Best Response Computation in Multiplayer Imperfect-Information Stochastic Games

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Abstract

Computing a best response is a fundamental task in game theory. One of its uses is to compute the degree of approximation error of an approximation of Nash equilibrium strategies. In many game classes best responses can be computed in polynomial time, but in imperfect-information stochastic games it is equivalent to solving a POMDP and is PSPACE-complete. Prior work has developed an algorithm for computing an approximation of Nash equilibrium strategies in a 4-player imperfect-information naval strategic planning problem which is modeled as a stochastic game. A heuristic approach was developed for computing best responses to determine the approximation error of these strategies, which was shown to have limited scalability. In this paper we present approaches that utilize parallelism to significantly speed up this computation allowing us to compute best responses in significantly larger games.

Introduction

Computing a best response is a fundamental task in games. Typically it can be performed efficiently (in polynomial time), while computing a Nash equilibrium (the standard game-theoretic solution concept) is PPAD-hard for games with more than two players and two-player non-zero-sum games. Since strategies are fixed for the opposing players computing a best response is just a single agent optimization problem, and is often significantly easier than computing solution concepts that involve reasoning about multiple players’ strategies. Computing a best response is important for three main reasons. First, if we are able to formulate a strategy for the opposing players (e.g., by using an opponent modeling algorithm), then computing a best response would allow us to obtain the highest possible expected payoff against this predicted model. Second, several algorithms for computing Nash equilibrium, such as fictitious play (Brown 1951), require the computation of best responses as a subroutine. And third, when an approximation of Nash equilibrium strategies has been computed in a large or complex game, computing a best response for each player allows us to quantify the degree of approximation error (ε). The value of ε denotes the largest amount that a player can gain by deviating from the strategy profile. In exact Nash equilibrium ε = 0, and so naturally our goal is to produce strategies with as small value of ε as possible. We say that the computed strategies constitute an ε-equilibrium. Formally, for a given candidate (mixed) strategy profile σ∗, define

$$\epsilon(\sigma^*) = \max_{i \in N} \max_{s_i \in S_i} \left[ u_i(s_i, \sigma^*_{-i}) - u_i(\sigma^*_i, \sigma^*_{-i}) \right].$$

Here N denotes the set of players, Si denotes the set of pure strategies for player i ∈ N, and ui denotes the utility function for player i. σ∗ is the component of σ∗ correspond to player i’s strategy, and σ∗i is the component of σ∗ corresponding to the strategies of the players excluding player i.

In a normal-form game (with any number of players) it is clear that a best response computation can be performed in polynomial time. The same is true for extensive-form games of sequential actions with either perfect or imperfect information. While extensive-form game trees can be used to model sequential actions of a known duration (e.g., repeating a simultaneous-move game for a specified number of iterations), they cannot model games of unknown duration, which can potentially contain infinite cycles between states. Such games must be modeled as stochastic games.

Definition 1. A stochastic game (aka Markov game) is a tuple (Q, N, A, P, r), where:

- Q is a finite set of (stage) games (aka game states)
- N is a finite set of n players
- A = A1 × ... × An, where Ai is a finite set of actions available to player i
- P : Q × A × Q → [0, 1] is the transition probability function; P(q, a, q′) is the probability of transitioning from state q to state q′ after action profile a
- R = r1, ..., rn, where ri : Q × A → R is a real-valued payoff function for player i

If strategies are fixed for all players excluding player i, then the best response problem for player i in a stochastic game is equivalent to solving a Markov decision process, and can be done in polynomial time. However, if the stochastic game has imperfect information (i.e., agents have private information that is not known to other agents), then computing a best response is equivalent to solving a partially observable Markov decision process (POMDP), and is PSPACE-complete. So there is no hope of finding an efficient algorithm for this problem in general; but there may...
be specialized algorithms that can efficiently solve specific problems of interest that may have a special structure.

Recently an algorithm has been developed for computing Nash equilibrium strategies in multiplayer stochastic games with imperfect information where the game states form a directed acyclic graph (Ganzfried 2021). This algorithm was used to approximate Nash equilibrium strategies in a 4-player imperfect-information naval strategic planning scenario that was constructed in consultation with a domain expert. In order to evaluate the quality of the computed strategy profile $\sigma^*$ (i.e., the $\epsilon$), we must compute a best response for each player to $\sigma^*$. An algorithm for doing this was developed that recursively calls itself for updated belief states corresponding to new states that can be transitioned to with horizon $t - 1$ (where we are interested in a time horizon of $t$ for the initial state). Unfortunately this algorithm was unable to converge in the initial version of the game, which had $K = 300$ non-terminal states, even using several heuristics for improved speed. However, the algorithm was able to solve a smaller version with $K = 150$ with use of these heuristics.

There have been recent applications of game-theoretic algorithms to important problems in national security. These game models and algorithms have differing levels of complexity. Typically these games have two players, and algorithms compute a Stackelberg equilibrium for a model where the “defender” acts as the leader and the “attacker” as the follower; the goal is to compute an optimal mixed strategy to commit to for the defender, assuming that the attacker will play a best response. Computing Stackelberg equilibrium is easier than Nash equilibrium; for example, for two-player normal-form general-sum games, optimal Stackelberg strategies can be computed in polynomial time (Conitzer and Sandholm 2006), while computing Nash equilibrium is PPAD-hard, and widely conjectured that no polynomial-time algorithms exist (Chen and Deng 2006; Daskalakis, Goldberg, and Papadimitriou 2009). Many realistic problems in national security involve more than two agents, sequential actions, imperfect information, probabilistic events, and/or repeated interactions of unknown duration. Several stochastic game models have been previously proposed for national security settings. For example, two-player discounted models of adversarial patrolling have been considered, for which mixed-integer program formulations are solved to compute a Markov stationary Stackelberg equilibrium (Vorobeychik and Singh 2012; Vorobeychik et al. 2014). One work has applied an approach to approximate a correlated equilibrium in a three-player threat prediction game model (Chen et al. 2006). However we are not aware of other prior research on settings with more than two players with guarantees on solution quality (or for computing Nash as opposed to Stackelberg or correlated equilibrium) for either perfect or imperfect information other than the very recent work that we build on (Ganzfried, Laughlin, and Morefield 2020; Ganzfried 2021).

**Imperfect-information naval strategic planning problem**

We will first review the perfect-information naval strategic planning problem that has been previously studied (Ganzfried, Laughlin, and Morefield 2020), and then describe the differences for the imperfect-information version. The game is based on a freedom of navigation scenario in the South China Sea where a set of blue players attempts to navigate freely, while a set of red players attempt to obstruct this from occurring (Figure 1). In our model there is a single blue player and several different red players which have different capabilities (we will specifically focus on the setting where there are three different red players). If a blue player and a subset of the red players happen to navigate to the same location, then a confrontation will ensue, which we call a Hostility Game.

![Figure 1: General figure for South China Sea scenario.](Image343x436 to 535x544)

In a Hostility Game, each player can initially select from a number of available actions (which is between 7 and 10 for each player). Certain actions for the blue player are countered by certain actions of each of the red players, while others are not (Figure 2). Depending on whether the selected actions constitute a counter, there is some probability that the blue player wins the confrontation, some probability that the red players win, and some probability that the game repeats. Furthermore, each action of each player has an associated hostility level. Initially the game starts in a state of zero hostility, and if it is repeated then the overall hostility level increases by the sum of the hostilities of the selected actions. If the overall hostility level reaches a certain threshold (300), then the game goes into kinetic mode and all players achieve a very low payoff (negative 200). If the game ends in a win for the blue player, then the blue player receives a payoff of 100 and the red players receive negative 100 (and vice versa for a red win). The game repeats until either the blue/red players win or the game enters kinetic mode. A subset of the game’s actions and parameters are given in Figure 3. Note that in our model we assume that all red players act independently and do not coordinate their actions. The game model and parameters were constructed from discussions with a domain expert.

**Definition 2.** A perfect-information hostility game is a tuple $G = (N, M, c, b^D, b^U, r^D, r^U, \pi, h, K, \pi^K)$, where

- $N$ is the set of players. For our initial model we will assume player 1 is a blue player and players 2–4 are red
The values of the type parameters affect the probabilities of each player’s success during a confrontation, with larger values leading to greater success probabilities. For example, suppose there is an encounter between a blue ship of type $t_b$ and red ship of type $t_r$, and suppose that blue player plays an action $a_b$ that is a counter-move to red’s action $a_r$. Then for the PIHG the probability of a blue success would be $p = b^{m_i}(a_b)$. In the IIHG we now have that the probability of a blue success will be $p' = p_t^{m_i}$. The other success/failure probabilities are computed analogously. Note that for $t_b = t_r$ we have $p' = p$ and the payoffs are the same as for the PIHG. If $t_r > t_b$ then $p' < p$, and similarly if $t_b < t_r$ then $p' > p$.

Prior research for approximating Nash equilibria in multiplayer imperfect-information stochastic games has focused on the case where the players’ private information is local and does not extend between game states (Ganzfried and Sandholm 2008; 2009). By contrast, the private information in the IIHG is persistent and extends throughout game play. It can also be observed that this problem has the special structure that the game states form a directed acyclic graph. This allowed them to devise a new algorithm that solves each game state sequentially, computes updated type distributions from the state equilibrium strategies at that state, and uses these updated type distributions for solving successive states within the current algorithm iteration. The best algorithm for approximating Nash equilibrium in this game class combines variants of fictitious play (Brown 1951) and policy iteration and is called Sequential Topological FIFP for Type-Dependent Values (ST-PiFP-TDV) (Ganzfried 2021). This was applied to approximate Nash equilibrium strategies in the IIHG.

Prior approach for best response computation

In order to evaluate the strategies computed from our algorithm, we need a procedure to compute the degree of Nash equilibrium approximation, $\epsilon$. For perfect-information stochastic games it turns out that there is a relatively straightforward approach for accomplishing this, based on the observation that the problem of computing a best response for a player is equivalent to solving a Markov decision process (MDP). We can construct and solve a corresponding MDP for each player, and compute the maximum that a player can obtain by deviating from our computed strategies for the initial state $G_0$. This approach is depicted in Algorithm 1, which applies a standard version of policy iteration (Puterman 2005). It turns out that Algorithm 1 can also be ap-
plied straightforwardly to stochastic games with local imperfect information. This algorithm was applied to compute the degree of Nash equilibrium approximation on the perfect-information hostility game (Ganzfried, Laughlin, and Morefield 2020) and a 3-player imperfect-information poker tournament (Ganzfried and Sandholm 2008; 2009).

Algorithm 1

Ex post check procedure

Create MDP $M$ from the strategy profile $s^*$

Run policy iteration on $M$ (using initial policy $s^*$) to get $\pi^*$

\[
\text{return } \max_{i \in N} \left[ v^1_i(s^1) \cdot \left( G_{0} \right) \right] - \left[ v^1_i(s^2) \cdot \left( G_{0} \right) \right]
\]

Unfortunately, Algorithm 1 cannot be applied to stochastic games with persistent imperfect information. For these games, computing the best response for each player is equivalent to solving a partially observable Markov decision processes (POMDP), which is significantly more challenging than solving an MDP. It turns out that computing the optimal policy for a finite-horizon POMDP is PSPACE-complete. The main algorithms are inefficient and typically require an amount of time that is exponential in the problem size (Cassandra, Kaelbling, and Littman 1994). Common approaches involve transforming the initial POMDP to an MDP with continuous (infinite) state space, where each state of the MDP corresponds to a belief state of the POMDP.

Due to the problem's intractability, a new procedure was devised for this setting that exploits domain-specific information to find optimal policies in the POMDPs which correspond to computing a best response. The algorithm is based on a recursive procedure, presented in Algorithm 2. The inputs to the procedure are a player $i$, a type $t_i$ for player $i$, a set of type distributions $\{\tau_j\}$ for the other players $j \neq i$, the strategies computed by our game-solving algorithm $\{s_j^*\}$, a game state $G_h$, and a time horizon $t$.

For the case where the prior type distribution is uniform (all values equal to $\frac{1}{4}$), which is what we use in our experi-
by $N_i$. Suppose we have $C^*$ cores to allocate between the entire subtree rooted at $v^*$. Let $b_{ij}$ be a binary variable that is 1 if core $j$ is used for node $v_i$. Let $Q_i$ denote the total number of cores that are allocated to node $i$. An integer program for determining the optimal allocation of cores to nodes is provided below. The first constraint ensures that the objective is to minimize the maximum amount of computation assigned to each core, where we assume that computation is divided equally between all nodes assigned to a given core. The second constraint ensures that we don’t assign more cores to a node than the number of nodes in its subtree. And the third constraint ensures that $Q_i$ is equal to the number of cores allocated to node $i$, as is its intended definition. Note that the first set of constraints involves division by the variables $Q_i$. We can define a new (continuous) variable $R_i$ by the constraint $R_i Q_i = 1$, and therefore the value of $R_i$ will equal $\frac{1}{Q_i}$, transforming the constraint into $u \geq \sum_{j=1}^{C^*} (N_i R_i b_{ij})$. This constraint now just involves the product of two variables, and is therefore quadratic. The overall formulation is now a mixed-integer quadratically-constrained program (QCP). Note that the first set of constraints is not convex, making the problem challenging to solve. Fortunately Gurobi has recently released an approach that is able to solve non-convex programs with quadratic objective and constraints (Gurobi Optimization 2019), which we can apply to our problem.

$$\begin{align*}
\min_{u} & \quad u \\
\text{s.t.} & \quad u \geq \sum_{i=1}^{C^*} N_i b_{ij} R_i \quad \text{for all } j \\
 & \quad Q_i \leq N_i \quad \text{for all } i \\
 & \quad \sum_{j=1}^{C^*} b_{ij} = Q_i \quad \text{for all } i
\end{align*}$$

We could apply this method recursively to determine the optimal allocation of cores to subtrees. Note that this approach assumes that we know in advance the values $N_i$ denoting the number of nodes in each subtree. If it is not possible to obtain these exactly, they could be approximated by repeatedly sampling from the tree.

Due to practical challenges of implementing this approach, we also consider a simpler approach for parallelizing Algorithm 2. When we are solving for player $i$ with type $t_i$ for the initial state $G_i$ with horizon $T$, we can assign the subtree corresponding to each initial action $a_i$ to a different core. This would require $|M_i|$ cores, and would result in a speedup by a factor of $|M_i|$ if all subtrees took the same amount of time.

Another improvement to Algorithm 2 can be made if we know the diameter of the tree in advance. Rather than computing $V_i$ for $t = 0, 1, 2, \ldots$ until it converges, we can just set $t = T^* - 1$, where $T^*$ is the diameter of the tree. Note also that the prior approach is not theoretically sound; it is possible that $V_i$ remains the same for several values of $t$ without the algorithm converging (for example, we may require $t = 50$ steps to reach a terminal state, and $V_i$ may be identical for $t = 15, 16, 17, \text{etc.}$).

To summarize, we can expect to obtain a speedup of approximately $\sum_{i=1}^{n} |T_i|$ by solving for all players and types in parallel, and a further speedup by a factor of $|M_i|$ if we assign a different core to each action at the initial node for
each of these computations. Assuming that all of these computations are roughly equal, this would result in a speedup of $\sum_{i=1}^{n} (|M_i||T_i|)$, or $n|T||M|$ if all $|T_i|$ equal $|T|$ and all $|M_i|$ equal $|M|$, assuming access to $\sum_{i=1}^{n} (|M_i||T_i|)$ cores.

Experiments

We ran experiments on the strategies computed by ST-PIF-P-TDV after 10 iterations for $K = 100$. We assume that $T_i = \{1, 2\}$ for all players $i$. For simplicity we compare the approaches for computing player 1’s best response with type 1 (note that the other calculations can be performed analogously). We first present results for Algorithm 2. Recall that the limitations of this algorithm are that it is sequential, and that it makes the assumption that it ignores transitions that have probability below some threshold $\delta$ (in prior experiments $\delta = 0.01$ was used). We computed that the diameter of the tree is $T^* = 13$, so we can just solve with $t = 12$.

In Table 1, we compare the runtimes of Algorithm 2 and the parallel approach that assigns different cores for each of the initial actions. Note that we are just considering player 1 with type 1 and $t = 12$, and that the parallel approach additionally benefits by solving for all $i, t_i$ in parallel. We are also ignoring the additional runtime of Algorithm 2 that would be due to iterating over different values of $t$. From the table, we can see that for $\delta = 0.00001$ we obtain a speedup by a factor of 4.4. (Note that $|M_2| = 9$.) The parallel approach is able to solve the problem exactly (using $\delta = 0$) in 26.55 hours, while the prior approach was not able to solve the problem within 45 hours. If we implemented the optimal parallelization based on the mixed-integer QCP described above we could obtain a significant further speedup.

| $\delta$  | Algorithm 2 runtime | Parallel runtime |
|----------|---------------------|-----------------|
| 0.01     | 3.247               | 1.306           |
| 0.001    | 5.183               | 2.211           |
| 0.0001   | 65.403              | 24.031          |
| 0.00001  | 14952.307           | 3457.727        |
| 0        | Did Not Finish      | 95589.746       |

Table 1: Running time in seconds for prior algorithm and new parallel algorithm for different values of $\delta$.

Conclusion

Computing a best response is a fundamental task in game theory. One important use is to evaluate the quality of Nash equilibrium approximations. While it can often be performed in polynomial time, in imperfect-information stochastic games it is equivalent to solving a POMDP, which is PSPACE-complete. We developed two new approaches for utilizing parallelization to improve the runtime of a prior approach; the first is an optimal approach based on a mixed-integer QCP, and the second is a simpler approach that parallelizes over actions taken at the initial game state. We show that the approaches lead to a significant improvement in runtime for a 4-player imperfect-information naval planning problem. We expect these approaches to be useful for significantly improving the runtimes for a variety of sequential-graph searching algorithms, including approaches for solving POMDPs.

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