Remarks on Renormalization of Black Hole Entropy

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Abstract

We elaborate the renormalization process of entropy of a nonextremal and an extremal Reissner-Nordström black hole by using the Pauli-Villars regularization method, in which the regulator fields obey either the Bose-Einstein or Fermi-Dirac distribution depending on their spin-statistics. The black hole entropy involves only two renormalization constants. We also discuss the entropy and temperature of the extremal black hole.

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Bekenstein [1] and Hawking [2] proposed an intrinsic entropy and temperature of black holes. The black hole entropy and temperature are purely geometric in that the former is proportional to the area of event horizon of each black hole and the latter is determined by the surface gravity at the event horizon. Entropy is, however, a thermodynamic quantity that is usually defined in terms of the classical or quantum mechanical number of states available to a system under consideration. The geometric nature of black hole entropy has thus raised the fundamental questions on the origin of black hole entropy over the last twenty years.

There have been various attempts to explain the black hole entropy thermodynamically. 't Hooft calculated the entropy of a scalar field in a black hole background and found that with a particular brick wall just outside the event horizon the thermodynamic entropy of the scalar field gives the correct geometric black hole entropy [3]. Quite recently Demers, Lafrance, and Myers (DLM) introduced a Pauli-Villars method to regularize the ultraviolet divergences of a Reissner-Nordström (RN) black hole entropy [4]. In the Pauli-Villars covariant regularization method, one introduces bosonic and fermionic regulator fields to regularize the divergences. The strong point of the method of Ref. [4] is that it does not necessitate a brick wall to avoid possible ultraviolet divergences and yields a one-loop renormalized entropy. They obtained the thermodynamic entropy of both the nonextremal and extremal black holes which involves only two renormalization constants. It was further shown that these two constants also appeared in the one-loop effective action of the scalar field coupled to gravity, and can be absorbed into the renormalization of the gravitational constant $G$ and the coupling constants of higher order curvatures in the effective action. The renormalization of black hole entropy in terms of the gravitational constant and the coupling constants was first discussed in [5] and also discussed in [6–11].

As usual, DLM used somewhat ad hoc arbitrary spin-statistics, the Bose-Einstein distribution, even for the fermionic regulator fields. In this paper we show that with the right
spin-statistics for the regulator fields the Pauli-Villars covariant regularization method can still work to yield a one-loop renormalized black hole entropy. We explicitly calculate the thermodynamic entropy of both the nonextremal and extremal RN black holes. The thermodynamic entropy computed with this method is shown to be of the same form as those in [4], but the two renormalization constants involved differ from those in [4].

Throughout this paper we adopt the units, $c = G = \hbar = k = 1$. The spacetime signature is $(-, +, +, +)$.

II. THE ENTROPY OF REISSNER-NORDSTRÖM BLACK HOLE

The thermodynamic entropy of a massive scalar field in an RN black hole background will be studied in this section. We shall follow mostly the method used by 't Hooft [3]. The RN black hole has the metric

$$ds^2 = - \left(1 - \frac{r_-}{r}\right) \left(1 - \frac{r_+}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_-}{r}\right) \left(1 - \frac{r_+}{r}\right)} + r^2 d\Omega^2,$$

(1)

where $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$, and $r_+ > r_-$ for the nonextremal RN black hole and $r_+ = r_-$ for the extremal RN black hole. The limiting case of $r_- = 0$ and $r_+ = 2M$ corresponds to the Schwarzschild black hole.

We consider a massive scalar field minimally coupled to the RN black hole background. The scalar field satisfies the (quantum) Klein-Gordon equation

$$\left[ -\frac{1}{(1 - \frac{r_-}{r})(1 - \frac{r_+}{r})} \frac{\partial^2}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left((r - r_-)(r - r_+)\frac{\partial}{\partial r}\right) + \frac{1}{r^2} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{\partial^2}{\partial \phi^2}\right) - m^2 \right] \phi(x) = 0.$$

(2)

Since the RN black hole has a timelike Killing vector field and the spherical symmetry, we expand the wave function by spherical harmonics, $\phi(x) = e^{-iEt} Y_{lm}(\theta, \phi) f(r)$. The remaining radial wave function of the Klein-Gordon equation satisfies the one-dimensional Schrödinger-like equation

$$\left[ \frac{1}{r^2} \frac{d}{dr} \left((r - r_-)(r - r_+)\frac{d}{dr}\right) + \frac{r^2 E^2}{(r - r_-)(r - r_+)} - \frac{l(l + 1)}{r^2} - m^2 \right] f(r) = 0.$$

(3)
The radial momentum, \( p_r = \frac{dS}{dr} \), can be found from the WKB wave function, \( f(r) = e^{iS(r)} \),

\[
p_r^2 = \frac{1}{(1 - \frac{r_-}{r})^2 (1 - \frac{r_+}{r})^2} \left[ E^2 - \left( 1 - \frac{r_-}{r} \right) \left( 1 - \frac{r_+}{r} \right) \left( \frac{l(l+1)}{r^2} + m^2 \right) \right].
\]

(4)

The scalar field suffers an infinite gravitational redshift when it passes from the event horizon to infinity.

We find the Helmholtz free energy from the number of states available and the statistical distribution. For a given \( E \) we count the action in the unit of Planck constant\(^1\) in order to get the number of states summed over the angular momentum states

\[
g(E) = \frac{1}{2\pi} \int_0^{\infty} dl (2l + 1) \int dr \frac{1}{(1 - \frac{r_-}{r}) (1 - \frac{r_+}{r})} \times \left[ E^2 - \left( 1 - \frac{r_-}{r} \right) \left( 1 - \frac{r_+}{r} \right) \left( \frac{l(l+1)}{r^2} + m^2 \right) \right]^{1/2}.
\]

(5)

### A. Ultraviolet Divergence

The radial momentum goes to infinity at the event horizon \( r = r_+ \) and it causes an ultraviolet divergence. To avoid the ultraviolet divergences ‘t Hooft introduced a brick wall of thickness \( h \) just outside the event horizon to regularize the divergences. The number of states after performing the angular momentum integration amounts to

\[
g(E) = \frac{1}{3\pi} \int_{r_+ + h} dr \frac{r^2}{(1 - \frac{r_-}{r}) (1 - \frac{r_+}{r})^2} \left[ E^2 - \left( 1 - \frac{r_-}{r} \right) \left( 1 - \frac{r_+}{r} \right) m^2 \right]^{3/2},
\]

(6)

and the Helmholtz free energy is

\[
F = \frac{1}{\beta} \sum_{N} \ln \left( 1 - e^{-\beta E_N} \right).
\]

(7)

For a continuous \( E \), we may take a continuum limit to compute the Helmholtz free energy

\(^1\)Here we adopted the convention of action \( \int dr p_r \) and the number of states \( \frac{1}{2\pi\hbar} \int dr p_r \) as in Ref. \cite{12}. 

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\[ F = - \int_0^\infty dE g(E) \frac{1}{e^{\beta E} - 1}. \]  

(8)

Substituting (5) into (8) and changing the variable to \( x = 1 - \frac{r_+}{r} \), we finally obtain

\[ F_{RN} = -\frac{r_+^3}{3\pi} \int_0^\infty dE \frac{1}{e^{\beta E} - 1} \int_\epsilon \frac{1}{x^2(1-x)^4(1-u+ux)^2} \left[ E^2 - m^2 x(1-u+ux) \right]^{3/2} \]  

(9)

where \( u = \frac{r_+}{r} \), \( \epsilon = \frac{h}{r_+ + \kappa} \), and \( u = 1 \) corresponds to the exremal RN black hole.

We focus on the possible ultraviolet divergences coming from the event horizon and expand the rational function part of integral (9) around \( x = 0 \). The Helmholtz free energy for the nonextremal black hole is then given by

\[ F_{n.ext} = -\frac{r_+^3}{3\pi} \sum_{n=0}^{\infty} C_n \int_0^\infty dE \frac{1}{e^{\beta E} - 1} \int_\epsilon dx x^{2+2n} \left[ E^2 - m^2 x(1-u+ux) \right]^{3/2} \]  

(10)

where

\[ C_0 = \frac{1}{(1-u)^2}, \quad C_1 = \frac{2(2-3u)}{(1-u)^3}, \]  

(11)

and for the extremal black hole by

\[ F_{ext} = -\frac{r_+^3}{3\pi} \sum_{n=0}^{\infty} D_n \int_0^\infty dE \frac{1}{e^{\beta E} - 1} \int_\epsilon dx x^{-4+n} \left[ E^2 - m^2 x^2 \right]^{3/2}, \]  

(12)

where

\[ D_0 = 1, \quad D_n = \frac{4 \cdot 5 \cdots (3+n)}{n!}, (n \geq 1). \]  

(13)

We show explicitly how the ultraviolet divergent terms in the nonextremal black hole emerge by performing integration of the first two terms in (10). By keeping only the divergent terms (13), we have

\[ F_{n.ext} \sim -\frac{r_+^3}{3\pi} \left[ \frac{\pi^4}{15(1-u)^2} \frac{1}{\beta^4} - \frac{\pi^2 m^2}{8(1-u) \beta^2} \ln(\epsilon^2) - \frac{\pi^4(2-3u)}{15(1-u)^3 \beta^4} \ln(\epsilon^2) \right], \]  

(14)

where we used the integral formula, for \( n = 0, 1 \) of (10),

\[ \int dx x E^2 - m^2 x(1-u+ux) = \frac{1}{E} \ln \left( \frac{\sqrt{E + a_+ x} - \sqrt{E + a_- x}}{\sqrt{E + a_+ x} + \sqrt{E + a_- x}} \right), \]  

\[ a_\pm = \frac{-(1-u)m^2 \pm \sqrt{(1-u)^2 m^4 + 4 u m^2 E^2}}{2E}. \]  

(15)
We note that $F_{\text{ext}}$ contains only a linearly divergent term $\frac{E^3}{\epsilon}$ and logarithmically divergent terms $E^3 \ln(\epsilon^2)$ and $Em^2 \ln(\epsilon^2)$ as $\epsilon \to 0$. The Helmholtz free energy of the scalar field thus diverges as the brick wall thickness $h$ (here $\epsilon$) collapses to zero at the event horizon.

The ultraviolet divergent terms in the extremal black hole similarly come from the first four terms in (12). Keeping only the divergent terms we get

$$F_{\text{ext}} \sim -\frac{r^3}{3\pi} \left[ \frac{\pi^4}{45\beta^4} + \frac{2\pi^4}{15\beta^4} \frac{1}{\epsilon^2} + \frac{2\pi^4}{3\beta^4} \frac{1}{\epsilon} - \frac{\pi^2 m^2}{6\beta^2} \frac{1}{\epsilon} - \frac{2\pi^4}{3\beta^4} \ln(\epsilon^2) + \frac{\pi^2 m^2}{2\beta^2} \ln(\epsilon^2) \right],$$

where the use has been made of the following integral formula (13), for $n = 1, 3$ of (12)

$$\int \frac{dx}{x\sqrt{E^2 - m^2 x^2}} = \frac{1}{2E} \ln \left( \frac{E - \sqrt{E^2 - m^2 x^2}}{E + \sqrt{E^2 - m^2 x^2}} \right).$$

The divergent terms are $\frac{E^3}{\epsilon^3}$, $\frac{E^3}{\epsilon^2}$, $\frac{E^3}{\epsilon}$, $\frac{Em^2}{\epsilon}$, $E^3 \ln(\epsilon^2)$, and $Em^2 \ln(\epsilon^2)$.

The main purpose of this paper is to see whether one may remove these divergent terms and may renormalize the free energy by introducing bosonic and fermionic regulator fields that obey the proper Bose-Einstein or Fermi-Dirac statistics. This point will be discussed in Sec. III.

**B. Brick Wall Model**

’t Hooft introduced a brick wall just outside the event horizon to cut off the ultraviolet divergence and suggested that the thermodynamic entropy of scalar fields might give the correct geometric entropy of black hole if a particular brick wall thickness is chosen. We reconsider his argument using the entropy of the nonextremal RN black hole including the logarithmically divergent terms.

First, the Schwarzschild black hole case is recovered from the limiting case of $r_- = 0$ and $r_+ = 2M$. Repeating the calculation, focusing only on the dominant divergent terms of the Helmholtz free energy, and using $S = \beta^2 \frac{\partial F}{\partial \beta}$, we find the thermodynamic entropy

$$S_{\text{Sch}} \sim \frac{r^3}{3\pi} \left[ \frac{4\pi^4}{15\beta^3} \frac{1}{\epsilon} + \frac{\pi^2 m^2}{2\beta} \ln(\epsilon) - \frac{16\pi^4}{15\beta^3} \ln(\epsilon) \right].$$

’t Hooft ascribed the black hole entropy only to the linearly divergent term and prescribed
\[ \epsilon = \frac{1}{180A}, \quad (19) \]

where \( A = 16\pi M^2 \) is the area of the event horizon of Schwarzschild black hole. He thus obtained \( S \sim \frac{A}{4} \). However, assigning \( \beta = 8\pi M \) one gets the on-shell black hole entropy

\[ S_{Sch} \sim \frac{1}{720} \frac{1}{\epsilon} + \frac{m^2 A}{96\pi} \ln(\epsilon) - \frac{1}{180} \ln(\epsilon). \quad (20) \]

It is to be noted that the second term of (18) gives rise to an entropy proportional to the area of the event horizon, so this term should not be neglected even for small \( \epsilon \). We should instead prescribe

\[ \epsilon \left( 1 - \frac{m^2}{24\pi} \ln(\epsilon) \right) = \frac{1 - 4\epsilon \ln(\epsilon)}{180A} \quad (21) \]

to get the black hole entropy

\[ S_{Sch} \sim \frac{A}{4}. \quad (22) \]

Similarly, we find the thermodynamic entropy of the nonextremal RN black hole

\[ S_{n.ext} \sim \frac{\pi^2}{3} \left[ \frac{4\pi^4}{15(1-u)^2} \frac{1}{\epsilon} + \frac{\pi^2 m^2}{2(1-u)\beta} \ln(\epsilon) - \frac{8\pi^4 (2-3u)}{15(1-u)^3 \beta^3} \ln(\epsilon) \right]. \quad (23) \]

The on-shell entropy, \( \beta = \frac{4\pi r_e}{1-u} \), is

\[ S_{n.ext} \sim \frac{1-u}{720} \frac{1}{\epsilon} + \frac{m^2 A}{96\pi} \ln(\epsilon) - \frac{2-3u}{360} \ln(\epsilon). \quad (24) \]

By prescribing the brick wall condition

\[ \epsilon \left( 1 - \frac{m^2}{24\pi} \ln(\epsilon) \right) = \frac{(1-u)(1-4\epsilon \ln(\epsilon))}{180A}, \quad (25) \]

we obtain the black hole entropy

\[ S_{n.ext} \sim \frac{A}{4}. \quad (26) \]

Thus we have seen that the same entropy formula is obtained by including the logarithmically divergent term that 't Hooft neglected. The contribution of logarithmically divergent term to the 2D black hole entropy was discussed in [3].
III. PAULI-VILLARS REGULARIZATION

It is our observation that by using the correct Bose-Einstein and Fermi-Dirac statistics for the regulator fields we can remove the same divergent terms for both the nonextremal and extremal RN black hole, because they contribute to the Helmholtz free energy with opposite signs, and the Bose-Einstein distribution

\[
\int_0^\infty dE \frac{E^{\nu-1}}{e^{\beta E} - 1} = \frac{\Gamma(\nu) \zeta(\nu)}{\beta^\nu}
\]

and the Fermi-Dirac distribution

\[
\int_0^\infty dE \frac{E^{\nu-1}}{e^{\beta E} + 1} = \left(1 - 2^{1-\nu}\right) \frac{\Gamma(\nu) \zeta(\nu)}{\beta^\nu}
\]

differ by only a constant factor. This suggests that in order to remove the ultraviolet divergences we may introduce bosonic and fermionic regulator fields that satisfy the Klein-Gordon equation (2) with arbitrarily large masses \(m_i\). The \(m_i\) will be determined later from the mass conditions that make all the ultraviolet divergent terms cancel each other.

Repeating the same calculation as for the bosonic massive scalar field, we get the number of states for each regulator field

\[
g_i(E) = \frac{1}{3\pi} \int_{r_{+}+h} dE \frac{r^2}{(1 - \frac{r_{+}}{r})^2 (1 - \frac{r_{+}}{r})^2} \left[ E^2 - \left(1 - \frac{r_{+}}{r}\right) \left(1 - \frac{r_{+}}{r}\right) m_i^2 \right]^{3/2},
\]

where \(i\) labels the species of regulator fields. Similarly, we obtain the Helmholtz free energy contribution of the \(i\)th regulator field

\[
F_i = \frac{1}{\beta} \sum_N \ln \left(1 \mp e^{-\beta E_N}\right),
\]

and for continuous \(E\)

\[
F_i = \frac{\pi^3}{3\pi} \int_0^\infty dE \frac{1}{e^{\beta E} + 1} \int dx \frac{1}{x^2 (1 - x)^4 (1 - u + ux)^2} \left[ E^2 - x(1 - u + ux) m_i^2 \right]^{3/2},
\]

where the upper (-) sign is used for bosonic fields and the lower (+) sign for fermionic fields, and we used the same variables and parameters as in Sec. II. As before, \(u = 1\) corresponds to the extremal black hole case.
A. Nonextremal RN Black Hole

The ultraviolet divergent terms of each regulator fields in the nonextremal RN black hole come from the singular behavior of the Helmholtz free energy (31) near $x = 0$. We compute the Helmholtz free energy around $x = 0$:

$$F_i = \mp \frac{r_+^3}{3\pi} \sum_{n=0}^{\infty} C_n \int_0^\infty dE \frac{1}{e^{\beta E} + 1} \int dx x^{-2+n} \left[ E^2 - m_i^2 x(1 - u + ux) \right]^{3/2}$$

(32)

where

$$C_0 = \frac{1}{(1 - u)^2}, \quad C_1 = \frac{2(2 - 3u)}{(1 - u)^3}. \quad (33)$$

Regardless of spin-statistics of regulator fields, the free energy contains a linearly divergent term $E_\epsilon^3$ and logarithmically divergent terms $E_\epsilon^3 \ln(\epsilon^2)$ and $E m_i^2 \ln(\epsilon^2)$ as $\epsilon \to 0$ as shown in Sec. II. We may remove these divergent terms by introducing the correct bosonic and fermionic regulator fields that contribute to the free energy (31) with opposite signs, and thereby regularize the free energy and entropy.

The linearly divergent contribution to the free energy is

$$-\frac{1}{\epsilon} \left( N_B - \frac{7}{8} N_F \right) \frac{\pi r_+^3}{45(1 - u)^2 \beta^4}, \quad (34)$$

where $N_B$ and $N_F$ are the number of the bosonic and fermionic fields, respectively. We can remove this term by introducing $7k$ ($k$ a positive integer) bosonic fields (including the original scalar field) and $8k$ fermionic regulator fields. In this paper we shall take the minimum number of regulator fields, that is, 7 bosonic and 8 fermionic fields.

Next, we consider the logarithmically divergent contributions

$$\left( N_B - \frac{7}{8} N_F \right) \frac{\pi^3(2 - 3u)r_+^3}{45(1 - u)^2 \beta^4} \ln(\epsilon^2),$$

$$- \left( \sum_B m_i^2 - \frac{1}{2} \sum_F m_i^2 \right) \frac{\pi r_+^3}{24(1 - u) \beta^2} \ln(\epsilon^2).$$

(35)

If we further impose a condition on the masses of the regulator fields

$$\sum_B m_i^2 - \frac{1}{2} \sum_F m_i^2 = 0, \quad (36)$$
then both the linearly divergent term and logarithmically divergent terms are made vanish. So we may remove the brick wall, $\epsilon = 0$, under the condition (36), and instead regularize the free energy by the large masses, $m_i$, of regulator fields.

We find the Helmholtz free energy in the limit $\epsilon = 0$. For this purpose we expand the right hand side of the integral (15), use the mass condition (36) to remove $\ln(\epsilon^2)$, and obtain the only nonvanishing integral

$$\int_0^\infty dx \frac{1}{x \sqrt{E^2 - x(1 - u + ux)m^2}} = \frac{1}{2E} \ln \left( \frac{(1 - u)^2 m_i^4 + 4um_i^2 E^2}{16E^4} \right). \tag{37}$$

Then the Helmholtz free energy is given by

$$F_{n,\text{ext}} = -\frac{r^3}{3\pi} \left[ \frac{3}{4(1 - u)} \sum_B m_i^2 \int_0^\infty dE \frac{E \ln \left( \frac{(1-u)^2 m_i^4 + 4um_i^2 E^2}{16E^4} \right)}{e^{\beta E} - 1} \right.$$  

$$- \frac{3}{4(1 - u)} \sum_F m_i^2 \int_0^\infty dE \frac{E \ln \left( \frac{(1-u)^2 m_i^4 + 4um_i^2 E^2}{16E^4} \right)}{e^{\beta E} + 1}$$

$$- \frac{2 - 3u}{(1 - u)^3} \sum_B \int_0^\infty dE \frac{E^3 \ln \left( \frac{(1-u)^2 m_i^4 + 4um_i^2 E^2}{16E^4} \right)}{e^{\beta E} - 1}$$

$$+ \frac{2 - 3u}{(1 - u)^3} \sum_F \int_0^\infty dE \frac{E^3 \ln \left( \frac{(1-u)^2 m_i^4 + 4um_i^2 E^2}{16E^4} \right)}{e^{\beta E} + 1} \right]. \tag{38}$$

For large masses of regulator fields, we expand the logarithmic terms in (38) in the inverse power of masses. The leading terms are $\frac{1}{E} \ln(m_i^2 - 2 \ln(E))$. Keeping only the surviving divergent contributions to the Helmholtz free energy as the regulator masses go to infinity we obtain

$$F_{n,\text{ext}} = -\frac{\pi r^3 B}{12(1 - u) \beta^2} - \frac{2\pi^3 r^3 (2 - 3u)A}{45(1 - u)^3 \beta^4}, \tag{39}$$

where

$$A = -\sum_B \ln(m_i^2) + \frac{7}{8} \sum_F \ln(m_i^2), \tag{40}$$

$$B = \sum_B m_i^2 \ln(m_i^2) - \frac{1}{2} \sum_F m_i^2 \ln(m_i^2)$$

$$- \frac{12}{\pi^2} \sum_B m_i^2 \int_0^\infty dt \frac{t \ln(t)}{e^t - 1} + \frac{12}{\pi^2} \sum_F m_i^2 \int_0^\infty dt \frac{t \ln(t)}{e^t + 1}. \tag{41}$$
From the inequality \( t > \ln(t) \geq -\frac{1}{e \alpha t} \) for \( 1 > \alpha > 0 \), we may find the bound for the integrals of (41):

\[
\Gamma(3) \zeta(3) > \int_0^\infty \frac{dt}{e^t - 1} \geq -\frac{\Gamma(2 - \alpha) \zeta(2 - \alpha)}{e \alpha},
\]

\[
\frac{3}{4} \Gamma(3) \zeta(3) > \int_0^\infty \frac{dt}{e^t + 1} \geq - \left( 1 - \frac{1}{2^{1 - \alpha}} \right) \frac{\Gamma(2 - \alpha) \zeta(2 - \alpha)}{e \alpha}.
\] (42)

The black hole entropy is then

\[
S_{n,\text{ext}} = \frac{\pi r_+^3 A}{6(1 - u) \beta} + \frac{8 \pi^3 r_+^3 (2 - 3u) A}{45(1 - u)^3 \beta^3}.
\] (43)

Note that if we had used the same spin-statistics even for the bosonic and fermionic regulator fields, the constants \( A \) and \( B \) would become

\[
A = - \sum_B \ln(m_i^2) + \sum_F \ln(m_i^2), \quad B = \sum_B m_i^2 \ln(m_i^2) - \sum_F m_i^2 \ln(m_i^2)
\] (44)

for three bosonic and fermionic fields, respectively, which are the same as those in [4].

The entropy obtained so far is off-shell and we substitute the Hawking temperature, \( \beta = \frac{4 \pi r_+}{1 - u} \), of the nonextremal RN black hole to get the on-shell entropy

\[
S_{n,\text{ext}} = \frac{B}{24 \pi} \frac{A}{4} + \frac{(2 - 3u) A}{180},
\] (45)

where \( A = 4 \pi r_+^2 \) is the surface area of the event horizon. It should be remarked that we obtained the exactly same form of entropy as in [4], but with different renormalization constants \( A \) and \( B \). \( A \) and \( B \) might be related with the renormalization of \( G \) and the coefficients of the effective action [5].

**B. Extremal RN Black Hole**

Now we turn to the exremal RN black hole, the case with \( u = 1 \). The Helmholtz free energy contribution of the \( i \)th regulator field is

\[
F_i = \frac{r_+^3}{3 \pi h} \int_0^\infty dE \frac{1}{e^{\beta E} + 1} \left[ E^2 - m_i^2 x^2 \right]^{3/2}.
\] (46)
We expand \( \frac{1}{(1-x)^4} \) around \( x = 0 \), and perform the integration

\[
F_i = \mp \frac{r_+^3}{3\pi\hbar} \sum_{n=0}^{\infty} D_n \int_0^{\infty} dE \frac{1}{e^{\beta E} + 1} \int_\epsilon dx x^{-4+n} \left[ E^2 - m_i^2 x^2 \right]^{3/2}.
\] (47)

The divergent terms are \( \frac{E_3}{\epsilon^2} \), \( \frac{E m_i^2}{\epsilon^2} \), \( E^3 \ln(\epsilon) \), and \( E m_i^2 \ln(\epsilon) \). Taking the correct spin-statistics of the bosonic and fermionic regulator fields into account, the ultraviolet divergences can be removed with the same condition on the number of regulator fields and the mass condition (36) as in the nonextremal RN black hole. We remove again the brick wall outside the event horizon, and let \( \epsilon = 0 \). Then the nonvanishing contribution to the free energy from the lower limit of integration is

\[
F_{\text{ext}} = - \frac{\pi r_+^3 B}{6\beta^2} - \frac{2\pi^3 r_+^3 A}{9\beta^4},
\] (48)

and the entropy is found to be

\[
S_{\text{ext}} = \frac{\pi r_+^3 B}{3\beta} + \frac{8\pi^3 r_+^3 A}{9\beta^3}.
\] (49)

The black hole entropy (49) has also the same form as in [4], but differs only by the renormalization constants \( A \) and \( B \).

IV. CONCLUSION AND DISCUSSION

In this paper, we obtained the thermodynamic entropy of the RN black holes using the Pauli-Villars regularization method. The primary difference of our method from that of DLM [4] is that we used the Bose-Einstein statistics for the bosonic regulator fields and the Fermi-Dirac statistics for the fermionic regulator fields, whereas they used the same Bose-Einstein statistics for both the bosonic and fermionic regulator fields. We confirm that the thermodynamic entropy of either the nonextremal or extremal RN black hole involves only two constants \( A \) and \( B \) as in [4]. But the regulator masses satisfy the mass condition (36) which is different from that in [4] due to the different spin-statistics of regulator fields. The renormalization constants of the thermodynamic entropy might be related with the renormalization of the gravitational constant \( G \) and coupling constants of higher order curvatures of the effective action.
The thermodynamic entropy of the extremal black hole has the same form as that of the nonextremal black hole. There still remains the problem of defining the temperature and the entropy. From our result we can infer several possibilities. First, as argued in many literatures the entropy in (49) goes to zero as the temperature approaches to zero. This fact is consistent with the argument in many literatures that both temperature and entropy of the extremal black hole must vanish. The second possibility is that the temperature is still inversely proportional to $r_+$ as for the Schwarzschild or nonextremal RN black holes and the area rule of black hole entropy remains valid. The third possibility is that the temperature is arbitrary [14] but it is related to the entropy as (49).

In [4] it was shown that the renormalization constants appearing in the entropy coincide with those from the coupling constants of the higher order curvatures in the one-loop effective action. It would be interesting to find the way to relate these renormalization constants with the correct spin-statistics for the regulator fields.

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