Isogeometric Analysis for Thin Square Bending Plate using Collocation Method

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Abstract. This paper discusses the evaluation of rectangular thin plate problem with uniform load using Isogeometric analysis (IGA) collocation method. The implementation details of the proposed method for problems with different boundary conditions are shown. The numerical results confirm that the proposed method represents an efficient formulation and shows good behavior for thin plate structures.

1. Introduction

Finite Element Analysis (FEA) is the most popular and powerful numerical method that is widely used in many engineering applications. The results of our research in Finite Element Method have been published in [1-13]. In general, finite element analysis (FEA) is unable to represent exact geometry of structural problems. It needs to approximate the geometry of structure to obtain representative results with a little deviation from the exact solutions. The shape functions and meshes which is used in FEA are the important factor that are able to increase the accuracy of FEA results. Shape functions of FEA can be either linear or quadratic which requires finer meshes to represent actual geometry. Meanwhile, isogeometric analysis uses a high-regularity property of its basis functions which makes IGA able to approximate the exact geometry faster and minimize the usage of finer meshes by standard FEA.

Isogeometric analysis (IGA) is proposed by Hughes and Cottrell [14] in 2005 by adapting model of geometry by using NURBS [15]. The basic IGA consists of the same basis functions used for geometry representations in CAD systems. In symmetrical problems such as rectangular plate, NURBS is the same as B-Splines. Development of IGA using MATLAB is also done by Nguyen et.al [16,17]. Auricchio et.al. [18] developed IGA collocation method. The basic idea consists of the discretization of the governing partial differential equations in strong form. Reali and Hughes [19] improved collocation approach and Reali and Gomez [20] using collocation method for Bernoulli-Euler beams and Kirchhoff plates. Beirão da Veiga et.al [21] have considered an IGA collocation approach for the approximation of initially straight planar Timoshenko beams. Kendl et.al. [22] proposed IGA collocation for the numerical simulation of Reissner–Mindlin plate problems. In 2017, the development of Isogeometric Galerkin based on UI approach have been proposed by Katili [23]

In this paper, we focus in discussing IGA collocation for the solution of thin plate structure described by fourth-order differential equations of Kirchhoff plate models. The numerical evaluation is focused on displacement and bending moment at the center of plate. The purpose of the numerical tests is to study the convergence speed of the method to the reference solutions by increasing the polynomial degree $p = 4, 6, 8, 10$ with only one element and increasing the number of elements with $p = 4$. In this paper three types of boundary condition will be evaluated. The implementation of IGA collocation for rectangular thin plates with several boundary conditions confirm the good behavior of the proposed formulations.
2. Principal of Kirchhoff Plate theory

In Kirchhoff plate theory the relation between vertical displacement and rotation is

\[
\{\beta\} = \begin{pmatrix}
\beta_x \\
\beta_y \\
\beta_{xy}
\end{pmatrix} = \begin{pmatrix}
-w_{,xx} \\
-w_{,yy}
\end{pmatrix}
\]

(1)

Where \( w \) is the total displacement (deflection) of the points on the middle plane \((z = 0)\), \( \beta_x \) is the rotation in the plane \( z-x \) and \( \beta_y \) is the rotation in the plane \( z-y \). The first derivative of total displacement with respect to \( x \) and \( y \) are denoted by \( w_x, w_y \) respectively. The curvatures

\[
\{\chi\} = \begin{pmatrix}
\chi_x \\
\chi_y \\
\chi_{xy}
\end{pmatrix} = \begin{pmatrix}
\beta_{x,xx} + \beta_{y,yy} \\
\beta_{x,yy} + \beta_{y,xx}
\end{pmatrix}
\]

(2)

Where \( \beta_{x,xx} \) and \( \beta_{x,yy} \) are the first derivative of \( \beta_x \) with respect to \( x \) and \( y \), respectively, while \( \beta_{y,xx} \) and \( \beta_{y,yy} \) are the first derivative of \( \beta_y \) with respect to \( x \) and \( y \), respectively. Substituting (1) into (2) gives

\[
\{\chi\} = \begin{pmatrix}
\chi_x \\
\chi_y \\
\chi_{xy}
\end{pmatrix} = \begin{pmatrix}
w_{,xx} \\
w_{,yy} \\
2w_{,xy}
\end{pmatrix}
\]

(3)

The bending moment \( \{M\} \) are defined as the following constitutive equations

\[
\{M\} = \begin{pmatrix}
M_x \\
M_y \\
M_{xy}
\end{pmatrix} = \begin{bmatrix}
H_b
\end{bmatrix} \begin{pmatrix}
\chi_x \\
\chi_y \\
\chi_{xy}
\end{pmatrix}
\]

(4)

Where \([H_b]\) is

\[
[H_b] = D_b \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1 - \nu^2 / 2
\end{bmatrix} \quad; \quad D_b = \frac{Eh^3}{12(1-\nu^2)}
\]

(5)

\( D_b \) is the generalized bending constitutive matrix, \( E \) is the Young’s Modulus, \( \nu \) is the Poisson’s ratio, and \( h \) is the plate thickness. The equilibrium equations of plate

\[
\begin{align*}
M_{x,xx} + M_{xy,yy} - T_x &= 0 \\
M_{y,yy} + M_{xy,xx} - T_y &= 0 \\
T_{xx} + T_{yy} + f_z &= 0
\end{align*}
\]

(6) (7) (8)

Where \( f_z \) is distributed load. Differentiating the shear forces in (6) and (7) with respect to \( x \) and \( y \) respectively, and substituting the derivatives into (8) gives

\[
M_{x,xxx} + 2M_{xy,xy} + M_{y,yyy} = f_z
\]

(9)

This is the second order of differential equation in Kirchhoff plate model. In order to use collocation method in isogeometric analysis, it is necessary to obtain relationship between uniform loading and displacement. The relationship is called strong form. Substituting equation (3) into equation (4) and then substituting the result into (9) give the fourth order differential equation relating to the Kirchhoff thin plate theory for isotropic material.

\[
D_b \left( w_{xxx} + 2w_{xyy} + w_{yyy} \right) = f_z
\]

(10)
Equation (10) is the strong form equation. Since uniform loading is fourth derivative of displacement, the minimum polynomial degree suitable for collocation method is 4. Boundary conditions are needed to solve the problem. The boundary conditions are

\[ \bar{w} = w \rightarrow \Gamma_w \]  
\[ \beta_x = -w_x \rightarrow \Gamma_{\beta_x} \]  
\[ \beta_y = -w_y \rightarrow \Gamma_{\beta_y} \]  
\[ \bar{M}_x = -D_b \left( w_{xx} + \omega w_{yy} \right) \rightarrow \Gamma_{M_x} \]  
\[ \bar{M}_y = -D_b \left( w_{yy} + \omega w_{xx} \right) \rightarrow \Gamma_{M_y} \]  
\[ \bar{T}_x = -D_b \left( w_{xxy} + w_{yyx} \right) \rightarrow \Gamma_{T_x} \]  
\[ \bar{T}_y = -D_b \left( w_{xyy} + w_{yxx} \right) \rightarrow \Gamma_{T_y} \]

3. **Isogeometric analysis**

3.1. **B-Splines function**

Basis function is the foundation of isogeometric analysis. B-splines as basis function are generated from knot vectors. Knot vectors value start from 0 and end with 1.

Knot vector is relying on two factors that is polynomial degree and number of elements. Polynomial degree affects knot vector in terms multiplicity. Multiplicity only applies at the first and the last knot value (knot zero and one) to amount of \( p+1 \) for each knot. Multiplicity has open knot vector behavior. Number of elements affects knot span value. Increasing the number of elements will increase the number of knot span and increase number of knots in knot vector.

B-splines is a recursive function meaning in order to obtain higher polynomial results, it is necessary to calculate using lower polynomial degree first. For polynomial degree of four (\( p = 4 \)), we need to calculate B-spline function from polynomial degree of zero and recalculate it up to polynomial degree of four. A knot vector in direction \( \xi \) and \( \eta \)

\[ \Xi_\xi = \left[ \xi_1, \xi_2, \ldots, \xi_{n+p+1} \right] \quad ; \quad \Xi_\eta = \left[ \eta_1, \eta_2, \ldots, \eta_{m+q+1} \right] \]

(18)

Where:

\( m \) and \( n \) are the number of B-Splines function in each direction.

\( p \) and \( q \) are polynomial degree of B-Splines in \( \xi \) and \( \eta \) directions, respectively.

In each direction, the number of knot in knot vector depends on both multiplicity and knot span. According to open knot vector properties applied in (18), there are multiplicity of the first (\( \xi_1 \) and \( \eta_1 \)) and the last knot vector (\( \xi_{n+p+1} \) and \( \eta_{m+q+1} \)), as many as \( (p+1) \) for \( \xi \) axis and \( (q+1) \) for \( \eta \) axis. On the other hand, knot span is a range of value between knot \( \xi_i \) and \( \xi_{i+1} \). The number of knot span depends on the number of element in each direction. When the number of element increases, the number of knot span increases, which also increases the number of knot in knot vector. The number of knot in knot vector is \( (n+p+1) \) for \( \Xi_\xi \) and \( (m+q+1) \) for \( \Xi_\eta \). From the number of knot in knot vector, the number of basis functions can be known.

The zeroth degree of B-Splines [15] in \( \xi \) direction is defined by:

\[ N_{\xi,0}(\xi) = \begin{cases} 1, & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0, & \text{otherwise} \end{cases} \]

(19)

The \( p-th \) degree of B-Splines in \( \xi \) direction is defined recursively using the relation

\[ N_{\xi,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{\xi,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{\xi+1,p-1}(\xi) \]

(20)
The zeroth degree of B-Splines \( \{M_j, 0\} \) in \( \eta \) direction is defined as
\[
M_{j,0}(\eta) = \begin{cases} 
1, & \text{if } \eta_i \leq \eta < \eta_{j+1} \\
0, & \text{otherwise}
\end{cases} \tag{21}
\]

The \( q \)-th degree of B-Splines in \( \eta \) direction is defined recursively using the relation
\[
M_{j,q}(\eta) = \frac{\eta - \eta_j}{\eta_{j+q} - \eta_j} M_{j,q-1}(\eta) + \frac{\eta_{j+q+1} - \eta}{\eta_{j+q+1} - \eta_{j+1}} M_{j+1,q-1}(\eta) \tag{22}
\]

Thereafter, in rectangular plate, we need B-Splines in two dimension. The two-dimension B-Splines can be defined by taking tensor products of their B-Splines in one-dimensional counterparts i.e. \( N_i(\bar{\xi}) \) in \( \xi \) direction and \( M_j(\eta) \) in \( \eta \) direction.
\[
R_k = R_{i,j}(\bar{\xi}, \eta) = N_i(\bar{\xi})M_j(\eta) \tag{23}
\]

Where:
\( i = 1, ..., n \); \( j = 1, ..., m \); \( k = i + n(j - 1) \); \( m \) and \( n \) is the number of basis function in \( \xi \) and \( \eta \) directions.

3.2. Displacement function and control variable

Control variable in isogeometric analysis is analogous to degree of freedoms in finite element analysis. However, the control variable values are not the real displacements but the values are required to obtain displacement function needed to get the real displacement value at specified point. This is described as:
\[
w(\bar{\xi}, \eta) = \{R(\bar{\xi}, \eta)\}\{\hat{\eta}_n\} \tag{24}
\]

where \( R(\bar{\xi}, \eta) \) is basis function for rectangular plate and \( \{\hat{\eta}_n\} \) are control variables. The function above shows us that basis function multiplies by control variable will give displacement function. This means that this function is the connection between isogeometric analysis and finite element.

3.3. Collocation method

The main idea of collocation method is discretizing strong form equations (10) and collocating them on a set of suitable points. For collocation points, we use Greville abscissae which are knot averages defined by knot vector and polynomial degrees. Collocation points are related to boundary of element in parametric space. The number of collocation points are the same as the number of control variables. This is described as:
\[
\bar{\xi}_i = \frac{1}{p} \sum_{j=1}^{p} \xi_{i+j} ; i = 1, ..., n \quad \bar{\eta}_j = \frac{1}{q} \sum_{k=1}^{q} \eta_{k+i} ; j = 1, ..., n \tag{25}
\]

Collocation points are stated in term of coordinate \((\bar{\xi}, \bar{\eta})\) to be used in strong form and boundary condition function. Using equation (10–25), we are able to obtain control variable which will be used to get real displacement, rotation, moment or shear values at any points.

4. Numerical Examples

This section shows the numerical results of IGA collocation in thin plate (for \( L/h = 100 \)). The numerical results are focused on displacement and bending moment at the center of plate. The purpose of the numerical tests is to study the convergence speed of the method to the reference solutions using two ways:

1. Increasing the polynomial degree \( (p = 4, 6, 8, 10) \) with only one element (NELT = 1×1)
2. Increasing the number of elements (NELT = 1×1, 2×2, 4×4, 8×8) with \( p = 4 \) three types of boundary conditions are considered:
   a. Fully clamped plate (CCCC)
   b. Simply supported plate (SSSS)
   c. Simply supported plate along \( x = 0 \) and \( x = L \) (SSFF).
4.1 Fully clamped thin plate (CCCC)

The first case is Fully clamped thin plate (CCCC) as in Figure 1. To solve this case, we need to apply some boundary conditions that are displacement, and rotations in x and y direction. From Figure 2 and Table 1, we are able to obtain convergence either by polynomial degree escalation or by number of element addition. Analyzing based on polynomial degree, \( p = 4 \) gives 38.14\% error compared to the reference solution given by Batista [24], and by increasing polynomial degree to \( p = 10 \) the error decreases to 1.47\%. Analyzing based on number of element addition, by increasing element to 8×8, we are able to obtain 0.94\% error with respect to the reference by Batista [24].

![Figure 1. CCCC plate](image)

**Table 1.** Displacement at the center of CCCC thin plate

| \( p \) Escalation (NELT = 1×1) | NELT Addition (\( p = 4 \)) |
|---|---|
| \( w \) | Error (%) | NELT | \( w \) | Error (%) |
| 4 | -782522.32 | 38.14 | 1×1 | -782522.32 | 38.14 |
| 6 | -1236782.82 | 28.30 | 2×2 | -988552.39 | 21.85 |
| 8 | -1268146.62 | 2.23 | 4×4 | -1232275.00 | 2.59 |
| 10 | -1267844.27 | 1.47 | 8×8 | -1276935.65 | 0.94 |

Ref. -1265000 (Batista [24])

![Figure 2. Convergence of normalized displacement at the center of CCCC thin plate](image)

**Table 2.** Moment at the center of CCCC thin plate

| \( p \) Escalation (NELT = 1×1) | NELT Addition (\( p = 4 \)) |
|---|---|
| \( M_x = M_y \) | Error (%) | NELT | \( M_x = M_y \) | Error (%) |
| 4 | -16250.00 | 29.65 | 1×1 | -16250.00 | 29.65 |
| 6 | -22571.68 | 2.29 | 2×2 | -20533.07 | 11.11 |
| 8 | -22916.49 | 0.79 | 4×4 | -23448.50 | 1.51 |
| 10 | -22904.58 | 0.85 | 8×8 | -23219.41 | 0.52 |

Ref. -23100 (Batista [24])
Figure 3 and Table 2 present the convergence of moment in the center of plate for the CCCC case. Convergence of moment can be obtained either by polynomial degree escalation or by number of element addition. Convergence by polynomial degree escalation is reached by \( p = 6 \) and higher. Satisfying results of number of element addition are reached when \( 4 \times 4 \) elements or finer meshes are applied.

**Figure 3.** Convergence of normalized moment at the center of CCCC thin plate

4.2 *Simply supported thin plate (SSSS)*

To solve this case as in Figure 4, we need to apply some boundary conditions that are displacement, and moment in \( x \) and \( y \) direction. The results of displacement in the center of plate are given below in Tables 2 – 3 and Figure 5 – 6.

**Figure 4.** SSSS plate

### Table 3. Displacement at the center of SSSS thin plate

| \( p \) Escalation (NELT = 1×1) | NELT Addition |
|---|---|
| \( p \) | \( w \) | Error (%) | \( \text{NELT} \) | \( w \) | Error (%) |
| 4 | -2960904 | 27.11 | 1×1 | -2960904 | 27.11 |
| 6 | -3958021 | 2.56 | 2×2 | -3289114 | 19.03 |
| 8 | -4057473 | 0.11 | 4×4 | -3866878 | 4.80 |
| 10 | -4062993 | 0.02 | 8×8 | -4042372 | 0.48 |

Ref. -4062000 (Batista [24])
Figure 5. Convergence of normalized displacement at the center of SSSS thin plate

We can see in Table 3 and Figure 5 the convergence of displacement in the center of plate. The good results given by polynomial escalation can be obtained from \( p = 6 \) with error 2.56% to \( p = 10 \) with error 0.02%. By number of elements addition, good results can be obtained starting from \( 4 \times 4 \) elements with error 4.8%. Based on the results, \( p \) escalation gives better convergence than number of element addition does.

Figure 6 and Table 5 present the convergence of moment in the center of plate for SSSS case. The results show the same behavior as the results of displacement in the center of plate. Degree of polynomial escalation gives better convergence than number of element addition does.

Table 4. Moment in the center of SSSS thin plate

| \( p \)   | \( p \) Escalation (NELT = 1×1) | NELT Addition (\( p = 4 \)) |
|---------|--------------------------------|-----------------------------|
|         | \( M_x = M_y \) | Error (%) | NELT | \( M_x = M_y \) | Error (%) |
| 4       | -36932             | 22.90     | 1×1  | -36932          | 22.90     |
| 6       | -46937             | 2.01      | 2×2  | -41026           | 14.35     |
| 8       | -47834             | 0.14      | 4×4  | -46526           | 2.87      |
| 10      | -47876             | 0.05      | 8×8  | -47799           | 0.21      |

Ref. -47900 (Batista [24])

Figure 6. Convergence of normalized moment at the center of SSSS thin plate

4.3 Thin plate with simply supported sides along \( x = 0 \) and \( x = L \) (SSFF)

To solve this case as in Figure 7, we need to apply some boundary conditions that are displacement, and zero moment in \( x \) and \( y \) direction for simply supported sides.
As can be seen in Figure 8 and Table 5, both results show that convergence can be obtained using even the lowest possible polynomial degree \( p = 4 \) with NELT = 1×1. Both \( p \) escalation and number of element addition show good results. However, increasing polynomial degree or NELT do not change the results.

**Table 5.** Displacement in the center of SSFF thin plate

| \( p \) Escalation (NELT = 1×1) | NELT Addition (\( p = 4 \)) |
|-------------------------------|----------------------------|
| \( p \) | \( w \) | Error (%) | NELT | \( w \) | Error (%) |
| 4    | -13024405 | 0.53 | 1×1 | -13024405 | 0.53 |
| 6    | -13024405 | 0.53 | 2×2 | -13024405 | 0.53 |
| 8    | -13024405 | 0.53 | 4×4 | -13024405 | 0.53 |
| 10   | -13024405 | 0.53 | 8×8 | -13024405 | 0.53 |

Ref. -13093680 (Batista [24])

**Figure 8.** Convergence of normalized displacement at the center of SSFF thin plate

Figures 9 – 10 and Tables 6 – 7 present the convergence of moment in the center of plate for SSFF case. Both \( M_x \) and \( M_y \) results show the same behavior as displacement in the center of plate. Convergence can be obtained with \( p = 4 \) and NELT = 1×1. Higher polynomial or higher number of elements still give good results.

**Table 6.** Moment \( M_x \) in the center of SSFF thin plate

| \( p \) Escalation (NELT = 1×1) | NELT Addition (\( p = 4 \)) |
|-------------------------------|----------------------------|
| \( p \) | \( M_x \) | Error (%) | NELT | \( M_x \) | Error (%) |
| 4    | -125000  | 2.25 | 1×1 | -125000  | 2.25 |
| 6    | -125000  | 2.25 | 2×2 | -125000  | 2.25 |
| 8    | -125000  | 2.25 | 4×4 | -125000  | 2.25 |
| 10   | -125000  | 2.25 | 8×8 | -125000  | 2.25 |

Ref. -122545 (Batista [24])
Figure 9. Convergence of normalized moment $M_y$ at the center of SSFF thin plate

Table 7. Moment $M_y$ in the center of SSFF thin plate

| $p$ Escalation (NELT = 1×1) | NELT Addition ($p = 4$) |
|-----------------------------|-------------------------|
| $p$ | $M_y$ | Error (%) | NELT | $M_y$ | Error (%) |
| 4  | -26250 | 3.06 | 1×1 | -26250 | 3.06 |
| 6  | -26250 | 3.06 | 2×2 | -26250 | 3.06 |
| 8  | -26250 | 3.06 | 4×4 | -26250 | 3.06 |
| 10 | -26250 | 3.06 | 8×8 | -26250 | 3.06 |

Ref. -27078 (Batista [24])

Figure 10. Convergence of normalized moment $M_y$ at the center of SSFF thin plate

5. Conclusions
This paper presents an IGA collocation method for thin structural plate governed by Kirchhoff model. Based on all numerical tests, we can conclude that IGA collocation method using both polynomial degree escalation and number of element addition are able to obtain good approximation of the exact solution even with relatively coarse mesh. The numerical results show the accuracy, stability, and robustness of the method.

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