COSMOLOGICAL EXPERIMENTS IN CONDENSED
MATTER SYSTEMS

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ABSTRACT

Topological defects are thought to be left behind by the cosmological phase transitions which occur as the universe expands and cools. Similar processes can be studied in the phase transitions which take place in the laboratory: “Cosmological” experiments in superfluid He$^4$ and in liquid crystals were carried out within the past few years, and their results shed a new light on the dynamics of the defect-formation process. The aim of this paper is to review the key ideas behind this cosmology - condensed matter connection and to propose new experiments which could probe heretofore unaddressed aspects of the topological defects formation process.

1. INTRODUCTION

“Big Bang” leaves a radiation-dominated universe at a temperature close to the Planck temperature. As the resulting primordial fireball expands, temperature falls, thus precipitating a sequence of symmetry-breaking phase transitions, the last of which converts quark-gluon plasma into nucleons. In the intervening period of expansion, many such transitions could have taken place. The exact number as well as the nature of these transitions is not known, but it is often suggested that the origin of the observed structures—galaxies, clusters, large-scale voids—is intimately related to the phase transitions which may have taken place very early in the history of the universe. For example, inflationary cosmology (Linde, 1990) traces the origins of the universe “as we know it” to the era in which the very first symmetry breaking in the GUT scheme was taking place. Indeed, in many models inflation happens in the course of the (usually second-order) phase transition (Guth, 1981; Albrecht and Steinhardt, 1982; Linde, 1982), leaving behind density perturbations which can act as seeds of the structures seen today in the universe.

In addition to inflation there may have been many other phase transitions, which could have left their imprints on the mass distribution of the universe. In particular, for
a large class of field theories the state left after the phase transition contains topological
defects—fragments of the original high-energy pre-transition (symmetric) “false vacuum”
which became locked in within the topologically stable configurations of the new, broken
symmetry “true vacuum” (Zeldovich, Kobzarev, and Okun, 1974; Kibble, 1976). Such
frozen-out relics of the distant past come in three basic varieties: monopoles, (cosmic)
strings, and domain walls. The kind of defects which will be left behind by the transition
depends on the interplay between the symmetry which is being broken during the phase
transition (more specifically, on the homotopy group of the manifold of the degenerate
broken symmetry vacua) and the symmetries of the space in which the evolution is taking
place (Kibble, 1976).

Monopoles are an inevitable outcome of the GUT-era phase transition and a major
threat to cosmological models. Their mass would dominate the universe and force it to
recollapse well before the present era. Inflationary cosmology was in part motivated by the
desire to dilute monopoles (Guth, 1981). Cosmic strings are “optional” and may result in
density perturbations not too different from those produced by inflation (Vilenkin, 1985).
Domain walls are again a threat—they are simply too massive to be consistent with either
the observed structures or the (relatively small) perturbations of the cosmic microwave
background.

The process of formation of topological defects—the main subject of this paper—is
perhaps the least understood (and most interesting) aspect of their evolution. The nature
of this process depends on the nature of the phase transition. In the course of the first-order
phase transitions, a new broken symmetry phase usually nucleates after the temperature
falls some distance below the critical temperature $T_c$. Separated islands of the new phase
form independently and expand, resulting in a local selection of the broken symmetry
vacuum. As a result of these independent selections, the final configuration may not be able
to get rid of the locked-out fragments of the original, pre-transition vacuum. First-order
phase transitions can also occur through the process known as spinodal decomposition
which by-passes the post-nucleation epoch of coexistence of the two phases.

Second-order phase transitions are still more interesting. There the phase transformation occurs (approximately) simultaneously throughout the volume. However, unless the critical temperature is traversed infinitesimally slowly, the resulting broken symmetry phase will contain many distinct regions with different choices of the local vacuum. This is because the selection of the new vacuum has to be somehow communicated if the same choice is to be made elsewhere, and the speed at which this information can be propagated is finite. Hence, appearance of the topological defects is—as was observed by Kibble in his seminal paper (Kibble, 1976)—a direct consequence of causality.

In the cosmological setting the ultimate causal limit on the size of the approximately uniform patches of the new vacuum is set by the size of the horizon at the time when the transition is taking place (Kibble, 1976). In the laboratory this ultimate limit is not relevant. Indeed, also in the cosmological context the process which will set the size of the uniform patches of the broken symmetry vacuum is likely to be associated with the dynamics of the order parameter rather than with the causal horizon (which can supply but an ultimate upper limit). In the original paper, Kibble appealed to the process of thermal activation, which is thought to occur in the course of the second-order phase transitions below the critical temperature, $T_c$, but above the so-called Ginzburg temperature $T_G$. When the temperature $T$ of the system falls in between $T_G$ and $T_c$, free energy required to “flip” locally uniform (correlation length-sized) volume of the new phase is less than the available thermal energy per degree of freedom, $\sim k_B T$. Hence, while locally the new vacuum has been selected, this choice is not really permanent—transitions between different possible vacua occur on a relaxation time scale, which can be much shorter than the time over which temperature traverses the region between $T_c$ and $T_G$. Consequently, it was originally postulated that the initial density of topological effects is set by the correlation length at the Ginzburg temperature, $\xi(T_G)$ (Kibble, 1976).
In the proposal for the early universe-like experiments suggested as a superfluid test of the cosmological scenario (Zurek, 1984; 1985; 1993) a different alternative was put forward. It focuses on the nonequilibrium aspect of the phase transition and predicts that the characteristic domain size should be set by the critical slowing down. Order parameter can adjust only on a relaxation time scale which diverges at $T = T_c$. Thus, as the critical temperature is approached from above, at a certain instant $\hat{t}$ evolution of the perturbations of the order parameter will become so sluggish that the time spent by the system in the vicinity of $T_c$ will be comparable with the relaxation time scale itself. The correlation length corresponding to such a “freezeout instant” will set the size of the regions over which the same vacuum can be selected. Hence, it will set the resulting density of the topological defects. This prediction (Zurek, 1984; 1985; 1993) has been now borne out by the experiment (Hendry et al., 1994; Hendry et al., 1995; Zurek, 1994).

The aim of this paper is to discuss the dynamics of the non-conserved order parameter during the course of the phase transitions in the wake of the “cosmological” quench experiments which were recently carried out in superfluid helium (Hendry et al., 1994; Hendry et al., 1995) and in liquid crystals (Chuang et al., 1991; Yurke, 1995; Bowick et al., 1994) and to propose further experiments which can shed a new light on the heretofore unaddressed aspects of the symmetry breaking. We shall focus on second-order phase transitions in superfluids and in superconductors, which constitute two experimentally attractive alternative implementations of symmetry breaking phase transitions in systems with a global and local gauge, respectively.

In the next section we shall briefly review “static” (equilibrium) aspects of symmetry breaking in field theories, in the superfluid helium, and in the superconductors. Section 3 will discuss topological defect formation in superfluid He$^4$, compare theoretical predictions with the recent experiment, and consider prospects for analogous experiments in superconductors. Section 4 will focus on phase transitions in annular geometry, in both superfluids and superconductors. In such geometry, interplay of the symmetry breaking with the torus
topology of the system results in particularly dramatic predictions. Section 5 contains an application of these considerations to the cosmological phase transitions. Conclusions are summed up in Section 6.

2. SPONTANEOUS SYMMETRY BREAKING IN HIGH ENERGY AND IN LOW TEMPERATURE PHYSICS

The ability to draw parallels between the high energy and condensed matter phenomena rests on the similarity of the behavior of free energy, especially in the vicinity of the phase transition. In particular, for the second-order phase transitions, potential contribution to the free energy is of the Landau-Ginzburg form:

\[ V(\varphi) = \alpha(T - T_c)|\varphi|^2 + \frac{1}{2}\beta|\varphi|^4, \]  

(2.1)

where \( \varphi \) is the order parameter or a field (see Fig. 1). The coefficients of the two forms have a well-prescribed dependence on the relative temperature:

\[ \epsilon = \frac{(T - T_c)}{T_c}. \]  

(2.2)

Thus:

\[ \alpha \simeq \alpha' \epsilon, \]  

(2.3)

and

\[ \beta = \text{const.} \]  

(2.4)

In the field theories the form of the potential energy, Eq. (2.1), is often simply postulated (although it can be also justified by the Gaussian approximation at a finite temperature; Aitchison and Hey, 1982). In the low-temperature (or, more generally, condensed matter) context, it is, on the other hand, usually derived in the mean field approximation from the underlying microscopic theory of the system in question. For instance, in the case for superconductors, the so-called Gorkov equations provide a link
between the microscopic BCS theory and the Landau-Ginzburg theory (Gorkov, 1959). In any case, the parallels between the equilibrium aspects of the phase transitions in the field theories relevant to high-energy physics and in the effective field theories emerging in the mean field description of condensed matter system have been appreciated for quite some time (Aitchison and Hey, 1982, and references therein).

Symmetry breaking arises when $T < T_c$, that is when the coefficient $\alpha$ in Eq. (2.1) becomes negative. Then the global minimum of the potential energy given by Eq. (2.1) moves from $\varphi = 0$ (where it resides at $T > T_c$) to the finite absolute value given by

$$\sigma = \sqrt{|\alpha|/\beta}.$$  \hspace{1cm} (2.5)

When $\varphi$ is two dimensional instead of just two alternatives ($+\sigma$ and $-\sigma$), degenerate minima form a circle of radius $\sigma$. For 3-D $\varphi$, a sphere of radius $\sigma$ constitutes the degenerate manifold of the alternative true vacua. The depth of the minimum of $\sigma$ is given by

$$\Delta V = V(0) - V(\sigma) = \frac{\alpha^2}{2\beta}.$$ \hspace{1cm} (2.6)

The tension between the long range order which is supposed to set in below the critical temperature and the relatively short range over which the choice of the broken symmetry vacuum can be communicated and in a finite time determined by the rate at which the phase transition is taking place is responsible for creation of the topological defects. The tension between the long range and locality constitutes therefore the primary focus of the dynamics of nonequilibrium phase transitions which are the subject of this paper.

**Field Theories**

To consider consequences of the symmetry breaking in field theories, we supplement potential energy, Eq. (2.1), with a kinetic term and consider a Lagrangian $L$:

$$L(\varphi) = \partial_\mu \varphi^* \partial^\mu \varphi - \alpha \varphi^* \varphi - \frac{\beta}{2} (\varphi^* \varphi)^2,$$ \hspace{1cm} (2.7)
where $\varphi$ is a complex field. When $\alpha < 0$, potential energy associated with Eq. (2.7) has a degenerate minimum—a circle at the radius $\sigma = \sqrt{|\alpha|/\beta}$. Thus, the simplest (static, space-independent) solution of Eq. (2.7) is given by

$$\varphi(x) = \sigma e^{i\theta},$$

(2.8)

where $\theta$ is the fixed phase. Small perturbations around such a uniform solution can be considered. That is, one may investigate the behavior of

$$\varphi(x, t) = \sigma + (u(x, t) + iv(x, t)) / \sqrt{2},$$

(2.9)

where $u, v \ll \sigma$. Above, $u$ is the perturbation of the absolute value of $\varphi$, and $v$ is the perturbation of its phase.

When $L$ is expressed in terms of $u$ and $v$ defined by Eq. (2.9) and the constant terms are ignored,

$$L = \frac{1}{2} (\partial_\mu u)^2 + \frac{1}{2} (\partial_\mu v)^2 - \beta \sigma^2 u^2 - \left(\frac{\beta}{\sqrt{2}}\right) \sigma u (u^2 + v^2) - \left(\frac{\beta}{8}\right) (u^2 + v^2)^2$$

(2.10)

obtains. One can regard it as a Lagrangian for two coupled fields, $u$ and $v$. Field $u$ has a positive mass given by $\beta \sigma^2 = -\alpha = |\alpha|$, and $v$—the so-called Goldstone boson—is massless.

Let us now consider a more interesting static solution of Eq. (2.10). To do this, we begin by deriving a dimensionless version of the Lagrangian $L$, which obtains when $\varphi$, the distances, and the time are expressed in the “natural” units:

$$\varphi \to \eta = \varphi/\sigma$$

(2.11)

$$\vec{r} \to \vec{\rho} = \vec{r}/\xi$$

(2.12)

$$t \to \tilde{t} = t/\tau$$

(2.13)

where $\sigma$ is given by Eq. (2.5), while $\xi$ and $\tau$ are the correlation length and relaxation time and are given by

$$\xi = \tau = \frac{1}{\sqrt{\alpha}}$$

(2.14)
in the convenient cosmological unit system ($\hbar = c = k_B = 1$) we are employing in this section. Now a time-independent solution will have to satisfy

$$\nabla^2 \eta = (|\eta|^2 - 1) \eta .$$

(2.15)

Apart from the trivial solution $|\eta|^2 = 1$, Eq. (2.15) has an axisymmetric solution of the form:

$$\eta = f(\varrho) \exp(in\phi) ,$$

(2.16)

where $(\varrho, \phi, z)$ are the cylindrical coordinates. Above, $n$ must be a whole number—otherwise, $\eta$ could not be single valued. The radial part of the above solution is regular near the origin ($f(\varrho) \simeq \varrho^n$ for $\varrho \ll 1$) and approaches the equilibrium value at large distances ($f^2(\varrho) \simeq 1 - n^2/\varrho^2$ for $\varrho \gg 1$). The phase of this solution is $\theta = n\phi$ on any $\varrho = \text{const.} > 0$ circle and is undefined for $\varrho = 0$, where the amplitude of the broken symmetry phase decreases to zero.

Equation (2.16) represents, of course, a string—topological defect—with the original symmetric vacuum locked out along the axis of symmetry by the broken symmetry phase in which a different vacuum is selected along the different radial directions in such a fashion that the transition to the minimum energy state, $\eta = 1$, is prohibited by an infinite energy barrier. The string tension—energy per unit length of the object defined by Eq. (2.16)—is logarithmically dependent on the cutoff in the distance. This solution was actually considered first by Ginzburg and Pitaevskii in the context of Landau-Ginzburg analysis of superfluid helium (Ginzburg and Pitaevskii, 1958), and we shall soon return to its interpretation as a superfluid vortex.

An even more common and interesting field-theoretic model corresponds to a Lagrangian:

$$L = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + [(\partial_\mu + ieA_\mu) \varphi^*] [(\partial_\mu - ieA_\mu) \varphi] - \alpha \varphi^* \varphi - \frac{\beta}{2} (\varphi^* \varphi)^2$$

(2.17)

with $A_\mu$ a massless gauge boson and

$$B_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

(2.18)
While the Lagrangian given by Eq. (2.10) is invariant under a global gauge symmetry (i.e., redefining the phase $\theta$ globally), Eq. (2.17) respects a more physically interesting local gauge transformation in which the phase is a function of space, that is

$$\varphi(x) \rightarrow e^{-i\theta(x)} \varphi(x)$$  \hspace{1cm} (2.19)

and

$$A_\mu(x) \rightarrow A_\mu(x) - e^{-1} \partial_\mu \theta(x).$$ \hspace{1cm} (2.20)

When the expansion around the local minimum of $|\varphi| = \sigma$ we have implemented before for the case of the global gauge with the help of the substitution, Eq. (2.9), is carried out for the locally gauge invariant theory of Eq. (2.17), we will find

$$L = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + e^2 \sigma^2 A_\mu A^\mu + \frac{1}{2} (\partial_\mu u)^2 + \frac{1}{2} (\partial_\mu v)^2 - \beta \sigma^2 u^2 - \sqrt{2} e \sigma A_\mu \partial_\mu v + \ldots$$ \hspace{1cm} (2.21)

The surprising (and famous) result of symmetry breaking is the term involving $A_\mu A^\mu$—it appears that the gauge vector field has acquired a mass (Aitchison and Hey, 1982).

The excitations of the amplitude of $\varphi$ around $\sigma$ are, of course, still massive. To see this, we can further simplify Eq. (2.21) by fixing the gauge so that $\theta(x)$ is the phase of the original complex field $\varphi(x)$. Then Eq. (2.9) reduces to

$$\varphi(x) = \sigma + u(x)/\sqrt{2},$$ \hspace{1cm} (2.22)

and the Lagrangian given by Eq. (2.21) becomes

$$L = \frac{1}{4} B'_{\mu\nu} B'^{\mu\nu} + e^2 \sigma^2 A'_\mu A'^\mu + \frac{1}{2} (\partial_\mu u)^2 - \beta \sigma^2 u^2 - \frac{1}{8} \beta u^4$$

$$+ \frac{1}{2} e^2 (A'_\mu) \left( \sqrt{2} \sigma u + u^2 \right).$$ \hspace{1cm} (2.23)

In this form it is apparent that $L$ describes interaction of the massive vector boson $A'_\mu$ with a real scalar field $u$, the Higgs boson, with the mass given by

$$m_H = \beta \sigma^2 = |\alpha|. \hspace{1cm} (2.24)$$
Vortex solution still exists (Nielsen and Olesen, 1973) for the Lagrangian given by Eq. (2.17), but its properties are now different than those given by the global string of Eq. (2.16). We shall consider it explicitly in a specific gauge below, in the course of the discussion of flux tubes in type II superconductors. Here, let us only anticipate one big difference: String tension associated with the local string is finite (rather than logarithmically divergent as was the case for global strings).

**Superfluid Helium 4**

The classic condensed matter example of a state with nontrivial broken symmetry phase is superfluid He\(^4\). Phase diagram of helium is shown in Fig. 2. Superfluid He\(\text{II}\) occupies the low temperature \((T \lesssim 2^\circ \text{K})\) low pressure \((p \lesssim 25 \text{ atm})\) corner of the phase diagram. Astonishing properties of superfluid helium (Tilley and Tilley, 1986) are associated with the Bose condensate of He\(^4\) atoms. Below the pressure dependent critical temperature \(T_c\) (which is also often called \(T_\lambda\) or “the \(\lambda\)-line” because of the behavior of specific heat in its vicinity), a temperature dependent fraction of atoms condenses out into a quantum fluid described by a Bose condensate wave function \(\Psi\):

\[
\Psi = |\Psi|e^{i\theta}.
\]

(2.25)

It is the phase \(\theta\) of \(\Psi\) which is responsible for many of the intriguing properties of superfluid helium. Some of them can be understood when we assume that \(\Psi\) obeys a Schrödinger equation of the form

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + \mu \Psi ,
\]

(2.26)

where \(m\) can be taken to be the mass of He\(^4\) atom, and \(\mu\) is the chemical potential—the energy gained by the system as a result of the addition of a single particle at constant volume and entropy.

The equation for superfluid flow can be obtained from the Schrödinger equation (2.26) by computing the rate of change of \(|\Psi|^2\) where this probability density is now interpreted as the density of the superfluid (Tilley and Tilley, 1986). Thus, for example, one can show
that the velocity $v$ of the superfluid and the phase $\theta$ of $\Psi$ are connected with a simple equation:

$$\vec{v} = \frac{\hbar}{m} \vec{\nabla} \theta . \quad (2.27)$$

The formal connection between superfluid helium and our field-theoretic considerations of the previous section can be established when it is assumed that the free energy density can be expanded in powers of $|\Psi|^2$ (which plays the role of the order parameter) and that it has a Landau-Ginzburg form:

$$f(\vec{r}) = \alpha |\Psi(\vec{r})|^2 + \frac{\beta}{2} |\Psi(\vec{r})|^4 + \frac{\hbar^2}{2m} |\vec{\nabla}\Psi(\vec{r})|^2 . \quad (2.28)$$

When the chemical potential $\mu$ is evaluated with the help of the above ansatz, the Schrödinger equation (2.26) we have written before acquires a form:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + \alpha \Psi + \beta |\Psi|^2 \Psi . \quad (2.29)$$

This relation is known as the Gross-Pitaevskii equation. It is generally treated with suspicion by the condensed matter theorists, who cite evidence that the so-called time-dependent Ginzburg-Landau (TDGL) equation (which has, on the left-hand side, a real rather than imaginary number multiplying the time derivative $\partial \Psi/\partial t$) is a better model for many systems (including superfluid He$^4$) in the vicinity of the phase transition (Hohenberg and Halperin, 1977).

Indeed, in the related case of superconductors TDGL equations have been rigorously derived from the microscopic theory (Gorkov, 1959). In that case the system is definitely overdamped, with the left-hand side of the equation for the order parameter dominated by $\Gamma \partial \Psi/\partial t$ where $\Gamma$ is real. Although the microscopic theory of He$^4$ superfluidity does not exist, the reason for overdamping (which in the superconductors is due to the coupling between the super and normal electrons) is certainly also present in superfluid helium. Thus, one can expect that approach to equilibrium will not be governed by Eq. (2.29), and this is indeed the case. The TDGL equation represents such relaxation quite accurately.
Nevertheless, Eq. (2.29) does have some interesting features. For example, the dispersion relation which obtains from the Gross-Pitaevskii equation does have both massless ($\omega \sim k$) and massive ($\omega \sim k^2$) components, although its form is not reminiscent (Bernstein and Dodelson, 1991) of the (second sound) phonon-roton dispersion relation which emerges only when the interatomic forces are taken into account (Vinen, 1969; Feynman, 1954).

We shall not depend on the Gross-Pitaevskii equation (2.29) in analyzing details of the rapid phase transition (the subject of the next section). It is nevertheless good to write it down and use it to extract (with the help of the Landau-Ginzburg free energy) several important pieces of information which turn out to be largely independent of this (rather suspect) method of their derivation. The form of Eq. (2.29) differs from the equation of motion one would obtain from the globally gauge invariant Lagrangian, Eq. (2.7), only in that it has a first (rather than second) time derivative on the left-hand side. As a result, when Eq. (2.29) is re-scaled to assure a dimensionless form,

$$i\dot{\eta} = -\nabla^2 \eta + (|\eta|^2 - 1) \eta , \quad (2.30)$$

the following three important quantities—relaxation time $\tau$, correlation length $\xi$, and equilibrium density of the superfluid $\sigma^2$—are employed:

$$\tau = \hbar / |\alpha| = \tau_0 / |\epsilon| , \quad (2.31)$$

$$\xi = \hbar / \sqrt{2m|\alpha|} = \xi_0 / \sqrt{|\epsilon|} , \quad (2.32)$$

and

$$\sigma^2 = -\alpha / \beta . \quad (2.33)$$

Scalings with temperature follow when we adopt the usual ansatz [Eqs. (2.2)-(2.4)] for the parameters $\alpha$ and $\beta$ (see Ginzburg and Pitaevskii, 1958, for numerical estimates of $\alpha'$ and $\beta'$):

$$\alpha = \alpha' \epsilon , \quad \alpha' \approx 10^{-16} [\text{erg}] ; \quad (2.34)$$

$$\beta = \text{const.} , \quad \beta \approx 4 \cdot 10^{-40} [\text{erg cm}^3] , \quad (2.35)$$
where $\epsilon = (T - T_c)/T_c$, Eq. (2.2), is the relative temperature.

When we restrict ourselves to the time-independent solutions of Eq. (2.30), we will be led back to Eq. (2.15) that we have already considered in the preceding subsection. In particular, the global string solution given by Eq. (2.16) for $n = 1$ is known as the Ginzburg-Pitaevskii vortex in the superfluid helium context (Ginzburg and Pitaevskii, 1958) (see Fig. 3). The flow around the axis of symmetry is caused by the phase gradient, so that at a distance $r$ away from the core of the vortex, its velocity is given by

$$v = \frac{\hbar}{m} \cdot \frac{1}{r}. \tag{2.36}$$

Vortices with the winding number $n > 1$ are unstable, as they are energetically more expensive than the $n = 1$ case. This can be seen by computing string tension $\varsigma$, which can be readily obtained from the kinetic energy of the flow for an arbitrary winding number $n$ out to some cutoff $L$:

$$\varsigma = \int_0^L 2\pi r \, dr \cdot \left( m|\Psi|^2 v^2 \right) / 2 \approx n^2 \cdot \frac{\pi h^2 \sigma^2}{m} \cdot \ln \left( \frac{L}{\xi} \right). \tag{2.37}$$

Clearly, topological constraint (i.e., a certain winding number $N$ within a closed path at some large radius) can be satisfied at a lesser energetic expense by $N$ vortices with $n = 1$ each rather than by a single vortex with $n = N$, for any $N > 1$.

The string tension, Eq. (2.37), can be re-written in a more suggestive way:

$$\varsigma = \left[ \pi \xi^2 \cdot \left( \frac{\alpha^2}{2\beta} \right) \right] \cdot 4 \ln \left( \frac{L}{\xi} \right). \tag{2.38}$$

This equation [obtained from Eq. (2.37) in the case of $n = 1$] lends itself to an intuitively appealing interpretation: String tension is approximately proportional to the energy of the symmetric vacuum which occupies the core (of radius $\xi$) of the vortex (and which is given by the term in square brackets).

We also note the logarithmic dependence of $\varsigma$ on the large-scale cutoff $L$. [The small-scale cutoff at $\xi$ appears naturally because of the absence of the superfluid condensate at small radii, where $|\Psi|^2 \sim \left( \frac{r}{\xi} \right)^2$, see discussion below Eq. (2.16).]
Existence of vortex lines was postulated nearly fifty years ago by Onsager (1949) and (independently, but somewhat later) by Feynmann (1954) to allow for superfluid rotation in spite of the requirement of single-valuedness of the Bose condensate wave-function. Vortex lines have since been studied both experimentally and theoretically (Tilley and Tilley, 1986; Donelly, 1991), although their possible origin in the “cosmological” scenario seems to have evaded attention of many of the low temperature scientists (see, for example, discussion of the vortex creation during the superfluid phase transition in Donelly, 1991, section 5.7).

In addition to the vortex lines, we shall consider “cosmological” implications of persistent superfluid flows in an annular container. This phenomenon is again a consequence of the crucial role of the quantum phase of the Bose condensate in HeII: When the phase \( \theta \) increases by \( 2\pi n \) as one follows a closed path along the (large) circumference \( C = 2\pi r \) of the torus containing superfluid He\(^4\), the velocity associated with the gradient of phase will be given by Eq. (2.27),

\[
v = \left( \frac{\hbar}{m} \right) \cdot \frac{2\pi n}{C} = \frac{\hbar}{m} \cdot \frac{n}{r}.
\]

Such flows are also stable (although for a somewhat different reason than the flow around the vortex line). Moreover, now \( n \) is essentially arbitrary [that is, until \( v \) reaches critical velocities at which superfluid flow begins to be dissipative (Tilley and Tilley, 1986)].

We end this subsection with a caveat: Landau-Ginzburg theory is only a qualitatively correct approximation for superfluid He\(^4\). Therefore, scaling relations for various quantities with the relative temperature \( \epsilon \) are somewhat different than one would infer from Eqs. (2.32)–(2.35). For instance, measurements show (Ahlers, 1976) that the correlation length \( \xi \) scales as \( |\epsilon|^{-\nu} \), where \( \nu = 2/3 \) and is consistent with the predictions of the renormalization group approach (see, e.g., Goldenfeld, 1992). We shall pay attention to the consequences of such differences in what follows. Here, let us only note that the discrepancy between the Landau-Ginzburg approach and the behavior of superfluid He\(^4\) is

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not unexpected. This is because, on the one hand, mean field theory which would allow one to apply the potential energy model of Eq. (2.21) to systems with atomic substructure is valid only when the correlation length $\xi$ is much larger than the interatomic spacing $a$,

$$\xi \gg a. \quad (2.40)$$

On the other hand, considerations based on the Landau-Ginzburg model are valid only when the effects of thermal fluctuations can be neglected. This means that the inequality:

$$E_G = \xi^3 \cdot \Delta V > \frac{1}{2} k_B T_c,$$  

(2.41)

where $\Delta V = \alpha^2/2\beta$, Eq. (2.6), known as the *Ginzburg condition* should be satisfied.

In superfluid $\text{He}^4$ the correlation length far away from $T_\lambda$ is of the order of 4–5 Ångstroms. Therefore Eq. (2.40) would demand $\epsilon \ll 1$. But (with the Landau-Ginzburg scaling $\xi \sim 1/\sqrt{\epsilon}$) the left-hand side of Eq. (2.41) scales as $\sqrt{\epsilon}$, and the inequality is satisfied only approximately $0.5^\circ K$ below $T_\lambda$.

Hence, the two conditions, Eq. (2.40) and Eq. (2.41), cannot be simultaneously satisfied. As a result, the Landau-Ginzburg theory can be only applied as a qualitative guide to the behavior of superfluid helium (Tilley and Tilley, 1986). We shall return to the condition of Eq. (2.41) below, as it defines Ginzburg temperature:

$$|\epsilon_G| = \left(\frac{\beta k_B T_c}{\xi_0^3 \alpha^2}\right)^2. \quad (2.42)$$

This relative temperature characterizes the demarcation line above which the potential difference between the minimum and the central peak is sufficiently small to make it easily traversable by the thermal fluctuations in the correlation length sized regions.

**Superconductors**

Superconducting phase transition can be modeled by a Landau-Ginzburg mean field theoretic model with the free energy given by (Tilley and Tilley, 1986; Tinkham, 1985; Werthamer, 1969)

$$F = \alpha |\Psi|^2 + \frac{1}{2} \beta |\Psi|^4 + \frac{1}{4m} \left( -i\hbar \vec{\nabla} - \frac{2e}{c} \vec{A}\right) \Psi)^2 + \frac{B^2}{8\pi} + E_0. \quad (2.43)$$
Above, $\Psi$ is the order parameter—wave function of the Bose condensate of Cooper pairs with mass $2m$ and charge $2e$—while $\vec{A}$ is the vector potential and

$$\vec{B} = \nabla \times \vec{A}.$$  \hfill (2.44)

Energetic contributions due to the applied (external) field are incorporated in the constant $E_0$.

Equation (2.43) can be justified rigorously by appealing to microphysics. The connection relies on the effective Gorkov equations (Gorkov, 1959) which can be derived from the BCS theory (Bardeen, Cooper, and Schrieffer, 1957) of superconductivity in the $\epsilon \ll 1$ region. Thus, in contrast to superfluid He$^4$, the Landau-Ginzburg theory is not just a phenomenological, qualitative approximation, but a microscopically justifiable quantitative tool (Tilley and Tilley, 1986; Tinkham, 1985). This fact will validate and simplify our discussion. On the other hand, instead of just one He$^4$, the list of superconducting materials is rather long, even if we were to ignore the new high-$T_c$ additions. Moreover, existence of the gauge field adds another characteristic length scale to the problem.

Thus, by the same token as before, we can define the relaxation time scale,

$$\tau = \hbar/|\alpha| = \tau_0/|\epsilon|,$$  \hfill (2.45)

where $\tau_0$ can be derived from the Gorkov equation

$$\tau_0 = \frac{\pi \hbar}{16 k_B T_c} \approx 1.5 \cdot 10^{-12} T_c^{-1} [s].$$  \hfill (2.47)

Correlation length:

$$\xi = \hbar/\sqrt{4m|\alpha|} = \xi_0/\sqrt{|\epsilon|}$$  \hfill (2.48)

characterizes variations of the order parameter while London’s penetration depth $\lambda$ is proportional to the correlation length,

$$\lambda(T) = \kappa \cdot \xi(T),$$  \hfill (2.49)
where $\kappa$ is the temperature-independent constant which distinguishes between the two kinds of superconductors: Type I (where $\kappa < 1/\sqrt{2}$) and type II (where $\kappa > 1/\sqrt{2}$).

For us this distinction is critically important, as the vortices can exist only in type II superconductors. There, the axially symmetric nontrivial solution which is a local minimum of Landau-Ginzburg free energy, Eq. (2.43), has the order parameter with the absolute value:

$$(\Psi/\sigma) \sim \tanh \frac{br}{\xi},$$

where $b$ is a constant close to unity. In the limit of extreme type II superconductors ($\kappa \gg 1$), the induction varies as

$$B(r) \approx \frac{\Phi_0}{2\pi\lambda^2} K_0 \left(\frac{r}{\lambda}\right),$$

where $K_0$ is a zero-order Hankel function of imaginary argument, and $\Phi_0$ is a flux quantum:

$$\Phi_0 = hc/2e.$$  \hspace{1cm} (2.52)

Qualitatively, $B(r)$ falls off as $e^{-r/\lambda}$ at large radii ($r > \lambda$) and can have significant values only in the shell $\xi < r < \lambda$ where it diverges as $\ln(\lambda/r)$. This shell disappears in type I superconductors (for which $\xi > \sqrt{2}\lambda$). Hence, type I superconductors do not allow for existence of vortices.

String tension of vortex lines is finite, and it is given by

$$\varsigma \approx \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \ln \kappa = \frac{H^2_c}{8\pi} \cdot 4\pi \xi^2 \ln \kappa,$$

where $H_c$ is the so-called critical field (which, when applied externally, expels superconducting phase from the sample). This formula is interesting because it demonstrates that the energetic cost of getting rid of the Bose condensate is given by the energy density associated with the symmetry breaking, $\Delta V$ of Eq. (2.6), so that

$$\Delta V = \alpha^2/2\beta = H^2_c/8\pi.$$  \hspace{1cm} (2.54)
Thus, Eq. (2.53) implies that the string tension corresponds to the energy of the condensate displaced by the tube of radius $r$ of normal metal times of proportionality factor of $4 \cdot \ln \kappa$. Similarly, one can check that $\varsigma$ is of the order of the energy of the field configuration associated with the flux line.

As was the case for superfluids, one can also consider loops of superconductors in an empty space. Single-valuedness of the order parameter requires that the phase accumulated along the circumference of the loop must correspond to an integer multiple of $2\pi$. Now, however—in contrast to the case of He$_4$—there are two sources of the phase gradient: In addition to the velocity $\vec{v}$ of the “superfluid” of Bose-condensed Cooper pairs, there is also the vector potential. Consequently, the quantization condition is

$$\frac{c}{2e} \oint_C \left( 2m\vec{v} + \frac{2eA}{c} \right) d\vec{l} = n\Phi.$$ (2.55)

In the limit in which the superconducting loop is wider than the Landau penetration depth $\lambda$ (so that there are closed paths $C$ along which the first term containing $\vec{v}$ is negligible), Eq. (2.55) implies quantization of the flux in units of $\Phi_0$. This need not always be the case, as we shall find out while analyzing phase transitions in the annular geometry. Then $\lambda(T)$ becomes large, and sufficiently near $T_c$ the resulting flux will be quantized in units smaller than $\Phi_0$, with the velocity term making up the difference of the phase of the Bose condensate.

The reliability of the Landau-Ginzburg theory for superconductors is due to the significant values of $\xi_0$ (typical $\xi_0 \gtrsim 1000 \text{Å}$, more than two orders of magnitude larger than in He$_4$). Thus, the Ginzburg energy which is approximately given by

$$E_G = \sqrt{\epsilon} : \xi_0^3 H_c^2(T = 0)/8\pi$$ (2.56)
[see Eq. (2.54)] is typically much larger than the corresponding quantity for the same $\epsilon$ in superfluid helium. To get an order of magnitude estimate of the relative Ginzburg temperature $\epsilon_G$, we compute

$$
\epsilon_G = 1.2 \times 10^{-5} \left( \frac{8\pi k_B T}{(\xi_0/1000 \, \text{Å})^3 (H_c/100 \, \text{Oe})^2} \right)^2.
$$

Thus, thermal fluctuations large enough to alter the configuration of the field are found only in the immediate vicinity of the critical temperature, even when a rather modest value for $H_c$ (which can be $\sim 10^3 \, \text{Oe}$) and $\xi_0$ (which is typically closer to a few thousand Ångstroms in type I superconductors) is adopted.

It is perhaps useful to point out that the above estimates do not apply to the high temperature superconductors, which have small $\xi_0$ and much more modest critical fields $H_c$ (Salamon, 1988).

Other Systems

The above two condensed matter phase transitions—the superfluid He$^4$ and superconductors—do not exhaust the list of the experimental possibilities, and the aim of this section is to point out—very briefly—that there are other systems in which topological defects in a nonconserved order parameter can be (or, indeed, have been) studied with the “cosmological” motivations in mind.

Liquid crystals were already mentioned in this context. They were the first system in which a version of the experiment suggested for superfluid He$^4$ was carried out (Chuang et al., 1991; Bowick et al., 1994). Nematic liquid crystals consist of rod-like molecules (see for example de Gennes, 1974). Above the phase transition temperature these molecules are oriented randomly, but as the temperature falls they tend to align. It is not difficult to imagine how this local order may lead to topological defects.

Formally, the order parameter in the liquid crystal is given by the director field $\vec{n}(\vec{r})$, which is just the local orientation of the rods. By definition $|\vec{n}(\vec{r})| = 1$. Moreover, since the molecules can be flipped by the angle $\pi$ without changing the physical configuration...
of the system, \( \vec{n}(\vec{r}) = -\vec{n}(\vec{r}) \). The vacuum manifold is therefore a two-sphere with the antipodal points identified.

Topologically stable defects which are expected in such a symmetry breaking scheme include strings (also known as disclination lines) and a monopole. Rod-like molecules are arranged in a “hedgehog” configuration around a monopole. And there are several distinct kinds of linear defects. Rods can, for example, simply point away from the core of the string. In that case, when the core is circumnavigated the director field changes its orientation by \( 2\pi \). But this is not the only possibility. Topologically stable defect can exist also when the director field changes its orientation by only \( \pi \), or by as much as \( 3\pi \). This is because \( \vec{n}(\vec{r}) \) and \(-\vec{n}(\vec{r})\) are identical. These topological defects can transform into each other in course of interactions.

Phase transition into the nematic liquid crystal is of the \textit{first} order. Following such a phase transition large densities of topological defects (especially the string-like disclination lines) were detected and studied (Chuang \textit{et al.}, 1991; Bowick \textit{et al.}, 1994). Disclination lines are visible under a microscope. Therefore, their scaling dynamics can be followed in detail. The disadvantage (from our point of view) is that the relevant phase transition is (weakly) first order. Thus, the most intriguing first stages of the development of the new broken symmetry phase can be understood in the relatively straightforward terms of the traditional nucleation paradigm, and do not shed any new light on the behavior of the order parameter in course of the second order critical dynamics.

\textbf{Superfluid He}^3 has even richer collection of topological defects. This is because the elementary particles which form the Bose condensate are pairs of He^3 atoms, and, as such, have a nontrivial internal structure. This results in additional degrees of freedom which are accessible even at the very low (milliKelvin) temperatures in which He^3 undergoes the superfluid phase transition. These additional degrees of freedom are associated with the non-zero spin of the Cooper pair (S=1) and with the orbital angular momentum of the Cooper pair around its center of mass. In the limit where the spin and angular momenta of
the Cooper pairs are strongly correlated, a liquid crystal like magnetic ordering will emerge. Thus, in addition to the broken gauge symmetry (present already in He\textsuperscript{4}) rotational symmetry of both spin and orbital degrees of freedom can be broken. Consequently, the order parameter of superfluid He\textsuperscript{3} is described by macroscopic wave functions like in He\textsuperscript{4}, but with \textit{nine} complex amplitudes. Moreover, there are two different phases of He\textsuperscript{3} superfluid: The quasi-isotropic phase B, which corresponds to Cooper pairs with no total (that is, spin plus orbital) angular momentum, as well as the anisotropic phase A, for which total angular momentum is non-zero when projected along some direction.

As a consequence of the rich structure of the order parameter, varied and interesting topological defects occur in both of the superfluid phases of He\textsuperscript{3} (Volovik, 1992). Moreover one He\textsuperscript{3} version of the quench experiment has been considered already some time ago (Mineev, Salomaa, and Lounasmaa, 1987; Salomaa, 1987; Salomaa and Volovik, 1987), and, as we shall discuss shortly, another one has been recently carried out (Ruutu \textit{et al.}, 1996).

Helium 3 has both advantages and disadvantages when compared with He\textsuperscript{4}. It is experimentaly more difficult to work with, and the variety of the topological defects may complicate the interpretation of the experimental results. Moreover, its correlation length is orders of magnitude larger than in He\textsuperscript{4}, so the density of the defects is expected to be significantly less. On the other hand, large correlation length implies that the Landau-Ginzburg theory is a much better model, so the analogy with the field theories is better justified. Moreover, vortices can be detected "one by one" using nucler magnetic resonance along with some other clever experimental techniques (Ruutu \textit{et al.}, 1996). Hence, lower initial vortex line density may turn out not to be a problem.

Last but not least, phase diagram of He\textsuperscript{3} allowes for a variety of quench trajectories, some of which raise the possibility of addressing other interesting issues such as the creation of topological defects in tuneably weak first order phase transformations, or even in a quick succession of the second and first order phase transitions.
3. QUENCHING OUT VORTEX LINES

The analogy between the static, equilibrium properties of the field theories and condensed matter systems implies—as we have seen it in the preceding section—existence of similar kinds of topological defects. The purpose of this section is to employ this analogy to devise laboratory experiments which would allow one to study defect formation in the course of rapid phase transitions. I shall focus on the condensed matter systems with non-conserved order parameter only and consider defect formation starting with the simplest case—superfluid helium. After a discussion of the freeze-out scenario, I shall go on to consider its consequences for the specific experiment already carried out by the Lancaster University group of Peter McClintock and his colleagues (Hendry et al., 1994; Hendry et al., 1995). The theoretical analysis will be repeated for the even more interesting (local gauge) case of a rapid phase transition to a superconductor, although it will be pointed out that the main obstacle for a successful study of a superconducting analog of the cosmological scenario is the difficulty of rapid detection of the vortices (flux lines). Thus, discussion of a more experimentally attractive (although more theoretically challenging) version of the superconducting experiment—which involves locking out magnetic flux in a loop—will be delayed to Section 4.

Quench Into Bulk Superfluid

Superfluid He⁴ has a number of experimental advantages which make it—in spite of the theoretical complications we have hinted at in the previous section—perhaps an ideal laboratory to study dynamics of phase transitions in systems with global symmetry. Thus, for example, one can obtain samples of He⁴ of almost unparalleled purity. Furthermore, He⁴ can accommodate only one kind of defect—vortex lines—simplifying the analysis. Moreover, detection of vortex lines—while not as straightforward as it is for defects in liquid crystals, where they are visible almost with a naked eye (Chuang et al., 1991; Bowick et al., 1994; Yurke, 1995)—is also relatively simple, as it relies on the well-tested techniques of second sound attenuation (Donnelly, 1991). We have already mentioned
that the superfluid transition is of the second order. And last, but definitely not least, phase transition in He$^4$ can be induced on a dynamical time scale by rapid decompression. Therefore, the critical temperature $T_\lambda$ will be reached throughout the bulk of the He$^4$ sample on a very short time scale limited by the first sound travel time ($\sim 220$ m/s in the vicinity of the critical temperature $T_\lambda$). This is orders of magnitude larger than the (second sound) velocity defined by the ratio of the correlation length and relaxation time scale:

$$s = \xi / \tau = \left( \frac{\xi_0}{\tau_0} \right) \sqrt{|\epsilon|},$$

which limits the speed with which different regions of the emerging superfluid become correlated. Above, $\xi_0 / \tau_0$ evaluates to approximately 70 m/s but second sound reaches this velocity only quite far from $T_\lambda$. In the vicinity of $T_\lambda$ $s$ is much less, so that the “second sound horizon” can be quite a bit smaller than the size of the sample. Second sound gives the rate of propagation of perturbations of the density perturbations of the superfluid. Hence, it also limits the rate at which the order parameter will be able to adjust. In this sense, one is able to reproduce the “acausal” nature of the cosmological phase transitions in the superfluid He$^4$ better than in the other systems, especially where the transport of heat is involved (as is the case in liquid crystals and will likely be the case in superconductors).

The schematic quench trajectory (which proceeds along the isentrope) is shown in Fig. 4. Figure 5 illustrates the transformation of the effective (Landau-Ginzburg) free energy, as well as its hypothetical effect on the configurations of the order parameter. The aim of this section is to estimate the density of the vortex lines left behind by the quench. To do this, we shall rely on the critical slowing down—the behavior of the relaxation time scale $\tau$ in the vicinity of $T_\lambda$. As we have seen in the previous section (and as is confirmed by the experiments), $\tau \sim 1/|\epsilon|$. Thus, in the course of the quench the time scale on which the order parameter can adjust to the new thermodynamic parameters (especially to the new value of $\epsilon$!) is becoming very long in the vicinity of the critical temperature. As a result, two regimes can be distinguished: (i) Sufficiently far from $T_\lambda$ the relaxation time scale $\tau$
is much smaller than the time on which the quench is proceeding. In this *adiabatic* regime order parameter will be characterized by an equilibrium configuration with the correlation length $\xi$ determined (to an excellent approximation) by the instantaneous value of $\epsilon$. By contrast, very near $T_\lambda$ the equilibrium relaxation time scale will be much larger than the time spent by the system with the corresponding value of $\epsilon$. As a consequence, we can define: (ii) The *impulse* region, in which $\tau$ is so large (because of the critical slowing down) that the configuration of the order parameter will be in effect immobilized on the timescale of interest.

The boundary between these two regimes will occur at the freeze-out time $\hat{t}$. We can compute it by assuming that, in the vicinity of $T_\lambda$, the relative temperature $\epsilon$ is approximately proportional to time:

$$\epsilon = t / \tau_Q . \quad (3.2)$$

Quench timescale $\tau_Q$ can be controlled by the rate at which the pressure is lowered. Critical temperature is reached at $t = 0$.

Freeze-out time is then set by the equality:

$$\tau(\hat{t}) = \hat{t} , \quad (3.3)$$

which, given Eq. (3.2) above, as well as $\tau = \tau_0 / |\epsilon|$, yields:

$$\hat{t} = \sqrt{\tau_0 \tau_Q} . \quad (3.4)$$

Consequently, the transition between the adiabatic and impulse regimes occurs (twice) during the quench at the relative temperature:

$$\dot{\epsilon} = \epsilon(\hat{t}) = \sqrt{\tau_0 / \tau_Q} . \quad (3.5)$$

Here, $\tau_0 \simeq \hbar / \alpha' \simeq 10^{-11}$ s, with more careful estimates yielding $\tau_0 = 8.5 \cdot 10^{-12}$ s. The fluctuating configuration of the order parameter is frozen out at $\dot{\epsilon} > 0$, above $T_\lambda$. When the
dynamics are “restarted” again with $\dot{\epsilon} < 0$, the parameter $\alpha$ is also negative, which means that the symmetry breaking has already occurred and topological defects are “frozen out.”

The characteristic correlation length $\hat{\xi}$ corresponding to $\ell$ and $\dot{\epsilon}$ above will decide the (initial) density of the vortex lines through the approximate formula:

$$\ell_0 = \hat{\ell} \approx \hat{\xi}^{-2} ,$$

which is based on the idea that typically a piece of the vortex line will fall within a $\hat{\xi}$-sized domain (Kibble, 1976; 1980; Vilenkin, 1981).

Correlation length diverges near the phase transition temperatures:

$$\xi(\epsilon) = \xi_0 / |\epsilon|^{\nu} .$$

In the mean field theory $\nu = 1/2$ [see Eq. (2.32)] and $\xi_0 = 5.6$ Å provide an acceptable fit to the data. However, experiments seem to point to a better accord with the renormalization group prediction for $\nu = 2/3$ and a somewhat different $\xi_0 = 4$ Å to go with it (Ahlers, 1976; Goldenfeld, 1992). Consequently, the estimated value of the initial vortex line density is given by

$$\ell_0 \approx \xi_0^{-2} \left( \frac{\tau_0}{\tau_Q} \right)^\nu .$$

The first obvious remark is that—because of the smallness of $\xi_0$—even for relatively slow quench time scales the anticipated density of vortex lines is enormous. The second comment is that this very large initial vorticity is likely to disappear on a rather rapid time scale. This is because the density of vortex lines decays approximately as (Vinen, 1957):

$$\frac{d\ell}{dt} = -\chi \frac{\hbar}{m} \ell^2 = -\gamma \ell^2 ,$$

where the Vinen parameter $\chi$ is a dimensionless constant. Thus, $\ell$ will decrease rather quickly, with the time-dependent density given by

$$\ell = \frac{\ell_0}{1 + \gamma \ell_0 t} .$$
This process is also reminiscent of the dynamics of cosmic string network, although absence
of cosmological expansion, the differences in the equation of motion of vortex lines, etc.,
make this a rather distant analogy.

Our derivation of the preferred scale $\hat{\xi}$ and the resulting initial density of topological
defects can be briefly summed up by noting that we are looking for the instant when the
quench becomes effectively instantaneous. The correlation length at that instant will be
obviously “frozen out” and will set the initial scale of the vortex line network in the regime
where the symmetry of the order parameter is broken.

There are two other ways of deriving the estimate of the characteristic scale in the
course of second-order phase transformations which appeal to somewhat different physical
input but lead to the same estimate of $\hat{\xi}$. We shall consider them now as they may be of
heuristic value to the readers.

To describe the first of them, we compare the velocity with which the correlation
length would have to increase to always maintain its equilibrium value,

$$v_\xi = \frac{d\xi}{dt} = \frac{d\xi}{d\epsilon} \dot{\epsilon}, \quad (3.11)$$

with the velocity of the second sound or, more generally, with the speed of propagation of
the perturbations of the order parameter,

$$s = \xi(\dot{\epsilon})/\tau(\dot{\epsilon}). \quad (3.12)$$

Simple algebra shows that $v_\xi$ would have to increase faster than $s$ when $\epsilon$ is smaller than
the characteristic value $\dot{\epsilon}$ which satisfies equation:

$$v_\xi(\dot{\epsilon}) = s(\dot{\epsilon}), \quad (3.13)$$

or

$$\xi_0 \dot{\epsilon}^{-3/2}/2\tau_0 Q \approx \xi_0 \dot{\epsilon}^{1/2}/\tau_0. \quad (3.14)$$

This last equality leads to Eq. (3.5), providing we ignore numerical corrections of order
unity. The physical implication of this way of obtaining $\dot{\epsilon}$ and $\hat{\xi}$ are straightforward: $\hat{\xi}$
is as large as the dynamics of the order parameter (characterized by $s$) and the quench timescale $\tau_Q$ allow it to become.

Freezeout temperature $\hat{\epsilon}$ can be also obtained by comparing $\xi$ with the size of the sonic horizon $h$—the distance over which perturbations of the order parameter can propagate in the course of a quench. This distance is set by

$$h = \int_0^t s(t) \, dt \simeq s(\epsilon) \cdot \epsilon \cdot \tau_Q .$$

The system will emerge in the broken symmetry phase with the correlation length no larger than the sonic horizon. This will be associated with the relative temperature given by

$$h(\hat{\epsilon}) = \xi(\hat{\epsilon}) ,$$

which, again, after some straightforward algebra, leads (approximately) to Eq. (3.5).

The three above arguments lead to the same conclusion. This is because they are essentially equivalent. The only difference between them is the manner in which they all appeal to the same concept of two different regimes of the quench—(i) slow, nearly adiabatic for $|\epsilon| > \hat{\epsilon}$ and (ii) almost instantaneous, nearly impulse for $|\epsilon| < \hat{\epsilon}$—by respectively comparing timescales, velocities, and distances characterizing dynamics of the order parameter in the vicinity of the critical temperature $T_c$.

It may perhaps also be useful to note that the Vinen equation, Eq. (3.9), can be reexpressed as an equation for the coherence scale of the order parameter phase, $D$, by simply defining

$$D(t) \simeq 1/\sqrt{\ell(t)} .$$

$D(t)$ is the size of the domains over which the order parameter has similar phase. Using Eqs. (3.9) and (3.17), one easily obtains

$$2 \frac{dD}{dt} = \frac{\gamma}{D} .$$
This is, of course, readily transformed into

\[ \frac{d}{dt}(D^2) = \gamma, \quad (3.19) \]

which immediately yields the solution

\[ D(t) = D(0) \cdot \sqrt{\gamma t}. \quad (3.20) \]

This rewriting of the equation for the decay of vortex lines in superfluid helium in the language of domain size growth offers a different perspective on the evolution of the order parameter in the post-quench period. It also establishes a connection between the evolution of the vortex line tangle in He\(^4\) (see Donnelly, 1991) and the more general problem of evolution of the domain size in the systems with nonconserved order parameter. There, on the basis of quite general considerations, one can establish Eq. (3.20) [except for the case of vortices/monopoles in two dimensions, when \( D(t) \sim (t/\ln t)^{1/2} \)] (Bray, 1994).

**Activation Mechanism**

The process we have ignored so far in the discussion—thermal activation—was initially believed to be essential in generating topological defects (Kibble, 1976). As we shall see below, this expectation does not seem to be borne out in the case of the superfluid phase transition in He\(^4\) [nor was it anticipated in the original experimental proposal (Zurek, 1984, 1985, 1993)]. To see why vortex line density is unlikely to reach values mandated by the thermal activation processes at the Ginzburg temperature, we consider the initial vortex line network laid down at \( \hat{t} \) by the freezeout of the order parameter. Approximately 70% of the vortex lines will be in a form of a long string, with only the remaining 30% in the form of loops, the smallest of which have size \( \hat{\xi} \). This follows from the numerical simulations (Vachaspati and Vilenkin, 1984), and approximately coincides with the fraction of nonintersecting random walks, although the precise numbers depend on the lattice as well as on the distribution of the sizes of domains in a way which is not yet understood (Hindmarsh and Kibble, 1995). Let us now suppose that—as will be typically the case—we have stopped the quench at some temperature \( T \) below the freeze-out temperature.
\( \hat{T} = t(\hat{t}) \), so that the original scale \( \hat{\xi} \) of the defect network substantially exceeds \( \xi(T) \).

Thermal fluctuations will generate loops of size \( \xi \), which will be typically smaller (much smaller) than the pre-existing structures. In particular, the long string can be modified due to the small-scale warping by the addition of perturbations on the scale \( \xi \); but this process has an essentially diffusive character, so it will proceed rather slowly. By contrast, the long string has a total energy which is much larger than the Ginzburg energy \( E_G \), Eq. (2.41):

\[
E_V = E_G \cdot (\ell/\xi) \cdot \mathcal{V} \approx \left( \xi/\hat{\xi} \right)^2 \cdot \left( \alpha^2/2\beta \right) \cdot \mathcal{V}.
\]

(3.21)

Above, \( \mathcal{V} \) is the volume of the container. This equation is easy to understand—it can be readily rewritten as

\[
E_V = L \cdot \xi^2 \cdot (\alpha^2/2\beta) \sim L \cdot \varsigma,
\]

(3.22)

where \( L \) stands for the total length of the string. Clearly, for any macroscopic-sized volume \( \mathcal{V} \) of the superfluid, energy \( E_V \) will be much in excess of the thermal energy. Consequently, while thermal activation energy may suffice to generate small-scale ringlets of the vortex line, it is unlikely to elongate the string. Indeed, if anything, the long string may begin to straighten before the quench is completed. This process will counteract—and will most likely even outweigh—thermal activation events which would act to increase vortex line density.

Consequently, it is plausible that the dominant vortex line formation process in the course of a quench will be the freezeout of the order parameter (Zurek, 1984, 1985, 1993).

**Comparison with the He\(^4\) Experiment**

In the recent experimental implementation (Hendry *et al.*, 1994, Hendry *et al.*, 1995) of the cosmologically inspired vortex line formation scenario (Zurek, 1984, 1985, 1993), McClintock and his colleagues of the Lancaster University have carried out the quench through the \( \lambda \)-line, approximately along the isentropic trajectories shown in Fig. 6. As
anticipated in the “freezout” scenario of defect formation, this led to a copious defect production with the initial line density, estimated to be

$$\ell_i \gtrsim 10^{13} \text{ m}^{-2} = 10^9 \text{ cm}^{-2}$$

(3.23)

in the quench which crossed the $\lambda$-line (Hendry et al., 1995). This is certainly a very large vortex line density. If a comparable density were to be generated by rotation of the ($\sim 1$ cm diameter) sample they have used, angular velocities of $\sim 4 \cdot 10^5$ radians per second would be required.

It is possible to estimate the quench time scale $\tau_Q$ from the change of relative temperature ($\Delta \epsilon \sim 0.1$) and from the time ($\Delta t \sim 3$ ms) it took to complete the pressure drop:

$$\tau_Q \approx \Delta t / \Delta \epsilon \approx 30 \text{ ms}.$$ 

(3.24)

Using this estimate of the quench time scale in the equation (3.8) we have just derived, we are led to predict

$$\ell_{LG}[m^{-2}] \approx 3 \cdot 10^{13} / (\tau_Q/100 \text{ ms})^{1/2},$$

(3.25)

when Landau-Ginzburg scaling is employed. A somewhat more modest

$$\ell_{RG}[m^{-2}] \approx 1.2 \cdot 10^{12} / (\tau_Q/100 \text{ ms})^{2/3}$$

(3.26)

obtains for the renormalization group scaling (Zurek, 1995). As it was already pointed out in the commentary (Zurek, 1994) on Hendry et al. (1994) [where the estimate was evaluated from Eq. (3.26) for $\tau_Q = 30$ ms and compared with the then available lower bound of $\ell_i \gtrsim 10^{11} \text{ m}^{-2}$], the agreement between the (order-of-magnitude) estimates offered by the theory and dramatic experimental results is impressive. Of course, more could be done on both theoretical and experimental fronts. Indeed, progress on the experimental front may be more rapid (for example, if the proposals listed in Hendry et al., 1995, are implemented) than in the theoretical treatment of the problem (since microphysical theory of superfluidity in He$^4$ is still missing).
A few remarks on both the experimental side (where I will largely echo discussion of Hendry et al., 1995) as well as on theoretical aspects of the treatment are nevertheless in order. In addition to: (1) the quench through the $\lambda$-line, McClintock and his colleagues have carried out two other types of quenches; (2) quenches far away from $T_\lambda$; and (3) quenches starting just below (few milli-Kelvins) the phase transition temperature. As we have already noted, copious production of vortex lines was detected in type (1) quenches. By contrast, no detectable vortex line production was detected in quenches of type (2) which never approached $T_\lambda$. This is reassuring, as it shows that vortex line creation can occur only where the transition crosses the $\lambda$-line and freezes-out the pre-existing fluctuations of the order parameter. However, quenches of type (3) did create detectable vorticity, although in the amounts which are significantly lower than for the $\lambda$-line crossing quenches of type (1).

This is an intriguing observation. At first, one might be tempted to appeal to thermally generated seed vortices which could exist in the “Ginzburg” regime, but—for the reasons I have already outlined—it seems unlikely that long (i.e., more than a few correlation length in diameter) vortex lines may exist a few milli-Kelvin below $T_\lambda$.

If vortices detected following the type (3) quench from just below the $\lambda$-line are not generated by thermal fluctuations alone, what else can they be? And why are they detected in type (3) quenches but do not appear when the quench starts far from the $\lambda$-line? I believe the crucial clue to the interpretation of this phenomenon may be the remark of Hendry et al. (1995), which attributes appearance of these extraneous vortices to the flows generated in He$^4$ by the pressure quench. The difference in the generated vorticity in type (2) and in type (3) quenches may be due to the different cost—measured in the vortex line tension $\varsigma$, Eqs. (2.37) and (2.38)—of creating topological defects. Thus, if the same energy $\Delta E_{\text{STIR}}$ were available in the stirred-up superfluid,

$$\ell_{\text{STIR}} \sim \Delta E_{\text{STIR}}/\varsigma$$  

(3.27)
of vortex line length could be created. This simple reasoning would predict that the stirred-up vortex line density should scale (to the leading order) as

$$\ell_{\text{STIR}} \sim 1/\rho_s ,$$  \hspace{1cm} (3.28)

where we have recognized that the vortex line string tension $\varsigma$ falls off with $\rho_s$ (and where we have ignored logarithmic terms; if they were taken into account, one would be led to the equation: $\ell_{\text{STIR}} \cdot \rho_s \sim (-\ln \ell_{\text{STIR}}\xi^2)^{-1}$, where both the superfluid density and the correlation length $\xi$ are evaluated near the initial point on the quench trajectory).

This simple discussion ignores many effects which could be relevant, but it does have an advantage of being testable: Quenches along the trajectories which differ only in the initial, near-$T_{\lambda}$ points, should allow one to verify (or prove wrong) Eq. (3.28) above, perhaps with the logarithmic corrections.

This discussion leads us to one further remark about the vortex line density generated in quenches which do cross $\lambda$-line. It is conceivable that the vorticity generated by the cosmological mechanism in the course of the quench may be further amplified by the inadvertent stirring. If this were indeed the case [and discussion in Hendry et al. (1995) does admit such possibility], then the direct comparison of the predictions of our discussion [e.g., Eq. (3.8)] and of the experimental results must be taken with a grain of salt.

It would be therefore quite important to have—in addition to the lower bound given by Hendry et al., 1995 [Eq. (3.23)]—to also have an experimental upper bound. In view of the rapid evolution of the vortex line density [Eq. (3.9)], it might be useful to attempt setting a thermodynamic upper limit on $\ell_i$. After all, decay of the vortex line density will convert string tension energy into the heat deposited in the superfluid, which should allow one—especially for $\ell_i \gg 10^{13} \text{ m}^{-2}$—to set limits on the initial $\ell_i$, simply by monitoring the temperature of the superfluid. The anticipated total temperature increase as a result of decay of the vortex line density $\ell$ can be estimated:

$$\Delta T = \ell \varsigma / Q .$$  \hspace{1cm} (3.29)
Above, $Q$ is the appropriate specific heat of He$^4$ at the temperature at which the decay of the network is taking place. Whether detecting corresponding temperature differences resulting from Eq. (3.19) may be possible on the scale rapid enough and with the accuracy sufficient to make the results useful is, of course, a rather difficult experimental question.*

If one were certain that the only way in which the quench could deposit heat in the superfluid was by dissipation of the string tension energy locked in the vortex line network generated in the course of the quench, one could amplify the effect by traversing the $\lambda$-line many times. Then the accumulated heat would be due to the small departures from the isentrope which are caused by depositing some of the energy in the ordered configurations of the Bose condensate, which—following each traversal of the $\lambda$-line—are dissipated in the destruction of the vortex line network. Among the assumptions which are made here are an excellent isolation of the sample as well as a sensible guess that the energy locked in the vortex network after each quench comes from the mechanical energy available in this nonequilibrium process rather than from thermal fluctuations (which, again, relates to the difference between freeze-out and thermal activation).

In summary, one should stress that all of the caveats listed above do not weaken the main conclusion of Hendry et al. (1994 and 1995). Very abundant vortex line creation

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* Using Eq. (3.29) above and a remark in Section 3 of Hendry et al. (1995) that “Fast expansions seem to follow the quasistatic ones to good approximation, but, from a common starting temperature, reach a final temperature which is typically several mK higher” we can set an upper limit on $\ell$:

$$\ell_{15}[m^{-2}] < 0.7\Delta T[mK].$$

Thus $\ell_{15} \lesssim 1$ (or $\ell \lesssim 10^{15} m^{-2}$) is a plausible upper limit on the vortex line density. Moreover in view of the inadvertant “stirring” in the course of the quench reported by Hendry et al. (1995), it might be a generous overestimate of the vortex line density attributable to “pure” quench.
is observed in quenches which cross the \( \lambda \)-line. Estimates based on the application of the cosmological scenario of topological defect creation are consistent with the detected vortex line density, providing that one relies on the idea of the freeze-out time \( \hat{t} \) rather than on thermal activation. Indeed, the final points of all the quenches are well within the Ginzburg region of He\(^4\), which would imply—if one were to rely on the vortex line creation by thermal activation—that the final vortex line density should not decay but, rather, that it should reach values set by the correlation length associated with the end-of-quench point. This is clearly not the case. Further experimental study of the freeze-out mechanism should involve limiting the contribution of “stirring up” of the superfluid in the course of the quench as well as varying the quench time scale to verify Eq. (3.8). This course of research has been already mapped out by the Lancaster University group (Hendry et al., 1995).

**Quenching out Defects in Superfluid He\(^3\)**

The first demonstration of defects formation in course of the phase transition into the B phase of superfluid He\(^3\) has been just reported by the Helsinki group and their collaborators (Ruutu et al., 1996). The transition occurs in a cigar-shaped, 100 \( \mu \)m sized volume in a rotating container of the B-phase of He\(^3\), which is heated above the thermal temperature by the decay products of a thermal neutron. As this volume cools and re-enters the broken symmetry phase, string-like defects form with the density which is consistent with the freezeout distance \( \hat{\xi} = \xi_0(\tau_Q/\tau_0)^{1/4} \) of approximately 1 \( \mu \)m. Ginzburg relative temperature for the superfluid He\(^3\) is very small, and corresponds to the size of the coherence length much larger than the volume of the “cigar” in which the phase transition occurs. Thus, again, the result of the experiment is inconsistent with the activation mechanism, but appears to conform with the freezeout scenario.

The superfluid is slowly rotating. As a result, sufficiently large vortex loops generated in the small volume that underwent the phase transition are stretched by the Magnus force. This eventually tends to transform them into rectilinear vortex lines parallel to the axis of
rotation. They are subsequently pulled into the center of the container, where they can be detected by NMR. The distribution of loop sizes obtained in this manner is consistent with the scale invariant distribution (see, e. g., Vachaspati and Vilenkin, 1984).

This very exciting experimental development complements work on superfluid He\textsuperscript{4} (where the density of vortex lines is typically much higher, but where they are also harder to detect). It also opens up the possibility of pressure quenches into either of the He\textsuperscript{3} phases. In particular, phase diagram of He\textsuperscript{3} allows for a variety of quench trajectories (see Fig. 7), some of which raise the possibility of addressing other interesting issues such as the creation of topological defects in tuneably weak first order phase transformations, or even in a quick succession of the second and first order phase transitions.

**Quench Into Bulk Superconductors**

Superconductors of type II have a Landau-Ginzburg form of the free energy, Eq. (2.43), and a negative surface energy (so that they can accommodate vortex lines). We can therefore repeat the argument we have put forward for the superfluid phase transition to compute the density of vortex (flux) lines which obtains as a result of a rapid quench. Again, we shall rely on the freeze-out time scale \( \hat{t} \) and the corresponding correlation length of the order parameter \( \hat{\xi} = \xi(\hat{t}) \). The differences between the case of He\textsuperscript{4} and superconductors will be not in the key ideas behind the estimates of the vortex line density but, rather, in the different experimental requirements, which will make superconducting bulk quenches more difficult to carry out. Moreover, the consequences of the superconducting quenches may not be easy to investigate (although, as we shall see in the next section, loop geometry may help!).

As before, we imagine the quench proceeding on a time scale \( \tau_Q \), with the distance away from the phase transition (measured by the relative temperature \( \epsilon \)) changing as \( \epsilon(t) \approx t/\tau_Q \), Eq. (3.2). As the relative temperature varies, the relaxation time scale is also changing, \( \tau = \tau_0/|\epsilon| \), so that sufficiently near the phase transition instant (where \( \epsilon = 0 \)), dynamics of the order parameter becomes too sluggish to adjust the correlation length \( \xi \).
to the equilibrium value given by $\xi_0 \sqrt{|\epsilon|}$, Eq. (2.48). The switch from the adiabatic to the impulse regime will happen (as it was the case for superfluids) at the freeze-out instant:

$$\hat{t} = \sqrt{\tau_0 \tau_Q},$$

(3.30)

where $\tau_0$ is the characteristic time given by Eq. (2.47). Consequently, we can evaluate

$$\hat{t} \cong 1.225 \sqrt{\tau_Q/T_c} \ [\mu s],$$

(3.31)

where $\tau_Q$ is in seconds and the critical temperature is expressed in Kelvins. The frozen-out correlation length is then given by

$$\hat{\xi} = 10^{-2} \left( \xi_0/1000 \text{ Å} \right) \tau_Q^{1/4} \text{[cm]}$$

(3.32)

and corresponds to the critical relative temperature:

$$\hat{\epsilon} = \epsilon(\hat{t}) = 1.225 \cdot 10^{-6} / \sqrt{T_c \tau_Q}.$$  

(3.33)

This $\hat{\epsilon}$ is rather small: Therefore, it might not be easy to go through the phase transition both sufficiently quickly and sufficiently uniformly to validate the standard prediction

$$\ell = 1/\hat{\xi}^2 = 10^4 (1000 \text{ Å}/\xi_0)^2 \tau_Q^{1/2} \text{[cm}^{-2}] .$$

(3.34)

The anticipated initial flux line density is therefore much smaller than was the case for superfluids. The decrease in estimated $\ell$ is mainly due to the increase of typical $\xi_0$, which, for a typical superconductor, is two orders of magnitude bigger than for the superfluid He$^4$. One could, of course, increase anticipated initial $\ell$ by selecting to work with high-temperature superconductors (which have much smaller $\xi_0 \sim 10$ Å). This strategy may be worth pursuing, although the switch to high-$T_c$ materials is likely to be accompanied by a specific set of problems (but, possibly, also by some advantages). In particular, the Landau-Ginzburg theory may not be as accurate for the reasons we have explored before.
Moreover, our estimates of $\tau_0$ will not apply, as they are based on the microscopic theory (BCS) which does not apply to high-$T_c$ superconductors.

Let us now return to the experimental complications. To begin with, the pressure quench will not result in as big a change of critical temperature as is the case for $T_\lambda$ in He$^4$. There is, of course, some sensitivity of critical temperature in superconductors on applied pressure (see, e.g., Lynton, 1963), but the effect is rather small and may be difficult to exploit experimentally. Critical temperature in high-$T_c$ material is more sensitive to pressure (Neumeier and Zimmerman, 1993; Neumeier, 1994). This may be another reason (in addition to smaller $\xi_0$) which could make them interesting for our purposes.

Temperature quench—rapid cooling of the sample—may be nevertheless the most practical way to precipitate the phase transition. Unfortunately, this is a diffusive process. Therefore, it is likely to be relatively slow. Moreover, heat transport is (of course) driven by temperature gradients. Thus, limiting the gradient of the relative temperature to much less than the value set by the ratio:

$$\hat{g} = \hat{\epsilon}/\hat{\xi},$$

may not be an easy task. Gradient $\hat{g}$ evaluates to:

$$\hat{g} \approx 1.225 \cdot 10^{-4} (1000 \text{Å}/\xi_0) T_c^{-1/2} \tau_Q^{-3/4} \text{[cm}^{-1} \text{]},$$

when Gorkov equations (Gorkov, 1959; Tinkham, 1985) apply. Again, one may be tempted to consider high-$T_c$ materials, but perhaps a more practical suggestion would be to work with thin layers of superconductors. This 2-D strategy allows one to transport the heat in the direction perpendicular to the plane in which the superconducting state will form. Thus, one may be able to violate the constraint imposed by Eqs. (3.35)–(3.36) without invalidating the reasoning which has led to the estimate of $\hat{\xi}$, Eq. (3.32). This two-dimensional sample strategy may have one additional advantage: The interior of the bulk superconductor is difficult to probe. However, flux lines can be detected when they emerge from the sample. In a sample with a thickness of the order of $\hat{\xi} (\sim 10^{-2} \text{ cm})$ or even
somewhat larger, all of the vortex lines can be “counted” where they pierce the surface. Moreover, using 2-D geometry may help in slowing down annihilation of vortex lines as one may be able to supply in a controlled fashion sufficiently many pinning sites to prevent or at least significantly impede vortex line migration.

The estimate of the vortex line density is now—in the 2-D case—given of course by Eq. (3.24) but with a somewhat revised interpretation. We are led to expect a flux line to emerge from within the area of the order of $\hat{\xi}$. Thus, the surface density of the vortex line endings is given by

$$\Sigma_i \approx 1/\hat{\xi}^2.$$  \hspace{1cm} (3.36)

Scattering of neutrons from a sample undergoing a superconducting phase transition may be a way to peer inside a truly 3-D fragment of material. Neutrons have a magnetic moment and will scatter from the network of flux-filled vortex lines, which should in principle allow one to deduce vortex line density and perhaps even get some information about their geometry.

It has also been suggested (Rudaz, Srivastava, and Varma, 1994) that an electron microscope could be used to image individual flux lines, although it is harder to imagine how one could use it to track a network of lines in the course of the phase transition.

So far we have ignored one more obvious complication. Superconducting vortices are associated with the gauge field ($\vec{B}$) which has to be screened in the experiments. Moreover, the relative importance of the order parameter and of the gauge field in the phase transitions with the local gauge symmetry breaking is not entirely obvious (Rudaz and Srivastava, 1993; Kibble and Vilenkin, 1995). This last remark may be regarded more as a motivating factor (rather than as a reason for discouragement): There are real, deep, and interesting questions and scenarios of defect formation in theories with local gauge symmetries. It is however also conceivable that transport of the field through the material on the verge of becoming a superconductor may complicate matters in a more mundane manner.
4. COSMOLOGICAL EXPERIMENTS IN ANNULAR GEOMETRY

Let us consider a system undergoing a symmetry-breaking phase transition. In the vicinity of the freeze-out instant, \( \hat{t} \), the fate of its order parameter is sealed: Phase differences in the neighboring domains are set. Hence, the distribution of the topological defects is also essentially decided. Each vertex defined by the domains with the phase differences which add to \( \sim 2\pi \) will become a section of a topological defect—a superfluid vortex, a flux line in a type II superconductor, a disclination line in a liquid crystal, or a cosmic string. This phase difference can be computed by adding the phases along a path which circumnavigates the vortex and which has a radius comparable to the size of the frozen-out domains.

The same postulate of the independent choice of the phase in independent domains also allows one to compute the total phase difference along a path with much larger circumference \( C \), which traverses many (\( N \)) domains. The total phase difference along such a path is given by

\[
\Delta \theta_C \approx \sqrt{N} = \sqrt{C/\hat{\xi}}. \tag{4.1}
\]

Already in the bulk experiments we have discussed so far this statement is of some interest, as it implies that the vortex lines will tend to be more anticorrelated than would be the case if their directions were assigned at random. This is easy to see. For, when the vortex line orientations are selected randomly, the net circulation along a path defined by a circle of size \( r \) (area \( \pi r^2 \)) would increase with the square root of the number of strings enclosed, and hence

\[
\Delta \theta \sim \sqrt{\pi r^2/\hat{\xi}^2}. \tag{4.2}
\]

This implies \( \Delta \theta \) increasing with the circumference rather than with the square root of the circumference [Eq. (4.1)], as the more careful analysis implies. This discrepancy between the two predictions, Eqs. (4.1) and (4.2), can be, of course, readily settled in favor of Eq. (4.1), not just by the discussion we have already carried out but by an additional
observation (Vachaspati and Vilenkin, 1984) that the string network generated in the course of a rapid phase transition will consist (∼ 70%) of a long string which will have to cross any surface spanned by $C$ in opposite directions more often than a random collection of vectors. The rest (30%) of the length is due to (mostly small) loops which will typically not contribute to $\Delta \theta_C$ unless $C$ passes through their centers (i.e., unless they are "strung" on $C$ like beads).

The purpose of this section is to discuss how Eq. (4.1) can be tested directly in both superfluids and in superconductors.

**Superfluid in an Annulus**

Consider quench in an annular container of superfluid He$^4$ (see Fig 8). The phase difference locked out in the course of the quench is given by Eq. (4.1) with

$$\hat{\xi} = \xi_0/|\hat{\epsilon}|\nu,$$  \hspace{1cm} (4.3)

where $\hat{\epsilon}$ is given by $\sqrt{\tau_0/\tau_Q}$, Eq. (3.5), and $\nu = 2/3$ in the renormalization group treatment of He$^4$. Therefore, in accord with the equation (2.27) which relates the phase gradient with the velocity of the superfluid, one expects (after the quench) a flow in a random direction with the velocity of

$$v = \frac{\hbar}{m} \cdot \frac{1}{C} \sqrt{\frac{C}{\hat{\xi}}} = \frac{\hbar}{m} \sqrt{\frac{1}{C\hat{\xi}}}. \hspace{1cm} (4.4)$$

This is obviously an intriguing prediction, especially since it evaluates to measurable velocities. To estimate the resulting $v$, we can further calculate

$$v = \frac{\hbar}{m} \frac{|\hat{\epsilon}|^{\nu/2}}{\sqrt{\xi_0 C}} \approx 0.8 \left(\frac{\tau_0/\tau_Q}{\sqrt{C}}\right)^{\nu/4} \text{[cm/s]}.$$  \hspace{1cm} (4.5)

Above, $\tau_0 \sim 8.5 \cdot 10^{-12}$ s, and $C$ is measured in centimeters. Thus, for $\tau_Q$ of the order of milliseconds and $\nu = 2/3$ one obtains

$$v \simeq 0.4 \left(\tau_Q[\mu s]\right)^{-1/6} / \sqrt{C[\text{cm}]} \text{[mm/s]}. \hspace{1cm} (4.6)$$

This is certainly a macroscopic (if not very large) velocity. Moreover, Eq. (4.6) is likely to be an underestimate, since Eq. (4.1) contains an implicit assumption that the phase
difference between the domains separated by $\hat{\xi}$ is approximately a radian, while the more usual assumption would be to set that difference at $\pi$ or $2\pi/3$, which would increase the estimate given by Eq. (4.6) above 1 mm/s. A still higher estimate of the quench-generated velocity (probably of the order of cm/s) would follow if we used a lower bound on the quench-generated vorticity implied by the experimental results of the Lancaster group (Hendry et al., 1995) to infer the relevant $\hat{\xi}$.

In any case, locking out a finite “phase around the loop” as a result of a rapid symmetry-breaking phase transformation is the essence of the cosmological scenario for the generation of topological defects. Performing the experiment in an annulus is bound to be a challenge, but the dramatic prediction we have outlined above makes it a worthwhile effort. Moreover, the locked-out phase in the loop has one additional advantage over the vortex creation in the bulk. The resulting winding number $n_w$:

$$n_w = \Delta \theta C / 2\pi$$

(4.7)

is [to an excellent approximation (Tilley and Tilley, 1986)] independent of time in the superfluid phase. Thus, quench can set up a persistent superflow with a macroscopic, measurable velocity. This velocity is directly related to the frozen-out domain size $\hat{\xi}$ through Eq. (4.4) and can be—in contrast to the rapidly decaying vortex line density $\ell$ [see Eqs. (3.8)–(3.10)]—measured at leisure long after the quench.

This discussion brings up a few questions. Let us start with an apparent paradox: How can a system which did not rotate before the phase transition acquire a finite angular momentum [corresponding to the velocity of $v$, Eqs. (4.4)–(4.6)] as a result of an axisymmetric pressure quench? A facile answer to this question would be to simply point out that the considerations concerning the order parameter of the superfluid above do not respect the law of conservation of angular momentum. Thus, for example, when an annular container with superfluid He$^4$ is cooled, the density of the superfluid is changing; and yet it is known that the phase difference (and, consequently, the velocity but not the
superfluid angular momentum) remains conserved (Tilley and Tilley, 1986; Donnelly, 1991). This is certainly a well-established fact, but it does not settle the basic issue underlying the angular momentum generation paradox I have sketched above. In other words, it does not say when and to what accuracy the winding number $n_w$ along some closed path $C$ will be a “good” (conserved) quantum number. For instance, in a bulk superfluid, for a fixed closed $C$, $n_w$ is certainly not conserved in the course of the evolution of the string network which follows the quench. This is because strings can move across $C$, a process which obviously alters $n_w$.

By contrast, in an annulus, cooling (or heating) of the superfluid will lead to a change of the superfluid angular momentum (as a result of a change of the Bose condensate density $\rho_s \sim |\Psi|^2$) but the global conservation law will be nevertheless respected, since the superfluid can “push off” the normal fluid, which will in turn dissipate the excess momentum on the walls of the container.

The first obvious remark in the wake of the above discussion is to note that, in order to be safe from the “bulk” decay of the vortex line network, one must choose the small radius of the annulus to be of the order of $\xi$. Otherwise, the resulting “phase around the loop” will be modified by the vortex lines enclosed within the torus. Thus, one will be forced to work with capillaries, as the typical values of $\xi$ are close to $\sim 10^4$ Å. This narrow annulus may suffer from the possibility of thermally activated transitions (especially near $T_c$), and this process may cause the decay of $n_w$. Perhaps the most likely mechanism for such slow decay of the superflow would be due to creation of vortex lines, which can then migrate across the annulus, thus changing $n_w$ by one unit.

The coincidence of the size of the small diameter of the annulus with the frozen-out correlation length $\hat{\xi}$ contains a crucial clue to the resolution of the angular momentum
generation paradox I have described at the beginning of this discussion. This is because we can now estimate the angular momentum involved in the superflow as

$$J_s = \left( \rho_s \cdot \frac{1}{4} \pi \xi^2 \cdot C \right) \cdot v \cdot r , \quad (4.8)$$

where \( r = C/2\pi \). The only conceivable source for angular momentum \( J_s \) (which, for a typical superfluid \( \rho_s \sim 0.1 \text{ g/cm}^3 \), \( \xi = 10^4 \text{ Å} \), and \( C = 1 \text{ cm} \) is, in accord with Eqs. (4.4)–(4.6), \( J_s \sim 5 \cdot 10^{-11} \text{ g cm}^2/\text{s} \)) would be the angular momentum of Brownian motion, which is given by

$$J_T = (k_B T \cdot M \cdot r^2)^{1/2} . \quad (4.9)$$

Here \( M \) is the mass of the involved material [which is given by \( M = \rho_s \cdot \pi \xi^2 \cdot C/4 \) in Eq. (4.8)]. The above equation can be readily derived when the kinetic energy of rotation (given by \( J_T^2/(2Mr^2) \)) is equated to the thermal \( k_B T/2 \) per degree of freedom. At \( T = T_\lambda = 2^\circ \text{K} \) for the same \( \rho_s \) and \( C \) we can estimate typical \( J_T \sim 10^{-13} \text{ g cm}^2/\text{s} \), which is much less (several orders of magnitude) than \( J_s \). Thus, at the first sight, it would seem unlikely that \( J_T \) could be the origin of \( J_s \).

However, the equation

$$J_s = J_T \quad (4.10)$$

would have to be satisfied not deep in the superfluid phase, where \( \epsilon \sim 1 \) and \( \rho_s \sim 0.1 \text{ [g/cm}^3 \)], as we have assumed above, but, rather, at the freeze-out instant \( \hat{t} \), for the corresponding relative temperature \( \hat{\epsilon} \ll 1 \). Now, it is well known that

$$\rho_s = \rho_s(0)\epsilon^\nu , \quad (4.11)$$

where \( \nu = 2/3 \) in the renormalization group theory and in the experiments (Ahlers, 1976), and \( \rho_0 \simeq 0.36 \text{ g/cm}^3 \). In accord with Eqs. (4.8)–(4.10) we get

$$\rho_s \left( \pi \xi \right) = 4 (m/h)^2 k_B T_\lambda , \quad (4.12)$$
which is to be satisfied at \( \dot{\xi} \). But the left-hand side of Eq. (4.12) is independent of \( \dot{\xi} \) as \( \dot{\xi} = \xi_0 \epsilon^{-\nu} \). Hence, we are led to an unexpected temperature-independent relation:

\[
\rho_s(0) \xi_0 = 4 \frac{m^2}{\hbar^2} \cdot k_B T_\lambda / \pi
\]

between the density of liquid He\(^4\), its correlation length (both at absolute zero), mass of the helium atom, fundamental physical constants \( \hbar \) and \( k_B \), and the critical temperature \( T_\lambda \). The left-hand side of Eq. (4.13), for the data cited by Ahlers (1976) is

\[
\rho_s(0) \xi_0 \approx 1.44 \cdot 10^{-8} [\text{g cm}^{-2}].
\]  

The right-hand side is

\[
4 \left( \frac{m^2}{\hbar^2} \right) k_B T / \pi \approx 1.4 \cdot 10^{-8} [\text{g cm}^{-2}].
\]

This coincidence is striking and inspires confidence in the Brownian motion explanation of the origin of the angular momentum locked out by the quench. Indeed, Eq. (4.13) has been postulated before in the discussion of healing length in superfluid He\(^4\) (Ahlers, 1976; Ferrel et al., 1969).

**Quench in a Superconducting Loop**

From the experimental point of view, trapping of the winding number \( n_w \) by a rapid quench in a loop of a superconducting wire may be simpler than the corresponding experiment in the superfluid He\(^4\). In particular, the detection of the effect—which, in a superfluid He\(^4\) involves measurement of small amounts of angular momentum—should be now much simpler, as it suffices to measure the magnetic field associated with the number of flux quanta trapped during the quench. These first impressions may be indeed valid as far as the experimental situation is concerned; but from the point of view of relating such experiments to cosmology, they are, I believe, too optimistic. As we shall see below, quench in a superconducting loop is complicated by the very presence of the magnetic (gauge) field, which makes the theoretical analysis more difficult and which will also complicate experiments.
We start by, in effect, repeating the “generic” prediction based on the cosmological scenario for the number of flux quanta generated in converse of a rapid quench:

\[ n_\Phi = n_w = \frac{\Delta \theta C}{2\pi} = (2\pi)^{-1} \sqrt{C/\xi}. \] (4.16)

Here \( n_\Phi \) stands for the number of trapped flux quanta. So far, the effect of the gauge fields is ignored. For a loop of radius \( r = 1 \) cm and a frozen-out correlation length \( \xi \sim 10^{-2} \) cm [see Eq. (3.22)] this yields \( n_\Phi \sim 3 \). The corresponding flux would be small but easily measurable by the techniques utilizing SQUID’s (Tinkham, 1985).

To arrive at the above prediction, Eq. (4.16), we have paid attention solely to the order parameter. But the evolution of the magnetic field trapped by the quench may play a role in the final outcome. To consider this possibility, let us first note that the energy associated with the trapped flux \( \Phi \) is nonnegligible:

\[ E_\Phi = \Phi^2 / 2L = n_\Phi^2 \cdot \Phi_0^2 / 2L = n_\Phi^2 E_0. \] (4.17)

Above, \( E_0 \) stands for the energy associated with a single flux quantum \( \Phi_0 = hc/2e \) [Eq. (2.52)] trapped in a loop with the inductance \( L \):

\[ E_0 = \Phi_0^2 / 2L. \] (4.18)

In the circular loop of wire the inductance (measured in Henrys) is given by

\[ L \cong 4\pi \cdot 10^{-9} r \ln(r/a) [H], \] (4.19)

where \( r \) is the radius given in centimeters and \( 2a \) is the diameter of the wire. This results in typical energies:

\[ E_0 \sim 2.5 \cdot 10^{-16} r^{-1} [\text{erg}], \] (4.20)

where we have set \( r/a = 1000 \). Energy of thermal excitations is—for comparison—given by

\[ E_T \sim \frac{1}{2} k_B T_c \sim 7 \times 10^{-17} \cdot T_c [\text{erg}] \] (4.21)
in the vicinity of the superconducting phase transition temperature $T_c$ (which is given in °K).

This comparison—$E_\Phi$ with $E_T$—is relevant because we would like to be able to trace the origin of the trapped quanta to either a “cosmological” freeze out of the order parameter or to the trapping of the thermal or other fluctuations of the electromagnetic field. This second possibility is obviously important, as it is well known that in the course of a slow phase transition in an external field the flux trapped by a superconducting loop is determined by that magnetic field. The above considerations allow us to at least differentiate between the experimentally testable consequences of the two alternatives. Predictions based on the “cosmological” Eq. (4.16) depend only on the quench rate (which determines $\hat{\xi}$) and on the circumference $C$ of the loop. By contrast, trapping of the thermal fluctuations of the field will be affected by the self-inductance $L$, which can be changed without altering $C$ (by, for example, coiling up the loop into a solenoid). Thus, if the trapped flux has its origin in thermal fluctuations of the trapped flux, the expected value of the flux will be given by

$$\frac{\delta \Phi_T^2}{2L} = \frac{1}{2} k_B T . \quad (4.22)$$

By contrast, freeze out of the order parameter leads to the prediction:

$$\left(\frac{\delta \Phi/\Phi_0}{2}\right)^2 = n_\Phi^2 = C/\hat{\xi} . \quad (4.23)$$

The number of trapped quanta should also slowly scale with the quench rate, since $n_\Phi$ in the Landau-Ginzburg theory varies as

$$n_\Phi \sim \tau_Q^{-1/8} . \quad (4.24)$$

The experiment should then be able to tell which of the two alternatives—the one driven by the fluctuations of the order parameter or the one in which the gauge field dominates—is realized “in practice.” This question is of cosmological interest since, as we have already indicated, the opinion concerning overwhelming importance of the order
parameter in the creation of topological defects is not unanimous (Rudaz and Srivastava, 1993; Kibble and Vilenkin, 1995). However, it should be noted that there are significant differences between the physics which is relevant in the cosmological (bulk) context and the one which will play an important role in the superconducting loop experiment considered here. In particular, the gauge field in the bulk phase transition acquires mass—and, therefore, becomes limited in its range to the penetration depth \( \lambda = \kappa \xi \)—as the phase transition is taking place. Thus, the range over which the order parameter may become causally connected will be in any case limited by an exponential to a finite distance, even if the phase information could propagate with the gauge field. By contrast, quench in a loop leaves the magnetic field inside the loop massless. Therefore, if gauge fields were to play a dominant role in setting up relative phases of the broken symmetry Bose condensate, the infinite range and the speed with which different sections of the loop can influence one another could prove significant.

Detailed consequences of this competition between the order parameter (Bose condensate) and the gauge (magnetic) field are difficult to predict in general. We shall explore some of the relevant physical phenomena below. We start by noting that, in addition to the relaxation time \( \tau \) and the quench time scale \( \tau_Q \) (which in turn decides the freeze-out instant \( \hat{t} \)), there is at least one more time scale which limits the speed with which changes in the magnetic field (or Bose condensate) configuration can take place. The loop is an \( R-L \) circuit, so the flux it encloses can change no faster than on a time scale:

\[
\tau_{RL} = L/R. \tag{4.25}
\]

Here \( R \) is the resistance. In the vicinity of the superconducting phase transition, this time scale is rapidly increasing, as \( R \) tends to zero. Moreover, it is dominated by the few sections of the loop which are still normal, because of, for example, slightly different critical temperature \( T_c \) in different sections of the wire.

This situation (and more generally, any configuration of a two-dimensional \( \Psi \) in an approximately 1-D space) can be represented in a diagram shown in Fig. 9. The quantity
of interest in our considerations—known as the fluxoid in the superconductor physics [see Eq. (2.55)]—can be used to define the gauge-invariant winding number:

\[ n_w = \oint_C \left( \nabla \theta - \frac{2\pi}{\Phi_0} \vec{A} \right) \, d\vec{l}. \]  

(4.26)

In effect, \( n_w \) is the number of times the phase of the order parameter wraps around along the loop \( C \).

In equilibrium, the amplitude of \( \Psi \) is set by \( \sigma \), Eq. (2.33). The turns are approximately equally spaced and can be thought of as regular (although their distribution is gauge-dependent). As the phase transition temperature is approached from below, the number of turns is constant, but \( \sigma \sim \sqrt{|\epsilon|} \) becomes smaller, which may eventually allow for changes in \( n_w \). We shall consider processes which can cause changes of \( n_w \) below. Let us begin with a situation where the loop \( C \) is superconducting except for a small section somewhere along its circumference. Then the winding number will not be fixed.

When a system exists in this state for a time long compared to the time scale on which the flux through the loop can change, the configurations of \( \Psi \) will explore distribution determined, on the one hand, by the energy of these configurations (set both by \( \Psi(x) \) and by the magnetic field associated with it) and by the external noise sources which can pump energy into the system both through the magnetic field and through \( \Psi \). When the temperature in the normal section of the wire is lowered (so that the two ends of \( \Psi(x) \) are joined together), the winding number cannot change any more.

An experiment which explored this regime (although with a somewhat different interpretation in mind) has been actually already carried out. Tate, Cabera, and Felch (1984) have heated up a small section of a superconducting niobium loop with a laser. After a laser was turned off, the number of trapped flux quanta inside the loop was measured. The random number of quanta had a dispersion corresponding to the temperature 6.78\(^\circ\)K, which is less than the critical temperature of pure niobium (\( T_c \approx 9.17\,^\circ\)K). They have interpreted this as a result of thermal excitations with the dispersion set by the local
critical temperature in the heated-up section of the loop. The discrepancy in the value of $T_c$ was explained as a result of local fluctuation of the critical temperature in the loop. In view of our preceding discussion, one might venture an alternative explanation. The lower temperature associated with the dispersion of the number of the locked out quanta may be a result of the coupling of the system (including the order parameter) with both the “hot spot” illuminated by the laser and with the heat bath (which was presumably at a still lower temperature than the reported 6.78$^\circ$K).

**Activation Processes in a Superconducting Loop**

The above discussion has prepared us to consider the problem of the phase transition in a loop from the new vantage point with the intuitive understanding based on Fig. 8. The first conclusion is already at hand. When the phase transition occurs so nonuniformly that the last “break” in the continuity of the Bose condensate disappears only well after the time scale needed for equilibration of the rest of the loop ($t > \tau_{RL}$) we can expect a distribution set (or, at least, significantly influenced) by the thermal and other fluctuations driving the system. One of these configurations becomes then topologically “trapped” when the whole loop becomes superconducting.

By contrast, when the phase transition occurs simultaneously along the loop, we recover the picture we have discussed in the cosmological context. Different sections of the loop reach the broken symmetry phase independently. Therefore, at the end of the quench, the flux trapped inside $C$ will be set by the winding number determined by Eqs. (4.1) and (4.16).

It should be noted that the value of the flux is not constrained to the multiple of the number of flux quanta but, rather, by the fluxoid quantization condition, Eqs. (2.55) and (4.26). The two are equivalent only in the limit where the penetration depth $\lambda$ is small compared with the thickness of the wire, $\lambda \ll 2a$. When this is not the case (i.e., when $\lambda > a$), the flux inside $C$ may be quantized in units smaller than $\Phi_0$. This is because in that limit both the velocity of the Cooper pairs and the vector potential are important
(Blatt, 1961; Bardeen, 1961). This can be investigated in the case where the current is approximately independent of the location inside the wire (which is the case when \( \lambda > 2a \)). The velocity of the Cooper pairs is then given by the gradient of the phase which is not compensated by the magnetic flux:

\[
v = \frac{hq}{m^*} = \frac{hq}{2m} .
\]

(4.27)

Here, \( m^* \) is the mass of a Cooper pair and \( q \) is the residual wave vector:

\[
q = \frac{2\pi}{C} \left( n_\Phi - \frac{\Phi}{\Phi_0} \right) .
\]

(4.28)

Above, the integer \( n_\Phi \) is chosen so that \( qC \) is in the interval \((-\pi, \pi)\).

Equilibrium values of the magnetic flux \( \Phi \) through \( C \) will be determined by the minima of the total energy associated with a certain flux and with the kinetic energy corresponding to the velocity of the charge carriers, Eqs. (4.27) and (4.28):

\[
E = \frac{\Phi^2}{2L} + N_C m^* v^2 / 2 .
\]

(4.29)

Here \( L \) is the inductance and \( N_C \) is the total number of Cooper pairs in the loop \( C \). It can be obtained from the (temperature-dependent) density of the Cooper pairs \( n_C^* = |\Psi|^2 \):

\[
N_C = C\pi a^2 \cdot |\Psi|^2 .
\]

(4.30)

Using the above equations, it is straightforward to show that \( E \) is minimized when the flux is given by

\[
\Phi = \Phi_0 \cdot n_\Phi / (1 + E_0/E_K) .
\]

(4.31)

Here, \( E_0 \) is defined by Eq. (4.18) and corresponds to a single trapped quantum of flux, while \( E_K \) is the kinetic energy of the charge carriers for the velocity corresponding to the gradient of phase given by \( 2\pi/C \):

\[
E_K = \frac{1}{2} N_C m^* \left( \frac{h}{m^*} \cdot \frac{2\pi}{C} \right)^2 .
\]

(4.32)
It follows that, in equilibrium, the flux in a loop of a given self-inductance $L$ and circumference $C$ will be quantized in units of

$$\tilde{\Phi}_0 = \Phi_0 / (1 + E_0 / E_K) .$$  \hspace{1cm} (4.33)

Furthermore, in the vicinity of $T_c$ the winding number $n_w = n_\Phi$ can be altered by thermal activation. The potential barrier which separates different integer values of the winding number is approximately given by

$$F_B \approx E_K / 2 ,$$ \hspace{1cm} (4.34)

for the small values of $n_w$. This energy can be easily small enough to be of the order of $k_B T_c$. Thus, in the course of the phase transition and already below $T_c$, $n_w$ can still change and may be driven towards a value set by thermal (or other) fluctuations which couple to the system.

The “unwinding” transition we have described above occurs along all of the circumference $C$. By contrast, one can imagine a localized thermal excitation involving a correlation-length sized section of the superconducting wire. The free energy barrier associated with such a transition is approximately given by the volume in which it takes place times the specific free energy of the symmetric state. The exact expression (see Chapter 7 of Tinkham, 1985) turns out to be

$$\Delta F = \frac{4\sqrt{2}}{3} \cdot \frac{\alpha^2}{\beta} \cdot A \xi ,$$ \hspace{1cm} (4.35)

where $A$ is the cross section of the wire. Typically $\Delta F$ [Eq. (4.35)] is significantly larger than $F_B$, Eq. (4.34). However, the time scale associated with the thermal excitations within one correlation length (given by $\tau$) is likely to be much smaller than the time (probably of the order of $\tau_{RL}$) which will set the “whole loop” rate of transitions guarded against by the barrier $F_B$, Eq. (4.34), we have discussed before. Thus, it is not clear which of these two processes will dominate or whether any of them will have a sufficient rate to reset the
number of trapped quanta from the value determined by the frozen-out fluctuations of the
order parameter, Eq. (4.23), to the estimate of Eq. (4.22), set by thermal fluctuations of
the flux.

Last but not least, we note that the study of the dynamics of the order parameter
in a superconducting loop with one (or more) “gaps” is an intriguing subject in its own
right. Thus, one could create a number of independent “domains” in the superconductor
by cutting up the order parameter (i.e., by heating up the loop locally). These
superconducting domains can be then welded together, and the resulting trapped winding
number (or the resulting flux) compared with theoretical expectations, which would have
to be based on some compromise between the order parameter freeze out and thermal
fluctuations driven activation process we have discussed above. One can imagine designing
experiments in which the individual domains are small enough so that \( n_w \) is given by
Eq. (4.16), or alternatively, where the flux is essentially thermal, with the expectation
value set by Eq. (4.22). It should be perhaps pointed out that such experiments could
also shed a new light on the fascinating questions about the nature of the phase of the
wave function of the Bose condensate raised by Anderson (1986) and Leggett (1980) and
intimately related to the interpretational issues of quantum theory (Wheeler and Zurek,
1983; Zurek, 1991).

5. COSMOLOGICAL IMPLICATIONS

The main difference between the dynamics of the phase transition in the condensed
matter and cosmological contexts arises from a different scaling of the relaxation time
scale with the relative temperature \( \epsilon \). Order parameters (fields) which are relevant for
cosmology obey the equation of motion with the second time derivative on the left-hand
side. Consequently, relaxation time is

\[
\tau = \frac{1}{\sqrt{|\alpha|}} \sim \frac{1}{\sqrt{|\epsilon|}}.
\]

(5.1)
(Above, we have returned to $\hbar = c = k_B = 1$.) One immediate consequence of this scaling is that the velocity with which perturbations of the order parameter can propagate, given by $\xi/\tau$, remains finite even at $T_c$, since

$$\xi = \frac{1}{\sqrt{|\alpha|}} \sim \frac{1}{\sqrt{|\epsilon|}}. \quad (5.2)$$

For the purpose of evaluating the initial density of the topological defects in the cosmological phase transitions, we should also note that $\alpha' = \alpha/\epsilon$ is approximately

$$\alpha' \approx \beta T_c^2 \quad (5.3)$$

in field-theoretic models.

In the early Universe the equation of state is dominated by radiation. Thus, the rate at which the phase transition happens will be determined by the relation

$$T^2 t = \Gamma M_{PL}, \quad (5.4)$$

which holds in the radiation-dominated Universe. Above $t$ is the time since the “Big Bang,” and $M_{PL}$ is the Planck mass. The coefficient $\Gamma$ depends on the effective number of different spin states $s$ of relativistic particles:

$$\Gamma = \frac{1}{4\pi} \cdot \sqrt{\frac{45}{\pi s}}. \quad (5.5)$$

The quench time scale can be then obtained directly from Eq. (5.4) as

$$\tau_Q = 1/\dot{\epsilon} = 2\Gamma M_{PL}/T_c^2. \quad (5.6)$$

Moreover, the relaxation time can be expressed as

$$\tau = \tau_0 / \sqrt{|\epsilon|}, \quad (5.7)$$

where $\tau_0$ can be evaluated with the help of Eqs. (5.1) and (5.3);

$$\tau_0 \sim \left(\sqrt{\beta} T_c\right)^{-1}. \quad (5.8)$$
We have all of the ingredients required to compute the cosmological freeze-out time \( \hat{t} \), which now obeys the equation

\[
\tau_0 \sqrt{\frac{\hat{t}}{\tau_Q}} = \hat{t}.
\] (5.9)

This is the relativistic version of Eq. (3.3) we have solved before for superfluids and superconductors, but now the solution reads:

\[
\hat{t} = \tau_0^{2/3} \tau_Q^{1/3}.
\] (5.10)

The difference with the previously obtained \( \hat{t} = \sqrt{\tau_0 \tau_Q} \), Eq. (3.4), stems from the different scaling of \( \tau \) with \( \epsilon \), Eq. (5.1).

It is now straightforward to evaluate

\[
\hat{t} = (2\Gamma M_{PL}/T_c)^{1/3} \left( \beta^{1/3} T_c \right)
\] (5.11)

and to obtain the corresponding \( \hat{\epsilon} \);

\[
\hat{\epsilon} = \beta^{-1/3} \left( \frac{T_c}{2\Gamma M_{PL}} \right)^{2/3}
\] (5.12)

We predict that the freeze out will take place at \( \hat{\epsilon} \) and that it will determine the overall structure of the initial configuration of cosmological defects.

The characteristic correlation length would be then of the order of

\[
\hat{\xi} = \left( \frac{2\Gamma M_{PL}}{T_c} \right)^{1/3} \cdot \frac{1}{\beta^{1/3} T_c}.
\] (5.13)

The corresponding density of topological defects obtains from the usual argument (i.e., one monopole/one \( \hat{\xi} \) section of a string/one \( \hat{\xi}^2 \) section of membrane per volume of \( \hat{\xi}^3 \)). This value of the freeze-out relative temperature \( \hat{\epsilon} \) is, of course, quite different from what obtains when the Ginzburg condition is employed. In that latter case:

\[
\epsilon_G \sim \beta.
\] (5.14)
This would in turn lead to the correlation length of

$$\xi_G = (\beta T_c)^{-1} ,$$  \hspace{1cm} (5.15)

which is, again, quite different from the estimate of Eq. (5.13). Indeed, the ratio

$$\hat{\xi}/\xi_0 = \beta^{2/3} \cdot \left( \frac{2\Gamma M_{PL}}{T_c} \right)^{1/3}$$  \hspace{1cm} (5.16)

shows that, for the usually assumed small $\beta$, $\hat{\xi}$ could be smaller than $\xi_G$ for very high temperature phase transitions. However, when $T_c \ll M_{PL}$—that is, for late phase transitions—$\hat{\xi}$ is likely to be much larger than $\xi_G$.

Whether the difference between the resulting initial densities of topological defects will be of great significance for a cosmological model is, of course, a separate question. For example, in the case of cosmic strings most of the observable consequences will be determined by the structure of the string network at late times and will be decided by the dynamics of string interactions, etc., rather than by the details of their initial configuration. On the other hand, when—as is sometimes proposed—topological defects are used as a catalyst in baryogenesis, their density will leave an immediate imprint on the Universe and will be critically important.

6. DISCUSSION

We have considered creation of topological defects in course of the second order phase transition involving non-conserved order parameter. This selection was dictated by the importance of such systems for cosmology as well as by the recent experiment in superfluid He$^4$ which afforded a laboratory test of the cosmological scenario. The original cosmological motivation notwithstanding, defect-forming dynamics of the second order phase transitions is of enormous interest in its own right.

The key question we have attempted to address concerned the initial density of the topological defects. This quantity is a witness to the symmetry breaking dynamics in course
of the phase transformation. Two paradigms were put forward to compute initial density
of topological defects. The older one appeals to the activation process and emphasizes the
relevance of the Ginzburg temperature—the temperature below which thermally activated
transitions become prohibitively expensive—on the initial density of the defects. This line
of reasoning would lead one predict a copious production of defects anywhere between the
Ginzburg temperature $T_G$ and the phase transition temperature $T_c$. Moreover, any rapid
quench trajectory which starts within that region would be expected to lead to more or less
the same density of defects (set by the correlation length at the Ginzburg temperature).

In contrast to the thermal activation mechanism sketched out above, one can consider
a scenario in which the density of defects is determined by the dynamics of the order
parameter in the immediate vicinity of the critical temperature $T_c$. The characteristic
scale (which sets the density of the topological defects) is decided at the instant when the
critical slowing down makes the order parameter so sluggish that it can no longer keep up
with the changes of the thermodynamic conditions induced by the quench. This happens
at the relative temperature $\hat{\epsilon}$ [see Eqs. (3.5) and (5.12) for the typical condensed matter
and cosmological cases, respectively].

Freezeout scenario is guaranteed to set the scale of the initial distribution of
cosmological defects when $\hat{\epsilon}$ is below the relative Ginzburg temperature $\epsilon_G$. This is because;

$$|\hat{\epsilon}| > |\epsilon_G|$$  \hspace{1cm} (6.1)

implies that the dynamics of the order parameter remains effectively frozen until $\hat{\epsilon}$ is
reached, so there is simply no time for the activation process. In the laboratory phase
transition this condition will in principle depend on the quench rate, but is in any case
likely to be satisfied in the superconductors, and violated in the superfluid He$^4$. In the
cosmological phase transitions it translates into a relatively simple inequality:

$$\beta^2 \gtrsim \frac{T_c}{2 \Gamma M_{PL}}.$$  \hspace{1cm} (6.2)
This condition follows from Eqs. (5.12), (5.14), and a demand that freezeout should persist as the Ginzburg temperature is being traversed during the quench. Thus there is certainly a range where the freezeout preempts activation “by default”.

It may seem more surprising that—as it was anticipated in the original proposal for “cosmology in the laboratory” (Zurek, 1984, 1985, 1993), and as it is apparently borne out by the experimental results (Hendry et al., 1994, 1995)—the freezeout scenario is valid even when inequality (6.1) is grossly violated, so that the activation events are expected to occur frequently. Apparently, in the superfluid (and, presumably, also in the cosmological phase transitions) the larger scale set by the critical slowing down is “remembered”, and its memory is never erased by the activation occurring on the smaller scales.

While this may have been a surprise, it was certainly not totally unexpected. For one thing, the energetic cost of creating large scale fluctuations is prohibitive. And only perturbations which have sizes comparable with the characteristic scale of the string laid down during the freezeout can appreciably alter initial density of the string network. Moreover, the bulk (∼70%) of network is expected to belong to a single long string. Its geometry may be effected by activation events on the smaller scale corresponding to the correlation length at the Ginzburg temperature, but the very fact that the phase transition has already taken place implies that such small scale fluctuations do not appreciably rearrange long range order.

Seen in this light, the apparent lack of importance of the Ginzburg temperature for the initial density of cosmological defects does not appear too surprising. The symmetry is broken—long range order sets in—at the critical temperature. Topological defects are, after all, imperfections of that long range order. Hence, persistence of the large scale structure imposed at the time when long range order comes into being for the first time can be reconciled with the small scale perturbations caused by thermal activation.

This is not to imply that the above arguments and the apparent accord between the experiment and the scenario which appeals to the freezeout (and an even more obvious
discord with the consequences of the activation scenario) settles all the issues which are worth settling. On the contrary, understanding of the dynamics of the process emergence of the long range order in the course of the second order phase transitions is only at its inception. Issues which have not been addressed in detail (such as the interplay between the freezeout and activation) can be perhaps tackled by numerical simulation (preliminary results are already in; Laguna and Zurek, in preparation). They also include mode dependence of the critical dynamics (Gill and Rivers, in preparation), the influence of the gauge fields, as well as a study of problems specifically designed to address experiments. Moreover, experiments themselves should not be restricted to just quenches through the critical region. Quenches from just below the critical temperature, as well as transitions from the broken symmetry phase to just above $T_c$ followed by a quick return to the broken symmetry phase should allow one to probe the timescale on which the memory of the broken symmetry is erased. One can also imagine performing quenches into the superfluid starting from within the solid phase of He$^4$ or He$^3$. The order parameter responsible for the existence of topological defects (presumably) disappears when superfluid turns into solid. However, second sound in the solid He$^4$ exists, which suggests that at least some aspect of the superfluid behavior may persist in the solid as well. It is of course difficult to imagine existence of vortices in the solid. But this (and other similar “out on the limb” speculations) can be experimentally addressed by checking that the topological defects can be created in course of the solid-superfluid pressure quench, and by finding out whether the topological defects imprinted on the superfluid can weather the transition into the solid phase. (If they did – which seems unlikely – then it should be possible to freeze out and then to “defrost” vortex lines.)

This paper was more a preview of the exciting possibilities rather than a review in the traditional sense. Its success should be measured by how much it contributes to its own obsolescence. For, it is hoped that the research carried out in the exciting areas outlined above will be rapid enough to surpass and eclipse the developments described here.
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Fig. 1.—Effective potential in Landau-Ginzburg model of second order phase transition, \( V(\varphi) = \alpha |\varphi|^2 + \frac{\beta}{4} |\varphi|^4 \). Above the phase transition temperature \( T_c \) (Fig. 1a) (coefficient \( \alpha > 0 \)), there is just a single minimum for \( V(\varphi) \) at \( \langle \varphi \rangle = 0 \). Below the phase transition the minimum is degenerate and corresponds to the expectation value of \( \langle \varphi \rangle = \sigma = \sqrt{\alpha/\beta} \), with \( \Delta V = V(0) - V(\sigma) = \alpha^2/2\beta \).

Fig. 2.—Phase diagram of He\(^4\). Liquid HeI is known as “normal,” while HeII is superfluid.

Fig. 3.—Structure of the superfluid vortex. (a) The density of the superfluid increases outward from a “normal” core on a scale set by the correlation length \( \xi \). Velocity of the superfluid falls off with the distance so that at every radius the topological constraint implied by Eqs. (2.27) and (2.36) is satisfied. This leads to one-dimensional vortex lines, shown in Fig. 3b.

Fig. 4.—Schematic trajectory of the pressure quench which can induce a rapid phase transition from normal He\(^4\) into the superfluid (a). As the \( \lambda \)-line is traversed, equilibrium relaxation timescale \( \tau \), Eq. (2.45) diverges (6). Thus, as the quench proceeds at a finite pace, the order parameter will not be able to adjust any more when the relaxation timescale becomes comparable to the time when the critical temperature is reached. This leads to a freezeout timescale \( \tilde{t} \); within the time interval \([ -\tilde{t}, \tilde{t} ]\) the quench is in effect instantaneous. Thus, the correlation length \( \tilde{\xi} \) corresponding to the instant \( \tilde{t} \) will be frozen out and will set the density of topological defects.

Fig. 5.—Formation of topological defects in the course of a rapid phase transition. Above the phase transition two-dimensional order parameter fluctuates around the minimum of the potential, assuming instantaneously values which “point” in approximately the same direction on the \((\Psi_x, \Psi_y)\) plane in the domains of correlation-length size \( \xi \). If this configuration was frozen out by an instantaneous phase transition, the symmetric vacuum would be trapped, resulting in formation of topological defects.

Fig. 6.—Actual quench trajectories in He\(^4\) (after Hendry et al., 1995). Trajectory (A) never crosses or even approaches the \( \lambda \)-line, and the corresponding quench does not lead to vortex line production. Trajectories which cross the \( \lambda \)-line [such as (C)] result in copious vortex line densities. Trajectories which do not cross the \( \lambda \)-line, but come very close to it [as is the case for (B)], also result in vortex line production, perhaps as a result of a freezeout of thermal population of vortices which may exist in the vicinity of the phase
transition temperature.

Fig. 7.—Phase diagram of He\textsuperscript{3} with a schematic indication of a few possible quench trajectories. In contrast to He\textsuperscript{4}, normal-superfluid quench in Helium 3 involves increase of pressure. One could also contemplate quenches from solid to superfluid for both isotopes of Helium.

Fig. 8.—Rapid quench in an annulus. Domain size $\hat{\xi}$ will now determine the typical velocity of the superflow. When $\hat{\xi} > 2r$, the problem is reduced to the evolution of a two-dimensional (i.e., complex) order parameter in a one-dimensional space and is illustrated in Fig. 9.

Fig. 9.—Evolution of the order parameter in the course of a quench in a thin annulus can be illustrated with a version of the Argand diagram shown above. Real and imaginary parts of the order parameter are plotted in the radial and in the vertical direction as a function of position along the circumference of the annulus. For a sufficiently thin annulus order parameter will depend on only one variable—it will vary only with the location along the circumference of the annulus, which is shown as a grey “doughnut”. An example of a possible instantaneous state is shown with the black line. Above the phase transition temperature order parameter $\Psi$ would fluctuate about the energetically favored $\Psi = 0$, changing its value significantly on a spatial scale given by the correlation length $\xi$. However, after the symmetry is broken typical value of $|\Psi|$ will be set by the minimum of the potential at $\sigma = \sqrt{|\alpha|/\beta}$, and the transitions through $\Psi = 0$ will become unlikely, as they require activation energy. Consequently, the initial configuration left by the quench will smooth out and be able to partially unwind (oppositely oriented twists of the spiral will cancel). The leftover winding number (given by the number of times the black line wraps around) will be stabilized and will result in a “permanent” superflow (in He\textsuperscript{4}) or supercurrent (in superconductors).