Magneto-optical Conductance of Pseudospin-1 Fermionic Gas

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Photon-like fermion (feroton) is the counterpart of the Weyl fermion with pseudospin-1 and can emerge in condensed matter systems. Due to the existence of a bunch of gapless modes associated with Landau levels, the magnetic transport property of feroton gas is very different from that of the Weyl semimetal. We calculate the magneto-optical conductance of the feroton gas. We find that these gapless modes will contribute to a series of resonant peaks near zero frequency. They are magnificently enhanced by a quadratic correction in the feroton spectrum. These resonant peaks also appear in the magneto-optical conductance of Kane fermion gas, the spinful feroton. Due to a quasi-one dimensional particle-hole symmetry, the magneto-optical conductance of Kane fermion is the double of the feroton. We hope these resonant phenomena would be experimentally observed in feroton materials candidates and Kane fermion materials.

I. INTRODUCTION

The discovery of Weyl semimetal\cite{1,2} opens a gate to observe new topological phenomena in condensed matter systems, such as the Fermi arcs, giant negative magneto-resistance, and three dimensional quantum anomalous Hall effect\cite{3}. These exotic properties are rooted from the non-trivial Berry’s phase of the Weyl nodes in spectrum: The linearly touched Weyl points serve as monopoles of the Berry’s curvature\cite{1,3}. One of the evidences of linear band touching structure of Weyl semimetals comes from the measurement of its magneto-optical conductance, i.e., the resonant frequency peaks between the extrema of Landau levels are proportional to the square root of the external magnetic field\cite{4,5,6}.

Besides the Weyl semimetal, new kinds of fermions have been proposed and observed in condensed matter systems\cite{9,10,11,12}. The triple degenerate node in the band structure of MoP is an example that does not have an analog in high energy systems\cite{13}. Such a fermionic quasiparticle emerges because these ”relativistic” quasiparticles in condensed matter systems are not confined by the ”Lorentz symmetry”. A more interesting triple degenerate fermion might be the photon-like fermion (feroton), the pseudospin-1 generalization of Weyl fermion. The feroton is proposed to exist in materials with space group 199 and 214, such as Pd\textsubscript{3}Bi\textsubscript{2}S\textsubscript{2} and Ag\textsubscript{3}Se\textsubscript{2}Au\cite{11}. Recently, APd\textsubscript{3}(A=Pb, Sn)\cite{14}, LaPtBi\textsuperscript{15}, and ZrTe\textsuperscript{16} are also proposed to have triple nodal points. More possible materials would be found by the method of symmetry indicators\cite{17,18}.

Although the topological properties of feroton are similar to those of Weyl fermion, e.g., the monopole charge of the feroton is twice as large as that of the Weyl fermion, there is a longitudinal photon-like flat band in free feroton gas which is absent in Weyl semimetal. After applying an external magnetic field, this flat band becomes a bunch of gapless modes in Landau levels, and they generate anomalous magnetic transport properties for feroton gas. The anomalous magnetic resistance and extra quantum oscillation of the density of states of the feroton gas were studied\cite{19}.

The magneto-optical measurement can also explore the physical effects of these gapless modes in Landau levels. In this paper, we will calculate the magneto-optical conductance of the feroton gas. We first calculate the magneto-optical conductance contributed by a single feroton node. We find that there are intriguing resonant peaks near the zero frequency in the magneto-optical conductance. These peaks are originated from the transition between the gapless Landau modes. Furthermore, the peak may split due to the resonance between the gapless Landau modes and the ordinary gapped ones, which won’t show up if the gapless Landau modes are exactly flat. Because of an emergent quasi-one dimensional particle-hole symmetry of the Hamiltonian, the contribution to the magneto-conductance from the feroton node with opposite chirality is the same as its chiral partner’s. Therefore, the magneto-optical conductance in a feroton gas can be obtained by toting-up that from each single feroton node. We also discuss the linear dependence between the resonant frequency \( \omega \) and the square root external magnetic field \( B \), which indicates the linear dispersion behavior of the gapless Landau modes.

The spinful counterpart of Weyl fermion is the massless Dirac fermion. Similarly, there is also the spinful counterpart of the feroton. It is the Kane fermion and has been recently experimentally observed in Hg\textsubscript{1–x}Cd\textsubscript{x}Te\cite{4}. The experimentally measured high frequency magneto-optical resonance is consistent with the characteristic of the massless Kane fermion. Recent experiment confirms these magneto-optical signature of massless Kane fermions in Cd\textsubscript{3}As\textsubscript{2}\cite{22}. Numerically, the magneto-optics of the Kane fermion was studied\cite{21,22}.

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However, for the gapless Kane fermion, the contributions from the gapless Landau modes to the magneto-optical conductance were not considered in the numerical calculations. Experimentally, the measurement for Cd$_3$As$_2$ does reach that low frequency while the data for Hg$_{1-x}$ Cd$_x$Te was well explained. Due to the quasi-one dimensional particle-hole symmetry of the gapless Kane fermion model, we find that the magneto-optical conductance of the gapless Kane fermion gas is simply the double of the feroton gas. Thus, our study can be directly applied to the gapless Kane fermion. Especially, we compare our calculation for the magneto-optical conductance near the zero frequency with the data for Hg$_{1-x}$ Cd$_x$Te in [9] and give a reasonable explanation to the low frequency peaks in the magneto-optical absorbance.

This paper was organized as follows: In Sec. II, we recall the Landau level structure of a single feroton node and calculate the magneto-optical conductance of the feroton gas. In Sec. III, we argued that the magneto-optical conductance of a single Kane fermion node is simply the double of a single feroton node because of the quasi-one dimensional particle-hole symmetry in the gapless Kane fermion model. Comparison of our numerical results with the experimental measurements is presented. In Sec. IV, we introduce a quadratic correction term to the spectrum of the feroton as well as the magneto-optical conductance. The last section is devoted to conclusions.

II. MAGNETO-OPTICAL CONDUCTANCE OF FEROTON GAS

A. Single feroton node

We consider a three dimensional lattice system. If there are feroton nodes, the band structure near a given feroton node can be described by the following effective Hamiltonian [10],

\[
H_0 = \hbar v_F \sum_{i=1,2,3} p_i S_i,
\]

where \((S_i)_{jk} = -i \epsilon_{ijk} \); \(v_F\) is the Fermi velocity and \(p_i\) is the Bloch wave vector. For later convenience, we choose \(\hbar = k_B = v_F = e = 1\) except explicitly shown. We apply a unitary transformation \(U\) on the basis with

\[
U = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & i/\sqrt{2} & 0 \end{pmatrix}. \tag{2}
\]

The Hamiltonian (1) becomes

\[
H = U H_0 U^\dagger = \begin{pmatrix} -p_3 & (p_1 + ip_2)/\sqrt{2} & 0 \\ (p_1 - ip_2)/\sqrt{2} & 0 & (p_1 + ip_2)/\sqrt{2} \\ 0 & (p_1 - ip_2)/\sqrt{2} & p_3 \end{pmatrix}. \tag{3}
\]

The Hamiltonian (3) has both inversion symmetry and time reversal symmetry, namely, \(O H(\vec{p}) O^\dagger = H(-\vec{p})^*\) with

\[
O = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4}
\]

However, there is no charge-conjugation symmetry for the Hamiltonian (3). The CPT theorem and the spin-statistics theorem are violated because the Lorentz symmetry is broken for the feroton. The spectrum of the feroton is a triply degenerate node with \(E_{\pm} = \pm |p|\) and \(E_0 = 0\).

Under a magnetic field \(B = \nabla \times A\) in the \(z\) direction, the momentum operator is given by \(D = -i \hbar \nabla + \frac{z}{2} \mathbf{A}\). Corresponding to (3), the Hamiltonian near the feroton node reads

\[
H_B = \begin{pmatrix} -p_z & a_l/\sqrt{2} & 0 \\ a_l & 0 & a_l/\sqrt{2} \\ 0 & a_l/\sqrt{2} & p_z \end{pmatrix}, \tag{5}
\]

with \(a = l_B (D_x - i D_y)/\sqrt{2}\) being the annihilation operator of Landau levels (The charge of feroton is set to

\[
\text{FIG. 1: A sketch diagram of Landau Level structure for feroton. We choose } l_B = 1 \text{ in Eq. (5). The green curves are the chiral lowest Landau level and the gapless ones, the blue curves are the 1st Landau level and the gapped ones with positive energies, and the red curves are the 1st Landau level and the gapped ones with negative energies. The black lines correspond to the chemical potential used in Fig. 2(a).}
\]
There are three eigenvalues $\alpha \geq 0$ for the $n$th Landau level. The eigen wave function $\Psi(n)$ of the $n$th Landau level has the form $(\sigma_1 \phi_n, \beta_n, -\phi_n, 0)^T$, where $\phi_n$ is the $n$th normalized Landau level wave function with respect to $a^\dagger a$ and $\alpha, \beta, \gamma$ are the coefficients. There are two special cases. $(0, 0, 0)^T$ and $(\alpha_1 \phi_1, \beta_0, 0, 0)^T$ correspond to the zeroth and the first Landau levels. We plot several Landau level structures in Fig. 1.

The zeroth and the first Landau levels have the dispersions $E_0 = -p_z$, and $E_1 = -p_z + \sqrt{p_z^2 + 4B}$. The robustness of these two Landau levels is guaranteed by the chiral anomaly. For the Landau level with $n \geq 2$, there are three eigenvalues

$$E_n^+ = \omega_+(-p_z/2 + \sqrt{\Delta})^\dagger + \omega_-(p_z/2 - \sqrt{\Delta})^\dagger,$$

$$E_n^- = \omega_+(-p_z/2 + \sqrt{\Delta})^\dagger + \omega_-(p_z/2 - \sqrt{\Delta})^\dagger,$$

$$E_n^0 = -(p_z/2 + \sqrt{\Delta})^\dagger + (-p_z/2 - \sqrt{\Delta})^\dagger,$$

where $\omega_{\pm} = -1 \pm \sqrt{3}$ and $\Delta = p_z^2/4 - (2n + 1 + p_z^2)^3/27$. In the $p_z \to 0$ limit, $E_n^+ \to \pm \sqrt{2n + 1}$ and $E_n^0 = 0$ which reduce to the two dimensional results.

$E_n^\pm$ can be thought as a pseudo-spin 1 counterpart of the Landau levels of Weyl fermion while $E_n^0$ emerge from the zero field flat band when the external magnetic field is applied. These modes are absent for Weyl fermion.

In a previous work, we showed the effects of the gapless Landau modes to the magnetic transport properties and the quantum oscillations of the density of states of the ferromagnet. We generalized the magneto-optical measurement is another possible way to explore the effects of the gapless Landau modes. Using the Kubo formula, we define the magneto-optical current

$$J_{\alpha} = \sigma_{\alpha\beta}E_{\beta},$$

where $\sigma_{\alpha\beta}(\omega) = \frac{-i}{2\pi l_B^2} \sum_{n,n',s,s'} \int \frac{dp_z}{2\pi} f(E_{ns}) - f(E_{n's'}) \langle \Psi_{ns}|J_{\alpha}|\Psi_{n's'}\rangle/\Delta(\omega + E_{ns} - E_{n's'} + i0^+),$

where $f(E_{ns}) = \frac{1}{\exp(E_{ns}/\mu) + 1}$ is the Fermi distribution. In the numerical calculation, we choose the temperature $T = 0.5/l_B$. $n$ is the Landau level index. $s = \pm 1$ are the band indices. Furthermore, the current matrices are $J_{\alpha} = \partial H/\partial p_{\alpha}$, namely,

$$J_x = US_1U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$J_y = US_2U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & i & 0 \\ 0 & -i & 0 & i \\ i & 0 & i & 0 \end{pmatrix}.$$

The expressions of the currents $J_x$ and $J_y$ give the selection rule $n \to n \pm 1$ for a given nth Landau level. We plot the absorptive part of the magneto-optical conductance, namely, the real part of $\sigma_{xx}/\omega$ and the imaginary part of $\sigma_{xy}$ in Fig. 2.

In the numerical calculation, we replace the delta function by a Gaussian distribution with a standard deviation of $\sigma = 0.01$.

For a small $p_z$, $E_n^0 \to -\frac{p_z^2}{2\mu}$ and when $|p_z| \to \infty$, $E_n^0 \to 0$. These behaviors suggest that the $E_n^0 > 0$ for gapless modes near $p_z = 0$ associated with the Landau level $n > 1$ (shorten as the gapless Landau modes). In the limit of $p_z \to 0$, the gapless Landau modes have opposite chirality to that of the zeroth Landau level.

Unlike the chiral mode in the zeroth Landau level which gives the negative magnetoresistance, the gapless Landau modes do not have a topological origin and cause a positive magnetoresistance without a quantized coefficient.

Instead of applying a strong magnetic field to project the system to the lowest Landau level, the gapless Landau modes will exist for arbitrary $B$. In a previous work, we showed the effects of the gapless Landau modes to the magnetic transport properties and the quantum oscillations of the density of states of the ferromagnet.
FIG. 2: (a) is the real part of $\sigma_{xx}$ with $\mu = 0.5/l_B$, $\mu = 1.5/l_B$ and $\mu = 2.5/l_B$ respectively. (b) is the imaginary part of $\sigma_{xy}$ with $\mu = 0.5/l_B$, $\mu = 1.0/l_B$ and $\mu = 1.5/l_B$ respectively. There are extra peaks near zero frequency which indicates the gapless Landau levels are not exactly flat bands.

FIG. 3: The linear behavior between the resonant peak $\omega$ and the square root of the external magnetic field $\sqrt{B}$ for $\mu = 0$. (a) The peaks between $E_{n,0}$ and $E_{n+1,0}$ bands. (b) The peaks between $E_{n,+}$ and $E_{n+1,+}$ bands. (c) The peaks between $E_{n,0}$ and $E_{n+1,-}$ bands. (d) The peaks between $E_{n,+}$ and $E_{n+1,0}$ bands.

From the numerical results (see Fig. 3), the resonant peak frequency $\omega$ of feroton modes is linearly dependent on the square root of external magnetic field $B$ at zero chemical potential. In the case of Weyl fermion, this behavior is exactly related to the linear band structure of the Weyl fermion [4, 5], while this may not be precise for feroton. The reason is that, for Weyl fermion, the gapped Landau levels are symmetric with respect to the $p_z = 0$ axis and hence the resonant transition occurs at $p_z = 0$. The resonant frequency of Weyl fermion is proportional to the energy difference between two adjacent Landau levels at $p_z = 0$, namely $\sqrt{B}$. This argument is not completely correct for feroton because the gapped Landau levels are slightly asymmetric to the $p_z = 0$ axis (see Fig. 1). Therefore the resonant transition does not occur exactly at $p_z = 0$, but some other $p_z$ that extremizes the energy difference between the Landau levels allowed by the selection rules. This indicates that the linear dependence of $\sqrt{B}$ on the energy difference of Landau levels at $p_z = 0$ is not rigorously related to the linear band structure of the feroton.
B. Feroton gas

In the Appendix A we generalize the fermion doubling theorem to the feroton gas, i.e., in a lattice model, the feroton nodes with opposite chirality emerge in pairs. All of the feroton pairs will contribute to the magneto-optical conductance. Notice that the Hamiltonian $H_B(p_z)$ has a quasi one-dimensional particle-hole symmetry. Namely, if we keep the two-dimensional Landau level structure unchanged,

$$CH_B(p_z)C^\dagger = -H_B(-p_z),$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (12)

Because of this symmetry, the eigen wave function $\Psi^s_n = (a^s_n, b^s_n, c^s_n)^T$ of Hamiltonian $H_B$ in [9] possesses the following property: If

$$H_B(p_z)(a^s_n, b^s_n, c^s_n)^T = E^s_n(p_z)(a^s_n, b^s_n, c^s_n)^T,$$

then,

$$-H_B(-p_z)(a^s_n, -b^s_n, c^s_n)^T = E^s_n(p_z)(a^s_n, -b^s_n, c^s_n)^T,$$

$$E^s_n(p_z) = -E^{-s}_n(-p_z),$$

where $s$ is the band index. For later convenience, we denote Hamiltonian $H^L$ near one feroton node as $H^L = H_B$, then the Hamiltonian $H^R$ of the other feroton with opposite chirality reads: $H^R = -H^L$, we have

$$E^s_n(p_z) = -E^{-s}_n(-p_z),$$

$$\Psi^s_n(p_z) = \Psi^{-s}_n(p_z).$$ \hspace{1cm} (17)

Thus

$$\sigma_{n\beta}^R = \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} f(E^s_n(k_z)) - f(E^{s'}_{n'}(k_z)) \langle \Psi^R_{n\alpha}(k_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(k_z) \rangle \langle \Psi^R_{n\alpha}(k_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(k_z) \rangle,$$

$$= -\int_{-\infty}^{\infty} \frac{d(-k_z)}{2\pi} f(E^s_n(k_z)) - f(E^{s'}_{n'}(k_z)) \langle \Psi^R_{n\alpha}(-k_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(-k_z) \rangle \langle \Psi^R_{n\alpha}(-k_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(-k_z) \rangle$$

$$= -\int_{-\infty}^{\infty} \frac{d(-k_z)}{2\pi} f(E^s_n(k_z)) - f(E^{s'}_{n'}(k_z)) \langle \Psi^R_{n\alpha}(-k_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(-k_z) \rangle \langle \Psi^R_{n\alpha}(-k_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(-k_z) \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} [ f(E^s_n(p_z)) - f(E^{s'}_{n'}(p_z)) ] \langle \Psi^R_{n\alpha}(p_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(p_z) \rangle \langle \Psi^R_{n\alpha}(p_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(p_z) \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} [ f(E^s_n(p_z)) - f(E^{s'}_{n'}(p_z)) ] \langle \Psi^R_{n\alpha}(p_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(p_z) \rangle \langle \Psi^R_{n\alpha}(p_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(p_z) \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} [ f(E^s_n(p_z)) - f(E^{s'}_{n'}(p_z)) ] \langle \Psi^R_{n\alpha}(p_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(p_z) \rangle \langle \Psi^R_{n\alpha}(p_z) | J^R_{\alpha} | \Psi^{-s'}_{n's'}(p_z) \rangle$$

$$= \sigma_{n\beta}^R.$$

III. MAGNETO-OPTICAL CONDUCTANCE OF KANE FERONM

A. Results for Kane Fermion

We study the magneto-optical conductance of the Kane fermion, which is the spinful feroton, similar to Weyl fermion versus Dirac fermion [9]. The effective Hamiltonian of a Kane fermion reads,

$$H_K = \left( \begin{array}{cc} p_i S_i & \delta \\ \delta^* & -p_i S_i \end{array} \right).$$  \hspace{1cm} (18)
$\delta$ is the mass matrix of Kane fermion. In the massless case, the effective Hamiltonian $H_K$ reduces to

$$H_K = H_\uparrow \oplus H_\downarrow, \quad (19)$$

with

$$H_\uparrow = p_i S_i, \quad H_\downarrow = -p_i S_i. \quad (20)$$

After the unitary transformation (24), the current operator $J_i$ becomes,

$$J_i = \frac{\partial H_K}{\partial p_i} = J^\uparrow_i \oplus J^\downarrow_i, \quad (21)$$

where

$$J^\uparrow_i = U S_i U^\dagger, \quad J^\downarrow_i = -U S_i U^\dagger. \quad (22)$$

One can easily check,

$$CH_{-i, i}^{\uparrow}(p_z)C^\dagger = H_{-i, i}^{\uparrow}(-p_z) = -H_{-i, i}^{\uparrow}(-p_z), \quad (23)$$

This means that the Hamiltonian $H_K$ is of a symmetry of the quasi one-dimensional particle-hole. The magneto-optical conductance reads,

$$\sigma^{K}_{\alpha\beta}(\omega) = \frac{-i}{2\pi F^2_B} \sum_{n,n',s,s'} \int d\xi \left\{ \frac{f(E^{s\dagger}_n) - f(E^{s'}_n)}{E^{s\dagger}_n - E^{s'}_n} \right\} \frac{\langle \Psi_{n,s} | J^\alpha \Psi_{n',s'} \rangle \langle \Psi_{n,s} | J^\beta \Psi_{n',s'} \rangle}{\omega + E^{s\dagger}_n - E^{s'}_n + i0^+} + \frac{f(E^{s\uparrow}_n) - f(E^{s\dagger}_n)}{E^{s\uparrow}_n - E^{s\dagger}_n} \frac{\langle \Psi_{n,s} | J^\uparrow \Psi_{n',s'} \rangle \langle \Psi_{n,s} | J^\dagger \Psi_{n',s'} \rangle}{\omega + E^{s\dagger}_n - E^{s\uparrow}_n + i0^+} \right. \right\}. \quad (24)$$

Because of the quasi one-dimensional particle-hole symmetry of $H_K(p_z)$, the magneto-optical conductance (24) of a Kane node is the double of that of a feroton node (8).

B. Comparing with experiments

The magneto-optical conductance of the Kane fermion was calculated in the literature (20, 21). However, the low frequency responses of the magneto-optics were ignored in previous calculations. Experimentally, the effective Kane model of Cd$_3$As$_2$ is not valid down to arbitrarily low energies (22).

For another candidate, Hg$_{1-x}$Cd$_x$Te, the magneto-optical absorption was also measured. We adapt Fig. 3 in (2) (see Fig. 4(a)) in order to compare with our study. These data can be reasonably explained by our numerical calculation, especially those low frequency peaks (see Fig. 4(b)). Taking the $\omega = 111$mev, 175mev, and 255mev peaks at $B=28$T in Fig. 4(a) as examples, their ratios are approximately 175/111 $\sim$ 1.58 and 255/111 $\sim$ 2.30. Similarly, in Fig. 4(b), we have three peaks around $\omega = 1.09/l_B$, 1.78/l_B, and 2.31/l_B, and the ratios are 1.78/1.09 $\sim$ 1.63 and 2.31/1.09 $\sim$ 2.12. The origin of the peaks all come from the transition between the gapless Landau level and the ordinary one ($E_{n\pm 1, 0} \leftrightarrow E_{n, 0}$). The peaks near the zero frequency in Fig. 4(a) were explained as phonon contribution (9). These peaks near zero frequencies in our calculation also appear (see Fig. 4(b)) but come from the transitions between the gapless Landau levels $E_{n, 0} \leftrightarrow E_{n+1, 0}$. Notice that there are weak peaks ranging from 50mev to 100mev at $B=24$T in Fig. 4(a). These peaks were not explained in Ref. (9). In Fig. 4(b), these peaks are not seen because of the suppression by the low temperature and a small chemical potential. Slightly raising the temperature and chemical potential, these peaks are shown in our calculation and they are corresponding to transitions between the ordinary Landau levels ($E_{n, +} \leftrightarrow E_{n+1, +}$), see Fig. 4(c). In Fig. 4, we see the peaks between the gapless Landau level and the ordinary one split, and the splitting is proportional to the maximal energy difference between $E_{n\pm 1, 0}$ bands.

IV. QUADRATIC CORRECTION

We may read out more informations from the $E^0_n$ bands by adding a quadratic term of the wave vector in the Hamiltonian $H_0$. This quadratic term will break the quasi one-dimensional particle-hole symmetry of $H_0$ and lift the degeneracy at $p_z = 0$ of $E^0_n$ Landau levels.

The perturbed Hamiltonian reads,

$$H_Q = p_i S_i + \frac{1}{2m^*} p_i^2, \quad (25)$$

where $m^*$ is the effective mass. Using the same unitary transformation (2), the Hamiltonian $H_Q$ under a constant magnetic field $B$ becomes (19),

$$H_{QB} = \begin{pmatrix}
-p_z + \frac{1}{2m^*}(p^2_z + 2B a^\dagger a + B) & \frac{1}{2m^*}(p^2_z + 2B a^\dagger a + B) & 0 \\
\sqrt{B} a^\dagger & \sqrt{B} a & \sqrt{B} a^\dagger \\
0 & \sqrt{B} a & \sqrt{B} a^\dagger 
\end{pmatrix}. \quad (26)$$

For a given eigen wave function of $H_B$, for instance,

$$\Psi(n), \quad H_{QB}(p_z)\Psi(n) = [H_B(p_z - \omega_c) + \frac{p_z^2}{2m^*} + \omega_c(n + \frac{1}{2})]\Psi(n), \quad (27)$$
with \( \omega_c = B/m^* \). This relation shows that, the quadratic correction does not change the eigen wave functions and the eigen energy satisfies

\[
E_{QB}(p_z, n, s) = E_B(p_z - \omega_c, n, s) + \frac{p_z^2}{2m^*} + \omega_c(n + \frac{1}{2}),
\]

where \( n \) is the Landau level index, and \( s = 0, \pm \) labels the bands within the \( n \)th Landau level. An example of the quadratic correction to the Landau level spectrum is shown in Fig. 5.

Furthermore, because of the quadratic correction, the currents \( J^Q_\alpha = \frac{\partial H}{\partial p_\alpha} \) become

\[
J^Q_x = j_x + \frac{1}{m^*} \sqrt{\frac{B}{2}} (a + a^\dagger), \quad (29)
J^Q_y = j_y + \frac{i}{m^*} \sqrt{\frac{B}{2}} (a - a^\dagger). \quad (30)
\]

Therefore the selection rules remain unchanged, namely, \( n \to n \pm 1 \). From the spectrum relation (28) and the unchanged selection rules, the resonant peaks will split to \( \omega \to \omega \pm \omega_c \).

We plot the magneto-optical conductance with quadratic correction to the spectrum in Fig. 5. We see that the mass term flattens the extrema of Landau levels, the absorption strength becomes stronger because the resonant frequency is determined by the difference between the extrema and the density of states becomes larger in the presence of the mass term. For a large enough mass \( m^* \), the peak splitting becomes obvious. For example, see the blue curve in Fig. 5(a). The magnitude of the peak splitting is \( 2\omega_c \). Furthermore, the quadratic term lifts the infinite degeneracy at \( p_z = 0 \) of the gapless Landau levels. The peaks near the zero frequency is enhanced magnificently. This fact provides a strong evidence for the existence of the gapless Landau levels instead of exact flat bands, which is experimentally detectable.

V. CONCLUSIONS

We studied the magneto-optical conductance of the feroton fermion and the Kane fermion. Besides the ordinary resonant peaks as in Weyl fermion, there are extra peaks stemming from the gapless Landau levels which are absent in the Weyl fermion case. Especially, there are peaks near zero frequency coming from the transition between the gapless Landau modes, and splitting of the peaks stemming from the transition between the gapless Landau levels and ordinary gapped ones, both of which show strong evidence for the existence of gapless modes instead of flat bands. We also considered the effects of quadratic correction. This term breaks the quasi one-dimensional particle-hole symmetry and splits the resonant peaks. The infinite degeneracy of the gapless Landau levels at zero momentum is also lifted by the quadratic correction and provides a magnicient enhancement in the peaks near the zero frequency. These phenomena would be experimentally detected in feroton material candidates. Applying our calculation to the Kane fermion, we gave the magneto-optical absorbance measurements for Hg_{1-x}Cd_xTe a reasonable explanation.

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Appendix A: Generalization of Nielsen-Ninomiya no-go theorem to the feroton semimetals

In 1981, Nielsen and Ninomiya published a series of papers proving a no-go theorem which stated the number of Weyl points with opposite chirality should be the same on a lattice satisfying four assumptions [20]:

(i) The Hamiltonian is local.
(ii) The lattice is translational invariant.
(iii) The Hamiltonian is Hermitian.
(iv) There is at least one quantized conserved charge \( Q \) for the fermion field.

In the following, we generalize Nielsen-Ninomiya no-go theorem to the feroton semimetals with an additional assumption, namely, only feroton points are involved in the lattice model near the Fermi surface.

Near a feroton point, the Hamiltonian can be expanded as,

\[
H^{(3)} = p_i V^a_i S^i + \mathcal{O}(p^2), \quad (A1)
\]

where \( (S^i)_{jk} = -i \epsilon_{ijk} \) being the spin-1 representation of the generators of the angular momentum algebra. The feroton point is right(left) handed if \( \det V > 0(\det V < 0) \). Define \( P_1 \equiv p_i V^a_i \), then the spectra and eigen wavefunctions in spherical coordinates read,

\[
E_0 = 0, \quad \psi_0 = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T,
E_+ = |P|, \quad \psi_+ = \left(\frac{-\cos \theta \cos \phi + i \sin \phi}{\sqrt{2}}, \frac{-i \cos \phi - \cos \theta \sin \phi}{\sqrt{2}}, \frac{\sin \theta}{\sqrt{2}}\right)^T,
E_- = -|P|, \quad \psi_- = \left(\frac{-\cos \theta \cos \phi - i \sin \phi}{\sqrt{2}}, \frac{i \cos \phi - \cos \theta \sin \phi}{\sqrt{2}}, \frac{\sin \theta}{\sqrt{2}}\right)^T. \quad (A2)
\]

As we discussed in our former paper on the feroton semimetal [19], for the free Hamiltonian, the zero energy flat band \( E_0 \) generates an emergent gauge symmetry. The reason is that, from the equation of motion of
Hamiltonian \[ A1 \],
\[ i\partial_t \psi_j = -i\partial_j S^{i}_{jk} \psi_k. \] (A3)
Since \( S^{i}_{jk} \) is totally anti-symmetric, then, multiplying both sides with \( \partial_j \),
\[ i\partial_t (\partial_j \psi_j) = 0, \] (A4)
which means \( \partial_j \psi_j = C(x_i) \). Notice that there is a zero energy wave function \( \psi_0 = \partial_t \Lambda(x_i) \) for the Hamiltonian \[ A1 \], and if we choose \( \Lambda(x_i) \) to satisfy \( \partial^2 \Lambda(x_i) = C(x_i) \), then we can consider a new state,
\[ \psi_j = \psi'_j + \partial_j \Lambda(x_i), \] (A5)
and,
\[ \partial_j \psi'_j = 0, \] (A6)
which is a constraint on the wave function \( \psi'_j \), namely, only two of the components in \( \psi'_j \) are independent. Therefore the flat band is a redundant gauge degree of freedom and unphysical which should be projected out, which is similar to the unphysical longitudinal photon \[ 19 \]. Later we will also prove that the flat band wave function is topologically trivial and will not contribute to the topological number which is associated with the chirality of the feroton point, namely, the exact flat band will not affect the no-go theorem.

In a proper basis, the Taylor expansion of the N-band lattice Hamiltonian near the feroton point reads \[ 20 \],
\[
H^{(3)}_N \sim p_a V^a = 
\begin{pmatrix}
(i - 1) & (i) & (i + 1) \\
S_{11}^i & S_{12}^i & S_{13}^i & (i - 1) \\
S_{21}^i & S_{22}^i & S_{23}^i & (i) \\
S_{31}^i & S_{32}^i & S_{33}^i & (i + 1)
\end{pmatrix}
\begin{pmatrix}
b_1 \\
0 \\
\vdots \\
0 \\
b_N
\end{pmatrix}
\begin{pmatrix}
(i - 1) & (i) & (i + 1) \\
0 & * & * & * \\
* & 0 & 0 & 0 & (i - 1) \\
* & 0 & 0 & 0 & (i) \\
* & 0 & 0 & 0 & (i + 1) \\
* & * & * & 0
\end{pmatrix}
+ \mathcal{O}(p^2),
\] (A7)
N-dimensional wave function being,
\[ |\omega_+(p_i)\rangle = \begin{pmatrix}
\psi_1^+ \\
\psi_2^+ \\
\psi_3^+ \\
0
\end{pmatrix} \begin{pmatrix}
i - 1 \\
i \\
i + 1 \\
0
\end{pmatrix}. \] (A8)
Now we define a normalized function
\[ f_+(\theta, \phi) = |\omega_+(\theta, \phi)\rangle \] (A9)
where the domain of \( f_+ \) is on an infinitesimal sphere \( S^2 \) around the feroton point. The map \( f_+ \) is an element in the homotopy group \( \pi_2(CP^{N-1}) \). The map \( f_+(\theta, \phi) \) restricted at the south pole \( \theta = \pi \) reads,
\[ f_+(\pi, \phi) = e^{i\phi} \begin{pmatrix}
0 \\
1/\sqrt{2} \\
-i/\sqrt{2} \\
0
\end{pmatrix} \begin{pmatrix}
i - 1 \\
i \\
i + 1 \\
0
\end{pmatrix}. \] (A10)
Notice that the south pole on \( S^2 \) can be viewed as the whole boundary of \( E^2 \) mapping to the same point, therefore the class \([ f \mid s^1 \in \pi_1(s^1) \) is the winding number which is +1 or -1. When \( \det V > 0 \), the coordinate system near the feroton point is right-handed, then the winding number is +1, and when \( \det V < 0 \), the coordinate system near feroton point is left-handed and the winding number is -1. Similarly, for the negative energy branch, when \( \det V < 0 \), the winding number is -1, and when \( \det V < 0 \), the winding number is +1.

For the exact flat band, the winding number is zero because \( f_0(\pi, \theta) = (0, 0, ..., 0, -1, 0, ..., 0)^T \). To sum, for the infinitesimal \( S^2 \) sphere surrounding a degenerate feroton point, the positive(negative) energy branch with positive(negative) helicity corresponds to +1 element of \( \pi_2(CP^{N-1}) \), and the positive(negative) energy branch with negative(positive) helicity corresponds to -1 element.

Because of the periodicity of the Brillouin zone and the additivity of the \( \pi_2(CP^{N-1}) \) group, we have \[ 26 \]
\[ \sum_i [ f_i ] = 0, \] (A11)
where \( f_{BS} \) imbeds the Brillouin zone surface \( S^2 \) into \( CP^{N-1} \), \([ f_{BS} \) denotes the corresponding element in \( \pi_2(CP^{N-1}) \), and the summation \( i \) runs over all the degenerate points \( i \). The Eq. \( A11 \) means the total winding number for the degenerate points between the \( i \)-th and \( (i + 1) \)-th or \( (i - 1) \)-th bands are zero (we have omitted the exact flat bands and assumed only feroton points are involved), namely,
\[ N_r(i, i + 1) - N_r(i - 1, i) = N_i(i, i + 1) - N_i(i - 1, i), \] (A12)
where \( N_r(i, i + 1) \) is the number of degenerate points between \( i \)-th and \( (i + 1) \)-th bands with the upper \( i \)-th
having positive helicity. Notice that for the highest band, \( N_r(0, 1) = N_l(0, 1) = 0 \), therefore we have,
\[
N_r(i, i + 1) = N_l(i, i + 1). \tag{A13}
\]

The Eq. (A13) proves our generalization of the Nielsen-Ninomiya no-go theorem, namely there are equal number of the left-handed feroton points and the right-handed ones in a lattice model.

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FIG. 4: (a) Landau levels (for \( p_z = 0 \)) and experimentally measured absorbance as a function of photon energy at a temperature of 1.8 K with \( B = 16T \) and 28T respectively in \( \text{Hg}_{1-x}\text{Cd}_x\text{Te} \). (Adapted from Fig. 3 in Ref. [9].) We plot the real part of \( \sigma_{xx} \) with \( l_B = 1 \) according to our calculations for (b) \( T = 0.01/l_B, \mu = 0.01/l_B, \mu = 0.03/l_B, \) and \( \mu = 0.05/l_B \); (c) \( T = 0.3/l_B, \mu = 0.1/l_B, \mu = 0.3/l_B, \) and \( \mu = 0.5/l_B \).
FIG. 5: A sketch diagram of Landau Level structure for feroton with quadratic corrections \( l_B = 1 \) and \( m^* = 0.02/l_B \). The meaning of the colored curves is similar to that in Fig. 1. The absorptions in the conductance are plotted in Fig. 6.
FIG. 6: (a) and (b) are the real part of $\sigma_{xx}$ and the imaginary part of $\sigma_{xy}$ with quadratic corrections. The masses are $1/m = 0$, $1/m = 0.005l_B$, and $1/m = 0.02l_B$ respectively. The chemical potential is chosen to be $\mu = 0.01l_B$. The peaks near zero frequency are magnificently enhanced by the quadratic terms. The other high energy peaks are also split by the correction term.