Tight Correlation-Function Bell Inequality for Multiparticle \(d\)-Dimensional System

Jing-Ling Chen\(^1\) and Dong-Ling Deng\(^1\)

\(^1\)Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, People’s Republic of China

(Dated: January 6, 2009)

We generalize the correlation functions of the Clauser-Horne-Shimony-Holt (CHSH) inequality to multiparticle \(d\)-dimensional systems. All the Bell inequalities based on this generalization take the same simple form as the CHSH inequality. For small systems, numerical results show that the new inequalities are tight and we believe this is also valid for higher dimensional systems. Moreover, the new inequalities are relevant to the previous ones and for bipartite system, our inequality is equivalent to the Collins-Gisin-Linden-Masser-Popescu (CGLMP) inequality.

PACS numbers: 03.65.Ud, 03.67.Mn, 03.65.-w

That local and realistic theories impose certain constraints in the form of some inequalities on statistical correlations of measurements on multiparticles was first shown by Bell in 1964 [1]. Bell pointed out that any kind of local hidden variable theory should obey these inequalities, while they can be violated easily in quantum mechanics. After Bell’s applaudable progress, extensive works on Bell inequalities have been done, including both theoretical analysis and experiment test. For instance, the Clauser-Horne-Shimony-Holt (CHSH) [2] inequality was proposed in 1969, which is more convenient for experiment to test the non-locality of two-dimensional (qubit) system. However, there exists a long-living open question: “What are the general inequalities for more complicated situations?” i.e., for more particles and higher-dimensional systems.

On the other hand, for higher dimensions of two particles, Collins et al. constructed a CHSH type inequality for arbitrary \(d\)-dimensional (qudit) systems in 2002, now known as the Collins-Gisin-Linden-Masser-Popescu (CGLMP) inequality [3]. This inequality was shown to be tight, i.e., it defines one of the facets of the convex polytope [4] of local-realistic (LR) models [5]. There are some other alternative forms of this inequality [6, 7], and the maximal quantum violation of this inequality was analyzed in [8], which showed that the maximal violation of this inequality occurs at the non-maximally entangled state. More recently, Seung Woo Lee and Dieter Jaksch introduced another tight Bell inequality which is maximally violated by maximally entangled states [9].

On the other hand, there are also various Bell inequalities for \(N\) \((N > 2)\) particles. In 1990, Mermin, in the first time, produced a series of two setting inequalities for arbitrary many qubits [10]. A complementary series of inequalities was introduced by Ardehali [11]. In the next step, Belinskii and Klyshko gave a series of two setting inequalities, which contained the tight inequalities of Mermin and Ardehali [12]. These inequalities, now known as Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities, are maximally violated by the Greenberger-Horne-Zeilinger (GHZ) states \(|\psi\rangle_{\text{GHZ}} = \frac{1}{\sqrt{2}}(|0\cdots0\rangle + |1\cdots1\rangle). But for the generalized GHZ states \(|\psi\rangle_{\text{GHZ}} = \cos\xi|0\cdots0\rangle + \sin\xi|1\cdots1\rangle\), and \(N\) odd, there exists one region \(\xi \in (0, \frac{\pi}{2} \arcsin(1/\sqrt{2N-1})]\) in which the MABK inequalities are not violated [13, 14]. Thus the MABK inequalities may not be the ‘natural’ generalizations of the CHSH inequality to more than two qubits, in the sense that the CHSH inequality violates all the pure states of two-qubit systems. In 2004, Chen et al. presented a two-setting Bell inequality for three qubits which can be seen numerically to be violated by any pure entangled state [15]. In [16], tight Bell inequalities for three particles with low dimension are presented. Nevertheless, up to now, there is no generic tight Bell inequality for arbitrary \(N\)-qubit systems, even for three qubits, no such inequality has been found. Since many of quantum communication schemes, such as multiparty key (secret) sharing [17] and quantum communication complexity problems [18], can be measured with multiparty Bell inequalities of some form [19], derivations of multiparty Bell inequalities are thus one of the most important and challenging subject in quantum theory.

The purpose of this paper is to present general Bell inequalities based on the correlation functions for \(N\)-qudit systems. These inequalities, obtained by using the same method in [6], are tight and relevant to the previous ones. What’s more, all the Bell inequalities based on this generalization take the same simple form as the CHSH inequality. For bipartite systems, our inequality is equivalent to the (CGLMP) inequality. However, to be honest, there are two disadvantages for these new inequalities: (i) The quantum violations of these inequalities are small and they are not as strong as the previous inequalities, namely, they are less resistant to noise; (ii) Some pure states do not violate these inequalities. It is interesting to note that these two disadvantages indicate that a tight Bell inequality may not always be the optimal one.

The approach to our new tight Bell inequalities for \(N\)-qudit systems is based on the Gedanken experiment. Consider \(N\) spatially separated parties and allow each of them to choose independently between two dichotomic...
observables. Let \( X^{[1]}_j, X^{[2]}_j \) \((j = 1, 2, \cdots, N)\) denote the two observables on the \( j \)-th party, each of them have \( d \) possible outcomes: \( x^{[1]}_j, x^{[2]}_j = 0, 1, \cdots, d - 1 \) \((j = 1, 2, \cdots, N)\). The joint probabilities are denoted by \( P(X^{[1]}_j, \cdots, X^{[N]}_j) \), which should satisfy the normalization condition:

\[
d - 1 \sum_{x^{[1]}_j, \cdots, x^{[N]}_j = 0} P(X^{[1]}_j = x^{[1]}_j, \cdots, X^{[N]}_j = x^{[N]}_j) = 1. \quad (1)
\]

For two-qudit systems, namely \( N = 2 \), Ref. \([3]\) introduced the correlation functions \( Q_{ij} \) in the following form:

\[
Q_{ij} = \frac{1}{S} \sum_{m=0}^{d-1} \sum_{n=0}^{d-1} f^{ij}(m, n) P(X^{[i]}_1 = m, X^{[j]}_2 = n), \quad (2)
\]

where \( S = (d - 1)/2 \) is the spin of the particle for the \( d \)-dimensional system. \( f^{ij}(m, n) \) is the correlation functions for multipartite \( d \)-dimensional systems, \( S = M[(\mod d) + 1] \)

\[
I^{[3]}_d = Q_{111} - Q_{222} + Q_{121} + Q_{212} \leq 2. \quad (5)
\]

Obviously, \( I^{[3]}_d \) is upper bounded by 4 since the extreme values of \( Q_{ijk} \) are \( \pm 1 \) and it can never reach this value because that the four functions in Eq.\((4)\) are strongly correlated. In fact, for hidden variable theories, it is easy to prove that the maximum value of \( I^{[3]}_d \) is 2. We use the same method as for two \( d \)-dimensional systems in Ref. \([3]\).

The essential idea of this proof is to enumerate all the possible relations between pairs of operators. Defining \( r_{111} \equiv X^{[1]}_1 + X^{[2]}_1 + X^{[3]}_1 \), \( r_{222} \equiv X^{[1]}_2 + X^{[2]}_2 + X^{[3]}_2 \), \( r_{121} \equiv X^{[1]}_1 + X^{[2]}_2 + X^{[3]}_3 \), and \( r_{212} \equiv X^{[1]}_2 + X^{[2]}_1 + X^{[3]}_3 \). Then the constraint follows immediately:

\[
r_{111} + r_{222} = r_{121} + r_{212}. \quad (6)
\]

For convenience, we define two functions: \( g_1(x) = \frac{S-M(x,d)}{S}, g_2(x) = \frac{M(x,d)-S-1}{S} \). Then, for a given choice of \( r_{111}, r_{222}, r_{121}, \) and \( r_{212} \), the correlation functions in Eq.\((4)\) can be rewritten as:

\[
Q_{111} = g_2(r_{111}), Q_{222} = g_1(r_{222}), Q_{121} = g_1(r_{121}), \) and \( Q_{212} = g_1(r_{212}). \) A direct calculation shows that:

\[
I^{[3]}_d = \frac{1}{S} \left[ M(r_{111}, d) + M(r_{222}, d) - M(r_{121}, d) - M(r_{212}, d) - 1 \right]. \quad (7)
\]

Now, we should enumerate all the possible cases according to the different values of \( r_{111}, r_{222}, r_{121}, \) and \( r_{212} \).

Case 1: Both \( r_{111} \) and \( r_{222} \) are less than \( d \). From \( (6) \), there are two cases for the rest: (i) none of \( r_{212} \) and \( r_{121} \) is larger than \( d \) then we have \( I^{[3]}_d = |r_{111}+r_{222}-(r_{212}+r_{121})-1|/S = -1/S \) (note that \( d = 2S + 1 \)); (ii) one of \( r_{212} \) and \( r_{121} \) is equal to or larger than \( d \). Then after some simple calculation, we get \( I^{[3]}_d = (d-1)/S = 2 \).

Case 2: \( r_{111} < d \) and \( d \leq r_{222} < 2d \) or \( d \leq r_{111} < 2d \) and \( r_{222} < d \). From \( (6) \), there are four cases for the rest: (i) both \( r_{212} \) and \( r_{121} \) are less than \( d \). then we have \( I^{[3]}_d = |r_{111}+r_{222}-(r_{212}+r_{121})-1|/S = -2(S+1)/S \); (ii) one of \( r_{212} \) and \( r_{121} \) is equal to or larger than \( d \). Then after some simple calculation, we get \( I^{[3]}_d = -1/S; (iii) \) Both \( r_{212} \) and \( r_{121} \) are larger than \( d \) and less than \( 2d \), then \( I^{[3]}_d = 2; (iv) \) one of \( r_{212} \) and \( r_{121} \) is less than \( d \), and the other is larger than \( 2d \), then we can also get \( I^{[3]}_d = 2 \).

Case 3: \( d \leq r_{111} < 2d \) and \( d \leq r_{222} < 2d \). From \( (6) \), there are four cases for the rest: (i) one of \( r_{212} \) and \( r_{121} \) is less than \( d \) and the other is larger than \( d \) and less than \( 2d \), then we have \( I^{[3]}_d = -2(S+1)/S \); (ii) both of them are larger than \( d \) and less than \( 2d \). Then obviously, \( I^{[3]}_d = -1/S; (iii) \) one of them is larger than \( 2d \) and the other is less than \( d \), then \( I^{[3]}_d = -1/s; (iv) \) one of them is larger than \( 2d \) and the other is larger than \( d \) and less than \( 2d \), then \( I^{[3]}_d = 2 \).

Case 4: Both \( r_{111} \) and \( r_{222} \) are equal to or larger than
2d. From [3], there are two cases for the rest:(i) one of \( r_{212} \) and \( r_{121} \) is larger than 2d and the other is larger than \( d \) and less than 2d. Then we have \( I_d = -2(S + 1)/S \); (ii) both of them are larger than 2d, then obviously, \( I_d^{[3]} = -1/S \).

Thus, we have proved that \( I_d^{[3]} \leq 2 \) for local realistic theories (Note that for \( X \) that this will generalize. If we set \( k \leq 1 \), then \( I_d^{[3]} \) has only two possible values \( \pm 2 \) since not all the possibilities enumerated above can occur). Moreover, we have found computationally that the inequality \( \Box \) is tight for \( d \leq 10 \), and suspect that this will generalize. If we set \( X_1^{[3]} = 0 \) and \( X_2^{[3]} = 0 \), then the inequality \( \Box \) reduces to a two qudits Bell inequality which is an alternative form of inequality \( \Box \) and equivalent to the CGLMP inequality.

Let us now focus on the quantum violation of the inequality \( \Box \). We will restrict the considerations to multi-port beamsplitters since the software takes too long to run on our computer if the most general measurements are employed. Actually, for low dimensional systems (\( d \leq 3 \)), we have used the most general measurements but no larger violations are found. In a Gedanken experiment [20], the matrix elements of an unbiased symmetric multi-port beamsplitter are given by \( U_{kl}(\varphi) = \frac{1}{\sqrt{d}} \alpha^k \exp(i\varphi l) \), here \( \alpha = \exp(\frac{2\pi i}{d}) \) and \( \varphi l = \frac{l}{d-1} \) are the settings of the appropriate phase shifters, for convenience we denote them as a \( d \) dimensional vector \( \vec{\varphi} = (\varphi^0, \varphi^1, \varphi^2, \cdots, \varphi^{d-1}) \). For state \( |\psi\rangle \) of three-qudit systems, the quantum prediction for the probabilities of obtaining the outcome \( (m,n,l) \) is then given by:

\[
P(X_1^i = m, X_2^i = n, X_3^i = l) = |\langle mnl|U(\vec{\varphi} X_1^i) \otimes U(\vec{\varphi} X_2^i) \otimes U(\vec{\varphi} X_3^i)|\psi\rangle|^2
\]

Substituting Eq. (8) into the inequality (5), one get the expression of \( I_d^{[3]} \) in quantum mechanics. For the generalized GHZ state of three qubits:

\[
|\psi_2^3\rangle = \cos \theta |000\rangle + \sin \theta |111\rangle,
\]

numerical results show that when we set \( \theta = \pi/4 \), \( \varphi X_1^1 = (0, -\pi/12), \varphi X_1^2 = (0, \pi/4), \varphi X_1^3 = (0, -\pi/6), \varphi X_2^1 = (0, \pi/3), \varphi X_3^1 = (0, 0), \varphi X_3^2 = (0, \pi/6) \), we get the maximal violation \( 2\sqrt{2} \), which is the same of the maximal violation of CHSH inequality for two qudits. For \( \theta \in (0, \pi/8) \), the state (3) does not violate the inequality. To measure the strength of violation of local realistic theories, we may consider the mixed state \( \rho(F) = (1-F)|\psi_2^3\rangle\langle \psi_2^3| + F/2 I \otimes I \otimes I \), where \( F \) is the amount of the noise present in the system [21] and \( I \) is a \( 2 \times 2 \) identity matrix. According to the proposal introduced in Ref. [21], there exists some threshold value of \( F \), denoted by \( F_{thr} \), such that for every \( F \leq F_{thr} \), local and realistic description does not exist. For inequality (5), the threshold fidelity is 0.29289, which is smaller than 1/2, the threshold fidelity for MABK inequality for three qubits. This indicate that our inequality is not as strong as the MABK inequality. Another set of states considered are the generalized W states: \( |\psi_2^3\rangle_W = \sin \beta |001\rangle + \sin \beta \cos \xi |010\rangle + \cos \beta |100\rangle \). The maximal violation of this set of states is also \( 2\sqrt{2} \). This result is surprising since for the previous inequalities, the violations of the generalized W states are always smaller than that of the GHZ states. Moreover, inequality (5) is relevant to three-qubit MABK inequality, i.e., there exist states which violate inequality (5) but do not violate the MABK inequality. For instance, one may check that the state: \( |\Psi\rangle = 0.169414(000) + 0.0461331(100) + 0.161369(010) + 0.193624(110) + 0.951652(111) \) do not violate the MABK inequality but it does violate inequality (5), and the violation is 2.00382.

For the generalized GHZ state of three qutrits:

\[
|\psi_3^3\rangle = \sin \theta_1 \sin \theta_2 |000\rangle + \sin \theta_1 \cos \theta_2 |111\rangle + \cos \theta_1 |222\rangle,
\]

numerical results shows that when we set \( \theta_1 = 0.9066, \theta_2 = 0.6663, \varphi X_1^1 = (0, -\pi/5, \pi/24), \varphi X_1^2 = (0, \pi/24, -5\pi/12), \varphi X_1^3 = (0, 0, \pi/12), \varphi X_2^1 = (0, \pi/3, -\pi/4, \pi/20), \varphi X_3^1 = (0, \pi/3, -\pi/4, \pi/20), \varphi X_3^2 = (0, \pi/8, \pi/6) \), we get the maximal violation 2.915, which is the same of the maximal violation of the CGLMP inequality for two qutrits. On the other hand, for the maximal entangled state for three qutrits, namely \( \theta_2 = \pi/4, \theta_1 = \arccos(1/\sqrt{3}) \), the quantum violation is 2.873, which is smaller than 2.915. This indicts that the maximal violation of inequality (5) occurs at the nonmaximally entangled state. For higher dimensions, our numerical results show that the maximal violation is similar to the CGLMP inequality and the inequality (5) is also relevant to the inequalities presented in Ref. [10].

The Bell inequalities can be easily generalized for arbitrary N-qudit systems. The correlation functions in this
case are in the following form:

\[ Q_{i_1, \ldots, i_N} = \frac{1}{2} \sum_{x_1[i_1]=0}^{d-1} \cdots \sum_{x_N[i_N]=0}^{d-1} f^{i_1 \cdots i_N}(x_1^{[i_1]}, \ldots, x_N^{[i_N]}) \times P(X_1^{[i_1]} = x_1^{[i_1]}, \ldots, X_N^{[i_N]} = x_N^{[i_N]}), \]

where \( S = (d - 1)/2 \), \( f^{i_1 \cdots i_N}(x_1^{[i_1]}, \ldots, x_N^{[i_N]}) = S - M((-1)^\chi(\sum_{j=1}^N x_j^{[i_j]}), d) \), and \( \chi = \prod_{j=1}^N i_j \).

Based on these correlation functions, the tight Bell inequality can be written as:

\[ I_{d}^{[2N]} = Q_{1 \cdots 1} + Q_{121 \cdots 12} + Q_{212 \cdots 21} - Q_{2 \cdots 2} \leq 2, \quad (10) \]
\[ I_{d}^{[2N+1]} = Q_{1 \cdots 1} + Q_{121 \cdots 21} + Q_{212 \cdots 12} - Q_{2 \cdots 2} \leq 2. \]

Using the same method as for the case of three qudits, one may check that the above inequalities (10) are valid for local hidden variable theory and they are tight.

For instance, we give two tight Bell inequalities for four qudits:

\[ I_{d}^{[4]} = Q_{1111} + Q_{1212} + Q_{2121} - Q_{2222} \leq 2, \quad (11) \]
\[ I_{d}^{[5]} = Q_{11111} + Q_{12121} + Q_{21212} - Q_{22222} \leq 2. \]

Numerical results show that when \( d = 2 \), the inequality (11) is maximally violated by the maximally entangled state: \( |\psi_2^{+}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \) if we set \( \varphi_{X_1}^{[1]} = (0, \pi/24), \varphi_{X_2}^{[2]} = (0, \pi/12), \varphi_{X_3}^{[3]} = (0, -\pi/6), \varphi_{X_4}^{[4]} = (0, \pi/3), \varphi_{X_5}^{[5]} = (0, 0) \), and \( \varphi_{X_6}^{[6]} = (0, 0) \). The violation is \( 2\sqrt{2} \), which is the same as that of inequality (3). Another example is the tight Bell inequality for five qudits:

\[ I_{d}^{[5]} = Q_{11111} + Q_{12121} + Q_{21212} - Q_{22222} \leq 2. \]

Numerical results show that when \( d = 2 \), the inequality (12) is maximally violated by the maximally entangled state: \( |\psi_2^{+}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |11111\rangle) \) when we set \( \varphi_{X_1}^{[1]} = (0, -\pi/12), \varphi_{X_2}^{[2]} = (0, -\pi/6), \varphi_{X_3}^{[3]} = (0, 0), \varphi_{X_4}^{[4]} = (0, 0), \varphi_{X_5}^{[5]} = (0, 0), \) and \( \varphi_{X_6}^{[6]} = (0, 0) \). The violation is also \( 2\sqrt{2} \).

From the quantum violations of inequalities (3), (11) and (12), we find that, different from the MABK, the quantum violations of our inequalities remain the same, rather than increase, with the increasing number of particles.

In summary, we have presented generic tight Bell inequalities for arbitrary \( N \)-qudit systems based on the generalized correlation functions. The new inequalities take the same simple form as the CHSH inequality and when \( N = 2 \) they reduce to the well-known CGLMP inequality. The new inequalities are not as strong as the MABK inequality and there exist some pure states that do not violate these inequalities, while they are the first tight general Bell inequalities for arbitrary \( N \)-qudit systems and they are relevant to the previous known Bell inequalities. Frankly speaking, we do not have a general proof of the tightness of these new inequalities. Indeed, we have only checked that for small systems (namely, three qudits for \( d \leq 10 \), four qudits for \( d \leq 7 \), and five qudits for \( d \leq 5 \)). Unfortunately, we have to leave this as an open question here and we shall investigate it subsequently.

Since the various use of Bell inequality in quantum information, our results may be very useful for the study of other Bell inequalities, quantum entanglement measurement, distillation protocols, etc.

This work was supported in part by NSF of China (Grant No. 10605013), Program for New Century Excellent Talents in University, and the Project-sponsored by SRF for ROCS, SEM.

* Electronic address: chenjl@nankai.edu.cn

[1] J. S. Bell, Physics (Long Island City, N. Y.) 1, 195 (1964).
[2] J. Clauser, M. Horne, A. Shimony, R. Holt, Phys. Rev. Lett. 23, 880 (1969).
[3] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. 88, 040404 (2002).
[4] A. Peres, Found. Phys. 29, 589 (1999).
[5] L. Masanes, Quant. Inf. Compt. 3, 345 (2002).
[6] L. B. Fu, Phys. Rev. Lett. 92, 130404 (2004).
[7] S. Zohren and R. D. Gill, Phys. Rev. Lett. 100, 120406 (2008).
[8] J. L. Chen, C. F. Wu, L. C. Kwek, C. H. Oh, and M. L. Ge, Phys. Rev. A 74, 032106 (2006).
[9] S. W. Lee and D. Jaksch, arxiv:quant-ph/0803.3097v1.
[10] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
[11] M. Ardehali, Phys. Rev. A 46, 5375 (1992).
[12] A. V. Belinskii and D. N. Klyshko, Phys. Usp. 36, 653 (1993).
[13] V. Scarani and N. Gisin, J. Phys. A 34, 6043 (2001).
[14] M. ˙Zukowski, ˇC. Brukner, W. Laskowski, and M. Wiesniak, Phys. Rev. Lett. 88, 210402 (2002).
[15] J. L. Chen, C. F. Wu, L. C. Kwek, and C. H. Oh, Phys. Rev. Lett. 93, 140407 (2004).
[16] J. L. Chen, C. F. Wu, L. C. Kwek, and C. H. Oh, arxiv:quant-ph/0506250v1.
[17] V. Scarani, and N. Gisin, Phys. Rev. Lett. 87, 117901 (2001); A. Acín, N. Gisin, and L. Masanes, Phys. Rev. Lett. 97, 120405 (2006); A. Acín, N. Gisin, and V. Scarani, Quant. Inf. Compt. 3, 563 (2003).
[18] Č. Brukner, M. ˙Zukowski, J. W. Pan, and A. Zeilinger, Phys. Rev. Lett. 92, 127901 (2004); Č. Brukner, M. ˙Zukowski, and A. Zeilinger, Phys. Rev. Lett. 89, 197901 (2002).
[19] A. Acín, N. Gisin, L. Masanes, and V. Scarani, Int. J. Quant. Inf. 2, 23 (2004).
[20] M. ˙Zukowski, A. Zeilinger, M. A. Horne, Phys. Rev. A 55, 2564 (1997).
[21] D. Kaszlikowski, P. Gniaciński, M. ˙Zukowski, W. Miklaszewski, and A. Zeilinger, Phys. Rev. Lett. 85, 4418 (2000).