Eigenwaves propagation in three-layer cylindrical viscoelastic shells with a filler non-uniform in thickness

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Abstract. The paper provides the statement of the problem, the development of a calculation method and an algorithm to solve the problems of propagation and absorption of non-axisymmetric natural waves in layered dissipative-inhomogeneous viscoelastic three-layer cylindrical bodies. A detailed analysis of well-known publications devoted to this problem is given. The paper posed the problem of non-axisymmetric natural wave propagation in three-layer cylindrical shells with fillers, taking into account the theory of hereditary viscoelasticity. Dispersion equations were obtained and the dependence of the phase velocity on the wavenumber for the three-layer shells was constructed. A calculation method was developed based on the Muller, Gauss, and orthogonal sweep methods. The program (in C++ language) was compiled based on the developed algorithm. Numerical results were obtained for the complex phase velocity depending on various wavenumbers and parameters of an axisymmetric cylindrical mechanical system for the Kirchhoff-Love and Timoshenko hypotheses. The change in the real and imaginary parts of the complex phase velocity under various parameters of the system was investigated for structurally homogeneous and inhomogeneous mechanical systems. The results obtained by the Kirchhoff-Love and Timoshenko hypotheses are in good agreement but differ in the wave absorbing abilities. It was found that a decrease in filler thickness at a rigid inner shell leads to a rapid increase in the real and imaginary parts of the vibration frequencies.

1. Introduction

Quite often, in problems of the dynamics of cylindrical shells with a filler or three-layer shells, the filler has a structural non-uniformity in thickness associated with the manufacturing technology or with the need for a more rational (optimal) use of the filler material while observing the limitations in strength and rigidity of the structure as a whole. When studying the dynamics problems for such structures, an exact analytical solution can be obtained only for a limited class of laws of change in mechanical parameters of the filler [1–3]. This is because there is a necessity to integrate systems of
differential equations with variable coefficients. Studies of the dynamic behavior of three-layer shells are of theoretical and applied importance. Such structures are widely used in aviation and rocketry, in nuclear power plant construction, in energy and chemical engineering and other areas of the national economy [4–6]. Wide implementation of such structures is explained, in particular, by the fact that three-layer shells formed by thin load-bearing outer layers (shells) and a filler (a polymeric material) of a large thickness have less weight at equal rigidity when compared to homogeneous structures. In addition, the middle layer can provide, for example, thermal insulation of structures [7]. State-of-the-art of the dynamic interaction of cylindrical shells with continuous media (gas, water, filler) was discussed in [8, 9], from which it follows that, despite numerous studies on these problems, a number of issues on the structural inhomogeneity of a mechanical system have not been sufficiently studied. In particular, it is very important for the practice to study the dynamic behavior of structures taking into account structural inhomogeneity under vibration influences. A significant number of publications were devoted to this issue [10–15]. However, up to now, general methods of calculating structurally inhomogeneous layered cylindrical shells surrounded or filled with a linear inhomogeneous continuous medium have not been developed. In [16], the problem of free wave propagation in three-layer plates was considered in a refined statement, when the motion of the filler was described by the Lamé equations with inertial terms, and the Kirchhoff–Love hypothesis was used for the lining. Dispersion equations were obtained and phase velocities were determined for symmetric and antisymmetric waves. Axisymmetric free waves in an infinitely long cylindrical shell with an elastic homogeneous filler were investigated in [17]. At the same time, the existing calculation methods within the framework of the known mathematical models do not allow a sufficiently accurate description of the dynamic behavior of the structure deformation [18–20].

Thus, the development of methods for dynamic calculation of three-layer shells based on structurally inhomogeneous mathematical models and their implementation in the form of computational algorithms represents a relevant problem in mechanics.

In contrast to other publications, this paper presents the statement, methods, algorithms and results of studying the problems of non-axisymmetric natural wave propagation in viscoelastic three-layer cylindrical shells with inhomogeneous fillers.

2. Methods

2.1 Statement of the problem and basic equations

Consider the problem of propagation of natural waves in two infinitely long three-layer viscoelastic cylindrical shells with viscoelastic and radially inhomogeneous fillers. The dynamic equations of a viscoelastic inhomogeneous cylindrical filler, in a cylindrical coordinate system \( r, \theta, z \), are written in the form [21, 22]:

\[
\bar{\mu}_s \nabla^2 \bar{u} + (\bar{\lambda}_s + \bar{\mu}_s) \text{grad} \text{div} \bar{u} = \rho_s \frac{\partial^2 \bar{u}}{\partial t^2},
\]

where \( \bar{u}(u_r, u_\theta, u_z) \) is the displacement vector of the filler points; \( \rho_s \) is the density of the filler material;

\[
\text{grad} \varphi \equiv \frac{\partial \varphi}{\partial r} \bar{k}_r + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \bar{k}_\theta + \frac{\partial \varphi}{\partial z} \bar{k}_z;
\]

\[
\text{div} \bar{u} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{1}{r} \frac{\partial [ru_r]}{\partial r} + \frac{\partial u_z}{\partial z};
\]

\[
\nabla^2 \bar{u} \equiv \frac{\partial}{\partial r} \left( \frac{r \partial \bar{u}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \bar{u}}{\partial \theta^2} + \frac{\partial^2 \bar{u}}{\partial z^2};
\]
\[
\bar{\lambda}_{s}(\xi)\varphi(t) = \lambda_{0s}, \bar{\lambda}_{s}(\xi) \int_{0}^{t} R_{\lambda s}(t-\tau)\varphi(\tau)d\tau,
\]
\[
\bar{\mu}_{s}(\xi)\varphi(t) = \mu_{0s}, \bar{\mu}_{s}(\xi) \int_{0}^{t} R_{\mu s}(t-\tau)\varphi(\tau)d\tau,
\]
\( R_{\lambda s}(t-\tau) \) and \( R_{\mu s}(t-\tau) \) are the relaxation kernels; \( \lambda_{0s}, \mu_{0s} \) are the instantaneous elastic moduli; \( \bar{\lambda}_{s}(\xi), \bar{\mu}_{s}(\xi) \) are the parameters that determine the inhomogeneity of the elastic modulus along the coordinates \( \xi(r, \theta, z) \); \( \varphi(t) \) is the time function; \( \vec{k}_{r}, \vec{k}_{\theta}, \vec{k}_{z} \) are the unit vectors in directions \( r, \theta, z \).

A number of assumptions were used in dynamic calculations of the equations of motion of three-layer shells of various kinds, primarily related to the nature of the filler strain [23]. In that work, when studying the problems of non-axisymmetric natural wave propagation in viscoelastic three-layer cylindrical shells with inhomogeneous fillers, the possibility of using the equations of three-layer shells with fillers obtained based on the dynamic theory of viscoelasticity (1) was checked. In this case, the bearing layers were considered as thin shells obeying the Kirchhoff-Love (or Timoshenko) hypotheses. The contact between the bearing layers and the filler can be rigid or sliding.

The equations of motion of shells (i.e., of the bearing layers) in displacements (in the symbolic vector-matrix form), are written in the form
\[
L_{ij} \ddot{U}_{k} - \int_{0}^{t} L_{ij}R_{Ek}(t-\tau)\ddot{U}_{k}(\xi, \tau)d\tau = \frac{1-v_{0k}}{\alpha_{0k}h_{0k}}q_{k} + \rho_{0k} \frac{1-v_{0k}}{\alpha_{0k}} \frac{\partial^{2} \ddot{U}_{k}}{\partial \tau^{2}} \quad (k = 1, 2)
\]

Here index \( k = 1 \) refers to the inner bearing layer, and \( k = 2 \) to the outer layer, \( \rho_{0k} \) is the density of the shell material, \( v_{0k} \) is the Poisson's ratio of the shell, \( \ddot{U}_{k} \) is the displacement vector of the points of the middle surface of the bearing layer; for Kirchhoff-Love shells it has a three-dimensional dimension, i.e. \( U_{1k} = u_{k}; U_{2k} = \theta_{k}; U_{3k} = w_{k} \). Therefore, for the shells obeying the Kirchhoff-Love hypothesis, we can write:

\[
L_{11} = \frac{\partial^{2}}{\partial z^{2}} + \frac{1-v_{k}}{2R_{k}} \frac{\partial^{2}}{\partial \theta^{2}} - \rho_{k} \frac{1-v_{k}}{2G_{k0}} \frac{\partial^{2}}{\partial t^{2}}; \quad L_{22} = \frac{1-v_{k}}{2R_{k}} \frac{\partial^{2}}{\partial z^{2}} + \rho_{k} \frac{1-v_{k}}{2G_{k0}} \frac{\partial^{2}}{\partial t^{2}};
\]
\[
L_{33} = L_{31} = \frac{\partial}{\partial z}; \quad L_{32} = \frac{1-v_{k}}{12} \frac{\partial^{2}}{\partial z^{2}} + \frac{1}{a_{k}^{2}} \frac{\partial^{2}}{\partial \theta^{2}} - \rho_{k} \frac{1-v_{k}}{2G_{k0}} \frac{\partial^{2}}{\partial t^{2}};
\]
\[
L_{33} = L_{31} = \frac{1}{a_{k}^{2}} \frac{\partial}{\partial \theta}; \quad L_{33} = \frac{h_{k}^{2}}{12} \frac{\partial^{2}}{\partial z^{2}} + \frac{1}{a_{k}^{2}} \frac{\partial^{2}}{\partial \theta^{2}} + \rho_{k} \frac{1-v_{k}}{2G_{k0}} \frac{\partial^{2}}{\partial t^{2}};
\]
\[
\nabla^{2} = \frac{\partial^{2}}{\partial z^{2}} + \frac{2}{a_{k}^{2}} \frac{\partial^{4}}{\partial z^{2} \partial \theta^{2}} + \frac{1}{a_{k}^{4}} \frac{\partial^{4}}{\partial \theta^{4}}
\]

Here \( h_{k}, a_{k} \) are the thickness and radius of the middle surface of the bearing layer; \( R_{Ek}(t-\tau) \) is the relaxation kernel; \( G_{k0} \) is the instantaneous modulus of elasticity.

The load vector components for the Kirchhoff-Love shell have the form
\[
\{P_{1k}, P_{2k}, P_{3k}\} = -\frac{1-v_{k}}{2G_{k0}h_{k}}\{p_{ck} + q_{ck}, p_{ok} + q_{ok}, p_{rk} + q_{rk}\}
\]
where the minus sign corresponds to the inner shell, and the plus sign - to the outer one; \( q_{ik}, q_{\theta k}, q_{z k} \) are the components of the filler response; \( p_{ik}, p_{\theta k}, p_{z k} \) - is the intensity of the given load in the corresponding directions.

For a sliding contact, when describing the motion of the bearing layers by the Kirchhoff-Love equations, the boundary conditions have the form

\[
\begin{align*}
\sigma_{rc} &= \sigma_{r\theta} = 0; \quad u_r = w_k (r = a_k); \\
\sigma_{rr} &= -q_{r1}; \quad (r = a_1); \quad \sigma_{rr} = -q_{r2} (r = a_2). 
\end{align*}
\]

It is assumed here that the shell contact with the filler occurs along the middle surfaces of the bearing layers: \( a_1, a_2 \) are the inner and outer radii of the filler: \( q_{ik} \) is the normal response of the filler to the shell vibrations.

At a rigid contact, the boundary conditions on the side surfaces of the filler are taken in the form

\[
\begin{align*}
&u_r = w_k; \quad u_\theta = G_k; \quad u_z = u_k; \\
&\sigma_{rr} = \mp q_k; \quad \sigma_{r\theta} = \mp q_{\theta k}; \quad \sigma_{r\theta} = \mp q_{z k}; 
\end{align*}
\]

The minus sign corresponds to \( k = 1 \), and the plus sign corresponds to \( k = 2 \).

The boundary conditions for the bearing layers described by equations of the Timoshenko type are written in a similar way.

When considering the problem of free wave propagation, the components of a given load \( p_{ik}, p_{\theta k}, p_{z k} \) are taken to be zero.

We assume that the integral terms in (1) and (2) are small. Then the function is \( \phi(t) = \psi(t)e^{-i\omega t} \), where \( \psi(t) \) is a slowly varying time function, \( \omega_R \) is the real constant. Further, applying the procedure of the freezing method [24–26], we replace relations (2) with approximate ones:

\[
\begin{align*}
&\overline{\lambda}_s(\zeta)[\phi(t)] = \lambda_{0s}(\zeta)[1 - \Gamma^c_{sk}(\omega_k) - i\Gamma^s_{sk}(\omega_k)]\phi(t); \\
&\overline{\mu}_s(\zeta)[\phi(t)] = \mu_{0s}(\zeta)[1 - \Gamma^c_{\mu s}(\omega_k) - i\Gamma^s_{\mu s}(\omega_k)]\phi(t); \\
&G_k(\zeta)[\phi(t)] = G_{sk}(\zeta)[1 - \Gamma^c_{sk}(\omega_k) - i\Gamma^s_{sk}(\omega_k)]\phi(t). 
\end{align*}
\]

Here \( G_{sk}(\zeta) \) is the parameter denoting the inhomogeneity of the elastic modulus along coordinates \( \zeta(r, \theta, z) \),

\[
\begin{align*}
\Gamma^c_{sk}(\omega_k) &= \int_0^\infty R_{sl}(\tau) \cos \omega_k \tau \, d\tau, \quad \Gamma^c_{\mu s}(\omega_k) = \int_0^\infty R_{\mu s}(\tau) \cos \omega_k \tau \, d\tau, \\
\Gamma^s_{sk}(\omega_k) &= \int_0^\infty R_{sl}(\tau) \sin \omega_k \tau \, d\tau, \quad \Gamma^s_{\mu s}(\omega_k) = \int_0^\infty R_{\mu s}(\tau) \sin \omega_k \tau \, d\tau,
\end{align*}
\]

are the cosine and sine of Fourier images of the material relaxation kernel, respectively. A three-parametric relaxation kernel \( R_{sk}(t) = R_{\mu s}(t) = Ae^{-Rt}/t^{1-\alpha} \) was taken in [22] as an example of the viscoelastic material. Then, instead of integro-differential equations (1) and (3), we obtain a system of partial differential equations with complex coefficients, in the form

\[
\begin{align*}
\bar{\mu}_s \nabla^2 \ddot{u} + (\overline{\lambda}_s + \bar{\mu}_s) \text{grad} \text{div} \ddot{u} &= \rho_s \frac{\partial^2 \ddot{u}}{\partial t^2}, \\
\bar{\nu}_k \ddot{U}_k &= \frac{1 - \nu_0^2}{G_0} \ddot{q}_k + \rho_0 \frac{1 - \nu_0^2}{G_0} \frac{\partial^2 \ddot{U}_k}{\partial t^2}.
\end{align*}
\]
Here \( \bar{L}^k_{ij}[\varphi(t)] = L^k_{ij} \left( 1 - (R^k_{ij} \delta^k_{ij})^{-1} \right) [\varphi(t)] \), \( \delta^k_{ij} \) are the Kronecker symbols, \( R^k_{ij} (R^k_{11} = R^k_{22} = R^k_{33} = \bar{G}_k[\varphi(t)]) \) is a diagonal matrix of the third order for the Kirchhoff-Love hypothesis, and of the fifth order for the Timoshenko hypothesis. The system of differential equations (11) is solved under boundary conditions (7) and (8). The studies in [27, 28] investigated the problem of eigenwave propagations (a spectral problem) in viscoelastic layered cylindrical bodies for the given values of wavenumbers.

2.2 Solution methods

In this work, to solve the system of partial differential equations (10) and (11), the method of orthogonal differential sweep by S.K.Godunov and the method of complex amplitudes [29, 30] were used. An algorithm for determining eigenfrequencies and constructing dispersion dependences for three-layer cylindrical shells with a viscoelastic filler non-uniform along the radial coordinate was given in [31].

Let us consider the problem of axially symmetric longitudinal-transverse wave propagation in two coaxial cylindrical shells of infinite length, connected, in the general case, by a viscoelastic inertial filler non-uniform in thickness. The Kirchhoff-Love equations describe the motion of viscoelastic shells; the equations of the theory of viscoelasticity are used for the filler (1). The contact between the shells and the filler is assumed to be rigid (or sliding), and the bond is two-way so that conditions (7) are satisfied. It is assumed that the modulus of elasticity changes over the filler thickness according to the law

\[
E_i[\varphi(t)] = E_0 f_i(r) \left[ 1 - \Gamma^c_{sE}(\omega_R) - i \Gamma^s_{sE}(\omega_R) \right] \varphi(t); \quad \rho_i = \rho_0 f_2(r); \quad v_s = \text{const}.
\]  

Here \( \Gamma^c_{sE}(\omega_R), i \Gamma^s_{sE}(\omega_R) \) are determined similar to (9), \( f_1(r) \) and \( f_2(r) \) are the given functions of the radial coordinate. The function \( f_2(r) \) can have an arbitrary form, it is assumed to be differentiable with respect to function \( f_1(r) \) and does not vanish at \( r \in [a_1, a_2] \). The equations of motion of a viscoelastic filler in displacements for an axisymmetric problem are obtained from the following differential equations of motion:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} \left( \sigma_{rr} - \sigma_{zz} \right) = \rho_i \frac{\partial^2 u_r}{\partial t^2};
\]

\[
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{zz} = \rho_i \frac{\partial^2 u_z}{\partial t^2}.
\]

Substituting the stress expressions into (13), considering (8) and (12), we find

\[
\left( \lambda_i + 2 \mu_i \right) \frac{\partial \varepsilon}{\partial r} + 2 \mu_i \frac{\partial \omega_u}{\partial z} + \frac{\partial \lambda_i}{\partial r} \varepsilon + 2 \frac{\partial \lambda_i}{\partial r} \frac{\partial u_z}{\partial r} = \rho_i \frac{\partial^2 u_r}{\partial t^2};
\]

\[
\left( \lambda_i + 2 \mu_i \right) \frac{\partial \varepsilon}{\partial z} - 2 \mu_i \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial \lambda_i}{\partial r} \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) = \rho_i \frac{\partial^2 u_z}{\partial t^2};
\]

\[
\varepsilon = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_z}{\partial r}; \quad 2 \omega_y = \frac{\partial u_z}{\partial z} - \frac{\partial u_r}{\partial r};
\]

\[
\lambda_i \varphi(t) = \frac{v_s E_i \varphi(t)}{(1 + v_s)(1 - 2v_s)}; \quad \bar{\mu}_i \varphi(t) = \frac{E_i \varphi(t)}{2(1 + v_s)}.
\]
If the filler is homogeneous, then \( f_1 = f_2 = 1 \) and \( \frac{\partial^2 \lambda_s}{\partial r^2} = \frac{\partial \mu_s}{\partial r} = 0 \). As a result, the well-known Lamé equations are obtained. The solutions of the equations of motion of the viscoelastic filler and shells (the bearing layers) are represented in exponential form

\[
\{u_s, u_s, w_k, u\} = \{u_{k,0}, U(r), w_{k,0}W(r)\} e^{-i(\alpha z - \omega t)},
\]

where \( \omega \) is the complex eigenfrequency; \( \alpha \) is the wavenumber.

Substituting (15) into the equations of motion of shells, we find the amplitudes of normal loads, transferred to the filler from the bearing layers:

\[
q_{r1}^0 = \frac{E_s h_s b_4}{(1 - v_s^2)} b_1 a_2^2; \quad q_{r2}^0 = \frac{E_s h_s b_4}{(1 - v_s^2)} b_1 a_2^2;
\]

\[
b_1 = \frac{k^2 \alpha_s^2}{12} - \delta^2; \quad b_2 = \frac{k^2}{12} (\delta^4 - \alpha_s^2) + 1; \quad b_3 = \frac{v_s^2 \delta^2 + b_1 b_2};
\]

\[
b_3 = \frac{v_s^2 \delta^2 + b_1 b_2 + b_3}{\varepsilon_i}; \quad \delta = \alpha a_z;
\]

\[
\omega_s^2 = \frac{12(1 - v_s^2)}{E_s h_s^2} \rho a_z^4 \omega^2
\]

Substituting (15) into (14), we obtain the following system of ordinary differential equations to determine \( U(r) \), \( W(r) \) with complex coefficients:

\[
\frac{d^2 W}{dr^2} + \varphi_1(r) \frac{dW}{dr} + \varphi_2(r) W = \frac{\delta}{2(1 - v_s)} \frac{dU}{dr} - \frac{v_s \delta}{1 - v_s} f_3(r) U = 0;
\]

\[
\frac{d^2 U}{dr^2} + \varphi_1(r) \frac{dU}{dr} + \varphi_2(r) U = \frac{\delta}{1 - 2v_s} \frac{dW}{dr} + \delta \varphi_3(r) W = 0;
\]

\[
\varphi_1 (r) = \frac{1}{r} + f_3 (r); \quad \varphi_3 (r) = \varphi_1 (r) + \frac{2v_s}{1 - 2v_s} \frac{1}{r};
\]

\[
\varphi_2 (r) = \frac{1}{r} \left( \frac{v_s r}{1 - v_s} f_3 (r) - 1 \right) + a_z^2 \frac{f_2 (r)}{f_1 (r)} - \frac{(1 - 2v_s) \delta^2}{2(1 - v_s)};
\]

\[
\varphi_3 (r) = \beta_z \frac{f_2 (r)}{f_1 (r)} - \frac{2(1 - v_s) \delta^2}{(1 - 2v_s)}; \quad f_3 (r) = \frac{\partial}{\partial r} \ln f_1 (r);
\]

Boundary conditions (7) considering (15) have the following form

\[
\frac{dW}{dr} + \frac{dU}{dr} = 0 \quad (r = 1, \varepsilon_i);
\]

\[
\left[ \frac{v_s b_1}{(1 - v_s) \varepsilon_i} - \frac{b_4}{(1 - v_s^2) E_s v_s f_1 (1)} \right] W - \frac{v_s \delta b_1}{1 - v_s} U + b_1 \frac{dW}{dr} = 0
\]

\[
\left[ \frac{v_s b_1}{(1 - v_s) \varepsilon_i} - \frac{b_4}{(1 - v_s^2) E_s v_s f_1 (\varepsilon_i)} \right] W - \frac{v_s \delta b_1}{1 - v_s} U + b_1 \frac{dW}{dr} = 0
\]

Frequency determination of free vibrations of cylindrical shells associated with a filler, non-uniform in thickness, is reduced to determining the eigenvalues for the boundary value problem (18) -
Let us note some special cases: if the inner shell is considered as absolutely rigid (a rigid core inside the filler), then condition (20) must be replaced by condition $W = 0$; the corresponding boundary conditions have the form

$$\sigma_\alpha^0 = 0; \quad \sigma_\nu^0 = 0; \quad (r = e_i)$$

(21)

and then instead of (20) we can write

$$\frac{v_i}{1-v_i} \frac{1}{\xi_i} W - \frac{v_i}{1-v_i} \frac{\delta}{U} dW = 0.$$  

(22)

An analytical determination of the eigenvalues for the boundary value problem in the case of arbitrary coefficients of system (14) could not be conducted; therefore, here the S.K. Godunov numerical algorithm was used. In this case, it is necessary to proceed from (14) - (16) to the boundary value problem for the system of differential equations of the first order, which in vector form can be written as

$$\frac{dX}{dr} = A(r) X; \quad BX(1) = 0; \quad CX(e_i) = 0.$$  

(23)

where $X$ is a four-dimensional solution vector, and matrices $A$, $B$ and $C$ have the form

$$A(r) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\phi_2(r) & \frac{v_i \delta}{1-v_i} f_1(r) & -\phi_1(r) & \frac{v_i \delta}{2(1-v_i)} \\ -\delta \phi_2(r) & -\phi_3(r) & -\frac{\delta}{1-2v} & -\phi_1(r) \end{pmatrix}; \quad B = \begin{pmatrix} v_i b_1 \\ 0 \\ b_i \\ 0 \end{pmatrix}; \quad \frac{1}{1-v_i} (1-v_i^2) E_r v_i f_1(1) \begin{pmatrix} v_i b_1 \\ 0 \\ b_i \\ 0 \end{pmatrix} 0; \\ C = \begin{pmatrix} \frac{v_i b_1}{1-v_i} + \frac{b_i}{1-v_i} & \frac{v_i \delta b_1}{1-v_i} & b_i & 0 \\ \frac{v_i b_1}{1-v_i} & \frac{v_i \delta b_1}{1-v_i} & b_i & 0 \\ \frac{\delta}{1-2v} & \frac{\delta}{1-2v} & b_i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3. Numerical results and discussion

To obtain numerical results in the problems under consideration, the integration interval $(e_i, 1)$ (23) is divided into $n$ sub-intervals by the orthogonalization nodes: $e_i = r_0, r_1, ..., r_n = 1$. Next, a system of linearly independent vectors $X_1, X_2, ..., X_n, \text{satisfying the conditions } CX_j(e_i) = 0 \text{ (} j = 1, 2, ..., n \text{), and a vector } k_j, \text{ which is a solution to the system } CX_j(e_i) = C_k, \text{ are sought. The following Cauchy problems at a node } r_j \text{ are solved using the Runge-Kutta method:}

$$\frac{dX_i}{dr} = A(r_i) X_i; \quad X_i(\xi_i) = X^0 \quad (i = 1/2/3).$$

(24)

where the vectors found above from conditions at $r_i = e_i$ are taken as initial values. The vectors $X_1, X_2, X_3$ obtained by integrating systems (24) are orthogonalized and normalized, which increases the solution accuracy [30]. Taking these vectors as initial data for the Cauchy problems at a node $r_j$, we find solutions to the latter at the next node, etc. In this way, we obtain an orthogonal system of
vectors $X_1, X_2, X_3$ at the end of the integration section at a point $r = 1$. Any solution to the boundary value problem that satisfies conditions $r = 1$ takes the value given in the form

$$X(l) = X_3 + a_1X_1 + a_2X_2. \quad (25)$$

Coefficients $a_1, a_2$ are determined from the condition

$$BX(l) = b \quad (26)$$

After $a_1$ and $a_2$ are found from condition (26), the solution to the boundary value problem at the solution nodes, which may not coincide with the nodes of orthogonalization selected above, is found by integrating the system

$$\frac{dX}{dr} = A(r)X \quad (27)$$

at initial conditions $X(l) = X'$ by the Runge-Kutta method. If the eigenvalue problem is solved, then all three vectors $X_1, X_2, X_3$ are found from the solution of the system $CX(\varepsilon_i) = 0$; representing the solution at $r = 1$ in the form (25), we arrive at system (29) with zero right-hand side.

When implementing the described algorithm on a computer, in addition to the master-program, the following subroutines were used: FUNN1 (finding the fundamental system of solutions to homogeneous $CX(\varepsilon_i) = 0$ and particular solutions of inhomogeneous system $CX(\varepsilon_i) = \varepsilon_k$ by the optimal elimination method) [32], the ORORM procedure (orthogonalization of a system of linearly-independent vectors by the reflection method) [32], the Runge-Kutta method (solution of the Cauchy problem for a system of ordinary differential equations). To assess the accuracy of results obtained by a numerical method, and to select the required number of orthogonalization nodes, in the limiting case of a homogeneous inertial deformable filler ($f_i(r) = 0$) in (25), the general solution of the system can be written in the form

$$U(r) = \frac{\delta}{a_2} Y_0(m_1) A - \frac{m_1^2}{a_2} Y_0(m_2) B + \frac{\delta}{a_2} J_0(m_3) C - \frac{m_2^2}{a_2} Y_0(m_2) D; \quad (28)$$

$$W(r) = -\frac{m_2}{a_2} Y_i(m_1) A - \frac{\delta m_2}{a_2} Y_i(m_2) B - \frac{m_1}{a_2} Y_i(m_1) C - \frac{\delta m_2}{a_2} Y_i(m_2) D;$$

$$m_1^2 = a^2 - \delta^2; \quad m_2^2 = \beta^2 - \delta^2.$$

Substituting (28) into the boundary conditions, we obtain a system of algebraic equations for $A, B, C, D$; from the condition of nontriviality of solutions, we find the characteristic equation

$$\det\begin{bmatrix} a_{ij} \end{bmatrix} = 0 \quad (i, j = 1, \ldots, 4);$$

$$\begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_10 & a_11 & a_12 \\ a_13 & a_14 & a_15 & a_16 \end{bmatrix}.$$
\[ a_{11} = a_{13} = 2\delta m_i; \quad a_{12} = a_{14} = t_3; \]
\[ a_{21} = m_i(2 - t_1) + t_6 s_j; \quad a_{22} = \delta(2 - t_1 - 2m_s s_6); \]
\[ a_{23} = m_i(2 - t_1) + t_6 s_j; \quad a_{24} = \delta(2 - t_1 - 2m_s s_{12}); \]
\[ a_{31} = 2\delta m_i s_j; \quad a_{32} = t_6 s_j; \quad a_{33} = 2\delta m_i s_j; \quad a_{34} = t_6 s_{10}; \]
\[ a_{41} = m_i \left( \frac{2}{\varepsilon_1} + t_2 \right) s_i + t_6 s_{12}; \quad a_{42} = \delta \left( \frac{2}{\varepsilon_1} + t_2 \right) s_i - 2m_s s_j; \]
\[ a_{43} = m_i \left( \frac{2}{\varepsilon_1} + t_2 \right) s_i + t_6 s_j; \quad a_{44} = \delta \left( \frac{2}{\varepsilon_1} + t_2 \right) s_i - 2m_s s_{11}; \]
\[ t_1 = \frac{2(1 + v)}{(1 - v^2)} E' s_i; \quad t_2 = t_4 = \frac{b_3}{b_3}; \quad t_3 = \delta^2 - m_s^2; \]
\[ s_i = \frac{J_1(m_i)}{J_1(m_i)}; \quad s_j = \frac{J_0(m_i)}{J_1(m_i)}; \quad s_3 = \frac{J_0(m_i)}{J_1(m_i)}; \]

\( s_4...s_6 \) is obtained from \( s_1...s_3 \) by replacing \( m_i \) with \( m_2 \), and \( s_7...s_{12} \) is obtained from \( s_1...s_6 \) by replacing function \( J_n(x) \) with function \( Y_n(x) \).

If to ignore the inertia of the filler, considering it weightless, then in equations (14) it is necessary to assume that \( \rho_c = 0 \). Then, instead of (29), the general solution of the system is written in the form

\[
U(r_i) = -\delta r_i K_1(\delta r_i) A - K_0(\delta r_i) B + \delta r_i I_1(\delta r_i) C + I_0(\delta r_i) D;
\]
\[
W(r_i) = \left[ -\delta r_i K_0(\delta r_i) + 4(1-v) K_1(\delta r_i) \right] A - K_1(\delta r_i) B +
\]
\[
+ \left[ -\delta r_i I_0(\delta r_i) - 4(1-v) I_1(\delta r_i) \right] C + I_1(\delta r_i) D.
\]

The characteristic equation is obtained similar to the case of an inertial filler and has the form

\[
\det |b_{ij}| = 0 \quad (i, j = 1, ..., 4);
\]

where
\[
b_{11} = \delta p_3 - 2(1 - v); \quad b_{12} = 1; \]
\[
b_{13} = \delta p_6 + 2(1 - v); \quad b_{14} = 1; \]
\[
b_{21} = \delta \left( t_5 + \frac{t_5 t_6}{2} \right) p_3 + t_6 \delta^2 + t_7 - 2(1 - v) t_6; \]
\[
b_{22} = \delta p_3 + 1 + \frac{t_5}{2} t_6; \quad b_{24} = \delta p_3 + 1 + \frac{t_5}{2} t_6; \]
\[b_{21} = \delta \left( t_5 + t_6 \frac{\varepsilon_1}{2} \right) p_6 - t_6 \delta^2 - t_5 + 2(1 - v_c) t_6;\]

\[b_{22} = p_1; \quad b_{31} = \delta \varepsilon_1 p_2 - 2(1 - v_c) p_1; \quad b_{32} = -p_2;\]

\[b_{33} = \delta \varepsilon_1 p_3 + 2(1 - v_c) p_3; \quad b_{34} = \left[ \frac{1}{\varepsilon_1} - \frac{t_2}{2} \right] p_1 t_6;\]

\[b_{41} = \delta \left( t_5 - \frac{\varepsilon_1 t_6}{2} \right) p_2 + \left[ t_6 \delta^2 - \frac{t_2}{\varepsilon_1} - 2(1 - v_c) t_6 \right] p_1;\]

\[b_{43} = \delta \left( t_5 - \frac{\varepsilon_1 t_6}{2} \right) p_4 - \left[ t_6 \delta^2 - \frac{t_2}{\varepsilon_1} + 2(1 - v_c) t_6 \right] p_4;\]

\[b_{44} = \left[ \delta p_4 + \frac{1}{\varepsilon_1} \right] t_6;\]

\[p_1 = \frac{I_1(\delta \varepsilon_1)}{I_1 \delta}; \quad p_2 = \frac{I_6(\delta \varepsilon_1)}{I_6 \delta}; \quad p_3 = \frac{I_3(\delta)}{I_3 \delta};\]

\[t_5 = 8v_c - 4v_c^2 - 3; \quad t_6 = 1 - 2v_c; \quad t_7 = 4\left(1 - 3v_c + 2v_c^2\right);\]

\[p_4 \ldots p_6 \] are obtained from \( p_1 \ldots p_4 \) by replacing functions \( I_n(x) \) with functions \( K_n(x) \).

If the inner shell is rigid, then the solution is obtained from equation (29) by replacing the elements of the fourth row of the determinant with the following ones:

\[ a_{41} = m_1 s_1; \quad a_{42} = \delta s_4; \quad a_{43} = m_5 s_7; \quad a_{44} = \delta^* s_{10};\]

For a shell with a hollow filler, the boundary conditions (31) hold, then the elements of the fourth row in the determinant are replaced:

\[ a_{41} = \frac{2m_1}{\varepsilon_1} s_1 + t_3 s_2; \quad a_{42} = 2\delta \left( \frac{s_4}{\varepsilon_1} - m_2 s_8 \right); \quad a_{43} = \frac{2m_7}{\varepsilon_1} s_7 + t_3 s_8; \quad a_{44} = 2\delta \left( \frac{s_{10}}{\varepsilon_1} - m_2 s_{11} \right).\]

Numerical results were obtained for the following values of dimensionless parameters:

\[ R(t) = A e^{-\beta t} / t^{1-a}; \quad A = 0,048; \quad \beta = 0,05; \quad a = 0,1; \quad k_1 = k_2 = k = 0,004; \quad v_1 = v_2 = v_c = 0,3; \quad \delta = 3; \quad E_c' = 0,8; \quad \rho_c^* = 0 \quad \text{and} \quad \rho_c = 1.\]

The dimensionless filler thickness \( \varepsilon_1 \) varied. First, the vibration frequencies for homogeneous inertial and inertia-free fillers were found as the reference ones using exact characteristic equations (32) with/without a rigid inner shell. Then, for a homogeneous filler and for the special cases, the results were found using a numerical algorithm (24) - (27). The optimal number of orthogonalization nodes was selected when compared to exact results.
The numerical results showed that for the number of intervals equal to 15, in all the cases considered, the numerical results practically coincide with the frequencies obtained using the analytical solution.

\[
f_1(r_i) = (1 + \xi_1 r_i)^n; \quad f_2(r_i) = (1 + \xi_2 r_i)^m; \tag{32}
\]

Figure 1 shows the dependence of the lower eigenfrequency of the system on the radius of the inner shell (the filler thickness) in the case of an inhomogeneous material \((n = 2; m = 2)\). The solid lines correspond to the real parts of eigenfrequencies, and the dashed lines - to the vibration damping coefficients for various modes of motion.

It was found that an account for the inertia of the filler could lead to an overestimation of the real parts of frequencies, the error being up to 11%.

For a relatively thick filler \((\varepsilon_i \leq 0,2)\), the boundary conditions on its inner surface have little effect on frequencies, but with a decrease in the filler thickness, due to the presence of a rigid inner shell, the real and imaginary parts of the vibration frequencies increase rapidly. On the contrary, a decrease in the filler thickness makes it possible, at \(\varepsilon_i > 0,8\), to use the model of a weightless filler with a permissible error.
Figure 2. Change in the real and imaginary parts of the first complex frequency versus the radius of the inner shell (at rigid contact).

Materials with power and exponential laws of variation of mechanical parameters in thickness are considered as examples of inhomogeneous filler:

\[ f_1(r) = e^{k_1r}; \quad f_2(r) = e^{k_2r} \quad (33) \]

Figure 2 shows the change in the real and imaginary parts of the first complex frequency versus the radius of the inner shell (at rigid contact). The solid lines correspond to the first radial mode of motion, and the dashed line is the coefficient of vibration damping for different modes of motion. As seen from the figure, an increase in variability \( \varepsilon_1 \) significantly increases the first frequency in comparison with the case when \( E_1 = const \), especially for a thick filler (the domain of small \( \varepsilon_1 \)). A similar result for a shell with a hollow filler was noted in [31], where the increase in \( E_0 \) was related not to the variability of the elastic modulus, but to the choice of a more rigid filler (by increasing \( E_0 \), which is constant). Similar studies of the effect of variability \( \rho_c \) have shown that with an increase in \( \rho_c \) according to the laws (32) or (33), the frequencies decrease.

Figure 3 shows the change in the real and imaginary parts of the first complex frequency depending on the wavenumber for different "n" (at rigid contact). Note that the developed numerical algorithm makes it possible to determine the frequencies of natural vibrations in the case when the mechanical parameters of the filler change abruptly (a multilayer filler with different characteristics of the layers; \( E_k, \rho_k \) within the layer could be constant or vary according to different laws). It is only necessary to select appropriately the conditions for the continuity of stresses and displacements.

An algorithm was developed, similar to the one described above for the problems of natural vibrations, for determining the propagation velocities of axisymmetric free waves in a system of two coaxial infinitely long cylindrical shells connected by a filler non-uniform in thickness (a three-layer shell). The bearing layers motion is described by equations that account for shear strains and inertia of rotation of the element; mechanical characteristics of the filler change according to (14). The solutions of the equations of motion of the bearing layers and the inhomogeneous filler are presented in the form (2). Then the amplitudes of normal loads transferred to the filler from the lining are expressed in terms of the amplitudes of normal displacements as follows:
\[
q^0_{rk} = G_k \kappa_k^2 \left[ \left( \kappa_0^2 - \frac{\kappa_0^4}{e_2} \frac{2}{3} c_{0k}^2 \right) \eta^2 + \frac{2}{1 - \nu_k} \frac{e_k^0}{\epsilon_k^2} \left( 1 - \frac{\nu_k^2}{e_1} \right) \right] \frac{\omega_{k,0}}{h_k}; \tag{34}
\]

Equations (30) maintain their form if \( \eta \) is everywhere substituted for \( \delta \). In accordance with representation (34), instead of (32) and (28), boundary conditions (19) - (21) are modified and the original problem is reduced to determining the eigenvalues of boundary value problem (25). In this case, the use of algorithm (24) - (38) brings the dispersion equation to an implicit form.

Equations (30) maintain their form if \( \eta \) is everywhere substituted for \( \delta \). In accordance with representation (34), instead of (32) and (28), boundary conditions (19) - (21) are modified and the original problem is reduced to determining the eigenvalues of boundary value problem (25). In this case, the use of algorithm (24) - (38) brings the dispersion equation to an implicit form.

Figure 3. The real and imaginary parts of the first complex frequency depending on the wavenumber for different "n" (at rigid contact)

\[\kappa_1 = \kappa_2 = \kappa = 0,02; \quad \nu_1 = \nu_2 = \nu_c = 0,3; \quad \kappa_0^2 = 2 / 3; \quad \epsilon_1 = 0,6; \quad E_c^* = 0,8; \quad \rho_c^* = 1,0.\]

Calculations were carried out for: \( \kappa_1 = \kappa_2 = \kappa = 0,02; \quad \nu_1 = \nu_2 = \nu_c = 0,3; \quad \kappa_0^2 = 2 / 3; \quad \epsilon_1 = 0,6; \quad E_c^* = 0,8; \quad \rho_c^* = 1,0.\) The laws of variation of the parameters of the filler inhomogeneity were considered according to the formula (34). In the case of a homogeneous filler, a good coincidence (when compared to the analytical results) of the phase velocities for different wavelengths was obtained already with ten orthogonalization nodes.

Figure 4 shows the change in the real and imaginary parts of the first complex phase velocity for various laws of variation of the parameters of the filler inhomogeneity (at rigid contact).
Figure 4. The real and imaginary parts of the first complex phase velocity for various laws of variation of parameters of the filler inhomogeneity (at rigid contact)

The solid lines (Fig. 4) refer to the real parts of the phase velocity, and the dashed lines correspond to the damping coefficients of the phase velocity. Curve 1 corresponds to a shell with a homogeneous filler ($f_1 = 1, R_e = 0$); for curve 2 - $\xi = 0,5; n = 2; k_1 = 0,5; f_1 \neq 1, R_e \neq 0$; for curve 3 - $\xi_1 = 1,0; n = 2; k_1 = 1,0$. As can be seen, the variability of $\tilde{E}_s$ has a significant effect on the phase velocities; with an increase in $\tilde{E}_s$ the shift of the minima of dispersion curves towards shorter waves is observed.

Studies of the effect of variability $\rho_s$ have shown that with an increase in $\rho_s$, according to the law (33) or (34), the phase velocities decrease.

4. Conclusions

1. A mathematical problem was posed on the non-axisymmetric natural wave propagation in three-layer cylindrical shells with a hereditary viscoelastic filler.

2. A calculation method was developed based on the Muller, Gauss, and orthogonal sweep methods. The program (in C++ language) was compiled based on the developed algorithm.

3. Dispersion equations to determine the phase velocity in three-layer shells with a filler non-uniform in thickness were obtained.

4. The dependence of the complex phase velocity was investigated using various hypotheses for the shell depending on the wavenumber and parameters of the axisymmetric cylindrical mechanical system.

5. It was found that the results obtained using the Kirchhoff-Love and Timoshenko hypotheses are in good agreement but the absorbing abilities of the waves differ.

6. It was revealed that at a decrease in the filler thickness, at a sufficiently rigid inner shell, the real and imaginary parts of the vibration frequencies of the system increase rapidly, and the decrease in the filler thickness allows the use of the model of a weightless filler.

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