Infinitesimal local operations and differential conditions for entanglement monotones

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Abstract

Much of the theory of entanglement concerns the transformations that are possible to a state under local operations with classical communication (LOCC); however, this set of operations is complicated and difficult to describe mathematically. An idea which has proven very useful is that of the entanglement monotone: a function of the state which is invariant under local unitary transformations and always decreases (or increases) on average after any local operation. In this paper we look on LOCC as the set of operations generated by infinitesimal local operations, operations which can be performed locally and which leave the state little changed. We show that a necessary and sufficient condition for a function of the state to be an entanglement monotone under local operations that do not involve information loss is that the function be a monotone under infinitesimal local operations. We then derive necessary and sufficient differential conditions for a function of the state to be an entanglement monotone. We first derive two conditions for local operations without information loss, and then show that they can be extended to more general operations by adding the requirement of convexity. We then demonstrate that a number of known entanglement monotones satisfy these differential criteria. Finally, as an application, we use the differential conditions to construct a new polynomial entanglement monotone for three-qubit pure states. It is our hope that this approach will avoid some of the difficulties in the theory of multipartite and mixed-state entanglement.

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I. INTRODUCTION

One of the most impressive achievements of quantum information theory is the theory of entanglement. The theory of entanglement concerns the transformations that are possible to a state under local operations with classical communication (LOCC). The paradigmatic experiment is a quantum system comprising several subsystems, each in a separate laboratory under control of a different experimenter: Alice, Bob, Cara, etc. Each experimenter can perform any physically allowed operation on his or her subsystem—unitary transformations, generalized measurements, indeed any trace-preserving completely positive operation—and communicate their results to each other without restriction. They are not, however, allowed to bring their subsystems together and manipulate them jointly. An LOCC protocol consists of any number of local operations, interspersed with any amount of classical communication; the choice of operations at later times may depend on the outcomes of measurements at any earlier time.

The results of Bennett et al. \[1, 2, 3\] and Nielsen \[4\], among many others \[5, 6, 7, 8, 9\], have given us a nearly complete theory of entanglement for bipartite systems in pure states. Unfortunately, great difficulties have been encountered in trying to extend these results both to mixed states and to states with more than two subsystems (multipartite systems). The reasons for this are many; but one reason is that the set LOCC is complicated and difficult to describe mathematically \[10\].

One mathematical tool which has proven very useful is that of the entanglement monotone: a function of the state which is invariant under local unitary transformations and always decreases (or increases) on average after any local operation. These functions were described by Vidal \[11\], and large classes of them have been enumerated since then.

We will consider those protocols in LOCC that preserve pure states as the set of operations generated by infinitesimal local operations: operations which can be performed locally and which leave the state little changed including infinitesimal local unitaries and weak generalized measurements. In Bennett et al. \[10\] it was shown that infinitesimal local operations can be used to perform any local operation with the additional use of local ancillary systems—extra systems residing in the local laboratories, which can be coupled to the subsystems for a time and later discarded. Recently we have shown that any local generalized measurement can be implemented as a sequence of weak measurements without the use of ancillas \[12\]. This implies that a necessary and sufficient condition for a function of the state to be a monotone under local operations that preserve pure states is the function to be a monotone under infinitesimal local operations.

In this paper we derive differential conditions for a function of the state to be an entanglement monotone by considering the change of the function on average under infinitesimal local operations up to the lowest order in the infinitesimal parameter. We thus obtain conditions that involve at most second derivatives of the function. We then prove that these conditions are both necessary and sufficient. We show that the conditions are satisfied by a number of known entanglement monotones and we use them to construct a new polynomial entanglement monotone for three-qubit pure states.

It is our hope that this approach will provide a new window with which to study LOCC, and perhaps avoid some of the difficulties in the theory of multipartite and mixed-state entanglement. By looking only at the differential behavior of entanglement monotones, we avoid concerns about the global structure of LOCC.

In section II, we define the basic concepts of this paper: LOCC operations, entanglement
monotones, and infinitesimal operations. In section III, we show how all local operations that preserve pure states can be generated by a sequence of infinitesimal local operations. In section IV, we derive differential conditions for a function of the state to be an entanglement monotone. There are two such conditions for pure-state entanglement monotones: the first guarantees invariance under local unitary transformations (LU invariance), and involves only the first derivatives of the function, while the second guarantees monotonicity under local measurements, and involves second derivatives. For mixed-state entanglement monotones we add a further condition, convexity, which ensures that a function remains monotonic under operations that lose information (and can therefore transform pure states to mixed states). In section V, we look at some known monotones—the norm of the state, the local purity, and the entropy of entanglement—and show that they obey the differential criteria. In section VI, we use the differential conditions to construct a new polynomial entanglement monotone for three-qubit pure states which depends on the invariant identified by Kempe [13]. Finally, in section VII we conclude. In the appendix, we show that higher derivatives of the function are not needed to prove monotonicity.

II. BASIC DEFINITIONS

A. LOCC

An operation (or protocol) in LOCC consists of a sequence of local operations with classical communication between them. Initially, we will consider only those local operations that preserve pure states: unitaries, in which the state is transformed

$$\rho \rightarrow \hat{U}_{} \rho \hat{U}^\dagger, \quad \hat{U}^\dagger \hat{U} = \hat{I},$$

(1)

and generalized measurements, in which the state randomly changes

$$\rho \rightarrow \rho_j = \hat{M}_j \rho \hat{M}_j^\dagger/p_j, \quad \sum_j \hat{M}_j^\dagger \hat{M}_j = \hat{I},$$

(2)

with probability $p_j = \text{Tr} \left\{ \hat{M}_j^\dagger \hat{M}_j \rho \right\}$, where the index $j$ labels the possible outcomes of the measurement. Note that we can think of a unitary as being a special case of a generalized measurement with only one possible outcome. One can think of this class of operations as being limited to those which do not discard information. Later, we will relax this assumption to consider general operations, which can take pure states to mixed states. Such operations do involve loss of information. Examples include performing a measurement without retaining the result, performing an unknown unitary chosen at random, or entangling the system with an ancilla which is subsequently discarded.

The requirement that an operation be local means that the operators $\hat{U}_{}$ or $\hat{M}_j$ must have a tensor-product structure $\hat{U} \equiv \hat{U}_{} \otimes \hat{I}$, $\hat{M}_j \equiv \hat{M}_j \otimes \hat{I}$, where they act as the identity on all except one of the subsystems. The ability to use classical communication implies that the choice of later local operations can depend arbitrarily on the outcomes of all earlier measurements. One can think of an LOCC operation as consisting of a series of “rounds.” In each round, a single local operation is performed by one of the local parties; if it is a measurement, the outcome is communicated to all parties, who then agree on the next local operation.
B. Entanglement monotones

For the purposes of this paper, we define an entanglement monotone to be a real-valued function of the state with the following properties: if we start with the system in a state \( \rho \) and perform a local operation which leaves the system in one of the states \( \rho_1, \ldots, \rho_n \) with probabilities \( p_1, \ldots, p_n \), then the value of the function must not increase on average:

\[
f(\rho) \geq \sum_j p_j f(\rho_j).
\]

(3a)

Furthermore, we can start with a state selected randomly from an ensemble \( \{\rho_k, p_k\} \). If we dismiss the information about which particular state we are given (which can be done locally), the function of the resultant state must not exceed the average of the function we would have if we keep this information:

\[
\sum_k p_k f(\rho_k) \geq f\left(\sum_k p_k \rho_k\right).
\]

(3b)

Some functions may obey a stronger form of monotonicity, in which the function cannot increase for any outcome:

\[
f(\rho) \geq f(\rho_j), \quad \forall j,
\]

(4)

but this is not the most common situation. Some monotones may be defined only for pure states, or may only be monotonic for pure states. In the latter case, monotonicity is defined as non-increase on average under local operations that do not involve information loss.

C. Infinitesimal operations

We call an operation infinitesimal if all outcomes result in only very small changes to the state. That is, if after an operation the system can be left in states \( \rho_1, \ldots, \rho_n \), we must have

\[
||\rho - \rho_j|| \ll 1, \quad \forall j.
\]

(5)

For a unitary, this means that

\[
\hat{U} = \exp(i\hat{\varepsilon}) \approx \hat{I} + i\hat{\varepsilon},
\]

(6)

where \( \hat{\varepsilon} \) is a Hermitian operator with small norm, \( ||\hat{\varepsilon}|| \ll 1 \), \( \hat{\varepsilon} = \hat{\varepsilon}^\dagger \). For a generalized measurement, every measurement operator \( \hat{M}_j \) can be written as

\[
\hat{M}_j = q_j (\hat{I} + \hat{\varepsilon}_j),
\]

(7)

where \( 0 \leq q_j \leq 1 \) and \( \hat{\varepsilon}_j \) is an operator with small norm \( ||\hat{\varepsilon}_j|| \ll 1 \).

Such measurements are called weak. The term weak measurement, however, is often taken to include measurements in which some of the outcomes change the state a great deal, but only with very low probability. We do not include such measurements in what follows. All outcomes must leave the state almost unchanged.
III. LOCAL OPERATIONS FROM INFINITESIMAL LOCAL OPERATIONS

In this section we show how any local operation that preserves pure states can be performed as a sequence of infinitesimal local operations. The operations that preserve pure states are unitary transformations and generalized measurements.

A. Unitary transformations

Every local unitary operator has the representation

\[ \hat{U} = e^{i\hat{H}}, \]

(8)

where \( \hat{H} \) is a local hermitian operator. We can write

\[ \hat{U} = \lim_{n \to \infty} (\hat{I} + i\hat{H}/n)^n, \]

(9)

and define

\[ \hat{\epsilon} = \hat{H}/n \]

for a suitably large value of \( n \). Thus, in the limit \( n \to \infty \), any local unitary operation can be thought of as an infinite sequence of infinitesimal local unitary operations driven by operators of the form

\[ \hat{U}_\epsilon \approx \hat{I} + i\hat{\epsilon}, \]

(11)

where \( \hat{\epsilon} \) is a small (\( \|\hat{\epsilon}\| \ll 1 \)) local hermitian operator.

B. Generalized measurements

Recently it has been shown [12] that any local measurement can be generated by a sequence of weak local measurements. Since a measurement with any number of outcomes can be implemented as a sequence of two-outcome measurements, it suffices to show this for generalized measurements with two outcomes.

If the initial state of the system has a density matrix \( \rho \), the two possible outcomes of a measurement with operators \( \hat{M}_1 \) and \( \hat{M}_2 \) are \( \hat{M}_1\rho\hat{M}_1^\dagger/p_1 \) and \( \hat{M}_2\rho\hat{M}_2^\dagger/p_2 \), where \( p_{1,2} = \text{Tr}(\hat{M}_{1,2}\rho\hat{M}_{1,2}^\dagger) \) are the corresponding probabilities. Using the polar decomposition, the two measurement operators can be written as \( \hat{M}_1 = \hat{U}_1\sqrt{\hat{M}_1^\dagger\hat{M}_1} \) and \( \hat{M}_2 = \hat{U}_2\sqrt{\hat{M}_2^\dagger\hat{M}_2} \), where \( \hat{U}_1 \) and \( \hat{U}_2 \) are unitary. Since we have already seen that we can do any local unitary transformation by a sequence of infinitesimal steps, if we can first measure the positive operators \( \sqrt{\hat{M}_1^\dagger\hat{M}_1} \) and \( \sqrt{\hat{M}_2^\dagger\hat{M}_2} \) by a series of infinitesimal steps, we can then apply \( \hat{U}_1 \) or \( \hat{U}_2 \) (conditional on the outcome), and can therefore measure \( \hat{M}_1 \) and \( \hat{M}_2 \) by infinitesimal steps as well. So without loss of generality, we consider only positive measurement operators: \( \hat{M}_j = \hat{M}_j^\dagger, I \geq \hat{M}_j \geq 0 \). Note that in this case, \( \hat{M}_1 \) and \( \hat{M}_2 \) commute: \( \hat{M}_1\hat{M}_2 = \hat{M}_2\hat{M}_1 \).

We now decompose this measurement into a series of weak measurements. We can think of the procedure as a random walk along a curve in state space; the position on this curve is indicated by a single parameter \( x \), with \( x = 0 \) being the initial state. The current state of the system at any point during the procedure can be written

\[ \hat{M}(x)\rho\hat{M}(x)/\text{Tr}(\hat{M}^2(x)\rho), \]

(12)
where
\[ \hat{M}(x) = \sqrt{\frac{\hat{I} + \tanh(x)(\hat{M}_2^2 - \hat{M}_1^2)}{2}}, \quad x \in \mathbb{R}. \] (13)

In the limit \( x \to \pm \infty \) the effective operator reduces to \( \hat{M}_{1,2} \).

In [12] it has been shown that depending on the current value of the parameter \( x \), one can perform a two-outcome measurement on the system with positive operators \( \hat{M}(x, \pm \epsilon) \) which satisfy
\[ [\hat{M}(x, \epsilon)]^2 + [\hat{M}(x, -\epsilon)]^2 = \hat{I}, \]
\[ \hat{M}(x, \pm \epsilon) \hat{M}(x) \propto \hat{M}(x \pm \epsilon). \] (14)

Since the state is normalized, the factor of proportionality is irrelevant; the two possible outcomes simply change the parameter \( x \) by \( +\epsilon \) or \( -\epsilon \). Thus the measurement procedure is a random walk along the curve \( \hat{M}(x) \), with a step size \( |\epsilon| \). We continue this walk until \( |x| \geq X \) for some \( X \) which is sufficiently large that \( \hat{M}(-X) \approx \hat{M}_1 \) and \( \hat{M}(X) \approx \hat{M}_2 \), to whatever precision we desire. It has been shown that the probabilities of the outcomes for this procedure are exactly the same as those for a single generalized measurement. The exact form of the operators \( \hat{M}(x, \pm \epsilon) \) is derived in [12]:
\[ \hat{M}(x, \pm \epsilon) = \sqrt{C_{\pm}} \hat{I} + \tanh(x \pm \epsilon)(\hat{M}_2^2 - \hat{M}_1^2)\hat{I} + \tanh(x)(\hat{M}_2^2 - \hat{M}_1^2), \] (15)
where the weights \( C_{\pm} \) are chosen to ensure that these operators form a generalized measurement:
\[ C_{\pm} = (1 \pm \tanh(\epsilon) \tanh(x))/2. \] (16)

For any finite \( x \), \( \hat{M}(x)^{-1} \) is well defined, so from (14) and (15) it is easy to see that if \( |\epsilon| \ll 1 \), we have \( \hat{M}(x, \epsilon) = \sqrt{1/2(\hat{I} + O(\epsilon))} \): the measurements are weak. Thus every measurement can be implemented as a sequence of weak measurements. Moreover, if the original measurement is local, the weak measurements are also local.

Clearly, the fact that infinitesimal local operations are part of the set of LO means that an entanglement monotone must be a monotone under infinitesimal local operations. The result discussed in this section implies that if a function is a monotone under infinitesimal local unitaries and generalized measurements, it is a monotone under all local unitaries and generalized measurements (the operations that do not involve information loss and preserve pure states). Based on this result, in the next section we derive necessary and sufficient conditions for a function to be an entanglement monotone.

IV. DIFFERENTIAL CONDITIONS FOR ENTANGLEMENT MONOTONES

Let us now consider the change in the state under an infinitesimal local operation. Without loss of generality, we assume that the operation is performed on Alice’s subsystem. In this case, it is convenient to write the density matrix of the system as
\[ \rho = \sum_{i,j,l,m} \rho_{ijlm} |i_A\rangle \langle l_A| \otimes |j_{BC\ldots}\rangle \langle m_{BC\ldots}|, \] (17)
where the set $\{|i_A\rangle\}$ and the set $\{|j_{BC\ldots}\rangle\}$ are arbitrary orthonormal bases for subsystem $A$ and the rest of the system, respectively. Any function of the state $f(\rho)$ can be thought of as a function of the coefficients in the above decomposition:

$$f(\rho) = f(\rho_{ijlm})$$

(18)

### A. Local unitary invariance

Unitary operations are invertible, and therefore the monotonicity condition reduces to an invariance condition for LU transformations. Under local unitary operations on subsystem $A$ the components of $\rho$ transform as follows:

$$\rho_{ijlm} \rightarrow \sum_{k,p} U_{ik} \rho_{kjpm} U^*_{lp},$$

(19)

where $U_{ik}$ are the components of the local unitary operator in the basis $\{|i_A\rangle\}$. We consider infinitesimal local unitary operations:

$$U_{lk} = (e^{i\hat{\varepsilon}})_{lk},$$

(20)

where $\hat{\varepsilon}$ is a local hermitian operator acting on subsystem $A$, and

$$\|\hat{\varepsilon}\| \ll 1.$$  

(21)

Up to first order in $\hat{\varepsilon}$ the coefficients $\rho_{ijlm}$ transform as

$$\rho_{ijlm} \rightarrow \rho_{ijlm} + i[\hat{\varepsilon}, \rho]_{ijlm}.$$  

(22)

Requiring LU-invariance of $f(\rho)$, we obtain that the function must satisfy

$$\sum_{i,j,l,m} \frac{\partial f}{\partial \rho_{ijlm}}[\hat{\varepsilon}, \rho]_{ijlm} = 0.$$  

(23)

Analogous equations must be satisfied for arbitrary hermitian operators $\hat{\varepsilon}$ acting on the other parties’ subsystems. In a more compact form, the condition can be written as

$$\text{Tr} \left\{ \frac{\partial f}{\partial \rho} [\hat{\varepsilon}, \rho] \right\} = 0,$$

(24)

where $\hat{\varepsilon}$ is an arbitrary local hermitian operator.

### B. Non-increase under infinitesimal local measurements

As mentioned earlier, a measurement with any number of outcomes can be implemented as a sequence of measurements with two outcomes, and a general measurement can be done as a measurement with positive operators, followed by a unitary conditioned on the outcome; therefore, it suffices to impose the monotonicity condition for two-outcome measurements with positive measurement operators. Consider local measurements on subsystem $A$ with
two measurement outcomes, given by operators \( \hat{M}_1^2 + \hat{M}_2^2 = \hat{I} \). Without loss of generality, we assume
\[
\hat{M}_1 = \sqrt{(\hat{I} + \hat{\varepsilon})/2}, \\
\hat{M}_2 = \sqrt{(\hat{I} - \hat{\varepsilon})/2},
\]
(25)
where \( \hat{\varepsilon} \) is again a small local hermitian operator acting on \( A \) (in the previous section we saw that any two-outcome measurement with positive operators can be generated by weak measurements of this type). Upon measurement, the state undergoes one of two possible transformations
\[
\rho \rightarrow \frac{\hat{M}_1 \rho \hat{M}_1}{p_1}, \\
\rho \rightarrow \frac{\hat{M}_2 \rho \hat{M}_2}{p_2},
\]
(26)
with probabilities \( p_{1,2} = \text{Tr} \left\{ \hat{M}_{1,2}^2 \rho \right\} \). Since \( \hat{\varepsilon} \) is small, we can expand
\[
\hat{M}_1 = \frac{1}{\sqrt{2}} (\hat{I} + \hat{\varepsilon}/2 - \hat{\varepsilon}^2/8 - \cdots), \\
\hat{M}_2 = \frac{1}{\sqrt{2}} (\hat{I} - \hat{\varepsilon}/2 - \hat{\varepsilon}^2/8 - \cdots).
\]
(27)
(28)
The condition for non-increase on average of the function \( f \) under infinitesimal local measurements is
\[
p_1 f(\hat{M}_1 \rho \hat{M}_1/p_1) + p_2 f(\hat{M}_2 \rho \hat{M}_2/p_2) \leq f(\rho).
\]
(29)
Expanding (29) in powers of \( \hat{\varepsilon} \) up to second order, we obtain
\[
\frac{1}{4} \text{Tr} \left\{ \frac{\partial f}{\partial \rho} [[\hat{\varepsilon}, \rho], \hat{\varepsilon}] \right\} + \text{Tr} \left\{ \frac{\partial^2 f}{\partial \rho \otimes^2} \left( \text{Tr}(\hat{\varepsilon} \rho) \rho - \frac{1}{2} \{\hat{\varepsilon}, \rho\} \right) \otimes^2 \right\} \leq 0,
\]
(30)
where \( \{\hat{\varepsilon}, \rho\} \) is the anti-commutator of \( \hat{\varepsilon} \) and \( \rho \). The inequality must be satisfied for an arbitrary local hermitian operator \( \hat{\varepsilon} \).

So long as (30) is satisfied by a strict inequality, it is obvious that we need not consider higher-order terms in \( \hat{\varepsilon} \). But what about the case when the condition is satisfied by equality? In the Appendix we will show that even in the case of equality, (30) is still the necessary and sufficient condition for monotonicity under local generalized measurements. There we also prove the sufficiency of the LU-invariance condition (24). This allows us to state the following

**Theorem:** A twice-differentiable function \( f(\rho) \) of the density matrix is a monotone under local unitary operations and generalized measurements, if and only if it satisfies (24) and (30).

Unitary operations and generalized measurements are the operations that preserve pure states. Other operations (which involve loss of information), such as positive maps, would in general cause pure states to evolve into mixed states. A measure of pure-state entanglement need not be defined over the entire set of density matrices, but only over
pure states. Thus a measure of pure-state entanglement, when expressed as a function of the density matrix, may have a significantly simpler form than its generalizations to mixed states. For example, the entropy of entanglement for bipartite pure states can be written in the well-known form $S_A(\rho) = -\text{Tr}(\rho A \log \rho_A)$, where $\rho_A$ is the reduced density matrix of one of the parties’ subsystems. When directly extended over mixed states, this function is not well justified, since $S_A(\rho)$ may have a different value from $S_B(\rho)$. Moreover, $S_A(\rho)$ by itself is not a mixed-state entanglement monotone, since it may increase under local positive maps on subsystem A (these properties of the entropy of entanglement will be discussed further in section V). One generalization of the entropy of entanglement to mixed states is the entanglement of formation $[3]$, which is defined as the minimum of $\sum_i \rho_i S_A(\rho_i)$ over all ensembles of bipartite pure states $\{\rho_i, \rho_i\}$ realizing the mixed state: $\rho = \sum_i \rho_i \rho_i$. This quantity is a mixed-state entanglement monotone. As a function of $\rho$, it has a much more complicated form than the above expression for the entropy of entanglement. In fact, there is no known analytic expression for the entanglement of formation in general. The problem of extending pure-state entanglement monotones to mixed states is an important one, since every mixed-state entanglement monotone can be thought of as an extension of a pure-state entanglement monotone. Note, however, that a pure-state entanglement monotone may have many different mixed-state generalizations. The relation between the entanglement of formation and the entropy of entanglement presents one way to perform such an extension (convex-roof extension). For every pure-state entanglement monotone $m(\rho)$, one can define a mixed-state extension $M(\rho)$ as the minimum of $\sum_i \rho_i m(\rho_i)$ over all ensembles of pure states $\{\rho_i, \rho_i\}$ realizing the mixed state: $\rho = \sum_i \rho_i \rho_i$. It is easy to verify that $M(\rho)$ is an entanglement monotone for mixed states. On the set of pure states the function $M(\rho)$ reduces to $m(\rho)$. As the example with the entropy of entanglement suggests, not every form of a pure-state entanglement monotone corresponds to a mixed-state entanglement monotone when trivially extended to all states – there are additional conditions that a mixed-state entanglement monotone must satisfy. On the basis of the above considerations, it makes sense to consider separate sets of differential conditions for pure-state and mixed-state entanglement monotones.

**Corollary 1:** A twice-differentiable function $f(\rho)$ of the density matrix is a pure-state entanglement monotone, if and only if it satisfies (24) and (30) for pure $\rho$.

For pure states $\rho = |\psi\rangle \langle \psi|$, the elements of $\rho$ are $\rho_{ij \ell m} = \alpha_{ij} \alpha^*_{\ell m}$, where the $\{\alpha_{ij}\}$ are the state amplitudes: $|\psi\rangle = \sum_{ij} \alpha_{ij} |i\rangle |j_{BC}\rangle$. Any function on pure states $f(\rho) \equiv f(|\psi\rangle)$ is therefore a function of the state amplitudes and their complex conjugates:

$$f(|\psi\rangle) = f(\{\alpha_{ij}\}, \{\alpha^*_{ij}\}).$$

(31)

By making the substitution $\rho_{ij \ell m} = \alpha_{ij} \alpha^*_{\ell m}$ into (24) and (30), we can (after considerable algebra) derive alternative forms of the differential conditions for functions of the state vector:

$$\sum_{i,j,k} \frac{\partial f}{\partial \alpha_{ij}} \varepsilon_{ik} \alpha_{kj} = \sum_{i,j,k} \frac{\partial f}{\partial \alpha^*_{ij}} \varepsilon_{ik} \alpha^*_{kj},$$

(32)

$$\sum_{i,j,k,l,m,n} \frac{\partial^2 f}{\partial \alpha_{ij} \partial \alpha_{mn}} (\varepsilon_{ik} \alpha_{kj} - \langle \hat{\varepsilon} \rangle \alpha_{ij}) (\varepsilon_{ml} \alpha_{\ell n} - \langle \hat{\varepsilon} \rangle \alpha_{mn}) + \text{c.c.} \leq 0.$$
Here $\hat{\epsilon}$ is a local hermitian operator acting on subsystem A. Analogous conditions must be satisfied for $\hat{\epsilon}$ acting on the other parties’ subsystems.

C. Monotonicity under operations with information loss

Besides monotonicity under local unitaries and generalized measurements, an entanglement monotone for mixed states should also satisfy monotonicity under local operations which involve loss of information. The most general transformation that involves loss of information has the form

$$\rho \rightarrow \rho_k = \frac{1}{p_k} \sum_j \hat{A}_{k,j} \rho \hat{A}_{k,j}^\dagger,$$  \hspace{1cm} (34)

where

$$p_k = \text{Tr}\left\{\sum_j \hat{A}_{k,j} \rho \hat{A}_{k,j}^\dagger\right\}$$  \hspace{1cm} (35)

is the probability for outcome $k$. The operators $\{\hat{A}_{k,j}\}$ must satisfy

$$\sum_{k,j} \hat{A}_{k,j}^\dagger \hat{A}_{k,j} = \hat{I}. \hspace{1cm} (36)$$

We can see that this includes unitary transformations, generalized measurements, and completely positive trace-preserving maps as special cases.

It occasionally makes sense to consider even more general transformations, where the operators need not sum to the identity:

$$\sum_{k,j} \hat{A}_{k,j}^\dagger \hat{A}_{k,j} \leq \hat{I}. \hspace{1cm} (37)$$

This corresponds to a situation where only certain outcomes are retained, and others are discarded; the probabilities add up to less than 1 due to these discarded outcomes. We say such a transformation involves postselection.

With or without postselection, we are concerned with the case where all operations are done locally, so that all the operators $\{\hat{A}_{k,j}\}$ act on a single subsystem. Every such transformation can be implemented as a sequence of local generalized measurements (possibly discarding some of the outcomes) and local completely positive maps. In operator-sum representation, a completely positive map can be written

$$\rho \rightarrow \sum_k \hat{M}_k \rho \hat{M}_k^\dagger,$$  \hspace{1cm} (38)

where

$$\sum_k \hat{M}_k^\dagger \hat{M}_k \leq \hat{I}. \hspace{1cm} (39)$$

Therefore, in addition to (24) and (30) we must impose the condition

$$f(\rho) \geq f\left(\sum_k \hat{M}_k \rho \hat{M}_k^\dagger\right). \hspace{1cm} (40)$$
for all sets of local operators \( \{\hat{M}_k\} \) satisfying (39).

Suppose the parties are supplied with a state \( \rho_k \) taken from an ensemble \( \{\rho_k, p_k\} \). Discarding the information of the actual state amounts to the transformation
\[
\{\rho_k, p_k\} \rightarrow \rho' = \sum_k p_k \rho_k.
\]
(41)

As pointed out in [11], discarding information should not increase the entanglement of the system on average. Therefore, for any ensemble \( \{\rho_k, p_k\} \), an entanglement monotone on mixed states should be *convex*:
\[
\sum_k p_k f(\rho_k) \geq f\left(\sum_k p_k \rho_k\right).
\]
(42)

Condition (42), together with condition (30) for monotonicity under local generalized measurements, implies monotonicity under local completely positive maps:
\[
f\left(\sum_k \hat{M}_k \rho \hat{M}_k^\dagger\right) \leq \sum_k p_k f\left(\frac{\hat{M}_k \rho \hat{M}_k^\dagger}{p_k}\right) \leq f(\rho).
\]
(43)

It is easy to see that if this inequality holds without postselection, it must also hold with postselection.

It follows that a function of the density matrix is an entanglement monotone for mixed states if and only if it is (1) a convex function on the set of density matrices and (2) a monotone under local unitaries and generalized measurements. Fortunately, there are also simple differential conditions for convexity. A necessary and sufficient condition for a twice-differentiable function of multiple variables to be convex on a convex set is that its Hessian matrix be positive on the interior of the convex set (in this case, the set of density matrices). Therefore, in addition to (24) and (30) we add the differential condition
\[
\text{Tr}\left\{\frac{\partial^2 f(\rho)}{\partial \rho^\otimes \sigma^\otimes 2}\sigma^\otimes 2\right\} \geq 0,
\]
(44)

which must be satisfied at every \( \rho \) on the interior of the set of density matrices for an arbitrary traceless hermitian matrix \( \sigma \).

**Corollary 2:** A twice-differentiable function \( f(\rho) \) of the density matrix is a mixed-state entanglement monotone, if and only if it satisfies (24), (30) and (44).

**V. EXAMPLES**

In this section we demonstrate how conditions (24), (30) and (44) can be used to verify if a function is an entanglement monotone. We show this for three well known entanglement monotones: the norm of the state of the system, the trace of the square of the reduced density matrix of any subsystem, and the entropy of entanglement. In the next section we will use some of the observations made here to construct a new polynomial entanglement monotone for three-qubit pure states.
A. Norm of the state

The most trivial example is the norm or the trace of the density matrix of the system:

\[ I_1 = \text{Tr}\{\rho\}. \tag{45} \]

Clearly \( I_1 \) is a monotone under LOCC, since all operations that we consider either preserve or decrease the trace. But for the purpose of demonstration, let us verify that \( I_1 \) satisfies the differential conditions.

The LU-invariance condition \([24]\) reads

\[ \text{Tr} \left\{ \frac{\partial I_1}{\partial \rho} [\hat{\epsilon}, \rho] \right\} = \text{Tr} \{ [\hat{\epsilon}, \rho] \} = 0. \tag{46} \]

The second equality follows from the cyclic invariance of the trace.

Since the trace is linear, the second term in condition \([30]\) vanishes, and we consider only the first term:

\[ \text{Tr} \left\{ \frac{\partial I_1}{\partial \rho} [[\hat{\epsilon}, \rho], \hat{\epsilon}] \right\} = \text{Tr} \{ [[\hat{\epsilon}, \rho], \hat{\epsilon}] \} = 0. \tag{47} \]

The condition is satisfied with equality, again due to the cyclic invariance of the trace, implying that the norm remains invariant under local measurements. The convexity condition \([44]\) is also satisfied by equality.

B. Local purity

The second example is the purity of the reduced density matrix:

\[ I_2 = \text{Tr}\left\{\rho_A^2\right\}, \tag{48} \]

where \( \rho_A \) is the reduced density matrix of subsystem \( A \) (which in general need not be a one-party subsystem). Note that this is an increasing entanglement monotone for pure states—the purity of the local reduced density matrix can only increase under LOCC.

It has been shown in \([14]\) that every \( m \)-th degree polynomial of the components of the density matrix \( \rho \) can be written as an expectation value of an observable \( \hat{O} \) on \( m \) copies of \( \rho \):

\[ f(\rho) = \text{Tr} \left\{ \hat{O} \rho \otimes m \right\}. \tag{49} \]

Here we have

\[ \text{Tr} \left\{ \rho_A^2 \right\} = \text{Tr} \left\{ \hat{C} \rho \otimes 2 \right\}, \tag{50} \]

where the components of \( \hat{C} \) are

\[ C_{lpsnkqm} = \delta_{jp} \delta_{mn} \delta_{lq} \delta_{ks}. \tag{51} \]

Therefore

\[ \text{Tr} \left\{ \frac{\partial I_2}{\partial \rho} [\hat{\epsilon}, \rho] \right\} = \text{Tr} \left\{ \hat{C} ([\hat{\epsilon}, \rho] \otimes \rho + \rho \otimes [\hat{\epsilon}, \rho]) \right\} \]

\[ = \text{Tr}_A \left\{ [\hat{\epsilon}, \rho] A \rho_A + \rho_A [\hat{\epsilon}, \rho] \right\} \]

\[ = 2 \text{Tr}_A \left\{ \rho_A [\hat{\epsilon}, \rho] \right\} , \tag{52} \]
where by $\hat{O}_A$ we denote the partial trace of an operator $\hat{O}$ over all subsystems except $A$. If $\hat{\epsilon}$ does not act on subsystem $A$, then $[\hat{\epsilon}, \rho]_A = 0$ and the above expression vanishes. If it acts on subsystem $A$, then $[\hat{\epsilon}, \rho]_A = [\hat{\epsilon}, \rho_A]$ and the expression vanishes due to the cyclic invariance of the trace.

Now consider condition (30). If $\hat{\epsilon}$ does not act on subsystem $A$, then $[\hat{\epsilon}, \rho]_A = [\hat{\epsilon}, \rho_A]$ and the expression vanishes due to the cyclic invariance of the trace.

From (30) we get

$$0 \leq \frac{1}{4} \text{Tr} \left\{ \frac{\partial I_2}{\partial \rho} [\hat{\epsilon}, \rho] \right\} + \text{Tr} \left\{ \frac{\partial^2 I_2}{\partial \rho \otimes_2} \left( \text{Tr} \{\hat{\epsilon}\rho\} \rho - \frac{1}{2} \{\hat{\epsilon}, \rho\} \right)^2 \right\}$$

$$= 2\text{Tr} \left\{ \left( \text{Tr} \{\hat{\epsilon}\rho\} \rho - \frac{1}{2} \{\hat{\epsilon}, \rho\} \right)^2 \right\}. \quad (54)$$

The inequality follows from the fact that $(\text{Tr} \{\hat{\epsilon}\rho\} \rho - (1/2) \{\hat{\epsilon}, \rho\})^2$ is a positive operator.

If $\hat{\epsilon}$ acts on $A$, we can use the fact that for pure states

$$\text{Tr} \{\rho_A^2\} = \text{Tr} \{\rho_B^2\}, \quad (55)$$

where $B$ denotes the subsystem complementary to $A$. Then we can apply the same argument as before for the function $\text{Tr} \{\rho_B^2\}$. Therefore $I_2$ does not decrease on average under local generalized measurements, and is an entanglement monotone for pure states.

What about mixed states? For increasing entanglement monotones the convexity condition (44) becomes a concavity condition—the direction of the inequality is inverted. In the case of $I_2$, however, we have

$$\text{Tr} \left\{ \frac{\partial^2 I_2(\rho)}{\partial \rho \otimes_2} \sigma^2 \right\} = 2\text{Tr} \{\sigma_A^2\} \geq 0, \quad (56)$$

i.e., the function is convex. This means that $\text{Tr}\{\rho_A^2\}$ is not a good measure of entanglement for mixed states. Indeed, when extended to mixed states, $I_2$ cannot distinguish between entanglement and classical disorder.

**C. Entropy of entanglement**

Finally consider the von Neumann entropy of entanglement:

$$S_A = -\text{Tr}(\rho_A \log \rho_A). \quad (57)$$

Expanding around $\rho_A = \hat{I}$, we get

$$S_A = -\text{Tr}[(\rho_A - \hat{I}) + \frac{1}{2}(\rho_A - \hat{I})^2 - \frac{1}{6}(\rho_A - \hat{I})^3 + ...]. \quad (58)$$

The LU-invariance follows from the fact that every term in this expansion satisfies (24). If we substitute the $n$-th term in the condition, we obtain

$$\text{Tr}([\hat{\epsilon}, \rho]_A(\rho_A - \hat{I})^{n-1}) = 0. \quad (59)$$
This is true either because $[\hat{\varepsilon}, \rho]_A = 0$ when $\hat{\varepsilon}$ does not act on $A$, or because otherwise $[\hat{\varepsilon}, \rho]_A = [\hat{\varepsilon}, \rho_A]$ and the equation follows from the cyclic invariance of the trace.

Now to prove that $S_A$ satisfies (30), we will first assume that $\rho_A^{-1}$ exists. Then we can formally write

$$\frac{\partial}{\partial \rho} \log \rho_A = \frac{\partial \rho_A}{\partial \rho} \frac{\partial}{\partial \rho} \log \rho_A = \frac{\partial \rho_A}{\partial \rho} \rho_A^{-1}. \quad (60)$$

Consider the case when $\hat{\varepsilon}$ does not act on $A$. Substituting $S_A$ in (30), we get

$$\frac{1}{4} \text{Tr} \left\{ \frac{\partial S_A}{\partial \rho} \left[ [\hat{\varepsilon}, \rho], \hat{\varepsilon} \right] \right\} + \text{Tr} \left\{ \frac{\partial^2 S_A}{\partial \rho \partial \rho^{\otimes 2}} \left( \text{Tr} \{ \hat{\varepsilon} \rho \rho - \frac{1}{2} \{ \hat{\varepsilon}, \rho \} \} \right)^{\otimes 2} \right\}$$

$$= 0 + \text{Tr} \left\{ \left( \frac{\partial}{\partial \rho} \otimes \left( - \log \rho_A \frac{\partial \rho_A}{\partial \rho} - \frac{\partial \rho_A}{\partial \rho} \right) \right) \left( \text{Tr} \{ \hat{\varepsilon} \rho \rho - \frac{1}{2} \{ \hat{\varepsilon}, \rho \} \} \right)^{\otimes 2} \right\}$$

$$= -\text{Tr} \left\{ \rho_A^{-1} \frac{\partial \rho_A}{\partial \rho} \frac{\partial \rho_A}{\partial \rho} \right\} \left( \text{Tr} \{ \hat{\varepsilon} \rho \rho - \frac{1}{2} \{ \hat{\varepsilon}, \rho \} \} \right)^{\otimes 2}$$

$$= -\text{Tr}_A \left\{ \rho_A^{-1} \left( \text{Tr} \{ \hat{\varepsilon} \rho \rho - \frac{1}{2} \{ \hat{\varepsilon}, \rho \} \} \right) \right\}$$

$$= -\text{Tr}_A \left\{ \left( \rho_A^{-1/2} \left( \text{Tr} \{ \hat{\varepsilon} \rho \rho - \frac{1}{2} \{ \hat{\varepsilon}, \rho \} \} \right) \right)^2 \right\} \leq 0. \quad (61)$$

If $\rho_A^{-1}$ does not exist, it is only on a subset of measure zero – where one or more of the eigenvalues of $\rho_A$ vanish. Therefore, we can always find an arbitrarily close vicinity in the parameters describing $\rho_A$, where $\rho_A^{-1}$ is regular and where (30) is satisfied. Since the condition is continuous, it cannot be violated on this special subset.

If $\hat{\varepsilon}$ acts on $A$, we can use an equivalent definition of the entropy of entanglement:

$$S_A = S_B = -\text{Tr} \{ \rho_B \log \rho_B \}, \quad (62)$$

and apply the same arguments. Therefore $S_A$ is an entanglement monotone for pure states.

The convexity condition is not satisfied, since

$$\text{Tr} \left\{ \frac{\partial^2 S_A}{\partial \rho \partial \rho^{\otimes 2}} \sigma^{\otimes 2} \right\} = -\text{Tr} \left\{ \rho_A^{-1} \sigma_A^{\otimes 2} \right\} \leq 0. \quad (63)$$

This reflects the fact that the entropy of entanglement, like $I_2$, does not distinguish between entanglement and classical randomness.

**VI. A NEW ENTANGLEMENT MONOTONE**

It has been shown [15] that the set of all entanglement monotones for a multipartite pure state uniquely determine the orbit of the state under the action of the group of local unitary transformations. For three-qubit pure states the orbit is uniquely determined by 5 independent continuous invariants (not counting the norm) and one discrete invariant [16, 17]. Therefore, for pure states of three qubits there must exist five independent continuous entanglement monotones that are functions of the five independent continuous invariants.
Any polynomial invariant in the amplitudes of a state

$$|\psi\rangle = \sum_{i,j,k} \alpha_{ijk} |i_A\rangle |j_B\rangle |k_C\rangle \cdots$$

is a sum of homogenous polynomials of the form [18]

$$P_{\sigma\tau\ldots}(|\psi\rangle) = \alpha_{i_1j_1k_1} \alpha^*_{i_1j_1k_1(1)} \cdots \alpha_{i_nj_nk_n} \alpha^*_{i_nj_nk_n(n)}$$

where $\sigma, \tau, \ldots$ are permutations of $(1,2,\ldots,n)$, and repeated indices indicate summation. A set of five independent polynomial invariants for three-qubit pure states is [18]

$$I_1 = P_{e, (12)}$$
$$I_2 = P_{(12), e}$$
$$I_3 = P_{(12), (12)}$$
$$I_4 = P_{(123), (132)}$$
$$I_5 = |\alpha_{i_1j_1k_1} \alpha_{i_2j_2k_2} \alpha_{i_2j_2k_2} \alpha_{i_4j_4k_4} \epsilon_{i_1i_2} \epsilon_{i_1i_2} \epsilon_{i_3i_4} \epsilon_{j_1j_2} \epsilon_{j_3j_4} |^2$$

In the last expression $\epsilon_{ij}$ is the antisymmetric tensor in 2 dimensions. The first three invariants are the local purities of subsystems C, B and A, $I_4$ is the invariant identified by Kempe [13] and $I_5$ is (up to a factor) the square of the 3-tangle identified by Coffman, Kundu and Wootters [19]. According to [15] the four known independent continuous entanglement monotones that do not require maximization over a multi-dimensional space are

$$\tau_{(AB)C} = 2(1 - I_1)$$
$$\tau_{(AC)B} = 2(1 - I_2)$$
$$\tau_{(BC)A} = 2(1 - I_3)$$
$$\tau_{ABC} = 2\sqrt{I_5}$$

and any fifth independent entanglement monotone must depend on $I_4$. Numerical evidence suggested that the tenth order polynomial $\sigma_{ABC} = 3 - (I_1 + I_2 + I_3)I_4$ might be such an entanglement monotone. However, no rigorous proof of monotonicity was given. Here, we will use conditions (24) and (30) to construct a different independent entanglement monotone, which is of sixth order in the amplitudes of the state and their complex conjugates.

Observe that in (64) the amplitudes have been combined in such a way that subsystem A is manifestly traced out. By appropriate rearrangement, one can write the same expression in a form where an arbitrary subsystem is manifestly traced out. Therefore, any polynomial invariant can be written entirely in terms of the components of $\text{Tr}_A \{\rho\}$ or $\text{Tr}_B \{\rho\}$, etc. This immediately implies that the LU-invariance condition (24) is satisfied, since if $\hat{\epsilon}$ acts on subsystem A, we can consider the expression in terms of $\rho_{BC\ldots}$, which, when substituted in (24), would yield zero because $[\hat{\epsilon}, \rho]_{BC\ldots} = 0$. It also implies that in order to prove monotonicity under local measurements we can only consider the second term in (30), since when $\hat{\epsilon}$ acts on subsystem A, we can again consider the expression for the function only in terms of $\rho_{BC\ldots}$ and the first term would vanish according to (53).

We will aim at constructing a polynomial function of three-qubit pure states $\rho$ which has the same form when expressed in terms of $\rho_{AB}$, $\rho_{AC}$, or $\rho_{BC}$, in order to avoid the necessity
for separate proofs of monotonicity under measurements on the different subsystems. It has been shown in [18] that
\[
I_4 = 3 \text{Tr} \left\{ \rho_{AB}(\rho_A \otimes \rho_B) \right\} - \text{Tr} \left\{ \rho_A^3 \right\} - \text{Tr} \left\{ \rho_B^3 \right\} \\
= 3 \text{Tr} \left\{ \rho_{AC}(\rho_A \otimes \rho_C) \right\} - \text{Tr} \left\{ \rho_A^3 \right\} - \text{Tr} \left\{ \rho_C^3 \right\} \\
= 3 \text{Tr} \left\{ \rho_{BC}(\rho_B \otimes \rho_C) \right\} - \text{Tr} \left\{ \rho_B^3 \right\} - \text{Tr} \left\{ \rho_C^3 \right\} .
\]
(74)

For local measurements on subsystem C it is convenient to use the first of the above expressions for \( I_4 \). The terms \( \text{Tr} \left\{ \rho_A^3 \right\} \) and \( \text{Tr} \left\{ \rho_B^3 \right\} \) are entanglement monotones by themselves. This can be easily seen by plugging them in condition (30):
\[
\frac{1}{4} \text{Tr} \left\{ \frac{\partial \text{Tr} \left\{ \rho_{A,B}^3 \right\}}{\partial \rho} \left[ [\hat{\varepsilon}, \rho], \varepsilon \right] \right\} + \text{Tr} \left\{ \frac{\partial^2 \text{Tr} \left\{ \rho_{A,B}^3 \right\}}{\partial \rho \circ^2} \left( \text{Tr} (\hat{\varepsilon} \rho) - \frac{1}{2} [\hat{\varepsilon}, \rho] \right) \circ^2 \right\} \\
= 0 + 6 \text{Tr} \left\{ \rho_{A,B} \left( \text{Tr} (\hat{\varepsilon} \rho) - \frac{1}{2} [\hat{\varepsilon}, \rho] \right) \circ^2 \right\} \geq 0.
\]
(75)

These terms, however, are not independent of the invariants \( I_2 \) and \( I_3 \). The term which is independent of the other polynomial invariants is \( \text{Tr} \left\{ \rho_{AB}(\rho_A \otimes \rho_B) \right\} \). When we plug this term into condition (30) we obtain an expression which is not manifestly positive or negative. Is it possible to construct a function dependent on this term, which similarly to \( \rho_{A,B}^3 \) would yield a trace of a manifestly positive operator when substituted in (30)?

It is easy to see that if the function has the form \( \text{Tr} \left\{ \hat{X}^3 \right\} \), where the operator \( \hat{X}(\rho_{AB}) \) is a positive operator linearly dependent on \( \rho_{AB} \), it will be an increasing monotone under local measurements on C (for simplicity we assume \( \hat{X}(0) = 0 \)):
\[
\frac{1}{4} \text{Tr} \left\{ \frac{\partial \text{Tr} \left\{ \hat{X}^3(\rho_{AB}) \right\}}{\partial \rho} \left[ [\hat{\varepsilon}, \rho], \varepsilon \right] \right\} + \text{Tr} \left\{ \frac{\partial^2 \text{Tr} \left\{ \hat{X}^3(\rho_{AB}) \right\}}{\partial \rho \circ^2} \left( \text{Tr} (\hat{\varepsilon} \rho) - \frac{1}{2} [\hat{\varepsilon}, \rho] \right) \circ^2 \right\} \\
= 0 + 6 \text{Tr} \left\{ \hat{X}(\rho_{AB}) \hat{X}^2 \left( \left( \text{Tr} (\hat{\varepsilon} \rho) - \frac{1}{2} [\hat{\varepsilon}, \rho] \right) \circ^2 \right) \right\} \geq 0.
\]
(76)

Since we want the function to depend on \( \text{Tr} \left\{ \rho_{AB}(\rho_A \otimes \rho_B) \right\} \), we choose \( \hat{X}(\rho_{AB}) = 2 \rho_{AB} + \rho_A \otimes I_B + I_A \otimes \rho_B \). This is clearly positive for positive \( \rho_{AB} \). Expanding the trace, we obtain:
\[
\text{Tr} \left\{ \hat{X}^3(\rho_{AB}) \right\} = 12 \text{Tr} \left\{ \rho_{AB}(\rho_A \otimes \rho_B) \right\} + 12 \text{Tr} \left\{ \rho_{AB}^2(\rho_A \otimes \rho_B) \right\} + 12 \text{Tr} \left\{ \rho_{AB}^2(\rho_A \otimes I_B) \right\} \\
+ 6 \text{Tr} \left\{ \rho_{AB}(\rho_A \otimes \rho_B)^2 \right\} + 6 \text{Tr} \left\{ \rho_{AB}(\rho_A \otimes I_B)^2 \right\} + 3 \text{Tr} \left\{ \rho_A \otimes \rho_B^2 \right\} \\
+ 3 \text{Tr} \left\{ \rho_A^2 \otimes \rho_B \right\} + \text{Tr} \left\{ I_A \otimes \rho_B^3 \right\} + \text{Tr} \left\{ \rho_A^3 \otimes I_B \right\} + 8 \text{Tr} \left\{ \rho_{AB}^3 \right\} \\
= 12 \text{Tr} \left\{ \rho_{AB}(\rho_A \otimes \rho_B) \right\} + 12 \text{Tr} \left\{ \rho_{AB}^2(\rho_A \otimes \rho_B) \right\} + 12 \text{Tr} \left\{ \rho_{AB}^2(\rho_A \otimes I_B) \right\} \\
+ 8 \text{Tr} \left\{ \rho_A^3 \right\} + 8 \text{Tr} \left\{ \rho_B^3 \right\} + 8 \text{Tr} \left\{ \rho_{AB}^3 \right\} + 3 \text{Tr} \left\{ \rho_A^3 \right\} + 3 \text{Tr} \left\{ \rho_B^3 \right\} .
\]
(77)

One can show that \( \text{Tr} \left\{ \rho_{AB}^2(\rho_A \otimes \rho_B) \right\} = \text{Tr} \left\{ \rho_{AC}(\rho_B \otimes \rho_C) \right\} \) and \( \text{Tr} \left\{ \rho_{AB}^2(\rho_A \otimes \rho_B) \right\} = \text{Tr} \left\{ \rho_{AC}(\rho_A \otimes \rho_C) \right\} \). We also have that \( \text{Tr} \left\{ \rho_{AB}^2 \right\} = \text{Tr} \left\{ \rho_{AB}^2 \right\} \). Using this and (74), we obtain
\[
\text{Tr} \left\{ \hat{X}^3(\rho_{AB}) \right\} = 12 I_4 + 16 \left( \text{Tr} \left\{ \rho_A^3 \right\} + \text{Tr} \left\{ \rho_B^3 \right\} + \text{Tr} \left\{ \rho_C^3 \right\} \right) + 3 \text{Tr} \left\{ \rho_A^3 \right\} + 3 \text{Tr} \left\{ \rho_B^3 \right\} .
\]
(78)
This expression is an increasing monotone under local measurements on $C$. If we add to it $3\text{Tr}\{\rho_{AB}^2\} = 3\text{Tr}\{\rho_C^2\}$, it becomes invariant under permutations of the subsystems. Since $\text{Tr}\{\rho_C^2\}$ is an increasing entanglement monotone, the whole expression will be a monotone under operations on any subsystem. We can define the closely related quantity

$$\phi_{ABC} = 69 - \text{Tr}\left\{(2\rho_{AB} + \rho_A \otimes I_B + I_A \otimes \rho_B)^3\right\} - 3\text{Tr}\{\rho_{AB}^2\}.$$  (79)

This is a decreasing entanglement monotone that vanishes for product states, which is more standard for a measure of entanglement. It depends on the invariant identified by Kempe and is therefore independent of the other known monotones for three-qubit pure states.

VII. CONCLUSIONS

We have derived differential conditions for a twice-differentiable function on quantum states to be an entanglement monotone. There are two such conditions for pure-state entanglement monotones—invariance under local unitaries and diminishing under local measurements—plus a third condition (overall convexity of the function) for mixed-state entanglement monotones. We have shown that these conditions are both necessary and sufficient. We then verified that the conditions are satisfied by a number of known entanglement monotones and we used them to construct a new polynomial entanglement monotone for three-qubit pure states.

It is our hope that this approach to the study of entanglement may circumvent some of the difficulties that arise due the mathematically complicated nature of LOCC. It may be possible to find new classes of entanglement monotones, for both pure and mixed states, and to look for functions with particularly desirable properties (such as additivity). There may also be other areas of quantum information theory where it will prove advantageous to consider general quantum operations as continuous processes. This seems a very promising new direction for research.

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After this paper was written, we became aware of the paper [20] by Plenio, which claims the existence of non-convex entanglement monotones. We would like to point out that such a conclusion arises from a different definition of entanglement monotones - namely, functions that obey (3a) but not necessarily (3b). At present, we do not know of any argument against imposing (3b), which was originally given in [11] and is briefly discussed in this paper. Since ultimately the requirement of (3b) is a matter of definition, just like condition (3a), we have conformed to the definition of entanglement monotones given in [11].

In this version of the paper we have corrected three minor mistakes that appear in the published article. The first one was a missing factor of 2 in condition (30), the second one was in the differential form of the convexity condition (44), and the third one was in the...
proof that the local purity is an entanglement monotone for pure states. Neither change alters the conclusions of the paper.

**Appendix: Proof of sufficiency**

The LU-invariance condition can be written as

\[ F(\rho, \hat{\varepsilon}) = 0, \]

(80)

where we define

\[ F(\rho, \hat{\varepsilon}) = f(e^{i\hat{\varepsilon}}\rho e^{-i\hat{\varepsilon}}) - f(\rho) \]

(81)

with \( \hat{\varepsilon} \) being a local hermitian operator. This condition has to be satisfied for every \( \rho \) and every \( \hat{\varepsilon} \). By expanding up to first order in \( \hat{\varepsilon} \) we obtained condition (24), which is equivalent to

\[ \text{Tr} \left\{ \frac{\partial F(\rho, \hat{\varepsilon})}{\partial \hat{\varepsilon}} \bigg|_{\hat{\varepsilon}=0} \right\} = 0. \]

(82)

This is a linear form of the components of \( \hat{\varepsilon} \) and the requirement that it vanishes for every \( \hat{\varepsilon} \) implies that

\[ \frac{\partial F(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij}} \bigg|_{\hat{\varepsilon}=0} = 0. \]

(83)

This has to be satisfied for every \( \rho \). Consider the first derivative of \( F(\rho, \hat{\varepsilon}) \) with respect to \( \varepsilon_{ij} \), taken at an arbitrary point \( \hat{\varepsilon}_0 \). We have

\[ \frac{\partial F(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij}} \bigg|_{\hat{\varepsilon}=\hat{\varepsilon}_0} = \frac{\partial F(\rho, \hat{\varepsilon}_0 + \hat{\varepsilon})}{\partial \varepsilon_{ij}} \bigg|_{\hat{\varepsilon}=0}. \]

(84)

But from the form of \( F(\rho, \hat{\varepsilon}) \) one can see that \( F(\rho, \hat{\varepsilon}_0 + \hat{\varepsilon}) = F(\rho', \hat{\varepsilon}) \), where \( \rho' = e^{i\hat{\varepsilon}_0}\rho e^{-i\hat{\varepsilon}_0} \). Therefore

\[ \frac{\partial F(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij}} \bigg|_{\hat{\varepsilon}=\hat{\varepsilon}_0} = \frac{\partial F(\rho', \hat{\varepsilon})}{\partial \varepsilon_{ij}} \bigg|_{\hat{\varepsilon}=0} = 0, \]

(85)

i.e., the first derivatives of \( F(\rho, \hat{\varepsilon}) \) with respect to the components of \( \hat{\varepsilon} \) vanish identically. This means that \( F(\rho, \hat{\varepsilon}) = F(\rho, 0) = 0 \) for every \( \hat{\varepsilon} \) and condition (24) is sufficient.

The condition for non-increase on average under local generalized measurements (29) can be written as

\[ G(\rho, \hat{\varepsilon}) \leq 0, \]

(86)

where

\[ G(\rho, \hat{\varepsilon}) = p_1 f(\hat{\mathsf{M}}_1 \rho \hat{\mathsf{M}}_1/p_1) + p_2 f(\hat{\mathsf{M}}_2 \rho \hat{\mathsf{M}}_2/p_2) - f(\rho). \]

(87)

The operators \( \hat{\mathsf{M}}_1 \) and \( \hat{\mathsf{M}}_2 \) in terms of \( \hat{\varepsilon} \) are given by (25), and the probabilities \( p_1 \) and \( p_2 \) are defined as before. As we have argued in section III, it is sufficient that this condition is satisfied for infinitesimal \( \hat{\varepsilon} \). By expanding the condition up to second order in \( \hat{\varepsilon} \) we obtained condition (30), which is equivalent to

\[ \text{Tr} \left\{ \frac{\partial^2 G(\rho, \hat{\varepsilon})}{\partial \hat{\varepsilon}^\otimes 2} \bigg|_{\hat{\varepsilon}=0} \right\} \leq 0. \]

(88)
Clearly, if this condition is satisfied by a strict inequality, it is sufficient, since corrections of higher order in \( \hat{\varepsilon} \) can be made arbitrarily smaller in magnitude by taking \( \hat{\varepsilon} \) small enough. Concerns about the contribution of higher-order corrections may arise only if the second-order correction to \( G(\rho, \hat{\varepsilon}) \) vanishes in some open vicinity of \( \rho \) and some open vicinity of \( \hat{\varepsilon} \) (we have assumed that the function \( f(\rho) \) is continuous). But the second-order correction is a real quadratic form of the components of \( \hat{\varepsilon} \) and it can vanish in an open vicinity of \( \hat{\varepsilon} \), only if it vanishes for every \( \hat{\varepsilon} \), i.e., if

\[
\frac{\partial^2 G(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \bigg|_{\hat{\varepsilon}=0} = 0. \quad (89)
\]

We will now show that if (89) is satisfied in an open vicinity of \( \rho \), there exists an open vicinity of \( \hat{\varepsilon} = 0 \) in which all second derivatives of \( G(\rho, \hat{\varepsilon}) \) with respect to \( \hat{\varepsilon} \) vanish identically. This means that all higher-order corrections to \( G(\rho, \hat{\varepsilon}) \) vanish in this vicinity and (86) is satisfied with equality.

Consider the two terms of \( G(\rho, \hat{\varepsilon}) \) that depend on \( \hat{\varepsilon} \):

\[
G_1(\rho, \hat{\varepsilon}) = p_1 f(\hat{M}_1 \rho \hat{M}_1/p_1), \quad (90)
\]

\[
G_2(\rho, \hat{\varepsilon}) = p_2 f(\hat{M}_2 \rho \hat{M}_2/p_2). \quad (91)
\]

They differ only by the sign of \( \hat{\varepsilon} \), i.e. \( G_1(\rho, \hat{\varepsilon}) = G_2(\rho, -\hat{\varepsilon}) \), and therefore

\[
\frac{\partial^2 G_1(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \bigg|_{\hat{\varepsilon}=0} = \frac{\partial^2 G_2(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \bigg|_{\hat{\varepsilon}=0} = \frac{1}{2} \frac{\partial^2 G(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \bigg|_{\hat{\varepsilon}=0}. \quad (92)
\]

If (89) is satisfied in an open vicinity of \( \rho \), we have

\[
\frac{\partial^2 G_1(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \bigg|_{\hat{\varepsilon}=0} = \frac{\partial^2 G_2(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \bigg|_{\hat{\varepsilon}=0} = 0 \quad (93)
\]

in this vicinity. Consider the second derivatives of \( G_1(\rho, \hat{\varepsilon}) \) with respect to the components of \( \hat{\varepsilon} \), taken at a point \( \hat{\varepsilon}_0 \):

\[
\frac{\partial^2 G(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \bigg|_{\hat{\varepsilon} = \hat{\varepsilon}_0} = \frac{\partial^2 G_1(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \bigg|_{\hat{\varepsilon} = \hat{\varepsilon}_0} + \frac{\partial^2 G_2(\rho, \hat{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \bigg|_{\hat{\varepsilon} = \hat{\varepsilon}_0} \quad (94)
\]

From the expression for \( G_1(\rho, \hat{\varepsilon}) \) one can see that \( \hat{\varepsilon} \) occurs in \( G_1(\rho, \hat{\varepsilon}) \) only in the combination \( \sqrt{\frac{\hat{I} - \hat{\varepsilon}}{2} - \rho} \). In \( G_1(\rho, \hat{\varepsilon} + \hat{\varepsilon}_0) \) it will appear only in \( \sqrt{\frac{\hat{I} - \hat{\varepsilon}_0 - \hat{\varepsilon}}{2} - \rho} \). But

\[
\sqrt{\frac{\hat{I} - \hat{\varepsilon}_0 - \hat{\varepsilon}}{2}} = \sqrt{\frac{\hat{I} - \hat{\varepsilon}'}{2}} \sqrt{\hat{I} - \hat{\varepsilon}_0}, \quad (95)
\]

where

\[
\hat{\varepsilon}' = \hat{\varepsilon}(\hat{I} - \hat{\varepsilon}_0)^{-1}. \quad (96)
\]
So we can write
\[ \sqrt{\frac{I - \hat{\varepsilon}_0 - \hat{\varepsilon}}{2}} \rho \sqrt{\frac{I - \hat{\varepsilon}_0 - \hat{\varepsilon}}{2}} = p' \sqrt{\frac{I - \hat{\varepsilon}'}{2}} \rho' \sqrt{\frac{I - \hat{\varepsilon}'}{2}}, \]  \tag{97}

where
\[ p' = p' G_1(\rho', \varepsilon'). \]  \tag{98}

and
\[ p' = \text{Tr} \left\{ \sqrt{I - \hat{\varepsilon}_0 \rho} \sqrt{I - \hat{\varepsilon}_0} \right\}. \]  \tag{99}

Then one can verify that
\[ G_1(\rho, \hat{\varepsilon}_0 + \hat{\varepsilon}) = p' G_1(\rho', \hat{\varepsilon}'). \]  \tag{100}

Similarly
\[ G_2(\rho, \hat{\varepsilon}_0 + \hat{\varepsilon}) = p'' G_2(\rho'', \hat{\varepsilon}''), \]  \tag{101}

where
\[ \hat{\varepsilon}'' = \hat{\varepsilon}(I + \hat{\varepsilon}_0)^{-1}, \]  \tag{102}

\[ p'' = \left( \frac{\sqrt{I + \hat{\varepsilon}_0 \rho} \sqrt{I + \hat{\varepsilon}_0}}{p'} \right). \]  \tag{103}

\[ p'' = \text{Tr} \left\{ \sqrt{I + \hat{\varepsilon}_0 \rho} \sqrt{I + \hat{\varepsilon}_0} \right\}. \]  \tag{104}

Note that \( \partial_{p,q} \partial_{\varepsilon_{ij}} \) and \( \partial_{p,q} \partial_{\varepsilon_{ij}} \) have no dependence on \( \hat{\varepsilon} \). Nor do \( p' \) and \( p'' \). Therefore we obtain
\[ \frac{\partial^2 G(\rho, \hat{\varepsilon})}{\partial_{\varepsilon_{ij}} \partial_{\varepsilon_{kl}}} |_{\hat{\varepsilon}=0} = p' \frac{\partial^2 G_1(\rho', \hat{\varepsilon}')}{\partial_{\varepsilon_{ij}} \partial_{\varepsilon_{kl}}} |_{\hat{\varepsilon}'=0} + p'' \frac{\partial^2 G_2(\rho'', \hat{\varepsilon}'')}{\partial_{\varepsilon_{ij}} \partial_{\varepsilon_{kl}}} |_{\hat{\varepsilon}''=0} + \sum_{p,q,r,s} \frac{\partial_{p,q} \partial_{\varepsilon_{rs}}}{\partial_{\varepsilon_{ij}} \partial_{\varepsilon_{kl}}} p' \frac{\partial^2 G_1(\rho', \hat{\varepsilon}')}{\partial_{\varepsilon_{ij}} \partial_{\varepsilon_{kl}}} |_{\hat{\varepsilon}'=0} + \sum_{p,q,r,s} \frac{\partial_{p,q} \partial_{\varepsilon_{rs}}}{\partial_{\varepsilon_{ij}} \partial_{\varepsilon_{kl}}} p'' \frac{\partial^2 G_2(\rho'', \hat{\varepsilon}'')}{\partial_{\varepsilon_{ij}} \partial_{\varepsilon_{kl}}} |_{\hat{\varepsilon}''=0}. \]  \tag{105}

We assumed that (23) is satisfied in an open vicinity of \( \rho \). If \( p' \) and \( p'' \) are within this vicinity, the above expression will vanish. But from (23) and (103) we see that as \( \|\hat{\varepsilon}_0\| \) tends to zero, the quantities \( \|\rho' - \rho\| \) and \( \|\rho'' - \rho\| \) also tend to zero. Therefore there exists an open vicinity of \( \hat{\varepsilon}_0 = 0 \), such that for every \( \hat{\varepsilon}_0 \) in this vicinity, the corresponding \( p' \) and \( p'' \) will be within the vicinity of \( \rho \) for which (13) is satisfied and
\[ \frac{\partial^2 G(\rho, \hat{\varepsilon})}{\partial_{\varepsilon_{ij}} \partial_{\varepsilon_{kl}}} |_{\hat{\varepsilon}=\hat{\varepsilon}_0} = 0. \]  \tag{106}

This means that higher derivatives of \( G(\rho, \hat{\varepsilon}) \) with respect to the components of \( \hat{\varepsilon} \) taken at points in this vicinity will vanish, in particular derivatives taken at \( \hat{\varepsilon} = 0 \). So higher order corrections in \( \hat{\varepsilon} \) to \( G(\rho, \hat{\varepsilon}) \) will also vanish. Therefore \( G(\rho, \hat{\varepsilon}) = 0 \) in the vicinity of \( \rho \) for
which we assumed that (30) is satisfied with equality, which implies that condition (30) is sufficient.