The Betti side of the double shuffle theory. III. Bitorsor structures

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Accepted: 17 November 2022 / Published online: 3 March 2023
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Abstract
In the first two parts of the series, we constructed stabilizer subtorsors of a ‘twisted Magnus’ torsor, studied their relations with the associator and double shuffle torsors, and explained their ‘de Rham’ nature. In this paper, we make the associated bitorsor structures explicit and explain the ‘Betti’ nature of the corresponding right torsors; we thereby complete one aim of the series. We study the discrete and pro-$p$ versions of the ‘Betti’ group of the double shuffle bitorsor.

Contents

Introduction ............................................... 2
Notation ............................................... 3
1 Torsors and bitorsors ......................................... 3
  1.1 Bitorsors ............................................. 3
  1.2 From torsors to bitorsors .................................... 4
  1.3 Some torsors and their interrelations .............................. 7
2 The bitorsor $G^{DR,B}(k)$ and its actions ................................ 7
  2.1 The group $(G^B(k), \otimes)$ . ................................ 7
    2.1.1 The group $(\mathcal{G}(V^B), \cdot)$ .................................. 7
    2.1.2 The automorphisms $\text{aut}_{\mu,g}^{V^B(1),B}$ and $\text{aut}_{\mu,g}^{V^B(10),B}$ ............... 8
    2.1.3 The group $(G^B(k), \otimes)$ .................................. 8
  2.2 Actions of the group $(G^B(k), \otimes)$ ................................ 8
  2.3 A bitorsor structure on $G^{DR,B}(k)$ ................................ 10
  2.4 Actions of the bitorsor $G^{DR,B}(k)$ ................................ 11
3 Subbitorsors of $G^{DR,B}(k)$ ...................................... 12

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3.1 $c_{\text{quad}}^{\text{DR,B}}(k)$ as a subbitorsor of $G_{\text{quad}}^{\text{DR,B}}(k)$ .............................................. 13
3.1.1 The group $(G_{\text{quad}}^{\text{B}}(k), \oplus)$ .............................................. 13
3.1.2 Bitorsor structure on $G_{\text{quad}}^{\text{DR,B}}(k)$ .............................................. 13
3.2 $\text{Stab}(\hat{\Lambda}_V,\text{DR/B})(k)$ as a subbitorsor of $G_{\text{DR,B}}^{\text{DR,B}}(k)$ .............................................. 14
3.3 $\text{Stab}(\hat{\Lambda}_M,\text{DR/B})(k)$ as a subbitorsor of $G_{\text{DR,B}}^{\text{DR,B}}(k)$ .............................................. 15
3.4 $\text{DMR}^{\text{DR,B}}(k)$ as a subbitorsor of $G_{\text{DR,B}}^{\text{DR,B}}(k)$ .............................................. 15
3.5 $\text{DMR}_B(k)$ as a subbitorsor of $G_{\text{DR,B}}^{\text{DR,B}}(k)$ .............................................. 16
3.6 $\text{M}(k)$ as a subbitorsor of $G_{\text{DR,B}}^{\text{DR,B}}(k)$ .............................................. 16
3.7 Groups corresponding to some torsors and their interrelations .............................................. 17
3.8 Scheme-theoretic and Lie algebraic aspects .............................................. 17
3.9 Relation with Hopf algebra and coalgebra isomorphism bitorsors .............................................. 19
4 Equivalent definitions of $\text{DMR}(k)$ and its Lie algebra .............................................. 22
4.1 Equivalent definition of $\text{DMR}^B(k)$ .............................................. 22
4.2 Equivalent definition of $\text{dmr}^B$ .............................................. 24
5 A discrete group $\text{DMR}^B$ .............................................. 25
5.1 Definition of $\text{DMR}^B$ .............................................. 25
5.2 Computation of $\text{DMR}^B$ .............................................. 25
6 $p$-pro aspects .............................................. 30
6.1 $p$-pro and prounipotent completions of discrete groups .............................................. 31
6.1.1 A morphism $\Gamma(p) \rightarrow \Gamma(\mathbb{Q}_p)$ .............................................. 31
6.1.2 Injectivity of $F_n(p) \rightarrow F_n(\mathbb{Q}_p)$ .............................................. 31
6.1.3 Exact sequences of $p$-pro completions .............................................. 32
6.1.4 Exact sequences of prounipotent completions .............................................. 33
6.1.5 Injectivity of $K_n(p) \rightarrow K_n(\mathbb{Q}_p)$ .............................................. 33
6.2 Results on GT$_p$ .............................................. 34
6.3 A $p$-pro analogue $\text{DMR}_p^B$ of the group scheme $\text{DMR}^B(-)$ .............................................. 36
References .............................................. 37

Mathematics Subject Classification 11M32 · 14G32

Introduction

This paper is a sequel of [4, 5]. The main result of this series of papers is the proof, independent of [6], of the inclusion of the torsor of associators $M(k)$ over a commutative $\mathbb{Q}$-algebra $k$ into the torsor $\text{DMR}^{\text{DR,B}}(k)$ of solutions of the double shuffle equations over the same algebra. This is obtained by constructing in [EF1] pairs of algebra coproducts $\hat{\Lambda}_V, ?$ and of module coproducts $\hat{\Lambda}_M, ?$, $? \in \{B, DR\}$, by studying their relation with associators, and by studying in [5] the relations of $M(k)$ and $\text{DMR}^{\text{DR,B}}(k)$ with the torsors of isomorphisms $\text{Stab}(\hat{\Lambda}_V,\text{DR/B})(k)$ and $\text{Stab}(\hat{\Lambda}_M,\text{DR/B})(k)$ related to these coproducts.

It is well-known that the category of torsors (i.e. pairs $(G, X)$, where $G$ is a group and $X$ is a set with a free and transitive action of $G$) is equivalent to that of bitorsors (i.e. triples $(G, X, H)$ where $(G, X)$ is a torsor and $H$ is a group with a right action on $X$, free, transitive and commuting with the action of $G$). An example of a bitorsor is the set of associators $M(k)$, where $G$ (resp. $H$) is the Grothendieck-Teichmüller group $\text{GT}(k)$ (resp. $\text{GRT}(k)$). Related examples of torsors are the ‘double shuffle’ pairs $(\text{DMR}_0^\text{DR}(k), \text{DMR}_0^\text{DR,B}(k))$ and $(\text{DMR}_0^\text{DR}(k), \text{DMR}_\mu^\text{DR,B}(k))$, with $\mu \in kn^\times$ (see [5, 10]).
The primary purpose of this paper is an explicit construction of the corresponding bitorsor structures. This is obtained in Theorem 3.13, (a), together with the analogous explicit constructions for the stabilizer torsors \( \text{Stab}(\hat{\Delta}^{\hat{W}, \text{DR}/B})(k) \) and \( \text{Stab}(\hat{\Delta}^{\hat{M}, \text{DR}/B})(k) \). In particular, the double shuffle counterpart \( \text{DMR}^B(k) \) of the group \( \text{GT}(k) \) is obtained in terms of a stabilizer of the coproduct \( \hat{\Delta}^{\hat{M}, B} \) (Lemma-Definition 3.9).

The other results of the paper are concerned with the study of the group \( \text{DMR}^B(k) \). We study its relation with the group \( \text{GT}(k) \) (Theorem 3.13, (b)). We give an equivalent definition of \( \text{DMR}^B(k) \) in terms of group-like elements for the coproduct \( \hat{\Delta}^{\hat{M}, B} \) (Sect. 4). Analogously to what is done in [2] for the group scheme \( k \mapsto \text{GT}(k) \), we construct and study discrete and pro-\( p \) versions of the group scheme \( k \mapsto \text{DMR}^B(k) \) (Sects. 5, 6).

The identification of the bitorsor structure of the torsor \( \text{DMR}^{\text{DR}, B}(k) \), obtained in Theorem 3.13, (a), is based on its relation with a stabilizer torsor (see [5], Section 3.1) and with the identification of the bitorsor of a stabilizer torsor (Lemma 1.10). The computation of the discrete analogue \( \text{DMR}^B \) of \( \text{DMR}^B(k) \) (Sect. 5.2) is based on the study of the discrete version \( \Delta^{\hat{M}, B} \) of the coproduct \( \hat{\Delta}^{\hat{M}, B} \). In order to establish the main properties of its pro-\( p \) version \( \text{DMR}^B_p \) (Sect. 6.3), we recall some basics on pro-\( p \) groups (Sect. 6.1), and prove statements of [2] on the pro-\( p \) analogue \( \text{GT}_p \) of \( \text{GT}(k) \) (Sect. 6.2).

The material is distributed as follows: Sect. 1 is devoted to basic material on torsors and bitorsors, Sect. 2 is devoted to the construction of an explicit bitorsor \( \text{G}^{\text{DR}, B}(k) \) related to the twisted Magnus group, Sect. 3 is devoted to the construction of several subbitorsors of \( \text{G}^{\text{DR}, B}(k) \) and to the first main result (Theorem 3.13), namely the construction of a bitorsor structure on \( \text{DMR}^{\text{DR}, B}(k) \) and of the group \( \text{DMR}^B(k) \), Sect. 4 is devoted to an equivalent definition of this group, Sect. 5 is devoted to the definition and computation of a discrete analogue of \( \text{DMR}^B(k) \), and Sect. 6 is devoted to the construction of its pro-\( p \) analogue.

**Notation**

In all the paper, \( k \) is a commutative and associative \( \mathbb{Q} \)-algebra with unit. For \( A \) an algebra, we denote by \( A^\times \) the group of its invertible elements. For \( a \in A \) (resp. \( u \in A^\times \)), we denote by \( \ell_a \) (resp. \( \text{Ad}_u \)) the self-map of \( A \) given by \( x \mapsto ax \) (resp. \( x \mapsto uxu^{-1} \)).

**1 Torsors and bitorsors**

In Sect. 1.1, we recall the formalism of torsors. We introduce the category of bitorsors in Sect. 1.2 and recall its equivalence with the category of torsors. In Sect. 1.3, we recall the main torsors of [5] and their interrelations.
1.1 Bitorsors

Recall from [5] the definitions of a (left) torsor, of a morphism between two torsors, of a subtorsor of a torsor (Definition 2.1), of the preimage of a subtorsor by a torsor morphism (Lemma 2.2), of the intersection of two subtorsors of a torsor (Lemma 2.3), of the trivial torsor \( G \) attached to a group \( G \) (Lemma 2.4), of the torsor injection \( \text{inj}_a : H \rightarrow G \) attached to a group inclusion \( H \subset G \) and an element \( a \in G \) (Lemma 2.5), of a stabilizer subtorsor (Lemma 2.6).

**Definition 1.1** (See [7], Chap. III, Definition 1.5.3)

(a) A *bitorsor* \( G \times_H \) is a triple \( (G, X, H) \), where \( X \) is a set and \( G \) and \( H \) are groups, equipped with commuting left and right actions of \( G \) and \( H \) on \( X \), which are both free and transitive.

(b) If \( G \times_H \) and \( G' \times_{H'} \), are bitorsors, a *bitorsor morphism* \( G \times_H \rightarrow G' \times_{H'} \) is the data of a map \( X \rightarrow X' \) and of compatible group morphisms \( G \rightarrow G' \) and \( H \rightarrow H' \).

(c) A *subbitorsor* of the bitorsor \( G' \times_{H'} \) is the data of a subset \( X \) of \( X' \) and of subgroups \( G, H \) of \( G', H' \), such that the inclusions build up a torsor morphism \( G \times_H \rightarrow G' \times_{H'} \), which is then a *bitorsor inclusion*.

We will use the expressions ‘the torsor \( X \)’ (resp. ‘the bitorsor \( X \)’) to designate a torsor \( G \times X \) (resp. a bitorsor \( G \times X \)).

**Lemma 1.2** Let \( G' \times_{H'} \rightarrow G \times_H \leftarrow G'' \times_{H''} \) be a diagram of a bitorsors. Let \( G''' \) (resp. \( X''', H''' \)) be the fibered product of \( G' \) (resp. \( X', H' \)) and \( G'' \) (resp. \( X'', H'' \)) above \( G \) (resp. \( X, H \)). Then either \( X''' \) is empty, or \( G''' \times_{H'''} \) is a bitorsor, called the fibered product of \( G' \times_{H'} \) and \( G'' \times_{H''} \) above \( G \times_H \), denoted \( X' \times_X X'' \) and fitting in a commutative diagram of bitorsors

\[
\begin{array}{ccc}
G''' \times_{H'''} & \longrightarrow & G'' \times_{H''} \\
\downarrow & & \downarrow \\
G' \times_{H'} & \longrightarrow & G \times_H
\end{array}
\]

If \( G''' \times_{H'''} \) is a subbitorsor of \( G' \times_{H'} \) and \( X'''' \) is nonempty, then \( G''' \times_{H'''} \) is a subbitorsor of \( G \times_H \), called the preimage of \( G'' \times_{H''} \) by the torsor morphism \( G \times_H \rightarrow G' \times_{H'} \).

**Proof** Obvious.

**Definition 1.3** We call a commutative square

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\]
of bitorsors Cartesian iff there is an isomorphism of bitorsors \( A \to B \times_D C \) such that the diagram

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \leftarrow & B \times_D C
\end{array}
\]

commutes.

**Lemma 1.4** Let \( G X_H \) be a bitorsor and let \( G' X'_H \) and \( G'' X''_H \) be subbitorsors. Then either \( X' \cap X'' \) is empty, or \( G' \cap G'' X' \cap X''_H \cap H \) is a subbitorsor of \( G X_H \), called the intersection of both bitorsors.

**Proof** Obvious. \( \square \)

**Lemma 1.5** A group isomorphism \( i : G' \to G \) gives rise to a bitorsor \( G G_G \), where \( G \) acts on the left on itself, and the right action of \( G' \) on \( G \) is given by \( g' \cdot g := gi(g') \). The bitorsor \( G G_G \) corresponding to \( i \) being the identity map of \( G \) is called the trivial bitorsor attached to \( G \).

**Proof** Obvious. \( \square \)

**Lemma 1.6** Let \( G \) be a group and \( H \) be a subgroup. For any \( a \in G \), there is a torsor inclusion \( \text{inj}_a : H H_H \to G G_G \) where the first (resp. second, third) component is the inclusion \( H \leftarrow G \) (resp. the map \( h \mapsto ha^{-1} \), the group morphism \( H \to G \), \( h \mapsto aha^{-1} \)).

**Proof** Obvious. \( \square \)

The map \( a \mapsto \text{inj}_a(G G_G) \) sets up a map \( G / H \to \{ \text{subbitorsors of } G G_G \} \).

If \( C \) is a category, we denote by \( \text{Iso}_C(O, O') \) the set of isomorphisms between two objects \( O \) and \( O' \) and by \( \text{Aut}_C(O) \) the group of automorphisms of an object \( O \).

**Definition 1.7** The bitorsor attached to a pair of isomorphic objects \( O, O' \) of a category \( C \) is \( \text{Bitor}(O, O') := \text{Aut}_C(O') \text{Iso}_C(O, O') \text{Aut}_C(O') \). An action of a bitorsor on the pair \((O, O')\) is a morphism from this bitorsor to \( \text{Bitor}(O, O') \).

**Lemma 1.8** Let \( C \) be a category and let \( O, O' \) be objects. Let \( i : G' \to G \) be a group isomorphism and let \( G \to \text{Aut}_C(O), G' \to \text{Aut}_C(O') \) be group morphisms, denoted \( g \mapsto g_0 \text{ and } g' \mapsto g'_0 \). Let \( i_{O',O} \in \text{Iso}_C(O', O) \) be such that for any \( g' \in G' \), one has \( i(g') o = i O' o g'_0 o (i O' o)^{-1} \). Then a bitorsor morphism \( G G_G' \to \text{Bitor}(O, O') \) is given by the above group morphisms and the map \( G \to \text{Iso}_C(O', O), g \mapsto g_0 o i O' o \).

**Proof** Obvious. \( \square \)

**Lemma 1.9** Let \( G X_H \) be a bitorsor acting on a pair \((O, O')\) of isomorphic objects of a \( k \)-linear tensor category \( C \). Then for any \( n, m \geq 0 \), \( \text{Hom}_C(O^\otimes n, O'^\otimes m) \), \( \text{Hom}_C(O'^\otimes n, O'^\otimes m) \) is a pair of isomorphic \( k \)-modules, equipped with an action of \( G X_H \).
Proof If the action on \((O, O')\) is denoted \(G \ni g \mapsto g_0 \in \text{Aut}_C(O)\), \(X \ni x \mapsto x'\) is isomorphic to \(X \times O'\), then the action on this pair of modules is as follows: \(g \in G\) acts by \(\text{Hom}_C(O^{\otimes n}, O^{\otimes m}) \ni f \mapsto x^{\otimes m} \circ f \circ (g^{\otimes n})^{-1}\), \(h \in H\) acts by \(\text{Hom}_C(O^{\otimes n}, O^{\otimes m}) \ni f' \mapsto h^{\otimes m} \circ f' \circ (h^{\otimes n})^{-1}\), \(x \in X\) by \(\text{Hom}_C(O^{\otimes n}, O^{\otimes m}) \ni f' \mapsto x^{\otimes m} \circ f' \circ (x^{\otimes n})^{-1} \in \text{Hom}_C(O^{\otimes n}, O^{\otimes m})\).

Lemma 1.10 Let \(G \times H\) be a bitorsor acting on a pair \((V, W)\) of isomorphic \(k\)-modules. If \((v, w) \in V \times W\), then either \(\text{Stab}_X(v, w) := \{x \in X | x \cdot w = v\}\) is empty, or \(\text{Stab}_G(v) \text{Stab}_X(v, w) \text{Stab}_H(w)\) is a subbitorsor of \(G \times H\), called the stabilizer bitorsor of \((v, w)\), where \(\text{Stab}_G(v) := \{g \in G | g \cdot v = v\}\) and \(\text{Stab}_H(w) := \{h \in H | h \cdot w = w\}\).

Proof Similar to that of [5], Lemma 2.6.

Lemma 1.11 Let \(T\) be a \(\mathbb{Q}\)-linear neutral Tannakian category and let \(\omega_1, \omega_2 : T \to \text{Vec}_\mathbb{Q}\) be two fiber functors. If it is nonempty, then the set \(\text{Iso}^{\otimes}(\omega_2, \omega_1)\) is a bitorsor under the left and right actions of \(\text{Aut}^{\otimes}(\omega_1)\) and \(\text{Aut}^{\otimes}(\omega_2)\).

Proof This is an example of the construction of Definition 1.7, where \(C\) is the category of fiber functors \(T \to \text{Vec}_\mathbb{Q}\).

Define a right torsor \(X_H\) to be a pair \((X, H)\) of a set \(X\) and a group \(H\) acting freely and transitively on \(X\) from the right. This gives rise to a torsor \(H^{\otimes n} X\). Let \(\text{Tor}\) and \(\text{Bitor}\) be the categories of torsors and bitorsors. Then there are two functors \(\text{Bitor} \to \text{Tor}\), defined on objects by \(G \times H \mapsto G\times X\) and \(G \times H \mapsto H^{\otimes n} X\).

1.2 From torsors to bitorsors

For \(G \times X\) a torsor, define \(\text{Aut}_G(X)\) to be subgroup of the permutation group of \(X\) acting on the right which commute with all the elements of \(G\). Then \(\text{Aut}_G(X)\) is a group, equipped with a right action on \(X\) which commutes with the left action of \(G\).

Lemma 1.12 (See [7], Chap. III, Proposition 1.5.1.)

(a) The assignment \(G \times X \mapsto \text{Aut}_G(X)\) defines a functor \(\text{Tor} \to \text{Gp}\) from the category of torsors to that of groups. The group \(\text{Aut}_G(X)\) will be called the ‘group attached to the torsor \(G \times X\).

(b) If \(G \times X\) is a torsor, then the right action of \(\text{Aut}_G(X)\) defines a bitorsor structure \(G \times \text{Aut}_G(X)\) on it. The assignment \(G \times X \mapsto G \times \text{Aut}_G(X)\) defines a functor \(\text{Tor} \to \text{Bitor}\), quasi-inverse to the functor \(\text{Bitor} \to \text{Tor}\), \(G \times H \mapsto G \times X\). In particular, if \(G \times H\) is a bitorsor, then there is a canonical isomorphism \(H \cong \text{Aut}(G)\).

Lemma 1.13 If \(G \times H\) is a bitorsor and if \(G' \times X', G'' \times X''\) are subbitorsors of \(G \times H\) such that \(G' \times X'\) is a subtorsor of \(G'' \times X''\), then \(H' \subset H''\) (inclusion of subgroups of \(H\)).

Proof Any element \(x \in X\) gives rise to a group isomorphism \(\text{Ad}_{x^{-1}} : G \to H\), defined by the identity \(g \cdot x = x \cdot \text{Ad}_{x^{-1}}(g)\) for any \(g \in G, x \in X\). If \(G' \times X'\) is a subbitorsor of \(G \times H\) and if \(x \in X'\), then \(\text{Ad}_{x^{-1}}\) restricts to a group isomorphism \(G' \to H'\). Likewise, \(\text{Ad}_{x^{-1}}\) restricts to a group isomorphism \(G'' \to H''\). Then the subgroup \(H'\) (resp. \(H''\)) of \(H\) is the image of the subgroup \(G'\) (resp. \(G''\)) of \(G\) by \(\text{Ad}_{x^{-1}}\). As \(G' \subset G''\), this implies \(H' \subset H''\).
1.3 Some torsors and their interrelations

In [5], we introduced a ‘twisted Magnus’ torsor \( G_{DR,B}(k) \) (Sect. 2.2), together with its subtorsors \( G_{quad,B}(k) \) (Sect. 2.3), \( \text{Stab}(\hat{\Delta}^{\mathcal{M},DR/B})(k) \) (\( \mathcal{M} \in \{ \mathcal{W}, \mathcal{M} \} \)) (Sects. 2.6, 2.7), \( M(k) \) (Sect. 2.4 and [2]), and \( \text{DMR}_{\mu,B}(k) \) (Sect. 2.5), which is constructed using the torsor \( \text{DMR}_{\mu}(k) \) for \( \mu \in k^\times ( [10] ) \) and contains it as a subtorsor.

In Theorem 3.1 from [5], we showed the following relations between these subtorsors:

1. Inclusion \( M(k) \hookrightarrow \text{DMR}_{DR,B}(k) \);
2. Equality \( \text{DMR}_{DR,B}(k) = \text{Stab}(\hat{\Delta}^{\mathcal{M},DR/B})(k) \cap G_{quad,B}(k) \);
3. Inclusion \( \text{Stab}(\hat{\Delta}^{\mathcal{M},DR/B})(k) \hookrightarrow \text{Stab}(\hat{\Delta}^{\mathcal{W},DR/B})(k) \).

The main purpose of this paper is the explicit computation of the groups attached to these torsors in the sense of Lemma 1.12, (a), as well as the study of their interrelations. This will make explicit the bitorsor structures attached to these torsors by Lemma 1.12, (b).

2 The bitorsor \( G_{DR,B}(k) \) and its actions

In Sect. 2.1, we construct a Betti counterpart \( (G_B(k), \otimes) \) of the variant \( G_{DR}(k) \) from [5] of the ‘twisted Magnus group’ from [10]. In Sect. 2.2, we construct actions of \( G_B(k) \), which are the Betti counterparts of the actions of \( G_{DR}(k) \) from [5], Section 1.6. In Sect. 2.3, we show how to use \( G_B(k) \) for upgrading the torsor structure \( G_{DR,B}(k) \) into a bitorsor one, and in Sect. 2.4, we show how the actions of Sect. 2.2 and of [5], Section 1.6 can be combined to construct an action of the bitorsor \( G_{DR,B}(k) \).

2.1 The group \( (G_B(k), \otimes) \)

2.1.1 The group \( (G(\hat{\mathcal{V}}^B), \cdot) \)

Let \( \mathcal{V}^B \) be the \( k \)-algebra introduced in [4], Section 2.1, and let \( (\mathcal{F}_i\mathcal{V}^B)_{i \geq 0} \) be the decreasing algebra filtration on it defined in [4], Section 2.1. Let \( \Delta^{\mathcal{V},B} : \mathcal{V}^B \to (\mathcal{V}^B)^{\otimes 2} \) be the morphism of filtered \( k \)-algebras introduced in [4], Section 2.3. Then \( (\mathcal{V}^B, \Delta^{\mathcal{V},B}) \) is a Hopf algebra. In [4], Section 2.5, we introduced the topological Hopf algebra \( (\hat{\mathcal{V}}^B, \hat{\Delta}^{\mathcal{V},B}) \) obtained by completion with respect to the filtrations.

Denote by \( G(\mathcal{V}^B) \) (resp. \( \hat{G}(\mathcal{V}^B) \)) the group of group-like elements of the Hopf algebra \( (\mathcal{V}^B, \Delta^{\mathcal{V},B}) \) (resp. \( (\hat{\mathcal{V}}^B, \hat{\Delta}^{\mathcal{V},B}) \)); we denote by \( \cdot \) its product. According to [4], Section 2.1, the Hopf algebra \( (\mathcal{V}^B, \Delta^{\mathcal{V},B}) \) is the group algebra \( kF_2 \), where \( F_2 \) is the free group with generators \( X_0, X_1 \), which leads to the equality \( G(\mathcal{V}^B) = F_2 \). Then the natural morphism \( \hat{G}(\mathcal{V}^B) \to G(\mathcal{V}^B) \) can be identified with the morphism from \( F_2 \) to its \( k \)-prounipotent completion.

\( \text{Birkhäuser} \)
2.1.2 The automorphisms $\text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(1)}$ and $\text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(10)}$

Let $\hat{\mathcal{V}}_1^B := 1 + F^1\hat{\mathcal{V}}_1^B \subset \hat{\mathcal{V}}^B$. For $a \in \hat{\mathcal{V}}_1^B$, $\log(a)$ may be expanded as a series in $a - 1$ and therefore makes sense as an element of $F^1\hat{\mathcal{V}}^B_1$. If now $\mu \in k$, $\exp(\mu \log(a))$ may be expanded as a series in $\mu \log(a)$ and therefore makes sense as an element of $\hat{\mathcal{V}}_1^B$. This defines a self-map of $\hat{\mathcal{V}}_1^B$, denoted $a \mapsto a^\mu$, which restricts to a self-map of $\mathcal{G}^B_1$.

For $(\mu, g) \in k^\times \times \mathcal{G}(\hat{\mathcal{V}}^B)$, one checks that there is a unique automorphism $\text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(1)}$ of the topological $k$-algebra $\hat{\mathcal{V}}^B$, such that

$$\text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(1)} : X_0 \mapsto g \cdot X_0^\mu \cdot g^{-1}, \quad X_1 \mapsto X_1^\mu$$

and a unique automorphism $\text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(10)}$ of the topological $k$-module $\hat{\mathcal{V}}^B$, such that

$$\forall a \in \hat{\mathcal{V}}^B, \quad \text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(10)}(a) = \text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(1)}(a) \cdot g.$$  \hfill (2.1.2)

2.1.3 The group $(G^B(k), \otimes)$

**Lemma-Definition 2.1** The product $\otimes$ given by

$$(\mu, g) \otimes (\mu', g') := (\mu \mu', \text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(10)}(g'))$$

defines a group structure on $G^B(k) := k^\times \times \mathcal{G}(\hat{\mathcal{V}}^B)$.

**Proof** When $(\mu, g) \in G^B(k)$, $\text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(1)}$ is an automorphism of the Hopf algebra $(\hat{\mathcal{V}}^B, \hat{\Delta}, \hat{\epsilon})$, which implies that for any $g' \in \mathcal{G}(\hat{\mathcal{V}}^B)$, $\text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(1)}(g') \in \mathcal{G}(\hat{\mathcal{V}}^B)$. It follows that $\text{aut}^{(\mu,g)}_{\hat{\mathcal{V}},(10)}(g') \in \mathcal{G}(\hat{\mathcal{V}}^B)$, and therefore that $\otimes$ is a well-defined map $G^B(k)^2 \to G^B(k)$, which restricts to a map $\mathcal{G}(\hat{\mathcal{V}}^B)^2 \to \mathcal{G}(\hat{\mathcal{V}}^B)$. The fact that $(\mathcal{G}(\hat{\mathcal{V}}^B), \otimes)$ is a group can be proved analogously to [10], Proposition 3.1.6. The group $k^\times$ acts on $\hat{\mathcal{V}}^B$ by $\mu \cdot X_i := X_i^\mu$ for $i = 0, 1$. This induces an action of $k^\times$ on $(\mathcal{G}(\hat{\mathcal{V}}^B), \otimes)$; one checks that $(G^B(k), \otimes)$ is the corresponding semidirect product, the injections $k^\times \hookrightarrow G^B(k)$ and $\mathcal{G}(\hat{\mathcal{V}}^B) \hookrightarrow G^B(k)$ being given by $\mu \mapsto (\mu, 1)$ and $g \mapsto (1, g)$. \hfill $\square$

2.2 Actions of the group $(G^B(k), \otimes)$

We denote by $k$-alg (resp. $k$-mod, $k$-alg-mod) the category of $k$-algebras (resp. $k$-modules, pairs $(A, M)$ of a $k$-algebra $A$ and a left $A$-module $M$). Recall that for $(A, M)$ an object in $k$-alg-mod, $\text{Aut}_{k}$-alg-mod$(A, M)$ is the set of pairs $(\alpha, \theta) \in \text{Aut}_{k}$-alg$(A) \times \text{Aut}_{k}$-mod$(M)$, such that for any $a \in A, m \in M$, one has $\theta(am) = \alpha(a)\theta(m)$; this is a subgroup of $\text{Aut}_{k}$-alg$(A) \times \text{Aut}_{k}$-mod$(M)$.
Lemma 2.2 For $(\mu, g) \in G^B(k)$, the pair $(\text{aut}_{(\mu,g)}^{Y^0}, \text{aut}_{(\mu,g)}^{Y_0})$ belongs to $\text{Aut}_{\text{alg-mod}}(\hat{\mathcal{V}}^B, \hat{\mathcal{V}}^B)$ (in which $\hat{\mathcal{V}}^B$ is viewed as the left regular module over itself). The map

$$(G^B(k), \otimes) \rightarrow \text{Aut}_{\text{alg-mod}}(\hat{\mathcal{V}}^B, \hat{\mathcal{V}}^B), \quad (\mu, g) \mapsto (\text{aut}_{(\mu,g)}^{Y^0}, \text{aut}_{(\mu,g)}^{Y_0})$$

is a group morphism.

**Proof** The proof of the first statement is similar to that of the corresponding statement in Lemma 1.10 of [5]. The fact that $(G^B(k), \otimes) \rightarrow \text{Aut}_{\text{alg}}(\hat{\mathcal{V}}^B), (\mu, g) \mapsto \text{aut}_{(\mu,g)}^{Y^0}$ (resp. $(G^B(k), \otimes) \rightarrow \text{Aut}_{k}(\hat{\mathcal{V}}^B), (\mu, g) \mapsto \text{aut}_{(\mu,g)}^{Y_0}$) is a group morphism is proved similarly to Lemma 1.8 (resp. Lemma 1.9) in [5]. These facts imply the second statement. \hfill \Box

**Lemma 2.3** (See [5], Lemma 1.2) Let $(A, M)$ be an object of $k\text{-alg-mod}$, and let $A_0$ be a subalgebra of $A_0$ and $M_0$ an $A$-submodule of $M$. There is a group diagram

$$\text{Aut}_{\text{alg-mod}}(A, M) \supset \text{Aut}(A, M|A_0, M_0) \rightarrow \text{Aut}_{\text{alg-mod}}(A_0, M/M_0)$$

where $\text{Aut}(A, M|A_0, M_0) \subset \text{Aut}_{\text{alg-mod}}(A, M)$ is the subset of pairs $(\alpha, \theta)$ such that $\alpha$ (resp. $\theta$) restricts to an automorphism of $A_0$ (resp. of $M_0$).

Let $\hat{\mathcal{M}}^B$ be the topological $k$-subalgebra of $\hat{\mathcal{M}}^B$ given by $\hat{\mathcal{M}}^B = kI \oplus \hat{\mathcal{M}}^B(X_1 - 1)$. View $\hat{\mathcal{M}}^B$ as the left regular topological module over itself; then $\hat{\mathcal{M}}^B(X_0 - 1)$ is a topological submodule. We denote the corresponding quotient topological $\hat{\mathcal{M}}^B$-module by $\hat{\mathcal{M}}^B$ and by $1_B$ the class of 1 in this module; according to [4], Section 2.5, the restriction of $\hat{\mathcal{M}}^B$ to $\hat{\mathcal{M}}^B$ is free of rank one, generated by $1_B$.

**Lemma-Definition 2.4** If $(\mu, g) \in G^B(k)$, then $(\text{aut}_{(\mu,g)}^{Y^0}, \text{aut}_{(\mu,g)}^{Y_0})$ is in

$$\text{Aut}(\hat{\mathcal{M}}^B, \hat{\mathcal{M}}^B(X_0 - 1)).$$

We denote by $(\text{aut}_{(\mu,g)}^{Y^0}, \text{aut}_{(\mu,g)}^{Y_0})$ the corresponding element of $\text{Aut}_{\text{alg-mod}}(\hat{\mathcal{M}}^B, \hat{\mathcal{M}}^B)$ (see Lemma 2.3). The map taking $(\mu, g)$ to this element is a group morphism $(G^B(k), \otimes) \rightarrow \text{Aut}_{\text{alg-mod}}(\hat{\mathcal{M}}^B, \hat{\mathcal{M}}^B)$.

**Proof** It follows from $\hat{\mathcal{M}}^B = kI \oplus \hat{\mathcal{M}}^B(X_1 - 1)$, from $\text{aut}_{(\mu,g)}^{Y^0}(X_1) = X_1^\mu$ for any $(\mu, g) \in G^B(k)$, and from $X_1^\mu - 1 = f_\mu(X_1 - 1) \cdot (X_1 - 1)$, where $f_\mu(t) := ((1 + t)^\mu - 1)/t \in k[[t]]$ for any $\mu \in k^\times$, that $\text{aut}_{(\mu,g)}^{Y^0}$ restricts to an automorphism of $\hat{\mathcal{M}}^B$ for any $(\mu, g) \in G^B(k)$.

If $(\mu, g) \in G^B(k)$ and $v \in \hat{\mathcal{M}}^B$, then $\text{aut}_{(\mu,g)}^{Y_0}(v \cdot (X_0 - 1)) = \text{aut}_{(\mu,g)}^{Y_0}(v \cdot (X_0 - 1)) \cdot g = \text{aut}_{(\mu,g)}^{Y_0}(v) \cdot g \cdot (X_0 - 1) \cdot (X_0 - 1)$; this implies that $\text{aut}_{(\mu,g)}^{Y_0}$ restricts to an automorphism of $\hat{\mathcal{M}}^B(X_0 - 1)$. \hfill \Box
Recall that $\hat{\mathcal{V}}^B$ is freely generated, as a topological algebra, by $\log X_0$ and $\log X_1$; for $g \in \hat{\mathcal{V}}^B$, we denote by $\{\log X_0, \log X_1\}^* \to k$ the map such that $g = \sum_{w \in \{\log X_0, \log X_1\}^*} (g|w) w$, where $\{\log X_0, \log X_1\}^*$ is the set of (possibly empty) words in $\log X_0, \log X_1$.

**Definition 2.5** For $g \in \hat{\mathcal{V}}^B$, $\Gamma_g \in k[[t]]^\times$ is the series given by

$$\Gamma_g(t) := \exp \left( \sum_{n \geq 1} (-1)^{n+1} (g|\log X_0)^{n-1} \log X_1 t^n / n \right).$$

**Definition 2.6** (See [5], Definition 1.1) If $A$ is a $k$-algebra and if $G \to \text{Aut}_{k\text{-alg}}(A)$, $g \mapsto \alpha_g$ is a group morphism, a cocycle of $G$ in $A$ equipped with $g \mapsto \alpha_g$ is a map $G \to A^\times$, $g \mapsto c_g$ such that $c_{gg'} = c_g \cdot \alpha_g(c_{g'})$ for any $g, g' \in G$.

**Lemma 2.7** The map $\Gamma : G^B(k) \to (\hat{\mathcal{V}}^B)^\times$, $(\mu, g) \mapsto \Gamma_g^{-1}(\log X_1)$, is a cocycle of $(G^B(k), \otimes)$ in $\hat{\mathcal{V}}^B$ equipped with $(\mu, g) \mapsto \text{aut}_{\hat{\mathcal{V}}^B}^{\mu, 1}$. This map corestricts to a map $\Gamma : G^B(k) \to (\hat{\mathcal{V}}^B)^\times$, which is a cocycle of the same group in $\hat{\mathcal{V}}^B$ equipped with $(\mu, g) \mapsto \text{aut}_{\hat{\mathcal{V}}^B}^{\mu, 1}$.

**Proof** The proof is similar to that of Lemmas 1.12 and 1.13 in [5].

**Lemma-Definition 2.8** For $(\mu, g) \in G^B(k)$, set

$$\Gamma_{\text{aut}}^{\mathcal{W}, 1} : = \text{Ad}_{\Gamma_g^{-1}(\log X_1)} \circ \text{aut}_{\hat{\mathcal{V}}^B}^{\mu, 1} \in \text{Aut}_{k\text{-alg}}(\hat{\mathcal{V}}^B)$$

$$\Gamma_{\text{aut}}^{\mathcal{M}, 10} : = \ell_{\Gamma_g^{-1}(\log X_1)} \circ \text{aut}_{\hat{\mathcal{V}}^B}^{\mu, 1} \in \text{Aut}_{k\text{-mod}}(\hat{\mathcal{M}}^B).$$

Then the pair $(\Gamma_{\text{aut}}^{\mathcal{W}, 1}, \Gamma_{\text{aut}}^{\mathcal{M}, 10})$ belongs to $\text{Aut}_{k\text{-alg-mod}}(\hat{\mathcal{V}}^B, \hat{\mathcal{M}}^B)$ and the map $(G^B(k), \otimes) \to \text{Aut}_{k\text{-alg-mod}}(\hat{\mathcal{V}}^B, \hat{\mathcal{M}}^B)$ is a group morphism.

**Proof** Lemma 1.4 in [5] gives conditions for a cocycle to be used for twisting a group morphism $G \to \text{Aut}_{k\text{-alg-mod}}(A, M)$. Lemmas 2.4 and 2.7 show that these conditions are fulfilled.

### 2.3 A bitorsor structure on $G^{DR, B}(k)$

The group $(G^{DR}(k), \otimes)$ is defined in [5], Section 1.6.1. Recall that $G^{DR}(k) = k^\times \times G(\hat{\mathcal{V}}^{DR})$ (equality of sets), where $(\hat{\mathcal{V}}^{DR}, \hat{\mathcal{M}}^{V, DR})$ is the topological Hopf algebras from [4], Section 1.3, and the product $\otimes$ is defined in [5], Section 1.6.1. In [4], Section 3.3, we defined an isomorphism $\text{iso}^V$ is defined in [5], Section 1.6.1. In [4], Section 3.3, we defined an isomorphism $\text{iso}^V : \hat{\mathcal{V}}^B \to \hat{\mathcal{V}}^{DR}$ of topological $k$-algebras.

**Lemma 2.9** $\text{iso}^V$ is an isomorphism of topological Hopf algebras.

**Proof** This follows from $\text{iso}^V : X_i \mapsto \exp(e_i)$ for $i = 0, 1$, and from the fact the $X_i$ (resp. $e_i$) is group-like (resp. primitive) for $\hat{\mathcal{V}}^B$ (resp. $\hat{\mathcal{V}}^{DR}$).

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Lemma 2.10 The map $\text{iso}^\mathcal{V}$ induces group isomorphisms $\text{iso}^G : (\mathcal{G}(\hat{\mathcal{V}}^B), \otimes) \to (\mathcal{G}(\hat{\mathcal{V}}^\text{DR}), \otimes)$ and $\text{iso}^G : (\mathcal{G}^B(\mathfrak{k}), \otimes) \to (\mathcal{G}^\text{DR}(\mathfrak{k}), \otimes)$, given by the map $G^B(\mathfrak{k}) = k^\times \times G(\hat{\mathcal{V}}^B) \to k^\times \times G(\hat{\mathcal{V}}^\text{DR}) = G^\text{DR}(\mathfrak{k})$.

**Proof** The fact that $\text{iso}^G$ and $\text{iso}^G$ are bijections follows from Lemma 2.9. The compatibility of these maps with the products $\otimes$ on both sides follows from comparison of their constructions, given respectively in Sect. 2.1.3 and [5], Section 1.6.1. \(\square\)

In [5], Def. 2.8, we introduced a torsor denoted $G^\text{DR},B(\mathfrak{k})$.

**Lemma-Definition 2.11** The underlying torsor of the bitorsor obtained by applying Lemma 1.5 to the group isomorphism from Lemma 2.10 is $G^\text{DR},B(\mathfrak{k})$. By abuse of notation, this bitorsor will also be denoted $G^\text{DR},B(\mathfrak{k})$.

**Proof** The underlying set of this torsor is $G^\text{DR}(\mathfrak{k})$ and the left action is that of $G^\text{DR}(\mathfrak{k})$ on itself. \(\square\)

The underlying set of this bitorsor is $G^\text{DR}(\mathfrak{k})$, the left action is that of $G^\text{DR}(\mathfrak{k})$ on itself, and the right action is that of $G^B(\mathfrak{k})$ on $G^\text{DR}(\mathfrak{k})$ given by $(\mu, g) \cdot (\mu', g') := (\mu, g) \otimes \text{iso}^G(\mu', g')$. The corresponding torsor coincides therefore with the torsor $G^\text{DR},B(\mathfrak{k})$ from [5], Definition 2.8.

### 2.4 Actions of the bitorsor $G^\text{DR},B(\mathfrak{k})$

In [4], Section 1.3, we defined a topological $\mathfrak{k}$-subalgebra of $\hat{\mathcal{V}}^\text{DR}$ by $\hat{\mathcal{V}}^\text{DR} := \mathfrak{k} \mathfrak{l} \oplus \mathcal{V}_1^\text{DR}$, a $\mathcal{V}_1^\text{DR}$-module $\mathcal{M}^\text{DR} := \mathcal{V}^\text{DR} / \mathcal{V}_0^\text{DR}$, and its element $1^\text{DR}$ defined as the class of 1, and showed that $\mathcal{M}^\text{DR}$ is freely generated by $1^\text{DR}$ as a $\hat{\mathcal{V}}^\text{DR}$-module. Recall from [4], Section 3.3 that the isomorphism $\text{iso}^\mathcal{V}$ induces compatible isomorphisms $\text{iso}^\mathcal{V}^B : \mathcal{V}^B \to \mathcal{V}^\text{DR}$ of topological $\mathfrak{k}$-algebras and $\text{iso}^\mathcal{M} : \mathcal{M}^B \to \mathcal{M}^\text{DR}$ of topological $\mathfrak{k}$-modules.

**Lemma 2.12** For $\omega \in \{B, DR\}$, the pair $(\mathcal{W}^\omega, \mathcal{M}^\omega)$, in which the second component is viewed as a left module over the first, is an object of $\mathfrak{k}$-alg-mod, which will be denoted $(\mathcal{W}, \mathcal{M})^\omega$. An isomorphism $\text{iso}^\mathcal{V},\mathcal{M} : (\mathcal{W}, \mathcal{M})^B \to (\mathcal{W}, \mathcal{M})^\text{DR}$ is given by the pair $(\text{iso}^\mathcal{V}, \text{iso}^\mathcal{M})$.

**Proof** The first statement follows from [4], Sections 1.3 and 2.5. The second statement follows from [4], Section 3.3. \(\square\)

In [5], Lemma 1.14, we defined a group morphism

$$(G^\text{DR}(\mathfrak{k}), \otimes) \to \text{Aut}_{\mathfrak{k}}\text{-alg-mod}(\mathcal{W}, \mathcal{M})^\text{DR}, \quad (\mu, g) \mapsto (\gamma_{\text{aut}_{\mathfrak{k}}^{\mathcal{W},(1),\text{DR}}}(\mu, g), \gamma_{\text{aut}_{\mathfrak{k}}^{\mathcal{M},(10),\text{DR}}}(\mu, g)).$$

**Lemma 2.13** The conditions of Lemma 1.8 are satisfied in case of the group isomorphism $\text{iso}^G : G^B(\mathfrak{k}) \to G^\text{DR}(\mathfrak{k})$, of the isomorphism $\text{iso}^\mathcal{V},\mathcal{M} : (\mathcal{W}, \mathcal{M})^B \to (\mathcal{W}, \mathcal{M})^\text{DR}$ in $\mathfrak{k}$-alg-mod, and of the group morphisms

$$G^\omega(\mathfrak{k}) \ni (\mu, g) \mapsto (\gamma_{\text{aut}_{\mathfrak{k}}^{\mathcal{W},(1),\omega}}(\mu, g), \gamma_{\text{aut}_{\mathfrak{k}}^{\mathcal{M},(10),\omega}}(\mu, g)) \in \text{Aut}_{\mathfrak{k}}\text{-alg-mod}(\mathcal{W}, \mathcal{M})^\omega$$
for $\omega \in \{B, DR\}$.

**Proof** This follows from the identities $\text{aut}_{\mathcal{V},(\alpha),B}(\mu,g) \circ \text{iso}_{\mathcal{V}}^{-1} = \text{iso}_{\mathcal{V}} \circ \text{aut}_{\mathcal{V},(\alpha),DR}(\mu,g)$ for any $(\mu,g) \in G^B(k)$, where $\alpha \in \{1, 10\}$, which follow from comparison of the definitions of $\text{aut}_{\mathcal{V},(\alpha),B}(\mu,g)$ and $\text{aut}_{\mathcal{V},(\alpha),DR}(\mu,g)$ in Sect. 2.1.2 and in [5], Section 1.6.2, and from $\Gamma_{\text{iso}_{\mathcal{V}}(g)}(t) = \Gamma_g(t)$ for $g \in G^B(k)$. \hfill $\square$

In [5], Definition 1.15, we defined, for any $(\mu, g) \in G^{DR}(k)$, isomorphisms

\[ \Gamma_{\text{comp}_{\mathcal{V},(1)}}^{\mathcal{V},(1)}(\mu,g) = \Gamma_{\text{comp}_{\mathcal{V},(1),DR}}^{\mathcal{V},(1),DR}(\mu,g) \circ \text{iso}_\mathcal{V} \in \text{Iso}_{k-\text{alg}}(\hat{\mathcal{V}}^B, \hat{\mathcal{V}}^{DR}) \]

and

\[ \Gamma_{\text{comp}_{\mathcal{M},(10)}}^{\mathcal{M},(10)}(\mu,g) = \Gamma_{\text{comp}_{\mathcal{M},(10),DR}}^{\mathcal{M},(10),DR}(\mu,g) \circ \text{iso}_\mathcal{M} \in \text{Iso}_{k-\text{mod}}(\hat{\mathcal{M}}^B, \hat{\mathcal{M}}^{DR}). \]

**Lemma 2.14** The map $G^{DR}(k) \to \text{Iso}_{k-\text{alg,mod}}((\hat{\mathcal{V}}, \hat{\mathcal{M}})^B, (\hat{\mathcal{V}}, \hat{\mathcal{M}})^{DR})$ underlying the bitorsor morphism $G^{DR,B}(k) \to \text{Bitor}((\hat{\mathcal{V}}, \hat{\mathcal{M}})^{DR}, (\hat{\mathcal{V}}, \hat{\mathcal{M}})^B)$ arising from Lemma 2.13 is

\[ (\mu, g) \mapsto (\Gamma_{\text{comp}_{\mathcal{V},(1)}}^{\mathcal{V},(1)}(\mu,g), \Gamma_{\text{comp}_{\mathcal{M},(1)}}^{\mathcal{M},(1)}(\mu,g)). \]

**Proof** Immediate. \hfill $\square$

### 3 Subbitorsors of $G^{DR,B}(k)$

In this section, we explicitly construct the bitorsors attached to the torsors of [5], by constructing Betti counterparts of their underlying groups; this is done in Sect. 3.1 (resp. Sects. 3.2, 3.3, 3.4, 3.5) for the 'linear and quadratic conditions’ torsor $G^{DR,B}_{\text{quad}}(k)$ (resp. the stabilizer torsors $\text{Stab}(\hat{\mathcal{V}}^{DR/B})$, $\text{Stab}(\hat{\mathcal{M}}^{DR/B})$, the double shuffle torsors $\text{DMR}^{DR,B}(k)$ and $\text{DMR}_\mu(k)$), and this construction is recalled from [2] in Sect. 3.6 for the case of the torsor of associators $M(k)$. In Sect. 3.7, we derive from the relations of the torsors of [5] certain relations between the Betti counterparts of the underlying groups. We make explicit the Lie algebras of these groups and their interrelations in Sect. 3.8. Finally, we make explicit the relations of the stabilizer bitorsors $\text{Stab}(\hat{\mathcal{V}}^{DR/B})$, $\text{Stab}(\hat{\mathcal{M}}^{DR/B})$ with coalgebra and Hopf algebra bitorsors (Sect. 3.9).
3.1 $G_{\text{quad}}^{\text{DR,B}}(k)$ as a subbitorsor of $G_{\text{quad}}^{\text{DR,B}}(k)$

3.1.1 The group $(G_{\text{quad}}^{\text{B}}(k), \otimes)$

Lemma-Definition 3.1 Set

$$G_{\text{quad}}^{\text{B}}(k) := \{(\mu, g) \in G^{\text{B}}(k) \mid (g|\log X_0) = (g|\log X_1) = 0, \mu^2 = 1 + 24(g|\log X_0\log X_1)\}.$$  

Then $G_{\text{quad}}^{\text{B}}(k)$ is a subgroup of $(G^{\text{B}}(k), \otimes)$.

Proof Define $G_{\text{lin}}^{\text{B}}(k)$ to be the subset of all pairs $(\mu, g)$ of $G^{\text{B}}(k)$ such that $(g|\log X_0) = (g|\log X_0) = 0$. Then $G_{\text{lin}}^{\text{B}}(k) = (\text{iso}^G)^{-1}(G_{\text{lin}}^{\text{DR}}(k))$, where $G_{\text{lin}}^{\text{DR}}(k)$ is the subgroup of $(G_{\text{quad}}^{\text{DR}}(k), \otimes)$ defined in [5], proof of Lemma 2.10, therefore $(G_{\text{lin}}^{\text{B}}(k), \otimes)$ is a subgroup of $(G^{\text{B}}(k), \otimes)$.

Recall from [5], proof of Lemma 2.10, the group $(k^\times \times k, \cdot)$, where $(\mu, c) \cdot (\mu', c') := (\mu\mu', c + \mu^2 c')$; it is the semidirect product corresponding to the action of $k^\times$ on $(k, +)$ by $\lambda \cdot a := \lambda^2 a$. The composed morphism $\varphi_{\text{quad}} \circ \text{iso}^G : G_{\text{lin}}^{\text{B}}(k) \to (k^\times \times k, \cdot)$ is given by $(\mu, g) \mapsto (\mu, (g|\log X_0\log X_1))$, where $\varphi_{\text{quad}} : G_{\text{lin}}^{\text{DR}}(k) \to (k^\times \times k, \cdot)$ is as in [5], proof of Lemma 2.10.

On the other hand, $\{(\mu, c) \mid c = (\mu^2 - 1)/24, \mu \in k^\times\}$ coincides with the subgroup $\text{Ad}_{(1, -1/24)}(k^\times)$ of $(k^\times \times k, \cdot)$, where $k^\times$ is the subgroup $\{c \mid c \in k^\times\}$ of $(k^\times \times k, \cdot)$. Its preimage by the above group morphism is therefore a subgroup of $(G_{\text{lin}}^{\text{B}}(k), \otimes)$. The result now follows from the fact that this preimage coincides with $G_{\text{quad}}^{\text{B}}(k)$. 

3.1.2 Bitorsor structure on $G_{\text{quad}}^{\text{DR,B}}(k)$

In [5], Definition 2.9, we introduced the subsets $G_{\text{quad}}^{\text{DR,B}}(k)$ and $G_{\text{quad}}^{\text{DR}}(k)$ of $G^{\text{DR}}(k)$, and in Lemma 2.10, proved that $G_{\text{quad}}^{\text{DR,B}}(k)$ is a subgroup of $G^{\text{DR}}(k)$, and that $G_{\text{quad}}^{\text{DR,B}}(k)$, equipped with the left action of this group, is a subbitorsor of $G^{\text{DR,B}}(k)$.

Lemma 3.2 The right action of $G^{\text{B}}(k)$ on $G^{\text{DR}}(k)$ restricts to a free and transitive action of $G_{\text{quad}}^{\text{B}}(k)$ on $G_{\text{quad}}^{\text{DR,B}}(k)$, so that this right action equips $G_{\text{quad}}^{\text{DR,B}}(k)$ with the structure of a subbitorsor of the bitorsor $G^{\text{DR,B}}(k)$.

Proof Let $i : G' \to G$ be a group isomorphism and let $H \subset G$ be a subgroup, and $H' := i^{-1}(H) \subset G'$. Then $i$ restricts to an isomorphism $H' \to H$. This gives rise to bitorsors $H H H'$ and $G G G'$ (see Lemma 1.5); the natural inclusions give rise to a bitorsor inclusion $H H H' \subset G G G'$. A group morphism $\varphi : H \to K$ gives rise to a group morphism $\varphi \circ i : H' \to K$ and to a bitorsor morphism $\varphi \circ \text{pri} : H H H' \to K K K$. Then a subgroup $L \subset K$ and an element $k \in K$ give rise to a bitorsor inclusion $\text{inj}_k : L L L \hookrightarrow K K K$ (see Lemma 1.6). Then $(\varphi \circ \text{pri})^{-1}(\text{inj}_k (L L L))$ is a subbitorsor of $H H H'$, hence of $G G G'$. One has $(\varphi \circ \text{pri})^{-1}(\text{inj}_k (L L L)) = A B C$, where $A := \varphi^{-1}(L) \subset G$. 

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C := i^{-1} \circ \varphi^{-1}(\text{Ad}_k(L)) \subset G', B := \varphi^{-1}(L \cdot k^{-1}) \subset G. This is summarized in the diagram of bitorsors

\[ \begin{array}{ccc}
(i \varphi_{\text{cot}})^{-1}(\text{inj}_k(LL_L)) & \xrightarrow{\text{inj}_k} & H H' \xrightarrow{\varphi_{\text{cot}}} GG' \\
L L_L & \quad & \quad \\
& \xrightarrow{\text{inj}_k} & \quad \quad \\
& & k \quad \\
\end{array} \]

When i := \text{iso}^G : G^B(k) \to G^{\text{DR}}(k), H := G^{\text{DR}}_{\text{lin}}(k), H' := G^B_{\text{lin}}(k), K := (k^x \times k, \cdot), \varphi := \varphi_{\text{quad}} (\text{see [5], proof of Lemma 2.3}), L := k^x = \{(\mu, 0) | \mu \in k^x\}, k := (1, -1/24), one has A = \varphi^{-1}_{\text{quad}}(k^x) = G^{\text{DR}}_{\text{quad}}(k) by [5], Lemma 2.3, C = (\text{iso}^V)^{-1} \circ \varphi^{-1}_{\text{quad}}(\text{Ad}(1, -1/24)(k^x)) = G^{\text{DR}}_{\text{quad}}(k) by Lemma-Definition 3.1. One checks that k^x \cdot k^{-1} = \{(\mu, c) | c = \mu^2/24\}, which shows that B = \varphi^{-1}_{\text{quad}}(k^x \cdot k^{-1}) = G^{\text{DR}, B}(k).

3.2 \text{Stab}(\hat{\Delta}^{\text{VR}, \text{DR}/B}(k)) as a subbitorsor of G^{\text{DR}, B}(k)

Lemma 3.3 The map \( G^B(k) \to \text{Aut}_{\text{k-mod}}(\text{Hom}_{\text{k-mod}_{\text{top}}}(\hat{\mathcal{V}}^B, (\mathcal{V}^B)^{\otimes 2}_{\text{\wedge}})) \) taking \((\mu, g)\) to \( F \mapsto (\Gamma_{\text{aut}}^{\hat{\mathcal{V}}, (1)}(B)) \otimes \circ F \circ (\Gamma_{\text{aut}}^{\mathcal{V}, (1)}(B))^{-1} \) defines an action of the group \( G^B(k) \) on the \( k \)-module \( \text{Hom}_{\text{k-mod}_{\text{top}}}(\hat{\mathcal{V}}^B, (\mathcal{V}^B)^{\otimes 2}_{\text{\wedge}}) \).

Proof Follows from Lemma-Definition 2.8.

Definition 3.4 Set \( \text{Stab}(\hat{\Delta}^{\text{VR}, B}(k)) := \text{Stab}_{G^B(k)}(\hat{\Delta}^{\text{VR}, B}). \) This is a subgroup of \( G^B(k, \otimes) \), equal to the set of \((\mu, g) \in G^B(k)\), such that \( (\Gamma_{\text{aut}}^{\hat{\mathcal{V}}, (1)}(B)) \otimes \circ \hat{\Delta}^{\text{VR}, B} = \hat{\Delta}^{\text{VR}, B} \circ (\Gamma_{\text{aut}}^{\mathcal{V}, (1)}(B))^{-1} \).

In [5], Definitions 2.14 and 2.15, we introduced the subsets \( \text{Stab}(\hat{\Delta}^{\text{VR}, \text{DR}/B}(k)) \) and \( \text{Stab}(\hat{\Delta}^{\text{VR}, B}(k)) \) of \( G^{\text{DR}}(k) \), and in Lemma 2.16 and Theorem 3.1, proved that \( \text{Stab}(\hat{\Delta}^{\text{VR}, \text{DR}/B}(k)) \) is a subgroup of \( G^{\text{DR}}(k) \), and that \( \text{Stab}(\hat{\Delta}^{\text{VR}, \text{DR}/B}(k)) \), equipped with the left action of this group, is a subtorsor of \( G^{\text{DR}, B}(k) \).

Lemma 3.5 The right action of \( G^{\text{DR}}(k) \) on \( G^{\text{DR}}(k) \) restricts to a free and transitive action of \( \text{Stab}(\hat{\Delta}^{\text{VR}, B}(k)) \) on \( \text{Stab}(\hat{\Delta}^{\text{VR}, \text{DR}/B}(k)) \), so that this right action equips \( \text{Stab}(\hat{\Delta}^{\text{VR}, \text{DR}/B}(k)) \) with the structure of a subbitorsor of the bitorsor \( G^{\text{DR}, B}(k). \)

Proof One derives from Sect. 2.4 an action of the bitorsor \( G^{\text{DR}, B}(k) \) on the pair of isomorphic topological \( k \)-algebras \((\hat{\mathcal{V}}^{\text{DR}}, \hat{\mathcal{V}}^B)\). Viewing this as a pair of objects of the tensor category \( \text{k-mod}_{\text{top}}, \) and using Lemma 1.9, one obtains an action of the bitorsor \( G^{\text{DR}, B}(k) \) on the pair of isomorphic \( k \)-modules \((\text{Hom}_{\text{k-mod}_{\text{top}}}(\hat{\mathcal{V}}^{\text{DR}}, (\mathcal{V}^{\text{DR}})^{\otimes 2}_{\text{\wedge}})), \text{Hom}_{\text{k-mod}_{\text{top}}}(\hat{\mathcal{V}}^B, (\mathcal{V}^B)^{\otimes 2}_{\text{\wedge}}))\). Recall that \( (\hat{\Delta}^{\text{VR}, \text{DR}}, \hat{\Delta}^{\text{VR}, B}) \) is a pair of elements of this pair of modules. By Lemma 1.10, this gives rise to the stabilizer subbitorsor \( \text{Stab}_{G^{\text{DR}}(k)}(\hat{\Delta}^{\text{VR}, \text{DR}}) \text{Stab}_{G^{\text{DR}}(k)}(\hat{\Delta}^{\text{VR}, B}) \text{Stab}_{G^{\text{DR}}(k)}(\hat{\Delta}^{\text{VR}, B}) \) of \( G^{\text{DR}, B}(k). \)

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3.3 \( \text{Stab}(\hat{\mathcal{M}}^\text{DR/B})(k) \) as a subbitorsor of \( G^\text{DR,B}(k) \)

**Lemma 3.6** The map \( G^B(k) \to \text{Aut}_k(\text{Hom}_{k}\text{-mod}_{\text{top}}(\mathcal{M}^B, (\mathcal{M}^B)^{\otimes 2})) \) taking \((\mu, g)\) to \( F \mapsto (\Gamma^\text{aut}_{(\mu, g)}(\mathcal{M}^{(10)}.B)^{\otimes 2} \circ F \circ (\Gamma^\text{aut}_{(\mu, g)}(\mathcal{M}^{(10)}.B))^{-1} \) defines an action of the group \( G^B(k) \) on the \( k \)-module \( \text{Hom}_{k}\text{-mod}_{\text{top}}(\mathcal{M}^B, (\mathcal{M}^B)^{\otimes 2}) \).

**Proof** Follows from Lemma-Definition 2.8. \( \square \)

**Definition 3.7** Set \( \text{Stab}(\hat{\mathcal{M}}, B)(k) := \text{Stab}_{G^B(k)}(\hat{\mathcal{M}}, B) \). This is a subgroup of \((G^B(k), \otimes)\), equal to the set of \((\mu, g) \in G^B(k)\), such that \((\Gamma^\text{aut}_{(\mu, g)}(\mathcal{M}^{(10)}.B)^{\otimes 2} \circ \hat{\mathcal{M}} \circ (\Gamma^\text{aut}_{(\mu, g)}(\mathcal{M}^{(10)}.B))^{-1} \).

In [5], Definitions 2.17 and 2.18, we introduced the subsets \( \text{Stab}(\hat{\mathcal{M}}, B)(k) \) and \( \text{Stab}(\hat{\mathcal{M}}, B)(k) \) of \( G^B(k) \), and in Lemma 2.19 and Theorem 3.1, proved that \( \text{Stab}(\hat{\mathcal{M}}, B)(k) \) is a subgroup of \( G^B(k) \), and that \( \text{Stab}(\hat{\mathcal{M}}, B)(k) \), equipped with the left action of this group, is a subtorisor of \( G^B(k) \).

**Lemma 3.8** The right action of \( G^B(k) \) on \( G^B(k) \) restricts to a free and transitive action of \( \text{Stab}(\hat{\mathcal{M}}, B)(k) \) on \( \text{Stab}(\hat{\mathcal{M}}, B)(k) \), so that this right action equips \( \text{Stab}(\hat{\mathcal{M}}, B)(k) \) with the structure of a subbitorsor of the bitorsor \( G^B(k) \).

**Proof** One derives from Sect. 2.4 an action of the bitorsor \( G^B(k) \) on the pair of isomorphic topological \( k \)-algebras \((\mathcal{M}^B, \hat{\mathcal{M}}^B)\). Viewing this as a pair of objects of the tensor category \( k \)-mod_{\text{top}}, and using Lemma 1.9, one obtains an action of the bitorsor \( G^B(k) \) on the pair of isomorphic \( k \)-modules \((\text{Hom}_{k}\text{-mod}_{\text{top}}(\mathcal{M}^B, (\mathcal{M}^B)^{\otimes 2})), \text{Hom}_{k}\text{-mod}_{\text{top}}(\mathcal{M}^B, (\mathcal{M}^B)^{\otimes 2})\)). Recall that \((\hat{\mathcal{M}}, \hat{\mathcal{M}}^B)\) is a pair of elements of this pair of modules. By Lemma 1.10, this gives rise to the stabilizer subbitorsor \( \text{Stab}_{G^B(k)}(\hat{\mathcal{M}}, B) \) of \( G^B(k) \).

3.4 \( DMR^{DR,B}(k) \) as a subbitorsor of \( G^{DR,B}(k) \)

**Lemma-Definition 3.9** Set \( DMR^B(k) := G^B_{\text{quad}}(k) \cap \text{Stab}(\hat{\mathcal{M}}, B)(k) \). Then \( DMR^B(k) \) is a subgroup of \((G^B(k), \otimes)\).

**Proof** This follows from Lemma-Definition 3.1 and from the fact that \( \text{Stab}(\hat{\mathcal{M}}, B)(k) \) is a subgroup of \((G^B(k), \otimes)\). \( \square \)

In [5], Definition 2.12, we introduced the subsets \( DMR^{DR,B}(k) \) of \( G^B(k) \), and in Lemma 2.13, proved that \( DMR^{DR,B}(k) \) is a subgroup of \( G^B(k) \), and that \( DMR^{DR,B}(k) \), equipped with the left action of this group, is a subtorisor of \( G^{DR,B}(k) \).

**Lemma 3.10** The right action of \( G^B(k) \) on \( G^B(k) \) restricts to a free and transitive action of \( DMR^B(k) \) on \( DMR^{DR,B}(k) \), so that this right action equips \( DMR^{DR,B}(k) \) with the structure of a subbitorsor of the bitorsor \( G^{DR,B}(k) \).
By Lemmas 1.4, 3.2 and 3.8, the intersection of the bitorsors $G_{\text{quad}}^{\text{DR},B}(k)$ and $\text{Stab}(\hat{\Delta}^{\mathcal{M},\text{DR}/B})(k)$ is a subbitorsor of $G_{\text{quad}}^{\text{DR},B}(k)$. The result then follows from $DMR_{\text{quad}}^{\text{DR}}(k) = G_{\text{quad}}^{\text{DR},B}(k) \cap \text{Stab}(\hat{\Delta}^{\mathcal{M},\text{DR}})(k) \subset G_{\text{quad}}^{\text{DR},B}(k)$ and $DMR_{\text{quad}}^{\text{DR},B}(k) = G_{\text{quad}}^{\text{DR},B}(k) \cap \text{Stab}(\hat{\Delta}^{\mathcal{M},\text{DR}/B})(k) \subset G^{\text{DR},B}(k)$, which follows from [5], Theorem 3.1, (a).

3.5 $DMR_{\mu}(k)$ as a subbitorsor of $G_{\mu}^{\text{DR},B}(k)$

Lemma-Definition 3.11 Set

$DMR_0^{B}(k) := \{ g \in G(\hat{\mathcal{Y}}^{B}) \mid (g|\log X_0) = (g|\log X_1) = 0 \text{ and } (\Gamma_{\text{aut}}^{\mathcal{M},(10),B}) \circ \hat{\Delta}^{\mathcal{M},B} = \hat{\Delta}^{\mathcal{M},B} \circ \Gamma_{\text{aut}}^{\mathcal{M},(10),B} \}$.

Then $DMR_0^{B}(k)$ is a subgroup of $(G(\hat{\mathcal{Y}}^{B}), \otimes)$.

Proof The map $(\mu, g) \mapsto \mu$ is a group morphism $G^{B}(k) \to k^{\times}$, whose kernel is equal to $(G(\hat{\mathcal{Y}}^{B}), \otimes)$. Then $DMR_0^{B}(k)$ is equal to the intersection of this kernel with $DMR^{B}(k)$, and is therefore the intersection of two subgroups of $G^{B}(k)$.

In [10], Racinet introduced the subsets $DMR_0(k)$ and $DMR_{\mu}(k)$ of $G(\hat{\mathcal{Y}}^{\text{DR}})$ for any $\mu \in k$, and proved that $DMR_0(k)$ is a subgroup of $(G(\hat{\mathcal{Y}}^{\text{DR}}), \otimes)$, and that $DMR_{\mu}(k)$, equipped with the left action of this group, is a subtorsor of $G(\hat{\mathcal{Y}}^{\text{DR}})$, viewed as a trivial torsor over itself. In [5], Lemma 2.14, we identified the torsor $DMR_{\mu}(k)$ with a subtorsor of $DMR_{\text{quad}}^{\text{DR},B}(k)$, viewed as a torsor under the left action of $DMR_{\text{quad}}^{\text{DR}}(k)$.

Lemma 3.12 The right action of $DMR^{B}(k)$ on $DMR_{\text{quad}}^{\text{DR}}(k)$ restricts to a free and transitive action of $DMR_0^{B}(k)$ on $DMR_{\mu}(k)$, so that this right action equips $DMR_0^{B}(k)$ with the structure of a subbitorsor of the bitorsor $DMR_{\text{quad}}^{\text{DR},B}(k)$.

Proof The maps $(\mu, g) \mapsto \mu$ set up a bitorsor morphism from $DMR_{\text{quad}}^{\text{DR}}(k)$ to $k^{\times}$, and $DMR_0(k)$ to $k^{\times}$. Then $\{[\mu]\}_{1}$ is a subbitorsor of the latter torsor. Its preimage is therefore a subbitorsor of the source bitorsor. The result follows from the identification of the preimage of the kernel of $DMR_{\text{quad}}^{\text{DR}}(k) \to k^{\times}$ (resp. $DMR_0(k) \to k^{\times}$) with $DMR_0(k)$ (resp. with $DMR_0^{B}(k)$) and of the preimage of $\mu$ under the map $DMR_{\text{quad}}^{\text{DR},B}(k) \to k^{\times}$ with $DMR_{\mu}(k)$, obtained in the proof of Lemma 2.14 in [5].

3.6 $M(k)$ as a subbitorsor of $G_{\mu}^{\text{DR},B}(k)$

It follows from [2] that the subgroup $G_{\mu}^{\text{op}}(k) \subset G_{\mu}^{B}(k)$ is the group attached to $M(k)$, viewed as a subbitorsor of $G_{\mu}^{\text{DR},B}(k)$ when equipped with the left action of $G_{\mu}^{\text{op}}(k)$. It follows that $M(k)$, equipped with the commuting left and right actions of $G_{\mu}^{B}(k)$ and $G_{\mu}^{\text{op}}$, is a subbitorsor of $G_{\mu}^{\text{DR},B}(k)$. 

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3.7 Groups corresponding to some torsors and their interrelations

Theorem 3.13  (a) The group attached, in the sense of Lemma 1.12, (a), to the subtor- 
sor $\text{Stab}(\hat{\Delta}^{V,DR/B}(k))$ (resp., $\text{Stab}(\hat{\Delta}^{M,DR/B}(k))$, $\text{DMR}^{DR,B}(k)$, $\text{DMR}_{\mu}(k)$) of the 
torsor $G^{DR,B}(k)$ is the subgroup $\text{Stab}(\hat{\Delta}^{V,B}(k))$ (resp., $\text{Stab}(\hat{\Delta}^{M,B}(k))$, $\text{DMR}^{B}(k)$, 
$\text{DMR}_{\mu}^{B}(k)$) of $G^{B}(k)$.

(b) The following inclusions hold between subgroups of $G^{B}(k)$:

$$\text{GT}^{\text{op}}(k) \subset \text{DMR}^{B}(k) \subset \text{Stab}(\hat{\Delta}^{M,B}(k)) \subset \text{Stab}(\hat{\Delta}^{V,B}(k)).$$

Proof  (a) follows from the fact that for any bitorsor $G \times H$, there is a group isomorphism $H \to \text{Aut}_{G}(X)$ (see Sect. 1.2). Then (b) follows from Lemma 1.13 from the present 
paper, Theorem 3.1 in [5] and Lemma-Definition 3.9 from the present paper.  \(\square\)

Remark 3.14  An explanation for the namings ‘de Rham’ and ‘Betti’ for the left and 
right sides of the bitorsor $M(k)$ is given in the two first paragraphs of the introduction 
of [4], which implicitly relies on Lemma 1.11.

3.8 Scheme-theoretic and Lie algebraic aspects

The assignments

$$\{\mathbb{Q} \text{-algebras}\} \ni k \mapsto G^{B}(k), (G(\hat{\mathcal{V}}^{B}), \otimes), \text{GT}(k)^{\text{op}}, \text{DMR}_{??}^{B}(k), \text{Stab}(\hat{\Delta}^{??,DR})(k),$$

(3.8.1)

where ? stands for no index or quad, ?? stands for no index or 0, ??? stands for $\mathcal{M}$ or 
$\mathcal{V}$, are $\mathbb{Q}$-group schemes. Recall that the Lie algebra of a $\mathbb{Q}$-group scheme $k \mapsto G(k)$ 
is defined as $\text{Ker}(G(\mathbb{Q}[\varepsilon]/(\varepsilon^{2})) \to G(\mathbb{Q}))$; it is a $\mathbb{Q}$-Lie algebra. Let

$$g^{B}, g_{0}^{B}, g^{\text{op}}, \text{dmr}^{B}, \text{stab(}\hat{\Delta}^{??,B}),$$

be the Lie algebras of the $\mathbb{Q}$-group schemes (3.8.1).

Denote with an index $\mathbb{Q}$ the objects and morphisms of Sect. 2.1 corresponding to 
k = $\mathbb{Q}$. The primitive part of $\hat{\mathcal{V}}^{B}_{\mathbb{Q}} = (\mathbb{Q}F_{2})^{\wedge}$ with respect to $\hat{\Delta}^{\mathcal{V},B}_{\mathbb{Q}}$ is the complete Lie 

algebra $\text{Lie} F_{2}(\mathbb{Q})$, topologically generated by the free family $\log x_{0}, \log x_{1}$, where 
log is the logarithm map $1 + (\hat{\mathcal{V}}^{B}_{\mathbb{Q}})_+ \to (\hat{\mathcal{V}}^{B}_{\mathbb{Q}})_+$.

Let $D$ be the derivation of $\text{Lie} F_{2}(\mathbb{Q})$ defined by $\log x_{i} \mapsto \log x_{i}$, for $i = 0, 1$. For 
x $\in \text{Lie} F_{2}(\mathbb{Q})$, let $D_{x}$ be the derivation of $\text{Lie} F_{2}(\mathbb{Q})$ defined by $\log x_{0} \mapsto [x, \log x_{0}], \log x_{1} \mapsto 0.$

Lemma 3.15  (see [2])  (a) $g^{B}$ is a complete Lie algebra, of which $g^{\text{op}}$ and $g_{0}^{B}$ are 
complete Lie subalgebras.

(b) $g^{B} = \mathbb{Q} \oplus \text{Lie} F_{2}(\mathbb{Q})$, with Lie bracket

$$((v, x), (v', x')) = v D(x') - v' D(x) + D_{x}(x') - D_{x'}(x) - [x, x'];$$
\[ \mathfrak{g}^B \text{ is the Lie subalgebra } \text{Lie} F_2(\mathbb{Q}). \]

(c) \( \mathfrak{g}^{op} \) is the subspace of \( \mathfrak{g}^B \) defined by relations (5.5) to (5.7) in [2].

For \( (v, x) \in \mathfrak{g}^B \), set
\[
der_{(v,x)}^{\mathcal{V},(1),B} := vD + D_x \in \text{End}_Q(\hat{\mathcal{V}}^B_Q), \quad \text{der}_{(v,x)}^{\mathcal{V},(10),B} := vD + D_x + r_x \in \text{End}_Q(\hat{\mathcal{V}}^B_Q).
\]

Then \( \text{der}_{(v,x)}^{\mathcal{V},(1),B} \) restricts to an endomorphism \( \text{der}_{(v,x)}^{\mathcal{W},(1),B} \in \text{End}_Q(\hat{\mathcal{W}}^B_Q) \), and \( \text{der}_{(v,x)}^{\mathcal{V},(10),B} \) induces an endomorphism \( \text{der}_{(v,x)}^{\mathcal{M},(10),B} \in \text{End}_Q(\hat{\mathcal{M}}^B_Q) \).

For \( x \in \text{Lie} F_2(\mathbb{Q}) \), set
\[
\gamma_x(t) := \sum_{n \geq 1} (-1)^{n+1}(x(\log X_0)^{n-1}\log X_1)t^n / n \in \mathbb{Q}[[t]]
\]
and for \( (v, x) \in \mathfrak{g}^B \), set
\[
\Gamma_{\text{der}}^{\mathcal{W},(1),B} := \text{der}_{(v,x)}^{\mathcal{W},(1),B} - \text{ad}_{\gamma_x(-\log X_1)} \in \text{End}_Q(\hat{\mathcal{W}}^B_Q), \quad \Gamma_{\text{der}}^{\mathcal{M},(10),B} := \text{der}_{(v,x)}^{\mathcal{M},(10),B} - \ell_{\gamma_x(-\log X_1)} \in \text{End}_Q(\hat{\mathcal{M}}^B_Q).
\]

**Lemma 3.16** (a) \( \mathfrak{g}^B_{\text{quad}}, \mathfrak{d} \mathfrak{m}^B_? \) and \( \text{stab}(\hat{\Delta}^{??,B}) \) are complete Lie subalgebras of \( \mathfrak{g}^B \).

(b) \( \mathfrak{g}^B_{\text{quad}} = \{(v, x) \in \mathfrak{g}^B | (x)\log X_0 = (x)\log X_1 = 0, v = 12(x)\log X_0\log X_1) \} \).

(c) \( \text{stab}(\hat{\Delta}^{\mathcal{W},B}) \) is the set of elements \( (v, x) \in \mathfrak{g}^B \) such that
\[
(\Gamma_{\text{der}}^{\mathcal{W},(1),B} \otimes \text{id} + \text{id} \otimes \Gamma_{\text{der}}^{\mathcal{W},(1),B}) \circ \hat{\Delta}^{\mathcal{W},B} = \hat{\Delta}^{\mathcal{W},B} \circ \Gamma_{\text{der}}^{\mathcal{W},(1),B}
\]

(equality in \( \text{Hom}_Q(\hat{\mathcal{W}}^B_Q, (\mathcal{W}_Q^B)^{\otimes 2,\wedge}) \)).

(d) \( \text{stab}(\hat{\Delta}^{\mathcal{M},B}) \) is the set of elements \( (v, x) \in \mathfrak{g}^B \) such that
\[
(\Gamma_{\text{der}}^{\mathcal{M},(10),B} \otimes \text{id} + \text{id} \otimes \Gamma_{\text{der}}^{\mathcal{M},(10),B}) \circ \hat{\Delta}^{\mathcal{M},B} = \hat{\Delta}^{\mathcal{M},B} \circ \Gamma_{\text{der}}^{\mathcal{M},(10),B}
\]

(equality in \( \text{Hom}_Q(\hat{\mathcal{M}}^B_Q, (\mathcal{M}_Q^B)^{\otimes 2,\wedge}) \)).

(e) \( \mathfrak{d} \mathfrak{m}^B = \mathfrak{g}^B_{\text{quad}} \cap \text{stab}(\hat{\Delta}^{\mathcal{M},B}) \) and \( \mathfrak{d} \mathfrak{m}^0 = \mathfrak{g}^0_B \cap \mathfrak{g}^B_{\text{quad}} \cap \text{stab}(\hat{\Delta}^{\mathcal{M},B}). \)

**Proof** (b), (c), (d) are obtained by linearization. (e) follows from the equalities \( \text{DMR}^B(k) = \mathfrak{g}^B_{\text{quad}}(k) \cap \text{Stab}(\hat{\Delta}^{\mathcal{M},\text{DR}/B})_k \) and \( \text{DMR}^0_B(k) = \text{DMR}^B(k) \cap (\hat{\mathcal{G}}(\hat{\mathcal{V}}^B), \otimes) \) (see Lemmas 3.9, 3.11). (a) follows from (b)-(e).

**Remark 3.17** The endomorphisms \( \Gamma_{\text{der}}^{??,B}_{(v,x)} \) are Lie algebraic analogues of the automorphisms \( \text{aut}^{??,B}_{(\mu,y)} \), \( \text{aut}^{??,B}_{(\nu,x)} \), where \( ? \) (resp. \( ?? \)) takes the values \( (\mathcal{V}, B), (\mathcal{M}, \alpha) \), with \( (\mathcal{X}, \alpha) \) in \( \{(\mathcal{V}, (1)), (\mathcal{V}, (10)), (\mathcal{W}, (1)), (\mathcal{M}, (10))\} \) (resp. in \( \{(\mathcal{V}, (1)), (\mathcal{M}, (10))\} \)), in particular, any of the maps taking \( (v, x) \) to one of these endomorphisms is a representation of \( \mathfrak{g}^B \).
Corollary 3.18 The following inclusions between Lie subalgebras of $\mathfrak{g}^B$ hold: (a) $\mathfrak{gl}^B \subset \mathfrak{d}^B$, (b) $\text{stab}(\Delta^M, B) \subset \text{stab}(\Delta^W, B)$.

Proof This follows directly from Theorem 3.13.

3.9 Relation with Hopf algebra and coalgebra isomorphism bitorsors

Recall that a module-coalgebra over a (topological) $k$-Hopf algebra $(W, \Delta_W)$ is a coassociative counital (topological) $k$-coalgebra $(M, \Delta_M)$, equipped with a map $\text{act}: W \otimes M \rightarrow M$ which both defines an action of $W$ (viewed as an associative $k$-algebra) on $M$ (viewed as a $k$-module) and a morphism of counital $k$-coalgebras (where $W \otimes M$ being equipped with the tensor product coalgebra structure); explicitly, $\text{act} \circ (m_W \otimes \text{id}_M) = \text{act} \circ (\text{id}_W \otimes \text{act})$ (equality of maps $W \otimes W \otimes M \rightarrow M$) and $\Delta_W \circ \text{act} = (\text{act} \otimes \text{act}) \circ (\text{id}_W \otimes \sigma \otimes \text{id}_M) \circ (\Delta_W \otimes \Delta_M)$ (equality of maps $W \otimes M \rightarrow M \otimes M$) where $m_W : W \otimes W \rightarrow W$ is the product map of $W$ and $\sigma : W \otimes M \rightarrow M \otimes W$ is the map of permutation of factors. Denote by $k$-HAMC the category of pairs $(W, \Delta_W), (M, \Delta_M)$ of a $k$-Hopf algebra and a module-coalgebra over it. It fits in a commutative diagram of forgetful functors

$$
\begin{array}{ccc}
\mathbf{k}\text{-coalg} & \xleftarrow{\text{forgetful}} & \mathbf{k}\text{-HAMC} \\
\downarrow & & \downarrow \\
\mathbf{k}\text{-mod} & \xleftarrow{\text{forgetful}} & \mathbf{k}\text{-alg-mod} \\
\mathbf{k}\text{-alg} & & \\
\end{array}
$$

(3.9.1)

where $k$-Hopf (resp. $k$-coalg, $k$-mod) is the category of $k$-Hopf algebras (resp. $k$-coalgebras, $k$-modules), in which the vertical functors are faithful. A pair of isomorphic objects in $k$-HAMC then induces a commutative diagram of isomorphism bitorsors, where the vertical maps are bitorsor inclusions.

Lemma 3.19 Let $((W^\omega, \Delta^W_\omega), (M^\omega, \Delta^M_\omega)), \omega \in \{B, DR\}$ be a pair of isomorphic objects in $k$-HAMC such that $M^B$ is free of rank one over $W^B$ with generator $1^B_M$ and such that $\Delta^B_M(1^B_M) \in ((W^B)^{\otimes 2}) \times (1^B_M)^{\otimes 2}$. Then the commutative diagram of bitorsors

$$
\begin{array}{ccc}
\text{Iso}_{k\text{-coalg}}((M, \Delta_M)^{DR/B}) & \xleftarrow{\text{forgetful}} & \text{Iso}_{k\text{-HAMC}}(((W, \Delta_W), (M, \Delta_M))^{DR/B}) \\
\downarrow & & \downarrow \\
\text{Iso}_{k\text{-mod}}((M^{DR/B}) & \xleftarrow{\text{forgetful}} & \text{Iso}_{k\text{-alg-mod}}((W, M)^{DR/B}) \\
\end{array}
$$

induced by the left square of (3.9.1) is Cartesian (see Definition 1.3).
Proof The argument follows that of [5], Section 3.4. Let \((c_W, c_M) \in \text{Iso}_k\text{-alg-mod} ((W, M)^{\text{DR}/B})\) be such that \(c_M \in \text{Iso}_k\text{-coalg} ((M, \Delta_M)^{\text{DR}/B})\). Then for \(w \in W^B\),

\[
c_{W}^{\otimes 2}(\Delta^B_W(w)) \cdot c_{M}^{\otimes 2}(\Delta^B_M(1_M^B)) = c_{W}^{\otimes 2}(\Delta^B_W(w) \cdot \Delta^B_M(1_M^B)) = c_{W}^{\otimes 2}(\Delta^B_M(1_M^B)) = \Delta^B_M(c_M(w \cdot 1_M^B)) = \Delta^B_M(c_M(w)) \cdot \Delta^B_M(c_M(1_M^B)) = \Delta^B_M(c_M(w)) \cdot c_{M}^{\otimes 2}(\Delta^B_M(1_M^B))
\]

by the various axioms. Then \(c_{W}^{\otimes 2}(\Delta^B_M(1_M^B)) \in c_{W}^{\otimes 2}((W^B)^{\otimes 2}) \cdot (1_M^B)^{\otimes 2} = c_{W}^{\otimes 2}((W^B)^{\otimes 2}) \cdot c_M(1_M^B)^{\otimes 2} \subset ((W^{\text{DR}})^{\otimes 2}) \cdot c_M(1_M^B)^{\otimes 2}\) which together with the fact that \(c_M(1_M^B)\) is a generator of \(M^{\text{DR}}\) as a free \(W^{\text{DR}}\)-module implies \(c_{W}^{\otimes 2}(\Delta^B_W(w))\) is a generator of \(W^{\text{DR}}\), therefore \(c_W^{\otimes 2} \circ \Delta^B_W \circ c_W^{-1} = \Delta^B_W\). It follows that the Hopf algebra structure on \(W^B\) pulled back from the Hopf algebra structure of \(W^{\text{DR}}\) has coproduct \(\Delta^B_W\). The uniqueness of the counit in a coalgebra in a symmetric monoidal category (proved by dualizing the standard argument proving the uniqueness of the unit in an algebra) and of the antipode in a bialgebra in such a category (based on the argument of [12], p. 71) then imply that the pull-back on \(W^B\) of the Hopf algebra structure of \(W^{\text{DR}}\) is the Hopf algebra structure of \(W^B\), therefore \(c_w \in \text{Iso}_k\text{-Hopf} ((W^{\text{DR}/B})\) so \((c_w, c_M) \in \text{Iso}_k\text{-HAMC} ((W, \Delta^B_W), (M, \Delta^B_M))^{DR/B}\).

Lemma 3.20 (a) For \(\omega \in \{B, DR\}, ((\hat{\mathcal{V}}^\omega, \hat{\Delta}^W_\omega), (\hat{\mathcal{M}}^\omega, \hat{\Delta}^M_\omega))\) is an object in \(k\text{-HAMC}\).

(b) The map \((\mu, \Phi) \mapsto (\Gamma \text{comp}_{(\mu, \Phi)}^{V,(1)}, \Gamma \text{comp}_{(\mu, \Phi)}^{M,(10)})\) defines a bitorsor morphism

\[
G^{\text{DR,B}}(k) \rightarrow \text{Iso}_k\text{-alg-mod}((\hat{\mathcal{V}}, \hat{\mathcal{M}})^{\text{DR/B}}).
\]

Its composition with the bitorsor morphism to \(\text{Iso}_k\text{-alg}((\hat{\mathcal{V}}^{\text{DR/B}})\) (resp. \(\text{Iso}_k\text{-mod}((\hat{\mathcal{M}}^{\text{DR/B}}))\) is \((\mu, \Phi) \mapsto \Gamma \text{comp}_{(\mu, \Phi)}^{V,(1)}\) (resp. \((\mu, \Phi) \mapsto \Gamma \text{comp}_{(\mu, \Phi)}^{M,(10)}\).

Proof The \(\omega = DR\) part of (a) follows from Sections 1.1 and 1.2 in [4], and its \(\omega = B\) part follows from Sections 2.1 and 2.4 in loc. cit. (b) follows from [5], Lemmas 1.21 to 1.24.

Lemma 3.20 leads to a diagram of bitorsors

\[
\begin{array}{cccc}
\text{Iso}_k\text{-coalg}((\hat{\mathcal{M}}, \hat{\Delta}^M)^{\text{DR/B}}) & \rightarrow & \text{Iso}_k\text{-HAMC}(((\hat{\mathcal{V}}, \hat{\Delta}^W_\omega), (\hat{\mathcal{M}}, \hat{\Delta}^M_\omega))^{\text{DR/B}}) & \rightarrow & \text{Iso}_k\text{-Hopf}((\hat{\mathcal{V}}, \hat{\Delta}^V_\omega)^{\text{DR/B}}) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Iso}_k\text{-mod}((\hat{\mathcal{M}})^{\text{DR/B}}) & \rightarrow & \text{Iso}_k\text{-alg-mod}((\hat{\mathcal{V}})^{\text{DR/B}}) & \rightarrow & \text{Iso}_k\text{-alg}((\hat{\mathcal{V}}^{\text{DR/B}}) \\
\downarrow & & \downarrow & & \downarrow \\
G^{\text{DR,B}}(k) & & & & \\
\end{array}
\]

(3.9.2)
which, by denoting each of the bitorsors \( \text{Isok}_C(X^{\text{DR/B}}) \) of (3.9.2) by \( T_C \) leads to a diagram of preimage subbitorsors of \( G^{\text{DR,B}}(k) \):

\[
G^{\text{DR,B}}(k) \times_{\text{mod}} T_{\text{coalg}} \supset G^{\text{DR,B}}(k) \times_{\text{alg-mod}} T_{\text{HAMC}} \subset G^{\text{DR,B}}(k) \times_{\text{alg}} T_{\text{Hopf}}
\]

(3.9.3)

**Lemma 3.21**  
(a) The subbitorsor \( G^{\text{DR,B}}(k) \times_{\text{mod}} T_{\text{coalg}} \) of \( G^{\text{DR,B}}(k) \) coincides with \( \text{Stab}(\hat{\Delta}^{M,\text{DR/B}})(k) \).

(b) The subbitorsor \( G^{\text{DR,B}}(k) \times_{\text{alg}} T_{\text{Hopf}} \) of \( G^{\text{DR,B}}(k) \) coincides with \( \text{Stab}(\hat{\Delta}^{W,\text{DR/B}})(k) \).

(c) The subbitorsors \( G^{\text{DR,B}}(k) \times_{\text{mod}} T_{\text{coalg}} \) and \( G^{\text{DR,B}}(k) \times_{\text{alg-mod}} T_{\text{HAMC}} \) of \( G^{\text{DR,B}}(k) \) are equal.

**Proof**  
(a) Let \((\mu, \Phi) \in G^{\text{DR,B}}(k)\). Then \((\mu, \Phi) \in G^{\text{DR,B}}(k) \times_{\text{mod}} T_{\text{coalg}} \) iff \( \Gamma \text{comp}_{\mu, \Phi}^{M,(10)} : (\hat{M}^B, \Delta^{M,B}) \to (\hat{\Delta}^{M,\text{DR}}, \hat{\Delta}^{M,\text{DR}}) \) is a coalgebra isomorphism. This implies the equality \( \Gamma \text{comp}_{\mu, \Phi}^{M,(10)} \otimes 2 \circ \Delta^{M,B} \circ (\Gamma \text{comp}_{\mu, \Phi}^{M,(10)})^{-1} = \Delta^{M,\text{DR}} \). Therefore \((\mu, \Phi) \in \text{Stab}(\hat{\Delta}^{M,\text{DR/B}})\).

Conversely, if \((\mu, \Phi) \in \text{Stab}(\hat{\Delta}^{M,\text{DR/B}})\), then the counital coalgebra structure on \( \hat{M}^B \) pulled back from that of \( \hat{\Delta}^{M,\text{DR}} \) has coproduct \( \Delta^{M,B} \). By the uniqueness of the counit in a coalgebra, the counit of the pulled back coalgebra structure necessarily coincides with that of \( \hat{M}^B \), therefore \( \Gamma \text{comp}_{\mu, \Phi}^{M,(10)} \) is a coalgebra isomorphism, so that \((\mu, \Phi) \in G^{\text{DR,B}}(k) \times_{\text{mod}} T_{\text{coalg}}\).

(b) The proof is similar to (a), using in addition to the uniqueness of the counit in a coalgebra, the uniqueness of the antipode in a bialgebra (see the argument of Lemma 3.19).

(c) Let \((\mu, \Phi) \in G^{\text{DR,B}}(k)\). Then \((\mu, \Phi) \in G^{\text{DR,B}}(k) \times_{\text{alg-mod}} T_{\text{HAMC}} \) iff \( (\Gamma \text{comp}_{\mu, \Phi}^{M,(10)}, \Gamma \text{comp}_{\mu, \Phi}^{W,(1)}) \) belongs to \( T_{\text{HAMC}} \). By Lemma 3.19, this condition is equivalent to the conjunction of

\[
(\Gamma \text{comp}_{\mu, \Phi}^{M,(10)}, \Gamma \text{comp}_{\mu, \Phi}^{W,(1)}) \in T_{\text{alg-mod}}
\]

and \( \Gamma \text{comp}_{\mu, \Phi}^{M,(10)} \in T_{\text{coalg}} \). Since \( (\Gamma \text{comp}_{\mu, \Phi}^{W,(1)}, \Gamma \text{comp}_{\mu, \Phi}^{M,(10)}) \) belongs to \( T_{\text{alg-mod}} \), this conjunction is equivalent to \( \Gamma \text{comp}_{\mu, \Phi}^{M,(10)} \in T_{\text{coalg}} \), i.e. \((\mu, \Phi) \in G^{\text{DR,B}}(k) \times_{\text{alg}} T_{\text{Hopf}}\).
morphism \( M(k) \to \text{Iso}_k\text{-HAMC}(((\hat{W}, \hat{M}), (\hat{\Delta}^W, \hat{\Delta}^M))^{DR/B}) \) such that the diagram

\[
\begin{array}{ccc}
M(k) & \longrightarrow & \text{Iso}_k\text{-HAMC}(((\hat{W}, \hat{M}), (\hat{\Delta}^W, \hat{\Delta}^M))^{DR/B}) \\
\downarrow & & \downarrow \\
G^{DR,B}(k) & \longrightarrow & \text{Iso}_{k}\text{-alg-mod}((\hat{W}, \hat{M})^{DR/B})
\end{array}
\]

commutes.

### 4 Equivalent definitions of \( \text{DMR}^B(k) \) and its Lie algebra

In this section, we prove that the group \( \text{DMR}^B(k) \) can be given by a definition, alternative to Lemma-Definition 3.9 (Sect. 4.1, Theorem 4.5). This result should be viewed as a ‘Betti’ counterpart to [3], Theorem 1.2 and [5], Lemma 3.8. In Sect. 4.2, we draw the Lie algebraic consequence of this result, which is a counterpart of [3], Theorem 3.1 and [5], Corollary 3.12, (b).

#### 4.1 Equivalent definition of \( \text{DMR}^B(k) \)

The following results are analogues of Lemmas 1.17 and 3.5 from [5].

**Lemma 4.1** The map \( (\mu, g) \mapsto \Gamma_{\text{aut}}^{M,(10),B}(\mu, g) \) exhibits the following compatibility with the map \( (\lambda, \varphi) \mapsto \Gamma_{\text{comp}}^{M,(10)}(\lambda, \varphi) \) and with the right action of \( G^B(k) \) on \( G^{DR}(k) \) (see Lemma-Definition 2.11): for \( (\lambda, \varphi) \in G^{DR}(k) \) and \( (\mu, g) \in G^B(k) \), one has

\[
\Gamma_{\text{comp}}^{M,(10)}(\lambda, \varphi) \circ \Gamma_{\text{aut}}^{M,(10),B}(\mu, g) = \Gamma_{\text{comp}}^{M,(10)}(\lambda, \varphi) \circ \Gamma_{\text{aut}}^{M,(10),B}(\mu, g)
\]

**Proof** One combines the identity \( \Gamma_{\text{aut}}^{M,(10),B}(\mu, g) = (\text{iso}^M)^{-1} \circ \Gamma_{\text{aut}}^{M,(10),DR}(\mu, g) \circ \text{iso}^M \) with the identity (1.7.2) from the proof of Lemma 1.17 in [5] and with the group morphism property of the map \( (G^{DR}(k), \oplus) \to \text{Aut}_{k}\text{-mod}(\hat{M}^{DR}) \), \( (\mu, g) \mapsto \Gamma_{\text{aut}}^{M,(10),DR}(\mu, g) \) proved there. \( \square \)

**Lemma 4.2** Let \( (\mu, g) \in G^B(k) \), then \( \Gamma_{\text{aut}}^{M,(10),B}(1_B) = (\Gamma_{g}(-\log X_1)^{-1} \cdot g) \cdot 1_B \).

**Proof** One has \( \Gamma_{\text{aut}}^{M,(10),B}(1_B) = g \) by (2.1.2), which by Lemma-Definition 2.4 implies \( \text{aut}^{M,(10),B}(1_B) = g \cdot 1_B \). Then Lemma-Definition 2.8 implies the result. \( \square \)

**Lemma 4.3** Let \( (\lambda, \varphi) \in M(k) \). If \( (\mu, g) \in G^B(k) \), then the topological \( k \)-module isomorphism \( \Gamma_{\text{comp}}^{M,(10)}(\lambda, \varphi) : \hat{M}^B \to \hat{M}^{DR} \) takes \( (\Gamma_{g}^{-1}(-\log X_1) \cdot g) \cdot 1_B \) to

\[
(\Gamma_{\varphi \circ (\lambda \cdot \text{iso}^G(g))}^{-1}(-e_1) \cdot (\varphi \oplus (\lambda \cdot \text{iso}^G(g)))) \cdot 1_B
\]

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(with the notation as in [5], Section 1.6).

**Proof** One has

\[
\Gamma^{-1}\varphi \oplus (\lambda \circ \text{iso}^G(g)) (-e_1) \cdot (\varphi \oplus (\lambda \circ \text{iso}^G(g))) \cdot 1_{DR} = \Gamma_{\text{comp}}^{\mathcal{M},(10)}(\lambda, \varphi), \text{iso}^G(\mu, g) 1_B = \Gamma_{\text{comp}}^{\mathcal{M},(10)}(\lambda, \varphi) \circ \Gamma_{\text{aut}}^{\mathcal{M},(10),B}(1_B) = \Gamma_{\text{comp}}^{\mathcal{M},(10)}((\Gamma_{-1}^{-1}(-\log X_1) \cdot g) \cdot 1_B),
\]

where the first equality follows from applying Lemma 3.5 from [5] to \((\lambda, \varphi) \oplus \text{iso}^G(\mu, g) = (\lambda, \mu, \varphi \oplus (\lambda \circ \text{iso}^G(g)))\) (see [5], (1.6.2)), the second equality follows from applying Lemma 4.1 to 1, and the third one follows from applying Lemma 4.2 to \((\mu, g)\).

**Lemma 4.4** Let \((\lambda, \varphi) \in M(k)\). The topological \(k\)-module isomorphism \(\Gamma_{\text{comp}}^{\mathcal{M},(10)} : \hat{\mathcal{M}}^B \sim \mathcal{M}_{\text{DR}}^B\) restricts to a bijection \(G(\hat{\mathcal{M}}^B) \sim G(\hat{\mathcal{M}}_{\text{DR}}^B)\), where \(G\) denotes the group-like parts of \(\hat{\mathcal{M}}^B\) (resp. \(\hat{\mathcal{M}}_{\text{DR}}^B\)) for \(\hat{\mathcal{M}}^B\) (resp. \(\hat{\mathcal{M}}_{\text{DR}}^B\)).

**Proof** This follows from the fact that \(\Gamma_{\text{comp}}^{\mathcal{M},(10)}(\lambda, \varphi)\) intertwines \(\hat{\mathcal{M}}^B\) and \(\hat{\mathcal{M}}_{\text{DR}}^B\), see [4], Theorem 3.1 (a), (b) and Definition 2.17. \(\square\)

**Theorem 4.5** DMR\(^B\) \((k)\) is equal to the subset of \(G^B(k)\) of pairs \((\mu, g)\) satisfying the following conditions:

1. \((1)\) \((\Gamma_{-1}^{-1}(-\log X_1) \cdot g) \cdot 1_B \in G(\hat{\mathcal{M}}^B),\) where \(\Gamma_g\) is as in Definition 2.5;
2. \((2)\) \((g|\log X_0) = (g|\log X_0) = 0\) and \(\mu^2 = 1 + 24(g|\log X_0|\log X_1)\).

**Proof** It follows from [2], Proposition 5.3 that \(\mathcal{M}_1(Q) = M(Q) \cap (\{1\} \times G(\hat{\mathcal{M}}_{\text{DR}}))\) is nonempty. Let us denote by \((1, \varphi) \in M(k)\) the image of an element of \(\mathcal{M}_1(Q)\). Then, by [5], Theorem 3.1 (a), (1, \varphi) \in DMR\(^{DR,B}\) \((k)\).

Let \((\mu, g) \in G^B(k)\). The right torsor property of DMR\(^{DR,B}\) \((k)\) under the action of \(\text{DMR}^B\) \((k)\) as in Lemma-Definition 2.11 (see Lemma 3.10) implies the equivalence

\[
((\mu, g) \in \text{DMR}^B(k)) \iff ((1, \varphi) \oplus \text{iso}^G(\mu, g) \in \text{DMR}^{DR,B}(k));
\]

the equality \((1, \varphi) \oplus \text{iso}^G(\mu, g) = (\mu, \varphi \oplus \text{iso}^G(g))\) (see [5], (1.6.2)) and [5], Definition 2.12 implies the equivalence

\[
((1, \varphi) \oplus \text{iso}^G(\mu, g) \in \text{DMR}^{DR,B}(k)) \iff ((a) \text{ and } (b)),
\]

where \((a), (b)\) are the following statements:

(a) \((\Gamma_{-1}^{-1}(-e_1) \cdot (\varphi \oplus \text{iso}^G(g))) \cdot 1_{DR} \in G(\hat{\mathcal{M}}_{\text{DR}})\)

(b) \((1, \varphi) \oplus \text{iso}^G(\mu, g) \in G^{DR,B}(k)\).

The fact that \(G^{B}(k)_{G^{B}(k)}\) is a subtorsor of the right torsor \(G^{DR}(k)_{G^{B}(k)}\) and the inclusions \((1, \varphi) \in \text{DMR}^{DR,B}(k) \subset G^{DR,B}(k)\) imply the equivalence

\[
(b) \iff ((\mu, g) \in G^{B}(k)).
\]
Lemmas 4.3 and 4.4 imply the equivalence

\[(a) \iff [(\Gamma_g^{-1}(-e_1) \cdot g) \cdot 1_B \in \mathcal{G}(\hat{M}^B)]\] .

The above equivalences combine into \((g \in \text{DMR}^B(\mathbf{k})) \iff (((\mu, g) \in G^B_{\text{quad}}(\mathbf{k}))\) and \((\Gamma_g^{-1}(-e_1) \cdot g) \cdot 1_B \in \mathcal{G}(\hat{M}^B))\), which by Lemma-Definition 3.1 yields the announced equivalence. \(\square\)

**Corollary 4.6** The subset of \(G^B(\mathbf{k})\) defined by the conditions (1) and (2) of Theorem 4.5 is a subgroup of \(G^B(\mathbf{k})\).

**Proof** By Theorem 4.5, this subset is equal to \(\text{DMR}^B(\mathbf{k})\), which is a subgroup of \(G^B(\mathbf{k})\) by Lemma-Definition 3.9. \(\square\)

**Remark 4.7** In [10], it is proved that a subset of \(G^{\text{DR}}(\mathbf{k})\), defined by de Rham analogues of conditions (1) and (2) of Theorem 4.5, is a subgroup. Corollary 4.6 is the ‘Betti side’ counterpart of this result; it relies on the ‘de Rham side’ results of [3, 10], as well as on the isomorphisms between ‘de Rham side’ and ‘Betti side’ objects. The authors do not know a ‘purely Betti-side’ proof of this statement (i.e., not relying on Betti-de Rham isomorphisms).

**Remark 4.8** By [4] Lemma 9.5 and Remark 9.6, we have

\[\Gamma_{\varphi}(t)\Gamma_{\varphi}(-t) = \frac{\mu t}{e^{\mu t/2} - e^{-\mu t/2}}\]

for \((\mu, \varphi) \in \text{DMR}^{\text{DR},B}(\mathbf{k})\). For \((\mu', \varphi') := (\mu, \varphi) \cdot (\lambda, g) \in \text{DMR}^{\text{DR},B}(\mathbf{k})\) with \((\mu, \varphi) \in \text{DMR}^{\text{DR},B}(\mathbf{k})\) and \((\lambda, g) \in \text{DMR}^B(\mathbf{k})\), we have \(\mu' = \mu \lambda\) and \((\varphi'\cdot e_{-1}^{k-1}e_1) = (\varphi\cdot e_{-1}^{k-1}e_1) + \mu k(g((\log X_0)^{k-1}((\log X_1))).\) Hence we have \(\Gamma_{\varphi'}(t)\Gamma_{\varphi}(-t) = \Gamma_{\varphi}(t)\Gamma_g(\mu t)\Gamma_{\varphi}(-t)\Gamma_g(-\mu t)\), from which we learn that

\[\Gamma_{\varphi}(t)\Gamma_{\varphi}(-t) = \frac{e^{t/2} - e^{-t/2}}{e^{\lambda t/2} - e^{-\lambda t/2}}\]

for any \((\lambda, g) \in \text{DMR}^B(\mathbf{k})\).

### 4.2 Equivalent definition of \(\text{drt}^B\)

For \(x \in \text{Lie}F_2(\mathbb{Q})\), set

\[\gamma_x(t) := \sum_{n \geq 1} (-1)^n (x|((\log X_0)^{n-1}((\log X_1))/\mathbb{Q}[t]).\]
Proposition 4.9 Set \( \mathcal{P}(\hat{\mathcal{M}}^B_Q) := \{ m \in \hat{\mathcal{M}}^B_Q | \Delta^{\mathcal{M},B}(m) = m \otimes 1_B + 1_B \otimes m \} \), then one has
\[
\varDelta^B = \{(v, x) \in g^B | (x|\log X_0) = (x|\log X_1) = 0, \ v = 12(x|\log X_0\log X_1), \\
(x + \gamma_x(-\log X_1)) \cdot 1_B \in \mathcal{P}(\hat{\mathcal{M}}^B_Q)\}.
\]

Proof This follows from the combination of Theorem 4.5 and the identification of \( \partial \mathfrak{m}^B \) with the kernel of the group morphism \( DMR^B(\mathbb{Q}[\epsilon]/(\epsilon^2)) \to DMR^B(\mathbb{Q}) \).

5 A discrete group \( DMR^B \)

In Sect. 5.1, we define a discrete group \( DMR^B \), which is an analogue of the discrete counterpart \( GT \) of the group scheme \( GT(-) \), and we show the equality of these groups in Sect. 5.2 (see Proposition 5.10).

5.1 Definition of \( DMR^B \)

Recall the group inclusions
\[
GT(\mathbb{Q})^\text{op} \hookrightarrow DMR^B(\mathbb{Q}) \hookrightarrow \mathbb{Q}^\times \ltimes F_2(\mathbb{Q}).
\]
The last of these groups contains the semigroup \( \{\pm 1\} \ltimes F_2 \). Then \( GT := GT(\mathbb{Q}) \cap ((\pm 1) \ltimes F_2)^\text{op} \) is a semigroup (see [2], Section 4), equal to \( \{\pm 1\} \) (see [2], Proposition 4.1).

Define similarly a semigroup
\[
DMR^B := DMR^B(\mathbb{Q}) \cap (\{\pm 1\} \ltimes F_2).
\]

5.2 Computation of \( DMR^B \)

Let \( ev^V : \mathcal{W}^B_Q \to \mathbb{Q}F_1 \) be the algebra morphism induced by \( X_0 \mapsto 1, X_1 \mapsto X \) (\( \mathbb{Q}F_1 \) being the algebra of the free group \( F_1 \) with one generator \( X \)). Since it takes \( X_0 - 1 \) to 0, it induces a module morphism \( ev^M : \mathcal{M}^B_Q \to \mathbb{Q}F_1 \) compatible with the algebra morphism \( ev^V \). We denote by \( ev^W : \mathcal{W}^B_Q \to \mathbb{Q}F_1 \) the restriction of \( ev^V \) to \( \mathcal{W}^B_Q \). Then \( ev^M \) is also compatible with the algebra morphism \( ev^W \).

Lemma 5.1 (a) \( ev^W \) is a Hopf algebra morphism, its source being equipped with \( \Delta^{\mathcal{W},B} \) and its target with the group Hopf algebra structure.

(b) \( ev^M \) is a coalgebra morphism, its source being equipped with \( \Delta^{\mathcal{M},B} \) and its target with the same structure as above.

Proof By [4], Propositions 2.3 and 2.4, \( \mathcal{W}^B_Q \) is generated by the elements \( Y_n^\pm, X_1^\pm, X_1^\pm, X_1^\pm, Y_n^\pm \to 0 \) and the coproducts are such that \( X_1^\pm \mapsto X_1^\pm \otimes X_1^\pm, Y_n^\pm \mapsto Y_n^\pm \otimes 1 + 1 \otimes \).

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\[ Y_n^\pm + \sum_{n' + n'' = n} Y_{n'}^\pm \otimes Y_{n''}^\pm \text{ and } X_{n}^\pm \mapsto X_{n+1}^\pm \otimes X_{n+1}^\pm \]; all this implies (a). (a) implies the commutativity of the right square in

\[
\begin{array}{ccc}
\mathcal{M}_Q^B & \xrightarrow{(-) \cdot 1^B} & W_Q^B \\
\Delta^\mathcal{M}.B & \downarrow & \Delta^W.B \\
(\mathcal{M}_Q^B) \otimes_2 & \xrightarrow{(-) \cdot 1^B \otimes_2} & (W_Q^B) \otimes_2 \\
\end{array}
\]

where \(\Delta_{Q,F_1}\) is the coproduct of \(QF_1\), while the commutativity of the left square follows from the definition of \(\Delta^\mathcal{M}.B\). (b) then follows from the combination of these squares, and the fact that \((-) \cdot 1^B\) is a \(Q\)-vector space isomorphism.

Since \(ev^V\) is the algebra morphism underlying a group morphism \(F_2 \rightarrow F_1\), it induces an algebra morphism \(ev^{V,\wedge} : \hat{V}_Q^B \rightarrow (QF_1)^\wedge\). It follows that the morphisms \(ev^\mathcal{X} : \mathcal{X} \in \{W, \mathcal{M}\}\) induce morphisms \(ev^{\mathcal{X},\wedge}\) between the completions of their sources and targets. We denote by \(iso : (QF_1)^\wedge \rightarrow Q[[t]]\) the isomorphism of topological Hopf algebras induced by \(X \mapsto e^t\).

**Lemma 5.2** If \(g \in F_2\) is such that \((\Gamma^{-1}_g(-\log X_1)) \cdot 1_B \in \mathcal{G}(\hat{\mathcal{M}}_Q^B)\), then there exists \(\lambda \in \mathbb{Q}\) such that \(\Gamma_g(t) = e^{\lambda t}\) and \(g \cdot 1_B \in \mathcal{G}(\mathcal{M}_Q^B)\).

**Proof** There exists a unique collection \(n, (a_i)_{i \in [1,n]}, (b_i)_{i \in [1,n]}, \alpha, \beta\), where \(n \geq 0\), the \(a_i, b_i\) are nonzero integers, and \(\alpha, \beta\) are integers, such that

\[ g = X_1^\alpha \prod_{i=1}^n (X_0^{a_i} X_1^{b_i}) \cdot X_0^\beta, \]

where \(\prod_{i=1}^n g_i := g_1 \cdots g_n\). One computes

\[ g \cdot 1_B = a \cdot 1_B, \text{ where } a := \sum_{i=1}^n X_1^{a_i} X_0^{b_i} \cdots X_0^{a_n_1} (X_1^{b_i} - 1) + X_1^\alpha \in W_Q^B. \]

Then \(ev^W(a) = ev^V(a) = X_1^{\alpha + b_1 + \cdots + b_n}\). Since \(ev^\mathcal{M}(g \cdot 1_B) = ev^W(a)\), one obtains

\[ iso \circ ev^\mathcal{M}(g \cdot 1_B) = e^{(\alpha + b_1 + \cdots + b_n) t}. \]

Then

\[ iso \circ ev^\mathcal{M}(\Gamma_g(-\log X_1)g \cdot 1_B) = iso \circ ev^W(\Gamma_g(-\log X_1)) \cdot iso \circ ev^\mathcal{M}(g \cdot 1_B) = \Gamma^{-1}_g(-t) e^{(\alpha + b_1 + \cdots + b_n) t}. \]

Since \(iso \circ ev^\mathcal{M}\) takes \(\mathcal{G}(\hat{\mathcal{M}}_Q^B)\) to the set \(\mathcal{G}(Q[[t]])\) of group-like elements of \(Q[[t]]\), \(\Gamma^{-1}_g(-t) \cdot e^{(\alpha + b_1 + \cdots + b_n) t} \in \mathcal{G}(Q[[t]])\), and therefore \(\Gamma^{-1}_g(-t) \in \mathcal{G}(Q[[t]])\), so that there exists \(\lambda \in \mathbb{Q}\) such that \(\Gamma_g(t) = e^{\lambda t}\).
Using the fact that \(e^{λ \log X_1} ∈ \hat{W}^B_\mathbb{Q}\) is group-like with respect to \(\hat{Δ}^{W,B}\), the module property of \(\hat{Δ}^{M,B}\) with respect to \(\hat{Δ}^{W,B}\) and \(γ^{-1}_g(-\log X_1)g \cdot 1_B ∈ G(\hat{M}^B_\mathbb{Q})\), we obtain \(g \cdot 1_B ∈ G(\hat{M}^B_\mathbb{Q})\) as \(g \cdot 1_B ∈ M^B_\mathbb{Q}\).

\[\square\]

Set for \(a ∈ \mathbb{Z}, Y_a := X_0^a(X_1 - 1)\). One derives from \([4], Proposition 2.3,\) that the algebra \(W^B_\mathbb{Q}\) is presented by generators \((Y_a)_{a ∈ \mathbb{Z} - \{0\}}, X_1^±, X_1^{-1} \cdot X_1 = 1\). It follows that \(W^B_\mathbb{Q}\) admits an algebra grading, for which \(\deg(Y_a) = 1\) for \(a ∈ \mathbb{Z} - \{0\}\) and \(\deg(X_1^±) = 0\). For \(n ≥ 0\), we denote by \(W_n\) the part of \(W^B_\mathbb{Q}\) of degree \(n\). Then

\[W^B_\mathbb{Q} = \oplus_{n ≥ 0} W_n.\]

We also set \(W_{≤n} := \oplus_{m ≤ n} W_m\).

**Lemma 5.3** One has for any \(a ≥ 1\),

\[Δ^{W,B}(Y_a) = Y_a ⊗ 1 + 1 ⊗ Y_a - \sum_{a' = 1}^{a-1} Y_a' ⊗ Y_{a-a'},\]

\[Δ^{W,B}(Y_{-a}) = Y_{-a} ⊗ X_1 + X_1 ⊗ Y_{-a} + \sum_{a' = 1}^{a-1} Y_{-a'} ⊗ Y_{a-a'} + \sum_{a' = 1}^{a-1} Y_{a'} ⊗ Y_{a-a} \quad (5.2.1)\]

**Proof** As remarked in \([EF1], proof of Proposition 2.4,\) the formal series \(s^±(t) := 1 + \sum_{k ≥ 1} t^k X^±_0 \) are group-like for \(Δ^{W,B}\). Then

\[s^±(t) = 1 + t X^±_0 \left[1 - t (X^±_0 - 1)^{-1} (1 - X^±_1)\right]^{-1} \left[1 - t X^±_0 (1 - X^±_1)\right]^{-1} = (1 + t - t X^±_0)^{-1} (1 + t - t X^±_0 X^±_1) = \tilde{s}^±(u),\]

where \(u := t/(1 + t)\) and \(\tilde{s}^±(u) := (1 - u X^±_0)^{-1} (1 - u X^±_0 X^±_1)\). It follows that the series \(\tilde{s}^±(u)\) are group-like. The expansions \(\tilde{s}^±(u) = 1 + \sum_{k ≥ 1} u^k X^±_0 (1 - X^±_1)\) imply \(\tilde{s}^+(u) = 1 - \sum_{k ≥ 1} u^k Y_k\), which implies the first part of \((5.2.1)\), and \(\tilde{s}^-(u) = 1 + \sum_{k ≥ 1} u^k Y_{-k} X_1^{-1}\), which together with the group-likeness of \(X_1\), implies its second part.

\[\square\]

**Remark 5.4** One can show that \((5.2.1)\) remains valid for any \(a ∈ \mathbb{Z}\) under the summation convention of \([4], (2.4.9)\).

**Lemma 5.5** The coproduct \(Δ^{W,B} : W^B_\mathbb{Q} → (W^B_\mathbb{Q})^2 \) is such that \(Δ^{W,B}(W_{≤n}) ⊂ (W_{≤n})^2\).

**Proof** This follows that the equalities \(Δ^{W,B}(X^±_1) = X^±_1 ⊗ X^±_1\) and \((5.2.1)\) for any \(a > 0\) (see \([4], Proposition 2.4)\).
Lemma 5.6 There is an algebra morphism
\[ \Delta^\mathcal{W},B_{\mod} : \mathcal{W}^B_Q \to (\mathcal{W}^B_Q) \otimes 2, \]
defined by \( X_1^\pm X_1^\pm \) and \( Y_{\pm a} \mapsto \sum_{a'=-1}^{a-1} Y_{\pm a'} \otimes Y_{\pm (a-a')} \) for \( a > 0 \).
Then \( \Delta^\mathcal{W},B_{\mod} (\mathcal{W}_n) \subset \mathcal{W}^\otimes 2_n \) for any \( n \geq 0 \), and the diagram
\[
\begin{array}{ccc}
\mathcal{W}_{\leq n} & \xrightarrow{\Delta^\mathcal{W},B_{\mod}} & (\mathcal{W}_{\leq n}) \otimes 2 \\
\downarrow{\text{pr}_n} & & \downarrow{\text{pr}_n^\otimes 2} \\
\mathcal{W}_n & \xrightarrow{\Delta^\mathcal{W},B_{\mod}} & \mathcal{W}_n \otimes 2 
\end{array}
\]
commutes, where \( \text{pr}_n : \mathcal{W}_{\leq n} \to \mathcal{W}_n \) is the projection on the highest degree component.

**Proof** Immediate. \( \square \)

For \( a_1, \ldots, a_n \in \mathbb{Z} - \{0\} \), set
\[ \mathcal{W}(a_1, \ldots, a_n) := \text{Span}_\mathbb{Q} \{ X_1^{b_0} Y_{a_1} X_1^{b_1} \cdots X_1^{b_{n-1}} Y_{a_n} X_1^{b_n} | b_0, \ldots, b_n \in \mathbb{Z} \} \subset \mathcal{W}_n. \]

Lemma 5.7 One has
\[ \mathcal{W}_n = \oplus_{a_1, \ldots, a_n \in \mathbb{Z} - \{0\}} \mathcal{W}(a_1, \ldots, a_n). \]

**Proof** This follows from the presentation of \( \mathcal{W}^B_Q \). \( \square \)

For \( a \in \mathbb{Z} - \{0\} \), set \( S(a) := \{ (a', a'') \in \mathbb{Z} - \{0\} | \text{sgn}(a') = \text{sgn}(a'') = \text{sgn}(a) \text{ and } a' + a'' = a \} \).

Lemma 5.8 For \( a_1, \ldots, a_n \in \mathbb{Z} - \{0\} \), one has
\[ \Delta^\mathcal{W},B_{\mod} (\mathcal{W}(a_1, \ldots, a_n)) \subset \oplus_{(a_1', a_1'') \in S(a_1), \ldots, (a_n', a_n'') \in S(a_n)} \mathcal{W}(a_1', \ldots, a_n') \otimes \mathcal{W}(a_1'', \ldots, a_n''). \]

**Proof** This follows from (5.3). \( \square \)

Lemma 5.9 Let \( g \in F_2 \). Then \( g \cdot 1_B \in G(\mathcal{M}^B_Q) \) iff there exist \( \alpha, \beta \in \mathbb{Z} \), such that \( g = X_1^\alpha X_0^\beta \).

**Proof** The group \( \mathbb{Z}^2 \) acts on the set \( F_2 \) by \( (\alpha, \beta) \cdot g := X_1^\alpha g X_0^\beta \). For \( n \geq 0 \), set
\[ (F_2)_n := \{ X_0^{a_1} X_1^{b_1} \cdots X_0^{a_n} X_1^{b_n} | a_1, \ldots, b_n \in \mathbb{Z} - \{0\} \} \subset F_2. \]
Then the composition
\[ \sqcup_{n \geq 0} (F_2)_n \to F_2 \to F_2 / \mathbb{Z}^2 \]
is a bijection.

Set \( S := \{ g \in F_2 \mid g \cdot 1_B \in \mathcal{G}(\mathcal{M}^B_{\mathbb{Q}}) \} \). One checks that \( S \) is stable under the action of \( \mathbb{Z}^2 \). It follows that
\[ S = \sqcup_{n \geq 0} \mathbb{Z}^2 \cdot ((F_2)_n \cap S). \] (5.2.3)

One has
\[ (F_2)_0 \cap S = \{ e \}. \] (5.2.4)

We now compute \( (F_2)_n \cap S \) for \( n > 0 \). Let \( g \in (F_2)_n \). Let \( a_1, b_1, \ldots, a_n, b_n \in \mathbb{Z} \setminus \{ 0 \} \) be such that \( g = X_0^{a_1} X_1^{b_1} \cdots X_0^{a_n} X_1^{b_n} \) and set
\[ w(g) := 1 + \sum_{i=1}^n X_0^{a_i} X_1^{b_1} \cdots X_0^{a_i} (X_1^{b_i} - 1) \in \mathcal{W}^B_{\mathbb{Q}}. \] (5.2.5)

The summand in the right-hand side of (5.2.5) corresponding to index \( i \) belongs to \( \mathcal{W}_{\leq i} \) and \( 1 \in \mathcal{W}_0 \). If follow that \( w(g) \in \mathcal{W}_{\leq n} \) and that
\[ w(g) \equiv Y_{a_1} \cdot \varphi_{b_1}(X_1) \cdots Y_{a_n} \cdot \varphi_{b_n}(X_1) \mod \mathcal{W}_{\leq n-1}. \] (5.2.6)

where for \( b \in \mathbb{Z} \setminus \{ 0 \} \), we set \( \varphi_b(t) := (t^b - 1)/(t - 1) \in \mathbb{Z}[t, t^{-1}] \). (5.2.6) and the fact that for \( b \neq 0 \), \( \varphi_b \neq 0 \) implies that
\[ \text{pr}_n(w(g)) \in \mathcal{W}(a_1, \ldots, a_n) - \{ 0 \}. \] (5.2.7)

Assume now that \( g \in (F_2)_n \cap S \). One has \( g \cdot 1_B = w(g) \cdot 1_B \), therefore \( g \cdot 1_B \in \mathcal{G}(\mathcal{M}^B_{\mathbb{Q}}) \) is equivalent to the group-likeness of \( w(g) \in \mathcal{W}^B_{\mathbb{Q}} \) for \( \Delta^{\mathcal{W}^B} \). Since \( w(g) \in \mathcal{W}_{\leq n} \), the diagram (5.2.2) implies
\[ \Delta_{\text{mod}}^{\mathcal{M}^B} (\text{pr}_n(w(g))) = \text{pr}_n(w(g))^{\otimes 2} \] (5.2.8)

(equality in \( \mathcal{W}_{\leq n}^{\otimes 2} \)). By Lemma 5.8, the left-hand side of (5.2.8) belongs to
\[ \bigoplus_{(a', a'') \in S(a_1), \ldots, (a'_n, a''_n) \in S(a_n)} \mathcal{W}(a'_1, \ldots, a'_n) \otimes \mathcal{W}(a''_1, \ldots, a''_n) \subset (\mathcal{W}_n)^{\otimes 2}, \]
while the right-hand side belongs to
\[ \mathcal{W}(a_1, \ldots, a_n) \otimes \mathcal{W}(a_1, \ldots, a_n) \subset (\mathcal{W}_n)^{\otimes 2}. \]

By the direct sum decomposition
\[ \mathcal{W}_n^{\otimes 2} = \bigoplus_{((a_1, b_1), \ldots, (a_n, b_n)) \in (\mathbb{Z} - \{0\})^2} \mathcal{W}(a_1, \ldots, a_n) \otimes \mathcal{W}(b_1, \ldots, b_n) \]
and since \( ((a_1, a_1), \ldots, (a_n, a_n)) \notin S(a_1) \times \cdots \times S(a_n) \) (as \( (a, a) \notin S(a) \) for any \( a \neq 0 \)), both sides of (5.2.8) should be zero, which contradicts (5.2.7). All this implies that \( (F_2)_n \cap S = \emptyset \) for \( n > 0 \). Together with (5.2.3) and (5.2.4), this implies Lemma 5.9.

\[ \square \]

**Proposition 5.10** There is an isomorphism \( \text{DMR}^B \cong \{ \pm 1 \} \).

**Proof** One obviously has \( \{ \pm 1, 1 \} \subset \text{DMR}^B \). Let us prove the opposite inclusion. Let
\[ (\mu, g) \in \text{DMR}^B(Q) \cap (\{ \pm 1 \} \times F_2). \]

By Lemmas 5.2 and 5.9, the condition that \( \Gamma^{-1}_g (-\log X_1) \cdot 1_B \in G(\hat{\Lambda}^B_Q) \) implies that for some \( \alpha, \beta \in \mathbb{Z} \), one has \( g = X_1^\alpha X_0^\beta \). The conditions \( (g|\log X_0) = (g|\log X_1) = 0 \) then imply \( \alpha = \beta = 0 \), therefore \( g = 1 \). This proves Proposition 5.10.

\[ \square \]

**Remark 5.11** Using the proof of Proposition 5.10, one can prove the stronger result \( \text{DMR}^B(Q) \cap (Q^\times \times F_2) = \{ \pm 1 \} \). Indeed, this proof implies that if \( (\mu, g) \) belongs to this intersection, then \( g = 1 \). The condition \( \mu^2 = 1 + 24(g|\log X_0log X_1) \) then implies that \( \mu = \pm 1 \).

**Remark 5.12** Proposition 5.10 is consistent with the conjectural equality of Lie algebras \( \mathfrak{grt}_1 = \mathfrak{dmt}_0 \). Indeed, this equality is equivalent to \( \text{DMR}^\text{DR}(\cdot) = \text{GRT}(\cdot) \), which via the isomorphism \( i_{(1, \Phi)} \), for \( \Phi \in \mathfrak{m}_1(Q) \), is equivalent to \( \text{DMR}^B(\cdot) = \text{GT}(\cdot) \), which upon taking rational points and intersecting with \( \{ \pm 1 \} \times F_2 \) implies the equality \( \text{DMR}^B = \text{GT} \), which is Proposition 5.10.

### 6 Pro-\( p \) aspects

The goal of this section is the definition of a pro-\( p \) analogue \( \text{DMR}^B_p \) of the group scheme \( \text{DMR}^B(\cdot) \). In this section, we first recall some material on the relation between the pro-\( p \) and prounipotent completions of discrete groups (Sect. 6.1), \( p \) being a prime number. In Sect. 6.2, we recall the definition of the pro-\( p \) analogue \( \text{GT}_p \) of the Grothendieck-Teichmüller group, and we use the results of Sect. 6.1 to prove a statement of [2] on the relations of \( \text{GT}_p \) with \( \text{GT}(\mathbb{Q}_p) \) (Corollary 6.14); we also make precise the relation between \( \text{GT}_p \) and the semigroup \( \text{GT}_p \) introduced in [2] (Proposition 6.15). We then define a group \( \text{DMR}^B_p \) (see Lemma-Definition 6.16) and show that it fits in a commutative diagram, which makes it into a natural pro-\( p \) analogue of the group scheme \( \text{DMR}^B(\cdot) \) (Sect. 6.3).
6.1 Pro-\(p\) and prounipotent completions of discrete groups

6.1.1 A morphism \(\Gamma^{(p)} \to \Gamma(\mathbb{Q}_p)\)

If \(\Gamma\) is a group, we denote by \(\Gamma^{(p)}\) its pro-\(p\) completion. If \(k\) is a \(\mathbb{Q}\)-algebra, we denote by \(\Gamma(k)\) the group of \(k\)-points of its prounipotent completion. We also denote by \(\text{Lie}(\Gamma)\) the Lie algebra of this prounipotent completion.

Lemma 6.1 ([8], Lemma A.7) Suppose that \(\Gamma\) is finitely generated discrete group, then there is a continuous homomorphism \(\Gamma^{(p)} \to \Gamma(\mathbb{Q}_p)\) compatible with the morphisms from \(\Gamma\) to its source and target.

When \(\Gamma\) is the free group \(F_n\), this gives a continuous homomorphism \(F_n^{(p)} \to F_n(\mathbb{Q}_p)\).

6.1.2 Injectivity of \(F_n^{(p)} \to F_n(\mathbb{Q}_p)\)

Let \(\Gamma\) be a group. Define the \(\mathbb{Z}_p\)-algebra of \(\Gamma\), denoted \(\mathbb{Z}_p[[\Gamma]]\), to be the inverse limit of the group algebras of the quotients of \(\Gamma\) which are \(p\)-groups with coefficients in \(\mathbb{Z}_p\). This is a topological Hopf algebra. (When \(\Gamma\) is a pro-\(p\) group, \(\mathbb{Z}_p[[\Gamma]]\) coincides with the object introduced in [11], p. 7.) If \(H\) is a (topological) Hopf algebra, we denote by \(G(H)\) the group of its group-like elements.

Lemma 6.2 The group \(G(\mathbb{Z}_p[[\Gamma]])\) is equal to \(\Gamma^{(p)}\).

Proof The group \(G(\mathbb{Z}_p[[\Gamma]])\) is the inverse limit of the groups of group-like elements of the group algebras \(\mathbb{Z}_p K\), where \(K\) runs over all the quotients of \(\Gamma\) which are \(p\)-groups. As \(G(\mathbb{Z}_p K)\) is equal to \(K\), \(G(\mathbb{Z}_p[[\Gamma]])\) is equal to the inverse limit of the finite quotients of \(\Gamma\) which are \(p\)-groups, therefore to \(\Gamma^{(p)}\). \(\square\)

Lemma 6.3 Let \(A(n) := \mathbb{Z}_p\langle\langle t_1, \ldots, t_n \rangle\rangle\) be the algebra of associative formal power series in variables \(t_1, \ldots, t_n\) with coefficients in \(\mathbb{Z}_p\), equipped with the topology of convergence of coefficients. Then \(A(n)\) has a Hopf algebra structure with coproduct given by \(t_i \mapsto t_i \otimes 1 + 1 \otimes t_i + t_i \otimes t_i\) for \(i = 1, \ldots, n\). Let \(F_n\) be the free group over generators \(X_1, \ldots, X_n\). There is an isomorphism

\[ F_n^{(p)} \simeq G(A(n)) \]

induced by \(X_i \mapsto 1 + t_i\) for \(i = 1, \ldots, n\).

Proof This follows from Lemma 6.2 combined with the isomorphism \(\mathbb{Z}_p[[F_n^{(p)}]] \simeq A(n)\), see [11], Section I.1.5, Proposition 7. \(\square\)

Lemma 6.4 The map \(F_n^{(p)} \to F_n(\mathbb{Q}_p)\) is injective.
Proof If $k$ is a $\mathbb{Q}$-algebra, there is an isomorphism $(kF_n)^\wedge \simeq k\langle \langle u_1, \ldots, u_n \rangle \rangle$, where each $u_i$ is primitive. Moreover, $F_n(k) = \mathcal{G}(kF_n)^\wedge$, therefore

$$F_n(k) = \mathcal{G}(k\langle \langle u_1, \ldots, u_n \rangle \rangle).$$

The result now follows from the specialization of this result for $k = \mathbb{Q}_p$, from the topological Hopf algebra inclusion $\mathbb{Z}_p\langle \langle t_1, \ldots, t_n \rangle \rangle \subset \mathbb{Q}_p\langle \langle u_1, \ldots, u_n \rangle \rangle$ given by $t_i \mapsto e^{u_i} - 1$, and from Lemma 6.3. \qed

6.1.3 Exact sequences of pro-$p$ completions

Lemma 6.5 ([9], see also [1]) Let $1 \to N \to G \to H \to 1$ be an exact sequence of discrete groups such that (i) $(G, N) = (N, N)$, and (ii) $N$ is a free group of finite rank greater than 1. Then the induced sequence of pro-$p$ completions $1 \to N^{(p)} \to G^{(p)} \to H^{(p)} \to 1$ is exact.

For $n \geq 3$, let $K_n$ be the Artin pure braid group with $n$ strands. It is presented by generators $x_{ij}, 1 \leq i < j \leq n$ and relations

$$(a_{ijk}, x_{ij}) = (a_{ijk}, x_{ik}) = (a_{ijk}, x_{jk}) = 1, \quad i < j < k, \quad a_{ijk} = x_{ij}x_{ik}x_{jk},$$

$$(x_{ij}, x_{kl}) = (x_{il}, x_{jk}) = 1, \quad (x_{ik}, x_{ij}^{-1}x_{jl}x_{ij}) \quad \text{if} \quad i < j < k < l.$$ For any $i \in \llbracket 1, n \rrbracket$, the elements $x_{i1}, \ldots, x_{in}$ of $K_n$ generate a subgroup isomorphic to $F_{n-1}$, and we have an exact sequence

$$1 \to F_{n-1} \to K_n \to K_{n-1} \to 1. \quad (6.1.1)$$

Lemma 6.6 If $n \geq 3$, then the exact sequence (6.1.1) induces a short exact sequence of pro-$p$ groups:

$$1 \to F_{n-1}^{(p)} \to K_n^{(p)} \to K_{n-1}^{(p)} \to 1.$$

Proof Let $P_{n+1}$ be the pure braid group of the sphere with $n + 1$ strings (cf. [4]). It is known that $P_{n+1}$ is isomorphic to the quotient $K_n/Z(K_n)^2$, where $Z(K_n)$ is the center of $K_n$.

In [9], Proposition 2.3.1, it is shown that for any $j \in \llbracket 1, n \rrbracket \backslash \{i\}$, $P_{n+1}$ is equal to the product $\langle x_{i1}, \ldots, x_{in} \rangle \cdot C(x_{ij})$, where the projection map $K_n \to P_{n+1}$ is denoted $g \mapsto \tilde{g}$, and where $C(x_{ij})$ is the centralizer subgroup of $x_{ij}$.

Since $Z(K_n)^2$ is contained in $C(x_{ij})$, $K_n$ is equal to the product $\langle x_{i1}, \ldots, x_{in} \rangle \cdot C(x_{ij})$, where $C(x_{ij})$ is the centralizer subgroup of $x_{ij}$.

Then any $k \in K_n$ can be expressed as $f \cdot c$, where $f \in \langle x_{i1}, \ldots, x_{in} \rangle$ and $c \in C(x_{ij})$. Then $(k, x_{ij}) = (f \cdot c, x_{ij}) = (f, x_{ij}) \in (F_{n-1}, F_{n-1})$. As this holds for any $j \in \llbracket 1, n \rrbracket \backslash \{i\}$, one obtains $(K_n, F_{n-1}) \subset (F_{n-1}, F_{n-1})$, therefore the equality of these subgroups of $K_n$ as the opposite inclusion is obvious.
One can therefore apply Lemma 6.5 to the exact sequence (6.1.1), which yields the result. \qed

6.1.4 Exact sequences of pronipotent completions

**Lemma 6.7** Let \( k \) be a \( \mathbb{Q} \)-algebra and let \( n \geq 3 \). The exact sequence (6.1.1) induces a short exact sequence of groups:

\[
1 \to F_{n-1}(k) \to K_n(k) \to K_{n-1}(k) \to 1.
\]

**Proof** According to [2], any associator \( \Phi \in M_1(\mathbb{Q}) \) and parenthesization \( P \) of a word with \( n \) identical letters gives rise to an isomorphism \( b^P_\Phi : \text{Lie}(K_n) \to \hat{t}_n \), where \( t_n \) is the graded \( \mathbb{Q} \)-Lie algebra with degree one generators \( t_{ij}, i \neq j \in \llbracket 1, n \rrbracket \) and relations \( t_{ji} = t_{ij}, [t_{ik} + t_{jk}, t_{ij}] = 0, \) and \( [t_{ij}, t_{kl}] = 0 \) for \( i, j, k, l \) all distinct, and where \( \hat{t}_n \) is its degree completion.

The morphisms of (6.1.1) induce Lie algebra morphisms \( \text{Lie}(F_{n-1}) \to \text{Lie}(K_n) \) and \( \text{Lie}(K_n) \to \text{Lie}(K_{n-1}) \). One has a commutative diagram

\[
\begin{array}{ccc}
\text{Lie}(K_n) & \xrightarrow{b^P_\Phi} & \hat{t}_n \\
\downarrow & & \downarrow \\
\text{Lie}(K_{n-1}) & \xrightarrow{b^{P^i}_\Phi} & \hat{t}_{n-1}
\end{array}
\]

where \( P^i \) is \( P \) with the \( i \)-th letter erased, and where the right vertical arrow is induced by the morphism \( t_n \to t_{n-1}, t_{ia} \mapsto 0, t_{ab} \mapsto t_{f(a)f(b)} \) for \( a, b \in \llbracket 1, n \rrbracket - \{i\}, f \) being the increasing bijection \( \llbracket 1, n \rrbracket - \{i\} \simeq \llbracket 1, n-1 \rrbracket \).

It follows from this diagram that \( b^P_\Phi \) restricts to a Lie algebra morphism \( \text{Lie}(F_{n-1}) \to \hat{t}_{n-1} \), where \( \hat{t}_{n-1} \) is the kernel of \( \hat{t}_n \to \hat{t}_{n-1} \), which is the degree completion of the kernel of \( t_n \to t_{n-1} \), a Lie algebra freely generated by the \( t_{ij}, j \in \llbracket 1, n \rrbracket - \{i\} \). The abelianization of this morphism can be shown to be an isomorphism, therefore \( \text{Lie}(F_{n-1}) \to \hat{t}_{n-1} \) is an isomorphism.

In the following diagram

\[
\begin{array}{cccccc}
0 & \xrightarrow{} & \hat{t}_{n-1} & \xrightarrow{} & \hat{t}_n & \xrightarrow{} & \hat{t}_{n-1} & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & \text{Lie}F_{n-1} & \xrightarrow{} & \text{Lie}K_n & \xrightarrow{} & \text{Lie}K_{n-1} & \xrightarrow{} & 0
\end{array}
\]

the top sequence is exact. Since the vertical arrows are isomorphisms and since the squares commute, it follows that the bottom sequence is exact. The result follows. \qed

6.1.5 Injectivity of \( K_n^{(p)} \to K_n(\mathbb{Q}_p) \)

**Lemma 6.8** For any \( n \geq 2 \), the map \( K_n^{(p)} \to K_n(\mathbb{Q}_p) \) is injective.
Proof The statement is obvious for \( n = 2 \) as \( K_2 \simeq \mathbb{Z} \). One then proceeds by induction over \( n \). Assume that the statement holds for \( n - 1 \), then we have a natural morphism between two exact sequences, which makes the following diagram commutative

\[
\begin{array}{cccccc}
1 & \longrightarrow & F_{n-1}^{(p)} & \longrightarrow & K_n^{(p)} & \longrightarrow & K_{n-1}^{(p)} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & F_{n-1}(\mathbb{Q}_p) & \longrightarrow & K_n(\mathbb{Q}_p) & \longrightarrow & K_{n-1}(\mathbb{Q}_p) & \longrightarrow & 1.
\end{array}
\]

The leftmost vertical map is injective by Lemma 6.4 and the rightmost vertical map is as well by the induction assumption, which shows that the middle vertical map is injective. \( \square \)

6.2 Results on \( \text{GT}_p \)

Let \( F_2 \) be the free group with generators \( X_0, X_1 \) (see Sect. 2.1.1).

A semigroup structure is defined on \( \mathbb{Z}_p \times F_2^{(p)} \) by \((\lambda_1, f_1) \circ (\lambda_2, f_2) = (\lambda_1 \lambda_2, f)\), where

\[
f(X_0, X_1) = f_2(f_1(X_0, X_1)X_0^{\lambda_1}f_1(X_0, X_1)^{-1}, X_1^{\lambda_1})f_1(X_0, X_1)
\]

(this is the opposite of the product of \([2], (4.11)\), with \( X_0, X_1 \) replacing \( X, Y \)).

Lemma 6.9 \((\mathbb{Z}_p^\times \times F_2^{(p)}, \circ)\) is a subgroup of \((G^B(\mathbb{Q}_p), \circ)\) (see Lemma 2.1).

Proof Recall the subsets \( F_2(\mathbb{Q}_p) = \mathcal{G}(\mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle) \) and \( 1 + \mathbb{Z}_p \langle \langle t_0, t_1 \rangle \rangle_0 \) of \( \mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle \times \). By Lemma 6.3, one has

\[
F_2^{(p)} = \mathcal{G}(\mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle) \cap (1 + \mathbb{Z}_p \langle \langle t_0, t_1 \rangle \rangle_0) \subset \mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle^\times,
\]

therefore

\[
\mathbb{Z}_p^\times \times F_2^{(p)} = (\mathbb{Z}_p^\times \times \mathcal{G}(\mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle)) \cap (\mathbb{Z}_p^\times \times (1 + \mathbb{Z}_p \langle \langle t_0, t_1 \rangle \rangle_0)) \subset \mathbb{Q}_p^\times \times \mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle^\times.
\]

Then \( \mathbb{Q}_p^\times \times \mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle^\times \) is equipped with the group structure

\[
(\lambda, a) \circ (\mu, b) := (\lambda \mu, a(t_0, t_1)b((1 + t_0)^\lambda - 1, a^{-1}((1 + t_1)^\lambda - 1)a)),
\]

and \( \mathbb{Q}_p^\times \times \mathcal{G}(\mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle) \) is then a subgroup, which identifies with \( G^B(\mathbb{Q}_p) \) under the identifications \( X_i = 1 + t_i, i = 0, 1 \).

The set \( \mathbb{Z}_p^\times \times (1 + \mathbb{Z}_p \langle \langle t_0, t_1 \rangle \rangle_0) \) is a sub-semigroup of \( (\mathbb{Q}_p^\times \times \mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle^\times, \circ) \).

Let \( (\mu, b) \in \mathbb{Z}_p^\times \times (1 + \mathbb{Z}_p \langle \langle t_0, t_1 \rangle \rangle_0) \) and let \( (1/\mu, a) \in \mathbb{Q}_p^\times \times \mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle^\times \) be its inverse. Then, one has

\[
a(t_0, t_1)b((1 + t_0)^{1/\lambda} - 1, a^{-1}[(1 + t_1)^{1/\lambda} - 1]a) = 1. \quad (6.2.1)
\]
One shows by induction on \( n \) that
\[
a \in 1 + \mathbb{Z}_p \langle \langle t_0, t_1 \rangle \rangle_0 + \mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle_{n+1}.
\] (6.2.2)

For \( n = 0 \), this follows from (6.2.1). Assume that (6.2.2) holds for \( n \). Then
\[
b((1 + t_0)^{1/\lambda} - 1, a^{-1}[(1 + t_1)^{1/\lambda} - 1]a) \in 1 + \mathbb{Z}_p \langle \langle t_0, t_1 \rangle \rangle_0 + \mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle_{n+1}.
\]
(6.2.1) then implies (6.2.2) with \( n \) replaced by \( n+1 \). Finally \( a \in 1 + \mathbb{Z}_p \langle \langle t_0, t_1 \rangle \rangle_0 \). It follows that \( \mathbb{Z}_p^\times \times (1 + \mathbb{Z}_p \langle \langle t_0, t_1 \rangle \rangle_0) \) is a subgroup of the group \( (\mathbb{Q}_p^\times \times \mathbb{Q}_p \langle \langle t_0, t_1 \rangle \rangle_0)^\times, \otimes \). It follows that \( \mathbb{Z}_p^\times \times F_2^{(p)} \) is the intersection of two subgroups of this group, which implies the result. □

**Lemma 6.10** The subgroup of invertible elements of \( (\mathbb{Z}_p \times F_2^{(p)}, \otimes) \) is \( \mathbb{Z}_p^\times \times F_2^{(p)} \).

**Proof** This follows from Lemma 6.9. □

Consider the morphisms from \( F_2 \) to various groups given by the following table:

| Name of morphism | \( \theta \) | \( \kappa \) | \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) | \( \alpha_4 \) | \( \alpha_5 \) |
|------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Target group     | \( F_2 \)   | \( F_2 \)   | \( K_4 \)   | \( K_4 \)   | \( K_4 \)   | \( K_4 \)   | \( K_4 \)   |
| Image of \( X_0 \) | \( X_1 \) | \( X_1 \) | \( x_{23}x_{24} \) | \( x_{12} \) | \( x_{23} \) | \( x_{34} \) | \( x_{12}x_{13} \) |
| Image of \( X_1 \) | \( X_0 \) | \( (X_0X_1)^{-1} \) | \( x_{12} \) | \( x_{23} \) | \( x_{34} \) | \( x_{13}x_{13} \) | \( x_{24}x_{24} \) |

The pro-\( p \) completions of these morphisms are denoted in the same way.

In [2], p. 846, \( \text{GT}_p \) is defined as the set of all \( (\lambda, f) \in (1 + 2\mathbb{Z}_p) \times F_2^{(p)} \) such that
\[
f \theta (f) = 1, \quad \kappa^2(f)(X_0X_1)^{-m} \kappa(f)X_1^m f X_0^m = 1,
\]
\[
\alpha_1(f)\alpha_3(f)\alpha_5(f)\alpha_2(f)\alpha_4(f) = 1,
\] (6.2.3)

where \( m = (\lambda - 1)/2 \) (equalities in \( F_2^{(p)} \) and \( K_4^{(p)} \)) and \( \text{GT}(\mathbb{Q}_p) \) is the set of all \( (\lambda, f) \in \mathbb{Q}_p^\times \times F_2^{(p)} \) such that the same identities hold in \( F_2(\mathbb{Q}_p) \) and \( K_4(\mathbb{Q}_p) \).

The subset \( \text{GT}_p \subset 2 \mathbb{Z}_p \times F_2^{(p)} \) is shown to be a sub-semigroup; it is equipped with the structure opposite to that induced by \( \mathbb{Z}_p \times F_2^{(p)} \). The group \( \text{GT}_p \subset \text{GT}_p \) is then defined to be the group of invertible elements in \( \text{GT}_p \).

**Corollary 6.11** \( \text{GT}_p = \text{GT}_p \cap (\mathbb{Z}_p^\times \times F_2^{(p)} \).\)

**Proof** This follows from Lemma 6.10 and the definition of \( \text{GT}_p \). □

**Proposition 6.12** \( \text{GT}_p = \text{GT}(\mathbb{Q}_p) \cap (\mathbb{Z}_p^\times \times F_2^{(p)} \).
Proof $\text{GT}(\mathbb{Q}_p)$ is the subset of $\mathbb{Q}_p^\times \times F_2(\mathbb{Q}_p)$ defined by the prounipotent versions of the conditions (6.2.3), while $\text{GT}_p$ is the subset of $((1+2\mathbb{Z}_p) \cap \mathbb{Z}_p^\times) \times F_2^{(p)} = \mathbb{Z}_p^\times \times F_2^{(p)}$ defined by the pro-$p$ versions of the same conditions (the equality follows from $1+2\mathbb{Z}_p = \mathbb{Z}_p^\times$ for $p = 2$, and $1+2\mathbb{Z}_p = \mathbb{Z}_p$ for $p \neq 2$). Moreover, the inclusion $K_4^{(p)} \subset K_4(\mathbb{Q}_p)$ and the compatibilities of the pro-$p$ and prounipotent completions of a given group morphism imply that an element of $\mathbb{Z}_p^\times \times F_2^{(p)}$ satisfies the pro-$p$ version of (6.2.3) iff its image in $\mathbb{Q}_p^\times \times F_2(\mathbb{Q}_p)$ satisfies its prounipotent version. □

Remark 6.13 Proposition 6.12 shows that $\text{GT}_p$ is the intersection of two subgroups of $\text{G}^B(\mathbb{Q}_p)^{\text{op}}$, and is therefore a group.

Corollary 6.14 ([2], p. 846) $\text{GT}_p \subset \text{GT}(\mathbb{Q}_p)$.

Proof This immediately follows from Proposition 6.12. □

Proposition 6.15 $\text{GT}_p = \text{GT}_p$ for $p = 2$ and $\text{GT}_p = \text{GT}_p \times \mathbb{Z}_p \mathbb{Z}_p^\times \mathbb{F}_2^{(p)}$ for $p > 2$.

Proof This follows from Corollary 6.11 together with $1+2\mathbb{Z}_p = \mathbb{Z}_p^\times$ for $p = 2$, and $1+2\mathbb{Z}_p = \mathbb{Z}_p$ for $p \neq 2$. □

6.3 A pro-$p$ analogue $\text{DMR}_p^B$ of the group scheme $\text{DMR}^B(\text{−})$

Lemma-Definition 6.16 One sets

$$\text{DMR}_p^B := \text{DMR}^B(\mathbb{Q}_p) \cap (\mathbb{Z}_p^\times \times F_2^{(p)}).$$

Then $\text{DMR}_p^B$ is a subgroup of $\text{G}^B(\mathbb{Q}_p) = (\mathbb{Q}_p^\times \times F_2(\mathbb{Q}_p), \circ)$. 

Proof This follows from the fact that both $\text{DMR}^B(\mathbb{Q}_p)$ and $\mathbb{Z}_p^\times \times F_2^{(p)}$ are subgroups of $(\mathbb{Q}_p^\times \times F_2(\mathbb{Q}_p), \circ)$. □

Proposition 6.17 The natural inclusions yield the following commutative diagram of groups, in which both squares are Cartesian

$$\begin{array}{ccc}
\text{GT}_p^{\text{op}} & \rightarrow & \text{DMR}_p^B \\
\downarrow & & \downarrow \\
\text{GT}(\mathbb{Q}_p)^{\text{op}} & \rightarrow & (\mathbb{Z}_p^\times \times F_2^{(p)}, \circ)
\end{array}$$

Proof The fact that the right square is Cartesian follows from the definition of $\text{DMR}_p^B$. Then

$$\text{GT}(\mathbb{Q}_p)^{\text{op}} \cap \text{DMR}_p^B = \text{GT}(\mathbb{Q}_p)^{\text{op}} \cap (\text{DMR}^B(\mathbb{Q}_p) \cap (\mathbb{Z}_p^\times \times F_2^{(p)})) \quad \text{(by the definition of } \text{DMR}_p^B)$$

$$= \text{GT}(\mathbb{Q}_p)^{\text{op}} \cap (\mathbb{Z}_p^\times \times F_2^{(p)}) \quad \text{(by the inclusion of group schemes } \text{GT}(\text{−})^{\text{op}} \subset \text{DMR}^B(\text{−}))$$

$$= \text{GT}_p^{\text{op}} \quad \text{(by Proposition 6.12).}$$
so that the left square is Cartesian.

\[
\begin{array}{c}
\end{array}
\]

Acknowledgements The collaboration of both authors has been supported by Grants JSPS KAKENHI JP18H01110 and HighAGT ANR-20-CE40-0016.

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