Thermodynamics of 5D dilaton-gravity

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Abstract. We calculate the free energy, spatial string tension and Polyakov loop of the gluon plasma using the dilaton potential of Ref. [1] in the dilaton-gravity theory of AdS/QCD. The free energy is computed from the Black Hole solutions of the Einstein equations in two ways: first, from the Bekenstein-Hawking proportionality of the entropy with the area of the horizon, and secondly from the Page-Hawking computation of the free energy. The finite temperature behaviour of the spatial string tension and Polyakov loop follow from the corresponding string theory in AdS5. Comparison with lattice data is made.

Keywords: Gauge-string correspondence, Black Holes, QCD Thermodynamics

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INTRODUCTION

The duality of string theory with 5D gravity is nowadays a powerful tool to study the strong coupling properties of gauge theories. In conformal $AdS_5$ the metric has a horizon in the bulk space at $r_T = \pi \ell^2 T$ where $\ell$ is the size of the AdS-space, and entropy scale like $s \propto r_T^2 \propto T^3$. To extend this duality to SU($N_c$) Yang-Mills theory, one of the first tasks is to control the breaking of conformal invariance. In this work we will study the 5D dilaton-gravity model introduced in Ref. [2], with the action

$$S = \frac{1}{16 \pi G_5} \int d^5 x \sqrt{-G} \left( R - \frac{4}{3} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) - \frac{1}{8 \pi G_5} \int \phi d^4 x \sqrt{-H} K .$$

The dilaton potential $V(\phi)$ is related to the $\beta$-function in a one-to-one relation. We assume the parameterization

$$\beta(\alpha) = -b_2 \alpha + \left( b_2 \alpha + \frac{b_2}{\alpha} - \beta_0 \right) \alpha^2 + \left( \frac{b_2}{2 \alpha^2} - \frac{\beta_0}{\alpha} - \beta_1 \right) \alpha^3 e^{-\alpha/\alpha} ,$$

which was proposed in Ref. [1] for the computation of the $Q\bar{Q}$ potential at $T = 0$ within this model. Eq. (2) has the standard behavior in the UV, $\beta(\alpha) \approx -\beta_0 \alpha^2 - \beta_1 \alpha^3 + \ldots$ for $\alpha \ll \bar{\alpha}$, and generates confinement in the IR. The optimum values to reproduce the $Q\bar{Q}$ potential at $T = 0$ are $b_2 = 2.3$, $\bar{\alpha} = 0.45$, and $\ell = 4.389$ GeV$^{-1}$ [1].

The model of Eq. (1) has two types of solutions. The thermal gas solution corresponds to the confined phase, and the black hole (BH) solution characterizes the deconfined phase and it has a horizon in the bulk coordinate at $z = z_h$ [2]. The BH metric takes the form

$$ds^2 = b^2(z) \left( f(z) d\tau^2 + dx_k dx^k + dz^2 / f(z) \right) ,$$

where $z = \ell^2 / r$ and $f(z)$ has the properties $f(z_h) = 0$ and $f'(z_h) = -4 \pi T$. Starting from the ultraviolet expansion of the $\beta$-function, one can solve analytically the equations of motion in the UV [2, 3, 4, 5]. One arrives at the following expansion for the metric factor up to $O(\alpha^2)$

$$b(\alpha) \approx \frac{4}{\pi} \ell \left[ 1 - \frac{4}{9} \beta_0 \alpha + \frac{2}{81} (22 \beta_0^2 - 9 \beta_1) \alpha^2 \right] .$$

FREE ENERGY

The phase transition from the confined glueball gas to the deconfined gluon plasma can be understood as follows. At low temperatures the physics is determined by the thermal gas metric with $f = 1$. At a minimal temperature the BH solution appears with a horizon at $z_h$ and $f \neq 1$, and becomes the preferred solution at the phase transition. So, the phase transition arises from the competition of the free energies computed in both metrics.

We show two different ways to get the thermodynamics of the 5D dilaton-gravity model of Eq. (1). One possibility is to use the Bekenstein-Hawking entropy formula which establishes the proportionality between the entropy and the area of the horizon of the BH. It reads:

$$s(T) = \frac{1}{4 G_5} b^3(z_h)$$

$$\alpha_0 \approx \frac{\pi^4 \ell^3}{4 G_5} \left[ 1 - \frac{4}{9} \beta_0 \alpha_0 + \frac{1}{9} (11 \beta_0^2 - 6 \beta_1) \alpha_0^2 \right] ,$$

where $\alpha_0 = \alpha(z_h)$. In the second equality we have used Eq. (4). From the entropy one can derive the pressure by solving $dp(T)/dT = s(T)$. Another possibility is to compute the free energy from the Einstein-Hilbert action. One has to regularize the action by introducing a cutoff $z = \epsilon$ in the integral over the on shell action Eq. (1), c.f.
The free energy is computed as the difference of the zero temperature one at finite temperature solution of the Einstein equations from the Bekenstein-Hawking entropy formula, c.f. Ref. [2]. The free energy is computed as the difference of the zero temperature one at finite temperature solution of the Einstein equations from the Bekenstein-Hawking entropy formula, c.f. Ref. [2].

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\[
\mathcal{F} = \frac{1}{b} \lim_{\beta \to 0} \left( \mathcal{F}_{\text{BH}}(\varepsilon) - \mathcal{F}_{\text{BH}}(\varepsilon) \right) = \frac{\text{Vol}(3)}{16 \pi G_5} \left[ 15 G - \frac{C_f}{4} \right],
\]

where \( C_f = 4 \pi T b^3(z_b) \) corresponds to an enthalpy contribution in \( \mathcal{F} \), and \( G = \frac{\pi G_5}{1} \beta(\alpha) \left( \langle \text{Tr} F_{\mu \nu}^2 \rangle_T - \langle \text{Tr} F_{\mu \nu}^2 \rangle_0 \right) \) is the gluon condensate up to normalization factors. This method to compute \( \mathcal{F} \) is more involved giving the fact that the computation of the gluon condensate \( G \) is rather subtle. The finite temperature solution of the Einstein equations differs from the zero temperature one at \( \mathcal{O}(z^4) \):

\[
\frac{b(z)}{b_0(z)} = 1 + \frac{G}{b} \varepsilon(z) \left( 1 + \frac{19}{12} \beta_0 \alpha_0(z) + \mathcal{O}(\alpha^2) \right) + \cdots.
\]

In Eq. (7) care has to be taken to keep track of logarithmic effects which are not usually considered, c.f. Ref. [2]. These higher order terms in \( \alpha_0 \) affect appreciably the extraction of \( G \) from a comparison of the numerical solutions of \( b \) and \( b_0 \). Only when at least the order \( \mathcal{O}(\alpha^2) \) is taken into account in Eq. (7), one gets good agreement of the thermodynamic quantities with the numerical results from the Bekenstein-Hawking entropy formula Eq. (5). We show in Fig. 1 the free energy obtained by using the Bekenstein-Hawking entropy formula, Eq. (5), and the Einstein-Hilbert action from Eq. (6), using NNLO terms in \( \alpha_0 \) in Eq. (7) to compute \( G \) as a function of \( T \). These plots correspond to a numerical solution of the equations of motion. See Ref. [5] for details. One recognizes in this figure the first order phase transition at \( T_c = 273 \text{MeV} \) for zero flavours, which is quite close to lattice simulations. The upper branch in Fig. 1 represents the small black holes.

**SPATIAL STRING TENSION AND POLYAKOV LOOP**

The spatial string tension is very useful to test AdS/QCD models. It is non-vanishing even in the deconfined phase, and it gives useful information about the non perturbative features of high temperature QCD. The computation of the correlation function of rectangular Wilson loops in the \((x,y)\) plane leads to a potential between quark and antiquarks which behaves linearly at large distances, i.e.

\[
(W[\ell]) \sim e^{-\gamma V(d)}, \quad V(d) \sim \sigma \cdot d.
\]

For details on the computation we refer to e.g. Ref. [6]. The spatial string tension takes the following form:

\[
\sigma_s(T) = \frac{1}{2\pi l_s^2} \alpha_h^{4\beta/3} b^2(z_b).
\]

From Eqs. (4) and (9), the ultraviolet expansion leads to

\[
\sigma_s(T) = \frac{\ell^2}{2 l_s^2 \pi T^2} \alpha_h^{4\beta/3} \times \left[ 1 - \frac{8}{9} \beta_0 \alpha_h + \frac{2}{81} (25 \beta_0^2 - 2 \beta_1) \alpha_h^2 + \mathcal{O}(\alpha^3) \right].
\]

We show in Fig. 2 a plot of \( T/\sqrt{\sigma_s} \) as a function of temperature including several orders in Eq. (10), and the full numerical computation from Eq. (9). One can see that the model reproduces very well the lattice data in the regime \( 1.10 < T/T_c < 4.5 \). A good fit to the lattice data taken from Ref. [7] is obtained by using \( l_s = 1.94 \text{GeV}^{-1} \). The vacuum expectation value of the Polyakov loop serves as an order parameter for the deconfinement transition in gluodynamics. The correlation function of two
Polyakov loops taken in the large distance limit leads to the vacuum expectation value of one single Polyakov loop squared. This means that the Polyakov loop is related to the free energy of a single quark $F_q$ as

$$\langle \bar{Q}(\vec{x}) \rangle = e^{-\frac{1}{T} F_q(\vec{x})}. \quad (11)$$

One can compute the Polyakov loop from the Nambu-Goto (NG) action of a string hanging down from a quark on the boundary into the bulk. The NG action reads [5, 8]

$$S_{\text{NG}}^{\alpha} = \frac{1}{2\pi l_s^2 T} \left[ \int_0^{z_c} dz \, \alpha^{4/3}(z) b^2(z) - \int_0^{z_c} dz \, \alpha_0^{4/3}(z) b_0^2(z) \right], \quad (12)$$

where we have regularized it by subtracting the action of the thermal gas solution up to a cutoff $z_c$. The renormalized Polyakov loop then writes

$$L_R(T) = e^{-S_{\text{NG}}^{\alpha}}. \quad (13)$$

We show in Fig. 3 the behavior of $L_R$ as a function of $T$ computed numerically from Eqs. (12)-(13). We perform a fit to lattice data by considering the string length $l_s$ and $z_c$ as free parameters. The best fit in the regime $T_c < T < 10 T_c$ leads to $l_s = 2.36 \text{GeV}^{-1}$, $z_c = 0.43 \text{GeV}^{-1}$. Our approach fits $L_R$ very well without a dimension two condensate, since a dimension two operator would show up in the UV expansion of the metric near $z = 0$, Eq. (7). This does not exclude that a good fit to the data exists as free parameters. The best fit in the regime

$$0.0903.4375$$

The UV asymptotic behavior of the Polyakov loop can be computed from Eqs. (4), (12) and (13). It reads

$$L_R(T) = \exp \left[ \frac{L^2}{2 l_s^2} \frac{4}{\alpha_h^2} \left( 1 + \frac{4}{9} \beta_0 \alpha_h + \beta'(\alpha_h^2) \right) \right]. \quad (14)$$

We plot in Fig. 3 also the UV asymptotics given by Eq. (14) and the pQCD result up to $\beta'(\alpha_T^2)$ [10, 11].

**CONCLUSIONS**

We study the thermodynamics of the 5D dilaton-gravity model by using a parameterization for the dilaton potential which was quite successful to reproduce the $Q\bar{Q}$ potential at $T = 0$ [1]. We compute analytically in the UV and numerically the entropy, free energy, spatial string tension and Polyakov loop. We demonstrate the numerical agreement between computations of the free energy from the Bekenstein-Hawking entropy formula and from the Einstein-Hilbert action. Both approaches lead to the same result, but the later is much more sensitive to numerical errors, and an accurate computation is only possible when one takes care of including logarithmic effects in an UV expansion of the scale factor at finite $T$.

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