Abstract

Manifestly consistent Fock representations of non-central (but “core-central”) extensions of the $\mathbb{Z}^N$-graded algebras of functions and vector fields on the $N$-dimensional torus $\mathbb{T}^N$ are constructed by a kind of renormalization procedure. These modules are of lowest-energy type, but the energy is not a linear function of the momentum. Modulo a technical assumption, reducibility conditions are proved for the extension of $\text{vect}(\mathbb{T}^N)$, analogous to the discrete series of Virasoro representations.

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1 Introduction

A well-understood representation theory for algebras of gauge transformations and vector fields exists only if the dimension $N$ of the base manifold is at most one. To obtain lowest-energy modules, the “quantum version” of an algebra must be considered. If $N = 1$ the quantum version is a central extension, i.e. the affine Kac-Moody and Virasoro algebras, respectively [1, 2, 3]. If $N = 0$, the current algebra is finite-dimensional, and it has no extension at all.

However, little is known about the higher-dimensional case [4] – [12]. There are two problems. A naïve attempt to construct Fock modules using normal ordering results in an infinite central extension, which clearly signals an inconsistency [9, 11]. Second, it is not clear what the quantum version is in higher dimensions. It can not be a central extension, because no interesting one exists [8, 10]. Therefore, we expect the quantum version to be something more complicated, and the obvious suggestion is that it is a non-central extension. These have appeared in physics in connection with anomalies [13], and some representations have been constructed [12].

The problem with non-central extensions is that they come in large numbers, but it is not easy to construct interesting modules with a given extension. Therefore, we suggest the opposite approach: construct interesting modules and determine the extension afterwards. In quantum physics relevant modules are characterized by the energy being bounded from below. We start with the Fock construction, but modify it where it goes wrong, which leads to a modification of the algebra itself. This can be regarded as a kind of renormalization. Indeed, the normal ordering prescription can be described in the same way: the original vacuum energy is infinite, but when it is assigned a finite value by hand, a central extension appears. Normal ordering is part of our prescription, but we must also make another redefinition to avoid other divergencies. The resulting module is manifestly consistent and the energy is bounded from below. The price is that we have a representation of a different algebra, but we will argue that it has the correct “classical limit”. To describe this algebra in terms of generators and brackets is very complicated and can probably not be done in closed form. Nevertheless, its representations in Fock space are easily described.

In this paper we are concerned with the algebras $\text{fun}(\mathbb{T}^N)$ and $\text{vect}(\mathbb{T}^N)$ of functions and vector fields on the $N$-dimensional torus $\mathbb{T}^N$, but the same methods can be used also for current, Poisson and Moyal algebras on $\mathbb{T}^N$. These algebras will collectively be referred to as torus algebras. The restriction to the torus is of technical nature; since every function can be decomposed into a Fourier series, these algebras possess a natural $\mathbb{Z}^N$-grading. However, since all manifolds of the same dimensions are locally diffeomorphic, any local result has wider applicability. Moreover, the formula

$$\exp(i(m_1x^1 + \ldots + m_Nx^N)) = (t^1)^{m_1}\ldots(t^N)^{m_N},$$

establishes an isomorphism between plane waves and monomials, so all results on
$T^N$ hold for Laurent polynomials as well.

Lowest-weight modules of course exist for $N^N$-graded algebras, such as the algebra of polynomials in $N$ variables, with the grading given by the total degree of monomials. However, this is not very interesting, because it is an algebra with a lowest root, and also because no involution can be naturally defined. These problems can be remedied by considering Laurent polynomials, but in that case the homogeneous subspaces are infinite-dimensional and we are back to our original problem. Rudakov studied representations of $N^N$-graded algebras long ago [4].

There might be alternatives to non-central extensions. Figuerido and Ramos [9] suggested that associativity should be abandoned, and presented some rather convincing arguments, but nothing seems to have come out of this.

This paper is organized as follows. In §2 we define the $Z^N$-graded Lie algebras of interest and recall their classical representations. In §3 the notion of a “core-central” (but non-central) extension is defined, which turns out to be the natural generalization of central extensions to more than one dimension. Explicit examples of core-central extensions are also presented, which precisely generalize the Virasoro and affine Kac-Moody algebras to $N > 1$. In §4 it is shown how to construct well-defined Fock modules of certain core-central extensions of torus algebras. The energy of these modules is bounded from below. It is natural to ask whether these modules are irreducible. To this end, we generalize in §5 Feigin’s and Fuks’ construction of singular vectors to higher dimensions, modulo a technical assumption. We thus have reducibility conditions for the extension of $vect(T^N)$. The modules admitting the maximal number of simultaneous singular vectors are especially important, but we have not managed to solve the polynomial equations characterizing them. Also, we have failed to obtain closed formulas for the eigenvalues of the Cartan subalgebra in terms of the Fock parameters, although it is clear that this can be done. The final section contains some comments.

2 The classical representations

Torus algebras act on fields, which can be expanded in a Fourier basis

$$\psi(x) = \sum_{n \in W} \psi(n)e^{-in \cdot x},$$

(2.1)

for some set of momenta $W$. This set is determined by the condition that $\psi(x + 2\pi \hat{j}) = \epsilon \psi(x)$, where $\hat{j}$ is the unit vector in direction $j$ and $\epsilon = +1$ for bosons and $\epsilon = -1$ for fermions.

Definition 2.1 Let $\mathbb{I}/2 = (1/2, 1/2, \ldots, 1/2) \in \mathbb{Z}^N$, $\Lambda = \mathbb{Z}^N \cup (\mathbb{Z}^N + \mathbb{I}/2)$, and $W = \mathbb{Z}^N + v \subset \Lambda$, where $v = 0$ for bosonic fields and $v = \mathbb{I}/2$ for fermionic fields.

$\Lambda$ is the weight lattice of $\text{fun}(T^N)$ and $\text{vect}(T^N)$, $\mathbb{Z}^N$ their root system and $W$ the set of momenta (weights) of the class of modules corresponding to $v$. 

3
Expanded in a Fourier basis, the semi-direct product between $\text{vect}(\mathbb{T}^N)$ and the current algebra $\text{map}(\mathbb{T}^N, g)$ takes the form

$$[L_\mu(m), L_\nu(n)] = n_\mu L_\nu(m+n) - m_\nu L_\mu(m+n),$$

$$[L_\mu(m), J^b(n)] = n_\mu J^b(m+n),$$

$$[J^a(m), J^b(n)] = f^{abc} J^c(m+n),$$  \hspace{1cm} (2.2)

where $m = (m^1, \ldots, m^N) \in \mathbb{Z}^N$ and $f^{abc}$ are the totally anti-symmetric structure constants of the finite-dimensional Lie algebra $g$ equipped with a Killing metric $\delta^{ab}$. $\text{fun}(\mathbb{T}^N)$ is a special case of $\text{map}(\mathbb{T}^N, g)$ with $g$ abelian.

An important class of $\text{vect}(\mathbb{T}^N)$ representations are tensor fields (or densities). Let

$$L_\mu(m) = e^{im \cdot x} (-i \partial_\mu + w_\mu + m_\sigma T^{\sigma}_\mu)$$  \hspace{1cm} (2.3)

where $w_\mu$ is a constant vector defined modulo $\mathbb{Z}^N$ and $(T^{\sigma}_\mu)_{\mu,\nu=1}^N$ satisfies $gl(N)$, i.e.

$$[T^\nu_\sigma, T^\mu_\tau] = \delta^\nu_\tau T^\mu_\sigma - \delta^\mu_\sigma T^\nu_\tau.$$

(2.4)

It is straightforward to prove that (2.3) satisfies (2.2). Hence there is a $\text{vect}(\mathbb{T}^N)$ representation for each $gl(N)$ representation. From a $gl(N)$ tensor with $p$ contravariant and $q$ covariant indices and weight $\lambda$, the corresponding tensor density is obtained. Denote by $\mathcal{T}^p_q(\lambda, w, v)$ the $\text{vect}(\mathbb{T}^N)$ module with basis $\{\psi^{\sigma_1 \cdots \sigma_p}_{\tau_1 \cdots \tau_q}(n)\}_{n \in W}$, with $W$ as in definition 2.1, and module action

$$[L_\mu(m), \psi^{\sigma_1 \cdots \sigma_p}_{\tau_1 \cdots \tau_q}(n)] = (n_\mu - w_\mu + (1 - \lambda)m_\mu)\psi^{\sigma_1 \cdots \sigma_p}_{\tau_1 \cdots \tau_q}(m+n)$$

$$+ \sum_{i=1}^p \delta^{\sigma_i}_\nu \psi^{\sigma_1 \cdots \nu \cdots \sigma_p}_{\tau_1 \cdots \tau_q}(m+n) - \sum_{j=1}^q m_\tau \psi^{\sigma_1 \cdots \sigma_p}_{\tau_1 \cdots \mu \cdots \tau_q}(m+n).$$

(2.5)

We write $\psi \in \mathcal{T}^p_q(\lambda, w, v)$ to indicate this formula. In particular, the action on a scalar field is

$$[L_\mu(m), \psi(n)] = (n_\mu - w_\mu + (1 - \lambda)m_\mu)\psi(m+n),$$

or in position space,

$$[L_\mu(m), \psi(x)] = -e^{im \cdot x} (-i \partial_\mu + w_\mu + \lambda m_\mu)\psi(x).$$

(2.7)

Define $\mathcal{T}^p_\lambda(\lambda) = \mathcal{T}^p_0(\lambda, 0, 0)$. The adjoint representation is $\mathcal{T}^0_1(1)$. The substitution

$$\psi^{\sigma_1 \cdots \sigma_p}_{\tau_1 \cdots \tau_q}(n) \longrightarrow \psi^{\sigma_1 \cdots \sigma_p}_{\tau_1 \cdots \tau_q}(n + v)$$

(2.8)

defines an isomorphism between the modules $\mathcal{T}^p_q(\lambda, w, v)$ and $\mathcal{T}^p_q(\lambda, w - v, 0)$, which thus may be identified.
There is a $\text{map}(\mathbb{T}^N, \mathfrak{g})$ representation for every $\mathfrak{g}$ irrep, which extends to $\text{vect}(\mathbb{T}^N) \times \text{map}(\mathbb{T}^N, \mathfrak{g})$.

$$[J^a(m), \psi(n)] = -M^a\psi(m + n),$$  \hspace{1cm} (2.9)

where $[M^a, M^b] = f^{abc}M^c$ and representation indices are suppressed.

The final classical representations are the connections; for $\text{vect}(\mathbb{T}^N)$:

$$[L_\mu(m), \Gamma^\sigma_{\tau\nu}(n)] = (m_\mu + n_\mu)\Gamma^\sigma_{\tau\nu}(m + n) - m_\tau\Gamma^\sigma_{\mu\nu}(m + n) - m_\nu\Gamma^\sigma_{\tau\mu}(m + n)$$

$$+ \delta^\sigma_{\mu\tau}m_\mu\Gamma^\nu_{\tau\nu}(m + n) + \delta^\sigma_{\mu\nu}m_\tau\delta(m + n)$$  \hspace{1cm} (2.10)

and for $\text{map}(\mathbb{T}^N, \mathfrak{g})$:

$$[J^a(m), A^b_\nu(n)] = f^{abc}A^c_\nu(m + n) + m_\nu\delta(m + n).$$  \hspace{1cm} (2.11)

3 Core-central extensions

**Definition 3.1** Let $\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}^N} \mathfrak{g}(m)$ be a $\mathbb{Z}^N$-graded Lie algebra. An extension of $\mathfrak{g}$ is a $\mathbb{Z}^N$-graded Lie algebra $\tilde{\mathfrak{g}}$ containing the brackets $[\mathfrak{g}(m), \mathfrak{g}(n)] \subset \mathfrak{g}(m + n) \oplus \mathfrak{h}(m, n)$, where $\mathfrak{h}(m, n)$ in general has non-zero brackets with both $\mathfrak{g}(s)$ and $\mathfrak{h}(s, t)$. The extension is $\mathbb{Z}^N$-graded by its total momentum $m + n$, and its core is the part with zero total momentum, i.e. $\bigoplus_{m \in \mathbb{Z}^N} \mathfrak{h}(m, -m)$. $\tilde{\mathfrak{g}}$ is a core-central extension of $\mathfrak{g}$ if its core is central.

The motivation for the name “core” is that it is the part of the extension “in the middle”. Being central, the core can be added to the Cartan subalgebra to characterize a representation. Of course, any central extension is core-central, but there are also non-central extensions with this property. As is well known, torus algebras have no interesting central extensions except in one dimension $\mathbb{Z}$. Therefore, the concept of a core-central extensions is a natural generalization of central extensions to higher dimensions. In this section some examples are given.

Every extension of $\text{vect}(\mathbb{T}^N)$ has the form

$$[L_\mu(m), L_\nu(n)] = n_\mu L_\nu(m + n) - m_\nu L_\mu(m + n) + R_{\mu\nu}(m, n),$$  \hspace{1cm} (3.1)

where $R_{\mu\nu}(m, n) = -R_{\nu\mu}(n, m)$. Every extension of $\text{map}(\mathbb{T}^N, \mathfrak{g})$ has the form

$$[J^a(m), J^b(n)] = f^{abc}J^c(m + n) + F^{ab}(m, n),$$  \hspace{1cm} (3.2)

where $F^{ba}(n, m) = -F^{ab}(m, n)$.

The generic extension is local in the sense that the total momentum is conserved, but it may depend in an essential way on $m$ and $n$ separately. However, there are examples of the form $f(m, n)S(m + n)$, where $f(m, n)$ is an ordinary function.
The result of \([11]\) is that the following abelian (i.e. \([R_{\mu\nu}(m,n), R_{\sigma\tau}(s,t)] = 0\)) but non-central extensions are consistent with the Jacobi identities.

\[
R_{\mu\nu}(m,n) = m_\mu n_\nu (n_\sigma - m_\sigma) S^\sigma (m + n) \tag{3.3}
\]

\[
R_{\mu\nu}(m,n) = m_\mu n_\nu m_\sigma n_\tau S^{\sigma\tau} (m + n) \tag{3.4}
\]

\[
R_{\mu\nu}(m,n) = m_\pi m_\rho n_\sigma n_\tau U^{\pi\rho\sigma\tau}_{\mu\nu} (m + n). \tag{3.5}
\]

where the brackets with \(L_\mu(m)\) are described by

\[
S^\sigma \in T^1_0(1), \quad S^{\sigma\tau} = -S^{\tau\sigma} \in T^2_0(1)
\]

and

\[
U^{\pi\rho\sigma\tau}_{\mu\nu} = -U^{\sigma\tau\pi\rho}_{\nu\mu} \in T^4_2(1), \text{ respectively.}
\]

Denote by \(C_p\) the \(T^p_0(1)\) submodule consisting of totally skew tensor densities, which may be identified with the \(p\)-chains on the torus. There is a \(vect(T^N)\) homomorphism

\[
\delta_p : C_p \rightarrow C_{p-1} \tag{3.6}
\]

\[
(\delta_p S)^{\nu_1 \cdots \nu_{p-1}}(n) = n_\sigma S^{\nu_1 \cdots \nu_{p-1} \sigma}(n).
\]

The extension \((3.4)\) may now be rewritten as

\[
m_\mu n_\nu m_\sigma n_\tau S^{\sigma\tau} (m + n) = m_\mu n_\nu m_\sigma (\delta_2 S)^\sigma (m + n), \tag{3.7}
\]

i.e. it is proportional to an exact one-chain. Similarly, we may consistently demand that the one-chain in \((3.3)\) is closed,

\[
(\delta_1 S)(n) = n_\sigma S^\sigma(n) = 0. \tag{3.8}
\]

Under the same assumptions, we have the following extension of \(map(T^N, g)\)

\[
F^{ab}(m,n) = \delta^{ab}(n_\sigma - m_\sigma) S^\sigma (m + n), \tag{3.9}
\]

\[
F^{ab}(m,n) = \delta^{ab} m_\mu n_\nu S^{\sigma\tau} (m + n), \tag{3.10}
\]

It is clear that \((3.3, 3.8, 3.9)\) together define non-central but core-central extensions. In one dimension, \((3.8)\) has only one solution: \(S(n) = \delta(n)\), and \((3.3)\) and \((3.9)\) reduce to the Virasoro and affine Kac-Moody algebras, respectively. Hence the notion of core-central extensions naturally generalize central extensions to more than one dimension.

We remark that some non-core-central extensions have appeared in physics \([12, 13]\), e.g.

\[
F^{ab}(m,n) = d^{abc} m_\mu n_\nu F^{\mu\nu\sigma}(m + n), \quad d^{abc} = \text{tr} \{J^a, J^b\} J^c. \tag{3.11}
\]
4 Fock modules

To construct Fock modules of \( g \) we must first define a decomposition of \( g \) into homogeneous components indexed by an integer. For symmetrically \( \mathbb{Z} \)-graded algebras this integer can simply be identified with the degree. Torus algebras have a natural \( \mathbb{Z}^N \)-grading but there is no canonical bijection between \( \mathbb{Z}^N \) and \( \mathbb{Z} \). Therefore, a non-canonical choice must be made.

**Definition 4.1** Let \( \Lambda \) be as in definition \([2,3]\). An energy function is a function \( e : \Lambda \rightarrow \mathbb{R} \), such that the following properties hold for every \( m,n \in \Lambda \).

(i) \( e(\cdot) \) is invertible.

(ii) The number of momenta \( s \in \Lambda \) with \( e(m) \leq e(s) < e(n) \) is finite.

(iii) \( e(0) = 0 \) and \( e(-m) = -e(m) \).

(iv) If \( e(m) > 0, e(n) > 0, \) then \( e(m + n) > 0 \).

However, the energy function is not additive: \( e(m) + e(n) \neq e(m + n) \) in general. Indeed, it is impossible to construct an additive energy function when \( N > 1 \). In one dimension the natural energy function is \( e(n) = n \), for every \( n \in \Lambda \). An example for \( N > 1 \) can be constructed as follows. Split \( \Lambda = \Lambda_{(-)} \cup \{0\} \cup \Lambda_{(+)} \) such that \( m \in \Lambda_{(+)} \Rightarrow -m \in \Lambda_{(-)} \). Assign to each momentum in \( \Lambda_{(+)} \) a unique positive integer \( e(m) \), and extend this function to \( \Lambda_{(-)} \) by \( e(-m) = -e(m) \). Clearly, this can be done in many ways.

**Definition 4.2** Denote by \( \prec \) the total order on \( \Lambda \) which is induced by the energy function \( e(\cdot) \): \( m \prec n \) iff \( e(m) < e(n) \).

\( \prec \) satisfies the axioms of an order relation (transitivity and trichotomy), but it is not assumed to be additive: \( m \prec n \) does not imply \( m + s \prec n + s \).

**Definition 4.3** Denote by \( |U| \) the number of momenta in a subset \( U \subset W \).

Consider the vector space of scalar fermion fields on \( \mathbb{T}^N \), with basis \( \{\psi(n)\}_{n \in W} \). Because \( \psi \) is fermionic, \( W = \mathbb{Z}^N + \mathbb{1}/2 \) and \( v = \mathbb{1}/2 \); this will henceforth always be the case. The canonical anti-commutation relations (CAR) are

\[
\{\psi(m), \psi^\dagger(n)\} = \delta(m + n),
\]

\[
\{\psi^\dagger(m), \psi^\dagger(n)\} = \{\psi(m), \psi(n)\} = 0,
\]

where \( \psi^\dagger \) is the conjugate of \( \psi \).

For a given energy function there is a unique representation of the CAR in a Fock space \( \mathcal{F}(e) \) with vacuum vector \( |e\rangle \), such that

\[
\psi(n)|e\rangle = \psi^\dagger(n)|e\rangle = 0, \quad \text{for } n \prec 0.
\]
The dual Fock space is defined by

\[ \langle e | \psi(n) = \langle e | \psi^\dagger(n) = 0 \quad \text{for } n \succ 0. \quad (4.3) \]

Each representation of a \( \mathbb{Z}^N \)-graded algebra can be embedded into the CAR algebra (the enveloping algebra of \((\mathbb{H})\)). For simplicity we discuss first \( g = fun(\mathbb{H}^N) \). Let

\[ J(m) = - \sum_{s \in W} \zeta \psi^\dagger(m - s) \psi(s). \quad (4.4) \]

It follows that

\[ [J(m), \psi(n)] = \zeta \psi(m + n), \quad [J(m), \psi^\dagger(n)] = -\zeta \psi^\dagger(m + n). \quad (4.5) \]

We thus have a representation of \( fun(\mathbb{H}^N) \) on the CAR algebra and a representation of the latter on \( F(e) \). If \( fun(\mathbb{H}^N) \) were finite-dimensional it would inherit the \( F(e) \) module, but this is not true here. The first problem is that the \( J(0) \) eigenvalue of the vacuum is infinite.

\[ J(0)|e\rangle = - \sum_{s \succ 0} \zeta \psi^\dagger(-s) \psi(s)|e\rangle = -\zeta |\{s : s \succ 0\}| |e\rangle. \quad (4.6) \]

This problem is avoided by normal ordering. Redefine \( J(0)|e\rangle = j|e\rangle \) for some finite number \( j \). In one dimension, this is sufficient to render the module consistent, but not so when \( N > 1 \). An attempt to define a Fock module by

\[ J(m)|e\rangle = \begin{cases} - \sum_{s \succ 0} \zeta \psi^\dagger(m - s) \psi(s)|e\rangle, & m \succ 0, \\ j|e\rangle, & m = 0, \\ 0, & m \prec 0, \end{cases} \quad (4.7) \]

is inconsistent because there is an infinite central extension.

\[ [J(m), J(-m)]|e\rangle = -\zeta^2 |\{s : s \succ 0m - s \succ 0\}| |e\rangle, \quad (4.8) \]

and there are infinitely many momenta \( s \) that satisfy both conditions, for every \( m \succ 0 \).

There is a remedy which seems natural, at least to this author. Change the summation domain by replacing the condition \( m - s \succ 0 \) by \( s \prec m \). Of course, the two conditions are equivalent if the energy is linear, i.e. in one dimension, but otherwise the change is quite dramatic. Namely, the energy function is defined so that the modified summation domain, \( 0 \prec s \prec m \), is finite, and hence no infinities can ever appear. This is the second “renormalization” needed to make the Fock construction consistent. From a mathematical point of view, it is no worse than normal ordering. In both cases the module is changed by hand to make it well defined, at the expense of changing the algebra as well.
There is also a physical argument why the modified algebra should have the right “classical limit”. A Fock vector is a superposition of states of the form
\[ \psi(n_1) \ldots \psi(n_k) \psi^\dagger(s_1) \ldots \psi^\dagger(s_\ell) |e\rangle. \] (4.9)
\( \mathcal{F}(e) \) is an infinite-dimensional vector space, and hence it is dominated by states where the number of quanta, i.e. \( k \) and \( \ell \), is very large. The action on such a state should depend mostly on the classical commutators (4.5) and not so much on the vacuum. Since (4.5) has the right form, we expect that the modified vacuum (and indeed, any reasonable vacuum) gives rise to a good quantum version of \( \text{fun}(\mathbb{T}^N) \).

This hand-waving argument does of course not prove that the resulting module is physically relevant, but it is at least well defined.

**Definition 4.4** Let \( n_q \in W \) be the \( q \):th momentum on \( W \), i.e. there are exactly \( q - 1 \) momenta \( n_j \subset W \) satisfying \( 0 \prec n_j \prec n_q \). Define shifted vacua as follows.
\[ |q; e\rangle = \psi(n_q) \ldots \psi(n_2) \psi(n_1) |e\rangle, \] (4.10)
if \( q \) positive, and
\[ | - q; e\rangle = \psi^\dagger(n_q) \ldots \psi^\dagger(n_2) \psi^\dagger(n_1) |e\rangle, \] (4.11)
otherwise.

To every energy function is associated a Hamiltonian \( H \), satisfying
\[ [H, \psi(n)] = e(n)\psi(n), \quad [H, \psi^\dagger(n)] = e(n)\psi^\dagger(n), \quad H|e\rangle = 0. \] (4.12)

Up to normal ordering,
\[ H = - \sum_{n \in W} e(n)\psi^\dagger(-n)\psi(n). \] (4.13)

The corresponding energy is positive for every state in \( \mathcal{F}(e) \). The energies of the shifted vacua \(|q; e\rangle\) and \(| - q; e\rangle\) are \( \sum_{i=1}^{q} e(n_i) > 0 \).

**Theorem 4.5** The following expression defines, together with (4.5), a well-defined representation in \( \mathcal{F}(e) \) of a certain core-central extension of \( \text{fun}(\mathbb{T}^N) \), denoted by \( \tilde{\text{fun}}(\mathbb{T}^N; e) \).
\[ J(m)|e\rangle = - \sum_{0 < s < m} \zeta \psi^\dagger(m - s)\psi(s)|e\rangle, \quad m > 0, \]
\[ = \ z|e\rangle, \quad m = 0, \] (4.14)
\[ = 0, \quad m < 0. \]

This module has a decomposition \( \mathcal{F}(e) = \bigoplus_{q=-\infty}^{\infty} \mathcal{F}(q; e) \) into sectors with fixed fermion number. \( \mathcal{F}(q; e) \) is a \( \tilde{\text{fun}}(\mathbb{T}^N; e) \) module with lowest energy (w. r. t. (4.13)), vacuum vector \(|q; e\rangle\), and \( J(0) \) eigenvalue \( j + q\zeta \).
Proof: By definition, the extension is $F(m, n) = [J(m), J(n)]$. Its brackets are completely specified by (4.14):

$$[F(m, n), \psi(s)] = [F(m, n), \psi^\dagger(s)] = 0,$$

$$F(m, n)|e\rangle = J(m)J(n)|e\rangle - J(n)J(m)|e\rangle.$$  

(4.15)

In particular, if $m \succ 0$ and $n \ll 0$,

$$F(m, n)|e\rangle = -\zeta^2 \sum_{A \setminus B} \psi^\dagger(m + n - s)\psi(s)|e\rangle,$$

$$A = \{s : 0 \prec s \prec mm + n - s \succ 0\},$$

$$B = \{s : 0 \prec s - n \prec ms \succ 0 m + n - s \succ 0\}.$$  

(4.16)

If the order is additive, the two sets are equal (they contain the momenta satisfying $0 \prec s \prec m + n$) and the extension vanishes. Since $F(m, n)$ does not otherwise vanish,

$$[J(m), F(s, t)] = G(m, s, t), \quad [F(m, n), F(s, t)] = H(m, n, s, t)$$

(4.17)

define two new extensions, whose brackets can be computed analogously. The module is well defined because (4.14), (4.13) and (4.3) only involve finite operations, and hence only finite linear combinations appear.

The extension is core-central:

$$[J(m), J(-m)]|e\rangle = J(-m) \sum_{0 \prec s \prec m} \zeta \psi^\dagger(m - s)\psi(s)|e\rangle$$

$$= \zeta^2 \sum_{0 \prec s \prec m} (-\psi^\dagger(-s)\psi(s) + \psi^\dagger(m - s)\psi(s - m))|e\rangle$$

(4.18)

$$-\zeta^2(|\{s : 0 \prec s \prec n\} - |\{s : 0 \prec s \prec ms - m \succ 0\}|)|e\rangle.$$

If the energy function is additive, the second term vanishes because the second condition is equivalent to $s \succ m$, which clearly has no overlap with the first condition.

$F(e)$ can be decomposed because each generator $J(m)$, $F(m, n)$, etc., preserves the fermion number (the difference between the number of $\psi$’s and $\psi^\dagger$’s). The $J(0)$ eigenvalue is calculated thusly

$$J(0)|q; e\rangle = [J(0), \prod_{i=1}^q \psi(n_i)]|e\rangle + \prod_{i=1}^q \psi(n_i)J(0)|e\rangle = (q\zeta + j)|e\rangle.$$  

(4.19)

We now turn to vect($\mathcal{V}N$).
Theorem 4.6 The following expressions define a well-defined representation in \( \mathcal{F}(e) \) of a core-central extension of \( \text{vect}(\mathbb{T}^N) \), denoted by \( \tilde{\text{vect}}(\mathbb{T}^N; e) \).

\[
L_\mu(m)|e\rangle = - \sum_{0 < s < m} (-\lambda m_\mu - w_\mu + s_\mu) \psi^\dagger(m - s) \psi(s)|e\rangle, \quad m > 0,
\]

\[
= h_\mu|e\rangle, \quad m = 0, \quad (4.20)
\]

\[
= 0, \quad m < 0,
\]

and \( \psi \in T^0_0(\lambda, w, 1/2) \), \( \psi^\dagger \in T^0_0(1 - \lambda, -w, 1/2) \). This module has a decomposition \( \mathcal{F}(e) = \bigoplus_{q=-\infty}^{\infty} \mathcal{F}(q; e) \) into sectors with fixed fermion number. \( \mathcal{F}(q; e) \) is a \( \tilde{\text{vect}}(\mathbb{T}^N; e) \) module with lowest energy (w. r. t. (4.13)), vacuum vector \( |q; e\rangle \), and \( L_\mu(0) \) eigenvalue \( h_\mu - qw_\mu + \sum_{i=1}^{q} n_{i\mu} \).

Proof: Analogous to theorem 4.5. □

Let us calculate the value the central core.

\[
[L_\mu(m), L_\nu(-m)]|e\rangle
\]

\[
= \sum_{0 < s < m} (-\lambda m_\mu - w_\mu + s_\mu)((1 - \lambda)m_\nu + w_\nu - s_\nu) \psi^\dagger(-s) \psi(s)|e\rangle
\]

\[
+ \sum_{0 < s < m} (-\lambda m_\mu - w_\mu + s_\mu)((\lambda - 1)m_\nu - w_\nu + s_\nu) \psi^\dagger(m - s) \psi(s - m)|e\rangle
\]

\[
= (-\alpha_{\mu\nu}(m) + \beta_{\mu\nu}(m))|e\rangle. \quad (4.21)
\]

where

\[
\alpha_{\mu\nu}(m) = \sum_{0 < s < m} (-\lambda m_\mu - w_\mu + s_\mu)((\lambda - 1)m_\nu - w_\nu + s_\nu),
\]

\[
\beta_{\mu\nu}(m) = \sum_{0 < s < m \atop m - s > 0} (-\lambda m_\mu - w_\mu + s_\mu)((\lambda - 1)m_\nu - w_\nu + s_\nu). \quad (4.22)
\]

If the energy function can be chosen additively, the second term vanishes because \( s > m \), but otherwise not.

We have failed to express these functions in closed form in general, but this may be easily achieved in the one-dimensional case.

\[
\alpha_{11}(m) = 2hm + \frac{c}{12}(m^3 - m),
\]

\[
\beta_{11}(m) = 0,
\]

\[
c = -2(6\lambda^2 - 6\lambda + 1) = 1 - 12(\lambda - \frac{1}{2})^2,
\]

\[
2h = (w - q)^2 - (\lambda - \frac{1}{2})^2. \quad (4.23)
\]
where \( m = m_1 \) is the only component of the vector \( m \) and \( h = h_1 \) is the \( L(0) \) eigenvalue of \(|q; e\rangle\). The core-central extension is then central (the Virasoro algebra),

\[
[L(m), L(n)] = (n - m)L(m + n) - \frac{c}{12}(m^3 - m)\delta(m + n).
\] (4.24)

Note that the value of \( h_\mu \) is arbitrary; in one dimension, it is fixed by demanding that the subalgebra generated by \{\( L(-1), L(0), L(1) \)\} has no extension. Similarly, the form of the functions in (4.22) depends on \( h_\mu \); it can be fixed e.g. by demanding that there is no extension in the \( sl(N+1) \) subalgebra generated by \{\( L_\nu(-\mu), L_\nu(\mu - \mu), \sum_{\sigma=1}^{N} L_\sigma(\mu) \}_{\mu\nu=1} \), where \( \mu \) denotes a unit vector in the \( \mu \) direction.

The modules constructed so far are manifestly consistent since only finite polynomials in Fock space occur. However, this may be too restrictive, because the physical condition is only that all matrix elements are finite. The following modules are well defined in this weaker sense.

**Theorem 4.7** Consider the \( \text{fun}(\mathbb{T}^N; e) \) and \( \text{vect}(\mathbb{T}^N; e) \) modules defined in theorems 4.3 and 4.6. Replace, for \( m > 0 \), the action of \( J(m) \) and \( L_\mu(m) \) on \(|e\rangle\) by

\[
J(m)|e\rangle = - \sum_{s>0 \atop m-s>0} f_{FD}(e(s) - e(m), \beta) \zeta \psi^\dagger(m - s)\psi(s)|e\rangle,
\]

\[
L_\mu(m)|e\rangle = - \sum_{s>0 \atop m-s>0} f_{FD}(e(s) - e(m), \beta) (-\lambda m_\mu - w_\mu + s_\mu)\psi^\dagger(m - s)\psi(s)|e\rangle,
\] (4.25)

where \( f_{FD}(\epsilon, \beta) = 1/(1 + \exp(\beta\epsilon)) \) is the Fermi-Dirac distribution function and \( \beta \) is a positive parameter. These expressions define representations in \( \mathcal{F}(e) \) of certain core-central extensions of \( \text{fun}(\mathbb{T}^N) \) and \( \text{vect}(\mathbb{T}^N) \), denoted by \( \text{fun}(\mathbb{T}^N; e, \beta) \) and \( \text{vect}(\mathbb{T}^N; e, \beta) \). Although infinitely many terms are created out of the vacuum, every matrix element is finite. The representations decompose into sectors with fixed fermion number and vacua \(|q; e\rangle\).

**Proof:** It is clear that the extensions are core-central and that the fermion number is conserved. Hence the only thing left to prove is that all matrix elements are finite. Consider a typical matrix element

\[
\langle e| J(m_1)\ldots J(m_k)|e\rangle = (-\zeta)^k \sum_{s_1\ldots s_k} f_{FD}(e(s_1) - e(m_1), \beta)\ldots \times f_{FD}(e(s_k) - e(m_k), \beta)\langle e|\psi^\dagger(m_1 - s_1)\psi(s_1)\ldots|e\rangle.
\] (4.26)

The only possible cause of divergence is that the sum runs over infinitely many momenta. However, the number of momenta with total energy \( \epsilon = e(s_1) + \ldots + e(s_k) \) grows only polynomially in \( \epsilon \), whereas \( f_{FD}(\epsilon, \beta) \approx \exp(-\beta\epsilon) \) falls off exponentially fast. The sum thus converges and the matrix element is finite. \( \blacksquare \)
If $\epsilon$ is kept fixed, the Fermi-Dirac distribution has the limits $f_{FD}(\epsilon, \beta) \to \theta(-\epsilon)$ when $\beta \to \infty$ and $1/2$ when $\beta \to 0$. Hence we have, formally, $\widetilde{fun}(\mathbb{T}^N; e) = \lim_{\beta \to \infty} \widetilde{fun}(\mathbb{T}^N; e, \beta)$, whereas $\lim_{\beta \to 0} \widetilde{fun}(\mathbb{T}^N; e, \beta)$ is the inconsistent module (1.7) with infinite central extension, up to normalization. We may thus view the family of Fock modules in Theorem 4.7 as a regularized version of the naïve Fock construction. All observables will depend analytically on $\beta$, except possibly in the limit $\beta \to 0$. However, if this limit exists, it can be used to define observables for $\beta = 0$. Analogous results hold for $\text{vect}(\mathbb{T}^N)$.

$\beta$ plays the role of an inverse temperature and $\epsilon(m)$ that of a chemical potential. It is not clear to us if this has a physical interpretation, because we introduced the Fermi-Dirac distribution solely as a mathematical trick to avoid divergencies.

Analogous Fock modules exist for the current algebra $\text{map}(\mathbb{T}^N, g)$. In this case the root system is $\mathbb{Z}^N \times \Phi_g$, where $\Phi_g$ is the root system of $g$. The Fock construction gives rise to a certain core-central extension, which in one dimension is central: the affine Kac-Moody algebra.

5 Reducibility conditions for $\text{vect}(\mathbb{T}^N)$

Feigin’s and Fuks’ celebrated construction of singular vectors in the Virasoro algebra consists of two steps: find invariants in the CAR algebra and apply these invariants to the vacuum. Their construction is generalized to $\widetilde{\text{vect}}(\mathbb{T}^N; e, \beta)$ in this section.

Lemma 5.1 If $\Psi \in \mathcal{T}_0^0(1, w, v)$, $\Psi(w)$ is an $\text{vect}(\mathbb{T}^N)$ invariant.

Proof: $[L_\mu(m), \Psi(n)] = (n_\mu - w_\mu)\Psi(m + n)$. ■

In view of this lemma, we must find a way to construct scalar densities with weight $\lambda = 1$ in order to construct invariants. To this end, we use that pointwise multiplication and the exterior derivative are $\text{vect}(\mathbb{T}^N)$ homomorphisms, the latter depending on the connection (2.10).

$$
* : \quad \mathcal{T}^{\nu_1}_{q_1}(\lambda_1, w_1, v_1) \otimes \mathcal{T}^{\nu_2}_{q_2}(\lambda_2, w_2, v_2) \longrightarrow \mathcal{T}^{\nu_1+\nu_2}_{q_1+q_2}(\lambda_1 + \lambda_2, w_1 + w_2, v_1 + v_2)
$$

$$
\nabla : \quad \mathcal{T}^p_q(\lambda, w, v) \longrightarrow \mathcal{T}^p_{q+1}(\lambda, w, v),
$$

(5.1)

E.g., the covariant derivative of a scalar field is

$$(\nabla \psi)_\mu(x) = (\partial_\mu + iw_\mu + \lambda \Gamma^{\nu}_{\sigma\nu}(x))\psi(x).$$

(5.2)

Definition 5.2 A composite field is the field obtained by multiplying various covariant derivatives of $\psi(x)$ at the same point $x$. The $k$:th shell of a composite field is the factor containing the $k$:th derivatives of $\psi(x)$. The occupation number of the $k$:th shell is the number $p_k$ of factors of $\psi$ in this shell.
The usefulness of this definition is that every composite field is a tensor field. The general expression for composite fields is rather cumbersome in \( N \) dimensions, but when \( N = 1 \) it reads
\[
\Phi^{(p_0,p_1,p_2,...)}(x) = \psi(x)^{p_0}(\nabla^2 \psi(x))^{p_1}(\nabla^2 \psi(x))^{p_2} \ldots.
\] (5.3)

The \( k \):th shell is thus the factor \((\nabla^k \psi(x))^{p_k}\), \( p_k \) is the occupation number of this shell, and only finitely many \( p_k \) are non-zero. Since \( \psi \) is fermionic \( p_k \leq 1 \); if \( p_k = 1 \) we say that the \( k \):th shell is filled, if \( p_k = 0 \) it is empty.

A composite field depends in general on the connection, but when the first \( p \) shells are filled and the rest empty (\( p_k = 1, k < p \), and \( p_k = 0, k \geq p \)), this is not the case. Denote this special composite field by \( \Psi^{(p)}(x) \). E.g.,
\[
\Psi^{(2)}(x) = \Phi^{(1,1,0,0,...)}(x) = \psi(x)(\partial + iw + \lambda \Gamma(x))\psi(x)
\]
\[
= \psi(x)\partial\psi(x) + (iw + \lambda \Gamma(x))\psi(x)^2,
\] (5.4)
and the second term vanishes because \( \psi(x)^2 = 0 \). \( \Psi^{(p)}(x) \) is simply the \( p \)-fermion Vandermonde determinant.

In higher dimensions the occupation numbers may be larger than one, because the covariant derivative has several components. We say that a shell is filled if its occupation number is maximal; this number depends on the shell. If only the \( p + 1 \) shells are non-empty and the first \( p \) shells are filled, the composite field is independent of the connection. In two dimensions, the composite field with \( p = 3 \) filled shells is
\[
\Psi^{(3)}_{\nu_1,\nu_2,\tau_{11},\sigma_{12},\tau_{22}}(x)
\]
\[
= \psi(x)(\nabla^3 \psi)_{\nu_1}(x)(\nabla^3 \psi)_{\nu_2}(x)(\nabla^2 \psi)_{\sigma_{11}}(x)(\nabla^3 \psi)_{\sigma_{12}}(x)(\nabla^2 \psi)_{\tau_{22}}(x)
\]
\[
= \psi(x)\partial_{\nu_1}\psi(x)\partial_{\nu_2}\psi(x)\partial_{\sigma_{11}}\psi(x)\partial_{\sigma_{12}}\psi(x)\partial_{\tau_{11}}\psi(x)\partial_{\tau_{22}}\psi(x).
\] (5.5)
To arrive at the last expression, we used that all references to the connection vanishes due to anti-symmetry, and thus all covariant derivatives can be replaced by ordinary ones.

**Lemma 5.3** There is a map
\[
\Lambda^{A_N(p)} T_0^0(\lambda, w, \mathbb{I}/2) \longrightarrow T^0_{B_N(p)}(A_N(p)\lambda, A_N(p)w, A_N(p)\mathbb{I}/2),
\] (5.6)
where
\[
A_N(p) = \binom{N - 1 + p}{N}, \quad B_N(p) = N \binom{N - 1 + p}{N + 1}.
\] (5.7)
Proof: Consider the composite field \( \Psi^{(p)}(x) \) with \( p \) filled shells. The occupation number of the \( k \)-th shell \( (k < p) \) is equal to the number of symmetric combinations of \( k \) indices which can take \( N \) different values, i.e.

\[
p_k = \binom{N-1+k}{k} = \binom{N-1+k}{N-1}.
\]  

(5.8)

Each \( \psi \) in the \( k \)-th shell contributes with \( k \) lower indices, wherefore

\[
\Psi^{(p)} \in T_{B_N(p)}^0(A_N(p)\lambda, A_N(p)w, A_N(p)\Pi/2),
\]

(5.9)

where

\[
A_N(p) = \sum_{k=0}^{p-1} p_k = \sum_{k=0}^{p-1} \binom{N-1+k}{N-1} = \binom{N-1+p}{N},
\]

\[
B_N(p) = \sum_{k=0}^{p-1} kp_k = \sum_{k=0}^{p-1} k \binom{N-1+k}{N-1} = N \sum_{j=0}^{p-2} \binom{N+j}{N}
\]

\[
= N \binom{N-1+p}{N+1}.
\]

(5.10)

The domain of the map is determined by the fact that \( A_N(p) \) is the total number of \( \psi \)'s in the \( p \)-shell composite field.

The range of this map is actually a submodule, characterized by certain symmetries. The composite field (5.5) is skew in \( \nu_1 \) and \( \nu_2 \), symmetric in \( \sigma_{ij} \) and \( \tau_{ij} \), and skew under interchange of any pairs \( \sigma_{ij} \tau_{ij} \leftrightarrow \sigma_{kl} \tau_{kl} \).

From lemma 5.3 we obtain a composite field, which has lower indices with certain symmetries. It can transformed into a scalar density by multiplication of some field with upper indices. There is a canonical choice: the permutation symbol may be regarded as a certain tensor density.

**Lemma 5.4** There is a totally skew constant field \( \epsilon^{\nu_1...\nu_N}(x) \in T^N_0(1) \), defined by \( \epsilon^{12...N}(x) \equiv 1 \).

**Proof:** By a direct computation, using the transformation law of \( T^N_0(1) \), skewness, and constancy, it is found that \([L_{\mu}(m), \epsilon^{\nu_1...\nu_N}(x)] = 0\). Hence the assumptions are consistent.

Dually, we may regard the permutation symbol as a constant field in the module \( T^0_N(-1) \). When \( N = 1 \), \( \epsilon(x) = 1 \) is the invariant of the module \( T^0_0(1) \) (lemma 5.4).

The desired scalar field is now formed by contracting the composite field in lemma 5.3 with the permutation symbol in a way which respects the symmetries. From (5.5), we obtain the following scalar field.

\[
\epsilon^{\nu_1\nu_2} \epsilon^{\sigma_1\sigma_2\phi} \epsilon^{\tau_1\tau_2\psi} \psi \partial_{\nu_1} \psi \partial_{\nu_2} \psi \partial_{\sigma_1} \partial_{\sigma_2} \psi \partial_{\tau_1} \partial_{\tau_2} \psi \partial_{\sigma_{12}} \partial_{\tau_{12}} \psi \frac{\propto}{\propto} \psi \partial_{\nu_1} \psi \partial_{\nu_2} \psi \partial_{\sigma_1} \psi \partial_{\sigma_2} \psi \partial_{\tau_1} \psi \partial_{\tau_2} \psi.
\]

(5.11)

This procedure is well defined for arbitrary composite fields with full shells.
Lemma 5.5 There is a map

\[ \Lambda^{A_N(p)} \mathbf{T}_0^0(\lambda, w, \frac{I}{2}) \longrightarrow \mathbf{T}_0^0(A_N(p)\lambda + \frac{1}{N}B_N(p), A_N(p)w, A_N(p)\frac{I}{2}). \] (5.12)

Proof: There are \( B_N(p) \) lower indices to contract. Since each \( e^{\nu_1...\nu_N} \) has \( N \) upper indices, a total of \( B_N(p)/N \) permutation symbols is needed, each contributing unity to the parameter \( \lambda \) and nothing to \( w \). \( \blacksquare \)

Proposition 5.6 There is an invariant in the module \( \Lambda^{A_N(p)} \mathbf{T}_0^0(\lambda, w, \frac{I}{2}) \) provided that \( A_N(p) \) is even and

\[ A_N(p)\lambda + \frac{1}{N}B_N(p) = 1, \]
\[ A_N(p)w_\mu = n_\mu, \] (5.13)

for some \( n \in \mathbb{Z}^N \).

Proof: Combine lemmas 5.1 and 5.5 and note that \( A_N(p) = 0 \mod 2 \). \( \blacksquare \)

The reducibility condition (5.13) can be cast into different equivalent forms.

\[ \lambda - \frac{1}{N+1} = \frac{1}{A_N(p)} - \frac{p}{N+1}, \]
\[ w_\mu = \frac{n_\mu}{A_N(p)}, \] (5.14)

because \( B_N(p) = N(p-1)A_N(p)/(N+1) \). Every solution to the second equation is parallel to \( w_\mu \). If we introduce \( \kappa = (N+1)\lambda - 1 \), the first equation takes the form

\[ (p + \kappa) \binom{N + p - 1}{N} = N + 1, \] (5.15)
i.e.

\[ p(p+1)\ldots(p + N - 1)(p + \kappa) = (N+1)!. \] (5.16)

This is a polynomial equation of degree \( N + 1 \), which generically has \( N + 1 \) complex solutions \( p_i \). The maximal number of invariants in this module is thus \( N + 1 \), which is obtained if all \( p_i \) are real and different and satisfy the simultaneous Diophantine equations

\[ w_\mu = \frac{n_{i\mu}}{A_N(p_i)}, \quad n_{i\mu} \in \mathbb{Z}^N, \quad i = 1, \ldots, N + 1. \] (5.17)

For concreteness we explicitly list the invariant conditions (5.16), (5.17) for \( N = 1, 2, 3 \). For \( N = 1 \),

\[ p\lambda + \frac{p(p-1)}{2} = 1, \quad pw = n, \] (5.18)
with the solutions
\[ p_{1,2} = -(\lambda - \frac{1}{2}) \pm \sqrt{(\lambda - \frac{1}{2})^2 + 2}. \] (5.19)

There are two different invariants provided that there are two integers \( n_1 \) and \( n_2 \) such that \( p_i = n_i x \) \((x = w^{-1})\), i.e.
\[ n_1 n_2 x^2 = -2, \quad (n_1 + n_2)x = -\kappa. \] (5.20)

This equation has a solution if
\[ n_1 n_2 < 0, \quad x = \sqrt{-\frac{2}{n_1 n_2}}, \quad \kappa^2 = -\frac{2(n_1 + n_2)^2}{n_1 n_2}. \] (5.21)

\( N = 2: \)
\[ 1 + \kappa = -(p_1 + p_2 + p_3), \]
\[ \kappa = p_1 p_2 + p_1 p_3 + p_2 p_3, \]
\[ 6 = p_1 p_2 p_3, \] (5.22)

where the \( p_i \) are related by
\[ \frac{p_i(p_i + 1)}{2} = n_i x, \quad n_i \in \mathbb{Z}. \] (5.23)

\( N = 3: \)
\[ 3 + \kappa = -\sum_{i=1}^4 p_i, \]
\[ 2 + 3\kappa = \sum_{1 \leq i < j \leq 4} p_i p_j \]
\[ 2\kappa = -\sum_{1 \leq i < j < k \leq 4} p_i p_j p_k \]
\[ 24 = -p_1 p_2 p_3 p_4, \] (5.24)

where the \( p_i \) are related by
\[ \frac{p_i(p_i + 1)(p_i + 2)}{6} = n_i x, \quad n_i \in \mathbb{Z}. \] (5.25)

We have not been able to find any solution to these simultaneous equations for \( N > 1 \).

The next step in the Feigin-Fuks procedure consists of applying an invariant to the Fock vacuum. The resulting Fock vector is singular.
**Definition 5.7** A singular vector $|\text{sing}\rangle \in \mathcal{F}(e)$ is annihilated by every generator with negative total momentum. It thus satisfies the same conditions as the vacuum, i.e. $L_\mu(m)|\text{sing}\rangle = 0$ for $m < 0$, etc.

**Conjecture 5.8** Let $\Psi^{(p)}(n) \in T_0^0(1, n, A_N(p)\mathbb{1}/2)$, where $n \in \mathbb{Z}^N$ and $A_N(p)$ is even, be the $\text{vect}(\mathbb{N}^N)$ invariant constructed in proposition 5.6. Then

$$|\text{sing}\rangle = (\Psi^{(p)}(n))^j |e\rangle$$

is a singular vector in the reducible $\tilde{\text{vect}}(\mathbb{N}^N; e, \beta)$ module $\mathcal{F}(jA_N(p); e)$.

**Remark:** Because $A_N(p)$ is even we can, at least formally, continue to half-integers and set $j = s/2, s \in \mathbb{N}$.

**Proof:** The vector is singular:

$$L_\mu(m)|\text{sing}\rangle = j[L_\mu(m), \Psi^{(p)}(n)](\Psi^{(p)}(n))^{j-1}|e\rangle + (\Psi^{(p)}(n))^j L_\mu(m)|e\rangle.$$  

The first term vanishes because $\Psi^{(p)}(n)$ is invariant. Hence $|\text{sing}\rangle$ is a singular vector because $|e\rangle$ is so. It is a vector in $\mathcal{F}(jA_N(p); e)$, which thus is reducible.

However, one difficulty remains, which is why we consider this statement to be a conjecture. The singular vector is a sum of infinitely many terms. Indeed, it can be written in the form

$$|\text{sing}\rangle = \left[ \sum_{s_i > 0} \Delta(s_1, \ldots, s_A(p)) \psi(s_1) \ldots \psi(s_A(p)) \right]^j |e\rangle,$$

for some function $\Delta$. Despite the constraint $\sum_i s_i = n$, the sum runs over infinitely many momenta $s_i > 0$, except in one dimension.

Nevertheless, we believe this difficulty to be of technical nature only. For example, a new norm in Fock space could be defined, unrelated to the inner product, such that $|\text{sing}\rangle$ is in the closure of $\mathcal{F}(e)$ relative to this norm. This can be done by damping the contribution from monomials with total momentum $m$ by $\exp(-|e(m)|)$.

The new norm can be relevant if we consider the algebra of smooth vector fields, $L_\xi = \sum_m \xi_\mu(-m)L_\mu(m)$, where $\xi_\mu(m)$ falls off exponentially with $|e(m)|$.

Assume that it is possible to make sense of conjecture 5.8. A large number of non-trivial modules can be constructed as follows. Introduce the equivalence relation $|\text{sing}\rangle \sim 0$. Then the factor module $\mathcal{F}_1 = \mathcal{F}(jA_N(p); e)/\sim$ is well-defined and non-trivial. There might exist another singular vector in $\mathcal{F}_1$; if so, $\mathcal{F}_1$ can be further reduced to $\mathcal{F}_2$ by equalling this vector to zero. This procedure can be carried on until all singular vectors are exhausted.

Unfortunately, we have not yet been able to arrive at explicit formulas for the main parameters characterizing the maximally reducible modules. One problem is that we lack solutions to (5.16) and (5.17). However, this is not so serious, because the modules which admit some but not all of the invariants are still exceptionally
small. Another difficulty is that the relation between $\lambda$ and $w$ and the functions (4.22) is complicated and depends on the energy function. This problem can be solved as follows. Proclaim that the extension should vanish in a certain finite-dimensional subalgebra $g$. Choose an order such that the generators of $g$ are first, which essentially amounts to a suitable choice of energy function. The restriction of $\tilde{\text{vect}}(\mathbb{T}^N; e)$ yields a proper representation of $g$ in Fock space, and the parameters $\lambda$ and $w$ can be related to the eigenvalues of the Cartan subalgebra. In particular, if we reinterpret $\text{vect}(\mathbb{T}^N)$ as the derivation algebra of Laurent polynomials, $g$ can be chosen as the conformal algebra. Hence a discrete spectrum of preferred critical exponents is obtained. We will address this issue in a future publication.

In one dimension we can proceed further. The reducibility conditions in terms of $\lambda$ and $w$ were calculated in (5.18), and the relation between these parameters and $h$ and $c$ follows from (4.23), where $q = jp = sp/2$.

$$c = 1 - 12\left(\frac{1}{q} - \frac{p}{2}\right)^2,$$

$$2h = \left(\frac{n}{p} - \frac{sp}{2}\right)^2 - \left(\frac{1}{p} - \frac{p}{2}\right)^2. \quad (5.29)$$

By eliminating $p$ we obtain for the special values of $\lambda$ given by (5.21),

$$c = 1 - \frac{6(k - l)^2}{kl},$$

$$h = \frac{(nk - sl)^2 - (k - l)^2}{4kl} \quad (5.30)$$

where $k = n_1$, $l = -n_2$, $k,l,n,s \in \mathbb{N}$. This is the discrete series of irreducible Virasoro representations which have many applications in physics [14].

6 Conclusion

In this paper manifestly consistent lowest-energy representations of core-central extensions of torus algebras have been constructed. The main new ingredient is the “renormalization”, which renders the modules finite. Previously the problem was that Fock modules could not be constructed. Now the problem is the opposite; there are too many of them. Different energy functions and renormalization prescriptions give rise to non-equivalent Fock modules. On the other hand, there are only a few main parameters such as $\lambda$ and $w$, and the reducibility conditions only depend on these. There is thus a kind of universality; a few universal quantities which are the same throughout every universality class. The main problem is clearly to calculate universal numbers in irreducible representations. We believe that such numbers appear in nature.

The present work can be extended in various ways. The form of the core-central extensions, or at least their cores, should be explicitly calculated for some specific
choice of energy function. Since $\mathfrak{V} \subset \mathbb{R}^2$, we expect that the Virasoro algebra is a subalgebra of $\tilde{\text{vect}}(\mathbb{T}^2; e, \beta)$ for some $\beta$, but this must be verified. Only Fock modules based on scalar densities have been considered, but the generalization to arbitrary tensor fields is straightforward. The construction of invariants and singular vectors also goes through with obvious modifications. However, one new feature arises. Consider the bosonic Fock module constructed from a metric $g^{\mu\nu} \in T^0_0(0)$, and let $G_{\mu\nu}(n)$ be the Fourier components of the corresponding Einstein tensor. The condition $G_{\mu\nu}(n)|e\rangle = 0$ for every $n$ can consistently be imposed, modulo technical problems analogous to conjecture 5.8. The factor module has a lowest energy and solves the Einstein equation in empty space, wherefore it might be relevant for quantum gravity.
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