. Let $Y = Y_1 \cap Y_2$, and denote maps:

\[
\begin{array}{ccc}
Y & \xrightarrow{i_1} & Y_1 \\
\downarrow i_2 & & \downarrow y_1 \\
Y_2 & \xrightarrow{y_2} & X
\end{array}
\]

Assume that $Y$ is compact, or more generally the support of a sheaf $\mathcal{H} \in \text{sh}(X)$ is compact.

Fix a positive contact flow on $S^*X$. For $i = 1, 2$, let $f_i : X \to \mathbb{R}_{>0}$ be associated neighborhood defining functions (Def. 2.4) for $Y_i$. Consider the map $f = f_1 \times f_2 : X \to \mathbb{R}_{>0}^2$.

Suppose for some $\epsilon$, over $0 < f_1, f_2 < \epsilon$, the intersection

\[
\text{span}(df_1, df_2) \cap \text{ss}(\mathcal{H}) \subset T^*X
\]

lies in the zero-section.
Then the natural map on global sections is an equivalence

\[ \Gamma(Y, i_1^*y_1 \mathcal{H}) \longrightarrow \Gamma(Y, i_2^*y_2 \mathcal{H}) \]

Proof. We compute the map on global sections after first pushing forward along \( f \). We assumed \( \text{supp}(\mathcal{H}) \) compact; in particular \( f \) is proper on it, hence we may invoke proper base-change to express (7) in the form

\[ i_*^*y_1 f_* \longrightarrow i_*^*y_2 f_* \]

The above hypothesis on \( \text{span}(df_1, df_2) \) ensures that singular support \( ss(f_* \mathcal{H}) \in T^*\mathbb{R}^2_{\geq 0} \) lies in the zero-section over \( \mathbb{R}^2_{\geq 0} \). This implies local constancy of this pushforward in this region. We now apply the prior lemma to \( f_* \mathcal{H} \) and conclude that (11), and hence (7), evaluated on \( \mathcal{H} \) is an equivalence.

Note that as we have only been interested in the geometry of \( X \) near \( Y \), one is free anywhere to replace \( X \) by a small neighborhood of \( Y \).

4.3. Second factor \( * \)-pullback of external hom. Here we will consider a very special case of our general nearby cycle setup. Consider manifolds \( V, W \). We write \( \pi_V \) and \( \pi_W \) for the operations of projecting out the \( V \) or \( W \) factor in various products. We write \( i : 0 \to \mathbb{R} \) for the inclusion, or anything base changed from it. Note by smooth base change we may freely commute most operations past any pushforward along \( \pi_V, \pi_W \).

Lemma 4.6. For \( \mathcal{F} \in \text{sh}(V) \) and \( \mathcal{G} \in \text{sh}(W \times \mathbb{R}) \), suppose that \( \mathcal{G}(W \times (-\varepsilon, \varepsilon)) \) is essentially constant. Then the natural map

\[ i^* : i^* \text{Hom}_{V \times W}(\pi_W^* \mathcal{F}, \pi_V^! \mathcal{G}) \to \text{Hom}_{V \times W}(\pi_W^* \mathcal{F}, \pi_V^! i^* \mathcal{G}) \]

induces an isomorphism on global sections.

Proof. Taking global sections means applying \( \pi_V^* \) and \( \pi_W^* \). After applying the natural adjunctions

\[ f_* \text{Hom}(f^* \cdot, \cdot) = \text{Hom}(\cdot, f^* \cdot) \quad f_* \text{Hom}(\cdot, f^! \cdot) = \text{Hom}(f^! \cdot, \cdot) \]

and various base changes, we are reduced to studying

\[ i^* \text{Hom}_{\mathbb{R}}(\pi_W^* \pi_V^! \mathcal{F}, \pi_W^* \mathcal{G}) \to \text{Hom}(\pi_V^! (\mathcal{F}), i^* \pi_W^! \mathcal{G}) \]

To clarify what is at stake, let us write \( \mathcal{F} = \pi_V^! \mathcal{F} \) for the relevant module and \( \underline{\mathcal{F}} \) for the constant sheaf on \( \mathbb{R} \) with this stalk. Then we are asking when the following map is an isomorphism

\[ \text{Hom}_{\mathbb{R}}(\underline{\mathcal{F}}, \pi_W^* \mathcal{G})_0 \to \text{Hom}(\mathcal{F}, (\pi_W^* \mathcal{G})_0) \]

By hypothesis, the projective system whose limit is \( (\pi_W^* \mathcal{G})_0 \) is in fact essentially constant, so the above identity would hold for any module \( A \).

Corollary 4.7. For \( \mathcal{F} \in \text{sh}(V) \) and \( \mathcal{G} \in \text{sh}(W \times \mathbb{R}) \), suppose that for any \( w \in W \), there is a basis of opens \( w \in W_{w, \alpha} \) such that for fixed \( \alpha \), the projective system \( \mathcal{G}(W_{w, \alpha} \times (-\varepsilon, \varepsilon)) \) is essentially constant.

Then the natural map

\[ i^* : i^* \text{Hom}_{V \times W}(\pi_W^* \mathcal{F}, \pi_V^! \mathcal{G}) \to \text{Hom}_{V \times W}(\pi_W^* \mathcal{F}, \pi_V^! i^* \mathcal{G}) \]

is an isomorphism.
Proof. It suffices to check that this is an isomorphism on stalks, hence on arbitrary $U \times W_{w,\alpha}$ for $U \subset V$ open. This is the same as checking that the original map is an isomorphism on global sections after replacing $V \to U$ and $W \to W_{w,\alpha}$. We conclude by Lemma 4.6. □

We now develop a criterion for verifying the hypothesis of the corollary. For a given positive flow on $S^*W$, for $x \in W$ and $r > 0$, let $B_r(x) \subset W$, $\overline{B}_r(x) \subset W$, $S_r(x) \subset W$ denote the respective open ball, closed ball and sphere around $x$ of size $r$ in the sense of 2.3.

Consider the inclusions of cylinders

$$c : C_{r,\delta}(x) = B_r(x) \times (-\delta, \delta) \to M \times \mathbb{R}$$
$$\overline{c} : \overline{C}_{r,\delta}(x) = \overline{B}_r(x) \times (-\delta, \delta) \to M \times \mathbb{R}$$

and their projections

$$p : C_{r,\delta}(x) = B_r(x) \times (-\delta, \delta) \to (-\delta, \delta)$$
$$\overline{p} : \overline{C}_{r,\delta}(x) = \overline{B}_r(x) \times (-\delta, \delta) \to (-\delta, \delta)$$

Note is $c$ is an open embedding and $\overline{p}$ is proper.

Lemma 4.8. Suppose $\Lambda \subset T^*(W \times \mathbb{R}_{\neq 0})$ is $\mathbb{R}_{\neq 0}$-noncharacteristic, and $\phi(\Lambda)^\infty \subset T^\infty(W \times 0)$ is pdfl.

Fix $x \in W$, and a positive flow displacing $S_x^*$ from $\phi(\Lambda)$. Then there exists $r(x) > 0$ and $\delta(x, r(x)) > 0$ so that for all $0 < r < r(x)$, $0 < \delta < \delta(x, r(x))$, the Legendrian at infinity $\Lambda^\infty \subset T^\infty(W \times \mathbb{R})$ is disjoint from the outward conormal Legendrian of the cylinder $C_{r,\delta}(x)$.

Proof. Note the closure $\overline{C}_{r,\delta}(x)$ is a manifold with corners and its outward conormal Legendrian is a union of several pieces: there are the two codimension one faces

$$\partial_r = S_r(x) \times (-\delta, \delta)$$
$$\partial_\delta = B_r(x) \times \{\pm \delta\}$$

and the corner

$$\partial_{r,\delta} = S_r(x) \times \{\pm \delta\}$$

The assertion for $\partial_\delta$ is immediate from the $\mathbb{R}_{\neq 0}$-noncharacteristic hypothesis.

We write $\Pi : T^*(W \times \mathbb{R}) \to (T^*W) \times \mathbb{R}$ for the projection to the relative cotangent. For fixed $r > 0$, if $\Lambda^\infty$ intersects the outward conormal along $\partial_r$, any point in the intersection defines a point in $\Pi(\Lambda)^\infty$. Similarly, by the $\mathbb{R}_{\neq 0}$-noncharacteristic hypothesis, if $\Lambda^\infty$ intersects the outward conormal along $\partial_{r,\delta}$, any point in the intersection also defines a point in $\Pi(\Lambda)^\infty$. Taking the limit of such points as $\delta \to 0$ gives an intersection point of $\phi(\Lambda)^\infty$ with the outward conormal of $B_r(x)$.

But these conormals were chosen disjoint from $\phi(\Lambda)^\infty$, this having been possible because this locus was assumed pdfl. □

Corollary 4.9. For $G \in \text{sh}(W \times \mathbb{R}_{\neq 0})$, suppose $ss(G) \subset T^*(W \times \mathbb{R})$ is $\mathbb{R}_{\neq 0}$-noncharacteristic and $\psi(ss(G))$ is pdfl.

Then for any $x \in W$, there exists $r(x) > 0$ and $\delta(x, r(x)) > 0$ so that for all $0 < r < r(x)$, $0 < \delta < \delta(x, r(x))$, the following restriction maps are isomorphisms.

$$\overline{p} \overline{c} j_* G \sim p_* e^* j_* G$$
$$\Gamma(\overline{C}_{r,\delta}(x), j_* G) \sim \Gamma(C_{r,\delta}(x), j_* G) \sim j_* G_x$$

Proof. Noncharacteristic propagation. □
Remark 4.10. Note that Corollary 4.9 implies the functor $p_\ast c^\ast$ commutes with standard operations: it commutes with $!$-pullbacks and $\ast$-pushforwards (since $c$ is smooth) and $\overline{\mathcal{F}}_\ast \overline{\mathcal{G}}^\ast$ commutes with $!$-pushforwards and $\ast$-pullbacks (since $\overline{\mathcal{F}}$ is proper).

Corollary 4.11. For $\mathcal{G} \in \text{sh}(W \times \mathbb{R}_{\neq 0})$ such that $\text{ss}(\mathcal{G}) \subset T^\ast(W \times \mathbb{R})$ is $\mathbb{R}_{\neq 0}$-noncharacteristic and $\psi(\text{ss}(\mathcal{G}))$ is pdfl, and for any $\mathcal{F} \in \text{sh}(V)$, the natural map

$$i^\ast : i^\ast \text{Hom}_{V \times W \times \mathbb{R}}(\pi_{W \times \mathbb{R}}^\ast \mathcal{F}, \pi_V^\ast \mathcal{G}) \to \text{Hom}_{V \times W}(\pi_W^\ast \mathcal{F}, \pi_V^\ast i^\ast \mathcal{G})$$

is an isomorphism.

Proof. We use Cor. 4.9 to verify the hypothesis of Cor. 4.7.

Remark 4.12. The conclusion of Cor. 4.11 also holds if $\mathcal{F}$ is cohomologically constructible. Indeed, in this case we may use Verdier duality:

$$(1_V \boxtimes i^\ast) \text{Hom}_{V \times W \times \mathbb{R}}(\pi_{W \times \mathbb{R}}^\ast \mathcal{F}, \pi_V^\ast \mathcal{G}) = (1_V \boxtimes i^\ast)(\mathbb{D} \mathcal{F} \boxtimes \mathcal{G})$$

Looking back through the logic of this section, recall at some moment we were to consider $\text{Hom}_{\mathbb{R}}(F, \pi_{W \times \mathbb{R}}^\ast \mathcal{G})_0 \to \text{Hom}(F, (\pi_{W \times \mathbb{R}}^\ast \mathcal{G})_0)$. Previously we argued it was an isomorphism because of some essential constancy of $\pi_{W \times \mathbb{R}}^\ast \mathcal{G}$ near zero. However, we could have also concluded that it was an isomorphism if $F$ were a compact object. But we do not want to impose any finiteness conditions on stalks, in particular because it would prevent us from later considering the category of ‘wrapped microlocal sheaves’.

4.4. The lower square of (4). For the lower square of (4), we do not need the gapped hypothesis, but do need displaceability from legendrians.

Lemma 4.13. Let $\mathcal{F}_U, \mathcal{G}_U$ be sheaves on $U = M \times \mathbb{R}_{\neq 0}$. Assume $\text{ss}(\mathcal{F}_U), \text{ss}(\mathcal{G}_U)$ are $\mathbb{R}_{\neq 0}$-noncharacteristic, and $\psi(\text{ss}(\mathcal{F}_U)), \psi(\text{ss}(\mathcal{G}_U))$ are pdfl.

Set $\mathcal{F} = j_\ast \mathcal{F}_U, \mathcal{G} = j_\ast \mathcal{G}_U$. Then the natural map

$$(12) \quad i^\ast \delta_{\mathbb{R}}^1 \text{H} \longrightarrow (i \times \text{id})^1(i \times i)^\ast \text{H}$$

arising from the lower square of (4) is an isomorphism.

Proof. We will check (12) is an isomorphism on the stalk at a point $(x_1, x_2) \in M \times M$.

Let $\Lambda \subset T^\ast(M \times M)$ denote the singular support of either side of (12). By standard estimates it is contained in $-\psi(\text{ss}(\mathcal{F}_U)) \boxtimes \psi(\text{ss}(\mathcal{G}_U))$. Because $\psi(\text{ss}(\mathcal{F}_U))$ and $\psi(\text{ss}(\mathcal{G}_U))$ are pdfl, the Legendrian at infinity $\Lambda^\infty \subset T^\infty(M \times M)$ is disjoint from the outward conormal Legendrian of the polyball $B_r(x_1) \times B_r(x_2)$, for all small $r > 0$.

Consequently, we can calculate (12) on stalks at $(x_1, x_2)$ by taking sections over the polyball $B_r(x_1) \times B_r(x_2)$ which is the special fiber of the polycylinder $C_{r, \delta}(x) \times C_{r, \delta}(x)$. We preserve from the previous subsection (see above Lemma 4.8) the notation for inclusion and projection of cylinders.

Writing this in terms of standard operations, we seek to show the induced map

$$(p \times p)_\ast (c \times c)^\ast i^\ast \delta_{\mathbb{R}}^1 \text{H} \longrightarrow (p \times p)_\ast (c \times c)^\ast (i \times \text{id})^1(i \times i)^\ast \text{H}$$

is an isomorphism.

For the right hand side, applying Lemma 4.8 and Corollary 4.8, in view of the identities of Remark 4.10, we find
\[(p \times p)_*(c \times c)^*(i \times id)^1(id \times i)^*\mathcal{H} = (i \times id)^1(id \times i)^*(p \times p)_*(c \times c)^*\mathcal{H} = (i \times id)^1(id \times i)^*\text{Hom}_{\text{R} \times \text{R}}(p c^f_*,j_*c^*\mathcal{F}_U,p_*c^*j_*\mathcal{G}_U) = (i \times id)^1(id \times i)^*\text{Hom}_{\text{R} \times \text{R}}(j_*p c^f_*,j_*p_*c^*\mathcal{G}_U)\]

Similarly, for the left hand side, repeating the arguments of Lemma 4.8 and Corollary 4.8, and thus deducing analogous identities to Remark 4.10, we find
\[(p \times p)_*(c \times c)^*\delta^1_B\mathcal{H} = i^*\delta^1_M\text{Hom}_{\text{R} \times \text{R}}(j_*p c^f_*,j_*p_*c^*\mathcal{G}_U)\]

Thus we seek to show the natural map
\[i^*\delta^1_M\text{Hom}_{\text{R} \times \text{R}}(j_*p c^f_*,j_*p_*c^*\mathcal{G}_U) \longrightarrow (i \times id)^1(id \times i)^*\text{Hom}_{\text{R} \times \text{R}}(j_*p c^f_*,j_*p_*c^*\mathcal{G}_U)\]
is an isomorphism.

A final application of Corollary 4.8, to replace \(p\) with the proper map \(\overline{p}\), shows by noncharacteristic propagation that \(pc^f_\overline{p} \mathcal{F}_U, p_*c^*\mathcal{G}_U\) are locally constant on \(((\delta, \delta) \setminus \{0\})^2\). Thus it remains to verify the assertion in the case when \(M\) is a point. This is an elementary exercise. \(\square\)

4.5. **The upper square of** (4). For the upper square of (4), we do not require the (pdfi) condition on microsupports or its “constructibility” consequences. However, we do need the gappedness of the pair \((\mathcal{F}, \mathcal{G})\).

**Lemma 4.14.** Let \(\mathcal{F}, \mathcal{G} \in \text{sh}(M \times B)\) be \(B\)-noncharacteristic, and assume the pair \((ss_\pi(\mathcal{F}), ss_\pi(\mathcal{G}))\) is gapped (Def. 2.9) over \(B \setminus \{0\}\). Then the natural map
\[\Gamma(M, i^*\delta^1_M(\delta^1_B\mathcal{H})) \rightarrow \Gamma(M, \delta^1_M i^*(\delta^1_B\mathcal{H}))\]
associated to
\[
\begin{array}{ccc}
M & \xrightarrow{i} & M \times B \\
\delta_M & & \delta_M \\
\downarrow & & \downarrow \\
M \times M & \xrightarrow{i} & (M \times M) \times B \\
\end{array}
\]
is an isomorphism.

**Proof.** Let \(m : T^\circ M \rightarrow \mathbb{R}\) be the linear conic Hamiltonian for the positive flow exhibiting gappedness (recall this Hamiltonian is assumed time independent). Let \(b\) be the conic hamiltonian for Reeb flow on \(T^*B \setminus B\) (i.e. the norm of the cotangent coordinate). On \(T^\circ(M \times M \times B)\) we consider the hamiltonian \(h := (m^2 \oplus m^2 \oplus b^2)^{1/2}\); being linear and conic it determines a contact flow on \(S^*(M \times M \times B)\).

We will apply Proposition 4.5 to the sheaf \(\delta^1_B\mathcal{H}\). In the notation there we should consider
\[X = (M \times M) \times B \quad Y_1 = M \times B \quad Y_2 = M \times M \quad Y = M\]

Let \(f_1\) and \(f_2\) be neighborhood defining functions for \(M \times B\) and \(M \times M\) respectively.

To verify the hypotheses of Proposition 4.5 we must show that for some \(\epsilon\), above \(\{f_1, f_2 \in (0, \epsilon)\}\), the locus \(ss(\delta^1_B\mathcal{H})\) is disjoint from the span of \(df_1, df_2\).

Evidently \(df_2\) is pulled back from the cotangent to \(B\), so the disjointness from \(df_2\) follows because \(\mathcal{F}, \mathcal{G}\) are \(B\)-noncharacteristic.
We now consider $df_1$. Because $df_2$ is contained in $T^*B$, it suffices to check that $ss(\delta_B^i \mathcal{H})$ is disjoint from $df_1$ in some neighborhood $f_1^{-1}(0, \epsilon)$ after projecting to $T^*X/T^*B$. That is, we should study the relative microsupport $ss_{\pi}(\delta_B^i \mathcal{H})$.

The relevant estimate is Cor. 3.15, which tells us that in the fiber over some particular point $b \in B$, i.e. inside $(T^*X/T^*B)_b = T^*(M \times M)$, we have

$$ss_{\pi}(\delta_B^i \mathcal{H})_b \subset -ss_{\pi}(\mathcal{F})_b \boxplus ss_{\pi}(\mathcal{G})_b$$

We therefore study intersections of $df_1$ with the RHS above. Per Lemma 2.19, these correspond to chords whose length is the value of $f_1$ at the intersection point. From the gapped hypothesis, we may once and for all choose $\epsilon$ small enough that there are no such in $f_1^{-1}(0, \epsilon)$.

5. **Microlocal sheaves**

Let $M$ be a manifold, and $\mathcal{C}$ a symmetric monoidal presentable\(^7\) stable\(^8\) $\infty$-category $\mathcal{C}$. ($\infty$-category. (The reader will not learn less from this article by taking $\mathcal{C}$ to be the dg derived category of modules over a commutative ring, say $\mathbb{Z}$.)

The category of sheaves on $M$ valued in $\mathcal{C}$ microlocalizes over the cotangent bundle $T^*M$ in the sense that it is the global sections of a sheaf of categories on $T^*M$. This sheaf of categories is defined as follows. Recall that for $V \subset T^*M$, we write $sh_V(M)$ for the category of sheaves on $M$ microsupported in $V$. For an open subset $\mathcal{U} \subset T^*M$, we set

$$\mu sh_{pre}(\mathcal{U}) := sh(M)/sh_{M \setminus \mathcal{U}}(M)$$

For $\mathcal{U} \subset \mathcal{V}$, we evidently have $sh_{M \setminus \mathcal{U}}(M) \supset sh_{M \setminus \mathcal{V}}(M)$, and thus there are restriction maps $\mu sh_{pre}(\mathcal{U}) \to \mu sh_{pre}(\mathcal{V})$; it is easy to see that these make $\mu sh_{pre}$ into a presheaf of symmetric monoidal presentable stable $\infty$-categories. We write $\mu sh$ for its sheafification (see Rem. 5.1 for a discussion of this construction in the present context).

While $\mu sh$ is sensible in the usual topology on $T^*M$, it is in fact pulled back from the conic topology, in which the open sets are all invariant under the $\mathbb{R}_{>0}$-action. We will often be interested in its restriction to the the complement of the zero section $T^*M \setminus M$, where it is pulled back from a sheaf on the cosphere bundle $(T^*M \setminus M)/\mathbb{R} = S^*M$, which we also denote by $\mu sh$.

For $F \in \mu sh(\mathcal{U})$, there is a well defined microsupport, $ss(F) \subset \mathcal{U}$. For $\Lambda \subset M$ we write $\mu sh_{\Lambda}(\mathcal{U})$ for the full subcategory of objects microsupported in $\Lambda \cap \mathcal{U}$. Note that $\mu sh_{\Lambda}$ is a subsheaf of $\mu sh$, and the pushforward of a sheaf on $\Lambda$.

**Remark 5.1.** Let us give some technical remarks regarding sheaves of $\infty$-categories, in particular sheafification; throughout, we rely upon the foundations provided by [32, 33]. To ease the exposition, when possible, we will say category in place of $\infty$-category.

In general, when discussing presheaves of categories, it would be necessary to specify in which category of categories we are working; specific natural choices would include the category $\text{Cat}$ of all categories, and also the categories $\text{Pr}^L$ and $\text{Pr}^R$ of presentable categories.

---

\(^7\)See [32, Ch. 5.5] for a foundational treatment of presentable $\infty$-categories. In particular: the adjoint functor theorem for presentable $\infty$-categories implies that being a left adjoint is equivalent to preserving colimits, and being a right adjoint is equivalent to being accessible and preserving limits. Note that a colimit preserving functor is certainly accessible, hence a functor which preserves colimits and limits has both adjoints.

\(^8\)See [33, Ch. 1] for a foundational treatment of stable $\infty$-categories.
with continuous or cocontinuous morphisms. On the other hand, to determine whether a presheaf is a sheaf, this is immaterial: limits in $Pr^L$ or $Pr^R$ exist and are computed by the corresponding limits in the category of categories [32, Chap. 5.5]. In particular, given a presheaf of presentable categories for which all restriction maps are continuous and cocontinuous, its sheafifications in $Pr^L$ and $Pr^R$ and $Cat$ all agree. Indeed, the universal property of sheafification guarantees natural maps between these. As sheafification preserves stalks and isomorphisms of sheaves is determined stalkwise, the sheafifications agree, and in particular this common sheafification has continuous and cocontinuous restriction maps.

Now let us return specifically to $\mu_{sh}$. As the microsupport of a limit or colimit is contained in the union of the microsupports of the terms, the full subcategory $sh_{M \cup_U}(M) \subset sh(M)$ is closed under limits and colimits. It follows (see e.g. [33, Prop A.8.20, Rem A.8.19]) that the quotient map $sh(M) \to sh(M)/sh_{M \cup_U}(M)$ is continuous and cocontinuous. Similarly, all restriction maps in $\mu_{sh}^{pre}$ are continuous and cocontinuous. Thus we need not concern ourselves with choosing between $Pr^b$, $Pr^R$, $Cat$ in defining $\mu_{sh}$, and the restriction maps for this sheaf of categories are continuous and cocontinuous. Similarly, since all categories in sight are stable and all functors exact, the sheafification is automatically a sheaf of stable categories.

Next, let us recall some properties of $\mu_{sh}$ available in the literature. While $\mu_{sh}$ is not explicitly considered in [29], $\mu_{sh}^{pre}$ appears in [29, Sec. 6.1], where $\mu_{sh}^{pre}(U)$ (with boundedness assumptions) is called $D^b(M;U)$. The stalks of $\mu_{sh}^{pre}$, which are also the stalks of $\mu_{sh}$, also appear (again with boundedness assumptions) in [29] under the name $D^b(M, p)$. In constructing a morphism of sheaves, it is enough to do so for their corresponding presheaves; to check properties of a morphism (in particular, when it is fully faithful or an isomorphism) it is enough to check on stalks. Thus the results of [29] serve well for these purposes. Most fundamentally, the $\mu_{hom}$ functor of [29] gives the sheaf of Homs of objects in $\mu_{sh}$. (Indeed, in [29, Theorem 6.1.2], it is shown that there is an isomorphism on stalks, and the proof first constructs a natural morphism on presheaves.) The map $\mu_{sh}^{pre}(U) \to \mu_{sh}(U)$ is not generally an isomorphism, but in case $U = T^*U$ one has $Sh(U) = \mu_{sh}^{pre}(T^*U) = \mu_{sh}(T^*U)$.

A basic tool to study $\mu_{sh}$ is the microlocal theory of quantized contact transformations developed in [29, Chap. 7]. A key result is that a contactomorphism induces local isomorphisms on $\mu_{sh}$. More precisely, a contactomorphism $\phi$ between a germ of $x \in S^*M$ and of $y \in S^*N$ induces an isomorphism\(^9\), respecting microsupports, of $\mu_{sh}_x$ and $\phi_\ast \mu_{sh}_y$ [29, Cor 7.2.2].

In order to globalize this, we note the evident:

**Lemma 5.2.** Suppose $\tilde{\Lambda} \subset T^*(M \times \mathbb{R})$ is the product of $\Lambda \subset T^*M$ and $T^*_0\mathbb{R}^n$. Then pullback along $i : M \times n \to M \times N$ induces an isomorphism $i^* : i^*\mu_{sh}_{\tilde{\Lambda}} \to \mu_{sh}_{\Lambda}$, and pullback along $\pi : M \times N \to M$ induces an isomorphism $\pi^* : \mu_{sh}_{\Lambda} \to \pi^*\mu_{sh}_{\Lambda}$. \(\square\)

By contact transformation we have:

**Lemma 5.3.** Suppose the germ of $\tilde{\Lambda} \subset S^*M$ is contactomorphic to the germ of $\Lambda \times N \subset U \times T^*N$ for some contact $U$, by a map restricting to $f : \tilde{\Lambda} \cong \Lambda \times N$.

Let $\pi : \Lambda \times N \to \Lambda$ be the projection. Then for $\lambda \in \Lambda$, there is an isomorphism

$$\mu_{sh}_{\Lambda}|_{\lambda} \cong \mu_{sh}_{\Lambda}|_{\pi \circ f(\lambda)}$$

\(^9\)The isomorphism is unique up to a choice of invertible object in the coefficient category.
Proof. The statement is local on $\tilde{\Lambda}$, so by contact transformation we are reduced to Lemma 5.2.

That is, $\mu sh_{\tilde{\Lambda}}$ is locally constant in the $N$ direction. In particular, by taking $\Lambda$ a point, one has:

Corollary 5.4. Let $X \subset S^*M$ be a smooth Legendrian. Then $\mu sh_X$ is locally isomorphic to the category of local systems on $X$.

That is, $\mu sh_X$ is a sheaf of categories of twisted local systems. The twistings are related to the Maslov obstruction and similar homotopical considerations, and are studied in [22, 27, 24, 26]. We will also consider them below.

We now consider contact isotopies. Recall that a hamiltonian isotopy $\phi_t$ on $S^*M$ determines a Lagrangian $\Phi \subset T^*M \times T^*M \times T^*R$. Using $\Phi$ as a correspondence gives a map (we also denote it $\Phi$) from subset of $T^*M$ to subsets of $T^*M \times T^*R$, and $\Phi$ is characterized by the property $\phi_t(X)$ is the symplectic reduction of $\Phi(X)$ over $t \in R$.

Definition 5.5. For a contact isotopy on $S^*M$, we obtain similarly a map $\Phi$ from subsets of $S^*M$ to subsets of $S^*M \times T^*R$ (e.g. by viewing it as a conic Hamiltonian isotopy). For $X \subset S^*M$ we term $\Phi(X)$ the symplectic reduction of $\Phi(X)$ over $t \in R$.

Lemma 5.6. Let $\phi_t$ be a contact isotopy on $S^*M$, and $\Lambda \subset S^*M$ any subset. Then there is an isomorphism $\phi_t \mu sh_{\Lambda} \cong \mu sh_{\phi_t(\Lambda)}$.

Proof. Note there is a contactomorphism $(\Phi(X), Nbd(\Phi(X))) \cong (X \times R, Nbd(X) \times T^*R)$, so we may apply Lemma 5.3 to conclude that $\mu sh_{\phi_t(\Lambda)}$ is constant in the $R$ direction. As $\Phi(\Lambda)$ is noncharacteristic for the inclusion of any $M \times t$, pullback along such an inclusion induces an isomorphism $\mu sh_{\phi_t(\Lambda)} |_{M \times t} \rightarrow \mu sh_{\phi_t(\Lambda)}$. □

We recall the stronger result:

Theorem 5.7. [23] For any contact isotopy $\phi_t$ of $S^*M$ there is a unique sheaf $K_\Phi \in sh_\Phi(M \times M \times R)$ such that $K_\Phi |_{M \times M \times 0}$ is the constant sheaf on the diagonal.

The relation of this theorem to the above discussion is that $K_\Phi |_{M \times M \times t}$ gives an integral kernel which on microsupports away from the zero section applies the contact transformation $\phi_t$. Its real strength has to do with the fact that one obtains equivalences of categories of sheaves, not just (as in Lemma 5.6) microsheaves away from the zero section.

As this result will be important to us, let us sketch the proof. Uniqueness can be seen as follows: consider the functor $sh_\Phi(M \times M \times R) \rightarrow \mu sh_\Phi(\Phi)$. Note the latter is a category of local systems. Thus if two candidate $K_\Phi$ are isomorphic at $M \times 0$, hence microlocally isomorphic along $\Phi|_0$, they must be microlocally isomorphic everywhere away from the zero section. Thus the cone between them is a local system; as it vanishes at $M \times 0$ it must be trivial. To show existence it suffices to show existence for small positive and negative isotopies (and then convolve the corresponding kernels). For a small positive or negative isotopy, arguing as Lemma 2.3 shows that the symplectic reduction of $\Phi$ at $t$ (for $t$ small) is a conormal to the boundary of a neighborhood of the diagonal. The corresponding $K_\Phi$ can be taken as the constant sheaf on this (open or closed according as the isotopy is positive or negative) neighborhood.

We will mainly use this result through its following consequence:
Corollary 5.8. Let \( \eta : M \times \mathbb{R} \to \mathbb{R} \) be the projection. Fix a contact isotopy \( \phi_t : S^*M \to S^*M \) and any \( X \subset S^*M \). Assume \( M \) is compact. Then the sheaf of categories (on \( \mathbb{R} \)) given by \( \eta_*sh_{\phi_t(X)} \) is locally constant. Pullback to \( t \in \mathbb{R} \) induces an equivalence \( (\eta_*sh_{\phi_t(X)})_t \cong sh_{\phi_t(X)}(M) \).

The hypothesis of compactness may as usual be replaced by an assumption that in the complement of a compact set, \( (M, X, \phi_t) \) are the product of some structures on a compact manifold with constant structures on a noncompact one.

6. Antimicrolocalization

Because \( \mu_{\text{sh}} \) is a quotient of sheaves of categories, hence suffers in its definition a sheafification, it is nontrivial to compute in \( \mu_{\text{sh}} \) directly. In particular, for \( X \subset S^*M \), it is not generally true that the map \( sh_X(M) \to \mu_{\text{sh}}X(X) \) is a quotient. One can often nevertheless reduce problems of microsheaf theory to problems of sheaf theory by finding some larger \( X' \subset S^*M \) for which the natural map \( sh_{X'}(M) \to \mu_{\text{sh}}X(X) \) has a right inverse. We term such an inverse an antimicrolocalization.

When \( X \) projects finitely to \( M \), a local version of this problem can be solved directly using the “refined microlocal cutoff” of [29]; we give an account in Section 6.1; similar results can be found in [53, 22].

When \( \Lambda \) is a smooth Legendrian in a jet bundle, \( \Lambda \subset J^1M \subset T^*(M \times \mathbb{R}) \), Viterbo observed that \( \Lambda \cup \Lambda' \) (the latter being a small Reeb pushoff) should provide an antimicrolocalization [51, 52]. His argument was Floer-theoretic: there is an (exact) Lagrangian \( L \cong \Lambda \times \mathbb{R} \) with \( \partial L = \Lambda \cup \Lambda' \); now \( \mu_{\text{sh}}L(L) \) is local systems on \( L \), and the map \( \mu_{\text{sh}}L \to sh_{\Lambda \cup \Lambda'}(M \times \mathbb{R}) \) is obtained by sending a given local system \( \mathcal{L} \) to the sheaf organizing the Floer theory of \( (L, \mathcal{L}) \) with cotangent fibers.

A direct sheaf-theoretical construction was later given by Guillermou [22]. Here we will prove an analogous result for a singular Legendrian \( \Lambda \) in an arbitrary cosphere bundle displaced by an arbitrary positive flow, and a generalization to the case when \( \Lambda \) is only locally closed. Even in the case of a jet bundle and smooth \( \Lambda \), the proof is new.\(^{10}\)

Remark 6.1. The fact that antimicrolocalization should exist in this generality is partially motivated by the results on “stop doubling” of [20, Ex. 8.6]. Indeed, these results show that given a Weinstein manifold \( W \) with skeleton \( \Lambda \), and an embedding as an exact hypersurface \( W \to S^*M \), then there is a fully faithful functor \( Fuk(W) \hookrightarrow Fuk(T^*M; \Lambda \cup \Lambda') \). In [21] it is shown that \( Fuk(T^*M; \Lambda \cup \Lambda') \cong sh_{\Lambda\cup\Lambda'}(M) \), and one could imagine running an analogue of Viterbo’s construction above – if one knew that \( \mu_{\text{sh}}(\Lambda) \cong Fuk(W) \). In fact [21] uses this idea in the reverse direction, applying the antimicrolocalization in order to reduce the general problem of showing \( \mu_{\text{sh}}(\Lambda) \cong Fuk(W) \) to the case of cotangent bundles. This result was originally conjectured in [36].

Remark 6.2. Note that antimicrolocalization is the special case of gapped specialization corresponding to taking the natural \( (\Lambda \times \mathbb{R}) \subset T^*M \) whose boundary is \( \Lambda \cup \Lambda' \), and flowing down by the Liouville flow.

\(^{10}\)Guillermou makes a local construction which he then proves glued. A similar construction will work for singular \( \Lambda \) in a jet bundle, but a difficulty of adapting this to the case of an arbitrary positive flow is that aside from the Reeb flow in the jet bundle, it is not clear how to construct a cover on the base \( M \) compatible with the effect of the contact flow on \( S^*M \). We instead provide a global construction.
6.1. Local antimicrolocalization. Let us recall from [29] the refined microlocal cutoff. The assertions are local so from the start we will work with \(X = \mathbb{R}^n\) a vector space and focus on the origin \(x_0 = 0\). Let \(X^* \simeq \mathbb{R}^n\) denote the dual vector space so that \(T^*X \simeq X \times X^*\).

Proposition 6.3. ([29, Proposition 6.1.4]) Let \(K \subset X^*\) be a proper closed convex cone, and \(U \subset K\) an open cone. Fix \(F \in \mathcal{S}(X)\) and \(W \subset X^*\) a conic neighborhood of \(K \cap \text{ss}(F)|_0 \setminus 0\). Then there exists \(F' \in \mathcal{S}(X)\) and a map \(u : F' \to F\) along with a neighborhood \(B \subset X\) of \(0 \in X\) such that

1. \(\text{ss}(F')|_B \subset B \times \overline{U}\) and \(\text{ss}(F')|_0 \subset W \cup 0\).
2. \(u\) induces an isomorphism in \(\mu \mathcal{S}h^\text{pre}(B \times U)\).\(^{11}\)

For further developments of the microlocal cutoff, in particular to treat the complex setting, see also [10, 53].

Corollary 6.4. In the situation above, after possibly shrinking \(B\), we have

1'. \(\text{ss}(F')|_B \subset B \times (W \cup 0)\).

Proof. If not, there exists a sequence \((x_i, \xi_i) \in \text{ss}(F')\), with \(x_i \to 0\), but \(\xi_i \not\in W \cup 0\). Since \(W\) and \(\text{ss}(F')\) are conic, we may assume \(|\xi_i| = 1\) and the sequence converges \(\xi_i \to \xi_\infty\). Since \(\text{ss}(F')\) is closed, we have \((0, \xi_\infty) \in \text{ss}(F')|_0 \subset W \cup 0\) with \(|\xi_\infty| = 1\), hence \(\xi_\infty \in W\). But \(W\) is open so ultimately \(\xi_i \in W\).

The proof given in [29, Proposition 6.1.4] involves \(F \in \mathcal{S}(X)\) only through its singular support \(\text{ss}(F) \subset T^*X\), and the sheaf \(F'\) is constructed functorially. In addition the cutoff functor makes sense for any \(F\); only its properties depend on the asserted condition on the microsupport of \(F\). In other words what their argument actually shows is:

Proposition 6.5. Suppose given

- \(K \subset X^*\) a proper closed convex cone
- \(U \subset K\) an open cone
- \(\Lambda \subset T^*X\) a closed conic subset
- \(W \subset X^*\) a conic neighborhood of \(K \cap \Lambda|_0 \setminus 0\).

Then there is a map \(\phi : \mathcal{S}(X) \to \mathcal{S}(X)\) with a natural transformation \(u : \phi \to \text{id}\), such that for a small enough neighborhoods \(B\) of \(0 \in X\)

1. For any \(F\), \(\text{ss}(\phi(F))|_B \subset B \times (W \cup 0)\).
2. If \(\text{ss}(F) \cap (B \times U) = \Lambda \cap (B \times U)\), then \(u : \phi(F) \to F\) induces an isomorphism in \(\mu \mathcal{S}h^\text{pre}(B \times U)\).

Corollary 6.6. In the situation of Proposition 6.5, suppose in addition \(W \subset U\). Then for any \(0 \in A \subset X\), and for any small enough \(B \supset 0\), the microlocal cutoff induces a functor

\[\phi : \mu \mathcal{S}h^\text{pre}_\Lambda(A \times U) \longrightarrow \mathcal{S}h_\Lambda(B)/\mathcal{S}h_{T^*_X B}(B)\]

commuting with the natural projections on both sides to \(\mu \mathcal{S}h^\text{pre}_\Lambda(B \times U)\).

Proof. We apply Proposition 6.5 with \(X\) replaced by its subset \(A\). We use the superscript \(\infty\) to denote the real projectivization of the complement of the zero section of a subset of \(T^*X\). Now for any \(F\), we have

\[\text{ss}^\infty(\phi(F))|_B \subset B \times W^\infty \subset B \times U^\infty\] by property (1)

\(^{11}\)In [29] it is stated “induces an isomorphism on \(U\)”. However from the proof, it is clear that the present assertion is what is meant.
If \( ss^\infty(\mathcal{F}) \cap (B \times U^\infty) = \Lambda \cap (B \times U^\infty) \) then also
\[
ss^\infty(\phi(\mathcal{F})) \cap (B \times U^\infty) = ss^\infty(\mathcal{F}) \cap (B \times U^\infty)
\]by property (2).

Thus under this microsupport condition, \( ss^\infty(\phi(\mathcal{F})) \cap (B \times U^\infty) = ss^\infty(\mathcal{F}) \cap (B \times U^\infty) \).

In particular if \( ss^\infty(\mathcal{F}) \cap (A \times U^\infty) = \emptyset \), then \( ss^\infty(\phi(\mathcal{F}))|_B = \emptyset \), so \( \phi \) factors through \( \mu\text{sh}^\text{pre}(A \times U) \) as stated. Additionally if \( ss^\infty(\mathcal{F}) \cap (A \times W) \subset \Lambda \) then \( ss^\infty(\phi(\mathcal{F}))|_B \subset B \times \Lambda \), giving the microsupport condition on the image.

Finally, the assertion that \( \phi \) commutes with the projections follows from the fact that \( u : \phi \rightarrow \text{id} \) induces an equivalence on \( B \times U \).

Now let us use this to prove the following. (See [53, Sect. 5.1] for similar arguments in the complex setting.)

**Lemma 6.7.** Fix a closed subset \( \Lambda \subset S^*X \). Suppose for a point \( x \in X \), the fiber \( \Lambda|_x = \Lambda \cap S^*_xX \) is a single point \( \lambda \). Then the natural map of sheaves of categories
\[
q : \text{sh}_{\Lambda}/\text{sh}_{T^*_XX} \longrightarrow \pi_*\mu\text{sh}_\Lambda
\]
is an isomorphism at \( x \).

**Proof.** The assertion is local so we may assume \( X = \mathbb{R}^n, x = 0 \). Thus \( T^*X \simeq X \times X^* \) and \( S^*X = X \times X^\infty \) where \( X^* \) is the dual of \( X \), and \( X^\infty = (X^* \setminus 0)/\mathbb{R}_{>0} \) is its projectivization.

We will Proposition 6.5 with \( U = W \subset X^* \) an open convex conic neighborhood of the coray \( \mathbb{R}_{>0} \cdot \lambda \subset X^* \), and \( K = \overline{U} = \overline{W} \subset X^* \) the closure. Note we are able to make these choices precisely due to the assumption that \( \Lambda|_x = \lambda \).

For any open neighborhood \( A \subset X \) of \( 0 \in X \), and small enough open neighborhood \( B \subset X \) of \( 0 \in X \), we have from Cor. 6.6 a functor
\[
\phi : \mu\text{sh}^\text{pre}_\Lambda(A \times U) \longrightarrow \text{sh}_\Lambda(B)/\text{sh}_{T^*_XX}(B)
\]

The rest of the argument follows the complex setting detailed in [53, Sect. 5.1]. Next, as in [53, Lemma 5.1.4], one observes the composed functor
\[
\mu\text{sh}^\text{pre}_\Lambda(A \times U) \xrightarrow{\phi} \text{sh}_\Lambda(B)/\text{sh}_{T^*_XX}(B) \xrightarrow{(\text{sh}_{\Lambda}/\text{sh}_{T^*_XX})|_x}
\]
factors through the natural map
\[
\mu\text{sh}^\text{pre}_\Lambda(A \times U) \longrightarrow \pi_*\mu\text{sh}^\text{pre}_\Lambda|_x
\]
For this, let us write \( \phi_{U,A} \) to convey its dependence on the choices. Given smaller \( U' \subset U \), \( A' \subset A \), by construction there is a natural map \( \phi_{U',A'} \rightarrow \phi_{U,A} \) which induces an isomorphism in \( (\text{sh}_{\Lambda}/\text{sh}_{T^*_XX})|_x \) since its cone must have singular support in the zero-section.

Finally, one checks the resulting map
\[
\phi_x : \pi_*\mu\text{sh}^\text{pre}_\Lambda|_x \longrightarrow (\text{sh}_{\Lambda}/\text{sh}_{T^*_XX})|_x
\]
is an inverse to \( q_x \). First, we have \( q_x \circ \phi_x \simeq \text{id} \) thanks to property (2). Second, we have \( \phi_x \circ q_x \simeq \text{id} \) since the cone of the natural morphism relating them must have singular support in the zero-section.

**Remark 6.8.** The \( \Lambda \) to which this is applied in practice will typically have enough tameness that \( \mu\text{sh}^\text{pre}_\Lambda|_x \) is computed by some particular \( \mu\text{sh}^\text{pre}_\Lambda(B \times U) \), i.e. one need not pass to the limit.
Now let us use the lemma to prove the following generalization.

**Proposition 6.9.** Fix a closed singular Legendrian $\Lambda \subset S^*X$. Suppose for a point $x \in X$, the fiber $\Lambda|_x = \Lambda \cap S^*_xX$ is a finite set of points $\lambda_1, \ldots, \lambda_k$. Then the natural map of sheaves of categories

$$q : sh_{\Lambda}/sh_{T^*_X X} \longrightarrow \pi_*\mu sh_{\Lambda}$$

is an isomorphism at $x$.

**Proof.** Let $\Lambda_1, \ldots, \Lambda_k \subset \Lambda$ be the connected components through the respective points $\lambda_1, \ldots, \lambda_k \in \Lambda|_x$. Note the evident direct sum decomposition

$$\pi_*\mu sh_{\Lambda}|_x \simeq \bigoplus_{i=1}^k \pi_*\mu sh_{\Lambda_i}|_x$$

Thus by the prior lemma, it suffices to show the natural map

$$\bigoplus_{i=1}^k sh_{\Lambda_i}/sh_{T^*_X X}|_x \longrightarrow sh_{\Lambda}/sh_{T^*_X X}|_x$$

is an equivalence.

Set $\Lambda' = \bigcup_{i=1}^{k-1} \Lambda_i, \Lambda'' = \Lambda_k$. By induction, it suffices to show the natural map

$$i' \oplus i'' : sh_{\Lambda'}/sh_{T^*_X X}|_x \oplus sh_{\Lambda''}/sh_{T^*_X X}|_x \longrightarrow sh_{\Lambda}/sh_{T^*_X X}|_x$$

is an equivalence. Let us first show the images of $i', i''$ are orthogonal to each other. For the natural map

$$q' \oplus q'' : sh_{\Lambda}/sh_{T^*_X X}|_x \longrightarrow \pi_*\mu sh_{\Lambda'|_x} \oplus \pi_*\mu sh_{\Lambda''}|_x$$

we have $q' \circ i', q'' \circ i''$ are equivalences, and $\text{im}(i') = \ker(q''), \text{im}(i'') = \ker(q')$. This implies the desired orthogonality since for example

$$\text{Hom}(i'F', i''F'') \simeq \text{Hom}(i'F', (q'')^r q''i''F'') \simeq \text{Hom}(q''i'F', q''i''F'') \simeq 0$$

where $(q'')^r$ denotes the right adjoint to $q''$.

Finally, it remains to show $i' \oplus i''$ is essentially surjective. For $F \in sh_{\Lambda}/sh_{T^*_X X}|_x$, the above identities imply we have a triangle

$$i'(i')^r F \longrightarrow F \longrightarrow (q'')^r q''F$$

where $(i')^r$ denotes the right adjoint to $i'$. Since $\text{im}((q'')^r) = \text{im}(i'')$, this implies $i' \oplus i''$ is essentially surjective. \qed

**Definition 6.10.** We say $\Lambda \subset S^*X$ is in **finite position** over $x \in X$ if it satisfies the hypothesis of Proposition 6.9 over $x$. When this is true at all $x$, we say $\Lambda$ is in **finite position**. If $\Lambda$ may be perturbed to have finite projection by a contact isotopy, we say $\Lambda$ is **perturbable to finite position**.

Finally, for $\Lambda$ a closed subset equipped with the germ of an embedding into a contact manifold, we say $\Lambda$ is **perturbable to finite position** if every codimension zero contact embedding of $\Lambda$ into a cosphere bundle is perturbable to finite position.

**Remark 6.11.** It is easy to see that a subset with an isotropic Whitney stratification is perturbable to finite position.
6.2. Cusps and doubling. Let $M$ be a manifold, and consider $\Lambda \subset S^*M$. Fix a positive contact isotopy $\phi_t$ on $S^*M$. For $s \geq 0$, let $\tilde{\phi}_s^{\pm} = \phi_{\pm s^{2/3}}$, and

$$\Lambda_{\pm s} := \tilde{\phi}_s^+(\Lambda) \subset S^*M \quad \Lambda^- = \tilde{\phi}_s^+(\Lambda) \cup \tilde{\phi}_s^-(\Lambda) \subset S^*(M \times \mathbb{R})$$

Note the contact reduction of $\Lambda^-$ over $s \geq 0$ is $\Lambda_s^+ \cup \Lambda_s^-$, and is empty for $s < 0$. Also observe the germ of $S^*(M \times \mathbb{R})$ around $\Lambda^-$ is locally contactomorphic to $\Lambda \times \mathbb{R} \subset \text{Nbd}(\Lambda) \times T^*\mathbb{R}$. In particular, $\mu_{\Lambda^-}$ is the pullback of $\mu_{\Lambda_s^+}$.

If $\Lambda$ was a point in $S^*\mathbb{R}$ and we take the Reeb flow, then the front projection of $\Lambda^-$ is the standard cusp $y^2 = x^3$. This example is studied in some detail by elementary arguments in [48, Sec. 3.3] and [40, Sec. 7.3.3].

We note the following key fact:

**Lemma 6.12.** Objects in $\text{sh}_{\Lambda^-}(M \times \mathbb{R})$ are locally constant over $M \times \mathbb{R}_{\leq 0}$.

**Proof.** This is obvious over $M \times \mathbb{R}_{\leq 0}$. Over $M \times 0$, note that $\Lambda^-$ has no conormals in the $ds$ directions, so noncharacteristic propagation shows that stalks agree with those over $M \times \mathbb{R}_{< 0}$. □

We write $\text{sh}_{\Lambda^-;0} \subset \text{sh}_{\Lambda^-}$ for the sheaf of full subcategories of sheaves (on $M \times \mathbb{R}$, micro-supported at infinity in $\Lambda^-$) with vanishing stalks near $M \times -\infty$ (and hence over $M \times \mathbb{R}_{\leq 0}$).

**Lemma 6.13.** Let $j : M \times \mathbb{R}_{> 0} \to M \times \mathbb{R}$ be the inclusion. Then the natural map of sheaves of categories

$$j^!j^! : \text{sh}_{\Lambda^-;0} \to j_*j^*\text{sh}_{\Lambda^-;0}$$

(induced on objects from the usual sheaf functor $j^!j^!$) is an equivalence.

**Proof.** There is nothing to prove except along $M \times 0$. Along this locus the result follows from the fact that the natural morphism on sheaves $j^!j^!F \to F$ is zero for any sheaf which vanishes over $M \times \mathbb{R}_{\leq 0}$. This vanishing holds for the sheaves in question by Lemma 6.12. □

We now consider the map $\eta : M \times \mathbb{R} \to M$ and the sheaf of categories on $\mathbb{R}$ given by $\eta_*\text{sh}_{\Lambda^-;0}$.

**Proposition 6.14.** The sheaf $\eta_*\text{sh}_{\Lambda^-;0}$ is zero on $(-\infty, 0)$ and constant on $[0, c_1)$, where $c_1$ is a (universal positive function of the) length of the shortest chord for the flow of $\Lambda$ under $\phi_t$.

**Proof.** Vanishing over $(-\infty, 0)$ is obvious. Over $[0, c_1)$ we may apply a contact isotopy to move $\Lambda_{\pm s}$ in order to trace out $\Lambda$; note this becomes impossible exactly at $c_1$. Thus local constancy follows from Cor 5.8. Finally, as we saw in Lemma 6.13, objects in $\text{sh}_{\Lambda^-;0}$ are all $!$-extensions of from objects in $\text{sh}_{\Lambda^-;0}(M \times (0, \infty))$. So constancy at zero follows from full faithfulness of the $!$-extension. □

**Corollary 6.15.** If $\Lambda$ is compact, then for any $0 < s' < s < c_1$, the following natural restriction maps are all isomorphisms.

$$\text{sh}_{\Lambda^-;0}(M \times \mathbb{R}_{\leq s}) \xrightarrow{\sim} \text{sh}_{\Lambda^-;0}(M \times \mathbb{R}_{< s}) \xrightarrow{\sim} \text{sh}_{\Lambda^-;0}(M \times \mathbb{R}_{< s'}) \xrightarrow{\sim} \text{sh}_{\Lambda^-;0}|_{M \times 0}(M)$$

In addition, restriction at $s$ gives a fully faithful map:

$$\text{sh}_{\Lambda^-;0}(M \times \mathbb{R}_{\leq s}) \hookrightarrow \text{sh}_{\Lambda \cup \Lambda_{-s}}(M)$$

**Proof.** Proper base change (of sheaves of categories). □
Let \( \pi : S^*M \to M \) be the projection, and likewise \( \tilde{\pi} : S^*(M \times \mathbb{R}) \to M \times \mathbb{R} \).

**Lemma 6.16.** Assume \( \Lambda \) is in finite position. Then the natural map of sheaves of categories on \( M \times \mathbb{R} \)
\[
sh_{\Lambda^<:0} \to \tilde{\pi}_* \mu sh_{\Lambda^<}
\]
is an equivalence along \( M \times 0 \).

**Proof.** It is enough to check at stalks. Evidently if \( \Lambda \) is in finite position, then \( \Lambda^< \) is in finite position over 0. By Prop 6.9, the stalk of \( \mu sh \) is the orthogonal complement to constant sheaves in a neighborhood. Along \( M \times 0 \), this orthogonal complement is \( sh_{\Lambda^<:0} \) by Lemma 6.13. \( \square \)

**Theorem 6.17.** Assume that \( \Lambda \) is perturbable to finite position. For \( s < c_1 \), the composition
\[
sh_{\Lambda^<:0}(M \times \mathbb{R}_\leq s) \to sh_{\Lambda^<:0}(M) \to \mu sh_{\Lambda^<:0}(\Lambda^\prec)
\]
is an equivalence. In particular, the second map has a right inverse.

**Proof.** By [23] the statement is invariant under contact perturbation (of \( \Lambda \), before forming \( \Lambda^\prec \)), so we may assume \( \Lambda \) is in finite position.

We have the commutative diagram
\[
\begin{array}{ccc}
sh_{\Lambda^<:0}(M \times \mathbb{R}_\leq s) & \xrightarrow{\text{Cor. 6.15}} & sh_{\Lambda^<:0}(M) \\
\downarrow \sim & & \downarrow \sim \\
sh_{\Lambda^<:0}|_{M \times 0}(M) & \xrightarrow{\text{Lemma 6.16}} & \tilde{\pi}_* \mu sh_{\Lambda^<}|_{M \times 0}(M) \\
\end{array}
\]

The labelled arrows are fully faithful or equivalences by the noted previous results, or in the case of comparisons between \( \mu sh \) categories, by Lemma 5.3. The result follows. \( \square \)

It follows from Thm 6.17 that the image of the map \( sh_{\Lambda^<:0}(M \times \mathbb{R}_\leq s) \to sh_{\Lambda^<:0}(M) \) is the subcategory generated by microstalks on \( \Lambda^\prec \). We now discuss how to give a more explicit characterization (which however we will not require in the sequel). By inspection of the [23] construction, the image is always contained in the locus of sheaves whose microsupport is contained in the locus
\[
U_s := \mathbb{R}_{>0}\Lambda_s \cup \mathbb{R}_{>0}\Lambda_{-s} \cup \bigcup_{-s \leq r \leq s} \pi(\Lambda_r) \subset T^*M
\]

**Example 6.18.** It is not always the case that the map to \( sh_{U_s} \) is surjective. Indeed, consider \( M = \mathbb{R} \) and \( \Lambda \) both conormals to 0. Now \( \mu sh_{\Lambda} \) is two copies of the coefficient category, while \( sh_{U_s} \) is representations of the \( A_3 \) quiver. An object not in the image is the skyscraper at 0.

The above is essentially the only difficulty, and in its absence we can characterize the image.

**Proposition 6.19.** Assume that \( \Lambda \) is in finite position, and \( c_1 > 0 \). Assume in addition that \( \Lambda \) is the closure of the locus of points where the front projection is an injection, or that \( M = \mathbb{N} \times \mathbb{R} \) and \( \Lambda \subset J^1 \mathbb{N} \subset S^*(\mathbb{N} \times \mathbb{R}) \). Then the map
\[
sh_{\Lambda^<:0}(M \times \mathbb{R}_\leq s) \to sh_{U_s}(M)
\]
is an isomorphism for small \( s \).
We define

\[ \Lambda = \Lambda \setminus \text{its closure} \]

This would not be useful as \( \Lambda \) is false for this \( \Lambda \) (the argument

\[ T \text{ onto } \pi \text{ also zero in the complement of } \Lambda. \]

relative doubling.

We fix some standard capping of the parabola in

\[ \text{ its image is a (smooth) hypersurface with ‘only one conormal’ direction. It follows that any } \]

\[ \text{ objects also must vanish along said image, hence be microsupported in the complement of a } \]

\[ \partial \text{ alize this to the case of locally closed } \Lambda. \]

We write

\[ \Lambda \subseteq \bigcup_{s \leq t \leq s} \pi(\Lambda_r) \]

this moreover evidently contains the corresponding locus for all smaller \( s \), we can cutoff the contact isotopy so it is constant above some locus in the complement of each component of \( \pi(\Lambda) \).

It follows that any object in the image of \( sh_{U_r}(M) \to sh_{\Lambda_{U_r}}(M \times R_{\leq s}) \) is also zero in the complement of \( \pi(\Lambda) \).

Since \( \Lambda \) is in generic position, then on a dense locus its image is a (smooth) hypersurface with ‘only one conormal’ direction. It follows that any objects also must vanish along said image, hence be microsupported in the complement of a dense set in the Legendrian subvariety \( \Lambda \), hence be zero.

### 6.3. Relative doubling.

Previously we have considered closed \( \Lambda \subset S^*M \).

We now generalize this to the case of locally closed \( \Lambda \).

We write \( \partial \Lambda := \overline{X} \setminus \Lambda \) (as opposed to \( \overline{X} \) minus its relative interior).

The need for a generalization is illustrated by the following example:

**Example 6.20.** Recall that the microsupport of a sheaf can never be a manifold with nontrivial boundary.

Consider \( M = R^2 \), and \( \Lambda \subset J^1R \subset S^*(R^2) \) with front projection the open interval \( (0,1) \times \{0\} \subset R^2 \).

Then while \( \mu sh_\Lambda(\Lambda) \) is the coefficient category, we have \( sh_{U_r}(R^2) = 0 \) by the above recollection. Thus the assertion of Theorem 6.17 is false for this \( \Lambda \) (the argument fails because Lemma 6.16 required \( \Lambda \) to be closed).

While we could apply Theorem 6.17 to the closure of \( \Lambda \), this would not be useful as \( \mu sh_{\overline{X}}(\Lambda) = 0 \) (again by the recollection above).

The solution is to cap off the boundaries of the doubled object.

**Definition 6.21.** A contact collar of a subset \( \Lambda \subset S^*M \) is a contact manifold \( U \) with a subset \( L \), and an embedding \( (L \times (0,\varepsilon); U \times T^*(-\varepsilon,\varepsilon)) \to (\Lambda, S^*M) \) carrying \( L \times 0 \) diffeomorphically onto \( \partial \Lambda \).

We say a contact flow on \( S^*M \) is compatible with the collar if its restriction to \( U \times T^*(-\varepsilon,\varepsilon) \) is the product of a flow on \( U \) and the constant flow on \( T^*(-\varepsilon,\varepsilon) \).

For collared \( \Lambda \), we define \( (\Lambda, \partial \Lambda)^\prec \subset S^*(M \times R) \) as the union of \( \Lambda^\prec \) with a ‘boundary capping off’ component contained entirely inside \( U \times T^*(-\varepsilon,\varepsilon) \times T^*R \). This component is constructed as follows. Note that \( \Lambda^\prec \) in this region is \( L \times [0,\varepsilon) \) times a parabola \( P \subset T^*R \). We fix some standard capping of the parabola in \( T^*R \) to a Legendrian half-paraboloid \( \tilde{P} \subset T^*(-\varepsilon,0] \times T^*R \), so that \( \tilde{P} \cup_P ([0,\varepsilon) \times P) \subset T^*(-\varepsilon,\varepsilon) \times T^*R \) is a manifold.\(^{12}\) We define

\[ \Lambda^\prec = \Lambda^\prec \cup (L \times \tilde{P}) \]

Note that by construction \( \Lambda^\prec \subset (\Lambda, \partial \Lambda)^\prec \), and in the fiber over \( M \times 0 \) the latter is the closure of the former.

\(^{12}\)It is enough for our purposes that this be a \( C^1 \) manifold; thus the construction can be made in the subanalytic category if desired.
Example 6.22. If $M = \mathbb{R}^2$ and $\Lambda$ is a Legendrian closed interval whose front projection is an embedding, then $(\Lambda, \partial \Lambda)^{\prec}$ is a disk whose front projection is (half of) a somewhat squashed flying saucer.

Remark 6.23. The reason for the delicate construction of the capping is that we want to verify local constancy of $\mu sh$ along the boundary by simply invoking Lemma 5.3. The local constancy would still be true with a less delicate capping, but one would have to argue for it directly.

Let us write $(\Lambda, \partial \Lambda)_s \subset S^*M$ for the contact reduction of $(\Lambda, \partial \Lambda)^{\prec} \subset S^*(M \times \mathbb{R})$ over $s \in \mathbb{R}$.

Lemma 6.24. The restriction of $(\Lambda, \partial \Lambda)^{\prec}$ to $S^*(M \times \mathbb{R} > 0)$ is the legendrian movie of a symplectic isotopy acting on $(\Lambda, \partial \Lambda)_s$.

Analogues of the results of the previous section hold for $(\Lambda, \partial \Lambda)^{\prec}$. Again writing $\eta : M \times \mathbb{R} \to M$, we have

Proposition 6.25. The sheaf $\eta_\ast sh((\Lambda, \partial \Lambda)^{\prec}; 0)$ is zero on $(-\infty, 0)$ and constant on $[0, c_1)$, where $c_1$ is the length of the shortest nonzero chord for the flow of $\Lambda$ under $\phi_t$.

Proof. Same as Proposition 6.14, with the additional observation that the capping off at the boundary adds no chords. □

Corollary 6.26. If $\Lambda$ is relatively compact, then for any $0 < s' < s < c_1$, the following natural restriction maps are all isomorphisms.

$$sh((\Lambda, \partial \Lambda)^{\prec}; 0)(M \times \mathbb{R} \leq s) \xrightarrow{\sim} sh((\Lambda, \partial \Lambda)^{\prec}; 0)(M \times \mathbb{R} < s) \xrightarrow{\sim} sh((\Lambda, \partial \Lambda)^{\prec}; 0)|_{M \times 0}(M)$$

In addition, restriction at $s$ induces a fully faithful functor

$$sh((\Lambda, \partial \Lambda)^{\prec}; 0)(M \times \mathbb{R} \leq s) \hookrightarrow sh((\Lambda, \partial \Lambda)_s)(M)$$

Proof. Proper base change (of sheaves of categories). □

Lemma 6.27. Assume $\Lambda$ is collared and in finite position. Then the natural maps of sheaves of categories on $M \times \mathbb{R}$

$$sh((\Lambda, \partial \Lambda)^{\prec}; 0) \to \tilde{\pi}_\ast \mu sh((\Lambda, \partial \Lambda)^{\prec}) \to \tilde{\pi}_\ast \mu sh_\Lambda^{\prec}$$

are both equivalences along $M \times 0$.

Proof. Note that since $\Lambda$ is in finite position, so is $(\Lambda, \partial \Lambda)^{\prec}$ over 0. Now the first map is an isomorphism by the same proof as Lemma 6.16. The second map is an isomorphism because (1) over $M \times 0$ the loci $\Lambda^{\prec} \subset (\Lambda, \partial \Lambda)^{\prec}$ agree except along the boundary where one is topologically $L \times (0, \epsilon)$ and the other $L \times [0, \epsilon)$. and (2) $\mu sh$ is locally constant along the $[0, \epsilon)$ direction by Lemma 5.3. □

Theorem 6.28. Assume $\Lambda$ is collared, relatively compact, and perturbable to finite position. For $s < c_1$, the composition

$$sh((\Lambda, \partial \Lambda)^{\prec}; 0)(M \times \mathbb{R} \leq s) \to sh((\Lambda, \partial \Lambda)_s)(M) \to \mu sh_{\Lambda^{-s}}(\Lambda^{-s})$$

is an equivalence. In particular, the second map has a right inverse.

Proof. Similar to the proof of Theorem 6.17. □
Proposition 6.29. Proposition 6.19 holds for any locally closed, relatively compact, and collared $\Lambda$, with

$$U_s := \mathbb{R}_{>0}(\Lambda, \partial \Lambda)_s \cup \mathbb{R}_{>0}(\Lambda, \partial \Lambda)_{-s} \cup \bigcup_{-s \leq r \leq s} \pi((\Lambda, \partial \Lambda)_r) \subset T^*M$$

Proof. Same as Proposition 6.19. \qed

6.4. Two sided double. Our previous constructions involved $M \times \mathbb{R}$ or $M \times [0, \epsilon]$, and microsupport going to infinity or the boundary. While these suffice for our purposes in this article, we give here a variant of the doubling construction which avoids this defect. This can be technically convenient when invoking theorems stated for compact manifolds; and will be the version of antimicrolocalization invoked in [21]. This subsection owes its existence to discussions with the authors of that article.

The prototype of the construction is to begin with a Legendrian point in the contact manifold $\mathbb{R}$, and produce the standard Legendrian unknot, rather than simply half of it as before. We just use two copies of our construction above: first begin forming $\Lambda^\prec$ or $(\Lambda, \partial \Lambda)^\prec$ as before, but near some time $s < c_1$, smoothly cutoff the Reeb pushoff so that in the region $t \in (s - \epsilon, s]$ one has $(\Lambda, \partial \Lambda)_{+t}$ independent of $t$. Finally, reverse the process so that the Legendrian for $t > s$ is just the reflection of the Legendrian when $t < s$.

We denote the resulting Legendrian as $(\Lambda, \partial \Lambda)^{\prec \succ} \subset S^*(M \times \mathbb{R})$. (Despite the notation, the corresponding front will be connected and rounded at the top and bottom.) Note this is supported over a compact subset of $\mathbb{R}$; fixing an inclusion $\mathbb{R} \subset S^1$ we may view $(\Lambda, \partial \Lambda)^{\prec \succ} \subset S^*(M \times S^1)$. As before we write $\text{Sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}(M \times S^1)_0$ for the full subcategory of sheaves with vanishing stalks over $S^1 \setminus \mathbb{R}$.

Theorem 6.30. The category $\text{Sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}(M \times S^1)_0$ is the (left and/or right) orthogonal complement to the category of local systems on $M \times S^1$. Moreover, the following natural morphisms are equivalences:

$$\text{Sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}(M \times S^1)_0 \xrightarrow{\sim} \mu \text{sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}((\Lambda, \partial \Lambda)^{\prec \succ}) \xrightarrow{\sim} \mu \text{sh}_{\Lambda}(\Lambda)$$

Proof. Let us first see that any $\mathcal{F} \in \text{Sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}(M \times S^1)_0$ is orthogonal to all local systems. By construction $\mathcal{F}$ is supported over some interval in $S^1$; thus morphisms between $\mathcal{F}$ and a local system will factor through some $\mathcal{G}$ which is a local system on $M$ times the constant sheaf on an open or closed interval $I$. Away from the zero section, the microsupport of $\mathcal{G}$ is entirely in the $dt$ direction ($t$ the coordinate on $S^1$), above $\partial I$. Shrinking $I$ to an interval entirely disjoint from the support of $\mathcal{F}$ and correspondingly propagating $\mathcal{G}$ is a noncharacteristic homotopy since $(\Lambda, \partial \Lambda)^{\prec \succ}$ is by construction $S^1$-noncharacteristic, i.e. disjoint from $dt$ covectors. Thus we see that $\text{Sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}(M \times S^1)_0$ is contained in both the left and right orthogonal complements to the local systems. For the converse inclusion, consider now any $\mathcal{F} \in \text{Sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}(M \times S^1)_0$. Let $\mathcal{L}_\mathcal{F}$ be the local system on $M$ given by restricting $\mathcal{F}$ to $M \times (S^1 \setminus \mathbb{R})$. Then the same noncharacteristic propagation argument shows that the identity induces morphisms in both directions between $\mathcal{F}$ and $\mathcal{L}_\mathcal{F}$, which are zero only if $\mathcal{L}_\mathcal{F}$ vanishes.

Regarding the morphisms in the second assertion of the proposition, the above orthogonality implies that the first is an isomorphism. Meanwhile the composite $\text{Sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}(M \times S^1)_0 \rightarrow \mu \text{sh}_{\Lambda}(\Lambda)$ factors through

$$\text{Sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}(M \times S^1)_0 \rightarrow \text{Sh}_{(\Lambda, \partial \Lambda)^{\prec \succ}}(M \times \mathbb{R}_{\leq \epsilon})_0 \xrightarrow{\sim} \mu \text{sh}_{\Lambda}(\Lambda)$$
where we learned that the second morphism is an isomorphism in Theorem 6.28. Thus it suffices to show the first morphism is an isomorphism. If \( (\Lambda, \partial \Lambda)^{\sim} \) is supported above \([0, T] \subset \mathbb{R} \subset S^1\), then the same argument as in Lemma 6.13 shows that \( Sh_{(\Lambda, \partial \Lambda)}^{\sim} (M \times S^1)_0 \) is restricts by an equivalence to \([0, T]\), and the same argument as in Corollary 6.15 shows that this is in turn restricts by an equivalence to \([0, \epsilon]\).

\[ \square \]

7. Gapped specialization of microlocal sheaves

We are interested here in limits of contact isotopies.

**Definition 7.1.** Let \( Y \) be a manifold and \( Z_t \) a family of subsets of \( Y \) defined for \( t \in (0, 1] \). We write

\[ Z_0 := \lim_{t \to 0} Z_t := \bigcup t \times Z_t \cap (0 \times Y) \]

where the closure and intersection are taken in \([0, 1] \times Y\).

**Remark 7.2.** We will be interested in the case when \( \phi_t \) is a contact isotopy, and the \( Z_t \) are \( \phi_t(Z) \). In this case we could also form a similar construction on the contact movie of \( \phi_t \).

The symplectic reduction of the zero fiber of the closure of the movie is contained in, but not in general equal to, the limit above.

Using the results on antimicrolocalization, we now give a microlocal version of the theorem on gapped specialization (Theorem 4.2).

**Theorem 7.3.** Consider \( \Lambda_1 \subset S^*M \), which is either compact or locally closed, relatively compact, and collared. Let \( \phi_t : S^*M \to S^*M \) be a contact isotopy for \( t \in (0, 1] \), compatible with the collar. Let \( \tilde{\Lambda} \subset S^*M \times T^*(0, 1] \subset S^*(M \times (0, 1]) \) be the contact movie, and let \( \Lambda_0 = \lim_{t \to 0} \phi_t(\Lambda_1) \subset S^*M \). Assume:

1. For some contact form on \( S^*M \), the family \( \Lambda_t \) (including \( t = 0 \)) is gapped.
2. Both \( \Lambda_0 \) and \( \Lambda_1 \) may be perturbed to project finitely to \( M \).
3. \( \Lambda_0 \) is positively displaceable from legendrians.

Then antimicrolocalization and nearby cycles induce a fully faithful functor

\[ \mu sh_{\Lambda_1}(\Lambda_1) \to \mu sh_{\Lambda_0}(\Lambda_0) \]

**Proof.** We will use the gapped hypothesis both in constructing the functor, and in proving full faithfulness. Let \( \eta_t \) be the Reeb flow for the given contact form. Let \( \tilde{\eta}_t \) be the lift of \( \eta_t \) to \( S^*M \times T^*(0, 1] \); this flow is positive (as recalled in Lemma 2.16).

As we assumed \( \phi \) compatible with the collar, \( \Lambda_0 \) is also collared (in a compatible way). We form \( (\tilde{\Lambda}, \partial \tilde{\Lambda})^{\sim} \subset S^*(M \times (0, 1] \times \mathbb{R}) \) and \( (\Lambda_0, \partial \Lambda_0)^{\sim} \subset S^*(M \times \mathbb{R}) \). Note that \( (\Lambda_0, \partial \Lambda_0)^{\sim} \) is the closure at zero of the projection of \((\tilde{\Lambda}, \partial \tilde{\Lambda})^{\sim}\) to \( S^*((M \times (0, 1]) \times \mathbb{R}) \). We have the diagram:

\[
\begin{array}{cccccc}
\mu sh_{\Lambda_1}(\Lambda_1) & \sim & \mu sh_{(\Lambda_1, \partial \Lambda_1)^{\sim};(0)}(M \times \mathbb{R} \leq \epsilon) & \sim & \mu sh_{(\Lambda_1, \partial \Lambda_1)^{\sim}}(M) \\
\sim & \uparrow & \sim & \sim & \\
\mu sh_{\tilde{\Lambda}}(\tilde{\Lambda}) & \sim & \mu sh_{(\tilde{\Lambda}, \partial \tilde{\Lambda})^{\sim};(0)}(M \times (0, 1] \times \mathbb{R} \leq \epsilon) & \sim & \mu sh_{(\tilde{\Lambda}, \partial \tilde{\Lambda})^{\sim}}(M \times (0, 1])
\end{array}
\]

\[ (16) \]

\[
\begin{array}{cccccc}
\mu sh_{\Lambda_0}(\Lambda_0) & \sim & \mu sh_{(\Lambda_0, \partial \Lambda_0)^{\sim};(0)}(M \times \mathbb{R} \leq \epsilon) & \sim & \mu sh_{(\Lambda_0, \partial \Lambda_0)^{\sim}}(M) \\
\downarrow \psi & & \downarrow \psi & & \\
\mu sh_{\Lambda_0}(\Lambda_0) & \sim & \mu sh_{(\Lambda_0, \partial \Lambda_0)^{\sim};(0)}(M \times \mathbb{R} \leq \epsilon) & \sim & \mu sh_{(\Lambda_0, \partial \Lambda_0)^{\sim}}(M)
\end{array}
\]
The upward arrows are isomorphism by \cite{23} (the leftmost one alternatively by Lemma 5.3). The top and bottom right horizontal arrows are the restriction at $\epsilon$, and are fully faithful by Cor. 6.26. The top and bottom left horizontal arrows are equivalences by Theorem 6.28. (If $\Lambda$ is closed we may use the easier Cor. 6.15 and Theorem 6.17.) Note we are using gappedness including at $\Lambda_0$ in order to apply antimicrolocalization to $\Lambda_0$ using the same $\epsilon$ as for the family.

The downward arrows are induced by nearby cycles, and the image sheaves have the stated microsupports by the standard estimate (Lemma 3.16). By Theorem 4.2, the right downward arrow is fully faithful; it follows formally that the middle is as well. Following around the diagram we find the desired fully faithful functor $\mu_{sh\tilde{\Lambda}}(\tilde{\Lambda}) \hookrightarrow \mu_{sh\Lambda_0}(\Lambda_0)$. \hfill $\Box$

**Remark 7.4.** In the above diagram, we use the top row to avoid quoting theorems for the middle row due to its noncompactness and also because we do not want to check hypotheses for $\tilde{\Lambda}$. We use the middle column to avoid explicitly characterizing the images in the right column; alternatively we could use Proposition 6.29. We use the right column because the gappedness is more evident there than in the middle column.

**Remark 7.5.** Note that gappedness of $\Lambda_t$ including at $t = 0$ is equivalent to gappedness of the family $\Lambda_t$ for $t \neq 0$ plus $\epsilon$-chordlessness of $\Lambda_0$ by itself, these all being considered for the same isotopy.

**Remark 7.6.** (On the isotropicity hypotheses.) The hypotheses (2) and (3) on $\Lambda_0$ roughly mean it is isotropic. We will see later that this comes at considerable cost: as a consequence we will be forced to consider only those Liouville manifolds with isotropic skeleton, and also only be able to prove invariance under homotopies of such manifolds, as opposed to arbitrary exact symplectomorphism. Since we know on other grounds (e.g. \cite{21}) that these latter constraints are in fact unnecessary, it is natural to expect that these hypotheses may be relaxed. Let us at least recall here precisely how the hypotheses arise, and speculate on how they may be relaxed. *This discussion will occupy the remainder of this section, and nothing in the present article depends upon it.*

Before we begin let us note that in contrast to hypotheses (2) and (3), the gappedness hypothesis (1) does *not* imply isotropicity (the core of any Liouville manifold would do) and additionally seems to be fundamental on philosophical grounds as discussed in the introduction.

The hypothesis (3) is inherited ultimately from Lemma 4.8 and Lemma 4.13, where it is used to control the limiting behavior of a family using only facts about the limit geometry. However, in our setting we are not fully ignorant of the nature of the family; plausibly this could be used in place of the hypothesis on the limit.

We will discuss hypothesis (2) in more detail. It comes from the fact that we use antimicrolocalization in order to define the specialization functor on microlocal sheaves, and antimicrolocalization ultimately relies on Lemma 6.7 which requires a finite front projection. On the one hand, perhaps this lemma may be improved – it derives from the refined microlocal cutoff \cite[Proposition 6.1.4]{29}, which has no explicit isotropicity hypotheses.

On the other hand, it may seem odd that we use antimicrolocalization to define the specialization functor at all: why not simply argue that nearby cycles respects microsupports hence factors through microlocalization? Let us develop this idea somewhat in order to expose a subtle difficulty. Let $\pi : E \to B$ be a smooth fiber bundle. The results of Section 3.2 make it natural to define a sheaf of categories of *relative microsheaves* on $T^*\pi$ as follows.
For an open \( U \subset T^*\pi \), take
\[
\text{Null}(U) = \{ F \in \text{sh}(E) \mid \text{ss}_\pi(F) \cap U = \emptyset \} \subset \text{sh}(E)
\]
and define \( \mu \text{sh}_\pi \) as the sheafification of the presheaf given by
\[
\mu \text{sh}_\pi^{\text{pre}}(U) = \text{sh}(E)/\text{Null}(U)
\]
One virtue of this construction is that for a submanifold \( A \subset B \), it follows from the estimates Lemma 3.13 and Remark 3.14, that there is a restriction map \( \mu \text{sh}_{T^*\pi}|_{T^*\pi_A} \to \mu \text{sh}_{T^*\pi_A} \). The same would not be true if we simply took the usual microsheaves \( \mu \text{sh}_{T^*E} \) on \( T^*E \) and pushed this sheaf of categories to \( \Pi_* \mu \text{sh}_{T^*E} \) on \( T^*\pi \). Indeed, while one has the equality of presheaves \( \mu \text{sh}_{\pi}^{\text{pre}} = \Pi_* \mu \text{sh}_{T^*E}^{\text{pre}} \), the sheafification does not commute with the pushforward: i.e. the natural map
\[
\Pi_* \mu \text{sh} |_{T^*\pi_A} \leftarrow \mu \text{sh}_{T^*\pi}|_{T^*\pi_A} \to \mu \text{sh}_{T^*\pi_A}
\]
To understand the interaction of relative microsheaves with nearby cycles, it therefore remains to consider an open \( j : B' \subset B \) and contemplate inducing a functor on relative microsheaves from \( j_* \).

Let \( \pi' : X' \to B' \) be the restriction, and similarly denote the various analogous maps with \( ' \). We want to consider the relationship between \( \mu \text{sh}_{T^*\pi'} \) and \( \mu \text{sh}_{T^*\pi} \). We write \( J_* \) for pushing forward sheaves or presheaves of categories by \( j \). We claim there always exists the following diagram:

\[
\begin{array}{ccc}
\mu \text{sh}_{T^*\pi}^{\text{pre}} & \xrightarrow{J_*} & \mu \text{sh}_{T^*\pi'}^{\text{pre}} \\
\downarrow & & \downarrow \\
\mu \text{sh}_{T^*\pi} & \xrightarrow{J_*} & (J_* \mu \text{sh}_{T^*\pi'}^{\text{pre}})^{\text{sh}} \\
\end{array}
\]

Above, \( j_* \) on sheaves induces the top line by microsupport estimates. The lower line is induced because sheafification is a functor, and a map from a presheaf to a sheaf factors through the sheafification. However note that we have not succeeded in constructing a functor \( j_* : J_* \mu \text{sh}_{T^*\pi'} \to \mu \text{sh}_{T^*\pi} \), since the natural map \((J_* \mu \text{sh}_{T^*\pi'}^{\text{pre}})^{\text{sh}} \to J_* \mu \text{sh}_{T^*\pi'}\) need not be an isomorphism. One difficulty arises by considering the possibility of sheaves with components of the microsupport accumulating near the boundary of \( B' \).

Plausibly by imposing appropriate hypotheses for the microsupport near \( \partial B' \) (e.g. fixing some \( \Lambda' \subset T^*\pi' \)), one can ensure that \( (J_* \mu \text{sh}_{T^*\pi'}^{\text{pre}})^{\text{sh}} \to J_* \mu \text{sh}_{T^*\pi'} \) is an isomorphism, and then use this construction to define the microlocal specialization functor. It would be of interest to determine for what \( \Lambda' \) this holds, and more generally to develop the relative microlocal sheaf theory.

8. Microlocal sheaves on polarizable contact manifolds

Following [45], we explain how a category of microlocal sheaves can be associated to polarizable contact manifolds or their closed subsets, such as the skeleton of a Weinstein manifold. We then apply Theorem 7.3 to show that Lagrangians define objects, and that the
category associated to (the skeleton of a) Weinstein manifold is invariant under deformation of the symplectic primitive. Note such deformations cause dramatic changes in the geometry of the skeleton.

The polarizability hypothesis is a very strong version of asking for Maslov obstructions to vanish. We will later relax the former hypothesis to the latter.

8.1. The embedding trick. Following [45], we use high codimension embeddings to define global categories of microlocal sheaves. The existence of the requisite embeddings follows from Gromov’s h-principle for contact embeddings, which implies in particular that for any contact manifold $U$, there’s a nonempty space of embeddings $U \subset R^{2N+1} \gg 0$, which can be made as connected as desired by increasing $N$.

Such an embedding gives $U$ its stable symplectic normal bundle $\nu_U$; as a stable symplectic bundle, it is the negative of the tangent bundle. We (symplectomorphically) identify a tubular neighborhood of $U$ in its embedding with neighborhood of the zero section in this normal bundle. By a thickening, we mean the preimage under this identification of the total space of a Lagrangian subbundle of $\nu_U$. Evidently this is determined by a section $\sigma$ of $LGr(\nu_U)$; we term such a section a “stable normal polarization”, and denote the corresponding thickening by $U^{\sigma}$.

Note the following facts about stable normal polarizations:

**Lemma 8.1.** Let $(M, d\lambda)$ be an exact symplectic manifold, and $(M \times \mathbb{R}, \lambda + dt)$ its contactization. The following are equivalent:

- A section of $LGr(TM \oplus \mathbb{R}^{2n}) \to M$ for $n \gg 0$.
- A section of $LGr(\ker(\lambda + dt) \oplus \mathbb{R}^{2n}) \to M \times \mathbb{R}$ for $n \gg 0$.

Let $(V, \eta)$ be any co-oriented contact manifold. Then the following are equivalent:

- A section of $LGr(\ker(\eta) \oplus \mathbb{R}^{2n}) \to V$ for $n \gg 0$.
- A section of the Lagrangian Grassmannian bundle of the stable normal bundle of $V$.

**Proof.** From $M$ one forms $M \to BU$ classifying its stable symplectic tangent bundle, and composes with the projection $BU \to B(U/O)$ to get the map $\phi : M \to B(U/O)$ classifying the Lagrangian Grassmannian. The Lagrangian Grassmanian of $\ker(\lambda + dt)$ over $M \times \mathbb{R}$ is likewise classified by a map $\tilde{\phi} : M \times \mathbb{R} \to B(U/O)$, which moreover satisfies $\tilde{\phi}|_{M \times 0} = \phi$.

Now if $(V, \lambda)$ is any contact manifold, and $\phi : V \to B(U/O)$ classifies $LGr(\ker(\eta) \oplus \mathbb{R}^{2n})$, then the Lagrangian Grassmannian of the stable normal bundle to $V$ is classified by $-\tilde{\phi}$.

As $U/O$ is an infinite loop space, hence a homotopy group, a section of the stable Lagrangian Grassmannian is the same as a trivialization of it, hence given by a null-homotopy of, respectively, $\phi, \tilde{\phi}, -\tilde{\phi}$ in the above cases. Evidently these are equivalent.

In any case, fixing a stable normal polarization, we may define microlocal sheaves on any contact manifold.

**Definition 8.2.** We write $\mu_{sh_{U, \sigma}} := \mu_{sh_{U^{\sigma}}}|_{U}$. 

**Remark 8.3.** Note that thickening by a lagrangian in the normal bundle ensures local constancy along the normal directions (Lemma 5.3).

Let us consider on what this invariant may depend. The h-principle implies that different embeddings are themselves isotopic, so by another application of Lemma 5.3 we see that $\mu_{sh_{U, \sigma}}$ does not depend on the embedding of $U$, save perhaps through the dimension $2N + 1$...
of the target. Similarly, homotopic choices of normal polarization give equivalent categories. (We will see in Section 10 below that in fact the category depends only on the image of the normal polarization under a certain map.)

To show independence of the embedding dimension, it suffices to observe that the category we defined is preserved under replacing $U$ by $U \times T^*[0, 1]$. The section of the Lagrangian Grassmannian is promoted by choosing a constant section in the $T^*[0, 1]$ direction, e.g. the zero section.

The objects representing objects in $\mu sh_U$ have a (micro)support in $U_\sigma$, constant in the normal bundle directions. We define their corresponding microsupport in $U$ to be the restriction to $U$. This newly defined microsupport is evidently co-isotropic (since the original was by [29]).

**Definition 8.4.** For $\Lambda \subset U$, we write $\mu sh_{\Lambda, \sigma}$ for the subsheaf of $\mu sh_U, \sigma$ consisting of full subcategories on the objects microsupported in $\Lambda$.

Obviously $\mu sh_{\Lambda, \sigma}$ only depends on the ambient $U$ through its germ along $\Lambda$.

**Remark 8.5.** When $U = S^*M$ is a cosphere bundle, there are now two notions of microlocal sheaves on $U$. The first is the original $\mu sh_{S^*M}$. The second is obtained from what we have just introduced, by observing that the cosphere fibers induce a polarization of the cosphere bundle, so in particular a stable normal polarization. It follows from the above discussion of invariance that the original $\mu sh_{S^*M}$ is canonically isomorphic to the new $\mu sh_{S^*M, \text{fibers}}$, as sheaves of categories on $S^*M$.

When, as for $S^*M$, the polarization is understood, or e.g. all polarizations in question are obtained by restricting some (possibly unspecified) fixed polarization on an ambient space, we may omit the polarization from the notation.

### 8.2. Quantization of Legendrians and Lagrangians

Given a set $X$ equipped with the germ of a contact embedding $X \to \hat{U}_X$, by a stabilized embedding of $X$ we mean a codimension zero embedding of $(X \times (0, 1)^n, \hat{U}_X \times T^*((0, 1)^n))$ into some contact manifold.

**Definition 8.6.** We say $X$ has a property ‘universally’ if every stabilized embedding of $X$ has the property.

**Theorem 8.7.** Let $U$ be a contact manifold equipped with a stable normal polarization $\tau$. Let $\Lambda_1 \subset U$ a compact Legendrian, $\phi_t : U \times (0, 1] \to \hat{U}_X$ a compactly supported contact isotopy, and $\phi_1 = \text{id}_U$. Let $\Lambda_t := \phi_t(\Lambda)$ and let $\Lambda_0 := \lim_{t \to 0} \Lambda_t \subset X$. Assume:

- The family $\Lambda_t$ (including at $t = 0$) is gapped (Def. 2.9) for some contact form on $U$.
- $\Lambda_0$ and $\Lambda_1$ are universally perturbable to finite position (Def. 6.10).
- $\Lambda_0$ is universally positively displaceable from legendrians (Def: 2.2).

Then there is a fully faithful functor

$$\psi : \mu sh_{\Lambda_1, \tau}(\Lambda_1) \to \mu sh_{\Lambda_0, \tau}(\Lambda_0)$$

More generally, the same holds for $\Lambda_1$ relatively compact and collared along its boundary, and $\phi_t$ compatible with the collar.

**Proof.** We will embed the problem and then appeal to Theorem 7.3. More precisely, fix an (high codimension) embedding $U \subset J^1(\mathbb{R}^N) \subset S^*(\mathbb{R}^{N+1})$, and thicken $U, X, \Lambda_1$, etc. in the direction of the polarization. Extend isotopies from $\tilde{U}$ to its neighborhood in the embedding
by pulling back the contact Hamiltonian by the projection of the tubular neighborhood = normal bundle to $\tilde{U}$. Note the resulting flows are compatible with thickening and do not introduce new chords in flows of thickenings. In particular $\Lambda^r_0$ remains gapped and $\Lambda^r_0$ is its limit. We have assumed that $\Lambda_0$ is universally pdfl, which by definition means that $\Lambda^0_0$ is pdfl. Likewise, $\Lambda^r_1, \Lambda^r_0$ are perturbable to finite position.

Thus we may appeal to Theorem 7.3 to deduce the existence and full faithfulness of some map $\psi: \mu sh_{\Lambda^r_1}(\Lambda^r_1) \to \mu sh_{\Lambda^r_0}(\Lambda^r_0)$. As the natural restriction maps $\mu sh_{\Lambda^r_1}(\Lambda^r_1) \to \mu sh_{\Lambda_1,\tau}(\Lambda_1)$ and $\mu sh_{\Lambda^r_0}(\Lambda^r_0) \to \mu sh_{\Lambda_0,\tau}(\Lambda_0)$ are equivalences (essentially by definition, and indeed before passing to global sections), we obtain a fully faithful functor $\psi: \mu sh_{\Lambda_1,\tau}(\Lambda_1) \to \mu sh_{\Lambda_0,\tau}(\Lambda_0)$. \hfill $\Box$

**Remark 8.8.** For $X \supset L_0$ we may compose with the natural inclusion to obtain a fully faithful map $\psi: \mu sh_{\Lambda_1,\tau}(\Lambda_1) \to \mu sh_{X,\tau}(X)$.

An exact symplectic manifold $(W, \lambda)$ (or any subset thereof) is canonically embedded in its contactization $W \times \mathbb{R}$, which we take with contact form $dz - \lambda$. In particular we may naturally speak of microsheaves on subsets of exact symplectic manifolds (with stable normal polarizations). Note also that if $v_{\lambda}$ is the Liouville flow, then $v_{\lambda} + z\frac{dz}{d\tau}$ is a contact vector field lifting $v_{\lambda}$; in case $W$ is Liouville, this contact vector field retracts the contactization to the core $c(W)$.

**Corollary 8.9.** Let $W$ be Liouville with stable normal polarization $\tau$, and let $\Lambda_1 \subset W \times \mathbb{R}$ be universally perturbable to finite position, and assume the projection $\Lambda_1 \to W$ is an embedding onto a subset $L$ which is either compact or has conic end.

Let $L_0$ be the limit of $L$ under the Liouville flow, and assume $L_0$ is universally pdfl and universally perturbable to finite position. Then there is a fully faithful functor:

$$\mu sh_{\Lambda_1,\tau}(\Lambda_1) \to \mu sh_{L_0,\tau}(L_0) \subset \mu sh_{c(W) \cup L_0,\tau}(c(W) \cup L_0)$$

**Proof.** Let $\Lambda_t$ be the flow of $\Lambda_1$ under $v_{\lambda} + z\frac{dz}{d\tau}$, parameterized to live over $t$ in $(0, 1]$. Then $\lim_{t \to 0} \Lambda_t = L_0$. We want to apply Theorem 8.7. Gappedness is automatic: the $\Lambda_t$ have no self-chords at all, as these would project to self-intersections of (the image under Liouville flow of) $\Lambda_1$, which we have assumed do not exist. The other hypotheses of the theorem hold by assumption. \hfill $\Box$

**Remark 8.10.** We expect that similar techniques can be applied in the case when $L$ has conic end which, rather than being constant, is itself varying in a family which is gapped with respect to some contact form on the contact end of $W$.

**Definition 8.11.** We say a subset of a contact manifold is *sufficiently isotropic* if it is universally pdfl and universally perturbable to finite position. We say a subset of an exact symplectic manifold is *sufficiently isotropic* if it is sufficiently isotropic in the contactization. (See defs. 2.2, 6.10, 8.6 for the terminology.)

**Remark 8.12.** A sufficient condition ensuring ‘sufficiently isotropic’ is the existence of a Whitney stratification by isotropics. Note also any subset of something sufficiently isotropic is sufficiently isotropic.

**Remark 8.13.** If $W$ has sufficiently isotropic core and $L$ is a smooth compact exact Lagrangian, then the hypotheses of Corollary 8.9 are readily verified. Indeed, $L$ is smooth Legendrian hence universally perturbable to finite position, and $L_0 \subset c(W)$, the latter being by assumption sufficiently isotropic.
In the corollary, we did not require that \( L \) was smooth. In particular, consider varying the symplectic primitive \( \lambda \) to some \( \lambda' = \lambda + df \). Denote the respective cores by \( c \) and \( c' \). If both give sufficiently isotropic cores, then using Corollary 8.9 for \( (W, \lambda) \), we obtain a fully faithful functor \( \mu sh_{c', r}(c') \hookrightarrow \mu sh_{c, r}(c) \); likewise using Corollary 8.9 for \( (W, \lambda') \) we obtain \( \mu sh_{c, r}(c) \hookrightarrow \mu sh_{c', r}(c') \).

**Definition 8.14.** We say a Liouville form \( \lambda \) on \( W \) is **sufficiently Weinstein** if the resulting core is sufficiently isotropic. We say \( \lambda, \lambda' \) are **sufficiently Weinstein cobordant** if there exists a symplectic primitive \( \eta \) for the stabilization\(^\text{13}\) \( W \times T^*\mathbb{R} \), such that \( \eta \) restricts to \( \lambda + pdq \) near \(-\infty\) and to \( \lambda' + pdq \) near \(+\infty\).

More generally, for \( \Lambda \subset W^\infty \), we say \( \lambda \) is **sufficiently Weinstein** for the pair \( (W, \Lambda) \) if the relative core is sufficiently isotropic; correspondingly we speak of sufficiently Weinstein cobordisms.

**Theorem 8.15.** Let \( \lambda, \lambda' \) be two sufficiently Weinstein forms for \( W \) which are sufficiently Weinstein cobordant. Then the functor \( \mu sh_{c, r}(\lambda(W)) \hookrightarrow \mu sh_{c', r}(\lambda'(W)) \) is an equivalence.

The same holds in the relative case, i.e. with \( W \) replaced by a pair \( (W, \Lambda \subset W^\infty) \).

**Proof.** The argument is identical in the absolute and relative cases, we content ourselves to discuss the former.

For notational clarity we write \( X = c_\lambda(W) \) and \( X' = c_{\lambda'}(W) \). We work inside \( W \times T^*\mathbb{R} \). We use \( q \) for the coordinate on \( \mathbb{R} \), and \( p \) for the cotangent coordinate. On \( W \times T^*\mathbb{R} \), we consider the three Liouville forms

\[
\tilde{\lambda} = \lambda + pdq \quad \tilde{\lambda}' = \lambda' + pdq \quad \eta
\]

The cores for these Liouville forms are evidently \( X \times T^*_\mathbb{R} \), \( X' \times T^*_\mathbb{R} \), and some \( Y \) which projects to \( T^*_\mathbb{R} \) under the projection \( W \times T^*\mathbb{R} \to T^*\mathbb{R} \) and interpolates between \( X \) for \( q \ll 0 \) and \( X' \) for \( q \gg 0 \).

Flowing by the appropriate Liouville forms give fully faithful maps

\[
\mu sh_{X \times T^*_\mathbb{R}}(X \times T^*_\mathbb{R}) \hookrightarrow \mu sh_{Y}(Y) \hookrightarrow \mu sh_{X' \times T^*_\mathbb{R}}(X' \times T^*_\mathbb{R})
\]

Restriction at \( q = -\infty \) gives canonical maps

\[
\mu sh_{X \times T^*_\mathbb{R}}(X \times T^*_\mathbb{R}) \sim \rightarrow \mu sh_{X}(X) \leftarrow \mu sh_{Y}(Y)
\]

These maps evidently commute with the \( \mu sh_{X \times T^*_\mathbb{R}}(X \times T^*_\mathbb{R}) \hookrightarrow \mu sh_{Y}(Y) \), since those maps are induced by flows constant near \(-\infty\). It follows formally that the composition \( \mu sh_{X \times T^*_\mathbb{R}}(X \times T^*_\mathbb{R}) \hookrightarrow \mu sh_{Y}(Y) \to \mu sh_{X \times T^*_\mathbb{R}}(X \times T^*_\mathbb{R}) \) is the identity, hence that the second map \( \mu sh_{Y}(Y) \to \mu sh_{X \times T^*_\mathbb{R}}(X \times T^*_\mathbb{R}) \) is essentially surjective. As we already knew it was fully faithful, it must be an equivalence; hence so must its left inverse. Likewise we learn that \( \mu sh_{Y}(Y) \sim \mu sh_{X' \times T^*_\mathbb{R}}(X' \times T^*_\mathbb{R}) \) are equivalences.

Finally, consider the commutative diagram

\[^{13}\text{Note this is a not a Liouville manifold, but rather an “open Liouville sector” in the sense of [19].}\]
\[ \mu sh_{X \times T^*_R \mathbb{R}}(X \times T^*_R \mathbb{R}) \longrightarrow \mu sh_Y(Y) \]

\[
\downarrow_{+\infty} \quad \downarrow_{+\infty}
\]

\[ \mu sh_X(X) \longrightarrow \mu sh_{X'}(X') \]

Here, the vertical maps are the restriction at \( q = +\infty \). The top horizontal map is the nearby cycles for flowing \( X \times T^*_R \mathbb{R} \) to \( Y \) using the Liouville form \( \eta \), and the lower horizontal map is the restriction of this to \( q = +\infty \). Note this agrees with the nearby cycle for flowing \( X \) to \( X' \) using \( \lambda' \), by construction. The diagram commutes because nearby cycles is a functor of sheaves of categories, hence commutes with restriction. Finally, we have seen that the top, left, and right arrows are equivalences, hence so is the bottom one. \( \Box \)

**Remark 8.16.** It is not completely obvious that the functor obtained by composing the flowdowns \( \mu sh_{X \times T^*_R \mathbb{R}}(X \times T^*_R \mathbb{R}) \rightarrow \mu sh_Y(Y) \rightarrow \mu sh_{X' \times T^*_R \mathbb{R}}(X' \times T^*_R \mathbb{R}) \) agrees with what would be obtained directly by flowing \( \mu sh_{X \times T^*_R \mathbb{R}}(X \times T^*_R \mathbb{R}) \rightarrow \mu sh_{X' \times T^*_R \mathbb{R}}(X' \times T^*_R \mathbb{R}) \). We did not use this assertion in the above argument, though it does follow from what we have shown.

**Remark 8.17.** Intuitively one would expect that if there is a one-parameter family of sufficiently Weinstein forms \( \lambda_t \) interpolating between \( \lambda \) and \( \lambda' \), then one can find a sufficiently Weinstein cobordism \( \eta \). We do not know if this is true as stated (or for what appropriate strengthening of “sufficiently isotropic” it is true). However if one assumes that the \( \lambda_t \) are all in fact Weinstein (for possibly generalized Morse function), then it is known that one can form \( \eta \) Weinstein as well [9]. Note in [21, Sec. 6.8] it is explained how to make Weinstein manifolds subanalytic; doing this as well will ensure sufficiently isotropic cores.

### 8.3. Conicity on symplectic hypersurfaces.

As the construction of Cor. 8.9 is built from the Liouville flow, the microsheaves it produces are automatically conic. Let us give some evidence that in fact all microsheaves on exact symplectic manifolds are automatically conic.

Recall the following stabilizations:

- If \((W, \lambda)\) is exact symplectic, then it canonically embeds as the zero section of its contactization \((W \times \mathbb{R}_z, dz - \lambda)\). Conversely, any codimension one symplectic hypersurface in a contact manifold has a local neighborhood with this local model.
- A contact manifold with contact form \((V, \alpha)\) determines an exact symplectic manifold \((V \times \mathbb{R}_t, e^t \alpha)\).

Recall we say a subset of a symplectic manifold \( Z \subset (W, \omega) \) is termed co-isotropic if \( T_z Z \) contains its orthogonal under the symplectic form, i.e. \( T_z Z^\perp \subset T_z Z \), and we say a smooth subset of a contact manifold \( Y \subset (V, \alpha) \) is co-isotropic if \( T_y Y^\perp \cap \ker \alpha \subset T_y Y \). (This notion depends on the choice of the contact form \( \alpha \) only through the contact distribution \( \ker \alpha \).)

**Lemma 8.18.** Let \((W, \lambda)\) be an exact symplectic manifold, and \( X \subset W \) a smooth submanifold.

- If \( \tau := TX^\perp_\omega \subset TW|_X \), then \( \tau \subset TX \) and \( \lambda|_\tau = 0 \).
- \( X \) is conic coisotropic.
- \( X \times 0 \subset (W \times \mathbb{R}_z, dz - \lambda) \) is contact coisotropic.
- \( X \times 0 \times \mathbb{R}_t \subset (W \times \mathbb{R}_x \times \mathbb{R}_t, e^t(dz - \lambda)) \) is coisotropic.
Proof. We will show the first property is equivalent to all others.

The submanifold \( X \subset W \) is coisotropic iff \( \tau \subset TX \). Conicity of \( X \subset W \) is equivalent to asking that the Liouville vector field \( v = (d\lambda)^{-1}\lambda \) is contained in \( TX \) along \( X \). Being contained in \( TX \) is the same as being orthogonal to \( \tau \), so a coisotropic is conic iff \( 0 = d\lambda(v, \tau) = \lambda|_{\tau} \).

That \( X \times 0 \) is contact coisotropic in \( (W \times \mathbb{R}_z, dz - \lambda) \) is asking that \( \ker(dz - \lambda) \cap \ker(d\lambda|_{X})_{d\lambda}^\perp \subset TX \). Evidently \( \ker(d\lambda|_{X})_{d\lambda}^\perp \subset W \times \mathbb{R} = TX \). As already noted, conic coisotropic whenever it is smooth.

Finally let us see what it means for \( X \) to be conic in \( (W \times \mathbb{R}_z, dz - \lambda) \), and any object \( \lambda \) that for a smooth manifold \( W \), there are two notions of tangent cone, \( C(X) \subset C(X, X) \subset TW|_{X} \). When \( X \) is smooth, \( C(X) = C(X, X) = TX \). If \( W \) is symplectic, with form \( \omega \), one says \( X \) coisotropic (involutive in \([29]\)) if

\[
C(X, X)_{\perp} \subset C(X)
\]

Meanwhile, conicity interacts well with \( C(X) \): according to \([29, \text{Lemma 6.5.3}]\), a vector field preserves \( X \) if it is contained in \( C(X) \) along \( X \).

However, at least the proof of Lemma 8.18 does not appear to generalize to arbitrary coisotropic subsets. More precisely, if we substitute \( \tau \to C(X, X)_{\perp}^\perp \) and elsewhere \( TX \to C(X) \), then the first and fourth conditions remain equivalent. However, \( \lambda|_{\tau} = 0 \) no longer seems to imply that \( X \) is conic. Indeed, \( \lambda|_{\tau} = 0 \) ensures that the Liouville vector field is in \( \tau^\perp = (C(X, X)_{\perp})^\perp \). But in general \( (C(X, X)_{\perp})^\perp \not\subset C(X) \), and it is in the latter space where we need the Liouville vector field to live.

Corollary 8.21. For a Liouville manifold \( W \) with normal polarization \( \tau \), and any object \( \mathcal{F} \in \text{ush}_{W, \tau}(W) \), the locus \( \text{ss}(\mathcal{F}) \) is conic coisotropic whenever it is smooth.

Proof. In an embedding \( W \times \mathbb{R} \to S^*M \), a local representative for \( \mathcal{F} \) has by \([29, \text{Theorem 6.5.4}]\) conic co-isotropic microsupport in \( T^*M \). By Lemma 8.18, since this microsupport is contained in \( W \times 0 \) at infinity, it must be conic in \( W \).

Remark 8.22. One could ask for even a stronger version of conicity, as follows. On the contactization \( (W \times \mathbb{R}_z, dz - \lambda) \) of \((W, \lambda)\), one has the contact vector field \( v + zd\lambda \) which lifts the Liouville vector field \( v \). By quantization of contact isotopy, one has an isomorphism \( \Phi_t : e^{tv}_{*} \text{ush} \cong \text{ush} \). One could ask for objects \( \mathcal{F} \) which are conic in the sense that there exist
isomorphisms $\mathcal{F} \cong \Phi_t(\mathcal{F})$. Note that the full subcategory spanned by these objects is the same the Liouville-equivariant microsheaves, since $\mathbb{R}$ is contractible.

In fact however any object with conic microsupport is also conic in this sense. Indeed, the quantization of the isotopy would produce an object in $W \times T^*\mathbb{R}$ whose microsupport was contained in $(\Phi_t(ss(\mathcal{F})), t, 0)$, the last factor being constant because the microsupport was already conic. Such an object is necessarily locally constant in the $t$ direction.

**Remark 8.23.** When $W = T^*M$ is a cotangent bundle, there are now two notions of microlocal sheaves on $W$. The first is the original $\mu sh_{T^*M}$. The second is what we have just defined, using the fiber polarization. In fact these are the same: $\mu sh_{T^*M} = \mu sh_{T^*M, \text{fibers}}$ as sheaves of categories. One can see this by considering the image of the natural map $\pi^* : sh(M) \rightarrow sh(M \times (0, \infty)) \rightarrow sh(M \times \mathbb{R})$. In particular all objects of the “new” $\mu sh$ are indeed conic.

9. The cotangent bundle of the Lagrangian Grassmannian

We will later want to universally trivialize Maslov obstructions by passing to the relative cotangent bundle of the Lagrangian Grassmannian bundle. In this section we discuss the relevant elementary symplectic geometry.

9.1. Symplectic structures on relative cotangent bundles. Let $\mathcal{f} : E \rightarrow M$ be a fiber bundle of smooth manifolds. We will denote points of $E$ by pairs $(m, q)$ where $m \in M$, and $q \in \mathcal{f}^{-1}(m)$.

Let $\pi : T^*E \rightarrow E$ denote the cotangent bundle. We will be interested in the relative tangent and cotangent bundles, which are bundles on $E$ characterized by the short exact sequences

$$0 \rightarrow T\mathcal{f} \rightarrow TE \xrightarrow{\mathcal{f}^*} \mathcal{f}^*TM \rightarrow 0$$

or equivalently, as a dual splitting of the second exact sequence

$$0 \rightarrow \mathcal{f}^*T^*M \xrightarrow{\mathcal{f}^*} T^*E \xrightarrow{\mathcal{f}^*} T^*\mathcal{f} \rightarrow 0$$

We write $\pi_\mathcal{f} : T^*\mathcal{f} \rightarrow E$ for the natural projection, and introduce the composition

$$\mathfrak{g} = \mathcal{f} \circ \pi_\mathcal{f} : T^*\mathcal{f} \rightarrow M$$

We denote points of $T^*\mathcal{f}$ by triples $(m, q, p)$ where $(m, q) \in E$, and $p \in \pi_{\mathcal{f}}^{-1}(m, q)$, and identify $E$ with the zero-section $\{p = 0\} \subset T^*\mathcal{f}$.

We are interested in having a “fiberwise canonical one-form” on the relative cotangent bundle $\mathfrak{g} : T^*\mathcal{f} \rightarrow M$. To construct this, we fix a connection on $\mathcal{f}$, which we will formulate as a splitting of the first exact sequence

$$0 \rightarrow T\mathcal{f} \xrightarrow{\mathcal{s}} TE \xrightarrow{\mathcal{f}^*} \mathcal{f}^*TM \rightarrow 0$$

or equivalently, as a dual splitting of the second exact sequence

$$0 \rightarrow \mathcal{f}^*T^*M \xrightarrow{\mathcal{f}^*} T^*E \xrightarrow{\mathcal{f}^*} T^*\mathcal{f} \rightarrow 0$$

Note that, by construction, the natural pairing splits: for $\xi \in T^*E$ and $v \in TE$, we have

$$\langle \xi, v \rangle_E = \langle \xi, (ts + t\mathfrak{g}_s)(v) \rangle_E = \langle v^\vee(\xi), s(v) \rangle_\mathcal{f} + \langle t^\vee(\xi), \mathfrak{g}_s(v) \rangle_M$$
where brackets with subscripts indicate the canonical pairings between dual bundles.

Let \( \theta_{\text{can}} \in \Omega^1(T^*E) \) denote the canonical one-form on the cotangent bundle \( T^*E \). Let \( \omega_{\text{can}} = d\theta_{\text{can}} \in \Omega^2(T^*E) \) denote the canonical symplectic form, and \( v_{\text{can}} \in \text{Vect}(T^*E) \) the Liouville vector field characterized by \( i_{v_{\text{can}}} (\omega_{\text{can}}) = \theta_{\text{can}} \). Recall that \( v_{\text{can}} \) is the Euler vector field generating the dilation.

**Lemma 9.1.** For any \( m \in M \), the restriction \( (s^\vee)^*\theta_{\text{can}}|_{T^*E_m} \) is the canonical one-form on \( T^*E_m \).

**Proof.** Consider the natural commutative diagram with compatible sections

\[
\begin{array}{ccc}
T^*E & \rightarrow & T^*\mathfrak{f} \\
\downarrow i & & \downarrow i \\
T^*E|_{E_m} & \rightarrow & T^*\mathfrak{f}|_{E_m} \\
\downarrow q & & \downarrow q \\
T^*E_m & \rightarrow & T^*\mathfrak{f}_m
\end{array}
\]

The canonical one-form \( \theta_{\text{can},m} \in \Omega^1(T^*E_m) \) is characterized by \( q^*\theta_{\text{can},m} = i^*\theta_{\text{can}} \). Using the section \( s^\vee \), it can be calculated as \( \theta_{\text{can},m} = (s^\vee)^*i^*\theta_{\text{can}} \). By commutativity of the diagram, \( (s^\vee)^*i^*\theta_{\text{can}} = i^*(s^\vee)^*\theta_{\text{can}} \).

We denote this “fiberwise canonical one-form” as

\[ \theta_{\mathfrak{f}} := (s^\vee)^*\theta_{\text{can}} \]

Note that \( \theta_{\text{can}} \) vanishes on all vectors based along the zero-section \( E \subset T^*E \), and so \( \theta_{\mathfrak{f}} \) vanishes on all vectors based along the zero-section \( E \subset T^*\mathfrak{f} \).

**Example 9.2.** When \( M = pt \), we have \( T^*\mathfrak{f} = T^*E \) is the absolute cotangent bundle, and \( \theta_{\mathfrak{f}} = \theta_{\text{can}} \) is the usual canonical one-form.

Next, consider the short exact sequence

\[
0 \rightarrow T^*E \rightarrow T(T^*\mathfrak{f}) \rightarrow \mathfrak{F}^*TM \rightarrow 0
\]

of bundles on \( T^*\mathfrak{f} \), where by definition \( T\mathfrak{F} := \ker(\mathfrak{F}_*) \).

**Lemma 9.3.** The two-form \( \omega_{\mathfrak{f}} := d\theta_{\mathfrak{f}} \in \Omega^2(T^*\mathfrak{f}) \) is non-degenerate on \( T\mathfrak{F} \subset T(T^*\mathfrak{f}) \). Its kernel \( K := \ker(\omega_{\mathfrak{f}}) \subset T(T^*\mathfrak{f}) \) provides a splitting of \( (18) \) in the sense that \( \mathfrak{F}_* \) restricts to an isomorphism

\[
\mathfrak{F}_*: K \overset{\sim}{\longrightarrow} \mathfrak{F}^*TM
\]

**Proof.** We have \( \omega_{\mathfrak{f}} = d((s^\vee)^*\theta_{\text{can}}) = (s^\vee)^*d\theta_{\text{can}} = (s^\vee)^*\omega_{\text{can}} \). We have \( T\mathfrak{F}|_{E_m} = T(T^*\mathfrak{f}|_{E_m}) = T(T^*E_m) \). From Lemma 9.1, the restriction of \( (s^\vee)^*\omega_{\text{can}} \) to each such fiber is the canonical symplectic form, hence non-degenerate. On the other hand, \( \omega_{\mathfrak{f}} \) clearly vanishes on \( \mathfrak{f}^*TM \). \( \square \)

**Lemma 9.4.** Suppose \( \mathfrak{f} : E \rightarrow M \) is a fiber bundle of smooth manifolds with splitting \( s^\vee : T^*\mathfrak{f} \rightarrow T^*E \) and associated one-form \( \theta_{\mathfrak{f}} := (s^\vee)^*\theta_{\text{can}} \in \Omega^1(T^*\mathfrak{f}) \).

If \( \omega_M \in \Omega^2(M) \) is a symplectic form, then \( \omega_{\mathfrak{f}} := \mathfrak{F}^*\omega_M + d\theta_{\mathfrak{f}} \in \Omega^2(T^*\mathfrak{f}) \) is a symplectic form. If \( L \subset M \) is Lagrangian, then so is \( E|_L \subset T^*\mathfrak{f} \).
Proof. We have the splitting \( T(T^*\mathfrak{g}) \cong \mathfrak{g}^*T\mathcal{M} \oplus K \) where \( K = \ker(d\theta) \). By the non-degeneracy of \( \omega_M \), we also have \( \mathfrak{g}^*T\mathcal{M} = \ker(\mathfrak{g}^*\omega_M) \). Since \( \omega_M \) is non-degenerate on \( K = \ker(d\theta) \), and \( d\theta \) is non-degenerate on \( \mathfrak{g}^*T\mathcal{M} = \ker(\mathfrak{g}^*\omega_M) \), we conclude that \( \omega_f \) is non-degenerate.

Finally, if \( L \subset M \) is Lagrangian, then \( \omega_M|_L = 0 \); on the other hand \( \theta|_E = 0 \), hence \( \omega_f|_{E|_L} = 0 \). \( \square \)

**Lemma 9.5.** Suppose \( \mathfrak{f} : E \to V \) is a fiber bundle of smooth manifolds with splitting \( s^\vee : T^*\mathfrak{f} \to T^*E \) and associated one-form \( \theta_f := (s^\vee)^*\theta_{can} \in \Omega^1(T^*\mathfrak{f}) \).

If \( \lambda_V \in \Omega^1(V) \) is a contact form, then \( \lambda_f := \mathfrak{g}^*\lambda_V + \theta_f \in \Omega^1(T^*\mathfrak{f}) \) is a contact form. If \( \Lambda \subset V \) is Legendrian, then so is \( E|_\Lambda \subset T^*\mathfrak{f} \).

Proof. We must show \( d\lambda_f = \mathfrak{g}^*d\lambda_V + d\theta_f \in \Omega^2(T^*\mathfrak{f}) \) is non-degenerate on \( \xi = \ker(\lambda_f) \subset T(T^*\mathfrak{f}) \), or other words, that \( \ker(d\lambda_f) \cap \xi = \{0\} \). Observe that \( \ker(d\lambda_f) \subset K = \ker(d\theta_f) \) since \( \omega_f = d\theta_f \) is non-degenerate on \( T\mathfrak{g} \subset T(T^*\mathfrak{f}) \). Recall also \( \mathfrak{g}_s|_K : K \cong \mathfrak{f}^*TM \), hence \( \ker(d\lambda_f) \subset K \cap \ker(\mathfrak{f}^*\lambda_M) \). But since \( d\lambda_M \) is non-degenerate on \( \ker(\lambda_M) \), we obtain the assertion. \( \square \)

Next, suppose \( (M, \omega_M = d\lambda_M) \) a Liouville manifold with \( v_M \in \text{Vect}(M) \) the Liouville vector field characterized by \( i_{v_M}(\omega_M) = \lambda_M \).

On the one hand, let \( \bar{v}_M \subset K = \ker(d\theta_f) \subset T^*\mathfrak{f} \) denote the lift of \( v_M \in TM \) under the isomorphism

\[
(20) \quad \mathfrak{g}_s|_K : K \cong \mathfrak{g}^*TM
\]

On the other hand, note that dilations of \( T^*E \) preserve the image \( s^\vee(T^*\mathfrak{f}) \subset T^*E \) of the bundle splitting \( s^\vee : T^*\mathfrak{f} \to T^*E \). Denote by \( \bar{v}_{can} \in \text{Vect}(T^*\mathfrak{f}) \) the vector field such that

\[
(s^\vee)_*\bar{v}_{can} = v_{can}|_{s^\vee(T^*\mathfrak{f})}
\]

Thus \( v_{can} \) lies in the kernel of \( \pi_{t^*} : T(T^*\mathfrak{f}) \to TE \), hence the kernel of \( \mathfrak{g}_s = f_* \circ \pi_{t^*} : T(T^*\mathfrak{f}) \to TM \).

Finally, recall that \( T^*\mathfrak{f} \) equipped with the one-form \( \lambda_f = \mathfrak{g}^*\lambda_V + \theta_f \in \Omega^1(T^*\mathfrak{f}) \) is an exact symplectic manifold. Denote by \( v_f \in \text{Vect}(T^*\mathfrak{f}) \) the Liouville vector field characterized by \( i_{v_f}(\omega_f) = \lambda_f \).

**Lemma 9.6.** The Liouville vector field \( v_f \in \text{Vect}(T^*\mathfrak{f}) \) is given by the formula

\[
u_f = \bar{v}_M + \bar{v}_{can}
\]

Proof. Note the identities

\[
i_{\bar{v}_M}(\mathfrak{g}^*d\lambda_M) = \mathfrak{g}^*i_{\mathfrak{g}^*\bar{v}_M}(d\lambda_M) = \mathfrak{g}^*i_{v_M}(d\lambda_M) = \mathfrak{g}^*\lambda_M
\]

with the latter due to the fact that \( \bar{v}_M \subset K = \ker(d\theta_f) \).

Note the additional identity

\[
i_{\bar{v}_{can}}(\mathfrak{g}^*d\lambda_M) = 0
\]

due to the fact that \( \mathfrak{g}_s\bar{v}_{can} = 0 \).

Thus we must show

\[
i_{\bar{v}_{can}}(d\theta_f) = \theta_f
\]
We calculate

\[ i_{\hat{\nu}_{\can}}(d\theta_{\parallel}) = i_{\hat{\nu}_{\can}}((s^\vee)^*d\theta_{\can}) = (s^\vee)^*i_{(s^\vee)_*\hat{\nu}_{\can}}(d\theta_{\can}) = (s^\vee)^*i_{\hat{\nu}_{\can}}(d\theta_{\can}) = (s^\vee)^*\theta_{\can} = \theta_{\parallel} \]

and we’re done. \[\square\]

Finally, when \((M, \omega_M = d\lambda_M)\) is a Liouville (resp. Weinstein) manifold, we have likewise that \((T^*\mathfrak{f}, \omega_{\parallel} = d\lambda_{\parallel})\) is a Liouville (resp. Weinstein) manifold.

**Proposition 9.7.** Let \((M, \omega_M = d\lambda_M)\) be a Liouville (resp. Weinstein) manifold.

Let \(\mathfrak{f} : E \to M\) be a proper fiber bundle of smooth manifolds, with relative cotangent bundle \(\pi_{\parallel} : T^*\mathfrak{f} \to E\), and total projection \(\mathfrak{f} = \mathfrak{f} \circ \pi_{\parallel} : T^*\mathfrak{f} \to M\).

For any splitting \(s^\vee : T^*\mathfrak{f} \to T^*E\), with associated one-form \(\theta_{\parallel} = (s^\vee)^*\theta_{\can} \in \Omega^1(T^*\mathfrak{f})\), we have the following:

1. The relative cotangent bundle \(T^*\mathfrak{f}\) equipped with the one-form \(\lambda_{\parallel} = \mathfrak{f}^*\lambda_M + \theta_{\parallel} \in \Omega^1(T^*\mathfrak{f})\) is a Liouville (resp. Weinstein) manifold.
2. The Liouville vector field \(v_{\parallel} \in \text{ Vect}(T^*\mathfrak{f})\) is given by \(v_{\parallel} = \hat{\nu}_M + \hat{\nu}_{\can}\), where \(\hat{\nu}_M \in \ker(d\theta_{\parallel})\) is the lift of the Liouville vector field \(v_M \in \text{ Vect}(M)\), and \(\hat{\nu}_M \in \ker(\pi_{\parallel})\) generates dilations.
3. If \(\mathbb{L} \subset M\) is the stable set of \((M, \omega_M = d\lambda_M)\), then \(E|_{\mathbb{L}} \subset T^*\mathfrak{f}\) is the stable set of \((T^*\mathfrak{f}, \omega_{\parallel} = d\lambda_{\parallel})\).
4. If \(L \subset M\) is (exact) Lagrangian, then so is \(E|_{\mathbb{L}} \subset T^*\mathfrak{f}\).
5. If \(v_M \in \text{ Vect}(M)\) satisfies \(d\varphi(v_M) > 0\) over an open \(U \subset M\) (resp. is gradient-like) for a function \(\varphi : M \to \mathbb{R}\), then \(v_{\parallel} \in \text{ Vect}(T^*\mathfrak{f})\) satisfies \(d\Phi(v_{\parallel}) > 0\) over the open \(\mathfrak{f}^{-1}(U) \subset T^*\mathfrak{f}\) (resp. is gradient-like) for the function \(\Phi = \mathfrak{f}^*\varphi + Q : T^*\mathfrak{f} \to \mathbb{R}\), for any positive definite quadratic form \(Q\) on the bundle \(\mathfrak{f} : T^*\mathfrak{f} \to E\).

**Proof.** In prior lemmas, we have seen \(\omega_{\parallel} = d\lambda_{\parallel}\) is a symplectic form with Liouville vector field \(v_{\parallel}\) as asserted in (2), with (exact) Lagrangians satisfying (4). The description of \(v_{\parallel}\) in (2) immediately implies (3) since we assume \(\mathfrak{f} : E \to M\) is proper, hence may lift any integral curve of \(v_M\) to an integral curve of \(\hat{\nu}_M\). The description of \(v_{\parallel}\) in (2) also immediately implies (5) which in turn implies (1). \(\square\)

**Example 9.8.** Let \((M, \omega_M = d\lambda_M)\) be a Liouville (resp. Weinstein) manifold with a compatible almost complex structure \(J_M\). Then we can take \(E\) to be the unitary frame bundle of \(M\) and choose any principal connection. Furthermore, for any compact manifold \(F\) with a unitary action, in particular any homogenous space, we can take \(E\) to be the associated bundle. For example, we can take \(E\) to be the Lagrangian Grassmannian bundle of \(M\).

### 9.2. Polarizations

**Lemma 9.9.** Let \((M, \omega)\) be a symplectic manifold, \(\mathfrak{f} : LGr(TM) \to M\) the bundle of Lagrangian Grassmannians of its symplectic tangent bundle, and \(T^*\mathfrak{f}\) the total space of the relative cotangent bundle. A choice of almost complex structure on \(M\) determines a symplectic structure on \(T^*\mathfrak{f}\) along with a canonical Lagrangian distribution, i.e. section of \(LGr(TT^*\mathfrak{f}) \to T^*\mathfrak{f}\).

**Proof.** We saw above that such a complex structure determines a connection on \(\mathfrak{f}\) and thence a symplectic structure on \(T^*\mathfrak{f}\). We also a splitting of bundles on \(T^*\mathfrak{f}\) (induced by the connection) \(T(T^*\mathfrak{f}) = T\mathfrak{f} \oplus \mathfrak{f}^*TM\), where \(\mathfrak{f} : T^*\mathfrak{f} \to M\) was the bundle of relative cotangent bundles. Note \(T\mathfrak{f}|_{E_m} = TT^*E_m\), and that \(T\mathfrak{f}\) contains as a sub-bundle \(\pi_\parallel T^*\mathfrak{f}\).
Recall we describe points of $T^*\mathfrak{f}$ as $(m, q, p)$ where $m$ is a point in $M$, $q$ is a point in the Lagrangian Grassmannian of $T_m M$, and $p \in \pi^{-1}(m, q)$ is a relative cotangent vector. Now over a point $(m, q, p)$, we can take the direct sum of the subspace of $\mathfrak{f}^*TM$ named by $q$ with the fibre of $\pi^*T^*\mathfrak{f}$ summand. Evidently this is isotropic for $\omega$, and half-dimensional, hence Lagrangian.

**Lemma 9.10.** Let $V$ be a contact manifold with contact form $\lambda$, let $\mathfrak{f} : LGr(\ker \lambda) \to M$ the bundle of Lagrangian Grassmannians of the contact distribution, and $T^*\mathfrak{f}$ the total space of the relative cotangent bundle. A choice of almost complex structure on $\ker(\lambda)$ determines a contact form $\lambda_\mathfrak{f}$ on $T^*\mathfrak{f}$ along with a canonical Lagrangian distribution in $\ker(\lambda_\mathfrak{f})$.

**Proof.** Similar to the previous lemma. \hfill \Box

10. **Maslov data and descent**

10.1. **A local system of categories over the Lagrangian Grassmannian.** If $M$ is any smooth manifold and $\tau$ a stable polarization on $T^*M$, then $\mu sh_{M, -\tau}$ is a locally constant sheaf of categories, locally isomorphic to the (symmetric monoidal) coefficient category $\mathcal{C}$. Note the restriction maps of $\mu sh$, being built ultimately from the six operations, are $\mathcal{C}$-linear. It follows that the monodromy is $\mathcal{C}$-linear, hence given by tensor product with some invertible element of $\mathcal{C}$. That is, the sheaf of categories $\mu sh_{M, -\tau}$ is classified by some map $-\tau : M \to BPic(C)$. The minus sign is to remind that we define $\mu sh$ using a stable normal polarization, which we can obtain by negating a tangent polarization.

**Remark 10.1.** Note that stable polarizations form a group, and that $Map(M, BPic(C))$ has an evident group structure. It will follow from our later considerations that the map from polarizations to this mapping space is a group homomorphism. Indeed, stable polarizations on $T^*M$ are classified by $Map(M, LGr)$, and we will now construct a group homomorphism $LGr \to BPic(C)$. Here $LGr$ is the stable Lagrangian Grassmannian.

We write $LGr(n)$ for Lagrangian Grassmannian, parameterizing Lagrangian subspaces in a 2n dimensional symplectic vector space. The exact symplectic manifold $T^*LGr(n)$ carries its natural polarization as a cotangent bundle. We may however create another stable polarization: inside $T^*LGr(n) \times \mathbb{R}^{2n}$, we may take the polarization given by the cotangent fiber in the first factor, and by the Lagrangian named by the given point in the second. We write $\chi$ for this polarization; there is a corresponding map $-\chi : LGr(n) \to BPic(C)$. On the other hand we can also consider the tautological Lagrangian $\mathcal{L}_n \subset T^*LGr(n) \times \mathbb{R}^{2n}$. Using now the trivial (cotangent times constant) polarization on $T^*LGr(n) \times \mathbb{R}^{2n}$, we consider $\mu sh_{\mathcal{L}_n}|_{LGr(n)}$. This defines another map $\chi_n : LGr(n) \to BPic(C)$. (Once we substantiate the above remark, it follows that these two maps are inverse as their notation indicates.)

There is a natural inclusion $LGr(n) \subset LGr(n + 1)$, given by direct sum with some fixed Lagrangian $\mathbb{R} \subset \mathbb{R}^2$. Observe that over a neighborhood of $LGr(n)$, the total Lagrangian $\mathcal{L}_{n+1}$ is a trivial thickening (i.e. a product with some $\mathbb{R}^k \subset T^*\mathbb{R}^k$) of $\mathcal{L}_n$. It follows that $\mu sh_{\mathcal{L}_{n+1}}|_{\mathcal{L}_n} \cong \mu sh_{\mathcal{L}_n}$ and therefore that $\chi_{n+1}|_{LGr(n)} \cong \chi_n$. This compatible family determines a map $\chi : LGr \to BPic(C)$

To show this map is a group homomorphism amounts to checking a series of compatibilities; we will describe the first but all the others are checked by the same principle. There is an
embedding \( LGr(n) \times LGr(m) \to LGr(n + m) \) induced from direct sum of Lagrangians. We should show there is a natural isomorphism \( \chi_{n+m}|_{LGr(n)\times LGr(m)} \cong \chi_n \boxtimes \chi_m \). Again, \( \mathcal{L}_{n+m}|_{LGr(n)\times LGr(m)} \) is a trivial thickening of \( \mathcal{L}_n \times \mathcal{L}_m \), so we have \( \mu sh_{\mathcal{L}_{n+m}}|_{\mathcal{L}_n \times \mathcal{L}_m} \cong \mu sh_{\mathcal{L}_n \times \mathcal{L}_m} \).

To conclude we need to know that this latter is \( \mu sh_{\mathcal{L}_n} \boxtimes \mu sh_{\mathcal{L}_m} \). But there is always a (fully faithful) morphism \( \mu sh_{\chi} \boxtimes \mu sh_{\chi} \to \mu sh_{\chi \times \chi} \), which by Lemma 5.3 is an isomorphism if either factor is a smooth Lagrangian.

We summarize this discussion in the following:

**Theorem 10.2.** Microsheaves on the tautological Lagrangian form a locally constant sheaf of categories classified by a group homomorphism \( \chi : LGr \to BPic(\mathcal{C}) \).

10.2. **Forgetting the polarization.** We now relax the requirement of the existence of a polarization \( \sigma \). Note that polarizations always exist locally, so for any contact \( U \) we should naturally obtain a sheaf of categories over \( LGr(\nu_U) \), locally constant in the Grassmannian direction. A clean way to see this is to form the Lagrangian Grassmannian of the contact distribution, \( LGr_U \), and the total space of its relative cotangent bundle, \( \tilde{U} \). Then per Lemma 9.10, \( \tilde{U} \) has a canonical polarization, thus we may consider \( \mu sh_{LGr_U} \). Similarly for \( \Lambda \subset U \), we have the full subsheaf of subcategories \( \mu sh_{LGr_A} \).

**Lemma 10.3.** Locally on \( \Lambda \), fix a choice of isomorphism of the contact distribution with a trivial symplectic bundle, and correspondingly an isomorphism \( LGr_{\Lambda} \cong \Lambda \times LGr \). Then \( \mu sh_{LGr_{\Lambda}} \cong \mu sh_{\Lambda} \boxtimes \chi \), where \( \chi \) is the local system of categories constructed in the previous section, and \( \mu sh_{\Lambda} \) is defined by a choice of local stable polarization (which locally exists and is unique up to contractible choice).

**Proof.** As noted above, there is always a natural morphism \( \mu sh_{\chi} \boxtimes \mu sh_{\chi} \to \mu sh_{\chi \times \chi} \), which is an isomorphism when one factor is a manifold inside its cotangent bundle.

In order to organize these local isomorphisms, we consider the following composite map.

\[
\mathcal{U} \to BU \to B^{\mathcal{L}}Gr \xrightarrow{B^{\chi}} B^{\mathcal{S}}Gr(\mathcal{C})
\]

Here the map \( \mathcal{U} \to BU \) classifies the symplectic tangent bundle, the map \( BU \to B^{\mathcal{L}}Gr \) is induced from the map sending a symplectic vector bundle to the fiberwise Lagrangian Grassmannian, and the final map is the delooping of the map \( \chi : LGr \to B^{\mathcal{S}}Gr(\mathcal{C}) \) constructed in the previous section (which exists and is unique because we showed that said map is a group homomorphism). We will write \( B^{\mathcal{S}}Gr(\mathcal{C})_{\mathcal{U}} \) for the \( B^{\mathcal{S}}Gr(\mathcal{C}) \) bundle classified by this composite.

**Lemma 10.4.** The sheaf of categories \( \mu sh_{LGr_{\mathcal{U}}} \) descends to a sheaf of categories \( \mu sh_{B^{\mathcal{S}}Gr(\mathcal{C})_{\mathcal{U}}} \) on \( B^{\mathcal{S}}Gr(\mathcal{C})_{\mathcal{U}} \), which is locally constant along the fiber.

**Proof.** The homomorphism \( \chi : LGr \to B^{\mathcal{S}}Gr(\mathcal{C}) \) classifies a character \( Pic(\mathcal{C}) \)-bundle on \( LGr \); to avoid confusion, we will denote it here by \( F_{\chi} \) where we previously wrote \( \chi \). (Informally speaking, the character structure means the pullback \( m^*F_{\chi} \) along the multiplication \( m : LGr \times LGr \to LGr \) is equipped with an isomorphism to \( F_{\chi} \boxtimes F_{\chi} \) along with higher coherence.) We also have the canonical character \( Pic(\mathcal{C}) \)-bundle \( F_{can} \) on \( B^{\mathcal{S}}Gr(\mathcal{C}) \) classified by the identity homomorphism \( B^{\mathcal{S}}Gr(\mathcal{C}) \to B^{\mathcal{S}}Gr(\mathcal{C}) \), along with a canonical equivalence of character bundles \( F_{\chi} \cong \chi^*F_{can} \).

We know by Theorem 10.2 and Lemma 10.3 that the sheaf \( \mu sh_{LGr_{\mathcal{U}}} \) is \( (LGr, F_{\chi}) \)-equivariant. (Informally speaking, the equivariance means the pullback \( a^*\mu sh_{LGr_{\mathcal{U}}} \) along the action \( a : \)}
$LGr \times LGr_{\mathcal{U}} \to LGr_{\mathcal{U}}$ is equipped with an isomorphism to $F_{\chi} \boxtimes \mu sh_{LGr_{\mathcal{U}}}$ along with higher coherence.) Such equivariance precisely means $\mu sh_{LGr_{\mathcal{U}}}$ descends to a $(BPic(\mathcal{C}), F_{can})$-equivariant sheaf on the induced $Pic(\mathcal{C})$-bundle $BPic(\mathcal{C})_{\mathcal{U}}$. In particular, the descent is implemented by taking $LGr$-invariants in the sheaf $F_{can} \boxtimes \mu sh_{LGr_{\mathcal{U}}}$ on $BPic(\mathcal{C}) \times LGr_{\mathcal{U}}$. □

**Definition 10.5.** We call the composite map $\mathcal{U} \to B^2Pic(\mathcal{C})$ the $(\mathcal{C}\text{-valued})$ Maslov obstruction. By Maslov data we mean a choice of a null-homotopy of this map.

A polarization on $\mathcal{U}$ is a null-homotopy of $\mathcal{U} \to BU \to BLGr$, so determines by composition a choice of Maslov data. Moreover, a choice of polarization $\tau$ gives a section $\tau$ of the bundle $LGr_{\mathcal{U}} \to \mathcal{U}$. By construction, $\tau^* \mu sh_{LGr_{\mathcal{U}}} \cong \mu sh_{\mathcal{U}, \tau}$.

More generally,

**Definition 10.6.** Given a choice of Maslov data $\tau$, we define $\mu sh_{\mathcal{U}, \tau} := \tau^* \mu sh_{BPic(\mathcal{U})}$, where $\tau : \mathcal{U} \to BPic(\mathcal{C})_{\mathcal{U}}$ is the section classified by the Maslov data.

Evidently if $\tau$ came from a polarization, $\mu sh_{\mathcal{U}, \tau}$ defined this way agrees with the previous definition. Moreover if two (possibly different) polarizations determine the same Maslov data, then they determine the same category.

**Proposition 10.7.** Theorem 8.7, Corollary 8.9 and Theorem 8.15 hold as stated with $\tau$ understood as Maslov data rather than as a polarization.

**Proof.** In the proof of those theorems, replace all objects with the Lagrangian Grassmannian bundles over them. □

**Remark 10.8.** Consider in particular the case of a Weinstein manifold $W$ and a smooth exact Lagrangian $L \subset W$. Fixing Maslov data $\tau$ on $W$, we have by Cor. 8.9 a fully faithful functor $\mu sh_{L, \tau}(L) \to \mu sh_{W, \tau}(W)$. Evidently this is similar to the statement that a Lagrangian (equipped with appropriate structures) determines an object in the Fukaya category. To make a more direct comparison, note that in a neighborhood (of the contact lift of) $L$, there is a canonical polarization $\ell$, coming from the structure as a jet bundle. We know that $\mu sh_{L, \ell}(L)$ is nothing other than the category of local systems on $L$. Thus an isomorphism $\ell \cong \tau$ of Maslov data determines

$$Loc(L) \cong \mu sh_{L, \ell}(L) \cong \mu sh_{L, \tau}(L) \to \mu sh_{W, \tau}(W)$$

We term such an isomorphism “secondary Maslov data”. (It has elsewhere been called ‘brane data’ [27].) Note the obstruction to the existence of this data is the difference $\ell - \tau \in [L, BPic(\mathcal{C})]$. In the case when the coefficient category $\mathcal{C}$ is the category of $\mathbb{Z}$-modules, $Pic(\mathcal{C}) = \mathbb{Z} \oplus B(\mathbb{Z}/2)$, and so $[L, BPic(\mathcal{C})] = H^1(L, \mathbb{Z}) \oplus H^2(L, \mathbb{Z}/2)$. When in addition $W = T^*M$ and $\tau$ is the fiber polarization, then the obstruction to the existence of secondary Maslov data would be the Maslov class of $L$ in the first factor, and $w_2(L, M)$ in the second, though in fact these are known to always vanish. However, the same calculation applies for Legendrians in $J^1(M)$, which can have nontrivial such classes.

**Remark 10.9.** Work of Jin [24, 26] shows that the map $\chi : LGr \to BPic(\mathcal{C})$ is in fact the topologists’ $J$-homomorphism in the universal case when $\mathcal{C}$ is the category of spectra. The specialization to the case of $\mathcal{C}$ the category of $\mathbb{Z}$ modules is essentially in [22]. We do not depend on these results for the abstract setup discussed above, but they are of course invaluable for actually constructing or characterizing Maslov obstructions and Maslov data in practice.
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