MASAS AND BIMODULE DECOMPOSITIONS OF II₁ FACTORS

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Abstract. The measure-multiplicity-invariant for masas in II₁ factors was intro-
duced in [10] to distinguish masas that have the same Pukánszky invariant. In
this paper we study the measure class in the measure-multiplicity-invariant. This is
equivalent to studying the standard Hilbert space as an associated bimodule. We
characterize the type of any masa depending on the left-right-measure using Baire
category methods (selection principle of Jankov and von Neumann). We present a
second proof of Chifan’s result [2] and a measure theoretic proof of the equivalence
of weak asymptotic homomorphism property (WAHP) and singularity that appeared
in [35].

1. Introduction

This is the first of a series of two papers developed by the author for his Phd the-
sis. The moral of this paper is : “The phenomena regularity, semiregularity, singu-
rity, weak asymptotic homomorphism property (WAHP) and asymptotic homomorphism
property (AHP) of masas in finite von Neumann algebras can all be explained by mea-
sure theory”. Throughout the entire paper ℳ will always denote a separable II₁ factor.
Let A ⊂ ℳ be a maximal abelian self-adjoint subalgebra henceforth abbreviated as a
masa. It is a theorem of von Neumann that A is isomorphic to $L^\infty([0, 1], dx)$. So the
study of masas in type II₁ factors is understanding its position (up to automorphisms
of the ambient von Neumann algebra). For a masa A ⊂ ℳ, Dixmier in [5] defined the
group of normalizing unitaries (or normaliser) of A to be the set
$$N(A) = \{u \in U(ℳ) : uAu^* = A\},$$
where $U(ℳ)$ denotes the unitary group of ℳ. He called
(i) A to be regular (also Cartan) if $N(A)^{''} = ℳ$,
(ii) A to be semiregular if $N(A)^{''}$ is a subfactor of ℳ,
(iii) A to be singular if $N(A) \subset A$.
He also exhibited the presence of all three kinds of masas in the hyperfinite II₁ factor.
Two masas A, B of ℳ are said to be conjugate if there is an automorphism $\theta$ of ℳ
such that $\theta(A) = B$. If there is an unitary $u \in ℳ$ such that $uAu^* = B$ then A and B
are called unitarily (inner) conjugate.

Feldman and Moore in [11], [12] characterized pairs A ⊂ ℳ, where A is a Cartan
subalgebra, as those coming from r-discrete transitive measured groupoids with a fi-
nite measure space X as base. It is a remarkable achievement of Connes, Feldman and
Weiss [3] that any countable amenable measured equivalence relation is generated by
a single transformation of the underlying space. When translated into the language of
operator algebras via the Feldman-Moore construction, this theorem together with a
theorem of Krieger [14] says that, if ℳ is any injective von Neumann algebra then any
two Cartan subalgebras are conjugate by an automorphism of ℳ. However it follows
from their theorem that, there are uncountably many equivalence classes of Cartan

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masas up to unitary conjugacy in the hyperfinite II$_1$ factor. See [23] for more examples. There exist II$_1$ factors with non conjugate Cartan masas (see [4]). These masas were distinguished with the presence or absence of nontrivial centralizing sequences. Recently Ozawa and Popa have exhibited examples of II$_1$ factors with no or at most one Cartan masa up to unitary conjugacy (see [22]).

The absence of Cartan masas in II$_1$ factors was first due to Voiculescu in [36]. In fact, it was his amazing discovery that, for any diffuse abelian algebra $A \subset L(\mathbb{F}_n)$, the standard Hilbert space $l^2(\mathbb{F}_n)$ as a $A, A$-bimodule contains a copy of $L^2(A) \otimes L^2(A)$. His result was improved by Dykema in [9] to rule out the presence of masas in free group factors with finite multiplicity.

Getting back to singular masas, in 1960 Pukánszky showed in [28] that there are countable non conjugate singular masas in the hyperfinite II$_1$ factor by introducing an algebraic invariant for masas in II$_1$ factors, today known as the Pukánszky invariant. In 1983 Popa [24] succeeded in showing that all separable continuous semifinite von Neumann algebras and all separable factors of type III$_\lambda$, $0 \leq \lambda < 1$ have singular masas. Although they exist, citing explicit examples is a very hard job. In this direction, Smith and Sinclair in [33] have given concrete examples of uncountably many non conjugate singular masas in the hyperfinite II$_1$ factor. White and Sinclair [32] have given explicit examples of a continuous path of non conjugate singular masas (Tauer masas) in the hyperfinite II$_1$ factor. All the masas in this path have the same algebraic invariant of Pukánszky. Subsequently, White in [37] proved that, any possible value of the Pukánszky invariant can be realized in the hyperfinite II$_1$ factor, and any McDuff factor which contains a masa of Pukánszky invariant $\{1\}$ contains masas of any arbitrary Pukánszky invariant.

Singularity is often quite hard to check (see [29]). In order to check if a masa is singular analytical properties “AHP” and “WAHP” were discovered in [30], [31]. Subsequently Smith, Sinclair, White and Wiggins in [35] characterized pairs $A \subset \mathcal{M}$, where $A$ is a singular masa in a II$_1$ factor $\mathcal{M}$ to be precisely those for which $A$ satisfies “WAHP”. All the theories that we have outlined have a common theme namely, “What is the structure of the standard Hilbert space as a $w^*$ $A, A$-bimodule.”

Although many invariants of masas in II$_1$ are known the first successful attempt to distinguish masas with a natural invariant, which have the same Pukánszky invariant was due to Dykema, Smith and Sinclair in [10]. We call this the measure-multiplicity-invariant. This invariant has two main components, a measure class and a multiplicity function. This invariant is not a new one and has existed in the literature for quite some time. For Cartan masas this invariant has very deep meaning and it is very hard to distinguish Cartan masas with this invariant. The term multiplicity in the measure-multiplicity-invariant is actually the Pukánszky invariant of the masa, making it a stronger invariant. A slightly different invariant was considered by Neshveyev and Størmer in [19].

Our intention is to study singular masas and distinguish them. In order to do so, it is necessary to think of singularity from a different point of view. The theory of Cartan masas and singular masas have so far been viewed from two different angles. While Cartan masas fit to the theory of orbit equivalence on one hand [12], singular masas fit to the intertwining techniques of Popa on the other [35]. But we would like to have an unique approach that explains all these phenomena. This is the primary goal of this paper. In this paper, we characterize masas by studying the structure of the standard Hilbert space as its associated bimodule.

Our second goal is to investigate that, after such a theory is outlined whether it is possible to obtain proofs of important theorems regarding masas that were obtained by a number of researchers by using different ideas. Many old theorems can indeed be
proved but we will mainly prove Chifan’s result on tensor products \[2\] and the equivalence of WAHP and singularity \[35\]. In fact, it seems that studying the bimodule is the most natural way to approach these problems as one can exploit a lot of results from Real Analysis.

In order to distinguish singular masas which have the same multiplicity understanding the measure in the measure-multiplicity-invariant is the most important task. So we study this invariant thoroughly throughout this article. The second paper will contain explicit calculations of the invariant and questions related to conjugacy of masas.

We have learned latter that Popa and Shylakhtenko in \[26\] has results of similar flavor in this direction. However our way of approaching is completely different. We think that what is really involved in understanding the types of masas are the measurable selection principle of Jankov and von Neumann and some generalized version of Dye’s theorem on groupoid normalisers. This is evident from \[8\], \[11\] and \[12\]. We present completely measure theoretic proofs based upon Baire category methods (selection principle). As an outcome of our approach many theorems related to structure theory of masas that were proved by different techniques just follows easily from our technique.

Singular masas are often constructed by considering weakly or strongly mixing actions of infinite abelian groups on finite von Neumann algebras. We will show that the definition of WAHP can be strengthened by considering Haar unitaries and Cesàro sums which exactly resembles the definition of weakly mixing actions. Weakly mixing actions are characterized by null sets of certain measures. The story for singular masas is also similar.

This paper is heavily measure theoretic. Much of the measure theory tools we require are scattered here and there in the literature. This article is organized as follows. In Sec. 2 we present some preliminaries of direct integrals, masas and define the measure-multiplicity-invariant. In Sec. 3 we study disintegration of measures and masas. Sec. 4 deals with generalized versions of Dye’s theorem. Sec. 5 contains the main result i.e the characterization theorem and a second proof of Chifan’s normaliser formula. This is a very technical section. Sec. 6 contains results on calculating certain two-norms and a second proof of the equivalence of WAHP and singularity. Sec. 4 uses the theory of \(L^1\) and \(L^2\) spaces associated to finite von Neumann algebras for which we have cited related results in that section without proofs. Appendix A contains structure theorems of measurable functions satisfying condition \((N)\) of Lusin which is used in Sec. 5.

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**2. Preliminaries**

The paper relies on the theory of direct integrals. So we have divided this section into three subsections. In the first subsection we give some well known results about direct integrals of Hilbert spaces with respect to an abelian von Neumann algebra. In the second part we give some preliminaries about masas in II\(_1\) factors and in the third subsection we will define the measure-multiplicity-invariant of masas in II\(_1\) factors.

**Notation:** Throughout the entire article \(\mathbb{N}_\infty\) will denote the set \(\mathbb{N} \cup \{\infty\}\).

**2.1. Direct Integrals.**

Let a separable Hilbert space \(\mathcal{H}\) be the direct integral of a \(\mu\)-measurable field of
Hilbert spaces \( \{H_x\}_{x \in X} \) over the base space \((X, \mu)\) where \(X\) is a \(\sigma\)-compact space and \(\mu\) is a positive, complete Borel measure.

**Definition 2.1.** An operator \( T \in B(H) \) is said to be **decomposable** relative to the decomposition \( H \cong \int_X H_x d\mu(x) \) if there exists a \(\mu\)-measurable field of operators \( T_x \in B(H_x) \), such that \( x \mapsto \|T_x\| \in L^\infty(X, \mu) \) and \( T = \int_X T_x d\mu(x) \).

If \( T_x = c(x)I_{H_x} \), where \( c(x) \in \mathbb{C} \) for almost all \( x \), then \( T \) is said to be **diagonalizable**.

It is easy to see that the fibres of a decomposable operator are uniquely determined up to an almost sure equivalence. The collection of diagonalizable and decomposable operators both form von Neumann subalgebras of \( B(H) \), with the later being the commutant of the former. Whenever there is no danger of confusion we will use the term measurable instead of \(\mu\)-measurable.

**Theorem 2.2.** Let \( A \subset B(H) \) be a diffuse abelian von Neumann algebra on a separable Hilbert space \( H \). Then there exists a measure space \((X, \mu)\), where \( X \) is a \(\sigma\)-compact space, \( \mu \) is a positive, Borel, non-atomic, complete measure on \( X \) and a measurable field of Hilbert spaces \( \{H_x\}_{x \in X} \), such that \( H \) is unitarily equivalent to,

\[
H \cong \int_X H_x d\mu(x)
\]

and \( A \) is (unitarily equivalent to) the algebra of diagonalizable operators on \( \int_X H_x d\mu(x) \) with respect to this decomposition.

The dimension function of the decomposition in Thm. 2.2 is defined as \( m : X \mapsto \mathbb{N}_\infty \) by, \( m(x) = \dim(H_x) \).

The dimension function \( m \) is \(\mu\)-measurable. Such results are known in greater generality. For a measure space \((X, \mu)\) we denote by \([\mu]\) the equivalence class of measures on \(X\) that are mutually absolutely continuous with respect to \(\mu\). This decomposition in Thm. 2.2 and hence the multiplicity function is unique up to measure equivalence from Thm. 3, 4 of Chapter 6 of [6].

We will be always working with finite measures. Since direct integrals of Hilbert spaces does not change when the measures are scaled, we will most of the time assume that the measures have total mass 1. Details of these facts can be found in [15], [20].

### 2.2. Basics on Masas in II\(_1\) factors.

**Definition 2.3.** Given a type I von Neumann algebra \( B \) we shall write \( \text{Type}(B) \) for the set of all those \( n \in \mathbb{N}_\infty \) such that \( B \) has a nonzero component of type \( I_n \).

Let \( M \) be a separable II\(_1\) factor with the faithful, normal, tracial state \( \tau \). This trace induces the two-norm \( \|x\|_2 = \tau(x^*x)^{1/2} \) on \( M \) and we write \( L^2(M) \) for the Hilbert space completion of \( M \) with respect to this norm. Let \( M \) act on \( L^2(M) \) via left multiplication. Let \( J \) denote the anti-unitary conjugation operator on \( L^2(M) \) obtained by extending the densely defined map \( J(x) = x^* \). Inclusions of von Neumann algebras will always be assumed to be unital until further notice.

Given a von Neumann subalgebra \( N \) of \( M \), let \( E_N \) be the unique trace preserving conditional expectation from \( M \) onto \( N \). This conditional expectation is obtained by restricting the orthogonal projection \( e_N \) from \( L^2(M) \) onto \( L^2(N) \) to \( M \).

Let \( A \subset M \) be a masa. Then the augmented algebra \( A = (A \cup JA')'' \) is an abelian algebra, with a type I commutant, the commutant being taken in \( B(L^2(M)) \) and the center of \( A' \) is \( A \). The Jones projection \( e_A \) onto \( L^2(A) \) lies in \( A \) [34]. Hence, \( A'(1 - e_A) \)
decomposes as,
\begin{equation}
(2.2) \quad \mathcal{A}'(1 - e_A) = \bigoplus_{n \in \mathbb{N}_\infty} \mathcal{A}'P_n
\end{equation}
where \( P_n \in \mathcal{A} \) are orthogonal projections summing up to \( 1 - e_A \) and \( \mathcal{A}'P_n \) is homogenous algebra of type \( n \) whenever \( P_n \neq 0 \).

**Lemma 2.4.** If \( A \subset \mathcal{M} \) be a masa and \( B \subset \mathcal{M} \) be any subalgebra, then \( (A \cup JB)^\prime \) is diffuse.

**Definition 2.5.** The *Pukánszky invariant* of a masa \( A \) in II\(_1\) factor \( \mathcal{M} \), denoted by \( \text{Puk}(A) \) (or \( \text{Puk}_\mathcal{M}(A) \) when the containing factor is ambiguous) is \{ \( n \in \mathbb{N}_\infty : P_n \neq 0 \} \) which is precisely Type(\( A'(1 - e_A) \)).

**Definition 2.6.** If \( A \) is an abelian von Neumann subalgebra of \( \mathcal{M} \), let \( \mathcal{GN}(A) \) or \( \mathcal{GN}(A, \mathcal{M}) \) be the *normalising groupoid*, consisting of those partial isometries \( v \in \mathcal{M} \) that satisfy \( v^*v, vv^* \in A \) and \( vAv^* = Avv^* = vv^*A \).

A theorem of Dye \[7\] says that, a partial isometry \( v \in \mathcal{GN}(A) \) if and only if there is an unitary \( u \in N(A) \) and a projection \( p \in A \) such that \( v = up = (upu^*)u \). Thus \( \mathcal{GN}(A)^\prime = N(A)^\prime \). Popa in \[25\] connected the *Pukánszky invariant* to the type of a masa showing that if \( 1 \notin \text{Puk}(A) \), then \( A \) is singular and that the *Pukánszky invariant* of a Cartan masa is \( \{1\} \).

Singularity is difficult to verify. The following two conditions were introduced in \[31\], \[30\] and \[35\] as they imply singularity and are often easier to verify in explicit situations.

**Definition 2.7.** (Smith, Sinclair) Let \( A \) be a masa in a II\(_1\) factor \( \mathcal{M} \).

(i) \( A \) is said to have the *asymptotic homomorphism property* (AHP) if there exists an unitary \( v \in \mathcal{M} \) such that
\[
\lim_{|u| \to \infty} \| E_A(xv^ny) - E_A(x)v^nE_A(y) \|_2 = 0 \quad \text{for all} \ x, y \in \mathcal{M}.
\]

(ii) \( A \) has the *weak asymptotic homomorphism property* (WAHP) if, for each \( \epsilon > 0 \) and each finite subset \( x_1, \ldots, x_n \in \mathcal{M} \) there is an unitary \( u \in \mathcal{M} \) such that
\[
\| E_A(x_iuax_j^*) - E_A(x_i)uE_A(x_j^*) \|_2 < \epsilon \quad \text{for} \ 1 \leq i, j \leq n.
\]

In \[35\] it was shown that singularity is equivalent to WAHP. We will prove in Sec. 6 that WAHP is indeed the most natural property. The next proposition is well known, we state it for completeness.

**Proposition 2.8.** Let \( \mathcal{N} \subset \mathcal{B}(\mathcal{H}) \) be a von Neumann algebra and let \( x_{i,j} \in \mathcal{N} \) and \( x'_{i,j} \in \mathcal{N}' \) for \( i, j = 1, 2, \ldots, n \). Then the following conditions are equivalent:

(i) \( \sum_{k=1}^{n} x_{i,k}x'_{k,j} = 0 \) for all \( 1 \leq i, j \leq n \).

(ii) There exist elements \( z_{i,j} \in \mathbb{Z}(\mathcal{N}) \), \( i, j = 1, 2, \ldots, n \) such that for all \( i, j \)
\[
\sum_{k=1}^{n} x_{i,k}z_{k,j} = 0, \quad \sum_{k=1}^{n} z_{i,k}x'_{k,j} = x'_{i,j}.
\]

### 2.3. Measure-Multiplicity-Invariant

We consider the *conjugacy invariant* for a masa \( A \) in a II\(_1\) factor \( \mathcal{M} \) derived from writing the direct integral decomposition of its left-right action. More precisely, we choose a compact Hausdorff space \( Y \) such that \( C(Y) \subset A \), is a norm separable unital \( C^* \) subalgebra and \( C(Y) \) is w.o.t dense in \( A \). \( \tau \) restricted to \( C(Y) \) gives rise to a probability measure \( \nu \) on \( Y \) so that \( A \) is isomorphic to \( L^\infty(Y, \widehat{\nu}) \), with \( \widehat{\nu} \) a completion of \( \nu \). For simplicity of notation we will use the same symbol \( \nu \) to denote its completion.
Now $a \otimes b \mapsto aJb^*J$, $a, b \in C(Y)$ extends to an injective $*$-homomorphism $\pi$ of $C(Y) \otimes C(Y)$ in $L^2(\mathcal{M})$. Indeed, as $\mathcal{M}$ is a factor so the map,
\[
\sum_{i=1}^n a_i \otimes b_i \mapsto \sum_{i=1}^n a_iJb_i^*J
\]
is injective by Prop. 2.8. Hence it induces a norm on $C(Y) \otimes_{alg} C(Y)$. Since abelian C$^*$-algebras are nuclear, this norm must be the min norm, and therefore $a \otimes b \mapsto aJb^*J$ extends to an injective representation of $C(Y) \otimes C(Y)$ in $L^2(\mathcal{M})$. Therefore $C(Y \times Y)$ is a w.o.t dense unital subalgebra of $\mathcal{A}$, so that $\mathcal{A}$ is isomorphic to $L^\infty(Y \times Y, \eta_{Y \times Y})$ for a complete, positive, Borel measure $\eta_{Y \times Y}$. By Lemma 2.4 $\eta_{Y \times Y}$ is non-atomic.

Remark 2.9. In general, if we allow $\mathcal{M}$ to be a finite von Neumann algebra that is not a factor then the map $\sum_{i=1}^n a_i \otimes b_i \mapsto \sum_{i=1}^n a_iJb_i^*J$ is never injective and the measure will be supported on smaller sets. See Rem. 5.17. This is the reason we consider factors, although most results of this article goes through even for finite von Neumann algebras.

Thus in view of the uniqueness of direct integrals with respect to an abelian algebra (see Thm. 2.2), $L^2(\mathcal{M})$ admits a direct integral decomposition $\{\mathcal{H}_{x,y}\}$ over the base space $(Y \times Y, \eta_{Y \times Y})$ so that $\mathcal{A} \cong L^\infty(Y \times Y, \eta_{Y \times Y})$ is the algebra of diagonalizable operators with respect to this decomposition. Let $m_Y$ denote the multiplicity function of the above decomposition. It is clear from the direct integral decomposition that, the Pukánszky invariant of $\mathcal{A}$ is the set of essential values of $m_Y$ (also check Cor. 3.2, [19]). We will call $[\eta_{Y \times Y}]$ the left-right-measure of $\mathcal{A}$. For reasons that will become clear, we will in most situation use the same terminology for the class of the measure $\eta_{Y \times Y}$ when restricted to the off diagonal. This will be clear from the context and will cause no confusion. A related invariant was considered by Neshveyev and Størmer in [19], which was a complete invariant for the pair $(\mathcal{A}, J)$.

Although the existence of such a measure is guaranteed we need an algorithm to figure out the left-right-measure. In order to do so fix a nonzero vector $\xi \in L^2(\mathcal{M})$. The cyclic projection $P_{\xi}$ with range $[\mathcal{A}\xi]$ is in $\mathcal{A}'$ and hence decomposable. For $f, g \in C(Y)$, there exists a complete positive measure $\mu_\xi$ (we complete it if necessary) on $Y \times Y$ such that
\[
\langle fJ^*J\xi, \xi \rangle_{L^2(\mathcal{M})} = \int_{Y \times Y} f(t)g(s)d\mu_\xi(t, s).
\]
$\mathcal{A}P_{\xi}$ is a diffuse abelian algebra in $\mathcal{B}(P_{\xi}(L^2(\mathcal{M})))$ with a cyclic vector, so is maximal abelian. Thanks to von Neumann, we have only one. Therefore,
\[
P_{\xi}(L^2(\mathcal{M})) \cong \int_{Y \times Y} C_{t,s}d\mu_\xi(t, s) \text{ where } C_{t,s} = C.
\]
Moreover $\mathcal{A}P_{\xi}$ is the diagonalizable algebra with respect to the decomposition in Eq. (2.4).

Two orthogonal cyclic subspaces $[\mathcal{A}\xi_1]$ and $[\mathcal{A}\xi_2]$ with cyclic vectors $\xi_1, \xi_2$ does not necessarily keep the fibres of its associated projections $P_{\xi_1}$ and $P_{\xi_2}$ orthogonal, neither does assert that they are direct integrals over disjoint subsets of $Y \times Y$. However, using the “gluing lemma” (Lemma 5.7, [10]) we single out a measure $\mu_{\xi_1,\xi_2}$ so that $(P_{\xi_1} + P_{\xi_2})(L^2(\mathcal{M}))$ has a direct integral decomposition with respect to $(Y \times Y, \eta_{\xi_1,\xi_2})$ and $\mathcal{A}(P_{\xi_1} + P_{\xi_2})$ is the diagonalizable algebra respecting that decomposition. This is the step where one will see the possible updates of the multiplicity function. Since we are working on a separable Hilbert space, after at most a countable infinite iterations
of this procedure we will finally find a measure \( \mu \) on \( Y \times Y \) so that

\[
L^2(\mathcal{M}) \cong \int_{Y \times Y}^{\oplus} \mathcal{H}_x d\mu(x)
\]

and \( \mathcal{A} \) is diagonalizable with respect to the decomposition in Eq. (2.4). Modulo the uniqueness of direct integrals we have found the measure. Needless to say, different choices of cyclic subspaces will produce same measure modulo the uniqueness. However for purpose of explicit computation to distinguish masas one learns, that nice choices of cyclic projections (vectors) is perhaps a little too costly.

For a set \( X \) we denote by \( \Delta(X) \) the set \( \{(x, y) \in X \times X : x = y\} \). The restriction of \( \tau \) to \( C(Y) \subset A \) gives rise to a Borel probability measure whose completion is denoted by \( \nu_Y \).

**Lemma 2.10.** The measure \( \eta_{Y \times Y} \) has the following properties:

(i) \( [\eta_{Y \times Y}] \) is invariant under the flip map \( \theta : (s, t) \mapsto (t, s) \) on \( Y \times Y \).

(ii) If \( \pi_1 \) and \( \pi_2 \) denote the coordinate projections from \( Y \times Y \) onto \( Y \) then,

\[
[(\pi_i)_* \eta_{Y \times Y}] = [\nu_Y] \text{ for } i = 1, 2.
\]

(iii) The subspace \( \int_\Delta(Y) \mathcal{H}_{t,s} d\eta_{Y \times Y}(t, s) \) is identified with \( L^2(A) \) and \( m_Y(t, t) = 1, \eta_{Y \times Y} \) a.e. on \( \Delta(Y) \).

(iv) The topological (closed) support of \( \eta_{Y \times Y} \) is \( Y \times Y \).

The multiplicity function \( m_Y \) has the property that

\[
m_Y(s, t) = m_Y(t, s)
\]

almost all \( \eta_{Y \times Y} \).

Lemma 2.10 is known so we omit its proof. Interested readers can consult [19] or [18]. In fact, it is possible to obtain a choice of \( \eta_{Y \times Y} \) such that \( \eta_{Y \times Y} = \theta_* \eta_{Y \times Y} \).

We are now almost ready to give the definition of the measure-multiplicity-invariant of a masa in a separable II\(_1\) factor. Let \( A \) be a masa in \( \mathcal{M} \). Let \( Y \) be any compact Hausdorff space such that the unital inclusion of \( C(Y) \) in \( A \) is w.o.t dense and \( C(Y) \) is norm separable. To each such \( Y \), we associate a quadruple \((Y, \nu_Y, [\eta_{Y \times Y}], m_Y)\). Define an equivalence relation on the quadruples \((Y, \nu_Y, [\eta_\Delta(Y)], m_Y)\) by

\[
(Y, \nu_Y, [\eta_{Y \times Y}], m_Y) \sim_{m.m} (Y', \nu_{Y'}, [\eta_{Y' \times Y'}], m_{Y'}) \text{ if and only if there exists a Borel isomorphism } F : Y \mapsto Y' \text{ such that, }
\]

\[
F_* \nu_Y = \nu_{Y'},
\]

\[
(F \times F)_* [\eta_{Y \times Y}] = [\eta_{Y' \times Y'}] \text{ and }
\]

\[
m_Y \circ (F \times F)^{-1} = m_{Y'}, \eta_{Y' \times Y'}, \text{ a.e.}
\]

We also have, \([\eta_{Y \times Y}] = [\eta_{\Delta(Y)}] + [\eta_{\Delta(Y')}]\).

Therefore if \((Y, \nu_Y, [\eta_{Y \times Y}], m_Y) \sim_{m.m} (Y', \nu_{Y'}, [\eta_{Y' \times Y'}], m_{Y'})\) then,

\[
(F \times F)_* [\eta_{\Delta(Y)}] = [\eta_{\Delta(Y')}],
\]

\[
m_{\Delta(Y')} \circ (F \times F)^{-1} = m_{\Delta(Y'), \eta_{\Delta(Y')}}, \text{ a.e.}
\]

**Lemma 2.11.** If \( C(Y_1) \subset C(Y_2) \subset A \subset \mathcal{M} \) be two w.o.t dense, unital, norm separable \( C^* \) subalgebras of \( A \) then \((Y_1, \nu_{Y_1}, [\eta_{Y_1 \times Y_1}], m_{Y_1}) \sim_{m.m} (Y_2, \nu_{Y_2}, [\eta_{Y_2 \times Y_2}], m_{Y_2})\).

**Proof.** The inclusion \( i : C(Y_1) \hookrightarrow C(Y_2) \) results from a continuous surjection \( \theta : Y_2 \mapsto Y_1 \). Therefore for all \( f \in C(Y_1) \),

\[
\tau(f) = \int_{Y_1} f d\nu_{Y_1} = \int_{Y_2} i(f) d\nu_{Y_2} = \int_{Y_2} (f \circ \theta) d\nu_{Y_2} = \int_{Y_1} f d(\theta_* \nu_{Y_2}).
\]
Therefore, \( \theta_s \nu_{Y_2} = \nu_{Y_1} \).

The inclusion \( i \) preserves least upper bounds at the level of continuous functions. So \( i \) extends to a surjective \(*\)-homomorphism \( \tilde{i} \) between \( L^\infty (Y_1, \nu_{Y_1}) \) and \( L^\infty (Y_2, \nu_{Y_2}) \) which is normal (Lemma 10.1.10 [15]). It is easy to see that \( \tilde{i} \) is also implemented by \( \theta \). That \( \tilde{i} \) is injective is obvious. So \( \theta \) is a Borel isomorphism between the underlying measure spaces.

Arguing similarly it is easy to see that \( \theta \times \theta : Y_2 \times Y_2 \rightarrow Y_1 \times Y_1 \) implements an isomorphism between \( L^\infty (Y_1 \times Y_1, \eta_{Y_1 \times Y_1}) \) and \( L^\infty (Y_2 \times Y_2, \eta_{Y_2 \times Y_2}) \). The statements regarding the measure classes now follows easily.

The statement about the multiplicity function is obvious from the uniqueness of direct integrals in Thm. 2.2 and the fact \( L^\infty (Y_1 \times Y_1, \eta_{Y_1 \times Y_1}) \cong L^\infty (Y_2 \times Y_2, \eta_{Y_2 \times Y_2}) \cong \mathcal{A} \).

**Proposition 2.12.** Let \( A \subset \mathcal{M} \) be a masa. The collection of quadruples \( (Y, \nu_Y, [\eta_{Y \times Y}], m_{Y}) \) for \( Y \) a compact Hausdorff space such that \( C(Y) \subset A \) is unit, norm separable and \( w.o.t \) dense in \( A \), under the equivalence relation \( \sim_{m.m} \) has exactly one equivalence class.

**Proof.** If \( C(Y_1), C(Y_2) \subset A \) be two \( w.o.t \) dense, unital, norm separable subalgebras of \( A \) then \( C^*(C(Y_1) \cup C(Y_2)) \cong C(Y_3) \) for a compact Hausdorff space \( Y_3 \), and \( C(Y_3) \) is unital, norm separable and \( w.o.t \) dense in \( A \). Therefore by Lemma 2.11 \((Y_3, \nu_{Y_3}, [\eta_{Y_3 \times Y_3}], m_{Y_3}) \sim_{m.m} (Y_i, \nu_{Y_i}, [\eta_{Y_i \times Y_i}], m_{Y_i}) \) for \( i = 1, 2 \).

**Definition 2.13.** Let \( A \subset \mathcal{M} \) be a masa. We define the **measure-multiplicity-invariant** of \( A \) as the equivalence class of the quadruples \( (Y, \nu_Y, [\eta_{\Delta(Y)^c}], m_{\Delta(Y)^c}) \) under \( \sim_{m.m} \) where,

(i) \( Y \) is a compact Hausdorff space such that \( C(Y) \) is an unit, norm separable and \( w.o.t \) dense subalgebra of \( A \).

(ii) \( \nu_Y \) is the completion of the probability measure obtained from restricting \( \tau \) on \( C(Y) \).

(iii) \([\eta_{\Delta(Y)^c}]\) is the equivalence class of the measure \( \eta_{Y \times Y} \) restricted to \( \Delta(Y)^c \).

(iv) \( m_{\Delta(Y)^c} \) is the multiplicity function restricted to \( \Delta(Y)^c \),

obtained from the **direct integral decomposition** of \( L^2(\mathcal{M}) \) over the base space \( (Y \times Y, \eta_{Y \times Y}) \) so that \( \mathcal{A} \) is the algebra of diagonalizable operators with respect to this decomposition.

The **measure-multiplicity-invariant** is an **invariant** for masas in the following sense.

If \( A \subset \mathcal{M} \) and \( B \subset \mathcal{N} \) are masas in \( \Pi_1 \) factors \( \mathcal{M}, \mathcal{N} \) respectively, and there is an unitary \( \mathcal{U} : L^2(\mathcal{M}) \mapsto L^2(\mathcal{N}) \) such that, \( \mathcal{U}A\mathcal{U}^* = \mathcal{B} \) and \( \mathcal{U}J_{\mathcal{M}}A\mathcal{J}_{\mathcal{M}}\mathcal{U}^* = \mathcal{J}_{\mathcal{N}}\mathcal{B}J_{\mathcal{N}} \), then for any choice of compact Hausdorff spaces \( Y_A, Y_B \) with \( C(Y_A)^{s.o.t} = A \) and \( C(Y_B)^{s.o.t} = B \), \( 1_M \in C(Y_A) \), \( 1_N \in C(Y_B) \) and \( C(Y_A), C(Y_B) \) norm separable, there exists a Borel isomorphism \( F_{Y_A, Y_B} : (Y_A, \nu_{Y_A}) \mapsto (Y_B, \nu_{Y_B}) \) such that,

\[
(F_{Y_A, Y_B})_* \nu_{Y_A} = \nu_{Y_B};
\]

\[
(F_{Y_A, Y_B} \times F_{Y_A, Y_B})_* [\eta_{\Delta(Y_A)^c}] = [\eta_{\Delta(Y_B)^c}] \quad \text{and} \quad m_{\Delta(Y_A)^c} \circ (F_{Y_A, Y_B} \times F_{Y_A, Y_B})^{-1} = m_{\Delta(Y_B)^c}, \quad \eta_{\Delta(Y_B)^c} \quad \text{a.e.}
\]

We will denote the **measure-multiplicity-invariant** of a masa \( A \) by \( m.m(A) \) (or \( m.m(\mathcal{A}) \) when the containing factor is ambiguous).
3. Conditional measures and Masas

As we will see latter, the measure-multiplicity-invariant contains substantial information of the masa. In order to extract more information we need to establish some house keeping results in measure theory.

Disintegration of measures is a very useful tool in ergodic theory, in the study of conditional probabilities and descriptive set theory. Measurable selection principle is a term closely linked with disintegration of measures and has been studied by a number of mathematicians in the last century. A detailed exposition of the existence of disintegration can be found in [1].

For the general definition of disintegration of measures we will restrict to the following set up. Let $T$ be a measurable map from $(X, \sigma_X)$ to $(Y, \sigma_Y)$ where $\sigma_X, \sigma_Y$ are $\sigma$-algebras of subsets of $X, Y$ respectively. Let $\lambda$ be a $\sigma$-finite measure on $\sigma_X$ and $\mu$ a $\sigma$-finite measure on $\sigma_Y$. Here $\lambda$ is the measure to be disintegrated and $\mu$ is often the push forward measure $T_*\lambda$, although other possibilities for $\mu$ is allowed.

**Definition 3.1.** We say that $\lambda$ has a disintegration $\{\lambda_t\}_{t \in Y}$ with respect to $T$ and $\mu$ or a $(T, \mu)$ disintegration if:

(i) $\lambda_t$ is a $\sigma$-finite measure on $\sigma_X$ concentrated on $\{T = t\}$ (or $T^{-1}\{t\}$), i.e. $\lambda_t(\{T \neq t\}) = 0$, for $\mu$-almost all $t$,

and for each nonnegative measurable function $f$ on $X$

(ii) $t \mapsto \lambda_t(f)$ is measurable.

(iii) $\lambda(f) = \mu'(\lambda_t(f)) = \int_Y \lambda_t(f) d\mu(t)$.

In probability theory the measures $\lambda_t$ are called the disintegrating measures and $\mu$ is the mixing measure. One also writes $\lambda(\cdot \mid T = t)$ for $\lambda_t(\cdot)$ on occasion.

When $\lambda$ and almost all $\lambda_t$ are probability measures one refers to the disintegrating measures as (regular) conditional distributions and $t \mapsto \lambda_t$ is called the transition kernel.

The reader should be cautious that “measurable” in Defn. 3.1 (ii), (iii) means measurable with respect to the $\sigma$-algebra of completion of $\lambda$.

**Theorem 3.2.** [1] (Existence Theorem) Let $\lambda$ be a $\sigma$-finite Radon measure on a metric space $X$ and $T$ be a measurable map into $(Y, \sigma_Y)$. Let $\mu$ be a $\sigma$-finite measure on $\sigma_Y$ such that $T_*\lambda \ll \mu$. If $\sigma_Y$ is countably generated and contains all singleton sets $\{t\}$, then $\lambda$ has a $(T, \mu)$ disintegration. The measures $\lambda_t$ are uniquely determined up to an almost sure equivalence: if $\lambda_t'$ is another $(T, \mu)$ disintegration then $\mu(\{t : \lambda_t \neq \lambda_t'\}) = 0$.

The condition $T_*\lambda \ll \mu$ in Thm. 3.2 is actually necessary for the disintegration to exist. The original version of Thm. 3.2 is due to von Neumann.

**Proposition 3.3.** Let $\lambda$ be a Radon measure on a compact metric space $X$ and $T$ be a measurable map into $(Y, \sigma_Y)$. Let $\mu$ be a $\sigma$-finite measure on $\sigma_Y$ such that $T_*\lambda \ll \mu$. Assume that $\sigma_Y$ is countably generated and contains all singleton sets. Let $t \mapsto \lambda_t$ denote the $(T, \mu)$ disintegration of $\lambda$. Let $X_a$ denote the set of atoms of $\{\lambda_t\}_{t \in Y}$ i.e.

$$X_a = \{x \in X \mid \exists t \in Y : \lambda_t(\{x\}) > 0\}.$$  

Then $X_a$ is a measurable set, measurable with respect to the $\sigma$-algebra of the completion of $\lambda$.

**Proof.** There is a measurable set $E \subseteq Y$ with $\mu(E^c) = 0$ such that for $t \in E$, $\lambda_t$ is concentrated on the set $\{T = t\}$. We can assume without loss of generality that $E = Y$. Now for $t \in Y$, the measure $\lambda_t$ is concentrated on $\{T = t\}$, so

$$\{x \in X \mid \exists t \in Y : \lambda_t(\{x\}) > 0\} = \{x \in X \mid \lambda_{T x}(\{x\}) > 0\}.$$
Let $B$ be a countable base for the topology on $X$. Then
\[
\{x \in X \mid \lambda_T(x) > 0\} = \bigcup_{n=1}^{\infty} X_a^{(n)}
\]
with
\[
X_a^{(n)} = \left\{ x \in X \mid \forall U \in B : x \in U \Rightarrow \lambda_T(U) \geq \frac{1}{n} \right\}
\]
Therefore, $\{x \in X \mid \exists t \in Y : \lambda_t(x) > 0\}$ is a measurable set by property (ii) of disintegration.

The next few lemmas are undoubtedly known to probablists but we lack the reference. So we record them for convenience. We will omit their proofs. For details check [18].

**Lemma 3.4.** Let $\lambda_1, \lambda_2$ be two Radon measures on a compact metric space $X$ and $T$ be a measurable map into $(Y, \sigma_Y)$. Let $\mu$ be a $\sigma$-finite measure on $\sigma_Y$ such that $T_*\lambda_1, T_*\lambda_2 \ll \mu$. Assume $\sigma_Y$ is countably generated and contains all singleton sets $\{t\}$. Let $\lambda_1^t, \lambda_2^t$ be the $(T, \mu)$ disintegration of $\lambda_1, \lambda_2$ respectively. Let $\lambda_1^0$ be the $(T, \mu)$ disintegration of $\lambda_1 + \lambda_2$. Then
\[
\lambda_1^0 = \lambda_1^t + \lambda_2^t \text{ a.e.}
\]

**Lemma 3.5.** Let $\lambda_1, \lambda_2$ be two Radon measures on compact metric spaces $X, Y$ and $T, S$ be measurable maps from $X, Y$ into $(Z, \sigma_Y), (W, \sigma_W)$ respectively. Let $\mu, \nu$ be $\sigma$-finite measures on $\sigma_Y, \sigma_W$ respectively such that $T_*\lambda_1 \ll \mu, S_*\lambda_2 \ll \nu$. Assume $\sigma_Y, \sigma_W$ are countably generated and contains all singleton sets $\{t\}, \{s\}$ respectively. Let $\lambda_1^t, \lambda_2^s$ be the $(T, \mu), (S, \nu)$ disintegration of $\lambda_1, \lambda_2$ respectively. Let $\lambda_{t,s}^0$ be the $(T \otimes S, \mu \otimes \nu)$ disintegration of $\lambda_1 \otimes \lambda_2$. Then
\[
\lambda_{t,s}^0 = \lambda_1^t \otimes \lambda_2^s \text{ a.e.}
\]

**Lemma 3.6.** Let $\lambda_1, \lambda_2$ be two Radon measures on a compact metric space $X$ and $T$ be a measurable map into $(Y, \sigma_Y)$. Let $\mu$ be a $\sigma$-finite measure on $\sigma_Y$ such that $T_*\lambda_1 \ll \mu$ and $T_*\lambda_2 \ll \mu$. Assume $\sigma_Y$ is countably generated and contains all singleton sets $\{t\}$. Let $\lambda_1^t, \lambda_2^t$ be the $(T, \mu)$ disintegrations of $\lambda_1, \lambda_2$ respectively.
(i) Assume that $\lambda_1 \ll \lambda_2 \ll \lambda_1$. Then for $\mu$ almost all $t$, $\lambda_1^t \ll \lambda_2^t \ll \lambda_1^t$. Moreover, if $g = \frac{d\lambda_1}{d\lambda_2}$ then $\frac{d\lambda_1^t}{d\lambda_2^t} = g_t$ a.e. $\mu$, where
\[
g_t = \begin{cases} g_{\{T=t\}} \text{ on } \{T=t\}, \\ 0 \text{ otherwise.} \end{cases}
\]
Conversely if $\lambda_1^t \ll \lambda_2^t \ll \lambda_1^t$ for $\mu$ almost all $t$ then $\lambda_1 \ll \lambda_2 \ll \lambda_1$.
(ii) If $\lambda_1 \perp \lambda_2$ then $\lambda_1^t \perp \lambda_2^t$ for $\mu$ almost all $t$.

**Lemma 3.7.** Let $\lambda$ be a Radon measure on $X \times X$ where $X$ is a compact metric space. Let $\mu$ be a $\sigma$-finite measure on $X$ such that $(\pi_i)_*\lambda \ll \mu$ where $\pi_i, i = 1, 2$ are coordinate projections onto $X$.
Assume that $\lambda$ is invariant under the flip of coordinates i.e. $\theta_*\lambda \ll \lambda \ll \theta_*\lambda$, where $\theta : X \times X \mapsto X \times X$ by $\theta(x, y) = (y, x)$. Let $\lambda_1^t, \lambda_2^t$ be the $(\pi_1, \mu), (\pi_2, \mu)$ disintegrations of $\lambda$ respectively. Then for $\mu$ almost all $t$,
\[
\lambda_1^t \ll \theta_*\lambda_2^t \ll \lambda_1^t.
\]
In particular, if for $\mu$ almost all $t$, $\lambda_2^t$ has an atom at $(s, t)$, then $\lambda_1^t$ has an atom at $(t, s)$ almost everywhere.
Theorem 3.8. Let $A \subset M$ and $B \subset N$ be masas in separable $\Pi_1$ factors $M, N$. Let $C(X_1) \subset A$, $C(X_2) \subset B$ be w.o.t dense, norm separable, unital subalgebras of $A, B$ respectively, where $X_i$ are compact metric spaces for $i = 1, 2$. Let $\nu_{X_i}$ denote the tracial measures with respect to the w.o.t dense subalgebras on $X_i$ respectively for $i = 1, 2$. Let $[\lambda_1], [\lambda_2]$ denote the left-right-measures of $A$ and $B$ respectively. All the mentioned measures are assumed to be complete. Suppose there is a unitary $U : L^2(M) \mapsto L^2(N)$ such that $UAU^* = B$ and $UJ_\mathcal{M}A J_\mathcal{M} U^* = J_N B J_N$.

Then there exists an isomorphism of measure spaces $F : X_1 \mapsto X_2$ such that, $F_* \nu_{X_1} = \nu_{X_2}$ and the following is true:

Denoting by $\lambda_1^{1,X_1}, \lambda_2^{1,X_1}$ the $(\pi_1, \nu_{X_1}), (\pi_2, \nu_{X_1})$ disintegrations of $\lambda_1$ respectively and $\lambda_2^{1,X_2}, \lambda_2^{2,X_2}$ the $(\pi_1, \nu_{X_2}), (\pi_2, \nu_{X_2})$ disintegrations of $\lambda_2$ respectively, one has

$$\lambda_1^{t,X_1} = [(F \times F)_* \lambda_1^{1,X_1}], \nu_{X_2} \text{ almost all } t',
\lambda_2^{s,X_2} = [(F \times F)_* \lambda_2^{2,X_1}], \nu_{X_2} \text{ almost all } s',
$$

where $\pi_1, \pi_2$ denotes the projection onto the first and second coordinates respectively.

If $(X, \sigma)$ be a measurable space and $\mu$ is a signed measure on $X$ then we denote by $\|\mu\|_{t.v.}$ to be the total variation norm of $\mu$. The next Lemma is used in this paper but it will be of significant use for computation in the next paper.

Lemma 3.9. Let $\lambda_0, \lambda, \lambda_0$ be Radon measures on a compact metric space $X$ such that, $\lambda_0 \neq 0, \lambda_n \ll \lambda$ for $n = 1, 2, \cdots, \lambda_0 \ll \lambda$ and $\lambda_n \rightarrow \lambda_0$ in $\|\|_{t.v.}$. Let $T$ be a measurable map into $(Y, \sigma_Y)$. Let $\mu$ be a $\sigma$-finite measure on $\sigma_Y$ such that $T_* \lambda \ll \mu$. Assume $\sigma_Y$ is countably generated and contains all singleton sets $\{t\}$. Let $\lambda^n_t, \lambda^0_t, \lambda_t$ be the $(T, \mu)$ disintegrations of $\lambda_n, \lambda_0, \lambda$ respectively.

(i) Then there is a $\mu$ null set $E$ and a subsequence $\{n_k\}$ ($n_k < n_{k+1}$ for all $k$) such that for all $t \in E$,

$$\sup_{A \subseteq \{t\}, \text{A Borel}} |\lambda^n_t(A) - \lambda^0_t(A)| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

(ii) Moreover, if for $\mu$ almost all $t$ one has $\lambda^n_t$ is completely atomic (or completely non-atomic) for all $n$, then so is $\lambda^0_t$ almost everywhere.

The proof is straightforward. We omit the proof. For details check [18].

For a masa $A \subset M$, fix a compact Hausdorff space $X$ such that $C(X) \subset A$ is an unital, norm separable and w.o.t dense $C^*$ subalgebra. For $\zeta \in L^2(M)$ let $\kappa_\zeta : C(X) \otimes C(X) \mapsto \mathbb{C}$ be the linear functional defined by

$$\kappa_\zeta(a \otimes b) = \langle a\zeta b, \zeta \rangle.$$

Then $\kappa_\zeta$ induces an unique Radon measure $\eta_\zeta$ on $X \times X$ given by

$$(3.1) \quad \kappa_\zeta(a \otimes b) = \int_{X \times X} a(t)b(s)d\eta_\zeta(t, s)$$

and $\|\eta_\zeta\|_{t.v.} = \|\kappa_\zeta\|.$

For $\zeta_1, \zeta_2 \in L^2(M)$ let $\eta_{\zeta_1, \zeta_2}$ denote the possibly complex measure on $X \times X$ obtained from the vector functional

$$(3.2) \quad \langle a\zeta_1 b, \zeta_2 \rangle = \int_{X \times X} a(t)b(s)d\eta_{\zeta_1, \zeta_2}(t, s), \text{ } a, b \in C(X).$$

We will write $\eta_{\zeta_1, \zeta_2} = \eta_\zeta$. Note that $\eta_\zeta$ is a positive measure for all $\zeta \in L^2(M)$. It is easy to see that the following polarization type identity holds:

$$(3.3) \quad 4\eta_{\zeta_1 + i\zeta_2} = (\eta_{\zeta_1 + \zeta_2} - \eta_{\zeta_1 - \zeta_2}) + i(\eta_{\zeta_1 + i\zeta_2} - \eta_{\zeta_1 - i\zeta_2}).$$
Note that the decomposition of $\eta_{\zeta_1, \zeta_2}$ in Eq. (3.3) need not be its Hahn decomposition in general, but
\[ |\eta_{\zeta_1, \zeta_2}| \leq (\eta_{\zeta_1+\zeta_2} + \eta_{\zeta_1-\zeta_2}) + (\eta_{\zeta_1+i\zeta_2} + \eta_{\zeta_1-i\zeta_2}) = 4(\eta_{\zeta_1} + \eta_{\zeta_2}). \]
So
\[ (3.4) \quad |\eta_{\zeta_1, \zeta_2}| \leq \eta_{\zeta_1} + \eta_{\zeta_2}. \]

**Lemma 3.10.** If $\zeta_n, \zeta \in L^2(\mathcal{M})$ be such that, $\zeta_n \rightarrow \zeta$ in $\|\cdot\|_2$ then
\[ \eta_{\zeta_n} \rightarrow \eta_{\zeta} \text{ in } \|\cdot\|_{t.v}. \]

**Proof.** Obvious. \qed

**Proposition 3.11.** Let $A \subset \mathcal{M}$ be a masa. Let $X$ be a compact Hausdorff space such that $C(X) \subset A$ is unital, norm separable and w.o.t dense in $A$ and let $\nu$ be the tracial measure. Let $0 \neq \zeta \in L^2(N(A)'' \mathcal{N})$. Then $\eta_{\zeta_1, \zeta_2}$ is completely atomic $\nu$ almost all $t, s$ where $\eta_{\zeta}$ is the measure defined in Eq. (3.1) and $\eta_{\zeta_1, \zeta_2}$ are $(\pi_1, \nu)$ and $(\pi_2, \nu)$ disintegrations of $\eta_{\zeta}$ respectively.

**Proof.** We only prove for the $(\pi_1, \nu)$ disintegration. If $\zeta = u$ where $u \in N(A)$ then the result is obvious as the measure $\eta_u$ will be concentrated on the automorphism graph. The span of $N(A)$ being s.o.t dense in $N(A)'' \mathcal{N}$ it suffices by Lemma 3.10 and 3.9 to prove the statement when $\zeta = \sum_{i=1}^{n} c_i u_i$ where $u_i \in N(A)$ and $c_i \in \mathbb{C}$ for $1 \leq i \leq n$. Now for $a, b \in A$
\[ \langle a(\sum_{i=1}^{n} c_i u_i)b, (\sum_{i=1}^{n} c_i u_i) \rangle = \sum_{i=1}^{n} |c_i|^2 \langle au_i b, u_i \rangle + \sum_{i \neq j} c_i c_j \langle au_i b, u_j \rangle. \]

The measures given by $a \otimes b \mapsto |c_i|^2 \langle au_i b, u_i \rangle$, $a, b \in C(X)$ are concentrated on the automorphism graphs implemented by $u_i$ and hence definitely disintegrates as atomic measures and so does their sum from Lemma 3.4. The measures given by $a \otimes b \mapsto c_i c_j \langle au_i b, u_j \rangle$, $a, b \in C(X)$ for $i \neq j$ are possibly complex measures. However Eq. 3.4 forces that these measures are also concentrated on the union of the automorphism graphs implemented by $u_i$ and $u_j$. Thus $\eta_{\zeta_n, \zeta_1, \zeta_2}$ is concentrated on the union of the automorphism graphs implemented by $u_i$, $1 \leq i \leq n$. Hence the result follows. \qed

**4. Fundamental Set and Generalized Dye’s Theorem**

This section is intended to characterize some operators in the normalizing algebra of a masa. Throughout this section $\mathcal{N}$ will denote a finite von Neumann algebra gifted with a faithful, normal, normalized trace $\tau$. $B \subset \mathcal{N}$ will denote a von Neumann subalgebra of $\mathcal{N}$.

As usual $\mathcal{N}$ will be assumed to be acting on $L^2(\mathcal{N}, \tau)$ by left multipliers. $L^2(\mathcal{N}, \tau)$ is a $B$-$B$ Hilbert $w^*$-bimodule for any von Neumann subalgebra $B \subset \mathcal{N}$. We know if $\mathbb{E}_B$ denotes the unique trace preserving conditional expectation onto $B$, then $\mathbb{E}_B$ is given by the Jones projection $e_B$ associated to $B$ via the formula $\mathbb{E}_B(x) \mathbb{1} = e_B(x \mathbb{1})$. For $b_1, b_2 \in B$ and $\zeta \in L^2(\mathcal{N}, \tau)$ one has
\[ (4.1) \quad e_B(b_1 \zeta b_2) = b_1 e_B(\zeta) b_2. \]
We will interchangeably use the symbols $\mathbb{E}_B$ and $e_B$.

**Definition 4.1.** For a subalgebra $B \subset \mathcal{N}$ define the fundamental set of $B$ to be
\[ N^f(B) = \{ x \in \mathcal{N} : Bx = xB \}. \]
Note that $x \in N^f(B)$ implies $x^* \in N^f(B)$. 

Definition 4.2. For a subalgebra $B \subset \mathcal{N}$ define the weak-fundamental set of $B$ to be
\[ N^f_2(B) = \{ \zeta \in L^2(\mathcal{N}, \tau) : B\zeta = \zeta B \}. \]

Note that $\zeta \in N^f_2(B)$ implies $\zeta^* \in N^f_2(B)$ and $N^f(B) \subset N^f_2(B)$. When $B$ is a masa, $\zeta \in N^f_2(B)$ implies $a\zeta, \zeta a \in N^f_2(B)$ for all $a \in B$.

To understand the normaliser of a masa the set $N^f_2(B)$ will naturally arise into the scene. However working with vectors in $L^2(\mathcal{N}, \tau)$ is always a technical issue. Polar decomposition of vectors and the theory of $L^1$ spaces are the tools we need, for which we will give a short exposition. For details check Appendix B of [34] and [18]. To keep it short we will omit most proofs. It is here, where one usually encounters unbounded operators. For results proved in this section we have borrowed ideas from Roger Smith.

The positive cone $L^2(\mathcal{N}, \tau)^+$ in $L^2(\mathcal{N}, \tau)$ is defined to be $\overline{\mathcal{N}^+ \| \cdot \|^2}$ i.e. the closure of the positive elements of $\mathcal{N}$ in $L^2(\mathcal{N}, \tau)$. It can be shown that $L^2(\mathcal{N}, \tau)$ is the algebraic span of $L^2(\mathcal{N}, \tau)^+$. For $x \in \mathcal{N}$ the equation $\|x\| = \tau(|x|)$ defines a norm on $\mathcal{N}$. The completion of $\mathcal{N}$ with respect to $\| \cdot \|_1$ is denoted by $L^1(\mathcal{N}, \tau)$. It can be shown that
\[ \|x\|_1 = \sup\{|\tau(xy)| : y \in \mathcal{N}, \|y\| \leq 1\}. \]

So $|\tau(x)| \leq \|x\|_1$. Thus by density of $\mathcal{N}$ in $L^1(\mathcal{N}, \tau)$, $\tau$ extends to a bounded linear functional on $L^1(\mathcal{N}, \tau)$ which will also be denoted by $\tau$. One can analogously define the positive cone of $L^1(\mathcal{N}, \tau)$ which we denote by $L^1(\mathcal{N}, \tau)^+$. Clearly $\|x\|_1 = \|x^*\|_1$. Consequently, the Tomita operator $J$ extends to a surjective anti-linear isometry to $L^1(\mathcal{N}, \tau)$ which will also be denoted by $J$. Moreover $J^2 = 1$. We will interchangeably use the notations $J\zeta$ and $\zeta^*$ for $\zeta \in L^1(\mathcal{N}, \tau)$.

Both the spaces $L^1(\mathcal{N}, \tau)$ and $L^2(\mathcal{N}, \tau)$ are unitary $\mathcal{N}$-$\mathcal{N}$ bimodules. The space $L^1(\mathcal{N}, \tau)$ can be identified with the predual of $\mathcal{N}$ and $L^2(\mathcal{N}, \tau)$ is dense in $L^1(\mathcal{N}, \tau)$. One also has $\tau(x\zeta) = \tau(\zeta x)$ for $x \in \mathcal{N}$ and $\zeta \in L^1(\mathcal{N}, \tau)$. Note that $\mathbb{E}_B$ is a contraction from $\mathcal{N}$ onto $B$. It can be shown that for $x \in \mathcal{N}$,
\[ \|\mathbb{E}_B(x)\|_1 \leq \|x\|_1. \]

Thus $\mathbb{E}_B$ has an unique bounded extension to a contraction from $L^1(\mathcal{N}, \tau)$ onto $L^1(B, \tau)$, which will as well be denoted by $\mathbb{E}_B$. This extension preserves the extension of the trace $\tau$, is $B$ modular, positive and faithful. The bilinear map $\Psi : \mathcal{N} \times \mathcal{N} \mapsto \mathcal{N}$ defined by $\Psi(x, y) = xy$ satisfies
\[ \|\Psi(x, y)\|_1 \leq \|x\|_2 \|y\|_2 \]
by Cauchy-Schwarz inequality. Therefore $\Psi$ lifts to a jointly continuous map from $L^2(\mathcal{N}, \tau) \times L^2(\mathcal{N}, \tau)$ into $L^1(\mathcal{N}, \tau)$. The extension is actually a surjection. Since $\Psi$ is the product map of operators at the level of von Neumann algebra one calls $\Psi(\zeta_1, \zeta_2)$ to be $\zeta_1\zeta_2$, for $\zeta_1, \zeta_2 \in L^2(\mathcal{N}, \tau)$.

Lemma 4.3. (B.5.1, [34]) Let $a, b \in \mathcal{N}$ be positives. Then
\[ \|a^{1\over 2} - b^{1\over 2}\|_2^2 \leq 2 \|a - b\|_1. \]

Elements of $L^1(\mathcal{N}, \tau)$ and $L^2(\mathcal{N}, \tau)$ can be regarded as unbounded operators on $L^2(\mathcal{N}, \tau)$. By using the unbounded operator theory for operators affiliated to $\mathcal{N}$, for each $\zeta \in L^1(\mathcal{N}, \tau)^+$ there exists an unique $0 \leq \zeta_0 \in L^2(\mathcal{N}, \tau)$ such that $\zeta_0^*\zeta_0 = \zeta^2 = \zeta$. In this case, $\zeta_0$ is said to be the square root of $\zeta$ and one writes $\zeta_0 = \sqrt{\zeta} = \zeta^{\frac{1}{2}}$. For $\zeta \in L^2(\mathcal{N}, \tau)$ one has $\zeta^*\zeta \in L^1(\mathcal{N}, \tau)$. From Eq. 4.3 and Lemma 4.3 it follows that $\zeta^*\zeta \in L^1(\mathcal{N}, \tau)^+$. In particular, $\sqrt{\zeta}\zeta \in L^2(\mathcal{N}, \tau)$ for any $\zeta \in L^2(\mathcal{N}, \tau)$ and the
square root of any positive in $L^1(\mathcal{N}, \tau)$ is an unique element of $L^2(\mathcal{N}, \tau)$. One also writes $|\zeta| = \sqrt{\zeta^\ast \zeta}$ for $\zeta \in L^2(\mathcal{N}, \tau)$. If $\zeta \in L^1(\mathcal{N}, \tau)$ be self adjoint i.e. $\zeta = \zeta^*$ then $\zeta = \zeta_+ - \zeta_-$ where $\zeta_+ \in L^1(\mathcal{N}, \tau)^+$ and this decomposition is unique by requiring that $\zeta_+ \frac{1}{2} \zeta_+ - \zeta_\mp = 0$.

Let $\zeta \in L^2(\mathcal{N}, \tau)$. Consider the projections $p, q$ in $\mathcal{B}(L^2(\mathcal{N}, \tau))$ whose ranges are $J \mathcal{N} J \sqrt{\zeta^* \zeta} = \mathcal{N}$ respectively. Since the ranges of $p, q$ are invariant subspaces of $J \mathcal{N} J = \mathcal{N}$ so $p, q$ lies in $\mathcal{N}$. Using unbounded operators one obtains polar decomposition of vectors (Eq. (4.7)) which we formalize below.

**Theorem 4.4.** There is an unique partial isometry $v \in \mathcal{N}$ with initial projection $p$ and final projection $q$ which satisfy the following condition:

\begin{equation}
(4.6) \quad v J x^* J \sqrt{\zeta^* \zeta} = J x^* J \zeta, \quad x \in \mathcal{N}.
\end{equation}

In particular,

\begin{equation}
(4.7) \quad v \sqrt{\zeta^* \zeta} = \zeta.
\end{equation}

(i) Let $B \subset \mathcal{N}$ be a masa, then $\zeta \in L^2(B, \tau)$ imply $p, q \in B$. 
(ii) For $\zeta \in L^2(\mathcal{N}, \tau)$ if $\zeta^* \zeta \in \mathcal{N}$ then $\zeta \in \mathcal{N}$.

For $\zeta \in L^2(\mathcal{N}, \tau)$ we define the left and right kernel of $\zeta$ to be respectively $\text{Ker}_l(\zeta) = \{x \in \mathcal{N} : \zeta x = 0\}$ and $\text{Ker}_r(\zeta) = \{x \in \mathcal{N} : x \zeta = 0\}$. Then $\text{Ker}_l(\cdot), \text{Ker}_r(\cdot)$ are subspaces of $\mathcal{N}$. $\text{Ker}_l(\cdot), \text{Ker}_r(\cdot)$ are w.o.t and s.o.t closed.

If $\zeta \in L^1(\mathcal{N}, \tau)$ then the left and the right kernels of $\zeta$ can be defined analogously. We will denote the kernels of the $L^1$ vectors by $\text{Ker}_l(\cdot), \text{Ker}_r(\cdot)$ as well. This is slight abuse of notation. In this case, they are norm closed subspaces of $\mathcal{N}$. 

For $\zeta \in L^2(\mathcal{N}, \tau)$ we have

\begin{equation}
(4.8) \quad \text{Ker}_l(\zeta) = \text{Ker}_l(\sqrt{\zeta^* \zeta}) = \text{Ker}_l(\zeta^* \zeta).
\end{equation}

However the righthand side is defined in $L^1$ sense. Therefore for $\zeta \in L^2(\mathcal{N}, \tau)$, $\text{Ker}_l(\zeta^* \zeta)$ (respectively $\text{Ker}_r(\zeta^* \zeta)$) are in fact w.o.t closed. Similar statements hold for $\text{Ker}_r(\cdot)$ as well.

For $\zeta \in L^2(\mathcal{N}, \tau)$ we define the left and right ranges of $\zeta$ to be respectively $\text{Ran}_l(\zeta) = \{\zeta x : x \in \mathcal{N}\}$ and $\text{Ran}_r(\zeta) = \{x \zeta : x \in \mathcal{N}\}$. Note that for $\zeta \in L^2(\mathcal{N}, \tau)$,

\begin{equation}
(4.9) \quad \{x \in \mathcal{N} : \zeta x = 0\} = \{x \in \mathcal{N} : \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{N}\} \\
\qquad = \{x \in \mathcal{N} : \langle x, \zeta y \rangle = 0 \text{ for all } y \in \mathcal{N}\}
\end{equation}

implies $\text{Ker}_l(\zeta) = \text{Ran}_l(\zeta) \perp$.

**Proposition 4.5.** Let $\zeta \in L^2(\mathcal{N}, \tau)$ and let $\zeta = v \sqrt{\zeta^* \zeta}$ be its polar decomposition. Then $v^* v$ is the projection from $L^2(\mathcal{N}, \tau)$ onto $\text{Ker}_l(\zeta) \perp$, and $v^* v$ is the projection onto $\text{Ran}_l(\zeta)$.

**Proposition 4.6.** Let $\zeta \in L^2(\mathcal{N}, \tau)$ and let $\zeta = v |\zeta|$ be its polar decomposition. Then $|\zeta| \frac{1}{2} k \to v^* v$ as $k \to \infty$ in $\|\cdot\|_2$.

The proof of Prop. (4.6) is a direct application of monotone convergence theorem.

**Lemma 4.7.** Let $A \subset \mathcal{N}$ be a masa. Let $\zeta \in L^1(\mathcal{N}, \tau)$ be a nonzero vector such that $a \zeta = \zeta a$ for all $a \in A$. Then $\zeta \in L^1(A, \tau)$.

**Proof.** First assume $\zeta \geq 0$. Then use uniqueness of square roots of $L^1$ vectors. In, the general case write $\zeta$ as a linear combination of four positives. We omit the details. \(\square\)
Proposition 4.8. Let $A \subset \mathcal{N}$ be a masa. Let $0 \neq \zeta \in L^1(\mathcal{N}, \tau)^+$ be such that $A\zeta = \zeta A$. Then $\zeta \in L^1(A, \tau)^+$.

Proof. Let $\mathcal{I} = \{a \in A : a\zeta = 0\}$. Then $\mathcal{I}$ is a weakly closed ideal (see Eq. (4.8) and related discussion) in $A$ and so has the form $A(1-p)$ for some projection $p \in A$. Then $p\zeta = \zeta$, so $\zeta = p\zeta$ by operating with extended Tomita’s involution operator. Thus $A\zeta = A(p\zeta) = p\zeta A$. For $a_1, a_2 \in A$ if $a_1p = a_2p$ then $\zeta(a_1 - a_2)p = 0$, so $\zeta(a_1^* - a_2^*) = 0$. Hence $p(a_1^* - a_2^*) \in \mathcal{I}$, but $1-p$ is the identity for $\mathcal{I}$. So $p(a_1^* - a_2^*) = 0$ and hence $a_1p = a_2p$. This means there is a well defined map $\psi : Ap \mapsto Ap$ such that $ap\zeta = \zeta \psi(ap)$ for $a \in A$.

Taking conditional expectation (see Eq. (4.3) and related discussion) one gets $(ap - \psi(ap))E_A(\zeta) = 0$ (the left and the right action by elements of $A$ coincides on $L^1(A, \tau)$). Suppose there is an operator $a \in A$ such that $ap - \psi(ap) \neq 0$. Write $ap - \psi(ap) = bp$ for $b \in A$. Then $pb^*bp E_A(\zeta) = 0$, so $E_A(pb^*bp\zeta) = 0$. Let $\zeta = \lim_n x_n$ in $\|\cdot\|_1$ where $x_n \in \mathcal{N}^+$. Therefore

$$\lim_n \tau(x_n^\frac{1}{2}(bp)^*bp x_n^\frac{1}{2}) = \lim_n \tau(pb^*bp x_n) = \lim_n \tau(E_A(pb^*bp x_n)) = 0.$$ 

The last statement follows from Eq. (4.2) and Eq. (4.3). So $\lim_n bp x_n^\frac{1}{2} = 0$ in $\|\cdot\|_2$ and hence $bp\zeta = \lim_n bp x_n = 0$, in $\|\cdot\|_1$ by Lemma 4.7 and Eq. (4.2). Thus $bp \in \mathcal{I}$ so $bp = bp(1-p) = 0$, a contradiction. Thus $\psi(ap) = ap$ for all $a \in A$.

Now $\zeta \in L^1(p\mathcal{N}p, \tau)$ and $Ap$ is a masa in $p\mathcal{N}p$, thus $\zeta \in L^1(A, \tau)$ as $ap\zeta = \zeta \psi(ap) = \zeta ap$ for all $a \in A$, from Lemma 4.7.

Theorem 4.9. (Generalized Dye’s theorem-L^2 form) Let $A \subset \mathcal{N}$ be a masa. Then $\zeta \in \mathcal{N}_2^2(A)$ if and only if $\zeta = v\xi$ for some $\xi \in L^2(A, \tau)$ and $v \in \mathcal{G}\mathcal{N}(A)$. In particular, $\text{span}\mathcal{N}(A)^\perp = L^2(N(A)^\vee, \tau)$.

Proof. Case 1: Assume $\zeta \in \mathcal{N}_2^2(A)$ and $\zeta \geq 0$ i.e. $\zeta \in \mathcal{N}^+\|\cdot\|_2$. Then $\zeta \in L^1(\mathcal{N}, \tau)^+$ as well. From Prop. 4.8 we get $\zeta \in L^1(A, \tau) \cap L^2(\mathcal{N}, \tau) = L^2(A, \tau)$.

Case 2: Let $\zeta \in \mathcal{N}_2^2(A)$. We may without loss of generality assume that $\|\zeta\|_2 = 1$. Then as $A\zeta = \zeta A$ we also have $A\zeta^* = \zeta^* A$. So $A\zeta^*\zeta = \zeta^* A\zeta = \zeta^*\zeta$. From Prop. 4.8

$$\zeta^*\zeta \in L^1(A, \tau)$$

and similarly we have $\zeta\zeta^* \in L^1(A, \tau)$. Then $\|\zeta\|_1 \leq 1$.

Arguing as in Prop. 4.8 there are projections $p_1, p_2 \in A$ such that $J_1 = \{a \in A : a\zeta = 0\} = A(1-p_1)$ and $J_2 = \{a \in A : a\zeta = 0\} = A(1-p_2)$. Therefore we have $p_1\zeta = \zeta$ and $p_2\zeta = \zeta$.

Then there is a well defined map (as explained before) $\psi : Ap_1 \mapsto Ap_2$ such that $ap_1\zeta = \zeta \psi(ap_1)$ for all $a \in A$.

Let $\zeta = v\sqrt{\zeta^*\zeta}$ be the polar decomposition of $\zeta$ from Thm. 4.4. Then $v$ is a partial isometry in $\mathcal{N}$ and the initial space of $v$ is $$\{\sqrt{\zeta^*\zeta}x : x \in \mathcal{N}\}^{-\|\cdot\|_2}$$ and the final space is $$\{|\zeta x : x \in \mathcal{N}|^{-\|\cdot\|_2}}.$$ Moreover the projections $v^*v$ and $vv^*$ are in $A$.

Indeed, by Prop. 4.5, $v^*v$ is the projection onto $\text{Ker}(\zeta)^\perp$ and $vv^*$ onto $\text{Ran}(\zeta)$. By
Prop. 4.6. \( v^*v \in A \). Replacing \( \zeta \) by \( \zeta^* \) and using \( \text{Ker}_i(\zeta) = \text{Ran}_i(\zeta^*) \) (see Eq. (4.9)), a similar argument will yield \( vv^* \in A \). Clearly \( v^*v = p_2 \) and \( vv^* = p_1 \). Then
\[
 ap_1 v \sqrt{\zeta^* \zeta} = v \sqrt{\zeta^* \zeta} \psi(ap_1).
\]
Now \( J_0 = \{ b \in A : ap_1 vb = vb \psi(ap_1) \text{ for all } a \in A \} \)
is a weakly closed ideal in \( A \) and its closure in \( \| \cdot \|_2 \) is precisely the set
\[
 J_0^{-\| \cdot \|_2} = \{ \xi \in L^2(A, \tau) : ap_1 v \xi = v \xi \psi(ap_1) \text{ for all } a \in A \}
\]
which contains \( \sqrt{\zeta^* \zeta} \).
Since the left and right action of \( A \) on \( L^2(A, \tau) \) agree, so \( \xi_0 \in J_0^{-\| \cdot \|_2} \) and \( a \in A \) implies that \( \xi_0 a, a \xi_0 \in J_0^{-\| \cdot \|_2} \).
Since the w.o.t closed ideal \( J_0 \) in \( A \) is just a cutdown of \( A \) by a projection from \( A \) any positive \( \zeta_0 \in J_0^{-\| \cdot \|_2} \) is a limit in \( \| \cdot \|_2 \) of an increasing sequence of positive operators from \( J_0 \). Now it follows that \( |\zeta_0|^k \in J_0^{-\| \cdot \|_2} \) for all \( k \in \mathbb{N} \). Therefore by Prop. 4.6 it follows that \( v^* v = p_2 \in J_0^{-\| \cdot \|_2} \) and hence \( p_2 \in J_0 \subseteq A \). Similarly arguing with \( \zeta^* \) one shows \( p_1 \in A \).
Therefore
\[
 ap_1 v p_2 = v p_2 \psi(ap_1) \text{ for all } a \in A.
\]
Then
\[
 v^* a v = (v p_2)^* a v p_2 = v^* a p_1 v = v^* v p_2 \psi(ap_1) = \psi(ap_1).
\]
Therefore \( v^* \) and hence \( v \) are groupoid normalisers. So
\[
 \zeta = v \xi.
\]
for \( v \in \mathcal{GN}(A) \) and \( \xi = |\zeta| \in L^2(A, \tau)^+ \). \( \square \)

5. Characterization by Baire Category Methods

The study of Cartan masas in \( \Pi_1 \) factors has received special attention by many experts. Our approach of studying measure-multiplicity-invariant was also considered implicitly by Popa and Shlyakhtenko in \([27]\). In this section we will use an alternative approach to characterize masas by their left-right-measure. As it turns out, many known theorems related to structure and normalisers of masas that were solved using different techniques can be solved by a single technique.

Let \( A = L^\infty(X, \nu_X), B = L^\infty(Y, \nu_Y) \) be two diffuse commutative von Neumann algebras, where \( \nu_X, \nu_Y \) are probability measures. Let \( C(A, B) \) denote the set of all \( A, B \)-bimodules. This set \( C(A, B) \) contains three distinguished subsets.

We will use the variable \( s \) to denote the first variable and \( t \) to denote the second variable. Following \([27]\) we define:

**Definition 5.1.** A discrete (respectively, diffuse, mixed) \( A, B \)-bimodule is a Hilbert space \( \mathcal{H} \) so that \( \mathcal{H} \cong \bigoplus_{i \in I} L^2(X \times Y, \mu_i) \) where for all \( i, \mu_i \) disintegrates as \( \mu_i(s, t) = \mu_i^{(i)}(s) \nu_Y(t) \) with \( \mu_i^{(i)} \) atomic (respectively non-atomic, a combination of both nonzero atomic part and nonzero non-atomic part) for \( \nu_Y \) almost all \( t \).

It is to be noted that in view of Lemma 4.6, the definition above only cares about the equivalence class of the measures \( \mu_i \) and not a particular member of the class. The definition forces \( \mu_i \) to be a non-atomic measure, and the existence of such a disintegration actually forces the push forward of \( \mu_i \)'s on the space \( Y \) to be dominated by \( \nu_Y \). We will restrict ourselves to the case \( I \) is countable. Let \( C_d(A, B), C_{n.a}(A, B), C_m(A, B) \) denote the set of all discrete, diffuse, mixed \( A, B \)-bimodules respectively.

Denote by \( C_d(A) \subset C_d(A, A) \subset C(A, A) \) the set of those bimodules \( \mathcal{H} \in C_d(A, A) \)
for which $\mathcal{H} \in C_d(A, A)$. Here $\mathcal{H}$ is the opposite Hilbert space of $\mathcal{H}$ with left and right actions interchanged. Bimodules in $C_d(A)$ are precisely those for which the associated measures $\mu_i$'s in Defn. 5.1 also have a completely atomic $\nu_X$ disintegration. Similarly define $C_{n,a}(A), C_m(A)$. Note that the spaces $C_d(A), C_{n,a}(A), C_m(A)$ are all closed with respect to taking sub bimodules.

When $A, B$ are masas in a $\text{II}_1$ factor $\mathcal{M}$ the standard Hilbert space $L^2(\mathcal{M})$ is naturally a $w^*$-continuous $A, B$ bimodule, meaning it carries a pair of mutually commuting normal representations of $A$ and $B$.

Note that when we deal with the left-right-measure of a masa, knowing the disintegration along the second variable enables us to know the disintegration along the first variable as well, by pushing forward the former with the flip map (see Lemma 3.7).

Before we proceed to the characterization of masas we will have to make few definitions and statements that are very valuable tools yet not appear in standard measure theory courses. For details see [14], [21].

**Definition 5.2.** Let $X$ be a Polish space. A subset $B$ of $X$ is said to have $\text{Baire property}$ if there is an open set $O \subset X$ and a comeager set $A \subset X$ such that $A \cap O = A \cap B$.

The collection of sets with $\text{Baire property}$ forms a $\sigma$-algebra which includes the $\text{Borel} \ \sigma$-algebra.

**Definition 5.3.** Let $X$ and $Y$ be Polish spaces. A function $f : X \rightarrow Y$ is said to be $\text{Baire measurable}$ if the inverse image of any open set has $\text{Baire property}$. The function $f$ is said to be $\text{universally Baire measurable}$ if given any Borel function $g$ into $X$ the function $f \circ g$ is Baire measurable.

Note that in particular every Borel function is $\text{Baire measurable}$.

**Definition 5.4.** A subset $E$ of a Polish space is said to be $\text{universally measurable}$ if it is measurable with respect to any $\text{complete Borel probability measure}$.

**Definition 5.5.** A subset $E$ of a Polish space $X$ is said to be $\Sigma^1_1$ or $\text{analytic}$, if there is a Polish space $Y$, a Borel subset $B$ of $Y$ and a Borel function $f : Y \rightarrow X$ such that $f(B) = E$. In other words, $\Sigma^1_1$ sets are Borel images of Borel sets.

**Remark 5.6.** The above definition of analytic sets is as per [14]. However in, [15] continuous images rather than Borel images are used. The two definitions are in fact equivalent.

A very nontrivial theorem of Lusin says the following.

**Theorem 5.7.** (Lusin) Every $\Sigma^1_1$ set has $\text{Baire property}$. Every $\Sigma^1_1$ set is universally measurable.

For a function $f : Y \rightarrow X$, the graph of $f$ will be denoted by $\Gamma(f) = \{(f(y), y) : y \in Y\}$. The next theorem is very crucial in all our analysis.

**Theorem 5.8.** (Selection Principle - Jankov, von Neumann) Let $X, Y$ be Polish spaces and let $E \subset X \times Y$ be in $\Sigma^1_1$. Then $E$ can be uniformized by a function that is both $\text{Baire and universally measurable}$, in the sense that for some $h : Y \rightarrow X$ we have $\Gamma(h|_{\pi_Y(E)}) \subseteq E$

with the property that $h^{-1}(U)$ has the $\text{Baire property}$ and is measurable with respect to any $\text{Borel probability measure}$ for all open $U \subseteq X$.

**Remark 5.9.** Let $\nu_X$ and $\nu_Y$ be any two Borel probability measures on $X,Y$ respectively. Let $\sigma_{\nu_X}$ and $\sigma_{\nu_Y}$ be the $\sigma$-algebras associated to the measures $\nu_X, \nu_Y$ respectively. If $h$ is the function in Thm. 5.8 then the inverse image of any Borel set in $X$
under \( h \) will lie in \( \sigma_{\nu_Y} \), because the collection of subsets of \( X \) whose inverse images fall in \( \sigma_{\nu_Y} \) is a \( \sigma \)-algebra and contains all open sets. If in addition, \( h \) satisfies the property that \( \nu_X(h(F)) = 0 \) if and only if \( \nu_Y(F) = 0 \), then \( h \) is \( (\sigma_{\nu_Y}, \sigma_{\nu_X}) \) measurable.

Let \( A \subset \mathcal{M} \) be a masa. Without loss of generality we assume that \( A = L^\infty([0, 1], \lambda) \) where \( \lambda \) is the Lebesgue measure on \([0, 1]\). Let \( [\eta_{[0,1] \times [0,1]}] \) denote the left-right-measure of \( A \). We are including the diagonal. Fix any member \( \eta_{[0,1] \times [0,1]} \) from the equivalence class. Since our base space is now fixed we will rename \( \eta_{[0,1] \times [0,1]} \) by \( \eta \) to reduce the notation. We assume that \( \eta \) is a finite measure.

Consider the set \( S_a = ([0, 1] \times [0, 1])_a \) as defined in Prop. \( 3.3 \) with respect to the disintegration along the \( y \)-axis i.e. the \( t \) variable. Then by Prop. \( 3.3 \) \( S_a \) is a \( [\eta] \)-measurable set, i.e. measurable with respect to the completion \( \sigma \)-algebra associated to \( \eta \). Define measures

\[
\eta_a = \eta([S_a \setminus \Delta([0,1])) \quad \text{and} \quad \eta_{n.a} = \eta(\{S_a \setminus \Delta([0,1]))\).
\]

Then

(i) \( \eta_\Delta([0,1]) = \eta_a + \eta_{n.a} \). \( \eta_a \perp \eta_{n.a} \).

(ii) Both \( \eta_a, \eta_{n.a} \) have disintegrations along the \( x, y \) axes with respect to \( \lambda \).

Note that the disintegration of the measure \( \eta_a \) along the \( x \) and \( y \)-axes must have at most countably many atoms almost all fibres (see Lemma \( 3.7 \)), otherwise \( \eta \) is an infinite measure. Since changing the measure \( \eta \) or \( \eta_a \) on a set of measure 0 does not change the measure class of \( \eta_a \) or \( \eta \), we can as well assume without loss of generality that, the disintegration of the measure \( \eta_a \) along \( y \)-axis (second variable) has at most countable number of atoms for all fibres. With this as set up we formalize the main theorem of this manuscript. Thm. \( 5.10 \) will be proved latter in this section.

**Theorem 5.10. (Classification of Types)** A masa \( A \subset \mathcal{M} \) is

(i) Cartan if and only if \( \eta_{n.a} = 0 \) equivalently \( L^2(A) \perp \in \mathcal{C}(A) \),

(ii) singular if and only if \( \eta_a = 0 \) equivalently \( L^2(A) \perp \in \mathcal{C}_{n.a}(A) \),

(iii) \( A \not\subset N(A)'' \not\subset \mathcal{M} \) if and only if \( \eta_a \neq 0, \eta_{n.a} \neq 0 \) equivalently \( L^2(A) \perp \in \mathcal{C}_{n}(A) \).

(iv) \( A \) is semiregular if and only if the closed support of \( \eta_a \)

is \([0, 1] \times [0, 1]\).

**Remark 5.11.** First of all, in view of Lemma \( 3.3 \) and \( 3.6 \) the characterization does not depend on any particular member of the left-right-measure.

Secondly, \( L^2(A) \) is always included in \( \mathcal{C}(A) \), the disintegration having one atom at each point of the diagonal. That is the reason one excludes \( L^2(A) \) from statements in Thm. \( 5.10 \).

Finally, from our discussion on direct integrals in Sec. 2, it follows that \( L^2(A) \perp \) is the direct integral over \([0, 1] \times [0, 1]\) with respect to the measure \( \eta_\Delta([0,1]) \), the measurable field of Hilbert spaces depending on \( m_{[0,1]} \) or the Pukánszky invariant. So the equivalent statements regarding the type of bimodules and measure in Thm. \( 5.10 \) are obvious statements.

The next technical lemma is the key to characterization of masas. There are several measures involved in its statement and proof. Since there is danger of confusion with measurability of objects involved we will always use phrases like “\( \mu \)-measurable”.

**Lemma 5.12.** Let \( \eta_a \neq 0 \). Let \( Y \subseteq (\Delta([0,1]))^c \) be a \( \eta \)-measurable set of strictly positive \( \eta_a \)-measure. There exists a \( \lambda \)-measurable set \( E^Y \subseteq [0, 1] \) with \( \lambda(E^Y) > 0 \) and a function \( h_Y : [0, 1] \mapsto [0, 1] \) such that

(i) \( h_Y \) is \( \lambda \)-measurable,
\((ii) \Gamma(h_Y)\) is a \(\eta\)-measurable set,
\((iii) \eta(\Gamma(h_Y)) > 0\) and \((h_Y(t), t) \in Y \cap S_a\) for \(t \in E^Y\),
\((iv)\) for \(E \subset [0, 1]\), \(\lambda(E) = 0\) if and only if \(\lambda(h_Y(E)) = 0\).

**Proof.** We have

\[\eta(S_a \cap Y) = \eta_h(S_a \cap Y) = \eta_a(Y) > 0.\]

Consider the disintegration of \(\eta_Y\) along the \(y\)-axis. There is a set \(F^Y \subseteq [0, 1]\) such that \(\lambda(F^Y) > 0\) and for each \(t \in F^Y\) the measure \(\eta_Y\) has atoms with at most countable number of atoms and for \(t \notin F^Y\) the same disintegration has no atoms. This is true because \(\eta\) is a finite measure, the set \(F^Y\) being \(\pi_y(S_a \cap Y)\), \(\pi_y\) denoting the projection on to the \(y\)-axis. The set \(S_a \cap Y\) is \(\eta\)-measurable, so \(S_a \cap Y = B \cup N\) where \(B\) is a Borel set in \([0, 1] \times [0, 1]\) and \(N\) is a \(\eta\)-null set. The set \(B\) is a continuous image of a Polish space by Thm. 14.3.5 of [15] and so is \(\pi_y(B)\). By Defn 3.1 \(\lambda(\pi_y(N)) = 0\).

So \(F^Y\) is \(\lambda\)-measurable set by Thm. 5.4. Throwing off another \(\lambda\)-null set from \(F^Y\) if necessary we can as well assume without loss of generality that \(F^Y\) is a Borel set.

Let \(F_a^Y = \{(Y \cap S_a) \cap ([0, 1] \times F^Y)\}\) which is \(\eta\)-measurable. Write \(F_a^Y = E_a^Y \cup N_1\) where \(N_1\) is a \(\eta\)-null set and \(E_a^Y\) is a Borel set. Then by Thm. 14.3.5 of [15], \(E_a^Y\) is in \(\Sigma_1\), in fact it is the continuous image of a Polish space. The hypothesis guarantees

\[\eta(E_a^Y) > 0.\]

Let \(E^Y = \pi_y(E_a^Y)\). Then \(E^Y\) is in \(\Sigma_1\) and hence \(E^Y\) is \(\lambda\)-measurable by Thm. 5.4.

Therefore by Defn 3.1 \(\lambda(E^Y) > 0\). By Thm. 5.8 applied to \(E_a^Y\), there exists a function \(h_Y: [0, 1] \mapsto [0, 1]\) that is both \(Baire\) and \(universally\ measurable\) in the sense of Thm. 5.8 such that \(\Gamma(\eta_Y|E^Y) \subseteq E_a^Y\).

The inverse image under \(h_Y\) of any Borel subset of \([0, 1]\) belongs to \(\sigma_\lambda\). Therefore given \(\epsilon > 0\), by Lusin’s theorem there is a closed subset \(G^Y \subseteq E^Y\) such that \(\lambda(E^Y \setminus G^Y) < \epsilon\) and \(h_Y|G^Y\) is continuous. Then \(h_Y|G^Y\) is Borel measurable. So by Cor. 2.11 of [20], \(\Gamma(\eta_Y|G^Y)\) is Borel measurable and hence \(\eta\)-measurable.

The disintegration along the \(y\)-axis of the measure \(\eta(\Gamma(h_Y|G^Y))\) is precisely the atom at the point \((h_Y(t), t)\) for each \(t \in G^Y\) of the measure \(\eta_t\). Outside \(G^Y\) we don’t care. If \(\eta(\Gamma(h_Y|G^Y)) = 0\) then by definition of disintegration

\[0 = \int_{G^Y} \eta_h(\Gamma(h_Y))d\lambda(t)\]

which implies that for \(\lambda\) almost all \(t \in G^Y\) the point \((h_Y(t), t)\) is not an atom of \(\eta_t\) and hence cannot be in \(S_a\). So \(\eta(\Gamma(h_Y|G^Y)) > 0\).

Clearly, \(h_Y|G^Y\) satisfies the property that for any \(E \subset G^Y\), \(\lambda(E) = 0\) if and only if \(\lambda(h_Y(E)) = 0\). Therefore by Thm. A.2 extend \(h_Y|G^Y\) to a continuous function \(\tilde{h}_Y\) which satisfies the property that for any \(E \subset [0, 1]\), \(\lambda(E) = 0\) if and only if \(\lambda(h_Y(E)) = 0\). So by Rem 5.9 \(\tilde{h}_Y\) is \((\sigma_\lambda, \sigma_\lambda)\) measurable. Rename \(\tilde{h}_Y\) to \(h_Y\) and \(G^Y\) to \(E^Y\). The rest is clear from construction. \(\square\)

**Lemma 5.13.** Let \(\eta_a \neq 0\). Let \((\Omega \subseteq (\Delta[0, 1])^e\) be a \(\eta\)-measurable set of strictly positive \(\eta_a\)-measure. Then \(\mathcal{U}(A) \subset N(A),\) where \(\mathcal{U}(A)\) denotes the unitary group of \(A\).

More precisely, there exists a subset \(F^Y\) of \([0, 1]\) such that \(\lambda(F^Y) > 0\), an invertible map \(h_Y: F^Y \mapsto h_Y(F^Y)\) and a nonzero vector \(\zeta_Y \in L^2(N(A)^\prime) \subset L^2(A)\) such that

\[(i)\) \(\zeta_Y = v_Y\rho_Y\) with \(v_Y \in \mathcal{G}N(A), \rho_Y \in L^2(A)\)
\[(ii) \quad A\zeta_Y A^\|\|_2 \ni \int_{\Gamma(h_Y)} C_{s,t} d\eta(s,t), \quad \text{where } C_{s,t} = C, \]

\[(iii) \quad \Gamma(h_Y) \subseteq Y \cap S_a, \]

\[(iv) \quad \eta(\Gamma(h_Y)) > 0, \]

\[(v) \quad 1 \in \text{Puk}(A). \]

**Proof.** Using Lemma 5.12, choose the function \(h_Y\) that satisfies the conclusion of that Lemma. Note that \(h_Y\) satisfies the conditions of Prop. [A.4] Apply Prop. [A.4] to the function \(h_Y\) and the set \(E^Y\) to extract a set \(F^Y \subseteq E^Y\) such that \(\lambda(F^Y) > 0\) and \(h_Y\) is one to one on \(F^Y\). So

\[h_Y : F^Y \mapsto h_Y(F^Y)\text{ is invertible.} \]

Note that as \(\lambda(F^Y) > 0\) so \(\eta(\Gamma(h_Y|F^Y)) > 0\). There is no information of the Pukánszky invariant yet. So assume that \(\text{Puk}(A) = \{n_i : n_i \in \mathbb{N}_\infty, i \in I\}\), where the indexing set \(I\) could be finite or countable. Let

\[E_{n_i} = \{(s,t) \in \Delta([0,1]^c) : m_{[0,1]}(s,t) = n_i\}, \]

where \(m_{[0,1]}\) denotes the multiplicity function of the direct integral decomposition of \(L^2(M)\) over \([0,1] \times [0,1]\) with respect to the measure \(\eta\). Then for each \(i \in I\) it is well known that \(E_{n_i}\) are \(\eta\)-measurable sets. Also

\[\int_{E_{n_i}} C_{n_i} d\eta(s,t) \cong L^2(E_{n_i}, \eta|E_{n_i}) \otimes C_{n_i} \text{ where } C_{n_i} = C_{n_i}, \text{ and} \]

\[\oplus_{i \in I} L^2(E_{n_i}, \eta|E_{n_i}) \otimes C_{n_i} \cong L^2(M) \otimes L^2(A). \]

In the above equation \(C^{\infty}\) stands for \(l^2(\mathbb{N})\). Fix orthonormal bases \(\{e_j^{(n_i)}\}_{1 \leq j \leq n_i}\) of \(C_{n_i}\) for all \(i \in I\).

Then

\[\sum_{i \in I} \chi_{\Gamma(h_Y|F^Y) \cap E_{n_i}} \otimes e_1^{(n_i)} \]

where \(\chi\) denotes the indicator function, can be identified with a vector \(\zeta_Y \in (1 - e_A)(L^2(M))\) such that

\[(5.1) \quad A\zeta_Y A = A\zeta_Y = \zeta_Y A. \]

Eq. [5.1] is easy to check, in fact one only uses that fact that \(h_Y\) is locally one to one and onto. That \(\zeta_Y \neq 0\) is due to the fact \(\eta(\Gamma(h_Y|F^Y)) > 0\). Then from Theorem 4.9 it follows that \(\zeta_Y = v_Y \rho_Y\) where \(\rho_Y = (\zeta_Y^* \zeta_Y)^{\frac{1}{2}} \in L^2(A)^+\) and \(v_Y \in \mathcal{G}N(A)\). Clearly, \(v_Y \not\in A\), as otherwise \(\overline{A\zeta_Y A^\|\|_2} \subseteq L^2(A)\) would become the direct integral of complex numbers over some subset of the diagonal with respect to the measure \(\Delta, \lambda\), where \(\Delta : [0,1] \mapsto [0,1] \times [0,1]\) is the map \(\Delta(x) = (x, x)\).

Thus \(\zeta_Y \in L^2(N(A)^\prime)\) and hence \(\overline{A\zeta_Y A^\|\|_2} \subseteq L^2(N(A)^\prime)\). Clearly

\[(5.2) \quad \overline{A\zeta_Y A^\|\|_2} \ni \int_{\Gamma(h_Y|F^Y)} C_{s,t} d\eta(s,t), \quad \text{where } C_{s,t} = C. \]

So \(\overline{A\zeta_Y A^\|\|_2} \perp L^2(A)\) and \(\overline{A\zeta_Y A^\|\|_2} \subseteq C_d(A)\). Since \(\overline{A\zeta_Y A^\|\|_2} \subseteq L^2(N(A)^\prime)\) so \(\eta(\Gamma(h_Y|F^Y) \cap E_{n_i}) = 0\) if \(n_i \geq 2\) from a result of Popa [25]. Thus \(1 \in \text{Puk}(A)\). \(\Box\)

Each partial isometry \(0 \neq v \in \mathcal{G}N(A)\) implements a measure preserving local isomorphism \(T : ([0,1], \lambda) \mapsto ([0,1], \lambda)\) such that \(vav^* = a \circ T^{-1}\) for all \(a \in A\). With
abuse of notation we will write \( v = T \). Then \( \Gamma(v) = \{(T(t), t) : t \in \text{Dom}(T)\} \), \( \text{Dom}(T) \)

denoting the domain of \( T \).

**Lemma 5.14.** Let \( \eta_a \neq 0 \). Let \( Y \subseteq (\Delta[0,1])^c \) be a \( \eta \)-measurable set of strictly positive \( \eta_a \)-measure. Then there is a nonzero partial isometry \( v \in \mathcal{G}\mathcal{N}(A) \) such that \( \Gamma(v) \subseteq Y \).

**Proof.** By Lemma 5.13, there exists a subset \( F' \subseteq [0,1] \) such that \( \lambda(F') > 0 \), a invertible map \( h_Y : F' \to h_Y(F') \) and a nonzero vector \( \zeta_Y \in L^2(\mathcal{G}\mathcal{N}(A)^\prime) \otimes L^2(A) \) such that \( \zeta_Y = v_\gamma \rho_Y \) with \( v_\gamma \in \mathcal{G}\mathcal{N}(A) \), \( \rho_Y \in L^2(A)^+ \) and satisfying property (ii), (iii), (iv) of Lemma 5.13.

Let \( \eta_{\zeta_Y}, \eta_{v_\gamma} \) be the measures on \([0,1] \times [0,1]\) defined in Eq. (3.1). Let \( q_\gamma = v_\gamma v_\gamma^* \in A \).

With abuse of notation we will regard \( q_\gamma \) as a measurable subset of \([0,1]\) as well. We claim that, \( \eta_{\zeta_Y} \ll \eta_{v_\gamma} \ll \eta_{\zeta_Y} \). Indeed for \( a, b \in C[0,1] \),

\[
\int_{[0,1] \times [0,1]} a(s)b(t) \, d\eta_{\zeta_Y}(s, t) = \int_{\Gamma(h_Y)} a(s)b(t) \, d\eta_{\zeta_Y}(s, t) = \tau(\rho_\gamma^* v_\gamma^* a v_\gamma \rho_\gamma b) = \tau(\rho_\gamma^* v_\gamma^* a v_\gamma \rho_\gamma b_\gamma) \quad (\text{as } \rho_\gamma b = b_\gamma) \]

\[
= \tau(v_\gamma^* a v_\gamma \rho_\gamma^* ) = \tau(v_\gamma^* a v_\gamma \rho_\gamma^* ) = \tau(v_\gamma^* a v_\gamma \rho_\gamma^* ) = \int_{q_\gamma} a(v_\gamma(t))b(t) \, |\rho_\gamma(t)|^2 \, d\lambda(t)
\]

\[
= \int_{\Gamma(v_\gamma)} a(s)b(t) \, |\rho_\gamma(t)|^2 \, d\eta_{v_\gamma}(s, t) = \int_{[0,1] \times [0,1]} a(s)b(t) \, |\rho_\gamma(t)|^2 \, d\eta_{v_\gamma}(s, t).
\]

In the above string of equalities we have used the facts that \( \tau \) extends to a trace like functional on \( L^1(A) \) and the left and right actions of \( A \) on \( L^2(A), L^1(A) \) coincides. Using Thm. 1.9 by standard arguments it follows that \( \eta_{\zeta_Y} \ll \eta_{v_\gamma} \ll \eta_{\zeta_Y} \). Thus the result follows with \( v = v_\gamma \).

Suppose \( \{v_j\}_{j \in J} \) is a family of partial isometries in \( \mathcal{G}\mathcal{N}(A) \) such that \( Av_j \perp Av_j' \) whenever \( j \neq j' \). Denote by 25

\[
\sum_{j \in J} Av_j = \left\{ x \in \mathcal{M} : x = \sum_{j \in J} a_j v_j, \text{ for } a_j \in A \text{ with } \sum_{j \in J} \|a_j v_j\|^2 _2 < \infty \right\}.
\]

**Theorem 5.15.** (Compare Cor. 2.5 25) Let \( \eta_a \neq 0 \). Then \( A \subseteq \mathcal{N}(A)' \). Moreover, there is a sequence \( \{v_n\}_{n=0}^\infty \subseteq \mathcal{G}\mathcal{N}(A) \) of nonzero partial isometries (with possibility that the sequence could be finite) with \( v_0 = 1 \) such that,

(i) \( \Gamma(v_n) \cap \Gamma(v_m) = \emptyset \) for \( n \neq m \),

(ii) \( \eta_a([0,1] \times [0,1]) = \sum_{n=1}^\infty \eta_a(\Gamma(v_n)) \),

(iii) \( \oplus_{n=0}^\infty Av_n \overset{\|\cdot\|_2}{\cong} \int_{\cup_{n=0}^\infty \Gamma(v_n)} \mathbb{C}_{s,t} d(\eta + \Delta s \lambda)(s, t) \cong L^2(\mathcal{N}(A)'') \),

(where \( \mathbb{C}_{s,t} = \mathbb{C} \text{ and } \Delta : [0,1] \mapsto [0,1] \times [0,1] \) by \( \Delta(x) = (x, x) \))
(iv) $N(A)'' = \sum_{n=0}^{\infty} Av_n,$

and $\mathcal{A}$ restricted to $\bigoplus_{n=0}^{\infty} Av_n$ is diagonalizable with respect to the decomposition in (iii).

Proof. First of all assuming that (i) in the statement is true it follows that $Av_n \perp Av_m$ whenever $n \neq m$. Indeed, $\overline{Av_n}$ is an abelian algebra with a cyclic vector, so it is maximal abelian. The projection $e_{Av_n}$ onto $\overline{Av_n}$ is the direct integral of complex numbers over a subset $X_n$ of $[0, 1] \times [0, 1]$ with respect to the measure $\eta$ and $\mathcal{A}$ restricted to $\overline{Av_n}$ is diagonalizable with respect to this decomposition. But $\eta(X_n \Delta \Gamma(v_n)) = 0$. Again $\eta_{\alpha, \beta}(\Gamma(v_n)) = 0$. So the direct integral as stated above is actually with respect to the measure $\eta_n + \Delta, \lambda$. The graphs being disjoint for $n \neq m$ forces the orthogonality of $Av_n$ and $Av_m$ whenever $n \neq m$. The sum in (iii) therefore makes sense.

Using Lemma 5.14 choose a maximal family $\{v_a\}_{a \in A} \subset \mathcal{G}(\mathcal{A})$, for some indexing set $\Lambda$, such that $\Gamma(v_a) \subset \Delta([0, 1])$ for all $\alpha \in \Lambda$ and $\Gamma(v_a) \cap \Gamma(v_\beta) = \emptyset$ whenever $\alpha \neq \beta$. Since $Av_\alpha \perp Av_\beta$ whenever $\alpha \neq \beta$ (by similar argument as above) so the indexing set must be countable by separability assumption of $L^2(\mathcal{M})$. So we index this maximal family by $\{v_n\}_{n=1}^{\infty}$. Let $v_0 = 1$. So (i) follows by construction.

If $\eta_{\alpha, \beta}(\Gamma(v_n)) > \sum_{n=1}^{\infty} \eta_{\alpha, \beta}(\Gamma(v_n))$ then $S_\alpha \setminus \bigcup_{n=1}^{\infty} \Gamma(v_n)$ is a set of strictly positive $\eta$ measure. A further application of Lemma 5.13 violates the maximality of $\{v_n\}_{n=1}^{\infty}$. This proves (ii).

By the argument of the first paragraph and Lemma 5.7 [10],

$$\bigoplus_{n=1}^{\infty} \overline{Av_n} \cong \int_{\bigcup_{n=1}^{\infty} \Gamma(v_n)} \mathcal{C}_{s,t} d\eta(s, t) \subseteq L^2(N(A))'' \oplus L^2(A)$$

and $\mathcal{A}$ restricted to $\bigoplus_{n=1}^{\infty} \overline{Av_n}$ is diagonalizable with respect to the decomposition in Eq. (5.3).

If $0 \neq \zeta = \zeta^* \in L^2(N(A))'' \oplus L^2(A)$ is such that $\zeta \perp Av_n$ for all $n \geq 1$ then $A\zeta A \perp Av_n$ for all $n \geq 0$. By arguments similar to the first paragraph, $\overline{A\zeta A}$ is the direct integral over a $\eta$-measurable set $X_\zeta$, of complex numbers with respect to the measure $\eta$ and $\mathcal{A}$ restricted to $\overline{A\zeta A}$ is diagonalizable respecting this decomposition. If $\zeta$ as a $L^2$ function stays nonzero on a set of positive $\Delta, \lambda$-measure then $\zeta$ cannot be perpendicular to $L^2(A)$. By Prop. 3.11 $\overline{A\zeta A}$ is in $C_d(A)$ and hence by Theorem 3.8 and Lemma 5.7 [10], we can assume $X_\zeta \subset S_\alpha \setminus \Delta([0, 1])$. Since $\zeta \neq 0$ so $\eta(X_\zeta) = \eta_a(X_\zeta) > 0$. Since $\eta_a$ is concentrated on $\bigcup_{n=1}^{\infty} \Gamma(v_n)$, so $X_\zeta \cap \Gamma(v_n)$ has strictly positive $\eta_a$ and hence $\eta$ measure for some $n \geq 1$. Note that $e_{N(A)''} \in \mathcal{A}$ and $Ae_{N(A)''} = A' e_{N(A)''}$ from [25]. On the other hand, by Lemma 5.7 [10], $L^2(N(A))'' \oplus L^2(A)$ will be expressed as a direct integral over some subset of $[0, 1] \times [0, 1]$ with respect to $\eta$, with multiplicity strictly bigger than 1 on a set of positive $\eta$-measure. This contradicts $Ae_{N(A)''}$ is maximal abelian. Thus

$$\bigoplus_{n=0}^{\infty} \overline{Av_n} \cong \int_{\bigcup_{n=0}^{\infty} \Gamma(v_n)} \mathcal{C}_{s,t} d(\eta_a + \Delta, \lambda)(s, t) \cong L^2(N(A))''$$.
with associated statements about diagonalizability of \( A \). Finally

\[
\sum_{n=0}^{\infty} Av_n = \left( \sum_{n=0}^{\infty} Av_n \right) \cap M = (L^2(N(A)'' \cap M = N(A)'').
\]

**Remark 5.16.** Thm. 5.15 generalizes Cor. 2.5 of [25]. In general we cannot hope to find unitaries as was the case in Cor. 2.5 [25]. The situation in Cor. 2.5 of [25] was completely different, where the assumption was that, the masa is Cartan. Assuming the masa is Cartan, forces the disintegration of the measure \( \eta \) to have at least one atom off the diagonal in almost every fibre. Such an assumption cannot be made for a general masa. For example consider the following situation. Let \( C \subset R \) be a Cartan masa and let \( S \subset R \) be a singular masa, where \( R \) denotes the hyperfinite II\(_1\) factor.

Then \( C \oplus S \subset R \oplus R \) is a masa, where the trace on \( R \oplus R \) is \( \frac{1}{2} \tau_R \oplus \frac{1}{2} \tau_R \), \( \tau_R \) denoting the unique, normal, faithful tracial state of \( R \). Then \( C \oplus S \subset (R \oplus R) \ast R \cong L(F_2) \) (from \([8]\)) is a masa for which such an assumption will fail from Prop. 5.10 \([10]\).

We will now present the proof of Thm. 5.10.

**Proof of 5.10.** Case (i). The necessary and sufficient condition for Cartan masas follows directly from Thm. 5.15.

Case (ii). The result for singular masas also follows from Thm. 5.15.

Case (iii). Let \( A \not\subset N(A)'' \not\subset M \). If \( \eta_A = 0 \) then, by conclusion of (ii), \( A \) would become singular. Therefore \( \eta_A \neq 0 \). If \( \eta_{n.a} = 0 \) then by conclusion of part (i), \( A \) would be Cartan. Therefore \( \eta_{n.a} \neq 0 \) as well.

Conversely, if \( \eta_{n.a} \neq 0 \) and \( \eta_A \neq 0 \), then by Theorem 5.15 \( A \not\subset N(A)'' \not\subset M \).

Case (iv). First assume that \( N(A)'' \) is a factor. From Thm. 5.15 it follows that

\[
L^2(N(A)'' \cong \int_{[0,1] \times [0,1]} \mathbb{C}_{s,t} \| \eta + \tilde{\Delta}_{x \lambda} \| (s,t), \text{ where } \mathbb{C}_{s,t} = \mathbb{C}
\]

and \( A \) restricted to this subspace is diagonalizable, where \( \tilde{\Delta} : [0,1] \rightarrow [0,1] \times [0,1] \) is defined by \( \Delta(x) = (x,x) \). Therefore \( \eta + \tilde{\Delta}_{x \lambda} \) is the left-right-measure for the inclusion \( A \subset N(A)'' \). If the closed support of \( \eta_A \) is strictly contained in \( [0,1] \times [0,1] \), then, there must be an open set \( U \subset [0,1] \times [0,1] \) such that \( \eta_A(U) = 0 \). It follows that the map \( a \otimes b \rightarrow aJ_{N(A)''}b^*J_{N(A)''}, \) where \( a, b \in C([0,1]) \) was not an injection. On the other hand, as \( N(A)'' \) is a factor the above map must be an injection by Prop. 2.8.

This contradiction proves that the closed support of \( \eta_A \) is \( [0,1] \times [0,1] \).

Conversely assume \( N(A)'' \) has a nontrivial center. Let \( p \in Z(N(A)'' \) be a projection which is different from 0 and 1. Then

\[
N(A)'' = N(A)''p \oplus N(A)''(1-p).
\]

So \( p \in A' \cap N(A)'' \) and hence \( p \in A \). So

\[
(5.4) \quad A = Ap \oplus A(1-p).
\]

It follows that there exist \( \lambda \)-measurable sets \( F_1, F_2 \subset [0,1] \) such that, \( F_1 \cup F_2 = [0,1], F_1 \cap F_2 = \emptyset, C(F_1), C(F_2) \) are w.o.t dense unital subalgebras of \( Ap \) and \( A(1-p) \) respectively and \( C(F_1) \oplus C(F_2) \) is w.o.t dense in \( A \). With respect to the Eq. (5.4) let \( a = a_1 \oplus a_2 \) and \( b = b_1 \oplus b_2 \) be the decompositions of \( a, b \in C([0,1]) \), where \( a_i, b_i \in C(F_i) \) for \( i = 1,2 \). For \( \zeta \in L^2(N(A)'' \) one has an analogous decomposition \( \zeta = \zeta_1 \oplus \zeta_2 \) with \( \zeta_1 = p\zeta \) and \( \zeta_2 = (1-p)\zeta \). The equation

\[
\langle a\zeta b, \zeta \rangle = \langle (a_1 \oplus a_2)(\zeta_1 \oplus \zeta_2)(b_1 \oplus b_2), (\zeta_1 \oplus \zeta_2) \rangle = \langle a_1\zeta_1 b_1, \zeta_1 \rangle + \langle a_2\zeta_2 b_2, \zeta_2 \rangle
\]
shows that the left-right-measure for the inclusion $A \subset N(A)''$ will be concentrated on $F_1 \times F_1 \cup F_2 \times F_2$. It follows that closed support of $\eta_a$ is strictly contained in $[0, 1] \times [0, 1]$. This completes the proof. \hfill \Box

Remark 5.17. The proof of Case (iv) actually shows that if $A \subset N$ is a masa where $N$ is a finite type II algebra with a nontrivial center then for any choice of compact Hausdorff space $X$ such that $C(X)$ is w.o.t dense, unital and norm separable subalgebra of $A$, the map

$$\sum_{i=1}^{n} a_i \otimes b_i \mapsto \sum_{i=1}^{n} a_i J b_i^* J$$

from $C(X) \otimes_{alg} C(X) \mapsto B(L^2(N))$ is not an injection for any choice of trace.

The following results about masas that were proved by experts in different ways are just easy consequences of the measurable selection principle as we have described in this section.

**Corollary 5.18.** If $A \subset M$ is a Cartan masa then $A \subset B$ is a Cartan masa for all von Neumann subalgebra $A \subset B \subset M$.

**Proof.** By Lemma 5.7 of [10] the left-right-measure of the inclusion $A \subset M$ is $[\eta_B + \eta_{B^\perp}]$ where $[\eta_B]$ is the left-right-measure of the inclusion $A \subset B$ and $\eta_B \perp \eta_{B^\perp}$. It follows that $\eta_B$ has atomic disintegration along both axes. The result is then immediate from Thm. 5.10 and Thm 5.15. \hfill \Box

**Corollary 5.19.** Let $A \subset M$ be a masa and let $Q$ be a finite von Neumann algebra such that $\dim(Q) \geq 2$. Then $N_{M\ast Q}(A) = N_M(A)$.

**Proof.** In this proof we consider left-right-measures restricted to the off diagonal. First of all it well known that $A \subset M \ast Q$ is a masa. Let $[\eta_M]$ denote the left-right-measure of the inclusion $A \subset M$. Write $\eta_M = \eta_1 + \eta_2$ where $\eta_1 \ll \lambda \otimes \lambda$ and $\eta_2 \perp \lambda \otimes \lambda$. Using Prop. 5.10 and Lemma 5.7 [10] it follows that the left-right-measure $[\eta_{M\ast Q}]$ of the inclusion $A \subset M \ast Q$ is given by

$$\eta_{M\ast Q} = \begin{cases} \eta_M + \lambda \otimes \lambda & \text{if } \eta_1 = 0, \\ \eta_2 + \lambda \otimes \lambda & \text{if } \eta_1 \neq 0. \end{cases}$$

The rest is obvious from Thm. 5.10 and Thm. 5.15. \hfill \Box

**Corollary 5.20.** Let $A \subset M$ be a Cartan masa and let $A \subset B \subset M$ be an intermediate subalgebra. Then there is a $v \in G\mathcal{N}(A)$ such that $v \perp B$.

**Proof.** By Lemma 5.7 [10], the left-right-measure of the inclusion $A \subset M$ is $[\eta_B + \eta_{B^\perp}]$ where $\eta_B \perp \eta_{B^\perp}$ and $[\eta_B]$ is the left-right-measure of the inclusion $A \subset B$. Note $\eta_{B^\perp} \neq 0$. Apply Lemma 5.14. \hfill \Box

We prove the next theorem in the context of $\Pi_1$ factors. But it can be easily generalized to finite von Neumann algebras. Let $\mathcal{M}_i$, $i = 1, 2$ be separably acting $\Pi_1$ factors with normal, faithful tracial states $\tau_i$ respectively. Let $\mathcal{M}_i$ act on $L^2(\mathcal{M}_i, \tau_i)$ by left multiplication. Let $A \subset \mathcal{M}_1$ and $B \subset \mathcal{M}_2$ be masas. Fix compact Polish spaces $X, Y$ such that $C(X) \subset A$ and $C(Y) \subset B$ are unital, norm separable and w.o.t dense. Let $\nu_X$ and $\nu_Y$ denote the tracial measures for $A, B$ respectively, which will be assumed to be complete. Let the left-right-measure of $A$ on $X \times X$ be $[\sigma_1]$ and that of $B$ on $Y \times Y$ be $[\sigma_2]$. Here we are allowing the diagonals, i.e. we are assuming $\sigma_1|_{\Delta(X)} = (\Delta_X)^* \nu_X$ and $\sigma_2|_{\Delta(Y)} = (\Delta_Y)^* \nu_Y$ where $\Delta_X : X \mapsto X \times X$ by $\Delta_X(x) = (x, x)$ and $\Delta_Y : Y \mapsto Y \times Y$ by $\Delta_Y(y) = (y, y).$
By Tomita’s theorem on commutants $A \overline{\otimes} B$ is a masa in $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$. $X \times Y$ is compact and Polish, and $C(X \times Y)$ is unital, norm separable and w.o.t dense in $A \overline{\otimes} B$. The standard Hilbert space and the Tomita’s involution operator for $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ is $L^2(\mathcal{M}_1, \tau_1) \otimes L^2(\mathcal{M}_2, \tau_2)$ and $J_{\mathcal{M}_1} \otimes J_{\mathcal{M}_2}$ respectively. The tracial measure for $A \overline{\otimes} B$ on $X \times Y$ is clearly $\nu_X \otimes \nu_Y$. With this as set up we formulate the next theorem which appeared in [2]. The same proof actually generalizes to infinite tensor products.

**Theorem 5.21.** (Chifan’s Normaliser Formula) Let $A \subset \mathcal{M}_1$ and $B \subset \mathcal{M}_2$ be masas in separably acting $\mathbb{II}_1$ factors $\mathcal{M}_1$ and $\mathcal{M}_2$. Then

$$N(A \overline{\otimes} B)'' = N(A)'' \overline{\otimes} N(B)''.$$  

*Proof.* Fix $\sigma_1$ and $\sigma_2$ from the aforesaid class of left-right-measures. The left-right-measure of $A \overline{\otimes} B$ on $(X \times Y) \times (X \times Y)$ which is denoted by $[\beta]$ is given by

$$d\beta(s_X, s_Y, t_X, t_Y) = d\sigma_1(s_X, t_X) d\sigma_2(s_Y, t_Y)$$

from Prop. 5.2 [11]. Here $s$ is the variable running along the first coordinate (horizontal direction) and $t$ along the second coordinate (vertical direction). Then from Lemma 3.5 it follows that the disintegration of $\beta$ along the $t$ variable (vertical direction) is given by

$$\beta_{t \times Y} = \sigma_{1,t} \otimes \sigma_{2,t}, \quad (t_X, t_Y) - a.e. \nu_X \otimes \nu_Y,$$

where $t_{X \times Y} = (t_X, t_Y) \in X \times Y$. The measure $\beta_{t \times Y}$ has an atom at $(s_X, s_Y, t_X, t_Y)$ if and only if $\sigma_{1,t}$ has an atom at $(s_X, t_X)$ and $\sigma_{2,t}$ has an atom at $(s_Y, t_Y)$. Therefore

$$(X \times Y) \times (X \times Y)_{a} = S_{2,3}((X \times X)_{a} \times (Y \times Y)_{a}),$$

where $S_{2,3}$ denotes the permutation (2, 3) on four symbols (see Prop. 3.3). Therefore,

$$\beta_{((X \times Y) \times (X \times Y))_{a}} = \sigma_{1,(X \times X)_{a}} \otimes \sigma_{2,(Y \times Y)_{a}}.$$  

Hence denoting $C_{s_X, s_Y, t_X, t_Y} = C$, $C_{s_X, t_X} = \mathbb{C} = C_{s_Y, t_Y}$ we have

$$L^2(N(A \overline{\otimes} B)'' \otimes \overline{\otimes} L^2(N(B)'' \otimes L^2(N(A)'' \otimes L^2(N(B)'' \otimes (X \times Y)_{a}) = \int_{(X \times Y)_{a}} C_{s_X, s_Y} d\sigma_1(s_X, t_X) \otimes \int_{(Y \times Y)_{a}} C_{s_Y, t_Y} d\sigma_2(s_Y, t_Y) = L^2(N(A)'' \otimes L^2(N(B)'')$$

from Thm. 5.15

Since the containment $N(A)'' \overline{\otimes} N(B)'' \subset N(A \overline{\otimes} B)''$ is obvious we are done. \hfill \Box

**Corollary 5.22.** [35] Let $A \subset \mathcal{M}_1$ and $B \subset \mathcal{M}_2$ be singular masas in separably acting $\mathbb{II}_1$ factors $\mathcal{M}_1$ and $\mathcal{M}_2$. Then $A \overline{\otimes} B$ is singular in $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$.

### 6. Asymptotic Homomorphism and Measure Theory

The equivalence of WAHP and singularity is a nontrivial theorem [35]. In this section we will give a direct proof of the equivalence of WAHP and singularity by using measure theoretic tools. We will also present partial results about AHP. In order to do so we will first have to relate certain norms to the left-right-measure. The measure theoretic tools described in this section will be used in a future paper for explicit calculation of left-right-measures.

Let $A \subset \mathcal{M}$ be a masa. Let $\lambda$ denote the Lebesgue measure on $[0, 1]$ so that $A \cong L^\infty([0, 1], \lambda)$. Then $\lambda$ is the tracial measure. Let $[\eta]$ denote the left-right-measure for $A$. We assume that $\eta$ is a probability measure on $[0, 1] \times [0, 1]$ and $\eta(\Delta([0, 1])) = 0$. Let $B[0, 1]$ denote the collection of all bounded measurable functions on $[0, 1]$.
Note that $\eta_i([0, 1] \times [0, 1]) = \tau([xx^*])$. From (ii) of Defn. 3.1 it follows that there exists a sequence of functions $f_n \in C[0, 1]$ such that $0 \leq f_n \leq 1$ and $f_n \to \chi_E$ pointwise. By dominated convergence theorem we have $\eta_x(f_n \otimes 1) \to \eta_x(\chi_E \otimes 1)$. On the other hand,

$$\eta_x(f_n \otimes 1) = \langle f_n x, x \rangle = \tau(f_n xx^*) = \tau(f_n \mathbb{E}_A(x x^*)) = \int_0^1 f_n(t) \mathbb{E}_A(x x^*)(t) d\lambda(t)$$

$$\to \int_0^1 \chi_E(t) \mathbb{E}_A(x x^*)(t) d\lambda(t), \text{ as } n \to \infty,$$

$$= \int_E \mathbb{E}_A(x x^*)(t) d\lambda(t).$$

From Defn. 3.1 again we have

$$\eta_x(\chi_E \otimes 1) = \int_0^1 \eta_x(t) \chi_E(t) d\lambda(t) = \int_E \eta_x(t) d\lambda(t).$$

Therefore for all Borel sets $E \subseteq [0, 1]$ we have

$$\int_E \eta_x([0, 1] \times [0, 1]) d\lambda(t) = \int_E \mathbb{E}_A(x x^*)(t) d\lambda(t).$$

Thus, $\eta_x([0, 1] \times [0, 1]) = \mathbb{E}_A(x x^*)$ for $\lambda$ almost all $t$.

\[ \square \]

\textbf{Lemma 6.2.} Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. Let $f \in B[0, 1]$. Then the functions $[0, 1] \ni t \mapsto \eta_x^i(1 \otimes f), [0, 1] \ni s \mapsto \eta_x^i(f \otimes 1)$ are in $L^\infty([0, 1], \lambda)$.

\[ \text{Proof.} \] We will only prove for the $(\pi_1, \lambda)$ disintegration. From Lemma 6.1, we know that $\eta_x$ admits a $(\pi_1, \lambda)$ disintegration. From Defn. 3.1, we also know that $[0, 1] \ni t \mapsto \eta_x^i(1 \otimes f)$ is measurable. Now of $0 \leq t \leq 1$

$$|\eta_x^i(1 \otimes f)| \leq \|f\| \eta_x^i([0, 1] \times [0, 1]).$$

Now use Lemma 6.1. \[ \square \]
Lemma 6.3. Let $x \in M$ be such that $E_A(x) = 0$. Let $b, w \in B[0, 1]$. Then
\[
\|E_A(bxwx^*)\|_2^2 = \int_0^1 |b(t)|^2 |\eta^*_x(1 \otimes w)|^2 d\lambda(t).
\]

Proof. We have noted before that $\eta_x$ admits $(\pi_i, \lambda)$ disintegrations for $i = 1, 2$. Secondly, as $b, w \in B[0, 1]$, so $[0, 1] \ni t \mapsto b(t)\eta^*_x(1 \otimes w)$ is in $L^\infty([0, 1], \lambda)$ from Lemma 6.2. Now
\[
\|E_A(bxwx^*)\|_2^2 = \sup_{a \in C[0, 1]} \|a\|_2 \leq 1 \left| \langle a, E_A(bxwx^*) \rangle \right|^2 = \sup_{a \in C[0, 1]} \|a\|_2 \leq 1 \left| \tau(abxwx^*) \right|^2 = \sup_{a \in C[0, 1]} \|a\|_2 \leq 1 \left| \int_0^1 a(t)b(t)w(s)d\eta_x(t, s) \right|^2 (\text{from Eq. (3.1)})
\]
\[
= \sup_{a \in C[0, 1]} \left| \int_{[0, 1] \times [0, 1]} a(t)b(t)w(\xi)d\eta_x(t, \xi) \right|^2 (\text{from Defn. 3.1})
\]
\[
= \int_0^1 |b(t)|^2 |\eta^*_x(1 \otimes w)|^2 d\lambda(t) (\text{from Lemma 6.2}).
\]

The following facts are well known, we just record them for completeness. For details we refer the reader to [16]. Recall that a subset $S \subseteq \mathbb{Z}$ is said to be of full density if
\[
\lim_{n \to \infty} \frac{|S \cap [-n, n]|}{2n + 1} = 1.
\]

Definition 6.4. A measure $\mu$ on $[0, 1]$ is called mixing (or sometimes Rajchman) if its Fourier coefficients $\hat{\mu}_n = \int_0^1 e^{2\pi int} d\mu(t)$ converge to 0 as $|n| \to \infty$.

By the Riemann-Lebesgue lemma any absolutely continuous measure is mixing. However there are many mixing singular measures as well. Atomic measures can never be mixing. The next proposition justifies why non-atomic measures are called weak (or weakly) mixing measures.

Proposition 6.5. (Wiener) A measure $\mu$ on $[0, 1]$ is non-atomic (diffuse) if and only if for a set $S \subseteq \mathbb{Z}$ of full density
\[
\lim_{n \to \infty} \hat{\mu}_n = 0.
\]

From Prop. 2.5 and Prop. 2.19 of [16], mixing and weakly mixing are just not properties of measures, they are in fact properties of equivalence class of measures.
We need the following fact from the calculus course. A bounded sequence of complex numbers \( \{a_n\}_{n \in \mathbb{Z}} \) converges to 0 strongly in the sense of Cesàro i.e.

\[
\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |a_n| = 0
\]

if and only if there is a set \( S \subseteq \mathbb{Z} \) of full density such that

\[
\lim_{n \in S,|n| \to \infty} |a_n| = 0.
\]

Let \( x, y \in \mathcal{M} \) be such that \( \mathbb{E}_A(x) = \mathbb{E}_A(y) = 0 \). Let \( a \in A \). Then the following polarization identity holds:

\[
4 \mathbb{E}_A(xay^*) = \mathbb{E}_A((x+y)a(x+y)^*) - \mathbb{E}_A((x-y)a(x-y)^*) + i \mathbb{E}_A((x+iy)a(x+iy)^*) - i \mathbb{E}_A((x-iy)a(x-iy)^*).
\]

Thus WAHP for a masa is equivalent to the following. For each finite set \( \{x_i\}_{i=1}^{n} \subset \mathcal{M} \) with \( \mathbb{E}_A(x_i) = 0 \) for all \( 1 \leq i \leq n \) and \( \epsilon > 0 \), there exists an unitary \( u \in A \) such that

\[
\|\mathbb{E}_A(x_iux_i^*)\|_2 \leq \epsilon \text{ for all } 1 \leq i \leq n.
\]

We will only prove the harder part of the equivalence of singularity and WAHP.

**Theorem 6.6.** Let \( A \subset \mathcal{M} \) be a masa such that \( L^2(A)^\perp \subset C_{n.a}(A) \). Then \( A \) has WAHP.

**Proof.** Suppose to the contrary \( A \) does not have WAHP. Then there is an \( \epsilon > 0 \) and operators \( 0 \neq x_i \in \mathcal{M}, 1 \leq i \leq n \) with \( \mathbb{E}_A(x_i) = 0 \) for all \( i \), such that

\[
\inf_{u \in \mathcal{U}(A)} \sum_{i=1}^{n} \|\mathbb{E}_A(x_iux_i^*)\|_2^2 \geq \epsilon,
\]

where \( \mathcal{U}(A) \) denotes the unitary group of \( A \). Note that for all \( 1 \leq i \leq n \), \( Ax_iA^* \in C_{n.a}(A) \) by Lemma 5.7 of [10] and Thm. 5.10. Equivalently, if \( t \mapsto \eta_{x_i}^t \) and \( s \mapsto \eta_{x_i}^s \) denote the \((\pi_1, \lambda)\) and \((\pi_2, \lambda)\) disintegrations respectively of \( \eta_{x_i} \), then for \( \lambda \) almost all \( t \), the measure \( \eta_{x_i}^t \) is completely non-atomic and similar statements hold for \( \eta_{x_i}^s \).

Let \( v \in A \) be the Haar unitary corresponding to the function \( t \mapsto e^{2\pi it} \). Then \( v \) generates \( A \). Now from Lemma 6.3 we have

\[
\sum_{i=1}^{n} \|\mathbb{E}_A(x_i^kv^*x_i^*)\|_2^2 = \int_{0}^{1} \sum_{i=1}^{n} |\eta_{x_i}^t(1 \otimes v^k)|^2 d\lambda(t) \geq \epsilon \text{ for all } k \in \mathbb{Z}.
\]

Throwing off a \( \lambda \)-null set \( F \) we assume that for \( t \in F^c \) the measures \( \eta_{x_i}^t \) are completely non-atomic, finite, concentrated on \( \{t\} \times [0, 1] \) and \( \eta_{x_i}^t([0, 1] \times [0, 1]) = \mathbb{E}_A(x_i^t) \) for all \( 1 \leq i \leq n \) (see Lemma 6.1). Let

\[
a_k(t) = \sum_{i=1}^{n} |\eta_{x_i}^t(1 \otimes v^k)|^2, \quad k \in \mathbb{Z}, t \in [0, 1].
\]

Then \( a_k \) is measurable for all \( k \in \mathbb{Z} \). For \( k \in \mathbb{Z} \) and \( t \in F^c \) we have

\[
a_k(t) = \sum_{i=1}^{n} \left| \int_{[0,1] \times [0,1]} e^{2\pi iks} d\eta_{x_i}^t(t', s) \right|^2 \leq \sum_{i=1}^{n} (\eta_{x_i}^t([0, 1] \times [0, 1]))^2.
\]
Then by Lemma 6.1, \(a_k(t) \leq \sum_{i=1}^n |E_A(x_i^*x_i^*)(t)|^2 < \infty\), for all \(t \in F^c\) and for all \(k \in \mathbb{Z}\). Define

\[s_N(t) = \frac{1}{2N+1} \sum_{k=-N}^{N} a_k(t), N \in \mathbb{N}.
\]

Then \(s_N\) is measurable for all \(N \in \mathbb{N}\). Since \(\eta_{x_i}^t\) is completely non-atomic for all \(1 \leq i \leq n\) and \(t \in F^c\) so

\[s_N(t) \to 0\] as \(N \to \infty\) for all \(t \in F^c\) from Eq. (6.1), (6.2) and Prop 6.5.

Again since \(s_N(t) \leq \sum_{i=1}^n |E_A(x_i^*x_i^*)(t)|^2\) for \(t \in F^c\) (from Lemma 6.1), so by dominated convergence theorem

\[\int_0^1 s_N(t)d\lambda(t) \to 0\] as \(N \to \infty\).

Therefore,

\[\int_0^1 s_N(t)d\lambda(t) = \frac{1}{2N+1} \sum_{k=-N}^{N} \int_0^1 \sum_{i=1}^n |\eta_{x_i}^t| (1 \otimes v^k)|^2 d\lambda(t)
\]

\[= \frac{1}{2N+1} \sum_{k=-N}^{N} \left( \sum_{i=1}^n \|E_A(x_i^*v^kx_i^*)\|_2 \right)^2 \to 0\] as \(N \to \infty\).

Consequently from Eq. (6.2) there is a set \(S \subseteq \mathbb{Z}\) of full density such that

\[\lim_{k \in S, |k| \to \infty} \sum_{i=1}^n \|E_A(x_i^*v^kx_i^*)\|_2^2 = 0.
\]

This is a contradiction to Eq. (6.4). So \(A\) must have WAHP. \(\square\)

The proof of Thm. 6.6 yields the following result.

**Theorem 6.7.** Let \(A \subseteq \mathcal{M}\) be a singular masa. Then given any finite set \(\{x_i\}_{i=1}^n \subseteq \mathcal{M}\) with \(E_A(x_i) = 0\) for all \(i\),

\[(6.5) \quad \frac{1}{2N+1} \sum_{k=-N}^{N} \left( \sum_{i=1}^n \|E_A(x_i^*v^kx_i^*)\|_2 \right)^2 \to 0\] as \(N \to \infty\).

where \(v\) is a Haar unitary generator of \(A\).

**Remark 6.8.** Thus the unitary in the definition of WAHP (Defn. ??) can always be chosen to be \(v^k\) where \(k\) is a large integer and \(v\) is a Haar unitary generator of the masa. This strengthens the definition of WAHP. Note that Eq. (6.5) is very closely related to definition of weakly mixing actions of abelian groups on finite von Neumann algebras.

The measures \(\eta_{x_i}^t, \eta_i^t\) are concentrated on \(\{t\} \times [0, 1]\) for \(\lambda\) almost all \(t\). We will denote by \(\tilde{\eta}_{x_i}^t, \tilde{\eta}_i^t\) the restriction of the measures \(\eta_{x_i}^t\) and \(\eta_i^t\) respectively on \(\{t\} \times [0, 1]\). Thus \(\tilde{\eta}_{x_i}^t, \tilde{\eta}_i^t\) can be regarded as measures on \([0, 1]\).

**Theorem 6.9.** Let \(A \subseteq \mathcal{M}\) be a masa. Let \([\eta]\) denote the left-right-measure for \(A\). If for \(\lambda\) almost all \(t\) the measures \(\tilde{\eta}_i^t\) are mixing, then \(A\) has AHP with respect to a Haar unitary generator of \(A\).
Proof. From Prop. 2.5 of [16] it follows that for $\lambda$ almost all $t$, any measure in the equivalence class $[\tilde{\eta}^t]$ is mixing. In view of Eq. (6.3), it is enough to show that for all $x \in \mathcal{M}$ with $\mathbb{E}_A(x) = 0$,
\[ \|\mathbb{E}_A(xv^n x^*)\|_2 \to 0 \text{ as } |n| \to \infty, \]
where $v \in A$ is a Haar unitary generator of $A$. Let $v \in A$ correspond to the function $s \mapsto e^{2\pi is}$. By Lemma 6.3
\[ \|\mathbb{E}_A(xv^n x^*)\|_2^2 = \int_0^1 |\tilde{\eta}^t_x(1 \otimes v^n)|^2 d\lambda(t). \]
From Lemma 5.7 [10] we know that $\eta_x \ll \eta$ and hence for $\lambda$ almost all $t$, $\eta^t_x \ll \eta^t$ from Lemma 3.6. So $\tilde{\eta}^t_x \ll \tilde{\eta}^t_x$ for $\lambda$ almost all $t$. Thus $\tilde{\eta}^t_x$ is mixing measure from Prop. 2.5 of [16] for $\lambda$ almost all $t$. Also from Lemma 6.1 the measures $\eta^t_x$ are finite for $\lambda$ almost all $t$. Use Lemma 6.1 and apply dominated convergence theorem to finish the proof. \qed

Appendix A. Structure of measurable functions

Making a measurable selection as we attempted in Lemma 5.12 is not enough. One likes to make a measurable selection so that the graph of the selection is an automorphism graph of the masa, the automorphism being implemented by an unitary in the factor. But this is a very delicate issue. We are not aware of such selection theorems. We can overcome this obstacle though. Structure theorems of continuous and measurable functions are what comes into play.

Definition A.1. Let $f : [0,1] \to \mathbb{R}$ be a function and $E$ be a subset of $[0,1]$. Then $f$ is said to satisfy condition (N) or null condition of Lusin relative to $E$ if $f(A)$ is a set of measure $0$ whenever $A \subset E$ is a set of measure $0$.

The definition implicitly assumes that there are two measures on $[0,1]$ and $\mathbb{R}$. For our purpose these measures will always be the Lebesgue measure, which we will denote by $\lambda$.

Proposition A.2. (Tietze’s Extension Type) Let $E \subseteq [0,1]$ be closed and let $f : E \to [0,1]$ be a continuous function that satisfy the property that for a measurable set $A \subset E$, $\lambda(A) = 0$ if and only if $\lambda(f(A)) = 0$. Then there exists a continuous function $F : [0,1] \to [0,1]$ such that
(i) $F|_E = f$,
(ii) $F$ satisfies the property that for a measurable set $A \subset [0,1]$, $\lambda(A) = 0$ if and only if $\lambda(F(A)) = 0$.

Proof. Since $E$ is closed it is a compact subset of $[0,1]$. Therefore $E$ has greatest and least members $m$ and $M$ respectively. If $m \neq 0$ or $M \neq 1$ then extend $f$ to a function $h$ on $E_1 = E \cup \{0\} \cup \{1\}$ by assigning the values $f(m)$ and $f(M)$ at the points $0$ and $1$ respectively. The function $h$ is continuous on $E_1$ and satisfies the same condition as $f$ relative to $E_1$. So without loss of generality we can assume $0,1 \in E$.

The complement of $E$ is a open set in $[0,1]$ and $E^c \subset (0,1)$. Then $E^c$ can be written as a countable disjoint union of intervals $\bigcup_{i=1}^{\infty} (a_i, b_i)$. Then note that $a_i, b_i \in E$ for all $i$. 

So we only have to define an extension on \((a_i, b_i)\). Define

\[
F(x) = \begin{cases} 
  f(x) & \text{if } x \in E, \\
  (1 - \lambda)f(x) + \lambda f(a_i) + (1 - \lambda)f(b_i) & \text{if } x = \lambda a_i + (1 - \lambda)b_i \in (a_i, b_i), \\
  \frac{2(1-f(a_i))}{b_i-a_i}(x-a_i) + f(a_i) & \text{if } a_i < x \leq \frac{a_i + b_i}{2} \text{ and } f(a_i) = f(b_i) < 1, \\
  \frac{2(1-f(b_i))}{a_i-b_i}(x-b_i) + f(b_i) & \text{if } \frac{a_i + b_i}{2} \leq x < b_i \text{ and } f(a_i) = f(b_i) < 1, \\
  \frac{2(x-a_i)}{a_i-b_i} + 1 & \text{if } a_i < x < b_i \text{ and } f(a_i) = f(b_i) = 1, \\
  \frac{2(b_i-x)}{b_i-a_i} + 1 & \text{if } a_i < x < b_i \text{ and } f(a_i) = f(b_i) = 1. 
\end{cases}
\]

The function \(F\) is now continuous, as it is a linear interpolation obtained from \(f\) and the construction satisfy the required conditions. \(\square\)

**Theorem A.3.** (Foran, [13]) A necessary and sufficient condition for a continuous function \(F : [0, 1] \mapsto [0, 1]\) to satisfy condition (N) relative to \([0, 1]\) is that there exists a sequence of measurable sets \(E_n \subseteq [0, 1], n = 0, 1, \ldots\), such that the following properties are true:

(i) \([0, 1] = \bigcup_{n=0}^{\infty} E_n\),

(ii) \(\lambda(F(E_n)) \leq n\lambda(E_n)\) for all \(n \geq 0\),

(iii) for each \(n > 0\), \(F\) is one to one on \(E_n\).

**Proposition A.4.** Let \(F : [0, 1] \mapsto [0, 1]\) be a measurable function such that for any measurable set \(A \subseteq [0, 1], \lambda(A) = 0\) if and only if \(\lambda(F(A)) = 0\). Then there exists a measurable set \(E \subseteq [0, 1]\) such that \(\lambda(E) > 0\) and \(F\) is one to one on \(E\).

Moreover, if \(Y_0 \subseteq [0, 1]\) is such that \(\lambda(Y_0) > 0\), then there exists \(Y_1 \subseteq Y_0\) with \(\lambda(Y_1) > 0\) such that \(F\) is one to one on \(Y_1\).

**Proof.** Let \(\epsilon > 0\). By Lusin’s theorem, choose a closed set \(H \subseteq [0, 1]\) such that \(\lambda([0, 1] \setminus H) < \epsilon\) and \(F\) is continuous relative to \(H\). Clearly, \(F|_H\) satisfies the property that \(A \subseteq H\), \(\lambda(A) = 0\) if and only if \(\lambda(F(A)) = 0\). By Prop. A.2 extend \(F\) to a continuous function \(\tilde{F} : [0, 1] \mapsto [0, 1]\) such that \(\tilde{F}\) has the property that for \(A \subseteq [0, 1]\), \(\lambda(A) = 0\) if and only if \(\lambda(\tilde{F}(A)) = 0\).

Now by Thm. A.3 choose measurable subsets \(E_n \subseteq [0, 1]\) such that \([0, 1] = \bigcup_{n=0}^{\infty} E_n\), \(\lambda(\tilde{F}(E_n)) \leq n\lambda(E_n)\) for all \(n = 0, 1, \ldots\), and for each \(n > 0\), \(\tilde{F}\) is one to one on \(E_n\). Since \(\lambda(\tilde{F}(E_0)) = 0\) so \(\lambda(E_0) = 0\). If \(\lambda(E_n \cap H) = 0\) for all \(n > 0\) then \(\lambda(H) = 0\), which is not the case. Therefore there is a \(n_0 > 0\) such that \(\lambda(E_{n_0} \cap H) > 0\). But \(\tilde{F}|_{E_{n_0} \cap H} = F|_{E_{n_0} \cap H}\) and clearly \(F\) is one to one on \(E_{n_0} \cap H\). Rename \(E = E_{n_0} \cap H\).

Suppose \(\lambda(Y_0) > 0\). By choosing \(\epsilon > 0\) small enough one can make sure that the closed set \(H\) in the first part of the proof satisfies \(\lambda(Y_0 \cap H) > 0\). The same argument as the first part applies, and there exists a \(n_0 > 0\) such that \(F\) is one to one on \(Y_1 = Y_0 \cap H \cap E_{n_0}\) and \(\lambda(Y_1) > 0\). \(\square\)

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