Cooperative Evasion by Translating Targets with Variable Speeds

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Abstract—We consider a problem of cooperative evasion between a single pursuer and multiple evaders in which the evaders are constrained to move in the positive $Y$ direction. The evaders are slower than the vehicle and can choose their speeds from a bounded interval. The pursuer aims to intercept all evaders in a given sequence by executing a Manhattan pursuit strategy of moving parallel to the $X$ axis, followed by moving parallel to the $Y$ axis. The aim of the evaders is to cooperatively pick their individual speeds so that the total time to intercept all evaders is maximized. We first obtain conditions under which evaders should cooperate in order to maximize the total time to intercept as opposed to each moving greedily to optimize its own intercept time. Then, we propose and analyze an algorithm that assigns evasive strategies to the evaders in two iterations as opposed to performing an exponential search over the choice of evader speeds. We also characterize a fundamental limit on the total time taken by the pursuer to capture all evaders when the number of evaders is large. Finally, we provide numerical comparisons against random sampling heuristics.

I. INTRODUCTION

We consider a single pursuer multi-evader pursuit evasion problem in which the aim of the pursuer is to intercept all of the evaders in a fixed given sequence. The evaders are constrained to move along the positive $Y$ direction. The pursuer moves with unit speed. The pursuer follows the Manhattan distance, i.e., moving parallel to the $X$-axis followed by moving parallel to the $Y$-axis. The aim of the evaders is to cooperatively maximize the total time to intercept all evaders. Such a set-up arises in riot control or border protection scenarios in which a ground or air vehicle would like to optimally visit mobile locations headed toward a boundary/asset, or in UAV monitoring of vehicles along a highway. This setup is also applicable in multiple robotic decoy deployment [1].

A. Related work

Since the seminal work by Isaacs in [2], much has been done in the field of pursuit evasion with a lot of focus on multi-agent pursuit evasion [3], [4], [5]. The case of a single pursuer and 2 evaders has been extensively analyzed [6], [7]. Protector-Prey-Predator [8] and Target-Attacker-Defender differential game [9] are some examples of this scenario. With more than two evaders, the complexity of the problem grows exponentially with number of evaders. The problem of successive pursuit with cooperative multiple evaders is considered in [10], [11], [12] and [13]. Our problem differs from [14], [15] as the pursuer follows a fixed strategy and the evaders are constrained to move in a fixed direction and can choose their individual speeds from a bounded interval to maximize the total intercept time. Thus, the evasive strategies are based on the range of evader speeds.

B. Contributions

We consider an optimal evasion problem between a single pursuer and $n$ evaders. The pursuer moves with unit speed. The evaders are constrained to move in the positive $Y$ direction such that their speeds $v_i, i \in \{1, \ldots, n\}$, lie in the interval $[u_{\text{min}}, u_{\text{max}}]$ with $0 < u_{\text{min}} < u_{\text{max}} < 1$. The evaders need to choose their speeds in order to maximize the total intercept time. We first present a complete solution to the optimal evasion problem for $n \leq 2$. We then show, for general $n$, that the optimal choice of the speed for each evader is one of the extremes, i.e., $u_{\text{min}}$ or $u_{\text{max}}$. Further we show that, by enforcing cooperation among evaders, they are able to maximize the total intercept time. In order to implement the cooperative strategies, it is important to determine the conditions under which cooperation is optimal. Such conditions are also provided in this paper. We present an algorithm which assigns the evasive strategies to the evaders in two iterations as opposed to performing an exponential search over the choice of evader speeds. For sufficiently large $n$, for which the global optimum is difficult to compute, we establish a fundamental upper bound to the total intercept time taken by the pursuer to capture all evaders. Finally, we provide comparisons through numerical results.

C. Organization

The paper is organized as follows. Section II comprises the formal problem definition. In section III, we derive an evasive strategy for multiple evaders and provide a Sequential-Greedy-Cooperation algorithm. Section IV establishes a fundamental upper bound on the total time to intercept all evaders. Section V presents the numerical simulations. Finally, section VI summarizes this paper and outlines directions for future work.

II. PROBLEM FORMULATION

We consider an optimal evasion problem played between a single pursuer with simple motion and $n$ mobile evaders. We denote the pursuer as $P$ and evaders as $E_i$, with $i \in \{1, \ldots, n\}$. The pursuer with initial location at $(X,Y)$ is assumed to be moving with unit speed either along the $X$ or the $Y$ axis. We term this pursuit strategy as Manhattan pursuit, and is formally defined as follows.
Definition 1 (Simple Manhattan pursuit) Given initial locations \((x_i, y_i)\) and \((X, Y)\) of an evader \(E_i\) and the pursuer \(P\) respectively, the pursuer

1) moves with unit speed along the positive or negative \(X\) direction until \(X(t) = x_i\) and then,
2) moves with unit speed along positive or negative \(Y\) axis to intercept the evader.

The evaders, initially located at \(\{(x_1, y_1), \ldots, (x_n, y_n)\}\), are constrained to move along the positive \(Y\) direction with simple motion such that their instantaneous speeds \(v_i, i \in \{1, \ldots, n\}\), lie in the interval \([u_{\text{min}}, u_{\text{max}}]\) with \(0 < u_{\text{min}} < u_{\text{max}} < 1\) (Fig. 1). The pursuer is said to intercept the \(i^{th}\) target when its location coincides with that of the \(i^{th}\) target. The game terminates when the pursuer intercepts the last evader. A strategy for an evader \(E_i\) is a measurable function, defined as \(v_i(\{X(t), Y(t)\}, \{x_i(t), y_i(t)\})_{t=1}^n \rightarrow [u_{\text{min}}, u_{\text{max}}]\), where the notation \(\{X(t), Y(t)\}\) denotes the set of all locations \((X(\tau), Y(\tau))\), \(\forall \tau \in [0, t]\). The goal is to solve the following problem.

Problem II.1 (Optimal evasion) Given that the pursuer follows a fixed order to intercept the evaders, determine strategies \(v_1^*, v_2^*, \ldots, v_n^*\) for the evaders that maximize the total time \(T_n\) taken by the pursuer to intercept all \(n\) evaders.

III. EVASIVE STRATEGY

We begin with the case of a single evader followed by two evaders, and then present the more general case. We start by defining the following simple Manhattan pursuit strategy.

A. Single Evader

In this section, we first consider the case of a single evader and a pursuer, located at \((x_1, y_1)\) and \((X, Y)\) respectively. We first present a result on the time taken to intercept a single evader. This will be used in deriving the optimal strategy for the evader. We denote \(\Delta x_i^{t-1} := |x_i - x_{i-1}|\), where \(x_0 = X\) and \(T_i^{t-1}(v_i)\) as the time taken by the pursuer to intercept evader \(i\) moving with speed \(v_i\) after intercepting evader \(i-1\). Specifically, \(T_i^0(v)\) is the time taken to intercept the first evader moving with speed \(v\). For brevity, we omit the proofs for the single evader case as they can be derived by following the steps in the proof of the general case presented later.

Lemma III.1 (Time to intercept a single evader) The time \(T_i^0(\nu)\) taken by \(P\) to intercept \(E_i\) is

\[
T_i^0(\nu) = \begin{cases} 
\frac{\Delta x_i^0 + y_i - Y}{1 + \nu}, & \text{if } \Delta x_i^0 > (Y - y_i)/\nu, \\
\frac{\Delta x_i^0 + y_i - Y}{1 + \nu}, & \text{otherwise}.
\end{cases}
\]

Lemma III.2 (Monotonicity of time to intercept) The time \(T_i^0(\nu)\) is a monotonically increasing function of \(\nu\) if \(\Delta x_i^0 > 1 + y_i - Y\). Otherwise, \(T_i^0(\nu)\) is a monotonically decreasing function of \(\nu\).

Remark 1 The time to intercept is monotonic even when the pursuer follows a Euclidean strategy, i.e., given the initial locations of \(E_i\) and \(P\), as \((x_1, y_1)\) and \((X, Y)\) respectively, the vehicle moves towards \((x_1 + y_1 + \nu T^0_1)\), where \(T^0_1\) is

\[
T^0_1(\nu) = \frac{(y_1 - Y)\nu}{1 - \nu^2} + \frac{\sqrt{(X - x_1)^2 - (Y - y_1)^2}}{(1 - \nu^2)^2}.
\]

Lemma III.2 characterizes the monotonic nature of \(T^0_i(\nu)\). This only means that the maximum is achieved at one of the extremes. The next theorem characterizes the evader’s optimal choice of speed.

Theorem III.3 (Single evader optimal strategy) Given the initial locations \((x_1, y_1)\) and \((X, Y)\) of the evader and the pursuer respectively, the optimal strategy \(\nu^*\) for the evader is

\[
\nu^* = \begin{cases} 
\nu_{\text{min}}, & \text{if } \nu_1 < Y - \Delta x_i^0 (\frac{u_{\text{min}} + u_{\text{max}}}{2 + u_{\text{min}} - u_{\text{max}}}) \\
u_{\text{max}}, & \text{otherwise}.
\end{cases}
\]

Proof: We provide only an outline. We find a location \((x_1, y_1')\) such that \(T_i^0(u_{\text{min}}) = T_i^0(u_{\text{max}})\), where \(T_i^0(u_{\text{min}})\) (resp. \(T_i^0(u_{\text{max}})\)) is the time to intercept when evader moves with \(u_{\text{min}}\) (resp. \(u_{\text{max}}\)). From Lemma III.1, \(T_i^0(u_{\text{min}}) = T_i^0(u_{\text{max}})\) \(\Rightarrow y' = Y - \Delta x_i^0 (\frac{u_{\text{min}} + u_{\text{max}}}{2 + u_{\text{min}} - u_{\text{max}}})\). This means that if \(\nu_1 = y'\), then from Lemma III.2, it would not matter if the evader moves with \(u_{\text{min}}\) or \(u_{\text{max}}\). From Lemma III.3, it would be best if \(T_i^0(\nu_{\text{min}})\) or \(T_i^0(\nu_{\text{max}})\) is the same. Assuming \(T_i^0(\nu_{\text{min}}) < T_i^0(\nu_{\text{max}})\) \(\Rightarrow y_1 > y'\) and thus, by contradiction, we get the result. The second case is analogous and we get the result.

We now consider the case of two evaders and derive the optimal evasion strategies for both evaders. We say that an evader \(E_i\) moves greedy if it moves with speed that maximizes its own intercept time. An evader cooperates if it moves with a speed that maximizes the total intercept time. We denote the greedy strategy of evader \(i\) as \(\nu_i^*\) and the cooperative strategy as \(\nu_i^c\).
**B. Two evaders**

Similar to previous section, we first derive an expression for the time taken to intercept the evaders followed by the optimal strategy for both evaders.

Let the first evader $E_1$ be located at $(x_1, y_1)$ and move with speed $v_1$ and the second evader $E_2$ be located at $(x_2, y_2)$ and move with speed $v_2$. Then, the following result summarizes the time to intercept $E_2$ after intercepting $E_1$. For ease of reference, we introduce the following condition:

$$\Delta x_2^2 > \frac{y_1 - y_2 + (v_1 - v_2)T_1^0(v_1)}{v_2}. \quad (1)$$

**Lemma III.4 (Time to intercept $E_2$)** The time $T_2^1(v_1, v_2)$ taken by $P$ to intercept $E_2$ after intercepting $E_1$ is

$$T_2^1(v_1, v_2) = \begin{cases} \frac{\Delta x_2^2 + y_2 - y_1 + (v_2 - v_1)T_1^0(v_1)}{v_2}, & \text{if (1) holds,} \\ \frac{\Delta x_2^2 + y_1 - y_2 - (v_1 - v_2)T_1^0(v_1)}{1 + v_2}, & \text{otherwise.} \end{cases}$$

**Proof:** Consider the case when condition (1) holds, after the intercept of $E_1$. This means that after the completion of stage (1) for the pursuit of $E_2$, the evader’s $Y$-coordinate strictly exceeds $Y = y_1 + v_1T_1^0(v_1)$. The additional time to intercept the second evader is

$$y_2 + v_2T_1^0(v_1) = y_1 - v_1T_1^0(v_1) + v_2\Delta x_2^2.$$  

Thus, total time to intercept $E_2$ after intercepting $E_1$ is

$$T_2^1(v_1, v_2) = \frac{\Delta x_2^2 + y_2 - y_1 + (v_2 - v_1)T_1^0(v_1)}{1 - v_2}.$$  

The second case can be derived analogously and this concludes the proof.

**Lemma III.5 (Monotonicity of time to intercept $E_2$)** Given that $E_1$ moves with $v_1$, the time $T_2^1(v_1, v_2)$ is monotonically increasing function of $v_2$ if condition (1) holds. Otherwise, $T_2^1(v_1, v_2)$ is a monotonically decreasing function of $v_2$.

**Proof:** From Lemma III.4, $\frac{dT_2^1(v_1, v_2)}{dv_2} > 0$ if condition (1) holds and $\frac{dT_2^1(v_1, v_2)}{dv_2} < 0$, otherwise.

We now characterize an optimal greedy strategy for $E_2$. In what follows, we denote $V := \frac{u_{\max} + u_{\min}}{2}.$

**Lemma III.6 (E2’s greedy strategy)** The greedy strategy $v_{2g}$ for $E_2$ for a greedy $E_1$ moving with $v_{1g}$ is

$$v_{2g} = \begin{cases} u_{\max}, & \text{if } y_2 \geq y_1 - \Delta x_2^2V + (v_{1g} - V)T_1^0(v_{1g}), \\ u_{\min}, & \text{otherwise.} \end{cases}$$

**Proof:** From Lemma III.5, $T_2^1(v_{1g}, v_2)$ is maximized at either $v_2 = u_{\min}$ or $v_2 = u_{\max}$. The aim is to find the critical location $y'_2$ such that if $E_2$ was located at $(x_2, y'_2)$, then the time $T_2^1(v_{1g}, u_{\max}) = T_2^1(v_{1g}, u_{\min})$. Note that this is possible only if condition (1) holds for $v_2 = u_{\max}$ and does not hold for $v_2 = u_{\min}$. From Lemma III.4, we get

$$y'_2 = y_1 - \Delta x_2^2V + (v_{1g} - V)T_1^0(v_{1g}).$$

We define that a point $A$, located at $(x_A, y_A)$, is above point $B$, located at $(x_B, y_B)$, if $y_A > y_B$ and we define point $A$ is below point $B$ if $y_A < y_B$.

**Lemma III.7 (Conditions on cooperation)** Given the initial locations of $E_1$, $E_2$, and $P$ as $(x_1, y_1)$, $(x_2, y_2)$, and $(X, Y)$ respectively, $E_1$ cooperates with $E_2$ if

(i) Case 1:

$$Y - \Delta x_1^0V \leq y_1 \leq Y - \Delta x_1^0u_{\min}, \quad \text{and}$$

$$y_2 \geq y_1 - \Delta x_2^1V + (v_{1g} - V)T_1^0(v_{1g}) + 2\left(\frac{u_{\min}\Delta x_1^0 - y_1 - Y}{2 + u_{\min} - u_{\max}}\right). \quad (2)$$

(ii) Case 2:

$$Y - \Delta x_1^0u_{\max} \leq y_1 \leq Y - \Delta x_1^0V, \quad \text{and}$$

$$y_2 \leq y_1 - \Delta x_2^1V + (v_{1g} - V)T_1^0(v_{1g}) + 2\left(\frac{u_{\max}\Delta x_1^0 + y_1 - Y}{2 + u_{\min} - u_{\max}}\right). \quad (3)$$

**Proof:** Let the initial location of $E_1$ satisfy $y_1 \geq Y - \Delta x_1^0V$ and initial location of $E_2$ satisfy $y_2 \geq y_1 - \Delta x_2^1V + y_2 = y_2'$, then irrespective of $E_2$’s choice of $u_{\min}$ or $u_{\max}$, the time to intercept $E_2$ will be the same and from Lemma III.5, the time to intercept $E_2$ will be maximum at both $u_{\min}$ and $u_{\max}$, given that $E_1$ moves greedy. Now, consider that the initial location of $E_2$ is such that $y_2 < y_2'$. From Lemma III.5, the time to intercept will be maximized only at either $u_{\min}$ or $u_{\max}$ and so assume that $T_2^1$ is maximized at $u_{\max}$, i.e.,

$$T_2^1(v_{1g}, u_{\max}) > T_2^1(v_{1g}, u_{\min}).$$

This implies $y_2 > y_2'$. This is a contradiction as $y_2 < y_2'$. This means that $T_2^1$ will be maximized if $E_2$ moves at $u_{\min}$. Similarly, when $y_2 \geq y_2'$, it can be shown that $T_2^1$ will be maximized if $E_2$ moves at $u_{\max}$ and has been omitted for brevity. In the case when condition (1) does not hold for $v_2 = u_{\max}$ or holds for $v_2 = u_{\min}$, then it implies that $y_2 < y_2'$ and $y_2 \geq y_2'$ respectively. This concludes the proof.

Fig. 2. Cases for cooperation. (a) The time (solid line) taken by the pursuer to intercept $E_2$ directly from $(X, Y)$ and time (dashed line) taken by the pursuer to intercept $E_1$ first then $E_2$ are the same. (b) $E_1$ and $E_2$ cooperate by moving at $u_{\min}$ and $u_{\max}$ respectively (c) $E_1$ and $E_2$ cooperate by moving at $u_{\max}$ and $u_{\min}$ respectively.
Thus, in order to increase the total time to intercept, $E_1$ and $E_2$ need to cooperate. One way to cooperate is that $E_1$ moves with a speed $v_1$ such that $E_1$ is intercepted below the pursuer, i.e., $\Delta x_1^0 < (Y - y_1)/v_1$ and $E_2$ moves greedily, i.e., with $u_{\text{max}}$ (see Fig 2 (b)). This is only possible if 

$$\Delta x_1^0 < (Y - y_1)/u_{\text{min}} \Rightarrow y_1 < Y - \Delta x_1^0 u_{\text{min}}.$$ 

If $y_1 > Y - \Delta x_1^0 u_{\text{min}}$, then for any speed $v_1 \in [u_{\text{min}}, u_{\text{max}}]$ for $E_1$, $\Delta x_1^0 > (Y - y_1)/v_1$, and so the total distance covered in the y-direction to intercept $E_1$ and $E_2$ will be the same as the total distance covered in the y-direction to intercept $E_2$, from the pursuer’s initial location $(X, Y)$, irrespective of $E_1$’s choice. The only other way in which $E_1$ and $E_2$ can cooperate in order to increase the total time intercept is when $E_1$ moves greedily, i.e., with $u_{\text{max}}$ and $E_2$ with speed $v_2$ such that $y_2 < y_1 - \Delta x_2 v_2 + (v_1 - v_2)T_1^0(v_1) < v_1$ such that intercept of $E_2$ is below the intercept location of $E_1$ (see Fig 2 (c)). Thus, to determine which of the two scenarios yield the greater time to intercept, we arrive at a condition 

$$T_1^0(u_{\text{min}}) + T_2^1(u_{\text{min}}, u_{\text{max}}) > T_1^0(u_{\text{max}}) + T_2^1(u_{\text{max}}, u_{\text{min}}).$$

Note that as $T_1^0(v_1)$ and $T_2^1(v_1, v_2)$ are monotonic in $v_1$ and $v_2$ respectively, so the above condition is checked only at the extreme values, i.e., $u_{\text{min}}$ and $u_{\text{max}}$. Thus, this means that $E_1$ should cooperate only if equation (2) holds. Furthermore, if the initial location of $E_1$ was such that $y_1 \leq Y - \Delta x_1^0 V$, then from Lemma III.3, $E_1$ moves greedily, i.e., with $u_{\text{min}}$. Since, this will already ensure that $\Delta x_1^0 < (Y - y_1)/u_{\text{min}}$ and so, there is no need for cooperation. Case 2 is analogous. This concludes our proof.

**Theorem III.8 (Optimal cooperative strategy for $E_1$)**

*Given the initial locations $(x_1, y_1)$, $(x_2, y_2)$, and $(X, Y)$ of $E_1$, $E_2$, and $P$, respectively, if the conditions for cooperation in Lemma III.7 hold, then the optimal cooperative strategy $v_{1c}$ for evader $E_1$ is 

$$v_{1c}^* = \begin{cases} u_{\text{min}}, & \text{for case 1 from Lemma III.7 or} \\ u_{\text{max}}, & \text{for case 2 from Lemma III.7.} \end{cases}$$

**Proof:** Consider that case 1 of Lemma III.7 holds. Then $E_1$ moves with speed $v_1$ such that $\Delta x_1^0 < (Y - y_1)/v_1$. We know from Lemma III.7 that the conditions on cooperation ensure that the total time to intercept during cooperation is higher than the greedy choice. Since $T_1^0(v_1)$ is monotonic in $v_1$, from Lemma III.2, $v_{1c} = u_{\text{min}}$. The second case is derived analogously. This concludes the proof.

In this subsection, we analyzed the case of 2 evaders, primarily to highlight the underlying problem structure. Next, we will consider the case of $n$ evaders. Similar to the two evader case, we will first present a result on the time taken to intercept the $k^{th}$ evader after intercepting the $k-1^{th}$ evader. Then we will present results on the greedy and cooperative strategies between $E_k$ and $E_{k-1}$.

**C. n Evaders**

For ease of presentation, we will denote $(y_j - y_i)$ as $\Delta y_j^i$ for some $i, j$ and for brevity, we denote $T_{i-1}^{-1}(v_1, \ldots, v_i)$ as $T_{i-1}^{-1}(v_{i-1}, v_i)$. We present the following condition for ease of reference.

$$\Delta x_k^{k-1} > \frac{\Delta y_{k-1}^{k-1} + (v_{k-1} - v_k) \sum_{i=1}^{k-1} T_{i-1}^{-1}}{v_k}.$$  (4)

**Lemma III.9 (Time to intercept $E_k$)** The time $T_{k-1}^{-1}(v_{k-1}, v_k)$ taken by $P$ to intercept $E_k$, moving with $v_k$, after intercepting $E_{k-1}$, moving with $v_{k-1}$, is

$$T_{k-1}^{-1} = \begin{cases} \Delta x_k^{k-1} + \Delta y_{k-1}^{k-1} + (v_{k-1} - v_k) \sum_{i=1}^{k-1} T_{i-1}^{-1}, & \text{if (4) holds,} \\ \Delta x_k^{k-1} + \Delta y_{k-1}^{k-1} + (v_{k-1} - v_k) \sum_{i=1}^{k-1} T_{i-1}^{-1}, & \text{otherwise.} \end{cases}$$

**Proof:** We establish this result using mathematical induction. Lemma III.4 yields the base of induction for $k = 2$. Assume that for some $k = \tilde{k}$, the result holds. Consider that the initial location of the next evader, $E_{\tilde{k}+1}$, is such that $\Delta x_{\tilde{k}+1}^{\tilde{k}+1} > (y_{\tilde{k}} - y_{\tilde{k}+1} + (v_{\tilde{k}} - v_{\tilde{k}+1}) \sum_{i=1}^{\tilde{k}} T_i^{-1} + v_{\tilde{k}+1})$. This means that after the completion of stage (1) of simple pursuit of $E_{\tilde{k}+1}$, the X-coordinate of the pursuer equals $x_{\tilde{k}+1}$ and at the same time, the evader’s y-coordinate strictly exceeds the pursuers y-coordinate ($y_{\tilde{k}} + v_{\tilde{k}} \sum_{i=1}^{\tilde{k}} T_i^{-1} + v_{\tilde{k}+1}$). Thus, the time to intercept $E_{\tilde{k}+1}$ after intercepting $E_{\tilde{k}}$ will be

$$\Delta x_{\tilde{k}+1}^{\tilde{k}+1} + y_{\tilde{k}+1} - y_{\tilde{k}} + (v_{\tilde{k}+1} - v_{\tilde{k}}) \sum_{i=1}^{\tilde{k}} T_i^{-1} \frac{1 - v_{\tilde{k}+1}}{1 - v_{\tilde{k}}}. $$

Thus, by induction the result holds for any value of $\tilde{k}$. The other case is derived analogously.

**Lemma III.10 (Monotonicity of time to intercept)**

Given that each $E_i$, $i \in 1, \ldots, k-1$ moves with $v_i$, the time $T_{k-1}^{-1}$ is monotonically increasing function of $v_k$ if condition (4) holds. Otherwise, $T_{k-1}^{-1}$ is a monotonically decreasing function of $v_2$. 

$(v_1^* - V)T_1^0(v_1^*)$. Then, from Theorem III.3, to maximize its own intercept time, $T_1^0(v_1^*)$, $E_1$ moves with $u_{\text{max}}$. Also, from Lemma III.6, $E_2$ moves with $u_{\text{max}}$ in order to maximize its own intercept time $T_2^1(v_1^*, v_2)$. As the pursuer follows the Manhattan pursuit strategy, the time taken to cover the path in the y-direction to intercept $E_1$ and then $E_2$ is the same as the time taken to cover the path in y-direction to intercept only $E_2$ from the initial pursuer location $(X, Y)$ (see Fig 2 (a)). Mathematically, 

$$T_1^0(u_{\text{max}}) + T_2^1(u_{\text{max}}, u_{\text{max}}) = \frac{\Delta x_1^0 + \Delta x_2^1 + y_2 - Y}{1 - u_{\text{max}}}. $$

$$\Delta x_1^0 < (Y - y_1)/u_{\text{min}} \Rightarrow y_1 < Y - \Delta x_1^0 u_{\text{min}}.$$
Lemma III.11 (Evader k’s greedy strategy) The greedy strategy $v_{k}^*$ for $E_k$, when each $E_i$, $i \in \{1, \ldots, k-1\}$ moves with $v_{1g}^*$ is:

$$v_{k}^* = \begin{cases} 
  u_{\text{max}}, & \text{if } y_k \geq y_{k-1} - \Delta x_{k-1}^{k-1}V + (v_{(k-1)g}^* - V) \sum_{i=1}^{k-1} T_i^{i-1}, \\
  u_{\text{min}}, & \text{otherwise}
\end{cases}$$

Proof: Suppose the result holds for some $k = \bar{k}$. Consider the next evader, $E_{\bar{k}+1}$. Similar to the proof of Lemma III.6, we find $y_{\bar{k}+1} = y_{\bar{k}} - \Delta x_{\bar{k}+1}^k V + (v_{k}^* - V) \sum_{i=1}^{\bar{k}} T_i^{i-1}$. If $y_{\bar{k}+1} < y_{\bar{k}+1}$, then, from Lemma III.10, the time will be maximized at either $u_{\text{min}}$ or $u_{\text{max}}$. Thus, assuming $T_{\bar{k}+1}^{\bar{k}}(v_{1g}^*, \ldots, v_{(k-1)g}^*, u_{\text{max}}) > T_{\bar{k}+1}^{\bar{k}}(v_{1g}^*, \ldots, v_{(k-1)g}^*, u_{\text{min}})$ yields $y_{\bar{k}+1} > y_{\bar{k}+1}$ which is a contradiction and so $u_{k}^* = u_{\text{min}}$. Moreover, by induction, the result holds for any value of $\bar{k}$. Case 2 is proved analogously. This concludes our proof.

The previous lemma presented a result on the greedy strategy of any evader $E_k$. This result is the first step of the Algorithm 1. As the second step of Algorithm 1 requires to check the conditions of cooperation between two consecutive evaders, we will now present a result on the conditions if two evaders should cooperate or not. We introduce the notation,

$$U := \frac{2+u_{\text{min}}-u_{\text{max}}}{2}.$$

**Lemma III.12 (Cooperation conditions for $E_{k-1}$)** Given the initial locations of $E_{k-1}$, $E_k$, and $P$ as $(x_{k-1}, y_{k-1})$, $(x_k, y_k)$, and $(X, Y)$ respectively, then $E_{k-1}$ will cooperate with $E_k$ if

(i) Case 1:

$$y_{k-2} + (v_{(k-2)a}^* - V) \sum_{i=1}^{k-2} T_i^{i-1} - \Delta x_{k-1}^{k-2}V \leq y_{k-1}$$

\[ \leq y_{k-2} + (v_{(k-2)a}^* - u_{\text{min}}) \sum_{i=1}^{k-2} T_i^{i-1} - \Delta x_{k-1}^{k-2}u_{\text{min}} \]

and

$$\Delta y_{k-1}^k > -\Delta x_{k-1}^k V + (v_{(k-1)g}^* - V) \sum_{i=1}^{k} T_i^{i-1} + U(u_{\text{min}}\Delta x_{k-1}^{k-2} + \Delta y_{k-2}^k - (v_{(k-2)a}^* - u_{\text{min}}) \sum_{i=1}^{k-2} T_i^{i-1})$$

(ii) Case 2:

$$y_{k-2} + (v_{(k-2)a}^* - u_{\text{max}}) \sum_{i=1}^{k-2} T_i^{i-1} - \Delta x_{k-1}^{k-2}u_{\text{max}}$$

\[ \leq y_{k-1} \leq y_{k-2} + (v_{(k-2)a}^* - V) \sum_{i=1}^{k} T_i^{i-1} - \Delta x_{k-1}^{k-2}V \]

and

$$\Delta y_{k-1}^k < -\Delta x_{k-1}^k V + (v_{(k-1)g}^* - V) \sum_{i=1}^{k} T_i^{i-1} + U(u_{\text{max}}\Delta x_{k-1}^{k-2} + \Delta y_{k-2}^k - (v_{(k-2)a}^* - u_{\text{max}}) \sum_{i=1}^{k-2} T_i^{i-1}),$$

where $v_{(k-2)a}^*$ determined by Algorithm 1.

**Proof:** Let us assume that this result holds for some $k = \bar{k} - 1$. The idea is to prove this result using induction by deriving the conditions for cooperation between $E_{\bar{k}}$ and $E_{\bar{k}+1}$. For brevity, we will reuse Figure 2 with the two evaders $E_1$ and $E_2$ in the figure corresponding to $E_{\bar{k}}$ and $E_{\bar{k}+1}$ respectively. Suppose that the initial location of $E_{\bar{k}}$ satisfies $y_{\bar{k}} \geq y_{\bar{k}-1} + (v_{(k-1)g}^* - V) \sum_{i=1}^{k-1} T_i^{i-1} - \Delta x_{k-1}^k V$ and the location of $E_{\bar{k}+1}$ satisfies $y_{\bar{k}+1} > y_{\bar{k}} - \Delta x_{\bar{k}+1}^\bar{k} V + (v_{(k-2)g}^* - V) \sum_{i=1}^{\bar{k}} T_i^{i-1}$. Then, from Lemma III.11, $E_{\bar{k}}$ moves with $u_{\text{max}}$ to maximize the component $T_{\bar{k}}^{\bar{k}}$ out of its intercept time and $E_{\bar{k}+1}$ moves with $u_{\text{max}}$ to maximize $T_{\bar{k}+1}^{\bar{k}+1}$. This implies that when $P$ completes stage 1 for the pursuit of $E_{\bar{k}}$, $P$ is below $E_{\bar{k}}$. As the pursuer follows the Manhattan pursuit strategy, the time taken to cover the path in the $y$-direction to intercept $E_{\bar{k}}$ and then $E_{\bar{k}+1}$ equals the time taken to cover the path in the $y$-direction to intercept only $E_{\bar{k}+1}$ from the pursuer’s location (2 (a)). Note that the pursuer is located at the intercept location of $E_{\bar{k}-1}$. So, to increase the total time to intercept, $E_{\bar{k}}$ and $E_{\bar{k}+1}$ need to cooperate which can occur in only two ways.
The first is that $E_k$ moves with speed $v_k$ satisfying
\[
\Delta x_{k-1}^k < (y_{k-1} - y_k + (v_{(k-1)a}^* - v_k)^2 T_{i-1}^{k-1})/v_k,
\]
which means that $P$ intercepts $E_k$ below the intercept point of $E_{k-1}$ and $E_{k+1}$ moves greedily, i.e., with $v_{\text{max}}$ (Fig. 2 (b)). This is possible only if $\Delta y_{k-1}^k \leq (v_{(k-1)a}^* - u_{\text{min}})^2 T_{i-1}^{k-1} \Delta x_{k-1}^k u_{\text{min}}$ holds. If $\Delta y_{k-1}^k > (v_{(k-1)a}^* - u_{\text{min}}^2 T_{i-1}^{k-1} - \Delta x_{k-1}^k u_{\text{min}}$, then for any speed $v_k$ for $E_k$, the total distance covered in the $Y$-direction to intercept $E_k$ and $E_{k+1}$ will be the same as the total distance covered to intercept $E_{k+1}$ from the pursuer’s location, irrespective of $E_k$’s choice.

The second case in which $E_k$ and $E_{k+1}$ cooperate is if $E_k$ moves greedily with $u_{\text{max}}$ and $E_{k+1}$ moves with speed $v_{k+1}$ such that $y_{k+1} < y_k - \Delta x_{k-1}^k v_{k+1} + (v_{k+1}^* - v_{k+1}) \sum_{i=1}^{k} T_i^{k-1}$, i.e., the intercept location of $E_{k+1}$ is below the intercept location of $E_k$ (Fig. 2 (c)). To determine which of the two scenarios yield greater intercept time, we arrive at the condition
\[
\sum_{i=1}^{k} T_i^{k-1} + T_{k+1}^k (u_{\text{max}}) > \sum_{i=1}^{k} T_i^{k-1} + T_{k+1}^k (u_{\text{min}})
\]
which yields the conclusion that $E_k$ should cooperate with $E_{k+1}$ only if equation (5) holds. This concludes our proof for case 1. Case 2 can be proved by following the steps for Case 1 and has been omitted for brevity.

Lemma III.12 establishes the conditions for cooperation between any two consecutive evaders. The next result characterizes the cooperative strategies of the evaders.

**Theorem III.13 (Cooperative strategy for $E_{k-1}$)** If the conditions on cooperation in Lemma III.12 hold, then the optimal strategy $v_{(k-1)c}^*$ for $E_{k-1}$ during cooperation with $E_k$ is

\[
v_{(k-1)c}^* = \begin{cases} 
  u_{\text{min}}, & \text{for case 1 of Lemma III.12}, \\
  u_{\text{max}}, & \text{for case 2 of Lemma III.12}.
\end{cases}
\]

**Proof:** Assume that the result holds for $k = k - 1$. The idea is to prove this result by induction by deriving this result for $E_k$. Suppose case 1 from Lemma III.12 holds for $E_k$ and $E_{k+1}$. From Lemma III.12, we know in order to cooperate with $E_{k+1}$, $E_k$ moves with a speed $v_k$ such that $\Delta x_{k-1}^k < (y_{k-1} - y_k + (v_{k-1}^* - v_k)^2 T_{i-1}^{k-1})/v_k$, i.e., $E_k$ is intercepted below the pursuer’s location. We also know from the same lemma that these conditions on cooperation ensure that the total time to intercept while cooperation is higher than total time to intercept when the evaders move greedy. Now, since $T_k^{k-1}$ is monotonic in $v_k$, $v_{(k-1)c}^* = u_{\text{min}}$. Similar steps can be followed for case 2.

**Remark 2 (Sandwiched evader)** For some $i \in \{1, \ldots, n\}$, if Lemma III.12 holds for evader $E_{i-1}$ and $E_i$ as well as $E_i$ and $E_{i+1}$, then evader $E_i$ moves greedy.

**IV. Fundamental Limit**

In the previous sections, we considered that the pursuer followed a fixed strategy to capture all evaders. We now establish a fundamental upper bound, for a large number of evaders, on the total time taken to intercept all evaders by the pursuer following any strategy. We first provide some existing results that will be useful in establishing the bound.

Given a set of $m$ points, a Euclidean minimum Hamiltonian path (EMHP) is the shortest path through $m$ points such that each point is visited exactly once. When the points are translating with some constant speed $v \in (0, 1)$, then the shortest tour though the points is called Translational minimum Hamiltonian path (TMHP) [16].

**Lemma IV.1 (Length of EMHP tour)** Given $m$ points in a $l \times h$ rectangle in the plane, where $h \in \mathbb{R}_{>0}$ and $l \in \mathbb{R}_{>0}$, there exists a path that starts from a unit length edge of the rectangle, passes through each of the $m$ points exactly once, and terminates on the opposite unit length edge, with length upper bounded by $\sqrt{2lm} + h + 2.5$

**Proof:** The proof is similar to the proof provided in [17] for a $1 \times h$ rectangle and thus, has been omitted.

To calculate the EMHP tour through translating points $s, s_1, \ldots, s_f$, $f$ that move with speed $v$, the points are scaled by defining a conversion map $C_v : \mathbb{R}^2 \to \mathbb{R}^2$ such that $C_v(x, y) = (\frac{s_x}{\sqrt{1-v^2}}, \frac{s_y}{\sqrt{1-v^2}})$ [16].

**Lemma IV.2 (Length of TMHP tour [16])** Let the initial and final point be denoted as $s = (x_s, y_s)$ and $f = (x_f, y_f)$ respectively, and $v \in (0, 1)$ denote a constant speed of all evaders, then the length of the TMHP tour is $\frac{(y_f - y_s)}{v} + L_E(C_v(s), C_v(s_1), \ldots, C_v(s_f), C_v(f))$, where $L_E(C_v(s), C_v(s_1), \ldots, C_v(s_f), C_v(f))$ denotes the length of the EMHP starting with point $s$, moving through points $s_1, \ldots, s_f$ and ending at point $f$.

The optimal order followed by the vehicle in the TMHP solution is the same as the optimal order followed by the vehicle in the EMHP solution.

Denote $n_{\text{max}} \in \mathbb{Z}^+_0$ as the total number of evaders that move with $u_{\text{max}}$ and $n_{\text{min}} = n - n_{\text{max}}$ as the total number of evaders that move with $u_{\text{min}}$. Let $A_{\text{max}}$ and $A_{\text{min}}$ denote the area of the smallest enclosing rectangular environment that the $n_{\text{max}}$ and $n_{\text{min}}$ evaders occupy initially. We assume that all of the evaders are initially located within a rectangular environment of area $A$. The pursuer’s strategy is to capture all the $n_{\text{max}}$ evaders first, followed by capturing all the evaders moving with $u_{\text{min}}$. This is because if the pursuer captures the $n_{\text{min}}$ evaders first then naturally, the evaders moving with $u_{\text{min}}$ will be further away from the pursuer.

Let $T_{\text{max}}$ be the time taken by the vehicle to capture all of the $n_{\text{max}}$ evaders and $T_{\text{min}}$ be the time taken to intercept the last evader that moves with $u_{\text{max}}$ and the first evader that moves with $u_{\text{min}}$ after capturing all of the $n_{\text{max}}$ evaders.
respectively. Let \( T_{n_{\text{fin}}} \) be the total time taken by the vehicle to capture all of the remaining \( n_{\text{fin}} \) evaders. The next result characterizes an upper bound on the time taken by the pursuer to capture all evaders following any strategy.

**Theorem IV.3 (Upper bound on intercept time)** Let \( \Delta y \) and \( \Delta x \) be the difference between the initial \( y \) and \( x \)-coordinate of the last evader captured moving with \( u_{\text{max}} \) and the first evader that is captured moving with \( u_{\text{min}} \). Then, from Lemma IV.1 and Lemma IV.2, the total time taken by the pursuer to capture all evaders is \( T = T_{n_{\text{fin}}} + T_{n_{\text{max}}} + T_{n_{\text{fin}}} \) where,

\[
T_{n_{\text{max}}} = \sqrt{\frac{2A_{n_{\text{max}}}}{1 - u_{\text{max}}^2}} \cdot n_{\text{max}} \quad \text{and} \quad T_{n_{\text{fin}}} = \sqrt{\frac{2A_{n_{\text{min}}}(n_{\text{min}} - 1)}{1 - u_{\text{min}}^2}} \cdot \frac{\Delta y + (u_{\text{min}} - u_{\text{max}})T_{n_{\text{fin}}}}{1 - u_{\text{min}}^2} + \frac{\Delta x^2}{(1 - u_{\text{min}}^2)^2} + \frac{\left( u_{\text{min}} - u_{\text{max}} \right)^2 T_{n_{\text{max}}}^2}{1 - u_{\text{min}}^2}.
\]

Moreover, for large \( n \), \( T \) is maximum for

\[
n_{\text{max}}^* = \left\lfloor \frac{(u_{\text{min}} - u_{\text{max}})^2 n}{(1 - u_{\text{min}}^2)^2(1 - u_{\text{max}}^2) + (u_{\text{min}} - u_{\text{max}})^2} \right\rfloor,
\]

where \( \lfloor x \rfloor \) denotes the integer nearest to \( x \).

**Proof:** The outline of the proof is as follows. The expression for \( T_{n_{\text{max}}} \) and \( T_{n_{\text{fin}}} \) follows directly from Lemma IV.1 and noting that \( n \) is large. The expression for \( T_{n_{\text{min}}} \) follows from [16]. Consider that the vehicle has just finished capturing all \( n_{\text{max}} \) evaders. Then, all the evaders moving with \( u_{\text{min}} \) would have translated \( u_{\text{min}}T_{n_{\text{max}}} \) in the \( y \) direction. Note that the area \( A_{n_{\text{min}}} \) will remain the same as it was initially. Since \( T_{n_{\text{max}}} \) is large for large \( n_{\text{max}} \), the distance between the vehicle after capturing the last evader moving with \( u_{\text{max}} \) and the first evader moving with \( u_{\text{min}} \) will be large. Furthermore, since \( u_{\text{max}} > u_{\text{min}} \) and \( T_{n_{\text{max}}} \) is large, the pursuer will always be above all evaders moving with \( u_{\text{min}} \) after capturing \( n_{\text{max}} \) evaders and thus, we get the expression for \( T_{n_{\text{max}}} \). The evaders can select \( n_{\text{max}} \) such that the total time \( T \) is maximized. Mathematically, \( n_{\text{max}}^* = \arg \max_{n_{\text{max}}} T(n_{\text{max}}) \). If we relax the requirement of \( n_{\text{max}} \) to be a real number then the function \( T : \mathbb{R} \to \mathbb{R} \) is concave with global maximum in the domain \( [0, n] \). This follows as \( \frac{dT}{dn_{\text{max}}} = 0 \) and then find the closest integer value that maximizes \( T \). By taking the derivative of \( T \) with respect to \( n_{\text{max}} \), we get

\[
\frac{dT_{n_{\text{max}}}}{dn_{\text{max}}} + \frac{dT_{n_{\text{max}}}}{dn_{\text{max}}} + \frac{dT_{n_{\text{min}}}}{dn_{\text{min}}} = 0
\]

where,

\[
\frac{dT_{n_{\text{max}}}}{dn_{\text{max}}} = \frac{\sqrt{\frac{2A_{n_{\text{max}}}}{1 - u_{\text{max}}^2}}}{n_{\text{max}}},
\]

\[
\frac{dT_{n_{\text{min}}}}{dn_{\text{min}}} = \frac{\sqrt{\frac{2A_{n_{\text{min}}}}{1 - u_{\text{min}}^2}}}{n_{\text{min}}} - \frac{\sqrt{\frac{2A_{n_{\text{max}}}}{1 - u_{\text{max}}^2}}}{n_{\text{max}}} + \frac{\Delta y + (u_{\text{min}} - u_{\text{max}})T_{n_{\text{fin}}}}{1 - u_{\text{min}}^2} + \frac{\Delta x^2}{(1 - u_{\text{min}}^2)^2} + \frac{(u_{\text{min}} - u_{\text{max}})^2 T_{n_{\text{max}}}^2}{1 - u_{\text{min}}^2}.
\]

In the worst case, the evader initial locations will cover the area \( A \) completely, i.e., \( A_{n_{\text{max}}} = A_{n_{\text{min}}} = A \). Adding and equating to zero, and noting that \( T_{n_{\text{min}}} \) is large, \( \Delta y \leq (u_{\text{min}} - u_{\text{max}})T_{n_{\text{min}}} \) and \( -u_{\text{min}}^2 < 0 \) at the critical point \( n_{\text{max}}^* \), \( n_{\text{max}}^* \) is indeed the point of maximum. This concludes the proof.

**V. Simulation Results**

We first present the numerical results for Algorithm 1. We compare the mean of the total time to intercept all evaders using Algorithm 1 to the mean of the total time to intercept all evaders by randomly sampling over the evader speeds of either \( u_{\text{min}} \) or \( u_{\text{max}} \) (see Figure 3). For each value of \( n \), we randomly generate the initial locations of the evaders and the pursuer and we consider 50 Monte Carlo trials. To select the best evader speeds, we choose \( 10n\ln(2/\delta) \) samples uniformly randomly over the set, which guarantees that the violation probability is less than a small quantity \( \delta \) [18], where \( \delta = 0.1 \). We compute the maximum over the samples and then report the mean value in Figure 3. We observe that Algorithm 1 outperforms random sampling.

Figure 4 shows a comparison when \( n_{\text{max}} \) is selected uniformly randomly to the upper bound obtained by \( n_{\text{max}}^* \) for given initial locations. To obtain the EMHP tour required for the time to intercept evaders, the linkern1 solver was used. We consider 50 Monte Carlo trials for each value of \( n \) and report the mean and standard deviation. It is observed that the total time to intercept all evaders by randomly selecting \( n_{\text{max}} \) is well below the upper bound obtained from \( n_{\text{max}}^* \). Thus, by performing an additional optimization to select \( n_{\text{max}} \) the evaders can reach the upper bound on time to intercept. This means that a strategy that only depends on \( n_{\text{max}} \) may be sub-optimal for the evaders.

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1The TSP solver linkern is freely available for academic research use at [http://www.math.uwaterloo.ca/tsp/concorde/](http://www.math.uwaterloo.ca/tsp/concorde/).
VI. CONCLUSIONS AND FUTURE WORK

An optimal evasion problem between single a pursuer and multiple evaders was addressed. It is shown that by enforcing cooperation among evaders, they are able to maximize the total interception time. Conditions where cooperation is optimal are also presented which are crucial to implement the cooperative strategies. An upper bound on the total time to intercept all evaders is also presented.

In subsequent work, a generalized setup of multiple pursuers and evaders will be considered. Constant factor approximations for both, the evaders and the pursuers will also be addressed. Identifying which evaders should move with $u_{max}$ is another possible extension.

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