Composite operators and form factors in $\mathcal{N} = 4$ SYM

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Abstract
We construct the most general composite operators of $\mathcal{N} = 4$ SYM in Lorentz harmonic chiral (≈twistor) superspace. The operators are built from the SYM supercurvature which is nonpolynomial in the chiral gauge prepotentials. We reconstruct the full nonchiral dependence of the supercurvature. We compute all tree-level MHV form factors via the LSZ reduction procedure with on-shell states made of the same supercurvature.

Keywords: form factors, harmonic superspace, composite operators, supertwistors, nonchiral supersymmetry, scattering amplitudes, supercurvature

(Some figures may appear in colour only in the online journal)

1. Introduction

In recent years, much effort has been devoted to the study of scattering amplitudes in the maximally supersymmetric $\mathcal{N} = 4$ gauge theory (SYM). This activity was to a large extent motivated by Witten’s approach to amplitudes based on twistor theory [1]. New techniques for the computation of amplitudes and beautiful mathematical structures were discovered. In particular, the superamplitudes have a remarkable Yangian symmetry [2–5] which strongly suggests that they are integrable.

Other very interesting objects in this conformal field theory are the correlation functions of gauge invariant composite operators, in particular of the protected stress-tensor multiplet. Being off-shell quantities, they have a much richer structure than the amplitudes. At the same time, they are finite objects with exact superconformal symmetry. A remarkable connection exists between scattering amplitudes and the singular light-like limit of correlators [6, 7]. This suggests that there is an intimate interplay between the two objects—one on-shell, the other
off-shell—and that the conjectured integrable structure of the former somehow extends to the latter.

There exists a third class of field theory objects, which interpolate between amplitudes and correlators—the form factors of gauge-invariant operators. The form factor in $\mathcal{N} = 4$ SYM is a quantity which describes the matrix element of a composite gauge-invariant operator $\mathcal{O}$ (or a supermultiplet of operators) and a final scattering state of particles constituting the vector multiplet of $\mathcal{N} = 4$ supersymmetry,

$$F_{\mathcal{O}}(1, 2, \ldots, n|x, \theta, \bar{\theta}) = \langle 1, 2, \ldots, n|\mathcal{O}(x, \theta, \bar{\theta})|0\rangle.$$  

It shares with the amplitude the presence of a number of external on-shell legs. At the same time, like the correlators, it involves an off-shell composite operator. Such a hybrid object is expected to inherit much of the remarkable simplicity of the $\mathcal{N} = 4$ SYM amplitudes, and at the same time to exhibit some of the non-trivial off-shell structure of the correlators.

Form factors have been extensively studied in a number of papers in the past, for example at weak coupling [8–11] and at strong coupling [12, 13]. We would like to mention in particular [14–16] where the form factors of the protected half-BPS operator have been examined and it has been shown that the $\mathcal{N} = 4$ supersymmetry Ward identities determine, to a large extent, the structure of the MHV form factors.

Very recently, a proposal as to how to compute the tree-level MHV form factors of all kinds of composite operators in $\mathcal{N} = 4$ SYM was put forward in [17]. The authors obtain the form factors from postulated effective operator vertices. These are non-local objects in twistors space, involving Wilson lines which connect the various constituents of the operator. The local expression is obtained by shrinking the Wilson loop to a point. Gauge invariance is restored only in this limit. The main result of the paper is a general formula for the tree-level MHV form factors.

In the present paper we give a step-by-step derivation of the tree-level MHV form factors from first principles. We employ the recently proposed formulation of $\mathcal{N} = 4$ SYM in Lorentz harmonic chiral (LHC) superspace [18, 19]. This is an alternative to the twistor space formulation of Mason et al [20–22]. It makes use of the conceptually simpler notion of harmonic superspace, first proposed for the formulation of theories with extended supersymmetry off shell [23, 24]. In it one uses harmonic fields having an infinite expansion on a coset of the R-symmetry group. This expansion provides the infinite sets of auxiliary and pure gauge fields needed to lift the theory off shell. The same concept was adapted in [25, 26] to the Lorentz group instead of the R-symmetry group. It was used in [26] to formulate the self-dual $\mathcal{N} = 4$ SYM theory of Siegel [27] in the form of a Chern–Simons action. In [18, 19] we extended this formulation to the full SYM theory and showed how to compute all non-chiral Born-level correlators of the stress-tensor multiplet, building upon the earlier work in [28].

In this paper we apply the formulation of [18, 19] to the construction of composite operators. Some simple examples appeared already in [18]. In [29] we explained the equivalence of our formulation with the alternative twistor construction in [30, 31]. Here we apply our method to the most general composite operators. We make use of the basic object of the theory, the $\mathcal{N} = 4$ SYM supercurvature $W_{AB}(x, \theta, \bar{\theta})$. The operators are obtained as local products of supercurvatures and their derivatives. Only in some special cases do we need to make the definition of the operator non-local in harmonic space (but not in space-time). In this we differ from the approach of [17] where all the effective operator vertices are non-local in twistor space.

An important ingredient in the calculation of form factors is the (super)momentum on-shell state. In the standard LSZ approach to amplitudes and form factors it corresponds to the amputation of the external legs. Here we carry out this procedure for the supersymmetric
propagators that we have found in [18]. We find a very simple and manifestly supersymmetric on-shell state, which we insert into our definition of the composite operators to obtain the tree-level MHV form factors. Our results agree with those of [17].

The paper is organized as follows. In section 2 we review the LSZ procedure for the calculation of form factors and indicate what is needed to supersymmetrize it. Our main point is to use (super)curvatures instead of gauge fields as external states. In section 3 we discuss the $\mathcal{N} = 4$ SYM supercurvature on shell and show how it can be converted into the standard Nair superstate. In section 5 we explain how to construct composite operators from the supercurvature. The LHC formulation is chiral, so we first consider the chiral truncation of the operators. Then we apply the on-shell $\bar{Q}$–supersymmetry rules found in [18] to reconstruct the full nonchiral operators. Section 6 is devoted to the LSZ amputation procedure of the super-propagator, which turns it into an on-shell superstate. In section 7 we insert the superstate into two simple operators, the stress-tensor and the Konishi multiplets, to obtain explicit examples of form factors. In section 8 we extend our construction to the most general operators with arbitrary spin and twist. This leads to the most general tree-level MHV form factors. We explain the role of the different gauge frames and the bridges between them in the LHC approach.

2. Form factors as on-shell limits of correlators

We study the form factors (1.1) using a superspace approach and a supergraph technique. In order to specify this quantity we need to construct the composite operator and the on-shell states of the scattering particles. We are going to express both of them in terms of the $\mathcal{N} = 4$ nonchiral supercurvature. At this point we slightly deviate from the traditional approach in the amplitude community to start with the chiral on-shell superstate (see (3.10)). Nevertheless, we can obtain the latter by a Grassmann half-Fourier transform, as explained in section 3.

Before we embark on the supersymmetric case, let us firstly illustrate our procedure on the simple example of pure YM theory. Consider the operator

$$O = \text{tr}(\tilde{F}_{\dot{\alpha} \dot{\beta}} \tilde{F}_{\dot{\alpha} \dot{\beta}})$$

where $\tilde{F}$ is the anti-self-dual part of the YM curvature in spinor notation. We wish to evaluate the simplest, tree-level form factor of $O$ and a final state with two positive helicity gluons,

$$\langle 0 | g^{(+1)} (p_1) g^{(+1)} (p_2) O(x) | 0 \rangle.$$ (2.1)

The standard LSZ reduction procedure for calculating the form factor makes use of the Green’s function $\langle A_{\alpha \dot{\alpha}}(p_1) A_{\beta \dot{\beta}}(p_2) O(x) \rangle$, in which the two gluon legs are amputated and the gluon states are projected with appropriate polarization vectors onto the required helicities. We prefer to replace the gluons by (self-dual) curvatures. This is commonly used in perturbative QCD calculations. The main advantage is that we maintain gauge invariance at all steps of the calculation.

So, at tree level we consider the following correlator of YM curvatures (we omit the color indices),

$$\langle F_{\alpha \dot{\beta}}(p_1) F_{\gamma \dot{\delta}}(p_2) \text{tr}(\tilde{F}_{\dot{\alpha} \dot{\beta}} \tilde{F}_{\dot{\gamma} \dot{\delta}})(x) \rangle_{\text{tree}}$$

$$= \int \frac{d^4 q}{(2\pi)^4} \delta^4(p_1 + p_2 - q) \langle F_{\alpha \beta}(p_1) \tilde{F}_{\dot{\alpha} \dot{\beta}}(-p_1) \rangle \langle F_{\gamma \delta}(p_2) \tilde{F}_{\dot{\gamma} \dot{\delta}}(-p_2) \rangle$$

$$= \frac{e^{i \hat{p}_1 \cdot p_2}}{p_1^2 p_2^2} \langle (\alpha \dot{\alpha})(p_1) (\beta \dot{\beta})(p_2) \rangle_{\text{chiral (p_1)}}.$$ (2.2)
It is obtained by multiplying two free propagators together \( \langle F_{\alpha\beta}(p)\bar{F}_{\alpha\beta}(-p) \rangle = p_{(\alpha}(p_{\beta)}\bar{p})/p^2 \). Notice that the presence of curvatures at the ends of the propagator makes this quantity gauge invariant.

The next step in the LSZ reduction is to remove the poles by multiplying (2.2) by \( p_i^2 p_j^2 \) and then taking the limit \( p_i^2 \to 0 \). Instead of doing this in the final expression for the correlator (2.2), we prefer to amputate each propagator separately, i.e. in the middle line of (2.2). We put the particle momentum on shell, \( p_{\alpha\alpha} = \lambda_{\alpha} \bar{\lambda}_{\alpha} \) and obtain

\[
\lim_{p_i \to 0} p_i^2 \langle F_{\alpha\beta}(p)\bar{F}_{\alpha\beta}(-p) \rangle = \lambda_{\alpha} \lambda_{\beta} \bar{\lambda}_{\alpha} \bar{\lambda}_{\beta}.
\]  

(2.3)

On shell, the self-dual curvature of the external state factorizes in a product of negative helicity spinors and the creation operator of a positive helicity gluon,

\[
F_{\alpha\beta}(p) = \lambda_{\alpha} \lambda_{\beta} g^{(+1)}(p).
\]  

(2.4)

This allows us to strip off the helicity spinors \( \lambda_{\alpha} \lambda_{\beta} \) from (2.3) (this step is equivalent to projecting out with a polarization vector). In this way we obtain the amputated leg

\[
\langle g^{(+1)}(p)\bar{F}_{\alpha\beta}(-p) \rangle = \bar{\lambda}_{\alpha} \bar{\lambda}_{\beta}.
\]  

(2.5)

Completing it with the momentum eigenstate wave function \( \delta^p \), we derive the substitution rule

\[
\bar{F}_{\alpha\beta}(p_i) \Rightarrow \delta^p \bar{\lambda}_{\alpha} \bar{\lambda}_{\beta}
\]  

(2.6)

for each field in the composite operator \( \mathcal{O} = \text{tr}(\bar{F}_{\alpha\beta} F^{\alpha\beta}) \). With this rule we find the form factor

\[
\langle 0 | g^{(+1)}(p_1)g^{(+1)}(p_2)|\mathcal{O}(x)|0 \rangle = [12]^2 e^{i(p_1+p_2)},
\]  

(2.7)

where \([12] = (\bar{\lambda}_{\alpha})_{\alpha} (\bar{\lambda}_{\beta})_{\beta}\).

Let us now come back to the supersymmetric case. The vector multiplet of \( \mathcal{N} = 4 \) SYM described by the supercurvature \( \mathbb{W}_{AB}(x, \theta, \bar{\theta}) = -\mathbb{W}_{BA} \) (with \( A, B = 1, \ldots, 4 \)). This is a nonchiral short (half-BPS) superfield. Its component expansion contains the physical fields: six real scalars \( \phi^{AB} = \frac{1}{2} \epsilon^{ABCD} \phi_{CD} \), four gluinos \( \psi^A_{\alpha} \) and four antigluinos \( \bar{\psi}_{\dot{\alpha}A} \), and the two halves of the curvature of the gluon field \( (F_{\alpha\beta}, \bar{F}_{\alpha\beta}) \). Half-BPS superfields are most naturally described in the R-symmetry harmonic \( \mathcal{N} = 4 \) superspace. One introduces harmonics \( w_{\pm} \) for the R-symmetry group SU(4). The projection \( \mathbb{W}_{++} \) of the supercurvature depends on half of the odd variables \( \theta_\pm = w_+ \cdot \theta, \bar{\theta}_\pm = \bar{w}_+ \cdot \bar{\theta} \) (for more details see section 3). Then we propose to consider the correlator

\[
G(x, \theta, \bar{\theta}) = \mathbb{W}_{++}^{\text{free}}(x, \theta_+, \bar{\theta}_+, w_+) \cdots \mathbb{W}_{++}^{\text{free}}(x_n, \theta_{n+}, \bar{\theta}_{n-}, w_n) \mathcal{O}(x, \theta, \bar{\theta})
\]  

(2.8)

as the generalization of the Green’s function (2.2). The role of the YM curvature, which generates the external on-shell states, is now played by the supercurvature \( \mathbb{W}_{++}^{\text{free}} \). It is taken in the free approximation because the nonlinear (interaction) terms in it do not create single-particle poles as in (2.2). This correlator is gauge invariant because the operator \( \mathcal{O} \) and the free (linearized) supercurvature \( \mathbb{W}_{++}^{\text{free}} \) are gauge invariant. The form factor arises as the residue of the correlator in the on-shell limit,

\[
\mathbb{W}_{++} \rightarrow e^{-i\Lambda} \mathbb{W}_{++} e^{i\Lambda} \text{ under a gauge transformation with parameter } \Lambda(x, \theta_+, \bar{\theta}_+). \]  

In the free case \( g = 0 \) it becomes invariant.

\[\begin{array}{c}
\end{array}\]
In section 3 we explain how the nonchiral off-shell odd variables \( \theta^+, \bar{\theta}^+ \) of \( \mathcal{W}^{\text{free}}_\perp \) reduce to the chiral on-shell Nair’s odd variables \( \eta_a \), which serve to assemble the particles of various helicities in the CPT self-conjugate vector multiplet. We also show explicitly that the R-symmetry harmonics \( w_{\pm} \) in (2.8) drop out in the on-shell regime. This procedure gives rise to the familiar chiral on-shell superstate (see (3.10)). We emphasize that the SYM theory is nonchiral, so the chirality of the on-shell superstate is not mandatory\(^5\). Further, in section 6 we carry out the amputation of the superspace propagator involving \( \mathcal{W}^{\text{free}}_\perp \) to obtain the analog of the on-shell state (2.5) and of the substitution rule (2.6). Finally, in sections 7 and 8 we apply these rules to the computation of various form factors.

### 3. From the nonchiral supercurvature to the chiral on-shell superstate

The supercurvature \( \mathcal{W}_{AB}(x, \theta, \bar{\theta}) \) is restricted by a number of constraints that put its component fields on shell \([33]\). The constraints can be (partially) solved in an \( SU(4) \) covariant manner in RH superspace\(^6\) \([34, 35]\) (for a recent review see \([36]\)). We introduce a set of harmonics \( w^A_{a A}, w^a_{-A} \) and their conjugates \( \bar{w}^{A}_{a a}, \bar{w}^{a}_{-a} \) on the R-symmetry group \( SU(4) \), projecting the index \( A = 1, \ldots, 4 \) of the (anti)fundamental irrep onto the subgroup \( SU(2) \times SU(2)' \times U(1) \) (indices \( a, a', \pm \))\(^7\):

\[
\begin{align*}
( w^A_{a A}, \bar{w}^{A}_{a a} ) & \in SU(4) : \\
& w^A_{a A} \bar{w}^{B}_{a a} = \delta^A_B, \quad w^A_{a A} \bar{w}^{A}_{a a} = 0, \quad w^A_{a A} \bar{w}^B_{a a} = 0, \quad w^A_{a A} \bar{w}^B_{a a} = \delta^A_B \\
& w^A_{a A} \bar{w}^{A}_{a a} + w^{A}_{a A} \bar{w}^{A}_{a a} = \delta^{A}_{A}.
\end{align*}
\]

The functions of the RHs are covariant with respect to the coset subgroup. In particular, this implies homogeneity in the \( U(1) \) charge. We use RHs to project the \( SU(4) \) indices carried by the odd variables and the fields. For example,

\[
\phi^{a++} = \frac{1}{2} \epsilon^{a'b'} \bar{w}^{A}_{a a} \bar{w}^{B}_{a b'} \phi_{AB}, \quad \psi^{a+a} = w^{A}_{a a} \psi^A, \quad \bar{\psi}^{a+a'} = \bar{w}^{A}_{a a'} \bar{\psi}^A.
\]

The constraints imposed on the supercurvature \( \mathcal{W}_{AB} \) can be partially solved in the linearized (or free) approximation. We introduce its RH projection onto the highest weight state of the irrep \([010] \) of \( SU(4) \), \( \mathcal{W}^{\text{free}}_\perp = \frac{1}{4} \epsilon^{a'b'} \bar{w}^{A}_{a a} \bar{w}^{B}_{a b'} \mathcal{W}_{AB} \). Then we interpret part of the constraints as an \( R \)-analyticity condition, which is an example of Grassmann analyticity \([23]\). This means that \( \mathcal{W}^{\text{free}}_\perp \) depends only on the \( \theta^+, \bar{\theta}^+ \) projections of the odd variables,

\[
\begin{align*}
\theta^{a}_{++} = w^{A}_{a A} \bar{w}^A, \quad \bar{\theta}^{a}_{+a'} = \bar{w}^{A}_{a a'} \bar{\theta}^A
\end{align*}
\]

\(^5\) See \([32]\) for an alternative formulation of the \( N = 4 \) SYM amplitudes with nonchiral superstates.

\(^6\) In this paper we employ two abbreviations, LH for Lorentz harmonics and RH for R-symmetry harmonics. The former parametrize a coset of the chiral half of the Euclidean Lorentz group \( SU(2)_L \times SU(2)_R \), the latter describe a coset of the R-symmetry group \( SU(4) \).

\(^7\) We raise and lower the R-symmetry indices \( a, a' \), as well as the Lorentz indices \( \alpha, \alpha' \) with the help of the two-dimensional Levi–Civita tensors \( \epsilon_{ab}, \epsilon^{ab'}, \epsilon_{a'\alpha}, \epsilon^{a'\alpha} \), etc. Our convention is \( \epsilon_{ab} = \delta_{ab}, \epsilon_{12} = 1 \).
but does not depend on their conjugates \(\tilde{\theta}_-, \tilde{\theta}_-\). In the free approximation, the supercurvature is an ultrashort superfield [35, 37], i.e. the expansion of \(\mathbb{W}^{\text{free}}_{++}\) in the odd variables contains only the terms \((\tilde{\theta}_+)^k(\theta_+)^m\) with \(k, m \leq 2\) [38],

\[
\mathbb{W}^{\text{free}}_{++}(x, \theta_+, \tilde{\theta}_+, \psi) = \phi_{++} + \theta_+^{a\alpha} \psi_{aa++} + (\theta_+^{b\beta} \theta_+^{\alpha\beta}) F_{\alpha\beta} + \theta_+^{a\alpha} \phi_{aalpha} + \ldots + \theta_+^{a\alpha} (\theta_+^{b\beta} \theta_+^{\alpha\beta}) \partial_{\alpha\lambda} \psi_{\lambda a'} - \\
+ (\tilde{\theta}_+^{a\lambda} \tilde{\theta}_+^{\beta\alpha}) \tilde{F}_{a\beta} + \ldots + (\tilde{\theta}_+^{a\lambda} \tilde{\theta}_+^{\beta\alpha}) \partial_{\alpha\lambda} \partial_{\beta\beta} \phi_{--}.
\]  

(3.4)

Each term in the expansion carries \(U(1)\) charge \((+2)^\delta\). The dots stand for other terms with derivatives of scalars and (anti)-gluinos. The physical fields carrying \(SU(4)\) indices are split up into a number of RH projections,

\[
\phi_{++}, \phi_{+--}, \phi_{+-}, \psi_{aa'}, \psi_{a+}, \psi_{a-a'}, \tilde{\psi}_{a-a'}
\]

(3.5)

which appear as various components of the ultrashort nonchiral multiplet \(\mathbb{W}^{\text{free}}_{++}\) (3.4).

The remaining constraint defines \(\mathbb{W}_{++}\) as a highest-weight state of \(SU(4)\) (harmonic analyticity). It restricts the RH dependence to polynomial and at the same time puts the component fields from (3.4) on shell.

On the mass shell we can pass from fields to momentum eigenstates. The gauge-covariant fields reduce to products of an on-shell state and helicity spinors carrying the Lorentz indices (see (2.4)),

\[
F_{\alpha\beta} = \lambda_\alpha \lambda_\beta g^{(+1)}, \quad \psi_+^A = \lambda_\alpha \psi^{(+1)A}, \quad \tilde{\psi}_-^A = \bar{\lambda}_\alpha \psi^{(-1)A}, \quad \tilde{F}_{\alpha\beta} = \lambda_\alpha \lambda_\beta g^{(-1)}.
\]  

(3.6)

Here the creation operators \(g^{(+1)}\) etc are supposed to act on the vacuum. After replacing the fields in (3.4) by the on-shell states (3.6), we see that the odd variables get projected by the helicity spinors,

\[
\chi_+^a \equiv \theta_+^{a\alpha} \lambda_\alpha, \quad \eta_+^{\mu} \equiv \bar{\theta}_+^{\mu\alpha} \bar{\lambda}_\alpha.
\]  

(3.7)

In this way the supercurvature \(\mathbb{W}^{\text{free}}_{++}\) is converted into the superstate

\[
\Phi_{++}(p, \chi_+, \eta_+, w) = \phi_{++} + \chi_+^a \psi_{a+}^{(+1)} + (\chi_+^2) g^{(+1)} + \eta_+^{\mu} \psi_{\mu a}^{(-1)} + (\eta_+^2) g^{(-1)} + \\
+ (\chi_+^2) \psi_{a a'}^{(+1)} + \ldots + (\chi_+^2) \eta_+^2 \phi_{--}.
\]  

(3.8)

where we have omitted the same terms as in (3.4). This is a nonchiral realization of the on-shell state which uses RHs to maintain the manifest \(SU(4)\) invariance (see [32]).

In order to rewrite the on-shell state (3.8) in the more familiar chiral form, we perform a Fourier transform (FT) from \(\chi_+^a\) to \(\eta_-\). The resulting change of variables \((\chi_+^a, \eta_+^a) \rightarrow (\eta_-, \eta_-^a) \equiv \eta_\alpha\) restores the \(SU(4)\) index,

\[
\Phi(p, \eta) \equiv \Phi(p, \eta_-, \eta_+) = \int d^2 \chi_+ e^{\eta_- \chi_+} \Phi_{++}(p, \chi_+, \eta_+, w).
\]  

(3.9)

As an illustration as to how the RHs drop out after the FT (3.9), consider the terms in (3.8) containing the projections of \(\psi^{(+\frac{1}{2})}\). They are transformed into \(\eta_-^a \psi_{a+}^{(+\frac{1}{2})} + \eta_-^{a'} \psi_{a-a'}^{(+\frac{1}{2})} = \eta_\alpha \psi^{(+\frac{1}{2})A}\) in view of the completeness relation for RHs (the third line in (3.1)). In the same way, in all the

\[\footnote{We use spinor units of charge, hence the charge \((+2)\) of the HWS of the vector irrep of \(SO(6) \sim SU(4)\).}
terms in (3.8) we reconstruct the representations of $SU(4)$ from their projections and obtain the familiar chiral on-shell superstate [39]

$$\Phi(p, \eta) = g^{(+1)} + \eta A \psi^{(+1)A} + \frac{1}{2} \eta \eta_{AB} \phi^{AB} + (\eta^3)^D \psi^D \eta^{(-1)} + (\eta)^4 g^{(-1)},$$  

(3.10)

where

$$(\eta)^4 = \frac{1}{3!} \epsilon^{ABCD} \eta_{AB} \eta_{CD}, \quad (\eta)^4 = \eta \eta_2 \eta_3 \eta_4.$$  

(3.11)

4. $\mathcal{N} = 4$ SYM in Lorentz harmonic chiral superspace

According to (2.9), the form factors are obtained as the on-shell residues of the correlator of the supercurvatures. In [18] we proposed a construction of the supercurvatures and the off-shell formulation of $\mathcal{N} = 4$ SYM in LHC superspace. In this formulation the chiral half of $\mathcal{N} = 4$ supersymmetry is realized off shell. We have developed a Feynman supergraph technique with manifest chiral supersymmetry. Having half of the supersymmetry off shell is possible due to the infinite number of auxiliary and pure gauge fields of arbitrarily high spin. Here we briefly review this formalism.

We work with the Euclidean Lorentz group $SO(4) \sim SU(2)_L \times SU(2)_R$. The left and right factors act on the undotted and dotted Lorentz indices of the space-time coordinates $x^{\mu \alpha} = x^\mu \tilde{\sigma}^{\alpha \nu}$, respectively. The LH variables $u^+_\alpha$ and $u^-_\alpha$ are a pair of spinors forming an $SU(2)_L$ matrix [24, 25]:

$$(u^+_\alpha, u^-_\alpha) \in SU(2)_L : u^+_+ u^-_\alpha = 1, \ (u^+_\alpha)^* = -u^-_\alpha, \ (u^+)^* = u^-.$$  

(4.1)

The LHs project the fundamental representation of $SU(2)_L$ onto the $U(1)$ subgroup, so that their indices denote the $U(1)$ charge. All expressions have to be homogeneous in the $U(1)$ charge. To distinguish the charges of the LHs from those of the RHs from section 3, we indicate the former upstairs and the latter downstairs.

Our (super)fields are LH functions defined by their infinite LH expansion on $S^2 \sim SU(2)_L/U(1)$. In it we find irreducible representations of $SU(2)_L$ of arbitrarily high spin (totally symmetric multispinors). For example, for a charge $(+2)$ LH field we have

$$f^{++}(x, u) = f^{\alpha \beta}(x) u^+_\alpha u^+_\beta + f^{(\alpha \beta \delta \epsilon)}(x) u^+_\alpha u^+_\beta u^+_\delta u^-_\epsilon + f^{(\alpha \beta \gamma \delta \epsilon \zeta)}(x) u^+_\alpha u^+_\beta u^+_\gamma u^+_\delta u^-_\epsilon u^-_\zeta + \ldots$$  

(4.2)

So, an LH field consists of an infinite set of ordinary multispinor fields $f^{(\alpha_1 \ldots \alpha_n)}(x)$.

The differential operators compatible with the normalization condition $u^+ u^- = 1$ (4.1) are the LH derivatives

$$\partial^{++} = u^+ \partial / \partial u^- \quad \partial^{--} = u^- \partial / \partial u^+.$$  

(4.3)

Acting on an LH function, they increase and decrease, respectively, its $U(1)$ charge. Supplementing them with the Cartan charge $\partial^0$ which counts the $U(1)$ charge of the LH functions we obtain the algebra of $SU(2)_L$:

$$[\partial^{++}, \partial^{--}] = \partial^0, \quad [\partial^0, \partial^{++}] = 2 \partial^{++}, \quad [\partial^0, \partial^{--}] = -2 \partial^{--}.$$  

(4.4)

The restriction to functions on $SU(2)_L$ with definite charge gives a particular realization of the LH coset $S^2 \sim SU(2)_L/U(1)$. 

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An LH function with $U(1)$ charge $q \geq 0$, which is the highest weight of a finite-dimensional irrep of $SU(2)_q$ of spin $q/2$, is a multispinor of rank $q$. This property can be equivalently formulated as an LH differential equation,

$$q \geq 0 : \partial^\pm f^{(q)}(u) = 0 \Rightarrow f^{(q)}(u) = f^{\alpha_1 \ldots \alpha_q} u^{+}_{\alpha_1} \ldots u^{+}_{\alpha_q}. \quad (4.5)$$

We also need an $SU(2)_L$ invariant LH integral on $S^2$. For the LH functions of nonzero $U(1)$ charge the integration gives zero, and for the chargeless LH functions having the LH expansion $f(u) = f^{(\alpha_0)} u^{+}_{\alpha_0} + \ldots$ the integral picks the singlet part, $\int du f^{(q)}(u) = f$. In particular, $\int du = 1$. This rule is compatible with integration by parts for the LH derivatives $f^{(q)}$. Alongside the regular LH functions that admit LH expansions on $S^2$ (see (4.2)), we also consider singular LH distributions. The LH delta function $\delta(u, v)$ is defined by the property

$$\int dv \delta(u, v) f^{(q)}(v) = f^{(q)}(u) \quad (4.6)$$

with a test function of $U(1)$ charge $q$.

The LHs are used to project the Lorentz indices of the odd variables and derivatives,

$$\theta^{\pm A} = u^{\pm A \alpha} \partial^{\alpha A}, \quad \partial^{\pm A} = u^{+ A \alpha} \partial^{\alpha A}, \quad \partial^{- A} = u^{+ A \alpha} \partial^{\alpha A}. \quad (4.7)$$

In this section we keep only the chiral odd variables $\theta^A$ and work with superfields which transform covariantly with respect to the $Q$-half of the $\mathcal{N} = 4$ supersymmetry algebra. The odd variables $\theta^A$ are absent, so the $\bar{Q}$-half of the supersymmetry is not manifest. We extensively use L-analytic harmonic superfields $\Phi(x, \theta^+, u)$ which depend only on half of the chiral odd variables. Equivalently we can formulate the L-analyticity of an LH superfield as $\partial^+ \Phi = 0$ (see (4.7)). L-analyticity is another form of Grassmann analyticity (see the $R$-analyticity from section 3).

In gauge theory we consider a gauge transformation whose parameter $\Lambda(x, \theta^+, u)$ is an L-analytic harmonic superfield of $U(1)$ charge zero in the adjoint representation of the gauge group $SU(N)$. This is the so-called analytic gauge frame. In it the flat derivatives $\partial^+ \Lambda$, $\partial^- \Lambda$, $\partial^+_A$, $\partial^-_A$ (see (4.3) and (4.7)) are extended to covariant derivatives $\nabla^+$, $\nabla^-$, $\nabla_A^+$, $\nabla_A^-$ by adding gauge connections. The derivatives $\partial^0$ and $\partial^+_A$ remain flat since $\partial^0 \Lambda = \partial^+_A \Lambda = 0$. A key role is played by the gauge connections $A^{++}(x, \theta^+, u)$ and $A^{++}_A(x, \theta^+, u)$, which are L-analytic superfields. In [18] we identified them as the dynamical fields of $\mathcal{N} = 4$ SYM. They are gauge prepotentials, i.e. the remaining gauge connections $A^{+-}$, $A^{-A}$, $A^+_A$ as well as the supercurvatures can be expressed in their terms. The latter are not L-analytic, so they depend on the full chiral odd variable $\theta^A$.

The crucial step in constructing all the gauge connections in terms of the dynamical fields $A^{++}$ and $A^{++}_A$ is finding $A^{-+}$. The $SU(2)_L$ algebraic structure of the LH derivatives provides the key. Indeed, the gauge connection $A^{-+}$ is present in the covariantized commutation relation (4.4),

$$[\nabla^{++}, \nabla^{-+}] = \partial^0. \quad (4.8)$$

This is an LH differential equation on $S^2$ with a unique solution for $A^{-+}$. The solution is a chiral superfield (not L-analytic) which is nonpolynomial in $A^{++}$ [24, 40].

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*In our conventions the gauge connections are accompanied by the YM coupling $g$. $\nabla \equiv \partial + g A$. The infinitesimal gauge transformations have the form $\delta_A A = \nabla A$ with an L-analytic parameter $\Lambda.$*
\[ A^{-}(x, \theta, u) = -\sum_{n=1}^{\infty} (-g)^{n-1} \int du_{1} \ldots du_{n} \frac{A^{++}(x, \theta \cdot u^{+}_{1}, u_{1}) \ldots A^{++}(x, \theta \cdot u^{+}_{n}, u_{n})}{(u^{+} u^{+_1}) \ldots (u^{+} u^{+_n})} \]  
\[ (4.9) \]

with \( \theta^{A} \cdot u^{+_k} = \theta^{\alpha A}(u^{+_k})_{\alpha} \). The right-hand side of (4.9) is local in \((x, \theta)\) space but nonlocal in LH space. It has zero \( U(1) \) charge with respect to the integration LH variables \( u_{1}, \ldots, u_{n} \) and charge \((-2)\) with respect to \( u \).

In the LHC formulation the \( N = 4 \) SYM action involves the prepotentials \( A^{++} \) and \( A^{+}_{\dot{+}} \). It consists of two terms, \( S_{\text{CS}} = S_{\text{CSW}} + S_{\text{Z}} \). The first term is a Chern–Simons-like action which describes the self-dual sector of the theory [1, 26, 27]. The second term involves only \( A^{+} \) and it is nonlocal in LH space (it looks very similar to (4.9)—see (6.25) below). The role of this term is to complete the self-dual sector to the full SYM theory [20]. The form of \( S_{\text{Z}} \) coincides with the \( N = 2 \) SYM action in RH superspace as given by Zupnik [40].

In quantum theory we need a gauge fixing condition. Following [21, 41] we choose the light-cone (or ‘axial’ or ‘CSW’) gauge \( \xi^{+\alpha} A^{++}_{\alpha} = 0 \). It is defined by the auxiliary LHs \( \xi^{\pm} \) for the \( SU(2)_{R} \) factor of the Euclidean Lorentz group (called ‘reference spinor’ in [21]). To distinguish them from the LHs \( u^{\pm} \) on \( SU(2)_{R} \) (see (4.1)) we denote their \( U(1) \) charge by \( \pm \). In this gauge the \( S_{\text{CS}} \) part of the action becomes free and all interactions are due to \( S_{\text{Z}} \). For the calculation of form factors we need propagators in the momentum representation. Repeating the argument in [18] (see also [42]), but this time in momentum space, we obtain

\[ \langle A^{+}_{\alpha}(p, \theta^{+}, u_{1}) A^{+\beta}(-p, 0, u_{2}) \rangle = 0, \]  
\[ (4.10) \]

\[ \langle A^{++}(p, \theta^{+}, u_{1}) A^{++}(-p, 0, u_{2}) \rangle = 4\pi \delta^{2}(p^{+} \cdot \cdot \cdot) \delta(u_{1}, u_{2}) \delta^{2}(\theta^{+}), \]  
\[ (4.11) \]

\[ \langle A^{+}_{\alpha}(p, \theta^{+}, u_{1}) A^{+\beta}(-p, 0, u_{2}) \rangle = 2i\xi^{\alpha}_{\alpha}/p^{+} \delta(u_{1}, u_{2}) \delta^{2}(\theta^{+}). \]  
\[ (4.12) \]

The LH delta function is defined in (4.6). The fermionic delta function \( \delta^{2}(\theta^{+}) \equiv \theta_{1}^{\alpha} \theta_{2}^{\beta} \theta_{3}^{\gamma} \theta_{4}^{\delta} \) carries \( U(1) \) charge \(+4\). The right-hand side of (4.11) also contains the complex delta function \( \delta^{2}(t) \equiv \delta(t, t) \) satisfying the relation \( \frac{\alpha}{\pi} \frac{1}{7} = \pi \delta^{2}(t) \). More specifically, we use \( \delta^{2}(p^{+}) = \delta(p^{+}, p^{+}) \) where \( p^{+\alpha} \equiv \xi^{\alpha}_{\alpha} p^{\alpha\alpha} u^{+}_{\alpha} \).

5. Operator supermultiplets in LHC superspace

We construct composite operators from the supercurvature \( \nabla^{\alpha}_{AB}(x, \theta, \bar{\theta}) \). It appears in the anticommutator of the covariantized spinor derivatives \( \{ \nabla^{\alpha}_{A}, \nabla^{\beta}_{B} \} = \epsilon^{\alpha\beta} \nabla^{A}_{A} \). In this section we consider several supermultiplets of composite operators which are of prime interest for the applications. These are the Konishi multiplet (and its higher twist generalizations) and the half-BPS multiplets (including the stress-tensor multiplet). We explain how to construct them in terms of the L-analytic superfields (see section 4). The construction is particularly simple for the chiral truncation of the multiplets. This is not a surprise since the constituent L-analytic prepotentials are chiral. The nonchiral sector of the multiplets is much more involved. We construct it acting on the chiral sector with the \( Q \)-supersymmetry generators. In sections 6 and 7 we show that the complications which arise in the nonchiral sector are removed if we restrict ourselves to the tree level MHV form factors. These simplest form factors do not capture all the sophisticated details of the nonchiral composite operators. In section 8 we generalize our construction to composite operators of arbitrary spin and twist.

\[ ^{10} \text{The vanishing of the symmetric in } \alpha, \beta \text{ part of the right-hand side defines } N = 4 \text{ SYM.} \]
5.1. Chiral truncation of the multiplets

Let us start with the chiral truncation of the supercurvature $W_{AB}(x, \theta) \equiv \nabla_{A} W_{B} |_{\theta = 0}$. In the analytic frame (see section 4) it is an LH superfield of $U(1)$ charge zero, so the defining anticommutation relation takes the form $\{\nabla_{A}, \partial_{B}\} = g W_{AB}(x, \theta, u)$ where $\nabla_{A} = \partial_{A} + gA_{A}(x, \theta, u)$. In [18] we expressed the gauge connection $A_{A}$ in terms of the gauge prepotential $A^{-+}$ and found a concise form of the chiral supercurvature (see (4.9)),

$$W_{AB}(x, \theta, u) = \partial_{A}^{+} \partial_{B}^{+} A^{-+-}. \quad (5.1)$$

It is covariant with respect to gauge transformations with an L-analytic parameter $\Lambda$, and it is covariantly LH independent,

$$W_{AB} \rightarrow e^{-g \Lambda(x, \theta^{+}, u)} W_{AB} e^{g \Lambda(x, \theta^{+}, u)}, \quad \nabla^{++} W_{AB}(x, \theta, u) = 0. \quad (5.2)$$

We can construct multiplets of operators multiplying together several chiral supercurvatures $W_{AB}$ and taking the trace over the adjoint representation of the gauge group $SU(N)$. For example, the chiral truncation of the Konishi multiplet is

$$K(x, \theta, u) = e^{ABCD} \text{tr} (W_{AB} W_{CD}). \quad (5.3)$$

The supercurvatures are defined in the analytic frame, i.e. they depend on the LHs. However, the gauge-invariant operators like (5.3) are LH independent. Indeed, due to (5.2) we have $\partial^{++} K(x, \theta, u) = 2 e^{ABCD} \text{tr} (W_{AB} \nabla^{++} W_{CD}) = 0$. Then in view of lemma (4.5) we have $K(x, \theta, u) = K(x, \theta)$.

Evidently, the $SU(4)$ singlet structure in (5.3) is not the only possibility. Another interesting subclass of operators are the chiral truncated half-BPS multiplets. Their bottom components transform in the irrep of the R-symmetry group with Dynkin labels $[0, k, 0]$, where $W_{++}(x, \theta, u, w) = \frac{1}{2} u^{a'b'} w^{A}_{+} w^{B}_{+} W_{AB}$ is the RH projection of the supercurvature (5.1) on the highest weight state of the $SU(4)$ irrep $[0, 1, 0]$, and $\partial_{a'}^{+} \equiv u^{+a'} \partial_{a} w^{a}_{+}$. In this example we have to deal with RHs of $SU(4)$ and LHs of $SU(2)_{L}$ simultaneously. The LH independence of the gauge-invariant operator (5.4) is established in the same way as for the Konishi multiplet. Let us note that $W_{++}$, contrary to the gauge invariant half-BPS operators (5.4), is not R-analytic (see section 3), i.e. $W_{++}$ depends on both $\theta_{+}$ and $\theta_{-}$. In fact the dependence on $\theta_{-}$ takes the form of a generalized gauge transformation, so it drops out in the gauge covariant combination (5.4). Also $\theta_{-}$ drops out from $W_{++}^{\text{free}}$ in the free approximation (recall (3.4)), which is insensitive to the choice of gauge frame.

The chiral truncated supercurvature contains only the scalars $\phi_{AB}$, the gluinos $\psi_{A}$, and the self-dual YM curvature $F_{\alpha \beta}$ (the first line in (3.4)). So only this subset of fields appears in the multiplets (5.3) and (5.4). The anti-gluinos $\bar{\psi}_{A}$ and the anti-self-dual YM curvature $\bar{F}_{\alpha \beta}$ reside in the nonchiral sector of the supercurvature (the last two lines in (3.4)).

5.2. Complete nonchiral multiplets

Since the odd $\theta$ variable is absent in the LHC formulation of $\mathcal{N} = 4$ SYM, we have to use $Q$-supersymmetry to restore the nonchiral sector of the supercurvature (5.1),

11 In this paper we consider only single-trace operators. The generalization to multi-trace operators is straightforward.

12 The reason is that in the analytic frame the $\theta_{-}$-independence condition involves covariant derivatives, $\partial_{a'}^{+} W_{++} = \nabla_{a'}^{+} W_{++} = 0$. This can be changed by going to another, R-analytic frame where $\nabla_{a'}^{+} = \partial_{a'}^{+}$ but there $\varphi_{-}$ becomes covariant.
Unlike \( Q \)-supersymmetry, \( \bar{Q} \)-supersymmetry is not manifest. It is realized on the dynamical fields \( A^+ \) and \( A^+_\alpha \) in the following way \cite{18} (our generators act on the fields, not on the coordinates),

\[
\bar{Q}_{\beta}^B A^+_\alpha = -2\theta^+{}^B \partial_{\beta} A^+_\alpha + \left( Q_{\alpha}^B A^+_\beta \right), \quad \bar{Q}_{\beta}^B A^{++} = -2\theta^+{}^B \left( \partial_{\beta} A^{++} + A^+_\beta \right). \tag{5.6}
\]

The \( \bar{Q} \)-variations mix up both dynamical fields \( A^{++} \) and \( A^+_\alpha \). They act highly nontrivially on \( A^+_\alpha \) due to the term \( A^{--} \) defined in (4.9), so it is nonpolynomial in \( A^{++} \). The supersymmetry algebra closes only on shell and modulo gauge transformations. The \( \bar{Q} \)-term is irrelevant for MHV tree-level form factors (see the appendix). In what follows we shall drop \( \bar{Q} \). The remaining part of the \( \bar{Q} \)-supersymmetry (5.6) corresponds to the self-dual sector of \( \mathcal{N} = 4 \) SYM \cite{1, 26, 27}. In this simplified case we can define the on-shell nonchiral superfield

\[
A^{++}(x, \theta^+, \bar{\theta}, u) = e^{\bar{\theta}^i \partial_i} A^{++}(x, \theta^+, u) \tag{5.7}
\]

where we repeatedly apply the \( \bar{Q} \)-variations (5.6),

\[
\bar{Q}_{\alpha_1} A_{\alpha_2} \ldots \bar{Q}_{\alpha_k} A^{++} = (-2)^{k+1} \theta^{++ A_1} \ldots \theta^{++ A_k} \left( \partial_{\alpha_1} \ldots \partial_{\alpha_k} A^{++} + k \partial_{\alpha_1} \ldots \partial_{\alpha_k} A^{++} \right) \tag{5.8}
\]

for \( k = 1, \ldots, 4 \). The fifth variation vanishes, since \( (\theta^+)^5 = 0 \). Using the notion of \( \mathcal{A}^{++} \) we recast the supercurvature (5.5) in the following form (recall (5.1))

\[
\mathcal{W}_{AB}(x, \theta, \bar{\theta}, u) = \partial_{\alpha} \partial_{\beta} \mathcal{A}^{++}, \tag{5.9}
\]

defining the nonchiral analog of \( A^{--} \) (4.9)

\[
\mathcal{A}^{--}(x, \theta, \bar{\theta}, u) = e^{\bar{\theta}^i \partial_i} A^{--}(x, \theta, u),
\]

\[
\mathcal{A}^{--}(x, \theta, \bar{\theta}, u) = -\sum_{n=1}^{\infty} (-2)^{n-1} \int du_1 \ldots du_n \frac{A^{++}(1) \ldots A^{++}(n)}{(u^+ u_1^+ \ldots u_n^+)(u_1^+ u_2^+ \ldots u_n^+) \ldots (u_1^+ u_2^+ \ldots u_n^+)} \tag{5.10}
\]

with \( \mathcal{A}^{++}(k) \equiv \mathcal{A}^{++}(x, \theta^+ u_1^+, \bar{\theta}, u_0) \).

The nonchiral completion of the half-BPS operators (5.4) is obtained from the RH projection of the nonchiral supercurvature (5.9),

\[
\mathcal{W}_{++}(x, \theta, \bar{\theta}, u, w) = 1/2 e^{\theta^i \bar{\theta}^j} \bar{w}_{++}^{B} w_{++}^{A} \mathcal{W}_{AB}(x, \theta, \bar{\theta}, u). \tag{5.11}
\]

Just like its chiral counterpart in (5.4), \( \mathcal{W}_{++} \) in (5.11) is not R-analytic (see section 3), i.e. it depends not only on \( \theta^+, \bar{\theta}^+ \) but also on \( \theta^-, \bar{\theta}^- \). The reason is that we work in the L-analytic frame and we have constructed \( \mathcal{W}_{++} \) from the L-analytic fields \( A^{++}, A^+_\alpha \). In (5.11) \( \theta^- \) and \( \bar{\theta}^- \) appear in the form of a generalized gauge transformation, so they drop out in gauge-invariant quantities.

6. The supercurvature \( \mathcal{W}_{++} \) as an on-shell state

Let us briefly recall the construction of the tree-level form factor in section 2. We used the amputated propagator (2.3) between the self-dual and anti-self-dual YM curvatures. It gave us the external on-shell state (2.5), which we substituted for each field \( F \) in the composite operator \( O = \text{tr}(\bar{F}F) \). The result was the tree-level form factor (2.7).
In this section we repeat the argument in the supersymmetric case. We start by replacing the curvature $F$ at the external legs by the free supercurvature (5.9) and (5.11)
\[
\mathcal{W}_{++}^{\text{free}} = (v^+ \partial_+)^2 \int \frac{du}{(v^+ u_1^+)^2} \mathcal{A}^{++}(u).
\] (6.1)

Then we evaluate the amputated propagator
\[
\lim_{p^2 \to 0} p^2 \langle \mathcal{W}_{++}^{\text{free}}(p) \mathcal{A}^{++}(-p) \rangle.
\] (6.2)

Notice that this time, the second end of the propagator is not a curvature as in (2.3) but the gauge prepotential itself. The reason is that our composite operators are made from supercurvatures which in turn are made from the prepotentials $A^{++}, A^+_\alpha$, see (5.8) and (5.9). At this stage the odd variables $\theta$ get projected with the negative helicity spinors $\lambda$, as explained in (3.7). Then we do the half-FT (3.9), which eliminates the RHs $w$ from the external on-shell state. Notice that the free super-curvature in (6.1) is in fact independent of the LH $v$, since it satisfies the constraint $\partial_+ \mathcal{W}_{++}^{\text{free}} = 0$\(^{13}\). The result is the supersymmetric analog of the state (2.5). It will be subsequently used in section 7 for the calculation of MHV form factors by a substitution rule which is the analog of (2.6).

6.1. Amputated chiral super-propagator

Let us first compute the chiral analog of (6.2), the amputated propagator of the chiral truncation $\mathcal{W}^{\text{free}}_{++}(\theta)$ of the super-curvature with the chiral prepotential $A^{++}$. The amputation of the external legs described in section 2 requires a pole $1/p^2$ in the propagator. Our LHC propagator $\langle A^{++}A^{++} \rangle$ has such a pole, as we show below.

We start by computing the propagator $\langle \mathcal{W}^{\text{free}}_{++}A^{++} \rangle$ with the help of (4.11):
\[
\langle \mathcal{W}^{\text{free}}_{++}(p, \theta_+, v, w) A^{++}(-p, 0, u) \rangle = (v^+ \partial_+)^2 \int \frac{du_1}{(v^+ u_1^+)^2} \langle A^{++}(u_1) A^{++}(u) \rangle
\]
\[
= (v^+ \partial_+)^2 \int \frac{du_1}{(v^+ u_1^+)^2} 4\pi \delta^4(p^- u^+) \delta(u_1, u) \delta^4(\theta \cdot u^+) = 4\pi \delta^2(p^- u^+) \delta^2(w^+ A^+ \theta^\alpha u^+).
\] (6.3)

The differentiation $(v^+ \partial_+)^2$ has been done by splitting $\delta^4(\theta \cdot u^+)$ with the help of RHs,
\[
(v^+ \partial_+)^2 \delta^4(\theta \cdot u^+) = (v^+ \partial_+)^2 [\delta^2(w^- \theta u^+) \delta^2(w^+ \theta u^+)] = (v^+ u^+)^2 \delta^2(w^+ \theta u^+).
\] (6.4)

We recall that in (6.3) $p^{-\alpha} = \xi_\alpha^+ p^{+\alpha}$ is the projection of the momentum with the light-cone-gauge-fixing parameter $\xi_\alpha^+$. As expected, the $W$ end of the propagator does not depend on the LH $v$ at that point, and it depends polynomially on the RH $w$.

This propagator has a pole $1/p^2$. To reveal it, we recall [18] that the bosonic delta function in (6.3) identifies
\[
u_\alpha^+ = p^+ \xi^\alpha_+ / \sqrt{p^2}, \quad u_\alpha^+ = -p^+ \xi^\alpha_+ / \sqrt{p^2}.
\] (6.5)

The LH $u$ will be integrated over, in expressions of the type
\[
\int du \delta^2(p^- u^+) \frac{P(u^+)}{Q(u^+)} = \frac{1}{\pi p^2} \frac{P(p^- \sqrt{p^2})}{Q(p^- \sqrt{p^2})} = \frac{1}{\pi p^2} \frac{P(p^-)}{Q(p^-)}.
\]

\(^{13}\) The covariant counterpart of this relation takes the form $\partial^+ W_{ab} + g[\mathcal{A}^{++}, W_{ab}] = 0$ (recall (5.2)).
Here $P, Q$ are homogeneous polynomials in $u^+$ of the same degree, so that their ratio has an LH charge of zero. Thus, after the integration the LH $u^+$ gets replaced by the projected momentum $p^+$. The presence of a pole allows us to do the amputation. On shell $p_\alpha\dot{\alpha} = \lambda_\alpha\dot{\lambda}_\alpha$, so $p_+ = [\xi^- \dot{\lambda}]\lambda_\alpha$. Once again, due to the vanishing LH charge we can drop the factor $[\xi^- \dot{\lambda}]$.

Thus, effectively (see (3.8))

$$\lim_{p^+ \to 0} p^+ (W^\text{free}_+ (\theta, w) A^+ (\theta_0, u))$$

$$\Rightarrow \langle \Phi_+ (\theta, w) A^+ (\theta_0, u) \rangle = \delta^2 (w_+ (\theta - \theta_0) | \lambda_\alpha ) \delta (\lambda, u),$$

(6.7)

where the bosonic delta function can be treated as a harmonic one (4.6), identifying $u^+ = \lambda$. We have also restored the dependence on $\theta_0$ by translation invariance.

What remains for us to do is to FT the Grassmann variable $\chi_+ = \langle \theta_+ \lambda \rangle$ (recall (3.7)) at the external leg as in (3.9),

$$\int d^2 \chi_+ e^{\eta^- \chi_+} \delta^2 (\chi_+ - w_+ \theta_0 | \lambda ) = e^{\eta^- w_+ \theta_0 | \lambda }.$$

(6.8)

So, finally, the on-shell state reads

$$\langle \Phi (\eta) A^+ (\theta_0) \rangle = \delta (\lambda, u) e^{\eta^- w_+ \theta_0 | \lambda }.$$

(6.9)

This result is intermediate—we still need to restore the dependence on the other half $\eta_+$ of the odd variables to complete the on-shell state.

6.2. The complete nonchiral on-shell state

In section 6.1 we used the chiral truncation of the supercurvature $W^\text{free}_+ (x, \theta_+)$ in the analytic frame. In section 5.2 we reconstructed the full nonchiral supercurvature $W^\text{free}_+ (x, \theta_+, \bar{\theta}_+)$ by working out the $\bar{Q}$-variations of the chiral supercurvature. Schematically,

$$W^\text{free}_+ (x, \theta_+, \bar{\theta}_+, u, w) = W^\text{free}_+ (\theta_+) + i \bar{\theta}_+^\alpha \bar{Q}_\alpha^{\prime \prime} W^\text{free}_+ - \frac{1}{2} (\bar{\theta}_+ \bar{Q}_-) W^\text{free}_+ + \cdots$$

(6.10)

Recall that the free supercurvature is manifestly R-analytic, i.e. it is annihilated by half of the super-momenta, $\bar{Q}_+ W^\text{free}_+ = 0$. This is why we only use $\bar{Q}_- W^\text{free}_+$ in the expansion (6.10).

The on-shell (amputated) $\bar{Q}$-variations of the propagator $\langle W^\text{free}_+ A^+ \rangle$ are given by (see (A.7))

$$\lim_{p^+ \to 0} p^2 (\bar{Q}_+^{\prime \prime} W^\text{free}_+ (\theta_+, w) A^+ (\theta_0, u)) = (-i) \bar{\lambda}_\alpha \bar{w}_\alpha^\prime \langle \theta_0 \lambda \rangle \langle \Phi_+ A^+ \rangle$$

(6.11)

$$\lim_{p^+ \to 0} p^2 (\bar{Q}_-^{\prime \prime} \bar{Q}_+^{\prime \prime} W^\text{free}_+ (\theta_+, w) A^+ (\theta_0, u)) = (-i)^2 \bar{\lambda}_\alpha \bar{w}_\alpha^\prime \langle \theta_0 \lambda \rangle \langle w_\alpha^\prime \theta_0 \lambda \rangle \langle \Phi_+ A^+ \rangle$$

(6.12)

with $\langle \Phi_+ A^+ \rangle$ from (6.7).

Substituting these variations in the antichiral expansion (6.10), we see that the odd coordinates $\bar{\theta}_+^\alpha$ get projected with the momentum helicity spinor $\bar{\lambda}_\alpha$, hence the expansion goes in the effective odd variables $\eta^+ = \bar{\theta}_+^{\prime \prime} \bar{\lambda}_\alpha$ (recall (3.7)). Then we find

$$\langle \Phi_+ (\theta, \bar{\theta}, w) A^+ (\theta_0, u) \rangle = [1 + \eta_+ w_+ \theta_0 | \lambda ] + \frac{1}{2} \eta_+ w_+ \theta_0 | \lambda \rangle \delta^2 (w_+ (\theta - \theta_0) | \lambda ) \delta (\lambda, u)$$

(6.13)
The FT with respect to $\chi_+ = \langle \bar{\theta} \lambda \rangle$ is performed as in (6.8), resulting in (from here on we drop the index 0 at the $A^{++}$ end)

$$\langle \Phi(\eta)A^{++} (\theta, u) \rangle = \delta(\lambda, u)e^{(\eta_{+W-} + \eta_{-W+})[\theta, \bar{\lambda}]} = \delta(\lambda, u)e^{\eta(\theta, \bar{\lambda})},$$

(6.14)

where we have used the completeness identity (3.1) for the RH $w$. As expected, the RH $w$ at the external end of the propagator has dropped out from the on-shell state.

Notice that the presence of the positive helicity spinor $\bar{\lambda}_\alpha$ in (6.11) explains why the on-shell anti-gluino becomes $\bar{\psi}_\alpha = \bar{\lambda}_\alpha e^{-\frac{1}{2}i}$ (see (3.6)). Similarly, the factor $\bar{\lambda}_\alpha \lambda_\beta$ in (6.12) explains why the negative helicity on-shell gluon is represented by $\tilde{F}_{\tilde{\alpha} \tilde{\beta}} = \bar{\lambda}_\alpha \lambda_\beta \bar{\delta}^{(-1)}$.

What we still need to do is to restore the $\bar{\theta}$ dependence at the $A^{++}$ end of the propagator, $A^{++} \rightarrow A^{++}$ (see (5.7)). Using the complete set of $\bar{Q}$–variations from (A.7) and repeating the above steps, we find that the variable $\bar{\eta}$ in the exponential in (6.14) is replaced by $\eta \rightarrow \eta + [\bar{\theta}]$. In this way we obtain the nonchiral on-shell state

$$\langle \Phi(\eta)A^{++} (\theta, \bar{\theta}, \bar{\lambda}, \bar{\theta}, u) \rangle = \delta(\lambda, u)e^{(\eta + [\bar{\theta}]) (\theta, \bar{\lambda})}.$$  

(6.15)

According to (2.6), we need to complete the on-shell state by the space-time factor $e^{iuP_\nu} = e^{i/2[\lambda x] \bar{\lambda}}$. So, (6.15) gives to the substitution rule

$$A^{++} (\lambda, \bar{\lambda}, \theta, \bar{\theta}, u) \Rightarrow \delta(\lambda, u) \exp \left\{ \frac{1}{2} \bar{\lambda}_\alpha \left( A^{\alpha\alpha} - 2 \theta \delta^A \theta^{\alpha A} \right) \lambda_\alpha + \eta_\alpha \theta^{\alpha A} \lambda_\alpha \right\}$$

(6.16)

for each $A^{++}$ in a composite operator made from these prepotentials. The applications of this rule will be discussed in detail in section 7.

We remark that the odd variables $\theta, \bar{\theta}$ appear in (6.16) projected with helicity spinors, as claimed in (3.7).

Our result (6.16) is the nonchiral generalization of the on-shell chiral state found in [22]. The approach of [22] is to start from the collection of component states in the Wess–Zumino gauge and perform a gauge transformation to the CSW gauge. Here we have given a direct derivation of the on-shell state from first principles. We have also shown explicitly how the on-shell state becomes independent of the gauge-fixing parameter $\xi^\alpha$.

### 6.3. The on-shell state and supersymmetry

Above we have derived the complete on-shell state by applying the amputation procedure to a supersymmetrized propagator. Here we wish to show that the main part of (6.16) can in fact be obtained by requiring invariance under the $\mathcal{N} = 4$ supersymmetry algebra with generators

$$Q_{\alpha A} = i \frac{\partial}{\partial \theta^{A \alpha}} + 2 \theta^{A \alpha} \frac{\partial}{\partial x^{\alpha A}} + i \lambda_\alpha \eta_\alpha, \quad \bar{Q}_A^\alpha = -i \frac{\partial}{\partial \bar{\theta}_A} - 2 \theta^{\alpha A} \frac{\partial}{\partial x^{\alpha A}} + 2i \bar{\lambda}_\alpha \frac{\partial}{\partial \eta_\alpha}$$

$$\{ Q_{\alpha A}, \bar{Q}_A^\beta \} = -2 \delta^\beta_\alpha \left( 2i \frac{\partial}{\partial x^{\alpha A}} + \lambda_\alpha \bar{\lambda}_\beta \right) = -2 \delta^\beta_\alpha P_{A \alpha}.$$  

(6.17)

Indeed, let us start with the bosonic state $exp \left\{ i/2 \bar{\lambda}_\alpha x^{\alpha A} \lambda_\alpha \right\}$. It is easy to see that this is the unique Lorentz invariant solution of the translation Ward identity $Pf(x, p) = 0$ with an on-shell momentum $p_{\alpha A} = \lambda_\alpha \bar{\lambda}_\beta$. Further, by adding to the bosonic variables $x, p$ the odd variables $\eta, \theta, \bar{\theta}$ in all allowed Lorentz and dilation invariant combinations, one can show that the unique solution of the supersymmetry Ward identities $Qf(x, p, \eta, \theta, \bar{\theta}) = Qf(x, p, \eta, \theta, \bar{\theta}) = 0$ is indeed the exponential factor in (6.16).
Remarkably, the combination

\[ x_{\text{ch}}^\alpha = x^\alpha - 2i\theta^\alpha \theta^\alpha \]  

(6.18)

which appears in (6.16) has the meaning of a basis shift from real superspace with space-time coordinate \( x \) to the (left-handed) chiral superspace with coordinate \( x_{\text{ch}} \). The latter transforms as follows:

\[ Q_{\alpha\beta} x_{\text{ch}}^\beta = 0, \quad \overline{Q}_{\alpha\beta} x_{\text{ch}}^\beta = -4\delta_{\alpha}^\beta \theta^\alpha \theta^\beta, \]  

(6.19)

i.e. it is inert under \( Q - \) supersymmetry. This result is not surprising because \( \Phi \) in (6.15) has been constructed as a chiral on-shell state—see (3.10). So, we can replace (6.16) by

\[ \hat{A}^{++}(\lambda, \bar{\lambda}, \theta, \bar{\theta}, u) \Rightarrow \delta(\lambda, u) \exp \left\{ \frac{i}{2} \hat{\lambda}_{\alpha} x_{\text{ch}}^\alpha \lambda_{\alpha} + \eta_{\alpha} \theta^\alpha \lambda_{\alpha} \right\}. \]  

(6.20)

In what follows we will make use of another basis, adapted to \( R \)-analytic superfields like the half-BPS operators (3.4), in particular the stress-tensor multiplet (7.1):

\[ x_{\text{an}}^\alpha = x^\alpha + 2i(\bar{\theta}^\alpha - \bar{\theta}^\alpha), \]  

(6.21)

where the odd variables are projected with RHs. In this basis the spinor derivatives \( D_+ \to \partial_+ \), \( D_- \to \partial_- \) become short. Consequently, the \( R \)-analyticity property (independence of \( \theta_-, \bar{\theta}_- \)) of the half-BPS operator \( T \) becomes manifest—see (7.1).

The relevance of the correct choice of basis in superspace becomes clear when we put a curvature at the second end of the amputated propagator (6.14), namely, \( \langle \Phi | W^{\text{free}}_{AB} | \Phi \rangle \). Now the ‘naive’ definition (5.9) has to be modified. Instead of partial spinor derivatives \( \partial_{\alpha}^A \) we have to use covariant ones. The latter are defined as operators anticommuting with the supersymmetry (5.9) has to be modified. Instead of partial spinor derivatives \( \partial_{\alpha}^A \) we have to use covariant ones. The latter are defined as operators anticommuting with the supersymmetry

\[ \{ \alpha, \beta \} = \{ \alpha, \beta \} = \{ \alpha, \beta \} = \{ \alpha, \beta \} = 0. \]  

(6.22)

So, we need to compute (recall (5.10) and (6.15))

\[ e^{i[\bar{\lambda}_{\alpha}] \langle \Phi | W^{\text{free}}_{AB} | \Phi \rangle} = D_A^+ D_B^+ \int \frac{dv}{(u^+ v)^2} e^{i[\bar{\lambda}_{\alpha}] \langle \Phi | A^{++} | v \rangle} \]

\[ = \frac{1}{(u^+ v)^2} D_A^+ D_B^+ e^{i[\bar{\lambda}_{\alpha}] \langle \Phi | B^{++} | v \rangle \rangle_{\lambda} \rangle} = \hat{\eta}_{\alpha} \eta_{\beta} \hat{e}^{i[\bar{\lambda}_{\alpha}] \langle \Phi | B^{++} | \eta_{\beta} \rangle \rangle_{\lambda} \rangle}, \]  

(6.23)

where we see the \( \hat{Q} \)-invariant combination

\[ \hat{\eta}_{\alpha} = \eta_{\alpha} + 2[\bar{\theta}\partial_+], \quad \hat{Q}^\beta \hat{\eta}_{\alpha} = 0. \]  

(6.24)

Here it was important to use the correct covariant derivatives \( D_A^+ \), with the super-torsion term \( \bar{\theta}\partial_+ \), in order to obtain the \( \hat{Q} \)-invariant (6.24).

### 6.4. MHV amplitude

As a very simple illustration, let us apply our substitution rule (6.20) to the \( n \) point MHV super-amplitude. The Zupnik (interaction) term in the \( \mathcal{N} = 4 \) SYM action has a form similar to (4.9) (see [18]),
Here the bilinear term in fact belongs to the free action, so the true interaction terms have \( n \geq 3 \). Now, consider the \( n \)-valent Zupnik vertex and substitute each chiral \( A^{++} \) by the chiral on-shell state (6.20). The delta functions remove the LH integrals and replace the LHs by negative helicity spinors \( \lambda \). The exponentials from (6.20), after the integration over the vertex point \( \int d^4x d^8\theta \), produce the complete super-momentum conservation delta function and we recover the familiar result for the \( n \)-particle amplitude [39]

\[
A_{\text{MHV}} = \frac{\delta^4(\sum \lambda_i \bar{\lambda}_i)\delta^8(\sum \lambda_i \eta_i)}{\langle 12 \rangle \ldots \langle n1 \rangle}.
\] (6.26)

7. **MHV form factors**

The construction of the full on-shell state (6.16) has led to the following very simple substitution rule for the computation of MHV form factors. Consider an operator in the form of a product of supercurvatures \( W_{AB}^{++} \) (or their derivatives—for details see section 8). Each \( W_{AB}^{++} \) is made from many \( A^{++} \)—see (5.9). In the form factor each \( A^{++} \) from a given vertex is connected to an external state by a free propagator. The substitution rule is to replace the \( i \)-th external leg at the vertex by the super-state (6.16). The bosonic delta function \( \delta(\lambda, u_l) \) removes the LH integral at the vertex and replaces the LH \( u^+ \) by the negative helicity spinor \( \lambda \).

In this section we consider in detail two simple examples of the application of this rule—the form factors of the stress-tensor and the Konishi multiplets.

7.1. **The stress-tensor multiplet**

The \( \mathcal{N} = 4 \) SYM stress-tensor multiplet is the simplest of the half-BPS operators defined in (5.4):

\[
T = \text{tr}(W_{++}^{++})^2(x_{an}, \theta_+, \bar{\theta}_+, w),
\] (7.1)

where the R-analytic basis coordinate \( x_{an} \) was defined in (6.21). We wish to evaluate its tree-level MHV form factor. To this end we need to first compute the Born-level correlation function

\[
\langle \mathcal{W}^{\text{free}}_{++}(1) \ldots \mathcal{W}^{\text{free}}_{++}(n) \text{tr}(W_{++}^{++})^2(x_{an}, \theta_+, \bar{\theta}_+, w) \rangle,
\] (7.2)

then amputate the external legs \( 1, \ldots, n \). We consider the color ordered part of (7.2), which implies cyclic ordering of the external legs.

The supercurvature \( \mathcal{W}_{++} \) is defined in (5.9) and (5.11). The Born-level correlation function (7.2) is obtained by connecting each external leg \( \mathcal{W}^{\text{free}}_{++}(l) \) with an \( A^{++} \) inside the composite operator \( \text{tr}(W_{++}^{++})^2 \) by a free propagator. The resulting expression is proportional to \( g^{n-2} \), as expected from the tree-level form factor. This procedure splits the correlator in two clusters. Let us choose two legs with labels \( i < j \). The first cluster contains the legs with \( i + 1 \leq l \leq j \), the second contains the legs with \( j + 1 \leq l \leq i \) (the legs are labeled in a cyclic way). The legs from the first cluster are contracted with the first factor \( \mathcal{W} \) from the composite operator, and the remaining legs are contracted with the second factor \( \mathcal{W} \). The complete correlator is obtained by summing over all values of \( 1 \leq i < j \leq n \) (see figure 1).
Let us examine the structure of a cluster made of \(k\) legs. After the amputation, the calculation amounts to applying the substitution rule (6.16) to each leg within a given cluster. The bosonic delta functions \(\delta(\lambda_l, u_l)\) remove the integrals in (5.10) and replace the LHs by the negative helicity spinors. The calculation basically repeats (6.23). The result for a cluster of \(k\) legs is simply
\[
\Gamma(1, 2, \ldots, k) \equiv \left( \prod_{l=1}^{k} \Phi(p_l, \eta_l) \cdot \Psi^{\pm +}(x_{an}, \theta, \bar{\theta}, w, u) \right)_{\text{tree MHV}}
\]
\[
= (u^+ D_+)^2 \sum_{l=1}^{k} \frac{e^{i z_l}}{(u^+ 1)\langle 12 \rangle \ldots \langle ku^+ \rangle} \left( \sum_{l=1}^{k} \langle u^+ l \rangle \tilde{\eta}_l^+ \right)^2 \frac{e^{i \sum_{l=1}^{k} z_l}}{(u^+ 1)\langle 12 \rangle \ldots \langle ku^+ \rangle},
\] (7.3)

where \(z_l = [l]_1 (x_{an} - 4i\bar{\theta} \theta - 4i\bar{\theta} \theta)_{\pm u_l^+} - 2i\eta_l^+ \langle \theta l \rangle\). Each \(\tilde{\eta}_{\pm u_l^+} = \tilde{w}_{\pm u_l^+}^\dagger \tilde{\eta}_{l A}\) is projected with the RH of the composite operator and we have made the appropriate change of basis.

We remark that this expression still depends on the LH \(u^+_l\) of the supercurvature. This dependence is due to the analytic gauge frame. It disappears once the gauge invariant composite operator has been reconstructed by combining together the two clusters:
\[
(1, 2, \ldots, n)T(x_{an}, \theta_+, \bar{\theta}_+, w)_{\text{tree MHV}} = \sum_{i<j} \Gamma(i+1, \ldots, j) \Gamma(j+1, \ldots, i)
\]
\[
= \frac{1}{\langle 12 \rangle \ldots \langle n1 \rangle} \delta^4 \left( \sum_{l=1}^{n} \tilde{\eta}_l^+ |l\rangle \right) \exp \left\{ \sum_{l=1}^{n} \left( \frac{i}{2} [l]_1 x_{an} |l\rangle + \eta_- (\theta_+ |l\rangle) \right) \right\},
\] (7.4)

where \(\tilde{\eta}\) was defined in (6.24). In deriving this expression we have applied the eikonal identity [43]
\[
\sum_{k=i}^{i+1} \frac{(k k + 1)}{(u^+ k + 1)} = \frac{\langle i j \rangle}{\langle u^+ j \rangle}
\] (7.5)
twice. This identity is responsible for the elimination of the LH \(u^+_l\) dependence at the operator point.

Notice that the final expression depends only on the RH projected odd variables \(\theta_+, \bar{\theta}_+\), as should be for the half-BPS operator \(T\) (7.1). It is annihilated by the generators of supersymmetry (6.17) adapted to the R-analytic basis (6.21).

Our result coincides with that of [14] where the form factor has not been computed but rather predicted from supersymmetry Ward identities. Here we have explained how the direct computation using our LHC field theory rules leads to the desired result.
We can obtain the same result more quickly if we use the alternative representation of the chiral truncation of the stress-tensor multiplet in terms of the interaction (Zupnik) term of the Lagrangian, $\mathcal{T} = (\partial_+)^4 L_Z$ (see [28]). The Lagrangian $L_Z$ can be read off from the Zupnik action (6.25). $S_Z = \int d^4x d^4\theta L_Z$. Doing the substitution (6.14) we obtain

$$
\langle 1, 2, \ldots, n|\mathcal{T}(x, \theta_+, w)|0\rangle_{\text{tree MHV}} = \frac{1}{\langle 12 \rangle \ldots \langle n1 \rangle} (\partial_+)^4 e^{\sum_{i=1}^n \left( \frac{1}{2} \bar{\phi}_i + n_i(0) \right)}
$$

which immediately gives the chiral truncation of (7.4). Then the $\bar{\phi}_-$-dependence can be restored by supersymmetry. Notice that in this derivation we do not need the eikonal identity (7.5). However, this shortcut is only possible for the stress-tensor multiplet. The other operators have to be constructed in terms of the supercurvature $\mathcal{W}_{AB}$, for example, the Konishi multiplet (5.3).

### 7.2. Konishi multiplet

We consider the full Konishi multiplet $K(x, \theta, \bar{\theta}) = \epsilon^{ABCD} \text{tr} (\mathcal{W}_{AB} \mathcal{W}_{CD})$, i.e. the nonchiral extension of the chiral truncation (5.3). The calculation of the MHV tree-level form factor for $K$ follows the pattern of the previous section. Firstly, we examine the contribution of a cluster containing $k$ external legs which are contracted by the amputated propagators with the nonchiral supercurvature $\mathcal{W}_{AB}$ (5.9). The substitution rule (6.16) gives rise to (recall (6.23))

$$
\Gamma_{AB}(1, 2, \ldots, k) \equiv \langle \prod_{l=1}^k \Phi(p_l, \eta_l) \rangle \mathcal{W}_{AB}(x, \theta, \bar{\theta}, w) |0\rangle_{\text{tree MHV}}
$$

$$
\Gamma_{AB}(1, 2, \ldots, k) = \left( \sum_{l=1}^k (u^+ l) \bar{\eta}_A \right) \left( \sum_{l=1}^k (u^+ l) \bar{\eta}_B \right) e^{\sum_{i=1}^k z_i} \left( u^+ 1 \right) \langle 12 \rangle \ldots \langle ku^+ \rangle,
$$

where $z_i = [\tilde{f}(x - 2i\bar{\theta} \theta)|l] - 2i\bar{\eta}_l(\theta l)$ and $\bar{\eta}$ was defined in (6.24). This amputated correlation function is gauge covariant, but not gauge invariant. Being defined in the analytic gauge frame, it depends on the LH $u$. This dependence disappears in the gauge-invariant form factor, again due to the eikonal identity (7.5)

$$
\langle 1, 2, \ldots, n|K(x, \theta, \bar{\theta}, w)|0\rangle_{\text{tree MHV}} = \epsilon^{ABCD} \sum_{i,j} \Gamma_{AB}(i+1, \ldots, j) \Gamma_{CD}(j+1, \ldots, i)
$$

$$
= \frac{\epsilon^{ABCD} \sum_{i,j} \bar{\eta}_A \bar{\eta}_B \bar{\eta}_C \bar{\eta}_D \langle jk \rangle |i\rangle}{\langle 12 \rangle \ldots \langle n1 \rangle} \sum_{i \leq j \leq k \leq l} (2 - \delta_{ij}) (2 - \delta_{kl}) \epsilon^{ABCD} \bar{\eta}_A \bar{\eta}_B \bar{\eta}_C \bar{\eta}_D \langle jk \rangle |i\rangle.
$$

Notice that the odd factor of degree 4 does not form a fermionic delta function, unlike the stress-tensor multiplet in (7.4).

### 8. Component operators

In this section we consider the composite operators in $\mathcal{N} = 4$ SYM made out of the scalar and (anti)-gluino fields, the YM curvatures and YM covariant derivatives. We describe their formulation in terms of the L-analytic prepotentials of section 4. We start with a series of examples of subclasses of operators admitting the simplest LHC formulation and then turn to the most general operators. In some cases the construction is rather involved. Nevertheless, we show in section 8.4 that the MHV tree-level form factors are blind to many of these subtle details.
8.1. Lowest twist operators

Instead of working with multiplets of operators, as in section 5, we can also construct their components one by one. To this end we need to extract component fields from the supercurvature $W_{AB}$. Taking derivatives $\partial_+^A$, which do not have a gauge connection in the analytic frame (see section 4), we define the LH fields

\[ \phi_{AB}(x, u) \equiv W_{AB}(x, \theta, u)|_{\theta=0} \]

\[ \psi^{+A}(x, u) \equiv 1/3! \epsilon^{ABCD} \partial_B^+ W_{CD}(x, \theta, u)|_{\theta=0} \]

\[ F^{++}(x, u) \equiv 1/4! \epsilon^{ABCD} \partial^+_A \partial^+_B W_{CD}(x, \theta, u)|_{\theta=0} \]  \hspace{1cm} (8.1)

carrying $U(1)$ charges 0, (+1), (+2), respectively. They transform covariantly under a gauge group with LH parameter $\lambda(x, u) = \Lambda(x, 0, u)$ (recall (5.2)). Equations (8.1) take the following form in terms of the L-analytic prepotential $A^{++}$ (recall (4.9))

\[ \phi_{AB}(x, u) = \partial_+^A \partial_+^B A^{--}|_{\theta=0}, \quad \psi^{+A}(x, u) = (\partial^{++})^3 A^{--}|_{\theta=0}, \quad F^{++}(x, u) = (\partial^{++})^4 A^{--}|_{\theta=0}. \]

Multiplying together several LH fields from (8.1) and taking the trace, we obtain a gauge-invariant operator\(^\text{14}\). For example,

\[ O^{++}+(x, u) = \text{tr} \left( F^{++} \psi^{+A} \right). \] \hspace{1cm} (8.2)

Unlike the LH-independent Konishi operator $K(x) = K(x, 0, u)$ (recall (5.3)), this operator depends on the LHs in a polynomial way, $O^{++}+(x, u) = u^{+\alpha} u^{+\beta} u^{+\gamma} O^{ABCD}_{\alpha\beta\gamma}(x)$. Indeed, from $\nabla^{++} W_{AB} = 0$ (recall (5.2)) and $[\nabla^{++}, \partial^{+\gamma}] = 0$ (L-analyticity of $A^{++}$) it follows that $\partial^{++} O^{++}+(x, u) = 0$. Then, lemma (4.5) implies that the LH-function is reduced to a single irrep of spin (3/2, 0). Inside the gauge-invariant operator (8.2) we can identify the LH fields (8.1) with the physical fields,

\[ O^A_{\alpha\beta\gamma}(x) = \text{tr} \left( F^{ABCD}_{(\alpha\beta\gamma)} \psi^C \right). \] \hspace{1cm} (8.3)

We can also easily include covariant YM derivatives $\nabla_{\alpha A} = \partial_{\alpha A} + g A_{\alpha A}$ in the game. In the analytic frame this amounts to acting with several covariant derivatives $\nabla^+_A = \partial^+_A + g A^+_A$ on the LH irreps (8.1) and then setting $\theta \rightarrow 0$. To obtain irreducible representations we symmetrize the dotted Lorentz indices. Then we form the product of several fields as before. The polynomial dependence on the LHs of the gauge-invariant operators follows again from lemma (4.5) and the equation of motion of $\mathcal{N} = 4$ SYM, $[\nabla^{++}, \nabla^{++}] = 0$ [18]. Eliminating the LHs we find the corresponding gauge-invariant operator in terms of the physical fields. For example,

\[ \text{tr} \left( \nabla_{(\alpha}^{+ A} \nabla^{\beta \gamma C} \phi_{AB} \nabla^{+ C} \psi^{+ B} \right)(x, u) \leftrightarrow u_{\alpha A} u_{\beta A} u_{\gamma A} \text{tr} \left( \nabla_{(\alpha}^{+ A} \nabla^{\beta \gamma C} \phi_{AB} \nabla^{+ C} \psi^{+ B} \right)(x). \] \hspace{1cm} (8.4)

Let us emphasize that we have to use covariant derivatives with the gauge connection $A^+_A$ to produce dotted Lorentz indices.

Thus, working in the analytic frame and using the chiral supercurvature $W_{AB}$ and the covariant derivatives $\partial_+^A$, $\nabla^+_A$ we are able to construct gauge-invariant operators made of the scalars $\phi_{AB}$, chiral gluino $\psi^+_A$, self-dual YM curvature $F_{\alpha\beta}$ and covariant YM derivatives $\nabla_{\alpha A}$. All undotted Lorentz indices of the gauge-invariant operator are symmetrized, which corresponds to the lowest twist.

\(^{14}\) As mentioned earlier, in this paper we consider only single trace operators.
In order to include the anti-chiral gluino $\bar{\psi}^\dagger_A(x,u)$ and the anti-self-dual YM curvature $\tilde{F}_{\alpha\beta}$ in the construction of composite operators, we have to apply the $\bar{Q}$-transformations (5.6) to the chiral supercurvature. We will need single and double $\bar{Q}$-variations since these component fields are accompanied by one and two $\theta$, respectively (the second and the third line in (3.4)).

In terms of the nonchiral supercurvature (5.9)$^{15}$

$$\bar{\psi}^\dagger_{\alpha\Lambda}(x,u) \equiv i\bar{Q}^\alpha_{\alpha\Lambda}W_{AB}(x,\theta,u)|_{\theta=\bar{\theta}=0} = \hat{\partial}^\alpha_{\alpha\Lambda}W_{AB}|_{\theta=\bar{\theta}=0},$$

$$\tilde{F}_{\alpha\beta}(x,u) \equiv i\bar{Q}^\alpha_\beta \bar{W}_{AB}(x,\theta,u)|_{\theta=\bar{\theta}=0} = \hat{\partial}^\alpha_\beta \bar{W}_{AB}|_{\theta=\bar{\theta}=0},$$

(8.5)

where $\hat{\partial}^\alpha_\beta \equiv \partial / \partial \theta^\alpha_\beta$. The LH independence of the gauge-invariant operators involving the LH fields (8.5) follows from lemma (4.5) and the property $[\nabla_+ +, \bar{Q}^\alpha_A]^+ \sim \theta$ of the $\bar{Q}$-transformation of $A_+^+$ (5.6). The $\bar{Q}$-transformations commute with the gauge transformations [18], so they do not spoil the gauge covariance of $W_{AB}$ (5.2).

Let us denote collectively by $\mathcal{W}$ the LH fields from (8.1) and (8.5), carrying a number of covariant derivatives $\nabla_+^A$ and taken at $\theta = 0$,

$$\mathcal{W}(x,u) \leftrightarrow \nabla_+^A \cdots \nabla_+^\alpha, A^+, \psi^\dagger_A, F_+^+, \bar{\psi}^\dagger_{\alpha\Lambda}, \bar{\tilde{F}}_{\alpha\beta} \right) (x,u).$$

(8.6)

We can rewrite this set in terms of the derivatives $\partial^+_A$ and $\bar{\partial}^+_A$ acting on $\mathcal{A}^-$ (5.10),

$$\mathcal{W}(x,u) \leftrightarrow \nabla_+^A \cdots \nabla_+^\alpha, \{ \partial_+^A, \partial_+^B, (\partial^+_+)^4, \bar{\partial}^+_A \partial_+^B \partial_+^B, \bar{\partial}^+_A \bar{\partial}^+_B \partial_+^B \} A^--|_{\theta=\bar{\theta}=0}. $$

(8.7)

where $(\partial^+_+)^4$ and $(\partial^+_+)^4$ are defined as in (3.11). The composite gauge-invariant operators are traces of their products$^{16}$

$$\text{tr} (\mathcal{W}_1(x,u) \ldots \mathcal{W}_n(x,u)).$$

(8.8)

The same argument as above shows that they are monomials in the LH $u^+$, i.e. irreps of the Lorentz group. This means that the undotted Lorentz indices are symmetrized, which corresponds to operators of the lowest twist. The LH fields transform covariantly under the gauge group,

$$\mathcal{W}(x,u) \rightarrow e^{-\lambda(x,u)}\mathcal{W}(x,u)e^{\lambda(x,u)}.$$  

(8.9)

Let us emphasize once more that both prepotentials $A_+^+$ and $A^+_\alpha$ appear in $\mathcal{W}$, and so they are indispensable in the construction of composite operators.

### 8.2. Central gauge frame

So far we have worked in the analytic frame, in which the gauge connections transform with an L-analytic parameter $A(x, \theta^+, u)$,

$$\delta_{\lambda} A = \nabla A \quad \text{where} \ A \ \text{denotes} \ A_+^+, A^-_{\alpha}, A^\dagger_{\alpha}.$$  

(8.10)

As a consequence, all the supercurvatures that we use depend on the LHs in a non-polynomial way, even though the gauge-invariant operators are polynomials in the LHs. This property can

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$^{15}$ Here we need only the self-dual part of the $\bar{Q}$-transformations (5.6), i.e. we can throw away $\bar{Q}_2$. Since $W_{AB}$ depends only on $A_+^+$, $\bar{Q}_2$ does not appear in the first $\bar{Q}$-variation. It does not arise in the second $\bar{Q}$-variation in (8.5) either, due to the index symmetrization $(\alpha\beta)$ (see the appendix).

$^{16}$ It is important to recall that in general such operators need diagonalization to become eigenstates of the dilatation operator (see, e.g. [11, 44]). We do not address this issue here.
be made manifest by switching to the so called central (or \(\tau\)-) frame \([20, 23]\). There the \(SU(2)_L\) algebra \((4.4)\) of the LH derivatives becomes flat but \(\partial_+^\alpha\) acquires a gauge connection instead,
\[
\{\nabla^+, \nabla^-, \partial^0, \partial^+_A, \nabla^\pm_A, \nabla^\pm\} \rightarrow \{\partial^{++}, \partial^{--}, \partial^0, u^{\alpha\alpha} \nabla_{\alpha\alpha}, u^{-\alpha} \nabla_{\alpha}, u^{+\alpha} \nabla_{\alpha} \}.
\] (8.11)

The ‘bridge’ relating the analytic and \(\tau\)-frames has the form of a generalized finite gauge transformation \(h(x, \theta, u)\) \([23, 45]\). In particular, \(h^{-1} \nabla^+ h = \partial^{++}\) and \(h^{-1} \partial^+_\alpha h = u^{+\alpha} A_{\alpha\lambda}(x)\).

In the \(\tau\)-frame the LH expansions like \((4.2)\) reduce to a single polynomial term, so the LHs can be stripped off. In the \(\tau\)-frame the gauge transformations of the super-connections are \(\delta_{\tau} A = \nabla \tau\) with a chiral LH-independent parameter \(\tau = \tau(x, \theta)\). The bridge \(h\) undergoes gauge transformation with respect to both the analytic and \(\tau\)-frames,
\[
h(x, \theta, u) \rightarrow e^{-g A(x, \theta^{+\alpha}), u(x, \theta, u)} e^{\text{gauge}(x, \theta)}.
\] (8.12)

The covariant LH-independence of the supercurvature \(W_{AB}(x, \theta, u)\) \((5.2)\) in the analytic frame is translated into the LH-independence of \(W_{AB}(x, \theta)\) in the \(\tau\)-frame,
\[
W_{AB}(x, \theta) = h^{-1}(x, \theta, u)W_{AB}(x, \theta, u)h(x, \theta, u).
\] (8.13)

Thus the bridge \(h\) effectively strips off the dependence on the LHs. The elimination of the LHs via equation \((8.13)\) is an unnecessary step if one is interested in constructing gauge-invariant objects out of \(W_{AB}\). Indeed, the bridge transformation \((8.13)\) drops out from gauge-invariant operators.

In the \(\tau\)-frame the fields \((8.1)\) (the constituents of the composite operators) take the form
\[
\phi_{AB}(x) = W_{AB}(x, \theta)|_{\theta = 0}, \quad \psi^A_\alpha(x) = 1/3! e^{ABCD} \nabla_{\alpha} W_{CD}(x, \theta)|_{\theta = 0}
\]
\[
F_{\alpha\beta}^A(x) = 1/4! e^{ABCD} \nabla^\alpha A^\beta W_{CD}(x, \theta)|_{\theta = 0}
\]
where we have stripped off the LHs. So in the \(\tau\)-frame the covariant derivatives \(\nabla^\alpha_A, \nabla^\alpha_{\alpha}\) are indispensable for producing undotted Lorentz indices.

### 8.3. Higher twist operators

Above we have presented the LHC construction of operators with totally symmetrized undotted Lorentz indices. Indeed the undotted Lorentz indices arise from acting with \(\partial^+_\alpha\) and \(\nabla^\alpha_{\alpha}\) on the supercurvature \(W_{AB}\) (recall \((8.1)\)), taking \(\theta \rightarrow 0\), forming a gauge-invariant operator out of them, and then stripping off the LHs. So all undotted Lorentz indices are contracted with the same LH \(u^\alpha_\alpha\) and hence are symmetrized.

Now we want to construct operators where some of the Lorentz indices can be contracted, which corresponds to higher twist. Some higher twist operators live in the supermultiplets considered in section 5. There are two ways to avoid the automatic symmetrization of the undotted Lorentz indices.

The first possibility is provided by the covariant derivatives \(\nabla^\alpha_A\) and \(\nabla^\alpha_{\alpha}\) written in the analytic frame
\[
\nabla_{\alpha A} = u^\alpha_\alpha \partial^+_A - u^\alpha_\alpha \nabla^+_A = \partial_{\alpha A} - u^\alpha_\alpha gA^+_\alpha
\]
\[
\nabla_{\alpha\alpha} = u^\alpha_\alpha \nabla^+_\alpha - u^\alpha_\alpha \nabla^-_\alpha = \partial_{\alpha\alpha} + u^\alpha_\alpha gA^+_\alpha - u^\alpha_\alpha gA^-_{\alpha}.
\] (8.14)

We can use them instead of \(\partial^+_\alpha\) and \(\nabla^\alpha_{\alpha}\) to produce Lorentz indices in \((8.1)\). The gauge connections from \((8.14)\) are expressed in terms of the prepotentials \(A^{++}, A^+_\alpha\) as follows: \(A^-_\alpha = -\partial^+_\alpha A^-\), \(A^-_\alpha = \partial^- A^+_\alpha - \partial^+_\alpha A^- + g[A^-\alpha, A^+_\alpha]\) with \(A^-\alpha\) given in \((4.9)\).
The second possibility is to consider each field $W$ (8.6) constituting the composite operator in its own analytic frame depending on its own LH $u$. Then the use of $\partial^+_\alpha$ and $\nabla^+_\alpha$ implies the symmetrization of the undotted Lorentz indices of each constituent field, but not between different fields. If we work with several analytic frames, we need bridge transformations relating them. Combining a pair of $h$ bridges, we obtain the transformation from the analytic frame with LHs $v$ to the analytic frame with LHs $u$,

$$U(x, \theta; u, v) = h(x, \theta, u)h(x, \theta, v)^{-1}.$$  

(8.15)

It is inert under the $\tau$-frame gauge transformations but transforms with respect to both analytic frames (see (8.12)),

$$U(x, \theta; u, v) \rightarrow e^{-g\Lambda(x, \theta^+, u)} U(x, \theta; u, v) e^{g\Lambda(x, \theta^+, v)}.$$  

(8.16)

In the twistor literature [20, 46] this object is called a ‘holomorphic frame’, and it is interpreted as a Wilson line in [47].

$\nabla^+ h = \partial^+ h$, with the boundary condition $U(u, u) = 1$ (see (8.15)). The explicit solution for $U$ in terms of the dynamical field $A^{++}$,

$$U(u, v) = 1 + \sum_{n=1}^{\infty} (-g)^n \int du_1 \ldots du_m (u^+ v^+) A^{++}(1) \ldots A^{++}(n) \frac{(u^+ u_1^+ \cdots u_m^+)}{(u_1^+ u_2^+ \cdots u_m^+)} (u^+ v^+),$$  

(8.17)

looks very similar to $A^{--}$ (4.9). In fact $A^{--}$ appears in the Taylor expansion $U(u, v) = 1 + (u^+ v^+) g A^{--} + \ldots$ as $v \rightarrow u$ [48].

Now we can form the gauge-invariant operators out of the LH fields $W_i$ (see (8.6)), each living in its own analytic frame specified by the LHs $u_i$, and we connect them by $U$-bridges taken at $\theta \rightarrow 0$.

$$O(x; u_1, \ldots , u_m) = \text{tr} (W_1(u_1)U(u_1, u_2)W_2(u_2)U(u_2, u_3)\ldots W_m(u_m)U(u_m, u_1)).$$  

(8.18)

We remark that this operator is nonlocal in the LH space but remains local in space-time. The gauge invariance of (8.18) follows from (8.9) and (8.16). The polynomiality of (8.18) with respect to $u_1, \ldots , u_m$ can be seen by switching to the central frame (see section 8.2). The $U$-bridge (8.17) is constructed out of $A^{++}$, but $W$ includes both prepotentials $A^{++}$ and $A^{+}_\alpha$. In the next section we prefer this formulation for the calculation of form factors, since it facilitates the combinatorics.

In conclusion, we repeat that this nonlocal construction of operators is only justified if we want to contract some of the undotted indices of the constituent fields. In all other cases the use of the $U$-bridge is superfluous.

8.4. Form factors

Having the composite operators formulated in terms of L-analytic superfields, the calculation of the MHV tree-level form factors is straightforward. We consider the form factor of the composite operator (8.18),

$$\langle 1, 2, \ldots , n|O(x; u_1, \ldots , u_m)|0\rangle^{\text{tree}}_{\text{MHV}}.$$  

(8.19)

When the LHs at each leg coincide, $u_1 = \ldots = u_m = u$, we are dealing with the lowest twist operators (8.8).

The calculation follows the scheme from section 7. We consider a cluster of $k$ scattering states and a $W$ (or $U$) constituting the operator (8.18). Then we substitute for each prepotential
A\sup{+} inside each $\mathcal{W}$ (or $U$) the on-shell states (6.16). In the case of the $U$-bridge (8.17) the result is especially simple. Indeed, $U$ is chiral and we take it at $\theta = 0$,

$$
\langle \prod_{i=1}^{k} \Phi(p_i, \eta) \cdot U(x; u, v)|\rangle_{\text{MHV}} = g^{k} \frac{(u^{+}v^{+})e^{\text{int}}}{(u^{+})^{(1)}(12)\ldots(kv^{+})}. \tag{8.20}
$$

where $P = p_1 + \ldots + p_k$. Turning to the LH fields $\mathcal{W}$ in (8.18) we first note that the covariant derivatives $\nabla_{\alpha}^{+} = \overline{\partial}^{\alpha} + gA^{+}_{\alpha}$ in (8.6) effectively reduce to $\overline{\partial}^{\alpha}$. The prepotential $A^{+}_{\alpha}$ does not contribute at the MHV tree level. Indeed, the propagator $\langle A^{+}_{\alpha}, A^{+} \rangle$ (4.12) does not contain a pole $1/p^2$, so $\lim_{p^2 \to 0} p^2 \langle \mathcal{W}(p, \eta)A^{+}_{\alpha} \rangle = 0$. Thus the contribution of the cluster of legs contracted with $\mathcal{W}$ takes the form

$$
\langle \prod_{i=1}^{k} \Phi(p_i, \eta) \cdot \mathcal{W}(x, u)|\rangle_{\text{MHV}} = g^{k-1} P_{\alpha_1}^{+} \ldots P_{\alpha_l}^{+} \frac{e^{\text{int}+\sum_{i=1}^{l}(-\eta+|\theta|)/0}}{(u^{+})^{(1)}(12)\ldots(kv^{+})}. \tag{8.21}
$$

where $P_{\alpha}^{+} = u^{+\alpha} P_{\alpha\alpha}$. Here we are acting on the right-hand side of (8.21) with a number of derivatives $\overline{\partial}^{\alpha}$ and $\overline{\partial}^{\alpha} \eta$ and then setting $\theta = \hat{\theta} = 0$ to specify $\mathcal{W}$ according to (8.7).

The form factor (8.19) is then obtained by multiplying the contributions together (8.20) and (8.21) of the $2m$ clusters (see (8.18)), and by summing over all ways of distributing $n$ on-shell states among $2m$ clusters preserving the cyclic ordering.

We know that the composite operators (8.18) are polynomial in the LHs $u^{+}_{1}, \ldots, u^{+}_{m}$. This is guaranteed by the invariance of (8.18) with respect to the gauge transformations in the analytic frame (8.10). The contributions of the clusters (8.20) and (8.21) are not polynomial in the LHs. The polynomiality applies only to the form factor (8.19), which is a gauge-invariant quantity, after assembling the contributions of all clusters. The explicit elimination of the spurious poles in the LHs is a purely algebraic problem that can be easily solved applying $2n$ times the eikonal identity (7.5). The combinatorics is the same as in [17] and the result is in their formula (4.6), which we do not reproduce here. One of our aims is to explain the field theory origin of the recipe given in [17] for the calculation of the MHV tree-level form factors.

9. Conclusions

In this paper we pursue two goals. Firstly, we formulate all composite gauge-invariant operators in terms of Lorentz harmonic chiral superfields. Besides the familiar physical fields of the $\mathcal{N} = 4$ vector multiplet, we introduce infinite sets of auxiliary and pure gauge fields which live in two gauge super-connections (harmonic superfields). These unphysical degrees of freedom enable us to realize the chiral half of $\mathcal{N} = 4$ supersymmetry off shell and to employ a chiral supergraph technique for perturbative calculations. The operators, which are polynomial in terms of the ordinary physical fields, become nonpolynomial in terms of the harmonic superfields. They are represented by infinite sums of vertices of arbitrarily high valence.

Secondly, we use the harmonic super-propagators for the calculation of MHV tree-level form factors. In this case the interaction is transferred from the Lagrangian to the infinite number of operator vertices. The form factors are obtained by the LSZ reduction procedure. It amounts to stretching amputated super-propagators between the on-shell states and the superfields at the operator vertices. The simplest form factors are for operators made from chiral supercurvatures only. We use the $Q$–transformations to reconstruct the full nonchiral multiplets of operators. The other fields from the vector multiplet are obtained by acting with spinor derivatives. After the insertion of the on-shell states in the operator the fields with dotted
spinor indices are equivalently represented by space-time derivatives. This reproduces the effective operator vertex prescription of [17]. Since we understand the complete construction of the composite operators (unrelated to the on-shell states), we see that the effective vertices of [17] can only work at the MHV tree level. At the $N^k$MHV level our supergraphs are equally applicable. For example, at the NMHV level we have to include a vertex from the Lagrangian and link it by a super-propagator to a superfield from the operator. The chiral truncation of the operators reproduces the CSW rules for form factors [14]. Our supergraph technique works equally well for the nonchiral operators.

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**Appendix. $\bar{Q}$—variations of the propagator**

In this appendix we derive the nonchiral completion of the amputated chiral propagator $\lim_{p^2 \to 0} p^2 \langle W_{++}^{\text{free}} A^{++} \rangle$ (6.7). We calculate the $\bar{Q}$—variations of the super-curvature $W_{++}$ and of the gauge connection $A^{++}$. This results in a finite $\bar{Q}$—transformation which restores the $\bar{\theta}$—dependence at both ends of the propagator. In this way we derive (6.15).

In view of (5.8) we can throw up to four $\bar{Q}$—variations on the propagator $\langle W_{++}^{\text{free}} A^{++} \rangle$ and distribute them arbitrarily between its two ends. Further we apply $\bar{Q}$—variations to the $\bar{A}^{++}$ end of the propagator. The calculation is similar to the chiral case (6.3). Acting with $k = 1, \ldots , 4$ $\bar{Q}$—variations (5.8) in the momentum representation, and using the propagators (4.11) and (4.12) we obtain

$$
\langle W_{++}^{\text{free}} (p, \theta^+, v, w) \bar{Q}_{\alpha_1} \ldots \bar{Q}_{\alpha_4} ^h A^{++} (-p, \theta^0, u) \rangle 
= 4(-i)^k \left[ kp_{\alpha_1} \cdots p_{\alpha_{2k}} \bar{\xi}_{\alpha_k} / p^{++} + \pi p_{\alpha_1} \cdots p_{\alpha_{2k}} \delta^2(p^{++}) \right] \theta^{h_1} \cdots \theta^{h_{2k}} (w_+ (\theta - \theta_0)^+).
$$

(A.1)

Here the LH $v$ has dropped out and all LH projections are done with $u$, i.e. $\theta^{+a} = u^a \theta^a$, $p_{\alpha} = u^{-a} p_{\alpha a}$, etc. We are going to amputate the propagator (A.1), so we need to reveal a pole $1/p^2$ in (A.1). The pole is due to the square bracket term in (A.1) which is a harmonic distribution. As in the chiral case (6.6), the pole emerges upon harmonic integration over $u$ of the distribution with a test function.

Firstly we show that the residue of the pole is independent of the auxiliary gauge-fixing spinor $\xi^-$. This spinor is a harmonic on the factor $SU(2)_g$ of the Euclidean Lorentz group (see section 4). Since (A.1) carries zero $SU(2)_g$ harmonic charge, by lemma (4.5) the $\xi^-$—independence of the residue is equivalent to showing that it is annihilated by $\partial^{--}$. We do the substitution $p_{\alpha} \to -\xi^+_{\alpha} p^{--}$ for the $p^{--}$'s accompanied by $\delta^2(p^{++})$ and act on the square brackets in (A.1) with $\partial^{--}$,

$$
\partial^{--} \left[ kp_{\alpha_1} \cdots p_{\alpha_{2k}} \bar{\xi}_{\alpha_k} / p^{++} + \pi \xi^+_{\alpha_1} \cdots \xi^+_{\alpha_k} (p^{--})^2(p^{++}) \right] 
= \pi \xi^+_{\alpha_1} \cdots \xi^+_{\alpha_k} \partial^{--} \left[ (p^{--})^2(p^{++}) \right],
$$

(A.2)

where we have used the following formula [18]
\[ \frac{\partial^\sim}{p^\sim} = \pi p^\sim \delta^2(p^\sim). \]  

(A.3)

Then we need to show that the residue of the distribution (A.2) at the pole \(1/p^2\) vanishes. To this end we integrate the right-hand side of (A.2) with a test function \(\varphi^{(\pm)}\), which is a holomorphic rational function of LH \(u^\pm\) of degree (+k). Using (6.6) we get

\[ \pi \partial^\sim \int du (-p^\sim)^k \delta^2(p^\sim)\varphi^{(\pm)}(u^\pm) = \frac{1}{p^2} \partial^\sim \varphi^{(\pm)}(p^\sim). \]  

(A.4)

On shell \(p_{\alpha\dot{\alpha}} = \lambda_{\alpha} \bar{\lambda}_{\dot{\alpha}}\), so \(\varphi^{(\pm)}(p^\sim) = [\xi^{\pm} \bar{\lambda}^{\dot{\alpha}} \varphi^{(\pm)}(\lambda)]\) is a polynomial in \(\xi^\pm\) annihilated by \(\partial^\sim\). Thus we proved that the residue of (A.1) does not depend on the gauge-fixing \(\xi^\pm\).

In order to see explicitly how \(\xi^\pm\) drops out in the amputated propagator (A.1) we need to integrate this distribution with a test function \(\varphi^{(\pm)}(u^\pm)\). This gives rise to a number of \(\xi^\pm\)-dependent terms. Then applying the Fierz identity multiple times we can check the cancellation of \(\xi^{\pm}\) among these terms. We prefer to take a shortcut here, by integrating the distribution over the LH \(\xi\). For the first term in square brackets in (A.1) we integrate by parts and take into account the formula (A.3),

\[ \int d\xi \frac{\xi^\pm}{p^{\pm \sim}} = \int d\xi \frac{\partial^\sim}{p^{\sim \pm}} \frac{\delta^2(p^\sim)}{p^2} = \pi p^\sim \int d\xi \delta^2(p^\sim) = \frac{1}{p^2} \delta^2(p^\sim). \]  

(A.5)

Here at the last step the delta function was integrated as in (6.6): \(\pi \int d\xi \delta^2(p^\sim) = 1/p^2\). This formula enables us to integrate the second term in the square brackets in (A.1). So the sum of two terms is equal to the propagator \((k + 1) p_{\alpha\dot{\alpha}} \ldots p_{\alpha\dot{\alpha}}/p^2\) averaged over \(\xi\). Then we amputate this propagator, choose the test function to be polynomial \(\varphi^{(\pm)}(u^\pm) = (u^\pm w^\pm)\ldots(u^\pm w^\pm)\), integrate over the LH \(u\), and go on shell:

\[(k + 1) \int du p_{\alpha\dot{\alpha}} \ldots p_{\alpha\dot{\alpha}} \varphi^{(\pm)}(u^\pm) = (p_{\alpha\dot{\alpha}} v^\pm_1) \ldots (p_{\alpha\dot{\alpha}} v^\pm_k) = \lambda_\alpha \ldots \bar{\lambda}_{\dot{\alpha}} \varphi^{(\pm)}(\lambda). \]  

(A.6)

Thus the amputated propagator (A.1) is the following distribution

\[ \lim_{p^2 \to 0} p^2 (W_{++}^\text{free} (p, \theta^+, v, w) \hat{Q}^A_{\alpha\dot{\alpha}} \ldots \hat{Q}^A_{\alpha\dot{\alpha}} A^{++} (-p, \theta_0, u)) = 4(-i)^k \lambda_\alpha \ldots \bar{\lambda}_{\dot{\alpha}} (\lambda \theta^A_{\alpha\dot{\alpha}}) \ldots (\lambda \theta^A_{\alpha\dot{\alpha}}) \delta(\lambda, u) \delta^2(w, -\theta_0 |\lambda|). \]  

(A.7)

This is the nonchiral deviation from the chiral formula (6.7). The \(\hat{Q}–\)variation at the \(A^{++}\) end of the amputated propagator (6.7) amounts to several factors \(\lambda_\alpha (\lambda \theta^A_{\alpha\dot{\alpha}})\). We remark that it is proportional to the momentum helicity spinor \(\lambda_\alpha\). Consequently, completing the variation with the antichiral odd variable \(\theta_0\), we see only the projection \([\theta_0 \lambda]\) appearing.

Some of the \(\hat{Q}–\)variations in (A.7) can be moved to the \(W_{++}\) end. There, only the RH projection \(\hat{Q}^\pm = w^\alpha_A \hat{Q}^A\) gives a non-trivial result, so we cannot move more than two \(\hat{Q}–\). The result is given by (A.7) with the right-hand side projected with RH \(w^\alpha_A\).

So far we have considered only the self-dual part of the \(\hat{Q}–\)variations. We have disregarded the \(\hat{Q}^\pm\) term in (5.6) for the following reason. It is expressed in terms of the nonpolynomial \(A^-\) (4.9), \(\hat{Q}^\pm \bar{F}^A_{\beta\alpha} = -2(\bar{\theta}^A)^\beta(\theta^- A^\alpha)\epsilon_{\alpha\dot{\beta}}\). At the \(W_{++}\) end of the propagator, the \(\hat{Q}^\pm\)-variation can appear only in \(\epsilon^{\alpha\dot{\beta}} \hat{Q}^\alpha_{\alpha\dot{\beta}} \hat{Q}^\beta_{\beta\dot{\beta}} W^\text{free}_{++} (p, \theta_+)\) with the complete \(\hat{Q}\) (5.6). There is no such state in the on-shell multiplet, and one can show that it does not contribute to the amputated propagator. We can also try to take \(\hat{Q}^\pm\) into account on the \(A^{++}\) end, i.e. to
reconstruct multiplets of operators by means of the complete $\bar{Q}$. Owing to the L-analytic projector $(\partial^+)^4$, each $\bar{Q}$-variation increases the Grassmann degree of the correlator by 4 units. So we do not need it at the MHV tree level.

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