The existence of null circular geodesics outside extremal spherically symmetric asymptotically flat hairy black holes

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Abstract The existence of null circular geodesics in the background of non-extremal spherically symmetric asymptotically flat black holes has been proved in previous works. An interesting question that remains, however, is whether extremal black holes possess null circular geodesics outside horizons. In the present paper, we focus on the extremal spherically symmetric asymptotically flat hairy black holes. We show the existence of the fastest circular trajectory around an extremal black hole. As the fastest trajectory corresponds to the position of null circular geodesics, we prove that null circular geodesics exist outside extremal spherically symmetric asymptotically flat hairy black holes.

1 Introduction

It is generally acknowledged that highly curved black hole spacetimes are characterized by the existence of null circular geodesics outside horizons [1,2]. Such geodesics are how massless particles can orbit the central black holes. They provide valuable information on the structure of the spacetime geometry, which is closely related to various remarkable properties of black hole spacetimes and has attracted considerable attention [3–5].

The presence of circular null geodesics has many important implications for astronomical observations, such as gravitational lensing and black hole shadow, as well as gravitational waves [4,6–15]. Circular null geodesics also provide a lower bound for the effective radius of scalar field hairs outside black holes [16–21]. The characteristic resonances of perturbed black holes can be determined by the instability properties of circular null geodesics [22–29]. Massless fields can accumulate on stable circular null geodesics, leading to nonlinear instabilities in highly curved spacetimes [4,30–36]. It has also been shown that the null circular geodesic corresponds to the fastest orbital trajectory around a central black hole [37].

The existence of null circular geodesics can be predicted by analyzing the non-linearly coupled Einstein-matter field equations. In non-extremal hairy black hole spacetimes, the existence of null circular geodesics has been proved in the asymptotically flat background [38]. Null circular geodesics also exist in the stationary axisymmetric non-extremal black hole spacetimes [39]. Hod raised an interesting question of whether null circular geodesics can also exist outside extremal black holes [40]. Motivated by the discussion of [40], our work and [42] independently provide alternative elegant proofs of the same result indicating that null geodesics do exist outside spherically symmetric extremal black holes. Specifically, [42] established remarkably compact proofs based on the dominant energy condition, and we will establish the proof that null circular geodesics provide the fastest orbital trajectory around a black hole [37].

In this work, we first introduce the extremal hairy black hole spacetimes. We prove the existence of null circular geodesics by analyzing the existence of the circular trajectory around central extremal hairy black holes. Finally, we summarize our main results and draw conclusions.

2 The existence of null circular geodesics in extremal black holes

We study the physical and mathematical properties of null circular geodesics in the background of extremal hairy black hole spacetimes. The spherically symmetric curved line element of a four-dimensional asymptotically flat hairy black hole is described by [18,41]

\[
ds^2 = -e^{-2\phi} \mu dr^2 + \mu^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{1}
\]
Here, $\delta$ and $\mu$ are metric solutions depending only on the Schwarzschild areal coordinate $r$. Angular coordinates are $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The black hole horizon $r_H$ is obtained from $\mu(r_H) = 0$. At the regular horizon, $\delta$ is finite. At the infinity, there is $\mu(\infty) = 1$ and $\delta(\infty) = 0$ in the asymptotically flat spacetimes.

We study null circular geodesics in the equatorial plane with $\theta = \pi/2$. In the asymptotically flat spacetimes. We study null circular geodesics in the equatorial plane with $\theta = \pi/2$. In this case, the orbital period can be obtained from $\mu(\infty)$.

The null circular geodesics usually correspond to the circular trajectory with the shortest orbital period [37]. In the following, we would like to search for the circular trajectory with the shortest orbital period as measured by asymptotic observers. In order to minimize the orbital period for a given radius $r$, one should move as close as possible to the speed of light. In this case, the orbital period can be obtained from $\mu(\infty)$.

The null circular geodesics equation is identical to the fastest circular trajectory equation (13). Thus, the extreme period circular radius equation and the null circular geodesics equation share the same roots. If we can prove the existence of the fastest circular trajectory, the null circular geodesics exist outside black holes. In fact, there are relations $T(r_H) = T(\infty) = \infty$ according to the formula (11). Thus, the function $T(r)$ must possess a minimum at some finite radius $r = r_{extrem}$. At this radius, Eqs. (10) and (13) hold, which means that $r = r_{extrem}$ corresponds to the position of the null circular geodesics. In other words, we prove the existence of null circular geodesics outside extremal black holes.

3 Conclusions

We have investigated the existence of null circular geodesics outside extremal spherically symmetric asymptotically flat hairy black holes. We first obtained the null circular geodesic characteristic equations. We then obtained the equation governing the fastest circular trajectory that particles can follow around the central extremal spherically symmetric asymptotically flat hairy black hole. We showed that the null circular geodesic equation is identical to the fastest circular trajectory equation. Thus, the extreme period circular radius equation and the null circular geodesic equation share the same roots. We found that solutions exist for the extreme period circular radius equation; therefore, we proved that the null circular geodesic exists outside extremal spherically symmetric asymptotically flat hairy black holes.

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It should be emphasized that an analytical proof for the existence of null circular geodesics in extremal black hole spacetimes has also recently been given in Ref. [42] (published in the arXiv in the same week as the current manuscript).

Data availability statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This work is based on analytical studies. So there is no numerical data and all analysis has been presented in the paper].

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