Generating Functions for Special Flows over the 1-Step Countable Topological Markov Chains

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Abstract

Let $Y$ be a topological Markov chain with finite leading and follower sets. Special flow over $Y$ whose height function depends on the time zero of elements of $Y$ is constructed. Then a formula for computing the entropy of this flow will be given. As an application, we give a lower estimate for the entropy of a class of geodesic flows on the modular surface. We also give sufficient conditions to guarantee the existence of a measure with maximal entropy.

1 Introduction

There are two main routines to compute the entropy of non-compact dynamical systems. The first is to use $(T, \varepsilon)$-spanning sets introduced by Bowen on metric spaces \cite{2, 7, 10}, and the second is to use the topological pressure from the thermodynamic formalism \cite{1, 5, 6, 9}. Our concern is the latter and in particular, we consider special flows over countable Markov chains. These flows are mainly associated with geodesic flows on non-compact manifolds with negative curvature. For instance in \cite{3}, it has been shown that the geodesic flows on the modular surface can be represented by special flow over countable alphabet. However, even in this case, depending on the properties, several definitions for entropies are given \cite{1, 5, 6, 9}.

In this paper we construct a special flow constructed from a certain class of topological Markov chains. Namely, we let the base $Y$ of the flow be taken from a 1-step topological Bernoulli scheme TBS with countable states so that the follower and leading sets are finite. This means that we partition the set of alphabet to $\{P_1, ..., P_m\}$ such that if $y = \{y_i\}_{i \in \mathbb{Z}}$, $y' = \{y'_i\}_{i \in \mathbb{Z}}$ are in $Y$ with
necessary for the results in [9], the topological entropy of the flow is 

\[h(T) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log |f_i(x)|\]

and [9], sufficient conditions for having a measure with maximal entropy for the 

flow has been given. Then in Section 5, we show that results 

in [8] can be deduced from our method. In Section 6, based on results in [8] 

and [9], sufficient conditions for having a measure with maximal entropy for the 

flow has been given.

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2 Notations and main definitions

Now we recall some notations and definitions many adopted from [8]. Let \( G \) be a directed graph with a countable vertex set \( V(G) \) and edge set 

\[E(G) \subseteq V(G) \times V(G)\]. If \( E(G) = V(G) \times V(G) \), then we denote \( G \) by \( G_0 \) which 

is called a complete graph. A path \( \gamma \) with length \( \ell(\gamma) = n \) in \( G \) from \( v_0 \) to \( v_n \) is 

the sequence \( \gamma = (v_0, v_1, ..., v_n) \), \( n \geq 1 \) of vertices of \( G \) such that \((v_k, v_{k+1}) \in E(G)\) 

for \( 0 \leq k \leq n - 1 \). A path \( \gamma = (v_0, ..., v_n) \), \( n \geq 1 \) in \( G \) is called a simple \( v \)-cycle 

if \( v_0 = v_n = v \) and \( v_i \neq v \) for \( 1 \leq i \leq n - 1 \). Denote by \( C(G; v) \) the set of all 

simple \( v \)-cycles in the graph \( G \).

Let \( Y(G) = \{(..., y_{i-1}, y_i, y_{i+1}, ...) : y_i \in V(G), (y_i, y_{i+1}) \in E(G), i \in \mathbb{Z}\} \) be the set of two-sided infinite paths in \( G \) and the shift transformation \( T : Y(G) \to Y(G) \) is defined as \( (T y)_i = y_{i+1} \), for \( y \in Y(G) \) and \( i \in \mathbb{Z} \). The system 

\((Y(G), T)\) is called a countable Topological Markov Chain TMC. In the case of 

complete graph, the dynamical system \((Y(G_0), T)\) is called countable Topological 

Bernoulli Scheme TBS. A TMC will be a local perturbation of a countable TBS if 

\( D = E(G_0) - E(G) \) is finite.

Let \((Y(G), T)\) be a given TMC. Consider on \( Y(G) \) a continuous positive 

function \( f : Y(G) \to (0, \infty) \) such that \( \sum_{k=1}^{\infty} f(T^k y) = \sum_{k=1}^{\infty} f(T^{-k} y) = \infty \), 

\( y \in Y(G) \). The set of such functions which depend only on zero coordinate \( y_0 \) 

of the sequence \( y \) is denoted by \( \mathcal{F}(Y(G)) \). A good candidate for such \( f \) is a 

special flow. To be more precise, let \( Y_f(G) = \{(y, u) : y \in Y(G), 0 \leq u \leq f(y)\} \) with the points \((y, f(y))\) and \((T y, 0)\) identified. For \( 0 \leq u, u + t \leq f(y) \) we let 

\( T f(y, t) = (y, u + t) \). The family \( T_f = \{T_f\}, t \in \mathbb{R} \) is a special flow constructed 

over its base \( Y(G) \).

Consider the following series 

\[F_{f, V}(x) = \sum_{v \in V} x^{f(v)}, \] (1)
which is defined for \( x \geq 0 \). This series is convergent at zero and we will call
\[
r(F_{f, V}) = \sup\{x \geq 0 : \sum_{v \in V} x f(v) \text{ is convergent} \}
\]
the radius of convergent of \( F_{f, V} \).

Let \( f \in F^0(Y(G)) \). The generating function of simple \( v \)-cycles with respect to the special flow \( T_f \) constructed over a TMC \( (Y(G), T) \) is defined to be the series
\[
\phi_{G, f, w}(x) = \sum_{\gamma \in C(G; w)} x f^*(\gamma), \quad x \geq 0
\]
where \( f^*(\gamma) = \sum_{i=0}^{n-1} f(v_i), \gamma = (v_1, \ldots, v_n) \). The radius of convergence \( r(\phi_{G, f, w}) \) in \([0, 1)\) is defined as was defined for \([1] \).

## 3 Computing Generating Function

Let \( V(G) \) and \( E(G) \) be as above. Consider a “weighted” adjacent matrix \( A_G = [a_{ij}] \), that is, a matrix where \( a_{ij} = x f(v) \) if \((v_i, v_j) \in E(G)\) and zero otherwise. For each \( v \in V(G) \), let \( V_v^+ = \{v' \in V(G) : (v, v') \in E(G)\} \) and \( V_v^- = \{v' \in V(G) : (v', v) \in E(G)\} \) be the follower and leading set for \( v \) respectively. Let \( \tilde{V} = \{v \in V(G) : \exists v' \in V(G) \ni (v, v') \notin E(G) \text{ or } (v', v) \notin E(G)\} \). Note that \( \tilde{V} \) may be infinite or even equal to \( V(G) \). Let \( \rho \) be an equivalence relation on \( V(G) \) defined by \( v \overset{\rho}{\sim} v' \Leftrightarrow (V_v^+ = V_{v'}^+ \text{ and } V_v^- = V_{v'}^-) \) and \( P \) be the associated partition. We are interested in cases where \(|P| < \infty\).

We alter a bit the above notations and will produce a quotient set for \( G \) which is again a connected directed graph. To achieve that fix \( w \in V(G) \) and let \( W_{\{w\}} = P \vee \{\{w\}, V(G) - \{w\}\} \) be the set of all non-empty intersections of \( P \) with the partition \( \{\{w\}, V(G) - \{w\}\} \). Then \(|W_{\{w\}}| < \infty \) and let \( V_0 = \{w\}, V_1, \ldots, V_m \) be the elements of \( W_{\{w\}} \).

For \( v \in V_i \) define the follower and leading sets for \( V_i \) as \( V_i^+ = V_v^+ \) and \( V_i^- = V_v^- \) respectively. Note that any \( V_i, V_i^+ \text{ or } V_i^- \) can be written as the union of some elements of \( W_{\{w\}} \). Therefore, a directed graph \( H \) arises with vertex set \( W_{\{w\}} \) and the edge set \( E(H) = \{(V_i, V_j) : (V_i, V_j) \in W_{\{w\}} \times W_{\{w\}}, v_i, v_j \in E(G), v_i, v_j \in V_i, v_j \in V_j\} \). We call \( H \) the quotient graph for \( G \). Graph \( H \) is connected because \( G \) is connected.

By reindexing the elements of \( W_{\{w\}} \), we may assume \( V_1, \ldots, V_k = V_{\{w\}}^+ \). Let \( H_{\{w\}} \) be a tree with root \( \{w\} \) and \( V_1, \ldots, V_k \) in its second level. We wish to extend \( H_{\{w\}} \) to a tree \( T_{\{w\}} \) whose first two levels are exactly \( H_{\{w\}} \) and any path starting from \( \{w\} \) ends at \( \{w\} \). By this we mean the third level of \( T_{\{w\}} \) consists of the follower sets of \( V_j \)'s, \( V_j \neq \{w\} \) and \( 1 \leq j \leq k \). Again the next level consists of the follower sets of vertices of third level which are not \( \{w\} \) and so on.

**Theorem 1.** Let \( k \) be the number of vertices at the second level of \( T_{\{w\}} \). Then all the elements of \( W_{\{w\}} \) appear at the vertices of \( T_{\{w\}} \) at most up to level \( m - k + 2 \).
Proof. If \( k = m \) we are done. So assume \( k < m \). Then the third level must have a vertex which is not in \( \{V_1, \ldots, V_m\} \). Otherwise, that vertex will not appear in any level which is in contradiction with the fact that \( H \) is connected. By the same reasoning, any higher level must have one new vertex until all of them have appeared.

This theorem justifies that such \( T(\omega) \) exists. Because by replacing \( \{w\} \) with any other vertex \( V \) in \( W(\omega) \) and using the same proof as the above theorem, \( \{w\} \) will appear at least once as vertex in a tree with root at \( V \).

The next lemma states that how in our case the computation of generating function can be simplified. Let \( \alpha_i = \alpha_i(x) = \sum_{v \in V_i} x^{f(v)} \) and set \( \alpha_{ij} = \alpha_{ij}(x) = \alpha_i(x) \) if \( (V_i, V_j) \in E(H) \) and zero otherwise.

**Lemma 1.** Suppose \((\gamma(G), T)\) is an RFT and \( f \in F^0(\gamma(G)) \) with \( r(F_{f,V}) > 0 \). Then there exist series \( A_i(x) \) which are the solution of the follower set of equations

\[
A_i(x) = \alpha_{i0}(x) + \alpha_{i1}(x)A_1(x) + \alpha_{i2}(x)A_2(x) + \ldots + \alpha_{im}(x)A_m(x),
\]

for \( 1 \leq i \leq m \) so that the generating function for the flow \( T_f \) is

\[
\phi_{G,f,w}(x) = \alpha_{00}(x) + \alpha_{01}(x)A_1(x) + \alpha_{02}(x)A_2(x) + \ldots + \alpha_{0m}(x)A_m(x).
\]

Here \( r(\phi_{G,f,w}) = \min \{r(A_1), \ldots, r(A_m)\} \leq r(F_{f,V}) \).

Proof. Let

\[
A_i(x) = \sum_{v \in V_i} \sum_{\gamma = (v, \ldots, w)} x^{f^*(\gamma)}
\]

be a series on all paths in \( G \) starting at a vertex \( v \in V_i \) and ending at \( w \). Then

\[
A_i(x) = \left( \sum_{v \in V_i} x^{f(v)} \right) \sum_{v_j \in V_i^+} \sum_{v_j' \in V_i} x^{f^*(\gamma)} = \alpha_{i0}(x) + \alpha_{i1}(x)A_1(x) + \alpha_{i2}(x)A_2(x) + \ldots + \alpha_{im}(x)A_m(x).
\]

Since \( V_i \cap V_j = \emptyset \) for \( i \neq j \), \( F_{f,V}(x) = \sum_{i=1}^m \alpha_i(x) \). Hence

\[
r(F_{f,V}) = \min \{r(\alpha_1), \ldots, r(\alpha_m)\}
\]

and since each \( A_i(x) \) is a rational map in variables \( \alpha_1(x), \ldots, \alpha_m(x) \), therefore \( \min \{r(A_1), \ldots, r(A_m)\} \leq r(F_{f,V}) \).

Also

\[
\phi_{G,f,w}(x) = \sum_{\gamma \in \gamma(G,w)} x^{f^*(\gamma)}
\]

for

\[
x^{f(w)} \sum_{v_i \in V_i^+} \sum_{v \in V_i} \sum_{\gamma = (v, \ldots, w)} x^{f^*(\gamma)} = \alpha_{00}(x) + \alpha_{01}(x)A_1(x) + \alpha_{02}(x)A_2(x) + \ldots + \alpha_{0m}(x)A_m(x).
\]

By the way \( A_i(x) \) is defined above, \( r(\phi_{G,f,w}) = \min \{r(A_1(x)), \ldots, r(A_m(x))\} \). \( \square \)
Set \( A(x) = (A_1(x), ..., A_m(x)) \) and \( \alpha(x) = (\alpha_1(x), ..., \alpha_m(x)) \) and consider them as column vectors when it applies and let

\[
M(x) = \begin{pmatrix}
\alpha_{11}(x) - 1 & \alpha_{12}(x) & \cdots & \alpha_{1m}(x) \\
\alpha_{21}(x) & \alpha_{22}(x) - 1 & \cdots & \alpha_{2m}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m1}(x) & \alpha_{21}(x) & \cdots & \alpha_{mm}(x) - 1
\end{pmatrix}.
\]

(5)

Then statement \( 4 \) in the conclusion of Lemma \( 1 \) implies

\[
M(x) A(x) = -\alpha(x).
\]

(6)

Consider \( 4 \) as a set of equations with unknown \( A(x) \). In the next theorem, we will find \( x \) such that \( A(x) \) satisfies \( 4 \) and \( A(x) \) is a solution of \( 3 \), that is, we will find \( r(\phi_{G,f,w}) \). In fact for \( x > 0 \), a solution of \( 4 \) is a solution of \( 3 \) if and only if \( A_i(x) > 0 \).

Set \( \tilde{x}_0 = r(F_{f,v}) \) if \( M(x) \) is invertible for \( 0 \leq x < r(F_{f,v}) \), or otherwise set \( \tilde{x}_0 = \inf\{x : 0 \leq x < r(F_{f,v})\} \) and \( \det(M(x)) \neq 0 \). Since \( M(0) \) is invertible and \( M(x) \) is continuous then clearly \( \tilde{x}_0 > 0 \) if and only if \( r(F_{f,v}) > 0 \).

**Theorem 2.** Suppose the hypothesis of Lemma \( 1 \) is satisfied and \( A(x), \ M(x) \) and \( \tilde{x}_0 \) are as above. Then \( r(\phi_{G,f,w}) = \tilde{x}_0 \) and if \( \tilde{x}_0 < r(F_{f,v}) \), \( \lim_{x \to \tilde{x}_0} A_i(x) = \lim_{x \to \tilde{x}_0} \phi_{G,f,w}(x) = \infty \).

**Proof.** Let \( e_i = (0, 0, ..., 1, 0, ..., 0) \) be the unit vector whose \( i \)-th entry is 1. Let \( P = \{ \sum_{i=1}^m t_i e_i : t_i \geq 0 \} \) and \( N = \{ \sum_{i=1}^m t_i e_i : t_i \leq 0 \} \). The boundary of \( P \) consists of \( m \) sets \( P_j = \{ \sum_{i=1}^m t_i e_i : t_i \geq 0 \}, 1 \leq j \leq m \). These are \( m-1 \) dimensional manifolds with boundaries. Let \( M_x : v(x) \mapsto M(x)v(x) \), \( S^+(x) = M_x(P) \) and \( S_j(x) = M_x(P_j) \). Then \( \partial P = \cup_{j=1}^m P_j \) and if \( M(x) \) is invertible, \( \partial S^+ = \cup_{j=1}^m S_j(x) \).

Recall that if \( W \) is a subspace of \( \mathbb{R}^m \) of codimension 1, then \( \mathbb{R}^m \setminus W \) consists of two unbounded components. But when \( M(x) \) is invertible, \( \partial S^+(x) \) is a homeomorphic image of such a \( W \) and hence \( \mathbb{R}^m \setminus \partial S^+(x) \) consists of two unbounded components as well. Also for any \( x \) and \( j \) we have \( S_j(x) \cap N^o = \emptyset \) where \( N^o \) is the interior of \( N \). That is because the \( j \)-th entry of a nonzero vector in \( S_j(x) \) which is equal to \( t_j \alpha_{j1}(x) + ... + t_{-1} \alpha_{(-1)j}(x) + t_{+1} \alpha_{(j+1)j}(x) + ... + t_m \alpha_{mj}(x) \) for nonzero \( t_j \)'s is never negative. Hence as far as \( M(x) \) is invertible, \( \partial N \) lies in one side of \( \mathbb{R}^m \setminus \partial S^+(x) \).

Note that \( M(0) = -Id \) and hence it is invertible and \( P = M_x^{-1}(N) \). Also by continuity, for small positive \( x \), \( M(x) \) remains invertible and in fact \( P \subseteq M_x^{-1}(N) \). It may happen that we have \( x \) such that \( M(x) \) is not invertible and set as above \( \tilde{x}_0 \) to be the smallest positive real such that \( \det(M(\tilde{x}_0)) = 0 \). Since the entries of \( M(x) \) does not decrease as \( x \) increases, \( S^+(x_1) \subseteq S^+(x_2) \) when \( x_1 < x_2 < \tilde{x}_0 \). Hence \( M^{-1}_x S^+(x_1) \subseteq M^{-1}_x S^+(x_2) \) and then \( M^{-1}_x(S^+(0)) = M^{-1}_x(N) \subseteq P \) for \( 0 < x < \tilde{x}_0 \).

In particular, \( A(x) = M^{-1}_x(-\alpha) \in P^o \) where by uniqueness of solutions \( A(x) \) will be the same as \( 4 \). We are done if we show that for each \( i \)

\[
\lim_{x \to \tilde{x}_0} A_i(x) = \infty.
\]

(7)
First note that since $A_i(x)$ is increasing, the limit exists and $\mathcal{M}_{x_0}(-\alpha) \in T := \mathcal{M}_{x_0}(\mathbb{R}^m)$ . The dimension of $T$ is at most $m - 1$ and we prove this by claiming that $T$ does not intersect $\mathcal{P}$ . If it was not the case, then by continuity there is $x_0$ such that $S^+(x_0)$ intersects $S^-(x_1)$ in at least one nonzero vector. That means there is $v \neq 0$ such that $v = \sum_{i \geq 0} t_i M(x_1) e_i = \sum_{i \geq 0} t_i M(x_1)(-e_i)$ or equivalently $\sum_{t_i, t_i' \geq 0} (t_i + t_i') M(x_1) e_i = 0$ which implies that $M(x)$ is nonsingular for $x_1 < x_0$ which is absurd. 

**Corollary 1.** Suppose $(Y(G), T)$ is an RFT and $f \in F^0(Y(G))$. Then $\phi_{G,f,w}(x)$ is $C^1$. 

**Proof.** This is clear if $r(\phi_{G,f,w}) = 0$. Otherwise the proof follows from the fact that $F_{f,V}(x)$ is $C^1$ (see proof of Theorem 2 in [5]). Because then $\alpha_{ij}(x)$ is $C^1$ and since $M(x)$ is invertible, $A_i(x)$ is a rational map on $\alpha_{ij}(x)$’s and hence $C^1$. Now since $\phi_{G,f,w}(x)$ is a polynomial on $\alpha_{ij}(x)$’s and $A_i(x)$’s, must be $C^1$. 

Still some other results may be interesting. For instance for $1 \leq i \leq m$ write $\alpha_{ij}(x)$ as 

$$ -\alpha_{0}(x) = \alpha_{1}(x) A_1(x) + ... + (\alpha_{ii}(x) - 1) A_i(x) + ... + \alpha_{im}(x) A_m(x), $$

and note that $\alpha_{ij}(x)$ and $A_i(x)$ are non-negative. This implies at least one coefficient of $A_i(x)$ on right, in our case $(\alpha_{ii}(x) - 1)$, must be negative or $\alpha_{ii}(x) < 1$. So if $\alpha_{ii}(x) > 0$ for all $i$, then $F_{f,V} = \sum_{i=1}^{m} \alpha_i(x)$ is uniformly bounded on the domain of $\phi_{G,f,w}(x)$. Though this last result holds anytime if $x_0 < r(F_{f,V})$. 

**Remark 1.** Let $(Y(G), T)$ be a topological Markov chain and $f \in F^0(Y(G))$ with generating function $\phi_{G,f,w}(x)$. Also suppose $h(T_f)$ the topological entropy of the flow $T_f$ on $Y(G)$ is not infinity. Then by a result in [2], for an arbitrary $w \in V(G)$

$$ h(T_f) = \inf \left\{ h \geq 0 : \sum_{\gamma \in C(G,w)} e^{-hf（\gamma）} \leq 1 \right\}. $$

Since $\sum_{\gamma \in C(G,w)} e^{-hf（\gamma）} = \phi_{G,f,w}(e^{-h})$, we have $h(T_f) = -\ln(\hat{x_f})$ where $\hat{x_f} = \sup \{x \geq 0 : \phi_{G,f,w}(x) \leq 1 \}$. Therefore, the problem of computing $h(T_f)$ reduces to find $\hat{x_f}$. By the fact that $\phi_{G,f,w}(x)$ is increasing, then $\hat{x_f}$ is either the unique solution of $\phi_{G,f,w}(x) = 1$ or $\hat{x_f} = r(\phi_{G,f,w})$. 

3 COMPUTING GENERATING FUNCTION

6
4 Applications

In this section we give three examples. The first two arise in the study of the geodesic flows in the modular surface. The first is a local perturbation of a TBS and our goal is to compare the algorithm given in [8] and the one deduced from Theorem 2. For this reason, we choose exactly the same example appearing as Example 1 in [8]. We only produce \( \phi_{G,f,w}(x) \) and refer the reader to [8] for the facts behind this example and also seeing how this function is applied to obtain an approximation for entropy of the respective dynamical system.

The second example with some more details is not a local perturbation of a countable TBS. For that example \( \phi_{G,f,w}(x) \) is obtained and an estimate of the entropy will be given. The last example illustrates a case where \( r(\phi_{G,f,w}) = r(F_{f,V}) \).

**Example 1.** Let \( V(G) = \{3, 4, 5, 6, \ldots\} \), \( w = 3 \) and take the set of forbidden edges to be
\[
D = \{(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)\}.
\]
This example occurs in the coding of geodesic flows on the modular surface. See Example 1 in [8] for a brief explanation. There the following formula is defined
\[
\phi_{G,f,3}(x) = \frac{x^{f(3)}(F_{f,V}(x) - x^{f(3)} - x^{f(4)} - x^{f(5)})}{1 + x^{f(3)} - F_{f,V}(x)}
\]
(8)
for the case when denominator is positive.

By the above method the relation \( \rho \) on \( V(G) \), is
\[
W_{(3)} = \{V_0 = \{3\}, V_1 = \{4, 5\}, V_2 = V(G) \setminus \{3, 4, 5\}\}.
\]
Hence \( m = 2 \) and \( \phi_{G,f,3}(x) = x^{f(3)}A_2(x) \). Statement [3] in the above theorem implies
\[
\begin{align*}
A_1(x) &= \alpha_{10}x + \alpha_{11}A_1(x) + \alpha_{12}A_2(x) = (x^{f(4)} + x^{f(5)})(A_1(x) + A_2(x)), \\
A_2(x) &= \alpha_{20}x + \alpha_{21}A_1(x) + \alpha_{22}A_2(x) = (\sum_{v \in V_2} x^{f(v)})(A_1(x) + A_2(x) + 1),
\end{align*}
\]
where all \( \alpha_{ij} = \alpha_{ij}(x) \) are real functions and in fact \( \alpha_{11} = \alpha_{12} = x^{f(4)} + x^{f(5)} \), \( \alpha_{21} = \alpha_{22} = \sum_{i=6}^{\infty} x^{f(i)} \) and \( \alpha_{20} = x^{f(3)} \). So, \( M = \begin{pmatrix} \alpha_{11} - 1 & \alpha_{12} \\ \alpha_{21} & \alpha_{22} - 1 \end{pmatrix} \) and if \( \det M \neq 0 \), then
\[
\begin{pmatrix} A_1(x) \\ A_2(x) \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ -\alpha_{20} \end{pmatrix}.
\]
Therefore, \( A_1(x) = \frac{\alpha_{11} - \alpha_{20}}{1 - \alpha_{11} - \alpha_{22}} \).
\( A_2(x) = \frac{\alpha_{20}(1-\alpha_{11})}{1-\alpha_{11} - \alpha_{22}} \). But \( 1 - \alpha_{11} - \alpha_{22} > 0 \), because \( A_i \)'s and \( \alpha_{ij} \)'s are positive by definition and \( 1 - \alpha_{11} > 0 \) by the results following Corollary [11]. That means \( \phi_{G,f,3}(x) = x^{f(3)}A_2(x)(1-\alpha_{11}(x)) \). By evaluating \( \alpha_{ij}(x) \), the formula [3] will be established. Note that \( \det M = 1 - \alpha_{11}(x) - \alpha_{22}(x) = 1 + x^{f(3)} - F_{f,V}(x) \) which is the denominator in [3].
Example 2. Recall from [3] that any bi-infinite sequence of non-zero integers \( \{\ldots, v_{-1}, v_0, v_1, \ldots\} \), \( |v_i| \neq 1 \) such that \( \frac{1}{v_i} + \frac{1}{v_{i+1}} \leq \frac{1}{2} \) is realized as a geometric code of an oriented geodesic on the modular surface. These codes are produced by choosing a suitable cross section, that is, a set which is hit infinitely many times in past and future by geodesic. So let \( V(G) = \{ v \in \mathbb{Z} : |v| \geq 2 \} \) and

\[
D_1 = \{(−3,−3),(−3,−4),(−3,−5),(−4,−3),(−5,−3), (3,3),(3,4),(3,5),(4,3),(5,3)\},
\]

\[
D_2 = \{(−v,−2),(−2,−v),(v,2),(2, v) : v \geq 2\}.
\]

Set \( D = D_1 \cup D_2 \), that is, \( (v, v′) \in E(G) \) if and only if \( |\frac{1}{v} + \frac{1}{v′} | \leq \frac{1}{2} \). Let \( X = \{\ldots, v_{−1}, v_0, v_1, \ldots\} : |v_i| \geq 2, |\frac{1}{v_i} + \frac{1}{v_{i+1}} | \leq \frac{1}{2} \}, \sigma(v_i) = v_{i+1} \) and \( (X, \sigma) \) the associated system. In fact, \( (X \cup \{\ldots, 1, −1, 1, −1, \ldots\}, \sigma) \) is the maximal 1-step countable topological Markov chain in the set of all admissible codes known as geometric codes [3, Theorem 2.3].

By putting \( w = 2, W_{[2]} = \{V_0 = \{2\}, V_1 = \{3\}, V_2 = \{4, 5\}, V_3 = \{6, 7, \ldots\}, V_4 = \{-2\}, V_5 = \{-3\}, V_6 = \{-4, −5\}, V_7 = \{−7, −6\}\}, \) one sees that \( (X, \sigma) \) is an RFT. Hence we apply our technique to give an estimation for the entropy of \( (X, \sigma) \).

Define \( f(\{\ldots, v_{−1}, v_0, v_1, \ldots\}) = 2 \ln |cv_0|, c = 1.25 \) and let \( \sigma_f \) be the special flow over \( X \) with the ceiling function \( f \). (To keep the continuity of argument, we later give some explanation to justify choosing such an \( f \).) We may assume \( f \) is defined on \( V(G) \) and \( f(v) = 2 \ln |cv| \).

Note that \( \phi_{G,f,2}(x) = x^{f(2)}(A_4(x) + A_5(x)) + A_6(x) + A_7(x) \). Also if \( \alpha_1 = \alpha_i(x) \) and \( A_i = A_i(x) \), then \( A_i \)'s are the solution of the follower set of equations.

\[
\begin{align*}
A_1 &= \alpha_1(A_3 + A_4 + A_5 + A_6 + A_7) \\
A_2 &= \alpha_2(A_2 + A_3 + A_4 + A_5 + A_6 + A_7) \\
A_3 &= \alpha_3(A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7) \\
A_4 &= \alpha_4(1 + A_1 + A_2 + A_3) \\
A_5 &= \alpha_5(1 + A_1 + A_2 + A_3 + A_7) \\
A_6 &= \alpha_6(1 + A_1 + A_2 + A_3 + A_6 + A_7) \\
A_7 &= \alpha_7(1 + A_1 + A_2 + A_3 + A_5 + A_6 + A_7).
\end{align*}
\]

Here \( \alpha_0 = \alpha_4 = x^{f(2)}, \alpha_1 = \alpha_5 = x^{f(3)}, \alpha_2 = \alpha_6 = x^{f(4)} + x^{f(5)} \) and \( \alpha_3 = \alpha_7 = \sum_{v=6}^{\infty} x^{f(v)} \). Therefore, the entropy will be \(-\ln \hat{x}_f = 0.8665 \) where \( \hat{x}_f \) is the unique solution of \( \phi_{G,f,2}(x) = 1 \). (We used the computer software Maple to perform the computations.)

Now we explain why such an \( f \) was chosen above. Let \( x = \{\ldots, v_{−1}, v_0, v_1, \ldots\} \) be a geometric code for an oriented geodesic \( \gamma \). Then \( w(x) = v_0(x) - \frac{1}{v_{−1}(x)} \) called minus continued fraction, represents the attractive end point of \( \gamma \). Let \( h \), the ceiling function, be the first return time function of oriented geodesic. This \( h \) records the time between two hits of cross sections by geodesic and is cohomologous to \( g(x) = 2 \ln |w(x)| \). Since two cohomologous ceiling functions give the same entropy for special flows on the same base space, take \( g \) to be the ceiling function over \( X \). Note that if \( |v_i| = 2 \) and \( |\frac{1}{v_i} + \frac{1}{v_{i+1}} | \leq \frac{1}{2} \), then
$v_i v_{i+1} < 0$ and $|v_{i+1}|$ can be arbitrary large. Therefore,

$$|w(x)| \leq |v_0(x)| + \left| \frac{1}{v_1(x) - \frac{1}{v_2(x) - \frac{1}{v_3(x) - \cdots}}} \right| \leq |v_0(x)| + \frac{1}{2} \leq |v_0(x)| + \frac{|v_0(x)|}{4}. $$

This in turn shows that $|w(x)| \leq 1.25|v_0(x)| = c|v_0(x)|$. But, this implies $g(x) \leq f(x)$ and hence $h(T_g) \geq h(T_f) \geq 0.8665$.

We like to mention that our estimate improves slightly the estimate obtained in [4]. There they consider $(\ldots, v_{i-1}, v_0, v_1, \ldots) : |v_i| \geq 3, v_i \in \mathbb{Z} \subset X$. Then $\sigma$ is invariant on $X$ and let $\sigma' = \sigma|_X$. Let $V'(G) = \{v \in V : |v_i| \geq 3\} \subseteq V(G)$ and $D' = D_1$. It is proved in [4] that the special flow associated to $(X', \sigma')$ is a local perturbation of a countable TBS and based on the results in [8], they estimate the entropy to be greater than 0.84171.

Recall that entropy of the geodesic flows in modular surface is 1 [3], if we roughly agree that bigger entropies of subsystems are due to richer dynamics, hence $(X \cup \{\ldots, 1, -1, -1, \ldots\}, \sigma)$ with entropy greater than 0.8665 is a fairly rich subsystem of the geodesic flows in modular surface.

**Example 3.** Let $G$ be a graph with vertex set $V(G) = \{v_0 = 1, v_1 = 2, \ldots\}$ and edge set $E(G) = \{(v_i, v_j) : v_i \neq v_j$ and either $v_i$ or $v_j$ is $v_0\}$. Let $w = \{1\}$. Then $W_{(1)} = \{V_0 = \{1\}, V_1 = V(G) - \{1\}\}$. So $M(x) = [-1]$ which is invertible for $0 \leq x \leq r(F_{f,V})$. Therefore, by Theorem 2 $r(\phi_{G,f,1}) = r(F_{f,V})$. Also $\phi_{G,f,1}(x) = x^{f(1)}(\sum_{v \in V(G) - \{1\}} x^{f(v)})$.

Now let $f$ be an increasing function such that takes the value 1 on $v_0 = 1$ and value $k \in \mathbb{N}$, $k \geq 2$ exactly $\left\lfloor \frac{r}{k} \right\rfloor$ times, where $\lfloor r \rfloor$ denotes the integer part of $r$. Then $r(F_{f,V}) = \frac{1}{2}$ and $\phi_{G,f,1}(x) = x^{1}(\sum_{v \in V(G) - \{1\}} x^{f(v)}) = x^{(\sum_{k=2}^{\infty} \left\lfloor \frac{r}{k} \right\rfloor) x^k} \leq \phi_{G,f,1}(\frac{1}{2}) < 0.85$. Hence $\hat{x}_f = \frac{1}{2}$. See Remark 1.

## 5 An equivalent formula for generating function

In this section by a rather new approach we give an explicit formula for $\phi_{G,f,w}(x)$ where in the special case of local perturbation of a TBS, the formula will exactly be the one given in [8]. (See corollary 2.) Let $W_{(w)} = \{V_0 = \{w\}, V_1, \ldots, V_m\}$ be the partition of $V(G)$ as before. Let $V_{\bar{G}} = \{v \in V : \exists v' \in V \ni (v, v') \notin E(G)\}$ \{w\}. We reindex $V_i$’s so that for some $1 \leq \ell \leq m$,

\[
\begin{align*}
V_{\bar{G}} = & \quad V_1 \cup \ldots \cup V_{\ell}; \\
\bigcup_{i=\ell+1}^{m-1} V_i = & \quad \{v \in V(G) : (v, v') \in E(G), v' \in V(G), (v'', v) \notin E(G), \exists v'' \in V(G)\}; \\
V_m = & \quad \{v \in V(G) : (v, v') \in E(G), (v', v) \in E(G), v' \in V(G)\}.
\end{align*}
\]
Let $M = M(x)$ be the matrix in (5) which is obtained from $W_{\{w\}} \setminus \{w\}$. Let $\ell$ be as in (9) and
\[
\begin{pmatrix}
\alpha_{11} - 1 & \alpha_{12} & \ldots & \alpha_{1\ell} \\
\alpha_{21} & \alpha_{22} - 1 & \ldots & \alpha_{2\ell} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{\ell 1} & \alpha_{\ell 2} & \ldots & \alpha_{\ell \ell} - 1
\end{pmatrix}
\] (10)
an $\ell \times \ell$ sub-matrix of $M$ on the upper left corner. Note that $B = C + Id$ represents a weighted adjacent matrix for the vertices of $V_G$. The next lemma shows when $\phi_{G,f,w}(x)$ is defined, then $C$ is invertible.

**Lemma 2.** Suppose $x_C$ is the smallest real number such that $\det C = 0$. Then $x_C \geq r(\phi_{G,f,w})$.

**Proof.** We have
\[
\sum_{n=0}^{\infty} [B(x)]_{V_i V_j}^{(n)} = \frac{(-1)^{e(V_i) + e(V_j)} \det((Id - B(x))_{e(V_i) e(V_j)})}{\det(Id - B(x))}
\] (11)
where $(Id - B(x))_{e(V_i) e(V_j)}$ is the sub-matrix of $Id - B(x) = -C$ obtained by deleting $V_j$th row and $V_i$th column. In fact (11) is the same identity as (2.18) in [8]. It holds because if we let $A = [a_{ij}]$ be a $p \times p$ matrix over $\mathbb{C}$ and if $a_{ij}$ be the 

Note that the right hand of (11) is by definition $[C^{-1}]_{V_i V_j}$, the $V_i V_j$th entry of $C^{-1}$. This also shows that the radius of convergence of the series on the left is $x_C$. But $\sum_{n=0}^{\infty} [B(x)]_{V_i V_j}^{(n)} = \sum x^{f(\gamma)}$ where the sum on the right is over the paths $\gamma$ from all $v_i$ in $V_i$ to all $v_j$ in $V_j$ with all possible lengths. If there is not any path from $V_i$ to $V_j$ in $\tilde{V}_G$ then $\sum_{n=0}^{\infty} [B(x)]_{V_i V_j}^{(n)} = 0$. Let $H$ be the reduced graph of $G$. Since the graph $H$ with vertices $W_{\{w\}}$ is connected so we can fix one path $\gamma_1$ from $w$ to $v_i \in V_i$ and one path $\gamma_2$ from $v_j \in V_j$ to $w$. Then $\gamma_1 \gamma_2 \in C(G; w)$ and $\sum_{V_i, V_j \in V_G} \sum_{\gamma \in [B(x)]_{V_i V_j}} x^{f(\gamma_1 \gamma_2)} \leq \phi_{G,f,w}(x)$ which implies $x_C \geq r(\phi_{G,f,w})$. \qed

Recall that $a_i = \alpha_i(x) = \sum_{v \in V_i} x^{f(v)}$, $1 \leq i \leq m$. Then $A_i$’s satisfy (3) or equivalently
\[
\begin{align*}
A_1 &= \alpha_{10} + \alpha_{11} A_1 + \ldots + \alpha_{1m} A_m \\
A_2 &= \alpha_{20} + \alpha_{21} A_1 + \ldots + \alpha_{2m} A_m \\
\vdots & \\
A_\ell &= \alpha_{\ell 0} + \alpha_{\ell 1} A_1 + \ldots + \alpha_{\ell m} A_m \\
A_{\ell + 1} &= \alpha_{\ell + 1}(1 + A_1 + A_2 + \ldots + A_m) \\
\vdots & \\
A_m &= \alpha_m(1 + A_1 + A_2 + \ldots + A_m).
\end{align*}
\] (12)
5 An Equivalent Formula for Generating Function

Note that \( \alpha_i(x) \neq 0 \) for \( i \geq m_0 \). Let \( \zeta = \zeta(\ell, m) := \alpha_{\ell+1} + ... + \alpha_{m-1} \) and \( F_{f, V, V_i} = F_{f, V, V_i}(x) = \sum_{v \in V_i^+ - V_G} x^{f(v)} \). Denote by \( \langle \cdot, \cdot \rangle \) the standard dot product of two vectors and \( \text{Row}_i(N) \) the \( i \)th row of matrix \( N \). Set

\[
\alpha_{H, f, w}(x) = \sum_{(w, V_i) \in \tilde{E}(H)} (\text{Row}_i(C^{-1}), [-\alpha_1 F_{f, V, V_1}, ... , -\alpha_{\ell} F_{f, V, V_\ell}])
\]

\[
\alpha_{H, f, \tilde{V}_G}(x) = \sum_{V_i \subseteq \tilde{V}_G} (\text{Row}_i(C^{-1}), [-\alpha_1 F_{f, V, V_1}, ... , -\alpha_{\ell} F_{f, V, V_\ell}])
\]

\[
\sigma_{H, f, w}(x) = \sum_{V_i \subseteq \tilde{V}_G} (\text{Row}_i(C^{-1}), [-\alpha_{10}, ... , -\alpha_{\ell 0}])
\]

\[
\tilde{\phi}_{H, f, w}(x) = \sum_{(w, V_i) \in \tilde{E}(H)} (\text{Row}_i(C^{-1}), [-\alpha_{10}, ... , -\alpha_{\ell 0}]) + \alpha_{00}.
\]

Theorem 3. Let \((Y(G), T)\) be an RFT and \( f \in \mathcal{F}^0(Y(G_0)) \). Then for \( w \in V(G) \)

\[
\phi_{G, f, w}(x) = x^{f(w)}(F_{f, V, w}(x) + \alpha_{H, f, w}(x)) \left( \frac{\sigma_{H, f, w}(x) + 1}{1 - \zeta(x) - \alpha_m(x) - \alpha_{H, f, \tilde{V}_G}(x)} \right)
\]

\[
+ \tilde{\phi}_{H, f, w}(x)
\]

for those \( 0 \leq x < r(F_{f, V}) \) where

\[
1 - \zeta(x) - \alpha_m(x) - \alpha_{H, f, \tilde{V}_G}(x) > 0.
\]

Proof. Let \( S_1 = S_1(x) := A_1(x) + ... + A_\ell(x) \) and \( S_2 = S_2(x) := A_{\ell+1}(x) + ... + A_{m-1}(x) \). Then \( S_2 = \zeta(1 + S_1 + S_2 + A_m) \) or equivalently \( S_2 = \frac{\zeta}{1 - \zeta - \alpha_m}(S_1 + 1) \). By Evaluating \( A_j, j > \ell \) in terms of \( S_1 \) and \( S_2 \), and then evaluating \( S_2 \) and \( A_m \) in terms of \( S_1, \zeta \), we will have

\[
C \begin{pmatrix} A_1 \\ \vdots \\ A_\ell \end{pmatrix} = \begin{pmatrix} -\alpha_{1(\ell+1)} A_{\ell+1} - ... - \alpha_{1(m-1)} A_{m-1} - \alpha_1 A_m - \alpha_{10} \\ \vdots \\ -\alpha_{\ell(\ell+1)} A_{\ell+1} - ... - \alpha_{\ell(m-1)} A_{m-1} - \alpha_{\ell} A_m - \alpha_{\ell 0} \end{pmatrix}
\]

\[
= \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_\ell \end{pmatrix} - \begin{pmatrix} \alpha_{10} \\ \vdots \\ \alpha_{\ell 0} \end{pmatrix}.
\]
AN EQUIVALENT FORMULA FOR GENERATING FUNCTION

where

\[ \xi_i = \sum_{k=\ell+1}^{m-1} -\alpha_{ik}A_k - \alpha_iA_m \]

\[ = \sum_{k=\ell+1}^{m-1} -\alpha_{ik}(\alpha_k + \alpha_kS_1 + \alpha_kS_2 + \alpha_kA_m) - \alpha_iA_m \]

\[ = -\sum_{k=\ell+1}^{m-1} (\alpha_{ik})\alpha_k(S_1 + S_2 + A_m + 1) - \alpha_iA_m \]

\[ = \frac{(S_1 + 1)(\sum_{k=\ell+1}^{m-1} \alpha_i(\alpha_{ik}))}{1 - \zeta - \alpha_m} \]

\[ = \frac{(S_1 + 1)(\alpha_iF_{f,V,V_i})}{1 - \zeta - \alpha_m}. \]

Therefore we have

\[ \phi_{G,f,w}(x) = \alpha_{01}A_1 + \ldots + \alpha_{0\ell}A_\ell + \alpha_{0(\ell+1)}A_{\ell+1} + \ldots + \alpha_{0m}A_m + \alpha_{00} \]

\[ + x^f(w)F_{f,V,w}(1 + S_1 + S_2 + A_m) + \alpha_{00}. \]

This leads to

\[ \phi_{G,f,w}(x) = x^f(w) \sum_{V_i \subseteq \tilde{V}_G} \langle Row_i(C^{-1}), [\xi_1, \ldots, \xi_\ell] - [\alpha_{10}, \ldots, \alpha_{\ell0}] \rangle \]

\[ + x^f(w)F_{f,V,w}\left(1 + S_1 + \frac{\alpha_m + \zeta}{1 - \zeta - \alpha_m}(1 + S_1)\right) + \alpha_{00} \]

\[ = x^f(w) \sum_{(w,V_i) \in \tilde{E}(H)} \langle Row_i(C^{-1}), [\xi_1, \ldots, \xi_\ell] \rangle \]

\[ + x^f(w) \sum_{(w,V_i) \in \tilde{E}(H)} \langle Row_i(C^{-1}), -[\alpha_{10}, \ldots, \alpha_{\ell0}] \rangle \]

\[ + x^f(w)\left(S_1 + 1\right)F_{f,V,w}\frac{1}{1 - \zeta - \alpha_m} + \alpha_{00} \]

\[ = x^f(w)\frac{(S_1 + 1)}{1 - \zeta - \alpha_m} \times \]

\[ \left( \sum_{(w,V_i) \in \tilde{E}(H)} \langle Row_i(C^{-1}), [-\alpha_1F_{f,V,V_1}, \ldots, -\alpha_\ell F_{f,V,V_\ell}] \rangle \right) \]

\[ + x^f(w)\left(S_1 + 1\right)F_{f,V,w}\frac{1}{1 - \zeta - \alpha_m} + \tilde{\phi}_{H,f,w}(x) \]

\[ = x^f(w)\frac{(S_1 + 1)}{1 - \zeta - \alpha_m} \left(\alpha_{H,f,w}(x) + F_{f,V,w} + \tilde{\phi}_{H,f,w}(x)\right). \]
But
\[
S_1 = A_1 + \ldots + A_\ell \\
= \sum_{i=1}^{\ell} \langle \text{Row}_i(C^{-1}), ([\xi_1, \ldots, \xi_\ell] - [\alpha_{10}, \ldots, \alpha_{10}]) \rangle \\
= \frac{S_1 + \ldots + \ell \sum_{i=1}^{\ell} \langle \text{Row}_i(C^{-1}), [-\alpha_1 F_{f,V_1}, \ldots, -\alpha_\ell F_{f,V_\ell}] \rangle}{1 - \zeta - \alpha_m} \\
+ \sigma_{H,f,w}(x) \\
= \frac{S_1 + \ldots + \ell \sum_{i=1}^{\ell} \langle \text{Row}_i(C^{-1}), [-\alpha_1 F_{f,V_1}, \ldots, -\alpha_\ell F_{f,V_\ell}] \rangle}{1 - \zeta - \alpha_m} \alpha_{H,f,\tilde{V}_G}(x) + \sigma_{H,f,w}(x).
\]
That means
\[
S_1 = \frac{\alpha_{H,f,\tilde{V}_G}(x) + \sigma_{G,f,w}(x)(1 - \zeta - \alpha_m)}{1 - \zeta - \alpha_m - \alpha_{H,f,\tilde{V}_G}(x)}
\]
where the denominator is positive. So
\[
\phi_{G,f,w}(x) = x^f(w)(F_{f,V,w} + \alpha_{H,f,w}(x)) \left( \frac{S_1 + \ldots + \ell \sum_{i=1}^{\ell} \langle \text{Row}_i(C^{-1}), [-\alpha_1 F_{f,V_1}, \ldots, -\alpha_\ell F_{f,V_\ell}] \rangle}{1 - \zeta - \alpha_m} \right) + \tilde{\phi}_{H,f,w}(x)
\]
\[
= x^f(w)(F_{f,V,w} + \alpha_{H,f,w}(x)) \left( \frac{\sigma_{G,f,w}(x) + 1}{1 - \zeta - \alpha_m - \alpha_{H,f,\tilde{V}_G}(x)} \right)
\]
\[
+ \tilde{\phi}_{H,w,f,w}(x).
\]
Note that all the arguments are reversible and the proof is established. \(\square\)

The following Theorem relates the condition (13) to the conclusion of Theorem 2.

**Theorem 4.** Let \(M\) and \(C\) be as before. Then
\[
\frac{\det M}{(-1)^{m-\ell} \det C} = 1 - \zeta - \alpha_m - \alpha_{H,f,\tilde{V}_G}(x).
\]
Note that the statement on right is the same statement appearing in (1).

**Proof.** For \(k \leq m\) let \(X_{(k)} = (x_1, \ldots, x_k)\) and \(b_k = (\alpha_{01}, \ldots, \alpha_{0k})\). Recall \(N_{(i)(j)}\) is the sub-matrix of \(N\) obtained by deleting its \(i\)th row and \(j\)th column. Also let \(N_{(j)}\) be the matrix obtained from an \(n \times n\) matrix \(N\) by replacing its \(j\)th column with \(b_n\). Consider \(N X_{(k)} = -b_k\). Then by using Cramer’s rule, we have
\[
x_i = \frac{\det N_{(i)}}{\det N} \quad \text{or} \quad \det N_{(i)} = x_i \det N, \quad 1 \leq i \leq n.
\]
The proof is based on the induction on the last \(m-\ell\) rows of \(M\). First let \(M = C\). Then \(\zeta, \alpha_m\) and \(\alpha_{H,f,\tilde{V}_G}(x)\) are all zero and the conclusion holds trivially. If we take \(m = \ell + 1\). Then by choosing the last row of \(M\) for computing
the determinant of $M$ and then using the Cramer’s rule we have

\[
\frac{\det M}{-\det C} = \frac{1}{-\det C} \left( (-1)^{m+1} \alpha_m \det M_{\varepsilon(m)\varepsilon(1)} + \cdots + (-1)^{m+\ell} \alpha_m \det M_{\varepsilon(m)\varepsilon(\ell)} + (-1)^{2m} (\alpha_m - 1) \det M_{\varepsilon(m)\varepsilon(m)} \right)
\]

\[
= \frac{\alpha_m}{-\det C} \left( (-1)^{m+1} (-1)^{m-1} \det C_{\varepsilon(1)} + \cdots + (-1)^{m+\ell} (-1)^{m-\ell} \det C_{\varepsilon(\ell)} \right) + (1 - \alpha_m)
\]

\[
= \frac{\alpha_m}{-\det C} (x_1 \det C + \cdots + x_\ell \det C) + (1 - \alpha_m)
\]

\[
= -\alpha_m \ell (\text{Row}_1 C^{-1}, -b_\ell) + \cdots + \langle \text{Row}_\ell C^{-1}, -b_\ell \rangle + (1 - \alpha_m)
\]

\[
= -\sum_{i=1}^\ell \langle \text{Row}_i C^{-1}, [-\alpha_1 F_{j,V,V_1}, \ldots, -\alpha_\ell F_{j,V,V_\ell}] \rangle + (1 - \alpha_m)
\]

\[
= -\alpha_{H,f,\bar{V}_G}(x) + (1 - \alpha_m).
\]

To emphasize the dependent of $\zeta$, $F_{j,V,V_i}$ and $\alpha_{H,f,\bar{V}_G}$ on $m$, we will show them by $\zeta^{(m)}$, $F^{(m)}_{j,V,V_i}$ and $\alpha^{(m)}_{H,f,\bar{V}_G}$. Now let $m_0 = \ell + k$, $k > 1$ and assume

\[
\det \frac{M}{-1}^{k+1} \det C = 1 - \zeta^{(m_0)} \alpha_{m_0} - \alpha^{(m_0)}_{H,f,\bar{V}_G}.
\]

(14)

We prove the formula for $m = m_0 + 1$. Again we apply Cramer’s rule, then

\[
\det \frac{M}{-1}^{k+1} \det C = \frac{(-1)^{2m}}{(-1)^{k+1} \det C} (\alpha_m \det M_{\varepsilon(m)\varepsilon(1)} + \cdots + \alpha_m \det M_{\varepsilon(m)\varepsilon(m_0)} + (\alpha_m - 1) \det M_{\varepsilon(m)\varepsilon(m)} \right)
\]

\[
= \frac{\alpha_m}{(-1)^{k+1} \det C} \left( x_1 + \cdots + x_{m_0} \right)
\]

\[
+ \frac{\det M_{\varepsilon(m)\varepsilon(m)} (-1)^{k+1} \det C}{(-1)^{k+1} \det C} (\alpha_m - 1).
\]

Compute $x_i$, $0 \leq i \leq m_0$ from the set of equations $M_{\varepsilon(m)\varepsilon(m)} X_{(m_0)} = -b_{m_0}$.

It is convenient to let $S_1 = x_1 + \cdots + x_\ell$ and $S_2 = x_{\ell+1} + \cdots + x_{m_0}$. The first $\ell$ equations can be written as

\[
C \begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix} = \begin{pmatrix} -\alpha_{01} - \alpha_{1(\ell+1)} x_{\ell+1} - \cdots - \alpha_{1m_0} x_{m_0} \\ \vdots \\ -\alpha_{0\ell} - \alpha_{\ell(\ell+1)} x_{\ell+1} - \cdots - \alpha_{\ell m_0} x_{m_0} \\ -\alpha_{01} - \alpha_1 (S_1 + S_2 + 1) F^{(m_0)}_{j,V,V_1} \\ \vdots \\ -\alpha_{0\ell} - \alpha_\ell (S_1 + S_2 + 1) F^{(m_0)}_{j,V,V_\ell} \end{pmatrix}.
\]
So
\[ S_1 = \langle \text{Row} ; C^{-1} , [-\alpha_0 , ..., -\alpha_0] \rangle \]
\[ + (S_1 + S_2 + 1) \sum_{i=1}^{\ell} \langle \text{Row} ; C^{-1} , [-\alpha_1 F_{V_1}^{(m_0)} , ..., -\alpha_\ell F_{V_\ell}^{(m_0)}] \rangle . \]  
(15)

The rest of equations are
\[ \begin{cases} \alpha_{\ell+1} x_1 + \cdots + (\alpha_{\ell+1} - 1) x_{\ell+1} + \cdots + \alpha_{\ell+1} x_{m_0} = -\alpha_{\ell+1} \\ \vdots \\ \alpha_{m_0} x_1 + \cdots + (\alpha_{m_0} - 1) x_{\ell+1} + \cdots + \alpha_{m_0} x_{m_0} = -\alpha_{m_0}. \end{cases} \]  
(16)

So
\[ (\zeta^{(m_0)})(x_1 + \ldots + x_\ell) + (\alpha_{\ell+1} + \ldots + \alpha_{m_0} - 1)(x_{\ell+1} + \ldots + x_{m_0}) = -\zeta^{(m_0)}. \]  
(17)

From (16) and (17) we have
\[ S_1 + S_2 = \frac{\langle \text{Row} ; C^{-1} , [-\alpha_0 , ..., -\alpha_0] \rangle + \zeta^{(m_0)}}{D} + \sum_{i=1}^{\ell} \frac{\langle \text{Row} ; C^{-1} , [-\alpha_1 F_{f,V_1}^{(m_0)} , ..., -\alpha_\ell F_{f,V_\ell}^{(m_0)}] \rangle}{D}, \]
where \( D = 1 - \zeta^{(m_0)} - \sum_{i=1}^{\ell} \langle \text{Row} ; C^{-1} , [-\alpha_1 F_{f,V_1}^{(m_0)} , ..., -\alpha_\ell F_{f,V_\ell}^{(m_0)}] \rangle \). Replace \( M_{e(m)\in(m)} \), for \( M \) in (14). Then
\[ \det M \frac{(-1)^{k+1}}{\det C} = \frac{\det M_{e(m)\in(m)}(\alpha_m(S_1 + S_2) + (\alpha_m - 1))}{\det C} \]
\[ = \sum_{i=1}^{\ell} \frac{\langle \text{Row} ; C^{-1} , [-\alpha_1 F_{f,V_1}^{(m_0)} , ..., -\alpha_\ell F_{f,V_\ell}^{(m_0)}] \rangle}{D} \]
\[ + (1 - \zeta^{(m)} - \alpha_m) \]
\[ = 1 - \zeta^{(m)} - \alpha_m - \alpha_m^{(m)} \]
\[ = 1 - \zeta^{(m)} - \alpha_m - \alpha_m^{(m)} \]
(17)

Now we are in a position to state that our result is identical to those of [3, §2] for a local perturbation of a TBS. Therefore, we recall some notations from [3]. Suppose \( (Y(G), T) \) is a local perturbation of a countable TMC. Fix \( w \in \tilde{V}_G \).

By our earlier notations, \( \tilde{V}_G \subseteq \tilde{V} \) and \( \tilde{V}_G \) is the union of some elements of \( W_w = \{ V_0 = \{ w \} , V_1 , ..., V_{\ell} \} \) which \( V_i , 1 \leq i \leq \ell \) all must have finite elements where \( \ell \) is defined as [3]. For \( v \in \tilde{V}_G \) let \( A_{G,v} = \{ \nu \in V : (v, \nu) \notin E(G) \} \).

Let \( \tilde{G}_w \) be the sub-graph of \( \tilde{G} \) with \( V(\tilde{G}_w) = \tilde{V}_G \setminus \{ w \} \) and \( E(\tilde{G}_w) = \{ (v, \nu) \in E(G) : v, \nu \in V(\tilde{G}_w) \} \). Define a matrix \( B^{\tilde{G}_w} (x) = (b^{\tilde{G}_w}_{\nu \nu'} (x)) , v, v' \in V(\tilde{G}_w) \) where \( b^{\tilde{G}_w}_{\nu \nu'} (x) \) is \( x^{f(v)} \) when \( (v, \nu') \in E(\tilde{G}_w) \) and zero otherwise. Denote by
$[B^G_w]_{n v}$ the entry corresponding to the $v$th row and $v'$th column of $n$th power of
the matrix $B^G_w$. Let $\beta^G_{v v'}(x) = \sum_{v=0}^{\infty} [B^G_w(x)]_{n v'}$, $v, v' \in (\overline{G}_w)$ and $F_{f, V, v}(x) = F_{f, V}(x) - \sum_{v' \in (\overline{V}_G \cup A_G)} x f(v')$ where $v \in V_G$ and the bar over a set is the complement operation. Note that $F_{f, V, v_i} = F_{f, V, V_j}$ for all $v_i \in V_j$. Finally for $U \subseteq \overline{V}_G$ let

$$
\alpha_{U, f, w}(x) = \sum_{v' \in U} \sum_{v'' \in V(\overline{G}_w)} \beta^G_{v v''}(x) x f(v'') F_{f, V, v''}(x)
$$

$$
\sigma_{G, f, w}(x) = \sum_{v' \in V(\overline{G}_w)} \sum_{v'' \in V(\overline{G}_w)} \beta^G_{v v''}(x) x f(v'')
$$

$$
\phi^G_{G, f, w}(x) = x f(w) \sum_{v' \in V(\overline{G}_w) \cup A_G, w} \sum_{v'' \in V(\overline{G}_w)} \beta^G_{v v''}(x) x f(v'') + I(w, w)
$$

where $I(w, w) = \alpha_{00}$.

Let $P = Id - B^\overline{G}_w$. Clearly, if $P$ is invertible then $C$ is invertible.

**Lemma 3.** Let $1 \leq i \leq \ell$. Set $k = 0$ if $i = 1$ and $k = |V_1| + \cdots + |V_{i-1}|$ if and
2 $\leq i \leq \ell$. Then

$$
\sum_{j=k+1}^{k+|V_i|} \langle Row_j(P^{-1}), [b_{10}^\overline{G}_w, \ldots, b_{n0}^\overline{G}_w] \rangle = \langle Row_i(C^{-1}), [-\alpha_{10}, \ldots, -\alpha_{00}] \rangle.
$$

**Proof.** Suppose

$$
C(s_1, \ldots, s_{\ell}) = (-\alpha_{10}, \ldots, -\alpha_{00}),
$$

$$
P(r_1, \ldots, r_{|V_i|} \ldots, r_{|V_1|}) = (b_{10}^\overline{G}_w, \ldots, b_{n0}^\overline{G}_w).$$

It suffices to show $r_1 + \cdots + r_{|V_i|} = s_i, 1 \leq i \leq \ell$.

For each $i, 1 \leq i \leq \ell$, $\alpha_{i1}s_1 + \cdots + (\alpha_{ii} - 1)s_i + \cdots + \alpha_{ii}s_{\ell} = -\alpha_{00}$ and

$$
\begin{cases}
P_{(k+1)1} r_1 + \cdots + (P_{(k+1)(k+1)} - 1)r_1 + \cdots + P_{(k+1)n} r_{|V_i|} = b_{(k+1)0}^\overline{G}_w \\
\vdots \\
P_{(k+|V_i|)1} r_1 + \cdots + (P_{(k+|V_i|)(k+|V_i|)} - 1)r_{|V_i|} + \cdots + P_{(k+|V_i|)n} r_{|V_i|} \\
= b_{(k+|V_i|)0}^\overline{G}_w
\end{cases}
$$

By summing up all the above equations we will have

$$
(P_{(k+1)1} + \cdots + P_{(k+|V_i|)1}) r_1 + \cdots + (P_{(k+1)(k+1)} + \cdots + P_{(k+|V_i|)(k+1)} - 1) r_1 \\
+ \cdots + (P_{(k+1)(k+|V_i|)} + \cdots + P_{(k+|V_i|)(k+|V_i|)} - 1) r_{|V_i|} + \cdots
$$

$$
+ (P_{(k+1)n} + \cdots + P_{(k+|V_i|)n}) r_{|V_i|} = b_{(k+1)0}^\overline{G}_w + \cdots + b_{(k+|V_i|)0}^\overline{G}_w
$$
and hence \((\alpha_i r^i_1 + \cdots + (\alpha_i - 1) r^i_1 + \cdots + (\alpha_i - 1) r^k), \cdots + (\alpha_i - 1) r^k V_i) = -\alpha_{i0} - \alpha_{i1} - \cdots - (\alpha_{in} r^k V_i)
\)
Therefore,
\[
\alpha_{i1}(r^1_1 + \cdots + r^1 V_i) + \cdots + (\alpha_{i1} - 1)(r^1_1 + \cdots + r^1 V_i) + \cdots + \alpha_{in}(r^1_1 + \cdots + r^1 V_i) = -\alpha_{i0}.
\]
Comparing this and (\ref{eq:1}) one has \(r^1_1 + \cdots + r^1 V_i = s_i\). \(\square\)

**Corollary 2.** Let \((Y(G), T)\) be a local perturbation of a topological Bernoulli scheme \((Y(G_0), T)\) with a countable set \(V\) and let \(f \in \mathcal{F}^0(Y(G))\) be a positive function. Then for \(w \in V_{G}\)

\[
\phi_{G,f}(x) = \phi_{G,f}(x) + \frac{x(f(w)(F_{f,V,w}(x) + \alpha_{V}(\tilde{G}_w)_{G,f,w}(x))(1 + \sigma_{G,f,w}(x))}{1 + \sum_{w' \in V_{G}} x(f(w)) - F_{f,V}(x) - \alpha_{V}(\tilde{G}_w),f,w}(x)}
\]
for those \(x \geq 0\) such that the denominator of last fraction is positive \([3, \text{Theorem } 2]\). \(\square\)

**Proof.** Let \(w \in V_{G}\) and \(W_{i} = w\) be the partition for vertices of \(V_{G}\). Notice that here \(|V_i| < \infty\), \(1 \leq i \leq \ell\) and \(F_{f,V,V_i} = F_{f,V,V_i}\) for all \(v_i \in V_j\). Also from (\ref{eq:1}), one has

\[
\beta_{v,v}(x) = ((-1)^{v_0(v') + v_0(\ell')}) \det((I - B_{v}(x))_{v_0(v')})/(\det(I - B_{v}(x)))
\]

\[= [P^{-1}]_{v_0(v')}.\]

Here \(Row_v(N)\) is the row corresponding to vertex \(v\). Using Lemma 2 \(\alpha_{H,f}(x)\) equals

\[
\sum_{V_i \subset \subset V_{G}} \langle Row_i(C^{-1}), [-\alpha_1 F_{f,V,V_i}, \cdots, -\alpha_{\ell} F_{f,V,V_i}] \rangle
\]

\[= \sum_{V_{i} \subset \subset V_{G}} \langle Row_{w'}(I - B_{V}(x))^{-1}, [x^{f(v)} F_{f,V,V_i}, \cdots, x^{f(v)} F_{f,V,V_i}] \rangle
\]

\[= \sum_{v' \in V_{G}} \sum_{v'' \in V_{G}} [(I - B_{V}(x))^{-1}]_{v'' v'} x^{f(v')} F_{f,V,V_i}(x)
\]

\[= \sum_{v' \in V_{G}} \sum_{v'' \in V_{G}} \beta_{v',v''}(x) x^{f(v')} F_{f,V}(x)
\]

\[= \alpha_{V}(\tilde{G}_w)_{G,f,w}(x).
\]

It is easy to see \(\alpha_{V}(\tilde{G}_w,f,w)(x), \sigma_{G,f,w}(x)\) and \(\phi_{G,f,w}(x)\) are equal to \(\alpha_{H,f}(x), \sigma_{H,f,w}(x)\) and \(\tilde{H}_{f,w}(x)\) respectively. Since \(\zeta + \alpha_{m} = F_{f,V} - \sum_{w \in V_{G}} x^{f(v)}\), the proof completes. \(\square\)
Remark 2. TBS is a special case for TMC where then,
\[ \phi_{G_0,f,v}(x) = \frac{x^{f(v)}}{1 + x^{f(v)} - F_{f,V}(x)}, \]
for \(1 + x^{f(v)} - F_{f,V}(x) > 0\) [3, Theorem 1]. □

6 Criteria for the Existence of a Measure with Maximal Entropy

This section is pretty short for there are similar results in [8] which can be used here directly.

Theorem 5. Let \((Y(G), T)\) be an RFT, \(f \in F_o(Y(G))\) and \(T_f\) the special flow constructed on \(Y(G)\). The following statements are equivalent:

i) \(h(T_f) < \infty\) and \(T_f\) has a (unique) measure with maximal entropy.

ii) There exists \(x_0 > 0\) such that \(\phi_{G,f,w}(x_0) = 1\).

Proof. By a result in [9], the existence of a measure with maximal entropy is guaranteed if and only if the following conditions are satisfied.

1) \(\sum_{\gamma \in C(G,w)} f^*(\gamma) e^{-h(T_f)f^*(\gamma)} < \infty\),

2) \(\sum_{\gamma \in C(G,w)} e^{-h(T_f)f^*(\gamma)} = 1\).

Condition (1) follows from the fact that \(\phi_{G,f,w}(x)\) is \(C^1\) by Corollary (1) and (2) means exactly that there must be \(x_0\) such that \(\phi_{G,f,w}(x_0) = 1\). It worths to mention that \(\phi_{G,f,w}(x)\) is an increasing function and if ever \(\phi_{G,f,w}(x_0) = 1\), then \(x_0\) must be unique. □

Corollary 3. Suppose the hypothesis of Theorem 5. Suppose either

1) \(\exists x > 0 \ni M(x) = 0\), or

2) \(\lim_{x \to r(F_{f,V})} F_{f,V}(x) = \infty\).

Then the existence of a measure with maximal entropy is guaranteed.

Proof. If (1) is satisfied, the proof is immediate from Theorem 2. Now suppose (2) is satisfied, and recall that \(F_{f,V}(x) = \sum_{i=1}^{m} \alpha_i(x) = \sum_{i=1}^{\ell} \alpha_i(x) + \sum_{i=\ell+1}^{m} \alpha_i(x)\). So \(\lim_{x \to r(F_{f,V})} - \sum_{i=1}^{\ell} \alpha_i(x) = \infty\) or \(\lim_{x \to r(F_{f,V})} - \sum_{i=1}^{m} \alpha_i(x) = \infty\). Then in Theorem 3 \(\lim_{x \to r(F_{f,V})} (\zeta + \alpha_m) = \infty\) or \(\lim_{x \to r(F_{f,V})} - \tilde{\phi}_{H,f,w}(x) = \infty\) which either implies the conclusion. □

Note that the first two examples in section 4 have measures with maximal entropy and the last one does not have such a measure.
References

[1] L. Barreira and G. Iommi, Suspension Flows over Countable Markov Shifts, *J. Stat. Phys.* **124** (2006), no. 1, 207230.

[2] R. Bowen, Topological Entropy for Non-compact Sets, *Trans. Amer. Math. Soc.*, **184**, (1973) 125-136.

[3] B. Gurevich, S. Katok, Arithmetic Coding and Entropy for the Positive Geodesic Flow on the Modular Surface, *Moscow Mathematical Journal*, **1** (2001), no.4, 569-582.

[4] S. Katok, I. Ugarcovici, Geometrically Markov Geodesics on the Modular Surface, *Moscow Mathematical Journal*, **5** (2005), no.1.

[5] T. Kempton, Thermodynamic Formalism for Suspension Flows over Countable Markov Shifts, preprint (2010).

[6] J. Jaerisch, M. Kessebohmer, S. Lamei, Induced Topological Pressure for Countable State Markov Shifts, preprint, (2010), arXiv:1010.2162v1.

[7] Y. B. Pesin, B. S. Pitskel, Topological Pressure and the Variational Principle for Non-compact Sets, *Funct. Anal. Appl.*, **18**, (1984), 307-318.

[8] A. B. Polyakov, On a Measure with Maximal Entropy for the Special Flow on a Local Perturbation of a Countable Topological Bernulli Scheme, *Sbornik: Mathematics** 192* (2001), 1001-1024.

[9] V. Savchenko, Special Flows Constructed from Countable Topological Markov Chains, *Funktsional. Anal. i Prilozhen.* **32**:1 (1998), 40-53; English transl. in *Functional Anal. Appl.* **32**:1 (1998).

[10] D. J. Thompson, Irregular Sets and Variational Principles in Dynamical Systems, Ph.D Thesis, The University of Warwick, March 2009.