IRREDUCIBILITY OF SOME NESTED HILBERT SCHEMES

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Abstract. Let $S$ be a smooth projective surface over $\mathbb{C}$. Let $S^{[n_1, \ldots, n_k]}$ denote the nested Hilbert scheme which parametrizes zero-dimensional subschemes $\xi_{n_1} \subset \ldots \subset \xi_{n_k}$ where $\xi_i$ is a closed subscheme of $S$ of length $i$. We show that $S^{[n,m]}$, $S^{[n,m+1]}$, $S^{[n,n+1,m]}$, $S^{[n,n+1,m+1]}$, $S^{[n,n+2,m]}$, and $S^{[n,n+2,m,m+1]}$ are irreducible.

1. Introduction

Let $S$ be a smooth projective surface over $\mathbb{C}$. The Hilbert scheme $S^{[n]}$ which parametrizes closed zero-dimensional subschemes of $S$ of length $n$ is a well studied space. It was shown by Fogarty in [Fog68, Theorem 2.4] that the Hilbert scheme $S^{[n]}$ is a smooth projective variety of dimension $2n$. A natural generalization of $S^{[n]}$ is the nested Hilbert Scheme about which far less is known. For an increasing tuple of positive integers $n_1 < \ldots < n_k$, the nested Hilbert scheme $S^{[n_1, \ldots, n_k]}$ parametrizes nested zero-dimensional subschemes $\xi_{n_1} \subset \ldots \subset \xi_{n_k}$ where $\xi_i$ is a subscheme of $S$ of length $i$.

The first nontrivial nested Hilbert scheme is $S^{[n,n+1]}$, which is in fact smooth and irreducible as shown in [Che98, Theorem 3.0.1]. In [GH04, Proposition 6], the authors show that the nested Hilbert scheme $S^{[n,n+2]}$ is irreducible of dimension $2n + 2$. The nested Hilbert scheme $S^{[1,n]}$ is irreducible of dimension $2n$ by [Fog73, Corollary 7.3]. In [RS21, Theorem 3.1], Ramkumar and Sammartano have shown that $S^{[2,n]}$ is irreducible of dimension $2n$. They further study the geometry of this space and show that it has rational singularities. In particular, it is normal and Cohen-Macaulay. In [Add16, §3.A] the irreducibility of $S^{[n,n+1,n+2]}$ is proved. In [RT22], Ryan and Taylor study the irreducibility, singularities and Picard groups of nested Hilbert schemes $S^{[n,n+1,n+2]}$. In [BE16], Bulois and Evain studied irreducible components of nested Hilbert schemes supported at a single point using the connection between nested Hilbert schemes and commuting varieties of parabolic subalgebras.

The following two results limit the collection of tuples $(n_1, \ldots, n_k)$ for which the nested Hilbert scheme $S^{[n_1, \ldots, n_k]}$ is irreducible. By [RT22, Theorem 3.15] the nested Hilbert scheme $S^{[1,2,\ldots,22,23]}$ is reducible. In [RS21, Proposition 3.7] the authors prove the existence of tuples $n_1 < \cdots < n_k$, for each $k \geq 5$, such that the nested Hilbert scheme $(\mathbb{A}^2)^{[n_1,\ldots,n_k]}$ is reducible. We refer the reader to [RT22], [RS21] and the references therein for more results related to irreducibility of nested Hilbert schemes.

If $E$ is a locally free sheaf on $S$, let $\text{Quot}(E,d)$ denote the Grothendieck Quot scheme of quotients of $E$ of length $d$. In [EL99, Theorem 1] it is proved that this Quot scheme is
irreducible. Our goal in this paper is to prove the following results on irreducibility of nested Hilbert schemes.

**Theorem (Theorem 3.8).** Let $n$ and $m$ be two positive integers such that $n < m$. Then $S^{[n,m,m+1]}$ and $S^{[n,m]}$ are irreducible.

**Theorem (Theorem 4.7).** Let $n$ and $m$ be two positive integers such that $n + 1 < m$. Then $S^{[n,n+1,m,m+1]}$ and $S^{[n,n+1]}$ are irreducible.

**Theorem (Theorem 5.2).** Let $n$ and $m$ be two positive integers such that $n + 2 < m$. Then $S^{[n,n+2,m,m+1]}$ and $S^{[n,n+2]}$ are irreducible.

The proofs of the above results proceed by combining some of the ideas in [EL99], [BE16] and [RT22].

## 2. Preliminaries

Let $S$ be a smooth projective surface over $\mathbb{C}$. For a pair of positive integers $n, m$ with $n < m$, the nested Hilbert scheme $S^{[n,m]}$ parametrizes nested subschemes $\xi_n \subset \xi_m$ of $S$, where $\xi_i$ is a finite scheme of length $i$. Recall that the scheme $S^{[n,m]}$ represents the functor of nested flat families $\text{hilb}^{[n,m]}_S$ $$\text{hilb}^{[n,m]}_S : \text{Sch}/\mathbb{C} \rightarrow \text{Sets},$$ where $\text{hilb}^{[n,m]}_S(T)$ is the set of isomorphism classes of $T$-flat subschemes $X_n \subset X_m \subset S \times T$ such that for each point $t \in T$, the length of the subscheme $X_n \otimes k(t)$ is $n$ and the length of the subscheme $X_m \otimes k(t)$ is $m$. In particular, we have universal nested families of closed subschemes $Z_n \subset Z_m \subset S \times S^{[n,m]}$. The closed points of $Z_n$ and $Z_m$ have the following descriptions:

$$Z_n = \{(p, \xi_n, \xi_m) \in S \times S^{[n,m]} \mid p \in \xi_n \subset \xi_m\},$$

$$Z_m = \{(p, \xi_n, \xi_m) \in S \times S^{[n,m]} \mid p \in \xi_m\}.$$

We have the projection map $$\pi_m : S^{[n,m]} \rightarrow S^{[m]}.$$ Let $\mathcal{I}_m$ denote the ideal sheaf of the universal subscheme inside $S \times S^{[m]}$. Consider the map $$\text{Id}_S \times \pi_m : S \times S^{[n,m]} \rightarrow S \times S^{[m]}.$$ Denote the pullback $$\hat{\mathcal{I}}_m := (\text{Id}_S \times \pi_m)^* \mathcal{I}_m.$$ Consider the projective bundle

$$\varphi : \mathbb{P}(\hat{\mathcal{I}}_m) \rightarrow S \times S^{[n,m]}.$$ (2.1)

On $\mathbb{P}(\hat{\mathcal{I}}_m)$ we have the tautological quotient $$\varphi^* \hat{\mathcal{I}}_m \rightarrow \mathcal{O}_{\mathbb{P}(\hat{\mathcal{I}}_m)}(1).$$
Let $\varphi_1$ denote the composite $\mathbb{P}(\tilde{I}_m) \xrightarrow{\varphi_1} S \times S^{[n,m]} \rightarrow S$, where the second map is the projection to $S$. Similarly, let $\varphi_2$ denote the composite $\mathbb{P}(\tilde{I}_m) \xrightarrow{\varphi_2} S \times S^{[n,m]} \rightarrow S^{[n,m]}$, where the second map is the projection to $S^{[n,m]}$. Consider the graph of $\varphi_1$,

$$\mathbb{P}(\tilde{I}_m) \xleftarrow{\iota} S \times \mathbb{P}(\tilde{I}_m).$$

Now consider the map

$$(\text{Id}_S \times \varphi_2): S \times \mathbb{P}(\tilde{I}_m) \rightarrow S \times S^{[n,m]}. $$

On $S \times \mathbb{P}(\tilde{I}_m)$ there is a canonical surjection

$$\delta: (\text{Id}_S \times \varphi_2)^* \tilde{I}_m \rightarrow \iota_\ast \iota^* (\text{Id}_S \times \varphi_2)^* \tilde{I}_m = \iota_\ast \varphi^* \tilde{I}_m \rightarrow \iota_\ast \mathcal{O}(\tilde{I}_m)(1).$$

Clearly, $\iota_\ast \mathcal{O}(\tilde{I}_m)(1)$ is flat over $\mathbb{P}(\tilde{I}_m)$.

We define a family $\mathcal{T}$ of quotients of length $m + 1$ by the push-out diagram below

$$0 \rightarrow \iota_\ast \mathcal{O}(\tilde{I}_m)(1) \xrightarrow{\delta} \mathcal{T} \xrightarrow{(\text{Id}_S \times \varphi_2)^* \mathcal{O}_m} 0$$

(2.2)

Clearly $\mathcal{T}$ is flat over $\mathbb{P}(\tilde{I}_m)$. This gives a nested family of quotients

$$\mathcal{O}_{S \times \mathbb{P}(\tilde{I}_m)} \rightarrow \mathcal{T} \rightarrow (\text{Id}_S \times \varphi_1)^* \mathcal{O}_m \rightarrow (\text{Id}_S \times \varphi_1)^* \mathcal{O}_n$$

on $S \times \mathbb{P}(\tilde{I}_m)$. Using the universal property for $S^{[n,m+1]}$ and the quotients

$$\mathcal{O}_{S \times \mathbb{P}(\tilde{I}_m)} \rightarrow \mathcal{T} \rightarrow (\text{Id}_S \times \varphi_1)^* \mathcal{O}_n$$

we get a map

$$\psi: \mathbb{P}(\tilde{I}_m) \rightarrow S^{[n,m+1]}.$$  

(2.3)

A pointwise description of this map is given as follows. Let $(p, \xi_n, \xi_m) \in S \times S^{[n,m]}$ be a closed point. So we have a short exact sequence

$$0 \rightarrow \mathcal{I}_{\xi_m} \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_{\xi_m} \rightarrow 0.$$  

A point in $\mathbb{P}(\tilde{I}_m)$ over $(p, \xi_n, \xi_m)$ is given by a quotient $\lambda: \mathcal{I}_{\xi_m} \rightarrow k(p)$. We get the quotient $\mathcal{O}_S \rightarrow \mathcal{O}_{\xi_m+1}$ by the push-out diagram below

$$0 \rightarrow k(p) \xrightarrow{\lambda} \mathcal{O}_{\xi_m+1} \xrightarrow{\mathcal{O}_{\xi_m}} \mathcal{O}_{\xi_m} \rightarrow 0$$

The map $\psi$ takes the point $(p, \xi_n, \xi_m, \lambda)$ of $\mathbb{P}(\tilde{I}_m)$ to the point $(p, \xi_n, \xi_{m+1})$. 


We note the following maps

\[
\begin{array}{ccc}
P(\tilde{\mathcal{I}}_m) & \xrightarrow{\psi} & S^{[n,m+1]} \\
\varphi & & \\
S \times S^{[n,m]} & & \\
\end{array}
\]  

(2.4)

For an \(\mathcal{O}_S\) module \(\mathcal{F}\), we shall denote by \(\mathcal{F}_p\) the localization \(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{S,p}\). Here \(\mathcal{O}_{S,p}\) is the local ring of \(S\) at the closed point \(p\).

**Lemma 2.5.** The map \(\psi\) is surjective on closed points.

**Proof.** A closed point in \(S^{[n,m+1]}\) corresponds to subschemes \(\xi_n \subset \xi_{m+1}\) with \(\text{length}(\xi_n) = n\) and \(\text{length}(\xi_{m+1}) = m + 1\). Let \(K\) denote the kernel of the map \(\mathcal{O}_{\xi_{m+1}} \rightarrow \mathcal{O}_{\xi_n}\). Then we may write

\[
K = \bigoplus_{p \in \text{Supp}(K)} K_p.
\]

Choose any map \(k(p) \rightarrow K_p\) of \(\mathcal{O}_{S,p}\) modules and form the diagram

\[
\begin{array}{ccc}
k(p) & \xrightarrow{\mu} & K \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_{\xi_{m+1}} \rightarrow \mathcal{O}_{\xi_n} \rightarrow 0 \\
\downarrow & & \downarrow \\
\mathcal{O}_{\xi_m} & \rightarrow & \mathcal{O}_{\xi_n} \rightarrow 0 \\
\end{array}
\]

Applying Snake Lemma to the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{I}_{\xi_{m+1}} \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_{\xi_{m+1}} \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{I}_{\xi_m} \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_{\xi_m} \rightarrow 0 \\
\end{array}
\]

one easily concludes that we have a short exact sequence of ideal sheaves

\[
0 \rightarrow \mathcal{I}_{\xi_{m+1}} \rightarrow \mathcal{I}_{\xi_m} \xrightarrow{\lambda} k(p) \rightarrow 0.
\]

One easily concludes that the closed point \((p, \xi_n, \xi_m, \lambda) \in P(\tilde{\mathcal{I}}_m)\) is mapped to the closed point \((\xi_n, \xi_{m+1}) \in S^{[n,m+1]}\) under \(\psi\). This completes the proof of the Lemma. \(\square\)

### 3. Irreducibility of \(S^{[n,m]}\)

Let \(W_{i,[n,m]}\) denote the following locus in \(S \times S^{[n,m]}\)

\[
W_{i,[n,m]} := \{(p, \xi_n, \xi_m) \in S \times S^{[n,m]} \mid \dim(\mathcal{I}_{\xi_m} \otimes k(p)) = i\}.
\]

In other words, it is the locus of points \((p, \xi_n, \xi_m)\) such that the ideal \(\mathcal{I}_{\xi_m}\) is generated by exactly \(i\) elements at the point \(p\). It is clear that \(W_{1,[n,m]}\) is the complement of the universal family \(Z_m\) in \(S \times S^{[n,m]}\). Define subsets \(W_{i',i,[n,m]} \subset W_{i,[n,m]}\) as follows.
Definition 3.1. Let $i \geq 2$. Let $W_{i,l',l,[n,m]} \subset W_{i,[n,m]}$ be the subset consisting of points $(p, \xi_n, \xi_m)$ such that $\text{length}(O_{\xi_m,p}) = l$ and $\text{length}(O_{\xi_n,p}) = l'$.

Notice that for the set $W_{i,l',l,[n,m]}$ to be nonempty we need that $0 \leq l' \leq n$, $0 \leq l' \leq l$ and $1 \leq l \leq m$. As $i \geq 2$, we have that $p \in \text{Supp}(\xi_m)$, which implies that $1 \leq l$. Clearly,

$$W_{i,[n,m]} = \bigcup_{l'} W_{i,l',l,[n,m]}.$$  

In the next lemma, using the sets $W_{i,l',l,[n,m]}$, we shall obtain a bound on the dimension of $W_{i,[n,m]}$. We need the following notations. By $S^{[0,m]}$ we mean $S^m$. Let $p \in S$ denote a closed point. Let $S_{p,l}$ denote the subset of $S^{[l]}$ corresponding to subschemes $\eta_l$ satisfying the following two conditions: $\text{Supp}(\eta_l) = \{p\}$ and $\dim(\mathcal{I}_{\eta_l} \otimes k(p)) = i$. Let $S_{p,l'}^{[l',\ell]}$ denote the subset of $S^{[l',\ell]}$ consisting of pairs $(\xi_{l'}, \xi_{l})$ satisfying the following two conditions: $\text{Supp}(\xi_{l}) = \{p\}$ and $\dim(\mathcal{I}_{\xi_{l}} \otimes k(p)) = i$. By $S_{p,l'}^{[\ell]}$ we mean $S_{p,l'}^{[l',\ell]}$.

Lemma 3.3. Fix integers $n < m$. Assume that $S^{[n-l',m-l]}$ is irreducible of dimension $2(m-l)$ for all pairs $(l',l)$ with $0 \leq n - l' \leq m - l$, $0 \leq l' \leq l$ and $1 \leq l$. Let $i \geq 2$. Then $\dim(W_{i,[n,m]}) \leq 2m + 2 - i$.

Proof. In view of (3.2) it suffices to show that if $W_{i,l',l,[n,m]}$ is nonempty then for $i \geq 2$ we have $\dim(W_{i,l',l,[n,m]}) \leq 2m + 2 - i$. The argument is similar to that of [RT22, Lemma 3.3], along with a key input from [BE16]. Consider the projection map $p_1 : W_{i,l',l,[n,m]} \rightarrow S$ which sends $(p, \xi_n, \xi_m) \mapsto p$. We shall find an upper bound for the dimension of the fiber over a closed point $p \in S$. Let $U$ denote the open subset $S \setminus \{p\}$. Given a point $(p, \xi_n, \xi_m) \in p_1^{-1}(p)$, we may write

$$O_{\xi_m} = O_{\xi_m,p} \bigoplus \left( \bigoplus_{q \in U} O_{\xi_m,q} \right), \quad O_{\xi_n} = O_{\xi_n,p} \bigoplus \left( \bigoplus_{q \in U} O_{\xi_n,q} \right).$$

The quotient $O_{\xi_m} \twoheadrightarrow O_{\xi_n}$ gives rise to quotients

$$O_{\xi_m,p} \twoheadrightarrow O_{\xi_n,p}, \quad \left( \bigoplus_{q \in U} O_{\xi_m,q} \right) \twoheadrightarrow \left( \bigoplus_{q \in U} O_{\xi_n,q} \right).$$

This gives rise to the following map which is an inclusion on closed points

$$p_1^{-1}(p) \rightarrow S_{p,l'}^{[l',\ell]} \times U^{[n-l',m-l]}.$$  

When $l' = 0$ the above map is

$$p_1^{-1}(p) \rightarrow S_{p,l}^{[l]} \times U^{[n,m-l]}.$$  

As $U^{[n-l',m-l]}$ is an open subset of $S^{[n-l',m-l]}$, and the latter is irreducible of dimension $2(m-l)$ by our hypothesis, it follows that $\dim(U^{[n-l',m-l]}) = 2(m-l)$. Next we estimate the dimension of $S_{p,l'}^{[l',\ell]}$.

First we consider the case $l' \neq 0$. Fix a point $\xi_l \in S_{p,l'}^{[l',\ell]}$. Let $M$ be a module over the local ring $O_{S,p}$ whose support is zero dimensional. By $\text{Soc}(M)$ we mean the space
\text{Hom}_{\mathcal{O}_\mathbb{P}}(k(p), M)$. It is easily checked that the space of subschemes $\xi_{l-1} \subset \xi_l$ is in bijective correspondence with $\mathbb{P}((\mathcal{O}_\xi)')$. By [EL99, Lemma 2], we have $\dim(\mathbb{P}((\mathcal{O}_\xi)')) = i - 2$. From this it follows that
\[
\dim(S_{p,i}^{[l-1,d]}) = \dim(S_{p,i}^{[l,d]}) + i - 2.
\]
In [BE16, Corollary 5.9] it is proved that $\dim(S_{p,i}^{[l-1,d]}) = l - 1$. As $S_{p,i}^{[l-1,d]} \subset S_{p,i}^{[l]}$ it follows that
\[
\dim(S_{p,i}^{[l]} + i - 2 = \dim(S_{p,i}^{[l-1,d]}) \leq \dim(S_{p,i}^{[l,d]}) = l - 1.
\]
Thus, it follows that
\[
(3.6) \quad \dim(S_{p,i}^{[l]} \leq l - i + 1.
\]
There is a natural map $S_{p,i}^{[l',d]} \rightarrow S_{p,i}^{[l]} \times S_{p}^{[l']}$ which is an inclusion on closed points. As $l' \geq 1$, $\dim(S_{p}^{[l']}) \leq l' - 1$, see [Bri77]. Thus, it follows that
\[
(3.7) \quad \dim(S_{p,i}^{[l',d]}) \leq l - i + 1 + l' - 1 = l + l' - i \leq 2l - i.
\]
Thus, using (3.4) it follows that
\[
\dim(p_1^{-1}(p)) \leq 2l - i + 2(m - l) = 2m - i,
\]
from which it follows that
\[
\dim(W_{i,l',l,[n,m]}) \leq 2m + 2 - i.
\]
Next we consider the case $l' = 0$. Using (3.5) and (3.6) we get
\[
\dim(p_1^{-1}(p)) \leq 2(m - l) + l - i + 1 = 2m - l - i + 1,
\]
from which it follows that
\[
\dim(W_{i,0,l,[n,m]}) \leq 2m - l - i + 3.
\]
As $l \geq \binom{i}{2}$, see [RT22, Lemma 3.2], it easily follows that $\dim(W_{i,0,l,[n,m]}) \leq 2m + 2 - i$. This completes the proof of the Lemma. \qed

\textbf{Theorem 3.8.} Let $n$ and $m$ be two positive integers such that $n < m$. Then $S^{[n,m,m+1]}$ and $S^{[n,m]}$ are irreducible.

\textbf{Proof.} Let $\mathcal{A}$ be the set of pairs of integers $(a, b)$ with $1 \leq a < b$ and $S^{[a,b]}$ reducible. Assume $\mathcal{A}$ is nonempty. By [Fog73, Corollary 7.3] for every $b \geq 2$ the pair $(1, b) \notin \mathcal{A}$. Similarly, by [Che98, Theorem 3.0.1] for every $a \geq 1$ the pair $(a, a + 1) \notin \mathcal{A}$. Consider the projection map to the first coordinate $\mathcal{A} \rightarrow \mathbb{Z}_{\geq 1}$, where $\mathbb{Z}_{\geq 1}$ denotes the set of positive integers. Let $n$ be the smallest integer in the image of this map. Clearly, $n > 1$. Among the set of integers $b$ such that $(n, b) \in \mathcal{A}$, let $b_0$ be the smallest. Clearly, $b_0 > n + 1$. Let $m = b_0 - 1$. Then $m \geq n + 1$. We conclude that for all pairs of integers $(a, b)$ with $1 \leq a < b$, if $a < n$ then $S^{[a,b]}$ is irreducible, and for all integers $b$ such that $n < b \leq m$, $S^{[n,b]}$ is irreducible. Further $S^{[n,m+1]}$ is reducible. Note that if $S^{[a,b]}$ is irreducible then its dimension is $2b$. We will arrive at a contradiction, which will prove that $\mathcal{A}$ is empty, and hence prove the theorem.
The method of proof is identical to the method in [EL99, Proposition 5]. Consider the map \( \varphi \) in (2.4). We can find locally free sheaves \( \mathcal{F} \) of rank \( r \) and \( \mathcal{E} \) of rank \( r+1 \) on \( S \times S^{[n,m]} \) which fit into a short exact sequence

\[
0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{I}_m \to 0
\]
on \( S \times S^{[n,m]} \). Let \( \pi : \mathbb{P}(\mathcal{E}) \to S \times S^{[n,m]} \) denote the projective bundle. It follows that \( \mathbb{P}(\mathcal{I}_m) \subset \mathbb{P}(\mathcal{E}) \) is the vanishing locus of the composite homomorphism \( \pi^*\mathcal{F} \to \pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \). As \( S \times S^{[n,m]} \) is irreducible, it follows that \( \mathbb{P}(\mathcal{E}) \) is irreducible of dimension \( 2m+2+r \).

As \( \mathbb{P}(\mathcal{I}_m) \) is locally cut out by \( r \) equations, it follows that each irreducible component of \( \mathbb{P}(\mathcal{I}_m) \) has dimension at least \( 2m+2 \).

Let \( i \geq 2 \). The hypothesis of Lemma 3.3 holds and so we get that \( \dim(W_i,[n,m]) \leq 2m+2-i \).

The dimension of the fiber of \( \varphi : \mathbb{P}(\mathcal{I}_m) \to S \times S^{[n,m]} \) over a point \( (p,\xi_n,\xi_m) \in W_i,[n,m] \) is \( i-1 \). Thus,

\[
\dim(\varphi^{-1}(W_i,[n,m])) \leq 2m+2-i+i-1 = 2m+1.
\]

Let \( T \) be an irreducible component of \( \mathbb{P}(\mathcal{I}_m) \). As \( \dim(T) \geq 2m+2 \), it follows that \( T \) cannot be contained in \( \varphi^{-1}(W_i,[n,m]) \) for any \( i \geq 2 \). Thus, \( T \) meets the set \( \varphi^{-1}(W_1,[n,m]) \). Note that \( W_1,[n,m] \) is the complement of \( Z_m \) and so is an open subset of \( S \times S^{[n,m]} \). Moreover, it is clear that

\[
\varphi : \varphi^{-1}(W_1,[n,m]) \to W_1,[n,m]
\]
is an isomorphism. Let \( \tilde{W}_1 \) denote the open and irreducible subset \( \varphi^{-1}(W_1,[n,m]) \). It follows that \( T \cap \tilde{W}_1 \) is open in \( T \) and so also dense in \( T \). It follows that \( T \) is contained in the closure of \( \tilde{W}_1 \). Thus, every irreducible component is contained in the closure of \( \tilde{W}_1 \). As \( \tilde{W}_1 \) is irreducible, so is its closure. It follows that every irreducible component of \( \mathbb{P}(\mathcal{I}_m) \) is contained in the closure of \( \tilde{W}_1 \). Thus, there is only one irreducible component, that is, \( \mathbb{P}(\mathcal{I}_m) \) is irreducible.

We saw in Lemma 2.5 that \( \psi \) is surjective. It follows that \( S^{[n,m+1]} \) is irreducible. This is a contradiction and so \( A \) is empty. As \( \mathbb{P}(\mathcal{I}_m) \cong S^{[n,m+1]} \), the above discussion also shows that \( S^{[n,m,m+1]} \) is irreducible. This completes the proof.

4. Irreducibility of \( S^{[n,n+1,m]} \)

For a tuple of positive integers \( a,b,c \) with \( a < b < c \), the nested Hilbert scheme \( S^{[a,b,c]} \) parametrizes nested closed subschemes \( \xi_a \subset \xi_b \subset \xi_c \) of \( S \), where \( \xi_i \) is a finite scheme of length \( i \). We have the universal nested family of closed subschemes \( Z_c \subset S \times S^{[a,b,c]} \). The closed points of \( Z_c \) have the following descriptions.

\[
Z_c = \{ (p,\xi_a,\xi_b,\xi_c) \in S \times S^{[a,b,c]} \mid p \in \xi_c \}.
\]

We have the projection map

\[
\pi_c : S^{[a,b,c]} \to S^{[c]}.
\]

Let \( \mathcal{I}_c \) denote the ideal sheaf of the universal subscheme inside \( S \times S^{[c]} \). Consider the map

\[
\text{Id}_S \times \pi_c : S \times S^{[a,b,c]} \to S \times S^{[c]}.
\]

Denote the pullback

\[
\tilde{\mathcal{I}}_c := (\text{Id}_S \times \pi_c)^*\mathcal{I}_c.
\]
Consider the projective bundle
\begin{equation}
\varphi : \mathbb{P}(\mathcal{J}_c) \longrightarrow S \times S^{[a,b,c]}.
\end{equation}
We define the map \(\psi : \mathbb{P}(\mathcal{J}_c) \longrightarrow S^{[a,b,c+1]}\) in the same way as defined in (2.3) in §2. We have the following maps
\begin{equation}
\begin{array}{ccc}
\mathbb{P}(\mathcal{J}_c) & \xrightarrow{\psi} & S^{[a,b,c+1]} \\
\downarrow \varphi & & \downarrow \\
S \times S^{[a,b,c]} & & 
\end{array}
\end{equation}

The pointwise description of the map \(\psi\) is similar to the one given in §2 and is left to the reader. By similar argument as in the proof of Lemma 2.5, we conclude that the map \(\psi\) is surjective on closed points.

As in the case of \(S^{[n,m]}\), here also we define the subsets \(W_{i,[n,n+1,m]}\) in a similar manner. Let \(W_{i,[a,b,c]}\) denote the locus in \(S \times S^{[a,b,c]}\) where the ideal sheaf \(\mathcal{I}_c\) of \(Z_c\) is generated by \(i\) elements, that is,
\[W_{i,[a,b,c]} := \{(p, \xi_a, \xi_b, \xi_c) \in S \times S^{[a,b,c]} \mid \dim(\mathcal{I}_{\xi_c} \otimes k(p)) = i\}.
\]
The set \(W_{1,[a,b,c]}\) is the complement of the universal family \(Z_c\) in \(S \times S^{[a,b,c]}\). Define subsets \(W_{i,l',l,[a,b,c]} \subset W_{i,[a,b,c]}\) as follows.

**Definition 4.3.** Let \(i \geq 2\). Let \(W_{i,l'',l',l,[a,b,c]} \subset W_{i,[a,b,c]}\) be the subset consisting of points \((p, \xi_a, \xi_b, \xi_c)\) such that \(\text{length}(\mathcal{O}_{\xi_a,p}) = l''\), \(\text{length}(\mathcal{O}_{\xi_b,p}) = l'\) and \(\text{length}(\mathcal{O}_{\xi_c,p}) = l\).

Notice that for the set \(W_{i,l'',l',l,[a,b,c]}\) to be nonempty we need that \(0 \leq a - l'' \leq b - l' \leq c - l\), \(0 \leq l'' \leq l' \leq l\) and \(1 \leq l\). As \(i \geq 2\), we have that \(p \in \text{Supp}(\xi_m)\), which implies that \(1 \leq l\). Clearly,
\begin{equation}
W_{i,[a,b,c]} = \bigcup_{l',l} W_{i,l'',l',l,[a,b,c]}.
\end{equation}
Let \(p \in S\) denote a closed point. Let \(S_{p,i}^{l'',l',l}\) denote the subset of \(S^{[l'',l',l]}\) consisting of the tuples \((\xi_{l''}, \xi_{l'}, \xi_l)\) satisfying the following two conditions: \(\text{Supp}(\xi_l) = \{p\}\) and \(\dim(\mathcal{I}_{\xi_l} \otimes k(p)) = i\).

**Lemma 4.5.** Let \(n\) and \(m\) be two positive integers such that \(n + 1 < m\). Assume that \(S^{[n-l'',n+1-l'-m-l]}\) is irreducible of dimension \(2(m-l)\) for all triples \((l'',l',l)\) with \(0 \leq n - l'' \leq n+1 - l' \leq m - l\), \(0 \leq l'' \leq l' \leq l\) and \(1 \leq l\). Let \(i \geq 2\). Then \(\dim(W_{i,[n,n+1,m]}) \leq 2m + 2 - i\).

**Proof.** It suffices to prove that for \(i \geq 2\), if \(W_{i,l'',l',l,[n,n+1,m]}\) is nonempty then
\[\dim(W_{i,l'',l',l,[n,n+1,m]}) \leq 2m + 2 - i.\]
The proof is very similar to the proof of Lemma 3.3 and so we omit some details. Consider \(p_1 : W_{i,l'',l',l,[n,n+1,m]} \longrightarrow S\) which sends \((p, \xi_n, \xi_{n+1}, \xi_m)\) to \(p\). We find an upper bound for
Thus, we get

\[ (\bigoplus_{q \in U} \mathcal{O}_{\xi_m,q}) \to (\bigoplus_{q \in U} \mathcal{O}_{\xi_{n+1},q}) \]

and the quotient \( \mathcal{O}_{\xi_{n+1}} \to \mathcal{O}_{\xi_n} \) gives rise to quotients

\[ (\bigoplus_{q \in U} \mathcal{O}_{\xi_{n+1},q}) \to (\bigoplus_{q \in U} \mathcal{O}_{\xi_n,q}) \]

This gives rise to the following map which is an inclusion on closed points

\[ p_1^{-1}(p) \to S[p^\prime]\times U_{[n-l''-n+1-l',m-l]} \]

We note that \( n + 1 - l' \geq n - l'' \), that is, \( l' \leq l'' + 1 \). If \( l'' = l' \) then by our hypothesis \( S[n-l''-n+1-l',m-l] \) is irreducible of dimension \( 2(m-l) \). If \( l'' = l' - 1 \) then \( S[n-l''-n+1-l',m-l] \) is same as \( S[n+1-l',m-l] \), which is irreducible of dimension \( 2(m-l) \) by Theorem 3.8. So it follows that \( \dim U_{[n-l'',n+1-l',m-l]} = 2(m-l) \).

Now we need to find an upper bound of \( \dim(S[p^\prime]) \). We have two cases: \( l'' = l' - 1 \) and \( l'' = l' \). We first consider the case \( l'' = l' - 1 \). There is a natural map

\[ S[p^\prime] \to S[l'] \times S[l''] \]

which is an inclusion on closed points. As \( l'' = l' - 1 \), by [BE16, Corollary 5.9] we have \( \dim(S[l']) = l' - 1 \). Also from (3.6), we get \( \dim(S[p^\prime]) \leq l + 1 - i \). So it follows that

\[ \dim S[p^\prime] \leq (l + 1 - i) + (l' - 1) \leq 2l - i \]

This gives

\[ \dim(p_1^{-1}(p)) \leq 2(m-l) + 2l - i = 2m - i \]

Thus, we get

\[ \dim(W_{i,l'',l',a,a+1,m}) \leq 2m - 2 - i \]

Next we consider the case \( l'' = l' \). In this case \( S[p^\prime] \) is same as \( S[l'] \) which has dimension at most \( 2l - i \) by (3.7). Thus again we get

\[ \dim(W_{i,l'',l',a,a+1,m}) \leq 2m - 2 - i \]

This proves the lemma. \( \square \)

**Theorem 4.7.** Let \( n \) and \( m \) be two positive integers such that \( n+1 < m \). Then \( S[n,n+1,m,m+1] \) and \( S[n,n+1,m] \) is irreducible.

**Proof.** We follow the same method as we used in the proof of Theorem 3.8. Let \( A \) be the set of pairs of integers \( (a,b) \) with \( 1 \leq a \), \( a + 1 < b \) and \( S[a,a+1,b] \) reducible. Assume that \( A \) is nonempty. By [RT22, Theorem 3.10] for every \( a \geq 1 \) the pair \( (a,a+2) \notin A \). Consider the projection map to the first coordinate \( A \to \mathbb{Z}_{\geq 1} \). Let \( n \) be the smallest integer in the image of this map. Among the set of integers \( b \) such that \( (n,b) \in A \), let \( b_0 \) be the smallest. Clearly, \( b_0 > n + 2 \). Let \( m = b_0 - 1 \). Then \( m \geq n + 2 \). We conclude that for all pairs of
integers \((a, b)\) with \(1 \leq a, a + 1 < b\), if \(a < n\) then \(S^{[a, a+1, b]}\) is irreducible and \(S^{[n, n+1, b]}\) is irreducible if \(b \leq m\). Further \(S^{[n, n+1, m+1]}\) is reducible. Note that if \(S^{[a, a+1, b]}\) is irreducible then its dimension is \(2b\). A similar argument as in the proof of Theorem 3.8, after replacing Lemma 3.3 with Lemma 4.5, concludes the proof of the Theorem. \(\square\)

5. Irreducibility of \(S^{[n, n+2, m]}\)

We begin with the following Lemma.

**Lemma 5.1.** Fix integers \(1 \leq n\) and \(n + 2 < m\). Assume that \(S^{[n-l''-n+2-l', m-l]}\) is irreducible of dimension \(2(m-l)\) for all triples \((l'', l', l)\) with \(0 \leq n-l'' \leq n+2-l' \leq m-l\), \(0 \leq l'' \leq l' \leq l\) and \(1 \leq l\). Let \(i \geq 2\). Then \(\dim(W_{i,[n,n+2,m]}) \leq 2m + 2 - i\).

**Proof.** From (4.4) we have

\[
W_{i,[n,n+2,m]} = \bigcup_{l''', l', l} W_{i,l''', l', l,[n,n+2,m]}.
\]

So it suffices to prove that \(\dim(W_{i,l''', l', l,[n,n+2,m]}) \leq 2m + 2 - i\) for \(i \geq 2\). Consider the projection \(p_1 : W_{i,l''', l', l,[n,n+2,m]} \rightarrow S\) which sends \((p, \xi_n, \xi_{n+2}, \xi_m)\) to \(p\). We find an upper bound for the dimension of the fiber over a closed point \(p \in S\). Let \(U\) be the open subset \(S \setminus \{p\}\). Given a point \((p, \xi_n, \xi_{n+2}, \xi_m) \in p_1^{-1}(p)\), the quotient \(O_{\xi_m} \rightarrow O_{\xi_{n+2}}\) gives rise to quotients

\[
O_{\xi_m, p} \rightarrow O_{\xi_{n+2}, p}, \quad \left( \bigoplus_{q \in U} O_{\xi_m, q} \right) \rightarrow \left( \bigoplus_{q \in U} O_{\xi_{n+2}, q} \right)
\]

and the quotient \(O_{\xi_{n+2}} \rightarrow O_{\xi_n}\) gives rise to quotients

\[
O_{\xi_{n+2}, p} \rightarrow O_{\xi_n, p}, \quad \left( \bigoplus_{q \in U} O_{\xi_{n+2}, q} \right) \rightarrow \left( \bigoplus_{q \in U} O_{\xi_n, q} \right).
\]

This gives rise to the following map which is an inclusion on closed points

\[
p_1^{-1}(p) \rightarrow S^{[l'', l', l]}_{p, i} \times U^{[n-l'', n+2-l', m-l]}.
\]

We note that \(n + 2 - l' \geq n - l''\), that is, \(l' \leq l'' + 2\). We consider three cases according to whether \(l'' = l'', l''+1, l''+2\). If \(l'' = l''+1\) then by our hypothesis \(S^{[n-l'', n+2-l', m-l]}\) is irreducible of dimension \(2(m-l)\). If \(l'' = l''+1\) then \(S^{[n-l'', n+2-l', m-l]}\) is the same as \(S^{[n-l'', n+1-l', m-l]}\), which is irreducible of dimension \(2(m-l)\) by Theorem 4.7. If \(l'' = l''+2\) then \(S^{[n-l'', n+2-l', m-l]}\) is same as \(S^{[n+2-l', m-l]}\), which is irreducible of dimension \(2(m-l)\) by Theorem 3.8. So it follows that \(\dim(U^{[n-l'', n+2-l', m-l]}) = 2(m-l)\).

Now we need to find an upper bound of \(\dim(S^{[l'', l', l]}_{p, i})\). We have three cases: \(l'' = l''-2, l'' = l''-1\) and \(l'' = l''\). We first consider the cases \(l'' = l''-2\) or \(l'' = l''-1\). There is a natural map

\[
S^{[l'', l', l]}_{p, i} \rightarrow S^{[l]}_{p, i} \times S^{[l', l]}_p
\]

which is an inclusion on closed points. If \(l'' = l''-2\) then we use [BE16, Corollary 7.5], and if \(l'' = l''-1\) then we use [BE16, Corollary 5.9], to conclude \(\dim(S^{[l', l']}_{p}) = l' - 1\). Also from
(3.6), we get $\dim(S_{p,i}^{[n]}) \leq l + 1 - i$. So it follows that

$$\dim S_{p,i}^{[n',l',l]} \leq (l + 1 - i) + (l' - 1) \leq 2l - i.$$ 

This gives

$$\dim(p_1^{-1}(p)) \leq 2(m - l) + 2l - i = 2m - i.$$ 

Thus we get

$$\dim(W_{i,l',l,[n,n+2,m]}) \leq 2m + 2 - i.$$ 

Next we consider the case $l'' = l'$. In this case $S_{p,i}^{[n',l',l]}$ is same as $S_{p,i}^{[l',l]}$ which has dimension at most $2l - i$ by (3.7). Thus again we get

$$\dim(p_1^{-1}(p)) \leq 2(m - l) + 2l - i = 2m - i$$

and hence

$$\dim(W_{i,l',l,[n,n+2,m]}) \leq 2m + 2 - i.$$ 

This proves the lemma. \hfill \Box

**Theorem 5.2.** Let $n$ and $m$ be two positive integers such that $n + 2 < m$. Then $S^{[n,n+2,m,m+1]}$ and $S^{[n,n+2,m]}$ are irreducible.

*Proof.* We follow the same method as we used in proof of Theorem 3.8. Let $\mathcal{A}$ be the set of pairs of integers $(a, b)$ with $1 \leq a$, $a + 2 < b$ and $S^{[a,a+2,b]}$ reducible. We prove that $\mathcal{A}$ is empty. Taking $(n, m) = (a, a + 2)$ in Theorem 4.7 shows that $S^{[a,a+1,a+2,a+3]}$ is irreducible and so it follows that $S^{[a,a+2,a+3]}$ is irreducible. Thus, it follows that for every $a \geq 1$ the pair $(a, a + 3) \notin \mathcal{A}$. Consider the projection map to the first coordinate $\mathcal{A} \rightarrow \mathbb{Z}_{\geq 1}$. Let $n$ be the smallest integer such in the image of this map. Among the set of integers $b$ such that $(n, b) \in \mathcal{A}$, let $b_0$ be the smallest. Clearly, $b_0 > n + 3$. Let $m = b_0 - 1$. Then $m \geq n + 3$. We conclude that for all pairs of integers $(a, b)$ with $1 \leq a$, $a + 2 < b$, if $a < n$ then $S^{[a,a+2,b]}$ is irreducible, and $S^{[n,n+2,m]}$ is irreducible if $b \leq m$. Further $S^{[n,n+2,m+1]}$ is reducible. A similar argument as in the proof of Theorem 3.8, after replacing Lemma 3.3 with Lemma 5.1, concludes the proof of the Theorem. \hfill \Box

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