Answer to a question of Alon and Lubetzky about the ultimate categorical independence ratio

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Abstract

Brown, Nowakowski and Rall defined the ultimate categorical independence ratio of a graph $G$ as $A(G) = \lim_{k \to \infty} i(G \times k)$, where $i(G) = \frac{\alpha(G)}{|V(G)|}$ denotes the independence ratio of a graph $G$, and $G \times k$ is the $k$th categorical power of $G$. Let $a(G) = \max\left\{ \frac{|U|}{|U| + |N_G(U)|} : U \text{ is an independent set of } G \right\}$, where $N_G(U)$ is the neighborhood of $U$ in $G$. In this paper we answer a question of Alon and Lubetzky, namely we prove that if $a(G) \leq \frac{1}{2}$ then $A(G) = a(G)$, and if $a(G) > \frac{1}{2}$ then $A(G) = 1$. We also discuss some other open problems related to $A(G)$ which are immediately settled by this result.

1 Introduction

The independence ratio of a graph $G$ is defined as $i(G) = \frac{\alpha(G)}{|V(G)|}$, that is, as the ratio of the independence number and the number of vertices. For two graphs $G$ and $H$, their categorical product (also called as direct or tensor product) $G \times H$ is defined on the vertex set $V(G \times H) = V(G) \times V(H)$ with edge set $E(G \times H) = \{(x_1, y_1), (x_2, y_2)\} : \{x_1, x_2\} \in E(G) \text{ and } \{y_1, y_2\} \in E(H)\}$. The $k$th categorical power $G \times k$ is the $k$-fold categorical product of $G$. The ultimate categorical independence ratio of a graph $G$ is defined as $A(G) = \lim_{k \to \infty} i(G \times k)$.

This parameter was introduced by Brown, Nowakowski and Rall in [2] where they proved that for any independent set $U$ of $G$ the inequality $A(G) \geq \frac{|U|}{|U| + |N_G(U)|}$ holds, where $N_G(U)$ denotes the neighborhood of $U$ in $G$. Furthermore, they showed that $A(G) > \frac{1}{2}$ implies $A(G) = 1$.

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Motivated by these results, Alon and Lubetzky \[1\] defined the parameters \(a(G)\) and \(a^*(G)\) as follows:

\[
a(G) = \max_{U \text{ is ind. set of } G} \frac{|U|}{|U| + |N_G(U)|} \quad \text{and} \quad a^*(G) = \begin{cases} a(G) & \text{if } a(G) \leq \frac{1}{2}, \\ 1 & \text{if } a(G) > \frac{1}{2}, \end{cases}
\]

and they proposed the following two questions.

**Question 1** (\[1\]). Does every graph \(G\) satisfy \(A(G) = a^*(G)\)? Or, equivalently, does every graph \(G\) satisfy \(a^*(G^\times 2) = a^*(G)\)?

**Question 2** (\[1\]). Does the inequality \(i(G \times H) \leq \max \{a^*(G), a^*(H)\}\) hold for every two graphs \(G\) and \(H\)?

The above results from \[2\] give us the inequality \(A(G) \geq a^*(G)\). One can easily see the equivalence between the two forms of Question 1; moreover it is not hard to show that an affirmative answer to Question 1 would imply the same for Question 2 (see \[1\]).

Following \[2\] a graph \(G\) is called self-universal if \(A(G) = i(G)\). As a consequence, the equality \(A(G) = a^*(G)\) in Question 1 is also satisfied for these graphs according to the chain inequality \(i(G) \leq a(G) \leq a^*(G) \leq A(G)\). Regular bipartite graphs, cliques and Cayley graphs of Abelian groups belong to this class \[2\]. In \[4\] the author proved that a complete multipartite graph is self-universal, except for the case when \(a(G) > \frac{1}{2}\), therefore the equality \(A(G) = a^*(G)\) is also verified for this class of graphs. (In the latter case \(A(G) = a^*(G) = 1\).) In \[1\] it is shown that the graphs which are disjoint union of cycles and complete graphs satisfy the inequality in Question 2.

In this paper we answer Question 1 affirmatively. Thereby a positive answer also for Question 2 is obtained. Moreover it solves some other open problems related to \(A(G)\). In the proofs we exploit an idea of Zhu \[3\] that he used on the way when proving the fractional version of Hedetniemi’s conjecture. In Section 2 this tool is presented. Then, in Section 3 first we prove the inequality

\[
i(G \times H) \leq \max \{a(G), a(H)\}, \quad \text{for every two graphs } G \text{ and } H,
\]

and give a positive answer to Question 2 (using \(a(G) \leq a^*(G)\)). Afterwards we prove that

\[
a(G \times H) \leq \max \{a(G), a(H)\}, \quad \text{provided that } a(G) \leq \frac{1}{2} \text{ or } a(H) \leq \frac{1}{2},
\]

and from this result we conclude the affirmative answer to Question 1. (If \(a(G) > \frac{1}{2}\) then \(a^*(G^\times 2) = a^*(G) = 1\). Otherwise applying the above result for \(G = H\) we get \(a(G^\times 2) \leq a(G)\), while the reverse inequality clearly holds for every \(G\). Thus we have \(a^*(G^\times 2) = a^*(G)\) for every graph \(G\).) Finally, in Section 4 we discuss further open problems which are solved by our result. For instance, we get a proof for the conjecture of Brown, Nowakowski and Rall, stating that \(A(G \cup H) = \max \{A(G), A(H)\}\), where \(G \cup H\) is the disjoint union of \(G\) and \(H\).
2 Zhu’s lemma

Recently Zhu [3] proved the fractional version of Hedetniemi’s conjecture, that is, he showed that for every graph $G$ and $H$ we have $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$, where $\chi_f(G)$ denotes the fractional chromatic number of the graph $G$. During the proof he showed the following result on the independent sets of categorical product of graphs. This will be the key idea also in our case.

Let $U$ be an independent set of $G \times H$. Zhu considered the partition $U$ into $U = A \sqcup B$, where

$$
A = \{(x, y) \in U : \exists (x', y) \in U \text{ s.t. } \{x, x'\} \subseteq E(G)\},
$$

$$
B = \{(x, y) \in U : \exists (x', y) \in U \text{ s.t. } \{x, x'\} \subseteq E(G)\}.
$$

(1)

In the sequel, we keep using the following notations for any $Z \subseteq V(G \times H)$.

For any $y \in V(H)$, let

$$
Z(y) = \{x \in V(G) : (x, y) \in Z\}.
$$

Similarly, for any $x \in V(G)$, let

$$
Z(x) = \{y \in V(H) : (x, y) \in Z\}.
$$

And, let

$$
N^G(Z) = \{(x, y) \in V(G \times H) : x \in N_G(Z(y))\}.
$$

In words, $N^G(Z)$ means that we decompose $Z$ into sections corresponding to the elements of $V(H)$, and in each section we pick those points which are neighbors of the elements of $Z(y)$ in the graph $G$.

Similarly, let

$$
N^H(Z) = \{(x, y) \in V(G \times H) : y \in N_H(Z(x))\}.
$$

Keep in mind, that $Z(y) \subseteq V(G)$ and $Z(x) \subseteq V(H)$, while $N^G(Z), N^H(Z) \subseteq V(G \times H)$.

**Lemma 1** [3]. The following holds:

1. For every $y \in V(H)$, $A(y)$ is an independent set of $G$. For every $x \in V(G)$, $B(x)$ is an independent set of $H$.

2. $A, B, N^G(A)$ and $N^H(B)$ are pairwise disjoint subsets of $V(G \times H)$.

For the sake of completeness we prove this lemma.

**Proof.** $A(y)$ is independent for every $y \in V(H)$ by definition. If for any $x \in V(G)$ the set $B(x)$ is not independent in $H$, that is $\exists y, y' \in B(x), \{y, y'\} \subseteq E(H)$, then from $(x, y') \in B$ we get that $\exists (x', y') \in U, \{x, x'\} \subseteq E(G)$. This is a contradiction, because $(x, y) \in B$ and $(x', y') \in U$ were two adjacent elements of the independent set $U$.

Now we show the second part of the lemma. By definition $A \cap B = \emptyset$. The first part of the lemma implies that the pair $(A, N^G(A))$ is also disjoint, as well as the pair $(B, N^H(B))$. 


For every two graphs $G$ and $H$ we have

$$i(G \times H) \leq \max\{a(G), a(H)\}.$$  

Proof. Let $U$ be a maximum-size independent set of $G \times H$, then we have

$$i(G \times H) = \frac{\alpha(G \times H)}{V(G \times H)} = \frac{|U|}{|V(G \times H)|}. \quad (2)$$

We partition $U = A \cup B$ according to $[\Pi]$. We also use the notations $A(y)$ for every $y \in V(H)$, $B(x)$ for every $x \in V(G)$, and $N^G(A)$, $N^H(B)$ defined in the previous section.

It is clear that $|U| = |A| + |B|$. From the second part of Lemma $[\Pi]$ we have that $|A| + |B| + |N^G(A)| + |N^H(B)| \leq |V(G \times H)|$. Observe that $|N^G(A)| = \sum_{y \in V(H)} |N_G(A(y))|$ and $|N^H(B)| = \sum_{x \in V(G)} |N_H(B(x))|$. Hence we get

$$\frac{|U|}{|V(G \times H)|} \leq \frac{|A| + |B|}{|A| + |B| + |N^G(A)| + |N^H(B)|} = \frac{\sum_{y \in V(H)} |A(y)| + \sum_{x \in V(G)} |B(x)|}{\sum_{y \in V(H)} (|A(y)| + |N_G(A(y))|) + \sum_{x \in V(G)} (|B(x)| + |N_H(B(x))|)}. \quad (3)$$

From the first part of Lemma $[\Pi]$ and by the definition of $a(G)$ and $a(H)$ we have $\frac{|A(y)|}{|A(y)| + |N_G(A(y))|} \leq a(G)$ for every $y \in V(H)$, and $\frac{|B(x)|}{|B(x)| + |N_H(B(x))|} \leq a(H)$ for every $x \in V(G)$, respectively.

3 Proofs

In this section we prove the statements mentioned in the Introduction. In Subsection 3.1 we give an upper bound for $i(G \times H)$ in terms of $a(G)$ and $a(H)$. In Subsection 3.2 we prove that the same upper bound holds also for $a(G \times H)$ provided that $a(G) \leq \frac{1}{2}$ or $a(H) \leq \frac{1}{2}$. Thereby we obtain that $A(G) = a^*(G)$ for every graph $G$.

3.1 Upper bound for $i(G \times H)$

As a simple consequence of Zhu’s result the following inequality is obtained.

Theorem 2. For every two graphs $G$ and $H$ we have

$$i(G \times H) \leq \max\{a(G), a(H)\}.$$
Using the fact that if \( \frac{t_1}{s_1} \leq r \) and \( \frac{t_2}{s_2} \leq r \) then \( \frac{t_1 + t_2}{s_1 + s_2} \leq r \), this yields
\[
\frac{\sum_{y \in V(H)} |A(y)| + \sum_{x \in V(G)} |B(x)|}{\sum_{y \in V(H)} (|A(y)| + |N_G(A(y))|) + \sum_{x \in V(G)} (|B(x)| + |N_H(B(x))|)} \leq \max\{a(G), a(H)\}. \tag{4}
\]
The inequalities (2), (3) and (4) together give us the stated inequality,
\[
i(G \times H) \leq \max\{a(G), a(H)\}.
\]
\]

As we stated in the Introduction, from Theorem 2 it follows that the answer to Question 2 is positive.

### 3.2 Answer to Question 1

In this subsection we answer Question 1 affirmatively. To show that \( a^*(G^{\times 2}) = a^*(G) \) holds for every graph \( G \) it is enough to prove that \( a(G^{\times 2}) \leq a(G) \) for every graph \( G \) with \( a(G) \leq \frac{1}{2} \). Because if \( a(G) > \frac{1}{2} \) then \( a^*(G^{\times 2}) = a^*(G) = 1 \), in addition every \( G \) satisfies \( a(G^{\times 2}) \geq a(G) \). The condition \( a(G) \leq \frac{1}{2} \) is necessary, since otherwise \( A(G) = 1 \) therefore \( i(G^{\times k}) \) and \( a(G^{\times k}) \) as well can be arbitrary close to 1 for sufficiently large \( k \). A bit more general, we prove the following theorem.

**Theorem 3.** If \( a(G) \leq \frac{1}{2} \) or \( a(H) \leq \frac{1}{2} \) then
\[
a(G \times H) \leq \max\{a(G), a(H)\}.
\]

**Proof.** We will show that for every independent set \( U \) of \( G \times H \) we have
\[
\frac{|U|}{|U| + |N_{G\times H}(U)|} \leq \max\{a(G), a(H)\}.
\]

First, let \( \hat{A}, \hat{B} \) and \( C \) be the following subsets of \( U \).

\[
\hat{A} = \{(x, y) \in U : \exists (x', y') \in U \text{ s.t. } \{x, x'\} \in E(G), \text{ but } \exists (x, y') \in U \text{ s.t. } \{y, y'\} \in E(H)\},
\]
\[
\hat{B} = \{(x, y) \in U : \exists (x', y') \in U \text{ s.t. } \{y, y'\} \in E(H), \text{ but } \exists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G)\},
\]
\[
C = \{(x, y) \in U : \exists (x', y') \in U \text{ s.t. } \{x, x'\} \in E(G), \text{ and } \exists (x', y') \in U \text{ s.t. } \{y, y'\} \in E(H)\}.
\]

It is clear that \( \hat{A}, \hat{B} \) and \( C \) are pairwise disjoint. In addition, there is no \( (x, y) \in U \) for which \( \exists (x', y') \in U \text{ such that } \{x, x'\} \in E(G) \text{ and } \{y, y'\} \in E(H) \), because \( \{(x', y), (x, y')\} \in E(G \times H) \) and \( U \) is an independent set. Hence \( U \text{ is partitioned into } U = \hat{A} \cup \hat{B} \cup C \). (The connection with the partition of Zhu defined in (1) is clearly the following, \( A = \hat{A} \cup C \) and \( B = \hat{B} \).)

Observe that the definition of \( a(G) \) can be rewritten as follows
\[
\min\left\{\frac{|N_G(U)|}{|U|} : U \text{ is independent in } G\right\} = \frac{1 - a(G)}{a(G)}.
\]
set \( b(G) = \frac{1-a(G)}{a(G)} \). It is enough to prove that \(|N_{G \times H}(U)| \geq \min\{b(G), b(H)\}|U|\). We shall give a lower bound for \(|N_{G \times H}(U)|\) in two steps.

In the first step we consider the elements of \( \hat{A} \) and \( C \) for every \( y \in V(H) \). By definition \((\hat{A} \cup C)(y)\) is independent in \( G \) for every \( y \in V(H) \), therefore \(|N_G(\hat{A} \cup C)(y))| \geq b(G)|\hat{A} \cup C|(y)|\). We partition \( N^G(\hat{A} \cup C) \) into two parts, let

\[
N_1 = N^G(\hat{A} \cup C) \cap N_{G \times H}(U) \quad \text{and} \quad M = N^G(\hat{A} \cup C) \setminus N_{G \times H}(U).
\]

(It is easy to see that \( N^G(\hat{A}) \subseteq N_{G \times H}(U) \). However \( N^G(C) \subseteq N_{G \times H}(U) \) is not necessarily true, that is why we make this partition.) Thus for \( N_1 \subseteq N_{G \times H}(U) \) we have

\[
|N_1| \geq b(G)(|\hat{A}| + |C|) - |M|. \tag{5}
\]

In the second step we consider the elements of \( \hat{B} \) and \( M \) for every \( x \in V(G) \). By the definition of \( \hat{A} \) and \( C \), \( \hat{B}(x) \) and \( M(x) \) are disjoint. Indeed, if \((x, y) \in M \subseteq N^G(\hat{A} \cup C)\) then \( \exists(x', y) \in \hat{A} \cup C, \{x, x'\} \in E(G) \) and so \((x, y)\) cannot be in \( \hat{B} \subseteq U \).

We claim that \((\hat{B} \cup M)(x)\) is independent in \( V(H) \). Clearly, \( \hat{B}(x) \) is independent. Furthermore, if \( y, y' \in M(x), \{y, y'\} \in E(H) \) then from \((x, y) \in M \) we get that \( \exists(x', y) \in \hat{A} \cup C, \{x, x'\} \in E(G) \), hence \((x, y') \in M \) is a neighbor of \((x', y) \in U \) which contradicts that \( M \cap N_{G \times H}(U) = \emptyset \). Similarly if \( y \in \hat{B}(x), y' \in M(x), \{y, y'\} \in E(H) \) then from \((x, y) \in \hat{B} \) it follows that \( \exists(x', y) \in U, \{x, x'\} \in E(G) \), but again, as \((x, y') \in M \) is a neighbor of \((x', y) \in U \) it is in a contradiction with the definition of \( M \). Therefore \(|N_H(\hat{B} \cup M(x))| \geq b(G)|\hat{B} \cup M|(x)|\). Let

\[
N_2 = N^H(\hat{B} \cup M).
\]

Considering the sum for all \( x \in V(G) \) we obtain

\[
|N_2| \geq b(H)(|\hat{B}| + |M|). \tag{6}
\]

We show that \( N_2 \subseteq N_{G \times H}(U) \). On the one hand, if \( y \in \hat{B}(x) \) and \( y' \) is a neighbor of \( y \) in \( H \), and so \((x, y') \in N^H(\hat{B}) \) then by the definition of \( \hat{B} \), \( \exists(x', y) \in U, \{x, x'\} \in E(G), \) hence \((x, y') \) is a neighbor of \((x', y) \in U \), that is, \((x, y') \in N_{G \times H}(U) \). On the other hand, if \( y \in M(x) \) and \( y' \) is a neighbor of \( y \) in \( H \), and so \((x, y') \in N^H(M) \) then by the definition of \( M \), \( \exists(x', y) \in \hat{A} \cup C, \{x, x'\} \in E(G), \) therefore \( \{(x', y), (x, y')\} \in E(G \times H) \), thus \((x, y') \in N_{G \times H}(U) \).

Next we prove that the neighborhood sets gotten in the two steps, \( N_1 \) and \( N_2 \) are disjoint. Suppose indirectly, that \((x, y) \in N_1 \cap N_2 \). Then \((x, y) \in N_1 \) implies that \( \exists(x', y) \in \hat{A} \cup C, \{x, x'\} \in E(G) \). While from \((x, y) \in N_2 \) we get that \( \exists(x, y') \in \hat{B} \) or \( \exists(x, y') \in M \) satisfying \( \{y, y'\} \in E(H) \). It is a contradiction since \((x', y) \) and \((x, y') \) are adjacent in \( G \times H \), but no edge can go between \( \hat{A} \cup C \) and \( \hat{B} \cup M \) by the independence of \( U \) and the definition of \( M \). As \( N_1, N_2 \subseteq N_{G \times H}(U) \) this yields

\[
|N_{G \times H}(U)| \geq |N_1| + |N_2|. \tag{7}
\]
If \( a(H) \leq \frac{1}{2} \), that is \( b(H) \geq 1 \), then

\[
\left( b(G)(|\\hat{A}| + |C|) - |M| \right) + \left( b(H)(|\\hat{B}| + |M|) \right) \geq \\
\geq \min\{b(G), b(H)\} \left( |\\hat{A}| + |\\hat{B}| + |C| \right) + (b(H) - 1)|M| \geq \min\{b(G), b(H)\} |U|.
\]

Combining the latter two inequalities we obtain \(|N_{G \times H}(U)| \geq \min\{b(G), b(H)\} |U|\), as desired.

If \( a(G) \leq \frac{1}{2} \) (and \( a(H) > \frac{1}{2} \)) we can change the role of \( G \) and \( H \) to get the same lower bound for \(|N_{G \times H}(U)|\), or we can argue as follows. We distinguish two cases. First, suppose \(|\\hat{A}| + |C| - \frac{|M|}{b(G)} \geq 0\).

By using \( b(G) \geq 1 \) this gives

\[
\left( b(G)(|\\hat{A}| + |C|) - |M| \right) + \left( b(H)(|\\hat{B}| + |M|) \right) = b(G) \left( |\\hat{A}| + |C| - \frac{|M|}{b(G)} \right) + b(H)(|\\hat{B}| + |M|) \geq \\
\geq \min\{b(G), b(H)\} \left( |\\hat{A}| + |\\hat{B}| + |C| + |M| \left( 1 - \frac{1}{b(G)} \right) \right) \geq \min\{b(G), b(H)\} |U|,
\]

finishing the inequality chain. While from \(|\\hat{A}| + |C| - \frac{|M|}{b(G)} < 0\) and \( b(G) \geq 1 \) it follows \(|\\hat{A}| + |C| < |M|\), hence we have

\[
|N_{G \times H}(U)| \geq |N_2| \geq b(H)(|\\hat{B}| + |M|) \geq \min\{b(G), b(H)\} |U|.
\]

Consequently, \(|N_{G \times H}(U)| \geq \min\left\{ \frac{1-a(G)}{a(G)}, \frac{1-a(H)}{a(H)} \right\} |U|\) in both cases, that is

\[
\frac{|U|}{|U| + |N_{G \times H}(U)|} \leq \max\{a(G), a(H)\},
\]

this completes the proof.

We mentioned in the Introduction that the two forms of Question 1 are equivalent. Hence from the equality \( a^*(G^{\times 2}) = a^*(G) \) for every graph \( G \) we obtain the following corollary. (Indeed, suppose on the contrary that \( G \) is a graph with \( a^*(G) < A(G) \) then \( \exists k \) such that \( a^*(G) < i(G^{\times k}) \leq a^*(G^{\times k}) \), and as the sequence \( \{a^*(G^{\times \ell})\}_{\ell=1}^{\infty} \) is monotone increasing, it follows that \( \exists m \) for which \( a^*(G^{\times m}) < a^*(G^{\times 2m}) \), giving a contradiction.)

**Corollary 4.** For every graph \( G \) we have \( A(G) = a^*(G) \).
4 Further consequences

Brown, Nowakowski and Rall in [2] asked whether \( A(G \cup H) = \max\{A(G), A(H)\} \), where \( G \cup H \) denotes the disjoint union of \( G \) and \( H \). From Corollary 4 we immediately receive this equality since the analogue statement, \( a^*(G \cup H) = \max\{a^*(G), a^*(H)\} \) is straightforward. In [1] it is shown that \( A(G \cup H) = A(G \times H) \), therefore we have

\[
A(G \cup H) = A(G \times H) = \max\{A(G), A(H)\}, \quad \text{for every graph } G \text{ and } H.
\]

The authors of [2] also addressed the question whether \( A(G) \) is computable, and if so what is its complexity. They showed that if \( G \) is bipartite then \( A(G) = \frac{1}{2} \) if \( G \) has a perfect matching, and \( A(G) = 1 \) otherwise. Hence for bipartite graphs \( A(G) \) can be determined in polynomial time. Moreover, it is proven in [1] that \( a(G) \leq \frac{1}{2} \) if and only if \( G \) contains a fractional perfect matching. Therefore given an input graph \( G \), determining whether \( A(G) = 1 \) or \( A(G) \leq \frac{1}{2} \) can be done in polynomial time. They also mentioned that deciding whether \( a(G) > t \) for a given graph \( G \) and a given value \( t \), is NP-complete. From Corollary 4 we can conclude that \( A(G) \) can be calculated, and the problem of deciding whether \( A(G) > t \) is NP-complete too.

Although any rational number in \((0, \frac{1}{2}] \cup \{1\}\) is the ultimate categorical independence ratio for some graph \( G \), as it is showed [2]. Here we remark that we obtained that \( A(G) \) cannot be irrational, solving another problem mentioned in [2].

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