Abstract—Randomization has shown catalyzing effects in linear algebra with promising perspectives for tackling computational challenges in large-scale problems. For solving a system of linear equations, we demonstrate the convergence of a broad class of algorithms that at each step pick a subset of \( n \) equations at random and update the iterate with the orthogonal projection to the subspace those equations represent. We identify, in this context, a specific degree-\( n \) polynomial that non-linearly transforms the singular values of the system towards equalization. This transformation to singular values and the corresponding condition number then characterizes the expected convergence rate of iterations. As a consequence, our results specify the convergence rate of the stochastic gradient descent algorithm, in terms of the mini-batch size \( n \), when used for solving systems of linear equations.

I. INTRODUCTION

Solving systems of linear equations is a fundamental computational problem with myriad applications in scientific computing and engineering. It is also core to many problems in optimization, signal processing, and machine learning that often require solutions to large-scale linear systems.

Iterative methods for solving \( Ax = b \) have long been considered attractive for large-scale problems where direct methods become computationally infeasible. Convergence of iterative methods to a satisfactory precision usually depends on the spectrum of \( A \) with the rate of convergence dependent on the spectral radius (e.g., Jacobi, SOR) or the ability of the algorithm to find polynomials that map the spectrum to as small a set of values as possible (e.g., Krylov methods) [21].

The Kaczmarz algorithm is an iterative method that pursues a succession of orthogonal projections to hyperplanes corresponding to individual rows of the system. Iterating over the rows, the pursuit converges to the intersection of all hyperplanes, thereby solving the system. By virtue of its small memory footprint and simplicity of iteration it is particularly attractive for large-scale problems where direct methods are computationally infeasible.

In their pioneering work, Strohmer & Vershynin [20] showed that randomizing the Kaczmarz algorithm, picking a row with probability proportional to its norm, can be proven to have exponential convergence in expectation. The significance of this development was that it demonstrated the convergence rate using the smallest singular value of \( A \) in its scaled condition number — quantities that bring the analysis of Kaczmarz closer to other iterative methods and relevant to practice. Their randomized Kaczmarz algorithm has since attracted significant attention with a number of variants, acceleration techniques and generalizations.

The Kaczmarz algorithm satisfies a single equation (corresponding to a row) at each iterate by an orthogonal projection to the hyperplane the equation represents. To scale up the convergence for practical applications, one can envision satisfying multiple equations corresponding to a block of rows from \( A \). There has been a long line of research investigating block methods for iterative solvers [3], including many block Kaczmarz methods [16, 13, 15]. The fundamental difficulty with these methods is that to show convergence one needs to form blocks that are all guaranteed to be well-conditioned. Forming uniformly well-conditioned blocks is possible when entries of \( A \) are randomly and independently drawn, but it is a hard computational problem for deterministic matrices that occur in applications (e.g., tomography, sampling theory).

Contributions

We provide analytical tools to eliminate this theoretical roadblock and demonstrate convergence rates that go beyond the bound associated with the smallest singular value with two classes of algorithms: At each iteration, the algorithm picks \( n \) rows randomly — either uniformly or nonuniformly with volume probabilities [4, 3] — and updates the iterate using the orthogonal projector to the intersection of hyperplanes that those rows represent. We develop tools to establish exponential convergence rates in both scenarios solely from the singular values of \( A \) and \( n \).

Specifically, the convergence rate of the single-row \( (n = 1) \) randomized Kaczmarz depends on the smallest singular value of \( A \). Our results show that the two-row pursuit \( (n = 2) \) transforms the singular values of \( A \) by the quadratic polynomial \( \Phi_2(x) = \|A\|_F^2 x - x^2 \). The convergence of the two-row pursuit depends on the smallest singular value after this transformation: \( \hat{\sigma}^2 := \Phi_2(\sigma^2) \), for all singular values \( \sigma \) of \( A \). More generally, for an \( n \)-row pursuit, we demonstrate that the singular values are transformed by the polynomial:

\[
\Phi_n(x) := \sum_{p=1}^{n} (-1)^{p-1} \text{vol}_{n-p} x^p, \tag{1}
\]
where \( \text{vol}_p \) denotes the sum of squared volumes of the parallelepipeds formed by any set of \( p \) rows of \( A \); \( \text{vol}_0 := 1 \) and \( \text{vol}_1 \) is the sum of individual row-norm squares amounting therefore to the Frobenius norm: \( \text{vol}_1 = \| A \|_F^2 \). Additionally, we establish that \( \text{vol}_p \) can be computed through a trace formula from the (transformed) squared singular values, avoiding what would otherwise be a combinatorial-large computation.

While randomized Kaczmarz methods have received considerable attention in the research community, in practical settings they require significantly large number of iterations making them not competitive with traditional (e.g., Krylov-based) iterative solvers [10]. Our results provide the theoretical foundations to level the playing-field by synthesizing multi-row generalizations of various randomized Kaczmarz algorithms that can leverage traditional (e.g., direct) methods at each step to boost the performance. This is especially relevant to practice as increasing \( n \) creates a unique opportunity to facilitate increased (parallelizable) computations per-iteration to reduce the number of iterations – with a reduction in total computation. Our experiments show that practical performance is shadowed by the convergence rates predicted for each \( n \) in our framework. In contrast to block-methods, our multi-row framework provides guarantees for increasing convergence rates with \( n \), without requiring an oracle, separate computation, assumptions on \( A \), or a preconditioner to form well-conditioned blocks.

**II. The Kaczmarz Algorithm and Randomization**

Given a matrix \( A \in \mathbb{R}^{M \times N} \) with \( M \geq N \) rows, we denote by \( a \in A \) a vector whose transpose, \( a^T \), is some row of \( A \), and by \( b_n \) the corresponding element of \( b \in \mathbb{R}^M \) on the right hand side of the system \( Ax = b \). The Kaczmarz algorithm is an iterative method that at each step picks an \( a \in A \) and updates the iterate by the orthogonal projection onto the hyperplane \( \langle a, x \rangle = b_n \) that the row represents:

\[
\begin{align*}
x_{k+1} := x_k + \frac{b_n - \langle a, x_k \rangle}{\|a\|^2} a. 
\end{align*}
\]

When the system is consistent, the pursuit converges to a point \( x^* \) at the intersection of all hyperplanes: \( \forall a \in A, \langle a, x^* \rangle = b_n \), solving the system.

Let \( P^a := a a^T / \| a \|^2 \) denote the rank-1 orthogonal projector onto the space spanned by \( a \). Then projection to a hyperplane orthogonal to \( a \) is accomplished in (2) by the rejection operator, \( I - P^a \), associated with the projector to the span. Recognizing that \( b_n = \langle a, x^* \rangle \), the Kaczmarz iterations can be viewed as:

\[
\begin{align*}
x_{k+1} = x_k - P^a(x_k - x^*).
\end{align*}
\]

Subtracting \( x^* \) from both sides allows us to quantify the gain (improvement) at each step of the pursuit:

\[
\begin{align*}
\|x_{k+1} - x^*\|^2 &= \| (I - P^a) (x_k - x^*) \|^2 \\
&= \langle x_k - x^* \rangle^T (I - P^a) (x_k - x^*),
\end{align*}
\]

since \( P^a \) is a symmetric projector. This shows that the greatest gain at a step is achieved by picking an \( a \) that is most aligned to \( x_k - x^* \), which is not an easy task and computationally expensive. Randomly picking a row with a probability \( pr(a) \) for each \( a \in A \) at each step allows us to quantify the expected gain from the expected projector:

\[
\mathbb{E}[P_1] = \sum_{a \in A} pr(a) P^a.
\]

The smallest eigenvalue of \( \mathbb{E}[P_1] \) can provide a lower bound for the expected gain at each step. Strohmer & Vershynin’s key observation was that picking rows with probabilities proportional to their squared norm, \( pr(a) = \| a \|^2 / \| A \|_F^2 \), allows one to identify this expected projector from which we recognize the Gram matrix associated with (the columns of) \( A \):

\[
\mathbb{E}[P_1] = \frac{1}{\| A \|_F^2} \sum_{a \in A} a a^T = A^T A / \| A \|_F^2 = \frac{1}{\| A \|_F^2} G.
\]

The significance of this observation is that it leads to a finite lower bound on the expected gain at each step based on the smallest eigenvalue of \( G \) which is \( \sigma_{\text{min}} \), the square of the smallest singular value of \( A \):

\[
\mathbb{E}[\|x_{k+1} - x^*\|^2] \leq \left(1 - \frac{\sigma_{\text{min}}^2}{\| A \|_F^2}\right) \|x_k - x^*\|^2. \tag{4}
\]

Based on independent draws at each step, this lower bound establishes exponential (geometric) convergence, in expectation, of the pursuit for every initialization \( x_0 \).

**III. Randomized Orthogonal Projectors**

To accelerate the convergence of iterations and open the door to parallel computation, one can pursue satisfying multiple equations at each step. For a pursuit that involves picking \( n \) rows from \( A \), for some \( 1 \leq n \leq N \), updating the iterate requires orthogonal projection to the intersection of the hyperplanes represented by the selected rows. Let \( a_1, \ldots, a_n \in A \) be an arbitrary set of \( n \) rows selected at a step. We denote by matrix \( A_n := [a_1, \ldots, a_n]^T \) the sub-matrix of \( A \) with the selected rows. When these vectors are linearly independent, the rank-\( n \) orthogonal projector to their span is formed using their \( n \times n \) Gram matrix:

\[
P_n := A_n^T G_n^{-1} A_n \quad \text{where} \quad G_n := A_n A_n^T. \tag{5}
\]

Generalizing (3) for \( n > 1 \), the orthogonal projection to the intersection of the \( n \) hyperplanes represented by the rows of \( A_n \) is achieved by the rejection operator \( I - P_n \), implying that the pursuit can be viewed as:

\[
x_{k+1} = x_k - P_n(x_k - x^*). \tag{6}
\]

Analysis of this pursuit is a challenge because random draws of a set of rows begetting \( A_n \) with some probability implies that the expected gain at each step is characterized by the smallest eigenvalue of the expected projector:

\[
\mathbb{E}[P_n] = \sum_{A_n \in A} pr(A_n) P_n. \tag{7}
\]

Besides the difficulty posed by linearly dependent subsets, averaging rank-\( n \) orthogonal projectors over all choices is a significant problem, characterizing the spectrum of which has materialized as a challenge in different contexts [8], [6].
Before addressing these challenges, we provide a recursive expansion for rank-\(n\) orthogonal projectors that to the best of our knowledge is new.

**Theorem 1** (Recursive Expansion of an Orthogonal Projector). Let \(U\) denote the span of \(n\) linearly independent vectors \(\{a_1, \ldots, a_n\}\). For each \(1 \leq s \leq n\), let \(P^s_1 := a_s a_s^T / \|a_s\|^2\) be the orthogonal projector to the subspace spanned by \(a_s\), and \(P^s_{n-1}\) be the orthogonal projector to the subspace \(U_s\) of \(U\), spanned by all but \(a_s\). Furthermore, let the angle between \(a_s\) and \(U_s\) be denoted by \(\theta_s\). Then the orthogonal projector to \(U\) has the expansion:

\[
P_n = \frac{1}{\sin^2 \theta_s} P^s_1 (I - P^s_{n-1}).
\]

(8)

**Proof.** Let \(L\) denote the expansion on the right hand side of (8). We prove \(P_n = L\). First, we note that for any \(t \neq s\),

\[
P^s_{n-1} a_t = a_t \quad (i.e., \text{if } t \neq s, \text{ then } \theta_s \neq \theta_t),
\]

which implies that \((I - P^s_{n-1}) a_t = 0\). We decompose \(a_t = a_t^s + a_t^\perp\) into its components in the subspace \(U_s\) orthogonal to that subspace: \(a_t^s = P^s_{n-1} a_s\) and \(a_t^\perp = (I - P^s_{n-1}) a_s\). The length of the orthogonal component is therefore \(\|a_t^\perp\| = \|a_s\| \sin \theta_s\).

We now use these observations to show that

\[
L a_t = a_t, \quad 1 \leq t \leq n
\]

Based on the above, the only term in the summation corresponding to \(L a_t\) that is non-zero is when \(s = t\), so that

\[
L a_t = \frac{1}{\sin^2 \theta_t} P^t_1 a_t^s = \frac{\langle a_s, a_t^s \rangle}{\sin^2 \theta_t \|a_s\|^2} a_t.
\]

Now \(\langle a_s, a_t^s \rangle = \|a_s\| \|a_t^s\| \cos(\angle(a_s, a_t^s))\). Since \(\theta_s = \angle(a_s, a_t^s)\), we get

\[
\langle a_s, a_t^s \rangle = \|a_s\| \|a_t^s\| \sin \theta_t. \quad \text{Plugging in } \|a_t^\perp\| = \|a_s\| \sin \theta_t \text{ then gives the result.}
\]

For any vector \(z\) orthogonal to \(U\), we have: \(P^s_{n-1} z = 0\) and \(P^s_n z = 0\) since \(z\) is orthogonal to all the \(n\) vectors \(a_1, \ldots, a_n\); therefore, \(L z = 0\).

This shows \(L = P_n\) is the orthogonal projector to \(U\), which is unique. \qed

**B. Quasi Projectors**

We expand on the notion of orthogonal projectors, from (5), and introduce a quasi projector that is well-defined even when the vectors are not linearly independent:

\[
Q_n := A_T^s \operatorname{adj}(G_n) A_n.
\]

(9)

The adjugate matrix, \(\operatorname{adj}(G_n)\), is also the cofactor matrix of \(G_n\) due to its symmetry. When the vectors are linearly dependent \(Q_n = 0\) (as we will see), otherwise the quasi projector is a scaled version of the orthogonal projector:

\[
Q_n = v_n^2 P_n \quad \text{where } v_n^2 := \det(G_n).
\]

(10)

Here \(v_n^2\) represents the (square of) \(n\)-volume of the parallelepiped formed by the rows chosen in \(A_n\). Using this volume definition, we can present a corollary to Theorem 1:

**Corollary 1** (Recursive Quasi Projector). Under the assumptions of the Theorem, let \(v_1^2 := \|a_s\|^2\) and \(v_{n-1}^2\) denote the volume of the parallelepiped formed by all but \(a_s\). Moreover, let \(Q^s_1 = a_s a_s^T\) and \(Q^s_{n-1}\) denote the quasi projectors corresponding to \(P^s_1\) and \(P^s_{n-1}\), respectively. Then we have:

\[
Q_n = \sum_{s=1}^{n} Q^s_1 \left( (v_n^2)^2 I - Q^s_{n-1} \right).
\]

(11)

**Proof.** The volume of the parallelepiped can be computed from the volume of any facet, \(v_{n-1}^2\), and the corresponding height, \(v_n^2 \sin \theta_s; \quad v_n = v_{n-1}^2 \sin \theta_s\). \qed

**Lemma 1.** For a linearly dependent set of vectors, \(\{a_1, \ldots, a_n\}\), the quasi projector \(Q_n = 0\), the zero matrix.

**Proof.** Based on (9), \(Q_n^T Q_n = 0\) since \(G_n \operatorname{adj}(G_n) = (\det G_n)I = 0\). This implies that all columns of \(Q_n\) have zero norm. Hence \(Q_n = 0\). \qed

**B. Expected Projector**

We can now go back and perform an expectation analysis for the iterations in (6). Since there are \(\binom{M}{n}\) possible choices to be made at each step. We introduce a choice function that indexes these possible choices:

\[
(i) \mapsto \{a_1, \ldots, a_n\} \quad \text{denotes the set of } n \text{ rows corresponding to the } i^{th} \text{ choice, } 1 \leq i \leq \binom{M}{n}.
\]

For example, \(v_{n}^2(i)\) denotes the volume of the parallelepiped formed by the vectors from the rows selected for the \(i^{th}\) choice, and likewise, \(Q_n(i)\) and \(P_n(i)\) denote the corresponding quasi projector and orthogonal projector.

We first establish the convergence rate using a probability for the \(i^{th}\) choice set proportional to \(v_n^2(i)\), and in the following section show corresponding results with the uniform distribution across choices. Recalling the definition of \(\text{vol}_n\), sum of squared volumes \(\text{vol}_n = \sum_i v_n^2(i)\), we have:

\[
pr(i) = \frac{v_n^2(i)}{\text{vol}_n}. \quad \text{What then needs to be characterized is the expected projector in (7), i.e., the average of the orthogonal projectors } \sum_i pr(i) P_n(i) = 1/\text{vol}_n \sum_i Q_n(i). \quad \text{To state our key result, we define a total quasi projector for a matrix } A:
\]

\[
\Phi_n := \sum_{i=1}^{\binom{M}{n}} Q_n(i).
\]

We now fully characterize \(\Phi_n\), specifically its spectrum, in our main result. When \(n = 1\) the total quasi projector coincides with the Gram matrix: \(\Phi_1 = G = \sum_i Q_1(i) = \sum_{a \in A} a a^T\). For a larger set of rows \(n > 1\) we show that \(\Phi_n\) is a degree-\(n\) polynomial of the Gram matrix.

**Theorem 2** (Total Quasi Projector). For \(n > 1\) rows we have:

\[
\Phi_n = G (\text{vol}_{n-1} I - \Phi_{n-1})
\]

(12)

where \(\Phi_1 = G\) and \(\text{vol}_n := \det(A) = \sum_{a_1 \in A} v_n^2(i).\)

**Proof.** We first expand the set of choices and define an operator that sums quasi projectors over all \(n\)-ordered choices of \(M\) rows with replacement for a total of \(M^n\) choices:

\[
\Phi_n := \sum_{j=1}^{M^n} Q_n(j) = \sum_{a_1 \in A} \sum_{a_2 \in A} \cdots \sum_{a_n \in A} Q_n.
\]
We denote the sum of volumes in the expanded setting as 
\[ \text{vol}_n := \sum_{j=1}^{M^n} v_n(j). \]
The expanded choices allow for summation over individual vectors that can sift through the recursion in (11). For example, the first term in (11) for \( s = 1 \) shows:

\[
\sum_{a_1 \in A} \sum_{a_2 \in A} \cdots \sum_{a_n \in A} Q_n^i \left( (v_{n-1}^i - Q_n^i) \right) = \sum_{a_1 \in A} \sum_{a_2, a_n \in A} \left( (v_{n-1}^i - Q_n^i) \right) = \sum_{a_1 \in A} \sum_{a_1, a_{n-1} \in A} \left( (v_{n-1}^i - Q_n^i) \right) = G \left( \text{vol}_{n-1} - \Phi_n \right).
\]

Observing that the result of this summation is independent of \( s \), allows us to establish:

\[ \Phi_n = nG \left( \text{vol}_{n-1} - \Phi_n \right). \]

Now we observe that in a particular choice of \( n \)-rows with replacement, if any row of \( A \) is selected more than once \( v_n^2(j) = 0 \) and \( Q_n = 0 \) for any \( n \). This means we can shrink the space of choices to \( n \)-permutations without replacement, with a total of \( M^n = n! \) choices, and still obtain the same \( \Phi_n \):

\[ \sum_{j=1}^{M^n} Q_n(j) = \Phi_n = nG \left( \text{vol}_{n-1} - \Phi_n \right). \]

To further shrink the space of choices to \( n \)-combinations, we note that permuting the order of the rows in a particular choice does not change the volume squared of the parallelepiped they form. This means \( \text{vol}_n = \text{vol}_n/n! \) for any \( n \). Moreover, permuting the rows in a particular choice does not change the orthogonal projector \( P_n \) and consequently \( Q_n \). This means \( \Phi_n = \Phi_n/n! \) for any \( n \):

\[ \Phi_n = \frac{n}{n!} G \left( (n-1)! \text{vol}_{n-1} - \Phi_n \right) = \frac{1}{(n-1)!} G \left( (n-1)! \text{vol}_{n-1} - (n-1)! \Phi_n \right) = G(\text{vol}_{n-1} - \Phi_n). \]

As a consequence, unwinding the recursion, we have:

**Theorem 3 (Volume Computation of all \( n \)-subsets).** Given a set of \( M \) vectors, arranged in rows of \( A \), the sum of squared volumes of parallelepipeds formed by size-\( n \) subsets is:

\[ \text{vol}_n = \sum_{i=1}^{M^n} v_n^2(i) = \frac{\text{tr} \Phi_n}{n}. \]

The trace formula escapes the combinatorially-large computation.

**Proof.** Using the fact that \( \text{tr} P_n = \text{rank}(P_n) = n \), and linearity of trace, we have \( \text{tr} Q_n = n \text{vol}_n \). Summing over all choices gives us:

\[ \text{tr} \Phi_n = \sum_{i=1}^{M^n} \text{tr} Q_n(i) = n \sum_{i=1}^{M^n} v_n^2(i) = n \text{vol}_n. \]

This shows the recursion in (11) can be written entirely in terms of the Gram matrix:

\[ \Phi_n = G \left( \frac{\text{tr} \Phi_{n-1}}{n-1} I - \Phi_{n-1} \right). \]

This implies \( \Phi_{n-1} \) provides a simple recursive form of a Cayley-Hamilton expansion for the inverse of the Gram matrix (or any symmetric positive semidefinite matrix), in terms of its powers and their traces.

**C. Condition Number and Convergence Rates**

The expected projector for the \( n \)-row pursuit in (6) can now be characterized as:

\[ \mathbb{E}[P_n] = \frac{\Phi_n}{\text{vol}_n}. \]

Eigenvalues of \( G \) (i.e., singular values of \( A \)) are then transformed by \( \Phi_n \) viewed as a polynomial in \( G \) as in (13).

**Corollary 3.** The expected error at each step of the \( n \)-row pursuit in (6) is at most:

\[ \mathbb{E} \left[ \| x_{k+1} - x^* \|^2 \right] \leq \left( 1 - \frac{\sigma_{\min}^2}{\text{vol}_n} \right) \| x_k - x^* \|^2, \]

where \( \sigma_{\min}^2 \) denotes the smallest value after all singular values \( \sigma \) of \( \mathbf{A} \) are transformed by (11):

\[ \sigma^2 := \Phi_n(\sigma^2) = \sum_{p=1}^{n} (-1)^{p-1} \text{vol}_{n-p} \sigma^{2p}. \]

**Proof.** Applying the expected projector on \( \| x_{k+1} - x^* \|^2 = (x_k - x^*)^T(I - P_n)(x_k - x^*) \) establishes the result.

This bound coincides with (11) for \( n = 1 \). The polynomial transformation of singular values defined in Corollary 3 can also be viewed from the perspective of condition number.

**Definition 1.** We define the grade-\( n \) condition number of a matrix \( \mathbf{A} \) as \( \kappa_n^2 = \kappa^2(\mathbf{A}, n) := \text{vol}_n/\sigma_{\min}^2 \).

This notion of grade-\( n \) condition number coincides with the scaled condition number for \( n = 1 \) first introduced by
Demmel [2]. With this notion of condition number, we can state the main result:

**Theorem 4** (Expected Exponential Convergence). The n-row pursuit in (6) converges in expectation with the exponential rate:

\[
\mathbb{E} \left[ \|x_k - x^*\|^2 \right] \leq \left( 1 - \frac{1}{\kappa_n^2} \right)^k \|x_0 - x^*\|^2
\]

with independent draws at each step, for any initial point \(x_0\).

This theorem provides a conservative bound on the convergence rate since it assumes that the gains made at each step are minimal all along. We will see that as iterations proceed, the expected rate can reach close to twice this bound, but can not exceed the factor of 2 (in the exponent).

The discrete dynamics of iterations is characterized by the expected projector \((I - \Phi_n/\text{vol}_n)\). Given a particular \(x_k\), (6) shows that the expected position of \(x_{k+1}\) is: \(\mathbb{E}[x_{k+1} - x^*] = (I - \Phi_n/\text{vol}_n)(x_k - x^*)\). Based on the independence of draws at each iteration, the position of \(x_k\) based on the initial point \(x_0\) is determined, similarly, in expectation:

\[
\mathbb{E}[x_k - x^*] = \left( I - \frac{\Phi_n}{\text{vol}_n} \right)^k (x_0 - x^*).
\]

This means that the pursuit acts, in expectation, as a power iteration using the expected projector. Based on (13), \(\Phi_n\) inherits the orthogonal eigenvectors of \(G\) (i.e., right singular vectors of \(A\)) while its eigenvalues are transformed from those of \(G\), according to Corollary 5. Given that the largest eigenvalue of \(I - \Phi_n/\text{vol}_n\) (i.e., \(1 - \sigma_{\text{min}}/\text{vol}_n\)) is strictly larger than other eigenvalues, and \(x_0 - x^*\) has a component along the corresponding eigenvector, iterations align \(\mathbb{E}[x_k - x^*]\) with that eigenvector, that we denote by \(v_{\text{min}}\). Under these conditions of the power method, the alignment with \(v_{\text{min}}\) occurs as a limiting behavior (independently discovered in [18] for \(n = 1\)). To study the ramifications of this alignment on convergence rate, we examine the iterations starting from \(x_0 - x^* = v_{\text{min}}\).

**Theorem 5.** The expected error in \(k\) steps of the n-row pursuit starting from a point \(x_0\) is at least:

\[
\mathbb{E}\left[ \|x_k - x^*\|^2 \right] \geq \left( 1 - \frac{1}{\kappa_n^2} \right)^{2k} \|x_0 - x^*\|^2,
\]

when \(x_0 - x^*\) is aligned with \(v_{\text{min}}\).

**Proof.** The norm squared of the expectation is \(\mathbb{E}\|x_k - x^*\|^2 = (1 - 1/\kappa_n^2)^{2k}\|x_0 - x^*\|^2\) which establishes the result with an application of Jensen’s Inequality.

We remark that the pursuit’s convergence rate reaches close to this bound since the gap is solely due to Jensen’s Inequality.

**IV. UNIFORMLY RANDOM DRAWS**

The randomized Kaczmarz algorithm analyzed thus far requires draws from a distribution that needs to be sampled (see (3)) by visiting all \(M\) rows – a task that can be a significant cost in large-scale problems (19) or in streaming settings. For \(n = 1\), the commonly-adopted approach is to scale the rows of \(A\) (and the corresponding elements of \(b\)) normalizing row norms prior to the iterations, so that uniform probabilities may be used. This, however, changes the singular values of \(A\) that then deviates from theory. In the multi-row pursuit this cost of computing probabilities becomes prohibitive quickly for \(n > 1\) rows. Moreover, the geometric nature of Kaczmarz pursuing projections to (intersections of) hyperplanes makes it agnostic to the algebraic representation of hyperplanes which renders the row-norm (or \(v_{\text{max}}^2(i)\)) based probabilities at odds with the geometry of the problem 11. Finally, in the stochastic gradient descent setting, uniform sampling (of rows) is often the only practical choice.

We now present a strategy for randomizing the Kaczmarz algorithm with a uniform distribution \(pr(i) = 1/\binom{M}{n}\). The key observation is that we can construct a similar expected operator by scaling the orthogonal projector update to fit the stipulation of the quasi projector. To incorporate this into the Kaczmarz algorithm, we identify a specific \(\mu_k \in (0, 2)\) for relaxing (6):

\[
x_{k+1} := x_k - \mu_k P_n(x_k - x^*),\quad \text{(14)}
\]

Leveraging the Pythagorean theorem while remaining in the span of \(A_n\), we get:

\[
\|x_{k+1} - x^*\|^2 = \|x_k - x^*\|^2 - \|P_n(x_k - x^*)\|^2 + \|P_n(x_k - x^*)\|^2.
\]

For the \(i\)th choice corresponding to \(P_n(i), \text{setting } \mu_k := 1 \pm \sqrt{1 - v_{\text{max}}^2(i)/v_{\text{max}}^2} \text{ with } v_{\text{max}}^2(i) := \max_i v_{\text{max}}^2(i), \text{we get:}

\[
\|x_{k+1} - x^*\|^2 = \|x_k - x^*\|^2 - \frac{1}{v_{\text{max}}^2} (x_k - x^*)^T Q_n(x_k - x^*).
\]

Defining \(v_{\text{max}}^2 := \binom{M}{n} v_{\text{max}}^2\), we can characterize the expected gain with uniform distribution \(pr(i) = 1/\binom{M}{n}:

\[
\mathbb{E}\|x_{k+1} - x^*\|^2 \leq \left( 1 - \frac{\sigma_{\text{min}}^2}{v_{\text{max}}^2} \right) \|x_k - x^*\|^2.
\]

While the smallest singular value, \(\sigma_{\text{min}}\), still controls the rate of convergence, the convenience of using the uniform distribution comes with some cost: \(\text{vol}_n = \sum_{i=1}^{\binom{M}{n}} v_{\text{max}}^2(i) \leq \binom{M}{n} v_{\text{max}}^2 = v_{\text{max}}^2\). The significance of uniform draws from a practical standpoint is that it eliminates the need to visit all rows a priori. Uniform draws, however, do need knowledge of \(v_{\text{max}}^2\) for computing \(\mu_k\) at each step of (14); however, \(v_{\text{max}}^2\) can be chosen to be the largest volume, \(v_{\text{max}}^2\), observed thus far in the iterations – providing a stable \(\mu_k \in (0, 2)\) that converges to its correct value.

**V. EXPERIMENTS**

Our first numerical experiment visualizes the transformation of a particular set of singular values through the map \(\Phi_n(\cdot)\) when iterating with various n-row pursuits, \(1 \leq n \leq N\). To relate this transformation to the corresponding bound obtained in Theorem 4, we normalize the square of transformed singular values by \(\text{vol}_n\). Fig. 1 shows these transformations for \(N = 8\). On the left the initial singular values are decaying linearly while on the right they have an exponential
aligning $\mathbf{x}_k - \mathbf{x}^*$ with $\mathbf{v}_{\min}$, the singular vector corresponding to $\sigma_{\min}$.

Fig. 3 shows an experiment where the pursuit picks $n$-rows uniformly at random as discussed in Section VI with iterations proceeding according to the scaled orthogonal projector in (14). The two solutions derived for $\mu_k$ present an undershoot and an overshoot scenario for the orthogonal projection. Fig. 3 shows that the undershoot scenario closely follows the convergence predicted in Theorem 4 when $\text{vol}_n$ is adjusted to $\text{vol}_n^{max}$. Our experiments show that the convergence in the overshoot scheme can be significantly faster, a well-known phenomena for over-determined systems [20]. While Fig. 3 presents a typical scenario observed in experiments for random choices of $\mathbf{A}$, the improvement of convergence rate by increasing $n$ can be non-monotonic in the uniform draws case. As discussed before, $\text{vol}_n \leq \text{vol}_n^{max}$ and the expected rate is influenced by the maximum volume of parallelepipeds $\sigma^2_{vol}(\text{max})$ present among the choices. Depending on the geometry of hyperplanes in $\mathbf{A}$, the maximum $n$-volume can affect the performance for a particular $n$.

VI. DISCUSSION

The framework provided in this paper allows for the investigation of randomized orthogonal projectors in different settings. For example, similar bounds can be obtained for least squares problems [22]. The relationship between randomized Kaczmarz and stochastic gradient descent (SGD) has been established before [17], [13]. Our results provide a characterization of convergence rates that go beyond the bound by the smallest singular value for any sample size $n$ for the mini-batch, which is of practical value.

The challenge in block methods has been the forming of blocks in a manner that the condition numbers across blocks are uniformly bounded. These bounds can then be used to characterize the convergence rates. For large-scale problems where Kaczmarz iterations become useful, even the cost of visiting all rows can be significant [19]. In these settings, forming effective blocks can be a significant additional cost. In such large-scale problems the pursuit with uniform draws provides an appealing approach with performance guarantees.

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(b) Upper bounds provided by Theorem 5 for each $n$.

Fig. 2: Ensemble of randomized pursuits for a matrix $A$ with $M = 15, N = 10$ and $\kappa^2(A, 1) = 175.16, \kappa^2(A, 2) = 73.56, \kappa^2(A, 3) = 40.44$ with $n$ rows sampled by volume probability. Slopes of dashed lines show the theoretical bounds on the expected convergence rate for each $n$: lower (left) and upper (right). An ensemble of 15 pursuits were generated for each $n$ and the sample mean of observed rates, at each iteration, are plotted in solid lines. Empirical convergence rates (slope of solid lines) are close to the respective upper bounds (slope of dashed lines in right panel) as power iterations, by the expected upper bound, align $E[\|x_k - x^*\|]$ with the singular vector corresponding to $\tilde{\sigma}_{\min}$.

Fig. 3: Ensemble of randomized pursuits for a matrix $A$ with $M = 15, N = 10$ where $n$ rows are picked uniformly at random. Dashed lines show the lower bound on the expected convergence rate provided by $(1 - \tilde{\sigma}_{\min}^2 / \text{Vol}_{\max})$ for each $n$. An ensemble of 15 pursuits were generated for each $n$ and the sample mean of observed rates, at each iteration, are plotted in solid lines.

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