The intermediate spectral form factor and the ‘t Hooft limit - a novel type of random matrix universality?

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Abstract

We extend the calculation of the intermediate spectral form factor (SFF) in \cite{1} to finite $N$. This is the SFF of the matrix model description of $U(pN^q)$ Chern-Simons theory on a three-dimensional sphere, which is dual to a topological string theory characterized by string coupling $g_s$. This matrix model displays level statistics characteristic of systems intermediate between chaotic and integrable, depending on the value of $g_s$. We check explicitly that taking $N \to \infty$ whilst keeping the string coupling $g_s$ fixed reduces the connected SFF to a pure linear ramp, thereby confirming the main result from \cite{1} for the intermediate ensemble. We then consider the ‘t Hooft limit, where $N \to \infty$ and $g_s \to 0$ such that $t = N g_s$ remains finite. In this limit, the SFF turns into a remarkable sequence of polynomials which, as far as the authors are aware, have not appeared in the literature thus far. The two-level statistics for the intermediate ensemble in the ‘t Hooft limit found here may be a representative of a novel random matrix universality.

1 Introduction

Random matrix theory provides phenomenological descriptions of a wide variety of systems appearing in physics and elsewhere, starting with nuclear physics \cite{2}, and later finding applications in quantum chaos \cite{3}, disordered electronic \cite{4} and mesoscopic \cite{5} systems, chromodynamics \cite{6}, models of 2d quantum gravity and string theory \cite{7}, economics \cite{8}, information theory \cite{9}, and number theory \cite{10}. One of the crucial concepts in the theory of random matrices is the Wigner-Dyson universality exhibited by the eigenvalue correlation properties of a wide variety of ensembles. For ensembles defined by the probability density with the weight function $w(x) = \exp(-V(x))$ where $V(H) \sim \text{tr}(H^2) + \ldots$
for Hermitian $N \times N$ matrix $H$, the density of eigenvalues $x$ has a famous, “universal” semi-circle law $\rho(x) = \sqrt{2N - x^2}/\pi$. This does not depend on the precise form of $V(x)$ for a broad range of potentials and the matrix size $N$. Rescaling the eigenvalues such that $\tilde{\rho}(u) = 1$ near the origin, typically called unfolding, leads to the two-level eigenvalue kernel function $G(u, v) = \sin[\pi(u - v)]/\pi(u - v)$. As a consequence, this leads to the “universal” form of the Spectral Form Factor (SFF), a Fourier transform of the density-density correlation function, $K(\xi) = 1 - \int G(x, 0)^2 \exp(ix\xi)dx$ resulting for $\xi > 0$ in the linear ramp until the Heisenberg “time” $\xi = \tau_H = 2\pi$ followed by a plateau for $\xi > \tau_H$. Integrable systems, on the other hand, display uncorrelated Poisson statistics leading to a constant SFF for all times. For systems with statistics intermediate between Poissonian and Wigner-Dyson, the SFF has mixed features. Typically, it is of a dip-ramp-plateau shape, where, for small $\xi$, the SFF dips until it reaches a minimum at Thouless time, after which it transitions into a linear ramp which saturates at a plateau at Heisenberg time. These features make the SFF a convenient tool for characterizing level statistics which has found extensive use in the RMT and quantum chaos community. We also mention here that the SFF has recently became popular in the string theory community, predominantly in the context of the AdS/CFT correspondence, as well as the SYK model and JT-gravity [11], [12], [13], [14], [15], [16]. We however not aware of studies related to intermediate matrix models in this context.

A class of matrix models with intermediate level statistics interpolating between Poisson and Wigner-Dyson behaviors has been proposed in [17], [18]. It was observed that Wigner-Dyson universality is lost for sufficiently shallow confining potentials, which asymptotically behave as $V(H) \sim g^{-1}_s \log^2 H$ for $|H| \gg 1$. Such potentials are associated with indeterminate moment problems, which is to say that the weight function $w(x)$ is not uniquely determined by its moments $m_j = \int x^j w(x)dx$ [19], [20]. A typical signature of this intermediate class of statistics is a lack of a simple translationally-invariant kernel of $\sin x/x$-form as before, which now becomes dependent on some parameter $q$. This opens up the possibility of non-unique (and more complicated) scaling limits, [21], [22]. This, in turn, has been associated with the spontaneous breaking of the $U(N)$-invariance of the ensemble [23].

The intermediate statistics exhibited by this class of matrix models are characteristic of ergodic-to-nonergodic transitions, such as disorder-induced (Anderson) localization [24], or pseudo-integrable billiards [25]. The connection of the intermediate matrix model with intermediate statistics can be seen to arise as follows. It was shown that taking $N \to \infty$ and then $q = e^{-g_s} \to 1$ leads to a kernel that is of the form $G(\theta, \phi) = g_s \frac{\sin[\pi(\theta - \phi)]}{\sinh[\pi(\theta - \phi)/2]}$ [17], [19]. This was found to be precisely the kernel for banded random matrix ensembles [18], which (phenomenologically) describe disordered conductors at the mobility edge, including eigenvector multifractality [26]. Another ensemble which shares the kernel is the Moshe-Neuberger-Shapiro (MNS) ensemble [27], where the unitary invariance of the ensemble is explicitly broken by a potential term involving a fixed matrix, typically chosen to be diagonal. The relation of the $(U(N)$-invariant) intermediate ensemble to the (non-$U(N)$-invariant) banded and MNS-ensembles has been argued to arise due to a spontaneous breaking of $U(N)$-invariance [28]. The intermediate RME reproduces both the nearest neighbor spacing and the level number variance at the same value of the parameter $q$ [28]. From the level number variance, one can determine certain properties of the eigenvector statistics as well, in particular their multifractal dimension(s).
further attesting to the suitability of the intermediate matrix model as a phenomenological
description of disordered conductors. The SFF was previously calculated using the limiting sinh-
kernel in [31]. The intermediate matrix model appears in unitary guise as a one-parameter dependent
generalization of the circular unitary ensemble (CUE), see e.g. [32] for a map between the unitary and
Hermitian versions. Denoting the parameter as \( q = e^{-\gamma} \), the unitary intermediate ensemble reduces
to the CUE for \( q \to 0 \) and produces Poissonian statistics for \( q \to 1 \), with intermediate statistics for
\( 0 < q < 1 \). This paper focuses on the unitary version of the intermediate matrix model.

The intermediate ensemble was later found by Mariño as a matrix model representation of type A
topological open string theory on the cotangent space of \( S^3 \) [33], which reduces to \( U(N) \) Chern-Simons
theory on \( S^3 \). The orthogonal polynomials associated to this ensemble are the Stieltjes-Wigert or
Rogers-Szegö polynomials in Hermitian and unitary description, respectively [20], [34]. Witten fa-
nomously showed that Wilson line expectation values in \( SU(2) \) Chern-Simons theory equal topological
(Jones) invariants of knots and links [35], which was later generalized to \( U(N) \) for general \( N \). In-
deed, the SFF \( \langle |\text{tr}\, U^n|^2 \rangle \) is (proportional to) the HOMFLY invariant of a \((2n,2)\)-torus link with one
Wilson line in the fundamental and the other in the antifundamental representation. As mentioned
above, the intermediate statistics are exhibited by systems undergoing ergodic-to-nonergodic transi-
tions. These transitions lack a local order parameter, a situation that is reminiscent from topological
phase transitions, in particular in the context of condensed matter physics. The absence of a local
order parameter combined with the topological nature of the matrix model that phenomenologically
describes ergodic-to-nonergodic transitions suggests that such transitions are topological in nature [1].

In the string theory literature on the intermediate matrix model, the ’t Hooft limit has been considered,
where one takes \( N \to \infty \) and simultaneously \( g_s \to 0 \) such that the ’t Hooft parameter \( t = N g_s \) remains
finite. This idea goes back to pioneering work by ’t Hooft in the context of \( U(N) \) gauge theories at
large \( N \). In the type A topological open string theory on \( T^*S^3 \) described by the intermediat matrix
model, certain large \( N \) dualities appear. In particular, it has been argued that the topological A-type
open string theory on \( T^*S^3 \) undergoes a conifold transition to a closed type A topological string theory
on the resolved conifold [36]. The magnitude of the B-field on the \( S^2 \) blowup of the conifold is given
by \( t = N g_s \), the ’t Hooft parameter. The mirror dual of the conifold geometry can be seen to arise
from the resolvent of the matrix model in the large \( N \) limit [37] [38], see also [39] [40]. These dualities
and related results have important applications in enumerative geometry and intersection theory.

As far as the authors are aware, the ’t Hooft limit has heretofore not been explicitly considered for
the intermediate ensemble in the RMT literature. In [41], a closely related limit was considered for
a Hermitian version of the \( q \)-deformed ensemble considered here, with the weak disorder (GUE) limit
corresponds to \( q = e^{-\gamma} \to 1 \) and \( N \to \infty \) such that \( \gamma N \to 0 \), while the strong disorder limit involves
\( \gamma N = \text{constant} \). In the latter limit, which is essentially the ’t Hooft limit, an approximate expression for
the parametric density correlation function was found in [41]. Further, a similar limit was considered
for another, closely related, \( q \)-deformed circular unitary ensemble in [32], see also [19]. It was found
that deviations from the CUE level density only persist in the infinite \( N \) limit if one simultaneously
scales \( q \) such that \( (1-q)N \) remains finite, which is essentially the ’t Hooft limit.
In [1], we calculated the spectral form factor (SFF) for infinite order \( N \to \infty \) invariant unitary matrix models satisfying the assumptions of Szegö’s theorem. In this limit, we found that for all such ensembles the SFF is of a surprisingly simple form, where the connected SFF always consists of an exact linear ramp and plateau, and the disconnected part constitutes a dip which consists of a squared simple power sum polynomial with of variables that can be read off from the weight function. In the present work we extend the calculation of the SFF for the intermediate ensemble to finite \( N \) and explore first how our previous results are recovered in the \( N \to \infty \). The present calculation proceeds from the expansion of the SFF in terms of Toeplitz minors, which are proportional to the product of two hook-shaped Schur polynomials of different specialization. The simplification that occurs for \( N \to \infty \) is not present here, which precludes us from generalizing these results to other matrix models as we did for infinite \( N \).

We then proceed to take the 't Hooft limit, where we find that the SFF turns into a remarkable sequence of polynomials, which are very close (or indeed equal) to straight lines. The approximate slope of the SFF lies between zero and one and is controlled by the 't Hooft parameter \( t = Ng \). Writing \( y = e^t \), we find that the SFF is symmetric around \( y = 1/2 \), with the SFF’s for \( y \) and \( 1 - y \) adding up to a linear ramp of unit slope. As far as the authors are aware, the polynomials that arise in the 't Hooft limit have not appeared before in the literature. Although the form of low-lying expansion coefficients can be found, the general structure of these polynomials (beyond those which we were able to calculate explicitly) remains elusive. The fact that deviations of the connected SFF from a linear ramp of unit slope persist only in the 't Hooft limit is similar to the findings for the level density of the \( q \)-RME that was considered in [42], which were mentioned above. Lastly, we consider the non-commutativity of the limits \( q \to 1 \) and \( N \to \infty \), which was already noted for the partition function and trace averages \( \langle \text{tr}U^n \rangle \) in [44].

The outline of the paper is as follows. In section 2, we review the calculation of \( U(N) \) integrals “twisted” by the insertion of \( U(N) \) characters (Schur polynomials), and we explicitly consider a few examples relevant for the calculation of the SFF. In section 3, we present the calculation of the SFF. In particular, in section 3.1 we calculate the SFF for general \( N \) and \( q \), and show how the connected SFF reduces to a linear ramp for \( q^N \to 0 \), thereby confirming one of the main results of [1]. We further demonstrate explicitly that the linear ramp emerges from \( U(N) \) averages over characters corresponding to identical (hook-shaped) representations, which, for \( N \to \infty \), is another finding of [1]. In section 3.2 we demonstrate how a plateau emerges sufficiently far from the origin and explain its arising from the properties of Schur polynomials \( s_\lambda \), in particular from the simple fact that \( s_\lambda(x_1, \ldots, x_N) = 0 \) if the number of non-empty rows in partition \( \lambda \) exceeds the number of variables \( N \). In section 3.3 we calculate the SFF in the ‘t Hooft limit and find that the connected SFF reduces to a sequence of polynomials which do not seem to have appeared in the literature thus far. We argue that this could constitute a novel RMT universality, different from the \( q \)-dependent universality described in [19]. Then, in section 3.4 we explore the non-commutativity of the limits \( N \to \infty \) and \( q \to 1 \) and calculate the SFF for small ‘t Hooft parameter. We finish this work by presenting our outlook and conclusions.
2 Twisted $U(N)$ integrals

In \cite{1}, we calculated the SFF for $N \to \infty$. In this limit, the weighted integral over a product of Schur polynomials simplifies greatly. Consider the weight function of some ensemble, expressed here as

$$f(z) = \sum_{k \in \mathbb{Z}} d_k z^k = \prod_{j=1}^{\infty} (1 + x_j z)(1 + x_j z^{-1}) = E(x; z)E(x; z^{-1}) ,$$

where $x = (x_1, x_2, \ldots)$ are the variables of $E(x; z)$, the generating function of elementary symmetric polynomials. Consider $U \in U(N)$ with eigenvalues $e^{i\phi_j}$. We write

$$\tilde{f}(U) = \prod_{j=1}^{N} f(e^{i\phi_j}) , \quad s_\lambda(U) = s_\lambda(e^{i\phi}) .$$

In the limit $N \to \infty$, the twisted $U(N)$ integral goes to \cite{45}

$$\frac{\int_{\mathbb{C}^N} \tilde{f}(U)s_\lambda(U^{-1})s_\mu(U)dU}{\int_{\mathbb{C}^N} \tilde{f}(U)dU} = \sum_{\nu} s_{(\nu/\nu)^\ell}(x)s_{(\mu/\nu)^\ell}(x) ,$$

where the superscript $^\ell$ denotes transposition and where the sum is over all partitions $\nu$ such that $\mu \geq \nu \subseteq \lambda$. Further, $x = (x_1, x_2, \ldots)$, the set of variables appearing in \cite{1}. If we have a finite number of distinct non-zero $x_j$, \cite{3} can be valid even for finite $N$ \cite{1}. We denote by $|x|$ the number of distinct, non-zero $x_j$. Then, \cite{3} remains valid as long as both $\ell(\lambda)$ and $\ell(\mu)$ are greater than $N - |x|$. However, we are most interested in intermediate matrix model, which has $x_j = q^{j-1/2}$ for all $j \in \mathbb{Z}^+$ so that we cannot apply \cite{3}.

We will therefore consider twisted $U(N)$ integrals for finite $N$ and general weight function. These can be expressed as a minors of a Toeplitz matrix of symbol $f(z) = \sum_{k \in \mathbb{Z}} d_k z^k$, i.e. a Toeplitz matrix with $d_k$ on the $k^{th}$ diagonal \cite{46},

$$D_{N-1}^{\lambda\mu}(f) = \det(d_{\lambda_j - j - \mu_k + k})_{j,k=1}^{N} = \int_{\mathbb{C}^N} \tilde{f}(U)s_\lambda(U^{-1})s_\mu(U)dU .$$

The weight function under consideration in the present work is the third theta function, which, for $0 < |q| < 1$, can be expressed as

$$\Theta_3(z) = \sum_{k \in \mathbb{Z}} q^{k^2/2}z^k = (q; q)_\infty \prod_{j=1}^{\infty} (1 + q^{j-1/2}z)(1 + q^{j-1/2}z^{-1})$$

$$= (q; q)_\infty E(q^{j-1/2}; z)E(q^{j-1/2}; z^{-1}) .$$

Note that, with this definition, $d_k = q^{k^2/2}$ rather than $d_k = q^k$, the latter being another common convention. We then have

$$D_{N-1}^{\lambda\mu}(\Theta_3) = \det \left( q^{(\Lambda_j - j - \mu_k + k)^2/2} \right)_{j,k=1}^{N} .$$
First taking $\lambda = \emptyset = \mu$, we have

$$D_{N-1}(\Theta_\emptyset) = \det \left( q^{(k-j)^2/2} \right)_{j,k=1}^{N} = \prod_{j<k} (1 - q^{k-j}) = \prod_{k=1}^{N-1} (1 - q^j)^{N-j} .$$

Taking only $\lambda = \emptyset$ gives

$$D_{N-1}^\emptyset(\Theta_\emptyset) = \det(d_{k-\mu_k-j})_{j,k=1}^{N} = \int_{U(N)} f(U)s_\mu(U)dU = \det \left( q^{(k-\mu_k-j)^2/2} \right)_{j,k=1}^{N} .$$

We have (see e.g. the appendix of [44]),

$$\det(q^{(\mu_k+k-j)^2/2})_{j,k=1}^{N} = q^{\sum_k \mu_k^2/2} \prod_{j>k} (1 - q^{\mu_j-\mu_k+k-j}) .$$

We define

$$W_{\lambda\mu} := \frac{D_{N-1}^{\lambda\mu}}{D_{N-1}} .$$

Up to a simple framing factor, this equals the Chern-Simons average over a pair of Wilson lines tied into a Hopf link, where one Wilson line carries a $U(N)$ representation $\lambda$ and the other carries $\mu$. For $\mu = \emptyset$, the Hopf link reduces to an unknot carrying rep $\lambda$, and vice versa for $\lambda = \emptyset$. Using (9), we then have

$$W_{\mu} = \frac{\det \left( q^{(k-\mu_k-j)^2/2} \right)}{\det \left( q^{(k-j)^2/2} \right)} = q^{\Sigma_j \mu_j^2/2} \prod_{j<k} (1 - q^{k-j-\mu_k+\mu_j}) = q^{-n(\mu)+\Sigma_j \mu_j^2/2}s_\mu(1,q,q^2,\ldots,q^{N-1}) ,$$

where $n(\mu) = \sum_{j=1}^{N} (j-1)\mu_j$. When we take $N \to \infty$, this equals [43]

$$W_{\mu}^{\infty} = s_\mu'(q^{-1/2}) = q^{(|\mu|/2+n(\mu'))-n(\mu)}s_\mu(q^{-1}) .$$

The power appearing in the prefactor is given by the sum of the content, $c(x)$, over the partition $\mu$. Specifically, $c(x) = j - i$ for $x = (i,j) \in \mu$, and [37]

$$\sum_{x \in \mu} c(x) = n(\mu') - n(\mu) ,$$

For general $\lambda$ and $\mu$, finite $N$, and $|q| < 1$, we have

$$W_{\lambda\mu} = \frac{1}{Z_N} \int s_\lambda(U^{-1})s_\mu(U)f(U)dU$$

$$= q^{\sum_{j=1}^{N}(\lambda_j^2/2+\mu_j^2/2-(j-1)(\lambda_j+\mu_j))}s_\mu(q^{-1})s_\lambda(q^{-\mu_1}, q^{1-\mu_2}, \ldots, q^{N-1-\mu_N})$$

$$= q^{-n(\lambda)+n(\mu)+\sum_{j=1}^{N}(\lambda_j^2/2+\mu_j^2/2)}s_\mu(q^{-1})s_\lambda(q^{-\mu_1}, q^{1-\mu_2}, \ldots, q^{N-1-\mu_N}) .$$
which, for \( N \to \infty \), goes to \[ W^\infty_{\lambda \mu} = \sum_\nu s_{(\lambda/\nu)}(q^{j-1/2}) s_{(\mu/\nu)}(q^{j-1/2}) . \] (15)

Consider, for example, \( \lambda = \square = \mu \). Then, using equation (5.9) in [47] 
\[ s_{\lambda}(x, y) = \sum_\mu s_{\lambda/\mu}(x) s_\mu(y) , \] (16)

and \( \lim_{N \to \infty} [N]_q = \lim_{N \to \infty} \frac{1 - q^N}{1 - q} \) for \( |q| < 1 \), we have
\[ W_{\square \square} = [N] + q^2 [N][N - 1] \lim_{N \to \infty} \frac{1}{1 - q} + \frac{q^2}{(1 - q)^2} . \] (17)

On the other hand,
\[ W^\infty_{\square \square} = \sum_\nu (s_{(\square/\nu)}(q^{j-1/2}))^2 = (s_{\square}(q^{j-1/2}))^2 + (s_{\square}(q^{j-1/2}))^2 \] (18)
\[ = q[N]^2 + 1 = \frac{q}{(1 - q)^2} + 1 = \lim_{N \to \infty} W_{\square \square} . \] (19)

As one can see, the fact that terms of the form \( q^N \) drop out as \( N \to \infty \) leads to the agreement between these expressions. Consider the Schur polynomial appearing in the unknot and Hopf link, given by the \( q \)-hook length formula,
\[ s_{\lambda}(q^{-1}) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{[N + c(x)]}{[h(x)]} = q^{n(\lambda)} \dim_q(\lambda) , \] (20)

where \( h(x) \) is the hook-length of \( x \in \lambda \). The quantity \( \dim_q(\lambda) \) is known as the \textit{quantum dimension}, or \( q \)-dimension. Its expression in (20) is simply the usual hook length formula where numbers are replaced by \( q \)-numbers \( [N] = \frac{1 - q^N}{1 - q} \). Using the \( q \)-hook-length formula, one finds that the hook-shaped Schur polynomial is given by
\[ s_{(a,1^b)}(x_i = q^{-1}) = q^{b(b+1)/2} \frac{[N + a - 1]}{[N - b - 1][a - 1][b][a + b]} . \] (21)

For \( c(x) \ll N, \forall x \in \lambda \) and \( N \to \infty \), (20) gives
\[ s_{\lambda}(q^{-1}) = \frac{q^{n(\lambda)}(1 - q)^{\lambda}}{N} \prod_{x \in \lambda} [h(x)]^{-1} = q^{n(\lambda)} \prod_{x \in \lambda} (1 - q^{h(x)})^{-1} . \] (22)

In particular, \( s_{\lambda}(q^{-1}) \) depends only on \( n(\lambda) \) and the hook lengths, so that e.g. \( \frac{[N + a - 1]}{[N - b - 1]} = (1 - q)^{-(a+b)} \). Therefore, for a partition \( \lambda \) for which
\[ \sum_{x \in \lambda} c(x) = n(\lambda^t) - n(\lambda) = 0 , \] (23)
the unknot \( W^\infty_N \) is invariant under taking \( \lambda \to \lambda^t \). One can clearly see from equation (20) that this is not the case for finite \( N \). These examples illustrate the simplification which occurs as \( q^N \to 0 \), which we will further comment on in the following section.

### 2.1 The ‘t Hooft limit

Alternatively, one may consider the following double scaling limit,

\[
N \to \infty , \quad g_s \to 0 , \quad \text{such that } t := Ng_s = \text{finite} ,
\]

known as the ‘t Hooft limit in the study of matrix models. For the matrix ensemble considered here, the ‘t Hooft limit has been considered in the context of topological string theory in e.g. \cite{36,38,39}. In this case, one has \( y := q^N = e^{-t} \neq 0 \). In this limit, \( q \) taken to a finite power will simply give 1, whereas \( q \) to the power of multiples of \( N \) will give powers of \( y \), which we need to keep track of to calculate the SFF. In the ‘t Hooft limit, the hook-shaped Schur polynomial in (21) goes to

\[
\lim_{q \to 1} \frac{1}{(a-1)!b!(a+b)} \left( \frac{1 - q}{1 - q} \right)^{a+b} ,
\]

which we write as a limit as it is a divergent quantity. However, we will find that the connected SFF is in fact not divergent for the explicit examples we were able to calculate. For the connected SFF not to be divergent, a precise cancellation between various powers of \((1 - q)^{-1}\) has to take place, which means that we cannot simply use (21) in this calculation. Instead, we will write,

\[
A_{a,b} := \lim_{q \to 1} \frac{q^{b(b+1)/2}}{(a-1)!b!(a+b)} \left[ \frac{N + a - 1}{N - b - 1} \right] \frac{\prod_{k=0}^{a+b-1} (1 - yq^{a-1-k})}{(1 - q)^{a+b}} ,
\]

where one should keep in mind that we take the limit \( q \to 1 \). In particular, for \( m \) finite, one can write

\[
\begin{bmatrix} N + m \\ k \end{bmatrix} = \frac{(yq^m; q^{-1})_k}{[k]!(1-q)^k} ,
\]

as a convenient way to extract factors of \( y \).

### 3 Spectral form factor

We proceed to calculate the SFF, which is given by

\[
K(n) := \frac{1}{N} \langle |\text{tr} U^n |^2 \rangle = \frac{1}{N} \sum_{r,s=0}^{n-1} (-1)^{r+s} \langle s_{(n-r,1^r)}^s s_{(n-s,1^s)} \rangle ,
\]
where we applied the expansion of the power sum polynomial,
\[
\text{tr}U^n = \sum_{j=1}^{N} e^{i\phi_j} = \sum_{r=0}^{n-1} (-1)^r s_{(n-r,1^r)}(e^{i\phi_j}).
\]  
(29)

Plugging equation (28) into (14), we have
\[
K(n) = \frac{q^{n^2}}{N} \sum_{r,s=0}^{n-1} (-1)^{r+s} q^{-n(r+s)} s_{(n-s,1^r)}(q^{j-1})s_{(n-r,1^s)}(q^{j-1})(q^{-s},1,\ldots,q^{s-1},q^{s+1},\ldots,q^{N-1}).
\]  
(30)

The first Schur polynomial appearing in the sum, \( s_{(n-s,1^r)}(q^{j-1}) = s_{(n-s,1^r)}(1,\ldots,q^{N-1}), \) is given in (21). The second Schur polynomial on the right hand side of (30), of the form \( s_\lambda(q^{j-\mu-1}) \) for hook-shaped \( \lambda \) and \( \mu \), is more complicated. Let us write \( \lambda = (a,1^b) \) and \( \mu = (c,1^d) \) for simplicity of notation. Defining the sets of variables \( x = q^{-c}, y = q^{d+1},\ldots,q^{N-1}, z = 1,\ldots,q^{d-1} \), we use the following expression [equation (5.10) in [17]],
\[
s_\lambda(x,y,z) = \sum_{\rho} s_{\lambda/\rho}(x)s_{\rho/\rho}(y)s_{\rho}(z),
\]  
(31)

where the sum runs over all partitions satisfying \( \nu \subset \rho \subset \lambda \). For \( \lambda = (a,1^b) \) and \( x,y,z \) as defined above, we get non-zero contributions only when \( \lambda/\rho \) is a horizontal strip, as \( x = q^{-c} \) consists only of a single variable. We then have to carefully distinguish between two types of partitions \( \rho \).

1. For \( \rho = (e,1^b) \), we again have \( \lambda/\rho = (a-e) \) which is obviously a horizontal strip, which gives
\[
s_{\lambda/\rho}(q^{-c}) = q^{-c(a-e)}. \]  
(32)

The requirement that \( \nu \subset \rho \) then gives \( \nu = (f,1^g) \) so that \( \rho/\nu = (e-f) \otimes (1^{b-g}) \) for \( \nu \neq \emptyset \) and \( \rho/\nu = \mu \) for \( \nu = \emptyset \).

2. For \( \rho = (e,1^{b-1}) \), we have \( \lambda/\rho = \emptyset \otimes (a-e) \) so that
\[
s_{\lambda/\rho}(q^{-c}) = q^{-c(a-e+1)}. \]  
(33)

Then, \( \nu = (f,1^g) \) so that \( \rho/\nu = (e-f) \otimes (1^{b-g-1}) \) and \( \rho/\nu = \rho \) for \( \nu = \emptyset \). This situation of course does not occur for \( b = 0 \), in which case we have a horizontal strip \( \lambda = (a) \).

Note that \( s_{(a)}(x) = h_a(x) \) and \( s_{(1^e)}(x) = e_a(x) \), the complete homogeneous and elementary symmetric polynomials of \( a \) boxes, respectively. We then have, for \( \rho/\nu = (e-f) \otimes (1^{b-g}) \)
\[
s_{\rho/\nu}(y) = h_{e-f}(y)e_{b-g}(y). \]  
(34)

We illustrate the two choices for \( \rho \) in equations (32) and (33) for \( \lambda = (4,1^2) \). For the sake of considering a specific example, we set \( e = 2 \), so that the first choice of \( \rho = (e,1^b) = (2,1^2) \). This gives \( \lambda/\rho = (4,1^2)/(2,1^2) = (2) \), which is obviously a horizontal strip. \( \lambda/\rho \) is represented in terms of Young
tableaux as follows.

\[
\begin{array}{c|c}
\begin{array}{ccc}
\hline
\hline
\end{array} & \begin{array}{ccc}
\hline
\hline
\end{array} \\
\hline
\end{array}
\begin{array}{c|c}
\begin{array}{ccc}
\hline
\hline
\end{array} & \begin{array}{ccc}
\hline
\hline
\end{array}
\end{array}
= \begin{array}{c|c}
\begin{array}{c}
\hline
\hline
\end{array} & \begin{array}{c}
\hline
\hline
\end{array}
\end{array}
\]

Consider now the second choice, \( \rho = (e, 1^{b-1}) = (2, 1) \), so that \( \lambda/\rho = (4, 1^2)/(2, 1) = (2) \times (1) \), represented in Young tableaux as follows.

\[
\begin{array}{c|c}
\begin{array}{ccc}
\hline
\hline
\end{array} & \begin{array}{ccc}
\hline
\hline
\end{array} \\
\hline
\end{array}
\begin{array}{c|c}
\begin{array}{c}
\hline
\hline
\end{array} & \begin{array}{c}
\hline
\hline
\end{array}
\end{array}
= \begin{array}{c|c}
\begin{array}{c}
\hline
\hline
\end{array} & \begin{array}{c}
\hline
\hline
\end{array}
\end{array} \times \begin{array}{c|c}
\begin{array}{c}
\hline
\hline
\end{array} & \begin{array}{c}
\hline
\hline
\end{array}
\end{array}
\]

The result is again a horizontal strip, consisting in this case of two disconnected components. As remarked above, the second choice for \( \mu \) clearly does not occur when \( b = 0 \). We consider these two choices for \( \mu \) and sum over all partitions \( \nu \) satisfying \( \nu \subset \rho \subset \lambda \) to calculate \( s_{(n-r,1^r)}(x,y,z) \) and, with that, the full SFF. We discuss the results of this calculation below.

### 3.1 Appearance of the linear ramp for \( n < N \)

For finite \( N > n \), we consider how the linear ramp appears for the connected SFF, which is found by subtracting the disconnected contribution, \( \langle \text{tr} U^n \rangle^2 \), from the full SFF. The disconnected contribution is thus given by the square of

\[
\langle \text{tr} U^n \rangle = q^{n^2/2} \sum_{r=0}^{n-1} (-1)^r q^{-nr} s_{(n-r,1^r)}(q^{j-1}) .
\]

(35)

For \( n = 1, 2, 3 \), this equals

\[
\langle \text{tr} U \rangle = \frac{q^{1/2}(1 - q^N)}{1 - q} = q^{1/2}[N]
\]

\[
\langle \text{tr} U^2 \rangle = \frac{(1 - q^N)(-q + q^N + q^{N+1} + q^{N+2})}{1 - q^2}
\]

\[
\langle \text{tr} U^3 \rangle = \frac{q^3 - q^N + (1 + q + q^2)^2 + q^2(1 + q + q^2)(1 + q + q^2)^2 - q^3N^2(1 + q^2)(1 + q + q^2 + q^3 + q^4)}{q^{3/2}(1 - q^3)}
\]

(36)

We write for convenience,

\[
F(n) = NK(n) , \quad F(n)_c = NK(n)_c .
\]

(37)

1In particular, we sum over \( g \) from 0 to \( \min(b, d-1) \) or to \( \min(b-1, d-1) \), corresponding to \( \rho = (e, 1^b) \) or \( \rho = (e, 1^{b-1}) \), respectively. We then sum over \( f \) from 1 to \( c \) and lastly over \( e \) from 0 to \( a \).
The connected SFF for small $n$ is then given by,

\[
F(1)_c = q^2 s_{(1)}(q^{-1})s_{(1)}(q^{-1}, q, q^2, \ldots) - q(s_{(1)}(1, q, q^2, \ldots))^2
\]

\[= q[N](1 - q) = 1 - q^N\]

\[
F(2)_c = \frac{(1 - q^N)(2q - q^N + q^2N - q^{2+N} + 2q^{1+2N} + q^{2+2N})}{q},
\]

\[
F(3)_c = -\frac{1}{q^4}(-3q^4 + q^{3N} - 2q^N + q^5N + q^{2+N} + 2q^{3+N} + 3q^{4+N} + 2q^{5+N} + q^{6+N} +
- 2q^{1+2N} - 4q^{2+2N} - 8q^{3+2N} - 8q^{4+2N} - 8q^{5+2N} - 4q^{6+2N} - 2q^{7+2N} +
+ 6q^{1+3N} + 10q^{2+3N} + 16q^{3+3N} + 18q^{4+3N} + 16q^{5+3N} + q^{6+3N} + 6q^{7+3N} +
+ q^{8+3N} - 6q^{1+4N} - 12q^{2+4N} - 16q^{3+4N} - 18q^{4+4N} - 16q^{5+4N} - 12q^{6+4N} +
- 6q^{7+4N} - 2q^{8+4N} + 2q^{1+5N} + 5q^{2+5N} + 6q^{3+5N} + 8q^{4+5N} + 6q^{5+5N} + 5q^{6+5N} +
+ 2q^{7+5N} + q^{8+5N})\] . \quad (38)

Examples of the SFF for higher $n$ are too long to print here. One thing one can see from (38), which persists for higher $n$, is that the SFF is of the form,

\[
F(n)_c = n + \mathcal{O}(q^A) , \quad A = N + \ldots . \quad (39)
\]

Therefore, for $N \to \infty$ and $q < 1$ fixed, $q^N \to 0$ and $F(n)_c \to n$. This reproduces the exact linear ramp that was found in [1].

### 3.1.1 Linear ramp from Schur bilinears

As mentioned above, as we take $N$ to infinity, the connected SFF for $n/N < 1$ is a linear ramp of unit slope [1]. In this limit, in the expansion of the form factor in terms of averages of bilinears of hook-shaped Schur polynomials,

\[
F(n) = \sum_{r,s=0}^{n-1} (-1)^{r+s} \langle s_{(n-r,1^r)}(U^{-1})s_{(n-s,1^s)}(U) \rangle , \quad (40)
\]

we get a contribution equal to 1 when considering two identical hook-shaped partitions $(n-r, 1^r) = (n-s, 1^s)$. Since there are $n$ hook-shaped partitions containing $n$ boxes, we get a contribution equal to $n$, which is the linear ramp. More details can be found in [1].

The above consideration leads us to conclude that, for finite $N$ and for $r = s \leq N - 1$, we should have that the summand of the SFF in (30) is of the following form

\[
A(N, n, q, r, r) := \langle s_{(n-r,1^r)}s_{(n-r,1^r)} \rangle
\]

\[
= q^{n-2nr}s_{(n-r,1^r)}(q^{-1})s_{(n-r,1^r)}(q^{-(n-r)}, 1, \ldots, q^{r-1}, q^{r+1}, \ldots, q^{N-1})
\]

\[
= 1 + \mathcal{O}(q) . \quad (41)
\]
Further, we should have

\[ A(N, n, q, r, s) = \mathcal{O}(q) \quad r \neq s. \]  \hfill (42)

There are two types of terms of \( \mathcal{O}(q) \) in the above expressions. First of all, there are terms of the form \( q^{N+\ldots} \), which go to zero as we take \( N \to \infty \) for fixed \( q < 1 \). Secondly, there are powers of \( q \) not containing factors of \( N \), which do not go to zero as \( N \to \infty \). Therefore, to recover the linear ramp as \( N \to \infty \), all the lower powers of \( q \) should mutually cancel out between the various terms in the sum in (30). It should be clear from equation (30) that the fact that such a cancellation occurs is a priori far from obvious. We start by verifying (41). Note that \( q \)-numbers and products thereof (such as \( q \)-factorials and \( q \)-binomials) are themselves of \( \mathcal{O}(1) \). For example,

\[ [N]_q = \frac{1 - q^N}{1 - q} = (1 - q^N) \sum_{k=0}^{\infty} q^k = 1 + q + q^2 + \cdots - q^N - q^{N+1} + \ldots. \]  \hfill (43)

Let us consider \( A(N, n, q, 0, 0) \). Plugging

\[ s_{(n)}(q^{-n}, q, \ldots, q^{N-1}) = q^{-n^2} + \mathcal{O}(q^{-n^2+1}). \]  \hfill (44)

into (41) leads to

\[ q^{n^2} s_{(n)}(q^{-1}) s_{(n)}(q^{-n}, q, \ldots, q^{N-1}) = q^{n^2} \left[ \frac{N + n - 1}{n} \right] \left( q^{-n^2} + \mathcal{O}(q^{-n^2+1}) \right) = 1 + \mathcal{O}(q), \]  \hfill (45)

where we use the aforementioned fact that \( q \)-binomials are of the form \( 1 + \mathcal{O}(q) \). When \( r = s \neq 0 \), the calculation is slightly more involved. First, we read off from (20) that

\[ s_{(n-r,1^r)}(q^{-1}) = q^{r(r+1)/2} (1 + \mathcal{O}(q)) \]  \hfill (46)

We then determine the lowest power of \( q \) appearing in

\[ s_{(n-r,1^r)}(q^{-n-r}, 1, \ldots, q^{-1}, q^{r+1}, \ldots, q^{N-1}) = \sum_{\mu, \nu} s_{(n-r,1^r)/\mu}(q^{-n-r}) s_{\mu/\nu}(1, \ldots, q^{-1}) s_{\nu}(q^{r+1}, \ldots, q^{N-1}), \]  \hfill (47)

where we used (31) on the right hand side. Consider the case where \( \mu = (1^r) \) and \( \nu = \emptyset \). This gives \( \lambda/\mu = (n-r-1) \times (1) \), so that \( s_{(n-r,1^r)/\mu}(q^{-n-r}) = q^{-(n-r)^2} \). Further, we have \( s_{\mu}(q, \ldots, q^{-1}) = q^{r(r-1)/2}, \) and \( s_{\nu} = s_{\emptyset} = 1. \) Therefore,

\[ s_{(n-r,1^r)}(q^{-(n-r)}, 1, \ldots, q^{-1}, q^{r+1}, \ldots, q^{N-1}) = q^{-(n-r)^2} q^{r(r-1)/2} (1 + \mathcal{O}(q)) \]  \hfill (49)
Plugging this into (41) gives
\[ A(N, n, q, r, r) = q^{n^2 - 2nr} q^{(r+1)/2} q^{-(n-r)^2} q^{r(r-1)/2} (1 + O(q)) = 1 + O(q) . \] (50)

One can readily check that any other choice of $\mu$ and $\nu$ leads to higher powers of $q$. For example, choosing $\nu = (1)$ increases the power of $q$ by 2, and choosing a different partition for $\mu$ either increases the power of $q$ by $n - r$ or gives zero (when $\ell((n - r, 1^r)/\mu) > 1$). This demonstrates equation (41).

Let us now consider the case where $r \neq s$, to derive equation (52). We take $r < s$ without loss of generality. Taking first $r = 0$, we have
\[ A(N, n, 0, s, q) = q^{n^2 - ns} s_{n-s,1^s} (q^j - 1)s_{j(n)} (q^{-(n-s)}, y, z) = q^{n^2 - ns} q^{s(s+1)/2} q^{-(n-s)} (1 + O(q)) = q^{s(s+1)/2} (1 + O(q)) = O(q) . \] (51)

where $y = (1, \ldots, q^{s-1})$ and $z = (q^{s+1}, \ldots, q^{N-1})$, as before. Lastly, we check the case where $0 \neq s \neq r \neq 0$, choosing again $r < s$ without loss of generality. Following the same procedure that lead to (49), we find
\[ s_{n-r,1^r} (q^{-(n-s)}, 1, \ldots, q^{s-1}, q^{s+1}, \ldots, q^{N-1}) = q^{-(n-r)^2} q^{r(r-1)/2} (1 + O(q)) , \] (52)
so that this term, too, appears with a positive power of $q$.
\[ A(N, n, r, s, q) = q^{((r-s)^2 + r+s)/2} (1 + O(q)) . \] (53)

We have thus shown that terms with $r = s$ contribute $1 + O(q)$, whereas Schur bilinears with $r \neq s$ contribute terms of $O(q)$. As mentioned above, powers of $q$ which do not contain a factor $N$ cancel out in the sum over $r$ and $s$, leaving only a linear ramp plus terms of the form $q^{(N+\ldots)}$, as can be seen in (38).

The calculations described here have a simple knot-theoretical interpretation. Indeed, remember that $A(N, n, r, s, q)$ is (proportional to) the HOMFLY invariant of a Hopf link, where the two components of the Hopf link carry $U(N)$ representations corresponding to partitions $(n - r, 1^r)$ and $(n - s, 1^s)$, respectively. We then see that Hopf links of Wilson lines carrying hook-shaped representations only give a contribution of (order) unity for two identical representations. Therefore, in the limit $q \to 0$, where the intermediate ensemble reduces to the CUE, all Hopf link invariants with different $n$-box hook-shaped partitions go to zero. On the other hand, those with identical $n$-box hook-shaped partitions go to one. This means, for example, that an unknot carrying $(a, 1^b)$ with $b \neq 0$ has invariant equal to zero, but if we tie two of these unknots together to form a Hopf link the resulting invariant equals one. On the other hand, an unknot carrying representation $(a)$ has invariant equal to one, but if we tie it to an unknot carrying $(a - b, 1^b)$ with $b \neq 0$, the result is again zero.
3.2 Relaxing the assumption that \( n < N \)

If we relax the assumption that \( n < N \), the SFF will eventually reach a plateau for large enough \( n \). A well-known, heuristic way to see that such a plateau is eventually reached is as follows. If we consider a diagonal matrix \( V = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_N}) \) with all \( \phi_j \) taking random values in \([0, 2\pi)\) and average over \( \phi_j \), we get

\[
\langle |\text{tr} V^n|^2 \rangle = \left\langle \sum_{k,l=1}^N e^{in(\phi_k - \phi_m)} \right\rangle = N .
\] (54)

Here, we use the fact that \( e^{in(\phi_k - \phi_m)} \) equals 1 for \( k = m \), whereas for \( k \neq m \) it is a random variable on \((2\pi)\) copies of) the complex unit circle, which averages to zero. A system with eigenphases distributed random across the unit circle therefore has a constant SFF, as was mentioned in the introduction.

On the other hand, random unitary matrices \( U \) display level repulsion with overwhelming probability, so that their eigenvalues tend to distribute more evenly across the complex unit circle. Therefore, \( \langle |\text{tr} U^n|^2 \rangle \) is much lower than \( N \) for small values of \( n \). However, for \( n \) close to \( N \), we have that \( n \) becomes of the order of the average spacing \( \phi_{k+1} - \phi_k \). In that case, \( e^{in(\phi_k - \phi_l)} \) is an approximately random element of \( T \) for all \( k \neq l \), so that these again average to zero and only the constant contribution \( N \) coming from \( k = l \) remains. This explains the origin of the plateau for \( n \geq N \).

Alternatively, the emergence of the plateau can be understood to arise from the fact that \( s_{\lambda}(x) = 0 \) for \( \ell(\lambda) > |x| \). In particular, the averages appearing on the right hand side of (28), are weighted matrix integrals written in (4) over Schur polynomials of the form \( s_{(n-r, 1^r)}(U) = s_{(n-r, 1^r)}(e^{i\phi_r}) \). If \( \ell(s_{(n-r, 1^r)}) = r + 1 > N \), then \( s_{(n-r, 1^r)}(U) = 0 \). Therefore, only Schur bilinears of the form

\[
\langle s_{(n-r, 1^r)} s_{(n-s, 1^s)} \rangle , \quad r, s \leq N - 1 ,
\] (55)

give a non-zero contribution to (28). It was demonstrated in the previous subsection that, for \( r, s \leq N - 1 \),

\[
\langle s_{(n-r, 1^r)} s_{(n-s, 1^s)} \rangle = \delta_{r,s} + \mathcal{O}(q) ,
\] (56)

from which arises the linear ramp. From equation (56), it follows that the linear ramp arising from \( r = s \leq N - 1 \), saturates at a plateau for \( n = N \). However, one should not that this does not take into account terms of \( \mathcal{O}(q^N) \), which will turn out to have a significant impact on the shape of the SFF, delaying the onset of the plateau as one increases \( q^N \).

To implement \( n > N \) in the expression for the SFF derived at the start of this section, one should take into account that

1. \( s_{(e, 1^b)}(y) = 0 \) for \( b > N - d - 2 \)

2. \( e_{b-d}(y) = 0 \) for \( g > d + b + 1 - N \).

where \( x = q^{-e} \), \( y = q^{d+1}, \ldots, q^{N-1} \), and \( z = 1, \ldots, q^{d-1} \), as before. Note that the functions above
both arise as $s_{ρ/ν}(y)$ for $ρ = (e, 1^b)$ in equation (31), where $e_b−q(y)$ is given in equation (34), whereas $s_{ρ/ν}(y)$ reduces $s_{(e,1^b)}(y)$ for $ν = ∅$. Below, we plot $K(n)$ resulting from this calculation for various choices of $N$ and $q$.

As one can see, the SFF is closest to a linear ramp for small values of $q$, which is to be expected as the limit $q \to 0$ corresponds to the CUE. Further, disconnected SFF becomes large for small $n$ as we increase $q$, leading to large, oscillating deviations close to the origin. Comparing figures 1 and 2 reveals that, for fixed $q$, deviation from a linear ramp decreases as we increase $N$. This suggests that a combination of $q$ and $N$ controls the approximate slope away from the origin.

The aforementioned points that, as we increase $q$, a dip emerges and the slope of the SFF decreases, are not unrelated. In particular, we find that the sum $\sum_{n=1}^{k} F(n)$ for large enough $k \geq N$ is almost independent of $q$. That is, as we increase $q$, we get positive contributions to $A(k)$ arising from the disconnected SFF which are compensated by a decrease in the slope of $F(n)$. We define, for $k > N$, the logarithm of the difference between the sum over the intermediate SFF and the CUE ($q \to 0$) SFF,

$$A(k) := \log \left[ \sum_{n=1}^{k} F(n) - N^2/2 - N(N - k) \right]. \quad (57)$$

We plot the results for $N = 10$ and $k = 10, \ldots, 20$ below. It is clear that the difference decreases quite rapidly with $k$ until it stabilizes around some small value. Further, we see that the difference decreases more slowly and acquires a larger minimum value as we increase $q$. Such measures as the sum (or integral, in the limit $n \to \infty$) can be used to quantify eigenvalue repulsion [48], which we intend to explore further in a future work.
Figure 2: The SFF for $N = 20$ and various values of $q$.

Figure 3: The logarithm of the difference between the sum over the intermediate SFF and the CUE ($q \rightarrow 0$) SFF, plotted for various values of $q$. We see that the difference is very small and decreases quite rapidly with $k$ but, conversely, increases with $q$.

3.3 The form factor in the 't Hooft-limit

Taking the 't Hooft limit, $N \rightarrow \infty$ and $g_s \rightarrow 0$ such that $t = Ng_s$ = finite leads to $q \rightarrow 1$ and $q^N = y$ with $0 \leq y < 1$. In this limit, the SFF turns into a remarkable sequence of polynomials of order $2n - 1$. 
We have calculated the connected SFF for $n = 1, \ldots, 11$, resulting in the following expressions.

\[
\begin{align*}
F(1)_c &= 1 - y, \\
F(2)_c &= 2 - 4y + 6y^2 - 4y^3, \\
F(3)_c &= 3 - 9y + 36y^2 - 84y^3 + 90y^4 - 36y^5, \\
F(4)_c &= 4 - 16y + 120y^2 - 560y^3 + 1420y^4 - 1968y^5 + 1400y^6 - 400y^7, \\
F(5)_c &= 5 - 25y + 300y^2 - 2300y^3 + 10150y^4 - 26880y^5 + 43400y^6 - 41800y^7 + 22050y^8 - 4900y^9, \\
F(6)_c &= 6 - 36y + 630y^2 - 7140y^3 + 47880y^4 - 200592y^5 + 544824y^6 - 974160y^7 + 1137780y^8 + \\
&1113928y^9 + 834960y^9 + 349272y^{10} - 63504y^{11}, \\
F(7)_c &= 7 - 49y + 1176y^2 - 18424y^3 + 173460y^4 - 1042524y^5 + 4187736y^6 - 11565624y^7 + \\
&+ 22246686y^8 - 29742020y^9 + 27087984y^{10} - 16024176y^{11} + 5549544y^{12} - 853776y^{13}, \\
F(8)_c &= 8 - 64y + 2016y^2 - 41664y^3 + 522480y^4 - 4237632y^5 + 23380896y^6 - 90830784y^7 + \\
&+ 253846296y^8 - 515838400y^9 + 762521760y^{10} - 81092736y^{11} + 604107504y^{12} + \\
&- 299065536y^{13} + 88339680y^{14} - 11778624y^{15}, \\
F(9)_c &= 9 - 81y + 3240y^2 - 85320y^3 + 1372140y^4 - 14394996y^5 + 103900104y^6 - 535847400y^7 + \\
&+ 2026445850y^8 - 5713765200y^9 + 12118597920y^{10} - 19364383584y^{11} + 23165382240y^{12} + \\
&- 20414698920y^{13} + 12853423440y^{14} - 5468226192y^{15} + 1407913650y^{16} - 165636900y^{17}, \\
F(10)_c &= 10 - 100y + 4950y^2 - 161700y^3 + 32406000y^4 - 42617520y^5 + 388588200y^6 + \\
&- 2556668400y^7 + 12488661900y^8 - 4620249200y^9 + 131172321280y^{10} + \\
&- 287919216000y^{11} + 489596250000y^{12} - 642659556000y^{13} + 644511582000y^{14} + \\
&- 484405727520y^{15} + 263957736900y^{16} - 98425126800y^{17} + 22457091800y^{18} + \\
&- 2363904400y^{19}, \\
F(11)_c &= 11 - 121y + 7260y^2 - 287980y^3 + 7031310y^4 - 113142744y^5 + 1269259992y^6 + \\
&- 10345746840y^7 + 63147440070y^8 - 295025713840y^9 + 1071727584928y^{10} + \\
&- 3059501029728y^{11} + 6907003486240y^{12} - 12358366232520y^{13} + 17490417431040y^{14} + \\
&- 19447530019632y^{15} + 16771920490182y^{16} - 10982054062980y^{17} + 5272925154640y^{18} + \\
&- 1749762036880y^{19} + 358415185128y^{20} - 34134779536y^{21}. 
\end{align*}
\]

To the best of the authors’ knowledge, the above polynomials have not appeared in the literature before. Their complicated form belies the fact that the SFF, when plotted, appears to be very close to a straight line for any $y$, with decreasing slope for increasing $q$. In fact, there are three choices of $y$
for which the SFF is a perfectly straight line. Writing \( F(n; y)_c \) to indicate dependence on \( y \), we have

\[
F(n; 0)_c = n,
F(n; 1/2)_c = \frac{n}{2},
F(n; 1)_c = 0. \tag{59}
\]

The fact that \( F(n; 0)_c = n \) was already mentioned in section 3.1 and is, in essence, one of the results derived in [1]. The last equality, written as a limit for \( y \), can easily be seen to be generally true and will be further commented on in section 3.4. The middle equality, on the other hand, is a priori completely unexpected (at least to the authors).

Figure 4: The connected SFF in the 't Hooft limit written in equation (58). As one can see, the resulting SFF is very close to a straight line for any choice of \( y \).

One can see from the above plot that the form factor appears to be symmetric around \( F(n; 1/2)_c = \frac{n}{2} \). Indeed, we find that

\[
F(n; y)_c + F(n; 1 - y)_c = n. \tag{60}
\]

To see how close the SFF is to a straight line, we fit a function \( I_b(x) = bx \) with \( 0 \leq b \leq 1 \) to the SFF for \( n = 1, \ldots, 11 \). We plot the difference between the SFF and the linear fit, \( F_c(n) - I_b(n) \) figure 5. The differences for \( y > 1/2 \) are the mirror images of those for \( y < 1/2 \), as can be seen from (60).
Figure 5: The difference between the SFF for various choices of $y$ and linear fits of the form $L_b(x) = bx$, $0 \leq b \leq 1$. The dashed lines are an interpolation to guide the eye. As one can see, the difference appears to be oscillating within an envelope that decreases with $n$. From (60), it follows that the differences are mirrored around $y = 1/2$.

Whilst, in [1], we found that taking only $N \to \infty$ leads to a universal connected SFF in the form of a linear ramp of unit slope, we see here that we get quite different behavior as we take the ‘t Hooft limit. Further, it is known that the intermediate matrix ensemble is a member of a family of ensembles exhibiting so called weak or soft confinement [49], which correspond to indeterminate moment problems and which have been argued to form their own one-parameter universality class [19], the parameter in question being $q$. In [1], we found that all invariant one-matrix models arising in topological string theory that are known (to us) belong to this family of soft confining ensembles.

It would be interesting to see whether the SFF found here exhibits a novel random matrix universality for weak confining RME’s in the ‘t Hooft limit. This would extend the one-parameter universality of the intermediate matrix model [19] to the ‘t Hooft limit, where the parameter controlling the statistics is $y = q^N$ instead of $q$. It appears that the ‘t Hooft limit should correspond to the strong disorder limit in disordered conductors, as was argued for related models in [41] and [42], since only in the ‘t Hooft limit do deviations of the connected SFF persist as we take $N \to \infty$.

3.3.1 Conjecture for the orthogonal polynomial structure

In the ‘t Hooft limit the polynomials of any order appearing in the expressions for the SFF in (38) gets factorized into the product of a factor $(1 - y)$ and polynomials $p_n(y)$ of order $2n - 2$. The first
few of them are given by

\[
\begin{align*}
p_0 & = 1, \\
p_1 & = 1 - y + 2y^2, \\
p_2 & = 1 - 2y + 10y^2 - 18y^3 + 12y^4, \\
p_3 & = 1 - 3y + 27y^2 - 113y^3 + 242y^4 - 250y^5 + 100y^6, \\
p_4 & = 1 - 4y + 56y^2 - 404y^3 + 1626y^4 - 3750y^5 + 4930y^6 - 3430y^7 + 980y^8, \\
p_5 & = 1 - 5y + 100y^2 - 1090y^3 + 6890y^4 - 26542y^5 + 64262y^6 - 98009y^7 \\
& \quad + 91532y^8 - 47628y^9 + 10584y^{10}, \\
p_6 & = 1 - 6y + 162y^2 - 2470y^3 + 22310y^4 - 126622y^5 + 471626y^6 - 1180606y^7 \\
& \quad + 1997492y^8 - 2251368y^9 + 1618344y^{10} - 670824y^{11} + 121968y^{12}, \\
p_7 & = 1 - 7y + 245y^2 - 4963y^3 + 60347y^4 - 469357y^5 + 2453255y^6 - 8900593y^7 + 22830194y^8 \\
& \quad - 41649606y^9 + 53665614y^{10} - 47700378y^{11} + 27813060y^{12} - 9570132y^{13} + 1472328y^{14}. 
\end{align*}
\]

The coefficients in front of \( y^n \) which we were able to identify are

\[
\begin{align*}
y^0 & : 1 \\
y^1 & : -n \\
y^2 & : n^2(n + 3)/2 \\
y^3 & : -n(n - 1)(2 + 10n + 6n^2 + n^3)/6 \\
y^4 & : n(n - 1)(-72 - 224n - 28n^2 + 87n^3 + 40n^4 + 5n^5)/144 \\
y^{2n-1} & : -C_n^{2n}2(2n - 1)/2(n + 1) \\
y^{2n} & : (C_n^{2n})^2/(n + 1),
\end{align*}
\]

where we have the binomial coefficient \( C_n^{2n} = (2n)!/(n!)^2 \). It seems that no further information about the expansion coefficients of \( p_n(y) \) can be obtained easily. We notice however that all the polynomials \( p_N(y) - 1 \) are divisible by \( y(2y - 1) \) which allow us to reduce their order. Denoting \( s_N(y) = (p_N(y) - 1)/(y(2y - 1)) \), we find that these can be expressed as functions of \( x \), which reduces the degree even further. Finally, considering the polynomials \( w_n(y) = s_{n+1}(y) - s_n(y) \) one can realize \(^2\)

that they have the following remarkable properties:

\begin{enumerate}
\item[a)] all the roots of \( w_n(y) \) are real;
\item[b)] they occupy the interval \([0, 1]\);
\item[c)] the roots have the interlacing property, meaning that the roots of lower order polynomials are located in between the roots of higher order polynomials.
\end{enumerate}

This and other observations suggest that the polynomials \( w_N(y) \) form a family of orthogonal polynomial.

\(^2\)We are very grateful to Dr. Denis Kurlov for pointing this fact out to us.
mials. However, the polynomials in (61) are only of even order, while the odd-order polynomials are missing. The weight function for this suspected family of orthogonal polynomials is not known to us at the moment.

3.4 Non-commutativity of the limit $q \to 1$ and $N \to \infty$

One can see from the expression of the SFF that the limits $q \to 1$ and $N \to \infty$ do not commute. Such non-commutativity has been discussed in the literature for decades, see [44]. In particular, if we take $q \to 1$ into expression (14) for finite $N$, the Schur polynomials simply give the dimension of the representation, that is

$$s_{\lambda}(1,1,\ldots,1) = \dim \lambda \quad (63)$$

Plugging this into (14) with $\lambda \vdash p n \backsim r, 1^r$ and $\mu \vdash p n \backsim s, 1^s$ shows that the SFF would be simply given by

$$\lim_{q \to 1} \langle \langle \text{tr}U^n \rangle^2 \rangle = \lim_{q \to 1} \langle \langle \text{tr}U^n \rangle \langle \text{tr}U^{-n} \rangle \rangle = \lim_{q \to 1} \langle \langle \text{tr}U^n \rangle \rangle^2 \quad (65)$$

for all $n$. In terms of knot theory, we see that taking $q \to 1$ for $N$ finite breaks the $(2n,2)$-torus link that is the SFF up into its separate $(n,1)$-torus knot components, as we have

$$\lim_{q \to 1} \langle \langle \text{tr}U^n \rangle^2 \rangle = \lim_{q \to 1} \langle \langle \text{tr}U^n \rangle \langle \text{tr}U^{-n} \rangle \rangle = \lim_{q \to 1} \langle \langle \text{tr}U^n \rangle \rangle^2 \quad (65)$$

The connected SFF then equals zero, as was the case in section 3.3. If, on the other hand, we first take $N \to \infty$, we should instead use (3). If we now take $q \to 1$, we again replace Schur polynomials by the dimensions of representations. Taking $\nu$ in (3) to be the empty partition gives the same expression as (64), but we get additional terms for $\nu$ non-empty. In particular, as shown in [1], the result is exactly given by a linear ramp, arising from the $n$ terms $(n-r, 1^r) = \nu = (n-s, 1^s)$ in (28).

Let us consider the case where $t = Ng_s \ll 1$ is very small. This allows us to use the following expansion [47, 1.3, example 10],

$$s_{\lambda}(1+x_1,1+x_2,\ldots,1+x_N) = \sum_{\mu} d_{\lambda\mu} s_{\mu}(x_1,\ldots,x_N) \quad (66)$$

where we sum over all $\mu \subseteq \lambda$ and where

$$d_{\lambda\mu} = \det \left( \lambda_i + n - 1 \atop \mu_j + n - j \right)_{1 \leq i,j \leq N} \quad (67)$$

Some simple examples are given by (see e.g. [50])

$$d_{\lambda\emptyset} = \dim \lambda \quad , \quad d_{\lambda\emptyset} = \dim \lambda \frac{c_1(\lambda)}{N} \quad (68)$$

where the first Casimir invariant is given by $c_1(\lambda) = |\lambda| = \sum \lambda_i$. For $q = e^{-g_s}$ close to 1 and $Ng_s \ll 1$,
we have

\[ s_{(a,1^b)}(q^{j-1}) \approx s_{(a,1^b)}(1, 1 - g_s, 1 - 2g_s, \ldots) = \sum_{\mu} d_{(a,1^b)}^{\mu} s_{\mu}(0, -g_s, -2g_s, \ldots). \]  

(69)

Expanding up to linear order in \( g_s \), we only get contributions for \( \mu = 0 \) and \( \mu = \emptyset \), which gives

\[ s_{(a,1^b)}(0, -g_s, -2g_s, \ldots) = \dim(a, 1^b) \left( 1 + \frac{a + b}{N} (-g_s - 2g_s - \ldots) \right) \]

\[ = \dim(a, 1^b) \left( 1 - \frac{(a + b)(N - 1)}{2} g_s \right). \]  

(70)

Further,

\[ s_{(a,1^b)}(q^{-c}, 1, q, \ldots, q^{d-1}, q^{d+1}, \ldots, q^{N-1}) \approx s_{(a,1^b)}(c g_s, -g_s, \ldots, -(d - 1) g_s, -(d + 1) g_s, \ldots) \]

\[ = \dim(a, 1^b) \left[ 1 + (a + b) \left( \frac{1 - N}{2} + \frac{c + d}{N} \right) g_s \right]. \]  

(71)

Plugging this into (14) for \( \lambda = (n - r, 1^r) \) and \( \mu = (n - s, 1^s) \) gives

\[ \langle W_{\lambda \mu} \rangle - \langle W_{\lambda} \rangle \langle W_{\mu} \rangle = \dim(n - r, 1^r) \dim(n - s, 1^s) \frac{n^2 g_s}{N} + O(g_s^2) \]  

(72)

We thus see that, to first order in \( N g_s \), the Wilson loop factorizes, so that

\[ F(n)_c = tn^2 + \mathcal{O}(t^2). \]  

(73)

If we now take \( N \to \infty \) in such a way that \( t \) remains small, we clearly get a very different result from the linear ramp \( F(n)_c = n \) that is found when taking \( q \to 1 \) after \( N \to \infty \). Indeed, one may check that the connected SFF in the 't Hooft limit for small \( n \) and \( y \approx 1 \) is very close to \( t n^2 \).

\section{Conclusion}

In this work, we calculate the SFF of the intermediate matrix model for general matrix size \( N \) and parameter \( q \) between zero and one. We find that, as \( y = q^N \to 0 \), we recover our previous results in [1]. That is, the connected SFF is given by a linear ramp of unit slope, with deviations from the linear ramp arising purely from the disconnected SFF. For \( y \) different from zero, we see that the (approximate) slope of the SFF becomes smaller than one as we increase \( q \). For \( n > N \), the SFF eventually saturates at a plateau, which takes longer for larger \( q \) due to the fact that the slope decreases with \( q \). The emergence of the linear ramp and its saturation at a plateau is shown to arise from the properties of Schur bilinears appearing in the character expansion of the SFF.

Taking the 't Hooft limit turns the SFF into a sequence of polynomials of order \( 2n - 1 \) that was calculated up to \( n = 11 \). As far as the authors are aware, this sequence has not appeared in the literature thus far. The SFF in the 't Hooft limit has remarkable properties more extensively described in the
main text, which, in the authors' view, warrant further mathematical investigation into their structure and their extension beyond $n = 11$. As mentioned in the introduction, the intermediate matrix model is equivalent to $U(N)$ Chern-Simons theory, and the SFF is proportional to the HOMFLY invariant of a $(2n,2)$-torus link with components carrying fundamental and antifundamental representations. The calculation of the SFF thus provides us with new expressions for link invariants, both for general $q, N$ as well as in the ‘t Hooft limit. Due to the appearance of $U(N)$ Chern-Simons theory at large $N$ in the form of various topological string theories described in the introduction, and the relation of these large $N$ dualities to enumerative geometry and intersection theory, the results derived here could be of mathematical interest beyond knot theory.

From a physical point of view, it seems that the ‘t Hooft limit leads to new and unexpected behavior of the SFF, such as it being mirrored around $y = 1/2$ for $y$ and $1 - y$. The widespread phenomenological applicability of various random matrix ensembles (see the examples in the introduction) begs the question which physical systems might be described by the intermediate matrix model in the ‘t Hooft limit. In the context of disordered conductors, it seems that the ‘t Hooft limit corresponds to strong disorder, as only in this limit does a spectral signature of non-ergodicity persist in the SFF. Similar conclusions were previously drawn for the approximate level density in [42], where the authors take a very similar limit in a $q$-deformed circular RME, see also [41]. Further, the intermediate matrix ensemble is a member of a family of soft confining ensembles which have been argued to form their own one-parameter universality class [19]. This suggests that the SFF in (58) may be a representative of a novel type of random matrix universality for the two-point correlation function of soft confining ensembles in the ‘t Hooft limit. We aim to return to the question of universality in a future work.

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References

[1] W.L. Vleeshouwers and V. Gritsev. Topological field theory approach to intermediate statistics. *SciPost Phys.*, 10:146, 2021.

[2] E.P. Wigner. *Characteristic Vectors of Bordered Matrices with Infinite Dimensions I*, pages 524–540. Springer Berlin Heidelberg, Berlin, Heidelberg, 1993.

[3] O. Bohigas, M.J. Giannoni, and C. Schmit. Characterization of chaotic quantum spectra and universality of level fluctuation laws. *Phys. Rev. Lett.*, 52:1–4, Jan 1984.

[4] Y. Imry and G. Grinstein. Directions in condensed matter physics, 1986.
[5] C.W.J. Beenakker. Random-matrix theory of quantum transport. *Reviews of Modern Physics*, 69(3):731–808, Jul 1997.

[6] J.J.M. Verbaarschot and T. Wettig. Random matrix theory and chiral symmetry in qcd. *Annual Review of Nuclear and Particle Science*, 50(1):343–410, Dec 2000.

[7] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin. 2d gravity and random matrices. *Physics Reports*, 254(1-2):1–133, Mar 1995.

[8] J.P. Bouchaud and M. Potters. Financial applications of random matrix theory: a short review. *arXiv.org, Quantitative Finance Papers*, 10 2009.

[9] A. Edelman and Y. Wang. *Random Matrix Theory and Its Innovative Applications*, pages 91–116. Springer US, Boston, MA, 2013.

[10] H.L. Montgomery. The pair correlation of zeros of the zeta function. In *Proc. Symp. Pure Math*, volume 24, pages 181–193, 1973.

[11] A.M. García-García and J.J.M. Verbaarschot. Spectral and thermodynamic properties of the sachdev-ye-kitaev model. *Physical Review D*, 94(12), Dec 2016.

[12] J.S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S.H. Shenker, D. Stanford, A. Streicher, and M. Tezuka. Black holes and random matrices. *Journal of High Energy Physics*, 2017(5), May 2017.

[13] P. Saad, S. H. Shenker, and D. Stanford. A semiclassical ramp in syk and in gravity, 2019.

[14] A. Altland and D. Bagrets. Quantum ergodicity in the syk model. *Nuclear Physics B*, 930:45–68, May 2018.

[15] A. Altland and J. Sonner. Late time physics of holographic quantum chaos. *SciPost Physics*, 11(2), Aug 2021.

[16] A. Belin, J. de Boer, P. Nayak, and J. Sonner. Generalized spectral form factors and the statistics of heavy operators, 2021.

[17] K.A. Muttalib, Y. Chen, M.E.H. Ismail, and V.N. Nicopoulos. New family of unitary random matrices. *Phys. Rev. Lett.*, 71:471–475, Jul 1993.

[18] V.E. Kravtsov and K.A. Muttalib. New class of random matrix ensembles with multifractal eigenvectors. *Phys. Rev. Lett.*, 79:1913–1916, Sep 1997.

[19] K.A. Muttalib, Y. Chen, and M.E.H. Ismail. q-random matrix ensembles. In Ismail M.E.H. Garvan F.G., editor, *Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics.*, pages 199–221. Springer, Boston, MA, 2001.

[20] M. Tierz. Soft matrix models and Chern-Simons partition functions. *Mod. Phys. Lett. A*, 19:1365–1378, 2004.
[21] K.A. Muttalib and M.E.H. Ismail. Power-law eigenvalue density, scaling, and critical random-matrix ensembles. *Phys. Rev. E*, 76(5), Nov 2007.

[22] Y. Takahashi and M. Katori. Oscillatory matrix model in chern-simons theory and jacobi-theta determinantal point process. *Journal of Mathematical Physics*, 55(9):093302, Sep 2014.

[23] C.M. Canali and V.E. Kravtsov. Normalization sum rule and spontaneous breaking of u(n) invariance in random matrix ensembles. *Phys. Rev. E*, 51:R5185–R5188, Jun 1995.

[24] F. Evers and A.D. Mirlin. Anderson transitions. *Rev. Mod. Phys.*, 80:1355–1417, Oct 2008.

[25] E.B. Bogomolny, U. Gerland, and C. Schmit. Models of intermediate spectral statistics. *Phys. Rev. E*, 59:R1315–R1318, Feb 1999.

[26] A.D. Mirlin, Y.V. Fyodorov, F. Dittes, J. Quezada, and T.H. Seligman. Transition from localized to extended eigenstates in the ensemble of power-law random banded matrices. *Phys. Rev. E*, 54:3221–3230, Oct 1996.

[27] M. Moshe, H. Neuberger, and B. Shapiro. Generalized ensemble of random matrices. *Phys. Rev. Lett.*, 73:1497–1500, Sep 1994.

[28] C.M. Canali. Model for a random-matrix description of the energy-level statistics of disordered systems at the anderson transition. *Phys. Rev. B*, 53:3713–3730, Feb 1996.

[29] J.T. Chalker, and V.E. Kravtsov, and I.V. Lerner. Spectral rigidity and eigenfunction correlations at the Anderson transition. *Soviet Journal of Experimental and Theoretical Physics Letters*, 64(5):386–392, Sep 1996.

[30] J.T. Chalker, I.V. Lerner, and R.A. Smith. Random walks through the ensemble: Linking spectral statistics with wave-function correlations in disordered metals. *Physical Review Letters*, 77(3):554–557, Jul 1996.

[31] C. Blecken, Y. Chen, and K.A. Muttalib. Transitions in spectral statistics. *Journal of Physics A: Mathematical and General*, 27(16):L563–L568, Aug 1994.

[32] T. Okuda. Derivation of Calabi-Yau crystals from Chern-Simons gauge theory. *JHEP*, 03:047, 2005.

[33] M. Marino. Chern-Simons theory, matrix integrals, and perturbative three manifold invariants. *Commun. Math. Phys.*, 253:25–49, 2004.

[34] Y. Dolivet and M. Tierz. Chern-Simons matrix models and Stieltjes-Wigert polynomials. *J. Math. Phys.*, 48:023507, 2007.

[35] E. Witten. Quantum field theory and the jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.

[36] R. Gopakumar and C. Vafa. On the gauge theory / geometry correspondence. *Adv. Theor. Math. Phys.*, 3:1415–1443, 1999.
[37] M. Aganagic, A. Klemm, and C. Vafa. Disk instantons, mirror symmetry and the duality web. Z. Naturforsch. A, 57:1–28, 2002.

[38] M. Aganagic, A. Klemm, M. Marino, and C. Vafa. Matrix model as a mirror of Chern-Simons theory. JHEP, 02:010, 2004.

[39] H. Ooguri and C. Vafa. Knot invariants and topological strings. Nucl. Phys., B577:419–438, 2000.

[40] H. Ooguri and C. Vafa. World sheet derivation of a large N duality. Nucl. Phys. B, 641:3–34, 2002.

[41] C. Blecken and K.A. Muttalib. Brownian motion model of a q-deformed random matrix ensemble. Journal of Physics A, 31:2123–2132, 1998.

[42] K.A. Muttalib and M.E.H. Ismail. Impact of localization on dyson’s circular ensemble. Journal of Physics A: Mathematical and General, 28(21):L541–L548, Nov 1995.

[43] D. García-García and M. Tierz. Matrix models for classical groups and toepitz ± hankel minors with applications to chern–simons theory and fermionic models. Journal of Physics A: Mathematical and Theoretical, 53(34):345201, aug 2020.

[44] E. Onofri. Su(n) lattice gauge theory with villain’s action. Il Nuovo Cimento A, 66, 12 1981.

[45] D. García-García and M. Tierz. Toeplitz minors and specializations of skew schur polynomials. Journal of Combinatorial Theory, Series A, 172:105201, 2020.

[46] D. Bump and P. Diaconis. Toeplitz minors. Journal of Combinatorial Theory, Series A, 97(2):252 – 271, 2002.

[47] I.G. Macdonald. Symmetric Functions and Hall Polynomials. Oxford classic texts in the physical sciences. Clarendon Press, 1998.

[48] W. Buijsman, V.V. Cheianov, and V. Gritsev. Sensitivity of the spectral form factor to short-range level statistics. Physical review. E, 102 4-1:042216, 2020.

[49] E. Bogomolny, O. Bohigas, and M.P. Pato. Distribution of eigenvalues of certain matrix ensembles. Phys. Rev. E, 55:6707–6718, Jun 1997.

[50] B. Fiol and F. Torrents. Exact results for Wilson loops in arbitrary representations. JHEP, 1:1029–8479, 2014.