Three-dimensional isolated quotient singularities in odd characteristic

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Abstract. Let a finite group $G$ act linearly on a finite-dimensional vector space $V$ over an algebraically closed field $k$ of characteristic $p > 2$. Suppose that the quotient space $V/G$ has an isolated singularity only. The isolated singularities of the form $V/G$ are completely classified in the case when $p$ does not divide the order of $G$, and their classification reduces to Vincent’s classification of isolated quotient singularities over $\mathbb{C}$. In the present paper we show that, if $\dim V = 3$, then the classification of isolated quotient singularities reduces to Vincent’s classification in the modular case as well (when $p$ divides $|G|$). Some remarks on quotient singularities in other dimensions and in even characteristic are also given.

Bibliography: 14 titles.

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§ 1. Introduction

Let $V$ be an algebraic variety defined over an algebraically closed field $k$ of characteristic $p$. Let $G$ be a finite group acting on $V$. Then there exist an algebraic variety $V/G$ and a morphism $\pi: V \to V/G$ such that the pair $(\pi, V/G)$ is a geometric factor for the action of $G$ on $V$. Let $P$ be a fixed point for the action of $G$ on $V$. We say that the point $Q = \pi(P) \in V/G$ is an isolated singularity if $V/G$ is singular at $Q$ (that is, the local ring $\mathcal{O}_{V/G, Q}$ is not regular) and there are no other singular points of $V/G$ in some Zariski open neighbourhood of $Q$. When $p$ does not divide the order $|G|$ of $G$, the isolated quotient singularities are completely classified up to formal, or, when $k = \mathbb{C}$ is the field of complex numbers, to analytic equivalence. The classification over $\mathbb{C}$ was obtained mainly by Vincent as a part of the classification of manifolds of constant positive curvature. This classification is presented in Ch. 5–7 of Wolf’s book [1] (see also our survey in [2] written from the point of view of singularity theory). It can readily be shown (see Theorem 3.13 in [2]) that Vincent’s classification can be extended almost verbatim also to isolated quotient singularities over an arbitrary algebraically closed field $k$ of characteristic 0 and of prime characteristic $p$, where $p$ does not divide $|G|$. Therefore, in what follows, by
Vincent’s classification we mean this generalized classification. The modular case $p \mid |G|$ remains open.

The first difficulty of the modular case is that the action of $G$ on $V$ is not linearizable in general in a formal neighbourhood of a fixed point $P$. This means that, in the general case, one cannot choose a system of parameters $x_1, \ldots, x_n$ in the complete local ring $\mathcal{O}_{V, P}$ for which $G$ acts on $\mathcal{O}_{V, P}$ by linear substitutions in the variables $x_1, \ldots, x_n$. At the same time, many isolated quotient singularities arise as quotients by a nonlinear action of a modular group (for example, see [3]). Thus, the classification problem is not reduced to a problem of the theory of linear representations (in contrast to the nonmodular case). In this paper, we do not consider nonlinear actions; therefore, we assume throughout that $V$ is a vector space and that $G$ is a subgroup of the general linear group $GL(V)$. By a classification of the isolated quotient singularities we mean a clearly described list $\{G_i\}$, $i \in I$, of subgroups of $GL(V)$ such that, if $Q = \pi(O) \in V/G$, $G < GL(V)$, is an isolated quotient singularity, then it is formally isomorphic to one of the singularities $\pi_i(0) \in V/G_i$, that is, the complete local ring of $Q \in V/G$ is isomorphic to the complete local ring of $\pi_i(O) \in V/G_i$ for some group $G_i$ in the given list. Under some additional assumptions ($G$ contains no pseudo-reflections and $k = \mathbb{C}$; for example, see Lemma 2.8 in [2]) one can even claim that every group $G$ giving an isolated quotient singularity is conjugate in $GL(V)$ to one of the groups $G_i$; however, we do not know whether or not a similar assertion holds in the modular case.

Another difficulty is that the inverse Chevalley-Shephard-Todd-Serre Theorem (see Theorem 1.1 below) fails to hold in the modular case. A linear map $g : V \to V$ is said to be a pseudo-reflection if it has finite order and the set of points fixed by the action of $g$ is a hyperplane. We denote by $S(V^*)$ the symmetric algebra of the space $V^*$ dual to $V$.

**Theorem 1.1** (see Theorem 7.2.1 in [4]). Let $V$ be a finite-dimensional vector space over a field $k$ of characteristic $p \geq 0$ and let $G$ be a subgroup of $GL(V)$. If the ring of invariants $S(V^*)^G$ of $G$ is polynomial (that is, $V/G$ is nonsingular), then $G$ is generated by pseudo-reflections. If $p$ does not divide the order of $G$, then the converse also holds.

In the modular case the quotient $V/G$ can be singular even for groups $G$ generated by pseudo-reflections, as the example of the symmetric group $S_6$ in its 4-dimensional irreducible 2-modular representation shows (see Example 2.2 in [5]). However, one can show that the quotient singularity $V/S_6$ is not isolated in this example. In fact, this is a general phenomenon, as follows from a remarkable result due to Kemper and Malle.

**Theorem 1.2** (see the main theorem in [5]). Let $V$ be a finite-dimensional vector space and let $G$ be a finite irreducible subgroup of $GL(V)$. Then $S(V^*)^G$ is a polynomial ring if and only if $G$ is generated by pseudo-reflections and the pointwise stabilizer in $G$ of any nontrivial subspace of $V$ has a polynomial ring of invariants.

If the irreducibility condition could be omitted in Theorem 1.2, the classification of isolated (linear) quotient singularities would reduce to the nonmodular case, that is, to Vincent’s classification. More precisely, in §3 we prove the following equivalence.
Theorem 1.3. The following assertions are equivalent:

1) (conjecture (S)) the Kemper-Malle Theorem, Theorem 1.2, holds for all (rather than only for irreducible) finite linear groups $G$;

2) let $G$ be a finite subgroup of $\text{GL}(V)$ and let $H$ be a normal subgroup of $G$ generated by pseudo-reflections. If the singularity of $V/G$ is isolated, then the variety $V/H$ is nonsingular, $p(= \text{char } k) \nmid |G/H|$ and

$$V/G \simeq (V/H)/(G/H),$$

that is, every isolated quotient singularity of $V/G$ is naturally isomorphic to a non-modular quotient singularity.

It should be noted that, since $G/H$ is nonmodular, its action can be linearized locally formally in a neighbourhood of a fixed point.

The ground field $k$ is not necessarily algebraically closed in the Kemper-Malle Theorem (Theorem 1.2). However, it can readily be seen that the theorem holds for a general field $k$ if and only if it holds for the algebraic closure $\overline{k}$ of $k$. Therefore, the assumption $k = \overline{k}$ leads to no loss of generality.

The assertion 1) of Theorem 1.3 was stated by several experts in the 1990s; we refer to it as conjecture (S) (from smooth). Kemper and Malle proved this conjecture not only for irreducible groups but also for groups $G$ acting on a 2-dimensional vector space $V$ (the conjecture was first proved for 2-dimensional groups in characteristic $p > 3$ by Nakajima in [6]) and for some indecomposable groups, and showed that it is sufficient to prove the conjecture in the indecomposable case (see [5]).

We are mainly interested in the classification problem for isolated quotient singularities, and we consider conjecture (S) as a key to the modular case. Indeed, as follows from Theorem 1.3, the meaning of the conjecture is that, if it holds, then, in essence, new modular isolated (linear) quotient singularities do not exist, that is, the existing singularities are in fact isomorphic to nonmodular singularities. On the other hand, if assertion 2) of Theorem 1.3 could be proved using methods of algebraic geometry, then this would also prove the conjecture. So far, we have succeeded only in dimension 3 and in odd characteristic $p > 2$. The method suggested by us uses the classification of 2-dimensional groups generated by transvections and cannot be generalized to higher dimensions. Our main result is the following.

Theorem 1.4. Let $V$ be a 3-dimensional vector space over an algebraically closed field of characteristic $p > 2$. Let $G$ be a finite subgroup of $\text{GL}(V)$ generated by pseudo-reflections. If the quotient variety $V/G$ is singular, then the singularity is not isolated; thus, conjecture (S) holds for these groups.

Theorem 1.4 implies an assertion on the formal classification of three-dimensional isolated quotient singularities in the general case (including the modular case).

Corollary 1.1. If $\dim V = 3$, $p > 2$, and $G$ is a finite subgroup of $\text{GL}(V)$ such that $0 \in V/G$ is an isolated singularity, then $0 \in V/G$ is formally isomorphic to one of the nonmodular isolated quotient singularities in Vincent’s classification.

In fact, our results are somewhat stronger than Theorem 1.4 and are applicable also to some groups in even characteristic; for the precise formulation, see §4.

The paper is organized as follows. In §2 we collect some material on isolated singularities which is used in the subsequent sections. In §3 we prove Theorem 1.3.
We also show that conjecture (S) holds for groups $G$ having a 1-dimensional invariant subspace, for example, for Abelian groups $G$. Section 4 is devoted to the proof of Theorem 1.4.

§ 2. Preliminaries

The results of this section have already been published in [7]. Lemma 2.1 was also known much earlier, although it is difficult for us to specify the source. For the convenience of the reader, we present these assertions using our notation and with complete proofs. Unless stated otherwise, in this section, $k$ stands for a field, of characteristic $p > 0$, which need not be algebraically closed.

Lemma 2.1. Let a finite group $G$ act linearly on a polynomial ring $R=k[x_1, \ldots, x_n]$. Then the ring of invariants $R^G$ is polynomial if and only if $R^G$ is regular at the maximal ideal $\mathfrak{m} \cap R^G$, where $\mathfrak{m} = (x_1, \ldots, x_n)$.

Proof. Only sufficiency needs a proof. Suppose that $R^G$ is regular at the ideal $\mathfrak{m} \cap R^G$ and $R^G$ is not polynomial. However, $R^G$ is a finitely generated $k$-algebra (see, for example, Theorem 1.3.1 in [4]); let $f_1, \ldots, f_m$ be a minimal family of generators of $R^G$. We denote by $g_1(y_1, \ldots, y_m), \ldots, g_r(y_1, \ldots, y_m)$ a generating set of all relations among $f_1, \ldots, f_m$. We note that the polynomials $f_i$, $i = 1, \ldots, m$, can be chosen to be homogeneous of positive degree, while the $g_j$, $j = 1, \ldots, r$, can be chosen to be weighted homogeneous with the condition

the weight of each variable $y_i$ is equal to $\deg f_i$.

This implies that $g_j(0, \ldots, 0) = 0$ for all $j = 1, \ldots, r$. Moreover, all monomials in the polynomial $g_j$ have degree $> 1$ because otherwise the set of generators $f_1, \ldots, f_m$ would not be minimal. Hence, by the Jacobian criterion, the ring

$$k[y_1, \ldots, y_m] \simeq R^G$$

is not regular at 0. A contradiction.

Consider now two algebras $A$ and $B$ without zero divisors over a field $k$, where $A$ is a subalgebra of $B$. Let $\mathfrak{m} \subset A$ and $\mathfrak{n} \subset B$ be maximal ideals such that $\mathfrak{n} \cap A = \mathfrak{m}$. We denote by $j: A \to B$ the inclusion map, by $\hat{A}$ and $\hat{B}$ the formal completions of $A$ at the ideal $\mathfrak{m}$ and of $B$ at the ideal $\mathfrak{n}$, respectively, and by $\hat{j}: \hat{A} \to \hat{B}$ the map induced by the inclusion $j$.

Lemma 2.2. Suppose that the following conditions hold:

1) the algebras $A$ and $B$ are Noetherian;
2) $A/\mathfrak{m} = B/\mathfrak{n} = k$;
3) $B$ is unramified at the ideal $\mathfrak{n}$ over $A$, that is, $\mathfrak{m}B_\mathfrak{n} = \mathfrak{n}B_\mathfrak{n}$, where $B_\mathfrak{n}$ stands for the localization of the ring $B$ with respect to $\mathfrak{n}$.

Then the map $\hat{j}$ is an isomorphism.

Proof. Let

$$j_n: \mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathfrak{n}^n/\mathfrak{n}^{n+1}, \quad n \geq 0,$$
be the natural map induced by the embedding $j$. By Lemma 10.23 in [8] it suffices to prove that $j_n$ is an isomorphism for every $n \geq 0$. We note that 
\[ n^n/n^{n+1} \simeq (nB_n)^n/(nB_n)^{n+1}, \]
and therefore the surjectivity of the morphism $j_n$ readily follows from condition 3). It remains to prove that $j_n$ is injective.

Suppose for a contradiction that $j_n$ is not injective for some $n \geq 0$. This means that there is an element $a \in m^n$ for which $a \notin m^{n+1} \cap B = n^{n+2} - 1$. Considering $a$ as an element of the ring $B_n$ and using condition 3) again, we can represent $a$ in the form 
\[ a = \sum b_i a_i, \]
where $b_i \in B_n$ and $a_i \in m^{n+1}$. Let us now use condition 2) and the fact that the field $k$ is contained both in $A$ and $B$ to write $b_i = b_i + b_i'$, where $b_i' \in k$ and $b_i' \in B_n$.

This enables us to represent $a$ in the form 
\[ a = \sum b_i^0 a_i + b', \]
where, on the other hand, $b'$ is an element of $A$ not belonging to $m^{n+1}$ (otherwise we would have $a \in m^{n+1}$); however, $b'$ belongs to $(nB_n)^{n+2} \cap B = n^{n+2}$. Moreover, $b' \neq 0$ because otherwise $a$ would not belong to the ideal $n^{n+1}$. Applying the same considerations to $b''$ we obtain $0 \neq b'' \in n^{n+3}$, $b'' \in m$ and $b'' \notin m^{n+1}$. This implies that $m^n \setminus m^{n+1}$ contains nonzero elements of the ideal $nN$ for arbitrarily large $N$.

The ideals $nN$ are $k$-vector subspaces of the algebra $B$. Condition 1) implies that $V = m^n/m^{n+1}$ is a finite-dimensional $k$-vector space. Thus, 
\[ V_N = (nN \cap m^n)/m^{n+1}, \quad N > n + 1, \]
is a descending sequence of subspaces of $V$. This sequence stabilizes at some subspace $W \subseteq V$. We have seen above that every term of the sequence has a nonzero element, and hence $W \neq 0$. On the other hand, $\bigcap_{n \geq 0} n^n = (0)$ by Krull’s Theorem, and thus we must have $W = 0$. This contradiction proves the injectivity of the morphism $j_n$ and completes the proof of the lemma.

The algebraic lemma, Lemma 2.2, can be applied to the quotient singularities as follows. Let $G$ be a finite group acting linearly on a vector space $V$. Let $P \in V$ be a (closed) point and let $H$ be the stabilizer of $P$ in $G$. Consider the quotient varieties $V/H = \text{Spec } S(V^*)^H$ and $V/G = \text{Spec } S(V^*)^G$. We denote by $Q \in V/H$ and $R \in V/G$ the images of $P$ with respect to the natural projections $p_H : V \to V/H$ and $p_G : V \to V/G$. The subgroup $H$ is not normal in $G$ in general, and thus there exists no natural group action on $V/H$; however, in any case, there exists a morphism $\varphi : V/H \to V/G$ making the following diagram commutative:

\[ V \xrightarrow{p_G} V/G \xrightarrow{\varphi} V/H \xrightarrow{p_H} V. \]
Let $\hat{\mathcal{O}}_{V/H,Q}$ and $\hat{\mathcal{O}}_{V/G,R}$ be the complete local rings of the points $Q \in V/H$ and $R \in V/G$, respectively.

**Lemma 2.3.** Suppose that the ground field $k$ is infinite. Then the map

$$\varphi: \hat{\mathcal{O}}_{V/H,Q} \to \hat{\mathcal{O}}_{V/G,R},$$

induced by the morphism $\varphi$ is an isomorphism.

**Proof.** Let $m_P$ be the maximal ideal of the point $P \in V$, let $m_Q = m_P \cap S(V^*)^H$ and $m_R = m_P \cap S(V^*)^G$. We apply now Lemma 2.2 to the algebras $A = S(V^*)^G$ and $B = S(V^*)^H$ and to the ideals $m_R$ and $m_Q$. We must verify condition 3) in Lemma 2.2, which claims that the algebra $B$ is unramified at the ideal $m_Q$ over $A$.

First, let us show that $m_Q$ is generated by polynomials $f_1, \ldots, f_n \in S(V^*)$ such that $f_i(P) = 0$ for all $i, 1 \leq i \leq n$, and $f_i(gP) \neq 0$ for all $g \in G, g \notin H$. Indeed, let us choose a linear function $l$ on $V$ such that $l(P) = 0$ and $l(gP) \neq 0$ for all $g \in G, g \notin H$. This choice is possible because the field $k$ is infinite. Consider the invariant

$$L(v) = \prod_{h \in H} l(hv)$$

of $H$. Suppose now that $f_1', \ldots, f_{n-1}'$ is an arbitrary generating set of $m_Q$. Then we can choose $c_1, \ldots, c_{n-1} \in k$ so that the system

$$f_1 = f_1' + c_1L, \quad \ldots, \quad f_{n-1} = f_{n-1}' + c_{n-1}L, \quad f_n = L$$

of generating sets of $m_Q$ has the desired property.

For every $i, 1 \leq i \leq n$, we consider now the polynomial

$$g_i(v) = N^G_H(f_i)(v) = \prod_g f_i(gv),$$

where $g$ ranges over a system of representatives of all right cosets of $H$ in $G$. By construction, $g_i$ is an invariant of $G$ belonging to $m_R$. Since $f_i$ is an invariant of $H$, it follows that

$$\frac{g_i}{f_i} = \prod_{g \notin H} f_i(gv),$$

where the product is taken over all representatives of nontrivial right cosets of $H$ in $G$, is also an invariant of $H$. Moreover, $g_i/f_i(Q) \neq 0$, and thus $g_i/f_i(Q)$ is an invertible element of the local ring $\mathcal{O}_{V/H,Q}$. Hence the polynomials $g_1, \ldots, g_n$ generate the ideal $m_Q \mathcal{O}_{V/H,Q}$.

**Lemma 2.4.** Let a finite group $G$ act linearly on a finite-dimensional vector space $V$. Suppose that $W$ is a subspace of $V$ that is fixed pointwise by the action of $G$. Let $P_1 \in W$ and $P_2 \in W$ be two (closed) points and let $Q_1 \in V/G$ and $Q_2 \in V/G$ be their images, respectively, with respect to the natural projection $p: V \to V/G$. Then the local rings of the points $Q_1$ and $Q_2$ are isomorphic,

$$\mathcal{O}_{V/G,Q_1} \simeq \mathcal{O}_{V/G,Q_2}.$$ 

**Proof.** Consider the vector $v$ of $V$ joining the points $P_1$ and $P_2$: $P_1 + v = P_2$. The shift by $v$ is an automorphism of $V$ viewed as a scheme. Since $v \in W$, this shift can be pushed forward along the projection $p$ to an automorphism of $V/G$ which takes the point $Q_1$ to $Q_2$. This completes the proof of the lemma.
§ 3. Conjecture (S) and isolated quotient singularities

In this section, the field $k$ is assumed to be algebraically closed. Let us prove Theorem 1.3. We consider the implication 2) $\Rightarrow$ 1) first. Let $G$ be a subgroup of $\text{GL}(V)$ generated by pseudo-reflections. By Lemma 2.1, the ring $S(V^*)^G$ is polynomial if and only if the variety $V/G$ is nonsingular at the image of the origin. However, then $V/G$ is nonsingular everywhere. Indeed, the singular set is closed, and $V/G$ can be singular only at the image of a linear subspace of $V$. Then our theorem follows from Theorem 1.1 (the Chevalley-Shephard-Todd-Serre Theorem) and Lemma 2.3.

If $W$ is a subspace of $V$, we denote by $\text{Fix}(W)$ the pointwise stabilizer of $W$ in $G$. In what follows, we refer to $\text{Fix}(W)$ as the fixator of $W$. Suppose that the ring $S(V^*)^{\text{Fix}(W)}$ is polynomial for each nontrivial subspace $W \subset V$. Then, by Lemma 2.3 the quotient $V/G$ is nonsingular everywhere, except possibly at the image of the origin. However, since $G$ is generated by pseudo-reflections, it follows that $V/G$ is nonsingular by part 2) of Theorem 1.3.

Let us now prove the implication 1) $\Rightarrow$ 2). Let $G$ be a finite subgroup of $\text{GL}(V)$ such that the singularity of $V/G$ is isolated and let $H$ be a subgroup of $G$ generated by pseudo-reflections. Then, by Lemma 2.3 and Theorem 1.1, the fixator $\text{Fix}(W)$ of every nontrivial subspace of $V$ is generated by pseudo-reflections and, moreover, $V/\text{Fix}(W)$ is nonsingular. We also note that $\text{Fix}(W)$ is contained in $H$. Then, by part 1), the quotient $V/H$ is nonsingular. Further, let $g$ be an element of $G$ of order $p^r$, $r > 0$, where $p = \text{char } k$. This element must fix a subspace $W$ of positive dimension. Thus, $g \in \text{Fix}(W)$; in particular, $g$ belongs to $H$. This implies that $p \nmid |G/H|$. The nonmodular group $G/H$ acts naturally on $V/H$ and

$$V/G \simeq (V/H)/(G/H).$$

This completes the proof of Theorem 1.3.

**Lemma 3.1.** The conjecture (S) holds for groups $G$ having a 1-dimensional invariant subspace.

**Proof.** In light of Theorem 1.1, we are to show that if $G$ is generated by pseudo-reflections and has a 1-dimensional invariant subspace and if the variety $V/G$ is singular, then the singularity of $V/G$ is not isolated. Let $W$ be a 1-dimensional invariant subspace of $G$. We note that, in this case, the fixator $\text{Fix}(W)$ is normal in $G$. If the variety $V/\text{Fix}(W)$ is singular, then the singularity is not isolated by Lemma 2.4. Hence, by Lemma 2.3 the singularity of $V/G$ is not isolated either.

Suppose that $V/\text{Fix}(W)$ is nonsingular. If $g$ is an element of $G$ of order $p^r$, $r > 0$, then the restriction of $g$ to $W$ is trivial. Thus, $g \in \text{Fix}(W)$. This implies that the quotient group $G/\text{Fix}(W)$ is nonmodular. The action of $\text{Fix}(W)$ on $V/\text{Fix}(W)$ can be linearized locally formally (see, for example, Lemma 2.3 of [2]), and since $G$ is generated by pseudo-reflections, the linearization of $G/\text{Fix}(W)$ is also generated by pseudo-reflections. Then it follows from the Chevalley-Shephard-Todd-Serre Theorem that the variety

$$V/G \simeq (V/\text{Fix}(W))/(G/\text{Fix}(W))$$

is nonsingular.
Corollary 3.1. The conjecture (S) holds for all Abelian groups generated by pseudo-reflections.

Proof. Indeed, a linear Abelian group always has a 1-dimensional invariant subspace.

Corollary 3.2. Let $G < \text{GL}(V)$ be a finite Abelian group such that the singularity of $V/G$ is isolated. Then $V/G$ is formally isomorphic to a nonmodular cyclic singularity.

Proof. We can get rid of pseudo-reflections and assume that $G$ is nonmodular. Then our assertion follows from Vincent’s classification of isolated quotient singularities (see [1] or [2]).

One can readily construct examples illustrating the assertion 2) of Theorem 1.3.

Example 3.1. Let $k$ be a field of characteristic 3 and let $V = k^2$. Let $G$ be the subgroup of $\text{GL}(2, k)$ generated by the elements

$$s = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$  

Then $G$ is isomorphic to the direct sum $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ and $t$ is a pseudo-reflection (a transvection: see §4) which generates a subgroup $H$ isomorphic to $\mathbb{Z}/3$. The basis invariants of $H$ are

$$f_1 = x(x + y)(x + 2y), \quad f_2 = y,$$

where $x$, $y$ is a basis of the space $V^*$. Therefore, the variety $V/H$ is nonsingular. The group $G/H \simeq \mathbb{Z}/2$ acts on $f_1$ and $f_2$ by the rule $f_1 \mapsto -f_1$, $f_2 \mapsto -f_2$. Thus, the singularity of $V/G$ is isolated and isomorphic to a quadratic cone.

§ 4. Proof of Theorem 1.4

Let us prove now our main result, Theorem 1.4. This theorem is a consequence of a more general theorem, Theorem 4.1. Let us first recall the terminology. A pseudo-reflection $g: V \to V$ is called a transvection if $g$ has a unique eigenvalue 1. Any transvection is necessarily of order $p$ equal to the characteristic of the ground field $k$. We note that, if dim $V = 2$, then every element of $\text{GL}(V)$ of order $p^r$, $r \geq 1$, is in fact of order $p$ and is a transvection.

Theorem 4.1. Let $V$ be a 3-dimensional vector space over an algebraically closed field of arbitrary characteristic $p$. Let $G$ be a finite subgroup of $\text{GL}(V)$ generated by pseudo-reflections. We denote by $G_p$ the normal subgroup of $G$ generated by all elements of order $p^r$, $r \geq 1$. Suppose that $G_p$ satisfies at least one of the following conditions:

1) $G_p$ is irreducible on $V$;
2) $G_p$ has a 1-dimensional invariant subspace $U$;
3) $G_p$ has a 2-dimensional invariant subspace $W$, and the restriction of $G_p$ to $W$ is generated by two noncommuting transvections (and thus is irreducible).

Then conjecture (S) holds for $G$, that is, if the variety $V/G$ is singular, then the singularity is not isolated. Moreover, if $G$ satisfies condition 3) or 2) and, in addition, the induced action of $G_p$ on the quotient space $V/U$ is generated by two noncommuting transvections, then $V/G$ is nonsingular.
Remark 4.1. To see how Theorem 1.4 follows from Theorem 4.1, we assume that \( p \) is odd. In case 3) of Theorem 4.1, we denote the restriction of \( G_p \) to \( W \) by \( H \). Since \( \dim W = 2 \), \( H \) is an irreducible group generated by transvections. As follows from representation theory, \( H \) is defined over a finite extension of the prime subfield of \( k \) (see Ch. XII of [9]). A classification of these groups has been known since the beginning of the 20th century (for example, see §1 of [10]); this classification implies that \( H \) is conjugate in \( \text{GL}(W) \) to the subgroup \( \text{SL}(2,q) \), \( q = p^n \), which is the group of \( 2 \times 2 \) matrices with determinant 1 and with entries in the Galois field \( \mathbb{F}_q \), or, if the characteristic is \( p = 3 \), then \( H \) can also be conjugate to the binary icosahedral group \( I^* \simeq \text{SL}(2,5) \) in its 3-modular representation. In the latter case, \( H \) is conjugate to the subgroup of \( \text{SL}(2,9) \) generated by the two transvections

\[
t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix},
\]

where \( \lambda^2 = -1 \). Every group \( \text{SL}(2,q) \) (for odd \( q \)) is also generated by two appropriate noncommuting transvections. In characteristic 2 every group generated by two noncommuting transvections is conjugate to an imprimitive group generated by the matrices

\[
t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix},
\]

where \( x \neq 0,1 \) is an element of the field \( \mathbb{F}_{2^n} \). Obviously, this group is isomorphic to the dihedral group \( D_n \). The group \( \text{SL}(2,2^n) \), \( n > 1 \), is not generated by two transvections, while, over an algebraically closed field, the group \( \text{SL}(2,2) \) is conjugate to an imprimitive group described above.

Proof. The case of irreducible groups \( G \) was studied by Kemper and Malle [5]. In case 2) the proof follows from Lemma 3.1. We concentrate here on the proof in case 3).

We note that if we show that the quotient variety \( V/G_p \) is nonsingular, then the nonmodular group \( G/G_p \) generated by pseudo-reflections acts naturally on the nonsingular variety \( V/G_p \). An action of this kind can be linearized locally formally, and therefore the quotient variety

\[
V/G \simeq (V/G_p)/(G/G_p)
\]

is not singular either. Thus, it suffices to prove our theorem in the case when \( G = G_p \).

We denote by \( N \) the kernel of the restriction map \( G \to H \) and consider the extension

\[
1 \to N \to G \to H \to 1.
\]

(1)

The group \( N \) is Abelian and consists of matrices of the form

\[
\begin{pmatrix}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix},
\]

where \( a, b \in k \) and a basis is chosen so that the invariant subspace \( W \) is generated by the first two vectors.
Lemma 4.1. In the assumptions of case 3) of Theorem 4.1, $G$ necessarily contains transvections whose restrictions to $W$ give (nontrivial) transvections of $H$.

Proof. According to the classification of irreducible groups generated by transvections, $H$ can be only one of the following groups: $\text{SL}(2, q)$ with $q$ odd, $I^* \simeq \text{SL}(2, 5)$ in the 3-modular representation described in Remark 4.1, and the imprimitive 2-modular group also described in Remark 4.1. In fact, our proof works also for the group $H = \text{SL}(2, 2^n)$, $n > 1$, and thus we will investigate this case too. Let us consider these possibilities in succession. In each case we assume that $G$ contains no transvections with nontrivial image in $H$ and arrive at a contradiction.

Case 1: $H = \text{SL}(2, q)$, where $q$ is odd. Then $H$ contains the matrices

$$
t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
$$

Let $\tilde{t}$ and $\tilde{s}$ be elements of $G$ projecting to $t$ and $s$, respectively. Suppose that $\tilde{t}$ and $\tilde{s}$ are not transvections. Then one can choose a basis in $V$ in which

$$
\tilde{t} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{s} = \begin{pmatrix} 1 & 0 & \mu \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

where $\mu \neq 0$ is an element of the field $k$. We have

$$
\tilde{t}^{-1} \tilde{s} \tilde{t} = \begin{pmatrix} 0 & -1 & \mu \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{s}^{-1} \tilde{t} \tilde{s} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},
$$

and

$$
u = \tilde{t}^{-1} \tilde{s} \tilde{t} \tilde{s}^{-1} \tilde{t} = \begin{pmatrix} 1 & 0 & \mu - 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in N.
$$

Then

$$
u^{-1} \tilde{s} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.
$$

This implies that we could assume from the start that $\mu \in \mathbb{F}_q$. Considering conjugates of $\nu$ by appropriate elements of $G$, one can obtain elements of $N$ with arbitrary vectors of the form $(a, b, 1)^T$, $a, b \in \mathbb{F}_q$, as the third column. In particular, $N$ contains an element $\nu_1$ with the third column $(0, -1, 1)^T$. The product $\nu_1 \tilde{t}$ is a transvection whose restriction is $\tilde{t} \in H$.

Case 2: $H = I^* < \text{SL}(2, 9)$. We represent the field $\mathbb{F}_9$ as the decomposition field of the polynomial

$$x^2 + x + 2$$

over $\mathbb{F}_3$. Then we can choose transvections generating $H$ in the form

$$
t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 \\ x + 2 & 1 \end{pmatrix}.$$
We denote again by $\tilde{t}$ and $\tilde{s}$ some ‘lifts’ of the elements $t$ and $s$, respectively, in $G$. If $\tilde{t}$ and $\tilde{s}$ are not transvections, then, in an appropriate basis, they have the matrices

$$
\tilde{t} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{s} = \begin{pmatrix} 1 & 0 & \mu \\ x + 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

where $\mu \neq 0$ is an element of the field $k$. A routine manipulation shows that

$$
\left(\tilde{t} \tilde{s}\right)^5 = \begin{pmatrix} 2 & 0 & 2x + 1 \\ 0 & 2 & \mu + 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \left(\tilde{s} \tilde{t}\right)^5 = \begin{pmatrix} 2 & 0 & 2x + 2\mu + 1 \\ 0 & 2 & \mu \\ 0 & 0 & 1 \end{pmatrix},
$$

$$
u = \left(\tilde{t} \tilde{s}\right)^5 \left(\tilde{s} \tilde{t}\right)^5 = \begin{pmatrix} 1 & 0 & \mu \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in N.
$$

This implies that

$$
\tilde{u} = u \tilde{s} = \begin{pmatrix} 1 & 0 & 0 \\ x + 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in G.
$$

Then we have

$$
\left(\tilde{t} \tilde{u}\right)^5 = \begin{pmatrix} 2 & 0 & 2x + 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \left(\tilde{u} \tilde{t}\right)^5 = \begin{pmatrix} 2 & 0 & x + 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

$$
u_1 = \left(\tilde{t} \tilde{u}\right)^5 \left(\tilde{u} \tilde{t}\right)^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in N.
$$

Thus, $(u_1^{-1}u)^{-1}\tilde{s}$ is a transvection whose restriction is $s \in H$.

**Case 3:** $H$ is imprimitive and the characteristic of $k$ is equal to 2. The generators $t$ and $s$ of $H$ are defined in Remark 4.1. Let us lift $t$ and $s$ to $G$ in the form

$$
\tilde{t} = \begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \mu^{-1} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{s} = \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

respectively; here $\mu \neq 0, 1$ is an element of $k$. We have

$$
\tilde{t}^2 = \begin{pmatrix} 1 & 0 & \mu + \mu^{-1} \\ 0 & 1 & \mu + \mu^{-1} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (\tilde{s} \tilde{t})^2 = \begin{pmatrix} 1 & 0 & \mu^{-1} + x\mu \\ 0 & 1 & \mu + x^{-1}\mu^{-1} \\ 0 & 0 & 1 \end{pmatrix} \in N.
$$

The group $N$ contains not only the elements $\tilde{t}^2$ and $(\tilde{s} \tilde{t})^2$ but also their products and conjugate elements. This implies readily that $N$ contains all matrices with the third column

$$(f(x)(x + 1)\mu, f(x^{-1})(x^{-1} + 1)\mu^{-1}, 1)^T$$

for all polynomials $f$ with coefficients in $\mathbb{F}_2$. If the minimal polynomial $g$ of $x$ over $\mathbb{F}_2$ is self-reciprocal, so that $g(x^{-1}) = 0$, then for some polynomial $h$ such that
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\[ h(x)(x + 1) = 1 \] we also have \( h(x^{-1})(x^{-1} + 1) = 1 \). Thus, \( N \) contains a matrix \( u \) with the third column \( (\mu, \mu^{-1}, 1)^T \). Then \( ut \) is a transvection whose restriction is \( t \in H \). If \( g \) is not self-reciprocal, then setting \( f = g \) we see that \( N \) contains a matrix with the third column

\[ (0, g(x^{-1})(x^{-1} + 1)\mu^{-1}, 1)^T. \]

Combinations of this matrix and one of its conjugates also give a matrix with the third column \( (\mu, \mu^{-1}, 1)^T \). Choosing \( f \) to be equal to the minimal polynomial of the element \( x^{-1} \), we also find a matrix in \( N \) with the third column \( (0, g(x^{-1})(x^{-1} + 1)\mu^{-1}, 1)^T \).

Case 4: \( H = \text{SL}(2, 2^n), n > 1 \). Since \( \text{SL}(2, 2) \) is a subgroup of each of the groups \( \text{SL}(2, 2^n) \), it suffices to prove the lemma for \( H = \text{SL}(2, 2) \). However, this case has already been proved because \( \text{SL}(2, 2) \) is conjugate to an imprimitive group.

**Lemma 4.2.** The group \( G \) contains transvections whose restrictions are generators of \( H \). The same holds also for \( H = \text{SL}(2, 2^n), n > 1 \).

**Proof.** The group \( \text{SL}(2, 2^n) \), as well as the imprimitive groups and the group \( I^* < \text{SL}(2, 9) \), has only one conjugacy class of transvections. Therefore, for these groups, our lemma follows immediately from Lemma 4.1. The group \( H = \text{SL}(2, q) \), where \( q \) is odd, has two conjugacy classes of transvections. Two transvections

\[ t' = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t'' = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \]

\( a, b \in \mathbb{F}_q \), are conjugate if and only if \( a \) and \( b \) are simultaneously squares or nonsquares in the field \( \mathbb{F}_q \). One can choose two noncommuting transvections \( t \) and \( s \) generating the group \( \text{SL}(2, q) \) in the form

\[ t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \]

where \( \lambda \) must be an element of \( \mathbb{F}_q \) belonging to no smaller field, and \( \lambda^2 \neq -1 \) for \( q = 9 \). We have already seen in the proof of Lemma 4.1 that \( t \) is the restriction of a transvection in \( G \). It is clear that \( \lambda \) (in the matrix \( s \)) can be chosen to be a square or a nonsquare of the field \( \mathbb{F}_q \), so that \( t \) and \( s \) will be conjugate.

**Lemma 4.3.** The extension (1) is a semidirect product, and \( H \) can be embedded in \( G \) in such a way that \( H \) acts on \( V \) by an indecomposable representation with two invariant subspaces of dimensions 1 and 2.

**Proof.** Let us lift two generating transvections \( t \) and \( s \) in \( H \) to transvections \( \tilde{t} \) and \( \tilde{s} \) of \( G \) and consider the subgroup generated by \( \tilde{t} \) and \( \tilde{s} \) in \( G \). The planes fixed under the action of \( \tilde{t} \) and \( \tilde{s} \) intersect along a line not contained in the invariant subspace \( W \). This gives the desired decomposition.

**Remark 4.2.** At least for \( H = \text{SL}(2, q) \), where \( q \) is odd, one could prove Lemma 4.3 using known results on the vanishing of the first and second cohomology groups of \( \text{SL}(2, q) \) with coefficients in the natural module (see [11] and [12]) or using the
complete reducibility of the low-dimensional modules over $\text{SL}(2, q)$ (see [13]). However, in this case, we would have to study the structure of $N$ as a module over $H$ in detail and to consider the cases of $H = I^*$ and of an imprimitive $H$ separately. For this reason we have decided to present the elementary general proof given above.

Throughout what follows we fix some splitting of the extension (1) and regard $H$ as a subgroup of $G$. Below we study the quotient variety $V/G$ in two steps: we first take the quotient $V/N$ and then consider the induced action of $H$ on $V/N$. By Theorem 3.9.2 of [14], the ring of invariants of $N$ is polynomial.

**Lemma 4.4.** The induced action of $H$ on $V/N \cong k^3$ is linear and decomposable, with invariant subspaces of dimensions 1 and 2.

**Proof.** Let us fix a basis of $V$ in which elements of $H$ are represented by block matrices of the form
\[
\begin{pmatrix}
  a & b & 0 \\
  c & d & 0 \\
  0 & 0 & 1
\end{pmatrix}.
\]
Let $x, y, z$ be the dual basis of $V^*$. Then $N$ acts on $x, y$ and $z$ by the transformations
\[
x \mapsto x + \lambda z, \quad y \mapsto y + \mu z, \quad z \mapsto z,
\]
$\lambda, \mu \in k$. It is clear that $f_3 = z$ is an invariant. Let $f_1$ and $f_2$ be two other invariants which, together with $f_3$, generate $S(V^*)^N$. One can choose $f_1$ and $f_2$ to be homogeneous and not containing the monomial $z^m$ with nonzero coefficient. Suppose first that $f_1$ and $f_2$ have distinct degrees, say, $\deg f_2 < \deg f_1$. Then $f_2$ must be semi-invariant with respect to the induced action of $H$. Indeed, the degree of the polynomial $f_2$ is preserved, and thus for $h \in H$ the polynomial $hf_2$ can be expressed in terms of $f_2$ and $f_3$. However, $h$ acts on $x$ and $y$ only, and therefore $hf_2$ can be expressed in terms of $f_2$ alone. Moreover, since $H$ is generated by elements of order $p$ only and the order of each element of the multiplicative group $\mathbb{F}_q^*$ is coprime to $p$, it follows that, in fact, the polynomial $f_2$ is an invariant of $H$. For the same reason, if
\[
hf_1 = \lambda f_1 + g(f_2, f_3),
\]
then $\lambda = 1$ for every $h \in H$. However, in this case, the subset of $V/N$ given by the equations $f_2 = f_3 = 0$ is pointwise fixed under the action of $H$. The natural projection $V \to V/N$ is $H$-equivariant. In turn, this implies that $G$ has an invariant line contained in the subspace $W = \{z = 0\}$, which contradicts the irreducibility of the group $H$ on $W$.

Thus, $\deg f_1 = \deg f_2$. Considerations similar to those used above show that $hf_1$ and $hf_2$ are linear combinations of $f_1$ and $f_2$. The group $H$ acts by block matrices on $V/N$, and hence the representation is decomposable. It must be that this lemma can also be proved by the direct evaluation of the invariants of $N$ and of the induced action of $H$ on the invariants.

Let $V/N = V_1 \oplus V_2$, $\dim V_1 = 2$ and $\dim V_2 = 1$, be a decomposition of $V/N$ into a sum of invariant $H$-modules. Since the group $H$ acting on $V$ is generated by transvections, it follows that the action of $H$ on $V_1$ and $V_2$ is also generated by transvections. In particular, the action on $V_2$ is trivial, and the ring of invariants
of $H$ acting on $V_1$ is polynomial (see, for example, Proposition 7.1 in [5]). Thus, the ring of invariants of $H$ acting on $V/N$ is also polynomial, and hence the ring of invariants of the group $G$ acting on $V$ is polynomial.

It remains to show that, if $G$ acts on $V$ with a 1-dimensional invariant subspace $U$ and if the induced action of $G$ on $W = V/U$ is irreducible and is generated by two noncommuting transvections, then $V/G$ is nonsingular again. In a sense, this situation is dual to that considered above. We denote by $H$ the natural image of $G$ in $\text{GL}(W)$ and by $N$ the corresponding kernel. We obtain an extension of the form (1) again. In a suitable basis, $N$ is an Abelian group of matrices of the form

$$
\begin{pmatrix}
1 & a & b \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

where $a, b \in k$. A consideration (which we omit here) similar to the above argument shows that $G$ contains transvections taken to the generators of $H$. The proof of the following lemma is simple, and we leave it to the reader.

**Lemma 4.5.** Every two transvections on a 3-dimensional vector space have a common 2-dimensional invariant subspace.

Now if $\tilde{t}$ and $\tilde{s}$ are transvections of $G$ taken to generators of $H$, we consider their common 2-dimensional invariant subspace. Obviously, it does not contain the subspace $U$, and thus can be identified with $W$. Hence, the extension (1) splits again.

In an appropriate basis $x, y, z$ of $V^*$ the group $N$ acts by transformations of the form

$$
x \mapsto x + ay + bz, \quad y \mapsto y, \quad z \mapsto z.
$$

The basis invariants of $N$ can readily be determined. These are

$$
f_1 = \prod_{g \in N} gx, \quad f_2 = y, \quad f_3 = z;
$$

in particular, the ring of invariants $S(V^*)^N$ is polynomial. It is also clear that, in this case, the induced action of $H$ on $V/N$ is linear and decomposable. As above, this implies that the variety $(V/N)/H$, and hence $V/G$ as well, is nonsingular. This completes the proof of Theorem 4.1.

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