When is the Bloch–Okounkov $q$-bracket modular?

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Received: 24 October 2018 / Accepted: 28 January 2019
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Abstract

We obtain a condition describing when the quasimodular forms given by the Bloch–Okounkov theorem as $q$-brackets of certain functions on partitions are actually modular. This condition involves the kernel of an operator $\Delta$. We describe an explicit basis for this kernel, which is very similar to the space of classical harmonic polynomials.

Keywords Modular forms · Partitions · Harmonic polynomials

Mathematics Subject Classification Primary 05A17 · Secondary 11F11 · 33C55

1 Introduction

Given a family of quasimodular forms, the question which of its members are modular often has an interesting answer. For example, consider the family of theta series

$$\theta_P(\tau) = \sum_{x \in \mathbb{Z}^r} P(x) q^{x_1^2 + \ldots + x_r^2} \quad (q = e^{2\pi i \tau})$$

given by all homogeneous polynomials $P \in \mathbb{Z}[x_1, \ldots, x_r]$. The quasimodular form $\theta_P$ is modular if and only if $P$ is harmonic (i.e. $P \in \ker \sum_{i=1}^r \frac{\partial^2}{\partial x_i^2}$) [10]. (As quasimodular forms were not yet defined, Schoeneberg only showed that $\theta_P$ is modular if $P$ is harmonic. However, for every polynomial $P$ it follows that $\theta_P$ is quasimodular by decomposing $P$ as in Formula (1).) Also, for every two modular forms $f$, $g$, one can consider the linear combination of products of derivatives of $f$ and $g$ given by

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Published online: 17 June 2019
\[
\sum_{r=0}^{n} a_r f^{(r)} g^{(n-r)} \quad (a_r \in \mathbb{C}).
\]

This linear combination is a quasimodular form which is modular precisely if it is a multiple of the Rankin–Cohen bracket \([f, g]_n\) [4,9]. In this paper, we provide a condition to decide which member of the family of quasimodular forms provided by the Bloch–Okounkov theorem is modular. Let \(\mathcal{P}\) denote the set of all partitions of integers and \(|\lambda|\) denote the integer that \(\lambda\) is a partition of. Given a function \(f : \mathcal{P} \to \mathbb{Q}\), define the \(q\)-bracket of \(f\) by

\[
\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda)q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}}.
\]

The celebrated Bloch–Okounkov theorem states that for a certain family of functions \(f : \mathcal{P} \to \mathbb{Q}\) (called shifted symmetric polynomials and defined in Sect. 2) the \(q\)-brackets \(\langle f \rangle_q\) are the \(q\)-expansions of quasimodular forms [2].

Besides being a wonderful result, the Bloch–Okounkov theorem has many applications in enumerative geometry. For example, a special case of the Bloch–Okounkov theorem was discovered by Dijkgraaf and provided with a mathematically rigorous proof by Kaneko and Zagier, implying that the generating series of simple Hurwitz numbers over a torus are quasimodular [5,7]. Also, in the computation of asymptotics of geometrical invariants, such as volumes of moduli spaces of holomorphic differentials and Siegel–Veech constants, the Bloch–Okounkov theorem is applied [3,6].

Zagier gave a surprisingly short and elementary proof of the Bloch–Okounkov theorem [13]. A corollary of his work, which we discuss in Sect. 3, is the following proposition:

**Proposition 1** There exists actions of the Lie algebra \(\mathfrak{sl}_2\) on both the algebra of shifted symmetric polynomials \(\Lambda^*\) and the algebra of quasimodular forms \(\tilde{M}\) such that the \(q\)-bracket \(\langle \cdot \rangle_q : \Lambda^* \to \tilde{M}\) is \(\mathfrak{sl}_2\)-equivariant.

The answer to the question in the title is provided by one of the operators \(\Delta\) which defines this \(\mathfrak{sl}_2\)-action on \(\Lambda^*\). Namely letting \(\mathcal{H} = \ker \Delta|_{\Lambda^*}\), we prove the following theorem:

**Theorem 1** Let \(f \in \Lambda^*\). Then \(\langle f \rangle_q\) is modular if and only if \(f = h + k\) with \(h \in \mathcal{H}\) and \(k \in \ker \langle \cdot \rangle_q\).

The last section of this article is devoted to describing the graded algebra \(\mathcal{H}\). We call \(\mathcal{H}\) the space of shifted symmetric harmonic polynomials, as the description of this space turns out to be very similar to the space of classical harmonic polynomials. Let \(\mathcal{P}_d\) be the space of polynomials of degree \(d\) in \(m \geq 3\) variables \(x_1, \ldots, x_m\), let \(||x||^2 = \sum_i x_i^2\), and recall that the space \(\mathcal{H}_d\) of degree \(d\) harmonic polynomials is given by \(\ker \sum_{i=1}^{r} \frac{\partial^2}{\partial x_i^2}\). The main theorem of harmonic polynomials states that every polynomial \(P \in \mathcal{P}_d\) can uniquely be written in the form

\[
P = h_0 + ||x||^2 h_1 + \ldots + ||x||^{2d} h_d.
\]
with $h_i \in \mathcal{H}_{d-2i}$ and $d' = \lfloor d/2 \rfloor$. Define $K$, the Kelvin transform, and $D^\alpha$ for $\alpha$ an $m$-tuple of non-negative integers by

$$f(x) \mapsto ||x||^{2-m} f\left(\frac{x}{||x||^2}\right) \quad \text{and} \quad D^\alpha = \prod_i \frac{\partial^\alpha_i}{\partial x_i^\alpha_i}.$$  

An explicit basis for $\mathcal{H}_d$ is given by

$$\left\{ KD^\alpha K(1) \mid \alpha \in \mathbb{Z}_m^m, \sum_i \alpha_i = d, \alpha_1 \leq 1 \right\},$$

see for example [1]. We prove the following analogous results for the space of shifted symmetric polynomials:

**Theorem 2** For every $f \in \Lambda^*_n$ there exists unique $h_i \in \mathcal{H}_{n-2i}$ ($i = 0, 1, \ldots, n'$ and $n' = \lfloor n/2 \rfloor$) such that

$$f = h_0 + Q_2 h_1 + \ldots + Q_2^{n'} h_n',$$

where $Q_2$ is an element of $\Lambda^*_2$ given by $Q_2(\lambda) = |\lambda| - \frac{1}{24}$.  

**Theorem 3** The set

$$\{ \text{pr} K \Delta_\lambda K(1) \mid \lambda \in \mathcal{P}(n), \text{all parts are } \geq 3 \}$$

is a vector space basis of $\mathcal{H}_n$, where $\text{pr}$, $K$, and $\Delta_\lambda$ are defined by (4), Definition 4, respectively, Definition 6.

The action of $\mathfrak{sl}_2$ given by Proposition 1 makes $\Lambda^*$ into an infinite-dimensional $\mathfrak{sl}_2$-representation for which the elements of $\mathcal{H}$ are the lowest weight vectors. Theorem 2 is equivalent to the statement that $\Lambda^*$ is a direct sum of the (not necessarily irreducible) lowest weight modules

$$V_n = \bigoplus_{m=0}^{\infty} Q_2^m \mathcal{H}_n \quad (n \in \mathbb{Z}).$$

## 2 Shifted symmetric polynomials

Shifted symmetric polynomials were introduced by Okounkov and Olshanski as the following analogue of symmetric polynomials [8]. Let $\Lambda^*(m)$ be the space of rational polynomials in $m$ variables $x_1, \ldots, x_m$ which are shifted symmetric, i.e. invariant under the action of all $\sigma \in \mathfrak{S}_m$ given by $x_i \mapsto x_{\sigma(i)} + i - \sigma(i)$ (or more symmetrically $x_i - i \mapsto x_{\sigma(i)} - \sigma(i)$). Note that $\Lambda^*(m)$ is filtered by the degree of the polynomials. We have forgetful maps $\Lambda^*(m) \to \Lambda^*(m-1)$ given by $x_m \mapsto 0$, so that we can define the space of shifted symmetric polynomials $\Lambda^*$ as $\lim_{\leftarrow m} \Lambda^*(m)$ in the category of
filtered algebras. Considering a partition $\lambda$ as a non-increasing sequence $(\lambda_1, \lambda_2, \ldots)$ of non-negative integers $\lambda_i$, we can interpret $\Lambda^*$ as being a subspace of all functions $\mathcal{P} \to \mathbb{Q}$.

One can find a concrete basis for this abstractly defined space by considering the generating series

$$w_\lambda(T) := \sum_{i=1}^{\infty} T^{\lambda_i-i+\frac{1}{2}} \in T^{1/2}\mathbb{Z}[T][[T^{-1}]]$$

for every $\lambda \in \mathcal{P}$ (the constant $\frac{1}{2}$ turns out to be convenient for defining a grading on $\Lambda^*$). As $w_\lambda(T)$ converges for $T > 1$ and equals

$$\frac{1}{T^{1/2} - T^{-1/2}} + \sum_{i=1}^{\ell(\lambda)} \left( T^{\lambda_i-i+\frac{1}{2}} - T^{-i+\frac{1}{2}} \right)$$

one can define shifted symmetric polynomials $Q_i(\lambda)$ for $i \geq 0$ by

$$\sum_{i=0}^{\infty} Q_i(\lambda)z^{i-1} := w_\lambda(e^z) \quad (0 < |z| < 2\pi).$$

The first few shifted symmetric polynomials $Q_i$ are given by

$$Q_0(\lambda) = 1, \quad Q_1(\lambda) = 0, \quad Q_2(\lambda) = |\lambda| - \frac{1}{4}. $$

The $Q_i$ freely generate the algebra of shifted symmetric polynomials, i.e. $\Lambda^* = \mathbb{Q}[Q_2, Q_3, \ldots]$. It is believed that $\Lambda^*$ is maximal in the sense that for all $Q : \mathcal{P} \to \mathbb{Q}$ with $Q \notin \Lambda^*$ it holds that $\langle \Lambda^*[Q] \rangle_q \not\subset \tilde{M}$.

**Remark 1** The space $\Lambda^*$ can equally well be defined in terms of the Frobenius coordinates. Given a partition with Frobenius coordinates $(a_1, \ldots, a_r, b_1, \ldots, b_r)$, where $a_i$ and $b_i$ are the arm and leg lengths of the cells on the main diagonal, let

$$C_\lambda = \left\{-b_1 - \frac{1}{2}, \ldots, -b_r - \frac{1}{2}, a_r + \frac{1}{2}, \ldots, a_1 + \frac{1}{2}\right\}.$$

Then

$$Q_k(\lambda) = \beta_k + \frac{1}{(k-1)!} \sum_{c \in C_\lambda} \text{sgn}(c)c^{k-1},$$

where $\beta_k$ is the constant given by

$$\sum_{k \geq 0} \beta_k z^{k-1} = \frac{1}{2 \sinh(z/2)} = w_\emptyset(e^z).$$
We extend $\Lambda^*$ to an algebra where $Q_1 \neq 0$. Observe that a non-increasing sequence $(\lambda_1, \lambda_2, \ldots)$ of integers corresponds to a partition precisely if it converges to 0. If, however, it converges to an integer $n$, Eqs. (2) and (3) still define $Q_k(\lambda)$. In fact, in this case

$$Q_k(\lambda) = (e^n \partial) Q_k(\lambda - n)$$

by [13, Proposition 1] where $\partial Q_0 = 0$, $\partial Q_k = Q_{k-1}$ for $k \geq 1$, and $\lambda - n = (\lambda_1 - n, \lambda_2 - n, \ldots)$ corresponds to a partition (i.e. converges to 0). In particular, $Q_1(\lambda) = n$ equals the number the sequence $\lambda$ converges to. We now define the Bloch–Okounkov ring $\mathcal{R}$ to be $\Lambda^*\left[Q_1\right]$, considered as a subspace of all functions from non-increasing eventually constant sequences of integers to $\mathbb{Q}$. It is convenient to work with $\mathcal{R}$ instead of $\Lambda^*$ to define the differential operators $\Delta_1$ and more generally $\Delta_\lambda$ later. Both on $\Lambda^*$ and $\mathcal{R}$, we define a weight grading by assigning to $Q_i$ weight $i$.

Denote the projection map by

$$\text{pr} : \mathcal{R} \to \Lambda^*. \quad (4)$$

We extend $\langle \cdot \rangle_q$ to $\mathcal{R}$.

The operator $E = \sum_{m=0}^{\infty} Q_m \frac{\partial}{\partial Q_{m+1}}$ on $\mathcal{R}$ multiplies an element of $\mathcal{R}$ by its weight. Moreover, we consider the differential operators

$$\partial = \sum_{m=0}^{\infty} Q_m \frac{\partial}{\partial Q_{m+1}} \quad \text{and} \quad \mathcal{D} = \sum_{k, \ell \geq 0} \binom{k + \ell}{k} Q_{k+\ell} \frac{\partial^2}{\partial Q_{k+1} \partial Q_{\ell+1}}.$$

Let $\Delta = \frac{1}{2}(\mathcal{D} - \partial^2)$, i.e.

$$2\Delta = \sum_{k, \ell \geq 0} \left( \binom{k + \ell}{k} Q_{k+\ell} - Q_k Q_\ell \right) \frac{\partial^2}{\partial Q_{k+1} \partial Q_{\ell+1}} - \sum_{k \geq 0} Q_k \frac{\partial}{\partial Q_{k+2}}.$$

In the following (antisymmetric) table, the entry in the row of operator $A$ and column of operator $B$ denotes the commutator $[A, B]$, for proofs see [13, Lemma 3].

|   | \Delta | $\partial$ | $E$ | $Q_1$ | $Q_2$ |
|---|---|---|---|---|---|
| $\Delta$ | 0 | 0 | 2$\Delta$ | 0 | $E - Q_1 \partial - \frac{1}{2}$ |
| $\partial$ | 0 | 0 | $\partial$ | 1 | $Q_1$ |
| $E$ | $-2\Delta$ | $-\partial$ | 0 | $Q_1$ | $2Q_2$ |
| $Q_1$ | 0 | $-1$ | $-Q_1$ | 0 | 0 |
| $Q_2$ | $-E + Q_1 \partial + \frac{1}{2}$ | $-Q_1$ | $-2Q_2$ | 0 | 0 |

**Definition 1** A triple $(X, Y, H)$ of operators is called an $sl_2$-triple if

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [Y, X] = H.$$
Let \( \hat{Q}_2 := Q_2 - \frac{1}{2} Q_1^2 \) and \( \hat{E} := E - Q_1 \partial - \frac{1}{2} \). The following result follows by a direct computation using the above table:

**Proposition 2** The operators \((\hat{Q}_2, \Delta, \hat{E})\) form an \(\mathfrak{sl}_2\)-triple. \(\square\)

For later reference, we compute \([\Delta, Q^n_2]\). This could be done inductively by noting that \([\Delta, Q^n_2] = Q^n_2 - Q^{n-1}_2 \partial + \Delta Q^n_2\) and using the commutation relations in the above table. The proof below is a direct computation from the definition of \(\Delta\).

**Lemma 1** For all \(n \in \mathbb{N}\), the following relation holds

\[
[\Delta, Q^n_2] = -\frac{n(n-1)}{2} Q^2_1 Q^{n-2}_2 - n Q_1 Q^{n-1}_2 \partial + n Q^{n-1}_2 (E + n - \frac{3}{2}).
\]

**Proof** Let \(f \in \mathbb{Q}[Q_1, Q_2], g \in \mathcal{R}\), and \(n \in \mathbb{N}\). Then

\[
\Delta(fg) = \Delta(f)g + \frac{\partial f}{\partial Q_2}(Eg - Q_1 \partial g) + f \Delta(g),
\]

\[
\Delta(Q^n_2) = n(n - \frac{3}{2}) Q^{n-1}_2 - \frac{n(n-1)}{2} Q^{n-2}_2 Q^2_1.
\]

By (5) and (6), we find

\[
\Delta(Q^n_2 g) = (n(n - \frac{3}{2}) Q^{n-1}_2 - \frac{n(n-1)}{2} Q^2_1 Q^{n-2}_2) g \\
+ n Q^{n-1}_2 (Eg - Q_1 \partial g) + Q^2_1 \Delta(g).
\]

\(\square\)

### 3 An \(\mathfrak{sl}_2\)-equivariant mapping

The space of quasimodular forms for \(\text{SL}_2(\mathbb{Z})\) is given by \(\tilde{M} = \mathbb{Q}[P, Q, R]\), where \(P\), \(Q\), and \(R\) are the Eisenstein series of weight 2, 4, and 6, respectively (in Ramanujan’s notation). We let \(\tilde{M}_k^{(\leq p)}\) be the space of quasimodular forms of weight \(k\) and depth \(\leq p\) (the depth of a quasimodular form written as a polynomial in \(P\), \(Q\), and \(R\) is the degree of this polynomial in \(P\)). See [12, Section 5.3] or [13, Section 2] for an introduction into quasimodular forms.

The space of quasimodular forms is closed under differentiation, more precisely the operators \(D = q \frac{d}{dq}, \partial = 12 \frac{d}{dp}, \) and the weight operator \(W\) given by \(Wf = kf\) for \(f \in \tilde{M}_k\) preserve \(\tilde{M}\) and form an \(\mathfrak{sl}_2\)-triple. In order to compute the action of \(D\) in terms of the generators \(P\), \(Q\), and \(R\), one uses the Ramanujan identities

\[
D(P) = \frac{P^2 - Q}{12}, \quad D(Q) = \frac{PQ - R}{3}, \quad D(R) = \frac{PR - Q^2}{2}.
\]

In the context of the Bloch–Okounkov theorem, it is more natural to work with \(\hat{D} := D - \frac{P}{24}\), as for all \(f \in \Lambda^*\) one has \(\langle Q_2 f \rangle_q = \hat{D}(f)_q\). Moreover, \(\hat{D}\) has the property that it increases the depth of a quasimodular form by 1, in contrast to \(D\) for which \(D(1) = 0\) does not have depth 1:
Lemma 2 Let \( f \in \tilde{M} \) be of depth \( r \). Then \( \hat{D} f \) is of depth \( r + 1 \).

Proof Consider a monomial \( P^a Q^b R^c \) with \( a, b, c \in \mathbb{Z}_{\geq 0} \). By the Ramanujan identities, we find
\[
D(P^a Q^b R^c) = \left( \frac{a}{12} + \frac{b}{3} + \frac{c}{2} \right) P^{a+1} Q^b R^c + O(P^a),
\]
where \( O(P^a) \) denotes a quasimodular form of depth at most \( a \). The lemma follows by noting that \( \frac{a}{12} + \frac{b}{3} + \frac{c}{2} - \frac{1}{24} \) is non-zero for \( a, b, c \in \mathbb{Z} \). \( \square \)

Moreover, letting \( \hat{W} = W - \frac{1}{2} \), the triple \((\hat{D}, \hat{d}, \hat{W})\) forms an \( \mathfrak{s}l_2 \)-triple as well. With respect to these operators, the \( q \)-bracket becomes \( \mathfrak{s}l_2 \)-equivariant. The following proposition is a detailed version of Proposition 1:

Proposition 3 (The \( \mathfrak{s}l_2 \)-equivariant Bloch–Okounkov theorem) The mapping \( \langle \cdot \rangle_q : \mathcal{R} \to \tilde{M} \) is \( \mathfrak{s}l_2 \)-equivariant with respect to the \( \mathfrak{s}l_2 \)-triple \((\hat{Q}_2, \Delta, \hat{E})\) on \( \mathcal{R} \) and the \( \mathfrak{s}l_2 \)-triple \((\hat{D}, \hat{d}, \hat{W})\) on \( \tilde{M} \), i.e. for all \( f \in \mathcal{R} \), one has
\[
\hat{D}(f)_q = \langle \hat{Q}_2 f \rangle_q, \quad \hat{d}(f)_q = \langle \Delta f \rangle_q, \quad \hat{W}(f)_q = \langle \hat{E} f \rangle_q.
\]

Proof This follows directly from [13, Equation (37)] and the fact that for all \( f \in \mathcal{R} \) one has \( \langle Q_1 f \rangle_q = 0 \). \( \square \)

4 Describing the space of shifted symmetric harmonic polynomials

In this section, we study the kernel of \( \Delta \). As \( [\Delta, Q_1] = 0 \), we restrict ourselves without loss of generality to \( \Lambda^* \). Note, however, that \( \Delta \) does not act on \( \Lambda^* \) as, for example, \( \Delta(Q_3) = -\frac{1}{2} Q_1 \). However, \( \text{pr} \Delta \) does act on \( \Lambda^* \).

Definition 2 Let
\[
\mathcal{H} = \{ f \in \Lambda^* \mid \Delta f \in Q_1 \mathcal{R} \} = \ker \text{pr} \Delta,
\]
be the space of shifted symmetric harmonic polynomials.

Proposition 4 If \( f \in Q_2 \Lambda^* \) is non-zero, then \( f \notin \mathcal{H} \).

Proof Write \( f = Q_2^n f' \) with \( f' \in \Lambda^* \) and \( f' \notin Q_2 \Lambda^* \). Then
\[
\text{pr} \Delta(f) = Q_2^{n-1} (n(n + k - \frac{3}{2}) f' + Q_2 \text{pr} \Delta f')
\]
by Lemma 1. As \( f' \) is not divisible by \( Q_2 \), it follows that \( \text{pr} \Delta(f) = 0 \) precisely if \( f' = 0 \). \( \square \)

Proposition 5 For all \( n \in \mathbb{Z} \), one has
\[
\Lambda^*_n = \mathcal{H}_n \oplus Q_2 \Lambda^*_{n-2}.
\]
Proof For uniqueness, suppose \( f = Q_2 g + h \) and \( f = Q_2 g' + h' \) with \( g, g' \in \Lambda_{n-2}^* \) and \( h, h' \in \mathcal{H}_n \). Then, \( Q_2(g - g') = h' - h \in \mathcal{H} \). By Proposition 4 we find \( g = g' \) and hence \( h = h' \).

Now, define the linear map \( T : \Lambda_n^* \to \Lambda_n^* \) by \( f \mapsto \text{pr}\Delta(Q_2 f) \). By Proposition 4 we find that \( T \) is injective, which by finite dimensionality of \( \Lambda_n^* \) implies that \( T \) is surjective. Hence, given \( f \in \Lambda_n^* \) let \( g \in \Lambda_{n-2}^* \) be such that \( T(g) = \text{pr}\Delta(f) \in \Lambda_{n-2}^* \). Let \( h = f - Q_2 g \). As \( f = Q_2 g + h \), it suffices to show that \( h \in \mathcal{H} \). That holds true because \( \text{pr}\Delta(h) = \text{pr}\Delta(f) - \text{pr}\Delta(Q_2 g) = 0 \).

Proposition 5 implies Theorem 2 and the following corollary. Denote by \( p(n) \) the number of partitions of \( n \).

**Corollary 1** The dimension of \( \mathcal{H}_n \) equals the number of partitions of \( n \) in parts of size at least 3, i.e.

\[
\dim \mathcal{H}_n = p(n) - p(n - 1) - p(n - 2) + p(n - 3).
\]

**Proof** Observe that \( \dim \Lambda_n^* \) equals the number of partitions of \( n \) in parts of size at least 2. Hence, \( \dim \Lambda_n^* = p(n) - p(n - 1) \) and the Corollary follows from Proposition 5.

**Proof of Theorem 1** If \( \langle f \rangle_q \) is modular, then \( \langle \Delta f \rangle_q = \partial \langle f \rangle_q = 0 \). Write \( f = \sum_{r=0}^{n'} Q_2^r h_r \) as in Theorem 2 with \( n' = \lfloor \frac{n}{2} \rfloor \). Then by Lemma 1 it follows that

\[
\text{pr}\Delta f = \sum_{r=0}^{n'} r(n - r - \frac{3}{2}) Q_2^{r-1} h_r.
\]

Hence,

\[
\sum_{r=1}^{n'} r(n - r - \frac{3}{2}) \hat{D}^{r-1} \langle h_r \rangle_q = 0. \tag{7}
\]

As \( \langle h_r \rangle_q \) is modular, either it is equal to 0 or it has depth 0. Suppose the maximum \( m \) of all \( r \geq 1 \) such that \( \langle h_r \rangle_q \) is non-zero exists. Then, by Lemma 2 it follows that the left-hand side of (7) has depth \( m - 1 \), in particular is not equal to 0. So, \( h_1, \ldots, h_{m'} \) \( \ker \langle \cdot \rangle_q \). Note that \( f \in \ker \langle \cdot \rangle_q \) implies that \( Q_2 f \in \ker \langle \cdot \rangle_q \). Therefore, \( k := \sum_{r=1}^{n'} Q_2^r h_r \in \ker \langle \cdot \rangle_q \) and \( f = h + k \) with \( h = h_0 \) harmonic.

The converse follows directly as \( \partial \langle h + k \rangle_q = \partial \langle h \rangle_q = \langle \Delta h \rangle_q = 0 \).

**Remark 2** A description of the kernel of \( \langle \cdot \rangle_q \) is not known.

Another corollary of Proposition 5 is the notion of **depth** of shifted symmetric polynomials which corresponds to the depth of quasimodular forms:

**Definition 3** The space \( \Lambda_k^{*(\leq p)} \) of shifted symmetric polynomials of depth \( \leq p \) is the space of \( f \in \Lambda_k^* \) such that one can write

\[
f = \sum_{r=0}^{p} Q_2^r h_r,
\]

with \( h_r \in \mathcal{H}_{k-2r} \).
Theorem 4 If \( f \in \Lambda^{*, \leq p}_k \), then \( \langle f \rangle_q \in \widetilde{M}^{\leq p}_k \).

**Proof** Expanding \( f \) as in Definition 3 we find

\[
\langle f \rangle_q = \sum_{k=0}^p \langle Q^k h_k \rangle_q = \sum_{k=0}^p \hat{D}^k \langle h_k \rangle_q.
\]

By Lemma 2, we find that the depth of \( \langle f \rangle_q \) is at most \( p \). \( \square \)

Next, we set up notation to determine the basis of \( \mathcal{H} \) given by Theorem 3.

Let \( \tilde{R} = R[Q^{-1/2}] \) and \( \tilde{\Lambda} = \Lambda^*[Q^{-1/2}] \) be the formal polynomial algebras graded by assigning to \( Q_k \) weight \( k \) (note that the weights are—possibly negative—integers). Extend \( \Delta \) to \( \tilde{\Lambda} \) and observe that \( \Delta(\tilde{\Lambda}) \subset \tilde{\Lambda} \). Also extend \( \mathcal{H} \) by setting

\[
\tilde{\mathcal{H}} = \{ f \in \tilde{\Lambda} | \Delta f \in Q_1 \tilde{R} \} = \ker \text{pr}\Delta|_{\tilde{\Lambda}}.
\]

**Definition 4** Define the partition-Kelvin transform \( K : \tilde{\Lambda}_n \to \tilde{\Lambda}_{3-n} \) by

\[
K(f) = Q^{3/2-n}_2 f.
\]

Note that \( K \) is an involution. Moreover, \( f \) is harmonic if and only if \( K(f) \) is harmonic, which follows directly from the computation

\[
\Delta K(f) = Q^{3/2-n}_2 \Delta f - \left( \frac{3}{2} - n \right) Q_1 Q^{1/2}_2 \partial f - \frac{1}{2} \left( \frac{3}{2} - n \right) Q_1 Q^{1/2}_2 Q^{-1/2}_2 f.
\]

**Example 1** As \( K(1) = Q^{3/2}_2 \), it follows that \( Q^{3/2}_2 \in \tilde{\mathcal{H}} \).

**Definition 5** Given \( \underline{i} \in \mathbb{Z}^n_{\geq 0} \), let

\[
|\underline{i}| = i_1 + i_2 + \ldots + i_n, \quad \partial_{\underline{i}} = \frac{\partial^n}{\partial Q_{i_1+1} \partial Q_{i_2+1} \cdots \partial Q_{i_n+1}}.
\]

Define the \( n \)th order differential operators \( \mathcal{D}_n \) on \( \tilde{R} \) by

\[
\mathcal{D}_n = \sum_{\underline{i} \in \mathbb{Z}^n_{\geq 0}} \left( |\underline{i}| \right) Q_{|\underline{i}|} \partial_{\underline{i}},
\]

where the coefficient is a multinomial coefficient.

This definition generalises the operators \( \partial \) and \( \mathcal{D} \) to higher weights: \( \mathcal{D}_1 = \partial, \mathcal{D}_2 = \mathcal{D}, \) and \( \mathcal{D}_n \) reduces the weight by \( n \).
Lemma 3 The operators \( \{D_n\}_{n \in \mathbb{N}} \) commute pairwise.

Proof Set \( I = |i| \) and \( J = |j| \). Let \( \hat{a}^k = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \). Then

\[
\left[ \left( I_{i_1}, i_2, \ldots, i_n \right) Q_I \partial_{i_1}, \left( J_{j_1}, j_2, \ldots, j_m \right) Q_J \partial_{j_1} \right] = \sum_{k=1}^n \delta_{i_k, J-1} J \left( i_1, i_2, \ldots, i_k, i_{k+1}, i_{k+2}, \ldots, i_n, j_1, j_2, \ldots, j_m \right) Q_I \partial_{i_k} \partial_{j_1} +
\sum_{l=1}^m \delta_{j_l, I-1} I \left( i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_{l-1}, \hat{j}_l, \ldots, j_m \right) Q_J \partial_{j_l} \partial_{j_1}.
\]

Hence, \( [D_n, D_m] = 0 \).

It does not hold true that \( [D_n, Q_1] = 0 \) for all \( n \in \mathbb{N} \). Therefore, we introduce the following operators:

Definition 6 Let

\[
\Delta_n = \sum_{i=0}^n (-1)^i \binom{n}{i} \mathcal{D}_{n-i} \partial^i.
\]

For \( \lambda \in \mathcal{P} \) let

\[
\Delta_{\lambda} = \left( \frac{1}{\lambda_1, \ldots, \lambda_{\ell(\lambda)}} \right) \prod_{i=1}^\infty \Delta_{\lambda_i}.
\]

(Note that \( \Delta_0 = \mathcal{D}_0 = 1 \), so this is in fact a finite product.)
Remark 3  By Möbius inversion

\[ \mathcal{D}_n = \sum_{i=0}^{n} \binom{n}{i} \Delta_{n-i} \partial^i. \]

The first three operators are given by

\[ \Delta_0 = 1, \quad \Delta_1 = 0, \quad \Delta_2 = \mathcal{D} - \partial^2 = 2\Delta. \]

Proposition 6  The operators \( \Delta_\lambda \) satisfy the following properties: for all partitions \( \lambda, \lambda' \)

(a) the order of \( \Delta_{|\lambda|} \) is \( |\lambda| \);
(b) \( [\Delta_\lambda, \Delta_{\lambda'}] = 0 \);
(c) \( [\Delta_\lambda, Q_1] = 0 \).

Proof  Property (a) follows by construction and (b) is a direct consequence of Lemma 3. For property (c), let \( f \in \tilde{\Lambda} \) be given. Then

\[
\Delta_n(Q_1 f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathcal{D}_{n-i} \partial^i (Q_1 f)
= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left( (n-i) \mathcal{D}_{n-i-1} \partial^i f + Q_1 \mathcal{D}_{n-i} \partial^i f + i \mathcal{D}_{n-i} \partial^{i-1} f \right)
= Q_1 \Delta_n(f) + \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left( (n-i) \mathcal{D}_{n-i-1} \partial^i f + i \mathcal{D}_{n-i} \partial^{i-1} f \right).
\]

Observe that by the identity

\[ (n-i) \binom{n}{i} = (i+1)(n+1), \]

the sum in the last line is a telescoping sum, equal to zero. Hence \( \Delta_n(Q_1 f) = Q_1 \Delta_n(f) \) as desired.

In particular, the above proposition yields \( [\Delta_\lambda, \Delta] = 0 \) and \( [\Delta_\lambda, \text{pr}] = 0 \).

Denote by \((x)_n\) the falling factorial power \((x)_n = \prod_{i=0}^{n-1} (x - i)\) and for \( \lambda \in \mathcal{P}_n \) define \( Q_\lambda = \prod_{i=1}^{\infty} Q_{\lambda_i} \). Let

\[ h_\lambda = \text{pr} K \Delta_\lambda K(1). \]

Observe that \( h_\lambda \) is harmonic, as \( \text{pr} \Delta \) commutes with \( \text{pr} \) and \( \Delta_\lambda \).
Proposition 7 For all $\lambda \in \mathcal{P}_n$ there exists an $f \in \Lambda_{n-2}^*$ such that

$$h_\lambda = \left(\frac{3}{2}\right)_n n! Q_\lambda + Q_2 f.$$

Proof Note that the left-hand side is an element of $\Lambda^*$ of which the monomials divisible by $Q_2^2$ correspond precisely to terms in $\Delta_\lambda$ involving precisely $n - i$ derivatives of $K(1)$ to $Q_2$. Hence, as $\Delta_\lambda$ has order $n$ all terms not divisible by $Q_2$ correspond to terms in $\Delta_\lambda$ which equal $\frac{\partial^n}{\partial Q_2^n}$ up to a coefficient. There is only one such term in $\Delta_\lambda$ with coefficient $\binom{|\lambda|}{\lambda_1, \ldots, \lambda_r} \lambda_1! \ldots \lambda_r! Q_\lambda$. $\square$

For $f \in \mathcal{R}$, we let $f^\vee$ be the operator where every occurrence of $Q_i$ in $f$ is replaced by $\Delta_i$. We get the following unusual identity:

**Corollary 2** If $h \in \mathcal{H}_n$, then

$$h = \text{pr} K h^\vee K(1) n! \left(\frac{3}{2}\right)_n. \quad (9)$$

Proof By Proposition 7, we know that the statement holds true up to adding $Q_2 f$ on the right-hand side for some $f \in \Lambda_{n-2}^*$. However, as both sides of (9) are harmonic and the shifted symmetric polynomial $Q_2 f$ is harmonic precisely if $f = 0$ by Proposition 4, it follows that $f = 0$ and (9) holds true. $\square$

Proof of Theorem 3 Let $\mathcal{B}_n = \{ h_\lambda \mid \lambda \in \mathcal{P}_n \text{ all parts are } \geq 3 \}$. First of all, observe that by Corollary 1 the number of elements in $\mathcal{B}_n$ is precisely the dimension of $\mathcal{H}_n$. Moreover, the weight of an element in $\mathcal{B}_n$ equals $|\lambda| = n$. By Proposition 7 it follows that the elements of $\mathcal{B}_n$ are linearly independent harmonic shifted symmetric polynomials. $\square$

Acknowledgements I would like to thank Gunther Cornelissen and Don Zagier for helpful discussions.

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Appendix: Tables of shifted symmetric harmonic polynomials up to weight 10

We list all harmonic polynomials $h_\lambda$ of even weight at most 10. The corresponding $q$-brackets $\langle h_\lambda \rangle_q$ are computed by the algorithm prescribed by Zagier [13] using SageMath [11].
When is the Bloch–Okounkov $q$-bracket modular?

| $\lambda$ | $h_\lambda$ | $(h_\lambda)_q$ |
|-----------|-------------|----------------|
| ()        | 1           | 1              |
| (4)       | $\frac{27}{4} \left( Q_3^3 + 2Q_4 \right)$ | $\frac{9}{4} 9Q$ |
| (6)       | $\frac{225}{4} \left( 63Q_6 + 9Q_2Q_4 + Q_2^3 \right)$ | $-\frac{55}{5} R$ |
| (3,3)     | $\frac{225}{4} \left( 63Q_3^2 - 108Q_2Q_4 + 2Q_2^3 \right)$ | $\frac{115}{2} R$ |
| (8)       | $\frac{19845}{16} \left( 3960Q_8 + 360Q_2Q_6 + 20Q_2^2Q_4 + Q_2^4 \right)$ | $\frac{19173}{2} Q^2$ |
| (5,3)     | $\frac{19845}{2} \left( 495Q_3Q_5 + 45Q_2Q_4^2 - 1350Q_2Q_6 - 50Q_2^2Q_4 + 2Q_2^4 \right)$ | $-\frac{2415}{2} Q^2$ |
| (4,4)     | $\frac{297675}{8} \left( 132Q_4^2 + 24Q_2Q_6^2 - 440Q_2Q_6 - 28Q_2^2Q_4 + Q_2^4 \right)$ | $-\frac{38241}{4096} Q^2$ |
| (10)      | $\frac{382725}{8} \left( 450450Q_{10} + 30030Q_2Q_8 + 1155Q_3Q_6 + 35Q_3^2Q_4 + Q_3^3 \right)$ | $-\frac{2053485}{4096} QR$ |
| (7,3)     | $\frac{1913625}{8} \left( 90090Q_3Q_7 + 6006Q_2Q_3Q_5 - 33636Q_2Q_8 + 231Q_2Q_8^2 + 12936Q_2^3Q_6 - 112Q_2^2Q_4 + 10Q_2^4 \right)$ | $\frac{11975985}{4096} QR$ |
| (6,4)     | $\frac{13395375}{8} \left( 12870Q_4Q_6 + 1716Q_2Q_3Q_5 + 858Q_2Q_4^2 - 96096Q_2Q_8 + 132Q_2^2Q_3^2 - 6501Q_2Q_6 - 89Q_2^2Q_4 + 5Q_2^4 \right)$ | $\frac{21255885}{4096} QR$ |
| (5,5)     | $\frac{8037225}{4} \left( 10725Q_5^2 + 1430Q_2Q_3Q_5 + 1430Q_2Q_4^2 - 10010Q_2Q_8 + 165Q_2^2Q_3^2 - 7700Q_2Q_6 - 120Q_2^2Q_4 + 6Q_2^4 \right)$ | $\frac{7758395}{1024} QR$ |
| (4,3,3)   | $\frac{13395375}{8} \left( 12870Q_2^2Q_4 - 34320Q_2Q_3Q_5 + 10296Q_2Q_4^2 + 363Q_2^2Q_3^2 + 55440Q_2^3Q_6 - 376Q_2^2Q_4 + 10Q_2^4 \right)$ | $-\frac{16583805}{4096} QR$ |

In case $|\lambda|$ is odd, the harmonic polynomials $h_\lambda$ up to weight 9 are given in the following table. The $q$-bracket of odd degree (harmonic) polynomials is zero, hence trivially modular.

| $\lambda$ | $h_\lambda$ |
|-----------|-------------|
| (3)       | $-\frac{9}{4} Q_3$ |
| (5)       | $-\frac{135}{16} (5Q_5 + Q_2Q_3)$ |
| (7)       | $-\frac{14175}{16} \left( 126Q_7 + 14Q_2Q_5 + Q_2^2Q_3 \right)$ |
| (4, 3)    | $-\frac{99225}{16} \left( 18Q_3Q_4 - 40Q_2Q_5 + Q_2^2Q_3 \right)$ |
| (9)       | $-\frac{297675}{8} \left( 7722Q_9 + 594Q_2Q_7 + 27Q_2^2Q_5 + Q_2^3Q_3 \right)$ |
| (6, 3)    | $-\frac{892025}{4} \left( 1287Q_3Q_6 + 99Q_2Q_3Q_4 - 4158Q_2Q_7 - 162Q_2^2Q_5 + 5Q_2^3Q_3 \right)$ |
| (5, 4)    | $-\frac{8037225}{8} \left( 286Q_4Q_5 + 66Q_2Q_3Q_4 - 1540Q_2Q_7 - 117Q_2^2Q_5 + 3Q_2^3Q_3 \right)$ |
| (3, 3, 3) | $-\frac{892025}{4} \left( 1287Q_3^3 - 3564Q_2Q_3Q_4 + 3240Q_2^2Q_5 + 10Q_2^3Q_3 \right)$ |
References

1. Axler, S., Bourdon, P., Wade, R.: Harmonic Function Theory. Graduate Texts in Mathematics, vol. 137, 2nd edn. Springer, New York (2011)
2. Bloch, S., Okounkov, A.: The character of the infinite wedge representation. Adv. Math. 149(1), 1–60 (2000)
3. Chen, D., Möller, M., Zagier, D.: Quasimodularity and large genus limits of Siegel–Veech constants. J. Am. Math. Soc. 31(4), 1059–1163 (2018)
4. Cohen, H.: Sums involving the values at negative integers of $L$-functions of quadratic characters. Math. Ann. 217(3), 271–285 (1975)
5. Dijkgraaf, R.: Mirror symmetry and elliptic curves. In: Dijkgraaf, R., Faber, C., van der Geer, G. (eds.) The Moduli Space of Curves (Texel Island, 1994), volume 129 of Progress-Mathematics, pp. 149–163. Birkhäuser Boston (1995)
6. Eskin, A., Okounkov, A.: Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials. Invent. Math. 145(1), 59–103 (2001)
7. Kaneko, M., Zagier, D.: A generalized Jacobi theta function and quasimodular forms. In: Dijkgraaf, R., Faber, C., van der Geer, G. (eds.) The Moduli Space of Curves (Texel Island, 1994), volume 129 of Progress-Mathematics, pp. 165–172. Birkhäuser Boston, Boston (1995)
8. Okounkov, A., Olshanski, G.: Shifted Schur functions. Algebra i Analiz 9(2), 73–146 (1997)
9. Rankin, R.A.: The construction of automorphic forms from the derivatives of a given form. J. Indian Math. Soc. 20, 103–116 (1956)
10. Schoeneberg, B.: Das verhalten von mehrfachen thetareihen bei modulsubstitutionen. Math. Ann. 116(1), 511–523 (1939)
11. The Sage Developers.: SageMath, the Sage Mathematics Software System (Version 8.0) (2017). http://www.sagemath.org
12. Zagier, D.: Elliptic modular forms and their applications. In: Ranestad, K. (ed.) The 1-2-3 of Modular Forms, Universitext, pp. 1–103. Springer, Berlin (2008)
13. Zagier, D.: Partitions, quasimodular forms, and the Bloch–Okounkov theorem. Ramanujan J. 41(1–3), 345–368 (2016)

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