Large distance asymptotic behavior of the emptiness formation probability of the $XXZ$ spin-$\frac{1}{2}$ Heisenberg chain

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Abstract

Using its multiple integral representation, we compute the large distance asymptotic behavior of the emptiness formation probability of the $XXZ$ spin-$\frac{1}{2}$ Heisenberg chain in the massless regime.
1 Emptiness formation probability at large distance

The Hamiltonian of the $XXZ$ spin-$\frac{1}{2}$ Heisenberg chain is given by

$$H = \sum_{m=1}^{M} \left( \sigma^x_m \sigma^x_{m+1} + \sigma^y_m \sigma^y_{m+1} + \Delta(\sigma^z_m \sigma^z_{m+1} - 1) \right).$$

(1.1)

Here $\Delta$ is the anisotropy parameter, $\sigma^x, \sigma^y, \sigma^z$ denote the usual Pauli matrices acting on the quantum space at site $m$ of the chain. The emptiness formation probability $\tau(m)$ (the probability to find in the ground state a ferromagnetic string of length $m$) is defined as the following expectation value

$$\tau(m) = \langle \psi_g | \prod_{k=1}^{m} \left( 1 - \frac{\sigma^z_k}{2} \right) | \psi_g \rangle,$$

(1.2)

where $| \psi_g \rangle$ denotes the normalized ground state. In the thermodynamic limit ($M \to \infty$), this quantity can be expressed as a multiple integral with $m$ integrations [1, 2, 3, 4, 5]. Recently, in the article [6], a new multiple integral representation for $\tau(m)$ was obtained. It leads in a direct way to the known answer at the free fermion point $\Delta = 0$ [9], in particular using a saddle point method, and to its first exact determination outside the free fermion point, namely at $\Delta = \frac{1}{2}$ [10].

The purpose of this letter is to present the evaluation of the asymptotic behavior of $\tau(m)$ at large distance $m$, in the massless regime $-1 < \Delta < 1$, via the saddle point method. We find

$$\lim_{m \to \infty} \frac{\log \tau(m)}{m^2} = \log \frac{\pi}{\zeta} + \log \frac{\pi}{\zeta} + \frac{1}{2} \int_{\mathbb{R} - i0} d\omega \frac{\sinh \frac{\pi}{2}(\pi - \zeta) \cosh \frac{\omega}{2}}{\sinh \frac{\omega}{2} \sinh \frac{\omega}{2} \cosh \omega \zeta},$$

(1.3)

where $\cos \zeta = \Delta$, $0 < \zeta < \pi$. If $\zeta$ is commensurate with $\pi$ (in other words if $e^{i\zeta}$ is a root of unity), then the integral in (1.3) can be taken explicitly in terms of $\psi$-function (logarithmic derivative of $\Gamma$-function). In particular for $\zeta = \frac{\pi}{2}$ and $\zeta = \frac{\pi}{3}$ (respectively $\Delta = 0$ and $\Delta = 1/2$) we obtain from (1.3)

$$\lim_{m \to \infty} \frac{\log \tau(m)}{m^2} = \frac{1}{2} \log 2, \quad \Delta = 0,$$

$$\lim_{m \to \infty} \frac{\log \tau(m)}{m^2} = \frac{3}{2} \log 3 - 3 \log 2, \quad \Delta = \frac{1}{2},$$

(1.4)

which coincides with the known results obtained respectively in [1, 8, 3] and in [11, 10]. For the particular case of the $XXX$ chain ($\Delta = 1, \zeta = 0$) the asymptotic behavior can be evaluated also by the saddle point method and it is given by

$$\lim_{m \to \infty} \frac{\log \tau(m)}{m^2} = \log \left( \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})} \right) \approx \log(0.5991),$$

(1.5)
which is in good agreement with the known numerical result \( \log(0.598) \), obtained in [12].

Below, we explain the main features of our method. A more detailed account of the proofs and techniques involved will be published later.

2 The saddle point method

The multiple integral representation for \( \tau(m) \) obtained in [6] can be written in the form

\[
\tau(m) = \left( \frac{i}{2 \zeta \sin \zeta} \right)^m \left( \frac{\pi}{\zeta} \right)^{m^2 - m/2} \int_{\mathcal{D}} d^m \lambda \cdot F(\{\lambda\}, m)
\times \prod_{a > b} \frac{\sinh \frac{\pi}{\zeta}(\lambda_a - \lambda_b)}{\sinh(\lambda_a - \lambda_b - i\zeta) \sinh(\lambda_a - \lambda_b + i\zeta)} \prod_{a=1}^m \left( \frac{\sinh(\lambda_a - \frac{i\zeta}{2}) \sinh(\lambda_a + \frac{i\zeta}{2})}{\cosh \frac{\pi}{\zeta} \lambda_a} \right)^m, \tag{2.1}
\]

with

\[
F(\{\lambda\}, m) = \lim_{\xi_1, \ldots, \xi_m \to -i\zeta} \frac{1}{\prod \sinh(\xi_a - \xi_b)} \det_m \left( \frac{-i \sin \zeta}{\sinh(\lambda_j - \xi_k) \sinh(\lambda_j - \xi_k - i\zeta)} \right). \tag{2.2}
\]

Here the integration domain \( \mathcal{D} \) is \(-\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \infty\).

Following the standard arguments of the saddle point method we estimate the integral (2.1) by the maximal value of the integrand. Let \( \{\lambda'\} \) be the set of parameters corresponding to this maximum. They satisfy the saddle point equations and for large \( m \) we assume that their distribution can be described by a density function \( \rho(\lambda') \):

\[
\rho(\lambda'_j) = \lim_{m \to \infty} \frac{1}{m(\lambda'_{j+1} - \lambda'_j)}. \tag{2.3}
\]

Thus for large \( m \), one can replace sums over the set \( \{\lambda'\} \) by integrals. Namely, if \( f(\lambda) \) is integrable on the real axis, then

\[
\frac{1}{m} \sum_{j=1}^m f(\lambda'_j) \to \int_{-\infty}^{\infty} f(\lambda) \rho(\lambda) d\lambda,
\]

\[
\frac{1}{m} \sum_{j \neq k}^m \frac{f(\lambda'_j)}{\lambda'_j - \lambda'_k} \to V.P. \int_{-\infty}^{\infty} \frac{f(\lambda)}{\lambda - \lambda'_k} \rho(\lambda) d\lambda, \tag{2.4}
\]

Due to (2.4) it is easy to see that in the point \( \lambda'_1, \ldots, \lambda'_m \) the products in the second line of (2.1) behave as \( \exp(c m^2) \).
Our goal is now to estimate the behavior of the term $F(\{\lambda'\}, m)$. To do this we factorize the determinant in (2.2) as follows for large $m$:

\[
\det_m \left( \frac{-i \sin \zeta}{\sinh(\lambda_j' - \xi_k) \sinh(\lambda_j' - \xi_k - i\zeta)} \right)
\]

\[
= (-2\pi i)^m \det_m \left( \delta_{jk} - \frac{K(\lambda_j' - \lambda_k')}{2\pi i \rho(\lambda_k')} \right) \det_m \left( \frac{i}{2\zeta \sinh(\lambda_j' - \xi_k)} \right),
\]

with

\[
K(\lambda) = \frac{i \sin 2\zeta}{\sinh(\lambda - i\zeta) \sinh(\lambda + i\zeta)}.
\]

Indeed, for $m \to \infty$ one has

\[
\det_m \left( \delta_{jk} - \frac{K(\lambda_j' - \lambda_k')}{2\pi i \rho(\lambda_k')} \right) \det_m \left( \frac{i}{2\zeta \sinh(\lambda_j' - \xi_k)} \right)
\]

\[
= \det_m \left( \frac{i}{2\zeta \sinh(\lambda_j' - \xi_k)} - \sum_{l=1}^{m} \frac{K(\lambda_j' - \lambda_l')}{2\pi i \rho(\lambda_l')} \frac{i}{2\zeta \sinh(\lambda_l' - \xi_k)} \right)
\]

\[
\to \det_m \left( \frac{i}{2\zeta \sinh(\lambda_j' - \xi_k)} - \int_{-\infty}^{\infty} \frac{K(\lambda_j' - \mu)}{2\pi i \rho(\lambda_j')} \frac{i d\mu}{2\zeta \sinh(\mu - \xi_k)} \right)
\]

\[
= \left( \frac{1}{2\pi} \right)^m \det_m \left( \frac{\sin \zeta}{\sinh(\lambda_j' - \xi_k) \sinh(\lambda_j' - \xi_k - i\zeta)} \right).
\]

Here we have used the fact that the function $i/2\zeta \sinh(\lambda_j' - \xi_k)$ solves the Lieb integral equation for the density of the ground state of the XXZ magnet \[13\] (and we have used the notations of \[6\]). The second determinant in the r.h.s. of (2.5) is a Cauchy determinant, hence,

\[
F(\{\lambda'\}, m) = (-i)^m \left( \frac{\pi}{\zeta} \right)^{m^2 + m} \prod_{a>b}^m \frac{\sinh \frac{\pi}{\zeta}(\lambda_a' - \lambda_b')}{\cosh \frac{\pi}{\zeta}\lambda_a'} \cdot \det_m \left( \delta_{jk} - \frac{K(\lambda_j' - \lambda_k')}{2\pi i \rho(\lambda_k')} \right).
\]

The behavior of the determinant in (2.8) can be estimated via Hadamard inequality

\[
|\det_m(a_{jk})| \leq (\max |a_{jk}|)^m m^\frac{m}{2}.
\]

applied to the above determinant and to the determinant of the inverse matrix, which shows that

\[
\lim_{m \to \infty} \frac{1}{m^2} \log \det_m \left( \delta_{jk} - \frac{K(\lambda_j' - \lambda_k')}{2\pi i \rho(\lambda_k')} \right) = 0.
\]
The last equation means that \( \det_m \left( \delta_{jk} - K(\lambda'_j - \lambda'_k) / 2\pi i m \rho(\lambda'_k) \right) \) does not contribute to the leading term of the asymptotics. Hence, it can be excluded from our considerations.

Thus, up to subleading corrections of the exponential type the emptiness formation probability behaves as

\[
\tau(m) \rightarrow \left( \frac{\pi}{\zeta} \right)^{m^2} e^{m^2 S_0}, \quad m \to \infty, \tag{2.11}
\]

with

\[
S_0 \equiv S(\{\lambda'\}) = \frac{1}{m^2} \sum_{a>b}^{m} \log \left( \frac{\sinh^2 \frac{\pi}{\zeta} (\lambda'_a - \lambda'_b)}{\sinh(\lambda'_a - \lambda'_b - i\zeta) \sinh(\lambda'_a - \lambda'_b + i\zeta)} \right) + \frac{1}{m} \sum_{a=1}^{m} \log \left( \frac{\sinh(\lambda'_a - i\zeta/2) \sinh(\lambda'_a + i\zeta/2)}{\cosh^2 \frac{\pi}{\zeta} \lambda'_a} \right). \tag{2.12}
\]

Here the parameters \( \{\lambda'\} \) are the solutions of the saddle point equations

\[
\frac{\partial S_0}{\partial \lambda'_j} = 0. \tag{2.13}
\]

In our case the system (2.13) has the form

\[
\frac{2\pi}{\zeta} \tanh \frac{\pi \lambda'_j}{\zeta} - \coth(\lambda'_j - i\zeta/2) - \coth(\lambda'_j + i\zeta/2)
\]

\[
= \frac{1}{m} \sum_{k=1}^{m} \left( \frac{2\pi}{\zeta} \coth \frac{\pi}{\zeta} (\lambda'_j - \lambda'_k) - \coth(\lambda'_j - \lambda'_k - i\zeta) - \coth(\lambda'_j - \lambda'_k + i\zeta) \right). \tag{2.14}
\]

Using (2.4) we transform (2.14) into the integral equation for the density \( \rho(\lambda) \)

\[
\frac{2\pi}{\zeta} \tanh \frac{\pi \lambda}{\zeta} - \coth(\lambda - i\zeta/2) - \coth(\lambda + i\zeta/2)
\]

\[
= V.P. \int_{-\infty}^{\infty} \left( \frac{2\pi}{\zeta} \coth \frac{\pi}{\zeta} (\lambda - \mu) - \coth(\lambda - \mu - i\zeta) - \coth(\lambda - \mu + i\zeta) \right) \rho(\mu) d\mu. \tag{2.15}
\]

Respectively the action \( S_0 \) takes the form

\[
S_0 = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \log \left( \frac{\sinh(\lambda - i\zeta/2) \sinh(\lambda + i\zeta/2)}{\cosh^2 \frac{\pi}{\zeta} \lambda} \right)
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} d\mu d\lambda \rho(\lambda) \rho(\mu) \log \left( \frac{\sinh^2 \frac{\pi}{\zeta} (\lambda - \mu)}{\sinh(\lambda - \mu - i\zeta) \sinh(\lambda - \mu + i\zeta)} \right). \tag{2.16}
\]
Since the kernel of the integral operator in (2.15) depends on the difference of the arguments, this equation can be solved via Fourier transform. Then

$$\hat{\rho}(\omega) = \int_{-\infty}^{\infty} e^{i\omega \lambda} \rho(\lambda) \, d\lambda = \frac{\cosh \frac{\omega \zeta}{2}}{\cosh \omega \zeta}. \quad (2.17)$$

Making the inverse Fourier transform we find

$$\rho(\lambda) = \frac{\cosh \frac{\pi \lambda}{2 \zeta}}{\sqrt{2} \cosh \frac{\pi \lambda}{2} \cosh \omega \zeta}. \quad (2.18)$$

which obviously satisfies the needed normalisation condition for density (integral on the real axis equals one). It remains to substitute (2.17), (2.18) into (2.16), and after straightforward calculations we arrive at

$$S_0 = \frac{1}{2} \int_{-\infty}^{\infty} d\omega \sinh \frac{\pi (\pi - \zeta)}{2} \cosh \frac{\omega \zeta}{2} \sinh \frac{\pi \omega}{2} \sinh \frac{\omega \zeta}{2} \cosh \omega \zeta. \quad (2.19)$$

Thus, we have obtained (1.3).

In the case of the XXX chain ($\Delta = 1$) one should rescale $\lambda_j \to \zeta \lambda_j$, $\xi_j \to \zeta \xi_j$ in the original multiple integral representation (2.1) for $\tau(m)$ and then proceed to the limit $\zeta \to 0$. The remaining computations are then very similar to the ones described above, therefore we present here only the main results. The behavior of $\tau(m)$ is now given by

$$\tau(m) \longrightarrow \pi m^2 e^{m^2 S_0}, \quad m \to \infty. \quad (2.20)$$

The action $S_0$ in the saddle point has the form

$$S_0 = \int_{-\infty}^{\infty} \log \left( \frac{(\lambda - i/2)(\lambda + i/2)}{\cosh^2 \pi \lambda} \right) \rho(\lambda) \, d\lambda$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} d\mu d\lambda \rho(\lambda) \rho(\mu) \log \left( \frac{\sinh^2 \pi (\lambda - \mu)}{(\lambda - \mu - i)(\lambda - \mu + i)} \right). \quad (2.21)$$

The analog of the integral equation (2.15) in the XXX case is

$$2\pi \tanh \pi \lambda - \frac{2\lambda}{\lambda^2 + \frac{1}{4}} = V.P. \int_{-\infty}^{\infty} \left( 2\pi \coth \pi (\lambda - \mu) - \frac{2(\lambda - \mu)}{(\lambda - \mu)^2 + 1} \right) \rho(\mu) \, d\mu. \quad (2.22)$$

The solution of this equation is

$$\rho(\lambda) = \frac{\cosh \frac{\pi \lambda}{2}}{\sqrt{2} \cosh \pi \lambda}. \quad (2.23)$$

Substituting (2.23) into (2.21) we finally arrive at (1.5).
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