ON SUBSCHEMES OF FORMAL SCHEMES

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Abstract. We think about what the subscheme of the formal scheme is. Differently form the ordinary scheme, the formal scheme has different notions of “subscheme”. We lay a foundation for these notions and compare them. We also relate them to singularities of foliations.

Introduction

In a foundation of the theory of formal schemes, it is a problem how to define a subscheme of a formal scheme. Grothendieck [EGA] defined a closed subscheme of a locally Noetherian formal scheme. However it is not definitive, and leads to a pathological phenomenon. The aim of this paper is to compare different notions of “subschemes”, and to complement the theory of formal schemes. We also relate the pathological phenomenon with singularities of foliations.

We generalize the notion of the closed subscheme of the locally Noetherian formal scheme defined in [EGA] to arbitrary ambient formal schemes and define a subscheme as a closed subscheme of an open subscheme. This notion has the advantage that a subscheme of a Noetherian formal scheme is Noetherian, and the disadvantage that a subscheme is not generally an open subscheme of a closed subscheme. B. Heinzer first found an example of the last pathological phenomenon [AJL, page 1]. We construct a more explicit one (Theorem 2.7).

As generalizations of subscheme, we define pre-subschemes and pseudo-subschemes. In particular, with pseudo-subschemes, the pathological phenomenon mentioned above does not occur: A pseudo-subscheme of a formal scheme with some mild condition is an open subscheme of a closed pseudo-subscheme. Unfortunately a pseudo-subscheme does not inherit the Noetherianity of the ambient formal scheme, but inherit instead a variant of Noetherianity, the ind-Noetherianity. A closed pseudo-subscheme $\mathcal{Y} \hookrightarrow \mathcal{X}$ coincides with a pseudo closed immersion of Alonso Tarrío, Jeremías López and Pérez Rodríguez [AJP], if $\mathcal{X}$ and $\mathcal{Y}$ are both locally Noetherian. Removing this restriction is essential and inevitable for our aim. Subschemes, pre-subschemes, pseudo-subschemes of formal schemes are all generalizations of subschemes of
ordinary schemes. Showing examples, we will see that subschemes, pre-subschemes and pseudo-subschemes are mutually different.

Formal schemes naturally appear in studies of algebraic foliations, thanks to Miyaoka’s formal Frobenius theorem [Miy]. Jouanolou [Jou] proved that there exist singular algebraic foliations on $\mathbb{C}^3$ without any formal separatrix at the origin. Applying this, we construct closed pseudo-subschemes of $\text{Spf} \, \mathbb{C}[w][[x, y, z]]$ that are not closed pre-subschemes nor Noetherian.

McQuillan [McQ] modified the definition of formal scheme. This modification is the right one, and we follow his definition with additional minor modifications.

**Conventions.** We denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{N}_0$ the set of non-negative integers. A *ring* means a commutative ring with unit. An *ordinary scheme* means a scheme, distinguished from a formal scheme that is not a scheme.

**Acknowledgement.** I would like to thank Fumiharu Kato for useful discussions.

1. **Preliminaries**

This section contains generalities on formal schemes. We adopt a slightly different definition of formal schemes from those in [EGA] and [McQ]. However most results in this section are found in [EGA] or [McQ], and similar arguments can apply.

1.1. **Admissible rings.** For a descending chain of ideals of a ring $A$,

$$I_1 \supseteq I_2 \supseteq \cdots,$$

there exists a unique topology on $A$ which makes $A$ a topological ring and for which the collection $\{I_i\}_{i \in \mathbb{N}}$ of ideals is a basis of (necessarily open) neighborhoods of $0 \in A$. We call this topology the $\{I_i\}$-topology. A *linearly topologized ring* is a topological ring with the $\{I_i\}$-topology for some descending chain $\{I_i\}_{i \in \mathbb{N}}$ of ideals.\(^1\)

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\(^1\)This definition is more restrictive than usual. One usually supposes only that there exists a (possibly uncountable) collection of ideals forming a basis of neighborhoods of 0. In most instances, our condition holds, and it makes arguments simpler.

We will put the corresponding assumption on complete modules and formal schemes. In [McQ], this condition is not supposed. However some results in op. cit, for instance, Claim 2.6 and Fact 3.3, seem valid only under this condition. For McQuillan uses in the proof the fact that if the projective system of short exact sequences satisfies the Mittag-Leffler condition, then its limit is also exact. It is true only if the system is indexed by $\mathbb{N}$. 
An open ideal $I$ of a topological ring is called an ideal of definition if every element $f \in I$ is topologically nilpotent (that is, $f^n \to 0$, as $n \to \infty$).\footnote{This definition is due to McQuillan \cite{McQ}. The one in \cite{EGA} is more restrictive: in op. cit., an ideal $I$ is an ideal of definition if for every open neighborhood $V$ of 0, there exists $n \in \mathbb{N}$ with $I^n \subseteq V$.} A linearly topologized ring is called an admissible ring if it is separated and complete, and admits an ideal of definition. Every admissible ring has the largest ideal of definition, the ideal of the topologically nilpotent elements.\footnote{This fails if we adopt the definition in \cite{EGA}.}

A descending chain of ideals of definition in a topological ring is called a basis of ideals of definition if it is a basis of neighborhoods of 0. If $A$ is an admissible ring with the $\{I_i\}$-topology and $J \subseteq A$ is an ideal of definition, then the collection $\{I_i \cap J\}$ of ideals is a basis of ideals of definition. Every admissible ring thus admits a basis of ideals of definition.

For a ring $A$ and its ideal $I$, the $I$-adic topology on $A$ is by definition the $\{I^n\}_{n \in \mathbb{N}}$-topology. An admissible ring $A$ is said to be adic if the topology on $A$ is identical to the $I$-adic topology for some ideal $I \subseteq A$.

**Example 1.1.** Let $k$ be a field and let $A := k[[x, y]]$ be endowed with the $\{(xy^n)\}_{n \in \mathbb{N}}$-topology. Then $A$ is admissible and $\{(xy^n)\}_{n \in \mathbb{N}}$ is a basis of ideals of definition. However $A$ is not adic. Indeed, for $i \geq 2$ and $n \geq 0$, $(x^iy^i)$ does not contain $(xy^n)$, and hence $(x^iy^i)$ is not open.

**Lemma 1.2.** A topological ring $A$ is admissible if and only if $A$ is isomorphic to the limit $\varprojlim A_i$ of some projective system of discrete rings,

$$A_1 \leftarrow A_2 \leftarrow \cdots,$$

such that for every $i$, the map $A_{i+1} \to A_i$ is surjective and every element of its kernel is nilpotent.

**Proof.** It is essentially the same statement as \cite{EGA} 0, Lem. 7.2.2]. The proof is parallel. \qed

**Corollary 1.3.** Let $A$ be an admissible ring and

$$I_1 \supseteq I_2 \supseteq \cdots$$

a descending chain of open ideals of $A$ (not necessarily a basis of neighborhoods of 0). Suppose that for every $i$, $\sqrt{I_i} = \sqrt{I_1}$. Then $B := \varprojlim A/I_i$ is an admissible ring and the natural map $A \to B$ is a continuous homomorphism.
Proof. The projective system of discrete rings
\[ A/I_1 \leftarrow A/I_2 \leftarrow \cdots \]
satisfies the condition in Lemma 1.2. Hence \( B \) is admissible. For each \( i \in \mathbb{N} \), put
\[ B \supseteq \hat{I}_i := \lim_{\substack{j \to \infty \atop j \geq i}} I_i/I_j. \]
Then the ideals \( \hat{I}_i \) form a basis of neighborhoods of \( 0 \in B \). The preimage of \( \hat{I}_i \) in \( A \) is \( I_i \), in particular, open. It follows that \( A \rightarrow B \) is continuous. \( \square \)

Definition 1.4. An admissible ring \( A \) is said to be pro-Noetherian if one of the following equivalent conditions holds:

1. For every open ideal \( I \), the quotient ring \( A/I \) is Noetherian.
2. For some basis \( \{I_i\}_{i \in \mathbb{N}} \) of ideals of definition and for every \( i \in \mathbb{N} \), the quotient ring \( A/I_i \) is Noetherian.

In particular, every Noetherian admissible ring is pro-Noetherian.

Lemma 1.5. Let \( A \) be a pro-Noetherian admissible ring and \( I \subseteq A \) an ideal of definition. Then for any neighborhood \( V \) of \( 0 \), there exists \( n \in \mathbb{N} \) with \( I^n \subseteq V \). (Namely, in the sense of \([EGA]\), \( I \) is an ideal of definition and \( A \) is admissible. See Footnote 1.1)

Proof. Since \( A \) is linearly topologized, we may suppose that \( V \) is an ideal. Then \( A/V \) is Noetherian. Therefore \( I(A/V) \) is finitely generated. Since every element of \( I(A/V) \) is nilpotent, so is \( I(A/V) \). This means that for some \( n \), \( I^n \subseteq V \). \( \square \)

Lemma 1.6. Every pro-Noetherian adic ring \( A \) is Noetherian. Furthermore for every ideal \( I \) of definition in \( A \), the topology on \( A \) is identical to the \( I \)-adic topology.

Proof. Let \( A \) be a pro-Noetherian adic ring and \( I \subseteq A \) an ideal such that \( \{I^n\}_{n \in \mathbb{N}} \) is a basis of ideals of definition. By definition, \( A/I \) and \( A/I^2 \) are Noetherian. Consequently \( I/I^2 \) is finitely generated, and from \([EGA\ 0, Cor. 7.2.6]\), \( A \) is Noetherian.

Let \( J \) be an arbitrary ideal of definition. Then for some \( m \in \mathbb{N} \), \( I^m \subseteq J \). Hence for every \( n \in \mathbb{N} \), \( I^{mn} \subseteq J^n \), and so \( J^n \) is open. Conversely, since \( J \) is finitely generated, for every \( n \in \mathbb{N} \), there exists \( m \in \mathbb{N} \) with \( J^m \subseteq I^n \). This proves the lemma. \( \square \)

Lemma 1.7. Let the notations be as in Corollary 1.3. Suppose that \( A \) is pro-Noetherian. Then \( B \) is also pro-Noetherian.
Proof. The admissible ring $B$ has a basis of ideals of definition $\{\hat{I}_i\}_i$, and the quotient rings $B/\hat{I}_i \cong A/I_i$ are Noetherian. Hence $B$ is pro-Noetherian. \qed

The following is required in §3.

**Proposition 1.8.** Let $A$ be a Noetherian adic ring and

$$I_1 \supseteq I_2 \supseteq \cdots$$

a descending chain of ideals of definition in $A$ (not necessarily a basis of ideals of definition), and let $B := \varprojlim A/I_i$. Suppose that $B$ is adic. Then the natural map $A \to B$ is surjective. Moreover the topology on $B$ is the $\{I^n_B\}$-topology.

**Proof.** From Lemmas 1.6 and 1.7, $B$ is also a Noetherian adic ring. Let

$$\hat{I}_i := \varprojlim_{j \geq i} I_i / I_j,$$

and $J := \hat{I}_1$. Since $J$ is an ideal of definition, from Lemma 1.6, the topology on $B$ is identical to the $J$-adic topology. Since $\{\hat{I}_i\}_{i \in \mathbb{N}}$ is also a basis of ideals of definition of $B$, for every $n$, there exists $i \in \mathbb{N}$ such that $\hat{I}_i \subset J^n$. Then we have

$$B/J^n = (B/\hat{I}_i) / (J^n / \hat{I}_i) = (A/I_i) / (I_1/I_i)^n.$$

For $i \gg m \geq n$, the kernel of $B/J^m \to B/J^n$ is

$$(I_1/I_i)^n / (I_1/I_i)^m = I^n_1 (B/J^m).$$

Besides $B/J = A/I_1$ is clearly a finitely generated $A$-module. As a result, the projective systems $\{A/I_i\}$ and $\{B/J\}$ satisfy the conditions of [EGA 0, Prop. 7.2.9], and hence

$$J^n = \text{Ker} (B \to B/J^n) = I^n_1 B.$$

This shows the second assertion.

The map $A/I_1 \to B/I_1 B$ is surjective and $B$ is separated for the $\{I^n_B\}$-topology. From [Mat] Th. 8.4, $A \to B$ is surjective. \qed

1.2. **Formal schemes.** We associate to each admissible ring $A$ a topologically ringed space $\text{Spf} A$, called the **formal spectrum** of $A$, as follows: The underlying topological space of $\text{Spf} A$ is the set of **open** prime ideals and identified with $\text{Spec} A/I$ for every ideal of definition $I$. The structure sheaf $\mathcal{O}_{\text{Spf} A}$ of $\text{Spf} A$ is a sheaf of topological rings. If $\{I_i\}_{i \in \mathbb{N}}$ is a basis of ideals of definition of $A$, then $\mathcal{O}_{\text{Spf} A}$ is defined to be the limit $\varprojlim \mathcal{O}_{\text{Spec} A/I_i}$ of sheaves $\mathcal{O}_{\text{Spec} A/I_i}$ of pseudo-discrete rings (for sheaves of pseudo-discrete rings, see [EGA]).
Let $A$ be an admissible ring with $\{ I_i \}$ a basis of ideals of definition and $x \in A$. Then we define the complete localization of $A$ by $x$, denoted $A_{\{ x \}}$, to be the projective limit $\varprojlim (A/I_i)_x$ of the localizations $(A/I_i)_x$ of $A/I_i$. Then $A_{\{ x \}}$ is also admissible. We call $\text{Spf} \ A_{\{ x \}}$ a distinguished open subscheme of $\text{Spf} \ A$. The distinguished open subschemes of an affine formal scheme form a basis of open subsets of the underlying topological space.

For an open prime ideal $p \subseteq A$, the stalk $\mathcal{O}_{\text{Spf} \ A, p}$ of the structure sheaf $\mathcal{O}_{\text{Spf} \ A}$ at $p$ is

$$\lim_{x \not\in p} A_{\{ x \}}.$$ We can see that it is a local ring as [EGA]. Thus the topologically ringed space $\text{Spf} \ A$ is a locally topologically ringed space.

After [McQ], we call the completion $\hat{\mathcal{O}}_{\text{Spf} \ A, p}$ of $\mathcal{O}_{\text{Spf} \ A, p}$ the fine stalk, which is isomorphic to

$$\lim_{\leftarrow \mathcal{O}_{\text{Spec} \ A/I_i, p/I_i}}.$$ Here $I_i$, $i \in \mathbb{N}$, form a basis of ideals of definition. In the [EGA] notation, if we put $S := A \setminus p$, then

$$\mathcal{O}_{\text{Spf} \ A, p} = A_{\{ S \}}$$ and $\hat{\mathcal{O}}_{\text{Spf} \ A, p} = A_{\{ S^{-1} \}}$.

From [EGA, 0, Prop. 7.6.17], the natural map $\mathcal{O}_{\text{Spf} \ A, p} \to \hat{\mathcal{O}}_{\text{Spf} \ A, p}$ is a local homomorphism.

An affine formal scheme is a topologically ringed space that is isomorphic to the formal spectrum of an admissible ring. A formal scheme is a topologically ringed space $(X, \mathcal{O})$ with $X$ the underlying topological space and $\mathcal{O}$ the structure sheaf such that there exists an open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, and for every $\lambda \in \Lambda$, the topologically ringed space $(U_\lambda, \mathcal{O}|_{U_\lambda})$ is an affine formal scheme.\(^4\) In particular, every formal scheme is locally topologically ringed space.

A morphism of formal schemes is a morphism as locally topologically ringed spaces: The maps of stalks are necessarily local homomorphisms, equivalently the maps of fine stalks are local homomorphisms.

A continuous homomorphism $A \to B$ of admissible rings induces a morphism of formal schemes, $\text{Spf} \ B \to \text{Spf} \ A$. Then the functor $A \mapsto \text{Spf} \ A$ from the category of admissible rings to the category of formal schemes is fully faithful.

\(^4\)This definition of formal scheme is different from those of [EGA] and [McQ], because of the difference of the definition of admissible ring. Moreover in [McQ], it is supposed the existence of basis of subschemes of definition.
1.3. Several Noetherianities. We now define some finiteness conditions on formal schemes.

**Definition 1.9.** Let $\mathcal{X}$ be a formal scheme.

1. $\mathcal{X}$ is *adic* if every point of $\mathcal{X}$ admits an affine open neighborhood $\text{Spf} \, A$ with $A$ adic.
2. $\mathcal{X}$ is *quasi-compact* (resp. *top-Noetherian*, *locally top-Noetherian*) if the underlying topological space of $\mathcal{X}$ is quasi-compact (resp. Noetherian, locally Noetherian).
3. $\mathcal{X}$ is *locally pre-Noetherian* (resp. *locally ind-Noetherian*, *locally Noetherian*) if every point of $\mathcal{X}$ admits an affine neighborhood $\text{Spf} \, A$ with $A$ Noetherian (resp. pro-Noetherian, Noetherian and adic).
4. $\mathcal{X}$ is *pre-Noetherian* (resp. *ind-Noetherian*, *Noetherian*) if $\mathcal{X}$ is locally pre-Noetherian (resp. locally ind-Noetherian, locally Noetherian) and quasi-compact.

We have the following implications among properties of a formal scheme:

- (locally) Noetherian $\iff$ (locally) ind-Noetherian + adic
- (locally) pre-Noetherian $\iff$ (adic)
- (locally) ind-Noetherian $\iff$
- (locally) top-Noetherian

The top horizontal arrow follows from Lemma 1.6.

1.4. Quasi-coherent sheaves.

**Definition 1.10.** Suppose that $A$ is a linearly topologized ring and $I_1 \supseteq I_2 \supseteq \cdots$ a basis of open ideals. An $A$-module $M$ endowed with a topology is said to be *complete* if

1. $M$ is a topological group with respect to the given topology and the addition,
2. there exists a basis of open $A$-submodules of $M$

\[M_1 \supseteq M_2 \supseteq \cdots\]
(that is, the collection \( \{ M_i \}_{i \in \mathbb{N}} \) is a basis of open neighborhoods of \( 0 \in M \) such that for every \( i \in \mathbb{N} \), \( I_i M \subseteq M_i \), and

(3) \( M \) is separated and complete.

We note that the second condition is independent of the choice of \( \{ I_i \} \).

Suppose that \( A \) is admissible with a basis \( \{ I_i \} \) of ideals of definition and that \( M \) is a complete \( A \)-module and \( \{ M_i \}_{i \in \mathbb{N}} \) is a basis of open submodules with \( I_i M \subseteq M_i \). Then \( M/M_i \) is an \( A/I_i \)-module. There exists a corresponding quasi-coherent sheaf \( M/M_i \) on \( \text{Spec} \, A/I_i \). The projective limit

\[
M^\Delta := \varprojlim M/M_i
\]

of sheaves \( M/M_i \) of pseudo-discrete groups is a complete \( \mathcal{O}_{\text{Spf} \, A} \)-module, that is, for every open subset \( U \subseteq \text{Spf} \, A \), \( M^\Delta(U) \) is a complete \( \mathcal{O}_{\text{Spf} \, A}(U) \)-module.

**Definition 1.11.** Let \( \mathcal{X} \) be a formal scheme. A complete \( \mathcal{O}_\mathcal{X} \)-module \( \mathcal{F} \) is said to be **quasi-coherent** if every point of \( \mathcal{X} \) has an affine neighborhood \( \text{Spf} \, A \) such that \( \mathcal{F}|_{\text{Spf} \, A} \cong M^\Delta \) for some complete \( A \)-module \( M \).

2. **Subschemes of formal schemes and their variants**

In this section, we see various kinds of “subschemes” of formal schemes, compare them, and prove their basic properties.

### 2.1. Subschemes

**Definition 2.1.** An **open subscheme** of a formal scheme \( \mathcal{X} \) is an open subset \( U \) of \( \mathcal{X} \) along with the restricted structure sheaf \( \mathcal{O}_\mathcal{X}|_U \). A morphism \( \mathcal{Y} \to \mathcal{X} \) is said to be an **open immersion** if it is an isomorphism onto an open subscheme of \( \mathcal{X} \).

**Lemma 2.2.** Let \( A \) be an admissible ring with a basis of open ideals \( \{ J_i \} \) and \( I \subseteq A \) a closed ideal. Give the quotient ring \( A/I \) the \( \{(J_i + I)/I\}\)-topology. Then \( A/I \) is admissible.

**Proof.** From [Mat], the middle of page 56, \( A/I \) is separated, and from Theorem 8.1 in op. cit, \( A/I \) is complete. (Recall that \( A \) is supposed to have a countable basis of open ideals.) If \( a \subseteq A/I \) is the ideal of topologically nilpotent elements, then its preimage in \( A \) contains the largest ideal of definition of \( A \), and open. Therefore \( a \) is also open (see the top of page 56 in op. cit.). We conclude that \( A/I \) is admissible. \( \Box \)

This definition is also due to McQuillan (see [McQ, §5]), and different from the [EGA] notion of quasi-coherence.
If $I$ is a closed ideal of an admissible ring $A$, then $I$ is a complete $A$-module for the induced topology. Hence we can define an ideal sheaf $I^\triangle \subseteq \mathcal{O}_{\text{Spf} A}$. The quotient sheaf $\mathcal{O}_{\text{Spf} A}/I^\triangle$ is canonically isomorphic to $\mathcal{O}_{\text{Spf} A/I}$.

Let $\mathcal{X}$ be a formal scheme. An ideal sheaf $I \subseteq \mathcal{O}_{\mathcal{X}}$ is said to be \textit{closed} if every point of $\mathcal{X}$ has an affine neighborhood $\text{Spf} A \subseteq \mathcal{X}$ such that $I|_{\text{Spf} A} = I^\triangle$ for some closed ideal $I \subseteq A$. For a closed ideal $I \subseteq \mathcal{O}_{\mathcal{X}}$, the topologically ringed space $Y = (\text{Supp} \mathcal{O}_{\mathcal{X}}/I, \mathcal{O}_{\mathcal{X}}/I)$ is a formal scheme.

\textbf{Definition 2.3.} For a formal scheme $\mathcal{X}$ and a closed ideal sheaf $I \subseteq \mathcal{O}_{\mathcal{X}}$, we call $(\text{Supp} \mathcal{O}_{\mathcal{X}}/I, \mathcal{O}_{\mathcal{X}}/I)$ the \textit{closed subscheme defined by $I$}. A morphism $Y \to \mathcal{X}$ of formal schemes is said to be a \textit{closed immersion} if it is an isomorphism onto a closed subscheme of $\mathcal{X}$. A morphism is said to be an \textit{immersion} if it is an open immersion followed by a closed immersion. A \textit{subscheme} of a formal scheme $\mathcal{X}$ is an equivalence class of immersions $Y \to \mathcal{X}$, where $f_i : Y_i \to \mathcal{X}$, $i = 1, 2$, are equivalent if there exists an isomorphism $g : Y_1 \to Y_2$ with $f_1 = f_2 \circ g$.

Consider the case where $\mathcal{X}$ is locally Noetherian. Since every ideal of a Noetherian adic ring is closed (see [ZS, page 264]), an ideal sheaf $I \subseteq \mathcal{O}_{\mathcal{X}}$ is closed if and only if $I$ is coherent in the sense of [EGA]. Therefore the definition above of the closed subscheme coincides with the one in [EGA] in this case.

\textbf{Lemma 2.4.} Every closed subscheme of an affine formal scheme $\text{Spf} A$ is defined by some closed ideal $I \subseteq A$.

\textit{Proof.} Let $I \subseteq \mathcal{O}_{\text{Spf} A}$ be a closed ideal sheaf and $\{J_i\}_{i \in \mathbb{N}}$ a basis of open ideals of $A$. Then for each $i$, there exists an ideal $I_i \subseteq A/J_i$ such that

\[ \tilde{I}_i = I + J^\triangle_i / J_i^\triangle \subseteq \mathcal{O}_{\text{Spec} A/J_i}. \]

Moreover the $I_i$ form a projective system and if we put $I := \varprojlim I_i$, then $I$ is a closed ideal of $A$. We easily see that $I = I^\triangle$. \hfill \Box

\textbf{Proposition 2.5.} (1) Let $P$ be any property in Definition 1.9 except the quasi-compactness, and let $\mathcal{X}$ be a formal scheme satisfying $P$. Then every subscheme of $\mathcal{X}$ satisfies $P$.

(2) If $g : Z \to Y$ and $f : Y \to \mathcal{X}$ are (closed) immersions of formal schemes, then $f \circ g$ is also a (closed) immersion.

(3) If $Y \to \mathcal{X}$ is an immersion of formal schemes and $Z \to \mathcal{X}$ is a morphism of formal schemes, then the projection $Y \times_{\mathcal{X}} Z \to Z$ is a immersion.
Proof. 1 and 2 are obvious. To prove 3, we may suppose that $\mathcal{X} = \text{Spf } A$, $\mathcal{Y} = \text{Spf } A/I$ and $\mathcal{Z} = \text{Spf } B$. Here $I \subseteq A$ is a closed ideal. Then $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} = \text{Spf } ((A/I) \widehat{\otimes}_A B)$. Let $\{I_i\}$ and $\{J_i\}$ be bases of ideals of definition in $A$ and $B$ respectively such that for each $i$, $I_i$ is contained in the preimage of $J_i$. Then we have

$$(A/I) \widehat{\otimes}_A B \cong \lim_{\leftarrow} (A/(I_i + I) \otimes_{A/I_i} B/J_i)$$

$\cong \lim_{\leftarrow} B/(IB + J_i)$$

$\cong B/\overline{IB}.$

Here $\overline{IB}$ is the closure of $IB$. Thus $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ is the closed subscheme defined by the closed ideal $\overline{IB}$. □

2.1.1. Pathological examples. As a consequence of a theorem in [HR], Bill Heinzer shown the following (see the first page of [AJL]):

**Theorem 2.6.** Let $k$ be a field. There exists a nonzero ideal $I \subseteq k[x^\pm, y, z][[t]]$ with $I \cap k[x, y, z][[t]] = (0)$.

The formal scheme $\text{Spf } k[x^\pm, y, z][[t]]$ is a distinguished open subscheme of $\text{Spf } k[x, y, z][[t]]$ and $\text{Spf } k[x^\pm, y, z][[t]]/I$ is a subscheme of $\text{Spf } k[x, y, z][[t]]$. In geometric terms, the theorem means that the smallest closed subscheme of $\text{Spf } k[x, y, z][[t]]$ containing a subscheme $\text{Spf } k[x^\pm, y, z][[t]]/I$ of $\text{Spf } k[x^\pm, y, z][[t]]$ is $\text{Spf } k[x, y, z][[t]]$ itself. In other words, the (scheme-theoretic) closure of $\text{Spf } k[x^\pm, y, z][[t]]/I$ in $\text{Spf } k[x, y, z][[t]]$ in a naive sense is $\text{Spf } k[x, y, z][[t]]$.

We find the following simpler and more explicit example:

**Theorem 2.7.** Consider an element of $\mathbb{C}[x^\pm, y][[t]]$

$$f := y + a_1x^{-1}t + a_2x^{-2}t^2 + a_3x^{-3}t^3 + \cdots, \ a_i \in \mathbb{C} \setminus \{0\}.$$  

Suppose that a function $i \mapsto |a_i|$ is strictly increasing and

$$\lim_{i \to \infty} \frac{|a_{i+1}|}{|a_i|} = \infty.$$  

Then

$$(f) \cap \mathbb{C}[x, y][[t]] = (0).$$

**Proof.** We prove the first assertion by contradiction. So we suppose that there exists $0 \neq g = \sum_{i \in \mathbb{N}_0} g_i t^i \in \mathbb{C}[x^\pm, y][[t]]$ with $g_i \in \mathbb{C}[x^\pm, y]$ such that $h := fg \in \mathbb{C}[x, y][[t]]$. If we write $h = \sum_{i \in \mathbb{N}_0} h_i t^i$ with $h_i \in \mathbb{C}[x, y]$, then for every $i \in \mathbb{N}_0$, we have

$$h_i = yg_i + \sum_{j=1}^{i} a_jx^{-j}g_{i-j} = yg_i + h'_i, \ (h'_i := \sum_{j=1}^{i} a_jx^{-j}g_{i-j}).$$
For each $i \in \mathbb{N}_0$, write
\[ g_i = \sum_{m \in \mathbb{Z}, n \in \mathbb{N}_0} g_{imn} x^m y^n, \quad g_{imn} \in \mathbb{C}. \]

We set
\[ d_i := \inf \{ m \in \mathbb{Z} | \exists n, g_{imn} \neq 0 \}, \]
\[ e_i := \inf \{ n \in \mathbb{N}_0 | g_{id_i, n} \neq 0 \}, \]
\[ D_i := \inf \{ d_{i-j} - j | 1 \leq j \leq i \} \]
\[ = \inf \{ d_{j'} - i + j' | 0 \leq j' \leq i - 1 \}, \]
\[ E_i := \inf \{ e_{i-j} | 1 \leq j \leq i, \ d_{i-j} - j = D_i \} \]
\[ = \inf \{ e_{j'} | 0 \leq j' \leq i - 1, \ d_{j'} - i + j' = D_i \}. \]

Here by convention, $\inf \emptyset = +\infty$. We easily see that for every $i' > i$,
\[ D_{i'} < D_i \text{ and } E_{i'} \leq E_i. \]

If for $i_0 \in \mathbb{N}$, $D_{i_0} < 0$ and if the coefficient of $x^{D_{i_0}} y^{E_{i_0}}$ in $h'_{i_0}$ is nonzero, then the coefficient of $x^{D_{i_0}} y^{E_{i_0}}$ in $y g_{i_0}$ is also nonzero. Moreover if either “$m < D_{i_0}$” or “$m = D_{i_0}$ and $n < E_{i_0}$”, then the coefficient of $x^m y^n$ in $y g_{i_0}$ vanishes. It follows that
\[ d_{i_0} = D_{i_0} \text{ and } e_{i_0} = E_{i_0}, \]
and that
\[ D_{i_0+1} = D_{i_0} - 1 \text{ and } E_{i_0+1} = E_{i_0} - 1, \]
and that the coefficient of $x^{D_{i_0+1}} y^{E_{i_0+1}}$ in $h'_{i_0+1}$ is again nonzero. As a result, $E_{i+1} = E_i - 1$ for every $i \geq i_0$. Since $E_i \in \mathbb{N}_0$ for every $i$, it is impossible.

Now it remains to show that for some $i \in \mathbb{N}$, $D_i < 0$ and the coefficient of $x^{D_i} y^{E_i}$ in $h_i'$ is nonzero. Suppose by contrary that for every $i \in \mathbb{N}$ with $D_i < 0$, the coefficient of $x^{D_i} y^{E_i}$ in $h_i'$ is zero. Since $i \mapsto D_i$ is strictly decreasing, there exists $i_1 \in \mathbb{N}$ such that for every $i \geq i_1$, $D_i < 0$. Then for every $i \geq i_1$, the coefficient of $x^{D_i} y^{E_i}$ in $y g_i$ must be zero. Therefore we have
\[ D_i = D_{i_1} - (i - i_1) \text{ and } E_i = E_{i_1}. \]

Let
\[ \Lambda := \{ j | \text{the coefficient of } x^{D_{i_1}} y^{E_{i_1}} \text{ in } x^{-j} g_{i_1-j} \text{ is nonzero} \} \subseteq \{ 1, 2, \ldots, i_1 \} \]
and let $0 \neq c_j \in \mathbb{C}$ be the coefficient of $x^{D_{i_1}} y^{E_{i_1}}$ in $x^{-j} g_{i_1-j}$, $j \in \Lambda$. For every $i \geq i_1$, the coefficient of $x^{D_i} y^{E_i}$ in $h_i'$ is
\[ \sum_{j \in \Lambda} a_{j+i-i_1} c_j = 0. \]
Let \( j_0 \in \Lambda \) be the largest element and \( j_1 \in \Lambda \) the second largest one. (Note that \( \#\Lambda \geq 2 \). From the assumption on the \( a_i \), for \( i \gg i_1 \), we have

\[
|a_{j_0+i-i_1}| - (\#\Lambda - 1)|a_{j_1+i-i_1}|(\max_{j \in \Lambda \setminus \{j_0\}}|c_j/c_{j_0}|) > 0.
\]

Therefore, for \( i \gg 0 \),

\[
0 = \left| \sum_{j \in \Lambda} a_{j+i-i_1}c_j \right| \\
\geq |c_{j_0}| \left( |a_{j_0+i-i_1}| - \sum_{j \in \Lambda \setminus \{j_0\}} |a_{j+i-i_1}c_j/c_{j_0}| \right) \\
\geq |c_{j_0}| \left( |a_{j_0+i-i_1}| - (\#\Lambda - 1)|a_{j_1+i-i_1}|(\max_{j \in \Lambda \setminus \{j_0\}}|c_j/c_{j_0}|) \right) \\
> 0
\]

This is a contradiction. We have proved the theorem. \( \square \)

If we remove one more variable, then any ideal as in Theorems 2.6 and 2.7 does not exist:

**Proposition 2.8.** Let \( k \) be a field. Then for any nonzero ideal \( I \) of \( k[x^\pm][[t]] \), \( I \cap k[x][[t]] \neq (0) \).

**Proof.** It suffices to prove the assertion in the case where \( I \) is principal, say \( I = (f) \), \( f \in k[x^\pm][[t]] \). Write

\[
f = \sum_{i \geq n} f_it^i \in k[x^\pm][[t]], \quad f_i \in k[x^\pm], \quad f_n \neq 0.
\]

Define \( g_i \in k[x^\pm, f_n^{-1}] \) inductively as follows;

\[
g_0 := f_n^{-1}, \quad g_{i+1} := -(\sum_{0 \leq j \leq i} g_jf_{n+i+1-j})/f_n.
\]

Then

\[
f \left( \sum_{i \geq 0} g_it^i \right) = \sum_{m \geq n} \left( (f_ng_{m-n} + \sum_{i+j=m \atop j < m-n} f_ig_j)t^m \right) \\
= t^n + \sum_{m > n} \left( -(\sum_{j < m-n} g_jf_{m-j} + \sum_{i+j=m \atop j < m-n} f_ig_j)t^m \right) \\
= t^n.
\]

Since \( \sum_{i \geq 0} g_it^i \) is invertible, ideals \((f)\) and \((t^n)\) of \( k[x^\pm, f_n^{-1}][[t]] \) are identical. Glueing \( \text{Spf} \ k[x^\pm][[t]]/(f) \) and \( \text{Spf} \ k[x, f_n^{-1}][[t]]/(t^n) \), we obtain a closed subscheme \( Z \) of \( \text{Spf} \ k[x][[t]] \). Since \( Z \) contains \( \text{Spf} \ k[x^\pm][[t]]/(f) \),
as an open subscheme, \( \mathcal{Z} \) is not identical to \( \text{Spf} \, k[[t]] \). From Lemma 2.4, it follows that \( \mathcal{Z} \) is defined by a nonzero ideal \( J \subseteq k[[t]] \). Therefore

\[
I \cap k[[t]] \supseteq J \neq (0).
\]

\[\square\]

2.2. Pre-subschemes. Recall that a morphism \( f : Y \to X \) of ordinary schemes is a closed immersion if and only if it is a closed embedding as a map of topological spaces and the map \( \mathcal{O}_X \to f_*\mathcal{O}_Y \) of sheaves is surjective. We may adopt this as the definition of closed immersions of ordinary schemes. However, concerning formal schemes, this condition leads to a different notion from the closed immersion defined above.

**Definition 2.9.** A closed pre-immersion of a formal scheme \( X \) is a morphism \( \iota : Y \to X \) such that the map of underlying topological spaces is a closed embedding and the map \( \mathcal{O}_X \to \iota_*\mathcal{O}_Y \) is surjective. An open immersion followed by a closed immersion is said to be a pre-immersion. A (closed) pre-subscheme is an equivalence class of (closed) immersions with respect to the equivalence relation in Definition 2.3.

The following is an example of a closed pre-subscheme that is not a closed subscheme.

**Example 2.10.** Let \( A := k[[x,y]] \) be endowed with the \( \{(xy^i)\}_{i \in \mathbb{N}} \)-topology as in Example 1.1 and \( A^{\text{adic}} \) the same ring endowed with the \( (xy) \)-adic topology. Then \( A \) and \( A^{\text{adic}} \) are both admissible rings. The identity map \( A^{\text{adic}} \to A \) is a continuous homomorphism.

The formal schemes \( \mathcal{X} := \text{Spf} \, A \) and \( \mathcal{X}^{\text{adic}} := \text{Spf} \, A^{\text{adic}} \) have the same underlying topological space, which consists of three open prime ideals, \((x,y)\), \((x)\) and \((y)\). The stalks of \( \mathcal{O}_X \) and \( \mathcal{O}_{\mathcal{X}^{\text{adic}}} \) at \((x,y)\) and \((y)\) are identical as rings, but not at \((x)\). We have

\[
\mathcal{O}_{\mathcal{X},(x)} = k((y))[[x]]/(x) = k((y)) \quad \text{and} \quad \mathcal{O}_{\mathcal{X}^{\text{adic}},(x)} = k((y))[[x]].
\]

It follows that the morphism \( \mathcal{X} \to \mathcal{X}^{\text{adic}} \) induced by the identity map \( A^{\text{adic}} \to A \) is a closed pre-immersion. However it is clear that this morphism is not a closed immersion.

2.3. Ordinary subschemes. If \( I \) is an open ideal of an admissible ring \( A \), then \( I \) is also closed and we can define a formal scheme \( \text{Spf} \, A/I \). Since \( A/I \) is discrete, \( \text{Spf} \, A/I \) is in fact canonically isomorphic to \( \text{Spec} \, A/I \). Conversely for a closed ideal \( I \) of an admissible ring \( A \), if \( \text{Spf} \, A/I \) is an ordinary scheme, then \( I \) is open.
Let $\mathcal{X}$ be a formal scheme. An ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_\mathcal{X}$ is said to be open if for every point of $\mathcal{X}$, there exists its affine neighborhood $\text{Spf } A \subseteq \mathcal{X}$ such that $\mathcal{I}|_{\text{Spf } A} = I^\triangle$ for some open ideal $I \subseteq A$. Every open ideal sheaf is a closed ideal sheaf. The closed subscheme of $\mathcal{X}$ defined by an open ideal is an ordinary scheme. Conversely a closed subscheme that is an ordinary scheme is defined by an open ideal sheaf.

**Definition 2.11.** Let $\mathcal{X}$ be a formal scheme. A (closed) subscheme $Y \hookrightarrow \mathcal{X}$ with $Y$ an ordinary scheme is said to be a (closed) ordinary subscheme.

**Proposition 2.12.** A pre-subscheme $Y \hookrightarrow \mathcal{X}$ with $Y$ ordinary subscheme is an ordinary subscheme.

**Proof.** Without loss of generality, we may suppose that $\mathcal{X}$ is affine, say $\mathcal{X} = \text{Spf } A$, and that $Y$ is a closed pre-subscheme. Then the underlying topological space of $Y$ is homeomorphic to that of an affine scheme. Therefore $Y$ is quasi-compact, and covered by finitely many affine schemes $\text{Spec } B_i$. The natural morphism $\text{Spec } B_i \to \text{Spf } A$ corresponds to a continuous homomorphism $A \to B_i$. Since $B_i$ is discrete, the kernel $J_i$ of $A \to B_i$ is open. Put $J = \bigcap_i J_i$. Then $J$ is an open ideal and $\iota$ factors as $Y \to \text{Spec } A/J \to \text{Spf } A$.

The $\alpha$ is a closed immersion of ordinary schemes and there exists an open ideal $I \supseteq J$ such that $Y \cong \text{Spec } A/I$. Thus $Y$ is a closed ordinary subscheme of $\mathcal{X}$. $\square$

2.4. **The closure of an ordinary subscheme.** If $Y$ is a subscheme of an ordinary scheme $X$ and if the inclusion map $Y \to X$ is quasi-compact, then from [EGA, Prop. 9.5.10], there exists a smallest closed subscheme $\bar{Y}$ of $X$ that contains $Y$ as an open subscheme. We say that $\bar{Y}$ is the (scheme-theoretic) closure of $Y$ in $X$. We can generalize this as follows.

**Proposition-Definition 2.13.** Let $\mathcal{X}$ be a formal scheme and $Y$ its ordinary subscheme. Suppose that the inclusion map of the underlying topological spaces is quasi-compact. Then there exists a smallest closed ordinary subscheme $\bar{Y}$ of $\mathcal{X}$ that contains $Y$ as an open subscheme. Moreover, $\bar{Y}$ is defined by the kernel of $\mathcal{O}_{\mathcal{X}} \to \iota_* \mathcal{O}_Y$, where $\iota$ is the inclusion. We call $\bar{Y}$ the closure of $Y$ in $\mathcal{X}$.

**Proof.** We first suppose that $\mathcal{X}$ is affine, say $\mathcal{X} = \text{Spf } A$. Then $Y$ is quasi-compact. Therefore there exists an open subscheme $\mathcal{U} \subseteq \mathcal{X}$ covered by finitely many distinguished open subschemes $\text{Spf } A_{(f_i)}$ of
such that $Y$ is a closed ordinary subscheme of $U$. For every $i$, $Y \cap \text{Spf } A_{f_i}$ is defined by an open ideal $J_i \subseteq A_{f_i}$. If $\{I_j\}_{j \in \mathbb{N}}$ is a basis of ideals of definition of $A$, and if for each $i, j$, $(I_j)_{f_i} := \lim_{j \geq j'} (I_j/I_j')_{f_i}$ is the complete localization of $I_j$ by $f_i$, then $\{(I_j)_{f_i}\}_{j \in \mathbb{N}}$ is a basis of ideals of definition of $A$. Hence for $j \gg 0$ and for every $i$, $(I_j)_{f_i} \subseteq J_i$. Then $Y$ is a subscheme of $\text{Spec } A/I_j$. From [EGA, Prop. 9.5.10], there exists a closure $\bar{Y}$ of $Y$ in $\text{Spec } A/I_j$, which is defined by the kernel of $O_{\text{Spec } A/I_j} \twoheadrightarrow \iota_* O_Y$. We can view $\bar{Y}$ as a closed ordinary subscheme of $\mathcal{X}$, which is defined by the kernel of $O_X \twoheadrightarrow O_{\text{Spec } A/I_j} \twoheadrightarrow \iota_* O_Y$. We have proved the assertion in this case.

In the general case, $\mathcal{X}$ is covered by affine open subschemes $\mathcal{X}_\lambda$, $\lambda \in \Lambda$. For each $\lambda$, there exists the closure $\bar{Y}_\lambda$ of $Y_\lambda := Y \cap \mathcal{X}_\lambda$ in $\mathcal{X}_\lambda$. Gluing $\bar{Y}_\lambda$, we obtain the closure $\bar{Y}$ of $Y$ in $\mathcal{X}$. □

2.5. Subschemes of definition.

Definition 2.14. Let $\mathcal{X}$ be a formal scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ an open ideal. We say that $\mathcal{I}$ is an ideal of definition if every point of $\mathcal{X}$ has an affine neighborhood $\text{Spf } A$ such that $\mathcal{I}|_{\text{Spf } A} = I^2$ for an ideal of definition $I \subseteq A$. The ordinary subscheme defined by an ideal of definition is called a subscheme of definition.

Proposition 2.15. An ordinary subscheme $Y$ of a formal scheme $\mathcal{X}$ is a subscheme of definition if and only if the underlying topological space of $Y$ is identical to that of $\mathcal{X}$.

Proof. The “only if” direction is trivial. Suppose that the underlying topological space of $Y$ is identical to that of $\mathcal{X}$. Without loss of generality, we may suppose, in addition, that $\mathcal{X}$ is affine, say $\mathcal{X} = \text{Spf } A$. Let $I \subseteq A$ be the open ideal defining $Y$. Then $\text{Spec } A/\sqrt{I}$ is a unique reduced subscheme of $\text{Spec } A$ whose underlying topological space is identical to that of $\text{Spf } A$. This shows that $\sqrt{I}$ must be the largest ideal of definition. Therefore $I$ consists of topologically nilpotent elements, and is an ideal of definition. □

Since every admissible ring $A$ admits a largest ideal of definition, every affine formal scheme admits a smallest subscheme of definition, which is the reduced subscheme of definition. Gluing the smallest subschemes of definition of affine open subschemes, we obtain a smallest subscheme of definition of an arbitrary formal scheme. In particular, every formal scheme has at least one subscheme of definition.
Definition 2.16. Let $\mathcal{X}$ be a formal scheme and 

$$I_1 \supseteq I_2 \supseteq \cdots$$

a descending chain of ideals of definition in $\mathcal{O}_\mathcal{X}$. We say that $\{I_i\}_{i \in \mathbb{N}}$ is a basis of ideals of definition if every point of $\mathcal{X}$ has an affine neighborhood $\text{Spf } A$ and there exists a basis $\{I_i\}$ of ideals of definition in $A$ such that for every $i$, $I_i|_{\text{Spf } A} = I_i^\wedge$. The ascending chain of subschemes of definition corresponding a basis of ideals of definition is called a basis of subschemes of definition.

Proposition 2.17. Every top-Noetherian formal scheme has a basis of subschemes of definition.

Proof. Let $\mathcal{X}$ be a top-Noetherian formal scheme and $\mathcal{X} = \bigcup_{i=1}^{\alpha} \mathcal{U}_i$ its finite affine covering. For each $i$, there exists a basis $\{I_{ij}\}_{j \in \mathbb{N}}$ of ideals of definition on $\mathcal{U}_i$ and the corresponding basis $\{Y_{ij}\}_{j \in \mathbb{N}}$ of subschemes of definition. For each $i, j$, we denote by $\bar{Y}_{ij}$ to be the closure of $Y_{ij}$ in $\mathcal{X}$ and by $\bar{I}_{ij} \subseteq \mathcal{O}_\mathcal{X}$ the corresponding ideal sheaf. For each $j \in \mathbb{N}$, set $\mathcal{J}_j := \bigcap_{i=1}^{\alpha} \bar{I}_{ij}$. The $\mathcal{J}_j$’s are open ideals. For each $i$, $\mathcal{J}_j|_{\mathcal{U}_i}$ is contained in $\mathcal{I}_{ij}$. It follows that for each $i$, $\{\mathcal{J}_j|_{\mathcal{U}_i}\}_{j \in \mathbb{N}}$ is a basis of ideals of definition and so is $\{\mathcal{J}_i\}_{i \in \mathbb{N}}$. □

Proposition 2.18. Every locally Noetherian formal scheme has a basis of subschemes of definition.

Proof. Let $\mathcal{X}$ be a locally Noetherian formal scheme and $\mathcal{I} \subseteq \mathcal{O}_\mathcal{X}$ the largest ideal of definition. Then $\{\mathcal{I}^n\}_{n \in \mathbb{N}}$ is a basis of ideals of definition. □

2.6. Pseudo-subschemes.

Definition 2.19. A pseudo-immersion of a formal scheme $\mathcal{X}$ is a morphism $\iota : \mathcal{Y} \to \mathcal{X}$ of formal schemes such that for every immersion $Y \hookrightarrow \mathcal{Y}$ with $Y$ ordinary scheme, the composition $Y \hookrightarrow \mathcal{Y} \xrightarrow{\iota} \mathcal{X}$ is an immersion. A pseudo-subscheme is an equivalence class of pseudo-immersions.

If $\mathcal{Y}$ is a pseudo-subscheme of $\mathcal{X}$ and if $Y$ is a subscheme of definition, then $Y$ is by definition an ordinary subscheme of $\mathcal{X}$. Therefore the underlying topological space of $\mathcal{Y}$ is a locally closed subset of that of $\mathcal{X}$.

If $\mathcal{Y} \to \mathcal{X}$ is a pre-immersion, then from Proposition 2.12 for every immersion $Z \hookrightarrow \mathcal{Y}$ with $Z$ ordinary scheme, the composition $Z \to \mathcal{Y} \to \mathcal{X}$ is an immersion. Hence $\mathcal{Y} \to \mathcal{X}$ is a pseudo-immersion.
When $\mathcal{Y}$ admits a basis $\{Y_i\}_{i \in \mathbb{N}}$ of subschemes of definition, then $\mathcal{Y} \to \mathcal{X}$ is a pseudo-immersion of $\mathcal{X}$ if and only if for every $i$, $Y_i \hookrightarrow \mathcal{Y} \to \mathcal{X}$ is an immersion.

**Definition 2.20.** A pseudo-immersion or a pseudo-subscheme is said to be closed if the map of underlying topological spaces is a homeomorphism onto a closed subset.

**Example 2.21.** Let $X$ be an ordinary scheme and $Y$ its closed subscheme. Then the completion $X/Y$ of $X$ along $Y$ is a closed pseudo-subscheme of $X$.

Let $Z_1 \subseteq Z_2 \subseteq \cdots$ be ordinary subschemes of a formal scheme $\mathcal{X}$, all of which have the same underlying topological space. Then, from Corollary 1.3, the inductive limit

$$Z := \lim_{\longrightarrow} Z_i$$

is a formal scheme and a pseudo-subscheme of $\mathcal{X}$.

**Example 2.22.** Suppose that $k$ is an algebraically closed field and that a ring $k[[x]][[t]]$ is endowed with the $(t)$-adic topology. Let $\mathcal{X} := \text{Spf} k[[x]][[t]]$. The underlying topological space of $\mathcal{X}$ is identified with that of $\mathbb{A}_k^1 = \text{Spec} k[x]$. For each $a \in k$, we define a subscheme of definition of $\mathcal{X}$,

$$Y_a := \text{Spec} k[[x]][[t]]/(t^2, (x - a)t).$$

It has an embedded point at a rational point $a \in \mathbb{A}_k^1$. For a finite subset $\{a_1, \ldots, a_n\}$ of $k$, we define $Y_{a_1, \ldots, a_n}$ to be the subscheme of definition of $\mathcal{X}$ that is isomorphic to $Y_{a_i}$ around $a_i$, $1 \leq i \leq n$, and to $\mathbb{A}_k^1$ around any point other than $a_1, \ldots, a_n$.

Let $\{a_i; i \in \mathbb{N}\}$ be a countable subset of $k$. Then we have an ascending chain of subschemes of $\mathcal{X}$,

$$Y_{a_1} \subseteq Y_{a_1, a_2} \subseteq Y_{a_1, a_2, a_3} \subseteq \cdots,$$

and obtain a closed pseudo-subscheme of $\mathcal{X}$,

$$\mathcal{Y} := \lim_{\longrightarrow} Y_{a_1, a_2, \ldots, a_n}.$$ 

Then

$$\mathcal{O}_{\mathcal{Y}, p} \cong \begin{cases} 
(k[x, y]/(y^2, xy))(x, y) & (p \in \{a_1, a_2, \ldots\}) \\
(k[x])_x & (p \in \mathbb{A}_k^1(k) \setminus \{a_1, a_2, \ldots\}) \\
k(x) & (p \text{ the generic point}).
\end{cases}$$
Thus all fine stalks of \( \mathcal{O}_Y \) are discrete. If \( Y \) is Noetherian, then it is impossible that infinitely many fine stalks of \( \mathcal{O}_Y \) have an embedded prime. Therefore \( Y \) is not a closed subscheme of \( X \). Moreover for every open subscheme \( U \subseteq X \), \( Y \cap U \) is not a closed subscheme of \( U \) either.

**Proposition 2.23.**  
(1) Every pseudo-subscheme of a (locally) ind-Noetherian formal scheme is (locally) ind-Noetherian.  
(2) If \( g : Z \to Y \) and \( f : Y \to X \) are pseudo-immersions, then \( f \circ g \) is also a pseudo-immersion.

(3) Let \( Y \to X \) be a pseudo-immersion of formal schemes and \( W \to \mathcal{X} \) a morphism of formal schemes. Then the projection \( Y \times_\mathcal{X} W \to W \) is a pseudo-immersion.

(4) Every closed pseudo-subscheme of an affine formal scheme is an affine formal scheme.

**Proof.**  
1. The assertion follows from Proposition 1.8.  
2. If \( W \twoheadrightarrow Z \) is an immersion with \( W \) ordinary scheme, then the natural morphism \( W \to X \) is an immersion. Therefore \( Z \to X \) is a pseudo-immersion.

3. Since the problem is local, we may suppose that \( Y, \mathcal{X} \) and \( W \) are affine. Let \( \{ Y_i \}, \{ X_i \} \) and \( \{ W_i \} \) be bases of subschemes of definition of \( Y, \mathcal{X} \) and \( W \) respectively such that for every \( i \), the natural morphisms \( Y_i \to \mathcal{X} \) and \( W_i \to \mathcal{X} \) factors through \( X_i \). Then \( \{ Y_i \times_\mathcal{X} W_i \} \) is a basis of subschemes of definition of \( Y \times_\mathcal{X} W \). Since \( Y_i \) is a subscheme of \( X_i \), the projection \( Y_i \times_\mathcal{X} W_i \to W_i \) is an immersion [EGA Prop. 4.4.1], the natural morphism \( Y_i \times_\mathcal{X} W_i \to W \) is also an immersion. Therefore \( Y \times_\mathcal{X} W \to W \) is a pseudo-immersion.

4. Let \( \mathcal{X} = \text{Spf} \ A \) and \( Y \) its closed pseudo-subscheme. Then the underlying topological space of \( Y \) is isomorphic to that of an affine scheme \( \text{Spec} \ R \). There exists an open covering \( \mathcal{Y} = \bigcup_{j=1}^n \mathcal{Y}_j \) such that for each \( j \), \( \mathcal{Y}_j \) is identified with a distinguished open subscheme \( \text{Spec} \ R_f \) as a topological space. Since the \( \mathcal{Y}_j \hookrightarrow \mathcal{Y} \) are quasi-compact, as in the proof of Proposition 2.17, we can show that \( \mathcal{Y} \) has a basis of subschemes of definition, \( \{ Y_i \}_{i \in \mathbb{N}} \). Then \( Y_i \) can be viewed as a closed ordinary subscheme of \( \text{Spf} \ A \). From Lemma 2.21 the \( Y_i \) are affine, say \( Y_i = \text{Spec} \ A_i \). It follows that \( \mathcal{Y} = \text{Spf} (\varprojlim A_i) \) is affine. \[\square\]

2.7. Chevalley’s theorem.

**Theorem 2.24.** Let \( \mathcal{X} \) be a Noetherian formal scheme. Every pseudo-subscheme of \( \mathcal{X} \) is a subscheme of \( \mathcal{X} \) if and only if the underlying topological space of \( \mathcal{X} \) is discrete.

**Proof.** The “if” direction is essentially due to Chevalley [Che]. To show this, we may suppose that the underlying topological space of \( \mathcal{X} \)
consists of a single point. Then for some Noetherian complete local ring \((A, \mathfrak{m})\) (with the \(\mathfrak{m}\)-adic topology), we have \(X \cong \text{Spec} \ A\). Let
\[
\mathcal{Y} = \lim_{\longrightarrow} \text{Spec} \ A/I_n
\]
be a pseudo-subscheme where
\[
A \supseteq I_1 \supseteq I_2 \supseteq \cdots
\]
is a descending chain of open ideals. Chevalley’s theorem \cite[Che, Lem. 7]{Chevalley} (see also \cite[Ch. VIII, §5, Th. 13]{ZS}) says that either
(1) for every \(n \in \mathbb{N}\), there exists \(i \in \mathbb{N}\) with \(I_i \subseteq \mathfrak{m}^n\), or
(2) \(\bigcap_i I_i \neq (0)\).
In the former case, the \(\{I_i\}\)-topology coincides with the \(\mathfrak{m}\)-adic topology, and so \(\mathcal{Y} = X\). In the latter case, replacing \(A\) with \(A/\bigcap_i I_i\), we can reduce to the former case. Consequently we see that \(\mathcal{Y} = \text{Spf} (A/\bigcap_i I_i)\) and that \(\mathcal{Y}\) is a subscheme of \(X\).

We now prove the “only if” direction. Suppose that the underlying topological space of \(X\) is not discrete. Then there exists a closed but not open point \(x\) of \(X\). Let \(\text{Spf} \ A \subseteq X\) be an affine neighborhood of \(x\). Then \(\text{Spf} \ A\) consists of at least two points. Let \(A_{\text{red}}\) be the reduced ring associated to \(A\), that is, the ring \(A\) modulo the ideal of nilpotent elements. Then \(\text{Spf} \ A\) and \(\text{Spf} A_{\text{red}}\) have the same underlying topological space. If \(\hat{A}_{\text{red}}\) is the \(\mathfrak{m}\)-adic completion of \(A_{\text{red}}\) with \(\mathfrak{m}\) the maximal ideal of \(x\), then \(\text{Spf} \hat{A}_{\text{red}}\) is a closed pseudo-subscheme of \(\text{Spf} A_{\text{red}}\) consisting of a single point, hence not isomorphic to \(\text{Spf} A_{\text{red}}\). Being injective, the natural map \(A_{\text{red}} \rightarrow \hat{A}_{\text{red}}\) does not factor as \(A_{\text{red}} \rightarrow A_{\text{red}}/J \cong \hat{A}_{\text{red}}\) for any nonzero ideal \(J\). Hence \(\text{Spf} \hat{A}_{\text{red}}\) is not any closed subscheme of \(\text{Spf} A_{\text{red}}\) or of \(\text{Spf} A\). \qed

2.8. The pseudo-closure of a pseudo-subscheme.

**Proposition-Definition 2.25.** Let \(X\) be a formal scheme and \(\mathcal{Y} \subset X\) its pseudo-subscheme. Suppose that the inclusion map of the underlying topological spaces is quasi-compact. Then there exists a smallest closed pseudo-subscheme \(\breve{\mathcal{Y}}\) of \(X\) that contains \(\mathcal{Y}\) as an open subscheme. We call \(\breve{\mathcal{Y}}\) the pseudo-closure of \(\mathcal{Y}\) in \(X\).

**Proof.** We first consider the case where \(X\) is quasi-compact and \(\mathcal{Y}\) admits a basis of subschemes of definition, say \(\{Y_i\}_{i \in \mathbb{N}}\). From Proposition-Definition 2.13 for each \(i\), there exists the closure \(\breve{Y}_i\) of \(Y_i\) in \(X\). Then we put
\[
\breve{\mathcal{Y}} := \lim_{\longrightarrow} \breve{Y}_i.
\]
Let \(\{Y'_j\}\) be another basis of subschemes of definition of \(\mathcal{Y}\). Then for every \(i \in \mathbb{N}\), there exists \(j \in \mathbb{N}\) such that \(\breve{Y}_i \subseteq \breve{Y}'_j\) and \(\breve{Y}'_i \subseteq \breve{Y}_j\).
Therefore $Y_i'$ can be viewed as closed subschemes of $\mathcal{Y}$ and also form a basis of subschemes of definition of $\mathcal{Y}$. It follows that $\mathcal{Y} = \lim \to Y_i'$. Thus $\mathcal{Y}$ is independent of the choice of $\{Y_i\}$. By construction, $\mathcal{Y}$ is an open subscheme of $\mathcal{X}$. Moreover $\bar{\mathcal{Y}}$ is a smallest closed pseudo-subscheme of $\mathcal{X}$ that contains $\mathcal{Y}$ as an open subscheme. Indeed if $\mathcal{Z}$ is another closed pseudo-subscheme of $\mathcal{X}$ containing $\mathcal{Y}$ as an open subscheme, then the $Y_i$ are also closed ordinary subschemes of $\mathcal{Z}$ and hence $\bar{\mathcal{Y}}$ is also a closed pseudo-subscheme of $\mathcal{Z}$.

We now consider the general case. Then there exists an open covering $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ such that for every $\lambda$, $X_\lambda$ is quasi-compact and $Y_\lambda := \mathcal{Y} \cap X_\lambda$ admits a basis of subschemes of definition. For each $\lambda$, there exists the pseudo-closure $\bar{Y}_\lambda$ of $Y_\lambda$ in $X_\lambda$. Let $X_{\lambda\mu} := X_\lambda \cap X_\mu$, $Y_{\lambda\mu} := \mathcal{Y} \cap X_{\lambda\mu}$ and $\bar{Y}_{\lambda\mu}$ the pseudo-closure of $Y_{\lambda\mu}$ in $X_{\lambda\mu}$. Then by the construction above of $\bar{Y}$, $\bar{Y}_{\lambda\mu} = Y_\lambda \cap X_{\lambda\mu}$. Therefore we can glue the $\bar{Y}_\lambda$ and obtain a closed pseudo subscheme $\bar{Y}$ of $\mathcal{X}$ that contains $\mathcal{Y}$ as an open subscheme. It is easy to see that $\bar{Y}$ is the smallest closed pseudo-subscheme with this property. \hfill \Box

**Example 2.26.** Let $\mathcal{Y}$ be a closed subscheme of $\text{Spf} \mathbb{C}[x^\pm, y][[t]]$ such that the only closed subscheme of $\text{Spf} \mathbb{C}[x, y][[t]]$ containing $\mathcal{Y}$ is $\text{Spf} \mathbb{C}[x, y][[t]]$. (Thanks to Theorem 2.7, such $\mathcal{Y}$ exists.) Then the pseudo-closure $\bar{Y}$ of $\mathcal{Y}$ in $\text{Spf} \mathbb{C}[x, y][[t]]$ is not a closed subscheme of $\text{Spf} \mathbb{C}[x, y][[t]]$.

**Remark 2.27.** Examples 2.22 and 2.26 are both pseudo-subschemes that are not subschemes, but have different flavors. It might be good to distinguish them, for example, by the following condition on a pseudo-subscheme $\mathcal{Y} \hookrightarrow \mathcal{X}$: For any ordinary subscheme $Z \hookrightarrow \mathcal{X}$, the fiber product $\mathcal{Y} \times_Z \mathcal{X}$ is an ordinary scheme. While Example 2.22 does not satisfy this, Example 2.26 does.

### 3. Formal separatrices of singular foliations

In this section, we see that a pathological phenomenon of formal schemes also comes from singularities of foliations.

**3.1. Formal separatrices.** Let $X$ be a smooth algebraic variety over $\mathbb{C}$, and $\Omega_X = \Omega_{X/\mathbb{C}}$ the sheaf of (algebraic) Kähler differential forms. A *(one-codimensional) foliation* on $X$ is an invertible saturated subsheaf $\mathcal{F}$ of $\Omega_X$ satisfying the integrability condition; $\mathcal{F} \wedge d\mathcal{F} = 0$. We say that a foliation $\mathcal{F}$ is *smooth* at $x \in X$ if the quotient sheaf $\Omega_X/\mathcal{F}$ is locally free around $x$, and that $\mathcal{F}$ is *singular* at $x$ otherwise. We say that $\mathcal{F}$ is *smooth* if $\mathcal{F}$ is smooth at every point. The pair $(X, \mathcal{F})$ of a smooth variety $X$ and a foliation on $X$ is called a *foliated variety*. 
Definition 3.1. Let \((X, \mathcal{F})\) be a foliated variety, \(x \in X(\mathbb{C})\), \(X_{/x} := \text{Spf} \hat{\mathcal{O}}_{X,x}\), \(Y \subseteq X_{/x}\) a closed subscheme of codimension one defined by \(0 \neq f \in \hat{\mathcal{O}}_{X,x}\), and \(\omega \in \Omega_{X,x}\) a generator of \(\mathcal{F}_x\). We say that \(Y\) is a formal separatrix (of \(\mathcal{F}\)) at \(x\) if \(f\) divides \(\omega \wedge df\).

Because of Leibniz rule, \(Y\) is a formal separatrix if and only if its associated reduced formal scheme \(Y_{\text{red}}\) is a formal separatrix. Frobenius theorem says that if \(\mathcal{F}\) is smooth at \(x\), there exists a unique smooth formal separatrix of \(\mathcal{F}\) at \(x\). Miyaoka [Miy] proved that the family of smooth formal separatrices at smooth points of a foliation form a formal scheme:

\[\text{Theorem 3.2. } [\text{Miy, Cor. 6.4}] \quad \text{Let } (X, \mathcal{F}) \text{ be a foliated variety. Suppose that } \mathcal{F} \text{ is smooth. Then there exists a closed subscheme } \mathcal{L} \text{ of } (X \times_{\mathbb{C}} X) / \Delta_X \text{ such that for every point } x \in X, p_2(p_1^{-1}(x)) \text{ is the smooth formal separatrix of } \mathcal{F} \text{ at } x. \text{ Here } \Delta_X \subseteq X \times_{\mathbb{C}} X \text{ is the diagonal and } (X \times_{\mathbb{C}} X) / \Delta_X \text{ is the completion of } X \times_{\mathbb{C}} X \text{ along } \Delta_X.\]

Let \((X, \mathcal{F})\) be a foliated variety and \(C \subseteq X\) a closed smooth subvariety of dimension 1. Suppose that \(C\) meets only at a single point \(o\) with the singular locus of \(\mathcal{F}\). Let \(U \subseteq X\) be the smooth locus of \(\mathcal{F}\) and \(\mathcal{L} \subseteq (U \times_{\mathbb{C}} U) / \Delta_U\) the family of formal separatrices as in the theorem. Then \(C \setminus \{o\}\) is a closed subvariety of \(U\). The fiber product

\[\mathcal{L}_{C \setminus \{o\}} := (C \setminus \{o\}) \times_{U, \text{pr}_1} \mathcal{L}\]

is the family of the smooth formal separatrices over \(C \setminus \{o\}\), and a subscheme of \((C \times_{\mathbb{C}} X) / \Delta_C\). Let \(\mathcal{L}_C := \overline{\mathcal{L}_{C \setminus \{o\}}}\) be the pseudo-closure of \(\mathcal{L}_{C \setminus \{o\}}\) in \((C \times_{\mathbb{C}} X) / \Delta_C\).

Proposition 3.3. The following are equivalent:

1. \(\mathcal{L}_C\) is Noetherian.
2. \(\mathcal{L}_C\) is adic.
3. \(\mathcal{L}_C\) is pre-Noetherian.
4. \(\mathcal{L}_C\) is a closed subscheme of \((C \times_{\mathbb{C}} X) / \Delta_C\).
5. \(\mathcal{L}_C\) is a closed pre-subscheme of \((C \times_{\mathbb{C}} X) / \Delta_C\).

Proof. 1 \(\implies\) 2 and 1 \(\implies\) 3: Trivial.
2 \(\implies\) 1: It follows from Lemma 1.6.
3 \(\implies\) 2: The underlying topological space of \(\mathcal{L}_C\) is identified with that of \(C\). Shrinking \(C\), we may suppose that \(\mathcal{L}_C\) is affine, say \(\mathcal{L}_C = \text{Spf} \ A\) with \(A\) a Noetherian admissible ring. Let \(I \subseteq A\) be the largest ideal of definition. This is a prime ideal and the symbolic powers \(I^{(n)}\) form a basis of ideals of definition in \(A\).
If \( f, g \in A \) are nonzero elements, then for \( n \gg 0 \), their images \( \bar{f}, \bar{g} \) in \( A/I^{(n)} \) are nonzero. Therefore the restrictions \( \bar{f}|_{C \setminus \{o\}} \) and \( \bar{g}|_{C \setminus \{o\}} \) of \( f \) and \( g \) are nonzero. Consequently the restriction \( (fg)|_{C \setminus \{o\}} \) of the product \( fg \) does not vanish, and the product \( fg \) does not neither. Thus \( A \) is a domain.

Let \( \mathfrak{m} \subseteq A \) be the maximal ideal of \( o \) and \( \hat{A} \) the \( \mathfrak{m} \)-adic completion of \( A \). We claim that \( \hat{A} \) is also a domain. To see this, we may suppose that \( (C \times_\mathbb{C} X)/\Delta_{C} \) is affine, say \( \text{Spf } B \). Let \( \mathfrak{n} \subseteq B \) be the maximal ideal of \( o \) and \( \hat{B} \) the \( \mathfrak{n} \)-adic completion of \( B \). Put

\[
J_{n} := \ker (\hat{B} \to \hat{A}/\hat{I}^{(n)}).
\]

Then \( \hat{A} \cong \varprojlim \hat{B}/J_{n} \). Since \( C \) is smooth, in particular, analytically irreducible, \( \hat{B}/J_{1} \) is a domain. Now we can prove the claim in the same way as above.

From [Zar] page 33, Lem. 3 (see also [ZS] Ch. VIII, §5, Cor. 5), the \( I^{(n)} \)-topology on \( A \) coincides with the \( I \)-adic topology.

2 \( \Rightarrow \) 4: It is a direct consequence of Proposition 1.8.

4 \( \Rightarrow \) 5: Trivial.

5 \( \Rightarrow \) 2: Put \( \mathcal{X} := (C \times_\mathbb{C} X)/\Delta_{C} \) and \( \mathcal{Y} := \mathcal{L}_{C} \). Identifying the underlying topological spaces of \( \mathcal{X} \) and \( \mathcal{Y} \) with \( C \), the point \( o \in C \) can be viewed as a point of \( \mathcal{X} \) and \( \mathcal{Y} \). Since \( \mathcal{X} \) is Noetherian, from [EGA 0, Cor. 7.6.18], the stalk \( \mathcal{O}_{\mathcal{X}, o} \) is Noetherian. Since the natural map \( \mathcal{O}_{\mathcal{X}, o} \to \hat{\mathcal{O}}_{\mathcal{Y}, o} \) is surjective, \( \mathcal{O}_{\mathcal{Y}, o} \) is also Noetherian. Let \( I \subseteq \mathcal{O}_{\mathcal{Y}, o} \) be the ideal of the topologically nilpotent elements. Then the symbolic powers \( I^{(n)} \) of \( I \) form a basis of open neighborhoods of \( \mathcal{O}_{\mathcal{Y}, o} \). (Note that \( \mathcal{O}_{\mathcal{Y}, o} \) is not a priori complete.) As in the proof of “3 \( \Rightarrow \) 2”, we can show that the topology on \( \mathcal{O}_{\mathcal{Y}, o} \) is identical to the \( I \)-adic topology. Hence if \( \mathcal{I} \subseteq \mathcal{O}_{\mathcal{Y}} \) is the ideal sheaf of topologically nilpotent sections, then its powers \( \mathcal{I}^{n} \) form a basis of ideals of definition, and \( \mathcal{Y} \) is adic.

When \( \mathcal{L}_{C} \) is a closed subscheme, it allows us to take the limit of smooth formal separatrices along \( C \):

**Theorem 3.4.** Suppose that one of the conditions in Proposition 3.3 holds. Then the fiber of \( \mathcal{L}_{C} \to C \) over \( o \) is a formal separatrix at \( o \).

**Proof.** We need to use complete modules of differentials of Noetherian formal schemes. For a morphism \( f : \mathcal{Y} \to \mathcal{X} \) of Noetherian formal schemes, we have a complete module of differentials, \( \hat{\Omega}_{\mathcal{Y}/\mathcal{X}} \), which is a quasi-coherent \( \mathcal{O}_{\mathcal{Y}} \)-module, and have a derivation \( \hat{d}_{\mathcal{Y}/\mathcal{X}} : \mathcal{O}_{\mathcal{Y}} \to \hat{\Omega}_{\mathcal{Y}/\mathcal{X}} \).

We refer to [AJP] for details.
If necessary, shrinking $X$, we can take a nowhere vanishing $\omega \in \mathcal{F}(X)$. Let
\[
\psi : \mathcal{X} := (C \times_{\mathbb{C}} X) / \Delta \to X.
\]
be the projection. Pulling back $\omega$, we obtain a global section $\psi^*\omega$ of $\hat{\Omega}_{X/C}$. Since $\mathcal{L}_C$ is a hypersurface in $\mathcal{X}$, it is defined by a section $f$ of $\mathcal{O}_X$. Since the restriction of $\mathcal{L}_C$ to $C \setminus \{o\}$ is the family of formal separatrices along $C \setminus \{o\}$, $f$ divides $\psi^*\omega \wedge \hat{d}_{X/C}f$.

Let $Y$ be the fiber of $\mathcal{L}_C \to C$ over $o$, which is a hypersurface of $X/o$ defined by the image $\bar{f} \in \hat{\mathcal{O}}_{X,o}$ of $f$. Then $\bar{f}$ divides $\omega \wedge \hat{d}_{X/o/C}\bar{f}$. Hence $Y$ is a formal separatrix. \hfill \Box

3.2. Jouanolou’s theorem and its application. We recall Jouanolou’s result on Pfaff forms. We refer to [Jou] for details.

An algebraic Pfaff form of degree $m$ on $\mathbb{P}^2_{\mathbb{C}}$ is a one-form
\[
\omega = \omega_1 dx + \omega_3 dy + \omega_3 dz
\]
such that $\omega_i$ are homogeneous polynomials of degree $m$ and the equation
\[
x \omega_1 + y \omega_2 + z \omega_3 = 0
\]
holds. A Pfaff equation of degree $m$ on $\mathbb{P}^2_{\mathbb{C}}$ is a class of algebraic Pfaff forms modulo nonzero scalar multiplications.

Let $\omega$ be an algebraic Pfaff form on $\mathbb{P}^2_{\mathbb{C}}$ and $[\omega]$ its Pfaff equation class. An algebraic solution of $\omega$ or $[\omega]$ is a class of homogeneous polynomials $f \in \mathbb{C}[x, y, z]$ modulo nonzero scalar multiplications such that $f$ divides $\omega \wedge df$.

Let $V_m$ be the vector space of the algebraic Pfaff forms of degree $m$ on $\mathbb{P}^2_{\mathbb{C}}$. Then the set of the Pfaff equations of degree $m$ on $\mathbb{P}^2_{\mathbb{C}}$ is identified with the projective space $\mathbb{P}(V_m) = (V_m \setminus \{0\}) / \mathbb{C}^*$. Define
\[
Z_m \subseteq \mathbb{P}(V_m)
\]
to be the set of the Pfaff equations that have no algebraic solution.

Theorem 3.5. [Jou, §4] Suppose $m \geq 3$. Then $Z_m$ is the intersection of countably many nonempty Zariski open subsets of $\mathbb{P}(V_m)$ and contains the class of the algebraic Pfaff form
\[
(x^{m-1}z - y^m)dx + (y^{m-1}x - z^m)dy + (z^{m-1}y - x^m)dz.
\]

From [Jou, page 4, Prop. 1.4], every algebraic Pfaff form $\omega$ on $\mathbb{P}^2_{\mathbb{C}}$ is integrable; $d\omega \wedge \omega = 0$. So $\omega$ defines also a foliation $\mathcal{F}_\omega$ on $\mathbb{C}^3$. From [Jou, page 85, Prop. 2.1], the only singular point of $\mathcal{F}_\omega$ is the origin. Accordingly we can define the family $\mathcal{L}_{\omega, C \setminus \{o\}}$ of formal separatrices along $C \setminus \{o\}$ and its pseudo-closure $\mathcal{L}_{\omega, C}$ for any line $C \subset \mathbb{C}^3$ through the origin.
Let $f = \sum_{i \geq n} f_i \in \mathbb{C}[[x, y, z]]$. Here $f_i$ is a homogeneous polynomial of degree $i$ and $f_n \neq 0$. Suppose that $f$ defines a formal separatrix at the origin, equivalently that $f$ divides $\omega \wedge df$. Then the class of $f_n$ is an algebraic solution of the Pfaff equation $[\omega]$. Hence if $[\omega] \in Z_m$, then $\mathcal{F}_\omega$ has no formal separatrix at the origin.

**Corollary 3.6.** For $[\omega] \in Z_m$ and a line $C \subseteq \mathbb{C}^3$ through the origin, a pseudo-subscheme $\mathcal{L}_{\omega, C}$ of $(C \times C \mathbb{C}^3)_{\Delta C} \cong \text{Spf} \mathbb{C}[w][[x, y, z]]$ is neither a closed pre-subscheme, pre-Noetherian nor adic.

**Proof.** If $\mathcal{L}_{\omega, C}$ is either a closed pre-subscheme, pre-Noetherian or adic, then from Theorem 3.4, the foliation $\mathcal{F}_\omega$ has a formal separatrix at the origin. Hence $[\omega] \notin Z_m$. \qedsymbol

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