An Exactly Solvable Model of Generalized Spin Ladder

Sergio Albeverio\textsuperscript{*}, Shao-Ming Fei\textsuperscript{*} and Yupeng Wang\textsuperscript{†}

\textsuperscript{*}Institut für Angewandte Mathematik, Universität Bonn, D-53115 Bonn
Fakultät für Mathematik, Ruhr-Universität Bochum, D-4478 Bochum
\textsuperscript{†}Institut für Physik, Universität Augsburg, 86135 Augsburg
Laboratory of Ultra-Low Temperature Physics, Chinese Academy of Science, Beijing 100080

Abstract

A detailed study of an $S = \frac{1}{2}$ spin ladder model is given. The ladder consists of plaquettes formed by nearest neighbor rungs with all possible $SU(2)$-invariant interactions. For properly chosen coupling constants, the model is shown to be integrable in the sense that the quantum Yang-Baxter equation holds and one has an infinite number of conserved quantities. The $R$-matrix and $L$-operator associated with the model Hamiltonian are given in a limiting case. It is shown that after a simple transformation, the model can be solved via a Bethe ansatz. The phase diagram of the ground state is exactly derived using the Bethe ansatz equation.

PACS numbers: 75.10.Jm

\textsuperscript{1}SFB 256; SFB 237; BiBoS; CERFIM (Locarno); Acc.Arch., USI (Mendrisio)
\textsuperscript{2}Institute of Physics, Chinese Academy of Science, Beijing.
\textsuperscript{3}AvH fellow.
Heisenberg spin ladders and generalized spin ladders have attracted considerable attention in recent years, due to the developing experimental results on ladder materials and the hope to get some insight into the physics of metal-oxide superconductors [1]. Especially generalized ladders including other couplings beyond the simplest case of rung and leg exchange interpolate among a variety of systems and exhibit a remarkably rich behavior [2-9]. In particular, it has been shown that the diagonal interactions may cause frustration and change the structure of the ground state [2, 3], while the biquadratic interactions, which can arise due to effective spin-spin interaction mediated by phonons in real magnetic systems [4], tend to produce dimerization and may lead to a phase transition into a “non-Haldane” spin liquid state with absence of magnon excitations [4, 5].

As spin ladders are generally not equivalent to spin chains with nearest neighbor interactions, till now little is known about integrable spin ladder models. In this letter we study a generalized $S = \frac{1}{2}$ spin ladder system with both isotropic exchange interactions and biquadratic interactions. Using ideas related to the quantum Yang-Baxter equations [10] we found in our systems some cases of integrable ladder systems, in the sense of models having an infinite number of conserved quantities with explicit $R$ matrices satisfying the Yang-Baxter equation. Properly choosing the spectral parameter, we get a Hamiltonian consisting of only nearest-neighbor and next-nearest-neighbor interactions. This model can be solved via an ordinary Bethe ansatz.

---

4The present work was submitted for publication in Euro. Phys. Lett. During the preparation of a revised version of this paper, another integrable ladder model without diagonal interactions was presented in [11].
We consider a symmetric $16 \times 16$ matrix:

\[
\begin{pmatrix}
    a_1 & 9a_2 & 3a_2 & 3b_2 \\
    9a_2 & a_3 & 3b_2 & 3a_2 \\
    3a_2 & 3b_2 & a_5 & a_2 \\
    3b_2 & 3a_2 & a_2 & a_5
\end{pmatrix}
\]

where $a_1 = 2(-1 + 9x)$, $a_2 = -b_2 = (-1 + x)$, $a_3 = 7 + 9x$, $a_4 = 2(3 + 5x)$, $a_5 = -1 + 17x$ and $x \in \mathbb{C}$.

Let $V$ denote a 4-dimensional complex vector space. $\tilde{R}(x)$ takes values in $\text{End}_\mathbb{C}(V \otimes V)$ and satisfies the quantum Yang-Baxter equation [10]:

\[
\tilde{R}_{12}(x)\tilde{R}_{23}(xy)\tilde{R}_{12}(y) = \tilde{R}_{23}(y)\tilde{R}_{12}(xy)\tilde{R}_{23}(x),
\]

where $\tilde{R}_{ij}$ denotes the matrix on the complex vector space $V \otimes V \otimes V$, acting as $\tilde{R}(x)$ on the $i$-th and the $j$-th components and as the identity on the other components. Namely $\tilde{R}_{12} = \tilde{R} \otimes 1$, $\tilde{R}_{23} = 1 \otimes \tilde{R}$ and $1$ is the identity operator on $V$.

Let us set $\mathcal{H} = V_1 \otimes V_2 \otimes \ldots \otimes V_N$, $N \in \mathbb{N}$. The corresponding $L$-operator acting on the
$i$-th space $V_i$, $i = 1, 2, \ldots, N$, is a $4 \times 4$ matrix with entries

$$(L_i(x))_{11} = \frac{1}{2} (4(3x - 1)\sigma_{1,i}^0 \sigma_{2,i}^0 + c_1 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0)$$

\[+ 6b_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0) + 2(1 + 3x)\sigma_{1,i}^0 \sigma_{2,i}^0)$$

$$(L_i(x))_{12} = \frac{1}{2} (a_5 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0) + a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0))$$

$$(L_i(x))_{13} = \frac{1}{2} (a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0) + a_5 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0))$$

$$(L_i(x))_{14} = 2c_1 \sigma_{1,i}^0 \sigma_{2,i}^0,$$

$$(L_i(x))_{21} = \frac{1}{2} (a_3 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0) + 9a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0))$$

$$(L_i(x))_{22} = \frac{1}{2} ((15x - 7)\sigma_{1,i}^0 \sigma_{2,i}^0 + 4(1 + x) (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0)$$

\[+ 8a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0) - c_1 \sigma_{1,i}^0 \sigma_{2,i}^0)$$

$$(L_i(x))_{23} = \frac{1}{2} (3b_2 (1 + \sigma_{1,i}^0 \sigma_{2,i}^0) - 2a_4 \sigma_{1,i}^0 \sigma_{2,i}^0 + 4a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0)$$

$$(L_i(x))_{24} = \frac{1}{2} (9a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0) + a_3 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0))$$

$$(L_i(x))_{31} = \frac{1}{2} (9a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0) + a_3 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0))$$

$$(L_i(x))_{32} = \frac{1}{2} (3b_2 (1 + \sigma_{1,i}^0 \sigma_{2,i}^0) + 4a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0)$$

\[+ 8a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0) - c_1 \sigma_{1,i}^0 \sigma_{2,i}^0)$$

$$(L_i(x))_{33} = \frac{1}{2} ((15x - 7)\sigma_{1,i}^0 \sigma_{2,i}^0 + 4(1 + x) (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0)$$

\[+ 8a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0) - c_1 \sigma_{1,i}^0 \sigma_{2,i}^0)$$

$$(L_i(x))_{34} = \frac{1}{2} (a_3 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0) + 9a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0))$$

$$(L_i(x))_{41} = 2c_1 \sigma_{1,i}^0 \sigma_{2,i}^0,$$

$$(L_i(x))_{42} = \frac{1}{2} (a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0) + a_5 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0))$$

$$(L_i(x))_{43} = \frac{1}{2} (a_5 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0) + a_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 - \sigma_{1,i}^0 \sigma_{2,i}^0))$$

$$(L_i(x))_{44} = \frac{1}{2} (4(3x - 1)\sigma_{1,i}^0 \sigma_{2,i}^0 - c_1 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0)$$

\[+ 6b_2 (\sigma_{1,i}^0 \sigma_{2,i}^0 + \sigma_{1,i}^0 \sigma_{2,i}^0) + 2(1 + 3x)\sigma_{1,i}^0 \sigma_{2,i}^0)$$

where $c_1 = a_1/2$, $\sigma_{\theta,i}^\pm$, $\sigma_{\theta,i}^R$, (resp. $\sigma_{\theta,i}^0$, $\theta = 1, 2$, are Pauli matrices (resp. $2 \times 2$ identity matrix) acting on the space $V_i$. A direct calculation shows that $L_i(x)$ satisfies

$$\hat{R} \left( \frac{x}{y} \right) [L_i(x) \otimes L_i(y)] = [L_i(y) \otimes L_i(x)] \hat{R} \left( \frac{x}{y} \right), \quad i = 1, 2, \ldots, N,$$

(2)

where $x, y \in C$, $y \neq 0$, $\otimes$ is the tensor product of matrices. Let

$$T(x) = L_N(x) L_{N-1}(x) \ldots L_1(x).$$

(3)

We have the fundamental commutation relations given by:

$$\hat{R} \left( \frac{x}{y} \right) [T(x) \otimes T(y)] = [T(y) \otimes T(x)] \hat{R} \left( \frac{x}{y} \right).$$

(4)
Let \( t(x) = Tr_0 T(x) \), where \( Tr_0 \) takes trace on the space of \( 4 \times 4 \) matrix. According to (1) a system with Hamiltonian of the form \( H_{x_0} = J \frac{d}{dx} \log t(x)|_{x=x_0} \) for some \( x_0 \in \mathbb{C} \) (such that \( \log t(x) \) is defined and differentiable, e.g. \( x_0 = 1 \), see (3)) and real constant \( J \), acting on \( \mathcal{H} \), has an infinite number of conserved quantities \( t(x) \):

\[
[H_{x_0}, t(x)] = 0, \quad \forall x \in \mathbb{C},
\]

where \([,]\) stands for the commutator. A system with Hamiltonian \( H_{x_0} \) is then by definition an integrable system. For arbitrary value of \( x_0 \), \( H_{x_0} \) generally describes integrable models with long range interactions. These models can be exactly solved by using the algebraic Bethe Ansatz method.

Let \( P \) denote the permutation matrix on the ladder and set \( R(x) = P \bar{R}(x) \). We see that \( R(x)|_{x=1} \) is proportional to the permutation matrix \( P \). Therefore for \( x_0 = 1 \), the Hamiltonian \( H \equiv H_1 \) describes a system with nearest-neighbor interactions:

\[
H = \frac{d}{dx} \log t(x)|_{x=1} \equiv \sum_{i=1}^{N} h_{i,i+1}
\]

\[
= \sum_{i=1}^{N} [5(S_{1,i} \cdot S_{2,i} + S_{1,i+1} \cdot S_{2,i+1}) + 3(S_{1,i} \cdot S_{1,i+1} + S_{2,i} \cdot S_{2,i+1})] -3(S_{1,i} \cdot S_{2,i+1} + S_{2,i} \cdot S_{1,i+1}) -12(S_{2,i} \cdot S_{1,i+1})(S_{1,i} \cdot S_{2,i+1})
\]

\[
+20(S_{1,i} \cdot S_{2,i})(S_{1,i+1} \cdot S_{2,i+1}) + 20(S_{1,i} \cdot S_{2,i}) + 20(S_{1,i} \cdot S_{1,i+1})(S_{2,i} \cdot S_{2,i+1}) + \frac{57}{4} 1 \otimes 1],
\]

where the periodic condition is assumed: \( S_{\theta,N+1} = S_{\theta,1} \), \( \theta = 1, 2 \) and \( S_{\theta,i} = (\sigma^x_{\theta,i}, \sigma^y_{\theta,i}, \sigma^z_{\theta,i})/2 \) is the spin vector operator on \( V_i \). Taking \( S_{1,i} \) (resp. \( S_{2,i} \)) to be the spin operator on the first (resp. second) leg of the \( i \)-th rung of a ladder, and \( V_i \) to be the tensor space for the action of these two spin operators, the Hamiltonian \( (6) \) describes an integrable spin ladder system with periodic boundary conditions. The model is \( SU(2) \)-symmetric, i.e., \([H, S^l] = 0\), where \( S^l = \sum_{i=1}^{N} (S^l_{1,i} + S^l_{2,i}) \), \( l = x, y, z \) are the total spin operators of the ladder.

Remark: Like the well known Affleck-Kennedy-Lieb-Tasaki model [12], the model \( (6) \) derived from the transfer matrix has no free parameters. Nevertheless, as we shall show below, it can be generalized to a model with two free parameters, i.e., the coupling constant along rungs and the coupling constant of the rung-rung biquadratic interactions without losing the integrability. The present model and the ones discussed in the previous papers [5, 9] have same interaction terms in Hamiltonian but with different coupling constants. The models in [5,9] with more free parameters are generally not integrable.
To exactly solve the model (6) the analytic algebraic Bethe Ansatz method may be applied. The reference state with all the spins up is an eigenstate. Some degenerate eigenstates can be obtained by applying the operator $S^- = S^x - iS^y$. The combinations of the products of $(T(x))_{12}$ and $(T(x))_{13}$ can be used to construct “Bethe Ansatz states” with an arbitrary number of spins down. The relations of (4) would then give the Bethe Ansatz equations. Some exact ground states can also be constructed using the theorem in [13].

In the following we use a simpler method to solve the model. Our Hamiltonian (6) can be rewritten as (up to an irrelevant constant term and a constant factor):

$$H = \frac{1}{2} \sum_{i=1}^{N} \left[ \frac{1}{2} + 2S_{1,i} \cdot S_{1,i+1} \right] \left[ \frac{1}{2} + 2S_{2,i} \cdot S_{2,i+1} \right] - \frac{1}{2} \sum_{i=1}^{N} \left[ \frac{1}{2} + 2S_{1,i} \cdot S_{2,i+1} \right] \left[ \frac{1}{2} + 2S_{2,i} \cdot S_{1,i+1} \right] - \frac{1}{2} \sum_{i=1}^{N} \left[ \frac{1}{2} + 2S_{1,i} \cdot S_{2,i} \right] \left[ \frac{1}{2} + 2S_{2,i+1} \cdot S_{1,i+1} \right] + \frac{5}{6} \sum_{j=1}^{N} \left[ \frac{1}{2} + 2S_{1,i} \cdot S_{2,i} \right] \left[ \frac{1}{2} + 2S_{2,i+1} \cdot S_{1,i+1} \right].$$

(7)

We define the rung states as

$$|0_i> = \frac{1}{\sqrt{2}} (|\uparrow_{1,i}, \downarrow_{2,i}> - |\downarrow_{1,i}, \uparrow_{2,i}>), \quad |1_i> = |\uparrow_{1,i}, \uparrow_{2,i}>,$$

$$|2_i> = \frac{1}{\sqrt{2}} (|\uparrow_{1,i}, \downarrow_{2,i}> + |\downarrow_{1,i}, \uparrow_{2,i}>), \quad |3_i> = |\downarrow_{1,i}, \downarrow_{2,i}>,$$

and the Hubbard operators $X^{\alpha\beta}_i \equiv |\alpha_i><\beta_i|$, $\alpha, \beta = 0, 1, 2, 3$. Hamiltonian (7) can be rewritten as

$$H = \frac{1}{2} \sum_{i=1}^{N} \left[ P^{rr}_{i,i+1} - P^{dd}_{i,i+1} + 20 \frac{1}{3} \left( \frac{1}{2} - X^{00}_{i} \right) \left( \frac{1}{2} - X^{00}_{i+1} \right) \right],$$

(8)

where

$$P^{rr}_{i,i+1} = \left( \frac{1}{2} + 2S_{1,i} \cdot S_{1,i+1} \right) \left( \frac{1}{2} + 2S_{2,i} \cdot S_{2,i+1} \right), \quad P^{dd}_{i,i+1} = \left( \frac{1}{2} + 2S_{1,i} \cdot S_{2,i+1} \right) \left( \frac{1}{2} + 2S_{2,i} \cdot S_{1,i+1} \right).$$

It can easily be checked that

$$P^{rr}_{i,i+1}|\alpha_i> |\beta_{i+1}> = |\beta_i> |\alpha_{i+1}>,$$

$$P^{dd}_{i,i+1}|\alpha_i> |\beta_{i+1}> = (-1)^{\epsilon(\alpha) + \epsilon(\beta)} |\beta_i> |\alpha_{i+1}>,$$

where $\epsilon(\alpha)$ is the parity of the state $|\alpha>$, $\epsilon(0) = 0$ and $\epsilon(1) = \epsilon(2) = \epsilon(3) = 1$. The $P$ operators can be expressed as

$$P^{rr}_{i,i+1} = \sum_{\alpha,\beta=0}^{3} X^{\alpha\beta}_i X^{\beta\alpha}_{i+1}, \quad P^{dd}_{i,i+1} = \sum_{\alpha,\beta=0}^{3} (-1)^{\epsilon(\alpha) + \epsilon(\beta)} X^{\alpha\beta}_i X^{\beta\alpha}_{i+1}.$$
Therefore, we can rewrite the Hamiltonian (6) or (7) as
\[
H = \sum_{i=1}^{N} \sum_{\alpha=1}^{3} (X_i^{\alpha 0} X_{i+1}^{0 \alpha} + X_i^{0 \alpha} X_{i+1}^{\alpha 0}) + U \sum_{i=1}^{N} \left( \frac{1}{2} - X_i^{00} \right) (\frac{1}{2} - X_{i+1}^{00}) - J \sum_{i=1}^{N} X_i^{00},
\]
with \( U = \frac{10}{3} \), and \( J = 0 \). In fact, the model (9) is integrable for arbitrary real constants \( U \) and \( J \). This corresponds to the case of an arbitrary rung exchange coupling and an arbitrary rung-rung biquadratic coupling. We shall discuss the generalized integrable case rather than the special case (7) in the following text.

Obviously, \( X_i^{\alpha \alpha} \) represents the number of the local state \( |\alpha_i> \) on the rung \( i \) and satisfies the hard-core condition \( \sum_{\alpha=1}^{3} X_i^{\alpha \alpha} = 1 \). If we choose \( |0> = |0_1 \otimes \cdots \otimes |0_N> \) as the vacuum state, \( X_i^{\alpha 0} \) (\( \alpha = 1, 2, 3 \)) can be looked upon the creation operators of \( \alpha \)-particles, i.e., \( X_i^{\alpha 0} |0_i> = |\alpha_i> \). The particle numbers \( N_\alpha = \sum_{i=1}^{N} X_i^{\alpha \alpha} \) are conserved quantities. Notice that only three of them are independent since \( \sum_{\alpha=0}^{3} N_\alpha = N \). In this sense we construct the Bethe states \( |N_1, N_2, N_3> \). As shown in Eq.(9), there is no hybridization among the states \( |1>, |2>, \) and \( |3> \) because of the absence of \( X_i^{12}, X_i^{23}, X_i^{13} \) and their conjugates in Eq.(9). That means that the excitations from \( |N_1, N_2, N_3> \) to \( |N'_1, N'_2, N'_3> \) are dispersionless (i.e. have zero excitation energy) as long as \( N_1 + N_2 + N_3 = N'_1 + N'_2 + N'_3 \). In fact, the quantities
\[
Y_{21} = \sum_{i=1}^{N} X_i^{21}, \quad Y_{31} = \sum_{i=1}^{N} X_i^{31},
\]
commute with the Hamiltonian, which means that the Bethe states are highly degenerate.

The general eigenstates can be constructed from \( |N_e, 0, 0> \):
\[
|N_1, N_2, N_3> = Y_{21}^{N_2} Y_{31}^{N_3} |N_e, 0, 0>, \quad N_e = N_1 + N_2 + N_3.
\]

Therefore, we need only to consider the Bethe state \( |N_e, 0, 0> \equiv |N_e> \). This state reads:
\[
|N_e> = \sum_{n_1, \cdots, n_{N_e}} \Psi(n_1, n_2, \cdots, n_{N_e}) \prod_{j=1}^{N_e} X_j^{10} |0>,
\]
where \( \Psi(n_1, n_2, \cdots, n_{N_e}) \) is the wave function, and \( n_j = 1, \cdots, N \) denotes the coordinate of the \( j \)-th triplet rung. Let \( \eta \) be defined by \( U = 2 \cosh \eta \). From an analysis similar to the one used in solving the XXZ spin chain (see, eg.\cite{14, 15}), we have the Bethe Ansatz equation (for \( \lambda_j, j = 1, \cdots, N_e \))
\[
\left[ \frac{\sin(\lambda_j - \frac{i}{2} \eta)}{\sin(\lambda_j + \frac{i}{2} \eta)} \right]^N = -\prod_{l=1}^{N_e} \frac{\sin(\lambda_j - \lambda_l - i\eta)}{\sin(\lambda_j - \lambda_l + i\eta)}, \quad (10)
\]
and the eigenenergy to (9) (up to an irrelevant additive constant)

\[ E = - \sum_{j=1}^{N} \left[ \frac{2 \sinh^2 \eta}{\cosh \eta - \cos 2\lambda_j} - J \right], \tag{11} \]

where \( \lambda_j \) are the rapidities of the triplet rungs. We note that a similar situation (namely mapping of a biquadratic spin-1 chain to an XXZ Heisenberg chain) was discussed in [16].

The phase diagram of the ground state spanned by \( J \) and \( U \) is almost the same to that of the XXZ Heisenberg spin chain with an effective magnetic field \( J \), in the sense that the triplet rungs and the singlet rungs serve as the up spins and down spins, respectively. We distinguish three regions, according to \( U > 2, -2 < U \leq 2, U \leq -2 \) respectively:

(i) \( U > 2 \): For \( |J| < J_c \), the ground state is a Mott-like “insulator” consisting of \( N/2 \) triplet rungs and \( N/2 \) singlet rungs with an energy gap (a gap at the lower end of the energy spectrum) \( \Delta = J_c - |J| \), where \( J_c \) is given by

\[ J_c = \frac{\pi \sinh \eta}{\eta} \sum_{n=-\infty}^{\infty} \text{sech} \frac{\pi^2}{2\eta} (1 + 2n). \tag{12} \]

For \( J > U + 2 \), the triplet rungs are unfavorable and the ground state is a rung-dimerized state (product of \( N \) singlet rungs) with an energy gap \( \Delta = J - (U + 2) \), while for \( J < -(U + 2) \), the ground state is a product of \( N \) triplet rungs with an energy gap \( \Delta = |J| - (U + 2) \). The latter two phases correspond to the completely polarized states in the XXZ spin chain. In the intermediate parameter region \( J_c \leq |J| \leq U + 2 \) one has a gapless phase.

(ii) \( -2 < U \leq 2 \): There is no Mott-like phase in this case. For \( J > U + 2 \), the ground state is still a rung-dimerized state (consisting of only singlet rungs) and for \( J < -(U + 2) \), the ground state is a product of \( N \) triplet rungs. For \( |J| \leq U + 2 \), the ground state is a spin liquid with gapless spinon excitations (cf. [14] for a discussion of phenomena of this type).

(iii) \( U \leq -2 \): There is no gapless phase except for \( J = 0, U = -2 \). The ground state is almost the same as that of a ferromagnetic spin chain (in the same sense as above). For \( J < 0 \), the ground state is a triplet-rung product while for \( J > 0 \) the ground state is a rung-dimerized state.

Acknowledgements: We would like to thank Dr. R.H. Yue (North-West University, Xian) for very helpful discussions. The DFG and SFB-237 support to the second author is gratefully acknowledged.
References

[1] For a review see, e.g., E. Dagotto and T. M Rice, Science 271, 618 (1996).

[2] I. Bose and S. Gayen, Phys. Rev. B 48, 10653 (1993);
Y. Xian, Phys. Rev. B 52, 12485 (1995);

[3] W.H. Zheng, V. Kotov, and J. Oitmaa, *Studies of 2-Chain Spin Ladder with Frustrating Second Neighbor Interactions*, cond-mat/9711006.

[4] A. A Nersesyan and A. M. Tsvelik, Phys. Rev. Lett. 78, 3939 (1997).

[5] A. K. Kolezhuk and H.-J. Mikeska, Phys. Rev. Lett. 80, 2709 (1998).

[6] S. Brehmer, A. K. Kolezhuk, H.-J. Mikeska and U. Neugebauer, J. Phys.: Condens. Matter 10, 1103 (1998).

[7] A. K. Kolezhuk and H.-J. Mikeska, Phys. Rev. B 56, R11 (1997).

[8] S. Brehmer, H.-J. Mikeska and U. Neugebauer, J. Phys: Condens. Matter 8, 7161 (1996).

[9] A.K. Kolezhuk and H.-J. Mikeska, Int. J. Mod. Phys. B 12, 2325 (1998).

[10] C.N. Yang, Phys. Rev. Lett. 19, 1312 (1967).
R.J. Baxter, *Exactly Solved Models in Statistical Physics*, Academic Press, New York 1982.

[11] Y. Wang, *Exact solution of a spin-ladder model*, cond-mat/9901168.

[12] I. Affleck, T. Kennedy, E H. Lieb and H. Tasaki, Commun. Math. Phys. 115, 477 (1988).

[13] S. Albeverio and S.M. Fei, *EuroPhys. Lett.* 41, 665 (1998).

[14] C.N. Yang and C.P. Yang, Phys. Rev. 147, 303 (1966); 150, 321 (1966); 150, 327 (1966); 151, 258 (1966).

[15] L.A. Takhatajyan and L.D. Faddeev, Russ. Math. Surveys 34:5, 11 (1979) [Usp. Mat. Nauk. 34:5, 13 (1979)].
[16] M.N. Barber and M.T. Batchelor, Phys. Rev. B 40, 4621 (1989).