The Grothendieck Ring of a Family of Spherical Categories

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Received: 4 January 2022 / Accepted: 1 July 2022
Published online: 22 August 2022 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract: We compute the fusion rule of a one-parameter family of spherical categories constructed by one author from the classification of singly generated Yang–Baxter planar algebras. The structure constant of the fusion rule is expressed in a closed-form formula of Littlewood–Richardson coefficients. We also compute the characters of the simple objects and their generating function in terms of symmetric functions with infinite variables.

1. Introduction

Jones introduced subfactor planar algebras to axiomatize the standard invariants of subfactors in [15]. Irreducible bimodules of factors form a ring under Connes’ fusion, which is a less complicated invariant. The notion of planar algebras is similar to the notion of spherical monoidal categories [1,10] and to Kuperberg’s spiders [16]. The fusion ring corresponds to the Grothendieck ring of a monoidal category. Constructing a planar algebra from a fusion ring, namely the categorification of a fusion ring, is equivalent to solving for $F$-symbols from pentagon equations. However, the difficulty of solving pentagon equations grows exponentially with respect to the rank of the fusion ring.

Skein theory provides an efficient way to present planar algebras by generators and relations with a consistent evaluation algorithm. It has been successful to study families of planar algebras with arbitrary large rank using skein theory. Remarkably, the skein theory of Jones’ family of type $A$ subfactor planar algebras [12] led to the discovery of the Jones polynomial [13], see [24] and extensive literature therein for further results related to quantum groups and 3-manifold invariants. As a cost, the fusion rule is no longer apparent from skein theory. For the representation category of a quantum group (at roots of unity), the fusion rule can be computed by the Verlinde formula [25] using the braided structure. It is challenging to compute the fusion rule for non-braided planar algebras with large rank. Actually known families of non-braided planar algebras are rare.
One author discovered a family of non-braided planar algebras \( \mathcal{C}(q) \) parameterized by a complex \( q \) from the classification of singly generated Yang–Baxter relation planar algebras in [17], inspired by early classifications on singly generated planar algebras with small dimensions [3–5]. The planar algebra \( \mathcal{C}(q) \) is semisimple for a generic complex parameter \( q \). Moreover, the canonical idempotent category of \( \mathcal{C}(q) \), denoted by \( \mathcal{C} q \), is a semisimple spherical monoidal category. It was shown in [17] that the Grothendieck ring \( G \) of \( \mathcal{C} q \) has simple objects \( X_\lambda \) labelled by all Young diagrams \( \lambda \).

The main purpose of this paper is computing the fusion rule of \( G \), corresponding to the fusion rule of irreducible bimodules in subfactor theory,

\[
X_\mu X_\nu = \sum_\lambda R^\lambda_{\mu, \nu} X_\lambda, \quad (1)
\]

where \( R^\lambda_{\mu, \nu} \in \mathbb{N} \) is called the structure constant for Young diagrams \( \mu, \nu, \lambda \). We first compute the fusion rule of \( G \) with respect to the fundamental representations in Theorem 3.12:

\[
X_{(1^r)} X_\mu = \sum_{i=0}^r \sum_{\nu: \mu - i \rightarrow \nu} \sum_{\lambda: \nu + 1 \rightarrow \lambda} X_\lambda. \quad (2)
\]

The multiplicity of \( X_\lambda \) is the number of ways of constructing \( \lambda \) from \( \mu \) by removing \( i \) cells, no two in the same column, and then adding \( r - i \) cells, no two in the same row. It is proved by constructing an explicit basis of the corresponding hom space through the skein theory of the Yang–Baxter relation planar algebra \( \mathcal{C} \). Then we compute the structure constant \( R^\lambda_{\mu, \nu} \) in a closed-form expression in Theorem 4.23,

\[
R^\lambda_{\mu, \nu} = \sum_{\alpha, \beta, \gamma} c^\mu_{\alpha, \beta} c^\nu_{\beta, \gamma} c^\lambda_{\alpha, \gamma}, \quad (3)
\]

where \( \gamma' \) is the Young diagram dual to \( \gamma \) and \( c_{\cdot, \cdot, \cdot} \) is the Littlewood–Richardson coefficient.

We establish a ring isomorphism \( \Phi_1: G \rightarrow \Lambda_1 \) in Definition 4.19, where \( \Lambda \) is the ring of symmetric polynomial with infinite variables. We compute the character \( \Phi_1(X_\lambda) \) of the simple object \( X_\lambda \) in a closed-form expression in Theorem 4.20

\[
\Phi_1(X_\lambda) = \sum_\mu (-1)^{|\mu|} s_{\lambda/2\mu}, \quad (4)
\]

where \( s_{\lambda/2\mu} \) is a skew-Schur polynomial. Moreover, we compute the generating function of the characters in a closed-form expression in Theorem 4.22,

\[
\sum_{\lambda} s_\lambda(x) \Phi_1(X_\lambda)(y) = \prod_{i_1 \leq i_2} \frac{1}{1 + x_i x_j} \prod_{i, j} \frac{1}{1 - x_i y_j}. \quad (5)
\]

Furthermore, we construct a basis of \( \text{hom}_G(X_\mu \otimes X_\nu, X_\lambda) \) interpreting Eq. (3), see Theorem 4.25 and Remark 4.26. In principle, each \( F \)-symbol for these bases could be computed using the evaluation algorithm of the Yang–Baxter relation. It remains challenging to compute these \( F \)-symbols in a closed-form expression.

For \( q = e^{\frac{\pi i}{2N}} \), \( N \in \mathbb{N} \), \( \mathcal{C}(q) \) is not semisimple, but its Markov trace is positive semi-definite, so its semisimple quotient is a subfactor planar algebra. Its canonical idempotent category is a unitary fusion category. A natural question is computing its fusion rule. We discuss the relation between this question and Xu’s open question on computing the fusion rule of quantum subgroups at the end of Sect. 4.
2. Yang–Baxter Relation Planar Algebras and Spherical Categories

Jones introduced planar algebras and studied their linear skein theory in [15], see basic definitions, properties of planar algebras and inspiring examples therein. One author introduced Yang–Baxter relation planar algebras and classified the ones generated by a 4-valent vertex in Theorem 1.3 in [17], inspired by previous classifications of singly generated planar algebras with small dimensions [3–5]. A surprising family of Yang–Baxter relation planar algebras $\mathcal{C}(q)$ was discovered from this classification. This family could be presented in terms of generators and relations in linear skein theory.

Definition 2.1 (Theorem 4.6, Definition 5.1 in [17]). Let $\mathcal{C}$ be the unshaded planar algebra over $\mathbb{C}(q)$ with circle parameter

$$\delta = q + q^{-1},$$

which is generated by $R = \begin{array}{c} \circ \end{array}$ with the following Yang–Baxter relations:

$$R \begin{array}{c} \circ \end{array} = i \begin{array}{c} \circ \end{array},$$

(6)

$$R \begin{array}{c} \circ \end{array} = 0,$$

(7)

$$R \begin{array}{c} \circ \end{array} R = \begin{array}{c} \circ \end{array} - \frac{1}{\delta} \begin{array}{c} \circ \end{array},$$

(8)

$$R \begin{array}{c} \circ \end{array} R \begin{array}{c} \circ \end{array} = \frac{i}{\delta^2} \left( \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} + \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} + \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \right)$$

$$- \frac{1}{\delta^2} \left( \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} + \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} + \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \right) + i \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array},$$

(9)

where $i = \sqrt{-1}$ and $q$ is a generic parameter.

2.1. Minimal idempotents and branching formula. We first recall the construction of minimal idempotents of the semisimple algebra $\mathcal{C}_n$ and the branching formula for minimal idempotents under the inclusion $\mathcal{C}_n \hookrightarrow \mathcal{C}_{n+1}$, which were proved in Theorem 6.5 in [17]. One may compare the planar algebra $\mathcal{C}$ with the Brauer algebra [9] and its two-parameter deformation, the Birman–Murakami–Wenzl (BMW) planar algebra [2,21]. As a filtered algebra, $\mathcal{C}$ is isomorphic to the string algebra on the Young’s lattice, see Definition 6.1 and Theorem 6.5 in [17]. (The Brauer algebra and the BMW algebra are isomorphic to the string algebra on the Young’s lattice as a filtered algebra as well.) In particular, $\mathcal{C}_n$ is semisimple and $\dim \mathcal{C}_n = (2n - 1)!!$.

For readers’ convenience, let us describe this filtered algebra $\mathcal{C}$ with more details here, keeping the notations in [17]. The vector space $\mathcal{C}_n$ consists of linear sums of $R$-labelled planar diagrams in a rectangle with $n$ boundary points on the top and $n$ boundary points at bottom modulo the Yang–Baxter relations (6)–(9). The multiplication $xy$ of $x$ and $y$ in $\mathcal{C}_n$ is stacking the diagram $y$ on the top of $x$. Under this multiplication, $\mathcal{C}_n$ forms an algebra. Furthermore, $\mathcal{C}_n$ is naturally embedded in $\mathcal{C}_{n+1}$ by adding a through string on the right. Therefore, $\mathcal{C} = (\mathcal{C}_n)_{n \in \mathbb{N}}$ is a filtered algebra. (Such 4-valent diagrams and the corresponding filtered algebra were studied by Brauer, where $R$ is replaced...
by a symmetric braiding in the Brauer algebra [9] and the Yang–Baxter relations are Reidemeister moves of type I, II, III of the braid.)

Now let us construct a basis for \( C_n \). (It is similar to the basis of the Brauer algebra.) Let us order the 2n boundary points of the rectangle as \( \{1, 2, \ldots, 2n\} \) from the left top clockwise. A pairing \( p \) of \( \{1, 2, \ldots, 2n\} \) is a bijection on \( \{1, 2, \ldots, 2n\} \), such that \( p^2 \) is the identity and \( p(j) \neq j, \forall 1 \leq j \leq 2n \). We call \( \{j, p(j)\} \) a pair of the paring \( p \). Let \( P_n \) be the set of pairings of 2n boundary points. Note that there are \( (2n-1)!! \) pairings.

For any pairing \( p \), we can construct a diagram in the rectangle which connects the \( n \) pairs of boundary points by \( n \) strings with a minimal number of crossings. (The minimal condition is equivalent to that any two strings intersect at up to one point transversally.) We label each crossing of the diagram by the generator \( R \), then we obtain an element in \( C_n \), denoted by \( \hat{p} \).

**Proposition 2.2.** The set \( B_n = \{ \hat{p} : p \in P_n \} \) is a basis of the vector space \( C_n \) over \( \mathbb{C}(q) \).

**Proof.** Applying the Yang–Baxter relation, any element in \( C_n \) is a linear sum of such \( \hat{p} \)'s, similar to the proof of Theorem 3.4 in [17]. On the other hand,

\[
\dim C_n = (2n - 1)!! = \#(\hat{p} : p \in P_n).
\]

Therefore, \( \{ \hat{p} : p \in P_n \} \) is a basis of \( C_n \).

**Remark 2.3.** Note that there are different ways to connect boundary points and there are four choices to label \( R \) at each crossing. We fix a choice at the beginning to define \( \hat{p} \) and the basis \( B_n \). For different choices of bases, we may change one to another using the Yang–Baxter relations.

**Example 2.4.** For example, \( \begin{array}{c} \includegraphics{example1} \end{array} \) and \( \begin{array}{c} \includegraphics{example2} \end{array} \) correspond to the same pairing. When we define the element \( \hat{p} \), we fix a choice. Either \( \begin{array}{c} \includegraphics{example3} \end{array} \) or \( \begin{array}{c} \includegraphics{example4} \end{array} \) with the 14 lower terms

\[
\begin{array}{c}
\includegraphics{example5} \\
\includegraphics{example6} \\
\includegraphics{example7} \\
\includegraphics{example8} \\
\includegraphics{example9} \\
\includegraphics{example10} \\
\end{array}
\]

form a basis of \( C_3 \).

The **tensor product** \( \otimes : C_n \otimes C_m \to C_{n+m} \) is a horizontal union of two diagrams. Let \( I_n \) be the identity of \( C_n \), represented by \( n \) vertical strings, and \( I_0 \) is the empty diagram of \( C_0 \). Then for any \( x \in C_n \) and \( y \in C_m \),

\[
x \otimes y = (x \otimes I_m)(I_n \otimes y) = (I_n \otimes y)(x \otimes I_m).
\]

We call \( I_a \otimes x \otimes I_b \) a **shift** of \( x \) in \( C_{a+n+b} \).

As shown in Sect. 5 in [17], the planar algebra \( C_n \) has a type A Hecke subalgebra \( H_n \) generated by

\[
\sigma = \begin{array}{c} \includegraphics{sigma} \end{array} = \frac{q - q^{-1}}{2} + \frac{q - q^{-1}}{2i} + \frac{q + q^{-1}}{2}.
\]

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\]
More precisely, \( H_n \) is algebraically generated by shifts of \( \sigma, \{ I_a \otimes \sigma \otimes I_b : a+2+b = n \} \), and

\[
\sigma - \sigma^{-1} = (q - q^{-1})I_2;
\]

\[
(I_1 \otimes \sigma)(\sigma \otimes I_1)(I_1 \otimes \sigma) = (\sigma \otimes I_1)(I_1 \otimes \sigma)(\sigma \otimes I_1).
\]

The planar algebra has a Markov trace \( tr_n \) on \( C_n \) by gluing the boundary point \( j \) on the top with the boundary point \( 2n+1 - j \) at the bottom from the right, for \( 1 \leq j \leq n \). The Markov trace of \( C_\bullet \) extends the Markov trace of the Hecke algebra in [14]. In particular,

\[
tr_2(\sigma) = \begin{array}{c}
\includegraphics{smiley}\n\end{array} = \delta r,
\]

where \( r = iq^{-1} \). The generic type A Hecke algebra has two parameters \( q \) and \( r \) without the condition \( qr = i \).

**Notation 2.5.** We denote \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), to be the Young diagram with \( n \) columns and the \( k^{th} \) column has \( \lambda_k \) cells. Denote \( |\lambda| = \sum_{k=1}^n \lambda_k \). When \( \lambda_k = 1, 1 \leq k \leq n \), denote \( \lambda = (1^n) \) or \( 1^n \) for short. When \( n = 1 \), denote \( \lambda = (\lambda_1) \) or \( \lambda_1 \) for short.

**Notation 2.6.** For a Young diagram \( \lambda \), we denote

(1) \( \lambda - 1 \) to be the set of Young diagrams which remove 1 cell from \( \lambda \);

(2) \( \lambda + 1 \) to be the set of Young diagrams which add 1 cell to \( \lambda \).

For any Young diagram \( \lambda \), \( |\lambda| = n \), a skein theoretical construction of a minimal idempotent \( y_\lambda \) in \( H_n \) was given in Sect. 2 of [29], such that \{ \( y_\lambda : |\lambda| = n \) \} represent all inequivalent minimal idempotents of \( H_n \). Moreover, the branching formula for \( H_\bullet \) is (e.g. Proposition 2.11 in [29])

\[
y_\lambda \otimes I_1 \sim \bigoplus_{\mu \in \lambda + 1} y_\mu,
\]

where \( \lambda + 1 \) is the set of Young diagrams which add 1 cell to \( \lambda \). In other words, the Bratteli diagram of the filtered algebra \( H_\bullet = (H_n)_{n \in \mathbb{N}} \) is Young’s lattice.

The idempotent \( e = \frac{1}{\delta} \bigcup_1^\infty \) in \( C_2 \) is called the Jones idempotent, corresponding to the Jones projection in operator algebras. The two-sided ideal of \( C_n \) generated by \( e \) is denoted by \( \mathcal{J}_n \), called the basic construction ideal. Recall that \( C_n \) is semisimple, so the maximal idempotent \( t_n \) of \( \mathcal{J}_n \) is central in \( C_n \).

**Proposition 2.7.** We define \( s_n = I_n - t_n \), then \( s_n \) is central in \( C_n \) and

\[
xs_n = 0, \forall x \in \mathcal{J}_n.
\]

\[
s_n(s_m \otimes s_{n-m}) = s_n, \forall m \leq n.
\]
Proof. Note that
\[ x I_n = x, \forall x \in \mathcal{I}_n. \]
So
\[ x s_n = 0, \forall x \in \mathcal{I}_n. \]
For \( m \leq n \), \( p_m \otimes I_{n-m} \) is in \( \mathcal{I}_n \), so
\[ s_n (p_m \otimes I_{n-m}) = 0. \]
Then
\[ s_n (s_m \otimes I_{n-m}) = s_n. \]
Similarly
\[ s_n (I_m \otimes s_{n-m}) = s_n. \]
Therefore,
\[ s_n (s_m \otimes s_{n-m}) = s_n, \forall m \leq n. \]
\[ \square \]
For \( k \in \mathbb{N} \), denote \( e \otimes^k \) in \( \mathcal{C}_{2k} \) to be the \( k^{th} \) tensor power of the Jones idempotent \( e \), and \( e^0 = I_0 \). The following result is a reformulation of Theorem 6.5 in [17]:

**Theorem 2.8.** For a Young diagram \( \lambda \), we define \( \tilde{y}_\lambda = s|\lambda|y_\lambda \). Then \( \tilde{y}_\lambda \) is a minimal idempotent. For any \( n \in \mathbb{N} \), \( \{ \tilde{y}_\lambda \otimes e \otimes^k : |\lambda| + 2k = n \} \) represent all inequivalent minimal idempotents of \( \mathcal{C}_n \). Moreover, we have the branching formula
\[ \tilde{y}_\lambda \otimes I_1 \sim \bigoplus_{\mu \in \lambda+1} \tilde{y}_\mu \otimes \bigoplus_{\nu \in \lambda-1} \tilde{y}_\nu \otimes e. \]

In other words, the principal graph of the planar algebra \( \mathcal{C} \) is Young’s lattice.

**Example 2.9.** For example, there are three inequivalent minimal idempotents in \( \mathcal{C}_2 \), \( \{ e, \tilde{y}_2, \tilde{y}_{12} \} \), where the subscript 2 of \( \tilde{y}_2 \) is the Young diagram with two cells in one row. Moreover,
\[ \tilde{y}_2 = y_2 - e = \frac{\sigma - q^{-1}}{q - q^{-1}} I_2 - e; \]
\[ \tilde{y}_{12} = y_{12} = \frac{q I_2 - \sigma}{q - q^{-1}}. \]
In particular,
\[ e \tilde{y}_{12} = 0; \quad \text{(13)} \]
\[ R \tilde{y}_{12} = - \tilde{y}_{12}. \quad \text{(14)} \]

**Proposition 2.10** (Proposition 3.5 in [17]). The algebra \( \mathcal{C}_n \) is generated by shifts of \( \mathcal{C}_2 \). In particular \( \mathcal{C}_n \) is algebraically generated by \( \{ I_n, I_a \otimes \sigma \otimes I_b, I_a \otimes e \otimes I_b : a + b + 2 = n \} \).
Proposition 2.11. For any \( x \in \mathcal{C}_n \), define \( \Psi(x) = s_n x \). Then \( \Psi \) induces an algebraic isomorphism

\[
H_n \cong s_n H_n = \Psi(\mathcal{C}_n). \tag{15}
\]

Moreover

\[
\mathcal{C}_n = \mathcal{I}_n \oplus H_n = \mathcal{I}_n \oplus s_n H_n. \tag{16}
\]

Proof. Recall that \( s_n \) is central in \( \mathcal{C}_n \), so \( \Psi \) is a homomorphism. Moreover, \( \{y_{\lambda}: |\lambda| = n\} \) represent inequivalent minimal idempotents of \( H_n \) and \( s_n y_{\lambda} = \tilde{y}_{\lambda} \neq 0 \) by Theorem 2.8, so \( \Psi \) induces an isomorphism \( H_n \cong s_n H_n \).

Moreover, \( I_a \otimes e \otimes I_b \in \mathcal{I}_n \), so \( s_n(I_a \otimes e \otimes I_b) = 0 \). Recall that \( s_n \) is central, by Proposition 2.10, \( s_n \mathcal{C}_n \) is algebraically generated by \( \{s_n, s_n(I_a \otimes \sigma \otimes I_b): a + b + 2 = n\} \).

As \( I_a \otimes \sigma \otimes I_b \in H_n \), we have that \( s_n H_n = s_n \mathcal{C}_n \).

Note that the kernel of \( \Psi \) is \( \mathcal{I}_n \), so \( \mathcal{C}_n = \mathcal{I}_n \oplus s_n H_n = \mathcal{I}_n \oplus H_n \). \( \Box \)

According to the decomposition \( \mathcal{C}_n = \mathcal{I}_n \oplus s_n H_n, n \in \mathbb{N} \), we decompose the set \( P_n \) of pairings into two subsets \( E_n \) and \( T_n \),

\[
E_n = \{p \in P_n: 1 \leq p(j) \leq n, \text{ for some } j, 1 \leq j \leq n\};
\]

\[
T_n = \{p \in P_n: n + 1 \leq p(j) \leq 2n, \ \forall 1 \leq j \leq n\}. \]

Proposition 2.12. For a pairing \( p \in P_n \), we have that \( p \in E_n \) iff \( \hat{p} \in \mathcal{I}_n \). Moreover, \( \{s_n \hat{p}: p \in T_n\} \) is a basis of \( s_n H_n \).

Proof. Obviously if \( p \in P_n \), then \( \hat{p} \in \mathcal{I}_n \) and \( s_n \hat{p} = 0 \). By Eq. (16), \( \{s_n \hat{p}: p \in T_n\} \) is a spanning set of \( s_n H_n \). By Eq. (15),

\[
\dim s_n H_n = \dim H_n = n! = \# T_n,
\]

so \( \{s_n \hat{p}: p \in T_n\} \) is a basis of \( s_n H_n \), and for any \( p \in T_n, s_n \hat{p} \neq 0 \), namely \( \hat{p} \notin \mathcal{I}_n \). \( \Box \)

2.2. Anti-symmetrizers and fundamental representations. Recall that \( 1^n \) denotes the Young diagram with one column and \( n \) cells. When \( q = 1 \), the minimal idempotent \( y_{1^n} \) corresponding to the alternating representation of \( S_n \) and the \( n^{th} \) fundamental representation of \( GL_m(\mathbb{C}) \), \( m \geq n \), in Schur–Weyl duality [22,26].

For a generic \( q \), the anti-symmetrizer \( y_{1^n} \) satisfy the following recursive formula

\[
y_{1^n+1} = y_{1^n} \otimes I_1 - \frac{q^n - q^{-n}}{q^{n+1} - q^{-n-1}}(y_{1^n} \otimes I_1)(q^{-1}I_{n+1} + \sigma \otimes I_{n-1})(y_{1^n} \otimes I_1).
\]

\( y_{1^0} = y_{\emptyset} = I_0 \) and \( y_{1^1} = y_1 = I_1 \), see e.g. Proposition 2.1.6 in [17] or Sect. 2 in [29]. Moreover, \( y_{1^n} \) is central in \( H_n \) and

\[
(I_a \otimes y_{1^k} \otimes I_b)y_{1^n} = y_{1^n}, \ a + k + b = n. \tag{17}
\]

Proposition 2.13. For any \( n \in \mathbb{N} \),

\[
y_{1^n} = \tilde{y}_{1^n}. \tag{18}
\]
Proof. By Eqs. (17) and (13), for $a + 2 + b = n$, we have

$$(I_a \otimes e \otimes I_b)y_{1^n} = (I_a \otimes e \otimes I_b)(I_a \otimes y_{1^2} \otimes I_b)y_{1^n} = (I_a \otimes ey_{1^2} \otimes I_b)y_{1^n} = 0.$$ 

Note that $y_{1^n}$ is a central, minimal idempotent of $H_n, C_n$. By Proposition 2.10,

$$xy_{1^n} = 0, \forall x \in \mathcal{I}_n.$$ 

In particular, $p_n y_{1^n} = 0$. So

$$\tilde{y}_{1^n} = s_n y_{1^n} = (I_n - p_n)y_{1^n} = y_{1^n}. \tag*{\square}$$

Notation 2.14. For $p_1 \in T_k$ and $p_2 \in T_{n-k}$, we define $p = p_1 \times p_2$ in $T_k$ as

$$p(j) = p_1(j) + 2n - 2k, \forall 1 \leq j \leq k;$$
$$p(j + k) = p_2(j) + k, \forall 1 \leq j \leq n - k.$$ 

We define $T_{k,n-k} = T_k \times T_{n-k} = \{ p \in T_n : p = p_1 \times p_2, p_1 \in T_k, p_2 \in T_{n-k} \}$.

For any pairing $p \in T_n$, we can identify $p$ with a permutation $p'$ on $n$ points,

$$p'(j) = 2n + 1 - p(j), 1 \leq j \leq n.$$ 

Let $|p|$ be the parity of the permutation $p'$. Any permutation $p'$ decomposes into a product of adjacent transpositions. It is called a minimal decomposition, if the number of adjacent transpositions is minimal. For $p = p_1 \times p_2 \in T_{k,n-k}$, if

$$p'_1 = \prod_a (m_a, m_a + 1); \tag{19}$$
$$p'_2 = \prod_b (m_b, m_b + 1), \tag{20}$$

are minimal decompositions of permutations $p'_1$ and $p'_2$, then

$$p' = \prod_a (m_a, m_a + 1) \prod_b (m_b + k, m_b + 1 + k) \tag{21}$$

is a minimal decomposition. By induction on $n$, we can choose a minimal decomposition for each $p$ in $T_n$, such that Eqs. (19)–(21) holds for $p = p_1 \times p_2$. Furthermore, we can choose

$$\hat{p} = \prod_k I_{k-1} \otimes R \otimes I_{n-k-1} \tag{22}$$

as the corresponding element in the basis $B_n$. Then for $p = p_1 \times p_2$, we have that

$$\hat{p} = \hat{p}_1 \otimes \hat{p}_2. \tag{23}$$

We call $B_n$ a good basis, if Eqs. (19)–(23) hold.
Lemma 2.15. For a good basis $B_n$, suppose that
\[
\tilde{y}_1^n = \sum_{p \in P_n} c_p \hat{p} \ , \ c_p \in \mathbb{C}(q).
\]
For any $0 \leq k \leq n$, we define
\[
\tilde{y}_1^n, k = \sum_{p \in T_{k,n-k}} c_p \hat{p} \ , \ c_p \in \mathbb{C}(q).
\]
Then
\[
\tilde{y}_1^n, k (\tilde{y}_1^k \otimes \tilde{y}_1^{n-k}) = c \tilde{y}_1^k \otimes \tilde{y}_1^{n-k},
\]
for some nonzero $c$ in $\mathbb{C}(q)$, and $\lim_{q \to 1} c = \left(\frac{n}{k}\right)^{-1}$.

Proof. By Eq. (17), we have
\[
(I_a \otimes y_1^2 \otimes I_b)y_1^n = y_1^n, \ a + 2 + b = n;
\]
By Eq. (14), $Ry_1^2 = -y_1^2$. So
\[
(I_a \otimes R \otimes I_b)y_1^n = (I_a \otimes Ry_1^2 \otimes I_b)y_1^n = -y_1^n.
\]
By Eq. (22), for any $p \in T_{k,n-k}$,
\[
\hat{p} (\tilde{y}_1^k \otimes \tilde{y}_1^{n-k}) = (-1)^{|p|} (\tilde{y}_1^k \otimes \tilde{y}_1^{n-k}).
\]
By definition $\tilde{y}_1^n, k = \sum_{p \in T_{k,n-k}} c_p \hat{p} \ , \ c_p \in \mathbb{C}(q)$, so
\[
\tilde{y}_1^n, k (\tilde{y}_1^k \otimes \tilde{y}_1^{n-k}) = \sum_{p \in T_{k,n-k}} (-1)^{|p|} c_p \tilde{y}_1^k \otimes \tilde{y}_1^{n-k} ,
\]
\[
c = \sum_{p \in T_{k,n-k}} (-1)^{|p|} c_p \tilde{y}_1^k .
\]

When $q \to 1$, the Hecke algebra $H$ specializes to the symmetric group algebra; the generator $\sigma$ becomes the symmetric braiding; $\sigma - R \to 0$; and $n!y_1^n$ becomes the alternating sum of permutations of the symmetric groups $S_n$. So for any $p \in T_n$,
\[
\lim_{q \to 1} c'_p = \frac{(-1)^{|p|}}{n!}.
\]
Then
\[
\lim_{q \to 1} c = \lim_{q \to 1} \sum_{p \in T_{i,n-1}} \frac{(-1)^{|p|}}{n!} c_p = \sum_{p \in T_{i,n-i}} \frac{1}{n!} = \left(\frac{n}{i}\right)^{-1}.
\]
Therefore, $c \neq 0$ in $\mathbb{C}(q)$. \qed
Notation 2.16. For Young diagrams \( \mu, v \) and \( \lambda \), \(|\mu| + |v| = |\lambda| = n\), we define

\[
\begin{align*}
\text{hom}_H(y_\mu \otimes y_v, y_\lambda) &= y_\lambda H_n(y_\mu \otimes y_v); \\
\text{hom}_\Psi(y_\mu \otimes y_v, y_\lambda) &= y_\lambda \mathcal{C}_n(y_\mu \otimes y_v).
\end{align*}
\]

Proposition 2.17. The map \( \Psi(x) = s_nx \) induces a linear isomorphism

\[
\text{hom}_H(y_\mu \otimes y_v, y_\lambda) \cong \text{hom}_\Psi(y_\mu \otimes y_v, y_\lambda).
\]

In particular, \( \dim \text{hom}_H(y_\mu \otimes y_v, y_\lambda) = \dim \text{hom}_\Psi(y_\mu \otimes y_v, y_\lambda) \).

Proof. By Eq. (12), \( s_n(y_\mu \otimes y_v) = s_n(y_\mu \otimes y_v) \) and \( s_n y_\lambda = y_\lambda \), so

\( s_n \text{hom}_H(y_\mu \otimes y_v, y_\lambda) \subseteq \text{hom}_\Psi(y_\mu \otimes y_v, y_\lambda). \)

By Proposition 2.11, \( \Psi \) is injective from \( \text{hom}_H(y_\mu \otimes y_v, y_\lambda) \) to \( \text{hom}_\Psi(y_\mu \otimes y_v, y_\lambda) \).

On the other hand, by Eq. 16, for any element \( z \) in \( \text{hom}_\Psi(y_\mu \otimes y_v, y_\lambda) \), we have a unique decomposition \( z = y + x \), such that \( y \in \mathcal{I}_n \) and \( z \in H_n \). Note that

\[
\begin{align*}
y_\lambda z(y_\mu \otimes y_v) &= z, \\
y_\lambda y(y_\mu \otimes y_v) &\in \mathcal{I}_n, \\
y_\lambda x(y_\mu \otimes y_v) &\in H_n.
\end{align*}
\]

By the uniqueness, \( y_\lambda x(y_\mu \otimes y_v) = x \). So \( x \in \text{hom}_H(y_\mu \otimes y_v, y_\lambda) \). Moreover, \( s_n x = s_n z = z \). So \( \Psi \) is surjective from \( \text{hom}_H(y_\mu \otimes y_v, y_\lambda) \) to \( \text{hom}_\Psi(y_\mu \otimes y_v, y_\lambda) \). Therefore, \( \text{hom}_H(y_\mu \otimes y_v, y_\lambda) \) and \( \text{hom}_\Psi(y_\mu \otimes y_v, y_\lambda) \) are isomorphic and they have the same dimension. \( \square \)

2.3. Spherical category. From the semisimple, spherical, unshaded planar algebra \( \mathcal{C} \), we obtain a \( \mathbb{Z}_2 \)-graded, spherical, monoidal category, denoted by \( \mathcal{C}^{q-1} \) in [17], following the standard construction of bimodule categories from a subfactor planar algebra, see e. g. [18].

More precisely, we extend the algebras \( \{\mathcal{C}_n\}_{n \in \mathbb{N}} \) to an algebroid \( \{\mathcal{C}_n^m\}_{m,n \in \mathbb{N}} \), where \( \mathcal{C}_n^m \) is the vector space spanned by \( R \)-labelled planar diagrams with \( m \) boundary points on the top and \( n \) boundary points at the bottom, modulo Yang–Baxter relations. Then \( \mathcal{C}_n = \mathcal{C}_n^1 \). By parity, \( m + n \) is even, and \( \mathcal{C}_n^m \) is isomorphic to \( \mathcal{C}_{\frac{m-n}{2}}^{n \oplus m} \) as a vector space.

For \( x \in \mathcal{C}_n^m \) and \( y \in \mathcal{C}_k^m \), the multiplication \( xy \) is stacking \( y \) on the top of \( x \). In this algebroid of a planar algebra \( \mathcal{C} \), a minimal idempotent \( p \) of \( \mathcal{C}_n \) is equivalent to a minimal idempotent \( q \) of \( \mathcal{C}_{n+2k} \) if and only if \( p \otimes e^{ot k} \) is equivalent to \( q \) in \( \mathcal{C}_{n+2k} \). In particular, the cap diagram \( \cap \in \mathcal{C}_0 \) and the cup diagram \( \cup \in \mathcal{C}_0^2 \) induce the equivalence between the Jones idempotent \( e \) and the unital idempotent \( I_0 \). By Theorem 2.8, \( \{\tilde{y}_\lambda : \text{Young diagram } \lambda\} \) represent all inequivalent minimal idempotents of the algebroid. From idempotent completion, we obtain a linear category with inequivalent simple objects \( \{\tilde{y}_\lambda : \text{Young diagram } \lambda\} \).

This category has a tensor functor \( \otimes : \mathcal{C}_{n_1}^{m_1} \otimes \mathcal{C}_{n_2}^{m_2} \rightarrow \mathcal{C}_{n_1+n_2}^{m_1+m_2} \) defined by a horizontal union of two diagrams. For Young diagrams \( \mu, v \) and \( \lambda \), the hom space is defined as

\[
\begin{align*}
\text{hom}_\Psi(y_\mu \otimes y_v, y_\lambda) &= y_\lambda \mathcal{C}_{|\lambda|+|\mu|+|\nu|}(y_\mu \otimes y_v).
\end{align*}
\]

This is non-zero only when \(|\mu| + |v| + |\lambda| \) is even. The parity of \( |\lambda| \) induces a \( \mathbb{Z}_2 \)-grading of the monoidal category.
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Notation 2.18. We denote $\cup_n$ in $C_{2n}^0$ as

$$\cup_n = \begin{array}{ccccccc}
  & & & & & & 1 \\
  & & & & & \cdots
  & & & & & \\
  & & & & & & n \\
  & & & & & \cdots
  & & & & & \\
  & & & & & & n+1 \\
  & & & & & \cdots
  & & & & & \\
  & & & & & & 2n
\end{array},$$

where the label $n$ aside the thick string in the first picture indicates the number of parallel strings in the second picture. Similarly we denote $\cap_n$ in $C_{2n}^0$ as

$$\cap_n = \begin{array}{ccccccc}
  & & & & & & n \\
  & & & & & \cdots
  & & & & & \\
  & & & & & & 1 \\
  & & & & & \cdots
  & & & & & \\
  & & & & & & n
\end{array}.$$

The dual object of a minimal idempotent $P$ in $C_n$ is the $180^\circ$ rotation of $P$, denoted by $\overline{P}$. The double dual is the $360^\circ$ rotation, which is identity. The evaluation map $\cap_P = (P \otimes P) \cap_n$ and the co-evaluation map $\cup_P = \cup_n(P \otimes P)$ satisfy

$$\left((\cup_P \otimes \overline{P})(\overline{P} \otimes \cap)\right) = \overline{P},$$

$$\left(P \otimes (\cap_P \otimes P)\right) = P.$$

The monoidal category is spherical, because the planar algebra is spherical. By Proposition 9.2 in [17],

$$\overline{\tilde{y}_n} = \tilde{y}_n . \quad (27)$$

Proposition 2.19. The dual object of $\tilde{y}_\lambda$ is equivalent to $\tilde{y}_{\lambda'}$, where $\lambda'$ is the reflection of $\lambda$ in the diagonal, called the Young diagram dual to $\lambda$.

Proof. This follows from the proof of Proposition 9.6 in [17]. For readers’ convenience, we recall the proof here. The duality map $\lambda \rightarrow \lambda'$ is a $\mathbb{Z}_2$ automorphism of the principal graph of the planar algebra $C$, which is Young’s lattice. This $\mathbb{Z}_2$ automorphism fixes the Young diagrams $\emptyset$ and 1, and switches $1^2$ and 2. Its composition with the reflection is an automorphism preserving $\lambda$ for $|\lambda| \leq 2$. By induction on $n = 2, 3, 4, \cdots$, if the automorphism preserves all Young diagrams with $n$ cell, then for $|\lambda| = n + 1$, the automorphism preserves the set $\lambda - 1$, which implies that the automorphism preserves $\lambda$. Therefore, the duality map is identical reflection on Young diagrams.

3. Fusion Rules of Fundamental Representations

A key difference between the planar algebra $C$ and the Brauer algebra (or BMW algebra) is Relation (6). This causes a big change of the behavior of the fusion rule. In Sect. 2.3, we described the spherical, monoidal category $C_{q,1}^q$ derived from the planar algebra $C$.

Notation 3.1. Let $G$ be the Grothendieck ring of $C_{q,1}^q$. Let $X_\lambda \in G$ correspond to the equivalence class of the minimal idempotent $\tilde{y}_\lambda$. Then $\{X_\lambda: \text{Young diagram } \lambda\}$ is a basis of $G$ representing simple objects. Moreover,

$$X_\mu X_\nu = \sum_{\lambda} R_{\mu,\nu}^\lambda X_\lambda,$$

where $R_{\mu,\nu}^\lambda \in \mathbb{N}$ is called the structure constants and

$$R_{\mu,\nu}^\lambda = \dim \text{hom}_C(y_\mu \otimes y_\nu, y_\lambda).$$
In this section, we compute the fusion rule for fundamental representations \( \tilde{y}_{1^n} \), namely \( R^\lambda_{\nu} \), in a closed form. We construct a basis for the corresponding hom space.

**Notation 3.2.** Let \( Y_\lambda \) be the element of \( G \) corresponding to the equivalence class of the idempotent \( y_\lambda \). Note that \( \tilde{y}_\lambda = s_{\lambda|\lambda} y_\lambda \), so \( y_\lambda - \tilde{y}_\lambda = p_n y_\lambda \), which is an idempotent in \( I_{\lambda|\lambda} \). Therefore,

\[
Y_\lambda = X_\lambda + \sum_{|\mu| < |\lambda|} n_{\lambda,\mu} X_\mu, \quad \text{for some } n_{\lambda,\mu} \in \mathbb{N}. \tag{28}
\]

We call \( n_{\lambda,\mu} \) the **extended constants**. Then we can solve for the \( X_\lambda \) recursively in terms of the \( Y_\lambda \), and

\[
X_\lambda = Y_\lambda + \sum_{|\mu| < |\lambda|} z_{\lambda,\mu} Y_\mu, \quad \text{for some } z_{\lambda,\mu} \in \mathbb{Z}. \tag{29}
\]

We call \( z_{\lambda,\mu} \) the **inverse extended constants**. By Eq. (18), for any \( n \geq 0 \),

\[
X_{1^n} = Y_{1^n}. \tag{30}
\]

**Theorem 3.3.** The Grothendieck ring \( G \) of \( C \) is the polynomial ring in the generators \( \{X_{1^n} : n > 0\} \). In particular, \( G \) is commutative.

**Proof.** Note that \( \{X_\lambda\} \) forms a basis of the Grothendieck ring \( G \). By Eqs. (28) and (29), \( \{Y_\lambda\} \) also forms a basis of \( G \). It is well-known that the set \( \{Y_\lambda\} \) is a basis of the polynomial ring in the generators \( \{Y_{1^n} : n > 0\} \). By Eq. (30), \( G \) is the polynomial ring in the generators \( \{X_{1^n} : n > 0\} \). \( \square \)

The ring generated by \( \{Y_\lambda\} \) is isomorphic to the ring \( \Lambda \) of symmetric polynomial with infinite variables and \( Y_\lambda \) corresponds to the Schur polynomial \( s_\lambda \). This correspondence is captured by Schur–Weyl duality [22, 26]. The idempotent \( Y_{1^n} \) corresponds the fundamental representation of \( GL_m(C) \), \( m \to \infty \). Its character, the Schur polynomial \( s_{1^n} \), is the \( n^{th} \) elementary symmetric polynomial. These elementary symmetric polynomials are generators of symmetric polynomials. We study this isomorphism in Sect. 4 and compute the symmetric polynomial corresponding to \( X_\lambda \).

**Notation 3.4.** For a Young diagram \( \lambda \), we define the following sets of Young diagrams:

1. \( \lambda - 1^n \) is the set of Young diagrams which remove \( n \) cells from \( \lambda \), and no two cells in the same row;
2. \( \lambda + 1^n \) is the set of diagrams which add \( n \) cells to \( \lambda \), and no two cells in the same row;
3. \( \lambda - n \) is the set of Young diagrams which remove \( n \) cells from \( \lambda \), and no two cells in the same column;
4. \( \lambda + n \) is the set of Young diagrams which add \( n \) cells to \( \lambda \), and no two cells in the same column.

The following result is well-known for the type A Hecke algebra. It can be derived from the fusion rule of the \( n^{th} \) fundamental representation of (quantum) \( GL_m(C) \), as \( m \to \infty \). The fusion rule can be characterized by Schur polynomials as well.

**Lemma 3.5.** Suppose \( \lambda \) and \( \mu \) are Young diagrams. If \( n = |\mu| - |\lambda| \geq 0 \), then

\[
\dim \hom_H (y_\lambda \otimes y_{1^n}, y_\mu) = \begin{cases} 1, & \forall \mu \in \lambda + 1^n; \\ 0, & \forall \mu \not\in \lambda + 1^n. \end{cases}
\]
Notation 3.6. The planar algebra $\mathcal{C}$ has a $\mathbb{Z}_2$ automorphism $\Omega$ mapping the generator $R$ to $-R$. By Proposition 9.5 in [17], the idempotent $\Omega(\tilde{y}_\lambda)$ is equivalent to $\tilde{y}_\lambda$. So $\Omega(X_\lambda) = X_{\lambda'}$. (The dual Young diagram $\lambda'$ is denoted by $\Omega(\lambda)$ in [17].) In particular, $n' = 1^n$. By Proposition 9.2 in [17], the dual object (or $180^\circ$ rotation) of $\tilde{y}_n$ is $\tilde{y}_n$.

Lemma 3.7. Suppose $\lambda$ and $\mu$ are Young diagrams. If $n = |\mu| - |\lambda| \geq 0$, then

$$\dim \text{hom}_C(\tilde{y}_\lambda \otimes \tilde{y}_1^n, \tilde{y}_\mu) = \begin{cases} 1, & \forall \mu \in \lambda + 1^n; \\ 0, & \forall \mu \notin \lambda + 1^n. \end{cases}$$

$$\dim \text{hom}_C(\tilde{y}_\lambda \otimes \tilde{y}_n, \tilde{y}_\mu) = \begin{cases} 1, & \forall \mu \in \lambda + n; \\ 0, & \forall \mu \notin \lambda + n; \end{cases}$$

If $n = |\lambda| - |\mu| \geq 0$, then

$$\dim \text{hom}_C(\tilde{y}_\lambda \otimes \tilde{y}_1^n, \tilde{y}_\mu) = \begin{cases} 1, & \forall \mu \in \lambda - n; \\ 0, & \forall \mu \notin \lambda - n; \end{cases}$$

$$\dim \text{hom}_C(\tilde{y}_\lambda \otimes \tilde{y}_n, \tilde{y}_\mu) = \begin{cases} 1, & \forall \mu \in \lambda - 1^n; \\ 0, & \forall \mu \notin \lambda - 1^n. \end{cases}$$

Proof. If $n = |\mu| - |\lambda| \geq 0$, then by Eq. (15), Proposition 2.17 and Lemma (3.5), we have

$$\dim \text{hom}_C(\tilde{y}_\lambda \otimes \tilde{y}_1^n, \tilde{y}_\mu) = \dim \text{hom}_H(y_\lambda \otimes y_{1^n}, y_\mu) = \begin{cases} 1, & \forall \mu \in \lambda + 1^n; \\ 0, & \forall \mu \notin \lambda + 1^n. \end{cases}$$

Note that $\mu \in \lambda + n$ iff $\mu' \in \lambda' + 1^n$. So

$$\dim \text{hom}_C(\tilde{y}_\lambda \otimes \tilde{y}_n, \tilde{y}_\mu) = \dim \text{hom}_C(\tilde{y}_{\lambda'}, \tilde{y}_{1^n}, \tilde{y}_{\mu'}) = \begin{cases} 1, & \forall \mu \in \lambda + n; \\ 0, & \forall \mu \notin \lambda + n. \end{cases}$$

If $n = |\lambda| - |\mu| \geq 0$, then by Frobenius reciprocity,

$$\dim \text{hom}_C(\tilde{y}_\lambda \otimes \tilde{y}_1^n, \tilde{y}_\mu) = \dim \text{hom}_C(\tilde{y}_\lambda, \tilde{y}_1^n \otimes \tilde{y}_n) = \begin{cases} 1, & \forall \mu \in \lambda - n; \\ 0, & \forall \mu \notin \lambda - n; \end{cases}$$

$$\dim \text{hom}_C(\tilde{y}_\lambda \otimes \tilde{y}_n, \tilde{y}_\mu) = \dim \text{hom}_C(\tilde{y}_\lambda, \tilde{y}_\mu \otimes \tilde{y}_n) = \begin{cases} 1, & \forall \mu \in \lambda - 1^n; \\ 0, & \forall \mu \notin \lambda - 1^n. \end{cases}$$



Notation 3.8. Suppose $a, b, c \in \mathbb{N}$, and $n = a + b + c$. Let $p_{a,b,c} \in P_n$ be the pairing

$$p_{a,b,c}(k) = \begin{cases} (2n + 1 - k), & \forall 1 \leq k \leq a \text{ or } 2n - a < k \leq 2n; \\ (2a + 2b + 1 - k), & \forall a < k \leq a + 2b; \\ (2n + 2b + 1 - k), & \forall n + b < k \leq 2n - a; \end{cases} \quad (31)$$

We can identify $\hat{p}_{a,b,c}$ as an element in $\mathcal{C}^{a+2b+c}_{a+c}$:

$$\hat{p}_{a,b,c} = a \quad (32)$$
where $a, b, c$ in the first picture indicate the number of parallel strings.

Suppose $\mu$ is a Young diagram, $|\mu| = a + b$. Take Young diagrams $\nu \in \mu - b$ and $\lambda \in \nu + 1^c$. By Lemma 3.7, there are non-zero morphisms $\rho_{1,v} \in \text{hom}_C(\tilde{y}_\mu, \tilde{y}_\nu \otimes \tilde{y}_b)$ and $\rho_{2,v} \in \text{hom}_C(\tilde{y}_\nu \otimes \tilde{y}_1^c, \tilde{y}_\lambda)$. We construct a morphism $\rho'_{\mu, v, \lambda} \in \text{hom}_C(\tilde{y}_\mu \otimes \tilde{y}_1^b \otimes \tilde{y}_1^c, \tilde{y}_\lambda)$ as

$$\rho'_{\mu, v, \lambda} := \rho_{2,v} \hat{p}_{a,b,c}(\rho_{1,v} \otimes \tilde{y}_1^b \otimes \tilde{y}_1^c).$$

We consider $\tilde{y}_{b+c}$ as a morphism in $\text{hom}_C(\tilde{y}_1^{b+c}, \tilde{y}_1^b \otimes \tilde{y}_1^c)$ and construct a morphism $\rho'_{\mu, v, \lambda} \in \text{hom}_C(\tilde{y}_\mu \otimes \tilde{y}_1^{b+c}, \tilde{y}_\lambda)$:

$$\rho_{\mu, v, \lambda} := \rho'_{\mu, v, \lambda}(\tilde{y}_\mu \otimes \tilde{y}_{b+c}) = \rho_{2,v} \hat{p}_{a,b,c}(\rho_{1,v} \otimes \tilde{y}_{b+c}).$$

Their pictorial representations are

\begin{align*}
\rho'_{\mu, v, \lambda} &:= \tilde{y}_v \\
\rho_{\mu, v, \lambda} &:= \tilde{y}_v
\end{align*}

Lemma 3.9. Suppose $a, b, c \in \mathbb{N}$. For any Young diagrams $\mu$ and $\lambda$, $|\mu| = a + b$, $|\lambda| = a + c$, the elements $\{\rho'_{\mu, v, \lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ are linearly independent in $\text{hom}_C(\tilde{y}_\mu \otimes \tilde{y}_1^b \otimes \tilde{y}_1^c, \tilde{y}_\lambda)$. 

Proof. By Frobenius reciprocity, for any $v$, $(\tilde{y}_v \otimes \cup_b)(\rho_{\mu, v} \otimes \tilde{y}_1^b) \neq 0$ in $\text{hom}_C(\tilde{y}_\mu \otimes \tilde{y}_1^b, \tilde{y}_v)$. As $C$ is semisimple, there is a morphism $\rho_{3,v} \in \text{hom}_C(\tilde{y}_v, \tilde{y}_\mu \otimes \tilde{y}_1^b)$, such that

$$(\tilde{y}_v \otimes \cup_b)(\rho_{\mu, v} \otimes \tilde{y}_1^b)\rho_{3,v} = \tilde{y}_v.$$ 

If

$$\sum_{v \in \mu - b, v \in \lambda - 1^c} c_v \rho'_{\mu, v, \lambda} = 0, c_v \in \mathbb{C}(q),$$

then $c_v = 0$ for all $v$. Therefore, the elements $\{\rho'_{\mu, v, \lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ are linearly independent.
then for any $v' \in \mu - b, v' \in \lambda - 1^c$,

$$\rho_{3,v'} \sum_{v \in \mu - b, v \in \lambda - 1^c} \rho'_{\mu,v,\lambda} = c_{v'} \rho_{2,v'} = 0.$$ 

So $c_{v'} = 0$. Therefore, $\{\rho'_{\mu,v,\lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ are linearly independent. 

**Lemma 3.10.** Suppose $a, b, c \in \mathbb{N}$. For any Young diagrams $\mu$ and $\lambda$ with $|\mu| = a + b$, $|\lambda| = a + c$, the morphisms $\{\rho_{\mu,v,\lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ form a spanning set of $\text{hom}_\mathcal{C}(\tilde{y}_\mu \otimes \tilde{y}_b^{b+c}, \tilde{y}_\lambda)$.

**Proof.** For any $p_1 \in P_{a+b}, p_2 \in P_{b+c}$ and $p_3 \in P_{a+c}$, we define

$$x_{p_1,p_2,p_3} = \hat{p}_3 \hat{p}_{a,b,c}(\hat{p}_1 \otimes \hat{p}_2),$$

(35)

where $\hat{p}_{a,b,c}$ is defined in Eq. (32). By Proposition 2.2, $\{x_{p_1,p_2,p_3} : p_1 \in P_{a+b}, p_2 \in P_{b+c}, p_3 \in P_{a+c}\}$ is a spanning set of $\mathcal{C}_n$, because any pairing in $P_n$ can be implemented by some diagram $x_{p_1,p_2,p_3}$ with a minimal number of crossings. Note that $\tilde{y}_{b+c}$ is a central minimal idempotent in $\mathcal{C}_{b+c}$. By Eq. (15), $\tilde{y}_{b+c}$ is a central minimal idempotent in $H_{b+c}$.

By Lemma 3.7, $\dim \text{hom}_\mathcal{C}(\tilde{y}_{b+c}, \tilde{y}_b \otimes \tilde{y}_c) = 1$, so $\tilde{y}_{b+c} \in \text{hom}_\mathcal{C}(\tilde{y}_{b+c}, \tilde{y}_b \otimes \tilde{y}_c)$ and $(\tilde{y}_b \otimes \tilde{y}_c) \tilde{y}_{b+c} = \tilde{y}_{b+c}$. We define

$$\tilde{x}_{p_1,p_2,p_3} = \tilde{y}_\lambda x_{p_1,p_2,p_3}(\tilde{y}_\mu \otimes \tilde{y}_{b+c}).$$

Then $\{\tilde{x}_{p_1,p_2,p_3} : p_1 \in P_{a+b}, p_2 \in P_{b+c}, p_3 \in P_{a+c}\}$ is a spanning set of $\text{hom}_\mathcal{C}(\tilde{y}_\mu \otimes \tilde{y}_{b+c}, \tilde{y}_\lambda)$. Recall that the $180^\circ$ rotation of $\tilde{y}_b$ is $\tilde{y}_b$. So

$$\tilde{x}_{p_1,p_2,p_3} = s_a$$

for some $\rho_{1,j} \in \text{hom}_{\tilde{y}_\mu, \tilde{y}_b \otimes \tilde{y}_b}, \rho_{2,j} \in \text{hom}_{\tilde{y}_b \otimes \tilde{y}_c, \tilde{y}_c},$ and $c_j \in \mathbb{C}(q)$. Precisely, the label $a$ is replaced by $s_a$ in the first equality by Eq. (12). Then $s_a$ is replaced by $\tilde{y}_v$ in the second equality by Eq. (15). Then we obtain the third equality by Lemma 3.7. Therefore, $\{\rho_{\mu,v,\lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ is a spanning set of $\text{hom}_\mathcal{C}(\tilde{y}_\mu \otimes \tilde{y}_{b+c}, \tilde{y}_\lambda)$.

**Lemma 3.11.** Suppose $a, b, c \in \mathbb{N}$ and $r = b + c$. For any Young diagrams $\mu$ and $\lambda$ with $|\mu| = a + b$, $|\lambda| = a + c$, the morphisms $\{\rho_{\mu,v,\lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ are linearly independent over $\mathbb{C}(q)$. 


Proof. Take $n = a + b + c$, and define

- $S_1 = \{k \in \mathbb{N} : 1 \leq k \leq a + b\}$,
- $S_2 = \{k \in \mathbb{N} : a + b < k \leq a + 2b + c\}$,
- $S_3 = \{k \in \mathbb{N} : a + 2b + c < k \leq 2n\}$,
- $S = \{p \in P_n : p \text{ has no pair in } S_i, \ i = 1, 2, 3\}$.

For any $p_1 \in T_{a+b}$, $p_2 \in T_{b+c}$ and $p_3 \in T_{a+c}$, we define

$$x_{p_1, p_2, p_3} = \hat{\rho}_{a,b,c}(\hat{\rho}_1 \otimes \hat{\rho}_2),$$

where $\hat{\rho}_{a,b,c}$ is defined in Eq. (32). Note that $x_{p_1, p_2, p_3} = \hat{\rho}$ for some $p \in S$.

Conversely, for any pairing $p \in S$, $p$ has $a$ pairs between $S_1$ and $S_3$; $b$ pairs between $S_1$ and $S_2$; and $c$ pairs between $S_2$ and $S_3$. So there is a unique $(p_1, p_2, p_3) \in T_{a+b} \times T_{b+c} \times T_{a+c}$, such that $x_{p_1, p_2, p_3}$ implies the pairing $p$. Therefore, we obtain a bijection $\iota : T_{a+b} \times T_{b+c} \times T_{a+c} \to S$, such that $p = \iota(p_1, p_2, p_3)$ is induced from

$$x_{p_1, p_2, p_3} = \hat{\rho}.$$

Note that $\mathcal{C}_{a+b+c}$ is isomorphic to $\mathcal{C}_n$. By Proposition 2.2, we identify $B_n = \{\hat{\rho} : p \in P_n\}$ of $\mathcal{C}_n$ as a basis of $\mathcal{C}_{a+b+c}$. Moreover, we choose $\hat{\rho}$, such that

$$\hat{\rho} = x_{p_1, p_2, p_3} \in B_n, \ \forall \ p = \iota(p_1, p_2, p_3) \in S,$$

and $\{\hat{\rho}_2 : p_2 \in T_{b+c}\}$ is a subset of a good basis $B_{b+c}$.

To prove the morphisms $\{\rho_{\mu,v,\lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ are linearly independent, we assume that

$$\sum_v c_{\mu,v,\lambda} \rho_{\mu,v,\lambda} = 0,$$

for some $c_{\mu,v,\lambda} \in \mathbb{C}(q)$. We need to show $c_{\mu,v,\lambda} = 0$.

Recall that $\rho'_{\mu,v,\lambda} \in \text{hom}_{\mathcal{C}}(\bar{y}_\mu \otimes \bar{y}_1 \otimes \bar{y}_c, \bar{y}_\lambda)$ and $\rho_{\mu,v,\lambda} \in \text{hom}_{\mathcal{C}}(\bar{y}_\mu \otimes \bar{y}_{1+b+c}, \bar{y}_\lambda)$ are defined in Eqs. (33) and (34). We identify the two hom spaces with subspaces of $\mathcal{C}_{a+b+c}$. As $B_n = \{\hat{\rho} : p \in P_n\}$ is a basis,

$$\rho_{\mu,v,\lambda} = \sum_{p \in P_n} b_{\mu,v,\lambda}(p) \hat{\rho} ;$$

$$\rho'_{\mu,v,\lambda} = \sum_{p \in P_n} b'_{\mu,v,\lambda}(p) \hat{\rho} ,$$

for some $b_{\mu,v,\lambda}(p), b'_{\mu,v,\lambda}(p) \in \mathbb{C}(q)$. By Eq. (37),

$$\sum_v \sum_{p \in P_n} c_{\mu,v,\lambda} b_{\mu,v,\lambda}(p) \hat{\rho} = 0 .$$

So

$$\sum_v c_{\mu,v,\lambda} b_{\mu,v,\lambda}(p) = 0, \ \forall p \in P_n .$$
Note that

\[ \rho_{\mu,v,\lambda} = \tilde{y}_v \]

On the other hand,

\[ \tilde{y}_{1+b+c,b} = \sum_{p \in T_{b,c}} c_p \hat{p} = \sum_{j} c_j \hat{p}_{1,j} \otimes \hat{p}_{2,j} \]

for some \( c_p, c_j \in \mathbb{C}(q) \), \( p_{1,j} \in T_b \) and \( p_{2,j} \in T_c \) as defined in Eq. (24). So

\[ \rho'_{\mu,v,\lambda}(\tilde{y}_\mu \otimes \tilde{y}_{1+b+c,b}) = \sum_{j} c_j \hat{p}_{1,j} \otimes \hat{p}_{2,j} \]

Take \( S_0 = t(T_{a+b} \times T_{b,c} \times T_{a+c}) \). Note that for any \( p \in S_0 \), if we express \( \tilde{y}_{1+b+c} \) in terms of a good basis \( B_{b+c} \), then only the components in \( T_b \times T_c \) contribute non-zero coefficients of \( p \). Recall that \( \tilde{y}_{1+b+c,b} \) is the sum of such components of \( \tilde{y}_{1+b+c} \) in \( T_b \times T_c \), so

\[ b'_{\mu,v,\lambda}(p) = b_{\mu,v,\lambda}(p) , \quad \forall \ p \in S_0 , \]

\[ b'_{\mu,v,\lambda}(p) = 0 , \quad \forall \ p \in S \setminus S_0 . \]

By Eq. (38),

\[ \sum_{p \in S_0} \sum_{v} c_{\mu,v,\lambda} b'_{\mu,v,\lambda}(p) \hat{p} = \sum_{p \in S_0} \sum_{v} c_{\mu,v,\lambda} b_{\mu,v,\lambda}(p) \hat{p} = 0. \]

Note that \( \tilde{y}_\mu = s_n \tilde{y}_\mu \). By Proposition 2.12, if \( p \) has a pair in \( S_1 \), then \( \hat{p}(\tilde{y}_\mu \otimes I_{b+c}) = 0 \). Similarly, if \( p \) has a pair in \( S_3 \), then \( \tilde{y}_\mu \hat{p} = 0 \). If

\[ b'_{\mu,v,\lambda}(p) \tilde{y}_\mu \hat{p}(\tilde{y}_\mu \otimes \tilde{y}_{1+b} \otimes \tilde{y}_{1+c}) \neq 0, \]
then $\rho$ has no pair in $S_1$ nor in $S_3$. Moreover, $b_{\mu,v,\lambda}'(p) \neq 0$, so that $p$ has to send the first $b$ points of $S_2$ to $S_1$ and to send the last $c$ points of $S_2$ to $S_3$. Then $p \in S_0$. Therefore,

$$b_{\mu,v,\lambda}'(p)\tilde{\rho}(\tilde{\nu}_\mu \otimes \tilde{\nu}_{1b} \otimes \tilde{\nu}_{1c}) = 0, \quad \forall \ p \notin S_0.$$ 

By Lemma 2.15, $\tilde{y}_{1b+c}'(\tilde{y}_{1b} \otimes \tilde{y}_{1c}) = c_0 \tilde{y}_{1b} \otimes \tilde{y}_{1c}$ for some $c_0 \neq 0$ in $\mathbb{C}(q)$. So

$$c_0 \sum_v c_{\mu,v,\lambda}'(\mu, \tilde{v}, \lambda) = c_0 \sum_v c_{\mu,v,\lambda}'(\mu, \tilde{v}, \lambda)(\tilde{\nu}_\mu \otimes \tilde{\nu}_{1b} \otimes \tilde{\nu}_{1c}) = \sum_v c_{\mu,v,\lambda}'(\mu, \tilde{v}, \lambda)(\tilde{\nu}_\mu \otimes \tilde{\nu}_{1b+c,b})(\tilde{\nu}_\mu \otimes \tilde{\nu}_{1b} \otimes \tilde{\nu}_{1c})$$

$$= \sum_v c_{\mu,v,\lambda}'(\mu, \tilde{v}, \lambda) \left( \sum_{p \in P_n} b_{\mu,v,\lambda}'(p) \tilde{\rho}(\tilde{\nu}_\mu \otimes \tilde{\nu}_{1b} \otimes \tilde{\nu}_{1c}) \right)$$

$$= \sum_v c_{\mu,v,\lambda}'(\mu, \tilde{v}, \lambda) \sum_{p \in S_0} b_{\mu,v,\lambda}'(p) \tilde{\rho}(\tilde{\nu}_\mu \otimes \tilde{\nu}_{1b} \otimes \tilde{\nu}_{1c})$$

$$= \tilde{\nu}_\lambda \left( \sum_{p \in S_0} \sum_v c_{\mu,v,\lambda}'(\mu, \tilde{v}, \lambda) b_{\mu,v,\lambda}'(p) \tilde{\rho}(\tilde{\nu}_\mu \otimes \tilde{\nu}_{1b} \otimes \tilde{\nu}_{1c}) \right) = 0.$$ 

By Lemma 3.9, we have that $c_{\mu,v,\lambda} = 0$ for all $v$.

Therefore, the elements $\{\rho_{\mu,v,\lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ are linear independent in $\mathcal{C}^{a+2b+c}_{a+c}$. In particular, $\rho_{\mu,v,\lambda} \neq 0$, whenever $v \in \mu - b, v \in \lambda - 1^c$. \hfill \Box

**Theorem 3.12.** Suppose $a, b, c \in \mathbb{N}$. For any Young diagrams $\mu$ and $\lambda$, $|\mu| = a + b, |\lambda| = a + c$, the elements $\{\rho_{\mu,v,\lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ form a basis of $\text{hom}_{\mathcal{G}}(\tilde{\nu}_{\mu} \otimes \tilde{\nu}_v, \tilde{\nu}_\lambda)$. In particular, we obtain the fusion for $X_{1^r}$ in a closed form:

$$X_{1^r}X_{\mu} = X_{\mu}X_{1^r} = \sum_{i=0}^{r} \sum_{\nu \in \mu - i} \sum_{\lambda \in \nu + 1^{r-i}} X_{\lambda}.$$ 

**Proof.** By Lemmas 3.10 and 3.11, $\{\rho_{\mu,v,\lambda} : v \in \mu - b, v \in \lambda - 1^c\}$ form a basis of $\text{hom}_{\mathcal{G}}(\tilde{\nu}_{\mu} \otimes \tilde{\nu}_{1^{b+c}}, \tilde{\nu}_\lambda)$. Take $b = i$ and $c = r - i$, then

$$R_{\mu,1^r}^\lambda = \dim \text{hom}_{\mathcal{G}}(\tilde{\nu}_{\mu} \otimes \tilde{\nu}_{1^r}, \tilde{\nu}_\lambda) = \# \{ v : v \in \mu - i, v \in \lambda - 1^{r-i} \}.$$ 

Note that $v \in \lambda - 1^{r-i}$ iff $\lambda \in v + 1^{r-i}$. So

$$X_{\mu}X_{1^r} = \sum_\lambda R_{\mu,1^r}^\lambda X_{\lambda} = \sum_{i=0}^{r} \sum_{\nu \in \mu - i} \sum_{\lambda \in \nu + 1^{r-i}} X_{\lambda}.$$ 

By theorem 3.3, $X_{\mu}X_{1^r} = X_{1^r}X_{\mu}$. \hfill \Box

We remove $i$ cells from $\mu$ (no two in the same column), and then we add $r - i$ cells (no two in the same row).
Corollary 3.13. Applying the automorphism $\Omega$, we obtain the fusion with $X_r$ in a closed form:

$$X_r X_\mu = X_\mu X_r = \sum_{i=0}^{r} \sum_{\nu \in \mu - 1^i} \sum_{\lambda \in \nu + (r-i)} X_\lambda.$$ 

We remove $i$ cells from $\lambda$ (no two in the same row), and then we add $r - i$ cells (no two in the same column).

4. Fusion Rules, Characters and the Generating Function

In this section, we compute the structure constants $R^G_{\lambda \mu, \nu}$ for simple objects in $G$. In principle, one can compute the fusion rule of $G$ recursively using the fusion rule of fundamental representations. However, the complexity grows exponentially w. r. t. the size of the Young diagrams. We observe that $G$ is isomorphic to the ring $\Lambda_1$ of symmetric polynomial with infinite variables. We establish a ring isomorphism $\Phi : G \rightarrow \Lambda$ in Definition 4.19 and compute $\Phi(X_\lambda)$ as the character of the simple object $X_\lambda$ of $G$. Moreover, we compute the generating function of the characters in a closed form. Using the generating function, we compute the structure constant in a closed form.

4.1. Symmetric functions. Recall that the ring of symmetric functions, $\Lambda$, is defined in the following way.

Definition 4.1. Let $n$ be a natural number, and $R_n = \mathbb{Z}[x_1, x_2, \ldots, x_n]^S_n$ be the ring of symmetric polynomial in $n$ variables. We write $R^k_n$ for the degree $k$ component of $R_n$. For each $k$, we have maps $\rho_n : R^k_n \rightarrow R^k_{n-1}$ defined by setting $x_n = 0$; these form an inverse system, so we may take the inverse limit $\lim_{\leftarrow} R^k_n$. Then, as an abelian group, we define:

$$\Lambda = \bigoplus_{k \geq 0} \lim_{\leftarrow} R^k_n.$$ 

The multiplication on $\Lambda$ is inherited from the multiplication $R^1_n \otimes R^2_n \rightarrow R^{1+2}_n$. We may complete $\Lambda$ with respect to the grading. In this case we obtain

$$\hat{\Lambda} = \prod_{k \geq 0} \lim_{\leftarrow} R^k_n.$$ 

We introduce some important elements of the ring of symmetric functions.

Proposition 4.2. We have the following facts about $\Lambda$:

1. The polynomials $\sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \in R^r_n$ define an element $e_r \in \lim_{\leftarrow} R^r_n$ called the $r^{th}$ elementary symmetric function. These $e_r$ freely generate $\Lambda$ as a polynomial ring: $\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$. We have the generating function $E(t) = \sum_r e_r t^r = \prod_i (1 + x_i t)$. 

Similarly, the polynomials \( \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \in R_n^k \) define an element \( h_r \in \lim_\Leftarrow R_n^k \) called the \( r \)th complete symmetric function. These \( h_r \) also freely generate \( \Lambda \) as a polynomial ring: \( \Lambda = \mathbb{Z}[h_1, h_2, \ldots] \). We have the generating function \( H(t) = \sum_i h_r t^r = \prod_i (1 - x_i t)^{-1} \).

The polynomials \( \sum_i x_i^r \in R_n^k \) define an element \( p_r \in \lim_\Leftarrow R_n^k \) called the \( r \)th power-sum symmetric function. They freely generate \( \mathbb{Q} \otimes \Lambda \) as a polynomial ring over \( \mathbb{Q} \) (but they do not generate \( \Lambda \) over \( \mathbb{Z} \)). We have the generating function \( P(t) = \sum_r p_{r+1} t^r = \sum_i \frac{x_i}{1 - x_i t} \).

The generating functions \( E(t) \) and \( H(t) \) satisfy the relation \( H(t) E(-t) = 1 \), and this equation encodes how to express the elementary symmetric functions in terms of the complete symmetric functions and vice versa. Similarly, we have \( H'(t)/H(t) = P(t) \), and \( E'(t)/E(t) = P(-t) \). In particular, we have the equations

\[
\sum_{r \geq 0} h_r t^r = \exp \left( \sum_{i \geq 1} \frac{p_i}{i} t^i \right),
\]

\[
\sum_{r \geq 0} e_r t^r = \exp \left( \sum_{i \geq 1} \frac{(-1)^{i-1} p_i}{i} t^i \right).
\]

Elements of \( \Lambda \otimes \Lambda \) may be viewed as polynomials in two sets of variables, say \( x_i \) and \( y_j \), symmetric in each separately. To indicate which variable set is being considered, we write \( f(x) \) or \( f(y) \). Given \( f \in \Lambda \), we write \( f(x, y) \) for the element of \( \Lambda \otimes \Lambda \) defined by the symmetric function \( f \) in the variable set \( \{x_i\} \cup \{y_j\} \). (This operation defines a comultiplication \( \Lambda \to \Lambda \otimes \Lambda \).)

Fix a Young diagram \( \lambda = (\lambda_1, \lambda_2, \ldots) \), adding trailing zeros if needed, so that \( \lambda \) has \( n \) parts (usually we do not distinguish between Young diagrams that differ by trailing zeros). The polynomials \( \det(x_i^{\lambda+j-n-j})/\det(x_i^{n-j}) \in R_n^{[\lambda]} \) define an element \( s_\lambda \in \lim_\Leftarrow R_n^{[\lambda]} \), called the Schur function associated to \( \lambda \).

We have that \( s_1 = e_r \) and \( s_r = h_r \).

**Example 4.3.** We have:

\[
\frac{p_1^2 + p_2}{2} = \frac{1}{2} \left( \sum_{i \neq j} x_i x_j + 2 \sum_i x_i^2 \right) = \sum_{i \leq j} x_i x_j = h_2.
\]

**Remark 4.4.** Schur functions may be viewed as the characters of irreducible representations of \( GL_n(\mathbb{C}) \) in the following sense. If \( M \in GL_n(\mathbb{C}) \) has eigenvalues \( x_i \), then the trace of the action of \( M \) on the irreducible representation of \( GL_n(\mathbb{C}) \) corresponding to the Young diagram \( \lambda \) is \( s_\lambda(x_i) \); the Schur function corresponding to \( \lambda \) evaluated at the eigenvalues \( x_i \). Note that this quantity is zero unless the Young diagram \( \lambda \) has at most \( n \) non-zero parts. This means we have a homomorphism \( \Lambda \to R_n \) whose kernel has basis \( s_\mu \) for Young diagrams \( \mu \) with more than \( n \) non-zero parts.

We now discuss a bilinear form on \( \Lambda \).
Proposition 4.5. The ring $\Lambda$ satisfies the following properties:

1. The Schur functions form a $\mathbb{Z}$-basis of $\Lambda$: $\Lambda = \mathbb{Z}\{s_\lambda \mid \lambda \text{ a Young diagram}\}$. In particular, there is a bilinear form $\langle - , - \rangle : \Lambda \otimes \Lambda \to \mathbb{Z}$ for which the Schur functions are orthonormal.

2. The adjoint to multiplication by $p_i$ is $i \frac{\partial}{\partial p_i}$ (where elements of $\Lambda$ are viewed as polynomials in the $p_i$ with possibly rational coefficients).

3. The adjoint to multiplication by $s_\mu$ (with respect to $\langle - , - \rangle$) is denoted $s_\perp^\mu$. The symmetric function $s_\perp^\mu(s_\lambda)$ is called a skew-Schur function, and denoted $s_{\lambda/\mu}$. It is non-zero if and only if $\mu_i \leq \lambda_i$ for all $i$.

4. Schur functions satisfy the following multiplication rule $s_\mu s_\nu = \sum_{\mu, \nu} c_{\lambda}^{\mu, \nu} s_\lambda$, where $c_{\lambda}^{\mu, \nu}$ are the Littlewood–Richardson coefficients (which are zero unless $|\mu| + |\nu| = |\lambda|$). They also satisfy $s_\lambda(x, y) = \sum_{\mu, \nu} c_{\lambda}^{\mu, \nu} s_\mu(x)s_\nu(y)$.

5. The identity $e_r(x, y) = \sum_{i=0}^r e_i(x)e_{r-i}(y)$ shows that $c_{\mu, \nu}^r$ is zero unless $\mu = 1^i$ and $\nu = 1^{r-i}$ for some $0 \leq i \leq r$, in which case it is equal to 1.

6. The Littlewood–Richardson coefficient $c_{\mu, \nu}^r$ is zero unless the diagram of $\lambda$ can be obtained by adding $r$ cells to the diagram of $\mu$, with no two cells in the same column; this is the Pieri rule. Similarly, $c_{\mu, \nu}^r$ is zero unless the diagram of $\lambda$ can be obtained by adding $r$ cells to the diagram of $\mu$, with no two cells in the same row; this is the dual Pieri rule.

There are two identities that will be important to us, which we now state.

Proposition 4.6. We have the following equations:

1. The following equality of series holds in a completion of $R_n \otimes R_n$ for each $n$, and therefore in a completion of $\Lambda$ (note that the homogeneous components of the right-hand side define elements of the inverse limits used to define the ring of symmetric functions):

$$\sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y) = \prod_{i,j} \frac{1}{1 - x_iy_j}.$$  

This is called the Cauchy Identity.

2. Similarly, we have the Dual Cauchy Identity:

$$\sum_{\lambda} s_{\lambda}(x)s_{\lambda'}(y) = \prod_{i,j} (1 + x_iy_j).$$  

Here, $\lambda'$ is the Young diagram dual to $\lambda$.

There is another operation on symmetric functions called plethysm.

Definition 4.7. Given symmetric functions $f$ and $g$ which are sums of monomials in the variables $x_i$ with coefficients in $\mathbb{Z}_{\geq 0}$, the plethysm of $g$ with $f$ is a symmetric function denoted $g[f]$. It may be calculated in the following way. Express $f(x_1, x_2, \ldots)$ as a sum of monomials (repeated according to their multiplicity) $f = \sum_{i} x_1^{a_1(i)} x_2^{a_2(i)} \cdots$. Then $g[f]$ is the symmetric function obtained by evaluating $g$ on the variable set given by
the monomials $x_1^{a_1(i)} x_2^{a_2(i)} \cdots$. It immediately follows that the map $\Lambda \to \Lambda$ defined by $g \mapsto g[f]$ is an algebra homomorphism (but this is not true for $f \mapsto g[f]$).

**Remark 4.8.** There is a way of generalizing the above definition to $f$ and $g$ for which are not necessarily a positive (or even integral, if one is prepared to base change $\Lambda$) sum of monomials. The most general definition is the one given in Chapter 1, Sect. 8 of [20].

**Remark 4.9.** Let $f = \sum \nu m_\nu s^{\nu}$ be the character of a representation $V$ of $GL_n(\mathbb{C})$ (where $n$ is taken to be sufficiently large), so $m_\nu \in \mathbb{Z}_{\geq 0}$, and all but finitely many $m_\nu$ are zero. Thus, $f$ encodes a homomorphism $\varphi_f : GL_n(\mathbb{C}) \to GL(V)$. Similarly, fix $g = \sum n_v s_v$ (with the same conditions on $n_v$ as on $m_\nu$), which uniquely defines a representation $W$ of $GL(V) = GL_{\dim(V)}(\mathbb{C})$, encoding a homomorphism $\varphi_g : GL(V) \to GL(W)$. Then, $W$ is a representation of $GL_n(\mathbb{C})$ via the composition $\varphi_g \circ \varphi_f$:

$$GL_n(\mathbb{C}) \xrightarrow{\varphi_f} GL(V) \xrightarrow{\varphi_g} GL(W).$$

The character of this representation is the plethysm $g[f]$. The value of $n$ used in this construction does not affect $g[f]$, provided it is large enough (e.g. $n = \deg(f) \deg(g)$ will suffice).

**Example 4.10.** We show that $e_1 = \sum_i x_i$ is a two-sided identity for plethysm. Note that by definition, $e_1[f]$ recovers the sum of the monomials of $f$, namely $f$ itself. On the other hand, $f[e_1]$ is the evaluation of $f$ on the variable set $\{x_i\}$ (the monomials of $e_1$), which again is $f$ itself. This is consistent with the formulation in terms of representations of general linear groups, where $\varphi_{e_1}$ represents the identity map $GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ (for any $n$).

**Remark 4.11.** For power-sum symmetric functions $p_r$, plethysm has some useful properties. In particular, $p_r[f] = f[p_r]$ for arbitrary $f$, because both sides are equal to the symmetric function obtained by multiplying the exponents of all monomials of $f$ by $r$. As a special case, we obtain $p_{r_1}[p_{r_2}] = p_{r_2}[p_{r_1}] = p_{r_1 r_2}$.

Ultimately, the result we need about plethysm is the following.

**Theorem 4.12.** We have the following equation:

$$h_r[h_2] = \sum_{|\lambda|=r} s_{2\lambda},$$

where, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then $2\lambda = (2\lambda_1, 2\lambda_2, \ldots, 2\lambda_k)$.

**Proof.** This can be found in Chapter 1, Sect. 8, Example 6 of [20]. Alternatively, see Example A2.9 of [23].

We now introduce a linear operator which will play an important role in what follows, and prove some properties that it satisfies.

**Definition 4.13.** Let $L$ be the linear operator on the completion of $\Lambda$ defined by multiplication by $\prod_{i \leq j} \frac{1}{1 + x_i x_j}$. ```
Proposition 4.14. The adjoint of $L$ with respect to $\langle -, - \rangle$ is:

$$L^\dagger = \sum_\mu (-1)^{\mu} s_{2\mu}^\perp.$$  

Proof. We recognise the product defining $L$ as the generating function of complete symmetric functions evaluated at $-1$, with variable set $\{x_i x_j\}_{i \leq j}$ (these are the monomials in $h_2$); the degree $2r$ component of this sum is precisely what is computed in Theorem 4.12. Thus,

$$\prod_{i \leq j} \frac{1}{1 + x_i x_j} = H(-1)[h_2] = \sum_{r \geq 0} (-1)^r h_r[h_2] = \sum_{r \geq 0} (-1)^r \sum_{|\mu| = r} s_{2\mu} = \sum_{\mu} (-1)^{|\mu|} s_{2\mu}.$$  

Noting that the adjoint of multiplication by $s_{2\mu}$ is $s_{2\mu}^\perp$, the proposition follows. \(\square\)

Notation 4.15. Let $\phi_2 : GL_n \to GL_{n(n+1)/2}$ be the symmetric square representation of $GL_n$ and $\phi_1^r$ be the $r$th antisymmetric power representation of $GL_{n(n+1)/2}$. Then $\phi_1^r \phi_2$ is a representation of $GL_n$. The multiplicity of the irreducible representation of $GL_n$ with highest weight $\lambda$ in $\phi_1^r \phi_2$ is denoted by $b_{n,r,\lambda}$. We define $b_{r,\lambda} = \lim_{n \to \infty} b_{n,r,\lambda}$. Then

$$e_r[h_2] = \sum_{\lambda} b_{r,\lambda} s_\lambda, \quad (39)$$

$$L^{-1} = \prod_{i \leq j} (1 + x_i x_j) = \sum_{r \geq 0} e_r[h_2] = \sum_{r \geq 0, \lambda} b_{r,\lambda} s_\lambda. \quad (40)$$

Lemma 4.16. Let $\theta_i = \frac{1 + (-1)^i}{2}$, so that $\theta_i$ is equal to 0 when $i$ is odd, and equal to 1 when $i$ is even. When expressed in terms of power-sum symmetric functions, $L$ has the following form:

$$L = \exp \left( \sum_i \frac{(-1)^i p_i^2 + 2(-1)^i/2}{2i} \right).$$

Proof. We write $L = H(-1)[h_2]$ (as in the proof of Proposition 4.14), where we express $H(-1)$ and $h_2$ in terms of power-sum symmetric functions. We use Remark 4.11 to manipulate the plethysm:

$$L = \exp \left( \sum_i \frac{(-1)^i p_i^2}{i} \left[ p_1^2 + p_2 \right] \right)$$

$$= \exp \left( \sum_i \frac{(-1)^i p_i^2}{i} \left[ p_1^2 + p_2 \right] \right)$$

$$= \exp \left( \sum_i \frac{(-1)^i p_i^2}{i} \left[ p_1^2 + p_2 \right] \right)$$
\[ = \exp \left( \sum_i \frac{(-1)^i p_i^2 + p_{2i}}{2i} \right) \]

We rearrange the sum so that all instances of \( p_i \) occur in the \( i^{th} \) summand. This means moving the term \((-1)^i p_{2i}/2i\) from the \( i^{th} \) summand to the \( 2i^{th} \) summand. Upon noting that only even index summands obtain a contribution in this way, we obtain the stated formula. \( \Box \)

**Proposition 4.17.** Consider \( \Lambda \otimes \Lambda \) as the set of symmetric functions in two sets of variables \( \{x_i^{(1)}\} \) and \( \{x_i^{(2)}\} \). Suppose that the symmetric function \( f \) satisfies \( f(x^{(1)}, x^{(2)}) = \sum_i g_i(x^{(1)})h_i(x^{(2)}) \). Then, we have the following equation:

\[
\left( \sum_{\lambda} s_{\lambda}(x^{(1)})s_{\lambda'}(x^{(2)}) \right) L(f)(x^{(1)}, x^{(2)}) = L(g_i)(x^{(1)})L(h_i)(x^{(2)}).
\]

**Proof.** In the definition of \( L \) (considered to have variable set \( \{x_i^{(1)}\} \cup \{x_i^{(2)}\} \)), products of pairs of variables take one of three forms: either both variables come from \( \{x_i^{(1)}\} \), or both variables come from \( \{x_i^{(2)}\} \), or one variable comes from each. Giving \( L \) a subscript to show its variable set, we obtain:

\[
L_{\{x_i^{(1)}\} \cup \{x_i^{(2)}\}} = \prod_{i_1 \leq i_2} \frac{1}{1 + x_i^{(1)} x_i^{(2)}}, \quad \prod_{i_1 \leq i_2} \frac{1}{1 + x_i^{(1)}_1 x_i^{(2)}_1}, \quad \prod_{i_1 \leq i_2} \frac{1}{1 + x_i^{(1)}_2 x_i^{(2)}_2}.
\]

Moving the last factor to the left-hand side, and using the Dual Cauchy Identity,

\[
\left( \sum_{\lambda} s_{\lambda}(x^{(1)})s_{\lambda'}(x^{(2)}) \right) L_{\{x_i^{(1)}\} \cup \{x_i^{(2)}\}} = L_{\{x_i^{(1)}\}}L_{\{x_i^{(2)}\}}.
\]

This is equivalent to the statement of the proposition. \( \Box \)

### 4.2. Characters, generating functions and fusion rules.

In this section, we recall some properties of the Grothendieck ring \( G \), and then study its structure using symmetric functions. Recall that \( G \) has basis \( \{Y_\lambda\} \) indexed by Young diagrams.

**Notation 4.18.** By Schur–Weyl duality, we obtain a ring isomorphism \( \Phi : G \to \Lambda \), such that \( \Phi(Y_\lambda) = s_\lambda \). Moreover,

\[ Y_\mu Y_\nu = \sum_\lambda c^\lambda_{\mu, \nu} Y_\lambda, \]

where \( c^\lambda_{\mu, \nu} \) are the Littlewood–Richardson coefficients.

Recall that \( s_{\lambda/2\mu} \) is a skew-Schur function, and \( 2\mu \) is the Young diagram obtained by doubling each part of \( \mu \).
Definition 4.19. By Theorem 3.3, we define a ring isomorphism \( \Phi : \mathcal{G} \to \Lambda \), such that
\[
\Phi(Y_\lambda) = s_\lambda, \ \forall \ \lambda.
\]

In particular,
\[
\Phi(X_{1^r}) = \Phi(Y_{1^r}) = s_1^r, \ \forall \ r \geq 0.
\]

Theorem 4.20. For any Young diagram \( \lambda \), we call \( \Phi(X_\lambda) \) the character of \( X_\lambda \) in \( \mathcal{G} \). Then
\[
\Phi(X_\lambda) = L^\dagger s_\lambda = \sum_\mu (-1)^{|\mu|/2} s_{\lambda/2\mu}; \tag{41}
\]
\[
X_\lambda = \sum_{\mu, \nu, 2|\mu| + |\nu| = |\lambda|} (-1)^{|\mu|} e_{2\mu, \nu} Y_\nu. \tag{42}
\]

Proof. Suppose \( \Phi' : \mathcal{G} \to \Lambda \) is linear extension of
\[
\Phi'(X_\lambda) = L^\dagger s_\lambda = \sum_\mu (-1)^{|\mu|} s_{\lambda/2\mu}.
\]

We need to prove \( \Phi(X_\lambda) = \Phi'(X_\lambda) \).

Note that \( \Phi(X_{1^r}) = e_r \). We first show that for any \( r \in \mathbb{N} \) and Young diagrams \( \mu \) and \( \lambda \),
\[
\Phi(X_{1^r}) \Phi'(X_\mu) = \sum_{i=0}^r \sum_{\nu \in \mu - i} \sum_{\lambda \in \nu + i} \Phi'(X_\lambda). \tag{43}
\]

When we encode the operations of removing and adding cells via the Pieri rule and dual Pieri rule, what we must prove becomes
\[
e_r L^\dagger (s_\lambda) = \sum_{i=0}^r L^\dagger (e_{r-i} h_i^\dagger s_\lambda).
\]

This is precisely the assertion of the following equality of operators: \( e_r L^\dagger = \sum_{i=0}^r L^\dagger e_{r-i} h_i^\dagger \). We prove the adjoint of this equality, namely \( L e_r^\dagger = \sum_{i=0}^r h_i e_{r-i}^\dagger L \).

To prove this statement for all \( r \) simultaneously, we multiply by \( t^r \) and sum over \( r \geq 0 \); it is equivalent to prove the following identity of (operator-valued) generating functions:
\[
L E(t)^\dagger = H(t) E(t)^\dagger L.
\]

We rewrite all quantities in terms of power-sum symmetric functions. We have:
\[
E(t)^\dagger = \exp \left( \sum_i \frac{(-1)^{i-1} p_i^\dagger}{i} t^i \right) = \exp \left( \sum_i (-1)^{i-1} \frac{\partial}{\partial p_i} t^i \right),
\]
\[
H(t) = \exp \left( \sum_i \frac{p_i}{i} t^i \right),
\]
\[
L = \exp \left( \sum_i \frac{(-1)^i p_i^2 + 2(-1)^{i/2} \theta_i p_i}{2i} \right).
\]
(Recall from Lemma 4.16 that $\theta_i$ is equal to 0 if $i$ is odd, and equal to 1 if $i$ is even.) We use an operator-theoretic version of Taylor’s theorem, namely

$$\exp\left(a \frac{\partial}{\partial x}\right) f(x) = f(x + a).$$

Applying this termwise to the composition of operators $E(t) \perp L$, we obtain:

$$E(t) \perp L
= \exp\left(\sum_i (-1)^{i-1} \frac{\partial}{\partial p_i} t_i\right) \exp\left(\sum_i \frac{(-1)^i p_i^2 + 2(-1)^{i/2} \theta_i p_i}{2i}\right)
= \exp\left(\sum_i \frac{(-1)^i p_i^2 + 2(-1)^{i/2} \theta_i (p_i + (-1)^{i-1} t_i)}{2i}\right)
\times \exp\left(\sum_i (-1)^{i-1} \frac{\partial}{\partial p_i} t_i\right)
= \exp\left(\sum_i \frac{(-1)^i p_i^2 - 2t_i p_i + (-1)^i t_i^2 + 2(-1)^{i/2} \theta_i (p_i + (-1)^{i-1} t_i)}{2i}\right)
\times E(t) \perp
= \exp\left(-\sum_i \frac{p_i t_i}{i}\right) \exp\left(\sum_i \frac{(-1)^i p_i^2 + 2(-1)^{i/2} \theta_i p_i}{2i}\right)
\times \exp\left(\sum_i \frac{(-1)^i t_i^2 + 2(-1)^{i/2} \theta_i (-1)^{i-1} t_i}{2i}\right) E(t) \perp.
$$

We recognise the first term as $H(t)^{-1}$, the second term as $L$, and the third term as 1 (noting that all powers of $t$ cancel out). Thus we have:

$$H(t)^{-1} LE(t) \perp = E(t) \perp L.$$

Therefore, Eq. (43) holds.

According to the fusion rule defined by Theorem 3.12, we have

$$\Phi(X_{1r}) \Phi'(X_{\mu}) = \Phi'(X_{1r}X_{\mu}), \ \forall \ r, \mu."
Furthermore,

\[
\Phi(X_\lambda) = \sum_\mu (-1)^{|\mu|} s_{\lambda/2\mu} \\
= \sum_{\mu, v \mid |\mu|+|v|=|\lambda|} (-1)^{|\mu|} e^\lambda_{2\mu,v} s_v \\
= \sum_{\mu, v \mid |\mu|+|v|=|\lambda|} (-1)^{|\mu|} e^\lambda_{2\mu,v} \Phi(Y_v).
\]

Recall that \(\Phi\) is an isomorphism, so

\[
X_\lambda = \sum_{\mu, v \mid |\mu|+|v|=|\lambda|} (-1)^{|\mu|} e^\lambda_{2\mu,v} Y_v.
\]

\[\Box\]

**Theorem 4.21.** For a Young diagram \(\lambda\), let us define \(\lambda_<\) to be set of proper sub Young diagrams \(\mu\), such that \(|\lambda| - |\mu| \in 2\mathbb{N}^+\). Then

\[
Y_\lambda = X_\lambda + \sum_{\mu \in \lambda_<} n_{\lambda,\mu} X_\mu,
\]

\[
\sum_\lambda n_{\lambda,\mu} s_\lambda = L^{-1} s_\mu = s_\mu \prod_{i \leq j} (1 + x_i x_j),
\]

\[
n_{\lambda,\mu} = \sum_{r \geq 0, v} b_{r,v} e^\lambda_{\mu,v}.
\]

**Proof.** We assume that

\[
Y_\lambda = \sum_\mu n_{\lambda,\mu} X_\mu, \quad \text{for some } n_{\lambda,\mu} \in \mathbb{N}.
\]

By Theorem 4.20,

\[
s_\lambda = \sum_\mu n_{\lambda,\mu} L^\dagger s_\mu.
\]

Then

\[
\langle L^{-1} s_v, s_\lambda \rangle = \sum_\mu n_{\lambda,\mu} \langle L^{-1} s_v, L^\dagger s_\mu \rangle = \sum_\mu n_{\lambda,\mu} \langle s_v, s_\mu \rangle = n_{\lambda,v}.
\]

By Eq. (40),

\[
\sum_\lambda n_{\lambda,\mu} s_\lambda = \sum_\lambda \langle L^{-1} s_\mu, s_\lambda \rangle s_\lambda = L^{-1} s_\mu = s_\mu \prod_{i \leq j} (1 + x_i x_j).
\]

Moreover,

\[
n_{\lambda,\mu} = \langle L^{-1} s_\mu, s_\lambda \rangle = \sum_{r \geq 0, v} b_{r,v} s_\mu s_v = \sum_{r \geq 0, v} b_{r,v} e^\lambda_{\mu,v}.
\]

\[\Box\]
Theorem 4.22. We have the following generating function for $\Phi(X_\lambda)$.

$$\sum_\lambda s_\lambda(x)\Phi(X_\lambda)(y) = \prod_{i_1 \leq i_2} \frac{1}{1 + x_i x_j} \prod_{i, j} \frac{1}{1 - x_i y_j}. $$

(Here $\Phi(X_\lambda)(y)$ means that the symmetric function $\Phi(X_\lambda)$ has variable set $\{y_j\}$).

Proof. Now we apply Theorem 4.20 to prove this theorem. We consider the first equation of Theorem 4.20 as a symmetric function variables $\{y_j\}$, and multiply by $s_\lambda(x)$. Summing over $\lambda$, we are required to show:

$$\sum_\lambda s_\lambda(x) \sum_\mu (-1)^{|\mu|} s_{\lambda/2\mu}(y) = \prod_{i_1 \leq i_2} \frac{1}{1 + x_i x_j} \prod_{i, j} \frac{1}{1 - x_i y_j}. $$

We now calculate:

$$\sum_\lambda s_\lambda(x) \sum_\mu (-1)^{|\mu|} s_{\lambda/2\mu}(y) = \sum_\lambda s_\lambda(x)L^\dagger(s_\lambda)(y)$$

$$= \sum_\lambda \sum_\rho \langle s_\rho, L^\dagger(s_\lambda) \rangle s_\lambda(x) s_\rho(y)$$

$$= \sum_\rho \sum_\lambda \langle L(s_\rho), s_\lambda \rangle s_\lambda(x) s_\rho(y)$$

$$= \prod_{i_1 \leq i_2} \frac{1}{1 + x_i x_j} \sum_\rho s_\rho(x) s_\rho(y)$$

$$= \prod_{i_1 \leq i_2} \frac{1}{1 + x_i x_j} \prod_{i, j} \frac{1}{1 - x_i y_j}. $$

$\Box$

Theorem 4.23. We have the following fusion rules:

$$R^\lambda_{\mu, \nu} = \sum_{\alpha, \beta, \gamma} c^\mu_{\alpha, \beta} c^\nu_{\beta', \gamma} c^\lambda_{\alpha, \gamma}. $$

(Here $\beta'$ is the Young diagram dual to $\beta$.)

Proof. Now we apply Theorem 4.22 to prove this theorem. To do this, we consider a suitable generating function for the $R^\lambda_{\mu, \nu}$, and express it in terms of two instances of the generating function in the second part of the theorem. We work with three variable sets: $\{x_i^{(1)}\}$, $\{x_i^{(2)}\}$, and $\{y_j\}$, and use Proposition 4.17.

$$\sum_{\mu, \nu, \lambda} R^\lambda_{\mu, \nu} s_\mu(x^{(1)}) s_\nu(x^{(2)}) \Phi(X_\lambda)(y)$$

$$= \sum_{\mu, \nu} s_\mu(x^{(1)}) s_\nu(x^{(2)}) \Phi(X_\mu)(y) \Phi(X_\nu)(y)$$
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Taking coefficient of $|\lambda|$, thus the equation in Theorem 4.23 is a finite sum.

At this point, we may take the coefficient of $\Phi(X_\lambda)(y)$ (these form a basis of $\Lambda$) to deduce

$$
\sum_{\mu, v} R^\lambda_{\mu, v} s_{\mu}(x^{(1)}) s_v(x^{(2)}) = 
\left( \sum_{\beta} s_{\beta}(x^{(1)}) s_{\beta'}(x^{(2)}) \right) s_\lambda(x^{(1)}, x^{(2)})
$$

$$
= \left( \sum_{\beta} s_{\beta}(x^{(1)}) s_{\beta'}(x^{(2)}) \right) \sum_{\mu, v} \sum_{\alpha, \gamma} c_{\alpha, \gamma} s_\alpha(x^{(1)}) s_\gamma(x^{(2)})
$$

Taking coefficient of $s_{\mu}(x^{(1)}) s_v(x^{(2)})$, we recover the formula for $R^\lambda_{\mu, v}$.

Note that if $|\alpha| = a$, $|\beta| = b$, $|\gamma| = c$, and

$$
c^\mu_{\alpha, \beta} c^v_{\beta', \gamma} c^\lambda_{\alpha, \gamma} \neq 0,
$$

then $|\mu| = a + b$, $|v| = b + c$, $|\lambda| = a + c$. Conversely, $a$, $b$, $c$ are determined by $|\mu|$, $|v|$, $|\lambda|$. Thus the equation in Theorem 4.23 is a finite sum.
We express $R_{\mu,v}^{\lambda}$ as fusion matrices for Young diagrams $\mu, v, \lambda$ with at most three cells ordered as $\{\emptyset, 1^2, 2, 1, 1^3, 3, [2, 1]\}$. Let $v$ be the column coordinate and $\lambda$ be the row coordinate.

\[
\begin{align*}
R_{\emptyset, v}^{\lambda} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
R_{1^2, v}^{\lambda} &= \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \\
R_{1^2, v}^{\lambda} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\end{align*}
\]

**Definition 4.24.** By Proposition 2.19, the dual object of $\tilde{y}_\gamma$ is $\overline{\tilde{y}_\gamma}$, which is equivalent to $\tilde{y}_\gamma'$. We denote $\cup_\beta$ to be a non-zero morphism in $\text{hom}_C(\tilde{y}_\beta \otimes \tilde{y}_\beta', \emptyset)$. For Young diagrams $\mu, v, \lambda, \alpha, \beta, \gamma$, with $|\mu| = a + b, |v| = b + c, |\lambda| = a + c, |\alpha| = a, |\beta| = b, |\gamma| = c$, we define the triangle map $\triangledown : \text{hom}_C(\tilde{y}_\mu, \tilde{y}_\alpha \otimes \tilde{y}_\beta') \otimes \text{hom}_C(\tilde{y}_\gamma, \tilde{y}_\alpha \otimes \tilde{y}_\beta \otimes \tilde{y}_\gamma') \otimes \text{hom}_C(\tilde{y}_\alpha \otimes \tilde{y}_\gamma, \tilde{y}_\lambda) \rightarrow \text{hom}_C(\tilde{y}_\mu \otimes \tilde{y}_v, \tilde{y}_\lambda)$ as
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\[ \nabla (\rho_1 \otimes \rho_2 \otimes \rho_3) = \rho_3 (\tilde{\gamma}_\mu \otimes \bigcup_\beta \tilde{\gamma}_\beta') (\rho_1 \otimes \rho_2) \]

\[ = \tilde{\gamma}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta' \tilde{\gamma}_\gamma \rho_1 \rho_2 \rho_3 \tilde{\gamma}_\lambda. \]

**Theorem 4.25.** The triangle map \( \nabla \) is injective and

\[
\text{hom}_\mathcal{C} (\tilde{\gamma}_\mu \otimes \tilde{\gamma}_\nu, \tilde{\gamma}_\lambda) = \bigoplus_{\alpha, \beta, \gamma} \nabla (\text{hom}_\mathcal{C} (\tilde{\gamma}_\mu, \tilde{\gamma}_\alpha \otimes \tilde{\gamma}_\beta') \otimes \text{hom}_\mathcal{C} (\tilde{\gamma}_\mu, \tilde{\gamma}_\beta \otimes \tilde{\gamma}_\gamma) \otimes \text{hom}_\mathcal{C} (\tilde{\gamma}_\alpha \otimes \tilde{\gamma}_\gamma, \tilde{\gamma}_\lambda)).
\]

**Proof.** Similarly to the proof of Lemma 3.10, we take

\[ \tilde{x}_{p_1, p_2, p_3} = \tilde{\gamma}_\lambda x_{p_1, p_2, p_3} (\tilde{\gamma}_\mu \otimes \tilde{\gamma}_v). \]

Then \( \text{hom}_\mathcal{C} (\tilde{\gamma}_\mu \otimes \tilde{\gamma}_v, \tilde{\gamma}_\lambda) \) is spanned by \( \{ \tilde{x}_{p_1, p_2, p_3} : p_1 \in P_a + b, p_2 \in P_b + c, p_3 \in P_{a+c} \} \). Note that the 180° rotation of \( s_b \) is \( s_b \). So

\[
\bigcup_{\alpha, \beta, \gamma} \nabla (\text{hom}_\mathcal{C} (\tilde{\gamma}_\mu, \tilde{\gamma}_\alpha \otimes \tilde{\gamma}_\beta') \otimes \text{hom}_\mathcal{C} (\tilde{\gamma}_\mu, \tilde{\gamma}_\beta \otimes \tilde{\gamma}_\gamma) \otimes \text{hom}_\mathcal{C} (\tilde{\gamma}_\alpha \otimes \tilde{\gamma}_\gamma, \tilde{\gamma}_\lambda))
\]

for some Young diagrams \( \alpha, \beta, \gamma \), with \( |\alpha| = a, |\beta| = b, |\gamma| = c \), and some morphisms \( \rho_1 \otimes \rho_2 \otimes \rho_3 \in \text{hom}_\mathcal{C} (\tilde{\gamma}_\mu, \tilde{\gamma}_\alpha \otimes \tilde{\gamma}_\beta') \otimes \text{hom}_\mathcal{C} (\tilde{\gamma}_\mu, \tilde{\gamma}_\beta \otimes \tilde{\gamma}_\gamma) \otimes \text{hom}_\mathcal{C} (\tilde{\gamma}_\alpha \otimes \tilde{\gamma}_\gamma, \tilde{\gamma}_\lambda) \). Therefore

\[
\bigcup_{\alpha, \beta, \gamma} \nabla (\text{hom}_\mathcal{C} (\tilde{\gamma}_\mu, \tilde{\gamma}_\alpha \otimes \tilde{\gamma}_\beta') \otimes \text{hom}_\mathcal{C} (\tilde{\gamma}_\mu, \tilde{\gamma}_\beta \otimes \tilde{\gamma}_\gamma) \otimes \text{hom}_\mathcal{C} (\tilde{\gamma}_\alpha \otimes \tilde{\gamma}_\gamma, \tilde{\gamma}_\lambda)).
\]
is a spanning set of $\text{hom}_H(\tilde{y}_\mu \otimes \tilde{y}_\nu, \tilde{y}_\lambda)$. By Proposition 2.17,  
\[ R^\lambda_{\mu, \nu} = \dim \text{hom}_H(\tilde{y}_\mu \otimes \tilde{y}_\nu, \tilde{y}_\lambda) \leq \sum_{\alpha, \beta, \gamma} \dim \text{hom}_H(\tilde{y}_\alpha \otimes \tilde{y}_\beta, \tilde{y}_\gamma) \times \dim \text{hom}_H(\tilde{y}_\mu \otimes \tilde{y}_\nu) \times \dim \text{hom}_H(\tilde{y}_\lambda) \]
\[ = \sum_{\alpha, \beta, \gamma} \dim \text{hom}_H(y_\mu \otimes y_\beta, y_\gamma) \times \dim \text{hom}_H(y_\mu \otimes y_\nu) \times \dim \text{hom}_H(y_\lambda) \]
\[ = \sum_{\alpha, \beta, \gamma} c^\mu_{\alpha, \beta} c^\nu_{\beta, \gamma} c^\lambda_{\alpha, \gamma}. \]

By Theorem 4.23, the equality holds. So the triangle map $\triangledown$ is injective and

\[ \text{hom}_H(\tilde{y}_\mu \otimes \tilde{y}_\nu, \tilde{y}_\lambda) = \bigoplus_{\alpha, \beta, \gamma} \triangle (\text{hom}_H(\tilde{y}_\mu \otimes \tilde{y}_\beta) \otimes \text{hom}_H(\tilde{y}_\mu \otimes \tilde{y}_\gamma) \otimes \text{hom}_H(\tilde{y}_\alpha \otimes \tilde{y}_\gamma, \tilde{y}_\lambda)). \]

\[ \square \]

**Remark 4.26.** Combining Theorem 4.25 and Proposition 2.17, we can construct a basis of $\text{hom}_H(\tilde{y}_\mu \otimes \tilde{y}_\nu, \tilde{y}_\lambda)$ using $S_n$ and a basis of the hom spaces $\text{hom}_H(\tilde{y}_\mu, y_\alpha \otimes y_\beta')$, $\text{hom}_H(y_\mu, y_\beta, y_\gamma)$, $\text{hom}_H(y_\alpha \otimes y_\gamma, y_\lambda)$ in the Hecke algebra $H$. Applying the evaluation algorithm of the Yang–Baxter relation, we obtain the $F$-symbols of $C$.

When the Young diagrams are small, the $F$-symbols can be computed by hand or by computer. We would not expect to compute $F$-symbols for large Young diagrams in this way, as the complexity of this algorithm grows exponentially w. r. t. the size of the Young diagrams. Even computing the $F$-symbols for $\text{Rep}(H(q))$ remains challenging.

We have computed the fusion rules for $C$. It was conjectured in [17] that the unitary quotient of $C$ is the centralizer algebra of the exceptional quantum subgroup of quantum $SU(N)_{N+2}$, because the planar algebra $C$ has a Hecke subalgebra $H(q, r)$, subject to $qr = i$. From this point of view, one can regard $C$ as the parameterization of a family of exceptional quantum subgroups.

This family of quantum subgroups were first constructed by Xu in [28] in 1998 through the $\alpha$-induction of the conformal inclusion $SU(N)_{N+2} \subseteq SU(N(N + 1)/2)_1$, $N \in \mathbb{N}$. Computing the fusion rules of the exceptional quantum subgroup remains an open question [27]. Böckenhauer, Evans and Kawahigashi developed the theory of $\alpha$-induction and they introduced a method to compute the NIM-rep fusion rules using modular invariance and quantum $F$ symbols in [6–8]. The computation is impractical when the rank is large.

Our method in this paper provides a different strategy of computing the fusion rule of quantum subgroups through their characters as an analogue of the Verlinde formula for quantum groups in [25]. The characters of simple objects of the unitary quotient of $C$ at roots of unity should be realized as the restriction of the characters of the simple objects of $C$ on certain spectrum. See [19] for an implementation of this idea on the fusion rule for quantum groups and its relation with the Verlinde formula.

It was conjectured in [17] that there is a continuous family of monoidal categories parameterizing the exceptional quantum subgroups from the $\alpha$-induction of any family
of conformal inclusions of quantum groups, or conformal pairs, of Lie type. Goddard, Nahm and Olive constructed such families of conformal pairs come from families of symmetric spaces of Lie types in [11]. It is worth mentioning that the key result about plethysm in Theorem 4.12 also works for other symmetric spaces of Lie types, see Chapter 1, Sect. 8 of [20]. We expect that our methods of computing fusion rules, characters and the generating function in this paper also apply to the other potential continuous families of monoidal categories parameterizing quantum subgroups of Lie types.

Acknowledgements. Zhengwei Liu was supported by Grant 100301004 from Tsinghua University and by Grant TRT 159 from Templeton Religion Trust and Grant 2020YFA0713000 from NKPs. Zhengwei Liu would like to thank Pavel Etingof and Feng Xu for helpful discussions and to thank Arthur Jaffe for the hospitality at Harvard University. Christopher Ryba would like to thank Pavel Etingof for useful conversations.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Communicated by Y. Kawahigashi