Statistics of extremes in eigenvalue-counting staircases

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**Unitary $\beta-$ ensembles and interacting fermions:**

Consider a unitary $N \times N$ matrix $U$ and denote the corresponding unimodular eigenvalues as $z_j = e^{i\theta_j}$, $j = 1, \ldots, N$, with phases $-\pi < \theta_i \leq \pi$. For any given $\beta > 0$ one can construct the so-called Circular $\beta$-Ensemble CUE$_\beta(N)$ in such a way that the expectation of a function $F \equiv F(\theta_1, \ldots, \theta_n)$ is given by

$$E(F) = c_N \prod_{j=1}^{N} \int_{-\pi}^{\pi} d\theta_i \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^{\beta} F$$

For $\beta = 2$ such matrices can be thought of as drawn uniformly according to the corresponding Haar’s measure on $U(N)$, whereas for a generic $\beta > 0$ the explicit construction is more involved, see Killip-Nenciu ’04. Such eigenvalues essentially behave as classical particles with 1-d logarithmic repulsion at inverse temperature $\beta > 0$. On the other hand, the r.h.s can be interpreted as the **quantum expectation** value of $F$ in the ground state of $N$ spinless fermions, of coordinates $\theta_i$ on the unit circle, described by the **Sutherland** Hamiltonian:

$$H = -\sum_i \frac{\partial^2}{\partial \theta_i^2} + \sum_{i<j} \frac{\beta(\beta-2)}{8 \sin^2(\frac{\theta_i - \theta_j}{2})}$$

Thus, for $\beta = 2$ the eigenvalues behave as **non-interacting** fermions, while for $\beta \neq 2$ the fermions interact, via an inverse square distance pairwise potential.
Define the number $\mathcal{N}_{\theta_A}(\theta)$ of eigenvalues $e^{i\theta_j}$ of a random unitary $N \times N$ matrix, drawn from CUE$_\beta(N)$, in the interval $\theta_j \in [\theta_A, \theta]$. As a function of $\theta$ this is a staircase with unit jumps upwards at random positions $\theta_j \in [\theta_A, \theta]$. The mean profile is $\mathbb{E}(\mathcal{N}_{\theta_A}(\theta)) = \frac{N(\theta - \theta_A)}{2\pi}$.

Constructing an instance of $\delta \mathcal{N}_0(\theta)$ for $\theta \in [0, \pi]$ for $\beta = 2$ and $N = 20$. Left: eigenvalues $\lambda = e^{i\theta_i}$. Right: counting staircase (top), with mean subtracted (bottom).
Staircase-deviation process:

In a given random matrix realization/sample one can define the deviation to the mean, 
\[ \delta \mathcal{N}_{\theta_A}(\theta) = \mathcal{N}_{\theta_A}(\theta) - \mathbb{E}(\mathcal{N}_{\theta_A}(\theta)), \]
and study it as a random process as a function of variable \( \theta \) for a fixed \( \theta_A \).

A single realization of \( \delta \mathcal{N}_{-\pi}(\theta) \) for the full circle \( \theta \in [-\pi, \pi] \) for \( \beta = 2 \) and \( N = 200 \).
Staircase-deviation process, non-local properties:

Outstanding: **Kolmogorov-Smirnov**-type statistics

\[
\max_{\theta \in [\theta_A, \theta_B]} |\delta N_{\theta A}(\theta) := N_{\theta A}(\theta) - \mathbb{E}(N_{\theta A}(\theta))| 
\]

Some recent results for \( \beta = 2 \) in: Clayes et al '19.

We are able to shed some light on ‘half’ of the problem by calculating **cumulants** of the distribution of the **one-sided maximum value** for the process \( \delta N_{\theta A}(\theta) \), i.e

\[
\delta N_m := \max_{\theta \in [\theta_A, \theta_B]} [\delta N_{\theta A}(\theta)]
\]

for any \( \beta > 0 \) and \( N \gg 1 \) at fixed \( \ell = |\theta_A - \theta_B| \).

**Note:** If we change above \( \max \mapsto \min \), the distribution remains the same, but two extremal values are expected to be highly **correlated**, cf. Cao-Le Doussal ’16

We also can characterize the **location** of the maximum:

\[
\theta_m := \text{argmax}_{\theta \in [\theta_A, \theta_B]} [\delta N_{\theta A}(\theta)]
\]
Relation to ‘log-correlated’ processes:

Naively, the process $\delta N_{\theta_A}(\theta)$ is given by the difference

$$\delta N_{\theta_A}(\theta) = \frac{1}{\pi} \text{Im} \log \xi_N(\theta) - \frac{1}{\pi} \text{Im} \log \xi_N(\theta_A)$$

where $\xi_N(\theta)$ is the characteristic polynomial defined as $\xi_N(\theta) = \det(1 - e^{-i\theta} U)$.

As has been shown in Hughes-Keating-O’Connell ’01 for $\beta = 2$ (see Chhaibi-Madaule-Najnudel ’18 for general $\beta > 0$) the joint probability density of $\text{Im} \log \xi_N(\theta)$ at any fixed distinct points $\theta_1 \neq \theta_2 \neq \ldots \neq \theta_k$ converges (in a suitable sense) as $N \to \infty$ to that of a mean-zero Gaussian process $W_\beta(\theta)$ with the covariance structure

$$\mathbb{E}(W_\beta(\theta_1)W_\beta(\theta_2)) = -\frac{1}{2\beta} \log \left[ 4 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) \right]$$

which is a particular instance of the 1D log-correlated Gaussian field. A large but finite $N$ provides a natural small-scale regularization, augmenting $W_\beta(\theta)$ with the finite variance: $\mathbb{E}(W_{N,\beta}(\theta)^2) = \beta^{-1} \log N + O(1)$.

Note that $\delta N_{\theta_A}(\theta = \theta_A) = 0$ in any realization, hence the relevant object is the pinned log-process closely related to fBm0. We shall however see that naively replacing $\delta N_{\theta_A}(\theta) \to \frac{1}{\pi} \left[ W_{N,\beta}(\theta) - W_{N,\beta}(\theta_A) \right]$ (the procedure closely related to the bosonization approach to fermionic problems) is not sufficient for characterizing the maximum of the process: it misses local ‘fermionic’ contributions.
Summary of the main results:

We predict that for any interval of the fixed length $\ell = \theta_B - \theta_A$ the mean value of the maximum deviation $\mathcal{N}_m = \max_{\theta \in [\theta_A, \theta_B]} [\mathcal{N}_\theta(\theta)]$ should exhibit, for $N \to \infty$, the universal behavior of the log-correlated processes

$$2\pi \sqrt{\frac{\beta}{2}} \mathbb{E}(\mathcal{N}_m) \simeq 2 \log N - \frac{3}{2} \log \log N + c_{\ell}(\beta)$$

where $c_{\ell}(\beta) = \mathcal{O}(1)$ is an unknown $\ell$-dependent constant.

The variance for the maximum $\mathcal{N}_m$ exhibits to the leading order the extensive universal logarithmic growth typical for pinned log-correlated field, on top of which we can evaluate the corrections of the order of unity:

$$\mathbb{E}^c(\mathcal{N}_m^2) \simeq \frac{2}{\beta (2\pi)^2} (2 \log N + \tilde{C}_2(\beta) + C_2(\ell))$$

Finally, the higher cumulants converge to a finite limit as $N \to \infty$:

$$\mathbb{E}^c(\mathcal{N}_m^k) \simeq \frac{2^{k/2}}{\beta^{k/2}(2\pi)^k} (\tilde{C}_k(\beta) + C_k(\ell)),$$

The constants $\tilde{C}_k(\beta)$ depend on $\beta$ but not on $\ell$, and reflect local fermion number statistics, whereas $C_k(\ell)$ depend on the length $\ell$, reflect log-correlated statistics and are known only for the full circle $\ell = 2\pi$ and mesoscopic $\Delta \ll \ell \ll 2\pi$. 
Local ‘fermionic’ contribution to max-cumulants:

All odd cumulants $\tilde{C}_{2p+1}$ vanish. All even cumulants for any $\beta > 0$ can be expressed in terms of functions $\psi^{(k)}(x) = \frac{d^{k+1}}{dx^{k+1}} \log \Gamma(x)$ as convergent infinite “dual” series. For the lowest cumulant we obtain:

$$\tilde{C}_{2}^{(\beta)} = 2\gamma_E + 2 \sum_{k=0}^{\infty} \left[ \frac{\beta}{2} \psi^{(1)} \left( 1 + \frac{\beta k}{2} \right) - \frac{1}{1+k} \right]$$

where $\gamma_E = \lim_{n \to \infty} \left( -\ln n + \sum_{k=1}^{n} \frac{1}{k} \right)$ is the Euler-Mascheroni constant.

For higher even cumulants $\tilde{C}_{2p}^{(\beta)}$ with $p \geq 2$ we find

$$\tilde{C}_{2p}^{(\beta)} = (-2)^{1-p} \beta^p \sum_{k=0}^{\infty} \psi^{(2p-1)} \left( 1 + \frac{\beta k}{2} \right) = (-2)^{p+1} \frac{1}{\beta^p} \sum_{m=1}^{\infty} \psi^{(2p-1)} \left( \frac{2m}{\beta} \right)$$

For rational $\beta = 2s/r$, with $s, r$ mutually prime and $k \geq 2$ we have an alternative representation: $\tilde{C}_{k}^{(\beta)} = \frac{d^k}{dt^k} |_{t=0} \log (A_{\beta}(t) A_{\beta}(-t))$, with

$$A_{\beta}(t) = r^{-t^2/2} \prod_{\nu=0}^{r-1} \prod_{p=0}^{s-1} \frac{G \left( 1 - \frac{\nu + it \sqrt{2/\beta}}{r} \right)}{G \left( 1 - \frac{\nu}{s} + \frac{\nu + it \sqrt{2/\beta}}{r} \right)} \prod_{\nu=0}^{r-1} \prod_{p=0}^{s-1} \frac{G \left( 1 - \frac{\nu + it \sqrt{2/\beta}}{r} \right)}{G \left( 1 - \frac{\nu}{s} + \frac{\nu + it \sqrt{2/\beta}}{r} \right)}$$

These expressions are intimately related to fermion number statistics.
Global ‘log-correlated’ contributions to max-cumulants:

By contrast the constants $C_k(\ell)$ are $\beta$-independent and can be directly related to the maximum of Fractional Brownian Motion with vanishing Hurst exponent (fBm0) (‘pinned’ at one end for the interval or at both ends in ‘bridge‘ version for the full circle) studied in YVF-Le Doussal ’16 and Cao-YVF-Le Doussal ’18. We find:

(i) maximum over the full circle $\ell = 2\pi$. In that case we find:

$$C_{k,\geq2}(2\pi) = (-1)^k \frac{d^k}{dt^k}|_{t=0} \log \left[ \frac{\Gamma(1+t)^2G(2-2t)}{G(2-t)^3G(2+t)} \right]$$

(ii) maximum over a mesoscopic interval $\Delta \ll \ell \ll 1$, where we obtain

$$C_k(\ell) \approx 2\log \ell \delta_{k,2} + (-1)^k \frac{d^k}{dt^k}|_{t=0} \left[ \frac{2\Gamma(1+t)^2G(2-2t)}{G(2+t)^2G(2-t)G(4-t)} \right]$$

Note that $\ell \rightarrow 0$ limit is expected to provide the $L \gg 1$ asymptotic for statistics of the maximum of $\delta \mathcal{N}_{\theta_A}(\theta)$ in intervals of the order $L\Delta$, comparable with the mean eigenvalue spacing. In particular, our mean $\mathbb{E}(\delta \mathcal{N}_m)$ agrees with one found in Holcomb-Paquette’18 for the large-$L$ asymptotics of $\text{Sine}_\beta$ process.
Distribution of the absolute maximum via Statistical Mechanics approach:

Given a random sequence \( \{V_i, i = 1, \ldots, M\} \) we are interested in finding the distribution of \( V_{(m)} = \max(V_1, \ldots, V_M) \) that is

\[
P(v) = \text{Prob}(V_{(m)} < v) = \text{Prob}(V_i < v, \forall i) = \mathbb{E} \left\{ \prod_{i=1}^{M} \chi(v - V_i) \right\}
\]

where \( \chi(v) = \begin{cases} 1, & v > 0 \\ 0, & v < 0 \end{cases} \) is the indicator function.

Next we use:

\[
\lim_{b \to \infty} \exp \left[ -e^{-b(v-V_i)} \right] = \begin{cases} 1, & v > V_i \\ 0, & v < V_i \end{cases} \equiv \chi(v - V_i)
\]

which immediately shows that:

\[
P(v) = \text{Prob}(V_{(m)} < v) = \lim_{b \to \infty} \mathbb{E} \left\{ \exp \left[ -e^{-bv} Z(b) \right] \right\}
\]

where \( Z(b) = \sum_{i=1}^{M} e^{bV_i} \) is a kind of Partition Function associated with the problem, with \( b = 1/T \) playing the role of inverse temperature.

For a random process \( V(\theta), \theta \in I \) similar method works with \( Z(b) = \int_I e^{bV(\theta)} d\theta \).
Sketch of the Calculation I:

**Method**: introduce the following “partition sum”:

\[ Z(b) = \frac{N}{2\pi} \int_{\theta_A}^{\theta_B} d\phi \ e^{2\pi b \sqrt{\beta/2} \delta N_{\theta A}(\phi)}, \]

thus mapping the search of the maximum to a **statistical mechanics** problem, with the “**inverse temperature**” equal to \(-2\pi b \sqrt{\beta/2}\). The **maximum** is retrieved from the “**free energy**” \( F \) for \( b \to +\infty \) as

\[ \delta N_m = \lim_{b \to +\infty} F, \quad F = \frac{1}{2\pi b \sqrt{\beta/2}} \log Z(b) \]

To study the statistics of the associated free energy we start with considering the integer moments of \( Z(b) \). Using the representation of the counting function given by

\[ N_{\theta A}(\theta) = \sum_{j=1}^{N} (\chi(\theta - \theta_j) - \chi(\theta_A - \theta_j)) , \quad \chi(u) = \begin{cases} 1, & u > 0 \\ 0, & u < 0 \end{cases} \]

we get

\[ \mathbb{E}[Z^n(b)] = \left(\frac{N}{2\pi}\right)^n \int_{\theta_A}^{\theta_B} e^{-b \sqrt{\beta/2} \sum_{a=1}^{n} N(\phi_a - \theta_A)} \mathbb{E}[\prod_{j=1}^{N} g(\theta_j)] \prod_{a=1}^{n} d\phi_a \]

where we defined

\[ \log g(\theta) = 2\pi b \sqrt{\beta/2} \sum_{a=1}^{n} (\chi(\phi_a - \theta) - \chi(\theta_A - \theta)) \]
Sketch of the Calculation II:

For $\beta = 2$

$$
\mathbb{E}[\prod_{j=1}^{N} g(\theta_j)] = \det_{1 \leq j, k \leq N} [g_j - g_k] \text{ - Toeplitz determinant}
$$

where $g_p = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-ip\theta} g(\theta)$ is the associated symbol, and $g(\theta)$ has $n$ jump singularities. The corresponding asymptotics as $N \to \infty$ is given by the famous Fisher-Hartwig formula proved rigorously by Deift-Its-Krasovsky '09 -'11. For a general rational $\beta$ the extension of FH formula has been conjectured by Forrester & Frenkel '04. Applying their formula to our case gives for $N \to +\infty$ and $nb^2 < 1$:

$$
\mathbb{E}[Z^n(b)] \sim (\frac{N}{2\pi})^n N^{b^2(n+n^2)} |A_{\beta}(b)|^{2n} |A_{\beta}(bn)|^2 I_C(n, b^2)
$$

where $I_C(n, b^2)$ is the so-called 'Coulomb integral' given by

$$
I_C := \int_{\theta_A}^{\theta_B} \prod_{1 \leq a < c \leq n} \left[ 1 - e^{i(\phi_a - \phi_c)} \right]^{-2b^2} \prod_{1 \leq a \leq n} \left[ 1 - e^{i(\phi_a - \theta_A)} \right]^{2nb^2} \prod_{a=1}^{n} d\phi_a
$$

Note: Had we used instead in our calculation an approximation replacing the difference $\delta N_{\theta_A}(\theta)$ in the large $-N$ limit with the logarithmically correlated Gaussian process $W_{\beta}(\theta)$ we would reproduce the Coulomb integral factor but completely missed the factors $A_{\beta}(b)$. Hence, this product encapsulates the residual non-Gaussianity of the process due to microscopic 'descrete' fermionic nature.
Sketch of the Calculation III:

Further progress is possible in the two cases (full circle case and mesoscopic interval case) when the Coulomb integrals $I_C(n, b^2)$ can be explicitly calculated by reducing them to Selberg integrals, for integer $n$ and $nb^2 < 1$. The same integrals appeared in the problem of maximum in logarithmically correlated fBm0 (Hurst index $H = 0$) Cao-YVF-Le Doussal’18.

Combining those results with analytical continuation of factors $|A_\beta(b)|^{2n}|A_\beta(bn)|^2$ we find that appropriately continued partition function moments depend on $b$ in the whole high-temperature phase $b < 1$ only via the combination $Q = b + \frac{1}{b}$, hence satisfy the conditions of the Freezing-Duality Conjecture (FDC).

This allows to perform $b \to \infty$ limit arriving at a generation function for cumulants for probability density of the maximum value, e.g. for the full circle:

$$
\mathbb{E}(e^{-2\pi \sqrt{\frac{\beta}{2}} \delta\mathcal{N}_m t}) \simeq N^{-2t^2 + t^2} e^{(\frac{3}{2} \ln \ln N + c)t} A_\beta(t) A_\beta(-t) \frac{\Gamma(1+t)^2 G(2-2t)}{G(2-t)^3 G(2+t)}
$$

Finally, addressing the question of the location of the maximum $\theta_m \in [\theta_A, \theta_B]$ of $\delta\mathcal{N}_{\theta_A}(\theta)$, let us define $y_m = (\theta_m - \theta_A)/\ell$. For the mesoscopic interval, we predict the PDF of $y_m$ to be symmetric around $\frac{1}{2}$, with $\mathbb{E}(y_m^2) = \frac{17}{50}$ and $\mathbb{E}(y_m^4) = \frac{311}{1470}$, thus deviating from the uniform distribution.