Boundary regularity of correspondences in $\mathbb{C}^n$

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Abstract. Let $M, M'$ be smooth, real analytic hypersurfaces of finite type in $\mathbb{C}^n$ and $\hat{f}$ a holomorphic correspondence (not necessarily proper) that is defined on one side of $M$, extends continuously up to $M$ and maps $M$ to $M'$. It is shown that $\hat{f}$ must extend across $M$ as a locally proper holomorphic correspondence. This is a version for correspondences of the Diederich–Pinchuk extension result for CR maps.

Keywords. Correspondences; Segre varieties.

1. Introduction and statement of results

1.1 Boundary regularity

Let $U, U'$ be domains in $\mathbb{C}^n$ and let $M \subset U, M' \subset U'$ be relatively closed, connected, smooth, real analytic hypersurfaces of finite type (in the sense of D'Angelo). A recent result of Diederich and Pinchuk [DP3] shows that a continuous CR mapping $f: M \to M'$ is holomorphic in a neighbourhood of $M$. The purpose of this note is to show that their methods can be adapted to prove the following version of their result for correspondences. We assume additionally that $M$ (resp. $M'$) divides the domain $U$ (resp. $U'$) into two connected components $U^+$ and $U^-$ (resp. $U'^\pm$).

Theorem 1.1. Let $\hat{f}: U^- \to U'$ be a holomorphic correspondence that extends continuously up to $M$ and maps $M$ to $M'$, i.e., $\hat{f}(M) \subset M'$. Then $\hat{f}$ extends as a locally proper holomorphic correspondence across $M$.

We recall that if $D \subset \mathbb{C}^p$ and $D' \subset \mathbb{C}^m$ are bounded domains, a holomorphic correspondence $\hat{f}: D \to D'$ is a complex analytic set $A \subset D \times D'$ of pure dimension $p$ such that $A \cap (D \times \partial D') = \emptyset$, where $\partial D'$ is the boundary of $D$. In this situation, the natural projection $\pi: A \to D$ is proper, surjective and a finite-to-one branched covering. If in addition the other projection $\pi': A \to D'$ is proper, the correspondence is called proper. The analytic set $A$ can be regarded as the graph of the multiple valued mapping $\hat{f} := \pi' \circ \pi^{-1}: D \to D'$.

We also use the notation $A = \text{Graph}(\hat{f})$.

The branching locus $\sigma$ of the projection $\pi$ is a codimension one analytic set in $D$. Near each point in $D \setminus \sigma$, there are finitely many well-defined holomorphic inverses of $\pi$. The symmetric functions of these inverses are globally well-defined holomorphic functions on $D$. To say that $\hat{f}$ is continuous up to $\partial D$ simply means that the symmetric functions extend
continuously up to \( \partial \mathcal{D} \). Thus in Theorem 1.1 the various branches of \( \hat{f} \) are continuous up to \( \hat{M} \) and each branch maps points on \( \hat{M} \) to those on \( \hat{M}' \).

We say that \( \hat{f} \) in Theorem 1.1 extends as a holomorphic correspondence across \( \hat{M} \) if there exist open neighbourhoods \( \hat{U} \) of \( \hat{M} \) and \( \hat{U}' \) of \( \hat{M}' \), and an analytic set \( \hat{A} \subset \hat{U} \times \hat{U}' \) of pure dimension \( n \) such that (i) Graph\( (\hat{f}) \) intersected with \( (\hat{U} \cap \hat{U}^-) \times (\hat{U}' \cap \hat{U}') \) is contained in \( \hat{A} \) and (ii) the projection \( \hat{\pi} : \hat{A} \to \hat{U} \) is proper. Without condition (ii), \( \hat{f} \) is said to extend as an analytic set. Finally, the extension of \( \hat{f} \) is a proper holomorphic correspondence if in addition to (i) and (ii), \( \hat{\pi} : \hat{A} \to \hat{U}' \) is also proper.

**COROLLARY 1.1.**

Let \( D \) and \( D' \) be bounded pseudoconvex domains in \( \mathbb{C}^n \) with smooth real-analytic boundary. Let \( \hat{f} : \hat{D} \to \hat{D}' \) be a holomorphic correspondence. Then \( \hat{f} \) extends as a locally proper holomorphic correspondence to a neighbourhood of the closure of \( D \).

The corollary follows immediately from Theorem 1.1 and \([\text{BS}]\) where the continuity of \( \hat{f} \) is proved. This generalizes a well-known result of \([\text{BR}]\) and \([\text{DF}]\) where the extension past the boundary of \( D \) is proved for holomorphic mappings.

### 1.2 Preservation of strata

Let \( M_i^+ \) (resp. \( M_i^- \)) be the set of strongly pseudoconvex (resp. pseudoconcave) points on \( M \). The set of points where the Levi form \( \mathcal{L}_\rho \) has eigenvalues of both signs on \( T^C(M) \) and no zero eigenvalue will be denoted by \( M^\pm \) and finally \( M^0 \) will denote those points where \( \mathcal{L}_\rho \) has at least one zero eigenvalue on \( T^C(M) \). \( M^0 \) is a closed real analytic subset of \( M \) of real dimension at most \( 2n - 2 \). Then

\[
M = M_i^+ \cup M_i^- \cup M^\pm \cup M^0.
\]

Further, let \( M_i^+ \) (resp. \( M_i^- \)) be the pseudoconvex (resp. pseudoconcave) part of \( M \), which equals the relative interior of \( M_i^+ \) (resp. \( M_i^- \)). For non-negative integers \( i, j \) such that \( i + j = n - 1 \), let \( M_{i,j} \) denote those points at which \( \mathcal{L}_\rho \) has exactly \( i \) positive and \( j \) negative eigenvalues on \( T^C(M) \). Each (non-empty) \( M_{i,j} \) is relatively open in \( M \) and semi-analytic whose relative boundary is contained in \( M^0 \). With this notation, \( M_{0,n-1} = M_i^- \) and \( M_{n-1,0} = M_i^+ \). Moreover, \( M^\pm \) is the union of all (non-empty) \( M_{i,j} \) where both \( i, j \) are at least 1 and \( i + j = n - 1 \). Note that points in \( M_i^- \), \( M_i^+ \) are in the envelope of holomorphy of \( U^- \). Following \([\text{H}]\), there is a semi-analytic stratification for \( M^0 \) given by

\[
M^0 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \quad (1.1)
\]

where \( \Gamma_4 \) is a closed, real analytic set of dimension at most \( 2n - 4 \) and \( \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \) is also a closed, real analytic set of dimension at most \( 2n - 3 \). Further, \( \Gamma_1, \Gamma_2, \Gamma_3 \) are either empty or smooth, real analytic manifolds; \( \Gamma_2, \Gamma_3 \) have dimension \( 2n - 3 \), and \( \Gamma_1 \) has dimension \( 2n - 2 \). Finally, \( \Gamma_2 \) and \( \Gamma_3 \) are CR manifolds of complex dimension \( n - 2 \) and \( n - 3 \) respectively. The set of points, denoted by \( \Gamma_i \) in \( \Gamma_1 \) where the complex tangent space to \( \Gamma_i \) has dimension \( n - 1 \) is semi-analytic and has real dimension at most \( 2n - 3 \), as otherwise there would exist a germ of a complex manifold in \( M \) contradicting the finite type hypothesis. Then \( \Gamma_1 \setminus \Gamma_1^0 \) is a real analytic manifold of dimension \( 2n - 2 \) and has CR dimension \( n - 2 \). Using the same letters to denote the various strata of \( M^0 \), there exists a refinement of (1.1), so that \( \Gamma_1, \Gamma_2, \Gamma_3 \) are all smooth, real analytic manifolds of dimensions
By the implicit function theorem, it is possible to choose neighbourhoods of the origin such that for any \( w \) in \( \Omega \), the symmetric point \( w' \) is denoted by \( w \). The set of all Segre varieties, and let \( \Omega \) be a discrete set of points. Thus \( \Omega \) is proper in a small neighbourhood of each point of \( M \).

For \( w \in U_1^+ \), the symmetric point \( w' \) is defined to be the unique point of intersection of the complex normal to \( M \) through \( w \) and \( Q_w \). The component of \( Q_w \cap U_2^+ \) that contains the symmetric point is denoted by \( Q_w' \).

Finally, for all objects and notions considered above, we simply add a prime to define their corresponding analogs in the target space.
3. Localization and extension across an open dense subset of $M$

In the proof of Theorem 1.1 in order to show extension of $\hat{f}$ as a holomorphic correspondence, it is enough to consider the problem in an arbitrarily small neighbourhood of any point $p \in M$. The reason is the following. Firstly, since the projection $\pi$: $\text{Graph}(\hat{f}) \to U^-$ is proper, the closure of $\text{Graph}(\hat{f})$ has empty intersection with $U^- \times \partial U'$. Therefore, by §20.1, to prove the continuation of $\hat{f}$ across $M$ as an analytic set, it is enough to do that in a neighbourhood of any point in $M$. Secondly, once the extension of $\hat{f}$ as a holomorphic correspondence in a neighbourhood of any point $p \in M$ is established, then globally there exists a holomorphic correspondence defined in a neighbourhood $\tilde{U}$ of $M$ which extends $\hat{f}$. To see that simply observe that if $F \subset \tilde{U} \times \tilde{U}'$ is an analytic set extending $\hat{f}$, then by choosing smaller $\tilde{U}$ we may ensure that the projection to the first component is proper, as otherwise there would exist a point $z$ on $M$ such that $\hat{F}(z)$ has positive dimension (here $\hat{F}$ is the map associated with the set $F$). This however contradicts local extension of $\hat{f}$ near $z$ as a holomorphic correspondence.

Since the projection $\pi$: $\text{Graph}(\hat{f}) \to U^-$ is proper, $\text{Graph}(\hat{f})$ is contained in the analytic set $A \subset U^- \times U'$, defined by the zero locus of holomorphic functions $P_1(z, z_1), P_2(z, z_2), \ldots, P_n(z, z_n)$ given by

$$P_j(z, z_j') = z_j^{l_j} + a_{j1}(z)z_j^{l_j-1} + \cdots + a_{jl}(z),$$

where $l$ is the generic number of images in $\hat{f}(z)$, and $1 \leq j \leq n$ (for details, see [C]). The coefficients $a_{\mu j}(z)$ are holomorphic in $U^-$ and extend continuously up to $M$. This is the definition of continuity of the correspondence $\hat{f}$ up to $M$ which is equivalent to that given in §1.1.

The discriminant locus is $\{R_j(z) = 0\}, 1 \leq j \leq n$, where $R_j(z)$ is a universal polynomial function of $a_{\mu j}(z)$ ($1 \leq \mu \leq l$) and hence by the uniqueness theorem, it follows that $\{R_j(z) = 0\} \cap M$ is nowhere dense in $M$, for all $j$. The set of points $S$ on $M$ which do not belong to $\{R_j(z) = 0\} \cap M$ for any $j$ is therefore open and dense in $M$. Near each point $p$ on $S$, $\hat{f}$ splits into well-defined holomorphic maps $f_1(z), f_2(z), \ldots, f_l(z)$ each of which is continuous up to $M$.

If $p \in S \cap (M^- \cup M^+)$, the functions $a_{\mu j}(z)$ extend holomorphically to a neighbourhood of $p$ and hence $\hat{f}$ extends as a holomorphic correspondence across $p$. It is therefore sufficient to show that $\hat{f}$ extends across an open dense subset of $S \cap M^+$. But this follows from Lemma 3.2 and Corollary 3.3 in [DP3]. We denote by $\Sigma \subset M$ the non-empty open dense subset of $M$ across which $\hat{f}$ extends as a holomorphic correspondence.

4. Extension as an analytic set

Fix $0 \in M$ and let $p'_1, p'_2, \ldots, p'_k \in \hat{f}(0) \cap M'$. The continuity of $\hat{f}$ allows us to choose irreducible and extend continuously up to $M$. Moreover, $\hat{f}_i(0) = p'_i$ for all $1 \leq i \leq k$. It will suffice to focus on one of the $\hat{f}_i$’s, say $\hat{f}_1$, and to show that it extends holomorphically across the origin. Abusing notation, we will write $\hat{f}_1 = \hat{f}, U_1' = U'$ and $p'_1 = 0'$. Thus $\hat{f}: U^-_1 \to U'$ is an irreducible holomorphic correspondence and $\hat{f}(0) = 0'$. Define

$$V^+ = \{(w, w') \in U_1^+ \times U': \hat{f}(Q_w^+) \subset Q_{w'}\}.$$
Then $V^+$ is non-empty. Indeed, $\hat{f}$ extends across an open dense set near the origin and $\mathbb{V}$ shows that the invariance property of Segre varieties then holds. Moreover, a similar argument as in $\mathbb{S}2$ shows that $V^+ \subset U_1^+ \times U'$ is an analytic set of dimension $n$ and exactly the same arguments as in Lemmas 4.2 – 4.4 of $\mathbb{D}P3$ show that: first, the projection $\pi: V^+ \rightarrow U^+ := \pi(V^+) \subset U_1^+$ is proper (and hence that $U^+ \subset U_1^+$ is open) and second, the projection $\pi': V^+ \rightarrow U'$ is locally proper. Thus, to $V^+$ is associated a correspondence $F^+: U^+ \rightarrow U'$ whose branches are $\hat{F}^+ = \pi' \circ \pi^{-1}$.

Let $a \in M$ be a point close to the origin, across which $\hat{f}$ extends as a holomorphic correspondence. If $\hat{f}$ is well-defined in the ball $B(a, r)$, $r \geq 0$ and $w \in B(a, r)^-$, it follows from Theorem 4.1 in $\mathbb{V}$ that all points in $\hat{f}(w)$ have the same Segre variety. By analytic continuation, the same holds for all $w \in U_1^-$. Using this observation, it is possible to define another correspondence $F^-: U_1^- \rightarrow U'$ whose branches are $\hat{F}^-(w) = (\lambda')^{-1} \circ \lambda' \circ \hat{f}(w)$.

Let $U := U_1^- \cup U^+ \cup (\Sigma \cap U_1)$. The invariance property of Segre varieties shows that the correspondences $\hat{F}^+, \hat{F}^-$ can be glued together near points on $\Sigma \cap U_1$. Hence, there is a well-defined correspondence $\hat{F}: U \rightarrow U'$ whose values over $U^+$ and $U_1^-$ are $\hat{F}^+$ and $\hat{F}^-$ respectively. Note that

$$F := \text{Graph}(\hat{F}) = \{(w, w') \in U \times U': w' \in \hat{F}(w)\}$$

is an analytic set in $U \times U'$ of pure dimension $n$, with proper projection $\pi: F \rightarrow U$. Once again, the invariance property shows that all points in $\hat{F}(w)$, $w \in U$, have the same Segre variety.

**Lemma 4.1.** The correspondence $\hat{F}$ satisfies the following properties:

(i) For $w_0 \in \partial U \cap U_1^-$, $\text{cl}_F(w_0) \subset \partial U'$.

(ii) $\text{cl}_F(0) \subset Q_0'$.

(iii) If $\text{cl}_F(0) = \{0'\}$, then $0 \in \Sigma$.

(iv) $F \subset (U_1 \setminus (M \setminus \Sigma)) \times U'$ is a closed analytic set.

**Proof.**

(i) Choose $(w_j, w'_j) \in F$ converging to $(w_0, w'_0) \in (\partial U \cap U_1^+) \times U'$. Then $\hat{f}(Q_{w_j}) \subset Q'_{w_j}$ for all $j$. If $w'_0 \in U'$, then passing to the limit, we get $\hat{f}(Q_{w_0}) \subset Q'_{w_0}$ which shows that $(w_0, w'_0) \in F$ and hence $w_0 \in U$, which is a contradiction. This proves (iv).

(ii) Choose $w_j \in U$ converging to 0. There are two cases to consider. First, if $w_j \in U_1^- \cup (\Sigma \cap U_1)$ for all $j$, it follows that $\hat{f}(w_j) \rightarrow 0'$. Moreover, for any $w'_j \in \hat{F}(w_j)$, $Q'_{w'_j} = Q'_{f(w_j)}$. If $U'$ is small enough, the equality $Q'_{w'_j} = Q'_{w'_0}$ implies that $w'_j = 0'$ and thus we conclude that $w_j \rightarrow 0' \in Q_0'$. Second, if $w_j \in U^+$ for all $j$, then $\hat{f}(Q_{w_j}) \subset Q'_{w'_j}$ for any $w'_j \in \hat{F}(w_j)$. Let $w'_j \rightarrow w_0' \in U'$. If $\zeta \in Q_{w'_0}$, then $\hat{f}(\zeta) \in Q'_{w'_0} \rightarrow Q'_{w'_j}$. But $w_j \rightarrow 0$ implies that $\text{dist}(Q_{w'_0}, 0) \rightarrow 0$ and hence $\hat{f}(\zeta) \rightarrow 0'$. Thus $0' \in Q'_{w'_0}$ which shows that $w'_0 \in Q_0'$.

(iii) If $\text{cl}_F(0) = \{0'\}$, then (i) shows that $0 \notin \partial U \cap U_1^+$. Let $B(0, r)$ be a small ball around the origin such that $B(0, r) \cap \partial U = \emptyset$. The correspondence $\hat{F}$ over $B(0, r)^+$ is the union of some components of the zero locus of a system of monic pseudo-polynomials whose coefficients are bounded holomorphic functions on $B(0, r)^+$. By
Trepureå’s theorem, all these coefficients extend holomorphically to \( B(0, r) \), and the extended zero locus contains the graph of \( \hat{f} \) near the origin since \( \Sigma \) is dense. It follows that \( 0 \in \Sigma \).

Following [S1], for any \( w_0 \in U \), it is possible to find a neighbourhood \( \Omega \) of \( w_0 \), relatively compact in \( U \) and a neighbourhood \( V \subset U_1 \) of \( Q_{w_0} \cap U_1 \) such that for \( z \in V \), \( \Omega_2 \cap \Omega \) is non-empty and connected. Associated with the pair \((\Omega, V)\) is

\[
\hat{F} := \hat{F}(w_0, \Omega, V) = \{(z, z') \in V \times U' : \hat{F}(Q_2 \cap \Omega) \subset Q'_2\}
\]

(4.1)

which (see [DP4]) is an analytic set of dimension at most \( n \). If \( w_0 \in \Sigma \), then Corollary 5.3 of [DP3], shows that \( F \cap (V \times U') \) is the union of irreducible components of \( \hat{F} \) of dimension \( n \). As in [DP3] we call \((w_0, z_0) \in U \times Q_{w_0}\) a pair of reflection if there exist neighbourhoods \( \Omega(w_0) \ni w_0 \) and \( \Omega(z_0) \ni z_0 \) such that for all \( w \in \Omega(w_0) \), \( \hat{F}(Q \cap \Omega(z_0)) \subset Q'_w \).

It follows from the invariance property of Segre varieties that the definition of the pair of reflection is symmetric. As an example we note that if the set \( \tilde{F} \) defined in (4.1) contains \( F \cap (V \times U') \), then \((w_0, z)\) is a point of reflection for any point \( z \) in a connected component of \( Q_{w_0} \cap U \) containing \( w_0 \).

Let \( w_0 \in U \), \( z_0 \in Q_{w_0} \cap \Sigma \) be a pair of reflection. Fix \( B(z_0, r) \), a small ball around \( z_0 \) where \( \tilde{F} \) is well-defined and let \( S(w_0, z_0) \subset F \cap (Q_{w_0} \cap U_1 \times U') \) be the union of those irreducible components that contain \( \text{Graph}(\tilde{f}) \) over \( Q_{w_0} \cap B(z_0, r) \). Note that \( S(w_0, z_0) \) is an analytic set of dimension \( n - 1 \) and is contained in \((Q_{w_0} \cap U_1) \times U'\) and moreover, the invariance property shows that

\[
S(w_0, z_0) \subset ((Q_{w_0} \cap U_1) \times (Q'_{F(w_0)} \cap U')).
\]

Furthermore, from the above considerations it follows that for any \( z \in \pi(S(w_0, z_0)) \) the point \((w_0, z)\) is a pair of reflection. Finally, let the cluster set of a sequence of closed sets \( \{C_j\} \subset \mathcal{D} \), where \( \mathcal{D} \) is some domain, be the set of all possible accumulation points in \( \mathcal{D} \) of all possible sequences \( \{c_j\} \) where \( c_j \in C_j \).

**PROPOSITION 4.1.**

Let \( \{z_v\} \in \Sigma \) converge to 0. Suppose that the cluster set of the sequence \( \{S(z_v, z_v)\} \) contains a point \((\zeta_0, \zeta_0') \in U \times U'\). Then \( \tilde{f} \) extends as an analytic set across the origin.

**Proof.** First, the pair \((z_v, z_v)\) is an example of a pair of reflection and hence \( S(z_v, z_v) \) is well-defined. Also, note that \((z_v, \tilde{f}(z_v)) \rightarrow (0, 0')\). Choose \((\zeta_v, \zeta_v') \in S(z_v, z_v) \) that converges to \((\zeta_0, \zeta_0') \in U \times U'\). It follows that \((\zeta_v, z_v)\) is a pair of reflection. Let \( \Omega, V \) be neighbourhoods of \( \zeta_0 \) and \( Q_{\zeta_0} \) as in the definition of \( \hat{F}(\zeta_0, \Omega, V) \). Since \( \zeta_0 \in U \), it follows that \( \hat{F}(\zeta_0, \Omega, V) \) is a non-empty, analytic set in \( V \times U'\). Shrinking \( U_1 \) if needed, \( Q_{\zeta_v} \cap U_1 \subset V \) and \( \zeta_v \in \Omega \) for all large \( v \). This shows that \( \hat{F}(\zeta_v, \Omega, V) = \hat{F}(\zeta_0, \Omega, V) \) for all large \( v \). Lemma 5.2 of [DP3] shows that \( \hat{F}(\zeta_v, \Omega, V) \) contains the graph of all branches of \( \tilde{f} \) near \( z_v \) and hence \( \hat{F}(\zeta_0, \Omega, V) \) contains the graph of \( \tilde{f} \) near \( (0, 0') \). Therefore, \( \hat{F}(\zeta_0, \Omega, V) \) extends the graph of \( \tilde{f} \) across the origin. \( \square \)

**Remarks.** First, as in [DP3] this proposition will be valid if the pair \((z_v, z_v)\) were replaced by a pair of reflection \((w_v, z_v) \in U \times \Sigma \) that converges to \((0, 0')\) and \( \tilde{f}(w_v) \) clusters at some point in \( U' \). Second, this proposition shows the relevance of studying the cluster set of a sequence of analytic sets (see [SV] also). In general, the hypothesis
that the cluster set of \( \{ S(z_v, z) \} \) (or \( S(w, z_v) \) in case \( (w, z_v) \) is a pair of reflection) contains a point in \( U \times U' \) cannot be guaranteed since the projection \( \pi: S(z_v, z) \to U \) is not known to be proper. However, the following version of Lemma 5.9 in [DP3]

holds.

Lemma 4.2. There are sequences \( (w, z_v) \in U \times \Sigma, w'_v \in \tilde{F}(w) \) and analytic sets \( \sigma_v \subset U \)
of pure dimension \( p \geq 1 \) (\( p \) independent of \( v \)) such that:

(i) \( (w, z_v) \to (0, 0) \) and \( (w, z_v) \) is a pair of reflection for all \( v \).

(ii) \( w'_v \to w'_v \) and \( z_v \in \sigma_v \subset \pi(S(w, z_v)) \).

Proof. Choose a sequence \( z_v \in \Sigma \) that converges to the origin. If the projections

\( \pi: S(z_v, z) \to U \) were proper for all \( v \), then for some fixed \( r > 0 \) and \( v \) large enough, let

\( \sigma_v := Q_{z_v} \cap B(z_v, r), w_v = z_v \) and \( w'_v \in \tilde{f}(z_v) \). It can be seen that the lemma holds with these choices. On the other hand, if \( \pi \) is not known to be proper on \( S(z_v, z) \), no fixed value of \( r \), as described above, exists. Hence, for arbitrarily small values of \( r' > 0 \), there exist

\( (w, w'_v) \in S(z_v, z) \cap (U \times U') \) such that \( w_v \to 0 \) and \( w'_v \to w'_v \) with \( |w'_v| = r' \). Since \( M' \)
is of finite type, we may assume that \( Q'_{w'_v} \neq Q'_w \). Moreover, note that \( w'_v \in Q'_{w'_v} \cap U' \) (which shows that \( v' \in Q'_{w'_v} \) and \( (w, z_v) \) is a pair of reflection for all \( v \). By making a small holomorphic perturbation of coordinates in the target space, if needed, it follows that \( v' \in Q'_{w'_v} \cap \{ v' \in U': v'_0 = \cdots = v'_{n-1} = 0, |v'| < \epsilon \} \)

(which is an analytic set of dimension 1 in \( U' \cap \{ |v'| < \epsilon \} \) containing the origin) has no limit points on \( \partial U' \cap \{ |v'| < \epsilon \} \). Let \( l \) be the multiplicity of \( \tilde{f}: U_1 \to U' \). Let

\( \tilde{f}(z_v) = \{ \xi_0, \xi_1, \ldots, \xi_l \} \), \( 1 \leq j \leq l \) counted with multiplicity. For large \( v \), the \( l \)
sets

\[ \left\{ v \in U': v'_0 = \cdots = v'_{n-1} = 0, |v'_{n-1}| < \epsilon \right\} \]

are analytic, of dimension 1, in \( U' \cap \{ |v'| < \epsilon \} \) without limit points on \( \partial U' \cap \{ |v'| < \epsilon \} \) and clearly contain \( \{ z_v, \xi_0, \xi_1, \ldots, \xi_l \} \). Since \( \pi' \in \text{cl}(S(w, z_v)) \subset Q'_{w'_v} \),

\[ s_{v, j} := S(w, z_v) \cap \{ z, z'_v = (\xi_j) \}, \quad 2 \leq k \leq n-1 \}

are analytic sets of dimension at least 1 in \( U_1 \times \{ z, |z'| < \epsilon \} \) for all \( 1 \leq j \leq l \). By

construction, the analytic sets \( s_{v, j} \) do not have limit points on \( \partial U' \cap \{ |z'| < \epsilon \} \) and hence

\( s_{v, j} \) do not have limit points on \( U_1 \times \{ |z'| < \epsilon \} \). By Lemma 4.1, \( \text{cl}_p(0) \subset Q'_{w'_v} \), \( \{ z_v = 0 \} \) and by shrinking \( U_1 \) if needed, this shows that \( s_{v, j} \) have no limit points on \( U_1 \times \{ |z'_v| = \epsilon \} \). Thus for large \( v \) and all \( j \), the projections \( \pi: s_{v, j} \to U_1 \) are proper and their images \( \sigma_{v, j} := \pi(s_{v, j}) \) are analytic sets in \( U_1 \) of dimension at least 1 and \( z_v \in \sigma_{v, j} \) for all \( v, j \). It remains to pass to subsequences if necessary to choose \( \sigma_{v, j} \) with constant dimension.

One conclusion that follows now is: if \( \tilde{f} \) does not extend as an analytic set across the

origin, then \( \text{cl}(\sigma_v) \subset M_1 \Sigma \). Indeed, if there exists \( \zeta_0 \in \text{cl}(\sigma_v) \cap (U_1 \setminus \{ M_1 \Sigma \}) \), let \( (\zeta_v, \zeta'_v) \in S(w, z) \) converge to \( (\zeta_0, \zeta'_0) \) in \( U_1 \times U' \). Proposition 4.1 now shows that \( \zeta'_0 \in \partial U \cap U_1 \). But since \( \zeta'_0 \notin M_1 \Sigma \), it follows from Lemma 4.1 that \( \zeta'_0 \in \partial U' \) which is a contradiction.

The goal will now be to show that \( \tilde{f} \) extends as an analytic set across the origin. For this, choose \( \{ z_v \} \subset \Sigma \) converging to the origin and consider the analytic sets \( S(z_v, z) \). By Proposition 4.1, it suffices to show that \( \text{cl}(S(z_v, z)) \cap U \neq \emptyset \). Let

\[ S' := \pi' \{ \text{cl}(S(z_v, z)) \cap (\{ 0 \} \times U') \} \subset Q'_{w'_v} \]
and let \( m \) be the dimension of \( \tilde{S} \) – the smallest closed analytic set containing \( S' \) (the so-called Segre completion of the analytic set across the origin). If \( m = 0 \), then \( 0' \) is an isolated point in \( S' \) and after shrinking \( U_1, U' \) suitably, it follows that \( \text{cl}(S(\zeta_v, \zeta_v)) \) has no limit points on \( U_1 \times \partial U' \).

Thus \( \pi: S(\zeta_v, \zeta_v) \to U_1 \) are proper projections and therefore \( \pi(S(\zeta_v, \zeta_v)) = Q_{\zeta_v} \cap U_1 \) for all large \( v \). Hence \( \pi(\text{cl}(S(\zeta_v, \zeta_v))) = Q_0 \cap U_1 \).

If \( \tilde{f} \) did not extend as an analytic set across the origin, the aforementioned remark shows that with \( \sigma := Q_{\zeta_v} \cap U_1 \), \( Q_0 \cap U_1 = \text{cl}(\sigma_v) \subset M \setminus \Sigma \subset M \).

This cannot happen as \( M \) is of finite type. Hence \( \tilde{f} \) extends as an analytic set across the origin in case \( m = 0 \). We may therefore suppose that \( m > 0 \). We recall the following lemma proved by Diederich and Pinchuk:

**Lemma 4.3.** ([DP3, Lemma 9.8]). Let \( S' \) be a subset of \( Q_{\nu}', 0' \in S' \) and \( m = \dim \tilde{S} \). Then after possibly shrinking \( U_1 \), there are points \( w^1, \ldots, w^k \in S' (k \leq n - 1) \) such that one of the following holds:

1. \( k = m \) and \( \dim(S' \cap Q_{\nu}') = 0 \);
2. \( k \geq 2m - n + 1 \) and \( \dim(S' \cap Q_{\nu}') = m - k \).

Thus there are two cases to consider.

**Case 1.** Choose \( (w_{1v}, w_{1v}'), (w_{2v}, w_{2v}'), \ldots, (w_{mv}, w_{mv}') \in S(\zeta_v, \zeta_v) \) so that \( w_{\mu v} \to 0 \) and \( w_{\mu v}' \to w_{\mu} \) for all \( 1 \leq \mu \leq m \). A generic choice of \( w_{\mu v} \) (see p. 136 in [DP3]) ensures that \( q_{\mu v} := Q_{\mu v} \cap Q_{\nu} \cap \cdots \cap Q_{\nu} \) has dimension \( n - m \). Each \( (w_{\mu v}, \zeta_v) \) is a pair of reflection and hence the analytic set

\[
S_{\nu}^m := \bigcap_{1 \leq \mu \leq m} S(w_{\mu v}, \zeta_v) \subset (q_{\mu v} \times q_{\nu}) \cap (U_1 \times U')
\]

is well-defined. If \( m = n - 1 \), then Lemma 9.7 of [DP3] shows that the germ of \( q^{(n-1)} \) at the origin has dimension 1. Moreover, \( \tilde{S}' = Q_{\mu} \) and Lemma 4.3 implies that \( q^{(m-1)} \cap \tilde{Q}_{\nu} \) contains \( 0' \) as an isolated point. Since \( \text{cl}(\tilde{f})(0) \subset Q_{\mu} \), it follows that \( 0' \) is an isolated point of

\[
\pi(\text{cl}(S_{\nu}^{m-1}) \cap \{0'\} \times U') \subset q^{(n-1)} \cap Q_{\mu} = \{0'\}.
\]

Shrinking \( U_1 \), the projection \( \pi: S_{\nu}^{m-1} \to U_1 \) becomes proper and \( \pi(S_{\nu}^{m-1}) = q^{n-1,v} \cap U_1 \).

By Theorem 7.4 of [DP3], there is a subsequence of \( q^{n-1,v} \cap U_1 \) that converges to an analytic set \( A \subset U_1 \) of positive dimension and as before this shows that \( \tilde{f} \) extends as an analytic set across the origin.

If \( m < n - 1 \), the dimension of \( S_{\nu}^{m} \cap S(\zeta_v, \zeta_v) \) is at least \( n - m - 1 > 0 \). Now

\[
\pi(\text{cl}(S_{\nu}^{m} \cap S(\zeta_v, \zeta_v)) \cap \{0'\}) \subset q^{m} \cap \tilde{S}' = \{0'\},
\]

the last equality following from Lemma 4.3. The projection \( \pi: S_{\nu}^{m} \cap S(\zeta_v, \zeta_v) \to U_1 \) is therefore proper for small \( U_1 \) and that \( \pi(S_{\nu}^{m} \cap S(\zeta_v, \zeta_v)) = q^{m} \cap q_{\zeta_v} \cap U_1 \). Again, by Theorem 7.4 of [DP3], there is a subsequence of \( q^{m} \cap q_{\zeta_v} \cap U_1 \) that converges to an analytic set \( A \subset U_1 \) of positive dimension and as before this shows that \( \tilde{f} \) extends as an analytic set across the origin.
Case 2. As before, choose \( (w_{1\nu}, w'_{1\nu}, (w_{2\nu}, w'_{2\nu}), \ldots, (w_{k\nu}, w'_{k\nu}) \in S(z_{\nu}, z'_{\nu}) \) such that \( w_{\mu\nu} \to 0 \) and \( w'_{\mu\nu} \to w'_{\mu}\) for all \( 1 \leq \mu \leq k \) and \( q_{\mu\nu} = Q_{w_{\mu\nu}} \cap Q_{w'_{\mu\nu}} \cap \cdots \cap Q_{w'_{k\nu}} \), \( \hat{q}^{kv} := Q_{z_{\nu}} \cap q^{kv} \) have dimension \( n - k \) and \( n - k - 1 \) respectively. Now note that \( \dim(S_{\nu} \cap S(z_{\nu}, z'_{\nu})) \geq n - k - 1 \). Indeed, the inequalities \( 2m - n + 1 \leq k < m \) show that \( m \leq n - 2 \) and hence \( k < n - 2 \). Since the dimension of \( S' \cap q'^{k} \) is \( m - k \), choose coordinates so that

\[
S' \cap q'^{k} \cap \{ z' \in U' : z'_1 = z'_2 = \cdots = z'_{m-k} = 0 \} = \{ 0' \}.
\]

Let \( \hat{f}(z_{\nu}) = \{ \xi'_{\nu} \}, 1 \leq j \leq l \), \( l \) being the multiplicity of \( \hat{f} \). The \( l \) sets

\[
T_{v,j} = \{ (z, z') \in S_{\nu} \cap S(z_{\nu}, z'_{\nu}) : z'_1 = (\xi'_{1})_{j}, z'_2 = (\xi'_{2}), \ldots, z'_{m-k} = (\xi'_{m-k})_{j} \},
\]

where \( 1 \leq j \leq l \) are analytic sets in \( U_{1} \times U' \) and have dimension at least \( n - k - 1 - (m - k) = m - n - 1 > 0 \). By construction,

\[
\pi'(\text{cl}(T_{v,j}) \cap \{ \{ 0 \} \times U' \}) \subset S' \cap q'^{k}
\]

\[
\cap \{ z' \in U' : z'_1 = z'_2 = \cdots = z'_{m-k} = 0 \} = \{ 0' \}
\]

and hence by shrinking \( U_{1}, U' \), the projections \( \pi : T_{v,j} \to U_{1} \) are proper and the images \( \sigma_{v,j} := \pi(T_{v,j}) \subset U_{1} \) are analytic and have dimension \( n - m - 1 \). Moreover \( \sigma_{v,j} \subset q^{kv} \), and since \( q^{kv} \) depend anti-holomorphically on the \( k \)-tuple defining it, Theorem 7.4 of [DP3] shows that \( \hat{q}^{kv} \) converges to an analytic set \( A \subset U_{1} \) of dimension \( n - k - 1 \), after passing to a subsequence. Working with this subsequence, we see that \( \text{cl}(\sigma_{v,j}) \subset \hat{A} \). On the other hand, since \( 2m - n + 1 \leq k \), it follows, as in [DP3], that

\[
\dim \hat{A} = n - k - 1 \leq 2(n - m - 1) = 2 \dim \sigma_{v,j}.
\]

Proposition 8.3 of [DP3] shows that \( \text{cl}(\sigma_{v,j}) \subset M \) and hence by Proposition 4.1 it follows that \( \hat{f} \) extends as an analytic set across the origin.

To complete the proof, it suffices to show that extension as an analytic set implies extension as a locally proper holomorphic correspondence. This is achieved in the next lemma.

**Lemma 4.4.** There exist neighbourhoods \( U_{0} \) of \( 0 \) and \( U'_{0} \) of \( 0' \) such that \( F \subset U \times U' \) is a proper holomorphic correspondence which extends \( \hat{f} \).

**Proof.** Extension as a holomorphic correspondence essentially follows from [DP4]. All nuances in the proof of Proposition 2.4 in [DP4] work in this situation as well provided the following two modifications are made. Let \( U, U' \) be neighbourhoods of \( 0, 0' \) respectively and suppose that \( F \subset U \times U' \) extends \( \hat{f} \) as an analytic set in \( U \times U' \). Then it needs to be checked that \( F \cap (U' \times U') \neq \emptyset \) and that there exists a sequence \( \{ z_{\nu} \} \subset M \) converging to \( 0 \) such that \( \hat{f} \) extends as a correspondence across each \( z_{\nu} \).

Suppose that \( F \cap (U' \times U') = \emptyset \). In this case, the proof of Proposition 3.1 (or even Proposition 4.1 in [SY]) shows that \( (0, 0') \) is in the envelope of holomorphy of \( U^{-} \times U' \). The coefficients \( a_{\mu\nu}(z) \) in (3.1) can be regarded as holomorphic functions on \( U^{-} \times U' \) (i.e., independent of the \( z' \) variables) and thus each \( a_{\mu\nu}(z) \) extends holomorphically across
(0, 0'). This extension must be independent of the $'z'$ variables by the uniqueness theorem and hence $a_{\mu
u}(z)$ extends holomorphically across the origin. This shows that $\hat{f}$ extends as a holomorphic correspondence across the origin. To show the existence of the sequence $\{z_r\}$ claimed above, let $\pi: F \rightarrow U$ be the natural projection and define

$$A = \{(z, z') \in F: \dim \left( \pi^{-1}(z) \right)_{(z, z')} \geq 1\},$$

where $(\pi^{-1}(z))_{(z, z')}$ denotes the germ of the fiber over $z$ at $(z, z')$. Then $A$ is a $\pi$-analytic subset of $F$, and since $F$ contains the graph of $\hat{f}$ over $U^-$, it follows that the dimension of $A$ is at most $2n - 1$. Since Lipschitz maps do not increase Hausdorff dimension, it follows that the Hausdorff dimension of $\pi(A)$ is at most $2n - 2$. Pick $p \in M \setminus \pi(A)$. The fiber $F \cap \pi^{-1}(p)$ is discrete and this shows that $\hat{f}$ extends as a holomorphic correspondence across $p$.

Finally, we show that $U'$ can be chosen so small that the projection $\pi': F \rightarrow U'$ is also proper. Indeed, for $'z' \in M'$, $\pi'^{-1}(z')$ is an analytic subset of $F$. Since $\pi$ is proper, it follows by Remmert's theorem that $\hat{F}^{-1}(z') = \pi' \circ \pi'^{-1}(z')$ is an analytic set. The invariance property of Segre varieties yields $\hat{F}(Q_{\sigma} \cap U) \subset Q'_{\sigma}$ for any $z \in \hat{F}^{-1}(z')$. Since $M$ is of finite type, the set $\cup_{z \in \hat{F}^{-1}(z')} Q_{\sigma}$ has Hausdorff dimension $n$, and therefore cannot be mapped by $\hat{F}$ into $Q'_{\sigma}$, which has dimension $n - 1$. This shows that projection $\pi'$ has discrete fibers on $M'$. It follows from the Cartan–Remmert theorem that there exists a neighbourhood $U'$ of $M'$ such that $\pi'$ has only discrete fibers, and therefore the projection $\pi'$ from $F$ to $U'$ will be proper.

This completes the proof of Theorem 1.1.

\[\square\]

5. Preservation of strata

Fix $p \in M$ and let $p_1', p_2', \ldots, p_k' \in \hat{f}(p) \subset M'$. Choose neighbourhoods $U, U'$ of $p, p_1'$ respectively and let $\hat{f}_1: U^- \rightarrow U'$ be a component of $\hat{f}$ such that $\hat{f}_1(p) = p_1'$. Then $\hat{f}_1$ extends as a holomorphic correspondence $\hat{F} \subset U \times U'$ and to prove Theorem 1.2, it suffices to focus on $\hat{f}_1$, which will henceforth be denoted by $\hat{f}$. The following two general observations can be made in this situation. First, the branching locus $\hat{\sigma}$ of $\hat{F}$ is an analytic set in $U$ and the finite-type assumption on $M$ shows that the real dimension of $\hat{\sigma} \cap M$ is at most $2n - 3$. The branching locus of $\hat{f}$ denoted by $\sigma$, is contained in $\hat{\sigma} \cap U^-$. Second, the invariance property of Segre varieties in \cite{DP1, V} shows that $\hat{F}$, the extended correspondence, preserves the two components $U^\pm$. That is, after possibly re-labelling $U^\pm$, it follows that $\hat{F}(U^\pm) \subset U'^\pm$ and $\hat{F}(M) \subset M'$. The same holds for $\hat{G} := F^{-1}: U'^+ \rightarrow U$.

\textbf{Proof of Theorem 1.2.} Let $p \in M^+$ and suppose that $\{\zeta_i'\} \in M'$ is a sequence converging to $p_1'$ with the property that the Levi form $\mathcal{L}_{\hat{g}}$ restricted to the complex tangent space to $M$ at $\zeta_{i_0}'$ has at least one negative eigenvalue. Fix $\zeta_{j_0}' \in U'$ for some large $j_0$. By shifting $\zeta_{j_0}'$ slightly, we may assume that $\zeta_{j_0}' \notin \hat{\sigma}' \cup \hat{F}(M^0 \cap U)$, where $\sigma'$ is the branching locus of $\hat{G}$, and at the same time retain the property of having at least one negative eigenvalue. Let $g_1$ be a locally biholomorphic branch of $\hat{G}$ near $\zeta_{j_0}'$. Then $g_1(\zeta_{j_0}')$ is clearly a pseudoconvex point and this contradicts the invariance of the Levi form. This shows that $\hat{f}(M^+) \subset M'^+$. The same arguments show that $\hat{f}(M^-) \subset M'^-$. 

Let $p \in M^+ \cap M^0$ and suppose that $p_1' \in M'^+$. The extending correspondence $\hat{F}: U \rightarrow U'$ satisfies the invariance property, namely $\hat{F}(Q_{\omega}) \subset Q'_{\omega}$ for all $(w, w') \in (U \times U') \cap
Graph(\(\hat{F}\)). But near \(p'_1\), the Segre map \(\lambda\) is injective and this shows that \(\hat{F}\), and hence \(\hat{f}\), is a single valued, proper holomorphic mapping, say \(f: U \to U'\) with \(f(p) = p'_1\). Two observations can be made at this stage: first, \(f\) cannot be locally biholomorphic near \(p\) due to the invariance of the Levi form. Second, if \(V_f \subset U\) is the branching locus of \(f\) defined by the vanishing of the Jacobian determinant of \(f\), then \(V_f\) intersects both \(U^\pm\). Indeed, suppose that \(V_f \cap U^\pm = \emptyset\). Choose a branch of \(f^{-1}\) near some fixed point \(d' \in U''\) and analytically continue it along all paths in \(U''\) to get a well-defined mapping, say \(g: U'' \to U'^-\). The analytic set \(\tilde{F} \subset U \times U'\) extends \(g\) as a correspondence and hence \(\text{[DP2]}\) \(g\) is a well-defined holomorphic mapping in \(U'\) and this must be the single valued inverse of \(f\). Thus \(f\) is locally biholomorphic near \(p\) and this is a contradiction. The same argument works to show that \(\tilde{V}_f\) must intersect \(U^+\) as well. Note that \(V_f \cap M\) has real dimension at most \(2n - 3\). If \(p \in \Gamma_1\), choose \(U\) so small that \(M^0 \cap U \subset \Gamma_1\). Then there exists \(q \in \Gamma_1 \setminus (V_f \cap M)\) near \(p\), where \(f\) is locally biholomorphic. Thus \(q\) is mapped locally biholomorphically to \(f(q)\) which is a strongly pseudoconvex point and this is a contradiction. If \(p \in \Gamma_3\), then again we shrink \(U\) so that \(M^0 \cap U \subset \Gamma_3\) and \((M \cap U) \setminus \Gamma_3 \subset M^+\). Then \(f\) is locally biholomorphic near all points in \((M \cap U) \setminus \Gamma_3\) and therefore \(V_f \cap U^-\) must cluster only along \(\Gamma_3\). Since the CR dimension of \(\Gamma_3 = n - 3 < (n - 1) - 1\) which is one less than the dimension of \(V_f\), it follows (Theorem 18.5 in [9]) that \(V_f \cap U^-\) is a closed, analytic set in \(U\). Thus \(V_f \cap U^-\) has two analytic continuations, namely \(V_f\) and \(\overline{V_f} \cap U^-\) and therefore they must be the same. This shows that \(V_f\) cannot intersect \(U^+\) which is a contradiction. The same argument works if \(p \in \Gamma_4\), the only difference being that \(\overline{V_f} \subset U^-\) is analytic because of Shiffman’s theorem. Thus if \(p \in M^+ \cap M^0\), then \(p'_1 \in M^+ \cap M^0\).

To study this further, suppose that \(p \in M^+ \cap \Gamma_1\) and \(p'_1 \in M^+ \cap \Gamma'_2\). Choose \(U, U'\) small enough so that \(M^0 \cap U \subset \Gamma_1\) and \(M^0 \cap U' \subset \Gamma'_2\). Pick \(q \in \Gamma_1 \setminus (\hat{\sigma} \cap M)\). Then \(\hat{f}\) splits near \(q\) into finitely many well-defined holomorphic mappings each of which extends across \(q\). Moving \(q\) slightly, if needed, on \(\Gamma_1 \setminus (\hat{\sigma} \cap M)\), each of these holomorphic mappings are even locally biholomorphic near \(q\). Working with one of these mappings, say \(f_1\), it follows that \(f_1(q) \notin M^+_1\) due to the invariance of the Levi form. This means that \(f_1(q) \in \Gamma_2\). In the same way, all points in \(\Gamma_2\) that are sufficiently near \(q\) are mapped locally biholomorphically by \(f_1\) to \(\Gamma_2\). This cannot happen as \(\Gamma_2\) has strictly smaller dimension than \(\Gamma_1\).

The same argument shows that \(p'_1 \notin \Gamma'_3 \cup \Gamma'_4\). Hence \(p'_1 \in M^+ \cap \Gamma'_1\).

Suppose that \(p \in M^+ \cap \Gamma_2\) and \(p'_1 \in M^+ \cap \Gamma'_1\). Considering \(\hat{f}^{-1}: U' \to U\), the arguments used in the preceding lines show that this cannot happen. The case when \(p'_1 \in \Gamma'_4\) can be dealt with similarly. Now suppose that \(p'_1 \in \Gamma'_3\). As always, \(U, U'\) will be small enough so that \(M^0 \cap U \subset \Gamma_2\) and \(M^0 \cap U' \subset \Gamma'_3\). The arguments used above show that the cluster set of points in \(M^+_1 \cap U\) is contained in \(M^+_1 \cap U'\) and hence \(\hat{f}\) splits into finitely well-defined mappings each of which is locally biholomorphic near points in \(M^+_1 \cap U\). This shows that the branching locus \(\sigma \subset U^-\) of \(f\) clusters only along \(\Gamma_2\). Then \(\hat{F}(\sigma)\) is an analytic set of dimension \(n - 1\) in \(U'\). There are two cases to consider: first, if \(\hat{F}(\sigma)\) clusters only along \(\Gamma'_3\), then arguing as above, \(\overline{\hat{F}(\sigma)} \subset U'^-\) is a closed, analytic set in \(U'\). The strong disk theorem shows that \(p'_1\) is in the envelope of holomorphy of \(U'^--\) and this is a contradiction. Second, if there are points in \(\overline{\hat{F}(\sigma) \cap M^+_1}\), this means that \((\hat{F}(\sigma) \cap M^1) \cap \Gamma'_3\) has real dimension at most \(2n - 4\). Pick \(q' \in \Gamma'_3 \setminus \overline{\hat{F}(\sigma) \cap M^1}\) and note that the continuity of \(\hat{f}\) implies that \(\hat{f}^{-1}(q') \subset M^+_1\). As seen above, this cannot happen. Thus \(p'_1 \in \Gamma'_3\) or \(M^+_1\). Similar arguments show that if \(p \in M^+ \cap \Gamma_3\) or \(M^+ \cap \Gamma_4\), then \(p'_1 \in M^+ \cap \Gamma'_3\) or \(M^+ \cap \Gamma'_4\) respectively.
By reversing the roles of \( U^\pm \), the same arguments used in the preceding paragraphs can be applied to show that
\[
\hat{f} (M^- \cap M^0) \subset M^- \cap M^0 \text{ with the preservation of } M^- \cap \Gamma_j \text{ for } j = 1, 3, 4.
\]
Finally, fix integers \( i, j \) both at least 1 such that \( i + j = n - 1 \) and suppose that \( p \in M_{i,j} \).
Then there exists a point \( p_0 \) in \( U \) (chosen so small that \( M \cap U \subset M_{i,j} \)) and arbitrarily close to \( p \), where all branches of \( \hat{f} \) are well-defined and locally biholomorphic. The invariance of the Levi form shows that the images of \( p_0 \) under any of the branches of \( \hat{f} \) should all be in \( M_{i,j} \). Note that each of these images is close to \( p_1' \). This cannot happen if \( p_1' \) is in \( M^+ \cap M^- \text{ or in } M'_{i,j} \text{ for } i \neq i' \text{ and } j \neq j' \). The only possibility is that \( p_1' \) is in the relative interior of \( M'_{i,j} \). The same argument works if \( p \) is in the relative interior of \( M_{i,j} \).

\[\square\]

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References

[BR] Baouendi M and Rothschild L, Germs of CR maps between real analytic hypersurfaces, *Invent. Math.* 93 (1988) 481–500
[B] Bedford E, Proper holomorphic mappings from domains with real analytic boundary, *Am. J. Math.* 106 (1984) 745–760
[BS] Berteloot F and Sukhov A, On the continuous extension of holomorphic correspondences, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 24 (1997) 747–766
[C] Chirka E M, Complex analytic sets (Dordrecht: Kluwer) (1990)
[DF] Diederich K and Fornaess J E, Proper holomorphic mappings between real analytic pseudoconvex domains in \( \mathbb{C}^n \), *Math. Ann.* 282 (1988) 681–700
[DP1] Diederich K and Pinchuk S, Proper holomorphic maps in dimension 2 extend, *Indiana Univ. Math. J.* 44 (1995) 1089–1126
[DP2] Diederich K and Pinchuk S, Reflection principle in higher dimensions, *Doc. Math. J. Extra Volume ICM* (1998) Part II, pp. 703–712
[DP3] Diederich K and Pinchuk S, Regularity of continuous CR maps in arbitrary dimension, *Mich. Math. J.* 51(1) (2003) 111–140
[DP4] Diederich K and Pinchuk S, Analytic sets extending the graphs of holomorphic mappings, *J. Geom. Anal.* 14(2) (2004) 231–239
[DW] Diederich K and Webster S, A reflection principle for degenerate real hypersurfaces, *Duke Math. J.* 47 (1980) 835–845
[S1] Shafikov R, Analytic continuation of germs of holomorphic mappings between real hypersurfaces in \( \mathbb{C}^n \), *Mich. Math. J.* 47(1) (2001) 133–149
[S2] Shafikov R, On boundary regularity of proper holomorphic mappings, *Math. Z.* 242(3) (2002) 517–528
[SV] Shafikov R and Verma K, A local extension theorem for proper holomorphic mappings in \( \mathbb{C}^2 \), *J. Geom. Anal.* 13(4) (2003) 697–714
[V] Verma K, Boundary regularity of correspondences in \( \mathbb{C}^2 \), *Math. Z.* 231(2) (1999) 253–299