A stochastic model for quantum measurement

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Abstract. We develop a statistical model of microscopic stochastic deviation from classical mechanics based on a stochastic process with a transition probability that is assumed to be given by an exponential distribution of infinitesimal stationary action. We apply the statistical model to stochastically modify a classical mechanical model for the measurement of physical quantities reproducing the prediction of quantum mechanics. The system+apparatus always has a definite configuration at all times, as in classical mechanics, fluctuating randomly following a continuous trajectory. On the other hand, the wavefunction and quantum mechanical Hermitian operator corresponding to the physical quantity arise formally as artificial mathematical constructs. During a single measurement, the wavefunction of the whole system+apparatus evolves according to a Schrödinger equation and the configuration of the apparatus acts as the pointer of the measurement so that there is no wavefunction collapse. We will also show that while the outcome of each single measurement event does not reveal the actual value of the physical quantity prior to measurement, its average in an ensemble of identical measurements is equal to the average of the actual value of the physical quantity prior to measurement over the distribution of the configuration of the system.

Keywords: stochastic particle dynamics (theory)
1. Motivation

Despite the superb empirical successes of quantum mechanics, with a broad range of practical applications in the development of technology, there are still many contradictory views on the meaning of the theory [1]. Such an absence of consensus on the conceptual foundation of quantum mechanics, after almost nine decades since its completion, might be due to the fact that the numerous postulates of standard quantum mechanics are highly ‘abstract and formal-mathematical’ with non-transparent physical and operational meaning. First, standard quantum mechanics does not provide a transparent explanation of the status of the wavefunction with regard to the state of the system under interest. Is the wavefunction physical or merely an artificial mathematical construct? Nor does it give a transparent physical and/or operational explanation why the physical observable quantities are represented by certain Hermitian operators, and which part of an experiment corresponds to their measurement. Equally importantly, one may also ask: when is a given Hermitian operator a representation of a meaningful or observable physical quantity? The above couple of problems are intimately related to the problem of the meaning and origin of the abstract and ‘strange’ [2] quantization procedures: canonical quantization, path integral, etc, via which quantum systems can be obtained from the corresponding classical systems.

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Further, the postulates of quantum mechanics give an algorithm to accurately predict/calculate the statistical results in an ensemble of identical measurement, rather than describe what is really happening physically and operationally in each single measurement event. This naturally raises several conceptual problems. First, one could ask: why measurement is dictated by a random, discontinuous and non-unitary time evolution, that of the postulate of wavefunction collapse, while when unobserved, a closed system follows a deterministic, continuous and unitary time evolution given by the Schrödinger equation. Providing a unified law for the time evolution of the system, observed or not, is the main issue of the infamous ‘measurement problem’. The algorithm also does not give us a clue as to the status or meaning of each single measurement result, and the statistics of the results in an ensemble of identical measurement, with regard to the properties of the system prior to measurement. Nor do the postulates tell us when and where the random collapse of a wavefunction occurs and what is needed for its occurrence (what constitutes a measurement?) It is also physically not clear why the statistical results follow Born’s rule. Moreover, to be precise, the apparatus is also composed of elementary particles so that the whole system + apparatus should be regarded as a legitimate closed quantum system which must evolve according to a Schrödinger equation. The linearity of the latter however allows the superposition of macroscopically distinguishable states of the pointer of the apparatus which, assuming that the wavefunction is physical and complete, leads to ‘the paradox of Schrödinger’s cat’.

To solve the above foundational problems, there is a growing interest recently in the program to reconstruct quantum mechanics. In this program, rather than directly pursuing interpretational questions on the abstract mathematical structures of quantum mechanics, one asks along with Wheeler: ‘why the quantum?’ [3], and wonders if the numerous abstract postulates of quantum mechanics can be derived from a concrete and transparent physical model. It is also of great interest if such a physical model can be singled out uniquely from a set of conceptually simple and physically transparent axioms [4]–[7]. Much progress along this line of approach has been made either within realist [8]–[11] or operational/information theoretical frameworks [12]–[23]. In the former, quantum fluctuations are assumed to be physically real and objective, and thus should be properly modeled by some stochastic processes. On the other hand, in the latter, quantum fluctuations are assumed to be fundamentally related to the concept of information and its processing. One then searches for a set of basic features of information processing which can be promoted as axioms to reconstruct quantum mechanics. One of the advantages of the reconstruction of quantum mechanics is that it might give useful physical insights to extend quantum mechanics either by changing the axioms or varying the free parameters of the physical model.

In the present paper, we shall develop a statistical model of stochastic deviation from classical mechanics in the microscopic regime based on Markovian stochastic processes to reconstruct quantum mechanics. This is done by assuming that the transition probability between two infinitesimally close spacetime points in configuration space via a random path is given by an exponential distribution of infinitesimal stationary action. We then apply the statistical model to stochastically modify a classical mechanical model of measurement so that the whole ‘system + apparatus’ is subjected to the stochastic fluctuations of infinitesimal stationary action. We shall show, by giving an explicit example
of the measurement of angular momentum, that the model reproduces the prediction of quantum mechanics.

Unlike canonical quantization, the system possesses a definite configuration all the time, as in classical mechanics, following a continuous trajectory randomly fluctuating with time. The configuration of the system should thus be regarded as the beable of the theory in Bell’s sense [24]. The Hermitian differential operator corresponding to the angular momentum and the wavefunction, on the other hand, arise formally and simultaneously as artificial convenient mathematical tools as one works in the Hilbert space representation. During a single measurement event, the wavefunction of the whole system + apparatus follows a Schrödinger equation and, as in classical mechanics, the configuration of part of the apparatus plays the role as the pointer so that there is no wavefunction collapse. We will also show that while the outcome of each single measurement event does not reveal the actual value of the angular momentum prior to measurement, its average in an ensemble of identical measurements is equal to the average of the actual value of the angular momentum prior to measurement over the distribution of the configuration of the system.

Without giving the technical detail, we shall also argue that the same conclusion carries over the measurement of position and angular momentum. In this paper, we shall confine the discussion to a system of spin-less particles.

2. A statistical model of microscopic stochastic deviation from classical mechanics

2.1. A class of stochastic processes in the microscopic regime based on random fluctuations of infinitesimal stationary action

There is a wealth of evidence that in the microscopic regime the deterministic classical mechanics suffers a stochastic correction. Yet, the prediction of quantum mechanics on the AB (Aharonov–Bohm) effect [25] and its experimental verification [26] suggest that the randomness in microscopic regime is inexplicable in terms of conventional random forces such as in Brownian motion. To describe such a microscopic randomness, let us develop the following stochastic model. Let us consider a system of particles whose configuration is denoted by \( q \) and whose evolution is parameterized by time \( t \). Let us assume that the Lagrangian depends on a randomly fluctuating variable \( \xi \):

\[
L = L(q, \dot{q}; \xi),
\]

whose origin is not our present concern. Let us assume that the time scale for the fluctuations of \( \xi \) is \( d\tau \).

Let us then consider two infinitesimally close spacetime points \((q; t)\) and \((q + dq; t + dt)\) such that \( \xi \) is constant. Let us assume that fixing \( \xi \), the principle of stationary action is valid to select a segment of path, denoted by \( J(\xi) \), that connects the two points. One must then solve a variational problem with fixed end points: \( \delta(L dt) = 0 \). This variational problem leads to the existence of a function \( A(q; t, \xi) \), Hamilton’s principal function, whose differential along the path is given by [27], for a fixed \( \xi \),

\[
dA = L dt = p \cdot dq - H dt,
\]

where \( p(\dot{q}) = \partial L/\partial \dot{q} \) is the momentum and \( H(q, p) = p \cdot \dot{q}(p) - L(q, \dot{q}(p)) \) is the Hamiltonian. Here we have made an assumption that the Lagrangian is not singular \( \det(\partial^2 L/\partial \dot{q}_i \partial \dot{q}_j) \neq 0 \). The above relation implies the following Hamilton–Jacobi equation:

\[
p = \partial_q A \\
- H(q, p) = \partial_t A,
\]

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which is valid during a microscopic time interval in which $\xi$ is fixed. Hence, $dA(\xi)$ is just the ‘infinitesimal stationary action’ along the corresponding short path during the infinitesimal time interval $dt$ in which $\xi$ is fixed.

Varying the value of $\xi$, the principle of stationary action will therefore pick up various different paths $J(\xi)$, all connecting the same two infinitesimally close spacetime points in configuration space, each with different values of infinitesimal stationary action $dA(\xi)$. $dA(\xi)$ thus is randomly fluctuating due to the fluctuations of $\xi$. Hence, we have a stochastic processes in which the system starting with configuration $q$ at time $t$ may take various different paths randomly to arrive at $q + dq$ at time $t + dt$. Assuming that the stochastic processes is Markovian, then it is completely determined by a ‘transition probability’ for the system starting with configuration $q$ at time $t$ to move to its infinitesimally close neighbor $q + dq$ at time $t + dt$ via a path $J(\xi)$, below denoted by

$$P((q + dq; t + dt)|\{J(\xi), (q; t)\}).$$

Since the stochastic processes is supposed to model a stochastic deviation from classical mechanics, then it is reasonable to assume that the transition probability is a function of a quantity that measures the deviation from classical mechanics.

It is natural to express the transition probability in terms of the stochastic quantity $dA(\xi)$ evaluated along the short segment of trajectory. To do this, let us first assume that $\xi$ is the simplest random variable with two possible values, a binary random variable. Without losing generality let us assume that the two possible values of $\xi$ differ from each other only by their signs, namely one is the opposite of the other, $\xi = \pm|\xi|$. Suppose that both realizations of $\xi$ lead to the same path so that $dA(\xi) = dA(-\xi)$. Since the stationary action principle is valid for both values of $\pm\xi$, then such a model must recover classical mechanics. Hence, the non-classical behavior should correspond to the case when the different signs of $\xi$ lead to different trajectories so that $dA(\xi) \neq dA(-\xi)$.

Now let us proceed to assume that $\xi$ may take continuous values. Let us assume that even in this case the magnitude of the difference of the value of $dA$ at $\pm\xi$,

$$Z(q; t, \xi) = dA(q; t, \xi) - dA(q; t, -\xi) = -Z(q; t, -\xi),$$

measures the non-classical behavior of the stochastic process, namely the larger the difference, the stronger is the deviation from classical mechanics. Hence $Z(\xi)$ is randomly fluctuating due to the fluctuations of $\xi$, and we shall use its distribution as the transition probability that we are looking for to construct the stochastic model:

$$P((q + dq; t + dt)|\{J(\xi), (q; t)\}) = P(Z(\xi)).$$

It is evident that the randomness is built into the statistical model in a fundamentally different way from that of classical Brownian motion. Unlike the latter in which the randomness is introduced by adding some random forces, the model is based on stochastic fluctuations of infinitesimal stationary action. Hence, we have assumed that the Lagrangian formalism based on energies is more fundamental than the Newtonian formalism based on forces. One may expect that this will explain the physical origin of the AB effect.
2.2. Exponential distribution of infinitesimal stationary action as the transition probability

How then is $Z(\xi)$ distributed? First, it is reasonable to assume that the transition probability must be decreasing as the non-classicality becomes stronger. Hence, the transition probability must be a decreasing function of the absolute value of $Z(\xi)$. There are infinitely many such probability distribution functions. Below, for a reason that will become clear later, we shall assume that $Z(\xi)$ is distributed according to the following exponential law:

$$P(Z) \propto N e^{(1/\lambda(\xi))Z(\xi)} = N e^{(1/\lambda(\xi))|dA(\xi) - dA(-\xi)|},$$

(6)

where $N$ is a factor independent of $Z(\xi)$, whose form is to be specified later, and $\lambda(\xi)$ is a non-vanishing function of $\xi$ with action dimensional, thus is randomly fluctuating. Note that, by definition, $Z(\xi)$ changes its sign as $\xi$ flips its sign: $Z(-\xi) = -Z(\xi)$. On the other hand, to guarantee the negative definiteness of the exponent in equation (6) for normalizability, $\lambda$ must always have the opposite sign of $Z(\xi)$. This demands that $\lambda$ must flip its sign as $\xi$ changes its sign. This fact therefore allows us to assume that both $\lambda(\xi)$ and $\xi$ always have the same sign. Hence the time scale for the fluctuations of the sign of $\lambda$ must be the same as that of $\xi$ given by $dt$.

However, it is clear that for the distribution of equation (6) to make sense mathematically, the time scale for the fluctuations of $|\lambda|$, denoted by $\tau_\lambda$, must be much larger than that of $|\xi|$, $\tau_\xi$. Let us further assume that $\tau_\lambda$ is much larger than $dt$. One thus has

$$\tau_\lambda \gg \tau_\xi \gg dt.$$  

(7)

Hence, in a time interval of length $\tau_\xi$, the absolute value of $\xi$ can be regarded constant while its sign may fluctuate randomly together with the sign of $\lambda$ in a time scale given by $dt$. Moreover, in a time interval $\tau_\lambda$, $|\lambda|$ is constant and $|\xi|$ fluctuates randomly so that the distribution of $Z(\xi)$ is given by the exponential law of equation (6) characterized by $|\lambda|$.

Next, let us introduce a new stochastic quantity $S(q; t, \xi)$ so that the differential along the path $J(\xi)$ is given by

$$dS(q; t, \xi) = \frac{dA(q; t, \xi) + dA(q; t, -\xi)}{2} = dS(q; t, -\xi).$$  

(8)

Subtracting $dA(q; t, \xi)$ from both sides, one has

$$dS(q; t, \xi) - dA(q; t, \xi) = \frac{dA(q; t, -\xi) - dA(q; t, \xi)}{2}.$$  

(9)

Using $dS$, the transition probability of equation (6) can thus be written as

$$P((q + dq; t + dt)|\{J(\xi), (q; t)\}) \propto N e^{-(2/\lambda)dS(q; t, \xi) - dA(q; t, \xi)} \equiv P_S(dS|dA).$$  

(10)

Since $dA(\xi)$ is just the infinitesimal stationary action along the path $J(\xi)$, then we shall refer to $dS(\xi) - dA(\xi)$ as a deviation from infinitesimal stationary action. One may therefore see the above transition probability to be given by an exponential distribution of deviation from infinitesimal stationary action $dS - dA$ parameterized by $|\lambda|$. It can also be regarded as the conditional probability density of $dS$ given $dA$, suggesting the use of the notation $P_S(dS|dA)$. The relevancy of such an exponential law to model microscopic stochastic deviation from classical mechanics was first suggested in [28].
Further, there is no *a priori* reason how the sign of \( dS(\xi) - dA(\xi) \), which is equal to the sign of \( Z(-\xi) = dA(-\xi) - dA(\xi) \) due to equation (9), should be distributed at any given spacetime point. The principle of insufficient reason (principle of indifference) [29] then suggests assuming that, at any spacetime point, there is equal probability for \( dS - dA \) to take positive or negative values. Since the sign of \( dS(\xi) - dA(\xi) \) changes as \( \xi \) flips its sign, then the sign of \( \xi \) must also be distributed equally probably. Hence, the probability density of the occurrence of \( \xi \) at any time, denoted below by \( P_H(\xi) \), must satisfy the following unbiased condition:

\[
P_H(\xi) = P_H(-\xi).
\] (11)

Let us note that \( P_H(\xi) \) may depend on time, thus it is in general not stationary. Since the sign of \( \lambda \) is always the same as that of \( \xi \), then the probability for the occurrence of \( \lambda \) must also satisfy the same unbiased condition

\[
P(\lambda) = P(-\lambda).
\] (12)

Moreover, from equation (8), one obtains, for a fixed value of \( \xi \), the following symmetry relations:

\[
\partial_q S(q; t, \xi) = \partial_q S(q; t, -\xi), \\
\partial_t S(q; t, \xi) = \partial_t S(q; t, -\xi).
\] (13)

Fixing \( |\lambda| \) in equation (10), then the average deviation from infinitesimal stationary action is given by

\[
|\lambda|/2.
\] (14)

One can then see that in the regime where the average deviation from infinitesimal stationary action is much smaller than the infinitesimal stationary action itself, namely \(|dA(\xi)|/|\lambda| \gg 1\), or formally in the limit \(|\lambda| \to 0\), equation (10) reduces to

\[
P_S(dS|dA) \to \delta(dS - dA),
\] (15)

or \( dS(\xi) \to dA(\xi) \), so that \( S \) satisfies the Hamilton–Jacobi equation of (2) by virtue of equation (1). Due to equation (9), in this regime one also has \( dA(\xi) = dA(-\xi) \). Hence such a limiting case must be identified to correspond to the macroscopic regime. This further suggests that \( |\lambda| \) must take a microscopic value.

### 2.3. Stochastic modification of Hamilton–Jacobi equation

Let us now derive a set of differential equations which characterizes the stochastic processes when the transition probability is given by equation (10). Let us consider a time interval of length \( \tau_\lambda \) in which \( |\lambda| \) is effectively constant. Recall that since \( \tau_\lambda \gg \tau_\xi \gg dt \), then within this time interval, \( dS(\xi) - dA(\xi) \) fluctuates randomly due to the fluctuations of \( \xi \), distributed according to the exponential law of equation (10) characterized by \( |\lambda| \).

Let us then denote the joint-probability density that at time \( t \) the configuration of the system is \( q \) and a random value of \( \xi \) is realized by \( \Omega(q, \xi; t) \). The marginal probability densities are thus given by

\[
\rho(q; t) = \int d\xi \Omega(q, \xi; t), \quad P_H(\xi) = \int dq \Omega(q, \xi; t).
\] (16)
To comply with equation (11), the joint-probability density must satisfy the following symmetry relation:

$$\Omega(q, \xi; t) = \Omega(q, -\xi; t).$$

(17)

Both equations (17) and (13) will play important roles later.

Let us then evolve $\Omega(q, \xi; t)$ along a time interval $\Delta t$ with $\tau_\xi \geq \Delta t \gg dt$ so that the absolute value of $\xi$ is effectively constant while its sign may fluctuate randomly. Given a fixed value of $\xi$, let us consider two infinitesimally close spacetime points $(q; t)$ and $(q + dq; t + dt)$. Let us assume that for this value of $\xi$, the two points are connected to each other by a segment of trajectory $\mathcal{J}(\xi)$ picked up by the principle of stationary action so that the differential of $S(\xi)$ along this segment is $dS(\xi)$, parameterized by $\xi$. Then for a fixed value of $\xi$, according to the conventional probability theory, the conditional joint-probability density that the system initially at $(q; t)$ traces the segment of trajectory $\mathcal{J}(\xi)$ and ends up at $(q + dq; t + dt)$, denoted below as $\Omega(\{(q + dq, \xi; t + dt),(q, \xi; t)\} | \mathcal{J}(\xi))$, is equal to the probability that the configuration of the system is $q$ at time $t$, $\Omega(q, \xi; t)$, multiplied by the transition probability between the two infinitesimally close points via the segment of trajectory $\mathcal{J}(\xi)$, which is given by equation (10). One thus has

$$\Omega(\{(q + dq, \xi; t + dt),(q, \xi; t)\} | \mathcal{J}(\xi))$$

$$= P(\{(q + dq; t + dt)| \mathcal{J}(\xi), (q; t)\}) \times \Omega(q, \xi; t)$$

$$\propto N e^{-\left(\frac{1}{\lambda(t)}\right)\left|dS(\xi) - dA(\xi)\right|} \times \Omega(q, \xi; t).$$

(18)

The above equation describing the dynamics of the ensemble of trajectories must give back the time evolution of classical mechanical ensemble of trajectories when $S$ approaches $A$. This requirement puts a constraint on the functional form of the factor $N$ in equation (10). To see this, let us assume that $N$ takes the following general form:

$$N \propto \exp(-\theta(S) dt),$$

(19)

where $\theta$ is a scalar function of $S$. Inserting this into equation (18), taking the limit $S \to A$ and expanding the exponential up to the first order one gets $\Omega(\{(q + dq, \xi; t + dt),(q, \xi; t)\} | \mathcal{J}(\xi)) \approx [1 - \theta(A) dt] \Omega(q, \xi; t)$, which can be further written as

$$d\Omega = -\theta(A) dt) \Omega,$$

(20)

where $d\Omega(q, \xi; t) = \Omega(\{(q + dq, \xi; t + dt),(q, \xi; t)\} | \mathcal{J}(\xi)) - \Omega(q, \xi; t)$ is the change of the probability density $\Omega$ due to the transport along the segment of trajectory $\mathcal{J}(\xi)$. Dividing both sides by $dt$ and taking the limit $dt \to 0$, one obtains $\dot{\Omega} + \theta(A) \Omega = 0$. To guarantee a smooth correspondence with classical mechanics, the above equation must be identified as the continuity equation describing the dynamics of an ensemble of classical trajectories. To do this, it is sufficient to choose $\theta(S)$ to be determined uniquely by the classical Hamiltonian as [28]

$$\theta(S) = \partial_q \cdot \left(\frac{\partial H}{\partial p} \bigg|_{p=q,S}\right),$$

(21)

so that, in the limit $S \to A$, it is given by the divergence of a classical velocity field.

Now, let us consider the case when $|\Delta S - 4A| / \lambda \ll 1$. Again, inserting equation (19) into (18) and expanding the exponential on the right hand side up to the first order, one

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Further, recalling that $\xi$ is fixed during the infinitesimal time interval $dt$, one can expand the differentials $d\Omega$ and $dS$ in equation (22) as $dF = \partial_t F \, dt + \partial_q F \cdot dq$. Using equation (1), one finally obtains the following pair of coupled differential equations:

$$
\begin{align*}
 p(\dot{q}) &= \partial_q S + \frac{\lambda}{2} \frac{\partial_q \Omega}{\Omega}, \\
 -H(q, p) &= \partial_t S + \frac{\lambda}{2} \frac{\partial_t \Omega}{\Omega} + \frac{\lambda}{2} \theta(S).
\end{align*}
$$

(23)

Some notes are in order. First, the above pair of relations are valid when $\xi$ is fixed. However, since, as discussed above, $P_S(dS|dA)$ is insensitive to the sign of $\xi$, which is always equal to the sign of $\lambda$, then the above pair of equations are valid in a microscopic time interval of length $\tau_\xi$ during which the magnitude of $\xi$, and also $\lambda$ due to equation (7), are constant while their signs may change randomly. To have an evolution for a finite time interval $\tau_\lambda > t > \tau_\xi$, one can proceed along the following approximation. First, one divides the time into a series of intervals of length $\tau_\xi$: $t \in [(k-1)\tau_\xi, k\tau_\xi)$, $k = 1, 2, \ldots,$ and attributes to each interval a random value of $\xi(t) = \xi_k$ according to the probability distribution $P_{\xi_k}(\xi_k) = P_{\xi_k}(-\xi_k)$. Hence, during the interval $[(k-1)\tau_\xi, k\tau_\xi)$, the magnitude of $\xi = \xi_k$ is kept constant while its sign may change randomly in a time scale $dt$, so that equation (23) is valid. One then applies the pair of equations in (23) during each interval of time with fixed $|\xi(t)| = |\xi_k|$, consecutively. Moreover, to have a time evolution for $t \geq \tau_\lambda$, one must now take into account the fluctuations of $|\lambda|$ with time.

It is evident that, as expected, in the formal limit $\lambda \rightarrow 0$, equation (23) reduces back to the Hamilton–Jacobi equation of (2). In this sense, equation (23) can be regarded as a generalization of the Hamilton–Jacobi equation due to the stochastic deviation from infinitesimal stationary action following the exponential law of equation (10). Unlike the Hamilton–Jacobi equation, in which we have a single unknown function $A$, however, to calculate the velocity or momentum and energy, one now needs a pair of unknown functions $S$ and $\Omega$. The relations in equation (23) must not be interpreted as the momentum and energy of the particles being determined causally by the gradient of the probability density $\Omega$ (or $\ln(\Omega)$), which is physically absurd. Rather it is the other way around, as shown explicitly by equation (22).

Let us then consider a system of two non-interacting particles whose configuration is denoted by $q_1$ and $q_2$. The Lagrangian is thus decomposable as $L(q_1, q_2, \dot{q_1}, \dot{q_2}) = L_1(q_1, \dot{q_1}) + L_2(q_2, \dot{q_2})$, so that the infinitesimal stationary action is also decomposable: $dA(q_1, q_2) = dA_1(q_1) + dA_2(q_2)$, and accordingly one has $dS(q_1, q_2) = dS_1(q_1) + dS_2(q_2)$ by virtue of equation (8). On the other hand, since the classical Hamilton $H$ is decomposable as $H(q_1, q_2, p_1, p_2) = H_1(q_1, p_1) + H_2(q_2, p_2)$, $p_i, i = 1, 2,$ is the classical momentum of the $i$-particle, then $\theta$ of equation (21) is also decomposable: $\theta(q_1, q_2) = \theta_1(q_1) + \theta_2(q_2)$. Inserting all these into equations (19) and (10), then one can see that the distribution of deviation from infinitesimal stationary action for the two non-interacting particles is separable as

$$
P_S(dS_1 + dS_2|dA_1 + dA_2) = P_S(dS_1|dA_1)P_S(dS_2|dA_2).$$

(24)
Namely, the joint-probability distribution of the deviations from infinitesimal stationary action of the two-particle system is separable into the probability distribution of the deviation with respect to each single particle. They are thus independent of each other, as intuitively expected for non-interacting particles. It is interesting to remark that the above statistical separability for non-interacting particles is unique to the exponential law. A Gaussian distribution of deviation from infinitesimal stationary action, for example, does not have such a property.

3. Stochastic processes for quantum measurement

In this section, we shall apply the above statistical model of microscopic stochastic deviation from classical mechanics to a classical mechanical model of measurement consisting of two interacting particles, one is regarded as the system whose physical properties are being measured and the other plays the role as the apparatus pointer. We have to admit that such a model of measurement apparatus by a single particle is too simple; it is inspired by rather than describes completely a realistic measurement. Nevertheless, we shall show that the model reproduces the prediction of standard quantum mechanics. We also believe that the approach can in principle be generalized to model realistic macroscopic apparatus. See for example [30] for richer models of quantum measurement with a realistic apparatus.

In the paper we shall only give the details for the measurement of angular momentum. The measurement of position and linear momentum can be done in exactly the same way.

For completeness of the presentation, in the appendix, we reproduce the application of the statistical model to a system of particles subjected to potentials with a Hamiltonian that is quadratic in momentum, as reported in [28]. Several subtle issues in [28] are clarified.

3.1. Measurement in classical mechanics

Let us first briefly discuss the essential points of a measurement model in classical mechanics consisting of two interacting particles. To do this, let us assume that the interaction classical Hamiltonian is given by

\[ H_1 = gO_1(q_1, p_1)p_2. \]  

(25)

Here \( g \) is an interaction coupling and \( O_1(q_1, p_1) \) is a physical quantity referring to the first particle. Let us further assume that the interaction is impulsive (\( g \) is sufficiently strong) so that the single-particle Hamiltonians of each particle are ignorable. The discussion on single-particle Hamiltonian is given in the appendix.

The interaction Hamiltonian above can be used as a classical mechanical model of measurement of the classical physical quantity \( O_1(q_1, p_1) \) of the first particle by regarding the position of the second particle as the pointer of the apparatus of measurement. To see this, first, in such a model \( O_1 \) is conserved: \( \dot{O}_1 = \{O_1, H_1\} = 0 \) where \( \{\cdot, \cdot\} \) is the usual Poisson bracket. The interaction Hamiltonian of equation (25) then correlates the value of \( O_1 \) with the momentum of the second particle \( p_2 \) while keeping the value of \( O_1 \) unchanged. On the other hand, one also has \( \dot{q}_2 = \{q_2, H_1\} = gO_1 \), which, by virtue of the fact that \( O_1 \)
is a constant of motion, can be integrated to give

\[ q_2(t_M) = q_2(0) + gO_1 t_M, \]  

(26)

where \( t_M \) is the time span of the measurement-interaction. The value of \( O_1 \) prior to the measurement can thus in principle be inferred from the observation of the initial and final values of \( q_2 \).

In this way, the measurement of the physical quantity \( O_1(q_1, p_1) \) of the first particle is reduced to the measurement of the position of the second particle \( q_2 \). In the model, \( q_2(t) \) therefore plays the role as the pointer of the apparatus of measurement. This is in principle what is actually done in experiment, either involving macroscopic or microscopic objects, where one reads the position of the needle in the meter or the position of the detector that ‘clicks’, etc. It is thus assumed that measurement of position can in principle be done straightforwardly. To have a physically and operationally smooth quantum–classical correspondence, we shall keep this ‘operationally clear’ measurement mechanism while we proceed below to subject the classical system to a stochastic fluctuations of infinitesimal stationary action according to the statistical model discussed in section 2.

It is also obvious from the above exposition that, in classical mechanics, each single measurement event reveals the value of the physical quantity under interest prior to the measurement up to the precision of position measurement of the pointer. In particular, there is a one to one mapping between the continuous values of the pointer \( q_2(t_M) \) and the continuous possible values of the physical quantity being measured \( O_1(q_1, p_1) \) prior to the measurement. We shall show below that, in the statistical model, this is in general no longer the case.

### 3.2. The Schrödinger equation in the measurement of angular momentum

Now let us apply the statistical model discussed in the previous section to stochastically modify the above classical mechanical model of measurement. Let us first consider a time interval of length \( \tau_\lambda \) in which the absolute value of \( \lambda \) is effectively constant while its sign is allowed to fluctuate randomly together with random fluctuations of the sign of \( \xi \). Let us then divide it into a series of microscopic time intervals of length \( \tau_\xi \), \( [(k-1)\tau_\xi, k\tau_\xi) \), \( k = 1, 2, \ldots \), and attribute to each interval a random value of \( \xi(t) = \xi_k \) according to the probability distribution \( P_{H_k}(\xi_k) = P_{H_k}(-\xi_k) \), so that in each interval the magnitude of \( \xi \) is constant while its sign is allowed to change randomly. During each time interval \( [(k-1)\tau_\xi, k\tau_\xi) \), the pair of equations in equation (23), each with constant value of \( |\xi_k| \), thus apply.

For concreteness, let us consider the measurement of the \( z \)-part of the angular momentum of the first particle. The classical interaction Hamiltonian of equation (25) then reads

\[ H_I = g l_{z_1} p_2, \quad \text{with} \quad l_{z_1} = x_1 p_{y_1} - y_1 p_{x_1}. \]  

(27)

Let us first consider a microscopic time interval \( [(k-1)\tau_\xi, k\tau_\xi) \). Using the above form of \( H_I \) to express \( \dot{q} \) in term of \( p \) via the (kinematic part of the) usual Hamilton equation
\( \dot{q} = \partial H/\partial p \), the upper equation of (23) becomes

\[
\dot{x}_1 = -gy_1 \left( \partial_{q_2} S + \frac{\lambda}{2} \frac{\partial_{q_2} \Omega}{\Omega} \right), \quad \dot{y}_1 = gx_1 \left( \partial_{q_2} S + \frac{\lambda}{2} \frac{\partial_{q_2} \Omega}{\Omega} \right),
\]

and \( \dot{z}_1 = 0 \). Assuming that the probability is conserved, one gets, after a simple calculation, the following continuity equation:

\[
0 = \partial_t \Omega + \dot{q} \cdot (q \Omega) = \partial_t \Omega - gy_1 \partial_x (\Omega \partial_{q_2} S) + gx_1 \partial_y (\Omega \partial_{q_2} S) + gx_1 \partial_q (\Omega \partial_{q_2} S).
\]

(28)

On the other hand, from equation (27), \( \theta(S) \) of equation (21) is given by

\[
\theta(S) = 2g(x_1 \partial_{q_2} \partial_{q_1} S - y_1 \partial_{q_2} \partial_{x_1} S).
\]

(30)

Substituting this into the lower equation of (23), one then obtains

\[
-H_1(q, p(\dot{q})) = \partial_t S + \frac{\lambda}{2} \frac{\partial_{q_2} \Omega}{\Omega} + g \lambda(x_1 \partial_y \partial_{q_2} S - y_1 \partial_x \partial_{q_2} S).
\]

(31)

Inserting the upper equation of (23) into the left-hand side of the above equation, and using (27), one has, after arrangement,

\[
\partial_t S + g(x_1 \partial_{q_2} S - y_1 \partial_{x_1} S) \partial_{q_2} S - g \lambda^2 \left( x_1 \frac{\partial_y \partial_{q_2} R}{R} - y_1 \frac{\partial_x \partial_{q_2} R}{R} \right) \\
+ \frac{\lambda}{2\Omega} \left( \partial_{q_2} \Omega - gy_1 \partial_x (\Omega \partial_{q_2} S) + gx_1 \partial_y (\Omega \partial_{q_2} S) + gx_1 \partial_q (\Omega \partial_{q_2} S) \right) \\
- gy_1 \partial_q (\Omega \partial_{x_1} S) + g \lambda(y_1 \partial_x \partial_{q_2} S - x_1 \partial_y \partial_{q_2} S) = 0,
\]

(32)

where \( R = \sqrt{\Omega} \) and we have used the identity:

\[
\frac{1}{4} \frac{\partial_{q_2} \Omega}{\Omega} = \frac{1}{2} \frac{\partial_{q_2} \Omega}{\Omega} - \frac{\partial_{q_2} \Omega}{\Omega}.
\]

(33)

Substituting equation (29), the last term of equation (32) in the bracket vanishes to give

\[
\partial_t S + g(x_1 \partial_{q_2} S - y_1 \partial_{x_1} S) \partial_{q_2} S - g \lambda^2 \left( x_1 \frac{\partial_y \partial_{q_2} R}{R} - y_1 \frac{\partial_x \partial_{q_2} R}{R} \right) = 0.
\]

(34)

One thus has a pair of coupled equations (29) and (34) which are parameterized by \( \lambda \). Recall that this pair of equations is valid in a microscopic time interval of length \( \tau_\xi \) during which the magnitude of \( \xi \) is constant while its sign is allowed to change randomly with equal probability. Averaging equation (29) for the cases \( \pm \xi \), one has, by virtue of equations (13) and (17),

\[
\partial_t \Omega - gy_1 \partial_x (\Omega \partial_{q_2} S) + gx_1 \partial_y (\Omega \partial_{q_2} S) + gx_1 \partial_q (\Omega \partial_{q_2} S) - gy_1 \partial_q (\Omega \partial_{x_1} S) = 0.
\]

(35)

Similarly, averaging equation (34) over the cases \( \pm \xi \), thus also over \( \pm \lambda \), will not change anything. We thus finally have a pair of equations (34) and (35) which are now

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parameterized by $|\lambda|$ valid for a microscopic time interval of duration $\tau_\xi$ characterized by a constant $|\xi|$.

Next, since $|\lambda|$ is non-vanishing, one can define the following complex-valued function:

$$\Psi = \sqrt{\Omega} \exp\left(\frac{i}{|\lambda|} S\right).$$

Using $\Psi$ and recalling that $|\lambda|$ is constant during the microscopic time interval of interest with length $\tau_\lambda$, the pair of equations (34) and (35) can then be recast into the following compact form:

$$i|\lambda|\partial_t \Psi = \frac{\lambda^2}{\hbar^2} \hat{H}_I \Psi.$$  \hfill (37)

Here $\hat{H}_I$ is a differential operator defined as

$$\hat{H}_I = g \hat{l}_z \hat{p}_2,$$  \hfill (38)

where $\hat{p}_i = -i\hbar \partial_{q_i}$, $i = 1, 2$ is the quantum mechanical momentum operator referring to the $i$-particle and $\hat{l}_z = x_1 \hat{p}_y - y_1 \hat{p}_x$ is the $z$-part of the quantum mechanical angular momentum operator of the first particle; all are Hermitian. Recall that equation (37) is valid only for a microscopic time interval $[(k-1)\tau_\xi, k\tau_\xi)$ during which $|\xi| = |\xi_k|$ is constant. For a finite time interval $t > \tau_\xi$, one must then apply equation (37) to each time interval, each of which is parameterized by a random value of $|\xi_k|$, $k = 1, 2, \ldots$, consecutively.

Let us then consider a specific case when $|\lambda|$ is given by the reduced Planck constant $\hbar$, namely $\lambda = \pm \hbar$, with equal probability for all the time, so that the average of the deviation from infinitesimal stationary action distributed according to the exponential law of equation (10) is given by

$$\hbar / 2.$$  \hfill (39)

Moreover, let us assume that $P_H(\xi)$ is stationary in time and the fluctuations of $|\xi|$ around its average are sufficiently narrow so that $\Omega(q, |\xi|; t)$ and $S(q; t, |\xi|)$ can be approximated by the corresponding zeroth order terms in their Taylor expansion around the average of $|\xi|$, denoted respectively by $\rho_Q(q; t)$ and $S_Q(q; t)$. In this specific case, the zeroth order approximation of equation (37) then reads

$$i\hbar \partial_t \Psi_Q(q; t) = \hat{H}_I \Psi_Q(q; t),$$

$$\Psi_Q(q; t) = \sqrt{\rho_Q(q; t)} e^{(i/\hbar)S_Q(q; t)}.$$  \hfill (40)

Unlike equation (37), equation (40) is now deterministically parameterized by the reduced Planck constant $\hbar$. Moreover, from equation (40), Born’s statistical interpretation of wavefunction is valid by construction

$$\rho_Q(q; t) = |\Psi_Q(q; t)|^2.$$  \hfill (41)

Equation (40) together with equation (38) is just the Schrödinger equation for the von Neumann model of measurement of angular momentum of the first particle using the second particle as the apparatus.

On the other hand, solving equation (37) for each time interval of length $\tau_\xi$ with a constant value of $|\xi|$, and inserting the modulus and phase of $\Psi$ into equation (28), we
obtain the time evolution of the velocities of both of the particles, which are randomly fluctuating due to the fluctuations of $\xi$. Recall again that equation (28) is valid in a time interval of length $\tau_\xi$ in which the absolute value of $\xi$ is effectively constant while its sign fluctuates randomly with equal probability. It is then tempting to define an ‘effective’ velocity as the average of the values of $\dot{q}$ at $\pm \xi$:
\[
\tilde{\dot{q}}(|\xi|) = \frac{\dot{q}(\xi) + \dot{q}(-\xi)}{2}.
\] (42)

When the actual velocities are given by equation (28), recalling that the sign of $\lambda$ is the same as that of $\xi$, one has, due to equations (13) and (17),
\[
\begin{align*}
\tilde{x}_1 &= -gy_1 \partial_{q_2} S_Q, \\
\tilde{y}_1 &= gx_1 \partial_{q_2} S_Q, \\
\tilde{q}_2 &= g(x_1 \partial_{y_1} S_Q - y_1 \partial_{x_1} S_Q),
\end{align*}
\] (43)

where we have counted only the zeroth order terms. Unlike equation (28), equation (43) is now deterministic, due to the deterministic time evolution of $S_Q$ given by the Schrödinger equation of (40).

3.3. A single measurement event, its ensemble and the Born’s rule

Let us now discuss the process of a single measurement event. From now on, we shall work with equation (40) instead of with equation (37). To do this, let $\phi_l(q_1)$ denote the eigenfunction of the angular momentum operator $\hat{l}_{z_1}$ belonging to an eigenvalue $\omega_l$: $\hat{l}_{z_1} \phi_l(q_1) = \omega_l \phi_l(q_1)$, $l = 0, 1, 2, \ldots$ $\{\phi_l\}$ thus makes a complete set of orthonormal functions. Then, ignoring the single-particle Hamiltonians for impulsive interaction, the Schrödinger equation of (40) has the following general solution:
\[
\Psi_Q(q_1, q_2; t) = \sum_l c_l \phi_l(q_1) \varphi(q_2 - g\omega_l t),
\] (44)

where $\varphi(q_2)$ is the initial wavefunction of the apparatus (the second particle), which is assumed to be sufficiently localized in $q_2$, $\{c_l\}$ are complex numbers, and
\[
\phi(q_1) \doteq \sum_l c_l \phi_l(q_1),
\] (45)

is the initial wavefunction of the system. $c_l$ is thus the coefficient of expansion of the initial wavefunction of the system in terms of the orthonormal set of the eigenfunctions of $\hat{l}_{z_1}$:
\[
c_l = \int dq_1 \phi_l^*(q_1) \phi(q_1).
\] (46)

Initially, the total wavefunction is thus separable as
\[
\Psi_Q(q_1, q_2; 0) = \left( \sum_l c_l \phi_l(q_1) \right) \varphi(q_2).
\] (47)

It evolves into an inseparable (entangled) wavefunction of equation (44) via the linear Schrödinger equation of (40) with the interaction quantum Hamiltonian given by equation (38). One can then see in equation (44) that for sufficiently large $g$, the set

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of wavefunctions
\[ \{ \varphi_l(q_2; t_M) = \varphi(q_2 - g\omega_l t_M) \}, \]  
are not overlapping for different \( l \) and each is correlated to a distinct \( \phi_l(q_1) \).

Further, to have a physically and operationally smooth quantum–classical correspondence, one must let \( q_2(t_M) \) have the same physical and operational status as the underlying classical mechanical system: namely, it must be regarded as the pointer of the measurement, the reading of our experiment. One may then infer that the ‘outcome’ of a single measurement event corresponds to the packet \( \varphi_l(q_2; t_M) \) whose support is actually entered by the apparatus particle. Namely, if \( q_2(t_M) \) belongs to the spatially localized support of \( \varphi_l(q_2; t_M) \), then we operationally admit that the outcome of the measurement is given by \( \omega_l \), the eigenvalue of \( \hat{l}_z \) whose corresponding eigenfunction \( \phi_l(q_1) \) is correlated with \( \varphi_l(q_2; t_M) \). The probability that the measurement yields \( \omega_l \) is thus equal to the frequency that \( q_2(t_M) \) enters the support of \( \varphi_l(q_2; t_M) \) in a large (in principle infinite) number of identical experiments.

It then remains to calculate the probability that \( q_2(t_M) \) belongs to the support of \( \varphi_l(q_2; t_M) \) given the initial wavefunction of the system \( \phi(q_1) = \sum_l c_l \phi_l(q_1) \). To do this, first, since for sufficiently large value of \( g \), \( \{ \varphi_l(q_2; t_M) \} \) in equation (44) does not overlap for different values of \( l \), then the joint-probability density that the first particle (system) is at \( q_1 \) and the second particle (apparatus) is at \( q_2 \) is, by virtue of equation (41), decomposed into
\[ \rho_Q(q; t_M) = |\Psi_Q(q; t_M)|^2 = \sum_l |c_l|^2 |\phi_l(q_1)|^2 |\varphi_l(q_2; t_M)|^2, \]  
namely the cross-terms are all vanishing. From the above equation, one can see that the joint-probability density that the first particle has coordinate \( q_1 \) and the second particle has coordinate \( q_2 \) inside the support of \( \varphi_l(q_2; t_M) \) is given by
\[ |c_l|^2 |\phi_l(q_1)|^2 |\varphi_l(q_2; t_M)|^2. \]  
The probability density that the second particle is inside the support of the wavepacket \( \varphi_l(q_2; t_M) \) regardless of the position of the first and second particles is thus
\[ P_{\omega_l} = \int dq_1 dq_2 |c_l|^2 |\phi_l(q_1)|^2 |\varphi_l(q_2; t_M)|^2 = |c_l|^2, \]  
which is just Born’s rule.

3.4. Discussion

As noted at the beginning of the section, in reality the above model of measurement with one-dimensional apparatus is oversimplified. Especially, the model excludes the irreversibility of the registration process, which can only be done by realistic apparatus plus a bath with large (macroscopic) degrees of freedom. See [30] for an elaborated discussion of quantum measurements having a realistic model of apparatus.

Now, notice that, as for the quantum Hamiltonian \( \hat{H}_I \), the quantum mechanical angular momentum operator \( \hat{l}_z \) appears formally when one works in Hilbert space by defining the wavefunction \( \Psi \) as in equation (36), or its zeroth order term \( \Psi_Q \), satisfying the linear Schrödinger equation of (40). Hence, the Hermitian operator and wavefunction
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are artificial convenient mathematical tools for calculation purposes with no fundamental ontology. Moreover, the Hermitian angular momentum operator \( \hat{L}_z \) emerges in the context of modeling a microscopic stochastic correction of the underlying classical mechanical model of measurement of angular momentum. The above results suggest that not all Hermitian operators are relevant for physics and conversely not all relevant and in principle observable physical quantities, such as position, time, mass etc, have to be represented by Hermitian operators.

To have physically and operationally smooth correspondence with the underlying classical mechanical model of measurement discussed at the beginning of the section, we have kept regarding the position of the second particle as the pointer reading of the measurement. Namely, the outcome of each single measurement is inferred operationally from the position of the pointer. We have shown, however, that unlike the classical mechanical case, in which the outcome of measurement may take arbitrary continuous values, the set of possible values of the outcome of the measurement of angular momentum in the statistical model is discrete, given by one of the eigenvalues of the angular momentum operator \( \hat{L}_z \). The statistical model thus explicitly describes the physical and operational origin of the quantum discreteness of measurement results of angular momentum.

Let us suppose that in a single measurement event \( q_2(t_M) \) belongs to the support of \( \varphi_l(q_2; t_M) \) so that we operationally infer that the measurement yields \( \omega_l \). If the measurement is not destructive, then immediately after the first measurement, \( q_2(t) \) will still belong to \( \varphi_l(q_2; t) \), such that repeating the measurement will naturally yield the same value as the previous one, \( \omega_l \). Moreover, in this case, right after the measurement, \( \Psi_{Q_i} = \phi_l(q_1) \varphi_l(q_2; t) \) is the effective or relevant wavefunction of the whole system + apparatus. This is due to the fact that \( q_2(t) \) is inside the support of \( \varphi_l(q_2; t) \) which is not overlapping with \( \varphi_l'(q_2; t) \), \( l' \neq l \), and \( q_2(t) \) cannot pass through the nodes of the wavefunction. This situation is what is effectively regarded as the projection of the initial wavefunction \( \phi(q_1) = \sum_i c_i \phi_i(q_1) \) of the system onto the corresponding eigenfunction \( \phi_i(q_1) \) of the measurement result \( \omega_l \), one of the eigenvalues of the angular momentum operator. We have thus an effective wavefunction collapse as one of the implications of the statistical model, rather than standing as an independent postulate as in standard quantum mechanics. Let us emphasize that, during the process of measurement, the wavefunction of the whole system + apparatus satisfies the Schrödinger equation of (40). There is thus no real wavefunction collapse. Hence, measurement is just a specific type of physical interaction following the same general law of dynamics and statistics.

In the statistical model, the ‘actual’ value of angular momentum prior to the measurement when the wavefunction is \( \phi(q_1) = \sum_i c_i \phi_i(q_1) \) can take any continuous real numbers given by

\[
l_z = x_1 p_y - y_1 p_x = x_1 \left( \partial_{y_1} S_{Q_1} + \frac{\lambda \partial_{y_1} \Omega_{Q_1}}{\Omega_{Q_1}} \right) - y_1 \left( \partial_{x_1} S_{Q_1} + \frac{\lambda \partial_{x_1} \Omega_{Q_1}}{\Omega_{Q_1}} \right),
\]

where \( \phi = \sqrt{\Omega_{Q_1}} \exp(i S_{Q_1} / \hbar) \), and we have used the upper equation in (23). By contrast, in the above statistical model, each single measurement event will yield only discrete possible real numbers, one of the eigenvalues of the angular momentum operator \( \hat{L}_z \). Hence, one may conclude that the outcome of each single measurement does not in general reveal the ‘actual’ value of angular momentum of the system prior to the measurement.
Next, calculating the average of the angular momentum of the first particle prior to measurement over the distribution of the configuration, one gets
\[
\langle l_z \rangle = \int dq_1 d\xi \left( x_1 \left( \partial_{q_1} S_{\xi,1} + \frac{\lambda}{2} \frac{\partial_{\Omega_{Q_1}}}{\Omega_{Q_1}} \right) - y_1 \left( \partial_{x_1} S_{\xi,1} + \frac{\lambda}{2} \frac{\partial_{\Omega_{Q_1}}}{\Omega_{Q_1}} \right) \right) \Omega_{Q_1} \\
= \int dq_1 \left( x_1 \partial_{q_1} S_{\xi,1} - y_1 \partial_{x_1} S_{\xi,1} \right) \Omega_{Q_1} \\
= \int dq_1 \phi^*(q_1) \hat{l}_z \phi(q_1), \tag{53}
\]
where in the first equality we have used equations (13) and (17), and taken into account the fact that the sign of \( \lambda \) is the same as that of \( \xi \). On the other hand, calculating the average of the results of measurement, one obtains
\[
\langle \hat{l}_z \rangle = \sum_l \omega_l P_{\omega_l} = \sum_l \omega_l |c_l|^2 = \int dq_1 \phi(q_1)^* \hat{l}_z \phi(q_1) = \langle l_z \rangle, \tag{54}
\]
where in the first equality we have used equation (51), in the second equality we have used equation (45) and the last equality is just equation (53). Hence, the average of measurement outcomes in an ensemble of identical measurement is equal to the average of the actual value of angular momentum of the system prior to the measurement over the distribution of the configuration.

Let us now ask: what is the actual value of the angular momentum of the first particle right after a measurement which yields \( \omega_l \)? Notice that, in this case, the relevant wavefunction is given by \( \phi_1 \). Writing in polar form \( \phi_l = \sqrt{\Omega_{Q_1}} \exp(i S_{\xi,1}^l) \), one then has
\[
l_z = x_1 \left( \partial_{q_1} S_{\xi_1}^l + \frac{\lambda}{2} \frac{\partial_{\Omega_{Q_1}}}{\Omega_{Q_1}} \right) - y_1 \left( \partial_{x_1} S_{\xi_1}^l + \frac{\lambda}{2} \frac{\partial_{\Omega_{Q_1}}}{\Omega_{Q_1}} \right), \tag{55}
\]
Averaging its values at \( \pm \xi \), the effective value reads
\[
\bar{l}_z(\xi) = \frac{l_z(\xi) + l_z(-\xi)}{2} = x_1 \partial_{q_1} S_{\xi,1}^l - y_1 \partial_{x_1} S_{\xi,1}^l = \frac{\text{Re} \{ \phi_l^* \hat{l}_z \phi_l \}}{|\phi_l|^2} = \omega_l, \tag{56}
\]
where in the first equality we have made use equations (13) and (17) and taken into account the fact that the sign of \( \lambda \) is the same as that of \( \xi \). Hence, when a measurement yields \( \omega_l \), the value of the effective angular momentum \( \bar{l}_z \) of the system right after the measurement is also given by \( \omega_l \).

Finally, let us mention without giving the technical details that the measurement of position and linear momentum can be done by exactly following all the steps for the measurement of angular momentum discussed above. One only needs to put \( O_1 = p_1 \) and \( O_1 = q_1 \) in equation (25) for the case of measurement of linear momentum and position, respectively. Repeating all the steps for the measurement of angular momentum, one will get a Schrödinger equation with the quantum Hamiltonian \( \hat{H}_l = g \hat{O}_1 \hat{P}_2 \), where \( \hat{O}_1 = \hat{p}_1 = -i \hbar \partial_{q_1} \) and \( \hat{O}_1 = \hat{q}_1 = q_1 \), respectively. All the qualitative and quantitative results for the measurement of angular momentum derived above then apply to the measurement of position and linear momentum. Note, however, that the spectrum of eigenvalues of \( \hat{p} \) and \( \hat{q} \) are continuous. The measurement of energy should be reduced to the measurement of position, linear and angular momentum.
4. Conclusion

We have developed a statistical model of microscopic stochastic deviation from classical mechanics based on stochastic processes with a transition probability between two infinitesimally close spacetime points along a random path that is given by an exponential distribution of infinitesimal stationary action. We then applied the stochastic model to a classical mechanical model for the measurement of angular momentum. In the statistical model, the system always has a definite configuration all the time, as in classical mechanics, following a randomly fluctuating continuous trajectory, regarded as the beable of the theory. On the other hand, we showed that the quantum mechanical Hermitian differential operator corresponding to the angular momentum arises formally together with the wavefunction as artificial convenient mathematical tools. In the model, the wavefunction is therefore neither physical nor complete.

Reading the pointer of the corresponding classical system to operationally infer the results of the measurement, the model reproduces the prediction of quantum mechanics that each single measurement event yields randomly one of the eigenvalues of the Hermitian angular momentum operator with a probability given by Born’s rule. Moreover, during a single measurement event, the wavefunction of the system + apparatus evolves according to the Schrödinger equation so that there is no wavefunction collapse. We have thus a physically and operationally smooth correspondence between measurement in macroscopic and microscopic worlds.

We have also shown that while the outcome of each single measurement event does not reveal the actual value of the angular momentum prior to measurement, its average in an ensemble of identical measurements is equal to the average of the actual value of the angular momentum prior to measurement over the distribution of the configuration. Moreover, we have shown that right after a single measurement, the effective value of angular momentum is equal to the outcome of the measurement.

Given the above results, it is then imperative to further ask, within the spirit of the reconstruction program, why the distribution of deviation from infinitesimal stationary action is given by the exponential law among the infinitude of possible distributions? Why Gaussian (say) will not work. It is then interesting to find a set of conceptually simple and physically transparent axioms which select uniquely the exponential law and to elaborate its relation with the characteristic traits of quantum mechanics.

Finally, recall that the predictions of quantum mechanics are reproduced as the zeroth order approximation of the stochastic model for a specific choice of the free parameter of the transition probability so that the average deviation from the infinitesimal stationary action is given by $\hbar/2$. It is then interesting to go beyond the zeroth order term, and to elaborate the case when the average deviation from the infinitesimal stationary action is deviating slightly from $\hbar/2$. These cases might therefore provide precision tests against quantum mechanics.

Appendix. Quantization of classical system of a single particle subjected to external potentials

To show the robustness of the model, we shall apply the statistical model to stochastically modify a classical system of a single particle subjected to external potentials so that the
classical Hamiltonian takes the following form:

$$H(q, p) = \frac{g^{ij}(q)}{2}(p_i - a_i)(p_j - a_j) + V,$$  \hspace{1cm} (A.1)

where $a_i(q), i = x, y, z$ and $V(q)$ are vector and scalar potentials respectively, the metric $g^{ij}(q)$ may depend on the position of the particle, and summation over repeated indices are assumed. This is a reproduction of the results reported in [28]. Note however that a couple of important assumptions put heuristically in [28], that of equations (13) and (17), are given physical argumentation in the present work. Below, we shall repeat all the steps we have taken to quantize the classical mechanical model for the measurement of angular momentum in the main text.

Let us first consider a time interval of length $\tau_\lambda$ during which the absolute value of $\lambda$ is effectively constant while its sign is allowed to fluctuate randomly together with the random fluctuations of the sign of $\xi$ in a time scale $d\tau$. Let us then divide it into a series of time intervals of length $\tau_\xi$, $[(k-1)\tau_\xi, k\tau_\xi), k = 1, 2, \ldots$, and attribute to each interval a random value of $\xi = \xi_k$ according to the probability distribution $P_{Hk}(\xi_k) = P_{Hk}(-\xi_k)$. Hence, in each interval, the magnitude of $\xi = \xi_k$ is constant while its sign is allowed to change randomly and the pair of equations in (23) with fixed $|\xi_k|$ apply.

Let us now consider a microscopic time interval $[(k-1)\tau_\xi, k\tau_\xi)$. Within this interval of time, using equation (A.1) to express $\dot{q}$ in term of $p$ via the Hamilton equation $\dot{q} = \partial H/\partial p$, one has, by the virtue of the upper equation of (23)

$$\dot{\xi}(\xi) = g^{ij}\left(\partial_{\eta_j}S(\xi) + \frac{\lambda(\xi)}{2} \frac{\partial_{\eta_j}\Omega(\xi)}{\Omega(\xi)} - a_j\right).$$  \hspace{1cm} (A.2)

Again, assuming the conservation of probability, which is valid for the closed system we are considering, one obtains the following continuity equation:

$$0 = \partial_t \Omega + \partial_q \cdot (q \Omega) = \partial_t \Omega + \partial_{\eta_i}(g^{ij}(\partial_{\eta_j}S - a_j)\Omega) + \frac{\lambda}{2} \partial_{\eta_i}(g^{ij}\partial_{\eta_j}\Omega).$$  \hspace{1cm} (A.3)

On the other hand, from equation (A.1), $\theta(S)$ of equation (21) is given by

$$\theta(S) = \partial_{\eta_i}g^{ij}(\partial_{\eta_j}S - a_j).$$  \hspace{1cm} (A.4)

The lower equation of (23) thus becomes

$$-H(q, p(\xi)) = \partial_t S + \frac{\lambda}{2} \frac{\partial_{\eta_i}\Omega}{\Omega} + \frac{\lambda}{2} \partial_{\eta_i}g^{ij}(\partial_{\eta_j}S - a_j).$$  \hspace{1cm} (A.5)

Plugging the upper equation of (23) into the left-hand side of equation (A.5) and using equation (A.1) one has, after an arrangement

$$\partial_t S + \frac{g^{ij}}{2}(\partial_{\eta_i}S - a_i)(\partial_{\eta_j}S - a_j) + V - \frac{\lambda^2}{2} \left(g^{ij}\frac{\partial_{\eta_i}R}{R} + g^{ij}\frac{\partial_{\eta_i}R}{R}\right)$$

$$+ \frac{\lambda}{2\Omega}\left(\partial_t \Omega + \partial_{\eta_i}(g^{ij}(\partial_{\eta_j}S - a_j)\Omega) + \frac{\lambda}{2} \partial_{\eta_i}(g^{ij}\partial_{\eta_j}\Omega)\right) = 0,$$  \hspace{1cm} (A.6)
where $R = \sqrt{\Omega}$ and we have again used the identity of equation (33). Inserting equation (A.3), the last line of equation (A.6) vanishes to give
\[
\partial_t S + \frac{g^{ij}}{2} (\partial_{q_i} S - a_i) (\partial_{q_j} S - a_j) + V - \frac{\lambda^2}{2} \left( \frac{g^{ij} \partial_{q_i} \partial_{q_j} R}{R} + \partial_{q_i} g^{ij} \partial_{q_j} R \right) = 0. \tag{A.7}
\]

We thus have a pair of coupled equations (A.3) and (A.7) which are parameterized by $\lambda(\xi)$. Recall that the above pair of equations is valid in a microscopic time interval of length $\tau_\xi$ during which the magnitude of $\xi$ is constant while its sign is allowed to change randomly with equal probability. Moreover, recall also that the sign of $\lambda$ is always the same as the sign of $\xi$. Keeping this in mind, averaging equation (A.3) for the cases $\pm \xi$, thus is also over $\pm \lambda$, one has, by virtue of equations (13) and (17),
\[
\partial_t \Omega + \partial_{q_i} (g^{ij}(\partial_{q_j} S - a_j) \Omega) = 0. \tag{A.8}
\]

Similarly, averaging equation (A.7) for the cases $\pm \xi$ will not change anything. We thus finally have a pair of coupled equations (A.7) and (A.8) which are now parameterized by a constant $|\lambda|$, valid during a microscopic time interval of length $\tau_\xi$ characterized by a constant $|\xi|$.

Using $\Psi$ defined in equation (36), and recalling the assumption that $|\lambda|$ is constant during the time interval of interest, the pair of equations (A.7) and (A.8) can then be recast into the following modified Schrödinger equation:
\[
i |\lambda| \partial_t \Psi = \frac{\hbar}{2} (-i |\lambda| \partial_{q_i} - a_i) g^{ij}(q)(-i |\lambda| \partial_{q_j} - a_j) \Psi + V \Psi. \tag{A.9}
\]

Notice that the above equation is valid only for a microscopic time interval $[(n-1)\tau_\xi, n\tau_\xi]$ during which the magnitude of $\xi = \xi_n$ is constant. For a finite time interval $t > \tau_\xi$, one must then apply equation (A.9) consecutively to each time interval of length $\tau_\xi$ with different random values of $[\xi_n], n = 1, 2, 3, \ldots$.

Let us again consider a specific case when $|\lambda| = \hbar$ so that the average of the deviation from infinitesimal stationary action distributed according to the exponential law of equation (10) is given by $\hbar/2$. The zeroth order approximation of equation (A.9) then reads
\[
\i \hbar \partial_t \Psi_Q(q; t) = \hat{H} \Psi_Q(q; t), \tag{A.10}
\]
where $\Psi_Q$ is defined as in equation (40) and $\hat{H}$ is the quantum Hamiltonian given by
\[
\hat{H} = \frac{1}{2} (\hat{p}_i - a_i) g^{ij}(q)(\hat{p}_j - a_j) + V. \tag{A.11}
\]
Unlike equation (A.9), equation (A.10) is now deterministically parameterized by $\hbar$. One can also see that unlike canonical quantization which, for the general type of $g^{ij}(q)$, suffers from the problem of operator ordering ambiguity, the resulting quantum Hamiltonian is unique in which $g^{ij}(q)$ is sandwiched by $\hat{p} - a$.

Solving the modified Schrödinger equation of (A.9) for each time interval of length $\tau_\xi$ with a fixed value of $|\xi|$, and inserting the modulus and phase of $\Psi$ into equation (A.2), one obtains the stochastic evolution of the velocity of the particle as $\Psi$ evolves with time. In this case, the effective velocity defined in equation (42) reads
\[
\tilde{q}^i(|\xi|) = \frac{\tilde{q}^i(\xi) + \tilde{q}^i(-\xi)}{2} = g^{ij}(\partial_{q_j} S(|\xi|) - a_j). \tag{A.12}
\]
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the zeroth order approximation of which gives

\[ \tilde{\dot{q}}^i = g^{ij} (\partial_{q^j} S_Q - a_j), \]  

(A.13)

which, unlike equation (A.2), is now deterministic, due to the deterministic time evolution of \( S_Q \) given by the Schrödinger equation of (A.10).

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