Convergence of space-discretised gKPZ via Regularity Structures

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Abstract

In this work, we show a convergence result for the discrete formulation of the generalised KPZ equation
\[ \partial_t u = (\Delta u) + g(u)(\nabla u)^2 + k(\nabla u) + h(u) + f(u)\xi_t(x), \]
where the \(\xi\) is a real-valued random field, \(\Delta\) is the discrete Laplacian, and \(\nabla\) is a discrete gradient, without fixing the spatial dimension. Our convergence result is established within the discrete regularity structures introduced by Hairer and Erhard [QW]. We extend with new ideas the convergence result found in [QX] that deals with a discrete form of the Parabolic Anderson model driven by a (rescaled) symmetric simple exclusion process. This is the first time that a discrete generalised KPZ equation is treated and it is a major step toward a general convergence result that will cover a large family of discrete models.

Keywords: Discrete models, Generalised KPZ equation, Regularity Structures, Stochastic PDE

MSC classification: 60L30, 60L40, 60H15

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1 Introduction

Martin Hairer’s theory of Regularity Structures [25] has been applied to construct a solution theory for a large subclass of singular stochastic PDEs of the form:

$$\partial_t u - \mathcal{L} u = F(u, \nabla u, \xi)$$  \hspace{1cm} (1.1)

where $\mathcal{L}$ is some differential operator and $F$ is some non-linearity affine in the noise $\xi$. Central to this solution theory is the idea of renormalisation which usually involves subtracting certain counterterms from the equation so as to deal with products of distributions that are undefined in the classical sense.

Since the original paper [25], the programme for generating the solution theory for a given SPDE has been automated. In [6], the authors explained how to extract rules from a given equation, and then proceduralised the construction of a renormalisation group that can affect renormalisation on the equation. In this programme, the choice of the aforementioned counter-terms comes from the classic BPHZ formalism [8, 25, 49]. In [13], the authors provide a functionally black-box like result that automatically produces the required stochastic estimates for the renormalised stochastic objects coming from [6]. The last step completed in [3] was to write the fixed point and the action of the renormalisation onto the right hand side of (1.1). Hence, the programme in the continuum has been automated by this series of papers.

Many equations of the form (1.1), arise from scaling limits of the microscopic models - consider [4, 56, 31] for derivations of Stochastic Burgers, KPZ, $\Phi^4$ and the parabolic Anderson model - and as such there is interest in discretisations of these equations. A classic example is from [4], where employing an approach based on the Cole-Hopf transform, the authors proved that a (rescaled) particle system converges to their notion of the solution of the KPZ. In [20, 21], the authors introduced a martingale type approach to the problem - the so-called energy solution. Further, they showed the limits of a sizeable class of suitably scaled interacting particle systems are also energy solutions. The uniqueness of the energy solution has been proved in [24] which implies various applications given in [22, 10]. These approaches that we have listed thus far have certain drawbacks - for the former the difficulty is that the Hopf-Cole transform is difficult to implement at a discrete level, and for the latter, knowledge and control of the invariant measure for the system is a requisite.
The use of regularity structures circumvents both of these problems and hence has been shown to have great promise for discrete problems such as these. In [28], the authors developed a framework to adapt the theory of regularity structure for certain spatial discretisations. Further generality was achieved in [17], where the authors do not fix any particular discretisation procedure. As applications to the discrete regularity structures constructed in these papers, in [28] a space-discretised KPZ was proven to converge to the solution of the generic KPZ, and then in [15] the same was proven for a space-time discretisation of the KPZ equation. Convergence of more general discretisations driven by interacting particle systems to stochastic PDEs in paper such as [18] and [24] was addressed. In the former, the author used multiple stochastic integrals with respect to martingales with particular properties that allow for certain moment bounds and expansions that are analogous to the Nelson’s Estimate and Wiener chaos expansions that are used in [25] in the case of (homogeneous) Gaussian Noise. In the latter, the authors begin with a discrete parabolic Anderson model driven by a symmetric simple exclusion process and show that the solutions to a suitably renormalised version of the equation converge in law to the solution of the same equation driven by a generalised Ornstein-Uhlenbeck process. This is achieved via estimates on joint cumulants of arbitrary large order for the exclusion process.

Another reason that drives interest in discrete regularity structures is that it may possibly provide a way to link invariant measures to their dynamics. See [5] for a conjecture connecting the Brownian loop measure to stochastic geometric heat equations, that falls short of being a theorem due to the lack of a convergence result in the way of [13] for discrete regularity structure. In [14], the authors study the stochastic quantisation for the Yang-Mills in the 2 and 3 dimensional Euclidean space and they have the same type of open problem for the invariant measure.

Apart from Regularity Structures, other theories have been successful in treating singular SPDEs. One such is via the use of paracontrolled calculus as in [19], which much in the way of Hairer, describes the spectral features of a function in terms of paracontrolled distributions. This methodology has also been used to look at discrete problems. In [4], for example, the author use the theory to approximate the Navier-Stokes equation, while in [22] the authors construct a piecewise linear approximation for the $\Phi_3^4$ model. In [21], the authors achieve a similar convergence result as we present here, for Sasamoto-Spohn type discretisations of the stochastic Burgers equation, in that they show that the limit of such discretisations solves (a version of) the continuous Burgers equations. In [12], the authors also look at the discrete PAM, but the tool for proving convergence is paracontrolled calculus and finally in [15] the authors develop a discrete version of paracontrolled distributions. In this paper, we intend to extend the work done in [18] by presenting a renormalisation procedure that is able to handle more complicated equations. We fix the following discretisation of the generalised KPZ equation:

$$\partial_t u = \Delta u + g(u) (\nabla u)^2 + k(u) \nabla u + h(u) + f(u) \xi_t (x) \quad \text{(gKPZ)}$$

to showcase our methodology. Here $t \geq 0, x \in \mathbb{Z}^d$, $\nabla$ is a discrete gradient and $\Delta$
is the discrete Laplacian defined:

$$\Delta u(t, x) = \sum_{y : x \sim y} [u(y, t) - u(x, t)]$$

where $x \sim y$ means that $x$ and $y$ are nearest neighbours with respect to the Euclidean norm on $\mathbb{Z}^d$ and $\xi$ is a $\mathbb{R}$-valued random field which we will not fix although we do assume that it is centred.

As in [18] we would like to prove that the solution of the gKPZ converges, in some sense to be specified later, to the solution $\bar{u}$ of the gKPZ equation driven by some space-time random field $Y$ defined on $\mathbb{R}_+ \times \mathbb{R}^d$:

$$\partial_t \bar{u} = \Delta \bar{u} + g(\bar{u})(\nabla \bar{u})^2 + k(\bar{u})\nabla \bar{u} + h(\bar{u}) + f(\bar{u})Y_t(x) \quad (1.2)$$

Towards this convergence the space is rescaled by reformulating our equation on $\mathbb{Z}_N^d = (\mathbb{Z}/2^N\mathbb{Z})^d$:

$$\partial_t \hat{u}^N = (\Delta_N \hat{u}^N) + g(\hat{u}^N)(\nabla_N \hat{u}^N)^2$$

$$+ k(\hat{u}^N)(\nabla_N \hat{u}^N) + h(\hat{u}^N) + 2^{N_d/2}\xi_N^t f(\hat{u}^N) \quad (1.3)$$

where $\Delta_N = 2^{2N}\Delta$, $\nabla_N = 2^N\nabla$ are the rescaled discrete Laplacian and discrete derivative defined on $2^N\mathbb{Z}^d$, with $x \in 2^{-N}\mathbb{Z}^d$, $t \geq 0$, $\xi_N^t(x) = \xi_{2^N}(2^N x)$. Recall that we did not fix any particular discrete derivative, but we do require from here on that $\nabla$ must be chosen so that we have $\|\nabla_N \varphi - \varphi'\|_\infty \to 0$.

Of course, with reference to the discussion in the opening regarding the ill-posedness of the products of distributions in general, it turns out that both the gKPZ and (1.2) fail to have any canonical meaning and instead of (1.3) one should be looking at:

$$\partial_t \hat{u}^N = (\Delta_N \hat{u}^N) + g(\hat{u}^N)(\nabla_N \hat{u}^N)^2$$

$$+ k(\hat{u}^N)(\nabla_N \hat{u}^N) + \bar{h}(\hat{u}^N) + 2^{N_d/2}\xi_N^t f(\hat{u}^N) \quad (1.4)$$

with $\bar{h}$ taking the form:

$$\bar{h}(u) = h(u) + \sum_{\tau \in \mathcal{T}_-} \frac{\Upsilon[\tau]}{S(\tau)} C_N(\tau).$$

where $\mathcal{T}_-$ is a finite set of decorated trees, $\Upsilon[\tau]$ are elementary differentials, $S(\tau)$ are symmetric factors and the $C_N(\tau)$ are renormalisation constants coming from the general theory which was developed in [3] and has since been used for stochastic geometric equations in [5]. For our purposes, the renormalisation constants need to be adjusted and we discuss this in section 2.2.

In application of the results we present here, the choice of the noise will be made according to the following assumption on its cumulants (refer to Appendix A for a primer on cumulants):
Assumption 1 There exists a space-time noise $\xi_t(x)$ defined on $\mathbb{Z}_N^d$, such that for $t \geq 0$ and some index set $|A| \geq 2$, one has the following bound:

$$E \left\{ \xi_{t_a}(x_a) : a \in A \right\} \lesssim \sum_{\sigma \in \sigma(A)} \prod_{i=1}^{|A|} \left( \| z_{\sigma(i+1)} - z_{\sigma(i)} \|_s \vee 2^{-N} \right)^{-\frac{3}{2}}$$

uniformly over $N > 0$ and over all collections of time-space points $z_a = (t_a, x_a)$ indexed by $a \in A$.

Then, our main result can be seen as an extension of the main result obtained in \[QXL\] Thm 1.1:

**Theorem 1.1 (Meta-theorem)** Denote by $u^N$ the solution of (1.4) defined on the rescaled torus $\mathbb{T}_N^d = 2^{-N} \mathbb{Z}_N^d$ and by $u$ the solution of the renormalised version of (1.2). If there exists a sequence of initial conditions $(u^N_0)_{N \in \mathbb{N}}$ such that there is $\eta \in (0, 1)$ and $u_0 \in \mathcal{C}^\eta$ for which it is true that

$$\lim_{N \to \infty} \| u_0; u^N_0 \|_\eta = 0,$$

and one has

$$\lim_{N \to \infty} \sup_{(t,x) \in [0,T] \times 2^{-N} \mathbb{S}} | \xi^{d,N}(t,x) - Y^d(t,x) | = 0$$

Then there exists a sequence of diverging constants such that the sequence of $u^N$ converges in law to $u$ as a member of $\mathcal{C}^\bar{\eta}_{\bar{N}}$ for all $\bar{\eta} \in (0, \frac{1}{2} \wedge \eta)$.

**Remark 1.2** The motivation behind this assumption and the convergence result comes from the work that Hairer and Erhard have done in [18], wherein they studied a discrete analogue of the parabolic Anderson model:

$$\partial_t u(x, t) = (\Delta u)(x, t) - \xi(x, t)u(x, t)$$

defined on $x \in \mathbb{Z}^d$. Unlike the usual (continuous) SPDE setup where the random noise comes from the space-time white noise, the authors in [18] instead solve a discretisation of the equation driven by a stationary symmetric simple exclusion process using discrete regularity structures, which they further show to converge to a parabolic Anderson model that is driven by a generalised Ornstein-Uhlenbeck process. The result they achieve is in $d = 3$, which is not surprising because for this spatial dimensionality, one expects the space-time white noise to be far too irregular to be amenable to the usual programme, while the Ornstein-Uhlenbeck process is more regular and hence susceptible to it.

**Remark 1.3** We adopt a flipping of the perspective. Instead of fixing a noise that drives the discretisation, we can begin with positing the bounds on the cumulants of the noise, for which the solution theory still works. A major work in [18] was to check that Assumption[1] holds for a stationary symmetric simple exclusion process. We assume these bounds were given to us.
Remark 1.4 As the noise no longer needs to be Gaussian, the second order moments are no longer sufficient in finding bounds on higher order moments and as such higher order cumulants are needed for the stochastic estimates central to the application of regularity structures in solving singular SPDEs. This means that more Feynman diagrams have to be controlled.

It is well established in the literature of (discrete) regularity structures that certain stochastic estimates are sufficient for ensuring convergence post renormalisation; one can refer to the prototypical Theorem 10.7 in [25] in the continuous setting. As we have remarked in the opening, these stochastic estimates have been automated for the continuum but in discrete settings such as these corresponding results are lacking. In lieu of those results, the authors in [18] appealed to the rather general bounds on labelled rooted binary trees presented in [30] Appendix A] to establish the stochastic estimates. The work presented here is motivated by the same but it turns out that there are certain estimates needed for the gKPZ that is not tractable under the existing framework. Indeed, one specific example is a singular integral with multiple points:

\[
I = \int f(x_0, x_1, \ldots, x_n) \prod_{i=1}^{n} K_i(x_i - x'_i) dx_1 \ldots dx_n
\]

where the \(x'_i\) are fixed and \(f\) is singular when the points \(x_0, \ldots, x_n\) are close. Our main new idea is to use a local transformation by introducing Taylor expansions via telescopic sums:

\[
I = \int f(x_0, x_1, \ldots, x_n) \left( K_1(x_1 - x'_1) - \sum_{k \leq n_f} \frac{(x_1 - x_0)^k}{k!} D^k K_1(x_0 - x'_1) \right) \prod_{i \neq 1} K_i(x_i - x'_i) dx_1 \ldots dx_n + \sum_{k \leq n_f} D^k K_1(x_0 - x'_1) \int f(x_0, x_1, \ldots, x_n) \frac{(x_1 - x_0)^k}{k!} \prod_{i \neq 1} K_i(x_i - x'_i) dx_1 \ldots dx_n.
\]

The first term contains a Taylor expansion that renormalised \(f\) up to its degree of divergence given by \(n_f\) whereas the second term is simpler as it contains less \(K_i\). Then, one iterates the process on all the \(K_i\) and ends up with terms of the form:

\[
\int f(x_0, x_1, \ldots, x_n) \prod_{i=1}^{n} (x_i - x_0)^k dx_1 \ldots dx_n
\]

that will be renormalisation constants by integrating out over the \(x'_i\) and if the term obtained is independent of \(x_0\). In the convergence theorem given in [30] [18], one can only treat the case with only one \(K_i\) which is enough for PAM but not sufficient for gKPZ as they are terms with two \(K_i\). Our contribution is to extend the [30]
result (compare our Ass. 2 with [30, Ass. A.1]) and couple it with a renormalisation procedure (see Section 5.3) that allows us to establish the erstwhile untractable stochastic estimates. Our approach is valid when one has to face subdivergences with no nested structures and with no overlap.

General estimates in the discrete setting will need a full extension of [13] or to push forward the approach advocated in this work, which we expect to be a very challenging task. Another route is to rely on a complete recursive proof of these estimates as it is performed with multi-indices in [38, 32, 33] or to use the local renormalisation approach developed in [9]. It has been used successfully in [11, 24] for writing the renormalised equation in a non-translation invariant setting.

**Overview of the paper:** In Section 1, we recall for the readers some major aspects of the theory of Discrete Regularity Structures; the presentation here follows very much that of [18]. We also briefly discuss the implementation of the renormalisation procedure, employing the recursive procedure found in [9].

In section 3, we present an extension of the bounds to be found in [30, Appendix A]. We show that one has certain bounds (see Theorem 3.1) on integrals over the so-called “Generalised Convolutions” (see (3.6)) under certain assumptions (see Assumption 2) and these bounds we prove via very general bounds (see Lemma 3.11) on rooted binary trees. The interest in these integrals stems from the fact that they can be specialised to integrals one needs to bound to get convergence in the Regularity Structure paradigm.

In Section 4, we present the notion of an "elementary graph" (see Definition 4.1) which is elementary in the sense that they can be morphed together to construct all the relevant graphs in our analysis. We also explain how to associate to the multiple integrals in $\Pi^N_0$ these elementary graphs. We will then present assumptions (see Assumption 4) under which Theorem 3.1 holds for the elementary graphs as well.

In Section 5, we use the bounds previously constructed to achieve the stochastic estimates to prove the convergence of the discrete gKPZ.

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## 2 Discrete Regularity Structures

A major tool in our analysis of the gKPZ is the theory of discrete Regularity Structures, as in [17]. In this section we recall the very basics to motivate the work we do in this paper. The presentation here is heavily inspired by [18, Section 2].

We start by recalling some notations. A scaling on $\mathbb{R}^{d+1}$ is the vector $\mathfrak{s} = \{s_0, s_1, \ldots, s_d\} \in \mathbb{N}^{d+1}_{\geq 1}$ and it allows the definition of a norm on $\mathbb{R}^{d+1}$ given by:
Discrete Regularity Structures

\[ \|z\|_s := \sup_{i \in \{0,\ldots,d\}} |z_i|^{1/s_i} \]

We also set \(|s| = \sum_{i=0}^d |s_i|\). In this paper it will always be the parabolic scaling, which is given by \(s = \{2, 1, \ldots, 1\}\). We will also find use of the following scaling function (and functional): for a \(\lambda > 0\), and \(\varphi : \mathbb{R}^{d+1} \to \mathbb{R}\), \(\delta^\lambda_{s_0}(z_0, z_1, \ldots, z_d) = (\lambda^{-s_0}z_0, \lambda^{-s_1}z_1, \ldots, \lambda^{-s_d}z_d)\) and \((\delta^\lambda_{s_0}\varphi)(y) = \lambda^{-|s|}\varphi(\delta^\lambda_{s_0}(y - z))\). Furthermore we denote by \((e_0, e_1, \ldots, e_d)\) the canonical basis of \(\mathbb{N}^{d+1}\).

For our purposes the role that H"{o}lder spaces play in the theory of regularity structures, will be played by the following discrete counter parts:

**Definition 2.1** For \(\eta \in (0, 1)\), we define discrete H"{o}lder spaces \(C^\eta_N(\mathbb{T}^d_N, \mathbb{R})\) as the space of all elements \(f \in \mathbb{R}^{d|\eta}_N\), with norm

\[ \|f\|_{C^\eta_N} := \sup_{x \in \mathbb{T}^d_N} |f(x)| + \sup_{x \neq y \in \mathbb{T}^d_N} \frac{|f(x) - f(y)|}{|x - y|^\eta}. \]

Let \(\eta \in (0, 1)\). To compare an element \(f \in C^\eta(\mathbb{T}^d, \mathbb{R})\) in the usual H"{o}lder space with an element \(f^N \in C^\eta_N(\mathbb{T}^d_N, \mathbb{R})\) we introduce the distance

\[ \|f - f^N\|_\eta := \sup_{x \in \mathbb{T}^d_N} |f(x) - f^N(x)| + \sup_{x \neq y \in \mathbb{T}^d_N} \frac{|(f(x) - f(y)) - (f^N(x) - f^N(y))|}{|x - y|^\eta} \]

\[ + \sup_{x, y \in \mathbb{R}^d; |x - y| < 2^{-N}} \frac{|f(x) - f(y)|}{|x - y|^\eta}. \]

To compare functions \(f \in C^\eta_0([0, T] \times \mathbb{T}^d, \mathbb{R})\) and \(f^N : [0, T] \times \mathbb{T}^d_N \to \mathbb{R}\), we define a “distance” by

\[ \|f - f^N\|_{C^\eta_0, T} := \sup_{(t, x) \in [0, T] \times \mathbb{T}^d_N} |f(t, x) - f^N(t, x)| + \sup_{(t, x), (s, y) \in [0, T] \times \mathbb{T}^d_N; \|(t, x) - (s, y)\| < 2^{-N}} \frac{|f(t, x) - f(s, y)|}{\|(t, x) - (s, y)\|_2^\eta} \]

\[ + \sup_{(t, x), (s, y) \in [0, T] \times \mathbb{T}^d_N; \|(t, x) - (s, y)\| \geq 2^{-N}} \frac{|(f(t, x) - f(s, y)) - (f^N(t, x) - f^N(s, y))|}{\|(t, x) - (s, y)\|_2^\eta}. \]

### 2.1 Discrete Models

**Definition 2.2 (Regularity Structures)** The pair \((\mathcal{F}, \mathcal{G})\) is called a regularity structure if one has the following:

- \(\mathcal{F}\) is a graded vector space \(\bigoplus_{\alpha \in A} T_\alpha\), where \(A\) is a locally finite and bounded from below index set, \(A \subset \mathbb{R}\), \(T_\alpha\) is a Banach space. To each element \(\tau_\alpha \in T_\alpha\), we associate the notion of a homogeneity: \(\|\tau\| = \alpha\).
**$G$** is a group of continuous linear operators defined on $T$ in such a manner that one has for every $\Gamma \in G$ and $\tau_\alpha \in T_\alpha$:

$$\Gamma \tau_\alpha - \tau_\alpha \in \bigoplus_{\beta < \alpha} T_\beta$$

We will always use **blue** for elements of any given regularity.

To construct the Regularity Structure $\mathcal{F}$ we will need, we begin with a bigger structure $\mathcal{F}$ first. This is defined recursively by beginning with $\{1, X_0, X_1, \ldots, X_d, \Xi \} \subset \mathcal{F}$. Here $(X_0, X_1, \ldots, X_d)$ corresponds to $(t, x_1, \ldots, x_d)$ with the parabolic scaling and for $k \in \mathbb{N}_{d+1}^+$, and we use the term $X^k$ to represent $X_0^{2k_0}X_1^{k_1} \cdots X_d^{k_d}$. Upon this set we require the existence of a product, i.e. $\tau_1, \ldots, \tau_n \in \mathcal{F}$, then $\tau_1 \cdots \tau_n \in \mathcal{F}$, which is assumed to be associative and commutative. $\Xi$ is included here to represent the noise. Finally we require that if $\tau \in \mathcal{F} \setminus \{1, X^k : k \in \mathbb{N}_{d+1}^+\}$ then $\{\mathcal{F}(\tau), \mathcal{F}_i(\tau), \ldots, \mathcal{F}_{\ell}(\tau)\} \subset \mathcal{F}$, where in general for $k \in \mathbb{N}_{d+1}^+$, $\mathcal{F}_i(\cdot)$ represents convolution with the kernel differentiated $k_i$ times in the $i$-th component and in particular $\mathcal{F} := \mathcal{F}_{\{0,0,\ldots\,0\}}$.

To each $\tau \in \mathcal{F}$, we associate a homogeneity $|\tau|_s$ which belongs to the reals. This is also defined recursively by setting $|\Xi|_s = -\frac{3}{2} - \kappa$, $|X_i| = 1$ unless $i = 0$ in which case $|X_0| = 2$, $|1|_s = 0$ and then requiring $|\tau_1 \cdots \tau_n|_s = |\tau_1|_s + \ldots + |\tau_n|_s$ and

$|\mathcal{F}_i(\tau)|_s = |\tau|_s + 2 - |k|_s$.

The recursive nature of this definition makes the space far too large. We instead restrict ourselves to those symbols that we expect to be able to extract from $\mathcal{F}$:

$$\mathcal{R} = \{X^k \mathcal{F}(\cdot)^\ell, X^k \mathcal{F}(\cdot)^j \mathcal{F}_i(\cdot), X^k \mathcal{F}(\cdot)^j \mathcal{F}_i(\cdot) \mathcal{F}_j(\cdot), X^k \mathcal{F}(\cdot)^j \Xi, \quad (2.3)
\}

k \in \mathbb{N}_{d+1}^+, \ell \in \mathbb{N}, i, j \in \{1, \ldots, d\}$$

Then we define:

$$\mathcal{H} = \{\tau \in \mathcal{F} : \tau = R(\tau_1, \ldots, \tau_n), R \in \mathcal{R} \text{ and } \tau_1, \ldots, \tau_n \in \mathcal{H} \text{ or } \tau = \Xi\}$$

and finally $\mathcal{F} = \text{Vec } \mathcal{H}$. $\mathcal{H}_\alpha$ is naturally the set of all symbols in $\mathcal{H}$ that have a homogeneity of $\alpha$, and then $\mathcal{F}_\alpha$ is then the span of $\mathcal{H}_\alpha$, whereby $A$ is the just the set of all of these $\alpha$.

It is known that for a small enough choice of $\kappa$, $\mathcal{F}_\alpha$ is finite dimensional for all $\alpha$. Finally we impose $\mathcal{F}(X^k) = 0$ for $k \in \mathbb{N}_{d+1}^+$ because the function such a convolution would realise would be smooth and we already have the polynomial regularity structure to describe smooth functions.

With a Regularity Structure fixed, one needs to define a model on it. One fixes first $\mathcal{X}_N = \mathcal{D}(\mathbb{R}, \mathbb{R}^{T_N^+})$, by which we mean the set of all càdlàg functions from $\mathbb{R}$ to $\mathbb{R}^{T_N^+}$. We also define an inclusion map $\iota_N : \mathcal{X}_N \rightarrow \mathcal{S}'(\mathbb{R}^{d+1})$:

$$(\iota_N f)(\varphi) = 2^{-dN} \sum_{x \in T_N^+} \int f(t, x) \varphi(t, x) \, dt. \quad (2.4)$$

On $\chi_N$ we define a family of seminorms $\| \cdot \|_{\alpha; k; \iota_N; z; N}$ in the following way:
||f||_{α; R_N; z; N} = \sup_{λ \in (0,2^{-N}]} \sup_{φ \in \Phi^N_{α, z}} λ^{-α} \left| \int f(s, x)(S^λ_{2, t} φ)(S) \right|

Here α ranges over IR, R_N ranges over all compact subsets in IR^{d+1} with diameter bounded by 2^{-N+1}, and z = (t, x) ∈ IR × TR_N is such that z ∈ R_N. With r > | min A| and λ ∈ (0, 2^{-N}] fixed, \Phi^N_{α, z} the set of all functions φ : IR → IR with \|φ\|_φ ≤ 1 and support contained in the unit ball so that supp \phi \succeq \{ s ∈ IR : (s, x) ∈ R_N \}. Further by \| · \|_φ we will denote the norm of the \ell-th component in \mathcal{T}. With these ingredients we define:

**Definition 2.3 (Discrete Model)** A discrete model consists of a collection of maps

z → Π^N_{z} ∈ \mathcal{L}(\mathcal{T}, \mathcal{X}_N) and Γ_N : IR^{d+1} × IR^{d+1} → \mathcal{Y} such that:

- \Gamma^N_{x z} ≡ id the identity operator, and the \Gamma^N_{x y} Γ^N_{y z} = Γ^N_{x z}, ∀ x, y, z ∈ IR.
- \Pi^N_z = \Pi^N_N Γ^N_{y z}, ∀ y, z ∈ IR^{d+1}

Such that the following estimates hold, for any compact set R ⊂ IR^d and every \tau ∈ \mathcal{T}_t:

\left| (λ \Pi^N_{z} τ)(S^λ_{2, t} φ) \right| \lesssim ||τ||_{\ell} λ^\ell, \quad ||\Pi^N_{z} τ||_{\ell; R_N; z; N} \lesssim ||τ||_{\ell}, \tag{2.5}

and

\left| \Gamma^N_{z z'} \tau \right|_m \lesssim ||τ||_{\ell} ||z - z'||_{s}^{\ell - m}, \quad ||z \mapsto \Gamma^N_{z z'} \tau||_{\ell; R_N; z; N} \lesssim ||τ||_{\ell} \tag{2.6}

for some fixed γ > 0, uniformly over λ ∈ (2^{-N}, 1], for all φ ∈ \Phi^N_{α, z}, all \ell ∈ A such that \ell < γ and m < \ell, all τ ∈ \mathcal{T}_t, and locally uniform over z, z' ∈ R such that \|z - z'||_s ∈ (2^{-N}, 1) and over all compact subsets R of diameter bounded by 2^{-N+1}.

**Remark 2.4** The norm || · ||_{\ell; R, N} has not yet been defined. Further below we replace it with another norm on functions of the form f : IR^{d+1} → IR > γ that we do explicitly state, when we discuss the “ℓ = 0”-hyperplane.

Central to the application of Regularity Structures to the study of semilinear SPDEs is the Green’s function of the differential operator of the equation. For the (KZP) this is ∂t - ΔN. It is known, from [10] Sec. A.1 say, that one can decompose the rescaled Green’s function of this operator as follows:

\[ 2^d N G^N = K^N + R^N, \tag{2.7} \]

where the function K^N can be decomposed as K^N = \sum_{n=1}^N K_n where each K_n has a support contained in the set \{ z ∈ IR^{d+1} : ||z|| \lesssim 2^{-n} \}, and annihilate polynomials of scaled degree less than or equal to two. For each n ∈ \{ 1, · · · , N - 1 \}, and each multi-index |k|_s ≤ 2,

\[ |D^k K_n(z)| \lesssim 2^n ||z||^{-2+|k|_s} \]
uniformly in all parameters. If \( n = N \), then the same estimate holds for \( k = 0 \).

What this inequality means that this function can be extended to a function in \( \mathbb{R}^{d+1} \), for which the estimate is true. \( R_N \) is compactly supported, is anticipative, and \( \| R_N \|_{q2} \) is bounded uniformly in \( N \), with the same interpretation for the bound as before. Let \( \zeta \in \mathbb{R} \) and \( n \leq N \), we define:

\[
(T_{n,\zeta+2}^N F)(z) = \sum_{|k|_d < \zeta + 2} \frac{X_k}{k!} \mathcal{Q}_k((T_{n,\zeta+2}^N F)(z)),
\]

where for \( z = (t, x) \in \mathbb{R}^{d+1}, n \in \{1, \ldots, N\} \), and \( |k|_d < \zeta + 2 \),

\[
\mathcal{Q}_k((T_{n,\zeta+2}^N F)(z)) = \begin{cases} 2^{-dN} \sum_{y \in \mathbb{T}_N^d} \int D_k^1 K_n(z, (s, y)) F(y, s) \, ds, & \text{if } n < N, \\ \delta_k = 2^{-dN} \sum_{y \in \mathbb{T}_N^d} \int K_N(z, (s, y)) F(y, s) \, ds, & \text{otherwise}, \end{cases}
\]

for all \( F \in \mathcal{X}_N \) for which the above expression makes sense.

**Definition 2.5 (Admissible Models)** Given \( K^N \) as above, we define the set of admissible models \( \mathcal{M} \) as consisting of all models such that for every multi-index \( k \):

\[
(\Pi^N_{(t,x)} X^k)(s, y) = (\langle s, y \rangle - (t, x))^k
\]

such that:

\[
(\Pi^N_{(t,x)} \mathcal{K}(\tau))(s, y) = 2^{-dN} \sum_{\bar{x} \in \mathbb{T}_N^d} \int K^N(\langle s, y \rangle, (\bar{s}, \bar{x}))(\Pi^N_{(t,x)} \tau)(\bar{s}, \bar{x}) \, d\bar{s} - \sum_{|k|_d < |\tau| + 2} \frac{((s, y) - (t, x))^k}{k!} \mathcal{Q}_k((T_{|\tau|+2}^N \Pi^N_{(t,x)} \tau)(t, x)).
\]

where the following shorthands have been employed:

\[
T_{\zeta+2}^N F = \sum_{n=1}^N T_{n,\zeta+2}^N F, \quad \text{and} \quad \mathcal{Q}_k((T_{n,\zeta+2}^N F)(\cdot)) = \sum_{n=1}^N \mathcal{Q}_k((T_{n,\zeta+2}^N F)(\cdot)).
\]

The standard way of constructing an admissible model \( [25] \) is, to begin with:

\[
(\Pi^N_{(t,x)} \Xi)(s, y) = \xi^N_s(y),
\]

for \( (t, x) \in \mathbb{R}^{d+1} \) and \( (s, y) \in \mathbb{R} \times \mathbb{T}_N^d \) and \( \Pi^N_{(t,x)} X^k \) as in \( [2.2] \) and the relationship:

\[
\Pi^N_{(t,x)} \tau^\tau = \Pi^N_{(t,x)} \tau \Pi^N_{(t,x)} \tau^\tau.
\]

To construct a solution theory, we extract from the equation an abstract fixed problem, and for that we need a notion of a \( \mathcal{T} \)-valued functions on our physical space. Recall that \( \Pi_{\zeta} \tau \) is to be understood as a local description of a possibly globally defined
distribution around \( z \) and motivated by this we would like our \( \mathcal{G} \)-valued function \( f \) to be defined in such a manner that \( \Pi_z f(z) \) is able to mimic a global distribution locally around \( z \) for all \( z \in \mathbb{R}^{d+1} \). To this end, we define first the “\( t = 0 \)”-hyperplane via \( P = \{(t, x) \in \mathbb{R}^{d+1} : t = 0\} \), and \( ||z||_x = 1 \wedge \inf_{y \in P} ||z - y||_x \) and \( ||y, z||_P = ||y||_P \wedge ||z||_P \), and finally:

\[
\mathcal{R}_P = \{(y, z) \in (\mathcal{R} \setminus P)^2 : y \neq z, \ ||y - z||_x \leq ||y, z||_P \}
\]

for some set \( \mathcal{R} \subset \mathbb{R}^{d+1} \).

Further for fixed \( \gamma > 0, f : \mathbb{R}^{d+1} \setminus P \to \mathcal{G}_{<\gamma} \). Then one defines for any compact set \( \mathcal{R} \subset \mathbb{R}^{d+1} \):

\[
\|f\|_{\gamma, \eta; \mathcal{R}; N} = \sup_{z \in \mathcal{R} \setminus P} \sup_{P \times \mathbb{T}^d_N, \gamma < \gamma} \|f(z)\|_{\beta} + \sup_{y, z \in \mathcal{R} \setminus P} \|f(z) - \Gamma_{\gamma, \eta} f(y)\|_{\beta},
\]

with \( \Pi_z f(z) \) being the lift of the smooth functions \( f, g, h, k, f \) as defined in \([17] \) Section 5.2, the abstract

\[
\mathcal{D}_N f(z) = (\Pi^N z f(z)),
\]

for \( z \in \mathcal{R} \times \mathbb{T}^d_N \).

With this diversion settled, we return to our fixed point equation. For \([gKPZ]\), the fixed point equation that is to be solved is given by:

\[
U^N = \mathcal{G}^N (F_\tau U^N 1_{\{t \geq 0\}}) + (G^N u_0^N)
\]

(2.16)

where \( \mathcal{G}^N \) is to be understood as the abstract version of convolution with the (rescaled) heat kernel \( G^N \) such that it satisfies the property:

\[
\mathcal{R}^N G^N = G^N
\]

(2.17)

and \( F_\tau (\tau) = (G(\tau) \mathcal{G} \tau)^2 + H \mathcal{D} \tau + K(\tau) + F(\tau) \Xi \) with \( G, H, K, F \) being the lifts of the smooth functions \( g, h, k, f \) as defined in \([17] \) Section 5.2, the abstract
derivative operator \( \mathcal{D} \) as seen in [17] Section 5.3 and the products of the objects in \( \mathcal{D}^{\gamma,\eta}_N \) made sense of as in [17] Section 5.1. Finally, following [17] Section 4 and the particular discretisation in [18], \( G^N u^N_0 \) is defined in the following manner:

\[
(G^N u^N_0)(t, x) = \sum_{|k|_s < \gamma} \frac{X^k}{k!} Q_k((G^N u^N_0)(t, x)),
\]

where \( Q_k((G^N u^N_0)(t, x)) \) is given by

\[
\begin{align*}
2^{-dN} \sum_{n < N} \sum_{y \in T^N_n} \int D^k_1[K_n + R^N_n](t, x, (s, y)) u^N_0(y) \, ds, & \quad \text{if } |k|_s > 0, \\
2^{-dN} \sum_{y \in T^N_n} \int [K^N_n + R^N_n](t, x, (s, y)) u^N_0(y) \, ds, & \quad \text{otherwise}.
\end{align*}
\]

That (2.16) is wellposed comes from (a slight modification of) the work in [17], where the authors introduce a notion of “coherence with non-linearity” in Definition 3.20, prove in Lemma 3.21 that being coherent with non-linearity for \( U^N \) is equivalent to \( U^N \) solving (2.16), and finally in [17] Section 2.8.1 check the coherence condition for (2KPZ). Moreover, it turns out that \( \mathcal{R}^N U^N \) solves (or perhaps more accurately coincides) with the solution of (2KPZ). At this juncture to show the convergence of the equations, one might think that we should just consider the convergence of the model, but it so transpires that the sequence fails to converge in space of admissible models. The remedy to this situation is an application of a set of continuous transformations \( \hat{\mathcal{M}}^N \) to the model \((\hat{\Pi}^N, \hat{\Gamma}^N) = (\mathcal{M}^N, \Pi^N, \Gamma^N)\) so that it has a renormalisation effect. A general construction of the underlying renormalisation group has been achieved in [17], and we will outline some of the features in the following section and show that our renormalisation procedure is completely consistent with the contraction extraction method they employ.

Before we explicate this renormalisation group, we remark that due to the recursive nature of an admissible model, a very natural graphical scheme is used in the literature as a shorthand to avoid cumbersome notation. For \( \Xi \) we use the notation \( \circ \). For the abstract integration of the previous symbol - \( \mathcal{F}(\Xi) \) - we add a downward facing edge to the previous diagram, \( \hat{\gamma} \), and for \( \mathcal{F}_i(\Xi) \), for \( i \in \{1, \ldots, d\} \), we signal the presence of a derivative by making the edge thicker \( \hat{\gamma} \) without distinguishing between which direction the derivative is taken in. For this reason we will use the notation \( \mathcal{F}'(\Xi) \) for \( \mathcal{F}_i(\Xi) \) with \( i \) unspecified. Multiplication of symbols is represented by joining them at the root, so for example one would have \( \mathcal{F}(\Xi) \mathcal{F}(\Xi) \) to represent \( \mathcal{F}(\Xi) \mathcal{F}(\Xi) \), except for multiplication of \( \tau \) by \( X \) in which case one uses the symbol \( \cdot \). If \( \Xi \) is in \( \tau \), one uses the symbol \( \bullet \) instead. In Table 1 we compile all the symbols with homogeneity less than or equal to naught whenever there’s a thick line in the symbol, it is meant to represent all of the \( d \) trees with the derivatives taken along different spatial dimensions.

### 2.2 Renormalised Model

In this subsection, we get into the definition of the renormalisation map announced in the previous section. The systematic study of the implementation of such a
renormalisation is due to [6], where (2.14) is redefined because in general we expect $(\Pi_\epsilon)^N(\tau)(\Pi_\epsilon)^N(\tau)$ to diverge. The work [6] uses heavily extended decorations which are not needed for the semi-general convergence theorem, we want to implement. Instead, we follow the approach with the local renormalisation map introduced in [9]. The works [11, 22] based on this approach give a more general proof of the renormalised equation introduced in [S].

To showcase the formulation of that result, we introduce decorated trees. Pick two symbols $\mathcal{J}$ and $\Xi$ and let $\mathcal{D} := \{\mathcal{J}, \Xi\} \times \mathbb{N}^{d+1}$ define the set of edge decorations. Decorated trees over $\mathcal{D}$ are of the form $T^n = (T, n, e)$ where $T$ is a non-planar rooted tree with node set $N_T$ and edge set $E_T$. The maps $n : N_T \to \mathbb{N}^{d+1}$ and $e : E_T \to \mathcal{D}$ are node, respectively edge, decorations. We denote the set of decorated trees by $\Sigma$. The tree product is defined by

$$(T', n', e') \cdot (T, n, e) = (T \cdot T', n + n', e + e'), \quad (2.20)$$

where $T \cdot T'$ is the rooted tree obtained by identifying the roots of $T$ and $T'$. The sums $n + n'$ mean that decorations are added at the root and extended to the disjoint union by setting them to vanish on the other tree. Each edge and vertex of both trees keeps its decoration, except the roots which merge into a new root decorated by the sum of the previous two decorations. We now make the connection with the symbolic notation introduced in the previous part.

- An edge decorated by $(\mathcal{J}, a) \in \mathcal{D}$ is denoted by $\mathcal{J}_a$. The symbol $\mathcal{J}_a$ is also viewed as the operation that grafts a tree onto a new root via a new edge with edge decoration $a$. The new root at hand remains decorated with $0$.
- An edge decorated by $(\Xi, 0) \in \mathcal{D}$ is denoted by $\Xi$.
- A factor $X^k$ encodes a single node $\bullet^k$ decorated by $k \in \mathbb{N}^{d+1}$. We write $X_i$, $i \in \{0, 1, \ldots, d\}$, to denote $X^{e_i}$. The element $X^0$ is identified with the empty
tree $\mathbb{1}$.

**Remark 2.7** Notice that the noise symbol $\Xi$ was depicted as a node in the previous section, but is seen as a decorated edge above. The result from [6] is of more general applicability, in particular to cases where one might have any finite number of noises attached to the same inner node, for which the notion of a noise edge is natural. For [eKPZ], each inner node is at most attached to one instance of noise, so for notational ease, we prefer illustrating the presence of the noise with a node. Of course for this work, the equivalence is obvious.

The degree $|\cdot|_s$ of a decorated tree is the one applied to its symbolic notation. The following is easily seen to be true:

$$|\tau_{\bar{\tau}}|_s = |\tau|_s + |\bar{\tau}|_s$$

We restrict ourselves to the trees that are generated by the graphical algorithm outlined in the last section from symbols residing only in $\mathcal{H}$. The collection of these trees is denoted by $\mathcal{T}_0$.

We say a tree $T \in \mathcal{T}$ is planted if either $T = \Xi$ or there exists $k \in \mathbb{N}^{d+1}$ and $\bar{T} \in \mathcal{T}$ such that $T = f_k(\bar{T})$, where $\bar{T}$ is the symbol associated to $\bar{T}$.

The set $\mathcal{T}_-$ is then the set of unplanted trees of negative degree and zero polynomial decoration at the root, which is to say:

$$\mathcal{T}_- := \{ \tau = T^n \in \mathcal{T}_0 : |\tau|_s < 0, n(\partial_T) = 0, \tau \text{ is not planted} \}.$$ 

We set $\mathcal{T}_- := \text{Vec} \mathcal{T}_-$ and then define the local extraction/contraction map $\Delta^-_{r}$ that will be central to the construction of the renormalisation map. It was first introduced in [9].

**Definition 2.8** We define for $T^n \in \mathcal{T}_0$

$$\Delta^-_{r} T^n_{T} = \sum_{A \subset T} \sum_{\varepsilon_A, n_A} \frac{1}{\varepsilon_A^!} \binom{n}{n_A} \pi_-(A, n_A + \pi \varepsilon_A, \varepsilon \upharpoonright E_A)$$

$$\otimes (T / A, [n - n_A]_A, \varepsilon + \varepsilon_A),$$

where:

- $\pi_- : \mathcal{T} \to \mathcal{T}_-$ which projects every $A \in \mathcal{T} \setminus \mathcal{T}_-$ to 0,
- For $C \subset D$, and $f : D \to \mathbb{N}^{d+1}$, the restriction of $f$ to $C$, is denoted by $f \upharpoonright C$.

Further one defines by $f! = \prod_{x \in D} f(x)!$ where $n! = \prod_{i=0}^d n_i!$ for $n \in \mathbb{N}^{d+1}$.

Given another $g : C \to \mathbb{N}^{d+1}$, we denote:

$$\binom{f}{g} = \prod_{x \in C} \binom{f(x)}{g(x)},$$

and unsurprisingly $\binom{n}{m} = \prod_{i=0}^d \binom{n_i}{m_i}$ for $k, n \in \mathbb{N}^{d+1}$.
• The outer sum is over all subtrees $A$ of $T$ which have the same root as $T$. The inner sum runs over all $n_A : N_A \to \mathbb{N}^{d+1}$ with $n_A \leq n$, and $e_A : \partial(A, T) \to \mathbb{N}^{d+1}$, where the boundary of $A$ in $T$ is defined by:

$$\partial(A, T) := \{(x, y) \in E_T \setminus E_A : x \leq y, x \in A\}$$

• For $f : E_T \to \mathbb{N}^{d+1}$, we set for every $x \in N_T$, $(\pi f)(x) = \sum_{(x,y) \in E_T} f(x,y)$

• We write $T/A$ for the tree obtained by contracting $A$ to the root $\varrho_T$ of $T$. For $f : N_T \to \mathbb{N}^{d+1}$ we define $[f]_A : N_{T/A} \to \mathbb{N}^{d+1}$ by $[f]_A(\varrho_T) = \sum_{y \in N_A} f(y)$ and $[f]_A(x) = f(x)$ for $x \in N_{T/A} \setminus \{\varrho_T\}$.

The definition above gives us a simple representation for the renormalisation map $R_g$ that we will use for the model:

$$R_g = (g \otimes \text{id})\Delta^-$$

with $g$ a linear map from $\mathcal{T}_-$ into $\mathbb{R}$.

**Example 2.9** Consider the simple tree given by: $T = \varrho^0$, for which one can check that $|T|_g = -1$. Then an easy computation gives:

$$\Delta^- \varrho^0 = 1 \otimes \varrho^0 + \varrho^0 \otimes 1, \quad R_g \varrho^0 = g(1) \varrho^0 - g(\varrho^0) 1$$

where we perform the following identification for the tensor product $a \otimes \tau = a\tau$

with $a \in \mathbb{R}$. As $g$ is a character one has directly that $g(1) = 1$ for any choice of $g$.

The BPHZ renormalisation given in [6] can be recovered by a specific choice of $g$:

$$g_{\text{BPHZ}}(\tau) = -\mathbb{E}(\langle \Pi^{N,g,\tau}(0) \rangle) \quad (2.22)$$

where one has:

$$\Pi^{N,g,\tau} = \hat{\Pi}^{N,g} R_g \tau, \quad \hat{\Pi}^{N,g,\bar{\tau}} = \hat{\Pi}^{N,g} \tau \hat{\Pi}^{N,g} \bar{\tau}.$$

$$(\hat{\Pi}^{N,g,\tau}(\pi))(s, y) = 2^{-dN} \sum_{x \in T_N^+} \int K^N((s, y), (\bar{s}, \bar{x}))(\Pi^{N,g,\tau})(\bar{s}, \bar{x}) d\bar{s}.$$}

This recursive formula has been introduced in [23] in a non-translation invariant setting. Then, the renormalised model is described recursively by a similar formula:

$$(\hat{\Pi}^{N,g}_{(t,x)} X^k)(s, y) = ((s, y) - (t, x))^k \Pi^{N,g}_{(t,x)\tau} = \hat{\Pi}^{N,g}_{(t,x)} R_g \tau,$$

$$(\hat{\Pi}^{N,g}_{(t,x)} \mathcal{F}(\pi))(s, y) = 2^{-dN} \sum_{x \in T_N^+} \int K^N((s, y), (\bar{s}, \bar{x}))(\Pi^{N,g}_{(t,x)\tau})(\bar{s}, \bar{x}) d\bar{s}$$

$$- \sum_{|k|_k < |\tau| + 2} \frac{(s, y) - (t, x))^k}{k!} Q_k((T^N_{|\tau|+2} \hat{\Pi}^{N,g}_{(t,x)\tau})(t, x)).$$
**Example 2.10** For \( \tau = \sigma^\rho \), let us compute \( g(\sigma^\rho) \) under the BPHZ paradigm. Due to (2.22) and the definition of \( \Pi^{N,g} \) following it, the quantity that we are interested in is:

\[
E[(\Pi^{N,g}_\sigma(\sigma^\rho))(0)] = E[ (\Pi^{N,g}_\sigma(\sigma^\rho))(\Pi^{N,g}_\sigma(\sigma^\rho))(0) ] = E[\int K^N(-z)\xi^N(z)\xi^N(0)\,dz] \\
= \int K^N(-z)E[\xi^N(z)\xi^N(0)]\,dz
\]

where we have used the shorthands \( z = (t, x) \in \mathbb{R}^{d+1} \) and \( \xi^N(z) = \xi^N(x) = \xi_{2N,d}(2^N x) \) and the integral is the semi-discrete integral i.e. integrals over space are actually the Riemann sums \( \sum_{x \in \mathbb{Z}_N^d} \). Notice that due to assumption of centrality on our noise and the properties of cumulants, we have \( E[\xi^N(z)\xi^N(0)] = E_c[\xi^N(z),\xi^N(0)] \). One can check that the constant \( c_N \) in \([QXL sec\,RNS]\) associated to \( \tau \) is the same as the quantity above and our renormalisation map achieves the same effect theirs does.

In the manner explained above one can compute the counter terms \( C_N \) associated with any \( \tau \in \mathcal{F}_\tau \). The following example explains the departure due to our noises being (generally) non-Gaussian from the usual homogeneous Gaussian case.

**Example 2.11** Consider the tree: \( \tau = \sigma^\rho_0 \). One checks then that:

\[
\Delta r^\rho_0 = 1 \otimes \sigma^\rho_0 + \sigma^\rho_0 \otimes \sigma^\rho + \sigma^\rho_0 \otimes 1
\]

We have already seen how to compute \( g(1) \) and \( g(\sigma^\rho) \), the new calculation at hand is \( g(\sigma^\rho_0) \). A very similar computation as in the previous example gives us:

\[
E[(\Pi^{N,g}_\sigma(\sigma^\rho_0))(0)] = \int K^N(-z_1)K^N(z_1 - z_2)E[\xi^N(z_1)\xi^N(z_2)\xi^N(0)]\,dz_1dz_2
\]

Were our noises centred Gaussian, the above expectation would have been zero due to the centredness and Wicks Formula. In the non-Gaussian case, the above integral is no longer guaranteed to vanish. As we have assumed control of the cumulants, we again use the assumption of centred noises to give \( E[\xi^N(z_1)\xi^N(z_2)\xi^N(0)] = E_c[\xi^N(z_1),\xi^N(z_2),\xi^N(0)] \). Hence we find that this constant is the same as \( c_N^{(1)} \) in \([QXL sec\,2.3]\).

We have thus seen how the counter-terms that renormalise our equation are generated for each of the symbols in Table[4] taking care to substitute the relevant cumulants for the expectation of the noise. In subtracting these counter terms, one cannot expect that our admissible model solves, in the manner described in the previous section, the equation \([KPZ]\) we stated in the introduction. The "renormalised equation" that it does solve, which is due to \([3\,Th.\,2.22]\), takes the form:

\[
\partial_t u = \Delta u + g(u)(\nabla_x u)^2 + k(u)(\nabla_x u) + \bar{h}(u) + f(u)\xi_t(x)
\]
where $h$ is given by:

$$h(u) = h(u) + \sum_{\tau \in \mathcal{F}} \frac{\Upsilon[\tau]}{S(\tau)} C_N(\tau).$$

Where the $C_N$ are found exactly in the manner we have explained above. The definitions of the symmetry factors and the elementary differential operators are directly from [3 Sec 2.7]. Let’s mention that [3] uses heavily decorated trees with extended decorations while [11 23] proposes a shorter proof valid for local renormalisation maps and without the extended decorations.

3 Labelled Graphs

It is known from the literature on Discrete Regularity Structure [17 28] that the sort of convergence we seek above relies on certain stochastic estimates on $\Pi^N$ and $\Pi$. As the general theory that exists in the continuum for getting these stochastic estimates from [13] remains out of reach for the discrete context, we follow the method in [18] build on the theory from [30]. Our contribution here is to extend the theory so that it is applicable to (gPZ).

3.1 Directed MultiGraph and Contracted Graphs

The main ingredient will be finite directed multigraphs $G = (V, E)$, with edges $e = (e_-, e_+) \in E$ that originate from $e_- \in V$, culminate at $e_+ \in V$, that carry the label $(a_e, r_e, v_e) \in \mathbb{R} \times \mathbb{Z} \times V$, and are associated with compactly supported kernels $K_e : \mathbb{R} \times \mathbb{T}_d \to \mathbb{R}$ that satisfy the following bound:

$$|D^k K_e(z)| \lesssim \|z\|_a^{-a_e - |k|}$$

uniformly over $\|z\|_a \leq 1$ and for all multi-indices $k$. In this sense, $a_e$ is the order of the singularity of the kernel $K_e$ associated to the edge. The quantity $r_e, v_e$ will be used to define the following Kernel:

$$\hat{K}_e(x_{e_-}, x_{e_+}) = K_e(x_{e_+} - x_{e_-}) - \sum_{|j|_s < r_e} \frac{(x_{e_+} - x_{v_e})^j}{j!} D^j K_e(x_{v_e} - x_{e_-})$$

In the above definition, one notices the first major departure from the [30] and all derivative literature. The only meaningful value for $r_e$ to take here is non-negative, whereas in [30] the handling of $r_e < 0$ is very non-trivial. Furthermore, in [30] the role of $v_e$ is played by the distinguished vertex 0 (see below), whereas we allow it to be any $v_e \in V$. We will show later how this can be used for a renormalisation effect. In the case that we have multiple edges with the same origin and destination vertices, we will assume only one has a non-zero $r_e$ label, and we will concatenate all the edges into one edge with the label $(\hat{a}_e, r_e)$ where $\hat{a}_e$ denotes the sum of all the $a_e$ for the edges that were concatenated. We require $G$ to have the distinguished vertex $0 \in V$ that will be connected by distinguished edges $e_{*,1}, \ldots, e_{*,M}$ to $M$.
We assume that the following assumptions hold on our graphs:

\[ \text{Assumption 2} \quad \text{For a given graph } G = (V, E) \text{ we require:} \]

- For every subset \( \bar{V} \subset V \), such that \( |\bar{V}| \geq 2 \), one has:
  \[
    \sum_{e \in E_{\text{out}}(\bar{V})} \hat{a}_e + \sum_{e \in E_{\text{in}}(\bar{V})} 1_{\{v_e \in \bar{V}\}}(\hat{a}_e + \hat{r}_e - 1) - \sum_{e \in E_{\text{in}}(\bar{V})} 1_{\{v_e \in \bar{V}\}} \hat{r}_e < |\bar{V}| - 1 \quad |s| \quad (3.3)
  \]

- For every non-empty subset \( \bar{V} \subset \hat{V} \setminus V_* \) one has:
  \[
    \sum_{e \in E_{\text{out}}(\bar{V})} \hat{a}_e + \sum_{e \in E_{\text{in}}(\bar{V})} (1_{\{v_e \in \hat{V} \setminus V_* = 0\}}(\hat{a}_e + \hat{r}_e - 1) - (\hat{r}_e - 1))
    + \sum_{e \in E_{\text{in}}(\bar{V})} (|\hat{a}_e + \hat{r}_e| - 1_{\{v_e \in \bar{V}\}} r_e) > |\bar{V}| |s|
  \]

An initial observation to make here is that these assumptions are different from the ones in [18] Assumption A.1. We forgot the milder conditions on integrability (i.e. Assumption A.1.1. and A.1.2. in [18]) but the "renormalisation" and "recentring" conditions we retain. The conditions are very similar but here we have to account for the fact that renormalisation is not only affected by keeping track of \( r_e \) but also \( v_e \). Why we need \( v_e \in \hat{V} \), will be clearer to the reader once we have explained our renormalisation procedure. Given our kernels \( K_e \), we will also assume the existence of the following particular decomposition:

**Assumption 3** Given \( K_e \) as above, we assume that there exist \( \{K_e^{(n)}\}_{0 \leq n \leq N} \) satisfying:

- \( K_e(z) = \sum_{0 \leq n \leq N} K_e^{(n)} \) for all \( z \neq 0 \).
- For all \( 0 \leq n < N \) the functions \( K_e^{(n)} \) are supported in the annulus \( 2^{-(n+2)} \leq \|z\|_s \leq 2^{-n} \) and \( \sup(K_e^{(N)}) \leq \{z : \|z\|_s \leq 2^{-N}\} \).
- For all \( p < \infty \) and some \( C < \infty \)

\[
    \sup_{0 \leq n \leq N - 1} 2^{-(a_e + |k|)n} |D^k K_e^{(n)}(z)| \leq C \quad \text{and} \quad 2^{-a_e N} |K_e^{(N)}(z)| \leq C; \quad (3.5)
\]
where the $N$ in the above definition is as it was fixed in $Z^d_N$. The largest $C$ that appears in \(3.5\) will be denoted by $\|K_e\|_{\alpha,p}$. For smooth test function $\varphi$ and $z = (t, x)$, we set

$$\varphi_{\lambda, \mu}(z) = \lambda^{-|s|s_1}|\mu|^{-|s_1|}\varphi(t/|s|, x/\lambda).$$

This is another departure from [SP], but this time towards more generality as seen in [18]. One has to take care to distinguish between the two regimes: $2^{-N} \leq \lambda = \mu$ or $\mu < 2^{-N} = \lambda$. In the former, $\mu = \lambda$ regime, the arguments are almost the same as in [18], so we explicate on the more demanding case.

The purpose of this section is to be able to bound Generalised Convolutions of the following form:

$$\mathcal{J}^G(\varphi_{\lambda, \mu}, K) := \int_{[\mathbb{R} \times T^d_N \setminus 0]} \prod_{e \in E} \hat{K}_e(z_{e_-, e_+}) \prod_{i=1}^M \varphi_{\lambda, \mu}(z_{v_{i}}) \, dz. \quad (3.6)$$

where $T^d_N = 2^{-N} Z^d_N$. When $\lambda = \mu$, we simply write $\mathcal{J}^G(\varphi, K)$. The main result (adapted to this setup) provides exactly this:

**Theorem 3.1** Let $G = (V, E)$ be a finite directed multigraph with labels $(a_e, r_e, v_e)$ and kernels $K_e$ with $e \in E$, such that Assumption 2 are satisfied and that the kernels satisfying Assumption 3. Then, there exists $C$ depending only on the structure of the graph $(V, E)$ and the labels $r_e$ such that

$$\mathcal{J}^G(\varphi_{\lambda, \mu}, K) \leq C\lambda^{\hat{a}}, \quad (3.7)$$

if either $2^{-N} \leq \lambda = \mu$ or $\mu < 2^{-N} = \lambda$. Here,

$$\hat{a} = |s||V \setminus V_s| - \sum_{e \in E} a_e. \quad (3.8)$$

We do not prove directly Theorem 3.1 but we rewrite it via a multi-scale decomposition of the kernels $\hat{K}_e$. We need to introduce some notations to present this expansion. The first building block is to define a cutoff function $\Psi(n)$ for every $n \in \mathbb{Z}$ using a smooth function supported on $[3/8, 1]$, $\psi : \mathbb{R} \rightarrow [0, 1]$ such that $\sum_{n \in \mathbb{Z}} \psi(2^n x) = 1$ for every $x \neq 0$. We set for every $n \in \mathbb{N}$: $\Psi(n) = \psi(2^n x)$. 

**Definition 3.2** For any $n_1 : E \setminus E_+ \mapsto \{0, 1, \ldots, N - 1, N\}^3$ and $e \in E \setminus E_+$, we define a function $\hat{K}_{e}^{(n_1)(e)}(z_1, z_2)$ as follows: if $r_e \leq 0$ then $\hat{K}_{e}^{(n_1)(e)} = 0$ unless $n_1 = (k, 0, 0)$ in which case $\hat{K}_{e}^{(n_1)(e)}(z_2 - z_1)$; if $r_e > 0$, $\hat{K}_{e}^{(n_1)(e)} = 0$, unless $n = (k, p, m) \in \{0, 1, \ldots, N - 1\}^3$, in which case we have:

$$\hat{K}_{e}^{(k,p,m)}(z_1, z_2) = \Psi(k)(z_2 - z_1)\Psi(p)(z_{v_e} - z_1)\Psi(m)(z_2 - z_{v_e})$$

$$\left(\hat{K}_e(z_2 - z_1) - \sum_{|j| < r_e} \frac{(z_2 - z_{v_e})^j}{j!} D^j K_e(z_{v_e} - z_1)\right)$$

or $n_1(e) = (N, 0, 0)$ in which case:

$$\hat{K}_{e}^{(n_1)(e)}(z_1, z_2) = \Psi(N)(z_2 - z_1)K_e(z_2 - z_1)$$
Note that the main difference in our treatment is in defining $\hat{K}_{e}^{(k,p,m)}$. Finally to avoid issues of differentiability at $t = 0$, we also define for a function $n_{2} : E_{*} \mapsto \{0,1,\ldots, \lceil \log_{2}(\mu) \rceil \}$ and $e \in E_{*}$, the kernel $\hat{K}_{e}^{(n_{2}(e))}$ as being zero unless $n_{2}(e) = (k,0,0)$ in which case $\hat{K}_{e}^{(n_{2}(e))} = \Psi^{(k)}(z_{2} - z_{1})K_{e}^{(k)}(z_{2} - z_{1})$. The multiplicity in the definitions allows the following definition:

$$\hat{K}^{(n)}(z) = \prod_{e \in E} \hat{K}^{(n_{e})}_{e}(z_{e_{-}}, z_{e_{+}}),$$  \hspace{1cm} (3.9)

where we have defined:

$$n_{e} = \begin{cases} n_{1}(e) & \text{if } e \in E \setminus E_{*} \\ n_{2}(e) & \text{if } e \in E_{*} \end{cases}$$

Whenever $n$ is given in such a form we write $n = n_{1} \cup n_{2}$. For $0 < \mu < \lambda = 2^{-N} \leq 1$, we set:

$$N_{\lambda,\mu} := \{ n = n_{1} \cup n_{2} : 2^{-|n_{2}(e_{i})|} \leq \mu, i = 1,\ldots,M \},$$  \hspace{1cm} (3.10)

And then finally:

$$\mathcal{F}^{G}_{\lambda,\mu}(K) := \sum_{n \in N_{\lambda,\mu}} \int_{(\mathbb{R}^{d} \times \mathbb{R}^{d}) \setminus \mathbb{V}_{0}} \hat{K}^{(n)}(z) \, dz .$$  \hspace{1cm} (3.11)

The benefit of the above definition is that the following theorem, when established will give us Theorem 3.1. Indeed assume that we know the following theorem to be true. Then the facts that test functions can be interpreted as kernels with $a_{e} = 0$ and that $\|K_{e}\|_{a_{e};p} \lesssim \lambda^{-|s_{i}|-|s_{1}|}$ allow us to conclude.

**Theorem 3.3** Under the same assumptions as in Theorem 3.1 the inequality

$$|\mathcal{F}^{G}_{\lambda,\mu}(K)| \leq C \lambda^{\alpha} \langle|s_{i}|2^{-N}|s_{1}|\rangle^{M} \prod_{e \in E} \|K_{e}\|_{a_{e};p},$$  \hspace{1cm} (3.12)

holds. If $\lambda = \mu$, we also have:

$$|\mathcal{F}^{G}_{\lambda,\mu}(K)| \leq C \lambda^{\alpha} \prod_{e \in E} \|K_{e}\|_{a_{e};p},$$  \hspace{1cm} (3.13)

In both cases $\alpha = |s| \|V_{0}\| - \sum_{e \in E} a_{e}$.

In the next section we prove Theorem 3.3 and hence Theorem 3.1.
3.2 Partition of Integral Domain

As in [30], we want to associate to every point \( z \in (\mathbb{R} \times \mathbb{T}_N^d)^V \) a rooted labelled binary tree, where by rooted tree we mean a connected graph without cycles, which has a distinguished node called the root, which is labelled in the sense that a mapping from the set of its edges into reals exists and finally binary in the usual graph theoretic way.

We denote by \( \mathcal{B}(V) \) the set of all labelled rooted binary trees which have \( V \) as their set of leaves. We further define a partial order by saying that for inner nodes \( \nu \) and \( \omega \) in a tree in \( \mathcal{B}(V) \), \( \nu \geq \omega \) means that \( \omega \) lies on the shortest path from the root to \( \nu \). We write \( \nu \wedge \omega \) for the most recent ancestor of \( \nu \) and \( \omega \). The main departure here from [30] which has been introduced in [18], is that when \( \nu \) is the most recent ancestor of two elements in \( V \) in lieu of the usual node labelling \( \ell \) we assign to it the pair of node labels \( (\ell_{\nu \wedge \omega}, \ell_{\nu}) = (\lceil |\log_2(\mu)| \rceil, N) \). This change is necessitated by the fact that under the scale \( 2^{-N} \) space does not matter and the time variable does not need to be controlled beyond scale \( \mu \). Now we may construct our trees so as to be able to impose:

\[
\ell^\alpha(v) \geq \ell^\beta(w) \text{ for } v \geq w,
\]

where the \( \alpha \) and \( \beta \) can either be empty (in which case we mean \( \ell \)) or it could be \( T \) (by which we mean \( \ell^T \)). The set of labelled trees that is constructed in this manner is denoted by \( \mathcal{T}(V) \) with a general element denoted by \((T, \ell, \ell^T)\).

**Example 3.4** Consider for example \( V = \{v_1, v_2, v_3\} \). The following diagram is of one possible example of a rooted tree on it (with \( r \) denoting the root).

Here for example we have \( \ell^\alpha_{v_1 \wedge v_2} \geq \ell^\beta_{v'} \) exactly because \( v' \) lies on the path from \( r \) to \( v_1 \wedge v_2 \).

The labelled trees \((T, \ell, \ell^T)\) partition the domain of integration and the labelling \( \ell \) is further such that for \( v, w \in V \), we have:

\[
\|z_v - z_w\|_b \sim 2^{\ell_{v \wedge w}}. \tag{3.14}
\]

For the above bound we have to replace \( \ell_{v \wedge w} \) by \( \ell^T_{v \wedge w} \) if \( v, w \in V_\ast \).

**Definition 3.5** Set \( c := \log|V| + 2 \). We define \( \mathcal{N}(T, \ell, \ell^T) \) as the collection of all functions \( n = n_1 \cup n_2 \) such that:

- for every edge \( e = (v, w) \in E \setminus E_\ast \), if one has \( n_e = (k, 0, 0) \) with \( |k - \ell_{v \wedge w}| \leq c \), alternatively if one has \( n_e = (k, p, m) \) with \( |k - \ell_{v \wedge w}| \leq c \), \( |p - \ell_{v \wedge w}| \leq c \), and \( |m - \ell_{v \wedge w}| \leq c \).
• for every edge \( e = (v, w) \in E_* \) with \( n_e = (k, 0, 0) \) one has \(|k - \ell_{v \wedge w}| \leq c\).

With this definition, one has the following result:

**Lemma 3.6** For every \( n : \tilde{E} \to \mathbb{N}^3 \) such that \( \tilde{K}^{(n)} \) as defined as before is non-vanishing, there exists an element \( (T, \ell, \ell^T) \in \mathcal{B}(\tilde{V}) \) with \( n \in N(T, \ell, \ell^T) \).

**Definition 3.7** Denote by \( \mathbb{T}_{\lambda, \mu}(V) \) those subsets in \( \mathbb{T}(V) \) such that \( 2^{-\ell_{v \wedge w}} \leq \mu \) for all \( v, w \in V_* \).

One can now use Lemma 3.6 (which is the extension of [30] Lemma A.9) and is proven in virtually the same manner) to turn the sum over \( N_\lambda \) into a sum over \( \mathbb{T}_{\lambda, \mu}(V) \):

**Lemma 3.8** For \( \mathcal{F}^G_{\lambda, \mu}(K) \) as in 3.11, we have the following bound

\[
|\mathcal{F}^G_{\lambda, \mu}(K)| \lesssim \sum_{(T, \ell, \ell^T) \in \mathbb{T}_{\lambda, \mu}(V)} \sum_{n \in N(T, \ell, \ell^T)} \left| \int_{\mathbb{T}(V_0)} \hat{K}^{(n)}(z) \, dz \right|.
\]  

(3.15)

For the proof of Lemma 3.8 one can refer to [18] Lemma A.9.

As is usual, we represent by \( T^o \) the interior nodes of \( T \), while setting \( T_n^o \) as the set of nodes \( \nu \) such that \( \nu = v_* \wedge w_* \) for some \( v_*, w_* \in V_* \), and \( \mathcal{D}(T, \ell, \ell^T) \subset (\mathbb{R} \times \mathbb{T}_N^o)^V \) such that \( \|z_v - z_w\|_2 \leq |V|2^{-\ell_{v \wedge w}} \), unless \( v, w \in T_0^o \) in which case we assume that \( \|z_v - z_w\|_2 \leq |V|2^{-\ell_{v \wedge w}} \). In the same vein, we are able to define \( N_{\lambda, \mu}(T^o) \), where instead of having a labelling fixed and the trees constituting the elements of the collection, we fix a tree and ask for all the labellings for which the previous construction holds true. With this notation we get the following volumetric bound:

**Lemma 3.9** With \( \mathcal{D}(T, \ell, \ell^T) \) as above, and given a collection of functions \( \hat{K}^{(n)} \)

indexed by \( (T, \ell, \ell^T) \) and \( n \in N(T, \ell, \ell^T) \), such that for each label:

\[
\text{supp}(\hat{K}^{(n)}) \subset \mathcal{D}(T, \ell, \ell^T), \quad \int_{(\mathbb{R} \times \mathbb{T}_N^o)^V} \hat{K}^{(n)}(z) \, dz = \int_{(\mathbb{R} \times \mathbb{T}_N^o)^V} \hat{K}^{(n)}(z) \, dz
\]

then the volume (in the lebesgue sense) of \( \mathcal{D}(T, \ell, \ell^T) \) is bounded:

\[
\mu(\mathcal{D}(T, \ell, \ell^T)) \lesssim \prod_{v \in T^o \setminus T^o_2} 2^{-\ell_{|v|}} \prod_{v \in T^o_2} 2^{-\ell_{|v|} - \ell_{|v|} |s_1|}
\]

**Proof (Sketch).** Firstly replace \( \hat{K}^N \) by \( \hat{K}^N \) and then \( \mathbb{R} \times \mathbb{T}_N^o \) by \( \mathcal{D}(T, \ell, \ell^T) \) in the domain of the integrals above. Then we recall that the elements of \( V \) are leaves to a suitably chosen \( T \) so one is able to find a spanning tree for \( V \) of the following form:

\[
\{(v_-, v_+) : v_-, v_+ \in V; v = v_- \wedge v_+; v \in T^o \}. \quad \text{By definition then} \quad \mathcal{D}(T, \ell, \ell^T) \text{ is then contained in the set of all } z, \text{ such that } \|z_v - z_{v_+}\| \leq |V|2^{-\ell_v}, \text{ with the edges } (v_-, v_+) \text{ coming exactly from the spanning tree. Now it is a matter of taking supremums and intergrating over all the coordinates turn by turn, to achieve the above bound.} \]
Remark 3.10 The construction of such a \( \tilde{K}^{(n)} \) is obviously a non-trivial point but we suppress that in this work. We assume that such a \( \tilde{K}^{(n)} \) has been found and \( \tilde{K}^{(n)} \) has been redefined to be that.

We digress here to introduce the general construction that will be pivotal for the bounds we are looking for. Given a rooted binary tree \( T \) with some distinguished leaves \( 0, v_*, 1, \ldots, v_*, M \) and a distinguished node \( \nu_\star \), we set \( T^0 \) as the set of inner nodes of \( T \) and \( \mathcal{N}_{\lambda, \mu}(T^0) \) has the same description as before. Given two functions \( \eta, \eta^T : T^0 \to \mathbb{R} \), we write:

\[
J_{\lambda, \mu}(\eta, \eta^T) = \sum_{(\ell, \ell^T) \in \mathcal{N}_{\lambda, \mu}(T^0)} \prod_{v \in T^0} 2^{-\ell_v \eta_v - \ell_v^T \eta_v^T}
\]

Set \( |\eta| = \sum_{v \in T^0} \eta_v \) and similarly for \( \eta^T \). Then we have:

Lemma 3.11 If \( \eta \) satisfies the following:

- For every \( v \in T^0 \), one has the \( \sum_{v \geq v} \eta_v > 0 \)
- For every \( v \in T^0 \) such that \( v \leq \nu^* \), one has \( \sum_{v \neq v} \eta_v < 0 \), if this sum contains at least one term.

and \( \eta^T \) is such that \( \eta_v^T \neq 0 \Leftrightarrow \ell_v^T > 0 \) and that necessarily \( \eta_v^T \) is positive in this case, then one has the bound \( J_{\lambda, \mu}(\eta, \eta^T) \lesssim \lambda^{|n|} |\mu| |\eta^T| \), uniformly over \( 0 < \mu < 2^{-N} = \lambda \).

Proof. Refer to [18, Lemma A.10]

Returning to the task at hand we notice that a consequence of Lemma 3.9 is the following bound:

\[
|J_{\lambda, \mu}(K)| \lesssim \sum_{(T, \ell, \ell^T) \in \mathcal{N}_{\lambda, \mu}(\mathcal{V})} \sum_{n \in \mathcal{N}(T, \ell, \ell^T)} \left( \prod_{v \in T^0 \setminus T^*_2} 2^{-\ell_v |s_v|} \prod_{v \in T^*_2} 2^{-\ell_v^T |s_v| - \ell_v |s_v|} \sup_z |\tilde{K}^{(n)}(z)| \right)
\]

With regards to the right hand side of the bound above, we can make a further observation:

\[
\sup_z |\tilde{K}^{(n)}(z)| \leq \prod_{e \in E} \sup_x |\tilde{K}_e^{(n_e)}(z_{e_-}, z_{e_+})|
\]

which means that we have essentially reduced our problem to bounding \( \tilde{K}_e^{(n_e)}(z_{e_-}, z_{e_+}) \).

It is here that we would like to use Lemma 3.11. To this end, we first make the following definitions. With \( e^\dagger := e_+ \wedge e_- \), define the following configuration: \( A^+, A \) and \( A^- \) as follows:
and by $A_0$ those edges such that $r_e = 0$.

On the interior of $T$ consider $\eta : T^o \to \mathbb{R}$ given by

$$\eta(v) = \sum_{e \in E} \eta_e(v)$$

where:

$$\eta_e(v) = \frac{1}{|v|} \sum_{e \in A} (a_e \mathbb{1}_{e_+}(v) + \mathbb{1}_{e_+}(r_e \mathbb{1}_{e_+ \wedge v_e}(v) - (a_e + r_e) \mathbb{1}_{e_+}(v))$$

and

$$\eta^T(v) = \begin{cases} |s_1| & \text{if } v \in T^o_* \\ 0 & \text{otherwise}. \end{cases}$$

and finally $\tilde{\eta}(v) = |s| + \eta(v)$.

**Lemma 3.12** With $\hat{K}^{(n_e)}$ defined as above and $\eta_e$ and $\eta^T$ as in (3.19), we have the following bound:

$$\sup_z \left| \prod_{e = (e_-, e_+)} \hat{K}^{(n_e)}(z_{e_-}, z_{e_+}) \right| \lesssim \prod_{v \in T_0} 2^{-\ell_v \eta(v)}$$

for every edge $(e_-, e_+) \in \hat{E}$.

**Proof.** Let $e \in E$, and $n_e = (k, p, m)$. Recall that:

$$|k - \ell_{ze_\wedge z_e}| \leq c; \quad |p - \ell_{ze_\wedge z_v}| \leq c; \quad |m - \ell_{ze_\wedge z_v}| \leq c$$

(3.21)

when $r_e > 0$. With reference to [25 Proposition A.1] one is able to write:

$$\hat{K}^{(n_e)}(z_{e_-}, z_{e_+}) = \Psi^{(k)}(z_{e_+} - z_{e_-}) \Psi^{(p)}(z_{v_+} - z_{v_-}) \sum_{|r| = r_e} \int_{\mathbb{R}^{d+1}} D^r K_e(y) Q_e^r(x, dy)$$
where the kernel has the property:
\[ Q_k^T(z, \mathbb{R}^{d+1}) \lesssim \| z_{e_+} - z_{e_0} \|_2^{r_e} \]
We can bound this in different ways, depending on how \( m \) and \( k \) compare. We consider the following possibilities with some constant \( C_0 \):

- If \( m \geq k + C_0 \), then:
  \[
  \sup_z |\hat{K}_e^{(n_e)}(z)| \lesssim 2^{-r_e m + (a_e + r_e)k} \sim 2^{-r_e (\ell_{e_+} \wedge s_{e_0}) + (a_e + r_e)\ell_{e_+} \wedge s_{e_0}}
  \]

- If \( k \geq m + C_0 \), then:
  \[
  \sup_z |\hat{K}_e^{(n_e)}(z)| \lesssim 2^{a_e k} + \sum_{|j| < r_e} 2^{-m|j| + (a_e + |j|)p} \lesssim 2^{a_e k}
  \]

- If \( k \sim m \) then:
  \[
  \sup_z |\hat{K}_e^{(n_e)}(z)| \lesssim 2^{a_e k} + \sum_{|j| < r_e} 2^{-m|j| + (a_e + |j|)p} \lesssim 2^{(a_e + r_e - 1)p - (r_e - 1)m}
  \]

When \( r_e = 0 \), the following bound is straightforward:
\[
\sup_z |\hat{K}_e^{(n_e)}(z_{e_-, z_{e_+}})| \lesssim 2^{a_e k}.
\]

Notice finally that these above three cases correspond to the three configurations in (3.18). Then we have the following bound:
\[
\sup_z |\hat{K}_e^{(n_e)}(z_{e_-, z_{e_+}})| \lesssim \prod_{v \in T^e} 2^{-\ell_v \eta_v(v)}.
\]

**Lemma 3.13** For \( \eta \) and \( \eta^T \) as given in (3.19) and (3.20), the hypothesis of Lemma 3.11 is satisfied.

**Proof.** The first comment we make is that the condition on \( \eta^T \) is certainly met. Indeed, if \( \eta^T > 0 \), then necessarily it is \( \| z_1 \| \) and the corresponding \( \ell^T \) is \( \| \log_2(\mu) \| \) which is strictly positive, because \( \mu < 2^{-N} \) (the other regime falls in the domain of the original Hairer-Quastel result). The other direction follows similarly. Now let \( v \in T^e \) and we consider \( L_v \subseteq V \) the leaves attached to \( v \). We have:
\[
\sum_{u \geq v} \eta(u) = |z|(|L_v| - 1) + \sum_{u \geq v} \left( \sum_{e \in A} -a_e 1_{e_1}(v) + \sum_{e \in A^+} \left( (r_e 1_{e_+} \wedge s_{e_0}) - (a_e + r_e) 1_{e_+}(v) \right) \right) - \sum_{e \in A^-} \left( (a_e + r_e - 1) 1_{e_+ \wedge s_{e_0}}(v) - (r_e - 1) 1_{e_1}(v) + \sum_{e \in A_0} (-a_e 1_{e_1}(v)) \right) \]
\[
= |z|(|L_v| - 1) - \sum_{e \in E_0(L_v)} a_e - \sum_{e \in E^+(L_v)} 1_{v_e \in L_v}(a_e + r_e - 1) + \sum_{e \in E^-(L_v)} 1_{v_e \in L_v}(r_e) > 0
\]
We can compute the contributions depending on what configuration the edge is in:

\[
\begin{array}{ccc}
A^+ & a_e & 0 \\
A & a_e & 0 \\
A^- & a_e & 1_{v_e \in \mathcal{V}(r_e)} (a_e + r_e - 1) \\
A_0 & a_e & 0
\end{array}
\]

To check the second condition, we fix some node \( v \in T^\circ \) such that \( \nu_* \geq v \). Denote by \( U_v = \{ u \in T^\circ : u \not\geq v \} \) and \( \mathcal{V} \) the set of leaves attached to \( U_v \). We must have \( |\mathcal{V}| = |U_v| \) and then it follows:

\[
\sum_{u \in U_v} \tilde{\eta}(u) = |\mathcal{V}||\mathcal{V}| + \sum_{u \in U_v} \left( \sum_{e \in A} -a_e \mathbb{1}_{e}(v) + \sum_{e \in A^+} (r_e \mathbb{1}_{e \supseteq v_e}(v) - (a_e + r_e) \mathbb{1}_{e}(v)) \right) \\
- \sum_{e \in A^-} ((a_e + r_e - 1) \mathbb{1}_{e \supseteq v_e}(v) - (r_e - 1) \mathbb{1}_{e}(v)) + \sum_{e \in A_0} (-a_e \mathbb{1}_{e}(v))
\]

As before we calculate the relevant cases:

\[
\begin{array}{ccc}
A^+ & a_e & a_e + r_e - \mathbb{1}_{v_e \in \mathcal{V}(r_e)} \\
A & a_e & a_e \\
A^- & a_e & 1_{v_e \in \mathcal{V}(a_e + r_e - 1) - (r_e - 1)} \\
A_0 & a_e & a_e
\end{array}
\]

Then:

\[
\sum_{u \in U_v} \tilde{\eta}(u) \leq |\mathcal{V}||\mathcal{V}| - \sum_{e \in A^0(\mathcal{V})} \sum_{e \in E^i(\mathcal{V})} \left( \sum_{v \in v_e \in \mathcal{V}(r_e)} (\hat{a}_e + \hat{r}_e - 1) - (\hat{r}_e - 1) \right) \\
+ \sum_{e \in E^i(\mathcal{V})} ((\hat{a}_e + \hat{r}_e) - \mathbb{1}_{v_e \in \mathcal{V}}) r_e < 0
\]

\[\square\]

We can now conclude this section by putting all of this together and proving Theorem 3.3.

**Proof of Theorem 3.3.** From Lemma 3.12 and 3.17, we have the following:

\[
\sup_{z} |\mathcal{K}^{n}(z)| \lesssim \prod_{v \in B^\circ} 2^{-\ell_v} \eta(v)
\]

Inserting the above bound into (3.16), and recalling that \( 2^{-\ell_v} = 2^{-N} \) for all \( v \in T^\circ \), gives us the following bound:

\[
|\mathcal{F}_{\mathcal{A},\mu}(K)| \lesssim \sum_{(T,\ell,\ell^T) \in \mathcal{Y}_{\mathcal{A},\mu}(G)} \sum_{n \in \mathcal{N}(T,\ell,\ell^T)} \prod_{v \in T^\circ} 2^{-\ell_v} \eta(v) - \ell^T_v \eta_T(v) 2^N|s_1|M
\]

(3.22)

with \( \eta(v) := |\mathcal{S}| + \eta(v) \). To conclude now we need to appeal to Theorem 3.11, the hypothesis of which we have already checked in Lemma 3.13. \[\square\]
4 Elementary Labelled Graphs

In the previous section, we presented our primary tool for obtaining the bounds we need on our stochastic integrals. We want to be able to prove that the graphs that show up in the analysis of the eKZ satisfy Assumption[2]. It turns out that while checking condition (3.3) is very non-trivial, one is able to link to the labelled graphs of the previous section a new construction called elementary labelled graphs, that always satisfy (a version of) (3.4). In this section, we will give a recursive construction of these elementary graphs.

4.1 Construction of Elementary Labelled Graphs

Definition 4.1 (Elementary Labelled Graph) An elementary labelled graph is the graph $G = (V, E)$ connected with two distinguished vertices $V_* = \{v_0, v_*\}$ with an edge label: $(a_e, r_e, v_e) \in \mathbb{R} \times \mathbb{N} \times V$ such that:

- it is almost a tree in the sense that $\tilde{T} = (\hat{V}, \hat{E}) = (V \setminus \{v_0\}, E \setminus E^\uparrow(\{v_0\}))$ is a rooted tree with root $v_*$. Furthermore, we can ascribe to it a labelled tree, $T_{n}^e$, as follows:
  - $E_T = \hat{E}$, $N_T = N_T$, and $L_T = L_T$
  - for every edge $e \in E_T$ $a_e = |s| + |\ell(e)|_s - |t|_s$
  - for every edge $e \in \{e : e_\in E_T\}$, $a_e = |s| + \kappa$, $|\ell(e)|_s = |\ell(e)|_s = 0$ and $|t(e)|_s = -|s|/2 - \kappa = \alpha$, with $\kappa > 0$
  - for every edge $v \in N_T$, $(a_{v_0, v}, r_{v_0, v}, v_{v_0, v}) = (\ell(v), 0, v_0)$

- For every edge of $\tilde{T}$, one has: $r_e = \max(|T_e|_s \setminus 0)$ where $T_e$ is the tree above edge $e$. In set-theoretic notation we may write $T_e = (V_e, E_0(V_e))$, where $V_e = \{v \in V \setminus \{v_0\} : e_+ \wedge v = e_+\}$

Definition 4.2 (Homogeneity) To each type $\ell \in \ell$ we associate a homogeneity $|\ell|_s \in \mathbb{R}$. The homogeneity of a labelled tree $T_{n}^e$ then is given by:

$$|T_{n}^e|_s = \sum_{u \in L_{\ell} \cup E_T} |\ell(u)|_s + \sum_{x \in N_T} |e(x)|_s - \sum_{e \in E_T} |\ell(e)|_s$$

For an elementary labelled graph, $G$, the homogeneity is defined as:

$$|G|_s = \left(|\hat{N}_G| + \frac{|L_G|}{2} - 2\right) |s| - \sum_{e \in G} a_e$$

The following proposition is an easy consequence of the definitions:

**Proposition 4.3** Let $G$ be an elementary labelled graph and $T_{n}^e$ the labelled graph associated to it. Then $|T_{n}^e|_s = |G|_s$

Recall that the regularity structure in Section[2] was built up recursively from: $\Xi$ and $X$. As we intend to codify $\Pi_0^N T_{n}^e$ with elementary graphs, we built up our space of elementary labelled graph similarly from graphs that encode $\Pi_0^N \Xi$ and $\Pi_0^N X$: 

**Definition 4.4** For $\Xi, X \in \mathcal{T}$, we define the elementary labelled graphs as:

$$G(\Xi) = \begin{array}{c}
\bullet \\
\cdot \\
\cdot \\
\cdot
\end{array} \quad G(X) = \begin{array}{c}
\bullet \\
\cdot
\end{array}$$

With reference to the relation $\Pi_2^N \tau \cdot \Pi_2^N \tau = \Pi_2^N \tau \cdot \Pi_2^N \tau$, we define the product of elementary graphs in the following way:

**Definition 4.5** Given two Elementary Labelled Graphs $G_1 = (E_1, V_2)$ and $G_2 = (E_2, V_2)$ define a new Elementary Labelled Graph $\bar{G} := G_1 \otimes G_2 = (\bar{E}, \bar{V})$, by $\bar{V} := V_1 \cup V_2$ and $\bar{E} := E_1 \cup E_2$. With $v_0^i$ and $v_\star$ for $i \in \{1, 2\}$ denoting the distinguished nodes coming from $G_i$ and $v_0, v_\star$ the distinguished nodes from $\bar{G}$, we identify $v_0 \sim v_0^1 \sim v_0^2$ and $v_\star \sim v_\star^1 \sim v_\star^2$.

**Example 4.6** Consider the product $\Xi X$. Putting the previous two definitions together gives us:

$$G(\Xi X) = \begin{array}{c}
\bullet \\
\cdot \\
\cdot
\end{array}$$

Further with reference to (2.11) we define the integration of graphs in the following manner:

**Definition 4.7** The integration of an elementary graph $G = (V, E)$ is the elementary graph $\bar{G} = \mathcal{J}_{\otimes}(G)$ given by: $(\bar{V} \cup \{v_\star^i\}, \bar{E})$ where $\bar{E}$ is defined by:

- if $a_{\alpha} \neq 0$ then $\bar{E}' = E \cup \{(v_\star^i, v_0), (v_\star^i, v_\star)\}$ with the label $(v_\star^i, v_0)$ given by $(0, 0)$.
- else $\bar{E}' = (E \setminus \{(v_0, v_\star)\}) \cup \{(v_0, v_\star^i), (v_\star, v_\star^i)\}$. where the edge label for $(v_0, v_\star^i)$ and $(v_\star, v_\star^i)$ are given by $(0, 0)$ and $(|s| - |\mathcal{J}(\cdot)|_{\alpha} + |n|_{\alpha}, 0 \vee |\bar{\mathcal{J}}(\cdot)|_{\alpha}, v_0)$.

Consider the following examples of this construction:
Elementary Labelled Graphs

\[ G(\mathcal{F}(\Xi)) = G(\mathcal{F}(\Xi)) \odot G(\Xi) = v \]

We differentiate between the "integration" of the graph and the "integration" of the abstract symbol by colouring the latter in blue.

**Definition 4.8 (Subtree)** Given a elementary labelled graph \( G = (V, E) \), a subtree \( T = (\bar{V}, \bar{E}) \) is defined via the inclusions \( \bar{V} \subset V \) and \( \bar{E} \subset E \) and one defines the homogeneity of \( \bar{T} \) as:

\[
|\bar{T}|_s = \left( |\bar{V}| - 1 \right) |\bar{V}| - \sum_{e \in E_0(\bar{V})} a_e - \sum_{e \in E_1(\bar{V})} a_e - \sum_{e \in E_1(\bar{V})} \mathbb{1}_{\{a_e < 0\}} a_e \quad (4.2)
\]

Recursively one could generate any number of graphs, but in our analysis of gKPZ, we will only be concerned with those that are constructed with the specific set of rules given in (2.3). So for example, the rules do not allow for any powers of \( \Xi \). We will denote by \( \mathcal{G}_R \) the set of elementary graphs built from \( R \) and \( G(\Xi) \) where all the symbols, products and integrations are replaced by the counterparts we have given in Definitions 4.4, 4.5 and 4.7. This recursive construction matches the recursive construction of the model \( \Pi^N \) as we have a one to one correspondence between a subclass of elementary graphs and the iterated integrals generated by \( \Pi^{N,\tau} \).

\( \mathcal{R} \) is a locally subcritical set of rules, which we recall means that for any subtree \( T \) of any elementary labelled graph \( G \in \mathcal{G}_R \) such that it is not \( \Xi \), we always have that \( |T|_s > |\Xi|_s = \alpha \).

### 4.2 Bounds on Elementary Labelled Graphs

To use the bounds of the last section on these elementary graphs, we will want to use Theorem 3.1. To this end we will suggest bounds on Elementary Labelled Graphs, that will give us the bounds (3.3) and (3.4) and hence satisfy the hypothesis of the theorem in question.

**Assumption 4** 1. For every subset \( \bar{V} \subset V_0 \) one has:

\[
\sum_{e \in E_0(\bar{V})} a_e \leq \left( |\bar{V}| - \frac{1}{2} \right) |\bar{V}| + \kappa \quad (4.3)
\]

2. For every subset \( \bar{V} \subset V \) one has for \( |\bar{V}| \geq 3 \):

\[
\sum_{e \in E_0(\bar{V})} a_e + \sum_{e \in E_1(\bar{V})} \mathbb{1}_{\{v \in \bar{V} \land r_e > 0\}} (a_e + r_e - 1) - \sum_{e \in E_1(\bar{V})} \mathbb{1}_{\{v \in \bar{V}\}} r_e
\]

\[
< \left( |\bar{V}| - 1 \right) |\bar{V}| \quad (4.4)
\]

we treat the vertex 0 as an inner node.
3. For every non-empty subset $\overline{V} \subsetneq V_0$, one has:

$$
\sum_{e \in E_0(\overline{V})} a_e + \sum_{e \in E^1(\overline{V})} (1_{\{v_e \in \overline{V} \vee r_e = 0\}}(a_e + r_e - 1) - (r_e - 1))
+ \sum_{e \in E^1(\overline{V})} ((a_e + r_e) - 1_{\{v_e \in \overline{V}\}} r_e + 1_{\{v_e \in \overline{V}\}} |G|_s > (|\overline{V}| - 1_{\{v_e \in \overline{V}\}}) |s|)
$$

(4.5)

if $\overline{V} = V_0$, we obtain the equality in the previous bound which is the definition of $|G|_s$.

The following result tells us that (4.5) is always satisfied for the graphs generated from $G_{\mathbb{R}}$.

**Proposition 4.9** Let $G = (V, E) \in G_{\mathbb{R}}$ and $\overline{T} = (\overline{V}, \overline{E})$ a sub-tree of $G$ such that $\overline{V} \subsetneq V_0$ then $\overline{V}$ satisfies the assumption (4.5).

**Proof.** Let $\rho$ be the root of $\overline{T}$, we denote by $T_\rho$ the tree above $\rho$. If $\rho \neq v_*$, we replace $T_\rho$ by $T_u$ where $(u, \rho) \in E$. To each edge $e = (e_-, e_+) \in E^1(\overline{V})$, we can associate a tree $T_e = T_{e_-}$, and notice that we have the following equalities:

$$
T_\rho = \overline{T} \cup \bigcup_{v \in E^1(\overline{V})} T_v,
|T_\rho|_s = |\overline{T}|_s + \sum_{e \in E^1(\overline{V})} |T_e|_s
$$

Then in view of (4.2) and the second equality above we get:

$$
|T_\rho|_s = (|\overline{V}| - 1_{\{v_e \in \overline{V}\}}) |s| - \sum_{e \in E_0(\overline{V})} a_e - \sum_{e \in E^1(\overline{V})} a_e - \sum_{e \in E^1(\overline{V})} |T_e|_s
$$

(4.6)

By definition, we have $\sum_{e \in E^1(\overline{V})} r_e > |T_\rho|_s$ if $\rho \neq v_*$, else $|T_\rho|_s = |G|_s$ and $E^1(\overline{V}) = \emptyset$. Moreover, if $e \in E^1(\overline{V})$, it follows that: provided $|T_e|_s > 0$ we would have $|T_e|_s > r_e - 1$, else by local subcriticality of the graph $G$, we would have $|T_e|_s > -\frac{\kappa}{2} > -a_e$. The previous bounds can then be put together with (4.6) to give the following:

$$
\sum_{e \in E_0(\overline{V})} a_e + \sum_{e \in E^1(\overline{V})} (a_e + r_e) + \sum_{e \in E^1(\overline{V})} (1_{\{r_e = 0\}}(a_e + r_e - 1) - (r_e - 1))
+ 1_{\{v_e \in \overline{V}\}} |G|_s > (|\overline{V}| - 1_{\{v_e \in \overline{V}\}}) |s|
$$

\[\square\]

Let $G = (V, E) \in G_{\mathbb{R}}$ and $\overline{V} \subsetneq V_0$. We consider $\overline{G} = (\overline{V}, \overline{E})$ where $\overline{E} = E_0(\overline{V})$. Then the graph $\overline{G}$ admits the following decomposition: $\overline{G} = \bigcup_{j \in K} T_j$ where $T_j = (V_j, E_j)$ are disjoint subtrees of $G$ and $K$ is a finite set. Using this characterisation we have:
Proposition 4.10 Let $G = (V, E) \in \mathcal{G}_R$ and let $\bar{V} \subset V \setminus V'$. Then $\bar{V}$ satisfies the assumption (4.3).

Proof. We decompose $\bar{V} = \bigcup_{j \in K} V_j$ where $V_j$ are disjoint sets and $T_j = (V_j, E_0(V_j))$ is a subtree of $G$. Then we apply the previous proposition on each of $V_j$ and by summing the bounds, we obtain the required result. \hfill \Box

Proposition 4.11 Let $G = (V, E)$ a labelled graph and $\bar{V} \subset V$ such that $v_0 \in \bar{V}$ and such that $\bar{V} = V \setminus \bar{V}$ satisfies (4.5) then $\bar{V}$ satisfies the assumption (4.4).

Proof. We suppose that $\bar{V}$ does not satisfy (4.4) which yields:

$$\sum_{e \in E_0(\bar{V})} a_e + \sum_{e \in E'(\bar{V})} \mathbf{1}_{\{r_e > 0\}} (a_e + r_e - 1) - \sum_{e \in E'(\bar{V})} \mathbf{1}_{\{r_e > 0\}} r_e \geq (|\bar{V}| - 1)|s|$$

On the other hand, $\bar{V} = V \setminus \bar{V}$ satisfies (4.5):

$$\sum_{e \in E_0(\bar{V})} a_e + \sum_{e \in E'(\bar{V})} \left( a_e + r_e - \mathbf{1}_{\{v_e \in \bar{V}\}} \right) + \sum_{e \in E'(\bar{V})} \left( \mathbf{1}_{\{v_e \in \bar{V}\}} (a_e + r_e - 1) - (r_e - 1) \right) + \mathbf{1}_{\{v_e \in \bar{V}\}} |G|_s \geq \left( |\bar{V}| - 1 \right) |s|$$

We notice that $E'(\bar{V}) = E'(\bar{V})$ and $E'(\bar{V}) = E'(\bar{V})$, and hence by summing the last two bounds, we get:

$$\sum_{e \in E} a_e + \mathbf{1}_{\{v_e \in \bar{V}\}} |G|_s > \left( |\bar{V}| - 1 - \mathbf{1}_{\{v_e \in \bar{V}\}} \right) |s| \quad (4.7)$$

We also have:

$$\frac{|s|}{2} - \kappa \leq |G|_s = (|V| - 2)|s| - \sum_{e \in E} a_e$$

and then:

$$\sum_{e \in E} a_e \leq \left( |V| - \frac{3}{2} \right) |s| + \kappa$$

But this of course contradicts (4.7) for all sufficiently small $\kappa > 0$. It follows then that $\bar{V}$ satisfies the condition (4.4). \hfill \Box

We are now able to prove that all graphs in $\mathcal{G}_R$ satisfy (4.4).

Proposition 4.12 Every $G = (V, E) \in \mathcal{G}_R$ satisfies the condition (4.4) for $\bar{V} \subset V$ and $v_0 \in \bar{V}$. 

Proof. Let \( \bar{V} = V \setminus \bar{V} \). One checks easily that the graph \( G \) satisfies the condition 4.5 on \( \bar{V} \). By the previous proposition, \( \bar{V} \) satisfies 4.4.

Proposition 4.13 Let \( G = (V, E) \in \mathcal{G}_{\mathcal{R}} \) and \( \bar{T} = (\bar{V}, \bar{E}) = \bigsqcup_{j \in K} T_j \) such that \( \bar{V} \subseteq V_0 \). Then

- If \( |K| \geq 3 \) or \( |\bar{T}|_\sigma > 0 \) then the condition (4.4) is satisfied.
- If \( |K| = 2 \) and there exists \( j' \in K \) such that \( |T_{j'}|_\sigma > -\frac{|\sigma|}{2} - \kappa \) then the condition (4.4) is satisfied but \( |\bar{T}|_\sigma < 0 \) then the condition (4.3) is satisfied but (4.4) is not.

Proof. For every \( j \in K \), we denote by \( V_{j,i} \), the inner nodes of the tree \( T_j \). By subcriticality we have that \( |T_j|_\sigma \geq -\frac{|\sigma|}{2} - \kappa \), which yields:

\[
(|V_{j,i}| - 1)|\sigma| - \sum_{e \in E_{\sigma}(V_j)} a_e \geq |T_j|_\sigma \geq -\frac{|\sigma|}{2} - \kappa
\]

Wherefore:

\[
\sum_{e \in E_{\sigma}(V)} a_e \leq \left( |V_{j,i}| - \frac{1}{2} \right) |\sigma| + \kappa
\]

By summing the previous bounds we have:

\[
\sum_{e \in E_{\sigma}(V)} a_e = \sum_{j \in K} \sum_{e \in E_{\sigma}(V_j)} a_e \leq \sum_{j \in K} \left( |V_{j,i}| - \frac{1}{2} + \frac{\kappa}{|\sigma|} \right) |\sigma|
\]

\[
= \left( |V_\sigma| - \sum_{j \in K} \left( \frac{1}{2} - \frac{\kappa}{|\sigma|} \right) \right)
\]

We may go case by case as follows:

- If \( |K| \geq 3 \), we have \( \sum_{j \in K} \left( \frac{1}{2} - \frac{|\sigma|}{2} \right) > 1 \) which gives the result.
- If \( |\bar{T}|_\sigma > 0 \), then \( \sum_{j \in K} |T_j|_\sigma > 0 \) which gives the required bound.
- We can conclude similarly as before when \( |K| = 2 \) and when one of the \( T_j \) satisfies \( |T_j|_\sigma > -\frac{|\sigma|}{2} - \kappa \).
- For the last assertion when \( |K| = 1 \) and \( |\bar{T}|_\sigma < 0 \), the fact that \( |T_j|_\sigma > -\frac{|\sigma|}{2} - \kappa \) proves the condition (4.3) but \( |\bar{T}|_\sigma < 0 \) is in contradiction with (4.4).

Proposition 4.14 Let \( G = (V, E) \in \mathcal{G}_{\mathcal{R}} \) and \( \bar{T} = (\bar{V}, \bar{E}) \) a subtree of \( G \). we suppose that there exists a leaf \( \ell \) in \( \bar{T} \) such that \( \ell \notin V_\ell \) then if \( |\bar{V}| > 1 \), the condition (4.3) is satisfied.

Proof. We have \( |\bar{T}|_\sigma > -\frac{|\sigma|}{2} - \kappa \) and we consider the tree \( \hat{\bar{T}} \) where we have replaced the node \( \ell \) by a leaf which will count for \( |\sigma| \). We always have \( |\hat{\bar{T}}|_\sigma > -\frac{|\sigma|}{2} - \kappa \) but now \( |\hat{\bar{T}}|_\sigma = |\bar{T}|_\sigma - \frac{|\sigma|}{2} > -\frac{|\sigma|}{2} - \kappa \), which gives \( |\hat{\bar{T}}|_\sigma > -\kappa \) and \( |\bar{T}|_\sigma > 0 \). Indeed, the set of homogeneities is a discrete set. Above \( -\kappa \), the next homogeneity is 0 which contains only the trivial tree. Finally the subtree \( \bar{T} \) satisfies (4.3).
5 Solving the Generalised KPZ Equation

5.1 Bounds on the Discrete Model

To simplify the task of finding the bounds, we employ the graphical shorthand that is already well-established in literature. In particular we will use "\(*\)" to denote a special node that represents the origin and "\(\bigcirc\)" for an instance of the noise. Furthermore "\(\bullet\)" will denote dummy variables that are to be integrated out. Arrows of the form "\(-a_e, r_e, v_e\rightarrow\)" will denote Kernels with the label \((a_e, r_e)\) when \(v_e = 0\) in the interest of brevity, we will simply write \((a_e, r_e)\). We assume for our analysis that we are given the kernels \(K^N(\cdot, \cdot)\) for which the Assumption \([\textbf{3}]\) hold true. Given such kernels, we define the following functions that are kernels in the same sense:

\[
K^N(t - s, y - x), \ K^N(t - s, y - x) - K^N(-s - x),
\]

\[
K^N(t - s, y - x) - K^N(-s - x) - \sum_{i=1}^{d} y_i \partial_i K^N(-s - x), \ \sum_{i=1}^{d} \partial_i K^N
\]  \hspace{1cm} (5.1)

These kernels come with the labels \((|s| - 2, 0), (|s| - 2, 1), (|s| - 2, 2), ((|s| - 2) + 1, 0)\). In \([\textbf{18}]\) the authors work in \(d = 3\) so we see kernels with weights \((3, 0), (3, 1), (3, 2), (4, 0)\). Furthermore, we use the arrow: \(\vdots\), to represent the test function \(\varphi^\lambda\). Finally, a red polygon with \(p\) dots inside will represent a joint cumulant \(p\)-th order of \(\xi^N\). Example: \(\bigcirc\); \(\bullet\); represents a 4th order cumulant.

Example 5.1 Consider the following graph:

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\downarrow \\
\bullet \\
\end{array}
\]

The integration variables here comprise \(z_1 = (t_1, x_1), z_2 = (t_2, x_2), z_3 = (t_3, x_3)\) and the integral it represents is:

\[
\int \varphi^\lambda(z') [K(z_1 - z_2)K(z_1 - z_3)] \mathbb{E}_c[\xi^N_2(x_2), \xi^N_3(x_3)] dz
\]

where by the integral we mean the semi-discrete integral we introduced in Section \([\textbf{2}]\) over all the variables.

To see why cumulant appears in the integral above consider first: \(\Pi^Q_0 \bigwedge^P 0\). With reference to \([\textbf{2.11}]\) and \([\textbf{2.14}]\), we expect to see the integral:

\[
\int \varphi^\lambda(z') [K(z_1 - z_2)K(z_1 - z_3)] \xi^N_1(x_1) \xi^N_3(x_2) dz
\]

The problem with analysing the integral above is that the singularity of the noises means that the product of the noises is not defined. The classical approach in quantum
physics (for white noise at least) is to use Wicks products to renormalise the product. One can refer to [39], where C. G. Wick first introduced the Wick product, and then [27] where the authors use a version of the Wick product in the Stochastic Analysis setup. In [37] finally, the authors extended it to stochastic distributions. We know then by Definition [A.3] that: \( \xi_t^N(x_1)\xi_t^N(x_2) = :\xi_t^N(x_1)\xi_t^N(x_2): + E_c[\xi_t^N(x_1), \xi_t^N(x_2)] \). Substituting this into the integral above we get the sum:

\[
\int \varphi_0^\lambda(z')[K(z_1 - z_2)K(z_1 - z_3)] :\xi_t^N(x_2), \xi_t^N(x_3): \, dz
+ \int \varphi_0^\lambda(z')K(z_1 - z_2)K(z_1 - z_3)E_c[\xi_t^N(x_2), \xi_t^N(x_3)] \, dz
\] (5.3)

The integrals in (5.3) are then represented via the graph notation as follows:

\[
\begin{align*}
\quad 
\end{align*}
\] (5.4)

Looking at our definition of a model one can infer that the integrals we will be interested in, will always take the following form:

\[
\int \varphi_0^\lambda(z) \prod_j \mathcal{K}_j^N(z^{(j,2)}, z^{(j,1)}): \prod_k \xi_{id}^N(x^{(k)}): \, dz
\] (5.5)

where \( \mathcal{K}_j^N \) stands for any of the four possibilities in (5.1). The quantities of interest then will be the \( p \)-th moment of (5.5):

\[
\int E_c \left[ \prod_{i=1}^p \varphi_0^\lambda(z_i) \prod_j \mathcal{K}_j^N(z_i^{(j,2)}, z_i^{(j,1)}): \prod_k \xi_{id}^N(x^{(k)}): \right] \, dz_i
\]

We want to use the bounds on the cumulants to bound these moments. For this, we use Lemma [A.5] to transition to:

\[
\sum \prod_{i=1}^p \varphi_0^\lambda(z_i) \prod_j \mathcal{K}_j^N(z_i^{(j,2)}, z_i^{(j,1)}): \prod_{B \in \pi} E_c\left( [\xi_t^N(x) : (t, x) \in B] \right) \, dz_i
\] (5.6)

The graphical shorthands we have defined in the beginning of this section are used to codify (5.6) (for some fixed \( p \)) are illustrated as graphs generated via the following algorithm:

- take \( p \) copies of \( G \),
- fix a partition \( \pi \) of \( V^p \) (set of all the leaves) such that each \( B \in \pi \) contains at least two elements that from different copies of \( G \),
• for each $B \in \pi$ draw a red polygon with $|B|$ dots inside and connect these $|B|$ dots to the elements of $B$,
• sum over all expression obtained in this way.

We illustrate this with an example:

**Example 5.2** Consider for example the term $\Xi \mathcal{J}(\Xi)$. In its treatment (the manipulation being the same as in [20]) but the contractions refer to cumulants arising as in Example [5.1] one sees the following graphs:

\[
(\Pi_0^N \Xi \mathcal{J}(\Xi))(\phi_0^\lambda) = z^{(1)} - z^{(2)},
\]

(5.7)

The first graph in the above depiction for example represents the following integral (in the sense of a Riemann sum over the space points and the usual sense over time):

\[
\int \phi_0^\lambda(z)[K^N(z^{(1)} - z^{(2)}) - K^N(-z^{(2)})] : \xi_{\phi_0^\lambda}(x^{(1)})\xi_{\phi_0^\lambda}(x^{(2)}) : dz_i
\]

(5.8)

while the second encodes:

\[
\int \phi_0^\lambda(z_i)K^N(z^{(1)})\mathbb{E}_c[\xi_{\phi_0^\lambda}(x^{(i)}): i \in \{1, 2\}] dz_i
\]

(5.9)

in both the $dz$ means integration over both $z^{(1)}$ and $z^{(2)}$.

Our aim is to be able to bound graphs such as these and for graphs like $\Xi$ in (5.7) this is a straightforward matter. One notices that it is non-random and as such it is a straightforward application of Theorem [3.7]. As per Assumption [1] we can replace each cumulant kernel in $\Xi$, by a cycle of weight $\frac{3}{2} + \kappa$ and then check Assumption [2].

So for example one may check that for the subtree $\{z^{(2)}, z^{(1)}\}$, condition [3.3] reads: $|s| - 2 < (2 - 1)|s| = |s|$ which is of course true for all $s$. Similarly for $\{z^{(2)}, z^{(1)}, 0\}$ one check that it becomes: $|s| - 2 + |s| - 2 = 2(|s| - 2) < (3 - 1)|s| = 2|s|$, which again holds true always. Condition [3.4] is checked similarly. From thence it is only a matter of invoking Theorem [3.1].

The arguments needed for $\Xi$ in (5.7) become more complicated because now we have a random object instead of a deterministic object. Now one applies the general program we expounded on before. The family of quantities (indexed by $p$) we are looking to bound is given by:

\[
\int \mathbb{E} \left[ \prod_{i=1}^p \phi_0^\lambda(z_i)[K^N(z_i^{(1)} - z_i^{(2)}) - K^N(-z_i^{(2)})] : \xi_{\phi_0^\lambda}(x^{(1)})\xi_{\phi_0^\lambda}(x^{(2)}) : dz_i \right]
\]

(5.10)
To be able to use Assumption 3 we need to transition from expectations to cumulants in (5.10). Via Lemma A.3 one gets:

$$
\sum \int \prod_{i=1}^{p} \varphi_{0}^\lambda(z_i)[K^{N}(z_{i}^{(1)}) - z_{i}^{(2)} - K^{N}(-z_{i}^{(2)}) \prod_{B \in \pi} \mathbb{E}_{e}(\{ \xi_{e}^{N}(x) : (t, x) \in B \})] dz
$$

with $\pi$ running over all partitions in $\mathcal{P}_{1,2}(\{1, 2\} \times \{1, 2, \ldots, p\})$. Let us now illustrate the algorithm for constructing illustrations of (5.11). Let us fix $p = 2$, which means we begin by taking two copies of $\textcircled{1}$ in (5.7). With the four possible leaves, the only partition that makes sense is the one where we pair one leaf from either copy with the other. This results in:

The next step would be to bound each of these diagrams. It was achieved via Proposition 4.7 in the original paper through a general criterion that when met checks automatically Assumption 2 for all the graphs generated for the $p$ moments. In the section that follows we present our version of that proposition.

### 5.2 General Criterion

Fix a graph $G$, like the first one in (5.7). As in [18] we would like to leverage the fact that the "cumulant" kernels appearing in the cumulant terms are replaced with cycles with some prescribed weight. To formulate the conditions we will use $V_\xi$ which we define as the subset of $V$ comprising interior nodes such that they are attached to an instance of the noise.

**Proposition 5.3** Starting with a graph $G = (V, E)$ for which we have replaced each cumulant term by a simple weighted cycle with weight $3/2 + \eta$, if the following bounds are met for every subset $\bar{V} \subseteq V_0$:

$$
\sum_{e \in E_0(\bar{V})} a_e + \frac{3}{2} |\bar{V} \cap V_\xi| \leq \left( |\bar{V}| - \frac{1}{4} \right) |s| + \kappa
$$

$$
\sum_{e \in E_0(\bar{V})} a_e \leq \left( |\bar{V}| - \frac{1}{2} \right) |s| + \kappa
$$

then all the $p$-th moments of $G$ satisfy (4.3).

Further if for every subset $\bar{V} \subseteq V$, such that $|\bar{V}| \geq 3$, one has the following bounds:

$$
\sum_{e \in E_0(\bar{V})} a_e + \sum_{e \in E^{-}(\bar{V})} \mathbb{1}_{\{v_e \in V \forall r_e > 0\}}(a_e + r_e - 1) - \sum_{e \in E^{-}(\bar{V})} \mathbb{1}_{\{v_e \in V \}} r_e + \frac{3}{2} |\bar{V} \cap V_\xi| < \left( |\bar{V}| - 1 \right) |s|
$$
\[
\sum_{e \in E_0(\bar{V})} a_e + \sum_{e \in E^1(\bar{V})} \mathbb{1}_{\{v_e \in \mathbb{V} \forall r_e > 0\}}(a_e + r_e - 1)
- \sum_{e \in E^1(\bar{V})} \mathbb{1}_{\{v_e \in \bar{V}\}} r_e < (|\bar{V}| - 1)|s|
\]  \hspace{1cm} (5.16)

then the \( p \)-th moments of the graph satisfy (4.4).

Finally if for every non-empty subset \( \mathbb{V} \subset \mathbb{V}_0 \), one has:

\[
\sum_{e \in E_0(\bar{V})} a_e + \sum_{e \in E^1(\bar{V})} \left( \mathbb{1}_{\{v_e \in \mathbb{V} \forall r_e > 0\}}(a_e + r_e - 1) - (r_e - 1) \right) + \sum_{e \in E^1(\bar{V})} ((a_e + r_e) - \mathbb{1}_{v_e \in \bar{V}} r_e) + \mathbb{1}_{\{v_e \in \bar{V}\}} |G|_s
+ \frac{3}{2}|\bar{V} \cap V| > \left( |\bar{V}| - 1\right)|s| \hspace{1cm} (5.17)
\]

**Proof.** In the case that \( p = 1 \), (5.13) directly gives us (4.3), so we may assume \( p \geq 2 \). We need to argue that for any subset \( \mathbb{V} \subset \mathbb{V}_0 \), which is the union of all the nodes in the \( p \) copies, satisfies (4.3). To this end we introduce the notation \( \mathbb{V}_j \) for the nodes in \( \mathbb{V} \) that come from the \( j \)-th copy; the decomposition \( \mathbb{V} = \bigcup_{j=1}^p \mathbb{V}_j \) is what we have in mind.

Now as in [13], we proceed by decomposing the sum \( \sum_{e \in E_0(\bar{V})} a_e \). The idea is that the nodes in \( \mathbb{V}_j \) that also lie in \( \mathbb{V}_\xi \), will lead to a cumulant cycle after contraction which under our strategy can be seen as an edge with weight \( 3/2 + \eta \). This means that it will add a weight of \( \frac{3}{2} |\mathbb{V}_j \cap \mathbb{V}_\xi| \) to the existing sum. Notice that here we have ignored the \( \eta > 0 \) and will do so in the proof of the other inequalities too. This is justified by noticing that apart from (4.3), the inequalities we are interested in are strict, and in the case of (4.3) we have an arbitrarily small constant, \( \kappa \), on the other side which can be adjusted in conjunction with \( \eta \). The nodes which are not attached to leaves, are unchanged and hence we are able to make the following calculation:

\[
\sum_{e \in E_0(\bar{V})} \hat{a}_e \leq \sum_{j=1}^p \left( \sum_{e \in E_0(\mathbb{V}_j)} a_e + \frac{3}{2} |\mathbb{V}_j \cap \mathbb{V}_\xi| \right) + \sum_{j=1}^p \sum_{e \in E_0(\mathbb{V}_j)} a_e
\]

\[
< |s| \sum_{j=1}^p \left( |\mathbb{V}_j| - \frac{1}{4} \right) + \kappa + |s| \sum_{j=1}^p \left( |\mathbb{V}_j| - \frac{1}{4} \right) + \kappa
\]

\[
< |s| \left( |\mathbb{V}| - \frac{1}{2} \right) + \kappa'
\]

where the last inequality follows from recalling \( p \geq 2 \) and setting \( \kappa := 2\kappa \), ultimately proving the claim.

Fix a graph \( G \), and assume we have a \( \mathbb{V} \) such that it has cardinality at least 3 and it satisfies the conditions in (5.15) and (5.16). The case for \( p = 1 \) is fulfilled directly by
so we may assume $p \geq 2$. Further assume that $\{0\} \notin \vec{V}$. Denote by $\vec{V}_j$ the vertices in the $j$-th copy of $G$, and then consider the decomposition $\vec{V} := \cup_{j=1}^p \vec{V}_j$.

Notice then that we have:

$$\sum_{e \in E_0(\vec{V})} \delta_e + \sum_{e \in E^1(\vec{V})} \mathbb{1}_{\{v_e \in \vec{V} \land r_e > 0\}}(a_e + r_e - 1) - \sum_{e \in E_1(\vec{V})} \mathbb{1}_{\{v_e \in \vec{V}\}} r_e$$

$$\leq \sum_{j=1}^p \sum_{e \in E_0(\vec{V}_j)} a_e + \sum_{j=1}^p \sum_{\vec{V}_j \cap \vec{V}_\xi = \emptyset} \left( \sum_{e \in E_0(\vec{V}_j)} a_e + \frac{3}{2} \vert \vec{V}_j \cap \vec{V}_\xi \vert \right)$$

$$+ \sum_{j=1}^p \left( \sum_{e \in E^1(\vec{V}_j)} \mathbb{1}_{\{v_e \in \vec{V}_j \land r_e > 0\}}(a_e + r_e - 1) - \sum_{e \in E_1(\vec{V}_j)} \mathbb{1}_{\{v_e \in \vec{V}_j\}} r_e \right)$$

$$< \vert \xi \vert \sum_{j=1}^p \left( \vert \vec{V}_j \vert - \frac{1}{2} \right) + \vert \xi \vert \sum_{j=1}^p \left( \vert \vec{V}_j \vert - \frac{1}{2} \right)$$

$$< \vert \xi \vert (\vert \vec{V} - 1 \vert)$$

where for (\circ) we have used the naive bound on the first term (recall that the worst $a_e$ can be is $\vert \xi \vert - 1$) and for the rest of the expression we have used (5.15).

If however $\{0\} \in \vec{V}$, we look at the decomposition $\vec{V} \setminus \{0\} = \cup_{j=1}^p \vec{V}_j$, so that none of the $\vec{V}_j$ contain $\{0\}$. The proof runs along the same line but with the adjustment that the summand over $\vec{V} \cap \vec{V}_\xi = \emptyset$ is bounded above by $\left( \vert \vec{V}_j \cup \{0\} \vert - 1 \right) \vert \xi \vert$ instead.

Finally for the last part, fix some $\vec{V} \subseteq \vec{V}_*$ and notice that for each $v \in \vec{V}_\xi$, there are two cumulant kernels emerging from it. This means that when we decompose $\vec{V} = \cup_{j=1}^p \vec{V}_j$, we can bound the required quantity from below as follows:

$$\sum_{e \in E_0(\vec{V})} a_e + \sum_{e \in E^1(\vec{V})} \mathbb{1}_{\{v_e \in \vec{V}_\mathrm{v} \land r_e > 0\}}(a_e + r_e - 1) - (r_e - 1)$$

$$+ \sum_{e \in E_1(\vec{V})} \mathbb{1}_{\{v_e \in \vec{V}_\mathrm{v}\}} r_e$$

$$\geq \sum_{j=1}^p \sum_{e \in E_0(\vec{V}_j)} a_e + \sum_{e \in E^1(\vec{V}_j)} \mathbb{1}_{\{v_e \in \vec{V}_j \land r_e > 0\}}(a_e + r_e - 1) - (r_e - 1)$$

$$+ \sum_{e \in E_1(\vec{V}_j)} \mathbb{1}_{\{v_e \in \vec{V}_j\}} r_e + \frac{3}{2} \vert \vec{V}_j \cap \vec{V}_\xi \vert + \mathbb{1}_{\{v_e \in \vec{V}_j\}} \mathbb{1}_{\{v_e \in \vec{V}_\mathrm{v}\}} \mathbb{1}_{\{v_e \in \vec{V}_\mathrm{v}\}} \mathbb{1}_{\{v_e \in \vec{V}_\mathrm{v}\}}$$

$$> \left( \vert \vec{V} \vert - \mathbb{1}_{\{v_e \in \vec{V}\}} \right) \vert \xi \vert$$

\[\square\]
5.3 Renormalisation Procedure

In this section, we will develop our renormalisation procedure that is general enough to deal with the gKPZ. Let \( G = (V, E) \) be the kind of diagrammatic representation we have seen already. Assume it has some “negative subtree” \( \bar{T} = (\bar{V}, \bar{E}) \) that requires renormalisation in the sense that it violates \((5.15)\) or \((5.16) - (5.17)\) is always satisfied while \((5.13)\) and \((5.14)\) are mild enough that we do not expect them to be contravened. We will effect this renormalisation by changing the label of some \( e \in E^1(\bar{V}) \) by replacing \( v_e = v_0 \) with a node of \( \bar{T} \) such that the new Taylor expansion point has a renormalisation effect on \( T \).

Let \( e = (v_1, v_2) \in E^1(\bar{V}) \) and \( v \in \bar{T} \) such that there exists a \( v' \) such that \( (v, v') \in E^1(\bar{V}) \). Diagrammatically we mean: 
\[
\begin{array}{c}
\bullet\rightarrow\bullet \\
\text{\( \bar{e} \)}
\end{array}
\]

where the symbol \( \bullet \rightarrow \bullet \) means that there exists a path between \( v_2 \) and \( v \). In most practical examples it will be an edge. The label of \( e \) is replaced by \( (a_e, r'_e, v) \). This transformation and the choice of the level \( r'_e \) will depend on the subtree \( \bar{T} \). For instance, we take \( r'_e = \max(\lceil -|\bar{T}|_{\text{max}} \rceil, r_e) \). Let’s start with \( v_e = v_0 \), and want to rewrite the Taylor expansion in the point \( v \). We proceed as follows:

\[
\begin{align*}
\hat{K}_e(x_{v_2} - x_{v_1}) &= K_e(x_{v_2} - x_{v_1}) - \sum_{|j|s < r_e} \frac{(x_{v_2} - x_{v_1})^j}{j!} K_e^{(j)}(-x_{v_1}) \\
&= K_e(x_{v_2} - x_{v_1}) - \sum_{|j|s < r'_e} \frac{(x_{v_2} - x_{v_1})^j}{j!} K_e^{(j)}(x_{v} - x_{v_1}) \\
&\quad + \sum_{|j|s < r'_e} \frac{(x_{v_2} - x_{v_1})^j}{j!} K_e^{(j)}(x_{v_1}) - \sum_{|j|s < r_e} \frac{(x_{v_2} - x_{v_1})^j}{j!} K_e^{(j+k)}(-x_{v_1}) \\
&= K_e(x_{v_2} - x_{v_1}) - \sum_{|j|s < r'_e} \frac{(x_{v_2} - x_{v_1})^j}{j!} K_e^{(j)}(x_{v} - x_{v_1}) \\
&\quad + \sum_{|j|s < r'_e} \frac{(x_{v_2} - x_{v_1})^j}{j!} K_e^{(j)}(x_{v_1}) - \sum_{|k|s < r_e - |j|s} \frac{(x_{v_1})^k}{k!} K_e^{(j+k)}(-x_{v_1})
\end{align*}
\]

Graphically, given a graph \( G \), we are effecting the following decomposition:

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\text{\( \bar{e} \)} & & \\
\text{\( v_{e} \)} & \rightarrow & \text{\( v_{e} \)} \\
\text{\( v_{2} \)} & & \\
\text{\( v \)} & + & \sum_{|j|s < r'_e} \frac{(x_{v_2} - x_{v_1})^j}{j!} \left( K_e^{(j)}(x_{v_1}) - \sum_{|k|s < r_e - |j|s} \frac{(x_{v_1})^k}{k!} K_e^{(j+k)}(-x_{v_1}) \right)
\end{array}
\]

where \( \gamma, \gamma_e, \gamma_j \) and \( \epsilon_j \) stand for the labels \((a_e, r_e, v_0), (a_e, r'_e, v_e), (a_e + |j|_s, \max(r_e - |j|_s, v_0), (-|j|_s, 0, v_0))\).

One is able to prove that in the case of the gKPZ, this renormalisation procedure is relatively mild, in that we do not expect it to create new divergences. We prove the following results in this direction:
**Proposition 5.4** If either of the conditions - (5.4), (5.17) - is satisfied in (5.18) for the terms with the labels \( \gamma \) and \( \gamma_e \), then that condition is also satisfied on the other terms on \( \bar{V} \subset V \) such that \( \bar{V} \cap \{v, v_1, v_2\} \neq \{v\} \).

**Proof.** We know that the conditions are satisfied on terms with \( \gamma \) and \( \gamma_e \), so if we are able to show that the sum of the contributions of \( \gamma_j \) and \( e_j \) are greater than the minimum of the previous they should also satisfy the same conditions. Let \( \bar{V} \subset V \), and consider \( \bar{V}_j = \bar{V} \cap \{v, v_1, v_2\} \):

| \( \bar{V}_j \) | \( \gamma_j \) | \( e_j \) | \( \gamma_e \) | \( \gamma \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \{v\}           | \( -1 \frac{(r_e - |j|s_a)}{|j|s_a} + 1 \frac{(r_e - |j|s_a)}{|j|s_a} |a_e + |j|s_a \) | \( -|j|s_a \) | 0 | 0 |
| \{v_1\}         | \( a_e + |j|s_a \) | \( -|j|s_a \) | \( -(r_e' - 1) \) | \( -(r_e - 1) \) |
| \{v_2\}         | \( a_e + |j|s_a + \max(r_e - |j|s, 0) \) | 0 | \( a_e + r_e' \) | \( a_e + r_e \) |
| \{v, v_1\}      | \( -1 \frac{(r_e - |j|s_a)}{|j|s_a} + 1 \frac{(r_e - |j|s_a)}{|j|s_a} |a_e + |j|s_a \) | \( -|j|s_a \) | \( -(r_e' - 1) \) | \( -(r_e - 1) \) |
| \{v, v_2\}      | \( a_e + |j|s_a \) | \( -|j|s_a \) | \( a_e \) | \( a_e + r_e \) |
| \{v_1, v_2\}    | \( a_e + |j|s_a + \max(r_e - |j|s, 0) \) | 0 | \( a_e + r_e' \) | \( a_e + r_e \) |
| \{v, v_1, v_2\} | \( a_e + |j|s_a \) | \( -|j|s_a \) | \( a_e \) | \( a_e \) |

The sum of the contributions of \( \gamma_j \) and \( e_j \) is greater than the minimum between \( \gamma_e \) and \( \gamma \) except for \( V_j = \{v\} \): when \( r_e - |j|s_a > 0 \), in which case \( -(r_e - 1) \leq 0 \).

**Proposition 5.5** If the condition (4.4) is satisfied in (5.18) for the terms with the labels \( \gamma \) and \( \gamma_e \) on some subset \( \bar{V} \), then this condition is satisfied for the other terms on the same subsets.

**Proof.** Again we have to consider the impact of the new edges \( e_j \) and \( \gamma \) for the terms in question. Let \( \bar{V} \subseteq \bar{V} \), if \( v_0 \in \bar{V} \) then from the previous proposition the conditions (4.4) are satisfied on \( \bar{V} \setminus \bar{V} \). To deduce the same for (4.4) on \( \bar{V} \), we suppose \( v_0 \notin \bar{V} \) and look at \( \bar{V} = \bar{V} \cap \{v, v_1, v_2\} \):

| \( \bar{V} \) | \( \gamma_j \) | \( e_j \) | \( \gamma_e \) | \( \gamma \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \{v\}           | 0 | 0 | 0 | 0 |
| \{v_1\}         | 0 | 0 | 0 | 0 |
| \{v_2\}         | 0 | 0 | 0 | 0 |
| \{v, v_1\}      | 0 | \( -|j|s_a \) | \( -r_e' \) | 0 |
| \{v, v_2\}      | \( a_e + |j|s_a \) | 0 | \( a_e + r_e' \) | \( a_e \) |
| \{v_1, v_2\}    | 0 | 0 | \( a_e \) | \( a_e \) |
| \{v, v_1, v_2\} | \( a_e + |j|s_a \) | \( -|j|s_a \) | \( a_e \) | \( a_e \) |

We see that the sum of the contributions from the edges in the terms in \( \gamma_j \) and \( e_j \) is smaller than the maximum between \( \gamma_e \) and \( \gamma \). □

After some number of contractions, one expects to see a graph with no instances of noises on it. If this graph still violates (5.15) or (5.16), we would like to use it to
motivate our definition of the renormalisation constant for our model. One has to be careful however that there are no Taylor expansions on some of the Kernels. To remedy this we would like to remove our Taylor expansions first, by which we mean a transformation of the form \((a_e, r_e, v_0) \rightarrow (a_e, 0)\). Let \(e = (v_1, v_2)\) and edge in \(\bar{T}\) with a label \((a_e, r_e, v_0)\) and \(r_e > 0\). We can perform the following decomposition to effect the previously stated transformation:

\[
\begin{align*}
v_2 &= a_e, 0 - \sum_{|k| < r_e} v_1 a_e + |j|_e - |j|_e, 0 \\
v_1 &= a_e, 0 - \sum_{|k| < r_e} v_1 a_e + |j|_e - |j|_e, 0 \\
0 &= a_e, 0 - \sum_{|k| < r_e} v_1 a_e + |j|_e - |j|_e, 0 
\end{align*}
\]

For the generalised KPZ equation we are able to prove that:

**Proposition 5.6** The previous terms depending on \(k\) satisfy the conditions 4.3 and 4.4 on \(V(\bar{T})\).

In the following list we compile the sort of negative subtrees we expect to see in our analysis:

\[
\begin{align*}
v_2 &\quad v_3 \\
v_2 &\quad v_1
\end{align*}
\]

\[
\begin{align*}
v_3 &\quad v_4 \\
v_2 &\quad v_1
\end{align*}
\]

\[
\begin{align*}
v_3 &\quad v_5 \\
v_2 &\quad v_1
\end{align*}
\]

\[
\begin{align*}
v_3 &\quad v_6 \\
v_2 &\quad v_1
\end{align*}
\]

The first two subdivergences tend to be similar in that they can be treated in a general fashion which we encapsulate in the propositions that follow. For the other divergences, we will present some ad-hoc methods in Section 5.4.

**Proposition 5.7** Let \(G\) be one of the first two graphs in 5.19. If \(G\) satisfies the condition 4.4 then the new graph \(G_e\) obtained from the transformation of the label of \(e = (v_2, v_1)\) satisfies the same condition.

**Proof.** We need only check that in changing to \(v_e\) from \(v_0\) in \(G\) the contribution of the edge \(e\) is preserved. Let \(\bar{V} \subset V\); we consider each case in turn:

- \(e \in E_0(\bar{V})\) then we still have the same contribution of \(a_e\).
we have to check that by changing case by case preserved and it has been improved in the case of subset and on conditions HSNSI for some subsets Proposition five. taboldstyle. eight. taboldstyle.

\[ a_{\gamma_1} + a_{\gamma_2} - 3 + \mathbb{1}_{v_\gamma \in \Psi}(a_{\gamma_2} - 3) \geq -|\bar{T}|_0 > (r'_e - 1) \]

Condition 4.5 is hence satisfied for \( \bar{V} = \{ v_1 \} \). For the second tree, if \( v_3 \not\in \bar{V} \), the contribution \(- (r'_{\gamma} - 1)\) is equal to zero. Otherwise, we may use the previous bound for \( \{ v_1 \} \) and the fact that \( r_{\gamma_1} \geq r_{\gamma} \) to conclude for \( \{ v_3 \} \).

- \( e \in E^{\dagger}(V) \), we have \( v_2 \in V \).
  - If \( v \not\in V \), we have a contribution bigger than \( a_v + r_v \) which is \( a_v + r'_v \).
  - Otherwise if \( v \in V \), we lose a factor \( r_v \) in the contribution. For the negative tree, we have \( r_{\gamma_3} = 0 \) and \( r_{\gamma_2} \) are not both strictly positive. Then for \( r_{\gamma_1} > 0 \), one calculates:
    \[ a_{\gamma_3} + a_{\gamma_1} + r_{\gamma_1} - 1 \leq 2|\delta| \]

a similar bound holds for \( r_{\gamma_2} \). Adding \( v_1 \) will make the bound sharper to check but similarly it is seen to be achieved.

Proposition 5.8 Let \( G \) be one of the two first graphs in (5.19). If \( G \) satisfies the conditions (3.3) for some subsets \( \bar{V} \neq \{ v, v_1 \} \) then the new graph obtained from the transformation of the label of \( e = (v_1, v_2) \) satisfies this condition on the same subset and on \( \{ v, v_1 \} \).

Proof. Let \( \bar{V} \subset G \), such that the condition (3.3) is satisfied or such that \( \bar{V} = \{ v, v_1 \} \).
We have to check that by changing \( v_e \) to \( v_0 \) in \( G \) the contribution of the edge \( e \) is preserved and it has been improved in the case of \( \bar{V} = G' \). As before we proceed case by case:
- \( e \in E_0(\bar{V}) \) then we still have the same contribution with \( a_e \)
- \( e \in E^{\dagger}(\bar{V}) \), we have \( v_2 \in V \).
  - If \( v \not\in \bar{V} \), the contribution does not change (from 0).
  - Otherwise if \( v \in \bar{V} \) then we can have a contribution \( a_e + r'_e - 1 \) which is bigger in general than \( a_e + r_e - 1 \). By adding \( v_1 \) into \( \bar{V} \) and using the fact that:
    \[ \sum_{i=1}^{2} a_{\gamma_i} - (|\bar{V}| - 1)|\delta| + r'_e - 1 > 0 \]
we obtain the desired bound.
• \( e \in E^\uparrow(\overline{V}) \), we have \( v_1 \in V \)
  - If \( v \in \overline{V} \) then the new contribution is \(-r'_e\) which in generally is less than or equal to \( r_e \).
  - If \( v \not\in \overline{V} \), then \( \overline{V} \) is not equal to \( G' \) and the condition (3.3) is satisfied.

\[ \square \]

**Proposition 5.9** For the graphs given in (5.19), if the condition (3.4) and (5.17) is satisfied on some subset \( \overline{V} \) in (5.18) for the terms with the labels \( \gamma \) and \( \gamma_e \), then this condition is satisfied on the other terms for the same subsets.

**Proof.** The result comes essentially from the proposition (5.4) except when we consider a subset \( V \) such that \( V \cap \{v, v_1, v_2\} = \{v\} \) and \( j < r_e \). In that case the node \( v \) has a negative contribution \(-(r_e - 1)\). But in all the examples we have \( r_e \leq 1 \), which gives the result. \( \square \)

We iterate the renormalisation procedure on each edge in \( E^\downarrow(\overline{V}) \) and we obtain a graph with no leaves which can be divergent. In the generalised KPZ terms, the diverging graph has the following form:

\[
\begin{align*}
  v_1 & \quad v_c = \gamma_2 & v_1 \\
  v & \quad v_c & v_{v_2} & v' + \sum_{k < r_e} v_{v_k} \\
  & \quad v_{v_k} & v_{v_{v_k}} & v_{v_{v_{v_k}}}
\end{align*}
\]

(5.20)

**Proposition 5.10** The condition (4.5) is satisfied for all the terms of the previous decomposition.

**Proof.** Let \( V \) be a subset of \( G \). We have for \( V' = V \cap \{v, v_1\} \):

| \{v, v_1\} | \{v\} | \{v_1\} | e | e' | (0, v) | (0, v_1) |
|-----------------|-----------------|-----------------|-----|-----|-----------------|-----------------|
|                  | \( a_e \)       | \( a_e \)       | \( r_e - 1 \) | \( -k \) | \( a_e + k \) | \( 0 \) |

For the first two rows of the previous table, the contribution of \( e' \) and the sum of \( (0, v) \) and \( (0, v_1) \) are greater than \( e \). The main difference occurs when \( V' = \{v_1\} \) and \( r_e > 0 \). In that case, we consider that \( v_c, v \in V \). The node \( v \) gives a contribution bounded by \( a_{\gamma} + r_{\gamma} \leq |s| \) where \( \gamma \in E^\uparrow(V) \) and the condition (4.5) is satisfied. By removing this node and keeping the other contribution different from \( \gamma \), we notice that \( a_e \) is sufficient for the required bound instead of \( a_e + r_e \). \( \square \)

**Proposition 5.11** The condition (4.4) is satisfied on each term \( G_k \) depending on \( k < r_e \) of the previous decomposition (5.20) and it is also satisfied for the first term \( G \) on subset \( V \neq \{v, v_c, v_1\} \).
Proof. Let $V \subset G$. If $0 \notin V$ then we can split the graph $G_k$ into two KPZ trees by splitting $v_c$ into two leaves and the condition (4.5) is satisfied on each tree which gives (4.4) on $V$. If $0 \in V$ the previous proposition gives the condition (4.5) on $G_k \setminus V$ which proves (4.4) on $V$.

If one compares the general renormalisation map in Section 2.2 (Definition 2.8) with the renormalisation procedure we have suggested, one finds that they are equivalent for these sorts of divergences. In fact, in our renormalisation procedure, we are moving errant edges to the root and excising the troublesome subtree through translation invariance. The extraction-contraction procedure of the renormalisation procedure amounts to the same. Indeed, for gKPZ, there are no nested neither overlapping divergences. One has to face only one subdivergence which is treated by the procedure described above and corresponds to the application of $R_g$ to a node in the tree. We apply the renormalisation procedure to a graph that comes from some $\Pi^N$ and it produces in the end a graph coming from $\Pi^N \delta \tau$ with $\tilde{g}$ equal to $g$ except on the empty tree, where it will be equal to zero. This forces the extraction of one subtree. In the end, we get:

$$\Pi^{N,g \tau} = \Pi^N \tau + \Pi^N \delta \tau$$

This suggests that we are computing the correct counter-terms with our renormalisation procedure.

Remark 5.12 For several disjoint subdivergences, the same procedure can be applied. Nested subdivergences could be handled in a recursive way by starting the procedure on the inner subdivergence and then proceeding to the outer. Overlapping divergences will require much more work and use similar techniques as in [13] concerning safe/unsafe forests.

5.4 Example Computations

In this section, we would like to work out some calculations to show how the divergences will look like in practice, and how they are then resolved - in particular we will show how the subdivergences that were listed in the previous section are dealt with. Not to clutter notation we fix the following labels upfront $\gamma_1 = (|s| - 2, 1, v_0)$, $\gamma_2 = (|s| - 1, 0, v_0)$, and $\gamma_3 = (|s| - 2, 0, v_0)$. Consider then the tree coming from $\tau = \mathcal{F}(\mathcal{F}(\Xi|\Xi)|\Xi)^2$. 

![Diagram]

\begin{itemize}
  \item \textbf{1}: $x_1$ \quad $v_3$ \quad $v_2$ \quad $v_1$
  \item \textbf{2}: $x_1$ \quad $v_3$ \quad $v_2$ \quad $v_1$
\end{itemize}
In our case however we will also see higher order contractions. Some possible examples are:

The quantities $(|V| - 1)|s|$ and $(|V| - \frac{1}{2})|s|$ will show up a lot in our analysis, so we define $A(n) \overset{\text{def}}{=} (n - \frac{1}{2})|s|$, $B(n) \overset{\text{def}}{=} (n - 1)|s|$. For $\otimes$, we notice that each edge carries at most the weight $|s| - 1$, while on the other side we have the recursive identities: $A(n + 1) = A(n) + |s|$, and $B(n + 1) = B(n) + |s|$. It is easy to see then, via an inductive argument for example, that (4.3) and (4.4) are satisfied for $\otimes$.

For $\otimes$ in the list above, consider the cumulant term in the subtree $\{v, v_1\}$. As per our assumption, it will carry the weight of 3 (the $\eta$ term is not really important here). Putting it into (4.3), one gets $2 - \kappa \leq |s|$. This gives us a first lower bound on $|s|$ for the theory to work. Consider now the subtree $\{v, v_*, v_1\}$, for which (4.4) gives us $3 + |s| - 1 + |s| - 1 = 2|s| + 1 \notin 2|s|$. To remedy this we apply our renormalisation procedure:
Here $\gamma_\star = (|s| - 2, 2, v_\star)$ and $\gamma_j = (|s| - 2 + j, 1 - j, 0)$. Our assumption on the cumulants gives us that can be replaced by a kernel of weight 3, and then it can be seen that all the above graphs satisfy (4.3) (where we have assumed the lower bound $2 - \kappa \leq |s|$) and (4.5). The benefit of the decomposition can be seen now in the fact that the tree with $\gamma_\star$ satisfies $4 - 3 + |s| - 1 + |s| - 1 + 0 - 2 = 2|s| - 1 < 2|s|$. For the terms with $\gamma_j$ the problem persists because notice that $3 + (|s| - 1) + (|s| - 1) - j = 2|s| + 1 - j \not\leq 2|s|$ for $j \in \{0, 1\}$. Fortunately a couple of clever decompositions can fix this as well. For the case $j = 1$, we may do:

where $\gamma_\star = (1, 1, v_\star)$ and $C_0^1 = 0$.

One can now check that for the first graph on the right hand side of the above equation one has: $3 + (|s| - 1) + (|s| - 1) + 0 - 1 - 1 = 2|s| - 1 < 2|s|$. When $j = 0$, we may consider:

\[ x_1 v \x_1 v_1 = x_1 v + C_0^2 \quad + C_0^2 v_\star 0 \quad + C_1 v_\star 0 \]
where $\gamma_* = (1, 2, v_*)$, and $C_0^2 = \gamma$ and $C_1 = \gamma$

Hence the renormalised term is given by:

$$
\begin{align*}
\int_0^\infty \mathcal{Z}
\gamma
\gamma
\gamma
\gamma

- C_1 \mathcal{Z}
\gamma
\gamma
\gamma
\gamma

- C_0^2 \mathcal{Z}
\gamma
\gamma
\gamma
\gamma
\end{align*}
$$

For $\mathcal{Z}$ and $\mathcal{Z}$ are satisfied for the same reason as before and additionally, we notice that the subdivergence caused by in the second tree, does not cause a problem here.

Consider now $\mathcal{Z}$. The subtree $\{x_1, v\}$ gives us a new construction to work with. As before the cumulant carries the weight 3. Adding this into the label of the existing edge gives us a new edge with weight $\hat{a}_e = (|s| - 2) + 3 = |s| + 1$. One can check that $\mathcal{Z}$ is satisfied if again we have $2 - \kappa \leq |s|$. The checks for $\mathcal{Z}$ are similar as before except for $\{x_1, v, v_*\}$ where we would have $3 + |s| - 2 + |s| - 1 = 2|s| < 2|s|$. When one moves on to $\{x_1, v, v_*, v_1\}$, the condition is no longer violated because the addition of the interior node adds a weight of $|s|$ to the RHS of the inequality, while the edge only adds a $|s| - 1$ to the LHS. Our respite is in the fact that the problematic set is not connected directly to the noises and as such one is able to "excise" it. Consider first the following decomposition:
Notice that the disappearance of the edge \( \{x_1, v\} \) from the second graph cures the divergence. The first graph on the left is rewritten via translation invariance.

As before, notice that due to the fact that \( K^N \) kills constants the second tree above on the right side is indeed zero, and so is the one on the left. In \( \otimes \) one can check that the subset \( \{x_1, v, v_*, v_1\} \) fails (3.3). Indeed one can calculate that \(|s| − 2 + |s| − 1 + |s| − 1 + 3/2(3) + 0 − 0 = 3|s| + 1/2 \not< 3|s|\). To remedy this, consider first the decomposition:

Notice that for the graph on the right the divergence has been cured because the edge \((x_1, v)\) (and the weight that it carries) is removed from the calculation. To deal with the left tree we begin by redirecting the edge \((v_2, v_1)\) to \(v_*\), which by our renormalisation procedure gives rise to the graphs:

where \( \gamma = (|s| − 2, 1, v_*) \). This means that the condition now becomes \(|s| − 2 + |s| − 1 + |s| − 1 + 3/2(3) − 1 = 3|s| − 1/2 < 3|s|\). One defines then \( C_1 := \)

and then renormalisation is a matter of removing a factor of \( C_1 v_* \).
Another example is furnished by the tree \( \circ \). The renormalisation map for the homogeneous Gaussian case gives us the following:

\[
\hat{\Pi}_0 \circ = - 2 + 2 + 2 - 2 - 2 - 2
\]

Then, our usual assumption on the second cumulant, allows us to check Proposition \([5,3]\) In our case however we have non-vanishing higher order cumulants, so we will see the graphs like:

In the above list, the tree \( \circ \) can be seen to be unproblematic. In \( \circ \) and \( \circ \), we see the subset \( \bar{V} = \{ v_4, v_2, v_3 \} \) we have needed to treat before but here that cumulant is of
order 3, and so the extra weight of $3/2^+$ associated to the cumulant terms attached to $v_4$ and $v_3$ will not be included in $E^\downarrow(\vec{V})$. For this reason these trees do not cause any problems.

The next tree we can look at comes from $\circ \circ$. The tree in consideration is then:

![Diagram](image)

It is an easy exercise in computations to check that this graph satisfies all the required condition. Some of the second order contractions we expect to see are:

![Diagram](image)

Checking the conditions on $\circ \circ$ and $\circ \circ$ will follow a similar line of reasoning as the uncontracted tree. The only difference stems from the second order contractions, which occurs in the subsets $\{v_3, v_1\}$ and $\{v_3, v_2\}$. Consider first the effect of $(v_3, v_2)$ in $\circ \circ$. As before we expect this to add a weight of 3 to the edge $(v_3, v_2)$ which is to say that...
\[ \hat{a}_{v_3, v_2} = |s| - 2 + 3 = |s| + 1, \]

but it is straightforward to see that conditions are satisfied for the same reasons as they were for the original tree. For \( \circ \), \( \{v_3, v_2, v_1\} \) is a new structure, but one checks that \( |s| - 2 + |s| - 2 + 3 = 2|s| - 1 < A(3) = 2|s| \).

The rest of the tree shares arguments with the original tree.

In \( \circ \), we again see the sort of sub-divergence that we have had to contend with before in \( \{v_1, v, v_*\} \). To deal with it we will, much like before, move the edge \((v_2, v_1)\) to \(v_*\) instead. This decomposition will take the following form:

\[
\begin{align*}
\gamma &= (1, 2, v_*) \\
\hat{a}_{v_3, v_2} &= |s| - 2 + 3 = |s| + 1
\end{align*}
\]

where \( \gamma = (1, 2, v_*) \) and as before this will cure the sub-divergence for the first graph on the right hand side. The treatment for the other terms on the right hand side will be the same as before.

The higher order cumulants will see take the form:

The subtree \( \{v_1, v_2, v_3\} \) appears to be problematic \( |s| - 2 + |s| - 2 + 3/2(3) = 2|s| + 1/2 \not< 2|s| \) but this is treated using the excision method we have already used. None of the other trees cause any trouble.

Next we look at the symbol: \( \circ \). The relevant tree is of the form:
It is an easy exercise to see that the above tree satisfies all the required conditions. The first order contractions one will see are as follows:

As before we expect the subset \(\{v_1, v_2, v_3\}\) to cause us problems, but this may be treated in our usual excision method, while the subset \(\{v, v_*\}\) does not cause any trouble. The constant at the end we do not expect to be zero, and as such will be removed by the renormalisation.

In the higher order cumulants, the problematic one that we will find happens to be:

One can calculate on \(\{v_2, v_3, v_1, v\}\) leads to \(|s| - 1 + |s| - 1 + |s| - 2 + \frac{3}{2} = 3|s| + \frac{1}{2} \neq (4 - 1)|s| = 3|s|\). This is dealt with the usual excision trick. No other combination (of three noises) causes any contravention of the rules. For example if either \(v_2\) or \(v_3\) is replaced in the previous set by \(v_*\), one cannot expect to see the
same divergence for an edge of weight $|s| - 1$ is replaced by an edge with weight $|s| - 2$, while the RHS remains the same. If one wants to consider the effect of the cumulant term generated by the noises attached to $v_2$, $v_3$, and $v_*$ we get, the set to be considered would be the entire tree except the origin and for this we will have $B(5) = 4|s|$.

5.5 Convergence of the model

In this section, we want to talk about how the methods developed so far are to be put together to give the convergence result we are after. With reference to [18], this section is comparable to Section 5 in that article, wherein the authors show that $u^N$ which solves the rescaled, discretised, renormalised PAM converges in probability to the solution of the renormalised PAM driven by the Ornstein-Uhlenbeck process in the space $C^0_{N,T}$ for $\tilde{\eta} \in (0, \frac{1}{2} \wedge \eta)$. In particular, we want to say what needs to be changed for our result to work.

Take $g$ to be any smooth, compactly supported function defined on $\mathbb{R}^{d+1}$ that integrates to one, and define for $\delta \in (2^{-N}, 1]$ the rescaled version $g^\delta(t, x) := \delta^{-5} g(\delta^{-2}, \delta^{-1}x)$ and also $\bar{g}^\delta,N := 2^{-dN} \int g^\delta(t, y)1\{|y-x| \leq 2^{-N-1}\}dy$. Then the semi-discrete convolution $g^\delta,N \star_N \xi^N$ defines a smooth discrete noise which we denote by $\xi^\delta,N$ and use to introduce the equation:

$$
\partial_t u^\delta,N = (\Delta u^\delta,N) + g(u^\delta,N)(\nabla u^\delta,N)^2 + k(u^\delta,N)(\nabla u^\delta,N) + \bar{h}(u^\delta,N) + \xi^\delta,N(u^\delta,N)
$$

defined on $\mathbb{R}_+ \times 2^{-N}S$ where $\bar{h}$ is as it was before and the constants are the same but with $\xi^N$ replaced by $\xi^\delta,N$. The point the authors take in [18] is that the setup above as it corresponds to the PAM is amenable to [17] Theorem 16 and hence one is able to argue that $\|u^N; u^\delta,N\|_{C_{\bar{\eta},T}}$ converges to zero in probability provided one takes $N \to \infty$ before the mollification is removed. We are not able to directly argue like this for our equation because the named theorem pertains to a fixed point with non-linearities only in $u^N$. We however anticipate, that due to the general result in [3], a similar conclusion for the (g)KPZ should also hold. The next step in the argument is to look to the equation:

$$
\partial_t u^\delta = (\Delta u^\delta) + g(u^\delta)(\nabla u^\delta)^2 + k(u^\delta)(\nabla u^\delta) + \bar{h}(u^\delta) + Y^\delta(u^\delta)
$$

defined on $\mathbb{R}_+ \times \mathbb{T}^d$, with $\bar{h}$ again as before, and $Y^\delta = g^\delta \star Y$ with $Y$ as in [1.2]. Then, with reference to the results in [17] and [6], one has that $u^\delta$ converges in probability in the space $\mathcal{C}_{\bar{\eta},T}^\delta$ to a limit $u$ as $\delta \to 0$. At this point it only remains to prove the convergence for our result to hold:

$$\lim_{\delta \to 0} \lim_{N \to \infty} \|u^\delta,N; u^\delta\|_{\mathcal{C}_{\bar{\eta},T}^\delta} = 0$$

Due to the smoothness of $u^\delta$, this follows as soon as we have:
\[
\lim_{N \to \infty} \sup_{(t,x) \in [0,T] \times 2^{-N} \mathbb{R}^2} |\xi^N(t,x) - Y^\delta(t,x)| = 0
\]
which we have assumed to be true.

Appendix A  Joint Cumulants

In this appendix, we recall the definition and basic properties of joint cumulants. Consider a collection of random variables \( \{X_a\}_{a \in \mathcal{A}} \) for some index set \( \mathcal{A} \). For the subsets \( B \subseteq \mathcal{A} \), we write \( X_B = \{X_a : a \in B\} \) and \( X^B = \prod_{a \in B} X_a \). One should be mindful of the fact that \( X_B \) is a collection of random variables while \( X^B \) is itself a random variable. Further, we write \( \mathcal{P}(B) \) for the set of all partitions of \( B \).

**Definition A.1**  Let \( B \subseteq \mathcal{A} \). The cumulant \( \mathbb{E}_c(X_B) \) is defined inductively over \( |B| \) by \( \mathbb{E}_c(X_B) = \mathbb{E}(X_a) \), if \( B \) is the singleton containing \( a \) and

\[
\mathbb{E}(X^B) = \sum_{\pi \in \mathcal{P}(B)} \prod_{B \in \pi} \mathbb{E}_c(X_B), \quad \text{if } |B| \geq 2 \quad (A.1)
\]

The usefulness of the notation \( X_B \) and \( X^B \) is in the fact that cumulants are defined for collections (say vectors) of random variables, while the moments can be taken of at most a product of random variables. Due to this reliance on the moment in its definition the following properties are natural:

- \( \mathbb{E}_c[ h_1 X_1, \cdots , h_k X_k ] = \prod_{i=1}^k h_i \times \mathbb{E}_c[X_1, \cdots , X_k] \)
- \( \mathbb{E}_c[X_B] = 0 \), if one has \( B = B' \cup B'' \), \( B' \cap B'' = \emptyset \), and \( X_{B'} \) and \( X_{B''} \) are independent
- For a jointly Gaussian collection of random variables \( X_B \), \( \mathbb{E}_c[X_B] = 0 \) if \( |B| > 2 \).

**Example A.2**  Consider the case when \( |B| = 2 \):

\[
\mathbb{E}_c[X_1, X_2] = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1] \mathbb{E}[X_2]
\]
that is to say, the cumulant of two random variables is its usual covariance.

When \( |B| = 3 \), one is able to calculate:

\[
\mathbb{E}_c[X_1, X_2, X_3] = \mathbb{E}[X_1 X_2 X_3] - \mathbb{E}[X_1 X_2] \mathbb{E}[X_3] - \mathbb{E}[X_1 X_3] \mathbb{E}[X_2] - \mathbb{E}[X_2 X_3] \mathbb{E}[X_1] + 2 \mathbb{E}[X_1] \mathbb{E}[X_2] \mathbb{E}[X_3]
\]

The Wick products, that have a central role in the renormalisation of SPDEs can then be formulated in the sense of definition \[A.1\].

**Definition A.3**  For \( A \subseteq \mathcal{A} \), the Wick product \( :X_A: \) is defined via \( :X_{\emptyset}: = 1 \) and then recursively:

\[
X^A = \sum_{B \subseteq A} :X_B: \sum_{\pi \in \mathcal{P}(A \setminus B)} \prod_{B \in \pi} \mathbb{E}_c(X_B) \quad (A.2)
\]
It is obvious that the above definition forces $E : X_A := 0$ whenever $A \neq \emptyset$, and moreover taking expectation on either side of the equality in Definition A.3 reduces to the equality in Definition A.1.

Example A.4 For a family of centred random variables $X_i$, one has $X_i := X_i, X_1X_2 := X_1X_2 - E(X_1X_2)$.

$$X_1X_2X_3 := X_1X_2X_3 - \sum_{i \neq j \neq k} X_iE(X_jX_k) - E(X_1X_2X_3)$$

We also define the following class of partitions: Let $M$ and $P$ be two sets and fix a subset $D \subseteq M \times P$. Then $\mathcal{P}_M(D)$ is the set of all partitions of $D$, such that for every $B \in \pi \in \mathcal{P}_M(D)$, there must exist $(i, k), (i', k) \in B$ such that $k \neq k'$.

Lemma A.5 For $m, p \in \mathbb{N}$. Set $M = \{i : 1 \leq i \leq m\}$ and $P = \{k : 1 \leq k \leq p\}$. Let $\{X_{(i,k)}\}_{i \in M, k \in P}$ be a collection of random variables with bounded moments of all orders. Then:

$$E\left(\prod_{k=1}^{p} \prod_{i=1}^{m} X_{(i,k)}\right) = \sum_{\pi \in \mathcal{P}_M(M \times P)} \prod_{B \in \pi} E_c(X_B)$$

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