High-Q Nanomechanics via Destructive Interference of Elastic Waves

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Mechanical dissipation poses an ubiquitous challenge to the performance of nanomechanical devices. Here we analyze the support-induced dissipation of high-stress nanomechanical resonators. We develop a model for this loss mechanism and test it on Si$_3$N$_4$ membranes with circular and square geometries. The measured $Q$-values of different harmonics present a non-monotonic behavior which is successfully explained. For azimuthal harmonics of the circular geometry we predict that destructive interference of the radiated waves leads to an exponential suppression of the clamping loss in the harmonic index. Our model can also be applied to graphene drums under high tension.

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Nanomechanical resonators offer great potential for practical device applications that exploit their ultra-low mass and high frequencies. Examples range from scanning probe force microscopy and mass-sensing down to the molecular scale to providing alternatives for radio frequency devices. These applications share the desirability of high mechanical quality-factor ($Q$) that, by virtue of narrow bandwidth, amounts to a better defined frequency thus enhancing performance. In turn, measurements of mechanical displacements with an imprecision below the standard quantum limit and the preparation of ultracold motional states have already been implemented with electromechanical and optomechanical systems. These breakthroughs foreshadow the possibility of realizing a “quantum optics” analogue involving a macroscopic mechanical degree of freedom which would set a new stage for fundamental tests and potential quantum devices. Once more mechanical dissipation, as determined by the $Q$-value, plays a critical role in such endeavors.

Though the mechanical $Q$-value may in general be influenced by various mechanisms, in a small suspended structure that is sufficiently clean and cold, internal losses induced by two-level fluctuators and radiation of elastic waves into the substrate are likely to play leading roles. In this respect, the reduction of the design-limited “clamping loss” induced by the coupling to the substrate will allow the use of nanomechanical devices to probe the internal losses and quantify the fundamental contributions of the constituent materials. Furthermore, with the advent of the use of stressed silicon nitride membranes, the high $Q$-values of these devices have demonstrated that it is indeed possible to attain low mechanical losses in nanoresonators.

In this letter, we present and test a model that captures the energy loss that occurs due to elastic wave radiation at the periphery of these high-stress resonators. We show that this mechanism is significant in state of the art structures and is strongly influenced by interference effects. We compare the results of our model to measurements of the resonant modes of two configurations, a single “drum resonator” and a composite array of drum resonators that effectively realizes a square membrane [cf. Fig. 1 (a) and (b)]. We examine the harmonics of these structures and accurately account for much of the variation in the corresponding $Q$-values. Our analysis reveals that certain types of modes are inherently resilient to clamping loss as a result of destructive interference of the radiated waves. Thus, we provide insight into resonators that might be realized and yield better $Q$-values in the future. On general grounds, the fact that the relevant stress at the resonator-support contact scales at least linearly with frequency combined with the 3D nature of the support, lead to the naive expectation that the dissipation $(1/Q)$ due to elastic-wave radiation should increase as one considers higher harmonics [cf. Eq. (1)]. In dramatic contrast, we find that for the harmonics of a circular membrane the clamping loss is exponentially suppressed as the number of radial nodal lines increases.

To derive an adequate model for the clamping losses, we adopt the phonon tunneling approach introduced in Ref. and start from the general weak coupling expression for the dissipation $1/Q$ in terms of the “overlaps” between the resonator mode and the free modes of the substrate (“support”).

\[
\frac{1}{Q} = \frac{\pi}{2\rho_s\rho_R\omega^2_R} \int \int_S \int_q \int_q \int_q d\bar{S} \cdot \left( \mathbf{\sigma}^{(0)}_q \cdot \bar{u}'_R - \mathbf{\sigma}'_R \cdot \bar{u}^{(0)}_q \right)^2 \times \delta(\omega_R - \omega(q)).
\]

(1)

Here $\mathbf{\sigma}'_R$ and $\bar{u}'_R$ are the stress and displacement fields associated with the normalized resonator mode, $\mathbf{\sigma}^{(0)}_q$ and $\bar{u}^{(0)}_q$ are the analogous fields for the continuum of support free modes labeled by $q$ [eigenfrequencies $\omega(q)$], and $\rho_s$ and $\rho_R$ are, respectively, the densities of the substrate and resonator materials. In our setting the resonator mode should satisfy clamped boundary conditions at the resonator-support contact area $S$ while the unperturbed support modes should satisfy free boundary conditions implying that only the second term in Eq. (1) contributes. The substrate is modelled as a half-space that contacts the membrane resonator at its rim $S$ — i.e. the underetched gap between the suspended structure
and the substrate is neglected when determining the support free mode23. We assume the “high stress” regime $t^2/D^2 \ll \sigma/E_R^2 \ll 1$, where $\sigma$ is the tensile stress in the membrane, $t$ its thickness, $D$ its large dimension (diameter or side) and $E_R$ the Young modulus of the resonator material. This implies that bending effects are negligible and one can use the classical wave equation adequate for a taut membrane.23 Thus for the drum’s eigenfrequencies we obtain [cf. Fig. 1 (a), (b)]: $\omega_{nm} = 2\zeta_{nm} c_R / D$ with $n = 0, 1, \ldots$ and $m = 1, 2, \ldots$; while the square’s eigenfrequencies are given by: $\omega_{nm} = \pi \sqrt{n^2 + m^2} c_R / D$ with $n, m = 1, 2, \ldots$ Here $c_R = \sqrt{2\sigma / \rho_R}$ is the phase velocity in the membrane, and $\zeta_{nm}$ is the nth zero of the Bessel function $J_n(x)$. In this context, the weak coupling approximation underpinning Eq. (1) reads $\omega_{nm} t / c_R \ll 1$. 

For the single drum we adopt support eigenmodes $\tilde{u}_{q,\theta,\gamma}^{(0)}(\vec{r})$ (with $l = 0, \pm 1, \ldots$) that have axial symmetry with respect to $z$ [cf. Fig. 1 (a)]. These are related to the plane wave eigenmodes $\tilde{u}_{q,\gamma}^{(0)}(\vec{r})$ by: $\tilde{u}_{q,\theta,\gamma}^{(0)}(\vec{r}) = [( - i )^n / \sqrt{2\pi}] \int_{-\pi}^{\pi} d\phi e^{i \rho \phi} \tilde{u}_{q,\gamma}^{(0)}(\vec{r})$; where $\gamma = l, t, s$ labels the different types of relevant modes [i.e. longitudinal ($l$), transverse SV ($t$), and SAW ($s$)] given that SH waves do not contribute] with velocities of propagation $c_\gamma$, and we use spherical coordinates for the incident wavevector $\vec{q} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ [cf. Eq. (A9) of Appendix A]. The expression of the matrix $\Sigma$ is given by the solution of the wave equation for the membrane and that the resonant wavevectors in the substrate satisfy $\omega_{nm} c_s / t \ll 1$, so that we can neglect the variation of $\tilde{u}_{q,\theta,\gamma}^{(0)}(\vec{r})$ across the thickness $t$. Thus, substitution of the support and resonator modes into Eq. (1) (cf. Appendix A) leads to 

$$ 1 \over Q_{nm} = \frac{4\pi^2 \zeta_{nm} \rho_R t}{\rho_s D} \sum_\gamma \sum_{\eta_\gamma} \eta_\gamma \tilde{u}_{n,\gamma}(\eta_\gamma \zeta_{nm}, \nu_s) . $$

Here we introduce the dimensionless functions $u_{l,\gamma}(\vec{q}, \nu_s) = \frac{2\pi}{10} \frac{\pi^2 / 2}{\rho \sin \theta} (\tilde{u}_{l,\gamma}^{(0)}(0, \nu_s))^2 J_l^2(\vec{q})$ and define $\eta_\gamma = c_R / c_s \sim \sqrt{\sigma / \rho_s E_s} - E_s$ where the prefactors of order unity, which depend on $\gamma$, are functions of the Poisson ratio for the substrate $\nu_s$. We note that $|\tilde{u}_{l,\gamma}^{(0)}(0, \nu_s)|^2$ solely depends on $\cos \theta$ and $\nu_s$ [cf. Eqs. (A9) of Appendix A].

In turn, for the square membrane an analogous proce-
This comparison takes into account three issues: (i) the

\[ \frac{1}{Q_{nm}} = \frac{16\pi^2 n^2 \rho R_t}{\sqrt{n^2 + m^2} \rho_s D} \sum_{l,\gamma} \eta_0^{\frac{1}{2}} w_{l,\gamma}^{n,m} \left( \sqrt{n^2 + m^2} \eta_0, \nu_s \right) \]

with \( w_{l,\gamma}^{n,m}(q, \nu_s) = |a_q^{(0)}(0, \nu_s)|^2 \tilde{f}_{nlm}(q), \ w_{l,\gamma}^{n,m}(q, \nu_s) = \int_0^{\pi/2} d\theta \sin \theta |a_{q,\nu_s}|^2 \tilde{f}_{nlm}(q \sin \theta), \) where \( l \geq 0 \) and we introduce \( f_{nlm}(x) = f_1(\pi x) \left[ Z_{nlm}(x) + Z_{mln}(x) \right] \)

\[ Z_{nlm}(x) \equiv \frac{z_n^{l}(x)}{n^3 (n^2 - x^2)^{3/2}} \left\{ 2n(n + 1) \sqrt{n^2 - x^2} + z_n^{l}(x) \right\} + \frac{16n^2 (n^2 - x^2) z_n^{l}(x)}{[z_n^{l}(x) + z_m^{l}(x)] [z_n^{l}(x) + z_m^{l}(x)]} \]

Here \( z_n^{l}(x) = 2n - x^2 + 2n \sqrt{n^2 - x^2} \) and the functions \( f_1(x) \) are given by: \( f_1(x) = |\delta_{01} - 2(-1)^n J_1(x) + (-1)^n J_4(\sqrt{2}x)| / (2^{1/2} x^{1/2}) \) for \( n + m \) even, and \( f_1(x) = |\delta_{01} - 2 \sin^2(\pi/2) J_2(x) - \cos(\pi/2) J_2(\sqrt{2}x)| / (2^{1/2} x^{1/2}) \) for \( n + m \) odd. Equation (3) is only valid for the case \( \min(n, m) > \eta_0 \sqrt{n^2 + m^2} \) which is satisfied for the resonances studied here — note that material properties always imply \( \eta_0 < \eta_0 < \eta_s \), and \( \sigma < E_s \) implies \( \eta_0 < 1 \).

We proceed to compare the predictions of our model [Eqs. (3)] with the dissipation measured in nanomechanical membrane resonators (cf. Fig. 1). These resonators are made of “stoichiometric” Si$_3$N$_4$ deposited by low pressure chemical vapor deposition on SiO$_2$. The nitride has an inherent stress of 1.2 GPa, as measured by a wafer bow technique and a density \( \rho_s = 2.7 \text{gcm}^{-3} \). After lithographic patterning to define access holes, the resonators are suspended by etching the underlying oxide through these holes, using buffered oxide etch (BOE) for the single drum and HF for the square membrane, and critical point dried. Thus a single access hole results in a circular drum geometry, while a square geometry is defined by a periodic square lattice of such holes (50 × 50 separated by 5 µm). Given the small size of the holes (< 1 µm) compared with the typical mode wavelength, we neglect them in our model. For the square array, the same consideration applies to the hole separation so that we use a square membrane model with uniform thickness \( t = 12.5 \text{nm} \) given by the average over the array and side \( D = \sqrt{A} = 253.2 \mu m \), where \( A \) is the suspended area. For the single drum (diameter \( D = 14.5 \mu m \)) the use of a BOE etch implies that the thickness is uniform and equal to the nitride thickness (110nm).

The mechanical resonances of the structures are characterized under vacuum and room temperature conditions, using a technique described in Ref. [11]. The resonators are actuated using a piezo disc that vibrates the chip in the out-of-plane direction and the motion is detected via a 633nm continuous wave laser. Figure 1 (c)–(f) compares the measured frequencies and \( Q \)-values of different harmonics for the two configurations, single drum and square array, with the predictions of our model. This comparison takes into account three issues: (i) the

release of the resonator leads to a local deformation of the wafer that lowers the membrane’s tensile stress with respect to the one in the nitride layer, (ii) in addition to clamping losses the resonator will be affected by internal dissipation, and (iii) the parameters for the half-space model of the substrate must be judiciously chosen.

To deal with (i) we determine the membrane phase velocity \( c_R \) from a suitable linear regression that uses as input the resonator size \( D \), the measured frequencies, and their mode indices which can be identified from the frequency ratios between the harmonics and the fundamental mode. We find an excellent correlation that yields \( c_R = 576.8 \text{ms}^{-1} (566.8 \text{ms}^{-1}) \) for the drum (square).

Our model is in excellent agreement with the observed trends providing the internal dissipation channel (ii) is frequency independent, and can be just added as a fit parameter (1/Q$_{mln}$) to the calculated dissipation. To elucidate (iii) one needs to compare the wavelengths of the resonant “support” modes with the thickness of the Si wafer (0.5mm). For the square the resonant frequencies are in the MHz range resulting in wavelengths in Si (4-8mm) much larger than the wafer’s thickness so that these modes are dominated by the properties of the underlying piezo and positioning system. Thus, we adopt \( \nu_s = 1/3 \) and leave the density \( \rho_s \) and Young modulus \( E_s \) as fit parameters. On the other hand for the drum the resonances studied lie in the 100 MHz range so that the elastic wave radiation is determined mostly by the anisotropic properties of crystalline Si. For this case we adopt \( \rho_s = 2.33 \text{gcm}^{-3} \) and \( \nu_s = 0.28 \) but leave \( E_s \) as a fit parameter given the isotropic nature of our model.

In both cases, drum and square geometry, we find a class of modes that consistently exhibit lower dissipation 1/Q$^2$ when compared to nearby modes [cf. Fig. 1 (c) and (d)]. Their measured Q remains approximately constant as the harmonic index is increased, leading to a growth in their fQ product that for the square reaches a maximum of 1.0 × 10$^{13}$Hz for the (6, 6) harmonic. These “special” classes of harmonics for the drum and square are, respectively, \( (n, 1) \rangle(n > 0) \) and \( (n, n) \rangle(n > 1) \) and correspond to the presence of nodal lines that intersect the periphery at evenly spaced points [cf. Fig. 1 (a) and (b)]. In contrast, for the square geometry the modes \( (n, 1) \), \( (1, n) \), where two of the sides do not intersect any nodal lines, tend to exhibit smaller frequency ratios between the harmonics and the fundamental mode. Thus, for the special modes the clamping loss is suppressed [cf. Fig. 1 (e) and (f)] due to destructive interference between the waves radiated by the different equivalent segments of the periphery, defined by the nodal lines, which have alternating \( \pi \)-phases. Concomitantly, unlike the fundamental mode, these special modes are associated to stress sources with vanishing total force.

A quantitative grasp of these striking features can be gained by exploiting the smallness of the \( \eta_0 \) underpin-
ning the aforementioned wavelength separation. For the drum, relevant harmonics satisfy the condition \( n_{\gamma} \ll \sqrt{n_1 + 1} \) which allows us to Taylor expand the Bessel functions in the \( \tilde{u}_{1,\gamma} \) yielding an approximation for Eq. 2 that implies the following:\[\]

\[ Q_{01} \approx \frac{\rho c_s^2}{2\pi^2 \sigma_R c_R^2 \omega_0 \tilde{u}_0(\nu_0)} = 0.029 \sqrt{\frac{\rho R}{\rho_s}} \left( \frac{E_s}{\sigma} \right)^3 \frac{D}{t} \]

\[ Q_{0n1} \approx n^{2n+1} \left( \frac{0.517 c_s}{c_R} \right)^{2n} \frac{Q_{n1}}{Q_{01}} \approx \left( \frac{\zeta_{n1}}{\sigma_{n1}} \right) \frac{Q_{n1}}{Q_{01}} \quad (4) \]

where \( \sigma_R = \rho_R t \) is the surface mass density of the membrane and \( \tilde{u}_0(\nu_0) = \sum (c_i/c_s)^3 \tilde{u}_{0,\gamma}(0, \nu_0) \). Thus, the clamping-loss limited \( Q \)-values of modes \( (n, 1) \) effectively grow exponentially — as the super-exponential factor plays a negligible role for relevant \( n \), in sharp contrast to series of modes for which \( m \) is increased while \( n \) is kept constant. These exhibit a decrease of \( Q_{\text{clamp}} \) for increasing frequency.\[\]

On the other hand, for the square geometry analogous considerations imply for \( m \sim n \gg g_0/2\pi \eta \), a rise in \( Q_{\text{clamp}} \) that is merely linear, with the damping rate tending to a constant value, as the harmonic indices are increased with their ratio \( m/n \) fixed. In turn, for our setting given the magnitude of \( 1/Q_{\text{int}} \) all the high-\( Q \) modes present roughly constant \( Q \)-values.

A comparison between the predictions [cf. Eqs. (2) and (3)] for the two geometries (with appropriate dimensions) also reveals that for “special” harmonics \([(n, 1)_{m>0} \text{ and } (n, n)_{n>1} \text{ with the same frequency and number of nodal lines the circular geometry always yields a higher } Q \). Finally, one should note that the scalings, embodied in Eq. (4), for the \( Q \)-values in terms of \( \rho_R/\rho_s \), \( E_s/\sigma \), and \( D/t \) are completely general and independent of the shape of the boundary. These, directly imply that the \( f Q_{\text{clamp}} \) product of a given harmonic is independent of \( D \). Furthermore, typical parameters yield for the fundamental mode \( f Q_{\text{clamp}} \sim 10^{12} \text{Hz} \), which is comparable to experimental values (cf. Fig. 1 (c) and (d), and Refs. 20, 21).

In conclusion, we find that the dissipation of different harmonics of a given membrane resonator exhibit a striking non-monotonic behavior which can be understood in terms of how the mode-shapes of different harmonics influence the clamping loss. We find classes of modes for which the measured \( Q \) remains approximately constant and substantially larger than for other modes with comparable frequency, and explain this phenomenon in terms of destructive interference between the radiated waves leading to a strong suppression of the clamping loss. Notably, our analysis implies that for modes \( (n, 1) \) of a circular geometry, the damping rate due to elastic-wave radiation vanishes exponentially in \( n \) rendering them “asymptotically mute”. Thus, for typical parameters, these azimuthal harmonics can be regarded as effectively clamping-loss free for moderate \( n \) (e.g. \( f Q_{\text{clamp}} \gtrsim 10^{11} \text{Hz} \) for \( n \gtrsim 5 \) and thickness \( t < 100 \text{nm} \)). Our results are relevant to state-of-the-art dispersive optomechanical setups20,21 and the model is also applicable to graphene nanodrums under tension21. Finally, we highlight that the interference effects we have unveiled will also be operational for the flexural modes of rigid plates.

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Appendix A: Circular geometry

In the case of the circular geometry, the axially symmetric support eigenmodes used are related to the plane wave eigenmodes \( \tilde{u}_{q,\gamma}^{(0)}(\vec{r}) \) by:

\[ \tilde{u}_{q,\theta,\phi,\gamma}^{(0)}(\vec{r}) = \frac{(-i)^n}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\varphi e^{in\varphi} \tilde{u}_{q,\theta,\phi,\gamma}^{(0)}(\vec{r}) \quad (A1) \]

Here \( \gamma = l, t, s \) labels the different types of relevant modes [i.e. longitudinal (l), transverse SV (t), and SAW (s)] given that SH waves do not contribute and \( \tilde{u}(q, \theta, \phi, \varphi) = q(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \) is the incident wavevector \([\theta = \pi/2 \text{ for } \gamma = s \text{ and } \theta \leq \pi/2 \text{ otherwise}] \). As the resonator material is prestressed, the stress \( \sigma_R^l \) in Eq. (1) is not the total stress but corresponds instead to the variation with respect to equilibrium. This, together with the validity of membrane theory23 implies that to linear order in the displacement field we have

\[ \tilde{d}S \cdot \sigma_R' \parallel \tilde{z} \quad (A2) \]

and

\[ \int_{-l/2}^{l/2} \tilde{z} \cdot \sigma_R' \cdot \tilde{r} dz = \sigma \sqrt{\frac{l}{\gamma R}} \]

\[ \gamma = \frac{\partial \phi_R}{\partial R} \quad , (A3) \]

which we will use to simplify Eq. (1). Here we introduce the normalized resonator eigenmodes \( \phi_R \rightarrow \phi_{nm} \), that satisfy the 2D classical wave equation with \( \phi_{nm} = 0 \) at the periphery of the membrane, and adopt cylindrical coordinates \( \vec{r} = (r \cos \phi, r \sin \phi, z) \). These eigenmodes read

\[ \phi_{nm}(r, \phi) = \sqrt{\frac{1}{\pi}} R_{nm}(r) \]

\[ \phi_{nm}(r, \phi) = \sqrt{\frac{2}{\pi}} R_{nm}(r) \cos n\phi \]

\[ \tilde{\phi}_{nm}(r, \phi) = \sqrt{\frac{2}{\pi}} R_{nm}(r) \sin n\phi \quad (A4) \]

with

\[ R_{nm}(r) = \frac{2 J_n(2\zeta_{nm}/D)}{D J_{n+1}(\xi_{nm})} \quad , (A5) \]

and have eigenfrequencies \( \omega_R \rightarrow \omega_{nm} \) given by

\[ \omega_{nm} = 2\zeta_{nm} \frac{cR}{D} \quad . (A6) \]
Here \( n = 0, 1, \ldots, m = 1, 2, \ldots, \) and the harmonics with \( n > 0 \) are doubly degenerate.

As the resonant wavevectors in the substrate satisfy
\[
\omega_{nm}/c_{s}t \ll 1
\]
we can neglect the variation of \( \bar{u}_{q,0,t,z}(\vec{r}) \) across the thickness \( t \) (i.e. the \( z \)-dependence at \( S \)). This approximation and Eq. (A2) imply that the support modes only enter into Eq. (1) through \( u_{q,0,t,z}(\vec{r}) \vert_{z=0} \). To determine the latter we exploit that reflection at the free surface preserves the tangential component of the wavevector implying
\[
u_{z} \frac{\partial}{\partial z} \vec{u}_{q,0,t,z}(\vec{r}) \vert_{z=0} = \nu_{z} \frac{\partial}{\partial t} \bar{u}_{q,0,t,z}(\vec{r}) \vert_{z=0} = \nu_{z} \frac{\partial}{\partial r} \bar{u}_{q,0,t,z}(\vec{r}) \vert_{z=0}
\]
Then, from Eqs. (A1) and (A7), using the Bessel integral
\[
J_{n}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \exp(\pi i n \phi - x \sin \phi) d\phi,
\]
we obtain
\[
u_{z} \frac{\partial}{\partial t} \bar{u}_{q,0,t,z}(\vec{r}) \vert_{z=0} = \sqrt{2\pi} \bar{u}_{q,0,t,z}(\vec{r}) \vert_{z=0}
\]
We now deploy Eq. (A2) and substitute Eqs. (A3)-(A5) and (A8) into Eq. (1). Subsequently, we use that here
\[
\int_{q} \sum_{l,\gamma} \int_{0}^{\infty} dq q^{\delta_{\gamma s} - 1} \left[ (1 - \delta_{\gamma s}) \int_{0}^{\pi/2} d\theta \sin \theta + \delta_{\gamma s} \theta \right] = \pi \left[ (1 - \delta_{1s}) \int_{0}^{\pi/2} d\theta \sin \theta + \delta_{1s} \theta \right]
\]
where \( d_{\gamma} \) is the dimensionality (i.e. \( d_{s} = 2 \) for \( \gamma = s \) and \( d_{s} = 3 \) for \( \gamma \neq s \), perform the substitution \( \omega = c_{s} \rho_{s} \) (for each \( \gamma \)), evaluate \( \frac{d\bar{u}_{q,0,t,z}(\vec{r})}{dt} \) using \( \bar{J}_{n}(\rho_{s}q_{nm}) = -J_{n+1}(\rho_{s}q_{nm}) \), and express \( \sigma \) in terms of \( c_{d} \) and \( \rho_{R} \). Finally, integration over \( \omega \) and \( \phi \), summation over \( l \) (all terms vanish except \( l = \pm n \)), and substitution of Eq. (A6) yields Eq. (2).

The \( z \)-displacements at the origin \( u_{q,0,t,z}(\vec{r}) \) only depend on \( \cos \theta \) and \( \nu_{s} \), and their absolute values squared are given by the following:
\[
\begin{align*}
|u_{q,0,t,z}(\vec{r})|^{2}_{\cos \theta = \nu_{s}} &= \frac{(1 - 2\alpha + 2\alpha v^{2})^{2} v^{2}}{2\pi [4\alpha^{3}/2(1 - \alpha + \alpha v^{2})v^{2} + (1 - 2\alpha + 2\alpha v^{2})^{2}]} \bigg{\vert}_{\alpha = \frac{1 - 2\nu_{s}}{2(1 - \nu_{s})}}, \quad (A9) \\
|u_{q,0,t,z}(\vec{r})|^{2}_{\cos \theta = \nu_{s}} &= \frac{2(1 - \alpha - \alpha v^{2})^{2}(1 - \alpha v^{2})^{2} v^{2}}{\pi^{3} [16(1 - \alpha - \alpha v^{2})(1 - \alpha v^{2})^{2} v^{2} + (2v^{2} - 1)^{4}]} \bigg{\vert}_{\alpha = \frac{1 - 2\nu_{s}}{2(1 - \nu_{s})}} \quad \text{for } 0 < v < \sqrt{1 - \alpha}, \quad (A10) \\
|u_{q,0,t,z}(\vec{r})|^{2}_{\cos \theta = \nu_{s}} &= \frac{2(\alpha - 1 + \alpha v^{2})^{2}(1 - \alpha v^{2})^{2} v^{2}}{\pi^{3} [16\alpha + 1 - \alpha v^{2}^{2} v^{2} + (2v^{2} - 1)^{2}]} \bigg{\vert}_{\alpha = \frac{1 - 2\nu_{s}}{2(1 - \nu_{s})}} \quad \text{for } \sqrt{1 - \alpha} < v < 1, \quad (A11) \\
|u_{q,0,t,z}(\vec{r})|^{2}_{\cos \theta = \nu_{s}} &= \frac{C^{2}(\alpha)}{2\pi^{2}} \left[ \sqrt{1 - \alpha^{2}(\alpha)} - \frac{1 - \alpha^{2}(\alpha)}{2} \right]^{2} \bigg{\vert}_{\alpha = \frac{1 - 2\nu_{s}}{2(1 - \nu_{s})}} \quad \text{for } \sqrt{1 - \alpha} < v < 1, \quad (A12)
\end{align*}
\]

Appendix B: Square geometry

For the square geometry we exploit that the whole structure still presents reflection symmetries with respect to the \( x-z \) and \( y-z \) planes. These are associated, respectively, with the operators \( \hat{R}_{x} \) and \( \hat{R}_{y} \) acting on the space of solutions of the elastic wave equations. Thus we can use normal modes of the decoupled support (elastic half-space) and of the resonator (i.e. square membrane) that are eigenvectors of \( \hat{R}_{x} \) and \( \hat{R}_{y} \). In the case of the support, for a given plane wave mode \( |u\rangle \equiv u_{q,0,t,z}(\vec{r}) \) of the elastic half-space one can generate modes \( |u_{+}\rangle \), \( |u_{-}\rangle \), \( |u_{+\gamma}\rangle \), and \( |u_{-\gamma}\rangle \) with the desired reflection properties by the following symmetrization procedure
\[
|u_{\mu\nu}\rangle = \hat{R}_{\mu\nu} |u\rangle \equiv \frac{1}{2} \left( |u\rangle + \mu \hat{R}_{x} |u\rangle + \nu \hat{R}_{y} |u\rangle + \mu \nu \hat{R}_{x} \hat{R}_{y} |u\rangle \right)
\]
with \( \mu, \nu = \pm \), which enforces
\[
\hat{R}_{x} |u_{\mu\nu}\rangle = \mu |u_{\mu\nu}\rangle, \\
\hat{R}_{y} |u_{\mu\nu}\rangle = \nu |u_{\mu\nu}\rangle.
\]
Naturally a complete basis is obtained by taking \( \bar{q} \) in the first quadrant, i.e. \( 0 \leq \varphi \leq \pi/2 \). As for the drum: (i) the smallness of \( t \) compared with the resonant wavelengths in the substrate implies that we can neglect in Eq. (1) the \( z \)-dependence of the symmetrized support modes at \( S \), and (ii) only their \( z \)-components are relevant. These
In turn, for the resonator modes we obtain

\[ u^{(0)}_{q_1} \left[ x, y, 0 \right] = 2u^{(0)}_{q_1} \left( 0 \right) \cos \theta \sin \varphi \cos qy \sin \theta \sin \varphi, \]  
(B3)

\[ u^{(0)}_{q_2} \left[ x, y, 0 \right] = 2u^{(0)}_{q_2} \left( 0 \right) \sin \theta \cos \varphi \sin qy \sin \theta \sin \varphi, \]  
(B4)

\[ u^{(0)}_{q_3} \left[ x, y, 0 \right] = 2u^{(0)}_{q_3} \left( 0 \right) \cos \theta \sin \varphi \sin qx \sin \theta \sin \varphi, \]  
(B5)

\[ u^{(0)}_{q_4} \left[ x, y, 0 \right] = 2u^{(0)}_{q_4} \left( 0 \right) \sin \theta \cos \varphi \sin qx \sin \theta \sin \varphi. \]  
(B6)

In turn, for the resonator modes we obtain

\[ \phi_{n,m} \left( x, y \right) = \frac{2}{D} \cos \frac{n\pi}{D} x \cos \frac{m\pi}{D} y \]  
for \( n, m \) odd \rightarrow \text{symmetric case},

(B7)

\[ \phi_{n,m} \left( x, y \right) = \frac{2}{D} \sin \frac{n\pi}{D} x \sin \frac{m\pi}{D} y \]  
for \( n, m \) even \rightarrow \text{antisymmetric case},

(B8)

\[ \phi_{n,m} \left( x, y \right) = \frac{2}{D} \cos \frac{n\pi}{D} x \sin \frac{m\pi}{D} y \]  
for \( n \) odd and \( m \) even \rightarrow \text{mixed symmetry case},

(B9)

\[ \phi_{n,m} \left( x, y \right) = \frac{2}{D} \sin \frac{n\pi}{D} x \cos \frac{m\pi}{D} y \]  
for \( n \) even and \( m \) odd \rightarrow \text{mixed symmetry case},

(B10)

with eigenfrequencies

\[ \omega_{nm} = \pi \sqrt{n^2 + m^2} \frac{c_R}{D} \quad n, m = 1, 2, \ldots \]  
(B11)

Naturally only modes with the same reflection symmetries are coupled via Eq. (1). We first treat the symmetric modes and then briefly outline the straightforward extension to the other cases.

1. Symmetric modes

We make use of

\[ \int_{-D/2}^{D/2} dx \cos \frac{n\pi}{D} x \cos q_x x = \frac{\sin \frac{n\pi + q_x D}{2}}{\frac{n\pi}{D} + q_x} + \frac{\sin \frac{n\pi - q_x D}{2}}{\frac{n\pi}{D} - q_x}. \]  
(B12)

Eq. \( \text{(A3)} \), and the symmetries of the square, to obtain from Eqs. \( \text{(B3)} \) and \( \text{(B7)} \) the following

\[ \int_0^{\pi/2} d\varphi \left| \int_S \frac{\hat{S} \cdot \sigma_R \cdot \hat{u}^{(0)}_{q_1}}{\sqrt{q}} \right|^2 = 256n^4 m^2 \frac{\sigma^2 t}{D^2} \left| u^{(0)}_{q_1} \left( 0 \right) \right|^2 \int_0^{\pi/2} d\varphi \cos^2 \left( \frac{qD}{2} \sin \theta \cos \varphi \right) \cos^2 \left( \frac{qD}{2} \sin \theta \sin \varphi \right) \times \left[ \left( n\pi \right)^2 - \left( qD \sin \theta \cos \varphi \right)^2 \right] \left[ \left( n\pi \right)^2 - \left( qD \sin \theta \sin \varphi \right)^2 \right]. \]  
(B13)

Subsequently we substitute into Eq. \( \text{(B13)} \) the decomposition

\[ 4 \cos^2 \left( \frac{qD}{2} \sin \theta \cos \varphi \right) \cos^2 \left( \frac{qD}{2} \sin \theta \sin \varphi \right) = 1 + \cos \left( qD \sin \theta \cos \varphi \right) + \cos \left[ qD \sin \theta \cos \left( \varphi - \frac{\pi}{2} \right) \right] + \frac{1}{2} \cos \left[ \sqrt{2}qD \sin \theta \cos \left( \varphi - \frac{\pi}{4} \right) \right] + \frac{1}{2} \cos \left[ \sqrt{2}qD \sin \theta \cos \left( \varphi + \frac{\pi}{4} \right) \right] \]

\[ = 1 + 2J_0(\sqrt{2}qD \sin \theta) + 4 \sum_{l=1}^{\infty} J_{2l}(qD \sin \theta) \cos 4l \varphi \]

\[ + J_0(\sqrt{2}qD \sin \theta) \sum_{l=1}^{\infty} (-1)^l J_{2l}(\sqrt{2}qD \sin \theta) \cos 4l \varphi, \]  
(B14)

which follows from the Jacobi-Anger expansion \( e^{i x \cos \varphi} = \sum_{n=-\infty}^{\infty} i^n J_n(x) e^{i n \varphi} \) and the relation \( J_{-n}(x) = (-1)^n J_n(x) \). Then, using the invariance of the integrand under \( \varphi \rightarrow \pi - \varphi \), which implies \( \int_0^{\pi/2} \rightarrow \frac{1}{2} \int_0^{\pi} \), and performing the
substitution \( \varphi' = 2 \varphi \) we obtain

\[
\int_0^{\pi/2} d\varphi \left| \int_{S} d\bar{S} \cdot \mathbf{\sigma}_{R} \cdot \bar{u}_{\gamma^{+++}}^{(0)} \right|^2 = 64n^2m^2 \frac{\sigma^2t}{D^2} \frac{1}{u_{\gamma^{+++}}^{(0)}(0)}^2 \left( \frac{\pi}{qD \sin \theta} \right)^4 \Re \left\{ \int_0^{2\pi} d\varphi \sum_{l=0}^{\infty} a_l(qD \sin \theta) e^{i2l\varphi} \right\} \]  

(B15)

with

\[
a_0(x) = 1 + 2J_0(x) + J_0(\sqrt{2}x), \quad a_l(x) = 2 \left[ 2J_l(x) + (-1)^l J_l(\sqrt{2}x) \right] \quad \text{for } l > 0. \]  

(B16)

The angular integral can be converted into a contour integral over the unit circle in the complex plane using \( z = e^{i\varphi} \). We focus on the case \( \min\{n, m\}/\sqrt{n^2 + m^2} > \eta_k \) in which for all \( \gamma \); the series in Eq. (B15) converges uniformly, so that the integral and the sum commute, and the denominator of the first (second) term in the last factor has one real root \( \tilde{z}_n < -\tilde{z}_m \) inside the unit circle and another one \( \tilde{z}_n > -\tilde{z}_m \) outside of it given by \( \tilde{z}_n \leq 2 \left( \frac{n\pi}{qD \sin \theta} \right)^2 - 1 \mp \frac{2n\pi}{qD \sin \theta} \sqrt{\left( \frac{n\pi}{qD \sin \theta} \right)^2 - 1} \).  

We now evaluate the resulting integral for each term in \( l \) by residues to obtain

\[
\int_0^{\pi/2} d\varphi \left| \int_{S} d\bar{S} \cdot \mathbf{\sigma}_{R} \cdot \bar{u}_{\gamma^{+++}}^{(0)} \right|^2 = 32\pi n^2m^2 \frac{\sigma^2t}{D^2} \frac{1}{u_{\gamma^{+++}}^{(0)}(0)}^2 \left( \frac{2\pi}{qD \sin \theta} \right)^4 \sum_{l=0}^{\infty} a_l(qD \sin \theta) \left[ \frac{(2l+1)\tilde{z}_{n<}^{2l}}{(\tilde{z}_{n<} - \tilde{z}_{n>})^2} - \frac{2\tilde{z}_{n<}^{2l+1}}{2\tilde{z}_{n<}^{2l+1} - \tilde{z}_{m<}^{2l+1}} \right] \]  

(B18)

where we have omitted the dependence on \( qD \sin \theta \) of the last factor. Subsequently, we regroup terms in the above using Eq. (B17) and substitute Eqs. (B18) and (B16) into Eq. (1). Then, we use that here \( \int_{-\pi}^{\pi} \) corresponds to

\[
\sum_{\gamma} \int_0^{\pi/2} d\varphi \int_0^\infty dq dq' \frac{1}{(-\delta_{\gamma})} \int_0^{\pi/2} d\theta \sin \theta + \delta_{\gamma} \]  

perform the substitution \( \omega = \sigma \cdot q \) (for each \( \gamma \)), and integrate over \( \omega \). Finally, we eliminate \( \sigma \) in favor of \( c_R \) and \( \rho_R \); introduce the functions \( f_l(x) \equiv a_l(x)/(2x^{2l}) \) and

\[
\tilde{z}_{n<} (x) \equiv \left( \frac{qD}{\pi} \sin \theta \right)^\frac{1}{2} \tilde{z}_{n<} \]  

(B19)

and substitute Eq. (B11), to obtain Eq. (3) specialized for \( n, m \) odd.

2. General case

For the other cases, antisymmetric and mixed-symmetry modes, analogous steps as for the symmetric modes (cf. (B12), allow us to obtain from Eqs. (A3), (B4), (B5), (B8), and (B9) expressions that differ from Eq. (B13) only in the first two factors of the R.H.S. integrand which are modified via

\[
\cos^2(q'_x) \cos^2(q'_y) \rightarrow \sin^2(q'_x) \sin^2(q'_y) \]  

(B20)

for \( n, m \) even (antisymmetric modes) and

\[
\cos^2(q'_x) \cos^2(q'_y) \rightarrow \cos^2(q'_x) \sin^2(q'_y) \]  

(B21)

for \( n \) odd and \( m \) even (mixed symmetry modes), where \( q'_x = \frac{qD}{2} \sin \theta \cos \varphi \) and \( q'_y = \frac{qD}{2} \sin \theta \sin \varphi \). The results for the other mixed symmetry case [Eqs. (B6), (B10)] can be obtained by interchanging the harmonic indices. Subsequently, we express these factors [Eqs. (B20), (B21)] as trigonometric series in \( \varphi \) using the Jacobi-Anger expansion and obtain again Eq. (B15) with

\[
a_0(x) = 1 - 2J_0(x) + J_0(\sqrt{2}x), \]  

(B22)

\[
a_l(x) = -2 \left[ 2J_l(x) - (-1)^l J_l(\sqrt{2}x) \right] \quad \text{for } l > 0. \]  

(B22)
in the antisymmetric case, and

\[
\begin{align*}
    a_0(x) &= 1 - J_0(\sqrt{2}x), \\
    a_l(x) &= -4J_{2l}(x) \quad \text{for } l \text{ odd}, \\
    a_l(x) &= -2(-1)^{l/2}J_{2l}(\sqrt{2}x) \quad \text{for } l > 0, \text{ even}
\end{align*}
\]  

in the case of mixed symmetry. Henceforth, the same steps followed in Sec. B 1 [cf. Eqs. (B15), (B17)-(B19)] lead to Eq. (3).

The series in \( l \), corresponding to the last dimensionless factor in Eq. (3), can be evaluated numerically using Eqs. (A9)-(A12). Without loss of generality, we assume \( n < m \) and find that, for ratios \( m/n \) not too large, the condition \( \eta \sqrt{1 + m^2/n^2} \ll 1 \) implies that prevalently \( |\bar{\omega}_{l+1,\gamma}/\bar{\omega}_{l,\gamma}| \sim \eta^2(1 + m^2/n^2)/4 \) ensuring rapid convergence.

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25 Equation (1) is also valid for degenerate resonator modes \( \bar{\omega}_l \) provided that: (i) the degeneracy (possibly split by disorder) is associated to a symmetry also satisfied by the support of the wave function or (ii) mode-mixing induced by disorder and/or interaction is negligible.
26 For our structures this gap was \( \lesssim 200\text{nm} \ll D \).
27 The average thickness of the membrane is inferred from modelling the isotropic etching process in each material.
28 The approximate scaling for \( Q_{n1}/Q_{b1} \) differs from the results of applying Eq. (2) by at most a factor of 2 for \( 0 < n < 15 \) in the relevant regime \( \eta_l < 0.3 \) and \( \nu = 1/3 \).
29 For such modes, \( \eta \zeta_{nm} \ll 1 \) eventually fails and for \( \eta \zeta_{nm} \gg |\alpha^2 - 1/4| \) one finds that \( 1/Q_{nm} \) exhibits oscillatory behavior dominated by the SAW contribution.
30 In the opposite case min\( \{n, m\}/\sqrt{n^2 + m^2} \leq \eta \) there are one or more types of modes \( \gamma \) for which at least one of the denominators in the last factor in Eq. (B15) has a pair of complex conjugate roots on the unit circle, so that this factor diverges for certain angles \( \phi \) and the convergence of the series is no longer uniform, rendering the decomposition used invalid.