A Strict Positivstellensatz for Enveloping Algebras

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Abstract

Let $G$ be a connected and simply connected real Lie group with Lie algebra $\mathfrak{g}$. Semialgebraic subsets of the unitary dual of $G$ are defined and a strict Positivstellensatz for positive elements of the universal enveloping algebra $\mathcal{E}(\mathfrak{g})$ of $\mathfrak{g}$ is proved.

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1 Introduction

Positive polynomials on semialgebraic sets have been intensively studied since E. Artin’s solution of Hilbert’s 17-th problem. In the last decade a number of new representation theorems for positive polynomials, usually called ”Positivstellensätze”, have been found (see e.g. [S1], [P], [R], [PV]). Excellent surveys are given in the recent books [PD], [M] and the article [S]. In a previous paper [S3] a variant of a non-commutative Positivstellensatz for the Weyl algebra was obtained. The aim of this paper is to prove a strict Positivstellensatz for enveloping algebras of finite dimensional Lie algebras.

Let $G$ be a connected and simply connected real Lie group with Lie algebra $\mathfrak{g}$ and let $\mathcal{E}(\mathfrak{g})$ be the complex universal enveloping algebra of $\mathfrak{g}$. The algebra $\mathcal{E}(\mathfrak{g})$ is a $\ast$-algebra with involution determined by $x^\ast = -x$ for $x \in \mathfrak{g}$. Let 1 denote
the unit element of $\mathcal{E}(\mathfrak{g})$. Let $\{x_1, \ldots, x_d\}$ be a basis of $\mathfrak{g}$ which will be fixed throughout this paper.

The algebra $\mathcal{E}(\mathfrak{g})$ has a canonical filtration $((\mathcal{E}_n(\mathfrak{g})))_{n \geq 0}$, where $\mathcal{E}_0(\mathfrak{g}) = \mathbb{C} \cdot 1$ and $\mathcal{E}_n(\mathfrak{g})$, $n \geq 1$, is the linear span of 1 and products $z_1, \ldots, z_r$ with $z_1, \ldots, z_r \in \mathfrak{g}$ and $r \leq n$ (see e.g. [D], 2.3). The associated graded algebra is the polynomial algebra $\mathbb{C}[t_1, \ldots, t_d]$, where the monomial $t_1^{k_1} \cdots t_d^{k_d}$ corresponds to the element $x_1^{k_1} \cdots x_d^{k_d}$ of $\mathcal{E}(\mathfrak{g})$. For an element $c \in \mathcal{E}(\mathfrak{g})$ of degree $n$, we denote by $c_n(t)$ the polynomial of $\mathbb{C}[t_1, \ldots, t_d]$ corresponding to the component of $c$ with degree $n$.

Set $x_0 := i \cdot 1$, where $i$ denotes the complex unit. Then we have $x_j^* = -x_j$ for $j = 0, \ldots, d$. Define

$$a := x_0^*x_0 + x_1^*x_1 + \cdots + x_d^*x_d = 1 - x_1^2 - \cdots - x_d^2. \quad (1)$$

Let $S$ be a right Ore subset of $\mathcal{E}(\mathfrak{g}) \setminus \{0\}$ containing $a$. That is, for any $s \in S$ and $z \in \mathcal{E}(\mathfrak{g})$ there are elements $s' \in S$ and $z' \in \mathcal{E}(\mathfrak{g})$ such that $sz' = zs'$. Note that such that a set $S$ exists since $\mathcal{E}(\mathfrak{g}) \setminus \{0\}$ is a (left and right) Ore set of $\mathcal{E}(\mathfrak{g})$ (see [D], 3.6). For instance, if $a$ belongs to the center of $\mathcal{E}(\mathfrak{g})$, then we may take the set of elements $a^n$, $n \in \mathbb{N}_0$, as $S$.

Let $f = (f_1, \ldots, f_r)$ be a finite set of hermitean elements of the enveloping algebra $\mathcal{E}(\mathfrak{g})$ such that $f_1 = 1$. Let $\mathcal{K}_f$ and $\mathcal{T}_f$ be the associated basic closed semialgebraic set and positive wedge, respectively, as defined by formulas (3) and (4) below.

The main result of this paper is the following

**Theorem 1.1** Suppose that $c$ is a hermitean element of the enveloping algebra $\mathcal{E}(\mathfrak{g})$ of even degree $2m$ satisfying the following assumptions:

(i) There exists $\varepsilon > 0$ such that $c - \varepsilon \cdot 1 \in \mathcal{T}_f$.

(ii) $c_{2m}(t) > 0$ for all $t \in \mathbb{R}^d$, $t \neq 0$.

If $m$ is even, there exists an element $s \in S$ such that $s^*cs \in \mathcal{T}_f$. If $m$ is odd, there is an $s \in S$ such that

$$\sum_{k=0}^{d} s^*x_k^*cx_k s \in \mathcal{T}_f.$$
Let $\mathcal{A}$ be a unital complex $*$-algebra. We denote by $\sum^2(\mathcal{A})$ the set of all finite sums of squares $x^*x$, where $x \in \mathcal{A}$. A subset of the hermitean part $\mathcal{A}_h := \{x \in \mathcal{A} : x^* = x\}$ is called an $m$-admissible wedge ([S2], p.22) if $\mathcal{C}$ is a wedge (that is, $x + y \in \mathcal{C}$ and $\lambda x \in \mathcal{C}$ if $x, y \in \mathcal{C}$ and $\lambda \geq 0$) such that the unit element is in $\mathcal{C}$ and $z^*xz \in \mathcal{C}$ for $x \in \mathcal{C}$ and $z \in \mathcal{A}$.

2 Unitary Representations and Semialgebraic Sets of the Dual

By a unitary representation of the Lie group $G$ we mean a strongly continuous homomorphism $U$ of $G$ into the group of unitary operators of a Hilbert space $\mathcal{H}(U)$. Let $\mathcal{D}^\infty(U)$ denote the vector space of $C^\infty$-vectors of $U$ and let $dU$ be the associated $*$-representation of the $*$-algebra $\mathcal{E}(g)$ on the dense domain $\mathcal{D}^\infty(U)$ of $\mathcal{H}(U)$ (see [S2], Chapter 10, or [Wa], Section 4.4, for details). For $f \in \mathcal{E}(g)$ we write $dU(f) \geq 0$ when $\langle dU(f)\varphi, \varphi \rangle \geq 0$ for all vectors $\varphi \in \mathcal{D}^\infty(U)$.

For later use we restate some classical results of E. Nelson and W. F. Stinespring [NS] and of E. Nelson [N] in

**Lemma 2.1** If $U$ is a unitary representation of $G$, then the closure $\overline{dU(a)}$ of the operator $dU(a)$ is self-adjoint and equal to $B := I - \sum_{k=1}^d \overline{dU(x_k)^2}$. Moreover,

$$\mathcal{D}^\infty(U) = \bigcap_{n=1}^\infty \mathcal{D}(\overline{dU(a)^n}) = \bigcap_{n=1}^\infty \mathcal{D}(B^n).$$

**Proof.** For the self-adjointness of $\overline{dU(a)}$ and the first equality of (2) see Corollaries 10.2.4 and 10.2.7 in [S2] or Theorems 4.4.3 and 4.4.4 in [Wa]. We prove that $\overline{dU(a)} = B$. By Lemma 10.4.5 in [S2] or by Lemma 4.4.4.8 in [Wa] there is constant $c > 0$ such that

$$\|dU(x_k)\varphi\| + \|dU(x_k)^2\varphi\| \leq c\|dU(a)\varphi\|, \varphi \in \mathcal{D}^\infty(U), k = 1, \ldots, d.$$

Since $B\varphi = dU(a)\varphi$ for $\varphi \in \mathcal{D}^\infty(U)$, the preceding inequality implies that $\mathcal{D}(\overline{dU(a)}) \subseteq \mathcal{D}(B)$ and $B\varphi = dU(a)\varphi$ for $\varphi \in \mathcal{D}(\overline{dU(a)})$. Since $B$ is a symmetric extension of the self-adjoint operator $\overline{dU(a)}$, we get $\overline{dU(a)} = B$. \hfill \Box

Let $\hat{G}$ denote the unitary dual of $G$, that is, $\hat{G}$ is the set of unitary equivalence classes of irreducible unitary representations of $G$. For each $\alpha \in \hat{G}$ we fix a representation $U_\alpha$ of the equivalence class $\alpha$. 

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**Definition:** A subset $K$ of $\hat{G}$ is called semialgebraic if $K$ is a finite Boolean combination (that is, using unions, intersections and complements) of sets $K_f = \{ \alpha \in \hat{G} : dU_\alpha(f) \geq 0 \}$ with $f \in \mathcal{E}(g)$.

For $r \in \mathbb{N}$ and an $r$-tuple $f = (f_1, \ldots, f_r)$ of elements $f_j \in \mathcal{E}(g)$ such that $f_1 = 1$, we define the basic closed semialgebraic set

$$
K_f = \{ \alpha \in \hat{G} : dU_\alpha(f_1) \geq 0, \ldots, dU_\alpha(f_r) \geq 0 \}
$$

and the associated positive wedges

$$
\mathcal{T}_f = \{ z = \sum_{j=1}^{k} \sum_{l=1}^{r} z_{jl}^* f_l z_{jl} ; z_{jl} \in \mathcal{E}(g), k \in \mathbb{N} \},
$$

$$
\mathcal{E}(g; f)_+ = \{ z \in \mathcal{E}(g)_h : dU_\alpha(z) \geq 0 \text{ for } \alpha \in K_f \}.
$$

Clearly, $\mathcal{T}$ and $\mathcal{E}(g; f)_+$ are $m$-admissible wedges of $\mathcal{E}(g)$ such that $\mathcal{T}_f \supseteq \mathcal{E}(g; f)_+$.

If $f = (1)$, then $K_f = \hat{G}$. In this case we denote $\mathcal{E}(g; f)_+$ by $\mathcal{E}(g)_+$. From decomposition theory (see e.g. [S2]) it follows that $\mathcal{E}(g)_+$ is the set of elements $x \in \mathcal{E}(g)$ such that $dU(x) \geq 0$ for all unitary representations $U$ of the group $G$.

**Example 1** $G = \mathbb{R}^d$

Then $g$ is the abelian Lie algebra $\mathbb{R}^d$ and $\mathcal{E}(g)$ is the polynomial algebra $\mathbb{C}[x_1, \ldots, x_d]$. For each point $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$ there is an irreducible unitary representation $U_t$ of the Lie group $\mathbb{R}^d$ on $\mathcal{H}(U_t) = \mathbb{C}$ such that $dU_t(x_j) = t_j, j = 1, \ldots, d$. Because these representations $U_t$ exhaust the dual of $\mathbb{R}^d$, we can identify the dual of the Lie group $\mathbb{R}^d$ with $\mathbb{R}^d$. In this manner the semialgebraic sets according to the definition given above are just the "ordinary" semialgebraic sets in semialgebraic geometry ([PD], p. 31). For an $r$-tuple $f = (f_1, \ldots, f_r)$ elements $f_j \in \mathbb{C}[x_1, \ldots, x_d]$ with $f_1 = 1$, let $\tilde{f}$ be the $\binom{r}{2} + 1$-tuple of elements $f_1$ and $f_i f_j, i \neq j, i, j = 1, \ldots, r$. Then we have $K_f = K_{\tilde{f}}$. The wedge $\mathcal{T}_{\tilde{f}}$ is the usual preorder from semialgebraic geometry and $\mathcal{E}(g; f)_+$ is the set of real polynomials which are nonnegative on $K_f$. Note that for noncommutative Lie groups products $f_i f_j$ cannot be added to the wedge $\mathcal{T}_f$ in general because the product of two noncommuting hermitean elements of $\mathcal{E}(g)$ is not even hermitean.

**Example 2** $G = SU(2)$

The Lie algebra of $SU(2)$ has a basis $\{x_1, x_2, x_3\}$ satisfying the commutation relations

$$
[x_1, x_2] = x_3, [x_2, x_3] = x_1, [x_3, x_1] = x_2.
$$

(5)
The element \( a = 1 - x_1^2 - x_2^2 - x_3^2 \) generates the center of the algebra \( \mathcal{E}(g) \). The unitary dual of \( G \) consists of equivalence classes of spin \( l \) representations \( U_l \), \( l \in \frac{1}{2}\mathbb{N}_0 \). We identify \( \hat{G} \) and \( \frac{1}{2}\mathbb{N}_0 \). Set \( H = -ix_1 \). Here we only need that the operators \( dU_l(H) \) and \( dU_l(a) \) act on an orthonormal basis \( e^l_j, j = -l, -l+1, \ldots, l \), of a \((2l+1)\)-dimensional Hilbert space by

\[
dU_l(H)e^l_j = 2je^l_j, \quad dU_l(a)e^l_j = (l^2 + l + 1)e^l_j.
\]

Clearly, a polynomial \( p(H) \) in the generator \( H \) is in \( \mathcal{E}(g)_+ \) if and only if \( p(n) \geq 0 \) for all \( n \in \mathbb{Z} \). Using the relations (5) it is easily shown that \( p(H) \in \sum^2(\mathcal{E}(g)) \) if and only if there is a polynomial \( q(H) \in \mathbb{C}[H] \) such that \( p(H) = q(H) \ast q(H) \). In particular, \( (H - c_1)(H - c_2) \notin \sum^2(\mathcal{E}(g)) \) if there is an integer \( n \) such that \( n \leq c_1 < c_2 \leq n + 1 \).

Let \( M = \{ l_1, \ldots, l_r \} \) be a finite set of \( \frac{1}{2}\mathbb{N}_0 \cong \hat{G} \). Put \( f_1 = g_1 = 1, f_{j+1} = 2l_j - H \) and \( g_{j+1} = -(a - l_j^2 - l_j - 1))^2 \) for \( j = 1, \ldots, r \). Then \( \mathcal{K}_f = \mathcal{K}_g = M \), so \( M \) is a semi-algebraic subset of \( \hat{G} \).

### 3 An Auxiliary *-Algebra

In what follows \( U \) denotes a fixed unitary representation of the Lie group \( G \) on a Hilbert space \( \mathcal{H}(U) \). In this section we define and study an auxiliary *-algebra \( \mathcal{X} \) depending on \( U \).

For notational simplicity we abbreviate

\[
X_k := dU(x_k), k = 0, \ldots, d, \quad \text{and} \quad A := dU(a).
\]

Clearly, \( \langle A\varphi, \varphi \rangle \geq \langle \varphi, \varphi \rangle \) for \( \varphi \in \mathcal{D}^\infty(U) \). Hence the inverse \( Y := A^{-1} \) exists and maps \( \mathcal{D}^\infty(U) \) onto \( \mathcal{D}^\infty(U) \) by Lemma 2.1.

Let \( \mathcal{X} \) denote the algebra of operators acting on the invariant dense domain \( \mathcal{D}^\infty(U) \) of \( \mathcal{H}(U) \) generated by the identity map \( I \equiv I_D \),

\[
Y_{kl} := X_kX_lY \quad \text{for} \quad Y_{-l,-k} := YX_lX_k \quad \text{and} \quad k, l = 0, \ldots, d. \quad (6)
\]

Clearly, \( \mathcal{X} \) is a *-algebra of operators with involution

\[
Y_{kl}^* = Y_{-l,-k}, \quad k, l = 0, \ldots, d. \quad (7)
\]
Let $\mathcal{X}_0$ denote two-sided $*$-ideal of $\mathcal{X}$ generated by $Y_{k0}, k = -d, \ldots, d$. Let $c_{ij}^k$ denote the structure constants of the Lie algebra $\mathfrak{g}$ and set $b_{ij}^k := c_{ij}^k + c_{ik}^j$. Then we have

$$X_iX_j - X_jX_i = \sum_{k=1}^{d} c_{ij}^k X_k. \quad (8)$$

The operators $Y_{kl}, X_j$ and $Y = -Y_{00}$ satisfy the following relations:

$$\sum_{k=0}^{d} Y_{k0}^* Y_{k0} = Y, \quad (9)$$

$$Y_{k0}^* Y_{l0} - Y_{l0}^* Y_{k0} = -i \sum_{j=1}^{d} c_{k}^j Y_{l0}^* Y_{j0} \text{ for } k, l = 1, \ldots, d, \quad (10)$$

$$Y_{k0}^* - Y_{k0} = i \sum_{j,l=1}^{d} b_{k}^j Y_{l0}^* Y_{j0} \text{ for } k = 1, \ldots, d, \quad (11)$$

$$\sum_{k,l=0}^{d} Y_{kl}^* Y_{kl} = I + i \sum_{j,k,l=1}^{d} b_{kl}^j Y_{j0}^* Y_{kl} = I - i \sum_{j,k,l=1}^{d} b_{kl}^j Y_{kl}^* Y_{j0}, \quad (12)$$

$$Y_{kl} - Y_{kl}^* \in \mathcal{X}_0, Y_{kl} - Y_{lk} \in \mathcal{X}_0 \text{ for } k, l = 0, \ldots, d, \quad (13)$$

$$Y_{x} - xY \in Y \mathcal{X}_0 = \mathcal{X}_0 Y \text{ for } x \in \mathcal{X}, \quad (14)$$

$$Y_{kl} Y_{ij} - Y_{ij} Y_{kl} \in \mathcal{X}_0 \text{ for } k, l, i, j = 1, \ldots, d, \quad (15)$$

$$Y_{kl} Y_{ij} - Y_{ki} Y_{lj} \in \mathcal{X}_0 \text{ for } k, l, i, j = 1, \ldots, d. \quad (16)$$

Note that relations (9) – (11) above have been found by W. Szymanski [Sz]. They are relations (3) – (5) in [Sz]. Setting $l = 0$ in (7) we get relations (1) – (2) in [Sz].

The relations (9) – (16) listed above are proved by straightforward algebraic manipulations using the Lie algebra commutation relation (8). Using (1), (8) and the abbreviations $b_{ij}^k = c_{ij}^k + c_{ik}^j, Y = dU(a)^{-1}$ we obtain

$$YX_k = X_k Y + \sum_{i,j=1}^{d} b_{kl}^j YX_i YX_j, \quad (17)$$

$$YX_k X_l = X_k X_l Y + \sum_{i,j=1}^{d} (b_{kl}^j YX_k X_i X_j Y + b_{kl}^j YX_i X_j X_l Y) \quad (18)$$

for $k, l = 1, \ldots, d$. All relations (9) – (16) are easily derived from (17) and (18) combined with (6). We omit the details of these verifications.
Lemma 3.1 For arbitrary numbers \( n \in \mathbb{N} \) and \( k_1, \ldots, k_{4n} \in \{0, \ldots, d\} \), we have
\[
Y^n X_{k_1} \cdots X_{k_{2n}} \in \mathcal{X} \\
Y^n X_{k_1} \cdots Y_{k_{4n}} Y^n - Y_{k_{1}k_{2}} \cdots Y_{k_{4n-1}k_{4n}} \in \mathcal{X}_0.
\]

Proof. Both equations are proved by induction on \( n \). We carry out the proof of (20). First let \( n = 1 \). By (6) and (13) we get
\[
Y X_{k_1} X_{k_2} X_{k_3} X_{k_4} Y - Y_{k_{1}k_{2}} Y_{k_{3}k_{4}} = (Y^*_{k_{2}k_{1}} - Y_{k_{1}k_{2}}) Y_{k_{3}k_{4}} \in \mathcal{X}_0.
\]
Suppose the assertion holds for \( n \). Using the abbreviations \( z_n = Y^n X_{k_1} \cdots X_{k_{4n}} Y^n \), \( z = Y X_{k_{4n+1}} \cdots X_{k_{4n+4}} Y^n \), \( y_n = Y_{k_{1}k_{2}} \cdots Y_{k_{4n-1}k_{4n}} \), \( y = Y_{k_{4n+1}k_{4n+2}} Y_{k_{4n+3}k_{4n+4}} \), \( Y = A^{-1} \) we compute
\[
Y^{n+1} X_{k_1} \cdots X_{k_{4n+4}} Y Y_{k_{1}k_{2}} \cdots Y_{k_{4n+3}k_{4n+4}} = Y z_n A^{n+1} Y^n - y_n y
\]
\[
= [Y, z_n] A^{n+1} [z, Y^n] + z_n A^n [z, Y^n] + [Y, z_n] A z + (z_n - y_n) z + y_n (z - y).
\]
(21)

From (14) we easily derive that \([z, Y^n] \in Y^n \mathcal{X}_0 \). Moreover, \([Y, z_n] \in Y \mathcal{X}_0 = \mathcal{X}_0 Y \) by (14) and \( z_n - y_n \in \mathcal{X}_0 \) and \( z - y \in \mathcal{X}_0 \) by the induction hypothesis. Using these facts and remembering that \( \mathcal{X}_0 \) is a two-sided ideal of \( \mathcal{X} \) it follows that the element in (21) belongs to \( \mathcal{X}_0 \). This proves the assertion for \( n + 1 \).

For \( z \in \mathcal{X} \) we set \( \text{Re } z := \frac{1}{2} (z + z^*) \) and \( \text{Im } z := \frac{1}{2} (z^* - z) \). Let \( \mathcal{X}_b \) be the set of all elements \( z \in \mathcal{X} \) for which there exists a positive number \( \lambda \) such that
\[
\lambda \cdot I \pm \text{Re } z \in \sum^2 (\mathcal{X}) \quad \text{and} \quad \lambda \cdot I \pm \text{Im } z \in \sum^2 (\mathcal{X}).
\]

From Lemma 2.1 ii) in [S3] it follows that a finite sum \( \sum_j z_j \) is in \( \mathcal{X}_b \) if and only if all \( z_j \) are in \( \mathcal{X}_b \). Moreover, \( \mathcal{X}_b \) is a *-algebra by Corollary 2.2 in [S3]. We shall use these two facts in the proof of Lemma 3.2 below. Following [S3] we say that the *-algebra \( \mathcal{X} \) is algebraically bounded if \( \mathcal{X} = \mathcal{X}_b \).

Lemma 3.2 The *-algebra \( \mathcal{X} \) defined above is algebraically bounded.

Proof. Recall that \( Y^* = Y = -Y_{\text{00}} \). Applying relation (9) we obtain
\[
\frac{1}{4} I - \left( \frac{1}{2} I - Y \right)^2 = Y - Y_{\text{00}}^2 = \sum_{k=1}^{d} Y_{k0} Y_{k0} \in \sum^2 (\mathcal{X}).
\]
Thus, \((\frac{1}{2}I - Y)^2 \in X_b\) and hence \(Y \in X_b\). Since \(Y \in X_b\), it follows from (12) that \(Y_{k0} \in X_b\) for \(k = 1, \ldots, d\). Using relation (12) and the fact that \(Y_{k0} = Y_{0k}\) we compute

\[
Y_{00}^2 + 2 \sum_{k=1}^{d} Y_{k0}^* Y_{k0} + \sum_{k,l=1}^{d} (Y_{kl} + \frac{i}{2} \sum_{j=1}^{d} b_{jl}^k Y_{j0})^* (Y_{kl} + \frac{i}{2} \sum_{j=1}^{d} b_{jl}^k Y_{j0})
\]

\[
= \sum_{k,l=0}^{d} Y_{kl}^* Y_{kl} - \frac{i}{2} \sum_{j,k,l=1}^{d} b_{jl}^k Y_{j0}^* Y_{kl} + \frac{i}{2} \sum_{j,k,l=1}^{d} b_{jl}^k Y_{kl}^* Y_{j0}
\]

\[
+ \frac{1}{4} \sum_{k,l=1}^{d} \left( \sum_{j=1}^{d} b_{jl}^k Y_{j0} \right)^* \left( \sum_{j=1}^{d} b_{jl}^k Y_{j0} \right)
\]

\[
= (r4) I + \frac{1}{4} \sum_{k,l=1}^{d} \left( \sum_{j=1}^{d} b_{jl}^k Y_{j0} \right)^* \left( \sum_{j=1}^{d} b_{jl}^k Y_{j0} \right).
\]

Since \(Y_{j0} \in X_b\) for \(j = 0, \ldots, d\) as just shown and \(X_b\) is a \(*\)-algebra, the right-hand side of the preceding equation belongs to \(X_b\). Therefore, \(Y_{kl} + \frac{i}{2} \sum_{j=1}^{d} b_{jl}^k Y_{j0} \in X_b\) and hence \(Y_{kl} \in X_b\) for \(k, l = 1, \ldots, d\). Hence all generators of \(X\) are in \(X_b\), so that \(X = X_b\). \(\square\)

Now we choose \(p \in \mathbb{N}\) such that \(4p \geq \text{degree } f_j\) for \(j = 1, \ldots, r\). Then, \(Y^p dU(f_j) Y^p \in X\) by (20). Let \(C_f\) denote the wedge of all finite sums of elements \(x^* x\) and \(z^* Y^p dU(f_l) Y^p z\), where \(x, z \in X\) and \(l = 1, \ldots, r\). The assertion of the next lemma is contained in [S3], Lemma 2.3. For completeness we include the short proof.

**Lemma 3.3** If \(z \in X\) is not in \(C_f\), then there exists a state \(F\) of the \(*\)-algebra \(X\) such that \(F(z) \leq 0\) and \(F(x) \geq 0\) for all \(x \in C_f\).

**Proof.** Since \(C_f \subseteq \bigoplus_{i=1}^{2}(X)\), the unit element \(I\) of \(X\) is an internal point of the wedge \(C_f\) in the real vector space \(X_h = \{x \in X : x = x^*\}\). By the separation theorem for convex sets [K], §17, (3), there is a linear functional \(G \not\equiv 0\) on \(X_h\) such that \(G(z) \leq 0\) and \(G(x) \geq 0\) for \(x \in C_f\). Since \(G \not\equiv 0\), we have \(G(I) > 0\). Take as \(F\) the extension of the \(\mathbb{R}\)-linear functional \(G(I)^{-1} G\) on \(X_h\) to a \(\mathbb{C}\)-linear functional on \(X\). \(\square\)
4 Representations of the Auxiliary $\ast$-Algebra

Since the $\ast$-algebra $\mathcal{X}$ is algebraically bounded by Lemma 3.2 for any $\ast$-representation of $\mathcal{X}$ all representation operators are bounded. Let $\pi$ be an arbitrary $\ast$-representation of $\mathcal{X}$ on a Hilbert space $\mathcal{H}$. By [14], $\mathcal{H}_\infty := \ker \pi(Y)$ is invariant and hence reducing for the bounded $\ast$-representation $\pi$. Let $\pi_\infty$ and $\pi_0$ denote the restrictions of $\pi$ to $\mathcal{H}_\infty$ and $\mathcal{H}_0 := \mathcal{H}_\infty^\perp$, respectively.

4.1

In this subsection we investigate the $\ast$-representation $\pi_0$. Since $\ker \pi_0(Y) = \{0\}$ and relations (9) – (11) hold, Lemma 1 in [Sz] applies. It is a reformulation of Nelson’s famous integrability theorem for Lie algebra representations ([N], see e.g. [S2], Theorem 10.5.6, or [Wa], Theorem 4.4.6.6) and states that there exists a $\pi_0$ such that (9) – (11) hold, Lemma 1 in [Sz] applies. It is a reformulation of Nelson’s famous integrability theorem for Lie algebra representations ([N], see e.g. [S2], Theorem 10.5.6, or [Wa], Theorem 4.4.6.6) and states that there exists a unitary representation $V$ of the simply connected Lie group $G$ on $\mathcal{H}_0$ such that

$$\overline{dV(x_k)} = -i\pi_0(Y_{k0})\pi_0(Y)^{-1}, \ k = 1, \ldots, d. \tag{22}$$

In the proof therein it is shown that $B := I - \sum_{k=1}^d \overline{dV(x_k)^2}$ is equal to the self-adjoint operator $\pi_0(Y)^{-1}$ on its domain $\pi_0(Y)\mathcal{H}_0$. By Lemma 2.1, $\mathcal{D}^\infty(V) = \cap_{n=1}^\infty \mathcal{D}(B^n) = \cap_{n=1}^\infty \pi_0(Y)^n\mathcal{H}_0$. Hence $\pi_0(Y)^{-1}$ maps $\mathcal{D}^\infty(V)$ onto $\mathcal{D}^\infty(V)$. Therefore, by (22) we have

$$dV(x_k)\varphi = -i\pi_0(Y_{k0})\pi_0(Y)^{-1}\varphi, \ \varphi \in \mathcal{D}^\infty(V), \ k = 1, \ldots, d. \tag{23}$$

Next we prove by induction on $n$ that

$$\pi_0(Y^n X_{k_1} \cdots X_{k_{2n}})\varphi = \pi_0(Y)^n dV(x_{k_1} \cdots x_{k_{2n}})\varphi, \ \varphi \in \mathcal{D}^\infty(V), \tag{24}$$

for $k_1, \ldots, k_{2n} \in \{0, \ldots, d\}$. Since $Y^n X_{k_1} \cdots X_{k_{2n}} \in \mathcal{X}$ by (15), the left hand side of (24) is well-defined. First let $n = 1$. For $\varphi \in \mathcal{D}^\infty(V)$ we set $\psi = \pi_0(Y)^{-1}\varphi$. Using (6) and (23) we compute

$$\pi_0(Y X_{k_1} X_{k_2})\varphi = \pi_0(Y X_{k_1} X_{k_2})\pi_0(Y)\psi = \pi_0(Y X_{k_1} Y)\pi_0(Y)^{-1}\pi_0(X_{k_2} Y)\psi$$

$$= -\pi_0(Y)\pi_0(Y_{k_1,0})\pi_0(Y)^{-1}\pi_0(Y_{k_2,0})\pi_0(Y)^{-1}\varphi$$

$$= \pi_0(Y)dV(x_{k_1})dV(x_{k_2})\varphi$$

which proves (24) for $n = 1$. Suppose now that (24) is true for $n$. Let $\varphi \in \mathcal{D}^\infty(V)$. Set $\psi = \pi_0(Y)^{-1}\varphi, z_n = X_{k_1} \cdots X_{k_{2n}}$ and $z = X_{k_2n+1} X_{k_{2n+2}}$. Since $\pi_0$ and...
Moreover, \( \epsilon t \) denote the sign of \( \epsilon t \) for all \( k \),\( l \). From (28) it follows that there is a \( \epsilon t \) such that degree \( \epsilon t \) for all \( k \),\( l \). Combining (28) and (29) we conclude that

\[
\pi_0(Y^n z_{n_0}) \psi = \pi_0(Y^n z_{n_0}) \pi_0(zY) \psi = \pi_0(Y^n z_{n_0}) \pi_0(zY) \psi
\]

which is equation (24) for \( n + 1 \). This completes the proof of (24). Applying the involution to both sides of (24) we obtain

\[
\pi_0(X_{k_1} \cdots X_{k_{4n}} Y^n) \psi = \pi_0(Y^n) \pi_0(x_{k_1} \cdots x_{k_{4n}}) \pi_0(Y^n) \psi, \psi \in D^\infty(V),
\]

Combining (24) and (25) we conclude that

\[
\pi_0(Y^n X_{k_1} \cdots X_{k_{4n}} Y^n) \psi = \pi_0(Y^n) \pi_0(x_{k_1} \cdots x_{k_{4n}}) \pi_0(Y^n) \psi, \psi \in D^\infty(V),
\]

for \( k_1, \ldots, k_{4n} \in \{0, \ldots, d\} \). This in turn implies that

\[
\pi_0(Y^n dU(x) Y^n) \psi = \pi_0(Y^n) \pi_0(Y^n) \psi, \psi \in D^\infty(V)
\]

for all \( x \in E(g) \) such that degree \( x \leq 4n \).

### 4.2

In this subsection we turn to the *-representation \( \pi_\infty \) of \( X \) on \( H_\infty = ker \pi(Y) \). Since \( \pi_\infty(Y) = 0 \), it follows from (7) and (9) that

\[
\pi_\infty(Y_{k_0}) = \pi_\infty(Y_{0k}) \text{ for } k = -d, \ldots, d.
\]

Moreover, \( \pi_\infty(X_0) = \{0\} \). Therefore, by (13), (15) and (16), \( y_{kl} := \pi_\infty(Y_{kl}) \), \( k, l = 1, \ldots, d \) and \( k, l = -d, \ldots, -1 \), are pairwise commuting bounded self-adjoint operators such that \( y_{kl} = y_{lk} \) and \( y_{-k,-l} = y_{kl} \). Let \( \chi \) be a character of the abelian unital \( C^* \)-algebra \( Y \) generated by these operators (or equivalently by \( \pi_\infty(X) \)). From (12), (27) and (16) we get

\[
\sum_{k,l=1}^d y_{kl}^2 = I \text{ and } y_{kl} y_{ij} = y_{ki} y_{lj}, i, j, k, l = 1, \ldots, d.
\]

From (28) it follows that there is a \( j \in \{1, \ldots, d\} \) such that \( \chi(y_{jj}) \neq 0 \). Let \( \epsilon \) denote the sign of \( \chi(y_{jj}) \). Take \( t_j \in \mathbb{R} \) such that \( t_j^2 = \epsilon \chi(y_{jj}) \) and put \( t_k := \chi(y_{jk}) \chi(y_{jj})^{-1} t_j, k = 1, \ldots, d \). By (28) we have

\[
\epsilon t_k t_l = \epsilon \chi(y_{jk}) \chi(y_{lj}) \chi(y_{jj})^{-1} t_j t_l = \epsilon \chi(y_{kl}) \chi(y_{jj})^{-1} t_j^2 = \chi(y_{kl})
\]

(29)
for \( k, l = 1, \ldots, d \) and hence
\[
\left( \sum_{k=1}^{d} t_k^2 \right)^2 = \sum_{k,l=1}^{d} \varepsilon \chi(y_{kk}) \varepsilon \chi(y_{ll}) = \sum_{k,l=1}^{d} \chi(y_{kl}^2) = \chi(I) = 1.
\]

Since all operators \( y_{kl} \) are self-adjoint, all numbers \( t_k \) are real. Thus, for each character \( \chi \) of \( \mathcal{Y} \) there exist \( \varepsilon \in \{-1, 1\} \) and a point \( t = (t, \ldots, t_d) \) of the unit sphere \( S^d \) of \( \mathbb{R}^d \) such that (29) holds. From the Gelfand theory it follows that there are reducing subspaces \( \mathcal{H}_\pi^{\pm} \) for \( \pi_\infty \) such that \( \mathcal{H}_\infty = \mathcal{H}_\pi^{+} \oplus \mathcal{H}_\pi^{-} \) and spectral measures \( E^\pm \) over \( S^d \) on \( \mathcal{H}_\infty \) such that for \( k, l = 1, \ldots, d \),
\[
\pi_\infty(y_{kl}) = \int_{S^d} t_k t_l dE^+(t) \oplus \int_{S^d} t_k t_l dE^-(t).
\]

5 Proof of Theorem 1

Suppose first that \( m \) is even, say \( m = 2n \). Since degree \( c = 4n \), it follows from formula (20) in Lemma 3.1 that \( Y^n dU(c) Y^n \in \mathcal{X} \). The crucial step of the proof is the assertion of the following

Lemma 5.1 \( Y^n dU(c) Y^n \) belongs to the wedge \( \mathcal{C}_f \) defined in Section 3

Proof. Assume the contrary. Then, by Lemma 3.4 there exists a state \( F \) of \( \mathcal{X} \) such that \( F(Y^n dU(c) Y^n) \leq 0 \) and \( F(x) \geq 0 \) for all \( x \in \mathcal{C}_f \). Let \( \pi_F \) be the *-representation of \( \mathcal{X} \) associated with \( F \) by the GNS construction. Then there is a cyclic vector \( \varphi_F \) such that \( F(x) = \langle \pi_F(x) \varphi_F, \varphi_F \rangle, x \in \mathcal{X} \). As shown in Section 4 \( \pi_F \) decomposes into a direct sum of representations \( \pi_0 \) and \( \pi_\infty \). If \( \varphi_0 \) and \( \varphi_\infty \) are the corresponding components of \( \varphi_F \), we have
\[
F(x) = \langle \pi_0(x) \varphi_0, \varphi_0 \rangle + \langle \pi_\infty(x) \varphi_\infty, \varphi_\infty \rangle, x \in \mathcal{X}.
\]

Our next aim is to derive inequality (33) below. Let \( V \) be the unitary representation of \( G \) from Subsection 4.1.

We prove that \( dV(f_i) \geq 0 \) for \( l = 1, \ldots, r \). Let \( \psi \in \mathcal{D}^\infty(V) \). Since the *-representation \( \pi_F \) is cyclic, there is a sequence \( \{b_n; n \in \mathbb{N}\} \) of elements \( b_n \in \mathcal{X} \) such that \( \pi_0(b_n) \varphi_0 \to \pi_0(Y)^{-p} \psi \) and \( \pi_\infty(b_n) \varphi_\infty \to 0 \). From (31) we obtain
\[
F(b_n Y^p dU(f_i) Y^p b_n) =
\langle \pi_0(Y^p dU(f_i) Y^p) \pi_0(b_n) \varphi_0, \pi_0(b_n) \varphi_0 \rangle + \langle \pi_\infty(Y^p dU(f_i) Y^p) \pi_\infty(b_n) \varphi_\infty, \pi_\infty(b_n) \varphi_\infty \rangle
\to \langle \pi_0(Y^p dU(f_i) Y^p) \pi_0(Y)^{-p} \psi, \pi_0(Y)^{-p} \psi \rangle = \langle dV(f_i) \psi, \psi \rangle,
\]

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where the last equality follows from equation (26). Since \( b_n^* Y^p dU(f_i) Y^p b_n \in C_f \) and hence \( F(b_n^* Y^p dU(f_i) Y^p b_n) \geq 0 \), we get \( (dV(f_i) \psi, \psi) \geq 0 \).

Since \( \pi_F \) and hence \( \pi_0 \) are cyclic representations, \( \mathcal{H}_0 \) is separable. Since the Lie group \( G \) is connected, \( G \) is separable. Therefore, the unitary representation \( V \) on \( \mathcal{H}_0 \) can be decomposed as a direct integral \( \int_\Lambda^\oplus U_\lambda d\mu(\lambda) \) of irreducible unitary representations (see [Ki], p.127). Now we need two (known) technical results from decomposition theory (see e.g. [S2], Chapter 12, pp. 343-344 and [Nu]).

The first one states that

\[
dV = \int_\Lambda^\oplus dU_\lambda d\mu(\lambda).
\]

For a unitary representation \( W \) of \( G \), let \( \tau_W \) denote the metric locally convex topology on \( \mathcal{D}_x^\infty(W) \) given by family of seminorms \( ||dW(x_1^{n_1} \cdots x_d^{n_d})|| \), where \( n_1, \ldots, n_d \in \mathbb{N}_0 \). Since \( \mathcal{H}_0 \) and hence \( \mathcal{D}_x^\infty(V) \) is separable, there is a countable dense subset \( \{\eta_n; n \in \mathbb{N}\} \) of \( \mathcal{D}_x^\infty(V) [\tau_V] \). The second result states that then \( \{\eta_\alpha(\lambda); n \in \mathbb{N}\} \) is dense in \( \mathcal{D}_x^\infty(U_\lambda) [\tau_{U_\lambda}] \) \( \mu \)-a.e.

For \( \zeta \in L_\infty(\Lambda, \mu) \), let \( M_\zeta \) denote the associated diagonalisable operator on \( \mathcal{H}_0 = \int_\Lambda^\oplus \mathcal{H}(U_\lambda)d\mu(\lambda) \). From (32) we obtain

\[
\langle dV(f_i) M_\zeta \eta_n, M_\zeta \eta_n \rangle = \int_\Lambda |\zeta(\lambda)|^2 \langle dU_\lambda(f_i)\eta_n(\lambda), \eta_n(\lambda) \rangle d\mu(\lambda)
\]

for all \( \zeta \in L_\infty(\Lambda, \mu) \). Since \( dV(f_i) \geq 0 \), the latter implies that there is a \( \mu \)-null set \( N_0 \) of \( \Lambda \) such that \( \langle dU_\lambda(f_i)\eta_n(\lambda), \eta_n(\lambda) \rangle \geq 0 \) for all \( n \in \mathbb{N} \) and all \( \lambda \in \Lambda \setminus N_0 \). From the density of \( \{\eta_n(\lambda); n \in \mathbb{N}\} \) in \( \mathcal{D}_x^\infty(U_\lambda)[\tau_{U_\lambda}] \) \( \mu \)-a.e. it follows that there is a \( \mu \)-null set \( N \) such that \( dU_\lambda(f_i) \geq 0 \) for all \( \lambda \in \Lambda \setminus N \) and \( l = 1, \ldots, r \). That is, the equivalence class of \( U_\lambda \) is in \( \mathcal{K}_f \) for all \( \lambda \in \Lambda \setminus N \). Since \( dU_\lambda(c - \varepsilon \cdot 1) \geq 0 \) for \( \alpha \in \mathcal{K}_f \) by assumption (i), from the latter and (32) we conclude that \( dV(c - \varepsilon \cdot 1) \geq 0 \).

Therefore, by (26) we have

\[
\langle \pi_0(Y^n dU(c) Y^n) \psi, \psi \rangle = \langle dV(c) \pi_0(Y)^n \psi, \pi_0(Y)^n \psi \rangle \geq \varepsilon \|\pi_0(Y)^n \psi\|^2
\]

for \( \psi \in \mathcal{D}_x^\infty(V) \) and hence

\[
\langle \pi_0(Y^n dU(c) Y^n) \varphi_0, \varphi_0 \rangle \geq \varepsilon \|\pi_0(Y)^n \varphi_0\|^2.
\]

Next we consider the second summand in (31) for \( x = Y^n dU(c) Y^n \). Let \( E(\cdot) \) denote the spectral measure \( E^+(\cdot) \oplus E^-(\cdot) \) on \( \mathcal{H}_\infty = \mathcal{H}_\infty^+ \oplus \mathcal{H}_\infty^- \). Since
\[ \pi_\infty(\mathcal{X}_0) = \{0\}, \] it follows from formulas \((16), (27)\) and \((30)\) that
\[
\langle \pi_\infty(Y^n dU(c)Y^n)\varphi_\infty, \varphi_\infty \rangle = \langle \pi_\infty(Y^n dU(c_{4n})Y^n)\varphi_\infty, \varphi_\infty \rangle 
= \int_{S^d} c_{4n}(t)d\langle E(t)\varphi_\infty, \varphi_\infty \rangle. \tag{34}
\]

By assumption (ii), \(c_{4n}(t) > 0\) for all \(t \in S^d\). Since \(F(Y^n dU(c)Y^n) \leq 0\), we conclude from \((31), (33)\) and \((34)\) that \(\pi_0(Y)^n\varphi_0 = 0\) and \(\langle E(\cdot)\varphi_\infty, \varphi_\infty \rangle = 0\). Therefore, \(\varphi_0 = 0\) and \(\varphi_\infty = 0\), so that \(F \equiv 0\) by \((31)\). Since \(F\) is a state on \(\mathcal{X}\), we have a contradiction. \hfill \Box

**Lemma 5.2** Let \(n \in \mathbb{N}_0\). For arbitrary elements \(z_1, \ldots, z_q \in \mathcal{X}\) there exists \(s \in S := dU(S)\) such that \(z_1 A^n s, \ldots, z_q A^n s \in \mathcal{U} := dU(\mathcal{E}(g)).\)

**Proof.** First we prove the assertion for single elements \(z \in \mathcal{X}\) of the form \(Y_{j_1} \cdots Y_{j_k}l_k\). We use induction on \(k\). Suppose that the assertion holds for \(k\).

Let \(z = Y_{j_l}w\), where \(w = Y_{j_l}t_l \cdots Y_{j_1}t_1\). By the induction hypothesis, there is \(s' \in S\) such that \(wA^n s' \in \mathcal{U}\). Assume that \(j \leq 0, l \leq 0\). Since \(S\) is a right Ore set containing \(a\) and \(dU(a) = A\), there are elements \(s'' \in S\) and \(v \in \mathcal{U}\) such that \(AV = X_j X_{l}(wA^n s')s''\). Set \(s = s's''\). Then \(zA^n s = A^{-1} X_j X_{l}wA^n s's'' = A^{-1} AV = v \in \mathcal{U}\). The case \(j > 0, l > 0\) and the case \(k = 1\) are treated similarly.

It suffices to prove the assertion of Lemma 5.2 for element \(z_j\) of the form \(Y_{j_l}t_l \cdots Y_{j_1}t_1\) because these elements and \(I\) span \(\mathcal{X}\). We proceed by induction on \(q\). For \(q = 1\) the assertion is proved in the preceding paragraph. Suppose that the assertion is true for \(q\). Let \(z_1, \ldots, z_{q+1} \in \mathcal{X}\). Then there exist elements \(s_1, s_2 \in S\) such that \(z_1 A^n s_1 \in \mathcal{U}\) for \(l = 1, \ldots, q\) and \(z_{q+1} A^n s_2 \in \mathcal{U}\). By the right Ore property of \(S\), there are \(s_3 \in S\) and \(u \in \mathcal{U}\) such that \(s_1 s_3 = s_2 u =: s\). Then, \(s \in S\), \(z_1 A^n s = (z_1 A^n s_1) s_3 \in \mathcal{U}\) for \(l = 1, \ldots, q\) and \(z_{q+1} A^n s = (z_{q+1} A^n s_2) u \in \mathcal{U}\). \hfill \Box

Now we are able to complete the proof of Theorem 1.1. By Lemma 5.1 there exist finitely many elements \(z_{jl} \in \mathcal{X}, l = 0, \ldots, d\), such that
\[
Y^n dU(c)Y^n = \sum_j z_{j_0}^* z_{j_0} + \sum_j \sum_{l=1}^d z_{jl}^* Y^n dU(f_l) Y^n z_{jl}.
\]
Let \(s = dU(s), s \in S\). Multiplying both sides by \(A^n s\) from the right and by \((A^n s)^*\) from the left we obtain
\[
dU(s^* cs) = \sum_j (z_{j_0} A^n s)^* z_{j_0} A^n s + \sum_j \sum_{l=1}^d (Y^n z_{jl} A^n s)^* dU(f_l) Y^n z_{jl} A^n s.
\]
By Lemma 5.2 we can find $s = dU(s)$ such that $z_j A^n s \in dU(E(g))$ and $Y^p P z_l A^n s \in dU(E(g))$ for all $j, l$. Then the right-hand side of the preceding equation is in $dU(T_f)$. Now we choose the unitary representation $U$ of $G$ such that the representation $dU$ of $E(g)$ is faithful (for instance, it suffices to take the regular representation of $G$). Then it follows from $dU(s^*cs) \in dU(T_f)$ that $s^*cs \in T_f$.

Finally, we suppose that $m$ is odd. Then $c' := \sum_{k=0}^d x_k^* c x_k$ satisfy assumptions (i) and (ii) with $m' = m + 1$ even, so the assertion of Theorem 1.1 follows from the previous case.  

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