Numerical study of the properties of the central moment lattice Boltzmann method

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SUMMARY

Central moment lattice Boltzmann method (LBM) is one of the more recent developments among the lattice kinetic schemes for computational fluid dynamics. A key element in this approach is the use of central moments to specify the collision process and forcing, and thereby naturally maintaining Galilean invariance, an important characteristic of fluid flows. When the different central moments are relaxed at different rates like in a standard multiple relaxation time (MRT) formulation based on raw moments, it is endowed with a number of desirable physical and numerical features. Because the collision operator exhibits a cascaded structure, this approach is also known as the cascaded LBM. While the cascaded LBM has been developed sometime ago, a systematic study of its numerical properties, such as the accuracy, grid convergence, and stability for well-defined canonical problems is lacking, and the present work is intended to fulfill this need. We perform a quantitative study of the performance of the cascaded LBM for a set of benchmark problems of differing complexity, viz., Poiseuille flow, decaying Taylor–Green vortex flow, and lid-driven cavity flow. We first establish its grid convergence and demonstrate second-order accuracy under diffusive scaling for both the velocity field and its derivatives, that is, the components of the strain rate tensor, as well. The method is shown to quantitatively reproduce steady/unsteady analytical solutions or other numerical results with excellent accuracy. The cascaded MRT LBM based on the central moments is found to be of similar accuracy when compared with the standard MRT LBM based on the raw moments, when a detailed comparison of the flow fields are made, with both reproducing even the small scale vortical features well. Numerical experiments further demonstrate that the central moment MRT LBM results in significant stability improvements when compared with certain existing collision models at moderate additional computational cost. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Early developments in the area of computational fluid dynamics (CFD) focused on the solution of the classical discretizations of the continuum description of the fluid motion. During the last two decades, there has been much interest and effort in the development of schemes that derive their basis on a more smaller scale picture involving particle motion, which may be classified as mesoscopic methods. One of the most promising of such approaches is the lattice Boltzmann method (LBM) [1–3]. Based on kinetic theory, it involves the solution of the lattice Boltzmann equation (LBE), which specifies the evolution of the particle populations along discrete directions that
LBM has found applications in the simulation of a wide range of fluid flow problems [1–3, 5–7]. Because of such competitive advantages, the LBM has found applications in the simulation of a wide range of fluid flow problems [1–3, 5–7].

Because the LBM is usually developed by means of a bottom-up strategy, there is certain level of flexibility in the construction of its various elements to recover the macroscopic fluid motion. In particular, the choice of a suitable collision model can have profound influence on the fidelity as well as the stability of the approach. As such, the construction of the collision step has been the subject of considerable attention since the inception of the LBM. The simplest among these is the so-called single-relaxation-time (SRT) model [8, 9], which is based on the Bhatnagar–Gross–Krook (BGK) approximation [10]. While it is popular, it has limitations in the representation of certain flow problems and is generally prone to numerical instability, particularly at high Reynolds numbers. A major development to address these aspects is the moment approach [11], which has been constructed based on multiple relaxation times (MRT) in particular to significantly improve the numerical stability [12]. While it is related to its precursor involving a more general relaxation approximation [13, 14], the characteristic difference being that it performs collision in an orthogonal moment space leading to an efficient and flexible numerical scheme. This moment approach, which is designated as the standard MRT formulation in this paper, has recently been studied and compared with some of the other collision models in detail [15]. A simpler version that is intermediate between the SRT and MRT model is the so-called two-relaxation-time (TRT) model [16], in which the moments of even and odd orders are relaxed to their equilibrium at different rates. This, along with the MRT model, can be adjusted such that it results in a minimization of undesirable discrete kinetic effects near walls for certain laminar flows. Another significant development is the so-called entropic LBM [17]. It involves an equilibria, which is based on a constrained minimization of a Lyapunov-type functional. By modulating the collision process through enforcing entropy involution locally, this approach aims to maintain nonlinear stability. This approach has resulted in a number of simplified variants recently [18, 19].

An important physical feature of the fluid motion is that its description be independent of any inertial frame of reference (e.g., [20]). This invariance property, which is termed as the Galilean invariance, should be satisfied by any model or numerical scheme for its general applicability. Furthermore, it has recently been shown that stabilization of classical schemes for compressible flow can be achieved when they are specifically constructed to respect this physical property [21–23]. Keeping these general notions in mind, Galilean invariance can be naturally prescribed in the LBM when its various elements are represented in terms of the central moments, that is, moments obtained by shifting the particle velocity by the local fluid velocity. That is, any dynamical changes because of the collision process and impressed forces can be represented in terms of suitable variations of a set of such central moments. In particular, a collision model based on the relaxation of central moments was constructed recently [24]. The model exhibits a cascaded structure, which was later shown to be equivalent to considering a generalized equilibrium in the lattice or rest frame of reference [25]. These central moments can be relaxed at different rates during collision leading to a cascaded MRT or central moment MRT formulation, whereas by contrast, the standard MRT formulation considers raw moments. A systematic derivation of this approach by including the effect of impressed forces based on central moments was presented in [26]. This leads to considering generalized sources, analogous to the generalized equilibrium in the rest frame of reference. They also presented a detailed Chapman–Enskog analysis of the cascaded MRT LBM for its consistency with the macroscopic fluid dynamical equations of motion. This approach was further extended to various lattice models in three-dimensions in [27], in the cylindrical coordinate system for axisymmetric flows in [28], and for accounting of non-equilibrium effects in [29].
Prior works on the cascaded LBM as discussed earlier have focused mainly on method developments or their mathematical analysis, with little attention towards its numerics except for few validation cases. In particular, a detailed numerical study of the properties of the cascaded LBM for established benchmark problems and also their performance against the other LBM approaches is lacking. The focus of the present work is intended to fill this gap by presenting a systematic study of the numerical properties of the cascaded LBM, viz., grid convergence, accuracy, and stability for various canonical problems of differing complexity in terms of flow features and temporal evolution [30]. Establishing the reliability and merits of the method in quantitative terms could provide confidence in their extension and applications to various complex flow problems of interest. To study the numerics of the cascaded LBM, we consider the Poiseuille flow, decaying Taylor–Green vortex flow, and the lid-driven cavity flow, for which either analytical solutions or detailed prior numerical results are available for comparison. Much of the literature on grid convergence studies of the LBM with other collision models have focused only on those for the velocity field. In this work, we present numerical results on the grid convergence of the cascaded LBM for the velocity field as well as its derivatives, that is, the strain rate tensor. Furthermore, an advantage of the kinetic schemes such as the LBM is that the strain rate tensor can be computed locally in terms of the non-equilibrium moments. In this work, we also present a direct comparison of the results obtained using the non-equilibrium moments of the cascaded LBM with those involving the finite differencing of the velocity field at various locations for the lid-driven cavity flow problem to assess their quantitative accuracy. It may be noted that a detailed comparative study of the SRT and the standard MRT models have recently been performed in [15]. Thus, in this work, we present a quantitative accuracy comparison between the standard MRT LBM and the cascaded or central moment MRT LBM for the lid-driven cavity flow. Finally, we will discuss the numerical stability performance of various LBM schemes for the aforementioned benchmark problem.

The paper is organized as follows. Section 2 provides an overview of the different lattice Boltzmann schemes used in this investigation, focusing, in particular, on the cascaded MRT LBM. Section 3 presents the accuracy and convergence study involving the Poiseuille flow and decaying vortex flow canonical problems. Subsequently, a quantitative study of the accuracy, convergence, and numerical stability of the cascaded MRT LBM and its comparison with the raw moment based standard MRT LBM for the more complex lid-driven cavity flow problem is provided in Section 4. In these case studies, detailed comparison with either analytical or other numerical benchmark solutions are made to quantify accuracy and convergence. Summary and conclusions of this work are given in Section 5.

2. VARIOUS LATTICE BOLTZMANN SCHEMES

2.1. Cascaded lattice Boltzmann method

We will first discuss the main features of the cascaded LBM as its characteristics is the main focus of this paper. Similar to the standard MRT LBM (see the next subsection), the cascaded MRT LBM also performs collisions in moment space, but these moments are obtained by shifting the particle velocity by the local fluid velocity, that is, using central moments. As a result, the approach can naturally maintain Galilean invariance. Central moment relaxation process was specified in [24], which was re-interpreted by considering generalized equilibrium in [25]. Its detailed mathematical consistency analysis in an MRT formulation with forcing was carried out in [26]. The computations of the cascaded LBM are actually performed after transforming the central moments into raw moments by means of a binomial formula. In this work, the specific formulation of the cascaded LBM given in [26], whose details are somewhat different from that given in [24], is used. This is briefly discussed in what follows.

A standard two-dimensional (2D), nine velocity (D2Q9) lattice is employed in this study. We consider the usual bra-ket notations in the description of the method as it provides a convenient representation. That is, we consider the depiction of vectors as $|$φ$|$ and $|$φ$\rangle$, where $|$φ$|$ represents a row vector of φ of any state in the corresponding direction $(\phi_0, \phi_1, \phi_2, \ldots, \phi_8)$ and $|$φ$\rangle$ represents a column vector $(\phi_0, \phi_1, \phi_2, \ldots, \phi_8)^T$. The inner product $\sum_{\alpha=0}^{8} \phi_\alpha \psi_\alpha$ is then denoted by $\langle |\phi\rangle|\psi\rangle$. Copyright © 2015 John Wiley & Sons, Ltd. Int. J. Numer. Meth. Fluids 2016; 82:59–90 DOI: 10.1002/fld
As the cascaded LBM is a moment approach, we need a set of nine linearly independent moment basis vectors for its specification. The (raw) moments of the distribution function \( f_\alpha \) of different orders can be defined as \( \sum_{\alpha=0}^{\infty} e_{ax}^m e_{ay}^n f_\alpha \). Here, \( \alpha \) is the discrete particle direction, and \( m \) and \( n \) are integers. Thus, a set of nine linearly independent nonorthogonal basis vectors obtained using the monomials \( e_{ax}^m e_{ay}^n \) in an ascending order can be written as

\[
|\rho| = |\tilde{e}_\alpha|^0 = (1, 1, 1, 1, 1, 1, 1, 1, 1)^T, \\
|e_{ax}| = (0, 1, 0, -1, 0, 1, -1, 1, -1)^T, \\
|e_{ay}| = (0, 0, 1, 0, 1, -1, -1, 1, -1)^T, \\
|e_{ax}^2 + e_{ay}^2| = (0, 1, 1, 1, 2, 2, 2, 2, 2)^T. \\
|e_{ax}^2 - e_{ay}^2| = (0, -1, -1, 1, 0, 0, 0, 0, 0)^T, \\
|e_{ax} e_{ay}| = (0, 0, 0, 0, 0, 1, -1, 1, -1)^T, \\
|e_{ax}^2 e_{ay}| = (0, 0, 0, 0, 1, 1, -1, -1, -1)^T, \\
|e_{ax}^2 e_{ay}^2| = (0, 0, 0, 0, 0, 0, 1, 1, 1)^T.
\]

This can be transformed by means of the Gram–Schmidt procedure into the following equivalent set of orthogonal basis vectors, which provides a computationally more efficient and convenient setting for the description of the method [26]:

\[
|K_0| = |\rho|, \\
|K_1| = |e_{ax}|, \\
|K_2| = |e_{ay}|, \\
|K_3| = 3|e_{ax}^2 + e_{ay}^2| - 4|\rho|, \\
|K_4| = |e_{ax}^2 - e_{ay}^2|, \\
|K_5| = |e_{ax} e_{ay}|, \\
|K_6| = -3|e_{ax}^2 e_{ay}| + 2|e_{ay}|, \\
|K_7| = -3|e_{ax} e_{ay}^2| + 2|e_{ax}|, \\
|K_8| = 9|e_{ax}^2 e_{ay}^2| - 6|e_{ax}^2 + e_{ay}^2| + 4|\rho|.
\]

Collecting the aforementioned set of vectors as a matrix \( \mathcal{K} \), it immediately follows that \( \mathcal{K} \mathcal{K}^T \) is a diagonal matrix, owing to orthogonality. This orthogonal matrix \( \mathcal{K} \) can be written in component form as

\[
\mathcal{K} = ([K_0], [K_1], [K_2], [K_3], [K_4], [K_5], [K_6], [K_7], [K_8]) = \\
\begin{bmatrix}
1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 4 \\
1 & 1 & 0 & -1 & 1 & 0 & 0 & 2 & -2 \\
1 & 0 & 1 & -1 & -1 & 0 & 2 & 0 & -2 \\
1 & -1 & 0 & -1 & 1 & 0 & 0 & -2 & -2 \\
1 & 0 & -1 & -1 & -1 & 0 & -2 & 0 & -2 \\
1 & 1 & 1 & 2 & 0 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & 2 & 0 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 2 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 2 & 0 & -1 & 1 & -1 & 1
\end{bmatrix}
\]

To specify the collision step and forcing, we need the central moments of the local equilibrium and sources, which can be obtained as follows. First, the local Maxwell–Boltzmann distribution function in the continuous particle velocity space \( (\xi_x, \xi_y) \) is written as \( f^M \equiv f^M(\rho, \bar{u}, \xi_x, \xi_y) = \frac{\rho}{2\pi c_s^2} \exp \left[ -\frac{(\bar{u} - \mu)^2}{2c_s^2} \right] \), where \( c_s \) is the speed of sound (typically, \( c_s^2 = 1/3 \)). Based on this, the continuous central moments of the local equilibrium distribution of order \( (m + n) \) can be defined as \( \hat{\Pi}^M_{xy} = \int f^M(\xi_x - u_x, \xi_y - u_y) d\xi_x d\xi_y \), which yields \( \hat{\Pi}^M_{x^n y^m} = (\hat{\Pi}^M_{x0} \hat{\Pi}^M_{x1} \hat{\Pi}^M_{x2} \hat{\Pi}^M_{xy} \hat{\Pi}^M_{xy} \hat{\Pi}^M_{xxy} \hat{\Pi}^M_{xyy} \hat{\Pi}^M_{xxyy})^T = (\rho, 0, 0, c_s^2 \rho, c_s^2 \rho, 0, 0, 0, c_s^4 \rho)^T \). Considering that the impressed forces only influence the fluid momentum, the central
moments of the sources of order \((m+n)\) because of a force field \((F_x, F_y)\) defined by
\[ \hat{\Gamma}_{x}^{x} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta f^{x}(\xi_x - u_x)^m(\xi_y - u_y)^n d\xi_x d\xi_y, \]
where \(\Delta f^{x}\) is the change in the distribution function because of force fields, can be simply written as [26]
\[ \hat{\Gamma}_{x}^{x} = \left( \hat{\Gamma}_0^{x}, \hat{\Gamma}_x^{x}, \hat{\Gamma}_y^{x}, \hat{\Gamma}_{xx}^{x}, \hat{\Gamma}_{xy}^{x}, \hat{\Gamma}_{xyy}^{x} \right)^T = \left( 0, F_x, F_y, 0, 0, 0, 0, 0 \right)^T. \]

Based on the aforementioned continuous central moments, the elements of the cascaded LBE can be formulated. Using the trapezoidal rule representation of the source term, the cascaded LBE can be written as [26]
\[
f_a(\vec{x} + \vec{e}_a \delta_t, t + \delta_t) = f_a(\vec{x}, t) + \frac{\Omega_a^{C}}{\alpha(\vec{x}, t)} + \frac{1}{2} \left[ S_a(\vec{x}, t) + S_a(\vec{x} + \vec{e}_a \delta_t + \vec{e}_b \delta_t) \right].
\] (4)

Here, the collision term \(\Omega_a^{C}\) can be represented as \(\Omega_a^{C} = \Omega_a^{C}(f, \hat{\vec{g}}) = (K \cdot \hat{\vec{g}})_a\), where \(f \equiv |f_a| = (0, f_0, f_1, \ldots, f_8)^T\) is the vector of distribution functions and \(\hat{\vec{g}} \equiv (\vec{g}_0, \vec{g}_1, \ldots, \vec{g}_8)^T\) is the vector of unknown collision kernel to be specified later. Owing to the cascaded nature of the central moment-based approach, it satisfies the following functional relation \(\hat{\vec{g}}_a = \hat{\vec{g}}_a(f, \hat{\vec{g}}_b)\). \(\beta = 0, 1, \ldots, \alpha - 1\). The discrete form of the source term \(S_a\) in the cascaded LBE given above represents the influence of the force field \((F_x, F_y)\) in the velocity space and is defined as \(S = |S_a| = (S_0, S_1, S_2, \ldots, S_8)^T\). Noting that Equation (4) is semi-implicit, by using the standard variable transformation \(\vec{f} = f_a - \frac{1}{2} S_a\), its implicitness can be effectively removed. This yields
\[
\vec{f}_a(\vec{x} + \vec{e}_a \delta_t, t + \delta_t) = \vec{f}_a(\vec{x}, t) + \Omega_a^{C}(\vec{x}, t) + S_a(\vec{x}, t).
\] (5)

The derivation of the collision term, that is, the collision kernel \(\hat{\vec{g}}\) and the source term \(S\) involves matching the discrete central moments and the continuous central moments of equilibria and sources, which are specified earlier, of all orders supported by the lattice set. We designate this step as the Galilean invariance matching principle. First, the discrete central moments of the distribution functions and sources of order \((m+n)\) can be defined, respectively, as \(\hat{\vec{\kappa}}_{x}^{m} = \langle (e_{ax} - u_x)^m(e_{ay} - u_y)^n | f_a \rangle\) and \(\hat{\vec{\sigma}}_{x}^{m} = \langle (e_{ax} - u_x)^m(e_{ay} - u_y)^n | S_a \rangle\). Also, in terms of the transformed distribution functions, we define \(\hat{\vec{\kappa}}^{eq}_{x} = \langle (e_{ax} - u_x)^m(e_{ay} - u_y)^n | \hat{\vec{f}}_a \rangle\), which satisfies \(\hat{\vec{\kappa}}^{eq}_{x} = \hat{\vec{\kappa}}_{x}^{m} - \frac{1}{4} \hat{\vec{\sigma}}_{x}^{m} \), and similarly for the local equilibria \(\hat{\vec{\kappa}}^{eq}_{x} = \langle (e_{ax} - u_x)^m(e_{ay} - u_y)^n | \hat{\vec{f}}_a \rangle\). Then, the Galilean invariance matching principle reads
\[
\hat{\vec{\kappa}}_{x}^{eq} = \hat{\vec{\Pi}}_{x}^{M}, \quad \hat{\vec{\sigma}}_{x}^{eq} = \hat{\vec{\Gamma}}_{x}^{f}.
\] (6)

This immediately specifies the various discrete central moments. Hence, we obtain
\[
\hat{\vec{\kappa}}_{x}^{eq} = \left( \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq} \right)^T
\] (7)

and
\[
\hat{\vec{\sigma}}_{x}^{eq} = \left( \hat{\vec{\sigma}}, \hat{\vec{\sigma}}, \hat{\vec{\sigma}}, \hat{\vec{\sigma}}, \hat{\vec{\sigma}}, \hat{\vec{\sigma}}, \hat{\vec{\sigma}}, \hat{\vec{\sigma}} \right)^T
\] (8)

and
\[
\hat{\vec{\kappa}}_{x}^{eq} = \left( \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq}, \hat{\vec{\kappa}}_{x}^{eq} \right)^T
\] (9)

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The next important step is to transform all the aforementioned discrete central moments in terms of the raw moments, which can be readily accomplished by means of the following binomial formula: 

$$
\langle (e_{ax} - u_x)^m (e_{ay} - u_y)^n \varphi \rangle = \langle e_{ax}^m e_{ay}^n \varphi \rangle + \left\{ \begin{array}{l}
\sum_{l=1}^{m} C_l^m (e_{ax}^{m-l} (u_x^l) \langle \varphi \rangle) + \left[ \sum_{i=1}^{n} C_i^n (e_{ay}^{n-i} (u_y^i) \langle \varphi \rangle) \right]
\end{array} \right.
$$

where $C_l^m = p^l (q^{m-l})$. Thus, we can obtain the following discrete central moments of sources $\hat{\sigma}_x^{m,y,n}$ as

$$
\hat{\sigma}_x' = (S_a | \rho) = 0, \quad \hat{\sigma}_x = (S_a | e_{ax}) = F_x, \quad \hat{\sigma}_y = (S_a | e_{ay}) = F_y,
$$

$$
\hat{\sigma}_{xx}' = (S_a | e_{ax}^2) = 2 F_x u_x, \quad \hat{\sigma}_{yy}' = (S_a | e_{ay}^2) = 2 F_y u_y, \quad \hat{\sigma}_{xy}' = (S_a | e_{ax} e_{ay}) = F_x u_y + F_y u_x,
$$

$$
\hat{\sigma}_{xyy}' = (S_a | e_{ax}^2 e_{ay}) = 2 F_x u_x u_y + 2 F_y u_y u_x.
$$

Based on the previous discussion, we now obtain the following source terms projected to the orthogonal moment basis vectors, that is, $\hat{m}_0 = (K_e | S_a)$, $\beta = 0, 1, 2, \ldots, 8$, which is needed to obtain the source terms in the velocity space subsequently:

$$
\hat{m}_0 = 0, \quad \hat{m}_1 = F_x, \quad \hat{m}_2 = F_y,
$$

$$
\hat{m}_3 = 6 (F_x u_x + F_y u_y), \quad \hat{m}_4 = 2 (F_x u_x - F_y u_y), \quad \hat{m}_5 = (F_x u_y + F_y u_x),
$$

$$
\hat{m}_6 = (2 - 3 u_x^2) F_y - 6 F_x u_x u_y, \quad \hat{m}_7 = (2 - 3 u_y^2) F_x - 6 F_y u_y u_x,
$$

$$
\hat{m}_8 = 6 \left[ (3 u_y^2 - 2) F_y u_x + (3 u_x^2 - 2) F_y u_y \right].
$$

Equivalently, this can be written in a matrix form as $\hat{K}^T \hat{S} = (K \cdot S)_a = (\hat{m}_0, \hat{m}_1, \hat{m}_2, \ldots, \hat{m}_8)^T \equiv |\hat{m}_0^2|$. By exploiting the orthogonal property of $K$, that is, $K^{-1} = K^T \cdot D^{-1}$, where the diagonal matrix is $D = \text{diag}(K_0 | K_0), (K_1 | K_1), (K_2 | K_2) \ldots (K_8 | K_8)$, we can invert the previous equation to obtain the source terms in velocity space $S_a$ as

$$
S_0 = \frac{1}{9} (-m_2^2 + m_8^2),
$$

$$
S_1 = \frac{1}{36} (6 m_1^2 - m_5^2 + 9 m_4^2 + 6 m_7^2 - 2 m_8^2),
$$

$$
S_2 = \frac{1}{36} (6 m_2^2 - m_3^2 - 9 m_4^2 + 6 m_6^2 - 2 m_8^2),
$$

$$
S_3 = \frac{1}{36} (-6 m_1^2 - m_3^2 + 9 m_4^2 - 6 m_7^2 - 2 m_8^2),
$$

$$
S_4 = \frac{1}{36} (-6 m_2^2 - m_3^2 - 9 m_4^2 - 6 m_6^2 - 2 m_8^2),
$$

$$
S_5 = \frac{1}{36} (6 m_1^2 + 6 m_2^2 + 2 m_3^2 + 9 m_4^2 - 3 m_6^2 - 3 m_7^2 + m_8^2),
$$

$$
S_6 = \frac{1}{36} (-6 m_1^2 + 6 m_2^2 + 2 m_3^2 - 9 m_4^2 - 3 m_6^2 + 3 m_7^2 + m_8^2),
$$

$$
S_7 = \frac{1}{36} (-6 m_2^2 - 6 m_1^2 + 2 m_3^2 + 9 m_4^2 + 3 m_6^2 + 3 m_7^2 + m_8^2),
$$

$$
S_8 = \frac{1}{36} (6 m_1^2 - 6 m_2^2 + 2 m_3^2 - 9 m_4^2 + 3 m_6^2 - 3 m_7^2 + m_8^2).
$$

The discrete raw moments of the transformed distribution functions $\hat{F}_x^{m,y,n}$, which will be needed in the evaluation of the collision kernel, can be conveniently written as follows:

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\[ \vec{\kappa}'_0 = (\overline{f_a}|\rho) = \rho \vec{\kappa}'_x = (\overline{f_a}|e_{ax}) = \rho u_x - \frac{1}{2} F_x, \vec{\kappa}'_y = (\overline{f_a}|e_{ay}) = \rho u_y - \frac{1}{2} F_y, \]

\[ \vec{\kappa}'_{xx} = (\overline{f_a}|e_{ax}^2) = \left( \sum_a \{1, 3, 5, 6, 7, 8\} \right) \times \overline{f_a}, \quad \vec{\kappa}'_{yy} = (\overline{f_a}|e_{ay}^2) = \left( \sum_a \{2, 4, 5, 6, 7, 8\} \right) \times \overline{f_a}, \]

\[ \vec{\kappa}'_{xy} = (\overline{f_a}|e_{ax}e_{ay}) = \left( \sum_a \{5, 7\} - \sum_a \{6, 8\} \right) \times \overline{f_a}, \quad \vec{\kappa}'_{xxyy} = (\overline{f_a}|e_{ax}^2e_{ay}^2) = \left( \sum_a \{5, 6, 7, 8\} \right) \times \overline{f_a}, \]

where we have used \((a \sum_a^A + b \sum_a^B + \cdots) \times \overline{f_a} = a(\overline{f}_{a_1} + \overline{f}_{a_2} + \overline{f}_{a_3} + \cdots) + b(\overline{f}_{b_1} + \overline{f}_{b_2} + \overline{f}_{b_3} + \cdots) + \cdots, \) with \(A = \{\alpha_1, \alpha_2, \alpha_3, \cdots\}, B = \{\beta_1, \beta_2, \beta_3, \cdots\}, \cdots, \) as a compact summation operator for ease of presentation. Furthermore, the raw moments of the collision kernels \(\sum_a (K \cdot \overline{g}_a)_{e_{ax}e_{ay}} = \sum_\beta (K_{\beta}e_{ax}e_{ay}) \overline{g}_\beta \) are needed in its construction. Collision invariants of conserved moments imply \(\overline{g}_0 = \overline{g}_1 = \overline{g}_2 = 0.\) Exploiting the orthogonal property of the matrix \(K,\) the non-conserved moments of \(\overline{g}_\beta\) at higher orders, that is, \(\beta = 3, 4, \ldots, 8\) can be obtained, which are detailed in [26].

Using the discussions earlier, the collision kernel \(\overline{g}_\beta,\) of the cascaded collision operator \(\Omega'_\alpha = \Omega'_\alpha(f, \overline{g}) = (K \cdot \overline{g})_{\alpha}\) can be obtained as follows. Starting from the lowest-order central moments that are non-collisional invariants (i.e., \(\vec{\kappa}'_{xx}\) and higher), their post-collision values are successively set equal to their local attractors based on the transformed equilibria. This step provides tentative expressions for \(\overline{g}_\alpha\) based on the equilibrium assumption, which is then modified to allow for a relaxation process during collision. That is, they are multiplied with the corresponding relaxation parameters [24]. In this step, care needs to be exercised to multiply the relaxation parameters only with those terms that are not yet in post-collision states (i.e., terms not involving \(\overline{g}_\beta, \beta = 0, 1, 2, \ldots, \alpha - 1\) for a given \(\overline{g}_\alpha\). See Geier et al. and Premnath and Banerjee [24, 26] for various details involved in this procedure. Here, we summarize the final expressions of the non-conserved collision kernels, which are given as follows:

\[ \overline{g}_3 = \frac{\omega_3}{12} \left\{ 2 \left[ \frac{2}{3} \rho + \rho \left( u_x^2 + u_y^2 \right) \right] - \left( \vec{\kappa}'_{xx} + \vec{\kappa}'_{yy} \right) - \frac{1}{2} \left( \vec{\sigma}'_{xx} + \vec{\sigma}'_{yy} \right) \right\}, \]

\[ \overline{g}_4 = \frac{\omega_4}{4} \left\{ \rho \left( u_x^2 - u_y^2 \right) - \left( \vec{\kappa}'_{xx} - \vec{\kappa}'_{yy} \right) - \frac{1}{2} \left( \vec{\sigma}'_{xx} - \vec{\sigma}'_{yy} \right) \right\}, \]

\[ \overline{g}_5 = \frac{\omega_5}{4} \left\{ \rho u_x u_y - \vec{\kappa}'_{xy} - \frac{1}{2} \vec{\sigma}'_{xy} \right\}, \]

\[ \overline{g}_6 = \frac{\omega_6}{4} \left\{ 2 pu_x^2 u_y + \vec{\kappa}'_{xxyy} - 2 u_x \vec{\kappa}'_{xy} - u_y \vec{\kappa}'_{xx} - \frac{1}{2} \vec{\sigma}'_{xxyy} \right\} - \frac{1}{2} u_y (3 \overline{g}_3 + \overline{g}_4) - 2 u_x \overline{g}_5, \]

\[ \overline{g}_7 = \frac{\omega_7}{4} \left\{ 2 pu_x u_y^2 + \vec{\kappa}'_{xxyy} - 2 u_y \vec{\kappa}'_{xy} - u_x \vec{\kappa}'_{yy} - \frac{1}{2} \vec{\sigma}'_{xxyy} \right\} - \frac{1}{2} u_x (3 \overline{g}_3 + \overline{g}_4) - 2 u_y \overline{g}_5, \]
\[ \tilde{g}_8 = \frac{\omega_8}{4} \left\{ \frac{1}{9} \rho + 3 \rho u_x^2 u_y - \left[ \tilde{w}_{xxx} - 2 \tilde{w}_{xxy} - 2 \tilde{w}_{xyy} + \tilde{w}_{xxy} + u_x^2 \tilde{w}_{yy} + u_y^2 \tilde{w}_{xx} \right] \right. \\
\left. + 4 u_x u_y \tilde{w}_{xy} \right\} - \frac{1}{2} \tilde{\sigma}_{xy} - \frac{1}{2} \tilde{\sigma}_{xx} - \frac{1}{2} u_x^2(3 \tilde{g}_3 + \tilde{g}_4) - \frac{1}{2} u_y^2(3 \tilde{g}_3 - \tilde{g}_4) - 4 u_x u_y \tilde{g}_5 - 2 u_y \tilde{g}_6 - 2 u_x \tilde{g}_7. \]

In the previous discussion, \( \omega_\beta \), where \( \beta = 3, 4, 5, \ldots, 8 \), are the relaxation parameters, satisfying the usual bounds \( 0 < \omega_\beta < 2 \). When a Chapman–Enskog expansion [31] is applied to the cascaded LBM, it can be shown to recover the Navier–Stokes equations with the relaxation parameters \( \omega_3 = \omega^k \) and \( \omega_4 = \omega_5 = \omega^{k'} \) controlling the bulk and shear viscosities, respectively (e.g., \( \nu = c_s^2 \left( \frac{1}{\omega^k} - \frac{1}{2} \right) \)) [26]. The rest of the parameters can be adjusted independently to improve the numerical stability. In this work, \( \omega_4 = \omega_5 = \frac{1}{4} \) is selected based on the specified kinematic viscosity; unless otherwise stated, the rest of the relaxation parameters for the higher order moments are set to 1 in this numerical study. The possibility of tuning the relaxation parameters for the higher order moments and its effect on the numerical stability will be explored in Section 4.3.

The cascaded LBE can now be re-written in the form of the usual stream-and-collide procedure, leading to the following two steps:

\[ \tilde{f}_a(x, t) = \tilde{f}_a(x, t) + \Omega_{a(x,t)} + S_{a(x,t)}, \]

\[ \tilde{f}_a(x + \tilde{c}_a, t + \delta t) = \tilde{f}_a(x, t). \]

where the symbol ‘tilde’ (\( \sim \)) in the earlier equations refers to the post-collision state of the distribution function. Expanding the collision term in the first step, the components of the post-collision distribution function can be explicitly written as

\[ \tilde{f}_0 = f_0 + [g_0 + 4(g_3 - g_8)] + S_0, \]

\[ \tilde{f}_1 = f_1 + [g_1 + g_3 + g_4 + 2(g_7 - g_8)] + S_1, \]

\[ \tilde{f}_2 = f_2 + [g_2 + g_3 + g_4 + 2(g_7 - g_8)] + S_2, \]

\[ \tilde{f}_3 = f_3 + [g_0 + g_1 + g_3 + g_4 + 2(g_7 + g_8)] + S_3, \]

\[ \tilde{f}_4 = f_4 + [g_0 + g_2 + g_3 + g_4 + 2(g_6 + g_8)] + S_4, \]

\[ \tilde{f}_5 = f_5 + [g_0 + g_1 + g_2 + 2g_3 + g_4 + 2g_7 + g_8] + S_5, \]

\[ \tilde{f}_6 = f_6 + [g_0 + g_1 + g_2 + 2g_3 + 2g_4 + 2g_6 + g_8] + S_6, \]

\[ \tilde{f}_7 = f_7 + [g_0 + g_1 + g_2 + 2g_3 + 2g_5 + 2g_6 + g_7 + g_8] + S_7, \]

\[ \tilde{f}_8 = f_8 + [g_0 + g_1 + g_2 + 2g_3 + 2g_5 + g_6 + g_7 + g_8] + S_8. \]

The hydrodynamic fields, that is, the fluid density and the velocity then follow from taking the zeroth and first moments of the distribution function yielding

\[ \rho = \sum_{a=0}^{8} \tilde{f}_a = \langle \tilde{f}_a | \rho \rangle, \]

\[ \rho u_i = \sum_{a=0}^{8} \tilde{f}_a e_{ai} + \frac{1}{2} F_i = \langle \tilde{f}_a | e_{ai} \rangle + \frac{1}{2} F_i, \quad i = x, y, \]

and the pressure \( p \) satisfies \( p = c_s^2 \rho \). A particularly useful feature of kinetic schemes such as the cascaded LBM is that the strain-rate tensor can be computed locally from a knowledge of the non-equilibrium moments. In fact, this can be shown by means of the Chapman–Enskog analysis, which
was performed on the cascaded LBE in [26]. Setting the components of the momentum as \( j_x = \rho u_x \) and \( j_y = \rho u_y \), such an analysis shows [26]
\[
\hat{f}_3^{(neq)} = -\frac{2}{3\omega_3} (\partial_x j_x + \partial_y j_y), \quad \hat{f}_4^{(neq)} = -\frac{2}{3\omega_4} (\partial_x j_x - \partial_y j_y) + \hat{f}_5^{(neq)} = -\frac{1}{3\omega_5} (\partial_x j_y + \partial_y j_x),
\]
where \( \hat{f}_\beta^{(neq)} \approx \hat{f}_\beta - \hat{f}_\beta^{eq} \) are the non-equilibrium raw moments. Specifically, \( \hat{f}_3 = \hat{\kappa}'_{xx} + \hat{\kappa}'_{yy} \), \( \hat{f}_4 = \kappa'_{xx} - \kappa'_{yy} \), and \( \hat{f}_5 = \kappa'_{xy} \), whose equilibria are \( \hat{f}_3^{eq} = 2/3\rho + \rho (u_x^2 + u_y^2) \), \( \hat{f}_4^{eq} = \rho (u_x^2 - u_y^2) \), and \( \hat{f}_5^{eq} = \rho u_x u_y \), respectively [26]. It thus follows that
\[
\partial_x j_x = -\frac{3\omega_3}{2} \left[ \sum_{a=0}^{8} f_a e_{ax}^2 - \left( \frac{1}{3}\rho + \rho u_x^2 \right) \right], \tag{23}
\]
\[
\partial_y j_y = -\frac{3\omega_4}{2} \left[ \sum_{a=0}^{8} f_a e_{ay}^2 - \left( \frac{1}{3}\rho + \rho u_y^2 \right) \right], \tag{24}
\]
\[
\partial_x j_y + \partial_y j_x = -\frac{3\omega_5}{2} \left[ \sum_{a=0}^{8} f_a e_{ax} e_{ay} - \rho u_x u_y \right]. \tag{25}
\]

These specific expressions will be exploited in the numerical study of the cascaded LBM in the remainder of this paper. In a later section, we will present the results obtained with the cascaded LBM for a set of benchmark problems and its comparison with other LBM schemes discussed in the succeeding sections in order to assess its numerical properties in terms of grid convergence, accuracy, and stability.

2.2. Multiple-relaxation-time lattice Boltzmann method based on raw moments

For the sake of completeness, we will now present the main features of the standard MRT LBM based on raw moments [11, 12]. Briefly, in the standard MRT LBM, the collision as well as the forcing are carried out in the raw moment space, while the streaming step is performed in the velocity space. Defining the vectors for the distribution function, its equilibrium and source terms in velocity space as \( \mathbf{f} = (f_0, f_1, \ldots, f_8) \), \( \mathbf{f}^{eq} = (f_0^{eq}, f_1^{eq}, \ldots, f_8^{eq}) \), and \( \mathbf{S} = (S_0, S_1, \ldots, S_8) \), respectively, the standard MRT LBM may be written as
\[
\mathbf{\tilde{f}}(\tilde{x}, t) = \mathbf{f}(\tilde{x}, t) + \mathbf{\varpi}(\tilde{x}, t), \tag{26}
\]
\[
f_a(\tilde{x} + \tilde{c}_a, t + \delta t) = \mathbf{\tilde{f}}_a(\tilde{x}, t), \tag{27}
\]
where
\[
\mathbf{\varpi}(\tilde{x}, t) = \mathbf{T}^{-1} \left[ -\hat{\Lambda} \left( \mathbf{\tilde{f}} - \mathbf{\tilde{f}^{eq}} \right) + \left( \mathbf{I} - \frac{1}{2}\hat{\Lambda} \right) \mathbf{S} \right] \tag{28}
\]
Here, \( \mathbf{\tilde{f}} = \mathbf{T} \mathbf{f} \), \( \mathbf{\tilde{f}^{eq}} = \mathbf{T} \mathbf{f}^{eq} \) and \( \hat{\mathbf{S}} = \mathbf{T} \mathbf{S} \) are the corresponding components of the distribution function, its equilibrium, and source terms, respectively, in moment space, where \( \mathbf{T} \) is the transformation matrix [12]. For example, the equilibrium moments are given as follows [12]:
\[
\begin{align*}
\hat{f}_0 &= \rho, & \hat{f}_1 &= e^{eq} = -2\rho + 3j \cdot \hat{j}/\rho, \\
\hat{f}_2 &= e^{eq}, & \hat{f}_3 &= j_x, & \hat{f}_4 &= q_x^{eq} = -j_x, & \hat{f}_5 &= j_y, & \hat{f}_6 &= q_y^{eq} = -j_y, \\
\hat{f}_7 &= \rho^x_{xx}^{eq} = (j_x^2 - j_y^2)/\rho, & \hat{f}_8 &= \rho^x_{xy}^{eq} = (j_x j_y)/\rho, \end{align*}
\]
where \( \hat{j} = (j_x, j_y) = (\rho u_x, \rho u_y) \) is the momentum vector. For source terms in moment space, see, for example [32]. In addition, \( \hat{\Lambda} = diag(s_0, s_1, \ldots, s_8) \) is the relaxation time matrix, in which \( s_7 = s_8 = 1/\tau \) is related to the shear kinematic viscosity, while the rest can be tuned to improve numerical stability. The standard sets of values for the various relaxation times prescribed in [12] will be used for the MRT LBM based on raw moments in this study. This standard MRT LBM, along with the cascaded formulation discussed earlier, will be used in Section 4.
2.3. Single-Relaxation-Time (SRT) Lattice Boltzmann method

The single-relaxation-time (SRT) LBM is the simplest among the various LBM schemes. In the SRT LBM, both collision and streaming steps are performed in the velocity space. All the modes are taken to relax at the same rate \( \tau \) during the collision process [9]. It may be written as

\[
\tilde{f}_\alpha(\tilde{x}, t) = f_\alpha(\tilde{x}, t) - \frac{1}{\tau} (f_\alpha - f_\alpha^{eq}) + \left(1 - \frac{1}{2\tau}\right) S_\alpha, \tag{29}
\]

\[
f_\alpha(\tilde{x} + \tilde{c}_\alpha, t + \delta_t) = \tilde{f}_\alpha(\tilde{x}, t), \tag{30}
\]

where \( f_\alpha^{eq} = w_\alpha \rho \left(1 + (\tilde{c} \cdot \tilde{u})/c_s^2 + (\tilde{c} \cdot \tilde{u})^2/(2c_s^4) - (\tilde{u} \cdot \tilde{u})/ (2c_s^2)\right) \) is the discrete equilibrium distribution function and \( w_\alpha \) are the weighting factors [9]. See, for example, Guo et al. [33] for an expression for the source term \( S_\alpha \). The SRT LBM, in addition to the other models, will be used in the case study involving an analysis of numerical stability in Section 4.3.

3. RESULTS AND DISCUSSION: NUMERICAL PERFORMANCE STUDY OF THE CASCADED LATTICE BOLTZMANN METHOD FOR POISEUILLE AND TAYLOR–GREEN VORTEX FLOWS

We first perform a numerical study involving the grid convergence and accuracy in the computation of the steady 2D Poiseuille flow and a time-dependent 2D decaying Taylor–Green vortex flow. In the various figures presented in this section and henceforth, the symbols represent the computed solution using the cascaded MRT LBM, the thin solid lines are the resulting slopes representing changes in the relative errors as the grid resolution increases, and the thick solid lines are the ideal slopes corresponding to second-order accuracy. In this work, a diffusive scaling is applied to perform the convergence tests [34]. According to this scaling, the errors due to compressibility effects decrease at the same rate as the errors due to grid discretization thus prescribing a consistent limit process to represent incompressible flow. That is, the velocity scales in the same proportion as the discretization length scales. Equivalently, this means that the ratio of the Mach number and the grid Knudsen number remains constant for different grid resolutions, that is, \( \text{Ma}/\text{Kn} = \text{constant} \).

3.1. Two-dimensional Poiseuille flow: grid convergence

The 2D Poiseuille flow is considered first. The flow is between two parallel plates of infinite length in the streamwise direction which is subjected to a constant body force. A periodic boundary condition is applied at the inlet and the outlet and a no-slip boundary condition at the solid boundaries by employing the standard half-way bounce back approach. The grid convergence is established by considering the following resolutions consisting of \( 3 \times 24 \), \( 3 \times 36 \), \( 3 \times 192 \) lattice nodes under diffusive scaling. The relaxation time for shear modes is set to \( \tau = 0.55 \) that specifies \( \omega_4 \) and \( \omega_5 \). The rest of the relaxation parameters in this case are set to unity. The flow is driven by a constant body force with the components \( F_x \) specified to yield the desired flow field (see succeeding sections) and with \( F_y = 0 \). This classical flow problem has the well-known parabolic profile as the analytical solution given by \( u(y) = u_{\text{max}} (1 - y^2/L^2) \), where \( u_{\text{max}} = F_x L^2/\nu \) is the maximum velocity occurring midway between the plates, \( \nu \) is the kinematic viscosity related to the relaxation time \( \tau \) as given in the previous section, and \( L \) denotes the half-width between the plates. Figure 1 illustrates the relative global errors between the computed solutions obtained using the cascaded MRT LBM and the analytical solutions for such flow at three different Reynolds numbers, that is, 100, 200, and 400. The relative global error, which quantifies the difference between the computed and analytical solutions, is defined as

\[
\text{Relative Error} = \frac{\sqrt{\sum_{i,j} [u_c(i,j) - u_a(i,j)]^2}}{\sqrt{\sum_{i,j} [u_a(i,j)]^2}}, \tag{31}
\]
where \( u_c(i, j) \) and \( u_a(i, j) \) are the computed and the analytical solutions, respectively, at the node \((i, j)\). We thus use the standard Euclidean norm or the \( L^2 \) norm in the aforementioned measurements, which is a common approach to estimate the relative error (see, e.g., [35] for its recent application of the LBM for cavity simulations). It is seen that the relative errors have slopes of almost equal to 2.00, which tells that the cascaded MRT LBM is well-posed second-order accurate for this problem. In addition, the relative errors are seen to slightly increase as the Reynolds number increases.

### 3.2. Two-dimensional decaying Taylor–Green vortex flow: grid convergence

The second problem considered is the decaying Taylor–Green vortex [36], which is a 2D unsteady flow induced by a prescribed initial vortex distribution and decaying because of fluid viscosity. The fluid domain is a square of side \( 2\pi \) with no inflow/outflow and wall boundaries. The initial condition is set to be periodic array of vortices in both \( x \) and \( y \) directions as follows:

\[
  u(x, y, 0) = -u_0 \cos(kx) \sin(ky),
\]

\[
  v(x, y, 0) = +u_0 \sin(kx) \cos(ky),
\]

\[
  p(x, y, 0) = p_0 \left[ 1 - \frac{u_0^2}{4c_s^2} (\cos(2kx) + \cos(2ky)) \right],
\]

where \( k = \frac{2\pi}{N} \) is the wavenumber and \( u_0 \) and \( p_0 \) are the initial values for the velocity and pressure, respectively. Here, \( N \) is the number of grid nodes in each direction. The temporal evolution has the characteristic time scale given by \( T = \frac{1}{2k^2\nu} \). Because there is no external energy supplied and because of the presence of fluid viscosity, the velocity field will decay with time because of viscous dissipation. There exists an analytical solution for this problem, which is a solution of the Navier–Stokes equations in a periodic domain and is given by

\[
  u(x, y, t) = -u_0 \cos(kx) \sin(ky)e^{-2k^2\nu t},
\]

\[
  v(x, y, t) = +u_0 \sin(kx) \cos(ky)e^{-2k^2\nu t},
\]
Furthermore, the components of the strain rate tensor also satisfy the following explicit analytical solution:

\[
P(x, y, t) = p_0 - \frac{u_0^2}{4} [\cos(2kx) + \cos(2ky)] e^{-4k^2vt}.\]  

(37)

In this test, the Reynolds number of the flow is set to \( Re = \frac{u_0 l}{v} = 3.77 \), where \( l = 2\pi \) is the length of the domain. A periodic boundary condition is applied to all the sides of the domain. We consider the following parameters in our grid convergence study: \( \tau = 0.55, k = 1, 2, \) and \( u_0 = 0.01 \). Applying the diffusive scaling, we obtain the relative global errors between the computed and the analytical solutions for the grid resolutions of \( 24^2, 48^2, 96^2, \) and \( 192^2 \) for a representative time \( t = 30.1T \). In Figure 2, the relative errors for the \( u \)-velocity component are shown, which have the slopes of 1.99 and 1.98 for the wavenumbers \( k = 1 \) and \( k = 2 \), respectively. Figure 3 shows the relative errors for the only independent strain rate tensor component \( S_{xx} = -S_{yy} \) with the slopes of 1.99 and 1.98 as well for the previous two wavenumbers. Thus, it is evident that the cascaded MRT LBM is second-order accurate not only for the velocity field but also for the components of the strain rates as well. This finding is consistent with a recent study with the SRT LBM for this problem [37]. Because the strain rate tensor components are related to the gradients of the velocity field, such observations of the second-order accuracy seem to be associated with \( H^1 \) type norm and not only the common norms such as the \( L^2 \) norm. This appears to be an interesting feature of the LBM in general, and the cascaded LBM, in particular.

Let us now make a more detailed comparison of the accuracy of the solutions computed using the cascaded LBM with prior analytical results for the two benchmark problems considered earlier.
3.3. Two-dimensional Poiseuille flow: accuracy

Figure 4 shows a comparison of the velocity profiles of the 2D Poiseuille flow between the results obtained using the cascaded LBM and the parabolic analytical solution at a constant Reynolds number of 200 with constant relaxation time $\tau = 0.515$ for different grid resolutions in the wall normal direction starting from 26 to 401. Here, diffusive scaling is employed in the selection of parameters. That is, as the resolution is doubled, the maximum flow velocity or the Mach number is decreased by a factor of two. The results are in excellent agreement with the analytical solution, in which the maximum relative error is less than 0.22%.
3.4. Two-dimensional decaying Taylor–Green vortex flow: accuracy

Using the same set of parameters specified for this time-dependent problem in the previous section, we now compare the computed $U$ — velocity and $V$ — velocity components along the vertical and horizontal centerlines, respectively, with the corresponding analytical solutions (Equations (35)–(36)) at three different representative instants. Figures 5 and 6 show such a comparison of the velocity components.

![Figure 5](image1.png)

Figure 5. Comparison of the $U$-velocity component in a decaying Taylor–Green vortex flow for $Re = 14.4$ at three different non-dimensional times $T$: $t = 6.55T$, $13.10T$, and $26.20T$. LBM, lattice Boltzmann method.

![Figure 6](image2.png)

Figure 6. Comparison of the $V$-velocity component in a decaying Taylor–Green vortex flow for $Re = 14.4$ at three different non-dimensional times $T$: $t = 6.55T$, $13.10T$, and $26.20T$. MRT LBM, multiple relaxation times lattice Boltzmann method.
components at times $t = 6.55T$, $13.10T$, and $26.20T$, where the characteristic time $T$ is defined in the previous section, reflecting the decaying of the initial vortex distribution. It is evident that the cascaded MRT LBM is in excellent agreement with the analytical solution at all times shown.

4. RESULTS AND DISCUSSION: NUMERICAL PERFORMANCE STUDY OF THE CASCADED LATTICE BOLTZMANN FOR TWO-DIMENSIONAL LID-DRIVEN CAVITY FLOWS

In this section, a numerical performance study of the 2D lid-driven cavity flow is considered, whose geometric simplicity is contrasted by various complex flow features. It is generally considered as a standard benchmark test for CFD methods and has been a subject of many investigations using a variety of methods (see, e.g., [38–42]). For example, Ghia et al. [38] have systematically studied this problem in much detail by employing a vorticity-stream function formulation of the 2D incompressible Navier–Stokes equations, which is solved by a multigrid method. Within the context of the LBM, this problem has been studied using different collision models (SRT and standard MRT) to assess their numerical performance by various researchers before (e.g., [15]). In this work, some of the numerical results presented by Ghia et al. [38] as well as those based on the spectral solutions by Botella and Peyret [43] will be used for making accuracy comparisons, focusing, in particular, on the more recent cascaded MRT LBM, which will be discussed in a later subsection. While the geometry is simple from the boundary condition implementation point of view, the flow contains singular points and becomes very complicated in terms of flow structures, particularly as the Reynolds number increases (see, e.g., [41] for a review). A schematic of the arrangement of the boundaries in a 2D lid-cavity flow is shown in Figure 7. Fluid is enclosed inside a square cavity of length $L$ and is set into motion by the moving upper wall that has a constant velocity $U_o$. The side and the bottom walls are considered to be stationary, which allows to implement a simple half-way bounce-back boundary condition on them. However, because the upper wall is in constant motion, a momentum correction needs to be added [44] to the regular bounce-back scheme for the upper boundary. This is implemented as $f_\alpha(i, N_y - 1) = f_\overline{\alpha}(i, N_y - 1) + 6\mu w_\alpha e_{xy} U_p$, where $f_\overline{\alpha}(i, N_y - 1)$ is the post-collision distribution function, for $\alpha = 4, 7, 8$, with $\overline{\alpha} = 2, 5, 6$ as the opposite directions of $\alpha$, and $w_\alpha$ is the weighting factor [44].

4.1. Two-dimensional lid-driven cavity flow: grid convergence

We first aim to analyze the grid convergence and an estimation of the order of accuracy of the cascaded MRT LBM for this flow problem. More detailed accuracy investigation of the various flow
features will be carried out in the next subsection. Because of the lack of analytical solutions, the computed solutions obtained by a relatively very fine grid resolution, that is, with $801^2$, are treated as the reference (‘analytical’) solutions. This is a standard practice in the CFD literature when exact analysis is not possible, especially for nonlinear problems, where the discretization error can be estimated from the difference between solutions obtained with the same method on systematically refined grids [45]. This approach can then be used to estimate the rate at which the error is reduced as the grid is refined [45]. As an example, Jothiprasad et al. [46] uses the results obtained with their higher order scheme for different step sizes to estimate its temporal order of accuracy; their estimates are found to be both below and above the formal order of accuracy for different quantities.

In the context of the LBM, Hou et al. [47] used the result of a very fine grid solution of the LBM as the exact solution to estimate its spatial order of accuracy for the lid-driven cavity flow problem. In addition, Lin and Lai [48] employ this approach as well as the discretization error order estimation method using successively finer grid results suggested in [45] for cavity flow simulations using the SRT LBM to determine the spatial order of accuracy. Hence, we adopt such an approach in this study. Nevertheless, we will also compare our results with a spectral solution for this problem [43] in a later section. In the present investigation, not only is the convergence of velocity fields tested but also the grid convergence of the components of the strain rate tensor is considered. It may be noted that the study involving the latter quantity has not so far received enough attention for this problem using the LBM.

The components of the velocity field and the strain rate tensor at the centerlines of the cavity in both vertical and horizontal directions are computed for a given Reynolds number once the solutions converge to a steady state. The solutions are considered to reach steady state convergence when the relative global errors are smaller than $10^{-15}$. Again, diffusive scaling is employed to set the parameters for different grid resolutions consisting of $13^2$, $19^2$, $25^2$, $31^2$, $37^2$, $49^2$, $61^2$, $85^2$, $97^2$, and $121^2$ nodes. Figure 8 shows the grid convergence of the U-component of the velocity field at a Reynolds number of 100. It is found that the best fit slopes are 2.11 and 2.19 along the vertical and the horizontal centerlines, respectively, for the $U$-velocity component. Likewise, the slopes are 2.18 and 2.11, respectively, along the vertical and the horizontal centerlines for the $V$-velocity component as shown in Figure 9. For the normal strain rate tensor component $\frac{\partial u}{\partial y}$, the slopes are found to be 1.81 and 1.95, respectively, for the vertical and the horizontal centerlines, which are shown in Figure 10. Furthermore, it is seen that along the vertical and horizontal centerlines, the strain rate tensor component $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ has the slopes of 2.12 and 2.07, respectively, for grid convergence (Figure 11). One reason why the slopes are either somewhat higher or lower than two, rather than the ideal value as seen with the other two problems discussed before, is that the reference solution

**Figure 8.** Grid convergence of the cascaded multiple relaxation times lattice Boltzmann method for the U-velocity component in a 2D lid-driven cavity flow for $Re = 100$.
Figure 9. Grid convergence of the cascaded multiple relaxation times lattice Boltzmann method for the V-velocity component in a 2D lid-driven cavity flow for $Re = 100$.

Figure 10. Grid convergence of the cascaded multiple relaxation times lattice Boltzmann method for the strain rate tensor component $\frac{\partial u}{\partial y}$ in a 2D lid-driven cavity flow for $Re = 100$.

for obtaining the relative error is taken to be that of the numerical solution with the very fine grid. This is often the practice as the ‘analytical’ solution does not exist for this problem, and such grid convergence estimation approaches are used, and the resulting observations are seen in the context of classical CFD schemes as well (e.g., [46]).

Overall, it is seen that the cascaded MRT LBM gives a very respectable second-order accuracy for a variety of flows, including the relatively simple Poiseuille flow and decaying Taylor–Green vortex flow, and for relatively complex flows such as the lid-driven cavity flow. The method is found to be second-order accurate not only for the velocity field but also for their derivatives for the aforementioned problems.

4.2. Two-dimensional lid-driven cavity flow: accuracy

Let us now consider the more detailed features of the lid-driven cavity flow problem at various Reynolds numbers in order to make quantitative accuracy comparisons of the flow fields. Figures 12 and 13 show the U-components and V-components of the velocity, respectively, along the center-
Figure 11. Grid convergence of the cascaded multiple relaxation times lattice Boltzmann method for the strain rate tensor component $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ in a 2D lid-driven cavity flow for $Re = 100$.

Figure 12. Comparison of the $U$-component of the velocity field along the vertical centerline of the cavity flow at various Reynolds numbers: $Re = 100, 400, 1000, 3200, 5000,$ and $7500$. Lines – cascaded multiple relaxation times lattice Boltzmann method and symbols – data by Ghia et al [38]. Grid resolution used is $401^2$.

lines of the square cavity at Reynolds numbers of 100, 400, 1000, 3200, 5000, and 7500 obtained using the cascaded MRT LBM along with the previous numerical data presented by Ghia et al [38]. The cascaded MRT LBM results corresponding to the finest grid considered earlier, that is, for the $401^2$ grid resolution are chosen to make comparison. In these figures, the solid lines represent the computed results obtained by the cascaded MRT LBM, and the symbols are the prior data provided by Ghia et al [38]. The velocities are normalized by the lid velocity $U_0$. Very good agreement is seen for all the Reynolds numbers considered.

In a previous work, it was established that the standard MRT LBM based on raw moments is superior when compared with the SRT LBM for the computation of lid-driven cavity flow [15]. Hence, it would be sufficient to make a direct comparison between the cascaded MRT LBM based on the central moments and the standard MRT LBM, which is based on raw moments, for various flow characteristics of this problem. First, in order to provide a global characteristics of the flow field, it would be interesting to compare the streamlines in the cavity at various Reynolds numbers.
Figure 13. Comparison of the $V$-component of the velocity field along the horizontal centerline of the cavity flow at various Reynolds numbers: $Re = 100, 400, 1000, 3200, 5000$, and $7500$. Lines – cascaded multiple relaxation times lattice Boltzmann method and symbols – data by Ghia et al [38]. Grid resolution used is $401^2$.

Figure 14. Comparison of the streamlines in a 2D lid-driven cavity flow at different Reynolds numbers computed with the cascaded (central moment) multiple relaxation times lattice Boltzmann method (MRT LBM) and standard (raw moment) MRT LBM: $Re = 100, 400, 1000, 3200, 5000$, and $7500$. Solutions obtained using a grid resolution of $201^2$ with both methods. (a) Cascaded MRT $Re = 100$; (b) standard MRT $Re = 100$; (c) cascaded MRT $Re = 400$; (d) standard MRT $Re = 400$; (e) cascaded MRT $Re = 1000$; (f) standard MRT $Re = 1000$; (g) cascaded MRT $Re = 3200$; (h) standard MRT $Re = 3200$; (i) cascaded MRT $Re = 5000$; (j) standard MRT $Re = 5000$; (k) cascaded MRT $Re = 7500$; (l) standard MRT $Re = 7500$.

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It is known that at a certain Reynolds number above 7500, the flow field becomes unsteady, and we restrict such comparisons for stationary state solutions only. Hence, Figure 14 shows the computed streamlines at Reynolds numbers of 100, 400, 1000, 5000, and 7500 using both the aforementioned methods. The streamlines computed by both these approaches are plotted side-by-side for comparison. It is found that the streamlines appear to be remarkably very similar for both the raw moment and central moment based approaches. At Reynolds numbers of 100, 400, and 1000, a major vortex appears around the geometric center of the cavity with two minor vortices around the lower corners. Because the lid is driven from left to right, the major vortex circulates in a clockwise direction, and the two minor vortices circulate in a counter-clockwise direction. In a recent study [15], it was shown that the standard MRT LBM was able to capture a third vortex (i.e., a second-secondary vortex) at the right bottom corner of the cavity for \( Re = 1000 \). While the cascaded MRT LBM results as shown in Figure 14 did not seem to reproduce this feature, upon a closer inspection, this was determined to be because of the choice of the grid resolution, which is \( 201^2 \). However, when we use a higher resolution of \( 401^2 \), the method is indeed able to reproduce this finer flow feature (Figure 15).

At Reynolds numbers of 3200 and 5000, in addition to the vortices that exist with the lower Reynolds number cases, there appears another minor vortex on the left upper corner, which circulates in a counter-clockwise direction. When the Reynolds number increases further to 7500, a fourth minor vortex is found on the right lower corner, which circulates in a clockwise direction. All the aforementioned flow features correspond to steady states. Furthermore, in order to provide a more detailed comparison, we present various secondary vortices that appear in the cavity at \( Re = 7500 \) in Figure 16. Again, remarkable similarity between the cascaded MRT LBM and the standard MRT LBM is found for these more detailed secondary flow structures.

In order to provide a more quantitative perspective, Figure 17 illustrates a comparison of the center of the primary vortex location in the cavity flow at different Reynolds numbers (i.e., \( Re = 100, 400, 1000, 3200, 5000, \) and 7500) between the cascaded and standard MRT LBM as well as the data by Ghia et al [38]. From the earlier streamline plots, it can be observed that the location of the primary vortex moves towards the geometric center of the cavity as the Reynolds number increases. The computed results using the cascaded MRT LBM and the standard MRT LBM are in excellent agreement (within 0.014%) with each other for all Reynolds numbers. In addition, they are both in very good agreement with the data by Ghia et al [38] to within 0.50% for all Reynolds numbers. These quantitative results for the primary vortex locations are enumerated in Table I. In addition, Table II presents a comparison between the aforementioned two methods and the prior numerical
Figure 16. Comparison of the streamlines of the secondary vortices in a 2D lid-driven cavity flow at \( Re = 7500 \) computed with the cascaded (central moment) multiple relaxation times lattice Boltzmann method (MRT LBM) and standard (raw moment) MRT LBM. (a) Cascaded MRT bottomright; (b) standard MRT bottomright; (c) cascaded MRT bottomleft; (d) standard MRT bottomleft; (e) cascaded MRT bottomright; (f) standard MRT bottomright.
data for the location of secondary vortices at different Reynolds numbers. Again, both the cascaded MRT LBM and the standard MRT LBM are in excellent quantitative agreement for the location of these detailed secondary vortical structures with the data by Ghia et al. [38].

Another useful global characteristic for comparison is the vorticity contours in the cavity at different Reynolds numbers. Figure 18 shows the vorticity contours computed using both the standard MRT LBM and the cascaded MRT LBM at three different Reynolds numbers (i.e., Re = 100, 400, and 1000). As Reynolds number increases, the vorticity contours become denser closer to the boundary walls. Overall, the vorticity distribution is found to be very similar using both the methods for all the Reynolds numbers considered thus corroborating the earlier results.

While the data comparisons in the previous discussions were made with the original results of Ghia et al. [38], which is of good quality and widely used in benchmarking numerical methods for incompressible flow simulations, it is of interest to also compare the results computed using the cascaded MRT LBM with those of more recent and high accuracy benchmark solutions obtained using spectral methods. For example, Figure 19 presents a comparison of the computed streamline contours obtained using the cascaded formulation with those of the spectral results by Botella and Peyret [43] for cavity flow at Re = 1000. Overall, good agreement between the two results are seen.

In addition, Table III provides a comparison of the numerical data involving the centerline velocity components computed using the cascaded MRT LBM with those based on the spectral solution [43] thereby quantifying accuracy and validating the former method. Moreover, Figure 20 shows the good agreement between the centerline velocity profiles obtained using the cascaded MRT LBM at two different grid resolutions of 201^2 and 401^2, which illustrates the grid convergence of the results.
Table II. Comparison of the location of various secondary vortices in a lid-driven cavity flow at different Reynolds numbers. Computed results are obtained using a grid resolution of $201^2$.

| $Re$   | Cascaded MRT LBM | Standard MRT LBM | Ghia et al (1982) [38] |
|--------|------------------|------------------|------------------------|
|        | First secondary vortex |                |                        |
|        | Top               | Bottom left      | Bottom right           |
| 100    | NA                | NA               | NA                     |
| 400    | NA                | NA               | NA                     |
| 1000   | NA                | NA               | NA                     |
| 3200   | (0.0547, 0.8976)  | (0.0546, 0.8973) | (0.0547, 0.8984)       |
| 5000   | (0.0644, 0.9081)  | (0.0641, 0.9076) | (0.0625, 0.9102)       |
| 7500   | (0.0676, 0.9102)  | (0.0677, 0.9099) | (0.0664, 0.9141)       |
| 100    | (0.0387, 0.0387)  | (0.0373, 0.0373) | (0.0313, 0.0391)       |
| 400    | (0.0533, 0.0493)  | (0.0530, 0.0494) | (0.0508, 0.0469)       |
| 1000   | (0.0842, 0.0791)  | (0.0842, 0.0791) | (0.0859, 0.0781)       |
| 3200   | (0.0821, 0.1207)  | (0.0821, 0.1207) | (0.0859, 0.1094)       |
| 5000   | (0.0740, 0.1378)  | (0.0740, 0.1378) | (0.0703, 0.1367)       |
| 7500   | (0.0654, 0.1536)  | (0.0654, 0.1536) | (0.0645, 0.1504)       |
| 100    | (0.9383, 0.0658)  | (0.9386, 0.0654) | (0.9453, 0.0625)       |
| 400    | (0.8833, 0.1243)  | (0.883, 0.1243)  | (0.8906, 0.1250)       |
| 1000   | (0.8631, 0.1128)  | (0.8631, 0.1128) | (0.8594, 0.1094)       |
| 3200   | (0.8229, 0.0853)  | (0.8229, 0.0852) | (0.8125, 0.0859)       |
| 5000   | (0.8037, 0.0739)  | (0.8037, 0.0739) | (0.8086, 0.0742)       |
| 7500   | (0.7892, 0.0663)  | (0.7893, 0.0663) | (0.7813, 0.0625)       |
|        | Second secondary vortex |                |                        |
|        | Bottom left       | Bottom right     |                        |
| 100    | NA                | NA               | NA                     |
| 400    | NA                | NA               | NA                     |
| 1000   | NA                | NA               | NA                     |
| 3200   | (0.0075, 0.0075)  | (0.0073, 0.0073) | (0.0078, 0.0078)       |
| 5000   | (0.0075, 0.0075)  | (0.0074, 0.0074) | (0.0117, 0.0078)       |
| 7500   | (0.0125, 0.0125)  | (0.0115, 0.0115) | (0.0117, 0.0117)       |
| 100    | NA                | NA               | NA                     |
| 400    | (0.9926, 0.0075)  | NA               | (0.9922, 0.0078)       |
| 1000   | (0.9923, 0.0076)  | (0.9928, 0.0073) | (0.9922, 0.0078)       |
| 3200   | (0.9875, 0.0113)  | (0.9885, 0.0115) | (0.9844, 0.0078)       |
| 5000   | (0.9775, 0.0200)  | (0.9771, 0.0193) | (0.9805, 0.0195)       |
| 7500   | (0.9508, 0.0429)  | (0.9509, 0.0429) | (0.9492, 0.0430)       |
|        | Third secondary vortex |                |                        |
|        | Bottom right      |                  |                        |
| 100    | NA                | NA               | NA                     |
| 400    | NA                | NA               | NA                     |
| 1000   | NA                | NA               | NA                     |
| 3200   | NA                | NA               | NA                     |
| 5000   | NA                | NA               | NA                     |
| 7500   | (0.9964, 0.0037)  | NA               | (0.9961, 0.0039)       |

MRT LBM, multiple relaxation times lattice Boltzmann method; NA, not applicable.

The discussion earlier involved computations up to a Reynolds number of 7500 as the flow patterns in such cases lead to stationary states allowing comparison of different methods to quantify accuracy using benchmark data from prior numerical solutions. However, the cascaded MRT LBM is capable of computing higher Reynolds number flows, where the flow field can become more complex and unsteady, among other features. For example, in order to demonstrate its capability in such cases, Figure 21 illustrates the instantaneous streamlines for $Re = 10,000$, which shows the presence of a multitude of vortical structures of different scales in the vicinity of the corners.
As discussed earlier, one of the useful features of kinetic schemes such as the cascaded MRT LBM is that the components of the strain rate tensor can be obtained locally from the components of the non-equilibrium moments of the distribution function (Equations (23)–(25)). The cavity flow being a shear driven problem generally has all the components of the strain rate tensor non-zero and whose magnitudes can dramatically change with the Reynolds number. Hence, this problem provides a good test for the evaluation of the accuracy of the computation of the strain rate tensor by kinetic theory considerations, that is, using non-equilibrium moments (Equations (23)–(25)). For the sake of comparison, we will make use of the standard second-order central differencing of the velocity field to obtain the usual direct estimation of the strain rate tensor components. In this regard, flow at two different Reynolds numbers are considered (i.e., $Re = 100$ and 1000), and the components of the
Figure 19. Comparison of the streamline contours computed for cavity flow at $Re = 1000$ using the cascaded multiple relaxation times lattice Boltzmann method with a grid resolution of $201^2$ with those of the benchmark data based on the spectral method results of Botella and Peryet [43]. The values given on the right-hand-side are taken from Botella and Peryet [43].

Table III. Comparison of the centerline velocity components in the cavity flow at $Re = 1000$ computed using the cascaded multiple relaxation times lattice Boltzmann method using a grid resolution of $201^2$ with the benchmark data based on the spectral method results by Botella and Peryet [43].

| $y/L$  | $u/U_0$     | Ref. [43] | $x/L$  | $v/U_0$     | Ref. [43] |
|--------|-------------|-----------|--------|-------------|-----------|
| 1.0000 | $-1.0000000$| $-1.0000000$ | 0.0000 | $0.0000000$ | $0.0000000$ |
| 0.9766 | $0.6795759$ | $0.6644227$ | 0.0312 | $0.2173188$ | $0.2279225$ |
| 0.9688 | $0.5927550$ | $0.5808359$ | 0.0391 | $0.2832045$ | $0.2936869$ |
| 0.9609 | $0.5255528$ | $0.5169277$ | 0.0469 | $0.3455434$ | $0.3553213$ |
| 0.9531 | $0.4781620$ | $0.4723329$ | 0.0547 | $0.4018385$ | $0.4103754$ |
| 0.8516 | $0.3381770$ | $0.3372212$ | 0.0937 | $0.5264241$ | $0.5264392$ |
| 0.7344 | $0.1893418$ | $0.1886747$ | 0.1406 | $0.4291873$ | $0.4264545$ |
| 0.6172 | $0.0572269$ | $0.0570178$ | 0.1953 | $0.3213972$ | $0.3202137$ |
| 0.5000 | $0.0621783$ | $0.0620561$ | 0.5000 | $0.0258072$ | $0.0257995$ |
| 0.4531 | $0.1084411$ | $0.1081999$ | 0.7656 | $0.3261843$ | $0.3253592$ |
| 0.2813 | $0.2812658$ | $0.2803696$ | 0.7734 | $0.3347824$ | $0.3339924$ |
| 0.1719 | $0.3883255$ | $0.3885691$ | 0.8437 | $0.3766197$ | $0.3769189$ |
| 0.1016 | $0.2973337$ | $0.3004561$ | 0.9062 | $0.3312864$ | $0.3304422$ |
| 0.0703 | $0.2193247$ | $0.2228955$ | 0.9219 | $0.3077586$ | $0.3099097$ |
| 0.0625 | $0.1986888$ | $0.2023300$ | 0.9297 | $0.2938585$ | $0.2962703$ |
| 0.0547 | $0.1775567$ | $0.1812881$ | 0.9375 | $0.2779576$ | $0.2807056$ |
| 0.0000 | $0.0000000$ | $0.0000000$ | 1.0000 | $0.0000000$ | $0.0000000$ |

strain rate tensor are obtained at five different locations within the cavity using the aforementioned two methods, which are enumerated in Table IV. As the Reynolds number is increased from 100 to 1000, the magnitudes of the strain rate tensor change significantly, which are quite well captured by the kinetic approach. Indeed, remarkably, the local computation using the non-equilibrium moments are in very good agreement with the finite-difference estimation at various locations in the cavity for both the Reynolds numbers, with the maximum difference within 2%. This further demonstrates the numerical fidelity of the approach. In particular, such non-equilibrium moments-based approach for the strain rate components can be used in the subgrid scale models for large eddy simulation of turbulent flows using the cascaded MRT LBM.
Figure 20. Comparison of the centerline velocity components for cavity flow at $Re = 1000$ for two different grid resolutions of $201^2$ and $401^2$ demonstrating grid convergence of the results obtained using the cascaded multiple relaxation times lattice Boltzmann method.

Figure 21. Instantaneous streamlines in an unsteady 2D lid-driven cavity flow at a higher Reynolds number of $Re = 10000$ computed using the cascaded multiple relaxation times lattice Boltzmann method with a $201^2$ grid resolution.

4.3. Two-dimensional lid-driven cavity flow: numerical stability

Characterization of the numerical stability of a numerical method is an important element from a practical point of view. Computation of the lid-driven cavity flow presents a stringent test case in this regard because it is a fully 2D problem with boundaries containing singularity, and the flow is shear driven. In fact, such a cavity flow problem was considered in detail to determine stability regimes of the SRT and the standard MRT collision models in a recent work [15]. Earlier, its three-dimensional counterpart was also considered from this viewpoint [49]. These studies have demonstrated the superiority of the use of the MRT in providing controlled additional numerical dissipation to enhance numerical stability on either coarser grids or at high Reynolds numbers when compared with the SRT model. Hence, it is appropriate to consider the 2D lid-driven cavity flow to establish the stability regime of the cascaded MRT LBM in the context of other collision models. We now make a direct comparison of the maximum threshold Reynolds number to maintain numerical stability of the SRT LBM, the standard MRT LBM, and the cascaded MRT LBM for this problem. With the cascaded MRT LBM, the relaxation parameters $\omega_4 = \omega_5 = 1/\tau$ are selected based on the specified kinematic
Table IV. Comparison of the components of the strain rate tensor computed using the local non-equilibrium moments (Equations (23)–(25)) and the finite-differencing (second-order central) of the velocity field with the cascaded multiple relaxation times lattice Boltzmann method at five different locations within the cavity for two different Reynolds numbers (i.e., $Re = 100$ and 1000).

| Location | Non-eqm. moments | Finite difference | Difference (%) |
|----------|------------------|------------------|----------------|
| $\frac{\partial u}{\partial y}$ | $\frac{\partial v}{\partial x}$ | $\frac{\partial u}{\partial y}$ | $\frac{\partial v}{\partial x}$ | $\frac{\partial u}{\partial y}$ | $\frac{\partial v}{\partial x}$ | $\frac{\partial u}{\partial y}$ | $\frac{\partial v}{\partial x}$ | $rac{\text{Difference}}{\%}$ |
| $\frac{1}{4} \cdot \frac{L}{4}$ | $\frac{1}{4} \cdot \frac{L}{4}$ | $2.416 \times 10^{-4}$ | $2.415 \times 10^{-4}$ | 0.044 |
| $\frac{1}{2} \cdot \frac{L}{4}$ | $\frac{1}{2} \cdot \frac{L}{4}$ | $1.711 \times 10^{-5}$ | $1.721 \times 10^{-5}$ | 0.613 |
| $\frac{1}{2} \cdot \frac{L}{2}$ | $\frac{1}{2} \cdot \frac{L}{2}$ | $1.850 \times 10^{-4}$ | $1.848 \times 10^{-4}$ | 0.102 |
| $\frac{3}{4} \cdot \frac{L}{4}$ | $\frac{3}{4} \cdot \frac{L}{4}$ | $-9.020 \times 10^{-5}$ | $-9.025 \times 10^{-5}$ | 0.057 |
| $\frac{3}{4} \cdot \frac{L}{2}$ | $\frac{3}{4} \cdot \frac{L}{2}$ | $-3.541 \times 10^{-4}$ | $-3.536 \times 10^{-4}$ | 0.125 |
| $\frac{1}{4} \cdot \frac{L}{4}$ | $\frac{1}{4} \cdot \frac{L}{4}$ | $3.516 \times 10^{-5}$ | $3.526 \times 10^{-5}$ | 0.300 |
| $\frac{1}{2} \cdot \frac{L}{4}$ | $\frac{1}{2} \cdot \frac{L}{4}$ | $-4.344 \times 10^{-4}$ | $-4.342 \times 10^{-4}$ | 0.045 |
| $\frac{1}{2} \cdot \frac{L}{2}$ | $\frac{1}{2} \cdot \frac{L}{2}$ | $-3.368 \times 10^{-4}$ | $-3.363 \times 10^{-4}$ | 0.135 |
| $\frac{3}{4} \cdot \frac{L}{4}$ | $\frac{3}{4} \cdot \frac{L}{4}$ | $4.599 \times 10^{-4}$ | $4.603 \times 10^{-4}$ | 0.093 |
| $\frac{3}{4} \cdot \frac{L}{2}$ | $\frac{3}{4} \cdot \frac{L}{2}$ | $-5.290 \times 10^{-4}$ | $-5.280 \times 10^{-4}$ | 0.198 |
| $\frac{1}{4} \cdot \frac{L}{4}$ | $\frac{1}{4} \cdot \frac{L}{4}$ | $4.220 \times 10^{-5}$ | $4.217 \times 10^{-5}$ | 0.050 |
| $\frac{1}{2} \cdot \frac{L}{4}$ | $\frac{1}{2} \cdot \frac{L}{4}$ | $2.196 \times 10^{-5}$ | $2.209 \times 10^{-5}$ | 0.596 |
| $\frac{1}{2} \cdot \frac{L}{2}$ | $\frac{1}{2} \cdot \frac{L}{2}$ | $5.017 \times 10^{-5}$ | $5.017 \times 10^{-5}$ | 0.008 |
| $\frac{3}{4} \cdot \frac{L}{4}$ | $\frac{3}{4} \cdot \frac{L}{4}$ | $2.370 \times 10^{-5}$ | $2.372 \times 10^{-5}$ | 0.073 |
| $\frac{3}{4} \cdot \frac{L}{2}$ | $\frac{3}{4} \cdot \frac{L}{2}$ | $6.397 \times 10^{-5}$ | $6.446 \times 10^{-5}$ | 0.750 |
| $\frac{1}{4} \cdot \frac{L}{4}$ | $\frac{1}{4} \cdot \frac{L}{4}$ | $-1.344 \times 10^{-4}$ | $-1.334 \times 10^{-4}$ | 0.782 |
| $\frac{1}{2} \cdot \frac{L}{4}$ | $\frac{1}{2} \cdot \frac{L}{4}$ | $8.137 \times 10^{-5}$ | $7.984 \times 10^{-5}$ | 1.914 |
| $\frac{1}{2} \cdot \frac{L}{2}$ | $\frac{1}{2} \cdot \frac{L}{2}$ | $-3.980 \times 10^{-5}$ | $-3.979 \times 10^{-5}$ | 0.021 |
| $\frac{3}{4} \cdot \frac{L}{4}$ | $\frac{3}{4} \cdot \frac{L}{4}$ | $2.291 \times 10^{-4}$ | $2.289 \times 10^{-4}$ | 0.049 |
| $\frac{3}{4} \cdot \frac{L}{2}$ | $\frac{3}{4} \cdot \frac{L}{2}$ | $-1.688 \times 10^{-4}$ | $-1.699 \times 10^{-4}$ | 0.626 |

viscosity, while the rest of relaxation parameters are set to unity for simplicity. For each approach, for a given grid resolution, the lid velocity was fixed, and the relaxation time $\tau$ was decreased gradually until the computation became unstable.

Figure 22 shows the maximum Reynolds number $Re = \frac{U_0 L}{v}$ that could be attained for each method before the computations became unstable, that is, when the relative global error increases rapidly or becomes exponentially large as the simulation progresses. Results are provided for different grid resolutions for these three approaches. It is clear that the cascaded MRT computations can reach Reynolds numbers that are about two or three times higher than that of the standard MRT approach, and the standard MRT computations can reach Reynolds numbers that are three or four
times higher than that of the SRT approach. The latter results are consistent with prior findings \[15, 49\]. Relaxation of different central moments at different rates provides a controlled additional numerical dissipation to maintain numerical stability. That is, maintaining frame invariance in conjunction with the use of multiple relaxation times further promotes the stability of the method. It may be noted that stabilization of certain classical methods have been achieved by constructing discretization operators that enforce Galilean invariance \[21–23\]. Hence, it may be expected that explicitly incorporating an invariance property could aid with other standard mechanisms of stabilization of the LBM.

As carried out in \[15\], we also perform an alternate stability test with the three approaches on a chosen coarse grid for this problem. In this test, the grid resolution is fixed at a relatively coarse resolution of \(26 \times 26\), and then the viscosity \(\nu\) (or equivalently \(\tau\)) is also set for all the three approaches.

We then intend to find the maximum lid velocity that can maintain the stability of computations for 50,000 time steps \[15\]. Figure 23 shows how the three methods behave for this test. For the cascaded MRT LBM, different choices of the relaxation parameter for the bulk viscosity \(\omega_3\) are considered while setting the rest of the relaxation parameters for the higher order moments to unity, that is, to their equilibrium state. It is seen that, in general, the parameter regime or the maximum lid velocity for stability is considerably larger with the cascaded MRT LBM when compared with the other approaches. When the relaxation parameter \(\omega_3\) is equal to 0.5, it results in a larger value of the bulk viscosity than that for the equilibrium case \((\omega_3 = 1)\). As a result, it damps the density perturbations or acoustic waves better leading to an improved numerical stability. This effect is seen to be more prominent in the lower ranges of the values for the shear relaxation parameter \((\omega_4 = \omega_5 = 1/\tau)\). Conversely, when \(\omega_3\) is set to 1.5, it leads a reduction in the bulk viscosity when compared with \(\omega_3 = 1\) and thereby a corresponding reduction in the numerical stability.

We then explore the possibility of tuning the relaxation parameters for the higher order moments \((\omega_6, \omega_7, \text{and} \omega_8)\) and its influence on numerical stability. Figure 24 presents the results of the stability test similar to the discussion earlier, but now by tuning the values of the relaxation parameters for the third-order \((\omega_6 \text{ and} \omega_7)\) and fourth-order \((\omega_8)\) moments, which yield the following interesting observations. As seen from Figure 24(a), for the equilibrium state of the fourth-order moment \((\omega_8 = 1)\), when \(\omega_6\) is varied for a fixed value of \(\omega_7 = 1\), improved numerical stability results when \(\omega_6 > 1\) than that for \(\omega_6 = 1\) for the lower ranges of the shear relaxation parameter \(1/\tau\); opposite trend is observed for the higher ranges of the shear relaxation parameter \(1/\tau\). On the other hand, in general, keeping the other third-order moment relaxation parameter \(\omega_7 < 1\) when both \(\omega_6 = \omega_8 = 1\) leads to a greater stability range than that for \(\omega_7 = 1\), especially for higher ranges of \(1/\tau\). In summary, if the interest is to simulate relatively higher \(Re\) flow, which can be carried out.
Figure 23. Alternative stability test to determine the maximum threshold lid velocity for different methods for a chosen coarse resolution of $26 \times 26$. For the cascaded multiple relaxation times (MRT) lattice Boltzmann method, the relaxation parameter for the bulk viscosity is varied while those for the higher order moments are set to unity. SRT, single relaxation time.

Figure 24. Alternative stability test using the cascaded multiple relaxation times lattice Boltzmann method to determine the effect of the (a) variation of the relaxation parameters for the third-order moments ($\omega_6$ and $\omega_7$) for a chosen coarse resolution of $26 \times 26$ and (b) variation of the relaxation parameter corresponding to the fourth-order moment ($\omega_8$).

by keeping $1/\tau$ close to 2, the parameter space for numerical stability seems enlarged when $\omega_6 > 1$ and $\omega_7 < 1$ when the fourth-order moment is in equilibrium state. It should be kept in mind that this observation is specific to the present case study involving the lid-driven cavity flow. Figure 24(b) illustrates the effect of varying the relaxation parameter for the fourth-order moment when the third-order moments are in equilibrium state. It is seen that setting $\omega_8 = 1$ is better for the intermediate ranges of $1/\tau$ and $\omega_8 < 1$ (i.e., under-relaxation) is more appropriate when $1/\tau$ is in the higher ranges, being close to 2. Again, it is reminded that these results are problem-dependent.

These results establish the merits of the use of MRT for central moment relaxation. Often, the stability of the CFD methods is characterized in terms of the grid or cell Reynolds number given by $Re_c = U_0 \Delta x / v$ (e.g., [50]). Thus, we also present maximum cell Reynolds number for the
numerical stability of the three approaches for this problem in Table V, where the cascaded MRT LBM uses the relaxation parameters for the higher order moments as 1 as the baseline case for the purpose of comparison. This summarized set of results further demonstrates the advantages of the cascaded MRT LBM.

In order to provide a flavor of the numerical results obtained near the stability limit, which depends on the nature of the flow and the choice of the relaxation parameters, we consider the simulation results for the coarsest grid considered in Table V, that is, $71 \times 71$, and use the cascaded MRT LBM to simulate the cavity flow at a relatively high $Re = 7500$ for which benchmark numerical results are available [38]. Figure 25 presents the centerline velocity profiles computed using the cascaded formulation as well the benchmark results [38]. Two different values of the relaxation parameters for the bulk viscosity are also considered with the other higher order relaxation parameters set to unity. It can be seen that the computed results are in fair agreement with the data, considering the coarseness of the grid employed for the relatively high Reynolds number. While this demonstrates the numerical effectiveness of the cascaded MRT LBM for this case study, in general, it can be expected to be problem-dependent.

Another important aspect is the computational cost. As shown previously, the cascaded MRT approach can be more stable with similar accuracy compared with the standard MRT for the lid-driven cavity flow. But if it is much more expensive for numerical computations than the standard MRT, its advantages will not be very useful. In this regard, we fully exploit all the optimization strategies that could be used with a moment approach, such as those specified in [51] for the cas-

Table V. Comparison of the maximum cell Reynolds number ($Re_c = U_0 \Delta x / v$) for numerical stability of different methods for simulation of the lid-driven cavity flow problem.

| Grid Resolution | SRT LBM | Standard MRT LBM | Cascaded MRT LBM |
|-----------------|---------|------------------|------------------|
| $71 \times 71$  | 14.42   | 54.42            | 128.57           |
| $101 \times 101$| 14.14   | 59.40            | 148.50           |
| $201 \times 201$| 14.21   | 62.18            | 165.83           |
| $401 \times 401$| 14.25   | 62.34            | 199.50           |

MRT LBM, multiple relaxation times lattice Boltzmann method; SRT LBM, single relaxation time lattice Boltzmann method.

Figure 25. Comparison of computed centerline velocity profiles using the cascaded multiple relaxation times lattice Boltzmann method for $Re = 7500$ with Ghia et al. [38] for a grid resolution of $71 \times 71$ with two values of the relaxation parameters for the bulk viscosity $\omega_3$. 

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caded MRT LBM. It is found that for the 2D lid-driven cavity flow problem, the cascaded MRT LBM takes about 11.6% longer than the standard MRT LBM, which is acceptable in view of the significant advantages in terms of numerical stability. It should be pointed out that these results pertain only to 2D problems. Additional work is required in three-dimensions to optimize the computational cost of the cascaded MRT LBM and also to optimize its relaxation parameters by means of a linear Fourier analysis.

5. SUMMARY AND CONCLUSIONS

Galilean invariance is one of the main physical attributes in the description of the fluid motion. This is naturally achieved by considering dynamical changes in terms of central moments in kinetic schemes, as was performed in the recently introduced cascaded LBM. Enforcing frame invariance is generally expected to have a positive influence on the numerical stability as seen in some recent work with other classical schemes. The use of the MRT in the central moment or cascaded LBM brings in the various flexibility associated with the standard MRT LBM based on raw moments. In particular, the relaxation of the different central moments at different rates introduces the additional dissipation as in the raw moment based approach, which can lead to enhanced stability.

In this paper, we discussed our results from systematic numerical studies on grid convergence, accuracy, and stability of the cascaded MRT LBM. We have chosen three commonly used 2D benchmark problems including the Poiseuille flow, the decaying Taylor–Green vortex flow, and the lid-driven cavity flow. In the grid convergence tests, the cascaded MRT approach has been found to be second-order accurate under diffusive scaling for all the benchmark problems considered. These results are shown to hold not only for the velocity field but also for the components of the strain rate tensors. Furthermore, comparisons of the numerical accuracy of the cascaded MRT LBM were made with other collision models and also with prior analytical or numerical results based on the solution of the Navier–Stokes equations. These demonstrated that the cascaded MRT LBM is in excellent agreement with the prior results, which includes high accuracy finite-difference or spectral numerical results, for the canonical problems considered. In particular, the detailed flow structures for the more complex lid-driven flow predicted by the cascaded MRT LBM are in very good quantitative agreement with the standard MRT LBM. In addition, the utility and the accuracy of the use of non-equilibrium moments with the cascaded MRT LBM for the computation of the components of the strain rate tensor is demonstrated. Finally, stability tests on a 2D lid-driven cavity flow problem was carried out, which showed substantial improvements in the numerical stability of the cascaded MRT LBM, with higher threshold Reynolds numbers, when compared with other models. With the use of proper optimization strategies, the 2D cascaded MRT LBM was found to be only about 10% to 20% more expensive when compared with the standard MRT LBM in terms of the computational time.

Future work could include further development of more optimized formulations of the three-dimensional cascaded LBM based on central moments with a view to maintain computational efficiency and their applications to unsteady multiscale problems such as turbulence. Optimization of the relaxation parameters by a linear Fourier analysis to introduce adequate additional dissipation for enhanced numerical stability while maintaining necessary physics with this approach is also desired.

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