Multipartite Nonlocal Quantum Correlations Resistant to Imperfections

Harry Buhrman  
CWI and University of Amsterdam, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands  

Peter Hoyer  and Hein Röhrig  
Department of Computer Science, University of Calgary, 2500 University Drive N.W., Calgary AB, Canada T2N 1N4  

Serge Massar  
Service de Physique Théorique, Université Libre de Bruxelles, C.P. 225, Bvd. du Triomphe, 1050 Bruxelles, Belgium  
(Dated: October 18, 2004)

We use techniques for lower bounds on communication to derive necessary conditions in terms of detector efficiency or amount of super-luminal communication for being able to reproduce with classical local hidden-variable theories the quantum correlations occurring in EPR-type experiments in the presence of noise. We apply our method to an example involving $n$ parties sharing a GHZ-type state on which they carry out measurements and show that for local-hidden variable theories, the amount of super-luminal classical communication $c$ and the detector efficiency $\eta$ are constrained by $\eta^2 - c/n = O(n^{-1/6})$ even for constant general error probability $\varepsilon = O(1)$.

PACS numbers: 03.67.Hk, 03.65.Ud

I. INTRODUCTION

Forty years ago, Bell [1] introduced the notion of “quantum nonlocality”: he showed that the correlations between the outcomes of measurements carried out on entangled quantum systems cannot be reproduced by a local classical theory (often called a local hidden variable theory). Since then, extensive work has been carried out on quantum nonlocality, both on the experimental and theoretical aspects. On the theory side research on quantum nonlocality has branched out into many different and complementary directions.

One important direction of investigation is the search for qualitatively different types of quantum nonlocality. Of particular interest is the discovery of the Greenberger-Horne-Zeilinger (GHZ) paradox [2, 3]. In this and related examples, correlations are characterized as nonlocal by the pattern of zero and nonzero joint probabilities. This property has been called “pseudo telepathy,” because in every run of the experiment, the parties appear to agree clandestinely on a subset of admissible outputs. It should be contrasted with other examples where it is the values of these joint probabilities which implies nonlocality.

Another important advance was to show that quantum nonlocality subsists even in the presence of noise as first demonstrated by Clauser, Horne, Shimony, and Holt [4]. This is essential since every experimental test will necessarily be affected by imperfections; the best experiments to date have error rates of the order of a few percent. Much additional work has been devoted to understanding the resistance of quantum nonlocality to imperfections.

In experiments involving entangled photons, there is one particular kind of imperfection that plays a central role, namely the small efficiency of single-photon detectors. A single-photon detector will register the presence of a photon with probability $\eta$, and will not register the presence of the photon with probability $1 - \eta$. For instance, as one goes from visible to infrared wavelengths, $\eta$ decreases from more than 50% to 10%. Detector inefficiency can be thought of as a specific type of noise. This imperfection was first discussed by Pearle [5] and remains to this day one of the major hurdles to overcome in order to carry out a loophole-free test of quantum nonlocality. Examples show that there are quantum correlations that are highly insensitive to detector inefficiency, but are much more sensitive to other kinds of noise, see Massar [6], and therefore this kind of imperfection should be studied independently of other kinds of noise.

Note that the complementary error, namely detectors clicking when they should not, can also occur. We consider this error as a general noise, as it cannot be distinguished from other types of noise such as non-maximally entangled states.

The development of quantum information theory over the past ten years (see [7] for a review) has brought a breath of fresh air to the study of quantum nonlocality, and important new questions have been raised. For instance Bell showed that the quantum correla-
tions could not be reproduced classically without “superluminal” communication between the parties. But Brassard et al. \cite{8} and Steiner \cite{9} initiated the study of how much super-luminal communication is required to reproduce the correlations. This question is closely related to quantum communication complexity, in which one enquires whether certain distributed communication tasks can be solved using less quantum communication than is required classically; see \cite{10} for a survey of quantum communication complexity.

Remarkably, the amount of classical communication required to reproduce the quantum correlations and the minimum detector efficiency required to close the detection loophole are closely related quantities as demonstrated by Gisin and Gisin \cite{11}, and Massar \cite{6}. In many cases, quantum correlations that require a lot of communication to reproduce classically cannot be simulated classically without communication, even when the actual detectors are very inefficient.

Another question that has been raised in the context of quantum information theory concerns the asymptotic limit when the size of the entangled system grows. Does the gap between classical and quantum correlations grow, and if so, at what rate? Brassard et al. \cite{8} showed that in the bipartite case the amount of communication required to classically reproduce the quantum correlations can increase exponentially with the number of entangled bits shared by the parties. And it follows from the results by Buhrman et al. \cite{12} that there are quantum correlations for \( n \) parties each holding a two-dimensional subsystem, so that the amount of communication that must be broadcast in a classical simulation increases logarithmically with the number of parties. Unfortunately these asymptotic results have only been proved in the total absence of noise.

The only prior asymptotic results in quantum communication complexity that hold in the presence of noise concern multi-round quantum communication protocols, such as the appointment-scheduling problem of Buhrman et al. \cite{13} or the example due to Raz \cite{14}. It appears that these results cannot be mapped to results concerning quantum nonlocality, whereas communication complexity problems with a single round of communication and nonlocal quantum correlations can generally be mapped one onto the other.

The present work lies at the intersection of these different lines of enquiry. Specifically we concentrate on the generalization of the GHZ paradox to \( n \) parties previously considered by Buhrman et al. \cite{12}; their bounds on the GHZ-inspired multiparty communication problem was only proved in the absence of noise. We extend it to the noisy case.

These GHZ-type correlations involve \( n \) parties. We suppose that there is a fixed (i.e., independent of \( n \)) nonzero probability \( \varepsilon \) for an error to occur. Denote by \( c \) the number of bits communicated (via a possibly superluminal channel) in order to reproduce the correlations. We show that

\[ c = \Omega(n \log n) . \]

Denote by \( \eta^* \) the maximum detector efficiency for which a local classical model exists. We show that

\[ \eta^* = O(n^{-1/6}) . \]

In fact, the superluminal communication and detection efficiency can be traded one for the other: we combine the above two results into the following bound:

\[ \eta^* 2^{-c/n} = O\left(n^{-1/6}\right) . \] (1)

This bound sheds new light on the relation between these two quantities, which was previously discussed in \cite{6,15}. Our result constitutes to our knowledge the first example in which the degree to which the quantum correlations are nonlocal increases with the size of the entangled system in the presence of noise and as such constitute a significant advance in our understanding of quantum communication complexity and of quantum nonlocality.

The present work builds upon the earlier results of \cite{12} and \cite{13}. As in these references we rely heavily on techniques and ideas from the field of communication complexity. The reader unfamiliar with these notions may consult the book by Kushilevitz and Nisan \cite{16} for an introduction to classical communication complexity.

The remainder of this article is organized as follows. In Section II we define precisely the main concepts of nonlocality used in this paper. In Section III we introduce the combinatorial notion of monochromatic rectangles and prove a general relation between \( \eta, c, \) and \( \varepsilon \), which depends on the maximum size of almost monochromatic rectangles. This general result is of interest in its own right and could be of use when studying other instances of quantum nonlocality that exhibit pseudo telepathy. In Section IV we apply the general bound to the GHZ paradox; the proof of Eq. (1) is based on an addition theorem for cyclic groups proved in Section V. Finally, we discuss our results and open problems in Section VI.

II. NONLOCALITY DEFINITIONS

Consider the following situation. There are \( n \) spatially separated parties; party \( i \) receives an input \( x_i \in \{1,\ldots,k\} \) and produces an output \( a_i \in \{1,\ldots,\ell\} \). With \( x = (x_1,\ldots,x_n) \) and \( a = (a_1,\ldots,a_n) \), let \( P(a|x) \) denote the probability of output \( a \) given input \( x \). The inputs are distributed according to the probability distribution \( \mu(x) \). We formalize this situation as follows.

Definition 1 An \((n,k,\ell)\) correlation problem with input distribution \( \mu \) is a family of probability distributions \( P(-|x) \) on the “inputs” \( \{1,\ldots,k\}^n \), for each “input” \( x \in \{1,\ldots,k\}^n \) with \( \mu(x) > 0 \). We denote the support of \( \mu \) by \( D := \{x : \mu(x) > 0\} \).
Note that in nonlocality experiments the distribution $\mu$ should be a product distribution, otherwise the parties would have trouble selecting $x$ according to $\mu$ when the measurements take place in timelike separated regions. On the other hand the mathematical proofs given below and in particular the example of Section 4 are based on non-product distributions. The way to get around this is the following: during the nonlocality experiment the inputs are distributed according to a product distribution $\mu_0$, for instance the uniform distribution. Then when analyzing the data one first throws away part of the data in such a way that, for the data that is kept, the inputs are distributed according to the desired distribution $\mu$. This can only make the task harder for the local hidden variable theory, since it does not know beforehand which runs will be kept and which will be thrown away. From now on we let $\mu$ be an arbitrary (possibly non-product) distribution.

We are interested in correlation problems obtained from measurements on multipartite entangled quantum states. We define these as follows.

**Definition 2** An $(n, k, \ell)$ measurement scenario is a correlation problem in which the parties share an entangled state $|\psi\rangle$; each input $x_1$ determines a positive operator valued measure (POVM) $\hat{x}_i = \{\hat{x}^1_i, \ldots, \hat{x}^\ell_i\}$ with $\sum_{j=1}^\ell \hat{x}^j_i = 1$. If the measurement of party $i$ produces outcome $\hat{x}^j_i$, then it outputs $a_i = j$. The probability $P_{QM}(a|x)$ to obtain outcome $a$ given input $x$ is

$$P_{QM}(a|x) = \langle \psi | \hat{x}^1_i \otimes \cdots \otimes \hat{x}^n_i | \psi \rangle.$$

Our aim is to study what classical resources are required to reproduce such measurement scenarios. Let us first consider classical models in which the parties cannot communicate after they have received the inputs. Such models are called local. The best the parties can do in this case is to randomly select in advance a deterministic strategy. This motivates the following definition.

**Definition 3** A deterministic local hidden variable (lhv) model is a family of functions $\lambda = (\lambda_1, \ldots, \lambda_n)$ from the inputs to the outputs: $\lambda_i : \{1, \ldots, k\} \rightarrow \{1, \ldots, \ell\}$. Each party outputs $a_i = \lambda_i(x_i)$. A probabilistic lhv model (or just lhv model) is a probability distribution $\nu(\lambda)$ over all deterministic lhv models for given $(n, k, \ell)$.

Thus in probabilistic lhv models the parties first randomly choose a deterministic lhv model $\lambda$ using the probability distribution $\nu$. Each party then outputs $a_i = \lambda_i(x_i)$.

We also consider classical models with communication. In such models, the parties may communicate over a possibly superluminal classical broadcast channel in order to reproduce the quantum correlations $P_{QM}$. Different communication models exist depending on whether the parties do not have access to randomness, possess local randomness only, or share randomness. These notions are adapted from the corresponding definitions in communication complexity.

**Definition 4** Consider $n$ parties who each receive an input $x_i \in \{1, \ldots, k\}$, communicate over a classical broadcast channel, and each produce an output $a_i \in \{1, \ldots, \ell\}$.

A deterministic classical model with communication is a rooted “communication protocol” tree $P$: each internal node $u$ is labeled with the party $i_u \in \{1, \ldots, n\}$ whose turn it is to broadcast a message; each edge $e$ from $u$ to a descendant is labeled with a set $X_e \subseteq \{1, \ldots, k\}$ such that the $X_e$ form a partition of $\{1, \ldots, k\}$; each leaf $v$ is labeled with a lhv model $\lambda_v$. An execution of the protocol on input $x$ starts at the root of tree; until a leaf is reached, the execution proceeds from node $u$ to the descendant of $u$ that is reached via the edge $e$ with $x_{i_u} \in X_e$. It is understood that the choice of the edge is broadcast to all parties so that all parties know at each moment at which node the execution is. When the execution has reached the leaf $v$, each party $i$ outputs $\lambda_{v,i}(x_i)$ and the execution terminates. If there are $m$ leaves and if the number of children of the nodes on the path from the root to the final leaf is $t_1, \ldots, t_m$, the number of bits broadcast is $c = \log t_1 + \cdots + \log t_m$.

A classical model with shared randomness is an arbitrary probability distribution $\nu(P)$ over deterministic classical models. An execution of such a model first probabilistically selects a deterministic model and then evaluates the deterministic model.

In a classical model with local randomness, the distribution $\nu(P)$ is constrained to be a product distribution of the individual strategies of the parties.

Of course, a classical model that always uses 0 bits of communication is just a lhv model.

**Definition 5** For a correlation problem $P$ with input distribution $\mu$, we denote by $D(P)$, $R(P)$, and $R^{prob}(P)$, respectively, the minimum number of bits that must be broadcast in order to perfectly reproduce the correlations $P$ when the parties are deterministic, have local randomness only, or have shared randomness.

Where the choice of the correlation problem $P$ is clear from the context, we drop it and write $D$, $R$, and $R^{prob}$. Clearly, $D(P) \geq R(P) \geq R^{prob}(P)$. Since the results of quantum measurements are inherently random, it is in general impossible to reproduce the quantum correlations using deterministic lhv models or using deterministic models with communication. Thus $D(P)$ is meaningless when trying to simulate quantum measurement scenarios. However, deterministic models are a very useful tool for studying the probabilistic models because properties of all deterministic models necessarily also hold for all probabilistic models, since the probabilistic models are just probabilistic mixtures of deterministic models. Note also that Massar et al. 17 showed that $R(P)$ can be infinite when $P$ arises from a quantum measurement.
scenario. In general, classical models cannot reproduce the quantum correlations \( P_{QM} \) unless communication is possible, the detector efficiency \( \eta \) is sufficiently small, or the error probability is sufficiently large.

Let us consider now the situation where the detectors are inefficient. In this case we enlarge the space of outputs to \( a_i \in \{1, \ldots, \ell\} \cup \{\perp\} \), where \( a_i = \perp \) is the event that the \( i \)-th detector does not produce an output (“click”). We suppose that each measurement \( x_i \) has probability \( \eta \) of giving a result and a probability \( 1 - \eta \) of not giving a result. Whether a detector clicks or does not click is independent of the other detectors. This affects the probabilities in a more structured way than simply decreasing the probability that all detectors click simultaneously. This issue has been discussed by Massar and Pironio \( \square \); for simplicity we will consider here only the two extreme cases, namely that all detectors click (which occurs with probability \( \eta^n \)) or that at least one detector does not click. We define detector efficiency accordingly.

**Definition 6** Let \( P(\cdot|x) \) be a fixed \( (n, k, \ell) \) correlation problem with input distribution \( \mu \). Let

\[
C := \{ a : \forall i \ a_i \neq \perp \}
\]

denote the output vectors where all detectors click. With slight abuse of notation, we also use \( C \) as the indicator random variable of the event \( a \in C \). We define the detection efficiency \( \eta \) of the correlations to be the expectation

\[
\eta := \left( E_{\mu} \left[ \sum_a P(a|x)C \right] \right)^{1/n}.
\]

Note that here the atomic events are tuples \((x, a)\) of an input and an output vector with a joint distribution of the form \( P[\text{input } x \text{ and output } a] = \mu(x)P(a|x) \). The expectation above is over the marginal distribution \( \mu \) of the inputs.

We are also interested in the possibility that the lhv model makes errors.

**Definition 7** Suppose that some classical model produces a probability distribution \( P(a|x) \), which should approximate the probability distribution produced by a measurement scenario \( P_{QM}(a|x) \). The total-variation distance is a measure for how much these two distributions differ:

\[
\varepsilon_{var} := E_{\mu} \left[ \sum_a |P_{QM}(a|x) - P(a|x)| \frac{C}{\eta^n} \right]
\]

The inclusion of the factor \( C/\eta^n \) takes care of the possible finite efficiency of the detectors, assumed to be the same for \( P_{QM}(a|x) \) and for \( P(a|x) \).

We will be particularly interested in quantum correlations that exhibit “pseudo telepathy”, i.e., such that \( P_{QM}(a|x) = 0 \) for some \( a \) and \( x \). For such correlations it is convenient to define the error probability as follows.

**Definition 8** Let \( F := \{ (a, x) : P_{QM}(a|x) = 0 \} \)

and again we also denote by \( F \) the indicator random variable of the event \( P_{QM}(a|x) = 0 \). The error probability is

\[
\varepsilon := E_{\mu} \left[ \sum_a P(a|x)F \frac{C}{\eta^n} \right].
\]

Thus \( \varepsilon \) is the probability to observe in one run an event that cannot occur in the quantum mechanical model. It is immediate to check that

\[
\varepsilon_{var} \geq \varepsilon.
\]

For an \((n, k, \ell)\) correlation problem \( P(\cdot|x) \) with input distribution \( \mu \), we denote by \( \eta^* \) the maximum detector efficiency of any lhv model that reproduces the quantum correlations, and by \( \eta_2^* \) the maximum detector efficiency that reproduces the quantum correlations up to error \( \varepsilon \). Similarly, we can define \( D_\varepsilon, R_\varepsilon, R_{\varepsilon \text{pub}} \) the amounts of communication required to reproduce the correlation problem \( P \) in the presence of error. We are interested in \( \eta^*_2 \) and by \( R_{\varepsilon \text{pub}} \).

We can map every communication model with \( c \) bits of communication with shared randomness into a model with inefficient detectors with efficiency \( \eta^n = 2^{-c} \); the shared randomness determines the conversation between the parties. Thus they all agree on the conversation. Each party \( i \) checks whether its input \( x_i \) is compatible with the conversation and, if yes, produces output \( a_i \) according to the communication model and otherwise produces no output, i.e., \( \perp \). The total probability that all detectors click is equal to the probability that \( x \) belongs to the conversation. Since each input belongs to one and only one conversation, the probability that all detectors click is equal to one over the number of conversations. Note that in this model the probability that a specific detector, say detector \( i \), clicks may depend on the input \( x_i \). However, the probability that all detectors click remains independent of the input.

**Theorem 1** Consider lhv models where the probability that all detectors click is independent of the input, but where the probability that each detector clicks, say detector \( i \), may depend on its input \( x_i \). Then there exists a lhv model if the probability \( \eta^* \) that all detectors click is at most \( 2^{-R_{\varepsilon \text{pub}}} \). This implies that in these models,

\[
(\eta^*)^n \geq 2^{-R_{\varepsilon \text{pub}}}.
\]

This result was given in \( \square \) in the absence of error, but it also holds when errors are present.

### III. COMBINATORIAL BOUNDS

We now introduce some definitions and notation, which allow us to state and then prove our result concerning a
general relation between $c$, $\eta$ and $\varepsilon$. We are concerned with pseudo-telepathy type correlations for which there are some $P(a|x)$ that vanish.

**Definition 9** Let $P(\cdot|x)$ be a fixed $(n,k,\ell)$ correlation problem with input distribution $\mu$. We define the sets of inputs that admit output $a$ as

$$\text{adm}(a) := \{ x : P(a|x) > 0 \}$$

for all $a \in C$. Moreover, for a set $S \subseteq \{1, \ldots, k\}^n$ of inputs and a specific output $a \in \{1, \ldots, \ell\}^n$, the $a$-advantage of $S$ is

$$\text{adv}_a(S) := \frac{\mu(S \cap \text{adm}(a))}{\mu(S)}$$

for all $a \in C$.

For sets $A_1, \ldots, A_n$, a subset $R$ of the Cartesian product $A_1 \times \cdots \times A_n$ is called a rectangle if there are $R_1 \subseteq A_1$, \ldots, $R_n \subseteq A_n$ such that $R = R_1 \times \cdots \times R_n$, i.e., $R$ is a Cartesian product itself. The importance of rectangles is that for a deterministic lhv model $\lambda = (\lambda_1, \ldots, \lambda_n)$, the set $R_\lambda(a) := \{ x : \lambda(x) = a \}$ of all inputs $x$ leading to output $a$ is a rectangle: $R_\lambda(a) = \lambda_1^{-1}(a_1) \times \cdots \times \lambda_n^{-1}(a_n)$.

**Theorem 2** Let $P$ be a fixed $(n,k,\ell)$ correlation problem with input distribution $\mu$. If for some $\delta$ ($0 \leq \delta \leq 1$), all rectangles $R$ with $\text{adv}_a(R) \geq \delta$ have $\mu(R) \leq r$ for every $a \in C$, then for every classical model $\nu(P)$ with $c$ bits of communication holds

$$\frac{1}{2^c} \eta^c(1 - \varepsilon) \left( 1 - \frac{1}{1 - \delta} \right) \leq \ell^c r.$$  

This shows the strong relation between the detection efficiency and the amount of classical communication required to reproduce the correlations. Indeed one quantity can be traded for the other.

**Proof of Theorem 2** Let $R_{\mathcal{P},v,a}$ denote the set of inputs $x$ for which the deterministic protocol $\mathcal{P}$ terminates in leaf $v$ and outputs $a$. Every $R_{\mathcal{P},v,a}$ is a rectangle. Let $L := \{(\mathcal{P}, v, a) : \text{adv}_a(R_{\mathcal{P},v,a}) \geq \delta \}$. Then

$$\eta^c(1 - \varepsilon) = \sum_{\mathcal{P},v,a} \nu(\mathcal{P}) \mu(x) C(1 - F)$$

$$= \sum_{\mathcal{P},v,a} \nu(\mathcal{P}) \mu(R_{\mathcal{P},v,a} \cap \text{adm}(a))$$

$$= \sum_{\mathcal{P},v,a} \nu(\mathcal{P}) \mu(R_{\mathcal{P},v,a}) \text{adv}_a(R_{\mathcal{P},v,a})$$

$$\leq \sum_{(\mathcal{P},v,a) \in L} \nu(\mathcal{P}) r + \sum_{(\mathcal{P},v,a) \in L} \nu(\mathcal{P}) \mu(R_{\mathcal{P},v,a}) \delta$$

$$\leq 2^c d^c r + \delta \sum_{(\mathcal{P},v,a) \in L} \nu(\mathcal{P}) \mu(R_{\mathcal{P},v,a})$$

where the $v$ range over the leaves of $\mathcal{P}$ and the $a$ over $\{1, \ldots, \ell\}^n$. Similarly,

$$\eta^c \varepsilon = \sum_{\mathcal{P},v,a} \nu(\mathcal{P}) \mu(x) CF$$

$$= \sum_{\mathcal{P},v,a} \nu(\mathcal{P}) \mu(R_{\mathcal{P},v,a} \cap \{1, \ldots, k\}^n \setminus \text{adm}(a))$$

$$= \sum_{\mathcal{P},v,a} \nu(\mathcal{P}) \mu(R_{\mathcal{P},v,a}) (1 - \text{adv}_a(R_{\mathcal{P},v,a}))$$

$$\geq 0 + \sum_{(\mathcal{P},v,a) \in L} \nu(\mathcal{P}) \mu(R_{\mathcal{P},v,a}) (1 - \delta)$$

$$= (1 - \delta) \sum_{(\mathcal{P},v,a) \in L} \nu(\mathcal{P}) \mu(R_{\mathcal{P},v,a})$$

Hence,

$$\eta^c(1 - \varepsilon) \leq 2^c \ell^c r + \frac{\delta}{1 - \delta} \eta^c \varepsilon,$$

which implies Theorem 2. \qed

**IV. APPLICATION TO THE GHZ CORRELATIONS**

In this measurement scenario each of the $n$ parties has a two-dimensional quantum system. The overall state of the $n$ qubits is

$$|\psi\rangle = \frac{|0^n\rangle + |1^n\rangle}{\sqrt{2}}$$  \hspace{1cm} (3)

where $|i^n\rangle = |i\rangle \otimes \cdots \otimes |i\rangle$ with $n$ terms in the product. Each party receives as input $x_i \in \{0, \ldots, k - 1\}$. Each party then measures his qubit in the basis

$$|\varphi_{\pm}\rangle = \frac{|0\rangle \pm e^{\pi i x_i / k} |1\rangle}{\sqrt{2}}$$  \hspace{1cm} (4)

If the qubit is projected onto state $|\varphi_{+}\rangle$, then party $i$ outputs $a_i = 0$ and if the qubit is projected onto state $|\varphi_{-}\rangle$, party $i$ outputs $a_i = 1$.

We call an input $x = (x_1, \ldots, x_n)$ valid if it satisfies

$$\left( \sum_{i=1}^{n} x_i \right) \mod k = 0$$  \hspace{1cm} (5)

and we let $D \subset \mathbb{Z}_k^n$ denote the set of all valid inputs. Let $F : \mathbb{Z}_k^n \rightarrow \{0, 1\}$ denote the Boolean function on the valid inputs defined by

$$F(x) = \frac{1}{k} \left[ \left( \sum_{i=1}^{n} x_i \right) \mod 2k \right].$$

The function $F$ can be viewed as computing the $(1 + \log k)$-th least significant bit of the sum of the $x_i$.

It is easy to check that the outputs of the quantum measurement are correlated as follows: if Eq. (5) holds, then

$$\left( \sum_{i=1}^{n} a_i \right) \mod 2 = \frac{1}{k} \left[ \left( \sum_{i=1}^{n} x_i \right) \mod 2k \right] = F(x).$$

Hence, if each party broadcasts its measurement outcome then each party can locally compute $F(x)$. 

Lemma 3 In the model with prior entanglement and classical broadcast communication, the communication complexity of computing $F(x)$ is $O(n)$.

Moreover, the above measurement scenario will exactly reproduce the following $(n, k, 2)$ correlation problem (see Definition 10):

Definition 10 Let $\mu(x)$ be a distribution on the inputs that gives zero weight to the invalid inputs $x$, which do not satisfy Eq. (4), and let

$$P(a|x) := \begin{cases} \frac{1}{2^n} & \text{if } F(x) = a_1 + \cdots + a_n \mod 2 \\ 0 & \text{otherwise.} \end{cases}$$

for all $a \in \{0, 1\}^n$ and $x \in D$.

A simple classical strategy for reproducing these correlations is for every party to broadcast its input. Hence, with $k = n^{1/6}$, the communication problem and the correlation problem can be solved exactly with $O(n \log n)$ bits of communication.

Note that for $n = 3$ and $k = 2$ the above correlations constitute the GHZ paradox as formulated by Mermin. The case $k = 2$ and arbitrary $n$ was studied by Mermin and recently revisited by Brassard et al.\cite{brassard1988}. In Buhrman et al.\cite{buhrman2001} and our earlier research\cite{buhrman2002}, the case where the number of settings $k$ is a power of two was considered. In\cite{buhrman2001} it was shown that the amount $c$ of classical communication which the parties must broadcast in order to reproduce exactly the correlations from Definition 10 is $c = O(n \log n)$ when $k = O(n)$. And in\cite{buhrman2002} it was shown that the maximum detector efficiency $\eta^*$ for which a local classical model can reproduce exactly the correlations from Definition 10 decreases as $1/n$. Furthermore the classical strategy described above shows that in the absence of noise these results are essentially optimal.

We will now show that this optimality continues to hold in the presence of noise and that classical strategy described above remains close to optimal in the presence of noise. Specifically we will show that

Theorem 4 Let $\mu$ be the uniform distribution on valid inputs. Then the number $c$ of bits broadcast, the efficiency $\eta$ and the error $\varepsilon$ of every lhv model $\nu$ are constrained by

$$\frac{1}{2^{c/\eta^*}} \left(1 - \varepsilon \left[2 + O\left(\frac{1}{n^{1/6}}\right)\right]\right)^{1/n} = O\left(\frac{1}{n^{1/6}}\right).$$

which, for fixed $\varepsilon$, $n$ large, implies Eq. (1). In particular we have

Corollary 1 Every bounded-error randomized public coin protocol for $F : \mathbb{Z}_T^k \to \{0, 1\}$ with $k \geq n^{1/6}$ requires $\Omega(n \log n)$ bits of communication.

We now turn to the proof of Theorem 4. We say a rectangle $R = A_1 \times \cdots \times A_n \subseteq \mathbb{Z}_T^n$ involves $m$ parties if at least $m$ of the $n$ subsets $A_i$ have size at least 2. Every rectangle involving at most $m$ parties can have size at most $k^m$.

Lemma 5 (Small rectangles are insignificant) Every rectangle $R$ involving at most $n^{5/6}$ parties satisfies $\log |R| \leq n^{5/6} \log k$.

We say a rectangle $R$ has bias at most $\delta$ if

$$|F^{-1}(1) \cap D \cap R| \leq (1 + \delta)|F^{-1}(0) \cap D \cap R|$$

and

$$|F^{-1}(0) \cap D \cap R| \leq (1 + \delta)|F^{-1}(1) \cap D \cap R|.$$

Note that for every $a$ we have $adm(a) \cap D = F^{-1}(a_1 + \cdots + a_n \mod 2) \cap D$. Therefore, if $\mu$ is a distribution that is uniform on $D$, then $R$ has bias at most $\delta$ if and only if it has $a$-advantage at most $(1 + \delta)/(2 + \delta)$ for every $a$. The next lemma expresses that every “large” rectangle is almost unbiased.

Lemma 6 (Large rectangles are almost unbiased) Every rectangle involving at least $n^{5/6}$ parties has bias at most $O(1/n^{1/6})$.

The proof of Lemma 6 is based on addition theorems for cyclic groups and is given in the next section.

Proof of Theorem 4. Let $\mathbb{Z}_T$ denote the additive cyclic group of order $T$. Let $\mu_A(x)$ denote the multiplicity of an element $x$ in the multiset $A$. For multisets $A$ and $B$ of $\mathbb{Z}_T$, let $A + B$ denote the multiset $\{a + b \mid a \in A, b \in B\}$.

Definition 11 We say a multiset $A$ of $\mathbb{Z}_T$ has bias at most $\varepsilon$ with respect to a subgroup $H \leq \mathbb{Z}_T$ if $\mu_A(a) \leq (1 + \varepsilon)\mu_A(a + h)$ for all $a \in A$ and all $h \in H$.

Theorem 7 (Addition Theorem) Let $A_1, \ldots, A_r$ be subsets of $\mathbb{Z}_T$, each of size at least 2, with $r \geq T^3$ and $T = 2^t$ a power of 2. Then the multiset $A_1 + A_2 + \cdots + A_r$ has bias at most $O(T^{3/2}/r^{1/2})$ with respect to the subgroup $\{0, 2^{t-1}\}$. 

V. AN ADDITION THEOREM

Let $\mathbb{Z}_T$ denote the additive cyclic group of order $T$. Let $\mu_A(x)$ denote the multiplicity of an element $x$ in the multiset $A$. For multisets $A$ and $B$ of $\mathbb{Z}_T$, let $A + B$ denote the multiset $\{a + b \mid a \in A, b \in B\}$.

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Definition 11 We say a multiset $A$ of $\mathbb{Z}_T$ has bias at most $\varepsilon$ with respect to a subgroup $H \leq \mathbb{Z}_T$ if $\mu_A(a) \leq (1 + \varepsilon)\mu_A(a + h)$ for all $a \in A$ and all $h \in H$.
Essentially, this theorem is derived by a sequence of simple reductions to the following observation: We may generate an almost uniformly distributed random number between 0 and \( K - 1 \) by flipping a fair coin \( K^2 \) times, and counting the number of heads modulo \( K \).

**Lemma 8** For multisets \( A \) and \( B \) over \( \mathbb{Z}_T \), if \( A \) has bias at most \( \varepsilon \) with respect to some subgroup \( H \), then so does \( A + B \). In particular, the multiset \( A \{ d \} \) has the same bias as \( A \).

**Lemma 9** Let \( f : \{0,1\}^s \to \mathbb{Z}_K \) be defined by
\[
f(a_1, \ldots, a_s) = \left( \sum_{i=1}^s a_i \right) \mod K.
\]
If \( s \geq K^2 \), then \( |f^{-1}(x)| \leq (1 + \frac{4K}{\sqrt{s}})|f^{-1}(y)| \) for all \( x, y \in \mathbb{Z}_K \).

**Proof.** First suppose \( x \leq y \). Then
\[
|f^{-1}(x)| = \sum_i \left( \sum_{x+iK < y/2}^s x + iK \right) + \sum_i \left( \sum_{y+iK \geq y/2}^s x + iK \right)
\]
\[
\leq \sum_i \left( \sum_{x+iK < y/2}^s y+iK \right) + \sum_i \left( \sum_{y+iK \geq y/2}^s y+iK \right) + \sum_i \left( \sum_{y+iK \geq y/2}^s x+iK + K \right) + \sum_i \left( \sum_{y+iK \geq y/2}^s x+iK + K \right)
\]
\[
= |f^{-1}(y)| + \left( \frac{s}{2} \right).
\]
Similarly, if \( x > y \), then still \( |f^{-1}(x)| \leq |f^{-1}(y)| + \left( \frac{s}{2} \right) \).

Thus, for all \( y \in \mathbb{Z}_K \), we have that \( |f^{-1}(y)| \) is within \( \left( \frac{s}{2} \right) \) of the average value of \( \frac{2s}{K} \). Hence,
\[
\left( \frac{s}{2} \right) \leq \frac{4s^2}{5K} \leq \frac{4}{5} \left( \frac{s}{2} \right) \frac{K}{\sqrt{s}} \cdot
\]
from which follows
\[
\left( \frac{s}{2} \right) \leq \frac{4}{5} \left( \frac{s}{2} \right) - 4|f^{-1}(y)|.
\]

**Proof of Theorem 7** Let \( B_1 \subseteq_R A_i \) be a random size-2 subset of \( A_i \), for each \( i \). By Lemma 11, the sub-rectangle \( R' = B_1 \times \cdots \times B_r \) is almost unbiased with respect to some nontrivial subgroup \( H' \). Since \( H' \) is nontrivial, it contains \( H = \{0,2^{t-1}\} \), and hence \( R' \) is also almost unbiased with respect to \( H \). By this selection process, every \( (a_1, \ldots, a_r) \in A_1 \times \cdots \times A_r \) has the same probability of being selected and, hence, \( R \) itself is almost unbiased with respect to \( H \).

**Proof of Lemma 6** Set \( t = \frac{1}{4} \log n \) and \( T = 2^t \). Consider any rectangle \( R = A_1 \times \cdots \times A_r \) involving at least \( r \geq n^{5/6} = T^5 \) parties. By the Addition Theorem, the multiset \( A_1 + \cdots + A_r \) has bias at most \( O(T^{3/2}/r^{1/2}) \) of \( O(1/n^{1/6}) \) with respect to \( \{0,2^{t-1}\} \). Hence, rectangle \( R \) has bias at most \( O(1/n^{1/6}) \), too.

**VI. CONCLUSIONS**

We studied experiments for validating quantum nonlocality in the presence of noise and with imperfect detectors. Specifically we concentrated on the generalization of the GHZ paradox to \( n \) parties previously considered as a quantum communication complexity problem.

There are several directions in which one may wish to improve the result Eq. 11. The first concerns the evaluation of the right-hand side of this relation. A detailed investigation of the proof shows that the right-hand side becomes nontrivial only for values of \( n \) that exceed a few hundred. Therefore our result will not be useful for the moderate values of \( n \), say, \( n \leq 10 \), which may be attainable by real-world experiments in the next few years. It would be interesting to try to improve Eq. 11 so as to
make it relevant for small values of $n$. Can the gap between the result in the absence of noise (when the right-hand side is $O(n^{-1})$) and the result in the presence of noise be closed?

Another question concerns our notion of error, which is not entirely appropriate to a multiparty setting: one expects that each party may induce an error independently of the other parties. Thus it would be more natural to consider that the probability of an error goes as $\varepsilon = 1 - \delta^n$. We do not know whether a constraint of the form Eq. (11) holds in this case.

Notwithstanding the above directions in which improvements are possible, there is a specific sense in which the above result can be shown to be close to optimal. Consider $n$ parties who share an entangled state $|\psi\rangle$ of dimension $2^n$. Each party’s system is two dimensional, i.e., each party has a single qubit. Fix a total-variation distance $\varepsilon_{\text{var}}$. Then for any measurement scenario involving local measurements on the quantum state $|\psi\rangle$, the amount of (super-luminal) communication required to reproduce these correlations up to total-variation distance $\varepsilon_{\text{var}}$ is at most of order $n \log n$, and the maximal detector efficiency $\eta^*$ for which these correlations are local is of order $n^{-c}$ for some constant $c$. This result will be reported elsewhere [22]. It shows that the example considered above is close to maximally nonlocal, at least if one restricts oneself to a large number of parties each possessing a single qubit.

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