Hierarchy wave functions—from conformal correlators to Tao-Thouless states

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Laughlin’s wave functions, describing the fractional quantum Hall effect at filling factors $\nu = 1/(2k+1)$, can be obtained as correlation functions in conformal field theory, and recently this construction was extended to Jain’s composite fermion wave functions at filling factors $\nu = n/(2kn+1)$ \textsuperscript{1}. Here we generalize this latter construction and present ground state wave functions for all quantum Hall hierarchy states that are obtained by successive condensation of quasi-electrons (as opposed to quasiholes) in the original hierarchy construction. By considering these wave functions on a cylinder, we show that they approach the exact ground states, the Tao-Thouless states, when the cylinder becomes thin. We also present wave functions for the multi-hole states, make the connection to Wen’s general classification of abelian quantum Hall fluids, and discuss whether the fractional statistics of the quasiparticles can be analytically determined. Finally we discuss to what extent our wave functions can be described in the language of composite fermions.

I. INTRODUCTION

Around 1990 it was noted that Laughlin’s wave functions \textsuperscript{1} for the fractional quantum Hall (QH) effect take the form of correlation functions in conformal field theories (CFT) \textsuperscript{2, 3, 4, 5}. It was conjectured that this is true for general QH-states, and that the same CFT that gives the ground state also describes the edge excitations \textsuperscript{3, 6}. Several other examples of wave functions that can be written as conformal correlators were given, and in particular the pfaffian which has quasiparticles that obey non-abelian fractional statistics was proposed \textsuperscript{6}. This wave function is now believed to describe the observed gapped state at $\nu = 5/2$ \textsuperscript{7}. Recently it was shown that the composite fermion wave functions in the Jain sequence $\nu = n/(2kn+1)$ \textsuperscript{8, 9}, $k,n = 1,2,\ldots$ can be constructed from correlators in a CFT with $n$ bosonic fields \textsuperscript{10}.

The Laughlin state at $\nu_1 = 1/t_1$, $t_1 = 3,5,\ldots$ has quasihole and quasi-electron excitations with fractional charge $e^* = \pm e/t_1$. According to the original hierarchy scheme \textsuperscript{11, 12, 13}, these quasiparticles may condense and form new fractional QH states at filling factors $\nu_2 = 1/(t_1 \pm 1/t_2)$, $t_2 = 2,4,6,\ldots$ in much the same way as the electrons condense to form the Laughlin states. In the case of quasihole condensation we must also include the $t_1 = 1$ parent state corresponding to a filled Landau level. The quasiparticles in these new QH states may then condense forming new states—a procedure that can be repeated ad infinitum, producing a unique QH state at each rational filling factor $\nu = p/q \leq 1$, where $q$ is odd. Such a hierarchy state is characterized by the set \{1,2,3,\ldots,$t_n$\}, where the level of the hierarchy $n = 1,2,\ldots$ is the number of condensates, $a_i = \pm 1$ depending on whether condensate $i$ consists of quasiholes or quasi-electrons. Note that $t_1$ is odd, whereas $t_i = 2,4,\ldots$, $i \geq 2$, are even.

In section \textsuperscript{14} we present candidate wave functions for all hierarchy states that are obtained by successive condensation of quasi-electrons, as opposed to quasiholes, \textit{i.e.} for the states with $a_i = -1$: \{1,2,\ldots,$t_n$\}. The wave functions are obtained using conformal field theory: a state at level $n$ is a correlation function in a theory with $n$ bosonic fields. If $a_i t_i = -2$ for $i = 2,\ldots,n$, we get the Jain sequence $\nu_n = n/(t_1 - 1)n + 1) = n/(2kn+1),k = 1,2,\ldots$, and the construction here reduces to the one presented in Ref. 10 where it was shown that these wave functions are identical to Jain’s composite fermion wave functions. The $t_i$’s determine the densities of the $n$ condensates that build up the state—the composite fermion state at level $n$ is the one where all but the first of these condensates have maximal density; when $t_i$ increases the density decreases. The wave functions presented here are thus obtained by a very natural generalization of the construction that gives the well-established ground state wave functions for the Jain fractions. Indeed, for general \{1,2,\ldots,$t_n$\}, the wave functions may alternatively be interpreted in terms of fractional quantum Hall states of composite fermions.

It has been noted that the QH problem is exactly solvable at any rational filling factor in a certain limit, and it has been argued that this solution is adiabatically connected to the experimentally realized fractional QH state \textsuperscript{14, 15, 16}. This limit can be obtained by considering the (polarized) electron gas on a cylinder or torus—this gives a natural mapping to a one-dimensional system—and letting the cylinder become thin. The electron-electron interaction then becomes purely electrostatic, and the ground state is a one-dimensional gapped crystal. Equivalently, one may consider this as the original infinite two-dimensional system with a modified electron-electron interaction. We call this limit the Tao-Thouless (TT) limit, and the ground states Tao-Thouless states, since the solution in this limit, for a Laughlin fraction, is the state proposed by Tao and Thouless as an explanation for the fractional QH effect \textsuperscript{17}. In the TT-limit the hierarchy
construction of QH states as successive condensates of quasiparticles is manifest.

We have candidate bulk wave functions for hierarchy states that consist of condensates of quasielectrons only, but states involving condensations of quasiholes also exist. In the TT-limit, there is a TT-ground state at each \( \nu = p/q, \) \( q \) odd, 
that we believe corresponds to a potential bulk fractional QH state. In the occupation number representation used in the TT-limit, the particle-hole symmetry of the lowest Landau level is manifest. This is not the case for the wave functions, and we believe that it is for this technical reason that we have not been able to obtain explicit wave functions for condensates involving quasiholes.

In section III we show that the hierarchy wave functions \( \{ t_1, t_2, \ldots, t_n \} \) that we construct using conformal techniques, reduce to the TT-states in the TT-limit, where they are constructed by \( n \) successive condensations of (quasi)electrons. In section IV we present the general quasihole wave functions and discuss their properties. Here we also explain how our wave functions fit into Wen’s classification of abelian quantum Hall fluids. Finally, in section V we discuss the relationship between our proposed hierarchy wave functions and the composite fermion scheme.

The new ground state wave functions discussed here have been briefly reported in Ref. 16. Several earlier approaches to hierarchy wave functions, in addition to Jain’s composite fermions, exist. In this context, we would like to refer to the construction of composite fermion wave functions directly in the lowest Landau level by Ginocchio and Haxton 18 and the work by Wojs, Quinn, and collaborators, see Ref. 19 and references therein. Hierarchy wave functions in terms of quasiparticle coordinates were obtained by Moore and Read using conformal techniques 20, whereas explicit wave functions were obtained by other methods by Greiter 20; a conformal approach was used by Flohr and Osterloh 21.

II. HIERARCHY WAVE FUNCTIONS

We here construct a unique ground state for each filling factor that is obtained, within the original hierarchy scheme, by successive condensation of quasielectrons. The state is obtained using conformal techniques generalizing the construction of the Laughlin states 22 and the Jain states 10.

First we recall the conformal field theory approach to the Laughlin states 22. On the plane, with complex coordinates \( z \), we introduce the vertex operator \( V_1(z) = : e^{i\varphi_1(z)} : \),

\[
V_1(z) = : e^{i\varphi_1(z)} : ,
\]

where the normal ordering symbol \( : : \) will be suppressed in the following. \( \varphi_1(z) \) is a free massless holomorphic bosonic field, normalized so that the propagator becomes

\[
\langle \varphi_1(z)\varphi_1(w) \rangle = -\ln(z-w) .
\]

This implies that vertex operators obey the relation

\[
e^{i\alpha\varphi_1(z)}e^{i\beta\varphi_1(w)} = e^{i(\alpha\varphi_1(z) + \beta\varphi_1(w))} = (z-w)^{i\alpha\beta}e^{i(\alpha+\beta)\varphi_1(w)} ,
\]

where the last line is the operator product expansion valid as \( z \to w \). Note that \( e^{i\alpha\varphi_1(z)} \) and \( e^{i\beta\varphi_1(w)} \) (anti)commute if \( \alpha \) is even (odd).

The Laughlin wave function is obtained as a correlation function in the CFT of a string of radially ordered operators

\[
\langle V_1(z_1) V_1(z_2) \ldots V_1(z_N) \rangle = \prod_{i<j} (z_i-z_j)^2 e^{-\sum_{i,j} \frac{|z_i|^2 - |z_j|^2}{(z_i-z_j)^2} ,}
\]

where \( \langle \ldots \rangle \) is an expectation value in a vacuum state with an appropriately chosen background charge. The crucial polynomial part follows directly from the operator product expansion in 3, whereas the gaussian factor is obtained from the background charge. In the following we will suppress all gaussian factors. Equation 4 gives the Laughlin wave functions at \( \nu = 1/t_1 \) provided

\[
\gamma_1 = \sqrt{t_1}, \ t_1 = 3,5, \ldots .
\]

We note that these values are precisely the ones that make the operator \( V_1 = e^{i\gamma_1\varphi_1(z)} \) anticommuting with itself; it is interpreted as an electron creation operator.

To obtain wave functions for hierarchy states at level \( n \) we define new operators recursively:

\[
V_{\alpha+1} = \partial V_{\alpha} e^{-i\varphi_\alpha/\gamma_\alpha} e^{i\gamma_{\alpha+1}\varphi_{\alpha+1}} ,
\]

for \( \alpha = 1,2,\ldots,n-1 \), where \( \varphi_{\alpha+1} \) is a new bosonic field obeying 2. Formally, \( V_{\alpha+1} \) are descendants of primary fields—the latter are exponentials of bosonic fields only, without derivatives. It can be shown analytically that, when going from any given level in the hierarchy to the next, it is necessary to introduce this additional partial derivative in order to produce non-zero wave functions. We shall elaborate on this point in section V. The vertex operators obey operator product expansions

\[
V_\alpha(z)V_\alpha(w) \sim' (z-w)^{s_\alpha} ,
\]

\[
V_\alpha(z)V_\beta(w) \sim' (z-w)^{s_{\alpha\beta}} ,
\]

where \( \sim' \) indicates that we have suppressed the derivatives. From 5 we find

\[
s_{\alpha+1} = s_\alpha + \gamma_{\alpha-2}^2 + s_{\alpha+1} - 2 , \quad s_{\alpha\beta} = s_{\beta\alpha} = s_\alpha - 1 , \quad \text{for } \beta > \alpha .
\]

From 3 we see that \( s_1 = t_1 = 3,5,\ldots \) is an odd positive integer. We require this to be true for all \( s_\alpha \) for the operators \( V_{\alpha} \) to anticommute; from 8 we see that this implies that \( \gamma_{\alpha-2}^2 + \gamma_{\alpha+1}^2 \) is an even positive integer \( t_{\alpha+1} \), thus

\[
\gamma_{\alpha+1} = \sqrt{t_{\alpha+1} - \gamma_\alpha^2}, \ t_{\alpha+1} = 2,4,6,\ldots .
\]
The exponents $s_\alpha, s_{\alpha \beta}$ in the operator product expansion \[7\] then become
\[
s_\alpha = s_{\alpha \beta} + 1 = \sum_{\gamma=1}^{\alpha} t_\gamma - 2 (\alpha - 1), \quad \beta > \alpha . \tag{10}
\]

The wave function is obtained as the conformal correlator
\[
\Psi = \mathcal{A} \left( \prod_{\alpha=1}^{n} \prod_{i=1; i\neq j}^{M_\alpha} V_\alpha (z_{i\alpha}) \right), \tag{11}
\]
where $\mathcal{A}$ denotes antisymmetrization, and $M_\alpha$ is the number of particles in subset $\alpha$.

It is straightforward to evaluate the correlators in \[11\], and the explicit wave functions are in fact fully determined by the short distance behavior coded in the operator product expansion \[7\] \[36\]. The coordinates are divided into $n$ sets with $M_\alpha$ particles each, and the exponents in the polynomials are $s_\alpha$ if both coordinates belong to set $\alpha$ and $s_{\alpha \beta}$ if the two coordinates belong to the different sets $\alpha$ and $\beta$. Equation \[10\] gives $s_\alpha$ and $s_{\alpha \beta}$ in terms of $t_\alpha$. At level $n$ of the hierarchy, the wave function is
\[
\Psi = \mathcal{A} \left\{ (1-1)^{s_1} \partial_1 (2-2)^{s_2} \cdots \partial_n^{n-1} (n-n)^{s_n} \right\} \times (1-2)^{s_1 (1-3)^{s_2} \cdots ((n-1) - n)^{s_n-1,n} \right\}, \tag{12}
\]
where
\[
\partial_n^{n-1} (\alpha-\alpha) s_\alpha \equiv \prod_{i=1}^{M_\alpha} \partial_{z_{i\alpha}}^{n-1} \prod_{i<j}^{M_\alpha} (z_{i\alpha} - z_{j\alpha})^{s_\alpha},
\]
and $z_{i\alpha}$ numbers the $M_\alpha$ coordinates in set $\alpha$. (The derivatives in \[13\] act on all of $\Psi$ in \[12\].)

From \[12\] and \[13\] we find that the highest power of a coordinate in subset $\alpha$ is
\[
s_\alpha (M_\alpha - 1) + \sum_{\beta \neq \alpha}^{n} s_{\alpha \beta} M_\beta -(\alpha - 1). \tag{14}
\]
The numbers of particles, $M_\alpha$, in the subsets are determined by requiring the highest power in each subset to be equal (up to terms of order one)
\[
s_\alpha M_\alpha + \sum_{\beta \neq \alpha}^{n} s_{\alpha \beta} M_\beta = \text{const.} , \tag{15}
\]
where the constant is independent of $\alpha$. This corresponds to the different sets of particles having the same size. Using \[10\], \[15\] implies
\[
M_\alpha = M_{\alpha+1} + (t_{\alpha+1} - 2) \sum_{\beta=\alpha+1}^{n} M_\beta , \tag{16}
\]
for $\alpha = 1, 2, \ldots n-1$, from which one easily obtains $M_\alpha$ in terms of $M_n$ for given $n$. The filling factor of $\Psi$ is
\[
\nu_n = \frac{1}{t_1 - \frac{1}{t_2 - \cdots - \frac{1}{n-1 - \frac{1}{M_n}}}}, \tag{17}
\]
in accordance with the hierarchy construction \[11\] \[13\]. This is determined in the TT-limit below but can also be obtained by counting the number of particles in the wave function \[11\] and calculating the area it covers.

### III. TAO-THOULESS LIMIT

To obtain the TT-limit of a wave function we first translate it to the cylinder \[22\] \[23\] and then let the circumference of the cylinder go to zero; for details see Ref. \[16\]. This may equivalently be viewed as changing the hamiltonian while keeping the two-dimensional space infinite. The first step is achieved by the replacement
\[
z_i \rightarrow \beta_i \equiv e^{2\pi i z_i/L_1}, \tag{18}
\]
in the polynomial part of the wave function. (In addition, the gaussian factor is changed according to: $e^{-\sum |z_i|^2/4\ell^2} \rightarrow e^{-\sum |y_i|^2/2\ell^2}$. This factor is unaffected by the limiting procedure.) $z_i = x_i + iy_i$ are now complex coordinates on a cylinder with circumference $L_1$ in the $y$-direction. A basis of lowest Landau level states is given by
\[
\psi_k = \pi^{-1/4} L_1^{-1/2} e^{2\pi i k x/L_1} e^{-(y+k2\pi \ell^2/L_1)^2/2\ell^2} \propto e^{-2(k^2\ell^2)/\beta^k}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{19}
\]
Since $\psi_k$ is centered at $y = -k2\pi \ell^2/L_1$, this maps the lowest Landau level onto a one-dimensional lattice model, where the momentum in the $x$-direction, $2\pi k/L_1$, numbers the sites. The next step is to translate the many-particle wave function to the occupation number basis by expressing it in terms of the single-particle wave functions $\psi_k$ using \[19\]. For a generic term in the polynomial we have (in the following we put $\ell = 1$)
\[
\prod_i \beta_i^{k_i} \propto e^{2(k^2\ell^2) \sum_i k_i^2 \prod_i \psi_k(z_i) ; \tag{20}
\]
hence, when $L_1 \rightarrow 0$, the wave function approaches the occupation number state which maximizes $\sum_i k_i^2$. The momentum $K = \sum_i k_i$ is conserved—all components of the wave function in the occupation number basis have the same $K$. Thus the task is to find the term in the polynomial \[12\] that has largest possible $\sum_i k_i^2$ for given $K = \sum_i k_i$. Since the state is fermionic, all $k_i$ will be different. In fact, all we have to do is to find one term that maximizes $\sum_i k_i^2$ among the terms where all $k_i$ are different; the partners giving the antisymmetrization will be there automatically, and the terms with equal $k_i$'s
cancel by antisymmetrization. The term with the obtained set \( \{ k_i \} \) gives the state in the TT-limit: the electrons occupy the sites \( \{ k_i \} \) on the one-dimensional lattice, and the wave function is the single Slater determinant \( \text{Det}[\psi_{k_i}(z_j)] \).

The term that maximizes \( \sum_i k_i^2 \) for given \( K = \sum_i k_i \) is obtained by first finding a coordinate with the largest \( k_i \), then among the terms with this coordinate and this \( k_i \) find a new coordinate with the next largest \( k_j \) and so on. (Note that the coordinates in the different sets have different maximal \( k_i \) and that we may assume that \( k_i > 0 \), for all \( i \), since shifting all \( k_i \)'s by the same constant just amounts to a rigid translation of the state along the cylinder.)

Subtracting the common constant in (15) from (14), we find that the highest power of a coordinate in subset \( \alpha \) is

\[
\kappa_\alpha \equiv -s_\alpha - (\alpha - 1) = -\sum_{\beta=1}^{\alpha} t_\beta + \alpha - 1 .
\]

(21)

We now pick a coordinate with the highest \( \kappa_\alpha \) and restrict to the terms that contain this coordinate to this power \( k_1 \equiv \max \kappa_\alpha \). We then determine the highest powers in the subsets \( \alpha \) of the remaining coordinates in these terms by inspecting the wave function in (12). These powers are obtained from the ones defined in (21) by \( \kappa_\alpha \to \kappa_\alpha + \delta \kappa_\alpha \), where

\[
\delta \kappa_\gamma = -s_\gamma , \quad \delta \kappa_\beta = -s_\beta \gamma , \quad \beta \neq \gamma ,
\]

(22)

where \( \gamma \) is the subset that had the highest power. We again pick a coordinate with the highest \( \kappa_\alpha \) and restrict to the terms that contain this coordinate to this power \( k_2 \equiv \max \kappa_\alpha \). Repeating this procedure we eventually find the set \( \{ k_1, k_2, \ldots, k_N \} \) that gives the state in the TT-limit.

To reveal the general structure of the hierarchy it is convenient to introduce the differences

\[
\Delta_\alpha \equiv \kappa_1 - \kappa_\alpha , \quad \alpha = 2, 3 \ldots n .
\]

(23)

From (21), we find their initial values

\[
\Delta_\alpha^{(0)} = \sum_{\beta=2}^{\alpha} t_\beta - (\alpha - 1) ,
\]

(24)

which are positive and increasing \( 0 < \Delta_\alpha < \Delta_{\alpha+1} \). From (22) we find how \( \Delta_\alpha \) changes. Letting \( \gamma \) be the subset with the highest power, we find

\[
\gamma = 1 : \quad \delta \Delta_\alpha = -1
\]

\[
\gamma \geq 2 : \quad \delta \Delta_\alpha = -2(\alpha - 1) , \quad \alpha < \gamma
\]

\[
\delta \Delta_\gamma = 1 + \sum_{\beta=2}^{\gamma} t_\beta - 2(\gamma - 1) + 1 ,
\]

\[
\delta \Delta_\alpha = \delta \Delta_\gamma - 1 , \quad \alpha > \gamma .
\]

(26)

We are now ready to determine the states in the TT-limit. At the first level of the hierarchy, \( n = 1 \), there is only one subset of coordinates in (12) and the highest power changes by \( \delta k_1 = -t_1 \) in each step, according to (22). Thus \( \{ k_1 \} = \{ 0, -t_1, -2t_1, -3t_1, \ldots \} \), where we have arbitrarily chosen \( k_1 = 0 \). In terms of the one-dimensional lattice this is a periodic system with one electron on every \( t_1 \)-th site, i.e., a crystal with unit cell

\[
C^{(1)} = 0_{t_1-1} , \quad \nu_1 = \frac{1}{t_1} .
\]

(27)

We use a notation where \( (0) \) denotes that a site is occupied (empty) and \( (0)_{t_1} = 001 \) etc. These are the states proposed by Tao and Thouless for the fractional quantum Hall effect (17); we see that they are the TT-limits of the Laughlin states (23).

From the relations

\[
\delta k_1 = -t_1 , \quad \gamma = 1
\]

\[
\delta k_1 = -t_1 + 1 , \quad \gamma \geq 2 ,
\]

(28)

which follow from (22), the initial values \( \Delta_\alpha^{(0)} \), and (26), we determine the sequence of \( \{ \kappa_1, \Delta_\alpha \}'s \) at higher levels of the hierarchy. These give the largest \( \kappa_\alpha \)'s, i.e., the \( k_i \)'s giving the state. It turns out that independent of the initial value, the sequence of \( \{ \Delta_\alpha \}'s \) is periodic, except for possible edge effects that we ignore, and that it contains the configuration \( \{ \Delta_\alpha \}'s \) when determining the sequence. In addition we set the initial largest \( \kappa_\alpha \), which is \( k_1 \), to zero.

At the second level, \( n = 2 \), we find the results in Table I. For clarity we have included \( k_2 = k_1 - \Delta_2 \) explicitly. The last column gives \( \gamma \), the group which has the largest \( \kappa_\alpha \). The state is periodic since \( \{ \Delta_2 \} \) returns to its initial value at the last step. The highest \( \kappa_\alpha \) at each step is shown in bold face, and these give

\[
\{ k_i \} = \{ 0, (t_1), 2t_1, 3t_1, \ldots, (t_2 - 1)t_1, t_2t_1 - 1 \}
\]

(29)

where the filling factor is obtained by simply counting the number of ones and sites in \( C^{(2)} \). Note that the unit cell at level one is repeated \( t_2 - 1 \) times in \( C^{(2)} \).
Here the TT-states at level three are

\[
C^{(3)} = \{0_{t_1-1}\}_{t_2-2} 0_{t_1-1}_{t_2-2} 0_{t_1-1}_{t_2-2} 0_{t_1-1}_{t_2-2} 0_{t_1-1}_{t_2-2}
\]

Table I: Level two

| $\kappa_1$ | $\kappa_2$ | $\Delta_2$ | $\gamma_0(\kappa_2 \text{ max})$ |
|----------|----------|-----------|-----------------|
| -1       | 0        | -1        | 2               |
| $-t_1$   | $-t_1-t_2+2$ | $t_2-2$   | 1               |
| $-2t_1$  | $-2t_1-t_2+3$ | $t_2-3$   | 1               |
|         |          |           |                 |
|         |          |           |                 |
| $-(t_2-1)t_1$ | $(t_2-1)t_1$ | $0$       | 1               |
| $-t_2t_1$ | $-t_2t_1+1$ | -1        | 2               |

Table II: Level three

| $\kappa_1$ | $\Delta_2$ | $\Delta_3$ | $\gamma_0(\kappa_2 \text{ max})$ |
|----------|-----------|-----------|-----------------|
| -1       | 0         | -1        | 3               |
| $-t_1$   | $t_2-2$   | $t_2+3-4$ | 1               |
|         |           |           |                 |
|         |           |           |                 |
| $-t_2t_1$ | -1        | $t_3-3$   | 2               |
| $-t_2t_1+1-t_1$ | $t_2-2$ | $t_2+3-5$ | 1               |
|         |           |           |                 |
|         |           |           |                 |
| $(t_3-1)\left(-t_2t_1+1\right)-t_1$ | $t_2-2$ | $t_2-2$ | 1               |
|         |           |           |                 |
| $-t_3t_2t_1+3-1$ | -1        | -1        | 2               |
| $-t_3t_2t_1+3-t_1$ | $t_2-2$ | $t_2-3$ | 1               |
|         |           |           |                 |
|         |           |           |                 |
| $-t_3t_2t_1+t_3-t_1$ | 0        | -1        | 3               |

At the third level of the hierarchy, $n = 3$, we obtain the sequences in Table II. By comparing $\{\kappa_1, \Delta_2, \gamma_0\}$ in the two tables, we see that the structure at level two is first repeated $t_3-1$ times at level three, thus $C^{(3)} = C^{(2)}_{t_3-1}a$. Here $a$ is obtained from the last part in Table II, it differs from the $C^{(2)}$-parts in that one $C^{(1)}$ is missing, $a = C^{(1)}_{t_2-2}0_{t_1-2}$. Thus the TT-states at level three are

\[
C^{(3)} = \{0_{t_1-1}\}_{t_2-1}0_{t_1-1}_{t_2-2}0_{t_1-1}_{t_2-2}0_{t_1-1}_{t_2-2}0_{t_1-1}_{t_2-2}
\]

By pondering the relation [26], one realizes that this structure extends to general level $n$, and that the unit cells for the states in the TT-limit obey the relation

\[
C^{(n)} = C^{(n-1)}_{t_1-1}C^{(n-2)}_{t_1-1}, \quad C^{(0)} = 0, \quad n = 1, 2, \ldots
\]

Here $C^{(n-1)}_{t_1-1}$ indicates that $C^{(n-1)}$ is repeated $t$ times and $C^{(n-2)}$ is the complement of $C^{(n-2)}$ in the unit cell $C^{(n-1)}$, i.e., $C^{(n-2)}C^{(n-2)} = C^{(n-1)}$. These unit cells are the ground states in the TT-limit[16], and the filling factors are the ones in [17]. Moreover, $C^{(n-2)}$ are the fractionally charged quasielectrons in the ground state with unit cell $C^{(n-1)}$, and hence the state with unit cell $C^{(n)}$ is, according to [21], a condensate of the quasielectrons in the state with the unit cell $C^{(n-1)}$. Thus the original hierarchy construction is manifest in the TT-limit.

IV. QUASIHOLES AND QUASIELECTRONS

In Ref. [10] it was shown how to construct quasihole wave functions for the Jain series, and how to relate these to Wen’s general classification[6] of abelian quantum Hall fluids. An abelian fluid (on a flat manifold) at level $n$ is specified by an $n \times n$ matrix $K$, an $n$-dimensional charge vector $t$, and $n$ distinct $n$-dimensional vectors $1^{(a)}$. Here we first generalize the quasihole construction to all the hierarchy states discussed above and give an explicit expression for the general multi-hole wave function. We then briefly discuss the quasielectron wave functions, but without giving explicit formulae. Next we make the connection to Wen’s classification, and give explicit formulae for the quantities $K$, $t$ and $1$. We end this section with a critical discussion of the status of fractional statistics in the hierarchy states.

A. Quasihole wave functions

First we express the $n$ electron operators [10], at level $n$, in the form

\[
V_\alpha = \partial^{\alpha-1} e^{iQ^{(\alpha)} \cdot \varphi}
\]

which is suitable for actually evaluating the correlators. Here $\varphi = (\varphi_1, \varphi_2, \ldots \varphi_n)$, and $Q^{(\alpha)}$ has components $Q^{(\alpha)}_\beta = q^{(\alpha)}_\beta / R_\beta$, where $q^{(\alpha)}_\beta$ are integers and $R_\beta$ is the compactification radius of the field $\varphi_\beta$. The $U(1)$ charge associated with the current $j(z) = i \sum_c c_\alpha \partial \varphi_\alpha (z) = ic \cdot \partial \varphi$, where the components of the "charge vector" $c$ are $c_\alpha = 1 / R_\alpha$, measures the "depletion" of the quantum Hall fluid, and is thus closely connected to the electric charge. The condition that the charges of the electron fields all equal one (we shall give charge in units of the electron charge $-e$) leads to the sum rule

\[
\sum_{\beta=1}^n Q^{(\alpha)}_\beta / R_\beta = 1.
\]

The explicit expressions for the integers $q^{(\alpha)}_\beta$, which will not be needed here, can be found in Ref. [24]. We also introduce the corresponding $n$ hole operators,

\[
H_\alpha = e^{iQ^{(\alpha)} \cdot \varphi},
\]
where the vectors $l^{(\alpha)}$, $\alpha = 1, \ldots, n$, are reciprocal to the vectors $Q^{(\alpha)}$,

$$l^{(\alpha)} \cdot Q^{(\beta)} = \delta_{\alpha \beta} \ .$$

The wave function for a state with $m$ holes at positions $\{\eta_i\}$ is given by the correlator

$$\Psi_h(\{\eta_i\}) = A \prod_{i=1}^{m} H_{\beta_i}(\eta_i) \prod_{\alpha=1}^{n} \prod_{i_\alpha=1}^{M_\alpha} V_\alpha(z_{i_\alpha}) \ ,$$

where antisymmetrization is performed over the electron coordinates $z_i$ only. Evaluating the correlators gives

$$\Psi_h = A \prod_{\alpha=1}^{n} M_\alpha \prod_{i_\alpha=1}^{M_\alpha} \prod_{\beta_i=1}^{m} \left( z_{i_\alpha} - \eta_i \right)^{i(\alpha),i(\beta)} \ ,$$

where $s_{\alpha \beta} = Q^{(\alpha)} \cdot Q^{(\beta)}$, $s_\alpha = s_{\alpha \alpha}$, and

$$f(\{\eta_i\}, \{z_{i_\alpha}\}) = \prod_{\alpha=1}^{n} \prod_{i_\alpha=1}^{M_\alpha} \prod_{\beta_i=1}^{m} \left( z_{i_\alpha} - \eta_i \right)^{i(\alpha),i(\beta)} \ .$$

As stressed by Moore and Read, and discussed in some detail in Ref. 10, there is no local operator that will create quasielectrons. The naive guess, $H_{\alpha}^{-1}$, will give correlators which are not analytic in the electron coordinates, and thus not acceptable lowest Landau level wave functions. Nevertheless, one can construct wave functions corresponding to many quasielectrons in specified angular momentum states. By forming coherent superpositions of such states one can then obtain localized multi-quasielectron wave functions. The explicit formulae are lengthy and not very illuminating, and we refer to Ref. 10 for some explicit examples. Although there is no local quasielectron operator, it is possible to construct a quasi-local operator, $P_\alpha(\eta)$, with the same charge and conformal dimension as $H_{\alpha}^{-1}(\eta)$, which directly creates the localized quasielectron wave functions.

### B. Topological classification

It is convenient to define the two matrices $Q$ and $L$ with components

$$Q_{\alpha \beta} = Q^{(\beta)}_{\alpha} \quad \text{and} \quad L_{\alpha \beta} = l^{(\beta)}_{\alpha} \ .$$

The condition then reads

$$c^T Q = (1,1,\ldots,1) \ ,$$

while the defining condition takes the form

$$L^T Q = 1 \ .$$

The charges of the quasiparticles $q = (q_1, q_2, \ldots, q_n)$ and the matrix $\Theta$ containing the (mutual) statistical angles $\theta_{\alpha \beta}$ between two particles in the groups $\alpha$ and $\beta$, respectively, are given by the relations

$$q = -c^T L$$

$$\Theta = \pi L^T L \ .$$

Also, as explained in Ref. 10, the filling fraction can be obtained from the background charges needed to make the correlators non-zero, and this yields the relation

$$\nu = c \cdot c \ .$$

The filling fraction $\nu$, together with the quasiparticle charges $q_i$ and the statistical angles $\Theta$, gives a complete macroscopic description of a quantum Hall fluid on an infinite plane. Using (40) and (41) the relations (44), (42), and (43) can be expressed in the more familiar form given by Wen [6],

$$\nu = t^T K^{-1} t$$

$$q_i = -t_i^T K^{-1} L_s$$

$$\Theta = \pi L_s^T K^{-1} L_s \ ,$$

if one defines

$$K = Q^T Q$$

$$t^T = (1,1,\ldots,1)$$

$$L_s = 1 \ .$$

The relations are precisely those introduced by Wen. Note that the columns of the unit vector $L_s$, which we denote as $l^{(\alpha)}$, are orthogonal unit vectors. They are Wen’s $l$-vectors characterizing the $n$ fundamental quasiholes; a generic (composite) quasiparticle can be expressed as a linear superposition of these. The values of the vectors $t$ and $l^{(\alpha)}$ correspond to his “symmetric basis”. Also note that the entries in the $K$-matrix only depend on the integer scalar products $Q^{(\alpha)} \cdot Q^{(\beta)}$, and thus on the powers of the Jastrow factors in the ground state wave function, but not on the specific choice of the vectors $Q^{(\alpha)}$.

A Laughlin hole, which is created by the insertion of a thin unit flux tube, amounts to a unit vortex in all the $n$ condensates, and is given by the operator $H_L = \prod_{\alpha=1}^{n} H_{\alpha}$, i.e., the $l$-vector $(1,1,\ldots,1)$. The expected values for statistics and charge, $\theta/\pi = -q = \nu$, follow from (44).

The quasiparticle states can also be constructed in the TT-limit and we suggest that a connection can be made to Wen’s classification.

### C. Fractional statistics of the quasiparticles

Above we tacitly assumed that the fractional statistics of the quasiholes could be read from the factors
arguments referred to above, the statistics of the elements by Kivelson and Rocek [29] that relate fractional charge be extracted [27, 28]. There are also general arguments quasihole wave function from which the Berry phase can be associated to the charge deficit [26], while Laughlin used the plasma analogy to evaluate the normalization of the two-electrons [27].

In the case of the Laughlin states, the Berry phase can be evaluated using several methods. In the original calculation by Arovas et al., it was directly related to the charge deficit [27], while Laughlin used the plasma analogy to evaluate the normalization of the two-quasihole wave function from which the Berry phase can be extracted [27, 28]. There are also general arguments by Kivelson and Rocek [29] that relate fractional charge and fractional statistics for quasiholes where the electron density goes to zero in the core. Furthermore, Su has argued that the statistics properties of the elementary quasiparticles follow from those of the Laughlin holes [30] if one assumes that the fractional statistics phase \( \theta_{AB} \) of a composite \( C = A + B \) is given by \( \theta_{C} = \theta_{A} + \theta_{B} + 2\theta_{AB} \), where \( \theta_{AB} \) is the mutual statistics phase [31].

That these very general arguments can be fallacious is demonstrated by the \( \nu = 1/2 \) Moore-Read pfaffian state. While the Laughlin hole with charge 1/2 has statistics \( \pi/2 \), the elementary holes carry charge 1/4 and obey non-abelian statistics, which is manifested only when four or more holes are present. In terms of the electronic wave function for \( \nu = 1/2 \) Moore-Read pfaffian state, the Berry phase is simply zero. The cause of this degeneracy of the electronic wave function for fixed positions of the quasiholes. This rather subtle effect is not at all obvious from the form of the pfaffian ground state wave function, but is readily seen in the structure of the quasiholes in the conformal field theory framework.

This last remark emphasizes a fact stressed by Nayak and Wilczek [32]: in all examples we know of, the expected fractional statistics is coded in the pertinent hole operators, \( i.e. \) the statistical phases—abelian or non-abelian—are simply the monodromies of the conformal blocks in question; the Berry phase is simply zero. They also proposed that this is true in general, and they gave some arguments to support this hypothesis (see also Ref. 33). From this perspective, the expression \( \Psi_{4/11} \) is interesting since the monodromies can be explicitly calculated (they are coded in the matrix \( \Theta \)) while we know of no analytic method to calculate the Berry phases. (The reason for this is that the monodromies are identical for all terms in the sum implied by the antisymmetrization, while the Berry phase involves cross terms that are difficult to handle.) However, assuming that the Berry phases are zero for the wave functions \( \Psi_{4/11} \), which is supported by numerical calculations at \( \nu = 2/5 \) as well as the general arguments referred to above, the statistics of the elementary holes in these hierarchy states is given by (45).

V. COMPOSITE FERMIONS

To illuminate the relationship between the wave functions (42), and those constructed using composite fermions, we first consider the \( n = 2 \) case,

\[
\Psi = A\{(1 - 1)^t_i \partial_2(2 - 2)^{t_i + t_j - 2}(1 - 2)^{t_i - 1}\},
\]

with groups 2 and 1 containing \( M_2 \) and \( M_1 = M_2(t_2 - 1) \) particles, respectively. The derivatives act on everything to their right. For \( t_2 = 2, \Psi \) are Jain's wave function at \( \nu = 2/(2t_1 - 1) \), while \( t_1 = 3, t_2 = 4 \) gives our proposal for the observed state at \( \nu = 4/11 \).

For general \( n \), if \( t_n = 2 \) for \( \alpha = 2, \ldots, n \), it follows from (17) that \( \nu_n = n/(t_1 - 1) n + 1 \), and the wave functions (47) are Jain's composite fermion wave functions as explained in Ref. [10]. The \( t_1 \)'s determine the densities of the \( n \) condensates that build up the state—the "principal" composite fermion states are the ones where all but the first of these condensates have maximal density; in the CF language, they correspond to integer quantum Hall states of composite fermions.

More generally, one can interpret all the hierarchy states (17) in terms of fractional quantum Hall states of composite fermions (34). This is most easily illustrated by the case of \( \nu = 4/11 \) obtained by taking \( t_1 = 3, t_2 = 4 \):

\[
\Psi_{4/11} = A\{(1 - 1)^3 \partial_3(2 - 2)^5(1 - 2)^2\} = A\{(1 - 1)^3 \partial_3(2 - 2)^3\} \prod_{i<j}(z_i - z_j)^2,
\]

where the squared Jastrow factor includes all pairs of particles. The first group of coordinates corresponds to the usual Slater determinant of the filled lowest CF Landau level, while the second describes the 1/3-filled second CF Landau level—it may be written as a sum of Slater determinants of the various allowed distributions of \( M_2 \) composite fermions in the 2\( M_2 \) states of the second CF Landau level. In other words, the 4/11 state can be viewed as a \( \nu' = 1 + 1/3 \) state of composite fermions, \( \nu' = \nu^{-1} + 2 = 11/4 \). This was first pointed out by Chang and Jain, who proposed a wave function which is similar to the one given here (34).

Note that the derivatives \( \partial_2 \) which are necessary in order for this wave function to be non-zero under antisymmetrization over the two groups [10], are the lowest Landau level projection of the anti-holomorphic coordinates \( \tilde{z}_2 \) characteristic of the second Landau level.

Similar reasoning can be applied at higher levels in the hierarchy, where a less trivial example is the \( \nu = 12/29 \) state at level \( n = 4 \), obtained from the set \( t_n = 3, 2, 4, 2 \), giving \( s_1 = s_2 = 3; s_3 = s_4 = 5; s_13 = s_23 = 2; s_34 = 4 \). Pulling out a full, squared Jastrow factor and rearranging derivatives, this wave function may be rewritten in the
following, suggestive way:
\[
\Psi_{12/29} = \mathcal{A}((1 - 1)^3 \partial_0 (2 - 2)^4 \partial_0^2 \partial_2^2) \times \left[ (3 - 3)^3 \partial_4 (4 - 4)^3 (3 - 4)^2 \right] \prod_{i<j} (z_i - z_j)^2.
\]

The CF interpretation of this state goes as follows: Groups 1 and 2 fill the two lowest CF Landau levels, while the part inside the square brackets has the form of a 2/5 CF state, consisting of the groups 3 and 4, cf (47). The double derivatives can be thought of as "lifting" this state to the third CF Landau level. So the complete state can be interpreted as the filling of the CF Landau levels, as wave functions have no notion of such a cyclotron gap. A more interesting question is whether the derivatives in (49) are all needed, especially since the topologically relevant quantities are all coded in the powers of the Jastrow factors. Here we only offered a partial answer: Within the hierarchy construction implied by (49), the derivative can not be omitted as argued above. Although we can not fully exclude the possibility of deleting derivatives from \(V_0\), when constructing \(V_{n+1}\), we find this possibility unlikely. The reason is that \(V_{n+1}\) is closely related to the quasi-electron operator which can be shown to be quasi-local, thus excluding long-range operators such as inverse derivatives.

The original hierarchy construction is manifest in the Tao-Thouless limit; however, a connection to composite fermions can be made also in this limit. The TT-states at filling factors \(1/3, 2/5\) and \(4/11\) are \(0_1, 0_{2101}, (0_21)_3\) respectively. \(0_{21}\) is the filled lowest effective (CF) Landau level. Quasielectrons \(0_1\) in a 1/3 ground state with \(N\) cells \((0_21)_N\) can be inserted in \(N\) equivalent places, which may be interpreted as an effective Landau level with \(N\) states. Filling all these gives the state with unit cell \(0_{2101}\), i.e., the 2/5 state, which thus can be interpreted as consisting of two filled effective Landau levels. If instead one third of the states in the second effective Landau level is filled, then the state with unit cell \((0_21)_3\), i.e., the 4/11 ground state, is obtained. This interpretation can be extended to other filling factors but becomes less natural at higher levels in the hierarchy.

VI. SUMMARY AND OUTLOOK

In this paper we constructed wave functions for the quantum Hall hierarchy ground states that are built by successive condensations of quasielectrons, and explicitly showed that, when put on a cylinder, they evolve into the Tao-Thouless ground states when the cylinder becomes thin. We also constructed the general quasi-hole states and showed, using standard assumptions about Berry phases, that they have all expected topological properties and fit into Wen’s general classification of QH fluids. Finally we pointed out that the successful composite fermion approach to QH physics fits naturally in our scheme.

There are many questions that have to be answered before we can claim to have a comprehensive understanding of the QH hierarchy states. First, our approach is so far limited to condensation of quasielectrons. It is not clear how to incorporate the condensation of quasiholes, but it is likely to involve products of holomorphic and non-holomorphic conformal blocks. These objects have nice topological properties, but it is less clear how to extract lowest Landau level wave functions.
Another interesting open question concerns the possibility of having non-abelian states with wave functions very similar to (12) and we already mentioned the one at $\nu = 2/7$.

Finally, it is of course crucial to establish the connection to experimentally observed states which are believed to be abelian but do not belong to the main Jain series. Here the state at $\nu = 4/11$ is perhaps the most interesting one\cite{23}, and it is important to test our proposed wave function numerically. To do this efficiently one must work on a finite geometry, which in practice means a sphere or a torus. Recently, we have shown how to formulate the wave functions (12) on a torus\cite{24}, and preliminary numerics on the $\nu = 4/11$ state is encouraging.

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\bibitem{36} This is not true on a topologically nontrivial manifold like the torus, where a careful evaluation of the correlators yields extra factors in addition to the periodized version of the wave functions presented here.
\bibitem{37} Working out a few examples, this is fairly evident, and we have not bothered to construct a formal proof.
\bibitem{38} For example, the TT-state at 12/29 is $(0_{12}01)_{31}(0_{210})_{201}$. As discussed in the text, two filled effective Landau levels, $\nu' = 1 + 1$, corresponds to $0_{12}01 \equiv a$; letting $b = 01$, we have $(0_{12}01)_{31}(0_{210})_{201} = a_{3}b_{0}a_{5}b$. Thus $b$ has been inserted in two out of five equivalent positions and hence one may interpret the $12/29$ state as a $\nu' = 1 + 1 + 2/5$ CF state. To minimize the energy, the $b$'s are as far separated as possible in the a–state.
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