Errata and Addenda to
“Anomaly Cancellation Condition in Lattice Gauge Theory”

Hiroshi Igarashi,* Kiyoshi Okuyama† and Hiroshi Suzuki‡

Department of Mathematical Sciences, Ibaraki University, Mito 310-8512, Japan

ABSTRACT

We correct some intermediate expressions and arguments in Nucl. Phys. B 585 (2000) 471–513. The main results do not change. We also mention some additional observations, including a constraint on a coefficient of the possible nontrivial anomaly which was not given in the paper.

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* igarashi@serra.sci.ibaraki.ac.jp
† okuyama@serra.sci.ibaraki.ac.jp
‡ hsuzuki@mito.ipc.ibaraki.ac.jp
The main result of Ref. [1] is the theorems stated in Section 3 which determine the general structure of gauge anomalies in lattice gauge theory. These theorems are based on local solutions to the consistency condition in abelian theory with the ghost number unity \( g = 1 \), Eq. (6.24). We found that, although the formula (6.24) for \( g = 1 \) and thus the theorems in Section 3 remain correct, some intermediate expressions and arguments for general \( g \) were wrong. Here we show how these must be corrected.

The field \( \tilde{\omega}_\mu^{a_0[a_1\cdots a_g]} \) in Eq. (6.8) is totally antisymmetric \( \tilde{\omega}_\mu^{a_0[a_1\cdots a_g]} = \tilde{\omega}_\mu^{[a_0\cdots a_g]} \) in nontrivial solutions as shown in Eq. (6.12). The argument to show this through Eqs. (6.10) and (6.11) is however wrong for general \( g \) and is corrected as follows. We consider the first three lines of Eq. (6.8). By making use of \( c_{a_0}^a(n + \hat{\mu}) = c_{a_0}^a(n) + \delta_B A_\mu(n) \), one sees that

\[
\begin{align*}
A_\mu^a(n) & c_{a_1}^a(n) \cdots c_{a_g}^a(n) \\
- c_{a_0}^a(n + \hat{\mu}) \sum_{i=1}^g \left( \begin{array}{c} g \\ i \end{array} \right) A_{\mu_i}^{a_i}(n) & \delta_B A_{\mu_i}^{a_i}(n) \cdots \delta_B A_{\mu_i}^{a_i}(n) c_{a_{i+1}}^a(n) \cdots c_{a_g}^a(n) \\
\end{align*}
\]

\[
- (\text{the totally antisymmetric part on } a_0, a_1, \cdots, a_g) \tilde{\omega}_\mu^{a_0[a_1\cdots a_g]}(n)
\]

\[
= \delta_B \left[ - \sum_{i=1}^g \frac{g!}{(i + 1)! (g - i)!} A_{\mu_0}^{a_0} A_{\mu_1}^{a_1} \delta_B A_{\mu_2}^{a_2} \cdots \delta_B A_{\mu_i}^{a_i} c_{a_{i+1}}^a \cdots c_{a_g}^a \tilde{\omega}_\mu^{a_0[a_1\cdots a_g]} \right],
\]

where the totally antisymmetric part of a quantity \( t^{a_0\cdots a_g} \) is defined by \( \sum_\sigma \varepsilon_\sigma^{t(a_0)\cdots (a_g)} / (g + 1)! \). Eq. (1) shows that only the totally antisymmetric part of \( A_\mu^a c_{a_1}^a \cdots c_{a_g}^a \cdots \) contributes to the nontrivial part. As the result, we can assume that \( \tilde{\omega}_\mu^{a_0[a_1\cdots a_g]} \) is totally antisymmetric in nontrivial solutions, as shown in Eq. (6.12).

The coefficient in Eq. (6.13) must be chosen as

\[
\tilde{\omega}_\mu^{[a_0\cdots a_g]}(n) = \frac{1}{3!} \varepsilon_{\mu\nu\rho\sigma} \frac{(-1)^g (g+1)/2}{g + 1} \tilde{\omega}_\mu^{[a_0\cdots a_g]}(n + \hat{\mu}),
\]

for the normalization of Eq. (6.14).

Eq. (6.23) which shows the symmetry of the coefficients \( B_2 \) and \( B_0 \) in Eq. (6.19) is wrong for general \( g \) and the derivation through Eqs. (6.20), (6.21) and (6.22) is replaced as follows.
We first note that Eq. (6.19) can be written as

\[ A d^4 x (d\theta)^g \simeq \sum_n \left[ \text{sym}(C^{a_1} \cdots C^{a_g}) L^{[a_1 \cdots a_g]} d^4 x + \text{sym}(C^{a_1} \cdots C^{a_g}) B_4^{[a_1 \cdots a_g]} \right. \]

\[ + \text{sym}(C^{a_1} \cdots C^{a_g} F^b) B_2^{[a_1 \cdots a_g] b} + \text{sym}(C^{a_1} \cdots C^{a_g} F^b F^c) B_0^{[a_1 \cdots a_g] (bc)} \]

\[ + \text{sym}(A^{a_0} C^{a_1} \cdots C^{a_g}) B_3^{[a_0 \cdots a_g]} + \text{sym}(A^{a_0} C^{a_1} \cdots C^{a_g} F^b) B_1^{[a_0 \cdots a_g] b} \right]. \]

Namely, the field strength 2-forms \(F^b\) can be put in the symmetrization symbol in spite of the noncommutativity of differential forms. If we substitute each coefficients in this expression by totally antisymmetrized ones including one of indices for the field strength 2-forms, \(B_2^{[a_1 \cdots a_g] b} \rightarrow B_2^{[a_1 \cdots a_g b]}\), \(B_0^{[a_1 \cdots a_g] (bc)} \rightarrow B_0^{[a_1 \cdots a_g b] c}\) and \(B_1^{[a_0 \cdots a_g] b} \rightarrow B_1^{[a_0 \cdots a_g b]}\), we see that

\[ \sum_n \text{sym}(C^{a_1} \cdots C^{a_g} F^{b_1} F^{b_2} \cdots F^{b_r}) B^{[a_1 \cdots a_g b_1] b_2 \cdots b_r} \]

\[ = s \sum_n \frac{g}{2} \text{sym}(A^{b_1} A^{a_1} C^{a_2} \cdots C^{a_g} F^{b_2} \cdots F^{b_r}) B^{[a_1 \cdots a_g b_1] b_2 \cdots b_r}, \] \hspace{1cm} (4)

and

\[ \sum_n \text{sym}(A^{a_0} C^{a_1} \cdots C^{a_g} F^{b_1} F^{b_2} \cdots F^{b_r}) B^{[a_0 \cdots a_g b_1] b_2 \cdots b_r} \]

\[ = s \sum_n \frac{g}{3!} \text{sym}(A^{b_1} A^{a_0} A^{a_1} C^{a_2} \cdots C^{a_g} F^{b_2} \cdots F^{b_r}) B^{[a_0 \cdots a_g b_1] b_2 \cdots b_r}. \] \hspace{1cm} (5)

As the result, the following antisymmetric parts of the coefficients \(B_2^{[a_1 \cdots a_g] b}\), \(B_0^{[a_1 \cdots a_g] (bc)}\) and \(B_1^{[a_0 \cdots a_g] b}\) in Eq. (3) can be set to zero because they contribute only to BRS trivial parts:

\[ B_2^{[a_1 \cdots a_g] b} = 0, \quad B_0^{[a_1 \cdots a_g] b c} = 0, \quad B_1^{[a_0 \cdots a_g] b} = 0. \] \hspace{1cm} (6)

The first two equations replace Eq. (6.23) and the last one gives rise to the constraint on \(B_1^{[a_0 \cdots a_g] b}\) which was not given in Ref. [1]. Actually, Eq. (6) is identical to the constraints for corresponding coefficients in the continuum theory. For solutions with the ghost number unity, \(g = 1\), the first two constraints in Eq. (6) are equivalent to Eq. (6.23). Therefore, Eq. (6.24) which is for \(g = 1\) holds as it stands.

A quick way to see the equivalence of Eq. (3) and Eq. (6.19) is to introduce the superspace derivative \(\tilde{d} = d + s\) and the superspace connection \(\tilde{A}^a = A^a + C^a\). We see that \(\tilde{d}\tilde{A}^a = dA^a = \)
$F^a$ (the horizontality condition) and $\tilde{d}F^a = 0$. With this language, we can write, for example,

$$\text{sym}(A^{a_0}C^{a_1} \ldots C^{a_g}F^{b_1} \ldots F^{b_r})B^{[a_0 \ldots a_g](b_1 \ldots b_r)} = \frac{1}{g+1}\text{sym}(\tilde{A}^{a_0}\tilde{A}^{a_1} \ldots \tilde{A}^{a_g}F^{b_1} \ldots F^{b_r})B^{[a_0 \ldots a_g](b_1 \ldots b_r)}\big|_{O(d\theta^g)}.$$  

(7)

Then we consider its difference to the combination

$$\frac{1}{g+1}\text{sym}(\tilde{A}^{a_0}\tilde{A}^{a_1} \ldots \tilde{A}^{a_g}F^{b_1} \ldots F^{b_r})F^{b_1} \ldots F^{b_r}B^{[a_0 \ldots a_g](b_1 \ldots b_r)}\big|_{O(d\theta^g)}.$$  

(8)

An exchange of $\tilde{A}^a$ and $F^b$, according to the noncommutative differential calculus [2], produces the commutator

$$[\tilde{A}^a, F^b] = \tilde{d}\varphi_{ab} + 2\varphi_{ab}^3.$$  

(9)

(see Eqs. (5.47), (5.48) and (5.52) of Ref. [1]) where $Y_{ab}^2$ and $\varphi_{ab}^3$ do not contain $d\theta$ and $\varphi_{ab}^3$ depends only on the field strength (see Eqs. (5.49) and (5.54)† of Ref. [1]). Under the summation $\sum_n$, one can do the integration by parts with respect to $\tilde{d}$ up to BRS trivial terms. After this integration by parts, $Y_{ab}^2$ in the commutator does not contribute to $O(d\theta^g)$-term in Eqs. (7) and (8), because $Y_{ab}^2$ and $\tilde{d}\tilde{A}^a = F^a$ do not contain $d\theta$ (recall that $\tilde{d}F^b = 0$).

Namely, under the summation $\sum_n$, we can neglect $Y_{ab}^2$ in the commutator, up to BRS trivial terms. $\varphi_{ab}^3$ in the commutator on the other hand cannot be neglected. However, it is easy to see that its contribution can be absorbed into the first term of the right hand side of Eq. (3) up to BRS trivial terms, because $\varphi_{ab}^3$ depends only on the field strength. In this way, we see the equivalence of Eq. (3) and Eq. (6.19). This superspace trick can also be applied to derive Eqs. (4) and (5).

The cumbersome proof of the covariant Poincaré lemma for $G = U(1)^N$ in Ref. [1] was limited for 4- or lower-dimensional lattice. It is however possible to give a simpler proof which works for arbitrary dimensional lattices. This proof, a detailed study of nontrivial local solutions to the consistency condition with an arbitrary ghost number and its applications will be given elsewhere [3].

* From this, one immediately sees that $[F^a, F^b] = 2\tilde{d}\varphi_{ab}^3 = 2d\varphi_{ab}^3$.

† Eq. (5.54) of Ref. [1] must be replaced by

$$\varphi_3(n) = \frac{1}{12}\left[ F_{\alpha\beta\gamma}(n)\tilde{F}_{\beta\gamma}\left(n\right) + 2F_{\alpha\beta\gamma}(n + \hat{\gamma})\tilde{F}_{\beta\gamma}^b(n) 
+ 2F_{\alpha\beta\gamma}(n)\tilde{F}_{\alpha\beta\gamma}(n + \hat{\alpha}) + F_{\alpha\beta\gamma}(n + \hat{\gamma})\tilde{F}_{\alpha\beta\gamma}(n + \hat{\alpha})\right]dx_\alpha\,dx_\beta\,dx_\gamma B_{[ab]}^{\hat{a}}.$$  

(10)
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