PBW THEORY FOR QUANTUM AFFINE ALGEBRAS

MASAKI KASHIWARA, MYUNGHO KIM, SE-JIN OH, AND EUIYONG PARK

Abstract. Let $U'_q(g)$ be a quantum affine algebra of arbitrary type and let $C^0_g$ be Hernandez-Leclerc’s category. We can associate the quantum affine Schur-Weyl duality functor $F_D$ to a duality datum $D$ in $C^0_g$. In this paper, we introduce the notion of a strong (complete) duality datum $D$ and prove that, when $D$ is strong, the induced duality functor $F_D$ sends simple modules to simple modules and preserves the invariants $\Lambda, \tilde{\Lambda}$ and $\Lambda_\infty$ introduced by the authors. We next define the reflections $S_k$ and $S_k^{-1}$ acting on strong duality data $D$. We prove that if $D$ is a strong (resp. complete) duality datum, then $S_k(D)$ and $S_k^{-1}(D)$ are also strong (resp. complete) duality data. This allows us to make new strong (resp. complete) duality data by applying the reflections $S_k$ and $S_k^{-1}$ from known strong (resp. complete) duality data. We finally introduce the notion of affine cuspidal modules in $C^0_g$ by using the duality functor $F_D$, and develop the cuspidal module theory for quantum affine algebras similar to the quiver Hecke algebra case. When $D$ is complete, we show that all simple modules in $C^0_g$ can be constructed as the heads of ordered tensor products of affine cuspidal modules. We further prove that the ordered tensor products of affine cuspidal modules have the unitriangularity property. This generalizes the classical simple module construction using ordered tensor products of fundamental modules.

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1. Introduction

Let \( q \) be an indeterminate and let \( \mathcal{C}_g \) be the category of finite-dimensional integrable modules over a quantum affine algebra \( U'_q(g) \). The category \( \mathcal{C}_g \) occupies an important position in the study of quantum affine algebras because of its rich structure. The simple modules in \( \mathcal{C}_g \) are indexed by using \( n \)-tuples of polynomials with constant term 1 (called Drinfeld polynomials) ([5, 6, 7] for the untwisted cases and [8] for the twisted cases). The simple modules can be obtained as the head of ordered tensor product of fundamental representations ([1, 25, 53]), and a geometric approach to simple modules was also studied in [45, 46, 53].

Let \( g_0 \) be a finite-dimensional simple Lie algebra of ADE type and let \( U'_q(g) \) be a quantum affine algebra of untwisted affine ADE type. Hernandez and Leclerc introduced the monoidal full subcategory \( \mathcal{C}_g^0 \) of \( \mathcal{C}_g \), which consists of objects whose all simple subquotients are obtained from the heads of tensor products of certain fundamental representations ([15]). Any simple module in \( \mathcal{C}_g \) can be obtained as a tensor product of suitable parameter shifts of simple modules in \( \mathcal{C}_g^0 \). For each Dynkin quiver \( Q \) of \( g_0 \) with a height function, Hernandez and Leclerc introduced a monoidal subcategory \( \mathcal{C}_Q \) of \( \mathcal{C}_g^0 \).
The category \( \mathcal{C}_Q \) is defined by using certain fundamental representations parameterized by vertices of the Auslander-Reiten quiver of \( Q \). It turns out that the complexified Grothendieck ring \( \mathbb{C} \otimes \mathbb{Z} \mathcal{K}(\mathcal{C}_Q) \) is isomorphic to the coordinate ring \( \mathbb{C}[N] \) of the unipotent group \( N \) associated with \( \mathfrak{g}_0 \) and, under this isomorphism, the set of isomorphism classes of simple modules in \( \mathcal{C}_Q \) corresponds to the upper global basis (or dual canonical basis) of \( \mathbb{C}[N] \) ([16]).

In [23, 34, 47, 48], the notion of the categories \( \mathcal{C}_0^\mathfrak{g} \) and \( \mathcal{C}_Q \) is extended to all untwisted and twisted quantum affine algebras. Suppose that \( U_q^+(\mathfrak{g}) \) is of an arbitrary affine type. We consider the set \( \sigma(\mathfrak{g}) := I_0 \times \mathbb{K}^\times / \sim \), where the equivalence relation is given by (2.7), with the arrows determined by the pole of \( \mathcal{R} \)-matrices between tensor products of fundamental representations \( V(\pi_i)_x ((i, x) \in \sigma(\mathfrak{g})) \). Let \( \sigma_0(\mathfrak{g}) \) be a connected component of \( \sigma(\mathfrak{g}) \). The category \( \mathcal{C}_0^\mathfrak{g} \) is defined to be the full subcategory of \( \mathcal{C}_0^\mathfrak{g} \) determined by \( \sigma_0(\mathfrak{g}) \) (see Section 2.5). Let \( \mathfrak{g}_{\text{fin}} \) be the simple Lie algebra of type \( X_\mathfrak{g} \) defined in (6.1). Note that, when \( \mathfrak{g} \) is of untwisted affine type \( ADE \), \( \mathfrak{g}_{\text{fin}} \) coincides with \( \mathfrak{g}_0 \). A \( Q \)-datum is a triple \( \mathcal{Q} = (\Delta, \sigma, \xi) \) consisting of the Dynkin diagram \( \Delta \) of \( \mathfrak{g}_{\text{fin}} \), an automorphism \( \sigma \) on \( \Delta \) and a height function \( \xi \) (see Section 6.2). When \( \mathfrak{g} \) is of untwisted affine type \( ADE \), \( \sigma \) is the identity and \( \mathcal{Q} \) is equal to a Dynkin quiver with a height function. To a \( Q \)-datum \( \mathcal{Q} \), the monoidal subcategory \( \mathcal{C}_\mathcal{Q} \) of \( \mathcal{C}_0^\mathfrak{g} \) was introduced in [16] for untwisted affine type \( ADE \), in [23] for twisted affine type \( A^{(2)} \) and \( D^{(2)} \), in [34, 48] for untwisted affine type \( B^{(1)} \) and \( C^{(1)} \), and in [47] for exceptional affine type. Similarly to the untwisted affine \( ADE \) case, the category \( \mathcal{C}_\mathcal{Q} \) categorifies the coordinate ring \( \mathbb{C}[N] \) of the maximal unipotent group \( N \) associated with \( \mathfrak{g}_{\text{fin}} \). The simple Lie algebra \( \mathfrak{g}_{\text{fin}} \) is more deeply related to the structure of the category \( \mathcal{C}_\mathfrak{g} \). It is proved in [29] that the simply-laced root system \( \Upsilon_\mathfrak{g} \) of \( \mathfrak{g}_{\text{fin}} \) arises form \( \mathfrak{g}_0 \) in a natural way and the block decompositions of \( \mathcal{C}_\mathfrak{g} \) and \( \mathcal{C}_0^\mathfrak{g} \) are parameterized by the lattice associated with the root system \( \Upsilon_\mathfrak{g} \). In the course of the proof, the new invariants \( \Lambda \) and \( \Lambda^\infty \) for \( \mathcal{C}_\mathfrak{g} \) introduced in [27] are used in a crucial way. These invariants are quantum affine algebra analogues of the invariants (with the same notations) for the quiver Hecke algebras (see [21, 24]).

Let \( R_C \) be a quiver Hecke algebra (or Khovanov-Lauda-Rouquier algebras) corresponding to a generalized Cartan matrix \( C \) and denote by \( R_C \)-gmod its finite-dimensional graded module category. The algebra \( R_C \) categorifies the half of the quantum group \( U_q(\mathfrak{g}) \) associated with \( \mathfrak{C} \) ([38, 39, 49]). The simple \( R_C \)-modules were studied and classified by using the structure of \( U_q^- \) via the categorification (see [2, 18, 37, 41, 42, 43, 52]). When \( R_C \) is symmetric and the base field is of characteristic 0, the set of isomorphism classes of simple \( R_C \)-modules correspond to the upper global basis of \( U_q^- \) ([50, 54]). Suppose that \( C \) is of finite type. One of the most successful construction for simple \( R_C \)-modules is the construction using cuspidal modules via the (dual) PBW theory for \( U_q^- \). For a reduced expression \( w_\mathfrak{b} \) of the longest element \( w_0 \) of the Weyl group \( \mathcal{W}_C \), one can define the associated cuspidal modules \( \{ \mathcal{V}_k \}_{k=1,\ldots,\ell} \), which correspond to the dual PBW vectors, and all simple \( R_C \)-modules are obtained as the simple heads of ordered tensor products of
cuspidal modules. The construction using Lyndon words was introduced in [41] (see also [18]) and the construction in a general setting with a convex order was studied in [37, 43]. It was also studied in [40, 44] for an affine case, and in [52] for a symmetrizable case with the viewpoint of MV polytopes.

The quantum affine Schur-Weyl duality [21] gives a connection between quiver Hecke algebras and quantum affine algebras. The quantum affine Schur-Weyl duality says that, for each duality datum $\mathcal{D} = \{L_i\}_{i \in J} \subset C_\mathfrak{g}$ associated with a generalized symmetric Cartan matrix $\mathcal{C}$, there exists a monoidal functor $\mathcal{F}_\mathcal{D}$, called a duality functor shortly, from the category $R_{\mathcal{C}}$-gmod to the category $C_\mathfrak{g}$. The duality functor is very interesting and useful, but it is difficult to handle it because the functor does not enjoy good properties in general. When $\mathcal{D}$ arises from a $Q$-datum, the duality functor $\mathcal{F}_\mathcal{D}$ enjoys good properties. It was shown in [20, 23, 34, 47, 11] that, for each choice of $Q$-datum $\mathcal{Q}$, the quantum affine Schur-Weyl duality functor $\mathcal{F}_\mathcal{Q}: R_{g_{\text{fin}}} \rightarrow C_\mathfrak{Q} \subset C_0\mathfrak{g}$ is exact and sends simple modules to simple modules, thus it induces an isomorphism at the Grothendieck ring level. Here $R_{g_{\text{fin}}}$ is the symmetric quiver Hecke algebra associated with $\mathfrak{g}_{\text{fin}}$. In this viewpoint, it is natural and important to ask which conditions for $\mathcal{D}$ provide the duality functor $\mathcal{F}_\mathcal{D}$ with such good properties, and what properties are preserved from $R_{\mathcal{C}}$-gmod to $C_\mathfrak{g}$ under the duality functor $\mathcal{F}_\mathcal{D}$.

This paper is a complete version of the announcement [31]. The main results of this paper can be summarized as follows:

(i) Let $U'_q(\mathfrak{g})$ be a quantum affine algebra of arbitrary type. We find a sufficient condition for a duality datum $\mathcal{D} = \{L_i\}_{i \in J}$ to provide the functor $\mathcal{F}_\mathcal{D}$ with good properties. We introduce the notion of strong duality datum by investigating root modules. We prove that the associated duality functor $\mathcal{F}_\mathcal{D}$ sends simple modules to simple modules and preserves the invariants $\Lambda, \tilde{\Lambda}$ and $\Lambda^\infty$. We also introduce the notion of complete duality datum, which can be understood as a generalization of the duality datum arising from a $Q$-datum. It turns out that the Cartan matrix $\mathcal{C}$ associated with a compete duality datum $\mathcal{D}$ is equal to the one of $\mathfrak{g}_{\text{fin}}$.

(ii) We introduce the reflections $\mathcal{S}_i$ and $\mathcal{S}_i^{-1}$ ($i \in J$) acting on strong duality data $\mathcal{D}$. We prove that if $\mathcal{D}$ is a strong (resp. complete) duality datum, then $\mathcal{S}_i(\mathcal{D})$ and $\mathcal{S}_i^{-1}(\mathcal{D})$ are also strong (resp. complete) duality data. This allows us to create new strong (resp. complete) duality data from known strong (resp. complete) duality data by applying a finite sequence of the reflections $\mathcal{S}_i$ and $\mathcal{S}_i^{-1}$. Indeed, the family $\{\mathcal{S}_i\}_{i \in J}$ satisfies the braid relations, etc. (see [30]). It will be discussed in a forthcoming paper.

(iii) We introduce the notion of affine cuspidal modules for the category $C_\mathfrak{g}^0$. Let $\mathcal{D}$ be a complete duality datum associated with a Cartan matrix $\mathcal{C}$. For a reduced expression
$w_0$ of the longest element of the Weyl group $W_C$, we define the affine cuspidal modules $\{S_k\}_{k \in \mathbb{Z}}$ for $\mathcal{G}_g^0$ by using the duality functor $F_D$, the right and left duals $\mathcal{D}$, $\mathcal{D}^{-1}$, and the cuspidal modules $\{V_k\}_{k=1, \ldots, t}$ of the quiver Hecke algebra $R_C$ associated with $w_0$. If $D$ arises from a $Q$-datum, then the affine cuspidal modules $\{S_k\}_{k \in \mathbb{Z}}$ consist of fundamental modules. But, in general, affine cuspidal modules are not fundamental. We prove that all simple modules in $\mathcal{C}_{g}^0$ can be obtained uniquely as the simple heads of the ordered tensor products $P_{D, w_0}(a)$, called standard modules, of cuspidal modules. We then show that the standard module $P_{D, w_0}(a)$ has the unitriangularity property. This generalizes the classical simple module construction taking the head of ordered tensor products of fundamental representations ([1, 25, 45, 46, 53]). The unitriangularity property allows us to define a monoidal subcategory $\mathcal{E}_{g}^{[a, b]} \otimes_{\mathbb{Z}} K$ of $\mathcal{C}_{g}^0$ for an interval $[a, b]$, which is a generalization of the subcategory $\mathcal{E}_l$ ($l \in \mathbb{Z}_{>0}$) introduced in [15]. This approach can be understood as a counterpart of the PBW theory for quiver Hecke algebras via the duality functor $F_D$. Hence we establish a base to answer the monoidal categorification conjecture for various monoidal subcategories of $\mathcal{C}_{g}^0$ in the same spirit of [24, 27]. The monoidal categorification conjecture will be discussed in a forthcoming paper ([32, 33]).

We remark that, when $U_q'(g)$ is of untwisted affine ADE type, it has been established by Hernandez-Leclerc that the complexified Grothendieck ring $\mathcal{C} \otimes_{\mathbb{Z}} K(\mathcal{E}_g^0)$ can be written as a product of copies of $\mathcal{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_Q) \simeq \mathbb{C}[N]$, where $N$ is the unipotent group associated with $g_0$ (see the proof of [16, Theorem 7.3]). When the orientation of the quiver $Q$ varies, one gets various copies of $\mathbb{C}[N]$ in $\mathcal{C} \otimes_{\mathbb{Z}} K(\mathcal{E}_g^0)$ and the basis of standard modules correspond to various PBW basis. The PBW theory developed in this paper explains this story transparently at the level of the module category.

Let us explain our results more precisely. Let $U_q'(g)$ be a quantum affine algebra of an arbitrary type. We first investigate several properties of root modules about the new invariants $\Lambda$, $\mathfrak{d}$, etc in Section 3. A root module is a real simple module $L$ such that

$$\mathfrak{d}(L, \mathcal{D}^k(L)) = \delta(k = \pm 1) \quad \text{for any} \; k \in \mathbb{Z}.$$  

Note that the name “root module” comes from Lemma 4.15. We prove several lemmas and propositions on root modules, which are used crucially in the proofs of the main results.

We next deal with the quantum affine Schur-Weyl duality. Let $\mathcal{D}$ be a duality datum associated with a generalized Cartan matrix $C = (c_{i,j})_{i,j \in J}$ of symmetric type. We study the affinizations of modules appeared in both of two categories $R_C$-gmod and $\mathcal{E}_g$ as pro-objects and modify slightly the definition of quantum affine Schur-Weyl duality in order that the duality functor $F_\mathcal{D}$ preserves the affinizations (see Theorem 4.2). This allow us to compare the invariants $\Lambda$, $\mathfrak{d}$, etc between quiver Hecke algebras and quantum affine algebras via the duality functor $F_\mathcal{D}$. When $D = \{L_i\}_{i \in J}$ is a strong duality datum of a Cartan matrix $C = (c_{i,j})_{i,j \in J}$ of simply-laced finite type (see Definition 4.7), we prove that
\( \mathcal{F}_D \) sends simple modules to simple modules (Theorem 4.10), i.e., \( \mathcal{F}_D \) is faithful (Corollary 4.11), and \( \mathcal{F}_D \) preserves the invariants: for any simple modules \( M, N \) in \( R_C\text{-gmod} \),

(i) \( \Lambda(M, N) = \Lambda(\mathcal{F}_D(M), \mathcal{F}_D(N)) \),
(ii) \( \mathfrak{b}(M, N) = \mathfrak{b}(\mathcal{F}_D(M), \mathcal{F}_D(N)) \),
(iii) \( (\text{wt } M, \text{wt } N) = -\Lambda^\infty(\mathcal{F}_D(M), \mathcal{F}_D(N)) \),
(iv) \( \mathfrak{b}(\\mathcal{D}^k\mathcal{F}_D(M), \mathcal{F}_D(N)) = 0 \) for any \( k \neq 0, \pm 1 \),
(v) \( \tilde{\Lambda}(M, N) = \mathfrak{b}(\\mathcal{D}\mathcal{F}_D(M), \mathcal{F}_D(N)) = \mathfrak{b}(\mathcal{F}_D(M), \mathcal{D}^{-1}\mathcal{F}_D(N)) \)

(see Theorem 4.12). The key part for the proof is to show that the invariants for determinantal modules \( \mathcal{D}(w\Lambda, \Lambda) \) (see Section 2.2) are preserved under \( \mathcal{F}_D \), which is given in Theorem 4.9. Corollary 4.14 says that the duality functor \( \mathcal{F}_D \) induces an injective ring homomorphism

\[
K_{q=1}(R_C\text{-gmod}) \hookrightarrow K(\mathfrak{c}_q),
\]

where \( K_{q=1}(R_C\text{-gmod}) \) is the specialization of the \( K(R_C\text{-gmod}) \) at \( q = 1 \). Interestingly, the \( \varepsilon_i \) and \( \varepsilon_i^* \) in the crystal theory for \( R_C\text{-gmod} \) can be interpreted in terms of the invariants \( \mathfrak{b} \) for \( \mathfrak{c}_q \) (see Corollary 4.13).

Let \( \mathcal{D} = \{ L_i \}_{i \in J} \) be a strong duality datum associated with a Cartan matrix \( C = (c_{i,j})_{i,j \in J} \) of simply-laced finite type, and define \( \mathfrak{c}_D \) to be the smallest full subcategory of \( \mathfrak{c}_q^0 \) such that

(a) it contains \( \mathcal{F}_D(L) \) for any simple \( R_C \)-module \( L \),
(b) it is stable by taking subquotients, extensions, and tensor products.

The induced map \( [\mathcal{F}_D] \) gives an isomorphism between \( K(\mathfrak{c}_D) \) and \( K_{q=1}(R_C\text{-gmod}) \) as a ring. We introduce the notion of unmixed pairs of modules in \( \mathfrak{c}_q \) (Definition 5.1) and investigate several properties. Lemma 5.5 says that if \( (M, N) \) are an unmixed pair of simple modules in \( R_C\text{-gmod} \), then \( (\mathcal{F}_D(M), \mathcal{F}_D(N)) \) is strongly unmixed. Let \( w_0 \) be the longest element of the Weyl group \( W_C \) of \( \mathfrak{g}_C \), and \( \ell \) the length of \( w_0 \). We define the affine cuspidal modules \( \{ S_k \}_{k \in \mathbb{Z}} \subset \mathfrak{c}_q^0 \) to be the simple \( U_q(\mathfrak{g}) \)-modules given by

(a) \( S_k = \mathcal{F}_D(V_k) \) for any \( k = 1, \ldots, \ell \),
(b) \( S_{k+\ell} = \mathcal{D}(S_k) \) for any \( k \in \mathbb{Z} \),

where \( \{ V_k \}_{k=1,\ldots,\ell} \subset R_C\text{-gmod} \) are the cuspidal modules associated with \( w_0 \). Note that the cuspidal module \( V_k \) corresponds to the dual PBW vectors associated with \( w_0 \) under the categorification using quiver Hecke algebras. We then prove that \( S_a \) is a root module for any \( a \in \mathbb{Z} \), and \( (S_a, S_b) \) is strongly unmixed for any \( a > b \), which tells us that the ordered tensor product \( S_{k_1}^{a_1} \otimes \cdots \otimes S_{k_l}^{a_l} \) has a simple head for any decreasing integers \( k_1 > \cdots > k_l \) and \( a_1, \ldots, a_l \in \mathbb{Z}_{\geq 0} \) (see Proposition 5.7). We next define the reflections \( \mathcal{J}_k \) and \( \mathcal{J}_k^{-1} \) on duality data (see (5.3)) and prove that the reflections preserve strong duality data with the same Cartan matrix (Proposition 5.9). Furthermore, we characterize simple modules in the intersections \( \mathfrak{c}_{\mathcal{J}_k(\mathcal{D})} \cap \mathfrak{c}_D \) and \( \mathfrak{c}_{\mathcal{J}_k^{-1}(\mathcal{D})} \cap \mathfrak{c}_D \) by using the cuspidal modules \( \{ V_k \}_{k=1,\ldots,\ell} \) (see Proposition 5.11).
We finally introduce the notion of a complete duality datum (see Definition 6.1). We prove that, if $\mathcal{D} = \{L_i\}_{i \in J}$ is a complete duality datum, then the associated Cartan matrix $C$ has the same type as that of $g_{\text{fin}}$. Note that the root system $\Upsilon_g$ of $g_{\text{fin}}$ provides the block decomposition of $\mathcal{C}_g$ (see [29]). The reflections $\mathcal{S}_k$ and $\mathcal{S}_k^{-1}$ preserve complete duality data with the same Cartan matrix (Theorem 6.3) and the duality datum $D_0$ arising from a $Q$-datum $D = (\Delta, \sigma, \xi)$ is complete (Proposition 6.5). By the definition, $C_{D_0}$ is equal to $C_{Q_0}$. Since a new complete duality datum can be constructed by applying the reflections to $D_0$, when $D$ is complete, the category $C_D$ can be viewed as a generalization of $C_Q$. We now assume that $D$ is complete. Let $\{S_k\}_{k \in \mathbb{Z}}$ be the affine cuspidal modules corresponding to $D$ and a reduced expression $w_0$, and set $Z := \mathbb{Z}_{\geq 0}$. We denote by $\prec$ the bi-lexicographic order on $Z$. For any $a = (a_k)_{k \in \mathbb{Z}} \in Z$, we define the standard module by

$$P_{Q,w_0}(a) := \cdots \otimes S_i^{\otimes a_1} \otimes S_0^{\otimes a_0} \otimes S_{-1}^{\otimes a_{-1}} \otimes \cdots,$$

and set $V_{Q,w_0}(a) := \text{hd}(P_{Q,w_0}(a))$. We prove that $V_{Q,w_0}(a)$ is simple for any $a \in Z$ and the set $\{V_{Q,w_0}(a) \mid a \in Z\}$ is a complete and irredundant set of simple modules of $\mathcal{C}_g$ up to isomorphisms (see Theorem 6.10). Furthermore, Theorem 6.12 says that, if $V$ is a simple subquotient of $P_{Q,w_0}(a)$ which is not isomorphic to $V_{Q,w_0}(a)$, then

$$a_{Q,w_0}(V) \prec a,$$

which means that the module $P_{Q,w_0}(a)$ has the unitriangularity property with respect to $\prec$. For an interval $[a, b]$, we define $\mathcal{C}_g^{[a,b],D,w_0}$ to be the full subcategory of $\mathcal{C}_g$ whose objects have all their composition factors $V$ satisfying

$$b \geq l(a_{D,w_0}(V)) \quad \text{and} \quad r(a_{D,w_0}(V)) \geq a,$$

where $l$ and $r$ are defined in (6.8). By the unitriangularity, the category $\mathcal{C}_g^{[a,b],D,w_0}$ is stable by taking tensor products, and it also enjoys the same properties (Theorem 6.16).

This paper is organized as follows. In Section 2, we give the necessary background on quiver Hecke algebras, quantum affine algebras, and the invariants related to $R$-matrices. In Section 3, we introduce the notion of root modules and investigate several properties. In Section 4, we study affinizations and the duality functor $F_D$, and prove that, when $D$ is strong, $F_D$ sends simple modules to simple modules and preserves the new invariants. In Section 5, we introduce the notions of affine cuspidal modules and reflections, and prove that the reflections preserve the strong duality data. In Section 6, we study the PBW theoretic approach to $\mathcal{C}_g$ using a complete duality datum and affine cuspidal modules.

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2. Preliminaries

Convention.
(i) For a statement $P$, $\delta(P)$ is 1 or 0 according that $P$ is true or not.
(ii) For a field $k$, $a \in k$ and $f(z) \in k(z)$, we denote by $\text{zero}_{z=a} f(z)$ the order of zero of $f(z)$ at $z = a$.
(iii) For a ring $A$, $A^\times$ is the set of invertible elements of $A$.

2.1. Quantum groups.

Let $I$ be an index set. A quintuple $(A, P, \Pi, P^\vee, \Pi^\vee)$ is called a (symmetrizable) Cartan datum if it consists of

(a) a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$,
(b) a free abelian group $P$, called the weight lattice,
(c) $\Pi = \{\alpha_i \mid i \in I\} \subset P$, called the set of simple roots,
(d) $P^\vee = \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$, called the coweight lattice,
(e) $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$, called the set of simple coroots

satisfying the following:
(i) $\langle h_i, \alpha_j \rangle = a_{ij}$ for $i, j \in I$,
(ii) $\Pi$ is linearly independent over $\mathbb{Q}$,
(iii) for each $i \in I$, there exists $\Lambda_i \in P$, called the fundamental weight, such that $\langle h_j, \Lambda_i \rangle = \delta_{j,i}$ for all $j \in I$.
(iv) there is a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $P$ satisfying

$$\langle \alpha_i, \alpha_i \rangle \in 2\mathbb{Z}_{\geq 0} \quad \text{and} \quad \langle h_i, \lambda \rangle = \frac{2\langle \alpha_i, \lambda \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$ 

We set $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ and $Q^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ and define $\text{ht}(\beta) = \sum_{i \in I} k_i$ for $\beta = \sum_{i \in I} k_i \alpha_i \in Q^+$. We define $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for any } i \in I\}$.

We write $\Phi^+$ for the set of positive roots associated with $A$ and set $\Phi^- := -\Phi^+$. Denote by $W$ the Weyl group, which is the subgroup of $\text{Aut}(P)$ generated by $s_i(\lambda) := \lambda - \langle h_i, \lambda \rangle \alpha_i$ for $i \in I$.

We denote by $U_q(g)$ the quantum group associated with $(A, P, P^\vee, \Pi, \Pi^\vee)$, which is a $\mathbb{Q}(q)$-algebra generated by $f_i$, $e_i$ $(i \in I)$ and $q^h$ $(h \in P^\vee)$ with certain defining relations (see [17, Chapter 3] for details). We denote by $U_q^+(g)$ (resp. $U_q^-(g)$) the subalgebra of $U_q(g)$ generated by $e_i$’s (resp. $f_i$’s). Set $A := \mathbb{Z}[q, q^{-1}]$ and write $U_q^+_A(g)$ for the $A$-lattice of $U_q^+(g)$, which is the $A$-subalgebra generated by $e_i^{(n)}$ (resp. $f_i^{(n)}$) for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$. We write the unipotent quantum coordinate ring

$$A_q(n) := \bigoplus_{\beta \in Q^-} A_q(n)_{\beta} \quad \text{where} \quad A_q(n)_{\beta} := \text{Hom}_{\mathbb{Q}(q)}(U_q^+(g)_{-\beta}, \mathbb{Q}(q)),$$

and denote by $A_q(n)_A$ the $A$-lattice of $A_q(n)$. Note that $A_q(n)$ is isomorphic to $U_q^-(g)$ as a $\mathbb{Q}(q)$-algebra [24, Lemma 8.2.2].
2.2. Quiver Hecke algebras.

Let $k$ be a field and let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum. Choose polynomials

$$Q_{i,j}(u, v) = \delta(i \neq j) \sum_{(p,q) \in \mathbb{Z}_{\geq 0}^2} t_{i,j;p,q} u^p v^q \in k[u, v]$$

with $t_{i,j;p,q} \in k$, $t_{i,j;p,q} = t_{j,i;p,q}$ and $t_{i,j;-a_{ij},0} \in k^\times$. Note that $Q_{i,j}(u, v) = Q_{j,i}(v, u)$ for $i, j \in I$. Let $S_n = \langle s_1, \ldots, s_{n-1} \rangle$ be the symmetric group on $n$ letters with the action of $S_n$ on $I^n$ by place permutation. For $\beta \in \mathbb{Q}^+$ with $ht(\beta) = n$, we set

$$I^\beta := \{ \nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta \}.$$  

**Definition 2.1.** Let $\beta \in \mathbb{Q}^+$ with $ht(\beta) = n$. The quiver Hecke algebra $R(\beta)$ associated with the parameters $\{Q_{i,j}\}_{i,j \in I}$ is the $k$-algebra generated by $\{e(\nu)\}_{\nu \in I^\beta}$, $\{x_k\}_{1 \leq k \leq n}$, $\{\tau_m\}_{1 \leq m \leq n-1}$ satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu \in I^\beta} e(\nu) = 1,$$

$$x_kx_m = x_mx_k, \quad x_k e(\nu) = e(\nu)x_k,$$

$$\tau_m e(\nu) = e(s_m(\nu))\tau_m, \quad \tau_k \tau_m = \tau_m \tau_k \quad \text{if} \ |k - m| > 1,$$

$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_k x_m - x_{s_k(m)} \tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if} \ m = k, \ \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if} \ m = k + 1, \ \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k)e(\nu) = \begin{cases} \frac{Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}}e(\nu) & \text{if} \ \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise}. \end{cases}$$

The algebra $R(\beta)$ has the $\mathbb{Z}$-grading defined by

$$\deg e(\nu) = 0, \quad \deg x_k e(\nu) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg \tau_k e(\nu) = -(\alpha_{\nu_k}, \alpha_{\nu_{k+1}}).$$

For a $\mathbb{Z}$-graded $k$-algebra $A$, we denote by $A\text{-}\mathbf{gMod}$ the category of graded left $A$-modules, and write $A\text{-}\mathbf{gproj}$ (resp. $A\text{-}\mathbf{gmod}$) for the full subcategory of $A\text{-}\mathbf{gMod}$ consisting of finitely generated projective (resp. finite-dimensional) graded $A$-modules. We set $R\text{-}\mathbf{gproj} := \bigoplus_{\beta \in \mathbb{Q}^+} R(\beta)\text{-}\mathbf{gproj}$ and $R\text{-}\mathbf{gmod} := \bigoplus_{\beta \in \mathbb{Q}^+} R(\beta)\text{-}\mathbf{gmod}$.

For $M \in R(\beta)\text{-}\mathbf{gMod}$ and $N \in R(\gamma)\text{-}\mathbf{gMod}$, we define their convolution product by

$$M \circ N := R(\beta + \gamma)e(\beta, \gamma) \otimes_{R(\beta) \otimes R(\gamma)} (M \otimes N),$$
where \( e(\beta, \gamma) = \sum_{\nu_1, \nu_2} e(\nu_1 \ast \nu_2) \). Here \( \nu_1 \ast \nu_2 \) is the concatenation of \( \nu_1 \) and \( \nu_2 \). We denote by \( M \triangledown N \) the head of \( M \circ N \) and by \( M \uparrow N \) the socle of \( M \circ N \). We say that simple \( R \)-modules \( M \) and \( N \) strongly commute if \( M \circ N \) is simple. A simple \( R \)-module \( L \) is real if \( L \circ L \) is simple. For \( i \in I \) and an \( R(\beta) \)-module \( M \), we define

\[
E_i(M) := e(\alpha_i, \beta - \alpha_i)M, \quad F_i(M) := R(\alpha_i) \circ M,
\]

and

\[
\begin{align*}
\text{wt}(M) &:= -\beta, \\
\varepsilon_i(M) &:= \max\{k \geq 0 \mid E_i^k(M) \neq 0\}, \\
\varphi_i(M) &:= \varepsilon_i(M) + (h_i, \text{wt}(M)).
\end{align*}
\]

For \( i \in I \), we denote by \( L(i) \) the self-dual 1-dimensional simple \( R(\alpha_i) \)-module. For a simple module \( M \), \( \tilde{f}_i(M) \) (resp. \( \tilde{e}_i(M) \)) is the self-dual simple \( R \)-module being isomorphic to \( L(i) \triangledown M \) (resp. \( \text{soc}(E_i(M)) \)). One also defines \( E_i^*, F_i^*, \varepsilon_i^* \), etc in the same manner as above if replacing the role of \( e(\alpha_i, \beta - \alpha_i) \) and \( R(\alpha_i) \circ - \) with the ones of \( e(\beta - \alpha_i, \alpha_i) \) and \(- \circ R(\alpha_i)\).

**Theorem 2.2 ([38, 39, 49]).** There exist \( A \)-bialgebra isomorphisms

\[
U^-_A(\mathfrak{g}) \xrightarrow{\sim} K(R\text{-gproj}) \quad \text{and} \quad A_q(n)_A \xrightarrow{\sim} K(R\text{-gmod}),
\]

where \( K(R\text{-gproj}) \) and \( K(R\text{-gmod}) \) are the Grothendieck groups of \( R(\text{gproj}) \) and \( R\text{-gmod} \).

**Definition 2.3.** The quiver Hecke algebra \( R(\beta) \) is said to be symmetric if \( Q_{i,j}(u,v) \) is a polynomial in \( u - v \) for any \( i, j \in I \).

When \( R \) is symmetric, the Cartan matrix \( A \) is of symmetric type. In this case we assume that \( (\alpha_i, \alpha_i) = 2 \) for all \( i \in I \).

In the sequel, we assume that \( R \) is symmetric.

Let \( z \) be an indeterminate with homogeneous degree 2. For an \( R(\beta) \)-module \( M \), we denote by \( M^{\text{aff}} \) the affinization of \( M \) (see [21, 35]). When \( R(\beta) \) is symmetric, \( M^{\text{aff}} = \mathbb{k}[z] \otimes_{\mathbb{k}} M \) and the \( R(\beta) \)-module structure of \( M^{\text{aff}} \) is defined by

\[
\begin{align*}
e(\nu)(f \otimes m) &= f \otimes e(\nu)m, \\
x_j(f \otimes m) &= (zf) \otimes m + f \otimes x_jm, \\
\tau_k(f \otimes m) &= f \otimes (\tau_k m)
\end{align*}
\]

for \( f \in \mathbb{k}[z], m \in M, \nu \in I^\beta \) and admissible \( j, k \). We sometimes write \( M_z \) instead of \( M^{\text{aff}} \) to emphasize \( z \).
Let $\beta \in \mathbb{Q}^+$ and $m = \text{ht}(\beta)$. For $k = 1, \ldots, m-1$ and $\nu \in I^\beta$, the intertwiner $\varphi_k \in R(\beta)$ is defined by
\[
\varphi_k(\nu) := \begin{cases} (\tau_k x_k - x_k \tau_k)e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \tau_k e(\nu) & \text{otherwise}. \end{cases}
\]
Note that $\{\varphi_k\}_{1 \leq k \leq m-1}$ satisfies the braid relation. Hence, we can define $\varphi_w$ for any $w \in \mathfrak{S}_m$. Let $M$ be an $R(\beta)$-module with $\text{ht}(\beta) = m$ and $N$ an $R(\beta')$-module with $\text{ht}(\beta') = n$. Let $w[n,m]$ be the element of $\mathfrak{S}_{m+n}$ which sends $k \mapsto k + m$ for $1 \leq k \leq n$ and $k \mapsto k - n$ if $n < k \leq m + n$. Then the $R(\beta) \otimes R(\beta')$-linear map $M \otimes N \to N \circ M$ defined by $u \otimes v \mapsto \varphi_{w[n,m]}(v \otimes u)$ can be extended to the $R(\beta+\beta')$-module homomorphism (up to a grading shift)
\[
R_{M,N} : M \circ N \to N \circ M.
\]
For non-zero $R$-modules $M$ and $N$, we set
\[
R_{M_z,N_{z'}}^\text{ren} := (z' - z)^{-s} R_{M_z,N_{z'}} : M_z \circ N_{z'} \to N_{z'} \circ M_z,
\]
where $s$ is the largest integer such that $R_{M_z,N_{z'}}(M_z \circ N_{z'}) \subset (z' - z)^s N_{z'} \circ M_z$. We call it the renormalized $R$-matrix. Then, we define
\[
r_{M,N} : M \circ N \to N \circ M
\]
as the specialization of $R_{M_z,N_{z'}}^\text{ren}$ at $z = z' = 0$ (up to a constant multiple), which never vanishes by the definition (see [21, Section 1] and [35, Section 2] for details).

**Definition 2.4.** Let $M$ and $N$ be simple $R$-modules. We set
\[
\Lambda(M, N) := \deg(r_{M,N}),
\]
\[
\tilde{\Lambda}(M, N) := \frac{1}{2}(\Lambda(M, N) + (\text{wt}(M), \text{wt}(N))),
\]
\[
b(M, N) := \frac{1}{2}(\Lambda(M, N) + \Lambda(N, M)).
\]

Many properties of $\Lambda$, $\tilde{\Lambda}$, and $b$ were obtained in [24, 26, 28].

We now define the monoidal subcategories $\mathcal{C}_w$, $\mathcal{C}_{w,v}$ and $\mathcal{C}_{w,v}$ of $R$-gmod for $w, v \in \mathcal{W}$. For $M \in R(\beta)$-gMod, we define
\[
\mathcal{W}(M) := \{ \gamma \in \mathbb{Q}^+ \cap (\beta - \mathbb{Q}^+) \mid e(\gamma, \beta - \gamma)M \neq 0 \},
\]
\[
\mathcal{W}^*(M) := \{ \gamma \in \mathbb{Q}^+ \cap (\beta - \mathbb{Q}^+) \mid e(\beta - \gamma, \gamma)M \neq 0 \}.
\]
For $w \in \mathcal{W}$, we denote by $\mathcal{C}_w$ the full subcategory of $R$-gmod whose objects $M$ satisfy
\[
\mathcal{W}(M) \subset \text{span}_{\mathbb{R}_{\geq 0}}(\Phi^+ \cap w \Phi^-).
\]
Similarly, for $v \in \mathcal{W}$, we define $\mathcal{C}_{w,v}$ to be the full subcategory of $R$-gmod whose objects $N$ satisfy
\[
\mathcal{W}^*(N) \subset \text{span}_{\mathbb{R}_{\geq 0}}(\Phi^+ \cap v \Phi^+).
Finally, we define \( C_{w,v} := C_w \cap C_{s,v} \).

When \( g \) is of finite type, we have \( C_{w_0} = R\text{-gmod} \) and
\[
M \in C_{s_i w_0} \text{ if and only if } \varepsilon_i(M) = 0,
M \in C_{s_i s_i} \text{ if and only if } \varepsilon_i^*(M) = 0,
\]
for any \( R\)-module \( M \) in \( R\)-gmod and \( i \in I \). Here \( w_0 \) denotes the longest element of \( W \) (see [26] for details).

Let \( w := s_{i_1} \cdots s_{i_l} \) be a reduced expression of \( w \in W \) and define
\[
(2.1) \quad \beta_k := s_{i_1} \cdots s_{i_{k-1}} (\alpha_{i_k}) \quad \text{for } k = 1, \ldots, l.
\]
Then we have \( \Phi^+ \cap w \Phi^- = \{ \beta_1, \ldots, \beta_l \} \) with the convex order \( \prec \) on \( \Phi^+ \cap w \Phi^- \), i.e., \( \beta_a \prec \beta_b \) for any \( a < b \). For \( \beta \in \Phi^+ \cap w \Phi^- \), a pair \( (\alpha, \gamma) \) is called a minimal pair of \( \beta \) if \( \beta = \alpha + \gamma \), \( \alpha \prec \gamma \) and there exists no pair \( (\alpha', \gamma') \) such that \( \beta = \alpha' + \gamma' \) and \( \alpha \prec \alpha' \prec \gamma' \prec \gamma \). The convex order provides the PBW vectors \( \{ E(\beta_k) \}_{k=1,\ldots,l} \) in \( U_A^+(\mathfrak{g}) \) and the dual PBW vectors \( \{ E^*(\beta_k) \}_{k=1,\ldots,l} \) in \( U_A^- (\mathfrak{g}) \). We set \( A_q(n(w)) \) to be the subalgebra of \( A_q(n) \) generated by \( E^*(\beta_k) \) for \( k = 1, \ldots, l \). The category \( C_w \) categorifies the algebra \( A_q(n(w)) \) ([24, 26]).

For \( k = 1, \ldots, l \), let \( V_k \) be the cuspidal module corresponding to \( \beta_k \) with respect to \( w \) (see [26, Section 2] for precise definition). Under the categorification, the cuspidal module \( V_k \) corresponds to the dual PBW vector \( E^*(\beta_k) \). It is known that the set
\[
\{ \text{hd}(V_i^{a_1} \circ \cdots \circ V_i^{a_l}) \mid (a_1, \ldots, a_l) \in \mathbb{Z}_{\geq 0}^l \}
\]
gives a complete set of pairwise non-isomorphic simple graded modules in \( C_w \), up to a grading shift (see [3, 37, 43, 52]). Note that, for a minimal pair \( (\beta_a, \beta_b) \) of \( \beta_k \), there exists an isomorphism
\[
(2.2) \quad V_a \nabla V_b \cong V_k
\]
(see [43, Lemma 4.2] and [3, Section 4.3].)

For \( \Lambda \in \mathbf{P}^+ \) and \( w, v \in W \) with \( w \geq v \), we denote by \( D(w \Lambda, v \Lambda) \) the determinantal module in \( R\text{-gmod} \) corresponding to the pair \( (w \Lambda, v \Lambda) \) (see [24, Section 10.2] and [26, Section 4] for precise definition). Under the categorification, the determinantal module \( D(w \Lambda, v \Lambda) \) corresponds to the unipotent quantum minor \( D(w \Lambda, v \Lambda) \) in \( A_q(n) \) ([26, Proposition 4.1]).

From now on, we assume that \( k \) is a field of characteristic 0 and that \( R \) is a symmetric quiver Hecke algebra of finite ADE type.

Note that, under the categorification by \( R\)-gmod, the upper global basis (or dual canonical basis) of \( A_q(n) \) corresponds to the set of isomorphism classes of simple \( R \)-modules [50, 54]. Then the reflection functor \( \mathcal{T}_i \) constructed in [37] gives an equivalence of categories:
\[
\mathcal{T}_i : C_{s_i w_0} \xrightarrow{\sim} C_{s_i s_i}.
\]
Note that \( T_i \) is denoted by \( T_i^* \) in [26]. Since, at the crystal level, this functor corresponds to the Saito crystal reflection (\((51)\)), we have
\[
T_i(M) \simeq \tilde{e}_i^*(M) \tilde{\varepsilon}_i^*(M)(M)
\]
for a simple module \( M \) with \( \varepsilon_i(M) = 0 \). For a reduced expression \( w := s_{i_1} \cdots s_{i_l} \), the cuspidal module \( V_k \) can be computed as follows (see [26, Section 5]):
\[
V_k \simeq T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(L(i_k))
\]
for \( k = 1, \ldots, l \).

2.3. Quantum affine algebras.

We assume that \( A = (a_{i,j})_{i,j \in I} \) is an affine Cartan matrix. Note that the rank of \( P \) is \(|I| + 1 \). We denote by \( \delta \in \mathbb{Q} \) the imaginary root and by \( c \) the central element in \( P^c \). Note that the positive imaginary root \( \Delta_{+}^m \) is equal to \( \mathbb{Z}_{>0} \delta \) and the center of \( \mathfrak{g} \) is generated by \( c \). We write \( P_{cl} := P/(P \cap \mathbb{Q}\delta) \), called the classical weight lattice, and take \( \rho \in P \) (resp. \( \rho^v \in P^v \)) such that \( \langle h_i, \rho \rangle = 1 \) (resp. \( \langle \rho^v, \alpha_i \rangle = 1 \)) for any \( i \in I \). We choose a \( \mathbb{Q} \)-valued non-degenerate symmetric bilinear form \((\ ,\ )\) on \( P \) satisfying
\[
\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad \langle c, \lambda \rangle = (\delta, \lambda)
\]
for any \( i \in I \) and \( \lambda \in P \). We define \( \mathfrak{g} \) to be the affine Kac-Moody algebra associated with \( A \). We shall use the standard convention in [19] to choose 0 \( \in I \) except \( A_{2n}^{(2)} \) type, in which we take the longest simple root as \( \alpha_0 \), and \( B_2^{(1)} \) and \( A_3^{(2)} \) types, in which we take the following Dynkin diagrams:

\[
A_{2n}^{(2)} : \begin{array}{cccccccc}
\circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\
\end{array} \\
B_2^{(1)} : \begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array} \\
A_3^{(2)} : \begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

Note that \( B_2^{(1)} \) and \( A_3^{(2)} \) in the above diagram are denoted by \( C_2^{(1)} \) and \( D_3^{(2)} \) respectively in [19].

Set \( I_0 := I \setminus \{0\} \).

Let \( q \) be an indeterminate and \( k \) the algebraic closure of the subfield \( \mathbb{C}(q) \) in the algebraically closed field \( \hat{k} := \bigcup_{m > 0} \mathbb{C}((q^{1/m})) \). For \( m, n \in \mathbb{Z}_{\geq 0} \) and \( i \in I \), we define \( q_i = q^{(\alpha_i, \alpha_i)/2} \) and
\[
[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^n [k]_i, \quad \left[ \begin{array}{c}
m \\ n \end{array} \right]_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}
\]

Let \( d \) be the smallest positive integer such that \( d^{(\alpha_i, \alpha_i)/2} \in \mathbb{Z} \) for all \( i \in I \).

**Definition 2.5.** The quantum affine algebra \( U_q(\mathfrak{g}) \) associated with an affine Cartan datum \((A, P, \Pi, P^c, \Pi^v)\) is the associative algebra over \( k \) with 1 generated by \( e_i, f_i \) \((i \in I)\) and \( q^h \) \((h \in d^{-1}P^v)\) satisfying following relations:

(i) \( q^0 = 1, q^h q^{h'} = q^{h+h'} \) for \( h, h' \in d^{-1}P^v \),
(ii) \( q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i \) for \( h \in d^{-1}P, i \in I \),

(iii) \( e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \) where \( K_i = q_i^{h_i} \),

(iv) \[
\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij})} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij})} f_j f_i^{(k)} = 0 \quad \text{for} \ i \neq j,
\]
where \( e_i^{(k)} = e_i^k/[k]! \) and \( f_i^{(k)} = f_i^k/[k]! \).

Let us denote by \( U'_q(\mathfrak{g}) \) the \( k \)-subalgebra of \( U_q(\mathfrak{g}) \) generated by \( e_i, f_i, K_i^{\pm1} \) \( i \in I \). Let \( \mathcal{C}_g \) be the category of finite-dimensional integrable \( U'_q(\mathfrak{g}) \)-modules, i.e., finite-dimensional modules \( M \) with a weight decomposition

\[ M = \bigoplus_{\lambda \in \mathbb{P}_{cl}} M_{\lambda} \quad \text{where} \ M_{\lambda} = \{ u \in M \mid K_i u = q_i^{(h_i,\lambda)} u \}. \]

Note that the trivial module \( 1 \) is contained in \( \mathcal{C}_g \) and the tensor product \( \otimes \) gives a monoidal category structure on \( \mathcal{C}_g \). The monoidal category \( \mathcal{C}_g \) is rigid. For \( M \in \mathcal{C}_g \), we denote by \( \mathcal{D} M \) and \( \mathcal{D}^{-1} M \) the right and the left dual of \( M \), respectively. Hence we have the evaluation morphisms

\[ M \otimes \mathcal{D} M \to 1 \quad \text{and} \quad \mathcal{D}^{-1} M \otimes M \to 1. \]

We extend this to \( \mathcal{D}^k \) for \( k \in \mathbb{Z} \). We set \( M^{\otimes k} := M \otimes \cdots \otimes M \) for \( k \in \mathbb{Z}_{\geq 0} \). For \( M, N \in \mathcal{C}_g \), we denote by \( M \nabla N \) the head of \( M \otimes N \) and by \( M \Delta N \) the socle of \( M \otimes N \). We say that \( M \) and \( N \) strongly commute if \( M \otimes N \) is simple. A simple \( U'_q(\mathfrak{g}) \)-module \( L \) is real if \( L \otimes L \) is simple.

A simple module \( L \) in \( \mathcal{C}_g \) contains a non-zero vector \( u \in L \) of weight \( \lambda \in \mathbb{P}_{cl} \) such that

(i) \( \langle h_i, \lambda \rangle \geq 0 \) for all \( i \in I_0 \), (ii) all the weight of \( L \) are contained in \( \lambda - \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \text{cl}(\alpha_i) \),

where \( \text{cl} : \mathbb{P} \to \mathbb{P}_{cl} \) is the canonical projection. Such a \( \lambda \) is unique and \( u \) is unique up to a constant multiple. We call \( \lambda \) the dominant extremal weight of \( L \) and \( u \) a dominant extremal weight vector of \( L \). For each \( i \in I_0 \), we set

\[ \varpi_i := \text{gcd}(c_0, c_i)^{-1}\text{cl}(c_0\Lambda_i - c_i\Lambda_0), \]

where the central element \( c \) is equal to \( \sum_{i \in I} c_i h_i \). For any \( i \in I_0 \), we denote by \( V(\varpi_i) \) the \( i \)-th fundamental representation. Note that the dominant extremal weight of \( V(\varpi_i) \) is \( \varpi_i \).

2.4. \textbf{R-matrices.} In this subsection we review the notion of R-matrices on \( U'_q(\mathfrak{g}) \)-modules and their coefficients (see [9], [1, Appendices A and B] and [25, Section 8] for details).

For a module \( M \in \mathcal{C}_g \), we denote by \( M^{\text{aff}} \) the affinization of \( M \) and by \( z_M : M^{\text{aff}} \to M^{\text{aff}} \) the \( U'_q(\mathfrak{g}) \)-module automorphism of weight \( \delta \). Note that \( M^{\text{aff}} \simeq k[z^{\pm1}] \otimes_k M \) with the action

\[ e_i(a \otimes v) = z^{h_i, \alpha_i} a \otimes e_i v \quad \text{for} \ a \in k[z^{\pm1}] \text{ and } v \in M. \]
We sometimes write $M_x$ instead of $M^\text{aff}$ to emphasize the endomorphism $z$. For $x \in k^\times$, we define

$$\begin{align*}
M_x := M^\text{aff} / (z_M - x) M^\text{aff}.
\end{align*}$$

We call $x$ a spectral parameter (see [25, Section 4.2] for details).

Take a basis $\{P_\nu\}_\nu$ of $U_q^+(\mathfrak{g})$ and a basis $\{Q_\nu\}_\nu$ of $U_q^- (\mathfrak{g})$ dual to each other with respect to a suitable coupling between $U_q^+(\mathfrak{g})$ and $U_q^- (\mathfrak{g})$. For $U_q'(\mathfrak{g})$-modules $M$ and $N$, we define

$$R_{M,N}^\text{univ}(u \otimes v) := q^{\text{wt}(u), \text{wt}(v)} \sum_\nu P_\nu v \otimes Q_\nu u \quad \text{for } u \in M \text{ and } v \in N,$$

so that $R_{M,N}^\text{univ}$ gives a $U_q'(\mathfrak{g})$-linear homomorphism $M \otimes N \to N \otimes M$, called the universal $R$-matrix, provided that the infinite sum has a meaning. As $R_{M,N}^\text{univ}$ converges in the $z$-adic topology for $M, N \in \mathscr{C}_q$, we have a morphism of $k((z)) \otimes U_q'(\mathfrak{g})$-modules

$$R_{M,N}^\text{univ} : k((z)) \otimes (M \otimes N_z) \longrightarrow k((z)) \otimes (N_z \otimes M).$$

Note that $R_{M,N}^\text{univ}$ is an isomorphism. For non-zero modules $M, N \in \mathscr{C}_q$, we say that the universal $R$-matrix $R_{M,N}^\text{univ}$ is rationally renormalizable if there exists $f(z) \in k((z))^\times$ such that

$$f(z) R_{M,N}^\text{univ} (M \otimes N_z) \subset N_z \otimes M.$$

In this case, we can choose $c_{M,N}(z) \in k((z))^\times$ such that, for any $x \in k^\times$, the specialization of $R_{M,N}^\text{ren} := c_{M,N}(z) R_{M,N}^\text{univ} : M \otimes N_z \to N_z \otimes M$ at $z = x$

$$R_{M,N}^\text{ren}_{\mid z = x} : M \otimes N_z \to N_z \otimes M$$

does not vanish. Note that $R_{M,N}^\text{ren}$ and $c_{M,N}(z)$ are unique up to a multiple of $k[z^\pm 1]^\times = \bigsqcup_{n \in \mathbb{Z}} k^\times z^n$. We call $R_{M,N}^\text{ren}$ the renormalized $R$-matrix and $c_{M,N}(z)$ the renormalizing coefficient. We denote by $r_{M,N}$ the specialization at $z = 1$

$$r_{M,N} := R_{M,N}^\text{ren} \big|_{z = 1} : M \otimes N \to N \otimes M,$$

and call it the $R$-matrix. The $R$-matrix $r_{M,N}$ is well-defined up to a constant multiple whenever $R_{M,N}^\text{univ}$ is rationally renormalizable. By the definition, $r_{M,N}$ never vanishes.

Let $M$ and $N$ be simple modules in $\mathscr{C}_q$ and let $u$ and $v$ be dominant extremal weight vectors of $M$ and $N$, respectively. Then there exists $a_{M,N}(z) \in k[[z]]^\times$ such that

$$R_{M,N}^\text{univ}(u \otimes vz) = a_{M,N}(z)(vz \otimes u).$$

Thus we have a unique $k(z) \otimes U_q'(\mathfrak{g})$-module isomorphism

$$R_{M,N}^\text{norm} := a_{M,N}(z)^{-1} R_{M,N}^\text{univ} \big|_{k(z) \otimes k[z^\pm 1]} : (M \otimes N_z) \to (N_z \otimes M),$$

from $k(z) \otimes k[z^\pm 1] (M \otimes N_z)$ to $k(z) \otimes k[z^\pm 1] (N_z \otimes M)$, which satisfies

$$R_{M,N}^\text{norm}(u \otimes vz) = vz \otimes u.$$
We call \( a_{M,N}(z) \) the *universal coefficient* of \( M \) and \( N \), and \( R_{M,N}^{\text{norm}} \) the *normalized \( R \)-matrix.*

Let \( d_{M,N}(z) \in \mathbb{K}[z] \) be a monic polynomial of the smallest degree such that the image of \( d_{M,N}(z)R_{M,N}^{\text{norm}}(M \otimes N_z) \) is contained in \( N_z \otimes M \), which is called the *denominator of \( R_{M,N}^{\text{norm}} \).* Then we have

\[
R_{\text{ren}}_{M,N} = d_{M,N}(z)R_{M,N}^{\text{norm}} : M \otimes N_z \longrightarrow N_z \otimes M \quad \text{up to a multiple of } \mathbb{K}[z^{\pm 1}]^\times.
\]

Thus

\[
R_{\text{ren}}_{M,N} = a_{M,N}(z)^{-1}d_{M,N}(z)R_{M,N}^{\text{univ}} \quad \text{and} \quad c_{M,N}(z) = \frac{d_{M,N}(z)}{a_{M,N}(z)}
\]

up to a multiple of \( \mathbb{K}[z^{\pm 1}]^\times \). In particular, \( R_{M,N}^{\text{univ}} \) is rationally renormalizable whenever \( M \) and \( N \) are simple.

The following proposition was one of the main results of [22].

**Proposition 2.6** ([22, Theorem 3.12]). Let \( M \) and \( N \) be simple modules, and assume that one of them is real. Then \( \text{Im}(r_{M,N}) \) is a simple module and it coincides with the head of \( M \otimes N \) and with the socle of \( N \otimes M \).

Let \( M \) and \( N \) be simple modules in \( \mathcal{C}_g \). Suppose that one of them is real. Thanks to Proposition 2.6, the diagram

\[
\begin{array}{cccccc}
M \otimes N & \xrightarrow{r_{M,N}} & M \triangledown N & \xrightarrow{\sim} & N \Delta M & \xrightarrow{r_{M,N}} & N \otimes M
\end{array}
\]

commutes. Here \( \xrightarrow{\sim} \) denotes the natural projection and \( \xrightarrow{\sim} \) denotes the embedding.

**Lemma 2.7** ([22, Corollary 3.13]). Let \( L \) be a real simple module. Then for any simple module \( X \), we have

\[
(L \triangledown X) \triangledown \mathcal{D}L \simeq X, \quad \mathcal{D}^{-1}L \triangledown (X \triangledown L) \simeq X,
\]

and

\[
L \triangledown (X \triangledown \mathcal{D}L) \simeq X, \quad (\mathcal{D}^{-1}L \triangledown X) \triangledown L \simeq X.
\]

**Lemma 2.8** ([22, Corollary 3.14]). Let \( X, Y \) and \( L \) be simple modules in \( \mathcal{C}_g \). Suppose that \( L \) is real.

(i) \( X \simeq L \triangledown Y \) if and only if \( X \triangledown \mathcal{D}L \simeq Y \),

(ii) \( X \simeq Y \triangledown L \) if and only if \((\mathcal{D}^{-1}L) \triangledown X \simeq Y \).

In the following theorem, we refer [25] for the notion of good modules. We only note that the fundamental module \( V(\varpi_i) \) is a good module.

**Theorem 2.9** ([1, 4, 25, 22]).

(i) For good modules \( M \) and \( N \), the zeroes of \( d_{M,N}(z) \) belong to \( \mathbb{C}[[q^{1/m}]]q^{1/m} \) for some \( m \in \mathbb{Z}_{>0} \).
For simple modules $M$ and $N$ such that one of them is real, $M_x$ and $N_y$ strongly commute to each other if and only if $d_{M,N}(z)d_{N,M}(1/z)$ does not vanish at $z = y/x$.

Let $M_k$ be a good module with a dominant extremal vector $u_k$ of weight $\lambda_k$, and $a_k \in k^x$ for $k = 1, \ldots, t$. Assume that $a_j / a_i$ is not a zero of $d_{M_i,M_i}(z)$ for any $1 \leq i < j \leq t$. Then the following statements hold.

(a) $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is generated by $u_1 \otimes \cdots \otimes u_t$.
(b) The head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is simple.
(c) Any non-zero submodule of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ contains the vector $u_t \otimes \cdots \otimes u_1$.
(d) The socle of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is simple.
(e) Let $r: (M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t} \to (M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$. Then the image of $r$ is simple and it coincides with the head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ and also with the socle of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$.

For any simple module $M \in \mathcal{C}_g$, there exists a finite sequence $\{(i_k, a_k)\}_{1 \leq k \leq t}$ in $\sigma(g)$ (see (2.7) below) such that $M$ has $\sum_{k=1}^t \omega_i$ as a dominant extremal weight and it is isomorphic to a simple quotient of $V(\omega_{i_1}) \otimes \cdots \otimes V(\omega_{i_t})$. Moreover, such a sequence $\{(i_k, a_k)\}_{1 \leq k \leq t}$ is unique up to a permutation.

We call $\sum_{k=1}^t (i_k, a_k) \in \hat{P} := Z^{\sigma(g)}$, the affine highest weight of $M$.

2.5. Hernandez-Leclerc categories. For $i \in I_0$, let $m_i$ be a positive integer such that

$$W \pi_i \cap (\pi_i + Z \delta) = \pi_i + Zm_i \delta,$$

where $\pi_i$ is an element of $P$ such that $cl(\pi_i) = \omega_i$. Note that $m_i = (\alpha_i, \alpha_i) / 2$ in the case when $g$ is the dual of an untwisted affine algebra, and $m_i = 1$ otherwise. Then, $V(\omega_i)_x \simeq V(\omega_i)_y$ if and only if $x^{m_i} = y^{m_i}$ for $x, y \in k^x$ (see [1, Section 1.3]). We define

$$\sigma(g) := I_0 \times k^x / \sim,$$

where the equivalence relation $\sim$ is given by

$$(i, x) \sim (j, y) \iff V(\omega_i)_x \simeq V(\omega_j)_y \iff i = j \text{ and } x^{m_i} = y^{m_j}.$$

We denote by $[(i, a)]$ the equivalence class of $(i, a)$ in $\sigma(g)$. When no confusion arises, we simply write $(i, a)$ for the equivalence class $[(i, a)]$. For $(i, x)$ and $(j, y) \in \sigma(g)$, we put $d$ many arrows from $(i, x)$ to $(j, y)$, where $d$ is the order of zeros of $d_{V(\omega_i), V(\omega_j)}(z_{V(\omega_i)}) / z_{V(\omega_i)} = y/x$. Thus, $\sigma(g)$ has a quiver structure.

We choose a connected component $\sigma_0(g)$ of $\sigma(g)$. Since a connected component of $\sigma(g)$ is unique up to a spectral parameter shift, $\sigma_0(g)$ is uniquely determined up to a quiver isomorphism. We define $\mathcal{C}_g^0$ to be the smallest full subcategory of $\mathcal{C}_g$ such that

(a) $\mathcal{C}_g^0$ contains $V(\omega_i)_x$ for all $(i, x) \in \sigma_0(g)$,
(b) $\mathcal{C}_g^0$ is stable by taking subquotients, extensions and tensor products.
For symmetric affine types, this category was introduced in [15]. Note that every simple module in $\mathcal{C}_g$ is isomorphic to a tensor product of certain spectral parameter shifts of some simple modules in $\mathcal{C}_g^0$ ([15, Section 3.7]).

### 2.6. Invariants related to R-matrices.

Let us recall the new invariants introduced in [27]. We set

$$\varphi(z) := \prod_{s=0}^{\infty} (1 - \tilde{p}^sz) = \sum_{n=0}^{\infty} \frac{(-1)^n \tilde{p}^{n(n-1)/2}}{\prod_{k=1}^{n}(1 - \tilde{p}^k)} z^n \in k[[z]],$$

where $p^* := (-1)^{\langle \rho^\vee, \delta \rangle} q^{\langle c, \rho \rangle}$ and $\tilde{p} := (p^*)^2 = q^{2\langle c, \rho \rangle}$. We consider the subgroup $G$ of $k((z))$ given by

$$G := \left\{ cz^m \prod_{a \in k^*} \varphi(az)^{\eta_a} \bigg| c \in k^*, m \in \mathbb{Z}, \eta_a \in \mathbb{Z} \text{ vanishes except finitely many } a's. \right\}.$$ 

For a subset $S$ of $\mathbb{Z}$, let $\tilde{p}^S := \{ \tilde{p}^k \mid k \in S \}$. We define the group homomorphisms

$$\text{Deg} : G \to \mathbb{Z} \quad \text{and} \quad \text{Deg}^\infty : G \to \mathbb{Z},$$

by

$$\text{Deg}(f(z)) = \sum_{a \in \tilde{p}^z \leq 0} \eta_a - \sum_{a \in \tilde{p}^z > 0} \eta_a \quad \text{and} \quad \text{Deg}^\infty(f(z)) = \sum_{a \in \tilde{p}^z} \eta_a$$

for $f(z) = cz^m \prod_{a \in k^*} \varphi(az)^{\eta_a} \in G$.

Note that

$$(2.8) \quad \text{Deg}(f(z)) = 2\text{zero}_{z=1} f(z) \quad \text{for } f(z) \in k(z)^{\times} \subset G$$

(see [27, Lemma 3.4]).

**Definition 2.10.** For non-zero $U'_q(\mathfrak{g})$-modules $M$ and $N$ such that $R_{M,N}^{\text{univ}}$ is rationally renormalizable, we define

$$\Lambda(M, N) := \text{Deg}(c_{M,N}(z)), \quad \Lambda^\infty(M, N) := \text{Deg}^\infty(c_{M,N}(z)), \quad \vartheta(M, N) := \frac{1}{2}(\Lambda(M, N) + \Lambda(N, M)).$$

Note that $\Lambda(M, N) \equiv \Lambda^\infty(M, N) \mod 2$.

**Proposition 2.11** ([27, Lemma 3.7, 3.8 and Corollary 3.23]). Let $M, N$ be simple modules in $\mathcal{C}_g$.

(i) $\Lambda^\infty(M, N) = -\text{Deg}^\infty(a_{M,N}(z)).$

(ii) $\Lambda^\infty(M, N) = \Lambda^\infty(N, M)$.

(iii) $\Lambda^\infty(M, N) = -\Lambda^\infty(\mathcal{D} M, N) = -\Lambda^\infty(M, \mathcal{D} N)$.

(iv) In particular, $\Lambda^\infty(M, N) = \Lambda^\infty(\mathcal{D} M, \mathcal{D} N)$. 

Proposition 2.12 ([27, Lemma 3.7 and Proposition 3.18]). Let $M, N$ be simple modules in $\mathcal{C}_g$.

1. $\Lambda(M, N) = \Lambda(N, \mathcal{D}M) = \Lambda(\mathcal{D}^{-1}N, M)$.
2. In particular,
   \[ \Lambda(M, N) = \Lambda(\mathcal{D}M, \mathcal{D}N) = \Lambda(\mathcal{D}^{-1}M, \mathcal{D}^{-1}N). \]

Proposition 2.13 ([27, Proposition 3.11]). Let $M, N$ and $L$ be non-zero modules in $\mathcal{C}_g$, and let $S$ be a non-zero subquotient of $M \otimes N$.

1. Assume that $R_{M,L}^{\text{univ}}z$ and $R_{N,L}^{\text{univ}}z$ are rationally renormalizable. Then $R_{S,L}^{\text{univ}}z$ is rationally renormalizable and
   \[ \Lambda(S, L) \leq \Lambda(M, L) + \Lambda(N, L) \quad \text{and} \quad \Lambda^\infty(S, L) = \Lambda^\infty(M, L) + \Lambda^\infty(N, L). \]
2. Assume that $R_{L,M}^{\text{univ}}z$ and $R_{L,N}^{\text{univ}}z$ are rationally renormalizable. Then $R_{L,S}^{\text{univ}}z$ is rationally renormalizable and
   \[ \Lambda(L, S) \leq \Lambda(L, M) + \Lambda(L, N) \quad \text{and} \quad \Lambda^\infty(L, S) = \Lambda^\infty(L, M) + \Lambda^\infty(L, N). \]

Proposition 2.14 ([27, Proposition 3.16]). Let $M$ and $N$ be simple modules in $\mathcal{C}_g$. Then we have

1. $\Lambda^\infty(M, N) = \sum_{k \in \mathbb{Z}} (-1)^k a_{M,N}(z) d_{N,M}(z^{-1})$,
2. $\Lambda(M, N) = \Lambda(N, M)$.

In particular, we have $\Lambda(M, N) \in \mathbb{Z}_{\geq 0}$.

Corollary 2.15 ([27, Corollary 3.17 and 3.20]). Let $M$ and $N$ be simple modules in $\mathcal{C}_g$.

1. Suppose that one of $M$ and $N$ is real. Then $M$ and $N$ strongly commute if and only if $\Lambda(M, N) = 0$.
2. In particular, if $M$ is real, then $\Lambda(M, M) = 0$.

Proposition 2.16 (i) and (ii) below were proved in [27, Proposition 3.22] and Proposition 2.16 (iii) is new. We add a whole proof of Proposition 2.16 for the reader’s convenience.

Proposition 2.16. For simple modules $M$ and $N$ in $\mathcal{C}_g$, we have the followings:

1. $\Lambda(M, N) = \sum_{k \in \mathbb{Z}} (-1)^{k+\delta(k<0)} b(M, \mathcal{D}^k N) = \sum_{k \in \mathbb{Z}} (-1)^{k+\delta(k>0)} b(\mathcal{D}^k M, N)$,
2. $\Lambda^\infty(M, N) = \sum_{k \in \mathbb{Z}} (-1)^k b(M, \mathcal{D}^k N)$,
3. $\lim_{z \to 1} c_{M, N}(z) = \sum_{k = 0}^{\infty} (-1)^k b(M, \mathcal{D}^k N)$.

Proof. We write $c_{M,N}(z) \equiv \prod_{\alpha \in k^\times} \varphi(az)^{n_\alpha} \mod k[z^{\pm 1}]^\times$. Then we have

\[ \frac{c_{M,N}(z)}{c_{M,N}(\tilde{z})} \equiv \prod_{\alpha \in k^\times} (1 - az)^{n_\alpha}, \]
which yields that
\[ \eta_{\tilde{p}^{-k}} = \text{zero}_{z = \tilde{p}^k} \left( \frac{c_{M,N}(z)}{c_{M,N} \left( \frac{\tilde{p}^k z}{\tilde{p} z} \right)} \right) = \text{zero}_{z = 1} \left( \frac{c_{M,N} \left( \tilde{p}^{-k} z \right)}{c_{M,N} \left( \frac{\tilde{p}^{-k} z}{\tilde{p}^{-k+1} z} \right)} \right) = \text{zero}_{z = 1} \left( \frac{c_{M,N} \left( \tilde{p}^{-k} z \right)}{c_{M,N} \left( \tilde{p}^k z \right)} \right) \]
\[ = \text{zero}_{z = 1} \left( \frac{d_{M,N\tilde{p}^k}(z) d_{N\tilde{p}^k,M}(z^{-1})}{d_{M,N\tilde{p}^k}(z) d_{N\tilde{p}^k,M}(z^{-1})} \right) = \mathfrak{b}(M, N_{\tilde{p}^k}) - \mathfrak{b}(\mathcal{O}^{-1} M, N_{\tilde{p}^k}) \]
\[ = \mathfrak{b}(M, \mathcal{O}^{-2k} N) - \mathfrak{b}(M, \mathcal{O}^{-2k-1} N), \]
where (*) follows from [27, Lemma 3.15] and (**) from Proposition 2.14. Therefore, we have
\[ \Lambda(M, N) = \sum_{k \in \mathbb{Z}} (-1)^{\delta(k > 0)} \eta_{\tilde{p}^k} = \sum_{k \in \mathbb{Z}} (-1)^{\delta(k < 0)} \eta_{\tilde{p}^{-k}} \]
\[ = \sum_{k \in \mathbb{Z}} (-1)^{\delta(k < 0)} \left( \mathfrak{b}(M, \mathcal{O}^{-2k} N) - \mathfrak{b}(M, \mathcal{O}^{-2k-1} N) \right) \]
\[ = \sum_{k \in \mathbb{Z}} (-1)^{k + \delta(k < 0)} \mathfrak{b}(M, \mathcal{O}^k N), \]
which imply the first assertion (i). Similarly, we obtain the second assertion (ii) as follows:
\[ \Lambda^\infty(M, N) = \sum_{k \in \mathbb{Z}} \eta_{\tilde{p}^k} = \sum_{k \in \mathbb{Z}} \left( \mathfrak{b}(M, \mathcal{O}^{-2k} N) - \mathfrak{b}(M, \mathcal{O}^{-2k-1} N) \right) \]
\[ = \sum_{k \in \mathbb{Z}} (-1)^k \mathfrak{b}(M, \mathcal{O}^k N). \]
Finally we obtain
\[ \text{zero}_{z = 1} c_{M,N}(z) = \sum_{k = 0}^{\infty} \eta_{\tilde{p}^{-k}} = \sum_{k = 0}^{\infty} \left( \mathfrak{b}(M, \mathcal{O}^{2k} N) - \mathfrak{b}(M, \mathcal{O}^{2k+1} N) \right) \]
\[ = \sum_{k = 0}^{\infty} (-1)^k \mathfrak{b}(M, \mathcal{O}^k N), \]
which gives the third assertion (iii). \( \square \)

**Proposition 2.17** ([27, Corollary 4.12]). Let \( L \) be a real simple module, and \( M \) a simple module. Assume that \( \mathfrak{b}(L, M) > 0 \). Then we have
\[ \mathfrak{b}(L, S) < \mathfrak{b}(L, M) \]
for any simple subquotient \( S \) of \( L \otimes M \) and also for any simple subquotient \( S \) of \( M \otimes L \).

The assumption in the following definition is slightly weaker than the one in [27, Definition 4.14]. Under this weak assumption, the same statements as in [27, Lemma 4.15 – 4.18] can be proved in the same manner.
Definition 2.18. Let $L_1, L_2, \ldots, L_r$ be simple modules such that they are real except for at most one. The sequence $(L_1, \ldots, L_r)$ is called a normal sequence if the composition of the $R$-matrices

$$r_{L_1, \ldots, L_r} := \prod_{1 \leq i < j \leq r} r_{L_i, L_j}$$

$$= (r_{L_{r-1}, L_r}) \circ \cdots \circ (r_{L_2, L_r} \circ \cdots \circ r_{L_2, L_3}) \circ (r_{L_1, L_r} \circ \cdots \circ r_{L_1, L_2})$$

$$: L_1 \otimes L_2 \otimes \cdots \otimes L_r \rightarrow L_r \otimes \cdots \otimes L_2 \otimes L_1.$$ does not vanishes.

Lemma 2.19 ([27, Lemma 4.15]). Let $(L_1, \ldots, L_r)$ be a normal sequence of simple modules such that they are real except for at most one. Then $\text{Im}(r_{L_1, \ldots, L_r})$ is simple and it coincides with the head of $L_1 \otimes \cdots \otimes L_r$ and also with the socle of $L_r \otimes \cdots \otimes L_1$.

Lemma 2.20 ([27, Lemma 4.16]). Let $L_1, L_2, \ldots, L_r$ be simple modules such that they are real except for at most one.

(i) If $(L_1, \ldots, L_r)$ is normal, then we have
(a) $(L_2, \ldots, L_r)$ and $(L_1, \ldots, L_{r-1})$ are normal.
(b) $\Lambda(L_1, \text{hd}(L_2 \otimes \cdots \otimes L_r)) = \sum_{j=2}^{r} \Lambda(L_1, L_j)$, and
$\Lambda(\text{hd}(L_1 \otimes \cdots \otimes L_{r-1}), L_r) = \sum_{j=1}^{r-1} \Lambda(L_j, L_r)$.
(ii) Assume that $L_1$ is real, $(L_2, \ldots, L_r)$ is normal and

$$\Lambda(L_1, \text{hd}(L_2 \otimes \cdots \otimes L_r)) = \sum_{j=2}^{r} \Lambda(L_1, L_j),$$

then $(L_1, \ldots, L_r)$ is normal.
(iii) Assume that $L_r$ is real, $(L_1, \ldots, L_{r-1})$ is normal and

$$\Lambda(\text{hd}(L_1 \otimes \cdots \otimes L_{r-1}), L_r) = \sum_{j=1}^{r-1} \Lambda(L_j, L_r),$$

then $(L_1, \ldots, L_r)$ is normal.

Lemma 2.21 ([27, Lemma 4.3 and Lemma 4.17]). Let $L, M, N$ be simple modules such that they are real except for at most one. If one of the following conditions
(a) $\mathfrak{b}(L, M) = 0$ and $L$ is real,
(b) $\mathfrak{b}(M, N) = 0$ and $N$ is real,
(c) $\mathfrak{b}(L, \mathcal{D}^{-1}N) = \mathfrak{b}(\mathcal{D}L, N) = 0$ and $L$ or $N$ is real
holds, then $(L, M, N)$ is a normal sequence, i.e.,

$$\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N), \quad \Lambda(L \nabla M, N) = \Lambda(L, N) + \Lambda(M, N).$$
Lemma 2.22 ([27, Corollary 4.18]). Let $L, M, N$ be simple modules. Assume that $L$ is real and one of $M$ and $N$ is real. Then $(L, M, N)$ is a normal sequence if and only if $(M, N, \mathcal{D}L)$ is a normal sequence.

In [27, Corollary 4.18], we proved it in a stronger condition, but the same proof still works without change.

Lemma 2.23. Let $L_1, L_2, \ldots, L_r$ be simple modules such that they are real except for at most one. Suppose that the sequence $(L_1, \ldots, L_r)$ is normal. For any $m \in \mathbb{Z}$, we have

$$D^m(\text{hd}(L_1 \otimes \cdots \otimes L_r)) \simeq \text{hd}(D^mL_1 \otimes D^mL_2 \otimes \cdots \otimes D^mL_r).$$

Proof. It suffices to prove the case for $m = \pm 1$. We assume that $m = 1$. Since the sequence $(D^1L_1, \ldots, D^1L_r)$ is normal, by Lemma 2.19, we have

$$\mathcal{D}(\text{hd}(L_1 \otimes \cdots \otimes L_r)) \simeq \text{soc}(DL_1 \otimes \cdots \otimes DL_1) \simeq \text{hd}(DL_1 \otimes \cdots \otimes DL_r).$$

The case for $m = -1$ can be proved in the same manner as above. □

Lemma 2.24. Let $L, M, N$ be simple modules. Assume that $L$ is real and one of $M$ and $N$ is real. Then $d(L, M \nabla N) = d(L, M) + d(L, N)$ if and only if $(L, M, N)$ and $(M, N, L)$ are normal sequences.

Proof. By the assumption, we have

$$2(b(L, M) + b(L, N) - b(L, M \nabla N)) = ((\Lambda(L, M) + \Lambda(L, N) - \Lambda(L, M \nabla N))$$

$$+ (\Lambda(M, L) + \Lambda(N, L) - \Lambda(M \nabla N, L))).$$

Since $\Lambda(L, M) + \Lambda(N, N) - \Lambda(L, M \nabla N)$ and $\Lambda(M, L) + \Lambda(N, L) - \Lambda(M \nabla N, L)$ are non-negative by Proposition 2.13, we have

$$\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N) \quad \text{and} \quad \Lambda(M \nabla N, L) = \Lambda(M, L) + \Lambda(N, N),$$

if and only if $b(L, M \nabla N) = b(L, M) + b(L, N)$. Then the assertion follows from Lemma 2.20. □

Corollary 2.25. Let $L$ and $M$ be real simple modules and $X$ a simple module.

(i) If $b(L, M) = b(D^{-1}L, M) = 0$, then we have $b(L, M \nabla X) = b(L, X)$.

(ii) If $b(L, M) = b(DL, M) = 0$, then we have $b(L, X \nabla M) = b(L, X)$.

Proof. (i) The triples $(L, M, X)$ and $(M, X, L)$ are normal by Lemma 2.21, and hence we have $b(L, M \nabla X) = b(L, M) + b(L, X) = b(L, X)$ by the preceding lemma.

(ii) can be proved similarly. □

The following lemma can be proved similarly to [24, Proposition 3.2.17], and we do not repeat the proof here.
Lemma 2.26. Let $M$ and $N$ be simple modules, and assume that one of them is real. If $d(M, N) = 1$, then $M \otimes N$ has length 2 and we have an exact sequence

$$0 \to N \nabla M \to M \otimes N \to M \nabla N \to 0.$$ 

The following lemma gives a criterion for a simple module to be real.

Lemma 2.27. Let $X$ be a simple module such that $d(X, X) = 0$ and $X \otimes X$ has a simple head. Then $X$ is real.

Proof. Since $d(X, X) = 0$, we have $R_{X, X}^{\text{ren}} \circ R_{X, X}^{\text{ren}} = f(z) \text{id}$ for some $f(z) \in k(z)$ which is invertible at $z = 1$. Thus we have $r_{X, X}^2 \in k^x \text{id}$. By normalizing, we may assume that $r_{X, X}^2 = \text{id}$. Then we have

$$X \otimes X = \text{Ker}(r_{X, X} - \text{id}) \oplus \text{Ker}(r_{X, X} + \text{id}).$$

Since $X \otimes X$ has a simple head, we conclude that $r_{X, X}$ should be $\pm \text{id}$, which implies the assertion by [22, Corollary 3.3 and Theorem 3.12].

□

Lemma 2.28. Let $M, N$ be real simple modules such that $d(M, N) = 1$. Then $M \nabla N$ is real.

Proof. It follows from Proposition 2.17 that

$$d(M, M \nabla N) < d(M, N) = 1, \quad d(N, M \nabla N) < d(M, N) = 1,$$

which implies that $d(M, M \nabla N) = d(N, M \nabla N) = 0$. We set $X := M \nabla N$. Since $d(M, X) = d(N, X) = 0$, we have $0 \leq d(X, X) \leq d(M, X) + d(N, X) = 0$, i.e.,

$$d(X, X) = 0.$$

Since $N$ is real and $X \otimes M$ is simple, $(X \otimes M) \otimes N$ has a simple head. Thus the surjection

$$(X \otimes M) \otimes N \to X \otimes X$$

tells us that $X \otimes X$ has a simple head. Thus the assertion follows from Lemma 2.27. □

Lemma 2.29. Let $M$ and $N$ be real simple modules such that $d(M, N) = 1$. Then we have

(i) $M \nabla N$ commutes with $M$ and $N$,

(ii) for any $m, n \in \mathbb{Z}_{\geq 0}$, we have

$$M^{\otimes m} \nabla N^{\otimes n} \cong \begin{cases} (M \nabla N)^{\otimes m} \otimes N^{\otimes (n-m)} & \text{if } m \leq n, \\
(M \nabla N)^{\otimes n} \otimes M^{\otimes (m-n)} & \text{if } m \geq n. \end{cases}$$

Proof. (i) It follows from $d(M, M \nabla N) \leq d(M, N) - 1 = 0$ and $d(N, M \nabla N) \leq d(N, M) - 1 = 0$. 

(ii) We shall prove only the first isomorphism. We shall argue by induction on \( m \leq n \). If \( m = 0 \) it is obvious. Assume that \( m > 0 \). Then we have
\[
M^{\otimes m} \otimes N^{\otimes n} \rightarrow M^{\otimes (m-1)} \otimes (M \nabla N) \otimes N^{\otimes (n-1)} \simeq (M \nabla N) \otimes M^{\otimes (m-1)} \otimes N^{\otimes (n-1)}
\]
\[
\rightarrow (M \nabla N) \otimes ((M \nabla N)^{\otimes (m-1)} \otimes N^{\otimes (n-m)}) \simeq (M \nabla N)^{\otimes m} \otimes N^{\otimes (n-m)}.
\]
Then the assertion follows from the fact that \((M \nabla N)^{\otimes m} \otimes N^{\otimes (n-m)}\) is a simple quotient of \(M^{\otimes m} \otimes N^{\otimes n}\) which has a simple head. \(\square\)

**Lemma 2.30.** Let \( M \) and \( N \) be real simple modules such that \( \mathfrak{v}(M, N) = 1 \). Then for any simple module \( X \), we have

- (i) The simple module \( M \nabla (N \nabla X) \) is isomorphic to either \((M \nabla N) \nabla X\) or \((N \nabla M) \nabla X\).
- (ii) The simple module \((X \nabla M) \nabla N\) is isomorphic to either \(X \nabla (M \nabla N)\) or \(X \nabla (N \nabla M)\).

**Proof.** Since the proof is similar, we prove only (i). Let us consider a commutative diagram with an exact row:

\[
\begin{array}{cccccc}
0 & \rightarrow & (N \nabla M) \otimes X & \rightarrow & M \otimes N \otimes X & \rightarrow & (M \nabla N) \otimes X & \rightarrow & 0 \\
& & f \downarrow & & \downarrow & & \downarrow & & \\
& & \rightarrow & & M \nabla (N \nabla X) & & \\
\end{array}
\]

The exactness follows from Lemma 2.26. By Lemma 2.28, \( M \nabla N \) and \( N \nabla M \) are real simple modules. If \( f \) does not vanish, then we have \((N \nabla M) \nabla X \simeq M \nabla (N \nabla X)\).

If \( f \) vanishes then there exists an epimorphism \((M \nabla N) \otimes X \rightarrow M \nabla (N \nabla X)\) and hence \((M \nabla N) \nabla X \simeq M \nabla (N \nabla X)\). \(\square\)

## 3. Root modules

In this section, we investigate properties of root modules.

**Definition 3.1.** A module \( L \in \mathcal{C}_g \) is called a root module if \( L \) is a real simple module such that

\[
\mathfrak{v}(L, \mathcal{D}^kL) = \delta(k = \pm 1) \quad \text{for any } k \in \mathbb{Z}.
\]

Note that for a root module \( L \), we have

\[
\Lambda^\infty(L, L) = -2
\]

by Proposition 2.16.

The name “root module” comes from Lemma 4.15 below.

**Example 3.2.** Using the denominators for fundamental modules (see [29, Appendix A] for example), one can easily prove that any fundamental module \( V(\varpi_i)_a \ (i \in I_0, a \in k^\times) \) is a root module.
3.1. Properties of root modules.

Lemma 3.3. Let $L$ be a root module and let $X$ be a simple module.

(i) For $k \in \mathbb{Z}$, we have

$$\mathfrak{b}(\mathcal{D}^k L, X) - \delta(k = 0, 2) \leq \mathfrak{b}(\mathcal{D}^k L, L \nabla X) \leq \mathfrak{b}(\mathcal{D}^k L, X) + \delta(k = \pm 1).$$

In particular, $\mathfrak{b}(\mathcal{D}^k L, L \nabla X) = \mathfrak{b}(\mathcal{D}^k L, X)$ for $k \neq -1, 0, 1, 2$, and

$$\mathfrak{b}(\mathcal{D}^k L, L \nabla X) - \mathfrak{b}(\mathcal{D}^k L, X) \in \begin{cases} \{0, 1\} & \text{for } k = \pm 1, \\ \{0, -1\} & \text{for } k = 0, 2. \end{cases}$$

Moreover, we have

$$(\mathfrak{b}(\mathcal{D}^{-1} L, L \nabla X) - \mathfrak{b}(\mathcal{D}^{-1} L, X)) + (\mathfrak{b}(L, X) - \mathfrak{b}(L, L \nabla X)) + (\mathfrak{b}(\mathcal{D} L, L \nabla X) - \mathfrak{b}(\mathcal{D} L, X)) + (\mathfrak{b}(\mathcal{D}^2 L, X) - \mathfrak{b}(\mathcal{D}^2 L, L \nabla X)) = 2.$$

(ii) For $k \in \mathbb{Z}$, we have

$$\mathfrak{b}(\mathcal{D}^k L, X) - \delta(k = -2, 0) \leq \mathfrak{b}(\mathcal{D}^k L, X \nabla L) \leq \mathfrak{b}(\mathcal{D}^k L, X) + \delta(k = \pm 1).$$

In particular, $\mathfrak{b}(\mathcal{D}^k L, X \nabla L) = \mathfrak{b}(\mathcal{D}^k L, X)$ for $k \neq -2, -1, 0, 1$, and

$$\mathfrak{b}(\mathcal{D}^k L, X \nabla L) - \mathfrak{b}(\mathcal{D}^k L, X) \in \begin{cases} \{0, 1\} & \text{for } k = \pm 1, \\ \{0, -1\} & \text{for } k = 0, -2. \end{cases}$$

Moreover, we have

$$(\mathfrak{b}(\mathcal{D}^{-2} L, X) - \mathfrak{b}(\mathcal{D}^{-2} L, X \nabla L)) + (\mathfrak{b}(\mathcal{D}^{-1} L, X \nabla L) - \mathfrak{b}(\mathcal{D}^{-1} L, X)) + (\mathfrak{b}(L, X) - \mathfrak{b}(L, X \nabla L)) + (\mathfrak{b}(\mathcal{D} L, X \nabla L) - \mathfrak{b}(\mathcal{D} L, X)) = 2.$$

Proof. (i) By [27, Proposition 4.2], we have

$$\mathfrak{b}(\mathcal{D}^k L, L \nabla X) \leq \mathfrak{b}(\mathcal{D}^k L, X) + \mathfrak{b}(\mathcal{D}^k L, L) = \mathfrak{b}(\mathcal{D}^k L, X) + \delta(k = \pm 1).$$

By the same reason, it follows from $X \simeq (L \nabla X) \nabla \mathcal{D} L$ (see Lemma 2.7) that

$$\mathfrak{b}(\mathcal{D}^k L, X) = \mathfrak{b}(\mathcal{D}^k L, (L \nabla X) \nabla \mathcal{D} L) \leq \mathfrak{b}(\mathcal{D}^k L, L \nabla X) + \mathfrak{b}(\mathcal{D}^k L, \mathcal{D} L) = \mathfrak{b}(\mathcal{D}^k L, L \nabla X) + \delta(k = 0, 2).$$

Hence we obtain the first assertion. Since

$$\Lambda^\infty(L, L \nabla X) = \Lambda^\infty(L, L) + \Lambda^\infty(L, X) = -2 + \Lambda^\infty(L, X),$$

it follows from (3.2) and Proposition 2.16 that

$$2 = \Lambda^\infty(L, L \nabla X) = \sum_{k \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k (\mathfrak{b}(\mathcal{D}^k L, X) - \mathfrak{b}(\mathcal{D}^k L, L \nabla X)).$$
\[
= (\mathfrak{b}(\mathcal{D}^{-1}L, L \nabla X) - \mathfrak{b}(\mathcal{D}^{-1}L, X)) + (\mathfrak{b}(L, X) - \mathfrak{b}(L, L \nabla X)) \\
+ (\mathfrak{b}(\mathcal{D}L, L \nabla X) - \mathfrak{b}(\mathcal{D}L, X)) + (\mathfrak{b}(\mathcal{D}^2L, X) - \mathfrak{b}(\mathcal{D}^2L, L \nabla X))\]

which yields the last assertion.

(ii) Using the fact that \(X \simeq (\mathcal{D}^{-1}L) \nabla (X \nabla L)\), it can be proved in the same manner as above. \(\square\)

**Lemma 3.4.** Let \(L\) be a root module and let \(X\) be a simple module. Suppose that \(\mathfrak{b}(L, X) > 0\).

Then we have the following:

(i) \(\mathfrak{b}(L, L \nabla X) = \mathfrak{b}(L, X) - 1\) and \(\mathfrak{b}(\mathcal{D}^{-1}L, L \nabla X) = \mathfrak{b}(\mathcal{D}^{-1}L, X)\),

(ii) \(\mathfrak{b}(L, X \nabla L) = \mathfrak{b}(L, X) - 1\) and \(\mathfrak{b}(\mathcal{D}L, X \nabla L) = \mathfrak{b}(\mathcal{D}L, X)\).

**Proof.** We shall prove only (i) since the proof of (ii) is similar.

Since \(\mathfrak{b}(L, X) > 0\), \(L\) does not commute with \(X\). By Proposition 2.17, we have

\[\mathfrak{b}(L, L \nabla X) < \mathfrak{b}(L, X)\].

On the other hand, Lemma 3.3 implies \(\mathfrak{b}(L, X) \leq \mathfrak{b}(L, L \nabla X) + 1\) which implies the first assertion.

Let us show the second equation in (i). By Lemma 3.3, we have \(\mathfrak{b}(\mathcal{D}^{-1}L, L \nabla X) = \mathfrak{b}(\mathcal{D}^{-1}L, X)\) or \(\mathfrak{b}(\mathcal{D}^{-1}L, X) + 1\). If

\[\mathfrak{b}(\mathcal{D}^{-1}L, L \nabla X) = \mathfrak{b}(\mathcal{D}^{-1}L, X) + 1\],

then Lemma 2.24 says that \((\mathcal{D}^{-1}L, L, X)\) is a normal sequence, and hence \((L, X, L)\) is also a normal sequence by Lemma 2.22, which implies that

\(L \nabla X \simeq X \nabla L\).

But it contradicts \(\mathfrak{b}(L, X) > 0\). Therefore we conclude that \(\mathfrak{b}(\mathcal{D}^{-1}L, L \nabla X) = \mathfrak{b}(\mathcal{D}^{-1}L, X)\). \(\square\)

**Lemma 3.5.** Let \(L\) be a root module and \(X\) a simple module.

(i) Assume one of the following conditions:
   (a) \(\mathfrak{b}(\mathcal{D}L, L \nabla X) > 0\),
   (b) \(\mathfrak{b}(\mathcal{D}L, X) > 0\),
   (c) \(\mathfrak{b}(\mathcal{D}^2L, X) = 0\).

Then we have

\[\mathfrak{b}(\mathcal{D}L, L \nabla X) = \mathfrak{b}(\mathcal{D}L, X) + 1\].

(ii) Assume one of the following conditions:
   (a) \(\mathfrak{b}(\mathcal{D}^{-1}L, X \nabla L) > 0\),
   (b) \(\mathfrak{b}(\mathcal{D}^{-1}L, X) > 0\),
   (c) \(\mathfrak{b}(\mathcal{D}^{-2}L, X) = 0\).
Then we have
\[ \mathfrak{b}(\mathcal{D}^{-1}L, X \nabla L) = \mathfrak{b}(\mathcal{D}^{-1}L, X) + 1. \]

**Proof.** We shall only prove (i) since the proof of (ii) is similar.

(a) Assume first that \( \mathfrak{b}(\mathcal{D}L, L \nabla X) > 0 \). Setting \( Y = L \nabla X \), we have \( X \simeq Y \nabla \mathcal{D}L \).

Hence Lemma 3.4 (ii) implies
\[ \mathfrak{b}(\mathcal{D}L, Y \nabla \mathcal{D}L) = \mathfrak{b}(\mathcal{D}L, Y) - 1. \]

(b) If \( \mathfrak{b}(\mathcal{D}L, X) > 0 \), then we have \( \mathfrak{b}(\mathcal{D}L, L \nabla X) \geq \mathfrak{b}(\mathcal{D}L, X) > 0 \) by Lemma 3.3.

(c) Finally, assume that \( \mathfrak{b}(\mathcal{D}^2L, X) = 0 \). If \( \mathfrak{b}(L, X) = 0 \), then we have
\[ \mathfrak{b}(\mathcal{D}L, L \nabla X) = \mathfrak{b}(\mathcal{D}L, X) + 1. \]

Suppose that \( \mathfrak{b}(L, X) > 0 \). Since \( \mathfrak{b}(\mathcal{D}^2L, X) = 0 \) and \( \mathfrak{b}(\mathcal{D}^2L, L \nabla X) = 0 \), we have
\[ \mathfrak{b}(\mathcal{D}^2L, L \nabla X) = 0. \]

Moreover, Lemma 3.4 tells us that
\[ \mathfrak{b}(\mathcal{D}L, L \nabla X) = \mathfrak{b}(\mathcal{D}L, L) + \mathfrak{b}(\mathcal{D}L, X) = \mathfrak{b}(\mathcal{D}L, X) + 1. \]

(3.3) \[ \mathfrak{b}(\mathcal{D}^2L, L \nabla X) = 0. \]

which yields the desired result. \( \Box \)

**Lemma 3.6.** Let \( L \) be a root module, \( X \) a simple module, and \( k \in \mathbb{Z}_{\geq 0} \).

(i) Suppose that one of the following conditions
\[(a) \quad \mathfrak{b}(\mathcal{D}L, L \oslash^k \nabla X) \geq k,
(b) \quad \mathfrak{b}(\mathcal{D}^2L, X) = 0
\]
is true. Then we have
\[ \mathfrak{b}(\mathcal{D}L, L \oslash^k \nabla X) = \mathfrak{b}(\mathcal{D}L, X) + k. \]

(ii) Suppose that one of the following
\[(a) \quad \mathfrak{b}(\mathcal{D}^{-1}L, X \nabla L \oslash^k) \geq k,
(b) \quad \mathfrak{b}(\mathcal{D}^{-2}L, X) = 0
\]
is true. Then we have
\[ \mathfrak{b}(\mathcal{D}^{-1}L, X \nabla L \oslash^k) = \mathfrak{b}(\mathcal{D}^{-1}L, X) + k. \]

**Proof.** This follows from the preceding lemma by induction on \( k \). Note that \( \mathfrak{b}(\mathcal{D}^2L, L \oslash^k \nabla X) = 0 \) as soon as \( \mathfrak{b}(\mathcal{D}^2L, X) = 0 \). \( \Box \)

**Proposition 3.7.** Let \( L \) be a root module and \( X \) a simple module. If \( \mathfrak{b}(L, X) \neq 1 \), then \( (L \nabla X) \nabla L \) is isomorphic to \( L \nabla (X \nabla L) \).
Proof. If \( \mathfrak{d}(L, X) = 0 \), then it is obvious. Hence we may assume that \( \mathfrak{d}(L, X) \geq 2 \).

Set \( Y := (L \nabla X) \nabla L \). Then, we have

\[
L \nabla X \simeq \mathcal{D}^{-1} L \nabla Y \quad \text{and} \quad X \simeq (\mathcal{D}^{-1} L \nabla Y) \nabla \mathcal{D} L.
\]

Lemma 3.4 says that \( \mathfrak{d}(L, L \nabla X) = \mathfrak{d}(L, X) - 1 > 0 \) and \( \mathfrak{d}(L, Y) = \mathfrak{d}(L, L \nabla X) - 1 \). Hence we obtain

\[
\mathfrak{d}(L, \mathcal{D}^{-1} L \nabla Y) = \mathfrak{d}(L, Y) + 1 = \mathfrak{d}(L, \mathcal{D}^{-1} L) + \mathfrak{d}(L, Y).
\]

Then Lemma 2.24 implies that \( (L, \mathcal{D}^{-1} L, Y) \) is a normal sequence, and \( (\mathcal{D}^{-1} L, Y, \mathcal{D} L) \) is also a normal sequence by Lemma 2.22. Hence we have

\[
(\mathcal{D}^{-1} L \nabla Y) \nabla \mathcal{D} L \simeq \mathcal{D}^{-1} L \nabla (Y \nabla \mathcal{D} L).
\]

Since \( X \simeq \mathcal{D}^{-1} L \nabla (Y \nabla \mathcal{D} L) \), we obtain \( Y \simeq L \nabla (X \nabla L) \). \( \square \)

3.2. Properties of pairs of root modules.

Let \( L \) and \( L' \) be root modules. Throughout this subsection, we assume that

\[
(3.5) \quad \mathfrak{d}(\mathcal{D}^k L, L') = \delta(k = 0) \quad \text{for } k \in \mathbb{Z}.
\]

Note that, by Proposition 2.16,

- \( \Lambda(L, L') = \Lambda^\infty(L, L') = 1 \).

Lemma 3.8. The simple module \( L \nabla L' \) is a root module.

Proof. Set \( L'' := L \nabla L' \). By Lemma 2.28, \( L'' \) is real.

It is obvious that \( \mathfrak{d}(\mathcal{D}^k L, L'') = \mathfrak{d}(\mathcal{D}^k L', L'') = 0 \) for \( k \neq 0, \pm 1 \). On the other hand, Lemma 3.4 implies that \( \mathfrak{d}(L, L'') = \mathfrak{d}(L', L'') = 0 \). Hence, we have \( \mathfrak{d}(\mathcal{D}^k L'', L'') = 0 \) for \( k \neq \pm 1 \). Now, we have

\[
\Lambda^\infty(L'', L'') = \Lambda^\infty(L, L) + 2\Lambda^\infty(L, L') + \Lambda^\infty(L', L') = (-2) + 2 + (-2) = -2.
\]

Then Proposition 2.16 implies that

\[
-2 = \sum_{k \in \mathbb{Z}} (-1)^k \mathfrak{d}(\mathcal{D}^k L'', L'') = -\mathfrak{d}(\mathcal{D} L'', L'') - \mathfrak{d}(\mathcal{D}^{-1} L'', L'').
\]

Since \( \mathfrak{d}(\mathcal{D} L'', L'') = \mathfrak{d}(\mathcal{D}^{-1} L'', L'') \), we obtain \( \mathfrak{d}(\mathcal{D}^{\pm 1} L'', L'') = 1 \). \( \square \)

Lemma 3.9. We have

\[
\mathfrak{d}(\mathcal{D}^k L, L \nabla L') = \delta(k = 1) \quad \text{and} \quad \mathfrak{d}(\mathcal{D}^k L, L' \nabla L) = \delta(k = -1).
\]

Proof. Since \( \mathfrak{d}(\mathcal{D}^k L, L \nabla L') \leq \mathfrak{d}(\mathcal{D}^k L, L) + \mathfrak{d}(\mathcal{D}^k L, L') \), we have \( \mathfrak{d}(\mathcal{D}^k L, L \nabla L') = 0 \) for \( k \neq -1, 0, 1 \). It follows from Lemma 3.4 that

\[
\mathfrak{d}(L, L \nabla L') = \mathfrak{d}(L, L') - 1 = 0,
\]

\[
\mathfrak{d}(\mathcal{D}^{-1} L, L \nabla L') = \mathfrak{d}(\mathcal{D}^{-1} L, L') = 0.
\]
On the other hand, since
\[ \vartheta(\mathcal{D}L, L \triangledown L') \leq \vartheta(\mathcal{D}L, L) + \vartheta(\mathcal{D}L, L') = 1, \]
we have \( \vartheta(\mathcal{D}L, L \triangledown L') \in \{0, 1\} \).

If \( \vartheta(\mathcal{D}L, L \triangledown L') = 0 \), then we have
\[ L' \simeq (L \triangledown L') \triangledown \mathcal{D}L \simeq (L \triangledown L') \otimes \mathcal{D}L, \]
which implies that
\[ 0 = \vartheta(\mathcal{D}^2L, L') = \vartheta(\mathcal{D}^2L, (L \triangledown L') \otimes \mathcal{D}L) = \vartheta(\mathcal{D}^2L, L \triangledown L') + \vartheta(\mathcal{D}^2L, \mathcal{D}L) \geq 1. \]
This is a contradiction. Hence, we conclude that \( \vartheta(\mathcal{D}L, L \triangledown L') = 1 \). Thus we obtained the first equality.

The second equality can be proved similarly. \( \square \)

**Lemma 3.10.** Let \( X \) be a simple module.

(i) If \( k \neq 0, 1 \), then
\[ \vartheta(\mathcal{D}^kL, L' \triangledown X) = \vartheta(\mathcal{D}^kL, X). \]
As for \( k = 0 \) and \( 1 \), one and only one of the following two statements is true.

(a) \( \vartheta(L, L' \triangledown X) = \vartheta(L, X) \) and \( \vartheta(\mathcal{D}L, L' \triangledown X) = \vartheta(\mathcal{D}L, X) - 1 \),
(b) \( \vartheta(L, L' \triangledown X) = \vartheta(L, X) + 1 \) and \( \vartheta(\mathcal{D}L, L' \triangledown X) = \vartheta(\mathcal{D}L, X) \).

(ii) If \( k \neq -1, 0 \), then
\[ \vartheta(\mathcal{D}^kL, X \triangledown L') = \vartheta(\mathcal{D}^kL, X). \]
As for \( k = -1 \) and \( 0 \), one and only one of the following two statements is true.

(a) \( \vartheta(L, X \triangledown L') = \vartheta(L, X) \) and \( \vartheta(\mathcal{D}^{-1}L, X \triangledown L') = \vartheta(\mathcal{D}^{-1}L, X) - 1 \),
(b) \( \vartheta(L, X \triangledown L') = \vartheta(L, X) + 1 \) and \( \vartheta(\mathcal{D}^{-1}L, X \triangledown L') = \vartheta(\mathcal{D}^{-1}L, X) \).

**Proof.** (i) By [27, Proposition 4.2], we have
\[ \vartheta(\mathcal{D}^kL, L' \triangledown X) \leq \vartheta(\mathcal{D}^kL, X) + \vartheta(k = 0), \]
\[ \vartheta(\mathcal{D}^kL, X) \leq \vartheta(\mathcal{D}^kL, L' \triangledown X) + \vartheta(k = 1), \]
where the second inequality follows from \( X \simeq (L' \triangledown X) \triangledown \mathcal{D}L' \). The above inequalities give the first assertion and
\[ \vartheta(L, L' \triangledown X) = \vartheta(L, X) \text{ or } \vartheta(L, X) + 1, \]
\[ \vartheta(\mathcal{D}L, L' \triangledown X) = \vartheta(\mathcal{D}L, X) \text{ or } \vartheta(\mathcal{D}L, X) - 1. \]

By the assumption (3.5), we have \( \Lambda^\infty(L, L') = 1 \), which implies
\[ 1 = \Lambda^\infty(L, L' \triangledown X) - \Lambda^\infty(L, X) = \sum_{k \in \mathbb{Z}} (-1)^k \left( \vartheta(\mathcal{D}^kL, L' \triangledown X) - \vartheta(\mathcal{D}^kL, X) \right) \]
\[ = \left( \vartheta(L, L' \triangledown X) - \vartheta(L, X) \right) + \left( \vartheta(\mathcal{D}L, X) - \vartheta(\mathcal{D}L, L' \triangledown X) \right). \]
Then (3.6) implies the second assertion.

(ii) can be similarly proved by using \( X \simeq \mathcal{D}^{-1}L' \triangledown (X \triangledown L') \). \( \square \)
Proposition 3.11. Let $X$ be a simple module.

(i) If $\mathfrak{b}(\mathcal{D}L, X) = 0$, then we have
$$\mathfrak{b}(L, L' \nabla X) = \mathfrak{b}(L, X) + 1, \quad \mathfrak{b}(\mathcal{D}L, L' \nabla X) = 0.$$  

(ii) If $\mathfrak{b}(\mathcal{D}^{-1}L, X) = 0$, then we have
$$\mathfrak{b}(L, X \nabla L') = \mathfrak{b}(L, X) + 1, \quad \mathfrak{b}(\mathcal{D}^{-1}L, X \nabla L') = 0.$$  

Proof. (i) Since $\mathfrak{b}(\mathcal{D}L, L' \nabla X) \leq \mathfrak{b}(\mathcal{D}L, L') + \mathfrak{b}(\mathcal{D}L, X) = 0$, Lemma 3.10 tells us that $\mathfrak{b}(L, L' \nabla X) = \mathfrak{b}(L, X) + 1$.

(ii) can be proved in the same manner as above. \qed

Corollary 3.12. Let $n \in \mathbb{Z}_{\geq 0}$ and let $X$ be a simple module.

(i) If $\mathfrak{b}(\mathcal{D}L, X) = 0$, then we have
$$\mathfrak{b}(L, L' \otimes^n \nabla X) = \mathfrak{b}(L, X) + n \quad \text{and} \quad \mathfrak{b}(\mathcal{D}L, L' \otimes^n \nabla X) = 0.$$  

(ii) If $\mathfrak{b}(\mathcal{D}^{-1}L, X) = 0$, then we have
$$\mathfrak{b}(L, X \nabla L' \otimes^n) = \mathfrak{b}(L, X) + n \quad \text{and} \quad \mathfrak{b}(\mathcal{D}^{-1}L, X \nabla L' \otimes^n) = 0.$$  

Proof. They easily follow from Proposition 3.11 by induction on $n$. \qed

Proposition 3.13. Let $m \in \mathbb{Z}_{\geq 0}$ and let $Y$ be a simple module. Set
$$X := L^\otimes \nabla Y.$$  

Suppose that
$$\mathfrak{b}(L, X) = 0, \quad \mathfrak{b}(\mathcal{D}L', Y) = 0, \quad \mathfrak{b}(\mathcal{T}L, Y) = 0 \quad \text{for } t = 1, 2.$$  

Then

(i) $\mathfrak{b}(\mathcal{D}L, X) = m$,

(ii) for any integer $k$ such that $0 \leq k \leq m$, we have
$$\mathfrak{b}(L, L'^\otimes k \nabla X) = 0, \quad \mathfrak{b}(\mathcal{D}L, L'^\otimes k \nabla X) = m - k,$$

(iii) for any integer $k \geq m$, we have
$$\mathfrak{b}(L, L'^\otimes k \nabla X) = k - m, \quad \mathfrak{b}(\mathcal{D}L, L'^\otimes k \nabla X) = 0.$$  

Proof. (i) As $\mathfrak{b}(\mathcal{T}L, Y) = 0$ for $t = 1, 2$, we have
$$\mathfrak{b}(\mathcal{D}L, X) = \mathfrak{b}(\mathcal{D}L, L^\otimes \nabla Y) = \mathfrak{b}(\mathcal{D}L, Y) + m = m$$
by Lemma 3.6 (i).

(ii) We have
- $L' \nabla L$ commutes with $L, L'$ and $\mathcal{D}L$ by Lemma 3.9,
- $\mathfrak{b}(\mathcal{D}L', X) = 0$ because $\mathfrak{b}(\mathcal{D}L', Y) = 0$ and $\mathfrak{b}(\mathcal{D}L', L) = 0$,
- the triple $(L'^\otimes a, A, X)$ is normal because $\mathfrak{b}(\mathcal{D}L', X) = 0$,
- the triple $(L'^\otimes a, B, L)$ is normal because $\mathfrak{b}(\mathcal{D}L', L) = 0$. 


the triples \((L'^a \otimes A, A, Y)\), \((L^a \otimes A, A, Y)\) and \(((L' \nabla L)^a \otimes A, A, Y)\) are normal because \(\mathfrak{d}(D' L', Y) = 0\) and \(\mathfrak{d}(D' L, Y) = 0\),

Here \(A\) is a real simple module, \(B\) is a simple module, and \(a\) is an arbitrary non-negative integer. We will use these fact freely in the subsequent arguments.

For any \(k\) such that \(0 \leq k \leq m\), we have

\[
L'^k \nabla X \simeq L'^k \nabla (L \otimes m \nabla Y) \\
\simeq (L'^k \nabla L) \nabla Y \\
\simeq ((L' \nabla L)^k \otimes L^{m-k}) \nabla Y \\
\simeq (L^{m-k} \otimes (L' \nabla L)^k) \nabla Y \\
\simeq L^{m-k} \nabla ((L' \nabla L)^k \nabla Y),
\]

where the third isomorphism follows from Lemma 2.29. Thus, we have, for \(1 \leq k \leq m\),

\[
L \nabla (L'^k \nabla X) \simeq L \nabla ((L^{m-k} \otimes (L' \nabla L)^k) \nabla Y) \\
\simeq (L^{m-k+1} \otimes (L' \nabla L)^k) \nabla Y \\
\simeq ((L' \nabla L) \otimes L^{m-k+1} \otimes (L' \nabla L)^{k-1}) \nabla Y \\
\simeq (L' \nabla L) \nabla ((L^{m-(k-1)} \otimes (L' \nabla L)^{k-1}) \nabla Y) \\
\simeq (L' \nabla L) \nabla (L'^{k-1} \nabla X),
\]

and

\[
(L'^k \nabla X) \nabla L \simeq L'^k \nabla (X \otimes L) \simeq L'^k \nabla (L \otimes X) \\
\simeq (L'^k \nabla L) \nabla X \\
\simeq ((L' \nabla L) \otimes L'^{k-1}) \nabla X \\
\simeq (L' \nabla L) \nabla (L'^{k-1} \nabla X).
\]

This tells us that

\[
L \nabla (L'^k \nabla X) \simeq (L'^k \nabla X) \nabla L,
\]

which implies that \(\mathfrak{d}(L, L'^k \nabla X) = 0\).

On the other hand, since \(\mathfrak{d}(\mathfrak{d}^2 L, Y) = 0\) and \(\mathfrak{d}(\mathfrak{d}^2 L, L' \nabla L) = 0\), we have

\[
\mathfrak{d}(\mathfrak{d}^2 L, (L' \nabla L)^k \nabla Y) = 0.
\]

Then, Lemma 3.6 implies that

\[
\mathfrak{d}(\mathfrak{d} L, L'^k \nabla X) = \mathfrak{d}(\mathfrak{d} L, L^{m-k} \nabla ((L' \nabla L)^k \nabla Y)) \\
= \mathfrak{d}(\mathfrak{d} L, (L' \nabla L)^k \nabla Y) + m - k \\
= m - k,
\]

where the last equality follows from \(\mathfrak{d}(\mathfrak{d} L, L' \nabla L) = 0\) and \(\mathfrak{d}(\mathfrak{d} L, Y) = 0\).
(iii) By (ii), we have
\[ \mathfrak{b}(L, L' \otimes^m \nabla X) = 0, \quad \mathfrak{b}(D L, L' \otimes^m \nabla X) = 0. \]
Since
\[ L' \otimes^k \nabla X \simeq L' \otimes^{k-m} \nabla (L' \otimes^m \nabla X), \]
we have the assertion by Corollary 3.12 (i). \[ \square \]

4. Quantum affine Schur-Weyl duality

Let \( \mathcal{D} := \{ L_i \}_{i \in J} \subset \mathcal{C}_g \) be a family of simple modules of \( \mathcal{C}_g \). The family \( \mathcal{D} \) is called a duality datum associated with a generalized Cartan matrix \( C = (c_{i,j})_{i,j \in J} \) of symmetric type if it satisfies the following:

(a) for each \( i \in J \), \( L_i \) is a real simple module,
(b) for any \( i, j \in J \) such that \( i \neq j \), \( \mathfrak{b}(L_i, L_j) = -c_{i,j} \).

Then one can construct a monoidal functor
\[ \mathcal{F}_\mathcal{D} : \mathcal{R}_C\text{-gmod} \rightarrow \mathcal{C}_g \]
using the duality datum \( \mathcal{D} \) (see [21, 35]).
The functor \( \mathcal{F}_\mathcal{D} \) is called a quantum affine Schur-Weyl duality functor or shortly a duality functor.

In § 4.2 below, we slightly modify the definition of quantum affine Schur-Weyl duality functor in order that it commutes with the affinization.

4.1. Affinizations.

4.1.1. Pro-objects. Let \( k \) be a base field and let \( \mathcal{C} \) be an essentially small \( k \)-abelian category. Let \( \text{Pro}(\mathcal{C}) \) be the category of pro-objects of \( \mathcal{C} \) (see [36] for details). One can show that
\[ \text{Pro}(\mathcal{C}) \simeq \{ \text{left exact } k\text{-linear functors from } \mathcal{C} \text{ to } k\text{-Mod} \}^{\text{opp}} \]
by the functor
\[ \left( \lim_{\longrightarrow} \right) M_i \mapsto \left( \mathcal{C} \ni X \mapsto \lim_{\longrightarrow} \text{Hom}_\mathcal{C}(M_i, X) \right). \]
Here, \( k\text{-Mod} \) is the category of vector spaces over \( k \), and “\( \lim \)” denotes the pro-lim (see [36, Section 2.6 and Proposition 6.1.7] for notations and details). Then, \( \text{Pro}(\mathcal{C}) \) is a \( k \)-abelian category which admits small projective limits. If no confusion arises, we regard \( \mathcal{C} \) as a full subcategory of \( \text{Pro}(\mathcal{C}) \), which is stable by extensions and subquotients. Any functor \( F : \mathcal{C} \rightarrow \mathcal{C}' \) extends to \( PF : \text{Pro}(\mathcal{C}) \rightarrow \text{Pro}(\mathcal{C}') \) which commutes with small filtrant projective limits:
\[ PF\left( \lim_{\longrightarrow} M_i \right) \simeq \left( \lim_{\longrightarrow} F(M_i) \right). \]
4.1.2. Affinization in quiver Hecke algebra case. Let $R$ be a symmetric quiver Hecke algebra. Note that

$$R(\beta)-\text{gMod} \to \text{Pro}(R(\beta)-\text{gmod}).$$

Recall that $R(\beta)-\text{gMod}$ is the category of graded $R(\beta)$-modules. Let

$$\text{Pro}(R) := \bigoplus_{\beta \in Q^+} \text{Pro}(R(\beta)-\text{gmod}),$$

which is a monoidal category. Let $z$ be an indeterminate of homogeneous degree 2, and we set

$$R(\beta)^{\text{aff}} := k[z] \otimes_k 1_z R(\beta),$$

which has the graded $R(\beta)$-bimodule structure. Here $1_z R(\beta)$ is a free right $R(\beta)$-module of rank one and the left module structure is given by

$$e(\nu) 1_z = 1_z e(\nu), \quad x_k 1_z = 1_z x_k + z 1_z \quad \text{and} \quad \tau_k 1_z = 1_z \tau_k.$$

Hence we have

$$1_z x_k = (x_k - z) 1_z. \quad (4.1)$$

For $X \in R(\beta)-\text{gmod}$, the affinization $X^{\text{aff}}$ of $X$ is isomorphic to $R(\beta)^{\text{aff}} \otimes R(\beta) X$. Since $X^{\text{aff}}$ is not in $R(\beta)-\text{gmod}$, we set

$$X^{\text{Aff}} := \lim_{\longleftarrow m} X^{\text{aff}} / z^m X^{\text{aff}} \in \text{Pro}(R(\beta)-\text{gmod}).$$

Note that

$$X^{\text{Aff}} \simeq k[[z]] \otimes_k X$$

as an object of $\text{Pro}(k-\text{mod})$ forgetting the action of $R(\beta)$. Here we regard $k[[z]]$ as the object of $\text{Pro}(k-\text{mod})$:

$$\lim_{\longleftarrow m} k[z] / k[z] z^m.$$

Similarly we set

$$R(\beta)^{\text{Aff}} := \lim_{\longleftarrow m} R(\beta)^{\text{aff}} / R(\beta)^{\text{aff}} (z, x_1, \ldots, x_{\text{ht}(\beta)})^m,$$

which is an object of $\text{Pro}(R(\beta)-\text{gmod})$ with a right $R(\beta)$-action. Here, $(z, x_1, \ldots, x_m)$ is the ideal of $k[z, x_1, \ldots, x_m]$ generated by $z, x_1, \ldots, x_m$. Then we have

$$M^{\text{Aff}} \simeq R(\beta)^{\text{Aff}} \otimes_{R(\beta)} M \quad \text{for any} \ M \in R(\beta)-\text{gmod}.$$

For $M, N \in R$-gmod, we have

$$M^{\text{Aff}} \circ N^{\text{Aff}} \simeq (M \circ N)^{\text{Aff}},$$

where

$$M^{\text{Aff}} \circ N^{\text{Aff}} := \text{Coker}(M^{\text{Aff}} \circ N^{\text{Aff}} \xrightarrow{z M - z N} M^{\text{Aff}} \circ N^{\text{Aff}}).$$
We remark that, in this paper, we use the language of pro-objects instead of the completion in [21, Section 3.1] and [10].

4.1.3. Affinization in quantum affine algebra case. Let $U'_q(g)$ be a quantum affine algebra and let $\mathcal{C}_g$ be the category of finite-dimensional integrable $U'_q(g)$-modules. We embed $\mathcal{C}_g$ into Pro($\mathcal{C}_g$). Note that Pro($\mathcal{C}_g$) is a $k$-abelian monoidal category. For $M \in \mathcal{C}_g$, let $M^{\text{aff}}$ be the affinization of $M$. Recall that

$$M^{\text{aff}} \simeq k[z_M^{\pm 1}] \otimes M$$

with the action

$$e_i(a \otimes v) = z_M^{d_i,0} a \otimes e_i v \quad \text{for } a \in k[z_M^{\pm 1}] \text{ and } v \in M.$$  

Here we use $z$ to distinguish from $z$ in the quiver Hecke algebra setting. We set

$$M_{\text{Aff}} := \lim_{\leftarrow m} M^{\text{aff}} / (z_M - 1)^m M^{\text{aff}} \in \text{Pro}(\mathcal{C}_g).$$

Note that there is a canonical algebra homomorphism

$$k[[z_M - 1]] \longrightarrow \text{End}_{\text{Pro}(\mathcal{C}_g)}(M^{\text{Aff}}).$$

For $M, N \in \mathcal{C}_g$, we have

$$M^{\text{Aff}} \otimes z N^{\text{Aff}} \simeq (M \otimes N)^{\text{Aff}},$$

where

$$M^{\text{Aff}} \otimes z N^{\text{Aff}} := \text{Coker}(M^{\text{Aff}} \otimes N^{\text{Aff}} \xrightarrow{z_M - z_N} M^{\text{Aff}} \otimes N^{\text{Aff}}).$$

For simple modules $M, N$ in $\mathcal{C}_g$, we can define the renormalized R-matrix

$$R^\text{ren}_{M,N}(z_N/z_M) : M^{\text{Aff}} \otimes N^{\text{Aff}} \longrightarrow N^{\text{Aff}} \otimes M^{\text{Aff}}.$$  

4.2. Quantum affine Schur-Weyl duality functor. We now consider a duality datum $D = \{L_i\}_{i \in J}$ associated with a symmetric generalized Cartan matrix $C = (c_{i,j})_{i,j \in J}$. For $i, j \in J$, we choose $c_{i,j}(x) \in k[[x]]$ such that

$$c_{i,j}(x)c_{j,i}(-x) = 1 \quad \text{and} \quad c_{i,i}(0) = 1.$$  

We set

$$P_{ij}(u, v) := c_{ij}(u - v) \cdot (u - v)^{d_{i,j}},$$

where $d_{i,j} := \text{zero}_{z=1}d_{L_i,L_j}(z)$.

Let $Q^+_C$ be the positive root lattice associated with $C$. For $\beta \in Q^+_C$ with $\ell = \text{ht}(\beta)$ and $\nu = (\nu_1, \ldots, \nu_\ell) \in J^\beta$, we set

$$\widehat{L}_{\nu} := L_{\nu_1}^{\text{Aff}} \otimes \cdots \otimes L_{\nu_\ell}^{\text{Aff}}$$

and

$$\widehat{L}(\beta) := \bigoplus_{\nu \in J^\beta} \widehat{L}_{\nu} \in \text{Pro}(\mathcal{C}_g).$$
The algebra \( R(\beta) \) acts \( \hat{L}(\beta) \) from the right as follows:

(a) \( e(\nu) \) is the projection to \( \hat{L}_\nu \).

(b) \( x_k \in R(\beta) \) acts by \( \log z_{L_{\nu k}} \), where \( \log z_{L_{\nu k}} \in k[[z_{L_{\nu k}} - 1]] \subset \text{End}(\hat{L}_\nu) \).

(c) \( e(\nu) \tau_k \) (\( 1 \leq k < \ell \)) acts on \( \hat{L}_\nu \) by

\[
\begin{cases}
R_{L_{\nu k}, L_{\nu k+1}}^{\text{norm}} \circ P_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) & \text{if } \nu_k \neq \nu_{k+1}, \\
(x_k - x_{k+1})^{-1} \left( R_{L_{\nu k}, L_{\nu k+1}}^{\text{norm}} \circ P_{\nu_k, \nu_{k}}(x_k, x_{k+1}) - \text{id}_{\hat{L}_\nu} \right) & \text{if } \nu_k = \nu_{k+1}.
\end{cases}
\]

Note that we used \( z - 1 \) instead of \( \log z \) in [21]. We have also relaxed the condition on \( c_{i,i}(u) \). We have changed the definition in order that we have Theorem 4.2 below.

Then \( \hat{L}(\beta) \) gives the monoidal functor

\[
\hat{F}_D: R\text{-gmod} \to \text{Pro}(\mathcal{C}_g)
\]

defined by

\[
\hat{F}_D(M) = \hat{L}(\beta) \otimes_{R(\beta)} M \quad \text{for } M \in R(\beta)\text{-gmod}.
\]

It extends to

\[
\hat{F}_D: \text{Pro}(R) \to \text{Pro}(\mathcal{C}_g)
\]

such that \( \hat{F}_D \) commutes with filtrant projective limits.

The following proposition can be proved in a similar manner to [21].

**Proposition 4.1.** \( \hat{F}_D \) is a monoidal functor and it induces a monoidal functor \( F_D: R\text{-gmod} \to \mathcal{C}_g \).

Then the following theorem tells us that the functor \( \hat{F}_D \) preserves affinizations.

**Theorem 4.2.** Functionally in \( M \in R\text{-gmod} \), we have an isomorphism

\[
\hat{F}_D(M^{\text{Aff}}) \simeq (F_D(M))^{\text{Aff}}.
\]

Moreover, we have

(i) the action of \( z_M \) on the left term coincides with \( \log z_{F_D(M)} \) on the right term,

(ii) for \( M, N \in R\text{-gmod} \), the following diagram commutes:

\[
\begin{array}{ccc}
\hat{F}_D(M^{\text{Aff}} \circ N^{\text{Aff}}) & \xrightarrow{\sim} & \hat{F}_D(M^{\text{Aff}}) \otimes \hat{F}_D(N^{\text{Aff}}) \\
\downarrow & & \downarrow \\
\hat{F}((M \circ N)^{\text{Aff}}) & \xrightarrow{\sim} & (F_D(M \circ N))^{\text{Aff}} \xrightarrow{\sim} (F_D(M))^{\text{Aff}} \otimes (F_D(N))^{\text{Aff}}.
\end{array}
\]

**Proof.** Let us show (i). Since \( \hat{F}_D(R(\beta)^{\text{Aff}}) \otimes_{R(\beta)} M \simeq \hat{F}_D(M^{\text{Aff}}) \) for any \( M \in R\text{-gmod} \), it is enough to show that

\[
\hat{F}_D(R(\beta)^{\text{Aff}}) \simeq (\hat{F}_D(R(\beta)))^{\text{Aff}}
\]
compatible with the right actions of $R(\beta)$. 

Set $\ell := \text{ht}(\beta)$ and $x_k := \log z_{L_{\nu_k}} \in \text{End}(\hat{L}(\beta))$ for $k = 1, \ldots, \ell$. Then we have $\hat{L}_\nu = k[[x_1, \ldots, x_\ell]] \otimes L_\nu$. Here we set
\[
L_\nu = L_{\nu_1} \otimes \cdots \otimes L_{\nu_\ell} \quad \text{and} \quad L(\beta) = \bigoplus_{\nu \in J_\beta} L_\nu.
\]
Then, $e_i$ acts on $\hat{L}_\nu$ by
\[
\sum_{k=1}^\ell e^{\delta_i, 0} x_k (e_i)_k.
\]
Here $e^x$ is the exponential function and $(e_i)_k$ denotes the action on $L_\nu$ given by
\[
\text{id} \otimes \cdots \otimes \text{id} \otimes e_i \otimes K_{i_1}^{-1} \otimes \cdots \otimes K_{i_\ell}^{-1}.
\]

Then we have
(i) $\hat{F}_D(R(\beta)^{\text{Aff}}) \simeq k[[z, x_1, \ldots, x_\ell]] \otimes L(\beta)$. Here $e_i$ acts by (4.2). The right action of $x_k \in R(\beta)$ is given by $x_k - z$ by (4.1).
(ii) $(\hat{F}_D(R(\beta)))^{\text{Aff}} \simeq k[[z, x_1, \ldots, x_\ell]] \otimes L(\beta)$. Here $e_i$ acts by
\[
e^{\delta_i, 0} \sum_{k=1}^\ell e^{\delta_i, 0} x_k (e_i)_k = \sum_{k=1}^\ell e^{\delta_i, 0} (x_k + z) (e_i)_k.
\]

The right action of $x_k \in R(\beta)$ is given by $x_k$.

Hence, the morphism
\[
f: \hat{F}_D(R(\beta)^{\text{Aff}}) \to (\hat{F}_D(R(\beta)))^{\text{Aff}}
\]
given by $a(z, x) \otimes v \mapsto a(z, x_1 + z, \ldots, x_\ell + z) \otimes v$ (with $a(z, x) \in k[[z, x_1, \ldots, x_\ell]]$ and $v \in L(\beta)$) gives an isomorphism in $\text{Pro}(C_0)$ and the right action of $x_k \in R(\beta)$ commutes. The compatibility of the right action of $\tau_k \in R(\beta)$ easily follows from the fact that $R_{ij}(u, v)$ is a function in $u - v$.

The second assertion (ii) is immediate.

4.3. Quantum affine Schur-Weyl duality with simply laced Cartan matrix. 

Hereafter, we assume that $C = (c_{i,j})_{i,j \in J}$ is a simply laced Cartan matrix of finite type.

Let $R_C$ be the symmetric quiver Hecke algebra associated with $C$. If no confusion arises, we simply write $R$ for $R_C$.

Let $D = \{L_i\}_{i \in J}$ be a duality datum associated with the Cartan matrix $C$.

**Proposition 4.3 ([21]).** We have
(i) $\hat{F}_D$ is an exact functor and it commutes with projective limits,
(ii) $F_D$ sends a simple module to a simple module or zero.
Lemma 4.4. Let $M \in R$ - mod be a real simple module, and assume that $\mathcal{F}_D(M)$ is simple. Then $\mathcal{F}_D(M)$ is also a real simple module.

Proof. Since $\mathcal{F}_D(M) \otimes \mathcal{F}_D(M) \simeq \mathcal{F}_D(M \circ M)$ and $M \circ M$ is simple, $\mathcal{F}_D(M) \otimes \mathcal{F}_D(M)$ is simple, i.e., $\mathcal{F}_D(M)$ is real. □

Lemma 4.5. Let $M, N \in R$ - mod be simple modules such that $\mathcal{F}_D(M)$ and $\mathcal{F}_D(N)$ are simple modules. Assume that one of $M$ and $N$ is real.

(i) $\mathfrak{b}(\mathcal{F}_D(M), \mathcal{F}_D(N)) \leq \mathfrak{b}(M, N)$.

(ii) The following conditions are equivalent:

(a) $\mathfrak{b}(\mathcal{F}_D(M), \mathcal{F}_D(N)) = \mathfrak{b}(M, N)$.

(b) $\mathcal{F}_D(M \nabla N)$ and $\mathcal{F}_D(N \nabla M)$ are simple.

If these conditions hold, then we have

(1) $\mathcal{F}_D(r_{M,N}) \neq 0$ and $\mathcal{F}_D(r_{N,M}) \neq 0$.

(2) $\mathcal{F}_D(M) \nabla \mathcal{F}_D(N) \simeq \mathcal{F}_D(M \nabla N)$ and $\mathcal{F}_D(N) \nabla \mathcal{F}_D(M) \simeq \mathcal{F}_D(N \nabla M)$.

Proof. Set $z = \log z$ and $d := \mathfrak{b}(M, N)$. By the definition of $\mathfrak{b}(M, N)$, we have the following commutative diagram (up to a constant multiple):

\[
\begin{array}{ccc}
M_z \circ N & \xrightarrow{\mathcal{F}_D(r_{M_z,N})} & N \circ M_z \\
\mathfrak{b}(\mathcal{F}_D(M), \mathcal{F}_D(N)) & \xrightarrow{\mathcal{F}_D(r_{M_z,N})} & \mathcal{F}_D(M) \circ \mathcal{F}_D(N).
\end{array}
\]

Applying $\hat{\mathcal{F}}$ to the above diagram, by Proposition 4.1 and Theorem 4.2, we obtain

\[
\begin{array}{ccc}
M_z \circ N & \xrightarrow{\mathcal{F}_D(r_{M_z,N})} & N \circ M_z \\
\hat{\mathcal{F}}(\mathcal{F}_D(r_{M_z,N})) & \xrightarrow{\hat{\mathcal{F}}(\mathcal{F}_D(r_{M_z,N}))} & \hat{\mathcal{F}}(\mathcal{F}_D(M)) \circ \mathcal{F}_D(N).
\end{array}
\]

Since $z^d$ id is non-zero, $\hat{\mathcal{F}}(\mathcal{F}_D(r_{M_z,N}))$ and $\hat{\mathcal{F}}(\mathcal{F}_D(r_{M_z,N}))$ are non-zero. Note that

\[
\begin{align*}
\text{Hom}_{k[[z]] \otimes U_q(\mathfrak{g})}(U \otimes V_z, V_z \otimes U) &= k[[z]]R_{U,V_z}, \\
\text{Hom}_{k[[z]] \otimes U_q(\mathfrak{g})}(U_z \otimes V, V \otimes U_z) &= k[[z]]R_{U,V_z},
\end{align*}
\]

for any simple modules $U, V \in \mathfrak{g}_0$ by [25, Proposition 9.5]. Hence, we have

$\hat{\mathcal{F}}_D(R_{M_z,N}) = z^a f(z) R_{\mathcal{F}_D(M)z,N}, \quad \hat{\mathcal{F}}_D(R_{N,z}) = z^b g(z) R_{N,\mathcal{F}_D(M)z}$

for some $a, b \geq 0$ and $f(z), g(z) \in k[[z]]^\times$. Hence it follows from (4.4) that

\[
d = a + b + \mathfrak{b}(\mathcal{F}_D(M), \mathcal{F}_D(N)).
\]

Hence, we have $\mathfrak{b}(\mathcal{F}_D(M), \mathcal{F}_D(N)) \leq d$.

Moreover $d = \mathfrak{b}(\mathcal{F}_D(M), \mathcal{F}_D(N))$ if and only if $a = b = 0$. Since $a = b = 0$ is equivalent to $\mathcal{F}_D(r_{M,N}) = \hat{\mathcal{F}}_D(R_{M_z,N})|_{z=0} \neq 0$ and $\mathcal{F}_D(r_{N,M}) = \hat{\mathcal{F}}_D(R_{N,M_z})|_{z=0} \neq 0$. The
last two conditions are equivalent to \( \text{Im}(\mathcal{F}_D(r_{M,N})) \simeq \mathcal{F}_D(M \nabla N) \neq 0 \) and \( \text{Im}(\mathcal{F}_D(r_{N,M})) \simeq \mathcal{F}_D(N \nabla M) \neq 0 \). □

Lemma 4.6. Let \( \mathcal{D} = \{ L_i \}_{i \in J} \) be a duality datum associated with a simply-laced finite Cartan matrix \( \mathcal{C} \). Let \( L, M, N \) be simple \( R_C \)-modules and \( S \) a simple subquotient of \( M \circ N \). Assume that \( \mathcal{F}_D(M), \mathcal{F}_D(N) \) and \( \mathcal{F}_D(S) \) are simple.

(i) Assume that \( \mathcal{F}_D(r_{M,L}) \) and \( \mathcal{F}_D(r_{N,L}) \) are non-zero. Then we have

\[
\Lambda(\mathcal{F}_D(M), \mathcal{F}_D(L)) + \Lambda(\mathcal{F}_D(N), \mathcal{F}_D(L)) - \Lambda(\mathcal{F}_D(S), \mathcal{F}_D(L)) \\
\geq \Lambda(M, L) + \Lambda(N, L) - \Lambda(S, L).
\]

The equality holds if and only if \( \mathcal{F}_D(r_{S,L}) \) does not vanish.

(ii) Assume that \( \mathcal{F}_D(r_{L,M}) \) and \( \mathcal{F}_D(r_{L,N}) \) are non-zero.

\[
\Lambda(\mathcal{F}_D(L), \mathcal{F}_D(M)) + \Lambda(\mathcal{F}_D(L), \mathcal{F}_D(N)) - \Lambda(\mathcal{F}_D(L), \mathcal{F}_D(S)) \\
\geq \Lambda(L, M) + \Lambda(L, N) - \Lambda(L, S).
\]

The equality holds if and only if \( \mathcal{F}_D(r_{L,S}) \) does not vanish.

Proof. Since the proof of (ii) is similar, we shall prove only (i).

As \( S \) is a simple subquotient of \( M \circ N \), there exists a submodule \( K \) of \( M \circ N \) such that \( S \) is a quotient of \( K \). We consider the following commutative diagram in \( R\text{-gmod} \)

\[
\begin{array}{ccc}
(M \circ N) \circ L_z & \xrightarrow{\text{Ren}_{M \circ N,L_z}} & L_z \circ (M \circ N) \\
| & & | \\
| & & | \\
K \circ L_z & \xrightarrow{\text{Ren}_{K \circ L_z}} & L_z \circ K \\
| & & | \\
S \circ L_z & \xrightarrow{\text{Ren}_{S \circ L_z}} & L_z \circ S
\end{array}
\]

for some \( c \in \mathbb{Z}_{\geq 0} \). Comparing the homogeneous degrees of morphisms in the above diagram, we have

\[
2c = \Lambda(M, L) + \Lambda(N, L) - \Lambda(S, L). \tag{4.5}
\]
We set \( z = \log z \). Applying the duality functor \( \hat{F}_D \) to the above diagram, we obtain

\[
\begin{array}{c}
(\tilde{M} \otimes \tilde{N}) \otimes \tilde{L}_z \xrightarrow{\hat{F}_D(R_{M,N,L}^{\text{ren}})} \tilde{L}_z \otimes (\tilde{M} \otimes \tilde{N}) \\
\tilde{K} \otimes \tilde{L}_z \xrightarrow{\hat{F}_D(R_{S,L}^{\text{ren}})} \tilde{L}_z \otimes \tilde{K} \\
\tilde{S} \otimes \tilde{L}_z \xrightarrow{\hat{F}_D(R_{S,L}^{\text{ren}})} \tilde{L}_z \otimes \tilde{S}
\end{array}
\]

where \( \tilde{X} \) denotes \( F_D(X) \) for a simple \( R_C \)-module \( X \). There exist \( a \in \mathbb{Z}_{\geq 0} \) and \( f(z) \in k[[z]] \) such that

\[
\hat{F}_D(R_{S,L}^{\text{ren}}) = z^a f(z) R_{S,L}^{\text{ren}}.
\]

Since \( F_D(r_{M,L}) \) and \( F_D(r_{N,L}) \) do not vanish, we have

\[
\hat{F}_D(R_{M,L}^{\text{ren}}) \equiv R_{M,L}^{\text{ren}}, \quad \hat{F}_D(R_{N,L}^{\text{ren}}) \equiv R_{N,L}^{\text{ren}}
\]

up to a multiple of \( k[[z]] \). The above diagram tells us that

\[
\frac{c_{M,L}(z)c_{N,L}(z)}{(z - 1)^{c+a} c_{S,L}(z)}
\]

is a rational function in \( z \) which is regular and invertible at \( z = 1 \). Hence, by [27, Lemma 3.4], we have

\[
\text{Deg} \left( \frac{c_{M,L}(z)c_{N,L}(z)}{c_{S,L}(z)} \right) = 2 \cdot \text{zero}_{z=1} \left( \frac{c_{M,L}(z)c_{N,L}(z)}{c_{S,L}(z)} \right) = 2(c + a).
\]

Therefore, by (4.5), we conclude that

\[
\Lambda(M, L) + \Lambda(N, L) - \Lambda(S, L) = 2c = \text{Deg} \left( \frac{c_{M,L}(z)c_{N,L}(z)}{c_{S,L}(z)} \right) - 2a
\]

\[
= \Lambda(\tilde{M}, \tilde{L}) + \Lambda(\tilde{N}, \tilde{L}) - \Lambda(\tilde{S}, \tilde{L}) - 2a.
\]

Hence we have

\[
\Lambda(M, L) + \Lambda(N, L) - \Lambda(S, L) \leq \Lambda(\tilde{M}, \tilde{L}) + \Lambda(\tilde{N}, \tilde{L}) - \Lambda(\tilde{S}, \tilde{L}).
\]

The equality holds if and only if \( a = 0 \) which is equivalent to \( F_D(r_{S,L}) \neq 0 \). \qed
4.4. Strong duality datum.

**Definition 4.7.** A strong duality datum \( \mathcal{D} = \{ L_i \}_{i \in J} \) is a duality datum associated with a simply-laced finite Cartan matrix \( C = (c_{i,j})_{i,j \in J} \) such that all \( L_i \)'s are root modules and

\[
\varepsilon(L_i, \mathcal{D}(L_j)) = -\delta(k = 0)c_{i,j}
\]

for any \( k \in \mathbb{Z} \) and \( i, j \in J \) such that \( i \neq j \).

In particular, we have

\[
\Lambda(L_i, L_j) = -c_{i,j} \text{ for } i \neq j,
\]

\[
\Lambda^\infty(L_i, L_j) = -c_{i,j} \text{ for all } i, j \in J.
\]

Let \( \mathcal{D} = \{ L_i \}_{i \in J} \) be a strong duality datum associated with a Cartan matrix \( C = (c_{i,j})_{i,j \in J} \) of finite ADE type. Let \( R_\mathcal{C} \) be the symmetric quiver Hecke algebra associated with \( C \). If no confusion arises, we simply write \( R \) for \( R_\mathcal{C} \). We denote by

\[
\mathcal{F}_\mathcal{D}: \mathcal{C}_g \to \mathcal{C}_g
\]

the duality functor arising from \( \mathcal{D} \). Recall that \( \mathcal{F}_\mathcal{D} \) sends simples to simples or zero. However if \( \mathcal{D} \) is strong, we can say more as we see below.

Throughout this subsection, we assume that \( \mathcal{D} \) is a strong duality datum.

**Lemma 4.8.** For \( w \in \mathcal{W}, \Lambda \in \mathcal{P}^+ \) and \( i \in J \), we have

\[
\varepsilon_i(D(w\Lambda, \Lambda)) = \begin{cases} 
-(\alpha_i, w\Lambda) & \text{if } s_iw < w, \\
0 & \text{if } s_iw > w,
\end{cases}
\]

\[
\varepsilon^*_i(D(w\Lambda, \Lambda)) = \begin{cases} 
(\alpha_i, \Lambda) & \text{if } w \geq s_i, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\varepsilon^*(L(i), D(w\Lambda, \Lambda)) = \begin{cases} 
0 & \text{if } s_iw < w, \\
(\alpha_i, w\Lambda) & \text{if } s_iw > w \text{ and } w \geq s_i, \\
(\alpha_i, w\Lambda - \Lambda) & \text{otherwise},
\end{cases}
\]

where \( D(w\Lambda, \Lambda) \) is the determinantal module appeared in §2.2.

**Proof.** The equality for \( \varepsilon_i \) is proved in [24, Proposition 10.2.4]. Let us show the equality for \( \varepsilon^*_i \). If \( w \geq s_i \), then we have \( w\Lambda \preceq s_i\Lambda \). Hence \( \varepsilon^*_i(D(w\Lambda, s_i\Lambda)) = 0 \) by the same proposition.

Since we have

\[
D(w\Lambda, \Lambda) \simeq D(w\Lambda, s_i\Lambda) \nabla D(s_i\Lambda, \Lambda) \simeq D(w\Lambda, s_i\Lambda) \nabla L(i)^{\diamond(\alpha_i, \Lambda)}
\]

by [24, Theorem 10.3.1], we have \( \varepsilon^*_i(D(w\Lambda, \Lambda)) = (\alpha_i, \Lambda) \).

Assume that \( w \npreceq s_i \). Then \( \Lambda - w\lambda \) does not contain \( \alpha_i \), and hence \( \varepsilon^*_i(D(w\Lambda, s_i\Lambda)) = 0 \).
The equality for $d$ immediately follows from
\begin{equation}
\mathfrak{b}(L(i), M) = \varepsilon_i(M) + \varepsilon_i^*(M) + (\alpha_i, \text{wt}(M))
\end{equation}
([26, Corollary 3.8]). \hfill \square

**Theorem 4.9.** Let $w \in W$, $\Lambda \in P^+$, and set
\[ V_w(\Lambda) := \mathcal{F}_D(D(w\Lambda, \Lambda)). \]
Then $V_w(\Lambda)$ is simple and
\[ d(L_i, V_w(\Lambda)) = d(L_i, D(w\Lambda, \Lambda)), \]
\[ \mathfrak{b}(\mathcal{D} L_i, V_w(\Lambda)) = \varepsilon_i(D(w\Lambda, \Lambda)), \]
\[ \mathfrak{b}(\mathcal{D}^2 L_i, V_w(\Lambda)) = 0. \]

**Proof.** First note that
\[ \mathfrak{b}(L_i, V_w(\Lambda)) = \mathfrak{b}(L_i, D(w\Lambda, \Lambda)), \]
\[ \mathfrak{b}(\mathcal{D} L_i, V_w(\Lambda)) = \varepsilon_i(D(w\Lambda, \Lambda)), \]
\[ \mathfrak{b}(\mathcal{D}^2 L_i, V_w(\Lambda)) = 0. \]

Once we prove that $V_w(\Lambda)$ is a simple module, we have $\mathfrak{b}(\mathcal{D}^2 L_i, V_w(\Lambda)) = 0$, since $\mathcal{D}^2 L_i$ commutes with all $L_j$’s.

Since $D(w\Lambda, \Lambda) \circ D(w\Lambda', \Lambda') \simeq D(w(\Lambda + \Lambda'), \Lambda + \Lambda')$ up to a grading shift for any $\Lambda, \Lambda \in P^+$ and $w \in W$ ([26, Proposition 4.2]), we may assume that $\Lambda = \Lambda_t$ for some $t \in J$. We may assume further that $\Lambda$ is $w$-regular: that is, $\ell(w) \leq \ell(w')$ for any $w' \in W$ such that $w' \Lambda = w \Lambda$. Then, by the preceding lemma, we have
\[ \mathfrak{b}(L(i), D(w\Lambda, \Lambda)) = \begin{cases} 0 & \text{if } s_iw < w, \\ (\alpha_i, w\Lambda) & \text{if } s_iw > w, \end{cases} \]
\[ \varepsilon_i(D(w\Lambda, \Lambda)) = \begin{cases} -\alpha_i, w\Lambda & \text{if } s_iw < w, \\ 0 & \text{if } s_iw > w, \end{cases} \]
if $w\Lambda \neq \Lambda$.

We shall argue by induction on $\ell(w)$.

If $\ell(w) = 0$, then there is noting to prove.

If $\ell(w) = 1$, then $V_w(\Lambda) = L_i$. Then it is straightforward that the assertion is true.

We now assume that $\ell(w) \geq 2$.

**(Case 1):** Assume that $s_iw < w$. We set
\[ w' = s_iw, \quad n := (\alpha_i, w'\Lambda) \in \mathbb{Z}_{\geq 0}. \]
Then $\Lambda$ is $w'$-regular and $w'\Lambda \neq \Lambda$. Hence, by the induction hypothesis, we have
\[ \mathfrak{b}(\mathcal{D} L_i, V_{w'}(\Lambda)) = 0, \quad \mathfrak{b}(L_i, V_{w'}(\Lambda)) = \mathfrak{b}(L(i), D(w'\Lambda, \Lambda)) = n. \]
Since $D(w\Lambda, \Lambda) \simeq L(i)^{\otimes n} \triangledown D(w'\Lambda, \Lambda)$, we have
\begin{equation}
V_w(\Lambda) \simeq L_i^\otimes n \triangledown V_{w'}(\Lambda)
\end{equation}
by Lemma 4.5. In particular, $V_w(\Lambda)$ is simple.
It follows from Lemma 3.4 that 
\[ \mathfrak{d}(L_i, V_w(\Lambda)) = 0. \]
Moreover, we have \[ \mathfrak{d}(\mathcal{D}^2 L_i, V_{w'}(\Lambda)) = 0. \] Applying Lemma 3.6 (i) to the setting \( L = L_i \) and \( X = V_{w'}(\Lambda) \), we obtain 
\[ \mathfrak{d}(\mathcal{D} L_i, V_w(\Lambda)) = \mathfrak{d}(\mathcal{D} L_i, V_{w'}(\Lambda)) + n = n, \]
which gives the assertion.

(Case 2): Assume that \( s_i w > w \). Since \( \ell(w) \geq 2 \), there exists \( j \in J \) such that \( s_j w < w \).
We set \( w' := s_j w, \quad n := (\alpha_j, w' \Lambda) \in \mathbb{Z}_{\geq 0}. \)
Note that \( \Lambda \) is \( w' \)-regular and \( w' \Lambda \neq \Lambda \). By (4.7), we have 
\[ V_w(\Lambda) \simeq L_j^\otimes n \nabla V_{w'}(\Lambda). \]
We set \( Z := V_w(\Lambda) \) and \( Z' := V_{w'}(\Lambda) \). Hence 
(4.8) \[ Z \simeq L_j^\otimes n \nabla Z'. \]

(1) Suppose that \( c_{i,j} = 0 \). Then \( s_i w' > w' \) since \( s_i s_j = s_j s_i \). By the induction hypothesis, we have 
\[ \mathfrak{d}(L_i, V_{w'}(\Lambda)) = (\alpha_i, w' \Lambda) = (s_j(\alpha_i), w \Lambda) = (\alpha_i, w \Lambda), \quad \mathfrak{d}(\mathcal{D} L_i, V_{w'}(\Lambda)) = 0. \]
Since \( \mathfrak{d}(\mathcal{D}^k L_i, L_j) = 0 \) for any \( k \in \mathbb{Z} \), it follows from (4.8) and Corollary 2.25 that 
\[ \mathfrak{d}(\mathcal{D}^k L_i, Z) = \mathfrak{d}(\mathcal{D}^k L_i, Z') \quad \text{for any } k \in \mathbb{Z}. \]
In particular, we have 
\[ \mathfrak{d}(L_i, Z) = \mathfrak{d}(L_i, Z') = (\alpha_i, w \Lambda), \]
\[ \mathfrak{d}(\mathcal{D} L_i, Z) = \mathfrak{d}(\mathcal{D} L_i, Z') = 0. \]

(2) We now assume that \( c_{i,j} = -1 \). Then we have two cases: \( s_i w' > w' \) and \( s_i w' < w' \).
(a) Assume that \( s_i w' > w' \). Then \( \mathfrak{d}(\mathcal{D} L_i, Z') = 0 \) by the induction hypothesis. Hence, 
by (4.8) and Corollary 3.12 (i), we have 
\[ \mathfrak{d}(\mathcal{D} L_i, Z) = 0 \]
and 
\[ \mathfrak{d}(L_i, Z) = \mathfrak{d}(L_i, Z') + n = (\alpha_i, w' \Lambda) + n \]
\[ = (\alpha_i, w \Lambda - n \alpha_j) = (\alpha_i, w \Lambda). \]
Here the second identity follows from the induction hypothesis.
(b) Assume that \( s_i w' < w' \). Letting \( w'' := s_i w' \), we have \( w = s_j s_i w'' \) and \( \ell(w) = 2 + \ell(w'') \). If \( s_j w'' < w' \), then \( \ell(w) = 3 + \ell(s_j w'') \) and \( w = s_j s_i (s_j w'') = s_i s_j s_i (s_j w'') \). This implies that \( s_i w < w \), which contradicts the assumption (Case 2). Hence we have \( s_j w'' > w'' \), which tells us that
\[
(\alpha_j, s_i w' \Lambda) = (\alpha_j, w'' \Lambda) \geq 0.
\]
Set \( m := (\alpha_i, s_i w' \Lambda) \in \mathbb{Z}_{\geq 0} \). Then, we have
\[
(\alpha_j, s_i w' \Lambda) = (\alpha_j, w' \Lambda + m \alpha_i) = n - m,
\]
which says that \( n - m \geq 0 \).
Set \( Z'' := V_{w''}(\Lambda) \). By the induction hypothesis, we have
\[
\mathfrak{b}(L_i, Z') = 0, \quad \mathfrak{b}(\mathcal{D} L_j, Z'') = 0, \quad \mathfrak{b}(\mathcal{D}^2 L_j, Z'') = 0.
\]
Applying Proposition 3.13 (iii) to the setting \( L := L_i, L' := L_j \) and \( X := Z', Y := Z'' \) and \( k := n \), we have
\[
\mathfrak{b}(L_i, Z) = \mathfrak{b}(L_i, L_j^{\otimes n} \nabla Z') = n - m,
\]
\[
\mathfrak{b}(\mathcal{D} L_i, Z) = 0.
\]
Since \( (\alpha_i, w \Lambda) = (\alpha_i, w' \Lambda - n \alpha_j) = -(\alpha_i, s_i w' \Lambda) + n = n - m \), we conclude that
\[
\mathfrak{b}(L_i, Z) = n - m = (\alpha_i, w \Lambda),
\]
which completes the proof. \( \square \)

**Theorem 4.10.** Let \( \mathcal{D} = \{ L_i \}_{i \in J} \) be a strong duality datum associated with a simply-laced finite Cartan matrix \( \mathcal{C} \). Then the duality functor \( \mathcal{F}_\mathcal{D} \) sends simple modules to simple modules.

**Proof.** Since the duality functor \( \mathcal{F}_\mathcal{D} \) sends a simple module to a simple module or zero, it suffices to show that \( \mathcal{F}_\mathcal{D}(X) \) is non-zero for any simple module \( X \in R\text{-}gmod \).

Let \( w_0 \) be the longest element of the Weyl group \( \mathcal{W} \) of \( \mathcal{C} \). Note that the category \( \mathcal{C}_{w_0} \) is equal to \( R\text{-}gmod \). For \( i \in J \), we set \( \mathcal{C}_i := \text{D}(w_0 \Lambda_i, \Lambda_i) \) and denote by \( (\mathcal{C}_i, R\mathcal{C}_i) \) the non-degenerate braider induced from R-matrices ([28, Proposition 4.1]). It is proved in [28, Section 5] that there is a localization \( \tilde{R} := R\text{-}gmod[\mathcal{C}_i^\circ\backslash | i \in J] \) of \( R\text{-}gmod \) by the braiders \( \mathcal{C}_i \). Moreover \( \tilde{R} \) is left rigid ([28, Corollary 5.11]). Thus, for any simple module \( X \in R\text{-}gmod \), there exists a module \( Y \in R\text{-}gmod \) and \( \Lambda \in \mathcal{P}^+ \) such that there exists a surjective homomorphism
\[
Y \circ X \rightarrow \text{D}(w_0 \Lambda, \Lambda).
\]
Applying the duality functor \( \mathcal{F}_\mathcal{D} \) to the above surjection, we have
\[
\mathcal{F}_\mathcal{D}(Y) \otimes \mathcal{F}_\mathcal{D}(X) \rightarrow \mathcal{F}_\mathcal{D}(\text{D}(w_0 \Lambda, \Lambda)).
\]
Since \( \mathcal{F}_\mathcal{D}(\text{D}(w_0 \Lambda, \Lambda)) \) is simple by Theorem 4.9, \( \mathcal{F}_\mathcal{D}(X) \) does not vanish. \( \square \)
Corollary 4.11. Let \( \mathcal{D} \) be a strong duality datum associated with a simply-laced finite Cartan matrix \( \mathbf{C} \). Then \( \mathcal{F}_D \) is faithful: i.e., for any non-zero morphism \( f \) in \( R_{\mathbf{C}}\text{-gmod}, \mathcal{F}_D(f) \) is non-zero.

Theorem 4.12. Let \( \mathcal{D} = \{L_i\}_{i \in J} \) be a strong duality datum associated with a simply-laced finite Cartan matrix \( \mathbf{C} = (c_{i,j})_{i,j \in J} \). Then, for any simple modules \( M, N \) in \( R_{\mathbf{C}}\text{-gmod} \), we have

(i) \( \Lambda(M, N) = \Lambda(\mathcal{F}_D(M), \mathcal{F}_D(N)) \),
(ii) \( \mathfrak{b}(M, N) = \mathfrak{b}(\mathcal{F}_D(M), \mathcal{F}_D(N)) \),
(iii) \( \text{wt} M, \text{wt} N = -\Lambda(\infty, \mathcal{F}_D(M), \mathcal{F}_D(N)) \),
(iv) \( \mathfrak{b}(\mathcal{D}^k \mathcal{F}_D(M), \mathcal{F}_D(N)) = 0 \) for any \( k \neq 0, \pm 1 \),
(v) \( \tilde{\Lambda}(M, N) = \mathfrak{b}(\mathcal{D} \mathcal{F}_D(M), \mathcal{F}_D(N)) = \mathfrak{b}(\mathcal{F}_D(M), \mathcal{D}^{-1} \mathcal{F}_D(N)) \).

Proof. Set \( \beta := -\text{wt}(M) \) and \( \gamma := -\text{wt}(N) \) and write \( m := \text{ht}(\beta) \) and \( n := \text{ht}(\gamma) \).

(i) We shall use induction on \( m + n \). If \( m = 0 \) or \( n = 0 \), then it is obvious. Hence we assume that \( m, n \geq 1 \).

If \( m + n = 2 \), then \( M = L(i) \) and \( N = L(j) \) for some \( i, j \in J \). Since the assertion is obvious in the case \( i = j \), we assume that \( i \neq j \). Since \( \mathcal{F}_D(M) \simeq L_i \) and \( \mathcal{F}_D(N) \simeq L_j \), we have

\[
\Lambda(L_i, L_j) = \mathfrak{b}(L_i, L_j) = -c_{i,j} = \Lambda(L(i), L(j)).
\]

Suppose that \( m + n \geq 3 \). If \( m \geq 2 \), then there exist simple modules \( M_1 \) and \( M_2 \) such that

(a) \( \text{wt}(M_1) \neq 0 \) and \( \text{wt}(M_2) \neq 0 \),
(b) one of \( M_1 \) and \( M_2 \) is real,
(c) \( M \simeq M_1 \nabla M_2 \).

Hence, by Lemma 4.6 together with Corollary 4.11, we obtain

\[
\Lambda(M_1, N) + \Lambda(M_2, N) - \Lambda(M, N) = \Lambda(\mathcal{F}_D(M_1), \mathcal{F}_D(N)) + \Lambda(\mathcal{F}_D(M_2), \mathcal{F}_D(N)) - \Lambda(\mathcal{F}_D(M), \mathcal{F}_D(N)).
\]

Since \( \Lambda(M_k, N) = \Lambda(\mathcal{F}_D(M_k), \mathcal{F}_D(N)) \) for \( k = 1, 2 \) by the induction hypothesis, we have

\[
\Lambda(M, N) = \Lambda(\mathcal{F}_D(M), \mathcal{F}_D(N)).
\]

The case where \( n \geq 2 \) can be similarly proved.

(ii) immediately follows from (i).

(iii) There exist sequences \( (i_1, \ldots, i_m) \) and \( (j_1, \ldots, j_n) \) in \( J \) such that \( M \) and \( N \) appear as quotients of \( L(i_1) \circ \cdots \circ L(i_m) \) and \( L(j_1) \circ \cdots \circ L(j_n) \), respectively. Note that \( \beta = -\sum_{p=1}^{m} \alpha_{i_p} \) and \( \gamma = -\sum_{q=1}^{n} \alpha_{j_q} \). Since \( \mathcal{F}_D \) is exact and \( \mathcal{F}_D(M) \) and \( \mathcal{F}_D(N) \) are simple,
\( \mathcal{F}_D(M) \) and \( \mathcal{F}_D(N) \) appear as quotients in \( L_{i_1} \otimes \cdots \otimes L_{i_m} \) and \( L_{j_1} \otimes \cdots \otimes L_{j_n} \), respectively. Therefore, by [27, Proposition 3.11], we have
\[
-\Lambda^\infty(\mathcal{F}_D(M), \mathcal{F}_D(N)) = -\sum_{p,q} \Lambda^\infty(L_{i_p}, L_{j_q}) = \sum_{p,q} c_{i_p,j_q}
= (\beta, \gamma).
\]
(iv) follows from \( \varepsilon(\mathcal{D}^k(L_i), L_j) = 0 \) for any \( i, j \) and \( |k| \geq 2 \).

(v) By (i), (iii) and (iv), we have
\[
\Lambda(M, N) = \varepsilon(\mathcal{F}_D(M), \mathcal{F}_D(N)) - \varepsilon(\mathcal{D}_D(M), \mathcal{D}_D(N)) + \varepsilon(\mathcal{D}_D(M), \mathcal{D}_D^{-1}(\mathcal{F}_D(N)) - \varepsilon(\mathcal{F}_D(M), \mathcal{D}_D^{-1}(\mathcal{F}_D(N)),
(\beta, \gamma) = -\varepsilon(\mathcal{F}_D(M), \mathcal{F}_D(N)) + \varepsilon(\mathcal{D}_D(M), \mathcal{D}_D^{-1}(\mathcal{F}_D(N)) + \varepsilon(\mathcal{D}_D(M), \mathcal{D}_D^{-1}(\mathcal{F}_D(N)).
\]
Thus we have
\[
\tilde{\Lambda}(M, N) = \frac{1}{2}(\Lambda(M, N) + (\beta, \gamma))
= \varepsilon(\mathcal{D}_D(M), \mathcal{D}_D^{-1}(\mathcal{F}_D(N)) = \varepsilon(\mathcal{D}_D(M), \mathcal{F}_D(N)). \tag*{\square}
\]

**Corollary 4.13.** Let \( \mathcal{D} = \{ L_i \}_{i \in J} \) be a strong duality datum. For any \( i \in J \) and any simple module \( M \in R_C\text{-gmod} \), we have

(i) \( \varepsilon_i(M) = \varepsilon(\mathcal{D}_L, \mathcal{F}_D(M)) \),
(ii) \( \varepsilon^*_i(M) = \varepsilon(\mathcal{D}_L^{-1}, \mathcal{F}_D(M)) \).

**Proof.** It follows from [26, Corollary 3.8] and Theorem 4.12 (v). \( \square \)

**Corollary 4.14.** Let \( \mathcal{D} = \{ L_i \}_{i \in J} \) be a strong duality datum associated with a simply-laced finite Cartan matrix \( C \). Then the duality functor \( \mathcal{F}_D \) induces an injective ring homomorphism
\[
K_{q=1}(R_C\text{-gmod}) \hookrightarrow K(C),
\]
where \( K_{q=1}(R_C\text{-gmod}) \) is the specialization of the \( K(R_C\text{-gmod}) \) at \( q = 1 \).

**Proof.** Thanks to Theorem 4.10, it is enough to show that \( \mathcal{F}_D(M) \not\cong \mathcal{F}_D(N) \) for any non-isomorphic pair of simple \( R \)-modules \( M \) and \( N \). Let \( M \) and \( N \) be simple \( R \)-modules such that \( \mathcal{F}_D(M) \cong \mathcal{F}_D(N) \).
We set \( \beta := -\text{wt}(M) \) and \( \gamma := -\text{wt}(N) \). We shall show \( M \cong N \).

We first assume that \( \mathcal{F}_D(M) = \mathcal{F}_D(N) = 1 \). Then we have \( (\beta, \beta) = -\Lambda^\infty(M, M) = 0 \), which implies \( \beta = 0 \). Hence \( M \cong 1 \). Similarly, we have \( N \cong 1 \).

We now assume that \( \mathcal{F}_D(M) \cong \mathcal{F}_D(N) \not\cong 1 \). Since \( M \not\cong 1 \), there exists \( i \in J \) such that \( \varepsilon_i(M) > 0 \). By Corollary 4.13, we have
\[
\varepsilon_i(M) = \Lambda(\mathcal{D}_L, \mathcal{F}_D(M)) = \Lambda(\mathcal{D}_L, \mathcal{F}_D(N)) = \varepsilon_i(N),
\]
which tells us that \( \tilde{e}_i(M) \neq 0 \) and \( \tilde{e}_i(N) \neq 0 \). Setting \( M' := \tilde{e}_i(M) \) and \( N' := \tilde{e}_i(N) \), we have

\[
L_i \nabla \mathcal{F}_D(M') \simeq \mathcal{F}_D(M) \simeq \mathcal{F}_D(N) \simeq L_i \nabla \mathcal{F}_D(N'),
\]

which implies that \( \mathcal{F}_D(M') \simeq \mathcal{F}_D(N') \) by Lemma 2.8. Thus, by the standard induction argument, we conclude that

\[
\tilde{e}_i(M) = M' \simeq N' = \tilde{e}_i(N),
\]

which yields that \( M \simeq N \). \( \square \)

**Lemma 4.15.** Let \( M \) be a real simple module in \( R\text{-}g\text{mod} \). Then \( \mathcal{F}_D(M) \) is a root module if and only if \( \text{wt}(M) \) is a root of \( g_{\text{fin}} \).

**Proof.** Set \( V = \mathcal{F}_D(M) \). Then we have \( \mathfrak{b}(\mathcal{D}^kV, V) = 0 \) for \( k \neq \pm 1 \). Hence we have \( (\text{wt}(M), \text{wt}(M)) = -\Lambda^\infty(V, V) = 2 \mathfrak{b}(\mathcal{D}V, V) \). Therefore we have

\[
V \text{ is a root module} \iff \mathfrak{b}(\mathcal{D}V, V) = 1 \iff (\text{wt}(M), \text{wt}(M)) = 2 \iff \text{wt}(M) \text{ is a root}.
\]

\( \square \)

5. **Strong duality datum and affine cuspidal modules**

5.1. **Unmixed pairs.**

The notion of an unmixed pair of modules over quiver Hecke algebras has an analogue for modules over quantum affine algebras.

**Definition 5.1.** Let \( (M, N) \) be an ordered pair of simple modules in \( \mathcal{C}_g \). We call it unmixed if

\[ \mathfrak{b}(\mathcal{D}M, N) = 0, \]

and strongly unmixed if

\[ \mathfrak{b}(\mathcal{D}^kM, N) = 0 \quad \text{for any} \ k \in \mathbb{Z}_{\geq 1}. \]

**Lemma 5.2.** Let \( M \) and \( N \) be simple modules in \( \mathcal{C}_g \). If \( (M, N) \) is strongly unmixed, then \( \Lambda^\infty(M, N) = \Lambda(M, N) \).

**Proof.** It follows from Definition 5.1 and Proposition 2.16 that

\[
\Lambda(M, N) = \sum_{k \in \mathbb{Z}} (-1)^{k+\delta(k>0)} \mathfrak{b}(\mathcal{D}^kM, N) = \sum_{k \in \mathbb{Z}} (-1)^k \mathfrak{b}(\mathcal{D}^kM, N) = \Lambda^\infty(M, N).
\]

\( \square \)

**Lemma 5.3.** Let \( L_1, \ldots, L_r \) be real simple modules in \( \mathcal{C}_g \) for \( r \in \mathbb{Z}_{>1} \). If \( (L_a, L_b) \) is unmixed for any \( a < b \), then \( (L_1, \ldots, L_r) \) is normal.
Proof. We shall argue by induction on $r$. Since it is obvious when $r = 2$, we assume that $r > 2$. By the induction hypothesis, $(L_1, \ldots, L_{r-1})$ is normal. Set $X = \text{hd}(L_2 \otimes \cdots \otimes L_{r-1})$. Then Lemma 2.20 implies that

$$\Lambda(L_1, X) = \sum_{k=2}^{r-1} \Lambda(L_1, L_k).$$

Since $(L_1, L_r)$ is unmixed, Lemma 2.21 implies that $(L_1, X, L_r)$ is normal. Hence we have

$$\Lambda(L_1, \text{hd}(L_2 \otimes \cdots \otimes L_r)) = \Lambda(L_1, X \nabla L_r) = \Lambda(L_1, X) + \Lambda(L_1, L_r),$$

which implies that

$$\Lambda(L_1, \text{hd}(L_2 \otimes \cdots \otimes L_r)) = \sum_{k=2}^{r} \Lambda(L_1, L_k).$$

Since $(L_2, \ldots, L_r)$ is normal, Lemma 2.20 implies that $(L_1, \ldots, L_r)$ is normal. □

5.2. Affine cuspidal modules. Let $D = \{L_i\}_{i \in J}$ be a strong duality datum in $C_0^g$ associated with a simply-laced finite Cartan matrix $C = (c_{i,j})_{i,j \in J}$. Let $R_C$ be the symmetric quiver Hecke algebra associated with $C$.

The category $C_D$ is defined to be the smallest full subcategory of $C_0^g$ such that

(a) it contains $F_D(L)$ for any simple $R_C$-module $L$,
(b) it is stable by taking subquotients, extensions, and tensor products.

Since $b(\mathcal{D}^k L_i, L_i) = 0$ for any $i, j \in J$ and $k \geq 2$, it follows from Theorem 4.12 that

(5.1) $b(\mathcal{D}^k M, N) = 0$ for any simple module $M, N \in C_D$ and $k \geq 2$.

For $k \in \mathbb{Z}$, let $D^k(C_D)$ be the full subcategory of $C_0^g$ whose objects are $D^k M$ for all $M \in C_D$.

Proposition 5.4. Let $k \in \mathbb{Z}$ with $k \neq 0$. If a simple module $M$ is contained in $C_D \cap D^k(C_D)$, then $M \cong 1$.

Proof. We may assume $k > 0$ without loss of generality. Let $M$ be a simple module in $C_D \cap D^k(C_D)$. By Theorem 4.10, there exists a simple module $V \in R_C\text{-gmod}$ such that $F_D(V) \cong M$. By Corollary 4.13 and Theorem 4.12 (iv), for any $i \in J$, we have

$$\varepsilon_i^*(V) = b(\mathcal{D}^{-1} L_i, M) = b(L_i, D M) = 0.$$  

Thus $V$ should be in $R_C(0)\text{-gmod}$, which says that $V \cong 1$. □

Lemma 5.5. Let $M, N$ be simple modules in $R_C\text{-gmod}$. If $(M, N)$ is unmixed, then $(F_D(M), F_D(N))$ is strongly unmixed.
Proof. By (5.1), we know that \( \mathfrak{e}(\mathcal{D}^k\mathcal{F}_D(M), \mathcal{F}_D(N)) = 0 \) for \( k \geq 2 \). It follows from [26, Proposition 2.12] that \( \Lambda(M, N) = -(\text{wt}(M), \text{wt}(N)) \), i.e., \( \widetilde{\Lambda}(M, N) = 0 \). Thus, by Theorem 4.12 (v), we obtain

\[
\mathfrak{e}(\mathcal{D}\mathcal{F}_D(M), \mathcal{F}_D(N)) = \widetilde{\Lambda}(M, N) = 0,
\]

which completes the proof. \( \square \)

Let \( \mathfrak{g}_C \) be the simple Lie algebra associated with \( C \). Let \( \Phi_C^+ \) be the set of positive roots of \( \mathfrak{g}_C \) and let \( \mathcal{W}_C \) be the Weyl group associated with \( \mathfrak{g}_C \). Let \( w_0 \) be the longest element of \( \mathcal{W}_C \), and \( \ell \) denotes the length of \( w_0 \). We choose an arbitrary reduced expression \( w_0 = s_{i_1} s_{i_2} \cdots s_{i_\ell} \) of the longest element \( w_0 \) of \( \mathcal{W}_C \). We extend \( \{i_k\}_{1 \leq k \leq \ell} \) to \( \{i_k\}_{k \in \mathbb{Z}} \) by

\[
i_{k+\ell} = (i_k)^* \quad \text{for any } k \in \mathbb{Z}.
\]

(Recall that, for \( i \in J \), \( i^* \) is a unique element of \( J \) such that \( \alpha_i^* = -w_0\alpha_i \).)

We can easily see that \( s_{i_{a+1}} \cdots s_{i_{a+\ell}} \) is also a reduced expression of \( w_0 \) for any \( a \in \mathbb{Z} \). Let

\[
\{V_k\}_{k=1, \ldots, \ell} \subset R_C\text{-gmod}
\]

be the cuspidal modules associated with the reduced expression \( w_0 \). Under the categorification, the cuspidal module \( V_k \) corresponds to the dual PBW vector \( E^*(\beta_k) \) corresponding to \( \beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Phi_C^+ \) for \( k = 1, \ldots, \ell \) (see Section 2.2).

We now introduce the notion of affine cuspidal modules for quantum affine algebras.

**Definition 5.6.** We define a sequence of simple \( U'_q(\mathfrak{g}) \)-modules \( \{S_k\}_{k \in \mathbb{Z}} \) in \( \mathcal{C}_B \) as follows:

(a) \( S_k = \mathcal{F}_D(V_k) \) for any \( k = 1, \ldots, \ell \), and we extend its definition to all \( k \in \mathbb{Z} \) by

(b) \( S_{k+\ell} = \mathcal{D}(S_k) \) for any \( k \in \mathbb{Z} \).

The modules \( S_k \) \( (k \in \mathbb{Z}) \) are called the affine cuspidal modules corresponding to \( \mathcal{D} \) and \( w_0 \).

**Proposition 5.7.** The affine cuspidal modules satisfy the following properties.

(i) \( S_a \) is a root module for any \( a \in \mathbb{Z} \).

(ii) For any \( a, b \in \mathbb{Z} \) with \( a > b \), the pair \( (S_a, S_b) \) is strongly unmixed.

(iii) Let \( k_1 > \cdots > k_l \) be decreasing integers and \( (a_1, \ldots, a_l) \in \mathbb{Z}_{\geq 0}^l \). Then

(a) the sequence \( (S_{k_1}^{a_1}, \ldots, S_{k_l}^{a_l}) \) is normal,

(b) the head of the tensor product \( S_{k_1}^{a_1} \otimes \cdots \otimes S_{k_l}^{a_l} \) is simple.

Proof. (i) follows immediately from Lemma 4.15.

(ii) Without loss of generality, we may assume that \( 1 \leq b \leq \ell \). We write \( a = \ell \cdot t + r \) for some \( t \in \mathbb{Z}_{\geq 0} \) and \( 1 \leq r \leq \ell \). By the definition, we have \( S_a = \mathcal{D}^t S_r \). If \( t \geq 1 \), then we have

\[
\mathfrak{e}(\mathcal{D}^k S_a, S_b) = \mathfrak{e}(\mathcal{D}^{k+t} S_r, S_b) = 0 \quad \text{for any } k \geq 1,
\]

by (5.1).
Suppose that \( t = 0 \). As \( \ell \geq a > b \geq 1 \), the pair \((V_a, V_b)\) is unmixed. Thus Lemma 5.5 says that \((S_a, S_b)\) is strongly unmixed. (iii) follows from Lemma 5.3 and Lemma 2.19. \(\square\)

**Example 5.8.** Let \( U'_q(\mathfrak{g}) \) be the quantum affine algebra of affine type \( A_2^{(1)} \), and let \( \mathcal{C}_q^0 \) be the Hernandez-Leclerc category corresponding to \( \sigma_0(\mathfrak{g}) = \{(1, (-q)^{2k}), (2, (-q)^{2k+1}) \mid k \in \mathbb{Z}\} \). For \( i \in I_0 \) and \( m \in \mathbb{Z}_{>0} \), we denote the Kirillov-Reshetikhin module by

\[
V(i^m) := \text{hd} \left( V(\varpi_i)(-q)^{m-1} \otimes V(\varpi_i)(-q)^{m-3} \otimes \cdots \otimes V(\varpi_i)(-q)^{m+1} \right).
\]

We simply write \( V(i) \) instead of \( V(i^1) \), which is the \( i \)th fundamental module \( V(\varpi_i) \).

Let \( L_1 := V(1) \) and \( L_2 := V(1)(-q)^2 \), and define \( \mathcal{D} := \{L_1, L_2\} \subset \mathcal{C}_q^0 \). Then \( \mathcal{D} \) is a strong duality datum (see [21, Section 4.1]). Let \( C \) be the Cartan matrix of finite type \( A_2 \). Then we have the duality functor \( \mathcal{F}_\mathcal{D}: R_C \text{-gmod} \rightarrow \mathcal{C}_q^0 \).

(i) We choose a reduced expression \( w_0 = s_1s_2s_1 \). Then we have

\[
\beta_1 := \alpha_1, \quad \beta_2 := s_1(\alpha_2) = \alpha_1 + \alpha_2, \quad \beta_3 = s_1s_2(\alpha_1) = \alpha_2,
\]

and the affine cuspidal modules corresponding to \( \mathcal{D} \) and \( w_0 \) are given as follows:

\[
\begin{align*}
S_1 &= \mathcal{F}_\mathcal{D}(L(1)) = L_1 = V(1), \\
S_2 &= \mathcal{F}_\mathcal{D}(L(1) \nabla L(2)) = L_1 \nabla L_2 = V(1) \nabla V(1)(-q)^2 = V(2)(-q), \\
S_3 &= \mathcal{F}_\mathcal{D}(L(2)) = L_2 = V(1)(-q)^2,
\end{align*}
\]

and \( S_{k+3} = \mathcal{D}(S_k) \) for \( k \in \mathbb{Z} \). Here \( L(i) \) be the self-dual 1-dimensional simple \( R(\alpha_i) \)-module. It is easy to see that the set \( \{S_k \mid k \in \mathbb{Z}\} \) of all affine cuspidal modules is equal to the set of all fundamental modules in \( \mathcal{C}_q^0 \).

(ii) We choose another reduced expression \( w'_0 = s_2s_1s_2 \). Then we have

\[
\begin{align*}
\beta_1' := \alpha_2, \quad \beta_2' := s_2(\alpha_1) = \alpha_1 + \alpha_2, \quad \beta_3' = s_2s_1(\alpha_2) = \alpha_1,
\end{align*}
\]

and the affine cuspidal modules corresponding to \( \mathcal{D} \) and \( w'_0 \) are given as follows:

\[
\begin{align*}
S_1' &= \mathcal{F}_\mathcal{D}(L(2)) = L_2 = V(1)(-q)^2, \\
S_2' &= \mathcal{F}_\mathcal{D}(L(2) \nabla L(1)) = L_2 \nabla L_1 = V(1)(-q)^2 \nabla V(1) = V(1^2)(-q), \\
S_3' &= \mathcal{F}_\mathcal{D}(L(1)) = L_1 = V(1),
\end{align*}
\]

and \( S'_{k+3} = \mathcal{D}(S'_k) \) for \( k \in \mathbb{Z} \). Note that the affine cuspidal modules \( S'_{2+3t} \) \( (t \in \mathbb{Z}) \) are not fundamental modules.

### 5.3. Reflections

For any \( k \in J \), we set

\[
\mathcal{S}_k(\mathcal{D}) := \{\mathcal{S}_k(L_i)\}_{i \in J} \quad \text{and} \quad \mathcal{S}^{-1}_k(\mathcal{D}) := \{\mathcal{S}^{-1}(L_i)\}_{i \in J},
\]

(5.3)
where
\[ I_k(L_i) := \begin{cases} \mathcal{D}L_i & \text{if } i = k, \\ L_k \nabla L_i & \text{if } c_{i,k} = -1, \\ L_i & \text{if } c_{i,k} = 0, \end{cases} \]
and
\[ I_k^{-1}(L_i) := \begin{cases} \mathcal{D}^{-1}L_i & \text{if } i = k, \\ L_i \nabla L_k & \text{if } c_{i,k} = -1, \\ L_i & \text{if } c_{i,k} = 0. \end{cases} \]

It is easy to see that \( I_k \circ I_k^{-1}(\mathcal{D}) = \mathcal{D} \) and \( I_k^{-1} \circ I_k(\mathcal{D}) = \mathcal{D} \) for any \( k \in J \).

**Proposition 5.9.** Let \( k \in J \).

(i) For any \( i \in J \), \( I_k(L_i) \) and \( I_k^{-1}(L_i) \) are root modules.

(ii) \( I_k(\mathcal{D}) \) and \( I_k^{-1}(\mathcal{D}) \) are strong duality data associated with the Cartan matrix \( C \).

**Proof.** We shall focus on proving the case for \( I_k \) since the case for \( I_k^{-1} \) can be proved in a similar manner.

Set \( L'_i := I_k(L_i) \) for \( i \in J \). For \( i, j \in J \), we write \( i \sim j \) if \( c_{i,j} = -1 \) and \( i \not\sim j \) if \( c_{i,j} = 0 \). Note that, for real simple modules \( L, M \) and \( N \), Lemma 2.21 says that if one of the following conditions

- \( \mathfrak{b}(L, M) = 0 \),
- \( \mathfrak{b}(M, N) = 0 \),
- \( \mathfrak{b}(L, \mathcal{D}^{-1}N) = \mathfrak{b}(\mathcal{D}L, N) = 0 \)

holds, then

\[ \Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N), \quad \Lambda(L \nabla M, N) = \Lambda(L, N) + \Lambda(M, N), \]

which will be used several times in the proof.

(i) follows from Lemma 3.8.

(ii) Thanks to (i), it suffices to prove that

\[ \mathfrak{b}(\mathcal{D}^t L'_i, L'_j) = -\delta(t = 0) c_{i,j} \quad \text{for } t \in \mathbb{Z} \text{ and } i \not\sim j. \]

Let \( i, j \in J \) with \( i \not\sim j \). We shall prove it case by case.

**Case 1:** If \( i \not\sim k \) and \( j \not\sim k \), then

\[ \mathfrak{b}(\mathcal{D}^t L'_i, L'_j) = \mathfrak{b}(\mathcal{D}^t L_i, L_j) = -\delta(t = 0) c_{i,j} \quad \text{for } t \in \mathbb{Z}. \]

**Case 2:** If \( i \not\sim k \) and \( j = k \), then \( c_{i,j} = 0 \) and

\[ \mathfrak{b}(\mathcal{D}^t L'_i, L'_j) = \mathfrak{b}(\mathcal{D}^t L_i, \mathcal{D} L_j) = \mathfrak{b}(\mathcal{D}^{-1} L_i, L_j) = 0 = -\delta(t = 0) c_{i,j}. \]

**Case 3:** Suppose that \( i \not\sim k \) and \( j \sim k \). Then we have

\[ \mathfrak{b}(\mathcal{D}^t L'_i, L'_j) = \mathfrak{b}(\mathcal{D}^t L_i, L_k \nabla L_j). \]

Then we have

\[ \Lambda(\mathcal{D}^t L_i, L_k \nabla L_j) = \Lambda(\mathcal{D}^t L_i, L_k) + \Lambda(\mathcal{D}^t L_i, L_j), \]

\[ \Lambda(L_k \nabla L_j, \mathcal{D}^t L_i) = \Lambda(L_k, \mathcal{D}^t L_i) + \Lambda(L_j, \mathcal{D}^t L_i), \]
where the first equality follows from \( b(\mathcal{D}^t L_i, L_k) = 0 \) and the second from \( b(\mathcal{D}^t L_k, \mathcal{D}^t L_i) = 0 \). Hence we obtain
\[
b(\mathcal{D}^t L_i', L_j') = b(\mathcal{D}^t L_i, L_k) + b(\mathcal{D}^t L_i, L_j) = b(\mathcal{D}^t L_i, L_j) = -\delta(t = 0)c_{i,j}.
\]

**Case 4:** Suppose that \( i \sim k \) and \( j \sim k \). Then, by Lemma 2.23,
\[
b(\mathcal{D}^t L_i', L_j') = b(\mathcal{D}^t (L_k \nabla L_i), L_k \nabla L_j) = b(\mathcal{D}^t L_k \nabla \mathcal{D}^t L_i, L_k \nabla L_j).
\]
Since \( C \) is of finite type, we have \( c_{i,j} = 0 \), i.e., \( b(\mathcal{D}^t L_i, L_j) = 0 \) for any \( t \in \mathbb{Z} \).

(a) If \( t \neq 0, \pm 1 \), then
\[
b(\mathcal{D}^t L_i', L_j') = b(\mathcal{D}^t (L_k \nabla L_i), L_k \nabla L_j) = 0
\]
since \( b(\mathcal{D}^t (L_a), L_b) = 0 \) for \( a, b = i, j, k \) by Theorem 4.12 (iv).

(b) Suppose that \( t = 0 \). Then
\[
\Lambda(L_k \nabla L_i, L_k \nabla L_j) = \Lambda(L_k \nabla L_i, L_k) + \Lambda(L_k \nabla L_i, L_j)
\]
\[
= -\Lambda(L_k, L_k \nabla L_i) + \Lambda(L_k, L_k \nabla L_i, L_j)
\]
\[
= -\Lambda(L_k, L_i) + \Lambda(L_k, L_j) + \Lambda(L, L_i, L_j).
\]
Here the first and second identities follow from \( b(L_k, L_k \nabla L_i) = 0 \) by Lemma 3.9 and the third follows from \( b(L_i, L_j) = 0 \). Exchanging \( i \) and \( j \), we have
\[
\Lambda(L_k \nabla L_j, L_k \nabla L_i) = -\Lambda(L_k, L_i) + \Lambda(L_k, L_j) + \Lambda(L_j, L_i),
\]
which tells us that
\[
b(L_i', L_j') = b(L_k \nabla L_j, L_k \nabla L_i) = b(L_i, L_j) = -c_{i,j}.
\]

(c) Suppose that \( t = \pm 1 \). We have
\[
\Lambda^\infty(L_i', L_j') = \Lambda^\infty(L_k \nabla L_i, L_k \nabla L_j)
\]
\[
= \Lambda^\infty(L_k, L_k) + \Lambda^\infty(L_k, L_j) + \Lambda^\infty(L_i, L_k) + \Lambda^\infty(L_i, L_j) = (-2) + 1 + 1 + 0 = 0.
\]
Hence we have
\[
0 = \sum_{t \in \mathbb{Z}} (-1)^t b(\mathcal{D}^t L_i', L_j') = -b(\mathcal{D} L_i', L_j') - b(\mathcal{D}^{-1} L_i', L_j').
\]
Hence \( b(\mathcal{D} L_i', L_j') = b(\mathcal{D}^{-1} L_i', L_j') = 0 \).

**Case 5:** Suppose that \( i \sim k \) and \( j = k \). Then, we have
\[
b(\mathcal{D}^t L_j', L_i') = b(\mathcal{D}^{t+1} L_k, L_k \nabla L_i) \quad \text{for} \ t \in \mathbb{Z},
\]
which is equal to \( \delta(t = 0) \) by Lemma 3.9. \( \square \)
Proposition 5.10. Let \( \{S_k\}_{k \in \mathbb{Z}} \) be the sequence of the affine cuspidal modules corresponding to \( D \) and a reduced expression \( w_0 = s_{i_1} \cdots s_{i_\ell} \) of \( w_0 \). Set \( S_k = S_{k+1} \) for \( k \in \mathbb{Z} \). Then \( \{S'_k\}_{k \in \mathbb{Z}} \) is the affine cuspidal modules corresponding to \( \mathcal{J}_i D \) and the reduced expression \( w'_0 = s_{i_2} \cdots s_{i_{\ell+1}} \) (see (5.2)).

Proof. Set \( i = i_1 \). We denote by \( \prec, \{\beta_k\}_{k = 1, \ldots, \ell} \) and \( \{V_k\}_{k = 1, \ldots, \ell} \) the convex order, the ordered set of positive roots and the cuspidal modules in \( R_{C\text{-}mod} \) corresponding to \( w_0 \) as in Section 2.2. Similarly, we write \( \prec', \{\beta'_k\}_{k = 1, \ldots, \ell} \) and \( \{V'_k\}_{k = 1, \ldots, \ell} \) for the ones corresponding to \( w'_0 \). It is enough to show that

\[
\mathcal{F}_{\mathcal{J}_i(D)}(V'_k) \simeq S_{k+1} \quad \text{for } 1 \leq k \leq \ell.
\]

It is easy to see that

\[
\begin{align*}
\beta_{k+1} &= s_i \beta'_k \quad \text{for } k = 1, \ldots, \ell - 1, \\
V_{k+1} &= T_i(V'_k) \quad \text{for } k = 1, \ldots, \ell - 1, \\
\alpha_i &= \text{is smallest (resp. largest) with respect to } \prec \text{ (resp. } \prec' \text{).}
\end{align*}
\]

It follows from (5.6) that \( V_1 \simeq L(i) \simeq V'_\ell \). Thus we have

\[
\mathcal{F}_{\mathcal{J}_i(D)}(V'_1) \simeq \mathcal{D} L_i \simeq \mathcal{D}(\mathcal{F}_D(V_1)) = S_{\ell+1}.
\]

It remains to prove that

\[
\mathcal{F}_{\mathcal{J}_i(D)}(V'_k) \simeq \mathcal{F}_D(V_{k+1}) \quad \text{for } k = 1, \ldots, \ell - 1.
\]

We shall use induction on \( \text{ht}(\beta'_k) \).

If \( \text{ht}(\beta'_k) = 1 \), then \( \beta'_k = \alpha_j \) for some \( j \in J \). Note that \( j \neq i \) because \( k < \ell \). Thus, we have \( \beta_{k+1} = s_i(\beta'_k) = s_i(\alpha_j) \) and

\[
V_{k+1} = \begin{cases} 
L(i) \nabla L(j) & \text{if } c_{i,j} = -1, \\
L(j) & \text{otherwise.}
\end{cases}
\]

By the definition of \( \mathcal{J}_i \), we have

\[
\mathcal{F}_{\mathcal{J}_i(D)}(V'_k) \simeq \mathcal{F}_{\mathcal{J}_i(D)}(L(j)) \simeq \mathcal{J}_i(L_j) \simeq \mathcal{F}_D(V_{k+1}).
\]

Suppose that \( \text{ht}(\beta'_k) > 1 \). We take a minimal pair \( (\beta'_a, \beta'_b) \) of \( \beta'_k \) with respect to \( \prec' \). It follows from (2.2) that

\[
V'_a \nabla V'_b \simeq V'_k.
\]

(Case 1): Suppose that \( b \neq \ell \). Applying \( T_j \) to (5.9), it follows from (5.5) that

\[
V_{a+1} \nabla V_{b+1} \simeq V_{k+1}.
\]

Applying \( \mathcal{F}_D \) to the above isomorphism (5.10) and using the induction hypothesis, we have

\[
\mathcal{F}_D(V_{k+1}) \simeq \mathcal{F}_D(V_{a+1}) \nabla \mathcal{F}_D(V_{b+1}).
\]
\[ F_{\mathcal{J}(D)}(V_a') \simeq F_{\mathcal{J}(D)}(V_b') \simeq F_{\mathcal{J}(D)}(V_k'). \]

**Case 2:** Suppose that \( b = \ell \). Since \( V_b' = L(i) \), by applying \( F_{\mathcal{J}(D)} \) to (5.9), we have
\[
F_{\mathcal{J}(D)}(V_a') \nabla \mathcal{D}L_i \simeq F_{\mathcal{J}(D)}(V_k').
\]
On the other hand, it follows from \( \tilde{f}_i(V_a') = V_a' \nabla L(i) \simeq V_k' \) that
\[
\varepsilon_i(V_a') + 1 = \varepsilon_i(V_k'), \quad \varphi_i(V_a') = \varphi_i(V_k') + 1.
\]
Thus, by (2.3) and (5.5), we have
\[
V_{a+1} = T_i(V_a') \simeq \tilde{f}_i^2(V_a') + 1 \varepsilon_i(V_a') + 1 \tilde{f}_i V_a' \nabla L(i) \tilde{f}_i(V_k') \simeq L(i) \nabla T_i(V_k') = L(i) \nabla V_{k+1},
\]
which implies that
\[
F_D(V_{a+1}) \simeq L_i \nabla F_D(V_{k+1}).
\]
Then, we have
\[
F_D(V_{k+1}) \simeq F_D(V_{a+1}) \nabla \mathcal{D}L_i \quad \text{by (5.12) and Lemma 2.7}
\]
\[
\simeq F_{\mathcal{J}(D)}(V_a') \nabla \mathcal{D}L_i \quad \text{by the induction hypothesis}
\]
\[
\simeq F_{\mathcal{J}(D)}(V_k') \quad \text{by (5.11)},
\]
which completes the proof for (5.8).

**Proposition 5.11.** Let \( i \in J \), and let \( S \) be a simple module in \( \mathcal{C}_b \).

(i) The following conditions are equivalent:

(a) \( S \in F_D(C_{s,s_i}) \),

(b) \( S \in \mathcal{C}_b \) and \( \mathcal{B}(\mathcal{D}^{-1}L_i, S) = 0 \),

(c) \( S \in \mathcal{C}_{\mathcal{J}(D)} \cap \mathcal{C}_D \).

(ii) The following conditions are equivalent:

(a) \( S \in F_D(C_{s_s u_0}) \),

(b) \( S \in \mathcal{C}_b \) and \( \mathcal{B}(\mathcal{D}L_i, S) = 0 \),

(c) \( S \in \mathcal{C}_{\mathcal{J}^{-1}(D)} \cap \mathcal{C}_D \).

Here, \( \mathcal{C}_{s,s_i} \) and \( \mathcal{C}_{s,s u_0} \) are the subcategories of \( R_C \text{-gmod} \) appeared in § 2.2.

**Proof.** We shall focus on proving the assertion (i) since the assertion (ii) can be proved in a similar manner.

Let us take a reduced expression \( u_0 = s_{i_1} \cdots s_{i_\ell} \) of \( u_0 \) such that \( i_1 = i \). Let \( \{V_k\}_{k=1,\ldots,\ell} \) be the cuspidal modules in \( R_C \text{-gmod} \) corresponding to \( u_0 \). Let \( \{S_k\}_{k \in \mathbb{Z}} \) be the affine cuspidal modules corresponding to \( \mathcal{D} \) and \( u_0 \). Set \( S_k = S_{k+1} \) or \( k \in \mathbb{Z} \). Then \( \{S_k\}_{k \in \mathbb{Z}} \) is the cuspidal modules corresponding to \( \mathcal{J}_i \mathcal{D} \) and \( u_0' = s_{i_2} \cdots s_{i_{\ell+1}} \) by Proposition 5.10.

Now we shall prove (i). It is known that
Example 5.12. We use the same notations given in Example 5.8.

(i) We shall apply $\mathcal{A}_1$ to the duality datum $\mathcal{D} = \{L_1, L_2\}$. Let

$\tilde{L}_1 := \mathcal{A}_1(L_1) = DL_1 = V(2)^{-q^3},$
$\tilde{L}_2 := \mathcal{A}_1(L_2) = L_1 \triangledown L_2 = V(2)^{-q}.$

Then we have $\mathcal{A}_1(\mathcal{D}) = \{\tilde{L}_1, \tilde{L}_2\}$. The affine cuspidal modules $\tilde{S}_k$ corresponding to $\mathcal{A}_1(\mathcal{D})$ and the reduced expression $s_2s_1s_2$ are given as follows:

$\tilde{S}_1 = \mathcal{F}_{\mathcal{A}_1, D}(L(2)) = \tilde{L}_2 = V(2)^{-q},$
$\tilde{S}_2 = \mathcal{F}_{\mathcal{A}_1, D}(L(2) \triangledown L(1)) = \tilde{L}_2 \triangledown \tilde{L}_1 = V(2)^{-q} \triangledown V(2)^{-q^3} = V(1)^{-q^2},$
$\tilde{S}_3 = \mathcal{F}_{\mathcal{A}_1, D}(L(1)) = \tilde{L}_1 = V(2)^{-q^3},$

and $\tilde{S}_{k+3} = D(\tilde{S}_k)$ for $k \in \mathbb{Z}$. Note that $\tilde{S}_k = S_{k+1}$ for any $k \in \mathbb{Z}$ (see Proposition 5.10).

(ii) We shall apply $\mathcal{A}_2$ to the duality datum $\mathcal{D} = \{L_1, L_2\}$. Let

$\hat{L}_1 := \mathcal{A}_2(L_1) = L_2 \triangledown L_1 = V(1)^{-q^2} \triangledown V(1) = V(1^2)^{-q},$
$\hat{L}_2 := \mathcal{A}_2(L_2) = DL_2 = V(2)^{-q^5}.$

Then we have $\mathcal{A}_2(\mathcal{D}) = \{\hat{L}_1, \hat{L}_2\}$. As you observe, the duality datum $\mathcal{A}_2(\mathcal{D})$ has a root module which is not fundamental.

The affine cuspidal modules $\hat{S}_k$ corresponding to $\mathcal{A}_2(\mathcal{D})$ and the reduced expression $s_1s_2s_1$ are given as follows:

$\hat{S}_1 = \mathcal{F}_{\mathcal{A}_2, D}(L(1)) = \hat{L}_1 = V(1^2)^{-q},$
\[ \hat{S}_2 = F_{\mathcal{D}}(L(1) \triangledown L(2)) = \hat{L}_1 \triangledown \hat{L}_2 = V(1^2) \triangledown V(2)(-q)^{5} \]

\[ \hat{S}_3 = F_{\mathcal{D}}(L(2)) = \hat{L}_2 = V(2)(-q)^{5} \]

and \( \hat{S}_{k+3} = \mathcal{D}(\hat{S}_k) \) for \( k \in \mathbb{Z} \). Note that \( \hat{S}_k = S_{k+1} \) for any \( k \in \mathbb{Z} \) (see Proposition 5.10).

6. PBW theoretic approach

6.1. Complete duality datum.

**Definition 6.1.** A duality datum \( \mathcal{D} \) is called complete if it is strong and, for any simple module \( M \in \mathcal{C}_g^0 \), there exists simple modules \( M_k \in \mathcal{C}_k \) \((k \in \mathbb{Z})\) such that

(a) \( M_k \simeq 1 \) for all but finitely many \( k \),
(b) \( M \simeq \text{hd}(\cdots \otimes \mathcal{P}^2 M_2 \otimes \mathcal{P} M_1 \otimes M_0 \otimes \mathcal{P}^{-1} M_{-1} \otimes \cdots) \).

In [29], we associate to the category \( \mathcal{C}_g^0 \) a simply laced finite type root system in a canonical way. For a simple module \( M \in \mathcal{C}_g^0 \), set \( E(M) \in \text{Hom}(\sigma(g), \mathbb{Z}) \) by

\[ E(M)(i, a) := \Lambda^\infty(M, V(\varpi_i)_a) \quad \text{for } (i, a) \in \sigma(g). \]

Let

\[ \mathcal{W}_0 := \{ E(M) \mid M \text{ is simple in } \mathcal{C}_g^0 \} \quad \text{and} \quad \Delta_0 := \{ s_{i, a} \mid (i, a) \in \sigma_0(g) \} \subset \mathcal{W}_0, \]

where we set \( s_{i, a} := E(V(\varpi_i)_a) \). Then \( \Upsilon_g := (\mathcal{W}_0, \Delta_0) \) forms a root system, and the type of \( \Upsilon_g \) is given as follows ([29, Theorem 3.6]):

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{Type of } g & A_n^{(1)} & B_n^{(1)} & C_n^{(1)} & D_n^{(1)} & A_{2n}^{(2)} & A_{2n-1}^{(2)} & D_{n+1}^{(2)} \\
\hline
\text{Type of } \Upsilon_g & A_n & A_{2n-1} & D_{n+1} & D_n & A_{2n} & A_{2n-1} & D_{n+1} \\
\hline
\text{Type of } g & E_6^{(1)} & E_7^{(1)} & E_8^{(1)} & F_4^{(1)} & G_2^{(1)} & E_6^{(2)} & D_4^{(3)} \\
\hline
\text{Type of } \Upsilon_g & E_6 & E_7 & E_8 & E_6 & D_4 & E_6 & D_4 \\
\hline
\end{array}
\]

We denote by \( X_g \) the type of \( \Upsilon_g \).

We define a symmetric bilinear form \( (\ , \ ) \) on \( \mathcal{W}_0 \) by \( (E(M), E(N)) = -\Lambda^\infty(M, N) \) for simple modules \( M \) and \( N \). Then \( (\ , \ ) \) is a Weyl group invariant positive definite bilinear form and \( \Delta_0 = \{ \alpha \in \mathcal{W}_0 \mid (\alpha, \alpha) = 2 \} \).

**Proposition 6.2.** Let \( \mathcal{D} := \{ L_i \}_{i \in J} \subset \mathcal{C}_g^0 \) be a complete duality datum associated with a simply-laced finite Cartan matrix \( C \). Then \( C \) is of type \( X_g \).
Proof. We denote by $Q_C$ and $\Phi_C$ the root lattice and the set of roots associated with $C$.

It follows from Proposition 2.11, Proposition 2.13, Theorem 4.10 and Definition 6.1, the abelian group $\mathcal{W}_0$ is generated by $E(M)$ for $M \in \mathcal{C}_D$. Moreover $E(\mathcal{F}_D(M))$ depends only on wt$(M)$ by Theorem 4.12 (iii). Hence the functor $\mathcal{F}_D$ induces the surjective additive map

$$[\mathcal{F}_D]: Q_C \to \mathcal{W}_0$$

given by $[\mathcal{F}_D](\alpha_i) = E(L_i)$ for $i \in J$. Moreover, $[\mathcal{F}_D]$ preserves the positive definite pairing $(-, -)$. Hence $[\mathcal{F}_D]$ is bijective. Since both of $\Phi_C$ and $\Delta_0$ are characterized by the condition $(X, X) = 2$ ([29, Corollary 3.8] and [19, Proposition 5.10]), the set $\{E(L_i)\}_{i \in J}$ becomes a basis of the root system $\Upsilon_g$. Since $c_{ij} = (\alpha_i, \alpha_j) = (E(L_i), E(L_j))$ for any $i, j \in J$ by Theorem 4.12 (iii), we conclude that the Cartan matrix $C = (c_{ij})_{i, j \in J}$ is of type $X_g$. \hfill $\square$

**Theorem 6.3.** Let $\mathcal{D} := \{L_i\}_{i \in J}$ be a complete duality datum. For any $i \in J$, $\mathcal{I}_i(\mathcal{D})$ and $\mathcal{I}^{-1}_i(\mathcal{D})$ are complete.

**Proof.** We shall focus on proving the case for $\mathcal{I}_i$ since the other case for $\mathcal{I}^{-1}_i$ can be proved in a similar manner. Since $\mathcal{I}_i(\mathcal{D})$ is strong by Proposition 5.9, it suffices to show that $\mathcal{I}_i(\mathcal{D})$ satisfies the conditions of Definition 6.1.

Let $i \in J$ and choose a reduced expression $w_0 = s_{i_1}s_{i_2}\cdots s_{i_r}$ of the longest element $w_0$ of $W_C$ with $i_1 = i$. Define $\{i_k\}_{k \in \mathbb{Z}}$ and the cuspidal modules $\{S_k\}_{k \in \mathbb{Z}}$ corresponding to $\mathcal{D}$ and $w_0$ as in § 5.2. Let $M$ be a simple module in $\mathcal{C}_g^0$. As $\mathcal{D}$ is complete, there exist simple modules $M_k \in \mathcal{C}_D$ ($k \in \mathbb{Z}$) such that $M_k \simeq 1$ for all but finitely many $k$ and

$$M \simeq \text{hd}\left(1 \otimes \mathcal{D}^2 M_2 \otimes \mathcal{D} M_1 \otimes M_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots\right).$$

For each $k \in \mathbb{Z}$, there exist $a_{k, 1}, \ldots, a_{k, \ell} \in \mathbb{Z}_{\geq 0}$ such that

$$M_k \simeq \text{hd}\left(S_\ell^{\otimes a_{k, \ell}} \otimes \cdots \otimes S_1^{\otimes a_{k, 1}}\right).$$

Set $c_{s+k\ell} = a_{k, s}$ for $1 \leq s \leq \ell$ and $k \in \mathbb{Z}$. Then, by Lemma 2.23, we have

$$\mathcal{D}^k M_k \simeq \text{hd}\left(S_{\ell+1}^{\otimes c_{k\ell+1}} \otimes \cdots \otimes S_{k\ell+1}^{\otimes c_{k\ell+1}}\right).$$

Hence we have

$$M \simeq \text{hd}\left(1 \otimes S_1^{\otimes c_1} \otimes S_0^{\otimes a_0} \otimes S_{-1}^{\otimes c_{-1}} \otimes \cdots\right).$$

Set

$$N_k = \text{hd}\left(S_{\ell+1}^{\otimes c_{k\ell+1}} \otimes \cdots \otimes S_{k\ell+2}^{\otimes c_{k\ell+2}}\right).$$

Then $N_k \in \mathcal{C}_{\mathcal{I}, D}$ by Proposition 5.10, and we have

$$\mathcal{D}^k N_k \simeq \text{hd}\left(S_{k\ell+1}^{\otimes c_{k\ell+1}} \otimes \cdots \otimes S_{k\ell+2}^{\otimes c_{k\ell+2}}\right).$$

Hence we obtain

$$M \simeq \text{hd}\left(1 \otimes N_1 \otimes \mathcal{D}^0 N_0 \otimes \mathcal{D}^{-1} N_{-1} \otimes \cdots\right).$$

$\square$
6.2. Duality datum arising from Q-datum.

The subcategory $\mathcal{C}_Q$ of $\mathcal{C}'_g$ was introduced in [16] for simply-laced affine type ADE, in [23] for twisted affine type $A^{(2)}$ and $D^{(2)}$, in [34, 48] for untwisted affine type $B^{(1)}$ and $C^{(1)}$, and in [47] for exceptional affine type. Let $\mathfrak{g}_{\text{fin}}$ be the simple Lie algebra of type $X_g$ defined in (6.1) and $I_{\text{fin}}$ the index set of $\mathfrak{g}_{\text{fin}}$. The category $\mathcal{C}_Q$ categorifies the coordinate ring $\mathbb{C}[N]$ of the maximal unipotent group $N$ associated with $\mathfrak{g}_{\text{fin}}$. This category is defined by a Q-datum. A Q-datum is a triple $\Delta := (\Delta, \sigma, \xi)$ consisting of the Dynkin diagram $\Delta$ of $\mathfrak{g}_{\text{fin}}$, an automorphism $\sigma$ on $\Delta$ and a height function $\xi$, which satisfy certain conditions (see [12] for details, and also [33, §6]). When $\mathfrak{g}$ is of untwisted affine type ADE, $\sigma$ is the identity and $\varnothing$ is equal to a Dynkin quiver with a height function. To a Q-datum $\Delta$, we can associate a subset $\sigma_\Delta(\mathfrak{g})$ of $\sigma_0(\mathfrak{g})$. This set $\sigma_\Delta(\mathfrak{g})$ is in a 1-1 correspondence to the set $\Phi^+_\text{fin}$ of positive roots of $\mathfrak{g}_{\text{fin}}$, which is denoted by

\begin{equation}
\phi_\Delta: \Phi^+_\text{fin} \xrightarrow{\sim} \sigma_\Delta(\mathfrak{g}).
\end{equation}

Set

$$
\mathcal{D}_\Delta := \{L_i\}_{i \in I_{\text{fin}}},
$$

where $L_i$ is the fundamental module corresponding to $\phi_\Delta(\alpha_i)$ for $i \in I_{\text{fin}}$. Then $\mathcal{D}_\Delta$ becomes a strong duality datum ([20, 23, 34, 47, 11, 12, 33]), which gives the duality functor $\mathcal{F}_\mathcal{D}_\Delta$. By the definition, we have $\mathcal{C}_\Delta = \mathcal{C}_{\mathcal{D}_\Delta}$. We simply write $\mathcal{F}_\Delta$ for $\mathcal{F}_{\mathcal{D}_\Delta}$:

$$
\mathcal{F}_\Delta: \mathcal{C}_{\mathfrak{g}_{\text{fin}}-\text{gmod}} \to \mathcal{C}_\mathfrak{g}.
$$

We refer the reader to [12, 48, 33] for the notion of (twisted) $\varnothing$-adapted reduced expressions of the longest element $w_0$ of the Weyl group of $\mathfrak{g}_{\text{fin}}$.

Let $\mathcal{W}_{\text{fin}}$ be the Weyl group of $\mathfrak{g}_{\text{fin}}$. For a $\varnothing$-adapted reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of the longest element $w_0$ of $\mathcal{W}_{\text{fin}}$, we define $\beta_k \in \Phi^+_\text{fin}$ ($1 \leq k \leq \ell$) by (2.1). Then there exist a sequence $\{(i_k, a_k)\}_{k \in \mathbb{Z}} \subset I_{\text{fin}} \times \mathbb{k}^\times$ and $\pi: I_{\text{fin}} \to I_0$ such that $(\pi(i_k), a_k) = \phi_{\varnothing}(\beta_k) \in \sigma_{\varnothing}(\mathfrak{g})$ for $k = 1, \ldots, \ell$ and

$$
(\pi(i_{s+m\ell}), a_{s+m\ell}) = \delta^m((\pi(i_s), a_s)) \quad \text{for } 1 \leq s \leq \ell \text{ and } m \in \mathbb{Z}.
$$

Here we set

$$
\delta^m((i, a)) := \begin{cases} (i, (p^*)^m a) & \text{if } m \text{ is even}, \\ (i^*, (p^*)^m a) & \text{if } m \text{ is odd}. \end{cases}
$$

(See [33, §6].)

We define the affine cuspidal modules $\{S_k\}_{k \in \mathbb{Z}}$ as in Definition 5.6.

Collecting results in [12, 16, 20, 23, 34, 47, 48], we have Proposition 6.4 below. In Proposition 6.4, the symmetric cases follow from [16, 20], the untwisted $B^{(1)}$ and $C^{(1)}$ cases follow from [48, 34], the twisted $A^{(2)}$ and $D^{(2)}$ cases follow from [23], and the exceptional cases follow from [47]. The uniform approach is given in [12]. See also [33, §6].

**Proposition 6.4** ([12, 16, 20, 23, 34, 47, 48]). Let $\varnothing$ be a Q-datum.
(i) \( \sigma_0(\mathfrak{g}) = \bigsqcup_{m \in \mathbb{Z}} \delta^m \sigma_2(\mathfrak{g}) \) (see [12, Proposition 4.21] for example).

(ii) There exists a \( \mathcal{D} \)-adapted reduced expression of \( w_0 \) (see [48, Section 3] for example).

(iii) For a \( \mathcal{D} \)-adapted reduced expression \( w_0 = s_{i_1}s_{i_2}\cdots s_{i_t} \) of \( w_0 \), let \( \{(i_k, a_k)\}_{k \in \mathbb{Z}} \) be the sequence as above, and let \( \{S_k\}_{k \in \mathbb{Z}} \) be the affine cuspidal modules corresponding \( \mathcal{D}_\mathcal{D} \) and \( w_0 \). We have

(a) \( \sum_k a_k \simeq V(\varpi_{i_k}) \),

(b) \( d_V(\varpi_{i_s}), V(\varpi_{i_t}) (a_t/a_s) \neq 0 \) for \( t, s \in \mathbb{Z} \) such that \( s > t \). Here, \( d \) is the denominator of the \( R \)-matrix.

(See [20, Theorem 4.3.4], [23, Theorem 5.1 and Lemma 5.2], [34, Theorem 6.3, 6.4] and [47, Section 6]).

Proposition 6.5. The duality datum \( \mathcal{D}_\mathcal{D} \) is a complete duality datum.

Proof. Recall that \( \sigma_0(\mathfrak{g}) = \{(\pi(i_k), a_k) \mid k \in \mathbb{Z}\} \). For a simple module \( M \) in \( \mathcal{C}_0^0 \), let \( \lambda = \sum_{k=1}^{t} (\pi(i_k), a_k) \) be the affine highest weight of \( M \) (see Theorem 2.9 (iv)). We may assume that \( \{k_s\}_{s \in \{1, \ldots, r\}} \) is a decreasing sequence. Then, by Proposition 6.4 and Theorem 2.9, we have \( M \simeq \text{hd}(S_{k_1} \otimes \cdots \otimes S_{k_r}) \). \( \square \)

Thanks to Theorem 6.3, we have the following.

Corollary 6.6. The duality datum obtained from \( \mathcal{D}_\mathcal{D} \) by applying a finite sequence of \( \mathcal{I} \) and \( \mathcal{I}^{-1} \ (i \in I_{\text{fin}}) \) is a complete duality datum.

Example 6.7. We use the same notations given in Example 5.8 and Example 5.12. Let \( \Delta \) be the Dynkin diagram of finite type \( A_2 \).

(i) Let \( \xi \) be the height function on \( \Delta \) defined by \( \xi(1) = 0 \) and \( \xi(2) = 1 \), and let \( \mathcal{D} \) be the \( \mathcal{Q} \)-datum consisting of \( \Delta \) and \( \xi \). Then \( \mathcal{D} \) is equal to the duality datum arises from the \( \mathcal{Q} \)-datum \( \mathcal{D} \), which says that \( \mathcal{D} \) is complete. The reduced expression \( s_1s_2s_1 \) is \( \mathcal{D} \)-adapted, but \( s_2s_1s_2 \) is not \( \mathcal{D} \)-adapted.

(ii) By Corollary 6.6, \( \mathcal{I}_1(\mathcal{D}) \) and \( \mathcal{I}_2(\mathcal{D}) \) are complete duality data. The duality datum \( \mathcal{I}_1(\mathcal{D}) \) arises from the \( \mathcal{Q} \)-datum consisting of \( \Delta \) and the height function \( \xi' \) defined by \( \xi'(1) = 2 \) and \( \xi'(2) = 1 \), but \( \mathcal{I}_2(\mathcal{D}) \) does not come from any \( \mathcal{Q} \)-datum.

6.3. PBW for quantum affine algebras.

In this subsection, we develop the PBW theory for \( \mathcal{C}_0^0 \) using a complete duality datum. This generalizes the ordinary standard modules and related results ([14, 25, 45, 46, 53]). Note that the ordinary standard modules are cyclic tensor products of fundamental modules.

Let \( \mathcal{C} = (c_{i,j})_{i,j \in J} \) be a simply-laced finite Cartan matrix. Throughout this subsection, we assume that

\( \mathcal{D} = \{L_i\}_{i \in J} \) is a complete duality datum associated with \( \mathcal{C} \).

Proposition 6.2 says that \( \mathcal{C} \) is of type \( \mathcal{X}_0 \) and \( J = I_{\text{fin}} \). Let \( \mathcal{W}_\mathcal{C} \) be the Weyl group associated with \( \mathcal{C} \). We fix a reduced expression \( \underline{w_0} = s_{i_1}s_{i_2}\cdots s_{i_t} \) of the longest element
$w_0$ of $W$, and let $S_k (k \in \mathbb{Z})$ be the affine cuspidal modules corresponding to $D$ and $w_0$. We define

$$Z := \mathbb{Z}_{\geq 0}^{\oplus \mathbb{Z}} = \{(a_k)_{k \in \mathbb{Z}} \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}} | a_k = 0 \text{ except finitely many } k's\}.$$  

We denote by $<$ the bi-lexicographic order on $Z$, i.e., for any $a = (a_k)_{k \in \mathbb{Z}}$ and $a' = (a'_k)_{k \in \mathbb{Z}}$ in $Z$, $a < a'$ if and only if the following conditions hold:

$$\begin{cases} 
(a) & \text{there exists } r \in \mathbb{Z} \text{ such that } a_k = a'_k \text{ for any } k < r \text{ and } a_r < a'_r, \\
(b) & \text{there exists } s \in \mathbb{Z} \text{ such that } a_k = a'_k \text{ for any } k > s \text{ and } a_s < a'_s.
\end{cases}$$

Similarly, we set $\prec_r$ (resp. $\prec_l$) to the right (resp. left) lexicographic order on $Z$, i.e., for any $a, a' \in Z$, $a \prec_r a'$ (resp. $a \prec_l a'$) if and only if the condition (a) (resp. (b)) in (6.5) holds. Hence, we have

$$a \prec a' \iff a \prec_l a' \text{ and } a \prec_r a'.$$

For $a = (a_k)_{k \in \mathbb{Z}} \in Z$, we define

$$P_{D, w_0}(a) := \bigotimes_{k = \pm \infty} S_k^{\otimes a_k} = \cdots \otimes S_2^{\otimes a_2} \otimes S_1^{\otimes a_1} \otimes S_0^{\otimes a_0} \otimes S_{-1}^{\otimes a_{-1}} \otimes S_{-2}^{\otimes a_{-2}} \otimes \cdots.$$ 

Here $P_{D, w_0}(0)$ should be understood as the trivial module $1$. We call the modules $P_{D, w_0}(a)$ standard modules with respect to the cuspidal modules $\{S_k\}_{k \in \mathbb{Z}}$.

**Lemma 6.8.** Let $k \in \mathbb{Z}$, $a \in \mathbb{Z}_{>0}$ and let $M$ be a simple module in $\mathcal{C}^0_D$.

(i) If $\mathcal{B}(D^t S_k, M) = 0$ for $t = 1, 2$, then $a = \mathcal{B}(D S_k, S_k^{\otimes a} \nabla M)$.

(ii) If $\mathcal{B}(D^t S_k, M) = 0$ for $t = -1, -2$, then $a = \mathcal{B}(D^{-1} S_k, M \nabla S_k^{\otimes a})$.

**Proof.** (i) Note that $S_k$ is a root module by Proposition 5.7. Applying Lemma 3.6 (i) to the setting $L := S_k$ and $X := M$, we have

$$\mathcal{B}(D S_k, S_k^{\otimes a} \nabla M) = a + \mathcal{B}(D S_k, M) = a.$$ 

(ii) can be proved in the same manner as above. \qed

**Lemma 6.9.** Let $m, l \in \mathbb{Z}$ with $m \geq l$ and $a_m, a_{m-1}, \ldots, a_l \in \mathbb{Z}_{>0}$. We set

$$M := \text{hd} \left( S_m^{\otimes a_m} \otimes S_{m-1}^{\otimes a_{m-1}} \otimes \cdots \otimes S_l^{\otimes a_l} \right).$$

(i) $\mathcal{B}(D S_k, M) = 0$ for any $k > m$.

(ii) Set $M_m := M$ and define inductively

$$d_k := \mathcal{B}(D S_k, M_k) \quad \text{and} \quad M_{k-1} := M_k \nabla D(S_k^{\otimes d_k}),$$ 

for $k = m, \ldots, l$. Then we have

$$d_k = a_k \quad \text{and} \quad M_k \simeq \text{hd} \left( S_k^{\otimes a_k} \otimes S_{k-1}^{\otimes a_{k-1}} \otimes \cdots \otimes S_l^{\otimes a_l} \right) \quad \text{for } k = m, \ldots, l.$$ 

(iii) $\mathcal{B}(D^{-1} S_k, M) = 0$ for any $k < l$. 

(iv) Set \( N_t := M \) and define inductively

\[
e_k := \mathfrak{b}(\mathcal{D}^{-1}S_k, N_k) \quad \text{and} \quad N_{k+1} := \mathcal{D}^{-1}(S_{k+1}^{\otimes e_k}) \nabla N_k,
\]

for \( k = l, \ldots, m \). Then we have

\[
e_k = a_k \quad \text{and} \quad N_k \simeq \text{hd} \left( S_m^{\otimes a_m} \otimes \cdots \otimes S_{k+1}^{\otimes a_{k+1}} \otimes S_{k}^{\otimes a_k} \right) \quad \text{for} \quad k = m, \ldots, l.
\]

**Proof.** (i) By Proposition 5.7 (ii), \((S_k, S_t)\) is strongly unmixed for any \( k > m \) and \( t = m, \ldots, l \). Thus we have \( \mathfrak{b}(\mathcal{D}S_k, S_t) = 0 \) for \( t = m, \ldots, l \), which implies that \( \mathfrak{b}(\mathcal{D}S_k, M) = 0 \).

(ii) By induction on \( k \), we may assume that \( k = m \). We set \( N := \text{hd} \left( S_{m-1}^{\otimes a_{m-1}} \otimes \cdots \otimes S_1^{\otimes a_1} \right) \).

By (i), we have \( \mathfrak{b}(\mathcal{D}S_m, N) = 0 \) for \( t = 1, 2 \). Proposition 5.7 (iii) tells us that \( M \simeq S_m^{\otimes a_m} \nabla N \). Thus, by Lemma 2.7 and Lemma 6.8, we have

\[
d_m = \mathfrak{b}(\mathcal{D}S_m, M) = \mathfrak{b}(\mathcal{D}S_m, S_m^{\otimes a_m} \nabla N) = a_m,
\]

\[
M \nabla \mathcal{D}(S_k^{\otimes a_m}) \simeq (S_m^{\otimes a_m} \nabla N) \nabla \mathcal{D}(S_k^{\otimes a_m}) \simeq N.
\]

The assertions (iii) and (iv) can be proved in the same manner as above. \( \square \)

**Theorem 6.10.**

(i) For any \( a \in \mathbb{Z} \), the head of \( \mathcal{P}_{\mathcal{D}, w_0}(a) \) is simple. We denote the head by

\[
\mathcal{V}_{\mathcal{D}, w_0}(a) := \text{hd} \left( \mathcal{P}_{\mathcal{D}, w_0}(a) \right).
\]

(ii) For any simple module \( M \in \mathcal{C}_g^0 \), there exists a unique \( a \in \mathbb{Z} \) such that

\[
M \simeq \mathcal{V}_{\mathcal{D}, w_0}(a).
\]

Therefore, the set \( \{ \mathcal{V}_{\mathcal{D}, w_0}(a) \mid a \in \mathbb{Z} \} \) is a complete and irredundant set of simple modules of \( \mathcal{C}_g^0 \) up to isomorphisms.

**Proof.** (i) follows from Proposition 5.7.

(ii) Let \( M \) be a simple module in \( \mathcal{C}_g^0 \). Since \( \mathcal{D} \) is complete, there exist simple module \( M_k \in \mathcal{C}_\mathcal{D} \) (\( k \in \mathbb{Z} \)) such that \( M_k \simeq 1 \) for all but finitely many \( k \) and

\[
M \simeq \text{hd} \left( \cdots \otimes \mathcal{D}^2 M_2 \otimes \mathcal{D} M_1 \otimes M_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots \right).
\]

Since \( M_k \in \mathcal{C}_\mathcal{D} \), there exist \( b_1^k, \ldots, b_{\ell}^k \in \mathbb{Z}_{\geq 0} \) such that \( M_k \simeq \text{hd} \left( S_{\ell}^{b_{\ell}^k} \otimes \cdots \otimes S_1^{b_1^k} \right) \), which yields that

\[
\mathcal{D}^k M_k \simeq \text{hd} \left( S_{\ell}^{b_{\ell}^k} \otimes \cdots \otimes S_1^{b_1^k} \right)
\]

by Lemma 2.23. For \( t \in \mathbb{Z} \), we define \( a_t := b_t^k \), where \( t = k \cdot \ell + r \) for some \( k \in \mathbb{Z} \) and \( r = 1, \ldots, \ell \), and set \( a := (a_t)_{t \in \mathbb{Z}} \). By Proposition 5.7, we have

\[
M \simeq \mathcal{V}_{\mathcal{D}, w_0}(a).
\]

The uniqueness for \( a \) follows from Lemma 6.9, which completes the proof. \( \square \)
The element $a \in \mathbb{Z}$ associated with a simple module $M$ in Theorem 6.10 (ii) is called the **cuspidal decomposition** of $M$ with respect to the cuspidal modules $\{S_k\}_{k \in \mathbb{Z}}$, and it is denoted by

$$(6.7) \quad a_{D,w_0}(M) := a.$$ 

**Lemma 6.11.** Let $L, M, N$ be simple modules in $C_g$ and assume that $L$ is real.

(i) If $(L, M)$ and $(L, N)$ are strongly unmixed and $L \nabla N$ appears in $L \otimes M$ as a subquotient, then we have $M \simeq N$.

(ii) If $(M, L)$ and $(N, L)$ are strongly unmixed and $N \nabla L$ appears in $M \otimes L$ as a subquotient, then we have $M \simeq N$.

**Proof.** (i) Since $(L, M)$ and $(L, N)$ are strongly unmixed,

$$\Lambda(L, M) = \Lambda^\infty(L, M) \quad \text{and} \quad \Lambda(L, N) = \Lambda^\infty(L, N)$$

by Lemma 5.2. Since $L \nabla N$ appears in $L \otimes M$, Proposition 2.13 tells us that

$$\Lambda(L, M) = \Lambda^\infty(L, M) = \Lambda^\infty(L, N) = \Lambda(L, N) = \Lambda(L, L \nabla N).$$

Thus it follows from [27, Theorem 4.11] that

$$L \nabla M \simeq L \nabla N,$$

which implies that $M \simeq N$ by Lemma 2.7.

(ii) can be proved in the same manner as above. \qed

For $c = (c_k)_{k \in \mathbb{Z}} \in \mathbb{Z}$, we set $l(c)$ (resp. $r(c)$) to be the integer $t$ such that

$$(6.8) \quad c_t \neq 0, \quad c_k = 0 \text{ for any } k > t \text{ (resp. } k < t).$$

**Theorem 6.12.** Let $a$ be an element of $\mathbb{Z}$. Then we have the following.

(i) The simple module $V_{D,w_0}(a)$ appears only once in $P_{D,w_0}(a)$.

(ii) If $V$ is a simple subquotient of $P_{D,w_0}(a)$ which is not isomorphic to $V_{D,w_0}(a)$, then we have

$$a_{D,w_0}(V) \prec a.$$

(iii) In the Grothendieck ring, we have

$$[P_{D,w_0}(a)] = [V_{D,w_0}(a)] + \sum_{a' \prec a} c(a') [V_{D,w_0}(a')],$$

for some $c(a') \in \mathbb{Z}_{\geq 0}$.

**Proof.** We focus on proving (ii) because (i) and (iii) follow from (ii).

Let $a = (a_k)_{k \in \mathbb{Z}}$ and set

$$l := l(a) \quad \text{and} \quad r := r(a).$$

Let $V$ be a simple subquotient of $P_{D,w_0}(a)$ which is not isomorphic to $V_{D,w_0}(a)$. We set

$$b = (b_k)_{k \in \mathbb{Z}} := a_{D,w_0}(V).$$
For $k > l$ and $r > t$, since $(S_k, S_l)$ and $(S_r, S_t)$ is strongly unmixed by Proposition 5.7, we have

$$b(S_k, P_{D, \omega_0}(a)) = 0, \quad b(S^{-1}_l, P_{D, \omega_0}(a)) = 0,$$

which implies that $b(S_k, V) = 0$ and $b(S^{-1}_t, V) = 0$ by [27, Proposition 4.2]. Thus, Lemma 6.9 tells us that

$$l \geq l(b) \quad \text{and} \quad r(b) \geq r.$$

We now shall prove $b \prec_l a$, where $\prec_l$ is the left lexicographical order on $\mathbb{Z}$. Note that, by Lemma 6.9, Proposition 5.7 and [27, Proposition 4.2], we have

$$b_l = b(S_l, V) \leq b(S_l, P_{D, \omega_0}(a)) = b(S_l, S_{a_l}) = a_l.$$

When either $l > l(b)$ or $l = l(b)$ and $b_l < a_l$, it is obvious that $b \prec_l a$ by the definition. We assume that $l = l(b)$ and $b_l = a_l$. Set

$$c := b_l = a_l, \quad a^- = (a^-_k)_{k \in \mathbb{Z}}, \quad \text{where} \quad a^-_k := \begin{cases} 0 & \text{if} \ k = l, \\ a_k & \text{otherwise}, \end{cases}$$

and

$$P^- := S_{l-1}^{\otimes a_l} \otimes \cdots \otimes S_r^{\otimes a_r}, \quad V^- := \text{hd} \left( S_{l-1}^{\otimes b_l} \otimes \cdots \otimes S_r^{\otimes b_r(b)} \right).$$

Note that

$$P^- = P_{D, \omega_0}(a^-), \quad P_{D, \omega_0}(a) = S_l^{\otimes c} \otimes P^-, \quad V \simeq (S_l^{\otimes c}) \nabla V^-,$$

where the third follows from proposition 5.7 (iii). As $V$ appears in $S_l^{\otimes c} \otimes P^-$ as a simple subquotient, there exist a simple subquotient $L$ of $P^-$ such that

$$V \text{ appears in } S_l^{\otimes c} \otimes L \text{ as a simple subquotient.}$$

By Proposition 5.7 (ii), we know that $(S_l, V^-)$ and $(S_l, L)$ are strongly unmixed. Hence, by Lemma 6.11, we conclude that

$$V^- \simeq L.$$

If $V^-$ is isomorphic to $\text{hd}(P^-)$, then we have $V \simeq \text{hd}(P_{D, \omega_0}(a))$ by (6.9), which contradicts the assumption. Hence $V^-$ is not isomorphic to $\text{hd}(P^-)$. Applying the standard induction argument to the setting $V^-$ and $P^-$, we obtain

$$a_{D, \omega_0}(V^-) \prec_l a^-,$$

which implies that $b \prec_l a$.

In the same manner as above, one can prove that $b \prec_r a$. Therefore it follows from (6.6) that

$$b \prec a.$$
Remark 6.13. Let $V$ be a simple subquotient of $P_{D,w_0}(a)$. Theorem 6.12 says that $a_{D,w_0}(V) \prec a$. There is another condition which $V$ should satisfy. By Proposition 2.13, we have

$$E(V) = E(V_{D,w_0}(a)),$$

where $E$ is given in Section 6.1. Thus they are in the same block of $\mathcal{C}_g$.

Remark 6.14. There is a well-known partial ordering, called Nakajima partial ordering, in the $q$-character theory. For simplicity, we assume that $U'_q(g)$ is of untwisted affine ADE type. Let $Y_{i,a}$ be an indeterminate for $i \in I_0$ and $a \in k^\times$. Then one can define a partial ordering $\leq$ on the set of monomials in $\mathbb{Z}[Y_{i,a}^\pm \mid i \in I_0, a \in k^\times]$ as follows: for monomials $m$ and $m'$, set $A_{i,a} := Y_{i,a}^{-1}Y_{i,a}\prod_{\alpha_i, \alpha_j}(-1)^{-1}Y_{j,a}^{-1}$. Then one can define a partial ordering $\preceq$ on the set of monomials in $\mathbb{Z}[Y_{i,a}^\pm \mid i \in I_0, a \in k^\times]$ as follows: for monomials $m$ and $m'$, if and only if $m^{-1}m'$ is a product of elements of $\{A_{i,a} \mid i \in I_0, a \in k^\times\}$ ([13, 46]). The simple modules and ordinary standard modules in $\mathcal{C}_g$ are parameterized by dominant monomials, which are denoted by $L(m)$ and $M(m)$ respectively for a dominant monomial $m$. Note that the fundamental module $V(\varpi_i)_a$ corresponds to $Y_{i,a}$. From the viewpoint of $(q,t)$-characters, it was shown in [45, 46] that

$$[M(m)] = [L(m)] + \sum_{m' < m} P_{m,m'}[L(m')]$$

in the Grothendieck ring $K(\mathcal{C}_g)$ and the multiplicity $P_{m,m'}$ can be understood as the specialization at $t = 1$ of an analogue $P_{m,m'}(t)$ of Kazhdan-Lusztig polynomial.

Let $\mathcal{Q}$ be a $\mathbb{Q}$-datum and let $w_0$ be a $\mathcal{Q}$-adapted reduced expression. In this case, the affine cuspidal modules $S_k$ are all fundamental modules in $\mathcal{C}_g$ and $P_{D,w_0}(a)$ are ordinary standard modules (see Example 5.8 (i) for instance). Let $m$ and $m'$ be dominant monomials and set $a := a_{D,w_0}(L(m))$ and $a' := a_{D,w_0}(L(m'))$. Considering the definition of $A_{i,a}$ and [33, Proposition 6.11], one can show that if $m \leq m'$ in the partial ordering, then $a \preceq a'$ in the ordering $(6.5)$. From this observation about two orders $\leq$ and $\preceq$, Theorem 6.12 is compatible with $(6.11)$. Since affine cuspidal modules are not necessary to be fundamental in general (see Example 5.8 (ii) for instance), Theorem 6.12 can be viewed as a generalization of $(6.11)$.

Remark 6.13 says that the condition $(6.10)$ holds when $V$ is a simple subquotient of $P_{D,w_0}(a)$. Thus it is interesting to ask under what conditions the ordering $(6.5)$ is equal to the ordering $\leq$.

For $a, b \in \mathbb{Z} \sqcup \{\pm \infty\}$, an interval $[a, b]$ is the set of integers between $a$ and $b$: $[a, b] := \{s \in \mathbb{Z} \mid a \leq s \leq b\}$.

If $a > b$, we understand $[a, b] = \emptyset$. 
For an interval \([a, b]\), we define \(C_{g[a,b],D,u_0}^{D,u_0}\) to be the full subcategory of \(C_g\) whose objects have all their composition factors \(V\) satisfying the following condition:
\[
b \geq l(a_{D,u_0}(V)) \quad \text{and} \quad r(a_{D,u_0}(V)) \geq a.
\]
(6.12)

Thanks to Theorem 6.12, we have the following proposition.

**Proposition 6.15.** The category \(C_{g[a,b],D,u_0}^{D,u_0}\) is stable by taking subquotients, extensions, and tensor products.

It is easy to show that the category \(C_{g[a,b],D,u_0}^{D,u_0}\) is equal to the smallest full subcategory of \(C_0\) satisfying the following conditions:

(i) it is stable under taking subquotients, extensions, tensor products and
(ii) it contains \(S_s\) for all \(a \leq s \leq b\) and the trivial module \(1\).

If there is no confusion arises, then we simply write \(C_{g[a,b]}\) instead of \(C_{g[a,b],D,u_0}^{D,u_0}\).

For an interval \([a, b]\), we set
\[
Z^{a,b} := \{ a = (a_k)_{k \in \mathbb{Z}} | a_k = 0 \text{ for either } k > b \text{ or } a > k \}
\]

Then the theorem below follows from Lemma 6.9, Theorem 6.10 and Theorem 6.12 directly.

**Theorem 6.16.** Let \([a, b]\) be an interval.

(i) The set \(\{V_{D,u_0}(a) \mid a \in Z^{a,b}\}\) is a complete and irredundant set of simple modules of \(C_g^{[a,b]}\) up to isomorphisms.

(ii) Let \(M\) a simple modules in \(C_g^0\). Then, \(M\) belongs to \(C_g^{[a,b]}\) if and only if
\[
v(DS_k, M) = 0 \text{ for } k > b \text{ and } v(D^{-1}S_k, M) = 0 \text{ for } k < a.
\]

(iii) For \(a \in Z^{a,b}\), the standard module \(P_{D,u_0}(a)\) is contained in \(C_g^{[a,b]}\) and, in the Grothendieck ring, we have
\[
[P_{D,u_0}(a)] = [V_{D,u_0}(a)] + \sum_{a' < a} c(a')[V_{D,u_0}(a')],
\]
for some \(c(a') \in \mathbb{Z}_{\geq 0}\).

**Example 6.17.** We use the same notations given in Example 5.8.

(i) We consider the affine cuspidal modules \(S_k\) given in Example 5.8 (i). Let \(l \in \mathbb{Z}_{\geq 0}\).

The category \(C_g^{[1,2(l+1)]}\) is determined by \(S_k\) for \(k \in [1, 2(l+1)]\). It follows from
\[
\{ S_k \mid k \in [1, 2(l+1)] \} = \{ V(1)_{(-t)\infty}, V(2)_{(-t)\infty+1} \mid t \in [0, l] \}
\]
that the category \(C_g^{[1,2(l+1)]}\) is equal to the Hernandez-Leclerc category \(C_i\) defined in [15, Section 3.8].
(ii) Let us take the affine cuspidal modules $S'_k$ given in Example 5.8 (ii). In this case, the category $\mathcal{C}_g^{[a,b]}$ is not equal to $\mathcal{C}_i$ in general. From this viewpoint, the category $\mathcal{C}_g^{[a,b],D_{ab}}$ is a generalization of the category $\mathcal{C}_i$.

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(M. Kashiwara) KYOTO UNIVERSITY INSTITUTE FOR ADVANCED STUDY, RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN & KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 02455, KOREA

Email address, M. Kashiwara: masaki@kurims.kyoto-u.ac.jp

(M. Kim) DEPARTMENT OF MATHEMATICS, KYUNG HEE UNIVERSITY, SEOUL 02447, KOREA

Email address, M. Kim: mkim@khu.ac.kr

(S.-j. Oh) DEPARTMENT OF MATHEMATICS, EWHA WOMANS UNIVERSITY, SEOUL 03760, KOREA

Email address, S.-j. Oh: sejin092@gmail.com

(E. Park) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SEOUL, SEOUL 02504, KOREA

Email address, E. Park: epark@uos.ac.kr