Critical base for the unique codings of fat Sierpinski gasket

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Abstract
Given $\beta \in (1, 2)$ the fat Sierpinski gasket $S_\beta$ is the self-similar set in $\mathbb{R}^2$ generated by the iterated function system (IFS) $f_d(x) = \frac{x + d}{\beta}$, $d \in A := \{(0, 0), (1, 0), (0, 1)\}$. Then for each point $P \in S_\beta$ there exists a sequence $(d_i) \in A^\mathbb{N}$ such that $P = \sum_{i=1}^{\infty} d_i/\beta^i$, and the infinite sequence $(d_i)$ is called a coding of $P$. In general, a point in $S_\beta$ may have multiple codings since the overlap region $O_\beta := \bigcup_{d \in A, d \neq f_d(\Delta_\beta) \cap f_{f_d(\Delta_\beta)}}$ has non-empty interior, where $\Delta_\beta$ is the convex hull of $S_\beta$. In this paper we are interested in the invariant set $\tilde{U}_\beta := \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \in S_\beta : \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \notin O_\beta \forall n \geq 0 \right\}$. Then each point in $\tilde{U}_\beta$ has a unique coding. We show that there is a transcendental number $\beta_c \approx 1.55263$ related to the Thue–Morse sequence, such that $\tilde{U}_\beta$ has positive Hausdorff dimension if and only if $\beta > \beta_c$. Furthermore, for $\beta = \beta_c$ the set $\tilde{U}_\beta$ is uncountable but has zero Hausdorff dimension, and for $\beta < \beta_c$ the set $\tilde{U}_\beta$ is at most countable. Consequently, we also answer a conjecture of Sidorov (2007). Our strategy is using combinatorics on words based on the lexicographical characterization of $\tilde{U}_\beta$.

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(Some figures may appear in colour only in the online journal)
1. Introduction

Given $\beta > 1$, let $S_\beta$ be the Sierpinski gasket in $\mathbb{R}^2$ generated by the iterated function system (IFS)

$$f_{\beta,d}(x) = \frac{x + d}{\beta}, \quad d \in A := \{(0,0),(1,0),(0,1)\}.$$ 

In other words, $S_\beta$ is the unique non-empty compact set in $\mathbb{R}^2$ satisfying $S_\beta = \bigcup_{d \in A} f_{\beta,d}(S_\beta)$. So for each point $P \in S_\beta$ there exists a sequence $(d_i) \in A^\mathbb{N}$ such that

$$P = \lim_{n \to \infty} f_{\beta,d_1 \cdots d_n}(0) = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = ((d_i))_\beta,$$

where $f_{\beta,d_1 \cdots d_n} := f_{\beta,d_n} \circ \cdots \circ f_{\beta,d_1}$ is the composition of $f_{\beta,d_1}, \ldots, f_{\beta,d_n}$, and $0 = (0,0)$ is the zero vector in $\mathbb{R}^2$. The infinite sequence $(d_i) \in A^\mathbb{N}$ is called a coding of $P$ with respect to the alphabet $A$. Therefore, the Sierpinski gasket $S_\beta$ can be rewritten as

$$S_\beta = \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} : d_i \in A \text{ for all } i \geq 1 \right\}.$$

When $\beta > 2$, it is easy to check that the overlap region (see figure 1)

$$O_\beta := \bigcup_{c,d \in A, c \neq d} f_{\beta,c}(\Delta_{\beta}) \cap f_{\beta,d}(\Delta_{\beta})$$

is empty, where $\Delta_{\beta}$ is the convex hull of $S_\beta$. In fact $\Delta_{\beta}$ is the triangle with vertices $(0,0),(1/(\beta-1),0)$ and $(0,1/(\beta-1))$. So for $\beta > 2$ the IFS $\{f_{\beta,d}(\cdot) : d \in A\}$ satisfies the strong separation condition, and then each point in $S_\beta$ has a unique coding (cf [1]). When $\beta = 2$, one can see that the overlap region $O_\beta$ consists of three points. Then the IFS $\{f_{\beta,d}(\cdot) : d \in A\}$ fails the strong separation condition, but still satisfies the open set condition. In this case, excluding countably many points in $S_\beta$ having precisely two codings all other points in $S_\beta$ have a unique coding.

However, when $\beta \in (1,2)$ the overlap region $O_\beta$ is non-trivial and it contains interior points. In this case we call $S_\beta$ a fat Sierpinski gasket, and the IFS $\{f_{\beta,d}(\cdot) : d \in A\}$ fails the open set condition [see ([2], remark 2.3)]. Furthermore, the set of points in $S_\beta$ with multiple codings has positive Hausdorff dimension. In particular, for $\beta \in (1,3/2]$ the Sierpinski gasket $S_\beta$ coincides with its convex hull $\Delta_{\beta}$. In this case Lebesgue almost every point in $S_\beta$ has a continuum of codings [see [2], theorem 3.5]. When $\beta \in (3/2,2)$ the structure of $S_\beta$ gets more complicated. Broomhead et al showed in [3] that for $\beta \leq \beta_*$, the set $S_\beta$ has non-empty interior, where $\beta_* \approx 1.54369$ is the appropriate zero of $x^3 - 2x^2 + 2x - 2$. Some recent progress in this direction can be found in [4].

When $\beta \in (1,2)$ the fat Sierpinski gasket $S_\beta$ has attracted much more attention in the past twenty years. Simon and Solomyak [5] showed that there exists a dense set of $\beta \in (1,2)$ such that $\dim_H S_\beta < \log 3/\log \beta$, where $\log 3/\log \beta$ is the self-similar dimension of $S_\beta$. We emphasize that $\dim_H S_\beta = \log 3/\log \beta$ for all $\beta \geq 2$. On the other hand, Jordan [6] showed that $\dim_L S_\beta = \log 3/\log \beta$ for Lebesgue almost every $\beta \in (3/4^{1/3},2) \approx (1.88988,2)$. Furthermore, he and Pollicott [7] proved that $S_\beta$ has positive Lebesgue measure for almost every $\beta \in (1,1.70853)$. Recently, Hochman ([8], theorem 1.16) made a great progress on the dimension of $S_\beta$ and showed that $\dim_H S_\beta = \min \{2, \log 3/\log \beta\}$ for $\beta \in (1,2)$ outside a set of zero packing dimension.
Let $\beta \in (1, 2)$. Then the overlap region $\mathcal{O}_\beta$ is nonempty. In this paper we are interested in an invariant subset of the fat Sierpinski gasket $S_\beta$ consisting of all points whose orbits never enter the overlap region $\mathcal{O}_\beta$. More precisely, we will investigate the intrinsic univoque set

$$\tilde{U}_\beta := \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \in S_\beta : \sum_{i=1}^{\infty} \frac{d_{n+i}}{\beta^i} \notin \mathcal{O}_\beta \text{ for all } n \geq 0 \right\}. $$

Observe that if a point $\sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \in S_\beta$ has multiple codings, then its orbit $\sum_{i=1}^{\infty} \frac{d_{n+i}}{\beta^i}$ must enter the overlap region $\mathcal{O}_\beta$ for some $n \geq 0$. This implies that each point in $\tilde{U}_\beta$ has a unique coding. Denote the univoque set by

$$U_\beta := \{ P \in S_\beta : P \text{ has a unique coding with alphabet } A \}.$$ 

Then $\tilde{U}_\beta \subseteq U_\beta$ for each $\beta \in (1, 2)$. When $\beta \in (1, 3/2]$ or $\beta$ is a multinacci number, the equality $\tilde{U}_\beta = U_\beta$ holds (see proposition 2.4 below). This explains why we call $\tilde{U}_\beta$ the ‘intrinsic univoque set’.

Let $\beta_G \approx 1.46557$ be the unique root in $(1, 2)$ of the equation $x^3 - x^2 - 1 = 0$. The following result was proved by Sidorov ([2], theorem 4.1).

**Theorem 1.1** ([Sidorov, 2007]. If $\beta \in (1, \beta_G]$, then

$$U_\beta = \left\{ (0, 0), \left( \frac{1}{\beta - 1}, 0 \right), \left( 0, \frac{1}{\beta - 1} \right) \right\}. $$

Furthermore, he conjectured in ([2], remark 4.3) that the univoque set $U_\beta$ is at most countable when $\beta \in (\beta_G, 3/2]$. Our first result answers his conjecture affirmatively.
Theorem 1. If $\beta \in (\beta_G, 3/2)$, then

$$U_\beta = \left\{ (0,0), \left( \frac{1}{\beta - 1}, 0 \right), \left( 0, \frac{1}{\beta - 1} \right) \right\} \cup \bigcup_{n=0}^{\infty} \bigcup_{P \in \mathcal{C}, Q \in \mathcal{D}} \left\{ P + \frac{Q}{\beta^n(\beta^3 - 1)} \right\},$$

where

$$\mathcal{C}_n := \left\{ (0,0), \left( \sum_{i=1}^{n} \frac{1}{\beta^i}, 0 \right), \left( 0, \sum_{i=1}^{n} \frac{1}{\beta^i} \right) \right\},$$

$$\mathcal{D} := \left\{ (1, \beta), (\beta, 1), (1, \beta^2), (\beta, \beta^2), (\beta^2, \beta) \right\}.$$

In order to describe the set $\tilde{U}_\beta$ for $\beta \in (3/2, 2)$, we introduce a Thue–Morse type sequence $(\lambda_i) \in \{0, 1\}^{\mathbb{N}}$. Let $\Theta$ be the block map defined on the set $\Omega := \{000, 001, 100, 101\}$ by

$$\Theta : \Omega \to \Omega; \quad 000 \mapsto 101, 001 \mapsto 100, 100 \mapsto 001, 101 \mapsto 000.$$ 

Then for a word $a = a_1 \cdots a_m \in \Omega^m$ we set $\Theta(a) := \Theta(a_1) \cdots \Theta(a_m)$ as the block obtained by concatenating blocks $\Theta(a_1), \ldots, \Theta(a_m)$. We emphasize that each digit $a_i$ is a block of length 3 from $\Omega$. Let $t_1 = 100 \in \Omega$, and for $n \geq 1$ we set

$$t_{n+1} := t^+_n \Theta(t^+_n),$$

where $t^+_n$ denotes the word by changing the last digit of $t_n$ from zero to one. For example,

$$t_2 = 101000, \quad t_3 = 1010010001000100100010001000000, \quad \ldots.$$ 

Then the sequence $(t_n)$ has a componentwise limit, denoted by $(\lambda_i)$. So $(\lambda_i)$ is an infinite sequence in $\{0, 1\}^{\mathbb{N}}$ related to the classical Thue–Morse sequence (cf [9]). One can check that the sequence $(\lambda_i)$ begins with $t^+_n$ for any $n \geq 1$ and $\lambda_{3k+2} = 0$ for all $k \geq 0$.

Let $\beta_c \approx 1.55263$ be the unique root in $(1, 2)$ of the equation

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{n^i} = 1. \tag{1.1}$$

In view of theorem 1.1, our second result describes the size of $\tilde{U}_\beta$ for $\beta \in (\beta_G, 2)$.

Theorem 2. The number $\beta_c$ is transcendental.

(a) If $\beta \in (\beta_G, \beta_c)$, then $\tilde{U}_\beta$ is countably infinite;

(b) If $\beta = \beta_c$, then $\tilde{U}_\beta$ is uncountable but has zero Hausdorff dimension;

(c) If $\beta \in (\beta_c, 2)$, then $\tilde{U}_\beta$ has positive Hausdorff dimension.

Remark 1.2.

- Theorem 2 can be viewed as an analogue of the main results of Glendinning and Sidorov [10] for the one dimensional unique $\beta$-expansions. Then $\beta_c$ is an analogue of the Komornik–Loreti constant first investigated by Komornik and Loreti in [11].

- Our proof of theorem 2 is using word combinatorics based on the lexicographical characterization of $U_\beta$ (see proposition 2.2). Our method allows us to give a complete description of $\tilde{U}_\beta$ for $\beta \in (1, \beta_c]$.
The rest of the paper is organized as follows. In section 2 we give the lexicographical characterization of the intrinsic univoque set $\mathcal{U}_\beta$. Based on this characterization we give an alternate proof of theorem 1.1 in section 3. Furthermore, we prove theorem 1 which provides an affirmative answer to a conjecture of Sidorov. In section 4 we investigate all possible admissible words in $\mathcal{U}_\beta$, based on the three types of Thue–Morse words with alphabet $A$. The proof of the main result theorem 2 is given in section 5. In the final section we pose some questions.

2. Characterization of $\mathcal{U}_\beta$ and a sufficient condition for $\mathcal{U}_\beta = \tilde{\mathcal{U}}_\beta$

Given $\beta \in (1, 2)$, recall that the Sierpinski gasket $S_\beta$ is the self-similar set in $\mathbb{R}^2$ generated by the IFS

$$f_{\beta,d}(x) = \frac{x + d}{\beta}, \quad d \in A = \{A, B, C\},$$

where $A := (0, 0), B := (1, 0)$ and $C := (0, 1)$ are the digits. Then for any point $P \in S_\beta$ there exists a sequence $(d_i) \in A^\mathbb{N}$ such that

$$P = (d_i)_{\beta} = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}. \quad (2.1)$$

Note that each point $P$ in $\mathbb{R}^2$ can be written as $P = (P^1, P^2)$, where $P^1$ and $P^2$ are the first and second coordinates of $P$. Similarly, each digit $d_i \in A$ can be written as $d_i = (d_i^1, d_i^2)$. Then (2.1) can be rewritten coordinately as

$$P^1 = (d_i^1)_{\beta} := \sum_{i=1}^{\infty} \frac{d_i^1}{\beta^i} \quad \text{and} \quad P^2 = (d_i^2)_{\beta} := \sum_{i=1}^{\infty} \frac{d_i^2}{\beta^i}.$$

In view of figure 1, the convex hull $\Delta_\beta$ of $S_\beta$ is the triangle with three vertices $(0, 0), (1/(\beta - 1), 0)$ and $(0, 1/(\beta - 1))$. Then $f_{\beta,A}(\Delta_\beta)$ is the left-bottom triangle, $f_{\beta,B}(\Delta_\beta)$ is the right-bottom triangle, and $f_{\beta,C}(\Delta_\beta)$ is the top triangle. Since $1 < \beta < 2$, the overlap region $O_\beta = O_A \cup O_B \cup O_C$ has non-empty interior, where

$$O_A := f_{\beta,B}(\Delta_\beta) \cap f_{\beta,C}(\Delta_\beta), \quad O_B = f_{\beta,A}(\Delta_\beta) \cap f_{\beta,C}(\Delta_\beta), \quad O_C = f_{\beta,A}(\Delta_\beta) \cap f_{\beta,B}(\Delta_\beta).$$

Indeed, each of the sets $O_A, O_B$ and $O_C$ has non-empty interior for $\beta \in (1, 2)$.

In order to describe the intrinsic univoque set $\mathcal{U}_\beta$, we need some notation from symbolic dynamics. For a word $e = c_1 \cdots c_n \in \{0, 1\}^*$ we mean a finite string of zeros and ones. For an integer $k \geq 1$ we denote by $e^k := ec \cdots e$ the $k$-times concatenation of $e$ with itself, and we write for $e^\infty$ the periodic sequence with periodic block $e$. For a word $e = c_1 \cdots c_n$ with $c_n = 0$ we denote by $e^+ := c_1 \cdots c_{n-1}(c_n + 1)$. For a sequence $(e_i) \in \{0, 1\}^\mathbb{N}$ we define its reflection by $(e_i^\top) := (1 - e_i)(1 - e_2) \cdots$. Throughout the paper we will use the lexicographical order between sequences and words. More precisely, for two sequences $(e_i), (d_i) \in \{0, 1\}^\mathbb{N}$ we write $(e_i) < (d_i)$ or $(d_i) > (e_i)$ if $c_i < d_i$, or there exists $k \geq 2$ such that $c_i = d_i$ for all $1 \leq i < k$ and $c_k < d_k$. Similarly, we write $(e_i) \preceq (d_i)$ or $(d_i) \succeq (e_i)$ if $(e_i) < (d_i)$ or $(e_i) = (d_i)$. Furthermore, for two words $e, d \in \{0, 1\}^*$ we say $e < d$ if $e^0 \preceq d^0 \preceq \cdots \preceq d^\infty$. Given $\beta \in (1, 2)$ let $\delta(\beta) = (\delta_i(\beta)) \in \{0, 1\}^\mathbb{N}$ be the quasi-greedy $\beta$-expansion of 1 (cf [12]), i.e., the lexicographically largest $\beta$-expansion of 1 not ending with a string of zeros. Based on the work of Parry [13] the following characterization of $\delta(\beta)$ is well known, see ([14], theorem 2.2) and ([15], lemma 2.1 and remark 2.2).
Lemma 2.1.

(a) The map $\beta \mapsto \delta(\beta)$ is a strictly increasing bijection from $(1, 2]$ onto the set $A$ of all sequences $(a_i) \in \{0, 1\}^\infty$ not ending with $0^\infty$ and satisfying

$$a_{n+1}a_{n+2} \cdots \preceq a_1a_2 \cdots$$

for all $n \geq 0$.

(b) The inverse map

$$\delta^{-1} : A \rightarrow (1, 2]; \quad (a_i) \mapsto \delta^{-1}(a_i)$$

is bijective and strictly increasing. Furthermore, $\delta^{-1}$ is continuous.

Recall that $U_\beta$ is the intrinsic univoque set consisting of all points $((d_i))_\beta \in S_\beta$ satisfying $((d_{n+i}))_\beta \notin O_\beta$ for all $n \geq 0$. Let

$$\tilde{U}_\beta := \left\{ (d_i) \in A^\infty : ((d_i))_\beta \in U_\beta \right\}$$

be the set of all codings of points from $\tilde{U}_\beta$. Note that $\tilde{U}_\beta \subseteq U_\beta$ for all $\beta \in (1, 2)$. This means each point in $\tilde{U}_\beta$ has a unique coding. So the map $(d_i) \mapsto ((d_i))_\beta$ is bijective from $U_\beta$ to $\tilde{U}_\beta$.

For a digit $d = (d^1, d^2) \in A$ we write $d^\oplus := d^1 + d^2$. Then $d^1, d^2, d^\oplus \in \{0, 1\}$. In the following we give a lexicographical characterization of $\tilde{U}_\beta$, or equivalently, of $U_\beta$.

**Proposition 2.2.** Let $\beta \in (1, 2)$. Then $(d_i) \in \tilde{U}_\beta$ if and only if $(d_i) \in A^\infty$ satisfies

$$\begin{cases} 
    d^1_{n+1}d^1_{n+2} \cdots \preceq \delta(\beta) & \text{whenever } d^1_n = 0, \\
    d^2_{n+1}d^2_{n+2} \cdots \preceq \delta(\beta) & \text{whenever } d^2_n = 0, \\
    d^\oplus_{n+1}d^\oplus_{n+2} \cdots \succeq \delta(\beta) & \text{whenever } d^\oplus_n = 1.
\end{cases}$$

Before proving the proposition we point out that each coding $(d_i) \in \tilde{U}_\beta$ is an infinite sequence of vectors from $A$, while the characterization is based on the projections of $(d_i)$ which are infinite sequences of zeros and ones.

**Proof.** First we prove the necessity. Take a sequence $(d_i) \in \tilde{U}_\beta$. We consider three cases.

Case 1. Suppose $d^1_0 = 0$ for some $n \geq 1$. Then $d_n = A$ or $d_n = C$. In view of figure 1 on page 2, it follows that

$$(d^1_0d^1_1 \cdots)_\beta \in (f_\beta A(\Delta_\beta) \cup f_\beta C(\Delta_\beta)) \setminus O_\beta,$$

which implies $(d^1_0d^1_1 \cdots)_\beta < 1/\beta$. Whence,

$$d^1_0d^1_1 \cdots \preceq \delta(\beta).$$

If $\delta(\beta)$ is not periodic, then $\delta(\beta)$ is also the greedy $\beta$-expansion of 1, which is the lexicographically largest $\beta$-expansion of 1 (cf [12]). So (2.2) implies

$$d^1_{n+1}d^1_{n+2} \cdots \preceq \delta(\beta)$$

as required.
If $\delta(\beta)$ is periodic, say $\delta(\beta) = (\delta_1 \cdots \delta_m)^\infty$ with $m$ the smallest period, then $\delta_m = 0$, and the greedy $\beta$-expansion of 1 is given by $\delta_1 \cdots \delta_m^\infty$. By (2.2) it follows that $d_{n+1}^1 d_{n+2}^1 \cdots \prec \delta_1 \cdots \delta_m^\infty$, which yields

\[ d_{n+1}^1 \cdots d_{n+m}^1 \prec \delta_1 \cdots \delta_m. \]

If the strict inequality holds, then (2.3) holds and we are done. Suppose $d_{n+1}^1 \cdots d_{n+m}^1 = \delta_1 \cdots \delta_m$. Then $d_{n+1}^1 = 0$. By the same argument as above, we either have (2.3) or $d_{n+1}^1 \cdots d_{n+2m}^1 = (\delta_1 \cdots \delta_m)^2$.

Repeating this procedure indefinitely we either have (2.3) or $d_{n+m+1}^1 d_{n+2}^1 \cdots = (\delta_1 \cdots \delta_m)^\infty = \delta(\beta)$. Note by (2.2) the second case can not occur. This proves (2.3) if $d_n^1 = 0$.

Case 2. Suppose $d_n^1 = 0$ for some $n \geq 1$. Then by a similar argument as in case 1 we can prove $d_{n+1}^2 d_{n+2}^2 \cdots \prec \delta(\beta)$.

Case 3. Suppose $d_{n}^1 = 1$, i.e., $d_n^1 = 1$ or $d_n^2 = 1$. By symmetry we may assume $d_n^1 = 1$. Then $d_n = (1, 0) = A$. In view of figure 1, it follows that $(d_{\alpha} d_{\beta} \cdots) \in f_{j, \beta}(\Delta) \setminus \mathcal{O}_\beta$, which implies

\[ (1 d_{n+1}^1 d_{n+2}^2 \cdots)_\beta = (d_{n+1}^1 d_{n+2}^1 \cdots)_\beta + (d_{n+1}^2 d_{n+2}^1 \cdots)_\beta \geq \frac{1}{\beta(\beta - 1)}. \]

This is equivalent to

\[ \frac{1}{\beta - 1} - (d_{n+1}^1 d_{n+2}^2 \cdots)_\beta < 1. \]

Whence,

\[ (d_{n+1}^1 d_{n+2}^2 \cdots)_\beta < 1. \]

Observe that $d_{n+1}^1 = 1 - d_{n+1}^2 \in \{0, 1\}$ for any $n \geq 1$, and $d_{n+1}^2 = 0$. Then by the same argument as in the case for $d_n^1 = 0$ we can show that

\[ d_{n+1}^1 d_{n+2}^2 \cdots \prec \delta(\beta). \]

In other words, $d_{n+1}^1 d_{n+2}^2 \cdots \prec \delta(\beta)$. This proves the necessity.

Now we turn to prove the sufficiency. Let $(d_i) \in A^\infty$ be the sequence satisfying the inequalities in the proposition. Then it suffices to prove

\[ (d_{\alpha} d_{\beta} \cdots) \in f_{j, \beta}(\Delta) \setminus \mathcal{O}_\beta \quad \text{for all } n \geq 1. \]  

(2.4)

Fix $n \geq 1$. Suppose $d_n = A = (0, 0)$. Then for $\ell \in \{1, 2\}$ we have

\[ d_{m+\delta}^\ell d_{m+\delta+2}^\ell \cdots \prec \delta(\beta) \quad \text{whenever } d_m^\ell = 0. \]  

(2.5)

Since $d_n^\ell = 0$, we claim that $(d_{n+1}^\ell d_{n+2}^\ell \cdots)_\beta < 1/\beta$.

Starting with $k_0 := n$ we define by recurrence a sequence of indices $k_0 < k_1 < \cdots$ satisfying for $j = 1, 2, \ldots$, the conditions

\[ d_{k_{j+1}}^j \cdots d_{k_{j+1}}^j = \delta_1(\beta) \cdots \delta_{k_{j+1}}(\beta) \quad \text{and} \quad d_{k_{j+1}}^j < \delta_{k_{j+1}}(\beta). \]
By (2.5) it follows that we have infinitely many indices \((k_j)\). Then

\[
(d_n^1, d_{n+1}^1, \ldots)_\beta = \beta^{n-1} \sum_{j=n+1}^\infty \sum_{i=1}^{k_j} \frac{d_i^1}{\beta^{k_{j-1}+1}} = \beta^{n-1} \sum_{j=1}^\infty \sum_{i=1}^{k_j-1} \frac{d_i^1}{\beta^{k_{j-1}+1}}
\]

\[
< \beta^{n-1} \sum_{j=1}^\infty \left( \sum_{i=1}^{k_j-1} \frac{\delta(\beta)}{\beta^{k_{j-1}+1}} - \frac{1}{\beta^{k_j}} \right) = \frac{\beta^{n-1}}{\beta^0} = \frac{1}{\beta}.
\]

So, for \(d_n = A\) we have

\[
(d_n^1, d_{n+1}^1, \ldots)_\beta < \frac{1}{\beta} \quad \text{and} \quad (d_n^2, d_{n+1}^2, \ldots)_\beta < \frac{1}{\beta}.
\]

In view of figure 1 this implies that \((d_n, d_{n+1}, \ldots)_\beta \in f_{\beta A}(\Delta, B)\setminus O_B\). Hence, (2.4) holds if \(d_n = A\).

Since the proof of (2.4) for \(d_n = C\) is similar to that for \(d_n = B\), it suffices to prove (2.4) for \(d_n = B = (1, 0)\). Note that

\[
d_{m+1}^1, d_{m+2}^1, \ldots \geq \delta(\beta) \quad \text{whenever} \quad d_m^1 = 1.
\]

This is equivalent to

\[
d_{m+1}^1, d_{m+2}^1, \ldots < \delta(\beta) \quad \text{whenever} \quad d_m^1 = 0.
\]

Observe that \(d_m^2 = 1 - d_m^1 = 0\). By the same argument as in the case for \(d_m^1 = 0\) it follows that

\[
\frac{1}{\beta} - (d_n^2, d_{n+1}^2, \ldots)_\beta = \frac{(d_n^2, d_{n+1}^2, \ldots)_\beta}{(d_n^1, d_{n+1}^1, \ldots)_\beta} < \frac{1}{\beta},
\]

which implies

\[
(d_n^1, d_{n+1}^1, \ldots)_\beta + (d_n^2, d_{n+2}^2, \ldots)_\beta = (d_n^2, d_{n+1}^2, \ldots)_\beta + (d_n^1, d_{n+1}^1, \ldots)_\beta > \frac{1}{\beta(\beta - 1)}.
\]

Since \(d_n^2 = 0\), by the above proof it follows that

\[
(d_n^1, d_{n+1}^1, \ldots)_\beta < \frac{1}{\beta}.
\]

In view of figure 1, we obtain by (2.6) and (2.7) that \((d_n, d_{n+1}, \ldots)_\beta \in f_{\beta B}(\Delta, B)\setminus O_B\). This proves (2.4) if \(d_n = B\), and then completes the proof.
Remark 2.3.

- Note by lemma 2.1 that the map $\beta \mapsto \delta(\beta)$ is strictly increasing in $(1, 2)$. Then proposition 2.2 implies that the set-valued map $\beta \mapsto \tilde{U}_\beta$ is increasing, i.e., for $1 < \beta_1 < \beta_2 < 2$ we have $\tilde{U}_{\beta_1} \subseteq \tilde{U}_{\beta_2}$. Moreover, the symbolic set $\tilde{U}_\beta$ is shift invariant, i.e., for any sequence $(d_i) \in \tilde{U}_\beta$ we have $\sigma((d_i)) := (d_{i+1}) \in \tilde{U}_\beta$.
- Observe that each point $P = (P^1, P^2) \in \tilde{U}_\beta$ has a unique coding with respect to the alphabet $\mathcal{A}$. However, this does not mean its projections $P^1$ and $P^2$ have a unique $\beta$-expansion with respect to the digits set $\{0, 1\}$. For example, take $\beta \in (1, 2)$ such that $\delta(\beta) = (101000)^\infty$. Then by proposition 2.2 one can check that the sequence

$$(d_i) = (BAC)^\infty \in \tilde{U}_\beta.$$ 

However, neither of its coordinate sequences $(d^1_i) = (100)^\infty$ and $(d^2_i) = (001)^\infty$ is a univoque sequence in the one dimensional sense (cf [16]).

At the end of this section we present a sufficient condition for which the intrinsic univoque set $\tilde{U}_\beta$ coincides with the univoque set $U_\beta$. A number $\beta \in (1, 2)$ is called a multinacci number if $\delta(\beta) = (1^m)^\infty$ for some positive integer $m$. So by lemma 2.1 the smallest multinacci number is the golden ratio $(1 + \sqrt{5})/2 \approx 1.61803$ with $\delta((1 + \sqrt{5})/2) = (10)^\infty$.

Proposition 2.4. If $\beta \in (1, 3/2]$ or $\beta$ is a multinacci number, then $\tilde{U}_\beta = U_\beta$.

Proof. Observe that $\tilde{U}_\beta \subseteq U_\beta$ for all $\beta \in (1, 2)$. Then it suffices to prove $U_\beta \subseteq \tilde{U}_\beta$ for $\beta \in (1, 3/2]$ and for $\beta$ being a multinacci number. If $\beta \in (1, 3/2]$, then $S_\beta = \Delta_\beta$. In view of figure 1 on page 2 it follows that

$$O_A \cap S_\beta = f_{3, A}(S_\beta) \cap f_{3, C}(S_\beta),$$
$$O_B \cap S_\beta = f_{3, A}(S_\beta) \cap f_{3, B}(S_\beta),$$
$$O_C \cap S_\beta = f_{3, A}(S_\beta) \cap f_{3, B}(S_\beta).$$

Therefore, any point in $O_\beta \cap S_\beta = (O_A \cup O_B \cup O_C) \cap S_\beta$ has at least two codings. For example, any point in $O_A \cap S_\beta$ has at least two codings: one begins with digit $B$ and the other begins with digit $C$. Hence, $U_\beta \subseteq \tilde{U}_\beta$ if $\beta \in (1, 3/2]$.

If $\beta$ is a multinacci number, then $S_\beta$ is a proper subset of $\Delta_\beta$. But we still have (2.8) since by ([3], theorem 3.3)

$$f_{3, d}(\Delta_\beta) \cap S_\beta = f_{3, d}(S_\beta) \quad \text{for any } d \in A.$$ 

Again by the same argument as above we conclude that any point in $O_\beta \cap S_\beta$ has at least two codings. This proves $U_\beta \subseteq \tilde{U}_\beta$ for $\beta$ being a multinacci number, and completes the proof. □

3. Description of $\tilde{U}_\beta$ when $S_\beta = \Delta_\beta$

In this section we will investigate the intrinsic univoque set $U_\beta$ when $S_\beta = \Delta_\beta$. As a result we give an alternate proof of theorem 1.1. In the second part of this section we prove theorem 1 which answers a conjecture of Sidorov [2] affirmatively.

Note that $S_\beta = \Delta_\beta$ if and only if $\beta \in (1, 3/2]$. Furthermore, by proposition 2.4 we have $U_\beta = \tilde{U}_\beta$ for any $\beta \in (1, 3/2]$. Hence, it suffices to describe the symbolic set $\tilde{U}_\beta$ for
\( \beta \in (1, 3/2] \). Recall that \( \beta_G \approx 1.46557 \) is the unique root in \((1, 2)\) of \( x^3 - x^2 - 1 = 0 \). Then
\[
\delta(\beta_G) = (100)^\infty.
\]

We will describe \( \tilde{U}_\beta \) for \( \beta \leq \beta_G \) and \( \beta \in (\beta_G, 3/2] \) separately.

### 3.1. Description of \( \tilde{U}_\beta \) for \( \beta \leq \beta_G \)

In this part we will investigate the intrinsic univoque set \( \tilde{U}_\beta \) for \( \beta \leq \beta_G \), and then give an alternate proof of theorem 1.1. We emphasize that Sidorov proved this result in [2] in a dynamical way, while our proof is using combinatorics on words based on the lexicographical characterization of \( \tilde{U}_\beta \) described in proposition 2.2. First we show that the symbolic set \( \tilde{U}_{\beta_G} \) does not contain any sequence with three consecutive distinct elements.

**Lemma 3.1.** Let \((d_i) \in \tilde{U}_{\beta_G}\). Then any three consecutive elements \(d_n, d_{n+1} \) and \(d_{n+2} \) cannot be all distinct.

**Proof.** Suppose there exists \( n \geq 1 \) such that \( d_n, d_{n+1} \) and \( d_{n+2} \) are distinct. Then
\[
d_n d_{n+1} d_{n+2} \in \{BAC, CBA, ACB; ABC, CAB, BCA\}.
\]

Since the proofs for different cases are similar, we may assume \( d_n d_{n+1} d_{n+2} = BAC \).

For convenience we represent the block \( d_n d_{n+1} d_{n+2} = BAC \) as three points located from the left to the right as in figure 2. Then any point in the first row has its first coordinate equaling 1, and any point in the third row has its second coordinate equaling 1. While any point in the middle row has both coordinates equaling 0.

First we claim that \( d_{n+3} = B \). Note that \( d_n d_{n+1} d_{n+2} = BAC \). If \( d_{n+3} = A \), then \( d_n^{(a)} \cdots d_{n+3}^{(a)} = 1010 \). This gives
\[
d_n^{(a)} = 1 \quad \text{and} \quad d_{n+1}^{(a)} d_{n+2}^{(a)} \cdots \prec (011)^\infty = \delta(\beta_G).
\]

By proposition 2.2 it follows that \( (d_i) \notin \tilde{U}_{\beta_G} \). If \( d_{n+3} = C \), then \( d_n^{(a)} d_{n+2}^{(a)} d_{n+3} = 011 \) which yields \( d_n^{(a)} = 0 \) and \( d_{n+1}^{(a)} d_{n+2}^{(a)} \cdots \prec (100)^\infty = \delta(\beta) \). Again by proposition 2.2 we have \( (d_i) \notin \tilde{U}_{\beta_G} \). Therefore, \( d_{n+3} = B \).

Next we claim \( d_{n+4} = A \). Note that \( d_n d_{n+1} d_{n+2} d_{n+3} = A CB \). Then \( d_n^{(a)} d_{n+1}^{(a)} d_{n+2}^{(a)} d_{n+3}^{(a)} = 011 \) if \( d_{n+4} = B \), and \( d_n^{(a)} \cdots d_{n+3}^{(a)} = 0101 \) if \( d_{n+4} = C \). In both cases we can deduce by proposition 2.2 that \( (d_i) \notin \tilde{U}_{\beta_G} \). So, \( d_{n+4} = A \).

Now we claim \( d_{n+5} = C \). Observe that \( d_n d_{n+1} d_{n+2} d_{n+3} = C BA \). If \( d_{n+5} = B \), then \( d_n^{(a)} d_{n+1}^{(a)} \cdots d_{n+5}^{(a)} = 0101 \). If \( d_{n+5} = A \), then \( d_n^{(a)} d_{n+1}^{(a)} \cdots d_{n+5}^{(a)} = 0101 \). In both cases we can infer from proposition 2.2 that \( (d_i) \notin \tilde{U}_{\beta_G} \). Hence, \( d_{n+5} = C \).

Observe that in the above arguments each three consecutive block \( d_{n+1} d_{n+2} d_{n+3} \) uniquely determines the next digit \( d_{n+4} \). Repeating this procedure indefinitely one can show that
\[
d_n d_{n+1} \cdots = (BAC)^\infty.
\]

Then \( d_{n+1}^2 = 0 \) and \( d_{n+2}^2 \cdots = (100)^\infty = \delta(\beta_G) \). However, by proposition 2.2 this implies that \( (d_i) \notin \tilde{U}_{\beta_G} \), leading to contradiction. Hence, any three consecutive elements of \( (d_i) \in \tilde{U}_{\beta_G} \) cannot be all distinct.

**An alternate proof of theorem 1.1.** By proposition 2.2 it follows that for any \( \beta \in (1, \beta_G] \) the sequences \( A^\infty, B^\infty \) and \( C^\infty \) all belong to \( \tilde{U}_\beta \). Observe by proposition 2.4 that \( U_\beta = \tilde{U}_\beta \).
for $\beta \in (1, \beta_G]$, and by lemma 2.1 and proposition 2.2 that the set-valued map $\beta \mapsto U_\beta$ is increasing on $(1, \beta_G]$. So it suffices to show that $\tilde{U}_{\beta_G}$ consists of the three sequences $A^\infty, B^\infty,$ and $C^\infty$.

Suppose on the contrary that there exists $(d_i) \in \tilde{U}_{\beta_G} \setminus \{A^\infty, B^\infty, C^\infty\}$. Then there is an integer $n \geq 1$ such that $d_{n+1} \neq d_n$. So,

$$d_nd_{n+1} \in \{AB, AC, BA, BC, CA, CB\}.$$ 

Since the proofs for different cases are similar, without loss of generality we may assume $d_{n+1} = AB$. By lemma 3.1 the next element $d_{n+2}$ must be equal to $A$ or $B$. We will finish the proof by showing that $(d_i) \notin \tilde{U}_{\beta_G}$.

Case 1. $d_{n+2} = A$. By lemma 3.1 we have $d_{n+3} \in \{A, B\}$. Then

$$d_n \cdots d_{n+3} = ABAA \quad \text{or} \quad d_n \cdots d_{n+3} = ABAB.$$ 

In the first case we have $d_n^3 d_{n+1}^3 d_{n+2} = 100$, and in the second case we have $d_n^3 \cdots d_{n+3}^3 = 0101$. In both cases, we can deduce by proposition 2.2 that $(d_i) \notin \tilde{U}_{\beta_G}$.

Case 2. $d_{n+2} = B$. Then $d_{n+1} d_{n+2} = ABB$, which gives $d_n^3 d_{n+1}^3 d_{n+2} = 011$. By proposition 2.2 this again yields $(d_i) \notin \tilde{U}_{\beta_G}$.

\[\square\]

3.2. Description of $\tilde{U}_\beta$ for $\beta_G < \beta \leq 3/2$

Let $\beta_2 \approx 1.5385 \in (\beta_G, 2)$ such that

$$\delta(\beta_2) = (101000)^\infty.$$ 

Then $3/2 \in (\beta_G, \beta_2)$. By theorem 1.1 it suffices to describe the difference set $\tilde{U}_{3/2} \setminus \tilde{U}_{\beta_G}$.

Proposition 3.2. Let $\beta \in (\beta_G, \beta_2]$. Then any sequence in $\tilde{U}_\beta \setminus \tilde{U}_{\beta_G}$ must end with $(ABC)^\infty$ or $(CBA)^\infty$.

First we show that for $\beta \leq \beta_2$ any block of the form ‘cdd’ cannot occur in sequences of $\tilde{U}_\beta$. In the following we prove this for $\beta$ in a larger range.

Lemma 3.3. Let $\beta \in (1, 2)$. If $\delta(\beta)$ begins with 10, then any block from the following set

$$\mathcal{F} = \{BAA, BCC, ABB, ACC, CBB, CAA\}$$

is forbidden in $\tilde{U}_\beta$.

Proof. Since the proofs for different cases are similar, without loss of generality we only prove that $BAA$ is forbidden in $\tilde{U}_\beta$. Suppose on the contrary there exists a sequence $(d_i) \in \tilde{U}_\beta$ such that $d_{n+1} d_{n+2} = BAA$ for some $n \geq 1$. Then $d_n^3 d_{n+1}^3 d_{n+2} = 100$. This implies $d_n^3 = 1$.

\[\square\]
and $d_{m+1}d_{m+2}d_{m+3}\cdots \notin \delta(\beta)$. By proposition 2.2 we have $(d_i) \notin \tilde{U}_{\beta}$, leading to a contradiction. So BAA is forbidden in $\tilde{U}_{\beta}$. □

In order to prove proposition 3.2 we also need the following lemma.

**Lemma 3.4.**  Let $\beta \in (1, 2)$ such that $\delta(\beta)$ begins with 101000. Suppose $(d_i) \in \tilde{U}_{\beta}$.

(a) If $d_{m+1}d_{m+2}d_{m+3} = BAB$ and $d_m \neq d_{m+1}$, then the next block $d_{m+4}d_{m+5}d_{m+6} = CAC$.
(b) If $d_{m+1}d_{m+2}d_{m+3} = CAC$ and $d_m \neq d_{m+1}$, then the next block $d_{m+4}d_{m+5}d_{m+6} = BAB$.
(c) If $d_{m+1}d_{m+2}d_{m+3} = ABC$ and $d_m \neq d_{m+1}$, then the next block $d_{m+4}d_{m+5}d_{m+6} = CBC$.
(d) If $d_{m+1}d_{m+2}d_{m+3} = CAC$ and $d_m \neq d_{m+1}$, then the next block $d_{m+4}d_{m+5}d_{m+6} = ABA$.
(e) If $d_{m+1}d_{m+2}d_{m+3} = ACA$ and $d_m \neq d_{m+1}$, then the next block $d_{m+4}d_{m+5}d_{m+6} = BCB$.
(f) If $d_{m+1}d_{m+2}d_{m+3} = BCB$ and $d_m \neq d_{m+1}$, then the next block $d_{m+4}d_{m+5}d_{m+6} = ACA$.

Before giving the proof we first explain (a)–(f) via figures 3–5. Firstly, (a) and (b) are represented in figure 3 which implies that if $(d_i) \in \tilde{U}_{\beta}$ begins with BAB, then $(d_i) = (BABCAC)^\infty$. Secondly, (c) and (d) are illustrated in figure 4 which yields that if $(d_i) \in \tilde{U}_{\beta}$ begins with ABA, then $(d_i) = (ABACBC)^\infty$. Finally, (e) and (f) are described in figure 5 which gives that if $(d_i) \in \tilde{U}_{\beta}$ begins with ACA, then $(d_i) = (ACABCB)^\infty$. The names ‘type-A’, ‘type-B’ and ‘type-C’ will be clarified in the next section.
Proof. Let \((d_i) \in \tilde{U}_\beta\). Since the proof of (b) is similar to (a), the proof of (e) is similar to (c), and the proof of (f) is similar to (d), we only prove the lemma for cases (a), (c), and (d).

(a) Suppose \(d_{m+1}d_{m+2}d_{m+3} = BAB\) and \(d_m \neq B\) (see figure 3). Then \(d_1 \cdots d_{m+3} = 01011\). Note that \(\delta(\beta)\) begins with 101000. By proposition 2.2 it follows that \(d_1d_3d_5d_7d_9 = 000\) which implies \(d_{m+4}d_{m+5}d_{m+6} = \{A, C\}\). Then by lemma 3.3 we have only two choices: either \(d_{m+4}d_{m+5}d_{m+6} = ACA\) or \(d_{m+4}d_{m+5}d_{m+6} =CAC\).

If \(d_{m+4}d_{m+5}d_{m+6} = ACA\), then \(d_{m+4} \cdots d_{m+6} = 1010011\). Observe that \(\overline{\delta(\beta)}\) begins with 010111. This implies that \(d_{i+1} = 1\) and \(d_{i+2}d_{i+3} \cdots \sim \overline{\delta(\beta)}\). By proposition 2.2 we have \(d_i \notin \tilde{U}_\beta\). Hence, \(d_{m+4}d_{m+5}d_{m+6} = \overline{CAC}\) as required.

(c) Suppose \(d_{m+1}d_{m+2}d_{m+3} = ABA\) and \(d_m \neq A\) (see figure 4). Then \(d_1d_3 = 1\) and \(d_5d_7d_9 = 010\). By proposition 2.2 it follows that \(d_1d_3d_5d_7d_9 = 000\) which gives \(d_{m+4}d_{m+5}d_{m+6} = \{B, C\}\). Using lemma 3.3 we have only two choices: either \(d_{m+4}d_{m+5}d_{m+6} = BCB\) or \(d_{m+4}d_{m+5}d_{m+6} = \overline{CBC}\).

If \(d_{m+4}d_{m+5}d_{m+6} = BCB\), then \(d_{m+4}d_{m+5}d_{m+6} = 0100101\), which implies \(d_{m+4} = 0\) and \(d_{m+5}d_{m+6} = \overline{\delta(\beta)}\). By proposition 2.2 this implies \(d_m \notin \tilde{U}_\beta\). Hence, \(d_{m+4}d_{m+5}d_{m+6} = \overline{CBC}\) as required.

(d) Suppose \(d_{m+1}d_{m+2}d_{m+3} = \overline{CBC}\) and \(d_m \neq \overline{C}\) (see figure 4 second column). Then \(d_1d_3 = 1\) and \(d_5d_7d_9 = \overline{010}\). Since \(\delta(\beta)\) begins with 101000, by proposition 2.2 it follows that \(d_1d_3d_5d_7d_9 = 000\) which yields \(d_{m+4}d_{m+5}d_{m+6} = \{A, B\}\). By lemma 3.3 we have only two choices: either \(d_{m+4}d_{m+5}d_{m+6} = ABA\) or \(d_{m+4}d_{m+5}d_{m+6} = \overline{BAB}\).

If \(d_{m+4}d_{m+5}d_{m+6} = ABA\), then \(d_{m+4} \cdots d_{m+6} = 01001011\), which implies \(d_{m+4} = 0\) and \(d_{m+5}d_{m+6} = \overline{\delta(\beta)}\). By proposition 2.2 we have \(d_m \notin \tilde{U}_\beta\). Hence, \(d_{m+4}d_{m+5}d_{m+6} = \overline{ABA}\). This completes the proof.

**Proof of proposition 3.2.** Take \(\beta \in (\beta_G, \beta_2)\). Note that \(\delta(\beta_G) = (100)^\infty\) and \(\delta(\beta_2) = (101000)^\infty\). Then by lemma 2.1 \(\delta(\beta)\) begins with 101000. By proposition 2.2 one can check that the sequences \((BAC)^\infty\) and \((ABC)^\infty\) belong to \(\tilde{U}_\beta\). Note that the set-valued map \(\beta \mapsto \tilde{U}_\beta\) is increasing. It suffices to show that any sequence in \(\tilde{U}_\beta \setminus \tilde{U}_G\) must end with \((ABC)^\infty\) or \((CBA)^\infty\).

Take \((d_i) \in \tilde{U}_\beta \setminus \tilde{U}_G\). Suppose on the contrary that \((d_i)\) ends with neither \((ABC)^\infty\) nor \((CBA)^\infty\). Note by theorem 1.1 that \(\tilde{U}_G = \{A^\infty, B^\infty, C^\infty\}\). Then by lemma 3.3 there exists \(m \geq 1\) such that

\[
d_m \neq d_{m+1} \quad \text{and} \quad d_{m+1}d_{m+2}d_{m+3} \in \{ABA, BAB, ACA, CAC, BCB, CBC\}.
\]

Since the proof for different cases are similar, we only consider the case \(d_{m+1}d_{m+2}d_{m+3} = ABA\) and \(d_m \neq A\). By lemmas 3.4(c) and (d) it follows that \(d_{m+1}d_{m+2} \cdots = (ABACBC)^\infty\). This implies that

\[
d_1d_3d_5d_7d_9 = 000\Rightarrow \delta(\beta_G).
\]

By proposition 2.2 we have \(d_m \notin \tilde{U}_G\), leading to a contradiction. This completes the proof.

**Proof of theorem 1.** Note by proposition 2.4 that for \(\beta \in (1, 3/2)\) we have \(U_\beta = \tilde{U}_\beta\). Furthermore, by theorem 1.1 we have \(U_G = \{A^\infty, B^\infty, C^\infty\}\). Therefore, the theorem follows by the proof of proposition 3.2 that for any \(\beta \in (\beta_G, \beta_2)\) the set \(\tilde{U}_G \setminus \tilde{U}_G\) consists of all sequences.
of the form
\[ d^n(BAC)\infty, \; d^n(CAB)\infty, \; d^n(CBA)\infty, \; d^n(ABC)\infty, \; d^n(ACB)\infty, \; d^n(BCA)\infty, \]
where \( d \in \mathcal{A} \) and \( n = 0, 1, 2, \ldots \). \( \square \)

4. Admissible words in \( \tilde{\mathcal{U}}_{\mathcal{E}} \)

In this section we will describe the admissible words in sequences of \( \tilde{\mathcal{U}}_{\mathcal{E}} \). Motivated by the previous section we will introduce three types of Thue–Morse words with respect to the alphabet \( \mathcal{A} = \{A, B, C\} \). First we recall from section 1 the Thue–Morse type words \( (t_n) \) defined on \( \Omega = \{000, 001, 100, 101\} \).

Let \( t_1 = 100 \), and for \( n \geq 1 \) we set
\[ t_{n+1} = t_n^+ \Theta(t_n^+), \]
where \( \Theta \) is the block map defined by
\[ \Theta : \Omega \to \Omega; \quad 000 \mapsto 101, \ 001 \mapsto 100, \ 100 \mapsto 001, \ 101 \mapsto 000. \]

Here for a word \( a = a_1 \cdots a_n \in \Omega^n \) we set \( \Theta(a) = \Theta(a_1) \cdots \Theta(a_n) \). Clearly, \( \Theta \circ \Theta(a) = a \).

Observe that \( t_n \) ends with 0 and has length \( 3 \cdot 2^{n-1} \). Furthermore, \( t_{n+1} \) begins with \( t_n^+ \) for all \( n \geq 1 \). This implies that the component-wise limit of the sequence \( (t_n) \) is well-defined, which is the Thue–Morse type sequence \( (\lambda_n) \).

So \( (\lambda_n) \) begins with \( t_n^+ \) for all \( n \geq 1 \). One can check that the first few terms of \( (\lambda_n) \) are given by
\[ 1010010001010001001010001001001000100100. \]

Similarly, the component-wise limit of the sequence \( (\Theta(t_n)) \) is also well-defined, denoted this limit by \( (\gamma_n) \). Then \( (\gamma_n) \) begins with \( \Theta(t_n^+) \) for all \( n \geq 1 \), and the first few terms of \( (\gamma_n) \) are given by
\[ 00010010001010001001001010001001001000. \]

Therefore, for any \( n \geq 1 \) we have \( \lambda_1 \cdots \lambda_{3 \cdot 2^n - 1} = t_n^+ \) and \( \gamma_1 \cdots \gamma_{3 \cdot 2^n - 1} = \Theta(t_n^+) \).

**Lemma 4.1.** For any \( n \geq 1 \) we have
\[ \gamma_1 \cdots \gamma_{3 \cdot 2^{n-1} - 1} \leq \lambda_{1+1} \cdots \lambda_{3 \cdot 2^{n-1} - 1} \leq \lambda_1 \cdots \lambda_{3 \cdot 2^{n-1} - 1} \]
(4.1)
\[ \gamma_1 \cdots \gamma_{3 \cdot 2^{n-1} - 1} \leq \gamma_{1+1} \cdots \gamma_{3 \cdot 2^{n-1} - 1} \leq \lambda_1 \cdots \lambda_{3 \cdot 2^{n-1} - 1} \]
(4.2)
for all \( 0 \leq i < 3 \cdot 2^{n-1} \).

**Proof.** Since the proof of (4.2) is similar, here we only prove (4.1).

Let \( (\tau_i)_{i=1}^{\infty} = 11010011 \cdots \) be the shifted classical Thue–Morse sequence (cf [11], see also [10]). By the definition of \( (\lambda_i) \) it follows that \( (\lambda_i) \) can be obtained from \( (\tau_i) \) by adding an extra zero between \( \tau_{2k+1} \) and \( \tau_{2k+2} \) for each \( k \geq 0 \). Similarly, the sequence \( (\gamma_i) \) can be reconstructed from \( (\gamma_i) \) by adding an extra zero between \( \gamma_{2k+1} \) and \( \gamma_{2k+2} \) for each \( k \geq 0 \), where \( \overline{0} = 1 \) and \( T = 0 \). In other words, for any \( k \geq 0 \) we have
\[ \lambda_{3k+1} = \overline{\tau_{2k+1}}, \quad \lambda_{3k+2} = 0, \quad \lambda_{3k+3} = \tau_{2k+2}, \]
\[ \gamma_{3k+1} = \overline{\tau_{2k+1}}, \quad \gamma_{3k+2} = 0, \quad \gamma_{3k+3} = \tau_{2k+2}. \]

(4.3)

Clearly, the inequalities in (4.1) holds for \( n = 1 \). Now let \( n \geq 2 \) and take \( 0 \leq i < 3 \cdot 2^{n-1} \).
We will prove (4.1) by considering three cases. 

(a) \(i = 3k\) with \(0 \leq k < 2^{n-1}\). Then by (4.3) it follows that 
\[
\lambda_{3k+1} \cdots \lambda_{3 \cdot 2^{n-1}} = \tau_{2k+1}0^{2} \tau_{2k+2}2 \tau_{2k+3}0^{\tau_{2k+4}} \cdots \tau_{2^{n}-3}0^{\tau_{2^{n}}}
\]

Then (4.1) follows by (4.3) and the property of \((\tau_j)\) that 
\[
t_1 \cdots t_{2^{n}-j} < t_{j+1} \cdots t_{2^{n}} \leq t_1 \cdots t_{2^{n}-j}
\]
for all \(0 \leq j < 2^{n}\) (cf [11]).

(b) \(i = 3k + 1\) with \(0 \leq k < 2^{n-1}\). Then by (4.3) we have 
\[
\lambda_{3k+2} \lambda_{3k+3} \lambda_{3k+4} = 0 \tau_{2k+2} \tau_{2k+3} = 0 \tau_{k+1}(1-\tau_{k+1}),
\]
where the last equality follows by using that \(\tau_{2j} = \tau_j\) and \(\tau_{2j+1} = 1 - \tau_j\) for all \(j \geq 1\) (cf [9]). Thus, (4.1) holds since \((\gamma_i)\) begins with 000 and \((\lambda_i)\) begins with 101.

(c) \(i = 3k + 2\) with \(0 \leq k < 2^{n-1}\). Again by (4.3) we have \(\lambda_{3k+3} \lambda_{3k+4} \lambda_{3k+5} = \tau_{2k+2} \tau_{2k+3}0 = \tau_{k+1}(1-\tau_{k+1})0\). By the same argument as in case (b) we prove (4.1).

Recall that \(\delta(\beta)\) is the quasi-greedy \(\beta\)-expansion of 1. By lemmas 2.1 and 4.1 it follows that each Thue–Morse type word \(t_n\) defines a unique base \(\beta_n \in (1, 2)\).

**Definition 4.2.** Set \(\beta_0 = 1\), and for \(n \geq 1\) let \(\beta_n \in (1, 2)\) such that \(\delta(\beta_n) = (t_n)^\infty\).

In terms of definition 4.2 we have \(\delta(\beta_1) = (100)^\infty\) and \(\delta(\beta_2) = (101000)^\infty\). Then \(\beta_1 = \beta_G\) and \(\beta_2 \approx 1.5385\) as defined in the previous section. Furthermore, by (1.1), lemmas 2.1 and 4.1 the sequence \((\lambda_i)\) is indeed the quasi-greedy \(\beta_i\)-expansion of 1, i.e.,
\[
\delta(\beta_i) = (\lambda_i) = t_i^n \Theta(t_n) \cdots = 101001000101000100101001 \cdots
\]
for any \(n \geq 1\). Observe that 
\[
t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots, \quad \text{and} \quad t_n \nearrow (\lambda_i) \quad \text{as} \quad n \to \infty.
\]

By lemma 2.1 and definition 4.2 it follows that 
\[
1 = \beta_0 < \beta_1 = \beta_G < \beta_2 < \cdots < \beta_n < \beta_{n+1} < \cdots, \quad \text{and} \quad \beta_n \nearrow \beta_c \quad \text{as} \quad n \to \infty.
\]
So these bases \((\beta_n)_{n=1}^\infty\) form a partition of the interval \((1, \beta_c)\).

In the following we describe all possible admissible words in \(\tilde{U}_\beta\) which are constructed by three types of Thue–Morse words in \(A^*\). In the next section we will show that the set-valued map \(\beta \mapsto \tilde{U}_\beta\) is constant on each subinterval \((\beta_n, \beta_{n+1}]\) for all \(n \geq 0\), and completely characterize the sets \(U_{\beta_n}\) and \(U_{\beta_c}\).

**4.1. Type-A Thue–Morse words in \(\tilde{U}_\beta\)**

Let \(\Phi_A\) be the bijective map on \(A\) defined by 
\[
\Phi_A: A \rightarrow A; \quad A \mapsto A, \quad B \mapsto C, \quad C \mapsto B.
\]
one can verify that

\[ \Phi \circ \Phi_A = \Phi_A \circ \Phi_A. \]

By definition 4.3 one can verify that

\[ u_1 = BABCAC, \quad u_3 = BABCAB CACBACBABCAC. \]

Clearly, for each \( n \geq 1 \) the length of \( u_n \) is \( 3 \cdot 2^{n-1} \) and each \( u_n \) ends with digit \( C \).

We will prove this by induction on \( n \). First we consider \( n = 1 \). Then \( u_1 = BAC \) and \( \Phi_A(u_1) = CAB \). This implies

\[ u_1 = \Phi_A(u_1) = u_1. \]

So, (4.5) holds for \( n = 1 \).

Now suppose (4.5) holds for \( n = k \), and we consider \( n = k + 1 \). Note that \( u_{k+1} = u_k \Phi_A(u_k) \) and \( u_k \) ends with digit \( C \). Recall that \( t_{k+1} = t_k \Theta(t_k) \), and then \( \Theta(t_{k+1}) = \Theta(t_k t_k^+) \). By using the induction hypothesis it follows that

\[ u_{k+1} = (u_k \Phi_A(u_k))^1 t_k \Theta(t_k) = t_{k+1}, \]

\[ u_k^2 = (u_k \Phi_A(u_k))^2 \Theta(t_k t_k^+) = \Theta(t_{k+1}). \]

This proves (4.5) for \( n = k + 1 \). Hence, the lemma follows by induction.

In the next section we will show that for \( \beta = \beta_c \) the type-A Thue–Morse words are all admissible in \( \tilde{U}_\beta \). The following proposition states that once the block \( u_k^\beta \) or \( \Phi_A(u_k^\beta) \) occurs in a sequence of \( \tilde{U}_\beta \), then the next block of length \( 3 \cdot 2^{k-1} \) is nearly determined.

**Proposition 4.5.** Let \( (d_i) \in \tilde{U}_\beta \). Then for any \( k \geq 1 \) the following statements hold.

(a) If \( d_{m+1} \cdots d_{m+3 \cdot 2^{k-1} - 1} = u_k^\beta \) and \( d_m \neq d_{m+1} \), then

\[ d_{m+3 \cdot 2^{k-1} + 1} \cdots d_{m+3 \cdot 2^k - 1} \in \{ \Phi_A(u_k^\beta), \Phi_A(u_k) \}. \]

(b) If \( d_{m+1} \cdots d_{m+3 \cdot 2^{k-1} - 1} = \Phi_A(u_k^\beta) \) and \( d_m \neq d_{m+1} \), then

\[ d_{m+3 \cdot 2^{k-1} + 1} \cdots d_{m+3 \cdot 2^k - 1} \in \{ u_k^\beta, u_k \}. \]
Since the proof of (b) is similar to (a), we only prove proposition 4.5(a). We do this now by induction on \( k \). First we consider the cases \( k = 1 \) and \( k = 2 \).

**Lemma 4.6.** Let \((d_i) \in \hat{U}_\beta\).

(a) If \(d_{m+1}d_{m+2}d_{m+3} = u_1^m\) and \(d_m \neq d_{m+1}\), then \(d_{m+4}d_{m+5}d_{m+6} \in \{\Phi_A(u_1^m), \Phi_A(u_1)\}\).

(b) If \(d_{m+1} \cdots d_{m+6} = u_2^m\) and \(d_m \neq d_{m+1}\), then \(d_{m+7} \cdots d_{m+12} \in \{\Phi_A(u_2^m), \Phi_A(u_2)\}\).

**Proof.** First we prove (a). Suppose \(d_{m+1}d_{m+2}d_{m+3} = u_1^m = BAB\) and \(d_m \neq B\). Then \(d_m^m \cdots d_{m+6} = 010110\) which implies \(d_m^{m+2}d_{m+3} = 010110 \cdots = \delta(\beta_m)\). By proposition 2.2 this follows (a).

Suppose \(d_{m+6} = A\). Then, in view of figure 6, \(d_{m+1}^m = 1\) and \(d_{m+2}^m \cdots d_{m+6}^m = 01010\), which implies \(d_m^{m+2}d_{m+3} \cdots = 010110 \cdots = \delta(\beta_m)\). By proposition 2.2 this implies \((d_i) \notin \hat{U}_\beta\).

Suppose \(d_{m+6} = B\) (see figure 6). Then \(d_{m+1}^m \cdots d_{m+6}^m = 0101001\). Observe by \((4.4)\) that \(\delta(\beta_m)\) begins with \(10100100\). Then by proposition 2.2 we have \(d_m^{m+3}d_{m+8}d_{m+9} = 000\). So by \(k = 3\) it follows that

\[
d_{m+7}d_{m+8}d_{m+9} \in \{ACA, CAC\}.
\]

If \(d_{m+7}d_{m+8}d_{m+9} = CAC\), then \(d_{m+1}^m \cdots d_{m+9}^m = 010101\). By proposition 2.2 and \((4.4)\) this yields \((d_i) \notin \hat{U}_\beta\). If \(d_{m+7}d_{m+8}d_{m+9} = ACA\), then \(d_{m+1}^m = 1\) and \(d_{m+2}^m \cdots d_{m+9}^m = 0101001\), which implies \(d_m^{m+2}d_{m+3} \cdots = 0101101 \cdots = \delta(\beta_m)\). By proposition 2.2 this again gives \((d_i) \notin \hat{U}_\beta\).

Therefore, \(d_{m+7}d_{m+8}d_{m+9} = CA\). By \(k = 3\) the next element \(d_{m+6} \in \{B, C\}\). Hence, either \(d_{m+4}d_{m+5}d_{m+6} = CAB = \Phi_A(u_1^m)\) or \(d_{m+4}d_{m+5}d_{m+6} =CAC = \Phi_A(u_1^m)\). This proves (a).

For (b) we assume \(d_{m+6}^m \cdots d_{m+9}^m = u_2^m = BABCA\) (see figure 7) and \(d_m \neq B\). Then \(d_{m+1}^m \cdots d_{m+6}^m = 0101001\). Observe by \((4.4)\) that \(\delta(\beta_m)\) begins with \(10100100\). Then by proposition 2.2 this implies that \(d_m^{m+3}d_{m+8}d_{m+9} = 000\). So by \(k = 3\) it follows that

\[
d_{m+7}d_{m+8}d_{m+9} \in \{ACA, CAC\}.
\]

If \(d_{m+7}d_{m+8}d_{m+9} = ACA\), then \(d_{m+1}^m = 1\) and \(d_{m+2}^m \cdots d_{m+9}^m = 01010\), which implies \(d_m^{m+2}d_{m+3} \cdots = \delta(\beta_m)\). By proposition 2.2 this gives \((d_i) \notin \hat{U}_\beta\).

Therefore, \(d_{m+7}d_{m+8}d_{m+9} = CAC = \Phi_A(u_1^m)\). By a similar argument as in the proof of (a) one can show that the next block \(d_{m+10}d_{m+11}d_{m+12} \in \{u_1^m, u_1\} = \{BAB, BAC\}\). Hence,

\[
d_{m+7} \cdots d_{m+12} = CACB = \Phi_A(u_2) \text{ or } d_{m+7} \cdots d_{m+12} = CACB = \Phi_A(u_2).
\]

This establishes (b).

**Proof of proposition 4.5.** Since the proof of (b) is similar, we only prove (a). This will be done by induction on \( k \). By \(k = 6\) it follows that the proposition holds for \( k = 1 \) and \( k = 2 \). Suppose the proposition holds for all \( k \leq n \) with \( n \geq 2 \), and we consider \( k = n+1 \).

Suppose \(d_{m+1}^m \cdots d_{m+3}^m = u_1^{n+1}^m\) and \(d_m \neq d_{m+1} = B\). Then by \(k = 5\) it follows that \(d_{m+4}^m \cdots d_{m+12}^m = t_{n+1}^m\). Observe by \((4.4)\) that \(\delta(\beta_m) = t_{n+1}^m \Theta(t_{m+1}) \cdots\). Then by proposition
where the equality follows by using \( t_{i+1} = t_i^{\uparrow} \Theta(t_i^{\downarrow}) \) for any \( i \geq 1 \) and \( \Theta^2(a) = a \) for any block \( a \in \Omega^+ \). Observe that

\[
d_{m+1} \cdots d_{m+3} = u_{n+1}^{\uparrow} = u_n^{\uparrow} \Phi_A(u_n) = u_n^{\uparrow} \Phi_A(u_{n-1}) u_{n-1}^{\downarrow}.
\]

Then \( d_{m+3} \cdots d_{m+3} \cdots = C \) is different from \( d_{m+3} \cdots d_{m+3} \cdots + 1 = B \). Furthermore, the block \( d_{m+3} \cdots d_{m+3} \cdots = u_{n-1}^{\uparrow} \). By the induction hypothesis it follows that the next block of length \( 3 \cdot 2^{n-2} \) is nearly determined, i.e., \( d_{m+3} \cdots d_{m+3} \cdots \in \{ \Phi_A(u_{n-1}^{\uparrow}), \Phi_A(u_{n-1}^{\downarrow}) \} \). By (4.6) and lemma 4.4 it follows that

\[
d_{m+3} \cdots d_{m+3} \cdots + 1 = \Phi_A(u_{n-1}^{\uparrow}).
\]

Note that \( d_{m+3} \cdots = B \) is not equal to \( d_{m+3} \cdots + 1 = C \). Again, by (4.7) and the induction hypothesis it follows that \( d_{m+3} \cdots d_{m+3} \cdots + 1 \cdots d_{m+3} \cdots + 1 \in \{ u_{n-1}^{\uparrow}, u_{n-1}^{\downarrow} \} \). Using lemma 4.4, (4.6) and (4.7) we obtain that

\[
d_{m+3} \cdots d_{m+3} \cdots + 1 = u_{n-1}^{\uparrow}.
\]

Therefore, by (4.7) and (4.8) it gives

\[
d_{m+3} \cdots d_{m+3} \cdots + 1 = \Phi_A(u_{n-1}^{\uparrow}) u_{n-1}^{\uparrow} = \Phi_A(u_{n-1}^{\uparrow}).
\]

Again, using the induction hypothesis the next block of length \( 3 \cdot 2^{n-1} \) is nearly determined, i.e., \( d_{m+3} \cdots d_{m+3} \cdots + 1 \cdots d_{m+3} \cdots + 1 \in \{ u_{n}^{\uparrow}, u_{n}^{\downarrow} \} \). Hence,

\[
d_{m+3} \cdots d_{m+3} \cdots + 1 \in \{ \Phi_A(u_{n}^{\uparrow}), \Phi_A(u_{n}^{\downarrow}) \} = \{ \Phi_A(u_{n+1}^{\uparrow}), \Phi_A(u_{n+1}^{\downarrow}) \}.
\]

This proves (a) for \( k = n + 1 \). Hence, (a) follows by induction. \( \square \)
4.2. Type-B and type-C Thue–Morse words in $\tilde{U}_{\beta}$

Similar to definition 4.3 we define the type-$B$ and type-$C$ Thue–Morse words recursively. First let $\Phi_B$ be defined by (see figure 4)

$$\Phi_B: A \mapsto A; \quad A \mapsto C, \quad B \mapsto B, \quad C \mapsto A.$$  

Accordingly, for a word $d = d_1 \cdots d_m \in A^m$ we put $\Phi_B(d) = \Phi_B(d_1) \cdots \Phi_B(d_m)$.

**Definition 4.7.** The type-$B$ Thue–Morse words $(v_n)$ are defined as follows. Let $v_0 := B, v_1 := CBA$, and for $n \geq 1$ we set

$$v_{n+1} := v_n^C \Phi_B(v_n^C),$$

where $v_n^C$ is the word by changing the last digit of $v_n$ to $C$.

Then by definition 4.7 one can check that $v_2 = CBCABA, v_3 = CBCABCABCBABA$, and $v_4 = CBCABCABBCABCBABA$. Furthermore, for each $n \geq 1$ the last digit of $v_n$ is $A$, and the length of $v_n$ equals $3 \cdot 2^{n-1}$.

Analogously, let $\Phi_C$ be the block map defined by (see figure 5)

$$\Phi_C: A \mapsto A; \quad A \mapsto B, \quad B \mapsto A, \quad C \mapsto C;$$

and for a word $d = d_1 \cdots d_m \in A^m$ we put $\Phi_C(d) = \Phi_C(d_1) \cdots \Phi_C(d_m)$.

**Definition 4.8.** The type-$C$ Thue–Morse words $(w_n)$ are defined as follows. Let $w_0 := C, w_1 := ACB$, and for $n \geq 1$ we set

$$w_{n+1} := w_n^A \Phi_C(w_n^A),$$

where $w_n^A$ is the word by changing the last digit of $w_n$ to $A$.

Then by definition 4.8 we obtain that $w_2 = ACABCB, w_3 = ACABCA BCABCB$ and $w_4 = ACABCABCABCABCABC$. Furthermore, for each $n \geq 1$ the last digit of $w_n$ is $B$, and the length of $w_n$ equals $3 \cdot 2^{n-1}$.

The following lemma describes the coordinate words of $v_n, \Phi_B(v_n), w_n$ and $\Phi_C(w_n)$.

**Lemma 4.9.** For any $n \geq 1$ we have

(a) $v_n^1 = (010)^{2^{n-1}}$ and $v_n^0 = t_n$.

(b) $\Phi_B(v_n)^1 = (010)^{2^{n-1}}$ and $\Phi_B(v_n)^2 = \Theta(t_n)$.

(c) $w_n^1 = \Theta(t_n)$ and $w_n^0 = (010)^{2^{n-1}}$.

(d) $\Phi_C(w_n)^1 = t_n$ and $\Phi_C(w_n)^2 = (010)^{2^{n-1}}$.

**Proof.** Since the proofs of (c) and (d) are similar to (a) and (b), we only prove the first two items. By the definition of $v_n$ it follows that $v_n^1 = (010)^{2^{n-1}}$ and $\Phi_B(v_n)^1 = (010)^{2^{n-1}}$. So it suffices to prove

$$v_n^2 = t_n \quad \text{and} \quad \Phi_B(v_n)^2 = \Theta(t_n). \quad (4.9)$$

We will prove this by induction on $n$.

For $n = 1$ we have $v_1 = CBA$ and $\Phi_B(v_1) = ABC$. Then

$$v_1^2 = 100 = t_1 \quad \text{and} \quad \Phi_B(v_1)^2 = 001 = \Theta(t_1).$$

For $n \geq 2$ we have $v_{n+1} = v_n^C \Phi_B(v_n^C)$ and $v_n^C = (010)^{2^{n-1}} + 1$. By induction $v_n^1 = (010)^{2^{n-1}}$. It follows that $v_n^2 = \Theta(t_n)$.

Therefore, $\Phi_B(v_n) = \Phi_B(v_n^1) \Phi_B(v_n^2) = \Phi_B((010)^{2^{n-1}}) \Theta(t_n) = (010)^{2^{n-1}} \Theta(t_n) = (010)^{2^n}$, and the proof is complete.
So, (4.9) holds for \( n = 1 \). Now suppose (4.9) holds for \( n = k \), and we consider \( n = k + 1 \). Note that

\[ v_{k+1} = v_k^C \Phi_B(v_k^C), \quad \Phi_B(v_{k+1}) = \Phi_B(v_k^C)v_k^C, \]

where for the second equality we use \( \Phi_B \circ \Phi_B(d) = d \) for any \( d \in \Omega^* \). Moreover, for any \( k \geq 1 \) the word \( v_k \) ends with digit \( A \) and \( \Phi_B(v_k) \) ends with digit \( C \). Therefore, by the induction hypothesis it follows that

\[ v_{k+1}^2 = (v_k^C)^2(\Phi_B(v_k^C))^2 = t_k^+ \Theta(t_k^+), \quad \Phi_B(v_{k+1})^2 = (\Phi_B(v_k^C))^2(\Phi_B(v_k))^2 = \Theta(t_k^+t_k^+) = \Theta(t_{k+1}). \]

This proves (4.9) for \( n = k + 1 \). Hence, by induction we establish (a) and (b).

For a word \( d = d_1 \cdots d_m \in \mathcal{A}^m \) we write

\[ d^\ominus := d_1^{(1)} \cdots d_m^{(1)} = (d_1^1 + d_1^2) \cdots (d_m^1 + d_m^2). \]

Then \( d^\ominus \) is a word of zeros and ones. Recall that for a word \( c = c_1 \cdots c_m \in \{0, 1\}^m \) its reflection is defined by \( \overline{c} = (1 - c_1) \cdots (1 - c_m) \). As a corollary of lemmas 4.4 and 4.9 we have the following.

**Corollary 4.10.** For any \( n \geq 1 \) we have

(a) \( u_{n}^{(1)} = \Phi_A(u_{n})^{(1)} = (101)^{n-1} \).

(b) \( v_{n}^{(1)} = \overline{\Theta(t_n)} \) and \( \Phi_B(v_{n})^{(1)} = \overline{t_n} \).

(c) \( w_{n}^{(1)} = \overline{t_n} \) and \( \Phi_C(w_{n})^{(1)} = \overline{\Theta(t_n)} \).

Similar to proposition 4.5 we show that the type-\( B \) and type-\( C \) Thue–Morse words are also forced in \( \tilde{U}_B \).

**Proposition 4.11.** Let \( (d_i) \in \tilde{U}_B \). Then for any \( k \geq 1 \) the following statements hold.

(a) If \( d_{m+1} \cdots d_{m+3} 2k-1 = v_k^C \) and \( d_m \neq d_{m+1} \), then

\[ d_{m+3}2k-1+1 \cdots d_{m+3}2k \in \{ \Phi_B(v_k^C), \Phi_B(v_k) \}. \]

(b) If \( d_{m+1} \cdots d_{m+3} 2k-1 = \Phi_B(v_k^C) \) and \( d_m \neq d_{m+1} \), then

\[ d_{m+3}2k-1+1 \cdots d_{m+3}2k \in \{ v_k^C, v_k \}. \]

(c) If \( d_{m+1} \cdots d_{m+3} 2k-1 = w_k^C \) and \( d_m \neq d_{m+1} \), then

\[ d_{m+3}2k-1+1 \cdots d_{m+3}2k \in \{ \Phi_C(w_k^C), \Phi_C(w_k) \}. \]

(d) If \( d_{m+1} \cdots d_{m+3} 2k-1 = \Phi_C(w_k^C) \) and \( d_m \neq d_{m+1} \), then

\[ d_{m+3}2k-1+1 \cdots d_{m+3}2k \in \{ w_k^C, w_k \}. \]

Since the proofs for different items in proposition 4.11 are similar, we only prove the first item. This will be done by induction on \( k \). First we consider the cases \( k = 1 \) and \( k = 2 \).
\[ B = (1, 0) \begin{array}{ccc} \cdot \cdot \cdot & d_{m+3} & d_{m+4} \\ \cdot \cdot \cdot & \cdot & \cdot \cdot \cdot \\ A = (0, 0) & d_{m+5} & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot & \cdot \cdot \cdot \\ C = (0, 1) & d_{m+1} & d_{m+3} & d_{m+4} & d_{m+6} \end{array} \]

Figure 8. The presentation of \( d_{m+1} \cdots d_{m+6} = CBCBA. \)

Lemma 4.12. Let \( (d_1) \in \tilde{U}_\beta. \)

(a) If \( d_{m+1} d_{m+2} d_{m+3} = \nu_1^\delta \) and \( d_m \neq d_{m+1}, \) then \( d_{m+4} d_{m+5} d_{m+6} \in \{ \Phi_B(\nu_1^\delta), \Phi_B(v_1) \}. \)

(b) If \( d_{m+1} \cdots d_{m+6} = \nu_2^\delta \) with \( d_m \neq d_{m+1}, \) then \( d_{m+7} \cdots d_{m+12} \in \{ \Phi_B(\nu_2^\delta), \Phi_B(v_2) \}. \)

Proof. Suppose \( d_{m+1} d_{m+2} d_{m+3} = \nu_1^\delta \) and \( d_m \neq d_{m+1}, \) then \( d_{m+4} d_{m+5} d_{m+6} \in \{ \Phi_B(\nu_1^\delta), \Phi_B(v_1) \}. \)

If \( d_{m+4} d_{m+5} d_{m+6} = \nu_2^\delta \) with \( d_m \neq d_{m+1}, \) then \( d_{m+7} \cdots d_{m+12} \in \{ \Phi_B(\nu_2^\delta), \Phi_B(v_2) \}. \)

As remarked before the proofs for different items are similar.

Proof of proposition 4.11. As remarked before the proofs for different items are similar.

So in the following we only prove (a) by using the same strategy as in the proof of proposition 4.5. We do this now by induction on \( k. \) First by lemma 4.12 it follows that (a) holds for \( k = 1 \) and \( k = 2. \) Now suppose (a) holds for all \( k \leq n \) with \( n \geq 2, \) and we consider \( k = n + 1. \)
Suppose $d_{m+1} \cdots d_{m+3} = v_{n+1}^C$ and $d_m \neq d_{m+1} = C$. Then $d_m^2 = 0$, and by lemma 4.9 we have $d_{m+1}^2 \cdots d_{m+3}^2 = (v_{n+1}^C)^2 = t_n^+$. Note by (4.4) that $d_{m+1}$ begins with $t_n^+$. By proposition 2.2 it follows that

$$d_{m+3}^2 \cdots d_{m+3}^2 = \Theta(t_{n+1}) = \Theta(t_{n+1}^+ t_{n+1}^+).$$  \hspace{1cm} (4.10)

Observe that $d_{m+1} \cdots d_{m+3} = v_n^C$. Then by the induction hypothesis it follows that $d_{m+3}^2 \cdots d_{m+3}^2 = \Theta(t_{n+1}^+ t_{n+1}^+ \cdots t_{n+1}^+)$. By (4.10) and lemma 4.9 it follows that

$$d_{m+3}^2 \cdots d_{m+3}^2 = \Phi_B(v_n^C).$$  \hspace{1cm} (4.11)

Again by the induction hypothesis the next block of length $3 \cdot 2^{n-2}$ is nearly determined, i.e., $d_{m+3}^2 \cdots d_{m+3}^2 = \Phi_B(v_{n-1}^C)$. In terms of lemma 4.9, by (4.10) and (4.11) we conclude that

$$d_{m+3}^2 \cdots d_{m+3}^2 = v_{n-1}^C,$$  \hspace{1cm} (4.12)

Therefore, by (4.11) and (4.12) we have $d_{m+3}^2 \cdots d_{m+3}^2 = \Phi_B(v_{n-1}^C) v_{n-1} = \Phi_B(v_n^C)$. Using the induction hypothesis the next block of length $3 \cdot 2^{n-1}$ is nearly determined, i.e., $d_{m+3}^2 \cdots d_{m+3}^2 = \Phi_B(v_n^C) v_n$. Hence,

$$d_{m+3}^2 \cdots d_{m+3}^2 \in \{ \Phi_B(v_n^C) v_n, \Phi_B(v_n^C) v_{n+1} \}.$$  \hspace{1cm} (4.4)

This proves (a) for $k = n + 1$. Hence, by induction this completes the proof. \hspace{1cm} $\square$

5. Proof of theorem 2

In this section we will show that the number $\beta_c$ defined in (1.1) is transcendental and it is the critical base for the intrinsic univoque set $U_c$, and then prove theorem 2. Motivated by the work of Allouche and Cosnard [17] (see also, [18]) we first prove the transcendental of $\beta_c$ by using the following well known result from Mahler [19].

**Theorem 5.1** \hspace{0.1cm} (Mahler [19], 1976). If $z$ is an algebraic number in the open unit disc, then the number

$$Z := \sum_{i=1}^{\infty} \tau_i z^i$$

is transcendental, where $(\tau_i)_{i=0}^{\infty}$ is the classical Thue–Morse sequence.

**Proposition 5.2.** $\beta_c$ is transcendental.
4.3) that the quasi-greedy expansion \( \delta(\beta_c) = (\lambda_i) \) satisfies

\[
\lambda_{3n+1} = \tau_{2n+1}, \quad \lambda_{3n+2} = 0 \quad \text{and} \quad \lambda_{3n+3} = \tau_{2n+2}
\]

for all \( n \geq 0 \), where \( (\tau_i)_{i=0}^{\infty} \) is the classical Thue–Morse sequence. Using that \( \tau_0 = 0 \) and \( \tau_{2n+1} = 1 - \tau_n, \tau_{2n+2} = \tau_{n+1} \) for any \( n \geq 0 \) it follows that

\[
1 = \sum_{i=1}^{\infty} \frac{\lambda_i}{\beta_c^i} = \sum_{n=0}^{\infty} \frac{\tau_{2n+1}}{\beta_c^i} + \sum_{n=0}^{\infty} \frac{\tau_{2n+2}}{\beta_c^{i+1}} = \sum_{n=0}^{\infty} \frac{1 - \tau_n}{\beta_c^{2n+1}} + \sum_{n=0}^{\infty} \frac{\tau_{n+1}}{\beta_c^{2n+2}}
\]

By rearrangement we have

\[
\sum_{n=1}^{\infty} \frac{\tau_n}{\beta_c^i} = \frac{\beta_c^3 - \beta_c^2 - 1}{(\beta_c - 1)(\beta_c^2 - 1)}
\]

If \( \beta_c \in (1, 2) \) is algebraic, then by theorem 5.1 the left-hand side of the above equation is transcendental, while the right-hand side is algebraic, leading to a contradiction. So \( \beta_c \) is transcendental.

Recall from definition 4.2 that the bases \( \beta_n \) strictly increases to \( \beta_c \) as \( n \to \infty \). In the following we show that for each \( n \geq 0 \) the periodic sequences \( (u_k)^\infty_n, (v_k)^\infty_n \) and \( (w_k)^\infty_n \) are all contained in \( \tilde{U}_{\beta_c} \).

**Lemma 5.3.** Let \( n \geq 1 \) and let \( \beta > \beta_n \). Then the following periodic sequences

\[
(u_k)^\infty_n, (v_k)^\infty_n, (w_k)^\infty_n, (\Phi_A(u_k))^\infty_n, (\Phi_B(v_k))^\infty_n, (\Phi_C(w_k))^\infty_n
\]

with \( 0 \leq k \leq n \), all belong to \( \tilde{U}_{\beta_c} \).

**Proof.** Note by theorem 1.1 and proposition 3.2 that the lemma holds for \( n = 0 \) and \( n = 1 \). Now we consider \( n \geq 2 \) and \( \beta > \beta_n \). Since the set-valued map \( \beta \mapsto \tilde{U}_\beta \) is increasing, and

\[
\begin{align*}
(u_k)^\infty_n &= u_k^{n-1} \Phi_A(u_k^{n-1}), \\
(v_k) &= v_k^{n-1} \Phi_B(v_k^{n-1}) \quad \text{and} \quad (w_k) &= w_k^{n-1} \Phi_C(w_k^{n-1})
\end{align*}
\]

for any \( k \geq 2 \), it suffices to prove that

\[
(u_k)^\infty_n, (v_k)^\infty_n, (w_k)^\infty_n \in \tilde{U}_{\beta_c} \quad \text{for all} \quad 2 \leq k \leq n.
\]

Take \( k \in \{2, 3, \ldots, n\} \). Since \( \beta > \beta_n \), by definition 4.2 and lemma 2.1 it follows that

\[
\delta(\beta) \succ (u_k)^\infty_n \succ t_k \succ (v_k)^\infty_n \succ (w_k)^\infty_n = \langle 101000 \rangle^\infty.
\]

Observe by lemma 4.4 and corollary 4.10 that

\[
u_k^1 = t_k, \quad u_k^2 = \Theta(t_k), \quad u_k^\vartheta = (101)^{k-1}.
\]

Then by lemma 4.1 and (5.1) it follows that

\[
\begin{align*}
\sigma'((u_k^1)^\infty_n) &= \sigma'((t_k)^\infty_n) \ll (t_k)^\infty_n \ll \delta(\beta), \\
\sigma'((u_k^2)^\infty_n) &= \sigma'(\Theta(t_k)^\infty_n) \ll (t_k)^\infty_n \ll \delta(\beta), \\
\sigma'((u_k^\vartheta)^\infty_n) &= \sigma'(101)^\infty_n \ll \delta(\beta).
\end{align*}
\]
Hence, by proposition 2.2 we conclude that \((u_k)^\infty \in \tilde{U}_\beta\).

Similarly, by lemma 4.9 and corollary 4.10 we have
\[
v_k^1 = (010)^{d_{k1}} - 1, \quad v_k^2 = t_k, \quad v_k^{d_k} = \Theta(t_k);
\]
\[
w_k^1 = \Theta(t_k), \quad w_k^2 = (010)^{d_{k1}} - 1, \quad w_k^{d_k} = \tilde{t}_k.
\]

Then by (5.1), lemma 4.1 and proposition 2.2 we can deduce that \((v_k)^\infty, (w_k)^\infty \in \tilde{U}_\beta\). □

In the following lemma we show that the sequences \((u_n)^\infty, (v_n)^\infty\) and \((w_n)^\infty\) are forbidden in \(U_{\beta_n}\). So the range for the parameter \(\beta\) in the previous lemma is critical.

**Lemma 5.4.** Let \((d_i) \in \tilde{U}_\beta\). If there exits \(m \in \mathbb{N}\) such that \(d_m \neq d_{m+1}\) and \(d_{m+1} \cdot \cdots \cdot d_{m+2n-3} \in \{u_{m-1}^B, \Phi_A(u_{m-1}^B), v_{m-1}^C, \Phi_B(v_{m-1}^C), w_{m-1}^A, \Phi_C(w_{m-1}^A)\}\), then \((d_i) \notin \tilde{U}_\beta\).

**Proof.** Since the proofs for different cases are similar, without loss of generality we assume on the contrary that \(d_{m+1} \cdot \cdots \cdot d_{m+2n-3} = u_{m-1}^B\) and \(d_m \neq d_{m+1} = B\). By proposition 4.5 it follows that the next block of length \(2n-2\) is nearly determined:
\[
d_{m+1} \cdot \cdots \cdot d_{m+2n-3} \in \{u_{m-1}^B, u_{m-1}^B\}.
\]

If \(d_{m+1} \cdot \cdots \cdot d_{m+2n-3} = u_{m-1}^B\), then by lemma 4.4 we have \((u_m)^B = t_m^+\). This implies
\[
d_{m+1}^4 \cdot d_{m+2}^4 \cdot \cdots \cdot (t_m)^\infty = \delta(\beta_n).
\]

Since \(d_{m+1}^4 = 0\), by proposition 2.2 this implies that \((d_i) \notin \tilde{U}_\beta\).

So \(d_{m+1} \cdot \cdots \cdot d_{m+2n-3} = u_{m-1}^B \cdot \Phi_A(u_{m-1}^B)\). Then by proposition 4.5 the next block of length \(2n-2\) is nearly determined, and thus
\[
d_{m+1} \cdot \cdots \cdot d_{m+2n-3} \in \\Phi_A(u_{m-1}^B)u_{m-2}^B, \Phi_A(u_{m-1}^B)u_{m-2}^B\}.
\]

If \(d_{m+1} \cdot \cdots \cdot d_{m+2n-3} = \Phi_A(u_{m-1}^B)u_{m-1}^B\), then by lemma 4.4 we have \((\Phi_A(u_{m-1}^B))^2 = t_m^+\). This implies \((d_i) \notin \tilde{U}_\beta\) by proposition 2.2. Therefore,
\[
d_{m+1} \cdot \cdots \cdot d_{m+2n-3} = u_{m-1}^B \cdot \Phi_A(u_{m-1}^B)u_{m-2}^B.
\]

Repeating the above arguments it follows that \(d_{m+1}d_{m+2} \cdot \cdots \cdot (u_{m-1}^B \Phi_A(u_{m-1}^B))^\infty = (u_m)^\infty\).

Then by lemma 4.4 we have
\[
d_{m}^4 = 0 \quad \text{and} \quad d_{m+1}^4 \cdot d_{m+2}^4 \cdot \cdots \cdot (t_m)^\infty = \delta(\beta_n),
\]
which again gives \((d_i) \notin \tilde{U}_\beta\) by proposition 2.2. This completes the proof. □

The next result is a generalization of proposition 3.2. Since \(\beta_n \not> \beta_c\) as \(n \to \infty\), as a consequence of the following proposition we establish theorem 2(a).

**Proposition 5.5.** Let \(n \geq 0\). Then for any \(\beta \in (\beta_n, \beta_{n+1}]\) we have \(\tilde{U}_\beta = \tilde{U}_{\beta_{n+1}}\). Furthermore, any sequence in \(\tilde{U}_{\beta_{n+1}}\) must end in
\[
\bigcup_{k=0}^{n} \{(u_k)^\infty, (v_k)^\infty, (w_k)^\infty\}.
\]
Proof. We will prove the proposition by induction on \( n \). By theorem 1.1 and proposition 3.2 it follows that the theorem holds for \( n = 0 \) and \( n = 1 \). Now take \( n \geq 2 \) and \( \beta \in (\beta_n, \beta_{n+1}] \). By lemma 5.3 it follows that the periodic sequences \((u_k)^\infty, (v_k)^\infty \) and \((w_k)^\infty \) with \( 0 \leq k \leq n \) all belong to \( \bar{U}_{\beta_{n+1}} \).

Let \((d_t) \in \bar{U}_\beta \backslash \bar{U}_{\beta_2} \). Observe by lemma 3.3 that any block of the form ‘\( cdd \)’ is forbidden in sequences of \((d_t)\). By theorem 1.1 and proposition 3.2 it follows that \((d_t)\) must contain one of the following blocks:

\[
\begin{align*}
  u_1^b &= BAB, \quad \Phi_d(u_1^b) = CAC; \\
  v_1^c &= CBC, \quad \Phi_d(v_1^c) = ABA; \\
  w_1^a &= ACA, \quad \Phi_d(w_1^a) = BCB.
\end{align*}
\]

Since the proofs for different cases are similar, we may assume \( d_{m+1} \cdots d_{m+3} = u_1^b \) and \( d_m \neq d_{m+1} \) for some smallest integer \( m \geq 1 \). By proposition 4.5 it follows that the next block of length 3 is nearly determined, i.e., \( d_{m+4}d_{m+5}d_{m+6} \in \{ \Phi_d(u_1^b), \Phi_d(u_1^1) \} \). Then

\[
d_{m+1} \cdots d_{m+6} = u_1^b \Phi_d(u_1^b) \quad \text{or} \quad d_{m+1} \cdots d_{m+6} = u_1^b \Phi_d(u_1^1) = u_2^b.
\]

Again, by proposition 4.5 we obtain the following recursive relation: for any \( \ell \geq 1 \)

- if \( d_{s+1} \cdots d_{s+3} = u_1^b \) and \( d_s \neq d_{s+1} \), then either \( d_{s+3} = u_1^b \) or \( d_{s+1} \cdots d_{s+3} = \Phi_d(u_1^b) \).
- if \( d_{s+1} \cdots d_{s+3} = u_1^b \) and \( d_s \neq d_{s+1} \), then either \( d_{s+3} = u_1^b \) or \( d_{s+1} \cdots d_{s+3} = u_1^b \).

Applying this recursive relation and using lemma 5.4 it follows that the sequence \( d_{m+1}d_{m+2} \cdots \) is nearly determined, and it eventually ends with \((u_{k-1}^b \Phi_d(u_{k-1}^b))^\infty = (u_k)^\infty \) for some \( 2 \leq k \leq n \).

Observe that our proof does not depend on the choice of \( \beta \in (\beta_n, \beta_{n+1}] \). So the set-valued map \( \beta \mapsto \bar{U}_\beta \) is constant in \((\beta_n, \beta_{n+1}], \) i.e., \( \bar{U}_\beta = \bar{U}_{\beta_{n+1}} \) for any \( \beta \in (\beta_n, \beta_{n+1}] \). This completes the proof. \( \square \)

Motivated by the recursive relation in the proof of proposition 5.5 and the analogues phenomenon occurs in one dimensional \( \beta \)-expansions (cf [10, 20]) we prove theorem 2(b).

**Proposition 5.6.** \( \bar{U}_\beta \) is uncountable and \( \dim_H \bar{U}_\beta = 0 \).

**Proof.** Note that \( \beta_n \nearrow \beta_c \) as \( n \to \infty \). By lemma 5.3 and the proof of proposition 5.5 it follows that \( U_\beta \) contains all of the following sequences

\[
\begin{align*}
  u_1^b u_2^b \cdots u_k^b \cdots,
\end{align*}
\]

where \( j_k \in \mathbb{N} \). Observe that for each \( k \geq 1 \) the block \( u_k \) ends with digit \( C \) and \( u_{k+1} = u_1^b \Phi_d(u_1^b) \). Then \( u_k \) cannot be written as concatenation of two or more blocks of the form \( u_{\ell} \) with \( \ell \leq k \).

This implies that \( \bar{U}_\beta \) is uncountable.

Furthermore, by the recursive relation described in the proof of proposition 5.5 and using propositions 4.5 and 4.11 it follows that any sequence in \( U_\beta \setminus \bar{U}_\beta \) is of the form
\[ a \left( u^R \phi_k(u^R) \right)^{f_i} \left( u^R \phi_k(u^R) \right)^{f_i} \left( u^R \phi_k(u^R) \right)^{f_i} \cdots; \]
\[ b \left( v^C \phi_B(v^C) \right)^{g_i} \left( v^C \phi_B(v^C) \right)^{g_i} \left( v^C \phi_B(v^C) \right)^{g_i} \cdots; \]
\[ c \left( w^A \phi_C(w^A) \right)^{h_i} \left( w^A \phi_C(w^A) \right)^{h_i} \left( w^A \phi_C(w^A) \right)^{h_i} \cdots, \]

where
\[ a, b, c \in \bigcup_{n=0}^{\infty} \mathbb{A}^n; \quad f_i, g_i \in \mathbb{N} \cup \{0, \infty\}, \quad h_i \in \{0, 1\} \]

and
\[ 1 \leq f_i < g_i \leq g_i < \cdots < g_i \leq f_i < f_i \leq \cdots. \]

Observe that the length of \( u_k, v_k \) and \( w_k \) is \( 2^{k-1}3 \) which grows exponentially fast. This implies that \( \dim_H \tilde{U}_{\beta_k} = 0 \).

We remark that for a detailed proof of \( \dim_H \tilde{U}_{\beta_k} = 0 \) we refer to ([21], theorem 2.9) by an easy adaption.

**Proof of theorem 2.** By propositions 5.2, 5.5 and 5.6 it suffices to prove that for any \( \beta > \beta_c \) we have \( \dim_H \tilde{U}_{\beta} > 0 \). We do this now by first constructing a sequence of bases \( (\beta_n) \) such that \( \beta_n \) strictly decreases to \( \beta_c \) as \( n \to \infty \), and then showing that for any \( \beta > \beta_n \) the set \( \tilde{U}_{\beta} \) contains a subshift of finite type with positive topological entropy which implies \( \dim_H \tilde{U}_{\beta} > 0 \).

For \( n \geq 1 \) let \( \beta_n \in (1, 2) \) such that \( \delta(\beta_n) = t^n_1(\Theta(t)_n)^\infty \). By lemmas 2.1 and 4.1 one can check that \( \beta_n \) is well-defined. Observe that \( \delta(\beta_n) \) begins with \( t^n_1 \Theta(t)_n \Theta(t)^n_\infty \) for any \( n \geq 1 \). Then
\[ \delta(\beta_1) > \delta(\beta_2) \cdots \quad \text{and} \quad \delta(\beta_n) \searrow (\lambda_n) = \delta(\beta_c) \quad \text{as} \quad n \to \infty. \]

So by lemma 2.1 we have
\[ \beta_1 > \beta_2 \cdots \quad \text{and} \quad \beta_n \searrow \beta_c \quad \text{as} \quad n \to \infty. \quad (5.2) \]

Now take \( \beta \in (\beta_n, \beta_{n-1}] \) with \( n \geq 2 \). Let \( (X_n, \sigma) \) be the subshift of finite type represented by the labeled graph \( G \) as in figure 10. Similar to the proof of lemma 5.3 one can prove by using lemmas 4.1, 4.4, 4.9 and corollary 4.10 that any sequence in the subshift of finite type satisfies the conditions in proposition 2.2. In other words, \( X_n \subseteq \tilde{U}_\beta \). Note that the blocks \( u^R_n, v^C_n, w^A_n \) and \( u_n \) have the same length \( 3 \cdot 2^{n-1} \). Then the topological entropy of \( X_n \) is given by (cf [22])
\[ h(X_n) = \frac{\log 2}{\log \left(2^{n-1}3\right)} \geq \frac{1}{n + 1}. \quad (5.3) \]

One can verify that the projection \( \{(d_i)_\beta : (d_i) \in X_n\} \) is a graph-directed set satisfying the open set condition (cf [23]). Then by (5.3) it follows that
\[ \dim_H \tilde{U}_\beta = \frac{h(X_n)}{\log \beta} \geq \frac{1}{(n + 1) \log \beta}. \]
This, together with (5.2), implies that \( \dim H_U > 0 \) for any \( \beta > \beta_c \). \( \Box \)

6. Open questions

Note that \( \tilde{U}_\beta \subseteq U_\beta \) for any \( \beta \in (1, 2) \). Furthermore, by proposition 2.4 the two sets \( \tilde{U}_\beta \) and \( U_\beta \) coincide if \( \beta \in (1, 3/2) \) or \( \beta \) is a multinacci number. Then it is natural to investigate the difference between \( U_\beta \) and \( \tilde{U}_\beta \) for other \( \beta \)s.

**Question 6.1.** Can we describe the set of \( \beta \in (3/2, 2) \) for which \( U_\beta = \tilde{U}_\beta \)? Is it true that \( \dim U_\beta = \dim \tilde{U}_\beta \) for all \( \beta \in (1, 2) \)?

Observe that the set-valued map \( \beta \mapsto \tilde{U}_\beta \) is constant on each interval \( (\beta_n, \beta_{n+1}] \subset (1, \beta_c) \).

Motivated by the work of de Vries and Komornik [16] we ask the following question.

**Question 6.2.** Is it true that the set-valued map \( \beta \mapsto \tilde{U}_\beta \) is locally constant for Lebesgue almost every \( \beta \in (1, 2) \)? If so, can we describe the bifurcation set

\[
V = \left\{ \beta \in (1, 2) : \tilde{U}_{\beta'} \neq \tilde{U}_\beta \text{ for any } \beta' > \beta \right\}.
\]

By theorem 2 we know that \( \dim \tilde{U}_\beta = 0 \) for all \( \beta \leq \beta_c \). For \( \beta > \beta_c \) the authors in [3] calculated the dimension of \( U_\beta \) only for \( \beta \) being a multinacci number. In view of the work by Komornik et al [24] and Alcaraz Barrera et al [25] we ask the following analogous question.

**Question 6.3.** Can we give a uniform formula for the Hausdorff dimension of \( \tilde{U}_\beta \) for \( \beta > \beta_c \)? Is it true that the entropy function \( \beta \mapsto h(\tilde{U}_\beta) \) is a Devil’s staircase, where \( h(\tilde{U}_\beta) \) denotes the topological entropy of \( \tilde{U}_\beta \)? If so, can we describe the bifurcation set

\[
B = \left\{ \beta \in (1, 2) : h(\tilde{U}_{\beta'}) \neq h(\tilde{U}_\beta) \text{ for any } \beta' > \beta \right\}.
\]

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