Supersymmetry and Membrane Interactions in M(atrix) Theory

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Abstract

We calculate the potential between various configurations of membranes and gravitons in M(atrix) theory. The computed potentials agree with the short distance potentials between corresponding 2-branes and 0-brane configurations in type IIA string theory, bound to a large number of 0-branes to account for the boost to the infinite momentum frame. We show that, due to the large boost, these type IIA configurations are almost supersymmetric, so that the short and long distance potentials actually agree. Thus the M(atrix) theory is able to reproduce correct long distance behavior in these cases.

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1 Introduction

Recently there has been a renewed effort towards understanding the quantum structure of M theory [1, 2], which is expected to underlie all the string theories that we know. One of the tools used is the kinematic improvements obtained by using an infinite momentum frame to view the excitations of the theory. It has been conjectured that in such a frame the dynamics can be captured by the Yang-Mills theory [3] that is obtained for 0-branes at short distances and low velocities. The gauge group would however be $SU(\infty)$, corresponding to the limit of an infinite number of 0-branes [4]. This Yang-Mills theory reproduces the action of the membrane [5] of 11-dimensional supergravity, which was one of the principal motivations for the conjecture. Recently it has been shown that the membrane constructed in this fashion exhibits the correct Berry’s phase when transported around a 5-brane of the theory [6].

If this conjecture can be verified then it would be a candidate for a non-perturbative formulation of M theory, and thus of string theory. Large string coupling corresponds to a large radius for the compactification radius of the coordinate $X^{11}$. When some of the directions of space are compact the $SU(\infty)$ Yang-Mills quantum mechanics is to be replaced by an $SU(\infty)$ Yang-Mills field theory in those directions [4, 7, 8, 9]. Another formulation of the conjecture can be found in [10].

In this paper we explore some of the consequences of the infinite momentum frame approach and the use of the Yang-Mills action to explore the amplitudes in M theory. We do the following:

(a) We examine the effect of using a large boost in the $X^{11}$ direction of M theory, as seen from the viewpoint of type IIA theory. The latter is the compactification of M theory on a circle in the direction $X^{11}$, and since we wish to compare to results in weakly coupled type IIA, we take the radius of this circle to be small. Suppose we wish to derive the potential between a 2-brane and an anti-2-brane at large distance. This configuration is not supersymmetric. But now suppose we boost the system in the direction $X^{11}$. From a type IIA string theory point of view, this amounts to adding a number $N$ of zero branes to each 2-brane. We show that this resulting configuration becomes close to being supersymmetric, as the number $N$ becomes large. We do this by mapping the configuration by a sequence of dualities to one where we have two clusters of zero branes, moving at a slow relative velocity. This latter system is known to be close to supersymmetric, and this is manifested in the fact that the interaction potential is the same at small and at large distances. The small distance potential is computed by using only the ground states of the open strings stretching between the clusters, while the long distance calculation used all the oscillator modes of the open strings. The miracle of supersymmetry is that the contribution of the higher oscillator
modes cancels out for the low velocity case. Since the boosted 2-brane - anti-2-brane system is related by exact dualities to the low velocity zero brane system, we see that the miracle of supersymmetry can work for the former system as well. Once we have the potential in the boosted frame, we use simple kinematics to find the potential of the 2-brane - anti-2-brane system at rest (i.e. no boost). We obtain exactly the correct known result for the potential at distances larger than string length.

(b) We find the potential between different branes in M(atrix) theory using the matrix technique postulated in [4]. It is useful to think of the matrix technique in the following way. The 2-brane bound to a number \( N \) of 0-branes can be described in two equivalent ways.

(1) As a 2-brane carrying a U(1) magnetic field of \( N \) units.

(2) As a SU(N) gauge theory defined with gauge fields and fundamentals in a twisted bundle, with the twist corresponding to the presence of the 2-brane. This latter approach was studied in [9].

We note that calculations of matrix theory correspond to using the latter language, and the usual string calculation corresponds to the former. We carry out calculations for the potential between (1) a zero brane and a 2-brane, with relative transverse velocity, (2) a 2-brane and an anti-2-brane, (3) two 2-branes moving with a relative transverse velocity, (4) two orthogonal branes moving with a transverse velocity. In all case we show that the above two approaches agree. Due to the large amount of zero-branes bound to the membrane these configuration become close to supersymmetric, and one expects to find that the short distance potential agrees with the leading long distance potential. We show that this is indeed the case. Thus the M(atrix) theory reproduces the correct results at long distance.

The plan of this paper is as follows. Section 2 explores the algebra of considering boosts in the \( X^{11} \) direction and regarding them as creating 0-branes in the type IIA theory, and shows that the brane configurations become more supersymmetric as the boost parameter is increased. In Section 3 we discuss in more details the effects of boosting in a compact direction, and in particular Lorentz invariance. In section 4 we give a detailed example of computing the long distance potential between a two-brane and an anti-two-membrane, using large boosts and short distance information. Section 5 discusses the examples of potentials between various branes in M(atrix) theory. Section 6 is a general discussion.

While we were writing up this work, a paper appeared with many similar results.
2 The relation between boosts and supersymmetry.

In this section we show that if we boost a non-supersymmetric configuration of branes then it becomes close to supersymmetric.

Suppose we have the following question. We have a 2-brane, wrapped on a 2-torus. We also have, at a large transverse distance, an anti-2-brane, which is just the 2-brane wrapped with the opposite orientation. We wish to know what is the leading potential at long distances between this pair of objects. Let us denote the potential as $V_2(r)$.

This calculation can be done through computing the 1-loop amplitude of open strings that start on one brane and end at the other. Since we are at long distances, which are in particular longer than string length, we expect that we will have to use all the oscillator modes of the open strings. Another way to say this is that since this configuration of branes breaks all supersymmetry, we do not have the miracle that happens for instance with the force between two zero branes at low relative velocity: the long distance force law calculation gives the same result as the short distance force law calculation, with the latter using only the ground states of the open strings stretched between the zero branes. But we shall see that after the boost in the direction $X^{11}$ the system does indeed become close to supersymmetric, so that the long distance potential can be found from the lowest modes of the open string.

2.1 The duality transformations.

We proceed to calculate the potential between the 2-brane and the anti-2-brane by a sequence of dualities, which will reduce the problem to the force between clusters of zero branes at low relative velocity. For the system of the 2-brane and anti-2-brane at rest, the interaction potential is known to be

$$V_2 = \frac{16\Gamma(5/2)(L^{(S)})^2L_1L_2}{2\sqrt{2}(2\pi)^{5/2}b^5}$$

at distances large compared to the string length. Our goal is to reproduce this result using only the knowledge of potentials that can be computed using only the ground state of the open string (for example the potential between slowly moving 0-branes).

(a) We assume that type IIA theory is just the compactification on $S^1$ of a Lorentz invariant 11-dimensional M theory. Instead of computing the potential between the branes at rest, we imagine that each brane is travelling at a speed close to that of light in the direction $X^{11}$. Thus each brane has a momentum

$$p_{11} = \frac{N}{R}$$
in the direction $X^{11}$. Here the length of the $X^{11}$ direction is

$$L_{11} = 2\pi R = gL^{(S)}$$  \hspace{1cm} (3)

$L^{(S)}$ is the length scale of the elementary string, which we will use as a reference length throughout this calculation. (The elementary string tension is $T^{(S)} = 2\pi L^{(S)^{-2}}$.) Under a T-duality, a circle of length $L = AL^{(S)}$ changes to a circle of length $L' = A^{-1}L^{(S)}$. Under S-duality the coupling $g$ changes to $g' = g^{-1}$ and a length $L = AL^{(S)}$ changes to a length $L' = AL^{(D)} = Ag^{1/2}L^{(S)}$. ($L^{(D)}$ is the length scale of the D-string.)

From the perspective of type IIA theory we have a 2-brane bound to $N$ 0-branes, and an anti-2-brane bound to $N$ zero branes. (The number of 0-branes is the same on both the 2-branes because they had the same mass before boosting, and thus at the same boost velocity will have the same amount of momentum.) Let the transverse distance between them be $b = AL^{(S)}$.

At this stage, the coupling constant of the type IIA theory is $g_1$, say. The sides of the torus on which the 2-brane is wrapped are

$$L_1^{(1)} = B_1L^{(S)}$$ \hspace{1cm} (4)
$$L_2^{(1)} = B_2L^{(S)}$$ \hspace{1cm} (5)

The third compactified direction and the transverse separation between the branes are

$$L_3^{(1)} = B_3L^{(S)}$$ \hspace{1cm} (6)
$$b_1 = AL^{(S)}$$ \hspace{1cm} (7)

(b) We T-dualise in one of the directions of the torus on which the 2-brane is wrapped, say the direction 2 (which has length $L_2$). The 2-brane then becomes a D-string, and the anti-2-brane becomes a D-string wrapped with the opposite orientation. The zero branes become D-strings, wrapped in a direction perpendicular to the above D-strings, with winding number $N$. For both the branes the direction of winding of this latter winding is the same. The bound state of the 2-branes with the zero branes thus becomes a D-string wrapped on a cycle of the torus that is $(1, N)$ for the 2-brane-zero brane bound state and $(-1, N)$ for the anti-2-brane - zero brane bound state. Note that at this stage the D-strings are close to being parallel, if $N$ is large. We know that this situation is close to being supersymmetric, since the case where the two D-strings are parallel is exactly supersymmetric, and the breaking of supersymmetry increases continuously with the angle between the D-strings. A consequence is that there will be an agreement between the potential computed at long distances and the potential computed at short distances for our present system.
3 The issue of Lorentz invariance.

In the above we have boosted the branes in the direction $X^{11}$. This was a compact direction though, not a noncompact one, so one may wonder what are the Lorentz kinematics of such a boost. One of the ideas of the M(atrix) approach is that we take the size of the compact direction to be large, so that we obtain an 11-dimensional M-theory rather than the dimensionally reduced 10 dimensional type IIA string description. Of course the larger the compact circle, the more the number of units of momentum that we wish to have, in order to approach the large momentum frame.

For a large size of the compact direction, if we have small momentum in that direction, we will of course approximate well the usual Lorentz kinematics obtained for non-compact spacetimes. But what happens if the boost is large? We argue below that once we boost to a momentum that will help us to enhance supersymmetry in the way referred to above, the Lorentz kinematics may not resemble that for a noncompact direction at all.

We argue as follows. The 2-brane and anti 2-brane were separated in a direction transverse to $X^{11}$, and if the Lorentz kinematics of noncompact space were to be a good approximation, then distances transverse to the boost direction do not shrink or expand. But in this system of 2-brane - anti-2-brane, we know that in the absence of any boost, there is a tachyonic instability, when the transverse separation becomes equal to string length. This instability can be seen in a divergence of the 1-loop potential between the branes, or more dynamically in a divergent phase shift in a scattering process, when the impact parameter approaches string distance.

If the Lorentz kinematics of non-compact space were to hold, then we would expect the tachyonic instability distance to not change when the boost in the direction $X^{11}$ were applied. But as we saw above, by a T-duality the boosted system of branes can be mapped to a system of D-strings that are almost parallel. (The noncompact directions were not affected in this change.) But with D-strings, we know that there is a tachyonic instability at a transverse separation of string length only when they are antiparallel; if we bring the relative orientation towards that of parallel D-strings then the tachyonic instability distance becomes shorter, and in fact becomes zero when the D-strings are exactly parallel.

Since the case at hand has mapped to almost parallel D-strings, we do have a big reduction in the tachyonic instability distance, when we compare the boosted and non-boosted cases. Note that this occurrence of almost parallel D-strings was exactly what helped us get the approach to supersymmetry, which as we will see below, allows us to use the Yang-Mills approximation to the full open string theory that governs that long distance interaction between the branes. Thus we conclude that if we boost enough to get the needed supersymmetry, then we cannot ignore the compactness of the direction $X^{11}$.
in the kinematics of the boost.

There are potential counters to the argument above; we do not know really the strong coupling physics of type IIA, so we do not know much about what happens for large $X^{11}$. The tachyonic instability argument is a 1-loop effect of open strings, and it may well be that other loops somehow contrive to cancel the change in tachyonic instability seen at this lowest order. But in any case this would be something that needs to be demonstrated; the limits of large radius for $X^{11}$ and large momentum in direction $X^{11}$ are not easily interchangeable.

4 Mapping to the 0-brane problem.

Thus we have shown the required approach to supersymmetry, but we continue the sequence of dualities to map the problem to the problem of slowly moving 0-branes, for which the potential was computed in [14].

At this stage the coupling is

\[ g_2 = g_1 B_2^{-1} \]  

(8)

The lengths of the sides of the torus are

\[ L_1^{(2)} = B_1 L^{(S)} \]  

(9)

\[ L_2^{(2)} = B_2^{-1} L^{(S)} \]  

(10)

The third compactified direction and the transverse separation are still

\[ L_3^{(2)} = B_3 L^{(S)} \]  

(11)

\[ b_2 = A L^{(S)} \]  

(12)

(c) We S-dualise this configuration to get elementary strings instead of D-strings in the same configuration as above. We have

\[ L_1^{(3)} = B_1 L^{(D)} = B_1 g_3^{1/2} L^{(S)} = B_1 g_1^{-1/2} B_2^{1/2} L^{(S)} \]  

(14)

\[ L_2^{(3)} = B_2^{-1} L^{(D)} = B_2^{-1/2} g_1^{1/2} L^{(S)} \]  

(15)

\[ L_3^{(3)} = B_3 g_1^{-1/2} B_2^{1/2} L^{(S)} \]  

(16)

\[ b_3 = A L^{(D)} = A g_1^{-1/2} B_2^{1/2} L^{(S)} \]  

(17)

(d) We T-dualise in the direction where the winding numbers were $\pm 1$, which is the direction 1. This converts the winding into momenta, so we have an $N$ times wound elementary string moving with one quantum of momentum transverse to itself on the torus, and a similar $N$ times wound elementary
string moving with one unit of momentum in the opposite direction, at the location of the other brane. We have

\[ g_4 = g_3[1 - B_1 1/2 B_2 1/2]^{-1} = g_1^{-1/2} B_1^{-1} B_2^{1/2} \]

(18)

\[ L_1^{(4)} = B_1^{-1} g_1^{1/2} B_2^{-1/2} L^{(S)} \]

(19)

\[ L_2^{(4)} = B_2^{-1/2} g_1^{-1/2} L^{(S)} \]

(20)

\[ L_3^{(4)} = B_3 g_1^{-1/2} B_2^{1/2} L^{(S)} \]

(21)

\[ b_4 = A g_1^{-1/2} B_2^{1/2} L^{(S)} \]

(22)

(e) We T-dualise in the direction 3, obtaining a type IIB theory

\[ g_5 = B_1^{-1} B_3^{-1} \]

(23)

\[ L_1^{(5)} = B_1^{-1} g_1^{1/2} B_2^{-1/2} L^{(S)} \]

(24)

\[ L_2^{(5)} = B_2^{1/2} g_1^{-1/2} L^{(S)} \]

(25)

\[ L_3^{(5)} = B_3 g_1^{-1/2} B_2^{1/2} L^{(S)} \]

(26)

\[ b_5 = A g_1^{-1/2} B_2^{1/2} L^{(S)} \]

(27)

(f) We perform an S-duality. We are still in type IIB theory. We have

\[ g_6 = B_1 B_3 \]

(28)

\[ L_1^{(6)} = B_1^{-1/2} B_3^{1/2} B_2^{-1/2} g_1^{1/2} L^{(S)} \]

(29)

\[ L_2^{(6)} = B_2^{-1/2} B_1^{1/2} B_3^{1/2} g_1^{-1/2} L^{(S)} \]

(30)

\[ L_3^{(6)} = B_1^{1/2} B_3^{-1/2} B_2^{-1/2} g_1^{1/2} L^{(S)} \]

(31)

\[ b_6 = A B_1^{1/2} B_2^{1/2} B_3^{1/2} g_1^{-1/2} L^{(S)} \]

(32)

(g) We perform a T-duality in the direction 2. Then we have

\[ g_7 = B_1^{1/2} B_2^{1/2} B_3^{1/2} g_1^{1/2} \]

(33)

\[ L_1^{(7)} = B_1^{-1/2} B_3^{1/2} B_2^{-1/2} g_1^{1/2} L^{(S)} \]

(34)

\[ L_2^{(7)} = B_2^{1/2} B_1^{-1/2} B_3^{1/2} g_1^{-1/2} L^{(S)} \]

(35)

\[ L_3^{(7)} = B_1^{1/2} B_3^{-1/2} B_2^{1/2} g_1^{1/2} L^{(S)} \]

(36)

\[ b_7 = A B_1^{1/2} B_2^{1/2} B_3^{1/2} g_1^{-1/2} L^{(S)} \]

(37)

We have now reduced the problem to that of slowly moving clusters of 0-branes at long distance.

The fact that the clusters are moving slowly follows from the fact that \( N \) branes share one unit of momentum, so that

\[ v \approx \frac{2\pi}{NL_1^{(7)} T_0} \]

(38)

where \( T_0 = 2\pi L^{(S)} g_7^{-1} \) is the mass of the zero brane (\( N \) is large).
4.1 Computing the potential.

4.1.1 The potential between the 0-brane clusters

We now discuss the potential between the clusters of 0-branes. If the spacetime were noncompact, the potential would be

\[ V = \frac{N^2 \Gamma(7/2)(2v)^4(L^{(S)})^6}{\sqrt{2}(2\pi)^{7/2}r^7} \tag{39} \]

where we have used that the relative velocity between the clusters is 2v.

If the space was a 2-torus with sides \( L_1, L_2 \), then to find the potential we note that we have to consider winding modes of the open strings around the cycles of the torus. An equivalent way to do this is to go to the covering space of the torus and thus find that each 0-brane at one location receives a contribution to its potential from an entire array of 0-branes at the other transverse location.

There are two cases where we can simplify further in the sense that we can replace the lattice sum by an integral. One is when the transverse distance \( b \) is much larger than the compactification radii of the torus. The other is when the density of 0-branes on the compact space is high. When there is one compact direction for instance, then the 0-branes form a bound state of size \( \sim g^{1/3}L^{(S)} \) when the compact direction is larger than this scale, but they behave as if they are spread out uniformly on the circle if the compact direction is much smaller than this scale [15]. We assume that the same phenomenon holds when we compactify on a 2-dimensional torus. We have the torus with an area which is much smaller than the scale of the bound state of zero branes for small \( g_1 \). Thus we will have the zero branes ‘smeared’ over the torus, and we will naturally get an integral over the position of the zero branes.

The lattice sum converted to an integral replaces the factor \( b^{-7} \) obtained in noncompact spacetime by

\[ \int (b^2 + z^2)^{-7/2} \frac{d^2z}{L_1L_2} = (b^5L_1L_2)^{-1} \frac{2\pi}{5} \tag{40} \]

where \( L_1, L_2 \) are the lengths of the compactified directions.

We will be interested in the case where one additional direction has been compactified, and the force is computed at long distances. The effect of this latter compactification is the same on both the computation of the force between the 2-branes and on the computation of the force using 0-branes. In each case a potential \( \sim b^{-5} \) is to be modified to

\[ \frac{1}{L_3} \int \frac{dx}{(b^2 + x^2)^{5/2}} = \frac{1}{L_3} \frac{4}{3b^4} \tag{41} \]
where $L_3$ is the length of the additional compactified direction.

Thus for the clusters of 0-branes with small transverse velocities, which we obtained by dualities from the initial problem concerning 2-branes, we get the potential

$$V_{\text{zero}} = \frac{N^2 \Gamma(7/2)(2v)^4 8\pi (L^{(S)})^6}{15\sqrt{2}(2\pi)^{7/2}b^4 L_1 L_2 L_3}$$

where now $L_1, L_2, L_3$ are the compactification lengths for the 0-brane case.

### 4.1.2 Potential between 2-branes

Let us now ask the question: What is the relation between the potential of the boosted 2-brane - anti-2-brane configuration and the potential of the same configuration without the boost. If the boost were in a noncompact direction, then we would have the following analysis. The energy of the moving configuration is

$$E = [P_{11}^2 + [2L_1 L_2 T^{(2)} + V_2]^2]^{1/2} \approx P_{11} + \frac{2L_1 L_2 T^{(2)} V_2}{P_{11}} + \frac{2(L_1 L_2 T^{(2)})^2}{P_{11}}$$

The middle term in the last expression is the potential energy that we attribute to the interaction between branes. Here $P_{11} = 2N/R$ is the total momentum of the system (arising because of the boost), the term $2L_1 L_2 T^{(2)}$ is the rest energy of the two branes when they are separated by infinite distance, and $V_2$ is the potential energy of interaction for the 2-brane anti-2-brane system at rest (i.e. without boost). Note that $V_2$ is proportional to $B_1 B_2$. Thus we would conclude that the interaction potential for the boosted configuration is

$$V_{\text{boost}} = \frac{2L_1 L_2 T^{(2)} V_2}{P_{11}}$$

Since the direction $X^{11}$ is actually compact, we again go to the covering space and note that each 2-brane receives a contribution to its potential energy from a 1-dimensional array of anti-2-branes, all spaced a distance $2\pi R$ apart in the $X^{11}$ direction. We wish to compare our results to a leading order in $g$ calculation of type IIA theory, so we assume the $R = g L^{(S)} / 2\pi$ is small, at least as compared to the transverse separation between the branes. Then the sum over the 1-dimensional array can be replaced by an integral. (It is the result of this lattice sum/integral which the potential $V_2$ describes for the case of the branes at rest.)

But if the branes are both moving rapidly in the direction $X^{11}$, then we have the following effect. The separation between branes is $2\pi R$ in the frame that we have been working with, but in the frame moving with the 2-brane, the spacing of the anti-2-branes is dilated to $2\pi R / \gamma$ where $\gamma^{-1} = \sqrt{1 - v_{11}^2}$. Note that for large boosts,

$$\gamma \approx \frac{p_{11}}{M} = \frac{N}{R L_1 L_2 T^{(2)}}$$
(M is the mass of the 2-brane.)

Thus when we view the 2-brane - anti-2-brane system as having just a large velocity in the $X^{11}$ direction, then we expect a potential energy changing with separation as

$$V_{\text{boost}} = \frac{2L_1 L_2 T^{(2)} V_2}{P_{11} \gamma}$$

where $V_2$ is the potential between the 2-brane anti-2-brane system at rest.

For large separations with just two directions compactified $V_2$ is known to be

$$V_2 = \frac{16\Gamma(5/2)(L^{(S)})^2 L_1 L_2}{2\sqrt{2}(2\pi)^{5/2} b^5}$$

(47)

With one additional direction compactified, at large separation we have

$$V_2 = \frac{32\Gamma(5/2)(L^{(S)})^2 L_1 L_2}{3\sqrt{2}(2\pi)^{5/2} L_3 b^4}$$

(48)

4.1.3 Comparing the 0-brane and the 2-brane potentials

We have performed several dualities in mapping the problem of the 2-branes boosted along $X^{11}$ to the problem of slowly moving 0-branes. To see if the potentials in the two cases agree, we must note the change of potential under duality. Let the potential between the boosted 2-branes be $V^{(1)} L^{(S)^{-1}}$, where $V^{(1)}$ is a dimensionless number. Under a T-duality a potential $VL^{(S)^{-1}}$ goes to $VL^{(S)^{-1}}$, while under an S-duality a potential $VL^{(S)^{-1}}$ goes to $VL^{(D)^{-1}} = VL^{(S)^{-1}} g^{-1/2}$. Following these changes we find that the potential between the 0-branes which we get at the end of our dualities, which we call $V^{(7)} L^{(S)^{-1}}$, will be given through

$$V^{(7)} = V^{(1)} g_1^{1/2} B_1^{-1/2} B_2^{-1/2} B_3^{-1/2}$$

(49)

For the 0-brane potential, in (42) we have to use the velocity (38). We have to use sides of the torus as $L_1^{(1)}, L_2^{(1)}, L_3^{(1)}$. For the boosted 2-brane potential we have to use (46), with $V_2$ given by (48) and with the sides of the torus being $L_1^{(1)}, L_2^{(1)}, L_3^{(1)}$. Substituting these values we find the exact agreement, using (49), between the potential (46) of the boosted 2-branes and the potential (42) implied by the moving 0-branes:

$$V_{\text{boost}} \equiv V^{(1)} L^{(S)^{-1}} = g_1^{-1/2} B_1^{1/2} B_2^{1/2} B_3^{1/2} V^{(7)} L^{(S)^{-1}} \equiv V_{\text{zero}}$$

(50)

where we have used that

$$V^{(1)} = \frac{(B_1 B_2)^3 \Gamma(5/2) 32}{B_3 N^2 3\sqrt{2}(2\pi)^{5/2} A^4}$$

(51)
\[ V^{(7)} = \frac{g_1^4 \Gamma(5/2) 32 L^{(8)11}}{b_1^4 [L_1^{(7)}]^5 L_2^{(7)} L_3^{(7)} N^2 3 \sqrt{2} (2\pi)^{5/2}} \]  

(52)

Thus we have shown that we can recover the long distance force between the 2-brane and anti-2-brane, by making a boost in the \( X^{11} \) direction, mapping to the problem of slowly moving 0-branes where short and long distance potentials agree, and boosting back.

### 4.2 A comment on boosts in compact directions.

Note that here we used relativistic kinematics in a domain where we may not have expected it to be valid. We have taken a very small size for the compact direction \( X^{11} \), since we took small coupling \( g_1 \); indeed this size is much smaller than the 11-dimensional Planck length. One may therefore wonder what are the kinematic rules for relating the potentials between frames boosted by large amounts in small compact directions. One effect of the compactness we have already seen: the potential is further scaled by a factor \( \gamma^{-1} \) due to fact that we have the potential coming from a sum over copies of the brane in the periodic direction, and the separation of copies was dilated by the boost.

There is a related effect that we describe as follows. Consider the potential between two 0-branes in type IIA theory. Let a pair of directions be compact. If we boost in the direction \( X^{11} \) then we get that each 0-brane is replaced by a large number \( N \alpha'/L_1L_2 \) of 0-branes. What is the potential between such 0-branes for low transverse velocities?

We know that for zero branes that are not boosted the potential is \( \sim v^4/b^7 \) for \( b < L_1, b < L_2 \) and is \( \sim v^4/b^5 \) for \( b > L_1, b > L_2 \). (Here \( L_1, L_2 \) are the compactification lengths.)

Now consider the boosted case. For a large boost, when the number \( N\alpha'/(L_1L_2) \) is sufficiently large, the 0-branes are smeared over the torus, as discussed above. Then we see that whether we have \( b < L_1, b < L_2 \) or \( b > L_1, b > L_2 \) we get the potential \( \sim v^4/b^5 \) in either case.

Thus we see that if we examine a system of branes after adding a high boost in a compact direction, then the potential that will result will agree with the ‘long distance’ potential between the branes. Here the term ‘long distance’ denotes the fact that if there are any compact directions in the spacetime, the potential will have the fall off pertinent to separations much larger than the compactification scale, and not to the potential at distances smaller than the compactification scale.

### 5 Potential between Branes

In the following subsections we will calculate the phase shift of an object when it is scattered from another object, and extract from it the potential
between them. We will consider configurations of a graviton scattering off a membrane, two parallel membranes moving with a relative velocity, two orthogonal membranes moving with a relative velocity, and membrane anti-membrane configuration (with no relative motion).

These calculations will be done in the framework of BFSS [4], which correspond to using only the ground states of the open strings between 0-branes. It is therefore not surprising that the results agree with the corresponding calculations carried out in the type IIA theory when the brane separations are short compared to string length. (We verify this agreement in several cases.) What may appear more surprising is that there is agreement with type IIA calculations where the velocities are small, even when the brane separation is large compared to the string length. But the reason for this, as mentioned in the introduction, is that these calculations all pertain to a situation where there is a large boost in the $X^{11}$ direction, and this brings the interacting system close to being supersymmetric. In such systems the potentials at long and short distances agree [14]. The potentials at large distances are what would characterise the interaction of membranes in the 11-dimensional supergravity theory [16, 17], and it may even be expected that non-renormalisation theorems protect the form of these potentials (for low velocities) as we go to strong coupling. If these expectations were to hold, then the M(atrix) calculations would tell us something about M theory in the sense that was hoped for in [4].

In the type IIA theory we are actually working at small coupling. The M(atrix) calculations are done at one loop, and thus also assume a small coupling. Non-renormalisation theorems may however carry over the results to larger coupling.

5.1 The set up

In this section we will set up the formalism for calculating the phase shift and the velocity dependent potential between two moving objects in M(atrix) theory. Let us start with the Lagrangian [4, 18, 19], we take the string length $l_s^2 = 2\pi$, the signature is $(-1, 1, \ldots, 1)$, and $D_t X = \partial_t X - i[A_0, X]$,

$$L = \frac{1}{2g} Tr \left[ D_t X_i D_t X^i + 2\theta^T D_t \theta - \frac{1}{2} [X^i, X^j]^2 - 2\theta^T \gamma_i [\theta, X^i] \right].$$  (53)

The supersymmetry transformations are

$$\delta X^i = -2\epsilon^T \gamma^i \theta,$$

$$\delta \theta = \frac{1}{2} \left[ D_t X^i \gamma_i + \frac{1}{2} [X^i, X^j] \gamma_{ij} \right] \epsilon + \epsilon',$$

$$\delta A_0 = -2\epsilon^T \theta.$$  (54)

We would like to calculate the phase shifts of a graviton scattering off various configurations and of membranes scattered off various configurations.
This can be done by calculating the vacuum energy of the Lagrangian (53) in a background corresponding to the configuration of objects that we are interested. This is done by giving expectation values to various matrices and expanding perturbatively all quadratic term around that background.

As in [14] we find it convenient to work in a background covariant gauge. For that let us go slightly back to the Yang mills theory from which the above Lagrangian was derived. Starting with

\[ L_1 = \text{Tr} \left\{ -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} - \frac{1}{2} (\bar{D}^\mu A_\mu)^2 + L_g \right\}. \] 

(55)

where \( \bar{D}^\mu A_\mu = \partial^\mu A_\mu - i [B^\mu, A_\mu] \), \( B_\mu \) is the expectation value of \( A_\mu \) and \( L_g \) is the corresponding ghost term. One can define as in [18, 19] the fields \( F_0^i = \partial_0 X_i - i [A_0, X_i] \) and \( F_{ij} = -i [X_i, X_j] \). If one chooses the \( B_\mu \) such that \( B_0 = 0 \) and the other \( B_i \) solve the equation of motion then we can expand (55) to quadratic order in the fluctuations around the background fields and find

\[ L_2 = \frac{1}{2} \text{Tr} \left\{ (\partial_0 Y_i)^2 - (\partial_0 A_0)^2 - 4i \dot{B}_i [A_0, Y^i] + \frac{1}{2} [B_i, Y_j]^2 + \frac{1}{2} [B_j, Y_i]^2 \right\} + [B_i, Y^j][Y^i, B^j] + [B_i, Y^i][B_j, Y^j] - [A_0, B_i]^2 + [B_i, B_j][Y^i, Y^j] + \partial_0 C^* \partial_0 C + [C^*, B^i][B_i, C]. \] 

(56)

This gauge is more convenient to work with as there will be only few and simple off diagonal terms in the mass matrix of the form \( [B_i, B_j][Y_i, Y_j] \) and \( 4 \dot{B}_i [A_0, Y^i] \), all other off diagonal terms cancel. To this Lagrangian we must add the fermionic terms from equation (53).

Now in order to compute scattering of two objects one has to insert the appropriate expectation values and integrate out the massive fields. As each object by itself will have zero vacuum energy one has only to consider fields that connect the two configurations, these will be the off diagonal matrix elements, which correspond in the language of strings to the degrees of freedom of the virtual strings stretched between the two objects. So we will take the following form for \( Y_i \) and \( \theta \).

\[ Y_i = \begin{pmatrix} 0 & \phi_i \\ \varphi_i & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & \psi \\ \chi & 0 \end{pmatrix} \]

Where the fermion are real sixteen component spinors, \( \varphi = \phi^i, \chi = \psi^T \) and these relationships should be understood in the matrix space. To calculate the phase shift one must calculate the one vacuum energy of the off diagonal components of the matrices. This is done at a one loop level by computing the determinants of the operators \( (\partial_0^2 + M^2) \) where \( M^2 \) is the mass matrix squared of all the fields including of course the fermions.

\[ ^1 \text{We would like to thank Dan Kabat for a very helpful discussion on this point} \]
5.2 Graviton Membrane scattering

In this section we will compute the one loop phase shift for a graviton scattered off a membrane as a function of the distance between them and the graviton transverse velocity. The graviton will actually be represented by one zero-brane with the understanding that to leading order we have to multiply the answer by $N_1$, the number of zero-branes the graviton is made off. The membrane is wrapped on a very large torus. From the phase shift we will extract the short distance behavior, the static potential between them and the velocity dependent potential at long distances.

The background configuration is

$$B_8 = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, B_9 = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, B_7 = \begin{pmatrix} b I & 0 \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} I v t & 0 \\ 0 & 0 \end{pmatrix}.$$  

$P, Q, I$ can be thought of as matrices, $[Q,P]=ic$. $\phi$ is a column vector while $\varphi$ is a row vector, and one should think of them as living in the adjoint representation of the gauge group. We are going later to calculate the vacuum energy in this configuration. For that we will have to calculate a large number of determinants, because the $P, Q, I$ matrices will be $N \times N$ matrices. we will first diagonalise the mass square matrix $M^2$, in the space of of $Y_i, A_0, C$ and then we will look at the spectrum in each one separately.

Plugging the $B$’s into the Lagrangian \((56)\) one finds that the mass term $(\varphi^i M^2 \phi_i)$ in the space of $(Y_2, \ldots, Y_7, C)$ is proportional to the identity with the proportionality constant being $2H$ and $H = P^2 + Q^2 + b^2 + v^2 t^2$.

In the space of $A_0, Y_1$ there are also off diagonal terms of $\pm 4iv$

$$M^2_{A_0 Y_1} = 2 \begin{pmatrix} -H & -2iv \\ 2iv & H \end{pmatrix}$$

In the space of $Y_8, Y_9$ one has also off diagonal terms $\pm 4ic$.

$$M^2_{Y_8 Y_9} = 2 \begin{pmatrix} H & -2ic \\ 2ic & H \end{pmatrix}$$

The way these results are derived is just to compute the trace of the matrices in equation \((56)\). For example

$$Tr[B_8, Y_i][B_9, Y_j] = -\begin{pmatrix} P \varphi^i \varphi^j Q & 0 \\ 0 & \varphi^i P Q \varphi^j \end{pmatrix}$$

Then

$$Tr[B_8, Y_i][B_9, Y_j] = -(P_{\alpha \beta} \varphi^i_{\beta} \varphi^j_{\alpha} Q_{\sigma \alpha} + \varphi^i_{\alpha} P_{\sigma \alpha} Q_{\alpha \beta} \varphi^j_{\beta})$$

$$= -\varphi^j_{\beta} (Q P)_{\beta \alpha} \varphi^i_{\alpha} - \varphi^i_{\beta} (P Q)_{\beta \alpha} \varphi^j_{\alpha}. \quad (57)$$
The ghost have the same mass $M^2 = 2H$ as some of the $Y$’s. We will eventually calculate the determinant of the operator $(\partial_0^2 + M^2)$ and then the ghost contribution will cancel some of the boson contribution. The determinant calculations will be done in Euclidean space, where $t = i\tau$ and $A_0 = -iA_r$, so the mass matrix in the $A_r, Y_1$ will be (remembering that the $Y_i$ kinetic term changes sign but that of $A_0$ does not)

$$M^2_{A_r Y_1} = 2 \begin{pmatrix} H & -2v \\ 2v & H \end{pmatrix}$$

Now in Euclidean space we will be left with four complex bosons with $M^2 = 2H$, one boson with $M^2 = 2H - 4c$, one with $M^2 = 2H + 4c$, one with $M^2 = 2H + 4iv$ and one with $M^2 = 2H - 4iv$. These complex bosons when represented in terms of real bosons will give twice the number of bosons but with half the mass squared.

Let us now turn to the fermions described by the matrix valued sixteen real spinor $\theta$. Inserting the $B$’s into the fermionic part on finds (remembering the fermions are real)

$$m_f = \gamma_8 P + \gamma_9 Q + \gamma_b + \gamma_1 vt.$$  

(58)

It is easier to convert the fermion determinants to the form $\det(\partial_0^2 + M_f^2)$, with

$$M_f^2 = H - ic\gamma_8\gamma_9 + v\gamma_1$$  

(59)

This can be diagonalised and we find: four fermions with $M_f^2 = H + c - iv$, four with $M_f^2 = H + c + iv$, four with $M_f^2 = H - c - iv$ and four with $M_f^2 = H - c + iv$.

We need to find the eigenvalues of the operator $H$. As $[Q, P] = ic$, The $Q, P$ operators can be represented as follows. On the space of $L_2$ functions of one variable $x$, $P$ acts as $-ic\partial_x$ and $Q$ acts as $x$. Then the spectrum of $H$ is just the spectrum of an harmonic oscillator with the $n$’th eigenvalue being $H_n = b^2 + v^2t^2 + c(2n + 1)$. The eigenfunctions are just the usual ones of a one dimensional harmonic oscillator. They are the wave functions of the off diagonal elements of the matrices.

The determinants are

$$\det(-\partial_r^2 + H_n)\det(-\partial_r^2 + H_n - 2c)\det(-\partial_r^2 + H_n + 2c)$$

$$\det(-\partial_r^2 + H_n - 2iv)\det(-\partial_r^2 + H_n + 2iv)\det(-\partial_r^2 + H_n + c + iv)$$

$$\det(-\partial_r^2 + H_n + c - iv)\det(-\partial_r^2 + H_n - c + iv)\det(-\partial_r^2 + H_n - c - iv)$$

We can now calculate the phase shift for the graviton scattered off the membrane, as in \cite{14} we use the proper time representation for the determinants. define: $\delta = \sum_{n} \delta_n$, $\gamma_{n}^2 = b^2 + c(2n + 1)$

\footnote{In our conventions $\gamma_{n}^2 = -1$}
then we find
\[ \delta_n = \frac{1}{2} \int \frac{ds}{s} e^{-r_n^2 s} \frac{1}{\sin sv}(4 + 2 \cosh 2cs + 2 \cos 2vs - 8 \cos vs \cosh cs) \] (60)

This can be integrated to give
\[ e^{i\delta_n} = \frac{\Gamma\left(\frac{ir^2}{2v} + \frac{1}{2} - \frac{ic}{2v}\right)\Gamma\left(\frac{ir^2}{2v} + \frac{1}{2} + \frac{ic}{2v}\right)\Gamma\left(\frac{ir^2}{2v} + \frac{1}{2}\right)}{\Gamma\left(\frac{ir^2}{2v} + 1 - \frac{ic}{2v}\right)\Gamma\left(\frac{ir^2}{2v} + 1 + \frac{ic}{2v}\right)\Gamma\left(\frac{ir^2}{2v} - \frac{ic}{2v}\right)} \] (61)

and
\[ |e^{i\delta_n}|^2 = \frac{(\cosh \frac{2r^2}{v} - \cosh \frac{2c}{v})^4}{\cosh \frac{2r^2}{2v}(\cosh \frac{2r^2}{v} + \cosh \frac{2c}{v})} \] (62)

For later comparison it is convenient to notice that
\[ 2 \sum_{n=0} e^{-cs(2n+1)} = \sinh^{-1} cs, \] (63)

then the expression for the phase shift becomes
\[ \delta = \frac{1}{2} \int \frac{ds}{s} e^{-b^2 s^4 + 2 \cosh 2cs + 2 \cos 2vs - 8 \cos vs \cosh cs} \frac{4 + 2 \cosh 2cs + 2 \cos 2vs - 8 \cos vs \cosh cs}{2 \sinh cs \sin sv} \] (64)

If we are interested in the potential it is defined through
\[ \delta = -\int dt V(b^2 + v^2 t^2). \] (65)

If \( b^2 \gg c \) we can expand equation (64) in powers of \( s \) to find a long range potential,
\[ V_{gm} = -\frac{N_1 \Gamma(5/2) (c^4 + 2v^2c^2 + v^4)}{4c^\sqrt{\pi} b^5}. \] (66)

where we have restored the factor of \( N_1 \).

Let us compare this to a string calculation of the phase shift of a zero-brane scattering off a bound state of a membrane and many zero-branes. From the string calculation this is just having a constant magnetic field \( F \) on the two brane \( \mathbb{Z} \). Then the phase shift using the technique in \[21, 22\], is \[17\] \( \tan(\pi \epsilon) = F, \tanh(\pi \nu) = v, \Theta(\rho) = \Theta(\rho, is) \),
\[ \delta_{IIA} = \frac{1}{2\pi} \int \frac{ds}{s} e^{-b^2 s} B \times J, \] (67)

\[ B = \frac{1}{2} f_1^{-6} \Theta_4^{-1}(i\epsilon s) \frac{\Theta'_4(0)}{\Theta_4(i\epsilon s)}, \]
\[ J = \{-f^6_2 \Theta_2(\nu s) \Theta_3(\nu s) + f^6_3 \Theta_2(\epsilon s) \Theta_3(\nu s) \Theta_4(0) + if^6_4 \Theta_4(\nu s) \Theta_4(\epsilon s)\}. \] (68)
Now there are many zero-branes on the membrane due to the fact that we originally were (almost) in the infinite momentum frame, and momentum in the eleventh direction is like having many zero-branes (for the classical solutions see [23, 24]), so $\epsilon \sim 1/2$. Let us take $\epsilon = \frac{1}{2} - c'$ (we assume $c'$ is small), and insert that to equation (68). Using the properties of the theta functions we find that when $F$ is large we get,

$$B = \frac{1}{2} f_1^{-6}(-i\Theta_1)^{-1}(ic's)\frac{\Theta'_1(0)}{\Theta_1(\nu s)},$$

$$J = \{-f_2^6 \Theta_2(\nu s) \Theta_2(0) + f_3^6 \Theta_3(i \nu s) \Theta_3(0) + f_4^6 \Theta_4(i \nu s) \Theta_4(0)\}.$$ (69)

One can compute the long range potential (that is expanding in powers of $s$) for the case of a zero-brane scattering off a bound state of a two-brane with many zero-branes from equations (69, 67). The long range potential can be then expanded to leading order in $v$ and $c$ (assuming they are small) to give,

$$V_{IIA} = -\frac{\Gamma(5/2)(2\nu^2(\pi c')^2 + (\pi c')^4 + \nu^4)}{4\pi c' \sqrt{\pi} b^5}.$$ (70)

exactly like the result from matrix theory when one identifies $c = \pi c'$, and multiplies by $N_1$. Now the magnetic field strength integrated over the two-brane is supposed to give $2\pi N$ where $N$ is the number of zero-branes on the two-brane, and also the size of the $P, Q$ matrix of the membrane. Using $F = 1/c$, we find $c = \frac{2\pi R_8 R_9}{N}$ where $R_8, R_9$ are the radiuses of the torus the membrane is wrapped on. Indeed for large enough $N$ $c$ is very small.

This gives an explanation from the point of view of type IIA theory of the parameter $c$. How is it that a configuration which is not supersymmetric in M-theory when calculating the long range force with the use of only the lightest open string mode reproduce the correct long range force.

Notice that equation (69) is actually equivalent up to an overall pre-factor to a calculation involving two relatively moving two-branes with a small magnetic field on one of them. As the non moving membrane configuration with no magnetic field is supersymetric, the velocity and small magnetic field only break the supersymmetry very softly. This is a situation when one expects that a long range potential could be reproduced by using only the lightest open string modes. In fact this is the case. The short distance behaviour of the type IIA configuration for large $F$ can be extracted by expanding equation (69) in the limit of large $s$. Then $(a_1$ is just a number)

$$B \times J = a_1 \frac{4 + 2 \cosh 2cs + 2 \cos 2\nu s - 8 \cosh cs \cos \nu s}{\sin \nu s \sinh cs}.$$ (71)
exactly like equation (64). In fact the matrix calculation is the short distance
Type II A calculation of a configuration of many zero branes and a membrane
bounded to many zero-branes.

This only happens because $N$ is large, so that from the point of view of
Type II A string theory there are almost infinite number of zero-branes on the
membrane, so its effect are almost swamped by the zero-branes. However if
we expand equation (68), rather than (69), in the lightest open string modes
and calculate the long range potential from that, we will find a very different
answer.

What happens when $c$ is not small. This is the situation in which we
have few zero-branes bounded to the membrane. Then the string calculation
using only the open string lowest modes will not reproduce the long distance
potential, because we are back to the expression (68). However the M(atrix)
calculation will still reproduce the correct long range force as long as $b^2 \gg c$.
So going to the infinite momentum frame has helped us to identify the
essential variables.

We now examine the issue of the tachyonic instability [25] present in
the type II A theory. As noticed in [17] the tachyonic instability starts at
a shorter scale when there are zero-branes on the two-brane. In the matrix
calculation from equation (62) one can see that there is a tachyonic instability
at $b^2 < c$, in agreement with the above statement. From our calculation
we can extract the wave function of the open string stretched between the
zero-brane and the membrane bounded to zero-brane system. The wave
functions are just $\Psi(x) = e^{-\frac{1}{2c}x^2} H_n(x/\sqrt{c})$ where $H_n(x)$ are just the Hermite
polynomials. $H_0(x) = 1$ gives the wavefunction for the tachyon.

If $c$ is very small compare to $v$ then from (62) one can see that there is
going to be a large imaginary part when $b^2 < v$ this is just the nucleation of
open string between the branes.

### 5.3 Membrane - anti-Membrane

An anti membrane is just a membrane with its orientation reversed. A mem-
brane parallel to an anti membrane stretched in the $X_8, X_9$ direction and
separated in the $X_7$ direction by a distance $b$ is described by the configura-

tion,

$$B_8 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \quad B_0 = \begin{pmatrix} Q_1 & 0 \\ 0 & -Q_2 \end{pmatrix}, \quad B_7 = \begin{pmatrix} 0 & 0 \\ 0 & bI \end{pmatrix}.$$  

Where $[Q_1, P_1] = ic$ and $[Q_2, P_2] = ic$. To see that this represents a mem-
brane anti-membrane configuration one can observe a few things. First each
configuration by itself breaks only half the supersymmetries in equation (44),
but together they break all the supersymmetries. Second using the results
of [8] one sees that the anti membrane has as a fermionic zero mode the
one with opposite chirality than the membrane, this then gives the opposite sign for the Berry phase and hence the opposite charge. Given the membrane configuration there is more that one configuration representing the anti-membrane (although they are all equivalent), for instance one can just take the membrane and exchange $P$ with $Q$ and vice versa, or multiply $P$ or $Q$ by $-1$. Multiplying both by $-1$ we get back a membrane.

Inserting this background to equation (56) one can read off the mass squared terms for the bosons and the fermions. Let us start with the bosons. In the space of $(Y_1, \ldots, Y_7, C)$ the matrix is diagonalised with diagonal entries all being $2H$ and,

\[ H = b^2 + (P_1 + P_2)^2 + (Q_1 + Q_2)^2 \]  

(72)

In $A_0$ space we get $-2H$ In the space of $Y_8, Y_9$ there are also off diagonal elements $\pm 8ic$. After diagonalization and subtracting the ghost contribution we end up in Euclidean space with: six bosons with $M^2 = 2H$ one with $M^2 = 2H - 8c$ and one with $M^2 = 2H + 8c$.

Turning now to the fermions one finds

\[ m_f = \gamma_8 (P_1 + P_2) + \gamma_9 (Q_1 + Q_2) - \gamma_7 b I \]  

(73)

So $M^2_f = H - 2icI\gamma_8\gamma_9 + Ib^2$. This gives eight fermions with $M^2_f = H + 2c$ and eight fermions with $M^2_f = H - 2c$.

We now investigate the spectrum of $H$. $H$ acts on a Hilbert space of functions of two variables. We are realizing $P$ as derivative operator and $Q$ as position operator. Lets label the two variables by $x, y$. then $H = -4c^2 \partial_x^2 \partial_y^2 + (x + y)^2$. so it will have eigenvalues of a harmonic oscillator but also a large number of functions with eigenvalue $b^2$ as $H f(x - y) = b^2 f(x - y)$. The eigenvalue of $H$ is $H_{n_+} = b^2 + 2c(2n_+ + 1)$, and there is a degeneracy which we will label as $N_-$. The degeneracy is expected to be $\sim N$.

Evaluating the determinants as before and defining $r^2_n = b^2 + 2c(2n + 1)$, the potential (not the phase shift) is

\[ V_n = N_- \frac{8}{\sqrt{\pi}} \int \frac{ds}{s} s^{-1/2} e^{-r^2_n s} \sinh^4 cs. \]  

(74)

Summing over $n$

\[ V = N_- \frac{8}{\sqrt{\pi}} \int \frac{ds}{s} s^{-1/2} e^{-r^2_n s} \sinh^4 cs \frac{2 \sinh 2cs}{2 \sinh 2cs}. \]  

(75)

The long range potential is then,

\[ V = N_- \frac{(2c)^3 \Gamma(5/2)}{4 \sqrt{\pi} b^5} \]  

(76)

We now turn to the type IIA configuration that matches the above configuration. It is just a two-brane and an anti two-brane with some magnetic
field $F$ on them. One can compute the potential between them as a function of the separation $b$ and the magnetic field $F$.

$$V_{IIA} = L^2 \int \frac{ds}{s} \frac{e^{-b^2 s}}{\sqrt{4\pi s}} B \times J, \quad (77)$$

$$B = \frac{2iF}{4\pi^2 f_1} \frac{\Theta'_1(0)}{\Theta_1(i\epsilon s)},$$

$$J = \frac{1}{2} \left\{-f_2^8 \frac{\Theta_2(i\epsilon s)}{\Theta_2(0)} + f_3^8 \frac{\Theta_3(i\epsilon s)}{\Theta_3(0)} \right\} \quad \text{and} \quad \left\{ -f_4^8 \frac{\Theta_4(i\epsilon s)}{\Theta_4(0)} \right\}. \quad (78)$$

Here $F = \tan \frac{\pi \epsilon}{2}$ and $L^2$ is the volume of the two-brane. Notice that for $F = 0$ one just gets the static potential between a two-brane and an anti two-brane, and of course we do not expect to get the full answer just from the lightest open string mode. However if $F$ starts becoming large then things change. As $F \to \infty$ we write $\epsilon = 1 - c'$, inserting that to equation (78) and using the transformation properties of the theta functions we get

$$B = \frac{2iF}{4\pi^2 f_1} \frac{\Theta'_1(0)}{\Theta_1(i\epsilon c)},$$

$$J = \frac{1}{2} \left\{-f_2^8 \frac{\Theta_2(i\epsilon c)}{\Theta_2(0)} + f_3^8 \frac{\Theta_3(i\epsilon c)}{\Theta_3(0)} \right\} \quad \text{and} \quad \left\{ -f_4^8 \frac{\Theta_4(i\epsilon c)}{\Theta_4(0)} \right\}. \quad (79)$$

Notice now the crucial sign change in the third term in $J$. Equation (79) now looks like two two-branes with some small magnetic field on one of them [27, 28], a configuration which we expect to have the long distance potential being reproduced by the light open string modes. This is the "magic" of the infinite momentum frame.

Expanding (79) for large $b$ one finds the long range potential

$$V_{IIA} = \frac{L^2 (\pi c')^2 \Gamma 5/2}{2\pi^{3/2} b^{-5}} \quad (80)$$

Since $F = \tan \frac{\pi \epsilon}{2}$ and $F = 1/c$ we see that $\pi c' = 2c$. Comparing equations (79) and (80), and using $c = \frac{L^2}{2\pi N}$ one sees that they agree if $N_- = 2N$. We will see a consistency check on this.

If we expand equation (79) in the limit where only the light open string modes contribute we find (with $a_2$ some coefficient)

$$B \times J = a_2 \frac{\sinh^4 \frac{\pi c'}{2}}{\sinh \pi c's}. \quad (81)$$
exactly like equation (74). This is another example how the M(atrix) calculation is just a short distance type IIA calculation. Notice that from equation (74) there is a tachyonic instability for $b^2 < 2c$.

5.4 Two moving membranes

Here we will consider two moving parallel membranes with relative velocity $v$. The background configuration is given by:

$$B_8 = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & -P_2 \end{pmatrix}, B_9 = \begin{pmatrix} Q_1 & 0 & 0 \\ 0 & -Q_2 \end{pmatrix}, B_7 = \begin{pmatrix} 0 & 0 & bI \\ 0 & 0 & vtI \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 \\ 0 & vtI \end{pmatrix}.$$

By now the derivation of the mass matrix is clear. We will first consider the situation where $[Q_1, P_1] = ic_1 = ic_2 = [Q_2, P_2]$. We look for the mass matrix in Euclidean space. For the bosons we find: six bosons with $M^2 = 2H$, one with $M^2 = 2H + 4iv$ and one with $M^2 = 2H - 4iv$. Where $H = (P_1 - P_2)^2 + (Q_1 + Q_2)^2 + I(b^2 + v^2t^2)$. For the fermions one finds: eight with $M^2_f = H + iv$ and eight with $M^2_f = H - iv$. the spectrum of $H$ is actually continuous in this case as $[(P_1 - P_2), (Q_1 + Q_2)] = 0$. Define $x = Q_1 + Q_2$, $y = Q_1 - Q_2$, then $(P_1 - P_2)^2$ is realized as $-4c^2\partial^2_x$ and $(P_1 + P_2)^2$ is realized as $-4c^2\partial^2_y$. Now $2dQ_1dQ_2 \rightarrow dx dy$, $2dP_1dP_2 \rightarrow dq dk$, and the correct phase space normalization has to include of course the $\frac{1}{2\pi}$. The spectrum of $H$ is $H_{xk} = b^2 + v^2t^2 + 4c^2k^2 + x^2$. Define:

$$r_{xk} = b^2 + 4c^2k^2 + x^2 \quad (82)$$

the phase shift is then

$$\delta = 8N_- \int dx dk \frac{d}{2\pi} \int ds \frac{e^{-r_{xk}}}{s \sin sv} \sin^4(sv/2). \quad (83)$$

This give a potential

$$V = N_- \frac{v^4\Gamma(5/2)}{8c\sqrt{\pi}} b^{-5} \quad (84)$$

Where the integral over $x, k$ was approximated by taking its limits to be $(-\infty, \infty)$.

What is the expected long range result from type IIA string theory? The configuration from the string theory point of view is that of two parallel moving membranes with a large world volume magnetic field on each one but which is the same (remember $c_1 = c_2$). It is well known that the one loop amplitude in this case is just the same as if there is no magnetic field, other than a multiple coefficient of the form $(1 + F^2)$ which is just the Born-Infeld action $[26, 27, 28]$. In our case $F = \frac{1}{L} = \frac{2\pi N_+}{L^2}$. Then the potential between two relatively moving two-branes is $[22, 16]$

$$V_{IIA} = \frac{L^2v^4\Gamma(5/2)}{8c^2(\pi)^{3/2}b^{-5}} \quad (85)$$

21
Equations (84) and (85) agree if we use the degeneracy that was used in the membrane anti-membrane case namely $2N$. This is an independent check on this degeneracy.

If we would have taken in this section $c_1 \neq c_2$ one would find a non zero force even at zero velocity and there would be a tachyonic instability at short distances $b^2 < (c_1 - c_2)$.

### 5.5 Two orthogonal moving membranes

Two orthogonal membranes is a supersymmetric configuration, but with only a quarter of the supersymmetries unbroken, thus we expect a $v^2$ term in the potential. The background for this configuration is

$$B_8 = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, B_9 = \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}, B_7 = \begin{pmatrix} 0 & 0 \\ 0 & bI \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & vtI \end{pmatrix}, B_5 = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}, B_9 = \begin{pmatrix} 0 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Performing the same calculations as before we find the spectrum of the bosons and fermions in Euclidean space. Define $H = P_1^2 + Q_1^2 + P_2^2 + Q_2^2 + b^2 + v^2 t^2$. For the complex bosons there are two with $M^2 = 2H$, one with $M^2 = 2H + 4iv$ one with $M^2 = 2H - 4iv$, one with $M^2 = 2H + 4c_1$, one with $M^2 = 2H - 4c_1$ one with $M^2 = 2H + 4c_2$ and one with $M^2 = 2H - 4c_2$. Remember that actually one has twice the number of real bosons with half the mass squared.

For the fermions we have: two with $M_f^2 = H + c_1 + c_2 + iv$, two with $M_f^2 = H + c_1 + c_2 - iv$, two with $M_f^2 = H - c_1 + c_2 + iv$, two with $M_f^2 = H - c_1 - c_2 - iv$, two with $M_f^2 = H + c_1 - c_2 + iv$, two with $M_f^2 = H + c_1 - c_2 - iv$, two with $M_f^2 = H - c_1 + c_2 + iv$, two with $M_f^2 = H - c_1 - c_2 - iv$, and two with $M_f^2 = H - c_1 + c_2 + iv$.

For $c_1 = c_2 = 0$ this is the result for two gravitons, and if $c_1 = 0 \neq c_2$ then this is the graviton membrane scattering.

Let us first take $c_1 = c_2 = c$. The spectrum of $H$ is $H_{n_1 n_2} = b^2 + v^2 t^2 + c(2n_1 + 2n_2 + 2)$. then the phase shift becomes

$$\delta = \int \frac{ds}{s} e^{-v^2 s} \frac{2 + 2 \cos 2vs + 4 \cosh 2cs - 4 \cos vs - 4 \sin vs \cosh 2cs}{8 \sinh^2 cs \sin vs},$$

(86)

Notice that for $v = 0$ this vanishes as expected. For large $b$ we can expand the integrand in powers of $s$, this gives

$$V = -\frac{(c^2 v^2 + \frac{1}{2} v^4) \Gamma(3/2)}{2c^2 \sqrt{\pi}} b^{-3}.$$

(87)

The string theory configuration is just two orthogonal two-branes with equal magnetic field on them. The phase shift is then given by the same
expression for a zero-brane scattered off the \((4 - 2 - 2 - 0)\) bound state in \([17]\). Let \(\epsilon = 1 - c'\), the phase shift takes the form \((\tanh \pi \nu = \nu)\)

\[
\delta_{IIA} = \frac{1}{2\pi} \int \frac{ds}{s} e^{-\nu s} B \times J.
\]  

\[88\]

\[
B = - \frac{1}{2} f_1^{-1} \Theta_1^{-2}(ic's) \frac{\Theta_1'(0)}{\Theta_1(\nu t)}.
\]

\[
J = \{ - f_2^4 \frac{\Theta_2(\nu s)}{\Theta_2(0)} \Theta_2^2(ic's) + f_3 \frac{\Theta_3^2(ic's)}{\Theta_3(0)} \Theta_4(\nu s) - f_4^4 \frac{\Theta_4(\nu s)}{\Theta_4(0)} \Theta_4^2(ic's) \}.
\]  

\[89\]

the long range potential is

\[
V_{IIA} = -\Gamma(3/2) \frac{2\left(1 - \cosh \pi \nu\right) \cos^2 \pi c' + \sinh^2 \pi \nu b^{-3}}{2\sqrt{\pi} \sin^2 \pi c'}
\]  

\[90\]

Expanding to lowest order in \(c'\) and \(\nu\), one finds exactly the result of \((87)\), with \(\pi c' = c\).

Again the expansion of the string result to the lowest modes of the open string will give exactly the M(atrix) result.

When \(c_1 \neq c_2\) then there is a non zero force even at \(\nu = 0\) and when \(b\) is small enough there is a tachyonic instability. Notice that there is a tachyonic instability when any two objects have different \(c'\)'s, that is when any two objects have different eleven dimensional velocities, very similar to string nucleation when two brane are relatively moving \([22, 28]\).

\section{Conclusions}

In this paper we have calculated the potentials between various configurations of 2-branes and gravitons by using the M(atrix) technique of \([4]\). We observed that the large boost made the configurations almost supersymmetric, and allowed the short distance answers obtained from Yang-Mills theory to reproduce also long range potentials, in accordance with one of the hopes of the conjecture in \([4]\). This fact is a more general version of the fact that in a boosted frame transverse velocities become small and nonrelativistic, which also brings us closer to a supersymmetric configuration.

One would, at the end, like to have potentials for branes that are not boosted, and thus are not in supersymmetric configurations. This requires us to relate the potentials obtained for the boosted branes to the potentials of the branes without the boost. In the case of the 2-brane - anti-2-brane system we saw in section 2 what form such a relation would take. The kinematics of
boosts in compact directions turned out to be somewhat different from the naive kinematics that would be expected for boosts in noncompact directions. It would be useful to have a more precise understanding of the kinematics in the compact case.

Following [9] we have noted that the 2-brane bound to many 0-branes can be described as a theory of the Yang-Mills for the 0-branes with a topological twist around the cycles where the two-brane is wrapped. This twist can be interpreted as a unit of magnetic field on the two dimensional space, thus characterised by the first chern class of the bundle in which the 0-brane variables take values. In this same spirit we expect that the 4-brane of type IIA theory, bound to many 0-branes, could be described by a similar treatment of zero branes moving on a 4-dimensional space with twists around the cycles corresponding to the presence of an instanton on the 4-dimesional space. We hope to study the interactions of four-branes with two-branes and gravitons in the future, using ideas similar to the ones used here for the interactions between two branes and gravitons.

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