SELF-SIMILAR PERFECT FLUIDS

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Space-times admitting an $r$-parameter Lie group of homotheties are studied for $r > 2$ devoting a special attention to those representing perfect fluid solutions to Einstein’s field equations.

1 Basic facts about homotheties

Throughout this paper $(M, g)$ will denote a space-time: $M$ then being a Hausdorff, simply connected, four-dimensional manifold, and $g$ a Lorentz metric of signature $(-,+,+,+)$. All the structures will be assumed smooth.

A global vector field $X$ on $M$ is called homothetic if either one of the following conditions holds on a local chart $L X g_{ab} = 2 \lambda g_{ab}$, $X_{a:b} = \lambda g_{ab} + F_{ab}$, (1)

where $\lambda$ is a constant on $M$, $L$ stands for the Lie derivative operator, a semi-colon denotes a covariant derivative with respect to the metric connection, and $F_{ab} = -F_{ba}$ is the so-called homothetic bivector. If $\lambda \neq 0$, $X$ is called proper homothetic and if $\lambda = 0$, $X$ is a Killing vector field (KV) on $M$. For a geometrical interpretation of (1) we refer the reader to [5], [6].

A necessary condition that $X$ be homothetic is

$L X R_{abcd} = L X R_{ab} = L X C_{abcd} = 0$, (3)

where $R_{abcd}$ are the components of the Riemann tensor in a coordinate chart, $R_{ab} = R_{acdb}$ and $C_{abcd}$ stand, respectively for the components of the Ricci and the conformal Weyl tensor.

The set of all HVFs on $M$ forms a finite dimensional Lie algebra under the usual bracket operation and will be referred to as the homothetic algebra, $H_r$, $r$ being its dimension. The set of all Killing vectors fields on $M$ forms a finite dimensional Lie algebra (dimension $s$) under the same bracket operation, and will be referred to here as the Lie algebra of isometries, $G_s$ which is contained...
in (i.e., is a subalgebra of) $\mathcal{H}_r$. Furthermore, it is immediate to see by direct computation that the Lie bracket of an HVF with a KV is always a KV. From these considerations it immediately follows that the highest possible dimension of $\mathcal{H}_r$ in a four-dimensional manifold is $r = 11$.

If $r \neq s$ then $s = r-1$ necessarily, and one may choose a basis $\{X_A\}_{A=1}^r \equiv \{X_1, \cdots, X_{r-1}, X\}$ for $\mathcal{H}_r$, in such a way that $X$ is proper homothetic and $X_1, \cdots, X_{r-1}$ are Killing vector fields spanning $\mathcal{G}_{r-1}$. If these vector fields in the basis of $\mathcal{H}_r$ are all complete vector fields, then each member of $\mathcal{H}_r$ is complete and Palais’ theorem guarantees the existence of an $r$-dimensional Lie group of homothetic transformations of $M$ ($\mathcal{H}_r$) in a well-known way; otherwise, it gives rise to a local group of local homothetic transformations of $M$ and, although the usual concepts of isotropy and orbits still hold, a little more care is required.

The following result will be useful: The orbits associated with $\mathcal{H}_r$ and $\mathcal{G}_{r-1}$ can only coincide if either they are four-dimensional or three-dimensional and null. (This result still holds if $\mathcal{H}_r$ is replaced by the conformal Lie algebra $\mathcal{C}_r$ and does not depend on the maximality of $\mathcal{H}_r$ or $\mathcal{C}_r$).

The set of zeroes of a proper HVF, i.e., $\{p \in M : X(p) = 0\}$ (fixed points of the homothety), either consists of topologically isolated points, or else is part of a null geodesic. The latter case corresponds to the well-known (conformally flat or Petrov type N) plane waves.

At any zero of a proper HVF on $M$ all Ricci and Weyl eigenvalues must necessarily vanish and thus the Ricci tensor is either zero or has Segre type $\{(2,11)\}$ or $\{(3,1)\}$ (both with zero eigenvalue), whereas the Weyl tensor is of the Petrov type $O$, $N$ or $III$ (see also for vacuum space-times).

## 2 Basic facts about perfect fluids admitting HVFs

The energy-momentum tensor for a perfect fluid is given by

$$T_{ab} = (\mu + p) u_a u_b + pg_{ab} \quad ,$$

where $\mu$ and $p$ are, respectively, the energy density and the pressure as measured by an observer comoving with the fluid, and $u^a$ ($u^a u_a = -1$) is the four-velocity of the fluid. If $X$ is an HVF then, from Einstein’s Field Equations (EFE) it follows that

$$\mathcal{L}_X T_{ab} = 0 \quad ,$$

and this implies in turn

$$\mathcal{L}_X u_a = \lambda u_a \quad , \quad \mathcal{L}_X p = -2\lambda p \quad , \quad \mathcal{L}_X \mu = -2\lambda \mu \quad .$$
Thus, the Lie derivatives of \( u_a \), \( p \) and \( \mu \) with respect to a KV vanish identically.

If a barotropic equation of state exists, \( p = p(\mu) \), and the space-time admits a proper HVF \( X \) then

\[
p = (\gamma - 1)\mu ,
\]

where \( \gamma \) is a constant (\( 0 \leq \gamma \leq 2 \) in order to comply with the weak and dominant energy conditions). Of particular interest are the values \( \gamma = 1 \) (pressure-free matter, “dust”) and \( \gamma = 4/3 \) (radiation fluid). In addition, the value \( \gamma = 2 \) (stiff-matter) has been considered in connection with the early Universe. Furthermore, values of \( \gamma \) satisfying \( 0 \leq \gamma < 2/3 \), while physically unrealistic as regards a classical fluid, are of interest in connection with inflationary models of the Universe. In particular, the value \( \gamma = 0 \), for which the fluid can be interpreted as a positive cosmological constant, corresponds to exponential inflation, while the values \( 0 < \gamma < 2/3 \) correspond to power law inflation in FRW models, but it is customary to restrict \( \gamma \) to the range \( 1 \leq \gamma \leq 2 \).

If the proper HVF \( X \) and the four-velocity \( u \) are mutually orthogonal (i.e., \( u^a X_a = 0 \)) and a barotropic equation of state is assumed, it follows that \( \gamma = 2 \), i.e., \( p = \mu \) stiff-matter, on the other hand, if \( X^a = \alpha u^a \) the fluid is then shear-free. Further information on this topic can be found in [16, 17, 18].

### 3 The “dimensional count-down”

In this section, the maximal Lie algebra of global HVF on \( M \) will be denoted as \( \mathcal{H}_r \) (\( r \) being its dimension), and it will be assumed that at least one member of it is proper homothetic.

The case of multiply transitive action is thoroughly studied in [11]. We summarize in the following table the results given there, which follow invariably from considerations on the associated Killing subalgebra and the fixed point structure of the proper HVF.

| Dimension | Nature | Type of Orbits | Petrov Type(s) | Interpretation | Perfect Fluid Solution |
|-----------|--------|----------------|----------------|----------------|------------------------|
| \( 0 \)   | Null   | \( N_2 \)      | \( N_2 \)      | Any            | Yes                    |
| \( 1 \)   | Timelike | \( T_2 \)    | \( T_2 \)      | Any            | Yes                    |
| \( 2 \)   | Spacelike | \( S_2 \)  | \( S_2 \)      | Any            | Yes                    |

The first entry in the table gives the dimension of the group of homotheties, the second and third entries stand for the nature and dimension of the homothetic and Killing orbits respectively (e.g.: \( N_2 \), \( T_2 \) and \( S_2 \) denote Null, Timelike and Spacelike two-dimensional orbits respectively, \( O_3 \) stands for three-dimensional orbits of either nature, timelike, spacelike or null), the fourth and fifth entries give the Petrov and Segre type(s) of the associated Weyl and Ricci tensors. Finally, the last two entries give respectively the possible interpretation whenever it is in some sense unique, and the existence or non-existence of perfect fluid solutions for that particular case, along with
some supplementary information; thus FRW stands for Friedmann-Robertson-Walker, LRS for Locally Rotationally Symmetric, and Bianchi refers to that family of perfect fluid solutions. The cases that cannot arise are labeled as “Not Possible”, and wherever no information is given on the Petrov and Segre types, it is to be understood that all types are possible in principle. The Segre type of the Ricci tensor of the case described in the last row, is unrestricted except in that it must necessarily have two equal (spacelike) eigenvalues; perfect fluid solutions of these characteristics constitute special cases of spherically, plane or hyperbolically symmetric perfect fluid space-times. For further information on LRS spacetimes, see 19, 20; for the case \( r = 4 \) transitive and null three-dimensional Killing orbits, see 21, 22. Regarding spatially homogeneous Bianchi models, see 22, 23, and for the last three cases occurring in the table, see 21, 24, 25.

| \( r \) | \( O_{\mu} \) | \( K_{\nu} \) | Petrov | Segre | Interpretation | PF info. |
|-----|-----|-----|-----|-----|-----|-----|
| 11  | \( M \) | \( M \) | \( O \) | \( 0 \) | Flat | \( \exists \) |
| 10  | \( M \) | \( M \) | - | - | Not Possible | \( \exists \) |
| 9   | \( M \) | \( M \) | - | - | Not Possible | \( \exists \) |
| 8   | \( M \) | \( M \) | \( O \) | \( \{(2, 11)\} \) | Gen. P. wave | \( \exists \) |
| 7   | \( M \) | \( M \) | \( N \) | \( 0 \), \( \{(2, 11)\} \) | Gen. P. wave | \( \exists \) |
| 7   | \( M \) | \( T_3 \) | \( O \) | \( \{(1, 11)\} \) | Tachyonic Fl. | \( \exists \) |
| 7   | \( M \) | \( N_3 \) | - | - | Not Possible | \( \exists \) |
| 7   | \( M \) | \( S_3 \) | \( O \) | \( \{(2, 11)\} \) | Gen. P. wave | \( \exists \) |
| 7   | \( M \) | \( S_3 \) | \( O \) | \( \{(1, 11)\} \) | Perfect Fluid | FRW |
| 6   | \( M \) | \( M \) | - | - | Not Possible | \( \exists \) |
| 6   | \( N_3 \) | \( N_3 \) | \( N \) | \( \{(2, 11)\} \) | Gen. P. wave | \( \exists \) |
| 5   | \( M \) | \( M \) | - | - | Not Possible | \( \exists \) |
| 5   | \( M \) | \( N_3 \) | - | - | - | \( \exists \) |
| 5   | \( N_3 \) | \( N_3 \) | - | - | Not Possible | \( \exists \) |
| 5   | \( M \) | \( T_3 \) | \( D, N, O \) | \( \{(1, 1(1))\}, \{(2, 11)\} \) | LRS | \( \exists \) |
| 5   | \( M \) | \( S_3 \) | \( D, O \) | \( \{(1, 1(1))\}, \{(2, 11)\} \) | LRS | \( \exists \) |
| 4   | \( M \) | \( N_3 \) | \( II, III, D, N, O \) | \( \{(1, 1(1))\}, \{(2, 11)\} \) | Plane waves | \( \exists \) |
| 4   | \( N_3 \) | \( N_3 \) | - | - | Not Possible | \( \exists \) |
| 4   | \( M \) | \( T_3 \) | - | - | Bianchi | \( \exists \) |
| 4   | \( M \) | \( S_3 \) | - | - | Bianchi | \( \exists \) |
| 4   | \( O_3 \) | \( N_2 \) | \( N, O \) | \( \{3, 1\}, \{2, 11\}, \{(1, 1)11\} \) | \( \exists \) |
| 4   | \( O_3 \) | \( T_2 \) | \( D, O \) | \( \{(1, 1)11\} \) | \( \exists \) |
| 4   | \( O_3 \) | \( S_2 \) | \( D, O \) | \( \{-(11)\} \) | \( \exists \) |

The case \( r = 3 \) has an associated Killing subalgebra \( G_2 \) and the respective dimensions of their orbits are 3 and 2 (see for instance 26, 27, 28, and references cited therein). When the Killing subalgebra has null orbits, the metric is of Kundt’s class 29 and perfect fluids are excluded. If the Killing orbits are time-like, the solutions can then be interpreted as special cases of axisymmetric stationary space-times 21, 28, and if they are spacelike as special cases of inhomogeneous cosmological solutions or cylindrically symmetric space-times. In
both cases, perfect fluid solutions have been found.

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