Abstract

We study a tentative generally covariant quantum field theory, denoted the T-Theory, as a tool to investigate the consistency of quantum general relativity. The theory describes the gravitational field and a minimally coupled scalar field; it is based on the loop representation, and on a certain number of quantization choices. Four-dimensional diffeomorphism-invariant quantum transition probabilities can be computed from the theory. We present the explicit calculation of the transition probability between two volume eigenstates as an example. We discuss the choices on which the T-theory relies, and the possibilities of modifying them.
1 Introduction

In this work we construct a generally covariant quantum field theory, based on the classical theory of the gravitational field interacting with a minimally coupled scalar field. We illustrate how quantum transition amplitudes can be computed from this theory without breaking four-dimensional diffeomorphism invariance, and we compute some of these transition amplitudes explicitly. Our first motivation is to explore diffeomorphism invariant quantum field theories with infinite degrees of freedom, in view of the quantum gravity puzzle [1]. Diffeomorphism invariance has far reaching consequences: since no quantity local in the space-time coordinates can be diffeomorphism invariant, a genuinely general covariant quantum field theory cannot be constructed in the conventional framework of local quantum field theory, as it is synthesized for instance in [2]. The physical meaning of diffeomorphism invariance consists in the fact that locality is only defined with respect to dynamical objects of the theory itself – we think that this idea captures General Relativity’s main discovery about Nature. The theory constructed in this paper is an attempt to incorporate this idea within a non-trivial quantum field theory.

The second motivation of this work is to carry on the development of the loop representation approach to quantum general relativity [3]. This research program has advanced in two main directions during the last few years. On the one hand, the credibility of the results obtained has been strengthened by the development of a mathematical-physics approach that has put these results on a firm ground [4]. On the other hand, a number of novel results [5, 6, 7, 8, 9, 10] and calculation tools [11] have moved the theory forward, to the point where one can begin to perform physical calculations and to address the issues of the theory’s consistency and physical implications. In its present form, however, the loop representation of quantum gravity is not a complete theory [12, 13, 14]; essential missing elements are the complete determination of the scalar product, the choice of the ordering of certain key operators, and a general recipe for defining and computing diffeomorphism invariant physical expectation values. Various ideas and various proposals for solving each of these problems have been put forward, but it is not clear whether there exists a combination of those proposals that yields a consistent quantum theory with the correct classical limit. The discussion on problems such as ordering, definition of diffeomorphism invariant quantum observables, or choice of scalar product, has traditionally been quite academic and ideas were tested only within over-
simplified finite dimensional models. The recent developments of the loop representation that we mentioned allow ideas to be tested in the realistic case. In this paper, we complete the definition of the loop representation theory by choosing a number of assumptions, listed below. Our intent is explorative \[15\] and we do not take these assumptions for granted in any sense: the theory defined by these assumptions is a "tentative" theory, likely to turn out either inconsistent or physically incorrect; accordingly, we denote it as the Tentative-theory, or T-theory. We present this theory as a possible candidate for a non trivial generally covariant quantum field theory and in order to begin exploring the possible ways of completing the loop representation.

The main ingredients of our construction are the following. We begin with the classical theory formed by General Relativity with cosmological constant term and a minimally coupled scalar matter field \(\phi(x)\). Following \[16\] and \[17\], we study this theory in the gauge \(\phi(\vec{x},t) \propto t\). In this gauge the local degree of freedom of the scalar field disappears and reappears as a new "longitudinal" degree of freedom of the gravitational field, a phenomenon analogous to the disappearance of the Higgs field that gives mass to gauge bosons. In the gauge fixed form, the theory has a genuine Hamiltonian (instead of a hamiltonian constraint as the non-gauge-fixed theory), which generates the evolution of the gravitational field as a function of the value of the spatially constant scalar field. In other words, we regard the scalar field as the independent variable for the temporal localization of events. We quantize this theory using the loop representation \[18\] and we make use of the result of reference \[10\], where the loop-representation hamiltonian operator was constructed and shown to be finite. We use the recently discovered spin network basis \[11\]. This basis, which was suggested by T. Thiemann \[19\] and independently discovered by J. Baez \[20\] and T. Foxon \[21\] (and perhaps others), is the eigenbasis of the volume operator \[9\]. We assume that this basis is orthonormal. This assumption fixes the scalar product. The computation of the action of the Hamiltonian on the spin network basis states has been recently completed \[22\]. The expression for the Hamiltonian contains the square root of a certain operator. The key technical step we take here is the explicit computation of this square root. This yields an explicit expression for the action of the Hamiltonian in the spin network basis. Then we can use perturbation theory, as first suggested by L. Smolin \[23\], to compute first order transition amplitudes between volume eigenstates. In particular, in this paper we shall compute probabilities, according to the T-theory, that if the system is in a volume eigenstate \(|s_i\rangle\) when the scalar
field has value $t$, it be later found in a volume eigenstate $|s_f>$ when the
scalar field has value $t + \Delta t$.

Let us list here the assumptions that define the T-theory. We will discuss
them, as well as the possibility of modifying them, in the conclusion.

- Choice of the scalar product: The Hilbert space structure is defined
  by the orthonormality of the spin network basis.

- Evolution: It make sense to deal with the temporal part of diffeo-
morphism invariance by considering quantum evolution with respect
to one of the dynamical variables –the scalar field. The scalar field
can loosely be interpreted as a clock field. We refer to the abundant
literature on the so called “Issue of Time” in quantum gravity for the
numerous alternative positions on this matter [24]. Here, we follow
essentially [25].

- An essential aspect of the theory below is the presence of the cosmo-
  logical constant. Physically, it is needed to balance the energy density
  of the clock field, so that the clock can run fast without crumpling the
  universe.

- Locality: No notion of locality is given in the theory in gene-
  ral. Quantum states are represented by abstract combinatorial and
  topological relations and have no space-time localization whatsoever. Spatial lo-
  calization emerges only approximately and is defined with respect to
  the quantum state itself [3].

- The ordering of the Hamiltonian is the one chosen in the references
  [10, 22].

The main message of the present paper is that the loop representation of
General Relativity, supplemented by these assumptions yields a quantum
theory of gravity, the T-theory, in which physical transitions amplitudes
can be computed. Thus, diffeomorphism invariant quantum field theoretical
calculations can be performed.

This paper is organized as follows. Section 2 presents the classical theory.
Section 3 describes the quantum theory and the Hamiltonian. In section 4
the square root is computed. In section 5 perturbation theory is introduced,
and a transition amplitude is explicitly computed. In section 6 we discuss
the assumptions on which the theory is based, present our conclusions and
indicate future directions of research.
2 Classical theory

We consider General Relativity, with a cosmological constant term, and a minimally coupled massless scalar field $\phi(x)$. The action is

$$S[g_{\mu\nu}, \phi] = - \int d^4x \sqrt{g} \left\{ \frac{R}{16\pi G} - \frac{\lambda}{16\pi G} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\}. \quad (1)$$

We follow [26] for notation and conventions. We put the velocity of light equal to one, but we indicate the Planck constant $\hbar$ and the Newton constant $G$ explicitly. We assume that the constant $\lambda$ is positive. With the signs in (1), this choice corresponds to a negative cosmological energy density. The equations of motion for the gravitational field are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{1}{2} \lambda g_{\mu\nu} = -8\pi G \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \phi \partial_\rho \phi \right). \quad (2)$$

We partially fix the gauge invariance of the theory as follows. We choose the time coordinate $t$ according to

$$\phi(x, t) = \mu t. \quad (3)$$

The constant $\mu$ has been inserted in order to adjust dimensions. Later we will choose a particularly convenient value for $\mu$. The scalar field has dimensions of $\sqrt{M/L}$ (M is mass and L is length, we have $c = 1$). Thus $\mu$ has dimensions $\sqrt{M/L^3}$. In fixing the gauge (3), we restrict the domain of application of the theory, because in a general solution of (2) the regions of constant $\phi$ may not be space-like hypersurfaces, as the $t = constant$ surfaces have to be. Thus, we must restrict the theory to those solutions and those space-time regions in which the surfaces $\phi(x, t) = constant$ define a space like foliation of space-time. Let us denote those regions as the “clock regime”. The classical theory is fully consistent in the gauge (3), provided we restrict to this regime. The physical interpretation of this gauge is simple: because of general covariance, General Relativity does not describe evolution in an external absolute time, but rather it describes the relative evolution of gravitational and matter variables with respect to each other. We thus pick the scalar field $\phi$, rather arbitrarily, as the independent physical variable with respect to which we consider the evolution of the other physical variables. In this very weak sense we may view the scalar field as a phenomenological description of a clock. Later, we will see that in appropriate circumstances the value of $\phi$ coincides with proper time. As any physical clock in General Relativity, the
clock $\phi$ ceases to behave as a good clock in certain physical regions. Since evolution can bring the gravitational system into these regions, the evolution equations in the gauge (3) may blow up. Such a blow up signals the exit from the domain of validity of the gauge fixed formalism: from the physical point of view it signals the fact that the object we are taking as clock has ceased to behave as a good clock. In particular, it could stay still, or run backwards. The first of the assumptions on which the quantum theory we are going to build relies is that this procedure, which is fully viable in the classical theory, is also viable (within the approximation we are going to take) in the quantum theory.

Gauges of the form (3) are well known in General Relativity, and commonly considered in discussing the intrinsic observable evolution; indeed, the idea of using dynamical matter to define preferred coordinates goes back to Einstein’s papers. See for instance [34]. In particular the use of a scalar field was considered in [17], where advantages and difficulties of this choice are discussed. Our treatment is almost identical to the one of Smolin in [16], where the gauge $\nabla \phi = 0$ was considered. For clarity, we discuss here the relation between the two gauges. The gauge choice considered by Smolin is time independent, and therefore well defined on the conventional ADM phase space. It reduces the four dimensional diffeomorphism group down to the product of the three dimensional diffeomorphism group times the reparametrizations of a single variable, and it fixes all the infinite degrees of freedom of the scalar field variable except one: its spatially constant, time dependent value $\Phi(t) = \phi(\vec{x}, t)$. Smolin then makes a “choice of intrinsic time”, by picking $\Phi$ as the independent variable with respect to which evolution of the rest of the variables is considered. The gauge choice (3) considered here, on the other hand, is stronger than Smolin’s gauge; it reduces the four dimensional diffeomorphism group fully down to the three dimensional one. Since there is no residual time reparametrization invariance in the gauge fixed theory, the theory has a genuine time evolution, and the (controversial?) problem of “choosing an intrinsic time” within the canonical scheme does not appear. The results, at the end of the day, are equivalent.

As a first step towards the quantum theory, we derive the hamiltonian formalism of the theory in the gauge (3). We follow the traditional non-covariant derivation, but in the present context it is worthwhile to recall the well known fact that the result of such a derivation, namely the hamiltonian formalism by itself, is fully covariant (see for instance [27]). The complete procedure is to first derive the hamiltonian formalism and then add the gauge
fixing; however, the same result is obtained in the present case by means of
the short-cut of directly inserting the gauge choice into the Lagrangian. This
procedure has the additional advantage of saving us from the complications
of the time dependent gauge fixing. If we do so, we obtain the lagrangian
density
\[ L = -\sqrt{-g} \left\{ \frac{R - \lambda}{16\pi G} + \frac{\mu^2}{2} g^{00} \right\}. \] (4)
The theory is still invariant under spatial diffeomorphisms, but not anymore
under time diffeomorphisms. Replacing the 4-dimensional metric variable
with the ADM variables \( N = (-g^{00})^{-1/2} \), \( N^a = g_{0a} \), \( q_{ab} = g_{ab} \), where
\( a, b... = 1, 2, 3 \), we obtain (using the Gauss-Codazzi equations), the ADM
form of the lagrangian density
\[ L = N \sqrt{q} \left\{ \frac{k_{ab}k^{ab} - k^2}{16\pi GN^2} + \frac{\lambda - R}{16\pi G} + \frac{\mu^2}{2N^2} \right\}, \] (5)
where
\[ k_{ab} = \frac{1}{2}(\dot{q}_{ab} - D_a N_b - D_b N_a) \] (6)
\((N^{-1} \text{ times the extrinsic curvature of the constant-} t \text{ surfaces})\), and \( R \) –from
now on– is the scalar curvature of the spatial metric \( q_{ab} \). Since the equations
of motion obtained by varying the Lapse function \( N \) can be solved for \( N \) itself, we can insert the solution back into the Lagrangian, obtaining
the same equations of motion, and there is no constraint associated with \( N \) (See
for instance [28]). The equation that we obtain varying the Lapse is
\[ N^2 = \frac{k_{ab}k^{ab} - k^2 + \mu^2 8\pi G}{\lambda - R}. \] (7)
Inserting this value for \( N \) back into the lagrangian density, we obtain
\[ L = -\sqrt{q} \left( \frac{\sqrt{2\mu}}{8\pi G} \right) \left( \frac{\lambda - R}{16\pi G q} \right) \]
which, as can be verified, yields the correct equations of motion. The hamil-
tonian formalism is then easily derived. The only hamiltonian constraint
is the usual diffeomorphism constraint \( C_a = D_b p^{ab} \), where \( p^{ab} \) is the
momentum conjugate to the three metric \( q_{ab} \), and a tedious but unproblematic
computation yields the Hamiltonian
\[ H = \sqrt{2\mu} \int d^3 x \sqrt{\frac{16\pi G \left( p_{ab}p^{ab} - p^2 \right)}{16\pi G q}} + \frac{\lambda - R}{16\pi G q}, \] (9)
We emphasize the fact that (9) is not a constraint, but a genuine Hamiltonian. The correctness of this direct derivation of the Hamiltonian formalism can be verified by checking the evolution equations that the formalism determines. These reproduce the Einstein equations in the gauge (3). Thus, the classical Hamiltonian theory in the particular gauge considered is defined by the three metric $q_{ab}$ and its conjugate variable $p^{ab}$, subjected to the conventional first class ADM diffeomorphism constraint and evolving under the evolution generated by the Hamiltonian (9). This evolution is to be interpreted as the evolution of the metric variable as a function of the field variable $\phi$. The Hamiltonian (9) can be written in term of the conventional ADM Hamiltonian constraint (with cosmological constant) $C_{ADM}^{(\lambda)}$ of the pure gravity theory as

$$H = \sqrt{2} \mu \int d^3 x \sqrt{q} \sqrt{-C_{ADM}^{(\lambda)}}.$$  (10)

Notice that in spite of the fact that only gravitational variables appear in this gauge, the theory has 3 degrees of freedom per space point (6 components of $q_{ab}(x)$ minus three first class constraints $C_a(x)$) which correspond to the 2 degrees of freedom per point of the gravitational field, plus the degree of freedom of the scalar field.

One way of understanding the Hamiltonian (10) is to consider the Hamiltonian constraint of the non-gauge-fixed theory, which is

$$qC_{ADM}^{(\lambda)} + \frac{1}{2} \Pi^2 = 0$$  (11)

where $\Pi$ is the momentum conjugate to the scalar field. In the regions in which the time derivative of the scalar field, and therefore its conjugate momentum $\Pi$, are positive definite, this is equivalent to the constraint

$$\Pi + \sqrt{2} \sqrt{q} \sqrt{-C_{ADM}^{(\lambda)}} = 0.$$  (12)

In particular, we may consider the evolution generated by the component of the Hamiltonian constraint obtained by integrating (12) in space

$$\mu \int \Pi + \sqrt{2} \mu \int \sqrt{q} \sqrt{-C_{ADM}^{(\lambda)}} = 0.$$  (13)

The first term evolves only the scalar field, yielding (for suitable initial data) $\phi(x,t) = \mu t$, which is our gauge choice; the second term evolves only the gravitational variables, and is equal to the Hamiltonian (10) we have derived.
This short re-derivation is incomplete by itself, but has some virtues. First, it confirms that every evolution generated by the Hamiltonian $H$ is indeed a solution of Einstein’s equation in the coordinates in which $\phi(x, t) = \mu t$. Second, it clarifies the origin of the curious $\sqrt{2}$ factor in (10). Third, and most importantly, it shows that on all physical solutions the term $-C_{ADM}^{(\lambda)}$ inside the square root in the Hamiltonian is always non-negative. This is clear from (11). Therefore, the exit of the system from the clock regime does not correspond to the Hamiltonian becoming imaginary, as one might have imagined, but to the vanishing of the Hamiltonian density.

Let us discuss some aspects of the classical physics of the theory we are considering, and its possible regimes. In particular we want to point out the existence of a regime of the theory to which we will make explicit reference in the last section. To this aim, consider the energy balance equation, namely the component of the equations of motion (2) normal to the constant-$t$ surface (or the scalar hamiltonian constraint of the non-gauge fixed theory). This equation –which is equivalent to equations (7) or (11)– can be written as

$$\rho_{gr} = \rho_{\text{matter}} - \rho_{\lambda}$$

(14)

where: $\rho_{\text{matter}}$ is the energy density of the scalar field

$$\rho_{\text{matter}} = N^{-2} T_{00} = \frac{N^{-2}}{2} \partial_0 \phi \partial_0 \phi = \frac{N^{-2}}{2} \mu^2;$$

(15)

$\rho_{\lambda}$ is the absolute value of the (negative) cosmological energy density

$$\rho_{\lambda} = \frac{\lambda}{16\pi G};$$

(16)

and $\rho_{gr}$ is the “gravitational energy density”, namely the ADM scalar constraint of pure gravity

$$\rho_{gr} = \frac{16\pi G}{q} (p_{ab} p^{ab} - p^2) + \frac{R}{16\pi G} = C_{ADM}.$$  

(17)

In (14), $\rho_{\lambda}$, as well as $\rho_{\text{matter}}$, are non-negative. Therefore, for suitable initial data, these two terms may cancel –or approximately cancel. We denote such set of initial data as the “balanced clock regime” or, if they cancel approximately, the “approximate balanced clock regime”. The cosmological

---

1Since the derivative of a square root is its inverse, the inverse of the square root appears in the denominator of the Hamilton equations generated by $H$, and thus vanishing of the square root yields divergences of time derivatives.
term induces Einstein’s cosmic repulsion, while the energy density of the “clock” field tends to crumple space-time. The two can be balanced. In the balanced clock regime the gravitational variables satisfy the constraint corresponding to pure gravity. In a sense (and for a short time interval), the gravitational field can be unaware of the existence of the clock field and of the cosmological constant: the two balance out. Notice that this situation is compatible with arbitrary initial conditions of the gravitational field corresponding to a pure gravity field. We will be particularly interested in this regime because it is the regime in which the theory we are studying better mimics pure gravity. In the “approximate balanced clock regime” the cosmological term in the Hamiltonian is, by definition, much larger than the pure gravity term, which is zero, or close to zero, and therefore we can view the Hamiltonian (18), which can be written as

\[ H = \sqrt{2\mu} \int d^3 x \sqrt{\bar{q}} \sqrt{\frac{\lambda}{16\pi G}} - C_{ADM}, \quad (18) \]

as formed by the “unperturbed term” given by the cosmological term plus a small perturbation given by \( C_{ADM} \): the second term in the square root is much smaller than the first.

In order to increase the readability of our equations, we now choose a value for the constant \( \mu \). A readjustment of this value determines only a rescaling of the coordinate time \( t \), which is of course arbitrary. It is clear that if we could fix \( \mu \) in such a way that the coordinate time \( t \) be precisely equal to the proper time, then the legibility of our results be greatly improved. It is not possible to achieve this result generically (because the relation between the flow of proper time and the flow of \( \phi \) is determined dynamically and we cannot impose it), but it is possible to choose \( \mu \) in such a way that \( t \) is the proper time in particularly interesting physical regimes. In fact, \( t \) is the proper time if the Lapse function is 1, or, using (7) if

\[ k_{ab}k^{ab} - k^2 + \mu^28\pi G = \lambda - R. \quad (19) \]

In the exact balanced clock regime this gives

\[ \mu^2 = \frac{\lambda}{8\pi G}, \quad (20) \]

which is the value of \( \mu \) that we shall definitely assume from now on. With this choice of the value of \( \mu \), we have that in the balanced clock regime the
Hamiltonian evolves the system in proper time. Inserting this value of $\mu$ in (18) we have the Hamiltonian

$$H = 2\rho_\lambda \int d^3 x \sqrt{q} \sqrt{1 - \frac{C_{ADM}}{\rho_\lambda}}.$$  \hspace{1cm} (21)

An observation that will play a major role below is that the unperturbed term

$$H_0 = \frac{\lambda}{8\pi G} \int \sqrt{q} = 2\rho_\lambda V$$  \hspace{1cm} (22)

is –up to a constant– the total volume $V$.

Before going to the quantum theory, we shift to the Ashtekar variables [29]. In terms of the Ashtekar variables, the theory we are considered is defined as follows. The phase space is coordinatized by the Ashtekar conjugate variables $A^a_i$ and $\tilde{E}^{ai}$, $i = 1, 2, 3$, where $A^a_i$ is the space projection of the selfdual spin connection and $\tilde{E}^{ai}$ is the densitized controvariant triad, subjected to the first class Ashtekar’s gauge and diffeomorphism constraints [29]. The Ashtekar’s hamiltonian constraint, which in the presence of the cosmological constant is

$$\tilde{C}_{Ashtekar}^{(\lambda)} = \frac{\epsilon_{ijk}}{16\pi G} \left( F_{aij} \tilde{E}^{aj} \tilde{E}^{bk} - \frac{1}{3!} \lambda \epsilon \tilde{E}^{ai} \tilde{E}^{bj} \tilde{E}^{ck} \right)$$  \hspace{1cm} (23)

is absent, and the dynamical evolution is generated by the Hamiltonian

$$H = \sqrt{2}\mu \int d^3 x \sqrt{\tilde{C}_{Ashtekar}^{(\lambda)}}.$$  \hspace{1cm} (24)
3 Quantum Theory

We now construct the quantum theory corresponding to General Relativity in the gauge described above. We refer to [12], for the definition of the classical loop variables $T[\alpha]$ and $T^a[\alpha](s)$ in terms of the Ashtekar variables, and for the definition of the corresponding quantum operators $\hat{T}[\alpha]$ and $\hat{T}^a[\alpha](s)$. We refer to [11], for the definition of the spin network basis (that solves the theory’s Mandelstam identities) and the s-knot states $|s\rangle$ that solve the diffeomorphism constraint, and to [3, 4, 22] for the construction of the quantum operator corresponding to the Hamiltonian (24). Here, we briefly review the results of these references, in order to fix conventions and notation, and for completeness. We start from the conventional vector Ashtekar variables $A^i_a$ and $\tilde{E}^{ai}$, as defined in [29, 30]. We denote the Pauli matrices as $\sigma_i$. The spinorial Ashtekar connection is defined by

$$A_a = -\frac{i}{2} A^i_a \sigma_i$$

(Ashtekar convention). The spinorial triad is defined by

$$\tilde{E}^a = -2i \tilde{E}^{ai} \sigma_i$$

(Iwasaki convention). Given a loop $\alpha : S_1 \to M$ with components $\alpha : s \in [0, 2\pi] \mapsto \alpha^a(s)$, the Ashtekar parallel propagator matrix $U_\alpha(t,s)$ along $\alpha$ is the path ordered exponential of the Ashtekar connection along the loop, namely the $SL(2,C)$ matrix defined by

$$\frac{d}{ds} U_\alpha(t,s) = U_\alpha(t,s) \frac{d\alpha^a(s)}{ds} A_a(\alpha(s)), \quad \lim_{s \to t^+} U_\alpha(t,s) = 1.$$ 

We indicate $\lim_{s \to t^-} U_\alpha(t,s)$ (namely the parallel transport all around the loop) as $U_\alpha(t)$. The loop observables are defined by

$$T[\alpha] = Tr[U_\alpha(0)]$$

$$T^a[\alpha](s) = Tr[U_\alpha(s)E^a(\alpha(s))]$$

$$T^{ab}[\alpha](s,t) = Tr[U_\alpha(s,t)E^a(\alpha(t))U_\alpha(t,s)E^b(\alpha(s))]$$

with obvious generalizations for the loop observables with three or more indices. (Notice the absence of the factor $\frac{1}{2}$ used in some papers; the $\frac{1}{2}$ factors are rather disturbing when using spin network states.) The loop
representation of the Poisson algebra of the loop variables is defined on a linear space of functionals \( \psi(\gamma) \) of multiple loops \( \gamma \) (set of a finite number of loops). The algebraic dual of this space (the bras) is spanned by the loop states \( \langle \gamma \rangle \), defined by

\[
\langle \gamma | \psi \rangle = \psi(\gamma).
\]  

(32)

The representation is given as follows

\[
\hat{T}[\alpha] \psi(\gamma) = \psi(\gamma \cup \alpha)
\]

(33)

\[
\hat{T}^a[\alpha](s) \psi(\gamma) = \Delta^a[\gamma, \alpha(s)] \left( \psi(\gamma \# \alpha) - \psi(\gamma \# \alpha^{-1}) \right)
\]

(34)

\[
\hat{T}^{ab}[\alpha](s,t) \psi(\gamma) = \Delta^a[\gamma, \alpha(s)] \Delta^b[\gamma, \alpha(t)] \sum_{i=1}^{4} \psi(\alpha^{#i}\gamma)
\]

(35)

where \( \gamma \# \alpha \) is the loop obtained going around \( \gamma \) and then around \( \alpha \), and the four loops \( \alpha^{#i}\gamma \) are obtained rerouting the two intersections (at \( s \) and \( t \)) in all possible ways. For more details, see [12]. The distributional factor is

\[
\Delta^a[\gamma, x] = \frac{l_p^2}{\sqrt{\hbar G}} \int ds d\gamma^a(s) \delta^3(\gamma(s), x)
\]

(36)

where \( l_p = \sqrt{\hbar G} \) is the Planck length. The loop basis is overcomplete [12]. A complete and non overcomplete basis is the spin network basis, defined in [11]. In this paper, we will only consider the sector of the quantum theory defined by the trivalent spin network states. The extension to higher order spin networks implies additional algebraic complexity, which has not been entirely worked out yet. A trivalent spin network \( S \) is an oriented colored trivalent (namely with three links per node) graph, imbedded into the space manifold, in which the coloring (assignment of non-negative integers to each link and each node of the graph) satisfies certain conditions at each node: the sum of the coloring of the three links incidents on each node is even, and none of the three coloring is larger than the sum of the other two [33]. These conditions are equivalent to the requirement that it is possible to decompose the graph into a family of closed loops as follows: replace each link colored \( l \) with \( l \) overlapping segments and pair-wise join the segments at each node, in such a way that no two segments belonging to the same link are joined. Such a decomposition is not unique, since in general there are many ways of joining segments, namely of routing the loops through the nodes of the graph. If \( \gamma_1, ..., \gamma_M \) are all the possible decompositions of a spin network \( S \), then the spin network state \( \langle S \rangle \) is defined by

\[
\langle S \rangle = \sum_j (-1)^{n_j+c_j+1} \langle \gamma_j \rangle
\]

(37)
where \( n_j \) is the number of single loops in \( \gamma_j \) and \( c_j \) is the number of crossings in an arbitrary planar representation of \( \gamma_j \) (so that the sum produces in fact an anti-symmetrization of the loops along each link; the overall sign is determined by the orientation of the spin network \([11]\)). It can be shown that the spin network states (including higher order spin networks) form a non-overcomplete basis \([11]\).

Since the spin network states form a basis, a ket state \(|\psi\rangle\) is completely characterized by the quantities \(\langle S|\psi\rangle\), which from now on we denote as \(\psi(S)\). Since the basis is not overcomplete, any assignment of quantities \(\psi(S)\) determines a state of the theory. In particular, we may define spin network characteristic (ket) states \(|\psi_S\rangle\) by

\[
\psi_S(S') = \begin{cases} 
1 & \text{if } S = S' \\
0 & \text{otherwise.}
\end{cases}
\]

(38)

The action of the elementary loop operators (35) on the spin network states is directly computed from (35) and (37). The diffeomorphism constraint can then be solved easily. The solution are labeled by the \(s\)-knots, namely by the diffeomorphism equivalent classes of imbedded spin networks, which we denote as \(s\). For each class \(s\), there is a (ket) state \(\psi_s\) that solves the diffeomorphism constraints, defined by

\[
\psi_s(S) = \begin{cases} 
1 & \text{if } S \in s \\
0 & \text{otherwise.}
\end{cases}
\]

(39)

We indicate the state \(\psi_s\) also as \(|s\rangle\). These states are labeled by knotted and linked (sets of) colored graphs. Each of these states represents an independent diffeomorphism invariant physical quantum state of the gravity+scalar field theory. Notice that the \(s\)-knots \(s\), which label the physical quantum states of the gravitational-scalar field system are not imbedded in space, they are abstract objects in the same sense in which knots of knot theory are. We recall here that Roger Penrose introduced spin networks precisely in an attempt to describe quantum geometry. In the present formalism, the abstract spin networks (\(s\)-knots) describe precisely the quantum states of the geometry. The \(s\)-knot states carry more information than in Penrose’s original version: first, they carry information about their knotting and linking, second, they may have intersections of order higher than three. The key addition with respect to Penrose’s construction, of course, is the knowledge of the set of quantum operators acting on the space spanned by these quantum states.
We now come to the second main assumption that define the T-theory. We promote the state space we have defined to an Hilbert space by choosing a scalar product. This is uniquely determined by requiring the s-knot states to be orthonormal

$$\langle s|s'\rangle = \delta_{ss'}.$$  \hfill (40)

We discuss this choice in the last section. We just notice here that, since the states $|s\rangle$ are linearly independent, the definition (40) is consistent, unlike earlier preliminary suggestions to postulate that the knot states be orthonormal $[12]$.

In order for an operator to be well defined on the space of the diffeomorphism invariant physical quantum states, this operator must be diffeomorphism invariant. The next step in the definition of the theory is thus to recognize diffeomorphism invariant observables, express them in terms of the loop operators, and compute their action on the s-knot states. Since diffeomorphism invariant quantities are in general non-trivial (and in particular non-linear) functions of the elementary fields, the construction faces the difficulty of defining operator products. Conventional regularization procedures fail, in general, in the present context, because they almost invariably break diffeomorphism invariance. Diffeomorphism invariant regularization techniques have then been developed in $[31, 5, 9, 10]$. These techniques can be viewed as diffeomorphism invariant analogs of normal-ordering prescriptions. Using these techniques, various operators of physical interest have been constructed. Once the regularization procedure has been chosen, the computation of the action of the operators on the states is a tedious but straightforward exercise.

In particular, the volume operator, which corresponds to the classical observable

$$V = \int d^3x \sqrt{\det q}$$ \hfill (41)

has been constructed in $[3]$, and shown to be diagonal in the spin network basis. It acts on the trivalent states as follows

$$\hat{V} |s\rangle = \frac{1}{4} f^3_p \sum_{i \in s} \sqrt{\hat{\nu}_i} |s\rangle;$$ \hfill (42)

here the index $i$ labels the nodes of $s$ and the operator $\hat{\nu}_i$ is defined by

$$\hat{\nu}_i |s\rangle = \nu_i |s\rangle$$ \hfill (43)

where

$$\nu_i = a_ib_ic_i + a_ib_i + b_ic_i + c_ia_i,$$ \hfill (44)
The integers \(a_i, b_i, c_i\) are defined by

\[
p_i = a_i + b_i, \quad q_i = b_i + c_i, \quad r_i = c_i + a_i,
\]

where \(p_i, q_i\) and \(r_i\) are the colors of the three links adjacent to the node \(i\). Geometrically, \(a_i\) is the number of segments routed through \(i\) between the \(p_i\) and the \(r_i\) link, \(b_i\) is the number of segments routed between the links \(p_i\) and \(q_i\), and so on. Therefore, an \(s\)-knot state \(|s\rangle\) is an eigenstate of the volume with eigenvalue

\[
V(s) = \frac{1}{4} l_p^3 \sum_{i \in s} \sqrt{a_i b_i c_i + a_i b_i + b_i c_i + c_i a_i}.
\]

The Hamiltonian operator \(\hat{H}\) corresponding to the Hamiltonian (24) has been constructed in [10]. Its action on the trivalent states is given by

\[
\hat{H}|s\rangle = \sqrt{\frac{2\mu^2}{16\pi G}} \sum_i \sqrt{l_p^4 Z \hat{M}_i + \frac{\lambda l_p^6}{16}} \hat{\nu}_i |s\rangle
\]

\[
= \rho \lambda l_p^3 \sum_i \sqrt{\hat{\nu}_i + \alpha \hat{M}_i} |s\rangle
\]

\[\alpha = \frac{16Z}{\lambda l_p^2}\tag{48}\]

\(Z\) is an arbitrary (finite) renormalization constant. Notice that \(\alpha\) is dimensionless. The action of the operator \(\hat{\nu}_i\) on a node \(i\) has been given above. The Hamiltonian node operator \(\hat{M}_i\) acts on a trivalent node as follows

\[
\hat{M}_i = \sum_{l=1,2,3} \sum_{\epsilon=1,-1} \sum_{\epsilon'=-1} A_{\epsilon\epsilon'}(p_l, q_l, r_l) \hat{D}_{i;\ell\ell'};
\]

the index \(l\) labels the three links adjacent to the node \(i\); \(r_l, p_l\) and \(q_l\) are the colors of these three links, in the following order: \(r_l\) is the color of the link \(l\); \(p_l\) the next one and \(q_l\) the last one, in the order given by the orientation of the spin network. Finally, \(\hat{D}_{i;\ell\ell'}\) is the operator that acts on an \(s\)-knot node by: (i) creating two additional nodes, one along each of the two links -different from \(l\)- adjacent to the node \(i\), (ii) creating a novel link, colored 1, joining these two nodes, (iii) assigning the coloring \(p_l + \epsilon\) and, respectively, \(q_l + \epsilon'\) to the links that join the new formed nodes with the node \(i\). This is illustrated in Figure 1.
The explicit computation of the $A_{eq'}$ coefficients has been recently concluded \cite{22}, giving

\begin{align}
A_{eq'}(p, q, r) &= \frac{B_{eq'}(p, q, r)}{(p + 1)(q + 1)}, \\
B_{++}(p, q, r) &= pq, \\
B_{+-}(p, q, r) &= -(q + r)pc, \\
B_{-+}(p, q, r) &= -(p + r)qa, \\
B_{--}(p, q, r) &= (p + 2)(q + 2)(pq + b - ac),
\end{align} \tag{50}

where $a$, $b$ and $c$ are defined as in \cite{15}.

While the general structure and the coefficients of the action of the operator $\hat{M}$ are determined by the classical Hamiltonian and by the regularization procedure, there is nevertheless freedom in defining the precise geometrical action of the $\hat{D}$ operator. This freedom is a conventional quantum mechanical ordering ambiguity. Some alternatives are ruled out immediately because they yield trivial or inconsistent hamiltonian operators; to some extent the remaining ambiguity is resolved by the requirement of preserving gauge invariance: the geometrical action must be well defined on knot classes. While no credible alternative ordering is known at the moment, there is no reason to believe that the one considered here is unique. Therefore, the choice of the ordering given by $\hat{D}$ represents another assumption in the process of constructing the diffeomorphism invariant T-theory.

There is a remaining open problem before having the explicit action of the hamiltonian operator on an arbitrary trivalent $s$-knot: to compute the operator square root in \cite{17}. The next section is devoted to solve this problem.
4 Square root

This section is rather technical. It derives the missing technical ingredient for the computation of transition amplitudes in the T-theory, namely the extraction of the square root in the Hamiltonian operator. To this aim, let us introduce an index $\mu = 1 \ldots 12$ as a collective index for the indices $l, \epsilon$ and $\epsilon'$ in (49). We also use $A^\mu(i) = A_{\epsilon \epsilon'}(p_l, q_l, r_l)$. We indicate a node $i$ as $|i\rangle$ and

$$|i, \mu\rangle = \hat{D}_{i, \mu} |i\rangle$$

(51)

indicates the portion of the s-knot that replaces $|i\rangle$ after the action of the operator. Adopting Einstein’s convention on repeated $\mu$–indices, we then can write (49) as

$$M_i |i\rangle = A^\mu(i) |i, \mu\rangle.$$  

(52)

Our task is to extract the square root in (47). We seek an operator $\hat{H}_i$ such that

$$\hat{H}_i^2 |i\rangle = v_i |i\rangle + \alpha A^\mu(i) |i, \mu\rangle,$$  

(54)

where the number $v_i$ is given in (44). The key observation is that $\hat{H}_i^2$ is –in a sense– lower triangular, and therefore we can seek for $\hat{H}_i$ of this same form. We thus assume the following form for $\hat{H}_i$

$$\hat{H}_i |i\rangle = a(i) |i\rangle + \alpha a^\mu(i) |i, \mu\rangle + \alpha^2 a^{\mu\nu}(i) |i, \mu\nu\rangle + \ldots$$  

(55)

where $|i, \mu\nu\rangle$ is given by

$$|i, \mu\nu\rangle = \hat{D}_{i, \nu} \hat{D}_{i, \mu} |i\rangle,$$  

(56)

and so on. We can further simplify the notation by dropping the index $i$ everywhere, since everything here is happening around a fixed intersection $i$. We now compute the coefficients $a = a_i$, $a^\mu = a^\mu(i)$, $a^{\mu\nu} = a^{\mu\nu}(i)$, ... by comparing (54) and (55). By acting twice on the state $|\rangle = |i\rangle$ with $\hat{H} = \hat{H}_i$, we obtain

$$\hat{H} \hat{H} |\rangle = \hat{H} \left[ a |\rangle + \alpha a^\mu |\mu\rangle + \alpha^2 a^{\mu\nu} |\mu\nu\rangle + \ldots \right]$$

$$= a \left( a |\rangle + \alpha a^\mu |\mu\rangle + \alpha^2 a^{\mu\nu} |\mu\nu\rangle + \ldots \right) + \alpha a^\mu (a |\mu\rangle + \alpha a^\nu (a |\mu\nu\rangle + \ldots) + \ldots)$$

18
\[ + \alpha^2 a^{\mu \nu}(\mu)v(\mu \nu)|\mu \nu\rangle \\
+ ... \\
= a^2|\rangle + (a + a(\mu)) \alpha a^\mu|\mu\rangle \\
+ \left[(a + a(\mu \nu)) \alpha^2 a^{\mu \nu} + \alpha^2 a^{\nu}(\mu)a^\mu\right]|\mu \nu\rangle + ... \quad (57) \]

Where \( v(\mu) \) indicates the value of \( v_i \) on the node \( i \) after the action of \( \hat{D}_{i,\mu} \) (the coloring of two of the links has been altered by a unit: \( \epsilon \) or \( \epsilon' \)); similarly, \( a^{\nu}(\mu) \) is the coefficient \( a^{\nu} \) acting on the same altered node. We now equate state by state with (54); notice that these equalities determine a sufficient but not necessary condition for (53) to hold, because the \( |\mu \nu...\rangle \)'s are not linearly independent in general. For the moment, however, we are not concerned with uniqueness. We obtain

\[ a = \sqrt{v} \quad (58) \]
\[ a^\mu = \frac{A^\mu}{a + a(\mu)} \quad (59) \]
\[ a^{\mu \nu} = \frac{a^{\mu}a^{\nu}(\mu)}{a + a(\mu \nu)}. \quad (60) \]

By repeating this procedure for higher orders, it is easy to conclude that the general form of the coefficients is

\[ a_{\mu_1...\mu_n} = \frac{a^{\mu_1...\mu_n}(i)}{a + a(\mu_1...\mu_n)} \]

Equations (58,59,61) express all the coefficients \( a^{\mu_1...\mu_n}(i) \) of the Hamiltonian, in terms of the quantities \( v(i) \) and \( A^\mu(i) \). The node Hamiltonian is then

\[ \hat{H}_i = \sum_{n=0}^{\infty} \alpha^n a^{\mu_1...\mu_n}(i) |i, \mu_1...\mu_n\rangle \quad (62) \]

And the full Hamiltonian is

\[ \hat{H}|s\rangle = \frac{\lambda}{8\pi G} \int_0^3 \sum_i \hat{H}_i |s\rangle \quad (63) \]

Since the evolution generated by the classical Hamiltonian breaks down at finite times, we do not expect the quantum Hamiltonian operator to be finite and well defined on all states. On which states is it ill defined? There are two potential sources of difficulties: one is of course the infinite sum
in the node Hamiltonian (62), which may not converge on certain states. But there is another one which we should discuss. In the definition (61) of the coefficients $a^{\mu_1...\mu_n}(i)$ we divide by the (square root of the) volume eigenvalues $v(i)$ of the intersection $i$. Looking at equation (44), we see that there is a degenerate case in which the $v(i)$ may vanish. This is the case in which two out of three numbers $a, b, c$ vanish (they cannot all vanish). In turn, this case corresponds to a (degenerate) trivalent intersection in which one of the three links has color zero, which is to say a “bivalent” intersection, or a point of non differentiability along the loop. These points were denoted “kinks” in early works on the loop representation, and cannot be discarded a-priori from the state space, because states with kinks are in the image of operators such as the hamiltonian itself. The operator (63) is ill defined on spin networks that contain such kinks. It is natural to suspect that states with kinks correspond to points were the clock regime breaks down, but such a conclusion is far from obvious, and more insight is needed. In particular, the ansatz that $H_i$ is lower diagonal could simply be wrong when kinks are present.

Finally, we recall that $\frac{\lambda}{16\pi G} = \frac{1}{2}\mu^2$ is the energy density $\rho_{\text{matter}}$ of the clock field; therefore if we denote the energy of the clock field per Planck volume as $E_0 = \frac{1}{2}\mu^2 l_p^3$, we may write

$$\hat{H} = 2E_0 \sum_i \sum_{n=0}^{\infty} \alpha^n a^{\mu_1...\mu_n}(i) \hat{D}_{\mu_1}...\hat{D}_{\mu_n}.$$

(64)

The sum in $i$ is over all the nodes of the spin network $s$. The coefficients $a^{\mu_1...\mu_n}(i)$ are explicitly given in equations (44,50,58,59,61), and are dimensionless functions of the colorings. The model is defined by two constants: $E_0$, which represents the energy of the clock per Planck volume, and the dimensionless constant $\alpha$, related to the ratio of the cosmological constant and the Planck energy. This completes the definition of quantum general relativity in the clock gauge.

5 Perturbation theory

By separating the first term from all the others in (64), we can rewrite the Hamiltonian as

$$\hat{H} = \frac{\lambda}{8\pi G} \hat{V} + \hat{W},$$

(65)
where $\hat{V}$ is the volume operator and $\hat{W}$ is

$$
\hat{W} = 2E_0 \sum_i \sum_{n=1}^{\infty} \alpha^n a_{\mu_1}^{i} \ldots a_{\mu_n}^{i} \hat{D}_{i;\mu_1} \ldots \hat{D}_{i;\mu_n}.
$$

(66)

In the balanced clock regime discussed at the end of section 2, the volume term dominates over the $\hat{W}$ term. We can therefore consider the possibility of viewing the term $\hat{W}$ as a perturbation. Notice that we do not need $\alpha$ to be small. This is a particularly interesting program for various reasons: first, the balanced clock regime is the regime in which our theory better approximates pure General Relativity; second, the eigenstates of the unperturbed Hamiltonian, namely the volume, are completely known. Indeed, they are precisely the spin network states that form the basis in which we are working. Developing a perturbation scheme is then straightforward. The idea of developing a perturbation scheme around the volume as unperturbed Hamiltonian was first suggested by L. Smolin [23]. In the unperturbed theory, the time evolution of the basis states is given by

$$
|s, t\rangle = e^{-i/\hbar E_{st}}|s\rangle
$$

(67)

where the energy $E_s$ of the $s$-knot state is (from (42) and (44))

$$
E_s = E_0 \sum_{i \in s} \sqrt{a_i b_i c_i + a_i b_i + b_i c_i + c_i a_i},
$$

(68)

and $t$ represents, thanks to the choice (20) of $\mu$, the proper time. We can define an interaction picture by expanding a generic time dependent state $|\psi, t\rangle$ in this time dependent $s$-knot basis

$$
|\psi, t\rangle = \sum_s \psi(s, t) e^{-i/\hbar E_{st} t} |s\rangle.
$$

(69)

Using conventional perturbation theory techniques, the amplitude of a transition from a state $|s_i\rangle$ to a (different) state $|s_f\rangle$ in a time $t$ is given by

$$
\langle s_f, t|s_i, 0 \rangle = \frac{-i}{\hbar} \int_0^t dt' W_{fi} e^{i/h(E_f - E_i)t}
\left[ \frac{-i}{\hbar} \right]^2 \int_0^t \int_0^{t'} dt' dt''
\times \sum_n W_{fn} W_{ni} e^{i/h(E_f - E_n)t'} e^{i/h(E_n - E_i)t''}
+ ... \right.
$$

(70)
where we introduced the notation

$$W_{ab} = \langle s_a | W | s_b \rangle$$  \hfill (71)

for the matrix elements of the perturbation term. Since these are time independent, we can perform the integrations, giving

$$\langle s_f, t | s_i, 0 \rangle = -W_{fi} \frac{e^{i/\hbar(E_f-E_i)t}}{E_f - E_i} + \sum_n W_{fn} W_{ni}$$

\begin{align*}
&\times \left[ \frac{e^{i/\hbar(E_f-E_n)t}}{(E_n - E_i)(E_f - E_i)} - \frac{e^{i/\hbar(E_f-E_n)t}}{(E_n - E_i)(E_f - E_n)} \right] \\
&+ \ldots \hfill (72)
\end{align*}

Up to now, we have made no approximations. For small time, we can expand the exponentials, giving

$$\langle s_f, t | s_i, 0 \rangle = -W_{fi} \frac{i}{\hbar} t + O(t^2)$$  \hfill (73)

So that, to first order in $t$, the transition probability is

$$P_{i \to f} = |\langle s_f, t | s_i, 0 \rangle|^2 = \frac{1}{\hbar^2} |W_{fi}|^2 t^2$$  \hfill (74)

If we further restrict our conditions and assume that $\alpha$ is small, then the first term in the sum (66) dominates over the others, and we can write

$$W_{fi} = \langle s_a | 2E_0 \alpha \sum_i a^\mu(i) \hat{D}_{i;\mu} | s_b \rangle + O(\alpha^2)$$

\begin{align*}
&= \langle s_a | 2E_0 \alpha \sum_i \frac{A^\mu(i)}{a(i) + a(i,\mu)} \hat{D}_{i;\mu} | s_b \rangle + O(\alpha^2) \hfill (75)
\end{align*}

The physical meaning of taking $\alpha$ small is that the cosmological constant energy density and the clock energy density (which, under our assumptions balance each other) are both large compared with the Planck energy density. This is perhaps a rather unrealistic assumption, and we consider it here only for illustrative purposes. As an example, let us compute a simple transition probability in the approximations considered. Let $s_i$ be the $s$-knot formed by two nodes, connected by three links, with colors 1,1 and 2. Let $s_f$ be the $s$-knot formed by 4 nodes (arranged as the vertices of a tetrahedron), with the couples of opposite (non adjacent) links having colors (1,1), (2,2) and (3,1). See Figure 2.

22
We begin by computing the action of the first term of the perturbation series on $|s_i\rangle$. The sum over the nodes gives only a factor of 2, for symmetry. On each node, there are three terms produced (one per couple of links). Two of these are equal, again by symmetry, and it is easy to see that the third fails to produce anything proportional to $s_f$. We thus have 4 equal terms (2 nodes $-$ 2 non-vanishing terms each). Of the four terms $(++, +-, --, -++)$ produced by each of these 4, it is the $(++)$ that yields a result proportional to $s_f$. We have in this case:

$$r = 1, \quad p = 2, \quad q = 1,$$

which yields:

$$a = 1, \quad b = 1, \quad c = 0.$$  \hfill (76) \hfill (77)

From which we have:

$$v(i) = \sqrt{abc + ab + bc + ca} = 1,$$

and:

$$A_{++} = \frac{(p + 2)(q + 2)}{(p + 1)(q + 1)}(pq + b^2 - ac) = 6$$  \hfill (79)

After the action of the operator, the node has colors:

$$r = 1, \quad p = 3, \quad q = 3,$$

which yields:

$$a = 1, \quad b = 2, \quad c = 0.$$  \hfill (80) \hfill (81)

From which we have:

$$v(i\mu) = \sqrt{abc + ab + bc + ca} = \sqrt{2},$$

23
Bringing everything together we have finally

\[ P_{i \rightarrow f} = \frac{1}{\hbar} 2E_0 \alpha 4 \frac{6}{1 + \sqrt{2}} t^2 = Z^2 \frac{2^{10}9}{(3 + 2\sqrt{2})\pi^2} \frac{c^2}{l_p^2} t^2, \]  

(83)

where we have reinserted the velocity of light \( c \neq 0 \) for clarity. A completely analogous calculation gives

\[ P_{i \rightarrow g} = Z^2 \frac{2^{10}9}{(3 + 2\sqrt{2})12^2 \pi^2} \frac{c^2}{l_p^2} t^2. \]  

(84)

for the transition probability to the state \(|s_g\rangle\) with the same graph as \(|s_f\rangle\), but with the couples of opposite (non adjacent) links having colors (1,1), (2,2) and (1,1). See Figure 2. Therefore the relative probability for \(|s_i\rangle\) to make a transition (in a short time) to \(|s_f\rangle\) or to \(|s_g\rangle\) is

\[ \frac{P_{i \rightarrow g}}{P_{i \rightarrow f}} = \frac{1}{144}. \]  

(85)

What is interesting here, of course, are not the computed values, but the fact that the above machinery allows us to compute qualitative predictions of decay ratios.

6 Discussion

Using non-perturbative techniques, and in particular the loop representation, we have a constructed a quantum theory describing two interacting fields: the gravitational field and a massless scalar field. The main ansatzs on which the theory relies are the following.

- Scalar product: The weakest assumption of the theory is the choice of the scalar product. The scalar product chosen satisfies some crucial requirements: the operator corresponding to the physical volume, for instance, is self-adjoint with respect to this scalar product.

On the other hand, it is far from clear, and perhaps dubious, that

\footnote{As made clear by the hamiltonian analysis, the theory has three degrees of freedom per space point: two gravitational and the scalar field’s one. In spite of its apparent disappearance in the gauge considered, therefore, the physical scalar field is alive and well, and its dynamics is fully taken into account. Despite immediate appearances, the quantum theory describes a quantum gravitational field and a quantum scalar field.}
the scalar product \( (40) \) could satisfy all the necessary conditions for the correct recovery of the classical limit. In particular, we recall that the choice of the scalar product corresponds to the implementation of the reality conditions in the quantum theory. The reality conditions on the connection are non-trivial in the Ashtekar formalism, and we do not know whether they are implemented by \( (40) \). The choice made may very well yield the Euclidean, rather than Lorentian, General Relativity in the classical limit – or none of the two. There is a strict relation between the problem of choice of the scalar product and possible self-adjointness requirements on the Hamiltonian discussed below; Smolin has suggested \( [23] \) that the scalar product \( (40) \) could be modified in order to make the Hamiltonian self-adjoint, and that this may be done order by order in the perturbation expansion (the zeroth order is already self-adjoint). The effect of the non-triviality of the Ashtekar reality conditions on the choice of the scalar product is being investigated by Ashtekar Lewandowski and collaborators \( [4] \), and we expect this investigation to shed light on this issue. Here we leave the problem open. On the other side, notice that the doubts about the correctness of the choice \( (40) \) do not bear on the issue of the consistency of the diffeomorphism invariant quantum theory we are constructing; what is at stake here is not the consistency of the quantum theory, but rather the possibility of recovering the correct classical limit.

Evolution: The idea that we can describe evolution in a covariant fashion by evolving with respect to an arbitrary variable is very old \( [34] \). A potential difficulty with its implementation in the quantum context is that with the ordering chosen, the Hamiltonian fails to be a densely defined self-adjoint operator. (This follows from the observation that the Hamiltonian may raise, but may not lower, the number of nodes, and therefore cannot be symmetric in the -orthogonal- spin network basis. With a different ordering, this is not necessarily the case. See below.). Self-adjointness of the Hamiltonian is not necessary for the consistency of the conventional probabilistic interpretation: non-unitary evolution is routinely employed in quantum mechanics to describe the dynamics of objects, for example, having a finite probability of decaying. In those cases the failure of the evolution to be unitary simply signals the probability that, say, the position of a particle may have no value at all at some later time, because the particle has decayed. In the
present case, the lack of unitarity of the evolution in the (rather arbitrary) independent variable signals the probability that the system will not reach a later value of such a variable: the scalar field may stop increasing. This has been discussed in detail elsewhere [25]. Whether or not such a procedure is viable is a much debated issue [24]; we want to suggest that a convincing solution of this issue might come from exploring the consequences of various proposed solutions within a reasonably realistic infinite dimensional theory as the one proposed here. A separate problem, we believe, is how the physical “flowing” time emerges from a generally covariant quantum field theory, in which no preferred time variable exists—the scalar field is an arbitrarily chosen independent variable, not at all necessarily connected with the “perceived flow” of physical time. We are convinced that this issue should be addressed in a different context [35], and is not relevant here.

• Ordering: Alternative orderings of the Hamiltonian have to be explored. The simplest alternative is to consider the symmetric part of the operator $\hat{H}$, which is a possible step towards self-adjointness. Taking the symmetric part of the Hamiltonian shouldn’t affect the classical limit of the theory, since the classical hamiltonian is real on the physical solutions. It would also be interesting to find uniqueness results, under the requirement of diffeomorphism invariance of the regularized operator. On the hamiltonian constraint see [36]. Similarly, it may be that alternative definitions of the square root are available.

• Diffeomorphism invariance. One should distinguish two different kinds of invariance requirements. On the one hand, the theory should be gauge invariant in the sense that physical quantities that are invariant under four-dimensional diffeomorphisms must be identified and the theory should yield expectation values for those quantities. The transition amplitudes computed here satisfy this requirement. In this sense, four dimensional diffeomorphism invariance is implemented in the theory in spite of the fact that the gauge has been fixed. This is like saying that QED predictions are gauge invariant, even if computed in the Lorentz gauge. On the other hand, the quantities computed here are transition amplitudes with respect to a specific evolution parameter, the scalar field, arbitrarily chosen. This fact raises the question of the relation between those quantities and quantities that represent evolution with respect to a different evolution parameter. This is a
very interesting issue, but we are not going to address it here.

- Use of loop observables as fundamental observables: Experience in quantum field theory indicates that field operators are too singular to be integrated in just one dimension, and one may suspect that loop operators cannot be defined in an interacting quantum field theory. This is a strong objection against the loop representation, but we believe that this objection overlooks an aspect the theory: The loop operators are not defined in the quantum theory. They play only an intermediate role at the unconstrained level of the non-diffeomorphism invariant state space. Any physical operator –as the hamiltonian operator considered in this paper– is integrated in three dimensions at least. Notice that the physical states, namely the $s$-knot states, do not have support on loops, but rather are formed by diffeomorphism invariant “extensions” of loop states, and can loosely be thought as a smearing over all space –or, better over all smooth deformations– of loop states supported in one dimension.

The diffeomorphism invariance of general relativity expresses the discovery that spatio-temporal location (locality) is physically meaningful only in reference to other dynamical components of the theory [37, 38, 34]. The incorporation of this aspect of General Relativity into quantum field theory requires a major step ahead with respect to local quantum field theory. The theory constructed in this paper is a crude attempt to take this step and to sketch such a general relativistic quantum field theory. The distance covered from local quantum physics is substantial: quantum states, operators and the very notion of evolution appear here in de-spatialized form: spatial, as well as temporal, location are only defined relationally in the T-theory. We emphasize in particular the unusual structure of the quantum states. They have no space-time (nor momentum space) dependence. Rather, a state is characterized by topological and combinatorial relations only. This feature derives from the strict implementation of three dimensional diffeomorphism invariance in the quantum field theory. Because of diffeomorphism invariance any genuine gauge invariant property expresses solely relative positions of physical structures (among these, the gravitational field); the combinatorial-topological structure of the $s$-knot quantum states expresses gauge invariant relative positions in a location-independent way. This is complemented by our treatment of time, in which temporal-location and evolution are replaced by temporal location and evolution relative to a physical field. In these two ways, the quantum theory we are
constructing is defined in a general relativistic manner. Most of the previous attempts to combine general covariance and quantum field theory (see for instance the various contributions in [39], and in [40]) have so far been carried on only in the restricted domain of theories with a finite number of degrees of freedom –topological quantum field theories [41] are the most interesting of these attempts. (This has produced the curious and obviously wrong popular belief that any fully diffeomorphism invariant field theory has a finite number of degrees of freedom.)

The extent to which the T-theory is successful in this attempt is far from clear to us. One may consider three criteria of evaluation of the theory: consistency, completeness, and classical limit. None of the three is clearly satisfied by the T-theory. Consider consistency. State space and hamiltonian operator of the T-theory are well defined (up to the incompleteness related to non-trivalent intersections). We have shown that it is not difficult to compute quantum transitions amplitudes to first order in some approximation. The potential problem is given by the divergences that can appear at higher orders. We do not know whether divergences appear, and, in the likely case in which they do, whether they can be controlled. We consider this as the most urgent problem to be investigated. Next, consider the classical limit issue. The procedure of starting from a classical theory and promoting it to a quantum theory by means of some quantization prescription is designed to yield a quantum theory with the desired classical limit. However, it is not clear whether in our construction we have followed all steps of a complete quantization prescription, and there is at least one key check that we failed to perform, which is the consistency between the Hilbert structure chosen and the full set of classical reality conditions. We just remark, once more, that even a –non-trivial– theory with the incorrect classical limit may have interest as an example of general relativistic quantum field theory. Finally, as far as completeness is concerned, we recall that all realistic physical quantum field theories presently utilized are badly incomplete: We know such theories either via a perturbation expansion that certainly does not cover the entire relevant physics, or by means of non-perturbative approaches, all of which provide approximations to this or that physical regime. Therefore, we do not expect completeness in the present context. The problem, is not so much completeness, but rather in which regimes (if any) is the theory predictive: the hope expressed here is that the T-theory might represent a tool for obtaining physical predictions at the Planck scale.

We conclude with a list of problems that we think deserve further investigation: (i) Study self-adjointness properties of other physical observables,
in order to check the scalar product (40) and study possible modifications of (40). (ii) Study alternative orderings for the Hamiltonian; search for a uniqueness result. (iii) Are there alternative ways of defining the square root? (iv) What is the physical interpretation of the kinks (bivalent intersections)? Can the definition of the square root be extended to incorporate them? (v) Compute eigenvalues of volume and the action of the Hamiltonian on higher valence intersections. (vi) Derive transition amplitudes in a different independent evolution parameter and compare the results. (vii) Study the physical meaning of the renormalization constant $Z$ in (48); is the low energy Newton constant related to the renormalized or unrenormalized Planck length? (vii) Study the convergence of the sum (64) that defines the node Hamiltonian and of the sum over intermediate states in second order transition amplitudes (see eq.(70)); are there uncontrollable divergences? (vii) Can the classical theory be reconstructed from the diffeomorphism invariant quantum theory? (viii) Extend the theory to fermions, along the lines of reference [3].

I thank Bernd Brügmann, Alain Connes, Jürgen Ehlers, Rudolf Haag, Gary Horowitz, Chris Isham, Ted Jacobson, Karel Kuchař, Ted Newman, Roger Penrose and Julius Wess, for various comments, discussions and difficult questions. I am particularly indebted to Abhay Ashtekar for discussions and criticisms, to Ranjeet Tate for important comments and a careful reading of the manuscript, and to Lee Smolin: many of the ideas on which this work is based emerged in discussions with Lee. This work was partially supported by the NSF grant PHY-9311465.

References

[1] For a recent brief overview of the present state of researches in quantum gravity, see CJ Isham: “Prima Facie Questions in Quantum Gravity” in Canonical Gravity: From Classical to Quantum. Proceedings of the 117th WE Heraeus Seminar, Bad Honnef, Germany, September 1993, J Ehlers and H Friedrich eds, Springer Verlag Berlin 1994

[2] R Haag: Local Quantum Physics, Springer Verlag, Berlin 1992

[3] C Rovelli, L Smolin: Phys Rev Lett 61 (1988) 1155; Nucl Phys B133 (1990) 80
[4] A Ashtekar, C Isham: Class and Quant Grav 9 (1992) 1069; A Ashtekar, J Lewandowski: in [39]; A Ashtekar, D Marolf, J Mourão: in The Proceedings of the Lanczos International Centenary Conference, ed D Brown et al SIAM Philadelphia 1994; A Ashtekar, J Lewandowski, D Marolf, J Mourão, T Thiemann: Quantum geometrodynamics, Journal of Mathematical Physics, to appear 1995; Coherent state transform on the space of connections, Journal of functional analysis, to appear 1995; J Baez in The Proceedings of the Lanczos International Centenary Conference, op cit; Diffeomorphism invariant generalized measures on the space of connections modulo gauge transformations, Proceedings of the conference on Quantum Topology, eds L Crane, D Yetter, hep-th/9301063, Link invariants, functional integration, and holonomy algebras hep-th/9301063.

[5] A Ashtekar, C Rovelli, L Smolin: Phys Rev Lett 69 (1992) 237

[6] H Morales-Tecotl, C Rovelli: Phys Rev Lett 72 (1994) 3642; “Fermions in Quantum Gravity”, Pittsburgh University preprint (1995)

[7] J Iwasaki C Rovelli: Int J of Mod Phys D1 (1993) 533; Class and Quantum Grav 11 (1994) 1653

[8] Rovelli C: Nuclear Physics B405 (1993) 797

[9] C Rovelli, L Smolin: Nucl Phys, to appear (1994)

[10] C Rovelli, L Smolin: Phys Rev Lett 72 (1994) 446

[11] C Rovelli, L Smolin: “Spin Networks in Quantum Gravity”, in preparation, 1995

[12] C Rovelli: Class Quant Grav 8 (1991) 1613

[13] B Brügmann: in Canonical Gravity: From Classical to Quantum. Proceedings of the 117th WE Heraeus Seminar, Bad Honnef, Germany, September 1993, J Ehlers and H Friedrich eds, Springer Verlag Berlin 1994

[14] A Ashtekar: in Canonical Gravity: From Classical to Quantum. Proceedings of the 117th WE Heraeus Seminar, Bad Honnef, Germany, September 1993, J Ehlers and H Friedrich eds, Springer Verlag Berlin 1994
[15] A Einstein, M Grossmann: “Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation”, Zeitschrift für Matematik und Physik 62 (1914) 225

[16] L Smolin: in the Brill Festschrift Proceedings, ed B-L Hu, T Jacobson, (CUP, Cambridge, 1993)

[17] K Kuchař, C Torre: Phys Rev D43 (1991) 419; D44 (1991) 3116; C Stone, K Kuchař, Class Quantum Gravity 9 (1991) 757

[18] R Gambini, A Trias: Phys Rev D23 (1981) 553; Lett al Nuovo Cimento 38 (1983) 497; Phys Rev Lett 53 (1984) 2359; Nucl Phys B238, 436 (1986); B278 (1986) 436; R Gambini, L Leal, A Trias: Phys Rev D39 (1989) 3127; R Gambini: Phys Lett B 255 (1991) 180

[19] T. Thiemann: unpublished

[20] J Baez: “Spin Network in Gauge Theory”, Advances in Mathematics, to appear 1995, gr-qc/941107

[21] TJ Foxon: “Spin networks, Turaev-Viro theory and the loop representation” Class and Quant Grav, to appear 1995, gr-qc/9408013

[22] R Borissov, C Rovelli, L Smolin: “Physical Evolution of Spin-network States”, in preparation

[23] L Smolin, unpublished notes

[24] CJ Isham: “Canonical Quantum Gravity and the Question of Time” in Canonical Quantum Gravity: from Classical to Quantum. Proceedings of the 117th WE Heraeus Seminar, Bad Honnef, Germany, September 1993, J Ehlers and H Friedrich eds, Springer Verlag Berlin 1994; K. Kuchař, Time and interpretations of quantum gravity in the Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, eds. G. Kunstatter, D. Vincent and J. Williams, World Scientific, Singapore, 1992

[25] C Rovelli: in Conceptual Problems of Quantum Gravity ed A Ashtekar and J Stachel, Birkhauser, Boston 1991; Phys Rev D 42 (1991) 2638; Phys Rev D 43 (1991) 442; Phys Rev D44 (1991) 1339; Class and Quantum Grav 8 (1991) 317
[26] S Weinberg: *Gravitation and Cosmology*, John Wiley & Sons, New York 1972

[27] A Ashtekar, L Bombelli, O Reula: in *Analysis, Geometry and Mechanics: 200 Years after Lagrange*, eds M Francaviglia and D Holm, North Holland, Amsterdam, 1990

[28] A Ashtekar, J Romano, R Tate: Phys Rev D40 (1989) 2572

[29] A Ashtekar: Phys Rev Lett 57 (1986) 2244; Phys Rev D 36 (1987) 1587

[30] A Ashtekar: *Non-perturbative canonical gravity, Lecture notes in collaboration with Ranjeet S Tate*, World Scientific, Singapore 1991

[31] L Smolin: in *Quantum Gravity and Cosmology*, World Scientific, Singapore, 1993

[32] A Ashtekar: in the Proceedings of the 1992 Les Houches lectures, B Julia ed, Elsevier, to appear

[33] R Penrose: in *Quantum Theory and beyond*, ed E A Bastin, Cambridge University Press 1971

[34] A Einstein: *Über die spezielle und die allgemeine Relativitätstheorie*, Vieweg, Braunschweig, 1920; PG Bergmann: Helv Phys Acta Suppl 4 (1956) 79; BS DeWitt in “Gravitation: an introduction to recent research”, ed L Witten, Wiley, New York 1962; J Stachel: Einstein search for general covariance 1912-1915 in Einstein and the History of General Relativity eds DHoward and J Stachel, Einstein studies, Volume 1, Birkhouser, Boston, 1989; see [17] for a historical account of this idea

[35] A Connes, C Rovelli: Class and Quant Grav 11 (1994); C Rovelli: Class and Quant Grav 10, 1549 (1993); (1993) 1567

[36] MP Blencowe: Nucl Phys B341 (1990) 213; V Husain: Nucl Phys B313 (1988) 711; B Brügmann, J Pullin: Nucl Phys B363 (1991) 221; B Brügmann: *The Hamiltonian Constraint in the Loop Representation* PhD Thesis 1992, unpublished; J Baez, JP Munian, D Piriz: *Quantum Gravity Hamiltonian for Manifolds with Boundary* [gr-qc/9501016]

[37] P G Bergmann: Rev of Mod Phys 33 (1961) 510
[38] C Rovelli: Class and Quantum Grav 8 (1991) 297

[39] Knots and Quantum Gravity ed J Baez, Oxford University Press, Oxford 1994

[40] “Quantum Geometry and Diffeomorphism Invariant Quantum Field Theory, Special Issue of the Journal of Mathematical Physics”, C Rovelli L Smolin eds, to appear 1995

[41] E Witten: Comm Math Phys 117 (1988) 353; 121 (1989) 351; Nuc Phys B311, 46 (1988); M Atiyah: The geometry and Physics of Knots, Cambridge University Press 1990