Tree-level unitarity in Gauge-Higgs Unification

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Abstract

We numerically estimate a scale $\Lambda_{\text{uni}}$ at which tree-level unitarity is violated in the $SO(5) \times U(1)_X$ gauge-Higgs unification model by evaluating amplitudes for scattering of the longitudinal W bosons. The scattering amplitudes take larger values in the warped spacetime than in the flat spacetime, and take maximal values when $\theta_H = \pi/2$, where $\theta_H$ is the Wilson line phase along the extra dimension. We take into account not only the elastic scattering but also possible inelastic scatterings in order to estimate $\Lambda_{\text{uni}}$. We found that $\Lambda_{\text{uni}} \simeq 1.3 m_{\text{KK}}$ in the warped spacetime, and $\Lambda_{\text{uni}} \simeq 140 m_{\text{KK}}$ in the flat spacetime, where $m_{\text{KK}}$ is the Kaluza-Klein mass scale. The tree-level unitarity is violated at $\mathcal{O}(1 \text{ TeV})$ for $\theta_H = \pi/2$ in the former case due to the vanishing $WWH$ coupling.

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1 Introduction

Extra dimensions are interesting candidates of the physics beyond the standard model (SM), and have been extensively investigated during the past decade. They open up new possibilities for various issues, such as the large hierarchy between the electroweak and Planck scales [1, 2] or among the fermion masses [3], a mechanism of gauge symmetry breaking [4], candidates of dark matter [5], and so on. Models with extra dimensions should be regarded as effective theories with cut-off energy scales because they are non-renormalizable and perturbative calculations will be invalid near those scales. Therefore it is important to estimate the cut-off scale of the model when we consider an extra-dimensional model. Tree-level unitarity provides a criterion for the perturbativity of a model at a given energy scale.

The tree-level unitarity is usually discussed by evaluating scattering amplitudes of the longitudinally polarized weak bosons \(W_L^\pm\) and \(Z_L\) at tree-level because they provide severer unitarity bound than other scattering processes. In SM, the Higgs boson plays an important role for the recovery of the unitarity. If it is sufficiently heavy and decoupled, the scattering amplitudes grow as \(E^2\), where \(E\) is the scattering energy, and exceed the unitarity bound at \(\mathcal{O}(1 \text{ TeV})\). This means that perturbative calculations are no longer reliable above the scale. In the five-dimensional (5D) Higgsless models [6], the tree-level unitarity is recovered by the Kaluza-Klein (KK) excitation modes of the gauge bosons instead of the Higgs boson in SM, and the unitarity violation delays up to \(\mathcal{O}(10 \text{ TeV})\) when the compactification scale is assumed to be around 1 TeV.

The situation is more complicated in the gauge-Higgs unification models [7]-[11] because they have the Higgs mode as well as the KK gauge bosons, both of which participate in the unitarization of the theory. The gauge-Higgs unification is an attractive scenario as a solution to the gauge hierarchy problem. Higher dimensional gauge symmetry protects the electroweak scale against quantum corrections. The Higgs boson whose vacuum expectation value (VEV) breaks the electroweak gauge symmetry is identified with one of extra-dimensional components of the higher dimensional gauge fields, which we refer to as the gauge-scalars in this paper. The electroweak symmetry breaking is characterized by the Wilson line phase \(\theta_H\) along the extra dimension, which is gauge invariant. In these models, coupling constants and the KK mass scale \(m_{KK}\) depend on \(\theta_H\) when we fix the W boson mass \(m_W\), and thus the scattering amplitudes for the weak bosons have nontrivial \(\theta_H\)-dependence. In particular in the models on the warped spacetime [12]-[16], the \(WWH\)
and $ZZH$ couplings ($H$ stands for the Higgs mode) deviate from the SM values and vanish at some specific values of $\theta_H$, such as $\pi$ or $\pi/2$, depending on the models [15, 16]. For such values of $\theta_H$, the Higgs mode cannot participate in the unitarization of the weak boson scattering, and the amplitudes grow until the KK gauge bosons start to propagate and unitarize the scattering processes. Therefore it is important to understand the $\theta_H$-dependence of the scattering amplitudes for the weak bosons in order to estimate the unitarity violation scale $\Lambda_{\text{uni}}$. This issue is discussed in Ref. [17] and some qualitative behaviors of the amplitudes are clarified.

In our previous work [18], we investigated it more quantitatively by numerical calculations of the scattering amplitude for the process: $W_L^+ + W_L^- \rightarrow Z_L + Z_L$ in the 5D $SU(3)$ gauge-Higgs unification model both in the flat and warped spacetimes. We found that the amplitude is enhanced for $\theta_H = O(1)$ in the warped case, which implies that the tree-level unitarity will be violated at a lower scale than that in the flat case. Although this result is expected to be common to the gauge-Higgs unification models, a specific value of $\Lambda_{\text{uni}}$ depends on the model. It is well-known that the $SU(3)$ model is not realistic because it gives a wrong value of the Weinberg angle $\theta_W$, i.e., $\sin^2 \theta_W = 3/4$. It is most interesting and useful to estimate $\Lambda_{\text{uni}}$ in a realistic model, such as the 5D $SO(5) \times U(1)_X$ model, which was first proposed in Ref. [13].

In this paper, we consider scattering of $W_L^+$ and $W_L^-$, investigate the $\theta_H$-dependence of the amplitudes, and numerically estimate $\Lambda_{\text{uni}}$ in the 5D $SO(5) \times U(1)_X$ model. Note that $\theta_H$ and the Higgs mass $m_H$ are dynamically determined by quantum effect once the whole field content of the model is given. In the following discussion, however, we do not specify the fermion sector and treat $\theta_H$ and $m_H$ just as free parameters because we are interested in the tree-level amplitudes. These parameters parameterize the radiatively induced effective potential in a model-independent way. We take into account not only the elastic scattering but also possible inelastic scattering to obtain a proper unitarity bound. For the tree-level S-wave amplitude for the elastic scattering of the W bosons, there is an infrared divergence originating a singularity at forward scattering. We show an appropriate treatment to regularize this divergence by taking into account the instability of the W bosons in the final state.

The paper is organized as follows. In Sec. 2, we briefly review the $SO(5) \times U(1)_X$ gauge-Higgs unification model and provide necessary ingredients to calculate the scattering
amplitudes for the weak bosons, which are extended versions of those used in Ref. [18] for the $SU(3)$ model. In Sec. 3, we provide explicit expressions of the scattering amplitudes and show their behaviors as functions of $E$ and $\theta_H$ in the flat and warped spacetimes. In Sec. 4, we estimate $\Lambda_{\text{uni}}$ from the unitarity condition by using the amplitudes calculated in Sec. 3. Sec. 5 is devoted to the summary. In Appendix A, we give definitions and explicit forms of the basis functions used in the text. In Appendix B, we derive the 5D propagators of the gauge fields. In Appendix C, we show a treatment of the singularity of the elastic scattering amplitude at forward scattering.

2 $SO(5) \times U(1)_X$ model

In this section, we review the $SO(5) \times U(1)_X$ gauge-Higgs unification model [13]. Most results in this section have been already obtained in the literature (see Ref. [16], for example), but we repeat the discussion to explain our notation and for later convenience.

2.1 Set-up

We consider an $SO(5) \times U(1)_X$ gauge theory compactified on $S^1/Z_2$. Arbitrary background metric with four-dimensional (4D) Poincaré symmetry can be written as

$$ds^2 = G_{MN}dx^Mdx^N = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2,$$

where $M, N = 0, 1, 2, 3, 4$ are 5D indices and $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1, 1)$. The fundamental region of $S^1/Z_2$ is $0 \leq y \leq L$. The function $e^{\sigma(y)}$ is a warp factor, which is normalized as $\sigma(0) = 0$. For example, $\sigma(y) = 0$ in the flat spacetime, and $\sigma(y) = ky$ ($0 \leq y \leq L$) in the Randall-Sundrum warped spacetime [2], where $k$ is the inverse AdS curvature radius.

The model has an $SO(5)$ gauge field $A_M$ and a $U(1)_X$ gauge field $B_M$. The former are decomposed as

$$A_M = \sum_{\alpha=1}^{10} A_M^{\alpha} T^\alpha = \sum_{a_L=1}^{3} A_M^{a_L} T^{a_L} + \sum_{a_R=1}^{3} A_M^{a_R} T^{a_R} + \sum_{\hat{a}=1}^{4} A_M^{\hat{a}} T^{\hat{a}},$$

where $T^{a_L,a_R}$ ($a_L, a_R = 1, 2, 3$) and $T^{\hat{a}}$ ($\hat{a} = 1, 2, 3, 4$) are the generators of $SO(4) \sim SU(2)_L \times SU(2)_R$ and $SO(5)/SO(4)$, respectively, and are normalized as

$$\text{tr}(T^\alpha T^\beta) = \frac{1}{2} \delta^{\alpha\beta}. $$

(2.3)
The 5D Lagrangian is
\[
\mathcal{L} = \sqrt{-G} \left[ -\text{tr} \left\{ \frac{1}{2} G^{M}{}^{L} G^{NP} F_{MN}^{(A)} F_{LP}^{(A)} + \frac{1}{\xi} \left( f_{gf}^{(A)} \right)^{2} \right\} 
- \left\{ \frac{1}{4} G^{M}{}^{L} G^{NP} F_{MN}^{(B)} F_{LP}^{(B)} + \frac{1}{2\xi} \left( f_{gf}^{(B)} \right)^{2} \right\} + \cdots \right], \tag{2.4}
\]
where \(\sqrt{-G} \equiv \sqrt{-\det(G_{MN})} = e^{-4\sigma}\), \(F_{MN}^{(A)} \equiv \partial_{M} A_{N} - \partial_{N} A_{M} - ig_{A} [A_{M}, A_{N}]\) \((g_{A}\) is the 5D gauge coupling constant for \(SO(5)\)), \(F_{MN}^{(B)} \equiv \partial_{M} B_{N} - \partial_{N} B_{M}\), and \(\xi\) is a dimensionless parameter. The ellipsis denotes the ghost and the matter sectors, which are irrelevant to the following discussion. The gauge-fixing function \(f_{gf}^{(A,B)}\) are chosen as
\[
f_{gf}^{(A)} = e^{2\sigma} \left\{ \eta_{\mu\nu} \partial_{\mu} A_{\nu} + \xi D_{y}(e^{-2\sigma} A_{y}) \right\},
D_{y} A_{M} \equiv \partial_{y} A_{M} - ig_{A} [A_{y}, A_{M}],
f_{gf}^{(B)} = e^{2\sigma} \left\{ \eta_{\mu\nu} \partial_{\nu} B_{\nu} + \xi \partial_{y}(e^{-2\sigma} B_{y}) \right\}, \tag{2.5}
\]
where \(A_{y}^{ix}(y)\) is the classical background of \(A_{y}(x, y)\).

The boundary conditions for the gauge fields are written as
\[
\begin{align*}
\left(A_{\mu}\right) \begin{pmatrix} x, y_{i} - y \end{pmatrix} &= Q_{i} \begin{pmatrix} A_{\mu} \end{pmatrix} \begin{pmatrix} x, y_{i} + y \end{pmatrix} Q_{i}^{-1}, \\
\left(B_{\mu}\right) \begin{pmatrix} x, y_{i} - y \end{pmatrix} &= \begin{pmatrix} B_{\mu} \end{pmatrix} \begin{pmatrix} x, y_{i} + y \end{pmatrix}, \tag{2.6}
\end{align*}
\]
where \(i = 0, L, y_{0} = 0, y_{L} = L\), and \(Q_{i} \in SO(5)\) are constant matrices satisfying \(Q_{i}^{2} = 1\).

In the present paper we take \(Q_{0} = Q_{L} = \text{diag}(1, 1, -1, -1)\) in the spinorial representation, or equivalently \(Q_{0} = Q_{L} = \text{diag}(-1, -1, -1, -1, 1)\) in the vectorial representation. Then the gauge symmetry is broken to \(SO(4) \times U(1)_{X}\) at both boundaries.

We assume that the residual \(SO(4) \times U(1)_{X} \sim SU(2)_{L} \times SU(2)_{R} \times U(1)_{X}\) is spontaneously broken to \(SU(2)_{L} \times U(1)_{Y}\) at \(y = 0\) by some dynamics on the boundary, which leads to the following boundary mass terms.
\[
\mathcal{L}_{bd} = 2\sqrt{-g} \left[ -\frac{M_{\pm}}{2} g^{\mu\nu} \left( A_{\mu}^{1R} A_{\nu}^{1R} + A_{\mu}^{2R} A_{\nu}^{2R} \right) - \frac{M_{0}}{2} g^{\mu\nu} A_{\mu}^{3\phi} A_{\nu}^{3\phi} \right] \delta(y) + \cdots, \tag{2.7}
\]
where \(g_{\mu\nu} = e^{-2\sigma} \eta_{\mu\nu}\), \(\sqrt{-g} \equiv \sqrt{-\det(g_{\mu\nu})} = e^{-4\sigma}\), \(M_{\pm}\) and \(M_{0}\) are boundary mass parameters, and
\[
\begin{pmatrix} A_{M}^{3\phi} \\ A_{M}^{3\phi} \end{pmatrix} \equiv \begin{pmatrix} c_{\phi} & -s_{\phi} \\ s_{\phi} & c_{\phi} \end{pmatrix} \begin{pmatrix} A_{M}^{3\phi} \\ B_{M} \end{pmatrix}. \tag{2.8}
\]
\end{align*}
\]
Table I: Boundary conditions for the gauge fields. The notation (D,N), for example, denotes
the Dirichlet boundary condition at \( y = 0 \) and the Neumann boundary condition at \( y = L \).

| \( A_{\mu}^{\text{pl}} \) | \( A_{\mu}^{1,2\text{R}} \) | \( A_{\mu}^{3\text{R}} \) | \( A_{\mu}^{Y} \) | \( A_{\mu}^{4} \) |
|----------------|----------------|----------------|----------------|----------------|
| (N,N)          | (D,N)          | (D,N)          | (N,N)          | (D,D)          |
| \( A_{y}^{\text{pl}} \) | \( A_{y}^{1,2\text{R}} \) | \( A_{y}^{3\text{R}} \) | \( A_{y}^{Y} \) | \( A_{y}^{4} \) |
| (D,D)          | (N,D)          | (N,D)          | (D,D)          | (N,N)          |

with

\[
c_{\phi} \equiv \frac{g_{A}}{\sqrt{g_{A}^{2} + g_{B}^{2}}}, \quad s_{\phi} \equiv \frac{g_{B}}{\sqrt{g_{A}^{2} + g_{B}^{2}}}. \tag{2.9}
\]

Here \( g_{B} \) is the 5D gauge coupling constant for \( U(1)_{X} \). The gauge symmetry broken by these
boundary mass terms can be recovered nonlinearly by introducing the Nambu-Goldstone (NG) modes localized at \( y = 0 \).

We do not specify the origin of the mass terms (2.7) because it is irrelevant to the
low-energy physics. We just assume that these masses are sufficiently heavier than the
compactification scale. Then the boundary conditions for \( A_{\mu}^{1,2\text{R}} \), \( A_{\mu}^{3\text{R}} \) and \( A_{\mu}^{Y} \) at \( y = 0 \) are
effectively changed from the Neumann-type to the Dirichlet-type. In such a case, those for
the gauge-scalars \( A_{y}^{1,2\text{R}} \), \( A_{y}^{3\text{R}} \) and \( A_{y}^{Y} \) correspondingly change from Dirichlet to Neumann.
The boundary degrees of freedom for the gauge-scalars at \( y = 0 \) are provided by the
boundary NG modes. \(^2\)

As a result, the effective boundary conditions for the gauge fields are tabulated in Table I.

Note that only \( (N,N) \) fields can have massless modes when perturbation theory is
developed around the trivial configuration \( A_{M} = B_{M} = 0 \). Thus the gauge symmetry
is broken to \( SU(2)_{L} \times U(1)_{Y} \) at tree-level. The zero-modes of the gauge-scalars form an
\( SU(2) \)-doublet 4D scalar \( (A_{y}^{1} + iA_{y}^{2}, A_{y}^{4} - iA_{y}^{3}) \), which plays a role of the Higgs doublet in SM whose VEV breaks \( SU(2)_{L} \times U(1)_{Y} \) to the electromagnetic symmetry \( U(1)_{EM} \). They yield non-Abelian Aharonov-Bohm phases (Wilson line phases) when integrated along the
fifth dimension. By using the residual \( SU(2)_{L} \times U(1)_{Y} \) symmetry, we can always push the
nonvanishing VEV into one component, say, \( A_{y}^{4} \). Then the Wilson line phase \( \theta_{H} \) is given by

\[
\theta_{H} = \frac{g_{A}}{\sqrt{2}} \int_{0}^{L} dy \ A_{y}^{4}(y). \tag{2.10}
\]

\(^2\) The equations of motion for the boundary NG modes relates them to the boundary values of the
gauge-scalars.
Table II: The structure constants for the generators $T^I$. For the other combinations of indices, $C^{IJK} = 0$.

According to the transformation properties under the unbroken $U(1)_{EM}$ and the rotation by a constant matrix $\Omega(L)$, the gauge fields are classified into the charged sector $(A_{+L}^1, A_{-L}^1, A_{0L}^1, A_{+R}^2, A_{-R}^2, A_{0R}^2, A_{+\hat{a}}^3, A_{-\hat{a}}^3)$, and the “Higgs” sector $A_{\hat{4}}^4$. Thus, in the following, we will use the index $I$ which run over both the $SO(5)$-part $\alpha = (a_L, a_R, \hat{a})$ and the $U(1)$-part as

$$I = I_+, I_-, I_0, \hat{4},$$

(2.11)

where $I_\pm = \pm L, \pm R, \pm$ and $I_0 = 3_L, 3_R, B, \hat{3}$. Then all the gauge fields are expressed in a matrix notation as

$$A_M \equiv \sum_I A^I_M T^I,$$

(2.12)

where $A^R_M \equiv B_M$. The generators are defined as

$$T^{\pm L} \equiv \frac{1}{\sqrt{2}} (T^{1L} \mp iT^{2L}), \quad T^{\pm R} \equiv \frac{1}{\sqrt{2}} (T^{1R} \mp iT^{2R}),$$

$$T^{\pm} \equiv \frac{1}{\sqrt{2}} (T^1 \mp iT^2), \quad T^B \equiv \frac{1}{\sqrt{2d}} 1_d,$$

(2.13)

where $d$ is a dimension of the representation. The orthonormal conditions for the generators are written as

$$\text{tr} \left(T^I T^J\right) = \frac{1}{2} \delta^{IJ},$$

(2.14)

where the index $\bar{J}$ runs as

$$\bar{J} = J_-, J_+, J_0, \hat{4}.$$  

(2.15)
2.2 Mode expansion

The expansion of the 5D gauge fields into 4D KK modes is performed in a conventional way (see Ref. [16], for example). We move to the Scherk-Schwarz basis, in which $\tilde{A}_{bg} = 0$. It is related to the original basis by the gauge transformation,

$$\tilde{A}_M = \Omega A_M \Omega^{-1} - \frac{i}{g_A} (\partial_M \Omega) \Omega^{-1}, \quad (2.16)$$

with

$$\Omega(y) \equiv P \exp \left\{ -ig_A \int_0^y dy' A_{bg}^4 (y') T^4 \right\}. \quad (2.17)$$

The symbol $P$ stands for the path-ordered operator from left to right.

For the following discussion, it is convenient to move to the momentum representation for the 4D part while remain the coordinate representation for the fifth dimension [19]. Then the 5D gauge fields are expanded into the KK modes as

$$\tilde{A}^I_\mu (p, y) = \sum_n u_n^I (y) A^{(n)}_\mu (p) + \sum_n w_n^I (y) p_\mu A^{(n)}_S (p),$$

$$\tilde{A}^I_y (p, y) = \sum_n v_n^I (y) \varphi^{(n)} (p). \quad (2.18)$$

Notice that $\tilde{A}^I_\mu (p, y)$ are decomposed into two parts, according to their polarization. In the above expression, $A^{(n)}_\mu (p)$ are polarized as $p_\mu A^{(n)}_\mu (p) = 0$ and include the transverse and the longitudinal modes, which are physical for the massive modes. On the other hand, $A^{(n)}_S (p)$ are unphysical scalar modes. The gauge-scalar modes $\varphi^{(n)} (p)$ are also unphysical besides the zero-mode.

By solving the mode equations with the boundary conditions shown in Table I, the mode functions are expressed by the basis functions $C_0 (y, m)$ and $S_0 (y, m)$ defined in Appendix A in the vector notation for the index $I$ as

$$\tilde{u}_n (y) = \mathcal{M}_0 (y, m_n) \vec{N}_n,$$

$$\tilde{w}_n (y) = \mathcal{M}_0 (y, \tilde{m}_n / \sqrt{\xi}) \vec{N}_n,$$

$$\tilde{v}_n (y) = \frac{d}{dy} \tilde{w}_n (y), \quad (2.19)$$

where the matrix $\mathcal{M}_0 (y, m)$ is a function defined by Eqs. (B.10) and (B.11), $m_n$ is a mass eigenvalue for $A^{(n)}_\mu$, and $\tilde{m}_n$ is a common mass eigenvalue for $A^{(n)}_S$ and $\varphi^{(n)}$. The constant vectors $\vec{N}_n$, $\vec{N}_n$ are determined by

$$\mathcal{W} (m_n) \vec{N}_n = 0, \quad \mathcal{W} (\tilde{m}_n) \vec{N}_n = 0, \quad (2.20)$$
where the matrix $W(m)$ is defined by Eq. (B.15), and by the orthonormal conditions
\[
\int_{0}^{L} dy \, \vec{u}_m(y) \cdot \vec{u}_n(y) = \delta_{mn},
\]
\[
\frac{\tilde{m}_m^2}{\xi} \int_{0}^{L} dy \, \vec{w}_m(y) \cdot \vec{w}_n(y) = \int_{0}^{L} dy \, e^{-2\sigma(y)} \vec{v}_m(y) \cdot \vec{v}_n(y) = \delta_{mn}.
\] (2.21)

The conditions that Eq. (2.20) has nontrivial solutions are
\[
\text{det } W(m_n) = 0, \quad \text{det } W(\tilde{m}_n) = 0,
\] (2.22)
which determine the mass eigenvalues $m_n$ and $\tilde{m}_n$.

Here we give explicit expressions of light modes. The $W$ boson is identified with the lightest mode in the charged sector. Its mass $m_W$ is determined as the lowest solution to
\[
C'_0(L, m_W) S_0(L, m_W) + \frac{m_W e^{\sigma(L)}}{2} \sin^2 \theta_H = 0,
\] (2.23)
and the corresponding mode function is calculated as
\[
u^+_{W}(y) = \sum_{J_+} \mathcal{M}_0^{ch_{I_+J_+}}(y, m_W) N_{W}^{J_+}, \quad u^+_{W}(y) = u^0_{W}(y) = u^{\tilde{3}}_{W}(y) = 0,
\] (2.24)
for the $W^+$ boson, and
\[
u^-_{W}(y) = \sum_{J_-} \mathcal{M}_0^{ch_{I_-J_-}}(y, m_W) N_{W}^{J_-}, \quad u^-_{W}(y) = u^0_{W}(y) = u^{\tilde{\bar{4}}}_{W}(y) = 0
\] (2.25)
for the $W^-$ boson. Here $\mathcal{M}_0^{ch}(y, m)$ is defined in Eq. (B.11) and
\[
N_{W} = \alpha_W \left( -S'_0(L, m_W), C'_0(L, m_W), \sqrt{2}C'_0(L, m_W) \cot \theta_H \right)^t.
\] (2.26)

The constant $\alpha_W$ is determined by the normalization condition (2.21).

The neutral sector has a zero-mode, which corresponds to the photon. Its mode function is a constant vector,
\[
u^0_{\gamma}(y) = \sqrt{\frac{1}{(1 + s^2_\phi)L}} \left( s_\phi, s_\phi, c_\phi, 0 \right), \quad u^{\|^+}_{\gamma}(y) = u^{\tilde{3}}_{\gamma}(y) = 0.
\] (2.27)

The $Z$ boson is identified with the second lightest mode in the neutral sector. Its mass $m_Z$ is determined as the lowest solution to
\[
C'_0(L, m_Z) S_0(L, m_Z) + \frac{m_Z e^{\sigma(L)}}{2} \frac{(1 + s^2_\phi)}{2} \sin^2 \theta_H = 0,
\] (2.28)
and the corresponding mode function is
\[ u^I_Z(y) = \sum_{J_0} M^0_{I J_0}(y, m_Z) N^J_0, \quad u^{I \pm}_Z(y) = u^{J \pm}_Z(y) = 0, \] (2.29)
where \( M^0_{I J_0}(y, m) \) is defined in Eq. (B.11) and
\[ N_Z = \alpha_Z \left( -S'_0, c^2 \phi C'_0 + s^2 \phi (S'_0 - C'_0), \sqrt{2} C'_0 \cot \theta_H \right)^t. \] (2.30)

The arguments in the right-hand side are \((L, m_Z)\), and the normalization constant \( \alpha_Z \) is determined by Eq. (2.21).

In the Randall-Sundrum spacetime \((\sigma(y) = ky)\), the basis functions are expressed by the Bessel functions as shown in Appendix A. When the warp factor \( e^{\sigma(L)} = e^{kL} \) is large enough, the masses of the \( W \) and \( Z \) bosons are approximated as
\[ m_W \simeq \frac{m_{KK}}{\pi} \sqrt{\frac{1}{kL}} |\sin \theta_H|, \quad m_Z \simeq \frac{m_{KK}}{\pi} \sqrt{\frac{1 + s^2 \phi}{kL}} |\sin \theta_H|, \] (2.31)
where
\[ m_{KK} \equiv \frac{k \pi}{e^{kL} - 1} \] (2.32)
is the KK mass scale. Thus the Weinberg angle \( \theta_W \) is expressed in terms of \( s_\phi \) as
\[ \tan \theta_W \simeq s_\phi. \] (2.33)

In the flat spacetime \((\sigma(y) = 0)\), the \( W \) and \( Z \) boson masses are expressed as
\[ m_W = \frac{1}{L} \sin^{-1} \left( \frac{1}{\sqrt{2}} \sin \theta_H \right), \quad m_Z = \frac{1}{L} \sin^{-1} \left( \sqrt{\frac{1 + s^2 \phi}{2}} \sin \theta_H \right). \] (2.34)
In contrast to the \( SU(3) \) model, the spectrum is not linear for the “Higgs VEV” \( \theta_H \) even in the flat spacetime \[16]. This stems from the fact that the mechanism of mass generation for the 4D gauge bosons involves not only 4D gauge fields in each KK level, but also fields in other KK levels. In the original basis, the \( W \) boson mass term comes from
\[
\mathcal{L} = g_A^2 e^{-2\sigma} \eta^{\mu\nu} \text{tr} \{ [A_\mu, A_y] [A_\nu, A_y] \} + \cdots \\
= -\frac{g_A^2 e^{-2\sigma}}{8} \left( A^4_y \right)^2 \eta^{\mu\nu} (A^{+L}_\mu A^{-L}_\nu - A^{-L}_\mu A^{+L}_\nu - A^{+R}_\mu A^{-L}_\nu + A^{+L}_\mu A^{-R}_\nu + A^{+R}_\mu A^{-R}_\nu + A^{+L}_\mu A^{-L}_\nu) + \cdots . \] (2.35)
In the flat spacetime, the profile of \( A^{4}_{y} \) is flat. Thus there would be no mixing among different KK levels due to the orthogonality of the mode functions if the mixing terms
between the $SU(2)_L$ and $SU(2)_R$ gauge fields were absent in Eq. (2.35), just like the case of the $SU(3)$ model. However, the KK level mixing actually occurs due to the presence of the mixing terms between the $SU(2)_L$ and $SU(2)_R$ gauge fields whose boundary conditions are different (see Table I). Then the lowest mode in each KK tower necessarily mixes with heavy KK modes when $\hat{A}_4^L$, or $\theta_H$, acquires a nonzero value. This mixing makes the $\theta_H$-dependence of the spectrum nonlinear.

### 2.3 5D propagators

For the purpose of calculating the scattering amplitude, it is convenient to use the 5D propagators $G_T(y, y', \sqrt{-p^2})$ defined in a mixed momentum/position representation [19]. It describes the propagation of the entire KK towers of excitations carrying the 4D momentum $p$ between two points $y$ and $y'$ in the extra dimension. This approach has an advantage that we need not explicitly calculate mass eigenvalues and mode functions for modes propagating in the internal lines of the Feynmann diagrams, nor sum over contributions from infinite (or large) number of KK modes. The definition and the derivation of the 5D propagator are given in Appendix B. It is expressed from Eq. (B.14) in the following block-diagonal form.

\[
G_T = \left(\begin{array}{ccc}
G_{T}^{ch} & G_{T}^{ch} & G_{T}^{nt} \\
G_{T}^{ch} & G_{T}^{nt} & G_{T}^{nt} \\
G_{T}^{nt} & G_{T}^{nt} & G_{T}^{nt}
\end{array}\right),
\]

(2.36)

where

\[
G_{T}^{ch}(y, y', |p|) = e^{2\sigma(L)}M_{0}^{ch}(y, |p|)W_{ch}^{-1}(|p|)M_{L}^{ch}(y', |p|)R_{ch}^{y},
\]

\[
G_{T}^{nt}(y, y', |p|) = e^{2\sigma(L)}M_{0}^{nt}(y, |p|)W_{nt}^{-1}(|p|)M_{L}^{nt}(y', |p|)R_{nt}^{y},
\]

\[
G_{T}^{44}(y, y', |p|) = \frac{e^{2\sigma(L)}S_{0}(y, |p|)S_{L}(y', |p|)}{|p| S_{0}(L, |p|)},
\]

(2.37)

and $|p| \equiv \sqrt{-p^2}$. The explicit forms of the matrices in the right-hand sides are given in Appendix B.

Using the mode equation and Eq. (B.1) with the boundary conditions, we can show the following relation.

\[
\bar{u}_n(y) = -(p^2 + m_n^2) \int_0^L dy' \ G_T(y, y', |p|)\bar{u}_n(y').
\]

(2.38)

---

3 This approach is also useful for models with continuum spectra [20].
Thus the 5D propagator can also be expressed as
\[
G_T(y, y', |p|) = - \sum_n \frac{\bar{u}_n(y) \bar{u}_n(y')}{p^2 + m_n^2}.
\] (2.39)

3 Weak boson scattering

Now we consider the scattering of the weak bosons. The scattering amplitudes are functions of the total energy \( E \) and the scattering angle \( \chi \) in the center-of-mass frame. Let us consider the scattering process: \( |p_1, \varepsilon_1, m\rangle + |p_2, \varepsilon_2, n\rangle \rightarrow |p_3, \varepsilon_3, l\rangle + |p_4, \varepsilon_4, k\rangle \), where \( p_i \) and \( \varepsilon_i \) \((i = 1, 2, 3, 4)\) denote the 4-momenta and the polarization vectors respectively, and \( m, n, \cdots \) labels the particle species including the KK levels.

3.1 Scattering amplitudes

As mentioned in Sec. 2.3, the scattering amplitudes are easily calculated by utilizing the 5D propagators. The tree-level amplitude \( \mathcal{A} \) for the vector boson scattering is expressed by
\[
\mathcal{A} = \mathcal{A}^C + \mathcal{A}^V + \mathcal{A}^S,
\] (3.1)
where \( \mathcal{A}^C \), \( \mathcal{A}^V \) and \( \mathcal{A}^S \) are contributions from the contact interactions, exchange of the vector modes and that of the gauge-scalar modes, respectively, and are given by

\[
\mathcal{A}_{mnlk}^C = -ig_A^2 \int_0^L dy \sum_I \left\{ U^I_{mn}(y)U^I_{lk}(y) + U^I_{mk}(y)U^I_{nl}(y) \right\} (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3^* \cdot \varepsilon_4^*)
\] (3.2)
\[
\mathcal{A}_{mnlk}^V = -ig_A^2 \sum_{I,J} \int_0^L dy \int_0^L dy' \left\{ U^I_{mn}(y)G_{T}^{IJ}(y, y', |p|)U^J_{lk}(y') \right\} P_{1234}
\] (3.3)
\[
\mathcal{A}_{mnlk}^S = ig_A^2 \sum_{I,J} \int_0^L dy e^{2\gamma(y)} \left\{ Y^I_{mn}(y)Y^J_{lk}(y) \frac{(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3^* \cdot \varepsilon_4^*)}{p_{12}^2}
\] (3.4)
where \( p_{12} \equiv p_1 + p_2, p_{13} \equiv p_1 - p_3, p_{14} \equiv p_1 - p_4, \)

\[
P_{1234} = 2(p_1 \cdot \varepsilon_2)\varepsilon_1 - 2(p_2 \cdot \varepsilon_1)\varepsilon_2 - (\varepsilon_1 \cdot \varepsilon_2)(p_1 - p_2) + \frac{p_{12}}{p_{12}^2} \eta_{\mu\nu} - \eta_{\mu\nu} \frac{p_{12}}{p_{12}^2} \]
\[
\times \{2(p_3^* \cdot \varepsilon_4^*)\varepsilon_3^* - 2(p_4^* \cdot \varepsilon_3^*)\varepsilon_4^* - (\varepsilon_3^* \cdot \varepsilon_4^*)(p_3 - p_4)\}^\nu,
\]
\[
P_{1324} = 2(p_1 \cdot \varepsilon_3^*)\varepsilon_1 + 2(p_3 \cdot \varepsilon_1^*)\varepsilon_3^* - (\varepsilon_1 \cdot \varepsilon_3^*)(p_1 + p_3) + \frac{p_{13}}{p_{13}^2} \eta_{\mu\nu} - \eta_{\mu\nu} \frac{p_{13}}{p_{13}^2} \]
\[
\times \{2(p_2 \cdot \varepsilon_4^*)\varepsilon_2 + 2(p_4 \cdot \varepsilon_2^*)\varepsilon_4^* - (\varepsilon_2 \cdot \varepsilon_4^*)(p_2 + p_4)\}^\nu,
\]
\[
P_{1423} = 2(p_1 \cdot \varepsilon_4^*)\varepsilon_1 + 2(p_4 \cdot \varepsilon_1^*)\varepsilon_4^* - (\varepsilon_1 \cdot \varepsilon_4^*)(p_1 + p_4) + \frac{p_{14}}{p_{14}^2} \eta_{\mu\nu} - \eta_{\mu\nu} \frac{p_{14}}{p_{14}^2} \]
\[
\times \{2(p_2 \cdot \varepsilon_3^*)\varepsilon_2 + 2(p_3 \cdot \varepsilon_2^*)\varepsilon_3^* - (\varepsilon_2 \cdot \varepsilon_3^*)(p_2 + p_3)\}^\nu, \tag{3.5}
\]

and the functions in the integrands are defined as

\[
U^I_{mn}(y) \equiv C^{JKL} u^J_m(y) u^K_n(y),
\]
\[
Y^I_{mn}(y) \equiv e^{-2\sigma(y)} C^{JKL} \left\{ (u^J_m)'(y) u^K_n(y) - u^J_m(y) (u^K_n)'(y) \right\}. \tag{3.6}
\]

Here we have used the relation \( p_i \cdot \varepsilon_i(p_i) = 0 \) (\( i = 1, 2, 3, 4 \)). The prime denotes derivative with respect to \( y \).

The first, second and third lines in Eq. (3.3) correspond to the \( s-, t- \) and \( u \)-channel diagrams exchanging the 4D vector modes, respectively. The above expression of the amplitude is a result of a cancellation between the gauge-dependent part \( G_S(y, y', |p|) \) in the propagator of the vector modes and the gauge-scalar propagator \( G_{yy}(y, y', |p|) \). This cancellation occurs due to the relation \[B:21\] and makes the resultant amplitude gauge-independent. The contribution \( A^S \) is a remnant of the cancellation.

The gauge invariance of the theory ensures the equivalence theorem \[21\], which states that the scattering of the longitudinally polarized vector bosons is equivalent to that of the (would-be) NG bosons eaten by the gauge bosons. In 5D models, the gauge-scalar modes \( \varphi^{(n)} \) coming from \( A_y \) play the role of the NG bosons in the equivalence theorem \[6\]
\[22\]. Namely, the following relation holds for the longitudinal vector modes \( A^{(n)}_L \):

\[
T(A^{(n)}_L, \cdots, A^{(n)}_L; \Phi) = C_l T(i\varphi^{(n)}_1, \cdots, i\varphi^{(n)}_L; \Phi) + O \left( \frac{M^2}{E^2} \right), \tag{3.7}
\]

where all external lines are directed inwards, \( \Phi \) denotes any possible amputated external physical fields, such as the transverse gauge boson, and \( M \) is the heaviest mass among the external lines. A constant \( C_l \) is gauge-dependent, but \( C_l = 1 \) at tree-level\(^4\). The correction

\(^4\) We can also take a gauge where \( C_l = 1 \) at all orders of the perturbative expansion \[22\].
term is $O(M^2/E^2)$ because of the 5D gauge invariance (see Ref. [24], for example). Eq.(3.7) is useful to discuss the high-energy behavior of the scattering amplitude $A$ because the corresponding NG boson amplitude does not have $O(E^4)$ contributions, which makes it easier to numerically calculate the amplitude thanks to the absence of cancellations between large numbers.

The scattering amplitude for the corresponding NG bosons comes only from diagrams exchanging the vector modes.

$$B_{mnk} = -ig_5^2 \sum_{I,J} \int_0^L dy \int_0^L dy' V_{mn}^I(y)(p_1 - p_2) \mu G_{\mu \nu}^{IJ}(p_{12}, y, y')(p_3 - p_4) \nu V_{ik}^J(y')$$

$$+ ig_5^2 \sum_{I,J} \int_0^L dy \int_0^L dy' V_{mj}^I(y)(p_1 + p_3) \mu G_{\mu \nu}^{IJ}(p_{13}, y, y')(p_2 + p_4) \nu V_{nk}^J(y')$$

$$+ ig_5^2 \sum_{I,J} \int_0^L dy \int_0^L dy' V_{mk}^I(y)(p_1 + p_4) \mu G_{\mu \nu}^{IJ}(p_{14}, y, y')(p_2 + p_3) \nu V_{nl}^J(y'), \quad (3.8)$$

where

$$V_{mn}^I(y) \equiv e^{-2\sigma(y)} C^{IJK} v_m^I(y) v_n^K(y). \quad (3.9)$$

### 3.2 Various behaviors of the amplitudes

Here we show various behaviors of the scattering amplitudes given in the previous subsection. For numerical calculation, we consider the flat ($\sigma(y) = 0$) and the Randall-Sundrum ($\sigma(y) = ky$) spacetimes, and choose the gauge parameter as $\xi = 1$, the 4D weak gauge coupling $g \equiv g_A/\sqrt{L}$ as $g^2 = 4\pi\alpha_{EM}/\sin^2 \theta_W = 0.4$. We take the $W$ boson mass $m_W$ as an input parameter. Then the size of the extra dimension $L$ becomes $\theta_H$-dependent after fixing $m_W$. (See Eqs.(2.31) and (2.34).) The KK mass scale $m_{K K} = \pi k/(e^{kL} - 1)$ also depends on $\theta_H$ for a given value of the warp factor $e^{kL}$. Thus the amplitudes are functions of the center-of-mass energy $E$, the Wilson line phase $\theta_H$ and the warp factor $e^{kL}$. The physical amplitude $A$ is of course gauge-independent, and the $\xi$-dependence of the gauge-scalar scattering amplitude $B$ is small in high-energy region as can be seen from Eq.(3.7).

Here let us comment on another advantage of using the 5D propagators. By using the relation (2.39), the scattering amplitudes given in the previous subsection are rewritten as

---

5 For the non-forward (non-backward) scattering, $O(E^2)$ contributions are also absent.

6 The Wilson line phase $\theta_H$ is dynamically determined at quantum level if we fix the whole matter content of the model.
the energy and the momentum of each particle are expressed as

\[ p_1 = (E_1, 0, 0, p_I) \]
\[ p_2 = (E_2, 0, 0, -p_I) \]
\[ p_3 = (E_3, p_F \sin \chi, 0, p_F \cos \chi) \]
\[ p_4 = (E_4, -p_F \sin \chi, 0, -p_F \cos \chi) \]
\[ \varepsilon_1(p_1) = (p_I, 0, 0, E_1)/m_m \]
\[ \varepsilon_2(p_2) = (p_I, 0, 0, -E_2)/m_n \]
\[ \varepsilon_3(p_3) = (p_F, E_3 \sin \chi, 0, E_3 \cos \chi)/m_l \]
\[ \varepsilon_4(p_4) = (p_F, -E_4 \sin \chi, 0, -E_4 \cos \chi)/m_k \]

Table III: The 4 momenta and the polarization vectors of the initial and the final states. The definitions of \( E_i \) \((i = 1, 2, 3, 4)\), \( p_I \) and \( p_F \) are given in Eq. (3.12), and \( \chi \) is the scattering angle in the center-of-mass frame.

more conventional forms in the KK analysis. For example, Eq. (3.3) is rewritten as

\[ \mathcal{A}_{mnkl}^{V} = i \sum_{r} \left\{ \frac{\lambda_{mnrl,k}}{p_{12}^2 + m_r^2} P_{1234} - \frac{\lambda_{mnlr,k}}{p_{13}^2 + m_r^2} P_{1324} - \frac{\lambda_{nlrk}}{p_{14}^2 + m_r^2} P_{1423} \right\}, \quad (3.10) \]

where

\[ \lambda_{mnrl,k} \equiv g_A \int_{0}^{L} dy \, C^{ijk} u_{r}^{I}(y) u_{m}^{J}(y) u_{n}^{K}(y) \quad (3.11) \]

is a 4D effective coupling constant among the KK modes. Below \( m_{KK} \), contributions of the heavy KK modes in the infinite sum are negligible because of suppression by large KK masses in the 4D propagators. Thus we can approximate \( \mathcal{A}_{mnkl}^{V} \) in a good accuracy by picking up only finite number of light modes in Eq. (3.10). However, above \( m_{KK} \), such an approximation becomes worse and much larger number of KK modes are necessary for summation in order to keep the accuracy of the approximation. Therefore this approximation is not practical for our purpose since we would like to see the behaviors of the scattering amplitudes beyond the KK mass scale. The 5D propagators enable us to calculate the amplitudes in high-energy region with sufficient accuracy.

The 4 momenta and the polarization vectors of the initial and final states are parameterized as in Table III. There, \( \chi \) is the scattering angle in the center-of-mass frame, and the energy and the momentum of each particle are expressed as

\[ E_1 = \frac{E}{2} + \frac{m_m^2 - m_n^2}{2E}, \quad E_2 = \frac{E}{2} + \frac{m_m^2 - m_m^2}{2E}, \]
\[ p_I = \sqrt{E_1^2 - m_m^2} = \sqrt{E_2^2 - m_m^2} = \left\{ \frac{E^2}{4} - \frac{m_m^2 + m_n^2}{2} + \frac{(m_m^2 - m_n^2)^2}{4E^2} \right\}^{1/2}, \]
\[ E_3 = \frac{E}{2} + \frac{m_l^2 - m_k^2}{2E}, \quad E_4 = \frac{E}{2} + \frac{m_k^2 - m_l^2}{2E}, \]
\[ p_F = \sqrt{E_3^2 - m_l^2} = \sqrt{E_4^2 - m_k^2} = \left\{ \frac{E^2}{4} - \frac{m_l^2 + m_k^2}{2} + \frac{(m_l^2 - m_k^2)^2}{4E^2} \right\}^{1/2}, \quad (3.12) \]

where \( E \) is the total energy in the center-of-mass frame.
3.2.1 Non-forward scattering

First we consider the non-forward (and non-backward) scattering. We choose the scattering angle as $\chi = \pi/3$ in the following. Let us consider the process: $W_L^+ + W_L^- \rightarrow Z_L + Z_L$, as an example. In this case, the mode functions in Eqs.(3.2)-(3.4) are taken as

$$u_m = (u_W^{+W}, u_W^{+R}, u_W^3, 0, 0, 0, 0, 0, 0, 0),$$

$$u_n = (0, 0, 0, u_W^{-W}, u_W^{-R}, u_W^3, 0, 0, 0, 0),$$

$$u_l = u_k = (0, 0, 0, 0, 0, u_W^{3L}, u_W^{3R}, u_W^0, 0),$$

where $u_W^{L}(y)$ and $u_W^{R}(y)$ are defined in Eqs.(2.24), (2.25) and (2.29), respectively. Then, Eqs.(3.2)-(3.4) are reduced to

$$A_{WWZZ}^C = -ig_A^2 \int_0^L dy \frac{U_{WWZ}(y)}{U_{WWZ}(y)} \left\{ 2(\varepsilon_1 \cdot \varepsilon_2) \left( \varepsilon_3 \cdot \varepsilon_4 \right) - (\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) - (\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) \right\},$$

$$A_{WWZZ}^V = ig_A^2 \int_0^L dy \int_0^L dy' \frac{U_{WWZ}(y) \cdot G_{T}^{eh}(y, y', |p_{13}|) \cdot U_{WWZ}(y')}{P_{13}}$$

$$+ ig_A^2 \int_0^L dy \int_0^L dy' \frac{U_{WWZ}(y) \cdot G_{T}^{eh}(y, y', |p_{14}|) \cdot U_{WWZ}(y')}{P_{14}}$$

$$A_{WWZZ}^S = ig_A^2 \int_0^L dy \frac{e^{2\sigma(y)}}{P_{12}^2} \left[ Y^4_{WW}(y) Y^4_{ZZ}(y) \left( \frac{(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4)}{P_{12}^2} + \frac{(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4)}{P_{13}^2} \right) \right],$$

where

$$U_{WW} \equiv \frac{1}{2} \left( 2u_W^{+L}u_{Z}^3 + u_W^{+R}u_{Z}^3, 2u_W^{+L}u_{Z}^3 + u_W^{+R}u_{Z}^3, u_W^{+L}u_{Z}^3 + u_W^{+R}u_{Z}^3 \right),$$

$$Y^4_{WW} \equiv e^{-2\sigma} \left\{ \left( u_W^{+L} - u_W^{+R} \right)^{\prime} u_W^{+L} - \left( u_W^{+L} - u_W^{+R} \right) \left( u_{Z}^3 \right)^{\prime} \right\},$$

$$Y^4_{ZZ} \equiv e^{-2\sigma} \left\{ \left( u_{Z}^3 - u_{Z}^{3R} \right)^{\prime} u_{Z}^3 - \left( u_{Z}^3 - u_{Z}^{3R} \right) \left( u_{Z}^3 \right)^{\prime} \right\},$$

$$Y_{WW} \equiv \frac{e^{-2\sigma}}{2} \left( 2(u_W^{+L})^{\prime} u_{Z}^3 + (u_W^{+R})^{\prime} u_{Z}^3 - 2u_W^{+L} (u_{Z}^3)^{\prime} - u_W^{+L} (u_{Z}^3)^{\prime} \right),$$

$$2 (u_W^{+L})^{\prime} u_{Z}^3 + (u_W^{+R})^{\prime} u_{Z}^3 - 2u_W^{+L} (u_{Z}^3)^{\prime} - u_W^{+L} (u_{Z}^3)^{\prime},$$

$$2 (u_W^{+L})^{\prime} u_{Z}^3 + (u_W^{+R})^{\prime} u_{Z}^3 - 2u_W^{+L} (u_{Z}^3)^{\prime} - u_W^{+L} (u_{Z}^3)^{\prime},$$

$$(u_W^{+L} + u_W^{+R})^{\prime} u_{Z}^3 + (u_W^{+L})^{\prime} (u_{Z}^3 + u_{Z}^{3R})$$

$$- (u_W^{+L} + u_W^{+R}) (u_{Z}^3)^{\prime} - u_W^{+L} (u_{Z}^3 + u_{Z}^{3R}) \right).$$

Fig. 1 shows the energy dependence of the scattering amplitudes in the unit of $m_W$. The solid and dashed lines represent the amplitudes for the vector modes $A$ and for the
gauge-scalar modes $\mathcal{B}$, respectively. We can explicitly see that the equivalence theorem holds both in the flat and warped cases, and $|\mathcal{B}| - |\mathcal{A}| = \mathcal{O}(m_W^2/E^2)$. In the warped case ($kL = 30$), the situation is similar to the $SU(3)$ toy model [18]. The amplitudes behave as $E^2$ and grow faster for larger values of $\sin^2 \theta_H$. In the flat case ($kL = 0$), on the other hand, the situation is quite different from the $SU(3)$ model. In contrast to the $SU(3)$ model, the amplitudes monotonically increase and depend on $\theta_H$. Again, they grow faster for larger values of $\sin^2 \theta_H$. This difference from the $SU(3)$ model stems from the mixing between different KK levels mentioned around Eq. (2.35). Note that $\theta_H = \mathcal{O}(1)$ is experimentally excluded in the flat spacetime because it leads to too light KK excitation modes. However, we will also plot the amplitudes for such values of $\theta_H$ in the following, in order to understand theoretical structure of the gauge-Higgs unification model.

These behaviors of the amplitudes reflect the $\theta_H$-dependences of the coupling constants among the gauge and Higgs modes and of $m_{KK}$. First of all, we should notice that the model reduces to SM when $\theta_H \ll 1$ irrespective of the 5D geometry. Every coupling constant in the gauge-Higgs sector takes almost the SM value and the KK modes are heavy enough to decouple. Thus the amplitude takes the same value as SM. Namely the amplitudes are almost constant for $E^2 \gg m_W^2$. When $\theta_H = \mathcal{O}(1)$, the coupling constants deviate from the SM values [15, 16]. In the flat spacetime, the $WWZ$ and $WWZZ$ couplings become smaller while the $WWH$ and $ZZH$ couplings take the SM values. In the warped spacetime, the latter couplings are suppressed by a factor $\cos \theta_H$ while the former couplings are almost unchanged from the SM values. Therefore the $\mathcal{O}(E^2)$ contributions miss to be cancelled.

Figure 1: The energy dependence of the amplitudes for $W_L^+ + W_L^- \rightarrow Z_L + Z_L$. The solid lines represent the vector mode scattering $\mathcal{A}$, and the dashed lines are the gauge-scalar mode scattering $\mathcal{B}$. The scattering angle is chosen as $\chi = \pi/3$. 

These behaviors of the amplitudes reflect the $\theta_H$-dependences of the coupling constants among the gauge and Higgs modes and of $m_{KK}$. First of all, we should notice that the model reduces to SM when $\theta_H \ll 1$ irrespective of the 5D geometry. Every coupling constant in the gauge-Higgs sector takes almost the SM value and the KK modes are heavy enough to decouple. Thus the amplitude takes the same value as SM. Namely the amplitudes are almost constant for $E^2 \gg m_W^2$. When $\theta_H = \mathcal{O}(1)$, the coupling constants deviate from the SM values [15, 16]. In the flat spacetime, the $WWZ$ and $WWZZ$ couplings become smaller while the $WWH$ and $ZZH$ couplings take the SM values. In the warped spacetime, the latter couplings are suppressed by a factor $\cos \theta_H$ while the former couplings are almost unchanged from the SM values. Therefore the $\mathcal{O}(E^2)$ contributions miss to be cancelled.
Figure 2: The energy dependence of the amplitude $B_{WWZZ}$ in the unit of $m_{KK}$. The solid, dotdashed, dotted and dashed lines correspond to $\theta_H = 0.1, 0.5, 1.0, 1.5$, respectively.

among the low-lying modes, and the amplitudes grows. For larger $\sin^2 \theta_H$, the deviation of the couplings become larger, and thus the amplitudes grow faster.

The remaining $\mathcal{O}(E^2)$ contribution is eventually cancelled by contributions from the KK modes. Namely, the amplitudes cease to increase and approach to constant values when the KK modes start to propagate. We can see this behavior by rescaling the unit of the horizontal axes in Fig. [1] to $m_{KK}$ (Fig. [2]). In the warped case, the $\theta_H$-dependence almost disappears in Fig. [2]. The $\theta_H$-dependence appearing in Fig. [1] is cancelled by that of $m_{KK}$. Thus the asymptotic constant value of the amplitude is almost determined only by the value of $kL$. (See Fig. 2 in Ref. [18].)

All the above behaviors can also be seen in other processes, such as the elastic scatterings: $W^+_L + W^-_L \rightarrow W^+_L + W^-_L$ and $W^+_L + Z_L \rightarrow W^+_L + Z_L$. In contrast to the process: $W^+_L + W^-_L \rightarrow Z_L + Z_L$, there are $s$-channel diagrams exchanging the KK vector bosons in these processes, which lead to the resonances. The tree-level amplitudes diverges there. In order to evaluate the amplitudes around the resonances, we have to include the widths of each states, which are obtained from one-loop correction of the 5D propagators.

3.2.2 Forward scattering

Next we consider the forward (backward) scattering, i.e., $\chi \approx 0 (\pi)$. Let us first consider the inelastic scattering process: $W^+_L + W^-_L \rightarrow Z_L + Z_L$. In this case, an $\mathcal{O}(E^2)$ contribution remains and the amplitude monotonically increases even above $m_{KK}$. This is because the power counting of $E$ for the amplitude changes around $\chi = 0$. For example, the brace part...
of $A^S$ in Eq. (3.14) is expanded (for nonzero $\sin \chi$) as

$$A_{tu} \equiv \frac{(\varepsilon_1 \cdot \varepsilon_3^*)(\varepsilon_2 \cdot \varepsilon_4^*)}{E^2} + \frac{(\varepsilon_1 \cdot \varepsilon_3^*)(\varepsilon_2 \cdot \varepsilon_3^*)}{P^2_{13}}$$

$$= \frac{E^2}{4m_W^2 m_Z^2} - \frac{m_W^2 + m_Z^2}{2m_W^2 m_Z^2} + \frac{2m_W^4 m_Z^2 + (m_W^4 + m_Z^2) \cos(2\chi)}{m_W^2 m_Z^2 E^2 \sin^2 \chi} + O(E^{-4}). \quad (3.16)$$

This means that the expansion becomes invalid when $\sin \chi \lesssim O(m_W/E)$. At $\chi = 0$, this quantity reduces to

$$A_{tu} = \frac{(m_W^4 + m_Z^4)E^2}{2m_W^2 m_Z^2(m_Z^2 - m_W^2)^2} - \frac{2(m_W^2 + m_Z^2)}{(m_Z^2 - m_W^2)^2}, \quad (3.17)$$

and the leading term for the high energy expansion changes. Therefore an $O(E^2)$ contribution is left in the total amplitude. Similar behavior of the amplitude is observed also in SM.

Next we consider the elastic scattering process: $W^+_L + W^-_L \rightarrow W^+_L + W^-_L$. The mode functions in Eqs. (3.2)-(3.4) are taken as

$$u_m = u_k = (u^+_W, u^-_W, u^+_W, 0, 0, 0, 0, 0, 0, 0),$$

$$u_n = u_l = (0, 0, 0, u^-_W, u^+_W, u^-_W, 0, 0, 0, 0). \quad (3.18)$$

Then the expression of the amplitude is reduced to

$$A^C_{WWW} = -i g^2_\alpha \int_0^L dy \ U_{WW}^2(y) \left\{ (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3^* \cdot \varepsilon_4^*) + (\varepsilon_1 \cdot \varepsilon_3^*)(\varepsilon_2 \cdot \varepsilon_3^*) - 2(\varepsilon_1 \cdot \varepsilon_3^*)(\varepsilon_2 \cdot \varepsilon_3^*) \right\},$$

$$A^V_{WWW} = -i g^2_\alpha \int_0^L dy \int_0^L dy' \ U_{WW}(y) \cdot G_T^m(y, y', |p_{12}|) \cdot U_{WW}(y) P_{1234},$$

$$+ i g^2_\alpha \int_0^L dy \int_0^L dy' \ U_{WW}(y) \cdot G_T^m(y, y', |p_{13}|) \cdot U_{WW}(y) P_{1324},$$

$$A^S_{WWW} = i g^2_\alpha \int_0^L dy \ \epsilon^{2\sigma(y)} \left\{ \frac{(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3^* \cdot \varepsilon_4^*)}{p^2_{12}} + \frac{(\varepsilon_1 \cdot \varepsilon_3^*)(\varepsilon_2 \cdot \varepsilon_4^*)}{p^2_{13}} \right\}, \quad (3.19)$$

where

$$U_{WW} \equiv \frac{1}{2} \left( 2(u^+_W)^2 + (u^-_W)^2, 2(u^+_W)^2 + (u^-_W)^2, 0, 2(u^+_W + u^-_W)u^+_W \right). \quad (3.20)$$

In this case, the amplitude $A_{WWW}(E, \chi)$ has a singularity at $\chi = 0$. This is due to the $t$-channel diagram exchanging the massless photon, which is proportional to $1/p^2_{13} = (E^2/2 - 2m_W^2)(1 - \cos \chi)^{-1}$. In any actual collider experiments, however, such forward scattering processes cannot be measured because they cannot be distinguished from the
ones that two particles pass by without interacting with each other. They are also irrelevant in the cosmological processes by the same reason. Therefore the divergence at $\chi = 0$ does not lead to any difficulties in most practical calculations. However we have to deal with this singularity in a proper manner when we estimate the unitarity bound. We will come back to this point in the next section.

4 Unitarity bound

4.1 Unitarity conditions

The unitarity bound originates from the unitarity condition of the $S$ matrix, $S^\dagger S = 1$, which, with the definition of $S = 1 + iT$, can be expressed as $T^\dagger T = 2\text{Im} T$. Taking the matrix element of both sides of the latter relation between identical 2-body states and inserting a complete set of intermediate states into the left-hand side, we obtain

$$\int_{PS_2} |T_{el}[2 \rightarrow 2]|^2 + \sum_N \int_{PS_N} |T_{inel}[2 \rightarrow N]|^2 = 2\text{Im} T_{el}[2 \rightarrow 2],$$  \hspace{1cm} (4.1)

where $T_{el}[2 \rightarrow 2]$ and $T_{inel}[2 \rightarrow N]$ denote amplitudes for a 2-body elastic scattering and for an inelastic scattering with $N$-body final state respectively, and $\int_{PS_N}$ denotes the $N$-body phase space integration. The right-hand side is evaluated in the forward direction. By performing the partial wave expansion for the scattering amplitudes for the $2 \rightarrow 2$ processes, Eq.(4.1) is rewritten as (see, for example, Ref. [25])

$$\sum_{j=0}^\infty (2j + 1) \left\{ \frac{1}{\rho_e} \left( \frac{\rho_e^2}{4} - \left| \eta_{elj}^{el} a_{j}^{el} - \frac{i\rho_e}{2} \right|^2 \right) - \sum_{N=2} \frac{\eta_{elj}^{el} \eta_{inelj}^{inel}}{\rho_i} \left| a_{j}^{inel} \right|^2 \right\} = \frac{\eta_{elj}^{el}}{32\pi} \sum_{N \neq 2} \int_{PS_N} |T_{inel}[2 \rightarrow N]|^2 > 0,$$  \hspace{1cm} (4.2)

where the symmetry factors $\rho_e$ and $\rho_i$ equal 1! (2!) if the 2-body final state consists of nonidentical (identical) particles for the elastic and inelastic scattering processes. The partial wave components of the amplitudes are defined as

$$a_{j}^{el} \equiv \frac{1}{32\pi} \int_{-1}^{1} d(\cos \chi) P_j(\cos \chi) T_{el}[2 \rightarrow 2],$$

$$a_{j}^{inel} \equiv \frac{1}{32\pi} \int_{-1}^{1} d(\cos \chi) P_j(\cos \chi) T_{inel}[2 \rightarrow 2],$$  \hspace{1cm} (4.3)

\footnote{Here we focus on the case that the two particles in the initial or final state have the same helicity.}
where \( P_j(x) \) are the Legendre polynomials. The factors \( \eta_i^{el} \) and \( \eta_i^{inel} \) are functions of the total energy and the masses of the final state particles defined as

\[
\eta(E, m_l, m_k) = \frac{2p_F}{E} = \left\{ 1 - \frac{2(m_l^2 + m_k^2)}{E^2} + \frac{(m_l^2 - m_k^2)^2}{E^4} \right\}^{1/2},
\]

evaluated for the elastic and inelastic scattering process, respectively. In the high energy region \( E^2 \gg m_l^2, m_k^2 \), these factors are approximately equal to one.

In the following we assume that the S-wave component \((j = 0)\) is dominant in Eq. (4.2). Then, for scattering of \( W^+ \) and \( W^- \), we obtain the following unitarity condition.

\[
\left| \eta_{WW}^{00} a_{00}^{00}[WW] - \frac{i}{2} \right|^2 + \frac{\eta_{WW}^{00} \eta_{ZZ}^{00}}{2} |a_{00}^{00}[ZZ]|^2
\]

\[+ \eta_{WW}^{00} \sum_{(l,k) \neq (0,0)} \left\{ \eta_{WW}^{lk} |a_{0k}^{lk}[WW]|^2 + \eta_{ZZ}^{lk} |a_{0k}^{lk}[ZZ]|^2 \right\} < \frac{1}{4}, \tag{4.5}\]

where \( a_{00}^{lk}[WW] \) and \( a_{00}^{lk}[ZZ] \) are the S-wave amplitudes for the processes to \( W^{+l}, W^{-k} \) and \( Z^{(l)}, Z^{(k)} \) in the final state respectively, and \( \eta_{WW}^{lk} \) and \( \eta_{ZZ}^{lk} \) are the corresponding factors defined in Eq. (4.4). Here \( W^{\pm l} \) and \( Z^{(l)} \) denote the \( l \)-th KK excitation modes in the charged and neutral sectors. \(^8\) The symmetry factor \( \rho_{lk} \) equals 1! (2!) when \( l \neq k \) \((l = k)\). We do not consider processes to fermions in the final state because we have not specified the matter sector.

Here we comment on contributions of the forward scattering to the S-wave amplitudes. Let us first consider the process: \( W_L^+ + W_L^- \rightarrow Z_L + Z_L \). Since \( iT_0[2 \rightarrow 2] = A_{WWZZ} \) at tree level, the S-wave amplitude is obtained as

\[
a_{00}^{00}[ZZ](E) = \frac{-i}{32\pi} \int_{-1}^{1} d(\cos \chi) A_{WWZZ}(E, \chi) = \frac{-i}{16\pi} \int_{0}^{1} d(\cos \chi) A_{WWZZ}(E, \chi). \tag{4.6}\]

In the last equality, we have used the relation \( A_{WWZZ}(E, \chi) = A_{WWZZ}(E, \pi - \chi) \). As mentioned in Sec. 3.2.2, the integrand grows as \( E^2 \) in the region \( 1 - |\cos \chi| \lesssim O(m_W^2/E^2) \) while it approaches to a constant for \( E^2 \gg m_{KK}^2 \) in the other region of \( \cos \chi \). Therefore, \( a_{00}^{00}[ZZ] \) behaves as \( O(E^0) \) at high energies. In fact, it grows logarithmically above \( m_{KK} \). (See Fig. 3 in Ref. [18].)

Next we consider the elastic scattering: \( W_L^+ + W_L^- \rightarrow W_L^+ + W_L^- \). As mentioned at the end of the previous section, the tree-level amplitude \( A_{WWWW}(E, \chi) \) diverges at \( \chi = 0 \).

\(^8\) In this notation, \( Z^{(l)} \) \((l = 0, 1, 2, \cdots)\) include the KK modes of the photon except for the massless photon. The lowest mode \( Z^{(0)} \) is identified with the Z boson.
Such divergence is smeared out by taking into account the instability of the $W$ bosons in the final state, as shown in Appendix C. The effect of the instability is translated into a cut-off for the $\cos \chi$-integral. Then the S-wave amplitude is calculated as

$$a_{00}^{00}[WW](E) = \frac{-i}{32\pi} \int_{-1}^{\text{cut}} d(\cos \chi) \mathcal{A}_{WWWW}(E, \chi),$$

(4.7)

where $x_{\text{cut}}$ is given by Eq. (C.14).

Notice that the Higgs boson is massless at tree-level in the gauge-Higgs unification scenario. Thus the $t$-channel diagram exchanging the Higgs boson is also singular at $\chi = 0$. Therefore the Higgs mass has to be incorporated in a proper manner in order to evaluate the S-wave amplitude. The consistent way to deal with the nonzero Higgs mass is to include quantum corrections, which is however beyond the scope of this paper. Instead, we introduce the Higgs mass parameter $m_H$ in the Higgs propagator appearing in the expressions of the amplitude, as a free parameter as discussed in the introduction. Namely, we modify the Higgs-propagator part of $A_{WWZZ}$ in Eq. (3.14) as

$$Y_{WW}^4(y) \frac{(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3^* \cdot \varepsilon_4^*)}{p_{12}^2} \rightarrow Y_{WW}^4(y) \frac{(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3^* \cdot \varepsilon_4^*)}{p_{12}^2 + m_H^2},$$

(4.8)

and of $A_{WWWW}$ in Eq. (3.19) as

$$\left(Y_{WW}^4(y)\right)^2 \left\{ \frac{(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3^* \cdot \varepsilon_4^*)}{p_{12}^2} + \frac{(\varepsilon_1 \cdot \varepsilon_3^*)(\varepsilon_2 \cdot \varepsilon_4^*)}{p_{13}^2} \right\} \rightarrow \left(Y_{WW}^4(y)\right)^2 \left\{ \frac{(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3^* \cdot \varepsilon_4^*)}{p_{12}^2 + m_H^2} + \frac{(\varepsilon_1 \cdot \varepsilon_3^*)(\varepsilon_2 \cdot \varepsilon_4^*)}{p_{13}^2 + m_H^2} \right\}. \quad (4.9)$$

This is a good approximation since the quantum corrections to the KK masses are sub-dominant and thus negligible. Fig. 3 shows the $m_H$-dependence of the S-wave amplitude $a_{00}^{00}[ZZ](E)$. We can see from these figures that the $m_H$-dependence disappears when $\theta_H = \pi/2$ in both the flat and warped cases. This is because the $WWH$ and $ZZH$ couplings vanish and the Higgs propagator does not contribute to the amplitude when $\theta_H = \pi/2$. For other values of $\theta_H$, the introduction of larger $m_H$ reduces the amplitude.

### 4.2 Unitarity bound from WW scattering

Now we estimate the unitarity bound. Let us define the summed amplitude $\bar{a}_0(E)$ as

$$\bar{a}_0 \equiv \left\{ \left(\eta_{WW}^{00} \text{Re} a_0^{00}[WW]\right)^2 + \frac{\eta_{WW}^{00} \eta_{ZZ}^{00}}{2} \left| a_0^{00}[ZZ]\right|^2 \right\}^{1/2}. \quad (4.10)$$
Figure 3: The S-wave amplitude for $W_L^+ + W_L^- \rightarrow Z_L + Z_L$. The solid (dashed) lines represent $a_0^{00}[ZZ]$ with $m_H = 2m_W (4m_W)$ for $\theta_H = 0.1, 0.5, 1.0, 1.5$ from bottom to top.

Then the following unitarity bound is obtained from Eq. (4.5).

$$\tilde{a}_0(E) < \frac{1}{2}.$$  \hspace{1cm} (4.11)

Notice that the left-hand side of Eq. (4.5) already saturates the unitarity bound if $\text{Im} a_0^{00}[WW] = 0$. Although the imaginary part of the S-wave amplitudes are zero at tree level, nonvanishing contribution comes out at loop level. This loop contribution can be large near $\Lambda_{\text{uni}}$ since perturbative expansion is less reliable there. Hence we should take it into account in order to obtain a nontrivial unitarity bound. However estimation of $\text{Im} a_0^{00}[WW]$ at loop level is beyond the scope of this paper. In this paper, we simply assume that there is enough contribution to $\text{Im} a_0^{00}[WW]$ at loop level to cancel $-i/2$ in the first term of the left-hand side of Eq. (4.5), and consider only the real part of the S-wave amplitudes to estimate the unitarity bound.

Fig. 4 shows $\tilde{a}_0(E)$ for various values of $\theta_H$ in the warped spacetime. The Higgs mass is chosen as $m_H = 2m_W$ in this plot. The dashed line represents the unitarity bound. From this figure, we can read off the (conservative) unitarity violation scale as $\Lambda_{\text{uni}} \approx 22m_W \approx 1.8 \text{ TeV}$ for $\theta_H = 1.5$, and $\Lambda_{\text{uni}} \approx 46m_W \approx 3.7 \text{ TeV}$ for $\theta_H = 0.5$. The unitarity is violated at $\mathcal{O}(1 \text{ TeV})$ when $\theta_H = \pi/2$ since the $WWH$ and $ZZH$ couplings vanish. We cannot see any KK resonances in Fig. 4 despite the fact that the amplitude $A_{WWWW}$ has divergent peaks at the resonances, which correspond to the KK gauge bosons. The reason for this is as follows. Such divergent peaks originate from the s-channel diagrams corresponding to a term proportional to $P_{1234}$ in Eq. (3.19). However this term will vanish after integrating for
Figure 4: The summed amplitude $\bar{a}_0$ defined in Eq. (4.10) in the warped spacetime. The Higgs mass is chosen as $m_H = 2m_W$. The dashed line represents the unitarity bound.

$\cos \chi$ over $[-1, 1]$ because it is proportional to $\cos \chi$. This fact can also be understood from the viewpoint of the spin composition. Since the longitudinal vector boson is a state with the angular momentum $(j, j_3) = (1, 0)$, intermediate KK vector boson states for the $s$-channel must also have the quantum number $(j, j_3) = (1, 0)$. When the orbital angular momentum is zero, however, it is impossible to create such a spin state by the composition of two states with $(j, j_3) = (1, 0)$. Therefore, the $s$-channel contribution to the S-wave amplitude is zero.

In the flat spacetime, the amplitude grows slowly and thus $\Lambda_{uni}$ is much higher than the warped case. In fact, the unitarity bound from Eq. (4.11) is determined by the logarithmic behavior of $\bar{a}_0(E)$ at high energies, which is mentioned below Eq. (4.6). In such a case, contributions of inelastic scattering involving the KK modes in the final state become important because a large number of scattering processes are kinematically allowed near $\Lambda_{uni}$. Therefore the summed amplitude $\bar{a}_0$ should be modified by including the contributions of such processes as

$$
\bar{b}_0^0 \equiv \left( \eta_{WW}^0 \text{Re} a_0^0 [WW] \right)^2 + \frac{\eta_{WW}^0 \eta_{ZZ}^0}{2} \left| a_0^0 [ZZ] \right|^2 \\
+ \eta_{WW}^0 \sum_{l \geq 1} \left( \eta_{WW}^l \left| a_l^0 [WW] \right|^2 + \frac{\eta_{ZZ}^l}{2} \left| a_0^l [ZZ] \right|^2 \right). \tag{4.12}
$$

Contributions of the scattering processes to different KK levels $a_l^k [WW]$ and $a_0^k [ZZ]$ ($l \neq k$) are generically small and can be neglected. The unitarity bound is written as

$$
\bar{b}_0^0 (E) < \frac{1}{4}. \tag{4.13}
$$

10 The cut-off $x_{cut}$ in Eq. (4.7) is introduced only the integral of the $t$-channel contribution, which corresponds to the term proportional to $P_{324}$ in Eq. (3.19).
Figure 5: The summed amplitudes in the flat spacetime. In the left figure, the solid lines denote $\bar{b}_0^2$ defined in Eq. (4.12), and the dashed lines denote $\bar{a}_0^2$ defined in Eq. (4.10), for $\theta_H = 0.2, 0.5, 1.0, 1.5$ from bottom to top. In the right figure, the solid, dotdashed, dotted and dashed lines correspond to $\theta_H = 0.2, 0.5, 1.0, 1.5$, respectively. The Higgs mass is chosen as $m_H = 2m_W$ in both figures.

The left plot in Fig. 5 shows the energy dependence of $\bar{b}_0$ (solid lines) and $\bar{a}_0$ (dashed lines). We can see from this plot that the summed amplitude $\bar{b}_0^2(E)$ asymptotically behaves as an increasing linear function, while $\bar{a}_0^2(E)$ does logarithmically. This is a consequence of the intrinsic nonrenormalizability of the higher dimensional gauge theory, as was pointed out in Ref. [27] in the context of the Higgsless models. The inclination of the asymptotic line vary over the values of $\theta_H$. It is mainly determined by the KK mass scale $m_{KK}$. For smaller values of $\theta_H$, the KK modes does not appear until higher energy scales, and the amplitude grows at a slower pace. This can be explicitly seen in the right plot of Fig. 5, in which the unit of the horizontal axis is rescaled to $m_{KK}$. In this plot, the inclination of the asymptotic line is almost independent of $\theta_H$. By extrapolating the asymptotic lines, we obtain the unitarity violation scale as $\Lambda_{uni} \simeq 140m_{KK}$, irrespective of the value of $\theta_H$.

In Ref. [27], it was found that $\Lambda_{uni}$ is roughly equal (up to a small numerical factor) to the cut-off scale of the 5D theory obtained from naive dimensional analysis (NDA) in the Higgsless model. In our model, the NDA cut-off scale $\Lambda_{NDA}$ is estimated as

$$\Lambda_{NDA} = \frac{24\pi^3}{g^2L} \simeq 592m_{KK} \simeq \frac{1.86 \times 10^3m_W}{\sin^{-1}\left(\frac{1}{\sqrt{2}}\sin\theta_H\right)} \quad (4.14)$$

Here we have used that $g^2 = g_A^2/L = 0.4$ is the 4D weak gauge coupling constant, and Eq. (2.34). Therefore $\Lambda_{uni}$ is lower than $\Lambda_{NDA}$ by about a factor of four in our model.

Finally we remark that $\Lambda_{uni}$ estimated here is a conservative one since we did not consider scattering processes to fermions in the final states, as mentioned in the introduction.
5 Summary

We have estimated a scale $\Lambda_{\text{uni}}$, at which the tree-level unitarity is violated, in the 5D $SO(5) \times U(1)_X$ gauge-Higgs unification model by evaluating amplitudes of the weak boson scattering. The 5D propagators are useful to evaluate the amplitudes because we need not explicitly calculate the KK mass eigenvalues and mode functions nor perform infinite summation over the KK modes propagating in the internal lines. In particular above the KK mass scale $m_{KK}$, they provide a practical method of evaluating the amplitudes. Although inelastic scattering processes to fermionic final states are not examined, the techniques illustrated in this article are also useful to evaluate them, and similar behaviors of the amplitudes are expected even when they are incorporated, while the numerical value of $\Lambda_{\text{uni}}$ would be somewhat reduced.

We have numerically checked the equivalence theorem between the amplitudes for the 4D longitudinal vector modes and for the gauge-scalar modes. The amplitude with nonzero scattering angle monotonically increases up to $m_{KK}$, and depends on the Wilson line phase $\theta_H$. It grows faster for larger values of $\sin^2 \theta_H$. In the warped spacetime, its value is enhanced for $\theta_H = \mathcal{O}(1)$ while it is reduced to that in the flat spacetime for $\theta_H \ll 1$. These behaviors can be understood by the $\theta_H$-dependences of the coupling constants among the gauge and Higgs modes and of $m_{KK}$. The growth of the amplitude stems from deviation of the coupling constants from the SM value. In the warped case, for example, the $WWH$ and $ZZH$ couplings are suppressed from the SM values by a factor of $\cos \theta_H$. Due to this deviation, the $\mathcal{O}(E^2)$ contributions of $\mathcal{A}^C$, $\mathcal{A}^V$ and $\mathcal{A}^S$ in Eqs. (3.2)-(3.4) miss to be cancelled among the light modes, and the amplitude grows as shown in Fig. 1. The remaining $\mathcal{O}(E^2)$ contribution depends on the deviation of the couplings and has the maximal value when $\theta_H = \pi/2$. It is eventually cancelled by the contributions from the KK gauge bosons. Then the amplitude ceases to increase and approaches to a constant value above $m_{KK}$. (See Fig. 2). These behaviors are also observed in the $SU(3)$ model \cite{18}, and are thought to be common to the gauge-Higgs unification models. In contrast to the $SU(3)$ model, however, the amplitude in our model grows and depends on $\theta_H$ even in the flat spacetime. This difference originates from the mixing between different KK levels mentioned around Eq. (2.35).

In Ref. \cite{17}, three separate scales that determine the dynamics of the scattering processes are introduced, \textit{i.e.}, the electroweak breaking scale $v$, the Higgs boson decay constant $f_h$ \cite{11} and the KK mass scale $m_{KK}$. In our notation, these scales are related to each other by

\footnote{This is the composite scale of the Higgs boson in the holographic dual picture.}
other as \( v = f_h \theta_H \) and \( f_h = \frac{\sqrt{2}}{(g_A \sqrt{L})} = \sqrt{2}m_{\text{KK}}/(\pi g) \) in the flat case, and \( v = f_h \sin \theta_H \) and \( f_h \simeq 2\sqrt{k} e^{-kL}/g_A \simeq 2m_{\text{KK}}/(\pi g \sqrt{kL}) \) in the warped case. In the terminology of Ref. [17], the case of \( \theta_H \ll 1 \) is referred to as the ‘Higgs limit’, and the case of \( \theta_H \simeq \pi/2 \) as the ‘Higgsless limit’. The Higgs boson mainly unitarizes the scattering processes in the former while it does not in the latter.

We have evaluated the S-wave amplitudes in order to estimate \( \Lambda_{\text{uni}} \). We considered the scattering of \( W_L^+ \) and \( W_L^- \) including possible inelastic scatterings. In order to evaluate the S-wave amplitude for the elastic scattering, we have to deal with the singularity of the amplitude at forward scattering in a proper manner. We have accomplished this by taking into account the instability of the W bosons in the final state. The results are depicted in Figs. 4 and 5. From Fig. 4, we can read off \( \Lambda_{\text{uni}} \simeq 1.3m_{\text{KK}} \simeq 7f_h \). The unitarity is violated at \( \mathcal{O}(1 \text{ TeV}) \) for \( \theta_H = \pi/2 \) in the warped case because the \( WWH \) and \( ZZH \) couplings vanish and the situation becomes similar to SM without the Higgs boson in such a case. For \( \theta_H = \mathcal{O}(0.1) \), the unitarity is maintained up to \( \mathcal{O}(20 \text{ TeV}) \). In the flat spacetime, \( \Lambda_{\text{uni}} \) becomes much higher than the warped case. In this case, a large number of inelastic scatterings to the KK modes become kinematically allowed around \( \Lambda_{\text{uni}} \), and thus we should take into account contributions from those scattering processes. The summed amplitude \( b_0^2(E) \) defined in Eq.(4.12) is approximately a linear function in the high energy region. (See Fig. 5.) This is a consequence of the intrinsic nonrenormalizability of the higher dimensional gauge theory. We have found that \( \Lambda_{\text{uni}} \simeq 140m_{\text{KK}} \), which is lower than the cut-off scale \( \Lambda_{\text{NDA}} \) from naive dimensional analysis by about a factor of four.

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12 These correspondence are essentially the same as in the \( SU(3) \) model in Ref. [18].

13 The unitarity bound from other processes, such as the scattering of \( W_L^+ \) and \( Z_L \) or of two Z bosons, is weaker.
A Bases of mode functions

Here we define bases of mode functions, following Ref. [28]. The functions $C_0(y, m)$ and $S_0(y, m)$ are defined as two independent solutions to

$$\left( \frac{d}{dy} e^{-2\sigma} \frac{d}{dy} + m^2 \right) f = 0, \quad (A.1)$$

with initial conditions

$$C_0(0, m) = 1, \quad C_0'(0, m) = 0,$$
$$S_0(0, m) = 0, \quad S_0'(0, m) = me^{-\sigma(L)}. \quad (A.2)$$

The prime denotes derivative in terms of $y$.

For the derivation of 5D propagators in Appendix B, it is convenient to define another basis functions $C_L(y, m)$ and $S_L(y, m)$ with initial conditions

$$C_L(L, m) = 1, \quad C_L'(L, m) = 0,$$
$$S_L(L, m) = 0, \quad S_L'(L, m) = me^{\sigma(L)}. \quad (A.3)$$

From the Wronskian relation, the above functions satisfy

$$C_0(y, m)S_0'(y, m) - S_0(y, m)C_0'(y, m) = C_L(y, m)S_L'(y, m) - S_L(y, m)C_L'(y, m) = me^{2\sigma(y) - \sigma(L)}. \quad (A.4)$$

The two bases are related to each other by

$$C_L(y, m) = \frac{e^{-\sigma(L)}}{m} \left\{ S_0'(L, m)C_0(y, m) - C_0'(L, m)S_0(y, m) \right\},$$
$$S_L(y, m) = - \left\{ S_0(L, m)C_0(y, m) - C_0(L, m)S_0(y, m) \right\}. \quad (A.5)$$

Flat spacetime

In the flat spacetime, i.e., $\sigma(y) = 0$, the basis functions are reduced to

$$C_0(y, m) = \cos(my), \quad S_0(y, m) = \sin(my),$$
$$C_L(y, m) = \cos \{ m(y - L) \}, \quad S_L(y, m) = \sin \{ m(y - L) \}. \quad (A.6)$$
Randall-Sundrum spacetime

In the Randall-Sundrum spacetime, \(i.e., \sigma(y) = ky\), the basis functions are written in terms of the Bessel functions as

\[
C_0(y, m) = \frac{\pi m}{2k} e^{ky} \left\{ Y_0 \left( \frac{m}{k} e^{ky} \right) J_1 \left( \frac{m}{k} e^{ky} \right) - J_0 \left( \frac{m}{k} \right) \frac{Y_1}{e^{ky}} \left( \frac{m}{k} e^{ky} \right) \right\},
\]

\[
S_0(y, m) = -\frac{\pi m}{2k} e^{ky} \left\{ Y_1 \left( \frac{m}{k} e^{ky} \right) J_1 \left( \frac{m}{k} e^{ky} \right) - J_1 \left( \frac{m}{k} \right) \frac{Y_1}{e^{ky}} \left( \frac{m}{k} e^{ky} \right) \right\},
\]

\[
C_L(y, m) = \frac{\pi m}{2k} e^{ky} \left\{ Y_0 \left( \frac{m}{k} e^{kL} \right) J_1 \left( \frac{m}{k} e^{ky} \right) - J_0 \left( \frac{m}{k} e^{kL} \right) \frac{Y_1}{e^{ky}} \left( \frac{m}{k} e^{ky} \right) \right\},
\]

\[
S_L(y, m) = -\frac{\pi m}{2k} e^{ky} \left\{ Y_1 \left( \frac{m}{k} e^{kL} \right) J_1 \left( \frac{m}{k} e^{ky} \right) - J_1 \left( \frac{m}{k} e^{kL} \right) \frac{Y_1}{e^{ky}} \left( \frac{m}{k} e^{ky} \right) \right\}.
\]

(B.7)

B Derivation of 5D propagators

Here we derive explicit forms of 5D propagators. We take the same strategy as in the appendix of Ref. [19]. Since the 4D vector part \(A_\mu\) and the gauge-scalar part \(A_y\) are decoupled at the quadratic level with our choice of the gauge-fixing function, the mixed components of the propagator \(\langle 0|T A_I^\mu(p, y) A_J^\nu(-p, y')|0 \rangle\) vanish. In this section, we work in the Scherk-Schwarz basis defined by Eqs. (2.10) and (2.17).

B.1 Vector propagator

The 5D propagator \(iG_{\mu\nu}^{IJ}(p, y, y') \equiv \langle 0|T A_I^\mu(p, y) A_J^\nu(-p, y')|0 \rangle\) satisfies

\[
\left\{ \partial_y^2 - 4 \sigma' \partial_y - e^{2\sigma} p^2 \left( \frac{1}{\xi} - 1 \right) p_\mu p_\nu \right\} G_{\mu\nu}^{IJ}(p, y, y') = e^{2\sigma} \eta_{\mu\nu} \delta^{IJ} \delta(y - y'),
\]

with boundary conditions,

\[
\partial_y G_{\mu\nu}^{IJ} = \partial_y \left( s_\phi G_{\mu\nu}^{3R,J} + c_\phi G_{\mu\nu}^{BJ} \right) = 0, \quad (I = \pm L, 3L)
\]

\[
G_{\mu\nu}^{IJ} = c_\phi G_{\mu\nu}^{3R,J} - s_\phi G_{\mu\nu}^{BJ} = 0, \quad (I = \pm R, \hat{3}, \hat{4})
\]

(B.2)

at \(y = 0\), and

\[
(R_\theta)^{IK} \partial_y G_{\mu\nu}^{KJ} = 0, \quad (I = \pm L, \pm R, 3L, 3R, B)
\]

\[
(R_\theta)^{IK} G_{\mu\nu}^{KJ} = 0, \quad (I = \pm, 3, \hat{4})
\]

(B.3)
where $\Omega$ is a rotation matrix for the indices of the adjoint representation corresponding to a transformation by $\Omega(L)$ defined in Eq. (2.17), i.e.,

$$(R_\theta)^I_J A^I_M = \left[\Omega^{-1}(L) A_M \Omega(L)\right]^I_J \equiv 2\text{tr}\left\{T^I \Omega^{-1}(L) A_M \Omega(L)\right\},$$

(B.4)

The explicit form of $R_\theta$ is given by

$$R_\theta = \begin{pmatrix} R^{\text{ch}}_\theta & R^{\text{ch}}_\theta \\ R^{\text{nt}}_{\bar{\theta}} & 1 \end{pmatrix},$$

(B.5)

where

$$R^{\text{ch}}_{\theta} = \begin{pmatrix} c^2_{\theta} & s^2_{\theta} & \sqrt{2}s_{\theta}c_{\theta} \\ s^2_{\theta} & c^2_{\theta} & -\sqrt{2}s_{\theta}c_{\theta} \\ -\sqrt{2}s_{\theta}c_{\theta} & \sqrt{2}s_{\theta}c_{\theta} & c^2_{\theta} - s^2_{\theta} \end{pmatrix}, \quad R^{\text{nt}}_{\bar{\theta}} = \begin{pmatrix} c^2_{\theta} & s^2_{\theta} & \sqrt{2}s_{\theta}c_{\theta} \\ s^2_{\theta} & c^2_{\theta} & -\sqrt{2}s_{\theta}c_{\theta} \\ -\sqrt{2}s_{\theta}c_{\theta} & \sqrt{2}s_{\theta}c_{\theta} & c^2_{\theta} - s^2_{\theta} \end{pmatrix}. $$

(B.6)

We can decompose $G_{\mu\nu}^{IJ}(p, y, y')$ into the following two parts.

$$G_{\mu\nu}^{IJ}(p, y, y') = \left(\eta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}\right) G_T^{IJ}(y, y', |p|) + \frac{p_{\mu}p_{\nu}}{p^2} G_S^{IJ}(y, y', |p|),$$

(B.7)

where $|p| \equiv \sqrt{-p^2}$. The first and second terms correspond to the propagators for $A^{(n)}_\mu$ and $A^{(n)}_S$, respectively. Writing $G_T^{IJ}(y, y', |p|)$ as

$$G_T^{IJ}(y, y', |p|) = \partial(y - y') G_T^{IJ}(y, y', |p|) + \partial(y' - y) G_T^{IJ}(y, y', |p|),$$

(B.8)

the solutions to Eq. (B.1) satisfying Eqs. (B.2) and (B.3) are given in the matrix notation for the indices $(I, J)$ by

$$G_{T<}^{IJ}(y, y', |p|) = \mathcal{M}_0(y, |p|) \alpha_{T<}(y', |p|),$$

$$R_\theta G_{T>}^{IJ}(y, y', |p|) = \mathcal{M}_L(y, |p|) \alpha_{T>}^{IJ}(y', |p|),$$

(B.9)

where

$$ \mathcal{M}_0 \equiv \begin{pmatrix} \mathcal{M}_0^{\text{ch}} & \mathcal{M}_0^{\text{ch}} \\ \mathcal{M}_0^{\text{nt}} & S_0 \end{pmatrix}, \quad \mathcal{M}_L \equiv \begin{pmatrix} \mathcal{M}_L^{\text{ch}} & \mathcal{M}_L^{\text{ch}} \\ \mathcal{M}_L^{\text{nt}} & S_L \end{pmatrix}. $$

(B.10)
The unknown matrix functions $\alpha_T(y', |p|)$ and $\alpha_{T'}(y', |p|)$ are determined by imposing the following matching conditions at $y = y'$. The continuity of $G_T$ at $y = y'$ leads to the condition

$$G_T(y, y, |p|) = G_T(y', y', |p|),$$

and we obtain from Eq.(B.1) the condition

$$\{ \partial_y G_T(y', y', |p|) - \partial_y G_T(y, y', |p|) \}_{y' \to y} = e^{2\sigma(y)}.$$

Using these conditions, we obtain the 5D propagators as

$$G_T(y, y', |p|) = e^{2\sigma(L)} M_0(y, |p|) W^{-1}(|p|) M_L(y', |p|) R_0,$$

$$G_{T'}(y, y', |p|) = \{ G_T(y', y, |p|) \}^t,$$

where

$$W(|p|) \equiv e^{-2\sigma(y)+2\sigma(L)} (M_L' R_0 M_0 - M_L R_0 M_0') (y, |p|)$$

$$= \begin{pmatrix} W_{ch}(|p|) \\ W_{nt}(|p|) \\ \hat{W}(|p|) \end{pmatrix}$$

is $y$-independent from the Wronskian relation (A.4). The explicit forms of the submatri-
ces $\mathcal{W}_{\text{ch}}, \mathcal{W}_{\text{int}}$ and $\mathcal{W}^{\text{I}\text{I}}$ are calculated as
\[
\mathcal{W}_{\text{ch}}(m) = -\begin{pmatrix}
c_0^2 C_0' & s_0^2 S_0' \\
-s_0^2 C_0' & c_0^2 S_0'
\end{pmatrix} \begin{pmatrix}
\frac{\sin\theta_H S_0'}{\sqrt{2}} \\
-\frac{\sin\theta_H S_0'}{\sqrt{2}} S_0
\end{pmatrix} = \begin{pmatrix}
c_0^2 C_0' & s_0^2 S_0' \\
-s_0^2 C_0' & c_0^2 S_0'
\end{pmatrix} \begin{pmatrix}
\frac{\sin\theta_H S_0'}{\sqrt{2}} \\
-\frac{\sin\theta_H S_0'}{\sqrt{2}} S_0
\end{pmatrix},
\]
\[
\mathcal{W}_{\text{int}}(m) = -\begin{pmatrix}
c_0^2 C_0' & s_0^2 (s_0^2 C_0' + c_0^2 S_0') \\
-s_0^2 C_0' & c_0^2 (s_0^2 C_0' + c_0^2 S_0')
\end{pmatrix} \begin{pmatrix}
\frac{\sin\theta_H S_0'}{\sqrt{2}} \\
-\frac{\sin\theta_H S_0'}{\sqrt{2}} S_0
\end{pmatrix} = \begin{pmatrix}
c_0^2 C_0' & s_0^2 (s_0^2 C_0' + c_0^2 S_0') \\
-s_0^2 C_0' & c_0^2 (s_0^2 C_0' + c_0^2 S_0')
\end{pmatrix} \begin{pmatrix}
\frac{\sin\theta_H S_0'}{\sqrt{2}} \\
-\frac{\sin\theta_H S_0'}{\sqrt{2}} S_0
\end{pmatrix},
\]
\[
\mathcal{W}^{\text{I}\text{I}}(m) = m e^\sigma S_0,
\]
where the right-hand sides are evaluated at $y = L$.

The scalar part $G_S(y, y', |p|)$ is obtained in a similar way, and related to $G_T(y, y', |p|)$ as
\[
G_S(y, y', |p|) = G_T(y, y', |p|/\sqrt{\xi}).
\]

### B.2 Gauge-scalar propagator

Next we consider the propagators for the gauge-scalar modes. The 5D propagator $iG_{yy}^{I\rangle}(y, y', |p|) \equiv \langle 0|TA_y^I(p, y)A^I_y(-p, y')|0 \rangle$ satisfies
\[
\{\xi \partial_y^2 e^{-2\sigma} - p^2\} G_{yy}^{I\rangle}(y, y', |p|) = e^{2\sigma} \delta^{I\rangle}(y - y'),
\]
with boundary conditions,
\[
G_{yy}^{I\rangle} = s_\phi G_{yy}^{3\rangle} + c_\phi G_{yy}^{BJ} = 0, \quad (I = \pm_1, 3_L)
\]
\[
\partial_y \left\{ e^{-2\sigma} G_{yy}^{I\rangle} \right\} = \partial_y \left\{ e^{-2\sigma} \left( c_\phi G_{yy}^{3\rangle J} - s_\phi G_{yy}^{BJ} \right) \right\} = 0, \quad (I = \pm_R, \pm, 3_L)
\]
at $y = 0$, and
\[
(R_\phi)^{I\rangle K} G_{yy}^{K\rangle} = 0, \quad (I = \pm_1, \pm_R, 3_L, 3_R, B)
\]
\[
(R_\phi)^{I\rangle K} \partial_y G_{yy}^{K\rangle} = 0, \quad (I = 3_L, 3_R, B)
\]
at $y = L$. These can be solved by the same manner as in the previous subsection. We find that $G_{yy}(y, y', |p|)$ is related to $G_S(y, y', |p|)$ as
\[
G_{yy\langle}(y, y', |p|) = -\frac{1}{p^2} \partial_y \partial_y G_S(\langle y, y', |p|),
\]
\[
G_{yy\rangle}(y, y', |p|) = -\frac{1}{p^2} \partial_y \partial_y G_S(\rangle y, y', |p|).
\]
C Treatment of the forward-scattering singularity

The S-wave amplitude for the elastic scattering: $W_L^+ + W_L^- \rightarrow W_L^+ + W_L^-$ logarithmically diverges because of the singularity of the amplitude $A_{WW\bar{W}}$ at $\chi = 0$. Here we show that this divergence is smeared out by taking into account the decay width of the W bosons in the final state. The instability of the W boson causes an ambiguity in the dispersion relation, which can be incorporated in the calculation by additional integrals, assuming a certain probability dispersion of the ambiguity. These additional integrals soften the divergence of the S-wave amplitude, as it is an infrared divergence.

We assume that the W bosons are exactly on-shell in the initial state while they can be slightly off-shell in the final state. The 4 momenta are parameterized as

\[ p_1 = (E/2, 0, 0, p_W), \]
\[ p_2 = (E/2, 0, 0, -p_W), \]
\[ p_3 = (E/2 + \delta E, (p_W + \delta p) \sin \chi, 0, (p_W + \delta p) \cos \chi), \]
\[ p_4 = (E/2 - \delta E, -(p_W + \delta p) \sin \chi, 0, -(p_W + \delta p) \cos \chi), \] (C.1)

where $p_W \equiv \sqrt{E^2/4 - m_W^2}$. Thus the invariant masses of the final state particles generically deviate from the W boson mass $m_W$, and are parameterized as

\[ p_3^2 = -(m_W + \delta m_3)^2, \quad p_4^2 = -(m_W + \delta m_4)^2. \] (C.2)

The parameters $\delta E$ and $\delta p$ in Eq.(C.1) are then expressed in terms of $\delta m_3$ and $\delta m_4$ as

\[ \delta E = \frac{m_W}{E} (\delta m_3 - \delta m_4) + \mathcal{O}(\delta m^2), \]
\[ \delta p = -\frac{m_W}{2p_W} (\delta m_3 + \delta m_4) + \mathcal{O}(\delta m^2). \] (C.3)

We assume that the distributions of $\delta m_3$ and $\delta m_4$ are given by the Gaussian profile,

\[ P(\delta m) = \frac{1}{\sqrt{2\pi}\Gamma_W} \exp \left( -\frac{\delta m^2}{2\Gamma_W^2} \right), \] (C.4)

where $\Gamma_W$ is the decay width of the W boson.

The S-wave amplitude $a_0^{00}[WW](E)$ is now expressed as

\[ a_0^{00}[WW](E) = \int_{-1}^{1} dx \int_{-\infty}^{\infty} d(\delta m_3) \int_{-\infty}^{\infty} d(\delta m_4) P(\delta m_3) P(\delta m_4) \frac{f(E,x)}{t + i\epsilon}, \] (C.5)
where \( x \equiv \cos \chi \), \( f(E, x) \) is a regular function of \( x \)\(^\text{14}\) and the Mandelstam variable \( t \) is given by
\[
t \equiv (p_1 - p_3)^2 = -\delta E^2 + \delta p^2 + 2p_W(p_W + \delta p)(1 - x). \tag{C.6}
\]
Now we will show the finiteness of the integral in Eq.(C.5). Let us divide the integral given by
\[
\text{integral (C.11) is calculated as}
\]
and take \( x_0 \) as
\[
\frac{|\delta E^2 - \delta p^2|}{2p_W^2} = \frac{8m_W^2 |\delta m_3\delta m_4|}{E^4} \lesssim \frac{16m_W^2 \Gamma_W^2}{E^4} \ll 1 - x_0 \ll 1. \tag{C.8}
\]
Here we have assumed that \( \delta m_{3,4} \lesssim \sqrt{2}\Gamma_W \)\(^\text{15}\). Then we can neglect the instability of the W boson and replace \( P(\delta m) \) with the delta function \( \delta(\delta m) \) in the first integral in Eq.(C.7).
For small \( \delta p \), the second integral is estimated as
\[
\int_{x_0}^{1} dx \frac{f(E, x)}{t + i\epsilon} \simeq -\frac{f(E, 1)}{2p_W^2} \ln \frac{|\delta E^2 - \delta p^2|}{2p_W^2 (1 - x_0)}. \tag{C.9}
\]
Here we have neglected the imaginary part of this integral coming from the principal value integral when \( \delta E^2 - \delta p^2 > 0 \) because it is not enhanced by large logarithm in contrast to the real part. Therefore, Eq.(C.5) is rewritten as
\[
a_0^{00}[WW](E) \simeq \int_{-1}^{x_0} dx \frac{f(E, x)}{2p_W^2} \frac{1}{1 - x} - \frac{f(E, 1)}{2p_W^2} I_{x_0}, \tag{C.10}
\]
where
\[
I_{x_0} \equiv \int d(\delta m_3)d(\delta m_4) P(\delta m_3)P(\delta m_4) \ln |\delta E^2 - \delta p^2| = \frac{2p_W^2}{2p_W^2 (1 - x_0)}. \tag{C.11}
\]
In the following, we will focus on the high energy region \( E^2 \gg m_W^2 \). Then the integral (C.11) is calculated as
\[
I_{x_0} \simeq \int d(\delta m_3)d(\delta m_4) P(\delta m_3)P(\delta m_4) \ln \frac{8m_W^2 |\delta m_3\delta m_4|}{E^4(1 - x_0)} \int_0^{2\pi} d\omega \int_0^\infty dr \frac{r}{2\pi \Gamma_W^2} \exp \left( -\frac{r^2}{2\Gamma_W^2} \right) \ln \frac{4m_W^2 r^2 |\sin 2\omega|}{E^4(1 - x_0)} = \frac{1}{2\pi} \int_0^{2\pi} d\omega \left\{ -\gamma_E + \ln \frac{8m_W^2 \Gamma_W^2 |\sin 2\omega|}{E^4(1 - x_0)} \right\} = -\gamma_E + \ln \frac{8m_W^2 \Gamma_W^2}{E^4(1 - x_0)} - \ln 2 = \ln \frac{4m_W^2 \Gamma_W^2}{e^{\gamma_E} E^4(1 - x_0)} \tag{C.12}
\]
\(^\text{14}\) The function \( f(E, x) \) also depends on \( \delta m_3 \) and \( \delta m_4 \) through the 4 momenta \( p_3, p_4 \) and the polarization vectors \( \varepsilon_3, \varepsilon_4 \). However such \( \delta m_{3,4} \)-dependences can be neglected in the following discussion because they provide only subdominant contributions.
\(^\text{15}\) Although larger values of \( \delta m_{3,4} \) are possible, their contributions to the integral in Eq.(C.5) are negligible due to the tiny probability \( P(\delta m) \ll 1 \).
Here we have moved to the polar coordinate \((\delta m_3, \delta m_4) = (r \cos \omega, r \sin \omega)\) in the second equality, and used the formulae

\[
\int_0^\infty dy \exp(-y) \ln y = -\gamma_E, \quad \int_0^{2\pi} d\omega \ln|\sin 2\omega| = -2\pi \ln 2. \tag{C.13}
\]

where \(\gamma_E = 0.577 \cdots\) is the Euler's constant.

Here we define

\[
x_{\text{cut}} = 1 - \frac{4m_W^2 \Gamma_W^2}{e^{\gamma_E} E^4} \simeq 1 - 1.6 \times 10^{-3} \frac{m_W^4}{E^4}. \tag{C.14}
\]

Then Eq.(C.10) with (C.12) is rewritten as

\[
a_{00}^{\infty}[W W](E) \simeq \int_{-1}^{x_0} dx \frac{f(E, x)}{2p_W^2 (1 - x)} + \frac{f(E, 1)}{2p_W^2} \ln \frac{1 - x_0}{1 - x_{\text{cut}}}
\]

\[
\simeq \int_{-1}^{x_{\text{cut}}} dx \frac{f(E, x)}{2p_W^2 (1 - x)}. \tag{C.15}
\]

We have used \(1 - x_{\text{cut}} \ll 1 - x_0 \ll 1\) at the last step. (See Eq.(C.8).) Note that the \(x_0\)-dependences are cancelled and the final result is independent of \(x_0\). This is a corollary of the fact that the division of the integral region (C.7) is just an artificial one. Eq.(C.15) means that the effect of the instability of the W bosons in the final state is translated into the cut-off \(x_{\text{cut}}\) in the \(x\)-integral, which regularizes the divergence as expected.

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