Cellular Stratified Spaces II: Basic Constructions

Dai Tamaki

Abstract

This is a sequel to [Tamb], in which a systematic study of cellular stratified spaces and related concepts was initiated. In this paper, we study important operations on cellular and stellar stratified spaces, including taking subspaces, subdivisions, and products. We also introduce a procedure which extends the classical duality for simplicial complexes. The results in this paper will be used to construct “Salvetti-type” models for configuration spaces.

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1 Introduction

This is a sequel to [Tamb], whose role is to lay foundations of the study of cellular stratified spaces introduced in a joint work [BGRT] with Basabe, González, and Rudyak and extend them to stellar stratified spaces.

One of the purposes of this project is to import techniques developed for studying complements of hyperplane arrangements and subspace arrangements to the study of configuration spaces. Our main tool is the following theorem proved in [Tamb].
Theorem 1.1. For a locally polyhedral cellular stratified space $X$, there exists a CW complex $\text{Sd}(X)$ and an embedding 
\[ \tilde{i} : \text{Sd}(X) \hookrightarrow X \]
whose image is a strong deformation retract of $X$.

When $X$ is the complement of a complexified hyperplane arrangement equipped with the cellular stratification described by Björner and Ziegler in [BZ92], $\text{Sd}(X)$ coincides with Salvetti’s construction [Sal87] as simplicial complexes.

Example 1.2. Let $\mathcal{A}$ be a real hyperplane arrangement in $\mathbb{R}^n$ and consider the induced cellular stratification $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$ on $\mathbb{R}^n \otimes \mathbb{R}^\ell$ defined in [BZ92]. The stratification $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$ induces a cellular stratification on the union of all affine subspaces in $\mathcal{A} \otimes \mathbb{R}^\ell$

\[ \text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell) = \bigcup_{i=1}^{k} H_i \otimes \mathbb{R}^\ell, \]

as well as on its complement

\[ M(\mathcal{A} \otimes \mathbb{R}^\ell) = \mathbb{R}^n \otimes \mathbb{R}^\ell - \text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell). \]

It turns out the CW complex $\text{Sd}(M(\mathcal{A} \otimes \mathbb{R}^\ell))$ is nothing but the simplicial version of the higher order Salvetti complex for $\mathcal{A} \otimes \mathbb{R}^\ell$. \hfill \square

The above example is a typical case we would like to consider. In this case, it is not hard to see that the induced stratification on the complement $M(\mathcal{A} \otimes \mathbb{R}^\ell)$ is a regular cellular stratification. In general, however, cellular structures do not behave very well with respect to taking subspaces and complements, although one of the most important features of stratified spaces is that the class of stratified spaces is closed under taking complements.

We need two more operations when we study configuration spaces; products and subdivisions. Recall that the configuration space of distinct $k$ points in a space $M$ is defined to be the complement of the discriminant set $\Delta_k(M) = \bigcup_{i<j} \{ (x_1, \ldots, x_k) \in M^k \mid x_i = x_j \}$ in the $k$-fold product of $M$;

\[ \text{Conf}_k(M) = M^k - \Delta_k(M). \]

When $M$ is a finite CW complex, $M^k$ has the product cell decomposition. For cellular stratified spaces in general, products might not inherit cellular structures. Even when the product stratification makes $M^k$ into a cellular stratified space, $\Delta_k(M)$ is not a stratified subspace of $M^k$. We need to subdivide $M^k$ in such a way $\Delta_k(M)$ is included as a stratified subspace.

With these constructions in mind, we study the following questions in this paper:

1. Find a condition under which a stratified subspace $A$ of a cellular stratified space $X$ inherits a cellular stratification.

   When $X$ satisfies one of “niceness conditions”, such as cylindrically normal or locally polyhedral, can the structure be restricted to $A$?

2. Given cellular stratified spaces $X$ and $Y$, when is the product stratification makes $X \times Y$ into a cellular stratified space?

   In particular, when both $X$ and $Y$ are locally polyhedral, does $X \times Y$ has a structure of locally polyhedral cellular stratified space?

3. Find a general criterion for a subdivision of a cellular stratified space $X$ to be a cellular stratified space.
We answer these questions by proving the following results:

1. Let $X$ be a cellular stratified space and $A$ a subspace.
   - If $X$ is cylindrically normal and all cell structure maps are hereditarily quotient, then $A$ inherits a structure of cylindrically normal cellular stratified space. (Proposition 3.10)
   - When $X$ is a locally polyhedral cellular stratified space, any locally finite cellular stratified subspace $A$ is locally polyhedral. (Proposition 3.13)

2. Let $X$ and $Y$ be cellular stratified spaces.
   - If $X$ and $Y$ are cylindrically normal cellular stratified spaces satisfying one of the following conditions:
     - locally polyhedral, or
     - all parameter spaces are compact.
     Then the product $X \times Y$ is a cylindrically normal cellular stratified space. (Theorem 3.18)
   - If both $X$ and $Y$ are locally finite locally polyhedral cellular stratified spaces, $X \times Y$ is a locally polyhedral cellular stratified space. (Corollary 3.21)

3. Let $X$ be a cylindrically normal cellular stratified space. Suppose that each domain $D_\lambda$ and each parameter space $P_{\mu, \lambda}$ have cellular stratifications in such a way that
   \[ b_{\mu, \lambda}: P_{\mu, \lambda} \times D_\mu \longrightarrow D_\lambda \]
   is a strict morphism of cellular stratified spaces. Then we obtain a cylindrically normal cellular stratification whose cells are cells in $\text{Int}(D_\lambda)$ for $\lambda \in P(X)$. (Proposition 3.38)

Another important construction discovered by Salvetti is to glue cells in his simplicial model for the complement of a complexified arrangement to obtain a cellular model with a small number of cells. For a general locally polyhedral cellular stratified space $X$, however, an analogous operation on $\text{Sd}(X)$ might not give rise to (globular) cells. We make use of stellar structures introduced in [Tamb] to define a dualization operation $D(X)$ under the assumptions that $X$ is locally polyhedral, the number of cells is finite, and all parameter spaces are compact. It turns out the double dual $D(D(X))$ can be defined directly for any cylindrically normal stellar stratified space. With this structure, Theorem 1.1 can be rephrased as follows.

Theorem 1.3 (Theorem 4.20). For a cylindrically normal stellar stratified space $(X, \pi_X)$, there exists a structure $\pi_{\text{Sal}(X)}$ of cylindrically normal stellar stratified space on $\text{Sd}(X)$ with which we have an embedding of cylindrically normal stellar stratified spaces

\[ i: (\text{Sd}(X), \pi_{\text{Sal}(X)}) \hookrightarrow (X, \pi_X). \]

When $X$ is a locally polyhedral cellular stratified space, the image of $i$ is a strong deformation retract of $X$ in the category of stellar stratified spaces. When all stellar cells in $X$ are closed, $\tilde{i}$ is an isomorphism of cylindrically normal stellar stratified spaces. Finally when $X$ is a finite locally polyhedral stellar stratified space with compact parameter spaces, $(\text{Sd}(X), \pi_{\text{Sal}(X)})$ can be identified with $D(D(X))$ as a stellar stratified space.

When $X$ is the complement of the complexification of a real hyperplane arrangement $\mathcal{A}$, the stellar stratified space $(\text{Sd}(X), \pi_{\text{Sal}(X)})$ coincides with the Salvetti complex of $\mathcal{A}$ as a cell complex. For this reason, $(\text{Sd}(X), \pi_{\text{Sal}(X)})$ is called the Salvetti complex of $X$ and is denoted by $\text{Sal}(X)$.

\[ \text{Definition A.3} \]
1.1 Organization

- We collect definitions and facts used in this paper in §2.
  We first recall definitions of stratifications and cells in §2.1. The definition of cellular stratified spaces and basic properties, especially their topology, are summarized in §2.2. Topological categories associated with good cellular stratified spaces and their classifying spaces are recalled in §2.3.

- For applications to configuration spaces, we need to understand subdivisions, subspaces, and their complements, which is the subject of §3.
  After summarizing basic properties of stratified subspaces, we study cell structures on stratified subspaces in §3.1. We obtain several reasonable conditions under which products of cell structures are cell structures in §3.2. Subdivisions of cells are studied in §3.3 and subdivisions of cylindrically normal and locally polyhedral cellular stratified spaces are studied in §3.4.

- In the last section §4, we introduce a couple of cellular and stellar stratifications on $Sd(X)$.
  The classical dual complex construction for finite regular cell complexes can be easily extended to finite locally polyhedral cellular stratified spaces. We review the construction in §4.1 and extend it to cylindrically normal stellar stratified spaces. The cellular stratification on $Sd(X)$ defined in §4.1 should be coarsened for practical applications. We define a stellar structure on $Sd(X)$ by gluing cells together and define the dual stellar stratified space $D(X)$ in §4.2. And then Theorem 1.3 (Theorem 4.20) is proved. We conclude this paper by showing that the barycentric subdivision of totally normal stellar stratified spaces corresponds to the barycentric subdivision of small categories through the face category functor in §4.3.

- General properties of quotient maps used in this paper are summarized in Appendix A.

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González suggested that the duality for totally normal cellular stratified spaces studied in [Tama] should be extended to locally polyhedral cellular stratified spaces. On the other hand, Iriye pointed out numerous ambiguities and gaps in [Tama] concerning quotient maps, products, and subspaces in a series of letters. The consideration on quotient maps is also influenced by discussions with Yuli Rudyak during the preparation of the joint paper [BGRT].

In order to incorporate their suggestions and comments, the author decided to extract portions in [Tama] concerning constructions of new cellular stratified spaces as a separate paper. The author really appreciates their suggestions and comments.

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2 Preliminaries

This section is preliminary. We recall definitions and basic properties concerning two main ingredients in this paper; cellular stratified spaces and topological categories.
2.1 Stratifications and Cells

A cellular stratified space is a stratified space in which each stratum is equipped with a cell structure satisfying a condition analogous to the one for cell complexes. We define a stratification by using a poset equipped with the Alexandroff topology.

Definition 2.1. Let Λ be a set equipped with a preorder ≤. The Alexanderoff topology on X is the topology in which closed sets are given by those subsets D satisfying the condition that, for any µ, λ ∈ Λ, λ ∈ D and µ ≤ λ imply µ ∈ D.

Definition 2.2. Let X be a topological space and Λ be a poset regarded as a topological space under the Alexandroff topology. A stratification of X indexed by Λ is an open continuous map

\[ \pi : X \rightarrow \Lambda \]

satisfying the condition that, for each λ ∈ Im π, π⁻¹(λ) is connected and locally closed.

Each π⁻¹(λ) is called a stratum with index λ. The set Im π is denoted by P(X, π) or P(X) and is called the face poset of (X, π).

The following fact implies that the above definition is equivalent to the one in [Tam1].

Lemma 2.3. Let π : X → Λ an continuous map from a topological space X to a poset Λ equipped with the Alexanderoff topology. Then π is an open map if and only if the condition π⁻¹(λ) ⊆ π⁻¹(Λ) is equivalent to λ ≤ λ'.

Proof. It is well-known that π is an open continuous map if and only if \[ \pi^{-1}(B) = \pi^{-1}(\overline{B}) \] for any subset B ⊆ Λ. Thus, when π is an open map, \[ π^{-1}(λ) ⊆ π^{-1}(Λ') \] if and only if \[ λ \in \{λ'\} \], which is equivalent to saying \[ λ ≤ λ' \].

Conversely suppose that \[ π^{-1}(λ) ⊆ π^{-1}(Λ') \] is equivalent to \[ λ ≤ λ' \]. For a subset B ⊆ Λ, we have \[ π^{-1}(B) ⊆ π^{-1}(\overline{B}) \] by the continuity of π. For \[ x ∈ π^{-1}(\overline{B}) \], \[ π(x) ∈ \overline{B} \]. By the definition of the Alexanderoff topology, there exists \[ λ ∈ B \] such that \[ π(x) ≤ λ \]. By assumption, this is equivalent to \[ π(x) ∈ π^{-1}(λ) \]. Thus \[ x ∈ π^{-1}(B) \] and we have shown that \[ π^{-1}(B) ⊇ π^{-1}(\overline{B}) \].

Remark 2.4. Our definition is inspired by a general notion of gradings by a comonoid object in a monoidal category [Tam1]. In other words, a stratification on X by a poset Λ is a grading of X by Λ in the monoidal category of topological spaces and open continuous maps.

Definition 2.5. Let \( (X, π_X) \) and \( (Y, π_Y) \) be stratified spaces. A morphism from \( (X, π_X) \) to \( (Y, π_Y) \) is a pair \( (f, f') \) of a continuous map \( f : X \rightarrow Y \) and a map \( f' : P(X) \rightarrow P(Y) \) of posets that are compatible with \( π_X \) and \( π_Y \).

When \( X \) is a subspace of \( Y \), \( f \) is the inclusion, and \( π_X \) is the restriction of \( π_Y \), \( (X, π_X) \) is said to be a stratified subspace of \( Y \). When \( π_X' = π_Y' \) for each \( λ \in P(X) \), \( X \) is said to be a strict stratified subspace of \( Y \).

When \( X = Y \) and \( f \) is the identity, \( (1_X, f) \) is called a subdivision. We also say that \( (X, π_X) \) is a subdivision of \( (Y, π_Y) \) or \( (Y, π_Y) \) is a coarsening of \( (X, π_X) \).

We usually impose the CW condition on stratified spaces.

Definition 2.6. A stratified space \( X \) is said to be CW if the following conditions are satisfied:

1. (Closure Finite) For each stratum \( e_λ \), the boundary \( \partial e_λ = \overline{π^{-1}(λ)} - e_λ \) is covered by a finite number of strata.
2. (Weak Topology) $X$ has the weak topology determined by the covering \( \{ \overline{e} \mid \lambda \in P(X) \} \).

We have verified the following fact in [Tamb].

**Proposition 2.7.** Any locally finite stratified space is CW.

**Corollary 2.8.** Let \((X, \pi)\) be a CW stratified space and \((X, \pi')\) be a subdivision. If each stratum \(e_\lambda\) in \((X, \pi)\) is subdivided into a finite number of strata in \((X, \pi')\), then \((X, \pi')\) is CW.

**Proof.** For each cell \(e_\lambda\) in \((X, \pi)\), \(\overline{e_\lambda}\) has the weak topology with respect to the covering \(\bigcup_{\lambda' \in P(X, \pi'), e_{\lambda'} \subset e_\lambda} \overline{e_{\lambda'}}\) because of the finiteness assumption. Thus \(X\) has the weak topology with respect to the covering \(X = \bigcup_{\lambda' \in P(X, \pi')} \overline{e_{\lambda'}} = \bigcup_{\lambda' \in P(X, \pi')} \left( \bigcup_{\lambda \in P(X, \pi'), e_{\lambda'} \subset e_\lambda} \overline{e_{\lambda'}} \right)\).

The closure finiteness condition also follows from the finiteness of the subdivision of each stratum. 

As is the case of CW complexes, metrizability implies local finiteness.

**Lemma 2.9.** Any metrizable CW stratified space is locally finite.

**Proof.** This fact is well known for CW complexes. The same argument can be used to prove this generalized statement. We give a proof for the convenience of the reader.

If \(X\) is not locally finite, there exists a point \(x \in X\) such that, for any open neighborhood \(U\) of \(x\), \(U\) intersects with infinitely many strata. For each \(n\), let \(U_n\) be the \(\frac{1}{n}\)-neighborhood of \(x\) and choose a stratum \(e_n\) with \(U_n \cap e_n \neq \emptyset\) and \(x \notin e_n\). Then the set \(A = \{ x_n \}_{n=1,2,...}\) is closed by the CW conditions. This contradicts to the fact that \(x \notin A\) and \(x \in \overline{A}\). Thus \(X\) is locally finite. 

A cell structure is a continuous map from a globular cell to a stratum satisfying a reasonable condition.

**Definition 2.10.** A *globular* \(n\)-cell is a subset \(D \subset D^n\) containing \(\text{Int}(D^n)\). We call \(D \cap \partial D^n\) the boundary of \(D\) and denote it by \(\partial D\). The number \(n\) is called the *globular dimension* of \(D\).

**Definition 2.11.** Let \(X\) be a topological space. For a non-negative integer \(n\), an *\(n\)-cell structure* on a subspace \(e \subset X\) is a continuous map \(\varphi : D \to X\) satisfying the following conditions:

1. \(\varphi(D) = e\) and the restriction \(\varphi|_{\text{Int}(D^n)} : \text{Int}(D^n) \to e\) is a homeomorphism onto \(e\).
2. \(\varphi : D \to \overline{e}\) is a quotient map.

For simplicity, we denote an \(n\)-cell structure \(\varphi : D \to e\) by \(e\) when there is no risk of confusion. The globular cell \(D\) is called the *domain* of \(e\). The number \(n\) is called the *globular dimension* of \(e\). The difference \(e - e\) is called the *boundary* of \(e\) and is denoted by \(\partial e\).

**Definition 2.12.** We say an \(n\)-cell \(e\) (more precisely, an \(n\)-cell structure \(\varphi : D \to e\)) in a topological space \(X\) is *closed* when the domain \(D\) of \(e\) is a closed \(n\)-disk \(D^n\).

**Remark 2.13.** The map \(\varphi\) is also called the *characteristic map* of \(e\) when \(X\) is a cell complex.
Remark 2.14. The requirement of being a quotient map on a cell structure $\varphi : D \to \overline{e}$ imposes some restrictions on the topology of $\overline{e}$. When a cell $e$ is closed, the “quotient map” condition on $\varphi$ is automatic because of the compactness of the domain. We recall important properties of quotient maps in Appendix A.

Definition 2.15. We say a cell structure $\varphi : D \to e$ is relatively compact if $\varphi^{-1}(y)$ is compact for each $y \in \overline{e}$. We also say that the cell $e$ is relatively compact.

We need stellar structures to define duality in §4.

Definition 2.16. A subset $S$ of $D^N$ is said to be an aster if $\{0\} \ast \{x\} \subset S$ for any $x \in S$, where $\ast$ is the join operation defined by connecting points by line segments and $0$ is the origin of $D^N$. The subset $S \cap \partial D^N$ is called the boundary of $S$ and is denoted by $\partial S$. The complement $S - \partial S$ of the boundary is called the interior of $S$ and is denoted by $\text{Int}(S)$.

We require the existence of a cellular stratification on the boundary in order to define the dimension.

Definition 2.17. A stellar cell is an aster $S$ in $D^N$ for some $N$ such that there exists a structure of regular cell complex on $S$ which contains $\partial S$ as a strict cellular stratified subspace.

When the (globular) dimension of $\partial S$ is $n - 1$, we define the stellar dimension of $S$ to be $n$ and call $S$ a stellar $n$-cell.

Definition 2.18. An $n$-stellar structure on a subset $e$ of a topological space $X$ is a continuous map $\varphi : S \to \overline{e}$ satisfying the following conditions:

1. $S$ is a stellar $n$-cell.
2. $\varphi(S) = \overline{e}$ and the restriction $\varphi|_{\text{Int}(S)} : \text{Int}(S) \to e$ is a homeomorphism onto $e$.
3. $\varphi : S \to \overline{e}$ is a quotient map.

A subspace $e \subset X$ equipped with an $n$-stellar structure is called a stellar $n$-cell in $X$. We say $e$ is thin if $S = \{0\} \ast \partial S$. When $e$ is thin and $S$ is compact, $e$ is called closed.

2.2 Topology of Cellular and Stellar Stratified Spaces

The definition of cellular stratified spaces introduced in [BGRT, Tamb] is very similar to the one of cell complexes.

Definition 2.19. Let $X$ be a Hausdorff space. A cellular stratification on $X$ is a pair $(\pi, \Phi)$ of a stratification $\pi : X \to P(X)$ on $X$ and a collection of cell structures

$$\Phi = \{\varphi_\lambda : D_\lambda \to \overline{e_\lambda}\}_{\lambda \in P(X)}$$

on strata $\{e_\lambda = \pi^{-1}(\lambda)\}_{\lambda \in P(X)}$ of $\pi$, satisfying the condition that, for each $n$-cell $e_\lambda$, the boundary $\partial e_\lambda$ is covered by cells of dimension less than or equal to $n - 1$.

A cellular stratified space is a triple $(X, \pi, \Phi)$ where $(\pi, \Phi)$ is a cellular stratification on $X$. As usual, we abbreviate it by $(X, \pi)$ or $X$, if there is no danger of confusion. When the underlying stratified space $(X, \pi)$ is CW, $(X, \pi, \Phi)$ is said to be CW. When all cell structures are relatively compact, $X$ is called relatively compact.

\[\text{See Definition 3.5.}\]

\[\text{Lemma A.1 says that this is a reasonable assumption.}\]
For cellular stratified spaces \((X, \pi_X, \Phi_X)\) and \((Y, \pi_Y, \Phi_Y)\), a morphism of cellular stratified spaces from \((X, \pi_X, \Phi_X)\) to \((Y, \pi_Y, \Phi_Y)\) consists of a morphism \(f : (X, \pi_X) \to (Y, \pi_Y)\) of stratified spaces and a family of maps \(\{f_\lambda : D_\lambda \to D_{f(\lambda)}\}_{\lambda \in P(X)}\) that are compatible with cell structures. A morphism of cellular stratified spaces is said to be strict if \(f\) is a strict morphism of stratified spaces and \(f_\lambda(0) = 0\) for each \(\lambda \in P(X)\).

The category of cellular stratified spaces is denoted by \(\text{CSSpaces}\). The subcategory of strict morphisms is denoted by \(\text{CSSpaces}^{\text{strict}}\).

As is the case of cell complexes, the finiteness implies the CW condition\(^4\) and a CW cellular stratified space has a good topology when it is relatively compact.

**Proposition 2.20.** Any relatively compact CW cellular stratified space \(X\) is paracompact.

This is a corollary to the following theorem due to K. Morita [Mor54].

**Theorem 2.21.** Let \(\{K_\alpha\}_{\alpha \in A}\) be a closed covering of a topological space \(X\). When each \(K_\alpha\) is paracompact Hausdorff and \(X\) has the weak topology with respect to \(\{K_\alpha\}_{\alpha \in A}\), then \(X\) is paracompact Hausdorff.

**Proof of Proposition 2.20.** By Lemma [A1], \(\overline{e_\lambda}\) is paracompact Hausdorff for each cell \(e_\lambda\). By the CW assumption, \(X\) has the weak topology with respect to the covering \(\{\overline{e_\lambda}\}_{\lambda \in P(X)}\). The result follows from Morita’s theorem. \(\Box\)

We introduced the following structure on cellular stratified spaces in [Tamb] in order to describe the combinatorial structure of cell structures in terms of acyclic categories.

**Definition 2.22.** A cylindrical structure on a normal cellular stratified space \((X, \pi)\) consists of

- a normal stratification on \(\partial D^n\) containing \(\partial D_\lambda\) as a stratified subspace for each \(n\)-cell \(\varphi_\lambda : D_\lambda \to \overline{e_\lambda}\),
- a stratified space \(P_{\mu, \lambda}\) and a morphism of stratified spaces
  \[ b_{\mu, \lambda} : P_{\mu, \lambda} \times D_\mu \longrightarrow \partial D_\lambda \]
  for each pair of cells \(e_\mu \subset \partial e_\lambda\), and
- a morphism of stratified spaces
  \[ c_{\lambda_0, \lambda_1, \lambda_2} : P_{\lambda_1, \lambda_2} \times P_{\lambda_0, \lambda_1} \longrightarrow P_{\lambda_0, \lambda_2} \]
  for each sequence \(\overline{e_{\lambda_0}} \subset \overline{e_{\lambda_1}} \subset \overline{e_{\lambda_2}}\) satisfying the following conditions:

1. The restriction of \(b_{\mu, \lambda}\) to \(P_{\mu, \lambda} \times \text{Int}(D_\mu)\) is a homeomorphism onto its image.
2. The following three types of diagrams are commutative:

\[
\begin{array}{c}
D_\lambda \overset{\varphi_\lambda}{\longrightarrow} X \\
\downarrow b_{\mu, \lambda} \quad \downarrow \varphi_\mu \\
P_{\mu, \lambda} \times D_\mu \overset{\text{pr}_2}{\longrightarrow} D_\mu
\end{array}
\]

\(^4\)Proposition 2.7
3. We have
\[ \partial D_\lambda = \bigcup_{e_\mu \subset e_\lambda} b_{\mu,\lambda}(P_{\mu,\lambda} \times \text{Int}(D_\mu)) \]
as a stratified space.

The space \( P_{\mu,\lambda} \) is called the parameter space for the inclusion \( e_\mu \subset e_\lambda \). When \( \mu = \lambda \), we define \( P_{\lambda,\lambda} \) to be a single point. A cellular stratified space equipped with a cylindrical structure is called a cylindrically normal cellular stratified space. When the map \( b_{\lambda,\mu} \) is an embedding for each pair \( e_\mu \subset e_\lambda \), the stratification is said to be strictly cylindrical.

Let \((X, \pi_X, \Phi_X)\) and \((Y, \pi_Y, \Phi_Y)\) be cylindrically normal cellular stratified spaces with cylindrical structures given by \( \{ b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \to D_\lambda \} \) and \( \{ b_{\alpha,\beta} : P_{\alpha,\beta} \times D_\alpha \to D_\beta \} \), respectively. A morphism of cylindrically normal cellular stratified spaces from \((X, \pi_X, \Phi_X)\) to \((Y, \pi_Y, \Phi_Y)\) is a morphism of stellar stratified spaces
\[ f = (f, f) : (X, \pi_X, \Phi_X) \to (Y, \pi_Y, \Phi_Y) \]
together with maps \( f_{\mu,\lambda} : P_{\mu,\lambda} \to P_{\lambda,\lambda} \) that are compatible with structure maps.

The category of cylindrically normal cellular stratified spaces is denoted by \( \text{CSSpaces}_cyl \).

Remark 2.23. Cylindrical structures on stellar stratified spaces are defined analogously. See [Tamb] for details.

Totally normal stellar stratified spaces form an important subclass of cylindrically normal stellar stratified spaces.

Definition 2.24. A totally normal cellular stratified space \((X, \pi)\) is a strictly cylindrically normal cellular stratified space whose parameter spaces \( P_{\mu,\lambda} \) are finite sets for all \( \mu, \lambda \in P(X, \pi) \).

Remark 2.25. The above definition is different from but is equivalent to the one in [Tamb].

For cylindrically normal cellular stratified spaces, the relative-compactness can be easily verified.

Lemma 2.26. Let \( X \) be a cylindrically normal cellular stratified space with parameter spaces \( \{ P_{\mu,\lambda} \}_{\mu \leq \lambda} \). A cell \( \varphi_\lambda : D_\lambda \to \overline{\text{Int}(X)} \) is relatively compact if and only if \( P_{\mu,\lambda} \) is compact for each \( \mu \leq \lambda \).
Proof. For \( y \in \partial e_\lambda \), there exists \( \mu \leq \lambda \) with \( y \in e_\mu \subset \partial e_\lambda \). In the commutative diagram

\[
\begin{array}{ccc}
D_\lambda & \xrightarrow{\varphi_\lambda} & e_\lambda \\
| & \downarrow{b_{\mu,\lambda}|_{P_{\mu,\lambda} \times \text{Int}(D_\mu)}} & \\
\text{Int}(D_\mu) & \xrightarrow{\varphi_\mu|_{\text{Int}(D_\mu)}} & \text{Int}(D_\mu)
\end{array}
\]

the restriction \( b_{\mu,\lambda}|_{P_{\mu,\lambda} \times \text{Int}(D_\mu)} \) is an embedding and \( \partial D_\lambda \) is covered by the disjoint union of such images. Thus we have \( \varphi_\lambda^{-1}(y) \cong P_{\mu,\lambda} \times \{y\} \) and the result follows from Corollary A.15 and the assumption.

Corollary 2.27. Let \( X \) be a CW cylindrically normal cellular stratified space. If all parameter spaces are compact, \( X \) is paracompact.

In order to make use of PL-topological techniques, we introduced the following structure in [Tamb].

Definition 2.28. A locally polyhedral stellar stratified space consists of

1. a CW cylindrically normal stellar stratified space \( X \),
2. a family of Euclidean polyhedral complexes \( \tilde{F}_\lambda \) indexed by \( \lambda \in P(X) \) and
3. a family of homeomorphisms \( \alpha_\lambda : \tilde{F}_\lambda \to D_\lambda \) indexed by \( \lambda \in P(X) \), where \( D_\lambda \) is the closure of \( D_\lambda \) in a disk containing \( D_\lambda \),

satisfying the following conditions:

1. For each cell \( e_\lambda \), \( \alpha_\lambda : \tilde{F}_\lambda \to \overline{D_\lambda} \) is a subdivision of a stratified space, where the stratification on \( \overline{D_\lambda} \) is defined by the cylindrical structure.

2. For each pair \( e_\mu < e_\lambda \), the parameter space \( P_{\mu,\lambda} \) is a locally cone-like space and the composition

\[
P_{\mu,\lambda} \times F_\mu \xrightarrow{1 \times \alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} D_\lambda \xrightarrow{\alpha_\lambda^{-1}} F_\lambda
\]

is a PL map, where \( F_\lambda = \alpha_\lambda^{-1}(D_\lambda) \).

Each \( \alpha_\lambda \) is called a polyhedral replacement of the cell structure of \( e_\lambda \). The collection \( A = \{ \alpha_\lambda \} \subseteq \lambda \subseteq P(X) \) is called a locally-polyhedral structure on \( X \).

A morphism \( f : X \to X' \) of locally polyhedral cellular stratified spaces consists of a morphism of cylindrically normal cellular stratified spaces and a family of PL maps \( f_{\lambda} : \tilde{F}_\lambda \to \tilde{F}'_{\lambda(\lambda)} \) for \( \lambda \in P(X) \) that are compatible with locally-polyhedral structures.

The category of locally polyhedral cellular stratified spaces is denoted by \( \text{CSSpaces}^{LP} \). The full subcategory of locally polyhedral cell complexes is denoted by \( \text{CellComplexes}^{LP} \).

We frequently encounter the following special case.

Lemma 2.29. Let \( X \) be a subspace of \( \mathbb{R}^N \) equipped with a structure of cylindrically normal CW cellular stratified space whose parameter spaces \( P_{\mu,\lambda} \) are locally cone-like spaces. Suppose, for each \( e_\lambda \in P(X) \), there exists a polyhedral complex \( \tilde{F}_\lambda \) and a homeomorphism \( \alpha_\lambda : \tilde{F}_\lambda \to D_{\dim e_\lambda} \) such that the composition

\[
P_{\mu,\lambda} \times F_\mu \xrightarrow{1 \times \alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} D_\lambda \xrightarrow{\varphi_\lambda} X \leftrightarrow \mathbb{R}^N
\]
is a PL map, where $F_\lambda = \alpha_\lambda^{-1}(D_\lambda)$. Suppose further that $\alpha_\lambda : \tilde{F}_\lambda \to D^{\dim e_\lambda}$ is a cellular subdivision, where the cell decomposition on $D^{\dim e_\lambda}$ is the one in the definition of cylindrical structure. Then the collection $\{\alpha_\lambda\}_{\lambda \in \mathcal{P}(X)}$ defines a local polyhedral structure on $X$.

Proof. See [Tamb].

**Definition 2.30.** A cellular stratified space satisfying the assumption of Lemma 2.29 is called a Euclidean locally polyhedral cellular stratified space.

The following fact is also proved in [Tamb].

**Lemma 2.31.** Any CW totally normal cellular stratified space has a locally polyhedral structure.

Any locally polyhedral cellular stratified space can be always embedded in a cell complex functorially.

**Lemma 2.32.** There exists a functor $U : \text{CSSpaces}^{LP} \to \text{CellComplexes}^{LP}$, whose restriction to $\text{CellComplexes}^{LP}$ is the identity.

Proof. For a locally polyhedral cellular stratified space $X$, a cell complex $U(X)$ containing $X$ as a cellular stratified subspace is defined in [Tamb] by extending the cell structures to the whole closed disks by using the PL condition. Since a morphism $X \to Y$ of locally polyhedral cellular stratified spaces are assumed to be PL on parameter spaces and cells, it extends to a morphism $U(f) : U(X) \to U(Y)$ of locally polyhedral cell complexes by Lemma A.38 in [Tamb].

**Definition 2.33.** For a locally polyhedral cellular stratified space $X$, the cell complex $U(X)$ is called the cellular closure of $X$.

The existence of cellular closure implies that cell structures of a locally polyhedral cellular stratified space are bi-quotient.

**Corollary 2.34.** Let $X$ be a locally polyhedral cellular stratified space. Then any cell structure $\varphi_\lambda : D_\lambda \to e_\lambda$ is bi-quotient.

Proof. For an $n$-cell $\varphi_\lambda : D_\lambda \to e_\lambda$ in $X$, let $\tilde{\varphi}_\lambda : D^n \to U(X)$ the extension. Since $\tilde{\varphi}_\lambda$ is proper, it is bi-quotient and hence is hereditarily quotient. By Lemma A.3, $\varphi_\lambda$ is also hereditarily quotient.

For $y \in e_\lambda \subset X$, the fiber $\varphi_\lambda^{-1}(y)$ can be identified with one of parameter spaces by the proof of Lemma 2.26 which is a cellular stratified subspace of a regular cell decomposition of $\partial D^n$. Thus the boundary $\partial \varphi_\lambda^{-1}(y)$ is compact. The result follows from 3 in Lemma A.4.

**Corollary 2.35.** Any locally polyhedral cellular stratified space is paracompact.

2.3 Topological Face Categories

We associate a topological category to a cylindrically normal stellar stratified space. Let us first fix notations for topological categories.

5Definition A.2
Definition 2.36. A topological category $C$ consists of topological spaces $C_0$ and $C_1$, and continuous maps
\[ s : C_1 \rightarrow C_0 \]
\[ t : X_1 \rightarrow C_0 \]
\[ \circ : N_2(C) = \{(u, v) \in C_1^2 \mid s(u) = t(v)\} \rightarrow C_1 \]
\[ \iota : C_0 \rightarrow C_1 \]
satisfying the following conditions:
1. $(u_3 \circ u_2) \circ u_1 = u_3 \circ (u_2 \circ u_1)$ for any $u_1, u_2, u_3 \in X_1$ with $s(u_3) = t(u_2)$ and $s(u_2) = t(u_1)$.
2. $\iota(y) \circ u = u = u \circ \iota(x)$ if $s(u) = x$ and $t(u) = y$.
Elements in $C_0$ are called objects and $u \in C_1$ with $s(u) = x$ to $t(u) = y$ is called a morphism from $x$ to $y$.

Remark 2.37. In this paper, we only consider the case when $C_0$ is equipped with the discrete topology, which forces a coproduct decomposition
\[ C_1 = \bigsqcup_{x,y \in C_0} C(x, y) \]
and we have
\[ N_2(C) = \bigsqcup_{x,y,z \in C_0} C(y, z) \times C(x, y). \]
Thus we may define a topological category $C$ (with $C_0$ discrete) in terms of $C_0$ and $\{C(x, y)\}_{(x,y) \in C_0^2}$.

Definition 2.38. Let $X$ be a cylindrically normal stellar stratified space. Define a category $C(X)$ as follows. Objects are cells in $X$; $C(X)_0 = P(X)$ with discrete topology.
For each pair $\dot{e}_\mu \subset \dot{e}_\lambda$, define
\[ C(X)(\dot{e}_\mu, \dot{e}_\lambda) = P_{\mu, \lambda}. \]
The composition of morphisms is given by
\[ C(X)(\dot{e}_\lambda, \dot{e}_\mu) : P_{\lambda_1, \lambda_2} \times P_{\lambda_0, \lambda_1} \rightarrow P_{\lambda_0, \lambda_2}. \]
The category $C(X)$ is called the cylindrical face category of $X$.

Definition 2.39. For a topological category $C$, the simplicial space $N(C) = \{N_n(C)\}_{n \geq 0}$ defined by
\[ N_n(C) = \text{Funct}([n], C) \]
is called the nerve of $C$, where $[n]$ is the poset $0 < 1 < \cdots < n$ regarded as a small category.
The geometric realization of $N(C)$ is denoted by $BC$ and is called the classifying space of $C$.

Remark 2.40. In the above definition, we regard a simplicial space $K$ as a functor
\[ K : \Delta^{op} \rightarrow \text{Spaces}, \]
where $\Delta$ is the category whose objects are $[n]$ for $n = 0, 1, 2, \ldots$ and whose morphisms are order preserving maps. The subcategory of $\Delta$ consisting of injective morphisms is denoted by $\Delta_{\text{inj}}$.
Recall that a functor
\[ K : \Delta_{\text{inj}}^{op} \rightarrow \text{Spaces} \]
is called a $\Delta$-space.

\[ \text{This is usually called a category internal to the category of topological spaces or a category object in the category of topological spaces.} \]
Lemma 2.41. For a cylindrically normal cellular stratified space $X$, the face category $C(X)$ is an acyclic topological category. When $X$ is totally normal, the topology is discrete.

Definition 2.42. For an acyclic category $C$, define a partial order $\leq$ on $C_0$ by

$$x \leq y \iff C(x, y) \neq \emptyset.$$ 

The resulting poset is denoted by $P(C)$.

Remark 2.43. For a cylindrically normal cellular stratified space $X$, we have $P(C(X)) = P(X)$ as posets.

When we study the classifying space of an acyclic category, we may remove all identity morphisms.

Definition 2.44. For an acyclic category $C$, define

$$\mathcal{N}_n(C) = N_n(C) - \bigcup_{i=0}^{n} s_i(N_{n-1}(C)).$$

The collection $\mathcal{N}(C) = \{\mathcal{N}_n(C)\}$ is called the nondegenerate nerve of $C$.

It is a well-known fact that the nondegenerate nerve of an acyclic topological category has a structure of $\Delta$-space.

Lemma 2.45. For an acyclic topological category $C$ (with discrete topology), the nondegenerate nerve $\mathcal{N}(C)$ has a structure of a $\Delta$-space. Furthermore we have

$$BC \cong \|\mathcal{N}(C)\|,$$

where the right hand side is the geometric realization of $\Delta$-space.

Hence, when the topologies on the morphism spaces $C(x, y)$ are discrete, the classifying space $BC$ is a totally normal cell complex.

Proof. The first half is a well-known property of acyclic categories. The last sentence follows from the fact that the geometric realization of a $\Delta$-set has a structure of totally normal cell complex. See [Tamb].

Cells in the geometric realization of a $\Delta$-set can be described as follows.

Definition 2.46. For a $\Delta$-set $K$, define a map

$$\pi_K : \|K\| \to \coprod_k K_k$$

by $\pi_K([x, t]) = x$ with $t \in \text{Int}(\Delta^k)$.

Lemma 2.47. For a $\Delta$-set $K$, the map $\pi_K$ defines a stratification on $\|K\|$. For $x \in K_k$, the stratum $\pi_K^{-1}(x)$ is homeomorphic to $\text{Int}(\Delta^k)$.

Proof. This is a restatement of the fact that the geometric realization of a $\Delta$-set has a structure of cell complex. The map $\pi_K$ is nothing but the stratification underlying the structure of cell complex on $\|K\|$.
Definition 2.48. For a cylindrically normal stellar stratified space $X$, the classifying space of $C(X)$ is denoted by $Sd(X)$ and is called the barycentric subdivision of $X$.

With this notation, Lemma 2.41 and Lemma 2.45 imply the following.

Corollary 2.49. For a totally normal stellar stratified space $X$, $Sd(X)$ has a structure of totally normal cell complex.

Vertices in $Sd(X)$ are in one-to-one correspondence to cells in $X$. We use the following notation.

Definition 2.50. Let $X$ be a cylindrically normal stellar stratified space. The image of $e^\lambda \in P(X)$, under

$$P(X) = \overline{N}_0(X) \hookrightarrow \coprod_k \overline{N}_k(X) \times \Delta^k \to Sd(X)$$

is denoted by $v(e^\lambda)$ and is called the barycenter of $e^\lambda$.

Lemma 2.51. Let $C$ be an acyclic topological category in which each morphism space $C(x,y)$ is equipped with a structure of cellular stratified space whose cell structures are bi-quotient. Then the classifying space $BC$ has a structure of cellular stratified space.

In particular when all morphism spaces are cell complexes, the classifying space $BC$ has a structure of cell complex.

Proof. Under the identification $BC \cong \|\overline{N}(C)\|$, any point in $BC$ can be represented by a pair $(x,t) \in \overline{N}_k(C) \times \text{Int}(\Delta^k)$ uniquely. In other words, we have a decomposition

$$BC = \coprod_{k=0}^\infty \overline{N}_k(C) \times \text{Int}(\Delta^k)$$

as sets. The bi-quotient assumption on cell structures on $C(x,y)$ implies that $\overline{N}_k(C)$ has a structure of cellular stratified space. Thus the above decomposition and the cellular stratification on each $\overline{N}_k(C)$ define a stratification on $BC$.

For each cell $\varphi : D \to \overline{e} \subset \overline{N}_k(C)$, let us denote the cell in $BC$ corresponding to $e \times \text{Int}(\Delta^k)$ by $e \times e^k$. The composition

$$D \times \Delta^k \xrightarrow{\varphi \times 1_{\Delta^k}} \overline{N}_k(C) \times \Delta^k \to \|\overline{N}(C)\| = BC$$

defines a cell structure on $e \times e^k$. $\square$

Definition 2.52. An acyclic topological category $C$ is said to be cellular if $C_0$ is discrete and $C_1$ has a structure of cellular stratified space whose cell structures are bi-quotient.

Remark 2.53. In the above definition, we assume that each $C(y,z) \times C(x,y)$ is a cellular stratified space under the product stratification. See Proposition 3.17 and discussions in §3.2 for conditions under which the bi-quotient assumption in Lemma 2.51 is satisfied.

When $C_0$ is not discrete it is not easy to define cellularity, since the pullback of stratified spaces over a stratified space may not be a stratified space in general.

When $X$ is locally polyhedral, the face category $C(X)$ has a cellular structure.

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7 See Definition 3.16 for product cellular stratifications
Lemma 2.54. When $X$ is a finite locally polyhedral stellar stratified space, the face category $C(X)$ has a structure of cellular category.

Proof. By definition $C(X)_0$ has the discrete topology. Since $X$ is locally polyhedral, the parameter space $P_{\mu,\lambda}$ is a locally cone-like space for each pair $e_{\mu} < e_{\lambda}$ of cells. It is well-known that locally cone-like spaces are stratified by simplices and simplicial cones. Thus $C(X)_1$ has a structure of regular locally polyhedral cellular stratified space. By Corollary 2.34, cell structures in $C(X)_1$ are bi-quotient. □

The following notations will be used in §4.2.

Definition 2.55. Let $C$ be an acyclic topological category and $x$ an object of $C$. The nondegenerate nerve $N(x \downarrow C)$ of the comma category $x \downarrow C$ is denoted by $\text{St}_{\geq x}(C)$ and is called the upper star of $x$ in $C$.

The full subcategory of $x \downarrow C$ consisting of $(x \downarrow C)_0 - \{1_x\}$ is denoted by $C_{>x}$. The nondegenerate nerve $N(C_{>x})$ is denoted by $\text{Lk}_{>x}(C)$ and is called the upper link of $x$ in $C$.

The functor induced by the target map in $C$ is denoted by

$$t_x : C_{>x} \subset x \downarrow C \rightarrow C.$$ 

The induced map of $\Delta$-spaces is also denoted by

$$t_x : \text{Lk}_{>x}(C) \subset \text{St}_{\geq x}(C) \rightarrow N(C).$$

Dually the nondegenerate nerves of the comma category $C \downarrow x$ and of its full subcategory $C_{<x}$ consisting of $(C \downarrow x)_0 - \{1_x\}$ are denoted by $\text{St}_{\leq x}(C)$ and $\text{Lk}_{<x}(C)$ and called the lower star and lower link of $x$ in $C$, respectively. We also have maps

$$s_x : C_{<x} \rightarrow C$$

$$s_x : \text{Lk}_{<x}(C) \rightarrow N(C)$$

induced by the source map.

Remark 2.56. The notation $C_{>x}$ and its definition is borrowed from Kozlov's book [Koz08]. Note that $\text{Lk}_{>x}(C)$ is different from the usual link of $x$ in $N(C)$ in general.

We have the following description.

Lemma 2.57. For an acyclic topological category $C$ and an object $x \in C_0$, we have the following identification

$$\text{Lk}_{>x}(C)_k \cong \begin{cases} \prod_{x \neq y} C(x, y), & k = 0 \\ \{ u \in N_{k+1}(C) \mid s(u) = x \}, & k > 0, \end{cases}$$

under which the face operators

$$d^{	ext{Lk}}_i : \text{Lk}_{>x}(C)_k \rightarrow \text{Lk}_{>x}(C)_{k-1}$$

are identified as $d^{	ext{Lk}}_i = d_{i+1}$, where $d_{i+1}$ is the face operator in $N(C)$.
Example 2.58. Consider the poset \([2] = \{0 < 1 < 2\}\) regarded as a category \(0 \to 1 \to 2\). The category \([2]_{\geq 1}\) has a unique object \(1 \to 2\) and no nontrivial morphism. Thus \(\text{Lk}_{>1}(2)\) is a single point. Under the map \(t_1 : \text{Lk}_{>1}(2) \to \overline{\mathcal{N}}(2) \cong \Delta^2\), \(\text{Lk}_{>1}(2)\) can be identified with the vertex \(2\) in \(\Delta^2\). On the other hand, the usual link of \(1\) in \(\Delta^2\) is the 1-simplex spanned by vertices 0 and 2. \(\square\)

Example 2.59. Consider the face category \(C(S^1; \pi_1)\) of the minimal cell decomposition of \(S^1\). \(C(S^1; \pi_1)_{>0,0}\) consists of two objects \(C(S^1; \pi_1)_{>0,0}) = C(S^1; \pi_1)(e^0, e^1) = \{b_-, b_+\}\) and no nontrivial morphisms. Thus \(\text{St}_{>0,0}(C(S^1; \pi_1))\) is the cell complex \([-1, 1] = \{-1\} \cup \{-1, 0\} \cup \{0\} \cup \{0, 1\} \cup \{1\}\) and \(\text{Lk}_{>0,0}(C(S^1; \pi_1))\) is \(S^0\). The map \(t_{e^0}\) maps the boundary \(\partial[-1, 1] = S^0\) to \(v(e^2)\) in \(\text{Sd}(S^1)\) and defines a 1-cell structure. \(\square\)

Note that, the comma category \(x \downarrow C\) has an initial object \(1_x\).

Lemma 2.60. For an acyclic topological category \(C\) and an object \(x \in C_0\), we have a homeomorphism

\[
\|\text{St}_{\geq x}(C)\| \cong \{1_x\} \ast \|\text{Lk}_{>x}(C)\|.
\]

Proof. Define a map \(h_x : \|\text{St}_{>x}(C)\| \to \{1_x\} \ast \|\text{Lk}_{>x}(C)\|\) as follows. For \([u, t] \in \|\text{St}_{>x}(C)\| = \|N(x \downarrow C)\|\), choose a representative \((u, t) \in N_k(x \downarrow C) \times \Delta^k\). Here we regard \(u\) as a sequence of composable \(k + 1\) morphisms in \(C\) starting from \(x\):

\[
u : x \xrightarrow{u_0} x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_k} x_k
\]

with \(u_1, \ldots, u_k\) non-identity morphisms. When \(u_0\) is not the identity morphism, \(u\) belongs to \(\text{Lk}_{>x}(C)_k\) and \([u, t]\) can be regarded as an element of \(\|\text{Lk}_{>x}(C)\|\). Define

\[
h_x([u, t]) = 01_x + 1([u, t]).
\]

When \(u_0 = 1_x\), write \(t = t_00 + (1 - t_0)t'\) under the identification \(\Delta^k \cong \{0\} \ast \Delta^{k-1}\) and define

\[
h_x([u, t]) = t_01_x + (1 - t_0)[(u', t')],
\]

where

\[
u' : x \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_k} x_k
\]

is the \((k - 1)\)-chain obtained from \(u\) by removing \(u_0\). Since \(u\) is a nondegenerate chain, \(u_1\) is not the identity morphism and \(u' \in \text{Lk}_{>x}(C)_{k-1}\).

Since the set of objects \(C_0\) has the discrete topology, the decomposition

\[
N_k(x \downarrow C) = \coprod_{x_1 < x_2 < \cdots < x_k} C(x, x_1) \times C(x, x_1) \times \cdots \times C(x_{k-1}, x_k)
\]

is a decomposition of topological spaces. The map \(h_x\) is continuous on each component of \(N_k(x \downarrow C) \times \Delta^k\) and defines a continuous map

\[
h_x : \|\text{St}_{>x}(C)\| \to \{1_x\} \ast \|\text{Lk}_{>x}(C)\|.
\]

It is easy to define an inverse to \(h_x\), and thus \(h_x\) is a homeomorphism. \(\square\)
3 Basic Constructions on Cellular Stratified Spaces

We study the following operations on cellular and stellar stratified spaces in this section:

- taking stratified subspaces, cellular stratified subspaces, and stellar stratified subspaces,
- taking products of cellular stratified subspaces, and
- taking subdivisions of cellular and stellar stratified spaces.

3.1 Stratified Subspaces

In this section, we consider the problem of restricting cellular stratifications to subspaces. Obviously the category of stratified spaces is closed under taking complements of stratified subspaces.

**Lemma 3.1.** Let $X$ be a stratified space and $A$ be a stratified subspace. Then the complement $X - A$ is also a stratified subspace of $X$.

This is one of the most useful facts when we work with stratifications on configuration spaces and complements of arrangements. On the other hand, when $X$ is a CW complex and $A$ is a subcomplex, $A$ is always closed. This is no longer true for cellular stratified spaces.

**Example 3.2.** In Example 1.2, $\text{Lk}(A \otimes \mathbb{R}^\ell)$ is a closed stratified subspace of $\mathbb{R}^n \otimes \mathbb{R}^\ell$. Its complement $M(A \otimes \mathbb{R}^\ell)$ is also a stratified subspace but is not closed in $\mathbb{R}^n \otimes \mathbb{R}^\ell$.

Let us consider cell structures on subspaces. Suppose $X$ is a cellular stratified space and $A$ is a stratified subspace. In order to incorporate cell structures, we need to specify a cell structure for each cell contained in $A$.

**Definition 3.3.** Let $X$ be a topological space and $A$ be a subset of $X$. For a hereditarily quotient $n$-cell structure $\varphi : D \to e \subset X$ with $e \subset A$, define an $n$-cell structure on $e$ in $A$ to be $(D_A, \varphi|_{D_A})$ where $D_A = \varphi^{-1}(e \cap A)$.

**Lemma 3.4.** Let $X$ be a topological space and $A$ be a subset of $X$. If $(D, \varphi)$ is a hereditarily quotient $n$-cell structure on $e \subset X$ and $e \subset A$, then $(D_A, \varphi|_{D_A})$ defined above is an $n$-cell structure on $e$ in $A$.

**Proof.** It suffices to check the maximality condition but the domain $D_A$ is chosen to guarantee the maximality.

**Definition 3.5.** Let $(X, \pi)$ be a cellular stratified space. A stratified subspace $(A, \pi|_A)$ of $X$ is said to be a cellular stratified subspace, provided cell structures on cells in $A$ are given as indicated in Definition 3.3. When the inclusion $A \hookrightarrow X$ is a strict morphism, $A$ is said to be a strict cellular stratified subspace.

**Remark 3.6.** In the above definition, all cell structure maps for cells in $A$ are assumed to be restrictions of hereditarily quotient cell structures in $X$.

We need to take cell decompositions of domains of cells into account for stellar stratified subspaces.

**Lemma 3.7.** Let $X$ be a topological space, $A$ a subspace, and $\varphi : D \to \pi$ a stellar $n$-cell of $X$ with $e \subset A$. When $D_A = \varphi^{-1}(e \cap A)$ is a stratified subspace of $D$ and $\varphi$ is hereditarily quotient, the restriction $\varphi|_{D_A} : D_A \to A$ defines a stellar $n$-cell structure on $e$ in $A$. 

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**Definition 3.8.** Let \((X, \pi)\) be a stellar stratified space and \(A\) a subspace of \(X\). If the assumption of Lemma 3.7 is satisfied for each cell in \(A\), \((A, \pi|_{A})\) is said to be a stellar stratified subspace of \(X\).

**Remark 3.9.** Let \(X\) be a cellular or stellar stratified space and \(A\) be a cellular or stellar stratified subspace. Then \(A\) is a strict stratified subspace of \(X\).

The following fact can be regarded as a generalization of the fact that any subcomplex of a regular cell complex is regular.

**Proposition 3.10.** Any stellar stratified subspace \(A\) of a cylindrically normal stellar stratified space \(X\) is cylindrically normal. Furthermore the parameter space for a pair of cells \(e_\mu < e_\lambda\) in \(A\) can be identified with the parameter space for the same pair when regarded as cells in \(X\).

**Proof.** Let \(e_\mu < e_\lambda\) be a pair of cells in \(A\). Let \(\varphi_\mu : D_\mu \rightarrow X\) and \(\varphi_\lambda : D_\lambda \rightarrow X\) be the cell structures for these cells in \(X\). When regarded as cells in \(A\), their cell structures are denoted by

\[
\varphi_{A,\mu} : D_{A,\mu} \rightarrow A \\
\varphi_{A,\lambda} : D_{A,\lambda} \rightarrow A
\]

respectively.

We need to show that the structure map

\[
b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \rightarrow D_\lambda
\]

of cylindrical structure for the pair in \(X\) can be restricted to

\[
b_{\mu,\lambda}|_{P_{\mu,\lambda} \times D_{A,\mu}} : P_{\mu,\lambda} \times D_{A,\mu} \rightarrow D_{A,\lambda}.
\]

This can be verified by the commutativity of the diagram

![Diagram](image)

and the definition of stellar structures on \(A\).

**Example 3.11.** Consider Example 1.2. The link \(\text{Lk}(\mathcal{A} \otimes \mathbb{R}^f)\) and the complement \(M(\mathcal{A} \otimes \mathbb{R}^f)\) are both cellular stratified subspaces of \((\mathbb{R}^n \otimes \mathbb{R}^f, \pi_{\mathcal{A} \otimes \mathbb{R}^f})\), which is regular, hence cylindrically normal.
Thanks to Corollary 2.34, cell structures in a locally polyhedral cellular stratified space $X$ are hereditarily quotient. Thus any cellular stratified subspace $A$ inherits a cylindrically normal structure with structure maps satisfying the PL conditions in the definition of locally polyhedral cellular stratification. The problem is the CW condition. It is easy to see that the closure finiteness condition can be restricted freely. The question is when a stratified subspace of a CW stratified subspace inherits the weak topology.

**Lemma 3.12.** A closed or an open stratified subspace $A$ of a CW stratified space $X$ is CW.

*Proof.* This follows from the corresponding property of weak topology. ☐

By Lemma 2.32 any locally polyhedral cellular stratified space $X$ can be embedded in a CW complex $U(X)$. In general, however, a cellular stratified subspace $A$ of $X$ is neither closed nor open in $U(X)$ and it is not easy to verify the weak topology condition. One of the practical conditions is the locally finiteness. The CW condition is guaranteed by Proposition 2.7.

**Proposition 3.13.** Let $X$ be a locally polyhedral cellular stratified space. Any locally finite cellular stratified subspace $A$ is locally polyhedral.

We end this section by an example which shows another difference between cellular stratified subspaces and subcomplexes. In the case of CW complexes, the colimit of an increasing sequence of finite subcomplexes

$$X_0 \subset X_1 \subset \cdots \subset \operatorname{colim}_n X$$

is automatically a CW complex. This is not true for cellular stratified spaces.

**Example 3.14.** Consider the space

$$X = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \cup \mathbb{Z} \times \{0\}.$$

The homeomorphism

$$p : D^2 - \{(0, 1)\} \longrightarrow \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$$

given by extending the stereographic projection $S^1 - \{(0, 1)\} \rightarrow \mathbb{R}$,

![Stereographic Projection Diagram]

defines a 2-cell structure on

$$e^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \subset X$$

by restricting $p$ to $D = p^{-1}(X)$.

Each $\{(n, 0)\} \subset \mathbb{Z} \times \{0\}$ can be regarded as a 0-cell $e^0_n$. And we have a cellular stratification on $X$

$$X = \left( \bigcup_{n \in \mathbb{Z}} e^0_n \right) \cup e^2.$$
$X$ is a colimit of

$$X_n = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \cup \{i \in \mathbb{Z} \mid |i| \leq n\} \times \{0\}.$$  

Each $X_n$ is a finite stratified subspace of $X$, hence is CW. But $X$ is not CW.

### 3.2 Products

In this section, we study products of stratifications, cell structures, and cellular stratified spaces and deduce a couple of conditions under which we may take products.

It is not difficult to define a stratification on the product of two stratified spaces.

**Lemma 3.15.** Let $(X, \pi_X)$ and $(Y, \pi_Y)$ be stratified spaces. The map

$$\pi_X \times \pi_Y : X \times Y \to P(X) \times P(Y)$$

defines a stratification on $X \times Y$.

**Proof.** The product of open maps is again open.

We have to be careful when we take products of cellular stratified spaces. Even in the category of CW complexes, there is a well-known difficulty in taking products. Given CW complexes $X$ and $Y$, we need to impose the local-finiteness on $X$ or $Y$ or to redefine a topology on $X \times Y$ in order to make $X \times Y$ into a CW complex.

In the case of cellular stratified spaces, we have another difficulty because of our requirement on cell structures. The product of two quotient maps may not be a quotient map.

**Definition 3.16.** Let $(X, \pi_X, \Phi_X)$ and $(Y, \pi_Y, \Phi_Y)$ be cellular stratified spaces and consider the product stratification

$$\pi_X \times \pi_Y : X \times Y \to P(X) \times P(Y)$$

on $X \times Y$ in Lemma 3.15. For cells $e_\lambda \subset X$ and $e_\mu \subset Y$, consider the composition

$$\varphi_{\lambda, \mu} : D_{\lambda, \mu} \cong D_\lambda \times D_\mu \frac{\varphi_\lambda \times \varphi_\mu}{e_\lambda \times e_\mu} \cong e_\lambda \times e_\mu \subset X \times Y,$$

where $D_{\lambda, \mu}$ is the subspace of $D^{\dim e_\lambda + \dim e_\mu}$ obtained by pulling back $D_\lambda \times D_\mu$ via the standard homeomorphism

$$D^{\dim e_\lambda + \dim e_\mu} \cong D^{\dim e_\lambda} \times D^{\dim e_\mu}.$$

When $\varphi_{\lambda, \mu}$ is a quotient map, $\varphi_{\lambda, \mu} : D_{\lambda, \mu} \to e_\lambda \times e_\mu$ is called the **product cell structure** on $e_\lambda \times e_\mu$. If $\varphi_{\lambda, \mu}$ is a quotient map for each pair of cells in $X$ and $Y$, the resulting cellular stratification is called the **product cellular stratification** on $X \times Y$.

The above definition is incomplete. Unless we have a general criterion for $\varphi_{\lambda, \mu}$ to be a quotient map, this definition is useless.

By Lemma A.8 and Proposition A.13, we can take products of bi-quotient cell structures. The question is when a cell structure is a bi-quotient map. By Corollary A.15 and Corollary 2.34, we obtain the following practical conditions under the assumption of cylindrical normality.
Proposition 3.17. Let $X$ and $Y$ be cylindrically normal cellular stratified spaces. If they satisfy one of the following conditions, any product $e_\lambda \times e_\mu$ has the product cell structure for $\lambda \in P(X)$ and $\mu \in P(Y)$:

1. All parameter spaces are compact.
2. Locally polyhedral.

Proof. If $X$ or $Y$ is locally polyhedral, all cell structures are bi-quotient by Corollary 2.21. By Lemma 2.26, each fiber of a cell structure in a cylindrically normal cellular stratified space can be identified with a parameter space. Thus, when all parameter spaces are compact, the cell structures are bi-quotient by Corollary A.15. 

If $X$ and $Y$ satisfy one of the above conditions, the product $X \times Y$ has a structure of cellular stratified space. It is reasonable to expect that $X \times Y$ is again cylindrically normal.

Theorem 3.18. Let $X$ and $Y$ be cylindrically normal cellular stratified spaces satisfying one of the conditions in Proposition 3.17. Then the product $X \times Y$ is a cylindrically normal cellular stratified space.

Proof. Let $\{\varphi^X_\lambda : D_\lambda \to e_\lambda^X\}$ and $\{\varphi^Y_\mu : D_\mu \to e_\mu^Y\}$ be cell structures on $X$ and $Y$, respectively. Cylindrical structures on $X$ and $Y$ are denoted by $\{b^X_{\lambda,\lambda'} : P^{X}_{\lambda,\lambda'} \times D_\lambda \to D_{\lambda'}\}$ and $\{b^Y_{\mu,\mu'} : P^{Y}_{\mu,\mu'} \times D_\mu \to D_{\mu'}\}$, respectively.

For a pair of cell structures $\varphi^X_\lambda : D_\lambda \to X$ and $\varphi^Y_\mu : D_\mu \to Y$, consider the product cell structure

$$\varphi_{\lambda,\mu} : D_{\lambda,\mu} \cong D_\lambda \times D_\mu \xrightarrow{\varphi^X_\lambda \times \varphi^Y_\mu} e^X_\lambda \times e^Y_\mu.$$ 

For $(\lambda, \mu) \leq (\lambda', \mu')$, define

$$P_{(\lambda,\mu),(\lambda',\mu')} = P^X_{\lambda,\lambda'} \times P^Y_{\mu,\mu'}.$$ 

When $(\lambda, \mu) < (\lambda', \mu')$, we have either $\lambda < \lambda'$ or $\mu < \mu'$ and thus the image of the composition

$$P_{(\lambda,\mu),(\lambda',\mu')} \times D_{\lambda,\mu} \cong P^X_{\lambda,\lambda'} \times P^Y_{\mu,\mu'} \times D_\lambda \times D_\mu \xrightarrow{\varphi^X_\lambda \times \varphi^Y_\mu} P^X_{\lambda,\lambda'} \times D_{\lambda} \times D_{\mu} \xrightarrow{b^X_{\lambda,\lambda'} \times b^Y_{\mu,\mu'}} D_{\lambda'} \times D_{\mu'} \cong D_{\lambda',\mu'}$$

lies in $\partial D_{\lambda,\beta} \cong (\partial D_\lambda \times \partial D_{\beta}) \cup (\partial D_{\lambda} \times D_{\beta})$. And we obtain a map

$$b_{(\lambda,\mu),(\lambda',\mu')} : P_{(\lambda,\mu),(\lambda',\mu')} \times D_{\lambda,\mu} \to \partial D_{\lambda',\mu'}.$$ 

The composition operations

$$P_{(\lambda_1,\mu_1),(\lambda_2,\mu_2)} \times P_{(\lambda_0,\mu_0),(\lambda_1,\mu_1)} \to P_{(\lambda_0,\mu_0),(\lambda_2,\mu_2)}$$

are defined in an obvious way.

It is straightforward to verify that these maps define a cylindrical structure on $X \times Y$ under the product stratification and the product cell structures. 

Let us consider the CW conditions on products next. The closure finiteness condition is automatic.
Lemma 3.19. If $X$ and $Y$ are stratified spaces satisfying the closure finiteness condition, then so is $X \times Y$.

As is the case of CW complexes [Dow52], the product of two CW stratifications may not satisfy the weak topology condition. In the case of CW complexes, the local finiteness of $X$ implies that $X \times Y$ is CW for any CW complex $Y$. The author does not know if an analogous fact holds for CW stratified spaces in general. The following obvious fact is still useful in many cases.

Lemma 3.20. Let $X$ and $Y$ be CW stratified spaces. If both $X$ and $Y$ are locally finite, then $X \times Y$ is a CW stratified space.

Proof. The product of locally finite stratified spaces is again locally finite. The result follows from Proposition 2.7. 

Thus, by Theorem 3.18, the product $X \times Y$ of locally finite cylindrically normal cellular stratified spaces $X$ and $Y$ is a CW cylindrically normal cellular stratified space. When $X$ and $Y$ are locally polyhedral, it is easy to verify that $X \times Y$ inherits a locally polyhedral structure.

Corollary 3.21. Let $X$ and $Y$ be locally polyhedral cellular stratified spaces. Suppose both $X$ and $Y$ are locally finite. Then the product $X \times Y$ is a locally polyhedral cellular stratified space.

Thus we may freely take finite products of Euclidean locally polyhedral cellular stratified spaces.

Corollary 3.22. For any Euclidean locally polyhedral cellular stratified spaces $X$ and $Y$, the product $X \times Y$ is locally polyhedral.

Proof. By Lemma 2.9, $X$ and $Y$ are locally finite. 

In Proposition 3.18 and Definition 3.16 we made an implicit choice of a homeomorphism $D^{m+n} \cong D^m \times D^n$.

This procedure can be avoided by using cubes as domains of cell structures, in which case it is a reasonable idea to require the structure maps to be compatible with cubical structures. Thus we introduce the following variant of locally polyhedral cellular stratified spaces.

Definition 3.23. Let $X$ be a CW cellular stratified space. We consider $I^n = (\Delta^1)^n$ as a stratified space (regular cell complex) under the product stratification of the simplicial stratification on $\Delta^1$. A cubical structure on $X$ consists of

- a cylindrical structure $\{b_{\mu,\lambda} : P_{\mu,\lambda} \times D_{\mu} \to \partial D_{\lambda}\}$, $\{c_{\lambda_0,\lambda_1,\lambda_2} : P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \to P_{\lambda_0,\lambda_2}\}$,
- a stratified subspace $Q_{\lambda}$ of $I^{\dim e_{\lambda}}$ for each $\lambda \in P(X)$ under a suitable regular cellular subdivision of $I^{\dim e_{\lambda}}$, and
- a homeomorphism $\alpha_{\lambda} : Q_{\lambda} \to D_{\lambda}$ for each $\lambda \in P(X)$,

satisfying the following conditions:

1. Each $P_{\mu,\lambda}$ is a stratified subspace of $I^{\dim e_{\lambda} - \dim e_{\mu}}$.
2. The composition

$$b_{\mu,\lambda} : P_{\mu,\lambda} \times Q_{\mu} \xrightarrow{1_{P_{\mu,\lambda}} \times \alpha_{\mu}} P_{\mu,\lambda} \times D_{\mu} \xrightarrow{b_{\mu,\lambda}} \partial D_{\lambda} \xrightarrow{\alpha_{\lambda}^{-1}} Q_{\lambda}$$

is a strict morphism of stratified spaces.
3. The map $\tilde{b}_{\mu, \lambda}$ is an affine embedding onto its image when restricted to each face.

The family of maps $\{\alpha_\lambda : Q_\lambda \to D_\lambda\}_{\lambda \in P(X)}$ is also called a **cubical structure**.

A cylindrically normal CW cellular stratified space equipped with a cubical structure is called a **cubically normal cellular stratified space**.

**Example 3.24.** The minimal cell decomposition of $S^n$ is cubically normal. The radial expansion $\alpha_n : I^n \to D^n$ defines a cubical structure. The parameter space between the 0-cell and the n-cell is $\partial I^n$ and is a stratified subspace of $I^n$.

**Example 3.25.** Recall that any 1-dimensional cellular stratified space is totally normal, hence is cylindrically normal. They are cubically normal for an obvious reason, if they are CW.

**Example 3.26.** Recall that $\mathbb{R}P^2$ can be obtained by gluing the edges of $I^2$ as follows.

![Diagram](image)

This can be regarded as a description of a cell decomposition of $\mathbb{R}P^2$, consisting of two 0-cells $e_0^0, e_0^2$, two 1-cells $e_1^1, e_1^2$ and a 2-cell $e^2$. This is obviously cubically normal.

**Remark 3.27.** When the domains of cells are not globular, the situation is more complicated and we do not discuss the product structure here.

### 3.3 Subdivisions of Cells

We defined the notion of cellular stratified subspace in Definition 3.5. It often happens that we need to subdivide cells before we take a stratified subspace. For example, the product cellular stratification on $(S^n)^k$ is subdivided before the complement of the discriminant set $\Delta_k(S^n)$ is taken in the study of configuration space $\text{Conf}_k(S^n)$ in [BGRT].

We have already defined subdivisions of stratified spaces in Definition 2.5. We impose the following “regularity condition” on the definition of subdivisions of cell structures.

**Definition 3.28.** Let $(\pi, \Phi)$ be a cellular stratification on $X$. A **cellular subdivision** of $(\pi, \Phi)$ consists of

- a subdivision of stratified spaces

$$s = (1_X, \tilde{s}) : (X, \pi') \to (X, \pi)$$

and

- a regular cellular stratification $(\pi_\lambda, \Phi_\lambda)$ on the domain $D_\lambda$ of each cell $e_\lambda$ in $(\pi, \Phi)$ containing $\text{Int}(D_\lambda)$ as a strict stratified subspace,

satisfying the following conditions:

---

9Definition 2.10
1. For each $\lambda \in P(X, \pi)$, the cell structure
\[ \varphi_\lambda : (D_\lambda, \pi_\lambda) \rightarrow (X, \pi') \]
of $e_\lambda$ is a strict morphism of stratified spaces.

2. The maps
\[ P(\varphi_\lambda) : P(\text{Int}(D_\lambda)) \rightarrow P(X) \]
induced by the cell structures $\{ \varphi_\lambda \}$ give rise to a morphism
\[ \prod_{\lambda \in P(X, \pi)} P(\text{Int}(D_\lambda), \pi_\lambda) \rightarrow P(X, \pi') \]
of posets, which is a bijection.

**Remark 3.29.** The morphism $s$ induces a surjective map
\[ P(s) : P(X, \pi') \rightarrow P(X, \pi), \]
which gives rise to a decomposition
\[ P(X, \pi') = \prod_{\lambda \in P(X, \pi)} P(s)^{-1}(\lambda) \]
of sets. The conditions in the definition of cellular subdivision imply that each cell structure $\varphi_\lambda$ induces an isomorphism of posets
\[ P(\varphi_\lambda) : P(\text{Int}(D_\lambda), \pi_\lambda) \cong P(s)^{-1}(\lambda), \]
under which we have an identification
\[ \prod_{\lambda \in P(X, \pi)} P(\text{Int}(D_\lambda), \pi_\lambda) \cong P(X, \pi'), \]
where the poset structure on the right hand side is given by $\mu' \leq \lambda'$ for $\mu' \in P(\text{Int}(D_\mu), \pi_\mu)$ and $\lambda' \in P(\text{Int}(D_\lambda), \pi_\lambda)$ if and only if $\varphi_\mu(e_{\mu'}) \subset \varphi_\lambda(e_{\lambda'})$. See the proof of Proposition 3.38.

**Remark 3.30.** Cellular subdivisions of stellar stratified spaces can be defined in a similar way. We may also define stellar subdivisions of cellular or stellar stratified spaces. We do not pursue such generalizations in this paper.

It is easy to verify that the category of cellular stratified spaces is closed under cellular subdivisions, when all cell structures are hereditarily quotient.

**Lemma 3.31.** Let $(X, \pi, \Phi)$ be a cellular stratified space whose cell structures are hereditarily quotient. Then any cellular subdivision of $(X, \pi, \Phi)$ defines a structure of cellular stratified space on $X$, under which, for $\lambda \in P(X, \pi)$ and $\lambda' \in P(\text{Int}(D_\lambda), \pi_\lambda)$, the composition
\[ D_{\lambda'} \xrightarrow{s_{\lambda'}} D_\lambda \xrightarrow{\varphi_\lambda} X \]
is the cell structure of the cell $e_{\lambda'}$ in $P(X, \pi')$, where $s_{\lambda'} : D_{\lambda'} \rightarrow D_\lambda$ is the cell structure of the cell in $P(D_\lambda, \pi_\lambda)$ indexed by $\lambda'$. 

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Proof. By assumption, each cell structure \( \varphi_\lambda \) is hereditarily normal and hence the composition \( \varphi_\lambda \circ s_\lambda : D_\lambda \to \mathbb{P}(X) \) is a quotient map. Each new stratum is connected, since \( e_\lambda' \) is connected. It is also locally closed, since \( \varphi_\lambda|_{\text{Int}(D_\lambda)} \) is a homeomorphism onto its image. Other conditions can be verified immediately. \( \square \)

**Remark 3.32.** The reader might want to define a subdivision of a cellular stratified space \( (X, \pi, \Phi) \) as a morphism

\[
s = (1_X, \{ s_\lambda \lambda' \lambda' \in P(X, \pi') \} : (X, \pi', \Phi') \to (X, \pi, \Phi)
\]

of cellular stratified spaces satisfying the condition that, for each cell \( e_\lambda \) in \( (\pi, \Phi) \), the stratification of the interior of the domain \( D_\lambda \) is indexed by \( s^{-1}(\lambda) \)

\[
\text{Int}(D_\lambda) = \bigcup_{\lambda' \in s^{-1}(\lambda)} s_\lambda'|\text{Int}(D_\lambda')
\]

However, we also need to specify the behavior of each cell structure on the boundary \( \partial D_\lambda \).

**Example 3.33.** Let \( A = \{ H_1, \ldots, H_k \} \) be a hyperplane arrangement in \( \mathbb{R}^n \) defined by affine 1-forms \( L = \{ \ell_1, \ldots, \ell_k \} \). The stratification \( \pi_{A\otimes \mathbb{R}^\ell} \) on \( \mathbb{R}^n \otimes \mathbb{R}^\ell \) in Example 1.2 is one of the coarsest cellular stratifications containing \( \bigcup_{i=1}^k H_k \otimes \mathbb{R}^\ell \) as a stratified subspace.

This efficiency is achieved by sacrificing symmetry. The stratification \( \pi_{A\otimes \mathbb{R}^\ell} \) is not compatible with the action of the symmetric group \( \Sigma_\ell \). As is stated by Björner and Ziegler in [BZ92], we may subdivide \( \pi_{A\otimes \mathbb{R}^\ell} \) by using the product sign vector \( S^1_\ell \). Define

\[
\pi_{A\otimes \mathbb{R}^\ell}^* : \mathbb{R}^n \otimes \mathbb{R}^\ell \to \text{Map}(L, S^1_\ell)
\]

by

\[
\pi_{A\otimes \mathbb{R}^\ell}^*(x_1, \ldots, x_\ell)(\ell_i) = (\text{sign}(\ell_i(x_1)), \ldots, \text{sign}(\ell_i(x_\ell))).
\]

Define a map \( c : S^1_\ell \to S_\ell \) of posets by

\[
c(\varepsilon_1, \ldots, \varepsilon_\ell) = \varepsilon_i e_i,
\]

where \( i = \max \{ j \mid \varepsilon_j \neq 0 \} \). This is surjective and the diagram

\[
\begin{array}{ccc}
\mathbb{R}^n \otimes \mathbb{R}^\ell & \text{Map}(L, S^1_\ell) \\
\pi_{A\otimes \mathbb{R}^\ell}^* \downarrow \quad \downarrow \pi_{A\otimes \mathbb{R}^\ell} \quad \downarrow \text{Map}(L, S_\ell) \\
\text{Map}(L, S^1_\ell) & \text{Map}(L, S_\ell)
\end{array}
\]

is commutative. Thus \( \pi_{A\otimes \mathbb{R}^\ell}^* \) is a subdivision of the stratification \( \pi_{A\otimes \mathbb{R}^\ell} \). The regularity of the cellular stratification \( \pi_{A\otimes \mathbb{R}^\ell} \) implies that this is a cellular subdivision. Note that \( (\mathbb{R}^n \otimes \mathbb{R}^\ell, \pi_{A\otimes \mathbb{R}^\ell}^*) \) is a \( \Sigma_\ell \)-cellular stratified space. \( \square \)

**Example 3.34.** Let \( \pi_{\min} \) be the minimal cell decomposition of \( S^2 \). By dividing the 2-cell by the equator, we obtain a subdivision \( \pi \) of \( \pi_{\min} \) as a stratified space. By dividing the domain of the cell structure of the 2-cell in \( \pi_{\min} \), this subdivision can be made into a subdivision of cellular stratified spaces, as is depicted in the following figure.
There is another way to make the stratification $\pi$ into a cellular stratified space. Choose a small disk $D$ in $D^2$ touching $\partial D^2$ at a point $p$.

The small disk $D$ is mapped homeomorphically onto the lower hemisphere via $\varphi'$. There is a map

$$\psi : D^2 \rightarrow D^2 - \text{Int}(D)$$

such that the composition $\varphi \circ \psi$ is a cell structure for the upper hemisphere. For example, such a map can be defined by identifying two points in the following figure to $p$.

The morphism of stratified spaces $\pi \rightarrow \pi_{\text{min}}$ also becomes a morphism of cellular stratified spaces with this cellular stratification on $\pi$, but it is not a subdivision of cellular stratified spaces.

We often define a cellular subdivision of a cellular stratified space $(X, \pi, \Phi)$ by defining subdivisions of domains of cell structures. The problem is of course the compatibility.

**Lemma 3.35.** Let $(X, \pi, \Phi)$ be a cellular stratified space whose cell structures are hereditarily quotient. Suppose that a regular cellular stratification $\pi_\lambda$ on each domain $D_\lambda$ is given. Define a stratification $\pi'$ on $X$ by

$$X = \bigcup_{\lambda \in P(X, \pi)} \bigcup_{\lambda' \in P(\text{Int}(D_\lambda))} \varphi_\lambda(e_{\lambda'}).$$

Suppose further that the following conditions are satisfied:

1. For each $\lambda \in P(X)$, $\text{Int}(D_\lambda)$ is a strict stratified subspace of $D_\lambda$.
2. For $\lambda' \in P(D_\lambda)$ and $\mu' \in P(D_\mu)$,

$$\varphi_\lambda \circ s_{\lambda'}(\text{Int}(D_{\lambda'})) \cap \varphi_\mu \circ s_{\mu'}(\text{Int}(D_{\mu'})) \neq \emptyset$$
implies
\[ \varphi_\lambda \circ s_\lambda' \left( \text{Int}(D_{\lambda'}) \right) = \varphi_\mu \circ s_\mu' \left( \text{Int}(D_{\mu'}) \right). \]

For \( \lambda' \in P(\text{Int}(D_\lambda)) \), define a map \( \psi_{\lambda'} : D_{\lambda'} \to X \) by the composition
\[ D_{\lambda'} \xrightarrow{s_{\lambda'}} D_\lambda \xrightarrow{\varphi_\lambda} X, \]
where \( s_{\lambda'} \) is the cell structure of \( e_{\lambda'} \). Then these structures define a cellular stratification on \( X \) with \( \{ \psi_{\lambda'} \} \) cell structure maps.

Proof. It suffices to verify that \( \varphi_\lambda \) is a strict morphism of cellular stratified spaces from \( (D_\lambda, \pi_\lambda) \) to \( (X, \pi') \).

By assumption,
\[ \varphi_\lambda \circ s_\lambda' \left( \text{Int}(D_{\lambda'}) \right) = \varphi_\mu \circ s_\mu' \left( \text{Int}(D_{\mu'}) \right) \]
and thus \( \varphi_\lambda : (D_\lambda, \pi_\lambda) \to (X, \pi') \) is a strict morphism of stratified spaces.

Definition 3.36. We say that the above cellular stratification is induced by the family \( \{ \pi_\lambda \}_{\lambda \in P(X)} \) of cellular stratifications on the domains of cells.

Example 3.37. Consider the following cellular stratification of an open annulus \( A = e^1 \cup e^2 \).

We use \( \text{Int}(I) \) and \( \text{Int}(I) \times I \) as the domains of the cell structures for the 1-cell and the 2-cell. The subdivisions of \( \text{Int}(I) \) in the middle and of \( \text{Int}(I) \times I \) by horizontal cut in the middle induce a subdivision of this stratification.

The following “tilted subdivision” of \( \text{Int}(I) \times I \), however, does not induce a subdivision of the annulus, since it does not satisfy the second condition of cellular subdivision.

By subdividing the boundary further, we obtain an induced subdivision, as is shown in the right figure.
3.4 Subdivisions of Cylindrical and Locally Polyhedral Structures

Example 3.37 says that, in order to obtain an induced subdivision of a cellular stratified space from subdivisions of domains of cells, the compatibility with lifts of cell structures is the key. In particular, if a cellular stratified space is equipped with a cylindrical structure, we may describe a condition for the existence of a cellular subdivision in terms of the cylindrical structure.

**Proposition 3.38.** Let \((X, \pi)\) be a cylindrically normal cellular stratified space whose cell structures are hereditarily quotient. Suppose that a regular cellular stratification \(\pi_\lambda\) on \(D_\lambda\) is given for each \(\lambda \in P(X)\) and that, for each pair \(\mu < \lambda\) in \(P(X)\), the structure map

\[ b_{\mu, \lambda} : P_{\mu, \lambda} \times D_\mu \longrightarrow D_\lambda \]

is a strict morphism of stratified spaces. Then cells in \(\text{Int}(D_\lambda)\) for \(\lambda \in P(X)\) give rise to a cellular subdivision of \(X\). Furthermore this is cylindrically normal.

**Proof.** Let us verify the condition of Lemma 3.35. We proceed by induction on the dimension of cells in \((X, \pi)\). When \(\text{dim} D_\lambda = 0\), the subdivision \(\pi_\lambda\) is trivial and there is nothing to prove. Suppose we have proved that the condition of Lemma 3.35 is satisfied for all cells in \(D_\mu\) with \(\text{dim} D_\mu < n\). Consider an \(n\)-cell \(\varphi_\lambda : D_\lambda \to X\) in \((X, \pi)\).

Each cell in \(\text{Int}(D_\lambda)\) is mapped homeomorphically onto its image by \(\varphi_\lambda\). Let \(e_\lambda \subset \partial D_\lambda\) be a cell. Since \(\partial D_\lambda\) is covered by the images of \(b_{\mu, \lambda}\) for \(\mu < \lambda\) and \(b_{\mu, \lambda}\) is a strict morphism of stratified spaces, there exist \(\mu \in P(X, \pi)\) and a stratum \(e \times e' \subset P_{\mu, \lambda} \times D_\mu\) with \(e_\lambda' = b_{\mu, \lambda}(e \times e')\).

The commutativity of the diagram

![Diagram](image)

implies that \(\varphi_\lambda(e_\lambda') = \varphi_\lambda(e')\). By the inductive assumption, \(\varphi_\mu(e')\) does not intersect with other cells nontrivially. Thus cells in \(D_\lambda\) also satisfy the condition of Lemma 3.35.

A cylindrical structure is defined as follows. For \((\mu, \mu') < (\lambda, \lambda')\) in \(P(X, \pi')\), if \(\mu = \lambda\), define \(P_{(\mu, \mu'), (\lambda, \lambda')}\) to be a single point, since the cellular stratification on \(D_\lambda\) is assumed to be regular. When \(\mu < \lambda\), define

\[ P_{(\mu, \mu'), (\lambda, \lambda')} = \{ x \in P_{\mu, \lambda} \mid b_{\mu, \lambda}(\{x\} \times s_{\mu'}(D_{\mu'})) \subset s_{\lambda'}(D_{\lambda'}) \} \]

Since \(b_{\mu, \lambda}\) is a strict morphism, \(P_{(\mu, \mu'), (\lambda, \lambda')}\) is a stratified subspace of \(P_{\mu, \lambda}\). The composition

\[ P_{(\mu, \mu'), (\lambda, \lambda')} \times D_{\mu'} \xrightarrow{\times s_{\mu'}} P_{\mu, \lambda} \times D_\mu \xrightarrow{b_{\mu, \lambda}} D_\lambda \]

has a unique lift

\[ b_{(\mu, \mu'), (\lambda, \lambda')} : P_{(\mu, \mu'), (\lambda, \lambda')} \times D_{\mu'} \longrightarrow D_{\lambda'} \]

because of the regularity of the cellular stratification on \(D_\lambda\). These maps define a cylindrical structure. \(\square\)
Example 3.39. Consider the minimal cell decomposition $\pi_{\text{min}}$ on $S^2$. As we have seen in [Tamb], $P_{0,2} = S^1$ and the canonical inclusion

$$b_{0,2} : S^1 \times D^0 \to \partial D^2 \subset D^2$$

defines a cylindrical structure on $(S^2, \pi_{\text{min}})$.

Divide the parameter space $P_{0,2} = S^1$ into three parts

$$P_{0,2}^+ = \{(x,y) \in S^1 \mid y > 0\}$$
$$P_{0,2}^0 = \{(x,y) \in S^1 \mid y = 0\}$$
$$P_{0,2}^- = \{(x,y) \in S^1 \mid y < 0\}.$$ 

Then $b_{0,2}$ becomes a strict morphism under the subdivision of $\pi_{\text{min}}$ in Example 3.34. And the subdivision is cylindrically normal.

Let us consider subdivisions of locally polyhedral structures. The following requirement should be reasonable.

Definition 3.40. Let $(X, \pi, \Phi)$ be a locally polyhedral cellular stratified space with polyhedral replacements $\alpha_\lambda : \tilde{F}_\lambda \to D_\lambda$. A locally polyhedral subdivision of $(X, \pi, \Phi)$ consists of

- a cellular subdivision $s = (\lambda, \{s_\lambda\}) : (\pi', \Phi') \to (\pi, \Phi)$ of $(\pi, \Phi)$,
- for each $\lambda \in P(X, \pi)$, a regular cell decomposition of $D^{\dim e_\lambda}$ and a subdivision of $\tilde{F}_\lambda$ via convex polytopes, and
- a subdivision of each parameter space $P_{\mu, \lambda}$ as a locally cone-like space,

satisfying the following conditions:

1. The subdivision of $D_\lambda$ under $s$ is a stratified subspace of $D^{\dim e_\lambda}$ for each $\lambda \in P(X, \pi)$.
2. The polyhedral replacement $\alpha_\lambda : \tilde{F}_\lambda \to D^{\dim e_\lambda}$ is an isomorphism of stratified spaces for each $\lambda \in P(X, \pi)$.
3. The structure map $b_{\mu, \lambda} : P_{\mu, \lambda} \times D_\mu \to D_\lambda$ is a strict morphism of stratified spaces.

Remark 3.41. Recall from Corollary 2.34 that cell structures in a locally polyhedral cellular stratified space are bi-quotient. Hence the “hereditarily quotient” condition in the definition of cellular subdivision is always satisfied.

For such a subdivision, we have a subcomplex $\tilde{F}_{\lambda'}$ of $\tilde{F}_{\lambda'}(X)$ corresponding to $D_{\lambda'}$ for each $\lambda' \in P(X, \pi')$. And the diagram

$$\begin{array}{cccccc}
F_{\lambda'} & \rightarrow & D_{\lambda'} & \rightarrow & \tilde{F}_{\lambda'}(X) & \rightarrow \\
\downarrow \quad & \quad & \downarrow \quad & \quad & \downarrow \quad & \\
\tilde{F}_{\lambda'} & \rightarrow & D_{\lambda'} & \rightarrow & \tilde{F}_{\lambda'}(X) & \\
\end{array}$$

is commutative, where $F_{\lambda'} = \alpha_{\lambda'}^{-1}(D_{\lambda'})$ and $F_{\lambda'} = \alpha_{\lambda'}^{-1}(s_{\lambda'}(D_{\lambda'}))$. Thus we obtain the following fact.
Lemma 3.42. Let \((X, \pi, \Phi)\) be a locally polyhedral cellular stratified space and \((\pi', \Phi')\) be a locally polyhedral subdivision of \((\pi, \Phi)\). Then there exists a locally polyhedral structure on \((X, \pi', \Phi')\) under which the subdivision \((X, \pi', \Phi') \to (X, \pi, \Phi)\) is a morphism of locally polyhedral cellular stratified spaces.

Proof. It remains to verify the CW condition, which is a consequence of Corollary 2.8. \qed

Remark 3.43. Recall that any CW totally normal cellular stratified space \(X\) can be made into a locally polyhedral cellular stratified space by Lemma 2.31. Subdivisions of \(X\) satisfying the conditions in Proposition 3.38 are locally polyhedral subdivisions of \(X\).

Proposition 3.38 provides us with a useful way to construct subdivisions that are compatible with cylindrical structures. For example, the stratification on the configuration space of spheres used in [BGRT] is obtained by taking subdivisions of the domain of the cell structure map for each cell in \((S^n)^k\) by the stratification \(\pi_{A_k^{-1}} \otimes \mathbb{R}^{n}\) associated with the braid arrangement \(A_k^{-1} \otimes \mathbb{R}^n\). The subdivisions are obtained by replacing the domain of the top cell by the cube of the same dimension \((I^n)^k \simeq (D^n)^k \to (S^n)^k\) and then by taking the subdivision of \((I^n)^k\) by the braid arrangement.

For a cubically normal\(^\text{10}\) cellular stratified space \(X\), we may define subdivisions of the domains of cells in the product stratification \(X^k\) by the stratification \(\pi_{A_k^{-1}} \otimes \mathbb{R}^n\). But the resulting subdivisions of domains might not be compatible with the cylindrical structure of \(X^k\). The main reason is that the automorphism group of the unit cube does not preserve the braid arrangement.

Definition 3.44. For \(\varepsilon \in \{\pm 1\}\) and \(1 \leq i < j \leq k\), define a hyperplane \(H^{\varepsilon}_{i,j}\) by

\[
H^{\varepsilon}_{i,j} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_i = \varepsilon x_j\}
\]

and define a hyperplane arrangement \(A^{\pm}_{k-1}\) in \(\mathbb{R}^k\) by

\[
A^{\pm}_{k-1} = \{H^{\varepsilon}_{i,j} \mid 1 \leq i < j \leq k, \varepsilon \in \{\pm 1\}\}.
\]

By using the symmetrized stratification\(^\text{11}\) \(\pi_{A^{\pm}_{k-1}} \otimes \mathbb{R}^n\) associated with this arrangement, we obtain a subdivision of \(X^k\).

Definition 3.45. Let \(X\) be a cubically normal finite cellular stratified space with cubical structure given by \(\{\alpha_{\lambda} : Q_{\lambda} \to D_{\lambda}\}\). Define a subdivision of \(X^k\) as follows.

Choose a linear extension of \(P(X)\). For each cell \(e_{\lambda_1} \times \cdots \times e_{\lambda_k}\) in the product stratification \(X^k\), choose a permutation \(\sigma \in \Sigma_k\) with

\[
(e_{\lambda_1} \times \cdots \times e_{\lambda_k})\sigma = (e_{\mu_1})^{m_1} \times \cdots \times (e_{\mu_{\ell}})^{m_{\ell}},
\]

where \(e_{\mu_1}, \ldots, e_{\mu_{\ell}}\) are distinct cells with \(\mu_1 < \cdots < \mu_{\ell}\).

The symmetrized stratification \(\pi_{A^{\pm}_{m_{\ell}-1}} \otimes \mathbb{R}^{\dim e_{\mu}}\) associated with \(A^{\pm}_{m_{\ell}-1}\) is a cellular stratification on \(\mathbb{R}^{m_1} \otimes \mathbb{R}^{\dim e_{\mu_1}}\). The inclusion

\[
(Q_{\mu_{\ell}})^{m_{\ell}} \subset (I^{\dim e_{\mu_{\ell}}})^{m_{\ell}} \subset \mathbb{R}^{m_{\ell}} \otimes \mathbb{R}^{\dim e_{\mu_1}}
\]

10Definition 3.23
11Example 3.33

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induces a cellular stratification on \((Q_\mu)^{m_1}\) and thus on the product \(Q_\mu^{m_1} \times \cdots \times Q_\mu^{m_k}\).

Define a stratification on \(e_{\lambda_1} \times \cdots \times e_{\lambda_k}\) by the composition

\[(\text{Int} Q_\mu)^{m_1} \times \cdots \times (\text{Int} Q_\mu)^{m_k} \xrightarrow{\alpha^{m_1}_1 \times \cdots \times \alpha^{m_k}_k} (e_\mu)^{m_1} \times \cdots \times (e_\mu)^{m_k} \xrightarrow{\sigma^{-1}} e_{\lambda_1} \times \cdots \times e_{\lambda_k},\]

which is a homeomorphism.

**Proposition 3.46.** For a cubically normal finite cellular stratified space \(X\), the above construction defines a locally polyhedral cellular stratification on \(X^k\) containing the discriminant set or the fat diagonal in \(X^k\)

\[\Delta_k(X) = \bigcup_{1 \leq i < j \leq k} \{(x_1, \ldots, x_k) \in X^k \mid x_i = x_j\}\]

as a stratified subspace. Thus it also contains \(\text{Conf}_k(X)\) as a cellular stratified subspace.

**Proof.** Let us verify the conditions in Proposition 3.38.

Let \(\{\alpha_\lambda : Q_\lambda \to D_\lambda\}\) be the cubical structure on \(X\). In the rest of the proof, we identify \(D_\lambda\) with \(Q_\lambda\) under \(\alpha_\lambda\). In particular, the structure map \(b_{\mu,\lambda}\) of the cylindrical structure on \(X\) is regarded as a map

\[b_{\mu,\lambda} : P_{\mu,\lambda} \times Q_\mu \to Q_\lambda,\]

which is an affine strict morphism of stratified spaces.

We need to show that for each sequence of pairs \(\mu_1 < \lambda_1, \ldots, \mu_k < \lambda_k\), there exists a cellular stratification on \(P_{\mu_1,\lambda_1} \times \cdots \times P_{\mu_k,\lambda_k}\) such that the composition

\[b_{\mu_1,\lambda_1} \times \cdots \times b_{\mu_k,\lambda_k} \quad \text{(equiv)} \quad (P_{\mu_1,\lambda_1} \times \cdots \times P_{\mu_k,\lambda_k}) \times (Q_{\mu_1} \times \cdots \times Q_{\mu_k}) \to (P_{\mu_1,\lambda_1} \times Q_{\mu_1}) \times \cdots \times (P_{\mu_k,\lambda_k} \times Q_{\mu_k}) \to Q_{\lambda_1} \times \cdots \times Q_{\lambda_k}\]

is a strict morphism of cellular stratified spaces, when \(Q_{\mu_1} \times \cdots \times Q_{\mu_k}\) and \(Q_{\lambda_1} \times \cdots \times Q_{\lambda_k}\) are equipped with the stratification defined in Definition 3.47. By the definition of the stratification, it suffices to show that there exists a cellular stratification on \(P_{\mu,\lambda}^{m}\) under which the composition

\[P_{\mu,\lambda}^{m} \times Q_\mu^{m} \xrightarrow{\approx} (P_{\mu,\lambda}^{m} \times Q_\mu^{m})^{b_{\mu,\lambda}} \xrightarrow{b_{\mu,\lambda}} Q_\lambda^{m}\]

is a strict morphism of cellular stratified spaces. By assumption \(P_{\mu,\lambda}\) is a stratified subspace of a cube \(I^N\). The stratification induced by the stratification on \((I^N)^m\) associated with the arrangement \(\pi_{A_{\lambda_1}}^{m} \otimes \cdots \otimes R^m\) does the job.

**Definition 3.47.** The stratification on \(X^k\) defined above is called the **symmetric braid stratification** and is denoted by \(\pi_{k}^{\br}\). The induced stratification on \(\text{Conf}_k(X)\) is also called the **symmetric braid stratification** and is denoted by \(\pi_{C_k}^{\br}\).

When coarser stratifications \(\pi_{A_{\lambda_1}}^{m} \otimes R^m, \pi_{A_{\lambda_1}}^{m} \otimes R^m, \pi_{A_{\lambda_1}}^{m} \otimes R^m\), or \(\pi_{A_{\lambda_1}}^{m} \otimes R^m\) define a cellular subdivision of the product stratification on \(X^k\), the resulting stratifications are denoted by \(\pi_{k}^{\br}, \pi_{k}^{\br}, \pi_{k}^{\br}, \) or \(\pi_{k}^{\br}\), respectively. The induced stratifications on \(\text{Conf}_k(X)\) are denoted by \(\pi_{C_k}^{\br}, \pi_{C_k}^{\br}, \pi_{C_k}^{\br}, \) or \(\pi_{C_k}^{\br}\), respectively. The coarsest stratification \(\pi_{k}^{\br}\) and its restriction \(\pi_{C_k}^{\br}\) are called the **braid stratifications.**
Example 3.48. The minimal cell decomposition of $S^n$ is cubically normal. Thus we have the symmetric braid stratification on $\text{Conf}_k(S^n)$. In this case, the braid stratification can be also defined.

Example 3.49. It is easy to see that any 1-dimensional finite cell complex $X$ is totally normal. It is also cubically normal, hence so is the product cellular stratification on $X^k$. Thus we obtain the symmetric braid stratification on $\text{Conf}_k(X)$, which coincides with the braid stratification.

Example 3.50. Consider the cubically normal cell decomposition of $\mathbb{R}P^2$ in Example 3.26. The symmetric braid stratification $\pi_{\pm}^{\text{br},s}$ defines a cellular stratification on $(\mathbb{R}P^2)^k$ by Proposition 3.46. However, we may use a coarser stratification $\pi_{\pm}^{\text{br},c}$, although the coarsest one $\pi_{\pm}^{\text{br},c}$ does not define a cellular stratification on $(\mathbb{R}P^2)^k$.

We shall pursue this idea of studying configuration spaces by defining a good subdivision of the product stratification in a separate paper.

4 Duality

In [Sal87], Salvetti first constructed a simplicial complex $Sd(M(\mathcal{A} \otimes \mathbb{C})) = BC(M(\mathcal{A} \otimes \mathbb{C}))$ modelling the homotopy type of the complement of the complexification of a real hyperplane arrangement $\mathcal{A}$ and then defined a structure of regular cell complex by gluing simplices. This process reduces the number of cells and allows us to relate the combinatorics of the arrangement $\mathcal{A}$ and the topology of Salvetti’s model $Sd(M(\mathcal{A} \otimes \mathbb{C}))$. The process is closely related to the classical concept of duality in PL topology.

In this section, we introduce an analogous process for locally polyhedral stellar stratified spaces. With stellar stratifications, we are able to extend both Salvetti’s construction and the duality in PL topology.

4.1 A Canonical Cellular Stratification on the Barycentric Subdivision

Let us first study stratifications on the barycentric subdivision $Sd(X)$ of a cylindrically normal cellular stratified space $X$.

By Corollary 2.49, we know that $Sd(X) = BC(X)$ has a structure of totally normal cell complex, when $X$ is a totally normal stellar stratified space. Let us extend this structure to a cellular stratification on $Sd(X)$ for a cylindrically normal stellar stratified space $X$.

Given a cylindrically normal stellar stratified space $X$, we have an acyclic topological category $C(X)$. By forgetting the topology, Lemma 2.44 and Lemma 2.46 give us a map

$$\pi_{Sd(X)} : Sd(X) \xrightarrow{\text{N}(C(X))} \bigoplus_k \text{N}_k(C(X)).$$

As we will see in Lemma 4.24, the set $\bigoplus_k \text{N}_k(C(X))$ can be identified with the barycentric subdivision of the acyclic category $C(X)$ and has a structure of poset. Let us verify that the map $\pi_{Sd(X)}$ defines a cellular stratification on $Sd(X)$.

Proposition 4.1. For a cylindrically normal stellar stratified space $X$, $\pi_{Sd(X)}$ defines a cellular stratification.

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12Definition 4.23
Proof. Let \( X \) be a cylindrically normal stellar stratified space. For each nondegenerate \( k \)-chain \( b \in N_k(C(X)) \), \( \pi_{Sd(X)}^{-1}(b) \) is homeomorphic to \( \text{Int}(\Delta^k) \). The closure of \( \pi_{Sd(X)}^{-1}(b) \) in \( Sd(X) \) is an identification space of \( \Delta^n \). Since the identification is defined only on the boundary, \( \pi_{Sd(X)}^{-1}(b) \) is open in its closure and is locally closed. For nondegenerate chains \( b \) and \( b' \), \( \pi_{Sd(X)}^{-1}(b) \subset \pi_{Sd(X)}^{-1}(b') \) if and only if \( \pi_{Sd(X)}^{-1}(b) \) is included in \( \pi_{Sd(X)}^{-1}(b') \) as a face. In other words, this holds if and only if there exists a sequence of face operators mapping \( b' \) to \( b \), which is equivalent to saying \( b \leq b' \). Thus \( \pi_{Sd(X)} \) is a stratification in the sense of Definition 2.2.

Under the standard homeomorphism \( \Delta^k \cong D^k \), we obtain a continuous map \( \varphi_b : D^k \cong \{b\} \times \Delta^k \hookrightarrow \coprod_k N_k(C(X)) \times \Delta^k \longrightarrow \|N(C(X))\| \cong Sd(X) \).

By the compactness of \( D^k \), \( \varphi_b \) is a quotient map onto its image, which is the closure of \( \pi_{Sd(X)}^{-1}(b) \). The boundary of \( \pi_{Sd(X)}^{-1}(b) \) consists of points in \( Sd(X) \) of the form \([b,t] \) with \( t \in \partial \Delta^k \). Under the defining relation, such a point is equivalent to a point in \( N_\ell(C(X)) \times \Delta^\ell \) for \( \ell < k \). Thus we obtain a cellular stratification.

Note that this cellular stratification is rarely a CW stratification, when parameter spaces have non-discrete topology.

Example 4.2. Consider the minimal cell decomposition \( \pi_n : S^n = e^0 \cup e^n \) of the \( n \)-sphere. We have

\[
\begin{align*}
N_0(C(S^n)) &= C(S^n)_0 = \{e^0, e^n\} \\
N_1(C(S^n)) &= C(S^n)(e^0, e^n) = S^{n-1} \\
\bigcap_{k \geq 2} N_k(C(S^n)) &= \emptyset.
\end{align*}
\]

Thus \( Sd(S^n; \pi_n) \) has only 0-cells and 1-cells. There two 0-cells \( v(e_0) \) and \( v(e^n) \) corresponding to \( e^0 \) and \( e^n \). There are infinitely many 1-cells parametrized by the equator \( S^{n-1} \). For \( x \in S^{n-1} \), the 1-cell \( e^1_x \) corresponding to \( x \) is given by the great half circle from the south pole to the north pole through \( x \) under the identification \( Sd(S^n; \pi_n) \cong S^n \).

4.2 Canonical Stellar Stratifications on the Barycentric Subdivision

As Example 4.2 shows, the cellular stratification on \( Sd(X) \) defined in the previous section is not very useful. However, Example 4.2 also suggests that by gluing cells in \( Sd(X) \) together, we may construct a good cellular stratification. In the classical PL topology, such a construction is called star.
Definition 4.3. Let $X$ be a stellar stratified space. For $x \in X$, define

$$\text{St}(x; X) = \bigcup_{x \in e} e_\lambda.$$ 

For a subset $A \subset X$, define

$$\text{St}(A; X) = \bigcup_{x \in A} \text{St}(x; X).$$

The stratified subspace $\text{St}(A; X)$ is called the star of $A$ in $X$.

Example 4.4. In the cellular stratification on $\text{Sd}(S^n; \pi_n)$ in Example 4.2, we recover the original cellular stratification on $S^n$ by

$$\text{Sd}(S^n) = v(e^0) \cup \text{St}(v(e^n); \text{Sd}(S^n; \pi_n)).$$

Note that we also have

$$\text{Sd}(S^n) = v(e^n) \cup \text{St}(v(e^0); \text{Sd}(S^n; \pi_n)).$$

The cellular stratifications in the above example can be described in terms of lower and upper stars defined in Definition 2.55.

Definition 4.5. Let $X$ be a cylindrically normal stellar stratified space. Define a map

$$\pi_{X^\text{op}} : \text{Sd}(X) \rightarrow P(X)$$

by the composition

$$\text{Sd}(X) \xrightarrow{\pi_{\text{Sd}(X)}} \coprod_k \overline{\mathcal{N}}_k(\mathcal{C}(X)) \overset{s}{\rightarrow} \mathcal{C}(X)_0 = P(X),$$

where $s$ is the map induced by the source map in the category $\mathcal{C}(X)$ and is also called the source map.

By definition, $\pi_{\text{Sd}(X)}$ is a subdivision of $\pi_{X^\text{op}}$ if $\pi_{X^\text{op}}$ defines a stratification on $\text{Sd}(X)$.

Each inverse image $\pi_{X^\text{op}}^{-1}(\lambda)$ has the following description.

Definition 4.6. For a cell $e_\lambda$ in a cylindrically normal stellar stratified space $X$, define

$$D_{\lambda}^{(X)} = \|\text{St}_{\geq e_\lambda}(\mathcal{C}(X))\|.$$

We also denote

$$D_{\lambda}^{(X), \circ} = \|\text{St}_{\geq e_\lambda}(\mathcal{C}(X))\| - \|\text{Lk}_{> e_\lambda}(\mathcal{C}(X))\|,$$

where $\|\text{Lk}_{> e_\lambda}(\mathcal{C}(X))\|$ is regarded as the bottom subspace of the cone under the identification

$$\|\text{St}_{\geq e_\lambda}(\mathcal{C}(X))\| \cong \{1_{e_\lambda}\} \ast \|\text{Lk}_{> e_\lambda}(\mathcal{C}(X))\|$$

in Lemma 2.60.

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13Definition 2.55

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Lemma 4.7. Let $X$ be a cylindrically normal stellar stratified space. In the stratification $\pi_{X^{op}}$, each stratum is given by the image of $D^\Lambda(X)^{op}$ under the map $t_\lambda = t_{e_\lambda} : \| \text{St}_{\geq e_\lambda}(C(X)) \| \to \text{Sd}(X)$ defined in Definition 4.5. Namely

$$
\pi_{X^{op}}^{-1}(\lambda) = t_\lambda \left( D^\Lambda(X)^{op} \right).
$$

Hence the closure of each stratum is given by

$$
\overline{\pi_{X^{op}}^{-1}(\lambda)} = t_\lambda \left( D^\Lambda(X) \right).
$$

Proof. Elements in $D^\Lambda(X)^{op}$ are those which are represented by $(u, t_00 + (1 - t_0)t')$, where $t_0 < 1$ and $u$ begins with the identity morphism $1_{e_\lambda}$. Therefore $s(t_\lambda(u)) = e_\lambda$ and $t_\lambda \left( D^\Lambda(X)^{op} \right) \subset \pi_{X^{op}}^{-1}(\lambda)$. Conversely, by choosing a representative $(u, t)$ of $[(u, t)] \in \pi_{X^{op}}^{-1}(\lambda)$ with $t \in \text{Int}(\Delta^k)$, we see $\pi_{X^{op}}^{-1}(\lambda) \subset t_\lambda \left( D^\Lambda(X)^{op} \right)$.

Let $p : \bigsqcup_k \mathcal{N}_k(C(X)) \times \Delta^k \to \text{Sd}(X)$ be the projection. Then the topology on $\text{Sd}(X)$ is the weak topology defined by the covering $\{ p(C(X)(e_{\lambda_k_0}, e_{\lambda_1}) \times \cdots \times C(X)(e_{\lambda_k-1}, e_{\lambda_k}) \times \Delta^k) \}$. Thus the closure of

$$
t_\lambda \left( D^\Lambda(X)^{op} \right) = p \left( \prod_k \prod_{\lambda \prec \lambda_k \subset \cdots \lambda_k} C(X)(e_{\lambda}, e_{\lambda_1}) \times \cdots \times C(X)(e_{\lambda_k-1}, e_{\lambda_k}) \times \Delta^k - d^0(\Delta^{k-1}) \right)
$$

is given by adding $C(X)(e_{\lambda}, e_{\lambda_1}) \times \cdots \times C(X)(e_{\lambda_k-1}, e_{\lambda_k}) \times d^0(\Delta^{k-1})$. And we have

$$
\overline{\pi_{X^{op}}^{-1}(\lambda)} = t_\lambda \left( D^\Lambda(X)^{op} \right) = t_\lambda \left( D^\Lambda(X) \right).
$$

Corollary 4.8. For a cylindrically normal stellar stratified space $X$, $\pi_{X^{op}}$ is a stratification whose face poset is the opposite $P(X)^{op}$ of $P(X)$.

Proof. The fact that $\pi_{X^{op}}^{-1}(\lambda)$ is locally closed for each $\lambda \in P(X)$ follows from the description in Lemma 4.7. It also says that $\pi_{X^{op}}^{-1}(\lambda)$ is connected, since it contains the barycenter $\nu(e_\lambda)$ of $e_\lambda$.

The compatibility with the partial order in $P(X)^{op}$ also follows from the description of the boundary in Lemma 4.7.

Definition 4.9. The stratification $\pi_{X^{op}}$ is called the stellar dual of $\pi_X$.

Thus when $\| \text{Lk}_{> e_\lambda}(C(X)) \|$ can be embedded in a sphere $S^{N-1}$ in such a way that the closure $\| \text{Lk}_{> e_\lambda}(C(X)) \|$ is a finite cell complex containing $\| \text{Lk}_{> e_\lambda}(C(X)) \|$ as a strict cellular stratified subspace, $\| \text{St}_{\geq e_\lambda}(C(X)) \|$ can be regarded as an aster in $D^N$.

Proposition 4.10. Let $X$ be a finite locally polyhedral cellular stratified space with compact parameter spaces. Then $\text{Sd}(X)$ has a structure of stellar stratified space whose underlying stratification is $\pi_{X^{op}}$ and the face poset is $P(\text{Sd}(X), \pi_{X^{op}}) = P(X, \pi_X)^{op}$.

Proof. For $\lambda \in P(X)$, consider the upper star $\text{St}_{\geq e_\lambda}(C(X))$ and the upper link $\text{Lk}_{> e_\lambda}(C(X))$ of $\lambda$ in $C(X)$. Since compact locally cone-like spaces can be expressed as a union of a finite number of simplices, each parameter space has a structure of a finite polyhedral complex. And the comma

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14 Theorem 2.11 in [RS72]
category $e_\lambda \downarrow C(X)$ is a cellular category whose morphism spaces are finite cell complexes. By Lemma 2.51 and the finiteness assumption, $\|St_{>e_\lambda}(C(X))\| = B(e_\lambda \downarrow C(X))$ is a finite cell complex and $\|Lk_{>e_\lambda}(C(X))\|$ is a subcomplex. Choose an embedding $\|Lk_{>e_\lambda}(C(X))\| \hookrightarrow S^{N-1}$. Then $D^{D(X)}_\lambda = \{e_\lambda\} * \|Lk_{>e_\lambda}(C(X))\|$ is embedded in $D^N$.

By definition, $t_{e_\lambda} : D^{D(X)}_\lambda \to \pi^{-1}_{X,\lambda}(\lambda) \subset Sd(X)$ is a quotient map. The fact that $t_{e_\lambda}$ is a homeomorphism onto $\pi^{-1}_{X,\lambda}(\lambda)$ when restricted to $\text{Int}(D^{D(X)}_\lambda) = D^{D(X)}_\lambda \circ$ follows easily from the description of elements in $D^{D(X)}_\lambda \circ$ in the proof of Lemma 4.7.

Remark 4.11. The three assumptions, i.e. local-polyhedrality, finiteness, and compactness of parameter spaces, are imposed only for the purpose of the existence of an embedding of $\|Lk_{>e_\lambda}(C(X))\|$ in a sphere. If we relax the definition of a stellar cell $\varphi_\lambda : D_\lambda \to \tau_\lambda$ by dropping the embeddability of the domain $D_\lambda$ in a disk, we do not need to require these conditions.

The next problem is to define a cylindrical structure for the stellar dual $(Sd(X), \pi_{X,\circ})$.

Theorem 4.12. Let $X$ be a finite locally polyhedral stellar stratified space. Suppose all parameter spaces $P_{\mu,\lambda}$ are compact. For $\lambda \leq^{op} \mu$ in $P(Sd(X), \pi_{X,\circ}) = P(X, \pi_X)^{op}$, define $P^{OP}_{\lambda,\mu} = P_{\mu,\lambda}$. Then the stellar structure in Proposition 4.10 and parameter spaces $\{P^{OP}_{\lambda,\mu}\}$ make $(Sd(X), \pi_{X,\circ})$ into a locally polyhedral stellar stratified space.

Proof. It remain to construct PL structure maps

$$b^{OP}_{\lambda,\mu} : P^{OP}_{\lambda,\mu} \times D^{D(X)}_\lambda \to D^{D(X)}_\mu$$

$$\circ^{OP} : P^{OP}_{\lambda_1,\lambda_0} \times P^{OP}_{\lambda_2,\lambda_1} \to P^{OP}_{\lambda_2,\lambda_0},$$

for $\lambda \leq^{op} \mu$ and $\lambda_2 \leq^{op} \lambda_1 \leq^{op} \lambda_0^{op}$.

The composition map $\circ^{op}$ is obviously given by the composition in $X$ under the identification

$$P^{OP}_{\lambda_1,\lambda_0} \times P^{OP}_{\lambda_2,\lambda_1} = P_{\lambda_0,\lambda_1} \times P_{\lambda_1,\lambda_2} \cong P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1}.$$  

The action $b^{OP}_{\lambda,\mu}$ of the parameter space is given by the following composition

$$P^{OP}_{\lambda,\mu} \times D^{D(X)}_\lambda = P_{\mu,\lambda} \times B(e_\lambda \downarrow C(X)) = C(X)(e_{\mu}, e_\lambda) \times B(e_\lambda \downarrow C(X))$$

$$= B(e_{\mu} \downarrow C(X)) \times C(X)(e_{\mu}, e_\lambda) \to B(e_{\mu} \downarrow C(X)) = D^{D(X)}_\mu,$$

where the last arrow is given by the composition in $C(X)$.

The compatibilities of $b_{\lambda,\mu}$ and $\circ$ in $X$ implies that $b^{OP}_{\lambda,\mu}$ and $\circ^{op}$ satisfy the conditions for cylindrical structure. Obviously this is locally polyhedral.

Definition 4.13. The stratified space $(Sd(X), \pi_{X,\circ})$ equipped with the stellar and locally polyhedral structures defined in Theorem 4.12 is called the stellar dual of $X$ and is denoted by $D(X)$.

The following fundamental example shows that cells in $D(X)$ are usually stellar and not cellular.
Example 4.14. Consider the standard regular cellular stratification on $\Delta^2$ as a simplicial complex. The barycentric subdivision $\text{Sd}(\Delta^2)$ is a simplicial complex depicted as follows.

The stellar stratification on $\text{Sd}(\Delta^2)$, i.e. $D(\Delta^2)$ is given by the following figure.

$D(\Delta^2)$ consists of one stellar 0-cell, three stellar 1-cells, and three stellar 2-cells. The stellar structure for the 2-cell at the left bottom in $D(\Delta^2)$ is indicated in the figure as $t_0$. Note that $\|\text{Lk}_{e_0}(C(\Delta^2))\|$ consists of two 1-simplices. It is embedded in $S^1$. The domain for this stellar structure map is the circular sector in the left. The middle point in the arc of the circular sector in $D_0^D(\Delta^2)$ is mapped to the barycenter of $\Delta^2$ and two radii in $D_0^D(\Delta^2)$ are mapped to the half edges touching the lower left vertex of $\Delta^2$.

The barycentric subdivision of $D(\Delta^2)$ can be easily seen to be isomorphic to $\text{Sd}(\Delta^2)$ as simplicial complexes. And we have $D(D(\Delta^2)) \cong \Delta^2$ as simplicial complexes.

For example, the 2-cell corresponding to the unique 0-cell in $D(\Delta^2)$ is the whole triangle. \qed

The next example shows that, when $X$ contains non-closed cells, the process of taking the double dual $D(D(X))$ slims down $X$ while retaining the stratification.

Example 4.15. Consider the 1-dimensional stellar stratified space $Y$ in the picture below.
It consists of a 0-cell $e^0$ and a stellar 1-cell $e^1$ whose domain $D_1$ is a graph of the shape of $Y$ with one vertex removed. The barycentric subdivision $Sd(Y)$ is the minimal regular cell decomposition of $S^1$. Both $D(Y)$ and $D(D(Y))$ are the minimal cell decomposition of $S^1$.

Note that the embedding $\tilde{i}$ in Theorem 1.1 embeds $D(D(Y))$ in $Y$ as a stellar stratified space. □

As the above example suggests, the embedding $\tilde{i}$ in Theorem 1.1 is an embedding of stellar stratified spaces if the domain is regarded as $D(D(X))$. Furthermore, when all cells are closed, we can always recover $X$ from this stellar structure.

Note that we may define this stellar stratification on $Sd(X)$ directly without using $D(-)$.

**Definition 4.16.** Let $X$ be a cylindrically normal stellar stratified space. For $\lambda \in P(X)$, define

$$D^\text{Sal}(X)_\lambda = \| St_{\leq e^0}(C(X)) \|$$
$$D^\text{Sal}(X)_{\lambda, o} = \| St_{\leq e^0}(C(X)) \| - \| Lk_{< e^0}(C(X)) \|.$$

The following fact is dual to Lemma 4.7. The proof is omitted.

**Lemma 4.17.** Let $X$ be a cylindrically normal stellar stratified space. Define a map

$$\pi^\text{Sal}(X) : Sd(X) \longrightarrow P(X)$$

by the composition

$$Sd(X) \xrightarrow{\pi^\text{Sal}(X)} \prod_k N^-_k(C(X)) \xrightarrow{t} C(X)_0 = P(X),$$

where $t$ is the target map. Then $\pi^\text{Sal}(X)$ is a stratification whose strata and their closures are given by

$$\pi^{-1}\text{Sal}(X)(\lambda) = s\lambda \left( D^\text{Sal}(X)_{\lambda, o} \right),$$
$$\pi^{-1}\text{Sal}(X)(\lambda) = s\lambda \left( D^\text{Sal}(X)_{\lambda} \right),$$

where $s\lambda = s_{e^0}$ is the map defined in Definition 2.55.

**Definition 4.18.** The stellar stratified space $(Sd(X), \pi^\text{Sal})$ defined above is called the Salvetti complex of $X$ and is denoted by $\text{Sal}(X)$.

**Remark 4.19.** When the three assumptions in Proposition 4.10 are satisfied, we have $\text{Sal}(X) = D(D(X))$ as stellar stratified spaces.

**Theorem 4.20.** Let $(X, \pi_X)$ be a cylindrically normal stellar stratified space. Then $\text{Sal}(X)$ has a structure of cylindrically normal stellar stratified space with which the map $\tilde{i} : Sd(X) \hookrightarrow X$ is an embedding of cylindrically normal stellar stratified spaces. When all cells in $X$ are closed, $\tilde{i}$ is an isomorphism of cylindrically normal stellar stratified spaces. When $X$ is a finite locally polyhedral stellar stratified space satisfying the assumptions of Proposition 4.10, we have $\text{Sal}(X) = D(D(X))$ as stellar stratified spaces.
**Definition 4.21.** When we regard $Sd(X)$ as $Sal(X)$, the embedding $\tilde{i}$ is denoted by $i_{Sal(X)} : Sal(X) \hookrightarrow X$.

We need a concrete description of $\tilde{i}$, which is rather complicated, in order to prove this fact. We refer to the proof of Theorem 4.15 in [Tamb] for a description of $\tilde{i}$.

**Proof of Theorem 4.20.** The proof of the fact that $\pi_{Sal}$ is a stellar stratification is analogous to that of Theorem 4.12. The point is that we do not need the finiteness and compactness assumptions in this case, since maps $z_e$ used in the construction of the embedding $\tilde{i}$ already provides embeddings of $D^{Sal(X)}_\lambda$ into $D_\lambda$.

In the proof of Theorem 4.15 in [Tamb], a map $z_e : N(\pi)^{-1}_k(e) \times \Delta^k \to D_\lambda$ is constructed for each $e : e_0 \leq \cdots \leq e_{k-1} \leq e_\lambda$ in $N_k(P(X))$ by extending the composition

$$N(\pi)^{-1}_k(e) \times \Delta^k = P_{k-1,\lambda} \times \Delta^k \xrightarrow{z_{d_\lambda}(e)} P_{k-1,\lambda} \times D_{k-1} \xrightarrow{b_{k-1}} D_\lambda.$$ 

by using the identification $\Delta^k = \Delta^k \ast \{k\}$ and the fact that $D_\lambda$ is an aster.

Under the identification

$$N_k(C(X) \downarrow e_\lambda) \cong \prod_{t(e) = e_\lambda} N(\pi)^{-1}_k(e),$$

we obtain a map

$$z_{k,\lambda} : N_k(C(X) \downarrow e_\lambda) \times \Delta^k \to D_\lambda.$$ 

In the proof of Theorem 4.15 in [Tamb], we have verified that these maps are compatible with the simplicial structure and we obtain a map

$$z_\lambda : |N(C(X) \downarrow e_\lambda)| \to D_\lambda.$$ 

Note that

$$St_{\leq e_\lambda}(C(X)) = \overline{N}(C(X) \downarrow e_\lambda)$$

by definition and we obtain a map

$$z_\lambda : D^{Sal(X)}_\lambda = \|St_{\leq e_\lambda}(C(X))\| = ||\overline{N}(C(X) \downarrow e_\lambda)|| = |N(C(X) \downarrow e_\lambda)| \to D_\lambda.$$ 

This is an embedding by construction.

The embedding $i : Sd(X) \hookrightarrow X$ is defined by gluing these maps by the cell structure maps under the decomposition

$$N_k(C(X)) = \coprod_{\lambda \in P(X)} \coprod_{t(e) = \lambda} N(\pi)^{-1}_k(e).$$

In other words, we have a commutative diagram

$$\begin{array}{ccc}
D^{Sal(X)}_\lambda & \xrightarrow{z_\lambda} & D_\lambda \\
\downarrow & & \downarrow \\
Sd(X) & \xrightarrow{i} & X.
\end{array}$$
By Lemma 4.10, the map
\[ s_{\lambda} : D_{\lambda}^{\text{Sal}(X)} = B(C(X) \downarrow e_{\lambda}) \rightarrow BC(X) = \text{Sd}(X) \]
induced by \( s_{\lambda} = s_{e_{\lambda}} \) defines a stellar structure on the stellar cell \( \pi_{\text{Sal}(X)}^{-1}(\lambda) \) indexed by \( \lambda \) and the domain \( D_{\lambda}^{\text{Sal}(X)} = \| \text{St}_{\leq \epsilon_{\lambda}} C(X) \| = \| \text{N}(C(X) \downarrow e_{\lambda}) \| \) is mapped into \( \text{St} \) via \( \tilde{\iota} \). Therefore the embedding \( i_{\text{Sal}} : \text{Sal}(X) \hookrightarrow X \) is a morphism of stratified spaces. The above diagram also says that \( \tilde{\iota} \) is a morphism of stellar stratified spaces.

The facts that the cylindrical structure on \( \text{Sal}(X) \) is given by the same parameter spaces \( P_{\mu,\lambda} \) as in \( X \) and that the structure maps are given by the concatenation of morphisms in \( C(X) \)
\[ b_{\mu,\lambda} : P_{\mu,\lambda} \times \| \text{St}_{\leq \epsilon_{\mu}} C(X) \| = P_{\mu,\lambda} \times \| \text{N}(C(X) \downarrow e_{\mu}) \| \rightarrow \| \text{N}(C(X) \downarrow e_{\lambda}) \| \]
can be easily verified as we have done in the proof of Theorem 4.12. The construction of \( z_{\lambda} \) also implies that \( b_{\mu,\lambda}^{\text{Sal}(X)} \) is compatible with the structure map \( b_{\mu,\lambda} \) in \( X \) and the embedding \( \tilde{\iota} \) is a morphism of cylindrically normal cellular stratified spaces.

When all cells in \( X \) are closed, \( \tilde{\iota} : \text{Sd}(X) \rightarrow X \) is a homeomorphism. The above argument implies that this map defines an isomorphism \( i_{\text{Sal}(X)} : \text{Sal}(X) \cong X \) of cylindrically normal stellar stratified spaces.

Suppose that \( X \) satisfies the assumption of Proposition 4.10. By Theorem 4.12 we have an isomorphism of categories \( C(D(X)) \cong C(X)^{\text{op}} \). Thus
\[ D_{\lambda}^{D(D(X))} = B(e_{\lambda} \downarrow C(D(X))) \cong B(e_{\lambda} \downarrow C(X)^{\text{op}}) \cong B(C(X) \downarrow e_{\lambda}) \]
and the stellar structure map \( t_{\lambda} : D_{\lambda}^{D(D(X))} \rightarrow \text{Sd}(D(X)) \) can be identified with \( s_{\lambda} : D_{\lambda}^{\text{Sal}(X)} \rightarrow \text{Sd}(X) \) under the identification \( \text{Sd}(D(X)) \cong B(C(X)^{\text{op}}) \cong B(C(X)) = \text{Sd}(X) \). And we have \( D(D(X)) \cong \text{Sal}(X) \).

The following example justifies the name for \( \text{Sal}(X) \).

**Example 4.22.** For a real hyperplane arrangement \( A \), the structure of regular cell complex on the Salvetti complex \( \text{Sal}(A) \) for the complexification of \( A \) defined in [Sal87] is nothing but \( D(D(M(A \otimes \mathbb{C}))) \). For example, in the case of the arrangement \( A = \{ \{0\} \} \) in \( \mathbb{R} \), the stratification on \( \mathbb{R} \) is
\[ \mathbb{R} = \{0\} \cup (-\infty, 0) \cup (0, \infty) \]
and the associated stratification on the complexification is
\[ \mathbb{C} = \{0, 0\} \cup (-\infty, 0) \times \{0\} \cup (0, \infty) \times \{0\} \cup \{x + iy \in \mathbb{C} | y > 0\} \cup \{x + iy \in \mathbb{C} | y < 0\} . \]
Then \( \text{Sd}(M(A \otimes \mathbb{C})), D(M(A \otimes \mathbb{C})), \) and \( D(D(M(A \otimes \mathbb{C}))) \) are given as follows:

\[
\begin{array}{ccc}
\text{Sd}(M(A \otimes \mathbb{C})) & D(M(A \otimes \mathbb{C})) & D(D(M(A \otimes \mathbb{C}))) \\
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array} & \\
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array} & \\
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array} & \\
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array}
\end{array}
\]
4.3 The Barycentric Subdivision of Face Categories

We conclude this paper by proving that the barycentric subdivision of a totally normal cellular stratified space corresponds to the barycentric subdivision of the face category.

Let us first recall the definition of the barycentric subdivision of a small category. We use a definition in Noguchi’s papers [Nog11, Nog]. See also the paper [dH08] by del Hoyo.

Definition 4.23. For a small category $C$, the barycentric subdivision $Sd(C)$ is a small category defined by

\[
Sd(C)_0 = \coprod_n N_n(C),
\]

\[
Sd(C)(f, g) = \{ \varphi : [m] \to [n] \mid g \circ \varphi = f \} / \sim
\]

for $f : [m] \to C$ and $g : [n] \to C$, where $\sim$ is the equivalence relation generated by the following relation: for functors $\varphi, \psi : [m] \to [n]$ with $g \circ \varphi = f$ and $g \circ \psi = f$, $\varphi \sim \psi$ if and only if

\[
g(\min\{\varphi(i), \psi(i)\}) \leq \max\{\varphi(i), \psi(i)\} \in C(g(\min\{\varphi(i), \psi(i)\}), g(\max\{\varphi(i), \psi(i)\}))
\]

is an identity morphism in $C$ for any $i$ in $[m]$.

The description can be simplified for acyclic categories as follows.

Lemma 4.24. Let $C$ be an acyclic small category. For $f, g \in Sd(C)_0$, the set of morphisms $Sd(C)(f, g)$ consists of a single point, if there exists $\varphi$ with $f = g \circ \varphi$, and an empty set otherwise. Therefore $Sd(C)$ is a poset.

Proof. Since $C$ is acyclic, $C(x, x) = \{1_x\}$ for any objects $x \in C_0$. This implies that for $\varphi, \psi : [m] \to [n]$ with $f = g \circ \varphi = g \circ \psi$, $\varphi \sim \psi$ if and only if $g(\varphi(i)) = g(\psi(i))$ for all $i \in [m]$. In other words, all elements in $\{ \varphi : [m] \to [n] \mid g \circ \varphi = f \}$ are equivalent to each other. Hence $Sd(C)(f, g)$ is a single point if the above set is nonempty.

In order to compare $C(Sd(X))$ and $Sd(C(X))$ for a totally normal stellar stratified space $X$, we need to understand the cellular stratification on $Sd(X)$. By Corollary 4.39 we know that $Sd(X)$ is a totally normal cell complex when $X$ is a totally normal stellar stratified space. Cells are parametrized by elements in $N(C(X))$. Let us denote the cell corresponding to $b \in N(C(X))$ by $c_b$. Cell structure maps are given as follows.

Lemma 4.25. For each $k$, fix a homeomorphism $D^k \cong \Delta^k$. Let $X$ be a totally normal stellar stratified space. For $b \in N_k(C(X))$, the composition

\[
D^k \cong \Delta^k \xrightarrow{B[k]} B(c_b) \xrightarrow{Bb} BC(X) = Sd(X)
\]

defines a cell structure on the cell corresponding to $b$, where we regard $b$ as a functor $b : [k] \to C(X)$.

Proof. The map $Bb : B[k] \to BC(X)$ is induced by the map $\overline{N}b : \overline{N}([k]) \to \overline{N}(C(X))$. As we have seen in the proof of Proposition 4.4 a cell structure map on the cell corresponding to $b$ is given by the composition

\[
\Delta^k \cong \{b\} \times \Delta^k \xrightarrow{\coprod_k N_k(C(X)) \times \Delta^k} \overline{N}(C(X)) \cong BC(X).
\]

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Since $N_k([k])$ consists of a single point, the above composition can be identified with

$$\Delta^k \simeq N_k([k]) \times \Delta^k \xrightarrow{N(b) \times 1} \prod_k N_k(C(X)) \times \Delta^k \rightarrow \|N(C(X))\| \simeq BC(X)$$

and the result follows. \hfill \Box

For simplicity, we use the standard simplices $\Delta^k$ as the domains of cells in $Sd(X)$. The cell structure map for $e_b$ is identified with $Bb$ by Lemma 4.25.

**Theorem 4.26.** For any totally normal stellar stratified space $X$, we have an isomorphism of categories

$$Sd(C(X)) \cong C(Sd(X)).$$

**Proof.** By definition, objects in $Sd(C(X))$ are elements of the nondegenerate nerve of $C(X)$. On the other hand, objects in $C(Sd(X))$ are in one-to-one correspondence with cells in $Sd(X) = BC(X)$. Under the stratification in Proposition 4.1, we obtain a bijection $C(Sd(X))_0 \cong Sd(X)_0$.

For $b \in N_k(C(X))$ and $b' \in N_m(C(X))$, we have

$$C(Sd(X); \pi_{Sd(X)}(e_b, e_b)) = \left\{ f : \Delta^m \rightarrow \Delta^k \mid BB' = Bb \circ f \right\}.$$

Since $BB'|_{\text{Int}(\Delta^m)}$ is injective, $f|_{\text{Int}(\Delta^m)}$ is also injective. The condition $BB' = Bb \circ f$ implies that $f|_{\text{Int}(\Delta^m)}$ is a PL map and hence $f$ is a PL map. Since $BB'|_{\text{Int}(\Delta^m)}$ is injective, such a PL map is unique if it were to exist. It is given by $f = B\varphi$ for some poset map $\varphi : [m] \rightarrow [k]$.

On the other hand, by Lemma 4.24, $Sd(C(X))(b', b)$ is nonempty (and a single point set) if and only if there exists a poset map $\varphi : [m] \rightarrow [k]$ with $b' = b \circ \varphi$. Thus the classifying space functor $B(-)$ induces an isomorphism of categories

$$B : Sd(C(X)) \rightarrow C(Sd(X)).$$

\hfill \Box

**Remark 4.27.** Note that we obtained an isomorphism of categories instead of an equivalence. Since $Sd(C(X))$ is a poset, it implies that $C(Sd(X))$ is also a poset. Thus the barycentric subdivision $Sd(X)$ of a totally normal stellar stratified space is a regular cell complex.

### A Generalities on Quotient Maps

In our definition of cell structure, we required the cell structure map $\varphi : D \rightarrow \overline{\tau}$ of a cell $e$ to be a quotient map. In order to perform operations on cellular stratified spaces, such as taking products and subspaces, we need to understand basic properties of quotient maps.

This quotient topology condition imposes some restrictions on the topology of $\overline{\tau}$, especially when $e$ is closed. For example, $\overline{\tau}$ is metrizable for any closed cell $e$. A proof can be found in a book [LW69] by Lundell and Weingram. Their proof can be modified to obtain the following extension of this fact.

**Lemma A.1.** Suppose $\varphi : D \rightarrow \overline{\tau} \subset X$ is an $n$-cell structure with $\varphi^{-1}(y)$ compact for each $y \in \overline{\tau}$. Then $\overline{\tau}$ is metrizable. In particular, it is Hausdorff and paracompact.

\[\text{Definition 2.11}\]
Proof. For \( y, y' \in \bar{\tau} \), define
\[
\overline{d}(y, y') = \min \{ d(x, x') \mid x \in \varphi^{-1}(y), x' \in \varphi^{-1}(y') \},
\]
where \( d \) is the metric on \( D^n \). By assumption, \( \varphi^{-1}(y) \) and \( \varphi^{-1}(y') \) are compact and \( \overline{d}(y, y') \) is defined. The compactness of \( \varphi^{-1}(y) \) and \( \varphi^{-1}(y') \) also implies that \( \overline{d} \) is a metric on \( \bar{\tau} \).

Let us verify that the topology defined by \( \overline{d} \) coincides with the quotient topology by \( \varphi \). The continuity of \( \varphi \) with respect to the metric topologies on \( D \) and \( \bar{\tau} \) implies that open subsets in the \( \overline{d} \)-metric topology are open in the quotient topology. Conversely let \( U \) be an open subset of \( \bar{\tau} \) with respect to the quotient topology. We would like to show that, for each \( y \in U \), there exists \( \delta > 0 \) such that the open disk \( U_\delta(y; \overline{d}) \) around \( y \) with radius \( \delta \) with respect to the metric \( \overline{d} \) is contained in \( U \). Let \( \delta \) be a Lebesgue number of the open covering \( \{ \varphi^{-1}(U) \} \) of the compact metric space \( \varphi^{-1}(y) \). For \( y' \in U_\delta(y; \overline{d}) \), there exist \( x \in \varphi^{-1}(y) \) and \( x' \in \varphi^{-1}(y') \) such that \( d(x, x') < \delta \). Thus \( x' \in U_\delta(x; d) \subset \varphi^{-1}(U) \) by the definition of Lebesgue number. Or \( y' = \varphi(x') \in U \). And we have \( U_\delta(y'; \overline{d}) \subset U \).

It is well-known that a metrizable space is paracompact. See Theorem 41.4 in [Mun00], for example. \( \square \)

In particular, when \( \varphi : D \to \bar{\tau} \) is proper (i.e. closed and each \( \varphi^{-1}(y) \) is compact), \( \bar{\tau} \) is metrizable. On the other hand, the properness of \( \varphi \) implies that \( \varphi \) is a bi-quotient map.

**Definition A.2.** A surjective continuous map \( f : X \to Y \) is said to be bi-quotient, if, for any \( y \in Y \) and any open covering \( U \) of \( f^{-1}(y) \), there exist finitely many \( U_1, \ldots, U_k \in U \) such that \( \bigcup_{i=1}^{k} f(U_i) \) contains a neighborhood of \( y \) in \( Y \).

Another important class of maps are hereditarily quotient maps.

**Definition A.3.** A surjective continuous map \( f : X \to Y \) is called hereditarily quotient if, for any \( y \in Y \) and any neighborhood \( U \) of \( f^{-1}(y) \), \( f(U) \) is a neighborhood of \( y \).

Michael [Mic68] proved that bi-quotient maps are abundant.

**Lemma A.4.** Any one of the following conditions implies that a surjective continuous map \( f : X \to Y \) is bi-quotient:

1. \( f \) is open.
2. \( f \) is proper.
3. \( f \) is hereditarily quotient and the boundary \( \partial f^{-1}(y) \) of each fiber is compact.

**Proof.** Proposition 3.2 in [Mic68]. \( \square \)

It is straightforward to verify that a hereditarily quotient map can be restricted freely.

**Lemma A.5.** Any hereditarily quotient map \( f : X \to Y \) is a quotient map. More generally, for any subspace \( A \subset Y \), the restriction \( f|_{f^{-1}(A)} : f^{-1}(A) \to A \) is hereditarily quotient, hence a quotient map.

**Proof.** Suppose \( f \) is hereditarily quotient. For a subset \( U \subset Y \), suppose that \( f^{-1}(U) \) is open in \( X \). For a point \( y \in U \), \( f^{-1}(U) \) is an neighborhood of \( f^{-1}(y) \). Since \( f \) is hereditarily quotient, \( f(f^{-1}(U)) = U \) is a neighborhood of \( y \) in \( Y \). Thus \( y \) is an interior point of \( U \) and it follows that \( U \) is an open subset of \( Y \).

Since the definition of hereditarily quotient map is local, \( f|_{f^{-1}(A)} \) is hereditarily quotient for any \( A \subset Y \). \( \square \)
Remark A.6. See also Arhangel’skii’s paper [Arh63] for hereditarily quotient maps.

The following fact is useful for the study of stratification in our sense.

**Lemma A.7.** Let \( f : X \to Y \) be a hereditarily quotient map. Then for any subset \( U \subset Y \),

\[
\overline{f^{-1}(U)} = f^{-1}(\overline{U}).
\]

**Proof.** It suffices to show that \( \text{Int}(f^{-1}(U)) = f^{-1}(\text{Int}(U)) \). The continuity of \( f \) implies \( f^{-1}(\text{Int}(U)) \subset \text{Int}(f^{-1}(U)) \). On the other hand, for \( x \in \text{Int}(f^{-1}(U)) \), let \( V \subset f^{-1}(U) \) be an open neighborhood of \( x \). Then \( f(V) \) is a neighborhood of \( f(x) \) with \( f(V) \subset U \). Thus \( f(x) \in \text{Int}(U) \) and \( x \in f^{-1}(\text{Int}(U)) \).

**Lemma A.8.** Any bi-quotient map is hereditarily quotient. In particular, it is a quotient map.

**Proof.** By definition.

Recall that a product of quotient maps may not be a quotient map. There exist a space \( X \) and a quotient map \( f : Y \to Z \) such that the product \( 1_X \times f : X \times Y \to X \times Z \) is not a quotient map. The following fact is a well-known result of J.H.C. Whitehead [Whi48].

**Lemma A.9.** For a locally compact Hausdorff space \( X \), \( 1_X \times f \) is a quotient map for any quotient map \( f \).

Unfortunately the domain of a cell structure may not be locally compact.

**Example A.10.** \( D^2 - \{(1,0)\} \) is locally compact, while \( \text{Int}(D^2) \cup \{(1,0)\} \) is not locally compact.

The domain \( D \) of an \( n \)-cell structure \( \varphi : D \to \mathcal{P} \) is often a stratified subspace of \( D^n \) under a normal cell decomposition of \( D^n \). In other words, \( D \) is obtained from \( D^n \) by removing cells. In the above example, \( D^2 \) is regarded as a cell complex \( D^2 = e^0 \cup e^1 \cup e^2 \). \( D^2 - e^0 \) is locally compact, while \( D^2 - e^1 \) is not. More generally we have the following characterization of locally compact subspaces in a CW complex.

**Proposition A.11.** Let \( X \) be a locally finite CW complex and \( A \) be a subcomplex, then \( X - A \) is locally compact.

This is an immediate corollary to the following fact, which can be found, for example, in Chapter XI of Dugundji’s book [Dug78] as Theorem 6.5.

**Lemma A.12.** Let \( X \) be a locally compact Hausdorff space. A subspace \( A \subset X \) is locally compact if and only if there exist closed subsets \( F_1, F_2 \subset X \) with \( A = F_2 - F_1 \).

**Proof of Proposition A.11.** Since \( X \) is locally finite, it is locally compact. The CW condition implies that \( A \) is closed in \( X \).

Let us go back to the discussion on products of quotient maps. The main motivation of Michael for introducing bi-quotient maps is that they behave well with respect to products.

**Proposition A.13.** For any family of bi-quotient maps \( \{f_i : X_i \to Y_i\}_{i \in I} \), the product

\[
\prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i
\]

is a bi-quotient map.
Proof. Theorem 1.2 in [Mic68].

The following property is also useful when we study cell structures.

**Lemma A.14.** Let \( f : X \to Y \) be a quotient map. Suppose that \( Y \) is first countable and Hausdorff and that, for each \( y \in Y \), \( \partial f^{-1}(y) \) is Lindelöf. Then \( f \) is bi-quotient.

**Proof.** Proposition 3.3(d) in [Mic68].

**Corollary A.15.** Let \( \varphi : D \to \tau \) be a relatively compact cell. Then \( \varphi \) is bi-quotient.

**Proof.** By Lemma A.1 \( \tau \) is first countable and Hausdorff. By assumption each fiber \( \varphi^{-1}(y) \) is compact and so is the boundary \( \partial \varphi^{-1}(y) \). The result follows from Lemma A.14.

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