On the structure of uniformly hyperbolic chain control sets

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Abstract

We prove the following theorem: Let $Q$ be an isolated chain control set of a control-affine system on a smooth compact manifold $M$. If $Q$ is uniformly hyperbolic without center bundle, then the lift of $Q$ to the extended state space $U \times M$, where $U$ is the space of control functions, is a graph over $U$. In other words, for every control $u \in U$ there is a unique $x \in Q$ such that the corresponding state trajectory $\varphi(t, x, u)$ evolves in $Q$.

Keywords: Nonlinear control; control-affine system; chain control set; uniform hyperbolicity

AMS Subject Classification (2010): 93C10; 93C15; 93B05; 37D05; 37D20

1 Introduction

The notion of a uniformly hyperbolic set, which axiomatizes the geometric picture behind the “horseshoe”, a general mechanism for producing complicated dynamics, was introduced by Smale in the 1960s. A uniformly hyperbolic set of a diffeomorphism $g : M \to M$ on a compact Riemannian manifold $M$ is a closed invariant set $\Lambda$ such that the tangent bundle over $\Lambda$ splits into two subbundles, $T\Lambda = E^s \oplus E^u$, invariant under the differential $dg$ with uniform exponential contraction (expansion) on $E^s$ ($E^u$). For a flow $(\phi_t)_{t \in \mathbb{R}}$, generated by an ordinary differential equation $\dot{x} = f(x)$, a uniformly hyperbolic set is defined differently, because for any trajectory bounded away from equilibria, the vector $f(x) \in T_x M$ is neither contracted nor expanded exponentially. In this case, a uniformly hyperbolic set is a closed invariant set $\Lambda$ such that $T\Lambda = E^s \oplus E^c \oplus E^u$ with three invariant subbundles, where additionally to the contracting and expanding bundles the one-dimensional center bundle $E^c$ corresponds to the flow direction. Without the center bundle $E^c$ in this definition, a flow could only have trivial uniformly hyperbolic sets, consisting of finitely many equilibria.

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The situation looks different for systems generated by equations with explicitly time-dependent right-hand sides. General models for such systems are skew-products, which are dynamical systems of the form $\Phi : \mathbb{T} \times \mathcal{B} \times \mathcal{M} \rightarrow \mathcal{B} \times \mathcal{M}$, $\Phi_t(b, x) = (\theta_t b, \varphi_t(x, b))$, with a time set $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$. The solutions of the equation are incorporated in the map $\varphi$, while $\theta$ is a ‘driving system’ on a base space $\mathcal{B}$ that models the time-dependency of the equation. Every non-autonomous difference equation $x_{t+1} = f(t, x_t)$ or differential equation $\dot{x} = f(t, x)$ with unique and globally defined solutions gives rise to a skew-product, where $\mathcal{B} = \mathbb{T}$ and $\theta_t(s) = t + s$. Other examples with less trivial base dynamics are random dynamical systems and control-affine systems. If $\mathcal{B}$ is a compact space, $\mathcal{M}$ a smooth manifold and $\Phi$ respects these structures, a uniformly hyperbolic set can be defined as a compact $\Phi$-invariant set $\Lambda \subset \mathcal{B} \times \mathcal{M}$ such that for every $(b, x) \in \Lambda$ the tangent space $T_x \mathcal{M}$ splits into subspaces $E^s_{b,x} \oplus E^u_{b,x}$ depending on $b$ and $x$. The invariance of the splitting now means that $d\varphi_t(b)(x) E^s_{b,x} E^u_{b,x}$, and contraction (expansion) rates should be uniformly bounded in $b$ and $x$. One major difference to the autonomous situation is that there can exist non-trivial uniformly hyperbolic sets (whose projection to $\mathcal{M}$ has nonempty interior) in the continuous-time case without the existence of a one-dimensional center sub-bundle. This, for instance, happens in random dynamical systems that arise as small time-dependent perturbations of a flow around a hyperbolic equilibrium (cf. [7] for the discrete-time case).

In this paper, we consider a special type of skew-product flow, namely the control flow generated by a control-affine system, i.e., a control system governed by differential equations of the form

$$\Sigma : \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^m u_i(t) f_i(x(t)), \quad u \in \mathcal{U}.$$ 

The set $\mathcal{U}$ of admissible control functions consists of all measurable $u : \mathbb{R} \rightarrow \mathbb{R}^m$ with values in a compact and convex set $\mathcal{U} \subset \mathbb{R}^m$, and $f_0, f_1, \ldots, f_m$ are $C^1$-vector fields on a smooth manifold $\mathcal{M}$. The set $\mathcal{U}$, endowed with the weak*-topology of $L^\infty(\mathbb{R}, \mathbb{R}^m)^*$, is a compact metrizable space. For each $u \in \mathcal{U}$ and $x \in \mathcal{M}$ a unique solution to the corresponding equation exists with initial value $x$ at time $t = 0$. Writing $\varphi(\cdot, x, u)$ for this solution and assuming that all such solutions exist on $\mathbb{R}$, one obtains a continuous skew-product flow $\Phi : \mathbb{R} \times \mathcal{U} \times \mathcal{M} \rightarrow \mathcal{U} \times \mathcal{M}$, $\Phi_t(u, x) = (\theta_t u, \varphi(t, x, u))$, where $\theta_t u(s) = u(t + s)$ is the shift flow on $\mathcal{U}$. There are remarkable relations between dynamical properties of $\Phi$ and control-theoretic properties of $\Sigma$, a comprehensive study of which can be found in [2]. In particular, the notions of control and chain control sets are to mention here. Control sets are the maximal subsets of $\mathcal{M}$ on which complete approximate controllability holds. Their lifts to $\mathcal{U} \times \mathcal{M}$ are maximal topologically transitive sets of $\Phi$. In contrast, chain control sets are the subsets of $\mathcal{M}$ whose lifts are the maximal invariant chain transitive
sets of Φ, and they can be seen as an outer approximation of the control sets, since under mild assumptions a control set is contained in a chain control set.

The purpose of this paper is to prove a theorem about the structure of a chain control set $Q$ with a uniformly hyperbolic structure without center bundle. We show that the lift of such $Q$, defined by

$$Q := \{(u, x) \in U \times M : \varphi(R, x, u) \subset Q\},$$

has the property that each fiber $\{x \in M : (u, x) \in Q\}$ is a singleton. In other words, $Q$ is the graph of a (necessarily continuous) function $U \to Q$. This simple structure can be seen as an analogue to the fact that a connected uniformly hyperbolic set of a flow without center bundle consists of a single equilibrium. Nevertheless, from the control-theoretic viewpoint uniformly hyperbolic chain control sets are not trivial, since they can have nonempty interior and in this case are the closures of control sets (cf. [1, 3]).

The paper is organized as follows. In Section 2 we review the shadowing lemma proved in [8] for uniformly hyperbolic sets of general skew-product maps. This is the main tool for the proof of our theorem, which is carried out in Section 3. The final Section 4 contains an application to invariance entropy.

2 A shadowing lemma for skew-product maps

In this section, we explain the contents of the shadowing lemma for skew-product maps proved in [8] by Meyer and Zhang. Let $M$ be a Riemannian manifold (with metric $d(\cdot, \cdot)$) and $B$ a compact metric space. Suppose that

$$\Phi : B \times M \to B \times M, \quad \Phi(b, x) = (\theta(b), \varphi(b, x)),$$

is a homeomorphism such that also $\theta : B \to B$ is a homeomorphism. For fixed $b \in B$ assume that $\varphi_b := \varphi(b, \cdot) : M \to M$ is a diffeomorphism whose derivative depends continuously on $(b, x)$. The orbit through $(b, x)$ is the set $O(b, x) = \{\Phi^k(b, x) : k \in \mathbb{Z}\}$. We write $\varphi(k, x, b)$ for the second component of $\Phi^k(b, x)$, i.e., $\Phi^k(b, x) = (\theta^k(b), \varphi(k, x, b))$. A sequence $(b_k, x_k)_{k \in \mathbb{Z}}$ in $B \times M$ is an $\alpha$-pseudo-orbit if

$$b_{k+1} = \theta(b_k) \quad \text{and} \quad d(\varphi(b_k, x_k), x_{k+1}) < \alpha \quad \text{for all } k \in \mathbb{Z}.$$  

A pseudo-orbit $(b_k, x_k)_{k \in \mathbb{Z}}$ is $\beta$-shadowed by an orbit $O(b, x)$ if

$$b = b_0 \quad \text{and} \quad d(\varphi(k, x, b), x_k) < \beta \quad \text{for all } k \in \mathbb{Z}.$$  

A set $\Lambda \subset M \times B$ is invariant if $\Phi(\Lambda) = \Lambda$. A closed invariant set $\Lambda$ is isolated if there exists a neighborhood $U$ of $\Lambda$ such that $\Phi^k(b, x) \in \text{cl} U$ for all $k \in \mathbb{Z}$ implies

\footnote{In [8], $\theta$ is assumed to be almost periodic. However, this is not used for the proof of the shadowing lemma.}
A closed invariant set \( \Lambda \) is \textit{uniformly hyperbolic} if there are constants \( C > 0, 0 < \mu < 1 \) and a continuous map \((b, x) \mapsto P(b, x) \in P(T_xM, T_xM)\), defined on \( \Lambda \), where \( P(T_xM, T_xM) \) denotes the space of all linear projections on \( T_xM \), such that

(i) \( P(\Phi(b, x))d\varphi_b(x) = d\varphi_b(x)P(b, x) \).

(ii) \( \|d\varphi_{k,b}(x)P(b, x)\| \leq C\mu^k \) for all \((b, x) \in \Lambda, k \geq 0\).

(iii) \( \|d\varphi_{k,b}(x)(I - P(b, x))\| \leq C\mu^{-k} \) for all \((b, x) \in \Lambda, k \leq 0\).

Here \( \varphi_{k,b} = \varphi(k, \cdot, b) \). A reduced version of the shadowing lemma \[8, \text{Lem. 2.11}\] reads as follows.

2.1 Lemma: Let \( \Lambda \subset B \times M \) be a compact invariant uniformly hyperbolic set. Then there is a neighborhood \( U \) of \( \Lambda \) such that the following holds:

(i) For any \( \beta > 0 \) there is an \( \alpha > 0 \) such that every \( \alpha \)-pseudo-orbit \((b_k, x_k)_{k \in \mathbb{Z}}\) in \( U \) is \( \beta \)-shadowed by an orbit \( \{\Phi^k(b_0, y) : k \in \mathbb{Z}\} \).

(ii) There is \( \beta_0 > 0 \) such that \( 0 < \beta < \beta_0 \) implies that the shadowing orbit in (i) is uniquely determined by the pseudo-orbit.

(iii) If \( \Lambda \) is an isolated invariant set of \( \Phi \), then the shadowing orbit is in \( \Lambda \).

3 The main result

3.1 Preliminaries and assumptions

Consider a control-affine system

\[
\Sigma: \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t)f_i(x(t)), \quad u \in \mathcal{U} = L^\infty(\mathbb{R}, U),
\]

on a compact Riemannian manifold \( M \) with distance \( d(\cdot, \cdot) \). The vector fields \( f_0, f_1, \ldots, f_m \) are assumed to be of class \( C^1 \) and the control range \( U \subset \mathbb{R}^m \) is compact and convex. The set \( \mathcal{U} \) of admissible control functions is endowed with the weak*-topology of \( L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^* \). We write

\[
\Phi: \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u)),
\]

for the associated control flow and \( \varphi_{t,u} = \varphi(t, \cdot, u) \). A chain control set is a set \( Q \subset M \) with the following properties:

(i) For every \( x \in Q \) there exists \( u \in \mathcal{U} \) with \( \varphi(\mathbb{R}, x, u) \subset Q \).
(ii) For each two \(x, y \in Q\) and all \(\varepsilon, T > 0\) there are \(n \in \mathbb{N}\), controls \(u_0, \ldots, u_{n-1} \in \mathcal{U}\), states \(x_0 = x, x_1, \ldots, x_{n-1}, x_n = y\) and times \(t_0, \ldots, t_{n-1} \geq T\) such that

\[d(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon, \quad i = 0, 1, \ldots, n - 1.\]

(iii) \(Q\) is maximal with (i) and (ii) in the sense of set inclusion.

Throughout the paper, we fix a chain control set \(Q\) and write

\[Q = \{(u, x) \in \mathcal{U} \times M : \varphi(R, x, u) \subset Q\}\]

for its full-time lift, which is a chain recurrent component of the control flow \(\Phi\) on \(\mathcal{U} \times M\) (cf. [2, Thm. 4.1.4]). We further assume that \(Q\) is an isolated invariant set for \(\Phi\), i.e., there exists a neighborhood \(N \subset \mathcal{U} \times M\) of \(Q\) such that \(\Phi(R, u, x) \subset N\) implies \((u, x) \in Q\). This, e.g., is the case if there are only finitely many chain control sets on \(M\), because then the chain recurrent components are the elements of a finest Morse decomposition. We further assume that \(T_xM = E^+_{u,x} \oplus E^-_{u,x}\) for all \((u, x) \in Q\), with subspaces \(E^\pm_{u,x}\) satisfying

(H1) \(d\varphi_{t,u}(x)E^\pm_{u,x} = E^\pm_{\varphi(t,u,x)}\) for all \((u, x) \in Q\) and \(t \in \mathbb{R}\).

(H2) There exist constants \(0 < c \leq 1\) and \(\lambda > 0\) such that for all \((u, x) \in Q\),

\[|d\varphi_{t,u}(x)v| \leq c^{-1}e^{-\lambda t}|v|\]

for all \(t \geq 0, \, v \in E^-_{u,x}\),

and

\[|d\varphi_{t,u}(x)v| \geq ce^{\lambda t}|v|\]

for all \(t \geq 0, \, v \in E^+_{u,x}\).

From (H1) and (H2) it follows that \(E^\pm_{u,x}\) depend continuously on \((u, x)\) (cf. [3 Lem. 6.4]). We define the \(u\)-fiber of \(Q\) by

\[Q(u) := \{x \in Q : (u, x) \in Q\}.\]

On \(\mathcal{U}\) we fix a metric, compatible with the weak*-topology, of the form

\[d_\mathcal{U}(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1 + |\int_\mathbb{R} \langle u(t) - v(t), x_n(t) \rangle dt|}{1 + |\int_\mathbb{R} \langle u(t) - v(t), x_n(t) \rangle dt|},\]

where \(\{x_n : n \in \mathbb{N}\}\) is a dense and countable subset of \(L^1(\mathbb{R}, \mathbb{R}^m)\) and \(\langle \cdot, \cdot \rangle\) is a fixed inner product on \(\mathbb{R}^m\) (cf. [2 Lem. 4.2.1]).
3.2 Statement of results and proofs

We observe that the time-1-map \(\Phi_1 : U \times M \to U \times M\), \((u, x) \mapsto (\theta_1 u, \varphi(1, x, u))\), of the control flow is a skew-product map and \(Q\) is a uniformly hyperbolic set for \(\Phi_1\) in the sense of Section 2. Moreover, \(Q\) is isolated for \(\Phi_1\), which easily follows from our assumption that \(Q\) is an isolated invariant set of the control flow. Continuous dependence of the derivative \(d\varphi_{1,u}(x)\) on \((u, x)\) is proved in [6, Thm. 1.1].

3.1 Proposition: For any \(u, v \in U\) the fibers \(Q(u)\) and \(Q(v)\) are homeomorphic.

Proof: The proof is subdivided into three steps.

Step 1. We claim that for every \(\varepsilon > 0\) there is \(\delta > 0\) such that for all \(u, v \in U\),
\[
\|u - v\|_{\infty} < \delta \implies d_U(\theta_t u, \theta_t v) < \varepsilon \quad \text{for all } t \in \mathbb{R},
\]
where \(\| \cdot \|_{\infty}\) is the \(L^\infty\)-norm. To prove this, choose \(N = N(\varepsilon) \in \mathbb{N}\) with
\[
\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}.
\]
Then put
\[
c(\varepsilon) := \max_{1 \leq n \leq N} \|x_n\|_1, \quad \delta := \frac{\varepsilon}{2c(\varepsilon)},
\]
where \(\| \cdot \|_1\) is the \(L^1\)-norm. Then, for every \(t \in \mathbb{R}\), \(\|u - v\|_{\infty} < \delta\) implies
\[
d_U(\theta_t u, \theta_t v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \int_{\mathbb{R}} (u(t + s) - v(t + s), x_n(s))ds \right| \leq \sum_{n=1}^{N} \frac{1}{2^n} \left| \int_{\mathbb{R}} (u(t + s) - v(t + s), x_n(s))ds \right| + \varepsilon = \varepsilon.
\]

Step 2. Consider the time-1-map \(\Phi_1\) of the control flow with the uniformly hyperbolic set \(Q\). Let \(\beta > 0\) be given and choose \(\alpha = \alpha(\beta)\) according to the shadowing lemma [21]. Then choose \(\varepsilon = \varepsilon(\alpha)\) such that
\[
d_U(u, v) < \varepsilon \quad \Rightarrow \quad d(\varphi(1, x, u), \varphi(1, x, v)) < \alpha, \quad (2)
\]
whenever \( u, v \in \mathcal{U} \) and \( x \in Q \). This is possible by uniform continuity of \( \varphi(1, \cdot, \cdot) \) on the compact set \( Q \times \mathcal{U} \). We claim that for all sufficiently small \( \beta \),

\[
\sup_{t \in \mathbb{R}} d_{\mathcal{U}}(\theta_t u, \theta_t v) < \varepsilon \quad \Rightarrow \quad Q(u) \text{ and } Q(v) \text{ are homeomorphic.} \tag{3}
\]

If \( Q(u) \) and \( Q(v) \) are both empty, there is nothing to show. Otherwise, we may assume that \( Q(u) \neq \emptyset \). Then choose \( x \in Q(u) \) arbitrarily and consider the doubly infinite sequence \( x_n := \varphi(n, x, u), n \in \mathbb{Z}, \) which is completely contained in \( Q \) and (by 2) satisfies

\[
d(\varphi(1, x_n, \theta_n v), x_{n+1}) = d(\varphi(1, x_n, \theta_n v), \varphi(1, x_n, \theta_n u)) < \alpha
\]

for all \( n \in \mathbb{Z} \). Hence, \( (x_n, \theta_n v)_{n \in \mathbb{Z}} \) is an \( \alpha \)-pseudo-orbit for \( \Phi_1 \). By the shadowing lemma there exists \( y \in M \) with

\[
d(\varphi(n, y, v), \varphi(n, x, u)) < \beta \quad \text{for all } n \in \mathbb{Z}.
\]

Since \( Q \) is isolated, Lemma 2.1 iii) implies \( (v, y) \in Q \), i.e., \( y \in Q(v) \). We claim that the map

\[
h_{uv} : Q(u) \to Q(v), \quad x \mapsto y,
\]

defined in this way, is a homeomorphism. If \( \beta \) is small enough, the shadowing orbit is unique by Lemma 2.1 ii), and hence \( h_{uv} \) is uniquely defined. Since the \( \beta \)-shadowing relation between \( u \)-orbits and \( v \)-orbits is symmetric, one can equivalently define a map \( h_{vu} : Q(v) \to Q(u) \), which by uniqueness must be the inverse of \( h_{uv} \). The proof for the continuity of \( h_{uv} \) is standard and will be omitted. (Continuity will also follow trivially from Proposition 3.)

**Step 3.** From convexity of \( U \) it follows that for arbitrary \( u, v \in \mathcal{U} \) the curve

\[
w : [0,1] \to \mathcal{U}, \quad \tau \mapsto w_\tau, \quad w_\tau(t) := (1 - \tau) u(t) + \tau v(t),
\]

is well-defined. It is continuous w.r.t. the \( L^\infty \)-topology on \( \mathcal{U} \), since

\[
\|w_{\tau_1} - w_{\tau_2}\|_\infty = \esssup_{t \in \mathbb{R}} |(1 - \tau_1)u(t) + \tau_1 v(t) - (1 - \tau_2)u(t) - \tau_2 v(t)|
\]

\[
= \esssup_{t \in \mathbb{R}} |(\tau_2 - \tau_1)u(t) - (\tau_2 - \tau_1)v(t)|
\]

\[
= \esssup_{t \in \mathbb{R}} |\tau_2 - \tau_1| \cdot |u(t) - v(t)| \leq |\tau_2 - \tau_1| \cdot \text{diam } U.
\]

Hence, for each \( \tau \in [0,1] \) we can pick a (relatively) open subinterval \( I_\tau \subset [0,1] \) containing \( \tau \) such that \( \|w_s - w_r\|_\infty \) is smaller than a given constant for all \( s, r \in I_\tau \). By Step 1 this implies that \( \sup_{t \in \mathbb{R}} d_{\mathcal{U}}(\theta_t w_s, \theta_t w_r) < \varepsilon \) for all \( s, r \in I_\tau \) with a given \( \varepsilon > 0 \). Choose \( \varepsilon \) according to Step 2 so that \( Q(w_s) \) and \( Q(w_r) \) are homeomorphic for any two \( s, r \in I_\tau \). By compactness, finitely many such intervals \( I_{\tau_1}, \ldots, I_{\tau_n} \) are sufficient to cover \([0,1] \). We may assume that \( 0 = \inf I_{\tau_i} < \inf I_{\tau_2} < \cdots < \inf I_{\tau_1} < \sup I_{\tau_i} = 1 \). To show that \( Q(u) \) and \( Q(v) \) are homeomorphic, we put \( t_0 := 0, t_1 := 1 \) and pick \( t_i \in I_{\tau_i} \cap I_{\tau_{i+1}} \neq \emptyset \) for \( i = 1, \ldots, l - 1 \). Then there exist homeomorphisms \( h_{t_{i+1}} : Q(w_{t_i}) \to Q(w_{t_{i+1}}) \) for \( 0 \leq i \leq l - 1 \). The composition of these homeomorphisms gives a homeomorphism from \( Q(u) = Q(w_0) \) to \( Q(v) = Q(w_1) \). \( \square \)
3.2 Corollary: If \(Q(u)\) is a singleton for one \(u \in U\), then \(Q\) is the graph of a continuous map from \(U\) to \(Q\).

Proof: By the proposition, \(Q(u)\) is a singleton for every \(u \in U\), so \(Q(u) = \{x(u)\}\). Consider the map \(u \mapsto (u, x(u))\), \(U \to Q\). This is an invertible map between compact metric spaces with (obviously) continuous inverse. Hence, it is a homeomorphism. This implies that the function \(u \mapsto x(u)\), \(U \to Q\), is continuous and \(Q\) is its graph. \(\square\)

The next proposition shows that the fibers \(Q(u)\) are finite.

3.3 Proposition: If \(u\) is a constant control function, then \(Q(u)\) consists of finitely many equilibria. Hence, there exists \(n \in \mathbb{N}\) such that \(Q(u)\) has precisely \(n\) elements for every \(u \in U\).

Proof: Let \(u \in U\) be a constant control function. Observe that \(Q(u)\) is a uniformly hyperbolic set for the diffeomorphism \(g := \varphi_{1,u} : M \to M\). It is well-known that a diffeomorphism is expansive on a uniformly hyperbolic set, i.e., there is \(\varepsilon > 0\) such that \(d(g^k(x), g^k(y)) < \varepsilon\) for all \(k \in \mathbb{Z}\) and \(x, y \in Q(u)\) implies \(x = y\) (see [5, Cor. 6.4.10]). If \(x \in Q(u)\) and \(w := f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \neq 0\), then the Lyapunov exponent \(l(w) := \limsup_{t \to \infty} (1/t) \log |d\varphi_{t,u}(x)w|\) vanishes if the trajectory \(\varphi(t,x,u)\) is bounded away from equilibria. A zero Lyapunov exponent, however, contradicts the existence of the uniformly hyperbolic splitting on \(Q\). Since the right-hand side of the system is bounded on compact sets, \(l(w) < 0\) follows, implying \(w \in E^{-}_{x,u}\). Writing \(f := f_0 + \sum_{i=1}^{m} u_i f_i\), this yields \(|d\varphi_{t,u}(x)w| = |f(\varphi(t,x,u))| \leq c^{-1}e^{-\lambda t}\) for \(t \geq 0\). Because of the uniform hyperbolicity, there can be at most finitely many equilibria in the compact set \(Q(u)\), and hence \(\varphi(t,x,u) \to z_\pm\) for some equilibrium \(z_\pm\). The same argumentation for the backward flow yields \(\varphi(t,x,u) \to z_-\) for an equilibrium \(z_-\). Choose \(t_0 > 0\) large enough so that

\[
d(\varphi(t,x,u), z_+) < \frac{\varepsilon}{2} \quad \text{if} \quad |t| \geq \frac{t_0}{2}.
\]

Then choose \(\delta > 0\) small enough so that

\[
d(x,y) < \delta \quad \Rightarrow \quad d(\varphi(t,x,u), \varphi(t,y,u)) < \varepsilon \quad \text{for all} \quad |t| \leq t_0.
\]

Finally, let \(\tau \in (0, t_0/2)\) be chosen so that

\[
d(x, \varphi(\tau,x,u)) < \delta.
\]

We let \(y := \varphi(\tau,x,u)\) and claim that \(d(\varphi(t,x,u), \varphi(t,y,u)) < \varepsilon\) for all \(t \in \mathbb{R}\), implying \(x = y\). Indeed, by [4] and [5] we have

\[
d(\varphi(t,x,u), \varphi(t,y,u)) < \varepsilon \quad \text{for all} \quad |t| \leq t_0.
\]

Now assume that \(t \geq t_0\). Then [4] yields

\[
d(\varphi(t,x,u), \varphi(t,y,u)) \leq d(\varphi(t,x,u), z_+) + d(z_+, \varphi(t+\tau,x,u)) < \varepsilon.
\]
If \( t < -t_0 \), we obtain \( t + \tau < -t_0 + \tau < -t_0 + t_0/2 = -t_0/2 \), and hence \([1]\) gives
\[
d(\varphi(t, x, u), \varphi(t, y, u)) \leq d(\varphi(t, x, u), z_-) + d(z_-, \varphi(t + \tau, x, u)) < \varepsilon.
\]
In particular, \( d(g^k(x), g^k(y)) < \varepsilon \) for all \( k \in \mathbb{Z} \), and hence \( x = y = z_+ = z_- \).

Consequently, \( Q(u) \) consists of finitely many equilibria. \( \square \)

The next theorem is our main result.

**3.4 Theorem:** Consider the control-affine system \( \Sigma \) with the uniformly hyperbolic chain control set \( Q \) with isolated lift \( \bar{Q} \). Assume that \( \text{int} \, U \neq \emptyset \) and let \( u_0 \) be a constant control function with value \( \text{in} \, \text{int} \, U \). Additionally suppose that the following hypotheses are satisfied:

(i) The vector fields \( f_0, f_1, \ldots, f_m \) are of class \( C^\infty \) and the Lie algebra generated by them has full rank at each point of \( Q \).

(ii) For each \( x \in Q(u_0) \) and each \( \rho \in (0, 1] \) it holds that \( x \in \text{int} \, \mathcal{O}^+_\rho(x) \), where
\[
\mathcal{O}^+_\rho(x) = \{ \varphi(t, x, u) : t \geq 0, \ u \in U^\rho \}
\]
with
\[
U^\rho = \{ u \in U : u(t) \in u_0 + \rho(U - u_0) \text{ a.e.} \}.
\]

Then \( Q \) is a graph of a continuous function \( \mathcal{U} \rightarrow Q \).

**3.5 Remark:** Before proving the theorem, we note that assumption (ii) is in particular satisfied if the system is locally controllable at \( (u_0, x) \) for each \( x \in Q(u_0) \) (using arbitrarily small control ranges around \( u_0 \)). A sufficient condition for this to hold, which is independent of \( \rho \), is the controllability of the linearization around \( (u_0, x) \).

**Proof:** Without loss of generality, we assume that \( u_0(t) \equiv 0 \). Let \( Q(u_0) = \{ x_1, \ldots, x_n \} \). We consider for each \( \rho \in (0, 1] \) the control-affine system
\[
\Sigma^\rho : \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^m u_i(t)f_i(x(t)), \quad u \in U^\rho.
\]

From the assumptions (i) and (ii) it follows by \([2]\) Cor. 4.1.7] that each \( x_i \) is contained in the interior of a control set \( D^\rho_i \) of \( \Sigma^\rho \). Each \( D^\rho_i \) is contained in a unique chain control set \( E^\rho_i \) of \( \Sigma^\rho \) (cf. \([2]\) Cor. 4.3.12]). By \([2]\) Cor. 3.1.14] the chain control sets depend upper semicontinuously on \( \rho \), hence \( E^\rho_i \subset Q = E^1_i \) for every \( 1 \leq i \leq n \) and \( \rho \in (0, 1] \). This implies that each \( E^\rho_i \) is uniformly hyperbolic. By \([1]\) Thm. 3] it follows that \( E^\rho_i = \text{cl} \, D^\rho_i \). If \( C_i \) denotes the chain recurrent component of the uncontrolled system \( \dot{x} = f_0(x) \) which contains the equilibrium \( x_i \), then \( C_i \subset E^\rho_i \) for each \( \rho \), because otherwise \( E^\rho_i \cup C_i \) would satisfy the first two properties of chain control sets, contradicting maximality of \( E^\rho_i \).

Since each chain recurrent component is connected and \( C_i \subset Q(u_0) \), we have \( C_i = \{ x_i \} \). By \([2]\) Cor. 3.4.10], the chain control set \( E^\rho_i \) shrinks to \( \{ x_i \} \) as \( \rho \searrow 0 \).
Hence, for small \( \rho \), the sets \( E_i^\rho \) are pairwisely disjoint. Since \( E_1^1 = Q \) for each \( i \), at some point the chain control sets have to merge as \( \rho \) increases. Since, by \cite{2} Thm. 3.1.12, the control sets \( D_i^\rho \) depend lower semicontinuously on \( \rho \), this is a contradiction if \( n > 1 \). It follows that \( n = 1 \) and Corollary \cite{3} yields the assertion. \( \Box \)

3.6 Remark: Of course, in many cases it will be easier to check directly that \( Q(u) \) is a single equilibrium for some constant control function \( u \) than verifying the conditions of the preceding theorem. We also note that the fact that \( Q \) is a graph over \( \mathcal{U} \) implies the existence of a topological conjugacy between the shift flow \( \theta \) on \( \mathcal{U} \) and the restriction of the control flow to \( Q \) (cf. \cite{3}).

4 Application to invariance entropy

The invariance entropy of a controlled invariant subset \( Q \) of \( M \) measures the complexity of the control task of keeping the state inside \( Q \). In general, it is defined as follows. A pair \((K, Q)\) of subsets of \( M \) is called admissible if \( K \) is compact and for every \( x \in K \) there is \( u \in \mathcal{U} \) with \( \varphi(\mathbb{R}_+, x, u) \subset Q \). In particular, if \( K = Q \), this means that \( Q \) is a compact and controlled invariant set. For \( \tau > 0 \), a set \( S \subset \mathcal{U} \) is \((\tau, K, Q)\)-spanning if for every \( x \in K \) there is \( u \in S \) with \( \varphi([0, \tau], x, u) \subset Q \). Then \( r_{inv}(\tau, K, Q) \) denotes the number of elements in a minimal such set and we put \( r_{inv}(\tau, K, Q) = \infty \) if no finite \((\tau, K, Q)\)-spanning set exists. The invariance entropy of \((K, Q)\) is

\[
h_{inv}(K, Q) := \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{inv}(\tau, K, Q),
\]

where \( \log \) is the natural logarithm. From \cite{3} Thm. 5.4 we can conclude the following result on the invariance entropy of admissible pairs \((K, Q)\), where \( Q \) is a uniformly hyperbolic chain control set. The difference to \cite{3} Thm. 5.4 is that we do not have to assume explicitly anymore that \( Q \) is a graph over \( \mathcal{U} \).

4.1 Theorem: Consider the control-affine system \( \Sigma \) with the uniformly hyperbolic chain control set \( Q \) with isolated lift \( \mathcal{Q} \). Let the assumptions (i) and (ii) of Theorem \cite{3} be satisfied, or alternatively, assume that \( Q(u) \) is a singleton for some \( u \in \mathcal{U} \). Then \( Q \) is the closure of a control set \( D \) and for every compact set \( K \subset D \) of positive volume the pair \((K, Q)\) is admissible and its invariance entropy satisfies

\[
h_{inv}(K, Q) = \inf_{(u, x) \in Q} \limsup_{\tau \to \infty} \frac{1}{\tau} \log \left| \det(d\varphi_{\tau, u}) \right|_{E^+_u, x} : E^+_u, x \to E^+_u \mathcal{Q}(u, x).
\]

4.2 Remark: The paper \cite{4} provides a rich class of examples for uniformly hyperbolic chain control sets that arise on the flag manifolds of a semisimple Lie group. The control-affine system in this case is induced by a right-invariant system on the group.
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