The group of diffeomorphisms of the circle: reproducing kernels and analogs of spherical functions

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The group of diffeomorphisms of the circle is an infinite dimensional analog of the real semisimple Lie groups $U(p, q)$, $Sp(2n, \mathbb{R})$, $SO^*(2n)$; the space $\Xi$ of univalent functions is an analog of the corresponding classical complex Cartan domains. We present explicit formulas for realizations of highest weight representations of $\text{Diff}(S^1)$ in the space of holomorphic functionals on $\Xi$, reproducing kernels on $\Xi$ determining inner products, and expressions ('canonical cocycles') replacing spherical functions.

1 Introduction

1.1. The purpose of the paper. The group $\text{Diff}(S^1)$ of orientation preserving diffeomorphisms of the circle has unitary projective highest weight representations. There is a well-developed representation theory of unitary highest weight representations of real semi-simple Lie groups (see, e.g., an elementary introduction in [29], Chapter 7, and further references in this book). In particular, this theory includes realizations of highest weight representations in spaces of holomorphic (vector-valued) functions on Hermitian symmetric spaces, reproducing kernels, Berezin–Wallach sets, Olshanski semigroups, Berezin–Guichardet–Wigner formulas for central extensions. All these phenomena exist for the group $\text{Diff}(S^1)$.

The purpose of this paper is to present details of a strange calculation sketched at the end of my paper [24], 1989. It also was included to my thesis [26], 1992. Later the calculation and its result never were exposed or repeated.

Let us consider a unitary highest weight representation $\rho$ of the group $\text{Diff}(S^1)$. Generally, it is a projective representation. Operators $\rho(g)$ are determined up to constant factors. We normalize them by the condition

$$\langle \rho(g)v, v \rangle = 1,$$

where $v$ is a highest weight vector. Then we have

$$\rho(g_1)\rho(g_2) = c(g_1, g_2) \rho(g_1g_2),$$

where the expression $c(g_1, g_2)$ (the canonical cocycle) is canonically determined by the representation $\rho$. In this paper, we derive a formula for $c(g_1, g_2)$.

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2Since the representation $\rho$ is projective, the operators $\rho(g)$ are defined up to scalar factors. If $\langle \rho(g)v, v \rangle \neq 0$, then we can write corrected operators

$$\tilde{\rho}(g) := (\langle \rho(g)v, v \rangle)^{-1}\rho(g).$$

The property $\langle \rho(g)v, v \rangle \neq 0$ holds for all $g \in \text{Diff}(S^1)$. This follows from explicit calculations given below. Also, it is possible a priori proof from formula (3.3). Of course, the new operators $\tilde{\rho}(g)$ are not unitary.
Remark. Let $G$ be a semisimple group, $K$ a maximal compact subgroup and $v$ a $K$-fixed vector, let $\varphi(g) = \langle \rho(g)v, v \rangle$ be the spherical function. Then

$$c(g_1, g_2) = \frac{\varphi(g_1 g_2)}{\varphi(g_1) \varphi(g_2)}.$$ 

In our case, spherical functions are not well-defined but the 'canonical cocycles', which are hybrids of central extensions and spherical functions, exist.

A formula for canonical cocycles implies some automatic corollaries: we can write explicit formulas for realizations of highest weight representations of $\text{Diff}(S^1)$ on the space univalent functions and for invariant reproducing kernels on the space of holomorphic functionals on the space of univalent functions.

1.2. Virasoro algebra and highest weight modules. For details, see, e.g., [27]. Denote by $\mathfrak{ Vect}_\mathbb{R}$ and $\mathfrak{ Vect}_\mathbb{C}$ respectively the Lie algebras of real (respectively complex) vector fields on the circle. Choosing a basis $L_n := e^{i\varphi} \partial/\partial \varphi$ in $\mathfrak{ Vect}_\mathbb{C}$, we get the commutation relations

$$[L_n, L_m] = (m - n)L_{m+n}.$$ 

Recall that the Virasoro algebra $\mathfrak{Vir}$ is a Lie algebra with a basis $L_n$, $\zeta$, where $n$ ranges in $\mathbb{Z}$, and commutation relations

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{n^3-n}{12} \delta_{n+m,0} \zeta, \quad [L_n, \zeta] = 0.$$ 

Let $h, c \in \mathbb{C}$. A module with the highest weight $(h, c)$ over $\mathfrak{Vir}$ is a module containing a vector $v$ such that

1) $L_0v = hv$, $\zeta v = cv$;
2) $L_{-n}v = 0$ for $n < 0$;
3) the vector $v$ is cyclic.

There exists a unique irreducible highest weight module $L(h, c)$ with a given highest weight $(h, c)$, also there is a universal highest weight module $M(h, c)$ (a Verma module), such that any module with highest weight $(h, c)$ is a quotient of the Verma module $M(h, c)$. In particular, if $M(h, c)$ is irreducible, we have $M(h, c) = L(h, c)$.

By [15], [6], a Verma module $M(h, c)$ is reducible if and only if $(h, c) \in \mathbb{C}^2$ satisfies at least one equation of the form

$$\left(h - \frac{1}{24}(\alpha^2 - 1)(c - 13) + \frac{1}{2}(\alpha^2 - 1)\right)\left(h - \frac{1}{24}(\beta^2 - 1)(c - 13) + \frac{1}{2}(\beta^2 - 1)\right) + \frac{1}{16}(\alpha^2 - \beta^2)^2 = 0,$$

where $\alpha, \beta$ range in $\mathbb{N}$.

If $M(h, c)$ is reducible, its composition series is finite or countable, it is described in [7], [8].
1.3. Unitarizability. A module \( L(h, c) \) is called \textit{unitarizable} if it admits a positive definite inner product such that \( L_n = -L_{-n}^* \). A module \( L(h, c) \) is unitarizable iff \( (h, c) \in \mathbb{R}^2 \) satisfies one of the following conditions:

1. (continuous series) \( h \geq 0, \quad c \geq 1; \quad (1.1) \)

2. (discrete series) \( c = 1 - \frac{6}{p(p+1)}; \quad h = \frac{(\alpha p - \beta(p+1))^2 - 1}{4p(p+1)}, \quad (1.2) \)

where \( \alpha, \beta, p \in \mathbb{Z}, p \geq 2, 1 \leq \alpha \leq p, 1 \leq \beta \leq p - 1. \)

For sufficiency of these conditions, see [11], for necessity, see [9], [21], [25].

Remark. 1) For \( h > 0, c > 1 \) we have \( L(h, c) = M(h, c) \). For \( (h, c) \) of the form (1.2) modules \( M(h, c) \) have countable composition series.

2) There are simple explicit constructions for representations \( L(h, c) \) for \( c \geq 1, h \geq (c-1)/24 \). The present paper is based on a such construction, for another (fermionic) construction, see [21], [25], Sect. 7.3.

3) There is an explicit construction for representations (1.2) with \( c = 1/2 \) (possible values of \( h \) are 0, 1/16, 1/2), see [20], [25], Sect. 7.3. The module \( L(0, 0) \) is the trivial one-dimensional module.

4) As far as I know, transparent constructions (that visualize the action of the algebra/group, the space of representation and the inner product) for the other points of discrete series and for the domain \( 0 \leq h < (c-1)/24 \) are unknown.

1.4. Standard boson realizations. Consider the space \( \mathcal{F} \) of polynomials of variables \( z_1, z_2, \ldots \). Define the \textit{creation and annihilation operators} \( a_n \), where \( n \) ranges in \( \mathbb{Z} \setminus \{0\} \), by

\[
a_n f(z) = \begin{cases} 
\sqrt{n}z_n f(z), & \text{for } n > 0; \\
\sqrt{(-n)} \cdot \frac{\partial}{\partial z_n} f(z), & \text{for } n < 0.
\end{cases}
\]

Next, we write formulas for representations of the Virasoro algebra in \( \mathcal{F} \). Fix the parameters \( \alpha, \beta \in \mathbb{C} \). For \( n \neq 0 \) we set

\[
L_n := \frac{1}{2} \sum_{k,l : k+l = n} a_k a_l + (\alpha + i n \beta)a_n;
\]

\[
L_0 := \sum_{n > 0} a_n a_{-n} + \frac{1}{2}(\alpha^2 + \beta^2);
\]

\[
\zeta := 1 + 12\beta^2.
\]

Then we have \( L_{-n} 1 = 0 \) for \( n < 0 \) and

\[
L_0 \cdot 1 = \frac{1}{2}(\alpha^2 + \beta^2), \quad \zeta \cdot 1 := 1 + 12\beta^2.
\]

In this way, we get a representation of \( \mathfrak{Vir} \), whose Jordan–Hölder series coincides with the Jordan–Hölder series of \( M(h, c) \) with

\[
h = \frac{1}{2}(\alpha^2 + \beta^2), \quad c = 1 + 12\beta^2.
\]
For points \((\alpha, \beta)\) in a general position we get a Verma module.

**Remarks.**
1) For formulas for singular vectors, see [19].
2) For \(\alpha\) and \(\beta\) in a general position representations with parameters \((\alpha, \beta)\) and \((\alpha, -\beta)\) are equivalent. As far as I know explicit formula for the intertwining operator remains to be unknown.

1.5. The group of diffeomorphisms of the circle. Denote by \(\text{Diff}(S^1)\) the group of smooth orientation preserving diffeomorphisms of the circle. Its Lie algebra is \(\text{Vect}_\mathbb{R}\).

Any unitarizable highest weight representation of \(\mathfrak{Vir}\) can be integrated to a projective unitary representation of \(\text{Diff}(S^1)\), see [13]. A module \(\mathcal{L}(h, c)\) with an arbitrary highest weight \((h, c)\in \mathbb{C}^2\) can be integrated to a projective representation of \(\text{Diff}(S^1)\) by bounded operators in a Frechet space, see [22]. This follows from the universal fermionic construction, see [28], Sect. VII.3.

1.6. The welding. Denote by \(\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\) the Riemann sphere. Denote by \(D^+\) the disk \(|z| \leq 1\) on \(\overline{\mathbb{C}}\), by \(D^-\) the disk \(|z| \geq 1\) on \(\overline{\mathbb{C}}\). Denote by \(D^\circ \pm\) their interiors. Denote by \(S^1\) the circle \(|z| = 1\).

We say that a function \(f : D^\pm \rightarrow \overline{\mathbb{C}}\) is univalent up to the boundary if \(f\) is an embedding, which is holomorphic in the interior of the disk and is smooth up to the boundary.

Let \(\gamma \in \text{Diff}(S^1)\). Let us glue the disks \(D^+\) and \(D^-\) identifying the points \(z \in S^1 \subset D^+\) with \(\gamma(z) \in S^1 \subset D^-\). We get a two-dimensional real manifold with complex structure on the images of \(D^+\) and \(D^-\). According [1] this structure admits a unique extension to the separating contour (in fact, the smoothness of \(\gamma\) is redundant, for a minimal condition, see [1]). Thus we get a one-dimensional complex manifold, i.e., a Riemann sphere \(\overline{\mathbb{C}}\), and a pair of univalent maps \(D^\pm \rightarrow \overline{\mathbb{C}}\).

Conversely, consider a Riemann sphere \(\overline{\mathbb{C}}\) and a pair of maps \(p_- : D^- \rightarrow \overline{\mathbb{C}}\), \(p_+ : D^+ \rightarrow \overline{\mathbb{C}}\) univalent up to a boundary such that

\[ p_-(D^2) \cap p_+(D^2) = \emptyset \quad p_-(D^-) \cup p_+(D^+) = \overline{\mathbb{C}}. \]

Then we have a diffeomorphism \(p_{-1} \circ p_+ : S^1 \rightarrow S^1\).

1.7. The semigroup \(\Gamma\). An element of the semigroup \(\Gamma\) is a triple \(\mathcal{R} := (\mathcal{R}, r^+, r^-)\), where \(\mathcal{R}\) is the Riemann surface (a one-dimensional complex manifold) equivalent to the Riemann sphere \(\overline{\mathbb{C}}\),

\[ r_- : D^- \rightarrow \mathcal{R}, \quad r_+ : D^+ \rightarrow \mathcal{R} \]

are univalent up to the boundary, and

\[ r_-(D^-) \cap r_+(D^+) = \emptyset. \]

Two triples \((\mathcal{R}, r^+, r^-)\) and \((\mathcal{R}', r'^+ , r'^-)\) are equivalent if there is a biholomorphic map \(\varphi : \mathcal{R} \rightarrow \mathcal{R}'\) such that \(r'_\pm = \varphi \circ r_\pm\).

\(^3\)This semigroup is an analog of Olshanski semigroups, see, e.g., [29], Add. to Chapter 3.
Define a product $\mathfrak{R} = \mathfrak{P} \mathfrak{Q}$ of two elements \( \mathfrak{P} := (P, p_+, p_-), \mathfrak{Q} := (Q, q_+, q_-) \).

We take surfaces (closed disks)

\[
P \setminus p_-(D^0_+), \quad Q \setminus q_+(D^0_+)\]

and glue them together identifying points \( p_-(z) \in P \setminus p_-(D^0_+) \) and \( q_+(z) \in Q \setminus q_+(D^0_+) \), where \( z \) ranges in \( S^1 \). We get a Riemann sphere and two univalent functions, i.e., an element of \( \Gamma \).

It is clear that diffeomorphisms \( \gamma \in \text{Diff}(S^1) \) can be regarded as limit points of \( \Gamma \). We also define the semigroup

\[
\Gamma = \Gamma \cup \text{Diff}(S^1).
\]

This semigroup was discovered in [23], [31]; for details, see [24], [28], Sect VII.4.

There are the following statements about representations of \( \Gamma \):

a) Any module \( L(h, c) \) can be integrated to a projective holomorphic representation of the semigroup \( \Gamma \) by bounded operators in a Frechet space.

b) Any unitarizable module \( L(h, c) \) can be integrated to a representation of \( \Gamma \), which is holomorphic on \( \Gamma \) and unitary up to scalars on \( \text{Diff}(S^1) \), see [24], [28], Sect. 7.4.

c) Standard boson representations of Virasoro algebra (see Subsect. 1.4) can be integrated to holomorphic representations of \( \Gamma \) by bounded operators, see [24]. Our calculation is based on explicit formulas for such representations.

1.8. The complex domain \( \text{Diff}(S^1)/\Gamma \). Next, consider the space \( \Xi \), whose points are pairs \((S, s, \sigma)\), where \( S \) is a Riemann surface equivalent to Riemann sphere and \( s : D^+ \to S \) is a map univalent up to the boundary, and \( \sigma \) is a point in \( S \setminus s(D^+\cdot) \). The semigroup \( \Gamma \) acts on \( \Xi \) in the following way. Let \( \mathfrak{P} := (P, p_+, p_-) \in \Gamma, \mathfrak{S} = (S, s) \). We glue \( P \setminus p_-(D^0_+) \) and \( S \setminus s(D^0_+) \) identifying points \( p_-(z) \in P \setminus p_-(D^0_+) \) and \( s(z) \in S \setminus s(D^0_+) \), where \( z \) ranges \( S^1 \). We get a Riemann sphere, an univalent map from \( D^+ \) to the Riemann sphere, and a distinguished point.

Below we assume

\[
S = \mathbb{C}, \quad s(0) = 0, \quad \sigma = \infty.
\]

Remark. The space \( \tilde{\Xi} \) of univalent functions \( s(z) = z + a_2 z^2 + \ldots \) from \( D^0_+ \) to \( \mathbb{C} \) was a subject of a wide and interesting literature (see books [12], [5]). In 1907 Koebe showed that this space is compact, in next 80 years extrema of numerous functionals on \( \tilde{\Xi} \) were explicitly evaluated. Unfortunately, this science declared the Bieberbach conjecture as the main purpose. The proof of the conjecture (De Branges, 1985) implied a decay of interest to the subject. Unfortunately, this happened before \( \Xi \) and \( \tilde{\Xi} \) became a topic of the representation theory [10, 23, 4, 24] in 1986-1989.

1.9. The cocycles \( \lambda \) and \( \mu \). Let

\[
r_- : D_- \to \mathbb{C} \setminus 0, \quad p_+ : D_+ \to \mathbb{C} \setminus \infty
\]
be functions univalent up to the boundary. We assume that \( r_-(\infty) = \infty, p_+(0) = 0 \) (there are no extra conditions).

Let us take a function \( r^+: D_+ \to \mathbb{C} \) such that \( r^+(0) = 0 \) and \( (\mathbb{C}, r_-, r^+) \) determines an element \( \text{Diff}(S^1) \), denote it \( \gamma_r \). Take a function \( p^-: D_- \to \mathbb{C} \) such that \( (\mathbb{C}, p_+, p^-) \) determines an element of \( \text{Diff}(S^1) \), denote it by \( \gamma_p \).

Consider the product \( \gamma_r \gamma_p \) and take the corresponding triple \( \gamma_r \gamma_p = (Q, q_+, q_-) \).

We can assume \( Q = \mathbb{C}, q_+(0) = 0, q_-(\infty) = \infty \).

We define two functions

\[
\lambda(r_-, p_+) = \int_{|z|=1} \left( \ln \frac{r_-(z)}{z} d\ln q_+(\gamma_r(z)) - \ln \frac{p_+(\gamma_p(z))}{\gamma_p(z)} d\ln \frac{q_-(z)}{p^-(z)} \right),
\]

\[
\mu(r_-, p_+) = \frac{1}{24} \int_{|z|=1} \left( -\ln r_-(z)' d\ln \frac{q_-'(\gamma_r(z))}{(r^+)'(\gamma_r(z))} + \ln p_+(\gamma_p(z)) d\ln \frac{q_-'(z)}{p^-'(z)}(z) \right).
\]

1.10. Canonical cocycles. Consider a highest weight representation \( L(h, c) \) of \( \mathfrak{Vir} \). Denote by \( N^{h,c}(\mathfrak{g}) \) the corresponding representation of the semigroup \( \Gamma \). Since our representation is projective, operators \( N^{h,c}(\mathfrak{g}) \) are defined up to scalar factors.

Denote by \( v \) a highest weight vector in the completed \( L(h, c) \), it is determined up to a scalar factor. The projection operator \( \pi \) to the line \( \mathbb{C}v \) is determined canonically (its kernel is a sum of all weight subspaces with weight different from \( h \)). The condition

\[
\pi \left( N^{h,c}(\mathfrak{g})v \right) = v
\]

determines an operator \( N^{h,c}(\mathfrak{g}) \) canonically. Now we have

\[
N^{h,c}(\mathfrak{g}) N^{h,c}(\mathfrak{h}) = \mathcal{C}_{h,c}(\mathfrak{g}, \mathfrak{h}) N^{h,c}(\mathfrak{g} \circ \mathfrak{h}),
\]

where a constant \( \mathcal{C}_{h,c}(\mathfrak{g}, \mathfrak{h}) \in \mathbb{C} \) is a canonically defined.

Theorem 1.1

\[
\mathcal{C}_{h,c}(\mathfrak{g}, \mathfrak{h}) = \exp\{ h\lambda(r_-, p_+) \} \cdot \exp\{ c\mu(r_-, p_+) \}.
\]

Remarks. 1) By construction, the functions \( \lambda, \mu: \text{Diff}(S^1) \times \text{Diff}(S^1) \to \mathbb{C} \) determine \( \mathbb{C} \)-central extensions of \( \text{Diff}(S^1) \). In other words they represent classes of the second group cohomologies \( H^2(\text{Diff}(S^1), \mathbb{C}) \).

\[4\]There is a one-parametric family of such diffeomorphisms

\[5\]The formulas in [24] contain a mistake in rational coefficients in the front of the integrals. This happened in manipulations with ‘logarithmic forms’ \( f(z) + \ln dz \). We have \( \ln dz = i\varphi + \ln d\varphi \), the summand \( i\varphi \) was lost.
2) The group of continuous cohomologies $H^2(\text{Diff}(S^1), \mathbb{R})$ is generated by two cocycles (see [10], Theorem 3.4.4). The first one is the Bott cocycle

$$c_1(\gamma_1, \gamma_2) = \frac{1}{2} \int_{|z|=1} \ln(\gamma'_1(\gamma_2(z))) \, d\ln \gamma'_2(z).$$

To define the second cocycle, we write the multivalued expression

$$c_2(\gamma_1, \gamma_2) = \ln(\gamma_1(\gamma_2(z))) - \ln(\gamma_2(z)) - \ln \gamma_1(z)$$

and choose its continuous branch on $\text{Diff}(S^1) \times \text{Diff}(S^1)$ such that $c_1(e, e) = 0$ (this cocycle can be reduced to a $\mathbb{Z}$-cocycle determining the universal covering group of $\text{Diff}(S^1)$). The cocycles $\mu, \lambda$ are equivalent to $c_1, c_2$ in the group $H^2(\text{Diff}(S^1), \mathbb{C})$.

3) I never met a theorem that all measurable $\mathbb{R}/2\pi\mathbb{Z}$-central extensions of $\text{Diff}(S^1)$ are reduced to cocycles $c_1, c_2$. In our context, cocycles are continuous by construction exposed below, see (2.16). However, the question has some interest from the point of view of representation theory.

1.11. Realization of highest weight representations in the space of holomorphic functionals on the domain $\Xi$.

Corollary 1.2 The formula

$$\rho_{h,c}(\mathcal{O}) f(\mathcal{O}) = f(\mathcal{O}) \exp\{h \lambda(q_-, s)\} \cdot \exp\{c \mu(q_-, s)\}$$

(1.7)
determines a representation of $\Gamma$ in the space of holomorphic functionals on $\Xi$. For $(h,c)$ in a general position (if the Verma module $M(h,c)$ is irreducible), the corresponding module over the Virasoro algebra is $M(h,c)$.

Remark. Formulas for the underlying action of the Lie algebra $\mathfrak{Vect}$ on $\Xi$ were firstly obtained in by Scheffer [30], 1948, and rediscovered in [18]. The action of $\mathfrak{Vir}$ corresponding [17] was present in [23] without a proof, for a proof, see [17]. Considering an action in the space of polynomials in Taylor coefficients of $p(z)$, we get a module dual to the Verma module. Some modifications of formulas are contained in [2].

Remark. Of course, we can invent many different definitions of holomorphic functions on $\Xi$ and of spaces of holomorphic functions. Our statement is almost independent on this. To avoid a scholastic discussion, we note that a construction [24], Subs. 4.12 gives an operator sending Fock space to the space of functions on $\Xi$.

1.12. Reproducing kernel. On Hilbert spaces determined by reproducing kernels see, e.g., [21], Sect. 7.1.

Corollary 1.3 Let $L(h,c)$ be unitarizable. Then the kernel

$$K^{h,c}(r,p) := \exp\{h \lambda(r^*, p)\} \cdot \exp\{c \mu(r^*, p)\}, \quad \text{where } r^*(z) := \overline{r(\overline{z})}^{-1},$$

(1.8)
determines a Hilbert space of holomorphic functions invariant with respect to the operators $\rho^{h,c}(\Omega)$ defined by (1.7). The corresponding representation of Virasoro algebra is $L(h,c)$. Moreover, for $q \in \text{Diff}(S^1)$ these operators are unitary.

2 Preliminaries. Explicit formulas for representations of $\Gamma$.

2.1. The Fock space. See [28], Sect. V.3, or [29], Sect. 4.1, for details of definition and proofs. The boson Fock space with $n$ degrees of freedom is a Hilbert space of holomorphic functions on $\mathbb{C}^n$ with inner product

$$(f,g) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-|z|^2} \prod_{j=1}^{n} d\text{Re} \, z_j \, d\text{Im} \, z_j.$$ 

For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ we define an element $\varphi_a^{(n)} \in F_n$ by

$$\varphi_a(z) := \exp\left\{ \sum_{j=1}^{n} z_j \bar{a}_j \right\}.$$ 

Then we have a reproducing property

$$f(a) = \langle f, \varphi_a \rangle_{F_n}.$$ 

Consider an isometric embedding $J_n: F_n \rightarrow F_{n+1}$ determined by

$$Jf(z_1, \ldots, z_n, z_{n+1}) = f(z_1, \ldots, z_n).$$ 

Next, consider the union of the chain of Hilbert spaces

$$\ldots \rightarrow F_n \rightarrow F_{n+1} \rightarrow \ldots.$$ 

Its completion is the Fock space $F_{\infty}$ with infinite number degrees of freedom.

For $a = (a_1, a_2, \ldots) \in \ell_2$ we define an element $\varphi_a \in \ell_2$ by

$$\varphi_a(z) := \exp\left\{ \sum_{j=1}^{n} z_j \bar{a}_j \right\} := \lim_{n \rightarrow \infty} \varphi(a_1, \ldots, a_n)(z_1, \ldots, z_n),$$

the limit is a limit in the sense of the Hilbert space $F_{\infty}$. Then for any element $h \in F_{\infty}$ we define a holomorphic function on $\ell_2$ by

$$f(u) = \langle f, \varphi_u \rangle_{F_{\infty}}.$$ 

We identify $F_{\infty}$ with this space of holomorphic functions.

Since any infinite-dimensional Hilbert space $H$ is isomorphic $\ell_2$, we can define a Fock space $F(H)$ as a space of functions on $H$ determined by the reproducing kernel

$$R(h_1, h_2) = \exp \langle h_1, h_2 \rangle_H.$$
On Hilbert spaces determined by reproducing kernels, see, e.g., [29], Sect. 7.1.

2.2. Symbols of operators. For details, see [28], Sect. V.3, VI.1, [29], Sect. 4.1, Sect. 7.1. Let \( A \) be a bounded linear operator in \( F_\infty \). Following F.A. Berezin, define its kernel by

\[
K(z, u) = \langle A\varphi_u, \varphi_z \rangle_{F_\infty}.
\]

Then for any function \( f \in F_\infty \), we have

\[
f(z) = \langle f, k_z \rangle_{F_\infty}.
\]

where \( k_z(u) := K(z, u) \).

Let \( A, B \) be bounded operators, let \( L, K \) be their kernels, let \( k_z(u) := K(z, u), l_z(u) := L(z, u) \) then the kernel \( M \) of \( BA \) is

\[
M(z, u) = \langle k_u, l_z \rangle_{F_\infty}.
\]

2.3. Gaussian operators. See [28], Sect. V.4, Sect. VI. 2-4, [29], Sect. 5-6. Consider a block symmetric matrix \( S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \) of size \( \infty + \infty \) and two vectors \( \lambda, \mu \in \ell_2 \). Define a Gaussian operator

\[
B = \begin{bmatrix} K & L \\ L^t & M \end{bmatrix}
\]

in \( F_\infty \) as an operator with the kernel

\[
\exp \left\{ \frac{1}{2} \left( z, \pi \right) \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \begin{pmatrix} z \cr \pi \end{pmatrix} + z\lambda^t + \pi\mu^t \right\}.
\] (2.1)

Here the vectors \( z, u, \lambda, \mu \in \ell_2 \) are regarded as vector-columns, \( z^t, u^t, \lambda^t, \mu^t \) are vector-columns. The expression in the curly brackets is a quadratic expression in \( z, \pi \).

If an operator \( B[\ldots] \) is bounded, then

1. \( \|S\| \leq 1, \|K\| < 1, \|M\| < 1 \);
2. \( K, M \) are Hilbert-Schmidt operators.

If also \( \|S\| < 1 \), then the operator \( B[S|\sigma] \) is bounded.

More on conditions of boundedness, see [28], Sect. V.2-4.

Product of two Gaussian operators is given by the formula

\[
B = \begin{bmatrix} K & L \\ L^t & M \end{bmatrix} B = \begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix} B = \begin{bmatrix} \lambda^t & \mu^t \\ \pi^t & \psi^t \end{bmatrix} = \sigma(M, P; \mu, \pi);
\]

\[
B = \begin{bmatrix} K + LP(1 - MP)^{-1}L^t \\ Q^t(1 - MP)^{-1}L \end{bmatrix} \begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix} \begin{bmatrix} \lambda^t + L(1 - PM)^{-1}(\pi^t + P\mu^t) \\ \pi^t + Q^t(1 - MP)^{-1}MQ \end{bmatrix}.
\] (2.2)

\[\]
where the constant $\sigma(\ldots)$ is given by

$$\sigma(M, P; \mu, \pi) = \det \left[ (1 - MP) \right]^{-1/2} \exp \left\{ \frac{1}{2} \left( \begin{array}{cc} \pi & \mu \\ • & • \end{array} \right) \left( \begin{array}{cc} -P & 1 \\ 1 & -M \end{array} \right)^{-1} \left( \begin{array}{c} \pi^t \\ \mu^t \end{array} \right) \right\}. \tag{2.3}$$

**Remark.** The conditions $\|M\| < 1$, $\|P\| < 1$ imply $\|MP\| < 1$. Hence we can evaluate $(1 - MP)^{-1/2}$ as

$$(1 - MP)^{-1/2} = 1 + \frac{1}{2} MP - \frac{3}{8} (MP)^2 + \ldots$$

Next, $M, P$ are Hilbert–Schmidt operators. Therefore $MP$ is a trace class operator. It is easy to show that the series for $-1 + (1 - MP)^{-1/2}$ converges in the trace norm, therefore the determinant in (2.3) is well-defined. \* \* \* 

**Remark.** The big matrix in the right-hand side of (2.2) admits a transparent geometric interpretation, see [28], Theorem 5.4.3, Sect. VI.5. \* \* \* 

2.4. Gaussian vectors. Define Gaussian vectors $b[P|\pi^t]$, where $P$ is a Hilbert–Schmidt symmetric matrix with $\|P\| < 1$ and $\pi \in \ell_2$ by

$$b[P|\pi^t](z) := \exp \left\{ \frac{1}{2} zPz^t + z\pi^t \right\}.$$

Then $b[P|\pi^t] \in F_{\infty}$. Also,

$$B \left[ \begin{array}{c} K \\ L^t \\ M \end{array} \right] = \lambda^t \mu^t \right\] b[P|\pi^t] = \sigma(M, P; \mu, \pi) \times$$

$$\times b[K + LP(1 - MP)^{-1}L^t|\lambda^t + L(1 - PM)^{-1}(\pi^t + P\mu^t)], \tag{2.4}$$

where $\sigma(\ldots)$ is given by (2.3). Notice that

$$K + LP(1 - MP)^{-1}L^t \quad \text{and} \quad \lambda^t + L(1 - PM)^{-1}(\pi^t + P\mu^t)$$

also are elements of the right-hand side of (2.2). Inner products of vectors $b[\ldots]$ are given by

$$(b[K|\mu^t], b[P|\pi^t])_{F_{\infty}} = \sigma(M, P; \mu, \pi'). \tag{2.5}$$

**Remark.** We have $b[O|0] = 1$. Obviously, for any Gaussian operator $B[S|\sigma]$, we have

$$\langle B[S|\sigma] \cdot 1, 1 \rangle = 1. \tag{2.6}$$

2.5. The Weil representation. Now consider the space $H = \ell_2 \oplus \ell_2$ and consider the group of bounded operators

$$\left( \begin{array}{cc} \Phi & \Psi \\ \overline{\Phi} & \overline{\Psi} \end{array} \right), \tag{2.7}$$

which are symplectic in the following sense:

$$\left( \begin{array}{cc} \Phi & \Psi \\ \overline{\Phi} & \overline{\Psi} \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} \Phi & \Psi \\ \overline{\Phi} & \overline{\Psi} \end{array} \right)^t = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right).$$
We denote by \( \text{Sp}(2\infty, \mathbb{R}) \) the group of such matrices satisfying an additional condition:

— \( \Psi \) is a Hilbert–Schmidt operator.

Next, we consider the group \( \text{ASp}(2\infty, \mathbb{R}) \) of affine transformations of \( \ell_2 \oplus \ell_2 \) generated by the group \( \text{Sp}(2\infty, \mathbb{R}) \) and shifts by vectors of the form \((h, \overline{h})\). According \( [3, \S 4, \text{Theorem 3}] \), the group \( \text{ASp}(2\infty, \mathbb{R}) \) has a standard projective unitary representation in \( F_\infty \) by unitary Gaussian operators. Elements of \( \text{Sp}(2\infty, \mathbb{R}) \) act by operators

\[
\tau \cdot B \begin{bmatrix} \Psi \Phi^{-1} & \Phi^{-1} & 0 \\ \Phi^{-1} & -\Phi^{-1} \Psi & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

The shifts \((h, \overline{h})\) act by the operators

\[
B \begin{bmatrix} 0 & 1 & h \\ 1 & 0 & -h \end{bmatrix}.
\]

Let \( g \in \text{ASp}(2n, \mathbb{R}) \), let \( B[S]\sigma \) be the corresponding Gaussian operator. By the formula \((2.2)\), the matrix \( S \) depends only on the linear part of \( g \).

### 2.6. The Hilbert space \( V \)

First, we consider a space \( V^{\text{smooth}} \) of smooth functions \( f(z) = \sum c_k z^k \) on the circle \( S^1 \) defined up to an additive constant. Define the inner product in \( V^{\text{smooth}} \) by

\[
(z^k, z^l)_{V^{\text{smooth}}} = |k|, \quad (z^k, z^l)_{V^{\text{smooth}}} = 0 \quad \text{if} \ l \neq k.
\]

Denote by \( V \) the completion of \( V^{\text{smooth}} \) with respect to this inner product. Notice that all elements of \( V \) are \( L^2 \)-functions on \( S^1 \). Denote by \( V_+ \) (resp. \( V_- \)) the subspace consisting of series \( \sum_{k>0} c_k z^k \) (resp. \( \sum_{k<0} c_k z^k \)). These subspaces consist of functions admitting holomorphic continuations to the disks \( D^o_+ \) and \( D^o_- \) respectively.

We define a skew-symmetric bilinear form on \( V \) by the formula

\[
\{f, g\} = \int_{|z|=1} f(z) \, dg(z).
\]

The subspaces \( V_+ \), \( V_- \) are isotropic and dual one two another with respect to this form. The projection operators to \( V_\pm \) are given by the Cauchy integral

\[
P_\pm f(z) = \pm \lim_{\varepsilon \to 0^\pm} \frac{1}{2\pi i} \int_{|u|=1+\varepsilon} \frac{f(u)}{u - z}.
\]

For a diffeomorphism \( \gamma \in \text{Diff}(S^1) \) we define the linear operator in \( V \) by

\[
T(\gamma) f(z) = f(\gamma^{-1}(z)).
\]

Evidently, our form is \( \text{Diff}(S^1) \)-invariant:

\[
\{T(\gamma)f_1, T(\gamma)f_2\} = \{f_1, f_2\} \quad \text{or} \quad \{T(\gamma)f_1, f_2\} = \{f_1, T(\gamma)^{-1}f_2\}.
\]
2.7. Construction of highest weight representations of \( \text{Diff}(S^1) \). Consider the Fock space \( F(V_-) \) corresponding to the Hilbert space \( V_- \). We wish to construct projective representations of \( \text{Diff}(S^1) \) in \( F(V_-) \) corresponding to the standard boson realization of representations of \( \mathfrak{Vir} \). Let us regard the circle as the quotient \( \mathbb{R}/2\pi \mathbb{Z} \) with coordinate \( \varphi \). Fix \( \alpha, \beta \in \mathbb{R} \). For \( \gamma \in \text{Diff}(S^1) \) we consider the affine transformation of \( V \) given by the formula

\[
T_{\alpha,\beta}(\gamma^{-1})f(\varphi) = f(\gamma(\varphi)) + \alpha(\gamma(\varphi) - \varphi) + \beta \ln \gamma'(\varphi).
\] (2.13)

Remark. Recall that \( \varphi, \gamma(\varphi) \) are elements of \( \mathbb{R}/2\pi \mathbb{Z} \). We take an arbitrary continuous \( \mathbb{R} \)-valued branch of \( \gamma(\varphi) \). Then the function \( \gamma(\varphi) - \varphi \) is well defined up to an additive constant and it is an element of \( V \) and we can multiply it by a constant \( \alpha \).

The formula (2.13) determines an embedding of the group \( \text{Diff}(S^1) \) to the group \( \text{ASp}(2\infty, \mathbb{R}) \). Restricting the Weil representation to \( \text{Diff}(S^1) \) we get a unitary projective representation of \( \text{Diff}(S^1) \). On the level of the Lie algebra we obtain the representation determined in Subsection 1.4. See [28], Sect. VII.2

Next, we wish to write the matrix (2.14). Applying (2.10) we represent the block \( \Phi \) in (2.7) as

\[
\Phi f(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z|=1+\varepsilon} f(\gamma^{-1}(z)) \frac{dz}{z-u}.
\]

Formally, the right-hand side is well-defined for real-analytic \( f \) and real analytic \( \gamma \in \text{Diff}(S^1) \). But it makes sense for any distribution \( f \) and any smooth \( \gamma \). For instance, we can decompose \( f(\gamma^{-1}(z)) \) into the Fourier series \( \sum_{k \in \mathbb{Z}} c_k z^k \) and take

\[
\Phi f(z) := \sum_{k>0} c_k z^k.
\]

In the same way, we write an expression for \( \Psi \).

In the next subsection, we will express the operators \( \Phi^{-1}, \Psi \Phi^{-1} \) in the terms of welding.

2.8. Formulas for action of \( \Gamma \). For details and proofs, see [24], [28], Sect. VII.4-5. Let \( \alpha, \beta \in \mathbb{C} \). The semigroup \( \Gamma \) acts in the space \( F(V_-) \) by Gaussian operators

\[
\mathcal{N}_{\alpha,\beta}(\mathcal{R}) = B \begin{bmatrix}
K(r_+) & L(r_+, r_-) \\
L(r_+, r_-)^t & M(r_-)
\end{bmatrix}
\begin{bmatrix}
-(\beta + i\alpha) \ell_1'(r_+) + \beta m_1'(r_+) \\
-(\beta + i\alpha) \ell_2'(r_-) + \beta m_2'(r_-)
\end{bmatrix},
\]

where operators

\[
K : V_- \to V_+, \quad L : V_- \to V_-, \quad L^t : V_+ \to V_+, \quad M : V_+ \to V_-
\]

and vectors

\[
\ell_1', m_1', \ell_2', m_2' \in V_+,
\]

will be defined now.
Consider a function \( f \in V_\pm \). Then the function \( f \circ (r_+)^{-1} \) is defined on the contour \( r_+(S^1) \). Let us decompose \( f \circ (r_+)^{-1} \) as \( F_1 + F_2 \), where \( F_1 \) is holomorphic in the domain \( r_+(D_0^+) \), and \( F_2 \) is holomorphic in \( \overline{\mathbb{C}} \setminus r_+(D_+) \). This decomposition is determined up to an additive constant, \( F_1 \mapsto F_1 + c, \ F_2 \mapsto F_2 - c \). We set
\[
Kf := F_1 \circ r_+, \quad Lf = F_2 \circ r_-
\]
In a similar way, we take a function \( f \in V_+ \), decompose \( f \circ (r_-)^{-1} \) as \( F_1 + F_2 \), where \( F_1 \) is holomorphic in \( r_-(D_0^-) \), and \( F_2 \) is holomorphic in \( C \setminus r_-(D_-) \). We set
\[
Mf := F_1 \circ r_-, \quad L^Tf = F_2 \circ r_+
\]
Recall that the subspaces \( V_+ \) and \( V_- \) are dual one to another with respect to the pairing \((2.9)\). It can be shown that the operators \( L : V_- \to V_- \) and \( L^T : V_+ \to V_+ \) are dual one to another. Also, the operators \( M : V_+ \to V_- \), \( K : V_- \to V_+ \) are dual to themselves.

Finally, we set
\[
\ell_1(r_+) = \ln \frac{r_+(z)}{z}, \quad \ell_2(r_-) = \ln \frac{r_-(z)}{z}, \quad (2.14)
\]
\[
m_1(r_+) = \ln r_+(z), \quad m_2(r_-) = \ln r_-^r(z). \quad (2.15)
\]

**Remark.** The operator \( K(r_+) \) is the Grunsky matrix of the univalent function \( r_+ \), it is a fundamental object of theory of univalent functions, see \([14], [12], [3]\). \( \square \)

It remains to explain the meaning of the expression \((2.1)\). For \( f_\pm \in V_\pm \), we set
\[
(f_- f_+) \begin{pmatrix} K & L^T \\ L & M \end{pmatrix} \begin{pmatrix} (f_-)_+^T \\ (f_+) \end{pmatrix} := \{f_-^T, K f_+^T\} + 2 \{f_+^T, L f_-^T\} + \{f_-^T, M f_+^T\},
\]
where the form \( \{\cdot, \cdot\} \) is defined by \((2.8)\), also
\[
(f_- f_+) \begin{pmatrix} \ell_1^T \\ \ell_2^T \end{pmatrix} := \{f_-^T, \ell_1^T\} + \{f_+^T, \ell_2^T\},
\]
we omit the similar expression with \( m_1, m_2 \).

Also we note that under this normalization we have
\[
\pi(N_{\alpha,\beta}(\mathbb{R})1) = 1, \quad (2.16)
\]
where \( \pi \) is the projection operator in \( F(V_-) \) to the line \( \mathbb{C} \cdot 1 \).
3 The calculation

3.1. Step 1. Consider

\[ \mathfrak{R} = (\mathfrak{C}_+, r_-), \quad \mathfrak{P} = (\mathfrak{C}_+, p_-) \in \Gamma \]

such that

\[ r_+(0) = 0, \quad r_-(\infty) = \infty, \quad p_+(0) = 0, \quad p_-(\infty) = \infty. \]

Next, we choose \( r^+, p^- \) such that \( (\mathfrak{C}, r^+, r_-), (\mathfrak{C}, p^+, p_-) \in \text{Diff}(S^1) \) as it was discussed in Subsection 1.9. Take the corresponding \( \gamma_r, \gamma_p \), and represent \( \gamma_r \gamma_p \) as \( (\mathfrak{C}, q^+, q_-) \).

According formulas (2.2), (2.3), we get

\[ N_{\alpha, \beta}(\mathfrak{R}) N_{\alpha, \beta}(\mathfrak{P}) = \kappa_{\alpha, \beta}(\mathfrak{R}, \mathfrak{P}) N_{\alpha, \beta}(\mathfrak{R} \circ \mathfrak{P}), \]

where the canonical cocycle \( \kappa_{\alpha, \beta} \) is given by

\[
\kappa_{\alpha, \beta}(\mathfrak{R}, \mathfrak{P}) = \det[(1 - MK)]^{-1/2} \times \\
\times \exp \left\{ \frac{1}{2} \left[ - (i\alpha + \beta)\ell_1 + \beta m_1 - (i\alpha + \beta)\ell_2 + \beta m_2 \right] \right. \\
\left. \times \left[ - (i\alpha + \beta)\ell_1 + \beta m_1 \right] \right\}. \tag{3.1}
\]

where

\[
K := K(p_+), \quad M := M(r_-), \\
\ell_1 := \ell_1(p_+), \quad \ell_2 := \ell_2(r_-), \quad m_1 := m_1(p_+), \quad m_2 := m_2(r_-).
\]

In particular, we see that the cocycle

\[ \kappa_{\alpha, \beta}(\mathfrak{R}, \mathfrak{P}) = \kappa_{\alpha, \beta}(r_-, p_+) \]

does not depend on \( p_-, q_+ \) and is holomorphic in \( \alpha, \beta \).

Keeping in the mind further manipulations we represent (3.1) in the form

\[
\kappa_{\alpha, \beta}(r_-, p_+) = \det[(1 - MK)]^{-1/2} \times \\
\times \exp \left\{ \frac{1}{2} \alpha^2 \left( \ell_1, \ell_2 \right) \left[ \frac{-K}{1 - M} \right]^{-1} \left[ \ell_1, \ell_2 \right] \right. \\
\left. - i\alpha \beta \left( \ell_1, \ell_2 \right) \left[ \frac{-K}{1 - M} \right]^{-1} \left[ -\ell_1 + m_1, -\ell_2 + m_2 \right] + \\
\right. \\
\left. + \frac{1}{2} \beta^2 \left[ -\ell_1 + m_1 - \ell_2 + m_2 \right] \left[ \frac{-K}{1 - M} \right]^{-1} \left[ -\ell_1 + m_1, -\ell_2 + m_2 \right] \right\}. \tag{3.2}
\]
3.2. Step 2. We also use the notation of Subsection 1.10. In particular, we have another notation for the canonical cocycles $x^{h,c}$. By definition,

$$x_{\alpha,\beta}(r_-,p_+) = x^{\frac{1}{2}(\alpha^2 + \beta^2), 1+12\beta^2}(r_-,p_+).$$

Setting $\alpha = \beta = 0$, we get

$$x_{0,0}(r_-,p_+) = x^{0,1}(r_-,p_+) = \det[(1 - K(p_+) M(r_-))^{-1/2}].$$

Obviously

$$x^{h_{1,c_1}(R,P)} x^{h_{2,c_2}(R,P)} = x^{h_{1+h_2,c_1+c_2}(R,P)}. \quad (3.3)$$

Keeping in the mind the holomorphy, we get that $x^{h,c}$ has the form

$$x^{h,c}(r_-,p_+) = \exp\{h\lambda(r_-,p_+) + c\mu(r_-,p_+)\},$$

where $\lambda(r_-,p_+), \mu(r_-,p_+)$ are some functions. In another notation,

$$x_{\alpha,\beta}(r_-,p_+) = \exp\left\{\frac{1}{2}(\alpha^2 + \beta^2)\lambda(r_-,p_+) + (1 + 12\beta^2)\mu(r_-,p_+)\right\} =$$

$$= \det[(1 - K(p_+) M(r_-))^{-1/2}] \times$$

$$\times \exp\left\{\frac{1}{2}(\alpha^2 + \beta^2)\lambda(r_-,p_+) + 12\beta^2\mu(r_-,p_+)\right\}. \quad (3.4)$$

3.3. Step 3. The exponentials in (3.3) and (3.4) must coincide. This implies that the term with $\alpha\beta$ in (3.2) is absent. This means that

$$\ell_1 \ell_2 \left(\begin{array}{cc} -K & 1 \\ 1 & -M \end{array} \right)^{-1} \left(\begin{array}{c} \ell_1' \\ \ell_2' \end{array} \right) = \ell_1 \ell_2 \left(\begin{array}{cc} -K & 1 \\ 1 & -M \end{array} \right)^{-1} \left(\begin{array}{c} m_1' \\ m_2' \end{array} \right). \quad (3.5)$$

Next, we transform the coefficient at $\beta^2$ in (3.2) using (3.3),

$$\frac{1}{2} (-\ell_1 + m_1) \ell_2 + m_2) \left(\begin{array}{cc} -K & 1 \\ 1 & -M \end{array} \right)^{-1} \left(\begin{array}{c} -\ell_1' + m_1' \\ -\ell_2' + m_2' \end{array} \right) =$$

$$= \frac{1}{2} (m_1 - m_2) \left(\begin{array}{cc} -K & 1 \\ 1 & -M \end{array} \right)^{-1} \left(\begin{array}{c} m_1' \\ m_2' \end{array} \right) - \frac{1}{2} \ell_1 \ell_2 \left(\begin{array}{cc} -K & 1 \\ 1 & -M \end{array} \right)^{-1} \left(\begin{array}{c} \ell_1' \\ \ell_2' \end{array} \right).$$

Therefore, we can reduce (3.2) to the form

$$x_{\alpha,\beta}(r_-,p_+) = \det[(1 - MK)]^{-1/2} \times$$

$$\times \exp\left\{-\frac{1}{2}(\alpha^2 + \beta^2) \ell_1 \ell_2 \left(\begin{array}{cc} -K & 1 \\ 1 & -M \end{array} \right)^{-1} \left(\begin{array}{c} \ell_1' \\ \ell_2' \end{array} \right) +$$

$$+ \frac{1}{2} \beta^2 (m_1 - m_2) \left(\begin{array}{cc} -K & 1 \\ 1 & -M \end{array} \right)^{-1} \left(\begin{array}{c} m_1' \\ m_2' \end{array} \right) \right\}. \quad (3.6)$$

\footnote{Let $(h_1, c_1), (h_2, c_2)$ be in the domain of unitarity (1.1). Let $v_1, v_2$ be the highest vectors in $L(h_1, c_1), L(h_2, c_2)$. The cyclic span of $v_1 \otimes v_2 \in L(h_1, c_1) \otimes L(h_2, c_2)$ is the module $L(h_1 + h_2, c_1 + c_2)$. This implies the statement for $(h_1, c_1), (h_2, c_2)$ satisfying (1.1). It remains to apply the holomorphy.}
Comparing this with (3.4), we come to
\[
\lambda(r_-, p_+) = - (\ell_1 \ell_2) \begin{pmatrix} -K & 1 \\ 1 & -M \end{pmatrix}^{-1} \begin{pmatrix} \ell_1^t \\ \ell_2^t \end{pmatrix}; \quad (3.7)
\]
\[
\mu(r_-, p_+) = \frac{1}{24} (m_1 \ m_2) \begin{pmatrix} -K & 1 \\ 1 & -M \end{pmatrix}^{-1} \begin{pmatrix} m_1^t \\ m_2^t \end{pmatrix}. \quad (3.8)
\]
Since
\[
\begin{pmatrix} -K & 1 \\ 1 & -M \end{pmatrix}^{-1} = \begin{pmatrix} M(1 - KM)^{-1} & (1 - MK)^{-1} \\ (1 - KM)^{-1} & K(1 - MK)^{-1} \end{pmatrix},
\]
we have
\[
\lambda(r_-, p_+) = \ell_1 \left[ M(1 - KM)^{-1} \ell_1^t + (1 - MK)^{-1} \ell_2^t \right] + \ell_2 \left\{ (1 - KM)^{-1} \ell_1^t + K(1 - MK)^{-1} \ell_2^t \right\} \quad (3.9)
\]
Also we have a similar expression for \(\mu(r_-, p_+)\).

3.4. Step 4. Next, we evaluate the following product by formula (2.2)
\[
N_{\alpha, \beta}(\gamma_r) N_{\alpha, \beta}(\gamma_p) = \text{const} \cdot N_{\alpha, \beta}(\gamma_r \gamma_p).
\]
The formula involves 3 operators of the type \(B[\ldots]\). This implies an identity involving 3 matrices \([\ldots]\). We substitute \(\beta = 0\) and write the expression for the last column in the matrix \([\ldots]\) corresponding to \(\gamma_r \gamma_p\):
\[
\ell_1^t(q_r) = \ell_1^t(r^+) + L(r^+, r_-) \left\{ (1 - K(p_+)) M(r_-) \right\}^{-1} \left( \ell_1^t(p_+) + K(p_+) \ell_2^t(r_-) \right); \quad (3.11)
\]
\[
\ell_2^t(q_r) = \ell_2^t(p^-) + L^t(p_+, p^-) \left\{ (1 - M(r_-) K(p_+)) \right\}^{-1} \left( \ell_2^t(r_-) + M(r_-) \ell_1^t(p_+) \right). \quad (3.12)
\]
Keeping in the mind identity
\[
M(1 - KM)^{-1} = (1 - MK)^{-1} M,
\]
we observe that the expressions in curly brackets in (3.11) and (3.10) coincide. We express the term in the curly bracket from (3.11)
\[
\left\{ \ldots \right\} = L(r^+, r_-)^{-1} (\ell_1^t(q_r) - \ell_1^t(r^+))
\]
and substitute to (3.11). Also, the expressions in the square brackets in (3.12) and (3.9) coincide. We express the term in the square brackets from (3.12) and substitute it to (3.9). In this way, we get
\[
\lambda(r_-, p_+) = \ell_2(r_-) L(r^+, r_-)^{-1} (\ell_1^t(q_r) - \ell_1^t(r^+)) + \\
+ \ell_1(p_+) L^t(p_+, p^-)^{-1} (\ell_2^t(q_r) - \ell_2^t(p^-)). \quad (3.13)
\]
3.5. **Step 5.** Now we wish to evaluate $L^{-1}(r^+, r_-)$. For this aim consider the Gaussian operator, corresponding to the diffeomorphism $\gamma_r$, 

$$B \begin{bmatrix} K(r^+) & L(r^+, r_-) \\ L^t(r^+, r_-) & M(r_-) \end{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix}$$

On the other hand applying formula (2.8) to the same $\gamma_r$, we get an expression of the form

$$B \begin{bmatrix} \Phi^{-1} & \Phi^{t-1} \\ \Phi^{-1} - \Phi^{-1} \Psi \end{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix}.$$ 

Therefore $L(r^+, r_-) = \Phi^{t-1}$, i.e. $L^{-1} = \Phi^t$. Thus,

$$L(r^+, r_-)^{t-1} \ell_2^t = P_+ T(\gamma_r) \ell_2^t,$$  \hspace{1cm} (3.14)

where $P_+$ is the projection operators to $V_+$. In the same way, we obtain

$$L(p_+, p_-)^{t-1} \ell_1^t = P_- T(\gamma_p) \ell_1^t.$$  \hspace{1cm} (3.15)

3.6. **Step 6.** Evidently, for $g \in V_-$, we have

$$\int_{|z|=1} g \bar{d}(P_+ f) = \int_{|z|=1} g \bar{d} = \{ f, g \}.$$  \hspace{1cm} (3.16)

In the same way, for $f \in V_+$,

$$\int_{|z|=1} g \bar{d}(P_- f) = \int_{|z|=1} g \bar{d} = \{ f, g \}.$$ 

Now we can transform two summands in formula (3.13) for $\lambda(r_-, p_+)$. The first summand equals to

$$\ell_2(q_+)L(r^+, r_-)^{-1}(\ell_1(q_+\gamma_r) - \ell_1^t(r^+)) = (\ell_1(q_+\gamma_r) - \ell_1^t(r^+))L(r^+, r_-)^{t-1} \ell_2^t(r_-) =$$

$$= (\ell_1(q_+) - \ell_1^t(r^+))P_+ T(\gamma_r) \ell_2^t(r_-) =$$

$$= \left\{ (\ell_1(q_+) - \ell_1^t(r^+)) , P_+ T(\gamma_r) \ell_2^t(r_-) \right\} =$$

$$= \left\{ (\ell_1(q_+) - \ell_1^t(r^+)) , T(\gamma_r) \ell_2^t(r_-) \right\} =$$

$$= \left\{ T(\gamma_r)^{-1}(\ell_1^t(q_+) - \ell_1^t(r^+)) \right\}.$$ 

We applied (3.14), (3.16), and the invariance (2.12).

A similar calculation gives us the second summand of (3.13),

$$\ell_1(p_+)L^t(p_+, p_-)^{-1}(\ell_2(q_-) - \ell_2^t(p_-)) =$$

$$= \left\{ \ell_1^t(p_+) , P_- T(\gamma_p) (\ell_2^t(q_-) - \ell_2^t(p_-)) \right\} =$$

$$= \left\{ \ell_1^t(p_+) , T(\gamma_p) (\ell_2(q_-) - \ell_2^t(p_-)) \right\} =$$

$$= \left\{ T(\gamma_p)^{-1}\ell_1^t(p_+) , (\ell_2^t(q_-) - \ell_2^t(p_-)) \right\}.$$
This gives us formula (1.3) for $\lambda(r_-, p_+)$.

3.7. Final remarks. The derivation of the formula (1.4) $\mu(r_-, p_+)$ is similar, we simply change $\ell_1, \ell_2$ to $m_1, m_2$.

Our calculation of $\lambda(r_-, p_+)$, $\mu(r_-, p_+)$ is valid for $c \geq 1$, $h \geq (c - 1)/24$.

Due to the holomorphy we extend the result to arbitrary $(h, c) \in \mathbb{C}^2$.

It remains to explain Corollaries 1.2–1.3 from the theorem.

3.8. The action of $\text{Diff}(S^1)$ in the space of holomorphic functions. For an univalent function $s$, se denote $S^0 := s(\mathbb{C})$. For any element $f$ of the Fock space $F(V_-)$ we assign a holomorphic functional $\Xi$ given by

$$F(s) := \left\langle f, b[K(s^*)|(-i\alpha + \beta)\ell(s^*) + \beta m(s^*)]\right\rangle_{F(V_-)}.$$ 

In this way, we send the Fock space to the space of holomorphic functionals on $\Xi$, this gives us the desired realization (see [24], Subs. 4.12).

3.9. Reproducing kernels. Let $c \geq 1$, $h \geq (c - 1)/24$. The problem can be easily reduced to evaluation of inner products

$$\langle N_{\alpha, \beta}(\gamma_1) \cdot 1, N_{\alpha, \beta}(\gamma_2) \cdot 1 \rangle_{F(V_-)}, \quad (3.17)$$

where $\gamma_1, \gamma_2 \in \text{Diff}(S^1)$. Since operators $N_{\alpha, \beta}(\gamma)$ are unitary up to a constant, we have

$$N_{\alpha, \beta}(\gamma)^* = s \cdot N_{\alpha, \beta}(\gamma^{-1}).$$

for some $s \in \mathbb{C}$. On the other hand,

$$\langle N_{\alpha, \beta}(\gamma) \cdot 1 \rangle_{F(V_-)} = \langle 1, N_{\alpha, \beta}(\gamma^{-1}) \cdot 1 \rangle_{F(V_-)} = 1$$

and this implies $s = 1$. Therefore, (3.17) equals to

$$\langle N_{\alpha, \beta}(\gamma_2^{-1}) N_{\alpha, \beta}(\gamma_1) \cdot 1, 1 \rangle_{F(V_-)}$$

and we reduce our problem to the evaluation of the canonical cocycle.

Next, we have apriory identity (see footnote 9) for unitary representations $L(h_1, c_1)$, $L(h_2, c_2)$:

$$K^{h_1, c_1}(r, p) K^{h_2, c_2}(r, p) = K^{h_1 + h_2, c_1 + c_2}(r, p).$$

Taking arbitrary $L(h_1, c_1)$, $c_2 > 1$, and sufficiently large $h_2$ we can obtain $K^{h_1, c_1}(r, p)$ as

$$K^{h_1, c_1}(r, p) = \frac{K^{h_1 + h_2, c_1 + c_2}(r, p)}{K^{h_2, c_2}(r, p)}.$$  

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