ON A MORELLI TYPE EXPRESSION OF COHOMOLOGY CLASSES OF TORIC VARIETIES

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Abstract. Let $X$ be a complete $\mathbb{Q}$-factorial toric variety of dimension $n$ and $\Delta$ the fan in a lattice $N$ associated to $X$. $\Delta$ is necessarily simplicial. For each cone $\sigma$ of $\Delta$ there corresponds an orbit closure $V(\sigma)$ of the action of complex torus on $X$. The homology classes $\{[V(\sigma)] \mid \dim \sigma = k\}$ form a set of specified generators of $H_{n-k}(X, \mathbb{Q})$. It is shown that, given $\alpha \in H_{n-k}(X, \mathbb{Q})$, there is a canonical way to express $\alpha$ as a linear combination of the $[V(\sigma)]$ with coefficients in the field of rational functions of degree 0 on the Grassmann manifold $G_{n-k+1}(N_\mathbb{Q})$ of $(n-k+1)$-planes in $N_\mathbb{Q}$. This generalizes Morelli’s formula [10] for $\alpha$ the $(n-k)$-th component of the Todd homology class of the variety $X$. Morelli’s proof uses Baum-Bott’s residue formula for holomorphic foliations applied to the action of complex torus on $X$ whereas our proof is entirely combinatorial so that it tends to more general situations.

1. Introduction

Let $X$ be a toric variety of dimension $n$ and $\Delta_X$ the fan associated to $X$. $\Delta_X$ is a collection of rational convex cones in $N_\mathbb{R} = N \otimes \mathbb{R}$ where $N$ is a lattice of rank $n$. For each $k$-dimensional cone $\sigma$ in $\Delta_X$, let $V(\sigma)$ be the corresponding orbit closure of dimension $n-k$ and $[V(\sigma)] \in A_{n-k}(X)$ be its Chow class. Then the Todd class $\mathcal{T}_{n-k}(X)$ of $X$ can be written in the form

$$\mathcal{T}_{n-k}(X) = \sum_{\sigma \in \Delta_X, \dim \sigma = k} \mu_k(\sigma) [V(\sigma)].$$

However, since the $[V(\sigma)]$ are not linearly independent, the coefficients $\mu_k(\sigma) \in \mathbb{Q}$ are not determined uniquely. Danilov [2] asks if $\mu_k(\sigma)$ can be chosen so that it depends only on the cone $\sigma$ not depending on a particular fan in which it lies.

The equality (1) has a close connection with the number $\#(P)$ of lattice points contained in a convex lattice polytope $P$ in $M_\mathbb{R}$ where $M$ is a dual lattice of $N$. For a positive integer $\nu$ the number $\#(\nu P)$ is expanded as a polynomial in $\nu$ (called Ehrhart polynomial):

$$\#(\nu P) = \sum_k a_k(P) \nu^{n-k}.$$

A convex lattice polytope $P$ in $M_\mathbb{R}$ determines a complete toric variety $X$ and an invariant Cartier divisor $D$ on $X$. There is a one-to-one correspondence between the cells $\{\sigma\}$ of $\Delta_X$ and the faces $\{P_\sigma\}$ of $P$. Then the coefficient $a_k(P)$ has an expression

$$a_k(P) = \sum_{\dim \sigma = k} \mu_k(\sigma) \text{vol} P_\sigma$$

with the same $\mu_k(\sigma)$ as in (1).

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We shall restrict ourselves to the case where $X$ is non-singular. Put $D_i = [V(\sigma_i)]$ for
the one dimensional cone $\sigma_i$, and let $x_i \in H^2(X)$ denotes the Poincaré dual of $D_i$. The
divisor $D$ is written in the form $D = \sum d_i D_i$ with positive integers $d_i$. Put $\xi = \sum d_i x_i$. It is known that
$$a_k(P) = \int_X e^\xi \mathcal{F}^k(X) \text{ and } \text{vol}_\sigma = \int_X e^\xi x_\sigma,$$
where $\mathcal{F}^k(X) \in H^{2k}(X)_\mathbb{Q} = H^{2k}(X) \otimes \mathbb{Q}$ is the $k$-th component of the Todd cohomology
class, the Poincaré dual of $\mathcal{F}_{n-k}(X)$, and $x_\sigma \in H^{2k}(X)$ is the Poincaré dual of $[V(\sigma)]$. The
cohomology class $x_\sigma$ can also be written as $x_\sigma = \prod_j x_j$ where the product runs over
such $j$ that $\sigma_j$ is an edge of $\sigma$. Then the equality (2) can be rewritten as
$$\int_X e^\xi \mathcal{F}^k(X) = \sum_{\dim \sigma = k} \mu_k(\sigma) \int_X e^\xi x_\sigma.$$
The reader is referred to [4] Section 5.3 for details and Note 17 there for references.

In his paper [10] Morelli gave an answer to Danilov’s question. Let $\text{Rat}(G_{n-k+1}(N_\mathbb{Q})))_0$
denote the field of rational functions of degree 0 on the Grassmann manifold of $(n-k+1)$-planes in $N_\mathbb{Q}$. For a cone $\sigma$ of dimension $k$ in $N_\mathbb{P}$ he associates a rational function $\mu_k(\sigma) \in \text{Rat}(G_{n-k+1}(N_\mathbb{Q})))_0$. With this $\mu_k(\sigma)$, the right hand side of (1) belongs to
$$\text{Rat}(G_{n-k+1}(N_\mathbb{Q})))_0 \otimes_\mathbb{Q} A_{n-k}(X)_\mathbb{Q},$$
and the equality (1) means that the rational function with values in $A_{n-k}(X)_\mathbb{Q}$ in the right hand side is in fact a constant function equal to $\mathcal{F}_{n-k}(X)$ in $A_{n-k}(X)_\mathbb{Q}$. In other
words this means that
$$\sum_{\sigma \in \Delta_X, \dim \sigma = k} \mu_k(\sigma)(E)[V(\sigma)] = \mathcal{F}_{n-k}(X)$$
for any generic $(n-k+1)$-plane $E$ in $N_\mathbb{Q}$.

Morelli gives an explicit formula for $\mu_k(\sigma)$ when the toric variety is non-singular using
Baum-Bott’s residue formula for singular foliations [1] applied to the action of $(\mathbb{C}^*)^n$ on $X$. He then shows that the function $\mu_k(\sigma)$ is additive with respect to non-singular subdivisions of the cone $\sigma$. This fact leads to (1) in its general form.

One can ask a similar question about general classes other than the Todd class whether it is possible to define $\mu(x, \sigma) \in \text{Rat}(G_{n-k+1}(N_\mathbb{Q})))_0$ for $x \in A_{n-k}(X)$ in a canonical way to satisfy
$$x = \sum_{\sigma \in \Delta_X, \dim \sigma = k} \mu(x, \sigma)[V(\sigma)].$$
When $X$ is non-singular one can expect that $\mu(x, \sigma)$ satisfies a formula analogous to (3)
$$\int_X e^\xi x = \sum_{\dim \sigma = k} \mu(x, \sigma) \int_X e^\xi x_\sigma$$
for any cohomology class $\xi = \sum_i d_i x_i$. In this sense the formula does not explicitly refer to convex polytopes. Fulton and Sturmfels [5] used Minkowski weights to describe intersection theory of toric varieties. For complete non-singular varieties or $\mathbb{Q}$-factorial varieties $X$ the Minkowski weight $\gamma_x : H^{2(n-k)}(X) \to \mathbb{Q}$ corresponding to $x \in H^{2k}(X)$ is defined by $\gamma_x(y) = \int_X xy$. Thus, if the $d_i$ are considered as variables in $\xi$, the formula (5) is considered as describing $\gamma_x$ as a linear combination of the Minkowski weights of $\gamma_{x_\sigma}$.
The purpose of the present paper is to establish the formula (5) by showing an explicit formula for $\mu(x, \sigma)$ when $X$ is a $\mathbb{Q}$-factorial complete toric variety, that is, when $\Delta$ is a complete simplicial fan. Moreover our proof is based on a simple combinatorial argument which can be generalized to the case of multi-fans introduced by Masuda [8]. Topologically the formula concerns equivariant cohomology classes on so-called torus orbifolds (see [6]). This would suggest that actions of compact tori equipped with some nice conditions admit topological residue formulas similar to Baum-Bott’ formula.

In Section 2 the definition of $\mu(x, \sigma)$ will be given and main results will be stated. In topological language, Theorem 2.1 states that (5) holds for any complete $\mathbb{Q}$-factorial toric varieties $X$. In Corollary 2.2 the explicit formula $\mu_k$ for the Todd class $x = \mathcal{O}(X)$ is given, generalizing that of Morelli. In Corollary 2.3 it will be shown that (4) holds for a complete multi-fans in Section 5 after some generalities about multi-fans and multi-polytopes are introduced in Section 4. Theorem 5.1 and Corollary 5.2 hold for (compact) torus orbifolds in general whereas Corollary 5.3 holds for torus orbifolds whose cohomology ring is generated by its degree two part like complete multi-fans.

Section 3 is devoted to the proof of Theorem 2.1 and Corollary 2.2. Corollary 2.2 will be proved in a generalized form in Section 5. The results in Section 2 are generalized to complete multi-fans in Section 5 after some generalities about multi-fans and multi-polytopes are introduced in Section 4. Theorem 5.1, Corollary 5.2 and Corollary 5.3 are the corresponding generalized results. Roughly speaking Theorem 5.1 and Corollary 5.2 hold for (compact) torus orbifolds in general whereas Corollary 5.3 holds for torus orbifolds whose cohomology ring is generated by its degree two part like complete $\mathbb{Q}$-factorial toric varieties.

## 2. Statement of Main Results

It is convenient to describe a fan $\Delta$ in a form suited for combinatorial manipulation. We assume that the fan $\Delta$ is simplicial, that is, every $k$-dimensional cone of $\Delta$ has exactly $k$ edges (one dimensional cones). Let $\Sigma^{(1)}$ denote the set of one dimensional cones. Then the set of cones of $\Delta$ forms a simplicial complex $\Sigma$ with vertex set $\Sigma^{(1)}$. The set of $(k-1)$-simplices, i.e. $k$-dimensional cones, will be denoted by $\Sigma^{(k)}$. The cone corresponding to a simplex $J \in \Sigma$ will be denoted by $C(J)$ and the fan $\Delta$ is denoted by $\Delta = (\Sigma, C)$. Note that $\Sigma^{(0)}$ consists of a unique element $o$ corresponding to the empty set as a subset of $\Sigma^{(1)}$ and $C(o) = 0$. (In [6] $\Sigma$ with $\Sigma^{(0)}$ added is called augmented simplicial set.)

For each $i \in \Sigma^{(1)}$ let $v_i$ be the primitive vector in $N \cap C(i)$. We put $\mathcal{V} = \{v_i\}_{i \in \Sigma^{(1)}}$ and $\mathcal{V}_J = \{v_j \mid j \in J\}$ for $J \in \Sigma$. Let $N_{J, v}$ be the sublattice generated by $\mathcal{V}_J$ and $N_J$ the minimal primitive (saturated) sublattice containing $N_{J, v}$. The quotient group $N_J/N_{J, v}$ is denoted by $H_J$. If $\sigma = C(J)$, $H_J$ is the multiplicity along the normal direction at a generic point of $V(\sigma)$ in the toric variety (orbifold) $X_\Delta$ corresponding to $\Delta$. The fan $\Delta$ is non-singular if and only if $H_I = 0$ for all $I \in \Sigma^{(n)}$, $n = \text{rank } \Delta$. In this case $H_J = 0$ for all $J \in \Delta$ since $H_J$ is contained in $H_I$ if $J \subset I$.

We denote the torus $N_K/N$ by $T$. $N$ can be identified with $\text{Hom}(S^1, T)$, and the dual $M = N^*$ with $\text{Hom}(T, S^1)$. Since $\text{Hom}(T, S^1)$ is identified with $H^1(T) = H^2(BT) = H^2_T(pt)$, $M$ is identified with $H^2_T(pt)$.

The Stanley-Reisner ring of the simplicial set $\Sigma$ is denoted by $H^*_T(\Delta)$. It is the quotient ring of the polynomial ring $\mathbb{Z}[x_i \mid i \in \Sigma^{(1)}]$ by the ideal generated by $\{x_K = \prod_{i \in K} x_i \mid K \subset \Sigma^{(1)}, K \notin \Sigma\}$. It is considered as a ring over $H^*_T(pt)$ (regarded as embedded in $H^*_T(\Delta)$) by the formula

$$u = \sum_{i \in \Sigma^{(1)}} \langle u, v_i \rangle x_i.$$

(6)
When $\Delta$ is the fan $\Delta_X$ associated to a complete $\mathbb{Q}$-factorial toric variety $X$, $H^*_T(\Delta_X) = H^*_T(\Delta_X) \otimes \mathbb{Q}$ can be identified with the equivariant cohomology ring of $X$ with respect to the action of compact torus $T$ acting on $X$ (see [8]).

For each $J \in \Sigma$ let $\{u^J_i\}$ be the basis of $N^*_I$. $N^*_J$ contains $N^*_I$. In particular $N^*_I$ contains $N^*_J$ for $I \in \Sigma(n)$ and will be considered as embedded in $M_Q = H^*_T(pt)_Q$. Define $\iota^*_I : H^*_T(\Delta) \rightarrow M_Q$ by

$$
\iota^*_I \left( \sum_{i \in \Sigma(I)} d_i x_i \right) = \sum_{i \in I} d_i u^I_i.
$$

$\iota^*_I$ extends to $H^*_T(\Delta_Q) \rightarrow H^*_T(pt)_Q$. It is an $H^*_T(pt)_Q$-module map, since $\iota^*_I(u) = u$ for $u \in H^*_T(pt)_Q$.

Let $S$ be the multiplicative set in $H^*_T(pt)_Q$ generated by non-zero elements in $H^*_T(pt)_Q$. The push-forward $\pi_* : H^*_T(\Delta_Q) \rightarrow S^{-1} H^*_T(pt)_Q$ is defined by

$$
\pi_*(x) = \sum_{I \in \Sigma(n)} \frac{\iota^*_I(x)}{|H_I| \prod_{i \in I} u^I_i}.
$$

It is an $H^*_T(pt)_Q$-module map, and lowers the degrees by $2n$. It is known [9] that, if $\Delta$ is a complete simplicial fan, then the image of $\pi_*$ lies in $H^*_T(pt)_Q$.

Assume that $\Delta$ is complete. Let $p_\ast : H^*_T(\Delta)_Q \rightarrow \mathbb{Q}$ be the composition of $\pi_\ast : H^*_T(\Delta)_Q \rightarrow H^*_T(pt)_Q$ and $H^*_T(pt)_Q \rightarrow H^*_T(pt)_Q = \mathbb{Q}$. Note that $p_\ast$ induces $\int_\Delta : H^*(\Delta)_Q \rightarrow \mathbb{Q}$ as noted in [9] where $H^*(\Delta)_Q$ is the quotient of $H^*_T(\Delta)_Q$ by the ideal generated by $H^*_T(pt)_Q$. Note that $H^*(\Delta)_Q$ is defined independently of $\mathcal{Y}$. If $\bar{x}$ denotes the image of $x \in H^*_T(\Delta)_Q$ in $H^*(\Delta)_Q$, then $\int_\Delta \bar{x} = p_\ast(x)$.

If $X = X_\Delta$ is the complete toric variety associated to $\Delta$, then $H^*(\Delta)_Q$ is identified with $H^*(X)_Q$. $\pi_\ast$ with the push-forward $H^*_T(X)_Q \rightarrow H^*_T(pt)_Q$ and $\int_\Delta$ with the ordinary integral $\int_X$ (see [9]).

Assume that $1 \leq k$. For $J \in \Sigma(k)$ let $M_J$ be the annihilator of $N_J$ and $\omega_J \in \bigwedge^{n-k} M$ be the element determined by $M_J$ with an orantient, namely $\omega_J = u_1 \wedge \ldots \wedge u_{n-k}$ with an oriented basis $\{u_1, \ldots, u_{n-k}\}$ of $M_J$. Define $f^J(x_i) \in \bigwedge^{n-k+1} M_Q$ by

$$
f^J(x_i) = \iota^*_I(x_i) \wedge \omega_J \quad \text{with } J \subset I \in \Sigma(n).
$$

$f^J(x_i)$ is well-defined independently of $I$ containing $J$. Let $S^*(\bigwedge^{n-k+1} M_Q)$ be the symmetric algebra over $\bigwedge^{n-k+1} M_Q$. $f^J : H^*_T(\Delta)_Q \rightarrow \bigwedge^{n-k+1} M_Q$ extends to $f^J : H^*_T(\Delta)_Q \rightarrow S^*(\bigwedge^{n-k+1} M_Q)$. For $x = \prod_i x_i^{a_i} \in H^*_T(\Delta)_Q$ we put

$$
f^J(x) = (f^J(x_i))^{a_i}.
$$

The definition of $f^J$ depends on the orientations chosen, but $\frac{f^J(x)}{f^J(x_j)}$ does not. It belongs to the fraction field of the symmetric algebra $S^*(\bigwedge^{n-k+1} M_Q)$ and has degree 0. Hence it can be considered as an element of $\text{Rat}(\mathbb{P}(\bigwedge^{n-k+1} N_Q)_0)$, the field of rational functions of degree 0 on $\mathbb{P}(\bigwedge^{n-k+1} N_Q)$. Let $\nu^* : \text{Rat}(\mathbb{P}(\bigwedge^{n-k+1} N_Q)_0) \rightarrow \text{Rat}(G_{n-k+1}(N_Q)_0$ be the induced homomorphism of the Plücker embedding $\nu : G_{n-k+1}(N_Q) \rightarrow \mathbb{P}(\bigwedge^{n-k+1} N_Q)$. The image $\nu^*(\frac{f^J(x)}{f^J(x_j)})$ will be denoted by $\mu(x, J)$.

Our first main result is stated in the following
Theorem 2.1. Let $\Delta$ be a complete simplicial fan in a lattice $N$ of rank $n$ and $x \in H^{2k}_T(\Delta)_Q$. For any $\xi \in H^2_T(\Delta)_Q$ we have

$$p_*(e^\xi x) = \sum_{J \in \Sigma(k)} \mu(x, J)p_*(e^{\xi_J}x_J) \text{ in } \text{Rat}(G_{n-k+1}(N_Q))_0.$$  

(9)

We say that $\xi = \sum_i d_i x_i \in H^2_T(\Delta)$ is $T$-Cartier if $\nu^*_T(\xi)$ belongs to $M = H^2_T(\text{pt})$ for all $I \in \Sigma(n)$. When $\Delta$ and $\xi$ come from a convex lattice polytope $P$, $\xi$ is $T$-Cartier if and only if $P$ is a lattice polytope. In this case we have

$$p_*(e^{\xi_J}x_J) = \frac{\text{vol } P_J}{|H_J|},$$

where $P_J$ is the face of $P$ corresponding to $J$ ($P_J = P_\sigma$ with $\sigma = C(J)$), cf. e.g. [4]. Furthermore it will be shown in Section 4 that there is an element $T(\Delta) \in H^2_T(\Delta)_Q$ such that

$$p_*(e^{\nu^*_T(\Delta)}) = \#(\nu P) = \sum_{k=0}^n a_k(P)\nu^{n-k}.$$  

Applying Theorem 2.1 to $x = T(\Delta)_k$ and the above $\xi$ the following Corollary will be obtained generalizing Morelli’s formula.

Corollary 2.2. Let $P$ be a convex simple lattice polytope, $\Delta$ the associated complete simplicial fan. Then we have

$$a_k(P) = \sum_{J \in \Sigma(k)} \mu_k(J) \text{vol } P_J$$

with

$$\mu_k(J) = \frac{1}{|H_J|} \sum_{h \in H_J} \nu^*(\prod_{j \in J} 1 - \chi(u_j^h, h)e^{-f_j^T(x_j)})_0$$

in $\text{Rat}(G_{n-k+1}(N_Q))_0$, where $\chi(u, h) = e^{2\pi \sqrt{-1}f_j^T(u, v(h))}$ for $u \in N^*_J$ and $v(h) \in N_J$ is a lift of $h \in H_J$ to $N_J$.

As another immediate corollary of Theorem 2.1 we obtain

Corollary 2.3. Let $\Delta$ be a complete simplicial fan and $x \in H^{2k}_T(\Delta)_Q$. Then

$$\bar{x} = \sum_{J \in \Sigma(k)} \mu(x, J)\bar{x}_J \text{ in } \text{Rat}(G_{n-k+1}(N_Q))_0 \otimes_Q H^{2k}_T(\Delta)_Q.$$  

Proofs of Theorem 2.1 and Corollary 2.3 will be given in the next section.

3. Proof of Theorem 2.1 and Corollary 2.3

For a primitive sublattice $E$ of $N$ of rank $n-k+1$ let $w_E \in \bigwedge^{n-k+1} N$ be a representative of $\nu(E) \in \mathbb{P}(\bigwedge^{n-k+1} N_Q)$. The equality (9) is equivalent to the condition that

$$p_*(e^\xi x) = \sum_{J \in \Sigma(k)} \frac{f^J(x)}{f^J(x_J)}(w_E)p_*(e^{\xi_J}x_J) \text{ holds for every generic } E.$$  

Let $E$ be a generic primitive sublattice in $N$ of rank $n-k+1$. The intersection $E \cap N_J$ has rank one for each $J \in \Sigma(k)$. Take a non-zero vector $v_{E,J}$ in $E \cap N_J$. (One can choose $v_{E,J}$ to be the unique primitive vector contained in $E \cap C(J)$. But any non-zero vector will suffice for the later use.) For $x \in H^{2k}_T(\Delta)$ and $J \in \Sigma(k)$ the value of $\nu^*_T(x)$ evaluated
on \(v_{E,J}\) for \(I \in \Sigma(n)\) containing \(J\) depends only on \(t^*_J(x)\) so that it will be denoted by \(t^*_J(x)(v_{E,J})\). Similarly we shall simply write \(\langle u^J_I, v_{E,J}\rangle\) instead of \(\langle u^I_I, v_{E,J}\rangle\).

**Lemma 3.1.** Put \(f^J_I = u^J_I \wedge \omega_J\). Then

\[
a(f^J_I, w_E) = \langle u^J_I, v_{E,J}\rangle,
\]

where \(a\) is a non-zero constant depending only on \(v_{E,J}\).

**Proof.** Take an oriented basis \(u_1, \ldots, u_{n-k}\) of \(M_J\). Take also a basis \(w_1, \ldots, w_{n-k+1}\) of \(E\) and write \(v_{E,J} = \sum_i c_i w_i\). Then, since \(\langle u_i, v_{E,J}\rangle = 0\),

\[
\sum_{i=1}^{n-k+1} c_i \langle u_i, w_i \rangle = 0, \quad \text{for} \quad i = 1, \ldots, n-k.
\]

The matrix \((a_{ij}) = (\langle u_i, w_j \rangle)\) has rank \(n-k\) and we get

\[
(c_1, \ldots, c_{n-k+1}) = a(A_1, \ldots, A_{n-k+1}), \quad a \neq 0,
\]

where

\[
A_t = (-1)^{t-1} \det \begin{pmatrix} a_{11} & \cdots & \widehat{a_{it}} & \cdots & a_{1n-k+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-k+1} & \cdots & \widehat{a_{nt}} & \cdots & a_{n-k-n-k+1} \end{pmatrix}.
\]

Then

\[
\langle u^J_I, v_{E,J}\rangle = \sum_{i=1}^{n-k+1} c_i \langle u^J_I, w_i \rangle = a \sum_{i=1}^{n-k+1} \langle u^J_I, w_i \rangle A_t
\]

\[
= a \det \begin{pmatrix} \langle u^J_I, w_1 \rangle & \cdots & \langle u^J_I, w_{n-k+1} \rangle \\ \langle u_1, w_1 \rangle & \cdots & \langle u_1, w_{n-k+1} \rangle \\ \vdots & & \ddots & & \vdots \\ \langle u_{n-k}, w_1 \rangle & \cdots & \langle u_{n-k}, w_{n-k+1} \rangle \\ \end{pmatrix}
\]

\[
= a \langle f^J_I, w_E \rangle
\]

where \(f^J_I = u^J_I \wedge u_1 \cdots \wedge u_{n-k}\) and \(w_E = w_1 \wedge \cdots \wedge w_{n-k+1}\). \(\Box\)

**Remark 3.1.** Let \(X\) be a non-singular complete toric variety of dimension \(n\) and \(\Delta\) the associated fan. Let \(T = T^n\) be the compact torus acting on \(X\). \(E \cap N_J\) determines a subcircle \(T_{E,J}^1\) of \(T\). Then \(T_{E,J}^1\) pointwise fixes an invariant complex submanifold \(X_J\). Hence it acts on the normal vector space at each generic point in \(X_J\). Then the numbers \(\langle u^J_I, v_{E,J}\rangle\) are weights of this action.

Lemma 3.1 implies that

\[
\frac{f^J(x)}{f^J(x)}(w_E) = \frac{t^*_J(x)}{\prod_{J \in J} w^*_J(v_{E,J})}.
\]

Then the equality \((\square)\) in Theorem holds if and only if

\[
p_*(e^x) = \sum_{J \in \Sigma(n)} \frac{t^*_J(x)}{\prod_{J \in J} w^*_J(v_{E,J})} p_*(e^x_J)
\]

for \(x \in X\).
This last expression is equal to \(p_x(e^\xi x) = 0\) and \(p_x(e^\xi x') = 0\) by Note after Lemma 3.2. Thus both sides of (10) for

\[\sum_{J \in \Sigma(k)} \xi_J(x)(v_{E,J})^*p_x(e^\xi x_J) = 0,\]

This last expression is equal to \(p_x(e^\xi x')\) since \(x'\) belongs to Case a). Furthermore \(p_x(e^\xi x') = 0\) and \(p_x(e^\xi x) = 0\) by Note after Lemma 3.2. Thus both sides of (10) for
$x = u_1 \cdots u_k u_{k+1} x_{j_{k+1}}$ are equal to 0. This completes the proof of Theorem 2.1 except for the proof of Lemma 3.3.

Proof of Lemma 3.3

Take a simplex $I \in \Sigma^{(n)}$ which contains $K$ and a simplex $K' \in \Sigma^{(k-1)}$ such that $K \subset K' \subset I$. Such a $K'$ exists since $k-k_1 \leq k-1$. Then there are exactly $n-k+1$ simplices $j^1, \ldots, j^{n-k+1} \in \Sigma^{(k)}$ such that $K' \subset j^i \subset I$. It is easy to see that the vectors $v_{E,j^1}, \ldots, v_{E,j^{n-k+1}}$ are linearly independent so that they span $E_Q$. Moreover $M_{K'Q}$ detects these vectors, that is, $M_{K'Q} \to M_Q \to E_Q^*$ is surjective. Since $M'_K \subset M_K \subset M$, $M_{KQ} \to E_Q^*$ is surjective.

Proof of Corollary 2.3

Fix a generic sublattice $E$ and put $x' = \sum_{J \in \Sigma^{(k)}} \mu(x, J)x_J$. Then

$$p_*(e^j x') = \sum_{J \in \Sigma^{(k)}} \mu(x, J)p_*(e^j x_J) = p_*(e^j x)$$

by Theorem 2.1. It follows that $p_*(e^j (x'-x)) = 0$. Thus, in order to prove Corollary 2.3 it suffices to show that $p_*(e^j y) = 0$, $\forall \xi \in H^1_\mathcal{F}(\Delta)_Q$, implies that $y$ belongs to the ideal $\mathcal{J}$ generated by $H^1_\mathcal{F}((p)t)_Q$. Since $H^*\mathcal{F}(\Delta)_Q = H^*\mathcal{F}(X_\Delta)_Q$ is a Poincaré duality space generated by $H^2(\Delta)_Q$, $p_*(e^j y) = 0$ implies that $p_*(y) = 0$, i.e. $y \in \mathcal{J}$.

4. Multi-fans and multi-polytopes

The notion of multi-fan and multi-polytope were introduced in [8]. In this article we shall be concerned only with simplicial multi-fans. See [8] [6] [7] for details.

Let $N$ be a lattice of rank $n$. A simplicial multi-fan in $N$ is a triple $(\Sigma, C, w)$ where $\Sigma = \bigsqcup_{k=0}^n \Sigma^{(k)}$ is an (augmented) simplicial complex, $C$ is a map from $\Sigma^{(k)}$ into the set of $k$-dimensional strongly convex rational polyhedral cones in the vector space $N_\mathbb{R} = N \otimes \mathbb{R}$ for each $k$, and $w$ is a map $\Sigma^{(n)} \to \mathbb{Z}$. $\Sigma^{(0)}$ consists of a single element $o = \emptyset$. The definition in [8] and [6] requires additional restriction on $w$.) We assume that any $J \in \Sigma$ is contained in some $I \in \Sigma^{(n)}$ and $\Sigma^{(n)}$ is not empty.

The map $C$ is required to satisfy the following condition; if $J \in \Sigma$ is a face of $I \in \Sigma$, then $C(J)$ is a face of $C(I)$, and for any $I$, the map $C$ restricted on $\Sigma(I) = \{ J \in \Sigma \mid J \subset I \}$ is an isomorphism of ordered sets onto the set of faces of $C(I)$. It follows that $C(I)$ is necessarily a simplicial cone and $C(o) = 0$. A simplicial fan is considered as a simplicial multi-fan such that the map $C$ on $\Sigma$ is injective and $w \equiv 1$.

For each $K \in \Sigma$ we set

$$\Sigma_K = \{ J \in \Sigma \mid K \subset J \}.$$ 

It inherits the partial ordering from $\Sigma$ and becomes a simplicial set where $\Sigma_K^{(j)} \subset \Sigma^{(j+|K|)}$. $K$ is the unique element in $\Sigma_K^{(0)}$. Let $N_K$ be the minimal primitive sublattice of $N$ containing $N \cap C(K)$, and $N^K$ the quotient lattice of $N$ by $N_K$. For $J \in \Sigma_K$ we define $C_K(J)$ to be the cone $C(J)$ projected on $N^K \otimes \mathbb{R}$. We define a function

$$w : \Sigma_K^{(n-|K|)} \subset \Sigma^{(n)} \to \mathbb{Z}$$

to be the restrictions of $w$ to $\Sigma_K^{(n-|K|)}$. The triple $\Delta_K = (\Sigma_K, C_K, w)$ is a multi-fan in $N^K$ and is called the projected multi-fan with respect to $K \in \Sigma$. For $K = o$, the projected multi-fan $\Delta_o$ is nothing but $\Delta$ itself.
A vector \( v \in N_R \) will be called generic if \( v \) does not lie on any linear subspace spanned by a cone in \( C(\Sigma) \) of dimension less than \( n \). For a generic vector \( v \) we set \( d_v = \sum_{\alpha \in C(I)} w(I) \), where the sum is understood to be zero if there is no such \( I \).

**Definition.** A simplicial multi-fan \( \Delta = (\Sigma, C, w) \) is called pre-complete if the integer \( d_v \) is independent of generic vectors \( v \). In this case this integer will be called the degree of \( \Delta \) and will be denoted by \( \deg(\Delta) \). It is also called the Todd genus of \( \Delta \) and is denoted by \( \text{Td}[\Delta] \). A pre-complete multi-fan \( \Delta \) is said to be complete if the projected multi-fan \( \Delta_K \) is pre-complete for every \( K \in \Sigma \).

A multi-fan is complete if and only if the projected multi-fan \( \Delta_J \) is pre-complete for every \( J \in \Sigma^{(n-1)} \).

Like a toric variety gives rise to a fan, a torus orbifold gives rise to a complete simplicial multi-fan, though this correspondence is not one to one. A torus orbifold is a closed oriented orbifold with an action of a torus of half the dimension of the orbifold itself with non-empty fixed point set and with some additional conditions on the isotopy groups (see 6). Most typical non-toric examples are given in 6. Cobordism invariants of torus orbifolds are encoded in the associated multi-fans.

In the sequel we shall often consider a set \( \mathcal{V} \) consisting of non-zero edge vectors \( v_i \) for each \( i \in \Sigma^{(1)} \) such that \( v_i \in N \cap C(i) \). We do not require \( v_i \) to be primitive. This has meaning for torus orbifolds (see 6). For any \( K \in \Sigma \) put \( \mathcal{V}_K = \{v_i\}_{i \in K} \). Let \( N_{K,\mathcal{V}} \) be the sublattice of \( N_K \) generated by \( \mathcal{V}_K \). The quotient group \( N_K/N_{K,\mathcal{V}} \) is denoted by \( H_{K,\mathcal{V}} \).

Let \( \Delta = (\Sigma, C, w) \) be a simplicial multi-fan in a lattice \( N \). We define the equivariant cohomology \( H^*_T(\Delta) \) of a multi-fan \( \Delta \) as the Stanley-Reisner ring of the simplicial complex \( \Sigma \) as in Section 1.

Let \( \mathcal{V} = \{v_i\}_{i \in \Sigma^{(1)}} \) be a set of prescribed edge vectors as before. Let \( \{w_i\}_{i \in K} \) be the basis of \( N_{K,\mathcal{V}}^* \) dual to \( \mathcal{V}_K \). We define a homomorphism \( M = N^* = H^*_T(pt) \rightarrow H^*_T(\Delta) \) by the same formula (6) as in the case of fans. Since this definition depends on the set \( \mathcal{V} \), the \( H^*_T(pt) \)-module structure of \( H^*_T(\Delta) \) also depends on \( \mathcal{V} \). To emphasize this fact we shall use the notation \( H^*_T(\Delta, \mathcal{V}) \). When all the \( v_i \) are taken primitive, the notation \( H^*_T(\Delta) \) is used.

For \( I \in \Sigma^{(n)} \) the map \( i_I^* : H^*_T(\Delta, \mathcal{V})_\mathbb{Q} \rightarrow H^*_T(pt)_\mathbb{Q} \) is defined by (7) as in the case of fans. On the other hand the definition of the push-forward is altered from (8) to

\[
\pi_*(x) = \sum_{I \in \Sigma^{(n)}} \frac{w(I) i_I^*(x)}{|H_I| \prod_{i \in I} w_i},
\]

cf. 6. If \( \Delta \) is complete the image of \( \pi_* \) lies in \( H^*_T(pt)_\mathbb{Q} \) as in the case of fans, and the map \( p_* : H^*_T(\Delta, \mathcal{V})_\mathbb{Q} \rightarrow \mathbb{Q} \) is also defined in a similar way.

Let \( K \in \Sigma^{(k)} \) and let \( \Delta_K = (\Sigma_K, C_K, w_K) \) be the projected multi-fan. The link \( \text{Lk} K \) of \( K \) in \( \Sigma \) is a simplicial complex consisting of simplices \( J \) such that \( K \cup J \in \Sigma \) and \( K \cap J = \emptyset \). It will be denoted by \( \Sigma_K' \) in the sequel. There is an isomorphism from \( \Sigma_K' \) to \( \Sigma_K \) sending \( J \in \Sigma_K' \) to \( K \cup J \in \Sigma_K \). We consider the polynomial ring \( R_K \) generated by \( \{x_i \mid i \in K \cup \Sigma_K^{(1)} \} \) and the ideal \( \mathcal{I}_K \) generated by monomials \( x_J = \prod_{i \in J} x_i \) such that \( J \notin \Sigma(K) \ast \Sigma_K' \) where \( \Sigma(K) \ast \Sigma_K' \) is the join of \( \Sigma(K) \) and \( \Sigma_K' \). We define the equivariant cohomology \( H^*_T(\Delta_K) \) of \( \Delta_K \) with respect to the torus \( T \) as the quotient ring \( R_K/\mathcal{I}_K \).

If \( \mathcal{V} \) is a set of prescribed edge vectors, \( H^*_T(pt) \) is regarded as a submodule of \( H^*_T(\Delta_K) \) by a formula similar to (6). This defines an \( H^*_T(pt) \)-module structure on \( H^*_T(\Delta_K) \) which will be denoted by \( H^*_T(\Delta_K, \mathcal{V}) \) to specify the dependence on \( \mathcal{V} \). The projection \( H^*_T(\Delta, \mathcal{V}) \rightarrow \)
$H_T^*(\Delta_K, \mathcal{V})$ is defined by sending $x_i$ to $x_i$ for $i \in K \cup \Sigma_{K}^{(1)}$ and putting $x_i = 0$ for
$i \notin K \cup \Sigma_{K}^{(1)}$. The restriction homomorphism $v_I^* : H_T^*(\Delta_K, \mathcal{V})_q \to H_T^*(pt)_q$ for $I \in \Sigma_{K}^{(n-k)}$
and the push-forward $\pi_+ : H_T^*(\Delta_K, \mathcal{V})_q \to S^{-1}H_T^*(pt)_q$ are also defined in a similar way as before.

Given $\xi = \sum_{i \in K \cup \Sigma_{K}^{(1)}} d_i x_i \in H_T^2(\Delta_K, \mathcal{V})_R$, $d_i \in \mathbb{R}$, let $A_k^*$ be the affine subspace
in the space $M_R$ defined by $\langle u, v_i \rangle = d_i$ for $i \in K$. Then we introduce a collection
$\mathcal{F}_K = \{ F_i \mid i \in \mathcal{S}_K^{(1)} \}$ of affine hyperplanes in $A_k^*$ by setting

$$F_i = \{ u \mid u \in A_k^*, \langle u, v_i \rangle = d_i \}.$$

The pair $\mathcal{P}_K(\xi) = (\Delta_K, \mathcal{F}_K)$ will be called a multi-polytope associated with $\xi$; see [7]. In case $K = \emptyset \in \mathcal{S}_K^{(0)}$, $\mathcal{P}_K(\xi)$ is simply denoted by $\mathcal{P}(\xi)$.

For $\xi = \sum_{i \in \mathcal{S}_K^{(1)}} d_i x_i$ and $K \in \mathcal{S}_K^{(k)}$ put $\xi_K = \sum_{i \in K \cup \Sigma_{K}^{(1)}} d_i x_i$ and $\mathcal{P}(\xi_K) = \mathcal{P}_K(\xi)$. It
will be called the face of $\mathcal{P}(\xi)$ corresponding to $K$.

For $I \in \Sigma_{K}^{(n-k)}$, i.e. $I \in \Sigma_{K}^{(n)}$ with $I \supset K$, we put $u_I = \cap_{i \in I} F_i = \cap_{i \in I \cap K} F_i \cap A_K^* \in A_K^*$. Note that $u_I$ is equal to $e_I^*(\xi)$. The dual vector space $(N_K^*)^*$ of $N_K^*$ is canonically identified with
the subspace $M_K^R$ of $M_R = H_T^*(pt)_R$. It is parallel to $A_K^*$, and $u_I^*$ lies in $M_K^R$ for $I \in \Sigma_{K}^{(n-k)}$ and $i \in I \setminus K$. A vector $v \in N_K^*$ is called generic if $\langle u_I^*, v \rangle \neq 0$ for any $I \in \Sigma_{K}^{(n-k)}$ and $i \in I \setminus K$. The image in $N_K^*$ of a generic vector in $N_R^*$ is generic. We take a generic vector $v \in N_R^*$, and define

$$(u_I^*)^+ := \left\{ \begin{array}{ll} u_I^* & \text{if } \langle u_I^*, v \rangle > 0 \\ -u_I^* & \text{if } \langle u_I^*, v \rangle < 0. \end{array} \right.$$

for $I \in \Sigma_{K}^{(n-k)}$ and $i \in I \setminus K$. We denote by $C_K^*(I)^*$ the cone in $A_K^*$ spanned by the
$(u_I^*)^+$, $i \in I \setminus K$, with apex at $u_I$, and by $\phi_I$ its characteristic function. With these understood, we define a function $D_H(\mathcal{P}_K(\xi))$ on $A_K^* \setminus \cup_i F_i$ by

$$D_H(\mathcal{P}_K(\xi)) = \sum_{I \in \Sigma_{K}^{(n-k)}} (-1)^f w(I) \phi_I.$$

As in [7] we call this function the Duistermaat-Heckman function associated with $\mathcal{P}_K(\xi)$.

When $K = \emptyset$, $D_H(\mathcal{P}(\xi))$ is defined on $M_R \setminus \cup_i F_i$.

Suppose that $\Delta$ is a simplicial fan. If all the $d_i$ are positive and the set

$$P = \{ u \in M_R \mid \langle u, v_i \rangle \leq d_i \}$$

is a convex polytope, then $D_H(\mathcal{P}(\xi))$ equals 1 on the interior of $P$ and 0 on other components of $M_R \setminus \cup_i F_i$.

The following theorem is fundamental in the sequel, cf. [7] Theorem 2.3 and [6] Corollary 7.4.

**Theorem 4.1.** Let $\Delta$ be a complete simplicial multi-fan. Let $\xi = \sum_{i \in K \cup \Sigma_{K}^{(1)}} d_i x_i \in H_T^2(\Delta_K, \mathcal{V})$ be as above with all $d_i$ integers and put $\xi_+ = \sum_i (d_i + \epsilon)x_i$ with $0 < \epsilon < 1$. Then

$$\sum_{u \in A_K^* \cap M} D_H(\mathcal{P}_K(\xi_+))(u) t^n = \sum_{I \in \Sigma_{K}^{(n-k)}} \frac{w(I)}{|H_{I,y}|} \sum_{h \in H_{I,y}} \prod_{I \in I \setminus K} (1 - \chi_I(u_I^*(\xi), h) \langle u_I^*(\xi), h \rangle^{1-e})$$

where $\chi_I(u, h)$ for $u \in N_I^*$ is defined as in Corollary [6]
which will be denoted by $H$

be denoted by $(13)$

$t = (t_1, \ldots, t_n)$. The equality shows that the right hand side, which is a rational function of \( t \), belongs to $R(T)$.

\[ \xi = \sum d_i x_i \in H^2_t(\Delta, \mathcal{V}) \]

This condition is equivalent to $u_f \in M$ for all $I \in \Sigma^{(n)}$. In this case $\mathcal{P}(\xi)$ is said lattice multi-polytope. If $\xi$ is $T$-Cartier, then $\chi_I(\xi_I(\xi), h) = 1$. Hence the above formula $(11)$ for $DH_{\mathcal{P}_K(\xi_{K^+})}$ reduces in this case to

\[ \sum_{u \in A^*_K \cap M} DH_{\mathcal{P}_K(\xi_{K^+})}(u) t^u = \sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_{I, r}|} \sum_{h \in H_{I, r}} \prod_{i \in I \setminus K} (1 - \chi_I(u_i, h)^{-1} t^{-u_i} e^{t^u (I)}}. \]

Let $H^{**}_T(\mathcal{P}_K)$ denote the completed equivariant cohomology ring. The Chern character $\text{ch}$ sends $R(T) \otimes \mathbb{R}$ to $H_T^{**}(pt) \mathbb{R}$ by $\text{ch}(t^u) = e^u$. The image of $(12)$ by $\text{ch}$ is given by

\[ \sum_{u \in A^*_K \cap M} DH_{\mathcal{P}_K(\xi_{K^+})}(u) e^u = \sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_{I, r}|} \sum_{h \in H_{I, r}} \prod_{i \in I \setminus K} (1 - \chi_I(u_i, h)^{-1} e^{t^u (I)}}. \]

Assume that $\xi = \sum d_i x_i \in H^2_t(\Delta, \mathcal{V})$ is $T$-Cartier. The number $\#(\mathcal{P}(\xi)_K)$ is defined by

\[ \#(\mathcal{P}(\xi)_K) = \sum_{u \in A^*_K \cap M} DH_{\mathcal{P}_K(\xi_{K^+})}(u). \]

It is obtained from $(13)$ by setting $u = 0$, that is, it is equal to the image of $(13)$ by $H_T^{**}(pt) \mathbb{R} \to H^{2}_T(pt) \mathbb{Q}$.

The equivariant Todd class $\mathcal{T}(\Delta, \mathcal{V})$ is defined in such a way that

\[ \pi_* (e^x \mathcal{T}(\Delta, \mathcal{V})) = \sum_{u \in M} DH_{\mathcal{P}_K(\xi_{K^+})}(u) e^u \]

for $\xi$ $T$-Cartier. In order to give the definition we need some notations.

For simplicity identify the set $\Sigma^{(1)}$ with $\{1, 2, \ldots, m\}$ and consider a homomorphism $\eta: \mathbb{R}^m = \mathbb{R}^{\Sigma^{(1)}} \to N$ sending $a = (a_1, a_2, \ldots, a_m)$ to $\sum_{i \in \Sigma^{(1)}} a_i v_i$. For $K \in \Sigma^{(k)}$ we define

\[ \tilde{G}_{K', \mathcal{V}} = \{a \mid \eta(a) \in N \text{ and } a_j = 0 \text{ for } j \notin K\} \]

and define $G_{K, \mathcal{V}}$ to be the image of $\tilde{G}_{K, \mathcal{V}}$ in $T = \mathbb{R}^m / \mathbb{Z}^m$. It will be written $G_K$ for simplicity. The homomorphism $\eta$ restricted on $\tilde{G}_{K, \mathcal{V}}$ induces an isomorphism

\[ \eta_K: G_K \cong H_{K, \mathcal{V}} \subset T = N / N. \]

Put

\[ G_\Delta = \bigcup_{I \in \Sigma^{(n)}} G_I \subset \tilde{T} \quad \text{and} \quad DG_\Delta = \bigcup_{I \in \Sigma^{(n)}} G_I \times G_I \subset G_\Delta \times G_\Delta. \]

Let $v(g) = a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ be a representative of $g \in \tilde{T}$. The factor $a_i$ will be denoted by $v_i(g)$. It is determined modulo integers. If $g \in G_I$, then $v_i(g)$ is necessarily a rational number. Define a homomorphism $\chi_i: \tilde{T} \to \mathbb{C}^*$ by

\[ \chi_i(g) = e^{2\pi \sqrt{-1} v_i(g)} \]

Let $g \in G_I$ and $h = \eta_I(g) \in H_{I, \mathcal{V}}$. Then $\eta(v(g)) \in N_I$ is a representative of $h$ in $N_I$ which will be denoted by $v(h)$. Then, for $g \in G_I$ and $i \in I$,

\[ v_i(g) \equiv \langle u'_i, v(h) \rangle \mod \mathbb{Z}, \]
and
\[ \chi_i(g) = e^{2\pi i (u_i^l, v)} = \chi_i(u^l_i, h). \]

Let \( \Delta \) be a complete simplicial multi-fan. Define
\[ \mathcal{T}_T(\Delta, \mathcal{V}) = \sum_{g \in G_\Delta} \prod_{i \in \Sigma^{(1)}} \frac{x_i}{1 - \chi_i(g)e^{-x_i}} \in H^{**}_T(\Delta, \mathcal{V})_\mathbb{Q}. \]

**Proposition 4.2.** Let \( \Delta \) be a complete simplicial multi-fan. Assume that \( \xi \in H^{2}_T(\Delta, \mathcal{V}) \) is \( T \)-Cartier. Then
\[ \pi_s(e^\xi \mathcal{T}_T(\Delta, \mathcal{V})) = \sum_{u \in M} \text{DH}_{\mathcal{P}(\xi)}(u)e^u. \]

Consequently
\[ p_s(e^\xi \mathcal{T}_T(\Delta, \mathcal{V})) = \#(\mathcal{P}(\xi)). \]

**Proof.** (cf. [6] Section 8). Let \( g \in G_\Delta \) and \( I \in \Sigma^{(n)} \). If \( g \notin G_I \), then there is an element \( i \notin I \) such that \( \chi_i(g) \neq 1 \); so
\[ \frac{x_i}{1 - \chi_i(g)e^{-x_i}} = (1 - \chi_i(g))^{-1}x_i + \text{higher degree terms} \]
for such \( i \). Hence \( i^*_I(\frac{x_i}{1 - \chi_i(g)e^{-x_i}}) = 0 \). Therefore, only elements \( g \) in \( G_I \) contribute to \( i^*_I(\mathcal{T}_T(\Delta, \mathcal{V})) \). Now suppose \( g \in G_I \). Then \( \chi_i(g) = 1 \) for \( i \notin I \), so \( i^*_I(\frac{x_i}{1 - \chi_i(g)e^{-x_i}}) = 1 \) for such \( i \). Finally, since \( i^*_I(x_i) = u^l_i \) for \( i \in I \), we have
\[ i^*_I(\mathcal{T}_T(\Delta, \mathcal{V})) = \sum_{g \in G_I} \prod_{i \in I} \frac{u^l_i}{1 - \chi_i(g)e^{-u^l_i}}. \]
This together with \([13]\) shows that
\[ \pi_s(e^\xi \mathcal{T}_T(\Delta, \mathcal{V})) = \pi_s\left(e^\xi \sum_{g \in G_\Delta} \prod_{i=1}^m \frac{x_i}{1 - \chi_i(g)e^{-x_i}}\right) \]
\[ = \sum_{I \in \Sigma^{(n)}} \frac{w(I)e^{\xi_I(\mathcal{V})}}{|H_I, \mathcal{V}|} \sum_{g \in G_I} \prod_{i \in I} \frac{1}{1 - \chi_i(g)e^{-u^l_i}} \]
\[ = \sum_{u \in M} \text{DH}_{\mathcal{P}(\xi)}(u)e^u. \]

\[ \square \]

More generally, for \( K \in \Sigma^{(k)} \), define \( \mathcal{T}_T(\Delta, \mathcal{V})_K \) by
\[ \mathcal{T}_T(\Delta, \mathcal{V})_K = \sum_{g \in G_{\Delta^K}} \prod_{i \in \Sigma^{(n)}(K)_{\chi_i}} \frac{x_i}{1 - \chi_i(g)e^{-x_i}} \in H^{**}_T(\Delta, \mathcal{V})_\mathbb{Q}. \]

Then the same proof as for Proposition 1.2 yields

**Proposition 4.3.** Let \( \Delta \) be a complete simplicial multi-fan. Assume that \( \xi \in H^{2}_T(\Delta, \mathcal{V}) \) is \( T \)-Cartier. Then
\[ \pi_s(e^\xi x_K \mathcal{T}_T(\Delta, \mathcal{V})_K) = \sum_{u \in A_K \cap M} \text{DH}_{\mathcal{P}(\xi_K)}(u)e^u. \]
for \( K \in \Sigma^{(k)} \), where \( x_K = \prod_{i \in K} x_i \). Consequently
\[ p_s(e^\xi x_K \mathcal{T}_T(\Delta, \mathcal{V})_K) = \#(\mathcal{P}(\xi)_K). \]
The lattice $M \cap A_K^*$ defines a volume element $dV_K$ on $A_K^*$. For $\xi = \sum_{i \in K} d_i x_i \in H^2_T(\Delta_K, V)$, the volume $\text{vol}_K(\xi)$ of $\mathcal{P}_K(\xi)$ is defined by

$$\text{vol}_K(\xi) = \int_{A_K} DH_{\mathcal{P}_K(\xi)} dV^*_K.$$ 

**Proposition 4.4.** For $\xi = \sum_{i \in \Sigma(1)} d_i x_i \in H^2_T(\Delta, V)$,

$$\frac{1}{|H_{K,V}|} \text{vol}_K(\xi) = p_*(e^\xi x_K).$$

**Proof.** We shall give a proof only for the case where $\xi$ is $T$-Cartier. The general case can be reduced to this case, cf. [6], Lemma 8.6. By Proposition 4.3

$$\#(P(\xi_K)) = p_*(e^\xi x_K T_T(\Delta, V)_K).$$

The highest degree term with respect to $\{d_i\}$ in the right hand side is nothing but

$$\text{vol}_K(\xi) = p_*(e^\xi x_K).$$

Hence

$$\text{vol}_K(\xi) = |G_K| p_*(\xi^{n-k}) = |H_{K,V}| p_*(e^\xi x_K).$$

\[\square\]

5. Generalization

The definition of $f^J(x)$ for simplicial fans can be also applied for simplicial multi-fans. Consequently $\mu(x, J) \in \text{Rat}(G_{n-k+1}(N_Q))_0$ is also defined. Theorem 2.1 is generalized in the following form.

**Theorem 5.1.** Let $\Delta$ be a complete simplicial multi-fan and $x \in H^2_T(\Delta, V)_Q$. For any $\xi \in H^2_T(\Delta)_Q$ we have

$$p_*(e^\xi x) = \sum_{J \in \Sigma(x)} \mu(x, J)p_*(e^J x_J) \text{ in } \text{Rat}(G_{n-k+1}(N_Q))_0.$$ 

Lemma 3.1, Lemma 3.2, Lemma 3.3 all hold in this new setting. Hence the proofs in Section 2 literally apply to prove Theorem 5.1.

As to Corollary 2.2, its generalization takes the following form.

**Corollary 5.2.** Let $\Delta$ be a complete simplicial multi-fan in a lattice of rank $n$. Assume that $\xi \in H^2_T(\Delta, V)$ is $T$-Cartier. Set

$$\#(\mathcal{P}(\nu \xi)) = \sum_{k=0}^{n} a_k(\xi) \nu^{n-k}.$$
Then we have
\[ a_k(\xi) = \sum_{J \in \Sigma(k)} \mu_k(J) \text{vol} \mathcal{P}(\xi)_J \]
with
\[ \mu_k(J) = \frac{1}{|H_{J,Y}|} \nu^* \left( \sum_{h \in H_{J,Y}} \prod_{j \in J} \frac{1}{1 - \chi(u_j^J, h)e^{-f^J(x_j)}} \right)_0 \]
in \text{Rat}(G_{n-k+1}(N\mathbb{C}))_0.

Note. It can be proved without difficulty that \( \mu_k(J) \) does not depend on the choice of \( \mathcal{V} \). Hence one has only consider the case where all the \( v_i \) are primitive.

Proof. By Proposition 4.3
\[ \#(\mathcal{P}(\nu \xi)) = p_*(e^{\nu \mathcal{P}}(\Delta, \mathcal{V})). \]
Put \( x = (\mathcal{P}(\Delta, \mathcal{V}))_k \in H^2(\Delta, \mathcal{V})_\mathbb{C} \). By (9) which is valid under the assumption of Theorem 5.1 too and by Proposition 4.4
\[ a_k(\xi) = \sum_{J \in \Sigma(k)} \nu^* \left( \frac{f^J(x)}{f^J(x_J)} \right) \frac{\text{vol} \mathcal{P}(\xi)_J}{|H_{J,Y}|}. \]
Thus it suffices to show that
\[ \frac{f^J(x)}{f^J(x_J)} = \left( \sum_{h \in H_{J,Y}} \prod_{j \in J} \frac{1}{1 - \chi(u_j^J, h)e^{-f^J(x_j)}} \right)_0, \]
or
\[ f^J(x) = \left( \sum_{h \in H_{J,Y}} \prod_{j \in J} \frac{f^J(x_j)}{1 - \chi(u_j^J, h)e^{-f^J(x_j)}} \right)_k. \]
Let \( g \in G_\Delta \). If \( g \not\in G_J \), then there is an element \( i \not\in J \) such that \( \chi_i(g) \neq 1 \), and, for such \( i \),
\[ f^J(\frac{x_i}{1 - \chi_i(g)e^{-x_i}}) = f^J((1 - \chi_i(g))^{-1}x_i + \text{higher degree terms}) = 0, \]
since \( f^J(x_i) = 0 \). Thus
\[ f^J \left( \prod_{i \in \Sigma(1)} \frac{x_i}{1 - \chi_i(g)e^{-x_i}} \right) = 0 \]
for \( g \not\in G_J \).
If \( g \in G_J \), then \( \chi_i(g) = 1 \) for \( i \not\in J \). Thus
\[ f^J(\frac{x_i}{1 - \chi_i(g)e^{-x_i}}) = f^J(1 + \frac{1}{2}x_i + \text{higher degree terms}) = 1 \]
for \( g \in G_J, i \not\in J \). It follows that
\[ f^J \left( \sum_{g \in G_\Delta, i \in \Sigma(1)} \frac{x_i}{1 - \chi_i(g)e^{-x_i}} \right) = \sum_{g \in G_J, i \in J} \frac{f^J(x_i)}{1 - \chi_i(g)e^{-f^J(x_i)}}. \]
This implies

\[ f^J(\mathcal{T}_\Delta, \mathcal{Y})_k = \left( \sum_{h \in H_\Delta \setminus \mathcal{Y}} \prod_{j \in J} \frac{f^J(x_j)}{1 - \chi_j(u_j^J, h)e^{-f^J(x_j)}} \right)_k. \]

□

As to Corollary 5.3 we need to put an additional condition on the multi-fan \( \Delta \). A simplicial complex \( \Sigma \) is said to be \( \mathbb{Q} \)-Gorenstein* if

\[ \tilde{H}_i(\text{Lk} \ J)_{\mathbb{Q}} = \begin{cases} \mathbb{Q}, & i = \dim \text{Lk} \ J \\ 0, & i < \dim \text{Lk} \ J \end{cases} \]

for all \( J \in \Sigma^{(k)}, \ 0 \leq k \leq n \). It is equivalent to say that the realization \( |\Sigma| \) of \( \Sigma \) is a \( \mathbb{Q} \)-homology manifold and has the same \( \mathbb{Q} \)-homology with the sphere \( S^{n-1} \). A complete simplicial multi-fan \( \Delta = (\Sigma, C, w) \) is called \( \mathbb{Q} \)-Gorenstein* if \( \Sigma \) is \( \mathbb{Q} \)-Gorenstein*.

**Corollary 5.3.** Let \( \Delta \) be a \( \mathbb{Q} \)-Gorenstein* simplicial multi-fan and \( x \in H^2k(\Delta)_{\mathbb{Q}} \). Then

\[ \bar{x} = \sum_{J \in \Sigma^{(k)}} \mu(x, J)\bar{x}_J \quad \text{in} \quad \text{Rat}(G_{n-k+1}(N_{\mathbb{Q}}))_0 \otimes_{\mathbb{Q}} H^2k(\Delta)_{\mathbb{Q}}. \]

We shall show that \( H^*(\Delta)_{\mathbb{Q}} \) is a Poincaré duality space and is generated by \( H^2(\Delta)_{\mathbb{Q}} \) if \( \Delta \) is \( \mathbb{Q} \)-Gorenstein*. Then the proof of Corollary 2.3 can be applied in this case too.

Our construction follows [3]. Assume that \( \Sigma \) is \( \mathbb{Q} \)-Gorenstein*.

Let \( \Sigma^* \) be the dual complex of \( \Sigma \). It is triangulated by the barycentric subdivision of \( \Sigma \). The set of dual cells are in one to one correspondence with \( \bigsqcup_{k=1}^n \Sigma^{(k)} \). The dual cell corresponding to \( K \in \Sigma^{(k)} \) is denoted by \( K^* \).

Let \( P \) be the cone over \( \Sigma^* \). \( P \) is itself a \( \mathbb{Q} \)-homology cell since \( \Sigma \) is \( \mathbb{Q} \)-Gorenstein*. It is considered as the dual cell \( o^* \) of \( o \). The dual cell \( K^* \) of \( K \in \Sigma^{(k)} \) has codimension \( k \) in \( P \). For \( p \in P \) define \( D(p) \) to be the minimal dual cell containing \( p \). The sublattice \( N_K \) determines a subtorus \( T_K \) of \( T \). We put \( T_p = T_K \) when \( D(p) = K^* \).

Then put \( \tilde{P} = T \times P / \sim \) where the equivalence relation \( \sim \) is defined by

\[ (g, p) \sim (h, q) \iff p = q, \ gh^{-1} \in T_p. \]

\( \tilde{P} \) has a natural \( T \)-action. It is easy to see that \( \tilde{P} \) is an orientable \( \mathbb{Q} \)-homology manifold.

Theorem 4.8 of [3] says that the cohomology ring \( H^*_T(\tilde{P})_{\mathbb{Q}} \) is isomorphic to \( H^*_T(\Delta)_{\mathbb{Q}} \). Theorem 5.10 of [9] tells us that \( H^*(\tilde{P})_{\mathbb{Q}} \) is isomorphic to \( H^*_T(\tilde{P})_{\mathbb{Q}} / J \) where \( J \) is the ideal generated by the image of \( H^*_T(pt)_{\mathbb{Q}} \) in \( H^*_T(\tilde{P})_{\mathbb{Q}} \). It follows that \( H^*(\Delta)_{\mathbb{Q}} \) is isomorphic to \( H^*(\tilde{P})_{\mathbb{Q}} \). Since \( \tilde{P} \) is an orientable \( \mathbb{Q} \)-homology manifold, \( H^*(\Delta)_{\mathbb{Q}} \) is a Poincaré duality space generated by its degree two part.

This finishes the proof of Corollary 5.3.

**Remark 5.1.** Let \( X \) be a torus orbifold and \( \Delta_X \) the associated multi-fan. It is known that, if the cohomology ring \( H^*(X)_{\mathbb{Q}} \) is generated by \( H^2(X)_{\mathbb{Q}} \), then \( H^*(X)_{\mathbb{Q}} \) is isomorphic to \( H^*(\Delta_X)_{\mathbb{Q}} \) ([8] Proposition 3.4) and \( \Delta_X \) is \( \mathbb{Q} \)-Gorenstein* ([9] Lemma 8.2). Hence Corollary 5.3 holds for \( x \in H^*_T(X)_{\mathbb{Q}} \) and \( \bar{x} \in H^*(X)_{\mathbb{Q}} \).
Remark 5.2. When $\Delta$ is the fan associated to a convex polytope $P$ and $\xi = D$, the Cartier divisor associated to $P$, we know (see, e.g. [4]) that

$$\mu_0(o) = 1, \ a_0(\xi) = \text{vol } \mathcal{P}(\xi), \ \mu_1(i) = \frac{1}{2}, \ a_1(\xi) = \frac{1}{2} \sum_{i \in \Sigma(1)} \text{vol } \mathcal{P}(\xi).$$

This is also true for simplicial multi-fans and $T$-Cartier $\xi$.

As to $a_n$ we have

$$a_n(\xi) = \text{Td}(\Delta).$$

In fact $a_n(\xi) = p_*(\mathcal{T}(\Delta, \mathcal{V})) = (\pi_*(\mathcal{T}(\Delta, \mathcal{V})))_0$. Thus the above equality follows from the following rigidity property:

Theorem 5.4. Let $\Delta$ be a complete simplicial multi-fan. Then

$$\pi_*(\mathcal{T}(\Delta, \mathcal{V})) = (\pi_*(\mathcal{T}(\Delta, \mathcal{V})))_0 = \text{Td}[\Delta].$$

See [6] Theorem 7.2 and its proof. Note that Td[$\Delta$] = 1 for any complete simplicial fan $\Delta$.

The explicit formula for $\pi_*(\mathcal{T}(\Delta, \mathcal{V}))$ is given by

$$\pi_*(\mathcal{T}(\Delta, \mathcal{V})) = \sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|H_I|} \sum_{h \in H_I, i} \prod_{i \in I} \frac{1}{1 - \chi_I(u_i^h, h)e^{-u_i^h}}.$$ 

This does not depend on the choice of $\mathcal{V}$ and is in fact equal to Td[$\Delta$].

Let $\Delta$ be a (not necessarily complete) simplicial fan in a lattice of rank $n$. Set

$$Td_T(\Delta) = \sum_{I \in \Sigma^{(n)}} \frac{1}{|H_I|} \sum_{h \in H_I, i} \prod_{i \in I} \frac{1}{1 - \chi_I(u_i^h, h)e^{-u_i^h}} \in S^{-1} H_T^*(pt)Q.$$ 

For a simplex $I$ let $\Sigma(I)$ be the simplicial complex consisting of all faces of $I$. For a fan $\Delta(I) = (\Sigma(I), C)$ $Td_T(\Delta(I))$ is denoted by $Td_T(I)$.

Theorem 5.5. $Td_T(I)$ is additive with respect to simplicial subdivisions of the cone $C(I)$. Namely, if $\Delta$ is the fan determined by a simplicial subdivision of $C(I)$, then the following equality holds

$$Td_T(\Delta) = Td_T(I).$$

For the proof it is sufficient to assume that $\Delta(I)$ and $\Delta$ are non-singular. In such a form a proof is give in [10]. The following corollary ensures that $\mu_k(J)$ can be defined for general polyhedral cones as pointed out by Morelli in [10].

Corollary 5.6. Let $\Delta(J) = (\Sigma(J), C)$ be a fan in a lattice $N$ of rank $n$ where $J$ is a simplex of dimension $k - 1$. Then $\mu_k(J) \in \text{Rat}(G_{n-k+1}(NQ))_0$ is additive with respect to simplicial subdivisions of $C(J)$.

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