On Abhyankar’s irreducibility criterion for quasi-ordinary polynomials

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Abstract

Let $f$ and $g$ be Weierstrass polynomials with coefficients in the ring of formal power series over an algebraically closed field of characteristic zero. Assume that $f$ is irreducible and quasi-ordinary. We show that if degree of $g$ is small enough and all monomials appearing in the resultant of $f$ and $g$ have orders big enough, then $g$ is irreducible and quasi-ordinary, generalizing Abhyankar’s irreducibility criterion for plane analytic curves.

1 Introduction

The paper is organized as follows. In the current section we introduce necessary notation and state our main result (Theorem 1.1). Its proof is in Section 3. Then, in Section 4, we introduce the notion of the logarithmic contact of irreducible Weierstrass polynomials and in Theorem 4.4 rewrite the main results of the paper in terms of the logarithmic contact. At the end of the paper we show that Abhyankar-Moh irreducibility criterion follows from Theorem 1.1.

Throughout the paper $K$ is an algebraically closed field of characteristic zero. We use the notation $K[[X]]$ for the ring $K[[X_1, \ldots, X_d]]$ of formal power series in $d$ variables with coefficients in $K$ and the notation $K[[X_1^{1/n}, \ldots, X_d^{1/n}]]$. In one variable case the elements of this ring are called Puiseux series. We will use a multi-index notation $X^{q} := X_{q_1}^{1/q_1} \cdots X_{q_d}^{1/q_d}$ for $q = (q_1, \ldots, q_d)$.

Let $f = Y^{n} + a_{n-1}(X)Y^{n-1} + \cdots + a_0(X) \in K[[X]][Y]$ be a unitary polynomial. Such a polynomial is called quasi-ordinary if its discriminant equals $u(X)X^q$ with $u(0) \neq 0$. We call $f$ a Weierstrass polynomial if $a_i(0) = 0$ for all $i = 0, \ldots, n$. The classical Abhyankar-Jung theorem (see [6]) states that every quasi-ordinary polynomial $f \in K[[X]][Y]$ has its roots in $K[[X^{1/m}]]$ for some positive integer $m$. Hence one can factorize $f$ to the product $\prod_{i=1}^{n}(Y - \alpha_i)$, where $\alpha_i \in K[[X^{1/m}]]$. We put $\text{Zer}f = \{\alpha_1, \ldots, \alpha_n\}$. Since the discriminant of a monic

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polynomial is a product of differences of its roots, we have $\alpha_i - \alpha_j = u_{ij}(X)X^{\lambda_{ij}}$ with $u_{ij}(0) \neq 0$. The $d$-tuple $d(\alpha_i, \alpha_j) := \lambda_{ij}$ of non-negative rational numbers will be called the contact between $\alpha_i$ and $\alpha_j$.

For irreducible $f$ the contacts $d(\alpha, \alpha')$ for $\alpha, \alpha' \in \text{Zer} f$, $\alpha \neq \alpha'$, are called the characteristic exponents of $f$.

Let us introduce a partial order in the set $Q^d_{\geq 0}$: $q \leq q'$ if and only if $q' - q \in Q^d_{\geq 0}$. Then the characteristic exponents can be set to the increasing sequence $(h_1, \ldots, h_s)$ (see [5, Lemma 5.6]). We call this sequence the characteristic of $f$ and denote it by $\text{Char}(f)$.

With the sequence of characteristic exponents we associate the increasing sequence of lattices $M_0 \subset M_1 \subset \cdots \subset M_s$ defined as follows: $M_0 = \mathbb{Z}^d$ and $M_i = \mathbb{Z}^d + \mathbb{Z}h_1 + \cdots + \mathbb{Z}h_i$ for $i = 1, \ldots, s$. We set $n_i = [M_i : M_{i-1}]$ for $i = 1, \ldots, s$, $n_{s+1} = 1$ and $c_i = n_{i+1} \cdots n_{s+1}$ for $i = 0, \ldots, s$. Then $\deg f = n_1 \cdots n_s$ (see [3, Remark 2.7]). Finally we set

$$q_i = \sum_{j=1}^{i} (e_{j-1} - e_j)h_j + e_i h_i$$

for $i = 1, \ldots, s$.

If $f, g \in K[[X]][Y]$, then the resultant of this polynomials is denoted by $\text{Res}(f, g)$.

We can now formulate our main result.

**Theorem 1.1.** Let $f \in K[[X]][Y]$ be a quasi-ordinary irreducible polynomial of characteristic $(h_1, \ldots, h_s)$ and let $g \in K[[X]][Y]$ be a Weierstrass polynomial of degree $\leq n_1 \cdots n_k$, where $1 \leq k \leq s$.

If all monomials appearing in $\text{Res}(f, g)$ have exponents greater than $(\deg g)q_k$ then

(i) $g$ is irreducible and quasi-ordinary of degree $n_1 \cdots n_k$ and characteristic $(h_1, \ldots, h_k)$;

(ii) for every $\gamma \in \text{Zer} g$ there exists $\alpha \in \text{Zer} f$ such that $\gamma - \alpha = \sum_{h > h_k} c_h X^h$.

Moreover, if $X^{(\deg g)q_{k+1}}$ divides $\text{Res}(f, g)$ then

(iii) $\text{Res}(f, g) = u(X)X^{(\deg g)q_{k+1}}$, where $u(0) \neq 0$;

(iv) for every $\gamma \in \text{Zer} g$ there exists $\alpha \in \text{Zer} f$ such that $\gamma - \alpha = c_{h_{k+1}} X^{h_{k+1}} + \sum_{h > h_{k+1}} c_h X^h$.

**Remark 1.2.** In the point (iv) of Theorem 1.1 the monomial $X^{h_{k+1}}$ does not appear in the power series $\gamma$. 
Example 1.3. Let \( f = Y^4 - 2X^3X_2^3Y^2 - 4X^5X_2^4Y - X_1^7X_2^6 + X_1^6X_2^4 \). The polynomial \( f \) is quasi-ordinary and irreducible in \( \mathbb{C}[[X_1, X_2]][Y] \) with the roots
\[
\alpha_1 = X_1^{3/2}X_2 + X_1^{7/4}X_2^{3/2} \\
\alpha_2 = X_1^{3/2}X_2 - X_1^{7/4}X_2^{3/2} \\
\alpha_3 = -X_1^{3/2}X_2 + \sqrt{-1}X_1^{7/4}X_2^{3/2} \\
\alpha_4 = -X_1^{3/2}X_2 - \sqrt{-1}X_1^{7/4}X_2^{3/2}
\]
and characteristic exponents \( h_1 = (\frac{3}{2}, 1) \) and \( h_2 = (\frac{7}{4}, \frac{3}{2}) \).

Let \( g = (Y^2 - X_1^4X_2^2)^2 - 4X_1^5X_2^3Y \). Then \( \text{Res}(f, g) = X_1^{28}X_2^{24} \). We have \( X^{(\deg g)\mathbb{Z}} = X_1^{28}X_2^{20} \), so according to Theorem 1.1, the polynomial \( g \) is irreducible and quasi-ordinary of characteristic \((h_1, h_2)\).

2 Auxiliary results

For \( g = \sum a_{\alpha}X^\alpha \in \mathbb{K}[[X^{1/m}]] \) we define the Newton polytope \( \Delta(g) \) as the convex hull of the set \( \bigcup_{a_{\alpha} \neq 0} (a + \mathbb{R}_{\geq 0}^d) \). The Newton polytope \( \Delta(f) \) of a polynomial \( f \in \mathbb{K}[[X^{1/m}]][Y] \) is the Newton polytope of \( f \) treated as an element of the ring \( \mathbb{K}[[X_1^{1/m}, \ldots, X_d^{1/m}, Y^{1/m}]] \). In two variable case Newton polytopes are called Newton polygons.

Let \( T \) be a single variable. The order of a fractional power series \( \gamma \in \mathbb{K}[[T^{1/m}]] \) will be denoted \( \text{ord} \gamma \). Note that for all \( \alpha, \beta, \gamma \in \mathbb{K}[[T^{1/m}]] \) we have \( \text{ord}(\alpha - \beta) \geq \min(\text{ord}(\alpha - \gamma), \text{ord}(\gamma - \beta)) \). We call this property the strong triangle inequality.

Lemma 2.1. Let \( g, \tilde{g} \in \mathbb{K}[[T^{1/m}]][Y] \) be Weierstrass polynomials such that \( \Delta(g) = \Delta(\tilde{g}) \). Then \( \{\text{ord} \gamma : \gamma \in \text{Zer}g\} = \{\text{ord} \gamma : \gamma \in \text{Zer} \tilde{g}\} \).

Proof. The Newton polygon of the product \( g = \prod_{i=1}^{\deg g} (Y - \gamma_i(T)) \) is the Minkowski sum of the Newton polygons of its factors and the shape of the Newton polygon of each factor \( Y - \gamma_i(T) \) determines the order of \( \gamma_i(T) \).

For a more detailed proof see [7, Theorem 2.1].

Let \( \mathbb{Q}_+ \) be the set of positive rational numbers. For a Newton polytope \( \Delta \subset \mathbb{R}_{\geq 0}^d \) and \( c \in \mathbb{Q}_+^d \) we define the face \( \Delta^c := \{ v \in \Delta : \langle c, v \rangle = \min_{w \in \Delta} \langle c, w \rangle \} \).

We will say that a condition depending on \( c \in \mathbb{Q}_+^d \) is satisfied for generic \( c \) if it holds in an open and dense subset of \( \mathbb{Q}_+^d \).

Lemma 2.2. Let \( \Delta \) be the Newton polytope of some nonzero fractional power series \( \gamma \in \mathbb{K}[[X^{1/m}]] \). Then for generic \( c \in \mathbb{Q}_+^d \), a face \( \Delta^c \) is a vertex of \( \Delta \).

Proof. Let \( V \) be the (finite) set of vertices of \( \Delta \). Then the set
\[
U = \{ c \in \mathbb{Q}_+^d : \forall v, w \in V (v \neq w \Rightarrow \langle c, v \rangle \neq \langle c, w \rangle) \}
\]
is open and dense in \( \mathbb{Q}_+^d \), and for every \( c \in U \) there is exactly one vertex \( v \) of \( \Delta \) such that \( \langle c, v \rangle = \min \{ \langle c, w \rangle : w \in V \} \).
With every $c = (c_1, \ldots, c_d) \in \mathbb{Q}_+^d$ we associate the monomial substitution $(X_1, \ldots, X_d) = (T^{c_1}, \ldots, T^{c_d})$ written $X = T^c$. Applying this substitution to $f = f(X, Y) \in \mathbb{K}[[X^{1/m}]]/[Y]$ we define $f^{[c]} := f(T^c, Y) \in \mathbb{K}[[T^{1/Nm}]]/[Y]$, where $N$ is a common denominator of coordinates of $c$.

**Lemma 2.3.** Let $\gamma_1, \gamma_2 \in \mathbb{K}[[X^{1/m}]]$ be nonzero fractional power series. If $\text{ord}_{\gamma_1} = \text{ord}_{\gamma_2}$ for generic $c \in \mathbb{Q}_+^d$, then $\Delta(\gamma_1) = \Delta(\gamma_2)$.

**Proof.** Suppose that $\Delta(\gamma_1) \neq \Delta(\gamma_2)$. Without loss of generality we may assume that $\Delta(\gamma_1) \setminus \Delta(\gamma_2)$ is nonempty. Since $\Delta(\gamma_2)$ is convex and closed, for any $v \in \Delta(\gamma_1) \setminus \Delta(\gamma_2)$ there exists $c \in \mathbb{R}_+^d$ such that $\langle c, v \rangle < \inf_{w \in \Delta(\gamma_2)} \langle c, w \rangle$.

Then by Lemma 2.2 there is a vertex $v_0$ of $\Delta(\gamma_1)$ and an open set $U \subset \mathbb{Q}_+^d$ such that $\Delta(\gamma_1)^* = \{v_0\}$ and $\langle c, v_0 \rangle < \inf_{w \in \Delta(\gamma_2)} \langle c, w \rangle$ for all $c \in U$. We get $\text{ord}_{\gamma_1} = \langle c, v_0 \rangle$ because in the fractional power series $\gamma_1^{[c]}$ there is no cancellation of the terms of order $(c, v_0)$ and $\text{ord}_{\gamma_2} > \langle c, v_0 \rangle$ (since all monomials appearing in $\gamma_2^{[c]}$ have orders bigger than $\langle c, v_0 \rangle$). Thus $\text{ord}_{\gamma_1} < \text{ord}_{\gamma_2}$ for $c \in U$. $\square$

**Lemma 2.4.** Let $f \in \mathbb{K}[[X^{1/m}]]/[Y]$ be a nonzero polynomial. Given $c \in \mathbb{Q}_+^d$ we define the linear mapping $L_c : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^2$, $L_c(x, y) = ((c, x), y)$. Then for generic $c \in \mathbb{Q}_+^d$

$$\Delta(f^{[c]}) = L_c(\Delta(f)).$$

**Proof.** Write $f = a_0(X)Y^n + a_{n-1}(X)Y^{n-1} + \cdots + a_0(X)$ and $f^{[c]} = \tilde{a}_n(T)Y^n + \tilde{a}_{n-1}(T)Y^{n-1} + \cdots + \tilde{a}_0(T)$. By Lemma 2.2 for generic $c \in \mathbb{Q}_+^d$ and for any nonzero $a_i(X)$ the polygon $\Delta(a_i(X))^c$ is a vertex of $\Delta(a_i(X))$. Denote this vertex by $v_i$. Then $\text{ord}_{\tilde{a}_i}(T) = \langle c, v_i \rangle$ because in the fractional power series $\tilde{a}_i(T) = a_i(T^c)$ there is no cancellation of the terms of the lowest order. Thus the vertices of $L_c(\Delta(f))$ belong to $\Delta(f^{[c]})$ which gives the desired equality. $\square$

**Remark 2.5.** Let $K$ (respectively $L$) be the field of fractions of the ring $\mathbb{K}[[X]]$ (respectively $\mathbb{K}[[X^{1/m}]]$). Denote by $\text{Gal}(L/K)$ the Galois group of the extension $K < L$. Then $L$ is normal over $K$ (as the splitting field of the family of polynomials $\{Y^m - X_i \in \mathbb{K}[[X]]/[Y] : i = 1, \ldots, d\}$) and every $\sigma \in \text{Gal}(L/K)$ is given by

$$\sigma\left(\sum_{a \in \mathbb{N}^d} c_a X^{a/m}\right) = \sum_{a \in \mathbb{N}^d} \varepsilon_a c_a X^{a/m}$$

for some $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$ with $\varepsilon_1^m = 1$. In particular, $\Delta(\sigma(\gamma)) = \Delta(\gamma)$ for all nonzero $\gamma \in \mathbb{K}[[X^{1/m}]]$.

For $\alpha \in \mathbb{K}[[T^{1/m}]]$ and a finite set $A \subset \mathbb{K}[[T^{1/m}]]$ we define the contact between $\alpha$ and $A$ as $\text{cont}(A, \alpha) := \max_{\gamma \in A} \text{ord}(\alpha - \gamma)$.

From now on up to the end of this section we work under the assumption that $f \in \mathbb{K}[[X]][Y]$ is a quasi-ordinary irreducible polynomial of characteristic $(h_1, \ldots, h_s)$ and $\text{Zer} f = \{\alpha_1, \ldots, \alpha_n\}$ is the set of its roots.

**Theorem 2.6.** Let $g \in \mathbb{K}[[X]][Y]$ be a Weierstrass polynomial. If $c \in \mathbb{Q}_+^d$ is generic, then for any $\beta, \beta' \in \text{Zer} f^{[c]}$ one has $\text{cont}(\text{Zer} g^{[c]}, \beta) = \text{cont}(\text{Zer} g^{[c]}, \beta')$. 

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Proof. For brevity we will write $\overline{p}$ instead of $p^{[c]}$ for every $p \in \mathbb{K}[[X^{1/m}]][Y]$.

Since $f = \prod_{i=1}^{n}(Y - \alpha_i)$, we get $f = \prod_{i=1}^{n}(Y - \overline{\alpha}_i)$ and consequently $\beta = \overline{\alpha}_i$, $\beta' = \overline{\alpha}_j$ for some $\alpha_i, \alpha_j \in \text{Zer} f$. The roots $\alpha_i, \alpha_j$ are conjugate by the Galois automorphism. Hence by Remark 2.4 the Newton polytopes of $\overline{g_i} = g(Y + \alpha_i)$ and $\overline{g_j} = g(Y + \alpha_j)$ are equal. By Lemma 2.7, the Newton polygons of $\overline{g_i}$ and $\overline{g_j}$ are also equal.

If $\text{Zer} \overline{g} = \{\gamma_1, \ldots, \gamma_k\}$ then $\text{Zer} \overline{g_i} = \{\gamma_1 - \beta, \ldots, \gamma_k - \beta\}$ and $\text{Zer} \overline{g_j} = \{\gamma_1 - \beta', \ldots, \gamma_k - \beta'\}$. Hence it follows from Lemma 2.1 that $\text{cont}(\text{Zer} \overline{g}, \beta) = \text{cont}(\text{Zer} \overline{g}, \beta')$. □

If $A$ is any set, then $\#A$ denotes the cardinality of $A$.

Lemma 2.7 (Contact structure of Zer $f$). For every $\tilde{\alpha} \in \text{Zer} f$ and $i \in \{1, \ldots, s\}$ we have $\#\{\alpha \in \text{Zer} f : d(\alpha, \tilde{\alpha}) > h_i\} = e_i$ and $\#\{\alpha \in \text{Zer} f : d(\alpha, \tilde{\alpha}) = h_i\} = e_{i-1} - e_i$.

Proof. See the proof of Proposition 3.1 from [3]. □

Fix $c \in \mathbb{Q}^d_+$ and for every $w \in \mathbb{Q}^d$ denote $\overline{w} := (\overline{c}, w)$. We set $\overline{h_0} = 0$, $\overline{h_{s+1}} = +\infty$, $\overline{h_{\overline{0}}} = 0$ and define a continuous function $\phi_c : [0, +\infty) \to [0, +\infty)$ such that

(i) $\phi_c(\overline{h_i}) = \overline{q_i}$ for $i = 0, \ldots, s$;

(ii) $\phi_c$ is linear in each interval $(\overline{h_i}, \overline{h_{i+1}})$ for $i = 0, \ldots, s$;

(iii) the graph of $\phi_c$ has slope 1 over the interval $(\overline{h_s}, +\infty)$.

Lemma 2.8. The function $\phi_c : [0, +\infty) \to [0, +\infty)$ is increasing. If $\tilde{\gamma}$ is a Puiseux series and $\text{cont}(\text{Zer} f^{[c]}, \tilde{\gamma}) = h$ then $\text{ord} f^{[c]}(\tilde{\gamma}) = \phi_c(h)$.

Proof. By equality (1) we get $\overline{q_{i+1}} = \overline{\gamma} + e_i(\overline{h_{i+1}} - \overline{h_i})$ for $i = 0, \ldots, s - 1$, hence the numbers $\overline{q_i}$ form an increasing sequence.

Let $h = \text{cont}(\text{Zer} f^{[c]}, \tilde{\gamma}) = \text{ord}(\overline{\gamma} - \overline{\alpha})$ for some $\alpha \in \text{Zer} f$. Assume that $h \in (\overline{h_r}, \overline{h_{r+1}})$. Then, by the strong triangle inequality and Lemma 2.7, we get

$$\text{ord} f^{[c]}(\overline{\gamma}) = \sum_{j=1}^{n} \text{ord}(\overline{\gamma} - \overline{\alpha_j}) = \sum_{\text{ord}(\overline{\alpha_j} - \overline{\gamma}) \leq h_r} \text{ord}(\overline{\gamma} - \overline{\alpha_j}) + \sum_{\text{ord}(\overline{\alpha_j} - \overline{\gamma}) > h_r} \text{ord}(\overline{\gamma} - \overline{\alpha_j}) = \sum_{\text{ord}(\overline{\alpha_j} - \overline{\gamma}) \leq h_r} \text{ord}(\overline{\alpha_j} - \overline{\gamma}) + \sum_{\text{ord}(\overline{\alpha_j} - \overline{\gamma}) > h_r} \text{ord}(\overline{\gamma} - \overline{\alpha_j}) = \sum_{i=1}^{r} (e_{i-1} - e_i) \overline{h_i} + e_r h = \overline{q_r} + e_r (h - \overline{h_r}).$$

□
Let $\alpha \in \text{Zer} f$ and $h_r$ be a characteristic exponent of $f$. By definition, the 
$h_r$–truncation of $\alpha$ is the fractional power series $\text{trunc}_r(\alpha)$ obtained from $\alpha$ by 
omitting all terms of order $\geq h_r$. We denote by $f_r$ the minimal polynomial of 
$\text{trunc}_r(\alpha)$ over the field $K$. As we will see in the lemma below, this polynomial 
does not depend on $\alpha$.

**Lemma 2.9.**

(i) $\text{Zer} f_r = \{ \text{trunc}_r(\alpha_j) : j = 1, \ldots, n \}$;

(ii) $f_r \in K[[X]]$ is monic, irreducible and quasi-ordinary;

(iii) $\deg f_r = n_1 \cdots n_{r-1}$;

(iv) $\text{Char}(f_r) = \{ h_1, \ldots, h_{r-1} \}$.

**Proof.** Since $\text{trunc}_r(\alpha)$ is not dependent on $\gamma$, all the roots of the polynomial $f_r$ are elements of $L$. It is easy to see that $\sigma(\text{trunc}_r(\alpha)) = \text{trunc}_r(\sigma(\alpha))$ for every $\sigma \in \text{Gal}(L/K)$.

The polynomial $f$ is irreducible over the field $K$, so $\text{Gal}(L/K)$ acts transitively on the set $\text{Zer} f$ and hence on the set $\text{Zer} f_r$, as well. This implies (i) and (ii).

If $d(\alpha, \alpha_j) \leq h_r$, then $d(\alpha, \alpha_j) = d(\text{trunc}_r(\alpha_i))$ and if $d(\alpha, \alpha_j) > h_r$, then $\text{trunc}_r(\alpha_i) = \text{trunc}_r(\alpha_j)$. Thus (iv) holds true and, as a consequence, we also obtain (iii). $\square$

### 3 Proof of Theorem 1.1

The proof will be organized as a sequence of claims. We denote the roots of $f$ (respectively the roots of $f_{k+1}$) by $\alpha_1, \ldots, \alpha_n$ (respectively by $\beta_1, \ldots, \beta_l$, where $l = n_1 \cdots n_k$). We will use the bar notation for polynomials and power 
series after the monomial substitution $X = T^c$.

Let $c \in \mathbb{Q}^+$ be generic in the sense that the conclusion of Theorem 2.6 for 
a polynomial $g$ and any $\bar{\beta}, \bar{\beta}' \in \text{Zer} \bar{f}_{k+1}$ is true.

**Claim 1.** For every $\bar{\beta} \in \text{Zer} \bar{f}_{k+1}$ there exists exactly one $\gamma \in \text{Zer} \bar{g}$ such that 
$\text{ord}(\gamma - \bar{\beta}) > h_k$.

**Proof.** By assumptions of the theorem we get $\text{ordRes}(\bar{g}, \gamma) = (\deg g)q_k$. If 
$\text{cont}(\text{Zer} \bar{f}, \gamma) \leq h_k$ for all the roots $\gamma$ of $\bar{g}$, then by Lemma 2.8 we obtain 
$\text{ordRes}(\bar{f}, \bar{g}) = \sum_{\gamma \in \text{Zer} \bar{g}} \text{ord} \bar{f}(\gamma) \leq (\deg \bar{g})q_k$. It follows that $\text{ord}(\bar{f} - \gamma) > h_k$ 
for some $\gamma \in \text{Zer} \bar{g}$ and $\alpha \in \text{Zer} f$.

Let $\beta = \text{trunc}_{k+1}(\alpha)$. Since $\text{ord}(\bar{f} - \gamma) > h_k$, we get $\text{ord}(\beta - \gamma) > h_k$ and consequently $\text{cont}(\text{Zer} \bar{g}, \beta) > h_k$. It follows from Theorem 2.6 that for every 
$\bar{\beta}' \in \text{Zer} \bar{f}_{k+1}$ there exists $\gamma \in \text{Zer} \bar{g}$ such that $\text{ord}(\gamma - \bar{\beta}') > h_k$.

Take any $\bar{\beta}, \bar{\beta}' \in \text{Zer} \bar{f}_{k+1}$ and $\gamma, \gamma' \in \text{Zer} \bar{g}$ such that $\text{ord}(\gamma - \bar{\beta}) > h_k$ and $\text{ord}(\gamma' - \bar{\beta}') > h_k$. Assume that $\bar{\beta} \neq \bar{\beta}'$. Then $\gamma \neq \gamma'$.

Indeed, if $\gamma = \gamma'$ then $\text{ord}(\bar{f} - \gamma) > h_k$ and we arrive at contradiction. From 
the above and the assumption $\deg g \leq n_1 \cdots n_k = \deg f_{k+1}$ we obtain that $\bar{g}$ 
has exactly $n_1 \cdots n_k$ roots, which completes the proof. $\square$
Using Claim 1 we may assume, without loss of generality, that \( \text{Zer} \mathcal{F} = \{ \gamma_1, \ldots, \gamma_l \} \), where \( \text{ord}(\gamma_i - \beta_i) > h_k \) for all \( 1 \leq i \leq l \). It follows immediately from the strong triangle inequality that
\[
\text{ord}(\gamma_i - \gamma_j) = \text{ord}(\beta_i - \beta_j) \quad \text{for all } 1 \leq i < j \leq l.
\] (2)

Hence the orders of the discriminants of polynomials \( \mathcal{F} \) and \( f_{k+1} \) are equal. Therefore, by Lemma 2.3, the Newton polytopes of the discriminants of \( g \) and \( f_{k+1} \) are equal too, so we conclude that \( g \) is quasi-ordinary.

Claim 2. Let \( v \) be a vertex of \( \Delta(\gamma - \gamma') \) for \( \gamma, \gamma' \in \text{Zer} \ g \). Then \( v \in \{ h_1, \ldots, h_k \} \).

Proof. Since \( g \) is quasi-ordinary, \( \Delta(\gamma - \gamma') \) has only one vertex. Thus for every \( c \in \mathbb{Q}_d^d \) we have \( \text{ord}(\gamma^c - \gamma'^c) = (c, v) \). It follows from (2) that \( (c, v) \in \{ (c, h_i) : 1 \leq i \leq k \} \). Observe that if \( v \neq h_i \), then the set \( \{ c' \in \mathbb{Q}_d^d : (c', v) = (c', h_i) \} \) is contained in a finite union of hyperplanes, hence is nowhere dense. This implies that \( v = h_i \) for some \( i \in \{ 1, \ldots, k \} \), since \( c \) is generic.

Claim 3. Let \( v \) be a vertex of \( \Delta(\gamma - \beta) \) for \( \gamma \in \text{Zer} \ g \) and \( \beta \in \text{Zer} f_{k+1} \). Then \( v \in \{ h_1, \ldots, h_k \} \) or \( v > h_k \).

Proof. By the strong triangle inequality and (2) we obtain \( \text{ord}(\gamma^c - \beta^c) \in \{ (c, h_i) : 1 \leq i \leq k \} \) or \( \text{ord}(\gamma^c - \beta^c) > (c, h_k) \).

Let \( v \) be a vertex of \( \Delta(\gamma - \beta) \) which is not in \( \Delta(X^{h_k}) \). Then there exists an open set \( U \subset \mathbb{Q}_d^d \) such that a face \( \Delta(\gamma - \beta)^c = \{ v \} \) and \( (c, v) < (c, h_k) \) for all \( c \in U \). Hence we have \( \text{ord}(\gamma^c - \gamma'^c) = (c, v) < (c, h_k) \). Using the same argument as in the proof of Claim 2, we conclude that \( v \in \{ h_1, \ldots, h_k \} \) which completes the proof.

Claim 4. For every \( \beta \in \text{Zer} f_{k+1} \) there exists exactly one \( \gamma \in \text{Zer} \ g \) such that \( \Delta(\gamma - \beta) \subseteq \Delta(X^{h_k}) \).

Proof. Let \( \Delta = \Delta(\gamma - \beta) \) for \( \gamma \in \text{Zer} \ g \) and \( \beta \in \text{Zer} f_{k+1} \). Then by Claim 3 two cases are possible: either some \( h_i \in \{ h_1, \ldots, h_k \} \) is a vertex of \( \Delta \) and \( \text{ord}(\gamma - \beta) = h_i \) or \( \Delta \nsubseteq \Delta(X^{h_k}) \) and \( \text{ord}(\gamma - \beta) > h_k \). To finish the proof it is enough to use Claim 1.

We will show that \( \text{Gal}(L/K) \) acts transitively on the set \( \text{Zer} \ g \). Indeed, take arbitrary \( \gamma, \gamma' \in \text{Zer} \ g \). By Claim 4 there exist unique \( \beta, \beta' \in \text{Zer} f_{k+1} \) such that \( \Delta(\gamma - \beta) \subseteq \Delta(X^{h_k}) \) and \( \Delta(\gamma' - \beta') \subseteq \Delta(X^{h_k}) \). Take \( \sigma \in \text{Gal}(L/K) \) such that \( \sigma(\beta) = \beta' \). Then by Remark 2.5 we have \( \Delta(\sigma(\gamma) - \beta') \subseteq \Delta(X^{h_k}) \), hence \( \sigma(\gamma) = \gamma' \).

It follows from the above that the polynomial \( g \) is irreducible. From (2) and Claim 2 we deduce that \( \{ h_1, \ldots, h_k \} \) is the characteristic of \( g \). Point (ii) of the theorem follows directly from Claim 4.

Now we prove statements (iii) and (iv) of Theorem 1.1. Assume that \( X^{(\deg g)q_{k+1}} \) divides \( \text{Res}(f, g) \). If the monomial \( X^{(\deg g)q_{k+1}} \) does not appear in \( \text{Res}(f, g) \) then
by the first part of the theorem we obtain \( \deg g = n_1 \cdots n_{k+1} \), which contradicts the assumption \( \deg g \leq n_1 \cdots n_k \). Thus \( \text{Res}(f, g) = u(X)X^{(\deg g)n_{k+1}} \), where \( u(0) \neq 0 \).

By Claim 4 for every \( \gamma \in \text{Zer} g \) there exists \( \beta \in \text{Zer} f_{k+1} \) such that \( \Delta(\gamma - \beta) \subseteq \Delta(X^{h_k}) \). Suppose that the Newton polytope \( \Delta = \Delta(\gamma - \beta) \) has a vertex which is not contained in \( \Delta(X^{h_{k+1}}) \). Then there exists \( c \in \mathbb{Q}_+^d \) such that \( \Delta' = \{v\} \) and \( \langle c, v \rangle < \langle c, h_{k+1} \rangle \). Thus \( \text{cont}(\text{Zer} f, \gamma) < h_{k+1} \). Since for any \( \sigma \in \text{Gal}(L/K) \) we have \( \Delta(\gamma - \beta) = \Delta(\sigma(\gamma) - \sigma(\beta)) \), the same is true for any \( \gamma' \in \text{Zer} g \). Then by Lemma 2.8 we get \( \text{ord}(\text{Res}(f, g)) = (\deg g)h_{k+1} \) and we arrive at contradiction.

We proved that \( \Delta(\gamma - \beta) \subseteq \Delta(X^{h_{k+1}}) \) which gives (iv) of Theorem 1.1.

4 Logarithmic distance

For any irreducible Weierstrass polynomials \( f, g \in K[[X]][Y] \) we define the Newton polytope \( \text{cont}_A(f, g) = \frac{1}{(\deg f)(\deg g)} \Delta(\text{Res}(f, g)) \) called the logarithmic distance between \( f \) and \( g \). We introduce the partial order in the set of Newton polytopes: \( \Delta_1 \geq \Delta_2 \) if and only if \( \Delta_1 \subseteq \Delta_2 \).

For any irreducible quasi-ordinary Weierstrass polynomials \( f, g, h \) the strong triangle inequality \( \text{cont}_A(f, g) \geq \inf \{\text{cont}_A(f, h), \text{cont}_A(h, g)\} \) holds true, where \( \inf \{A, B\} \) denotes the Newton polytope spanned by the union of \( A \) and \( B \). Let us prove it now.

For \( \alpha = \sum_{a \in \mathbb{Q}_+^d} c_a X^a \in K[[X^{1/n}]] \) and \( \omega \in \mathbb{R}_+^d \) we define a weighted order \( \text{ord}_\omega(\alpha) := \min\{\langle \omega, a \rangle : c_a \neq 0\} \) and a weighted contact between quasi-ordinary polynomials \( f, g \in K[[X]][Y] \) as follows:

\[
\text{cont}_\omega(f, g) := \frac{1}{\deg f \deg g} \sum_{\alpha \in \text{Zer} f \atop \beta \in \text{Zer} g} \text{ord}_\omega(\alpha - \beta) = \frac{1}{\deg f \deg g} l(\omega, \Delta(\text{Res}(f, g))),
\]

where \( l(\omega, \Delta(\text{Res}(f, g))) := \min\{\langle \omega, a \rangle : a \in \Delta(\text{Res}(f, g))\} \).

For every irreducible quasi-ordinary polynomials \( f, g \in K[[X]][Y] \) and for any \( \gamma, \gamma' \in \text{Zer} g \) we have \( \text{ord}_\omega(f(\gamma)) = \text{ord}_\omega(f(\gamma')) \). Therefore, using the same method as in the proof of [2] Proposition 2.2, we get for any irreducible quasi-ordinary polynomials \( f, g, h \in K[[X]][Y] \) a strong triangle inequality \( \text{cont}_\omega(f, g) \geq \min\{\text{cont}_\omega(f, h), \text{cont}_\omega(h, g)\} \).

Lemma 4.1. Assume that \( \Delta_1, \Delta_2, \Delta_3 \) are Newton polytopes. If

\[
l(\omega, \Delta_1) \geq \min\{l(\omega, \Delta_2), l(\omega, \Delta_3)\}
\]

for all \( \omega \in \mathbb{R}_+^d \), then \( \Delta_1 \geq \inf\{\Delta_2, \Delta_3\} \).

Proof. Suppose that the inequality \( \Delta_1 \geq \inf\{\Delta_1, \Delta_2\} \) is false. Therefore there exists \( v \in \Delta_1 \setminus \text{conv}(\Delta_2 \cup \Delta_3) \) and then we can find a linear form \( L \) such that \( L(v) < L(x) \) for all \( x \in \text{conv}(\Delta_2 \cup \Delta_3) \). It means that \( \langle \omega, v \rangle < \langle \omega, x \rangle \), \( x \in \text{conv}(\Delta_2 \cup \Delta_3) \), for some \( \omega \in \mathbb{R}_+^d \). Thus \( l(\omega, \Delta_1) < l(\omega, \text{conv}(\Delta_1 \cup \Delta_2)) \) and the inequality 3 does not hold. \( \square \)
The above lemma implies immediately the strong triangle inequality for the logarithmic distance of irreducible quasi-ordinary Weierstrass polynomials.

Unfortunately, the strong triangle inequality does not extend to a wider class of irreducible Weierstrass polynomials as the following examples show.

**Example 4.2.** Let \( f = Y, \ g = Y - X_1 - X_2^2, \ h = Y^2 - (X_1 + X_2)Y + 2X_1^3 + X_2^3 \). The polynomials \( f, \ g, \ h \) are irreducible in \( K[[X_1, X_2]][Y] \). We have \( \text{Res}(f, g) = -X_1 - X_2^2, \ \text{Res}(f, h) = 2X_1^3 + X_2^3, \ \text{Res}(g, h) = X_1X_2^2 - X_1X_2 + 2X_1^3 + X_2^4 \), hence there is no strong triangle inequality between \( \cont_A(f, g), \ \cont_A(f, h) \) and \( \cont_A(h, g) \) as illustrated in the following picture.

![Diagram](cont_A(f,g), cont_A(f,h), cont_A(g,h))

**Example 4.3.** Let \( f = Y - 2X_1^3, \ g = (Y - X_1)(Y - X_1^3 - X_1^4) + X_2, \ h = Y - X_1^3 \). Then \( \text{Res}(f, g) = -2X_1^7 + 2X_1^6 + X_1^5 - X_1^4 + X_2, \ \text{Res}(f, h) = X_1^3, \ \text{Res}(g, h) = -X_1^7 + X_1^5 + X_2 \). The polynomials \( f, \ g, \ h \) are irreducible, \( fh \) is quasi-ordinary and the inequality \( \cont_A(f, g) \geq \inf \{ \cont_A(f, h), \cont_A(h, g) \} \) does not hold (see the picture below).

![Diagram](cont_A(f,g), cont_A(f,h), cont_A(g,h))

The results of the first section can be reformulated in terms of the logarithmic distance.

**Theorem 4.4.** Let \( f \in K[[X]][Y] \) be a quasi-ordinary irreducible Weierstrass polynomial of characteristic \( (h_1, \ldots, h_s) \) and let \( g \in K[[X]][Y] \) be a Weierstrass polynomial such that \( \deg g \leq \deg f_{k+1} \) and \( \cont_A(f, g) > \cont_A(f, f_k) \). Then \( g \) is an irreducible quasi-ordinary polynomial of characteristic \( (h_1, \ldots, h_k) \) and \( \deg g = \deg f_{k+1} \). Moreover, if \( \cont_A(f, g) \geq \cont_A(f, f_{k+1}) \) then \( \cont_A(f, g) = \cont_A(f, f_{k+1}) \).
5 The Abhyankar-Moh irreducibility criterion

In this section we will show that our main result is a generalization of the well-known Abhyankar-Moh irreducibility criterion (see e.g. [4, Theorem 1.2]). At the beginning let us recall the classical Weierstrass preparation theorem for the ring $\mathbb{K}[[X, Y]]$.

**Theorem 5.1** (Weierstrass). Assume that $f = \sum_{i=0}^{\infty} a_i Y^i \in \mathbb{K}[[X, Y]]$ and there exists $m > 0$ such that $a_i(0) = 0$ for $i < m$ and $a_m(0) \neq 0$. Then there exist unique $u_f, f_1 \in \mathbb{K}[[X, Y]]$ such that $f = u_ff_1$, $u_f(0) \neq 0$ and $f_1$ is a Weierstrass polynomial with respect to the variable $Y$.

The polynomial $f_1$ from the above theorem is called the *Weierstrass polynomial* of $f$. If we assume additionally that $f \in \mathbb{K}[[X, Y]]$ is irreducible, then its Weierstrass polynomial is irreducible in the ring $\mathbb{K}[[X]][Y]$ and the characteristic of this polynomial will be denoted by $\text{Char}(f)$.

For $f, g \in \mathbb{K}[[X, Y]]$ we define the *intersection multiplicity number* $i_0(f, g)$ as the dimension of the $\mathbb{K}$–vector space $\mathbb{K}[[X, Y]]/(f, g)$.

**Theorem 5.2** (Abhyankar-Moh). Let $f, g \in \mathbb{K}[[X, Y]]$. Assume that $f$ is irreducible, $i_0(f, X) = n < +\infty$ and $\text{Char}(f) = \{h_1, \ldots, h_s\}$. If $i_0(g, X) = n$ and $i_0(f, g) > nq_s$, then $g$ is irreducible and $\text{Char}(g) = \text{Char}(f)$.

**Proof.** Let $f_1$ and $g_1$ be the Weierstass polynomials of $f$ and $g$. Then $\deg f_1 = i_0(f, X) = n$, $\deg g_1 = i_0(g, X) = n$ and $\text{ord } \text{Res}(f_1, g_1) = i_0(f, g) > nq_s$. Since $f$ is irreducible, the polynomial $f_1 \in \mathbb{K}[[X]][Y]$ is also irreducible. Therefore Theorem [1,1] with $k = s$ implies that $g_1$ is irreducible of characteristic $(h_1, \ldots, h_s)$ and the theorem follows.

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