Can Lévy noise induce coherence and stochastic resonances in a birhythmic van der Pol system?

Raoul Mbakob Yonkeu\textsuperscript{1}, René Yamapi\textsuperscript{2,a}, Giovanni Filatrella\textsuperscript{3}, and Jürgen Kurths\textsuperscript{4,5}

\textsuperscript{1} Fundamental Physics Laboratory, University of Bamenda, Faculty of Sciences, Department of Physics, P.O. Box 39 Bambili, NWR, Cameroon
\textsuperscript{2} Fundamental Physics Laboratory, Physics of Complex System Group, Department of Physics, Faculty of Science, University of Douala, Box 24 157 Douala, Cameroon
\textsuperscript{3} INFN Gruppo Collegato Salerno and Department of Sciences and Technologies University of Sannio, Via F. De Sanctis, 82100 Benevento, Italy
\textsuperscript{4} Potsdam Institute for Climate Impact Research (PIK), 14 473 Potsdam, Germany
\textsuperscript{5} Department of Physics, Humboldt University, 12 489 Berlin, Germany

Abstract. The analysis of a birhythmic modified van der Pol type oscillator driven by periodic excitation and Lévy noise shows the possible occurrence of coherence resonance and stochastic resonance. The frequency of the harmonic excitation in the neighborhood of one of the limit cycles influences the coherence of the dynamics on the time scale of the oscillations. The autocorrelation function, the power spectral density and the signal-to-noise-ratio used in this analysis are shown to be maximized for an appropriate choice of the noise intensity. In particular, a proper adjustment of the Lévy noise intensity enhances the output power spectrum of the system, that is, promotes stochastic resonance. Thus, the resonance, as examined using standard measures, seems to occur also in the presence of nonstandard noise. The initial selection of the attractor seems to have an influence on the coherence, while the skewness parameter of the Lévy noise has not a notable impact on the resonant effect.

1 Introduction

Stochastic dynamical systems arise as mathematical models for complex phenomena in almost all scientific areas, for the presence of noise, due to internal or external fluctuations, is inevitable in real world systems. Special phenomena such as coherence or stochastic resonance (CR or SR) \cite{1-3}, noise-induced transitions \cite{4,5}, noise enhanced stability \cite{6,7}, and stochastic bifurcations \cite{8-12} have attracted a great deal of attention in various fields including physics, chemistry, biology, and engineering. SR, a term coined by Benzi et al. \cite{13} to explain the cyclical variation of the warm and the cold climate in paleo-climatology, has been broadly applied to describe a counter-intuitive phenomenon where the presence of a suitable amount of noise can optimize the output signal quality in a nonlinear system \cite{14,15}.

Noise can be grouped into two main broad categories: Gaussian noise (GN) and non-Gaussian noise. GN is an ideal model to describe random fluctuations with finite variance; however, non-GN, infinite variance fluctuations are also observed in various areas, such as biology, seismology, electrical engineer, and finance \cite{16}. Lévy noise (LN) is a class of stable non-GN that exhibits long heavy tails of its distribution, for infinite variance distribution are characterized by large, potentially infinite, jumps. However, LN is encountered in nature if long jumps are associated with a complex structure of the environment \cite{17}. Lévy stable distributions are a rich class of probability distributions and have many intriguing mathematical properties.

The archetypal phenomena of noise-induced ordering are robust and can be detected also in systems driven by Markovian, non-Gaussian, heavy-tailed fluctuations with infinite variance \cite{18}, and has been investigated in diverse systems as Josephson junction \cite{19-21} and in biophysics to help detect faint signals \cite{22-24}. For instance a LN model more accurately describes how the neuron’s membrane potential evolves than a simpler diffusion model because the more general Lévy model includes not only pure-diffusion and pure-jump models but jump-diffusion models as well \cite{25}. Stochastic Lévy processes describe transport processes with anomalous diffusion, as characterized by an anomalous mean squared displacement (\(\Delta x\) represents the independent increment), i.e.,

\[
\langle (x(t) - \langle x(t) \rangle)^2 \rangle = \langle (\Delta x)^2 \rangle \propto t^\alpha,
\]  \(1\)
with $\kappa = 3 - \gamma \neq 1.0$, at variance with the linear dependence $\langle (\Delta x)^2 \rangle \propto t$ that characterizes Gaussian diffusion, is connected with the LN index $\gamma$. Therefore, $\kappa$ determines the anomalous diffusion exponent that specifies the process, either sub-diffusive (with $0 < \kappa < 1$), super-diffusive (1 < $\kappa < 2$), whereas the ordinary Gaussian case corresponds to the ballistic case ($\kappa = 2$) [26,27]. Hence, LN can be considered as a standard stochastic process that describes in the simplest fashion the effect of fast surroundings [28].

LN driven birhythmic systems are known to manifest interesting physical properties. The response of birhythmic systems to noise is peculiar, in that the system possesses two stable attractors, each characterized by two frequencies. The presence of two stable attractors make the birhythmic van der Pol similar to standard bistable systems, albeit it possesses only a quasi-or pseudo-potential [29]. Recently, it has been shown that LN can induce transitions between attractors and enhanced stability in a birhythmic van der Pol system [30]. The theory of escape time from one attractor to another quantifies the overall stability of the attractors of this system, also in the presence of LN. Numerical simulations have demonstrated that in the presence of LN, the induced escapes from an attractor to another are similar to the escapes between stable points in an ordinary potential. Comparing the escapes under the influence of LN with the Gaussian case, it is evident that the differences are more pronounced for large values of the Lévy index $\gamma$. In this work we consider both the effects of periodic signal and Lévy noise in the birhythmic self-sustained system. The present aim is to check if the Lévy noise can induce some forms of coherence, using the tools employed to quantify CR [1] and SR [13]. The occurrence of SR phenomenon implies that these measures will exhibit a well-marked maximum at a particular noise level [31]. The CR and SR can be determined by using various measures, including residence time distributions, Power Spectrum Density (PSD), Signal-to-Noise Ratio (SNR), spectral power amplification, input/output cross-correlation measures, and probability of detection [32]. In this paper we will use some measures to highlight different effects of LN on birhythmic systems.

The paper is organized as follows. Section 2 describes the birhythmic van der Pol system driven by LN and periodic signal, and the algorithm of the numerical simulations. This section start by recalling the birhythmic properties on the free birhythmic van der Pol system, and the parameter region where birhythmicity appears; the Lévy process and the algorithm of numerical simulations are also presented. Section 3 deals with the diagnostic tools to detect coherence and stochastic resonances between noise and deterministic oscillations in the stochastic birhythmic van der Pol system. The purpose is to formalize the main tools employed in this work. Section 4 studies coherence and stochastic resonances-like phenomena in the birhythmic van der Pol system. By means of the numerical tools it is possible to quantify the degree of coherence (or anti-coherence), as described in Section 4. In the same section, is also discussed the occurrence of SR for some sets of parameters, which depend highly on the noise intensity. The last section presents the conclusions.

2 The driven birhythmic van der Pol system with Lévy noise modeling

This section is devoted to the model description and to summarize the main physical properties of the birhythmic van der Pol type oscillator, Section 2.1, and the numerical method, Section 2.2.

2.1 The driven birhythmic van der Pol system with Lévy noise

The system under study is a model which can be used to describe, for instance, a biological enzyme-substrate system with a ferroelectricity behavior in brain wave [11]. Assuming that the influence of external factors on the enzymes amounts to a combination of a random term (LN type) and a deterministic periodic excitation, the stochastic evolution can be described by an equation for the activated enzyme molecules, akin to the physical description used by Fröhlich [33,34]. Thus, a model is the following stochastic driven well-known variation of the van der Pol equation [35–37]:

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6)x + x = \Gamma(t) + \eta(t),$$

(2)

in the enzymatic-substrate reactions interpretation of equation (2) $x$ is proportional to the population of enzyme molecules in the excited polar state, $\Gamma(t) = E\sin \omega t$ is the periodic drive, $E$ and $\omega$ are the amplitude and frequency, respectively. The random term $\eta(t)$ denotes the LN generalized Wiener process, obeying the Lévy distribution $L_{\gamma, \sigma}(\eta, \sigma, \mu_1)$. In fact, $\eta(t)$ is the time derivative of the Lévy process $\zeta$. The overdot denotes the derivative with respect to time, $\alpha$ and $\beta$ are positive parameters which indicate the system behavior to a ferroelectric instability compared with its electrical resistance and $\mu$ is a non linear parameter related to nonlinear damping [38]. The autonomous version of equation (2) without LN and periodic force:

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6)x + x = 0$$

(3)

describes a multi-limit cycle oscillator [35] that exhibits self-sustained oscillations with more than one limit cycle, which is the condition for birhythmicity. Following reference [38], the periodic solution of equation (3) can be approximated by

$$x(t) = A \cos(\Omega t),$$

(4)

where $A$ represents the amplitude and $\Omega$ the frequency of the autonomous oscillations. The amplitude equation is as follows [38]:

$$\frac{5\beta}{64} \frac{A^6}{4} - \frac{\alpha}{8} A^4 + \frac{1}{4} A^2 - 1 = 0,$$

(5)
which determines a codimension two saddle-node bifurcations. The frequency of the oscillations $\Omega$ is approximately given by:

$$\Omega = 1 + \mu^2 \Omega_2 + o(\mu^3)$$

where

$$\Omega_2 = \frac{93\beta^2}{65536} A^{12} - \frac{69\alpha \beta}{16384} A^{10} + \left( \frac{67\beta}{8192} + \frac{3\alpha^2}{1024} \right) A^8 - \left( \frac{73\beta}{2048} + \frac{\alpha}{96} \right) A^6 + \left( \frac{1}{128} + \frac{\alpha}{24} \right) A^4 - \frac{3}{64} A^2.$$ 

Figure 1 describes the bifurcation lines that enclose the region of birhythmicity in the two parameters domain $(\alpha, \beta)$. In the gray region are observed three limit cycles (and hence the system is birhythmic for the presence of two stable orbits), while one limit cycle appears in the white region [9,39]

The orbits’ stability can be characterized by means of a pseudo-potential function $U(A)$ [10]. In the sole presence of a harmonic drive, the pseudo-potential can be approximated as follows:

$$-U(A) = \frac{\mu}{128} \left[ 32A^2 - 4A^4 + \frac{4}{3} \alpha A^6 - \frac{5}{8} \beta A^8 \right] - \frac{\mu E}{2\omega} A.$$  

(7)

The pseudo-potential (or quasi-potential) is shown in Figure 2 for parameters $\alpha$ and $\beta$ such that it appears almost symmetrical for $E = 0.0$ (Fig. 2a) or $E = 0.2$ (Fig. 2b) as a function of the amplitude for several values of the frequency of the external drive. It appears that the effect of periodic force is still visible also for $\omega = 1.5$ and for $E = 0.2$; for $\omega > 1.0$, its effect is no longer visible. The periodic force destroys the symmetry of the potential.

In the interpretation of the pseudo-potential as a bona fide potential, the system appears to be alike to an ordinary bistable system, which offers the possibility for a particle to periodically roll from one potential well to the other.

In the birhythmic region four cases have been selected, as reported in Table 1, in which [39]: (i) the potential is asymmetric and the frequencies are almost identical ($S_1$),

(ii) the potential is asymmetric but the frequencies are different ($S_2$), (iii) and (iv) the potential is symmetric and the frequencies are almost identical ($S_3$ and $S_4$).

### 2.2 The Lévy noise process and the algorithm for numerical simulations

Lévy distributions are a class of probability distributions with many intriguing mathematical properties [40]. Lévy noise, that is noise which follows the Lévy distribution, is characterized by four parameters [41,42]: Stability index $\gamma$ ($0 < \gamma \leq 2$), skewness parameter $b$ ($-1 \leq b \leq 1$), mean parameter $\mu_1$ ($\mu_1 \in \mathbb{R}$) and scale parameter $\sigma$ ($\sigma > 0$) [43]. If $\zeta$ obeys to Lévy distribution $L_{\gamma,b}(\zeta,\sigma,\mu_1)$, the characteristic function reads [31]:

$$\Phi(k) = \int_{-\infty}^{+\infty} d\eta e^{ik\eta} L_{\gamma,b}(\zeta,\sigma,\mu_1).$$

(8)

Therefore

$$\Phi(k) = \exp \left[ i\mu_1 k - \sigma \gamma \left( 1 - ib \text{sgn}(k) \tan \left( \frac{\pi \gamma}{2} \right) \right) \right]$$

if $\gamma e(0;1) \cup (1;2]$,  

(9a)
of the random electric field. The parameter $LN$ has the physical meaning of a measure of the intensity exists. The parameter $D$ value of the distribution, if the mean of the distribution the center, or location, parameter $\sigma$ denotes the mean value of the distribution, if the mean of the distribution exists. The parameter $D = \sigma \gamma$ is the noise intensity [44], in LN has the physical meaning of a measure of the intensity of the random electric field. The parameter $\gamma$ embodies GN, for $\gamma = 2$, and non-GN for $0 < \gamma < 2$. Thus, varying $\gamma$ the model covers thermal fluctuations and non-thermal fluctuations. The prominent characteristic feature of the distributions $L_{\gamma,b}(\zeta, \sigma, \mu_1)$ is the presence of moments up to the order $\gamma$, i.e. the integral:

$$\int_{-\infty}^{+\infty} L_{\gamma,b}(\zeta, \sigma, \mu_1) \zeta^\gamma d\zeta$$

is finite. The only stable distribution with a second moment is the Gaussian ($\gamma = 2$); for all the other values of $\gamma$ the variance of a stable distribution diverges.

In this paper, we use the Janicki-Weron algorithm [31] to generate the Lévy distribution:

For $\gamma \neq 1$, the Lévy distribution is simulated as:

$$\zeta = D_{\gamma,b,\sigma} \frac{\sin [\gamma(r + C_{\gamma,b})]}{[\cos(r)]^{\frac{\gamma}{\sigma}}} \left[ \cos(r - \gamma (r + C_{\gamma,b})) \right]^{\frac{1-\gamma}{2}} \mu_1,$$

(10)

with $C_{\gamma,b} = \frac{\arctan[b \tan (\frac{\pi}{2})]}{\gamma}$ and $D_{\gamma,b,\sigma} = \sigma \left( \cos(\arctan[b \tan (\frac{\pi}{2})]) \right)^{\frac{1}{\gamma}}$.

For $\gamma = 1$, it is simulated as:

$$\zeta = \frac{2\sigma}{\pi} \left( \frac{\pi}{2} + br \right) \tan(r) - b \ln \left( \frac{\pi}{2} + br \right) + \mu_1.$$  

(11)

Here $r$ and $w$ are independent random variables, with $r$ uniformly distributed in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $w$ is standard exponential distribution. The parameters are chosen fixing $\mu_1 = 0$. In Figure 3, the Lévy density function $L_{\gamma,b}(\zeta, \sigma, \mu_1)$ under $\gamma$ and $b$ are simulated by using Janicki-Weron algorithm [31] with the parameters $\mu_1 = 0$ and $\sigma = 1$. From these pictures, it can be seen that the Lévy distribution becomes symmetric when $b = 0$ and reduces to the Gaussian distribution for $\gamma = 2$. Considering the case where $\gamma < 1$, the Lévy function $L_{\gamma,b}(\zeta, \sigma, \mu_1)$ is left-skewed for $b < 0$ and right-skewed for $b > 0$; the inverse is true when $\gamma > 1$. The noise intensity is generally smaller than the periodic signal, which plays a leading role. Simulations are performed with a time step $\Delta t = 0.001$, the initial values $x(0)$ and $\dot{x}(0)$ are taken on one stable limit cycle.

3 Tools to quantify coherence and stochastic resonances

In this section we describe some measures of the coherence of the oscillator, to quantify the effect of noise and drive on the birhythmic oscillator. Coherence resonance occurs if noise is large enough to cause excursions, or evasions, from the original orbit but the system regularity is enhanced. In the following subsections some measures of the degree of coherence are described.

3.1 Auto-correlation function analysis

The Auto Correlation Function (ACF) characterizes the long time-scale behavior in which transitions can occur from an attractor to the other. We are particularly interested to the analysis of effects of the LN index $\gamma$ on the...
ACF as characterized by the decay rate \( \text{Cor} \) [45], defined as follows:

\[
\text{Cor}(\varrho) = \frac{\langle \tilde{A}(t)\tilde{A}(t-\varrho) \rangle}{\langle \tilde{A}^2 \rangle},
\]

where \( \tilde{A} = A - \langle A \rangle \), \( \varrho \) is a normalized time. For \( \varrho = 0 \) the ACF reads \( \text{Cor}(0) = 1 \). Noise induces decoherence. Therefore for large \( \varrho \) values, one expects a decrease of \( \text{Cor}(\varrho) \) as an effect of the LN. The average value is obtained as follows:

\[
\langle \tilde{A}^2 \rangle = \frac{1}{T_M} \int_0^{T_M} \tilde{A}^2 dt.
\]

The time \( T_M \gg \varrho \) is taken equal to \( 10^{20} \).

To summarize the induced coherence, it is convenient to define the autocorrelation time \( \tau \):

\[
\tau = \frac{1}{100T_M} \int_0^{100T_M} \text{Cor}^2(\varrho) d\varrho.
\]

Informally, this time characterizes the similarity between any two points in the dynamics as a function of time lag between them. In other words, time \( \tau \) summarizes the correlation between points separated by various time lag. Autocorrelation is a mathematical representation of the degree of similarity between a given time series and the lagged versions. Therefore, if there exists an optimum value of noise which enhances the coherence of the system response, it can be detected through the autocorrelation time.

3.2 Power spectral density analysis

The Power spectral density is useful in the case of a short time scale, that is, in the weak noise regime, when the random disturbance is weak enough to forbid jumps from a meta-stable state to the other in the observed time. The power spectrum is retrieved through the numerical Fourier transform:

\[
H(\omega) = \int x(t)e^{i\omega t} dt.
\]

The power \( P(\omega) \) can in turn be computed as

\[
P(\omega) = |H(\omega)|^2 + |H(-\omega)|^2.
\]

The discrete noise power spectrum is numerically estimated using a Fast Fourier Transform (FFT) algorithm with \( n = 2^{12} \) sampled data; the average power spectrum is obtained with 25000 realizations to achieve a better accuracy.

SNR can be conveniently expressed in decibels:

\[
\text{SNR} = 10\log_{10} \left( \frac{S}{N} \right),
\]

where \( S \) and \( N \) correspond to the average power spectra (16) with an applied deterministic drive (“Signal”, if \( x(t) \) is retrieved from Eq. (2) with \( E \neq 0 \)), and the purely noisy system (“Noise”, if \( x(t) \) is retrieved from Eq. (2) with \( E = 0 \)), respectively. The signal \( S \) and the noise \( N \) values are choose around a peak in the spectrum at a frequency \( \omega \). The average power spectrum of the signal \( S \) is defined as follow:

\[
S = \frac{1}{2\Delta\Omega} \int_{\Omega-\Delta\Omega}^{\Omega+\Delta\Omega} P(\omega)d\omega,
\]

where \( \Delta\Omega = 0.01 \), denotes a convenient interval of frequencies around the central frequency \( \Omega \). The numerical definition (17) has to be adapted to the discrete nature of the numerical FFT to include in the noise signal \( S \) the necessary bins. Moreover, to obtain reliable results, the system must explore the phase space; in particular if the system is perturbed with very weak noise, one
should ensure that the results do not depend upon the observation time or the initial conditions.

The estimate (17) of the SNR from the power spectral density measures the response of the system to a combination of deterministic drive and noise. The structure is very rich, for the self oscillatory system is influenced by the attractors frequencies, the drive period, and the average escape time. The drive induces a sharp peak in the PSD located at the forcing frequency, which emerges from the noise background. The frequency of the peaks changes with the LN index $\gamma$; moreover, for a given signal, the frequency corresponding the maximum of $|H(\omega)|^2$ is little affected by the behavior of the noise intensity $D$. In the numerical solution of equation (2), because of the heavy tails, discontinuity and irregular jumps of the LN may cause the numerically generated sample paths to diverge decreasing $\gamma$. To circumvent this problem, some methods have been used, for example, Zhang and Song [46] impose a constraint on the value of the solution $x(t)$.

### 4 The coherence resonance

In this section, we investigate the possible appearance of CR, using the auto-correlation function and the power spectral density tools, described in the previous Section 3. Figure 4 shows the auto-correlation function $\text{Cor} \rho$ equation (12), versus the correlation time $t$. The figures show the Lévy noise index $\gamma$ (for increasing values from $\gamma = 0.1$ to $\gamma = 1.5$) at a fixed value of the noise intensity.
Fig. 5. (a1, a2) The autocorrelation time $\tau$ versus $D$. (b1, b2) Variation of the autocorrelation function versus $D$ for both transitions with $\rho = 1.5$. The other parameters are: $S_1$ for (a1, b1) and $S_2$ for (a2, b2) (refer to Tab. 1), $\mu = 0.01$, and $b = 0.0$. $D = 0.1$. The behavior of Cor($\rho$) together with the Gaussian case $\gamma = 2$ are insensitive to the initial conditions, for the transitions starting from the inner ($A_1$) or the outer ($A_3$) attractors are very similar.

Figures 5a1 and 5a2 show the dependence of the autocorrelation time $\tau$ versus $D$, while in Figures 5b1 and 5b2 it is displayed the dependence of Cor versus $D$ for several different values of $\gamma$. The data are collected for both transitions, $A_1 \Rightarrow A_3$ or $A_3 \Rightarrow A_1$. The statistical average is independent of the initial conditions. One observes for both transitions that, $D$ causes a steady-state response and increases the autocorrelation time (for $\gamma = 0.5$). As expected, in both transitions the autocorrelation time $\tau$, defined in equation (15), increases with the noise intensity $D$, while the ACF Cor decreases. For both transitions, the behavior is qualitatively the same, but quantitatively changes if the control parameters are changed, see for example Figures 5b1 and 5b2.

The PSDs of the van der Pol system (2) for few values of the noise intensity $D$ are displayed in Figure 6. The system is initialized on the inner attractor, and therefore the power spectrum refers to fluctuations around the orbits $A_1$. It is evident that there is a qualitative effect of the noise on the peaks of the spectrum. In Figure 7 the PSD is evaluated for the other attractor, $A_3$. It is evident that the behavior of PSD strongly depends also on the Lévy distribution index $\gamma$. Increasing the Lévy index $\gamma$ from 0.15, one observes a small peak around a frequency $\Omega = 0.14$, while the main peak stays around the frequency $\Omega = 0.2$. This main peak increases if the index $\gamma$ is raised up to $\gamma = 0.45$, while the lower peak disappears. The amplitude of one of the harmonic first increases to reach the maximum and then decreases. The peak of the PSD at the modulation frequencies is not suppressed as $\gamma$ of the Lévy noise spectrum becomes large enough and the signal amplitude diminishes for various values of $\gamma$. In all figures, the PSD exhibits a sharp peak at the frequency $\omega$, which indicates that there is a stronger signal at frequency $\omega = 0.2$. The value of this peak is significant to indicate the possibility of the occurrence of SR, implying that noise can enhance the output power spectrum of the system at some special frequency, and thus resembles stochastic resonance, that is an optimal output with a proper adjustment of the noise amplitude $D$. For the attractor $A_3$ (see for example Fig. 7), the peaks appear in general less evident. In the two figures, it seems that the behavior of PSD depends on the central frequency, on the nearby attractor, and the portion of the PSD under consideration. Figure 8 shows the effects of the LN parameter $\gamma$ on the PSD in the presence non-GN and periodic external forcing, for the oscillations around $A_3$.

The Figures demonstrate that SR in this system has certain connections with the nonconventional SR, although only a very rough match between the driving frequency and the noise-tuned frequency peak has been observed. The nonconventional SR in this underdamped bistable van der Pol oscillator [10] should reflect the complexity of the quasi- or pseudo-potential. This complexity has
been reported from the viewpoint of the dynamical states of the moving particle subject to random disturbance [9,10,39,47], as will be examined in the next section.

5 Stochastic resonance detection

In this section the SNR given by equations (16) and (17) is employed to quantify the occurrence of SR in the birhythmic system (2). The occurrence of SR may be detected through the presence of a peculiar nonlinear behavior of the response of bistable systems exposed to increasing intensity of the noise level. In fact, at low noise level, the particle oscillates at the bottom of one of the potential wells for a long time, and rarely switches to the other potential wells. However, if the noise intensity is increased, the frequency of the switches grows and can become coherent with some natural period of the system. As noise is further increased, the switches from an attractor to another become purely random, and thus the coherence is eventually lost.

5.1 Stochastic resonance under Gaussian noise

In this section, a numerical method of moment based on non perturbation expansion is presented to observe SR under Gaussian noise (that is equivalent to Lévy noise with \( \gamma = 2.0 \)). Therefore, our aim here is to make a comparison of the two numerical methods to make sure that the results are consistent with reference [11]. The results summarized in Table 2 confirm that the simulations of GN and of LN for \( \gamma = 2 \) hold the same results.

5.2 Stochastic resonance under Lévy noise

Stochastic resonance induced by LN (i.e., \( \gamma < 2 \)) in the system (2) is investigated. The role of the Lévy index \( \gamma \) on SR is highlighted through the analysis of the PSD, to retrieve the SNR per equation (17), as illustrated in Section 2.2. The external drive frequency \( \omega \) is fixed to focus on the role of the orbits’ period and of the symmetry of the pseudo-potential. The maximum in the SNR, detected adjusting the noise intensity, is the signature of the presence of SR. Figure 9 shows the effects of the control parameters \( \alpha \) and \( \beta \) on the SNR dependence of the noise intensity. It is found that stochastic resonance does not appear with the control parameters: \((\alpha, \beta) = (0.0675, 0.0009)\), curve (c) in Figure 9.

The SNR for different values of the control parameters \( \alpha \) and \( \beta \) and of the noise index \( \gamma \) is computed in Figure 10, that shows the SNR versus \( D \) for the asymmetric \((S_1)\) and symmetric \((S_3)\) system (the initial conditions are selected on the outer attractor \( A_3 \)). For few values of \( \gamma \), from 0.4 to 1.92, the SNR increases with \( D \), and no maximum is observed. One concludes that presumably SR does not occur in this case.

In Figures 11–13, for initial conditions chosen on the inner \( A_1 \) attractor, it is shown the SNR versus \( D \) for
Fig. 7. Effect of $\gamma$ on the variation of PSD in the presence of LN excitation, periodic external force for four values of $\gamma$ ((a) $\gamma = 0.15$, (b) $\gamma = 0.45$, (c) $\gamma = 0.8$, (d) $\gamma = 1.0$, (e) $\gamma = 1.5$, (f) $\gamma = 2.0$). The parameters of the system are $\alpha = 0.114$, $\beta = 0.003$, $E = 1/2$, $D = 0.01$, $\mu = 0.01$, $\omega = \sqrt{5}/3$. The initial conditions are chosen on the outer orbit of radius $A_3$.

Table 2. Comparison between the results obtained for the birhythmic system under GN and LN with $\gamma = 2.0$. The results show that the numerical method employed to generate LN is consistent with a straightforward generation of Gaussian noise.

| $S_i = (\alpha, \beta)$ | $D_{SR}$ for GN | $SNR_{\text{max}}$ for GN | $D_{SR}$ under LN | $SNR_{\text{max}}$ under LN |
|------------------------|----------------|---------------------------|-----------------|---------------------------|
| (0.12; 0.0032)         | 0.4138         | 2.8752                    | 0.4063          | 2.8747                    |
| (0.1476; 0.0053)       | 4.2347         | 2.6891                    | 4.2748          | 2.6808                    |
| (0.0675; 0.0009)       | Nothing        | Nothing                   | Nothing         | Nothing                   |
| (0.1547; 0.006)        | 4.6874         | 2.7799                    | 4.6881          | 2.7802                    |
| (0.145; 0.005)         | 2.3043         | 2.8260                    | 2.3202          | 2.7876                    |

different values of the skewness parameter $b$ (Fig. 11) and the Lévy index $\gamma$ (Figs. 12 and 13). The skewness parameter $b$ has a marginal effect, see Figure 11. In the same figure it is evident that as the noise intensity $D$ increases, the SNR value also increases up to a maximum; such a noise intensity value is named $D_{SR}$. A further increase in the noise intensity $D$ leads to a decline of the SNR, that is precisely the signature of the presence of SR. Figures 12

Fig. 8. Effects of the LN parameter $\gamma$ on the PSD in the presence non-Gaussian noise and periodic external forcing for $\gamma = 0.45$ (a), $\gamma = 0.8$ (b), $\gamma = 1.3$ (c), and $\gamma = 1.9$ (d). The initial conditions are chosen on the attractor $A_3$ and the control parameters in the birhythmic region: $\alpha = 0.114$, $\beta = 0.003$, $E = 1/2$, $D = 0.01$, $\mu = 0.01$, $\omega = \sqrt{5}/3$.

Fig. 9. The SNR versus noise intensity $D$ for few control parameters. The data correspond to the dynamical states for initial conditions chosen on the inner attractor $A_1$ and the control parameters in the birhythmic region: (a) $\alpha = 0.12$, $\beta = 0.0032$; (b) $\alpha = 0.1476$, $\beta = 0.0053$; (c) $\alpha = 0.0675$, $\beta = 0.0009$; (d) $\alpha = 0.1547$, $\beta = 0.006$; (e) $\alpha = 0.145$, $\beta = 0.005$. All data refer to $\mu = 0.001$, $E = 0.1$, $\omega = 0.2$.

and 13 display the effects of noise intensity at different values of the index $\gamma$ for four sets of parameters of Table 1. A maximum of the SNR occurs for these values, and therefore SR is present in the system. In Figures 14 and 15 are displayed the noise intensity $D_{SR}$ at which a peak of SNR the noise occurs and the amplitude of the SNR at the peak, respectively, as a function of the Lévy noise index $\gamma$. The results of Figure 14 indicate that increasing the parameter $\gamma$ the peak occurs at higher noise intensity for an attractor ($S_1$), and at a lower noise intensity for the other ($S_4$). The analysis of SNR for different $\gamma$ in Figure 15 indicates that the SNR curves have the same trend as a function of the parameter $\gamma$. In general, thus, one can conclude that the peak value of SNR decreases and moves to larger noise intensity decreasing the index $\gamma$. As a general conclusion about the role of the parameter $\gamma$, we feel it appropriate to state that the data demonstrate a strong dependence of the SNR upon the Lévy index.

6 Conclusion

We have addressed the problem of the coherence and stochastic resonance in a non linear birhythmic, high order van der Pol system [8–12] driven by Lévy noise and a periodic signal. The system is bistable, in the sense that it can oscillate at two different frequencies. However, being a dissipative system, it only possesses a quasi- or pseudo-potential [29,35]. Preliminarily, numerical simulations of stochastic trajectories under the effect of LN, have demonstrated through ACF and SNR, the existence of CR and SR of the system. The influence of the Lévy index $\gamma$ on the features of SR, measured by inspection of SNR, in van der Pol birhythmic oscillator is influenced by the features of the double-well quasi-potential.
Fig. 10. Effects of $\gamma$ on the variation of SNR versus $D$ for the asymmetric ($S_1$) and symmetric ($S_3$) system, Table 1. These figures correspond to the dynamical states when the initial conditions are choose on $A_3$ and the frequencies of the two stable attractors are both $\simeq 1.0$. All data refer to $\mu = 0.01$, $E = 0.25$, $\omega = 0.5$. (a) corresponds to $(0.114, 0.003)$ and (b) corresponds to $(0.12, 0.0016)$.

Fig. 11. Effects of the skewness parameter $b$ and of the LN index $\gamma$ on the SNR versus $D$. The panels correspond to the dynamical states resulting from initial conditions on the inner attractor $A_1$. The frequencies of the two stable attractors are both $\simeq 1.0$. Data refer to (a) parameter $S_1$, (b) parameter $S_3$ (see Tab. 1). The other parameters read: $E = 0.25$, $\omega = 0.5$, $\gamma = 1.7$.

Fig. 12. Effects of $\gamma$ on the variation of SNR versus $D$ for the asymmetric system. These figures correspond to the dynamical states when the initial conditions are choose on $A_1$ and the frequencies of the two stable attractors are both $\simeq 1.0$. Data refer to (a) parameter $S_1$, (b) parameter $S_2$ (see Tab. 1). The other parameters read: $\mu = 0.01$, $E = 0.25$, $\omega = 0.5$. 
Fig. 13. Effects of $\gamma$ on the variation of SNR versus $D$ for the symmetric system. These figures correspond to the dynamical states when the initial conditions are choose on $A_1$ and the frequencies of the two stable attractors are both $\simeq 1.0$. Data refer to (a) parameter $S_1$, (b) parameter $S_2$ (see Tab. 1). The other parameters used are defined in Figure 12.

Fig. 14. Variation of $D_{SR}$ (the $D$ at which SNR is maximum) versus $\gamma$ for initial conditions choose on $A_1$ and for the set of parameters $S_1$ and $S_4$. Data refer to (a) parameter $S_1$, (b) parameters $S_4$ (see Tab. 1). The other parameters used are defined in Figure 12.

Fig. 15. Variation of $SNR_{\text{max}}$ (the maximum of SNR) versus $\gamma$ for initial conditions choose on $A_1$ and for the set of parameters $S_1$ and $S_4$ (see Tab. 1 for details). Data refer to (a) parameter $S_1$, (b) parameter $S_4$. The other parameters used are defined in Figure 12.
The behavior of the SR quantifiers is a clear indication that also Lévy noise, as ordinary Gaussian noise, can cooperate with the spontaneous self sustained cycles, as well as with the external periodic drive. When SR occurs, the index $\gamma$ systematically leads to an increase of the peak of the SNR; In these circumstances, it is possible to find a value of noise intensity, for which the SR phenomenon is observed. Generally, SR determines that noise and the periodic drive affect the van der Pol’s output power spectrum, that reaches a peak value at a certain noise intensity. The results indicate that the decrease of $\gamma$, that leads to larger fluctuations and heavier tails of the noise, causes the switches between the two potentials wells to become more irregular, and consequently weakens the SR phenomenon. However, for a fixed value of the Lévy noise index $\gamma$, the birhythmic van der Pol systems’ response is analogous to ordinary bistable systems.

By way of conclusions, the mechanism of SR seems robust enough to be at work in spite of the contemporary presence of two variations respect to the standard case: non Gaussian and birhythmicity. The findings offer new support to SR as a general phenomenon, that can be applied to such a diverse system.

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**Author contribution statement**

This paper was proposed by R. Yamapi and G. Filatrella. R. Mbakob Yonkeu is the main author of the paper, while J. Kurths was the supervisor of this work.

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