Finite Temperature Quantum Field Theory with Impurities

M. Mintchev\textsuperscript{a} and P. Sorba\textsuperscript{b}

\textsuperscript{a} INFN and Dipartimento di Fisica, Università di Pisa, Via Buonarroti 2, 56127 Pisa, Italy

\textsuperscript{b} LAPTH, 9, Chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux cedex, France

Abstract

We apply the concept of reflection-transmission (RT) algebra, originally developed in the context of integrable systems in 1+1 space-time dimensions, to the study of finite temperature quantum field theory with impurities in higher dimensions. We consider a scalar field in \((s+1)+1\) space-time dimensions, interacting with impurities localized on \(s\)-dimensional hyperplanes, but without self-interaction. We discuss first the case \(s=0\) and extend afterwards all results to \(s>0\). Constructing the Gibbs state over an appropriate RT algebra, we derive the energy density at finite temperature and establish the correction to the Stefan-Boltzmann law generated by the impurity. The contribution of the impurity bound states is taken into account. The charge density profiles for various impurities are also investigated.
1 Introduction

Impurities (defects) exhibit a number of intriguing physical and mathematical features. Due to the rapid progress in building nanoscale quantum devices, they find nowadays interesting physical applications. These facts have motivated several investigations in the context of quantum mechanics [1] – [5] and quantum field theory (QFT) [6] – [13].

In some recent papers [14] – [16] we have proposed an algebraic framework for dealing with defects in 1+1 space-time dimensions, introducing the so called reflection-transmission (RT) algebras. We have shown that these algebras represent a powerful approach to integrable systems with impurities, allowing to reconstruct the total scattering operator from the basic scattering data, namely the two-body bulk $S$-matrix and the reflection and transmission amplitudes of a single particle interacting with the impurity.

In the present article we pursue further our analysis, extending the framework of RT algebras in two directions. We show that RT algebras are well adapted for treating impurity problems in higher space-time dimensions. Moreover, we generalize the framework to finite temperatures. Both these generalizations are essential for realistic applications to condensed matter physics. The most natural way to perform the extension to finite temperature is to substitute the Fock vacuum, used in scattering theory, with a Gibbs grand canonical equilibrium state at given (inverse) temperature $\beta$. The Gibbs state is required to satisfy the Kubo-Martin-Schwinger (KMS) condition and defines a new (non-Fock) representation of the underlying RT algebra.

We start the paper by performing the quantization of a scalar field in the background of a general point-like defect in 1+1 space-time dimensions. For this purpose we use an appropriate RT algebra $\mathcal{C}$. In section 3 we construct the finite temperature representation of $\mathcal{C}$ and derive the thermal real time correlation functions, obeying the KMS condition. In section 4 we compute the energy density and establish the corrections to the Stefan-Boltzmann law generated by the defect. The study of $j_\mu$ reveals on the other hand that the impurity affects also the charge distribution in the Gibbs state, but does not induce a persistent current.

We consider the case without self-interactions and focus essentially on two observables – the energy-momentum tensor $\theta_{\mu\nu}$ and the conserved $U(1)$-current $j_\mu$. Computing the expectation value of $\theta_{00}$ in the Gibbs state, we derive the thermal energy density and establish the corrections to the Stefan-Boltzmann law generated by the defect. The study of $j_\mu$ reveals on the other hand that the impurity affects also the charge distribution in the Gibbs state, but does not induce a persistent current.
the previous sections about point-like impurities in 1+1 space-time dimensions to $s$-dimensional hyperplane-defects in $(s + 1) + 1$ dimensions. We discuss in particular plane-defects in the physical 3+1 dimensions. The last section collects our conclusions and comments about the further research in the subject.

2 General point-like defects

Employing the concept of RT algebra, we extend in this section some basic results [1]–[5] on point-like impurities in quantum mechanics to QFT in 1+1 dimensions. We consider an impurity localized at $x = 0$ and a hermitian scalar quantum field $\varphi(t, x)$ satisfying

$$[\partial_t^2 - \partial_x^2 + m^2] \varphi(t, x) = 0, \quad x \neq 0,$$

with the standard initial conditions:

$$[\varphi(0, x_1), \varphi(0, x_2)] = 0, \quad [(\partial_t \varphi)(0, x_1), \varphi(0, x_2)] = -i \delta(x_1 - x_2).$$

(2.2)

It is worth stressing that the equation of motion (2.1) is not imposed on the impurity. For this reason the solution of eqs. (2.1,2.2) is not unique, contrary to the case in which eq. (2.1) holds for any $x \in \mathbb{R}$. The physical problem of describing all admissible point-like impurities is equivalent to the mathematical problem of classifying the possible solutions of eqs. (2.1,2.2). In order to solve the latter, one has to analyse the operator $-\partial_x^2$, defined on the space $C_0^\infty(\mathbb{R} \setminus \{0\})$ of smooth functions with compact support separated from the origin $x = 0$. This operator is not self-adjoint, but its closure admits self-adjoint extensions. It has been shown in [1, 2] that all of them are parametrized by

$$\{(a, b, c, d; \varepsilon) : ad - bc = 1, \varepsilon = 1, a, ..., d \in \mathbb{R}, \varepsilon \in \mathbb{C}\}.$$  

(2.3)

Since $\varphi$ is hermitian, in our case $\varepsilon \in \mathbb{R}$ and therefore $\varepsilon = \pm 1$. The value $\varepsilon = -1$ can be absorbed in $(a, b, c, d)$ and we are thus left with

$$\Gamma = \{\gamma = (a, b, c, d) : ad - bc = 1, a, ..., d \in \mathbb{R}\}.$$  

(2.4)

Each $\gamma \in \Gamma$ defines a unique solution of eqs. (2.1,2.2), which obeys

$$\left( \begin{array}{c} \varphi(t, +0) \\ \partial_x \varphi(t, +0) \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} \varphi(t, -0) \\ \partial_x \varphi(t, -0) \end{array} \right)$$

(2.5)

for any $t \in \mathbb{R}$. We observe that the impurity boundary conditions (2.5) can be implemented, coupling the field $\varphi$ to an external potential with support in $x = 0$. The quadruple

$$\gamma_\eta = (1, 0, 2\eta, 1), \quad \eta \in \mathbb{R},$$

(2.6)
for instance, corresponds to the potential

\[ V(x) = 2\eta \delta(x), \tag{2.7} \]

known \[ \Pi \] as \( \delta \)-impurity. The conventional free scalar field is obtained for \( \eta = 0 \). A potential incorporating all parameters \[ \square \] has been recently proposed in \[ 13 \].

A fundamental input in the quantization of \( \varphi \) is the spectrum of the self-adjoint extension \( \{-\partial_x^2, \gamma \} \). It is convenient to distinguish the following three subdomains of \( \Gamma \):

\[ \Gamma_0 = \{b < 0, r_+ \geq 0\} \cup \{b = 0, r_0 \geq 0\} \cup \{b > 0, r_- \geq 0\}, \tag{2.8} \]

\[ \Gamma_1 = \{b < 0, r_+ < 0 \leq r_-\} \cup \{b = 0, r_0 < 0\} \cup \{b > 0, r_- < 0 \leq r_+\}, \tag{2.9} \]

\[ \Gamma_2 = \{b < 0, r_- < 0\} \cup \{b > 0, r_+ < 0\}, \tag{2.10} \]

where

\[ r_0 = \frac{c}{a + d}, \quad r_\pm = \frac{a + d \pm \sqrt{(a - d)^2 + 4}}{2b}. \tag{2.11} \]

By construction

\[ \Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2, \tag{2.12} \]

the index \( n \) of \( \Gamma_n \) indicating the number of bound states present in the spectrum of \( \{-\partial_x^2, \gamma \in \Gamma_n\} \). We devote section 6 to the case \( \gamma \in \Gamma_1 \cup \Gamma_2 \), focusing in the rest of the paper on the case without bound states.

A complete orthonormal system of scattering states for \( \gamma \in \Gamma_0 \) is given by

\[ \psi^+_k(x) = \theta(-k) \left\{ \theta(-x)T^+_k(k)e^{ikx} + \theta(x) \left[ e^{ikx} + R^+_k(-k)e^{-ikx} \right] \right\}, \tag{2.13} \]

\[ \psi^-_k(x) = \theta(k) \left\{ \theta(x)T^-_k(k)e^{ikx} + \theta(-x) \left[ e^{ikx} + R^-_k(-k)e^{-ikx} \right] \right\}, \tag{2.14} \]

where \( \theta \) denotes the standard Heaviside function and

\[ R^+_k(k) = \frac{bk^2 + i(a - d)k + c}{bk^2 + i(a + d)k - c}, \quad T^+_k(k) = \frac{2ik}{bk^2 + i(a + d)k - c}, \tag{2.15} \]

\[ R^-_k(k) = \frac{bk^2 + i(a - d)k + c}{bk^2 - i(a + d)k - c}, \quad T^-_k(k) = \frac{-2ik}{bk^2 - i(a + d)k - c}, \tag{2.16} \]

are the reflection and transmission coefficients from the impurity. It is easily verified that the reflection and transmission matrices, defined by

\[ \mathcal{R}(k) = \begin{pmatrix} R^+_k(k) & 0 \\ 0 & R^-_k(k) \end{pmatrix}, \quad \mathcal{T}(k) = \begin{pmatrix} 0 & T^+_k(k) \\ T^-_k(k) & 0 \end{pmatrix}, \tag{2.17} \]
satisfy hermitian analyticity
\[ \mathcal{R}(k)\dagger = \mathcal{R}(-k), \quad \mathcal{T}(k)\dagger = \mathcal{T}(k), \quad (2.18) \]
and unitarity
\[ \mathcal{T}(k)\mathcal{T}(k) + \mathcal{R}(k)\mathcal{R}(-k) = \mathbb{I}, \quad (2.19) \]
\[ \mathcal{T}(k)\mathcal{R}(k) + \mathcal{R}(k)\mathcal{T}(-k) = 0. \quad (2.20) \]

Following [15], for \( \gamma \in \Gamma_0 \) we introduce the decomposition
\[ \varphi(t, x) = \varphi_+(t, x) + \varphi_-(t, x), \quad (2.21) \]
setting
\[ \varphi_\pm(t, x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi \sqrt{2\omega(k)}} \left[ a^{\mp\dagger}(k) \psi_\pm^\dagger(x) e^{i\omega(k)t} + a_\pm(k) \psi_\pm^\dagger(x) e^{-i\omega(k)t} \right]. \quad (2.22) \]
where \( \omega(k) = \sqrt{k^2 + m^2} \) and \( \{ a^{\xi\dagger}(k), a_\xi(k) : \xi = \pm, k \in \mathbb{R} \} \) generate the bosonic RT algebra \( \mathcal{C} \) with identity element \( 1 \):
\[ \begin{align*}
  a_\xi(k_1) a_\xi(k_2) - a_\xi(k_2) a_\xi(k_1) &= 0, \quad (2.23) \\
  a^{\xi\dagger}(k_1) a^{\xi\dagger}(k_2) - a^{\xi\dagger}(k_2) a^{\xi\dagger}(k_1) &= 0, \quad (2.24) \\
  a_\xi(k_1) a^{\xi\dagger}(k_2) - a^{\xi\dagger}(k_2) a_\xi(k_1) &= \\
  \left[ \delta^\xi_{\xi_1} + \mathcal{T}^\xi_{\xi_1}(k_1) \right] 2\pi \delta(k_1 - k_2) \mathbf{1} + \mathcal{R}^\xi_{\xi_1}(k_1) 2\pi \delta(k_1 + k_2) \mathbf{1}. \quad (2.25) 
\end{align*} \]
The presence of the impurity is captured by the reflection and transmission matrices \( (2.17) \), which appear in the right-hand side of eq. \( (2.25) \) and by the constraints
\[ \begin{align*}
  a_\xi(k) &= \mathcal{T}_\xi^\eta(k) a_\eta(k) + \mathcal{R}_\xi^\eta(k) a_\eta(-k), \quad (2.26) \\
  a^{\xi\dagger}(k) &= a^{\*\eta}(k) \mathcal{T}_\eta^\xi(k) + a^{\*\eta}(-k) \mathcal{R}_\eta^\xi(-k), \quad (2.27) 
\end{align*} \]
imposed on the generators of \( \mathcal{C} \). Taking into account \( (2.26, 2.27) \), the fields \( \varphi_\pm \) can be rewritten as
\[ \varphi_\pm(t, x) = \theta(\pm x) \int_{-\infty}^{+\infty} \frac{dk}{2\pi \sqrt{2\omega(k)}} \left[ a^{\pm\dagger}(k) e^{i\omega(k)t - ikx} + a_\pm(k) e^{-i\omega(k)t + ikx} \right]. \quad (2.28) \]
The fact that \( \varphi_\pm \) satisfy the defect boundary conditions \( (2.5) \) is less explicit in this form, which is suitable however for the generalization to higher space-time dimensions described in section 7.
All details about the structure defined by equations (2.23)–(2.27) and their origin are given in [15]. The reflection and transmission matrices, appearing in the right hand side of (2.25), capture the interaction with the impurity. The constraints (2.26, 2.27) are a consequence of unitarity (2.19, 2.20) and of a peculiar (called reflection-transmission) automorphism of the RT algebra $C$, established in [15]. The fields $\varphi_{\pm}$, defined by (2.28), are elements of $C$. In order to obtain quantum fields acting in a Hilbert space, one must take a representation of $C$. In [15] we have constructed the Fock representation $F(C)$, showing that the asymptotic in- and out-spaces of $\varphi$ coincide with $F(C)$. The basic correlator of $\varphi$ in $F(C)$ is the two-point vacuum expectation value. It is obtained employing (2.25) and the fact that $a_\xi(k)$ annihilates the vacuum state $\Omega \in F(C)$. One gets

$$\langle \varphi(t_1, x_1) \varphi(t_2, x_2) \rangle_\Omega = \int_{-\infty}^{+\infty} \frac{dk}{4\pi \omega(k)} e^{-i\omega(k)t_{12}} E(k; x_1, x_2; \gamma),$$

(2.29)

where

$$E(k; x_1, x_2; \gamma) = \theta(x_1)\theta(-x_2)T_-^+(k)e^{ikx_{12}} + \theta(-x_1)\theta(x_2)T_+^-(k)e^{ikx_{12}} + \theta(x_1)\theta(x_2) \left[e^{ikx_{12}} + R_+^+(k)e^{ik\tilde{x}_{12}} \right] + \theta(-x_1)\theta(-x_2) \left[e^{ikx_{12}} + R_-^-(k)e^{ik\tilde{x}_{12}} \right],$$

(2.30)

with $x_{12} = x_1 - x_2$ and $\tilde{x}_{12} = x_1 + x_2$. As expected, for the free field $\gamma_0 = (1, 0, 0, 1)$ the above expression gives

$$E(k; x_1, x_2; \gamma_0) = e^{ikx_{12}}.$$

The symmetry content of our system can be deduced from (2.29, 2.30). First of all (2.29) is invariant under time translations, which implies energy conservation. In spite of the relativistic dispersion relation $\omega(k)^2 = k^2 + m^2$, each non-trivial ($\gamma \neq \gamma_0$) impurity violates Lorentz and space translation invariance. The reflection $x \mapsto -x$ leaves invariant eqs. (2.24, 2.25), but not always (2.26). In fact, (2.29) is invariant under spatial reflections if and only if

$$a = d.$$

(2.31)

This condition selects the parity preserving impurities.

Let us observe in conclusion that the Hamiltonian $H$, generating the time evolution

$$\varphi(t, x) = e^{itH} \varphi(0, x)e^{-itH},$$

(2.32)

is given by the familiar quadratic expression

$$H = \sum_{\xi = \pm} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \omega(k)a^\dagger_\xi(k)a_\xi(k),$$

(2.33)
which will be adopted in the next section for the construction of the Gibbs state over the RT algebra $C$.

## 3 The Gibbs state over $C$

Our goal here is to construct a thermal representation $G_\beta(C)$ of $C$. For this purpose we introduce $K \in C$ defined by

$$K = H - \mu N, \quad \mu \in \mathbb{R},$$

(3.1)

where $H$ is the Hamiltonian (2.33), $\mu$ is the chemical potential and $N$ is the number operator

$$N = \sum_{\xi = \pm} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \xi(k) a_\xi(k).$$

(3.2)

We choose as a cyclic vector, determining $G_\beta(C)$, the Gibbs grand canonical equilibrium state corresponding to $K$ at inverse temperature $\beta \equiv T^{-1}$. Differently from the vacuum $\Omega \in \mathcal{F}(C)$, the Gibbs state is not annihilated by $a_\xi(k)$ and represents an appropriate idealization of a thermal bath, keeping our system in equilibrium. The expectation value of a generic polynomial $P$ in this state is defined by

$$\langle P(a_\xi^*(p_1), a_\zeta(q_2)) \rangle_\beta = \frac{\text{Tr} \left[ e^{-\beta K} P(a_\xi^*(p_1), a_\zeta(q_2)) \right]}{\text{Tr} e^{-\beta K}}. \quad (3.3)$$

Assuming the normalization condition

$$\langle 1 \rangle_\beta = 1 \quad (3.4)$$

for the identity element $1 \in C$, the non-vanishing two-point correlation functions in the Gibbs state are given by

$$\langle a_\xi^*(p_1) a_\zeta(p_2) \rangle_\beta = \frac{e^{-\beta[\omega(p_1) - \mu]}}{1 - e^{-\beta[\omega(p_1) - \mu]}},$$

$$\left\{ \delta_\xi + T_\xi(p_1) \right\} 2\pi \delta(p_1 - p_2) + R_\xi(p_1) 2\pi \delta(p_1 + p_2), \quad (3.5)$$

$$\langle a_\xi(p_1) a_\zeta^*(p_2) \rangle_\beta = \frac{1}{1 - e^{-\beta[\omega(p_1) - \mu]}},$$

$$\left\{ \delta_\zeta + T_\zeta(p_1) \right\} 2\pi \delta(p_1 - p_2) + R_\zeta(p_1) 2\pi \delta(p_1 + p_2). \quad (3.6)$$

Notice that the Bose-Einstein distribution appears as a factor in the right hand side of (3.5, 3.6).
It is instructive to sketch here a simple derivation of the correlators \( \xi \) \( \beta \), generalizing the argument in [17] for the case without impurities. Since \( \xi \) \( \beta \) are related by the commutator \( [a_\xi, a_\xi^\dagger] = \delta_\xi \), it is enough to concentrate on \( \xi \) \( \beta \) for instance. Using the identity
\[
e^{-\beta K} a_\xi^\dagger(k) = e^{-\beta \omega(k) - \mu} a_\xi^\dagger(k) e^{-\beta K},
\] (3.7)
the cyclicity of the trace and the commutation relation \( [\xi, a_\xi^\dagger] = 0 \), one finds
\[
\langle a_\xi^\dagger(p_1) a_\xi^\dagger(p_2) \rangle_\beta = \frac{\Tr [a_\xi^\dagger(p_1) e^{-\beta K} a_\xi^\dagger(p_2)]}{\Tr e^{-\beta K}} e^{-\beta \omega(p_1) - \mu} = e^{-\beta \omega(p_1) - \mu} \langle a_\xi^\dagger(p_1) a_\xi^\dagger(p_2) \rangle_\beta + e^{-\beta \omega(p_1) - \mu} \left\{ [\delta_\xi + \mathcal{T}_\xi(p_1)] 2\pi \delta(p_1 - p_2) + \mathcal{R}_\xi(p_1) - (p_1) 2\pi \delta(p_1 + p_2) \right\},
\] (3.8)
which represents an equation for \( \langle a_\xi^\dagger(p_1) a_\xi^\dagger(p_2) \rangle_\beta \). The solution of this equation gives \( \xi \) \( \beta \). In general, any correlator \( \xi \) \( \beta \) can be evaluated by iteration, applying the identity
\[
\langle \prod_{i=1}^m a_\xi^\dagger(p_i) \prod_{j=1}^n a_\xi(p_j) \rangle_\beta = \delta_{mn} \sum_{k=1}^m \langle a_\xi^\dagger(p_1) a_\xi(p_2) \rangle_\beta \langle \prod_{i=2}^m a_\xi^\dagger(p_i) \prod_{j=1}^n a_\xi(p_j) \rangle_\beta,
\] (3.9)
and the commutation relations \( [\xi, a_\xi^\dagger] = 0 \). Like in the Fock representation \( F(C) \), all correlators in \( G_\beta(C) \) with different number of creation and annihilation operators vanish.

We have thus shown that eqs. \( \xi \) \( \beta \) completely determine the thermal representation \( G_\beta(C) \) of the RT algebra \( C \). One can analyse at this point the KMS condition in \( G_\beta(C) \). This condition relates the two-point correlators \( \xi \) \( \beta \) according to
\[
\langle [\alpha_\xi a_\xi(p_1)] a_\xi^\dagger(p_2) \rangle_\beta = \langle a_\xi^\dagger(p_2) [\alpha_\xi a_\xi(p_1)] \rangle_\beta,
\] (3.10)
\( \alpha_\xi \) being the automorphism on \( C \)
\[
\alpha_\xi a_\xi(k) = a_\xi^\dagger(k) e^{i\omega(k) - \mu}, \quad \alpha_\xi a_\xi(k) = a_\xi(k) e^{-i\omega(k) - \mu},
\] (3.11)
generated by \( K \). The identity \( \xi \) \( \beta \) can be verified directly, using the explicit form of the correlators. Eq. \( \xi \) \( \beta \) then extends the validity of the KMS condition to the whole \( G_\beta(C) \).

The decomposition \( \xi \) \( \beta \) allows to determine all correlation functions of \( \varphi \) in \( G_\beta(C) \). The two-point function reads
\[
\langle \varphi(t_1, x_1) \varphi(t_2, x_2) \rangle_\beta =
\]
\[
\int_{-\infty}^{+\infty} \frac{dk}{4\pi \omega(k)} \left\{ \frac{e^{-\beta[\omega(k) - \mu]} + e^{-i\omega(k)t_{12}} + e^{-i\omega(k)t_{12}}}{1 - e^{-\beta[\omega(k) - \mu]}} \right\} E(k; x_1, x_2; \gamma). \tag{3.12}
\]

and comparing with eq. (2.29), we see that the passage from zero to finite temperature is equivalent to the substitution

\[
e^{-i\omega(k)t_{12}} \mapsto \frac{e^{-\beta[\omega(p_1) - \mu]} + e^{-i\omega(k)t_{12}} + e^{-i\omega(k)t_{12}}}{1 - e^{-\beta[\omega(p_1) - \mu]}} \tag{3.13}
\]

in the integrand of (2.29). We observe also that in the limit \(\beta \to \infty\) one recovers from (3.12) the Fock space correlator (2.29), provided that \(\mu \leq m\).

Summarizing, we investigated above a hermitian scalar field interacting with a general point-like impurity in 1+1 space-time dimensions. We derived in explicit form the real time finite temperature correlation functions, which satisfy the KMS condition. Using this result, we will analyse in what follows the physical properties of the system under consideration. For simplicity we will consider mostly the case \(\mu = 0\). Non-vanishing chemical potentials can be dealt with along the same lines.

**4  Energy density and Stefan-Boltzmann law**

In order to derive the energy density in the Gibbs state, we first express the Hamiltonian (2.33) in terms of \(\varphi\). One has

\[
H = \int_{-\infty}^{0} dx \, \theta_{00}(t, x) + \int_{0}^{+\infty} dx \, \theta_{00}(t, x), \tag{4.1}
\]

where \(\theta_{00}\) is the energy density operator

\[
\theta_{00}(t, x) = \frac{1}{2} \left[ : \partial_t \varphi \partial_t \varphi : (t, x) - : \varphi \partial_x^2 \varphi : (t, x) + m^2 : \varphi \varphi : (t, x) \right] \tag{4.2}
\]

and \(\cdot : \cdot :\) denote the normal product in the algebra \(\mathcal{C}\). Using the conditions (2.25) at \(x = \pm 0\), one can check directly that \(H\) given by (4.1) is time independent. In this aspect, the position of the derivative \(\partial_x^2\) in the second term of \(\theta_{00}\) is essential for matching the boundary terms, arising at \(x = \pm 0\) from the integration by parts. The expectation value

\[
\mathcal{E}(x, \beta; \gamma) = \langle \theta_{00}(t, x) \rangle_{\beta}, \quad x \neq 0, \tag{4.3}
\]

represents the thermal energy density we are looking for. \(\mathcal{E}\) is \(t\)-independent because the Gibbs state is invariant under time translations.
Because of the normal product in (4.2), $E$ can be expressed in terms of the correlator (3.5). One finds

$$E(x, \beta; \gamma) = \varepsilon_{S-B}(\beta) + \varepsilon(x, \beta; \gamma)$$  \hspace{1cm} (4.4)

with

$$\varepsilon_{S-B}(\beta) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\omega(k)e^{-\beta\omega(k)}}{1 - e^{-\beta\omega(k)}} ,$$  \hspace{1cm} (4.5)

and

$$\varepsilon(x, \beta; \gamma) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\omega(k)e^{-\beta\omega(k)}}{1 - e^{-\beta\omega(k)}} \left[ \theta(x) R_+^+(k) + \theta(-x) R_-^-(k) \right] e^{2ikx} .$$  \hspace{1cm} (4.6)

$E$ depends explicitly only on the reflection coefficients, but one should keep in mind the unitarity constraints (2.19,2.20), which involve the transmission coefficients as well. Taking into account

$$|R_+^+(k)| \leq 1 , \quad |R_-^-(k)| \leq 1 , \quad \forall k \in \mathbb{R} ,$$  \hspace{1cm} (4.7)

one easily derives from (4.6) the estimate

$$|\varepsilon(x, \beta; \gamma)| \leq \varepsilon_{S-B}(\beta) ,$$  \hspace{1cm} (4.8)

which implies

$$E(x, \beta; \gamma) \geq 0 .$$  \hspace{1cm} (4.9)

The term (4.5) is present even without impurity and is actually the familiar Stefan-Boltzmann (S-B) contribution. One has for instance

$$\varepsilon_{S-B}(\beta)|_{m=0} = \frac{\pi}{6\beta^2} ,$$  \hspace{1cm} (4.10)

which is the thermal energy density for a massless hermitian scalar field in 1+1 space-time dimensions. The term (4.6) is of special interest because it describes the correction to the S-B law due to the defect. For getting a more precise idea about this correction, let us derive $\varepsilon$ for the $\delta$-impurity $\gamma_\eta^+ \equiv (1, 0, 2\eta > 0, 1)$. Taking for simplicity $m = 0$, one has

$$\varepsilon(x, \beta; \gamma_\eta^+) = -\frac{i\eta}{2\pi} \int_{-\infty}^{+\infty} dk \frac{|k|e^{-\beta|k|} e^{2ik|x|}}{1 - e^{-\beta|k|} k + i\eta} ,$$  \hspace{1cm} (4.11)

or equivalently, using the Feynman parameter $\alpha$ for representing the denominator $k + i\eta$,

$$\varepsilon(x, \beta; \gamma_\eta^+) = -\frac{i\eta}{2\pi} \int_{-\infty}^{+\infty} dk \int_{0}^{+\infty} d\alpha \frac{|k|e^{-\eta\alpha - \beta|k|}}{1 - e^{-\beta|k|}} e^{ik(2|x|+\alpha)} .$$  \hspace{1cm} (4.12)
The exponential factor $e^{-\eta \alpha - \beta |k|}$ in the integrand of (4.12) allows to exchange the integrals in $k$ and $\alpha$. Integrating first over $k$ and then over $\alpha$ one gets [18]

$$
\varepsilon(x, \beta; \gamma_\eta^+) = \frac{\eta}{2} \int_0^{+\infty} d\alpha e^{-\eta \alpha} \left\{ \frac{\pi}{\beta^2 \sinh^2[\pi \beta^{-1}(2|x| + \alpha)]} - \frac{1}{\pi(2|x| + \alpha)^2} \right\} = \\
\frac{2\pi \eta e^{-4\pi |x|}}{\beta(2\pi + \beta \eta)} \, _2F_1 \left[ 2, 1 + \frac{\beta \eta}{2\pi}, 2 + \frac{\beta \eta}{2\pi}; e^{-4\pi |x|} \right] - \frac{\eta}{4\pi |x|} - \frac{\eta^2 e^{2|x|\eta}}{2\pi} \text{Ei}(-2|x|\eta), \quad (4.13)
$$

$$_2F_1$$ and Ei being the hypergeometric and exponential-integral functions respectively. The study of (4.13) shows that the correction to the S-B law generated by the $\delta$-impurity is relevant close to the impurity. Moreover, $\varepsilon$ vanishes in both limits $\beta \to \infty$ and $|x| \to \infty$, as illustrated in Fig.1 and Fig.2 respectively.

![Figure 1: Plot of $\varepsilon(0.1, \beta; \gamma_\eta=1)$](image1)

![Figure 2: Plot of $\varepsilon(x, 10; \gamma_\eta=1)$](image2)

Combining (4.10) with (4.13), one gets

$$
\mathcal{E}(x, \beta; \gamma_\eta^+) = \frac{\pi}{6\beta^2} +
$$
\[
\frac{2\pi \eta e^{-4\pi |x\eta|}}{\beta(2\pi + \beta \eta)} \ _2F_1 \left[ 2, 1 + \frac{\beta \eta}{2\pi}; 2 + \frac{\beta \eta}{2\pi}; e^{-4\pi |x\eta|} \right] - \frac{\eta^2 e^{2|x\eta|}}{2\pi} \mathrm{Ei}(-2|x\eta|). \tag{4.14}
\]

By construction \( \mathcal{E} \) collects only the thermal energy contributions and thus vanishes in the limit \( T \to 0 \). Technically, this feature is a consequence of the normal product in (4.2). Concerning the total energy density, one has
\[
\mathcal{E}_{\text{tot}}(x, \beta; \gamma) = \mathcal{E}(x, \beta; \gamma) + \mathcal{E}_C(x; \gamma), \tag{4.15}
\]
where \( \mathcal{E}_C \) is the Casimir energy density at \( T = 0 \), namely
\[
\mathcal{E}_C(x; \gamma) = \int_{-\infty}^{+\infty} \frac{dk}{4\pi} \omega(k) \left[ \theta(x) R_+^+(k) + \theta(-x) R_-^-(k) \right] e^{2ikx} \tag{4.16}
\]

Eqs. (4.15,4.16) can be derived from (4.2), adopting the point splitting regularization instead of the normal product and subtracting the vacuum energy density of the free scalar field without impurity. For example, the Casimir energy density of the massless scalar field with the \( \gamma^+_\eta \)-impurity equals
\[
\mathcal{E}_C(x; \gamma^+_\eta) = \frac{\eta}{4\pi |x|} + \frac{\eta^2 e^{2|x\eta|}}{2\pi} \mathrm{Ei}(-2|x\eta|), \tag{4.17}
\]
which leads to the following total energy density
\[
\mathcal{E}_{\text{tot}}(x, \beta; \gamma^+_\eta) = \frac{\pi}{6\beta^2} + \frac{2\pi \eta e^{-4\pi |x\eta|}}{\beta(2\pi + \beta \eta)} \ _2F_1 \left[ 2, 1 + \frac{\beta \eta}{2\pi}; 2 + \frac{\beta \eta}{2\pi}; e^{-4\pi |x\eta|} \right]. \tag{4.18}
\]

Finally, it is instructive to compare the total energy density with impurity to the energy density of the free scalar field \( \gamma_0 = (1, 0, 0, 1) \). Defining the pure impurity contribution by
\[
\mathcal{E}_{\text{imp}}(x, \beta; \gamma) \equiv \mathcal{E}_{\text{tot}}(x, \beta; \gamma) - \mathcal{E}_{\text{tot}}(x, \beta; \gamma_0), \tag{4.19}
\]
one gets
\[
\mathcal{E}_{\text{imp}}(x, \beta; \gamma) = \int_{-\infty}^{+\infty} \frac{dk}{4\pi} \omega(k) \frac{1 + e^{-\beta\omega(k)}}{1 - e^{-\beta\omega(k)}} \left[ \theta(x) R_+^+(k) + \theta(-x) R_-^-(k) \right] e^{2ikx}. \tag{4.20}
\]
For the \( \delta \)-impurity one finds
\[
\mathcal{E}_{\text{imp}}(x, \beta; \gamma^+_\eta) = \frac{2\pi \eta e^{-4\pi |x\eta|}}{\beta(2\pi + \beta \eta)} \ _2F_1 \left[ 2, 1 + \frac{\beta \eta}{2\pi}; 2 + \frac{\beta \eta}{2\pi}; e^{-4\pi |x\eta|} \right], \tag{4.21}
\]
which concludes our investigation of the energy density.
5 Charge and current densities

In this section we consider a complex scalar field

$$\varphi(t, x) = \frac{1}{\sqrt{2}} [\varphi_1(t, x) + i\varphi_2(t, x)] ,$$

(5.1)

where both $\varphi_1$ and $\varphi_2$ are Hermitian scalar fields satisfying (2.1, 2.2) and (2.5) with the same $\gamma \in \Gamma_0$. Our goal will be to derive the charge and current densities of the $U(1)$ conserved current

$$j_\mu(t, x) = -i [: (\partial_\mu \varphi^*) \varphi : (t, x) - : \varphi^*(\partial_\mu \varphi) : (t, x)] .$$

(5.2)

in the Gibbs state constructed in sect. 3. Using the two-point function (3.12), one obtains

$$\rho(x, \beta; \gamma) = \langle j_0(t, x) \rangle_\beta =$$

$$\int_{-\infty}^{+\infty} \frac{dk}{\pi} \frac{e^{-\beta \omega(k)}}{1 - e^{-\beta \omega(k)}} \left[ 1 + \left[ \theta(x) R_+^+(k) + \theta(-x) R_-^-(k) \right] e^{2ikx} \right] ,$$

(5.3)

$$\langle j_1(t, x) \rangle_\beta = 0 .$$

(5.4)

The coexistence of non-trivial charge density with vanishing current density is not surprising because the impurity breaks down Lorentz symmetry. Like for the energy density $E$, one easily shows that

$$\rho(x, \beta; \gamma) \geq 0 .$$

(5.5)

The temperature dependence of $\rho$ is simple: $\rho$ vanishes in the limit $T \to 0$ and diverges for $T \to \infty$. Concerning the space dependence, it is instructive to compare the $x$-distribution of the charge at a given temperature for different defects. Besides the already familiar $\delta$-impurity, we will consider

$$\gamma_\xi^+ = (1, 2\xi, 0, 1), \quad \xi > 0 ,$$

(5.6)

and

$$\gamma_\zeta^+ = (2, 0, \zeta/2, 1/2), \quad \zeta > 0 .$$

(5.7)

Inserting (2.6, 5.6, 5.7) in (5.3) one finds

$$\rho(x, \beta; \gamma_\eta^+) = \int_{0}^{+\infty} \frac{dk}{\pi} \frac{2}{e^{\beta \omega(k)} - 1} \left[ 1 + \frac{[\eta k \sin(2k|x|) - \eta^2 \cos(2k|x|)]}{k^2 + \eta^2} \right] ,$$

(5.8)
\[
\rho(x, \beta; \gamma_{\xi}^+) = \int_{0}^{+\infty} \frac{dk}{\pi} \frac{e^{\beta \omega(k)} - 1}{\pi^2} \left\{ 1 + \frac{[\xi k \sin(2k|x|) + \xi^2 k^2 \cos(2k|x|)]}{\xi^2 k^2 + 1} \right\}, \quad (5.9)
\]

and
\[
\rho(x, \beta; \gamma_{\xi}^+) = \int_{0}^{+\infty} \frac{dk}{\pi} \frac{e^{\beta \omega(k)} - 1}{\pi^2} \left\{ 1 + \theta(x) \frac{[2\zeta k \sin(2kx) + (k^2 \zeta^2 + 15) \cos(2kx)]}{k^2 \zeta^2 + 25} + \theta(-x) \frac{[-8\zeta k \sin(2kx) + (k^2 \zeta^2 - 15) \cos(2kx)]}{k^2 \zeta^2 + 25} \right\}, \quad (5.10)
\]

respectively. The densities (5.8, 5.9) are even functions in \(x\) in agreement with the fact that the impurities \(\gamma_{\eta}^+\) and \(\gamma_{\xi}^+\) are parity preserving. The defect \(\gamma_{\chi}^+\) instead violates parity and gives rise to a distribution which is asymmetric with respect to the origin.

![Figure 3: Plot of \(\rho(x, 10; \gamma_{\eta}=1)\) for \(m = 0.1\).](image)

![Figure 4: Plot of \(\rho(x, 10; \gamma_{\xi}=1)\) for \(m = 0.1\).](image)

To integrate in (5.8, 5.10) over \(k\) analytically is a hard job, but the numerical study gives a precise idea about the charge distribution. The Figures 3–5 display
the shape of $\varrho$ as a function of $x$. We see that the profile of $\varrho$ strongly depends on the type of impurity. For the $\delta$-defect the charge density (Fig.3) is minimal at the impurity. For $\gamma_\xi$ the distribution $\varrho$ has (Fig.4) two maxima localized away from the impurity. Finally, for $\gamma_\zeta$ one finds (Fig.5) that $\varrho$ is discontinuous in $x=0$ and reaches a minimum (maximum) when $x \to 0$ from the left (right).

The possibility to design the charge density profile by choosing appropriate $\gamma \in \Gamma$ looks very attractive for potential applications.

6 The contribution of impurity bound states

This section collects our results about the case $\gamma \in \Gamma_1 \cup \Gamma_2$. The spectrum of $\{-\partial_x^2, \gamma \in \Gamma_1\}$ involves one bound state. A complete orthonormal system of eigenstates is given by (2.14), supplemented by

$$\chi(x) = \begin{cases} \sqrt{|r_+|} e^{i|x|r_+}, & b < 0, \\ \sqrt{|r_0|} e^{i|x|r_0}, & b = 0, \\ \sqrt{|r_-|} e^{i|x|r_-}, & b > 0. \end{cases} \quad (6.1)$$

When $\gamma \in \Gamma_2$ one has two bound states, namely

$$\chi_\pm(x) = \sqrt{|r_\pm|} e^{i|x|r_\pm}. \quad (6.2)$$

The states (6.1, 6.2) give rise to new quantum degrees of freedom for $\varphi$, which are represented by the last term in the decomposition

$$\varphi(t, x) = \varphi_+(t, x) + \varphi_-(t, x) + \varphi_b(t, x) \quad (6.3)$$
where $\varphi_\pm$ are given by (2.28). We emphasize that the contribution $\varphi_b$ is uniquely fixed by the initial condition (2.2). If $\gamma \in \Gamma_1$ for instance, we must introduce the creation and annihilation operators $\{b^*, b\}$, which commute with $C$ and satisfy

$$[b, b] = [b^*, b^*] = 0, \quad [b, b^*] = 1. \quad (6.4)$$

One has in this case

$$\varphi_b(t, x) = \frac{1}{\sqrt{2\omega}} \left( b^* e^{it\omega} + b e^{-it\omega} \right) \chi(x), \quad (6.5)$$

with

$$\omega = \begin{cases} \sqrt{m^2 - r_+^2}, & b < 0, \\ \sqrt{m^2 - r_0^2}, & b = 0, \\ \sqrt{m^2 - r_-^2}, & b > 0. \end{cases} \quad (6.6)$$

The two-point correlator reads

$$w(t_{12}, x_1, x_2) \big|_{\gamma \in \Gamma_1} = w(t_{12}, x_1, x_2) \big|_{\gamma \in \Gamma_0} + \frac{1}{2\omega} e^{-it_{12}\omega} \chi(x_1) \chi(x_2). \quad (6.7)$$

Analogously, in the case $\gamma \in \Gamma_2$ one needs two oscillators $\{b^\pm, b\pm\}$, both satisfying (6.4) and commuting with each other and with $C$. Now

$$\varphi_b(t, x) = \sum_{\sigma=\pm} \frac{1}{\sqrt{2\omega}} \left( b^\sigma e^{it\omega_\sigma} + b_\sigma e^{-it\omega_\sigma} \right) \chi_\sigma(x), \quad \omega_\pm = \sqrt{m^2 - r_\pm^2}, \quad (6.8)$$

and

$$w(t_{12}, x_1, x_2) \big|_{\gamma \in \Gamma_2} = w(t_{12}, x_1, x_2) \big|_{\gamma \in \Gamma_0} + \sum_{\sigma=\pm} \frac{1}{2\omega} e^{-it_{12}\omega_\sigma} \chi_\sigma(x_1) \chi_\sigma(x_2). \quad (6.9)$$

It follows from (6.6) that in order to avoid imaginary energies and the associated quantum field instabilities, we must restrict further $\Gamma_1 \cup \Gamma_2$ to

$$\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 = \{ \gamma \in \Gamma_1 \cup \Gamma_2 : \vert r_0 \vert \leq m, \vert r_+ \vert \leq m \}. \quad (6.10)$$

These restrictions are a consequence of the relativistic dispersion relation for $\varphi$ and are not present in the quantum mechanical context [1–5]. Notice in particular that there are no defect bound state contributions for $m = 0$, because both $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are empty in this case.
In the construction of the Gibbs state for $\gamma \in \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$ one must take into account the defect bound state degrees of freedom described by the oscillator algebras $\{b, b^*\}$ and $\{b_\pm, b^*_\pm\}$. The Hamiltonian is

$$H = \sum_{\xi = \pm} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \omega(k) a^{*\xi}(k) a_\xi(k) + h_b,$$

where

$$h_b = \begin{cases} \omega b^* b, & \gamma \in \tilde{\Gamma}_1, \\ \sum_{\sigma = \pm} \omega_\sigma b^{*\sigma} b_\sigma, & \gamma \in \tilde{\Gamma}_2. \end{cases}$$

For the two-point functions at temperature $\beta$ and chemical potential $\mu$, one obtains

$$w_\beta(t_{12}, x_1, x_2)|_{\gamma \in \tilde{\Gamma}_1} = w_\beta(t_{12}, x_1, x_2)|_{\gamma \in \Gamma_0} + \frac{e^{-\beta(\omega - \mu) + i\omega t_{12}} + e^{-i\omega t_{12}}}{2\omega[1 - e^{-\beta(\omega - \mu)}]} \chi(x_1) \chi(x_2),$$

and

$$w_\beta(t_{12}, x_1, x_2)|_{\gamma \in \tilde{\Gamma}_2} = w_\beta(t_{12}, x_1, x_2)|_{\gamma \in \Gamma_0} + \sum_{\sigma = \pm} \frac{e^{-\beta(\omega_\sigma - \mu) + i\omega_\sigma t_{12}} + e^{-i\omega_\sigma t_{12}}}{2\omega_\sigma[1 - e^{-\beta(\omega_\sigma - \mu)}]} \chi_\sigma(x_1) \chi_\sigma(x_2).$$

At this point we are ready to derive the impurity bound states correction to the S-B law. Instead of (4.4) one has

$$\mathcal{E}(x, \beta; \gamma) = \varepsilon_{S-B}(\beta) + \varepsilon(x, \beta; \gamma) + \varepsilon_b(x, \beta; \gamma),$$

where

$$\varepsilon_b(x, \beta; \gamma) = \begin{cases} \frac{e^{-\beta\omega}}{1 - e^{-\beta\omega}} \omega \chi(x)^2, & \gamma \in \tilde{\Gamma}_1, \\ \sum_{\sigma = \pm} \frac{e^{-\beta_\sigma\omega_\sigma}}{1 - e^{-\beta_\sigma\omega_\sigma}} \omega_\sigma \chi_\sigma(x)^2, & \gamma \in \tilde{\Gamma}_2 \end{cases}$$

is the bound state contribution. In view of (6.1,6.2) this contribution decays exponentially with the distance from the impurity. The same property has the bound state correction to the charge density $\rho$.

7 Defects in higher space-time dimensions

Our goal below is to show that all the results obtained in the previous sections can be actually generalized to higher space-time dimensions. For this purpose we consider
satisfy the following RT algebra commutation relations and the initial conditions

\[ t, x \in \mathbb{R} \quad \text{and} \quad (s, \text{the impurity boundary condition} \]

Eqs. (7.1-7.4) have a unique solution. For \( \gamma \in \Gamma_0 \) one has

\[
\varphi(t, x, y) = \varphi_+(t, x, y) + \varphi_-(t, x, y),
\]

Here \( \omega(k, p) = \sqrt{k^2 + p^2 + m^2} \) and \( \{a^\xi(k, p), a_\xi(k, p) : \xi = \pm, k \in \mathbb{R}, p \in \mathbb{R}^s\} \) satisfy the following RT algebra commutation relations

\[
\begin{align*}
& a^\xi_1(k_1, p_1) a^\xi_2(k_2, p_2) - a^\xi_2(k_2, p_2) a^\xi_1(k_1, p_1) = 0, \quad (7.7) \\
& a^\xi_2(k_1, p_1) a^\xi_2(k_2, p_2) - a^\xi_2(k_2, p_2) a^\xi_2(k_1, p_1) = 0, \quad (7.8) \\
& a^\xi_1(k_1, p_1) a^\xi_2(k_2, p_2) - a^\xi_2(k_2, p_2) a^\xi_1(k_1, p_1) = \left\{ [\delta^\xi_1 + T^\xi_1(k_1)] \delta(k_1 - k_2) + R^\xi_1(k_1) \delta(k_1 + k_2) \right\} (2\pi)^{s+1} \delta(p_1 - p_2) \mathbf{1} \quad (7.9)
\end{align*}
\]

and the constraints

\[
\begin{align*}
& a_\xi(k, p) = T_\xi^\eta(k) a_\eta(k, p) + R_\xi^\eta(k) a_\eta(-k, p), \quad (7.10) \\
& a^*\xi(k, p) = a^*\eta(k, p) T_\eta^\xi(k) + a^*\eta(-k, p) R_\eta^\xi(-k). \quad (7.11)
\end{align*}
\]
The construction of the Gibbs state for this algebra follows that of section 3 and gives

\[ \langle \alpha^* \xi_1(k_1, p_1) \alpha \xi_2(k_2, p_2) \rangle_\beta = \frac{e^{-\beta[\omega(k_1, p_1)-\mu]}}{1 - e^{-\beta[\omega(k_1, p_1)-\mu]}}. \]

\[ \left\{ \delta^\xi_2 + \sigma^\xi_2(k_1) \right\} \delta(k_1 - k_2) + R^\xi_2(-k_1) \delta(k_1 + k_2) \right\} (2\pi)^{s+1} \delta(p_1 - p_2), \quad (7.12) \]

\[ \langle \alpha \xi_1(k_1, p_1) \alpha^* \xi_2(k_2, p_2) \rangle_\beta = \frac{1}{1 - e^{-\beta[\omega(k_1, p_1)-\mu]}}. \]

\[ \left\{ \delta^\xi_2 + \sigma^\xi_2(k_1) \right\} \delta(k_1 - k_2) + R^\xi_2(k_1) \delta(k_1 + k_2) \right\} (2\pi)^{s+1} \delta(p_1 - p_2). \quad (7.13) \]

The expectation values (7.12, 7.13) determine the finite temperature two-point function

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dk \, dp \, e^{-ipy_{12}}}{(2\pi)^{s+1} 2\omega(k, p)} \left\{ \frac{\langle \varphi(t_1, x_1, y_1) \varphi(t_2, x_2, y_2) \rangle_\beta}{1 - e^{-\beta[\omega(k, p)-\mu]}} \right\} E(k; x_1, x_2; \gamma). \quad (7.14) \]

One can easily derive at this point the energy and the charge densities

\[ E(x, \beta; \gamma) = \langle \theta_{00}(t, x, y) \rangle_\beta, \quad \theta(x, \beta; \gamma) = \langle j_0(t, x, y) \rangle_\beta, \quad (7.15) \]

which are both \( t \) and \( y \)-independent because of the symmetry properties of the Gibbs state. Setting \( \mu = 0 \), one finds

\[ E_s(x, \beta; \gamma) = \int_{-\infty}^{+\infty} \frac{dk \, d^sp \, \omega(k, p)e^{-\beta\omega(k, p)}}{(2\pi)^{s+1} 1 - e^{-\beta\omega(k, p)}} \left\{ 1 + \left[ \theta(x) R^+(k) + \theta(-x) R^-(k) \right] e^{2ikx} \right\}, \quad (7.16) \]

and

\[ \theta_s(x, \beta; \gamma) = \int_{-\infty}^{+\infty} \frac{dk \, d^sp \, e^{-\beta\omega(k, p)}}{(2\pi)^{s+1} 1 - e^{-\beta\omega(k, p)}} \left\{ 1 + \left[ \theta(x) R^+(k) + \theta(-x) R^-(k) \right] e^{2ikx} \right\}. \quad (7.17) \]

Since \( \omega(k, p) \) depends actually on \( |p| \), the integration over the angular variables in \( p \)-space is easily performed and gives the area of the unit sphere in \( s \) dimensions. In this way one is left with the integrals in \( k \) and \( |p| \).

The case of major interest is \( s = 2 \), which corresponds to a plane-defect in 3+1 space-time dimensions. The identity in the curly brackets under the integral (7.16) precisely reproduces the S-B energy

\[ \varepsilon_{s-B}(\beta) |_{s=2, m=0} = \frac{\pi^2}{30\beta^4} \quad (7.18) \]
of the free boson field in 3+1 space-time dimensions. The correction generated by the \( \delta \)-impurity reads

\[
\varepsilon_2(x, \beta; \gamma^+_\eta) = \int_0^{+\infty} dk \int_0^{+\infty} dp |p| \sqrt{k^2 + |p|^2} \left[ \frac{\eta k \sin(2k|x|) - \eta^2 \cos(2k|x|)}{2\pi^2(k^2 + \eta^2)} \right].
\]

(7.19)

It is instructive to compare now the corrections to the S-B law for \( s = 0 \) and \( s = 2 \), given by (4.11) and (7.19) respectively. For this purpose we have plotted \( \varepsilon_2 \) in Fig. 6 for the same values of \( \beta \) and \( \eta \) for which \( \varepsilon \) is displayed in Fig. 2. We see that the profiles are the same, but from the numerical values on the vertical axes we deduce that \( \varepsilon_2 \) is shifted and squeezed along this axis with respect to \( \varepsilon \). This phenomenon takes place for the charge densities as well. In fact, inserting the data (2.6, 5.6, 5.7) in the general formula (7.17) and plotting the resultant expressions, one gets Fig.7–9. The comparison between Fig.4–6 and Fig.7–9 confirms this observation.

An aspect which deserves further investigation is the introduction of self-interaction, e.g. a term \( g\varphi^3 \) in the equation of motion (7.1). The above setup provides the basis
for a perturbative investigation of this case. Let us mention in this respect that some non-perturbative results about the non-relativistic $\phi^4$-theory at zero temperature have been recently obtained in [19, 20], using the RT algebra technique.

Another interesting issue concerns the quantum fields induced on the defect. Following [21, 22], one can show that the limits

$$\lim_{x \to \pm 0} \phi(t, x, y) = \Phi_{\pm}(t, y)$$

exist and correctly define two quantum fields $\Phi_{\pm}(t, y)$, which propagate in the defect. $\Phi_+$ and $\Phi_-$ coincide only for parity preserving impurities.

Summarizing, we have shown in this section that the RT algebra approach to impurities has a natural generalization to higher dimensions, where impurities have a more realistic physical interpretation.
8 Outlook and conclusions

We studied above the family of impurities defined by all possible self-adjoint extensions of the operator $-\partial^2_x$ on functions in $\mathbb{R} \setminus \{0\}$, showing that they can be described in a purely algebraic way. The relevant structure is an appropriate RT algebra, which translates in simple algebraic terms the solution of the analytic problem defined by eqs. (2.1, 2.2, 2.5). Constructing the Gibbs state over this algebra, we were able to investigate the finite temperature behavior of the associated physical systems. Motivated by potential applications in condensed matter physics, we derived in this framework the energy and charge densities of the systems in the Gibbs state, discussing the dependence of these quantities on the impurity type. We also computed in this context the correction to the Stefan-Boltzmann law, corresponding to a generic impurity from the above family. The contribution of the impurity bound states was discussed as well. Self-interactions can be treated in perturbation theory with the propagator determined by eq. (7.14).

We started our investigation with point-like defects in 1+1 space-time dimensions, demonstrating later on that the RT algebra framework actually applies to $s$-dimensional hyperplane-defects in $(s + 1) + 1$ dimensions for any $s \geq 0$. Since our hyperplane-defects both reflect and transmit, they generalize the concept of brane, which is usually assumed to reflect only. A challenging open question is if strings give raise to such more general configurations. Another interesting issue is the study of the fields induced according to (7.20) on impurities and their role in the construction of quantum field theories localized on defects.

RT algebras emerged in the context of factorized scattering theory for integrable models in 1+1 dimensions. The results of this paper show that they have actually a wider radius of application, representing an efficient tool for dealing with impurities at finite temperatures and in higher space-time dimensions.

References

[1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn ans H. Holden, Solvable models in quantum mechanics (Springer-Verlag, Berlin, 1988).

[2] S. Albeverio, L. Dabrowski and P. Kurasov, Lett. Math. Phys. 45 (1998) 33.

[3] F. A. B. Coutinho, Y. Nogami and L. Tomio, J. Phys. A 32 (1999) 4931. arXiv:quant-ph/9903098.

[4] F. A. B. Coutinho, Y. Nogami and J. F. Perez, J. Phys. A 30 (1997) 3937.
[5] A. G. M. Schmidt, B. K. Cheng and M. G. E. da Luz, Phys. Rev. A 66 (2002) 062712.

[6] G. Delfino, G. Mussardo and P. Simonetti, Nucl. Phys. B 432 (1994) 518 [arXiv:hep-th/9409076].

[7] R. Konik and A. LeClair, Nucl. Phys. B 538 (1999) 587 [arXiv:hep-th/9703085].

[8] H. Saleur, “Lectures on Non-perturbative field theory and quantum impurity problems”, [arXiv:cond-mat/9812110].

[9] H. Saleur, “Lectures on Non-perturbative field theory and quantum impurity problems II”, [arXiv:cond-mat/0007309].

[10] O. Castro-Alvaredo and A. Fring, Nucl. Phys. B 649 (2003) 449 [arXiv:hep-th/0205076].

[11] P. Bowcock, E. Corrigan and C. Zambon, “Classically integrable field theories with defects”, [arXiv:hep-th/0305022].

[12] P. Bowcock, E. Corrigan and C. Zambon, “Affine Toda field theories with defects”, [arXiv:hep-th/0401020].

[13] M. Hallnäs, E. Langmann, “Exact solutions of two complementary 1D quantum many-body systems on the half-line”, [arXiv:math-ph/0404023].

[14] M. Mintchev, E. Ragoucy and P. Sorba, Phys. Lett. B 547 (2002) 313 [arXiv:hep-th/0209052].

[15] M. Mintchev, E. Ragoucy and P. Sorba, J. Phys. A 36 (2003) 10407 [arXiv:hep-th/0303187].

[16] M. Mintchev and E. Ragoucy, J. Phys. A 37 (2004) 425 [arXiv:math.qa/0306084].

[17] O. Bratteli and D. W. Robinson, Operator Algebras and Quantum Statistical Mechanics, Vol. 2 (Springer-Verlag, Berlin, 1996).

[18] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, Vol. 1 (Gordon and Breach, New York, 1986).

[19] V. Caudrelier, M. Mintchev and E. Ragoucy, “The quantum non-linear Schrödinger model with point-like defect”, [arXiv:hep-th/0404144].
[20] V. Caudrelier, M. Mintchev and E. Ragoucy, “Solving the quantum non-linear Schrödinger equation with delta-type impurity”, arXiv:math-ph/0404047.

[21] M. Mintchev and L. Pilo, Nucl. Phys. B 592 (2001) 219 arXiv:hep-th/0007002.

[22] M. Mintchev, Class. Quant. Grav. 18 (2001) 4801 arXiv:hep-th/0103259.