Generalized Alignment Chain: Improved Converse Results for Index Coding

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Abstract—In this paper, we study the information-theoretic converse for the index coding problem. We generalize the definition for the alignment chain, introduced by Maleki et al., to capture more flexible relations among interfering messages at each receiver. Based on this, we derive improved converse results for the single-server index coding problem. Compared to those identified in the literature, related result where we identify a smaller single-server index coding problem. We also present a separate, but the generally tighter polymatroidal bound is computationally impractical. We then extend these new bounds to the multi-server index coding problem. We also present a separate, but related result where we identify a smaller single-server index coding instance, compared to those identified in the literature, for which non-Shannon-type inequalities are necessary to give a tighter converse.

I. INTRODUCTION

Index coding, introduced by Birk and Kol in [1], investigates the broadcast rate of $n$ messages from a server to multiple receivers with side information. Despite the substantial progress achieved during the past two decades, the index coding problem remains open in general. See [2] and the references therein.

In contrast to the single-server centralized index coding (CIC) problem, in the more general distributed index coding (DIC) problem, different subsets of the messages are stored at multiple servers. See [3]–[5] and the references therein.

In this paper, we study the information-theoretic converse for both CIC and DIC problems. For the CIC problem, the maximum acyclic induced subgraph (MAIS) bound was presented in [7], [8]. Both bounds have been extended to the DIC problem [4], [5]. The PM bound is generally tighter than the MAIS bound. However, it has a much higher computational complexity, which can be forbidding for large problems.

Therefore, it is of interest to find bounds that are strictly tighter than the MAIS bound, and at the same time, not as computationally intensive as the PM bound. The internal conflict bound for the CIC problem, introduced in [9], [10] based on the alignment chain model, can sometimes be useful. However, it does not subsume the MAIS bound in general.

In this paper, we first generalize the internal conflict bound for the CIC problem by extending the alignment chain model. We prove that the new converse results are no looser than the internal conflict bound and the MAIS bound. We show by examples that they can sometimes be strictly tighter. We then generalize these results to the DIC problem. We also present a separate result. That is, we identify a smaller CIC problem in terms of the number of messages, compared to those previously identified in [11], [2, Section 5.3], where Shannon-type inequalities are insufficient to give a tight converse.

Notation: For non-negative integers $a$ and $b$, $[a]$ denotes the set $\{1, 2, \ldots, a\}$, and $[a : b]$ denotes the set $\{a, a+1, \ldots, b\}$. If $a > b$, $[a : b] = \emptyset$. For a set $S$, $|S|$ denotes its cardinality.

II. SYSTEM MODEL

Assume that there are $n$ messages, $x_i \in \{0, 1\}^{w_i}, i \in [n]$. For brevity, when we say message $i$, we mean message $x_i$. Let $X_i$ be the random variable corresponding to $x_i$. We assume that $X_1, \ldots, X_n$ are independent and uniformly distributed. For any $S \subseteq [n]$, set $S^c = [n] \setminus S$, $x_S = \{x_i, i \in S\}$, and $X_S = \{X_i, i \in S\}$. By convention, $x_0 = X_0 = \emptyset$.

There are $n$ receivers, where receiver $i \in [n]$ wishes to obtain $x_i$ and knows $X_{A_i}$ as side information for some $A_i \subseteq [n] \setminus \{i\}$. The set of indices of interfering messages at receiver $i$ is denoted by the set $B_i = (A_i \cup \{i\})^c$.

To avoid redundancy, we describe the remaining system model for the DIC problem only, in which there are $2^n - 1$ servers, each containing a unique nonempty subset of the $n$ messages. The server indexed by $J \in \mathcal{N}$ contains messages $x_J$, where $\mathcal{N} = \{J \subseteq [n] : J \neq \emptyset\}$. Every server is connected to all receivers via its own noiseless broadcast channel with finite link capacity $C_J \geq 0$. Clearly, the DIC problem is a special case of the DIC problem with $C_{[n]} = 1$ (normalized) and $C_J = 0$ otherwise. Let $y_J \in \{0, 1\}^{\mathcal{N}}$ be the output of server $J$ to be broadcast, which is a function of $x_J$, and $y_J$ be the corresponding random variable.

For any DIC problem, we define a $(u, r) = ((u_i, i \in [n]), (r_J, J \in \mathcal{N}))$ distributed index code by

- $2^n - 1$ encoders, one for each server $J \in \mathcal{N}$, such that $\phi_J : \prod_{i \in J} \{0, 1\}^{w_i} \to \{0, 1\}^{r_J}$ maps the messages $x_J$ in server $J$ to an $r_J$-bit sequence $y_J$.
- $n$ decoders, one for each receiver $i \in [n]$, such that $\psi_i : \prod_{J \in \mathcal{N}} \{0, 1\}^{r_J} \times \prod_{k \in A_i} \{0, 1\}^{w_k} \to \{0, 1\}^{w_i}$ maps the received sequences $(y_J, J \in \mathcal{N})$ and the side information $x_{A_i}$ to $\hat{x}_i$.

We say that a rate–capacity tuple $(R, C) = ((R_i, i \in [n]), (C_J, J \in \mathcal{N}))$ is achievable if for every $\epsilon > 0$, there exist a $(u, r)$ code and a positive integer $r$ such that $R_i \leq \frac{w_i}{r}, i \in [n], C_J \geq \frac{a_r}{n}, J \in \mathcal{N}$, and that $|\mathcal{P}(\{\hat{X}_1, \ldots, \hat{X}_n\} \neq (X_1, \ldots, X_n))| \leq \epsilon$.

For a given link capacity tuple $C$, the capacity region $\mathcal{C}(C)$ is the closure of the set of all rate tuples $R$ such that $(R, C)$ is
achievable. The symmetric capacity is defined as $C_{\text{sym}}(C) = \max\{R_{\text{sym}} : (R_{\text{sym}}, \ldots, R_{\text{sym}}) \in \mathcal{C}(C)\}$. The centralized index code, the achievable rate tuple $R$, the capacity region $\mathcal{C}$, and the symmetric capacity $C_{\text{sym}}$ can be defined accordingly.

Any CIC or DIC problem can be represented by a sequence $(i[j \in A_i], i \in [n])$. For example, for $A_1 = \emptyset$, $A_2 = \{3\}$, and $A_3 = \{2\}$, we write $(1\rightarrow, (2\rightarrow), (3\rightarrow))$. It can also be represented by a side information graph $\mathcal{G}$ with $n$ vertices, in which vertex $i$ represents message $i$, and a directed edge $(i, j)$ means that $i \in A_j$. For any nonempty message subset $S \subseteq [n], \mathcal{G}|S$ denotes the subgraph of $\mathcal{G}$ induced by $S$. If $\mathcal{G}|S$ is acyclic, we simply say that $S$ is acyclic.

III. PRELIMINARIES

We briefly review the MAIS bound [6] and the internal conflict bound [9], [10].

Proposition 1 (MAIS bound, [6]): For the CIC problem $(i[A_i]), i \in [n]$, if $R_{\text{sym}}$ is achievable, then

$$R_{\text{sym}} \leq \min_{S \subseteq [n], \mathcal{G}|S \text{ is acyclic}} \frac{1}{|S|}.$$

For the internal conflict bound, we first re-state the definition of the alignment chain [9] in our notation as follows.

Definition 1 (Alignment Chain, [9]): For the CIC problem $(i[A_i]), i \in [n]$, messages $i(1), i(2), \ldots, i(m), i(m + 1)$ and $k(1), k(2), \ldots, k(m)$ constitute an alignment chain of length $m$ denoted as

$$i(1) \xrightarrow{k(1)} i(2) \xrightarrow{k(2)} i(3) \cdots \xrightarrow{k(m)} i(m + 1) \quad (1)$$

if the conditions listed below are satisfied,

1) $i(1) \in B_i(m+1)$ or $i(m + 1) \in B_i(1)$;
2) for any $j \in [m]$, we have $\{i(j), i(j + 1)\} \subseteq B_{k(j)}$.

For any alignment chain or any weighted alignment chain to be proposed later, we call the edge between $i(j)$ and $i(j + 1)$ edge $j$. In Definition 1, the two terminals $i(1)$ and $i(m + 1)$ are underlined to indicate that $\{i(1), i(m + 1)\}$ is acyclic. Note that Definition 1 does not depend on the server setup, and thus also works for the DIC problem.

For a CIC or DIC problem, if there exits at least one alignment chain, then we say there is an internal conflict between the two terminal messages $i(1)$ and $i(m + 1)$ of the alignment chain and that the problem is internally conflicted. The symmetric capacity for the CIC problems that are not internally conflicted is known [2], [9], [12]. For the internally conflicted CIC problems, the following bounds hold.

Proposition 2 (Internal conflict bound, [9], [10]): For the internally conflicted CIC problem $(i[A_i]), i \in [n]$, if $R_{\text{sym}}$ is achievable, then $R_{\text{sym}} \leq \frac{1}{1 + 2\Delta}$, where $\Delta$ denotes the minimum length of alignment chains for the problem.

In the rest of this paper, whenever we say a CIC or a DIC problem, we assume that the problem is internally conflicted.

IV. MAIN RESULTS
A. Improved Necessary Conditions for CIC

We start by introducing a simple structure which will play a crucial role as the basic building block in the generalized alignment chains to be developed henceforth.

Definition 2 (Basic Tower): For the CIC problem $(i[A_i]), i \in [n]$, messages $i(1), i(2) \in [n]$ and $k_1(1), \ldots, k_h(1) \in [n]$ constitute the following basic tower $B_1$,

$$k_h(1) \xrightarrow{i(1)} k_{h-1}(1) \xrightarrow{i(2)},$$

if $\{(i(1), i(2), k_1(1), \ldots, k_{h-1}(1)) \subseteq B_{k(h-1)}\}$ for any $\ell \in [h_1]$.

A visualization of the above definition is given in Figure 1(a). In the basic tower $B_1$, messages $i(1)$ and $i(2)$ are placed horizontally at the ground level of the tower, and message $k_1(1)$ is placed on the $\ell$-th floor for any $\ell \in [h_1]$, where $h_1$ is called the height or the weight of the tower. The receiver $k_1(1)$ who wants message $k_1(1)$ on the $\ell$-th floor cannot know any of the $k$-labeled messages on the lower floors nor the two $i$-labeled messages at the ground level as its side information.

Figure 1. Schematic graphs for Definitions 2 and 3: (a) a basic tower $B_1$, and (b) a singleton weighted alignment chain including $m$ concatenated basic towers. To help with understanding, we draw dashed arrows such that if there is a directed path of dashed arrows from message $a$ to $b$, then $b \in B_a$.

Now we propose the first generalization of the alignment chain, namely, the singleton weighted alignment chain.

Definition 3 (Singleton Weighted Alignment Chain): For the CIC problem $(i[A_i]), i \in [n]$, we have the following singleton weighted alignment chain,

$$k_h(1) \xrightarrow{i(1)} k_{h-1}(1) \xrightarrow{i(2)} \cdots \xrightarrow{i(m)} i(m+1),$$

denoted compactly as $i(1) \xrightarrow{\mathcal{I} \mathcal{K}} i(m+1)$, where $\mathcal{I} = \{i(j) : j \in [m+1]\}$, and $\mathcal{K} = \{k_h(1) : j \in [m], \ell \in [h_j]\}$, if the conditions listed below are satisfied:

1) $i(1) \in B_i(m+1)$ or $i(m + 1) \in B_i(1)$;
2) for any $j \in [m]$, $\ell \in [h_j]$, $\{i(j), i(j + 1), k_1(1), \ldots, k_{h-1}(1)\} \subseteq B_{k_i(j)}$, i.e., messages $i(j), i(j + 1), k_1(j), \ldots, k_{h_j}(j)$ form a basic tower $B_j$.

For simpler notation, in the rest of the paper, we use $K_{l,j}(i)$ to denote the message sequence $k_l(j)$, $\ell \in L$ for some $j \in [m], L \subseteq [h_j]$, e.g., $\{K_{h_1}(j)\} = \{k_1(1), \ldots, k_{h_1}(j)\}$.

The singleton weighted alignment chain can be seen as a horizontal concatenation of $m$ basic towers, $B_1, \ldots, B_m$, such that the terminal message set of the chain, $\{i(1), i(m + 1)\}$ is acyclic. See Figure 1(b) for visualization.

Theorem 1: For the CIC problem $(i[A_i]), i \in [n]$, if $R_{\text{sym}}$ is achievable, then for any of its singleton weighted alignment chains $i(1) \xrightarrow{\mathcal{I} \mathcal{K}} i(m+1)$ we have

$$R_{\text{sym}} \leq \frac{m}{|\mathcal{I}| + |\mathcal{K}|} = \frac{m}{1 + m + \sum_{j \in [m]} h_j}.$$ (2)
As Theorem 1 can be directly implied by Theorem 2 to be presented later, its proof is omitted due to limited space.

Towards further generalization of the alignment chain, we introduce the following structure.

**Definition 4 (Crossing Tower):** For the CIC problem \((i|A_i), i \in [n]\), we have the following crossing tower,

\[
\begin{align*}
\frac{k_{h_i}(1)}{k_i(1)} & \leftarrow \cdots \leftarrow \frac{k_{h_q}(q)}{k_i(q)} & i(1) & \leftarrow \cdots \leftarrow i(q + 1),
\end{align*}
\]

denoted as \(X_j\), if the conditions listed below are satisfied:

1. For any \(j' \in [q] \setminus \{j\}\), the message group \(\{i(j'), i(j' + 1)\}\) constitutes a basic tower \(B_{i(j')}\); and
2. For any \(\ell \in [h_j]\), \([K_{\ell - 1}(j)]\) constitutes a basic tower \(B_{k_i(j)}\).

The disjoint weighted alignment chain can be seen as a horizontal concatenation of the crossing towers \(X_j, j \in M\) and the basic towers \(B_{i(j')}, j' \in M'\), such that \(i(1), i(m + 1)\) is acyclic.

For the crossing tower \(X_j\), we call edge \(j\) the central edge, and the message group \(\{i(j), i(j + 1), K_{h_i}(j)\}\) the core. Every other edge \(j' \in [q] \setminus \{j\}\) corresponds to a basic tower \(B_{i(j')}\). In the core, message \(k_i(j)\) on the \(\ell\)-th floor has messages \(i(s_{\ell,j})\) and \(i(t_{\ell,j})\) to start and terminate its coverage, respectively. The coverage of a message on a lower floor is within the range of the coverage of any message on a higher floor. We call the coverage of the message \(k_i(j)\) on the top floor the total coverage of the crossing tower, and define \(G_j = \{s_{h,j} : t_{h,j} - 1\}\) denoting the set of edges located within the total coverage. Note that any basic tower \(B_{i(j')}\) can be seen as a special crossing tower with \(s_{\ell,j'} = j'\) and \(t_{\ell,j'} = j' + 1\) for any \(\ell \in [h_j]\), and hence \(G_{j'} = \{j'\}\). If \(j\) and \(j'\) are not adjacent, then we say a crossing tower we assume that it is not a basic tower.

We present our most general alignment chain model below.

**Definition 5 (Disjoint Weighted Alignment Chain):** For the CIC problem \((i|A_i), i \in [n]\), we have the following disjoint weighted alignment chain,

\[
\begin{align*}
\frac{k_{h_i}(1)}{k_i(1)} & \rightarrow \frac{k_{h_2}(2)}{k_2(2)} \rightarrow \cdots \rightarrow \frac{k_{h_m}(m)}{k_m(m)} & i(1) & \rightarrow \cdots \rightarrow i(m + 1),
\end{align*}
\]

denoted as \(i(1) \rightarrow i(m + 1)\), if the conditions listed below are satisfied,

1. \(i(1) \in B_{i(m + 1)}\), or \(i(m + 1) \in B_{i(1)}\);
2. for every \(j \in [m]\), the message group \(\{i(j), i(j + 1), K_{h_i}(j)\}\) constitutes either a basic tower \(B_{i(j)}\) or the core of a crossing tower \(X_j\);
3. set \(M = \{j \in [m] : |G_j| \geq 2\}\) denote the set of central edges of the crossing towers. Then for any \(j_1 \neq j_2 \in M\), \(G_{j_1} \cap G_{j_2} = \emptyset\).

We remove any subscripts for the edges in the horizontal chain in Definition 5 since the positions of the basic and crossing towers are flexible in general. For a specific example of Definition 5, see Figure 2. To help understanding, dashed arrows from \(k_i(j)\) of some edges to the corresponding \(i(s_{\ell,j})\) and \(i(t_{\ell,j})\) are drawn. The dashed arrow is purple if the edge \(j\) is in set \(M\), and blue otherwise. Definitions 4 and 5 jointly ensure that two purple dashed arrows can never cross-cross.

Define \(M' = [m] \setminus (\bigcup_{j \in M} G_j)\) as the set of edges located outside the total coverage of any crossing tower. Then the disjoint weighted alignment chain can be seen as a horizontal concatenation of the crossing towers \(X_j, j \in M\) and the basic towers \(B_{i(j')}, j' \in M'\), such that \(i(1), i(m + 1)\) is acyclic.

**Theorem 2:** For the CIC problem \((i|A_i), i \in [n]\), if \(R_{\text{sym}}\) is achievable, then for any of its disjoint weighted alignment chains \(i(1) \rightarrow i(m + 1)\) we have

\[
R_{\text{sym}} \leq \frac{m}{|Z| - |K|} - \frac{1}{1 + m + \sum_{j \in [m]} h_j}. \tag{3}
\]

For a brief proof see Appendix A. The full proof is in [13].

**Proposition 3:** \(R_{\text{DW}} \leq R_{\text{SW}} \leq R_\Delta\), and \(R_{\text{MAIS}}\) denote the upper bounds given by Theorem 2, Theorem 1, Proposition 2, and Proposition 1, respectively.

**Proposition 3:** \(R_{\text{DW}} \leq R_{\text{SW}} \leq R_\Delta\), and \(R_{\text{MAIS}}\) denote the upper bounds given by Theorem 2, Theorem 1, Proposition 2, and Proposition 1, respectively.

**Proof:** Any alignment chain can be seen as a singleton weighted alignment chain and any singleton weighted alignment chain can be seen as a disjoint weighted alignment chain. Hence, \(R_{\text{DW}} \leq R_{\text{SW}} \leq R_\Delta\). Set \(s = 1/R_{\text{MAIS}}\), then there exists an acyclic message set of size \(s\), say \(\{i(1), i(2), \ldots, i(s)\}\). In other words,

\[
\{i(1), \ldots, i(\ell - 1)\} \subseteq B_i(\ell), \quad \forall \ell \in [s]. \tag{4}
\]

Therefore, we have the following one-edge singleton weighted alignment chain,

\[
i(1) \rightarrow i(2), \tag{5}
\]

and thus by Theorem 1, we have \(R_{\text{SW}} \leq 1 = R_{\text{MAIS}}\). The relationships in Proposition 3 can sometimes be strict.

**Example 1:** Consider the 6-message CIC problem

\[
\begin{align*}
(1 & [2, 3, 4, 6], \quad (2 | 4, 5, 6), \quad (3 | 1, 2, 4, 5, 6) \quad (4 | 1, 2, 6), \quad (5 | 2, 3, 4, 6), \quad (6 | -).
\end{align*}
\]

For this problem, \(R_\Delta = R_{\text{MAIS}} = \frac{1}{3}\). However, we have the following singleton weighted alignment chain,

\[
\begin{align*}
\frac{6}{1} & \rightarrow \frac{6}{2} \rightarrow \frac{6}{3} \rightarrow \frac{6}{4} \rightarrow \frac{6}{5}.
\end{align*}
\]

and thus by Theorem 1, we have \(R_{\text{sym}} \leq \frac{2}{7} = \frac{2}{5}\), which matches the composite coding lower bound [8] on the symmetric capacity. Therefore, \(C_{\text{sym}} = R_{\text{DW}} = R_{\text{SW}} = \frac{2}{7} < R_\Delta = R_{\text{MAIS}} = \frac{1}{3}\). Note that in the above chain, message 6 appears twice in two different basic towers, which is allowed.
The new bounds can solve some large problems, for which the more general PM bound is computationally infeasible.

**Example 2:** Consider the 17-message CIC problem as follows, denoted by \((i|B_i), i \in [n]\) rather than \((i|A_i), i \in [n]\), \((1,6),(2,7,8),(3,8,11,17), (4,\ldots),(5,\ldots),(6,1),(7,1,2), (8,1,2,3,4,7), (9,2,3), (10,1,4,9), (11,3,4,8), (12,5,6), (13,4,5), (14,4,6,7,13,15,17), (15,\ldots),(16,5,6,12), (17,8)\).

For this problem, \(R_{DA} = 2 R_{MASS} = 2 R_{SW} = \frac{1}{2}\). However, we have the following disjoint weighted alignment chains,

\[
\frac{1}{8} \xrightarrow{4} 3 \xrightarrow{16} 5 \xrightarrow{12} 6, \quad 10 \xrightarrow{12} 2 \xrightarrow{8} \cdot \xrightarrow{16} 5 \xrightarrow{12} 6,
\]

and thus by Theorem 2 we have \(R_{sym} \leq R_{DW} \leq \frac{5}{2} < \frac{1}{2}\), which matches the composite coding lower bound on \(C_{sym}\).

**B. Improved Necessary Conditions for DIC**

Definitions 1-5 also work for the DIC problem. In the following, we extend Theorems 1 and 2 to Theorems 3 and 4, respectively. Recall that \(G_j = \{s_{h_j \cdot t_{h_j - 1}}\}, M = \{j \in [m] : G_j \geq 2\}, M' = \{m \in [m] \setminus (\bigcup_{j \in M} G_j)\}.

**Theorem 3:** For the DIC problem \((i|A_i), i \in [n]\) with link capacity \(C\), if \(R_{sym}\) is achievable, then for any of its singleton weighted alignment chains \(i(1) \xrightarrow{L} s_{i(m + 1)}\) we have

\[
R_{sym} \leq \frac{1}{|Z| + |K|} \sum_{j \in [m]} \left( \sum_{i \in [m]} \sum_{j \in [m]} \left( \sum_{i \in [m]} \sum_{j \in [m]} C_j \right) \right)\]

**Theorem 4:** For the DIC problem \((i|A_i), i \in [n]\) with link capacity \(C\), if \(R_{sym}\) is achievable, then for any of its disjoint weighted alignment chains \(i(1) \xrightarrow{L} s_{i(m + 1)}\) we have

\[
R_{sym} \leq \frac{1}{|Z| + |K|} \left( \sum_{j \in [m]} \sum_{t \in \mathbb{G}_j} \left( \sum_{j \in [m]} \sum_{t \in \mathbb{G}_j} C_j \right) \right) + \sum_{j \in [m]} \sum_{t \in \mathbb{G}_j} C_j, \quad (6)
\]

where

\[
T_1 = \{i(s_{h_j}, j), i(t_{h_j}, j), K_{h_j}(j)\},
T_2 = \{i(s_{h_j}, j), i(m + 1), K_{h_j}(j)\},
T_3 = \{i(j'), i(j' + 1), K_{h_j}(j')\},
T_4 = \{i(j'), i(s_{t_j - 1}), K_{h_j}(j')\},
T_5 = \{i(j'), i(t_{j_j}), K_{h_j}(j')\}.
\]

The proofs for Theorems 3 and 4 are presented in [13].

**Example 3:** Consider the following DIC problem with \(n = 5\) messages and equal link capacities \(C_j = 1, j \in N^\prime, (1,2,3,4,5), (2,1,3,4,5), (3,2,4,5), (4,3,5), (5,1,4)\).

For this problem, there exists a singleton weighted alignment chain as \(1 \xrightarrow{1} 2 \xrightarrow{1} 3 \xrightarrow{1} 4 \xrightarrow{1} 5 \xrightarrow{1} 3\), and thus by Theorem 3,

\[
R_{sym} \leq \frac{1}{5} \left( \sum_{j \in [1,2,4]} 1 + \sum_{j \in [4,3,5]} 1 \right) = \frac{2}{5},
\]

which matches the distributed composite coding lower bound [4] on the symmetric capacity.

C. A 9-Message CIC Problem Which Needs Non-Shannon-Type Inequalities for Capacity Characterization

We present a 9-message CIC problem, which is the CIC problem with the smallest number of messages \(n\) identified so far, where non-Shannon-type inequalities are necessary.

**Example 4:** Consider the following 9-message CIC problem, denoted by \((i|B_i), i \in [n]\), 

\[
(1,2), (2,1,5,8), (3,\ldots),(4,\ldots),(5,2,4,8),
(6,1,3), (7,3,4), (8,2,3,5), (9,1,4,6).
\]

The PM bound\(^1\) gives \(\sum_{i \in [n]} R_i < 19/6\). However, applying Zhang-Yeung non-Shannon-type inequalities [15] to the problem, the upper bound can be further tightened to \(\sum_{i \in [n]} R_i \leq 25/8\), the proof for which is presented in [13].

**APPENDIX A**

**Proof of Theorem 2**

For the CIC problem \((i|A_i), i \in [n]\), define \(g(S) = \frac{1}{2} H(Y[n]|X_{\mathcal{S}^c})\), \(S \subseteq [n]\). The set function \(g(S)\) is monotonically non-decreasing, \(g(S) \leq g(S')\) for \(S \subseteq S' \subseteq [n]\), and submodular, \(g(S \cap S') + g(S \cup S') \leq g(S) + g(S')\), \(\forall S, S' \subseteq [n]\). Also, \(g(\emptyset) = 0\), and \(g(S) \leq 1, \forall S \subseteq [n]\). We sometimes use \(g(i, i \in S)\) to denote \(g(S)\).

The following lemmas hold.

**Lemma 1** ([14]): For any \(i \in [n]\), we have

\[
R_i + g(B) = g(B \cup \{i\}), \quad \forall B \subseteq B_i. \quad (7)
\]

Particularly, when \(B = \emptyset, g(B) = 0\), and thus \(R_i = g(i)\).

**Lemma 2:** For any basic tower \(B_j\) constituted by the message group \(\{i(j), i(j + 1), K_{h_j}(j)\}\), we have

\[
g(i(j), i(j + 1)) + \sum_{\ell \in [h_j]} R_{k(i(j)} \leq 1.\]

One can show Lemma 2 via repeated application of Lemma 1 for \(\ell \in [h_j]\), and using \(g(\{i(j), i(j + 1), K_{h_j}(j)\}) \leq 1\).

**Lemma 3:** For any crossing tower \(X_j\), we have

\[
\sum_{j \in G_j} R_{k(i(j)} + \sum_{j \in G_j} \sum_{i \in [h_j]} R_{k(i(j)} + g(i(s_{h_j}(j)), i(t_{h_j}(j))) \leq |G_j|. \quad (8)
\]

**Proof:** According to Definition 4, there is a basic tower \(B_j\) for any \(j \in G_j\). Hence by Lemma 2, we have

\[
\sum_{j \in G_j} \left( \sum_{i \in [h_j]} R_{k(i(j)} + g(i(j'), i(j' + 1)) \right) \leq 1 - |G_j|. \quad (9)
\]

For the core of \(X_j\), consider any \(\ell \in [2 : h_j]\). Using monotonicity and submodularity of \(g(S)\), we write

\[
g(K_{[\ell - 1]}(j), i(s_{\ell - 1}), i(t_{\ell - 1})) \leq g(K_{[\ell - 1]}(j), i(s_{\ell - 1}), i(t_{\ell - 1}))
\]

\[
\leq g(K_{[\ell - 1]}(j), i(s_{\ell - 1}), i(t_{\ell - 1}), i(s_{\ell - 1}), i(t_{\ell - 1})),
\]

\[
\leq g(K_{[\ell - 1]}(j), i(s_{\ell - 1}), i(t_{\ell - 1})) - g(i(s_{\ell - 1}), i(t_{\ell - 1})), \quad (10)
\]

\(^1\)The PM bound, including all decoding constraints, is as tight as the bound utilizing all Shannon-type inequalities. See the full arXiv version of [14].
where the equality is due to Lemma 1 and Definition 4. By Lemma 1, Definition 4, and the fact that $g(K_{h_1}(j), i(s_{h_1}, j), i(t_{h_1}, j)) \leq 1$, we have
\[
R_{h_1}(j) + g(K_{h_1}(j), i(s_{h_1}, j), i(t_{h_1}, j)) \leq 1. \tag{11}
\]
Summing up (10) for all $\ell \in [2 : h_j]$ and adding (11) gives
\[
\sum_{\ell \in [h_j]} g(i(s_{\ell,j}), i(t_{\ell,j})) + \sum_{\ell \in [h_j]} R_{h_1}(j) \\
\leq 1 + \sum_{\ell \in [2 : h_j]} g(i(s_{\ell,j}), i(t_{\ell,j}), i(s_{\ell-1,j}), i(t_{\ell-1,j})). \tag{12}
\]
Consider any $a \leq b \in G_j$. For any $c \in [a + 1 : b]$, by the submodularity and monotonicity of $g(S)$, and Lemma 1,
\[
g(i(a), i(c)) + g(c, i(c + 1)) \geq g(i(a), i(c + 1)).
\]
Summing up the above inequality for all $c \in [a + 1 : b]$ yields
\[
\sum_{c=a}^{b} g(i(c), i(c + 1)) \geq \sum_{c=a+1}^{b} R_{i(c)} + g(i(a), i(b + 1)).
\]
Consider any $\ell \in [2 : h_j]$. Using the above and the submodularity of $g(S)$, together with Lemma 1, we obtain
\[
\sum_{j' \in [s_{\ell,j}, s_{\ell-1,j} - 1]} g(i(j'), i(j' + 1)) + g(i(s_{\ell-1,j}), i(t_{\ell-1,j})) \\
\geq \sum_{j' \in [s_{\ell,j}, s_{\ell-1,j} - 1]} R_{i(j')} + \sum_{j' \in [s_{\ell-1,j}, s_{\ell-1,j} - 1]} R_{i(j')} \\
+ g(i(s_{\ell,j}), i(s_{\ell-1,j}), i(t_{\ell-1,j}), i(t_{\ell-1,j})). \tag{13}
\]
Definition 4 states that for any $\ell \in [2 : h_j]$, $s_{\ell,j} \leq s_{\ell-1,j} \leq s_{1,j} = j$, and $j + 1 = t_{\ell,j} = t_{\ell-1,j} \leq t_{\ell,j}$. Hence, adding up (13) for all $\ell \in [2 : h_j]$ and rearranging yields
\[
\sum_{j' \in [s_{h_j}, t_{h_j} - 1]} R_{i(j')} - \sum_{j' \in G_j \setminus \{j\}} g(i(j'), i(j' + 1)) \\
- \sum_{\ell \in [2 : h_j]} g(i(s_{\ell-1,j}), i(t_{\ell-1,j})) \\
\leq \sum_{j' \in [s_{h_j}, s_{h_j} + 1 : t_{h_j} - 1]} R_{i(j')} + g(i(s_{h_j}), i(s_{h_j} - 1), i(s_{h_j} - 1), i(t_{h_j})). \tag{14}
\]
Finally, summing up (9), (12), and (14) yields (8).

Proof: Given any disjoint weighted alignment chain $i(1) \longleftrightarrow \cdots \longleftrightarrow i(m + 1)$, by Lemma 1 and Definition 5, we have
\[
R_{i(1)} + R_{i(m + 1)} = g(i(1), i(m + 1)). \tag{15}
\]
Any disjoint weighted alignment chain is a concatenation of the crossing towers $X_j$, $j \in M$ and the basic towers $B_j$, $j' \in M'$. By Lemmas 2 and 3 we have
\[
\sum_{j' \in M'} \left( \sum_{\kappa \in [h_{j'}]} R_{k_{j'}}(j') + g(i(j'), i(j' + 1)) \right) \\
+ \sum_{j \in M} \left( \sum_{\ell \in [h_{j}]} \sum_{j' \in [s_{j}, t_{j}] + 1 : t_{h_j} - 1]} R_{k_{j'}}(j') \right) \\
+ g(i(s_{h_j}), i(t_{h_j}))) \leq 1 + \sum_{j' \in M} |G_j|. \tag{16}
\]
Note that any basic tower $B_j'$, $j' \in M'$ is a special crossing tower with $s_{h_j}, j' = j'$, $t_{h_j}, j' = j' + 1$. Hence,
\[
\sum_{j \in M} g(i(s_{h_j}), j', i(t_{h_j})) + \sum_{j' \in M'} g(i(j'), i(j' + 1)) \\
\geq \sum_{j \in M} R_{i(j')} \\
+ \sum_{j' \in [s_{h_j}, s_{h_j} + 1 : t_{h_j} - 1]} R_{i(j')} \\
+ g(i(s_{h_j}), i(t_{h_j}))) \\
\geq \sum_{j' \in [s_{h_j}, s_{h_j} + 1 : t_{h_j} - 1]} R_{i(j')} \tag{17}
\]
where the first inequality follows from the submodularity of $g(S)$ and Lemma 1, and the second inequality follows from the monotonicity of $g(S)$, and (15), and that there exist some $j_1, j_2 \in M \cup M'$ such that $s_{h_1}, j_1 = 1$, $t_{h_2}, j_2 = m + 1$.

Given (17), we can bound the LHS of (16) as
\[
LHS \geq \sum_{j \in [m]} \sum_{\ell \in [h_j]} R_{i(j')} + \sum_{j \in M} \sum_{j' \in [s_{j}, t_{j} + 1 : t_{h_j} - 1]} R_{i(j')} \\
+ \sum_{j' \in [s_{h_j}, s_{h_j} + 1 : t_{h_j} - 1]} R_{i(j')} \\
\geq |K| + |J| \cdot \sum_{j \in [m]} |G_j| \tag{18}
\]
By (16) and (18), $R_{\text{sym}} \leq \frac{M' + \sum_{j \in M} |G_j|}{|J| + |K|}$.

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