Asymptotic Behavior of Coarse-grained Models for Opinion Dynamics on Large Networks

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May 11, 2014

Abstract

In this paper, we propose a general mathematical framework to represent many multi-agent signalling systems in recent works. Our goal is to apply previous results in monotonicity to this class of systems and study their asymptotic behavior. Hence we introduce a suitable partial order for these systems and prove nontrivial extensions of previous results on monotonicity. We also derive a convenient sufficient condition for a signalling system to be monotone and test our condition on the Naming Games, NG and K-NG on complete networks both with and without committed agents. We also give a counter example which fails to satisfy our condition. Next we further extend our conclusions to systems on sparse random networks. Finally we discuss several meaningful consequences of monotonicity which narrows down the possible asymptotic behavior of signalling systems in mathematical sociology and network science.
1 Introduction

We introduce a mathematical framework comprehensive enough to represent many of the multi-agent signalling networks in recent works such as the generalized Naming Games (NG) [1] [6] [10] [5] and the Voter models used in modeling of ad-hoc wireless networks, agreement on a small list of tags in the WWW and social opinion dynamics in forums [3] and elections.

Our aim is to show that the asymptotic behavior of a proper subset of these signalling games on very large networks (where the effects of demographic noise can be safely neglected), can be described rigorously as a system of nonlinear coupled ODEs with the Monotonicity Property (MP) in phase-space consisting of the coarse-grained population fractions of each type. The nonempty complement of the monotone signalling systems include two main types, namely, (I) the signalling systems where the associated random walk models are Martingales, that is, the deterministic drifts are zero everywhere in phase space; in such systems, which includes all the Voter models, the asymptotic behavior is diffusion-driven [2], [3], and (II) signalling systems which have nonzero drift almost everywhere in phase space but whose nonlinear ODEs are not monotone; we will give such a counterexample below.

Previous works on Monotonicity include the works of [8] [9] which are mainly used in applications to population genetics and mathematical ecology [15]. In this paper, our main aim is to extend the applications of monotonicity to information sharing and opinion dynamics on large social networks of interacting (signalling) agents consisting of a finite, but possibly large, number of opinion or information types. The mathematical framework given in detail below includes all known multi-agent binary-signalling networks, that
is, where the signals are restricted to the binary symbols $A, B$. There are many ways to see why the binary-signals case represents the end-game in the final stages of signalling networks based on more than two symbols. Concrete evidence of this has been discussed in [4, 3] where expected times to multi-consensus for a symmetric multi-opinions Voter model was calculated in terms of the bottleneck times when exactly two opinions remain viable in the game [3].

In the process of working out the mathematical and socio-physical consequences [11] [16] of monotonicity [8], we will employ new versions of partial order that are explicitly designed for signalling networks. In particular, we focus on a new way to overcome a previous obstacle to the direct applications of Monotonicity results, namely, a mismatch in the cardinality of the set of local (edge or node) spin types and the number of independent (edge or node) population types. This mismatch arises specifically in attempts to apply the traditional Monotonicity results to signalling systems on large social (especially random) networks other than the complete graph. Along the way, we will also prove nontrivial extensions of the original monotonicity theorems in [8, 9].

2 Framework-General signalling system and its macrostate space

Consider a signalling network consisting of $N$ agents, each of which is assigned with a spin $s_i$ taking value from the spin state space $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_K\}$. The (micro)state of the system is fully described by the vector of spins $\vec{S} = (s_1, s_2, \ldots, s_N)$.

In each time step, a speaker and a listener are randomly selected. The
speaker sends a message containing 1 bit of information to the listener, and the listener changes its state according to the message. In this context, the message can be “A” or “B”. The probability for the selected speaker in state $\gamma_k$ sending a message “A” is $\alpha_k$. Let $\vec{\alpha} = (\alpha_1, ..., \alpha_K)^T$ ($\alpha_k \in [0,1]$).

When listener receive a message A (resp. B), the transition matrix of the listener’s state is $G_A$ (resp. $G_B$), the $i-j$th entry of which is $P(s_i \rightarrow s_j | A)$ for $G_A$. Both $G_A$ and $G_B$ are constant matrices and the signalling dynamics of the network is fully governed by $\vec{\alpha}$, $G_A$ and $G_B$.

A natural macrostate representation of this system is given by $\vec{n} = (n_1, ..., n_k, ..., n_K)^T$ where $n_k$ is the fraction of nodes in spin state $\gamma_k$ ($k = 1, ..., K$). The space of all possible macrostates is denoted by $M$, which is a simplex:

\[
\begin{cases}
  \sum_{i=1}^{d} n_i = 1 \\
  n_i \geq 0 \ (i = 1, ..., d)
\end{cases}
\]  

(1)

We refer to the vertices of $M$ as pure macrostates in which all nodes of the system stay in the same spin state. For later use, let $\sigma : \Gamma \rightarrow M$ map a spin state to its corresponding pure macrostate. We rewrite the macrostate as a linear combination of pure macrostates.

$$\vec{n} = (n_1, ..., n_k, ..., n_K) = \sum_{k=1}^{K} n_k \sigma(\gamma_k)$$

3 Partial order

We begin with $\{\Gamma, \prec\}$, the spin state space ordered by a partial order relation “$\prec$”, satisfying:

\[
\begin{cases}
  \text{reflexivity} : \gamma \prec \gamma \\
  \text{symmetry} : (\gamma \prec \gamma') \wedge (\gamma' \prec \gamma) \Rightarrow \gamma = \gamma' \ (\text{areequal}) \\
  \text{transitivity} : (\gamma_1 \prec \gamma_2) \wedge (\gamma_2 \prec \gamma_3) \Rightarrow \gamma_1 \prec \gamma_3
\end{cases}
\]  

(2)
In later sections we will discuss what further conditions this partial order should satisfy. We induce a partial order relation between the pure macrostates from \( \{\Gamma, \prec\} \) through \( \sigma: \Gamma \to M \):

\[
\sigma(\gamma) \prec \sigma(\gamma') \iff \gamma \prec \gamma'
\]

We use the same notation for the partial orders in both spaces with no ambiguity. Next we extend the partial order over \( \Gamma \) through affine combination. We require

\[
\vec{n}_1 \prec \vec{n}_1', \vec{n}_2 \prec \vec{n}_2' \Rightarrow L(\vec{n}_1, \vec{n}_2) \prec L(\vec{n}_1', \vec{n}_2').
\]

Here \( L(\vec{n}_1, \vec{n}_2) = a\vec{n}_1 + b\vec{n}_2 \) is an affine combination of \( \vec{n}_1, \vec{n}_2 \) which satisfies \( 0 \leq a, b \leq 1 \) and \( a + b = 1 \) so that \( L(\vec{n}_1, \vec{n}_2) \in M \) whenever \( \vec{n}_1, \vec{n}_2 \in M \). Now the extended partial order, \( \vec{n} \prec \vec{n}' \) if and only if \( \vec{n}' - \vec{n} \) can be represented as

\[
\vec{n}' - \vec{n} = \sum_i \lambda_i (\sigma(\gamma'_i) - \sigma(\gamma_i))
\]

The above summation is over all possible ordered pure macrostate pairs \( \sigma(\gamma_i) \prec \sigma(\gamma'_i) \) and \( \lambda_i \geq 0 \) for all \( i \). Note that some terms in the summation may be further decomposed into combination of other ordered pairs, so we consider the independent set

\[
B = \{ \sigma(\gamma'_i) - \sigma(\gamma_i) | \gamma_i \prec \gamma'_i, \text{ and } \not\exists \gamma'' \text{ st. } \gamma_i \prec \gamma'' \prec \gamma'_i \}.
\]

Let \( TM \) be the tangent space of macrostate space \( M \). Then \( C_+ \subset TM \) is the cone determined by the non-negatively linear combination of set \( B \). The definition of partial order here can be described as

\[
\vec{n} \prec \vec{n}' \iff \vec{n}' - \vec{n} \in C_+ \tag{3}
\]
If $|B|$, the cardinality of $B$, equals $\dim(TM)$, the dimension of $TM$, the partial order we described here can be considered as a standard type K partial order in Cartesian coordinate system ($x < y \Leftrightarrow y - x \in \mathbb{R}_+^n$) after suitable linear transformation [8]. However, it is possible that $|B| \neq \dim(TM)$ regarding some specific “≺” on $\Gamma$ (we will provide some examples later), therefore our partial order given by $C_+$ is a more general definition than the standard ones in the literature. It is clear that this more general form is needed for the applications in this paper to some well-known network games [4].

4 Macrostate dynamics of signalling system on complete networks

On a complete network, assume at time step $t$, the macrostate is $\vec{n}(t)$, then at time step $t + 1$, the expected change of macrostate is

$$E[\vec{n}(t + 1) - \vec{n}(t)] = \frac{1}{N} [pG_A + (1 - p)G_B - I] \vec{n}(t),$$

where $p$ is the overall probability for a message to be “A”, given by

$$p = \vec{\alpha}^T \vec{n}.$$

With standard time scaling $dt = 1/N$, we obtain the mean field equation for the evolution of signalling system:

$$\frac{d\vec{n}}{dt} = f(\vec{n}) = Q(\vec{n})\vec{n}(t) = [pG_A + (1 - p)G_B - I] \vec{n}(t).$$

Here $f$ maps $M$ to its tangent space $TM$. Denote $\phi_t$ as the semiflow which give the solution of the ODE, $\vec{n}(t) = \phi_t(\vec{n}_0)$. 
A system is said to be order-preserving or monotone, if for \( \forall \vec{n} < \vec{n}' \) and \( \forall t > 0 \), we have \( \phi_t(\vec{n}) < \phi_t(\vec{n}') \).

We now give a condition for monotonicity, which is an analogue of the Kamke condition [8]. Assume \( B = \{ \vec{e}_1, \ldots, \vec{e}_{|B|} \} \). \( f \) is said to be type C if for each \( k \in \{1, \ldots, |B|\} \), for \( \forall \vec{n} < \vec{n}' \) such that \( \vec{n}' - \vec{n} = \sum_{i \neq k} a_i \vec{e}_i \) (\( a_i \geq 0 \)), there exists a representation of \( f(\vec{n}') - f(\vec{n}) = \sum_{i=1}^{|B|} b_i \vec{e}_i \) in which \( b_k \geq 0 \). Note that when \( |B| \) is greater than \( dim(TM) \), the representation here may not be unique, but when \( |B| < dim(TM) \), the representations may not exist regardless the sign of the coefficients.

**Proposition 1** (type C condition): The system \( \frac{d\vec{n}}{dt} = f(\vec{n}) \) is monotone if and only if it is type C.

The proof is straightforward by contradiction and very similar to that of the Kamke condition. If the type C condition does not hold, then by continuity of \( f \), there exists an \( \epsilon > 0 \) such that \( \phi_{\epsilon}(\vec{n}) \succ \phi_{\epsilon}(\vec{n}') \) which violates the monotone property.

As the Kamke condition [19, 8] can be expressed in terms of partial derivatives, the type C condition has an expression in terms of directional derivatives.

**Proposition 2:** The system \( \frac{d\vec{n}}{dt} = f(\vec{n}) \) is monotone if and only if (A): for \( \forall \vec{n} \in Int(M) \) and \( \forall k \in \{1, \ldots, |B|\} \), there exists a representation \( \frac{d}{dt} f(\vec{n} + \epsilon_k \vec{e}_k) = \sum_{i=1}^{|B|} b_i \vec{e}_i \) s.t. for \( \forall i \neq k \), \( b_i \geq 0 \).

**Proof:** For \( \forall \vec{n} < \vec{n}' \) such that \( \vec{n}' - \vec{n} = \sum_{i \neq k} a_i \vec{e}_i \) (\( a_i \geq 0 \)) and the representation of \( f(\vec{n}') - f(\vec{n}) = \sum_{i=1}^{|B|} b_i (\vec{n}) \vec{e}_i \) holds with the sign of \( b_i \) undecided,

\[
f(\vec{n}') - f(\vec{n}) = \int_0^1 \frac{d}{d\lambda} f(\vec{n} + \lambda \sum_{i \neq k} a_i \vec{e}_i)d\lambda
\]  

(4)
\begin{align}
\sum_{j \neq k} \int_0^1 a_i \frac{d}{d\epsilon} \tilde{f}(\bar{n} + \lambda \sum_{i \neq k} a_i \bar{e}_i + \epsilon \bar{e}_j) d\lambda
\end{align}

(a) (A) ⇒ type C: If condition (A) holds, then 
\[
\frac{d}{d\epsilon} \tilde{f}(\bar{n} + \sum_{i \neq k} a_i \bar{e}_i + \epsilon \bar{e}_j) = \sum_{i=1}^{[B]} b_i^{(j)}(\lambda) \bar{e}_i
\]
such that for \( \forall i \neq j, b_i \geq 0 \). We point-wisely replace the directional derivative by its linear representation, therefore
\[
f(\bar{n}') - f(\bar{n}) = \sum_{j \neq k} \int_0^1 a_i \sum_{i=1}^{[B]} b_i^{(j)}(\lambda) \bar{e}_i d\lambda.
\]

Summing all the coefficients before \( \bar{e}_k \) we get 
\[
\sum_{j \neq k} a_k \int_0^1 b_k^{(j)}(\lambda) d\lambda \geq 0,
\]
so \( f \) is type C.

(b) Type C ⇒ (A): Suppose for some \( \bar{n}_0 \in Int(M) \), 
\[
\frac{d}{d\epsilon} f(\bar{n}_0 + \epsilon \bar{e}_j) = \sum_{i=1}^{[B]} b_i \bar{e}_i, \quad b_k < 0 \quad (k \neq j).
\]
By continuity of \( f \), there exists a small enough \( \delta_0 > 0 \) such that \( b_k < 0 \) also holds for \( \bar{n} = \bar{n}_0 + \epsilon \bar{e}_j \) when \( 0 < \delta \leq \delta_0 \).
Applying Eq. (6) \[
f(\bar{n}') - f(\bar{n}) = \delta_0 \sum_{i=1}^{[B]} \left( \int_0^1 b_i^{(j)}(\lambda) d\lambda \right) \bar{e}_i.
\]
Since \( b_k^{(j)}(\lambda) < 0 \) for \( 0 \leq \lambda \leq 1 \), the coefficient before \( \bar{e}_k \) is \( \delta_0 \sum_{i=1}^{[B]} \left( \int_0^1 b_i^{(j)}(\lambda) d\lambda \right) < 0 \) contradicting the type C condition. QED

For the signalling system on a complete network, we have 
\[
f(\bar{n}) = [pG_A + (1 - p)G_B - I] \bar{n}(t),
\]
thus,
\[
\frac{d}{d\epsilon} f(\bar{n} + \epsilon \bar{e}_k) = (pG_A + (1 - p)G_B - I) \bar{e}_k + \left[ \frac{dp(\bar{n} + \epsilon \bar{e}_k)}{d\epsilon} \left( \frac{dp}{d\epsilon} (pG_A + (1 - p)G_B)\bar{n} \right) \right]
\]
\[
= pG_A \bar{e}_k + (1 - p)G_B \bar{e}_k - \bar{e}_k + (\bar{e}_k^T \bar{e}_k)(G_A - G_B)\bar{n}
\]

Now we derive a sufficient conditions for condition (A). According to Proposition 2, only \( b_i \) \( (i \neq k) \) affects the monotonicity, so the third term in the last expression \( -\bar{e}_k \) can be neglected. If we require the other three terms be positive in terms of \( \prec \), we obtain the following theorem.
Theorem 1: A signalling dynamics on complete network governed by \( \vec{\alpha}, G_A, G_B \) is monotone if it satisfies

\[
\begin{align*}
(a)&: \forall \gamma \in \Gamma, \ G_B \sigma(\gamma) \prec G_A \sigma(\gamma) \\
(b)&: \gamma \prec \gamma' \Rightarrow \vec{\alpha}^T \sigma(\gamma) < \vec{\alpha}^T \sigma(\gamma') \\
(c)&: \gamma \prec \gamma' \Rightarrow G_A \sigma(\gamma) < G_A \sigma(\gamma'), G_B \sigma(\gamma) < G_B \sigma(\gamma')
\end{align*}
\]

This theorem gives a convenient sufficient condition on the essential components of the signalling system in order for it to be monotone. In this theorem, (a) fixes the preferred (greater) one between A and B to orient the partial order, that is, the listener will switch to a greater state according to this partial order when receiving A than when receiving B. Condition (b) says the speaker in a greater state has more probability to send a message A. One can switch the roles of A, B in the conditions (a), (b). Condition (c) says \( G_A \) and \( G_B \) preserve the partial order. In other words, we can now determine directly whether a signalling system is monotone according to the properties of its three governing elements \( \vec{\alpha}, G_A, G_B \).

For particular applications of these sufficient conditions to social networks of signalling agents, we can imbue the partial order with a moral value system or a utility function [11].

5 Examples of binary signalling system on complete graph

5.1 Binary Listener-only Naming Game (LO-NG) with and without committed agents

In this case [1], [6], the spin states \( \Gamma = \{ A, AB, B \} \), and a macrostate is given by the corresponding populations \( \vec{n} = (n_A, n_{AB}, n_B)^T \). Here, the governing
elements are given by the following vector which fixes the probabilities of
sending the symbol $A$ when in the the associated sin states and a pair of
transition matrices, which define the transition probabilities upon receiving
a $A$ (resp. $B$) symbol:

$$\vec{\alpha} = (1, 1/2, 0)^T,$$

$$G_A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, G_B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$  

Thus, the governing elements of a signalling system of networked agents have
two natural parts, namely, (I) the probabilities for sending a symbol, and
(II) the transition probabilities to the next local state (opinion) on receiving
a symbol.

After choosing the opinion $A$ to be ”morally superior” say, and inducing
from the implicit order in $\vec{\alpha}$, the suitable partial order over $\Gamma$ that satis-
fies condition (b) in the main theorem 1, should be $B \prec AB \prec A$ which .
Therefore, the independent subset of pairs, identified in the general mathem-
atical framework given above, is $B = \{\sigma(A) - \sigma(AB), \sigma(AB) - \sigma(B)\} =
\{(1, -1, 0), (0, 1, -1)\}$, where the second equality should obviously mean that
the negative ones denote the second element / term of the differences in the
independent set $B$, that is, the last set of 3-vectors are incidence vectors
for the differences in $B$ viewed as directed edges. Thus, this partial order
can be represented by a directed graph, in which $\Gamma$ gives the vertices and $B$
gives the directed links, as in the figure below.

Next we check the conditions (a) and (c) in Theorem 1. In Table I comparing
two rows, we verify (a) $G_B(\sigma(X)) \leq G_A(\sigma(X)) (X \in \{A, AB, B\})$;
comparing three columns, we verify (c) $G_X(\sigma(B)) \prec G_X(\sigma(AB)) \prec G_X(\sigma(A))$
$(X \in \{A, B\})$.  

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In the case with committed agents both on A and B, \( \Gamma = \{C_A, A, AB, B, C_B\} \); \( C_A \) and \( C_B \) denote the spin state committed in A and in B respectively and the populations macrostate is given by \( \vec{n} = (n_{C_A}, n_A, n_{AB}, n_B, n_{C_B})^T \). The governing elements in this case are

\[
\vec{\alpha}_c = (1, 1, 1/2, 0, 0)^T,
\]

\[
G_{A,c} = \begin{pmatrix} 1 & G_A \\ G_A & 1 \end{pmatrix}, \quad G_{B,c} = \begin{pmatrix} 1 & G_B \\ G_B & 1 \end{pmatrix}.
\]

The suitable partial order is shown in the directed graph below (just add two disconnected points to that of the non-committed or symmetric LO-NG case). Committed agents never changes its state, so \( G_A(\sigma X) = G_B(\sigma X) = \sigma X \ (X \in \{C_A, C_B\}) \) which therefore satisfy condition (a) in Theorem 1. Conditions (b) and (c) only affect the pairs ordered by the partial order; since the committed case does not introduce any new ordered pairs w.r.t. the non-committed case, and \( \vec{\alpha}_c, G_{A,c}, G_{B,c} \) restricted to the non-committed pure macrostates are exactly \( \vec{\alpha}, G_A, G_B \), hence conditions (b) and (c) n Theorem 1 are satisfied.
From here, we can easily get the following corollary which has applications to the scenario in [13] and [14]:

**Corollary:** If a signalling system without committed agents is monotone, adding committed agents into this signalling system does not change the monotonicity.

**Figure 2:** Partial order of NG with committed agents

5.2 K-NG

In the one-parameter family of listener-only Naming Games, K-NG [10], where $K$ presents the stubbornness of agents to full conversion from $B$ to the $A$ opinion, there are $K+1$ spin states $\Gamma = \{0, 1, ..., K\}$. As in the original LO-NG case which corresponds to the value $K = 2$ in the $K-NG$ family of models, we firstly find a suitable partial order satisfying condition (b) in Theorem 1, that agrees with the given probability vector for sending the $A$ symbol, $\bar{\alpha} = (0, 1/K, ..., i/K, ..., 1)^T$. This partial order is shown in the Fig. 3. Next we find the independent subset $B = \{\sigma(k + 1) - \sigma(k) | k = 0, ..., K - 1\}$.

**Figure 3:** Partial order of K-NG

Since the $K-NG$ family of NG models have the following property:

$$\sigma(max(k-1,0)) \prec G_B(\sigma(k)) \prec \sigma(k) \prec G_A(\sigma(k)) \prec \sigma(max(k+1,K)),$$
conditions (a) and (c) in Theorem 1 follow easily.

5.3 Counter-example which is not monotone

Next we provide an explicit example of a binary signalling system which is not monotone, thus establishing that the above definitions in our mathematical framework has non-vacuous complement. This example of a signalling system on two symbols has the same spin state space $\Gamma$ and macrostate representation as the LO-NG model discussed previously. However, its governing elements (the probabilities for sending $A$ and the spin transition probabilities upon receiving the symbols $A$ or $B$) differ from the LO-NG in the matrix $G_B$:

$$\vec{\alpha} = (1,1/2,0)^T,$$

$$G_A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We claim there does not exist a non-trivial partial order in this example, where by nontrivial we mean that the partial order has a nonempty independent subset $B \neq \emptyset$. The proof of this claim follows:

Proof: According to condition (b) in Theorem 1, there are at most three possible elements in $B$ for this example, i.e. $B \subset \{\vec{e}_1 = \sigma(A) - \sigma(B), \vec{e}_2 = \sigma(AB) - \sigma(B), \vec{e}_3 = \sigma(A) - \sigma(AB)\}$. Since $G_A\vec{e}_1 = \vec{e}_3, G_A\vec{e}_2 = \vec{e}_3$, by (c) in Theorem 1, $\vec{e}_1 \in B \Rightarrow \vec{e}_3 \in B$ and $\vec{e}_2 \in B \Rightarrow \vec{e}_3 \in B$. If $B \neq \emptyset$, then $\vec{e}_3 \in B$. However, $G_B\vec{e}_3 = \sigma(B) - \sigma(AB)$, by (c) again, $\sigma(AB) \prec \sigma(B)$ which contradicts condition (b) in Theorem 1. QED

The solution trajectories of this system mapped into 2D space $(n_A, n_B)$ is shown below. This figure gives us a concrete idea of one type of non-
monotone signalling dynamics on two symbols.

Figure 4: Trajectories of counter example

6 On sparse random networks

Under a homogeneous pairwise mean field assumption, the coarse-grained dynamics of the binary Naming Game on a random sparse network with average degree \(< k >\) is governed by the 6-dimensional ODEs [12] [17]:

\[
\frac{d\vec{l}}{dt} = 2 \left[ \frac{1}{< k >} D + (\frac{< k > - 1}{< k >}) R \right] \vec{l}.
\]

(8)

for macrostate \(\vec{l} = [l_{A-A}, l_{A-B}, l_{A-AB}, l_{B-B}, l_{B-AB}, l_{AB-AB}]^T\) of link-types population fractions. For a general signalling system of the above type, we can obtain a similar ODE system using exactly the same approach as in [17]. The type of a link in this ODEs is ultimately given by the opinions or node-spins of its two ends \((\gamma_i - \gamma_j)\) regardless of the order but it is more convenient to work with link-based macrostates; a transformation between the link-based macrostate and the node-based macrostate \(\vec{n}\) is given in [17].
Changes of $\vec{l}$ come from two parts: the direct change and the related change. In each time step, a realized listener-speaker pair of agents and therefore a link or edge in the underlying random graph is selected. Then

**Direct change** is the change of the selected link and is given by $D\vec{l}$ where $D$ is the probability transition matrix of the selected link - the $(i, j)$ entry of $D$, $D_{ij}$, is the probability that a link of type $j$ changes into type $i$ in one step given that the selected link is of type $j$. $D$ is a constant matrix given by $\vec{\alpha}$, $G_A$, $G_B$ and the way of selecting listeners and speakers.

Consider a link type $\gamma_1 - \gamma_2$ and its corresponding pure macrostate $\sigma(\gamma_1 - \gamma_2)$ which, according to the general mathematical framework in this paper, are now the basic elements on which a partial order is defined. Then according to this theory and the above 6-dimensional ODEs (with effectively a 5-dim tangent space after reduction by one degrees of freedom),

$$D\sigma(\gamma_1 - \gamma_2) = P(\gamma_1 \text{ is listener})[p_1 \sigma(G_A \gamma_1 - \gamma_2) + (1 - p_1)\sigma(G_B \gamma_1 - \gamma_2)] + P(\gamma_2 \text{ is listener})[p_2 \sigma(\gamma_1 - G_A \gamma_2) + (1 - p_2)\sigma(\gamma_1 - G_B \gamma_2)],$$

where $p_1 = \vec{\alpha}^T \sigma(\gamma_2)$, $p_2 = p_1 = \vec{\alpha}^T \sigma(\gamma_1)$. The notations $\sigma(G_A \gamma_1 - \gamma_2)$ is defined by

$$\sigma(G_A \gamma_1 - \gamma_2) = \sum_{i=1}^{K} P(G_A \sigma(\gamma_1) = \sigma(\gamma_i))\sigma(\gamma_i - \gamma_2)$$

The related links are those incident at the listener other than the selected link. Then the **Related change** is the change of the related links when the listener changes and is given by $(< k > - 1)R(\vec{l})\vec{l}$, where $< k > - 1$ is the expected number of related links. $R$ is a probability transition matrix that varies according to the current macrostate. For Naming Game, $R$ is given explicitly in [17]. For general signalling systems, $R$ is given in detail later.
A natural partial order of link-based macrostates is induced from that of node-based macrostates: the link states

\[ X - Y \prec X' - Y' \iff X \prec X' \text{ and } Y \prec Y'. \]

The independent set of ordered pairs \( B \) is thus

\[ B = \{ \sigma(X - Y') - \sigma(X - Y) | Y \prec Y' \}. \]

For example, the partial order and the set \( B \) for NG on a sparse random network is shown in the following figure. Note that it is an example in which \( |B| = 6 \geq \dim(TM) = 5 \).

Figure 5: Partial order of NG on random sparse networks

The partial order in the link-based macrostate space is stronger than that in the node-based macrostate space, i.e. if two node-based macrostates are \( \vec{n}, \vec{n}' \) and their respective linked based macrostates \( \vec{l}, \vec{l}' \), then \( \vec{l} \prec \vec{l}' \Rightarrow \vec{n} \prec \vec{n}' \) and the reverse does not hold.

In Section 4, we proved a sufficient condition for a signalling dynamics to be monotone on a complete network graph. We will show in the following
theorem, that the signalling dynamics satisfying this condition will also be monotone on a random sparse network.

**Theorem 2:** If a signalling dynamics satisfies the conditions in Theorem 1, then it is monotone on a random sparse network.

**Proof:** Firstly, we show that given the conditions in (6), the direct change part $D\vec{e}_k$ satisfies condition (a). For each $\vec{e}_k = \sigma(X - Y') - \sigma(X - Y)$,

$$D\vec{e}_k = D\sigma(X - Y') - D\sigma(X - Y)$$

$$= P(\text{X is listener})[\vec{\alpha}^T\sigma(Y)\left(\sigma(G_A X - Y') - \sigma(G_A X - Y)\right)$$

$$+ (1 - \vec{\alpha}^T\sigma(Y))\left(\sigma(G_B X - Y') - \sigma(G_B X - Y)\right)$$

$$+ (\vec{\alpha}^T\sigma(Y') - \vec{\alpha}^T\sigma(Y))\left(\sigma(G_A X - Y') - \sigma(G_B X - Y')\right)]$$

$$+ P(Y \text{ or } Y' \text{ is listener})[\vec{\alpha}^T\sigma(X)\left(\sigma(X - G_A Y') - \sigma(X - G_A Y)\right)$$

$$+ (1 - \vec{\alpha}^T\sigma(X))\left(\sigma(X - G_B Y') - \sigma(X - G_B Y)\right)]$$

By the definition of link-based partial order, $\sigma(G_A X - Y') - \sigma(G_A X - Y) \succ 0$, $\sigma(G_B X - Y') - \sigma(G_B X - Y) \succ 0$. According to condition (6), $\vec{\alpha}^T\sigma(Y') - \vec{\alpha}^T\sigma(Y) > 0$, $\sigma(X - G_A Y') - \sigma(X - G_A Y) \succ 0$, $\sigma(X - G_B Y') - \sigma(X - G_B Y) \succ 0$. Therefore $D\vec{e}_k$ preserve the order of $l$

For the related change part, $R\vec{e}_k$, we represent $\vec{e}_k$ as a suitably weighted symmetric adjacency matrix $M = M(\vec{l})$ labeled by the node-based spin types,

$$M_{ii} = l_{\gamma_i - \gamma_i},$$

$$M_{ij} = \frac{1}{2}l_{\gamma_i - \gamma_j} (i \neq j).$$

where the row sum and column sum of $M(\vec{l})$ are the node-based macrostate, $\vec{n}$. It is obvious that, $M(\vec{l})$ is a $1 - 1$ presentation for $\vec{l}$. $R(\vec{l})$ is given by the following

$$M(R(\vec{l})\vec{e}_k) = W(\vec{l})M(\vec{l})W(\vec{l})^T,$$
where $W(\vec{l})$ is the transition matrix of spin states, i.e. the entry of $W(\vec{l})$, $W_{ij}$ is the probability that a node in spin state $\gamma_j$ changes into $\gamma_i$ given the link-based macrostate $\vec{l} = (l_1, ..., l_k, ..., l_K)$,

$$W_{ij} = \sum_{k=1}^{K} P(\gamma_j \rightarrow \gamma_i| \text{link type } k) l_k.$$ 

Then we prove the following lemma.

**Lemma 1:** There exists a unique decomposition of $M(\vec{l})$, $M(\vec{l}) = u + v$. $u \in U = \{u|u = u^T, \exists \text{column vector } \vec{m} \text{ s.t. } u = \vec{m} \otimes \vec{m}^T\}$, $\otimes$ is the kronecker product. $v \in V = \{v|v = v^T, v\vec{1} = \vec{0}, \vec{1}^Tv = 0\}$, where $\vec{1}$ is a column vector with all 1 entries.

**Proof:** Taking $\vec{m} = \vec{n}$ the node-based population fractions, $u = \vec{n} \otimes \vec{n}^T$ is symmetric by construction. The row sum and column sum of $u$ are both $\vec{n}$, the same as $M(\vec{l})$. Therefore the row sum and column sum of $v$ are 0. As $M(\vec{l})$ and $u$ are symmetric, so is $v$.

**Lemma 2:** $U$ and $V$ are invariant space of the operator $R(\vec{l})$ for any macrostate $\vec{l}$.

**Proof:** $\forall u = \vec{m}^T \otimes \vec{m} \in U$, $WuW^T = (W\vec{m}) \otimes (W\vec{m})^T \in U$. $\forall v \in V$, $v\vec{1} = 0$, since $W^T\vec{1} = \vec{1}$, $WvW^T\vec{1} = \vec{1}^TWvW^T = 0$, therefore $WvW^T \in V$.

According to Lemma 1 and 2, $WM(\vec{l})W^T$ restricted on $U$ is just $(Q\vec{n}) \otimes (Q(\vec{n}))^T$, where $Q$ is the transition matrix of the corresponding dynamics on complete network, $\vec{n} = M(\vec{l})\vec{1}$ is the corresponding node-based macrostate. Besides, $WM(\vec{l})W^T$ restricted on $V$ has zero effect on the dynamics of $\vec{n}$. So $WM(\vec{l})W^T$ preserve the partial order of $\vec{n}$. This completes the proof of theorem 2.
7 Application

The many significant consequences of monotone dynamical system, especially low-dimensional ones have been explored in [18, 8, 9]. In particular this substantial reduction in complexity of the phase trajectories and the organization of the phase space into hyperbolic equilibria and the heteroclinic orbits that connect them have clear implications for mathematical sociology and network science.

For application of monotonicity, one additional property approximate from below (above) is important.

Definition: $x$ can be approximated from below (above), if there is a sequence $\{x_n\}$ satisfying $x_n \prec x_{n+1} \prec x$ ($x_n \succ x_{n+1} \succ x$) for $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow \infty$.

With the partial order for signalling system we discussed in this paper, every point $x$ in macrostate space can be approximated from below and above except for the two consensus state, since the approximating sequence is given by $x + \epsilon_n \vec{e}_i$ with $\vec{e}_i \succ 0$ and $\epsilon_n \rightarrow 0_-(0_+)$. Besides, the macrostate space is a simplex therefore is finite-dimensional, convex and compact.

Considering the properties of signalling system mentioned above, the discussion in [8] implies the following relevant consequences:

**Theorem 3 - Global asymptotic stability** [8, 9]: If a monotone signalling system contains exactly one equilibrium $e$, then every initial macrostate converge to $e$.

**Theorem 4 - Tipping point** [8, 9]: If a monotone signalling system has two equilibria $x \prec y$, $[x, y]$ denotes the set of all the points $z$ that $x \prec z \prec y$, then one of the following holds:

1) $y$ is stable, every point except $x$ converges to $y$. 
2) $x$ is stable, every point except $y$ converges to $x$.

3) $x, y$ are both stable, and there exists another equilibrium $z \in [x, y]$, $z \neq x, y$.

According to the above two theorems, the global convergence of a monotone signalling system is simply decided by the equilibria. Theorem 3 guarantees the global convergence without knowing the stability of the equilibrium. Theorem 4 is especially relevant to the “tipping point” phenomenon found in 2-word Naming Game \cite{6,17,1}. It predicts all possible global structures of the 2-word NG dynamics from only their monotonicity property instead of detailed inter-agent rules. Therefore the previous results obtained for the NG \cite{6,17,1,5} can now be qualitatively generalized to any monotone binary-signalling systems with two consensus states. If the signalling system contains more than two ordered stable equilibria, say $x < y < z$, then Theorem 4 can be applied on the domains (called attracting basins) $[x, y]$ and $[y, z]$ separately.

**Theorem 5 - Convergence** \cite{8,9} For a monotone signalling system, $M = \text{Int}(C)$. Here $M$ is the macrostate space, $C$ is the set of points that will eventually converge. In another word, $\text{Int}(\overline{C}) = \emptyset$, the set of points that do not converge to anywhere is a nowhere dense set.

Considering that the signalling systems in the real world always contain some noise, it is impossible for the trajectory to stay inside a nowhere dense set even with infinitely small noise, therefore a monotone signalling system starting from any initial state will eventually go to a stable equilibrium state.

Taken together, these theorems on Monotonicity and the new results in this paper are applicable to many of the convergence, coherence or synchrony questions that arise in mathematical biology and ecology \cite{15}.
Acknowledgement

This work was supported in part by the Army Research Office Grant No. W911NF-09-1-0254 and W911NF-12-1-467054. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Office or the U.S. Government.
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