An operational interpretation for global multipartite entanglement

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We introduce an operational interpretation for pure-state global multipartite entanglement based on quantum estimation. We show that the estimation of the strength of low-noise locally depolarizing channels, as quantified by the regularized quantum Fisher information, is directly related to the Meyer-Wallach multipartite entanglement measure. Using channels that depolarize across different partitions, we obtain related multipartite entanglement measures. We show that this measure is the sum of expectation values of local observables on two copies of the state.

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Schrödinger, inspired by the EPR paper, described an entangled state in the following terms: “the best possible knowledge of the whole does not include the best possible knowledge of its parts” \textsuperscript{1}. Entanglement has ever since played a prominent role in foundational studies of quantum mechanics because of the relation between entanglement and stronger than classical nonlocal correlations. Nevertheless, only in recent years has a formal theory of quantum entanglement been developed. The reason is that entanglement is a prerequisite for performing paradigmatic tasks in quantum information, such as teleportation, dense coding, or pure-state quantum computation \textsuperscript{2}. More precisely, the most common bipartite entanglement and stronger than classical nonlocal correlations. Nevertheless, only in recent years has a formal theory of quantum entanglement been developed. The reason is that entanglement is a prerequisite for performing paradigmatic tasks in quantum information, such as teleportation, dense coding, or pure-state quantum computation \textsuperscript{2}. More precisely, the most common bipartite entanglement measures have an operational interpretation in terms of a quantification of the available resources for a specific task \textsuperscript{3}. In this Letter a similar operational interpretation for global multipartite entanglement is derived.

Entanglement is only a necessary but not sufficient condition for computational speedups, as shown by the Gottesman-Knill theorem \textsuperscript{4}. On the other hand, the figure of merit of any quantum information processing task can be used to define an entanglement monotone if the initial state is optimized with local operations and classical communication (LOCC) preprocessing \textsuperscript{5}. One particular measure, localizable entanglement, arose from the study of the suitability of a given state to perform quantum communication with quantum repeaters. It quantifies the amount of entanglement attainable between two specific parties after performing LOCC on the rest. When there is enough symmetry, localizable entanglement can be analytically computed and indicates multipartite entanglement \textsuperscript{6}.

Entanglement also improves the precision of quantum measurements, a task itself related to quantum computation \textsuperscript{7}. Instead of focusing on computational tasks, here we will present an interpretation of entanglement as a resource for a specific parameter estimation problem. A good candidate for a parameter invariant under local unitaries is the strength of a locally depolarizing channel, i.e., a tensor product of depolarizing channels which mimics the tensor structure that defines locality for the given parties. It has been noted that entanglement helps, as expected, in estimating the parameters of a quantum channel \textsuperscript{8,9}. In the specific case of a two-qubit locally depolarizing channel, maximally entangled states achieve the best precision in the estimation for some range of depolarization. On the other hand, entanglement is not useful for all values of the depolarization strength, and mixed entangled states tend to perform worse than separable states \textsuperscript{10}. Finally, Fisher information, a concept central to the quantification of estimation sensitivity, as we will see, has been found to be proportional to the logarithmic negativity, in the context of dense coding for squeezed states and some particular two-qubit states \textsuperscript{11}.

The keystone of quantum parameter estimation is the so-called quantum Cramér-Rao bound (QCRB) \textsuperscript{12}. To understand its meaning we first draw an analogy with the theory of statistical estimation. A statistical model $M$ is a parametrized family of probability distributions $M = \{p_\epsilon(x); \epsilon \in \Theta\}$. Estimators $\hat{\epsilon}$ are functions of the outcomes $x$ onto the parameter space $\Theta$. An estimator is unbiased if $\sum_x \hat{\epsilon}(x)p_\epsilon(x) = \epsilon$. The single-parameter Camér-Rao bound \textsuperscript{12} for unbiased estimators is $\text{Var}_\epsilon[\hat{\epsilon}]I_\epsilon \geq 1$, where $I_\epsilon$ is the Fisher information of the model $M$, and $\text{Var}_\epsilon[\hat{\epsilon}]$ is the variance of the estimator,

$$I_\epsilon = \sum_x \left( \frac{\partial \log p_\epsilon(x)}{\partial \epsilon} \right)^2 p_\epsilon(x).$$

(1)

Note that $I_\epsilon$ provides a measure of distinguishability.

Quantum-mechanically, the statistical model is replaced by the quantum model, i.e., a parameterized family of quantum states $\mathcal{M} = \{\rho_\epsilon; \epsilon \in \Theta\}$. While the classical Fisher information provided by a measurement depends on the measurement itself, the quantum Fisher information (QFI) $J_\epsilon$ (defined below) does not. The single-parameter QCRB is $\text{Var}_\epsilon[\hat{\epsilon}]J_\epsilon \geq 1$, and is attainable asymptotically in the number of measurements. When an estimator $\hat{\epsilon}$ attains the QCRB it is said to be efficient. The QFI $J_\epsilon$ provides the quantum model with a geometric structure of operational significance. It can also be

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shown that the QFI is proportional to the Hessian matrix of the quantum fidelity \[12\], \(F(\rho, \sigma) = \text{tr}[\sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}}]\), \(F(\rho_\epsilon, \rho_\epsilon + \phi) = 1 - \frac{1}{2} \phi^2\), hence showing that the quantum Fidelity has a clear interpretation in terms of distinguishability.

Entanglement is a fragile resource under local noise. It is this feature that gives entangled states their useful-ness in loss estimation. A quantification of this usefulness would entail, in principle, a means of quantifying the amount of entanglement. It turns out that, since entangled states are the ones that decohere faster, above some threshold value of \(\epsilon\) their sensitivity drops below that of a separable state. This is the transition effect found by Fujiwara \[10\] and shown in Fig. (1), which plots the regularized QFI, \(\epsilon J(\rho_\epsilon)\), as a function of the strength of the channel \(\epsilon\). A similar effect is found when analyzing phase estimation in the presence of decoherence \[13\]. When one takes a pure-state (separable or not) through a low-noise channel, the state becomes slightly mixed. The noise parameter becomes related to the entropy of the state itself. It turns out that the QFI diverges when one approaches the boundary of pure states from the set of mixed states, a fact closely related to the divergence of the Bures metric for pure states, as well as some surprising results in amplitude damping channel estimation \[14\]. This divergence becomes intuitive when one considers the problem of estimating the parameter \(p\) of a binomial distribution. When such parameter approaches zero, the variance of the estimation \(p(1-p)\) also does at the same rate, and the QFI also diverges as \(1/p\). This is a key signature of Poissonian statistics. The discussion above shows that it will be necessary to appropriately regularize the divergence.

In the case of two qubits there is only one kind of entanglement, given by the Schmidt coefficients, so before the crossover maximally entangled states are optimal, and after the crossover separable states are optimal. In the case of three qubits the entanglement of the W states is not the same as the entanglement of the GHZ states, and the crossover occurs at a slightly different point, so for some range of \(\epsilon\), W states outperform GHZ states (see Fig. (1)). This can be explained by the rate of change of the von Neumann entropy, initially higher for GHZ states, but later higher for W states. Incidentally, GHZ states are also optimal for local phase estimation without decoherence. On the other hand, the entanglement in GHZ states is more persistent than in W states \[15\]. The von Neumann entropy is flat for GHZ states around \(\epsilon = .5\), which corresponds to the totally depolarizing channel according to our choice for channel parametrization -see Eq. (2) and (3)-, and GHZ states provide no QFI in that region. That is not the case for W states. In all cases the rate of change of the von Neumann entropy diverges when \(\epsilon \to 0\), which also explains the divergence of the QFI. Finally, the QFI increases as the channel becomes less noisy \[16\], but separable states still outperform entangled states in that region.

We now show that the QFI for a low-noise locally depolarizing channel is an entanglement measure. To avoid the difficulties discussed so far when the state becomes too mixed, we will take the limit as \(\epsilon \to 0\), i.e., the channel is close to the identity channel. We will use the renormalized QFI, \(\epsilon J(\rho_\epsilon)\), to cancel the divergence of the Bures metric in the transition from pure to mixed states. Due to this divergence, we do not expect our proposed measure to be a good candidate for detection of entanglement with mixed states: pure states, even without any entanglement, will in general perform better than initially mixed states.

The trace preserving channel that commutes with all unitaries can be written as \[16\]
\[
\mathcal{E}_\epsilon(\rho) = (1 - d\epsilon)\rho + \epsilon \mathbb{I} \text{tr} \rho ,
\]
where \(d\) is the dimension of the Hilbert space on which the channel acts. Up to a choice of \(\epsilon\), the channel is unique. The channel is completely positive in the range \(0 \leq \epsilon \leq d/(d^2 - 1)\). To first order in \(\epsilon\), the channel for \(n\) parties acting on state \(\rho\) gives
\[
\rho_\epsilon \equiv \mathcal{E}_\epsilon^{\otimes n}(\rho) = \left(1 - \epsilon \sum_j d_j\right)\rho + \epsilon \sum_j \mathbb{I}_j \otimes \text{tr}_j \rho ,
\]
where \(\mathbb{I}_j\) denotes the identity in party \(j\) and \(\text{tr}_j\) denotes the partial trace with respect to party \(j\).

To obtain the QFI, the standard procedure starts by solving for the symmetric logarithmic derivative \(\Lambda_\epsilon\), defined as any Hermitian operator that satisfies the equation \(\Lambda_\epsilon \rho_\epsilon + \rho_\epsilon \Lambda_\epsilon = 2 \partial_\rho \rho_\epsilon\). The QFI does not depend on the particular choice of \(\Lambda_\epsilon\), and is given by (note the clear analogy with Eq. (1))
\[
J(\rho_\epsilon) = \text{tr}[\rho_\epsilon \Lambda_\epsilon^2] = \text{tr}[(\partial_\rho \rho_\epsilon)\Lambda_\epsilon] .
\]

The output state can be expanded as \(\rho_\epsilon = \rho - \epsilon \rho' + O(\epsilon^2)\), where \(\rho' = -[\partial_\rho \rho_\epsilon]_{\epsilon=0}\). Because of the \(1/\epsilon\) divergence in the frontier of pure states, a solution for \(\Lambda_\epsilon\), for initial pure states, is \[4\] \(\Lambda_\epsilon = (1 - \rho)/\epsilon - \rho' + O(\epsilon)\). Substituting in Eq. (4), the QFI reads, to leading order,
\[
J(\rho_\epsilon) = \frac{1}{\epsilon} \text{tr}[\rho \rho'] + O(1) .
\]
In the limit $\epsilon \to 0$ the problem becomes classical since the optimal measurement is independent of $\epsilon$. In fact, the projection-valued measurement $\{O_\alpha\}$ with $O_0 = \rho$, $O_1 = 1 - \rho$, together with $\epsilon(x) = x/\text{tr}[\rho \rho']$, provide an unbiased and efficient estimator to leading order near $\epsilon = 0$, with $\text{Var}_\epsilon[\hat{\epsilon}] = \epsilon (\text{tr}[\rho \rho' \nu^{-1}] + O(\epsilon^2))$, where $\nu$ is the number of samples measured.

We define the entanglement measure as

$$E(\rho) \equiv K + \lim_{\epsilon \to 0} \epsilon J(\rho) = K + \text{tr}[\rho \rho'] , \quad \text{(6)}$$

where $K = \sum_j (1 - d_j)$ is a constant, depending only on the dimensions of the parties $\{d_1, \ldots, d_n\}$, to ensure that for separable states $E(\rho_{\text{sep}}) = 0$. Another interpretation of this measure can be given by rewriting Eq. (6) as $E(\rho) = K - \lim_{\epsilon \to 0} \epsilon \partial \mathcal{F}(\rho, \rho')^2$, where $\mathcal{F}(\rho, \sigma) = \text{tr}[(\sqrt{\sigma} \rho \sqrt{\sigma})]$ is the fidelity. This confirms that the entanglement measure corresponds to the rate at which the state $\rho$ moves away from the initial state under the action of a low-noise locally depolarizing channel. Usually the QFI will correspond to the second derivative of the fidelity, and the fidelity would have a local maximum for $\epsilon = 0$. In this case, though, the channel is unphysical for $\epsilon < 0$, and the first derivative of the fidelity at $\epsilon = 0$ does not vanish. This is captured by the divergence of the QFI.

To proceed, we get an expression for $\rho'$ from Eq. (3), $\rho' = \sum_j (d_j \rho - 1_j \otimes \rho_j)$, where $\rho_j = \text{tr}_\rho \rho$. Plugging back into the definition of the entanglement measure Eq. (6) we obtain

$$E(\rho) = \sum_j \left(1 - \text{tr}[\rho (1_j \otimes \rho_j)]\right) = \sum_j \left(1 - \text{tr}[\rho_j^2]\right) . \quad \text{(7)}$$

The final entanglement measure is just the sum of local linear entropies. Up to normalization, this is the Meyer-Wallach multipartite entanglement measure, itself a special case of Generalized Entanglement [17]. We have shown that the precision of the estimation of the strength of a low-noise locally depolarizing channel is given by the global multipartite entanglement of the initial state. Notice, though, that this procedure does not detect genuine multipartite entanglement [18].

Different entanglement measures can be derived using channels with different tensor structures. For a selection of parties $\alpha = \{\alpha_1, \ldots, \alpha_k\}$, consider the depolarizing channel for those parties $\mathcal{E}\alpha = (1 - c_{\alpha}) \rho + c_{\alpha} \rho_\alpha \otimes \rho_\alpha$. The corresponding QFI is, up to additive constants, $J_\alpha(\rho) \approx 1 - \text{tr}[\rho_\alpha^2]$. When composing channels which depolarize with respect to different partitions, the channels commute to first order in $\epsilon$, so the order of the composition is not important, and the QFI is, up to constants, the sum of the corresponding local linear entropies. For instance, the composition of the depolarizing channels for all partitions $\mathcal{E}\alpha = \mathcal{E}_{\alpha_1} \circ \cdots \circ \mathcal{E}_{\alpha_k}$, gives an entanglement measure

$$E_p(\rho) = K_p + \lim_{\epsilon \to 0} \epsilon J_p(\rho) = \sum_{\alpha} \left(1 - \text{tr}[\rho_\alpha^2]\right) , \quad \text{(8)}$$

where $\alpha$ runs over all partitions. This measure is proportional to a generalization of the Meyer-Wallach entanglement measure [19]. Similar measures have been used in the context of quantum phase transitions [20]. Here we will not consider this extensions any further, but the following analysis applies trivially.

Pure-state entanglement measures can be extended to mixed states by the convex roof,

$$E(\rho) \equiv \min_{\{p_j, \rho_j\}} \left\{ \sum_j p_j E(|\rho_j\rangle) \right\} , \quad \text{(9)}$$

where $\rho = \sum_j p_j |\rho_j\rangle \langle \rho_j|$. The convex roof extension can be understood as the solution to a zero-sum two-player game: system and “environment”. Let the parties share a mixed state of the system, $\rho$, and Eve holds a purification of $\rho$. The parties want to optimize their estimation of the channel while Eve aims at minimizing the amount of information. Eve is allowed to perform any rank 1 measurement on her purification but has to communicate the classical outcome to the parties. Let $|\Psi_j\rangle$ be the state that the parties are left with after Eve’s measurement, with probability $p_j$. The expected QFI obtained by the parties is $\sum_j p_j E(|\Psi_j\rangle)$. On the other hand, by virtue of the HJW theorem [21], Eve can prepare any ensemble $\{p_j, |\Psi_j\rangle\}$ such that $\rho = \sum_j p_j |\Psi_j\rangle \langle \Psi_j|$. The minimization performed by Eve will result in an expected QFI which immediately translates into Eq. (9).

We proceed to note some properties of this entanglement measure. Invariance under local unitaries follows from the symmetry of the channel. It is also invariant when adding a pure local ancilla. Strong monotonicity means that $E(\rho) \geq \sum_j p_j E(\sigma_j)$, where $\{p_j, \sigma_j\}$ is any ensemble obtained from $\rho$ with LOCC [3]. This prevents $E(\rho)$ from increasing with LOCC. It is also desirable that the entanglement measure does not increase when information is lost, that is, for any ensemble $\sum_j p_j \tau_j = \rho$, $E(\rho) \leq \sum_j p_j E(\tau_j)$. Mulitpartite convex roof extensions derived from bipartite pure-state entanglement measures it is sufficient to verify that the local bipartite function is concave in order to prove the above properties. The concavity of the local linear entropy has already been shown [16].

For bipartite states, the entanglement measure given by Eq. (6) is known as the tangle [10]. The tangle is the convex roof of the square of a generalization of the concurrence [22], derived through the universal inverter $S \otimes P_\mathbb{I} - I$, where $P_\mathbb{I}$ is proportional to the projection superoperator onto the identity operator, and $I$ is the identity superoperator. For pure states the tangle is, up to additive constants, $\text{tr}[\rho S^{\otimes 2}(\rho)] \approx -\text{tr}[\rho (P_\mathbb{I} \otimes I + I \otimes P_\mathbb{I})(\rho)]$. Now, for the depolarizing channel, $\partial_\epsilon \mathcal{E}(\rho)_{|\epsilon=0} = P_\mathbb{I} \otimes I + I \otimes P_\mathbb{I}$, and $E(\rho) = K - \text{tr}[\rho \mathcal{E}(\rho)_{|\epsilon=0}]$, where $K$ fixes the relevant constants. This shows the relation between the QFI of the locally depolarizing channel and the universal inverter.

We now introduce an observable whose expectation value gives, up to normalization, the quantity $E(\rho)$. Let
us assume that the parties have access to many copies of the same pure state. Further, we assume that they can perform repeated collective measurements on pairs of states. Because $E(\rho)$ is a quadratic function, this will be enough [13, 23]. In particular, generalizing the expression for the bipartite tangle from [24], we can write $1 - \text{tr}[\rho_j^2] = 2\langle \Psi | (\Psi | P_j^+ | \Psi) | | \Psi \rangle$, where $P_j^+$ is the projector onto the antisymmetric subspace of the $j$th local Hilbert space of the two copies. The sum of linear entropies is then $E(\rho) = \sum_{j=1}^n 2\langle \Psi | (\Psi | P_j^+ | \Psi) | | \Psi \rangle$, showing that $E(\rho)$ is a sum of expectation values of local observables, where locality refers to the parties (not the copies). This measure has been implemented experimentally for two-qubit states [24].

In conclusion, while the bipartite entanglement of a state has a quantitative operational interpretation as the number of qubits that can be teleported using that state, a similarly clear interpretation has been lacking for multipartite entanglement. In this Letter we have proposed a quantitative operational interpretation for global multipartite entanglement as the enhancement on the estimation of the strength of a low-noise locally depolarizing channel. The estimation is, by construction, invariant under local unitaries, and embodies the appropriate tensor structure. The variance of the estimation is related to the rate of change of the von Neumann entropy, and, therefore, to decoherence. Technical considerations show that the right interpretation is derived from the regularized quantum Fisher information in the low-noise limit. This gives an entanglement monotone proportional to the Meyer-Wallach entanglement measure. Low-noise depolarizing channels with different tensor structures give related entanglement measures. The Meyer-Wallach entanglement measure reduces to the sum of the averages of local projectors, and might be implementable with current technology, as has been done already for the bipartite case.

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