The Magnetosphere of Oscillating Neutron Stars in General Relativity

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ABSTRACT
Just as a rotating magnetised neutron star has material pulled away from its surface to populate a magnetosphere, a similar process can occur as a result of neutron-star pulsations rather than rotation. This is of interest in connection with the overall study of neutron star oscillation modes but with a particular focus on the situation for magnetars. Following a previous Newtonian analysis of the production of a force-free magnetosphere in this way Timokhin et al. (2000), we present here a corresponding general-relativistic analysis. We give a derivation of the general relativistic Maxwell equations for small-amplitude arbitrary oscillations of a non-rotating neutron star with a generic magnetic field and show that these can be solved analytically under the assumption of low current density in the magnetosphere. We apply our formalism to toroidal oscillations of a neutron star with a dipole magnetic field and find that the low current density approximation is valid for at least half of the oscillation modes, similarly to the Newtonian case. Using an improved formula for the determination of the last closed field line, we calculate the energy losses resulting from toroidal stellar oscillations for all of the modes for which the size of the polar cap is small. We find that general relativistic effects lead to shrinking of the size of the polar cap and an increase in the energy density of the outflowing plasma. These effects act in opposite directions but the net result is that the energy loss from the neutron star is significantly smaller than suggested by the Newtonian treatment.

Key words: stars: magnetic field – stars: neutron – stars: oscillations – pulsars: general

1 INTRODUCTION
Study of the internal structure of neutron stars (NSs) is of fundamental importance for subatomic physics since these objects provide a laboratory for studying the properties of high-density matter under very extreme conditions. In particular, there is the intriguing possibility of using NS oscillation modes as a probe for constraining models of the equation of state of matter at supranuclear densities. It was suggested long ago that if a NS is oscillating, then traces of this might be revealed in the radiation which it emits (Pacini & Ruderman 1974; Tsygan 1973; Boriakoff 1976; Bisnovatyi-Kogan 1995; Ding & Cheng 1997; Duncan 1998). Recently, a lot of interest has been focused on oscillations of magnetized NSs because of the discovery of gamma-ray flare activity in Soft Gamma-Ray Repeaters (SGRs) which are thought to be the very highly magnetised NSs known as magnetars (for recent review on the SGRs see Woods & Thompson 2006; Watts & Strohmayer 2007). The giant flares in these objects are thought to be powered by global reconfigurations of the magnetic field and it has been suggested that the giant flares might trigger starquakes and excite global seismic pulsations of the magnetar crust.

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Indeed, analyses of the observations of giant flares have revealed that the decaying part of the spectrum exhibits a number of quasi-periodic oscillations (QPOs) with frequencies in the range from a few tens of Hz up to a few hundred Hz (Israel et al. 2005; Strohmayer & Watts 2006). There is substantial evidence that the observed SGR QPOs are caused by neutron star pulsations, there is a great deal of uncertainty about how stellar surface motion gets translated into the observed features of the X-ray radiation (Strohmayer & Watts 2006; Timokhin et al. 2007). To make progress with this, it is necessary to develop a better understanding of the processes occurring in the magnetospheres of oscillating neutron stars.

Standard pulsars typically have magnetic fields of around $10^{12}$ G while magnetars may have fields of up to $10^{14} - 10^{15}$ G near to the surface. Rotation of a magnetized star generates an electric field:

$$E_{\text{rot}} \sim \frac{\Omega R}{c} B,$$

(1)

where $B$ is the magnetic field strength, $c$ is the speed of light and $\Omega$ is the angular velocity of the star with radius $R$. Depending on the rotation velocity and the magnetic field strength, the electric field may be as strong as $10^{10}$ V cm$^{-1}$ and it has a longitudinal component (parallel to $B$) which can be able to pull charged particles away from the stellar surface, if the work function is sufficiently small and accelerate them up to ultra-relativistic velocities. This result led Goldreich & Julian (1969) to suggest that a rotating NS with a sufficiently strong magnetic field should be surrounded by a magnetosphere filled with charge-separated plasma which screens the accelerating electric field and thus hinders further outflow of charged particles from the stellar surface. Even if the binding energy of the charged particles is sufficiently high to prevent them being pulled out by the electric field, the NS should nevertheless be surrounded by charged particles produced by plasma generation processes (Sturrock 1971; Ruderman & Sutherland 1975), which again screen the longitudinal component of the electric field. These considerations led to the development of a model for pulsar magnetospheres which is frequently called the “standard model” (an in depth discussion and review of this can be found in, e.g., Michel 1991; Beskin et al. 1993; Beskin 2003).

Timokhin, Bisnovatyi-Kogan & Spruit (2000) (referred to as TBS from here on) showed that an oscillating magnetized NS should also have a magnetosphere filled with charge-separated plasma, even if it is not rotating, since the vacuum electric field induced by the oscillations would have a large radial component which can be of the same order as rotationally-induced electric fields. One can show this quantitatively by means of the following simple arguments. To order of magnitude, the radial component of the vacuum electric field generated by the stellar oscillations is given by

$$E_{\text{osc}} \sim \frac{\omega \xi}{c} B,$$

(2)

where $\omega$ is the oscillation frequency and $\xi$ is the displacement amplitude. Using this together with Eq. (1), it follows immediately that the electric field produced by oscillations will be stronger than the rotationally induced one for sufficiently slowly-rotating neutron stars, having

$$\Omega \lesssim \frac{\omega \xi}{R}.$$  
(3)

For stellar oscillations with $\xi/R \sim 0.001$ and $\omega \sim 1$ kHz, the threshold is $\Omega \sim 1$ Hz. Within this context, TBS developed a formalism extending the basic aspects of the standard pulsar model to the situation for a non-rotating magnetized NS undergoing arbitrary oscillations. This formalism was based on the assumption of low current densities in the magnetosphere, signifying that the influence of currents outside the NS on electromagnetic processes occurring in the magnetosphere is negligibly small compared to that of currents in the stellar interior. This assumption leads to a great simplification of the Maxwell equations, which then can be solved analytically. As an application of the formalism, TBS considered toroidal oscillations of a NS with a dipole magnetic field, and obtained analytic expressions for the electromagnetic field and charge density in the magnetosphere. (Toroidal oscillations are thought to be particularly relevant for magnetar QPO phenomena.) They found that the low current density approximation (LCDA) is valid for at least half of all toroidal oscillation modes and analyzed the energy losses due to plasma outflow caused by these modes for cases where the size of the polar cap (the region on the stellar surface that is crossed by open magnetic field lines) is small, finding that the energy losses are strongly affected by the magnetospheric plasma. For oscillation amplitudes larger than a certain critical value, they found that energy losses due to plasma outflow were larger than those due to the emission of the electromagnetic waves (assuming in that case that the star was surrounded by vacuum). Recently, Timokhin (2007) considered spheroidal oscillations of a NS with a dipole magnetic field, using the TBS formalism, and found that the LCDA again holds for at least half of these modes. Discussion in Timokhin (2007) also provided some useful insights into the role of rotation for the magnetospheric structure of oscillating NSs.

The TBS model was a very important contribution and, to the best of our knowledge, remains the only model for the magnetosphere of oscillating NSs available in the literature. However, it should be pointed out that it does not include several ingredients that a fully consistent and realistic model ought to include. Most importantly, it does not treat the magnetospheric...
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currents in a fully consistent way: although it gives a consistent solution for around half of the oscillation modes, the remaining solutions turn out to be unphysical and, as TBS pointed out, this is a symptom of the LCDA failing there. Also, rotation and the effects of general relativity can be very relevant; in particular, several authors have stressed that using a Newtonian approach may not give very good results for the structure of NS magnetospheres (see, e.g., [Beskin 1992; Moz & Ahmedov 2004; Morozova et al. 2008]). However, a more realistic model would naturally be more complicated than the TBS one whose relative simplicity can be seen as a positive advantage when using it as the basis for further applications.

The aim of the present paper is to give a general relativistic reworking of the TBS model so as to investigate the effects of the changes with respect to the Newtonian treatment. We derive the general relativistic Maxwell equations for arbitrary small-amplitude oscillations of a non-rotating spherical NS with a generic magnetic field configuration and show that they can be solved analytically within the LCDA as in Newtonian theory. We then apply this solution to the case of toroidal oscillations of a NS with a dipole magnetic field and find that the LCDA is again valid for at least half of all toroidal oscillation modes, as in Newtonian theory. Using an improved formula for the determination of the last closed field line, we calculate the energy losses resulting from these oscillations for all of the modes for which the size of the polar cap is small and discuss the influence of GR effects on the energy losses.

The paper is organized as follows. In Section 2 we introduce some definitions and derive the quasi-stationary Maxwell equations in Schwarzscild spacetime as well as the boundary conditions for the electromagnetic fields at the stellar surface. In Section 3 we sketch our method for analytically solving the Maxwell equations for arbitrary NS oscillations with a generic magnetic field configuration. In Section 4 we apply our formalism to the case of purely toroidal oscillations of a NS with a dipole magnetic field and also discuss the validity of the LCDA and the role of GR effects. In Section 5 we calculate the energy losses due to plasma outflow caused by the toroidal oscillations. Some detailed technical calculations related to the discussion in the main part of the paper are presented in Appendices A-C.

We use units for which $c = 1$, a space-like signature $(-, +, +, +)$ and a spherical coordinate system $(t, r, \theta, \phi)$. Greek indices are taken to run from 0 to 3 while Latin indices run from 1 to 3 and we adopt the standard convention for summation of repeated indices. We indicate four-vectors with bold symbols (e.g. $u$) and three-vectors with an arrow (e.g. $\vec{u}$).

2 GENERAL FORMALISM

2.1 Quasi-stationary Maxwell equations in Schwarzschild spacetime

The study of electromagnetic processes related to stellar oscillations in the vicinity of NSs should, in principle, use the coupled system of Einstein-Maxwell equations. However, such an approach would be overly complicated for our study here, as it is for many other astrophysical problems. Here we simplify the problem by neglecting the contributions of the electromagnetic fields, the NS rotation and the NS oscillations to the spacetime metric and the structure of the NS, noting that this is expected to be a good approximation for small-amplitude oscillations. Indeed, for a star with average mass-energy density $\bar{\rho}$, mass $M$ and radius $R$, the maximum fractional change in the spacetime metric produced by the magnetic field is typically of the same order as the ratio between the energy density in the surface magnetic field and average mass-energy density of the NS, i.e.,

$$\frac{B^2}{8\pi \bar{\rho} c^2} \approx 10^{-7} \left( \frac{B}{10^{15} \text{G}} \right)^2 \left( \frac{1.4M_{\odot}}{M} \right) \left( \frac{R}{10 \text{ km}} \right)^3.$$ \hspace{1cm} (4)

The corresponding fractional change in the metric due to rotation is of order

$$0.1 \left( \frac{\Omega}{\Omega_K} \right)^2 \approx 10^{-7} \left( \frac{\Omega}{1 \text{ Hz}} \right)^2 \left( \frac{1 \text{ kHz}}{\Omega_K} \right)^2$$ \hspace{1cm} (5)

where $\Omega_K$ is the Keplerian angular velocity at the surface of the NS. Moreover, in the case of magnetars, which we consider in our study, the oscillations are thought to be triggered by the global reconfiguration of the magnetic field. Due to this reason, the corrections due to the oscillations should not exceed the contribution due to the magnetic field itself given by estimate (4). Therefore, we can safely work in the background spacetime of a static spherical star, whose line element in a spherical coordinate system $(t, r, \theta, \phi)$ is given by

$$ds^2 = g_{00}(r)dt^2 + g_{11}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

while the geometry of the spacetime external to the star (i.e. for $r \geq R$) is given by the Schwarzschild solution:

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$ \hspace{1cm} (7)

1 Several authors have, in fact, studied the equilibrium configurations of magnetars by solving the Einstein-Maxwell equations in full general relativity ([Bocquet et al. 1993; Bonazzola et al. 1996; Cardall et al. 2001]) or by using perturbative techniques ([Colaiuda et al. 2008; Haskell et al. 2008]).
where \( N \equiv (1 - 2M/r)^{1/2} \) and \( M \) is the total mass of the star. For the part of the spacetime inside the star, we represent the metric in terms of functions \( \Lambda \) and \( \Phi \) as

\[
g_{00} = -e^{2\Phi(r)}, \quad g_{11} = e^{2\Lambda(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1},
\]

where \( m(r) = 4\pi \int_0^r r'^2 \rho(r') dr' \) is the volume integral of the total energy density \( \rho(r) \) over the spatial coordinates. The form of these functions is given by solution of the standard TOV equations for spherical relativistic stars (see, e.g., Shapiro & Teukolsky [1983]) and they are matched continuously to the external Schwarzschild spacetime through the relations

\[
g_{00}(r = R) = N^2_R, \quad g_{11}(r = R) = N^{-2}_R,
\]

where \( N_R \equiv (1 - 2M/R)^{1/2} \). Within the external part of the spacetime, we select a family of static observers with four-velocity components given by

\[
(u^\alpha)_{\text{obs}} \equiv N^{-1}(1, 0, 0, 0).
\]

and associated orthonormal frames having tetrad four vectors \( \{e_\mu \} = \{e_0, e_r, e_\theta, e_\phi \} \) and 1-forms \( \{\omega^\mu \} = \{\omega^0, \omega^r, \omega^\theta, \omega^\phi \} \), which will become useful when determining the “physical” components of the electromagnetic fields. The components of the vectors are given by equations (6)-(9) of Rezzolla & Ahmedov (2004) (hereafter Paper I).

The general relativistic Maxwell equations have the following form (Landau & Lifshitz [1987])

\[
3F_{[\alpha\beta,\gamma]} = F_{\alpha\beta\gamma} + F_{\gamma\alpha\beta} + F_{\beta\gamma\alpha} = 0,
\]

\[
F^{\alpha\beta}_{\;\beta\alpha} = 4\pi J^\alpha,
\]

where \( F^{\alpha\beta} \) is the electromagnetic field tensor and \( J^\alpha \) is the electric-charge 4-current. We consider the region close to the star (the near zone), at distances from the NS much smaller than the wavelength \( \lambda = 2\pi c/\omega \). In the near zone the electromagnetic fields are quasi-stationary, therefore we neglect the displacement current term in the Maxwell equations. Once expressed in terms of the physical components of the electric and magnetic fields, equations (11) and (12) become (see Section 2 of Paper I for details of the derivation)

\[
\sin \theta \partial_\theta \left(r^2 B^\phi\right) + N^{-1} r \partial_\theta \left(\sin \theta B^\phi\right) + N^{-1} r \partial_\theta B^\phi = 0,
\]

\[
\left(\sin \theta\right) \frac{\partial B^\phi}{\partial t} = N \left[ \partial_\theta E^\theta - \partial_\phi \left(\sin \theta E^\phi\right) \right],
\]

\[
\left( N^{-1} r \sin \theta \right) \frac{\partial B^\phi}{\partial t} = -\partial_\theta E^\theta + \sin \theta \partial_\phi \left(r N E^\phi\right),
\]

\[
\left( N^{-1} r \right) \frac{\partial E^\phi}{\partial t} = -\partial_\phi \left(r N E^\theta\right) + \partial_\theta E^\theta,
\]

\[
N \sin \theta \partial_\theta \left(r^2 E^\phi\right) + r \partial_\theta \left(\sin \theta E^\phi\right) + r \partial_\phi E^\phi = 4\pi \rho_e r^2 \sin \theta,
\]

\[
\left[ \partial_\theta \left(\sin \theta B^\phi\right) - \partial_\phi B^\phi \right] = 4\pi r \sin \theta J^\phi,
\]

\[
\partial_\theta B^\phi - \sin \theta \partial_\phi \left( r N B^\phi \right) = 4\pi r \sin \theta J^\phi,
\]

\[
\partial_\phi \left(Nr B^\phi\right) - \partial_\theta B^\phi = 4\pi r J^\phi,
\]

where \( \rho_e \) is the proper charge density. We further assume that the force-free condition,

\[
\vec{E}_{\text{SC}} \cdot \vec{B} = 0,
\]

is fulfilled everywhere in the magnetosphere, implying that the magnetosphere of the NS is populated with charged particles that cancel the longitudinal component of the electric field. The charge density \( \rho_{SC} \) responsible for the electric field \( \vec{E}_{\text{SC}} \) (cf. equation 17) is the characteristic charge density of the force-free magnetosphere; this is appropriate for describing the charge density in the inner parts of the NS magnetosphere. We will refer to \( \vec{E}_{\text{SC}} \) as the space-charge (SC) electric field, while to \( \rho_{SC} \) as the SC charge density.

Finally, we introduce the perturbation of the NS crust in terms of its four-velocity, with the components being given by

\[
w^\alpha \equiv e^{-\Phi} \left(1, e^{-\Lambda} \delta v^r, \frac{\delta v^\theta}{r}, \frac{\delta v^\phi}{r \sin \theta}\right),
\]
where $\delta v^i = dx^i/dt$ is the relative oscillation three-velocity of the conducting stellar surface with respect to the unperturbed state of the star.

### 2.2 Boundary conditions at the surface of star

We now begin our study of the internal electromagnetic field induced by the stellar oscillations. We assume here that the material in the crust can be treated as a perfect conductor and the induced electric field then depends on the magnetic field and the pulsational velocity field according to the following relations (see Paper I for details of the derivation):

\[
E_{\text{in}}^\rho = -e^{-\Phi} \left[ \delta \rho^\rho B^\rho - \delta \nu^\rho B^\rho \right],
\]

\[
E_{\text{in}}^\theta = -e^{-\Phi} \left[ \delta \rho^\theta B^\theta - \delta \nu^\theta B^\theta \right],
\]

\[
E_{\text{in}}^\phi = -e^{-\Phi} \left[ \delta \rho^\phi B^\phi - \delta \nu^\phi B^\phi \right].
\]

Boundary conditions for the magnetic field at the stellar surface ($r = R$) can be obtained from the requirement of continuity for the radial component, while leaving the tangential components free to be discontinuous because of surface currents:

\[
B_{\text{ex}}^\rho |_{r=R} = B_{\text{in}}^\rho |_{r=R},
\]

\[
B_{\text{ex}}^\theta |_{r=R} = B_{\text{in}}^\theta |_{r=R} + 4\pi \dot{\nu}^\theta,
\]

\[
B_{\text{ex}}^\phi |_{r=R} = B_{\text{in}}^\phi |_{r=R} - 4\pi \dot{\nu}^\phi,
\]

where $\dot{i}$ is the surface current density. Boundary conditions for the electric field at the stellar surface are obtained from requirement of continuity of the tangential components, leaving $E_{\text{ex}}^i$ to have a discontinuity proportional to the surface charge density $\Sigma_s$:

\[
E_{\text{ex}}^\rho |_{r=R} = E_{\text{in}}^\rho |_{r=R} + 4\pi \Sigma_s = -N^{-1}_R \left[ \delta \rho^\rho B^\rho - \delta \nu^\rho B^\rho \right] |_{r=R} + 4\pi \Sigma_s,
\]

\[
E_{\text{ex}}^\theta |_{r=R} = E_{\text{in}}^\theta |_{r=R} = -N^{-1}_R \left[ \delta \rho^\theta B^\theta - \delta \nu^\theta B^\theta \right] |_{r=R},
\]

\[
E_{\text{ex}}^\phi |_{r=R} = E_{\text{in}}^\phi |_{r=R} = -N^{-1}_R \left[ \delta \rho^\phi B^\phi - \delta \nu^\phi B^\phi \right] |_{r=R},
\]

where $\Sigma_s$ is the surface charge density.

### 2.3 The low current density approximation

The low current density approximation was introduced by TBS, and in the present section we present a brief introduction to it for completeness. Close to the NS surface, the current flows along the magnetic field lines, and so in the inner parts of the magnetosphere it can be expressed as

\[
\vec{J} = \alpha(r, \theta, \phi) \cdot \vec{B},
\]

where $\alpha$ is a scalar function. The system of equations (23–25) and (29–31) forms a complete set but is overly complicated for solving in the general case. However, within the LCDA these equations can, as we show below, be solved analytically for arbitrary oscillations of a NS with a generic magnetic field configuration.

The LCDA scheme is based on the assumption that the perturbation of the magnetic field induced by currents flowing in the NS interior is much larger than that due to currents in the magnetosphere, which are neglected to first order in the oscillation parameter $\xi = \xi/R$:

\[
\frac{4\pi}{c} \frac{\vec{J}}{\xi} \ll \nabla \times \vec{B},
\]

and

\[
\nabla \times \vec{B}^{(1)} = 0,
\]

where $\vec{B}^{(1)}$ is the first order term of the expansion in $\xi$. This also implies that the current density satisfies the condition
where $\rho_{SC}(R)$ is the SC density near to the surface of the star. Here we have used the relation $\rho_{SC}(R) \approx B^{(0)} \eta / c R$, where $\eta$ is the velocity amplitude of the oscillation and $\omega$ is its frequency.

In regions of complete charge separation, the maximum current density is given by

$$J \approx \frac{\rho_{SC} c \omega \gamma}{\omega r}.$$  \hspace{1cm} (35)

For a more detailed discussion of the LCDA and its validity, we refer the reader to Sections 2.3 and 3.2.1 of TBS.

3 THE LCDA SOLUTION

3.1 The electromagnetic field in the magnetosphere

We now begin our solution of the Maxwell equations, assuming that the LCDA condition (35) is satisfied everywhere in the magnetosphere. Within the LCDA, equations (18)–(20) for the magnetic field in the magnetosphere take the form

$$\begin{align*}
\partial_\theta \left( \sin \theta B^\phi \right) - \partial_\phi B^\theta &= 0, \\
\partial_\theta B^\phi - \sin \theta \partial_\theta \left( r N B^\phi \right) &= 0, \\
\partial_r \left( N r B^\phi \right) - \partial_\theta B^\theta &= 0.
\end{align*}$$

(37) (38) (39)

As demonstrated in Paper I, the components of the magnetic field $B^r, B^\theta$ and $B^\phi$ can be expressed in terms of a scalar function $S$ in the following way:

$$\begin{align*}
B^r &= \frac{1}{r \sin \theta} \left[ \sin \theta \partial_\theta (\sin \theta \partial_\phi S) + \partial_\phi \partial_\theta S \right], \\
B^\theta &= \frac{N}{r} \partial_\theta \partial_r S, \\
B^\phi &= \frac{N}{r \sin \theta} \partial_\phi \partial_r S.
\end{align*}$$

(40) (41) (42)

Substituting these expressions into the Maxwell equations (13)–(16), we obtain a system of equations for the electric field components which has the following general solution

$$\begin{align*}
E^r_{\text{SC}} &= -\partial_r (\Psi_{\text{SC}}), \\
E^\theta_{\text{SC}} &= -\frac{1}{N r \sin \theta} \partial_\theta \partial_\phi S - \frac{1}{N r} \partial_\phi (\Psi_{\text{SC}}), \\
E^\phi_{\text{SC}} &= \frac{1}{N r} \partial_\theta \partial_\phi S - \frac{1}{N r \sin \theta} \partial_\phi (\Psi_{\text{SC}}),
\end{align*}$$

(43) (44) (45)

where $\Psi_{\text{SC}}$ is an arbitrary scalar function. The terms proportional to the gradient of $\Psi_{\text{SC}}$ are responsible for the contribution of the charged particles in the magnetosphere. The vacuum part of the electric field is given by the derivatives of the scalar function $S$. Substituting (13)–(16) into equation (17), we get an expression for the SC charge density in terms of $\Psi_{\text{SC}}$:

$$\rho_{\text{SC}} = \frac{1}{4 \pi r^2} \left[ N \partial_\theta (r^2 \partial_\phi \Psi_{\text{SC}}) + \frac{1}{N} \bigtriangleup \omega \Psi_{\text{SC}} \right],$$

(46)

where $\bigtriangleup \omega$ is the angular part of the Laplacian:
\[ \Delta \Omega = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\phi. \]  

(47)

3.2 The equation for \( \Psi_{SC} \)

Substituting expressions (40)–(42) and (43)–(45) for the components of the electric and magnetic fields into the force-free condition (41), we get the following equation for \( \Psi_{SC} \):

\[ \frac{1}{\sin \theta} (\sin \theta \partial_\theta (\sin \theta \partial_\theta S) + \partial_\phi \partial_\phi S) \partial_\theta (\Psi_{SC}) - \frac{1}{\sin \theta} [\partial_\theta \partial_\theta S \partial_\theta \partial_\theta S - \partial_\phi \partial_\phi S \partial_\theta \partial_\phi S] \]

\[ \quad - \partial_\theta \partial_\theta S \partial_\theta (\Psi_{SC}) - \frac{1}{\sin^2 \theta} \partial_\phi \partial_\phi S \partial_\theta (\Psi_{SC}) = 0. \]  

(48)

If the amplitude of the NS oscillations is suitably small \((\xi \ll 1)\), the function \( S \) can be series expanded in terms of the dimensionless perturbation parameter \( \xi \) and can be approximated by the sum of the two lowest order terms

\[ S(t, r, \theta, \phi) = S_0(t, \theta, \phi) + \delta S(t, r, \theta, \phi). \]  

(49)

Here the first term \( S_0 \) corresponds to the unperturbed static magnetic field of the NS, while \( \delta S \) is the first order correction to it. At this level of approximation, equation (48) for \( \Psi_{SC} \) takes the form

\[ \frac{1}{\sin^2 \theta} [\sin \theta \partial_\theta (\sin \theta \partial_\theta S_0) + \partial_\phi \partial_\phi S_0] \partial_\theta (\Psi_{SC}) = 0. \]  

(50)

Next we expand \( S \) in terms of the spherical harmonics:

\[ S = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} S_{\ell m}(t, r) Y_{\ell m}(\theta, \phi). \]  

(51)

where the functions \( S_{\ell m} \) are given in terms of Legendre functions of the second kind \( Q_\ell \) by \[ \text{Rezzolla et al. 2001}\]

\[ S_{\ell m}(t, r) = -\frac{r^2}{M^2} \frac{d}{dr} \left[ r \left( 1 - \frac{2M}{r} \right) \frac{d}{dr} Q_\ell \left( 1 - \frac{r}{M} \right) \right] s_{\ell m}(t). \]  

(52)

Note that all of the time dependence in (52) is contained in the integration constants \( s_{\ell m}(t) \) which, as we will see later, are determined by the boundary conditions at the surface of the star. We now series expand the coefficients \( S_{\ell m}(t, r) \) and \( s_{\ell m}(t) \) in terms of \( \xi \)

\[ S_{\ell m}(t, r) = S_{0\ell m}(t, r) + \delta S_{\ell m}(t, r), \quad s_{\ell m}(t) = s_{0\ell m} + \delta s_{\ell m}(t), \]  

(53)

where all of the time dependence is now confined within the coefficients \( \delta S_{\ell m}(t, r) \) and \( \delta s_{\ell m}(t) \), while the coefficients \( S_{0\ell m} \) and \( s_{0\ell m} \) are responsible for the unperturbed static magnetic field of the star. Using these results, we can also express \( S \) and \( \delta S \) in terms of a series in \( Y_{\ell m}(\theta, \phi) \) in the following way

\[ S_0 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} S_{0 \ell m}(r) Y_{\ell m}(\theta, \phi), \]  

(54)

\[ \delta S = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \delta S_{\ell m}(r) Y_{\ell m}(\theta, \phi). \]  

(55)

The variables \( r \) and \( t \) in the functions \( S_{\ell m}(t, r) \) and \( S_{\ell m}(t, r) \) can be separated using relation (52):

\[ S_{0\ell m}(r) = -\frac{r^2}{M^2} \frac{d}{dr} \left[ r \left( 1 - \frac{2M}{r} \right) \frac{d}{dr} Q_\ell \left( 1 - \frac{r}{M} \right) \right] s_{0\ell m}. \]  

(56)

\[ S_{\ell m}(t, r) = -\frac{r^2}{M^2} \frac{d}{dr} \left[ r \left( 1 - \frac{2M}{r} \right) \frac{d}{dr} Q_\ell \left( 1 - \frac{r}{M} \right) \right] \delta s_{\ell m}(t). \]  

(57)

3.3 The boundary condition for \( \Psi_{SC} \)

We now derive a boundary condition for \( \Psi_{SC} \) at the stellar surface using the behaviour of the electric and magnetic fields in that region. Following TBS, we assume that near to the stellar surface the interior magnetic field has the same behaviour as
the exterior one:
\[ B^\ell = -\frac{C_1}{r^2 \sin^2 \theta} \left[ \sin \theta \partial_\theta (\sin \theta \partial_\theta S) + \partial_\phi S \right], \]  
\[ B^\phi = C_1 \frac{e^{-\Lambda}}{r} \partial_\theta \partial_\theta S, \]  
\[ B^\theta = C_1 \frac{e^{-\Lambda}}{r \sin \theta} \partial_\phi \partial_\phi S. \]  

Using the continuity condition for the normal component of the magnetic field \( [B^r] = 0 \) at the stellar surface (Pons & Geppert 2007) together with the condition \( e^{-\Lambda} |_{r=R} \equiv N_R \), one finds that the integration constant \( C_1 \) is equal to one. The interior electric field components can then be obtained by substituting (58) – (60) (with \( C_1 = 1 \)) into (23) – (25):
\[ E^\ell_{\text{in}} = -\frac{e^{-(\Phi+\Lambda)}}{r \sin \theta} \left\{ \delta^\ell_{\theta} \partial_\theta \partial_\theta S - \sin \theta \delta^\ell_\theta \partial_\phi \partial_\phi S \right\}, \]  
\[ E^\phi_{\text{in}} = \frac{e^{-(\Phi+\Lambda)}}{r} \left\{ \frac{\delta^\ell_\theta}{r \sin \theta} \left[ \sin \theta \partial_\theta (\sin \theta \partial_\theta S) + \partial_\phi S \right] + \frac{N \delta^\ell_\phi}{\sin \theta} \partial_\phi \partial_\phi S + \frac{1}{\sin \theta} \partial_\phi \partial_\theta S \right\}, \]  
\[ E^\theta_{\text{in}} = \frac{e^{-(\Phi+\Lambda)}}{r} \left\{ \frac{\delta^\ell_\phi}{r \sin \theta} \left[ \sin \theta \partial_\theta (\sin \theta \partial_\theta S) + \partial_\phi S \right] + \frac{N \delta^\ell_\theta}{\sin \theta} \partial_\theta \partial_\theta S + \frac{1}{\sin \theta} \partial_\theta \partial_\phi S \right\}. \]

The continuity condition for the \( \theta \) component of the electric field across the stellar surface \( [E^\theta] \) gives a boundary condition for \( \partial_\phi \Psi_{\text{SC}} |_{r=R} \):
\[ \Psi_{\text{SC}, \theta} |_{r=R} = -\left\{ \frac{\delta^\ell_\theta}{R \sin^2 \theta} \left[ \sin \theta \partial_\theta (\sin \theta \partial_\theta S) + \partial_\phi \partial_\phi S \right] + \frac{N \delta^\ell_\phi}{\sin \theta} \partial_\phi \partial_\phi S + \frac{1}{\sin \theta} \partial_\phi \partial_\theta S \right\} \big|_{r=R}, \]  
while the continuity condition for \( E^\phi \) \( [E^\phi] \) gives a boundary condition for \( \partial_\phi \Psi_{\text{SC}} |_{r=R} \):
\[ \Psi_{\text{SC}, \phi} |_{r=R} = \left\{ \frac{\delta^\ell_\phi}{R \sin \theta} \left[ \sin \theta \partial_\theta (\sin \theta \partial_\theta S) + \partial_\phi \partial_\phi S \right] + \frac{N \delta^\ell_\theta}{\sin \theta} \partial_\theta \partial_\theta S + \frac{1}{\sin \theta} \partial_\theta \partial_\phi S \right\} \big|_{r=R}. \]

Integration of equation (62) or equation (63) over \( \theta \) or \( \phi \) respectively, gives a boundary condition for \( \Psi_{\text{SC}} \). We will use the result of integrating equation (62) over \( \theta \). Assuming that the perturbation depends on time \( t \) as \( e^{-i\omega t} \), we obtain the following condition, correct to first order in \( \xi^\ell \):
\[ \Psi_{\text{SC}} |_{r=R} = -\int_0^\theta \left\{ \frac{\delta^\ell_\phi}{R \sin^2 \theta} \left[ \sin \theta \partial_\theta (\sin \theta \partial_\theta S_0) + \partial_\phi \partial_\phi S_0 \right] + \frac{N \delta^\ell_\theta}{\sin \theta} \partial_\theta \partial_\theta S_0 + \frac{1}{\sin \theta} \partial_\theta \partial_\phi (\delta S_0) \right\} d\theta |_{r=R} + e^{i\omega t} F(\phi), \]  
where \( F(\phi) \) is a function only of \( \phi \) which we will determine below.

The components of the stellar-oscillation velocity field are continuously differentiable functions of \( r, \theta \) and \( \phi \). The boundary conditions for the electric field (60) – (64) imply that the tangential components of the electric field \( E_{\text{SC}} \) must be finite. The vacuum terms on the right-hand side of (16) - (19) and the terms on both sides of equation (45) are also finite. Consequently, the term
\[ -\frac{\partial_\phi (\Psi_{\text{SC}})}{\sin \theta} |_{r=R} \]  
should also be finite. Hence we obtain that \( \partial_\phi (\Psi_{\text{SC}}) |_{\theta=0, \pi, r=R} = 0 \) and so the function \( F(\phi) \) in the expression for boundary condition (60) must satisfy the condition \( \Psi_{\text{SC}} |_{\theta=0, \pi, r=R} = C e^{-i\omega t} \), where \( C \) is a constant. Using gauge invariance, we choose
\[ \Psi_{\text{SC}} |_{\theta=0, r=R} = 0, \]  
and from this and equation (66), we obtain our expression for the boundary condition for \( \Psi_{\text{SC}} \) at the stellar surface:
\[ \Psi_{\text{SC}} |_{r=R} = -\int_0^\theta \left\{ \frac{\delta^\ell_\phi}{R \sin^2 \theta} \left[ \sin \theta \partial_\theta (\sin \theta \partial_\theta S_0) + \partial_\phi \partial_\phi S_0 \right] + \frac{N \delta^\ell_\theta}{\sin \theta} \partial_\theta \partial_\theta S_0 + \frac{1}{\sin \theta} \partial_\theta \partial_\phi (\delta S_0) \right\} d\theta |_{r=R}. \]

### 4 TOROIDAL OSCILLATIONS OF A NS WITH A DIPOLE MAGNETIC FIELD

As an important application of this formalism, we now consider small-amplitude toroidal oscillations of a NS with a dipole magnetic field. For toroidal oscillations in the \((\ell', m')\) mode, a generic conducting fluid element is displaced from its initial
The corresponding magnetic field components have the form
where the derivatives of the SC potential (64)-(65), we find that

\[ \begin{align*}
\delta v^r &= e^{-i\omega t}\eta(r) \left( \frac{1}{\sin\theta} \partial_\theta Y_{\ell m}(\theta, \phi) \right), \\
\delta v^\theta &= \frac{d\xi^\theta}{dt} = -e^{-i\omega t}\eta(r) \partial_\theta Y_{\ell m}(\theta, \phi), \\
\delta v^\phi &= \frac{d\xi^\phi}{dt} = \frac{d\delta \psi}{dt},
\end{align*} \]

where \( \omega \) is the oscillation frequency and \( \eta(r) \) is the transverse velocity amplitude. Note that in the above expressions (60), the oscillation mode axis is directed along the \( z \)-axis. We use a prime to denote the spherical harmonic indices in the case of the oscillation modes.

4.1 The unperturbed exterior dipole magnetic field

If the static unperturbed magnetic field of the NS is of a dipole type, then the coefficients \( s_{0,10} \) involved in specifying it have the following form (see eq. 117 of Paper I)

\[ \begin{align*}
s_{0,10} &= -\frac{\sqrt{3\pi}}{2} \mu \cos \chi, \\
s_{0,11} &= \frac{\sqrt{3\pi}}{2} \mu \sin \chi,
\end{align*} \]

where \( \mu \) is the magnetic dipole moment of the star, as measured by a distant observer, and \( \chi \) is the inclination angle between the dipole moment and \( z \)-axis. Substituting expressions (71) into (65) and then the latter into (54), we get

\[ S_0 = -\frac{3\mu^2}{8M^2} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] (\cos \theta \cos \chi + e^{i\phi} \sin \theta \sin \chi) \]

The corresponding magnetic field components have the form

\[ \begin{align*}
B_0^r &= -\frac{3\mu}{4M^2} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] (\cos \chi \cos \theta + \sin \chi \theta e^{i\phi}), \\
B_0^\theta &= \frac{3\mu N}{4M^2 r} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right] (\cos \chi \sin \theta - \sin \chi \cos \theta e^{i\phi}), \\
B_0^\phi &= \frac{3\mu N}{4M^2 r} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right] (\sin \chi e^{i\phi}).
\end{align*} \]

At the stellar surface, these expressions for the unperturbed magnetic field components become

\[ \begin{align*}
B_R^r &= f_R B_0 (\cos \chi \cos \theta + \sin \chi \theta e^{i\phi}), \\
B_R^\theta &= h_R B_0 (\cos \chi \sin \theta - \sin \chi \cos \theta e^{i\phi}), \\
B_R^\phi &= -i h_R B_0 (\sin \chi e^{i\phi}),
\end{align*} \]

where \( B_0 \) is defined as \( B_0 = 2\mu/R^3 \). In Newtonian theory \( B_0 \) would be the value of the magnetic strength at the magnetic pole but this becomes modified in GR. The GR modifications are contained within the parameters

\[ \begin{align*}
h_R &= \frac{3R^2 N_R}{8M^2} \left[ \frac{R}{M} \ln N^2 + \frac{1}{N^2} + 1 \right], \\
f_R &= -\frac{3R^3}{8M^3} \left[ \ln N^2 + \frac{2M}{R} \left( 1 + \frac{M}{R} \right) \right].
\end{align*} \]

For a given \( \mu \), the magnetic field near the surface of the NS is stronger in GR than in Newtonian theory, as already noted by [Ginzburg & Ozerov (1964)].

4.2 The equation for \( \Psi_{SC} \)

Substituting \( S_0 \) from (72) into equation (59), we obtain a partial differential equation containing two unknown functions \( \Psi_{SC} \) and \( \delta S \) for arbitrary oscillations of a NS with a dipole magnetic field

\[ -2r^2 q_1(r) \left( \cos \theta \cos \chi + e^{i\phi} \sin \theta \sin \chi \right) \partial_r(\Psi_{SC}) + \partial_r \left[ r^2 q_1(r) \right] \left( \sin \theta \cos \chi - e^{i\phi} \cos \theta \sin \chi \right) \partial_\theta(\Psi_{SC}) \]

\[ -\partial_r \left[ r^2 q_1(r) \right] \frac{e^{i\phi} \sin \theta}{\sin \theta} \partial_\phi(\Psi_{SC}) + \partial_r \left[ r^2 q_1(r) \right] \left( \sin \theta \cos \chi - e^{i\phi} \cos \theta \sin \chi \right) \partial_\phi \partial_\phi(\delta S) + ie^{i\phi} \sin \chi \sin \theta \partial_\theta \partial_\phi(\delta S) \right] = 0,
\]

where we have introduced a new function \( q_1(r) \) for simplicity of notation [see Eq. (A2) for the definition of \( q_1(r) \)].

From (70), the boundary condition for \( \Psi_{SC} \) at the stellar surface is

\[ \Psi_{SC}|_{r=R} = \int_0^\theta \left( B_0 R f_R d\theta \left( \cos \theta \cos \chi + e^{i\phi} \sin \theta \sin \chi \right) - \frac{1}{\sin \theta} \partial_\theta \partial_\phi(\delta S) \right) d\theta|_{r=R}. \]

Using the expressions for the velocity field of the toroidal oscillations (70) and for the boundary conditions for the partial derivatives of the SC potential (69)-(65), we find that \( \partial_\delta S \) is given by (see Appendix A for details of the derivation)

\[ \partial_\delta S(r, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{B_0 R f_R \tilde{d}_\ell \ell^2 q_\ell(R)}{R^2 q_\ell(R)} \]

\[ \left( \frac{1}{(\ell + 1)} \right) \]
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\[ \times \int_{4\pi} \left[ \partial_\theta Y_{\ell m} \left( \sin \theta \cos \chi - e^{i\phi} \cos \theta \sin \chi \right) + ie^{i\phi} \partial_\phi Y_{\ell m} \sin \theta \sin \chi \right] \frac{Y_{\ell m}^* (\theta, \phi)}{\sin \theta} d\Omega. \]

From here on, for simplicity, we will consider only the case with \( \chi = 0 \). Although our solution depends on the angle between the magnetic field axis and the oscillation mode axis, focusing on the case \( \chi = 0 \) does not actually imply a loss of generality because any mode with its axis not aligned with a given direction can be represented as a sum of modes with axes along this direction. We have developed a MATHEMATICA code for analytically solving equation (50) and hence obtaining analytic expressions for the electric and magnetic fields and for the SC density.

The solution of equation (85) for the case \( \chi = 0 \) is given in Appendix B, where we show that the general solution has the following form

\[ \Psi_{SC} = \frac{1}{2} \frac{m'^2}{\ell' (\ell' + 1)} B_0 R f_\ell \tilde{\eta}_R \int_{R}^{r} \frac{\partial_{\ell' r'} q_{11}(r')}{q_{1}(r')} \frac{q_{\ell \ell'}(r')}{R^2 q_\ell(R)} \frac{Y_{\ell' \ell'}^*(\theta(r'), \phi)}{\cos \theta(r')} dr' + \Phi_2 \left[ \sqrt{-r^2 q_1(r)} \sin \theta, \phi, t \right] , \]

where \( r' \) is the integration variable. In order to solve this integral, the function \( \theta(r') \) is expressed in terms \( r' \) and a constant \( \varphi_2 \) through the characteristic equation (B6) and, after performing the integration, \( \varphi_2 \) is removed again using (B6). The unknown function \( \Phi_2 \) is determined using the boundary condition for \( \Phi_2 |_{r=R} \) given by (B13). Once the integral on the right-hand side of (B13) has been evaluated, we then express all of the trigonometric functions resulting from the integral, in terms of \( \sin \theta \).

This \( \theta \) is the value at \( r = R \). To obtain an expression for the value of \( \Phi_2 \) at a general radius, we write this \( \theta \) (at \( r = R \)) in terms of the value of \( \theta \) at a general point (with \( r > R \)) using the characteristic relation (B6), i.e.,

\[ \sin \theta \to \frac{\sqrt{r^2 q_1(r)}}{[R^2 q_1(R)]} \times \sin \theta , \]

so that

\[ \cos \theta \to \left[ 1 - \frac{r^2 q_1(r)}{R^2 q_1(R)} \sin^2 \theta \right]^{1/2} \sin \theta , \]

where \( \text{sign}(x) \) is defined such that \( \text{sign}(x) = +1 \) if \( x > 0 \), and \( \text{sign}(x) = -1 \) if \( x < 0 \). There are then different expressions for \( \Psi_{SC} \) in the two regions \( \theta \in [0, \pi/2] \) and \( \theta \in [\pi/2, \pi] \). If these two expressions do not coincide at the equatorial plane for \( r > R \), then there will be a discontinuity in \( \Psi_{SC} \) at \( \theta = \pi/2 \), and quantities that depend on \( \partial_\ell \Psi_{SC} \) will become singular there. As shown by TBS, the function \( \Psi_{SC} \) is indeed discontinuous at \( \theta = \pi/2 \) for some oscillation modes and, as we discussed in Section 2.3 above, this unphysical behaviour indicates that the LCDA ceases to be valid for those modes. In these cases, the accelerating electric field cannot be canceled without presence of strong currents which may become as large as (36) in some regions of the magnetosphere. The occurrence of such singularities was explained by TBS and the reader is referred to Section 3.2 of their paper for a detailed discussion.

Next we discuss how GR effects contribute to our solution. As discussed above, for a given magnetic moment \( \mu \) (as measured by a distant observer) the strength of the unperturbed magnetic field near to the surface of the NS is larger in GR than in Newtonian theory. Due to the linearity of the Maxwell equations, a perturbation of a stronger magnetic field should produce a larger electric field for the same oscillation parameters. This in turn should lead to a larger absolute value of the SC density in GR, since the SC density takes the value necessary to cancel the electric field. In the next Section, we will give a more quantitative analysis of the GR contribution in our solution.

We point out that the function \( \Psi_{SC} \) does not depend on \( \ell > 1 \) perturbations to the stellar magnetic field in the case of axisymmetric \( (m' = 0) \) toroidal modes. This can be seen from the fact that these perturbations are confined within the \( \delta S \) terms which enter equation (78) for the function \( \Psi_{SC} \) only through a derivative with respect to \( \phi \); hence vanish for the axisymmetric modes. Therefore, the only perturbation to the magnetic field is due to the \( \ell = 1 \) term, and the solution for these modes is much simpler than that for non-axisymmetric \( (m' \neq 0) \) modes. It is then convenient to discuss separately the axisymmetric and non-axisymmetric cases.

The solution (81) for case the \( m' = 0 \) modes at \( r = R \) has the following form

\[ \Psi_{SC} (r, \theta, \phi, t) |_{r=R} = -B_0 R f_\ell \tilde{\eta}_R \int_0^\theta \cos \theta \partial_\theta Y_{\ell m}(\theta, \phi) \, d\theta . \]

Using the properties of the spherical harmonics, we can express \( \Psi_{SC} (r, \theta, \phi, t) |_{r=R} \) for odd \( \ell' \) modes in the general form

\[ \Psi_{SC} (r, \theta, \phi, t) |_{r=R} \sim \sum_{n=1}^{N} A_{2n} \sin^{2n} \theta , \]

while for even \( \ell' \) modes, the general form of \( \Psi_{SC} \) is

\[ \Psi_{SC} (r, \theta, \phi, t) |_{r=R} \sim (A + B \cos \theta) \sum_{n=1}^{N} A_{2n} \sin^{2n} \theta , \]

where the coefficients \( A \), \( B \) and \( A_{2n} \) do not depend on \( r \) and \( \theta \). The value of \( N \) equals \( \ell'/2 + 1 \) for even \( \ell' \) and \( (\ell' + 1)/2 \) for
odd $\ell'$. As we discussed above, in order to obtain the solution for $\Psi_{SC}$ for $r > R$, one has use $\sin \theta \rightarrow \sqrt{r^2 q_1 (r)/R^2 q_1 (R)} \sin \theta$ on the right-hand sides of (\ref{eq:PsiGR}) and (\ref{eq:PsiNewt}). Thus the GR effects contribute to the solution for $m'=0$ modes only through terms $f_R [r^2 q_1 (r)/R^2 q_1 (R)]^n$, where $n \geq 1$. Note that the factor $f_R$ in this term appears due to the boundary condition at the surface of the star, namely from the continuity of the tangential components of the electric field, while the factor $[r^2 q_1 (r)/R^2 q_1 (R)]^n$ appears due to the presence of charged particles in the magnetosphere. The second factor is equal to 1 at the stellar surface and approaches its Newtonian value $(R/r)^n$ at large $r$ and small $M/R$. Since $f_R > 1$, the absolute value of $\Psi_{SC}$ should be greater in GR than in Newtonian theory. For example, in the case of small $\theta$, the only term which is important is that with $n = 1$ and hence we get $(\Psi_{SC})_{GR}/(\Psi_{SC})_{Newt} = f_R r^3 q_1 (r)/R^3 q_1 (R)$). This quantity is shown in Figure 1 (left panel), where we can see that, near to the stellar surface, the function $\Psi_{SC}$ is larger in GR than in Newtonian theory, while at larger $r$ it asymptotically approaches its Newtonian value.

Analysis of the GR contribution to the solution in the case of non-axisymmetric modes is more complicated because in this case the solution depends not only on $\ell = 1$ perturbations to the magnetic field but also on $\ell > 1$ perturbations, which are contained in the term $\partial_t \partial \phi \partial \phi S$ of equation (\ref{eq:PsiGR}), and contribute to the solution due to the integral in (\ref{eq:PsiGR}). Nevertheless, some rough estimates of the GR effects can be made in the following way. Near to the stellar surface the integral in (\ref{eq:PsiGR}) can be approximated as

$$f_R \int_{r/R} \partial_t \partial \phi \partial \phi S_{\text{GR}} \sim f_R r^2 q_1 (r)/R^2 q_1 (R) \frac{Y_{\text{lm}}(\theta)}{\cos \theta} \sin \theta.$$

Close to the star, $(r^2 q_1 (r))/R^2 q_1 (R) \simeq 1$ and so the leading GR contribution comes from the factor $f_R$ which increases the absolute value of $\Psi_{SC}$ with respect to the Newtonian case. Further away from the star, $r^2 q_1$ is approximately proportional to $r^{-\ell}$ and so the integral in (\ref{eq:PsiGR}) can be approximated as $\sim f_R (R/r)^{\ell-1+m}/2$ to leading order in $R/r$. Therefore, while this integral makes an important contribution to $\Psi_{SC}$ near to the star, it becomes negligibly small for $\ell > 1$ at $r > R$ as compared with $\Phi_Q$. The GR effects contribute to $\Phi_Q$ through the terms $f_R [r^2 q_1 (r)/R^2 q_1 (R)]^n$ in a similar way to their contribution to $\Psi_{SC}$ for the axisymmetric modes discussed above. This increase of the function $\Psi_{SC}$ due to the GR effects also lead to an increase in the absolute values of the SC density $\rho_{SC}$ near to the star, as shown in Figures 1 (right panel) and 2 for some toroidal oscillation modes.

5 ENERGY LOSSES

It was shown by TBS that the kinetic energy of the stellar oscillations should be lost through being passed to plasma near to the stellar surface which then flows out along the open magnetic field lines. Note that within the framework of the TBS model, electromagnetic fields are considered only in the near zone and so the existence of the plasma outflow cannot be shown explicitly; however, qualitatively, the mechanism for the plasma outflow should be the following. The charged particles that were accelerated to high energies by the longitudinal electric field move along the magnetic field lines in the near zone. If the kinetic energy density of the plasma at the equator becomes comparable to the energy density of the magnetic field at some point, then the field line which crosses the equator at that point becomes open. Plasma flowing along open field lines forms an electromagnetically driven wind which closes at infinity. There is then an electric current flowing along the stellar surface.
between positive and negative emission regions. Because this current must cross the magnetic field lines at the stellar surface, it exerts a braking torque on the NS oscillations and thus reduces their kinetic energy (see Section 3.2.2 of TBS for more details).

In the following, we carry out a GR calculation of the energy lost by toroidal stellar oscillations due to plasma outflow. First we calculate the energy losses due to the outflow of a particle along an open field line from a given point on the stellar surface. For this purpose, we start by considering the motion of the charged particle along the $\theta$-direction on the stellar surface (where it crosses the magnetic field lines, hence exerts a braking torque on the stellar oscillations). The equation of motion for a test particle of mass $m$ in a generic electromagnetic field has the general form (Landau & Lifshitz 1987)

$$m \frac{Dw^\alpha}{d\tau} = eF^{\alpha\beta}w_\beta,$$

(88)

where $D/d\tau$ is a comoving derivative, $w^\alpha$ is the four-velocity of the particle given by

$$w_\alpha = \frac{u_\alpha + v_\alpha}{\sqrt{1 - v^2}},$$

(89)

$u^\alpha$ is the 4-velocity of the static observer (10), and $v^\alpha$ is the velocity of the particle relative to the static observer.

Because of time-invariance, there exists a timelike Killing vector $\xi^\alpha$ such that $\xi^\alpha \xi_\alpha = -N^2$. The four-velocity of the static observer can be expressed in terms of $\xi^\alpha$ as $u^\alpha = N^{-1} \xi^\alpha$, and therefore the energy of the particle is given by

$$E = -p_\alpha \xi^\alpha = -mw_\alpha \xi^\alpha.$$

(90)

Contracting the equation of motion (88) with the Killing vector $\xi^\alpha$ gives

$$\xi_\alpha mw_\beta w^\beta = -eF^{\alpha\beta} \frac{u_\alpha + v_\alpha}{\sqrt{1 - v^2}} \xi_\beta,$$

(91)

The right-hand side of this can be rewritten as

$$-eF^{\alpha\beta} \frac{u_\alpha + v_\alpha}{\sqrt{1 - v^2}} \xi_\beta = eF^{\alpha\beta} \frac{v_\beta \xi_\alpha}{\sqrt{1 - v^2}} = eF^{\alpha\beta} N \frac{u_\beta v_\alpha}{\sqrt{1 - v^2}} = e \frac{E^\alpha v_\alpha N}{\sqrt{1 - v^2}},$$

(92)

while the left-hand side can be transformed as

$$\xi_\alpha mw_\beta w^\beta = m(w_\alpha \xi^\alpha)_\beta w^\beta - m\xi_\alpha w^\alpha w^\beta,$$

(93)

The second term on the right-hand of this equation vanishes due to antisymmetry of the tensor $\xi_{\alpha;\beta}$. Therefore, the projection of the equation of motion onto the Killing vector can be written as

$$\frac{dE}{d\tau} = eN \frac{E^\alpha v_\alpha N}{\sqrt{1 - v^2}}.$$  

(94)

For particles moving along the $\theta$-direction, this equation takes the form

$$dE = \frac{E^\alpha RN^2}{\sqrt{1 - v^2}} d\theta.$$  

(95)
Integrating this over the interval \((0, \theta)\), we get
\[
\Delta E = cRN^2_R \int_0^\theta E^\phi d\theta.
\] (96)

Note that in deriving this last equation, we have used the fact that \(v \ll 1\). The quantity \(\Delta E\) measures the energy that would be carried away by a particle that leaves the surface of the star from a point with coordinates \((R, \theta, \phi)\). In order to calculate the energy loss per unit time through a given surface element \(dS\), \(\Delta E\) needs to be multiplied by the the current density, \(j^\mu\) and integrated over the surface element \(n_\alpha dS\), where \(n_\alpha\) is the unit spacelike vector orthogonal to the surface \(r = R\). The energy loss per unit time due to plasma emission through the surface element \(dS\) is then given by
\[
dL = \Delta E(\theta, \phi) j^\phi (R, \theta, \phi) n_\phi \Delta r \sin \theta d\theta d\phi,
\] (97)
while the total energy loss \(L\) from the NS is obtained by integrating \(dL\) over the entire open field line region on the stellar surface.

Now we determine the angle \(\theta_0\) at which the last closed magnetic field line intersects the stellar surface. Following TBS, we define the last closed line as being that for which the kinetic energy density of the outflowing plasma at the equator becomes equal to the corresponding energy density of the NS magnetic field. We now derive a mathematical condition for this. The energy-momentum tensor of the electromagnetic field is \(\text{[Landau & Lifshitz, 1987]}\)
\[
T^{\alpha\beta}_{\text{em}} = 4\pi \left( F^{\alpha\sigma} F_{\sigma\beta} - \frac{1}{4} \delta^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right).
\] (98)

Using this expression, one can obtain an expression for the energy density of the electromagnetic field
\[
\varepsilon_{\text{em}} = N T^{\alpha\beta} u_\alpha u_\beta = N \frac{B^2 + E^2}{8\pi}.
\] (99)

The method for the calculation of the last closed field line proposed by TBS [equation (65) of TBS] is based on an implicit assumption that the plasma flows out from the star isotropically and its kinetic energy is distributed uniformly over the surface of a sphere with radius \(r = R_\alpha\), where \(R_\alpha\) is the radial coordinate of the point where the last closed field line crosses the equatorial plane. However, according to the definition of the last closed field line, the outflowing plasma should move along the field lines throughout the region \(r < R_\alpha\) and so its energy cannot be distributed uniformly over the sphere \(r = R_\alpha\). In the following, we derive an alternative formula for the calculation of \(\theta_0\) that takes into account the correction due to the anisotropic plasma outflow along the magnetic field lines.

If the stellar magnetic field is dipolar, then the density of the field lines decreases with distance from the centre of the star. Since the charged particles move along the field lines, this implies that the kinetic energy density of a comoving element of plasma should decrease monotonically with \(r\) as the field lines separate (we are assuming here that the plasma speed along the field lines is roughly constant). Since for dipole magnetic field lines \(f_r \sin^2 \theta/r = \text{const.}\), it is easy to show the energy density of the plasma when it reaches the equatorial plane is smaller by a factor of 0.5\(R_\alpha^4/(R_\alpha f_r)\) compared to its value at the stellar surface (assuming that the angle \(\theta\) is small). Therefore, the kinetic energy density of the outflowing plasma at the point \((R_\alpha, 0, \phi)\) is given by
\[
\varepsilon_{pl}(R_\alpha, 0, \phi) = \frac{1}{2f_r N R_\alpha R_\alpha^3} \Delta E j^\phi n_\phi,
\] (100)
where \(N R_\alpha = \sqrt{1 - 2M/R_\alpha}\) and the factor \(N R_\alpha^5\) accounts for the gravitational redshift of the energy of the plasma. The energy density of the magnetic field lines at the same point is determined from (99) and has the following form
\[
\varepsilon_{em}(R_\alpha, 0, \phi) = \frac{N R_\alpha R_\alpha^4}{32\pi R_\alpha^5 B_{\text{em}}}.
\] (101)

Note that \(R_\alpha \gg R\) for small \(\theta_0\), and so in this case \(N R_\alpha \approx 1\) to good accuracy; from here on we will consider only the case of small \(\theta_0\) and take \(N R_\alpha = 1\). Also \(j^r \gg j^\theta\) for small \(\theta\), and so in the following we will neglect the \(\theta\) component of the current above the polar cap. The last closed field line is determined by the conditions
\[
\varepsilon_{pl}(R_\alpha, 0, \phi) = \varepsilon_{em}(R_\alpha, 0, \phi).
\] (102)
Substituting (101) and (100) into (102), one can obtain an algebraic equation
\[
16\pi N R \Delta E \rho_{\text{em}} = f_r^4 R_\alpha^2 \theta_0^6
\] (103)
for determining \(\theta_0\) after expressing \(\Delta E\) in terms of \(\theta_0\). Once we know \(\theta_0\), we can calculate the total energy of the outflowing plasma by integrating (97) over the entire open field line region of the stellar surface:
\[
L = \int_0^{2\pi} d\phi \int_0^{\theta_0} d\theta \left| j^\phi (R, \theta, \phi) \Delta E(\theta, \phi) \right| R^2 \sin \theta.
\] (104)
The motion of charged particles in strong magnetic fields can be approximated as a relativistic motion along the field lines. This is a reasonable approximation because the electrons are estimated to become relativistic at a height of a few centimeters above the stellar surface. Therefore in the frame of the static observer we take Muslimov & Tsygan (1992)

\[
j^\ell = \rho \mathbf{B}^\ell \tag{105}\]

where \(\rho(R, \theta, \phi)\) is the SC density.

TBS solved the Newtonian version of (103) and calculated \(L\) for three modes: \((1,1)\), \((2,0)\) and \((3,0)\). They also estimated the order of magnitude of the energy loss \(L\) and the angle \(\theta_0\) for all of the oscillation modes for which \(\theta_0\) is small. They found that \(\theta_0\) is small for modes with \(m' < 3\) and, as we will see later, this result also holds in GR. We will next solve the relativistic equation (103) and calculate \(\theta_0\) and \(L\) for all of the modes \((\ell', m')\) for which the angle \(\theta_0\) is small. We will present elsewhere a study of the case for large \(\theta_0\).

In order to solve equation (103) and calculate \(L\) in the case of small \(\theta_0\), we use the following procedure. First, we approximate equation (102) and the boundary condition (111) for small \(\theta\) by expanding all of the functional dependence on \(\theta\) in a Taylor series, taking into account only terms of the two lowest orders in \(\theta\). Secondly, we solve this equation and calculate \(\rho_{\text{SC}}\) and \(\Delta \mathcal{E}\) using the approximate solution for \(\Psi_{\text{SC}}\). Then, we calculate the angle \(\theta_0\) by substituting \(\rho_{\text{SC}}\) and \(\Delta \mathcal{E}\) into (103). Finally, substituting the expressions for \(\theta_0\), \(\rho_{\text{SC}}\) and \(\Delta \mathcal{E}\) into (103), gives us the expression for \(L\). Since the methods for solving equation (102) for \(m' = 0\) and \(m' \neq 0\) exploit a similar technique, we present just the solution for \(m' = 0\) below, in this Section, while the solution for non-axisymmetric modes is presented in Appendix C.

For small angles \(\theta\), the spherical harmonic \(Y_{\ell m}(\theta, \phi)\) can be approximated by the sum of the first two lowest order terms in the expansion in terms of \(\theta\)

\[
Y_{\ell m}(\theta, \phi) \approx A^{(1)}_{\ell m}(\phi) \theta^m + A^{(2)}_{\ell m}(\phi) \theta^{m+2}. \tag{106}\]

In this case, the characteristics (102) take the form

\[
\varphi = \sqrt{-r^2 q_1(r)} \theta. \tag{107}\]

Specialising to small \(\theta\) in the boundary condition (111) for \(\Psi_{\text{SC}}\), gives the following expression:

\[
\Psi_{\text{SC}}|_{\varphi=0} = -B_0 R f_R \tilde{\eta}_R A^{(2)}_{\ell m} \theta^2. \tag{108}\]

In order to calculate \(\Psi_{\text{SC}}\) for arbitrary \(r\), one has to replace \(\theta\) in this equation by \(\sqrt{[r^2 q_1(r)]/[R^2 q_1(R)]} \theta\), following the argument given earlier for \(\sin \theta\), and this then gives the solution:

\[
\Psi_{\text{SC}} = -B_0 R f_R \tilde{\eta}_R \frac{r^2 q_1(r)}{R^2 q_1(R)} A^{(2)}_{\ell m} \theta^2. \tag{109}\]

Substituting this into formula (106), we obtain the expression for the SC density:

\[
\rho_{\text{SC}} = \frac{B_0 R f_R \tilde{\eta}_R}{\pi N_R R} \frac{r^2 q_1(r)}{R^2 q_1(R)} A^{(2)}_{\ell m}. \tag{110}\]

Substituting \(\Psi_{\text{SC}}\) as given by (109) into (112), we obtain the expression for electric field \(E_{\text{SC}}^\phi\). Substituting this into (106) gives \(\Delta \mathcal{E}\):

\[
\Delta \mathcal{E}(\theta, \phi) = B_0 R f_R \tilde{\eta}_R N_R A^{(2)}_{\ell m} \theta^2. \tag{111}\]

Using this expression for \(\Delta \mathcal{E}\) in equation (103), we obtain an algebraic equation for \(\theta_0\), which has the following solution:

\[
\theta_0 = 2 N_R^{1/4} \left[ \frac{\tilde{\eta}_R A^{(2)}_{\ell m}}{f_R} \right]^{1/2}. \tag{112}\]

Substituting (111) and (110) into (112), gives us the following expression for the total energy loss:

\[
L_{\ell} = \frac{1}{2} \left[ B_0 R f_R \tilde{\eta}_R A^{(2)}_{\ell m} \right]^2 \theta_0^4, \tag{113}\]

and then using (114) in equation (112), gives

\[
L_{\ell} = 8 N_R \left[ B_0 R \tilde{\eta}_R (A^{(2)}_{\ell m})^2 \right] \theta_0^4. \tag{114}\]

\[\text{2} \quad \text{It should be noted here that, as we mentioned above, TBS uses a slightly different formula for the determination of } \theta_0 \text{ which does not take into account the anisotropy of the plasma outflow.}\]
The energy losses for non-axisymmetric modes are calculated in Appendix C. The angle $\theta_0$ and the energy loss $L$ are given by expressions (C7) and (C9) in that case. Note that determining $\theta_0$ using the direct relativistic extension of the TBS method would give a value that is smaller than (112) by a constant factor of $2^{3/5}$, causing $L$ to be smaller by a factor of 4. For the $m’ = 1$ modes, these correction factors are $(2\pi)^{0.25}$ and $2\pi$ respectively, while for the $m’ = 2$ modes they are $(3\pi)^{0.5}$ and $(3\pi)^{3}$.

We now continue our discussion with analysis of the contribution of the effects of GR to the quantities $\theta_0$ and $L_{\ell\theta}$. The ratio of $\theta_0$ in GR to its Newtonian counterpart for a given oscillation amplitude [i.e., for $(\tilde{\eta}_R)_{GR} = (\tilde{\eta}_R)_{Newton}$] can be obtained using equation (112):

$$\frac{\theta_0^{GR}}{\theta_0^{Newton}} = \frac{N_R^3}{f_R^{3/2}}.$$  

(115)

This ratio is shown in Figure 3 (left panel) plotted as a function of the stellar compactness parameter $M/R$. It can be seen that in GR, the angle $\theta_0$ is smaller than in Newtonian theory (it approaches its Newtonian value as $M/R \to 0$), meaning that GR effects lead to the polar cap being smaller. The reason for this can be understood in the following way. The angle $\theta_0$ is determined from equation (103) and it is convenient to analyze the relativistic effects by looking at this equation. The contribution from the curvature is contained here in three terms: i) the factor $f_0^4$ on the right-hand side, ii) the factor $N_R$ on the left-hand side and iii) a factor $f_0^2$ on the left-hand side that is contained implicitly within the term $\Delta \xi_{polar}$. The first of these three terms accounts for the modification of the geometry of the background dipole magnetic field lines due to the curvature. As we discussed above, the magnetic field lines obey the relation $f_0 \sin \theta/r^2 = \text{const}$, with $f_0 = 1$ in the Newtonian case but $f_0 > 1$ in GR. This means that a magnetic field line with a given energy density at $r = R_0$ on the equatorial plane, crosses the stellar surface closer to the pole in GR than in Newtonian theory. The second relativistic factor accounts for the redshift of the kinetic energy of the plasma due to the displacement from $r = R$ to $r = R_0$. Finally, the third factor is responsible for the amplification of the energy density of the outflowing plasma caused by the increase of the magnetic field strength (for a given magnetic moment) due to GR effects. The last of these would obviously lead to an increase of $\theta_0$ in GR if it were acting alone. However, this effect is counteracted by the change of the geometry of the magnetic field lines and the gravitational redshift and those two effects are substantially stronger, giving the result that the angle $\theta_0$ is smaller in GR than in Newtonian theory. Note that, as expected, the ratio $(\theta_0)^R / (\theta_0)^{Newton}$ for the axisymmetric toroidal modes does not depend on $\ell’$, while for the general $m’ \neq 0$ modes, there is dependence on $\ell’$. This is because, as discussed above, the function $\Psi_{SC}$ for $m’ = 0$ depends only on the lowest $\ell = 1$ perturbation to the magnetic field since the higher $\ell$ perturbations are contained in the term $\partial_\theta \partial_\phi S$, which disappears in axisymmetry.

The ratio of the angle $\theta_0$ in GR to its Newtonian counterpart, for $m’ \neq 0$ modes, can be obtained using equation (C7) (see Appendix C for details of the derivation):

$$\frac{(\theta_0)^{GR}}{(\theta_0)^{Newton}} = \left(\frac{N_R}{f_R^2}\right)^{1/2} \left(\frac{|D^\ell \ell'|_{GR}}{|D^\ell \ell'|_{Newton}}\right)^{1/4 - 2m’}.$$  

(116)

Figure 3 (central and right panels) shows the ratio of angle $\theta_0$ in GR to its Newtonian equivalent for the modes $m’ = 1$, 2 for several values of $\ell’$. As can be seen in these plots, $\theta_0$ is smaller in GR than in Newtonian theory also for the $m’ = 1$ and $m’ = 2$ modes. The reason for this is similar to that for the $m’ = 0$ modes which we discussed above. We have made the analysis for values of $\ell’$ up to $\ell’ = 10$ and the ratio $(\theta_0)^{GR} / (\theta_0)^{Newton}$ was found to be rather insensitive to the values of $\ell’$. (The dependence on $\ell’$ is due to the fact that $(\rho_{polar})_{GR} / (\rho_{polar})_{Newton}$ is generally larger for higher $\ell’$ at small $\theta_0$.)

The ratio of the energy losses for the $m’ = 0$ modes in GR and in Newtonian theory is equal to $N_R$, as one can obtain using equation (114), and this is plotted in the left panel of Figure 3. The GR modification is caused by the gravitational redshift of the plasma energy; the other “magnetic” GR effects do not influence this, because the shrinking of the polar cap and the increase in the plasma energy density exactly compensate each other.

The ratio of the energy losses for the $m’ \neq 0$ modes in GR and in Newtonian theory can be straightforwardly obtained using equation (C9):

$$\frac{(L^\ell \ell’)^{GR}}{(L^\ell \ell’)^{Newton}} = f_0^2 N_R^{3/2} \left(\frac{|D^\ell \ell’|_{GR}}{|D^\ell \ell’|_{Newton}}\right)^{1/4 - 2m’}.$$  

(117)

This is also shown in Figure 4 where one can see that the energy loss of the $m’ \neq 0$ modes is smaller in GR than in Newtonian theory. The reason for this is the same as for the $m’ = 0$ modes discussed above, i.e., the increase in the energy density of the outflowing plasma cannot compensate the shrinking of the polar cap. Moreover, for the $m’ = 2$ modes, the energy density of the outflowing plasma is proportional to $\theta^{2m’}$ (see eqs. (10) and (38)) and so the total energy losses are very sensitive to the size of the polar cap. Because of this, the ratios of the total energy losses for the $m’ = 2$ modes in GR and in Newtonian theory are much smaller than those for the $m’ = 0$ and $m’ = 1$ modes.
6 SUMMARY

In this paper, we have described our general relativistic model for the force-free magnetosphere of an oscillating, non-rotating neutron star. Our approach is based on the previous Newtonian model developed by TBS and focuses on toroidal modes which are thought to be particularly relevant for magnetar QPO phenomena. We have taken the spacetime geometry to be spherically symmetric and have neglected any modifications of it caused by the electromagnetic fields, the stellar oscillations and the magnetospheric plasma. Within this context, we have derived the general relativistic Maxwell equations for arbitrary small-amplitude oscillations of a neutron star with a generic magnetic field configuration and have shown that, as in the Newtonian case, they can be solved analytically for the force-free configuration of the electromagnetic fields (i.e. $E_{||} \ll B$) under the low current density approximation (LCDA). We have applied our formalism to small-amplitude toroidal oscillations of a neutron star with a dipole magnetic field and have found that the LCDA is valid for at least half of these modes in GR, as in the Newtonian calculations of TBS. We have also discussed the contribution of the GR effects to our solution, finding that they lead to an increase in the absolute values of the electromagnetic fields and the space charge (SC) density near to the stellar surface.

We have calculated the energy losses due to plasma outflow resulting from these oscillations, focusing on cases where the size of the polar cap is small so that one can expand the Maxwell equations as Taylor series in powers of $\theta$, retaining only the two lowest order terms. This approach leads to a great simplification and allowed us to perform a thorough analysis of the solution. We have found that in GR, the polar cap is smaller than in Newtonian theory and have shown that this is due to the change in the geometry of the dipole magnetic field and the gravitational redshift of the energy of the outflowing plasma. Also, we found that the oscillation modes which have small $\theta_0$, all have $m' < 3$ as in the Newtonian case.

The total energy loss resulting from the stellar oscillations causing plasma outflow through the polar cap region, is
determined through an integral over the whole polar cap area, and so it depends on both the kinetic energy density of the
outflowing plasma and the surface area of the polar cap. Although GR effects lead to some increase in the energy density of
the outflowing plasma (due to the increase in the surface magnetic field strength for a given magnetic moment), the area of
the polar cap is smaller in GR and we have found that the increase in the energy density of the outflowing plasma cannot
compensate for the shrinking in size of the polar cap. Therefore the total energy losses for the toroidal oscillation modes are
significantly smaller in GR than in Newtonian theory.

In conclusion, we point out that while our calculations represent an advance with respect to previous ones, they still do
not include a number of very important aspects which would be necessary for a realistic description of these phenomena. Most
importantly, as noted above, they do not take account of electric currents flowing in the magnetosphere. Inclusion of these in a
consistent way would require solving a version of the non-linear “pulsar equation” (Michel 1973) for oscillating neutron stars,
possibly by adopting a numerical approach similar to that of Contopoulos et al. (1999) (see also Gruzinov 2005; Timokhin
2006) or performing time-dependent simulations of the magnetosphere (Spitkovsky 2006; Komissarov 2006; McKinney 2006).
This will be the subject of future investigations.

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REFERENCES
Beskin V. S., 1990, Soviet Astron. Lett., 16, 286
Beskin V. S., Gurevich A. V., Istomin Ya. N., 1993, Physics of the Pulsar Magnetosphere, Cambridge University Press
Beskin V. S., 2005, ”Oesimmetrichnye Stacionarnye Techeniya v Astrofizike” (Axisymmetric Stationary Flows in
Astrophysics), Moscow, Fizmatlit (in Russian)
Bisnovatyi-Kogan G. S., 1995, ApJS, 97, 185
Bocquet, M., Bonazzola, S., Gourgoulhon, E., Novak, J., 1995, A & A, 301, 757
Bonazzola S., Gourgoulhon E., 1996, A & A, 312, 675
Boriakoff V., 1976, ApJ, 208, L43
Cardall C., Y., Prakash M., Lattimer J. M., 2001, Apj, 554, 322
Colaiuda A., Ferrari V., Gualtieri L., Pons J., 2008, MNRAS, 385, 2080
Contopoulos I., Kazanas D., Fendt C., 1999, ApJ, 511, 351
Courant R., Hilbert D., 1962, Methods of Mathematical Physics, John Wiley & Sons
Ding K. Y., Cheng K. S., 1997, MNRAS, 287, 671
Duncan R. C., 1998, ApJ, 498, L45
Ginzburg V. L., Ozernoy L. M., 1964, Zh. Eksp. Teor. Fiz., 47, 1030
Glampedakis K., Samuelsson L., Andersson N., 2006, MNRAS, 371, L74
Goldreich P., Julian W. H., 1969, ApJ, 157, 869
Gruzinov A., 2005, PRL, 94, 021101
Hasskell B., Samuelsson L., Glampedakis K., Andersson N., 2008, MNRAS, 385, 531
Israel G., et al, 2005, ApJ, 628, L53
Komissarov S. S., 2006, MNRAS, 367, 19
Landau L., Lifshitz E. M., 1987, The Classical Theory of Fields, 4th ed., Pergamon Press, Oxford
Levin Y., 2007, MNRAS, 377, 159
McKinney J. C., 2006, MNRAS, 368, L30
Michel, F. C., 1973, ApJ, 180, L133
Michel F. C., 1991, Theory of Neutron Star Magnetospheres, The University of Chicago Press
Mofiz U. A., Ahmedov B. J., 2000, ApJ, 542, 484
Morozova V. S., Ahmedov B. J., Kagransanova V. G., 2008, ApJ, in press [arXiv:0806.2370]
Muslimov A. G., Tsygan A. I., 1992, MNRAS, 255, 61
Pacini F., Ruderman M., 1974, Nat., 251, 399
Pons J. A., Geppert U., 2007, A&A, 470, 303
Rezzolla L., Ahmedov B. J., 2004, MNRAS, 352, 1161
Rezzolla L., Ahmedov B. J., Miller J. C., 2001a, MNRAS, 322, 723; Erratum 338, 816 (2003)
Ruderman M., Sutherland P. G., 1975, ApJ, 196, 51
Samuelsson L., Andersson N., 2007, MNRAS, 374, 256
Schwartz S. J., Zane S., Wilson, R. J., et al., 2005, ApJ, 627, L129
Shapiro, S. L., Teukolsky, S. A., 1983, Black holes, White Dwarfs and Neutron Stars. Wiley, New York
Sotani H., Kokkotas K. D., Stergioulas N., 2007, MNRAS, 375, 261
Sotani H., Kokkotas K. D., Stergioulas N., 2008, MNRAS, 385, L5
Spitkovsky A., 2006, ApJ, 648, L51
Strohmayer T. E., 2008, AIPC, 968, 85S
Strohmayer T. E., Watts A. L., 2006, ApJ, 653, 593
Sturrock P. A., 1971, ApJ, 164, 529
Thompson C., Duncan R. C., 1995, MNRAS, v275, 255
Thompson C., Duncan R. C., 2001, ApJ, 561, 980
Timokhin A. N., Bisnovatyi-Kogan G. S., Spruit H. C., 2000, MNRAS, 316, 734
Timokhin A. N., 2006, MNRAS, 368, 1055
Timokhin A. N., 2007, Ap & SS, 308, 345
Timokhin A. N., 2007, Ap & SS, 308, 345
Tsygan A. I., 1975, A&A, 44, 21
Unno W., Osaki Y., Ando H., Sato H., Shibahashi H., 1989, Nonradial Oscillations of Stars, 2nd ed., University of Tokyo Press
Watts A. L., Strohmayer T. E., 2006, ApJ, 637, L117
Watts A. L., Strohmayer T. E., 2007, AdSR, 40, 10, 1446
Woods P. M., Thompson C., 2006, in Lewin W. H. G., van der Klis M., eds, Compact Stellar X-ray Sources, Cambridge Univ. Press, Cambridge

APPENDIX A: CALCULATION OF $\delta S$

The derivative with respect to $\phi$ of the right hand side of equation (64) must be equal to the derivative with respect to $\theta$ of the left-hand side of (65). Using this condition and expression (49) for $S$, we obtain

$$
\Delta \Omega (\partial_t \delta S)|_{r=R} = - \left[ (N\delta v^\ell \partial_\ell \Delta \Omega S_0 + \frac{\delta v^\ell}{r} \partial_\ell \Delta \Omega S_0 + \frac{\delta v^\ell}{r \sin \theta} \partial_\ell \Delta \Omega S_0) 
+ \frac{\Delta \Omega S_0}{r \sin \theta} \left( \partial_\ell \left( \sin \theta \delta v^\ell \right) + \partial_\ell \delta v^\ell \right) + N \partial_\ell \partial_\theta S_0 \partial_\ell \delta v^\ell + \frac{N}{\sin^2 \theta} \partial_\ell \partial_\theta S_0 \partial_\ell \delta v^\ell \right] \right|_{r=R}. \quad (A1)
$$

We now introduce a new function for shorthand:

$$
q_\ell(r) = -M \frac{\partial}{\partial r} \left[ r \left( 1 - \hat{r} \frac{2M}{r} \right) \frac{d}{dr} Q_\ell \left( 1 - \frac{r}{M} \right) \right]. \quad (A2)
$$

Using this notation, one can rewrite equations (57) and (55) in the following way

$$
\delta S_{\ell m}(t) = \frac{r^2 q_\ell(r)}{M^3} \delta S_{\ell m}(t), \quad (A3)
$$

$$
\delta S(t, r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{r^2 q_\ell(r)}{M^3} \delta S_{\ell m}(t) Y_{\ell m}(\theta, \phi). \quad (A4)
$$

Substituting the right hand side of the last equation into the left hand side of (A1), we get

$$
\frac{r^2}{M^3} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} q_\ell(r) \partial_\ell \delta S_{\ell m}(t) \ell(\ell + 1) Y_{\ell m}|_{r=R} = \left[ (N\delta v^\ell \partial_\ell \Delta \Omega S_0 + \frac{\delta v^\ell}{r} \partial_\ell \Delta \Omega S_0 + \frac{\delta v^\ell}{r \sin \theta} \partial_\ell \Delta \Omega S_0) 
+ \frac{\Delta \Omega S_0}{r \sin \theta} \left( \partial_\ell \left( \sin \theta \delta v^\ell \right) + \partial_\ell \delta v^\ell \right) + N \partial_\ell \partial_\theta S_0 \partial_\ell \delta v^\ell + \frac{N}{\sin^2 \theta} \partial_\ell \partial_\theta S_0 \partial_\ell \delta v^\ell \right] \right|_{r=R}. \quad (A5)
$$
The characteristics of this system are

\[ \Psi \]

APPENDIX B: SOLUTION OF THE \( \Psi_{SC} \) EQUATION FOR \( \chi = 0 \)

Substituting the right hand side of equation (A10) and expressions (70) into (B8), we get the following expression for \( \partial_t \delta s_{\ell m}(t) \) for a NS with a dipole magnetic field:

\[ \partial_t \delta s_{\ell m}(t) = \frac{1}{\ell (\ell + 1)} B_0 R f_R M^3 \tilde{\eta}_h \]

\[ \times \int_{4\pi} \left[ \partial_\theta Y_{\ell m} \left( \sin \theta \cos \chi - e^{i\phi} \cos \theta \sin \chi \right) + ie^{i\phi} \partial_\phi Y_{\ell m} \sin \theta \sin \chi \right] \frac{Y^{*}_{\ell m} (\theta, \phi)}{\sin \theta} d\Omega, \]

where \( \tilde{\eta}_h = \eta_h e^{-ir_{\ell m}} \). Substituting this into (A4), we obtain the expression for \( \delta S \) for a toroidal oscillation mode \((\ell', m')\) of a NS with dipole magnetic field:

\[ \partial_t \delta S(r, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{B_0 R f_R \tilde{\eta}_h}{\ell (\ell + 1)} \frac{r^2 q_\ell(r)}{R^2 q_\ell(R)} \int_{4\pi} \left[ \partial_\theta Y_{\ell m} \left( \sin \theta \cos \chi - e^{i\phi} \cos \theta \sin \chi \right) + ie^{i\phi} \partial_\phi Y_{\ell m} \sin \theta \sin \chi \right] \frac{Y^{*}_{\ell m} (\theta, \phi)}{\sin \theta} d\Omega. \]

If the magnetic dipole moment is aligned with the oscillation mode axis \((\chi = 0)\), one can easily show that

\[ \partial_t \delta s_{\ell m}(t) = \frac{im}{\ell (\ell + 1)} B_0 R f_R M^3 \tilde{\eta}_h \]

\[ \times \int_{4\pi} \left[ \partial_\theta Y_{\ell m} \left( \sin \theta \cos \chi - e^{i\phi} \cos \theta \sin \chi \right) + ie^{i\phi} \partial_\phi Y_{\ell m} \sin \theta \sin \chi \right] \frac{Y^{*}_{\ell m} (\theta, \phi)}{\sin \theta} d\Omega. \]

where \( \delta_{\ell \ell' \ell m} \) is the Kronecker tensor.

APPENDIX B: SOLUTION OF THE \( \Psi_{SC} \) EQUATION FOR \( \chi = 0 \)

Substituting the right hand side of equation (A10) and expressions (70) into (B9), we obtain a boundary condition for the function \( \Psi_{SC} \) for toroidal oscillations of a NS, for the case \( \chi = 0 \):

\[ \Psi_{SC} |_{r=R} = -B_0 R f_R \tilde{\eta}_h \int_0^\theta \left[ \cos \theta \partial_\theta Y_{\ell m'}(\theta, \phi) - \frac{m'^2}{\ell' (\ell' + 1)} \frac{Y^{*}_{\ell m'}(\theta, \phi)}{\sin \theta} \right] d\theta, \]

where \( \theta \) is the integration variable. Substituting the expression for \( \partial_t \delta S \) given by (A10) into equation (B8), we obtain an equation for the SC potential, \( \Psi_{SC} \), for a toroidal oscillation mode \((\ell', m')\) of a NS having a dipole magnetic field aligned with the oscillation mode axis:

\[ -2r^2 q_1(r) \cos \theta \partial_\theta \psi_{SC} + \partial_r \left[r^2 q_1(r) \sin \theta \partial_\theta \psi_{SC}\right] - \frac{m'^2}{\ell' (\ell' + 1)} B_0 R f_R \tilde{\eta}_h \partial_r \frac{r^2 q_1(r)}{R^2 q_\ell(R)} Y_{\ell m'}(\theta, \phi) = 0. \]

This is a first-order partial differential equation. According to a well-known theorem from the theory of such equations, (B2) is equivalent to the following system of first-order ordinary differential equations (see, for example, Chapter II of Volume II of Courant & Hilbert [1962] for a thorough discussion):

\[ \frac{d\theta}{2r^2 q_1(r) \cos \theta} = \frac{d\phi}{m'^2 B_0 R f_R \tilde{\eta}_h \partial_r \frac{r^2 q_1(r)}{R^2 q_\ell(R)} Y_{\ell m'}(\theta, \phi)} = \frac{dt}{\partial_\theta \psi_{SC}} = \frac{dr}{\partial_\phi \psi_{SC}} = \frac{d\theta}{0} = \frac{dt}{0}. \]

The characteristics of this system are

\[ \varphi_0 = t, \]

\[ \varphi_1 = \phi, \]

\[ \varphi_2 = \sqrt{-r^2 q_1(r) \sin \theta}, \]

\[ \varphi_3 = \psi_{SC} + \frac{m'^2}{2 \ell' (\ell' + 1)} B_0 R f_R \tilde{\eta}_h \int \frac{\partial_r \frac{r^2 q_1(r)}{q_\ell(r)} q_\ell(r) Y_{\ell m'}(\theta, \phi)}{R^2 q_\ell(R) \cos \theta(r)} dt, \]
where \( \theta(r) \) depends on the variable \( r \) due to (B7):

\[
\theta(r) = \arcsin \left( \frac{\varphi_2}{\sqrt{-r^2 q_1(r)}} \right). \tag{B8}
\]

The integral of equation (B2) is an arbitrary function of \( \varphi_0, \varphi_1, \varphi_2 \) and \( \varphi_3 \):

\[
\Gamma(\varphi_0, \varphi_1, \varphi_2, \varphi_3) = 0. \tag{B9}
\]

Using this, one can obtain an expression for the general solution of equation (B2):

\[
\Psi_{SC} = \left\{ -\frac{1}{2} \frac{m^2}{l}(l'+1) B_0 R f_R \bar{\eta}_R \right\} \int_{R}^{\ell} \frac{\partial_r \left[ r^2 q_1(r) \right]}{\cos \theta(r)} \frac{q_e(r)}{R^2 q_e(R)} \cos \theta(r) \, dr + \Phi_1 \left[ \sqrt{-r^2 q_1(r)} \sin \theta, \phi, t \right], \tag{B10}
\]

where \( \Phi_1 \) is an arbitrary function that has to be determined from the boundary condition (B1). Using (B1), we can obtain a boundary condition for \( \Phi_1 \):

\[
\Phi_1 \left[ \sqrt{-r^2 q_1(r)} \sin \theta, \phi, t \right] \bigg|_{r=R} = \Psi_{SC} \bigg|_{r=R} + \left. \frac{1}{2} \frac{m^2}{l}(l'+1) B_0 R f_R \bar{\eta}_R \right\} \int_{R}^{\ell} \frac{\partial_r \left[ r^2 q_1(r) \right]}{\cos \theta(r)} \frac{q_e(r)}{R^2 q_e(R)} \cos \theta(r) \, dr \bigg|_{r=R}, \tag{B11}
\]

where \( \Psi_{SC} \bigg|_{r=R} \) is given by (B1). Now using equations (B10) and (B11), we can write a final expression for the general solution of equation (B2) in the following form

\[
\Psi_{SC} = \left\{ -\frac{1}{2} \frac{m^2}{l}(l'+1) B_0 R f_R \bar{\eta}_R \right\} \int_{R}^{\ell} \frac{\partial_r \left[ r^2 q_1(r) \right]}{\cos \theta(r)} \frac{q_e(r)}{R^2 q_e(R)} \cos \theta(r) \, dr + \Phi_2 \left[ \sqrt{-r^2 q_1(r)} \sin \theta, \phi, t \right], \tag{B12}
\]

where \( r' \) is the integration variable. The function \( \theta(r) \) under this integral must be substituted by (B8) and, after performing the integration, the function \( \phi_2 \) should be replaced by (B7). The unknown function \( \Phi_2 \left[ \sqrt{-r^2 q_1(r)} \sin \theta, \phi, t \right] \) is determined using boundary condition (B11):

\[
\Phi_2 \left[ \sqrt{-r^2 q_1(r)} \sin \theta, \phi, t \right] \bigg|_{r=R} = -B_0 R f_R \bar{\eta}_R \int_{0}^{\theta} \left[ \cos \vartheta \partial_{\vartheta} Y_{\ell m}(\vartheta, \phi) - \frac{m^2}{l}(l'+1) \frac{Y_{\ell m}(\vartheta, \phi)}{\sin \vartheta} \right] d\vartheta. \tag{B13}
\]

In order to obtain \( \Phi_2 \) for arbitrary \( r \), one has to express the right hand side of this equation in terms of \( \sin \theta \), and then replace \( \sin \theta \) by \( \sqrt{\left| r^2 q_1(r) \right|} / \left| R^2 q_1(R) \right| \times \sin \theta \). We then have our analytical expression for the function \( \Psi_{SC} \) for toroidal oscillations of a NS.

**APPENDIX C: ENERGY LOSSES FOR \( M' \neq 0 \) MODES**

In this Section we solve equation (B2) and calculate \( \theta_0 \) together with the total energy losses for non-axisymmetric toroidal oscillation modes for small \( \theta_0 \). Taking the limit of small \( \theta \) in the boundary condition (B1) for \( \Psi_{SC} \), we obtain the following approximate expression for \( \Psi_{SC} \bigg|_{r=R} \):

\[
\Psi_{SC} \bigg|_{r=R} = -B_0 R f_R \bar{\eta}_R \left\{ \left[ 1 - \frac{m'}{l}(l'+1) \right] A_{l m}^{(1)} \theta^{m'} + \left[ A_{l m}^{(2)} - \frac{m'^2}{l^2} \frac{(A_{l m}^{(1)} / 6 + A_{l m}^{(2)})}{(m'+2)(m+2)} \right] \theta^{m'+2} \right\}. \tag{C1}
\]

Using this result, one can show that the general solution of equation (B2) for small \( \theta \) has the following form

\[
\Psi = -B_0 R f_R \bar{\eta}_R g_1(r) m'/2 \left\{ \frac{m^2}{l}(l'+1) \int_{R}^{\ell} g_1(r) g_1(r)^{m'/2+1} \, dr + \frac{1}{g_1(R)^{m'/2}} \left[ 1 - \frac{m'}{l}(l'+1) \right] A_{l m}^{(1)} \theta^{m'} \right. \\
+ \left[ \frac{m'}{6} \left[ A_{l m}^{(1)} + A_{l m}^{(2)} \right] g_1(r) \int_{R}^{\ell} g_1(r) g_1(r)^{m'/2} \, dr + \frac{m'^2}{l^2} \frac{(A_{l m}^{(1)} / 6 + A_{l m}^{(2)})}{(m'+2)(m+2)(l'+1)} \right] \theta^{m'+2} \right\}, \tag{C2}
\]

where we have introduced a new function \( g_1(r) = r^2 q_1(r) \) for simplicity of notation. Using (14) and (50), we obtain an expression for \( \Delta \varepsilon \):

\[
\Delta \varepsilon = -B_0 R f_R \bar{\eta}_R N_R \int_{0}^{\theta} \cos \theta \partial_{\theta} Y_{\ell m}(\theta) d\theta. \tag{C3}
\]
Substituting the expansion of \( Y_{l'm'} \) for small \( \theta \) given by (106) into (C3), we get the following expression for \( \Delta \mathcal{E} \) for small \( \theta \):

\[
\Delta \mathcal{E} = -B_0 R \tilde{f} \tilde{h}_R N_R \left[ A_{l'm'}^{(1)} \theta^{m'} + \left( A_{l'm'}^{(2)} - \frac{m'}{2(m' + 2)} A_{l'm'}^{(1)} \right) \theta^{m' + 2} \right].
\]

(C4)

Now substituting (C2) into (46), we get an expression for \( \rho_{sc}(R, \theta, \phi) \):

\[
\rho_{sc}(R, \theta, \phi) = -\frac{B_0 \tilde{f} \tilde{h}_R}{\pi N_R R} D_{l'm'}(R, M) \theta^{m'},
\]

(C5)

where we have introduced a new quantity \( D_{l'm'}(R, M) \) for simplicity of notation:

\[
D_{l'm'}(R, M) = \frac{m'}{48} \left\{ 4A_{l'm'}^{(1)}(1 + m') \left[ 1 - \frac{m'}{\ell' (\ell' + 1)} \right] + \frac{3}{\ell' (\ell' + 1)} \left[ 8m'(1 + m') \left[ \frac{A_{l'm'}^{(1)} \ell' m'}{2(2 + m')} + \frac{(A_{l'm'}^{(1)} + 6A_{l'm'}^{(2)}) m'}{6(\ell' + \ell'^2)(2 + m')} \right] - A_{l'm'}^{(2)} \right] + \frac{A_{l'm'}^{(1)}}{g_1(R)^2 g_{l'}(R)} \left[ \partial_r g_1(\ell') g_{l'}(R)(4g_1(R) + (m' - 2)R \partial_r g_1(r) + 2mRg_1(R) \partial_r g_{l'}(r) \right) \right.

\]

\[
+ 2\ell' (\ell' + 1)Rg_1(R)g_{l'}(R) \partial_{r, r'} g_1(r) \right\} \bigg|_{r = R}.
\]

(C6)

Substituting (C4) and (C5) into (104), we obtain an algebraic equation for \( \theta_0 \) for \( m \neq 0 \) toroidal oscillation modes, which has the solution

\[
\theta_0 = \left[ \frac{16 N_R \tilde{\eta}_R^2 |A_{l'm'}^{(1)} D_{l'm'}(R, M)|}{f_R^2} \right]^{\frac{1}{m' + 2}}.
\]

(C7)

We can see from this expression that \( \theta_0 \) is small for modes with \( m' < 3 \), in agreement with the estimate of TBS. Remarkably, \( \theta_0 \) does not depend on \( B_0 \) as in the case of \( m' = 0 \). Substituting (C4), (C5) and (105) into (104) gives

\[
L_{l'm'} = \frac{2}{(m' + 1)\pi} (B_0 R \tilde{f} \tilde{h}_R)^2 |A_{l'm'}^{(1)} D_{l'm'}(R, M)| \theta_0^{2m' + 2}.
\]

(C8)

Replacing \( \theta_0 \) in this equation by the right hand side of equation (C7), we obtain

\[
L_{l'm'} = \frac{B_0^2 R^2 f_R^4 \tilde{\eta}_R^4}{8\pi (m' + 1)} \left[ \frac{16 N_R \tilde{\eta}_R^2 |A_{l'm'}^{(1)} D_{l'm'}(R, M)|}{f_R^2} \right]^{\frac{1}{m' + 2}}.
\]

(C9)