THE COBORDISM DISTANCE BETWEEN A KNOT AND ITS REVERSE

CHARLES LIVINGSTON

Abstract. We consider the question of how knots and their reverses are related in the concordance group C. There are examples of knots for which K \neq K' \in C. This paper studies the cobordism distance, d(K, K'). If K \neq K' \in C, then d(K, K') > 0 and it elementary to see that for all K, d(K, K') \leq 2g_4(K). It is known that d(K, K') can be arbitrarily large. Here we present a proof that for non-slice knots satisfying g_4(K) = g_4(K), one has d(K, K') \leq 2g_4(K) − 1. This family includes all strongly quasi-positive knots and all non-slice genus one knots. We also construct knots K of arbitrary four-genus for which d(K, K') = g_4(K). Finding knots for which d(K, K') > g_4(K) remains an open problem.

1. Introduction

This work explores concordance relationships between knots and their reverses. Recall that if a knot K is defined formally as an oriented pair (S^3, K), then K' is shorthand for (S^3, −K). The results here apply equally in the smooth and in the topological locally flat category.

The problem of distinguishing a knot K from K' did not receive much attention until Fox noted the difficulty of the problem in [4] Problem 10. However, from the perspective of the classification of knots, the distinction is essential: the connected sum of knots is not well-defined outside of the oriented category. Advances in the study of knot reversal include [9,14,20].

Distinguishing the concordance classes of K and K' is more difficult; see [13] for an example in which Casson-Gordon invariants, realized as twisted Alexander polynomials, are applied to the problem. Infinite families of topologically slice knots for which K \neq K' in smooth concordance are produced in [12]. The issue appears in current research concerning satellite operations. Given a knot P \subset S^1 \times B^2, there is a self-map of the concordance group C given by using P to form a satellite, often denoted K → P(K), where here K and P(K) denote smooth concordance classes. In general, P(K) \neq P(K'), P(K) \neq P'(K), and P'(K') \neq P(K).

Being independent in C is only one measure of knots being distinct. Here we consider the cobordism distance, which is a metric on C: d(K, J) is the minimum genus of a cobordism from K to J; it equals the four-ball genus, g_4(K \# −J). Research about the cobordism distance includes a careful analysis of torus knots. For instance, for distinct positive parameters, Litherland [15] proved that the torus knots T(p, q) and T(p', q') are linearly independent. On the other hand, Feller-Park [6] analyzed the question of determining for which pairs d(T(p, q), T(p', q')) = 1, resolving all but one case: d(T(3,14), T(5,8)). Related work includes [1,5].

The triangle inequality states that for all K and J, d(K, J) \leq g_4(K) + g_4(J). It seems possible that in some generic sense, one usually has an equality. In particular, one might suspect that in general d(K, K') = 2g_4(K). Of course, this wouldn’t appear for low crossing number knots, most of which are reversible. Here we present a simple construction to prove that for a large class of knots this equality does not hold.

Theorem 1. If K is nontrivial, then d(K, K') < 2g_4(K).

Corollary 2. If g_4(K) = g_4(K) \neq 0, then d(K, K') < 2g_4(K).

Here are a few observations about Corollary 2.

This work was supported by a grant from the National Science Foundation, NSF-DMS-1505586.

1
(1) The hypothesis hold for all strongly quasi-positive knots [19].
(2) The hypothesis holds for all non-slice knots for which $g_3(K) = 1$.
(3) The theorem and corollary can be strengthened by replacing the condition $g_3(K) = g_4(K)$ with the condition on the concordance genus, $g_c(K) = g_4(K)$.
(4) There are no known examples of a non-slice knot for which $d(K, K^r) = 2g_4(K)$. In fact, there are no examples known for which $d(K, K^r) > g_4(K)$.

The proof of Theorem 1 is in Section 2. It generalizes an observation made in [11] about a specific family of genus one knots. In Section 3 we discuss extensions of this result.

Section 4 presents examples of knots with arbitrarily large four-genus for which $d(K, K^r) = g_4(K)$, here stated formally:

**Theorem 3.** For every integer $g \geq 0$, there exists a knot $K$ for which $d(K, K^r) = g_4(K) = g$.

The proof uses techniques from the work of Gilmer [8] that applied Casson-Gordon theory to study the four-genus of knots. It slightly refines a result in [11] where it was shown that $g_4(K \# -K^r)$ can be arbitrarily large.

**Acknowledgments.** Thanks to JungHwan Park for providing valuable feedback on an early version of this paper.

### 2. Proof of Theorem 1

A basic knot theory reference, such as the text by Rolfsen [18], is sufficient for the results used in the proof here. A schematic illustration for the proof is presented in Figure 1.

**Figure 1.** A schematic diagram of the proof of Theorem 1, illustrating the band sum that creates the surgery curve on $F \# F^*$ having boundary $K \# -K^r$. In this example, $g_3(K) = 1$. The thin (red) curve represents $\alpha \#_b \alpha^*$ in the notation of the proof.

Let $F$ be a minimal genus Seifert surface for $K$. Denote its genus by $g$. In Figure 1 the genus of $F$ is 1. Let $\alpha$ be an oriented simple closed curve on $F$ representing a nontrivial homology class. We will also think of $\alpha$ as representing a knot. (In the figure, $\alpha$ has the knot type of $J_1$.)

There is a natural choice of Seifert surface for $K \# -K^r$, which we will denote $F \# F^*$. We denote by $\alpha^*$ the curve of $F^*$ that corresponds to $\alpha$. One must specify an orientation on $\alpha^*$, but that choice will not be relevant in the proof. What is essential is the elementary observation that with some orientation, $\alpha^* = -\alpha$ as a knot.

If $V$ is the Seifert form for $F \# F^*$, then it is straightforward to show $V(\alpha, \alpha) = -V(\alpha^*, \alpha^*)$. Notice that reversing a knot on a Seifert surface does not change its self-linking.
Consider the abstract surface $\hat{G}$ formed by cutting $F \# F^*$ along $\alpha$ and $\alpha^*$. It has five boundary components: one that corresponds to $K \# -K^*$; a pair $\{\alpha_1, \alpha_2\}$ corresponding to the $\alpha$ cut, and a pair $\{\alpha_1', \alpha_2'\}$ that corresponds to the $\alpha'$ cut. Choose simple paths $\gamma_i$ from $\alpha_1$ to $\alpha_2'$. Each $\gamma_i$, viewed as a curve in $S^3$, can be used to form the band connected sum of $\alpha$ and $\alpha^*$ with the band in $F \# F^*$. Orientations can be chosen so that one is $\alpha \# b \alpha^*$ and the other is $\alpha \# b^* \alpha^*$. Thus, we can choose one of the bands to find $\alpha \# b - \alpha$ embedded on $F \# F^*$ as a homologically nontrivial curve.

The curve $\alpha \# b - \alpha$ has Seifert framing 0. If it is slice, then the Seifert surface $F \# F^*$ can be surgered in the four-ball to yield a surface of genus $2g - 1$ bounded by $K \# -K^*$ in $B^4$. The proof can now be completed in one of two ways. One option is to observe that for a natural choice of band, $\alpha \# b - \alpha = \alpha \# -\alpha$. As an alternative, one can use the following result of Miyazaki, for which we include a short summary proof.

**Lemma 4.** If $\{K, J\}$ is a split two component link, then the concordance class of the band connected sum $K \#_b J$ is independent of the choice of band.

**Proof.** The connected sum of knots is a special case of the band connected sum, so the result will be proved by showing that $(K \#_b J) \# -(K \# J)$ is slice. In this sum, one can slide the knot $-K$ over the band to see that

$$(K \#_b J) \# -(K \# J) \cong (K \# -K) \#_b (J \# -J).$$

It is now evident that this knot is slice: a band move (dual to the band $b$) splits it into two components forming an split link consisting of $K \# -K$ and $J \# -J$, both of which are slice. \hfill $\square$

### 3. Generalizations

The proof of Theorem 1 could have been based on the following lemma, which can also be applied in some cases to reduce the genus of the bounding surface to be less than $2g_3(K) - 1$. The proof of Lemma 5 is much the same as that of Lemma 4, so we do not include it here. Similarly, the proof of Corollary 6 follows along the lines of that of Lemma 4 and Corollary 2.

**Lemma 5.** Suppose that $\{J_1, \ldots, J_n\}$ and $\{J_1', \ldots, J_n'\}$ are split links. Then for any set of bands $\{b_i\}$ joining each $J_i$ to $J_i'$, the band connected sum is concordant to the split link with components $J_i \# J_i'$.

**Corollary 6.** Suppose that $g_3(K) = g_4(K) \neq 0$ and that $F$ is a minimum genus Seifert surface for $K$. If a rank $k$ subgroup of $H_1(F)$ is represented by a set of $k$ disjoint simple closed curves on $F$ that form a split link, then $d(K, K') \leq 2g_4(K) - k$.

As an example we use pretzel knots, which include the first knots shown to not be reversible [20].

**Example 7.** Consider the $n = 2k + 1$ stranded pretzel knot, with $n$ odd: $P = P(p_1, \ldots, p_n)$. Assume that $p_i > 0$ and each $p_i$ is odd. Then $g_3(P) = g_4(P) = k$ and $d(P, P') \leq k$. To prove this, one considers the Seifert matrix $V_P$ of dimension $2k \times 2k$. Since the $p_i$ are positive, the matrix is easily seen to be diagonally dominant, and thus the signature satisfies $\sigma(P) = 2k$. It follows that $g_4(P) \geq k$, but clearly $g_3(P) \leq k$. Using the fact that $g_4(P) \leq g_3(P)$ leads to the observation that $g_3(P) = g_4(P) = k$. There is an obvious set of $k$ curves on the standard Seifert surface for $P$ that represent independent classes in homology, and these form an unlink.

**Example 8.** More complicated examples can be built from the knot $P(p_1, \ldots, p_n)$ by tying arbitrary knots in the bands of its canonical Seifert surface.

### 4. Examples of $g_4(K \# -K^*) = g_4(K)$

In this section we prove Theorem 3. Our approach modifies examples built in [11], and again our proof depends on results of Gilmer [8] that demonstrated that Casson-Gordon invariants offer bounds on the four-genus of knots. The result in [11] is sufficient to find knots for which $g_4(K \# -K^*) \geq g_4(K)$; the added feature here is that for the current examples the four-genus $g_4(K)$ is exactly identified.
Figure 2 is a schematic representation of a knot we denote $K(A, B)$. In the figure there is a link $(K, \alpha, \beta)$. If we replace neighborhoods of $\alpha$ and $\beta$ with the complements of knots $A$ and $B$ (identifying the meridian and longitudes of $A$ and $B$ with the longitudes and meridians of $\alpha$ and $\beta$, respectively), the resulting manifold is diffeomorphic to $S^3$. The knot $K$ now represents the knot we call $K(A, B)$. In effect, the knots $A$ and $B$ are being tied in the two bands of the obvious Seifert surface.

![Diagram of knot K(A, B)](image)

**Figure 2. Knot K(A, B)**

We will prove the following.

**Theorem 9.** For every $n \geq 0$ there is a pair of knots $A$ and $B$ such that

$$g_4(nK(A, B)) = n = g_4(nK(A, B) \# -nK(A, B)^r).$$

The difficult work is in showing that $n$ is a lower bound for both invariants. We begin with the simple fact that it is an upper bound.

**Lemma 10.** For any choice of $A$ and $B$, $g_4(nK(A, B)) \leq n$ and $g_4(nK(A, B) \# -nK(A, B)^r) \leq n$.

**Proof.** We have that $g_3(K(A, B)) = 1$. Thus $g_4(K(A, B)) \leq 1$ and it follows that $g_4(nK(A, B)) \leq n$. By Theorem 1, $g_4(K(A, B) \# -K(A, B)^r) \leq 1$, and thus $g_4(nK(A, B) \# -nK(A, B)^r) \leq n$. \hfill $\square$

The remainder of this section is devoted to reversing these inequalities.

### 4.1. Signature bounds related to $g_4(nK(A, B))$.

In [8] Gilmer considered the case of $A = J = B$ for some $J$. However, his result [8, Theorem 1] can be applied directly to achieve the next result.

**Theorem 11.** If $g_4(nK(A, B)) = g < n$, then for some $a$ and $b$ satisfying $0 \leq a \leq n$, $0 \leq b \leq n$ and at least one of $a$ and $b$ are nonzero, one has:

$$\left| \sum_{i=1}^{n} \sigma_{\lambda}(K, \chi_i) + 2a\sigma_{1/3}(A) + 2b\sigma_{1/3}(B) \right| \leq 4g.$$

We will not define the quantity $\sigma_{\lambda}(K, \chi_i)$; all that is needed is that it is independent of the choice of $A$ and $B$, that there are only a finite number of possible $\chi_i$ that can arise, and that for each $\chi$, the values of $\sigma_{\lambda}(K, \chi_i)$ are uniformly bounded by some constant as functions of $\lambda \in S^1 \subset \mathbb{C}$.

### 4.2. Signature bounds related to $g_4(nK(A, B) \# -K(A, B)^r)$.

The proof of the main theorem in [11, Section 5] yields the following result, stated in terms of the classical Levine-Tristram signatures.

**Theorem 12.** If $g_4(nK(A, B) \# -K(A, B)^r) = g < n$, then there exists a pair of integers $a$ and $b$ satisfying $0 \leq a \leq n$, $0 \leq b \leq n$, and at least one of $a$ and $b$ are positive, such that

$$|a(\sigma_{1/7}(A) + \sigma_{2/7}(A) + \sigma_{3/7}(A)) - b(\sigma_{1/7}(B) + \sigma_{2/7}(B) + \sigma_{3/7}(B))| \leq 6g.$$

**Proof.** In the proof that appears in [11], there is a knot $J$ such that $A = J = -B$. The sum that appears is simpler, just a positive multiple of $\sigma_{1/7}(J) + \sigma_{2/7}(J) + \sigma_{3/7}(J)$. The proof however carries over almost verbatim in the current situation. \hfill $\square$
4.3. **Putting the results together.** According to [2], the values of the functions \( \{\sigma_{1/7}, \sigma_{2/7}, \sigma_{1/7}, \sigma_{1/3}\} \) are linearly independent on the concordance group and can be chosen to be arbitrary even integers. Thus, to ensure that Theorem 11 provides the desired bound on \( g_4(nK(A, B)) \), we first select \( A \) so that \( \sigma_{1/3}(A) \) is large enough that

\[
\left| \sum_{i=1}^{n} \sigma_{3/7}(K, \chi_i) + 2a\sigma_{1/3}(A) \right| > 4g
\]

for all possible values of \( \sum_{i=1}^{n} \sigma_{3/7}(K, \chi_i) \) and for all \( 0 < a \leq n \). We then select \( B \) so that \( \sigma_{1/3}(B) \) is large enough to ensure that \( \left| \sum_{i=1}^{n} \sigma_{3/7}(K, \chi_i) + 2a\sigma_{1/3}(A) + 2b\sigma_{1/3}(B) \right| > 4g \) for all possible values of \( \sum_{i=1}^{n} \sigma_{3/7}(K, \chi_i) \) and for all \( a \) satisfying \( 0 \leq a \leq n \).

Similarly, to ensure that Theorem 12 provides the desired bound on \( g_4(nK(A, B) \# -nK(A, B)^r) \), we first select \( A \) so that in addition to satisfying the condition of \( \sigma_{1/3}(A) \), it satisfies \( |\sigma_{1/7}(A) + \sigma_{2/7}(A) + \sigma_{3/7}(A)| > 6n \). We then select \( B \) so that, it satisfies

\[
|\sigma_{1/7}(B) + \sigma_{2/7}(B) + \sigma_{3/7}(B)| > n(|\sigma_{1/7}(A) + \sigma_{2/7}(A) + \sigma_{3/7}(A)| + 6n)
\]

while maintaining the condition on \( \sigma_{1/3}(B) \).

### 5. SUMMARY REMARKS

In summary, for an arbitrary knot \( K \), \( 0 \leq d(K, K^r) \leq 2g_4(K) \). We know that for reversible knots \( d(K, K^r) = 0 \) and that for all knots \( K \) with \( g_4(K) = 1 \) we have \( d(K, K^r) = 1 \); more generally, we have shown here that for many knots, \( d(K, K^r) \leq 2g_4(K) - 1 \). In the reverse direction, we have constructed for each \( g \geq 0 \) a knot for which \( g_4(K) = g = d(K, K^r) \). Although we would conjecture that in some generic sense, \( d(K, K^r) = 2g_4(K) \), we have been unable to solve the following problem.

**Problem 1.** Find a knot \( K \) for which \( d(K, K^r) > g_4(K) \).

In the paper [12], topologically slice knots \( K \) were constructed for which \( d(K, K^r) = 1 \). The following is open.

**Problem 2.** Find a topologically slice knot \( K \) for which \( d(K, K^r) \geq 2 \).

Our choice of the knots \( A \) and \( B \) depends on \( g \). Is this necessary?

**Problem 3.** Find a knot \( K \) such that \( g_4(nK) = n = d(nK, nK^r) \) for all \( n \).

Finally, as is usual, our proof relied on the careful construction of particular knots that meet all our needs. Are there more natural examples? As an example, the knot \( 8_{17} \) is not concordant to its reverse \( 13 \), so \( d(8_{17}, 8_{17}^r) \geq 1 \) and it is known that \( g_4(8_{17}) = 1 \). The challenging case is for knots \( K \) with \( g_4(K) \geq 2 \).

**Problem 4.** Are there any low-crossing number knots \( K \) for which \( g_4(K) \geq 2 \) and \( d(K, K^r) \geq g_4(K) \)?

KnotInfo [10] lists 360 prime knots \( K \) of 12 or fewer crossings which are not reversible and which satisfy \( g_4(K) \geq 2 \). Presumably one can show many satisfy \( d(K, K^r) \geq 1 \). I am aware of no techniques that can applied to show that any have \( d(K, K^r) \geq 2 \).

### REFERENCES

1. Sebastian Baader, *Scissor equivalence for torus links*, Bull. Lond. Math. Soc. **44** (2012), no. 5, 1068–1078. MR2975163
2. Jae Choon Cha and Charles Livingston, *Knot signature functions are independent*, Proc. Amer. Math. Soc. **132** (2004), no. 9, 2809–2816. MR2054808
3. Tim Cochran and Shelly Harvey, *The geometry of the knot concordance space*, Algebr. Geom. Topol. **18** (2018), no. 5, 2509–2546. MR3848393
4. Tim D. Cochran, Christopher W. Davis, and Arumina Ray, *Injectivity of satellite operators in knot concordance*, J. Topol. **7** (2014), no. 4, 948–964. MR3286894
5. Peter Feller, *Optimal cobordisms between torus knots*, Comm. Anal. Geom. **24** (2016), no. 5, 993–1025. MR3622312
6. Peter Feller and JungHwan Park, *Genus one cobordisms between torus knots*, arXiv e-prints (October 2019), available at [arxiv.org/abs/1910.01672](https://arxiv.org/abs/1910.01672)
[7] R. H. Fox, Some problems in knot theory, Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), 1962, pp. 168–176. MR0140100
[8] Patrick M. Gilmer, On the slice genus of knots, Invent. Math. 66 (1982), no. 2, 191–197. MR656619
[9] Richard Hartley, Identifying noninvertible knots, Topology 22 (1983), no. 2, 137–145. MR683753
[10] Matthew Hedden and Juanita Pinzon-Caicedo, Satellites of Infinite Rank in the Smooth Concordance Group, arXiv e-prints (September 2018), available at arxiv.org/abs/1809.04186
[11] Se-Goo Kim and Charles Livingston, Knot mutation: 4-genus of knots and algebraic concordance, Pacific J. Math. 220 (2005), no. 1, 87–105. MR2195064
[12] Tahee Kim and Charles Livingston, Knot reversal acts non-trivially on the concordance group of topologically slice knots, arXiv e-prints (April 2019), available at arxiv.org/abs/1904.12014
[13] Paul Kirk and Charles Livingston, Twisted knot polynomials: inversion, mutation and concordance, Topology 38 (1999), no. 3, 663–671. MR1670424
[14] Greg Kuperberg, Detecting knot invertibility, J. Knot Theory Ramifications 5 (1996), no. 2, 173–181. MR1395778
[15] R. A. Litherland, Signatures of iterated torus knots, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), 1979, pp. 71–84. MR547456
[16] Charles Livingston and Allison H. Moore, Knotinfo: Table of knot invariants, Current Year.
[17] Allison N. Miller and Lisa Piccirillo, Knot traces and concordance, J. Topol. 11 (2018), no. 1, 201–220. MR3784230
[18] Dale Rolfsen, Knots and links, Publish or Perish, Inc., Berkeley, Calif., 1976. Mathematics Lecture Series, No. 7. MR0515288
[19] Lee Rudolph, Quasipositivity as an obstruction to sliceness, Bull. Amer. Math. Soc. (N.S.) 29 (1993), no. 1, 51–59. MR1193540
[20] H. F. Trotter, Non-invertible knots exist, Topology 2 (1963), 275–280. MR0158395

Charles Livingston: Department of Mathematics, Indiana University, Bloomington, IN 47405
Email address: livingst@indiana.edu