New and refined bounds for expected maxima of fractional Brownian motion

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Abstract

For the fractional Brownian motion \( B^H \) with the Hurst parameter value \( H \) in \((0,1/2)\), we derive new upper and lower bounds for the difference between the expectations of the maximum of \( B^H \) over \([0,1]\) and the maximum of \( B^H \) over the discrete set of values \( in^{-1}, i = 1,\ldots,n \). We use these results to improve our earlier upper bounds for the expectation of the maximum of \( B^H \) over \([0,1]\) and derive new upper bounds for Pickands’ constant.

Key words and phrases: fractional Brownian motion, convergence rate, discrete time approximation, Pickands’ constant.

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1 Introduction

Let \( B^H = (B^H_t)_{t \geq 0} \) be a fractional Brownian motion (fBm) process with Hurst parameter \( H \in (0,1) \), i.e. a zero-mean continuous Gaussian process with the covariance function \( \mathbb{E}B^H_sB^H_t = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}) \), \( s, t \geq 0 \). Equivalently, the last condition can be stated as \( B^H_0 = 0 \) and

\[
\mathbb{E}(B^H_s - B^H_t)^2 = |s-t|^{2H}, \quad s, t \geq 0.
\] (1)

Recall that the Hurst parameter \( H \) characterizes the type of the dependence of the increments of the fBm. For \( H \in (0, \frac{1}{2}) \) and \( H \in (\frac{1}{2}, 1) \), the increments of \( B^H \) are respectively negatively and positively correlated, whereas the process \( B^{1/2} \) is the standard Brownian motion which has independent increments. The fBm

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processes are important construction blocks in various application areas, the ones with \( H > \frac{1}{2} \) being of interest as their increments exhibit long-range dependence, while it was shown recently that fBm’s with \( H < \frac{1}{2} \) can be well fitted to real life telecommunications, financial markets with stochastic volatility and other financial data (see, e.g., [2, 3]). For detailed exposition of the theory of fBm processes, we refer the reader to [4, 8, 9] and references therein.

Computing the value of the expected maximum

\[ M^H := \mathbb{E} \max_{0 \leq t \leq 1} B^H_t \]

is an important question arising in a number of applied problems, such as finding the likely magnitude of the strongest earthquake to occur this century in a given region or the speed of the strongest wind gust a tall building has to withstand during its lifetime etc. For the standard Brownian motion \( B^{1/2} \), the exact value of the expected maximum is \( \sqrt{\pi/2} \), whereas for all other \( H \in (0, 1) \) no closed-form expressions for the expectation are known. In the absence of such results, one standard approach to computing \( M^H \) is to evaluate instead its approximation

\[ M_n^H := \mathbb{E} \max_{1 \leq i \leq n} B^H_{i/n}, \quad n \geq 1, \]

(which can, for instance, be done using simulations) together with the approximation error

\[ \Delta_n^H := M^H - M_n^H. \]

Some bounds for \( \Delta_n^H \) were recently established in [5]. The main result of the present note is an improvement of the following upper bound for \( \Delta_n^H \) obtained in Theorem 3.1 of [5]: for \( n \geq 2^{1/H} \),

\[ \Delta_n^H \leq \frac{2(\ln n)^{1/2}}{n^H} \left( 1 + \frac{4}{n^H} + \frac{0.0074}{(\ln n)^{3/2}} \right). \quad (2) \]

Lower bound for \( \Delta_n^H \) is obtained as well and we study for which \( H \) and \( n \) upper and lower bounds hold simultaneously. We also obtain a new upper bound for the expected maximum \( M^H \) itself and some functions of it, which refines previously known results (see e.g. [5, 12]), and use it to derive an improved upper bound for the so-called Pickands’ constant, which is the basic constant in the extreme value theory of Gaussian processes.

The paper is organized as follows: Section 2 contains the results, with comments and examples, and Section 3 contains the proofs.

2 Main results

From now on, we always assume that \( H \in (0, \frac{1}{2}) \). The next theorem is the main result of the note. As usual, \([x]\) and \(\lceil x \rceil\) denote the floor and the ceiling of the real number \(x\).
Theorem 1. 1) For any $\alpha > 0$ and $n \geq 2^{1/\alpha} \vee (1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)}$ one has

$$\frac{\Delta_n^H}{n^{-H} (\ln n)^{1/2}} \leq \frac{(1 - \lfloor n^{\alpha} \rfloor^{-1})^H (1 + \alpha)^{1/2}}{1 - \lfloor n^{\alpha} \rfloor^{-H} (1 + \alpha/(1 + \alpha))^{1/2}}.$$  \hspace{1cm} (3)

2) For any $n \geq 2$ one has

$$\frac{\Delta_n^H}{n^{-H} (\ln n)^{1/2}} \geq n^H \left( \frac{L}{(\ln n^H)^{1/2}} - 1 \right)^+,$$  \hspace{1cm} (4)

where $L = 1/\sqrt{4\pi e \ln 2} \approx 0.2$ and $\alpha^+ = a \vee 0$.

Remark 1. Note that inequality (4) actually holds for all $H \in (0, 1)$.

Remark 2. Let us study for which $H$ and $n$ upper and lower bounds (3) and (4) hold simultaneously under assumption that (4) is non-trivial. For non-triviality we need to have $n < \exp \frac{L^2}{H}$. In order to have $2^{1/\alpha} \leq \exp \frac{L^2}{H}$ we restrict $\alpha$ to $\alpha \geq \frac{H \ln 2}{L^2}$. In order to have $(1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)} \leq \exp \{ \frac{L^2}{H} \}$, or, what is equivalent,

$$\left( 1 + \frac{\alpha}{1 + \alpha} \right)^{1/\alpha} \leq \exp (2L^2),$$  \hspace{1cm} (5)

we note that the function $q(\alpha) = (1 + \frac{\alpha}{1+\alpha})^{1/\alpha}$ continuously strictly decreases in $\alpha \in (0, \infty)$ from $e$ to $1$, and taking into account the value of $L$, we get that there is a unique root $\alpha^* \approx 7.48704$ of the equation $(1 + \frac{\alpha}{1+\alpha})^{1/\alpha} = \exp (2L^2)$ and for $\alpha \geq \alpha^*$ we have that (5) holds. Therefore for $\alpha > \alpha^*$, $H < \frac{\alpha^* L^2}{\ln 2} \approx 0.456$ and $\exp \{ \frac{L^2}{H} \} > n > 2^{1/\alpha} \vee (1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)}$ we have that lower bound (3) holds and is non-trivial. Moreover, $2^{1/\alpha} < 2^{1/\alpha^*} (\approx 1.097) < \exp \{ \frac{L^2}{H} \}$, $(1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)} < (1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha^* H)} = \exp \{ \frac{L^2}{H} \}$, so the interval $(2^{1/\alpha} \vee (1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)}, \exp \{ \frac{L^2}{H} \})$ is non-empty and for such $n$ upper bound (3) holds. The only question is if this interval contains the integers. If it is not the case we can increase the value of $\alpha$.

For example, put $H = 0.01$, $\alpha = 16$, then it holds that the interval $(2^{1/\alpha} \vee (1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)}, \exp \{ \frac{L^2}{H} \}) = (1.044 \vee 7.534, 20.085) = (7.534, 20.085)$.

Remark 3. Consider the sequence $\alpha = \alpha(m) \to 0$ slowly enough as $m \to \infty$ (take, e.g., $\alpha(m) = (\ln \ln m) / \ln m$). Then for sufficiently large enough $m$ we have that $m \geq 2^{1/\alpha(m)} \vee (1 + \frac{\alpha(m)}{1+\alpha(m)})^{1/(2\alpha(m)H)}$ therefore for such $m$ the upper bound (3) holds. Returning to standard notation $n$ for the argument, we obtain from the upper bound in (3) that, for any fixed $H \in (0, \frac{1}{2})$, one has

$$\Delta_n^H \leq n^{-H} (\ln n)^{1/2} (1 + o(1)), \quad n \to \infty,$$  \hspace{1cm} (6)

which refines (2).
Remark 4. Recall that, in the case of the standard Brownian motion \((H = \frac{1}{2})\), the exact asymptotics of \(\Delta_n^{1/2}\) are well-known:

\[
\Delta_n^{1/2} = n^{-1/2}(\beta + o(1)), \quad n \to \infty,
\]

where \(\beta = -\zeta(1/2)/\sqrt{2\pi} = 0.5826\ldots\) and \(\zeta(\cdot)\) is the Riemann zeta function (see [13]). Comparing it with (6), we see that now we have additional logarithmic multiplier.

The next simple assertion enables one to use the upper bound obtained in Theorem 1 to get an upper bound for the approximation rate of the expectation of a function of the maximum of an fBm. Such a result is required, for instance, for bounding convergence rates when approximating Bayesian estimators in irregular statistical experiments (see, e.g., [7, 10]).

Set

\[
\mathcal{B}_H := \max_{0 \leq t \leq 1} B_H^t, \quad \mathcal{B}_{H,n} := \max_{1 \leq i \leq n} B_H^{i/n}, \quad \Delta_n^{H,f} := \mathbb{E}f(B_H^1) - \mathbb{E}f(B_{H,n}^n)
\]

and, for a function \(f : \mathbb{R} \to \mathbb{R}\), denote its continuity modulus by

\[
\omega_{h,\delta}(f) := \sup_{0 \leq s < t \leq (s + \delta) \wedge h} |f(s) - f(t)|, \quad h, \delta > 0.
\]

Corollary 1. Let \(f \geq 0\) be an arbitrary non-decreasing function on \(\mathbb{R}\) such that \(f(x) = o\left(\exp\left((x - M^H)^2/2\right)\right)\) as \(x \to \infty\). Then, for any number \(M > M^H\),

\[
\Delta_n^{H,f} \leq \omega_{\Delta_n^{H},M}(f) + \int_{M}^{\infty} f(x)(x - M^H) \exp\left\{-\left(x - M^H\right)^2/2\right\} dx.
\]

To roughly balance the contributions from the two terms in the bound, one may wish to choose \(M\) so that \(\exp\left\{-\left(M - M^H\right)^2/2\right\}\) would be of the same order of magnitude as \(\Delta_n^{H}\) (as for regular functions \(f\) that are mostly of interest in applications are locally Lipschitz, so that \(\omega_{h,\delta}(f)\) admits a linear upper bound in \(\delta\)). To that end, one can take \(M := M^H + (-2 \ln \Delta_n^{H})^{1/2} + \text{const}\) (assuming that \(n\) is large enough so that \(\Delta_n^{H} < 1\)). We will illustrate that in two special cases where \(f\) is the exponential function (this case corresponds to the above-mentioned applications from [7, 10]) and a power function, respectively.

Example 1. Assume that \(f(x) = e^{ax}\) with a fixed \(a > 0\), and that \(\Delta_n^{H} < 1\). Choosing \(M := M^H + a + |2 \ln \Delta_n^{H}|^{1/2}\) we get

\[
\omega_{\Delta_n^{H},M}(f) \leq e^{aM} \Delta_n^{H} = \exp\{aM^H + a^2 + a|2 \ln \Delta_n^{H}|^{1/2}\} \Delta_n^{H}
\]

and, setting \(y := x - M^H\) and using the well-known bound for the Mills’ ratio for
the normal distribution, obtain that
\[
\int_M^\infty f(x)(x - M^H) \exp\{-(x - M^H)^2/2\} \, dx = e^{aM^H} \int_M^\infty ye^{-y^2/2+ay} \, dy
\]
\[
= e^{aM^H+a^2/2} \left[ \int_M^{\infty} (y - a)e^{-(y-a)^2/2} \, dy + a \right] \int_M^{\infty} e^{-(y-a)^2/2} \, dy
\]
\[
\leq e^{aM^H+a^2/2} \left( 1 + \frac{a}{M - M^H - a} \right) e^{-(M - M^H - a)^2/2}
\]
\[
= e^{aM^H+a^2/2} \left( 1 + \frac{a}{|2 \ln \Delta_n^{H}|^{1/2}} \right) \Delta_n^{H}.
\]
Therefore
\[
\Delta_n^{H,f} \leq e^{aM^H+a^2/2} \left( 1 + e^{a^2/2+|2 \ln \Delta_n^{H}|^{1/2}} + \frac{a}{|2 \ln \Delta_n^{H}|^{1/2}} \right) \Delta_n^{H}.
\]

**Example 2.** For the function \( f(x) = x^p, \ p \geq 1 \), one clearly has
\[
\Delta_n^{H,f} \leq pMp^{-1} \Delta_n^{H} + \int_M^{\infty} x^p(x - M^H) \exp\{-(x - M^H)^2/2\} \, dx.
\]
Observe that \( x^p = (x - M^H)^p \left( 1 + \frac{M^H}{x - M^H} \right)^p \leq (x - M^H)^p \left( \frac{M^H}{M - M^H} \right)^p \) for \( x \geq M \),
while, for any \( A > 0 \),
\[
\int_M^{\infty} z^p e^{-z^2/2} \, dz = - \int_M^{\infty} z^p e^{-z^2/2} \, dz = A^p e^{-A^2/2} + p \int_M^{\infty} z^{p-1} e^{-z^2/2} \, dz,
\]
where the last integral does not exceed \( A^{-2} \int_M^{\infty} z^p e^{-z^2/2} \, dz \), so that
\[
\int_M^{\infty} z^p e^{-z^2/2} \, dz \leq \frac{A^p e^{-A^2/2}}{1 - pA^{-2}} \quad \text{for} \quad A^2 > p.
\]
Hence, choosing \( A := M - M^H = |2 \ln \Delta_n^{H}|^{1/2} \), we obtain that, for \( \Delta_n^{H} < e^{-p/2} \),
\[
\Delta_n^{H,f} \leq (M^H + |2 \ln \Delta_n^{H}|^{1/2}) \left( p + \frac{M^H + |2 \ln \Delta_n^{H}|^{1/2}}{1 - p|2 \ln \Delta_n^{H}|^{-1}} \right) \Delta_n^{H}.
\]

Finally, in the next corollary we use Theorem 2 to improve the known upper bound \( M^H < 16.3H^{-1/2} \) for the expected maximum \( M^H \) from Theorem 2.1(ii) in [5].

**Corollary 2.** Assume that \( H \) is such that \( 2^{2/H} \) is integer. Then
\[
M^H < 1.695H^{-1/2}.
\]

**Remark 5.** If \( 2^{2/H} \) is not integer then, in the above formula, one can use instead of \( H \) the largest value \( \tilde{H} < H \) such that \( 2^{2/\tilde{H}} \) is integer, i.e. \( \tilde{H} = 2/ \log_2 [2^{2/H}] \). This is so since it follows from Sudakov–Fernique’s inequality (see e.g. Proposition 1.1 and Section 4 in [5]) that the expected maximum \( M^H \) is a non-increasing function of \( H \).
Remark 6. Our new upper bound for $M^H$ can be used to improve Shao’s upper bound from [12] for Pickands’ constant $H$, which is a basic constant in the extreme value theory of Gaussian processes and is of interest in a number of applied problems. That constant appears in the asymptotic representation for the tail probability of the maxima of stationary Gaussian processes in the following way (see e.g. [11]).

Assume that $(X_t)_{t \geq 0}$ is a stationary Gaussian process with zero mean and unit variance of which the covariance function $r(v) := \mathbb{E}X_tX_{t+v}$, satisfies the following relation: for some $C > 0$ and $H \in (0,1]$, one has $r(t) = 1 - C|t|^{2H} + o(|t|^{2H})$ as $t \to 0$. Then, for each fixed $T > 0$ such that $\sup_{\varepsilon \leq t \leq T} r(t) < 1$ for all $\varepsilon > 0$,

$$
P \left( \sup_{0 \leq t \leq T} X_t > u \right) = C^{1/(2H)} \mathcal{H}_H (2\pi)^{-1/2} e^{-u^2/2} u^{1/H-1}(T + o(1)), \quad u \to \infty. $$

It was shown in [12] that, for $H \in (0,1/2]$,

$$
\mathcal{H}_H \leq (2^{1/2} eHM^H)^{1/H}.
$$

Using our Corollary 2 we obtain the following new upper bound for Pickands’ constant:

$$
\mathcal{H}_H < (42.46H)^{1/(2H)}, \quad H \in (0,1/2],
$$

which is superior to Shao’s bound

$$
\mathcal{H}_H \leq \left\{ 1.54H + 4.82H^{1/2} (4.4 - H \ln(0.4 + 1.25/H))^{1/2} \right\}^{1/H}, \quad H \in (0,1/2]
$$

(see (1.5) in [12]; there the notation $a := 2H$ is used). For example, the ratio of our bound to Shao’s equals 0.344 when $H = 0.45$ and is 0.046 when $H = 0.15$.

3 Proofs

Proof of Theorem 7. First we will prove (3). Let $n_k := nm^k$, $k \geq 0$, where we set $m := \lfloor n^a \rfloor \geq 2$. It follows from the continuity of $B^H$ and monotone convergence theorem that

$$
\Delta_n^H = \sum_{k=0}^{\infty} (M_{n_{k+1}}^H - M_{n_k}^H). \quad (7)
$$

Although this step is common with the proof of Theorem 3.1 in [5], the rest of the argument uses a different idea. Namely, we apply Chatterjee’s inequality ([6]; see also Theorem 2.2.5 in [1]) which, in its general formulation, states the following. For any $N$-dimensional Gaussian random vectors $X = (X_1, \ldots, X_N)$, $Y = (Y_1, \ldots, Y_N)$ with common means: $\mathbb{E}X_i = \mathbb{E}Y_i$ for $1 \leq i \leq N$, one has

$$
|\mathbb{E} \max_{1 \leq i \leq N} X_i - \mathbb{E} \max_{1 \leq i \leq N} Y_i| \leq (\gamma \ln N)^{1/2}, \quad \gamma := \max_{1 \leq i < j \leq N} |d_{ij}(X) - d_{ij}(Y)|, \quad (8)
$$

where, for a random vector $Z \in \mathbb{R}^N$, we set $d_{ij}(Z) := \mathbb{E}(Z_i - Z_j)^2$, $1 \leq i,j \leq N$. 


To be able to apply inequality (8) to the terms in the sum on the right-hand side of (7), for each \( k \geq 0 \) we introduce auxiliary vectors \( X^k, Y^k \in \mathbb{R}^{n_{k+1}} \) by letting

\[
X_i^k := B^H_{i/n_{k+1}}, \quad Y_i^k := B^H_{i/m}/n_k, \quad 1 \leq i \leq n_{k+1}.
\]

Note that \( M_{n_{k+1}}^H = E \max_{1 \leq i \leq n_{k+1}} X_i^k \) and \( M_{n_k}^H = E \max_{1 \leq i \leq n_{k+1}} Y_i^k \), so that now (8) is applicable. Next we will show that

\[
\gamma^k := \max_{1 \leq i < j \leq n_{k+1}} |d_{ij}(X^k) - d_{ij}(Y^k)| \leq n_k^{-2H} (1 - m^{-1})^{2H}.
\]

Indeed, one can clearly write down the representations \( i = a_i m + b_i, \ j = a_j m + b_j \) with integer \( a_j \geq a_i \geq 0 \) and \( 1 \leq b_i, b_j \leq m \), such that \( b_j > b_i \) when \( a_i = a_j \). Then it follows from (7) that

\[
d_{ij}(X^k) = \left( \frac{(a_j - a_i) m + b_j - b_i}{n_{k+1}} \right)^{2H}, \quad d_{ij}(Y^k) = \left( \frac{(a_j - a_i) m}{n_{k+1}} \right)^{2H}.
\]

Since for \( 2H \leq 1 \) the function \( x \mapsto x^{2H}, \ x \geq 0 \), is concave, it is also sub-additive, so that \( x^{2H} - y^{2H} \leq (x - y)^{2H} \) for any \( x \geq y \geq 0 \). Setting \( x := d_{ij}(X^k) \lor d_{ij}(Y^k) \) and \( y := d_{ij}(X^k) \land d_{ij}(Y^k) \), this yields the desired bound

\[
|d_{ij}(X^k) - d_{ij}(Y^k)| \leq \left( \frac{|b_i - b_j|}{n_{k+1}} \right)^{2H} \leq \left( \frac{m - 1}{n_{k+1}} \right)^{2H} = \frac{1}{n_k^{2H}} \left( 1 - \frac{1}{m} \right)^{2H}.
\]

Now it follows from (8) that

\[
M_{n_{k+1}}^H - M_{n_k}^H \equiv E \max_{1 \leq i \leq n_{k+1}} X_i^k - E \max_{1 \leq i \leq n_{k+1}} Y_i^k
\]

\[
\leq (\gamma^k \ln n_{k+1})^{1/2} \leq \left( \frac{1 - m^{-1}}{n^{H} m^{kH}} \right) \left( \ln n + (k + 1) \ln m \right)^{1/2}
\]

\[
\leq \left( \frac{\ln n}{n^H} \right)^{1/2} (1 - m^{-1})^H \frac{(1 + \alpha + \alpha k)^{1/2}}{m^{kH}}.
\]

The last bound together with (7) leads to

\[
\Delta_n^H \leq \frac{\left( \frac{\ln n}{n^H} \right)^{1/2}}{(1 - m^{-1})^H} \sum_{k=0}^{\infty} \frac{(1 + \alpha + \alpha k)^{1/2}}{m^{kH}}.
\]

The sum of the series on the right hand side is exactly \( \alpha^{1/2} \Phi(m^{-H}, -\frac{1}{2}, 1 + \alpha^{-1}) \), where \( \Phi \) is the Lerch transcendent function. For our purposes, however, it will be convenient just to use the elementary bound \( (1 + \alpha + \alpha k)^{1/2} \leq (1 + \alpha)^{1/2} (1 + \alpha/(1 + \alpha))^{k/2} \), to get

\[
\Delta_n^H \leq \frac{\left( \frac{\ln n}{n^H} \right)^{1/2}}{(1 - m^{-1})^H} \cdot \frac{(1 - m^{-1})^H (1 + \alpha)^{1/2}}{1 - m^{-H} (1 + \alpha/(1 + \alpha))^{1/2}}.
\]

The right inequality in (3) is proved. To establish the left one, note that, on the one hand, it was shown in Theorem 2.1 [5] that \( M^H \geq LH^{-1/2} \) for all \( H \in (0, 1) \).
On the other hand, it follows from Sudakov–Fernique’s inequality (see e.g. Proposition 1.1 in [5]) that, for any fixed \( n \geq 1 \), the quantity \( M^H_n \) is non-increasing in \( H \), and it follows from Lemma 4.1 in [5] that

\[
M^0_n := \lim_{H \to 0} M^H_n = 2^{-1/2} E \xi_n, \quad \xi_n := \max_{1 \leq i \leq n} \xi_i,
\]

where \( \xi \) are i.i.d. \( N(0, 1) \)-distributed random variables. Furthermore, the last expectation admits the following upper bound:

\[
E \xi_n \leq \sqrt{2 \ln n}, \quad n \geq 1.
\] (9)

Although that bound has been known for some time, we could not find a suitable literature reference or stable Internet link for it. So we decided to include a short proof thereof for completeness’ sake. By Jensen’s inequality, for any \( s \in \mathbb{R} \),

\[
e^{sE \xi_n} \leq E e^{s \xi_n} = E \max_{1 \leq i \leq n} e^{s \xi_i} \leq E \sum_{1 \leq i \leq n} e^{s \xi_i} = n e^{s^2/2},
\]

so that \( E \xi_n \leq s^{-1} \ln n + s/2 \). Minimizing in \( s \) the expression on the right-hand side yields the desired bound (9).

From the above results, we obtain that

\[
M^H - M^0_n \geq M^H - M^0_n \geq LH^{-1/2} - (\ln n)^{1/2}
\]

\[
= n^{-H}(\ln n)^{1/2} \cdot n^H (L(H \ln n)^{-1/2} - 1),
\]

which completes the proof of Theorem [1]. \( \square \)

**Proof of Corollary [1].** Since \( f \geq 0 \), for any \( M > M^H \) we have

\[
\Delta^H_n f \leq E \left( f(B^H_n) - f(B^H_{n, n}); B^H_1 \leq M \right) + E \left( f(B^H_1); B^H_1 > M \right)
\]

\[
\leq \omega_{\Delta^H,M}(f) + \int_M^\infty f(x) dF(x),
\] (10)

where \( F(x) := P(B^H_1 \leq x) \). From the well-known Borell–TIS inequality for Gaussian processes (see, e.g., Theorem 2.1.1 in [1]) it follows that, for any \( u > 0 \),

\[
P \left( B^H_1 - M^H > u \right) \leq e^{-u^2/2}.
\]

Therefore, for any \( M > M^H \), integrating by parts, using the assumed property that \( f(x) \exp \{-(x - M^H)^2/2\} \to 0 \) as \( x \to \infty \), and then again integrating by parts, we can write

\[
\int_M^\infty f(x) dF(x) = f(M)(1 - F(M)) + \int_M^\infty (1 - F(x)) df(x)
\]

\[
\leq f(M) \exp \{-(M - M^H)^2/2\} + \int_M^\infty \exp \{-(x - M^H)^2/2\} df(x)
\]

\[
= - \int_M^\infty f(x) d \exp \{-(x - M^H)^2/2\}.
\]

Together with (10) this establishes the assertion of Corollary [1]. \( \square \)
Proof of Corollary. Using Chatterjee’s inequality with the zero vector \( Y \), we get for any \( n \geq 1 \) the bound \( M_n^H \leq ((1 - n^{-2H}) \ln n)^{1/2} \), so that we obtain from Theorem that

\[
M^H \leq \Delta_n^H + ((1 - n^{-2H}) \ln n)^{1/2} < \left[ \frac{n^{-H}(1 + \alpha)^{1/2}}{1 - m^{-H}(1 + \alpha/(1 + \alpha))^{1/2}} + (1 - n^{-2H})^{1/2} \right] (\ln n)^{1/2}.
\]

Now choosing \( n := 4^{1/H} \) (which was assumed to be integer) and \( \alpha := 2 \), we get \( m = n^\alpha = 4^{2/H} \) and

\[
M^H < H^{-1/2} \left[ \frac{4^{-1}3^{1/2}}{1 - 16^{-1}(5/3)^{1/2}} + (1 - 16^{-1})^{1/2} \right] (\ln 4)^{1/2} < 1.695H^{-1/2}.
\]

\[\square\]

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