Research article

Multiple solutions of Kirchhoff type equations involving Neumann conditions and critical growth

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Abstract: In this paper, we consider a Neumann problem of Kirchhoff type equation

\[
\begin{cases}
-\left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u + u = Q(x)|u|^4u + \lambda P(x)|u|^{q-2} u, & \text{in } \Omega, \\
\frac{\partial u}{\partial v} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with a smooth boundary, \( a, b > 0, 1 < q < 2, \lambda > 0 \) is a real parameter, \( Q(x) \) and \( P(x) \) satisfy some suitable assumptions. By using the variational method and the concentration compactness principle, we obtain the existence and multiplicity of nontrivial solutions.

Keywords: Kirchhoff type equation; Neumann problem; critical growth; variation methods; nontrivial solution

Mathematics Subject Classification: 35B33, 35B35, 35J33

1. Introduction and main results

We study the following Neumann problem of Kirchhoff type equation with critical growth

\[
\begin{cases}
-\left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u + u = Q(x)|u|^4u + \lambda P(x)|u|^{q-2} u, & \text{in } \Omega, \\
\frac{\partial u}{\partial v} = 0, & \text{on } \partial \Omega,
\end{cases}
\]  
(1.1)

where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with a smooth boundary, \( a, b > 0, 1 < q < 2, \lambda > 0 \) is a real parameter. We assume that \( Q(x) \) and \( P(x) \) satisfy the following conditions:

(\(Q_1\)) \( Q(x) \in C(\bar{\Omega}) \) is a sign-changing;
There exists \( x_M \in \Omega \) such that \( Q_M = Q(x_M) > 0 \) and
\[
|Q(x) - Q_M| = o(|x - x_M|) \quad \text{as} \quad x \to x_M;
\]
(Q) there exists \( 0 \in \partial \Omega \) such that \( Q_m = Q(0) > 0 \) and
\[
|Q(x) - Q_m| = o(|x|) \quad \text{as} \quad x \to 0;
\]
(P) \( P(x) \) is positive continuous on \( \overline{\Omega} \) and \( P(x_0) = \max_{x \in \Omega} P(x) \);
(P2) there exist \( \sigma > 0, R > 0 \) and \( 3 - q < \beta < \frac{6-q}{2} \) such that \( P(x) \geq \sigma |x - y|^{-\beta} \) for \( |x - y| \leq R \), where \( y \) is \( x_M \in \Omega \) or \( 0 \in \partial \Omega \).

In recent years, the following Dirichlet problem of Kirchhoff type equation has been studied extensively by many researchers
\[
\begin{cases}
-\left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\tag{1.2}
\]
which is related to the stationary analogue of the equation
\[
u_t = \left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u - V(x)u = f(x, u)
\tag{1.3}
\]
proposed by Kirchhoff in [13] as an extension of the classical D’Alembert’s wave equation for free vibrations of elastic strings. Kirchhoff’s model takes into account the changes in length of the string produced by transverse vibrations. In (1.2) and (1.3), \( u \) denotes the displacement, \( b \) is the initial tension and \( f(x, u) \) stands for the external force, while \( a \) is related to the intrinsic properties of the string (such as Youngs modulus). We have to point out that such nonlocal problems appear in other fields like biological systems, such as population density, where \( u \) describes a process which depends on the average of itself (see Alves et al. [2]). After the pioneer work of Lions [18], where a functional analysis approach was proposed. The Kirchhoff type Eq (1.2) with critical growth began to call attention of researchers, we can see [1, 9, 14, 17, 23, 24, 28, 30] and so on.

Recently, the following Kirchhoff type equation has been well studied by various authors
\[
\begin{cases}
-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u + V(x)u = f(x, u), & \text{in } \mathbb{R}^3, \\
u > 0, u \in H^1(\mathbb{R}^3),
\end{cases}
\tag{1.4}
\]
There has been much research regarding the concentration behavior of the positive solutions of (1.4), we can see [10–12, 25, 33]. Many papers studied the existence of ground state solutions of (1.4), for example [5, 8, 15, 16, 21, 22, 24]. In addition, the authors established the existence of sign-changing solutions of (1.4) in [20, 31]. In papers [27, 32] proved the existence and multiplicity of nontrivial solutions of (1.4) by using mountain pass theorem.

In particular, Chabrowski in [6] studied the solvability of the Neumann problem
\[
\begin{cases}
-\Delta u = Q(x)|u|^{2^* - 2}u + \lambda f(x, u), & \text{in } \Omega, \\
\frac{\partial u}{\partial v} = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $2^* = \frac{2N}{N-2} (N \geq 3)$ is the critical Sobolev exponent, $\lambda > 0$ is a parameter. Assume that $Q(x) \in C(\overline{\Omega})$ is a sign-changing function and $\int_{\Omega} Q(x) dx < 0$, under the condition of $f(x,u)$. Using the space decomposition $H^1(\Omega) = \text{span}1 \oplus V$, where $V = \{v \in H^1(\Omega) : \int_{\Omega} vdx = 0\}$, the author obtained the existence of two distinct solutions by the variational method.

In [14], Lei et al. considered the following Kirchhoff type equation with critical exponent

$$
\begin{cases}
-\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = u^5 + \frac{\lambda u^{q-1}}{|x|^q}, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $a, b > 0$, $1 < q < 2$, $\lambda > 0$ is a parameter. They obtained the existence of a positive ground state solution for $0 \leq \beta < 2$ and two positive solutions for $3 - \beta < 2$ by the Nehari manifold method.

In [34], Zhang obtained the existence and multiplicity of nontrivial solutions of the following equation

$$
\begin{cases}
-\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u + u = \lambda|u|^{\beta-2}u + f(x,u) + Q(x)u^5, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega$ is an open bounded domain in $\mathbb{R}^3$, $a, b > 0$, $1 < q < 2$, $\lambda \geq 0$ is a parameter, $f(x,u)$ and $Q(x)$ are positive continuous functions satisfying some additional assumptions. Moreover, $f(x,u) \sim |u|^{p-2}u$ with $4 < p < 6$.

Comparing with the above mentioned papers, our results are different and extend the above results to some extent. Specially, motivated by [34], we suppose $Q(x)$ changes sign on $\Omega$ and $f(x,u) \equiv 0$ for (1.5). Since (1.1) is critical growth, which leads to the cause of the lack of compactness of the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$, we overcome this difficulty by using P.Lions concentration compactness principle [19]. Moreover, note that $Q(x)$ changes sign on $\Omega$, how to estimate the level of the mountain pass is another difficulty.

We define the energy functional corresponding to problem (1.1) by

$$I_\lambda(u) = \frac{1}{2}||u||^2 + \frac{b}{4} \left(\int_{\Omega} |\nabla u|^2 dx\right)^2 - \frac{1}{6} \int_{\Omega} Q(x)|u|^6 dx - \frac{\lambda}{q} \int_{\Omega} P(x)|u|^q dx.$$ 

A weak solution of problem (1.1) is a function $u \in H^1(\Omega)$ and for all $\varphi \in H^1(\Omega)$ such that

$$\int_{\Omega} (a \nabla u \nabla \varphi + u \varphi) dx + b \int_{\Omega} |\nabla u|^2 dx \int_{\Omega} \nabla u \nabla \varphi dx = \int_{\Omega} Q(x)|u|^4 \varphi dx + \lambda \int_{\Omega} P(x)|u|^{q-2} u \varphi dx.$$

Our main results are the following:

**Theorem 1.1.** Assume that $1 < q < 2$ and $Q(x)$ changes sign on $\Omega$. Then there exists $\Lambda_0 > 0$ such that for every $\lambda \in (0, \Lambda_0)$, problem (1.1) has at least one nontrivial solution.

**Theorem 1.2.** Assume that $1 < q < 2$, $3 - \beta < \frac{q}{2}$ and $Q(x)$ changes sign on $\Omega$, there exists $\Lambda_* > 0$ such that for all $\lambda \in (0, \Lambda_*)$. Then problem (1.1) has at least two nontrivial solutions.

Throughout this paper, we make use of the following notations:

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Lemma 2.1. There exist constants \( r \) and there exists a constant \( C \) such that the functional \( I_\lambda(u) \) has a mountain pass geometry.

**Lemma 2.1.** There exist constants \( r, \rho, \Lambda_0 > 0 \) such that the functional \( I_\lambda \) satisfies the following conditions for each \( \lambda \in (0, \Lambda_0) \):

(i) \( I_{\lambda}\big|_{u=\rho} \geq r > 0 \); \( \inf_{u \in \rho} I_\lambda(u) < 0 \).

(ii) There exists \( e \in H^1(\Omega) \) with \( \|e\| > \rho \) such that \( I_\lambda(e) < 0 \).

**Proof.** (i) From \((P_1)\), by the Hölder inequality and the Sobolev inequality, for all \( u \in H^1(\Omega) \) one has

\[
\int_\Omega P(x)|u|^q \, dx \leq P(x_0) \int_\Omega |u|^q \, dx \leq P(x_0)|\Omega|^{\frac{6-q}{p}} S^{-\frac{2}{q}} \|u\|^q, \tag{2.1}
\]

and there exists a constant \( C > 0 \), we get

\[
\left| \int_\Omega Q(x)|u|^q \, dx \right| \leq C \int_\Omega |u|^6 \, dx \leq CS^{-3} \|u\|^6. \tag{2.2}
\]

Hence, combining (2.1) and (2.2), we have the following estimate

\[
I_\lambda(u) = \frac{1}{2} \|u\|^2 + \frac{b}{4} \left( \int_\Omega |\nabla u|^2 \, dx \right)^2 - \frac{1}{6} \int_\Omega Q(x)|u|^6 \, dx - \frac{\lambda}{q} \int_\Omega P(x)|u|^q \, dx \\
\geq \frac{1}{2} \|u\|^2 - \frac{C}{6} \int_\Omega |u|^6 \, dx - \frac{\lambda}{q} P(x_0)|\Omega|^{\frac{6-q}{p}} S^{-\frac{2}{q}} \|u\|^q \\
\geq \|u\|^q \left( \frac{1}{2} \|u\|^{2-q} - \frac{C}{6} S^{-3} \|u\|^{6-q} - \frac{\lambda}{q} P(x_0)|\Omega|^{\frac{6-q}{p}} S^{-\frac{2}{q}} \right).
\]
Set \( h(t) = \frac{t}{2} \frac{2}{3} - \frac{C}{t} S^{-1} t^{-q} \) for \( t > 0 \), then there exists a constant \( \rho = \left( \frac{3(2-q)S}{c(6-q)} \right)^{\frac{1}{q}} > 0 \) such that \( \max_{t>0} h(t) = h(\rho) > 0 \). Letting \( \Lambda_0 = \frac{aS}{\rho} \), there exists a constant \( r > 0 \) such that \( I_{\text{loc}} \geq r \) for every \( \lambda \in (0, \Lambda_0) \). Moreover, for all \( u \in H^1(\Omega) \setminus \{0\} \), we have

\[
\lim_{t \to 0} \frac{I_{\lambda}(tu)}{t^q} = \frac{-\lambda}{q} \int_{\Omega} P(x)|u|^q dx < 0.
\]

So we obtain \( I_{\lambda}(tu) < 0 \) for every \( u \neq 0 \) and \( t \) small enough. Therefore, for \( ||u|| \) small enough, one has

\[
m = \inf_{u \in B_\rho} I_{\lambda}(u) < 0.
\]

(ii) Let \( v \in H^1(\Omega) \) be such that \( \text{supp } v \subset \Omega^+ \), \( v \neq 0 \) and \( t > 0 \), we have

\[
I_{\lambda}(tv) = \frac{t^2}{2} ||v||^2 + \frac{b t^4}{4} \left( \int_{\Omega} |\nabla v|^2 dx \right)^2 - \frac{t^6}{6} \int_{\Omega} Q(x)|v|^6 dx - \frac{\lambda t^q}{q} \int_{\Omega} P(x)|v|^q dx \to -\infty
\]
as \( t \to \infty \), which implies that \( I_{\lambda}(tv) < 0 \) for \( t > 0 \) large enough. Therefore, we can find \( e \in H^1(\Omega) \) with \( ||e|| > \rho \) such that \( I_{\lambda}(e) < 0 \). The proof is complete.

Denote

\[
\begin{align*}
\Theta_1 &= \frac{aS^3_0}{4Q_m} + \frac{b^3 S^6_0}{24Q^2_m} + \frac{aS_0 \sqrt{b^2 S^4_0 + 4aS_0 Q_M}}{6Q_M} + \frac{b^2 S^4_0 \sqrt{b^2 S^4_0 + 4aS_0 Q_M}}{24Q^2_M}, \\
\Theta_2 &= \frac{aS^3_0}{16Q_m} + \frac{b^3 S^6_0}{384Q^2_m} + \frac{aS_0 \sqrt{b^2 S^4_0 + 16aS_0 Q_M}}{24Q_m} + \frac{b^2 S^4_0 \sqrt{b^2 S^4_0 + 16aS_0 Q_M}}{384Q^2_m}.
\end{align*}
\]

Then we have the following compactness result.

**Lemma 2.2.** *Suppose that \( 1 < q < 2 \). Then the functional \( I_\lambda \) satisfies the (PS)\(_{c_\lambda}\) condition for every \( c_\lambda < c_* = \min \{ \Theta_1 - D \lambda \frac{2}{3}, \Theta_2 - D \lambda \frac{2}{3} \} \), where \( D = \frac{2-q}{3q} \left( \frac{6-q}{4} P(x_0) S^{-\frac{2}{q}} |\Omega|^{-\frac{2}{q}} \right)^{\frac{2}{q}} \).*

**Proof.** Let \( \{u_n\} \subset H^1(\Omega) \) be a (PS)\(_{c_\lambda}\) sequence for

\[
I_{\lambda}(u_n) \to c_\lambda \quad \text{and} \quad I'_{\lambda}(u_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.3}
\]

It follows from (2.1), (2.3) and the H"older inequality that

\[
c_\lambda + 1 + o(||u_n||) \geq I_{\lambda}(u_n) - \frac{1}{6} \langle I'_{\lambda}(u_n), u_n \rangle \\
\geq \frac{1}{3} ||u_n||^2 + \frac{b}{12} \left( \int_{\Omega} |\nabla u_n|^2 dx \right)^2 \\
- \lambda \left( \frac{1}{q} - \frac{1}{6} \right) P(x_0) S^{-\frac{q}{2}} |\Omega|^\frac{6-q}{2} ||u_n||^q \\
\geq \frac{1}{3} ||u_n||^2 - \frac{\lambda(6-q)}{6q} P(x_0) S^{-\frac{q}{2}} |\Omega|^\frac{6-q}{2} ||u_n||^q.
\]
Therefore \( \{u_n\} \) is bounded in \( H^1(\Omega) \) for all \( 1 < q < 2 \). Thus, we may assume up to a subsequence, still denoted by \( \{u_n\} \), there exists \( u \in H^1(\Omega) \) such that

\[
\begin{aligned}
&u_n \rightharpoonup u, \quad \text{weakly in } H^1(\Omega), \\
u_n \rightarrow u, \quad \text{strongly in } L^p(\Omega) \ (1 \leq p < 6), \\
u_n(x) \rightarrow u(x), \quad \text{a.e. in } \Omega,
\end{aligned}
\]  

(2.4)
as \( n \to \infty \). Next, we prove that \( u_n \to u \) strongly in \( H^1(\Omega) \). By using the concentration compactness principle (see [19]), there exist some at most countable index set \( J, \delta_{x_j} \) is the Dirac mass at \( x_j \subset \bar{\Omega} \) and positive numbers \( \{\nu_j\}, \{\mu_j\}, j \in J \), such that

\[
|u_n|^6 dx \rightharpoonup dv = |u|^6 dx + \sum_{j \in J} \nu_j \delta_{x_j},
\]

\[
|\nabla u_n|^2 dx \rightharpoonup d\mu = |\nabla u|^2 dx + \sum_{j \in J} \mu_j \delta_{x_j}.
\]

Moreover, numbers \( \nu_j \) and \( \mu_j \) satisfy the following inequalities

\[
\frac{S_0 \nu_j^{\frac{1}{3}}}{\delta_j^2} \leq \mu_j \quad \text{if } x_j \in \Omega,
\]

\[
\frac{S_0 \nu_j^{\frac{1}{3}}}{\delta_j^2} \leq \mu_j \quad \text{if } x_j \in \partial \Omega.
\]  

(2.5)

For \( \varepsilon > 0 \), let \( \phi_{\varepsilon,j}(x) \) be a smooth cut-off function centered at \( x_j \) such that \( 0 \leq \phi_{\varepsilon,j} \leq 1, |\nabla \phi_{\varepsilon,j}| \leq \frac{2}{\varepsilon} \), and

\[
\phi_{\varepsilon,j}(x) = \begin{cases} 1, & \text{in } B(x_j, \frac{\varepsilon}{2}) \cap \bar{\Omega}, \\ 0, & \text{in } \Omega \setminus B(x_j, \varepsilon). \end{cases}
\]

There exists a constant \( C > 0 \) such that

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_\Omega P(x)|u_n|^q \phi_{\varepsilon,j} dx \leq C_{\varepsilon} \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{B(x_j, \varepsilon)} |u_n|^q dx = 0.
\]

Since \( |\nabla \phi_{\varepsilon,j}| \leq \frac{2}{\varepsilon} \), by using the Hölder inequality and \( L^2(\Omega) \)-convergence of \( \{u_n\} \), we have

\[
\begin{aligned}
&\lim_{\varepsilon \to 0} \lim_{n \to \infty} \left( a + \int_\Omega |\nabla u_n|^2 dx \right) \int_\Omega \langle \nabla u_n, \nabla \phi_{\varepsilon,j} \rangle u_n dx \\
\leq &\ C \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left( \int_\Omega |\nabla u_n|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega |u_n|^2 |\nabla \phi_{\varepsilon,j}|^2 dx \right)^{\frac{1}{2}} \\
\leq &\ C \lim_{\varepsilon \to 0} \left( \int_{B(x_j, \varepsilon)} |u|^6 dx \right)^{\frac{1}{3}} \left( \int_{B(x_j, \varepsilon)} |\nabla \phi_{\varepsilon,j}|^3 dx \right)^{\frac{1}{3}} \\
\leq &\ C \lim_{\varepsilon \to 0} \left( \int_{B(x_j, \varepsilon)} |u|^6 dx \right)^{\frac{1}{3}} \left( \int_{B(x_j, \varepsilon)} \left( \frac{2}{\varepsilon} \right)^3 dx \right)^{\frac{1}{3}} \\
\leq &\ C \lim_{\varepsilon \to 0} \left( \int_{B(x_j, \varepsilon)} |u|^6 dx \right)^{\frac{1}{3}} \\
\leq &\ 0,
\end{aligned}
\]
where $C_1 > 0$, and we also derive that
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^2 \phi_{e,j} dx \geq \lim_{\varepsilon \to 0} \int_{\Omega} |\nabla u|^2 \phi_{e,j} dx + \mu_j = \mu_j,
\]
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} Q(x)|u_n|^6 \phi_{e,j} dx = \lim_{\varepsilon \to 0} \int_{\Omega} Q(x)|u|^6 \phi_{e,j} dx + Q(x)\nu_j = Q(x)\nu_j,
\]
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} u_n^2 \phi_{e,j} dx = \lim_{\varepsilon \to 0} \int_{\Omega} u^2 \phi_{e,j} dx \leq \lim_{\varepsilon \to 0} \int_{\Omega} u^2 dx = 0.
\]
Noting that $u_n \phi_{e,j}$ is bounded in $H^1(\Omega)$ uniformly for $n$, taking the test function $\varphi = u_n \phi_{e,j}$ in (2.3), from the above information, one has
\[
0 = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \langle u_n', u_n \phi_{e,j} \rangle
\]
\[
= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left\{ \left( a + b \int_{\Omega} |\nabla u_n|^2 dx \right) \left( \int_{\Omega} \langle \nabla u_n, \nabla (u_n \phi_{e,j}) \rangle \right) dx + \int_{\Omega} u_n^2 \phi_{e,j} dx 
\right.
\]
\[
- \int_{\Omega} \left( \int_{\Omega} Q(x) |u_n|^6 \phi_{e,j} dx - \lambda \int_{\Omega} |u_n|^4 \phi_{e,j} dx \right)
\]
\[
= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left\{ \left( a + b \int_{\Omega} |\nabla u|^2 dx + b \mu_j \right) \left( \int_{\Omega} |\nabla u|^2 \phi_{e,j} dx + \mu_j \right) 
\right.
\]
\[
- \int_{\Omega} \left( \int_{\Omega} Q(x) |u|^6 \phi_{e,j} dx - Q(x)\nu_j \right)
\]
\[
\geq \left( a + b \mu_j \right) \mu_j - Q(x)\nu_j,
\]
so that
\[
Q(x)\nu_j \geq \left( a + b \mu_j \right) \mu_j,
\]
which shows that $\{u_n\}$ can only concentrate at points $x_j$ where $Q(x_j) > 0$. If $\nu_j > 0$, by (2.5) we get
\[
\nu_j \geq \frac{bS_0^2 + \sqrt{b^2S_0^4 + 4aS_0^3Q_M}}{2Q_M} \quad \text{if} \quad x_j \in \Omega,
\]
\[
\nu_j \geq \frac{bS_0^2 + \sqrt{b^2S_0^4 + 16aS_0^3Q_M}}{2^2Q_M} \quad \text{if} \quad x_j \in \partial\Omega.
\]
From (2.5) and (2.6), we have
\[
\mu_j \geq \frac{bS_0^3 + \sqrt{b^2S_0^6 + 4aS_0^3Q_M}}{2Q_M} \quad \text{if} \quad x_j \in \Omega,
\]
\[
\mu_j \geq \frac{bS_0^3 + \sqrt{b^2S_0^6 + 16aS_0^3Q_M}}{8Q_M} \quad \text{if} \quad x_j \in \partial\Omega.
\]
To proceed further we show that (2.7) is impossible. To obtain a contradiction assume that there exists \( j_0 \in J \) such that \( \mu_{j_0} \geq \frac{bS_0^3 + \sqrt{b^2S_0^4 + 4aS_0Q_M}}{2Q_M} \) and \( x_{j_0} \in \Omega \). By (2.1), (2.3) and (2.4), one has

\[
c_A = \lim_{n \to \infty} \left\{ I_A(u_n) - \frac{1}{6} (I'_A(u_n), u_n) \right\} \\
= \lim_{n \to \infty} \left( \frac{a}{3} \int_{\Omega} |\nabla u_n|^2 dx + \frac{b}{12} \left( \int_{\Omega} |\nabla u_n|^2 dx \right)^2 \right. \\
\left. + \frac{1}{3} \int_{\Omega} u_n^2 dx - \frac{6 - q}{6q} \int_{\Omega} P(x)|u_n|^q dx \right) \\
\geq \frac{a}{3} \left( \int_{\Omega} |\nabla u|^2 dx + \sum_{j \in J} \mu_j \right) + \frac{b}{12} \left( \int_{\Omega} |\nabla u|^2 dx + \sum_{j \in J} \mu_j \right)^2 \\
\left. + \frac{1}{3} \int_{\Omega} u^2 dx - \frac{6 - q}{6q} P(x_0)S^{-\frac{q}{2}} |\Omega|^{\frac{m-q}{m}} |u|^q \right) \\
\geq \frac{a}{3} \mu_{j_0} + \frac{b}{12} \mu_{j_0}^2 + \frac{1}{3} |u|^2 - \frac{6 - q}{6q} P(x_0)S^{-\frac{q}{2}} |\Omega|^{\frac{m-q}{m}} |u|^q. \\
\]

Set

\[
g(t) = \frac{1}{3} t^2 - \frac{6 - q}{6q} P(x_0)S^{-\frac{q}{2}} |\Omega|^{\frac{m-q}{m}} t^q, \quad t > 0, \\
\]

then

\[
g'(t) = \frac{2}{3} t - \frac{6 - q}{6} P(x_0)S^{-\frac{q}{2}} |\Omega|^{\frac{m-q}{m}} t^{q-1} = 0, \\
\]

we can deduce that \( \min_{t \geq 0} g(t) \) attains at \( t_0 > 0 \) and

\[
t_0 = \left( \frac{6 - q}{4} P(x_0)S^{-\frac{q}{2}} |\Omega|^{\frac{m-q}{m}} \right)^{\frac{1}{q-1}}. \\
\]

Consequently, we obtain

\[
c_A \geq \frac{abS_0^3}{4Q_M} + \frac{b^3S_0^6}{24Q_M^2} + \frac{aS_0 \sqrt{b^2S_0^4 + 4aS_0Q_M}}{6Q_M} \\
\left. + \frac{b^2S_0^4 \sqrt{b^2S_0^4 + 4aS_0Q_M}}{24Q_M^2} - D \lambda^{\frac{2}{q-1}} \right) \\
= \Theta_1 - D \lambda^{\frac{2}{q-1}}, \\
\]

where \( D = \frac{2-q}{3q} \left( \frac{6-q}{4} P(x_0)S^{-\frac{q}{2}} |\Omega|^{\frac{m-q}{m}} \right)^{\frac{q}{q-1}} \). If \( \mu_{j_0} \geq \frac{bS_0^3 + \sqrt{b^2S_0^4 + 16aS_0Q_M}}{8Q_m} \) and \( x_{j_0} \in \partial \Omega \), then, by the similar calculation, we also get

\[
c_A \geq \frac{abS_0^3}{16Q_M} + \frac{b^3S_0^6}{384Q_m^2} + \frac{aS_0 \sqrt{b^2S_0^4 + 16aS_0Q_M}}{24Q_m} \\
\left. + \frac{b^2S_0^4 \sqrt{b^2S_0^4 + 16aS_0Q_M}}{384Q_m^2} - D \lambda^{\frac{2}{q-1}} \right) \\
= \Theta_2 - D \lambda^{\frac{2}{q-1}}. \\
\]
Let $c_* = \min\{\Theta_1 - D\lambda^{\frac{1}{2q}}, \Theta_2 - D\lambda^{\frac{1}{2q}}\}$, from the above information, we deduce that $c_* \geq c_*$. It contradicts our assumption, so it indicates that $\nu_j = \mu_j = 0$ for every $j \in J$, which implies that
\[
\int \Omega |u_n|^6 dx \to \int \Omega |u|^6 dx
\] (2.8)
as $n \to \infty$. Now, we may assume that $\int \Omega |\nabla u_n|^2 dx \to A^2$ and $\int \Omega |\nabla u|^2 dx \leq A^2$, by (2.3), (2.4) and (2.8), one has
\[
0 = \lim_{n \to \infty} \langle I'(u_n), u_n - u \rangle
= \lim_{n \to \infty} \left[ a + b \int \Omega |\nabla u_n|^2 dx \right] \left( \int \Omega |\nabla u_n|^2 dx - \int \Omega \nabla u_n \nabla u dx \right)
+ \int \Omega u_n(u_n - u) dx - \int \Omega Q(x)|u_n|^5(u_n - u) dx - \lambda \int \Omega P(x)|u_n|^{q-1}(u_n - u) dx
= (a + bA^2) \left( A^2 - \int \Omega |\nabla u|^2 dx \right)
\]
Then, we obtain that $u_n \to u$ in $H^1(\Omega)$. The proof is complete. □

As well known, the function
\[
U_{\varepsilon, y}(x) = \frac{(3\varepsilon^2)^{\frac{1}{2}}}{(\varepsilon^2 + |x - y|^2)^{\frac{1}{2}}}, \text{ for any } \varepsilon > 0,
\]
satisfies
\[
-\Delta U_{\varepsilon, y} = U_{\varepsilon, y}^5 \text{ in } \mathbb{R}^3,
\]
and
\[
\int_{\mathbb{R}^3} |\nabla U_{\varepsilon, y}|^2 dx = \int_{\mathbb{R}^3} |U_{\varepsilon, y}|^6 dx = S_0^\frac{2}{3}.
\]
Let $\phi \in C^1(\mathbb{R}^3)$ such that $\phi(x) = 1$ on $B(x_M, \frac{R}{2})$, $\phi(x) = 0$ on $\mathbb{R}^3 - B(x_M, R)$ and $0 \leq \phi(x) \leq 1$ on $\mathbb{R}^3$, we set $v_\varepsilon(x) = \phi(x)U_{\varepsilon, y}(x)$. We may assume that $Q(x) > 0$ on $B(x_M, R)$ for some $R > 0$ such that $B(x_M, R) \subset \Omega$. From [4], we have
\[
\left\|
\begin{array}{l}
\nabla v_\varepsilon \\
v_\varepsilon \\
v_\varepsilon
\end{array}
\right\|_2^2 = S_0^{\frac{3}{2}} + O(\varepsilon),
\]
\[
\left\|
\begin{array}{l}
\nabla v_\varepsilon \\
v_\varepsilon \\
v_\varepsilon
\end{array}
\right\|_6^6 = S_0^3 + O(\varepsilon^3),
\]
\[
\left\|
\begin{array}{l}
\nabla v_\varepsilon \\
v_\varepsilon \\
v_\varepsilon
\end{array}
\right\|_2^2 = O(\varepsilon),
\]
\[
\left\|
\begin{array}{l}
\nabla v_\varepsilon \\
v_\varepsilon \\
v_\varepsilon
\end{array}
\right\|_2^2 = aS_0^{\frac{3}{2}} + O(\varepsilon). \quad (2.9)
\]
Moreover, by [28], we get
\[
\left\|
\begin{array}{l}
\nabla v_\varepsilon \\
v_\varepsilon \\
v_\varepsilon
\end{array}
\right\|_2^2 \leq S_0^3 + O(\varepsilon),
\]
\[
\left\|
\begin{array}{l}
\nabla v_\varepsilon \\
v_\varepsilon \\
v_\varepsilon
\end{array}
\right\|_6^6 \leq S_0^6 + O(\varepsilon),
\]
\[
\left\|
\begin{array}{l}
\nabla v_\varepsilon \\
v_\varepsilon \\
v_\varepsilon
\end{array}
\right\|_2^2 \leq S_0^9 + O(\varepsilon). \quad (2.10)
\]
Then we have the following Lemma.

**Lemma 2.3.** Suppose that $1 < q < 2$, $3 - q < \beta < \frac{6-q}{2}$, $Q_M > 4Q_m$, $(Q_1)$ and $(Q_2)$, then $\sup_{t \geq 0} I_1(tv_\varepsilon) < \Theta_1 - D\lambda^{\frac{1}{2q}}$. 
Proof. By Lemma 2.1, one has $I_\lambda(t\nu_\varepsilon) \to -\infty$ as $t \to \infty$ and $I_\lambda(t\nu_\varepsilon) < 0$ as $t \to 0$, then there exists $t_\varepsilon > 0$ such that $I_\lambda(t_\varepsilon \nu_\varepsilon) = \sup_{t > 0} I_\lambda(t\nu_\varepsilon) \geq r > 0$. We can assume that there exist positive constants $t_1, t_2 > 0$ and $0 < t_1 < t_\varepsilon < t_2 < +\infty$. Let $I_\lambda(t_\varepsilon \nu_\varepsilon) = \beta(t_\varepsilon \nu_\varepsilon) - \lambda \psi(t_\varepsilon \nu_\varepsilon)$, where
\[
\beta(t_\varepsilon \nu_\varepsilon) = \frac{t_\varepsilon^2}{2} ||\nu_\varepsilon||^2 + \frac{bt_\varepsilon^4}{4} ||\nabla \nu_\varepsilon||^2 - \frac{t_\varepsilon}{6} \int_\Omega Q(x)|\nu_\varepsilon|^6 dx,
\]
and
\[
\psi(t_\varepsilon \nu_\varepsilon) = \frac{t_\varepsilon^2}{q} \int_\Omega P(x)|\nu_\varepsilon|^q dx.
\]
Now, we set
\[
h(t) = \frac{t^2}{2} ||\nu||^2 + \frac{bt^4}{4} ||\nabla \nu||^2 - \frac{t}{6} \int_\Omega Q(x)|\nu|^6 dx.
\]
It is clear that $\lim_{t \to 0} h(t) = 0$ and $\lim_{t \to +\infty} h(t) = -\infty$. Therefore there exists $T_1 > 0$ such that $h(T_1) = \max_{t \geq 0} h(t)$, that is
\[
h'(t)|_{t_1} = T_1 ||\nu||^2 + bT_1^3 ||\nabla \nu||^2 - T_1^3 \int_\Omega Q(x)|\nu|^6 dx = 0,
\]
from which we have
\[
||\nu||^2 + bT_1^3 ||\nabla \nu||^2 = T_1^4 \int_\Omega Q(x)|\nu|^6 dx.
\]
By (2.11) we obtain
\[
T_1^2 = \frac{b||\nabla \nu||^2}{2 \int_\Omega Q(x)|\nu|^6 dx} + \frac{\sqrt{b^2||\nabla \nu||^8 + 4||\nu||^2 \int_\Omega Q(x)|\nu|^6 dx}}{2 \int_\Omega Q(x)|\nu|^6 dx}
\]
In addition, by (2.2), for all $\eta > 0$, there exists $\rho > 0$ such that $|Q(x) - M_\eta| < \eta|x-x_M|$ for $0 < |x-x_M| < \rho$, for $\varepsilon > 0$ small enough, we have
\[
\left| \int_\Omega Q(x)\nu_\varepsilon^6 dx - \int_\Omega M_\eta \nu_\varepsilon^6 dx \right| \leq \int_\Omega |Q(x) - M_\eta|\nu_\varepsilon^6 dx
\]
\[
< \int_{B(x_M, \rho)} \eta|x-x_M| \left( \frac{(3\varepsilon^2)^3}{\varepsilon^2 + |x-x_M|^2} \right)^3 dx
\]
\[
+ C \int_{\Omega\setminus(B(x_M, \rho))} \left( \frac{(3\varepsilon^2)^3}{\varepsilon^2 + |x-x_M|^2} \right)^3 dx
\]
\[
\leq C\varepsilon^3 \int_0^\rho \frac{r^3}{(\varepsilon^2 + r^2)^3} dr + C\varepsilon^3 \int_\rho^R \frac{r^2}{(\varepsilon^2 + r^2)^3} dr
\]
\[
\leq C\varepsilon \int_0^{\rho/\varepsilon} \frac{t^3}{(1+t^2)^3} dt + C \int_{\rho/\varepsilon}^{R/\varepsilon} \frac{t^2}{(1+t^2)^3} dt
\]
\[
\leq C_1 \varepsilon + C_2 \varepsilon^3,
\]
where $C_1, C_2 > 0$ (independent of $\eta, \varepsilon$). From this we derive that
\[
\limsup_{\varepsilon \to 0} \left| \int_\Omega Q(x)\nu_\varepsilon^6 dx - \int_\Omega M_\eta \nu_\varepsilon^6 dx \right| \leq C_1 \varepsilon.
\]
Then from the arbitrariness of $\eta > 0$, by (2.9) and (2.12), one has

$$
\int_{\Omega} Q(x)|v^6_x|dx = Q_M \int_{\Omega} |v^6_x|dx + o(\varepsilon) = Q_M S_0^{\frac{3}{2}} + o(\varepsilon).
$$

(2.13)

Hence, it follows from (2.9), (2.10) and (2.13) that

$$
\beta(t_0 v_x) \leq h(T_1)
$$

$$
= T_1^2 \left( \frac{1}{3} ||v^3_x||^2 + \frac{b T_1^\frac{2}{3}}{12} ||\nabla v^3_x||^2 \right)
$$

$$
= \frac{b ||\nabla v^6_x||^2 ||v^6_x||^2}{4 \int_{\Omega} Q(x)|v^6_x|dx} + \frac{b^3 ||\nabla v^6_x||^2}{24 \int_{\Omega} Q(x)|v^6_x|dx^2}
$$

$$
\int_{\Omega} v^6_x \sqrt{b^2 ||\nabla v^6_x||^2 + 4 ||v^6_x||^2} \int_{\Omega} Q(x)|v^6_x|dx
$$

$$
+ \frac{b^2 ||\nabla v^6_x||^2}{24 \int_{\Omega} Q(x)|v^6_x|dx^2}
$$

$$
\leq \frac{b(S_0^3 + O(\varepsilon))(aS_0^3 + O(\varepsilon))}{4(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))} + \frac{b^3(S_0^6 + O(\varepsilon))}{24(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))^2}
$$

$$
+ \frac{(aS_0^3 + O(\varepsilon)) \sqrt{b^2(S_0^6 + O(\varepsilon)) + 4(aS_0^3 + O(\varepsilon))(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))}}{6(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))}
$$

$$
+ \frac{b^2(S_0^6 + O(\varepsilon)) \sqrt{b^2(S_0^6 + O(\varepsilon)) + 4(aS_0^3 + O(\varepsilon))(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))}}{24(Q_M S_0^{\frac{3}{2}} + o(\varepsilon))^2}
$$

$$
\leq \frac{abS_0^3}{4Q_M} + \frac{b^3 S_0^6}{24 Q_M^2} + \frac{aS_0 \sqrt{b^2 S_0^4 + 4aS_0 Q_M}}{6Q_M}
$$

$$
+ \frac{b^2 S_0^4 \sqrt{b^2 S_0^4 + 4aS_0 Q_M}}{24 Q_M^2} + C_3 \varepsilon
$$

$$
= \Theta_1 + C_3 \varepsilon,
$$

where the constant $C_3 > 0$. According to the definition of $v_x$, from [29], for $\frac{R}{2} > \varepsilon > 0$, there holds

$$
\psi(t_0 v_x) \geq \frac{1}{q} \frac{1}{R^q} \int_{B_R(x_M, \frac{R}{2})} \frac{\sigma \rho^{q/2} \varepsilon^2}{(\varepsilon^2 + |x - x_M|^2)^{\frac{q}{2}} |x - x_M|^q} dx
$$

$$
\geq C \varepsilon^2 \int_0^{R/2} \frac{r^2}{(\varepsilon^2 + r^2)^{\frac{q}{2}} r^q} dr
$$

$$
= C \varepsilon^2 \int_0^{R/2 \varepsilon} \frac{t^2}{(1 + t^2)^{\frac{q}{2}} t^q} dt
$$

(2.14)
Lemma 2.4. Suppose that $\partial_\nu \lambda > 0$ (independent of $\varepsilon, \lambda$). Consequently, from the above information, we obtain

$$I_3(t_\varepsilon \nu) = \beta(t_\varepsilon \nu) - \lambda \psi(t_\varepsilon \nu) \leq \Theta_1 + C_3 \varepsilon - C_4 A \frac{6 - q}{\varepsilon^{2 - \beta}}$$

$$\leq \Theta_1 - D \lambda \frac{2}{\varepsilon^2}.$$  

Here we have used the fact that $\beta > 3 - q$ and let $\varepsilon = \lambda^{\frac{2}{3}}$, $0 < \lambda < \Lambda_1 = \min\{1, \frac{C + B}{C_4} \}$, then

$$C_3 \varepsilon - C_4 A \frac{6 - q}{\varepsilon^{2 - \beta}} = C_3 \lambda^{\frac{2}{3}} - C_4 A \frac{6 - 2q - 2\beta}{\varepsilon^{2 - \beta}}$$

$$\leq \lambda^{\frac{2}{3}} (C_3 - C_4 \lambda^{\frac{6 - 2q - 2\beta}{6 - 2q}}) \quad (2.15)$$

$$< -D \lambda^{\frac{2}{3}}.$$  

The proof is complete. □

We assume that $0 \in \partial \Omega$ and $Q_m = Q(0)$. Let $\varphi \in C^1(\mathbb{R}^3)$, $\varphi(0) = 1$ on $B(0, \frac{\varepsilon}{3})$, $\varphi(0) = 0$ on $\mathbb{R}^3 - B(0, R)$ and $0 \leq \varphi(x) \leq 1$ on $\mathbb{R}^3$. Let $u_\varepsilon(x) = \varphi(x) u(x)$, the radius $R$ is chosen so that $Q(0) > 0$ on $B(0, R) \cap \Omega$. If $H(0)$ denotes the mean curvature of the boundary at 0, then the following estimates hold (see [6] or [26])

$$\left\{ \begin{array}{l}
\|u_\varepsilon\|_2^2 = O(\varepsilon), \\
\|\nabla u_\varepsilon\|_2^2 \leq \frac{C_0}{\lambda^2} - A_3 H(0) \varepsilon \log \frac{1}{\varepsilon} + O(\varepsilon),
\end{array} \right. \quad (2.16)$$

where $A_3 > 0$ is a constant. Then we have the following lemma.

**Lemma 2.4.** Suppose that $1 < q < 2$, $3 - q < \beta < \frac{6 - q}{2}$, $Q_m \leq 4Q_m$, $H(0) > 0$, $Q$ is positive somewhere on $\partial \Omega$, $(Q_1)$ and $(Q_3)$, then $\sup_{t \geq 0} I_3(t \nu u_\varepsilon) < \Theta_2 - DA \frac{2}{\varepsilon^2}.$

**Proof.** Similar to the proof of Lemma 2.3, we also have by Lemma 2.1, there exists $t_\varepsilon > 0$ such that $I_3(t_\varepsilon u_\varepsilon) = \sup_{t \geq 0} I_3(t \nu u_\varepsilon) \geq r > 0$. We can assume that there exist positive constants $t_1, t_2 > 0$ such that $0 < t_1 < t_\varepsilon < t_2 < +\infty$. Let $I_3(t \nu u_\varepsilon) = A(t \nu u_\varepsilon) - \lambda B(t \nu u_\varepsilon)$, where

$$A(t \nu u_\varepsilon) = \frac{t^2}{2} \|u_\varepsilon\|^2 + \frac{br^4}{4} \|\nabla u_\varepsilon\|_2^4 - \frac{t^6}{6} \int_\Omega Q(x)|u_\varepsilon|^6 dx,$$

and

$$B(t \nu u_\varepsilon) = \frac{t^q}{q} \int_\Omega P(x)|u_\varepsilon|^q dx.$$  

Now, we set

$$f(t) = \frac{t^2}{2} \|u_\varepsilon\|^2 + \frac{br^4}{4} \|\nabla u_\varepsilon\|_2^4 - \frac{t^6}{6} \int_\Omega Q(x)|u_\varepsilon|^6 dx.$$  

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Therefore, it is easy to see that there exists $T_2 > 0$ such that $f(T_2) = \max_{f \geq 0} f(t)$, that is

$$f'(t)|_{t_2} = T_2||u_0||^2 + bT_2^2\|\nabla u_0\|^2 - T_2^3\int_{\Omega} Q(x)|u_0|^6\,dx = 0. \quad (2.17)$$

From (2.17) we obtain

$$T_2^2 = \frac{b\|\nabla u_0\|^2 + \sqrt{b^2\|\nabla u_0\|^8 + 4||u_0||^2 \int_{\Omega} Q(x)|u_0|^6\,dx}}{2 \int_{\Omega} Q(x)|u_0|^6\,dx}.$$

By the assumption $(Q_1)$, we have the expansion formula

$$\int_{\Omega} Q(x)|u_0|^6\,dx = Q_m \int_{\Omega} |u_0|^6\,dx + o(\epsilon). \quad (2.18)$$

Hence, combining (2.16) and (2.18), there exists $C_5 > 0$, such that

$$A(t, u_0) \leq f(T_2)$$

$$= T_2^2 \left( \frac{1}{3} ||u_0||^2 + \frac{bT_2^2}{12} \|\nabla u_0\|^4 \right)$$

$$= \frac{b\|\nabla u_0\|^2 ||u_0||^2}{4 \int_{\Omega} Q(x)|u_0|^6\,dx} + \frac{b^3\|\nabla u_0\|^2}{24(\int_{\Omega} Q(x)|u_0|^6\,dx)^2} \int_{\Omega} Q(x)|u_0|^6\,dx$$

$$+ \frac{\|u_0||^2 \sqrt{b^2\|\nabla u_0\|^8 + 4||u_0||^2 \int_{\Omega} Q(x)|u_0|^6\,dx}}{6 \int_{\Omega} Q(x)|u_0|^6\,dx}$$

$$+ \frac{b^2\|\nabla u_0\|^8}{24(\int_{\Omega} Q(x)|u_0|^6\,dx)^2} \int_{\Omega} Q(x)|u_0|^6\,dx$$

$$\leq \frac{ab}{4Q_m} \left( \int_{\Omega} |u_0|^6\,dx \right) + O(\epsilon) + \frac{b^3}{24Q_m^2} \left( \int_{\Omega} |u_0|^6\,dx \right)^2 + O(\epsilon)$$

$$+ \frac{a}{6Q_m} \left( \int_{\Omega} |u_0|^6\,dx \right) \sqrt{\frac{b^2}{\int_{\Omega} |u_0|^6\,dx} + \frac{4aQ_m}{\int_{\Omega} |u_0|^6\,dx} + O(\epsilon)}$$

$$+ \frac{b^2}{24Q_m^2} \left( \int_{\Omega} |u_0|^6\,dx \right) \left( \int_{\Omega} |u_0|^6\,dx \right) \sqrt{\frac{b^2}{\int_{\Omega} |u_0|^6\,dx} + \frac{4aQ_m}{\int_{\Omega} |u_0|^6\,dx} + O(\epsilon)}$$

$$\leq \frac{abS_0^3}{16Q_m} + \frac{b^3S_0^6}{384Q_m^2} + \frac{aS_0}{24Q_m} \sqrt{b^2S_0^4 + 16aS_0Q_m}$$

$$+ \frac{b^2S_0^4}{384Q_m^2} \sqrt{b^2S_0^4 + 16aS_0Q_m} + C_5\epsilon \quad = \Theta_2 + C_5\epsilon.$$
Consequently, by (2.14) and (2.15), similarly, there exists $\Lambda_2 > 0$ such that $0 < \lambda < \Lambda_2$, we get
\[
I_\lambda(t_n u_n) = \Lambda(t_n u_n) - \lambda B(t_n u_n) \\
\leq \Theta_2 + C_\varepsilon \varepsilon - C_\varepsilon \varepsilon^{\frac{6}{2} - \beta} \\
< \Theta_2 - D \varepsilon^{\frac{6}{2} - \beta},
\]
where $C_\varepsilon > 0$ (independent of $\varepsilon, \lambda$). The proof is complete. \hfill \Box

**Theorem 2.5.** Assume that $0 < \lambda < \Lambda_0$ ($\Lambda_0$ is as in Lemma 2.1) and $1 < q < 2$. Then problem (1.1) has a nontrivial solution $u_1$ with $I_\lambda(u_1) < 0$.

**Proof.** It follows from Lemma 2.1 that
\[m \doteq \inf_{u \in \overline{B}_0} I_\lambda(u) < 0.\]

By the Ekeland variational principle [7], there exists a minimizing sequence \(\{u_n\} \subset \overline{B}_0\) such that
\[
I_\lambda(u_n) \leq \inf_{u \in \overline{B}_0} I_\lambda(u) + \frac{1}{n}, \quad I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n}||v - u_n||, \quad v \in \overline{B}_0.
\]

Therefore, there holds $I_\lambda(u_n) \to m$ and $I'_\lambda(u_n) \to 0$. Since $\{u_n\}$ is a bounded sequence and $\overline{B}_0$ is a closed convex set, we may assume up to a subsequence, still denoted by $\{u_n\}$, there exists $u_\lambda \in \overline{B}_0 \subset H^1(\Omega)$ such that
\[
\begin{align*}
u_n &\to u_\lambda, \quad \text{weakly in } H^1(\Omega), \\
u_n &\to u_\lambda, \quad \text{strongly in } L^p(\Omega), \quad 1 \leq p < 6, \\
u_n(x) &\to u_\lambda(x), \quad \text{a.e. in } \Omega.
\end{align*}
\]

By the lower semi-continuity of the norm with respect to weak convergence, one has
\[
m \geq \liminf_{n \to \infty} I_\lambda(u_n) - \frac{1}{6} \langle I'_\lambda(u_n), u_n \rangle
\]
\[
= \liminf_{n \to \infty} \left[ \frac{1}{3} \int_{\Omega} (a|\nabla u_n|^2 + u_n^2) \, dx + \frac{b}{12} \left( \int_{\Omega} |\nabla u_n|^2 \, dx \right)^2 \\
+ \lambda \left( \frac{1}{6} - \frac{1}{q} \right) \int_{\Omega} P(x)|u_n|^q \, dx \right] \\
\geq \frac{1}{3} \int_{\Omega} (a|\nabla u_\lambda|^2 + u_\lambda^2) \, dx + \frac{b}{12} \left( \int_{\Omega} |\nabla u_\lambda|^2 \, dx \right)^2 \\
+ \lambda \left( \frac{1}{6} - \frac{1}{q} \right) \int_{\Omega} P(x)|u_\lambda|^q \, dx \\
= I_\lambda(u_\lambda) - \frac{1}{6} \langle I'_\lambda(u_\lambda), u_\lambda \rangle = I_\lambda(u_\lambda) = m.
\]

Thus $I_\lambda(u_\lambda) = m < 0$, by $m < 0 < c_3$ and Lemma 2.2, we can see that $\nabla u_n \rightharpoonup \nabla u_\lambda$ in $L^2(\Omega)$ and $u_\lambda \neq 0$. Therefore, we obtain that $u_\lambda$ is a weak solution of problem (1.1). Since $I_\lambda(|u_\lambda|) = I_\lambda(u_\lambda)$, which suggests that $u_\lambda \geq 0$, then $u_\lambda$ is a nontrivial solution to problem (1.1). That is, the proof of Theorem 1.1 is complete. \hfill \Box
Theorem 2.6. Assume that $0 < \lambda < \Lambda_0$ ($\Lambda_0 = \min\{\Lambda_0, \Lambda_1, \Lambda_2\}$), $1 < q < 2$ and $3 - q < \beta < \frac{6 - q}{2}$. Then the problem (1.1) has a nontrivial solution $u_1 \in H^1(\Omega)$ such that $I_\lambda(u_1) > 0$.

Proof. Applying the mountain pass lemma [3] and Lemma 2.2, there exists a sequence $\{u_n\} \subset H^1(\Omega)$ such that

$$I_\lambda(u_n) \to c_\lambda > 0 \quad \text{and} \quad I_\lambda'(u_n) \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in C([0, 1], H^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}.$$

According to Lemma 2.2, we know that $\{u_n\} \subset H^1(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_1$ in $H^1(\Omega)$ as $n \to \infty$,

$$I_\lambda(u_1) = \lim_{n \to \infty} I_\lambda(u_n) = c_\lambda > r > 0,$$

which implies that $u_1 \neq 0$. Therefore, from the continuity of $I_\lambda'$, we obtain that $u_1$ is a nontrivial solution of problem (1.1) with $I_\lambda(u_1) > 0$. Combining the above facts with Theorem 2.5 the proof of Theorem 1.2 is complete. \hfill \Box

3. Conclusions

In this paper, we consider a class of Kirchhoff type equations with Neumann conditions and critical growth. Under suitable assumptions on $Q(x)$ and $P(x)$, using the variational method and the concentration compactness principle, we proved the existence and multiplicity of nontrivial solutions.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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