Natural oscillation shapes for a heavy fluid in an elliptic vessel

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Abstract. The problem of constructing of shapes of the natural oscillations in a vertical cylindrical basin (vessel) with an elliptic cross-section in linearized form of an ideal incompressible fluid is considered. Only the gravity force is acting on the fluid. It is required to define the frequencies and shapes of the natural oscillations of the fluid as a function of the system’s parameters. The solution algorithm represented here is based on the method of separation of variables in elliptic coordinates, and the special developed method for the high-accuracy solution of the boundary problem. It used the variational principle and the procedure of continuation in the parameter. This approach shows its effectiveness and obtains the almost complete solution of the problem. For five lower oscillation modes, the natural shapes are determined with high degree of accuracy for a wide range of the eccentricity values.

1. Introduction

The problem of the natural oscillations in a vertical cylindrical basin (vessel) with an elliptic cross-section [1–4] in linearized form of an ideal incompressible fluid is considered. On the fluid acts the gravity force with the acceleration $g$. The dependence of the natural oscillation shapes and frequencies on the parameters of the vessel is to determine: the vessel depth $h$ and the ellipse semi-axes $a$ and $b$. An approximate numerical-analytic investigation of the problem in the case $h \ll a$ (and $a > b$) was carried out in [1], with the representations of the Mathieu functions. The lowest natural frequency was calculated for $e = 0.8$ (the ellipse aspect ratio $a:b = 5:3$). It is corresponding to the nodal line along the minor $b$ axis. For $e \to 1$ the higher oscillation modes symmetric about the minor axis were estimated. Unfortunately, this approach is unsuitable for calculating such shapes which nodal line coincides with one of the confocal ellipses.

To avoid this inconvenience the special numerical-analytic method was developed in [4-6]. With it help the solution of the problem for an arbitrary vessel parameters and for arbitrary oscillation mode can be obtain. Here, the five lowest natural shapes and frequencies of the oscillations of a fluid in an elliptic vessel will calculate for $0 \leq e \leq 0.9$.

2. The problem formulation

Let it be the Cartesian coordinate system $xyz$ such that the $x$ and $y$ axes are directed along the major and minor axes, respectively, and the origin at the centre of the ellipse. The fluid occupies domain $D$ in the undisturbed state. It can be described by the expressions

$$D = \left\{ x, y, z : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, -h \leq z \leq 0 \right\}$$

(1)
Let us determine the potential $u(x,y,z,t)$ of the fluid flow velocities in the linear approximation as the $t$-periodic solution of the boundary value problem [2]

$$\Delta_t u = 0, \quad (x,y,z) \in D \setminus \partial D$$

$$(x,y) \in E, \quad -h \leq z \leq 0, \quad \left(\Delta_3 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Here, $\partial D$ is the vertical boundary of the elliptic cylinder. According to (1), $E$ is the elliptic domain in the $xy$ plane and $\partial u/\partial n$ is the normal derivative. Denote the elevation of the free surface as $Z = \eta(x,y,t)$ and found it using the potential $u$:

$$\eta(x,y,t) = -\frac{1}{g} \left. \frac{\partial u}{\partial t} \right|_{z=0}$$

We can separate the variables [3] in the domain (1) for the boundary value problem (2), (3). In the form of trigonometric functions of $t$ the unknown $u$ and $\eta$ are constructed with the variable $z$ is separated. So we can obtain the boundary value problems where the boundary of the ellipse is $\partial E$:

$$Z'' - \lambda Z = 0, \quad gZ'(0) - \omega^2 Z(0) = 0, \quad Z'(-h) = 0 \quad (4)$$

$$\Delta_2 U + \lambda U = 0, \quad x, y \in E, \quad \left. \frac{\partial U}{\partial n} \right|_{E} = 0$$

$$\left\{ \Delta_3 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad u = U(x,y)Z(z)e^{i\omega t}, \quad \eta = \gamma(x,y)e^{i\omega t} \right\}$$

The separation constants (parameters) $\lambda$ and $\omega^2$ are unknown; along with the functions $U$ and $Z$, which can be obtain as solutions of the problem (4), (5). The relation between the constants $\omega^2$ and $\lambda$, as well as an expression for $Z$ follows from the boundary value problem (4)

$$Z(z) = c_z \sqrt{\lambda}(z + h), \quad c_z = \text{const}, \quad \omega^2 = \sqrt{g} \lambda$$

It follows from (6) that the frequency $\omega$ is related to $\lambda$ by the expression $\omega = \sqrt{g} \lambda$ for $\sqrt{\lambda} \ll 1$. In the opposite case $\sqrt{\lambda} \gg 1$, we have the approximate expression $\omega \approx g\sqrt{\lambda}$ since $\theta\left(\sqrt{\lambda}h\right) \approx 1 - 2\exp\left(-2\sqrt{\lambda}h\right)$. Thus, we will need to investigate the plane problem (5) with Neumann boundary conditions for the elliptic domain $E$.

3. **Sturm-Liouville problem with Neumann boundary conditions**

Let us introduce a variational isoperimetric problem with Neumann boundary condition [3] by representing the problem (6).

$$J[U] = \int_E \left[ \left( \frac{\partial U}{\partial x} \right)^2 + \left( \frac{\partial U}{\partial y} \right)^2 \right] dxdy \rightarrow \min, \quad \left. \frac{\partial U}{\partial n} \right|_{E} = 0$$

$$I[U] = \int_E U^2 dxdy = 1$$

$$(x,y) \in E = \left\{ x, y : x^2 + \frac{y^2}{1-e^2} \leq 1 \right\}$$

Here we re-denoting $\lambda a^2 \rightarrow \lambda$ and set $a = 1$ and $b = \sqrt{1-e^2}$ ($0 \leq e \leq 1$ is the eccentricity). It can be achieved by dividing $x$ and $y$ by $a$. For mutually orthogonal eigenfunctions $U_k, k = 0, 1, \ldots$ we have
\[ \Phi[U_k, U_{k'}] = \int E U_k U_{k'} \, dx \, dy = 0, \quad k \neq k', \quad k, k' = 0, 1, 2, \ldots \] (8)

The orthogonal elliptic coordinate system \((\xi, \phi)\), can be introduce:
\[
\begin{align*}
x + iy &= \sigma (\xi + i\phi), \quad i = \sqrt{-1}, \quad 0 \leq \xi \leq \infty, \quad 0 \leq \phi < 2\pi \\
x &= \sigma \xi \cos \phi, \quad y = \sigma \xi \sin \phi \\
dl_x &= \sigma \sqrt{\delta} d\xi, \quad dl_y = \sigma \sqrt{\delta} d\phi, \\
dS &= dl_x dl_y = \sigma^2 \delta d\xi d\phi \quad (dS = dx \, dy) \\
\delta &= \cosh^2 \xi - \cos^2 \phi \geq 0
\end{align*}
\] (9)

Here, we introduce a constant \(\sigma > 0\) which is determined by the eccentricity of the ellipse.

To obtain the modified functional \(J'[U]\), we introducing for the function \(U\) in \(\phi\) the condition of periodicity, the Lagrangian multiplier \(\lambda\) and the Neumann boundary condition in the form:
\[
J'[U] = \int E \left[ \left( \frac{\partial U}{\partial \xi} \right)^2 + \left( \frac{\partial U}{\partial \phi} \right)^2 - \frac{\lambda e^2}{2} (\cosh 2\xi - \cos 2\phi) U^2 \right] \, d\xi d\phi \to \min
\]
\[
U = U(\xi, \phi) \equiv U(\xi, \phi + 2\pi), \quad \left. \frac{\partial U}{\partial \xi} \right|_{\xi = \xi_0} = 0
\] (11)

The Euler-Lagrange equation in the form of a boundary value problem for the absolute extremum of functional (11) in elliptic coordinates will be the next:
\[
\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \phi^2} + \mu (2\xi - \cos 2\phi) U = 0, \quad \left. \frac{\partial U}{\partial \xi} \right|_{\xi = \xi_0} = 0
\] (12)

For the unknown function \(U = \Phi(\phi) A(\xi)\) the separation of the variables \(\xi, \phi\) permit us to derive a system of equations
\[
\Phi''(\nu - \mu \cos 2\phi) \Phi = 0, \quad A''(\mu \cosh 2\xi - \nu) A = 0, \quad |\nu| \leq \infty, \quad \mu > 0
\] (13)

where \(\nu\) is the constant of separation. Here we have a canonical Mathieu equation for \(\Phi(\phi)\) and associated (or modified) Mathieu equation for \(A(\xi)\). The solutions of the Mathieu equation \(\Phi(\phi)\) (symmetric and asymmetric) correspond to certain functions \(A(\xi)\)
\[
\Phi'(0) = \Phi'(\pi) = 0, \quad A'(0) = A'(\xi_0) = 0
\] (14)

\[
\Phi(0) = \Phi(\pi) = 0, \quad A(0) = A(\xi_0) = 0
\] (15)

For a nonzero solution of the boundary value problems (13) – (15) we must find corresponding values of the constants \(\nu(e)\) and \(\mu(e)\). Introducing \(e\) as the family parameter of symmetric and asymmetric solutions of the Mathieu equation (14), (15) we constructing the scheme of solution
\[
\{ \nu_n^s(\mu), \quad \Phi_n^s(\phi, \mu); \quad \nu_n^a(0, \mu) = \Phi_n^a(\pi, \mu) = 0, \quad n = 0, 1, 2, \ldots \} \quad (16)
\]
\[
\{ \nu_n^i(\mu), \quad \Phi_n^i(\phi, \mu); \quad \nu_n^i(0, \mu) = \Phi_n^i(\pi, \mu) = 0, \quad n = 1, 2, \ldots \} \quad (17)
\]

Here \(\mu\) and \(n\) are the numbers of the nodal lines in \(\phi\). The quantities \(\nu_n^s(\mu), \nu_n^a(\mu)\), calculated by this scheme are substituted in (13) and we have two generalized problems for the eigenvalues and eigenfunctions
\[ A'' + (\mu \chi 2\xi - \nu''_n (\mu))A = 0, \quad 0 \leq \xi \leq \xi_0(e) \]  
\[ A'(0) = A'(\xi_0) = 0; \quad A''(0) = A''(\xi_0) = 0 \]  
(18)

Now using the special developed method [4-6], mentioned above, we construct for a fixed value of \( e \in (0,1) \) the solutions of the boundary value problems (18)

\[ \{ \mu''_n(e), \quad A''_n(\xi, e); \quad m = 1,2,\ldots \} \]  
(19)

At last, the unknown solutions of the problems (13) – (15) completely determines by substitution of the eigenvalues (19) in (16), (17).

For another value of the eccentricity \( e \), differ from the previous by small variation, we can find the new parameters \( \mu \) and \( \nu \) numerically, using the residual for the abscissas \( \xi \) and \( \varphi \) by the procedure for solving a system for the symmetric and asymmetric cases:

\[ \nu = \nu''_n(\mu), \quad \mu = \mu''_n(\nu, e); \quad \nu = \nu''_n(e); \quad n = 0,1,2,\ldots, \quad m = 1,2,\ldots \]  
(20)

\[ \nu = \nu''_n(\mu), \quad \mu = \mu''_n(\nu, e); \quad \nu = \nu''_n(e); \quad n, m = 1,2,\ldots \]  
(21)

We need the known values of the quantities \( \nu''_n(e) \) and \( \mu''_n(e) \) for the integrating the Cauchy problem for eq. (13) with the initial conditions (see (14), (15)). So we can obtain the eigenfunctions \( \Phi''_{n}(\varphi, e) \) and \( A''_n(\xi, e) \)

\[ \Phi''(0) = 1, \quad \Phi'(0) = 0; \quad A''(0) = 1, \quad A'(0) = 0 \]  
(22)

\[ \Phi''(0) = 0, \quad \Phi'(0) = 1; \quad A''(0) = 0, \quad A'(0) = 1 \]  
(23)

In accordance with (9) we can recalculate the functions \( \Phi''_{n}, \quad A''_{n}, \) and \( U''_{n} \) in the Cartesian coordinate system \( xy \):

\[ \xi = \sqrt{\zeta}, \quad \zeta = \sqrt{\xi - 1} \]

\[ \cos \varphi = \frac{x}{e \sqrt{\zeta}}, \quad \sin \varphi = \frac{y}{e \sqrt{\zeta - 1}} \]  
(24)

\[ \xi = \frac{e^2 + r^2}{2e^2} + \frac{1}{2e^2} \sqrt{(e^2 + r^2)^2 - 4e^2 x^2} \]

\[ r^2 = x^2 + y^2, \quad 0 \leq \varphi \leq 2\pi \]

Below, we will show the general view of the eigenfunctions \( U''_{n} \), \( (x, y) \in E \) in three-dimensional space in an isometric projection, using the level lines \( U''_{n} = \text{const} \).

### 4. Numerical calculations of the natural shapes

For \( e = 0 \) the solution of the problem is well known in cylindrical domain with a circular cross-section of radius \( a = 1 \)

\[ U''_{n} = C''_{n} \cos n\varphi J_{n} (\gamma_{nm} r), \quad n = 0,1,\ldots, \quad m = 1,2,\ldots \]

\[ U'_{n} = C'_{n} \sin n\varphi J_{n} (\gamma'_{nm} r), \quad n = 1,2,\ldots, \quad m = 1,2,\ldots \]  
(25)

Here, \( C''_{n,m} \) taken from the normalization condition (7) and \( \gamma'_{nm} \) is the \( m \)-th root of the function \( J_{n}'(\gamma) \). Because the eigenvalues \( \lambda''_{nm} = \gamma''_{nm} \) doubly degenerate for \( n = 1,2,\ldots \) the functions \( U''_{n} \) must be the orthogonal.

Now we can illustrate the numerical scheme and for the lower oscillation modes plot shapes of oscillations for \( e = 0.5 \). Above we introduced the eigenvalues \( \mu''_{n}(e) \) and \( \nu''_{n}(e) \) of the related boundary value problems (13) – (15). They can be find by using the scheme (16) – (19) and the procedure of continuation in the eccentricity \( e \).
The unknown eigenvalues $\lambda_{nm}^c(e)$ can be calculated on the basis of the known quantities $\mu_{nm}^c(e)$ in accordance with (12) and represented in the dimensional form

$$\lambda_{nm}^c(e) = (2/e^2)\mu_{nm}^c(e), \quad \lambda \to \lambda/a^2$$

(26)

The first two shapes includes the nodal lines $\varphi_{11}^c = \pi/2$ and $\varphi_{11}^s = 0$ for the symmetric (figure 1) and asymmetric (figure 2) modes. This is because of rotation the surface about the major semiaxis $a$ (the $x$ axis) or the minor semiaxis $b$. In the variable $\xi$, $0 \leq \xi \leq \xi_0(e)$ these surfaces have no nodal lines.

The subsequent oscillation modes $U_{21}^c$ have more complex shapes. They are significantly depends on $e$, and $\varphi_{21}^c = 0$, $\pi/2$ and correspond to two cross-shaped nodal lines $\varphi_{21}^c$, $\pi - \varphi_{21}^c$ and $\varphi_{21}^c = \arg \Phi_{21}$.
These surfaces are represented on figures 3, 4 and have no nodal lines in the coordinate $\xi$. The third shape $U^c_{21}$ (figure 3) has a complex form. The lateral edges at $y \approx \pm b$ are folded downward and the end edges of the ellipse at $x \approx \pm 1$ are folded upward. For this surface the nodal lines form an oblique cross. The angle between the arms is $2(\pi/2 - \varphi_{21})$ and the arms approach the minor semiaxis as $e \to 1$. On Figure 4 represented the fourth shape $U^s_{21}$. For this shape the left upper and the right lower parts are bent in the negative direction ($U^s_{21} < 0$), while the left lower and right upper parts of the ellipse — in the positive. The nodal lines coincide with the semiaxes of the ellipse and form a straight cross.

The fifth oscillation mode $U^0_{02}$ is represented on figure 5. There are no nodal lines in $\varphi$ for it. The oscillation shape has a form of hat. There is a single maximum at $x = y = 0$. The shape of $U^0_{02}$ is reproduce by cross-sections in the parameter $\xi$, where $0 \leq \xi \leq \xi_0(e)$. The boundary segment of all shapes correspond to $\xi = \xi_0(e)$ and the centre segment corresponds to the value $\xi = 0$.
5. Conclusions
The solution algorithm represented here is based on the method of separation of variables in elliptic coordinates, and the special developed method for the high-accuracy solution of the boundary problem (13). It used the variational principle and the procedure of continuation in the parameter. This approach shows its effectiveness and obtains the almost complete solution of the problem. It allows to constrain the five lowest oscillation shapes of the natural oscillations of a heavy fluid in an elliptic vessel of finite depth. The lack of high-accuracy calculations for this problem may be explain that the calculations of the eigenvalue and eigenfunction for an elliptic domain with Neumann conditions are more unstable and difficult than in the case of Dirichlet conditions [1, 3, 5, 7].

6. References
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