THE NEARLY SINGULAR BEHAVIOR OF THE 3D NAVIER–STOKES EQUATIONS

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ABSTRACT. Whether the 3D incompressible Navier–Stokes equations can develop a finite time singularity from smooth initial data is one of the most challenging problems in nonlinear PDEs. In this paper, we present some new numerical evidence that the 3D incompressible axisymmetric Navier–Stokes equations with smooth initial data of finite energy develop nearly singular solutions at the origin. This nearly singular behavior is induced by a potential finite time singularity of the 3D Euler equations that we reported in [20]. One important feature of the potential Euler singularity is that the solution develops nearly self-similar scaling properties that are compatible with those of the 3D Navier–Stokes equations. We will present numerical evidence that the 3D Navier–Stokes equations develop nearly singular scaling properties with maximum vorticity increased by a factor of $10^7$. Moreover, the nearly self-similar profiles seem to be very stable to the small perturbation of the initial data. However, the 3D Navier–Stokes equations with our initial data do not develop a finite time singularity due to the development of a mild two-scale structure in the late stage, which eventually leads to viscous dominance over vortex stretching. To maintain the balance between the vortex stretching term and the diffusion term, we solve the 3D Navier–Stokes equations with a time-dependent viscosity roughly of order $O(\log(T-t)^{-3})$ in the late stage. We present strong numerical evidence that the 3D Navier–Stokes equations with such time-dependent viscosity develop a finite time singularity.

1. Introduction

The three-dimensional (3D) incompressible Navier–Stokes equations in fluid dynamics govern the motion of viscous incompressible flows. They have been used to model ocean currents, weather patterns, and other fluids related phenomena. The question regarding the global regularity of the 3D Navier–Stokes equations with smooth initial data of finite energy is one of the most important fundamental questions in nonlinear partial differential equations and is one of the seven Millennium Problems posted by the Clay Mathematics Institute [17]. The main difficulty associated with the global regularity of the 3D Navier–Stokes equations is that the nonlinearity due to vortex stretching is super-critical. A closely related problem is the global regularity of the 3D incompressible Euler equations [36]. It is generally believed that the 3D Euler equations are more likely to develop a finite time singularity than the 3D Navier–Stokes equations due to the lack of viscous regularization.

In this paper, we present numerical evidence that the 3D incompressible axisymmetric Navier–Stokes equations with smooth initial data of finite energy develop nearly singular solutions at the origin. The nearly singular behavior of the Navier–Stokes equations is induced by the potential finite time singularity of the 3D Euler equations that we reported in a companion paper [20]. This singularity scenario is different from the Luo-Hou scenario [34, 35], which occurs on the boundary. An important feature of our new blowup scenario is that the solution of the 3D Euler equations develops nearly self-similar scaling properties that are compatible with those of the 3D Navier–Stokes equations. To investigate whether the 3D Navier–Stokes equations would develop potential singularity, we solve the Navier–Stokes equations with a relatively large viscosity $\nu = 5 \cdot 10^{-3}$. Surprisingly, this relatively strong viscous regularization enhances nonlinear alignment of vortex stretching and the 3D Navier–Stokes equations develop nearly singular scaling properties with maximum vorticity increased by a factor of $10^7$. To the best of our knowledge, such a large growth rate of maximum vorticity has not been reported in the literature for the 3D Navier–Stokes equations.
Despite the rapid growth of the maximum vorticity and other quantities, the solution of the Navier–Stokes equations develops a mild two-scale structure in the late stage, which eventually leads viscous dominance over vortex stretching and prevents the Navier–Stokes equations from blowing up in a finite time. To maintain the balance between vortex stretching and diffusion, we solve the Navier–Stokes equations with a time-dependent viscosity roughly of order $O(\log(T - t))^{-3}$ in the late stage. Our numerical results suggest that the 3D Navier–Stokes equations with such time-dependent viscosity develop a finite time singularity with maximum vorticity increased by a factor of $10^9$.

A potential blowup of the Navier–Stokes equations with such time-dependent viscosity raises an interesting question whether the Navier–Stokes equations with a fractional diffusion operator slightly weaker than the standard diffusion operator may develop a finite time singularity. If this is the case, the non-blowup of the 3D Navier–Stokes equations for our initial condition would be a borderline case. Due to the nonlocal nature of the fractional diffusion operator, it is much more difficult to design a fast algorithm using adaptive mesh to solve the Navier–Stokes equations with a fractional diffusion operator. We choose to solve the Navier–Stokes equations with such time-dependent viscosity as an alternative approach to study the competition between the destabilizing effect of the vortex stretching term and the stabilizing effect of the diffusion term.

We consider the 3D axisymmetric Navier–Stokes equations in a periodic cylindrical domain. We impose a no-slip no-flow boundary condition at $r = 1$ for the Navier–Stokes equations. We use a periodic boundary condition in the axial variable $z$ with period $1$. Let $u^\theta$, $\omega^\theta$, and $\psi^\theta$ be the angular components of the velocity, the vorticity, and the vector stream function, respectively. In [23], Hou and Li introduced the following change of variables:

$$u_1 = u^\theta / r, \quad \omega_1 = \omega^\theta / r, \quad \psi_1 = \psi^\theta / r,$$

and transformed the Navier–Stokes equations into the form

$$u_{1,r} + u^r u_{1,r} + u^z u_{1,z} = 2u_1 \psi_{1,z} + \nu \left( u_{1,rr} + \frac{3}{r} u_{1,r} \right) + \nu u_{1,zz}, \quad (1.1a)$$

$$\omega_{1,r} + u^r \omega_{1,r} + u^z \omega_{1,z} = 2u_1 u_{1,z} + \nu \left( \omega_{1,rr} + \frac{3}{r} \omega_{1,r} \right) + \nu \omega_{1,zz} \quad (1.1b)$$

$$- \left( \frac{\partial^2}{r^2} + \frac{3}{r} \frac{\partial}{r} + \frac{\partial^2}{z^2} \right) \psi_1 = \omega_1, \quad (1.1c)$$

where $u^r = -r \psi_1, \quad u^z = 2\psi_1 + r \psi_{1,r}$. We will solve the 3D Navier–Stokes equations using the initial condition defined in (2.3). We choose a constant viscosity $\nu = 5 \cdot 10^{-4}$ from $t = 0$ to $t_0 = 0.00227375$, then increase the viscosity to $\nu = 5 \cdot 10^{-3}$ for $t \geq t_0$.

### 1.1. The near singular behavior of the Navier–Stokes equations

The solution of the Navier–Stokes equations with our initial condition shares several attractive features of the two-scale traveling wave solution studied by Hou-Huang in [21]. In particular, the oddness of angular vorticity induce two antisymmetric (with respect to $z$) vortex dipoles, which generate a hyperbolic flow structure near the symmetry axis $r = 0$. The antisymmetric vortex dipoles produce a strong shear layer for the axial velocity, which transports the solution towards $z = 0$. Moreover, the 2D velocity field $(u^r(t), u^z(t))$ in the $rz$-plane forms a closed circle right above $(R(t), Z(t))$. The corresponding streamlines are trapped in the circle region in the $rz$-plane and a significant portion of $u_1$ and $\omega_1$ are retained in this circle region instead of being transported upward along the $z$-direction. This is an important mechanism that leads to a sustainable growth of the solution. The induced antisymmetric local convective circulation further pushes the solution near $z = 0$ towards the symmetry axis $r = 0$.

One interesting feature of our initial condition is that $\psi_{1,z}$ becomes large, positive and relatively flat in a local region near the origin. This local region is characterized by a rectangular domain with $0 \leq r \leq 0.9 R(t)$ and $0 \leq z \leq 0.5 Z(t)$. Beyond this local domain, $\psi_{1,z}(r, z, t)$ decays rapidly and
becomes negative near the tail region. The large value of $\psi_{1,z}$ in this local region near the origin generates a large growth of $u_1$ through the vortex stretching term and the small or negative value of $\psi_{1,z}$ in the tail region generates a slower growth rate of $u_1$. This difference in the growth rates in the local region and the tail region produces a one-scale traveling wave solution approaching the origin. The traveling wave induced by the strong nonlinear alignment of vortex stretching is so strong that it overcomes the stabilizing effect of advection in the upward $z$-direction (see e.g. [23, 22]).

Although the angular vorticity is set to zero initially, the oddness of $u_1$ in $z$ generates a large positive gradient $u_{1,z}$, which contributes positively to the rapid growth of $\omega_1$ through the vortex stretching term $2u_1u_{1,z}$ in (1.1b). The rapid growth of $\omega_1$ in turn feeds back to the rapid growth of $\psi_{1,z}$, leading to even faster growth of $u_{1,z}^2$. The whole coupling mechanism described above forms a positive feedback loop.

After a short transition time, the solution of the 3D Navier–Stokes equations seems to develop nearly singular scaling properties. Denote by $(R(t), Z(t))$ the location in the $rz$ plane at which $u_1$ achieves its maximum. If we introduce $\xi = (r - R(t))/Z(t)$ and $\zeta = z/Z(t)$ as the dynamically rescaled variables, we observe that the rescaled solutions of the 3D Navier–Stokes equations have nearly self-similar profiles in terms of $(\xi, \zeta)$. Moreover, the nearly self-similar profiles are very stable with respect to the small perturbation of the initial data. Our study also indicates that $Z(t)$ seems to scale like $O((T-t)^{1/2})$, which is consistent with the scaling properties of the Navier–Stokes equations.

During the rapid transient growth period, the solution of the 3D Navier–Stokes equations develops nearly self-similar scaling properties. Both $R(t)$ and $Z(t)$ scale roughly like $O((T-t)^{1/2})$. The maximum vorticity seems to grow like $O((T-t)^{-1})$ and the maximum velocity grows like $O((T-t)^{-1/2})$. Another quantity of interest is pressure $p$. According to [39], if there is a blowup for the Navier–Stokes equations, the pressure must tend to negative infinity at the singularity time. Our study seems to indicate that $\| - p \|_{L^\infty}$ grows like $O((T-t)^{-1})$. Moreover, $\| \frac{1}{2} |u|^2 + p \|_{L^\infty}$ seems to scale like $O((T-t)^{-1})$. This provides support for the nearly singular behavior of the 3D Navier–Stokes equations.

1.2. The non-blowup of the Navier-Stokes equations with our initial condition. Despite the nearly singular behavior of maximum vorticity, maximum velocity and the negative pressure, the Navier–Stokes equations with our initial condition do not develop a finite time blowup. The reason for the non-blowup is due to the development of a mild two-scale structure roughly in the form $R(t)/Z(t) \sim |\log(T-t)|^{-1/2}$ in the late stage. Such mild two-scale structure eventually leads to viscous dominance over vortex stretching and prevents a finite time singularity. When we apply the non-blowup criteria in terms of $\|u\|_{L^{3,\infty}}$ [16], we observe that $\|u(t)\|_{L^3}$ first experiences a mild transient growth and then decreases in the late stage. This provides a strong evidence that the 3D Navier–Stokes equations with our initial condition do not develop a finite time singularity. We remark that the non-blowup criterion using the $\|u\|_{L^{3,\infty}}$ estimate is based on a compactness argument. As a result, the bound on $\max_{0 \leq t \leq T} \|u(t)\|_{L^3}$ does not provide a direct estimate on the dynamic growth rate of the solution (e.g. the maximum vorticity) of the Navier–Stokes equations during the time interval $[0, T]$.

To provide further evidence for the non-blowup of the Navier–Stokes equations with our initial condition, we examine the growth rate of the enstrophy $\|\omega(t)\|_{L^2}^2$. A linear fitting seems to suggest that the enstrophy grows roughly like $O((T-t)^{-1/2})$. Moreover, $\int_0^T \|\omega(s)\|_{L^2}^4 ds$ experiences a rapid growth. However, a more careful study of $\int_0^T \|\omega(s)\|_{L^2}^4 ds$ using a rescaled time reveals that $\int_0^T \|\omega(s)\|_{L^2}^4 ds$ seems to saturate around $1.6 \cdot 10^8$. A standard energy estimate implies that if $\int_0^T \|\omega(s)\|_{L^2}^4 ds$ is bounded up to time $T$, the solution of the Navier–Stokes equations will remain regular up to time $T$.

In theory, all non-blowup criteria are equivalent. If the $\|u(t)\|_{L^3}$ is bounded in time, all other quantities, including the maximum vorticity, the maximum velocity and negative pressure should not blow up in a finite time. Therefore, the growth rates of the maximum vorticity, the maximum velocity and negative pressure based on linear fitting reflect only the rapid transient growth of these quantities. If we could afford to use a much finer mesh to push the computation further in time, the growth of...
these quantities should eventually saturate, just as we observed for the growth rate of the enstrophy. The fact that the solution of the Navier–Stokes equations with our initial condition can experience such a rapid transient growth and yet avoid forming a finite time singularity seems quite surprising.

1.3. Navier–Stokes equations with a time-dependent viscosity. To avoid viscous dominance, we solve the Navier–Stokes equations with a time-dependent viscosity in the late stage. More precisely, we introduce a time-dependent factor $g(t) = \|\psi_{1,z}(t)\|_\infty^2 Z(t)^2$. We first solve the 3D Navier–Stokes equations using our initial condition with a constant viscosity described in the previous subsections. At the time $t = t_1$ when the maximum vorticity has grown by a factor of $10^6$, we change the viscosity from $\nu = 5 \cdot 10^{-3}$ to $\nu = 5 \cdot 10^{-3} g(t)/g(t_1)$. A simple scaling analysis suggests that $\|\psi_{1,z}\|_\infty$ should scale like $O((T - t)^{-1})$ if there is a locally self-similar blowup. This is also consistent with our numerical results. If $Z(t) = O((T-t)^{1/2})$, then $g(t)$ should remain $O(1)$. However, we observe that $Z(t)$ seems to decay faster that $O((T-t)^{1/2}$ by a factor of $(\log(1/(T-t)))^{3/2}$ in the late stage. Our numerical fitting also indicates that $g(t)$ decays roughly like $O(\log(T-t)^{1/3})$ in the late stage. A simple scaling analysis indicates that the decaying factor $g(t) = \|\psi_{1,z}(t)\|_\infty^2 Z(t)^2$ would ensure the balance between the vortex stretching term and the viscous term.

The solution of the Navier–Stokes equations with such time-dependent viscosity and the solution of the Navier–Stokes equations with a constant viscosity share many similar features, including the scaling properties and the tornado singularity structure. One significant difference is that the vortex stretching term dominates the diffusion term throughout the computation for the Navier–Stokes equations with such time-dependent viscosity. This leads to a very stable and robust blowup with maximum vorticity increased by a factor more than $10^9$. We will present strong numerical evidence that the 3D Navier–Stokes equations with such time-dependent viscosity develop a finite time singularity. The time dependent viscosity $\nu(t)$ decreases very slowly in the late stage from $5 \cdot 10^{-3}$ at $t = t_1$ to $4.154 \cdot 10^{-4}$ by the end of the computation. Such a slow decay in the viscosity reflects the logarithmic nature of the decay rate, which tends to decay very slowly at the beginning. The decay of $\nu(t)$ will accelerate as we get closer to the potential singularity time and $\nu(t)$ will vanish at the potential singularity time.

1.4. Comparison with the two-scale traveling wave singularity [21]. Despite some similarity in the solution behavior between the new initial condition considered in this paper and the initial condition considered in [21], there are some important differences between the two potential blowup scenarios. One important difference is that our solutions essentially have a one-scale structure instead of a two-scale structure observed in [21]. Moreover, our solutions of the 3D Euler equations seem to develop nearly self-similar scaling properties that are compatible with those of the 3D Navier–Stokes equations. This important property is critical for the nearly singular behavior of the 3D Navier-Stokes equations. In comparison, the maximum vorticity of the 3D Navier–Stokes solution with a constant viscosity $\nu = 10^{-5}$ reported in [21] has grown by a factor less than 2.

Another important difference is that the solutions of the 3D Euler equations considered in [21] develops a three-scale structure and the thickness of the sharp front does not seem to settle down a stable scale. Without viscous regularization, we cannot resolve the Euler solution to obtain convincing blowup scaling properties of the Euler solution. It is essential to apply degenerate variable diffusion coefficients of order $O(r^2) + O(z^2)$ to select a two-scale solution structure with $R(t) \sim (T - t)^{1/2}$ and $Z(t) \sim (T - t)$. For the new initial condition considered in this paper, the Euler solution has essentially a one-scale structure and we can obtain nearly self-similar scaling properties of the potential blowup solution. Moreover, the Navier–Stokes equations with degenerate diffusion coefficients reported in [21] develop strong shearing instability in the tail region. We need to apply some low pass filtering to stabilize this shearing instability. In comparison, our solutions are very stable and do not suffer from the shearing instability in the tail region. Thus, there is no need to apply any low pass filtering.
1.5. **Numerical Methods.** We use a similar adaptive mesh strategy developed in [21] by constructing two adaptive mesh maps for $r$ and $z$ explicitly. The solutions of the Navier–Stokes equations with our initial data are much smoother than those considered in [21] due to the relative large viscous regularization. More details on how to construct the adaptive mesh will be provided in the Appendix. We use a second order finite difference method to discretize the spatial derivatives, and a second order explicit Runge–Kutta method to discretize in time. An adaptive time-step size is used according to the standard time-stepping stability constraint with the smallest time-step size of order $O(10^{-15})$. The overall method is second order accurate. We have performed careful resolution study and confirm that our method indeed gives second order accuracy in the maximum norm.

1.6. **Review of previous works.** For the 3D Navier–Stokes equations, the partial regularity result due to Caffarelli–Kohn–Nirenberg [3] is one of the best known results (see also [32]). This result implies that any potential singularity of the axisymmetric Navier–Stokes equations must occur on the symmetry axis. There have been some very interesting theoretical developments regarding the lower bound on the blow-up rate for axisymmetric Navier-Stokes equations [5, 4, 30]. Another interesting development is a result due to Tao [40] who proposed an averaged three-dimensional Navier–Stokes equation that preserves the energy identity, but blows up in finite time.

There have been a number of theoretical developments for the 3D incompressible Euler equations, including the Beale–Kato–Majda non-blowup criteria [36], the geometric non-blowup criteria due to Constantin–Fefferman–Majda [11] and its Lagrangian analog due to Deng-Hou-Yu [12]. In 2019, Elgindi [14] (see also [15]) proved an exciting result: the 3D axisymmetric Euler equations develop a finite time singularity for a class of $C^{1,\alpha}$ initial velocity with no swirl. There have been a number of interesting theoretical results inspired by the Hou–Lou blowup scenario [34, 35], see e.g. [29, 9, 10, 28, 6, 8, 7] and the excellent survey article [27].

There have been relatively few papers on the numerical study regarding the potential blowup of the 3D Navier–Stokes equations, although there were a number of attempts to look for potential Euler singularities numerically, see [19, 13, 26, 24, 34, 35, 2]. The work by Boratav and Pelz in [1] has generated some interests. By solving the 3D Navier-Stokes equations with increasing Reynolds numbers, they conjectured that the Euler equations with Kida’s high-symmetry initial data would develop a finite time singularity. In [25], Hou and Li reproduced the computation of [1] and found that the potential singularity reported in [1] is likely an artifact due to under-resolution. We refer to a review article [18] for more discussions on potential Euler singularities.

The rest of the paper is organized as follows. In Section 2, we describe the setup of the problem. In Section 3, we describe the nearly singular solutions of the 3D Navier–Stokes equations using a relatively large viscosity. Section 4 is devoted to the Navier–Stokes equations with a time-dependent viscosity. Some concluding remarks are made in Section 5. Some technical details regarding the construction of our adaptive mesh for the 3D Navier–Stokes equations will be deferred to the Appendix.

2. **Description of the Problem**

In this paper, we study the 3D incompressible Navier–Stokes equations. Define $\omega = \nabla \times u$ as the 3D vorticity vector. The 3D incompressible Navier–Stokes equations in the vorticity stream function formulation are given below:

$$\omega_t + u \cdot \nabla \omega = \omega \cdot \nabla u + \nu \Delta \omega.$$  \hspace{1cm} (2.1)

The velocity $u$ is related to the vorticity via the vector-valued stream function $\psi$:

$$-\Delta \psi = \omega, \quad u = \nabla \times \psi.$$

The vorticity equations have a simpler form for the axisymmetric Navier–Stokes equations. We decompose the radially symmetric velocity field as follows

$$u(t, r, z) = u^r(t, r, z)e_r + u^\theta(t, r, z)e_\theta + u^z(t, r, z)e_z.$$
\[ e_r = \frac{1}{r} (x, y, 0)^T, \quad e_\theta = \frac{1}{r} (-y, x, 0)^T, \quad e_z = (0, 0, 1)^T. \]

Let \( \omega^\theta \) be the angular vorticity and \( \psi^\theta \) be the angular stream function. By making the change of variables, \( u_1 = u^\theta/r, \ \omega_1 = \omega^\theta/r, \ \psi_1 = \psi^\theta/r \), Hou and Li [23] derived the following equivalent system of equations:

\[ u_{1,t} + u^r u_{1,r} + u^z u_{1,z} = 2u_1 \psi_1 + \nu \left( u_{1,rr} + \frac{3}{r} u_{1,r} \right) + \nu u_{1,zz}, \quad (2.2a) \]

\[ \omega_{1,t} + u^r \omega_{1,r} + u^z \omega_{1,z} = 2u_1 u_{1,z} + \nu \left( \omega_{1,rr} + \frac{3}{r} \omega_{1,r} \right) + \nu \omega_{1,zz} \quad (2.2b) \]

\[- \left( \frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \psi_1 = \omega_1, \quad (2.2c) \]

\[ u^r = -r \psi_{1,z}, \quad u^z = 2 \psi_1 + r \psi_{1,r}. \quad (2.2d) \]

This reformulation has the advantage of removing the \( 1/r \) singularity from the cylindrical coordinates.

Our smooth initial condition has a very simple form and is given below:

\[ u_1(0, r, z) = \frac{12000(1 - r^2)^{18} \sin(2\pi z)}{1 + 12.5 \sin(\pi z)^2}, \quad \omega_1(0, r, z) = 0. \quad (2.3) \]

The nontrivial part of the initial data lies only in the angular velocity \( u^\theta = ru_1 \). Thus the flow is completely driven by large swirl initially. Note that \( u_1 \) is an odd and periodic function of \( z \) with period 1. The oddness of \( u_1 \) induces the oddness of \( \omega_1 \) dynamically although \( \omega_1 \) is set to zero initially. Note that \( u_1 \) decays rapidly as \( r \) approaches the boundary \( r = 1 \). The denominator breaks the even symmetry of \( \sin(2\pi z) \) with respect to \( z = 1/4 \) along the \( r \) direction with a bias towards \( z = 0 \). This specific form of the initial condition is designed in such a way that the solution has comparable scales along the \( r \) and \( z \) directions, leading to a one-scale traveling solution moving towards the origin. We will show in Section 3.6 that the solution behavior is very stable to a small perturbation of the initial data.

Our initial condition does not favor alignment of vortex stretching in the very early stage. In fact, the maximum of \( u_1 \) actually decreases in the very early stage. After a short transition time, some favorable solution structure develops dynamically and we observe strong nonlinear alignment of vortex stretching. The nonlinear alignment of vortex stretching remains strong throughout the computation.

We will solve the transformed equations (2.2)-(2.3) in a periodic cylinder

\[ \mathcal{D} = \{(r, z) : 0 \leq r \leq 1 \}. \]

We will impose a periodic boundary condition in \( z \) with period 1 and the odd symmetry of \( u_1, \omega_1 \) and \( \psi_1 \) as a function of \( z \). According to [33], \( u^\theta, \omega^\theta, \psi^\theta \) must be an odd function of \( r \) as long as the solution remains smooth. Consequently, \( u_1, \omega_1, \psi_1 \) must be even functions of \( r \), which imposes the following pole conditions:

\[ u_{1,r}(t, 0, z) = \omega_{1,r}(t, 0, z) = \psi_{1,r}(t, 0, z) = 0. \quad (2.4) \]

For the Navier–Stokes equations, the velocity satisfies a no-slip no-flow boundary condition on the solid boundary \( r = 1 \). The no-flow boundary condition is given by

\[ \psi_1(t, 1, z) = 0 \quad \text{for all } z, \quad (2.5) \]

and the no-slip boundary condition is given by:

\[ u^\theta(t, 1, z) = u^z(t, 1, z) = 0, \quad \text{for all } z. \quad (2.6) \]
In view of (2.2d) and (2.5), this further leads to \( \psi_{1,r}(t, 1, z) = 0 \). Therefore, the no-slip boundary in terms of the new variables \( u_1, \omega_1, \psi_1 \) reads
\[
  u_1(t, 1, z) = 0, \quad \omega_1(t, 1, z) = -\psi_{1,r}(t, 1, z), \quad \text{for all } z.
\]
(2.7)
We will enforce the no-slip boundary condition (2.7) numerically. The periodicity and the odd symmetry of the solution imply that we only need to solve equations (2.2) in the half-period domain
\[
  \mathcal{D}_1 = \{(r, z) : 0 \leq r \leq 1, 0 \leq z \leq 1/2\},
\]
and \( u^r \) and \( u^z \) satisfy the following conditions
\[
  u^r = -r\psi_{1,z} = 0 \quad \text{on } r = 0, 1 \quad \text{and} \quad u^z = 2\psi_1 + r\psi_{1,r} = 0 \quad \text{on } z = 0, 1/2.
\]
Thus the boundaries of \( \mathcal{D}_1 \) behave like “impermeable walls”. To numerically compute the potential singularity formation of the equations (2.2)-(2.3), we adopt the numerical methods developed in my recent joint work with Dr. De Huang [21]. In particular, we design an adaptive mesh by constructing two adaptive mesh maps for \( r \) and \( z \) explicitly. The computation is performed in the transformed domain using a uniform mesh. When we map back to the physical domain, we obtain a highly adaptive mesh with the smallest mesh size of order \( O(10^{-10}) \). We will provide more details how to construct the adaptive mesh for the 3D Euler and Navier–Stokes equations in the Appendix. The detailed descriptions of the overall numerical methods can be found in Appendix A in [21].

3. Nearly singular behavior of the 3D Navier–Stokes equations

In this section, we will investigate the nearly singular behavior of the 3D Navier–Stokes equations. It turns out that the choice of the viscosity coefficient plays a crucial role in generating a stable and sustainable growth of the maximum vorticity. On one hand, if the viscosity is too large, it would destroy the mechanism that leads to the potential Euler singularity. On the other hand, if the viscosity is too small, then it is not strong enough to stabilize the shearing instability generated by the 3D Euler equations. After performing many experiments, we find that first solving the Navier–Stokes equations with viscosity \( \nu = 5 \cdot 10^{-4} \) up to a short time \( t = 0.00227375 \) and then increasing \( \nu \) to \( 5 \cdot 10^{-3} \) seem to give the nearly optimal growth rate. Using \( \nu = 5 \cdot 10^{-4} \) for the early stage from \( t = 0 \) to \( t = 0.00227375 \) enables us to preserve the main mechanism leading to the potential Euler singularity. Using a larger viscosity \( \nu = 5 \cdot 10^{-3} \) beyond \( t = 0.00227375 \) enables us to stabilize the fluid dynamic instability induced by the 3D Euler equations. This choice of viscosity coefficient produces a relatively long stable phase of nonlinear alignment of vortex stretching and nearly self-similar scaling properties.

3.1. Numerical Results. We have numerically solved the 3D axisymmetric Navier–Stokes equations (2.2)-(2.3) on the half-period cylinder \( \mathcal{D}_1 = \{(r, z) : 0 \leq r \leq 1, 0 \leq z \leq 1/2\} \) using meshes of size \( (n_1, n_2) = (256p, 256p) \) for \( p = 2, 3, \ldots, 6 \). In this subsection, we first present the major features of the nearly singular solution of the Navier–Stokes equations using our initial condition. In Section 3.3, we carry out a careful resolution study of the numerical solutions. Then we investigate the nearly singular scaling properties in Section 3.4.

3.1.1. Profile evolution. In this subsection, we investigate how the profiles of the solution evolve in time. We will use the numerical results computed on the adaptive mesh of size \( (n_1, n_2) = (1536, 1536) \). We have computed the numerical solution up to time \( t = 0.0022768453 \) when it is still well resolved.

In Figure 3.1, we present the 3D solution profiles of \( u_1, \omega_1 \) at 3 different times \( t_1 = 0.0022861547, t_2 = 0.0022867812 \) and \( t_3 = 0.0022868453 \). We can see that the magnitudes of \( u_1, \omega_1 \) grow very rapidly in time. The maximum vorticity has increased by a factor of \( 10^4, 10^5 \) and \( 10^6 \), respectively at these three instants. We observe that the singular support of the profiles travels towards the origin (note that the domain size and the amplitude have been rescaled). Due to the strong viscous regularization, the profile of \( u_1 \) remains relatively smooth near \((R(t), Z(t))\), the maximum location of \( u_1 \). Moreover,
the thin structure for \( \omega_1 \) that we observed for the 3D Euler equations in [20] becomes much smoother. The tail part of \( u_1 \) and \( \omega_1 \) is quite smooth and decays rapidly into the far field. This is quite different from the long tail that we observed for the two-scale traveling wave singularity reported in [21].

3.2. **Trajectory and alignment.** The maximum location \((R(t), Z(t))\) of \( u_1 \) lies on \( r = 0 \) initially and travels downward. Then it moves outward almost horizontally. After a short time, \((R(t), Z(t))\) turns around and propagates towards the origin, see Figure 3.2 (a). Figure 4.4(a) gives a global view of the trajectory. Due to the viscous regularization, the solution becomes smoother and is more stable. We are able to compute up to a time when \((R(t), Z(t))\) is very close to the origin. This is something we could not achieve for the 3D Euler equations [20]. We also observe a positive alignment between \( \psi_{1z} \) and \( u_1 \) and the alignment becomes stronger and stronger in time, see Figure 3.2 (c)-(d). It is worth noting that there is a relatively long stable phase in which the alignment between \( u_1 \) and \( \psi_{1z} \) is relatively flat. This indicates that the 3D Navier–Stokes equations enter a relatively stable scaling relationship period. This is also something that we did not observe for the 3D Euler equations [20].

In the Figure 3.2 (b), we observe that the ratio \( R(t)/Z(t) \) decays most of the time and has a very mild growth in the late stage of our computation. This property is very different from the two-scale traveling wave singularity reported in [21]. In our case, we have an essentially one-scale solution structure for the majority of the time. The mild growth of \( R(t)/Z(t) \) in the late stage signals the formation of a mild two-scale structure.

3.2.1. **Rapid growth.** We observe that there is a rapid growth of the solution and report the maximum of \( |u_1|, |\omega_1| \) and \( |\omega| \) as a function of time in Figure 3.3. We can see that these variables grow extremely rapidly in time. In the second row in Figure 3.3, we plot that the double logarithm of \( ||u_1||_{L^\infty}, ||\psi_{1z}||_{L^\infty} \) and \( ||\omega||_{L^\infty} \) as a function of time. We can see that the solution grows much faster than a double-exponential rate. Compared with the growth of the corresponding quantities for the 3D Euler equations, the growth rate of these quantities for the 3D Navier–Stokes equations is much larger in magnitude and is more stable.
Figure 3.2: (a) the trajectory of \((R(t), Z(t))\) approaching to the origin, (b) the ratio \(R(t)/Z(t)\), (c) the alignment between \(\psi_{1z}/u_1\) and \(\psi_{1x}\) in the second stage with \(0.00227375 \leq t \leq 0.0022866\). (d) the alignment \(\psi_{1z}/u_1\) in the late stage with \(0.0022866 \leq t \leq 0.002286845\).

We also compute the relative growth of maximum vorticity \(\|\omega(t)\|_\infty/\|\omega(0)\|_\infty\) and \(\int_0^t \|\omega(s)\|_\infty\,ds\) in Figure 3.4. The final time of this computation is at \(t = 0.0022868502\). We observe that \(\|\omega(t)\|_\infty/\|\omega(0)\|_\infty\) has increased by a factor of \(10^7\) by the end of the computation. To best of our knowledge, such a large growth rate of the maximum vorticity has not been reported for the 3D incompressible Navier–Stokes equations in the literature. The rapid growth of \(\int_0^t \|\omega(s)\|_\infty\,ds\) provides another evidence that the 3D Navier–Stokes equations develop nearly singular behavior.

3.2.2. Velocity field. In this subsection, we investigate the feature of the velocity field. We first study the 3D velocity field by looking at the induced streamlines. In Figure 3.5, we plot the streamlines induced by the velocity field \(u(t)\) at a relatively early time \(t = 0.00227375\). This is the time by which we end our computation using viscosity \(\nu = 5 \cdot 10^{-4}\) and switch to the larger viscosity \(\nu = 5 \cdot 10^{-3}\). By this time, the ratio between the maximum vorticity and the initial maximum vorticity, i.e. \(\|\omega(t)\|_\infty/\|\omega(0)\|_\infty\), has increased by a factor of \(498.42\) only. We can also regard the solution at \(t = 0.00227375\) as the new initial condition for the subsequent computation using the larger viscosity \(\nu = 5 \cdot 10^{-3}\).

Interestingly the induced streamlines look qualitatively the same as those obtained for the 3D Euler equations [20]. In Figure 3.5, we plot the streamlines induced by the velocity field \(u(t)\) at \(t = 0.00227375\) in a macroscopic scale (the whole cylinder domain \(D_1 \times [0, 2\pi]\)) for different initial points with (a) \((r_0, z_0) = (0.8, 0.2)\), (b) \((r_0, z_0) = (0.5, 0.1)\), and (c)-(d) \((r_0, z_0) = (0.1, 0.01)\). The velocity field resembles that of a tornado spinning around the symmetry axis (the green pole). In Figure 3.5(a), we observe that the streamlines from the starting point first spin around the symmetry axis near \(z = 0\), at the same time travel towards the symmetry axis. As they get very close to the symmetry axis, they move upward towards the \(z = 1/2\) (the “ceiling”) and then move away from...
Figure 3.3: First row: the growth of $\|u_1\|_{L^\infty}$, $\|\omega_1\|_{L^\infty}$ and $\|\omega\|_{L^\infty}$ as functions of time. Second row: log log $\|u_1\|_{L^\infty}$, log log $\|\omega_1\|_{L^\infty}$ and log log $\|\omega\|_{L^\infty}$.

Figure 3.4: Left plot: the amplification of maximum vorticity relative to its initial maximum vorticity, $\|\omega(t)\|_{L^\infty}/\|\omega(0)\|_{L^\infty}$ as a function of time. Right plot: the time integral of maximum vorticity, $\int_0^t \|\omega(s)\|_{L^\infty} ds$ as a function of time. The solution is computed using $1536 \times 1536$ grid. The final time instant is $t = 0.0022868502$.

the symmetry axis. If we move $z_0$ a bit lower towards $z = 0$, we observe in Figure 3.5(b) that the streamlines do not spin as much as they approach the symmetry axis. They travel in an “inward-upward-outward-downward” cycle in the $r_z$-coordinates. On the other hand, if the initial point is very close to $z = 0$ as in Figure 3.5(c), the streamlines will spin around the symmetry axis and do not travel upward. We see this effect better in a zoomed view presented in Figure 3.5(d).

Next, we study the streamlines at the late stage of the computation. In Figure 3.6, we plot the streamlines at time $t = 0.002286845$ for different initial points near the maximum location $(R(t), Z(t))$ of $u_1(t)$. By this time, $\|\omega(t)\|_{L^\infty}/\|\omega(0)\|_{L^\infty}$ has increased by a factor of $10^6$. The red ring represents the location of $(R(t), Z(t))$, and the green pole is the symmetry axis $r = 0$. The first two settings of $(r_0, z_0)$ are exactly the same as the zoom-in case for the 3D Euler equations [20]. We make a small modification for the last case. More specifically, we have (a) $(r_0, z_0) = (2R(t), 0.01Z(t))$, (b)
The nearly singular behavior of the 3D Navier-Stokes equations

\( r_0 = 0.8, z_0 = 0.2 \)

\( r_0 = 0.5, z_0 = 0.1 \)

\( r_0 = 0.1, z_0 = 0.01 \)

\( r_0 = 0.1, z_0 = 0.01 \)

\( r_0 = 0.1, z_0 = 0.5 R(t), 2Z(t) \) and (c)-(d) \( (r_0, z_0) = (0.5 R(t), 1.25 Z(t)) \). Surprisingly, we observe the same qualitative behavior as we did for the 3D Euler equations \[20\]. In some sense, the Navier–Stokes equations preserve the same qualitative features of the 3D Euler equations. The relatively large viscosity enables us to compute the solution for much longer time and we observe a much more rapid growth of the maximum vorticity.

**Figure 3.5**: The streamlines of \((u_r'(t), u_\theta'(t), u_z'(t))\) at time \(t = 0.00227375\) with initial points given by (a) \((r_0, z_0) = (0.8, 0.2)\), (b) \((r_0, z_0) = (0.5, 0.1)\), (c) \((r_0, z_0) = (0.1, 0.01)\) (3D view), (d) \((r_0, z_0)\) is the same as (c), a zoom view. The green pole is the symmetry axis \(r = 0\).

\( (r_0, z_0) = (1.05R(t), 2Z(t)) \)

The 2D flow. To understand the phenomena in the most singular region as shown in Figure 3.6, we study the 2D velocity field \((u', u_\theta')\). In Figure 3.7(a)-(b), we plot the dipole structure of \(\omega_1\) in a local symmetric region and the hyperbolic velocity field induced by the dipole structure in a local microscopic domain \([0, R_b] \times [0, Z_b]\) at two different times, \(t_1 = 0.0022861547\) and \(t_2 = 0.0022868453\). The dipole structure for the 3D Navier–Stokes equations look qualitatively similar to that of the Euler equations except that the dipole structure for the Navier–Stokes equations is much smoother than that for the Euler equations \[20\]. By the time \(t = t_1\), \(\|\omega(t)\|_{L^\infty}/\|\omega(0)\|_{L^\infty}\) has increased by a factor of \(10^4\) and by time \(t = t_2\), \(\|\omega(t)\|_{L^\infty}/\|\omega(0)\|_{L^\infty}\) has increased by a factor of \(10^6\).

The 2D velocity field shares the same qualitative features as those of the 3D Euler equations. As in the case of the Euler equations, the negative radial velocity near \(z = 0\) induced by the antisymmetric vortex dipoles pushes the solution towards \(r = 0\), then move upward away from \(z = 0\). This is one of the driving mechanisms for a potential singularity on the symmetry axis. Since the value of \(u_1\) becomes very small near the symmetry axis \(r = 0\), the streamlines almost do not spin around the symmetry axis, as illustrated in Figure 3.6(a).
Figure 3.6: The streamlines of \((u^r(t), u^\theta(t), u^z(t))\) at time \(t = 0.0022868453\) with initial points given by (a) \((r_0, z_0) = (2R(t), 0.01Z(t))\), (b) \((r_0, z_0) = (1.05R(t), 2Z(t))\), (c) \((r_0, z_0) = (0.5R(t), 1.25Z(t))\) (3D view), (d) \((r_0, z_0)\) is the same as (c), a top view. \((R(t), Z(t))\) is the maximum location of \(u_1(t)\), indicated by the red ring. The green pole is the symmetry axis \(r = 0\).

Moreover, the velocity field \((u^r(t), u^z(t))\) also forms a closed circle right above \((R(t), Z(t))\). The corresponding streamlines are trapped in the circle region in the rz-plane. The fluid spins fast around the symmetry axis \(r = 0\), see Figure 3.6(b). As in the case of the 3D Euler equations, this local circle structure of the 2D velocity field is critical in stabilizing the blowup process.
3.2.4. The effect of viscous regularization. To study the effect of viscous regularization, we plot the velocity contours in Figure 3.8. While they share the same qualitative features as those of the 3D Euler equations, we notice that the velocity field becomes much smoother due to the strong viscous regularization. Thus, the sharp front does not have a rapidly decreasing thickness as in the case of the 3D Euler equations. We have a stable scaling relationship for a much longer time.

**Figure 3.8:** The level sets of $u'$ (left) and $u^2$ (right) at $t = 0.002286845$. The red point is the maximum location $(R(t), Z(t))$ of $u_1(t)$.

Although we apply a relatively large viscosity to stabilize the potential Euler singularity, the main mechanism that leads to the potential Euler singularity is still preserved by the Navier–Stokes equations during the rapid transient growth period. The relatively strong viscous regularization enhances
nonlinear alignment of vortex stretching. In Figure 3.9(a)-(b), we demonstrate the alignment between \( \psi_{1,z} \) and \( u_1 \) at two different times \( t_1 = 0.002286155 \) and \( t_2 = 0.002286845 \). The maximum vorticity has increased by a factor of \( 10^4 \) at \( t_1 \) while the maximum vorticity has increased by a factor of \( 10^6 \) at \( t_2 \). Although the maximum vorticity has grown so much by these times, the local solution structures have remained qualitatively the same. In particular, we do not observe the sharp drop in the \( \psi_{1,z}(t, r, Z(t)) \) as a function of \( r \) near \( r = R(t) \) as we observed for the 3D Euler equation in the late stage \([20]\). This shows that the viscous effect has a strong stabilizing effect that enhances the nonlinear alignment of vortex stretching. We also observe that \( \psi_{1,z} \) is relatively flat in the region \( \{ (r, z)| 0 \leq r \leq 0.9R(t), 0 \leq z \leq 0.5Z(t) \} \). This property is critical for \( u_1 \) to remain large between the sharp front and \( r = 0 \), thus avoiding the formation of a vacuum region and a two-scale structure.

Due to the odd symmetry of \( u_1 \) as a function of \( z \), the large, positive, and relative flat \( \psi_{1,z}(t, R(t), z) \) near \( z = 0 \) induces a large growth of \( u_1 \) through the vortex stretching term \( 2\psi_{1,z}u_1 \) in the \( u_1 \)-equation \((2.2a)\). Due to the oddness of \( u_1 \) as a function of \( z \), we have \( u_1 = 0 \) at \( z = 0 \). Thus, the large growth of \( u_1 \) near \( z = 0 \) generates a large positive gradient of \( u_1^2 \) in the \( z \)-direction between \( z = 0 \) and \( z = Z(t) \). This in turn induces a rapid growth of \( \omega_1 \) through the vortex stretching term \( 2(u_1^2)_z \) in the \( \omega_1 \)-equation \((2.2b)\). The antisymmetric dipole structure generated by \( \omega_1 \) produces a strong negative radial velocity in between the dipole, see Figure 3.7, which leads to a rapid growth of \( \psi_{1,z} \) near \( z = 0 \) (recall \( \psi_{1,z} = -u_1'/r \)). The rapid growth of \( \psi_{1,z} \) induces an even faster growth of \( u_1 \) and a larger positive gradient of \( u_1^2 \) in the \( z \)-direction, leading to the rapid growth of \( \omega_1 \). The whole coupling mechanism forms a positive feedback loop.

3.3. Numerical Results: Resolution Study. In this subsection, we perform resolution study and investigate the convergence property of our numerical methods. In particular, we will study the effectiveness of the adaptive mesh in Section 3.3.1, and the convergence of the solutions as \( h_\rho, h_\eta \to 0 \) in Section 3.3.2.

3.3.1. Effectiveness of the adaptive mesh. Since we solve the Navier–Stokes equations in the transformed \((\rho, \eta)\) coordinates, we would like to see if the solution remains smooth in the \((\rho, \eta)\) coordinates. In Figure 3.10(a), we plot the 3D profile of \( u_1 \) at \( t = 0.002286845 \) in the original \( rz \)-plane. This plot seems to suggest that the solution develops a focusing and nearly singular solution at the origin. In Figure 3.10(b), we plot the profile of \( u_1 \) at the same time in the \( \rho\eta \)-plane. We can see clearly that the solution is quite smooth in the \((\rho, \eta)\) coordinates and our adaptive mesh resolves the nearly singular structure of the solution in the \((\rho, \eta)\) coordinates.

![Figure 3.10: The adaptive mesh resolves the solution in the \( \rho\eta \)-plane. (a) shows the focusing nearly singular profile of \( u_1 \) at \( t = 0.002286845 \) in the \( rz \)-plane on the whole computational domain \( \mathcal{D}_1 \). (b) plots the profile of \( u_1 \) in the \( \rho\eta \)-plane.](image-url)
In Figure 3.11, we show the top views of the profiles of \(u_1, \omega_1\) in a local domain at \(t = 0.002286845\). This figure demonstrates how the mesh points are distributed in different phases of the adaptive mesh. As we can see, we have the most mesh points in phase 1 in both directions, and the adaptive mesh resolves the most singular part of the solution.

![Figure 3.11: The adaptive mesh has different densities in different regions.](image)

Inspired by my recent joint work with De Huang in [21], we define the mesh effectiveness functions \(ME_\rho(v), ME_\eta\) with respect to some solution variable \(v\) as follows:

\[
ME_\rho(v) = \frac{h_\rho v_\rho}{\|v\|_{L^\infty}}, \quad ME_\eta(v) = \frac{h_\eta v_\eta}{\|v\|_{L^\infty}}.
\]

We further define the corresponding mesh effectiveness measures (MEMs) as follows:

\[
ME_{\rho,\infty}(v) = \|ME_\rho(v)\|_{L^\infty}, \quad ME_{\eta,\infty}(v) = \|ME_\eta(v)\|_{L^\infty}.
\]

As indicated in [21], the MEMs quantify the largest relative growth of a function \(v\) in one single mesh cell. If the MEMs are small, the adaptive mesh has done a good job in resolving the function \(v\) effectively. Thus, the MEMs provide a good measure on the effectiveness of our adaptive mesh.

In Figure 3.12, we plot the mesh effectiveness functions of \(u_1, \omega_1\) at time 0.002286845 on the mesh of size \((n_1, n_2) = (1536, 1536)\). We can see that these functions are all uniformly bounded in absolute value by a relatively small value.

Table 3.1 reports the MEMs of \(u_1, \omega_1\) at \(t = 0.002286845\) on meshes of different sizes. We observe that the MEMs decrease as the grid sizes \(h_\rho, h_\eta\) decrease. This is reasonable since the MEMs are proportional to \(h_\rho, h_\eta\). Table 3.2 reports the MEMs of \(u_1, \omega_1\) at different times using the same mesh size \((n_1, n_2) = (1536, 1536)\). We can see that the MEMs remain relatively small throughout this time interval. This study implies that our adaptive mesh strategy is effective in resolving the nearly singular solution of the 3D Navier–Stokes equations over the entire computational domain \(\Omega_1\).

3.3.2. Resolution study. In this subsection, we perform resolution study on the numerical solutions of the initial-boundary value problem (2.2) at various time instants \(t\). We will estimate the relative error of a solution variable \(f_p\) computed on the 256\(p\) \times 256\(p\) mesh by comparing it to a reference variable \(\hat{f}\) that is computed at the same time instant on the finest mesh of size 1536 \times 1536.

We first study the sup-norm error of the solution, which is the most important measure on the accuracy of our numerical method. Tables 3.3–3.6 report the sup-norm relative errors and numerical orders of different solution variables at times \(t = 0.00227375\) and \(t = 0.002286781\), respectively. The first time instant \(t = 0.00227375\) corresponds to the time at which we stop the computation using viscosity \(\nu = 5 \cdot 10^{-4}\). After this time, we increase the viscosity to \(\nu = 5 \cdot 10^{-3}\). We will use the
Figure 3.12: First row: the mesh effectiveness functions of $u_1$ at $t = 0.002286845$ with mesh dimension $(n_1, n_2) = (1536, 1536)$. Second row: the mesh effectiveness functions of $\omega_1$ in the same setting.

| Mesh size     | MEMs on mesh at $t = 0.002286845$ |
|---------------|-----------------------------------|
| $512 \times 512$ | $0.025$ $0.015$ $0.050$ $0.027$ |
| $768 \times 768$ | $0.016$ $0.010$ $0.035$ $0.023$ |
| $1024 \times 1024$ | $0.013$ $0.008$ $0.027$ $0.018$ |
| $1280 \times 1280$ | $0.011$ $0.005$ $0.022$ $0.011$ |
| $1536 \times 1536$ | $0.009$ $0.004$ $0.017$ $0.009$ |

Table 3.1: MEMs of $u_1, \omega_1$ at $t = 0.002286845$ on the meshes of different sizes.

| Time           | MEMs on mesh $(n_1, n_2) = (1536, 1536)$ |
|----------------|----------------------------------------|
| $0.00227375$   | $0.006$ $0.006$ $0.013$ $0.029$       |
| $0.002286155$  | $0.007$ $0.006$ $0.018$ $0.017$       |
| $0.002286781$  | $0.008$ $0.005$ $0.018$ $0.011$       |
| $0.002286845$  | $0.009$ $0.004$ $0.017$ $0.009$       |

Table 3.2: MEMs of $u_1, \omega_1$ at different times on the mesh of size $(n_1, n_2) = (1536, 1536)$.

The solution computed by our finest resolution $1536 \times 1536$ at this time as a new initial condition for the coarser meshes. Thus it is important to make sure that we have resolved the Navier–Stokes equations accurately up to this time. The convergence results at both $t = 0.00227375$ and $t = 0.002286781$ look qualitatively the same although the errors at $t = 0.002286781$ have increased modestly. This seems to be quite reasonable since the maximum vorticity has increased by a factor of $10^5$ by $t = 0.002286781$. 

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In comparison, the maximum vorticity has increased only by a factor of 498.42 by \( t = 0.00227375 \). These results confirm that our method is at least 2nd-order accurate.

| Mesh size | Sup-norm relative error at \( t = 0.00227375 \) for 3D Navier–Stokes equations |
|-----------|--------------------------------------------------------------------------------|
| \( 512 \times 512 \) | \( 1.6181 \times 10^{-1} \) | \( 4.9486 \times 10^{-1} \) | \( 4.2245 \times 10^{-2} \) |
| \( 768 \times 768 \) | \( 6.3639 \times 10^{-2} \) | \( 2.302 \) | \( 2.1788 \times 10^{-1} \) | \( 2.023 \) | \( 1.5929 \times 10^{-2} \) | \( 2.405 \) |
| \( 1024 \times 1024 \) | \( 2.7017 \times 10^{-2} \) | \( 2.978 \) | \( 9.4386 \times 10^{-2} \) | \( 2.908 \) | \( 6.6958 \times 10^{-3} \) | \( 3.013 \) |
| \( 1280 \times 1280 \) | \( 9.6517 \times 10^{-3} \) | \( 4.613 \) | \( 3.3906 \times 10^{-2} \) | \( 4.588 \) | \( 2.3826 \times 10^{-3} \) | \( 4.631 \) |

**Table 3.3:** Sup-norm relative errors and numerical orders of \( u_1, \omega_1, \psi_1 \) at \( t = 0.00227375 \) for the 3D Navier–Stokes equations.

| Mesh size | Sup-norm relative error at \( t = 0.00227375 \) for 3D Navier–Stokes equations |
|-----------|--------------------------------------------------------------------------------|
| \( 512 \times 512 \) | \( 1.6238 \times 10^{-1} \) | \( 3.6640 \times 10^{-1} \) | \( 5.0751 \times 10^{-1} \) |
| \( 768 \times 768 \) | \( 6.4154 \times 10^{-2} \) | \( 2.290 \) | \( 1.4711 \times 10^{-1} \) | \( 2.251 \) | \( 2.2110 \times 10^{-1} \) | \( 2.049 \) |
| \( 1024 \times 1024 \) | \( 2.7213 \times 10^{-2} \) | \( 2.981 \) | \( 6.2551 \times 10^{-2} \) | \( 2.973 \) | \( 9.5467 \times 10^{-2} \) | \( 2.919 \) |
| \( 1280 \times 1280 \) | \( 9.7121 \times 10^{-3} \) | \( 4.617 \) | \( 2.2333 \times 10^{-2} \) | \( 4.615 \) | \( 3.4244 \times 10^{-2} \) | \( 4.595 \) |

**Table 3.4:** Sup-norm relative errors and numerical orders of \( u^r, u^s, \omega \) at \( t = 0.00227375 \) for 3D Navier–Stokes equations.

| Mesh size | Sup-norm relative error at \( t = 0.002286781 \) for 3D Navier–Stokes equations |
|-----------|--------------------------------------------------------------------------------|
| \( 512 \times 512 \) | \( 2.9842 \times 10^{-1} \) | \( 5.4093 \times 10^{-1} \) | \( 7.4719 \times 10^{-2} \) |
| \( 768 \times 768 \) | \( 1.1109 \times 10^{-1} \) | \( 2.437 \) | \( 2.1295 \times 10^{-1} \) | \( 2.299 \) | \( 2.6937 \times 10^{-2} \) | \( 2.516 \) |
| \( 1024 \times 1024 \) | \( 5.4326 \times 10^{-2} \) | \( 2.486 \) | \( 1.0462 \times 10^{-1} \) | \( 2.471 \) | \( 1.3168 \times 10^{-2} \) | \( 2.488 \) |
| \( 1280 \times 1280 \) | \( 2.2040 \times 10^{-2} \) | \( 4.043 \) | \( 4.2516 \times 10^{-2} \) | \( 4.035 \) | \( 5.3224 \times 10^{-3} \) | \( 4.060 \) |

**Table 3.5:** Sup-norm relative errors and numerical orders of \( u_1, \omega_1, \psi_1 \) at \( t = 0.002286781 \) for the 3D Navier–Stokes equations.

| Mesh size | Sup-norm relative error at \( t = 0.002286781 \) for 3D Navier–Stokes equations |
|-----------|--------------------------------------------------------------------------------|
| \( 512 \times 512 \) | \( 2.5750 \times 10^{-1} \) | \( 4.3600 \times 10^{-1} \) | \( 5.6599 \times 10^{-1} \) |
| \( 768 \times 768 \) | \( 9.4107 \times 10^{-2} \) | \( 2.482 \) | \( 1.6258 \times 10^{-1} \) | \( 2.433 \) | \( 2.1966 \times 10^{-1} \) | \( 2.334 \) |
| \( 1024 \times 1024 \) | \( 4.5528 \times 10^{-2} \) | \( 2.524 \) | \( 7.9726 \times 10^{-2} \) | \( 2.477 \) | \( 1.0734 \times 10^{-1} \) | \( 2.489 \) |
| \( 1280 \times 1280 \) | \( 1.8425 \times 10^{-2} \) | \( 4.054 \) | \( 3.2303 \times 10^{-2} \) | \( 4.049 \) | \( 4.3522 \times 10^{-2} \) | \( 4.045 \) |

**Table 3.6:** Sup-norm relative errors and numerical orders of \( u^r, u^s, \omega \) at \( t = t = 0.002286781 \) for the 3D Navier–Stokes equations.

We will also study the convergence of some variables as functions of time. In particular, we report the convergence of the quantities \( \|u_1\|_{L^\infty}, \|\omega_1\|_{L^\infty}, \|\omega\|_{L^\infty} \), and the kinetic energy \( E \). We will report the relative errors and the numerical orders in three different time intervals. In Figures 3.13, we
show the relative errors and numerical orders of \( \|u_1\|_{L^\infty}, \|\omega\|_{L^\infty} \) and \( \|\psi_1\|_{L^\infty} \) in the early time interval \([0, 0.00227375]\) during which we use \( \nu = 5 \cdot 10^{-4} \). We observe that the relative maximum errors are small and we obtain second order accuracy for this early stage computation. In Figure 3.14, we plot the relative errors and numerical orders of \( \|\omega\|_{L^\infty}, \|\psi_1\|_{L^\infty} \), and the kinetic energy \( E \) as functions of time during the second time interval \([0.00227375, 0.002286781]\). We also observe that our method is 2nd-order in \( h, h_\eta \) during this time interval.

In Figure 3.15-3.16, we plot the relative errors and the numerical orders of \( \|u_1\|_{L^\infty}, \|\omega_1\|_{L^\infty}, \|\psi_1\|_{L^\infty}, \|\omega\|_{L^\infty} \), and the kinetic energy \( E \) in the third time interval between \( t = 0.002286781 \) and \( t = 0.002286845 \). This corresponds to the late stage of our computation. Due to the strong viscous regularization, we observe that the errors become quite stable in the late stage of our computation. This is very different from the situation for the 3D Euler equations [20]. It also shows that our adaptive mesh strategy is very effective for the computation of the 3D Navier–Stokes equations with \( \nu = 5 \cdot 10^{-3} \) up to the late stage of the computation.

We remark that there are two sources of errors, the discretization error and the interpolation error when we change from one adaptive mesh to another adaptive mesh. A finer mesh tends to change
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2.274 2.276 2.278 2.28 2.282 2.284 2.286
10^{-3}

2.274 2.276 2.278 2.28 2.282 2.284 2.286
10^{-6}

2.274 2.276 2.278 2.28 2.282 2.284 2.286
10^{-5}

2.274 2.276 2.278 2.28 2.282 2.284 2.286
10^{-4}

2.274 2.276 2.278 2.28 2.282 2.284 2.286
10^{-3}

2.274 2.276 2.278 2.28 2.282 2.284 2.286
10^{-2}

2.274 2.276 2.278 2.28 2.282 2.284 2.286
10^{-1}

Figure 3.14: First row: relative error and numerical order of $||\omega(t)||_{L^\infty}$. Second row: relative error and numerical order of $||\psi_1(t)||_{L^\infty}$. Third row: relative error and numerical order of the kinetic energy, $E(t)$. The computation is between $t = 0.00227375$ and $t = 0.002286781$.

meshes more frequently. Thus, it is possible that the interpolation error may dominate the discretization error in the early stage, especially for a fine mesh. We can only observe the expected order of accuracy when the discretization error dominates the interpolation error.

3.4. Scaling Properties of the solution. In this subsection, we will examine the scaling properties of the solution of the 3D Navier–Stokes equations. We will provide some qualitative numerical evidences that the solution of the 3D Navier–Stokes equations enjoys nearly self-similar scaling properties.

3.4.1. Fitting of the growth rate. We now study the scaling properties the numerical solutions obtained from $1536 \times 1536$ resolution. Figure (3.17) shows the fitting results for the quantities $||u_1(t)||_{L^\infty}$ and $||\psi_{1,z}||_{L^\infty}$ on the time interval $[t_1, t_2] = [0.002275858596, 0.002286850216]$. We would like to emphasize that the maximum vorticity relative to its initial maximum vorticity, i.e. $||\omega(t_2)||_{L^\infty}/||\omega(0)||_{L^\infty}$, has increased by a factor of $10^7$ while we only have $||\omega(t_1)||_{L^\infty}/||\omega(0)||_{L^\infty} = 514.58$. During this time interval, the viscosity remains the same and is equal to $\nu = 5 \cdot 10^{-3}$.

We can see that

$$||u_1(t)||_{L^\infty} \sim \frac{1}{(T-t)}$$

$$||\psi_{1,z}||_{L^\infty} \sim \frac{1}{(T-t)}$$
where we have neglected the viscous effect since we have vortex stretching dominating diffusion during the asymptotic analysis is consistent with our linear fitting results.

Next, we study the growth of the maximum vorticity \( \| \omega(t) \|_{L^\infty} \). In Figure 3.18 (a), we show the linear fitting of \( \log(\| \omega(t) \|_{L^\infty}) \) as a function of time on the time interval \([t_1, t_2] = [0.002277464739, 0.002286850216] \). This \( t_1 \) is slightly larger than the \( t_1 \) that we use for the fitting of \( u_1 \) and \( \psi_{1z} \) since \( \omega \) enters the asymptotic self-similar regime slightly later than that for \( u_1 \) and \( \psi_{1z} \). The end time \( t_2 \) is still the same as for \( u_1 \) and \( \psi_{1z} \). We observe that \( \| \omega(t) \|_{L^\infty} = O(\| \psi_{1z}(t) \|_{L^\infty})/(T-t)) \) has good linear fitness with R-Square values very close to 1. Since we observe \( \| \psi_{1z}(t) \|_{L^\infty} = O(1/(T-t)) \),
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Figure 3.16: First row: relative error and numerical order of $||\omega(t)||_{L^\infty}$. Second row: relative error and numerical order of $E(t)$. The time period shown in the figure is from $t = 0.002286781$ to $t = 0.002286845$.

Figure 3.17: The linear regression of (a) $||u_1(t)||_{L^3}^{-1}$ vs $t$, (b) $||\psi_{12}(t)||_{L^2}^{-1}$ vs $t$. The blue points are the data points obtained from our computation, and the red lines are the linear models. The solution is computed using $1536 \times 1536$ grid. The final time instant is $t = 0.002286850216$.

This seems to imply that

$$||\omega(t)||_{L^\infty} \sim \frac{\log(T - t)}{(T - t)}.$$ 

If this scaling relationship holds up to time $T$, the Beale-Kata-Majda blowup criteria would imply that the 3D Navier–Stokes equations develop a finite time singularity. In Figure 3.18 (b), we also plot the linear fitting of $||\omega_1(t)||_{L^\infty}^{2/3}$ as a function of time on the same time interval for $u_1$ and $\psi_{12}$. We see that

$$||\omega_1(t)||_{L^\infty}^{2/3} \sim \frac{1}{(T - t)^{3/2}}$$
Figure 3.18: The linear regression of (a) $\log(||\psi_{1z}||_\infty)||\omega(t)||_{L^1}^{-1}$ vs $t$, (b) $\|\omega_1(t)\|_{L^{2/3}}^{-2/3}$ vs $t$. The blue points are the data points obtained from our computation, and the red lines are the linear models. The solution is computed using $1536 \times 1536$ grid. The final time instant is $t = 0.002286850216$.

Figure 3.19: The linear regression of (a) $\|u(t)\|_{L^\infty}^{-2}$ vs $t$, (b) $Z(t)^2$ vs $t$. The blue points are the data points obtained from our computation, and the red lines are the linear models. The solution is computed using $1536 \times 1536$ grid. The final time instant is $t = 0.002286850216$.

has good linear fitness with R-Square values very close to 1.

To further illustrate the nearly singular behavior of the Navier–Stokes equations, we perform linear fitting for the maximum velocity and $Z(t)$. For the fitting of maximum velocity, the fitting time interval is the same as that for $u_1$ and $\psi_{1z}$. For $Z(t)$, we use the same time interval as that for $\omega$, i.e. \([t_1, t_2] = [0.002277464739, 0.002286850216]\). In Figure 3.19 (a), we observe that $\|u(t)\|_{L^\infty}^{-2} = O((T - t))$ has good linear fitness with R-Square values very close to 1, which implies that $\|u(t)\|_{L^\infty}$ has the form of an inverse power law

$$\|u(t)\|_{L^\infty} \sim \frac{1}{(T - t)^{1/2}}.$$ 

The scaling properties of the maximum vorticity and maximum velocity seem to suggest that the small scale of the solution, which is characterized by $Z(t)$, should have the form: $Z(t) \sim (T - t)^{1/2}$. Indeed, in Figure 3.19(b), we observe that

$$Z(t) \sim (T - t)^{1/2},$$
has good linear fitness. The fitting is not as good as that for the maximum velocity. We can see that the fitting for $Z(t)$ seems to deteriorate towards the end and the R-Square values are not as close to 1 as those for other quantities. This is due to the mild two-scale structure that is developed in the late stage.

To capture this mild change of the scaling relationship between $R(t)$ and $Z(t)$, we use a shorter time interval $[t_1, t_2]$ with a larger $t_1 = 0.0022865621$ while keeping $t_2$ the same as before. We obtain a slightly different fitting that captures the two-scale structure in the late stage. In Figure 3.20, we plot the linear fitting of $(\log(\|\psi_{12}\|_{\infty}))^2 R(t)^2$ and $(\log(\|\psi_{12}\|_{\infty}))^3 Z(t)^2$ versus $t$. We obtain excellent linear fitting with R-Square values very close to 1. This seems to imply that

$$R(t) \sim \frac{(T - t)^{1/2}}{\log(T - t)}$$
$$Z(t) \sim \frac{(T - t)^{1/2}}{\log(T - t)^{3/2}}.$$  

![Figure 3.20](image)

(a) linear regression of $(\log(\|\psi_{12}\|_{\infty}))^2 R(t)^2$

(b) $(\log(\|\psi_{12}\|_{\infty}))^3 Z(t)^2$

The solution is computed using $1536 \times 1536$ grid. The final time instant is $t = 0.002286850216$.

Another important indicator for a potential singular behavior of the 3D Navier–Stokes equations is the growth of negative pressure, i.e. the pressure should approach to negative infinity at the singularity time, see [39]. In Figure 3.21 (a), we plot the linear fitting of the inverse of the maximum norm of negative pressure as a function of time. We obtain excellent linear fitting

$$\| - p(t) \|_{L^\infty} \sim \frac{1}{(T - t)},$$

with R-Square values very close to 1. We also perform linear fitting for another related quantity: $\| \frac{1}{2} |\mathbf{u}|^2 + p \|_{L^\infty}$ as a function of time. We also obtain excellent linear fitting

$$\| \frac{1}{2} |\mathbf{u}|^2 + p \|_{L^\infty} \sim \frac{1}{(T - t)},$$

with R-Square values very close to 1, see Figure 3.21 (b).

To give a better idea how the nearly singular pressure profile looks like, we plot the 3D pressure profile at $t = 0.0022868452680$ in Figure 3.22. By this time, the maximum vorticity has increased by a factor of $10^6$. We can see that the pressure profile is still very well resolved by our adaptive mesh. The pressure develops a nearly singular structure with the minimum of the pressure roughly of the order $-10^3$. In Figure 3.22 (b), we plot a top view of the same pressure profile. We can see a focusing nature of the nearly singular pressure profile.

3.4.2. Numerical evidence of locally self-similar profiles. To study the nearly self-similar scaling properties of the 3D Navier–Stokes equations, we study the solution in the dynamically rescaled variables $(\xi, \zeta)$
as follows:

$$u_1(t, r, z) = \max(u_1)\overline{U}(t, \xi, \zeta), \quad \omega_1(t, r, z) = \max(\omega_1)\overline{\Omega}(t, \xi, \zeta),$$

where

$$\xi = \frac{r - R(t)}{Z(t)}, \quad \zeta = \frac{z}{Z(t)},$$

are the dynamically rescaled variables.

In Figure 3.23, we compare the level sets of \(u_1\) at different time instants. In the first row of Figure 3.23, we plot the level sets of \(u_1\) in a local domain. We plot the profiles in a short time interval at three different times with \(t_1 = 0.002286850057, t_2 = 0.002286850158\) and \(t_3 = 0.002286850216\). By the time \(t = t_3\), \(\|\omega(t_3)\|_{L^{\infty}}/\|\omega(0)\|_{L^{\infty}}\) has grown by a factor of \(10^7\). As we can see, the singular support of the profile shrinks in space and travels towards the origin. Compared with the solution of the 3D Euler equations, the sharp front has been regularized by the relatively large viscosity. In the second row of Figure 3.23, we can see that the rescaled profile \(\overline{U}\) (in the \(\xi\zeta\)-plane) is almost static during this time interval. This observation suggests that there exists an approximate self-similar profile \(U(\xi, \zeta)\).

In Figure 3.24, we compare the level sets of \(\omega_1\) and the level sets of the spatially rescaled function \(\overline{\Omega}\) in a similar manner. Again, we can see that although the profile of \(\omega_1\) has a noticeable change during this time interval, there seems to exist an approximate self-similar profile \(\Omega(\xi, \zeta)\).
The nearly singular behavior of the 3D Navier-Stokes equations

Figure 3.23: Comparison of the level sets of $u_1$ at different time instants. First row: original level sets of $u_1$ in the domain $(r, z)$ in different times. Second row: rescaled level sets of $u_1$ as a function of $(\xi, \zeta)$ in the domain $(\xi, \zeta)$. The computation is performed using the $1536 \times 1536$ grid. Here $t_1 = 0.002286850057$, $t_2 = 0.002286850158$ and $t_3 = 0.002286850216$.

Next, we compare the cross sections of the solution at different time instants to study the nearly self-similar scaling properties. In Figure 3.25(a) and (c), we present the evolution of the cross sections of $u_1$ through the point $(R(t), Z(t))$ in both directions. The length scale of the profile shrinks in both directions, and the sharp front along the $r$-direction travels towards $r = 0$. For comparison, Figure 3.25(b) and (d) plot the corresponding cross sections of the rescaled profile $\tilde{U}$ in terms of the dynamically rescaled variables $(\xi, \zeta)$. We can see that the rescaled profiles seem to be very close to a limiting profile. These results support the existence of an approximate self-similar profile.

3.5. The non-blowup of the Navier–Stokes equations with our initial condition. Despite the nearly singular scaling properties that we have demonstrated in the previous subsection, the 3D Navier–Stokes equations with our initial condition manage to escape a finite time singularity. The main reason is due to the development of a mild two-scale structure in the late stage, which eventually leads to viscous dominance over vortex stretching. To demonstrate the non-blowup of the 3D Navier–Stokes equations, we study the $L^{3, \infty}$ norm of the velocity field, i.e. $\|u\|_{L^{3, \infty}}$. As shown in [16], the Navier–Stokes equations cannot blow up at time $T$ if $\|u(t)\|_{L^3}$ is bounded up to time $T$. In Figure 3.26 (a), we plot the dynamic growth of $\|u(t)\|_{L^3}$ as a function of time. We observe that after a mild growth in the early time, $\|u(t)\|_{L^3}$ begins to decrease in the late stage and remains bounded throughout the computation. This
Figure 3.24: Comparison of the level sets of $\omega_1$ at different time instants. First row: original level sets of $u_1$ in the domain $(r, z)$ in different times. Second row: rescaled level sets of $\omega_1$ as a function of $(\xi, \zeta)$ in the domain $(\xi, \zeta)$. The computation is performed using the $1536 \times 1536$ grid. Here $t_1 = 0.002286850057$, $t_2 = 0.002286850158$ and $t_3 = 0.002286850216$.

provides a strong evidence that the 3D Navier–Stokes equations with our initial condition do not develop a finite time singularity. We remark that the non-blowup criterion using the $\|u\|_{L^3}$ estimate is based on a compactness argument. As a result, the bound on $\max_{0 \leq t \leq T} \|u(t)\|_{L^3}$ does not provide a direct estimate on the dynamic growth rate of the Navier–Stokes solution up to $T$. In fact, we observe that $\|u(t)\|_{L^3} \leq 46.86$ throughout our computation, but the maximum vorticity has already grown by a factor of $10^7$ by the end of the computation.

Another important non-blowup result is the lower bound on the growth rate of the maximum velocity for the axisymmetric Navier–Stokes equations. The results in [5, 4, 30] imply that the 3D axisymmetric Navier–Stokes equations cannot develop a finite time singularity if the maximum velocity field is bounded by $\|u\|_{L^\infty} \leq C(T-t)^{1/2}$, provided that $|ru(t, r, z)|$ remains bounded for $r \geq r_0$ for some $r_0$. From our numerical experiments, both conditions seem to hold true. Thus, we do not expect to have a finite time blowup.

To provide further evidence on the non-blowup of the Navier–Stokes equations with our initial condition, we study the growth rate of the enstrophy, $\|\omega(t)\|_{L^2}^2$. A simple energy estimate would imply that if $\int_0^T \|\omega(t)\|_{L^2}^4 dt$ is bounded up to $T$, then the solution of the 3D Navier–Stokes equations would remain smooth up to time $T$. In Figure 3.26 (b), we plot the growth of $\|\omega(t)\|_{L^2}^2$ as a function of time.
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Figure 3.25: Cross sections and rescaled cross sections of $u_1$ through the point $R(t), Z(t)$ in both directions at different time instants. (a) Cross sections in the $r$ direction. (b) Rescaled cross sections in the $r$ directions. (c) Cross sections in the $z$ direction. (d) Rescaled cross sections in the $z$ directions. The computation is performed using the $1536 	imes 1536$ grid. Here $t_1 = 0.002286849845$, $t_2 = 0.002286850057$, $t_3 = 0.002286850158$ and $t_4 = 0.002286850216$.

Figure 3.26: (a) growth of $\|u(t)\|_{L^3}$ vs $t$. (b) growth of $\|\omega(t)\|_{L^2}^2$ vs $t$. The solution is computed using $1536 \times 1536$ grid. The final time instant is $t = 0.002286850216$.

We observe a rapid growth of the enstrophy. In Figure 3.27 (a), we plot the growth of $\int_{t_0}^{t} \|\omega(s)\|_{L^2}^4 ds$ as a function of time. Here $t_0 = 0.00227375$ is the time when we switch the viscosity from $\nu = 5 \cdot 10^{-4}$ to
\[ \nu = 5 \cdot 10^{-3} \] and we use the same viscosity \( \nu = 5 \cdot 10^{-3} \) beyond \( t = t_0 \). We can see that \( \int_{t_0}^{t} \| \omega(s) \|_{L^2}^4 \, ds \) seems to grow without bound.

Figure 3.27: (a) \( \int_{t_0}^{t} \| \omega(s) \|_{L^2}^4 \, ds \) vs \( t \). (b) \( \int_{t_0}^{t} \| \omega(s) \|_{L^2}^4 \, ds \) vs \( \log(\| \psi_{1z}(t) \|_{\infty}) \). The solution is computed using 1536 \( \times \) 1536 grid. Here \( t_0 = 0.00227375 \). The final time instant is \( t = 0.002286850216 \).

Figure 3.28: (a) growth of \( \| (\log(r)^2 \Gamma(t)) \|_{L^\infty} \) vs \( t \). (b) growth of \( \| (\log(r)^4 \Gamma(t)) \|_{L^\infty} \) vs \( t \). The solution is computed using 1536 \( \times \) 1536 grid. The final time instant is \( t = 0.002286850216 \).

However, a more careful study reveals that the growth of \( \int_{t_0}^{t} \| \omega(s) \|_{L^2}^4 \, ds \) actually slows down and becomes saturated in the late stage. To demonstrate this subtle change in its growth behavior, we plot \( \int_{t_0}^{t} \| \omega(s) \|_{L^2}^4 \, ds \) as a function of the rescaled time given by \( \tilde{t} = \log(\| \psi_{1z}(t) \|_{\infty}) \). We can see that the growth slows down and seems to saturate around \( 1.6 \cdot 10^8 \) as the rescaled time increases.

We also examine another non-blowup criteria. According to a recent result by Lei and Zhang in [31], the 3D axisymmetric Navier–Stokes equations cannot blow up in a finite time if the following bound is satisfied:

\[
\sup_{r \leq r_0, z} |(\log(r))^2 \Gamma(r, z, t)| \leq C_1,
\]

for some constants \( C_1 > 0 \) and \( r_0 > 0 \). Here \( \Gamma = ru \) is the total circulation, which enjoys the maximum principle for the axisymmetric Navier–Stokes equations. In Figure 3.27 (a), we plot \( \| (\log(r)^2 \Gamma(t)) \|_{L^\infty} \) as a function of time. We observe that after a rapid transient growth in time, \( \| (\log(r))^2 \Gamma(t) \|_{L^\infty} \) begins to decrease in the late stage. This provides another evidence that the 3D Navier–Stokes equations with our initial condition escape a finite time singularity in the late stage.
Out of curiosity, we also monitor the growth of a slightly more singular quantity, $\|(\log(r))^4\Gamma(t)\|_{L^\infty}$ by changing $(\log(r))^2$ to $(\log(r))^4$. We observe that $\|(\log(r))^4\Gamma(t)\|_{L^\infty}$ grows much faster, see Figure 3.27 (b). Since we do not expect to have a finite time blowup, this quantity should eventually decrease in time after achieving a transient maximum growth rate, just like what we observed for the rapid transient growth of the enstrophy. As we will see in Figure 4.2, the diffusion term will eventually dominate the vortex stretching term in the late stage. This explains why the rapid transient growth of the maximum vorticity, the maximum velocity, the negative pressure and the enstrophy is not sustainable due to the development of the mild two-scale structure in the late stage.

3.6. **Stability of the nearly self-similar profile to small perturbation of initial data.** In this subsection, we study whether the observed nearly self-similar profile is stable with respect to a small perturbation of the initial condition. We will solve the 3D Navier–Stokes equations using four different initial data defined in (3.2)-(3.5). Our study shows that the approximate self-similar profile seems to be very stable to small perturbation of the initial data.

In our stability study, we consider the four different initial data given below.

**Case 1.** We choose the same initial condition given in (2.3), i.e.

$$u_1(0, r, z) = \frac{12000(1 - r^2)^{18} \sin(2\pi z)}{1 + 12.5 \sin(\pi z)^2}, \quad \omega_1(0, r, z) = 0.$$

**Case 2.** We choose the initial condition as a small perturbation to the initial condition defined in (3.2)

$$u_1(0, r, z) = \frac{12000(1 - r^2)^{18} \sin(2\pi z)}{1 + 12.5 \sin(\pi z)^2} + \frac{(1 - r^2)^{10} \sin(6\pi z)}{1 + 12.5 \sin(\pi z)^2}, \quad \omega_1(0, r, z) = 0.$$

**Case 3.** We choose the initial condition as a larger perturbation to the initial condition defined in (3.2)

$$u_1(0, r, z) = \frac{12000(1 - r^2)^{18} \sin(2\pi z)}{1 + 12.5 \sin(\pi z)^2} + \frac{42(1 - r^2)^6 \sin(10\pi z)}{1 + 12.5 \sin(\pi z)^2}, \quad \omega_1(0, r, z) = 0.$$

**Case 4.** We choose the initial condition that is $O(1)$ perturbation to the initial condition defined in (3.2), i.e.

$$u_1(0, r, z) = \frac{12000(1 - r^2)^{18} \sin(4\pi z)}{1 + 12.5 \sin(\pi z)^2}, \quad \omega_1(0, r, z) = 0.$$

The relative size of the perturbation in Case 2 is approximately $1.928 \cdot 10^{-4}$ while the relative size of the perturbation in Case 3 is about $10^{-2}$. We remark that the decay of the perturbation along the $r$-direction is slower than the original unperturbed initial condition and the perturbation along the $z$-direction is more oscillatory. In Case 4, we just change $\sin(2\pi z)$ in the numerator in the original initial condition to $\sin(4\pi z)$. Everything else is the same. With this change, $u_1$ becomes negative for $z$ near 0.5. This induces an $O(1)$ structural change to $u_1$.

For the Case 1 and Case 2 initial data, we solve the Navier–Stokes equations by using viscosity $\nu = 5 \cdot 10^{-4}$ from $t = 0$ to $t_1 = 0.00227375$, and then switch to $\nu = 5 \cdot 10^{-3}$. For the Case 3 initial data, we solve the Navier–Stokes equations by using viscosity $\nu = 5 \cdot 10^{-4}$ from $t = 0$ to $t_2 = 0.0022266053$ and then switch to $\nu = 5 \cdot 10^{-3}$. We choose this time $t_2$ so that $\|\omega(t_2)\|_{L^\infty}/\|\omega(0)\|_{L^\infty}$ matches exactly $\|\omega(t_1)\|_{L^\infty}/\|\omega(0)\|_{L^\infty} \approx 493.08619$ obtained from the first initial data. Due to the relative large perturbation in the Case 3 initial data, the growth rate of the solution is quite different from that for the first two initial data. So we cannot switch the viscosity at the same time for all three initial data.

We solve the Navier–Stokes equations for these three different initial data for a very large number of time steps using a $1024 \times 1024$ grid. For the Case 1 initial data, we solve the Navier–Stokes equations...
for 370,000 time steps and stop at $T_1 = 0.002286851153$. By this time, the maximum vorticity has increased by a factor of $1.03 \cdot 10^7$. For the Case 2 initial data, we solve the the Navier–Stokes equations for 320,000 time steps and stop at $T_2 = 0.002286395676$. By this time, the maximum vorticity has increased by a factor of $9.66 \cdot 10^6$. For the Case 3 initial data, we solve the the Navier–Stokes equations for 280,000 time steps and stop at $T_3 = 0.002239211579$. The maximum vorticity has increased by a factor of $3.188 \cdot 10^6$. One may expect that the nearly singular solution for these three initial data would behave very differently after solving the Navier–Stokes equations for so many time steps. To our surprise, the nearly singular profile seems to be quite stable with respect to the small perturbation of the initial data.

Figure 3.29: (a) Rescaled $\xi$-cross section of $u_1$ at $z = Z(t)$ for the Case 1 and Case 2 initial data. (b) Rescaled $\zeta$-cross section of $u_1$ at $r = R(t)$ for the Case 1 and Case 2 initial data. (c) Rescaled $\xi$-cross section of $u_1$ at $z = Z(t)$ for the Case 1 and Case 3 initial data. (d) Rescaled $\zeta$-cross section of $u_1$ at $r = R(t)$ for the Case 1 and Case 3 initial data.

In Figure 3.29 (a)-(b), we plot the rescaled solution $u_1$ using the Case 1 and Case 2 initial conditions. The rescaled solution $U$ is defined below:

$$u_1 \equiv \max(u_1) U(t, \xi, \zeta), \quad \xi = (r - R(t))/Z(t), \quad \zeta = z/Z(t).$$

We observe that the rescaled profiles $U$ as a function of $(\xi, \zeta)$ for the first two initial conditions almost collapse on each other. This shows that the nearly self-similar profile is very stable with respect to a small perturbation. In Figure 3.29 (c)-(d), we compare the rescaled profiles obtained using the first and the third initial data. We observe a small difference between the rescaled profile $U$ obtained from the first initial data and that obtained from the third initial data. As we mentioned earlier, the difference between the first and the third initial data is relatively large, about 1%. The time at which
we switch the viscosity is also quite different. The small difference that we observe in the rescaled profile is compatible with the size of the perturbation. This suggests that the nearly singular profile of the solution is relatively stable to this moderate size (1%) perturbation of the initial condition.

We also solve the Case 4 initial data using the same viscosity $\nu = 5 \cdot 10^{-4}$ for all time. We do not switch the viscosity due to the fact that this initial condition leads to a two-scale solution structure as we observed in the case of the 3D Euler equations [20]. We see a completely different behavior. The solution behaves qualitatively similar to the two-scale traveling wave solution reported in [21]. In particular, there is a vacuum region for $u_1$ developed dynamically between the sharp front and the symmetry axis and a compact support of $u_1$ along the $z$-direction, see Figure 3.30 (a)-(b). The maximum vorticity grows much slower than that for the first three initial data. The maximum vorticity relative to its initial maximum vorticity grows only by a factor of 2078 after solving the Navier–Stokes equations for 300,000 time steps. For the same number of time steps, the maximum vorticity for the Case 1 initial data has grown by a factor of $4.186 \cdot 10^6$. The slow growth rate of the maximum vorticity for the Case 4 initial data is due to the two-scale structure and the viscous dominance in the late stage. Moreover, as we can see from Figure 3.30(c)-(d), the rescaled profile of the Case 4 initial data looks completely different from that of the Case 1 initial data.

**Figure 3.30**: (a) The $r$-cross section of $u_1$ and $\psi_{1z}$ at $z = Z(t)$ for the Case 4 initial data in the original physical space. (b) The $z$-cross section of $u_1$ and $\psi_{1z}$ at $r = R(t)$ for the Case 4 initial data in the original physical space. (c) Rescaled $\xi$-cross section of $u_1$ for the Case 1 and Case 4 initial data. (d) Rescaled $\zeta$-cross section of $u_1$ for the Case 1 and Case 4 initial data.
4. **3D Navier–Stokes equations with a time-dependent viscosity**

In this section, we will investigate the 3D Navier–Stokes equations with a time-dependent viscosity. As we mentioned in the previous section, the solution of the Navier–Stokes equations with a constant viscosity develops a mild two-scale structure in the late stage, which leads to viscous dominance over vortex stretching and prevents the formation of a finite time singularity. This mild two-scale structure is characterized by the logarithmic correction in the linear fitting of $R(t)$ and $Z(t)$ in the late stage of the computation.

Based on the nearly self-similar properties that we study in the previous section, we know that the small scale of the solution is characterized by $Z(t)$ through the dynamically rescaled variables $(\xi, \zeta) = ((r-R(t))/Z(t), z/Z(t))$. Since both $u_1$ and $\psi_{1z}$ scale like $(T-t)^{-1}$, if $Z(t) \sim (T-t)^{1/2} \log(T-t)^{-3/2}$, then the viscous term will eventually dominate the vortex stretching term $2\psi_{1z}u_1$ in the $u_1$-equation. Similar observation also applies to the $\omega_1$-equation.

4.1. **A time-dependent viscosity.** In order to maintain the balance between the vortex stretching term and the diffusion term, it is necessary to apply a weaker diffusion term. We can either apply a weaker diffusion operator like a fractional diffusion operator, or apply a time-dependent viscosity. However, it is much more difficult to solve the Navier–Stokes equations with a fractional diffusion operator using an adaptive mesh due to the nonlocal nature of the fractional diffusion operator. Therefore, we will adopt the second approach of applying a time-dependent viscosity with the standard diffusion operator.

To maintain the balance between the vortex stretching term and the diffusion term, we choose such time-dependent viscosity of the form

$$\nu(t) = 5 \cdot 10^{-3} g(t)/g(t_1), \quad g(t) = \|\psi_{1z}\|_{L^\infty} Z(t)^2,$$

where $t_1 = 0.002286845268$. We first solve the 3D Navier–Stokes equations with our initial condition using $\nu = 5 \cdot 10^{-3}$ and a $1536 \times 1536$ grid as we described in the previous section. At the time $t = t_1$ when the maximum vorticity has grown by a factor of $10^6$, we change the viscosity from $\nu = 5 \cdot 10^{-3}$ to $\nu = 5 \cdot 10^{-5} g(t)/g(t_1)$ and use the solution of the Navier–Stokes equations at $t_1$ as our initial condition for the time-dependent viscosity computation. Since the vortex stretching term for the $u_1$-equation is given by $2\psi_{1z}u_1$ and $Z(t)$ characterizes the small scale of the solution, a simple scaling analysis implies that such time-dependent viscosity $\nu(t)$ defined above maintains the balance between vortex stretching and the diffusion term for the $u_1$-equation.

![Figure 4.1](image_url): The ratio between vortex stretching and diffusion for the time-dependent viscosity $\nu = 5 \cdot 10^{-3} g(t)/g(t_1)$, here $g(t) = \|\psi_{1z}(t)\|_{L^\infty} Z(t)^2$. The blue curve corresponds to the ratio between the vortex stretching term and the diffusion term evaluated at $(R(t), Z(t)$ and while the red curve corresponds to the ratio between the vortex stretching term and the diffusion term evaluated at the position where $\omega_1$ achieves its maximum. The solution is computed using $1536 \times 1536$ grid. The initial time is $t_1 = 0.002286845268$ and the final time is $t = 0.002286850566$. 


In Figure 4.1, we plot the ratio between the vortex stretching term \(2\psi_{1z}u_1\) and \(\nu(t)\Delta u_1\) evaluated at \((R(t), Z(t))\) (see the blue curve) for the Navier–Stokes equations with such time-dependent viscosity defined in (4.1). We also plot the ratio between the vortex stretching term \(2\psi_{1z}u_1z\) and \(\nu(t)\Delta \omega_1\) evaluated at the location where \(\omega_1\) achieves its maximum (see the red curve). We can see that vortex stretching term for the \(u_1\)-equation dominates the diffusion terms throughout the computation with a strong upward trend towards the end. The same observation also applies to the \(\omega_1\)-equation although the vortex stretching dominance seems to weaken toward the end.

As a comparison, we also compute the same ratio between vortex stretching and the diffusion term for the Navier–Stokes equations with a constant viscosity \(\nu = 5 \cdot 10^{-3}\) in Figure 4.3. We can see that the ratio between vortex stretching and the diffusion term for the \(u_1\)-equation has a downward trend, while the ratio between vortex stretching and the diffusion term for the \(\omega_1\)-equation remains relatively flat and has a slightly upward trend toward the end. Since the potential blowup is driven by large swirl \(\mu^0\), the downward trend in the ratio between vortex stretching and the diffusion term for the \(u_1\)-equation implies that the growth of \(u_1\) would eventually lose steam. We believe that this is the main reason for the Navier–Stokes equations to escape the finite time blowup.

To see how fast the time-dependent viscosity decays in time, we plot the inverse of \(g(t)\) versus the rescaled time \(\tilde{t} = \log(||\psi_{1z}||_{L^\infty})\). We first perform the fitting for the Navier–Stokes equations with a constant viscosity \(\nu = 5 \cdot 10^{-3}\) using the data from the late stage of the computation. The linear fitting for \(Z(t)^2\) in Figure 3.20 seems to imply that \(g(t)\) should decay roughly of the order \((\log(||\psi_{1z}||_{L^\infty}))^{-3}\). Indeed, Figure 4.3 (a) confirms that a cubic polynomial fitting for \(g(t)^{-1}\) as a function of \(\log(||\psi_{1z}||_{L^\infty})\) gives excellent fitting with R-Square values very close to 1. In Figure 4.3 (b), we investigate the growth rate of \(g(t)^{-1}\) as a function of \(\log(||\psi_{1z}||_{L^\infty})\) for the time-dependent viscosity \(\nu(t)\) using the data from the late stage of the computation. We observe that the growth rate of \(g(t)^{-1}\) as a function of \(\log(||\psi_{1z}||_{L^\infty})\) is also proportional to \((\log(||\psi_{1z}||_{L^\infty}))^2\).

In Figure 4.4, we plot the trajectory of \((R(t), Z(t))\) as a function of time. The left plot is the trajectory in a global view while the right plot show the evolution of the trajectory after we change the constant viscosity \(\nu = 5 \cdot 10^{-3}\) to the time-dependent viscosity. We can see that using such time-dependent viscosity, we can compute the solution much longer and the trajectory is getting very close to the origin, the point of potential finite time singularity.

4.2. Rapid growth of the solution. In this subsection, we present some numerical results that demonstrate the rapid dynamic growth of the solution using such time-dependent viscosity. In Figure 4.5 (the first row), we plot the dynamic growth of \(||u_1||_{L^\infty}, ||\omega_1||_{L^\infty}, \text{ and } ||\omega||_{L^\infty}\) as a function of time. We can see that all three quantities grow extremely fast in time, especially in the late stage. In the
Figure 4.3: (a) A cubic polynomial fitting of $\|\psi_{12}\|_{L^\infty}Z(t)^2$ vs $\log(\|\psi_{12}\|_{L^\infty})$ for the solution of the Navier–Stokes equations with a constant viscosity, $b_3 = 0.03773$, $b_2 = -2.12$, $b_1 = 40.08$, $b_0 = -253.1$. (b) A cubic order polynomial fitting of $\|\psi_{12}\|_{L^2}(t)^2$ vs $\log(\|\psi_{12}\|_{L^\infty})$ for the time-dependent viscosity, $b_3 = 0.346$, $b_2 = -23.25$, $b_1 = 522.7$, $b_0 = -3924$. The solution is computed using $1536 \times 1536$ grid. The fitting time for (a) is from $t = 0.002286820098176$ to $t = 0.002286850216035$, and the fitting time for (b) is from $t = 0.002286850016175$ to $t = 0.002286850555065$.

Figure 4.4: (a) the trajectory of $(R(t), Z(t))$, a global view, (b) the trajectory of $(R(t), Z(t))$ in late stage when we apply a time-dependent viscosity. In Figure (a), the trajectory first travels downward along $r = 0$, then moves outward horizontally, finally turns around to approach the origin.

second row of the same figure, we plot $\log(\log(\|u_1\|_{L^\infty}))$, $\log(\log(\|\omega_1\|_{L^\infty}))$, and $\log(\log(\|\omega\|_{L^\infty}))$ as a function of time. We can see that they also grow rapidly in time, certainly faster than linear growth in time, which implies that $\|u_1\|_{L^\infty}$, $\|\omega_1\|_{L^\infty}$, and $\|\omega\|_{L^\infty}$ all grow faster than double exponentially in time.

To quantify the growth rate of maximum vorticity relative to its initial maximum vorticity, we plot $\|\omega(t)\|_{L^\infty}/\|\omega(0)\|_{L^\infty}$ as a function of time in Figure 4.6 (a). We observe that by the time $t = 0.002286850566$, $\|\omega(t)\|_{L^\infty}/\|\omega(0)\|_{L^\infty}$ has already increased by a factor of more than $10^9$ and the time-dependent viscosity has decreased from $\nu = 5 \times 10^{-3}$ to $\nu = 4.154 \times 10^{-4}$. In Figure 4.6 (a), we compute the time integral of the maximum vorticity as a function of time, i.e. $\int_0^t \|\omega(s)\|_{L^\infty}ds$. We can see that the time integral $\int_0^t \|\omega(s)\|_{L^\infty}ds$ seems to blow up in a finite time as well. According to the Beale–Kato–Majda blowup criteria, this provides a strong evidence that the solution of the Navier–Stokes equations with such time-dependent viscosity seems to develop a finite time singularity.

In Figure 4.7, we show the 3D plots of $u_1$ and $\omega_1$ in a local domain at $t = 0.002286850566$ to illustrate the potential singular solution profile. We can see that the solution profile looks qualitatively
Figure 4.5: First row: the growth of $\|u_1\|_{L^\infty}$, $\|\omega_1\|_{L^\infty}$ and $\|\omega\|_{L^\infty}$ as functions of time. Second row: log log $\|u_1\|_{L^\infty}$, log log $\|\omega_1\|_{L^\infty}$ and log log $\|\omega\|_{L^\infty}$. The solution is computed using 1536 x 1536 grid for the time-dependent viscosity. The final time instant is $t = 0.002286850566$.

Figure 4.6: Left plot: the amplification of maximum vorticity relative to its initial maximum vorticity, $\|\omega(t)\|_{L^\infty}/\|\omega(0)\|_{L^\infty}$ as a function of time. Right plot: the time integral of maximum vorticity, $\int_0^t \|\omega(s)\|_{L^\infty} ds$ as a function of time. The solution is computed using 1536 x 1536 grid for the time-dependent viscosity. The final time instant is $t = 0.002286850566$.

similar to what we observed for the 3D Navier–Stokes equations with constant viscosity. But due to the decay of the time-dependent viscosity, the profile becomes more singular, but is still well resolved by our adaptive mesh.

We have also studied the 3D streamlines of the solution at time $t = 0.002286850566$ in Figure 4.8. By this time, the maximum vorticity has grown by a factor of more than $10^9$ compared with its initial maximum vorticity. We use a set of starting points similar to those that we used for the Navier–Stokes equations with a constant viscosity $\nu = 5 \cdot 10^{-3}$ in Figure 3.6 except for a small modification for cases (b) and (c). We observe essentially the same qualitative behavior as that for the Navier–Stokes equations with a constant viscosity, i.e. a tornado like singularity.

In Figure 4.9, we study the 2D velocity field and the dipole structure of the flow. In the left plot, we show the 2D velocity field with the contours of $\omega_1$ as a background for the solution of the Navier–Stokes
equations with a constant viscosity $\nu = 5 \cdot 10^{-3}$ at $t = 0.0022868452680$. We use the solution of the Navier–Stokes equations at this time as our initial condition for the Navier–Stokes equations with the time-dependent viscosity. In the right plot, we show the 2D velocity field with the contours of $\omega_1$ as a background for the solution of the Navier–Stokes equations with the time-dependent viscosity at time $t = 0.002286850566$. We can see that the solution with the time-dependent viscosity preserves the same qualitative dipole structure as the solution using a constant viscosity. On the other hand, the time-dependent viscosity also has a significant effect on the solution in the sense that $\omega_1$ develops a thinner structure. The solution has grown a lot in magnitude and has moved much closer to the origin by a factor of $20$.

The strong and stable growth of the solution is a consequence of the strong nonlinear alignment between $\psi_{1z}$ and $u_1$, as shown in Figure 4.10(a). In fact, we observe an upward trend of the alignment in the late stage. This strong nonlinear alignment between $\psi_{1z}$ and $u_1$ is also supported by the cross section plots along the $r$ and $z$ directions, see Figure 4.11. Again, we observe that there is a relatively flat region of $\psi_{1z}$ at $r = R(t)$ between $z = 0$ and $z = 0.5Z(t)$ in the $z$-cross section. This property generates a traveling wave for $u_1$ that propagates towards $z = 0$, overcoming the upward transport along the $z$-direction. Similarly, we observe that the maximum of $\psi_{1z}$ is located to the left hand side of the maximum position of $u_1$ along the $r$-cross section at $z = Z(t)$. This desirable property not only enhances the alignment between $\psi_{1z}$ and $u_1$, it also induces a traveling wave propagating towards $r = 0$. Notice that the two antisymmetric dipoles induce a negative radial velocity field that pushes the solution towards $r = 0$.

Finally, we observe in Figure 4.10(b) that the mild two-scale structure continues to persist for the Navier–Stokes equations with the time-dependent viscosity. The ratio $R(t)/Z(t)$ maintains a similar growing trend as we observe for the Navier–Stokes equations with constant viscosity. We will show later that both $R(t)$ and $Z(t)$ seem to have the same qualitative fitting as that for the Navier–Stokes equations with a constant viscosity.

4.3. Resolution study. In this subsection, we will perform resolution study for the Navier–Stokes equations with the time-dependent viscosity. We will follow the same strategy as we used for the Navier–Stokes equations with a constant viscosity.

4.3.1. Effectiveness of the adaptive mesh. In this subsection, we study the effectiveness of our adaptive moving mesh. To see how well the adaptive mesh resolves the solution, we first visualize how it transforms the solution from the $rz$-plane to the $\rho\eta$-plane. Figure 4.12(a) shows the function $u_1$ at $t = 0.002286850566$ in the original $rz$-plane. This plot suggests that the solution seems to develop a focusing singularity at the origin $(r, z) = (0, 0)$. For comparison, Figure 4.12(b) plots the profile of $u_1$
THE NEARLY SINGULAR BEHAVIOR OF THE 3D NAVIER-STOKES EQUATIONS

3D streamlines, $r_0 = 2R(t), z_0 = 0.01Z(t)$

(a) $r_0 = 2R(t), z_0 = 0.01Z(t)$

3D streamlines, $r_0 = 0.8R(t), z_0 = 0.75Z(t)$

(c) $r_0 = 0.8R(t), z_0 = 0.75Z(t)$, 3D view

(b) $r_0 = 1.05R(t), z_0 = 1.5Z(t)$

(d) same as (c), a top view

Figure 4.8: The streamlines of $(u^1(t), u^2(t), u^3(t))$ using $\nu(t)$ at time $t = 0.002286850566$ with initial points given by (a) $(r_0, z_0) = (2R(t), 0.01Z(t))$, (b) $(r_0, z_0) = (1.05R(t), 1.5Z(t))$, (c) $(r_0, z_0) = (0.8R(t), 0.75Z(t))$ (3D view), (d) $(r_0, z_0)$ is the same as (c), a top view. $(R(t), Z(t))$ is the maximum location of $u_1(t)$, indicated by the red ring. The green pole is the symmetry axis $r = 0$.

Velocity field $(u^r, u^\theta)$ at $t = 0.0022868452689$

Velocity field $(u^r, u^\theta)$ at $t = 0.002286850566$

Figure 4.9: The dipole structure of $\omega_1$ and the induced local velocity field at two different times, $t = 0.0022868452680$ (left plot) and $t = 0.002286850566$ (right plot), respectively. The solution is computed using $1536 \times 1536$ grid for the time-dependent viscosity. The red point is the maximum location $(R(t), Z(t))$ of $u_1(t)$.

at the same time in the $\rho\eta$-plane. We can see that our adaptive mesh resolves the potentially singular solution in the $(\rho, \eta)$ coordinates.

In Figure 4.13, we show the top views of the profiles of $u_1, \omega_1$ in a local domain at $t = 0.002286850566$. This figure demonstrates how the mesh points are distributed in different phases of
Figure 4.10: (a) the alignment between $\psi_{1z}$ and $u_1$ at $R(t), Z(t)$, (b) the ratio $R(t)/Z(t)$ for the time-dependent viscosity.

Figure 4.11: (a) the $r$-cross section of $u_1$ and $\psi_{1z}$, (b) the $z$-cross section of $u_1$ and $\psi_{1z}$ for the time-dependent viscosity.

Figure 4.12: The adaptive mesh resolves the solution in the $\rho\eta$-plane. (a) shows the focusing nearly singular profile of $u_1$ at $t = 0.002286850566$ in the $rz$-plane on the whole computational domain $\mathcal{D}_1$. (b) plots the profile of $u_1$ in the $\rho\eta$-plane. The solution is computed using $1536 \times 1536$ grid for the time-dependent viscosity.

the adaptive mesh. As we can see, the adaptive meshes have the most mesh points in phase 1 in both directions, and resolve the most singular part of the solution.

To obtain a quantitative measure on the effectiveness of our adaptive mesh, we again study the quantities $ME_\rho(\nu), ME_\eta(\nu)$. Figure 4.14 plots the mesh effectiveness functions of $u_1, \omega_1$ at time
Figure 4.13: The adaptive mesh has different densities in different region. Left: adaptive mesh for $u_1$. Right: adaptive mesh for $\omega_1$. The solution is computed using $1536 \times 1536$ grid for the time-dependent viscosity. The final time instant is $t = 0.002286850566$.

Figure 4.14: First row: the mesh effectiveness functions of $u_1$ at $t = 0.002286850566$. Second row: the mesh effectiveness functions of $\omega_1$ in the same setting. The solution is computed using $1536 \times 1536$ grid for the time-dependent viscosity.

t = 0.002286850566 on the mesh of size $(n_1, n_2) = (1536, 1536)$. We can see that these functions are all uniformly bounded in absolute value by a small number.

4.3.2. Resolution study. In this subsection, we perform resolution study on the numerical solutions of the Navier–Stokes equations with the time-dependent viscosity. We will estimate the relative error of a solution variable $f_p$ computed on the $256p \times 256p$ mesh by comparing it to a reference variable $f$ that is computed at the same time instant on the finest mesh of size $1536 \times 1536$. 
We will also study the convergence of $\|u_1\|_{L^\infty}$, $\|\omega_1\|_{L^\infty}$, $\|\omega\|_{L^\infty}$, and the kinetic energy $E$ as functions of time. Figures 4.15-4.16 plot the relative errors and numerical orders of these quantities as functions
of time. The results further confirm that our method is 2nd-order in \( h_p, h_\eta \). We also observe that the errors tend to have a growing trend near the end of the computation. This is in part due to the fact that the viscosity has decayed from \( \nu = 5 \cdot 10^{-3} \) at \( t = 0.002286845268 \) to \( \nu = 4.154 \cdot 10^{-4} \) by the end of the computation with \( t = 0.002286850566 \). By this time, the solution has developed a highly focusing potential singularity very close to the origin. The ratio between the maximum vorticity and the initial maximum vorticity has grown by a factor of \( 10^9 \). The solution becomes much more singular compared with the solution of the Navier–Stokes equations with a constant viscosity \( \nu = 5 \cdot 10^{-3} \). Our adaptive mesh strategy is beginning to lose its power due to the extremely high adaptive mesh ratio.

4.4. **Asymptotical scaling analysis.** In this subsection, we study the scaling properties of the solution of the Navier–Stokes equations with the time-dependent viscosity. All fittings are performed using the time interval \([0.002277465739, 0.002286850566]\) except for the late stage fitting of \( R(t) \) and \( Z(t) \). In Figure 4.17, we perform linear fitting for \( \|u_1\|_L^∞ \) and \( \|\psi_{12}\|_L^∞ \) as a function of time. We obtain an excellent linear fitting for these two quantities with R-Square values very close to 1. This suggests that we have

\[
\|u_1\|_L^∞ \sim \frac{1}{T-t}, \quad \|\psi_{12}\|_L^∞ \sim \frac{1}{T-t}.
\]

![Figure 4.17](image)

(a) linear regression of \( \|u_1(t)\|_L^∞ \) vs \( t \), (b) linear regression of \( \|\psi_{12}(t)\|_L^∞ \) vs \( t \) for the time-dependent viscosity. The blue points are the data points obtained from our computation, and the red lines are the linear models. The solution is computed using \( 1536 \times 1536 \) grid for the time-dependent viscosity. The fitting interval is \([0.002277465739, 0.002286850566]\).

Next, we study the growth of the maximum vorticity \( \|\omega\|_L^∞ \), which is an important indicator of a finite-time singularity. Figure 4.18 (a) shows the linear fitting of \( \log(\|\psi_{12}\|_L^∞)\|\omega(t)\|_L^∞ \) as a function of time. We see that \( \|\omega(t)\|_L^∞ = O(\log(\|\psi_{12}\|_L^∞)/(T-t)) \) has good linear fit with R-Square values very close to 1. Since we observe \( \|\psi_{12}(t)\|_L^∞ = O(1/(T-t)) \), this seems to imply that

\[
\|\omega(t)\|_L^∞ \sim \frac{\log(T-t)}{(T-t)}.
\]

If we apply the Beale-Kata-Majda blowup criteria, this would imply that the 3D Navier–Stokes equations with the time-dependent viscosity develop a finite time singularity. In Figure 4.18 (b), we also plot the linear fitting of \( \log(\|\psi_{12}\|_L^∞)\|\omega_1(t)\|_L^{2/3} \) as a function of time. We see that

\[
\|\omega_1(t)\|_L^∞ \sim \frac{\log(T-t)^{3/2}}{(T-t)^{3/2}}
\]

has excellent linear fit with R-Square values very close to 1.
To further illustrate the singular scaling properties of the Navier–Stokes equations with the time-dependent viscosity, we perform the linear fitting on the maximum velocity and $Z^1_t$. In Figure 4.18(b), we observe that $Z^1_t$ has good linear fitness. Such scaling property is consistent with the scaling properties of the maximum vorticity and maximum velocity. We observe that the fitting is not as good as for the maximum velocity. This is due to the fact that the solution develops a mild two-scale structure in the late stage.

In order to better understand the mild two-scale structure developed in the late stage, we find another fitting for $Z^1_t$ and $R^1_t$ by using the data only in the late stage. In Figure 4.19(b), we observe that $Z(t) \sim (T - t)^{1/2}$ has good linear fitness. Such scaling property is consistent with the scaling properties of the maximum vorticity and maximum velocity. We observe that the fitting is not as good as for the maximum velocity. This is due to the fact that the solution develops a mild two-scale structure in the late stage.
a linear fitting using the data from the time interval [0.002286562097, 0.002286850566], which is much later than the starting time \(t_1 = 0.002277464739\) when we turn on the time-dependent viscosity. We observe that both \(R(t)\) and \(Z(t)\) seem to decay faster than \((T - t)^{1/2}\). More specifically, we obtain

\[
R(t) \sim (T - t)^{1/2} / |\log(T - t)|^{-1}, \quad Z(t) \sim (T - t)^{1/2} / |\log(T - t)|^{-3/2}.
\]

The logarithmic correction in \(Z(t)\) may explain why we observe that the decaying factor \(g(t) = c_0 \|\psi_{12}\|_{L,\infty}^2 Z(t)^2\) seems to decay like \(O(|\log(T - t)|^{-3})\). This scaling relationship also explains the mild two-scale structure that we observe for \(R(t)/Z(t)\) in the late stage of the computation.

4.4.1. Numerical evidence of locally self-similar profiles. To study the nearly self-similar properties of the 3D Navier–Stokes equations with the time-dependent viscosity, we again study the solution in the dynamically rescaled variables \((\xi, \zeta)\) as follows:

\[
u_1(t,r,z) = \max(u_1)\overline{U}(t,\xi,\zeta), \quad \omega_1(t,r,z) = \max(\omega_1)\overline{\Omega}(t,\xi,\zeta),
\]

where

\[
\xi = \frac{r - R(t)}{Z(t)}, \quad \zeta = \frac{z}{Z(t)}.
\]

In Figure 4.21, we compare the level sets of \(u_1\) at different time instants. The first row of Figure 4.21 plots the level sets of \(u_1\) in a local domain. We plot the profiles in a short time interval at three different times with \(t_1 = 0.002286850559010\), \(t_2 = 0.002286850561332\) and \(t_3 = 0.002286850563531\). By the time \(t = t_3\), the maximum vorticity relative to its initial maximum vorticity has grown by a factor of \(10^9\). We can see that the singular support of the profile shrinks in space and travels towards the origin. In the second row of Figure 4.21, we can see that the rescaled profile \(\overline{U}\) (in the \((\xi,\zeta)\)-plane) is almost static during this time interval. This seems to suggest that there exists an approximate self-similar profile \(U(\xi,\zeta)\).

In Figure 4.22, we compare the level sets of \(\omega_1\) and the level sets of the spatially rescaled profile \(\overline{\Omega}\) in a similar manner. Again, we can see that although the profile of \(\omega_1\) has changed noticeably, the rescaled profile \(\overline{\Omega}(t,\xi,\zeta)\) seems to be very close to a limiting profile \(\Omega(\xi,\zeta)\).

We further compare the cross sections of the solution at different time instants to study the potential self-similar blowup. In Figure 4.23(a) and (c), we present the evolution of the cross sections of \(u_1\) through the point \((R(t),Z(t))\) in both directions. The length scale of the profile shrinks in both
Figure 4.21: Comparison of the level sets of $u_1$ at different time instants. First row: original level sets of $u_1$ in the domain $(r, z)$ in different times. Second row: rescaled level sets of $u_1$ as a function of $(\xi, \zeta)$ in the domain $(\xi, \zeta)$. The computation is performed using the $1536 \times 1536$ grid. Here $t_1 = 0.002286850559010$, $t_2 = 0.002286850561332$ and $t_3 = 0.002286850563531$.

directions, and the sharp front along the $r$-direction travels towards $r = 0$. For comparison, Figure 4.23(b) and (d) plot the corresponding cross sections of the rescaled profile $\tilde{U}$ in the dynamically rescaled variables $(\xi, \eta)$. We observe that the rescaled profile $\tilde{U}$ is very close to a limiting profile. These results support the existence of an approximate self-similar profile of the solution. Note that the maximum vorticity has increased by a factor of $10^9$ by the final time.

5. Concluding Remarks

In this paper, we presented numerical evidences that the 3D axisymmetric Navier–Stokes equations with our initial condition develop nearly singular solutions at the origin. This nearly singular behavior of the Navier–Stokes equations is induced by the potential Euler equations at the origin [20]. The solution of the Navier–Stokes equations with our initial condition preserves many essential features of the Euler solution using the same initial condition. An important feature of the potential Euler singularity is that it has singular scaling properties that are compatible with those of the 3D Navier–Stokes equations. Unlike the two-scale traveling wave solution studied in [21], our solutions for the 3D Euler and Navier–Stokes equations have essentially a one-scale structure. The relatively large viscosity regularizes the sharp front in the Euler solution and enhances the nonlinear alignment of vortex stretching. We
observed a relatively long stable phase of strong nonlinear alignment of vortex stretching, which is responsible for the rapid growth of the maximum vorticity and the nearly self-similar scaling properties. Moreover, we showed that the rescaled solution profiles in terms of the dynamically rescaled variable \((\xi, \zeta)\) seem to be very stable with respect to the small perturbation of the initial data.

Despite the rapid transient growth of the solution, the 3D Navier–Stokes equations with our initial condition do not develop a finite time singularity due to the development of a mild two-scale structure in the late stage of the computation, which eventually leads to the viscous dominance over vortex stretching. We observed that the \(L^3\) norm of the velocity field remains bounded throughout the computation. This provides a strong evidence that the 3D Navier–Stokes equations with our initial condition manage to escape a finite time singularity.

To maintain the balance between the vortex stretching term and the diffusion term, we solved the Navier–Stokes equations with a time-dependent viscosity. The decay of the viscosity is roughly of the order \(O(|\log(T - t)|^{-3})\). The solution of the Navier–Stokes equations with such time-dependent viscosity preserves almost all major features of the solution of the Navier–Stokes equations with a constant viscosity. We presented strong numerical evidence that the Navier–Stokes equations with such time-dependent viscosity using our initial condition seem develop a finite time singularity.

Figure 4.22: Comparison of the level sets of \(\omega_1\) at different time instants. First row: original level sets of \(u_1\) in the domain \((r, z)\) in different times. Second row: rescaled level sets of \(\omega_1\) as a function of \((\xi, \zeta)\) in the domain \((\xi, \zeta)\). The computation is performed using the 1536 \(\times\) 1536 grid. Here \(t_1 = 0.002286850559010\), \(t_2 = 0.002286850561332\) and \(t_3 = 0.002286850563531\).
Figure 4.23: Cross sections and rescaled cross sections of \( u_1 \) through the point \( R(t), Z(t) \) in both directions at different time instants. (a) Cross sections in the \( r \) direction. (b) Rescaled cross sections in the \( r \) direction. (c) Cross sections in the \( z \) direction. (d) Rescaled cross sections in the \( z \) directions. The computation is performed using the 1536 \( \times \) 1536 grid for the time-dependent viscosity. Here \( t_1 = 0.002286850555065 \), \( t_2 = 0.002286850559010 \), \( t_3 = 0.002286850561332 \) and \( t_4 = 0.002286850563531 \).

For the axisymmetric Navier–Stokes equations, the diffusion term for the total circulation \( \Gamma \) has the form \( \Delta - \frac{2}{r} \partial_r \). Thus, the diffusion term introduces an additional transport term \( \nu \frac{2}{r} \Gamma_r \) in the \( \Gamma \)-equation, which transports \( \Gamma \) away from \( r = 0 \). Since \( \Gamma = 0 \) at \( r = 0 \), this additional transport term has a stabilizing effect of keeping \( \Gamma \) small near \( r = 0 \) and counteracting the negative radial velocity field that generates a sharp front and pushes the solution towards \( r = 0 \). This could be one of the mechanisms that prevents a finite time singularity for the axisymmetric Navier–Stokes equations. According to [31], the rate at which \( \Gamma \rightarrow 0 \) as \( r \rightarrow 0 \) is critical for the non-blowup of the Navier–Stokes equations.

It would be very interesting to further investigate whether the 3D Navier–Stokes equations without imposing the axisymmetry would develop a finite time singularity. A first step would be to find out whether there exists a potential singular solution of the non-axisymmetric 3D Navier–Stokes equations that is very close to the one that we have computed in this paper. An effective way to investigate this question is to solve the 3D Navier–Stokes equations using the dynamic rescaling formulation (see e.g. [37, 6, 8, 7]). We can use the approximate self-similar profile that we obtained in this paper as an initial condition. Since the solution of the Navier–Stokes equations with our initial condition does not have exactly self-similar scaling properties (see also [38, 41]), it is very likely that the dynamically rescaled Navier–Stokes equations would not converge to a steady state as the rescaled time tends to infinity. However, as long as there exists a nontrivial approximate steady state with a uniformly
negative scaling exponent $c_\omega$ (see e.g. [8, 6]) as the rescaled time tends to infinity, the solution of the 3D Navier–Stokes equations will develop a finite time singularity. It is possible that such approximate self-similar profile may have some unstable modes in the sense that some of the eigenvalues of the linearized operator around the approximate self-similar profile may have positive real parts. If this is the case, we may not be able to obtain an approximate steady state by solving the dynamically rescaled Navier–Stokes equations as an initial value problem. Instead, we can use an iterative method such as the Newton Method to solve the steady state equation directly by using the approximate self-similar profile that we obtained in this paper as an initial guess. This may enable us to obtain an approximate steady state with a small residual. By analyzing the spectrum property of the linearized operator around this approximate steady state, we can get some insight on the stability of the potential singular solution. We would like to explore this line of research in our future work.

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Appendix A. Construction of the adaptive mesh

In this appendix, we describe our adaptive mesh strategy to resolve the singularity formation near the origin $(r, z) = (0, 0)$. We will use the method described in Appendix B of [21] to construct our adaptive mesh maps $r = r(\rho)$ and $z = z(\eta)$. We will discretize the equations in the transformed variables $(\rho, \eta)$ with $n_1$ grid points along the $z$ direction and $n_2$ grid points along the $r$-direction.

A.1. The adaptive (moving) mesh algorithm. To effectively and accurately compute the potential blowup, we have carefully designed a special meshing strategy that is dynamically adaptive to the more and more singular structure of the solution. The adaptive mesh covering the half-period computational domain $\mathcal{D}_1 = \{(r, z) : 0 \leq r \leq 1, 0 \leq z \leq 1/2\}$ is characterized by a pair of analytic mesh mapping functions

$$r = r(\rho), \quad \rho \in [0, 1]; \quad z = z(\eta), \quad \eta \in [0, 1].$$

These mesh mapping functions are both monotonically increasing and infinitely differentiable on $[0, 1]$, and satisfy $r(0) = 0$, $r(1) = 1$, $z(0) = 0$, $z(1) = 1/2$. In particular, we construct these mapping functions by carefully designing their Jacobians/densities

$$r_\rho = r'(\rho), \quad z_\eta = z'(\eta),$$

using analytic functions that are even functions at 0. The even symmetries ensure that the resulting mesh can be smoothly extended to the full-period cylinder $\{(r, z) : 0 \leq r \leq 1, -1/2 \leq z \leq 1/2\}$. The density functions contain a small number of parameters, which are dynamically adjusted to the solution. Once the mesh mapping functions are constructed, the computational domain is covered with a tensor-product mesh:

$$\mathcal{G} = \{(r_i, z_j) : 0 \leq i \leq n_2, 0 \leq j \leq n_1\},$$

where $r_i = r(ih_\rho)$, $h_\rho = 1/n_2$; $z_j = z(jh_\eta)$, $h_\eta = 1/n_1$. The precise definition and construction of the mesh mapping functions are described in Appendix B of [21].

Figure A.1 gives an example of the densities $r_\rho, z_\eta$ (in log scale) we use in the computation. We design the densities $r_\rho, z_\eta$ to have three phases:
Phase 1 covers the inner profile of the smaller scale near the sharp front; Phase 2 covers the outer profile of the larger scale of the solution; Phase 3 covers the (far-field) solution away from the symmetry axis $r = 0$.

We add a phase 0 in the density $r_\rho$ to cover the region near $r = 0$ and also add a phase 0 in the density $z_\eta$ to cover the region near $z = 0$ in the late stage. In our computation, the number (percentage) of mesh points in each phase are fixed, but the physical location of each phase will change in time, dynamically adaptive to the structure of the solution. Between every two neighboring phases, there is also a smooth transition region that occupies a fixed percentage of mesh points.

In the second time period, we use the following parameters $r_1 = 0.001$, $r_2 = 0.05$, $r_3 = 0.2$ and $s_{p_1} = 0.001$, $s_{p_2} = 0.5$, $s_{p_3} = 0.85$ to construct the mapping $r = r(\rho)$ using a four-phase map. Similarly, we use the following parameters $s_1 = 0.1$, $s_2 = 0.25$ and $s_{\eta_1} = 0.5$, $s_{\eta_2} = 0.85$ to construct the mapping $z = z(\eta)$ using a three-phase map. We then update the mesh $z = z(\eta)$ dynamically using $s_1 = 2\xi(I)$ and $s_2 = 10\xi(I)$ with $s_{\eta_1} = 0.6$, $s_{\eta_2} = 0.9$ when $I < 0.2n_1$, but keep $r = r(\rho)$ unchanged during this early stage. Here $I$ is the grid point index along the $r$-direction at which $\omega_1$ achieves its maximum.

In the second time period, we use the following parameters $s_{p_1} = 0.05$, $s_{p_2} = 0.6$, $s_{p_3} = 0.9$, $r_2 = r(J) + 2dr$, $r_1 = \max((s_{p_1}/s_{p_2})r_2, r(J_r) - 5dr)$, and $r_3 = \max(3r(J), (r_2 - r_1)(s_{p_3} - s_{p_2})/(s_{p_2} - s_{p_1}) + r_2)$, where $J$ is the grid index at which $u_1$ achieves its maximum along the $r$-direction, $J_r$ is the grid index at which $u_{1,r}$ achieves its maximum along the $r$-direction, and $dr = r(J_r) - r(J_r)$. We update the mapping $r(\rho)$ dynamically when $J_r < 0.2n_2$. The adaptive mesh map for $z(\eta)$ in the second time period remains the same as in the first time period.

In the third time period, due to the viscous regularization, we do not need to allocate as many grid points to resolve the sharp front. Instead we allocate more grid points to cover the intermediate region in phase 2. We use the following parameters $s_{p_1} = 0.05$, $s_{p_2} = 0.5$, $s_{p_3} = 0.9$, $r_2 = r(J) + 8dr$, $r_1 = \max((s_{p_1}/s_{p_2})r_2, r(J_r) - 3dr)$, and $r_3 = \max(4.5r(J), (r_2 - r_1)(s_{p_3} - s_{p_2})/(s_{p_2} - s_{p_1}) + r_2)$. To construct the mesh map $z(\eta)$, we use the following parameters $s_{\eta_1} = 0.05$, $s_{\eta_2} = 0.5$, $s_{\eta_3} = 0.85$, $z_2 = z(I_w) + 2dz$, $z_1 = \max((s_{\eta_1}/s_{\eta_2})z_2, z(I_{wz}) - 6dz)$, and $r_3 = \max(4.3z(I_w), (z_2 - z_1)(s_{\eta_3} - s_{\eta_2})/(s_{\eta_2} - s_{\eta_1}) + s_2)$, where $I_w$ is the grid index at which $\omega_1$ achieves its maximum along the $z$-direction, $I_{wz}$ is the grid index at which $\omega_2$ achieves its maximum along the $z$-direction, and $I_{wz} = I_w + n_z$, $n_z$ is the number of time steps equal to 45000. The second time period corresponds to the time interval $t_1 = 0.002191729$ and $t_2 = 0.002261605$ with $\| \omega(t_2) \|_{L^\infty} / \| \omega(0) \|_{L^\infty} \approx 139.5777$ for the 1536 $\times$ 1536 grid and the number of time steps equal to 60000. The third time period is for $t \geq t_2$.

A.2. Adaptive mesh for the 3D Navier–Stokes equations. We also use three different adaptive mesh strategies for three different time periods. The first time period corresponds to the time interval between $t = 0$ and $t_1 = 0.001$ to $t = 0.002191729$ with $\| \omega(t_1) \|_{L^\infty} / \| \omega(0) \|_{L^\infty} \approx 20.5235$ for the 1536 $\times$ 1536 grid and the number of time steps equal to 45000. The second time period corresponds to the time interval between $t_1 = 0.002191729$ and $t_2 = 0.002261605$ with $\| \omega(t_2) \|_{L^\infty} / \| \omega(0) \|_{L^\infty} \approx 139.5777$ for the 1536 $\times$ 1536 grid and the number of time steps equal to 60000. For the first time period, since we use a very smooth initial condition whose support covers the whole domain, we use the following parameters $r_1 = 0.001$, $s_{p_1} = 0.001$, $s_{p_2} = 0.5$, $s_{p_3} = 0.85$ to construct $r = r(\rho)$ using a four-phase map. Similarly, we use the following parameters $s_1 = 0.1$, $s_2 = 0.25$ and $s_{\eta_1} = 0.5$, $s_{\eta_2} = 0.85$ to construct the mapping $z = z(\eta)$ using a three-phase map. We then update the mesh $z = z(\eta)$ dynamically using $s_1 = 2\xi(I)$ and $s_2 = 10\xi(I)$ with $s_{\eta_1} = 0.6$, $s_{\eta_2} = 0.9$ when $I < 0.2n_1$, but keep $r = r(\rho)$ unchanged during this early stage. Here $I$ is the grid point index along the $z$-direction at which $\omega_1$ achieves its maximum.

In the second time period, we use the following parameters $s_{p_1} = 0.05$, $s_{p_2} = 0.6$, $s_{p_3} = 0.9$, $r_2 = r(J) + 2dr$, $r_1 = \max((s_{p_1}/s_{p_2})r_2, r(J_r) - 5dr)$, and $r_3 = \max(3r(J), (r_2 - r_1)(s_{p_3} - s_{p_2})/(s_{p_2} - s_{p_1}) + r_2)$, where $J$ is the grid index at which $u_1$ achieves its maximum along the $r$-direction, $J_r$ is the grid index at which $u_{1,r}$ achieves its maximum along the $r$-direction, and $dr = r(J_r) - r(J_r)$. We update the mapping $r(\rho)$ dynamically when $J_r < 0.2n_2$. The adaptive mesh map for $z(\eta)$ in the second time period remains the same as in the first time period.

In the third time period, due to the viscous regularization, we do not need to allocate as many grid points to resolve the sharp front. Instead we allocate more grid points to cover the intermediate region in phase 2. We use the following parameters $s_{p_1} = 0.05$, $s_{p_2} = 0.5$, $s_{p_3} = 0.9$, $r_2 = r(J) + 8dr$, $r_1 = \max((s_{p_1}/s_{p_2})r_2, r(J_r) - 3dr)$, and $r_3 = \max(4.5r(J), (r_2 - r_1)(s_{p_3} - s_{p_2})/(s_{p_2} - s_{p_1}) + r_2)$.

Figure A.1: The mapping densities $r_\rho$ (left) and $z_\eta$ (right) with phase numbers labeled.
index at which $\omega_{1,z}$ achieves its maximum along the $z$-direction, and $dz = z(I_w) - r(I_wz)$. We will update $r(\rho)$ dynamically when $J_r < 0.2n_2$ and update $z(\eta)$ when $I_z < 0.2n_1$.

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