Large spin systematics in CFT

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In the last few years there has been an increasing interest in conformal field theories in space time dimensions higher than two. Much of this interest is due to the effectiveness of the conformal bootstrap program. [Rattazzi, Rychkov, Tonni, Vichi]

This approach consists in constraining the CFT data by requiring consistency of higher correlation functions, namely crossing-symmetry, together with basic properties of well behaved CFT’s, such as unitarity and the structure of the OPE.

In this talk I will propose how to use these properties and symmetries in order to constrain CFT data for operators with large spin.

Strongly coupled CFTs become free for operators with infinite spin – new perturbation expansion in inverse powers of spin. What is the systematics of this expansion?

Our results hold for a vast family of theories!
Motivation

The problem
In this talk we will consider operators

$$O \partial_{\mu_1} \ldots \partial_{\mu_\ell} O$$

in a Conformal Field Theory (CFT) and study their conformal dimensions $\Delta$ and three-point couplings for large values of the spin $\ell$.

The motivation

- They form the leading twist sector of many theories.
- Important role in QCD analysis of deep inelastic scattering.
- Related to divergencies of Wilson loops with cusps and scattering amplitudes.
- Fundamental role in applying integrability to AdS/CFT.
Leading twist operators and reciprocity

General four dimensional gauge theory:

Leading twist operators built from scalars and derivatives

\[ \mathcal{O}_\ell = \text{Tr} (\phi \partial_{\mu_1} \ldots \partial_{\mu_\ell} \phi) + \ldots \]

- At large values of the spin \( \ell \), they acquire a logarithmic anomalous dimension \([\text{Gross, Wilczek}]\)

\[ \Delta_\ell - \ell = f(g) \log \ell + \ldots \]

- This behaviour is valid to all orders in perturbation theory!

- \( f(g) \) is the cusp anomalous dimension – appears in many computations.

What can we say about subleading corrections?

\[ \Delta_\ell - \ell = f(g) \log \ell + f^{(0)}(g, \log \ell) + \frac{f^{(1)}(g, \log \ell)}{\ell} + \frac{f^{(2)}(g, \log \ell)}{\ell^2} + \ldots \]
Reciprocity

Reciprocity principle

Odd powers of $1/\ell$ are fixed in terms of the even ones!

\[ \gamma_\ell \equiv F(\ell + \frac{1}{2} \gamma_\ell) \rightarrow F(\ell) = a_0 (\log J_0) + \frac{a_2 (\log J_0)}{J_0^2} + \frac{a_4 (\log J_0)}{J_0^4} + \ldots \]

where $\gamma_\ell = \Delta_\ell - \ell - 2$ and $J_0^2 = \ell (\ell + 1)$.

- First observed in QCD
- Checked for many other theories, including MSYM
- But a proof was still missing!

More convenient formulation

Define the full Casimir:

\[ J^2 = (\ell + \gamma_\ell/2)(\ell + \gamma_\ell/2 + 1) \]

Reciprocity principle

\[ \gamma_\ell = a_0 (\log J) + \frac{a_2 (\log J)}{J^2} + \frac{a_4 (\log J)}{J^4} + \ldots \]

Let us prove this for a generic four dimensional CFT!
Correlation functions in conformal field theories

- **Two-point functions** of conformal primary scalar operators:

  \[ \langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{\delta_{12}}{|x_{12}|^{2\Delta_1}}. \]

- **Three-point functions** of conformal primary scalar operators:

  \[ \langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle = \frac{c_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_1+\Delta_3-\Delta_2}}. \]

- All higher-point correlators can be obtained from **Operator Product Expansion** (OPE)

  \[ \phi_1(x) \times \phi_2(0) = \sum_{\mathcal{O}} c_{12\mathcal{O}} \left( \frac{x^{\mu_1} \ldots x^{\mu_\ell}}{|x|^{\Delta_1+\Delta_2-\Delta_\mathcal{O}+\ell}} \mathcal{O}_\mu(0) + \text{descendants} \right). \]

**CFT data**

Conformal dimensions and OPE coefficients completely determine given conformal field theory

Higher-point correlators → consistency conditions for the CFT data.
Conformal bootstrap equation

- **Four-point correlator** (of identical scalar operators with dimension $d$) from conformal symmetry:

\[
\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{G(u, v)}{x_{12}^{2d}x_{34}^{2d}}.
\]

$G(u, v)$ – function of **cross-ratios**

\[
u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2} \quad v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}
\]

- **OPE → conformal partial wave decomposition**

\[
G(u, v) = 1 + \sum_{\Delta, \ell} c_{\Delta, \ell}^2 G_{\Delta, \ell}(u, v) = \]

$G_{\Delta, \ell}(u, v)$ – **conformal blocks** resumming all contributions from conformal descendants.

- Crossing symmetry: correlator symmetric under exchange of any two space-time points.

Conformal bootstrap equation
Conformal blocks

Properties of conformal blocks

- Conformal blocks repack all the contributions of all descendants of a given primary.
- In four dimensions they are known in a closed form. [Dolan, Osborn]
- Small $u$ behaviour:
  \[
  G_{\Delta, \ell}(u, v) \sim u^{\frac{1}{2}(\Delta - \ell)} k_{\Delta, \ell}(1 - v)
  \]
  
  \[
  k_{\Delta, \ell}(1 - v) = (1 - v)_{\ell} {}_{2}F_{1}\left(\frac{\Delta - \ell}{2}, \frac{\Delta - \ell}{2}, \Delta - \ell, 1 - v\right)
  \]
  
  The power of $u$ is controlled by the twist $\tau = \Delta - \ell$.

- Small $u$ behaviour is independent of the space-time dimension.
**At the tree-level:**

- \( \gamma_\ell = 0 \) and \( a_\ell = a_\ell^{(0)} \)
  \[
  \sum_{\ell=0,2,\ldots} a_\ell^{(0)} k_{\ell+2,\ell}(1 - v) = \frac{1}{v} + 1
  \]

- We can solve for \( a_\ell^{(0)} \) and obtain
  \[
  a_\ell^{(0)} = 2 \frac{\Gamma(\ell + 1)^2}{\Gamma(2\ell + 1)}
  \]

- Each term in the sum **diverges logarithmically** as \( v \to 0 \).

- We need an **infinite number** of terms to reproduce the divergence on the right hand side!

- The divergence comes solely from the region \( \ell \gg 1 \).

- Large \( \ell \) behaviour of \( a_\ell^{(0)} \) is **fixed** by the divergence of the right hand side.
In perturbation theory

\[ \tau_\ell = \tau_0 + \gamma_\ell(g), \quad a_\ell = a_\ell^{(0)} \hat{a}_\ell(g) \]

\[ \mathcal{G}(u, v) = \mathcal{G}^{(0)}(u, v) + g \mathcal{G}^{(1)}(u, v) + \ldots \]

- To any order in perturbation theory

\[
\frac{1}{v} + 1 \rightarrow \frac{1}{v} h^{(0)}(\log u, \log v) + h^{(1)}(\log u, \log v) + v h^{(2)}(\log u, \log v) + \ldots
\]

- Only integer powers of \( v \) can appear due to analitycity!

- We want to understand how to evaluate the divergence in \( v \) starting from the partial wave sum as \( v \rightarrow 0 \).

- The divergence will come from the region of large \( \ell \).
The method and perturbative results

Consider

$$\sum_{\ell} a_{\ell} u^{\gamma_{\ell}/2} k_{\ell+2+\gamma_{\ell}, \ell}(1 - v) = \frac{1}{v} h^{(0)}(\log u, \log v) + h^{(1)}(\log u, \log v) + \ldots$$

Assume a general large $\ell$ behaviour

$$\gamma_{\ell} = p_0(\log J) + \frac{p_1(\log J)}{J} + \frac{p_2(\log J)}{J^2} + \ldots$$

$$\hat{a}_{\ell} = q_0(\log J) + \frac{q_1(\log J)}{J} + \frac{q_2(\log J)}{J^2} + \ldots$$

where $J^2 = (\ell + \gamma_{\ell}/2)(\ell + \gamma_{\ell}/2 + 1)$.

Results

- $\gamma(\ell)$ expanded for large $\ell$ contains **only even powers** of $1/J$
- $\frac{\hat{a}(\ell)}{2+\gamma'(\ell)}$ expanded for large $\ell$ contains **only even powers** of $1/J$

These results are valid at **any loop order** in perturbation theory!

They extend to **arbitrary dimensions**, and general **non-scalar operators**!
Generalizations

Non-perturbative CFT

- Given $\mathcal{O}$ of dimension $\Delta_\mathcal{O}$ there are double trace operators $\mathcal{O}\partial^\ell\mathcal{O}$ of dimension

$$\Delta_\ell - \ell = 2\Delta_\mathcal{O} + \gamma_\ell, \quad \gamma_\ell = -\frac{c}{\ell\tau_{\text{min}}} + \ldots$$

$\tau_{\text{min}}$: Twist of the minimal twist operator in OPE of $\mathcal{O} \times \mathcal{O}$.

- Our method applies also to this case:

$$\gamma_\ell = -\frac{c}{J\tau_{\text{min}}} \left(1 + \frac{p_2}{J^2} + \frac{p_4}{J^4} + \ldots\right)$$

- Extensively checked for the critical $O(N)$ model and theories with holographic dual.

- Similar techniques allow to find analytic expressions for conformal dimensions of operators in the 3D Ising model, compatible with the predictions from numerical bootstrap!

[Alday, Maldacena, Fitzpatrick et.al, Komargodski et. al.]
Non-conformal theories

- We can consider a theory with non-vanishing beta function.
- The anomalous dimension will depend on the scheme. Using DREG with $D = 4 - 2\epsilon$
  $$\beta_\epsilon = -2\epsilon + \beta(g)$$
- Beta function vanishes at $\epsilon_{cr} = \beta(g)/2$.
- Only even powers appear in the expansion in terms of the corrected Casimir
  $$J_\beta^2 = (\ell + \gamma_\ell/2 - \beta/2)(\ell + \gamma_\ell/2 - \beta/2 + 1)$$
- We have checked this in the case of QCD and $\mathcal{N} = 0, 1, 2$ SYM theories!
Conclusions

- We have derived an infinite number of constraints for the large spin expansion for operators with large spin
  - anomalous dimension → proof of reciprocity for leading twist operators
  - OPE coefficients → new reciprocity principle for structure constants
- The derivation relied solely in CFT arguments and symmetries, and applies to a large class of theories.

Outlook:

- Reciprocity served as a powerful assumption to make progress in the computation of the dimension of twist-two operators in MSYM. Can we use this to make progress in the computation of structure constants?
- Can we use the full power of crossing symmetry?
- Can we extend our analysis to higher orders in $u$ expansion – subleading twist?
Thank you.
The method:

Focus in the small $v \leftrightarrow$ large $\ell$ region:

$$v = \epsilon, \quad \ell = \frac{x}{\epsilon^{1/2}}, \quad \sum_{\ell} \to \frac{1}{2} \int_{0}^{\infty} dx$$

Introduce the rescaled Casimir and perform a change of variables $x \to j$:

$$\frac{j^2}{\epsilon} = \left( \frac{x}{\epsilon^{1/2}} + \gamma \ell \right) \left( \frac{x}{\epsilon^{1/2}} + \gamma \ell + 1 \right)$$

Use integral representation for the hypergeometric function and the saddle point method:

$$\frac{4}{\epsilon} \int_{0}^{\infty} q_{0} u_{p_{0}/2}^{0} j K_{0}(2j) dj + \frac{1}{\epsilon^{1/2}} \int_{0}^{\infty} u_{p_{0}/2}^{0} (q_{0} p_{1} \log u + 2q_{1} - q_{0} p_{0}') K_{0}(2j) dj + \text{finite}$$

Claim: all divergent terms (not only the leading one) are captured!

From the right hand side: $v^{-1} h^{(0)}(\log u, \log v) + v^{0} h^{(1)}(\log u, \log v) + \ldots$

$$\int_{0}^{\infty} u_{p_{0}/2}^{0} (q_{0} p_{1} \log u + 2q_{1} - q_{0} p_{0}') K_{0}(2j) dj = 0$$

[Alday, Maldacena; Fitzpatrick et al; Komargodski et al, ...]
At any order in perturbation theory $p_i, q_i$ are polynomials in $\log j^2/\epsilon$

$$\int_0^\infty P(\log j^2/\epsilon)K_0(2j)dj = 0 \implies P(\log j^2/\epsilon) = 0 \implies p_1 = 0, \quad q_1 = \frac{1}{2} q_0 p_0'$$

Acting with $\mathcal{D}$ multiplies the integrand by $\frac{f^2}{\epsilon}$ and increases the power of divergence by one $\rightarrow$ the right hand side is still a function of integer powers of $v$ only!

We obtain a new constraint, involving the higher order terms in the expansions!