Stability and control of a confined 1D quantum system with time-dependent delta potentials

Andrea Mantile

CPT-CNRS, UMR 6207, Université du Sud, Toulon-Var, B.P. 20132, 83957 La Garde Cedex, France

E-mail: andrea.mantile@cpt.univ-mrs.fr

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Abstract

The evolution problem for a quantum particle confined in a 1D box and interacting with one fixed point through a time-dependent point interaction is considered. Under suitable assumptions of regularity for the time profile of the Hamiltonian, we prove the existence of strict solutions to the corresponding Schrödinger equation. The result is used to discuss the stability and steady-state local controllability of the wavefunction when the strength of the interaction is used as a control parameter.

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1. Introduction

We consider the evolution problem for a quantum particle confined in a 1D bounded domain and interacting with one fixed point through a delta-shaped potential whose strength varies in time. The Hamiltonian associated with this class of potentials, denoted in the following with $H_{\alpha(t)}$, is defined in terms of a time-dependent parameter, $\alpha(t)$, which describes the time profile of the interaction. Under suitable regularity assumptions, we study the dynamical system defined by $H_{\alpha(t)}$, its stability properties and the local controllability of the dynamics when $\alpha$ is used as a control function.

General conditions for the solution to the quantum evolution equation related to nonautonomous Hamiltonians, $H(t)$, have been investigated for many years (e.g. in [18, 20] and [12]). When the operator’s domain $D(H(t))$ depends on time (as in the case of time-dependent point interactions), the Cauchy problem

$$\begin{cases}
\frac{d}{dt} \psi = -iH(t)\psi \\
\psi_{t=0} = \psi_0
\end{cases}$$

(1)

was explicitly considered in [15] by Kisynski using coercivity and $C^2_{\text{loc}}$-regularity of $t \rightarrow H(t)$. A similar assumption, $\alpha \in C^2_{\text{loc}}(t_0, +\infty)$ in the above notation, is used in the works of Yafaev
[21, 22] and [23] (with Sayapova) to prove the existence of a strongly differentiable time propagator for the scattering problem with time-dependent delta interactions in \( \mathbb{R}^3 \). Such a condition, however, can be relaxed by exploiting the explicit character of point interactions. The dynamics associated with this class of operators is essentially described by the evolution of a finite-dimensional variable related to the values taken by the regular part of the state in the interaction points (e.g. in [2, 4, 11]). This leads to simplified evolution equations allowing rather explicit estimates under the Fourier transform. Using this approach, the quantum evolution problem for a 1D time-dependent delta interaction has been considered in [14]. In particular, it is shown that \( \alpha \in H^2(0, T) \) allows us to define a strongly continuous dynamical system in \( L^2(\mathbb{R}) \), while \( \alpha \in H^\frac{3}{2}(0, T) \) leads to the existence of strong solutions to

\[
\begin{aligned}
\frac{d}{dt} \psi &= -iH_{\alpha(t)} \psi \\
\psi_{t=0} &= \psi_0 \in D(H_{\alpha(0)}).
\end{aligned}
\]  

The same strategy has also been adopted to study the diffusion problem with nonautonomous delta potentials in \( \mathbb{R}^3 \) [11], and the quantum evolution problem for a 1D nonlinear model where the parameter \( \alpha \) is assigned as a function of the particle’s state [2].

The techniques used in these works can be adapted to the 1D confining case. However, the lack of a simple explicit expression for the free propagator kernel and the use of eigenfunction expansions, which replaces the Fourier transform analysis in this setting, necessarily require some additional efforts to obtain a result. In the perspective of the stability and controllability analysis, we restrict ourselves to the case \( \alpha \in H^1(0, T) \) and \( H_\alpha = -\frac{d^2}{dx^2} + \alpha \delta \) with Dirichlet conditions on the boundary of \( I = [-\pi, \pi] \). Under these assumptions, we prove that the evolution problem (2) admits a strongly differentiable time propagator preserving the regularity of the initial state. This turns out to be a key point to study the controllability of the dynamics.

The controllability of a quantum dynamics through an external field has attracted increasing attention due to possible applications in nuclear magnetic resonance, laser spectroscopy, photochemistry and quantum information. This problem has been considered for confining Schrödinger operators with regular potentials of the form \( H = -\Delta_\Omega^D + V(x) + u(t)W(x) \), where \( \Delta_\Omega^D \) is the Dirichlet Laplacian in the bounded domain \( \Omega \subset \mathbb{R}^m \), while \( u \) is used as a control function. The particular setting \( m = 1, V = 0 \) and \( W(x) = \chi \), corresponding to a quantum particle confined in a 1D box and moving under the action of a time-dependent uniform electric field, has been considered in [6]. The exact controllability of the quantum state was proved, in this case, in \( H^2 \)-neighbourhoods of the steady states by using \( u \in L^2(0, T) \) with \( T > T_m > 0 \) (actually, a simpler version of this proof holding for all \( T > 0 \) is given in [8]). For the same system, the exact controllability between neighbourhoods of any couple of eigenstates is discussed in [7]. In the more general setting, with \( V, W \in C^\infty(\Omega, \mathbb{R}^m) \) and \( m \) being any space dimension, different approximate controllability results in \( L^2 \) have been presented in [10] and [16]. An example concerned with singular pointwise potentials is presented in [1], where point interaction Hamiltonians are used to construct a general scheme allowing the system to be steered between the eigenstates of a drift operator \( H_0 \) (the 1D Dirichlet Laplacian). The idea is to introduce adiabatic perturbations of the spectrum of \( H_0 \) producing eigenvalues’ intersection, while controlling the state’s evolution through the adiabatic theorem. The result is the approximate state to state controllability in infinite time. The particular case of 1D point interactions is also related to control problems on quantum graphs where these Hamiltonians naturally arise.

In section 4, the nonlinear map \( \alpha \to \psi_T \), associating with the function \( \alpha \) the solution at time \( T \) of (2), is considered for a fixed initial state \( \psi_0 \in D(H_{\alpha(0)}) \). This can be regarded as a control system where the control parameter is \( \alpha \) and the target state is \( \psi_T \). Using the
regularity properties of the time propagator, we show that for \( \psi_0 \in H^2 \cap H_0^1(I) \), the map \( \alpha \to \psi_T \) is of class \( C^1(\mathbb{H}^1(0, T), H^2 \cap H_0^1(I)) \). Then, the local controllability relies on the surjectivity of the corresponding linearized map according to an inverse function argument. This point is considered in section 4.2 where a general condition for the solution of the linearized control problem is given in (102), (103) and proposition 17. When \( \psi_0 \) coincides with an even eigenstate of the Dirichlet Laplacian, this scheme provides with a result of local steady state controllability in finite time. Possible extensions of this result according to the location of the interaction point are discussed in section 4.3.

2. The model

Point interactions form a particular class of singular perturbations of the Laplacian supported by finite set of points. In \( \mathbb{R}^d \), \( d \leq 3 \), these Hamiltonians have been rigorously defined using the theory of self-adjoint extensions of symmetric operators (e.g. in [4]). The definition easily extends to the case of bounded regions by taking into account the Dirichlet conditions on the boundary for the functions in the operator’s domain (e.g. in [9]). In what follows, we consider a delta perturbation of the Dirichlet Laplacian, centered in the origin of the interval \( I = [-\pi, \pi] \). In terms of quadratic forms, this family of Hamiltonians acts on \( H_0^1(I) \) as

\[
H_\alpha = -\Delta_D^I + \alpha \delta, \quad (3)
\]

where \( \Delta_D^I \) is the Dirichlet Laplacian on the interval, \( \delta \) is the Dirac distribution centered in the origin, while \( \alpha \) is a real parameter. The operator’s domain \( D(H_\alpha) \) extends to all those vectors \( \psi \in L^2(I) \), such that \( H_\alpha \psi \in L^2(I) \). The description of \( D(H_\alpha) \) is strictly related to the properties of Green’s kernel, \( G_\alpha \), for the resolvent of the unperturbed operator \( (-\Delta_D^I + z)^{-1} \).

Fix \( x' \in I \); whenever \( z \) does not belong to the spectrum of \( \Delta_D^I \), \( G_\alpha \) satisfies the equation

\[
(-\Delta_D^I + z)G_\alpha(x, x') = \delta(x - x'), \quad z \in \mathbb{C} \setminus \sigma_D^I. \quad (4)
\]

The solution to (4) can be represented as the sum of the ‘free’ Green’s function plus an additional term taking into account the conditions at the boundary

\[
G_\alpha(x, x') = \frac{e^{-\sqrt{z}|x-x'|}}{2\sqrt{z}} - h(x, x', z) \quad (5)
\]

\[
\begin{align*}
(-\Delta_D^I + z)h(\cdot, x', z) &= 0 \\
h(\cdot, x', z)|_{x=\pm\pi} &= \frac{e^{-\sqrt{z}|x'|}}{2\sqrt{z}} |_{x=\pm\pi}.
\end{align*} \quad (6)
\]

The solution of (6) is

\[
G_\alpha(x, x') = \frac{e^{-\sqrt{z}|x-x'|}}{2\sqrt{z}} - \frac{1}{\sqrt{z}} \frac{e^{-\pi\sqrt{z}}}{e^{2\pi\sqrt{z}} - e^{-2\pi\sqrt{z}}} \times [e^{\sqrt{z}\sinh(\sqrt{z}(\pi + x'))} + e^{-\sqrt{z}\sinh(\sqrt{z}(\pi - x'))}]. \quad (7)
\]

Another useful representation of \( G_\alpha \) is in terms of the Fourier expansion w.r.t. Laplacian’s eigenfunctions. Denoting with \( \psi_k \) and \( \lambda_k \) respectively the eigenfunctions and eigenvalues of \( -\Delta_D^I \), we have

\[
\psi_k(x) = \begin{cases} \\
1 \sin \frac{k}{2}x & k \text{ even} \\
1 \cos \frac{k}{2}x & k \text{ odd}
\end{cases} \quad ; \quad \lambda_k = \frac{k^2}{4}, \quad k \in \mathbb{N} \quad (8)
\]
\[ \mathcal{G}_0(x, x') = \sum_{k \in \mathbb{N}} \frac{1}{\lambda_k + z} \psi_k^*(x') \psi_k(x), \quad z \notin \sigma_{\Delta_0}. \]  

Using (3) and (4), a straightforward calculation shows that \( H_\alpha \psi \in L^2(I) \) whenever \( \psi \) has the form

\[ \psi = \phi + q \mathcal{G}_0(\cdot, 0), \quad \phi \in D(\Delta_0^D), \quad -q = \alpha \psi(0). \]  

This is a general characterization of \( D(H_\alpha) \). It can be derived by using the von Neumann theory of self-adjoint extensions to identify \( H_\alpha \) as the extension of the symmetric operator \( H_0 \)

\[ \begin{align*}
D(H_0) &= \{ \psi \in H^2 \cap H^1_0(I) \mid \psi(0) = 0 \} \\
H_0 \psi &= -\frac{d^2}{dx^2} \psi
\]  

to those vectors \( \psi \in D(H^*_0) \) fulfilling the self-adjoint boundary conditions (e.g. in [5] and [18])

\[ \begin{align*}
\psi(0^+) - \psi(0^-) &= 0 \\
\psi'(0^+) - \psi'(0^-) &= \alpha \psi(0).
\]  

The following proposition is a rephrasing of the result in [4] in the bounded domain case.

**Proposition 1.** Let \( \alpha \in \mathbb{R}, \lambda \in \mathbb{C}\setminus\sigma_{\Delta_0} \) and denote with \( H_\alpha \) the family of self-adjoint extensions of \( H_0 \) associated with the boundary condition (12). The following representation holds:

\[ \begin{align*}
D(H_\alpha) &= \{ \psi \in H^2 \cap H^1_0(I) \mid \psi = \phi \lambda + q \mathcal{G}_\lambda(\cdot, 0); \quad \phi \lambda \in H^2 \cap H^1_0(I); \quad -q = \alpha \psi(0) \} \\
H_\alpha \psi &= -\frac{d^2}{dx^2} \phi \lambda - \lambda q \mathcal{G}_\lambda(\cdot, 0).
\]  

Moreover, for \( z \in \mathbb{C}\setminus\mathbb{R}, \psi \in L^2(I), \) the resolvent \((H_\alpha + z)^{-1}\) is written as

\[ (H_\alpha + z)^{-1} \psi = \left( -\Delta_0^D + z \right)^{-1} \psi - \frac{\alpha}{1 + \alpha \mathcal{G}_0(0)} \left( -\Delta_0^D + z \right)^{-1} \psi(0) \mathcal{G}_0(\cdot, 0). \]  

**Remark 2.** Although the representation \( \psi = \phi \lambda + q \mathcal{G}_\lambda(\cdot, 0) \) may change with different choices of \( \lambda \), the operator \( H_\alpha \) depends only on the value of the parameter \( \alpha \) related to the interaction’s strength.

Next we assign \( \alpha \) as a function of time and consider the nonautonomous quantum system defined by \( H_{\alpha(t)} \)

\[ \begin{align*}
\frac{d}{dt} \psi(x, t) &= H_{\alpha(t)} \psi(x, t), \\
\psi(x, 0) &= \psi_0(x) \in D(H_{\alpha(0)}).
\]  

The mild solutions of (16) are defined by the integral equation

\[ \begin{align*}
\psi(t, \cdot) &= e^{i t \Delta_0^D} \psi_0 + i \int_0^t q(s) e^{i(t-s) \Delta_0^D} \delta ds, \\
-q(t) &= \alpha(t) \psi(0, t),
\]  

(17)
where $e^{it\Delta}$ is the time propagator associated with $-\Delta_f$, while the relation $-q(t) = \alpha(t)\psi(0, t)$ fixes the boundary condition of the operator's domain at time 't'. The action of the free time propagator $e^{i\Delta_f}$ on $L^2(I)$ is

$$e^{i\Delta_f} f = \sum_{k \in \mathbb{N}} (\psi_k, f)_{L^2(I)} e^{-ik\sqrt{\lambda}} \psi_k \quad \forall f \in L^2(I),$$

(18)

where the scalar product in $L^2$ is defined according to $(u, v)_{L^2} = \int_{-\pi}^\pi \bar{u} v$. This relation suggests to replace the distributional part on the rhs of (17) with

$$\frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{N}} \int_0^t q(s) e^{-ik\sqrt{\lambda}(t-s)} ds \psi_k.$$

It follows that

$$\psi(\cdot, t) = e^{i\Delta_f} \psi_0 + \frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{N}} \int_0^t q(s) e^{-ik\sqrt{\lambda}(t-s)} ds \psi_k$$

(19)

and

$$q(t) = -\alpha(t) \left[ e^{i\Delta_f} \psi_0(0) + \frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{N}} \int_0^t q(s) e^{-ik\sqrt{\lambda}(t-s)} ds \psi_k \,ight].$$

(20)

The essential information about the dynamics described in (19) and (20) is contained in the auxiliary variable $q$, usually referred to as the charge of the particle (e.g. in [4]). In what follows, the solution of the above problem is considered under suitable regularity assumptions for $\alpha$. The equivalence with the original Cauchy problem (16) is further addressed. Finally, the dependence of this solution from $\alpha$, considered as a control parameter, is investigated. Our result is exposed in the following theorem.

**Theorem 3.** Assume $\alpha \in H^1(0, T)$, $\psi_0 \in D(H_{10}(0))$ and let $\psi_T(\psi_0, \alpha)$, $q(t, \psi_0, \alpha)$ respectively, denote the solution at time $T$ and the charge as functions of the initial state and of the parameter $\alpha$. The following properties hold.

(1) The system (19)–(20) admits a unique solution $\psi_t \in C(0, T; H^1(I) \cap C^1(0, T; L^2(I)))$ with $\psi_t \in D(H_{10})$ and $i\partial_t \psi_t = H_{10(0)} \psi_t(\cdot, t)$ at each $t$.

(2) The map $\alpha \to \psi_T(\psi_0, \alpha)$ is $C^1(H^1(0, T), H^1_0(I))$ in the sense of Fréchet. Moreover, the 'regular part' of the solution at time $T$, defined by $\psi_T(\psi_0, \alpha, \alpha) = q(T, \psi_0, \alpha)G_{\alpha},$ and considered as a function of $\alpha$, is of class $C^1(H^1(0, T), H^2 \cap H^1_0(I))$. If $\alpha \in H^1_0(0, T)$, then $\psi_T(\psi_0, \alpha)$ coincides with its regular part and $\alpha \to \psi_T(\psi_0, \alpha)$ in $C_0^1(H^1_0(0, T), H^2 \cap H^1_0(I)).$

(3) Let $\alpha \in H^1_0(0, T)$ and $W$ be the subspace of $H^2 \cap H^1_0(I)$ generated by the system $\{\psi_k, \; k \text{ odd}\}$. If $\psi_0 \in W$, then $\alpha \to \psi_T(\psi_0, \alpha) \in C^1(H^1_0(0, T), W)$. If $\psi_0 = \psi_k$ for a fixed $k$ odd and $T \geq \pi$, then there exists an open neighbourhood $V \times \mathbb{R} \subseteq H^1_0(0, T) \times W$ of the point $(0, \psi_k)$ such that $\alpha \to \psi_T(\psi_0, \alpha)|_V$: $V \to \mathbb{R}$ is surjective.

The proof is developed in propositions 10, 13, 15 and in theorem 18 of sections 3 and 4. A possible extension of theorem 3 to the more general setting $H_{\infty}(x_0) = -\Delta_f + \alpha \delta (\cdot - x_0)$ is discussed in section 4.3.

**Notation.** The Sobolev spaces are denoted with $H^\nu$ or $H^\nu_0$ in the case of Dirichlet boundary conditions. The notation '≈' is to be intended as '≤' where $C$ is a suitable positive constant. The set $\mathcal{D}_0$ is defined by

$$\mathcal{D}_0 = \{k \in \mathbb{N}, \; k \text{ odd}\}.$$
3. The existence of the dynamics

In what follows, the existence of solutions to (16) is discussed under the assumption \( \alpha \in H^1(0, T) \). Our strategy consists in using the mild approach described by (19) and (20). It articulates in two steps. First we prove that (20) admits a unique solution in \( H^1(0, T) \) for any finite time \( T \) and any \( \alpha \in H^1(0, T) \). This result is then used to obtain estimates for the solution of (16).

3.1. The charge equation

Consider the linear map \( U \):

\[
U q = \sum_{k \in D_0} \int_0^T q(s) e^{-ik(t-s)} ds, \quad D_0 = \{ k \in \mathbb{N}, k \text{ odd} \}, \quad \lambda_k = \frac{k^2}{4}. \tag{21}
\]

To study its properties we make use of the following remark and the next auxiliary lemma.

**Remark 4.** Depending on the value of \( T \), some of the functions \( e^{-i\lambda_k t} \) may belong to the standard basis \( \{ e^{i\omega n t}; \ \omega = \frac{2\pi}{T} \} \) of the space \( L^2(0, T) \). In particular, for \( T = 8\pi N \), \( N \in \mathbb{N} \), it follows that

\[
\{ e^{-i\lambda_k t} \}_{k \in \mathbb{N}} \subset \{ e^{i\omega n t}; \ \omega = \frac{2\pi}{T} \} \quad n \in \mathbb{Z}. \tag{22}
\]

Equation (22) will often be used as an auxiliary condition in the forthcoming analysis.

**Lemma 5.** Let \( D_{x_0}, S_{x_0} \) denote the sets

\[
D_{x_0} = \{ k \in \mathbb{N}, \psi_k(x_0) \neq 0 \}; \quad S_{x_0} = \{ n \in \mathbb{Z}, \lambda_k + \omega n \neq 0 \ \forall \ k \in D_{x_0} \}, \tag{23}
\]

and \( P_{x_0} \) the operator: \( \ell^2(S_{x_0}) \rightarrow \ell^2(S_{x_0}) \)

\[
P_{x_0} f_n = r_n f_n, \quad r_n = \sum_{k \in D_{x_0}} \frac{|\psi_k(x_0)|^2}{\lambda_k + \omega n}, \quad n \in S_{x_0}. \tag{24}
\]

where the \( \psi_k(\cdot) \) are defined in (8). For any \( x_0 \in I \), \( P_{x_0} \) is compact.

**Proof.** According to relations (7) and (9) for \( z \in \mathbb{C} \setminus \sigma_D \), we have

\[
G_\beta(x_0, x_0) = \sum_{k \in D_{x_0}} \frac{|\psi_k(x_0)|^2}{\lambda_k + z} \tag{25}
\]

and

\[
\sum_{k \in D_{x_0}} \frac{|\psi_k(x_0)|^2}{\lambda_k + z} = \frac{1}{2\sqrt{\pi}} \frac{e^{-\pi z}}{\sqrt{z}} - \frac{1}{2\sqrt{\pi}} \left( e^{-\pi z} - e^{-2\pi z} \right)
\]

\[
\times \left[ e^{\sqrt{z_0}} \sinh(\sqrt{z}(\pi + x_0)) + e^{-\sqrt{z_0}} \sinh(\sqrt{z}(\pi - x_0)) \right]. \tag{26}
\]

It follows that

\[
\sum_{k \in D_{x_0}} \frac{|\psi_k(x_0)|^2}{\lambda_k + \omega n} \sim \mathcal{O} \left( \frac{1}{\sqrt{|\omega n|}} \right), \quad n \in S_{x_0}. \tag{27}
\]

Using this asymptotic condition, the sequence of finite rank maps \( P_{x_0, N} \)

\[
\{ P_{x_0, N} f_n \}_{n \in S_{x_0}} = \begin{cases} f_n \left( \sum_{k \in D_{x_0}} \frac{|\psi_k(x_0)|^2}{\lambda_k + \omega n} \right), & n \leq N \\ 0, & n > N \end{cases} \tag{28}
\]
converges to $P_{\omega t}$, as $N \to +\infty$, in the $\ell_2(S_0)$-operator norm. The statement of the lemma follows (we refer to theorem VI.12 [19]).

Next, $U$ is considered as a map on the Hilbert space $\mathcal{H}_T$: \[
\mathcal{H}_T = \{ q \in H^1(0, T), \; q(0) = 0 \}
\]
equipped with the $H^1$-norm.

**Lemma 6.** The map $U$ (21) is bounded in $\mathcal{H}_T$ with the estimate \[
\|Uq\|_{\mathcal{H}_T} \lesssim (T + 1) \|q\|_{\mathcal{H}_T}.
\]

**Proof.** We use the standard basis $\{e^{i\omega nt}; \omega = \frac{2\pi}{T}\}_{n \in \mathbb{Z}}$ of $L^2(0, T)$. The Fourier coefficients of $\int_0^T q(s) e^{-i\lambda_k(t-s)} dt$ w.r.t. the vectors $e^{i\omega nt}$ are written as \[
\int_0^T \int_0^t ds q(s) e^{-i\lambda_k(t-s)} e^{-i\omega nt} = \int_0^T ds q(s) e^{i\omega nt} \int_s^T dt e^{-i(\lambda_k+\omega n)t}.
\]
Assume, at first, that $T$ fulfills condition (22) and, consequently, $\{e^{i\omega nt}; \omega = \frac{2\pi}{T}\}_{n \in \mathbb{Z}}$. This choice allows us to avoid small divisors; with this assumption, the above expression is written as \[
\int_0^T \int_0^t ds q(s) e^{-i\lambda_k(t-s)} e^{-i\omega nt} = \left[ \int_0^T ds q(s) e^{i\omega nt} \int_0^T dt e^{-i(\lambda_k+\omega n)t} \right] \lambda_k + \omega n = 0,
\]
for $\lambda_k + \omega n \neq 0$ and the expansion \[
\int_0^T ds q(s) e^{-i\lambda_k(t-s)} = (T q_{-Nk^2} - (tq)_{-Nk^2}) e^{-i\lambda_k t} + \sum_{n \in \mathbb{Z}, \lambda_k + \omega n \neq 0} \frac{i}{\lambda_k + \omega n} \{q_{-Nk^2} - q_n\} e^{i\omega nt}
\]
follows, $q_n$ denoting the $n$th Fourier coefficient of $q$ (in particular $q_{-Nk^2} = (q, e^{-i\lambda_k t})_{L^2(0,T)}$).
Let $S_0 = \{n \in \mathbb{Z} : \omega n \notin \{-\lambda_k\}_{k \in \mathbb{Z}}\}$, we have \[
\int_0^T (te^{-i\lambda_k t}) e^{-i\omega nt} = \frac{iT}{\lambda_k + \omega n}, \quad \forall \; n \in S_0.
\]
This allows us to write \[
\sum_{n \in \mathbb{Z}, \lambda_k + \omega n \neq 0} \frac{i}{\lambda_k + \omega n} q_{-Nk^2} e^{i\omega nt} = q_{-Nk^2} \Pi_{S_0} \left( \frac{T}{T} e^{-i\lambda_k t} \right),
\]
where $\Pi_{S_0}$ is the projector over the subspace spanned by $\{e^{i\omega nt} \}_{n \in S_0}$. Using (31) and the above relation, the Fourier expansion of $Uq$ is written as \[
Uq = \sum_{k \in D_0} (T q_{-Nk^2} - (tq)_{-Nk^2}) e^{-i\lambda_k t} + \sum_{k \in D_0} q_{-Nk^2} \Pi_{S_0} \left( \frac{T}{T} e^{-i\lambda_k t} \right) = \sum_{k \in D_0} \sum_{n \in D_0} \frac{iq_n}{\lambda_k + \omega n} e^{i\omega nt}.
\]
Consider the contributions to $Uq$. 

\[
= I + II + III.
\]
(I) The norm of $I$ on the rhs of (32) is bounded by
\[
\left\| \sum_{k \in D_0} (T q_{-N k^2} - (t q)_{-N k^2}) e^{-i k t} \right\|_{L^2(0,T)} \leq 2T \|q\|_{L^2(0,T)}.
\] (33)

(II) For the second term, one has
\[
\left\| \sum_{k \in D_0} q_{-N k^2} \Pi_{D_0} \left( \frac{t}{T} e^{-i k t} \right) \right\|_{L^2(0,T)} = \left\| \Pi_{D_0} \sum_{k \in D_0} q_{-N k^2} \left( \frac{t}{T} e^{-i k t} \right) \right\|_{L^2(0,T)} \leq \left( \sum_{k \in D_0} \|q_{-N k^2}\|^2 \right)^\frac{1}{2} \leq \|q\|_{L^2(0,T)}.
\] (34)

(III) The remaining term is a superposition of $L^2$-functions: $\sum_{n \in S_0} \frac{i q_n}{\lambda_n + \omega n} e^{i k t}$ parametrized by the index $k \in D_0$. Its $n$th Fourier coefficient is formally expressed by $\sum_{k \in D_0} \frac{q_n}{\lambda_n + \omega n}$ and the $L^2$-norm is
\[
\left\| \sum_{k \in D_0} \sum_{n \in S_0} \frac{i q_n}{\lambda_k + \omega n} e^{i k t} \right\|_{L^2(0,T)}^2 = \left\| \sum_{n \in S_0} \sum_{k \in D_0} \frac{q_n}{\lambda_k + \omega n} \right\|_{L^2(0,T)}^2 = \left\| \sum_{k \in D_0} \frac{q_n}{\lambda_k + \omega n} \right\|_{L^2(S_0)}^2.
\] (35)

The result of lemma 5 yields
\[
\left\| \sum_{k \in D_0} \frac{q_n}{\lambda_k + \omega n} \right\|_{L^2(S_0)}^2 = \|P_0(q_n)\|_{L^2(S_0)}^2 \leq \|q_n\|_{L^2(S_0)}^2
\] (36)
and
\[
\left\| \sum_{k \in D_0} \sum_{n \in S_0} \frac{i q_n}{\lambda_k + \omega n} e^{i k t} \right\|_{L^2(0,T)} \lesssim \|q\|_{L^2(0,T)}.
\] (37)

Condition (22) and the estimates (33), (34), (37) lead us to
\[
\|Uq\|_{L^2(0,T)} \lesssim (T + 1) \|q\|_{L^2(0,T)}.
\] (38)

According to remark 5, (22) holds for $T = 8\pi N$, $N \in \mathbb{N}$. For $q \in L^2(0, \tilde{T})$, $\tilde{T} \in (0, +\infty)$, let $N_T \in \mathbb{N}$ be the smaller integer $\tilde{T} < 8\pi N_T$ and set $T = 8\pi N_T$; then, $\tilde{T} \in (T - 8\pi, T)$; using (38) we obtain
\[
\|Uq\|_{L^2(0,T)} = \|U_{1_{(0,\tilde{T})}} q\|_{L^2(0,T)} \leq (T + 1) \|q\|_{L^2(0,T)} \leq (\tilde{T} + 8\pi + 1) \|q\|_{L^2(0,\tilde{T})} \leq C(\tilde{T} + 1) \|q\|_{L^2(0,\tilde{T})},
\] (39)
where $1_{(0,T)}$ is the characteristic function of the interval, while $C > 0$ is independent of $T$.

Thus, the constraint (22) about the Fourier frequencies of $L^2(0, T)$ is not a necessary condition and the estimate (38) actually holds independently from it.

Next, consider $q \in \mathcal{H}_T$. An integration by part of (21) gives
\[
Uq = \sum_{k \in D_0} \frac{1}{i \lambda_k} q(t) - \sum_{k \in D_0} \frac{1}{i \lambda_k} \int_0^t q(s) e^{-i \lambda_k (t-s)} \, ds.
\] (40)
Due to the asymptotic behaviour of the coefficients $\frac{1}{\lambda_k}$, this sum uniformly converges to a continuous function of $t \in [0, T]$, with

$$ \frac{d}{dt} U q = \sum_{k \in \mathbb{Z}_0} \int_0^t q'(s) e^{-i \lambda_k (t-s)} \, ds. $$

(41)

From (38), we obtain

$$ \left\| \frac{d}{dt} U q \right\|_{L^2(0,T)} \lesssim (T+1) \| q' \|_{L^2(0,T)} $$

(42)

and

$$ \| U q \|_{H^1(0,T)} \lesssim (T+1) \| q \|_{H^1(0,T)}. $$

(43)

In the next lemma we give an iteration scheme for the solution of charge-like equations.

Lemma 7. Let $f, \varphi \in H^1(0,T)$, $\varphi$ real valued, and $G_0^\lambda(\cdot,0)$ be Green’s function defined in (3) and (6) with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The equation

$$ v = f - \varphi \left( v(0) e^{i t \Delta_p} G_0^\lambda(\cdot,0) \right)_{x=0} + \frac{i}{\pi} U v $$

has a unique solution in $H^1(0,T)$ allowing the estimate

$$ \| v \|_{H^1(0,T)} \lesssim \| f \|_{H^1(0,T)} (1 + \| \varphi \|_{H^1(0,T)} + \mathcal{P}(\| \varphi \|_{H^1(0,T)})), $$

(45)

where $\mathcal{P}(\cdot)$ is a suitable positive polynomial.

Proof. Since lemma 6 provides us with an estimate of $U f$ in the $\mathcal{H}_T$-norm, we introduce the variable $w(t) = v(t) - v(0)$. In this setting, our problem is written as

$$ w = R - \frac{1}{\pi} U w, $$

(46)

$$ R(t) = f(t) - v(0) - \varphi(t) \left( v(0) e^{i t \Delta_p} G_0^\lambda(\cdot,0) \right)_{x=0} + \frac{i}{\pi} U [v(0)](t) . $$

(47)

Next, the regularity of $R(t)$ is considered. Using the definition of $U$, and the explicit expansion

$$ e^{u \Delta_p} G_0^\lambda(0,0) = \frac{1}{\pi} \sum_{k \in \mathbb{Z}_0} \frac{e^{-i \lambda_k t}}{\lambda_k + \lambda}, $$

(48)

$R(t)$ is written as

$$ R(t) = f(t) - v(0) - v(0) \varphi(t) \left( e^{u \Delta_p} G_0^\lambda(\cdot,0) \right)_{x=0} + \frac{1}{\pi} \sum_{k \in \mathbb{Z}_0} \frac{1}{\lambda_k} (1 - e^{-i \lambda_k t}) $$

$$ = f(t) - v(0) - v(0) \varphi(t) \frac{1}{\pi} \sum_{k \in \mathbb{Z}_0} \left( \frac{1}{\lambda_k} - e^{-i \lambda_k t} \frac{\lambda}{\lambda_k + \lambda} \right), $$

(49)

where the last term on the rhs is $H^1(\mathbb{R})$ for any $-\lambda \notin \sigma_{-\Delta_p}$. Recalling that $H^1(0,T)$ is an algebra w.r.t. the product, the above relation and $f, \varphi \in H^1(0,T)$ imply $R \in H^1(0,T)$. Taking into account the initial condition $R(0) = 0$ (following from the definition of the rescaled variable and the operator $U$ in (46)), we obtain $R \in \mathcal{H}_T$ (see (29)).

Next consider the term $\frac{i}{\pi} U w$ in (46). For $w \in \mathcal{H}_T$ and $\varphi \in H^1(0,T)$, lemma 6 yields

$$ \left\| \frac{i}{\pi} U w \right\|_{\mathcal{H}_T} \leq C \frac{(1+t)}{\pi} \| \varphi \|_{H^1(0,T)} \| w \|_{\mathcal{H}_T}. $$

(50)
for any $t \in [0, T]$ and a suitable $C > 0$. To construct a global solution, we note that, for fixed $T, C > 0$, $\phi \in H^1(0, T)$, there exist $\delta > 0$ and a finite set $\{t_j\}_{j=1}^N$, $t_0 = 0$ and $t_j > t_{j-1}$, fulfilling the conditions

$$[0, T] = \bigcup_{j=1}^N [t_{j-1}, t_j], \quad t_j - t_{j-1} < \delta,$$

(51)

$$C(1 + \delta)\|\phi\|_{H^1(0, t_j)} < \pi c_\delta \quad \text{for all } j,$$

(52)

with $c_\delta < 1$, depending on $\delta$. According to (50) and (52), with $j = 1, \phi \frac{i}{\pi} U$ is the contraction map in $\mathcal{H}_1$ and equations (46) and (47) admit a unique solution, $w_1 = 1_{[0,t_j]} w$, bounded by

$$\|w_1\|_{\mathcal{H}_1} \lesssim \|R\|_{\mathcal{H}_1} \lesssim \|\phi\|_{H^1(0, t_j)} + |v(0)| (1 + \|\phi\|_{H^1(0, t_j)}).$$

Using (44), the initial value of $v$ if formally related to $f(0)$ by $v(0) (1 + \phi(0) G_0^j(0,0)) = f(0)$; this allows us to define $v(0)$ provided that $(1 + \phi(0) G_0^j(0,0))$ is not null. According to the resolvent’s formula (15), the condition $(1 + \phi(0) G_0^j(0,0)) = 0$, with $z \neq \lambda e^{i\theta}$, corresponds to the eigenvalue equation for the Hamiltonian $H_\phi$. Since this is a self-adjoint model, the assumption $\lambda \in \mathbb{C} \setminus \mathbb{R}$ implies $(1 + \phi(0) G_0^j(0,0)) \neq 0$. It follows that $v(0) = (1 + \phi(0) G_0^j(0,0))^{-1} f(0)$ and $|v(0)| \lesssim |f(0)|$. This leads to

$$\|w_1\|_{\mathcal{H}_1} \lesssim \|f\|_{H^1(0, t_j)} (1 + \|\phi\|_{H^1(0, t_j)}).$$

The definition $w_1 = 1_{[0,t_j]} (v - v(0))$ provides with a similar estimate for the function $v$

$$\|v\|_{H^1(0, t_j)} \lesssim \|f\|_{H^1(0, t_j)} (1 + \|\phi\|_{H^1(0, t_j)}).$$

(53)

This can be extended to larger times by the following iteration scheme:

$$w_{j+1}(t) = v(t) - v(t_j), \quad t \in [t_j, t_{j+1}], \quad j = 1, \ldots, N - 1$$

(54)

$$w_{j+1}(t' + t_j) = R_{j+1}(t' + t_j) - \phi(t' + t_j) \frac{i}{\pi} [U w_{j+1}(\cdot + t_j)](t'), \quad t' \in [0, t_{j+1} - t_j]$$

(55)

$$R_{j+1}(t) = f(t) - v(t_j) - \phi(t) \left( \left( v(0) e^{-i\lambda t} G_0^j(\cdot, 0) \right)_{t=0} - \frac{i}{\pi} U[v(t_j)] + \sum_{k \in \mathbb{Z}_0} \int_0^t (v(s) - v(t_j)) e^{-\lambda_k (t-s)} ds \right),$$

(56)

with the source term $R_{j+1}$ depending at each step on the past solution $1_{[0,t_j]} q$. From this definition, it follows that $R_{j+1}(t_j) = 0$, while an integration by part gives

$$R_{j+1}(t) = f(t) - v(t_j) - \frac{\phi(t)}{\pi} \sum_{k \in \mathbb{Z}_0} \left( \frac{v(t_j)}{\lambda_k} + v(0) \left( \frac{-\lambda e^{-i\lambda t}}{\lambda_k^2 + \lambda_k} \right) - \frac{1}{\lambda_k} \int_0^t v(s) e^{-\lambda_k (t-s)} ds \right),$$

(57)

where (48) has been taken into account. If $v \in H^1(0, t_j)$, all the contributions on the rhs of (57) are $H^1(0, T)$ and $R_{j+1} \in \mathcal{H}_{1, \lambda - t_j}$. Then, the contractivity property of the operator, expressed by (55), implies the existence of a unique solution $w_{j+1} \in \mathcal{H}_{1, \lambda - t_j}$ with the bound

$$\|w_{j+1}\|_{\mathcal{H}_{1, \lambda - t_j}} \lesssim \|R_{j+1}\|_{\mathcal{H}_{1, \lambda - t_j}} \lesssim \|f\|_{H^1(0, t_j)} + \|\phi\|_{H^1(0, t_j)} \|\|\|_{H^1(0, t_j)}.$$  

(58)

Starting from (53), an induction argument based on the iterated use of (58) leads to

$$\|w_j\|_{H^1(0, T)} \lesssim \|f\|_{H^1(0, T)} (1 + \|\phi\|_{H^1(0, T)}) + \mathcal{P}(\|\|_{H^1(0, T)}),$$

(59)

where $\mathcal{P}(\cdot)$ is a positive polynomial.

□

A first application of this lemma gives an $H^1$-bound for the solutions to the charge equation.
Corollary 8. Let $\alpha \in H^1(0, T; \mathbb{R})$ and $\psi_0 \in D(H_w(0))$ and denote with $\phi^0_0$ the regular part of $\psi_0$ defined according to representation (13). Then: (i) equation (20) admits a unique solution $q \in H^1(0, T)$ such that

$$
\|q\|_{H^1(0, T)} \leq C_{\lambda, \phi^0_0} \|\alpha\|_{H^1(0, T)}(1 + \mathcal{P}(\|\alpha\|_{H^1(0, T)})).
$$

(60)

where $\mathcal{P}()$ is a positive polynomial and $C_{\lambda, \phi^0_0}$ is a positive constant depending on $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\phi^0_0$. (ii) For a fixed $\psi_0 \in D(H_w(0))$, the map $\alpha \to q$ is locally Lipschitzian in $H^1(0, T)$.

Proof.

(i) With the notation introduced in (21), the charge equation is

$$
q = -\alpha e^{i\Delta \mathcal{P}} \psi_0(0) - \alpha \frac{1}{\pi} U q.
$$

(61)

Using the decomposition $\psi_0 = \phi^0_0 + q(0)G^0_{\alpha}(\cdot, 0)$, $\phi^0_0 \in H^2 \cap H^1_0(I)$, holding for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (we refer to proposition 1), this equation rephrases as follows:

$$
q = -\alpha e^{i\Delta \mathcal{P}} \phi^0_0 - \alpha \left( q(0) e^{i\Delta \mathcal{P}} G^0_{\alpha}(\cdot, 0) \right|_{\lambda = 0} + \frac{i}{\pi} U q ,
$$

(62)

Consider $e^{i\Delta \mathcal{P}} \phi^0_0(0)$. As a consequence of Stone’s theorem (e.g. in [20]), the operator $e^{i\Delta \mathcal{P}}$ defines a continuous flow on $H^2 \cap H^1_0(I)$ strongly differentiable in $L^2(I)$. For $\phi^0_0 \in H^2 \cap H^1_0(I)$, we use the Fourier expansion $\phi^0_0 = \sum_{k \in \mathbb{N}} a_k \psi_k$, where the coefficients $a_k = (\phi^0_0, \psi_k)_{L^2(I)}$ are characterized by $a_k \in L^2(\mathbb{N})$, $\lambda k a_k \in L^2(\mathbb{N})$ (as follows integrating by parts and exploiting the boundary conditions of $\psi_k$). The time propagator $e^{i\Delta \mathcal{P}}$ acts like $e^{i\Delta \mathcal{P}} \phi^0_0 = \sum_{k \in \mathbb{N}} a_k e^{-\lambda k t} \psi_k$. In particular, we have

$$
e^{i\Delta \mathcal{P}} \phi^0_0(0) = \frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{N}} a_k e^{-\lambda k t}.
$$

(63)

According to $\lambda k a_k \in L^2(\mathbb{N})$, it follows that $\partial_t e^{i\Delta \mathcal{P}} \phi^0_0(0) = -i \sum_{k \in \mathbb{N}} a_k \lambda k e^{-\lambda k t}$, in the weak sense, and $e^{i\Delta \mathcal{P}} \phi^0_0(0) \in H^1(0, T)$. Then the first statement follows as an application of lemma 7 with $f = -\alpha e^{i\Delta \mathcal{P}} \phi^0_0(0)$ and $\varphi = \alpha$.

(ii) Let $\alpha, \tilde{\alpha} \in H^1(0, T)$, $\psi_0$ and $\tilde{\psi}_0$ be defined with the same regular part

$$
\psi_0 = \phi^0_0 + q(0)G^0_{\alpha}(\cdot, 0) , \quad \tilde{\psi}_0 = \phi^0_0 + \tilde{q}(0)G^0_{\alpha}(\cdot, 0)
$$

(64)

and consider the corresponding solutions to (61) $q$ and $\tilde{q}$. The initial values depend on $\alpha$ and $\tilde{\alpha}$ through the conditions

$$
q(0) = \frac{\phi^0_0(0)}{1 + \alpha(0)G^0_{\alpha}(0, 0)} , \quad \tilde{q}(0) = \frac{\phi^0_0(0)}{1 + \tilde{\alpha}(0)G^0_{\alpha}(0, 0)}
$$

(65)

while the difference $u = q - \tilde{q}$ solves the equation

$$
u = S - \alpha \left( u(0) e^{i\Delta \mathcal{P}} G^0_{\alpha}(\cdot, 0) \right|_{\lambda = 0} + \frac{i}{\pi} U u ).
$$

(66)

$$
S(t) = - (\alpha - \tilde{\alpha}) \left[ e^{i\Delta \mathcal{P}} \phi^0_0(0) - \frac{\tilde{q}}{\tilde{\alpha}} \right].
$$

(67)

From the previous point, we have $e^{i\Delta \mathcal{P}} \phi^0_0(0), q, \tilde{q} \in H^1(0, T)$. The same inclusion holds for the ratios $\frac{q}{\alpha}, \frac{\tilde{q}}{\tilde{\alpha}}$, according to the equation’s structure. An application of lemma 7 with $f = S$ and $\varphi = \alpha$ gives

$$
\|u\|_{H^1(0, T)} \leq \|\alpha - \tilde{\alpha}\|_{H^1(0, T)}
$$

(68)

\[ \square \]
3.2. Solution of the evolution problem

Next we consider the system (19)–(20) with
\[ \alpha \in H^1(0, T), \quad \psi_0 \in D(H_{\alpha(0)}). \]

Due to the result of lemma 7, equation (20) has a unique solution \( q \in H^1(0, T) \). This can be used to show that the corresponding evolution, defined by (19), is \( C([0, T], H_0^1(I)) \cap C^1([0, T], L^2(I)) \) and solves the Cauchy problem (16). Fix \( \lambda \in \mathbb{C}\setminus\sigma_D \); from the definition of the operator’s domain (see (13)), any \( \psi \in D(H_{\alpha(t)}) \) can be represented as the sum of a ‘regular’ part plus Green’s function
\[ \psi(x, t) = \phi(x, t) + q(t)G(t, x). \]

Next we consider the operator
\[ F(q, t) = \frac{i}{\sqrt{\pi}} \sum_{k \in \mathbb{N}_0} \left( \int_0^t q(s) e^{-i\lambda_k (t-s)} \right) \psi_k. \]

Lemma 9. For \( q \in H^1(0, T) \), the map \( F(q, t) \) is bounded in \( C([0, T], H_0^1(I)) \).

Proof. Let \( q_t = 1_{[0, t)} q \) and \( N_T \) be the smallest integer such that \( 8\pi N_T \geq T \). By definition (see (22)), \( \{e^{-i\lambda_k t}\}_{k \in \mathbb{N}} \) forms a subset of the standard basis in \( L^2(0, 8\pi N_T) \). The Fourier coefficients of \( q_t \) along these frequencies are
\[ \int_0^{8\pi N_T} q_t(s) e^{i\lambda_k t} \, ds = \int_0^t q(s) e^{i\lambda_k t} \, ds = c_k(t) \]
and the inequality
\[ \frac{1}{8\pi N_T} \sum_{k \in \mathbb{N}} |c_k(t)|^2 \leq \|q_t\|_{L^2(0, 8\pi N_T)}^2 = \int_0^t |q(s)|^2 \, ds \]
holds. It follows that
\[ \|F(q, t)\|_{L^2(I)}^2 = \frac{1}{\pi} \sum_{k \in \mathbb{N}_0} |c_k(t)|^2 \leq 8N_T \int_0^t |q(s)|^2 \, ds \]
and
\[ \sup_{t \in [0, T]} \|F(q, t)\|_{L^2(I)}^2 \leq 8N_T \int_0^T |q(s)|^2 \, ds = 8N_T \|q\|_{L^2(0, T)}^2. \]

For \( q \in H^1(0, T) \), an integration by part gives
\[ F(q, t) = \frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{N}_0} \frac{1}{\lambda_k} \left( q(t) - q(0) e^{-i\lambda_k t} - \int_0^t q'(s) e^{-i\lambda_k (t-s)} \, ds \right) \psi_k, \]
while, using the relation \( \frac{d}{dx} \psi_k = \lambda_k^2 \psi_k \), the weak derivative \( \frac{d}{dx} F(q, t) \) can be written as
\[ \frac{d}{dx} F(q, t) = \frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{N}_0} \frac{1}{\lambda_k^2} \left( q(t) - q(0) e^{-i\lambda_k t} - \int_0^t q'(s) e^{-i\lambda_k (t-s)} \, ds \right) \psi_k. \]
For the first term on the rhs of (78), the asymptotic behaviour of the coefficients $\lambda_k^2 = \frac{2}{l}$ and the assumption $q \in H^1(0, T)$ implies $\sum_{k \in \mathbb{N}} \lambda_k^2 (q(t) - q(0) e^{-ikt}) \psi_k \in C((0, T], L^2(I))$. For the remaining term, we proceed as before: let $q'_i = 1_{(0,\tau)} q'$ and denote with $c'_k$ the Fourier coefficients of $q'_i$ along the frequencies $\{e^{-ikv}\}_{k \in \mathbb{N}}$.

\[
c'_k = \int_0^\tau q'(s) e^{-ikv(t-s)} \, ds.
\]

It follows that

\[
\left\| \sum_{k \in \mathbb{N}} \frac{1}{\lambda_k^2} \int_0^\tau q'(s) e^{-ikv(t-s)} \, ds \psi_k \right\|_{L^2(I)}^2 = \sum_{k \in \mathbb{N}} \frac{|c'_k|^2}{\lambda_k^2} \leq 8\pi N_T \int_0^\tau |q'(s)|^2 \, ds
\]

and

\[
\sup_{t \in [0, T]} \left\| \sum_{k \in \mathbb{N}} \frac{1}{\lambda_k^2} \int_0^\tau q'(s) e^{-ikv(t-s)} \, ds \psi_k \right\|_{L^2(I)}^2 \leq 8\pi N_T \| q' \|_{L^2(0, T)}^2.
\]

The above estimates lead us to

\[
\sup_{t \in [0, T]} \| F(q, t) \|_{H^0_0(I)} \lesssim \| q \|_{H^1(0, T)}.
\]

**Proposition 10.** Let $\alpha \in H^1(0, T)$ and $\psi_0 \in D(H_\alpha(0))$. The system (19)–(20) has a unique solution $\psi_t \in C([0, T], H^1_0(I)) \cap C^1([0, T], L^2(I))$ such that $\psi_t \in D(H_\alpha(I))$ and $i\partial_t \psi_t = H_\alpha(I) \psi_t$ at each $t$.

**Proof.** The proof articulates in two steps. We first consider the conditions $\psi_t \in C([0, T], H^1_0(I))$ and $\psi_t \in C^1([0, T], L^2(I))$. Then we discuss the equivalence of the system (19)–(20) with the initial problem.

(1) Using the notation introduced in (72), the solution $\psi_t$ of (19) and (20) is written as

\[
\psi_t = e^{i\Delta_p} \psi_0 + F(q, t)
\]

with $q \in H^1(0, T)$ solving the charge equation (20), and $\psi_0 \in D(H_\alpha(0))$. Due to the domain’s structure, $\psi_0 \in H^1_0(I)$: in this case, the term $e^{i\Delta_p} \psi_0$ defines a $C([0, T], H^1_0(I))$ map (as a consequence of the Stone’s theorem, e.g. in [20]). Moreover, following the result of lemma 9, one has $F(q, t) \in C([0, T], H^1_0(I))$. In order to study the $C^1([0, T], L^2(I))$ regularity of $\psi_t$, one can use the decomposition stated in proposition 1: $\psi_0 = \phi_0 + q(0) G_0(\cdot, 0)$, with $\phi_0 \in H^2 \cap H^1_0(I)$ and $\lambda \in \mathbb{C} \setminus \sigma_{\Delta_p}$. For $w(t) = q(t) - q(0)$, $\psi_t$ is written as

\[
\psi_t = e^{i\Delta_p} \phi_0^0 + F(w, t) + Z(t),
\]

\[
Z(t) = q(0) [e^{i\Delta_p} \phi_0^0 (\cdot, 0) + F(1, t)].
\]

Due to the regularity of $\phi_0^0$, one has $e^{i\Delta_p} \phi_0^0 \in C^1([0, T], L^2(I))$. Next consider $F(w,t)$: an integration by part gives

\[
F(w, t) = \frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{N}} \frac{1}{\lambda_k} \left( w(t) - \int_0^t w'(s) e^{-i\lambda k(t-s)} \, ds \right) \psi_k.
\]
and
\[ \frac{d}{dt} F(w, t) = i \frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{D}_0} \int_0^1 w'(s) e^{-i\lambda k(t-s)} \, ds \psi_k = F(w', t). \] (84)

Using estimate (76) in lemma 9, this relation gives \( F(w, t) \in C^1([0, T], L^2(I)) \).

Concerning the term \( Z(t) \), the formula
\[ e^{i\mu \Delta_t^\mu} G^0_0(\cdot, 0) = \frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{D}_0} e^{-i\lambda_k t} \psi_k \]
and an explicit computation lead to
\[ Z(t) = \frac{q(0)}{\sqrt{\pi}} \sum_{k \in \mathbb{D}_0} \frac{1}{\lambda_k} \psi_k = \frac{q(0)}{\sqrt{\pi}} \sum_{k \in \mathbb{D}_0} \frac{\lambda e^{-i\lambda_k t}}{\lambda_k (\lambda_k + \lambda)} \psi_k. \] (85)

It follows that \( Z(t) \in C^1([0, T], H^1_0(I)) \) and \( \psi_t \in C([0, T], H^1_0(I)) \cap C^1([0, T], L^2(I)) \).

(2) The regular part of \( \psi_t \) is written as
\[ \phi_t^\mu = e^{i\mu \Delta_t^\mu} \psi_0 + F(q, t) - q(t) G^0_0(\cdot, 0). \] (86)

Using once more the decomposition \( \psi_0 = \phi_0^\mu + q(0) G^0_0(\cdot, 0), \phi_0^\mu \in H^2 \cap H^1_0(I), \lambda \in \mathbb{C} \setminus \sigma_{\mu} \) and \( w(t) = q(t) - q(0) \), we obtain
\[ \phi_t^\mu = e^{i\mu \Delta_t^\mu} \phi_0^\mu + F(w, t) - w(t) G^0_0(\cdot, 0) + q(0) Q(t) \]
\[ Q(t) = \left[ \left( e^{i\mu \Delta_t^\mu} - 1 \right) G^0_0(\cdot, 0) + F(1, t) \right]. \] (87)

The regularity of \( \phi_0^\mu \) yields \( e^{i\mu \Delta_t^\mu} \phi_0^\mu \in C([0, T], H^2 \cap H^1_0(I)) \), while, according to the explicit form of \( e^{i\mu \Delta_t^\mu} G^0_0(\cdot, 0) \) and \( F(1, t), Q(t) \) is
\[ Q(t) = \frac{1}{\sqrt{\pi}} \sum_{k \in \mathbb{D}_0} \frac{1}{\lambda_k (\lambda_k - \lambda)} \psi_k. \] (88)

This yields \( Q(t) \in C([0, T], H^2 \cap H^1_0(I)) \). For the remaining term, \( F(w, t) - w(t) G^0_0(\cdot, 0) \), it has already been noticed that \( F(w,t) \) is continuously embedded into \( H^1_0(I) \); the same holds for \( w(t) G^0_0(\cdot, 0) \), since \( G^0_0(\cdot, 0) \in H^1_0(I) \) and \( w \) is continuous. Moreover, a direct computation of the Fourier coefficients of \( \frac{d^2}{dt^2} \left[ F(q, t) - q(t) G^0_0(\cdot, 0) \right] \) w.r.t. the system \( \{ \psi_k \}_{k \in \mathbb{N}} \) gives
\[ b_k = \begin{cases} \frac{1}{\sqrt{\pi}} \int_0^1 w'(s) e^{-i\lambda_k (t-s)} \, ds - \lambda w(t) (G^0_0(\cdot, 0), \psi_k)_{L^2(I)} & \text{if } k \text{ odd} \\ 0 & \text{otherwise} \end{cases} \] (90)
from which it follows that
\[ -\frac{d^2}{dt^2} \left[ F(w, t) - w(t) G^0_0(\cdot, 0) \right] = iF(w', t) + \lambda w(t) G^0_0(\cdot, 0). \] (91)

Then, estimate (76) in lemma 9 yields \( \frac{d^2}{dt^2} \left[ F(q, t) - q(t) G^0_0(\cdot, 0) \right] \in C([0, T], L^2(I)) \) and \( \left[ F(w, t) - w(t) G^0_0(\cdot, 0) \right] \in C([0, T], H^2 \cap H^1_0(I)) \), which implies
\[ \phi_t^\mu \in C([0, T], H^2 \cap H^1_0(I)). \]

This, together with the boundary condition \( -q(t) = \alpha(t) \psi_t(0) \) (arising from equation (20)), ensures that \( \psi_t \in D(H_{\alpha(t)}) \) at each \( t \).
Next we discuss the last point of the proposition. The continuity in time of the map $\psi_t$ and (19) allow us to write $\psi(.,t=0)=\psi_0$. According to (83)–(85), the derivative $i\partial_t\psi_t$ is

$$i\frac{d}{dt}\psi_t = -\frac{d^2}{dx^2}e^{i\Delta\lambda_0}\phi_0^k + iF(w',t) + iZ'(t)$$

$$iZ'(t) = -\frac{q(0)}{\sqrt{\pi}} \sum_{k\in D_0} \frac{\lambda e^{-i\lambda t}}{\lambda_k + \lambda} \psi_k. \tag{92}$$

On the other hand, from definition (14) one has

$$H_{\alpha(t)}\psi_t = -\frac{d^2}{dx^2}\phi_t^k - \lambda q(t)G_0^k (.,0). \tag{93}$$

Making use of (87)–(89) and (91), this is written as

$$H_{\alpha(t)}\psi_t = -\frac{d^2}{dx^2}e^{i\Delta\lambda_0}\phi_0^k + iF(w',t) + \lambda q(0)G_0^k (.,0) - q(0)\frac{d^2}{dx^2}Q(t)$$

with

$$-q(0)\frac{d^2}{dx^2}Q(t) = iF(w',t) + \lambda q(0)G_0^k (.,0).$$

Combining the above relations, it follows that

$$i\frac{d}{dt}\psi_t - H_{\alpha(t)}\psi(x,t) = 0 \text{ in } L^2(I).$$

### 4. Stability and local controllability

The evolution problem (16) defines a map $\Gamma(\alpha,\psi_0)$ associating with the coupling parameter, $\alpha$, and the initial state, $\psi_0$, the solution at time $T$. According to the notation introduced in (21) and (72), it is written as

$$\Gamma(\alpha) = e^{iT\Delta}\psi_0 + F(V(\alpha),T), \tag{94}$$

$$V(\alpha) = q : q(t) = -\alpha(t) \left( e^{i\Delta\lambda_0}\psi_0(0) + \frac{i}{\pi}Uq(t) \right). \tag{95}$$

The local controllability in time $T$ of the dynamics described by (16), when $\alpha$ is considered as a control parameter, is connected with the following question: for $\psi_0 \in D(H_{\alpha(t)})$, does $\alpha$ exist such that $\Gamma(\alpha,\psi_0) = \psi_T$ for arbitrary $\psi_T$ in a neighbourhood of $\psi_0$? In what follows, we fix the initial state $\psi_0$ and discuss the regularity of the map $\alpha \rightarrow \Gamma(\alpha,\psi_0)$. Hence, this control problem will be considered in the local setting where $\psi_0$ coincides with a steady state of the Dirichlet Laplacian and $\alpha$ is small in $H^1_0(0,T)$. To simplify the notation, we use $\Gamma(\alpha)$ instead of $\Gamma(\alpha,\psi_0)$.

#### 4.1. Regularity of $\Gamma$

Recall a standard result in the theory of differential calculus in Banach spaces. Let $X$ and $Y$ denote two Banach spaces, $W$ an open subset of $X$, and $d\alpha F$ and $d^G_\alpha F$, respectively, the Fréchet and the Gâteaux derivatives of $F : W \rightarrow Y$ evaluated in the point $\alpha$. A functional $F : W \rightarrow Y$ is of class $C^1$ if the map

$$F' : W \rightarrow \mathcal{L}(X,Y), \quad F'(\alpha) = d\alpha F \tag{96}$$

is continuous.
Theorem 11 (theorem 1.9 in [3]). Suppose \( F : W \to Y \) is Gâteaux-differentiable in \( U \). If the map
\[
F'_G : W \to \mathcal{L}(X, Y), \quad F'_G(\alpha) = d^G_\alpha F
\]
is continuous at \( \alpha^* \in W \), then \( F \) is Fréchet-differentiable at \( \alpha^* \) and its Fréchet derivative evaluated in \( \alpha^* \) results
\[
d_{\alpha^*}F = d^G_{\alpha^*}F.
\]

Lemma 12. The map \( V \) defined by (95) is \( C^1(H^1(0, T); H^1(0, T)) \).

Proof. The continuity of \( V \) has been discussed in lemma 7. Next we consider the differential map \( V' : H^1(0, T) \to \mathcal{L}(H^1(0, T), H^1(0, T)) \). Due to theorem 11, it is enough to prove that \( \alpha \to d^G_\alpha V \) is continuous w.r.t. the operator norm in \( \mathcal{L}(H^1(0, T), H^1(0, T)) \). The action of \( d^G_\alpha V \) on \( u \in H^1(0, T) \) is
\[
d^G_\alpha V(u) = q,
\]
where
\[
q = -\frac{V'(\alpha)}{\alpha} - \alpha \left( u(0) e^{i \Delta G_\alpha} \right)_{x=0} + \frac{i}{\pi} U_q.
\]

Then, the \( H^1 \)-bound
\[
\| q \|_{H^1(0, T)} \lesssim \| u \|_{H^1(0, T)}
\]
and the continuous dependence from \( \alpha \) follows from lemma 7 and proceeding as in corollary 8.

Proposition 13. For \( \psi_0 \in D(H_0^1(0)) \), the map \( \alpha \to \Gamma(\alpha) \) is of class \( C^1(H^1(0, T), H^1_0(I)) \). Moreover, the regular part of the solution at time \( T \), \( \Gamma(\alpha) = V(\alpha)(T) \hat{G}_0 \), considered as a function of \( \alpha \) is of class \( C^1(H^1(0, T), H^1_0(I)) \).

Proof. The continuity of \( \Gamma(\alpha) \) follows directly from the continuity of the map \( \alpha \to V(\alpha) \) and estimate (81) in lemma 9. Next consider the differential map \( \Gamma' : H^1(0, T) \to \mathcal{L}(H^1(0, T), H^1_0(I)) \). Our aim is to prove that \( \Gamma' \) is continuous in the operator norm. Due to theorem 11, it is enough to prove that \( \alpha \to d^G_\alpha \Gamma \) is continuous. The action of \( d^G_\alpha \Gamma \) on \( u \in H^1(0, T) \) is \( d^G_\alpha \Gamma(u) = F(d^G_\alpha V(u), T) \); for \( \alpha, \bar{\alpha} \in H^1(0, T) \) the difference \( d^G_\alpha \Gamma - d^G_{\bar{\alpha}} \Gamma \) is expressed by
\[
d^G_\alpha \Gamma - d^G_{\bar{\alpha}} \Gamma = F(d^G_\alpha V(u) - d^G_{\bar{\alpha}} V(u), T).
\]
Using (81), one has
\[
\| d^G_\alpha \Gamma - d^G_{\bar{\alpha}} \Gamma \| = \sup_{\| u \|_{H^1(0, T)} = 1} \| F(d^G_\alpha V(u) - d^G_{\bar{\alpha}} V(u), T) \|_{H^1_0(I)}
\]
\[
\lesssim \sup_{\| u \|_{H^1(0, T)} = 1} \| d^G_\alpha V(u) - d^G_{\bar{\alpha}} V(u) \|_{H^1(0, T)} = \| d^G_\alpha V - d^G_{\bar{\alpha}} V \|,
\]
then, the continuity of \( \alpha \to d^G_\alpha \Gamma \) follows from the continuity of \( \alpha \to d^G_\alpha V \) proved in lemma 12.

Let us introduce the map
\[
\mu(\alpha) = \Gamma(\alpha) - V(\alpha)(T) \hat{G}_0(\cdot, 0)
\]
representing the regular part of the state at time \( T \). Combining the result of lemma 12 and the first part of this proof, one has \( \mu(\alpha) \in C^1(H^1(0, T), H^1_0(I)) \). To achieve our result it remains
to prove that $-\partial_2^2 \mu(\alpha)$ belongs to $C^1(H^1(0,T), L^2(I))$. According to (87) and (88), this can be written as

$$\frac{d^2}{dx^2} \left[ e^{iT\Delta} \phi_0^\alpha + F(w,T) - w(T)G_0^\alpha(\cdot,0) + [V(\alpha)](0)Q(T) \right].$$

where $w(t) = [V(\alpha)](t) - [V(\alpha)](0)$, while the regularity of the terms $-\frac{d^2}{dx^2} e^{iT\Delta} \phi_0^\alpha$ and $Q(T)$ is discussed in proposition 10. For $\alpha, \tilde{\alpha} \in H^1(0,T)$, the difference $\partial_2^2 (\mu(\alpha) - \mu(\tilde{\alpha}))$ is

$$\partial_2^2 (\mu(\alpha) - \mu(\tilde{\alpha})) = \frac{d^2}{dx^2} \left[ (F(w,T) - F(\tilde{w},T)) - (w(T) - \tilde{w}(T))G_0^\alpha \right]$$

$$- (\nabla) [V(\alpha)](0) - [V(\tilde{\alpha})](0) \frac{d^2}{dx^2} Q(T).$$

Using (91), one has

$$\partial_2^2 (\mu(\alpha) - \mu(\tilde{\alpha})) = i(F(w',T) - F(\tilde{w}',T)) + \lambda(w(T) - \tilde{w}(T))G_0^\alpha$$

$$- (\nabla) [V(\alpha)](0) - [V(\tilde{\alpha})](0) \frac{d^2}{dx^2} Q(T),$$

with $w'$ and $\tilde{w}'$ respectively denoting the time derivatives of the functions $V(\alpha)$ and $V(\tilde{\alpha})$. The continuity of $\partial_2^2 \mu(\alpha)$, then, follows from the continuity of $F(\cdot, T)$ (expressed by (81)) and of the map $V(\cdot)$. The Gâteaux derivative $d_G^\alpha \mu$ acts on $u \in H^1(0,T)$ as

$$d_G^\alpha \mu(u) = -\frac{d^2}{dx^2} \left[ F(d_G^\alpha w(u),T) - \left[ d_G^\alpha w(u) \right](T)G_0^\alpha(\cdot,0) + \nabla [d_G^\alpha V(\alpha)](0)Q(T) \right].$$

Proceeding as before, the difference $d_G^\alpha \mu(u) - d_G^\alpha \mu(u)$ is

$$d_G^\alpha \mu(u) - d_G^\alpha \mu(u) = i \left( F \left( \partial_t d_G^\alpha w(u) - \partial_t d_G^\alpha w(u),T \right) \right)$$

$$+ \lambda \left( \left[ d_G^\alpha w(u) \right](T) - \left[ d_G^\alpha w(u) \right](T) \right) G_0^\alpha - \left( \left[ d_G^\alpha w(u) \right](0) - \left[ d_G^\alpha w(u) \right](0) \right)$$

$$\times \frac{d^2}{dx^2} Q(T).$$

Using (81) and recalling, from lemma 12, that $\alpha \rightarrow d_G^\alpha V$ is continuous in the $L(H^1(0,T), H^1(0,T))$ operator norm, this relation leads to the continuity of $\alpha \rightarrow d_G^\alpha \mu$ in the $L(H^1(0,T), L^2(I))$-norm. □

4.2. The linearized map

Next we discuss the surjectivity of the linearized map $d_G^\alpha \Gamma$ for $\alpha \in H^1_0(0,T)$. Since the transformations $V$ and $d_G^\alpha V$ preserve Dirichlet boundary conditions, in this framework the result of lemma 12 is rephrased as
Lemma 14. The map $V$ defined by (95) is $C^1(H^1_0(0, T); H^1_0(0, T))$.

This also implies that the state at the initial and final times possesses only the regular part. Thus, proposition 13 becomes

Proposition 15. For $\psi_0 \in H^2 \cap H^1_0(I)$, the map $\alpha \rightarrow \Gamma(\alpha)$ is of class $C^1(H^1_0(0, T), H^2 \cap H^1_0(I))$.

It is worthwhile to note that, when $\psi_0$ is given by a linear superposition of eigenstates of odd kind (i.e. $\psi_k = \sin k\pi x$, $k$ even), the source term in (95), $e^{i\Delta t} \psi_0(0)$, is null at each $t$, and the charge, $V(\alpha)$ in the above notation, is identically zero. In these conditions, the particle does not ‘feel’ the interaction and the evolution is simply determined by the free propagator $e^{i\Delta t}$. This implies $\Gamma(\alpha) = e^{i\Delta T} \psi_0$. In order to avoid this situation, we assume the initial state $\psi_0$ to be fixed in the subspace of even functions

$$\mathcal{W} = \left\{ \varphi \in H^2 \cap H^1_0(I) \mid \varphi = \sum_{k \in D_0} c_k \psi_k \right\}.$$

This choice also implies that $\Gamma$ takes values in $\mathcal{W}$, as can be checked by its explicit formula. Due to proposition 15, one has $\Gamma \in C^1(H^1_0(0, T), \mathcal{W})$.

The linearized map $d^G\Gamma$ is defined according to

$$d^G\Gamma(u) = \frac{1}{\sqrt{\pi}} \sum_{k \in D_0} \left( \int_0^T q(s) e^{-i\lambda s} ds \right) \psi_k - q(t) = \frac{1}{\pi} \alpha(t) U q(t) + u(t) \left( e^{i\Delta} \psi_0(0) + \frac{1}{\pi} U [V(\alpha)] (t) \right).$$

Let $\psi_T \in \mathcal{W}$ be a target state for the linear transformation (102)–(103). We address the following question: does $u \in H^1_0(0, T)$ exist such that $d^G\Gamma(u) = \psi_T$?

Consider first the equation

$$\psi_T = \frac{1}{\sqrt{\pi}} \sum_{k \in D_0} \left( \int_0^T \rho(s) e^{-i\lambda s} ds \right) \psi_k$$

w.r.t. the variable $\rho$. Denoting with $c_k$ the Fourier coefficients of $\psi_T$ in the basis $\{\psi_k\}_{k \in D_0}$, the above equation is equivalent to

$$c_k = \frac{1}{\sqrt{\pi}} \int_0^T \rho(s) e^{-i\lambda s} ds, \quad k \in D_0.$$

This is a trigonometric moment problem; the existence of solutions is established by Beauchard and Laurent in [8] for arbitrary values of $T > 0$ according to the following result.

Theorem 16. Let $T > 0$ and $\{\omega_k\}_{k \in \mathbb{N}_0}$, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be an increasing sequence of $[0, +\infty)$ such that $\omega_0 = 0$ and

$$\lim_{k \rightarrow +\infty} (\omega_{k+1} - \omega_k) = +\infty.$$

For every $c \in \ell^2(\mathbb{N}_0, \mathbb{C})$, there exist infinitely many $v \in L^2((0, T), \mathbb{C})$ solving

$$c_n = \int_0^T v(s) e^{i\omega_n s} ds, \quad n \in \mathbb{N}_0.$$

Proof. The proof rephrases the one given in corollary 1 of [8], where the authors discuss the existence in $L^2((0, T), \mathbb{R})$ under the particular condition $c \in \ell^2(\mathbb{N}_0, \mathbb{C})$, $c_0 \in \mathbb{R}$. For the sake of completeness we give a sketch of it.
Let us define $\omega_n = -\omega_n$ for $n \in \mathbb{N}_0$. According to the result of [13] (referred as theorem 1 in [8]), the family $\{e^{i\omega t}\}_{n \in \mathbb{Z}}$ is a Riesz basis of the subspace $\mathcal{F}$ defined by the closure in $L^2(0, T)$ of the $\text{Span}(e^{i\omega t})_{n \in \mathbb{Z}}$. This condition is equivalent to the existence of an isomorphism $J$ (we refer to proposition 20 in [8]):

\[ J : \mathcal{F} \to \ell^2(\mathbb{Z}, \mathbb{C}) \]

\[ \psi(t) \to \int_0^T \psi(s)e^{i\omega t}ds. \]

Given $\epsilon \in \ell^2(\mathbb{N}_0, \mathbb{C})$, we consider $\hat{\epsilon} \in \ell^2(\mathbb{Z}, \mathbb{C})$ such that $\hat{\epsilon}_k = \epsilon_k$ for $k \in \mathbb{N}_0$. To any choice of $\hat{\epsilon}$ there corresponds a unique solution to (106) defined by $J^{-1}(\hat{\epsilon})$. □

When the $\lambda_k$ coincide with the frequencies of the standard basis $\{e^{i\omega t}\}_{n \in \mathbb{Z}}$ in $L^2(0, T)$, equation (105) can also be interpreted as a constraint over the Fourier coefficients of the function $\rho$. In this case, it is possible to give a solution of the problem respecting Dirichlet boundary conditions. These remarks are summarized in the following proposition.

**Proposition 17.** The following properties hold. (1) For $\{c_k\} \in \ell_2(D_0)$ and $T > 0$, (105) admits infinitely many solutions $\rho \in L^2(0, T)$. (2) Let $T \geq 8\pi$. For any $\psi_T \in \mathcal{W}$, it is possible to find at least one $\rho \in H^1_0(0, T)$ solving (104).

**Proof.**

(1) Let $\psi_T = \sum_{k \in D_0} c_k \psi_k$, and consider the corresponding moment equation (105). For $k \in D_0$, we set $k = 2n - 1, \omega_n = \lambda_{2n-1}$ with $n \in \mathbb{N}$, and $\omega_0 = 0$; with this notation, our problem is written as

\[ \hat{\epsilon}_{n,T} = \int_0^T \hat{\rho}(s)e^{i\omega nt}ds, \quad n \in \mathbb{N}_0 \]

(107)

\[ \hat{\rho} = \frac{i}{\sqrt{\pi}} \rho, \quad \hat{\epsilon}_{n,T}|_{n \in \mathbb{N}} = e^{i\lambda_n}c_{2n-1} \quad \text{and} \quad \hat{\epsilon}_0 = 0. \]  

(108)

According to theorem 16, (107) admits infinitely many solutions $\hat{\rho} \in L^2(0, T)$, which are determined by fixing the extensions of the families $e^{i\omega nt}$ and $\hat{\epsilon}_{n,T}$ to $n \in \mathbb{Z}$.

(2) For $T = 8\pi$, the functions $\{e^{i\lambda_n t}\}_{k \in D_0}$ form a subsystem of the standard basis $\{e^{i\omega t}\}_{n \in \mathbb{Z}}$ in $L^2(0, T)$. Thus, equation (105) partially determines the Fourier coefficients of the function $\rho$. A particular solution $\rho_0$ is obtained by a superposition respecting Dirichlet conditions on the boundary

\[ \frac{i}{\sqrt{\pi}} \rho_0(t) = \sum_{k \in D_0} e^{i\lambda_n}c_k (e^{-i\lambda_n t} - e^{i\lambda_n t}). \]

(109)

Since the regularity of $\psi_T \in \mathcal{W}$ implies $\{k^2 c_k\} \in \ell_2(D_0)$, it follows that $\rho \in H^1_0(0, T)$. For $T > 8\pi$, the target $\psi_T$ is attained by $\rho$ such that $\rho|_{t \in (0,8\pi)} = \rho_0$ is a solution of (104) in $H^1_0(0, 8\pi)$ and $\rho(t \geq 8\pi) = 0$. □

In order to solve the inverse problem related to (102) and (103), we proceed as follows: for a target state $\psi_T \in \mathcal{W}$, proposition 17 allows us to determine (at least one) $q \in H^1_0(0, T)$ solving (102); then, this function is replaced in (103) in the attempt of finding a suitable $u \in H^1_0(0, T)$ such that

\[ q(t) = -\frac{1}{\pi} \varphi(t)U q(t) + u(t) \left( e^{i\Delta} \psi_0(0) + \frac{i}{\pi} U \{V(\alpha)\} (t) \right). \]

(110)

The solvability of this equation w.r.t. $u$ is strictly connected with the specific choice of the initial state, $\psi_0$, and of the linearization point $\alpha$. In particular, if the function
Consider a simplified framework where $\psi_0$ is a Laplacian's eigenstate in $\mathcal{W}$ and $x = 0$. In this setting, equation (103) is written as:

$$-q(t) = \frac{1}{\sqrt{\pi}} u(t) e^{-\frac{1}{2}t^2}$$

and the inverse problem

$$\psi_T = -\frac{i}{\pi} \sum_{k \in \mathbb{Z}_0} \left( \int_0^T u(s) e^{-i\lambda k s} e^{-i\lambda k (T-s)} ds \right) \psi_k$$

is solved in $H^1_0(0, T)$ by the last point of proposition 17. This result and the regularity of the map $\Gamma$ lead to the local steady state controllability of the system (94)–(95) around the point $\alpha = 0$.

**Theorem 18** (Local controllability). Let $T \geq 8\pi$ and assume (112). There exists an open neighbourhood $V \times P \subseteq H^1_0(0, T) \times \mathcal{W}$ of the point $(0, \psi_k)$ such that $\Gamma|_V : V \to P$ is surjective.

**Proof.** As remarked at the beginning of this section, for $\psi_0 \in \mathcal{W}$, the map $\Gamma$ is $C^1(H^1_0(0, T), \mathcal{W})$. Moreover, under our assumptions, $d_\alpha \Gamma$ is surjective for $\alpha = 0$. Then, by the local surjectivity theorem (e.g., [17]) we know that there exists an open neighbourhood $V \times P \subseteq H^1_0(0, T) \times \mathcal{W}$ of $(0, \Gamma(0))$ such that the restriction $\Gamma|_V : V \to P$ is surjective. By definition, $\Gamma(0) = e^{-ib \lambda T} \psi_k$ and $P$ is a neighbourhood of $\psi_k$ in $\mathcal{W}$. □

### 4.3. Further remarks: the control problem and the interaction’s location

The use of a symmetric setting, with a delta interaction centered in the origin of the potential well, is a particular configuration of the more general quantum system defined by the operators

$$H_\alpha(x_0) = -\Delta^P + \alpha \delta(\cdot - x_0)$$

with $x_0 \in (-\pi, \pi)$, $\alpha \in \mathbb{R}$. The representation of proposition 1 rephrases in this case as follows: for $\lambda \in \mathbb{C}\setminus\sigma_{\Delta^P}$

$$D(H_\alpha(x_0)) = \{ \psi \in H^2(I \setminus \{0\}) \cap H^1_0(I) \mid \psi = \phi + qG_\alpha(\cdot, x_0); \phi \in H^2 \cap H^1_0(I); \phi(0) = \phi(1) \}$$

$$-q = \alpha \phi(x_0)$$

while the corresponding resolvent admits the Krein-like formula

$$(H_\alpha(x_0) + z)^{-1} \psi = (-\Delta^P + z)^{-1} \psi - \frac{\alpha}{1 + \alpha G^2_\alpha(x_0, x_0)} (-\Delta^P + z)^{-1} \psi(0) G^2_\alpha(\cdot, x_0)$$

holding with $z \in \mathbb{C}\setminus\mathbb{R}$, $\psi \in L^2(I)$. For $\alpha = \alpha(t)$, the regularity and stability properties of the dynamical system

$$\begin{cases}
\frac{d}{dt} \psi(x, t) = H_{\alpha(t)}(x_0) \psi(x, t), \\
\psi(x, 0) = \psi_0(x) \in D(H_{\alpha(t)}(x_0))
\end{cases}$$

(116)
can be discussed by adapting the techniques introduced in sections 3 and 4. To this concern we note that the mild form of (116) is

$$\psi(. , t) = e^{i \lambda t} \psi_0 + i \sum_{k \in \mathbb{N}} \psi_k^*(x_0) \int_0^t q(s) e^{-i \lambda _k (t-s)} \, ds \, \psi_k$$  \hspace{1cm} (117)$$

$$q(t) = -\alpha(t) [e^{i \lambda t} \psi_0 + i[U_{D_0},q](t)]$$ \hspace{1cm} (118)$$

with

$$U_{D_0} q = \sum_{k \in \mathbb{D}_0} |\psi_k(x_0)|^2 \int_0^t q(s) e^{-i \lambda _k (t-s)} \, ds,$$

$$\mathcal{D}_{x_0} = \{ k \in \mathbb{N}, \psi_k(x_0) \neq 0 \},$$

$$\lambda_k = \frac{k^2}{4}.$$  

Lemma 5 and the uniform bound $|\psi_k(x_0)| \leq \frac{1}{\sqrt{n}}$ allow us to extend the estimate (30) to this modified map (actually, it is sufficient to replace $\mathcal{D}_0$, $S_0$ and $\frac{1}{2}$ with $\mathcal{D}_{x_0}$, $S_{x_0}$ and $|\psi_k(x_0)|^2$ in the proof of lemma 6 to obtain the result). Then the analysis of equations (117) and (118) arises from a straightforward rewriting of lemmas 7, 9 and propositions 10, 13. This allows us to extend the results of the points (1) and (2) of theorem 3 to the modified problem (116).

In particular, for each fixed $\psi_0 \in H^2 \cap H^1_0 (I)$, the map $\Gamma_{x_0}(\alpha)$, associated with the coupling parameter, $\alpha$, the solution at time $T$, is of class $C^1(H^1_0 (0, T), H^2 \cap H^1_0 (I))$.

Next consider the controllability of this dynamics w.r.t. $\alpha \in H^1_0 (0, T)$. In this new setting, the relevant subspace is $\mathcal{W}_{x_0}$:

$$\mathcal{W}_{x_0} = \left\{ \psi \in H^2 \cap H^1_0 (I) \left| \psi = \sum_{k \in \mathcal{D}_{x_0}} c_k \psi_k \right. \right\},$$  \hspace{1cm} (120)$$

while, for any initial state $\psi_0(x) \in D(\Delta^D_0) \backslash \mathcal{W}_{x_0}$, (116) is solved by $e^{i \lambda t} \psi_0$. The linearized problem for $\alpha = 0$ consists in finding $u \in H^1_0 (0, T)$ solving

$$\begin{cases}
\psi_T = i \sum_{k \in \mathcal{D}_{x_0}} \psi_k^*(x_0) \left( \int_0^T q(s) e^{-i \lambda_k (t-s)} \, ds \right) \psi_k \\
- q(t) = u(t) e^{i \lambda \psi_0(x_0)}
\end{cases}$$  \hspace{1cm} (121)$$

for a fixed target state $\psi_T$ in $\mathcal{W}_{x_0}$. Let $\psi_T = \sum_{k \in \mathcal{D}_{x_0}} c_k \psi_k$; the equation

$$\psi_T = i \sum_{k \in \mathcal{D}_{x_0}} \psi_k^*(x_0) \left( \int_0^T q(s) e^{-i \lambda_k (t-s)} \, ds \right) \psi_k$$  \hspace{1cm} (122)$$

is equivalent to the modified moment problem

$$\frac{c_k}{\psi_k^*(x_0)} = i \int_0^T \rho(s) e^{-i \lambda_k (t-s)} \, ds,$$ \hspace{1cm} (123)$$

To solve (122), one needs to control the product $c_k(\psi_k^*(x_0))^{-1}$ in $\ell^2(\mathcal{D}_{x_0})$. Adapting the arguments used in the proof of proposition 17, it is possible to show that if the multiplication map $a_k \rightarrow a_k(\psi_k^*(x_0))^{-1}$ is bounded in $\ell^2(\mathcal{D}_{x_0})$, then, for any $\psi_T \in \mathcal{W}_{x_0}$, there exists at least one solution $\rho \in H^1_0 (0, T)$ solving (122), provided that $T \geq 8 \pi$.

Taking into account the explicit form of the eigenstates $\psi_k$, it is possible to exhibit examples where the coefficients $\psi_k^*(x_0)$ are bounded from below by a positive constant
uniformly w.r.t. $k \in D_{x_0}$. For instance, take $\frac{x_0}{2} = \frac{\pi}{|M|}$, with the integer $M > 2$; then, the functions $\psi_k\left(\frac{\pi}{|M|}\right)$ are periodic w.r.t. $k \in \mathbb{N}$ and the lower bound is fixed by $c_M > 0$:

$$c_M = \min_{k \in \mathbb{N}} \left\{ \left| \psi_k\left(\frac{\pi}{|M|}\right)\right|, \psi_k\left(\frac{\pi}{|M|}\right) \neq 0 \right\}.$$  

(As an explicit example consider $\frac{x_0}{2} = \frac{\pi}{6}$. In this case one has $\mathbb{N} \setminus D_{\frac{\pi}{6}} = \{k = 3 + 6n, n \in \mathbb{N}_0\} \cup \{k = 6 + 6n, n \in \mathbb{N}_0\}$, while the corresponding lower bound is $\left| \psi_k\left(\frac{\pi}{6}\right)\right| \geq \frac{1}{2}$.)

Proceeding as in section 4.2, such a condition allows us to prove the local and finite-time controllability around eigenstates in $W_{x_0}$.

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