A MATHEMATICAL MODEL FOR HEPATITIS B WITH INFECTION-AGE STRUCTURE

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Abstract. A model with age of infection is formulated to study the possible effects of variable infectivity on HBV transmission dynamics. The stability of equilibria and persistence of the model are analyzed. The results show that if the basic reproductive number $R_0 < 1$, then the disease-free equilibrium is globally asymptotically stable. For $R_0 > 1$, the disease is uniformly persistent, and a Lyapunov function is used to show that the unique endemic equilibrium is globally stable in a special case.

1. Introduction. Hepatitis B is a potentially life-threatening liver infection caused by the hepatitis B virus (HBV), which is a circular genome composed of partially double-stranded DNA and can hardly be cleared after infection because of the formation of cccDNA. It is estimated that approximately 2 billion people have serological evidence of past or present HBV infection. Over 350 million people are chronic carriers of HBV and every year about 600,000 people die from HBV-related liver disease or hepatocellular carcinoma (HCC). Although there is no widely available treatment for chronic HBV carriage, the infection can be safely and effectively prevented by vaccination, which includes passive immunoprophylaxis and after birth active immunization[1].

HBV can cause an acute illness and a chronic liver infection which is characterized by persistent serum level of HBV surface antigen (HBsAg), IgG anti-core antigen(anti-HBc) and HBV DNA [3]. Chronic infection may later develop into cirrhosis of the liver or liver cancer[11]. The incapability to clear infection and the subsequent development of the carrier state is almost certainly due to host factors [4]. It has become increasingly acknowledged that the probability of becoming chronically infected is dependent on age of the host[11, 4, 12, 6]. In general, the average age at which individuals become infected is largely determined by the prevalence of infection in the population. Mathematical model with age structure may be more reasonable to study the important consequences of age for the HBV infection.

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In many epidemiological models, it has been assumed that all infected individuals are equally infectious during their infectivity period, which is proved to be reasonable for some diseases such as influenza. Actually, the infectivity of HBV individuals varies at different age of infection, thus it is necessary to formulate age structure models to describe the heterogeneity in infectious individuals, which will lead to a system of partial differential equations. In general, there are two different age structures in disease models, i.e., biological age and infection age. Although their dynamical analysis is particularly challenging, the epidemic models with age structures have been extensively studied recently[9, 17, 2, 5]. Since the progress of the acute and chronic stages of HBV is complicated, depending on the time that has passed since the moment of infection, it is imperative to develop models with the infection age structure to show its influence on transmission dynamics of HBV infection.

Several recent studies have focused on modeling HBV transmission dynamics [13, 25, 20, 16]. These models mainly investigated the influence of prevention and control measures including vaccination and antivirus treatment in some regions and countries, with the form of ordinary equations. Age-structured models have also been developed to study the epidemiology of HBV[4, 23, 24]. Medley et al observed a feedback mechanism that determined the prevalence of HBV infection, relating the rate of transmission, average age at infection and age-related probability becoming carriers[12]. Based on sero-survey data, Zhao et al constructed an age-structured model to predict the dynamics of HBV and to evaluate the long-term effectiveness of the vaccination programme [23]. Zou et al in [24] proposed a model that incorporated age structure into individuals to study the transmission dynamics of HBV, analysing the existence and stability of the disease-free and endemic steady state solutions. They used numerical simulations to illustrate the optimal strategies for controlling the disease.

In order to study the possible effects of variable infectivity on HBV transmission, we develop a mathematical model with infection-age structure to describe the HBV infection dynamics, which extends the existing one of ordinary equations[21]. In this model, the infectivity is allowed to depend on the age of infection. The parameters, such as the transmission coefficient, the rate moving from acute to chronic stage, and the rate moving from carrier to immunized class, are age-dependent. A detailed analysis of stability and uniform persistence is conducted. The dynamic behaviors of the system are determined by the basic reproductive number $R_0$, and a special case is studied by constructing a Lyapunov function to analyse the global stability of endemic equilibrium.

The organization of the remaining part is as follows. In section 2, we formulate the model with infection-age structure for HBV transmission. In Section 3, we study the existence and stability of steady states. Section 4 focuses on the analysis of uniform persistence. A special case is presented in section 5 to illustrate the global stability of endemic equilibrium. Section 6 gives the summary and conclusions.

2. Formulation of the model. The transmission begins when a susceptible subject acquires an acute HBV infection through effective contact with a temporary or a chronic HBV carrier, shifts to the latent period, and then the individual becomes an acute HBV for 3 months on average. If the acute infection does not progress to a chronic one, the host clears HBV, recovers, and becomes immune. A chronic HBV carrier that lasts for many years can also follow acute infection. A few chronic
of HBV transmission, we construct the following age-structured model for HBV risk of becoming chronic carriers is extremely high (90%) for the neonates who horizontal transmission. We assume that all the newborns are vaccinated at the birth. It can be seen that the infection age of an infected individual may be important for the transmission of HBV among population. Two major transmission routes are included in the model, that is, perinatal and horizontal transmission. We assume that all the newborns are vaccinated at the same efficacy, and all the vertical infected infants are the chronic carriers since the risk of becoming chronic carriers is extremely high (90%) for the neonates who acquire HBV infection perinatally. Based on these characteristics and assumptions of HBV transmission, we construct the following age-structured model for HBV

\[
\begin{align*}
S'(t) &= b_\omega (1 - \int_{a_1}^{a_2} v(a)c(a,t)da) - (\int_{a_1}^{a_2} \beta(a)i(a,t)da + \alpha \int_{a_1}^{a_2} \beta(a)c(a,t)da)S - (\mu + p)S, \\
E'(t) &= (\int_{a_1}^{a_2} \beta(a)i(a,t)da + \alpha \int_{a_1}^{a_2} \beta(a)c(a,t)da)S - (\mu + \sigma)E, \\
\frac{\partial c(a,t)}{\partial a} + \frac{\partial c(a,t)}{\partial t} &= -(\mu + \gamma_1(a))i(a,t), \quad 0 < a < a_1 \tag{S_1} \\
R'(t) &= b(1 - \omega) + \int_{a_1}^{a_2} (1 - q(a))\gamma_1(a)i(a,t)da + \int_{a_1}^{a_2} \gamma_2(a)c(a,t)da - \mu R + pS,
\end{align*}
\]

with the boundary conditions

\[
i(0, t) = \sigma E(t), \quad c(a_1, t) = \int_{a_1}^{a_2} q(a)\gamma_1(a)i(a,t)da + b_\omega \int_{a_1}^{a_2} v(a)c(a,t)da,
\]

and initial conditions

\[
S(0) = S_0, \quad E(0) = E_0, \quad i(0) = i_0(a), \quad c(0) = c_0(a), \quad R(0) = R_0.
\]

Here, \(S(t), E(t)\) and \(R(t)\) denote population of susceptible, exposed and immunized at time \(t\), respectively. \(i(a, t)\) and \(c(a, t)\) denote the density of acute infections and chronic HBV carriers with infection age \(a\) at time \(t\). The population of acute HBV is \(I(t) = \int_{a_1}^{a_2} i(a,t)da\), and \(C(t) = \int_{a_1}^{a_2} c(a,t)da\) represents the chronic HBV carriers, where the critical infection age \(a_1\) equals to 3 months, since an acute HBV lasts for 3 months on average, and then the chronic stage begins. The definitions of the parameters in system \((S_1)\) are listed in Table 2.1.

**Table 1.** Definitions of parameters used in model \((S_1)\)

| Symbol | Definition |
|--------|------------|
| \(b\)  | birth rate |
| \(\mu\) | death rate |
| \(\omega\) | proportion of children that is unsuccessfully immunized at birth |
| \(p\)  | vaccination rate of susceptible children and adults |
| \(\alpha\) | infectiousness of carriers relative to acute infections |
| \(\sigma\) | rate of transfer from exposed to acute infection |
| \(v(a)\) | proportion of children developing to HBV carriers born to carrier mothers of infection age \(a\) |
| \(\beta(a)\) | age-dependent transmission coefficient |
| \(\gamma_1(a)\) | age-dependent rate moving from acute to chronic or immunized class |
| \(\gamma_2(a)\) | age-dependent rate moving from carrier to immunized |
| \(q(a)\)  | rate leaving acute infection and progressing to carrier with age \(a\) |
| \(\theta(a)\) | HBV induced death rate with age-dependence |

In our analysis, the functions \(v(a), \beta(a), \gamma_1(a), \gamma_2(a), q(a)\) and \(\theta(a)\) are assumed to be nonnegative, bounded and integrable in their definition intervals. The function \(\beta(a)\) describes the variable probability of infectiousness as the disease progresses...
within an infected individual. The other constant parameters are nonnegative, and the initial conditions satisfy \( S_0 \geq 0, E_0 \geq 0, R_0 \geq 0, i_0(a) \in L^1(0, \infty) \) and \( c_0(a) \in L^1(0, \infty) \).

To simplify expressions, we introduce the following notations

\[
\begin{align*}
\pi_1(a) &= e^{-\int_0^a (\mu + \gamma_1(s)) \, ds}, \quad a \in [0, a_1], \\
\pi_2(a) &= e^{-\int_{a_1}^{a} (\mu + \gamma_2(s) + \theta_s) \, ds}, \quad a \in [a_1, \infty), \\
W_1 &= \int_{a_1}^{a} \beta(a) \pi_1(a) \, da, \quad W_2 = \int_{a_1}^{a} q(a) \gamma_1(a) \pi_1(a) \, da, \\
W_3 &= \int_{a_1}^{a} \beta(a) \pi_2(a) \, da, \quad W_4 = \int_{a_1}^{a} \gamma_2(a) \pi_2(a) \, da, \\
W_5 &= \int_{a_1}^{a} v(a) \pi_2(a) \, da,
\end{align*}
\]

here \( \pi_1(a) \) is the age-specific survival probability of an acute infected individual, and \( \pi_2(a) \) is that of a chronic one. With the given boundary and initial conditions, integrating \( i(a, t) \) and \( c(a, t) \) along the characteristic lines \((t - a = \text{constant})\) yields

\[
i(a, t) = \begin{cases} i(0, t - a) \pi_1(a) & t > a, \quad a \in [0, a_1] \\
i_0(a - t) \pi_1(a) & t \leq a, \quad a \in [0, a_1]. \end{cases}
\]

and

\[
c(a, t) = \begin{cases} c(a_1, a_1 + t - a) \pi_2(a) & t + a_1 > a, \quad a \in [a_1, \infty) \\
c_0(a - t + a) \pi_2(a) & t + a_1 \leq a, \quad a \in [a_1, \infty). \end{cases}
\]

Using classical existence and uniqueness results for functional differential equations, it is seen that the integro-differential system \((S_1)\) has a unique solution, in which \( i(a, t) \) and \( c(a, t) \) are substituted with the expression \((1)\) and \((2)\), respectively. By \((1)\) and \((2)\), it is easy to see that \( i(a, t) \) and \( c(a, t) \) remains nonnegative for any nonnegative initial value. Further, if there exists \( t^* \) such that \( S(t^*) = 0 \) and \( S(t) > 0 \) for \( 0 < t < t^* \), then from the first equation of \((S_1)\) we can get \( S'(t^*) = b\omega(1 - \int_{a_1}^{a_1} v(a)c(a,t) \, da) > 0 \), which implies that \( S(t) \geq 0 \) for all \( t \geq 0 \). Similarly, it can be shown that \( E(t) \geq 0 \) and \( R(t) \geq 0 \) for all \( t \geq 0 \) and for all nonnegative initial values.

For epidemic model, the basic reproductive number is an important threshold parameter, which gives the expected number of secondary cases produced in a completely susceptible population by a typical infective individual. As for the HBV transmission model \( S_1 \), we define the basic reproductive number as

\[
R_0 = \frac{b\omega}{\mu + p} \frac{\sigma}{\sigma + \mu} \left( W_1 + \frac{\alpha W_2 W_3}{1 - b\omega W_5} \right).
\]

\( W_1 \) gives the number of individuals infected by one acute infection, and \( W_2 \) is that of leaving acute infection and progressing to chronic stage. The number of infections produced by an HBV carrier is expressed by \( \alpha W_3 \). \( b\omega W_5 \) is the quantity of vertical infected infants.

3. Steady states and their stability. System \( S_1 \) always has a disease-free equilibrium \( P_0 = (S_0, 0, 0, 0, R_0) \) where \( S_0 = \frac{b\omega}{\mu + p}, \quad R_0 = \frac{b(\mu + p - \omega)}{\mu(\mu + p)} \). There may exist an endemic equilibrium \( P^* = (S^*, E^*, i^*(a), c^*(a), R^*), \) where \( i^*(a) \) and \( c^*(a) \) satisfy the following equations

\[
i^*(a) = \sigma E^* \pi_1(a), \\
c^*(a) = c^*(a_1) \pi_2(a).
\]

Substituting \( i^*(a) \) and \( c^*(a) \) into the boundary conditions about yields

\[
c^*(a_1) = \sigma E^* W_2 + b\omega c^*(a_1) W_5,
\]
so it follows that

\[ c^*(a_1) = \frac{\sigma E^* W_2}{1 - b_0 W_5}. \]

Solving the second and first equations of system \( S_1 \) in terms of \( S \) and \( E \) gives

\[ S^* = \frac{(1 - b_0 W_5)(\sigma + \mu)}{\sigma W_1 (1 - b_0 W_5) + \alpha \sigma W_2 W_5}, \]

and

\[ E^* = \frac{(b_0 - (\mu + p) S^*)(1 - b_0 W_5)}{(\sigma + \mu)(1 - b_0 W_5) + b_0 \sigma W_2 W_5}. \]

System \( S_1 \) has a unique equilibrium \( P^* = (S^*, E^*, i^*(a), c^*(a), R^*) \) if and only if \( R_0 > 1 \), where

\[ S^* = \frac{b_0}{(\sigma + \mu)(1 - b_0 W_5)}, \quad E^* = \frac{b_0 (1 - b_0 W_5)}{(\sigma + \mu) (1 - b_0 W_5) + b_0 \sigma W_2 W_5}, \]

\[ i^*(a) = \sigma E^* \pi_1(a), \quad c^*(a) = \frac{\sigma E^* W_2 \pi_2(a)}{1 - b_0 W_5}, \]

\[ R^* = \frac{1}{\mu} b (1 - \omega) + \int_0^{a_1} (1 - q(a)) \gamma_1(a) i^*(a) da + \int_0^{\infty} \gamma_2(a) c^*(a) da + p S^*. \]

Notice that the variable \( R \) does not appear in equations for other variables, thus the equation of \( R \) can be ignored when studying the dynamics of HBV infection. We denote the system of the remaining equations as \( S_2 \), which will be studied in the rest part of the paper.

We consider the stability of disease-free equilibrium \( P_0 \), and have the following result

**Theorem 3.1.** The disease-free equilibrium \( P_0 \) is globally stable when \( R_0 < 1 \) and it is unstable when \( R_0 > 1 \).

**Proof.** Firstly, we use the following transformation to study the local stability of \( P_0 \)

\[ \begin{align*}
S(t) &= s(t) + \bar{S}_0, \quad E(t) = e(t), \quad i(a, t) = u_1(a, t), \quad c(a, t) = u_2(a, t).
\end{align*} \]

Using these expressions and noticing that \( \bar{S}_0 = \frac{b_0}{\mu + p} \), we linearize \( S_2 \) at \( P_0 \), which leads to the following linear system

\[ \begin{align*}
&\begin{cases}
    s'(t) = -(\mu + p) s(t) - b_0 \int_{a_1}^{\infty} v(a) u_2(a, t) da - \bar{S}_0 \left( \int_{a_1}^{\infty} \beta(a) u_1(a, t) da + \alpha \int_{a_1}^{\infty} \beta(a) u_2(a, t) da \right), \\
    e'(t) = \bar{S}_0 \left( \int_{a_1}^{\infty} \beta(a) u_1(a, t) da + \alpha \int_{a_1}^{\infty} \beta(a) u_2(a, t) da \right) - (\mu + \sigma) e(t), \\
    u_1(a, t) = -\left( \mu + \gamma_1(a) \right) u_1(a, t), \\
    u_2(a_1, t) = 0, \\
    u_2(0, t) = \sigma e(t), \\
    u_2(a_1, t) = \int_{a_1}^{\infty} q(a) \gamma_1(a) u_1(a, t) da + \mu \omega \int_{a_1}^{\infty} v(a) u_2(a, t) da,
\end{cases}
\end{align*} \]

Now we consider the solutions of the form \( s(t) = e^{zt} s_0, \quad e(t) = e^{zt} e_0, \quad u_1(a, t) = e^{zt} u_1(a), \quad u_2(a, t) = e^{zt} u_2(a) \), and let

\[ \begin{align*}
&W_1(z) = \int_{a_1}^{\infty} \beta(a) e^{- \int_{0}^{z} (z + \mu + \gamma_1(\rho)) d\rho} da, \\
&W_2(z) = \int_{a_1}^{\infty} q(a) \gamma_1(a) e^{- \int_{0}^{z} (z + \mu + \gamma_1(\rho)) d\rho} da, \\
&W_3(z) = \int_{a_1}^{\infty} \beta(a) e^{- \int_{0}^{z} (z + \mu + \gamma_2(\rho) + \theta(\rho)) d\rho} da, \\
&W_4(z) = \int_{a_1}^{\infty} \gamma_2(a) e^{- \int_{0}^{z} (z + \mu + \gamma_2(\rho) + \theta(\rho)) d\rho} da, \\
&W_5(z) = \int_{a_1}^{\infty} v(a) e^{- \int_{0}^{z} (z + \mu + \gamma_2(\rho) + \theta(\rho)) d\rho} da.
\end{align*} \]

Since

\[ u_1(0, t) = \bar{u}_1(0) e^{zt}, \]
and

\[ u_1(0, t) = \sigma e(t) = \sigma \bar{e}e^{zt}, \]

it follows that

\[ \bar{u}_1(0) = \sigma \bar{e}. \]

By the equation of \( u_1(a, t) \), we have

\[ e^{zt} \frac{du_1(a)}{da} + ze^{zt} \bar{u}_1(a) = -(\mu + \gamma_1(a))e^{zt} \bar{u}_1(a), \]

thus

\[ \bar{u}_1(a) = \bar{u}_1(0)e^{-\int_0^a (\zeta + \mu + \gamma_1(\rho))d\rho}. \]

Similarly,

\[ \bar{u}_2(a) = \bar{u}_2(1)e^{-\int_0^a (\zeta + \mu + \gamma_2(\rho) + \theta(\rho))d\rho}. \]

Note that

\[ u_2(a_1, t) = \bar{u}_2(a_1)e^{zt}, \]

and

\[ u_2(a_1, t) = \int_0^{a_1} q(a)\gamma_1(a)\bar{u}_1(a)da + \mu \omega \int_0^{a_1} v(a)\bar{u}_2(a)da, \]

hence

\[ \bar{u}_2(a_1) = \int_0^{a_1} q(a)\gamma_1(a)\bar{u}_1(a)da + \mu \omega \int_0^{a_1} v(a)\bar{u}_2(a)da \]
\[ = \int_0^{a_1} q(a)\gamma_1(a)\bar{u}_1(0)e^{-\int_0^a (\zeta + \mu + \gamma_1(\rho))d\rho}da \]
\[ + \mu \omega \int_0^{a_1} v(a)\bar{u}_2(a_1)e^{-\int_0^a (\zeta + \mu + \gamma_2(\rho) + \theta(\rho))d\rho}da \]
\[ = \bar{u}_1(0)W_2(z) + \mu \omega \bar{u}_2(a_1)W_5(z) \]

it follows that

\[ \bar{u}_2(a_1) = \frac{\bar{u}_1(0)W_2(z)}{1 - \mu \omega W_5(z)}. \]

Solving the equation of \( e(t) \) yields

\[ \bar{e} = \frac{\bar{S}_0(\int_0^{a_1} \beta(a)\bar{u}_1(a)da + \alpha \int_0^{a_1} \beta(a)\bar{u}_2(a)da)}{z + \sigma + \mu} \]
\[ = \bar{S}_0 u_1(0) \int_0^{a_1} \beta(a)e^{-\int_0^a (\zeta + \mu + \gamma_1(\rho))d\rho}da \]
\[ + \bar{S}_0 \alpha \bar{u}_2(a_1) \int_0^{a_1} \beta(a)e^{-\int_0^a (\zeta + \mu + \gamma_2(\rho) + \theta(\rho))d\rho}da \]
\[ = \bar{S}_0(\bar{u}_1(0)W_1(z) + \alpha \bar{u}_2(a_1)W_3(z)) \]
\[ = \frac{\bar{S}_0[\sigma eW_1(z) + \alpha \bar{u}_2(a_1)W_3(z)]}{z + \sigma + \mu}, \]

which leads to the following characteristic equation

\[ 1 = \frac{\bar{S}_0\sigma[(1 - \mu \omega W_5(z))W_1(z) + \alpha W_2(z)W_3(z)]}{(z + \sigma + \mu)(1 - \mu \omega W_5(z))}, \]  \( \text{(3)} \)

When \( z = 0 \), equation (3) becomes \( R_0 = 1 \). Denote the right side of (3) with \( G(z) \), so \( G(0) = R_0 \), \( G(+\infty) = 0 \). Since \( W_i(z)(i = 1, 2, 3, 4, 5) \) are decreasing functions
of $z$, it can be easily verified that $G'(z) < 0$, which implies that equation (3) has a unique real root $z^*$. Thus if $R_0 < 1$, then $z^* < 0$, and if $R_0 > 1$, then $z^* > 0$.

Supposing $z = x + iy$ is the complex root of (3), then

$$1 = G(z) = |G(x + iy)| = G(x),$$

which implies that $z^* > x$, so equation (3) has characteristic root with negative real part if and only if $R_0 < 1$.

Next, we will deal with the global attraction of $P_0$.

Substituting (1) and (2) into the boundary condition about $c(a_1, t)$, we get

$$c(a_1, t) = \int_0^a q(s) \gamma_1(s) i(s, t)ds + b\omega \int_0^\infty v(s)c(s, t)ds$$

$$= \int_0^a q(s) \gamma_1(s) \sigma E(t - s)\pi_1(s)ds + \int_0^a q(s) \gamma_1(s) i(s, t)ds$$

$$+b\omega[\int_1^a \gamma_1(s) v(s)c(a_1, t + a_1 - s)\pi_2(s)ds + \int_0^\infty v(s)c(s, t)ds]$$

Observe that when time $t \to \infty$, age $a$ in the integral is fixed at $a_1$, i.e., $a = a_1$. Denoting $f(t) = \lim_{t \to \infty} sup f(t)$, by the above equality, we have

$$\lim_{t \to \infty} sup c(a_1, t) \leq \sigma E^\infty W_2 + b\omega \lim_{t \to \infty} sup c(a_1, t)W_5$$

Letting $c^\infty(a_1) := \lim_{t \to \infty} sup c(a_1, t)$, it follows that

$$c^\infty(a_1) \leq \frac{\sigma E^\infty W_2}{1 - b\omega W_5}. \quad (4)$$

Since

$$E'(t) = S[\int_0^a \beta(s) i(s, t)ds + \alpha \int_a^\infty \beta(s) c(s, t)ds] - (\mu + \sigma)E$$

$$= S[\int_0^a \beta(s) \sigma E(t - s)\pi_1(s)ds + \int_a^\infty \beta(s) i(s, t)ds]$$

$$+S\alpha[\int_1^a \beta(s)c(a_1, t + a_1 - s)\pi_2(s)ds + \int_0^\infty \beta(s)c(s, t)ds]$$

$$- (\mu + \sigma)E$$

then by the results in [18], there exists a sequence $\{t_n\}$ as $t_n \to \infty$ such that $E(t_n) \to E^\infty$ and $E'(t_n) \to 0$, which implies

$$0 \leq \tilde{S}_0 \sigma E^\infty W_1 + \tilde{S}_0 \alpha c^\infty(a_1)W_3 - (\mu + \sigma)E^\infty, \quad (5)$$

by (4) and (5), we obtain

$$E^\infty \leq \frac{\tilde{S}_0 \sigma W_1(1 - b\omega W_5) + \tilde{S}_0 \alpha \sigma W_5}{(\mu + \sigma)(1 - b\omega W_5)} E^\infty = R_0 E^\infty.$$ 

Hence if $R_0 < 1$, then $E^\infty = 0$, which implies that $\lim_{t \to \infty} i(a, t) = 0$, since $i(0, t) = \sigma E(t)$. By the equation of $c(a, t)$ and its initial condition, it can be easily inferred that $\lim_{t \to \infty} c(a, t) = 0$, and $\lim_{t \to \infty} S(t) = \frac{b\omega}{\mu}$. \hfill \Box

**Theorem 3.2.** The endemic equilibrium $P^*$ is locally stable when $R_0 > 1$.

*Proof.* Let

$$S(t) = s(t) + S^*, \quad E(t) = e(t) + E^*, \quad i(a, t) = u_1(a, t) + i^*(a),$$

$$c(a, t) = u_2(a, t) + c^*(a),$$

and

$s(t) = e^{t\tilde{s}}, \quad e(t) = e^{t\tilde{e}}, \quad u_1(a, t) = e^{t\tilde{u}_1(a)}, \quad u_2(a, t) = e^{t\tilde{u}_2(a)}.$

Then the equation of $i(a, t)$ satisfies

$$\frac{di^*(a)}{da} + \frac{\partial u_1(a, t)}{\partial t} + \frac{\partial u_1(a, t)}{\partial a} = -(\mu + \gamma_1(a))(i^*(a) + u_1(a, t)),$$
thus
\[ i^*(a) = \sigma E^* \pi_1(a), \]
\[ \tilde{u}_1(a) = \sigma \tilde{e} e^{-\int_0^z (z+\mu+\gamma_1(p))d\rho} =: \sigma \tilde{e} \pi_1(z,a). \]

And similarly,
\[ c^*(a) = \frac{\sigma W_2 \pi_2(a) E^*}{1 - b\omega W_5}, \]
\[ \tilde{u}_2(a) = \frac{\sigma \tilde{e} W_2(z)}{1 - b\omega W_5(z)} e^{-\int_{a_1}^z (z+\mu+\gamma_2(\rho)+\theta(\rho))d\rho} =: \frac{\sigma \tilde{e} W_2(z)}{1 - b\omega W_5(z)} \pi_2(z,a). \]

By the first equation in system (S2), we get the following linear equation
\[ z\tilde{s} = -(\mu + p) \tilde{s} - b\omega \int_{a_1}^\infty v(a) \tilde{u}_2(a) da - S^*(\int_{a_1}^\infty \beta(a) \tilde{u}_1(a) da + \alpha \int_{a_1}^\infty \beta(a) \tilde{u}_2(a) da) \]
\[ - \tilde{s}(\int_{a_1}^\infty \beta(a) i^*(a) da + \alpha \int_{a_1}^\infty \beta(a) c^*(a) da), \]
so
\[ z\tilde{s} = \frac{-b\omega \int_{a_1}^\infty v(a) \tilde{u}_2(a) da - S^*(\int_{a_1}^\infty \beta(a) \tilde{u}_1(a) da + \alpha \int_{a_1}^\infty \beta(a) \tilde{u}_2(a) da)}{z + \mu + p + \int_{a_1}^\infty \beta(a) i^*(a) da + \alpha \int_{a_1}^\infty \beta(a) c^*(a) da}. \]

Similarly,
\[ \tilde{e} = \frac{S^*(\int_{a_1}^\infty \beta(a) \tilde{u}_1(a) da + \alpha \int_{a_1}^\infty \beta(a) \tilde{u}_2(a) da) + \tilde{s}(\int_{a_1}^\infty \beta(a) i^*(a) da + \alpha \int_{a_1}^\infty \beta(a) c^*(a) da)}{z + \mu + \sigma}, \]

Take the expressions of \( \tilde{u}_1(a), \tilde{u}_2(a), i^*(a), c^*(a) \) and \( \tilde{s} \) into \( \tilde{e} \), we have
\[ 1 = \frac{S^*(\sigma W_1(z) + \alpha \sigma W_2(z) W_3(z)) (1 - \sigma E^* W_1 - \alpha \sigma W_2 E^* W_3)}{1 - b\omega W_5(z)} \]
\[ - \frac{b\omega \sigma W_2(z) W_3(z)}{1 - b\omega W_5(z) (\sigma E^* W_1 + \alpha \sigma W_2 E^* W_3)} \frac{z + \mu + \sigma}{z + \mu + \sigma}, \]

If \( z = 0 \), then
\[ 1 = \frac{S^*(\sigma W_1 + \alpha \sigma W_2 W_3)}{1 - b\omega W_5} (1 - \sigma E^* W_1 - \alpha \sigma W_2 E^* W_3) \]
\[ - \frac{b\omega \sigma W_2 W_3 (\sigma E^* W_1 + \frac{\mu + \sigma}{1 - b\omega W_5})}{1 - b\omega W_5} \frac{\mu + \sigma}{\mu + \sigma}, \]
\[ = \frac{S_0 R_0}{S_0 (1 - R_0 E^* \frac{1}{S_0}) - b\omega \sigma W_2 W_3 R_0 E^* \frac{1}{S_0}} \frac{1}{S_0} \]
\[ = 1 - (1 + b\omega \frac{\sigma W_2 W_3}{1 - b\omega W_5}) \frac{R_0 E^*}{S_0} \frac{\mu (R_0 - 1)(1 - b\omega W_5)}{(\sigma + \mu)(1 - b\omega W_5) + b\omega \sigma W_2 W_3 E^*}, \]

which leads to a contradiction when \( R_0 > 1 \).

If \( \Re z > 0 \), where \( \Re z \) denotes the real part of \( z \), then
\[ 1 < \left| \frac{S^*(\sigma W_1(z) + \alpha \sigma W_2(z) W_3(z)) (1 - \sigma E^* W_1 - \alpha \sigma W_2 E^* W_3)}{1 - b\omega W_5(z)} \right| \]
\[ < \frac{S_0 \sigma W_1 (1 - b\omega W_5) + \alpha \sigma W_2 W_3 (1 - \sigma W_1 (1 - b\omega W_5) + \alpha \sigma W_2 W_3 E^*)}{\mu (\sigma + \mu)(1 - b\omega W_5) + b\omega \sigma W_2 W_3 E^*} \]
\[ < \frac{1}{\frac{b + \sigma}{S_0} E^* R_0} \]
\[ < 1 \]
which is a contradiction, so the characteristic equation (6) has eigenvalues only with negative real part.

\[ \square \]

4. Uniform persistence. In this section, we establish the uniform persistence for the model when \( R_0 > 1 \). We reformulate the system \( S_2 \) as an abstract Cauchy problem, and let

\[ u = (S, E, i, e)^T \in \dot{X} = \mathbb{R}^2 \times L^1((0, +\infty), \mathbb{R}^2), \]

which is endowed with the following norm

\[ \| u \| = |S| + |E| + \int_0^a |i(a, t)| \, da + \int_{a_1}^{\infty} |c(a, t)| \, da. \]

In order to take into account the boundary condition, we extend the state space and set

\[ \dot{X}_+ = \mathbb{R}^2_+ \times L^1((0, +\infty), \mathbb{R}^2), \]
\[ X = \dot{X} \times \mathbb{R}^2, \]
\[ X_+ = \dot{X}_+ \times \mathbb{R}^2_+, \]
\[ X_0 = \dot{X} \times \{0\} \times \{0\}, \]
\[ X_{0+} = X_0 \cap X_+ \]

For any \( u = (u_1, u_2, u_3, u_4)^T \in \dot{X} \), \( v = (u, 0, 0) \in X_0 \), let \( A : \text{Dom}(A) \to X \) be the linear operator defined by

\[ A v = \begin{pmatrix} -\mu u_1 \\ -(\mu + \sigma) u_2 \\ -\left(\frac{\beta}{\alpha} + \mu + \gamma_1(a)\right) u_3 \\ -\left(\frac{\beta}{\alpha} + \mu + \gamma_2(a) + \theta(a)\right) u_4 \\ -u_3(0) \\ -u_4(0) \end{pmatrix}, \]

with \( \text{Dom}(A) = \{ v \in X_{0+} \mid u_3(\cdot), u_4(\cdot) \in W^{1,1}[0, +\infty) \} \), where \( W^{1,1} \) denotes a Sobolev space.

Define nonlinear operator \( F : X_0 \to X \) as

\[ F(v) = \begin{pmatrix} b\omega(1 - \int_0^a v(a) u_4(a, t) \, da - \int_0^{a_1} \beta(a) u_3(a, t) \, da + \alpha \int_0^a \beta(a) u_4(a, t) \, da) u_1 \\ \int_0^a \beta(a) u_3(a, t) \, da - \int_0^{a_1} \beta(a) u_4(a, t) \, da u_1 \\ \int_0^{a_1} \beta(a) u_3(a, t) \, da + b\omega \int_0^a v(a) u_4(a, t) \, da \\ \int_0^{a_1} \beta(a) u_4(a, t) \, da - \int_0^{a_1} \beta(a) u_4(a, t) \, da u_1 \\ \int_0^{a_1} \beta(a) u_4(a, t) \, da \\ \int_0^{a_1} \beta(a) u_4(a, t) \, da \end{pmatrix}. \]

Then we rewrite the system \( S_2 \) as the following abstract Cauchy problem

\[ \frac{dv(t)}{dt} = Av + F(v) \quad (7) \]

with \( v(0) = v_0 \in X_{0+} \) for any \( t \geq 0 \). By Theorem 2 and Theorem 3 in [19], it can be verified that there exists a unique solution semiflow \( U(t) : X_{0+} \to X_{0+} \) defined by system (7).

Let \( N(t) \) denote the total population size, then we have

\[ N'(t) = S'(t) + E'(t) + \frac{d}{dt} \int_0^a \beta(a) i(a, t) \, da + \frac{d}{dt} \int_0^{a_1} c(a, t) \, da + R'(t) \]

\[ \leq b - \mu N(t) + \int_0^{a_1} \theta(a) c(a, t) \, da - i(a_1, t) \]

For the system \( \frac{d}{dt} = b - \mu y \), the equilibrium \( \frac{b}{\mu} \) is globally asymptotically stable. By the comparison principle, it follows that

\[ \lim_{t \to \infty} \sup N(t) \leq \frac{b}{\mu}, \]
which implies that all solutions of system $S_2$, and correspondingly, the solutions of system (7) are ultimately bounded. Moreover, when $N(t) > \frac{b}{\mu}$, we have $\frac{dN(t)}{dt} < 0$, which implies that all solutions are uniformly bounded. Therefore, the solution semiflow $U(t) : X_0+ \rightarrow X_0+$ is point dissipative. And it holds that the set

$$B = \{(u_1, u_2, u_3, u_4, 0,0) \in X_0+: u_1 + u_2 + \int_0^{a_1} u_3 da + \int_0^{\infty} u_4 da < \frac{b}{\mu}\}$$

is positively invariant absorbing under the semiflow $U(t)$ on $X_0+$, and $U(t)$ maps any bounded set to a precompact set in $X_0+$. Consequently, by Theorem 3.6.1 in [7], $U(t)$ is compact (completely continuous) for any $t > 0$. Thus, Theorem 1.1.3 in [22] implies that $U(t)$ has a compact global attractor in $X_0+$. Consequently, we have the following result.

**Lemma 4.1.** The model system (7) has a unique solution semiflow $U(t)$ in $X_0+$. Furthermore, there is a compact global attractor $A$ for $U(t)$.

Let

$$\dot{M} = \{(i,c)^T \in L^1((0,\infty),\mathbb{R}^2) : \int_0^{a_1} i(a) da > 0 \text{ or } \int_0^{\infty} c(a) da > 0\},$$

$$M = \mathbb{R}^2_+ \times M \times \{0\} \times \{0\},$$

$$\partial M = X_+ \times \{0\} \times \{0\} \setminus M.$$

**Lemma 4.2.** The subset $\partial M$ is positively invariant under the semiflow $\{U(t)\}_{t \geq 0}$, i.e., $U(t)\partial M \subset \partial M$. Moreover $U(t)x \rightarrow \bar{x} = (S_0, 0, 0, l_1, 0, l_1, 0, 0)$ for each $x \in \partial M$ as $t$ is large enough.

**Proof.** Take $(S_0, 0, i_0, c_0, 0, 0) \in \partial M$ and consider the system

$$\begin{align*}
E'(t) &= (\int_0^{a_1} \beta(a)i(a,t)da + \alpha \int_0^{a_1} \beta(a)c(a,t)da)S - (\mu + \sigma)E, \\
\frac{\partial i(a,t)}{\partial a} + \frac{\partial i(a,t)}{\partial t} &= -(\mu + \gamma_1(a))i(a,t), \quad 0 < a < a_1 \\
\frac{\partial c(a,t)}{\partial a} + \frac{\partial c(a,t)}{\partial t} &= -(\mu + \gamma_2(a) + \theta(a))c(a,t), \quad a_1 < a < \infty \\
i(0,t) &= \sigma E(t), \\
c(a_1,t) &= \int_0^{a_1} q(a)\gamma_1(a)i(a,t)da + b\omega \int_0^{\infty} v(a)c(a,t)da,
\end{align*}$$

$$E(0) = 0, i(a,0) = i_0(a), c(a,0) = c_0(a).$$

Since $S(t) \leq \frac{b}{\mu}$, by the comparison principle, we deduce that

$$E(t) \leq \bar{E}(t), i(a,t) \leq \bar{i}(a,t), c(a,t) \leq \bar{c}(a,t),$$

where $(\bar{E}(t), \bar{i}(a,t), \bar{c}(a,t))$ is a solution of the following system

$$\begin{align*}
\bar{E}'(t) &= (\int_0^{a_1} \beta(a)i(a,t)da + \alpha \int_0^{a_1} \beta(a)c(a,t)da)S - (\mu + \sigma)\bar{E}, \\
\frac{\partial \bar{i}(a,t)}{\partial a} + \frac{\partial \bar{i}(a,t)}{\partial t} &= -(\mu + \gamma_1(a))\bar{i}(a,t), \quad 0 < a < a_1 \\
\frac{\partial \bar{c}(a,t)}{\partial a} + \frac{\partial \bar{c}(a,t)}{\partial t} &= -(\mu + \gamma_2(a) + \theta(a))\bar{c}(a,t), \quad a_1 < a < \infty \\
\bar{i}(0,t) &= \sigma \bar{E}(t), \\
\bar{c}(a_1,t) &= \int_0^{a_1} q(a)\gamma_1(a)\bar{i}(a,t)da + b\omega \int_0^{\infty} v(a)\bar{c}(a,t)da,
\end{align*}$$

$$\bar{E}(0) = 0, \bar{i}(a,0) = i_0(a), \bar{c}(a,0) = c_0(a).$$

From system (8), we can get the following expression.
\begin{equation}
\begin{aligned}
\dot{i}(a,t) &= \begin{cases}
\lambda_1(t-a)\pi_1(a), & t > a, \quad a \in [0,a_1] \\
i_0(a-t)\pi_1(a), & t \leq a, \quad a \in [0,a_1]
\end{cases} \\
\dot{c}(a,t) &= \begin{cases}
\lambda_2(a_1+t-a)\pi_2(a), & t + a_1 > a, \quad a \in [a_1,\infty) \\
c_0(a-t-a_1)\pi_2(a), & t + a_1 \leq a, \quad a \in [a_1,\infty)
\end{cases}
\end{aligned}
\end{equation}

where \( \lambda_1(t) \) and \( \lambda_2(t) \) are the unique solution of the following system

\begin{align*}
\lambda_1(t) &= \frac{b}{\mu} \int_0^t e^{-(\mu+\sigma)(t-s)} \left\{ \int_0^a \beta(\tau)\lambda_1(s-\tau)\pi_1(\tau) d\tau + \int_0^{a_1} \beta(\tau)i_0(\tau-s)\pi_1(\tau) d\tau + \alpha \int_{a_1}^{a_1+\tau} \beta(\tau)\lambda_2(a_1+s-\tau-\tau)\pi_2(\tau) d\tau \right\} ds \\
&\quad + \int_0^{a_1} \beta(\tau)c_0(\tau-s-a_1)\pi_2(\tau) d\tau + \omega \int_0^{a_1} v(s)\lambda_2(a_1+\tau-s-\tau)\pi_2(\tau) d\tau + \omega \int_0^{a_1} v(s)\lambda_2(a_1+\tau-s-\tau)\pi_2(\tau) d\tau \\
\lambda_2(t) &= \int_0^t q(s)\gamma_1(s)\pi_1(s) ds + \int_0^{a_1} q(s)\gamma_1(s)\pi_1(s) ds + \int_0^{a_1+\tau} v(s)\lambda_2(a_1+\tau-s-\tau)\pi_2(s) ds + \omega \int_0^{a_1+t} v(s)c_0(\tau-s-a_1)\pi_2(\tau) d\tau
\end{align*}

Noticing that \((i_0, c_0) \in L_1((0,\infty), \mathbb{R}^2) \setminus \bar{M} \), it holds that

\begin{align}
&\int_0^{a_1} \beta(\tau)i_0(\tau-s)\pi_1(\tau) d\tau + \alpha \int_0^{a_1} \beta(\tau)c_0(\tau-s-a_1)\pi_2(\tau) d\tau = 0,
&\int_0^{a_1} q(s)\gamma_1(s)i_0(\tau-s)\pi_1(\tau) ds + \omega \int_0^{a_1+t} v(s)c_0(\tau-s-a_1)\pi_2(\tau) d\tau = 0.
\end{align}

Then \( (10) \) becomes

\begin{align}
\lambda_1(t) &= \frac{b}{\mu} \int_0^t e^{-(\mu+\sigma)(t-s)} \int_0^a \beta(\tau)\lambda_1(s-\tau)\pi_1(\tau) d\tau ds \\
&\quad + \alpha \int_0^{a_1} \beta(\tau)\lambda_2(a_1+s-\tau)\pi_2(\tau) d\tau ds, \\
\lambda_2(t) &= \int_0^t q(s)\gamma_1(s)\pi_1(s) ds + \omega \int_0^{a_1+t} v(s)\lambda_2(a_1+\tau-s-\tau)\pi_2(s) ds,
\end{align}

which has a unique solution

\( \lambda_1(t) = 0, \quad \lambda_2(t) = 0, \quad \forall t \geq 0. \)

It follows that \( i(a,t) = 0 \) for \( 0 \leq a \leq t \) and \( c(a,t) = 0 \) for \( a_1 + t > a \). When \( a > t \), we have

\[ \|\dot{i}(a,t)\|_{L^1} = \|i_0(a-t)\pi_1(a)\|_{L^1} \leq e^{-\mu t} \|i_0\|_{L^1}, \]

which implies that \( \dot{i}(a,t) \to 0 \) as \( t \to \infty \). When \( a_1 + t \leq a \), we have

\[ \|\dot{c}(a,t)\|_{L^1} = \|c_0(a-t-a_1)\pi_2(a)\|_{L^1} \leq e^{-\mu t} \|c_0\|_{L^1}, \]

which implies that \( \dot{c}(a,t) \to 0 \) as \( t \to \infty \). Thus for any \( x \in \partial M \), it follows that \( U(t)x \to \bar{x} = (S_0, 0,0,L_1,0,0,0,0) \in \partial M \), which completes the proof. \( \square \)

By the results in \([15]\), we can obtain the following theorem.

**Theorem 4.3.** When \( \mathcal{R}_0 > 1 \), the semiflow \( \{U(t)\}_{t \geq 0} \) is uniform persistence with respect to \((M, \partial M)\), i.e., there exists \( \eta > 0 \) such that the solution of system (7) with initial value \((S_0, E_0, i_0, c_0, 0, 0) \in M \) satisfies

\[ \lim_{t \to \infty} \inf E(t) > \eta, \quad \lim_{t \to \infty} \inf \|i(.,t)\| > \eta, \quad \lim_{t \to \infty} \inf \|c(.,t)\| > \eta. \]
Moreover, there exists $A_0$ a compact subset of $M$ which is a global attractor for $\{U(t)\}_{t \geq 0}$ in $M$.

Proof. It is shown in Lemma 4.2 that $\tilde{x} = (\tilde{S}_0, 0, 0, \tilde{c}_0, 0, 0)$ is globally asymptotically stable in $\partial M$. In the following we will study the behaviour of the solutions starting in $M$ in some neighbourhood of $\tilde{x}$. It is sufficient to prove that for any $\epsilon > 0$ and $x = (S_0, E_0, i_0, c_0, 0, 0) \in \{y \in M : \|\tilde{x} - y\| \leq \epsilon\}$, there exists $t_0 \geq 0$ such that $\|\tilde{x} - U(t_0)x\| > \epsilon$, i.e., $W^s(\{\tilde{x}\}) \cap M = \emptyset$.

By contradiction, suppose that for each $n \geq 0$, we can find

$$x_n = (S^n_0, E^n_0, i^n_0, c^n_0, 0, 0) \in \{y \in M : \|\tilde{x} - y\| \leq \frac{1}{n+1}\}$$

such that

$$\|\tilde{x} - U(t)x_n\| \leq \frac{1}{n+1}, \quad \forall t \geq 0. \quad (12)$$

Set $U(t)x_n = (S^n(t), E^n(t), i^n(\cdot, t), c^n(\cdot, t), 0, 0)$, then we have

$$\|S^n(t) - \tilde{S}_0\| \leq \frac{1}{n+1}, \quad \forall t \geq 0.$$

Consider the following system

$$\begin{aligned}
\frac{dE^n(t)}{dt} & = (\int_0^{a_1} \beta(a)v^n(a, t)da + \alpha \int_{a_1}^{\infty} \beta(a)c^n(a, t)da)S^n - (\mu + \sigma)E^n,
\frac{dE^n(t)}{dt} & = -\gamma_1(a)(c^n(a, t)), \quad 0 < a \leq a_1
\frac{dc^n(a, t)}{dt} & = -\mu c^n(a, t), \quad a_1 < a < \infty
\end{aligned}$$

By the comparison principle, we deduce that

$$E^n(t) \geq \hat{E}^n(t), \quad i^n(\cdot, t) \geq \hat{i}^n(\cdot, t), \quad c^n(\cdot, t) \geq \hat{c}^n(\cdot, t), \quad (13)$$

where $(\hat{E}^n(t), \hat{i}^n(\cdot, t), \hat{c}^n(\cdot, t))$ is a solution of the following system

$$\begin{aligned}
\frac{d\hat{E}^n(t)}{dt} & = (\int_0^{a_1} \beta(a)i^n(a, t)da + \alpha \int_{a_1}^{\infty} \beta(a)c^n(a, t)da)(\hat{S}_0 - \frac{1}{n+1}) - (\mu + \sigma)\hat{E}^n,
\frac{d\hat{E}^n(t)}{dt} & = -\gamma_1(a)(\hat{c}^n(a, t)), \quad 0 < a \leq a_1
\frac{d\hat{c}^n(a, t)}{dt} & = -\mu \hat{c}^n(a, t), \quad a_1 < a < \infty
\end{aligned}$$

Since $R_0 > 1$, similar to (3), we deduce that for all $n \geq 0$ large enough, the dominant eigenvalue of system (14) satisfies the characteristic equation

$$\hat{G}(z_n) = 1, \quad (15)$$

where

$$\hat{G}(z_n) = (\hat{S}_0 - \frac{1}{n+1}) - \sigma[(1 - \beta \omega W_1(z_n))W_1(z_n) + \alpha W_2(z_n)W_3(z_n)],$$

Noticing that $\hat{G}(z_n)$ has the similar property of $G(z)$ in Theorem 3.1, it follows that equation (15) has a unique real root $z_n^* > 0$ for all $n \geq 0$ large enough.

Let $w(t) = (\hat{E}^n(t), \hat{i}^n(\cdot, t), \hat{c}^n(\cdot, t))$ and $\Delta(z_n) = 1 - \hat{G}(z_n)$. We define a linear system corresponding to system (14) as
\[
\frac{dw(t)}{dt} = \mathcal{B}w(t)
\]

with
\[
\mathcal{B}w(t) = \begin{pmatrix}
-(\mu + \sigma) \tilde{E}^n(t) \\
-\frac{d\gamma}{da} - (\mu + \gamma_1(a))\tilde{c}^n \\
-\frac{d\bar{\gamma}}{d\bar{\alpha}} - (\mu + \gamma_2(a) + \theta(a))\bar{c}^n
\end{pmatrix},
\]

and
\[
\mathcal{D}(\mathcal{B}) = \{ (\tilde{E}^n(t), \tilde{c}^n(t), \bar{c}^n(t), t) \in \mathbb{R} \times W^{1,1}((0, +\infty), \mathbb{R}^2) : \}
\]
\[
\tilde{E}^n(0) = (\bar{S}_0 - \frac{1}{\alpha + 1})\left(\int_0^a \beta(a)\tilde{E}^n(a, t)da + \alpha \int_a^\infty \beta(a)\tilde{c}^n(a, t)da\right),
\]
\[
\tilde{c}^n(0) = \sigma \tilde{c}^n, \quad \bar{c}^n(0) = \int_0^a q(a)\tilde{c}^n(a, t)da + b\omega \int_a^\infty v(a)\tilde{c}^n(a, t)da.
\]

Thus the operator \((\mathcal{B}, \mathcal{D}(\mathcal{B}))\) can generate a solution semiflow, which is strongly continuous, denoted by \(\{U(t)\}_{t \geq 0}\). For \(x_n \in M\), let \(\Pi_n\) be the projector on the eigenspace associated to the dominant eigenvalue \(z_n^*\). By the results in [14] and [10], we have the expression
\[
\Pi_n U(t) \begin{pmatrix}
E^n_0 \\
i^n_0 \\
c^n_0
\end{pmatrix} = e^{s_n^* t} \Pi_n \begin{pmatrix}
E^n_0 \\
i^n_0 \\
c^n_0
\end{pmatrix} = e^{s_n^* t} \lim_{z \to z^*_n} (z - z^*_n)(zI - \mathcal{B})^{-1} \begin{pmatrix}
E^n_0 \\
i^n_0 \\
c^n_0
\end{pmatrix}
\]

here
\[
\varphi_1 = \frac{\bar{S}_0 - \frac{1}{\alpha + 1}}{\frac{d\gamma}{da} - (\mu + \gamma_1(a))\tilde{c}^n} \left(\int_0^a q(a)\gamma_1(a)\int_0^a e^{-\int_0^s(z_n^* + \mu + \gamma_1(t))dt}i_0^a dsda\right)
\]
\[
+ b\omega \int_a^\infty v(a)\int_0^a e^{-\int_0^s(z_n^* + \mu + \gamma_1(t))dt}c_0^a dsda + \alpha \int_a^\infty \beta(a)\int_0^a e^{-\int_0^s(z_n^* + \mu + \gamma_1(t))dt}c_0^a dsda,
\]
\[
\varphi_2 = \sigma \varphi_1 e^{-\int_0^a (z_n^* + \mu + \gamma_1(t))dt},
\]
\[
\varphi_3 = \frac{\lambda - b\omega i_0^a}{\frac{d\bar{\gamma}}{d\bar{\alpha}} - (\mu + \gamma_2(a) + \theta(a))\bar{c}^n} \left[\varphi_2 \mathcal{W}_2(z_n^*) + \int_0^a q(a)\gamma_1(a)\int_0^a e^{-\int_0^s(z_n^* + \mu + \gamma_1(t))dt}i_0^a dsda\right]
\]
\[
+ b\omega \int_a^\infty v(a)\int_0^a e^{-\int_0^s(z_n^* + \mu + \gamma_1(t))dt}c_0^a dsda + \alpha \int_a^\infty \beta(a)\int_0^a e^{-\int_0^s(z_n^* + \mu + \gamma_1(t))dt}c_0^a dsda.
\]

By using the fact \(\frac{d\gamma}{da} - (\mu + \gamma_1(a))\tilde{c}^n > 0\) and \((i_0^a, c_0^a) \in \hat{M}\), we deduce that
\[
\lim_{t \to +\infty} \tilde{E}^n(t) = +\infty, \quad \lim_{t \to +\infty} \|\tilde{c}^n(t)\| = +\infty, \quad \lim_{t \to +\infty} \|\bar{c}^n(t)\| = +\infty,
\]
and by using (13) we obtain
\[
\lim_{t \to +\infty} E^n(t) = +\infty, \quad \lim_{t \to +\infty} \|i^n(t)\| = +\infty, \quad \lim_{t \to +\infty} \|c^n(t)\| = +\infty,
\]
which is a contradiction with (12). Theorem 4.2 in [8] implies that the semiflow \(\{U(t)\}_{t \geq 0}\) is uniformly persistent with respect to \((M, \partial M)\) and the results follows.
5. A special case. In this section, we will study the global stability of endemic equilibrium $P^*$ in a special case $\theta(a) = 0$.

**Theorem 5.1.** If $\theta(a) = 0$ and $R_0 > 1$, the endemic equilibrium $P^*$ is globally asymptotically stable.

**Proof.** Define functions $L_i(t) (i = 1, 2, 3, 4)$ as follows

\[
L_1(t) = S(t) - S^* - S^* \ln \frac{S(t)}{S^*}, \\
L_2(t) = E(t) - E^* - E^* \ln \frac{E(t)}{E^*}, \\
L_3(t) = \int_0^{a_1} \Phi(a)G(i(a, t), i^*(a))da, \\
L_4(t) = \int_0^{a_1} \Psi(a)G(i(a, t), i^*(a))da,
\]

where

\[
\Psi(a) = \int_0^\infty S^* \alpha \beta(s)e^{-\int_s^t (\mu + \gamma_2(\rho) + \theta(\rho))d\rho}ds, \\
\Phi(a) = \int_0^{a_1} [\beta(s)S^* + \Psi(a_1)q(s)\gamma_1(s)]e^{-\int_s^t (\mu + \gamma_1(\rho))d\rho}ds, \\
G(x, x^*) = x - x^* - x^* \ln \frac{x}{x^*},
\]

for $x, x^* > 0$. The derivatives of $L_1$ and $L_2$ along $S_2$ are

\[
L_1'(t) = -\frac{d^2}{dt^2} (S - S^*)^2 - S(\int_0^{a_1} \beta(a) i(a, t) da + \alpha \int_0^{a_1} \beta(a) c(a, t) da) \\
+ S^*(\int_0^{a_1} \beta(a)i^*(a) da + \alpha \int_0^{a_1} \beta(a)c^*(a) da) \\
+ S^*(\int_0^{a_1} \beta(a)i(t) da + \alpha \int_0^{a_1} \beta(a)c(t) da) \\
- \frac{(S^*)^2}{S}(\int_0^{a_1} \beta(a)i^*(a) da + \alpha \int_0^{a_1} \beta(a)c^*(a) da)
\]

and

\[
L_2'(t) = S(\int_0^{a_1} \beta(a)i(a, t) da + \alpha \int_0^{a_1} \beta(a)c(a, t) da) \\
- \frac{d^2}{dt^2} S(\int_0^{a_1} \beta(a)i(a, t) da + \alpha \int_0^{a_1} \beta(a)c(a, t) da) - (\sigma + \mu)(E - E^*)
\]

Due to the above equalities, we have

\[
\Psi(a) = \int_0^\infty S^* \alpha \beta(s)e^{-\int_s^t (\mu + \gamma_2(\rho) + \theta(\rho))d\rho}ds = S^* \alpha W_3, \\
\Phi(a) = -\beta(a)S^* + (\mu + \gamma_2(a) + \theta(a))\Psi(a), \\
\Phi(0) = \int_0^{a_1} [\beta(s)S^* + \Psi(a_1)q(s)\gamma_1(s)]e^{-\int_s^t (\mu + \gamma_1(\rho))d\rho}ds \\
= S^* W_1 + \Psi(a_1)W_2 = \frac{S^* \alpha}{\sigma}, \\
G(x, x^*) = xG_x(x, x^*) + x^*G_x^*(x, x^*).
\]

Function $L_3(t)$ can be rewritten as

\[
L_3(t) = \int_0^t \Phi(a)G(i(0, t - a)\pi_1(a), i^*(a))da + \int_0^t \Psi(a)G(i_0(a, t - a)\pi_1(a), i^*(a))da \\
= \int_0^t \Phi(t - s)G(i_0(s)\pi_1(t - s), i^*(t - s))ds + \int_0^t \Psi(t - s)G(i_0(s)\pi_1(t - s), i^*(t - s))ds
\]

then differentiating of $L_3(t)$ along the solutions of $S_2$ yields

\[
L_3'(t) = \Phi(0)G(i(0, t), i^*(0)) + \int_0^t \Phi'(t - s)G(i_0(s)\pi_1(t - s), i^*(t - s))ds \\
- \int_0^t \Phi(t - s)(\mu + \gamma_1(t - s))i_0(s)\pi_1(t - s)G_x(i_0(s)\pi_1(t - s), i^*(t - s))ds \\
- \int_0^t \Phi(t - s)(\mu + \gamma_1(t - s))i^*(t - s)G_x^*(i_0(s)\pi_1(t - s), i^*(t - s))ds \\
+ \int_0^t \Phi'(t - s)G(i_0(s)\pi_1(t + s), i^*(t + s))ds \\
- \int_0^t \Phi(t + s)(\mu + \gamma_1(t + s))i_0(s)\pi_1(t + s)G(i_0(s)\pi_1(t + s), i^*(t + s))ds \\
- \int_0^t \Phi(t + s)(\mu + \gamma_1(t + s))i^*(t + s)G(i_0(s)\pi_1(t + s), i^*(t + s))ds
\]

\[
= \Phi(0)G(i(0, t), i^*(0)) + \int_0^t \Phi'(a) - (\mu + \gamma_1(a))\Phi(a)G(i(a, t), i^*(a))da,
\]
Adding Construct a Lyapunov function $L$

Note that From the equalities

$$
\int_0^\infty \beta(a)S[a, t] - c^*(a) + c^*(a) \ln \frac{c(a,t)}{c(a,0)} da,
$$

Construct a Lyapunov function $L(t)$ as

$$
L(t) = L_1(t) + L_2(t) + L_3(t) + L_4(t).
$$

Adding $L_i(t)(i = 1, 2, 3, 4)$ together gives

$$
L(t) = -\frac{s^*}{s} \left[ (S - S^*)^2 + S^* \int_0^a \beta(a) \gamma_1(a) \right] \int_0^a \beta(a) i(a) da + \alpha \int_0^\infty \beta(a) c^*(a) da
$$

so

$$
\frac{d}{dt} \left[ (S - S^*)^2 + S^* \int_0^a \beta(a) i(a) da + \alpha \int_0^\infty \beta(a) c^*(a) da \right]
$$

Note that

$$
S^* \int_0^a \beta(a) i(a) da + S^* \alpha \int_0^\infty \beta(a) c^*(a) da
$$
Since the function \( f_1 \) are determined by the basic reproductive number \( R_0 \) and \( R_L \). They are studied analytically in this paper. A detailed analysis of stability and uniform limits of epidemic models that incorporate infection-age structure to describe the effects because of its incidence of new cases, the prevalence of carriers and the burden of disease dynamics. Since the infection of HBV is characterized by replicative and persistence is conducted, which shows that the dynamic behaviors of the system are determined by the basic reproductive number \( R_0 \). Specifically, by linearizing the system at the steady states, it is proved that the disease-free equilibrium is stable when \( R_0 < 1 \), and the endemic equilibrium is locally stable when \( R_0 > 1 \).

Furthermore, in the case \( \theta(a) = 0 \), it can be seen that the endemic equilibrium is globally asymptotically stable by constructing a Lyapunov function. When \( R_0 > 1 \), the model system is described by an abstract Cauchy problem and the uniform
persistence for the system is established. Maybe the possible effects of infection-age on the transmission dynamics of HBV can be illustrated numerically by using some reasonable parameters relating age of infection in our future research.

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