$B \to \rho \bar{\nu}_\ell \bar{\nu}_\ell$ Form Factors

Damir Becirevic

Laboratoire de Physique Théorique et Hautes Energies
Université de Paris XI, Bâtiment 211, 91405 Orsay Cedex, France

ABSTRACT

The bounds on the form factors for $B \to \rho \bar{\nu}_\ell \bar{\nu}_\ell$ decay are studied. Constrained by lattice data and a constrained conformal mapping, the more informations can be obtained for $A_1(q^2)$ form-factor which dominates the decay rate at large $q^2$. Specifically, we confirm a moderately increasing behavior of this form factor.

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\[\text{damir@qcd.th.u-psud.fr}\]
1 Introduction

In the Standard Model, the couplings of quarks and $W$-boson are given in the form of Cabibbo-Kobayashi-Maskawa ($CKM$) matrix elements. In this picture, quark mixing and $CP$ - violation are closely related to each other. In order to check whether this picture really describes $CP$ - violation and, more generally, whether the Standard Model gives a proper description of weak decays of hadrons, one must determine precise values of the $CKM$-matrix elements - $|V_{ij}|$. One of the most poorly known among them is $|V_{ub}|$. Naturally it is to be extracted from experimental data for the semileptonic $b \rightarrow u$ transition. Experimentally, these processes are very difficult to measure, because of the dominant $b \rightarrow c$ transition. Thus, $b \rightarrow u$ are clearly observable in a small fraction of the phase space, i.e. beyond the lepton momentum spectrum for $b \rightarrow c$. Theoretical description of the inclusive $B \rightarrow X_u \ell \bar{\nu}_\ell$ decay is tremendously difficult and suffers from large uncertainties. On the other side, there is a general hope that $|V_{ub}|$ can be extracted from the exclusive channels, particularly from $B \rightarrow \rho \ell \bar{\nu}_\ell$. For reliable extraction of $|V_{ub}|$, we must have precise experimental data on branching ratios as well as a hadronic matrix element accurately calculated from $QCD$. Recently, CLEO collaboration [1, 2] observed and measured the branching ratios for this decay. The reported values are:

$$Br(\bar{B}^0 \rightarrow \rho^+ \ell^- \bar{\nu}_\ell) = \begin{cases} 
(3.88 \pm 0.54 \pm 1.01) \times 10^{-4} & WSB \\
(2.28 \pm 0.36 \pm 0.59) \times 10^{-4} & ISGW 
\end{cases}$$

(1)

This branching ratio depend upon theoretical input for the efficiency calculation. They used the quark model predictions ($WSB$ [11] and $ISGW$ [12]). We see that differences between two results are due to the models they used, i.e. theoretical uncertainties are important source of error. Theoretically, the problem is how to calculate the hadronic matrix element and the corresponding weak transition amplitude since it receives hard non-perturbative (low-energy) $QCD$ contributions. Unfortunately, up to now there is no theoretical tool by which one can calculate the corresponding form factors exactly in the whole physical region for this decay.

Most of the problems with quark model calculations are related to a lack of a fully relativistic treatement of quark spins. In fact, they calculate a specific form factor at one physical point and then assume the functional dependence on $q^2$ in the whole physical region. The most popular assumption is the polelike one. A step forward in calculating form factors was the chiral perturbation theory of heavy hadrons, where the chiral and heavy quark symmetries were used to construct a phenomenological lagrangian [26, 27]. However, only the small range of $q^2$ close to $q^2_{max}$ can be covered, and again the ansatz on functional dependence of form factors must be adopted. Different versions of the $QCD$ sum rules were employed as well. These calculations allow to conclude about the functional dependence, except when close to $q^2_{max}$. Still, the various $QSR$ give different results. Finally, there are lattice $QCD$ simulations which, along with the $QCD$ sum rules, are the only methods to treat non-perturbative $QCD$ in a consistent way (the nice review of the heavy mesons phenomenology from lattice $QCD$ can be found in Ref.[31]). But, due to the $UV$-cutoff, the relevant matrix elements are calculated for $m_Q \sim m_c$ and $q^2 \lesssim m^2_c$. $HQET$ is then used to control the extrapolation to $m_b$ and
consequently the values of accessible $q^2$ are restricted to a small range close to $q^2_{\text{max}}$. However, this is the phase-space region which is expected to contribute to a substantial fraction of the decay events and is above the endpoint for charm production in the decay. For branching ratio, we need the values of the hadronic matrix element at small $q^2$ too. Then the corresponding form factors must be extrapolated to this region. To perform this, we need an extrapolation law which is always the hypothesis we make. The form factors values after extrapolation to small $q^2$ are extrapolation-law dependent. In this respect, the situation is not very different from the quark-model approach. Still by lattice results we may constrain the unitarity bounds which is the subject of this letter, and hence try to reduce the possible choice on the scaling laws for extrapolation to small $q^2$.

The authors of Ref. [16] proposed another model-independent way to extrapolate $|V_{ub}|$ using the double Grinstein type ratio and more specifically the following decay modes: $B \rightarrow \rho \tilde{\nu}_\ell$, $B \rightarrow K^* \nu \tilde{\nu}$, $D \rightarrow \rho \nu \tilde{\nu}$ and $D \rightarrow K^* \tilde{\nu}_\ell$. However, the rare decays $B \rightarrow K^* \nu \tilde{\nu}$ and $D \rightarrow K^* \tilde{\nu}_\ell$ have not been observed and there is much to be learned on form factors.

In Sec.2 of this paper, we give the necessary definitions and motivations for this analysis. In Sec.3 we generate the unitarity bounds on the form factors which are constrained by lattice results in Sec.4. In Sec.5 we examine the functional dependence $A_1(q^2)$ by additionally constrained bounds. Concluding remarks are given in Sec.5 and 6.

## 2 Form factors

The hadronic matrix element for the $B \rightarrow \rho \tilde{\nu}_\ell$ decay is parametrised as:

$$
< \rho(p')|J^\mu|B(p)> = \frac{2V(q^2)}{M+m} q^{\mu\nu\alpha\beta} p'_\alpha p'_\beta \epsilon_{\epsilon r} - i(M+m)A_1(q^2)\epsilon^*\epsilon_r + i A_2(q^2) (p+p')^{\mu} \epsilon^*\epsilon_r - i \frac{2m}{q^2} (\epsilon^*p) A(q^2)(p-p')^{\mu}
$$

(2)

where $J^\mu = (V^\mu - A_\mu) = \bar{u} \gamma^\mu (1 - \gamma^5)b$; $M$ and $m$ are the masses of the $B$ and $\rho$ mesons, $p$ and $p'$ are their momenta respectively; $q = p - p'$ is the momentum transferred to the leptons and $\epsilon^r$ is the polarization vector of the $\rho$-meson. The form factor $A(q^2)$ does not contribute to the decay rate in the limit of massless leptons. In this case (i.e. $\ell = e, \mu$ is a very good approximation), we have:

$$
\frac{d\Gamma}{dq^2}(B^0 \rightarrow \rho^+ \ell^- \tilde{\nu}_\ell) = \frac{G^2|V_{ub}|^2}{192\pi^3 M^3} q^2 \lambda^{1/2}(q^2) \left[ |H_+(q^2)|^2 + |H_-(q^2)|^2 + |H_0(q^2)|^2 \right].
$$

(3)

$\lambda(t) = (t + M^2 - m^2)^2 - 4M^2m^2$ is the usual triangular function. $H_0$ and $H_\pm$ are the helicity amplitudes which come from the longitudinally and transversely polarized $\rho$-mesons and are given by:

$$
H_0(q^2) = \frac{M+m}{2m\sqrt{q^2}} \left[ (M^2 - m^2 - q^2) A_1(q^2) - \frac{\lambda(q^2)}{(M+m)^2} A_2(q^2) \right]
$$
The dynamics of this decay is described by $A_1(q^2)$, $A_2(q^2)$ and $V(q^2)$ - Lorentz invariant form factors, which are obviously functions of $q^2$ ($\equiv t$). The physical region $0 \leq q^2 \leq q_{\max}^2$ (20:3 GeV$^2$) is very large. In dispersion relations approach, the form factor $A_{1,2}(t)$ can be associated to $J^P = 1^+$, while $V(t)$ to $J^P = 1^-$ intermediate state. This suggests a nearest pole dominance assumption on the behaviour of the form factors. ELC[5] and APE[3] used this approach to extrapolate to $q^2 = 0$ for all form factors:

$$f_i(q^2) = \frac{f_i(0)}{1 - \frac{q^2}{M_{pole}^2}}$$  \hspace{1cm} (5)

where $f_i = V, A_1, A_2$.

In Ref.[8, 9], it was found that $A_1(t)$ decreases with $t$, while $A_2(t)$ moderately increases. Ref.[7] suggested that $V(t)$ is consistent with the pole dominance, $A_1(t)$ decreases and $A_2(t)$ can be fitted with the pole behavior. Light cone sum rules were used in Ref.[10] and [29] and they conclude that $V(t)$ is steeper than pole, $A_1(t)$ increases but is flatter than pole and $A_2(t)$ is compatible with pole behavior. UKQCD[4] in their analysis show that the pole behavior for $A_1(t)$ is preferred (but with $m_{pole} = t_{\pm 2}^{-1/2}$ GeV). Casalbuoni et al. [27] keep the nearest-pole dominance for $V(t)$ and $A_2(t)$, while $A_1(t)$ is consisted of polar and polynomial terms, so that their effective lagrangian approach leads to the so-called 'soft scaling':

$$V, A_2(q_{\max}^2) \sim \frac{M_H + M_V}{\sqrt{M_H}} \; ; \; A_1(q_{\max}^2) \sim \frac{\sqrt{M_H}}{M_H + M_V}.$$  \hspace{1cm} (6)

The common choice is the so-called 'hard scaling' which comes from HQET at leading order $1/M_H$ and in infinite heavy quark mass limit:

$$V, A_2(q_{\max}^2) \sim \sqrt{M_H} \; ; \; A_1(q_{\max}^2) \sim \frac{1}{\sqrt{M_H}}.$$  \hspace{1cm} (7)

These scaling laws (up to $O(M_H^{-2})$ and log corrections) are used by lattice groups for the extrapolation from $m_Q \sim m_c$ to $m_b$. A very nice discussion about the scaling laws can be found in Refs.[27, 28]. The other way is to extrapolate first to $q^2 = 0$ and then to heavy masses. The advantage is evident since the physical region is smaller and the results after extrapolation to $q^2 = 0$ are not that much affected by the ansatz assumed for the form factors’ behaviour. But the HQET scaling laws are valid only for the $q^2 \simeq q_{\max}^2$. The way out was recently pointed out in ref.[29] which states that one can extract from QCD that all form factors decrease at $q^2 = 0$ with heavy mass as $\sim M_H^{-3/2}$. We hope that this would help us to reduce the errors in future lattice analyses. There were also several quark models employed for a prediction of $B \to \rho \ell \bar{\nu}_\ell$ form factors. For instance, in the framework of the light-cone formalism, a relativistic treatment of spin was proposed in Ref.[15], but the form factor values were accessible for $q^2 \leq 0$. 
Dispersion formulation of this quark model was used to relate form factors from $q^2 \leq 0$ region to the physical region by performing the analytic continuation [14]. A relativistic quark model based on the quasipotential approach was discussed in Ref.[13]. From this small list of results, in spite of the evident progress of QSR and lattice results, we see that no definite conclusion on the functional dependence of the form factors can be drawn. Let us derive unitarity bounds for $V(t)$ and $A_1(t)$. In this letter, we concentrate to these two form factors and more particularly to $A_1(t)$ which dominates the decay rate at large $t$. This sort of analysis was started by authors of Ref.[21], and applied to $K \to \pi \nu \ell \nu$ decay in Ref.[19]. In $B$-physics, Refs.[18, 20, 23, 24, 25] used it for the heavy-to-heavy meson semileptonic decays. The idea to employ the method for heavy-to-light decays was first discussed in Ref.[17]. So, most of the material discussed in this paper can be found in the above references.

3 Unitarity bounds

The starting point is the two-point function:

$$\Pi_{\mu \nu}^{V,A} \equiv i \int d^4xe^{iqx} < 0|T(J_{\mu}^{V,A}(x)J_{\nu}^{V,A}(0))|0> = (q^\mu q^\nu - q^2 g^{\mu \nu})\Pi^{V,A}_T(q^2) + g^{\mu \nu}\Pi^{V,A}_L(q^2)$$

In QCD, both sides of this equation satisfy once subtracted dispersion relations:

$$\chi_{T,L}^{V,A}(Q^2) = \frac{\partial \Pi_{T,L}^{V,A}(q^2)}{\partial q^2}|_{q^2=-Q^2} = \frac{1}{\pi} \int_0^\infty \frac{Im\Pi_{T,L}^{V,A}(t)}{(t+Q^2)^2} dt$$

(9)

For $Q^2 = 0$, we are far from the region where the currents can create resonances $((m_Q + m_q)\Lambda_{QCD} \ll (m_Q + m_q)^2 + Q^2)$, so that $\chi_{T,L}^{V,A}$ can be reliably calculated by means of perturbative QCD. The spectral functions $Im\Pi_{T,L}^{V,A}$ can be obtained from the unitarity relation:

$$(q^\mu q^\nu - q^2 g^{\mu \nu})Im\Pi_{T,A}^{V,A}(t+i\epsilon) + g^{\mu \nu}Im\Pi_{L,A}^{V,A}(t+i\epsilon) =$$

$$\frac{1}{2} \sum_{\Gamma} \int d\rho(2\pi)^4\delta(q-p\Gamma) < 0|J_{\mu}^{V,A}(0)|\Gamma><\Gamma|J_{\nu}^{V,A}(0)|0>$$

(10)

where $\Gamma$ are all possible hadron states with appropriate quantum numbers, and the integration goes over the phase space allowed to each intermediate state. We proceed as in Ref.[18]. For $\mu = \nu$, this is the sum of positive definite terms. By concentrating on the $B\rho$-intermediate state, we obtain the strict inequalities. From crossing symmetry, we know that the $B\rho \to \text{vacuum}$ matrix element is described by the same set of the form factors as in (2), but in the different region: $((M + m)^2 \leq q^2 \leq \infty)$, i.e. on the cut. Taking the space-space components of the unitarity relation, we obtain the same combinations of the form factors as those entering the decay rate expression. So, it suffices to take:

$$\chi_{V,A} = \left(\chi_{V,A}^{T}(Q^2) + \frac{1}{2} \frac{\partial}{\partial Q^2} \chi_{V,A}^{L}(Q^2)\right)_{Q^2=0}$$

(11)
Finally, we have,

\[ \text{Im} \Pi_{\text{V}}^{ii}(t) \geq \frac{2}{3\pi(M+m)^2} \frac{\lambda^2(t)}{t} |V(t)|^2 \theta(t-t_+) \]  

(12)

\[ \text{Im} \Pi_{\text{A}}^{ii}(t) \geq \frac{1}{12\pi t} \lambda^2(t) \left[ 2(M+m)^2 |A_1(t)|^2 + |H_0(t)|^2 \right] \theta(t-t_+) \]

(13)

where \( t_\pm = (M \pm m)^2 \). Inserting these functions in the dispersion relation at \( Q^2 = 0 \):

\[ \chi_{\text{V,A}} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi_{\text{V,A}}^{ii}(t)}{t^3} \]

(14)

we obtain the set of inequalities:

\[ \frac{1}{12\pi \chi_{\text{V}} t_+} \int_{t_+}^\infty dt \frac{\lambda^2(t)}{t^4} |V(t)|^2 \leq 1 \]  

(15)

\[ \frac{1}{12\pi \chi_{\text{A}}} \int_{t_+}^\infty dt \frac{\lambda^2(t)}{t^4} \{ 2t_+ |A_1|^2 + |H_0|^2 \} \leq 1 \]

(16)

For bounds on the form factors, we satisfy ourselves by calculating \( \chi_{\text{V,A}} \) at leading order:

\[ \chi_{\text{V,A}} = \frac{N_c}{4(2\pi)^{b/2}} \Gamma(3-D/2) \int_0^1 dx \frac{x^2(1-x)^2[(D+2)(m_Q \mp m_q)(m_Q x + m_q(1-x)) + 8m_qm_Q]}{(m^2_Q x + m^2_q(1-x))^{4-D/2}} \]

(17)

which in our case (\( m_u = 0 \)) gives: \( \chi_V = \chi_A = \frac{3}{4\pi m^2_q} \).

Actually, we have three inequalities which constrain form factors in the unphysical kinematic region. To obtain the bounds on the physically interesting form factors for \( B \to \rho \ell\bar{\nu}_\ell \) transition, we perform the conformal mapping:

\[ \frac{1+z}{1-z} = \sqrt{\frac{t_+ - t}{t_+ - t_-}} \]  

(18)

by which the whole complex \( t \)-plane is mapped onto the unit disc \( |z| \leq 1 \). More specifically: \( t_- \leq t \leq t_+ \) is mapped into the segment of the real axis \(-1 < z \leq 0\), while \( 0 \leq t \leq t_- \) is mapped into \( 0 \leq z < 1 \). In other words, the physical region for \( B\rho \to \text{vacuum} \) transition lies on the unit circle, and for \( B \to \rho \ell\bar{\nu}_\ell \) decay on the right segment of the real axis inside the unit circle. In the \( z \)-plane, the inequalities (16,17) become:

\[ \frac{1}{2\pi i} \int_C \frac{dz}{z} |\Phi_i(z)f_i(z)|^2 \leq 1. \]

(19)

Here, we wrote generically \( f_i = V, A_1 \) and their corresponding functions \( \Phi_i \); \( C \) is the unit circle. The procedure for obtaining the functions \( \Phi_i(z) \) is well-known and is the solution
of Dirichlet’s boundary problem ([21, 24]): its value is known on the circle \( i.e. |\Phi_i(e^{i\theta})|^2 \). Solutions are:

\[
\Phi_V(z) = \sqrt{\frac{2}{3\pi X_V}} \frac{32M^2m^2}{(M + m)^5} \frac{(1 + z)^2}{(1 - z)^2} \left( 1 + \frac{2\sqrt{Mm}}{M + m} \frac{1 + z}{1 - z} \right)^{-4} 
\]

\[
\Phi_{A_1}(z) = \sqrt{\frac{1}{3\pi X_V}} \frac{8Mm}{(M + m)^3} \frac{1 + z}{(1 - z)^2} \left( 1 + \frac{2\sqrt{Mm}}{M + m} \frac{1 + z}{1 - z} \right)^{-4} 
\]

The functions \( \Phi_i(z) \) are analytic everywhere inside the unit circle. From the point of view of analyticity, the problems arise with the form factors. All singularities situated above threshold \( t \) can be absorbed in the phase that can be added in redefinition of \( |\Phi_i(e^{i\theta})| \) and eventually will not contribute to our bounds. But the singularities inside the gap \( t_- \leq t \leq t_+ \) are important. First of all, there are two poles, one at \( t = (5.32 \text{ GeV})^2 \rightarrow z_{pole1} = -0.1666 \), contributing to \( V(t) \) and the other at \( t = (5.73 \text{ GeV})^2 \rightarrow z_{pole2} = -0.3514 \), contributing to \( A_1(t) \). Since we do not know the residua of the form factors at these poles, we will simply remove them. This can be achieved by introducing the Blaschke factors:

\[
P_V = \frac{z - z_{pole1}^*}{1 - z_{pole1}^*} \quad P_{A_1} = \frac{z - z_{pole2}^*}{1 - z_{pole2}^*} \]

Since the Blaschke factors \( P_V \) and \( P_{A_1} \) are unimodular, after inserting them into (20) \( (\Phi_i(z) \rightarrow P_i(z)\Phi_i(z)) \) the inequalities remain the same. In fact, this biases our analysis: if we knew the values of residua, the form factors would be precisely determined for the large values of \( q^2 \), or at least our bounds would be very narrowed in the whole physical region.

There is also a problem to incorporate subthreshold singularities. They are expected in the analysis of the \( A_1(t) \) form factor at \( (M_{P^*} + nm^*)^2 \). So, there are two branch points below the threshold \( (z_1 = -0.4671, z_2 = -0.7061) \). Their effect is negligible in our case, as it can be verified by applying the models discussed in Refs. [24] and [18]. It turns out that the bounds on \( A_1(t) \) would be relaxed by no more than 1%. The last step in deriving the bounds is the construction of the inner product:

\[
(g_i, g_j) = \int_C \frac{dz}{2\pi i z} g_i^*(z)g_j(z).
\]

and choose \( g_1(z) = \Phi_i(z)P_i(z)f_i(z) \) and \( g_2(z) = (1 - zz_1^*)^{-1} \). Then, from the positivity of the inner product, we obtain that the determinant of the \((g_i, g_j)\) matrix is positive, \( i.e. \):

\[
\left| \begin{array}{c}
1 \\
\frac{f_i(z_1)\Phi_i^*(z_1)P_i^*(z_1)}{1-|z_1|^2}
\end{array} \right| \geq 0, \quad \forall z_1 \in \text{IntC} \]

(24)
or
\[ |V(z)| \leq \frac{1}{P_V(z) \Phi_V(z)} \frac{1}{1 - |z|^2}, \quad |A_1(z)| \leq \frac{1}{P_{A_1}(z) \Phi_{A_1}(z)} \frac{1}{1 - |z|^2} \tag{25} \]

We see from the picture (Fig.1) that such bounds are not at all restrictive \(|V(t)| \leq 16, |A_1(t)| \leq 18.3\). Our nice exercise did not lead to any reasonable restriction on the form factors. To constrain them more we can use some of the form factor values that we have on our disposal. Similar analysis was performed for the heavy-to-light transition but for the case of \(B \to \pi l \bar{\nu}_l\) in Refs.\[22, 32\].

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{fig1.png}
  \caption{Unitarity bounds on the form factors \(A_1(q^2)\) and \(V(q^2)\).}
\end{figure}

4 Lattice constrained bounds

To incorporate \(n\) lattice results in our analysis, we define \(g_i(z) = (1 - z z_i^*)^{-1} \) \((i = 1, \ldots, n)\) and construct the matrix \((n + 2) \times (n + 2)\) whose determinant is positive, \(i.e.\):

\[
\begin{vmatrix}
1 & f_i^*(z)\Phi_i^*(z) & f_i^*(z_1)\Phi_i^*(z_1) & \ldots & f_i^*(z_n)\Phi_i^*(z_n) \\
\frac{1}{1 - |z_i|^2} & \frac{1}{1 - z_i z_i^*} & \ldots & \frac{1}{1 - z_i z_n^*} \\
\frac{1}{1 - z_n z_i^*} & \frac{1}{1 - z_n z_i} & \ldots & \frac{1}{1 - |z_n|^2}
\end{vmatrix} \geq 0
\tag{26}
\]
For our purpose, we take three (or two) lattice data of the form factors i.e. the results for the $B$-meson at rest, which are more precise. Since the $UKQCD$ data are very accurate, we take their results (all technical details can be found in Ref.\[4\]). In the table below, the lattice results of three collaborations are given ²:

|                  | $q^2 \, [GeV^2]$ | $A_1(q^2)$ | $A_2(q^2)$ | $V(q^2)$ |
|------------------|-------------------|------------|------------|----------|
| **UKQCD\[4\]**  |                   |            |            |          |
| $24^3 \times 48$ lattice | 20.3             | 0.46$^{+2}_{-3}$ | -           | -        |
| $\beta = 6.2$    | 17.5$^{+2}_{-2}$  | 0.43$^{+2}_{-2}$ | 0.8$^{+2}_{-2}$ | 1.6$^{+1}_{-1}$ |
| Clover Action    | 15.3$^{+3}_{-3}$  | 0.39$^{+3}_{-3}$ | 0.7$^{+2}_{-1}$ | 1.2$^{+1}_{-1}$ |
|                  | 16.7$^{+2}_{-2}$  | 0.38$^{+3}_{-3}$ | 0.6$^{+3}_{-1}$ | 1.5$^{+1}_{-1}$ |
|                  | 14.4$^{+3}_{-3}$  | 0.39$^{+6}_{-5}$ | 0.7$^{+3}_{-2}$ | 1.4$^{+3}_{-2}$ |
| **APE\[3\]**    |                   |            |            |          |
| $18^3 \times 64$ lattice | 20.3             | 0.43 $\pm .08$ | -           | -        |
| $\beta = 6.0$    | 17.6              | 0.48 $\pm .16$ | 0.51 $\pm .50$ | 1.6 $\pm .6$ |
| Clover Action    | 16.6              | 0.70 $\pm .37$ | 0.55 $\pm .70$ | 1.2 $\pm 2.2$ |
|                  | 13.5              | 1.06 $\pm .98$ | 1.05 $\pm 1.10$ | 1.2 $\pm 4.1$ |
| **ELC\[5\]**    |                   |            |            |          |
| $24^3 \times 60$ lattice | 20.3             | 0.60(6)       | -           | -        |
| $\beta = 6.4$    | 15.8              | 0.53(9)       | 0.49(22)    | 0.77(16) |
| Wilson Action    | 12.5              | 0.43(25)      | 0.29(37)    | 0.56(20) |

We see from the table, that the lattice results are quite far from being satisfactory when a behavior of form factors is to be studied. Results quoted above are concentrated in the vicinity of $q^2_{\text{max}}$, and must be extrapolated to small $q^2$. With our constrained bounds, we want to restrict the values of the form-factors in the region of small and intermediate $q^2$. With lattice data incorporated, the constrained bounds are obtained from (27) (thus, we take first three/two results from the table above). Resulting bounds are shown on the pictures (Fig.2).

²The Wuppertal group \[6\] also studied this decay by extrapolating first to $q^2 = 0$ and then to the heavy mass.
Fig. 2: Lattice constrained unitarity bounds on the form factors $A_1(q^2)$ and $V(q^2)$ plotted with lattice UKQCD results.

From the pictures, we see immediately how stronger the bounds are (for instance, $-0.22 \leq A_1(0) \leq 1.14$ and $-0.17 \leq V(0) \leq 0.72$). They are far more restrictive than the previous ones but the problem of the form factors behavior remains unsolved. Beside statistical, we did not incorporate other errors in this analysis\textsuperscript{3}. This will be done in the next section. If we try to see typical ansätze taken for extrapolations to $q^2 = 0$, we see that most of them fall inside the region of allowed values for the form factors at hand. Still, we can try to modify our analysis in the step (19).

5 More constrained bounds

Again, we perform the conformal mapping:

$$
\frac{1 + z}{1 - z} = \sqrt{\frac{t_+ - t}{N(t_+ - t_-)}}
$$

(27)

where $N$ is a constant. The endpoints of the physical region are mapped as;

$$
t = 0 \quad \leftrightarrow \quad z_{\text{max}} = \frac{\sqrt{t_+} - \sqrt{N(t_+ - t_-)}}{\sqrt{t_+} + \sqrt{N(t_+ - t_-)}}
$$

(28)

$$
t = q^2_{\text{max}} \quad \leftrightarrow \quad z_{\text{min}} = -\left(\frac{\sqrt{N} - 1}{\sqrt{N} + 1}\right)
$$

(29)

\textsuperscript{3}The complete treatment of errors is discussed in Ref.\textsuperscript{22}
The inequalities that we derived in Sec.2 remain the same, but the functions (21,22) now become:

\[ \Phi_N^V = \frac{16M^2m^2N}{\sqrt{3\pi\chi'}(M+m)^5}(1+z)^{\frac{7}{2}}(1-z)^{-\frac{9}{2}} \left(1 + \frac{1}{\sqrt{N(1+z)}}\right)^{\frac{3}{2}} \left(1 + \frac{2\sqrt{NMm}}{M+m} \frac{1+z}{1-z}\right)^{-4} \]  
(30)

\[ \Phi_N^{A_1} = \frac{4\sqrt{2}MmN}{\sqrt{3\pi\chi}(M+m)^3}(1+z)^{\frac{3}{2}}(1-z)^{-\frac{5}{2}} \left(1 + \frac{1}{\sqrt{N(1+z)}}\right)^{\frac{3}{2}} \left(1 + \frac{2\sqrt{NMm}}{M+m} \frac{1+z}{1-z}\right)^{-4} \]  
(31)

Of course, for \( N = 1 \) we recover (21,22).

Let us choose \( N \) in such a way that \( z_{\text{max}} = -z_{\text{min}} \). This gives \( N \approx 1.5 \) and consequently \( z_{\text{max}} = -z_{\text{min}} = 0.1011 \). This means that \( z \) is a small kinematic parameter in the whole physical region, and that we can Taylor expand the functions \( \Phi_i(z)P_i(z)f_i(z) \), around \( z = 0 \) i.e.:

\[ f_i(z) = \frac{1}{P_i(z)\Phi_i(z)} \sum_{n=0}^{\infty} a_n z^n \]  
(32)

and from inequalities (17) we extract the additional constraint:

\[ \sum_{n=0}^{\infty} |a_n|^2 \leq 1 \]  
(33)

Actually, the coefficients in the series \( a_i \) can only be obtained from the data (In the case of \( B \to D^{(*)} \bar{\nu}_\ell \) one coefficient is obtained by the help of HQS which gives the absolute normalization of the form factors at \( q^2_{\text{max}} \). In our case, we do not have such an advantage.). It implies that we have to truncate our series. By taking the first \( k \) terms, and using the Schwartz inequality and condition (34), we can estimate the truncation error as:

\[ \Delta_{tr}[f_i] = \max |f_i(z) - f_i^k(z)| \leq \max \frac{1}{|P_i(z)\Phi_i(z)|} \sqrt{\sum_{n=k+1}^{\infty} |a_n|^2 z^{2n}} \]  
\[ < \max \frac{1}{|P_i(z)\Phi_i(z)|} \sqrt{z^{k+1}} \]  
(34)

For the truncation errors in our case, we have:

| \( k \) | \( \Delta_{tr}[V(t)] \) | \( \Delta_{tr}[A_1(t)] \) | \( \Delta_{tr}[H_0(t)] \) |
|---|---|---|---|
| 0 | 1.761 | 2.672 | 32.226 |
| 1 | 0.178 | 0.269 | 3.267 |
| 2 | 0.018 | 0.027 | 0.330 |
As it was already mentioned, for conservative bounds we take three/two lattice data for the form factors $A_1(t) \ (k = 2) \ / \ V(t) \ (k = 1)$. We display the resulting plot for $V(t)$ even though the truncation error is 'large'. In Ref.[18], it was noticed that we can incorporate other uncertainties by choosing the constant on the r.h.s. of (34) greater than one. Namely, in calculation of $\chi_{V,A}(0)$, we may include $O(\alpha)$ corrections which contribute $\sim 18\%$ of the one loop contribution. This corrections, and uncertainties on $m_b = 4.8^{+2}_{-2} \text{GeV}$, as well as contributions of the subthreshold singularities relax the constraint (34) by no more than $20\%$, i.e. $\sum_{n=0}^{\infty} |a_n|^2 \leq 1.2$. By this condition, we can determine the value of the coefficient $a_3(a_2)$ in the expansion for $A_1(z)(V(z))$. Since we consider the bounds, we take the sign which leads to more conservative bounds. With this in mind, we obtain:

| $A_1(z)$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|-----------|-------|-------|-------|-------|
| upper     | 0.0178| -0.0393| -0.3474| -1.0379|
| lower     | 0.0141| -0.0956| -0.7279| 0.8129 |

| $V(z)$ | $a_0$ | $a_1$ | $a_2$ |
|--------|-------|-------|-------|
| upper  | 0.0155| -0.0412| -1.0946|
| lower  | 0.0123| -0.0589| 1.0938 |

Besides statistical errors, in Ref.[4] the systematic errors were estimated. Apart from quenching, they quote $11\%$ for $A_1(t)$ and $15\%$ for $V(t)$ of systematic errors. We relax the resulting bounds by the value of these errors. Final bounds are plotted on Fig.3.

![Figure 3](image_url)

**Fig.3:** More constrained bounds on the form factors $A_1(q^2)$ and $V(q^2)$ (see text) and lattice UKQCD results (bounds are relaxed by the value of systematic errors).

From these figures, we may see first that $A_1(t)$ increases with $t$. We also notice that $A_1(0)$ is small and $A_1(0) \leq 0.18$. On Fig.4, we plot some predictions on functional depen-

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4We used also *ELC* and *APE* results to generate the bounds. Naturally, these bounds are much weaker.
dence $A_1(t)$ against our bounds: On Fig.4c, the comparison with the nearest pole behavior \[11\] shows that $A_1$ must be flatter. Lattice groups ($UKQCD$, $APE$ and $ELC$) tried to fit with the pole behavior but with the masses $m_{pole} = 7^{+5}_{-3}$ $GeV$; $5.99\pm0.62GeV$; $5.62\pm0.11GeV$ respectively. If we want $A_1(t)$ to fall within our bounds, the pole mass should be $m_{pole} \geq 6.5 \, GeV$ which is quite bigger than the nearest pole mass. This can be interpreted as if radial and orbital excitations had more impact on the form factor behavior in the physical region. For $B \to \pi\ell\bar{\nu}_\ell$ in Ref.[30], the authors tried to study such effects by taking into account the first few orbital excitations (uncertainties are large since the values of $f_{B_i}$ are almost unknown). On Fig.4b, we plot $UKQCD$ prediction by taking $m_{pole} = 7 \, GeV$. In Ref.[27], they obtain that $A_1(t)$ should be the nearest pole plus a polynomial term and more specifically, they take a polynomial term to be a constant. This is plotted on Fig.4a and it seems that linear term should be included too. Finally, on Fig.4d the functional dependence $A_1(t)$ predicted by light cone sum rules is plotted.
Fig. 4: a) Effective lagrangian (CHPTHH, Ref. [27]) prediction of $A_1(q^2)$; b) UKQCD pole fit (see text and Ref. [4]); c) Nearest pole dominance [11]; d) Light-cone QCD sum rules prediction of $A_1(q^2)$; are plotted against the bounds from Sec.4 (dotted) and Sec.5 (dashed).

Note that the bounds generated in Sec.4 are relaxed by the value of the systematic errors.
6 Conclusion

In this paper, we studied the unitarity bounds on the form factors for $B \to \rho \ell \bar{\nu}_\ell$ decay. Form factors and perturbative calculation of a two-point function are related to each other by crossing symmetry and a dispersion relation. By conformal mapping we obtain the 'unrestrictive' bounds on the physically interesting form factors. The presence of poles was taken into account by corresponding Blaschke factors which simply remove them, since we do not know the residua i.e. $g_{\rho BB_{pole}} f_{pole}$. The bounds obtained in this way are constrained by the lattice results (more specifically, results obtained by UKQCD). We took only these values where the heavy meson was at rest, and with errors included to make our bounds more conservative. Besides the bounds obtained in this way, we wanted to study the functional dependence of the form factors. With more constrained analysis, we see that with present data we can study only $A_1(t)$ which dominates the decay rate for large values of $t$. We confirm that $A_1(t)$ cannot decrease, but moderately increases, i.e. it is flatter than the nearest pole dominance hypothesis would give. The value of $A_1(0)$ bounded in this way, is surprisingly small ($A_1(0) \leq 0.18$). In these final bounds, we did not include the errors of quenching. If we take them into account and relaxe our bounds by conservative 10%, we would have $A_1(0) \lesssim 0.20$. For $V(t)$ we can not conclude whether it behaves like pole or it is steeper due to the truncation error that we have. In fact both, pole and double-pole functional dependence, are fully inside the region allowed by our bounds. The analysis performed in this paper is even more needed for $B \to K^*\gamma$ form factors. This work is in progress.

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