Intersections of moving fractal sets

I. Mandre and J. Kalda

Institute of Cybernetics at the Tallinn University of Technology - Akadeemia tee 21, 12618, Tallinn, Estonia, EU

Received 15 May 2013; accepted 2 July 2013

PACS 05.45.Df – Fractals
PACS 05.40.Jc – Brownian motion
PACS 05.45.Tp – Time series analysis

Abstract – Intersection of a random fractal or self-affine set with a linear manifold or another fractal set is studied, assuming that one of the sets is in a translational motion with respect to the other. It is shown that the mass of such an intersection is a self-affine function of the relative position of the two sets. The corresponding Hurst exponent is a function of the scaling exponents of the intersecting sets. A generic expression for is provided, and its proof is offered for two cases —intersection of a self-affine surface with a line, and of two fractal sets. The analytical results are tested using Monte Carlo simulations.

Copyright © EPLA, 2013

There is a wide spectrum of problems which can be reduced to finding and studying intersections of fractal sets. For instance, rain intensity is a multifractal function of space and time [1,2]; rainfall at a given point on Earth’s surface is a time-integral of this function —a measure of the intersection of the rain intensity field with a line, parallel to the time axis. Next, the Doppler absorption spectra depend on how many points of the flow move with a certain velocity in a certain direction; in the case of fully turbulent flows, velocity is a self-affine function of coordinates: the velocity in a certain direction; in the case of fully turbulent, the dry surface height is defined by the size of the tire surface described by its height \( z \).

We start by deriving the dependance \( h = h(H) \) for the case of a fB curve intersecting with a line in two-dimensional space. Let us consider a finite-length segment \( x \in [0, L] \) of a fB curve \( u = f(x) \) with zero mean. Then, typically, the curve varies from \( u \sim -L^H \) to \( u \sim L^H \) (here “\( \sim \)” means “is of the order of”). The fractal dimension of the fB curve is \( 2 - H \) [6], and the dimension of its

\[
\left\langle [N(u,v) - N(u',v')]^2 \right\rangle \propto \left((u - u')^2 + (v - v')^2\right)^h, \quad (1)
\]

and the same scaling law holds for \( g(x,y) \). Here, angular braces denote averaging over an ensemble of surfaces, and \( H \) is the Hurst exponent, \( 0 < H < 1 \). Let us assume that the lower cut-off scale of this scaling law is unity, and that the gradient components become smooth below that scale. Furthermore, we assume that the wavelength of the incident light is much smaller than one. The functions \( f \) and \( g \) define a two-dimensional random self-affine surface (the gradient surface) \( u = f(x,y), v = g(x,y) \) in four-dimensional space \( x,y,u,v \). The propagation direction of incident light from a point on the surface \( z \) is determined by its gradient components at that point. Therefore, the intensity of light reflected at a given direction from the entire surface \( z \) is proportional to the number of intersection points \( N \) of the gradient surface with a two-dimensional linear manifold \( u = u_0, v = v_0 \). This number is a random function of the propagation direction, \( N = N(u,v) \) —the light intensity fluctuates as the observation direction is changed. We will show that the function \( N(u,v) \) can be described by another Hurst exponent \( h = h(H) \):

\[
\left\langle [N(u,v) - N(u',v')]^2 \right\rangle \propto \left((u - u')^2 + (v - v')^2\right)^h. \quad (2)
\]
intersection with a line \( u = u_0 \) is \( d_f = 1 - H \) \[7,8\]. The intersection at level \( u = 0 \) is also known as the zero set.

Since the lower cut-off scale is unity, the number of intersection points at some fixed \( u_0 \) is estimated as \( N(u_0) \sim L^{d_f} = L^{1-H} \). Let us denote the change in the number of intersection points when changing from “altitude” \( u_0 \) to \( u_0 + \Delta u \) as \( \Delta N(\Delta u) \equiv N(u_0) - N(u_0 + \Delta u) \). We now make use of a scale-decomposition of the function \( f(x) \) by introducing coarse-grained functions \( F_a(x) = \sum_{2^i > a} f_2(x) \), where \( f_2(x) \) are the scale components that can be obtained, for instance, via a forward and reverse Fourier transform of \( f(x) \), where only the wavelengths between \( 2^{i-1} \) and \( 2^i \) are kept. Let us denote the number of intersection points of the line \( u = u_0 \) with the coarse-grained curve \( u = F_a(x) \) as \( N_a(u_0) \sim (L/a)^{1-H} \) and the change in the intersection points due to displacement \( \Delta u \) as \( \Delta N_a(\Delta u) \equiv N_a(u_0) - N_a(u_0 + \Delta u) \). As we increase the level difference \( \Delta u \), the line \( u = u_0 + \Delta u \) will cross from time to time the extrema of the function \( u = F_a(x) \). By each crossing, the number of intersections \( N_a(u_0 + \Delta u) \) will change by two — increase in the case of a minimum, and decrease in the case of a maximum. When \( \Delta u \ll a^H \), that is it is well below the vertical characteristic scale, these changes are purely incidental — they are caused by uncorrelated extrema that are separated by large distances. Therefore, at \( \Delta u \ll a^H \), the value \( \Delta N_a(\Delta u) \) is a compound Poisson process, that is \( \Delta N_a(\Delta u) = \sum_{i=1}^{N(\Delta u)} D_i \), where \( \{ D(i) : \Delta u \geq 0 \} \) is a Poisson process with rate \( \lambda \) and \( \{ D_i : i \geq 1 \} \) are independent random values drawn with equal probability from \( \{-2, +2\} \). The variance of the compound Poisson process \( \lambda \) has zero mean, we conclude that

\[
\langle \Delta N_a(\Delta u)^2 \rangle = 4\lambda \Delta u \quad (\Delta u \ll a^H). \tag{3}
\]

We estimate the density of extrema for \( N_a \) as \( \lambda \sim (L/a) / L^H = L^{1-H} a^{-1} \) — the number of peaks is \( L/a \) and they are distributed quasi-homogeneously in the range \(-L^H \) to \( L^H \). We note that eq. (3) can also be used to estimate the scaling exponent for displacements below the lower cut-off scale of \( f(x) \), that is for \( \Delta u \ll 1 \), we obtain a super-universality \( h(H) = 1/2 \).

Around each intersection point with the coarse-grained curve, the line \( u = u_0 \) also intersects with the fine-scaled structure \( f(x) - F_a(x) \). But as the intersection points with the coarse-grained curve are typically spaced at greater distances than \( a \), the number of intersections with the fine-scaled structure around each such point are uncorrelated (as correlations in the fine-scaled structure only extend to distances around \( a \)). We denote the average number of such intersections around each point as \( n_a \sim a^{1-H} \) and conclude that the total number of intersections with the whole curve \( u = f(x) \) is \( N(u_0) \sim n_a N_a(u_0) \sim a^{1-H} N_a(u_0) \). As we move the intersecting line from level \( u_0 \) to \( u_0 + \Delta u \), the number of intersections with \( u = f(x) \) changes. When the displacement \( \Delta u \) is smaller than \( a^H \), the contributions from the fine-scaled intersections are highly correlated, but at displacement \( \Delta u \gg a^H \) they are basically uncorrelated. Consequently,

\[
\Delta N(\Delta u) \sim a^{1-H} \Delta N_a(\Delta u) \quad (\Delta u \gg a^H). \tag{4}
\]

At the marginally applicable limit \( \Delta u = a^H \), eqs. (3) and (4) combine into

\[
|\Delta N(a^H)| \sim a^{1-H} |\Delta N_a(a^H)| \sim L^{1-H} 2 a^{1-H}. \tag{5}
\]

To estimate \( |\Delta N(\Delta u)| \), we choose \( a = \Delta u^{1/H} \), yielding

\[
|\Delta N(\Delta u)| \sim L^{1-H} 2^{1/H} \Delta u^{1/H}, \tag{6}
\]

and so the scaling exponent \( h \) for the intersection of a fB curve and a moving line is

\[
h = \frac{1 - H}{2H} \quad (1 \ll \Delta u \ll L^H). \tag{7}
\]

It should be noted that for \( H < \frac{1}{3} \), this equation yields \( h > 1 \). Result \( h > 1 \) means that large-scale fluctuations are so strong that the gradients of large-scale components dominate over the gradients caused by small-scale fluctuations. In that case, eq. (2) would yield the Hurst exponent \( h = 1 \). However, using wavelet or Fourier analysis, it is possible to generalize eq. (2) and reveal scaling laws with \( h > 1 \).

Returning to the case of the intensity of light reflected by the sea surface, where we have an intersection of a self-affine surface and a flat surface in 4D space, the scaling law (2) can be derived in a similar fashion, resulting in \( h = \frac{2-2H}{2H} \), where \( 2-2H \) is the fractal dimension of the intersection studied.

We have run a series of Monte Carlo simulations to test the result (7). At each calculation point \( H \) we generated 1000 fractional Brownian curves \( f(x) \) with length \( L = 2^{27} \) \[10–12\]. Samples of the intersection functions \( N(u) \) can be seen in fig. 1. The data was analyzed using the continuous wavelet transform and the Mexican hat wavelet \[13\]. The results follow the predicted relationship \( h = \frac{1-H}{2H} \) quite closely except at greater values of \( H \) (fig. 1(d)). The discrepancy is due to distortions in the function \( N(u) \) — as \( H^H \) grows, the density of intersections falls and the function \( N(u) \) starts to experience large ranges where it is of constant small value (see fig. 1(c) for a sample \( N(u) \) at \( H = 0.7 \)). This is a finite-size effect — to overcome it one would have to calculate at much greater length \( L \). We also did some calculations for \( H < \frac{1}{3} \). The results were as expected with \( h > 1 \).

We now turn our attention to general statistically self-similar fractal sets. It is easy to imagine that the interactions (changes in the intersection) of a line and a random fractal set at displacements well below the lower scaling length of the fractal are completely random. We have also found, that for the intersections of a fB curve or surface with a line or a plane, the analytically derived scaling exponents all came out as \( h = d_f/(2H) \), where \( d_f \) is the
fractal dimension of the intersection. Considering all this, one can conjecture that this relation also applies to intersections of general random fractals. We proceed to make this claim more specific.

Let us have two fractal sets \( F \) and \( X \) with corresponding fractal dimensions \( d_F \) and \( d_X \). Let the set \( X \) be translatable in some direction \( \hat{u} \), with the position identified by coordinate \( u \). Further, we assume that it is self-similar and with finite scaling range \([1, L]\). This set may also have a topological dimension that is equal to its fractal dimension (for example, it may be a simple line or a plane). We assume that the fractal set \( F \) is random, that is it is only statistically self-similar (we will clarify the nature of this randomness further along the way). Let the fractal \( F \) have the same scaling range as \( X \) in the directions perpendicular to \( \hat{u} \) but let it be possibly self-affine in the direction \( \hat{u} \) with scaling range \([1, L^{H\hat{u}}]\).

We denote the fractal dimension of the intersection of the two sets as \( \delta \) (for many cases \( \delta = d_F + d_X - D \), where \( D \) is the dimension of the surrounding space). As the set \( X \) moves, the total fractal mass of this intersection \( M(u) \) (the number of points, the surface area, the volume, or other such measure that is suitable for the given fractal depending on its topological dimension) will change. We fix \( u = u_0 \) and denote this change at translation to \( u = u_0 + \Delta u \) as \( \Delta M(\Delta u) \equiv M(u_0) - M(u_0 + \Delta u) \). We conjecture that the function \( M(u) \) is fractional-Brownian-moion-like, that is it can be described by

\[
\left\langle \Delta M(\Delta u)^2 \right\rangle \propto |\Delta u|^{2h},
\]

with the Hurst exponent \( h \) as

\[
h = \frac{d_F}{2H\hat{u}},
\]

where \( H\hat{u} \) describes the scaling of the fractal \( F \) in the direction \( \hat{u} \) (\( H\hat{u} \) is unity for a self-similar fractal set).

We will now continue with a derivation leading to this result for the case of self-similar fractals (with \( H\hat{u} = 1 \)). For this we will first approximate the fractals by the use of a ball cover —this results in a “coarse-grained” version of the fractal at a specific grain size. Then, we will derive how \( M(u) \) scales at movements either much smaller or much greater than the length used at the ball cover. Finally, we bring these two estimates together to yield the exponent \( h \).

In case the set \( F \) is self-similar, the fractal mass of the intersection \( F \cap X \) can be estimated as \( M(u_0) \sim L^\delta \), where \( d_F = d_F + d_X - D \). Let us assume that we can find minimal covers for both sets \( F \) and \( X \) with \( D \)-dimensional closed balls of diameter \( a \), where \( 1 \ll a \ll L \). The number of balls in either cover can be estimated as \( N_F(a) \sim (L/a)^{d_F} \) and \( N_X(a) \sim (L/a)^{d_X} \).

At a location where two balls, each from a different set, intersect, the fractal sets themselves usually intersect, with the average fractal mass of the intersection (assuming \( F \) is random, for example the balls can’t be globally aligned) estimated as \( m_a \sim a^\delta \). The total number of such intersections is \( N_F(a_0) \sim M(u_0)/m_a \sim (L/a)^\delta \). We move the set \( X \) in the direction \( \hat{u} \) by distance \( \Delta u \). This will cause the cover of the set \( X \) also move. As a ball from that cover moves, it penetrates or exits balls covering the standing set \( F \). As a result, the value \( N_F(u_0) \) will increase or decrease by one. We denote the total change in the number of intersecting balls as \( \Delta N_F(\Delta u) \equiv N_F(u_0) - N_F(u_0 + \Delta u) \).

In case the movement is much greater than \( a \), that is \( a \ll \Delta u \ll L \), a moving ball that is penetrating a standing ball will exit it. We assume that \( F \) is random in such a way that the masses of the sub-fractals contained in individual standing balls separated by distances much greater than \( a \) are uncorrelated. In such a case

\[
\Delta M(\Delta u) \sim m_a \Delta N_F(\Delta u) \quad (a \ll \Delta u \ll L).
\]  

In case the movement is much smaller than \( a \), that is \( \Delta u \ll a \), a moving ball that is intersecting a standing ball will rarely exit it. Also, it has very little chance to interact with other standing balls or the correlations in their placement (defined by the structure of the fractal). With small movement individual moving balls have no chance to interact with the fractal structure of the ball cover. And assuming the fractal \( F \) is random, that is the balls are not globally aligned, we can ignore their interactions as a group. In such a case the change in the number of balls intersected can be approximated as a compound Poisson process, that is \( \Delta N_F(\Delta u) = \sum_{i=1}^{P(\Delta u)} D_i \), where \( \{P(\Delta u) : \Delta u \geq 0\} \) is a Poisson process with rate \( \lambda \), and \( \{D_i : i \geq 1\} \) are independent random values drawn with equal probability from \([-1, +1]\). The variance of the compound Poisson process is \( \Delta u \left(\bar{D}^2\right) \); but as \( \Delta N_F(\Delta u) \) has zero mean, we conclude that

\[
\left\langle \Delta N_F(\Delta u)^2 \right\rangle = \lambda \Delta u \quad (\Delta u < a).
\]
Fig. 2: Wavelet based scaling exponent fitting for the intersections of random percolation cluster (h_c) and hull (h_h); the randomized Sierpinski carpet (h_{rand}); the self-affine randomized Sierpinski carpet (h_{rand1} and h_{rand2}); and the intersection of a percolation cluster with a deterministic Sierpinski carpet (h_{n}). Results for different fractals have been moved up or down to fit on the same graph. Uncertainties are given with 0.95 significance. The predicted values $h = d_f/2H$ are in parentheses. Only filled points were used for the fits.

We now estimate the Poisson process rate $\lambda$. At displacement $\Delta u$ the moving balls cover the volume $V_X \sim N_X(a)D^{-1}\Delta u \sim L^dx aD^{-1-d}x \Delta u$. Assuming the standing balls are distributed quasi-homogeneously (the fractal $F$ is random), their density per unit of space is $\rho_F \sim N_F(a)/L^D \sim L^d a^{-d}x \Delta u$. The number of balls encountered during movement $\Delta u$ must then be $N_{\Delta u} \sim \rho_F V_X \sim L^{d_f} a^{-1-d_f} \Delta u$. The rate of balls encountered is $\lambda \sim N_{\Delta u}/\Delta u \sim L^{d_f} a^{-1-d_f}$.

At the marginally applicable limit $\Delta u = a$ of the two expressions (10) and (11), we estimate the change in the mass as

$$|\Delta M(a)| \sim m_u a^{1/2} \sim L^{d_f} a^{1/2}.$$  

Since the ball cover size $a$ can be freely chosen between 1 and $L$, we can pick $a = 1/2$, confining conjecture (8) with the Hurst exponent $h = d_f/2$.

For the case $H_0 \neq 1$, one would have to take into account that the correlations in the self-similar structure of the fractal scale at a different rate in the direction $\hat{u}$.

The intersection of two fractals may have a dimension less than 0. Previously, it has been interpreted as how “empty” the intersection is [8,14]. In eq. (9) this would result in negative $h$. This is not necessarily a pathological case, as a negative $h$ can be used when instead of (8) the scaling is given through the Fourier power spectrum, that is through the relation $|\hat{\psi}(k)|^2 \propto |k|^{-2h-1}$. However, we have not tested this numerically.

To test the relation (9) we ran Monte Carlo simulations for the following cases: two-dimensional random bond percolation cluster and hull intersected with a horizontal line, with predicted $h_c = (91/48 + 1 - 2)/2$ and $h_h = (7/4 + 1 - 2)/2$ for the cluster and hull respectively; randomized $3 \times 3$ Sierpinski carpet [15–19] (with one cell cleared randomly at each construction step) intersected with a horizontal line, with predicted $h_s = (\log_3 8 + 1 - 2)/2$; self-affine randomized $4 \times 3$ Sierpinski carpet (with one cell cleared randomly at each construction step) intersected with vertical and horizontal lines, with the carpet’s box dimension $d_{x,3} = \log_3 (3^{1-\log_3 3} 1^{\log_3 3})$ and predicted scaling exponents $h_{n1} = \frac{\log_3 11}{2 \log_3 4}$ and $h_{n2} = \frac{d_{x,3} + 1 - 2}{2 \log_3 3}$; percolation cluster intersected with a deterministic $3 \times 3$ Sierpinski carpet, with predicted $h_{n} = (91/48 + \log_3 5 - 2)/2$. The results from the Monte Carlo simulations are all very close to the predicted values (fig. 2). As we increased calculation lattice sizes we saw improvement across the board, indicating that the small discrepancies are due to the finite size effects.

To conclude, it is now easy to see that the flow rate of the river Nile, famously studied by Harold Edwin Hurst [20], is an integral quantity of the fractal structure of precipitation [1,2] over its drainage basin, and as confirmed by the analytical relation we have found, is self-affine. This analytical relation should be applicable in both predictive and descriptive capacity for many problems, from the matter distribution of the universe to the formation of 1/f-like noise in semiconductor devices.

***

This work was supported by Estonian Science Targeted Project No. SF0140077s08, and EU Regional Development Fund Centre of Excellence TK124. The authors would like to thank the Department of Computer Engineering at the Tallinn University of Technology for providing computational resources.

REFERENCES

[1] LOVEJOY S. and MANDELBROT B. B., Tellus A, 37A (1985) 209.
[2] LOVEJOY S. and SCHERTZER D., J. Geophys. Res., 95 (1990) 2021.
[3] FOSSUM J. O., BERGENE H. H., HANSEN A. O., ORourke B. and MANIFICAT G., Phys. Rev. E, 69 (2004) 036108.
[4] ZAKHAROV V. E. and FILONENKO N. N., J. Appl. Mech. Tech. Phys., 8 (1967) 37.
[5] DYACHENKO A. I., KOROTKEVICH A. O. and ZAKHAROV V. E., Phys. Rev. Lett., 92 (2004) 134501.
[6] OREY S., Z. Wahrscheinlichkeitstheor. Verw. Geb., 15 (1970) 249.
[7] MANDELBROT B. B., The Fractal Geometry of Nature (Freeman, New York) 1982.
[8] MANDELBROT B. B., J. Stat. Phys., 34 (1984) 895.
[9] TEKHS H. C., A First Course in Stochastic Models (John Wiley & Sons Ltd., Chichester, UK) 2003.
[10] DAVIES R. B. and HARTO D. S., Biometrika, 74 (1987) 95.
[11] DIETRICH C. R. and NEWSAM G. N., SIAM J. Sci. Comput., 18 (1997) 1088.
[12] Wood A. T. A. and Chan G., J. Comput. Graph. Stat., 3 (1994) 409.

[13] Simonsen I., Hansen A. and Nes O. M., Phys. Rev. E, 58 (1998) 2779.

[14] Mandelbrot B. B., Physica A, 163 (1990) 306.

[15] Sierpiński W., C. R. Acad. Sci. Paris, 162 (1916) 629.

[16] McMullen C., Nagoya Math. J., 96 (1984) 1.

[17] Bedford T. J., Crinkly curves, Markov partitions and dimension, PhD, University of Warwick (1984).

[18] Peres Y., Math. Proc. Cambridge Philos. Soc., 115 (1994) 437.

[19] Gui Y. and Li W., Nonlinearity, 21 (2008) 1745.

[20] Hurst H. E., Trans. Am. Soc. Civ. Eng., 116 (1951) 770.