JORDAN GROUPS, CONIC BUNDLES AND ABELIAN VARIETIES

TATIANA BANDMAN AND YURI G. ZARHIN

Abstract. A group $G$ is called Jordan if there is a positive integer $J = J_G$ such that every finite subgroup $B$ of $G$ contains a commutative subgroup $A \subset B$ such that $A$ is normal in $B$ and the index $[B : A] \leq J$ (V.L. Popov). In this paper we deal with Jordaness properties of the groups $\operatorname{Bir}(X)$ of birational automorphisms of irreducible smooth projective varieties $X$ over an algebraically closed field of characteristic zero. It is known (Yu. Prokhorov - C. Shramov) that $\operatorname{Bir}(X)$ is Jordan if $X$ is non-uniruled. On the other hand, the second named author proved that $\operatorname{Bir}(X)$ is not Jordan if $X$ is birational to a product of the projective line $\mathbb{P}^1$ and a positive-dimensional abelian variety.

We prove that $\operatorname{Bir}(X)$ is Jordan if (uniruled) $X$ is a conic bundle over a non-uniruled variety $Y$ but is not birational to $Y \times \mathbb{P}^1$. (Such a conic bundle exists if and only if $\dim(Y) \geq 2$.) When $Y$ is an abelian surface, this Jordaness property result gives an answer to a question of Prokhorov and Shramov.

1. Introduction

In this paper we deal with the groups of birational and biregular automorphisms of algebraic varieties in characteristic zero.

If $X$ is an irreducible algebraic variety over a field $K$ of characteristic zero then we write $\mathcal{O}_X$ for the structure sheaf of $X$, $\operatorname{Aut}(X) = \operatorname{Aut}_K(X)$ (resp. $\operatorname{Bir}(X) = \operatorname{Bir}_K(X)$) for the group of its biregular (resp. birational) automorphisms and $K(X)$ for the field of rational functions on $X$. We have

$$\operatorname{Aut}(X) \subset \operatorname{Bir}(X) = \operatorname{Aut}_K(K(X))$$

where $\operatorname{Aut}_K(K(X))$ is the group of $K$-linear field automorphisms of $K(X)$. We write $\operatorname{id}_X$ for the identity automorphism of $X$, which is the identity element of the groups $\operatorname{Aut}(X)$ and $\operatorname{Bir}(X)$. If $n$ is a positive integer then we write $\mathbb{P}_K^n$ (or just $\mathbb{P}^n$ when it does not cause a confusion) for the $n$-dimensional projective space over $K$.

If $X$ is smooth projective then we write $q(X)$ for its irregularity. For example, if $X$ is an abelian variety then $q(X) = \dim(X)$.

In what follows we write $k$ for an algebraically closed field of characteristic zero. We write $\cong$ and $\sim$ for an isomorphism and birational isomorphism of algebraic varieties respectively. If $A$ is a finite commutative group then we call its rank the smallest
possible number of its generators and denote it by \( \text{rk}(A) \). If \( B \) is a finite group then we write \( |B| \) for its order.

1.1. **Jordan groups.** Recall (Popov [21, 22]) the following definitions that were motivated by the classical theorem of Jordan about finite subgroups of the complex matrix group \( \text{GL}(n, \mathbb{C}) \) [3, §36].

**Definition 1.1.** Let \( G \) be a group.
- \( G \) is called *Jordan* if there is a positive integer \( J = J_G \) such that every finite subgroup \( B \) of \( G \) contains a commutative subgroup \( A \subset B \) such that \( A \) is normal in \( B \) and the index \( [B : A] \leq J \).
- We say that \( G \) has *finite subgroups of bounded rank* if there is a positive integer \( m = m_G \) such that any finite abelian subgroup \( A \) of \( G \) can be generated by at most \( m \) elements [20, 23].
- We call a Jordan group \( G \) *strongly Jordan* if there is a positive integer \( m = m_G \) such that any finite abelian subgroup \( A \) of \( G \) can be generated by, at most, \( m \) elements [20]. In other words, \( G \) is strongly Jordan if it is Jordan and has finite abelian subgroups of bounded rank.
- We say that \( G \) is *bounded* [22, 23] if there is a positive integer \( C = C_G \) such that the order of every finite subgroup of \( G \) does not exceed \( C \). (A bounded group is Jordan and even strongly Jordan.)

One may introduce similar definitions for families of groups [21, 23].

**Definition 1.2.** Let \( \mathcal{G} \) be a family of groups.
- We say that \( \mathcal{G} \) is *uniformly Jordan* (resp. *uniformly strongly Jordan*) if there is a positive integer \( J = J_\mathcal{G} \) such that each \( G \in \mathcal{G} \) enjoys Jordan property (resp. strong Jordan property) with \( J_G = J \) (resp. with \( J_G = J \) and \( m_G = M \)).
- We say that \( \mathcal{G} \) is *uniformly bounded* if there is a positive integer \( C = C_\mathcal{G} \) such that the order of every finite subgroup of every \( G \) from \( \mathcal{G} \) does not exceed \( C \). (See [23, Remark 2.9 on p. 2058].)

**Remark 1.1.** In the terminology of [23, p. 2067], a family \( \mathcal{G} \) is uniformly strongly Jordan if and only if it is uniformly Jordan and has finite subgroups of uniformly bounded rank.

1.2. **Jordan properties of \( \text{Bir}(X) \) and \( \text{Aut}(X) \).** Let \( X \) be an irreducible quasiprojective variety over \( k \). There is the natural group embedding

\[
\text{Aut}(X) \hookrightarrow \text{Bir}(X)
\]

that allows us to view \( \text{Aut}(X) \) as a subgroup of \( \text{Bir}(X) \). In particular, the Jordan property (resp. the strong Jordan property) for \( \text{Bir}(X) \) implies the same property for \( \text{Aut}(X) \). However, the converse is not necessarily true. More precisely:
- It is known that \( \text{Aut}(X) \) is Jordan if \( \dim(X) \leq 2 \) [21, 29, 1].
• It is also known (Popov [21]) that Bir(X) is Jordan if \( \dim(X) \leq 2 \) and \( X \) is not birational to a product \( E \times \mathbb{P}^1 \) of an elliptic curve \( E \) and the projective line \( \mathbb{P}^1 \). (The Jordan property of the two-dimensional Cremona group Bir(\( \mathbb{P}^2 \)) was established earlier by J.-P. Serre [27].)

• In the remaining case Bir(\( E \times \mathbb{P}^1 \)) is not Jordan. More generally, if \( A \) is an abelian variety of positive dimension over \( k \) and \( n \) is a positive integer, then Bir(\( A \times \mathbb{P}^n \)) is not Jordan [28].

Notice that Bir(\( A \)) coincides with Aut(\( A \)) and is strongly Jordan [21]. Actually, if \( A_d \) is the family of groups Bir(\( A \)) when \( A \) runs through the set of all \( d \)-dimensional abelian varieties over \( k \) then \( A_d \) is uniformly strongly Jordan [23, Corollary 2.15 on p. 2058].

• In higher dimensions, a recent result of Meng and Zhang [16] asserts that Aut(\( X \)) is Jordan if \( X \) is projective.

For groups of birational automorphisms in higher dimensions Prokhorov and Shramov [23] proved the following strong result [23, Th. 1.8].

Theorem 1.2. (1) If \( X \) is non-uniruled then Bir(\( X \)) is Jordan.

(2) If \( X \) is non-uniruled and \( q(X) = 0 \) then Bir(\( X \)) is bounded.

Remark 1.3. Prokhorov and Shramov [23, Remark 6.9 on p. 2065] noticed that if \( X \) is non-uniruled then Bir(\( X \)) has finite subgroups of bounded rank. This means that in the non-uniruled case Bir(\( X \)) is strongly Jordan.

In addition, in dimension 3 Prokhorov and Shramov proved [23] that:

• If \( q(X) = 0 \) then Bir(\( X \)) is Jordan.

• If \( X \) is rationally connected then Bir(\( X \)) is strongly Jordan. Even better, if \( X \) varies in the set of rationally connected threefolds then the corresponding family of groups Bir(\( X \)) is uniformly strongly Jordan [24, Th. 1.7 and 1.10].

Actually, they proved all these assertions in arbitrary dimension \( d \), assuming that the well-known conjecture of A. Borisov, V. Alexeev and L. Borisov (BAB conjecture [2]) about the boundedness of families of \( d \)-dimensional Fano varieties with terminal singularities is valid in dimension \( d \).

In light of their results it remains to investigate Jordaness properties of Bir(\( X \)) when \( X \) is uniruled with \( q(X) > 0 \). According to Prokhorov and Shramov [23, p. 2069], it is natural to start with a conic bundle \( X \) over an abelian surface \( A \) when \( X \) is not birational to a product \( A \times \mathbb{P}^1 \).

In this work we prove that in this case Bir(\( X \)) is Jordan. Actually, we prove the following more general statement.

Theorem 1.4. Let \( X \) be an irreducible smooth projective variety of dimension \( d \geq 3 \) over \( k \) and \( f : X \to Y \) be a surjective morphism over \( k \) from \( X \) to a \( (d-1) \)-dimensional abelian variety \( Y \) over \( k \). Let us assume that the generic fiber of \( f \) is a genus zero smooth projective irreducible curve \( X_f \) over \( k(Y) \) without \( k(Y) \)-points. Then Bir(\( X \)) is strongly Jordan.
We deduce Theorem 1.4 from the following more general statement.

**Theorem 1.5.** Let \( d \geq 3 \) be a positive integer, \( X \) and \( Y \) are smooth irreducible projective varieties over \( k \) of dimension \( d \) and \( d - 1 \) respectively. Let \( f : X \to Y \) be a surjective morphism, whose generic fiber is a genus zero smooth projective irreducible curve \( X_f \) over \( k(Y) \) without \( k(Y) \)-points. Assume that \( Y \) is non-uniruled. Then \( \text{Bir}(X) \) is strongly Jordan.

The following assertion is a variant of Theorem 1.5

**Theorem 1.6.** Let \( d \geq 3 \) be a positive integer, \( X \) and \( Y \) are smooth irreducible projective varieties over \( k \) of dimension \( d \) and \( d - 1 \) respectively. Let \( f : X \to Y \) be a surjective morphism. Suppose that there exists a nonempty open subset \( U \) of \( X \) such that for all \( y \in U(k) \) the corresponding fiber \( X_y \) of \( f \) is \( k \)-isomorphic to the projective line over \( k \) (i.e. the general fiber \( X_y \cong \mathbb{P}_k^1 \)).

Assume that \( Y \) is non-uniruled and \( f \) does not admit a rational section \( Y \dashrightarrow X \). Then \( \text{Bir}(X) \) is strongly Jordan.

The paper is organized as follows. Section 2 deals with Jordaness of groups. In Section 3 we remind basic properties of conic bundles over non-uniruled varieties. In Section 4 we describe finite subgroups of automorphisms group of a conics without rational points. We prove main results of the paper in Section 5.

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## 2. Group Theory

We will need the following useful result of Anton Klyachko [23, Lemma 2.8 on p.2057]. (See also [29, Lemma 6.2].)

**Lemma 2.1.** Let \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be families of groups such that \( \mathcal{G}_1 \) is uniformly bounded and \( \mathcal{G}_2 \) is uniformly strongly Jordan. Let \( \mathcal{G} \) be a family of groups \( G \) such that \( G \) sits in an exact sequence

\[
\{1\} \to G_1 \to G \to G_2 \to \{1\},
\]

where \( G_1 \in \mathcal{G}_1 \) and \( G_2 \in \mathcal{G}_2 \). Then \( \mathcal{G} \) is uniformly Jordan.

Actually, we will use the following slight refinement of Lemma 2.1.

**Lemma 2.2.** In the notation and assumptions of Lemma 2.1 the family \( \mathcal{G} \) is uniformly strongly Jordan.

**Proof.** The assertion follows readily from Lemma 2.1 combined with Lemma 2.7 of [23, p. 2057] about extensions of groups with uniformly bounded ranks. \( \square \)
Corollary 2.3. Let \( d \) be a positive integer. Let \( \mathcal{G}_1 \) be a uniformly bounded family of groups. Let \( \mathcal{A}_d \) be the family of groups \( \text{Bir}(A) \) where \( A \) runs through the set of all \( d \)-dimensional abelian varieties over \( k \). Let \( \mathcal{G} \) be a family of groups \( G \) such that there exists an exact sequence

\[
\{1\} \to G_1 \to G \to G_2 \to \{1\},
\]

where \( G_1 \in \mathcal{G}_1 \) and \( G_2 \in \mathcal{A}_d \). Then \( \mathcal{G} \) is uniformly Jordan.

Proof. One has only to recall that \( G_2 := \mathcal{A}_d \) is uniformly strongly Jordan [23, Corollary 2.15 on p. 2058] and apply Lemma 2.2. \( \square \)

Lemma 2.4. Let \( G \) be a strongly Jordan group and let \( H \) be a subgroup of \( G \). Suppose that there exists a positive integer \( N_H \) such that every periodic element in \( H \) has order that does not exceed \( N_H \).

Then \( H \) is bounded.

Proof. Let \( J_G \) be the Jordan index of \( H \). We know that there is a positive integer \( m_G \) such that every finite abelian subgroup in \( G \) is generated by, at most, \( m_G \) elements.

Let \( \mathcal{B} \) be a finite subgroup of \( H \). Clearly, \( \mathcal{B} \) is a subgroup of \( G \) as well. Then \( \mathcal{B} \) contains a finite abelian subgroup \( \mathcal{A} \) with index \( [\mathcal{B} : \mathcal{A}] \leq J_G \). The abelian group \( \mathcal{A} \) is generated by, at most, \( m_G \) elements, each of which has order \( \leq N_H \). This implies that \( |\mathcal{A}| \leq N_H^{m_G} \) and therefore

\[
|\mathcal{B}| \leq J_H \cdot |\mathcal{A}| \leq J_G \cdot N_H^{m_G}.
\]

Recall [20] that the matrix group \( \text{GL}(n, \mathbb{C}) \) is strongly Jordan and its every finite abelian subgroup is generated by, at most, \( n \) elements. This implies that for any field \( K \) of characteristic zero the matrix group \( \text{GL}(n, K) \) is a strongly Jordan group with Jordan index \( J_{\text{GL}(n, \mathbb{C})} \) (see [22] Sect. 1.2.2 on p. 187); in addition, its every finite abelian subgroup is generated by, at most, \( n \) elements. Combining this observation with Lemma 2.4 and the last formula of its proof, we obtain the following assertion that may be of independent interest.

Theorem 2.5. Let \( K \) be a field of characteristic zero and \( n \) a positive integer. Suppose that \( H \) is a subgroup of \( \text{GL}(n, K) \) and \( N \) is a positive integer such that every periodic element in \( H \) has order \( \leq N \). Then there exist a positive integer \( N = N(n, N) \) that depends only on \( n \) and \( N \), and such that every finite subgroup in \( H \) has order \( \leq N \). In particular, \( H \) is bounded.

3. Conic bundles

Let \( f : X \to Y \) be a surjective morphism of smooth irreducible projective varieties of positive dimension over \( k \). Since \( X \) and \( Y \) are projective, \( f \) is a projective morphism. It is well known [9, Ch. III, Sect. 10, Cor. 10.7] that there is an open Zariski dense subset \( U = U(f) \) of \( Y \) such that the restriction \( f^{-1}(U) \to U \) is smooth ([9] Ch.
III, Sect. 10, Cor. 10.7]) and flat ([13, Lect. 8, 2°], [7, Theorem 6.9.1]). Thus the generic fiber $\mathcal{X} := \mathcal{X}_f$ is a smooth projective variety over $k(Y)$ and all its irreducible components have dimension $\dim(X) - \dim(Y)$. ([9, Ch. III, Sect. 9, Corollary 9.6]), ([9, Ch. III, Sect. 10, Prop. 10.1]). In addition, if $y$ is a closed point of $U$ then the corresponding fiber $X_y$ of $f$ is a smooth projective variety over the field $k(y) = k$ and all its irreducible components have dimension $\dim(X) - \dim(Y)$.

Notice that dominant $f$ defines the field embedding

$$f^* : k(Y) \hookrightarrow k(X)$$

that is the identity map on $k$. Further we will identify $k(Y)$ with its image in $k(X)$. The field of rational functions of $\mathcal{X}_f$ coincides with $k(X)$ and the group of birational automorphisms $\text{Bir}_{k(Y)}(\mathcal{X}_f)$ coincides with (sub)group

$$(1) \quad \text{Aut}(k(X)/k(Y)) \subset \text{Aut}(k(X)/k) = \text{Bir}_k(X)$$

that consists of all automorphisms of the field $k(X)$ leaving invariant every element of $k(Y)$.

We say that $X$ is a conic bundle over $Y$ if the generic fiber $\mathcal{X} := \mathcal{X}_f$ is an absolutely irreducible genus 0 curve over $k(Y)$. (See [22, 26].) In particular, $\dim(X) - \dim(Y) = 1$ and therefore the general fiber of $f$ is a (smooth projective) curve.

**Remark 3.1.** As usual, by the general fiber of $f$ we mean the fiber $X_y$ of $f$ over a point $y$ from some nonempty open subset of $Y$. If the generic fiber is an irreducible smooth projective curve then there is an open nonempty subset $U$ of $Y$ such that for all closed points $y \in U$ the corresponding (closed) fibers $X_y$ are irreducible smooth projective curves over $k(y) = k$ as well ([8, Corollary 9.5.6, Proposition 9.7.8]). Semi-continuity Theorem ([8, Ch III, Theorem 12.8], [17, Corollary, p.47] implies that the general fiber has genus zero if and only if the same is true for the generic fiber. Thus, the condition that the generic fiber is a smooth irreducible curve of genus zero is in our setting equivalent to the same condition for the general fiber.

**Remark 3.2.** If the genus 0 curve $\mathcal{X}_f$ has a $k(Y)$-rational point then it is birational to the projective line over $k(Y)$ ([10, Th. A.4.3.1 on p. 75]). This implies that $X$ is $k$-birational to $Y \times \mathbb{P}_1$. It follows from [28] that if $Y$ is an abelian variety (of positive dimension) then $\text{Bir}(X)$ is not Jordan.

**Example 3.3.** Let us consider a smooth projective plane quadric

$$\mathcal{X}_q = \{a_1T_1^2 + a_2T_2^2 + a_3T_3^2 = 0\} \subset \mathbb{P}_2^k(Y)$$

over the field $K := k(Y)$ where all $a_i$ are nonzero elements of $k(Y)$ such that the nondegenerate ternary quadratic form

$$q(T) = a_1T_1^2 + a_2T_2^2 + a_3T_3^2$$

in $T = (T_1, T_2, T_3)$ is anisotropic over $k(Y)$, i.e., $q(T) \neq 0$ if all $T_i \in k(Y)$ and, at least, one of them is not 0 in $k(Y)$. (It follows from [12, Th. 1 on p. 155] that such a form exists if and only if $2^{\dim(Y)} \geq 3$, i.e., if and only if $\dim(Y) \geq 2$.) Clearly, $\mathcal{X}_q$
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is an absolutely irreducible smooth projective curve of genus 0 over $K$ that does not have $K$-points.

We want to construct a conic bundle with generic fiber $X_q$ without $K$-rational point. First, let us consider the field $K(X_q)$ of the rational functions on $X_q$. It is finitely generated over $K$ and has transcendence degree 1 over it. This implies that $K(X_q)$ is finitely generated over $k$ and has transcendence degree $\dim(Y)+1$ over it. Since $X_q$ is absolutely irreducible over $K$, the latter is algebraically closed in $K(X_q)$.

By Hironaka’s results, there is an irreducible smooth projective variety $X$ over $k$ of dimension $\dim(Y)+1$ with $k(X) = K(X_q)$ such that the dominant rational map $f : X \to Y$ induced by the field embedding $k(Y) = K \subset K(X_q) = k(X)$ is actually a morphism. Clearly, the generic fiber $X_f$ is a smooth projective variety, all whose irreducible components have dimension 1. Since $K$ is algebraically closed in $K(X_q)$, the latter is algebraically closed in $K(X_q) = k(X_q)$. By the very definition of the generic fiber. Since $K(X_f) = k(X) = K(X_q)$, the $K$-curves $X_f$ and $X_q$ are birational. Taking into account that both curves are smooth projective and absolutely irreducible over $K$, we conclude that $X_f$ and $X_q$ are biregularly isomorphic over $K$. This implies that $X_f$ has genus zero and has no $k(Y)$-rational points. Thus $f : X \to Y$ is the conic bundle we wanted to construct.

**Lemma 3.4.** Let $X$ and $Y$ be smooth irreducible projective varieties of positive dimension over $k$, $f : X \to Y$ be a surjective morphism, such that the general fiber $F_y = f^{-1}(y)$ is isomorphic to $\mathbb{P}^1_k$. Let us identify $k(Y)$ with its image in $k(X)$. Assume additionally that $Y$ is non-uniruled. (E.g., $Y$ is an abelian variety.)

Then every $k$-linear automorphism $\sigma$ of the field $k(X)$ leaves invariant $k(Y)$, i.e.,

$$\sigma(k(Y)) = k(Y), \ \forall \sigma \in \text{Aut}(k(X)).$$

In addition, there is exactly one birational automorphism $u_Y$ of $Y$, whose action on $k(Y)$ coincides with $\sigma$.

**Proof.** There is a birational automorphism $u_X$ of $X$ that induces $\sigma$ on $k(X)$. Let $\tilde{X} \xrightarrow{\pi} X$ be a resolution of indeterminancy of $u_X$, i.e., we consider a smooth irreducible projective $k$–variety $\tilde{X}$, and birational morphisms

$$\pi, \tilde{u}_X : \tilde{X} \to X$$

that enjoy the following properties.

- $\pi^{-1} : X \dashrightarrow \tilde{X}$ is an isomorphism outside the indeterminancy locus of $u_X$;
- the following diagram commutes:
Consider morphisms $\tilde{f} = f \circ \pi : \tilde{X} \to Y$ and $g = f \circ \tilde{u}_X : \tilde{X} \to Y$.

Let $\Sigma_1 \subset X$, $\Sigma_2 \subset X$ be the loci of indeterminancy of $\pi^{-1}$ and $\tilde{u}_X^{-1}$ respectively.

Since $\text{codim}_X(\Sigma_1) \geq 2$ and $\text{codim}_X(\Sigma_2) \geq 2$, we obtain that $\text{codim}_Y(f(\Sigma_1)) \geq 1$ and $\text{codim}_Y(f(\Sigma_2)) \geq 1$. (Recall that $\dim(X) = \dim(Y) + 1$.)

This implies that there is a nonempty open subset $U \subset Y \setminus (f(\Sigma_1) \cup f(\Sigma_2))$ such that

$$\tilde{F}_y := \tilde{f}^{-1}(y) = \pi^{-1}(F_y) \cong F_y \cong \mathbb{P}^1$$

and

$$G_y := g^{-1}(y) = u_X^{-1}(F_y) \cong F_y \cong \mathbb{P}^1$$

for all $y \in U(k)$.

Since $Y$ is non-uniruled, $g(\tilde{F}_y)$ and $\tilde{f}(G_y)$ are points for every $y \in U(k)$ (see, e.g. [14, Chapter IV, Proposition 1.3, (1.3.4), p. 183]).

It follows from Kawamata’s Lemma [11, Lemma 10.7 on pp. 314–315] applied (twice) to morphisms

$$\tilde{f}, g : \tilde{X} \to Y$$

that there exist rational maps $h_1, h_2 : Y \to Y$ such that

$$g = h_1 \circ \tilde{f}, \quad \tilde{f} = h_2 \circ g.$$

This implies that $h_1$ and $h_2$ are mutually inverse birational automorphisms of $Y$. Let us put

$$u_Y := h_1 \in \text{Bir}(Y).$$

Then $u_Y$ may be included into the commutative digram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & \tilde{u}_X \\
\downarrow & \searrow & \downarrow \\
X & \xrightarrow{u_X} & X \\
\downarrow & \downarrow & \downarrow \\
f & \quad & f \\
Y & \xrightarrow{u_Y} & Y
\end{array}$$

For the corresponding embeddings $k(Y) \hookrightarrow k(X)$ of fields of rational functions we have: $f^* \circ (u_Y)^* = \sigma \circ f^*$, thus

$$\sigma(k(Y)) = k(Y).$$
Remark 3.5. Lemma 3.4 follows from the Theorem on Maximal Rational Connected Fibrations [14, Chapter IV, Theorem 5.5, p.223]. However, our particular case is much easier, so we were tempted to provide a simple proof rather than to use the powerful theory.

The next statement follows immediately from Lemma 3.4.

Corollary 3.6. Keeping the notation and assumptions of Lemma 3.4, the map
\[ u_X \mapsto u_Y \]
gives rise to the group homomorphism \( \text{Bir}(X) \to \text{Bir}(Y) \), whose kernel is \( \text{Aut}(k(X)/k(X)) = \text{Bir}_{k(Y)}(\mathcal{X}_f) \) (see (1)) where \( \mathcal{X}_f \) is the generic fiber of \( f \). In particular, we get an exact sequence of groups
\[ \{1\} \to \text{Bir}_{k(Y)}(\mathcal{X}_f) \to \text{Bir}(X) \to \text{Bir}(Y). \]

Remark 3.7. The special case of Corollary 3.6 when \( Y \) is an abelian surface may be deduced from [25, Cor. 1.7].

Corollary 3.8. Keeping the notation and assumptions of Lemma 3.4, suppose that \( \text{Bir}(\mathcal{X}_f) \) is bounded. Then \( \text{Bir}(X) \) is strongly Jordan.

Proof. By results of Prokhorov and Shramov (Theorem 1.2 and Remark 1.3), \( \text{Bir}(Y) \) is strongly Jordan, because \( Y \) is non-uniruled. More precisely, they proved in [23, Corollary 6.8] that group \( \text{Bir}(Y) \) is Jordan provided \( Y \) is non-uniruled. In [23, Remark 6.9] they claim that actually if \( Y \) is non-uniruled then \( \text{Bir}(Y) \) has finite subgroups of bounded rank (and therefore is strongly Jordan): the proof has to be based on the same arguments as the proof of Cor. 6.8 of [23] but is not presented. Let us provide the needed mild modifications of the proof of Cor. 6.8 of [23] in order to obtain that non-uniruledness of \( Y \) implies having finite subgroups of bounded rank. Indeed, for \( Y \) non-uniruled and for a finite commutative subgroup \( G \subset \text{Bir}(Y) \) one has (using [23, Proposition 6.2] and its notation except replacing \( X \) by \( Y \) and \( X_{nu} \) by \( Y_{nu} \)) the following.

- \( Y_{nu} \simeq Y \) (thanks to Lemma 4.4 and Remark 4.5 of [23, Lemma 4.4 on p. 2061]) and the group \( G_{nu} \) is isomorphic to \( G \). In particular, \( G_{nu} \) is also finite and commutative.
- There are short exact sequences (23, p. 2063, 6.5])
\[ 1 \to G_{alg} \to G \to G_N \to 1, \]
\[ 1 \to G_L \to G_{alg} \to G_{ab} \to 1, \]
where
1. The finite groups \( G_{alg} \) and \( G_N \) are commutative.
2. \( \text{rk}(G_{ab}) \leq m_1 \) where \( m_1 \) depends only on \( q(Y) \).
(3) \( |G_L| \leq n = n(Y)\), i.e. \( G_L \) is finite and its order is bounded from above by a number \( n \) that depends on \( Y \) but not on \( G \). (This follows from Lemma 5.2 of [23, p. 2062].)

(4) It follows from (2) and (3) combined with the exact sequence (6.5) of [23, p. 2063] that \( \text{rk}(G_{alg}) \leq n + m_1 \).

(5) \( |G_N| \leq b := b(Y) \) is bounded from above by a number \( b(Y) \) that depends only on \( Y \). (It follows from Cor. 2.14 of [23, p. 2058].)

It follows from the exact sequences that

\[
\text{rk}(G) = \text{rk}(G_{nu}) \leq \text{rk}(G_{alg}) + b(Y) \leq (n + m_1) + b(Y) =: m(Y)
\]

where the bound \( m(Y) \) depends on \( Y \) but does not depend on \( G \). This proves that Bir(\( Y \)) has finite subgroups of bounded rank.

So, we know that Bir(\( Y \)) is strongly Jordan. Now the desired result follows readily from Corollary 3.6 combined with Lemma 2.2. \( \square \)

In the next section we prove that Bir(\( X_f \)) is bounded if \( X_f \) is not the projective line over \( k(\bar{Y}) \).

4. Linear Algebra

Throughout this section \( K \) is a field of characteristic 0 that contains all roots of unity. Let \( V \) be a vector space over \( K \) of finite positive dimension \( d \). We write \( 1_V \) for the identity automorphism of \( V \). As usual, End\(_K(V)\) stands for the algebra of \( K \)-linear operators in \( V \) and

\[
\text{Aut}_K(V) = \text{End}_K(V)^*\]

for the group of linear invertible operators in \( V \). We write

\[
\det = \det_V : \text{End}_K(V) \to K
\]

for the determinant map. It is well known that \( \text{Aut}_K(V) \) consists of all elements of \( \text{End}_K(V) \) with nonzero determinant and

\[
\det : \text{Aut}_K(V) \to K^*
\]

is a group homomorphism.

Since \( K \) has characteristic zero and contains all roots of unity, every periodic (\( K \)-linear) automorphism \( u \in \text{Aut}_K(V) \) of \( V \) admits a basis of \( V \) that consists of eigenvectors of \( u \), because \( u \) is semisimple and all its eigenvalues lie in \( K \).

Let

\[
\phi : V \times V \to K
\]

be a nondegenerate symmetric \( K \)-bilinear form that is anisotropic, i.e. \( \phi(v, v) \neq 0 \) for all nonzero \( v \in V \). The form \( \phi \) defines the involution of the first kind

\[
\sigma = \sigma_\phi : \text{End}_K(V) \to \text{End}_K(V)
\]
characterized by the property
\[ \phi(ux, y) = \phi(x, \sigma(u)y) \forall x, y \in V \]
(see [13 Ch. 1]). It is known [13, Ch. 1, Sect. 2, Cor. 2.2 on p.14 and Prop. 2.19 on p. 24] that
\[ \det(u) = \det(\sigma(u)) \forall u \in \text{End}_K(V). \]

We write \( \text{GO}(V, \phi) \subset \text{Aut}_K(V) \) for the (sub)group of similitudes of \( \phi \). In other words, a \( K \)-linear automorphism \( u \) of \( V \) lies in \( \text{GO}(V, \phi) \) if and only if there exists \( \mu = \mu(g) \in K^* \) such that
\[ \phi(ux, uy) = \mu \cdot \phi(x, y) \forall x, y \in V. \]
If this is the case then
\[ \sigma(u)u = \mu \cdot 1_V. \]
Clearly,
\[ K^* \cdot 1_V \subset \text{GO}(V, \phi). \]
We have
\[ \text{SO}(V, \phi) \subset \text{O}(V, \phi) \subset \text{GO}(V, \phi) \]
where
\[ \text{O}(V, \phi) = \{ u \in \text{Aut}_K(V) \mid \phi(ux, uy) = \phi(x, y) \forall x, y \in V \} \]
while \( \text{SO}(V, \phi) \) consists of all elements of \( \text{O}(V, \phi) \) with determinant 1. (Recall that elements of \( \text{O}(V, \phi) \) have determinant 1 or \(-1\). In particular, \( \text{SO}(V, \phi) \) is a normal subgroup of index 2 in \( \text{O}(V, \phi) \).) Clearly,
\[ \text{O}(V, \phi) = \{ u \in \text{Aut}_K(V) \mid \sigma(u)u = 1_V \}. \]
It is also clear that
\[ \text{GO}(V, \phi) \to K^*, \ u \mapsto \mu(u) \]
is a group homomorphism, whose kernel coincides with \( \text{O}(V, \phi) \); in particular, \( \text{O}(V, \phi) \) is a normal subgroup of \( \text{GO}(V, \phi) \). It is well known (and may be easily checked) that
\[ \text{O}(V, \phi) \bigcap [K^* \cdot 1_V] = \{ \pm 1_V \}; \]
in addition, if \( d = \dim(V) \) is odd then
\[ \text{SO}(V, \phi) \bigcap [K^* \cdot 1_V] = \{ 1_V \}. \]
We denote by \( \text{PGO}(V, \phi) \) the quotient group \( \text{GO}(V, \phi)/(K^* \cdot 1_V) \).

**Remark 4.1.** The importance of the group \( \text{PGO}(V, \phi) \) is explained by the following result [5] Sect. 69, Corollary 69.6 on p. 310]. Let
\[ q(v) := \phi(v, v) \]
be the corresponding quadratic form on \( V \) and let
\[ X_q \subset \mathbb{P}(V) \]
be the projective quadric defined by the equation $q(v) = 0$, which is a smooth projective irreducible $(d - 2)$-dimensional variety over $K$. Then the groups $\text{Aut}(X_q)$ and $\text{PGO}(V, \phi)$ are isomorphic.

**Remark 4.2.** Restricting the surjection
$$\text{GO}(V, \phi) \to \text{GO}(V, \phi)/(K^* \cdot 1_V) = \text{PGO}(V, \phi)$$
to the subgroup $\text{O}(V, \phi)$, we get a group homomorphism
$$\text{O}(V, \phi) \to \text{PGO}(V, \phi),$$
whose kernel is a finite subgroup $\{\pm 1_V\}$. This implies that if $u$ is an element of $\text{O}(V, \phi)$, whose image in $\text{PGO}(V, \phi)$ has finite order then $u$ itself has finite order.

**Lemma 4.3.** Let $u$ be an element of finite order in $\text{O}(V, \phi)$. Then $u^2 = 1_V$.

**Proof.** Let $\lambda$ be an eigenvalue of $u$. Then $\lambda$ is a root of unity and therefore lies in $K$. This implies that there is a (nonzero) eigenvector $x \in V$ with $ux = \lambda x$. Since $u \in \text{O}(V, \phi)$,
$$\phi(ux, ux) = \phi(x, x).$$
Since $ux = \lambda x$,
$$\phi(ux, ux) = \phi(\lambda x, \lambda x) = \lambda^2 \phi(x, x)$$
and therefore $\lambda^2 \phi(x, x) = \phi(x, x)$. Since $\phi$ is anisotropic, $\phi(x, x) \neq 0$ and therefore $\lambda^2 = 1$. In other words, every eigenvalue of $u^2$ is 1 and therefore (semisimple) $u^2 = 1_V$. \hfill \Box

**Corollary 4.4.** Let $G$ be a finite subgroup of $\text{O}(V, \phi)$. If $G$ does not coincide with $\{1_V\}$ then it is a commutative group of exponent 2, whose order divides $2^d$. If, in addition, $G$ lies in $\text{SO}(V, \phi)$ then its order divides $2^{d-1}$.

**Proof.** By Lemma 4.3 every $u \in G$ satisfies $u^2 = 1_V$. This implies that $G$ is commutative. In addition, $G$ is a 2-group, i.e., its order is a power of 2. The commutativeness of $G$ implies that there is a basis of $V$ such that the matrices of all elements of $G$ become diagonal with respect to this basis. Since all the diagonal entries are either 1 or $-1$, the order of $G$ does not exceed $2^d$ and therefore divides $2^d$. If, in addition, all elements of $G$ have determinant 1 then the order of $G$ does not exceed $2^{d-1}$ and, therefore, divides $2^{d-1}$. \hfill \Box

**Corollary 4.5.** Let $u$ be an element of finite order in $\text{PGO}(V, \phi)$. Then $u^4 = 1$.

**Proof.** Choose an element $u$ of $\text{GO}(V, \phi)$ such that its image in $\text{PGO}(V, \phi)$ coincides with $u$. Then there is $\mu \in K^*$ such that
$$\phi(ux, uy) = \mu \phi(x, y) \ \forall \ x, y \in V.$$ 
This implies that $u_2 := \mu^{-1}u^2$ lies in $\text{O}(V, \phi)$. Clearly, the image $\bar{u}_2 \in \text{PGO}(V, \phi)$ of $u_2$ coincides with $u^2$ and therefore has finite order. By Corollary 4.2, $u_2$ has finite order. It follows from Lemma 4.3 that $u_2^2 = 1_V$. This implies that $u^2$ has order 1 or 2 and therefore the order of $u$ divides 4. \hfill \Box
Corollary 4.6. If $\mathcal{B}$ is a finite subgroup of $\text{PGO}(V, \phi)$ then it sits in a short exact sequence

$$\{1\} \to A_1 \to \mathcal{B} \to A_2 \to \{1\}$$

where both $A_1$ and $A_2$ are finite elementary commutative 2-groups and $|A_1|$ divides $2^{d-1}$. In particular, each finite subgroup $\mathcal{B}$ of $\text{PGO}(V, \phi)$ is a finite 2-group such that $[[\mathcal{B}, \mathcal{B}], [\mathcal{B}, \mathcal{B}]] = \{1\}$.

Proof. Let $A_1$ be the subgroup of all elements of $\mathcal{B}$ that are the images of elements of $\text{O}(V, \phi)$. Since $\text{O}(V, \phi)$ is normal in $\text{GO}(V, \phi)$, the subgroup $A_1$ is normal in $\mathcal{B}$. It follows from the proof of Corollary 4.5 that for each $u \in \mathcal{B}$ its square $u^2$ lies in $A_1$. This implies that all the elements of the quotient $A_2 := \mathcal{B}/A_1$ have order 1 or 2. It follows that $A_2$ is an elementary abelian 2-group. We get a short exact sequence $\{1\} \to A_1 \to \mathcal{B} \to A_2 \to \{1\}$.

Let $\tilde{A}_1$ be the preimage of $A_1$ in $\text{O}(V, \phi)$. Clearly, $\tilde{A}_1$ is a subgroup of $\text{O}(V, \phi)$ and $|\tilde{A}_1| = 2 \cdot |A_1|$. On the other hand, it follows from Corollary 4.4 that $\tilde{A}_1$ is an elementary abelian 2-group, whose order divides $2^d$. Since $\tilde{A}_1$ maps onto $A_1$, the latter is also an elementary abelian 2-group and its order divides $\frac{1}{2}2^d = 2^{d-1}$. Since $|\mathcal{B}| = |A_1| \cdot |A_2|$, the order of $\mathcal{B}$ is a power of 2, i.e., $\mathcal{B}$ is a finite 2-group. On the other hand, since $\mathcal{B}_2 = \mathcal{B}/A_1$ is abelian, $[[\mathcal{B}, \mathcal{B}], [\mathcal{B}, \mathcal{B}]] \subset [\mathcal{A}_1, \mathcal{A}_1] = \{1\}$, i.e., $[[\mathcal{B}, \mathcal{B}], [\mathcal{B}, \mathcal{B}]] = \{1\}$. \hfill $\square$

In the case of odd $d$ we can do better. Let us start with the following observation.

Lemma 4.7. Suppose that $d = 2\ell + 1$ is an odd integer that is greater or equal than 3. Then every $u \in \text{GO}(V, \phi)$ can be presented as

$$u = \mu_0 \cdot u_0$$

with $u_0 \in \text{SO}(V, \phi)$ and $\mu_0 \in K^*$

Example 4.8. If $u$ is an element of $\text{O}(V, \phi)$ with determinant $-1$ then

$$u = (-1) \cdot (-u), \quad (-u) \in \text{SO}(V, \phi).$$

Proof of Lemma 4.7. Recall that there is $\mu \in K^*$ such that

$$\phi(u x, u y) = \mu \cdot \phi(x, y) \quad \forall x, y \in V$$

and therefore

$$\sigma(u) u = \mu \cdot 1_V.$$

Now let $\gamma \in K^*$ be the determinant of $u$. Since $\det(\sigma(u)) = \det(u)$, we obtain that

$$\mu^{2\ell+1} = \mu^d = \det(\mu \cdot 1_V) = \det(\sigma(u) u) = \det(\sigma(u)) \det(u) = \gamma^2.$$
This implies that
\[ \gamma^2 = \mu^d = \mu^{2\ell + 1}. \]
Let us put
\[ \mu_0 = \frac{\gamma}{\mu^\ell}, \quad u_0 = \mu_0^{-1} \cdot u. \]
Then
\[ \mu_0^2 = \mu, \quad \gamma = \mu_0^{2\ell + 1} = \mu^d, \]
\[ u = \mu_0 \cdot u_0, \quad \det(u_0) = \mu_0^{-d}, \quad \det(u) = \gamma^{-1} \gamma = 1. \]
We also have
\[ \phi(u_0 x, u_0 y) = \phi(\mu_0^{-1} u x, \mu_0^{-1} u y) = \mu_0^{-2} \cdot \phi(u x, u y) = \mu^{-1} \cdot \phi(x, y) = \phi(x, y). \]
This implies that \( u_0 \in O(V, \phi) \). Since \( \det(u_0) = 1 \),
\[ u_0 \in SO(V, \phi). \]

**Corollary 4.9.** Suppose that \( d = 2\ell + 1 \) is an odd integer that is greater or equal than 3. Then the group homomorphism
\[ \text{prod} : K^*1_V \times SO(V, \phi) \to GO(V, \phi), \quad (\mu_0 \cdot 1_V, u_0) \mapsto \mu_0 \cdot u_0 \]
is a group isomorphism. In particular, the group \( PGO(V, \phi) = GO(V, \phi)/\{K^*1_V\} \) is canonically isomorphic to \( SO(V, \phi) \).

**Proof.** Since \( d \) is odd,
\[ \text{SO}(V, \phi) \cap \{K^* \cdot 1_V\} = \{1_V\}, \]
which implies that \text{prod} is injective. Its surjectiveness follows from Lemma 4.7. \( \square \)

**Theorem 4.10.** Suppose that \( K \) is a field of characteristic zero that contains all roots of unity, \( d \geq 3 \) is an odd integer, \( V \) is a \( d \)-dimensional \( K \)-vector space and
\[ \phi : V \times V \to K \]
a nondegenerate symmetric \( K \)-bilinear form that is anisotropic, i.e. \( \phi(v, v) \neq 0 \) for all nonzero \( v \in V \).

Let \( G \) be a finite subgroup in \( PGO(V, \phi) \). Then \( G \) is commutative, all its non-identity elements have order 2 and the order of \( G \) divides \( 2^{d-1} \).

**Proof.** The result follows readily from Corollary 4.9 combined with Corollary 4.4. \( \square \)

**Corollary 4.11.** Suppose that \( K \) is a field of characteristic zero that contains all roots of unity, \( d \geq 3 \) an odd integer, \( V \) a \( d \)-dimensional \( K \)-vector space and a quadratic form such that \( q(v) \neq 0 \) for all nonzero \( v \in V \). Let us consider the projective quadric \( X_q \subset \mathbb{P}(V) \) defined by the equation \( q = 0 \), which is a smooth projective irreducible \( (d - 2) \)-dimensional variety over \( K \). Let \( Aut(X_q) \) be the group of birational automorphisms of \( X_q \). Let \( G \) be a finite subgroup in \( Aut(X_q) \). Then \( G \) is commutative, all its non-identity elements have order 2 and the order of \( G \) divides \( 2^{d-1} \).
Proof. Let $\phi : V \times V \to K$ be the symmetric $K$-bilinear form such that $\phi(v, v) = q(v) \forall v \in V$.

Namely, for all $x, y \in V$

$$\phi(x, y) := \frac{q(x + y) - q(x) - q(y)}{2}.$$  

Clearly, $\phi$ is nondegenerate. In the notation of [5, Sect. 69, p. 310], $GO(q) = GO(V, \phi), \ PGO(q) = PGO(V, \phi)$.

By Corollary 69.6 of [5, Sect. 69], the groups $Aut(X_q)$ and $PGO(q)$ are isomorphic.

Now the result follows from Theorem 4.10. □

Corollary 4.12. Suppose that $K$ is a field of characteristic zero that contains all roots of unity. Let $C$ be a smooth irreducible projective genus 0 curve over $K$ that is not biregular to $\mathbb{P}^1$ over $K$.

Let $\text{Bir}_K(C)$ be the group of birational automorphisms of $C$. Let $G$ be a finite subgroup in $\text{Bir}_K(C)$. Then $G$ is commutative, all its non-identity elements have order 2 and the order of $G$ divides 4. In other words, if $G$ is nontrivial then it is either a cyclic group of order 2 or is isomorphic to a product of two cyclic groups of order 2.

Proof. Since $C$ has genus zero, it is $K$-biregular to a smooth projective plane quadric

$$\mathcal{X} = \{a_1T_1^2 + a_2T_2^2 + a_3T_3^2 = 0\} \subset \mathbb{P}^2$$

where all $a_i$ are nonzero elements of $K$. Since $C$ is not biregular to $\mathbb{P}^1$, the set $\mathcal{X}(K)$ is empty ([10, Th. A.4.3.1 on p. 75], [5, Sect. 45, Prop. 45.1 on p. 194]), which means that the nondegenerate ternary quadratic form

$$q(T) = a_1T_1^2 + a_2T_2^2 + a_3T_3^2$$

is anisotropic. We may view $q$ as the quadratic form on the coordinate three-dimensional $K$-vector space $V = K^3$. Then (in the notation of Corollary 4.11) $d = 3$ and $\mathcal{X} = X_q$. Since $\mathcal{X}$ is a smooth projective curve, its group $\text{Bir}_K(\mathcal{X})$ of birational automorphisms coincides with the group $\text{Aut}(\mathcal{X})$ of biregular automorphisms. Now the result follows from Corollary 4.11. □

The rest of this section deals with the case of even $d$; its results will not be used elsewhere in the paper.

Theorem 4.13. Suppose that $K$ is a field of characteristic zero that contains all roots of unity, $d \geq 2$ is an even positive integer, $V$ is a $d$-dimensional $K$-vector space, and $\phi : V \times V \to K$

a nondegenerate symmetric $K$-bilinear form that is anisotropic, i.e. $\phi(v, v) \neq 0$ for all nonzero $v \in V$. Then the group $PGO(V, \phi)$ is bounded. More precisely, there is a positive integer $n = n(d)$ that depends only on $d$ and such that every finite subgroup of $PGO(V, \phi)$ has order dividing $2^{n(d)}$. 
Proof. We deduce Theorem 4.13 from Theorem 2.5. Let $\text{Aut}_K(\text{End}_K(V))$ be the group of automorphisms of the $K$-algebra $\text{End}_K(V)$. We write $V_2$ for $\text{End}_K(V)$ viewed as the $d^2$-dimensional $K$-vector space and $\text{Aut}_K(V_2)$ for its group of $K$-linear automorphisms. We have

$$\text{Aut}_K(\text{End}_K(V)) \subset \text{Aut}_K(V_2).$$

Let us choose a basis $\{e_1, \ldots, e_{d^2}\}$ of $V_2$. Such a choice gives us a group isomorphism $\text{Aut}_K(V_2) \cong \text{GL}(d^2, K)$.

Let us consider a group homomorphism $\text{Ad} : \text{GO}(V, \phi) \subset \text{Aut}_K(V) \to \text{Aut}_K(\text{End}_K(V))$, $u \mapsto \{w \mapsto uwu^{-1} \forall w \in \text{End}_K(V)\}$ for all $u \in \text{GO}(V, \phi) \subset \text{Aut}_K(V)$. Clearly,

$$\ker(\text{Ad}) = K^* \cdot 1_V.$$

This gives us an embedding

$$\text{PGO}(V, \phi) = \text{GO}(V, \phi)/\{K^* \cdot 1_V\} \hookrightarrow \text{Aut}_K(\text{End}_K(V)) \hookrightarrow \text{Aut}_K(V_2) \cong \text{GL}(d^2, K).$$

This implies that $\text{PGO}(V, \phi)$ is isomorphic to a subgroup of $\text{GL}(d^2, K)$. By Corollary 4.5, every periodic element in $\text{PGO}(V, \phi)$ has order dividing 4. This implies (thanks to First Sylow Theorem) that the order of every finite subgroup of $\text{PGO}(V, \phi)$ is a power of 2. In other words, all finite subgroups in $\text{PGO}(V, \phi)$ are 2-groups. Now the desired result follows from Theorem 2.5 (applied to $n = d^2$ and $N = 4$).

Combining Theorem 4.13, Corollary 4.6 and Remark 4.2, we obtain the following assertion.

**Theorem 4.14.** Suppose that $K$ is a field of characteristic zero that contains all roots of unity, $d \geq 2$ an even integer, $V$ a $d$-dimensional $K$-vector space. Let $q : V \to K$ be a quadratic form such that $q(v) \neq 0$ for all nonzero $v \in V$. Let us consider the projective quadric $X_q \subset \mathbb{P}(V)$ defined by the equation $q = 0$, which is a smooth projective irreducible $(d - 2)$-dimensional variety over $K$. Let $\text{Aut}(X_q)$ be the group of birational automorphisms of $X_q$.

Then:

1. $\text{Aut}(X_q)$ is bounded. More precisely, there is a positive integer $n = n(d)$ that depends only on $d$ and such that every finite subgroup of $\text{Aut}(X_q)$ has order dividing $2^{n(d)}$.

2. If $B$ is a finite subgroup of $\text{Aut}(X_q)$ then it is a finite 2-group that sits in a short exact sequence

$$\{1\} \to A_1 \to B \to A_2 \to \{1\}$$

where both $A_1$ and $A_2$ are finite elementary abelian 2-groups and $|A_1|$ divides $2^{d-1}$. In particular,

$$[[B, B], [B, B]] = \{1\}.$$
We hope to return to a classification of finite subgroups of $\text{Aut}(X_q)$ (for even $d$) in a future publication.

5. Jordaness properties of Bir

Proof of Theorem 1.3. Let us put $K = k(Y)$. Then $\text{char}(K) = 0$. Since $K$ contains algebraically closed $k$, it contains all roots of unity. In the notation of Corollary 4.12 let us put $C = X_f$. Since $X_f$ has no $K$-points, it is not birational to $\mathbb{P}^1$ over $K$.

It follows from Corollary 4.12 that $\text{Bir}(X_f)$ is bounded. Now Corollary 3.8 implies that $\text{Bir}(X)$ is strongly Jordan.

Proof of Theorem 1.6. It follows from Remark 3.1 that $f : X \to Y$ is a conic bundle. In particular, the generic fiber $X = X_f$ is an absolutely irreducible smooth projective genus zero curve over $K := k(Y)$.

Since each $K$-point of $X$ gives rise to a rational section $X \to X_f$ of $f$, there are no $K$-points on $X$. Now the result follows from Theorem 1.5.

Example 5.1. Let $Y$ be a smooth irreducible projective variety over $k$ of dimension $\geq 2$. Let $a_1, a_2, a_3$ be nonzero elements of $k(Y)$ such that the ternary quadratic form

$$q(T) = a_1T_1^2 + a_2T_2^2 + a_3T_3^2$$

is anisotropic over $k(Y)$. Example 3.3 gives us a a smooth irreducible projective variety $\tilde{X}_q$ and a surjective regular map $f : \tilde{X}_q \to Y$, whose generic fiber is the quadric

$$\{a_1T_1^2 + a_2T_2^2 + a_3T_3^2 = 0\} \subset \mathbb{P}^2_{k(Y)}$$

over $k(Y)$ without $k(Y)$-points. Now Theorem 1.5 tells us that $\text{Bir}(\tilde{X}_q)$ is strongly Jordan if $Y$ is non-uniruled.

Remark 5.2. Recall (Example 3.3) that if $\text{dim}(Y) \geq 2$ then there always exists an anisotropic ternary quadratic form over $k(Y)$. (A theorem of Tsen implies that such a form does not exist if $\text{dim}(Y) = 1$.)

Proof of Theorem 1.4. Recall that an abelian variety $Y$ does not contain rational curves; in particular, it is not uniruled. Now Theorem 1.4 follows from Theorem 1.5.

Theorem 5.3. Let $d \geq 3$ be an integer. Let $\mathcal{G}$ be the collection of groups $\text{Bir}(X)$ where $X$ runs through the set of smooth irreducible projective $d$-dimensional varieties that can be realized as conic bundles $f : X \to Y$ over a $(d - 1)$-dimensional abelian variety $Y$ but $X$ is not birational to $Y \times \mathbb{P}^1$. Then $\mathcal{G}$ is uniformly strongly Jordan.

Proof. It follows from Remark 3.2 that the generic fiber $X_f$ of $f$ has no $k(Y)$-rational points. It follows from Corollary 4.12 that the collection of groups of the form $\text{Bir}(X_f)$ is uniformly bounded - actually, the order of every finite subgroup in $\text{Bir}(X_f)$ divides 4. Recall (Corollary 3.6) that there is an exact sequence

$$\{1\} \to \text{Bir}(X_f) \to \text{Bir}(X) \to \text{Bir}(Y).$$
Now the result follows from Corollary 2.3.

\[ \square \]

**Remark 5.4.** The condition that \( k \)-varieties \( X, Y \) in Theorem 1.4, Theorem 1.5, and Theorem 1.6 are smooth, is non-essential. Indeed, let \( X, Y \) be irreducible projective varieties of dimensions \( d \) and \( d - 1 \), respectively, endowed with a surjective morphism \( f : X \to Y \). Let \( Y \) be non-uniruled. Due to the resolution of singularities (see, for example, [15, Chapter 3, section 3.3]) one can always find two smooth projective irreducible varieties \( \tilde{X}, \tilde{Y} \), the birational morphisms \( \pi_X : \tilde{X} \to X \), \( \pi_Y : \tilde{Y} \to Y \), and a morphism \( \tilde{f} : \tilde{X} \to \tilde{Y} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\pi_X \downarrow & & \pi_Y \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

Then the following properties are valid:

1. \( \tilde{Y} \) is non-uniruled, since it is birational to the non-uniruled \( Y \) (see, for example, [3, Chapter 4, Remark 4.2]).

   Indeed, there is an open dense \( V' \subset \tilde{Y} \) such that
   
   \(-\pi_Y \) is an isomorphism of \( V' \) to \( \pi_Y(V') \);
   
   for every \( v \in V' \) the exceptional set \( S_X \) of morphism \( \pi_X \) intersects the fiber \( \tilde{F}_v \) of \( \tilde{f} \) only at an empty or a finite set of points. It is valid, because \( \dim(S_X) \leq \dim(\tilde{Y}) = d - 1 \), hence the restriction of \( \tilde{f} \) onto an irreducible component of \( S_X \) is either non-dominant, or generically finite.

   Thus, \( \tilde{F}_v \cap (\tilde{X} \setminus S_X) \) is open and dense in \( \tilde{F}_v \) for point \( v \in V' \) (because every irreducible component of \( \tilde{F}_v \) has dimension at least 1). On the other hand, \( \tilde{F}_v \cap (\tilde{X} \setminus S_X) \) is isomorphic via \( \pi_X \) to \( F_{\pi_Y(v) \cap (X \setminus \pi_X(S_X))} \), which is an open and dense subset of irreducible \( F_{\pi_Y(v)} \), if \( \pi_Y(v) \in U \). Hence, for all \( v \in V' \cap \pi_Y^{-1}(U) \) the fibers \( \tilde{F}_v \) and \( F_{\pi_Y(v)} \) are birational.

2. If the general fiber \( F_u := f^{-1}(u) \), \( u \in U \subset Y \) is irreducible, then so is the general fiber \( \tilde{F}_v := \tilde{f}^{-1}(v) \), \( v \in V \subset \tilde{Y} \) of \( \tilde{f} \) (here \( U, V \) are open dense subsets of \( Y, \tilde{Y} \), respectively).

Indeed, there is an open dense \( V' \subset \tilde{Y} \) such that

\(-\pi_Y \) is an isomorphism of \( V' \) to \( \pi_Y(V') \);

for every \( v \in V' \) the exceptional set \( S_X \) of morphism \( \pi_X \) intersects the fiber \( \tilde{F}_v \) of \( \tilde{f} \) only at an empty or a finite set of points. It is valid, because \( \dim(S_X) \leq \dim(\tilde{Y}) = d - 1 \), hence the restriction of \( \tilde{f} \) onto an irreducible component of \( S_X \) is either non-dominant, or generically finite.

Thus, \( \tilde{F}_v \cap (\tilde{X} \setminus S_X) \) is open and dense in \( \tilde{F}_v \) for point \( v \in V' \) (because every irreducible component of \( \tilde{F}_v \) has dimension at least 1). On the other hand, \( \tilde{F}_v \cap (\tilde{X} \setminus S_X) \) is isomorphic via \( \pi_X \) to \( F_{\pi_Y(v) \cap (X \setminus \pi_X(S_X))} \), which is an open and dense subset of irreducible \( F_{\pi_Y(v)} \), if \( \pi_Y(v) \in U \). Hence, for all \( v \in V' \cap \pi_Y^{-1}(U) \) the fibers \( \tilde{F}_v \) and \( F_{\pi_Y(v)} \) are birational.

3. If the generic fiber \( X_f \) of \( f \) is absolutely irreducible, so is generic the fiber \( X_{\tilde{f}} \) of \( \tilde{f} \). Indeed, in this case the general fiber of \( f \) is irreducible (see [8, Proposition 9.7.8]). According to property [2], the general fiber of \( \tilde{f} \) is also irreducible, and, hence, so is \( X_{\tilde{f}} \) (ibid).

4. The general and generic fibers \( \tilde{F}_v \) and \( X_{\tilde{f}} \) of \( \tilde{f} \) are smooth, since \( \tilde{f} \) is a surjective morphism between smooth projective varieties.

It follows that if the generic (respectively, general) fiber of \( f \) is a rational curve, then the generic (respectively, general) fiber of \( \tilde{f} \) is a smooth rational curve. According to Theorem 1.5 (respectively, Theorem 1.6, ) \( \text{Bir}(X) = \text{Bir}(\tilde{X}) \) has to be Jordan.
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Department of Mathematics, Bar-Ilan University, 5290002, Ramat Gan, ISRAEL
E-mail address: bandman@math.biu.ac.il

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA
E-mail address: zarhin@math.psu.edu