The singular field used to calculate the self-force on non-spinning and spinning particles

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The singular field of a point charge has recently been described in terms of a new Green’s function of curved spacetime. This singular field plays an important role in the calculation of the self-force acting upon the particle. We provide a method for calculating the singular field and a catalog of expansions of the singular field associated with the geodesic motion of monopole and dipole sources for scalar, electromagnetic and gravitational fields. These results can be used, for example, to calculate the effects of the self-force acting on a particle as it moves through spacetime.

I. INTRODUCTION

Detailed knowledge of the evolution of a binary system consisting of a stationary supermassive black hole (of mass of the order of $10^6 M_\odot$) and a stellar-mass neutron star or black hole would be an important aid for the analysis of data from space-based gravitational wave detectors such as LISA [4]. In general, the stellar-mass neutron star or black hole, usually modelled as a point-like particle, inspirals toward the central black hole due to radiation reaction. Radiation reaction is only one consequence of the self-force on a particle, namely the force which results from the interaction of the particle with its own gravitational field.

Generally, the spin of the stellar-mass compact object is expected to have an effect on the gravitational waveforms by such a system. That effect is expected to be small and does not need to be taken into account in template waveforms aimed at detection of such binaries. However it will have to be taken into account for accurate estimation of the physical parameters of the binary system. For that reason, self-force calculations for combined monopole and dipole sources are required. This paper is a step toward self-force calculations for spinning sources.

A. Self-Force Calculation

One method for calculating the self-force on a particle generating an electromagnetic field was suggested by DeWitt and Brehme [2]. That method involves decomposing the retarded field at point $p$, generated by the charged particle at point $p'$, into its direct part (which comes from the part of the Green’s function with support only on the null cone of point $p$) and its tail part (which comes from the part of the Green’s function with support only within the null cone of point $p$), and then using the tail part of the electromagnetic field in the Lorentz force law. Similar analysis by Mino, Sasaki and Tanaka [8] described the gravitational self-force on a massive particle in terms of the tail part of the gravitational field. Axiomatic approaches were used by Quinn for the scalar field [4] and by Quinn and Wald for the electromagnetic and gravitational fields [5] and arrived at similar conclusions.

The actual calculation of the self-force on a particle in geodesic motion around a Schwarzschild black hole, in terms of the tail part of the field, has been successfully performed by a variety of groups [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. In these applications, the retarded field is first calculated via separation of variables coupled with a sum over modes. Analytic methods are used to calculate the part of the self-force from each mode of the direct part of the field. Then, for each mode, the part of the force from the direct part is subtracted from the part of the force from the retarded part. Finally, this difference is summed over all modes and this sum provides the self-force which results from the tail part of the field. It is clear, then, that the tail part provides a useful calculational avenue to the self-force. However, a few shortcomings remain on a fundamental level [16]. The Green’s function of the tail part has support inside the past null cone of the field point, which implies that this description of the self-force formally depends upon the entire past history of the particle. Also, in some circumstances the tail part is finite but not differentiable at the location of the particle and averaging of the derivative over different directions of approach to the particle is required to determine the actual self-force. Perhaps most critically, the tail part of the field is not a solution of the field equations. Thus, the tail part of the field of a particle can be described mathematically, but it cannot by itself be considered a field.

1. Self-force calculation using the singular field

A different method for calculating the self-force was proposed by Detweiler and Whiting [16] for the scalar, electromagnetic and gravitational fields. That method uses a new Green’s function for curved spacetime to de-
termine the singular field of the particle. The singular field was first defined in [16] where it was shown by construction to be the part of the retarded field of the particle that exerts no force on the particle itself. The properties of the singular field were detailed in section 13.5 of [17]. For completeness, we give here the definition and properties of the singular field. For a more detailed discussion, including the motivation for the definition of the singular field, we point the reader to [16] and [17].

Assume a particle of scalar charge \( q \) at point \( p' \) in a background spacetime. We want to calculate the retarded field of that charge at point \( p \) in spacetime. The retarded Green’s function \( G^{\text{ret}} \) obeys the inhomogeneous equation

\[
\nabla^2 G^{\text{ret}}(p, p') = -4\pi \delta(p, p')
\]

where \( \nabla \) denotes differentiation with respect to the background metric. The retarded Green’s function generates a field which is the physically relevant solution, because it obeys appropriate outgoing-wave boundary conditions at infinity.

The singular Green’s function \( G^S \) is defined \([16, 17]\) as the function that satisfies the same inhomogeneous equation as the retarded Green’s function

\[
\nabla^2 G^S(p, p') = -4\pi \delta(p, p'),
\]

with the additional requirements that it is symmetric in its arguments \( p \) and \( p' \) and that it vanishes if \( p \) is within the past or future null cones of \( p' \). By definition, the singular Green’s function has the same singular behavior at the location of the particle as the retarded Green’s function: the divergence of the field near the particle is insensitive to the boundary conditions at infinity. It has been shown that such a function always exists in the local convex neighborhood of \( p' \) (see chapter 13 of [17]) and that the singular field that results from this does not exert any force on the particle itself. As explained in [17], because the singular Green’s function is symmetric in its arguments, it does not distinguish between past and future and it produces a field that contains equal amounts of outgoing and incoming radiation at infinity.

Removing \( G^S \) from the retarded Green’s function \( G^{\text{ret}} \) has the effect of removing the singular behavior of the field without that affecting the motion of the particle.

The radiative Green’s function, also referred to as the regular remainder, is defined by

\[
G^R(p, p') = G^{\text{ret}}(p, p') - G^S(p, p').
\]

This definition implies that the radiative Green’s function obeys the homogeneous equation

\[
\nabla^2 G^R(p, p') = 0,
\]

that it is equal to the retarded Green’s function if \( p \) is within the future null cone of \( p' \) and that it vanishes if \( p \) is in the past null cone of \( p' \). The radiative Green’s function produces a field that is smooth and differentiable on the world line of the particle and which properly encodes the outgoing-wave boundary conditions at infinity.

As has already been mentioned, after the singular field is subtracted from the retarded field, the radiative field is entirely responsible for the self-force. Specifically for the scalar case

\[
\psi^R(p) = \psi^{\text{ret}}(p) - \psi^S(p)
\]

and the self-force is calculated by using the radiative field in the self-force equation

\[
F^a = \lim_{p \to p'} \nabla^a \psi^R(p) = \lim_{p \to p'} \nabla^a [\psi^{\text{ret}}(p) - \psi^S(p)].
\]

As was mentioned in [16], for the self-force calculation for the electromagnetic case knowledge of the singular electromagnetic potential is required, while for the self-force calculation for the gravitational case knowledge of the singular gravitational field is required. The corresponding equations are given in [16].

We argue here that the method for calculating the self-force that uses the singular and radiative fields is preferable to the one that uses the direct and tail fields. For the scalar, electromagnetic and gravitational cases, the singular field obeys the inhomogeneous Poisson, Maxwell and Einstein equations respectively and has support only outside the null cone of the particle. In that sense, knowledge of the entire past history of the particle is not required, as is the case for the tail field, and the singular field needs to be calculated locally in the neighborhood of the point particle. Additionally, the radiative field obeys the corresponding homogeneous equation and thus is finite and differentiable everywhere along the world line of the particle. Consequently, this method for calculating the self-force does not present the difficulties of interpretation inherent in the method involving the direct and tail fields.

A recent self-force calculation using the singular and radiative fields \([18]\) showed that this method gives identical results to the ones derived using the direct and tail fields. The results were also used \([19]\) to predict the self-force effects on various orbits of scalar particles in a Schwarzschild background. Finally, a different approach to the self-force calculation described in \([20]\) uses the singular field in order to extract the contribution of that field to the Weyl scalar \( \Psi_0^S \) (or \( \Psi_0^S \)), subtract it from \( \Psi_0^{\text{ret}} \) (\( \Psi_0^{\text{ret}} \)) resulting from the retarded field of the particle and use the renormalized \( \Psi_0^{\text{ren}} \) (\( \Psi_0^{\text{ren}} \)) to calculate the perturbations on the background spacetime.

### B. Outline

In this paper we establish a method for calculating the singular field for monopole and dipole sources of scalar, electromagnetic and gravitational fields and we present a catalog of the expansions of the singular field for those sources. The coordinates that are used to calculate the
expansion of the singular field are the Thorne-Hartle-Zhang coordinates, (abbreviated as THZ coordinates). Those coordinates were initially introduced by Thorne and Hartle 21 and later extended by Zhang 22. A short discussion of them is presented in Sec. II.

Expansions are given for the singular fields associated with all monopole and dipole sources for scalar and electromagnetic fields and the monopole gravitational field in Sec. IIIA, III B and III C respectively. A short discussion of the results is given in Sec. IV and future work is outlined in Sec. V.

II. THZ COORDINATES

The Poisson, Maxwell and Einstein equations for the singular field assume a relatively simple form written in a coordinate system in which the background spacetime looks as flat as possible. In the following, it is assumed that the particle is moving on a geodesic $\Gamma$ in a vacuum background described by the metric $g_{ab}$. Also, $R$ is a representative length scale of the background geometry, the smallest of the radius of curvature, the scale of inhomogeneities of the background and the time scale of curvature changes along the geodesic $\Gamma$.

For a geodesic $\Gamma$, locally inertial coordinates $x^a = (t, x, y, z)$ can be found so that, on the geodesic, the metric and its first derivatives coincide with the Minkowski metric. They are of inhomogeneities of the background and the time scale of curvature changes along the geodesic $\Gamma$. In this and the following, $a, b, c$ and $d$ denote spacetime indices. The indices $i, j, k, l, n, m, p$ and $q$ are spatial indices and, to the order up to which the calculations are performed, they are raised and lowered by the 3-dimensional flat space metric $\delta_{ij}$. The dot denotes differentiation with respect to the time $t$ along the geodesic. Also, $\epsilon_{ijk}$ is the 3-dimensional flat space antisymmetric Levi-Civita tensor.

The tensors $F$ and $B$ are spatial, symmetric and trace-free and their components are related to the Riemann tensor on the geodesic $\Gamma$ by

\[ F_{ij} = R_{itlj} \]  \hspace{1cm} \text{(11)}

\[ B_{ij} = 1/2 \varepsilon_{lpq} R_{pqlt} \]  \hspace{1cm} \text{(12)}

\[ F_{ijk} = \left[ \nabla_k R_{itlj} \right]_{\text{STF}} \]  \hspace{1cm} \text{(13)}

\[ B_{ijk} = 3/8 \left[ (\varepsilon_{lpq} \nabla_k R_{pqlt}) \right]_{\text{STF}} \]  \hspace{1cm} \text{(14)}

where STF means to take the symmetric, trace-free part with respect to the spatial indices $i, j$ and $k$. The components $F_{ij}$ and $B_{ij}$ are of $O(1/R^2)$ and their time derivatives are of $O(1/R^3)$. The components $F_{ijk}$ and $B_{ijk}$ are also of $O(1/R^3)$.

If $H_{ab}$ consists of the terms given in Eq. (9), the coordinates are second-order THZ coordinates and are well defined up to the addition of arbitrary functions of $O(\rho^3/R^3)$ 23. Notice that in that case, only the Riemann tensor components and their time derivatives appear in the metric. If $H_{ab}$ also includes the terms given in Eq. (10), the coordinates are third-order THZ coordinates and are well defined up to the addition of arbitrary functions of $O(\rho^3/R^3)$ 23. In this case, only spatial derivatives of the Riemann tensor components appear in the metric.

The contributions to the metric components of $2H_{ab}$ that contain $F_{ij}$ and $B_{ij}$ are of $O(\rho^3/R^3)$, as are the contributions to the metric components of $3H_{ab}$ that contain $F_{ijk}$ and $B_{ijk}$. Therefore, any calculation that includes the $F_{ij}$ and $B_{ij}$ terms of $2H_{ab}$ must also include the $F_{ijk}$ and $B_{ijk}$ terms of $3H_{ab}$ as well.

Using the simple symmetry properties of the tensors $F$ and $B$, the following relationships can be shown for their components and the spatial THZ coordinates $(x, y, z)$:

\[ \epsilon_{ijk} F_{ik} + \epsilon_{ikl} F_{kj} - \epsilon_{jkl} F_{ik} = 0 \]  \hspace{1cm} \text{(15)}

\[ \epsilon_{ijk} B_{ki} + \epsilon_{ikl} B_{kj} - \epsilon_{jkl} B_{ki} = 0 \]

and

\[ \epsilon_{ijk} (F_{nk} x_{li} F_{nl} x^{lk} + \epsilon_{nk} F_{nl} x_{li} F_{nl} x^{lk} x^{nk} x^{lk} = 0 \]

\[ \epsilon_{ijk} (B_{nk} x_{li} B_{nl} x^{lk} + \epsilon_{nk} B_{nl} x_{li} B_{nl} x^{lk} x^{nk} x^{lk} = 0. \]  \hspace{1cm} \text{(16)}
These relationships are used in Sec. III where the singular fields for different sources are calculated, in order to simplify the expressions for those fields.

Before we proceed to the calculation of the singular fields, we need to make an important point on the notation and language that we use. In the following, the subscripts (or superscripts) (0), (2), (3) and (4) are used to indicate the order of significance of each term or component, for the singular fields, the covariant derivatives with respect to the THZ metric and the components of the Riemann tensor that show up in the Poisson, Maxwell and Einstein equations below. The subscript (0) refers to the most dominant contribution (resulting from the part of the THZ metric that is of order 1). The subscripts (2) and (3) refer to the next two more significant corrections (resulting from the parts of the THZ metric that are of order \( \rho^2/R^2 \) and \( \rho^3/R^3 \) respectively), which are calculated for the singular fields. The subscript (4) refers to the next correction (resulting from the \( \rho^4/R^4 \) part of the metric). The subscript (1) is not used due to the fact that there are no \( \rho/R \) terms in the metric. The terms “first correction”, “second correction” and “third correction” refer to the correction coming from the \( O(\rho^2/R^2) \), \( O(\rho^3/R^3) \) and \( O(\rho^4/R^4) \) parts of the metric respectively. The fact that they are labeled with indices (2), (3) and (4) (instead of the intuitive choice (1), (2) and (3), respectively) should not cause any confusion, since the reason for it is justified.

### III. CALCULATION OF THE SINGULAR FIELDS

The singular fields for a variety of sources are calculated below. The calculations were done using the program GrTENSOR running under MAPLE.

It should be noted that in all cases it is assumed that the characteristic of the moving source that is responsible for its own field (i.e. charge, dipole moment, mass) is small compared to the background curvature. That assumption allows us to safely consider the effects of the self-force on the particle. A similar argument can be used to rule out the presence of at least regular quadratic and cubic terms as well.

The derivation of each singular field follows a similar pattern. To illustrate that pattern we use the simple example of the scalar monopole field. In THZ coordinates the monopole scalar field is assumed to have \( q/\rho \) as the leading term in the expansion in powers of the distance away from the point source. This singular behavior correctly accounts for the \( \delta \)-function source at the origin. However, the \( \nabla^2 \) operator in curved spacetime, in THZ coordinates, differs from its flat space counterpart by terms involving the \( E_{ij} \) and \( B_{ij} \) multipole moments; these have dimensions of \( 1/(\text{length})^2 \) and are of \( O(1/R^2) \). It is obvious that the first correction to the singular monopole scalar field must behave as \( q(E_{ij} \text{ or } B_{ij})x^ix^j/\rho \) in order to have the proper index structure and dimension. The correction must have no free indices, and the two indices on either of the the tracefree \( E_{ij} \) or \( B_{ij} \) require precisely the \( x^i \) and \( x^j \). The power of \( \rho \) in the denominator is determined by the need to have the correct overall dimension. With a similar argument it follows that the succeeding correction to the singular monopole field should generally be of the form \( q(E_{ij} \text{ or } B_{ij})x^ix^jx^k/\rho \).

This picture of the expansion is consistent with the general description of the Hadamard expansion of the Green’s function in curved spacetime for the singular field. The issue that remains is to verify that this expansion does not surreptitiously include some of what ought to be considered the “regular remainder” \( R \)-part of the retarded field, which is responsible for the self-force. Assume, for the moment, that the expansion of the correct singular monopole field does include a regular piece; such a regular piece must be a homogeneous solution of the field equations in the vicinity of the particle because the \( \delta \)-function source is already accounted for in the expansion which was just described. Thus, the general form of such a correction must be expandable in THZ coordinates about the particle, and must behave as

\[
\text{constant} + \text{linear term} + \text{quadratic} + \text{cubic} + \ldots
\]  

But there is no “something linear” in \( x^i \) which has the proper dimensions and index structure. Thus, while the singular monopole field constructed above might, in principle, include an extra “homogeneous part,” the extra part could not be linear in \( x^i \) and could not affect the actual self-force on the particle. A similar argument can be used to rule out the presence of at least regular quadratic and cubic terms as well.

It will be seen in the following that the order up to which we have to calculate the singular fields (in order to be able to calculate the self-force) depends on the particle in question. In some cases, it is sufficient to calculate the first correction, namely that which results from the \( O(\rho^2/R^2) \) parts of the THZ metric. In the other cases, we must also know the second correction that comes from the \( O(\rho^3/R^3) \) parts of the metric. For completeness, we calculate all these corrections for all the sources that we examine and we explicitly state in which cases that is not strictly necessary.

In the cases in which calculating the second correction is not strictly necessary, we can still significantly benefit from such a calculation. Firstly, the more accurate approximation of the singular field, the more differentiable the radiative field. For example, as explained in
A. Scalar Field

Assume that a particle moving on a background geodesic creates a scalar source \( \varrho \). The scalar field generated by the particle can be expanded as

\[
\Psi^S = \Psi^S_{(0)} + \Psi^S_{(2)} + \Psi^S_{(3)} + \ldots
\]

and it obeys Poisson’s equation

\[
\nabla^2 \Psi^S = \nabla^2_{(0+2+3+\ldots)}(\Psi^S_{(0)} + \Psi^S_{(2)} + \Psi^S_{(3)} + \ldots) = -4\pi \varrho.
\]  

Regardless of the form of \( \varrho \), the leading term in the expansion is the scalar field generated by the particle when it is stationary at the origin of a Cartesian coordinate system and it obeys the lowest order differential equation derived from Eq. (20)

\[
\nabla^2_{(0)} \Psi^S_{(0)} = -4\pi \varrho.
\]

The first correction obeys the differential equation derived from Eq. (20) by keeping only the lowest-order terms, namely

\[
\nabla^2_{(2)} \Psi^S_{(2)} + \nabla^2_{(2)} \Psi^S_{(0)} = 0
\]

which means that the \( \nabla^2_{(2)} \) acting on the zeroth-order part of the field is equal to minus the source term in the scalar differential equation for the first correction to the field. In general, the source term in this equation is expected to contain the components of the tensors \( E \) and \( B \) and to give the first correction coming from the particle’s motion on the background geodesic. From this equation it can be inferred that

\[
\Psi^S_{(2)} \sim [O(\Psi^S_{(0)}) \times O(\frac{\rho^3}{R^2})].
\]

The second correction obeys the differential equation derived from Eq. (20) by keeping the \( O(\rho^3/R^3) \) terms of the metric, namely

\[
\nabla^2_{(3)} \Psi^S_{(3)} + \nabla^2_{(3)} \Psi^S_{(0)} = 0.
\]

The source term in this case is identically equal to minus the \( O(\rho^3/R^3) \)-part of \( \nabla^2 \) acting on \( \Psi^S_{(0)} \) and in principle involves \( E_{ijk} \) and \( B_{ijk} \) as well as the time-derivatives of \( E_{ij} \) and \( B_{ij} \). From this equation it can be inferred that

\[
\Psi^S_{(3)} \sim [O(\Psi^S_{(0)}) \times O(\frac{\rho^3}{R^3})].
\]

Notice that \( \Psi^S_{(2)} \) is not included in the equation for \( \Psi^S_{(3)} \) because it shows up in the term \( \nabla^2_{(2)} \Psi^S_{(2)} \), a term of order \( O(\Psi^S_{(0)}) \times O(\rho^4/R^3) \), which must be included in the calculation of the third correction.

The source for the third correction contains both \( \Psi^S_{(0)} \) and \( \Psi^S_{(2)} \). Specifically, the differential equation derived from Eq. (20) by keeping all metric terms of \( O(\rho^4/R^3) \) is

\[
\nabla^2_{(0)} \Psi^S_{(4)} + \nabla^2_{(2)} \Psi^S_{(2)} + \nabla^2_{(4)} \Psi^S_{(0)} = 0
\]

in which the second and third terms are effectively the source terms. That gives that the third correction is

\[
\Psi^S_{(4)} \sim [O(\Psi^S_{(0)}) \times O(\frac{\rho^4}{R^3})].
\]

1. Monopole Field

Assume that the particle in question is carrying a scalar charge \( q \) and is moving on a geodesic. The scalar field generated by that particle is the scalar monopole field. The detailed calculation of that field in the THZ coordinates was presented in [18] and the result is given here for completeness.

The zeroth-order term is the Coulomb-like potential \( (q/\rho) \). Both the first and second order corrections can be shown to be equal to 0 due to symmetries, so the result is

\[
\Psi^S = \frac{q}{\rho} + O(\frac{\rho^3}{R^2}).
\]

Disregarding the fact that the first and second corrections to the field are equal to zero, Eqs. (23) and (25) imply that one would only need to calculate up to the first correction of the scalar monopole singular field in order to be able to calculate the self-force on the scalar charge. That is because the second correction, were it not zero, would be of \( O(\rho^2/R^3) \) which would give no contribution to the self-force on the particle after the first derivative followed by the limit to the location of the particle were taken.

2. Dipole Field

Assume that the particle is carrying a scalar dipole moment and is moving on a geodesic of the given background. The dipole moment is assumed to have a random orientation and its THZ components are denoted as \( K_a = (0, K_x, K_y, K_z) \).
The leading term in the scalar field series expansion is the scalar field generated by a dipole that is stationary at the origin of a Cartesian coordinate system and is

$$\Psi_S^{(0)} = \frac{K_i x^i}{\rho^3}. \quad (29)$$

For the first correction, explicit evaluation of Eq. (22) shows that $\nabla^2 (2) \Psi_S^{(0)} = 0$, which results in

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \Psi_S^{(2)} = 0, \quad (30)$$

so the first correction to the field can be set equal to a constant, or, for self-force calculations, equal to 0.

Eq. (24) gives that the second correction to the singular field obeys a differential equation that relates the second derivatives of $\Psi_S^{(3)}$ to terms of the form

$$K \hat{B} \cdot \frac{x x y x x x}{\rho^7},$$

where the dots denote appropriately contracted indices for each term. The tensors $\epsilon_{ijk}$ or $B_{ijk}$ do not show up in the equation. The fact that there is a large number of terms in the differential equation makes it very inconvenient to present the equation explicitly here. In order to give the reader an idea of what the equation looks like, we explicitly write a few terms of it (28), namely

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \Psi_S^{(3)} + K_x \hat{B}_y x^2 y^2 z^2 + \ldots = 0. \quad (31)$$

Using arguments similar to the ones mentioned at the beginning of Sec. (III) for the scalar monopole field we conclude that $\Psi_S^{(3)}$ must be proportional to

$$A \epsilon_{jkl} K^j \hat{B}^k x^l x^i, \quad (32)$$

where $A$ is a constant. Substituting this expression into the differential equation for $\Psi_S^{(3)}$ gives that $A = -1/7$, or that

$$\Psi_S^{(3)} = -\frac{1}{7} \epsilon_{jkl} K^j \hat{B}^k x^l x^i \rho. \quad (33)$$

Eq. (27) gives that the third correction is of $O(\rho^3/R^4)$. This correction is not necessary for self-force calculations. Differentiation of it would give terms proportional to the coordinates, which would give zero after the limit to the location of the particle is taken.

Finally, the singular scalar field of a dipole moving on a geodesic is equal to

$$\Psi_S = \frac{K_i x^i}{\rho^3} - \frac{1}{7} \epsilon_{jkl} K^j \hat{B}^k x^l x^i + O\left(\frac{\rho^2}{R^3}\right). \quad (33)$$

In this case, knowledge of the second correction is necessary in order to calculate the self-force. Differentiating the second correction gives a term of order $O(\rho^0/R^4)$ which is non-zero when the limit to the location of the particle is taken.

### B. Electromagnetic Potential

Assume that a particle carries an electric charge, an electric dipole moment or a magnetic dipole moment. The source associated with it is denoted by $J$. In this case, the self-force can be calculated using the singular electromagnetic potential, which can be expanded as

$$A_S^a = A_S^{(0)} + A_S^{(2)} + A_S^{(3)} + \ldots \quad (34)$$

In the vacuum background the Ricci tensor is equal to zero. That means that the electromagnetic potential obeys the Maxwell’s equations

$$\nabla^2 (0+2+3+\ldots) A_S^a (0+2+3+\ldots) = -4\pi J^a. \quad (35)$$

Regardless of the exact form of the source, the leading term in the expansion (34) is the electromagnetic potential that would be generated by the particle if it were stationary at the origin of a Cartesian coordinate system. It obeys the equation

$$\nabla^2 (0) A_S^a (0) = -4\pi J^a. \quad (36)$$

The first correction obeys the differential equation derived from Eq. (35), namely

$$\nabla^2 (0) A_S^a (2) + \nabla^2 (2) A_S^a (0) = 0. \quad (37)$$

The $O(\rho^2/R^2)$-part of $\nabla^2$ acting on the zeroth-order electromagnetic potential is the source term for the first correction. From this equation it can also be inferred that the order of the first correction to the electromagnetic potential is

$$A_S^a (2) \sim [O(\rho^2/R^2)] \times O(\rho^2/R^2). \quad (38)$$

According to Eq. (35), the second correction obeys the differential equation

$$\nabla^2 (0) A_S^a (3) + \nabla^2 (3) A_S^a (0) = 0 \quad (39)$$

meaning that the $O(\rho^3/R^3)$-part of $\nabla^2$ acting on the zeroth order electromagnetic potential acts as the source term in the differential equation for $A_S^a (3)$. In the scalar case, $A_S^a (2)$ does not show up in this equation, but will show up in the equation for the third correction. The equation for $A_S^a (3)$ indicates that

$$A_S^a (3) \sim [O(\rho^4/R^4)] \times O(\rho^3/R^3) \quad (40)$$

The equation for the third correction as derived from Eq. (35) is

$$\nabla^2 (0) A_S^a (4) + \nabla^2 (2) A_S^a (2) + \nabla^2 (4) A_S^a (0) = 0 \quad (41)$$

which gives the order of the third correction

$$A_S^a (4) \sim [O(\rho^5/R^5)] \times O(\rho^4/R^4). \quad (42)$$
1. Monopole Potential

If the particle is endowed with an electric charge $q$, Eq. (30) gives that the zeroth order potential is the Coulomb electromagnetic potential

$$ A_{S(0)}^q = \left( \frac{q}{\rho}, 0, 0, 0 \right). $$

(43)

Substituting the THZ components of $A_{S(2)}^q$

$$ A_{S(2)}^q = (A_{S(2)}^x, A_{S(2)}^y, A_{S(2)}^z) $$

into Eq. (37) results in four differential equations, one for each of these components. Each equation relates the sum of second derivatives of a component to a sum of terms of the form:

$$ qE \cdot \frac{x \cdot x \cdot x \cdot x}{\rho^5} $$

for the $t$-component and

$$ qB \cdot x \cdot x \cdot x \cdot x $$

for the spatial components, where the dots again denote appropriate indices.

The form of the solution can be predicted based on the dimensionality and the index structure as before. One must take into account the fact that the $t$-component must have no free indices and that each spatial component must have one free index. Thus it can be derived that the solution should be proportional to

$$ qE_{ij} \frac{x^i x^j}{\rho} $$

for the $t$-component and to

$$ qe_{p,ij} B_{ijkl} x^j x^k \rho $$

for the $p$-spatial component, each term multiplied by an appropriate constant. Substituting these expressions into the four differential equations results in algebraic equations for the constants. Finally, the second correction to the electromagnetic potential is

$$ A_{S(3)}^t = -\frac{1}{6} qE_{i,j} \frac{x^i x^j x^k}{\rho}, $$

$$ A_{S(3)}^p = \frac{1}{2} qe_{p,ij} B_{ijkl} \frac{x^j x^k}{\rho}. $$

(45)

The second correction to the electromagnetic potential comes from the part of the $\nabla^2$ that is of $O(\frac{1}{\rho^2})$ acting on the zeroth-order electromagnetic potential. Assuming that the THZ components of $A_{S(3)}^q$ are

$$ A_{S(3)}^q = (A_{S(3)}^x, A_{S(3)}^y, A_{S(3)}^z, A_{S(3)}^z) $$

(46)

and substituting into Eq. (39) gives a differential equation for each component of $A_{S(3)}^q$. Each equation relates a sum of second derivatives of each component to terms of the form

$$ qE \cdot \frac{x \cdot x \cdot x \cdot x}{\rho^5} $$

for the $t$-component, and

$$ qE \cdot \frac{x \cdot x \cdot x}{\rho^3} $$

for the spatial components. The $t$-component in this case is expected to be proportional to

$$ qE_{ij} \frac{x^i x^j x^k}{\rho} $$

and the $p$-spatial component is expected to be a sum of the terms

$$ qE_{i,j} x^i x^j x^k \rho, qE_{ij} \frac{x^i x^j}{\rho}, qe_{p,ij} B_{ijkl} \frac{x^j x^k}{\rho}, qe_{p,ij} B_{ijkl} \frac{x^j x^k}{\rho}, qe_{i,j} B_{ijkl} \frac{x^j x^k}{\rho}, qe_{i,j} B_{ijkl} \frac{x^j x^k}{\rho} $$

each term multiplied by an appropriate constant. Substituting these expressions into the four differential equations gives algebraic equations for the constants. Finally, the second correction to the electromagnetic potential is

$$ A_{S(3)}^t = -\frac{1}{6} qE_{i,j} \frac{x^i x^j x^k}{\rho}, $$

$$ A_{S(3)}^p = \frac{1}{2} qe_{p,ij} B_{ijkl} \frac{x^j x^k}{\rho}. $$

(47)

According to Eq. (42), the order of the third correction to the electromagnetic potential of a charge is

$$ A_{S(4)}^q \sim O(\frac{1}{\rho^3}). $$

(48)

and it is not necessary for self-force calculations.

Finally, the components of the singular electromagnetic potential of a charge $q$ that is moving on a geodesic are equal to

$$ A_{S}^t = \frac{q}{\rho} - \frac{1}{6} qE_{i,j} \frac{x^i x^j x^k}{\rho} + O(\frac{1}{\rho^3}), $$

$$ A_{S}^p = \frac{1}{2} e_{p,ij} qB_{ijkl} \frac{x^j x^k}{\rho} $$

(49)

$$ -\frac{1}{18} qE_{i,j} x^i x^j \rho + O(\frac{1}{\rho^3}) $$

$$ + \frac{2}{9} qe_{p,ij} B_{ijkl} \frac{x^j x^k x^l}{\rho} + O(\frac{1}{\rho^3}). $$

(49)
In this case, the second correction is not necessary for the calculation of the self-force. After differentiation, that correction will give terms proportional to the coordinates which, after taking the limit to the location of the particle, will give zero.

2. Electric Dipole Potential

Assume that a particle carrying an electric dipole moment is moving on a geodesic $\Gamma$. The dipole moment is assumed to point at some random direction and its THZ components are $q^a = (0, q^x, q^y, q^z)$.

According to Eq. \(36\) the leading term in the expansion of the singular electromagnetic potential is the electromagnetic potential generated by an electric dipole that is stationary at the origin of the Cartesian coordinate system

$$A^a_{S(0)} = \left(\frac{q_l x^i}{\rho^3}, 0, 0, 0\right). \quad (50)$$

The THZ components of the first correction are

$$A^a_{S(2)} = (A^x_{S(2)}, A^y_{S(2)}, A^z_{S(2)}). \quad (51)$$

If substituted into Eq. \(37\), the differential equation for $A^a_{S(2)}$ is broken into a set of four second-order differential equations for these components. Each differential equation relates the sum of second derivatives of a component to a sum of terms of the form

$$q_i \mathcal{E}_{ij} \frac{x^j x^k}{\rho^3}, \frac{q_i \mathcal{E}_{ij} x^j}{\rho}$$

for the $t$-component,

$$q_i \mathcal{B}_{ij} \frac{x^j x^k}{\rho^3}$$

for the spatial components.

In this case we can deduce that the solution should be equal to a sum of the terms

$$q_i \mathcal{E}_{jk} \frac{x^i x^j x^k}{\rho^3}, \frac{q_i \mathcal{E}_{ij} x^j}{\rho}$$

for the $t$-component and a sum of the terms

$$\epsilon^{ipq}_{ijkl} \mathcal{B}^{jl}_{ijk} \frac{x^l}{\rho}, \epsilon^{ipq}_{ijkl} \mathcal{B}^{jl}_{ijk} \frac{x^l}{\rho^3}, \epsilon^{ipq}_{ijkl} \mathcal{B}^{jl}_{ijk} \frac{x^l}{\rho^3},$$

for the $p$-spatial component, each term multiplied by an appropriate constant so that the differential equations are satisfied. Additionally, using Eq. \(15\) and \(16\), the last two terms that are expected to show up in the solution for the $p$-component can be eliminated in favor of the remaining four. Substituting the sums of the remaining terms into the four differential equations gives simple algebraic equations for the multiplicative constants. The final expressions for the components of $A^a_{S(2)}$ are

$$A^t_{S(2)} = -\frac{1}{2} \xi^i \mathcal{E}_{jk} \frac{x^i x^j x^k}{\rho^3},$$

$$A^p_{S(2)} = \frac{1}{2} \epsilon_{ijkl} \mathcal{B}^{jl}_{ijk} \frac{x^l}{\rho} + \frac{1}{2} \epsilon_{ijkl} \mathcal{B}^{jl}_{ijk} \frac{x^l}{\rho^3} + \frac{1}{2} \epsilon_{ijkl} \mathcal{B}^{jl}_{ijk} \frac{x^l}{\rho^3}. \quad (52)$$

The second correction to the singular electromagnetic potential obeys Eq. \(39\). By substituting its THZ components

$$A^a_{S(3)} = (A^x_{S(3)}, A^y_{S(3)}, A^z_{S(3)}) \quad (53)$$

into \(39\) we get a set of four second-order differential equations for each one of those components. Each equation relates the second derivatives of one component to terms of the form

$$q_i \mathcal{E}_{ijk} \frac{x^1 x^2 x^3}{\rho^3}, q_i \mathcal{E}_{ijk} \frac{x^1 x^2 x^3}{\rho^3}, q_i \mathcal{E}_{ijk} \frac{x^1 x^2 x^3}{\rho^3},$$

for the $t$-component, and to terms of the form

$$q_i \mathcal{B}_{ijkl} \frac{x^1 x^2 x^3}{\rho^3},$$

for each spatial component.

In this case the solution must be a sum of the terms

$$\epsilon_{ijkl} \mathcal{B}^{jl}_{ijk} \frac{x^l}{\rho},$$

$$q_i \mathcal{E}_{ijkl} \frac{x^1 x^2 x^3}{\rho^3}, q_i \mathcal{E}_{ijkl} \frac{x^1 x^2 x^3}{\rho^3},$$

for the $t$-component, and a sum of terms

$$q_i \mathcal{E}_{ijkl} \frac{x^1 x^2 x^3}{\rho^3}, q_i \mathcal{E}_{ijkl} \frac{x^1 x^2 x^3}{\rho^3},$$

for the $p$-spatial component, each term multiplied by an appropriate constant so that the differential equations are satisfied. We substitute these expressions into the differential equations and solve the resulting systems of algebraic equations for those constants. Thus, the second
correction to the electromagnetic potential is

$$A_S^f (3) = - \frac{1}{7} iq_{ijk} B^j_i x^k x^j x^l \rho - \frac{1}{9} q_i \mathcal{E}_{jkl} x^k x^j x^l \rho^3$$

$$A_S^p (3) = - \frac{1}{18} q^2 \dot{E}_{ij} x^l \rho - \frac{5}{18} q_{ij} \dot{E}_l x^i x^j \rho^3 + \frac{2}{9} q^l \dot{E}_{ij} x^i x^j x^k \rho^3$$

$$+ \frac{1}{18} q^l \dot{E}_{ij} x^i x^j x^k \rho^3 + \frac{7}{18} q^l \dot{E}_{ij} x^i x^j x^k \rho^3$$

$$+ \frac{2}{9} q^l \dot{E}_{ij} x^i x^j x^k \rho^3 + 2 \frac{q^l \dot{E}_{ij} B^m_{jkl} x^k x^j x^m \rho^3}{9}.$$  (54)

According to Eq. (42), the third correction is

$$A_S^3 (4) \sim O(\frac{\rho^2}{\rho^3})$$  (55)

which is clearly not necessary for self-force calculations, because taking the first derivative and the limit to the location of the particle gives zero.

Finally, the singular electromagnetic potential for an electric dipole moving on a geodesic is equal to

$$A_S^e = q_i x^i \rho^2 - \frac{1}{2} q_i \dot{E}_{jkl} x^k x^j x^l \rho^3$$

$$- \frac{1}{7} q_{ijkl} B^j_i x^k x^j x^l \rho^3 - \frac{1}{9} q_i \mathcal{E}_{jkl} x^k x^j x^l \rho^3 + O(\frac{\rho^2}{\rho^3})$$

$$A_S^p = \frac{1}{2} \epsilon_{ijq} \dot{q} B^j_i x^l \rho + \frac{1}{2} \epsilon_{ijq} \dot{q} B^j_i x^l \rho^3$$

$$+ \frac{1}{2} \epsilon_{ijq} \dot{q} B^j_i x^l \rho$$

$$- \frac{1}{18} q^2 \dot{E}_{ij} x^l \rho - \frac{5}{18} q_{ij} \dot{E}_l x^i x^j \rho^3 + \frac{2}{9} q^l \dot{E}_{ij} x^i x^j x^k \rho^3$$

$$+ \frac{1}{18} q^l \dot{E}_{ij} x^i x^j x^k \rho^3 + \frac{7}{18} q^l \dot{E}_{ij} x^i x^j x^k \rho^3$$

$$+ \frac{2}{9} q^l \dot{E}_{ij} B^m_{jkl} x^k x^j x^m \rho^3 + \frac{2}{9} \epsilon_{ijkl} \dot{q} B^p_{jkl} x^k x^l \rho^3 + O(\frac{\rho^2}{\rho^3}).$$  (56)

In this case the second correction is necessary for the self-force calculation. That correction is proportional to the first power of the coordinates, differentiation of which gives a constant term that does not vanish when the limit to the location of the particle is taken.

### 3. Magnetic Dipole Potential

Assume that a particle with a given magnetization is moving on the geodesic $\Gamma$. The magnetization $m^a$ is assumed to point at some random direction and its THZ components are $m^a = (0, m^x, m^y, m^z)$.

Eq. (56) gives that the leading term in the expansion of the electromagnetic potential is the potential generated by a magnetic dipole that is stationary at the origin of a Cartesian coordinate system. Its THZ components are

$$A_S^{0(0)} = (0, \epsilon^x_i m^1 x^2 \rho^3, \epsilon^y_i m^1 x^2 \rho^3, \epsilon^z_i m^1 x^2 \rho^3).$$  (57)

The first correction $A_S^{0(2)}$ has THZ components:

$$A_S^{(2)} = (A_S^{(2)} x^2, A_S^{(2)} x^2, A_S^{(2)} x^2).$$  (58)

When those components are substituted into Eq. (37) the result is a set of four second-order differential equations, one for each one of those four components. Each equation relates a sum of the second derivatives of a component to a source term that consists of terms of the form

$$m_B x^i x^j \rho^5$$

for the $t$-component, and terms of the form

$$m E x^i x^j x^k \rho^5$$

for the spatial components, where the dots denote appropriate indices.

The form of the solution can be predicted as previously. In this case, the $t$-component is expected to be a sum of the terms

$$m_i B^j_{ij} x^j \rho, m_i B_{ik} x^j x^k \rho$$

and the $p$-spatial component a sum of the terms

$$\epsilon^x_i m^1 E_{jkl} x^j x^k x^l \rho^3, \epsilon^y_i m^1 E_{jkl} x^j x^k x^l \rho^3, \epsilon^z_i m^1 E_{jkl} x^j x^k x^l \rho^3$$

with appropriate constants multiplying each term so that the differential equations are satisfied. Using Eq. (15) and (16), the first and last terms expected to show up in the sum for the $p$-component can be eliminated, since they can be expressed as linear combinations of the remaining four terms. Substituting these expressions into the differential equations gives simple systems of algebraic equations for those constants. The result is that the first correction to the electromagnetic potential has components:

$$A_S^{f(2)} = \frac{1}{6} m_i B_{jkl} x^i x^j x^k \rho^4 - \frac{2}{3} m_i B_{lj} x^j \rho^4$$

$$A_S^{p(2)} = \epsilon^x_i m^1 E_{jkl} x^j x^k x^l \rho^3 - \epsilon^y_i m^1 E_{jkl} x^j x^k x^l \rho^3 - \frac{1}{2} \epsilon_{ijkl} m^1 E_{jkl} x^j x^k x^l \rho^5.$$  (59)

The second correction to the magnetic dipole potential can be calculated using Eq. (59). Substituting the THZ components of that correction:

$$A_S^{e(3)} = (A_S^{e(3)} x^2, A_S^{e(3)} x^2, A_S^{e(3)} x^2)$$  (60)
to Eq. [39] we get four differential equations, one for each component, which relate second derivatives of those components to terms of the form

$$m \ddot{\mathbf{E}} \cdot \frac{x^i x^j x^k}{\rho^3}, m \mathbf{B} \cdot \frac{x^i x^j x^k}{\rho^3}$$

for the \( t \)-component and terms of the form

$$m \dot{\mathbf{B}} \cdot \frac{x^i x^j x^k}{\rho^3}, m \dot{\mathbf{E}} \cdot \frac{x^i x^j x^k}{\rho^3}$$

for the spatial components. The terms expected to show up in the solution can be constructed as before. We expect a sum of the terms

$$\epsilon_{ijk} m^i \dot{\mathbf{E}}_j \frac{x^k x^l}{\rho}, m^i \mathbf{B}_{ijk} \frac{x^j x^k x^l}{\rho^3}$$

for the \( t \)-component and a sum of the terms

$$m^i \dot{\mathbf{B}}_{ij} \frac{x^j x^l}{\rho}, m^i \mathbf{B}^i_{kl} \frac{x^j x^k x^l}{\rho}, m^i \dot{\mathbf{B}}_{ij} \frac{x^j x^l}{\rho}, m^i \mathbf{B}^i_{kl} \frac{x^j x^k x^l}{\rho^3},$$

$$e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l}{\rho}, e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l}{\rho^3}, e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l}{\rho^3}, e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l}{\rho^3}$$

for the \( p \)-spatial component, each term multiplied by an appropriate constant. Substituting those sums into the differential equations, we get systems of algebraic equations for those constants. Solving those we get that the second correction to the electromagnetic potential is

$$A_{S(3)}^t = \frac{1}{3} \epsilon_{ijk} m^i \dot{\mathbf{E}}_j \frac{x^k x^l}{\rho} - \frac{1}{3} m^i \mathbf{B}_{ijk} \frac{x^j x^k x^l}{\rho^3} + \frac{1}{9} m_i \dot{\mathbf{B}}_{ijk} \frac{x^j x^k x^l}{\rho^3}$$

$$A_{S(3)}^p = \frac{37}{252} m^p \dot{\mathbf{B}}_{ij} \frac{x^j x^l}{\rho} + \frac{25}{126} m^i \mathbf{B}^i_{kl} \frac{x^j x^l}{\rho} + \frac{5}{63} m^i \dot{\mathbf{B}}_{ij} \frac{x^j x^l}{\rho^3}$$

$$- \frac{4}{63} m^i \dot{\mathbf{B}}_{ij} \frac{x^j x^k x^l}{\rho^3} + \frac{1}{6} e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l}{\rho^3} - \frac{1}{6} e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l}{\rho^3} - \frac{1}{6} e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l}{\rho^3} + O\left(\frac{\rho^2}{R^4}\right).$$

The order of the third correction can be predicted as usual, by Eq. (11) and it is

$$A_{S(4)}^t \sim O\left(\frac{\rho^2}{R^4}\right).$$

These correction terms are not needed for self-force calculations. As in the case of the electric dipole electromagnetic potential, differentiating these terms would result in terms that are proportional to the coordinates and taking the limit to the location of the particle gives zero.

Finally, the singular electromagnetic potential generated by a magnetic dipole moving on a geodesic \( \Gamma \) has components equal to

$$A_f^t = \frac{1}{6} m_j B_{jk} x^i x^j x^k - \frac{2}{3} m^i B_{ij} \frac{x^j}{\rho} + \frac{1}{6} \epsilon_{ijk} m^i \dot{\mathbf{E}}_j \frac{x^k x^l}{\rho^3} - \frac{1}{3} m^i B_{ijk} \frac{x^j x^k x^l}{\rho^3} + O\left(\frac{\rho^2}{R^4}\right)$$

$$A_f^p = e^p_{ij} m^i \frac{x^j}{\rho^3} + e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l}{\rho^3} - \frac{1}{2} \epsilon_{ijk} m^i \mathbf{E}^{jkl} \frac{x^k x^l}{\rho^3} - \frac{37}{126} m^i \dot{\mathbf{B}}_{ij} \frac{x^j x^l}{\rho} + \frac{25}{126} m^i \mathbf{B}^i_{kl} \frac{x^j x^l}{\rho} + \frac{5}{63} m_i \dot{\mathbf{B}}_{ij} \frac{x^j x^l}{\rho^3}$$

$$- \frac{1}{63} m_i \dot{\mathbf{B}}_{ij} \frac{x^j x^k x^l}{\rho^3} + \frac{1}{6} e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l x^m}{\rho^3} - \frac{1}{6} e^p_{ij} m^i \mathbf{E}^{jkl} \frac{x^k x^l x^m}{\rho^3} + O\left(\frac{\rho^2}{R^4}\right).$$

It is clear that in the case of the magnetic dipole, it is necessary to know the second correction to the singular electromagnetic potential in order to be able to calculate the self-force.

**C. Gravitational Field**

We proceed by looking into the more interesting (and computationally more complicated) case of a particle moving on a background geodesic \( \Gamma \) and producing a gravitational field. The singular gravitational field can be expanded as

$$h_{ab} = h_{ab}^{(0)} + h_{ab}^{(2)} + h_{ab}^{(3)} + \ldots$$

For the calculation of the gravitational singular field it is more convenient to use the trace-reversed version of \( h_{ab} \) which is defined as

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2} g_{ab} h_c^c.$$
The singular field obeys the linearized Einstein equations which, in the harmonic gauge, become
\[
\nabla^2_{(0+2+3+\ldots)} h_S^{(0+2+3+\ldots)}_{ab} + 2R_{(0+2+3+\ldots)a}^{\phantom{(0+2+3+\ldots)}b} d_t h_S^{(0+2+3+\ldots)}_{cd} = -16\pi T_{ab}.
\]  
(67)

In Eq. (67) \( T_{ab} \) is the stress-energy tensor associated with the particle.

Regardless of the exact form of the stress-energy tensor, the leading term in the expansion of the gravitational field is the field that the particle would produce if it were stationary at the origin of a Cartesian coordinate system and it obeys the equation
\[
\nabla^2_{(0)} h_S^{(0)}_{ab} + 2R_{(0)a}^{\phantom{(0)b}} d_t h_S^{(0)}_{cd} = -16\pi T_{ab}.
\]  
(68)

The first correction obeys the differential equation derived from Eq. (67) by keeping only terms that come from the \( O(\rho^2/R^2) \) part of the metric, namely
\[
\nabla^2_{(2)} h_S^{(2)}_{ab} + 2R_{(2)a}^{\phantom{(2)b}} d_t h_S^{(2)}_{cd} = -\nabla^2_{(2)} h_S^{(2)}_{ab} - 2R_{(2)a}^{\phantom{(2)b}} d_t h_S^{(2)}_{cd}.
\]  
(69)

The \( \nabla^2 \) and Riemann tensors of \( O(\rho^2/R^2) \) acting on the zeroth order gravitational field act as the source terms in the differential equation of the first correction. Eq. (69) also implies that the first correction is
\[
h_S^{(2)}_{ab} \sim [O(h_S^{(0)}_{ab}) \times O(\rho^2/R^2)].
\]  
(70)

Similarly, the second correction obeys the differential equation derived from Eq. (67) by keeping only terms that come from the \( O(\rho^3/R^3) \) part of the metric, namely
\[
\nabla^2_{(3)} h_S^{(3)}_{ab} + 2R_{(3)a}^{\phantom{(3)b}} d_t h_S^{(3)}_{cd} = -\nabla^2_{(3)} h_S^{(3)}_{ab} - 2R_{(3)a}^{\phantom{(3)b}} d_t h_S^{(3)}_{cd}.
\]  
(71)

Eq. (71) also implies that the second correction is
\[
h_S^{(3)}_{ab} \sim [O(h_S^{(0)}_{ab}) \times O(\rho^3/R^3)].
\]  
(72)

The order of the third correction to the gravitational field can also be predicted, based on the differential equation that could potentially yield that correction. Specifically, Eq. (67) gives
\[
\nabla^2_{(4)} h_S^{(4)}_{ab} + 2R_{(4)a}^{\phantom{(4)b}} d_t h_S^{(4)}_{cd} = -\nabla^2_{(4)} h_S^{(4)}_{ab} - 2R_{(4)a}^{\phantom{(4)b}} d_t h_S^{(4)}_{cd} -\nabla^2_{(2)} h_S^{(2)}_{ab} - 2R_{(2)a}^{\phantom{(2)b}} d_t h_S^{(2)}_{cd}.
\]  
(73)

which implies that the third correction is of order
\[
h_S^{(4)}_{ab} \sim [O(h_S^{(0)}_{ab}) \times O(\rho^4/R^4)].
\]  
(74)

1. Massive Particle

Assume that the particle that generates the gravitational field is a massive point particle of mass \( m \). The calculation for the first correction to the gravitational singular field for this case was first shown in [24], where a different coordinate system was used. Here we perform the calculation of the singular field in the THZ coordinate system, including up to the second correction to it.

The leading term in the expansion of the singular gravitational field is the gravitational field generated by a massive particle that is stationary at the origin of the Cartesian coordinate system. We consider only the terms that are of first-order in the mass \( m \). Thus
\[
h_S^{(0)}_{ab} = \frac{2m}{\rho} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]  
(75)

The first correction to this gravitational field obeys Eq. (69) and must be a symmetric tensor. Substituting its components into Eq. (69) results in ten second-order differential equations. There is one independent set of four differential equations for the four diagonal components, each equation containing all four diagonal components. There is also one differential equation for each of the \( t-i \) components and one differential equation for each of the spatial \( i-j \) components with \( i \neq j \). In each equation a sum of second derivatives of components is related to a sum of terms of the form
\[
mE_{ij} \frac{x^ix^j}{\rho^2},
\]
for the \( t-t \) and \( p-q \) components and a sum of terms
\[
mB_{ij} \frac{x^ix^j}{\rho^2},
\]
for the \( t-p \) components.

Solving the differential equations in this case is more complicated than in the previous cases, mainly because of the fact that four of them involve all diagonal components rather than only one of them. However, the process that was described at the beginning of Sec. III for predicting the form of the solution is applicable in this case as well. It is clear that the \( t-t \) component must have no free indices, each of the \( t-p \) components must have one free index and each purely spatial component must have two free indices. We conclude that the \( t-t \) component of the first correction to the singular field must be proportional to
\[
mE_{ij} \frac{x^ix^j}{\rho},
\]
each \( t-p \) component must be proportional to
\[
mE_{pJ} B_{ik} \frac{x^i x^j}{\rho},
\]
where
and each purely spatial $p - q$ component must be a sum of the terms
\[ m\mathcal{E}_{pq}\rho, \quad m\mathcal{E}_{pi}x_q x^i/\rho \]
\[ m\mathcal{E}_{ij}x_p x^i x^j/\rho^2, \quad m\eta_{pq}\epsilon_{ij}/\rho \]
with appropriate constants multiplying each term so that the differential equations are satisfied. Eq. (15) and \(16\) can be used to eliminate the second and third terms in favor of the remaining two, for the expression for the $p - q$ spatial components. Substituting into the differential equations we get simple algebraic equations for the multiplicative constants. The result is that the first correction to the singular field has components
\[ h^{(2)}_{ij} = 2mE_{ij} x^j x^j/\rho \]
\[ h^{(2)}_{ip} = -\frac{2}{3}m\epsilon_{pik} B^k x^j x^j/\rho \]
\[ h^{(2)}_{pq} = -4m\epsilon_{pq}\rho + 2m\eta_{pq}\epsilon_{ij} x^j x^j/\rho. \]

The second correction is also a symmetric tensor and must obey the differential equations derived from Eq. (11). Substituting the components of the second correction results in ten differential equations. As in the case of the first correction, four of these equations involve second derivatives of the four diagonal components. Each one of the other six differential equations contains the second derivatives of the six non-diagonal components of the second correction. The four equations for the diagonal components and the three equations for the non-diagonal purely spatial components relate derivatives of those components to terms of the form
\[ m\mathcal{B}_{ik} x^k x^i/\rho, \quad m\mathcal{E}_{ik} x^k x^i/\rho. \]
The equations for the $t - p$ components relate their derivatives to terms of the form
\[ m\mathcal{E}_{ik} x^k x^i/\rho, \quad m\mathcal{B}_{ik} x^k x^i/\rho. \]

We can predict the form of the solution as previously. The $t - t$ component must have no free indices. It can be easily deduced that all possible terms that contain $\mathcal{B}$ and satisfy that requirement are identically equal to zero, so the $t - t$ component is expected to contain only the $\mathcal{E}_{ijk}$ tensor and be proportional to
\[ m\mathcal{E}_{ijk} x^i x^j x^k/\rho. \]
The $t - p$ components are expected to be a sum of the terms
\[ m\mathcal{E}_{ip} x^i, \quad m\mathcal{E}_{ij} x^i x^j x^p/\rho, \]
\[ m\epsilon_{pik} B^k x^j x^j/\rho, \]
and the $p - q$ spatial components are expected to be a sum of the terms
\[ m\epsilon_{ij(p} B_{q)} x^j x^p/\rho, \quad m\epsilon_{ij(p} x_{q)} B^k x^j x^k/\rho, \]
\[ \eta_{pq} m\mathcal{E}_{ijk} x^i x^j x^k/\rho, \quad m\mathcal{E}_{pq} x^i x^i/\rho, \quad m\mathcal{E}_{ij(p} x_{q)} x^i x^j/\rho, \]
each term multiplied by an appropriate factor so that the differential equations are satisfied. The parantheses around two indices denote the usual symmetrization convention with respect to those indices, for example
\[ \epsilon_{ij(p} x_{q)} x^i x^j = \frac{1}{2} (\epsilon_{ij} x^i x^j + \epsilon_{jq} x^i x^j). \]

Substituting these expressions into the differential equations gives simple algebraic equations for the multiplicative constants, which can be easily solved. The second correction is then equal to
\[ h^{(3)}_{ij} = \frac{2}{3}m\mathcal{E}_{ijk} x^j x^j x^k/\rho \]
\[ h^{(3)}_{ip} = -\frac{10}{63}m\epsilon_{pik} B^k x^j x^j/\rho - \frac{38}{63}m\mathcal{E}_{ij} x^i x^j x^3/\rho - \frac{2}{9}m\epsilon_{pik} B^k x^j x^j x^3/\rho \]
\[ h^{(3)}_{pq} = \frac{10}{7}m\epsilon_{pik} B_{q} x^i x^j x^k/\rho - \frac{10}{21}m\epsilon_{ij(p} x_{q)} B^k x^i x^j x^k/\rho - \frac{2}{3}\eta_{pq} m\mathcal{E}_{ijk} x^i x^j x^k/\rho - 2m\mathcal{E}_{pq} x^i x^i/\rho. \]

The order of the next correction can be predicted by Eq. (13) and it is
\[ h^{(4)} = O\left(\frac{\rho^3}{R^4}\right). \]
Evaluating these terms is clearly not necessary for self-force calculations because they will result in zero contribution after the first derivative and the limit to the location of the particle is taken.

Finally, the singular gravitational field of a point par-
ticle of mass $m$ moving on a geodesic is equal to

\[
\begin{align*}
    h_{stt} &= \frac{2m}{\rho} + 2m\mathcal{E}_{ij} \frac{x^i x^j}{\rho} + \frac{2}{3}m\mathcal{E}_{ijk} \frac{x^i x^j x^k}{\rho} + O(\frac{\rho^3}{R^4}) \\
    h_{stp} &= -\frac{2}{3}m\varepsilon_{pjk} B_k^i \frac{x^j x^k}{\rho} \\
    &- \frac{10}{63}m\dot{\mathcal{E}}_{pi} \frac{x^i}{\rho} - \frac{38}{63}m\dot{\mathcal{E}}_{ij} \frac{x^i x^j}{\rho} \\
    &- \frac{2}{9}m\varepsilon_{pjk} B_k^i \frac{x^j x^k}{\rho} + O(\frac{\rho^3}{R^4}) \\
    h_{spq} &= \eta_{pq} \frac{2m}{\rho} - 4m\mathcal{E}_{pq} \frac{x^i x^j}{\rho} \\
    &+ \frac{10}{21}m\varepsilon_{ijkl} \dot{B}^i_q x^j x^k - \frac{10}{21}m\varepsilon_{ij(p} x_q) B^i_k \frac{x^j x^k}{\rho} \\
    &- \frac{2}{3}\eta_{pq} m\mathcal{E}_{ij} \frac{x^i x^j x^k}{\rho} - 2m\mathcal{E}_{ppi} \frac{x^i}{\rho} + O(\frac{\rho^3}{R^4}).
\end{align*}
\]

Direct substitution into Eq. (66) shows that this expression for the singular field is consistent with the harmonic gauge condition.

It is clear that, in this case, the second correction to the singular gravitational field is not necessary for the calculation of the self-force on the massive particle. After differentiating, that correction will give terms proportional to the coordinates which, after taking the limit to the location of the particle, will give zero.

IV. DISCUSSION

There is a very obvious categorization of the particles mentioned above, based on the order up to which it is necessary to calculate the singular fields in order to be able to calculate the self-force. For the particles that carry no intrinsic spin, namely the scalar monopole particle, the charged particle and the non-spinning massive particle, it is sufficient to calculate the singular fields/potentials up to, and including, first corrections. For the particles that do carry intrinsic spin, namely the scalar dipole particle and the particle carrying an electric or a magnetic dipole moment, it is necessary to include the second correction to the singular fields, in order to be able to calculate the self-force.

That can be explained quite easily by considering the fields generated by each particle and we use here the simple example of the scalar fields. Let us compare the scalar monopole field at a point $\vec{r}'$

\[
\psi(\vec{r}') = \frac{q}{|\vec{r}' - \vec{r}_0|}
\]

of a scalar charge $q$ at point $\vec{r}_0$ to the dipole field at a point $\vec{r}'$

\[
\psi(\vec{r}') = \frac{K_i (x^i - x_{\nu}^i)}{|\vec{r}' - \vec{r}_0|^3}
\]

of a dipole moment $\vec{K}$ at point $\vec{r}_0$. The monopole field is less singular at the location of the particle than the dipole field is. The former falls off as $1/(|\vec{r}' - \vec{r}_0|)$ while the latter falls off as $1/(|\vec{r}' - \vec{r}_0|)^2$, in the limit $\vec{r}' \rightarrow \vec{r}_0$. Differently said, the field of the non-spinning particle is “less singular” in the limit $\vec{r}' \rightarrow \vec{r}_0$ than the field of the spinning particle is.

V. FUTURE WORK

The author and collaborators are currently working on a self-force calculation for a particle of mass $m$ in the vicinity of a Schwarzschild black hole, as that calculation is described in [20]. The ultimate goal is to generalize the calculation to the more interesting (for LISA data analysis) case of a particle moving in the vicinity of a Kerr black hole. Knowledge of either the singular gravitational field given in Eq. (79) or of the direct contribution to the retarded gravitational field is necessary for those calculations.

The author is also working on calculating the part of the self-force that is due to the spin of a particle. The calculation of the singular gravitational field of a spinning particle will be presented in a future paper. Even though we intend to follow the general method described here, that calculation is more complicated than those described in this paper. At that time we will also present the spherical harmonic decomposition of the singular fields, which is necessary for the self-force calculation as was first described in [2] and outlined in Sec. III in this paper. We will first carry out the spherical harmonic decomposition for the toy-problem of the scalar dipole field given in Eq. (33) to try and identify any issues that arise from the presence of intrinsic spin and then we will move on to performing the same calculation for the gravitational field of the spinning particle.

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14

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