Maximal $L^2$ regularity for Ornstein-Uhlenbeck equation in convex sets of Banach spaces

G. Cappa*

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Abstract

We study the elliptic equation $\lambda u - L^\Omega u = f$ in an open convex subset $\Omega$ of an infinite dimensional separable Banach space $X$ endowed with a centered non-degenerate Gaussian measure $\gamma$, where $L^\Omega$ is the Ornstein-Uhlenbeck operator. We prove that for $\lambda > 0$ and $f \in L^2(\Omega, \gamma)$ the weak solution $u$ belongs to the Sobolev space $W^{2,2}(\Omega, \gamma)$. Moreover we prove that $u$ satisfies the Neumann boundary condition in the sense of traces at the boundary of $\Omega$. This is done by finite dimensional approximation.

1 Introduction

Let $X$ be a separable Banach space, let $\gamma$ be a centered non-degenerate Gaussian measure in $X$ with covariance $Q$, and let $H = Q^{1/2}(X)$ be the associated Cameron-Martin space. In this paper we consider the equation

$$\lambda u - L^\Omega u = f \quad \text{in } \Omega,$$

where $\lambda > 0$ and $f \in L^2(\Omega, \gamma)$ are given, $\Omega$ is an open convex set of $X$ and $L^\Omega$ is the Ornstein-Uhlenbeck operator associated to the quadratic form

$$E_{\Omega, \gamma}(u, v) := \int_\Omega \langle \nabla_H u, \nabla_H v \rangle_H d\gamma \quad \text{for } u, v \in W^{1,2}(\Omega, \gamma).$$

Precise definition of the Sobolev spaces $W^{1,2}(\Omega, \gamma)$, $W^{2,2}(\Omega, \gamma)$, and of the $H$-gradient $\nabla_H$ are in the next section. As usual a function $u \in W^{1,2}(\Omega, \gamma)$ is called weak solution to (1) if

$$\int_\Omega (\lambda u \varphi + \langle \nabla_H u, \nabla_H \varphi \rangle_H) d\gamma = \int_\Omega f \varphi d\gamma \quad \forall \varphi \in W^{1,2}(\Omega, \gamma).$$

It is not hard to see that for every $\lambda > 0$ and $f \in L^2(\Omega, \gamma)$, problem (1) has a unique weak solution $u$. In this paper we prove a maximal regularity result for

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* Dipartimento di Matematica e Informatica, Università degli Studi di Parma, Parco Area delle Scienze 53/A, PARMA, Italy. E-mail address: gianluca.cappa@nemo.unipr.it
the weak solution $u$ of (1), that is for every $f \in L^2(\Omega, \gamma)$ the weak solution $u$ belongs to $W^{2,2}(\Omega, \gamma)$ and there exists $C > 0$ independent of $f$ such that
\[
\|u\|_{W^{2,2}(\Omega, \gamma)} \leq C\|f\|_{L^2(\Omega, \gamma)}.
\] (2)

It is sufficient to have that (2) holds if $F$ is a cylindrical smooth bounded function (see Section 2), because the space of such functions is dense in $L^2(\Omega, \gamma)$. In this case, we define a sequence of functions $\{u_n\}_{n \in \mathbb{N}}$, by using the cylindrical approximation $\{\Omega_n\}_{n \in \mathbb{N}}$ of $\Omega$ made in [8]. In particular,
\[
u_n = \varphi_n \circ \pi_n
\] where $\pi_n(X)$ is a finite dimensional subspace of $H$, identified in an obvious way with $\mathbb{R}^q$ with $q = q(n, f)$. So $\pi_n(\Omega_n)$ is identified with an open subset $\mathcal{O}_n$ of $\mathbb{R}^q$, and $\varphi_n : \mathcal{O}_n \subset \mathbb{R}^q \to \mathbb{R}$ solves
\[
\begin{cases}
\lambda \psi - \mathcal{L}^{\Omega_n} \psi = \tilde{f} & \text{in } \mathcal{O}_n \subset \mathbb{R}^q, \\
\frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial \mathcal{O}_n
\end{cases}
\] (3)

where $\tilde{f}$ is a suitable smooth bounded function. Here, the reference measure is the standard Gaussian measure $\mathbb{N}_0, I$, and $\nabla_H$ is the usual gradient. For the finite dimensional problems (3) we prove dimension free $W^{2,2}$ estimates. Therefore the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W^{2,2}(\Omega, \gamma)$, and a subsequence weakly converges to $u \in W^{2,2}(\Omega, \gamma)$. Eventually we prove that $u$ is a weak solution of (1).

Moreover, under some regularity assumption on the boundary of $\Omega$, we prove that the weak solution of (1) satisfies
\[
\langle \nabla_H u, \nabla_H g \rangle_H = 0
\] on $\partial \Omega$, in the sense of traces. This identity plays the role of the Neumann boundary condition. We use the same sequence $\{u_n\}_{n \in \mathbb{N}}$ defined above, and we show that
\[
\int_\Omega (\lambda u_n - \mathcal{L}^{\Omega_n} u_n) \varphi \, d\gamma = \int_\Omega f \varphi \, d\gamma,
\] for all smooth cylindrical functions $\varphi$, where $\mathcal{L}^{\Omega_n}$ is the Ornstein-Uhlenbeck operator associated to the quadratic form $\mathcal{E}_{\Omega_n, \gamma}$. Applying the integration by parts formula (6) we get
\[
\int_\Omega \lambda \varphi u_n \, d\gamma + \int_\Omega \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma = \int_\Omega f \varphi \, d\gamma + \int_{\partial \Omega} \langle \nabla_H u_n, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho,
\] here $g : X \to \mathbb{R}$ is a suitable convex function such that $g^{-1}(0) = \partial \Omega$ and $\rho$ is the surface measure associated to the Gaussian measure, see [7]. Taking the limit along a weakly convergent subsequence, we obtain
\[
\int_\Omega \lambda \varphi u \, d\gamma + \int_\Omega \langle \nabla_H u, \nabla_H \varphi \rangle_H \, d\gamma = \int_\Omega f \varphi \, d\gamma + \int_{\partial \Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho,
\] for all smooth cylindrical functions $\varphi$. Since $u$ is the weak solution of (1) then we can conclude that
\[
\int_{\partial \Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho = 0
\]
for all smooth cylindrical functions $\varphi$, that is equivalent to (4).

The maximal $L^p$ regularity for Ornstein-Uhlenbeck equations was established in [11] by Meyer when $\Omega$ is the whole space $X$ for $1 < p < \infty$. When $\Omega \subsetneq X$ and $p = 2$ the maximal regularity problem was also studied in Hilbert spaces by Da Prato and Lunardi in [5] with Dirichlet boundary condition and in [6] with Neumann boundary condition for a different class of differential operators that doesn’t contain the classical Ornstein-Uhlenbeck operator. Also, the proof in [6] is different from ours because it uses a penalization method approaching the weak solution by a sequence of solutions of problems on whole $X$.

In finite dimension more results are available. Maximal $L^p$ regularity, for $p \in (1, \infty)$, was studied by Metafune, Pruess, Rhandi, and Schnaubelt in [10] when $\Omega = \mathbb{R}^n$ for a class of second order differential operators with unbounded coefficients that contains symmetric Ornstein-Uhlenbeck operators. Maximal $L^2$ regularity in open convex sets of $\mathbb{R}^n$, with Neumann boundary condition, was established in [4] by methods different from ours.

2 Preliminaries and definitions

In this section we recall some basic definitions and notations. Hereafter $h_i$ will denote the $i$-th element of an orthonormal basis of $H$; for every $k \in \mathbb{N}$ set $\hat{h}_k = Q^{-1}(h_k)$ (see [1, p. 39-40]). If $x_i \in \mathbb{R}^n$ we denote by $D_{x_i}$ the directional derivative in the direction of $x_i$ while by $\partial_i$ we denote the directional derivative in the direction of $h_i$.

Definition 1. $FC^k_b(X)$ is the space of cylindrical functions of the form

$$f(x) = \varphi(l_1(x), \ldots, l_n(x)),$$

with $\varphi \in C^k_b(\mathbb{R}^n)$, $l_1, \ldots, l_n \in X^*$ and $n \in \mathbb{N}$.

Definition 2. $W^{1,2}(\Omega, \gamma)$ is the Sobolev space defined as the completion of the restriction to $\Omega$ of the elements of space $FC^1_b(X)$ with respect to the norm

$$\|f\|_{W^{1,2}(\Omega, \gamma)}^2 = \int_\Omega \left( f^2 + |\nabla_H f|_{H}^2 \right) d\gamma,$$

where $\nabla_H$ is the gradient along the direction of $H$.

Definition 3. $W^{2,2}(\Omega, \gamma)$ is the Sobolev space defined as the completion of the restriction to $\Omega$ of the elements of space $FC^2_b(X)$ with respect to the norm

$$\|f\|_{W^{2,2}(\Omega, \gamma)}^2 = \int_\Omega \left( f^2 + |\nabla_H f|_{H}^2 + \|D_H^2 f\|_H^2 \right) d\gamma,$$

where $D_H^2$ is the $H$-Hessian operator and $\| \cdot \|_H$ is the Hilbert-Schmidt norm.

Definition 4 (Weak solution). The function $u \in W^{1,2}(\Omega, \gamma)$ is a weak solution of (1) if

$$\int_\Omega \lambda u \varphi \ d\gamma + \int_\Omega \langle \nabla_H u, \nabla_H \varphi \rangle_H d\gamma = \int_\Omega f \varphi \ d\gamma \quad \forall \varphi \in FC^1_b(X) \quad (5)$$

or equivalently for all $\varphi \in W^{1,2}(\Omega, \gamma)$. 

3
Assumption 1. We suppose that $\Omega = g^{-1}(-\infty, 0)$, where $g : X \to \mathbb{R}$ is a continuous function such that

- $g \in W^{2,p}(X, \gamma)$ for all $p > 1$;
- there exists $\delta > 0$ such that $\frac{1}{|\nabla_H g|_H} \in L^p(g^{-1}(-\delta, \delta), \gamma)$ for all $p > 1$.

These conditions allow the definition of traces of Sobolev functions at $g^{-1}(0) = \partial \Omega$, see [3].

Let $\varphi, \psi \in W^{1,2}(\Omega, \gamma)$, we recall the integration by parts formula:

$$\int_{\Omega} \partial_k \varphi \psi \, d\gamma = - \int_{\Omega} \varphi \partial_k \psi \, d\gamma + \int_{\Omega} \varphi \psi \nabla k \, d\gamma + \int_{\partial \Omega} \text{Tr} \varphi \text{Tr} \psi \frac{\partial_k g}{|\nabla_H g|_H} \, dp.$$  \hfill (6)

where in the last integral $\rho$ is the surface measure associated to the Gaussian measure and $\text{Tr} \varphi, \text{Tr} \psi$ are the traces of the function $\varphi, \psi$ (see [3]).

In [2] the Logarithmic-Sobolev inequality is proved:

$$\int_{\Omega} f^2 \log(f^2) \, d\mu \leq \int_{\Omega} |\nabla_H f|^2_H \, d\gamma + \|f\|^2_{L^2(\Omega, \gamma)} \log(\|f\|^2_{L^2(\Omega, \gamma)}),$$ \hfill (7)

that holds for every $f \in W^{1,2}(\Omega, \gamma)$.

For $u, v \in W^{1,2}(\Omega, \gamma)$ let

$$\mathcal{E}_{\Omega, \gamma}(u, v) := \int_{\Omega} \left\langle \nabla_H u, \nabla_H v \right\rangle_H \, d\gamma$$

be the quadratic form associated to $\nabla_H$; we set

$$D(L^\Omega) = \left\{ u \in W^{1,2}(\Omega, \gamma) : \exists f \in L^2(\Omega, \gamma) \text{ s.t. } \mathcal{E}_{\Omega, \gamma}(u, v) = - \int_{\Omega} f v \, d\gamma, \forall v \in W^{1,2}(\Omega, \gamma) \right\}$$ \hfill (8)

and we put $L^\Omega u := f$.

Let $\mathcal{O}$ be a smooth convex set of $\mathbb{R}^n$ and let $\mu$ be the standard Gaussian measure in $\mathbb{R}^n$. Let $L^\mathcal{O}$ be the Ornstein-Uhlenbeck operator associated to the quadratic form $\mathcal{E}_{\mathcal{O}, \mu}$. It is known, see [3], that

$$D(L^\mathcal{O}) = \left\{ f \in W^{2,2}(\mathcal{O}, \mu) : \langle x, \nabla f \rangle \in L^2(\mathcal{O}, \mu) \text{ and } \frac{\partial f}{\partial \nu} = 0 \right\}$$ \hfill (9)

where $\nu(x)$ is the exterior normal vector to $\partial \mathcal{O}$ at $x$. Moreover

$$L^\mathcal{O} f(x) = \Delta f(x) - \langle x, \nabla f(x) \rangle \text{ for every } f \in D(L^\mathcal{O}).$$ \hfill (10)

We recall the finite dimensional logarithmic Sobolev inequality

$$\int_{\mathcal{O}} f^2 \log(f^2) \, d\mu \leq \int_{\mathcal{O}} |\nabla_H f|^2_H \, d\mu + \|f\|^2_{L^2(\mathcal{O}, \mu)} \log(\|f\|^2_{L^2(\mathcal{O}, \mu)}),$$ \hfill (11)

that holds for each $f \in W^{1,2}(\mathcal{O}, \mu)$, see [4].
3 Finite-dimensional estimates

Let $O$ be an open smooth convex subset of $\mathbb{R}^n$, with fixed $n$. We assume that

$$O = \{ x \in \mathbb{R}^n : g(x) < 0 \}$$

where $g$ is a smooth convex function whose gradient does not vanish at the boundary $\partial O$. We denote by $\nu(x)$ the exterior normal vector to $\partial O$ at $x$, $\nu(x) = \frac{\nabla g(x)}{|\nabla g(x)|}$. Let $\mu$ be the standard Gaussian measure in $\mathbb{R}^n$ and let $L^O$ be the associated Ornstein-Uhlenbeck operator, that is

$$L^O \psi(x) = \sum_{i=1}^{n} D_{i} \psi(x) - \sum_{i=1}^{n} x_i D_{i} \psi(x)$$

for $\psi \in D(L^O)$.

In this section we consider the following problem

$$\begin{cases}
\lambda \psi - L^O \psi = f & \text{in } O \subset \mathbb{R}^n, \\
\frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial O
\end{cases}$$

(12)

where $f \in L^2(O, \mu)$ and $\lambda > 0$.

Let us introduce a weighted surface measure on $\partial O$:

$$d\sigma = N(x) dH^{n-1}$$

where $N(x) = (2\pi)^{-n/2} \exp(-|x|^2/2)$ is the Gaussian weight and $dH^{n-1}$ is the usual Hausdorff $(n-1)$ dimensional measure. Using the surface measure $d\sigma$ the integration by parts formula reads as:

$$\int_{O} \nabla \varphi \psi \ d\mu = -\int_{O} \varphi \nabla \psi \ d\mu + \int_{O} \nabla_\nu \varphi \psi \ d\mu + \int_{\partial O} \nabla g \varphi \psi \ d\sigma,$$

for each $\varphi, \psi \in W^{1,2}(O, \mu)$ one of which with bounded support, so the boundary integral is meaningful. Indeed $W^{1,2}(O, \mu) \subset W^{1,2}_{loc}(O, dx)$ and the trace at the boundary of any function in $W^{1,2}(O, \mu)$ belongs to $L^2_{loc}(\partial O, dH^{n-1}) = L^2_{loc}(\partial O, d\sigma)$.

Applying (13) with $\varphi$ replaced by $D_k \varphi$ and summing up, we find

$$\int_{O} L^O \varphi \psi \ d\mu = -\int_{O} \langle \nabla \varphi, \nabla \psi \rangle d\mu + \int_{\partial O} \frac{\langle \nabla \varphi, \nabla \psi \rangle}{|\nabla \psi|} \psi \ d\sigma,$$

(14)

for every $\varphi \in W^{2,2}(O, \mu)$, $\psi \in W^{1,2}(O, \mu)$ one of which with bounded support.

Now we give dimension free estimates for the weak solution $u \in W^{1,2}(O, \mu)$ to (12) with $\lambda > 0$ and $f \in L^2(O, \mu)$. We can suppose $f \in C^\infty_c(O)$ because $C^\infty_c(O)$ is dense in $L^2(O, \mu)$. In this case, thanks to classical results about elliptic equations with smooth coefficients we know that the weak solution $u$ of (12) belongs to $C^\infty(\overline{O}) \subset W^{1,2}_{loc}(O, \mu)$. Since $O$ can be unbounded, we introduce a smooth cutoff function $\theta : \mathbb{R}^n \to \mathbb{R}$ such that

$$0 \leq \theta(x) \leq 1, \ \ |\nabla \theta(x)| \leq 2 \ \forall x \in \mathbb{R}^n, \ \ \theta \equiv 1 \ \text{in } B(0,1), \ \ \theta \equiv 0 \ \text{outside } B(0,2)$$

and we set, for $R > 0$

$$\theta_R(x) = \theta(x/R), \ \ x \in \mathbb{R}^n.$$
For the $W^{1,2}$ estimates we take $u$ as a test function in the definition of weak solution and we get

$$
\lambda \int_{\Omega} u^2 \, d\mu + \int_{\Omega} |\nabla u|^2 \, d\mu = \int_{\Omega} f \, d\mu,
$$

(15)

then

$$
\int_{\Omega} u^2 \, d\mu \leq \frac{1}{\lambda^2} \|f\|^2_{L^2(\Omega, \mu)}, \quad \int_{\Omega} |\nabla u|^2 \, d\mu \leq \frac{1}{\lambda^2} \|f\|^2_{L^2(\Omega, \mu)}.
$$

(16)

The following lemma takes into the account the convexity of $\Omega$.

**Lemma 1.** If $u \in C^2(\overline{\Omega})$ satisfies $\langle \nabla u, \nu \rangle = 0$ on $\partial \Omega$ then

$$
\langle D^2 u \cdot \nabla u, \nu \rangle \leq 0 \quad \text{on} \quad \partial \Omega.
$$

**Proof.** We recall that $\partial \Omega = g^{-1}(0)$ where $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function. Let $\tau \in \mathbb{R}^n$ such that $\langle \tau, \nu(x) \rangle = 0$ for $x \in \partial \Omega$, then we have

$$
\frac{\partial \nu}{\partial \tau}(x, \tau) \geq 0 \quad \forall x \in \partial \Omega.
$$

(17)

Indeed

$$
\frac{\partial \nu}{\partial \tau} = \frac{\partial}{\partial \tau} \left( \frac{\nabla g}{|\nabla g|} \right) = \frac{1}{|\nabla g|} \frac{\partial}{\partial \tau} \langle \nabla g \rangle + \frac{\partial}{\partial \tau} \left( \frac{1}{|\nabla g|} \right) \nabla g
$$

$$
= \frac{1}{|\nabla g(x)|} D^2 g \cdot \tau + \langle \nabla \left( \frac{1}{|\nabla g(x)|} \right) \rangle \nabla g,
$$

then for $x \in \partial \Omega$ we have

$$
\frac{\partial \nu}{\partial \tau}(x, \tau) = \frac{1}{|\nabla g(x)|} \langle D^2 g(x) \cdot \tau, \tau \rangle + \langle \nabla \left( \frac{1}{|\nabla g(x)|} \right) \rangle \langle \nabla g(x), \tau \rangle
$$

$$
= \frac{1}{|\nabla g(x)|} \langle D^2 g(x) \cdot \tau, \tau \rangle \geq 0
$$

since $D^2 g$ is a positive semi-definite symmetric matrix. Now we recall that $\langle \nabla u, \nu \rangle = 0$ on $\partial \Omega$ therefore

$$
\frac{\partial}{\partial \tau}(\langle \nabla u(x), \nu(x) \rangle) = \langle D^2 u(x) \cdot \tau, \nu(x) \rangle + \langle \nabla u(x), \frac{\partial \nu}{\partial \tau}(x) \rangle = 0, \quad x \in \partial \Omega
$$

for each $\tau \in \mathbb{R}^n$ such that $\langle \tau, \nu \rangle = 0$ on $\partial \Omega$. If we take $\tau = \nabla u(x)$ then we get

$$
\langle D^2 u(x) \cdot \nabla u(x), \nu(x) \rangle = -\langle \tau, \frac{\partial \nu}{\partial \tau}(x) \rangle \leq 0, \quad x \in \partial \Omega.
$$

The following lemma takes into the account the convexity of $\Omega$. 

**Lemma 1.** If $u \in C^2(\overline{\Omega})$ satisfies $\langle \nabla u, \nu \rangle = 0$ on $\partial \Omega$ then

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\langle D^2 u \cdot \nabla u, \nu \rangle \leq 0 \quad \text{on} \quad \partial \Omega.
$$

**Proof.** We recall that $\partial \Omega = g^{-1}(0)$ where $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function. Let $\tau \in \mathbb{R}^n$ such that $\langle \tau, \nu(x) \rangle = 0$ for $x \in \partial \Omega$, then we have

$$
\frac{\partial \nu}{\partial \tau}(x, \tau) \geq 0 \quad \forall x \in \partial \Omega.
$$

(17)

Indeed

$$
\frac{\partial \nu}{\partial \tau} = \frac{\partial}{\partial \tau} \left( \frac{\nabla g}{|\nabla g|} \right) = \frac{1}{|\nabla g|} \frac{\partial}{\partial \tau} \langle \nabla g \rangle + \frac{\partial}{\partial \tau} \left( \frac{1}{|\nabla g|} \right) \nabla g
$$

$$
= \frac{1}{|\nabla g(x)|} D^2 g \cdot \tau + \langle \nabla \left( \frac{1}{|\nabla g(x)|} \right) \rangle \nabla g,
$$

then for $x \in \partial \Omega$ we have

$$
\frac{\partial \nu}{\partial \tau}(x, \tau) = \frac{1}{|\nabla g(x)|} \langle D^2 g(x) \cdot \tau, \tau \rangle + \langle \nabla \left( \frac{1}{|\nabla g(x)|} \right) \rangle \langle \nabla g(x), \tau \rangle
$$

$$
= \frac{1}{|\nabla g(x)|} \langle D^2 g(x) \cdot \tau, \tau \rangle \geq 0
$$

since $D^2 g$ is a positive semi-definite symmetric matrix. Now we recall that $\langle \nabla u, \nu \rangle = 0$ on $\partial \Omega$ therefore

$$
\frac{\partial}{\partial \tau}(\langle \nabla u(x), \nu(x) \rangle) = \langle D^2 u(x) \cdot \tau, \nu(x) \rangle + \langle \nabla u(x), \frac{\partial \nu}{\partial \tau}(x) \rangle = 0, \quad x \in \partial \Omega
$$

for each $\tau \in \mathbb{R}^n$ such that $\langle \tau, \nu \rangle = 0$ on $\partial \Omega$. If we take $\tau = \nabla u(x)$ then we get

$$
\langle D^2 u(x) \cdot \nabla u(x), \nu(x) \rangle = -\langle \tau, \frac{\partial \nu}{\partial \tau}(x) \rangle \leq 0, \quad x \in \partial \Omega.
$$

Now we can give an estimate of the second order derivatives of $u$.

**Proposition 1.** For every $f \in C^\infty_c(\Omega)$ and $\varepsilon > 0$ there exists $R_0 > 0$ such that for $R > R_0$ the solution $u$ to (12) satisfies

$$
(1 - \varepsilon) \int_{\Omega} \theta_R^2 \text{Tr}[(D^2 u)^2] \, d\mu \leq \left( 2 + \frac{\varepsilon}{\lambda} \right) \|f\|^2_{L^2(\Omega, \mu)}.
$$

(18)
Proof. Recall that $u \in C^\infty(O)$; differentiating (12) with respect to $x_h$ yields

$$\lambda D_h u - \Delta D_h u - \langle x, \nabla (D_h u) \rangle + D_h u = D_h f.$$  

Multiplying by $D_h u \theta_R^2$ we obtain

$$(\lambda + 1)(D_h u)^2 \theta_R^2 - \Delta D_h u \cdot D_h u \theta_R^2 - \langle x, \nabla (D_h u) \rangle D_h u \theta_R^2 = D_h f D_h u \theta_R^2.$$  

Integrating over $O$ and using (14) yields

$$\int_O (\lambda + 1)|(\nabla f u|^2 \theta_R^2 \, d\mu + \int_O |\nabla (D_h u)|^2 \theta_R^2 \, d\mu + 2 \int_O \theta_R \langle \nabla (D_h u), \nabla \theta_R \rangle D_h u \, d\mu
= \int_{\partial O} \frac{\langle \nabla (D_h u), \nabla g \rangle D_h u}{|\nabla g|} \theta_R^2 \, d\sigma + \int_O D_h f D_h u \theta_R^2 \, d\mu.$$  

Summing over $h$ we obtain

$$\int_O (\lambda + 1)|\nabla u|^2 \theta_R^2 \, d\mu + \int_O \text{Tr}[(D^2 u)^2] \theta_R^2 \, d\mu + 2 \int_O (D^2 u \cdot \nabla u, \nabla \theta_R) \theta_R \, d\mu
= \int_{\partial O} \frac{(D^2 u \cdot \nabla u, \nabla g)}{|\nabla g|} \theta_R^2 \, d\sigma + \int_O (\nabla f, \nabla u) \theta_R^2 \, d\mu.$$  

Since $f$ has compact support, for $R$ large enough $\theta_R \equiv 1$ on the support of $f$. For such $R$ we obtain

$$\left| \int_O (\nabla f, \nabla u) \theta_R^2 \, d\mu \right| = \left| - \int_O L^2 u f \, d\mu \right| = \left| \int_O (\lambda u - f) f \, d\mu \right| \leq 2 \|f\|_{L^2(O, \mu)}^2.$$  

Moreover

$$\left| \int_O (D^2 u \cdot \nabla u, \nabla \theta_R) \, d\mu \right| \leq \int_O \sum_{i,j=1}^n |D_{ij} u| |D_{ij} \theta_R| \theta_R \, d\mu
\leq \frac{1}{2} \sum_{i,j=1}^n |D_{ij} u|^2 |D_{ij} \theta_R| \theta_R \, d\mu + \frac{1}{2} \sum_{i,j=1}^n |D_{ij} u|^2 |D_{ij} \theta_R| \theta_R \, d\mu
\leq \frac{1}{2} \frac{\|\nabla \theta\|_{L^\infty}^2}{R} \int_O \theta_R^2 \, d\mu + \frac{1}{2} \frac{\|\nabla \theta\|_{L^\infty}^2}{R} \int_O \|\nabla u\|^2 \, d\mu
\leq \left( \frac{1}{2} \int_O \text{Tr}[(D^2 u)^2] \theta_R^2 \, d\mu + \frac{1}{2} \frac{\|\nabla \theta\|_{L^\infty}^2}{R} \right) \frac{\|\nabla \theta\|_{L^\infty}^2}{R}.$$  

Taking $R$ large enough, such that $\|\nabla \theta\|_{L^\infty}/R \leq \varepsilon$, we get

$$(1 - \varepsilon) \int_O \theta_R^2 \text{Tr}[(D^2 u)^2] \, d\mu \leq \left( 2 + \frac{\varepsilon}{\lambda} \right) \|f\|_{L^2(O, \mu)}^2 + \int_{\partial O} \frac{\theta_R^2 (D^2 u \cdot \nabla u, \nabla g)}{|\nabla g|} \, d\sigma.$$  

By using Lemma 1 the statement follows. \hfill \Box

**Theorem 1.** For each $\lambda > 0$ there exists $C = C(\lambda) > 0$, independent of $n$ and $O$, such that for each $f \in L^2(O, \mu)$ the weak solution $u$ of problem (12) belongs to $W^{2,2}(O, \mu)$, and satisfies

$$\|u\|_{W^{2,2}(O, \mu)} \leq C\|f\|_{L^2(O, \mu)}.$$  

(19)
Proof. Let \( f \in C_c^\infty(\mathcal{O}) \). Taking the limit as \( R \to \infty \) in (18) and using the monotone convergence theorem, we get
\[
(1 - \varepsilon) \int_\mathcal{O} \text{Tr}[(D^2u)^2]d\mu \leq \left( 2 + \frac{\varepsilon}{\lambda} \right) \|f\|_{L^2(\mathcal{O},\mu)}^2.
\]

Now taking the limit as \( \varepsilon \to 0 \) we get
\[
\int_\mathcal{O} \text{Tr}[(D^2u)^2]d\mu \leq 2 \|f\|_{L^2(\mathcal{O},\mu)}^2.
\]

Taking into account (15), (16), and (20) we obtain
\[
\|u\|_{W^{2,2}(\mathcal{O},\mu)}^2 = \|u\|_{L^2(\mathcal{O},\mu)}^2 + \|\nabla u\|_{L^2(\mathcal{O},\mu)}^2 + \|\text{Tr}[(D^2u)^2]\|_{L^2(\mathcal{O},\mu)}
\leq \left( \frac{1}{\lambda^2} + \frac{1}{\lambda} + 2 \right) \|f\|_{L^2(\mathcal{O},\mu)}^2
\]
which is (19) with \( C(\lambda) = \frac{1}{\lambda^2} + \frac{1}{\lambda} + 2 \). For \( f \in L^2(\mathcal{O},\mu) \) the statement follows approaching it by a sequence of functions belonging to \( C_c^\infty(\mathcal{O}) \).

Now we get a characterization of the domain of \( L^2 \). We recall that (9) holds, and we prove that the condition \( \langle \cdot, \nabla f \rangle \in L^2(\mathcal{O},\mu) \) can be omitted.

**Proposition 2.** If \( f \in W^{2,2}(\mathcal{O},\mu) \) then \( \langle x, \nabla f \rangle \in L^2(\mathcal{O},\mu) \), moreover the map
\[
f \mapsto \langle \cdot, \nabla f \rangle
\]
is continuous from \( W^{2,2}(\mathcal{O},\mu) \) to \( L^2(\mathcal{O},\mu) \).

**Proof.** Let \( f \in W^{2,2}(\mathcal{O},\mu) \), then
\[
\int_\mathcal{O} |\langle \nabla f, x \rangle|^2d\mu = \int_\mathcal{O} \sum_{i=1}^n (D_i f x_i)^2d\mu
\]
by assumption \( D_i f \in W^{1,2}(\mathcal{O},\mu) \) and if \( c < 1/4 \), by using (11), we have
\[
\int_\mathcal{O} (D_i f(x))^2 x_i^2 e^{-|x|^2/2}dx
= \int_{\{x \in \mathcal{O} : cx_i^2 > \log |D_i f(x)|\}} (D_i f(x))^2 x_i^2 e^{-|x|^2/2}dx
+ \int_{\{x \in \mathcal{O} : cx_i^2 \leq \log |D_i f(x)|\}} (D_i f(x))^2 x_i^2 e^{-|x|^2/2}dx
\leq \int_\mathcal{O} e^{2cx_i^2} x_i^2 e^{-|x|^2/2}dx + \int_\mathcal{O} \frac{1}{c} |D_i f|^2 \log |D_i f| e^{-|x|^2/2}dx
\leq C_1 + \frac{1}{c} \left( \int_\mathcal{O} |\nabla D_i f|d\mu + \frac{1}{2} \int_\mathcal{O} (D_i f)^2d\mu \log \left( \int_\mathcal{O} (D_i f)^2d\mu \right) \right).
\]
Summing over \( i \) from 1 to \( n \) we have \( \langle \nabla f, x \rangle \in L^2(\mathcal{O},\mu) \).
4 Approximation by cylindrical functions

Now we consider the infinite dimensional problem. Let $\Omega \subset X$ be an open convex set and let $\{\Omega_n\}$ be a sequence of cylindrical open convex sets, defined in [3], of the form $\Omega_n = \pi_n^{-1}(\Omega_n)$ where $\Omega_n \subset F_n$, $F_n$ a is finite dimensional subspace of $Q(X^*) \subset H$ with $\dim F_n = j(n) \leq n$, $F_n \subset F_{n+1}$ for $n \in \mathbb{N}$, and $\pi_n : X \to F_n$ is the projection defined below. Let $\{h_n\}_{n \in \mathbb{N}} \subset Q(X^*)$ be an orthonormal basis of the Cameron-Martin space $H$ such that $F_n = \text{span}\{h_1, \ldots, h_{j(n)}\}$. Therefore

$$\pi_n(x) = \sum_{i=1}^{j(n)} \hat{h}_i(x)h_i.$$ 

Moreover $\Omega_{n+1} \subset \Omega_n$, $\partial \Omega_n$ is smooth, $\Omega \subset \Omega_n$ and

$$\Omega = \bigcap_{n \in \mathbb{N}} \Omega_n, \quad \gamma \left( \bigcap_{n \in \mathbb{N}} \Omega_n \setminus \Omega \right) = 0.$$ 

We recall that since $\Omega$ and $\Omega_n$ are open convex sets, then $\gamma(\partial \Omega) = \gamma(\partial \Omega_n) = 0$.

Now we show that the restriction to $\Omega$ of cylindrical continuous smooth functions is dense in $L^2(\Omega, \gamma)$. Let $\psi \in L^2(\Omega, \gamma)$, then the zero extension outside $\Omega$, $\overline{\psi}$, belongs to $L^2(X, \gamma)$. We have from [1, Corollary 3.5.2] that there exists a sequence of $L^2(X, \gamma)$ cylindrical functions $\psi_n$ that converges to $\overline{\psi}$ in $L^2(X, \gamma)$. In its turn, each $\psi_n$ may be approached by a sequence of $C^\infty_0$ functions.

Therefore we suppose $f \in FC^\infty_0(X)$. Then, for some $k \in \mathbb{N}$,

$$f(x) = w(l_1(x), \ldots, l_k(x))$$

where $w \in C^\infty_0(\mathbb{R}^k)$, $l_i \in X^*$ for $i = 1, \ldots, k$.

Let $G = G(n, f) := \text{span}\{F_n, Q(l_1), \ldots, Q(l_k)\}$. Then $G$ is a subspace of $H$ of dimension $q = q(n, f) \leq j(n) + k$; setting $d := q - j(n)$ let $\mathcal{O} = \mathcal{O}(n, f) := \mathcal{O}_n \times \mathbb{R}^d$. If we denote by

$$\pi_G(x) = \sum_{i=1}^{q} \hat{h}_i(x)h_i$$

then

$$f(x) = \tilde{f}(\pi_G(x))$$

where $\tilde{f} \in C^\infty_0(G)$. Let $\gamma_G$ be the induced measure $\gamma \circ \pi_G^{-1}$ in $G$; if $G$ is identified with $\mathbb{R}^q$ through the isomorphism $x \mapsto (\hat{h}_1(x), \ldots, \hat{h}_q(x))$ then $\gamma_G$ is the standard Gaussian measure in $\mathbb{R}^q$.

We recall that $L^\Omega_n$ is the Ornstein-Uhlenbeck operator associated to the quadratic form $\mathcal{E}_{\partial \gamma}$ while $L^\mathcal{O}$ is Ornstein-Uhlenbeck operator associated to the quadratic form $\mathcal{E}_{\gamma_G}$.

**Proposition 3.** Let $v$ be the weak solution of the finite dimensional problem

$$\lambda v - L^\partial v = f|\mathcal{O} \quad \text{in } \mathcal{O}$$

Then $u(x) := v(\pi_G(x))$ is the weak solution of

$$\lambda u - L^\Omega_n u = f|\Omega_n \text{ in } \Omega_n.$$
Proof. We remark that the space $X$ can be split as $X = G \times \tilde{X}$ where $\tilde{X} = (I - \pi_G)(X)$, and $\gamma = \gamma_G \otimes \bar{\gamma}$ where $\bar{\gamma} = \gamma \circ (I - \pi_G)^{-1}$ is the measure induced on $\tilde{X}$ by the projection $I - \pi_G$. Let $\varphi \in W^{1,2}(\Omega_n, \gamma)$, then

\[
\int_{\Omega_n} (\lambda u(x) \varphi(x) + (\nabla_H u(x), \nabla_H \varphi(x))_H) \gamma(dx) \\
= \int_{\Omega_n} \lambda v(\pi_G(x)) \varphi(\pi_G(x) + (I - \pi_G)(x)) \\
+ (\nabla_H v(\pi_G(x)), \nabla_H \varphi(\pi_G(x) + (I - \pi_G)(x))_H \gamma(dx) \\
= \int_{\Sigma \times \tilde{X}} \lambda v(\xi) \tilde{\varphi}(\xi + y) \\
+ (\nabla v(\xi), \nabla \tilde{\varphi}(\xi + y))_G(d\xi) \tilde{\gamma}(dy) \quad \text{(where $\tilde{\varphi}(\cdot + y) \in W^{1,2}(\Sigma, \gamma_G)$)} \\
= \int_{\Sigma \times \tilde{X}} \tilde{f}(\xi) \tilde{\varphi}(\xi + y) \gamma_G(d\xi) \tilde{\gamma}(dy) \\
= \int_{\Omega_n} f(x) \varphi(x) \gamma(dx),
\]
and the statement follows. \hfill \Box

**Proposition 4.** The function $u$ satisfies

$$
\|u\|_{W^{2,2}(\Omega, \gamma)} \leq K
$$

where

$$
K := C\|f\|_{L^2(\Omega_1, \gamma)}
$$

and $C$ is the constant of Theorem 1.

Proof. We recall that $u(x) = v(\pi_G(x))$. Then

\[
\|u\|^2_{W^{2,2}(\Omega, \gamma)} \leq \|u\|^2_{W^{2,2}(\Omega_n, \gamma)} \\
= \int_{\Omega_n} |u(x)|^2 + \sum_{i=1}^{\infty} |D_i u(x)|^2 + \sum_{i,j=1}^{\infty} |D_{ij} u(x)|^2 \gamma(dx) \\
= \int_{\Sigma} \left( |v(\xi)|^2 + \sum_{i=1}^{q} |\partial_i v(\xi)|^2 \right) \mu(d\xi) \quad \text{(by using Theorem 1)} \\
\leq C^2 \|\tilde{f}\|^2_{L^2(\Sigma, \mu)} = C^2 \|f\|^2_{L^2(\Omega_1, \gamma)} \leq C^2 \|f\|^2_{L^2(\Omega_n, \gamma)}
\]

\hfill \Box

If we consider the sequence $\{u_n\}_{n \in \mathbb{N}}$ of weak solutions of the problems

$$
\lambda \psi - L^{\Omega_n} \psi = f_{|\Omega_n} \quad \text{in } \Omega_n.
$$
By Proposition 4 it follows
\[ \|u_n\|_{W^{2,2}(\Omega, \gamma)} \leq K. \]

Possibly replacing \( u_n \) by a subsequence, there exists \( u \in W^{2,2}(\Omega, \gamma) \) such that \( u_n \to u \) in \( W^{2,2}(\Omega, \gamma) \).

**Proposition 5.** The function \( u \) is the weak solution of (1).

**Proof.** We know that for all \( \varphi \in FC^1_b(X) \)
\[ \int_{\Omega_n} \lambda u_n \varphi \, d\gamma + \int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma = \int_{\Omega_n} f \varphi \, d\gamma. \]

We claim that
\[ \lim_{n \to \infty} \int_{\Omega_n} \lambda u_n \varphi \, d\gamma = \int_{\Omega} \lambda u \varphi \, d\gamma. \]

Indeed,
\[ \int_{\Omega_n} \lambda u_n \varphi \, d\gamma = \int_{\Omega} \lambda u_n \varphi \, d\gamma + \int_{\Omega \setminus \Omega_n} \lambda u_n \varphi \, d\gamma; \quad (21) \]
by the weak convergence
\[ \lim_{n \to \infty} \int_{\Omega_n} \lambda u_n \varphi \, d\gamma = \int_{\Omega} \lambda u \varphi \, d\gamma \]
while
\[ \left| \int_{\Omega \setminus \Omega_n} \lambda u_n \varphi \, d\gamma \right| \leq \lambda \left( \int_{\Omega \setminus \Omega_n} |u_n|^2 \, d\gamma \right)^{1/2} \left( \int_{\Omega \setminus \Omega_n} |\varphi|^2 \, d\gamma \right)^{1/2} \leq \lambda K \left( \int_{\Omega \setminus \Omega_n} |\varphi|^2 \, d\gamma \right)^{1/2} \]
that goes to zero as \( n \to \infty \) by the absolute continuity of the integral. Now we claim that
\[ \lim_{n \to \infty} \int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma = \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H \, d\gamma. \]
In fact,
\[ \int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma = \int_{\Omega} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma + \int_{\Omega \setminus \Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma. \]
By the weak convergence in \( W^{1,2}(\Omega, \gamma) \)
\[ \lim_{n \to \infty} \int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma = \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H \, d\gamma \]
while
\[ \left| \int_{\Omega \setminus \Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma \right| \leq \lambda \left( \int_{\Omega \setminus \Omega_n} |\nabla_H u_n|^2 \, d\gamma \right)^{1/2} \left( \int_{\Omega \setminus \Omega_n} |\nabla_H \varphi|^2 \, d\gamma \right)^{1/2} \leq \lambda K \left( \int_{\Omega \setminus \Omega_n} |\nabla_H \varphi|^2 \, d\gamma \right)^{1/2}. \]
that goes to zero as $n \to \infty$.

Moreover,

$$\lim_{n \to \infty} \int_{\Omega_n} f \varphi \, d\gamma = \int_{\Omega} f \varphi \, d\gamma.$$ 

Therefore letting $n \to \infty$ in (21) we get that $u$ satisfies (5). 

Finally we give the maximal regularity estimate.

**Theorem 2.** If $u$ is the weak solution of $\lambda u - L\Omega u = f$ on $\Omega$ then $u \in W^{2,2}(\Omega, \gamma)$

and

$$\|u\|_{W^{2,2}(\Omega, \gamma)} \leq C \|f\|_{L^2(\Omega, \gamma)}$$

**Proof.** By Proposition 4 it follows

$$\|u_n\|_{W^{2,2}(\Omega, \gamma)} \leq C \|f\|_{L^2(\Omega_n, \gamma)} \quad (22)$$

where $C = C(\lambda)$ is the constant of the Theorem 1.

We remark that

$$\lim_{n \to \infty} \|f\|_{L^2(\Omega_n, \gamma)} = \|f\|_{L^2(\Omega, \gamma)}$$

since $\gamma(\Omega_n \setminus \Omega) \to 0$.

By the weak convergence of $u_n$ to $u$ we have

$$\|u\|_{W^{2,2}(\Omega, \gamma)} \leq \limsup_{n \to \infty} \|u_n\|_{W^{2,2}(\Omega, \gamma)}.$$ 

Letting $n \to \infty$ in (22) we get our claim. 

5 **The Neumann boundary condition**

In this section we put under Assumption 1 and we prove that the weak solution $u$ of (1) satisfies a Neumann type boundary condition.

First we prove a useful lemma.

**Proposition 6.** If $u \in L^p(\partial \Omega, \rho)$ and

$$\int_{\partial \Omega} u \varphi \, d\rho = 0 \quad \forall \varphi \in FC^1_{\delta}(X),$$

then $u = 0 \rho-a.e. \ in \ \partial \Omega$.

**Proof.** Since the map

$$v \mapsto \int_{\partial \Omega} uv \, d\rho$$

is continuous from $W^{1,q}(\Omega, \gamma)$ to $\mathbb{R}$ for all $q > p'$, and $FC^1_{\delta}(X)$ is dense in $W^{1,q}(\Omega, \gamma)$, it follows that

$$\int_{\partial \Omega} u \psi \, d\rho = 0 \quad \forall \psi \in W^{1,q}(\Omega, \gamma).$$

In particular, since the restrictions to $\Omega$ of the Lipschitz continuous and bounded functions $\psi : X \to \mathbb{R}$ belong to $W^{1,q}(\Omega, \gamma)$, we have

$$\int_{\partial \Omega} u \psi \, d\rho = 0 \quad \forall \psi \in Lip_b(X).$$
Lemma 3 yields
\[ \int_{\partial \Omega} u \psi \, d\rho = 0 \quad \forall \psi \in L^2(\partial\Omega, \rho) \]
and this implies that \( u = 0 \) \( \rho \)-a.e.. \( \square \)

Now we are ready to prove that the weak solution of (1) satisfies a boundary condition similar to the Neumann boundary condition.

Proposition 7. If \( u \) is the weak solution of \( \lambda u - Lu = f \) on \( \Omega \) then
\[ \langle \nabla_H u(x), \frac{\nabla_H g(x)}{|\nabla_H g(x)|_H} \rangle_H = 0 \quad \rho - \text{a.e.} \quad x \in \partial \Omega. \] (23)

Proof. We fix \( \varphi \in \mathcal{FC}_b(X) \). We denote by \( u_n \) the solution to
\[ \lambda \varphi_n - L^{\Omega_n} \psi = f|_{\Omega_n} \text{ in } \Omega_n. \] (24)

We recall that \( u_n \) is a cylindrical function and, thanks to the result of Section 4, we have \( u_n \in W^{2,2}(\Omega_n, \gamma) \). We multiply the differential equation (24) by \( \varphi \) and we integrate on \( \Omega \), getting
\[ \int_{\Omega_n} (\lambda u_n - L^{\Omega_n} u_n) \varphi \, d\gamma = \int_{\Omega} f \varphi \, d\gamma. \]

We recall that \( L^{\Omega_n} u_n \) is cylindrical, then
\[ L^{\Omega_n} u_n(x) = \sum_{i=1}^q \partial_i u_n(x) - \hat{h}_i(x) \partial_i u_n(x). \]

Therefore, by using (23), we obtain
\[ \int_{\partial \Omega} \lambda \varphi_n^H \varphi_n \, d\gamma + \int_{\Omega} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma = \int_{\Omega} f \varphi \, d\gamma + \int_{\partial \Omega} \langle \nabla_H u_n, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho, \]
where
\[ \langle \nabla_H u_n, \nabla_H \varphi \rangle_H = \sum_{i=1}^q \partial_i u_n \partial_i \varphi, \]
and
\[ \langle \nabla_H u_n, \nabla_H g \rangle_H = \sum_{i=1}^q \partial_i u_n \partial_i g. \]

As in the previous section we have
\[ \lim_{n \to \infty} \int_{\Omega_n} \lambda \varphi_n \, d\gamma = \int_{\Omega} \lambda \varphi u \, d\gamma, \]
and
\[ \lim_{n \to \infty} \int_{\partial \Omega} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma = \int_{\partial \Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H \, d\gamma. \]

We claim that the map
\[ v \mapsto \int_{\partial \Omega} \langle \nabla_H v, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho \]
from $W^{2, 2}(\Omega, \gamma)$ to $\mathbb{R}$ belongs to $(W^{2, 2}(\Omega, \gamma))'$. Indeed, the function
\[ x \mapsto \langle \nabla H v(x), \frac{\nabla H g(x)}{|\nabla H g(x)|} \rangle_H \varphi(x) =: F(x) \]
belongs to $W^{1, q}(\Omega, \gamma)$ for all $q \in (1, 2)$. Moreover $\|F\|_{W^{1, q}(\Omega, \gamma)} \leq \tilde{C}\|v\|_{W^{2, 2}(\Omega, \gamma)}$, and the trace operator is linear and continuous from $W^{1, q}(\Omega, \gamma)$ to $L^1(\partial \Omega, \rho)$. Therefore, since $u_n \rightharpoonup u$ in $W^{2, 2}(\Omega, \gamma)$,
\[ \lim_{n \to \infty} \int_{\partial \Omega} \langle \nabla H u_n, \frac{\nabla H g}{|\nabla H g|} \rangle_H \varphi d\rho = \int_{\partial \Omega} \langle \nabla H u, \frac{\nabla H g}{|\nabla H g|} \rangle_H \varphi d\rho. \]
Then we have
\[ \int_{\Omega} \lambda u \varphi \, d\gamma + \int_{\Omega} \langle \nabla H u, \nabla H \varphi \rangle_H \, d\gamma = \int_{\Omega} f \varphi \, d\gamma + \int_{\partial \Omega} \langle \nabla H u, \frac{\nabla H g}{|\nabla H g|} \rangle_H \varphi d\rho \]
and since $u$ is a weak solution of (1) we get
\[ \int_{\partial \Omega} \langle \nabla H u, \frac{\nabla H g}{|\nabla H g|} \rangle_H \varphi d\rho = 0 \]
for all $\varphi \in FC^1_b(X)$. By using Proposition 6 we obtain (23). 

Therefore, if $u \in D(L)$ then $u \in W^{2, 2}(\Omega, \gamma)$ and $u$ satisfies the Neumann boundary condition (23).

A Density properties

In this appendix we show some density results for which we thank Simone Ferrari. Let $(Y, d)$ be a complete metric space and let $\rho$ be a finite Radon measure defined on the Borel sets of $Y$. Let $BUC(Y)$ be the set of real valued uniformly bounded continuous functions and let $Lip_b(Y)$ be the set of Lipschitz bounded functions.

Lemma 2. Let $f : Y \to \mathbb{R}$ be a bounded $\rho$–measurable function. Then for all $\varepsilon > 0$ there exists $g \in BUC(Y)$ such that
\[ \rho(\{x \in Y : f(x) \neq f_\varepsilon(x)\}) < \varepsilon \]
and
\[ \sup_{x \in Y} |g(x)| \leq 2 \sup_{x \in Y} |f(x)|. \]

Proof. We fix $\varepsilon > 0$. Since $\rho$ is a Radon measure then there exists $K_0$, compact subset of $Y$, such that $\rho(Y \setminus K_0) < \varepsilon$. By the Lusin theorem there exists a function $f_0 \in C(K_0) = BUC(K_0)$ such that:
\[ \rho(\{x \in K_0 : f_0(x) \neq f|_{K_0}(x)\}) < \varepsilon \]
and
\[ \sup_{x \in K_0} |f_0(x)| \leq \sup_{x \in K_0} |f(x)| \leq \sup_{x \in Y} |f(x)|. \]
We define the following function, studied in [9],

$$g(x) = \begin{cases} f(x) & \text{if } x \in K_0 \\ \inf_{y \in K_0} f_0(y) \frac{d(x,y)}{d(x,K_0)} & \text{if } x \notin K_0 \end{cases}$$

then $g$ is a BUC extension of $f_0$ to the whole $Y$. We remark that for $x \notin K_0$ there exists $y_\varepsilon \in K_0$ such that

$$d(x,K_0) = \inf_{y \in K_0} d(x,y) \geq d(x,y_\varepsilon) - \varepsilon,$$

therefore for $x \notin K_0$ we have

$$|g(x)| = \left| \inf_{y \in K_0} f_0(y) \frac{d(x,y)}{d(x,K_0)} \right| \leq \sup_{x \in Y} |f(x)| \frac{d(x,y_\varepsilon)}{d(x,K_0)} \leq \sup_{x \in Y} |f(x)| \frac{d(x,K_0) + \varepsilon}{d(x,K_0)}$$

for all $\varepsilon$. Then for all $x \notin K_0$ we have

$$g(x) \leq \sup_{y \in Y} |f(y)|.$$

Finally

$$\sup_{x \in Y} |g(x)| = \sup_{x \in Y} |g_{|K_0}(x) + g_{|Y \setminus K_0}(x)| \leq \sup_{x \in K_0} |g(x)| + \sup_{x \in Y \setminus K_0} |g(x)|$$

$$= \sup_{x \in K_0} |f_0(x)| + \sup_{x \in Y \setminus K_0} |g(x)| \leq 2 \sup_{x \in Y} |f(x)|.$$

Moreover

$$\rho(\{x \in Y : g(x) \neq f(x)\}) \leq \rho(\{x \in K_0 : g(x) \neq f(x)\}) + \rho(\{x \in Y \setminus K_0 : g(x) \neq f(x)\})$$

$$\leq \rho(\{x \in K_0 : f_0(x) \neq f(x)\}) + \rho(Y \setminus K_0) < 2\varepsilon.$$

\[\square\]

**Lemma 3.** The subspace $\text{Lip}_0(Y)$ is dense in $L^p(Y,\rho)$ with respect the norm $\|\cdot\|_{L^p(Y,\rho)}$.

**Proof.** Let $f \in L^p(Y,\rho)$. For $k \in \mathbb{N}$ we put

$$f_k(x) = \begin{cases} k & \text{if } f(x) > k \\ f(x) & \text{if } f(x) \in [-k,k] \\ -k & \text{if } f(x) < -k \end{cases}$$

so that $f_k(x)$ is bounded and measurable. Then by **Lemma 2** there exists $\tilde{f}_k \in \text{BUC}(Y)$ such that

$$\rho(\{x \in Y : \tilde{f}_k(x) \neq f_k(x)\}) \leq \frac{1}{2^k}$$

Then by [12] there exists $g_k \in \text{Lip}_0(Y)$ such that

$$\|g_k - \tilde{f}_k\|_{L^\infty(Y)} \leq \frac{1}{2^k}.$$
Now we estimate
\[ \|g_k - f\|_{L^p(Y, \rho)} \leq \|g_k - \tilde{f}_k\|_{L^p(Y, \rho)} + \|\tilde{f}_k - f_k\|_{L^p(Y, \rho)} + \|f_k - f\|_{L^p(Y, \rho)}, \]
where
\[ \|g_k - \tilde{f}_k\|_{L^p(Y, \rho)} = \left( \int_Y |g_k(x) - \tilde{f}_k(x)|^p \rho(dx) \right)^{1/p} \]
\[ \leq \|g_k - \tilde{f}_k\|_{L^\infty(Y)} \rho(Y)^{1/p} \leq \frac{\rho(Y)^{1/p}}{2k}. \]
Concerning the second term we recall that
\[ \sup_{x \in Y} |\tilde{f}_k(x)| \leq 2 \sup_{x \in Y} |f_k(x)| = 2k \]
then
\[ \|\tilde{f}_k - f_k\|_{L^p(Y, \rho)} = \left( \int_Y |\tilde{f}_k(x) - f_k(x)|^p \rho(dx) \right)^{1/p} \]
\[ = \left( \int_{\{x \in Y : \tilde{f}_k(x) \neq f_k(x)\}} |\tilde{f}_k(x) - f_k(x)|^p \rho(dx) \right)^{1/p} \]
\[ \leq 3k \rho(\{x \in Y : \tilde{f}_k(x) \neq f_k(x)\})^{1/p} \leq \frac{3k}{2k^{1/p}}. \]
Finally we remark that since \( f_k \to f \) \( \rho \)-a.e. for \( k \to \infty \), and \( |f_k(x)| \leq |f(x)| \in L^p(Y, \rho) \), then the Lebesgue theorem yields
\[ \|f_k - f\|_{L^p(Y, \rho)} \to 0, \ k \to \infty. \]

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