Dependent coordinates in the Lagrange-Poincaré equations for mechanical systems with symmetry

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Abstract

The Lagrange–Poincaré equations for the mechanical system describing the motion of a scalar particle on a Riemannian manifold with a given free and isometric action of a compact Lie group is obtained. In an arising principle fibre bundle, the total space of which serves as a configuration space of the considered mechanical system, the local description of the reduced motion is done in terms of dependent coordinates. In obtaining of the equations we use the variational principle developed by Poincaré for the mechanical systems with a symmetry.

1 Introduction

The main methods used at present for the study of mechanical systems with symmetries are based on the reduction theory [1, 2]. The theory gives us a necessary instrument for revealing an existing internal motion in the original system. This is achieved by “removing” symmetry out of the system which leads to a new mechanical system defined on the reduced space.

In the reduction theory, the dynamical behaviour of the system is described with the help of the Lagrange–Poincaré equations (the reduced Euler–Lagrange equations) [3]. In this system of equation consisting of two equations, the first equation, known as the “horizontal equation”, represents the local evolution given on the reduced space. The second equation is related to

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the local evolution of the group variable (or more precisely, to the evolution of the variable in the Lie algebra of the symmetry group.)

The study of the reduction in mechanical systems, in addition to their own interests, are motivated by the possibility to use the obtained methods in other dynamic systems, for example, such as those that are associated with the mechanics of fluids, with the various field-theoretical models and etc. Of course, it should be done with some care since we are trying to apply the methods obtained for the finite-dimensional systems to the systems with infinite-dimensional degrees of freedom.

As for the field theory, we are very interested in methods borrowed from the finite-dimensional dynamical systems which can be used to study the reduction in the various models of gauge fields.

It is known that in mechanics there is a simple finite-dimensional dynamical system which can serve for this purpose. This system represents a classical motion of the scalar particle on a smooth (compact) Riemannian manifold on which a free proper the isometric smooth action of the (compact) Lie group is given. Due to the symmetry, the configuration space of this mechanical system, an original Riemannian manifold, can be viewed as the total space of the principal fibre bundle. This principal bundle carries the natural connection known by the name of the “mechanical” connection.

Although the Lagrange–Poincaré equations for this system have been obtained earlier in [4, 5], but one of the main questions, which is important for the gauge field theories, was not considered in this paper. The question is related to the description of the evolution of the system in the reduced space. In gauge theory, the evolution of the gauge fields on the orbit space (on the base of the principal bundle) can not be represented explicitly. This evolution is described with the help of the dependent coordinates given on appropriate gauge surface. (This surface is determined by a chosen gauge.)

However, in the cited papers the possibility of using dependent coordinates in the Lagrange–Poincaré equations was not considered. Note that in the equations of Lagrange-Poincaré obtained for field theories in [6], this aspect of the reduced evolution also was not investigated.

In this paper, our aim is to clarify the issues arising in a local description of the reduced motion in terms of the dependent coordinates in a mechanical systems with a symmetry and to get the Lagrange–Poincaré equations for the mechanical system which is mentioned above. In obtaining of this equations we are based on the variational methods developed by Poincaré for the mechanical systems with a symmetry.

The plan of the paper is as follows. In Section 2 we give a short introduction into the geometry of our problem. Section 3 is devoted to the derivation of the Lagrange–Poincaré equations. In the last Section we discuss the ob-
We consider a motion of scalar particle on a smooth (compact) finite-dimensional Riemannian manifold \( \mathcal{P} \). It is assumed that there is a free, proper, isometric and smooth action of a compact Lie group \( \mathcal{G} \) on this manifold. This right action can be written in coordinates as \( \tilde{Q}^A = F^A(Q^B, a^\alpha) \), where \( Q^A, A = 1, \ldots, N_P \), are the coordinates given on \( \mathcal{P} \) and \( a^\alpha, \alpha = 1 \ldots N_G \), are the coordinates of a group element.

From the general theory it follows that the original manifold \( \mathcal{P} \) has structure of the total space of a principal bundle \( \pi : \mathcal{P} \rightarrow \mathcal{P}/\mathcal{G} = \mathcal{M} \). This means that one can introduce the local coordinates connected with the coordinates of the fibre bundle on the original manifold \( \mathcal{P} \). To perform this we will follow to the methods given in our papers [7–10]. These methods are generalization of those that have been proposed for finite-dimensional systems in [11] and gauge field theories in [12].

In accordance with these methods the coordinates can be introduced as follows. The bundle coordinates are given by the use of the local sections, which in our case are determined by a local “gauge” surface \( \Sigma \), a local submanifold of \( \mathcal{P} \). The submanifold \( \Sigma \) is given by a set of equations \( \chi^\alpha(Q) = 0, \alpha = 1, \ldots, N_G \). It is required that this local submanifold has a transversal intersection with the orbits of the group \( \mathcal{G} \) in \( \mathcal{P} \). The set of the coordinates \( Q^A, A = 1, \ldots, N_P \), satisfying the equations \( \chi^\alpha(Q) = 0 \) are called the dependent coordinates. They are denoted by \( Q^*^A \) (that is, we have \( \chi^\alpha(Q^*) = 0 \)).

The dependent coordinates \( Q^*^A \) together with the group coordinates \( a^\alpha \) are used to set the coordinates of an arbitrary point \( p \) given on the principal bundle. If a point \( p \) has a coordinates \( Q^A \), then the coordinates \( a^\alpha(Q) \) are determined from the equation

\[
\chi^\alpha(F^A(Q, a^{-1}(Q))) = 0.
\]

After that, the coordinates \( Q^*^A(Q) \) can be found by moving the point \( p \) to the submanifold \( \Sigma : Q^*^A = F^A(Q, a^{-1}(Q)) \).

Note that since the trivial principal bundle \( \Sigma \times \mathcal{G} \rightarrow \Sigma \) is locally isomorphic to the principal fibre bundle \( \mathcal{P}(\mathcal{M}, \mathcal{G}) \), we can also use the dependent coordinates \( Q^*^A \) for description of the motion given on the orbit space \( \mathcal{M} \).
The interconnection between the coordinates $Q^A$ and $(Q^*^A, a^a)$ for the same point gives us a rule by which we can perform the replacement of the coordinates on the original manifold $\mathcal{P}$. As a consequence of this replacement we have the following transformation of the coordinate vector fields:

\[
\frac{\partial}{\partial Q^B} = F_B^C(F(Q^*, a), a^{-1})N_C^A(Q^*) \frac{\partial}{\partial Q^A} + F_B^E(F(Q^*, a), a^{-1})\chi_E^\mu(Q^*)(\Phi^{-1})_\mu^\beta(Q^*)v_\beta^\alpha(a) \frac{\partial}{\partial a^\alpha}.
\]

Here $F_B^C(Q, a) \equiv \frac{\partial F^C}{\partial Q^B}(Q, a)$, $\chi_E^\mu(Q) = (\Phi^{-1})_\mu^\beta(Q)$ – the matrix which is inverse to the Faddeev – Popov matrix:

\[
(\Phi)_\mu^\beta(Q) = K_\mu^\alpha(Q) \frac{\partial \chi_\alpha(Q)}{\partial Q^A}.
\]

$(K_\mu$ are the Killing vector fields for the Riemannian metric $G_{AB}(Q)$, the matrix $v_\beta^\alpha(a)$ is the inverse of the matrix $u_\beta^\alpha(a)$. $(u_\beta^\alpha(a) and v_\beta^\alpha(a)$ are the auxiliary functions for the group $\mathcal{G}$.)

$N_C^A(Q)$ is the projection operator $(N_B^A N_C^B = N_C^A)$ onto the subspace which is orthogonal to the Killing vector field subspace:

\[
N_C^A(Q) = \delta_C^A - K_\alpha^A(Q)\Phi_\mu^\alpha(Q)\chi_\mu^\beta(Q).
\]

Being restricted to the submanifold $\Sigma$, it is equal to $N_C^A(Q^*)$.

After performing the replacement of the coordinates to a new coordinate basis $(\frac{\partial}{\partial Q^A}, \frac{\partial}{\partial a^a})$, we come, as in [11], to the following representation for the original metric $G_{AB}(Q)$ of the manifold $\mathcal{P}$:

\[
\hat{G}_{AB}(Q^*, a) = \begin{pmatrix}
G_{CD}(Q^*)(P_\perp)_A^C(P_\perp)_B^D & G_{CD}(Q^*)(P_\perp)_A^D K^C_\mu \bar{u}_\mu^\alpha(a) \\
G_{CD}(Q^*)(P_\perp)_A^C K^D_\nu \bar{u}_\nu^\beta(a) & \gamma_{\mu\nu}(Q^*) \bar{u}_\mu^\alpha(a) \bar{u}_\nu^\beta(a)
\end{pmatrix},
\]

where $G_{CD}(Q^*) \equiv G_{CD}(F(Q^*, e))$, $(e$ is an identity element of the group $\mathcal{G}$), $\gamma_{\mu\nu}$ is the metric given on the orbit of the group action. It is defined by the following relation $\gamma_{\mu\nu} = K_\mu^\beta G_{AB} K^B_\nu \cdot P_\perp$ is a projection operator on the tangent plane to the submanifold $\Sigma$ given by the gauges $\chi$:

\[
(P_\perp)_A^B = \delta_B^A - \chi_\alpha^A(\chi^\top)^{-1}_\alpha^B(\chi^\top)_B^\alpha.
\]

Here $(\chi^\top)_B^\alpha$ is a transposed matrix to the matrix $\chi_B^\alpha$:

\[
(\chi^\top)^A_\mu = G^{AB}_\gamma \gamma_\mu^\nu \gamma_\nu^\nu = K_\mu^A G_{AB} K^B_\nu.
\]
The above projection operators have the following properties:

\[(P_\perp)_B^C A_N^C = (P_\perp)_B^C, \quad N_B^A (P_\perp)_B^C = N_B^C.\]

Note that in the formula (1), \(K^A_\mu\) and \((P_\perp)_B^A\) are given on \(\Sigma\).

The pseudoinverse matrix \(\tilde{G}^{AB}(Q^*, a)\) to matrix (1) is as follows:

\[
\begin{pmatrix}
  G^{EF} N_E^C N_F^B \\
  G^{CB} \chi_C^\gamma (\Phi^{-1})_{\gamma}^\beta N_B^\alpha \tilde{v}_\beta^\alpha \\
  G^{SB} N_S^C \chi_D^\mu (\Phi^{-1})_{\mu}^\nu \tilde{v}_\nu^\mu
\end{pmatrix},
\]

where \(\tilde{v}_\nu^\sigma \equiv \tilde{v}_\nu^\sigma (a)\) and other components depend on \(Q^*\).

The pseudoinversion of \(\tilde{G}_{BC}\) means that

\[\tilde{G}^{AB} \tilde{G}_{BC} = \begin{pmatrix}
  (P_\perp)_B^C & 0 \\
  0 & \delta_\beta^\alpha
\end{pmatrix}.
\]

3 The Lagrangian in the horizontal lift basis

We assume that the considered mechanical system has an invariant Lagrangian which in local coordinates \(Q^A\) can be written as follows:

\[\mathcal{L} = \frac{1}{2} G_{AB}(Q) \dot{Q}^A \dot{Q}^B - V(Q), \quad (3)\]

where \(G_{AB}(Q)\) is an invariant metric (under the action of the group \(G\)) and \(V\) is an invariant potential: \(V(F(Q, a)) = V(Q)\).

The replacement of the coordinates \(Q^A\) for \((Q^{*B}, a^\alpha)\), with \(Q^A = F^A(Q^{*B}, a^\alpha)\), leads to the transformation of the velocities \(\dot{Q}^A(t)\):

\[\dot{Q}^A(t) \equiv \frac{dQ^A}{dt} = F^C_\alpha (P_\perp)_D^C \frac{dQ^{*D}}{dt} + F^A_\alpha \frac{da^\alpha}{dt}.\]

We note that as an operator, the vector field \(\frac{\partial}{\partial Q^*^{\alpha}}\) is defined in its action on functions by the rule

\[\frac{\partial}{\partial Q^{*\alpha}} \varphi(Q^*) = (P_\perp)_A^{B}(Q^*) \frac{\partial \varphi(Q)}{\partial Q^B} \bigg|_{Q^* = Q^{*}}.\]

Since \(F^A_\alpha = F^A_C K^C_\beta \tilde{u}_\alpha^\beta\), we can rewrite the right-hand side of the expression for \(\dot{Q}^A(t)\) to get

\[\dot{Q}^A(t) = F^A_C \left( (P_\perp)_D^C \frac{dQ^{*D}}{dt} + K^C_\beta (Q^*) \tilde{u}_\alpha^\beta (a) \frac{da^\alpha}{dt} \right)\]
\[ F^A_C \left( \frac{dQ^*}{dt} + K^C_Q(Q^*) \bar{u}_\alpha(a) \frac{da^\alpha}{dt} \right). \]

The last transition has been made by means of the identity \((P_\perp)^C_D \frac{dQ^*}{dt} = \frac{dQ^*}{dt}\). The identity is due to the fact that the velocity vector belongs to the tangent plane to the surface \(\Sigma\).

As a result of the replacement of the coordinates we come to the following representation for the Lagrangian:

\[
\mathcal{L} = \frac{1}{2} G_{CD}(Q^*) \left( \frac{dQ^*}{dt} + K^C_a \bar{u}_\alpha(a) \frac{da^\alpha}{dt} \right) \left( \frac{dQ^*}{dt} + K^D_a \bar{u}_\beta(a) \frac{da^\beta}{dt} \right) - V(Q^*). \tag{4}
\]

Before proceeding to the derivation of the Lagrange–Poincaré equations, we have to make another replacement of the coordinate vector fields on the manifold \(\mathcal{P}\). Namely, we change the basis vector fields \(\frac{\partial}{\partial Q^*}, \frac{\partial}{\partial a^\mu}\) for the horizontal lift basis \((H_A, L_\alpha)\) introduced in [13] as a generalisation of the diagonal lift basis used in [14].

In a new basis, \(L_\alpha = v^\mu_\alpha(a) \frac{\partial}{\partial a^\mu}\) are the left-invariant vector fields with the commutation relations

\[
[L_\alpha, L_\beta] = c^\gamma_{\alpha\beta} L_\gamma,
\]

where the \(c^\gamma_{\alpha\beta}\) are the structure constants of the group \(G\).

The horizontal vector fields \(H_A\) are determined as follows

\[
H_A = N^E_A(Q^*) \left( \frac{\partial}{\partial Q^*} - \omega^\alpha_E L_\alpha \right),
\]

where \(\omega^\alpha_E(Q^*, a) = \bar{\rho}^\alpha_\mu(a) \omega^\mu_E(Q^*)\). The matrix \(\bar{\rho}^\alpha_\mu\) is inverse to the matrix \(\rho^\mu_\alpha\) of the adjoint representation of the group \(G\), and \(\omega^\mu_E = \gamma^\mu\nu K^R_\mu G_{RP}\) is the mechanical connection defined in our principal fiber bundle \(P(\mathcal{M}, \mathcal{G})\).

The commutation relation of the horizontal vector fields

\[
[H_C, H_D] = (\Lambda^\gamma_D N^P_D - \Lambda^\gamma_D N^P_C) K^{\gamma P}_\gamma H_S - N^E_C N^P_D \tilde{F}_EP L_\alpha,
\]

with \(\Lambda^\gamma_D = (\Phi^{-1})^\gamma_D \lambda_D\), the curvature \(\tilde{F}_EP\) of the connection \(\omega\), which is given by

\[
\tilde{F}_EP = \frac{\partial}{\partial Q^*} \omega^\alpha_P - \frac{\partial}{\partial Q^*} \omega^\alpha_E + c^\alpha_\nu \omega^\nu_E \omega^\sigma_P,
\]

\((\tilde{F}_EP(Q^*, a) = \bar{\rho}^\alpha_\nu(a) \tilde{F}_{E P}(Q^*))\), can be rewritten as the commutation relations of the nonholonomic basis:

\[
[H_C, H_D] = \mathcal{C}^A_C H_A + \mathcal{C}^\alpha_{CD} L_\alpha \tag{5}
\]
with the structure constants

\[ \mathcal{C}_{CD}^A = (\Lambda^\gamma_C K^A_{\gamma D} - \Lambda^\gamma_D K^A_{\gamma C}) \]  

(6)

and

\[ \mathcal{C}_{CD}^\alpha = -N^S_C N^P_D \tilde{F}^\alpha_{SP}. \]  

(7)

In our basis \((H_A, L_\alpha)\), \(L_\alpha\) commutes with \(H_A\):

\[ [H_A, L_\alpha] = 0. \]

And the metric \((\ref{metric})\) has the following diagonal representation:

\[ \tilde{G}_{AB} = \left( \begin{array}{cc} G^H_{AB} & 0 \\ 0 & \tilde{\gamma}_{\alpha\beta} \end{array} \right), \]  

(8)

where the “horizontal metric” \(G^H\) is defined by the projection operator \(\Pi^A_B = \delta^A_B - K^A_\mu K_\nu^D G_{DB}\) as follows: \(G^H_{DC} = \Pi^B_D \Pi^C_B G_{DC}\).

Note that the projection operator \(\Pi^A_B\) satisfies the properties: \(\Pi^A_C L^B_N = \Pi^A_C\) and \(\Pi^L_B N^A_C = N^A_B\).

The pseudoinverse matrix \(\tilde{G}^{AB}\) to the matrix \(\tilde{G}_{BC}\) is defined by the following orthogonality condition:

\[ \tilde{G}^{AB} \tilde{G}_{BC} = \left( \begin{array}{cc} N^A_C & 0 \\ 0 & \delta^\alpha_\beta \end{array} \right), \]

and can be written as

\[ \tilde{G}^{AB} = \left( \begin{array}{cc} G^{EF} N^A_E N^B_F & 0 \\ 0 & \tilde{\gamma}^{\alpha\beta} \end{array} \right). \]

It can be shown that in the horizontal lift basis \((H_A, L_\alpha)\), the Lagrangian \((\ref{lagrangian})\) becomes

\[ \hat{\mathcal{L}} = \frac{1}{2} G^H_{CD} \omega^C \omega^D + \frac{1}{2} \tilde{\gamma}_{\mu\nu} \omega^\mu \omega^\nu - V, \]

(9)

where we have introduced the following variables connected with the velocities:

\[ \omega^E = (P_\perp)_{\overline{B}}^E \frac{dQ^B}{dt} = \frac{dQ^E}{dt} \]  

(10)

and

\[ \omega^\alpha = v^\alpha_\sigma \frac{da^\sigma}{dt} + \omega^D \tilde{\omega}^\alpha_D. \]  

(11)

Note also that

\[ \frac{da^\beta}{dt} = v^\beta_\alpha \omega^\alpha - \omega^D v^\beta_\alpha \tilde{\omega}_D^\alpha. \]
4 The relationship between partial derivatives of velocities and deformations in the Poincaré variational principle

The variational principle proposed Poincaré for mechanical systems is to use the variations of paths that are associated with independent vector fields provided they exist on the configuration space. It is important that vector fields may also be nonholonomic vector fields.

For example, suppose we have the vector fields \( v_1, \ldots, v_n \) on a some smooth manifold. And these (nonholonomic) vector fields form a basis. Then, the commutator of the vector fields is expanded over this basis: \( [v_i, v_j] = c_{ij}^k(q) v_k \). If we have a some smooth path \( q(t) \) on the considered manifold, then the time derivative of a smooth function \( f \), given on this path \( q(t) \), can be presented as

\[
\frac{df(q(t))}{dt} = \frac{\partial f}{\partial q^i} \frac{dq^i}{dt} = \sum_i v_i(f) \omega^i,
\]

where \( v_i(f) \) is the directional derivative of \( f \) along the vector field \( v_i \). The variables \( \omega^i \) are called the quasi-velocities, as they are linear functions of the velocities \( v_i \).

Then, as is done in the usual calculus of variations, it is necessary to introduce the deformation \( q(u, t) \) of the path \( q(t) \). (These deformations have the standard properties. For the variations with the fixed ends, they are given, for example, in [17].) But now the derivative of the function given on the deformation is calculated in accordance with the following formula

\[
\frac{\partial f(q(u, t))}{\partial u} = \sum_i v_i(f) w^i(u, t),
\]

where the introduced variations \( w^i(u, t) \) are independent within the time interval \([t_1, t_2]\) which is used for consideration of the variational problem. And at the ends of the time interval, they satisfy \( w^k(u, t_1) = 0 \) and \( w^k(u, t_2) = 0 \).

The variation of the functional \( F(q(t)) \) in this variational calculus is defined as usual, i.e., as

\[
\delta F = \left. \frac{dF(q(u, t))}{du} \right|_{u=0}.
\]

In our case, we are given two sets of the basis vector fields, \( \{H_A\} \) and \( \{L_\alpha\} \), and we know that vector fields of these sets are independent between themselves: \( [H_A, L_\alpha] = 0 \). So, we can apply the Poincaré variational principle to the action functional

\[
S = \int_{t_1}^{t_2} \hat{L} \, dt,
\]
where the Lagrangian $\hat{L}$ is given by (9).

Before applying this variational principle to the functional (12), it is necessary to find a relationship between the derivations of the velocities $\omega^A$ and $\omega^\alpha$ that are in the Lagrangian (9) and the variations $w^A$ and $w^\alpha$.

They are followed from the expansion of the time-derivative of the function $f(Q^*, a)$ in the horizontal lift basis:

$$
\frac{df(Q^*, a)}{dt} = (P_\perp)_B^E \frac{dQ^B}{dt} H_E(f) + (P_\perp)_B^\alpha \omega^{\alpha}_B L_\alpha(f) \frac{dQ^B}{dt} + \frac{\partial f}{\partial a^\alpha} \frac{da^\alpha}{dt} = \omega^E H_E(f) + \omega^\alpha L_\alpha(f),
$$

(13)

where $\omega^E$ and $\omega^\alpha$ are as in (10) and (11), correspondingly, and by $H_E(f)$ we denote the action of the vector field $H_E$ on the function $f$. A similar notation is used for $L_\alpha(f)$.

First we consider the relation between the derivatives of the functions $\omega^A$ and the variations $w^A$. Taking $f = Q^*A$ in (13), we get

$$
\frac{dQ^*A(t)}{dt} = \omega^E H^A_E(Q^*(t)),
$$

where

$$
H^A_E(Q^*) \equiv H_E(Q^{*A}) = N^A_E(Q^*).
$$

The previous equality for the time derivative of $Q^*$ can be generalized to a similar equality for the deformation $Q^{*A}(u, t)$ of the path $Q^{*A}(t)$:

$$
\frac{dQ^{*A}(u, t)}{dt} = H_E(Q^{*A}(u, t)) \omega^E(Q^*(u, t)).
$$

(14)

On the other hand, for the partial derivative of $Q^{*A}(u, t)$ with respect to $u$, we suppose the following equation:

$$
\frac{\partial Q^{*A}(u, t)}{\partial u} = H_E(Q^{*A}(u, t)) w^E(Q^*(u, t)),
$$

(15)

where we have introduced the variation $w^E(Q^*(u, t))$.

Now taking the partial derivative of (14) with respect to $u$, we obtain

$$
\frac{\partial}{\partial u} \frac{dQ^{*A}(u, t)}{dt} = \frac{\partial H^A_E(Q^*)}{\partial Q^B} \frac{\partial Q^B}{\partial u} \omega^E + H^A_E \frac{\partial w^E(Q^*(u, t))}{\partial u} = \frac{\partial H^A_E}{\partial Q^B} H^B_P w^P \omega^E + H^A_E \frac{\partial w^E(Q^*(u, t))}{\partial u}.
$$

(16)
But if we perform the differentiation of (15) with respect to \( t \), we get
\[
\frac{\partial}{\partial t} \frac{dQ^A(u,t)}{du} = \frac{\partial H_A^E(Q^*)}{\partial Q^B} H_P^B \omega^P w^E + H_E^A \frac{\partial w^E}{\partial t}.
\] (17)

Since (16) and (17) are equal, then from their equality we obtain the following equation:
\[
\left( \frac{\partial H_A^E}{\partial Q^B} H_P^B - \frac{\partial H_A^E}{\partial Q^B} H_E^B \right) \omega^P w^E + H_E^A \frac{\partial w^E}{\partial u} - H_E^A \frac{\partial w^E}{\partial t} = 0.
\] (18)

Taking into account the commutation relation between \( H_A \) and \( H_B \) (5), and the expressions for the structure constants (6) and (7), we come to the following equation:
\[
H_R^A \left( \frac{\partial \omega^R}{\partial u} - \frac{\partial w^R}{\partial t} + C_{PE}^R \right) = 0.
\] (Note that \( H_R^A \) coincides with the projection operator \( N_R^A(Q^*) \).)

The second equation connecting the derivatives of \( \omega^A \) and the variations \( w^A \) can be obtained in the Appendix A. It is the equation (A.5) and looks as follows:
\[
\frac{\partial \omega^\beta}{\partial u} = \frac{\partial w^\beta}{\partial t} + c_{\alpha' \mu}^\beta \omega^{\alpha'} w^\mu + N_E^C N_P^C \tilde{f}_{\beta C}^\gamma \omega^E w^P.
\]

Now we can proceed to derivation of the Lagrange-Poincaré equations.

5 The Lagrange-Poincaré equation

To obtain the equations of motions by means of the variational principle from an action functional it is necessary to replace the paths in the Lagrangian by their deformations and then to calculate the variation of the action functional. At first we must to calculate the derivative of the functional \( S \) with respect to the variable that is connected with the deformation of the paths. For the functional (12) with the Lagrangian (9), this derivative is given as follows:
\[
\frac{dS}{du} = \int_{t_1}^{t_2} \left( \frac{\partial \hat{L}}{\partial \omega^C} \frac{\partial \omega^C}{\partial u} + \frac{\partial \hat{L}}{\partial \omega^\mu} \frac{\partial \omega^\mu}{\partial u} + \frac{\partial \hat{L}}{\partial Q^B} \frac{\partial Q^B}{\partial u} + \frac{\partial \hat{L}}{\partial a^\alpha} \frac{\partial a^\alpha}{\partial u} \right) dt.
\] (19)

In (19) the first term in the integrand can be rewritten as
\[
\frac{\partial \hat{L}}{\partial \omega^C} \frac{\partial \omega^C}{\partial u} = G_{CD}^H \omega^D \frac{\partial \omega^C}{\partial u}.
\]
Our next transformation of this term consists in replacing the partial derivative of \( \omega \) with respect to \( u \) for the partial derivative of the variation \( w \) with respect to \( t \). For this we make use of the previously obtained equation (18)

\[
N^e C^e \frac{\partial \omega}{\partial u} = N^e C^e \frac{\partial w^e}{\partial t} + N^e C^e C^e E P \omega^e w^P,
\]

which we multiply by \( G^H_{C^C D} \). Because of the identity \( G^H_{C^C E N^E K} = G^H_{C^C K} \) we will have

\[
G^H_{C^C D} \frac{\partial \omega}{\partial u} = G^H_{C^C D} \left( \frac{\partial w^e}{\partial t} + C^e E P \omega^e w^P \right).
\]

Therefore, using this expression for the first term in the integral (19) and integrating it by parts, we obtain

\[
\left( G^H_{C^C D} \omega^D w^e \right) \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left( \frac{d}{dt} \left( G^H_{C^C D} \omega^D \right) w^e - G^H_{C^C D} \omega^D C^e E P \omega^e w^P \right) dt.
\]

This can also be rewritten as follows:

\[
\left( G^H_{C^C D} \omega^D w^e \right) \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left( \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \omega^e} \right) \omega^e + \frac{\partial \hat{L}}{\partial \omega^e} C^e E P \omega^e w^P \right) dt.
\]

Similarly, the second term can be transformed by the following way

\[
\frac{\partial \hat{L}}{\partial \omega^e} \omega^e = \gamma^e \omega^e \left( \frac{\partial w^e}{\partial t} + c^e \omega^e \omega^\mu w^\mu + N^e C^e C^e F^e \omega^e w^P \right).
\]

Here we have used the equation (A.5). Substituting such a representation in the integral, and then integrating it by parts, we will have

\[
\left( G^H_{C^C D} \omega^D w^e \right) \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \omega^e} \right) \omega^e + \frac{\partial \hat{L}}{\partial \omega^e} c^e \omega^e \omega^\mu w^\mu + N^e C^e C^e F^e \omega^e w^P \right) dt.
\]

The last terms in the integrand of (19) can be transformed as

\[
\frac{\partial \hat{L}}{\partial Q^B} \frac{\partial Q^B}{\partial u} + \frac{\partial \hat{L}}{\partial a^e} \frac{\partial a^e}{\partial u} = \frac{\partial \hat{L}}{\partial Q^B} N^e B^e w^e + \frac{\partial \hat{L}}{\partial a^e} \left( H_E (a^e) w^e + L_\beta (a^e) \omega^\beta \right).
\]

Since

\[
H_E (a^e) = -N^e C^e A^e e_\beta, \quad L_\beta (a^e) = e^{\alpha}_\beta,
\]

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the last terms can also be written as follows:

$$H_E(\hat{\mathcal{L}})w^E + L_\alpha(\hat{\mathcal{L}})w^\alpha.$$  

Because of the independence of the variations $w^E$ and $w^\alpha$ we come to the system of two equations, the Lagrange-Poincaré equations. The first of the equations, the horizontal equation (after an appropriate changing of the notation of the indices) is given by

$$-\frac{d}{dt}\left(\frac{\partial \hat{\mathcal{L}}}{\partial \omega^E}\right) + \frac{\partial \hat{\mathcal{L}}}{\partial \omega^C}C^P_{CE}w^C + \frac{\partial \hat{\mathcal{L}}}{\partial \omega^\alpha}N^C_E\tilde{F}^\alpha_{CB}\omega^B + H_E(\hat{\mathcal{L}}) = 0. \quad (20)$$

Here we have used the identity $N\tilde{P}\omega^\nu = \omega^\nu$.

The second of the Lagrange-Poincaré equation, the “vertical” equation, is as follows:

$$-\frac{d}{dt}\left(\frac{\partial \hat{\mathcal{L}}}{\partial \omega^\alpha}\right) + \frac{\partial \hat{\mathcal{L}}}{\partial \omega^\nu}\epsilon^\nu_{\gamma\mu\beta}\omega^\gamma\omega^\mu + L_\alpha(\hat{\mathcal{L}}) = 0. \quad (21)$$

These equations can be rewritten in more explicit form:

$$-\frac{d}{dt}\left(G^H_{ED}\omega^D\right) + C^R_{CE}G^H_{RD}\omega^D\omega^C + \tilde{\gamma}_{\alpha\beta}\omega^\beta N^B_E\tilde{F}^\alpha_{BE}\omega^A + H_E(\hat{\mathcal{L}}) = 0 \quad (22)$$

and

$$-\frac{d}{dt}(\tilde{\gamma}_{\alpha\beta}\omega^\beta) + \epsilon^{\nu}_{\alpha\beta\gamma}\omega^\gamma\omega^\nu + L_\alpha(\hat{\mathcal{L}}) = 0. \quad (23)$$

It remains to calculate the $H_E(\hat{\mathcal{L}})$-terms in the obtained equations. We recall that the horizontal vector field $H_E$ is $H_E = N_S^E(Q^*)\left(\frac{\partial}{\partial Q^S} - \mathcal{A}_S^\alpha L_\alpha\right)$. Its action on our Lagrangian $\hat{\mathcal{L}}$ is given by

$$H_E(\hat{\mathcal{L}}) = \frac{1}{2}N^S_E\frac{\partial}{\partial Q^S}(G^H_{AB}\omega^A\omega^B) + \frac{1}{2}N^S_E\frac{\partial}{\partial Q^S}(\tilde{\gamma}_{\mu\nu})\omega^\mu\omega^\nu - \frac{1}{2}N^S_E\tilde{\gamma}_{\mu\nu}L_\alpha(\hat{\mathcal{L}}) - N^S_E\frac{\partial}{\partial Q^S}(V(Q^*)). \quad (24)$$

First note that the second and the third terms of $H_E(\hat{\mathcal{L}})$ can be rewritten as

$$\frac{1}{2}N^S_E\tilde{\gamma}_{\mu\nu}L_\alpha(\hat{\mathcal{L}})\omega^\mu\omega^\nu = -\frac{1}{2}N^S_E(\mathcal{A}_S^\alpha\gamma_{\sigma\rho}^\mu\gamma_{\beta\delta}^\nu\omega^\sigma\omega^\rho)\omega^\nu\omega^\delta.$$

Then we rewrite the equation (22) in the following form:

$$-G^H_{ED}\omega^D - \frac{d}{dt}(G^H_{ED})\omega^D + C^R_{CE}G^H_{RD}\omega^D\omega^C + \tilde{\gamma}_{\alpha\beta}\omega^\beta N^B_E\tilde{F}^\alpha_{BE}\omega^A + H_E(\hat{\mathcal{L}}) = 0. \quad (25)$$
Note that the derivation of $G^H_{ED}$ with respect to time can be performed in accordance with the following rule:

$$\frac{d}{dt}(G^H_{ED}(Q^*)) = (P_\perp)_S^M \left( \frac{\partial G^H_{ED}(Q)}{\partial Q^M} \right)_{Q=Q^*} \omega^S = \frac{\partial G^H_{ED}}{\partial Q^M} \omega^M.$$  

(Also note that here we have used the identity $(P_\perp)_S^M \omega^S = \omega^M$.)

The first term of $H_E(\mathcal{L})$ can be presented as

$$N_E \frac{\partial}{\partial Q^S} (\frac{1}{2} G^H_{AB} \omega^A \omega^B) = \frac{1}{2} N_E (P_\perp)_S^M \frac{\partial G^H_{AB}(Q)}{\partial Q^M} \bigg|_{Q=Q^*} \omega^A \omega^B \quad = \frac{1}{2} N_E^M \frac{\partial G^H_{AB}}{\partial Q^M} \omega^A \omega^B.$$  

We combine it with the term of horizontal equation (25) which contains the derivative with respect to time of $G^H_{ED}$:

$$(-G^H_{ED,M} \omega^M \omega^D + \frac{1}{2} N_E^M G^H_{AB,M} \omega^A \omega^B),$$

or

$$(-G^H_{EA,B} + \frac{1}{2} N_E^M G^H_{AB,M}) \omega^A \omega^B.$$  

But

$$N_E^M G^H_{MA,B} = G^H_{EA,B} - K^M_\alpha \Lambda^\alpha_E G^H_{MA,B}.$$  

So

$$G^H_{EA,B} = N_E^M G^H_{MA,B} + K^M_\alpha \Lambda^\alpha_E G^H_{MA,B}.$$  

Using this representation for $G^H_{EA,B}$, we come to

$$- \left( N_E \left( \frac{1}{2}(G^H_{MA,B} + G^H_{MB,A} - G^H_{AB,M}) + K^M_\alpha \Lambda^\alpha_E G^H_{MA,B} \right) \omega^A \omega^B, $$

or

$$- \left( N_E^M G^H_{MP} \Gamma^P_{AB} + K^M_\alpha \Lambda^\alpha_E G^H_{MA,B} \right) \omega^A \omega^B.$$  

Notice that the second term of this expression is mutually concealed with the “$C^R_{CE}$ - term” of (25). It can be done as follows.

The two terms can be rewritten as

$$(G^H_{RA} \gamma^R_{BE} - K^M_\alpha \Lambda^\alpha_E G^H_{MA,B}) \omega^A \omega^B.$$  

Because of $\Lambda^\gamma_B \omega^B = 0$,

$$C^R_{BE} \omega^A \omega^B = (\Lambda^\gamma_B K^R_{\gamma E} - \Lambda^\gamma_E K^R_{\gamma B}) \omega^A \omega^B = - \Lambda^\gamma_E K^R_{\gamma B} \omega^A \omega^B.$$
And we come to
\[ \Lambda^E \left( -G^H_{RA} K^R_{\alpha B} - K^R_{\gamma B} G^H_{RA,B} \right) \omega^A \omega^B. \]

If one takes the partial derivative of the following equality
\[ K^R_{\gamma C^H_{RA}} = 0, \]

one obtains
\[ \frac{\partial}{\partial Q^*}(K^R_{\gamma C^H_{RA}}) = (P_{\perp})^S_B (K^R_{\gamma S} G^H_{RA} + K^R_{\gamma G^H_{RA,S}}) = 0. \]

Since \( (P_{\perp})^S_B \omega^B = \omega^S \), it follows that
\[ K^R_{\gamma G^H_{RA,S}} \omega^S = -K^R_{\gamma G^H_{RA}} \omega^S, \]

and we have
\[ \Lambda^E \left( -G^H_{RA} K^R_{\alpha B} + K^R_{\gamma B} G^H_{RA} \right) \omega^A \omega^B = 0. \]

As a result of our transformation we get
\[ -\frac{d}{dt} (\tilde{\gamma}^A_\epsilon \omega^\epsilon) + c^\mu_{\mu A} \tilde{\gamma}^\mu_\nu \omega^\nu + L_\alpha \tilde{L} = 0. \]
in which
\[ L_{\alpha}(\mathcal{L}) = \frac{1}{2} (c_{\alpha \nu} \tilde{\gamma}_{\mu} + c_{\alpha \nu} \tilde{z}_{\mu}) \omega^\nu \omega^\xi = c_{\alpha \nu} \tilde{\gamma}_{\mu} \omega^\nu \omega^\xi. \]

It follows that the second term of the equation cancels the third one.

The first term can be rewritten as follows:
\[ \frac{d}{dt}(\rho^\nu_{\alpha} p_{\nu}) = \rho^\nu_{\alpha} \frac{d}{dt} p_{\nu} + \frac{\partial \rho^\nu_{\alpha}}{\partial a^\mu} \left( \frac{da^\mu}{dt} \right) p_{\nu}. \]

But
\[ \frac{da^\mu}{dt} = \nu^\mu_{\alpha} \omega^\sigma - \omega^C \tilde{\gamma}_C^\mu v^\sigma. \]

Substituting this expression for \( \frac{da^\mu}{dt} \), we get
\[ \rho^\nu_{\alpha} \frac{d}{dt} p_{\nu} + \nu^\mu_{\alpha} \frac{\partial \rho^\nu_{\alpha}}{\partial a^\mu} \omega^\sigma p_{\nu} - \nu^\mu_{\alpha} \frac{\partial \rho^\nu_{\alpha}}{\partial a^\mu} \tilde{\gamma}^\sigma_C \omega^C p_{\nu} = 0. \]

In this equation, we have \( L_{\sigma}(\rho^\nu_{\alpha}) = c_{\sigma \alpha} \rho^\nu_{\alpha} \) and \( c_{\sigma \alpha} \rho^\nu_{\alpha} = c_{\sigma \alpha} \rho^\nu_{\alpha} \). So we can multiply the equation by \( \tilde{\rho}^\beta_{\alpha} \) to obtain
\[ \frac{d}{dt} p_{\beta} + c_{\epsilon \beta} \rho^\epsilon_{\sigma} \omega^\sigma p_{\nu} - c_{\epsilon \beta} \rho^\epsilon_{\sigma} \tilde{\gamma}^\sigma_C \omega^C p_{\nu} = 0. \]

The equation may be rewritten in the final following form:
\[ \frac{d}{dt} p_{\beta} + c_{\epsilon \beta} \gamma^\epsilon_{\sigma} p_{\sigma} p_{\nu} - c_{\epsilon \beta} \gamma^\epsilon_{\sigma} \tilde{\gamma}^\sigma_C \omega^C p_{\nu} = 0. \] (28)

Thus, the Lagrange-Poincaré equations for our Lagrangian (9) are given by the horizontal equation (27) and the vertical equation (28).

As an important consequence of the obtained equations are the equations for the relative equilibrium:
\[ G^{DE} N^L_D N^S_E \left( \frac{1}{2}(\mathcal{O}_{\gamma} \mathcal{S}) p_\kappa p_\sigma + \frac{\partial}{\partial Q^S} V(Q^S) \right) = 0, \]
\[ c_{\epsilon \beta} \gamma^\epsilon_{\sigma} p_{\sigma} p_{\nu} = 0. \] (29)

6 Conclusion

We have obtained the local Lagrange-Poincaré equations defined in some chart of the principle fibre bundle. If the principle bundle is a trivial one, these equations may be considered as a global equations. In general, to determine the equations on the whole manifold it is necessary to know how they are changed under the transition from one chart to another.
Earlier, these equations were derived in our paper [18] as the geodesic equations in the horizontal lift basis on a manifold. The horizontal equations of this paper coincide with the one from our previous work, but the vertical equations are slightly differed. This is connected with the different definition of the variable $p$. The previous definition was without the mechanical connection.

Note that the horizontal Lagrange-Poincaré equation is known by the name Wong’s equation [19] in physical literature. But in reduction theory, the mechanical connection in the Lagrange-Poincaré equations is not an arbitrary as in the Wong’s equations, but is determined by the geometry of the reduced problem. Therefore, it would be interested in to study the questions related to the behaviour of the Lagrange-Poincaré equation in field theories.

Note also that generalization of the equation of the relative equilibrium to the field-theoretical equations could be useful for studies of the possible stable configurations that can be used in perturbative calculations.

Appendix A

Relationship between derivatives of velocities $\omega^\alpha$ and variations $w^\alpha$

The velocity $\frac{da^\alpha}{dt}$ can be expressed in terms of the vector fields of the horizontal lift basis:

$$ \frac{da^\alpha}{dt} = \omega^E H_E(a^\alpha) + \omega^\beta L_\beta(a^\alpha), $$

where

$$ H_E(a^\alpha) = - N^C_E \delta^\beta_\gamma \nu^\alpha_{\beta \gamma}, \quad \delta^\beta_\gamma(Q^*, a) = \bar{\rho}_\gamma^\beta(a) \omega^\mu_{(Q^*)} $$

and $L_\mu(a^\beta) = v^\beta_\mu$.

(Note that we have the following action of the vector field $L_{\alpha}$ on the matrix of the adjoint representation of the group Lie: $L_{\mu} \bar{\rho}_{\alpha}^\gamma(a) = - c_{\mu \nu}^\beta \bar{\rho}_{\gamma}^\beta(a)$.)

The velocities of deformations $a^\alpha(u, t)$ of the path $a^\alpha(t)$ have a similar expansion over the basis $(H_A, L_{\alpha})$, which is taken now on deformed paths:

$$ \frac{da^\alpha(u, t)}{dt} = H_E(a^\alpha(u, t)) \omega^E(Q^*(u, t)) + L_{\beta}(a^\alpha(u, t)) \omega^\beta(Q^*(u, t), a(u, t)). $$

(A.1)

Assuming the same structure for the partial derivative of $a^\alpha(u, t)$ with respect to $u$ and introducing the variations, instead of the velocities $\omega^A$ and
\[ \omega^\beta, \text{ we write this partial derivative of } a^\alpha \text{ as} \]
\[ \frac{\partial a^\alpha(u,t)}{\partial u} = H_E(a^\alpha(u,t)) w^E(Q^*(u,t)) + L_\beta^\prime(a^\alpha(u,t)) w^\beta(Q^*(u,t), a(u,t)). \]

(A.2)

Taking the partial derivative of (A.1) with respect of \( u \), we get
\[ \frac{\partial}{\partial u} \frac{da^\alpha}{dt} = \frac{\partial H_P(a^\alpha)}{\partial Q^*} H_P^B(Q^*) w^P + \frac{\partial H_P(a^\alpha)}{\partial a^\beta} \left( H_P(a^\beta) w^P + L_\alpha (a^\beta) \omega^\alpha \right) \omega^E 
\]
\[ + H_E(a^\alpha) \frac{\partial \omega^E}{\partial u} + \frac{\partial L_\mu(a^\alpha)}{\partial a^\beta} \left( H_P(a^\beta) w^P + L_\alpha(a^\beta) \omega^\alpha \right) w^\mu + L_\mu(a^\alpha) \frac{\partial \omega^\mu}{\partial t}. \] 

(A.3)

The partial derivative of (A.2) with respect to \( t \) gives us
\[ \frac{d}{dt} \frac{da^\alpha}{dt} = \frac{\partial H_P(a^\alpha)}{\partial Q^*} H_P^B(Q^*) w^P + \frac{\partial H_P(a^\alpha)}{\partial a^\beta} \left( H_E(a^\beta) \omega^E + L_\mu(a^\beta) \omega^\mu \right) w^P 
\]
\[ + H_E(a^\alpha) \frac{\partial w^E}{\partial t} + \frac{\partial L_\mu(a^\alpha)}{\partial a^\beta} \left( H_P(a^\beta) \omega^P + L_\alpha(a^\beta) \omega^\alpha \right) w^\mu + L_\mu(a^\alpha) \frac{\partial w^\mu}{\partial t}. \] 

(A.4)

The equality of (A.3) and (A.4) leads us to the following equation:
\[ \left( H_P^B(Q^*) \frac{\partial H_E(a^\alpha)}{\partial Q^*} - H_E^B(Q^*) \frac{\partial H_P(a^\alpha)}{\partial Q^*} \right) \omega^E w^P \]
\[ + \left( H_P(a^\beta) \frac{\partial H_E(a^\alpha)}{\partial a^\beta} - H_E(a^\beta) \frac{\partial H_P(a^\alpha)}{\partial a^\beta} \right) \omega^E w^P \]
\[ + \frac{\partial H_E(a^\alpha)}{\partial a^\beta} L_\mu(a^\beta)(w^\mu \omega^E - \omega^\mu w^E) + \left( H_E(a^\alpha) \frac{\partial \omega^E}{\partial u} - H_E(a^\alpha) \frac{\partial w^E}{\partial t} \right) \]
\[ + \left( H_P(a^\beta) \frac{\partial L_\alpha(a^\alpha)}{\partial a^\beta} w^P \omega^\alpha - H_P(a^\beta) \frac{\partial L_\mu(a^\alpha)}{\partial a^\beta} \omega^P w^\mu \right) \]
\[ + \frac{\partial L_\alpha(a^\alpha)}{\partial a^\beta} L_\mu(a^\beta) w^\mu \omega^\alpha - \frac{\partial L_\mu(a^\alpha)}{\partial a^\beta} L_\alpha(a^\beta) \omega^\alpha w^\mu \]
\[ + L_\mu(a^\alpha) \frac{\partial \omega^\mu}{\partial u} - L_\mu(a^\alpha) \frac{\partial w^\mu}{\partial t}. \]

After fulfilling the corresponding changes, these nine terms of the obtained equation can be rewritten to give
\[ 1+2 \]
\[ N_E^C N_P^C \tilde{F}_{CC'}^\beta v_\beta^\alpha \omega^E w^P + \left( N_E^C \frac{\partial}{\partial Q^*} \left( N_P^C \right) \omega^E C' - N_P^C \frac{\partial}{\partial Q^*} \left( N_E^C \right) \omega^E C' \right) \omega^E w^P \]
\[ 3 \]
Note that it can be shown that the sum of the third and the fifth terms is equal to zero.

By making use of the equation (18) of the main text of the paper in the fourth term, one can show that the resulting expression cancels with the second term standing in the sum of the first and second terms, i.e., in (1+2)-term.

Finally, after all transformations one can obtain the following equation which relates the partial derivative of $\omega^\beta$ and the partial derivative of the variation $w^\beta$:

$$\frac{\partial \omega^\beta}{\partial u} = \frac{\partial w^\beta}{\partial t} + c^\beta_{\alpha'\mu} \omega^\alpha w^\mu + N^C_E N^C_P \bar{\mathcal{F}}^\beta_{C'C} \omega^E w^P. \tag{A.5}$$

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