BASIC AUTOMORPHISM GROUPS OF COMPLETE CARTAN FOLIATION COVERED BY FIBRATIONS

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We prove a theorem that gives a sufficient condition for the full basic automorphism group of a complete Cartan foliation to admit a unique (finite-dimensional) Lie group structure in the category of Cartan foliations. Emphasize that the transverse Cartan geometry may not be effective. Some estimates of the dimension of this group depending on the transverse geometry are found. Further, we investigate Cartan foliations covered by fibrations. When the global holonomy group of that foliation is discrete, we obtain the explicit new formula for determining its basic automorphism Lie group. Examples of computing the full basic automorphism group of complete Cartan foliations are constructed.

Keywords: foliation; Cartan foliation; Lie group; basic automorphism; automorphism group, foliated bundle.

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1 Introduction. Main results

The automorphism group is associated with every object of a category. Among central problems there is the question whether the automorphism group can be endowed with a (finite-dimensional) Lie group structure [9].

In the theory of foliations with transverse geometries, morphisms are understood as local diffeomorphisms mapping leaves onto leaves and preserving transverse geometries. The group of all automorphisms of a foliation $(M, F)$ with transverse geometry is denoted by $A(M, F)$. Let $A_L(M, F)$ be the normal subgroup of $A(M, F)$ formed by automorphisms mapping each leaf onto itself. The quotient group $A(M, F)/A_L(M, F)$ is called the full basic automorphism group and is denoted by $A_B(M, F)$.

In the investigation of foliations $(M, F)$ with transverse geometry it is natural to raise the above problem of the existence of a Lie group structure for the full group $A_B(M, F)$ of basic automorphisms of $(M, F)$.

J. Leslie [11] was the first who to solve a similar problem for smooth foliations on compact manifolds. For foliations with complete transversal projectable affine connection this problem was raised by I.V. Belko [2].

Foliations $(M, F)$ with transverse rigid geometries were investigated by the first author [18], where an algebraic invariant $g_0 = g_0(M, F)$, called the structural Lie algebra of $(M, F)$, was constructed and it was proved that $g_0 = 0$ is a sufficient condition for the existence of a unique Lie group structure in the full basic automorphism group of this foliation. In the case where $(M, F)$ is a Riemannian foliation, the concept of the structural Lie algebra was introduced previously by P. Molino [12].
The leaf space $M/F$ of the foliation is a diffeological space, and the group $A_B(M, F)$ can be considered as a subgroup of the diffeological Lie group $Diff(M/F)$. For Lie foliations with dense leaves on a compact manifold, the diffeological Lie groups $Diff(M/F)$ were computed by Hector and Macias-Virgos [8].

Spaces which we call Cartan geometries were introduced by Elie Cartan in the 1920s and were called by him espaces généralisés. The investigation of Cartan geometries (see definition 2.1) gives us the possibility to consider different geometry structures from the unified viewpoint.

We use the notion of Cartan foliation in the sense of Blumenthal [3]. We emphasize that parabolic, conformal, Weil, projective, pseudo-Riemannian, Lorentzian, Riemannian foliations and foliations with transverse linear connection belong to the class of Cartan foliations. Therefore, proved by us Theorems 1.1–1.5 are valid for all these foliations. Let us denote by $\mathcal{C}\mathcal{F}$ the category of Cartan foliations (the definition is given in subsection 2.2).

If a Cartan foliation $(M, F)$ is trivial, that is $M = L \times N$, where $N$ is a connected smooth manifold endowed with a Cartan geometry $\xi$, then $A_B(M, F) = Aut(N, \xi)$ is the Lie group of all automorphisms of Cartan geometry in the category $\mathcal{C}\mathcal{F}$. Therefore the problem of the existence of a Lie group structure in the basic automorphism group $A_B(M, F)$ may be reformulated in the following way:

How complicated may be the construction of a Cartan foliation admitting a Lie group structure in the basic automorphism group depending on the transverse geometry?

In subsection 2.3 we remind the notion of the effective Cartan geometry. It was shown by the first author ([17], Proposition 1) that a Cartan foliation modelled on a noneffective Cartan geometry $\xi = (P(N, H), \omega)$ of type $(G, H)$ admits an effective transversal Cartan geometry of the type $(G', H')$ where $G' = G/K$, $H' = H/K$ and $K$ is the kernel of the pair $(G, H)$, that is the maximal normal subgroup of $G$ belonging to $H$. Due to this fact we may construct the associated foliated bundle for any Cartan foliation in the sense of Blumenthal. Note that in ([3], Proposition 3.1) this construction is not correct in general.

By the structural Lie algebra $g_0 = g_0(M, F)$ of a complete Cartan foliation $(M, F)$ we mean the structural Lie algebra of $(M, F)$ considered with the associated effective transversal Cartan geometry indicated above.

Let us denote by $A(M, F)$ the group of all the automorphisms of the Cartan foliation $(M, F)$ in the category $\mathcal{C}\mathcal{F}$ and by $A_B(M, F)$ the full basic automorphism group.

We have the following theorem about a sufficient condition for the existence a unique Lie group structure in the group of basic automorphisms of complete Cartan foliations and some exact estimates of its dimension.

**Theorem 1.1.** Let $(M, F)$ be a complete Cartan foliation modelled on a Cartan geometry of type $\mathfrak{g}/\mathfrak{h}$. If the structural Lie algebra $g_0 = g_0(M, F)$ is zero, then the basic automorphism group $A_B(M, F)$ of this foliation is a Lie group whose dimension satisfies the inequality

$$\dim A_B(M, F) \leq \dim(\mathfrak{g}) - \dim(\mathfrak{k}),$$

where $\mathfrak{k}$ is the kernel of the pair $(\mathfrak{g}, \mathfrak{h})$, that is, the maximal ideal of the Lie algebra $\mathfrak{g}$ belonging to $\mathfrak{h}$, and the Lie group structure in $A_B(M, F)$ is unique.

Moreover,

(a) if there exists an isolated closed leaf or if the set of closed leaves is countable, then

$$\dim A_B(M, F) \leq \dim(\mathfrak{h}) - \dim(\mathfrak{k}).$$

Theorem 1.1. Let $(M, F)$ be a complete Cartan foliation modelled on a Cartan geometry of type $\mathfrak{g}/\mathfrak{h}$. If the structural Lie algebra $g_0 = g_0(M, F)$ is zero, then the basic automorphism group $A_B(M, F)$ of this foliation is a Lie group whose dimension satisfies the inequality

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Moreover,

(a) if there exists an isolated closed leaf or if the set of closed leaves is countable, then

$$\dim A_B(M, F) \leq \dim(\mathfrak{h}) - \dim(\mathfrak{k});$$

Let us denote by $A_B(M, F)$ the full basic automorphism group.

(b) if the set of closed leaves is countable and dense, then
\[ \dim A_B(M, F) = 0. \]

The estimates (1), (2) are exact and the case of (b) is realized.

In other words, if the associated lifted foliation \((\mathcal{R}, \mathcal{F})\) is formed by fibres of a locally trivial fibration, then the basic automorphism group of \((M, F)\) is a Lie group.

Examples 9.1 – 9.3 show the exactness of estimates (1) and (2). In Example 9.5 we construct the foliation with the countable dense set of closed leaves and show the realization of the case (b) of Theorem 1.1.

Recall that a leaf \(L\) of a foliation \((M, F)\) is proper if \(L\) is an embedded submanifold in \(M\). A foliation is called proper \([16]\) if all its leaves are proper. A leaf \(L\) is said to be closed if \(L\) is a closed subset of \(M\).

The following assertion contains sufficient conditions in terms of topology of leaves and their holonomy groups for the basic automorphism group of a Cartan foliation to be a Lie group.

**Corollary 1.1.** Let \((M, F)\) be a complete Cartan foliation. If at least one of the following conditions holds:

(i) there exists a proper leaf \(L\) with discrete holonomy group in the sense of definition 3.2,

(ii) there is a closed leaf \(L\) with discrete holonomy group;

(iii) there exists a proper leaf \(L\) with finite holonomy group;

(iv) there is a closed leaf \(L\) with finite holonomy group,

then the basic automorphism group \(A_B(M, F)\) admits a Lie group structure of dimension at most \(\dim(\mathfrak{h}) - \dim(\mathfrak{t})\), and this structure is unique.

In particular, we have

**Corollary 1.2.** If \((M, F)\) is a proper complete Cartan foliation, then the basic automorphism group \(A_B(M, F)\) admits a unique Lie group structure of dimension at most \(\dim(\mathfrak{g}) - \dim(\mathfrak{t})\).

**Remark 1.1.** I.V. Belko ([2], Theorem 2) stated that the existence of a closed leaf of a foliation \((M, F)\) with complete transversally projectable affine connection is sufficient for the fact that the basic automorphism group \(A_B(M, F)\) to admit a Lie group structure.

Example 9.4 shows that this statement is not true in general.

Due to the remark above about the existence of the associated effective Cartan geometry, we may assume further without loss of generality that all Cartan foliations are modelled on effective Cartan geometries.

**Definition 1.1.** Let \(\kappa : \widetilde{M} \to M\) be the universal covering map. We say that a smooth foliation \((M, F)\) is covered by fibration if the induced foliation \((\widetilde{M}, \widetilde{F})\) is formed by fibres of a locally trivial fibration \(\widetilde{r} : \widetilde{M} \to B\).

Further we investigate Cartan foliation covered by fibration. First we describe the global structure entering the holonomy groups of the Cartan foliations covered by fibrations.
Corollary 1.4. Under conditions of Theorem \(\Psi\) closure of task and to prove the following statement.

**Theorem 1.2.** Let \(r : \tilde{M} \to B\) where \(\tilde{\kappa} : \tilde{M} \to M\) is the universal covering map. Then

1. there exists a regular covering map \(\kappa : \tilde{M} \to M\) such that the induced foliation \(\tilde{F}\) is made up of fibres of the locally trivial bundle \(r : \tilde{M} \to B\) over a simply connected Cartan manifold \((B, \eta)\);
2. a group \(\Psi\) of automorphisms of the Cartan manifold \((B, \eta)\) and epimorphism \(\chi : \pi_1(M, x) \to \Psi\) of the fundamental group \(\pi_1(M, x), x \in M\), onto \(\Psi\) is determined;
3. for all points \(y \in M\) and \(z \in \kappa^{-1}(y)\) the restriction \(\kappa|_L : \tilde{L} \to L\) to the leaf \(\tilde{L} = \tilde{L}(z)\) of the foliation \((\tilde{M}, \tilde{F})\) is a regular covering map onto the leaf \(L = L(y)\), and the group of deck transformations of \(\kappa|_L\) is isomorphic to the stationary subgroup \(\Psi_b\) of the group \(\Psi\) at the point \(b = r(z) \in B\). Moreover, the subgroup \(\Psi_b\) is isomorphic to the holonomy group \(\Gamma(L, y)\) of the leaf \(L\);
4. the group of deck transformation of \(\kappa : \tilde{M} \to M\) is isomorphic to \(\Psi\).

**Definition 1.2.** The group \(\Psi = \Psi(M, F)\) satisfying Theorem 1.2 is called the **global holonomy group** of the Cartan foliation \((M, F)\) covered by fibration.

We recall the notion of an Ehresmann connection (subsection 4.2). The following theorem shows that the class of Cartan foliations covered by fibrations is large.

**Theorem 1.3.** Let \((N, \eta)\) be any connected Cartan manifold and \(\Psi\) be any subgroup of the automorphism group \(\text{Aut}(N, \eta)\) of the Cartan manifold \((N, \eta)\). Suppose that \((M, F)\) is a foliation defined by \(N\)-cocycle \(\{U_i, f_i, \{\gamma_{ij}\}_{i,k \in J}\}\) where every \(\gamma_{ij}\) is a restriction of some transformation from \(\Psi\). If \((M, F)\) admits an Ehresmann connection, then it is a Cartan foliation covered by fibration and statements of Theorem 1.2 are valid for it.

**Corollary 1.3.** If the transverse curvature of a complete Cartan foliation \((M, F)\) is equal to zero, then \((M, F)\) is covered by fibration.

**Remark 1.2.** The first author proved ([19], Theorem 5) that any complete non-Riemannian conformal foliation of codimension \(q \geq 3\) is covered by fibration.

The application of Theorem 7 proved by the first author in [18] to Cartan foliations gives us the following interpretation of the structural Lie algebra of Cartan foliations covered by fibrations.

**Theorem 1.4.** Let \((M, F)\) be a complete Cartan foliation covered by the fibration \(\tilde{r} : \tilde{M} \to B\) where \(\tilde{\kappa} : \tilde{M} \to M\) is the universal covering map. Then the structural Lie algebra \(\mathfrak{g}_0 = \mathfrak{g}_0(M, F)\) is isomorphic to the Lie algebra of the Lie group \(\hat{\Psi}\), which is the closure of \(\Psi\) in the Lie group \(\text{Aut}(B, \eta)\), where \((B, \eta)\) is the induced Cartan geometry.

**Corollary 1.4.** Under conditions of Theorem 1.4 the structural Lie algebra \(\mathfrak{g}_0(M, F)\) is zero if and only if the global holonomy group \(\Psi\) is a discrete subgroup of the Lie group \(\text{Aut}(B, \eta)\) where \(\eta\) is the induced Cartan geometry.

Our next objective is to give rigorous proof of the explicit formula for computing basic automorphism groups for the investigated class of foliations. Application of the foliated bundle over \((M, F)\) and Theorems 1.1, 1.2 and 1.4 allow us to accomplish this task and to prove the following statement.
Theorem 1.5. Let \((M, F)\) be a complete Cartan foliation modelled on a Cartan geometry \(\xi = (P(N, H), \omega)\) covered by fibration \(r : \widetilde{M} \to B\), where \((B, \eta)\) is the induced Cartan geometry on simply connected manifold \(B\) determined in Theorem \([12]\). Suppose that a global holonomy group \(\Psi\) is discrete subgroup in the Lie group \(\text{Aut}(B, \eta)\). Let \(N(\Psi)\) be the normalizer of \(\Psi\) in \(\text{Aut}(B, \eta)\). Then the group of basic automorphisms \(\text{Aut}_B(M, F)\) in the category of Cartan foliations \(\mathfrak{CF}\) is a Lie group, and it is equal to the Lie quotient group \(N(\Psi)/\Psi\), i.e.,

\[
\text{Aut}_B(M, F) = N(\Psi)/\Psi, \tag{4}
\]

if the group \(\text{Aut}_B(M, F)\) is not countable, otherwise \(\text{Aut}_B(M, F)\) is a discrete Lie group.

Corollary 1.5. If \((M, F)\) is a Cartan foliation on a simply connected manifold \(M\) and \((M, F)\) is formed by fibres of a locally trivial fibration \(r : M \to B\), then the basic automorphism group of the Cartan foliation \((M, F)\) is isomorphic to the full automorphism group \(\text{Aut}(B, \eta)\) of the induced Cartan geometry \((B, \eta)\), i.e. \(\text{Aut}_B(M, F) \cong \text{Aut}(B, \eta)\).

Remark 1.3. In the case where \(\text{Aut}_B(M, F)\) is discrete, we have not proved formula \((3)\), but we believe that formula \((3)\) is true in that case also.

Notations We denote by \(\mathfrak{X}(N)\) the Lie algebra of smooth vector fields on a manifold \(N\). If \(\mathfrak{M}\) is a smooth distribution on \(M\), then \(\mathfrak{X}_{\mathfrak{M}}(M) := \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{M}_u \quad \forall u \in M\}\). If in addition \(f : K \to M\) is a submersion, then \(f^*\mathfrak{M}\) is the distribution on the manifold \(K\) such that \((f^*\mathfrak{M})_z := \{X \in T_zK \mid f_*(X) \in \mathfrak{M}_{f(z)}\}\) where \(z \in K\).

Let \(\mathfrak{F}\) be the category of foliations where morphisms are smooth maps transforming leaves into leaves.

Following notations of \([10]\) we denote by \(P(N, H)\) a principal \(H\)-bundle over manifold \(N\).

Let \(\cong\) denote a group isomorphism.

2 The category of Cartan foliations

2.1 Determination of foliations by \(N\)-cocycles

Let \(M\) be a smooth \(n\)-dimension manifold. Let \(N\) be a smooth \(q\)-dimensional manifold the connectivity of which is not assumed. We will call an \(N\)-cocycle on \(M\) a family \(\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}\) satisfying the following conditions:

1) \(\{U_i \mid i \in J\}\) is a covering of the manifold \(M\) by open connected subsets \(U_i\) of \(M\), and \(f_i : U_i \to N\) is a submersion with connected fibres;

2) if \(U_i \cap U_j \neq \emptyset, i, j \in J\), then a diffeomorphism \(\gamma_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)\) is defined, and \(\gamma_{ij}\) satisfies the equality \(f_i = \gamma_{ij} \circ f_j\) on \(U_i \cap U_j\);

3) \(\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}\) if \(U_i \cap U_j \cap U_k \neq \emptyset\) for all \(x \in U_i \cap U_j \cap U_k, i, j, k \in J\).

Two \(N\)-cocycles are called equivalent if their union is an \(N\)-cocycle. Let \(\{\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}\}\) be the equivalence class of \(N\)-cocycles on manifold \(M\) containing a cocycle \(\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}\). Denote by \(\Sigma\) the set of fibres of all the submersions \(f_i\) of this equivalence class. Note, that \(\Sigma\) is the base of some new topology \(\tau\) in \(M\). The linear connected components of the topological space \((M, \tau)\) form a partition \(F := \{L_\alpha \mid \alpha \in \Sigma\}\) of the manifold \(M\) which is called the foliation of the codimension \(q\). \(L_\alpha\) are called its leaves and \(M\) is the foliated manifold. It is said that foliation \((M, F)\) is determined by \(N\)-cocycle \(\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}\). Further we denote the foliation by the pair \((M, F)\).
2.2 The category of Cartan foliations

We recall here the definition of Cartan geometries, see [9] and [15].

Let \( G \) be a Lie group and \( H \) is a closed subgroup of \( G \). Denote by \( \mathfrak{g} \) and \( \mathfrak{h} \) the Lie algebras of Lie groups \( G \) and \( H \) relatively.

**Definition 2.1.** Let \( N \) be a smooth manifold. A Cartan geometry on \( N \) of type \((G, H)\) is the principal right \( H \)-bundle \( P(N, H) \) with the projection \( p : P \to N \) together with a \( \mathfrak{g} \)-valued 1-form \( \omega \) on \( P \) satisfying the following conditions:

1. \( \omega \) is an isomorphism of vector spaces for each \( w \in P \);
2. \( R_h^\omega = Ad_G(h^{-1}) \omega \) for each \( h \in H \), where \( Ad_G : H \to GL(\mathfrak{g}) \) is the joint representation of the Lie subgroup \( H \) of \( G \) in the Lie algebra \( \mathfrak{g} \);
3. \( \omega(A^*) = A \) for every \( A \in \mathfrak{h} \), where \( A^* \) is the fundamental vector field determined by \( A \).

The \( \mathfrak{g} \)-valued form \( \omega \) is called the **Cartan connection**. This Cartan geometry is denoted by \( \xi = (P(N, H), \omega) \). The pair \((N, \xi)\) is called a Cartan manifold.

Let \( \xi = (P(N, H), \omega) \) and \( \xi' = (P'(N', H), \omega') \) be two Cartan geometries with the same structure group \( H \). The smooth map \( \Gamma : P \to P' \) is called a morphism from \( \xi \) to \( \xi' \) if \( \Gamma^* \omega' = \omega \) and \( R_a \circ \Gamma = \Gamma \circ R_a, \ a \in H \). If \( \Gamma \in \text{Mor}(\xi, \xi') \), then the projection \( \gamma : N \to N' \) is defined such that \( p' \circ \gamma = \gamma \circ p \), where \( p : P \to N \) and \( p' : P' \to N' \) are the projections of the corresponding \( H \)-bundles. The projection \( \gamma \) is called an automorphism of Cartan manifold \( (N, \xi) \). Denote by \( \text{Aut}(N, \xi) \) the full automorphism group of \((N, \xi)\) and by \( \text{Aut}(\xi) \) the full automorphism group of \( \xi \). The category of Cartan geometries is denoted by \( \mathcal{C} \text{ar} \). Thus, every automorphism \( \Gamma \) of \( \xi \) is the automorphism of the parallelizable manifold \((P, \omega)\) such that \( \Gamma \circ R_a = R_a \circ \Gamma, \ a \in H \).

Let \( A(P, \omega) := \{ \Gamma \in \text{Diff}(P) | \Gamma^* \omega = \omega \} \) be the automorphism group on the parallelizable manifold \((P, \omega)\) and \( A^H(P) = \{ \Gamma \in A(P, \omega) | \Gamma \circ R_a = R_a \circ \Gamma \} \) is its closed Lie subgroup. Then \( \text{Aut}(\xi) = A^H(P) \) is the automorphism group of Cartan geometry \( \xi \). The Lie group epimorphism \( \sigma : A^H(P, \omega) \to \text{Aut}(N, \xi) : \Gamma \mapsto \gamma \), where \( \gamma \) is the projection of \( \Gamma \), is defined.

2.3 Effectiveness of Cartan geometries

Remind the notion of *effective Cartan geometry*. Consider a pair Lie groups \((G, H)\), where \( H \) is closed subgroup of \( G \). Let \((\mathfrak{g}, \mathfrak{h})\) be the appropriate pair of Lie algebras. The maximal ideal \( \mathfrak{k} \) of the algebra \( \mathfrak{g} \) which is contained in \( \mathfrak{h} \) is called the *kernel* of pair \((\mathfrak{g}, \mathfrak{h})\). If \( \mathfrak{k} = 0 \), then the pair \((\mathfrak{g}, \mathfrak{h})\) is called *effective*. Maximal normal subgroup \( K \) of the group \( G \) belonging to \( H \) is called the kernel of pair \((G, H)\). As it is known, the Lie algebra of \( K \) is equal \( \mathfrak{k} \). The Cartan geometry \( \xi = (P(M, H), \omega) \) of the type \( \mathfrak{g}/\mathfrak{h} \) modelled on pair of the Lie group \((G, H)\), is called effective if the kernel \( K \) of the pair \((G, H)\) is trivial. As it was proved in ([15], Theorem 4.1), the Cartan geometry \( \xi = (P(M, H), \omega) \) of type \( \mathfrak{g}/\mathfrak{h} \) is effective if and only if the pair of Lie algebras \((\mathfrak{g}, \mathfrak{h})\) is effective and group

\[
N := \{ h \in H | Ad_G(h) = id_\mathfrak{g} \}
\]

is trivial.

**Remark 2.1.** We emphasize that the defined above group epimorphism \( \sigma : A^H(P) \to \text{Aut}(N, \xi) \) is isomorphism for any effective Cartan geometry \( \xi \).
2.4 Cartan foliations in the sense of Blumenthal

**Definition 2.2.** Let \((M, F)\) be a foliation determined by \(N\)-cocycle \([U_i, f_i, \{\gamma_{ij}\}]\) and every \(\gamma_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)\) from this cocycle there exists an isomorphism \(\Gamma_{ij} : P_{f_j(U_i \cap U_j)} \to P_{f_i(U_i \cap U_j)}\) of induced Cartan geometries \(\xi_{f_j(U_i \cap U_j)}\) and \(\xi_{f_i(U_i \cap U_j)}\) having the projection \(\gamma_{ij}\), then \((M, F)\) is called a Cartan foliation (in the sense of Blumenthal [3]) modelled on the Cartan geometry \(\xi = (P(N, H), \omega)\).

Note that \(\xi = (P(N, H), \omega)\) is also called the transverse Cartan geometry of the foliation \((M, F)\).

**Remark 2.2.** The first author introduced a different notion of Cartan foliation in [17] that is equivalent to the notion of Cartan foliation in the sense of Blumenthal if and only if the transverse Cartan geometry is effective.

2.5 Morphisms in the category of Cartan foliations

Let \((M, F)\) and \((M', F')\) are Cartan foliations defined by an \((N, \xi)\)-cocycle \(\eta = \{U_i, f_i, \{\gamma_{ij}\}\}\) and an \((N', \xi')\)-cocycle \(\eta' = \{U'_i, f'_i, \{\gamma'_{ij}\}\}\) respectively. All objects belonging to \(\eta'\) are distinguished by prime. Let \(f : M \to M'\) be a smooth map which is a local isomorphism in the category \(\mathfrak{Fol}\). Hence for any \(x \in M\) and \(y := f(x)\) there exist neighborhoods \(U_k \ni x\) and \(U'_s \ni y\) from \(\eta\) and \(\eta'\) respectively and a diffeomorphism \(\varphi : V_k \to V'_s\), where \(V_k := f_k(U_k)\) and \(V'_s := f'_s(U'_s)\), satisfying the relations \(f(U_k) = U'_s\) and \(\varphi \circ f_k = f'_s \circ f|_{U_k}\). Further we shall use the following notations: \(P_k := P|_{V_k}\), \(P'_s := P'|_{V'_s}\), \(p_k := p|_{P_k}\), \(p'_s := p|_{P'_s}\).

We say that \(f\) preserves transverse Cartan structure if every such diffeomorphism \(\varphi : V_k \to V'_s\) is an isomorphism of the induced Cartan geometries \((V_k, \xi_{V_k})\) and \((V'_s, \xi'_{V'_s})\), that is if there exists an isomorphism \(\Phi : P_k \to P'_s\) in the category \(\mathfrak{Fol}\) having the projection \(\varphi\), and the diagram

![Diagram](image)

is commutative. We emphasize that the indicated above isomorphism \(\Phi : P_k \to P'_s\) is not unique if the transverse Cartan geometries are not effective. This notion is well defined, i. e., it does not depend of the choice of neighborhoods \(U_k\) and \(U'_s\) from the cocycles \(\eta\) and \(\eta'\).

**Definition 2.3.** By a morphism of two Cartan foliations \((M, F)\) and \((M', F')\) we mean a local diffeomorphism \(f : M \to M'\) which transforms leaves to leaves and preserves transverse Cartan structure.

The category \(\mathfrak{Fol}\) objects of which are Cartan foliations, morphisms are their morphisms, is called the category of Cartan foliations.
3 The foliated bundle associated with a Cartan foliation

3.1 Associated foliated bundles

The following statement is important for further, and it was proved by the first author in the work ([17], Proposition 1).

**Proposition 3.1.** Let $(M,F)$ be a Cartan foliation in the sense of Blumenthal with the transverse Cartan geometry $\tilde{\xi} = (\tilde{P}(N,\tilde{H}),\tilde{\omega})$ of type $\tilde{g}/\tilde{h}$ modeled on a pair of Lie groups $(\tilde{G},\tilde{H})$ with kernel $K$. Then,

(i) there exists an effective Cartan geometry $\xi = (P(N,H),\omega)$ of type $g/h$, modeled on the pair of Lie groups $(G,H)$, where $G = \tilde{G}/K$, $H = \tilde{H}/K$, $g = \tilde{h}/k$, $h = \tilde{h}/k$, and $k$ is the kernel of the pair of Lie algebras $(\tilde{g},\tilde{h})$;

(ii) the original foliation $(M,F)$ is a Cartan foliation with an effective transverse Cartan geometry $\xi = (P(N,H),\omega)$.

Proposition 3.1 allows us to construct the foliated bundle for an arbitrary Cartan foliation in the sense of Blumenthal $(M,F)$ with noneffective, in general, transverse Cartan geometry $\tilde{\xi}$. Because for effective transverse Cartan geometries the notions of Cartan foliations in the sense of Blumenthal and in the sense of [17] are equivalent, we apply ([17], Proposition 2) to the associated transverse Cartan geometry $\xi$ and have the following assertion.

Remind that a Cartan foliation of type $g/o$ is named transversally parallelizable or e-foliation.

**Proposition 3.2.** Let $(M,F)$ be a Cartan foliation modelled on Cartan geometry $\tilde{\xi} = (\tilde{P}(N,\tilde{H}),\tilde{\omega})$ of type $\tilde{g}/\tilde{h}$ and $\xi = (P(N,H),\omega)$ is the associated transverse Cartan geometry of type $(G,H)$, where $G = \tilde{G}/K$, $H = \tilde{H}/K$, $K$ is the kernel of the pair $(\tilde{G}/\tilde{H})$. Then there exists a principal $H$-bundle with a projection $\pi : R \rightarrow M$, $H$-invariant foliation $(R,F)$ and $g$-valued $H$-equivariant 1-form $\beta$ on $R$ which satisfy the following conditions:

(i) $\beta(A^*) = A$ for any $A \in h$;

(ii) the mapping $\beta_u : T_uR \rightarrow g \forall u \in R$ is surjective, and $ker(\beta_u) = T_uF$;

(iii) the foliation $(R,F)$ is transversally parallelizable;

(iv) the Lie derivative $L_X\beta$ is equal to zero for every vector field $X$ tangent to the foliation $(R,F)$.

The statement (iv) of Proposition 3.2 is true due to transversal projectability of the 1-form $\beta$.

**Definition 3.1.** The principal $H$-bundle $R(M,H)$ satisfying Proposition 3.2 is said to be the associated foliated bundle. The foliation $(R,F)$ is called the associated lifted foliation for the Cartan foliation $(M,F)$.

We denote by $\Gamma(L,x)$ the germ holonomy group of a leaf $L$ of the foliation usually used in the foliation theory [16], Proposition 3 about different interpretations of the holonomy groups of any complete Cartan foliation follows from ([18], Theorem 4).
**Proposition 3.3.** Let \((M, F)\) be a complete Cartan foliation, \(L = L(x)\) be an arbitrary leaf of this foliation and \(\mathcal{L} = \mathcal{L}(u), \ u \in \pi^{-1}(x)\), be the corresponding leaf of the lifted foliation. Then the germ holonomy group \(\Gamma(L, x)\) of leaf \(L\) is isomorphic to each of following two groups:

(i) the group of deck transformations of the regular covering map \(\pi|_\mathcal{L} : \mathcal{L} \to L\);

(ii) the subgroup \(H(\mathcal{L}) = \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}\) of the Lie group \(H\)

If we change \(u\) by an other point \(\tilde{u} \in \pi^{-1}(x)\), then \(H(\mathcal{L})\) is changed by the conjugate subgroup \(H(\tilde{\mathcal{L}})\), where \(\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\tilde{u})\), in the group \(H\).

Due to Proposition 3.3, the following definition is correct.

**Definition 3.2.** The holonomy group of a complete Cartan foliation \((M, F)\) is called discrete if the corresponding group \(H(\mathcal{L})\) is a discrete subgroup of the Lie group \(H\).

### 4 Ehresmann connections and completeness of Cartan foliations

#### 4.1 Completeness of Cartan foliations

Let \((M, F)\) be an arbitrary smooth foliation on a manifold \(M\) and \(TF\) be the distribution on \(M\) formed by the vector spaces tangent to the leaves of the foliation \(F\). The vector quotient bundle \(TM/TF\) is called the transverse vector bundle of the foliation \((M, F)\). Let us fix an arbitrary smooth distribution \(\mathfrak{M}\) on \(M\) that is transverse to the foliation \((M, F)\), i.e., \(T_xM = T_xF \oplus \mathfrak{M}_x, \ x \in M\), and identify \(TM/TF\) with \(\mathfrak{M}\).

Let \((M, F)\) be a Cartan foliation and \((\mathcal{R}, F)\) be the lifted foliation. It is natural to identify the transverse vector bundle \(TR/TF\) with the distribution \(\tilde{\mathfrak{M}} := \pi^*\mathfrak{M}\) on \(\mathcal{R}\).

**Definition 4.1.** A Cartan foliation \((M, F)\) is said to be \(\mathfrak{M}\)-complete if any transverse vector field \(X \in \mathfrak{X}_{\tilde{\mathfrak{M}}}(\mathcal{R}, F)\) such that \(\beta(X) = \text{const}\) is complete. A Cartan foliation \((M, F)\) of arbitrary codimension \(q\) is said to be complete if there exists a smooth \(q\)-dimensional transverse distribution \(\mathfrak{M}\) on \(M\) such that \((M, F)\) is \(\mathfrak{M}\)-complete [17].

In other words, \((M, F)\) is an \(\mathfrak{M}\)-complete foliation if the lifted \(e\)-foliation \((\mathcal{R}, F)\) is complete with respect to the distribution \(\tilde{\mathfrak{M}} = \pi^*\mathfrak{M}\) in the sense of Conlon [7]. Remark that a complete \(e\)-foliation in the sense of Conlon is also complete in the sense of Molino [12].

#### 4.2 Ehresmann connections for foliations

Let \((M, F)\) be a foliation of codimension \(q\) and \(\mathfrak{M}\) be a smooth \(q\)-dimensional distribution on \(M\) that is transverse to the foliation \(F\). The piecewise smooth integral curves of the distribution \(\mathfrak{M}\) are said to be horizontal, and the piecewise smooth curves in the leaves are said to be vertical. A piecewise smooth mapping \(H\) of the square \(I_1 \times I_2\) to \(M\) is called a vertical-horizontal homotopy if the curve \(H|_{(s)}\) is vertical for any \(s \in I_1\) and the curve \(H|_{(t)}\) is horizontal for any \(t \in I_2\). In this case, the pair of paths \((H|_{(1)}\times I_2)\) is called the base of \(H\). It is well known that there exists at most one vertical-horizontal homotopy with a given base. A distribution \(\mathfrak{M}\) is called an Ehresmann connection for a foliation \((M, F)\) (in the sense of Blumenthal and Hebda [4]) if, for any pair of paths \((\sigma, h)\) in \(M\) with a common initial point \(\sigma(0) = h(0)\),
where \( \sigma \) is a horizontal curve and \( h \) is a vertical curve, there exists a vertical-horizontal homotopy \( H \) with the base \( (\sigma, h) \).

For a simple foliation \( F \), i.e., such that it is formed by the fibers of a submersion \( r: M \to B \), a distribution \( \mathcal{M} \) is an Ehresmann connection for \( F \) if and only if \( \mathcal{M} \) is an Ehresmann connection for the submersion \( r \), i.e., if and only if any smooth curve in \( B \) possesses horizontal lifts.

The following statement was proved by the first author ([17], Proposition 3).

**Proposition 4.1.** If \((M, F)\) is an \( \mathcal{M} \)-complete Cartan foliation, then \( \mathcal{M} \) is an Ehresmann connection for this foliation.

### 4.3 Structural algebras Lie of Lie foliations with dense leaves

Let \((M, F)\) be a Lie foliation with dense leaves. It is the Cartan foliation of a type \( g_0/\mathfrak{g} \). J. Leslie [11] was the first who observed that the Lie algebra \( g_0 \) of that foliation is invariant in the category of foliations \( \mathcal{F}_{\text{ol}} \).

**Definition 4.2.** The Lie algebra \( g_0 \) of the Lie foliation \((M, F)\) with dense leaves is called the **structural Lie algebra** of \((M, F)\).

### 4.4 Structural Lie algebras of Cartan foliations

Applying of the relevant results of P. Molino [12] on complete e-foliations we obtain the following theorem.

**Theorem 4.1.** Let \((M, F)\) be a complete Cartan foliation and \((\mathcal{R}, F)\) be the associated lifted e-foliation. Then:

(i) the closure of the leaves of the foliation \( F \) are fibers of a certain locally trivial fibration \( \pi_b: \mathcal{R} \to W \);

(ii) the foliation \((\mathcal{L}, F|_{\mathcal{L}})\) induced on the closure \( \mathcal{L} \) is a Lie foliation with dense leaves with the structural Lie algebra \( g_0 \), that is the same for any \( \mathcal{L} \in F \).

**Definition 4.3.** The structural Lie algebra \( g_0 \) of the Lie foliation \((\mathcal{L}, F|_{\mathcal{L}})\) is called the **structural Lie algebra** of the complete foliation \((M, F)\) and is denoted by \( g_0 = g_0(M, F) \).

If \((M, F)\) is a Riemannian foliation on a compact manifold, this notion coincides with the notion of the structural Lie algebra in the sense of P. Molino [12].

**Definition 4.4.** The fibration \( \pi_b: \mathcal{R} \to W \) satisfying Theorem 4.1 is called the **basic fibration** for \((M, F)\).

### 5 Basic automorphisms of Cartan foliations

#### 5.1 Groups of basic automorphisms of Cartan foliations

**Definition 5.1.** Let \( A(M, F) \) be the full automorphism group of a Cartan foliation \((M, F)\) in the category of Cartan foliation \( \mathcal{C}_F \). The group

\[
A_L(M, F) := \{ f \in A(M, F) \mid f(L_\alpha) = L_\alpha \ \forall L_\alpha \in F \}
\]

is a normal subgroup of \( A(M, F) \) which is called the **leaf automorphism group** of \((M, F)\). The quotient group \( A(M, F)/A_L(M, F) \) is called the **basic automorphism group** and is denoted by \( A_B(M, F) \).
Let \((M, F)\) be a Cartan foliation. Let \(M/F\) be the leaf space of \((M, F)\), and \(q : M \to M/F\) be the natural projection onto the leaf space which maps any \(x \in M\) to the leaf \(L(x)\) considered as a point \([L(x)]\) in \(M/F\). Each \(f \in A(M, F)\) maps an arbitrary leaf \(L\) of \(F\) onto some leaf of this foliation. Hence the equality \(f([L]) = [f(L)]\) defines a mapping \(\tilde{f}\) of the leaf space \(M/F\) onto itself such that the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{q} & M/F \\
\downarrow{f} & & \downarrow{f} \\
M & \xrightarrow{q} & M/F.
\end{array}
\]

is commutative. Since \(q\) is an open and continuous mapping, (5.1) implies that \(\tilde{f}\) is a homeomorphism of the leaf space \(M/F\). Denote by \(A(M/F)\) the set of such homeomorphisms of \(M/F\). Then

\[
\tilde{q} : A(M, F) \to A(M/F) : f \mapsto \tilde{f}
\]

is a group epimorphism with the kernel \(\ker \tilde{q} = A_L(M, F)\). Therefore the basic automorphism group \(A_B(M, F)\) is canonically isomorphic to \(A(M/F)\). Thus, the basic automorphism group \(A_B(M, F)\) can be considered as a subgroup \(A(M/F)\) of the homeomorphism group \(Homeo(M/F)\) of the leaf space \(M/F\) of this foliation.

Let us emphasize, that the basic automorphism group \(A_B(M, F)\) of a Cartan foliation \((M, F)\) is an invariant of this foliation in the category \(\mathcal{F}\).

### 5.2 Properties of basic automorphism groups of Cartan foliations

For a Cartan foliation with effective transverse Cartan geometry Proposition 5.1 follows from ([18], Proposition 9).

**Proposition 5.1.** Let \((M, F)\) be a Cartan foliation modelled on an effective Cartan geometry. Let \(A^H(\mathcal{R}, \mathcal{F}) := \{h \in A(\mathcal{R}, \mathcal{F}) | R_a \circ h = h \circ R_a \forall a \in H\}\), \(A^H_B(\mathcal{R}, \mathcal{F}) := \{h \in A_L(\mathcal{R}, \mathcal{F}) | R_a \circ h = h \circ R_a \forall a \in H\}\) and \(A^H_B(\mathcal{R}, \mathcal{F})\) be the quotient group \(A^H(\mathcal{R}, \mathcal{F})/A^H_L(\mathcal{R}, \mathcal{F})\). Then there exists natural group isomorphism \(\delta : A^H_B(\mathcal{R}, \mathcal{F}) \to A_B(M, F)\) satisfying the commutative diagram

\[
\begin{array}{ccc}
A^H(\mathcal{R}, \mathcal{F}) & \xrightarrow{\mu} & A(M, F) \\
\alpha^H \downarrow & & \downarrow \alpha \\
A^H_B(\mathcal{R}, \mathcal{F}) & \xrightarrow{\delta} & A_B(M, F).
\end{array}
\]

where \(\alpha^H\) and \(\alpha\) are the group epimorphisms onto the indicated quotient groups.

We accentuate that the problem of the existence a Lie group structure in \(A_B(M, F)\) is reduced to the same problem for the group \(A^H_B(\mathcal{R}, \mathcal{F})\) due to Proposition 5.1.

The following statement deepens Proposition 11 from [18] and is essentially used by us in the sequel.

**Proposition 5.2.** Let \((M, F)\) be a complete Cartan foliation with an effective transverse geometry, with \(g_0 = g_0(M, F) = 0\) and \(\dim(A^H(W)) \geq 1\). Then there exists a Lie group isomorphism

\[
\nu : A^H_B(\mathcal{R}, \mathcal{F}) \to A^H(W) : h \cdot A^H_L(\mathcal{R}, \mathcal{F}) \mapsto \tilde{h},
\]
where \( h \in A^H(\mathcal{R}, \mathcal{F}) \) and \( \tilde{h} \) is the projection of \( h \) with respect to the basic fibration \( \pi_b : \mathcal{R} \to W \).

Consequently, \( \varepsilon = \nu \circ \delta^{-1} : A_B(M, F) \to A^H(W) \) is a Lie group isomorphism.

In the case \( A^H(W) = 0 \) the group \( A_B(M, F) \) is a discrete Lie group.

\[ \text{Доказательство.} \] By condition \( g_0(M, F) = 0 \) and the lifted foliation \( (\mathcal{R}, \mathcal{F}) \) is formed by fibres of the submersion \( \pi_B : \mathcal{R} \to W \). Then every \( h \in A^H(\mathcal{R}, \mathcal{F}) \) induces \( \tilde{h} \in A^H(W) \), and the map \( \rho : A^H(\mathcal{R}, \mathcal{F}) \to A^H(W) \) is defined. It is clear that \( \rho \) is a group homomorphism with the kernel \( \text{Ker}(\rho) = A^H(\mathcal{R}, \mathcal{F}) \).

As \( A^H(\mathcal{R}, \mathcal{F}) \) is the normal subgroup of \( A^H(\mathcal{R}, \mathcal{F}) \), there exists a group monomorphism \( \nu : A^H(\mathcal{R}, \mathcal{F}) \to A^H(W) \) satisfying the equality \( \rho := \nu \circ \alpha^H \), where \( \alpha^H : A^H(\mathcal{R}, \mathcal{F}) \to A^H(W) \) is the natural projection onto the quotient group \( A^H(\mathcal{R}, \mathcal{F}) = A^H(\mathcal{R}, \mathcal{F})/A^H(\mathcal{R}, \mathcal{F}) \).

Therefore \( A^H(W) \) is a discrete Lie group, then \( A^H(\mathcal{R}, \mathcal{F}) \) is also discrete Lie group.

Further we assume that \( \dim(A^H(W)) \geq 1 \).

It is enough to prove that the monomorphism \( \nu : A^H(\mathcal{R}, \mathcal{F}) \to A^H(W) \) is surjective or, that is equivalent, the homomorphism \( \rho : A^H(\mathcal{R}, \mathcal{F}) \to A^H(W) \) is surjective.

Let \( a \) be the Lie algebra of the Lie group \( A^H(W) \). Let \( A^* \) be the fundamental vector field defined by \( A \in a \). Hence \( X := A^* \) is a complete vector field on \( W \) which defines an 1-parameter group \( \phi_t^X \), \( t \in (-\infty, \infty) \), of transformations from \( A^H(W) \).

Case 1. Let \( f \) be any element from the identity component \( A^H(W) \) of the Lie group \( A^H(W) \). Then there exists \( A \in a \) and \( t_0 \in (-\infty, +\infty) \) such that \( f = \phi_{t_0}^X \) where \( X = A^* \). Since \( \pi_B : \mathcal{R} \to W \) is the submersion with the Ehresmann connection \( \mathfrak{M} \), where \( \mathfrak{M} = 
abla \mathfrak{M} \), there exists a unique vector field \( Y \in \mathfrak{X}_{\mathfrak{M}}(\mathcal{R}) \) such that \( \pi_B Y = X \). The completeness of the vector field \( Y \) implies the completeness of the vector field \( X \).

Hence \( Y \) defines a 1-parameter group \( \psi_t^Y \), \( t \in (-\infty, \infty) \), of diffeomorphisms of the manifold \( \mathcal{R} \). The direct check made in details by the first author in the proof of Proposition 11 in [18] shows that \( \psi_t^Y \in A^H(\mathcal{R}, \mathcal{F}) \) for all \( t \in (-\infty, \infty) \), and \( f := \psi_{t_0}^Y \) lies over \( f \) relatively \( \pi_B \), i.e. \( \rho(f) = f \).

Case 2. Suppose now that \( f \notin A^H(W) \), then there is \( \Gamma \in A^H(W) \) such that \( f \in \Gamma \circ A^H(W) \). Therefore \( f = \Gamma \circ \phi_{t_0}^X \) for some \( t_0 \in (-\infty, +\infty) \), where \( \phi_t^X \) is 1-parametric group from \( A^H(W) \).

The vector field \( Y := \Gamma X \) generates a smooth 1-parametric group \( \phi_t^Y = \Gamma \circ \phi_t^X \circ \Gamma^{-1} \), and \( f = \phi_{t_0}^Y \). Let \( \tilde{Y} \) be \( \mathfrak{M} \)-lift of \( Y \) to \( \mathcal{R} \). Then \( \tilde{Y} \) is a complete vector field on \( \mathcal{R} \). Hence \( \tilde{Y} \) defines an 1-parametric group \( \psi_t^\tilde{Y} \), and \( f := \phi_{t_0}^\tilde{Y} \). By analogy with the case 1 we show that \( \psi_t^\tilde{Y} \in A^H(\mathcal{R}, \mathcal{F}) \) for all \( t \in (-\infty, \infty) \), and \( f := \psi_{t_0}^\tilde{Y} \) lies over \( f \) relatively \( \pi_B \), i.e. \( \rho(f) = f \).

Consequently \( \nu \) is surjection.

Thus, \( \varepsilon = \nu \circ \delta^{-1} : A_B(\mathcal{R}, \mathcal{F}) \to A^H(W) : f \mapsto \nu(f) = \nu \circ \delta^{-1} = \nu \circ \delta^{-1} \) is the isomorphism of Lie groups.

\[ \square \]

6 Proof of Theorem 1.1 and Corollaries

6.1 Proof Theorem 1.1

Let \( (M, F) \) be a Cartan foliation modelled on a Cartan geometry \( \tilde{\xi} = (\tilde{P}(N, H), \tilde{\omega}) \) of type \( \tilde{g}/\tilde{h} \) of type \( (\tilde{G}, \tilde{H}) \) and the Lie group \( K \) be the kernel of the pair Lie groups \( (\tilde{G}, \tilde{H}) \), \( \mathfrak{g} \) be the Lie algebra of \( K \). Then the associated effective Cartan geometry \( \xi = (P(M, H), \omega) \) of type \( (G, H) \), where \( G = \tilde{G}/K, H = \tilde{H}/K \), is defined. Here \( \omega \) is the \( g \)-valued 1-form on \( P \), where \( g = \tilde{g}/\mathfrak{g} \). According to Proposition 3.2 the associated
foliated bundle $\mathcal{R}(M, H)$ with the lifted foliation $(\mathcal{R}, \mathcal{F})$ and the projection $\pi : \mathcal{R} \to M$ are defined.

Assume that the structural Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$ is zero for a complete Cartan fibration $(M, F)$. Then the lifted foliation $(\mathcal{R}, \mathcal{F})$ is formed by fibres of the locally trivial fibration $\pi_B : \mathcal{R} \to W$ and the $\mathfrak{g}$-valued 1-form $\beta$ on $\mathcal{R}$ is determined according to Proposition 3.2. In compliance with (17), Proposition 4) the map $W \times H \to W : (w, a) \mapsto \pi_B(R_a(u)) \quad \forall (w, a) \in W \times H, u \in \pi_B^{-1}(w),$ defines a locally free action of the Lie group $H$ on the basic manifold $W$, and the orbits space $W/H$ is homeomorphic to the leaf space $M/F$. Identify $W/H$ with $M/F$. Connected components of the orbits of this action form a regular foliation $(W, F_H^\beta)$. The equality $\pi_B^*\beta := \beta$ defines an $\mathfrak{g}$-valued 1-form $\beta$ on $W$ such that $\beta(A_W^*v) = A$, where $A_W^*$ is the fundamental vector field on $W$ defined by $A \in \mathfrak{h} \subset \mathfrak{g}$.

Denote by $A(W, \beta)$ the Lie group of automorphisms of the parallelizable manifold $(W, \beta)$, i.e., $A(W, \beta) = \{f \in Diff(W) \mid f^*\beta = \beta\}$.

Let $A^H(W) = \{f \in A(W, \beta) \mid f \circ R_a = R_a \circ f\}$. Then $A^H(W)$ and its unity component $A^H(W)_e$ are Lie groups as closed subgroups of $A(W, \beta)$.

If $\dim(A^H(W)) = 0$, then according to Proposition 5.2, $A_B(M, F)$ is a discrete Lie group, hence the estimates (1) and (2) are valid.

Suppose now that $\dim(A_B(M, F)) \geq 1$.

According to Proposition 5.2 $\varepsilon : Aut_B(M, F) \to A^H(W)$ is the group isomorphism onto the Lie group $A^H(W)$. Therefore the basic automorphism group $A_B(M, F)$ admits a Lie group structure of the dimension not more than $\dim(W) = \dim(\mathfrak{g})$. Because $\mathfrak{g} = \mathfrak{g}/\mathfrak{t}$ we have $\dim(\mathfrak{g}) = \dim(\mathfrak{g}) - \dim(\mathfrak{t})$. Hence the dimension of the Lie group $Aut_B(M, F)$ satisfies the following inequality $\dim(A_B(M, F)) \leq \dim(\mathfrak{g}) - \dim(\mathfrak{t})$. As the Lie group $A_B(M, F)_e$ is realized as a closed subgroup of the automorphism Lie group $A(W, \beta)$ of a parallelizable manifold, then it admits a unique topology and a unique smooth structure that make it into a Lie group (11, Proposition 1). The same is valid for the group $A_B(M, F)$.

(a) Let $s : W \to W/H$ be the projection onto the orbit space. Assume now that there exists an isolated closed leaf $L$ of the foliation $(M, F)$. Let $x \in L, u = \pi^{-1}(x)$ and $w = \pi_B(u) \in W$. Denote by $A^H(W)_e$ the unity component of $A(W)$. Then $q(L) = s(w)$ and the orbit $A^H(W)_e \cdot w$ belongs to $s^{-1}(s(w))$. Therefore $\dim(A_B(M, F)) = \dim(A^H(W)_e \cdot w) \leq \dim(H) = \dim(\mathfrak{h}) = \dim(\mathfrak{h}) - \dim(\mathfrak{t})$. Thus $\dim(A_B(M, F)) \leq \dim(\mathfrak{h}) - \dim(\mathfrak{t})$.

(b) Now we suppose that the set of closed leaves of $(M, F)$ is countable (nonempty). Consider any 1-parametric group $\varphi_t, t \in (-\infty, \infty), \text{from } A^H(W) \cong A_B(M, F).$ Let $L = L(x)$ be any leaf, $u = \pi^{-1}(x)$ and $w = \pi_B(u) \in W$. Let $w \cdot H$ be the orbit of $w$ relatively $H$. Remark that $[L] = q(L) = s(w \cdot H)$ is a point of the leaf space $M/F = W/H$, and $w \cdot H$ is a closed orbit if and only if $L$ is a closed leaf. Thus, the set of closed orbits is countable. Observe that any automorphism $f \in A^H(W)$ transforms a closed orbit of $H$ to a closed orbit. Hence $\varphi_t(w \cdot H) = w \cdot H$ for all $t$, because otherwise the set of closed orbits $\{\varphi_t(w \cdot H) \mid t \in (-\infty, \infty)\}$ is the set of continuum and we get contradiction. By analogy with the previous case we have the estimate (2).

Assume that $\dim(A_B(M, F)) \geq 1$. Let $\varphi_t, t \in (-\infty, \infty)$, be any 1-parametric subgroup of the automorphism group $A^H(W) \cong A_B(M, F).$ As it was proved above, it is necessary $\varphi_t(w_n \cdot H) = w_n \cdot H$ for all $t \in (-\infty, \infty)$ and $n \in \mathbb{N}$. Denote by $\tilde{\varphi}_t$ the induced 1-parametric group of homeomorphisms of the leaf space $M/F$ (see the diagram 5.1).
Therefore, for each \( t \in (-\infty, \infty), \ t \neq 0 \), the homeomorphism \( \tilde{\varphi}_t \) has dense subset 
\[ \{ [L_n] = s(w_n \cdot H) | \ n \in \mathbb{N} \} \] of fixed points in the leaf space \( M/F = W/H \).

Due to continuity and openness of the projection \( q : M \to M/F \), the leaf space \( M/F \) is a first-countable space, that is every its point has a countable neighbourhood basis. Then \( \tilde{\varphi}_t \) is sequentially continuous. Therefore the existence of dense subset of fixed points of the homeomorphism \( \tilde{\varphi}_t \) implies \( \tilde{\varphi}_t = id_{M/F} \). Hence \( \varphi_t = id_W \) that contradicts with the assumption.

Thus, \( \dim(A_B(M, F)) = 0 \) and (3) is proved.

6.2 Proof of Corollary 1.1

As it was proved by the first author ([18], Proposition 12), the existence of a proper leaf \( L \) with discrete holonomy group guarantees the equality \( g_0(M, F) = 0 \).

Remark that any closed leaf of a foliation is proper and each finite holonomy group is a discrete one. Hence we have implications (iv) \( \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i) \). Thus, applying Theorem 1.5 we get the required assertion.

6.3 Proof of Corollary 1.2

It is well known that any foliation has leaves without holonomy. Therefore, the Corollary 1.2 follows from the item (iii) of Corollary 1.1.

7 The structure of Cartan foliations covered by fibrations

7.1 \((G, B)\)-foliations

Let \( B \) be a connected smooth manifold and \( G \) be a Lie group of diffeomorphisms of \( B \). The group \( G \) is said to act quasi-analytically on \( B \) if, for any open subset \( V \) in \( B \) and an element \( g \in G \) the equality \( g|_V = id_V \) implies \( g = e \), where \( e \) is the identity transformation of \( B \).

**Definition 7.1.** Assume that the group \( \text{Lie} \ G \) of diffeomorphisms of a manifold \( B \) acts on \( N \) quasi-analytically. A foliation \((M, F)\) defined by an \( B \)-cocycle \( \{U_i, f_i, \{\gamma_{ij}\}_{i,j \in J}\} \) is called a \((G, B)\)-foliation if for any \( U_i \cap U_j \neq \emptyset, i, j \in J \), there is an element \( g \in G \) such that \( \gamma_{ij} = g|_{f_j(U_i \cap U_j)} \).

7.2 Proof of Theorem 1.2

The following lemma will be useful for us.

**Lemma 7.1.** Let \( \xi = (P(B, H), \omega) \) be an effective Cartan geometry on a connected manifold \( B \) and \( \Phi \) be a group of automorphisms of \((B, \xi)\). Then the group \( \Phi \) acts quasi-analytically on \( B \).

**Доказательство.** Suppose that there are \( \gamma \in \Phi \) and an open subset \( U \subset B \) such that \( \gamma|_U = id_U \). Then there exists a unique \( \Gamma \in Aut(\xi) \) lying over \( \gamma \). Let \( p : P \to B \) be the projection of the \( H \)-bundle \( P(B, H) \). Observe that any connected component of \( P \) is a connected component of some point \( v \in p^{-1}(U) \), i.e. it may be represented in the form \( P_v \). The effectiveness of the Cartan geometry \( \xi \) implies \( \Gamma|_{p^{-1}(U)} = id_{p^{-1}(U)} \). Hence \( \Gamma \) preserves each connected component \( P_v \) of \( P \). Because \( \Gamma \) is an isomorphism
of the connected parallelizable manifold \((P_v, \omega|_{P_v})\) and \(\Gamma(v) = v\), then it is necessary \(\Gamma|_{P_v} = id_{P_v}\). Therefore \(\Gamma = id_P\) and \(\gamma = id_B\).

Thus, the group \(\Phi\) acts quasi-analytically on \(B\). \(\square\)

Suppose that a Cartan foliation \((M, F)\) modelled on the effective Cartan geometry \(\xi = (P(N, H), \omega)\) is covered by a fibration \(\tilde{r} : \tilde{M} \to B\), where \(\tilde{\kappa} : \tilde{M} \to M\) is the universal covering map. For an arbitrary point \(b \in B\) take \(y \in r^{-1}(b)\) and \(x = \tilde{\kappa}(y)\).

Without loss generality we assume that there is a neighbourhood \(U_i, x \in U_i\), from the \(\mathcal{N}\)-cocycle \(\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in \mathcal{J}}\) which defines \((M, F)\) and a neighbourhood \(\tilde{U}_i, y \in \tilde{U}_i\), such that \(\tilde{\kappa}|_{\tilde{U}_i} : \tilde{U}_i \to U_i\) is a diffeomorphism.

Let \(\tilde{V}_i := \tilde{r}(\tilde{U}_i)\). Then there exists a diffeomorphism \(\phi : \tilde{V}_i \to V_i\) satisfying the equality \(\phi \circ \tilde{r} = f_i \circ \tilde{\kappa}|_{\tilde{U}_i}\). The diffeomorphism \(\phi\) induces the Cartan geometry \(\eta|_{\tilde{V}_i}\) on \(\tilde{V}_i\) such that \(\phi\) becomes the isomorphism \((\tilde{V}_i, \eta|_{\tilde{V}_i})\) and \((V_i, \xi|_{V_i})\) in the category \(\mathcal{C}_\mathfrak{G}\) of Cartan geometries. The direct check shows that by this way we define the Cartan geometry \(\eta\) on \(B\), and \(\eta|_{\tilde{V}_i} = \eta|_{\tilde{V}_i}, i \in \mathcal{J}\).

Let us fix points \(x_0 \in M\) and \(y_0 \in \tilde{\kappa}^{-1}(x_0) \in \tilde{M}\). Then the fundamental group \(\pi_1(M, x_0)\) acts on the universal covering space \(\tilde{M}\) as a deck transformation group \(\tilde{G} \cong \pi_1(M, x_0)\) of \(\tilde{\kappa}\). Since \(\tilde{G}\) preserves the inducted foliation \((\tilde{M}, \tilde{F})\) formed by fibres of the fibration \(\tilde{r} : \tilde{M} \to B\), then every \(\tilde{\psi} \in \tilde{G}\) defines \(\psi \in Diff(B)\) satisfying the relation \(\tilde{r} \circ \tilde{\psi} = \psi \circ \tilde{r}\). The map \(\tilde{\chi} : \tilde{G} \to \Psi : \tilde{\psi} \to \psi\) is the group epimorphism. Observe that \(\tilde{\Psi}\) is a subgroup of the automorphism group \(Aut(\tilde{M}, \tilde{F})\) of \((\tilde{M}, \tilde{F})\) in the category \(\mathcal{C}_\mathfrak{G}\). Therefore \(\Psi\) is a subgroup of the automorphism group \(Aut(\tilde{B}, \eta)\) in the category \(\mathcal{C}_\mathfrak{G}\) of Cartan geometries. The kernel \(ker(\tilde{\chi})\) of \(\tilde{\chi}\) determines the quotient manifold \(\tilde{M} := \tilde{M}/ker(\tilde{\chi})\) with the quotient map \(\tilde{\kappa} : \tilde{M} \to \tilde{M}\) and the quotient group \(G := \tilde{G}/ker(\tilde{\chi})\) such that \(M \cong \tilde{M}/G\). The quotient map \(\kappa : \tilde{M} \to M\) is the required regular covering map, with \(G\) acts on \(\tilde{M}\) as a deck transformation group. The map \(G \to \Psi : \tilde{\psi}Ker(\tilde{\chi}) \mapsto \tilde{\chi}(\tilde{\psi})\) is a group isomorphism.

Remark that the induced foliation \((\tilde{M}, \tilde{F}), \tilde{F} = \kappa^*F\), is covered by a foliation \(r : \tilde{M} \to B\) such that \(\tilde{r} = r \circ \tilde{\kappa}\).

Note the assertion (3) of Theorem 1.3 is easy proved with an application of Lemma 7.1.

### 7.3 Proof of Theorem 1.3

Let \((N, \eta)\) be an effective Cartan geometry on a connected manifold and \(\Psi\) be a subgroup of the automorphism Lie group \(Aut(N, \eta)\) in category \(\mathcal{C}_\mathfrak{G}\) satisfying the conditions of Theorem 1.3. Then \((M, F)\) is a Cartan foliation modelled on \((N, \eta)\). Denote by \(\Phi\) the closure of the group \(\Psi\) in the Lie group \(Aut(N, \eta)\), then \(\Phi\) is a Lie subgroup of \(Aut(N, \eta)\). In compliance with Lemma 7.1 the Lie group \(\Phi\) acts on \(N\) quasi-analytically. Thus \((M, F)\) is a \((\Phi, N)\)-foliation. By condition of Theorem 1.3 this foliation admits an Ehresmann connection. Thus, Theorem 2 proved by the first author in 19 is applicable to \((M, F)\), and according to this theorem the foliation \((M, F)\) is covered by fibration and satisfies Theorem 1.2.
8 The structure of basic automorphism groups

8.1 Properties of regular covering maps

Definition 8.1. Let $f : M \to B$ be a submersion. It is said that $\hat{h} \in Diff(M)$ lying over $h \in Diff(B)$ relatively $f$ if $h \circ f = f \circ \hat{h}$.

Theorems 28.7, 28.9, and 28.10 which were proved by H. Busemann for $G$-spaces in [5] are valid also for the smooth case. We reformulate them in the following way.

Busemann’s theorem. Let $\tilde{\kappa} : \tilde{K} \to K$ be the universal covering map with the deck transformations group $\Gamma$, where $K$ and $\tilde{K}$ are smooth manifolds. Then

(i) For any $h \in Diff(K)$ there exists $\hat{h} \in Diff(\tilde{K})$ such that $\hat{h}$ lying over $h$.

(ii) A diffeomorphism $\hat{h} \in Diff(\tilde{K})$ lies over some diffeomorphism $h \in Diff(K)$ if and only if it satisfies the equality $\hat{h} \circ \Gamma = \Gamma \circ \hat{h}$.

(iii) The set of all diffeomorphisms lying over id$_K$ is coincided with $\Gamma$.

(iv) Let $\Phi$ be a group of diffeomorphisms of the manifold $K$. The set of all diffeomorphisms of $\tilde{K}$ lying over diffeomorphisms from $\Phi$ forms a group $\check{\Phi}$ which is isomorphic to the quotient group $\Phi / \Gamma$.

Remark 8.1. It is well known that the statement (i) of the Busemann’s theorem is not true in the case of regular covering maps in general case.

The following statement is proved by us for regular covering maps by analogy with the statements (ii), (iii) and (iv) of the Busemann’s theorem.

Proposition 8.1. Let $\kappa : \tilde{K} \to K$ be a smooth regular covering map with the deck transformation group $\Gamma$ and $\check{\kappa} : \tilde{K} \to K$ be the universal covering map with the deck transformation group $\hat{\Gamma}$. Then

(1) For $h \in Diff(K)$ there exists $\hat{h} \in Diff(\tilde{K})$ lying over $h$ if and only if there is $\bar{h}$ lying over $h$ relatively $\kappa : \tilde{K} \to K$ satisfying the equality $\hat{h} \circ \hat{\Gamma} = \hat{\Gamma} \circ \bar{h}$, where $\hat{\Gamma}$ is the deck transformation group of the universal covering map $\check{\kappa} : \tilde{K} \to K$.

(2) The set of all diffeomorphisms lying over id$_K$ relatively $\kappa : \tilde{K} \to K$ is coincided with the deck transformation group of $\Gamma \cong \hat{\Gamma} / \bar{\Gamma}$.

(3) Let $G$ be a group of diffeomorphisms of the manifold $K$ such that for every $g \in G$ there exists $\hat{g} \in Diff(\tilde{K})$ lying over $g$. Let $\hat{\Gamma}$ be the full group of $\hat{g} \in Diff(\tilde{K})$ lying over transformations from $G$. Then $G$ is isomorphic to the quotient group $\hat{\Gamma} / \bar{\Gamma}$.

Доказательство. (1) Suppose that for $g \in Diff(K)$ there exists $\hat{g} \in Diff(\tilde{K})$ lying over $g$. Consider the universal covering map $\hat{\kappa} : \tilde{K} \to \hat{K}$. It is well known that there is the universal covering map $\kappa : \tilde{K} \to K$ satisfying the equality $\kappa \circ \hat{\kappa} = \kappa$. According to the assertion (i) of the Busemann’s theorem there exists $\hat{g} \in Diff(\tilde{K})$ over $\hat{g}$ relatively $\hat{\kappa}$. By the assertion (ii) of the Busemann’s theorem, $\hat{g}$ satisfies both equalities $\hat{g} \circ \hat{\Gamma} = \hat{\Gamma} \circ \hat{g}$ and $\hat{g} \circ \hat{\Gamma} = \hat{\Gamma} \circ \hat{g}$, and $\hat{\Gamma}$ is the normal subgroup of the group $\hat{\Gamma}$.

Converse. Suppose that for $h \in Diff(K)$ there exists $\hat{h} \in Diff(\tilde{K})$ lying over $h$ relatively $\kappa : \tilde{K} \to K$. Then by the statement (i) of the Busemann’s theorem there is $\hat{h} \in Diff(\tilde{K})$ lying over $\hat{h}$ relatively the universal covering map $\hat{\kappa} : \tilde{K} \to \hat{K}$ that is
\( \hat{\kappa} \circ \hat{h} = \hat{h} \circ \hat{\kappa} \). In according to assertion (i) of Busemann’s theorem, \( \hat{h} \) satisfies both equalities \( \hat{h} \circ \hat{\Gamma} = \hat{\Gamma} \circ \hat{h} \) and \( \hat{h} \circ \hat{\Gamma} = \hat{\Gamma} \circ \hat{h} \). Therefore, applying the equality \( \kappa \circ \hat{\kappa} = \hat{\kappa} \), for each \( \hat{x} \in \hat{K} \), we get the chain of equalities

\[
\begin{align*}
\kappa \circ \hat{h}(\hat{x}) &= \kappa \circ (\kappa \circ \hat{h})(\hat{\kappa}^{-1}(\hat{x})) = ((\kappa \circ \hat{\kappa}) \circ \hat{h})(\hat{\kappa}^{-1}(\hat{x})) = (\hat{\kappa} \circ \hat{h})(\hat{\kappa}^{-1}(\hat{x})) = \\
&= (h \circ \kappa \circ \hat{\kappa})(\hat{\kappa}^{-1}(\hat{x})) = (h \circ (\kappa \circ \hat{\kappa}))(\hat{\kappa}^{-1}(\hat{x})) = (h \circ \kappa)(\hat{\kappa}^{-1}(\hat{x})) = h \circ \kappa(\hat{x}).
\end{align*}
\]

Hence, \( \kappa \circ \hat{h} = h \circ \kappa \), i.e. \( \hat{h} \) lies over \( h \) relatively \( \kappa \).

The statement (2) is obvious.

(3) Let \( G \) be the group of projections of \( \hat{G} \) and \( f : \hat{G} \to G : \hat{h} \mapsto h \), where \( h \) is the projection of \( \hat{h} \), is a group epimorphism, since \( f \) is surjective by the condition. In according with previous statement \( \text{Ker}(f) = \hat{\Gamma} \), then \( G \cong \hat{G}/\hat{\Gamma} \). \( \square \)

### 8.2 Proof of Theorem 1.5

Suppose that a Cartan foliation \((M, F)\) is covering by fibration. By definition 6.1 the induced foliation \((\hat{M}, \hat{F})\) on the space of the universal covering \( \hat{\kappa} : \hat{M} \to M \) is defined by a locally trivial fibration \( \hat{\tau} : \hat{M} \to B \). Due to Theorem 1.2 the regular covering map \( \kappa : \hat{M} \to M \) and locally trivial fibration \( r : \hat{M} \to B \) are defined, where \( B \) is a simply connected manifold with the inducted Cartan geometry \( \eta \). Let \( \Psi \) be the global holonomy group of \((M, F)\), then and \( \Psi \) is isomorphic to the deck transformations group \( G \) of \( \kappa : \hat{M} \to M \). Since the manifold \( \hat{M} \) is simply connected, then there exists the universal covering map \( \hat{\kappa} : \hat{M} \to \hat{M} \) satisfying the equality \( \kappa \circ \hat{\kappa} = \hat{\kappa} \). Let \( \hat{G}, G \) and \( \hat{G} \) be the deck transformation groups of the covering maps \( \hat{\kappa}, \kappa \) and \( \hat{\kappa} \) relatively, with \( \Psi \cong G \cong \hat{G}/\hat{\Gamma} \).

Let us consider following preimages of \( \mathcal{R} \), relatively \( \hat{\kappa} \) and \( \kappa \)

\[
\hat{\mathcal{R}} := \{(\hat{x}, u) \in \hat{M} \times \mathcal{R} | \hat{\kappa}(\hat{x}) = \pi(u)\} = \hat{\kappa}^* \mathcal{R} \quad \text{and} \quad \hat{\mathcal{R}} := \{(\hat{x}, u) \in \hat{M} \times \mathcal{R} | \kappa(\hat{x}) = \pi(u)\} = \kappa^* \mathcal{R}.
\]

Remark that the maps

\[
\begin{align*}
\hat{\theta} : \hat{\mathcal{R}} \to \mathcal{R} : (\hat{x}, u) \mapsto (\hat{\kappa}(\hat{x}), u) & \forall (\hat{x}, u) \in \hat{\mathcal{R}}, \\
\theta : \mathcal{R} \to \mathcal{R} : (x, u) \mapsto (\kappa(x), u) & \forall (x, u) \in \mathcal{R}, \\
\hat{\theta} : \hat{\mathcal{R}} \to \hat{\mathcal{R}} : (\hat{x}, u) \mapsto (\hat{\kappa}(\hat{x}), u) & \forall (\hat{x}, u) \in \hat{\mathcal{R}},
\end{align*}
\]

are regular covering maps with the deck transformation groups \( \hat{\Gamma}, \Gamma \) and \( \hat{\Gamma}, \) relatively, which are isomorphic to the relevant groups \( \hat{G}, G \) and \( \hat{G}, \) i.e. \( \hat{\Gamma} \cong \hat{G}, \Gamma \cong G \) and \( \hat{\Gamma} \cong \hat{G} \).

Let \((\mathcal{R}, \hat{F})\) and \((\hat{\mathcal{R}}, \hat{F})\) be the corresponding lifted foliations. Since \((\hat{M}, \hat{F})\) and \((\hat{M}, \hat{F})\) are simple foliations and are formed by locally trivial fibrations \( \hat{\pi}_B : \hat{\mathcal{R}} \to \hat{W} \) and \( \hat{\pi}_B : \hat{\mathcal{R}} \to \hat{W} \), then \((\mathcal{R}, \hat{F})\) and \((\hat{\mathcal{R}}, \hat{F})\) are also simple foliations, hence \( g_0(\mathcal{R}, \hat{F}) = 0 \), \( g_0(\hat{\mathcal{R}}, \hat{F}) = 0 \) and \( \hat{W} = \hat{\mathcal{R}} \cap \hat{\mathcal{F}} \), \( \hat{W} = \hat{\mathcal{R}} \cap \hat{\mathcal{F}} \).

Since the fibrations \( \hat{\tau} : \hat{M} \to B \) and \( r : \hat{M} \to B \) have the same base \( B \), each leaf of the foliation \((\hat{M}, \hat{F})\) is invariant relatively the group \( \hat{G} \), i.e. \( \hat{G} \subset A_{L}(\hat{M}, \hat{F}) \). Therefore \( \hat{\Gamma} \subset A_{L}(\hat{\mathcal{R}}, \hat{\mathcal{F}}) \) and the leaf spaces \( \hat{\mathcal{R}}/\hat{\mathcal{F}} = \hat{W} \) and \( \mathcal{R}/\mathcal{F} = \hat{W} \) are coincided, i.e. \( \hat{W} = W \). Consequently basic automorphism groups \( A_{B}(\mathcal{R}, \hat{F}) \) and \( A_{B}(\hat{\mathcal{R}}, \hat{F}) \) may be identified. Further we put \( A_{B}(\mathcal{R}, \hat{F}) = A_{B}(\hat{\mathcal{R}}, \hat{F}) \).

According to the conditions of Theorem 1.5 \( \Psi \) is a discrete subgroup of the Lie group \( A(B, \eta) \). Let \( N(\Psi) \) be the normalizer of \( \Psi \) in the Lie group. Hence, \( N(\Psi) \) is a
closed Lie subgroup of the Lie group Aut(B,η) and the quotient group N(Ψ)/Ψ is also a Lie group.

Let \( \pi : \mathcal{R} \to M \) be the projection of the foliated bundle of \((M,F)\). Due to Theorem 1.4 the discreteness of the global holonomy group \( \Psi \) implies that the structural Lie algebra \( \mathfrak{g}_0 \) of the Cartan foliations \((M,F)\) is zero. Therefore the lifted foliation \((\mathcal{R},\mathcal{F})\) is formed by fibres of the basic fibration \( \pi_B : \mathcal{R} \to W \).

Observe that there exists a map \( \tau : \hat{W} \to W \) satisfying the quality \( \tau \circ \hat{\pi}_B = \theta \circ \pi_B \). It is easy to show that \( \tau : \hat{W} \to W \) is a regular covering map with the deck transformations group \( \Phi, \Phi \subset A^H(\hat{W}) \), which is naturally isomorphic to the groups \( \Psi, G \) and \( \Gamma \).

Denote by \( \eta = (P(B,H),\omega) \) the Cartan geometry with the projection \( p : P \to B \) onto \( B \) determined in the proof of Theorem 2. Remark that \( \hat{W} = P \) is the space of the \( H \)-bundle of the Cartan geometry \( \eta \).

Since \( \kappa : \hat{M} \to M \) and \( \pi : \mathcal{R} \to M \) are morphisms of the following foliations \( \kappa : (\hat{M},\hat{F}) \to (M,F) \) and \( \pi : (\mathcal{R},\mathcal{F}) \to (\mathcal{R},\mathcal{F}) \) in the category of the foliations \( \mathfrak{fol} \), then maps \( \hat{\tau} : B \to M/F \) and \( s : W \to W?H = M/F \) are defined, and the following diagram

\[
\begin{array}{ccc}
P = \hat{W} & \xrightarrow{\pi_B} & W \\
p \downarrow & & \downarrow s \\
\hat{M}/\hat{T} = B & \xrightarrow{\hat{\tau}} & M/F \\
\end{array}
\]

is commutative.

Due to Proposition 5.2 there are the Lie group isomorphisms

\[ \varepsilon : A_B(M,F) \to A^H(W) \] and \[ \hat{\varepsilon} : A_B(\hat{M},\hat{F}) = A_B(\tilde{M},\tilde{F}) \to A^H(\hat{W}). \]

Let us show that for any \( h \in A^H(W) \) there exists an automorphism \( \tilde{h} \in A^H(\hat{W}) \) lying over \( h \) relatively \( \tau : \hat{W} \to W \). Denote \( \tilde{\varepsilon}^{-1}(h) \in A_B(M,F) \) by \( f \cdot A_L(M,F) \in A_B(M,F) \), where \( f \in A(M,F) \). Because \( \tilde{k} : \hat{M} \to M \) is the universal covering map, then by the statement \( (i) \) of the Busemann’s theorem there exists \( \tilde{f} \in Diff(\hat{M}) \) lying over \( f \) relatively \( \tilde{k} \). It not difficult to see that \( \tilde{f} \in A(\hat{M},\hat{F}) \). Hence \( \tilde{f} \circ A_L(\hat{M},\hat{F}) \in A_B(\tilde{M},\tilde{F}) \). Consider \( \tilde{h} := \tilde{\varepsilon}^{-1}(\tilde{f} \cdot A_L(\tilde{M},\tilde{F})) \in A^H(\hat{W}) \). The direct check shows that \( \tilde{h} \) lies over \( h \).

Applying the statement \((3)\) of Proposition 5.1 we get that the group of all automorphisms \( \tilde{h} \in A^H(\hat{W}) \) lying over automorphisms \( h \in A^H(W) \) is equal the normalizer \( N(\Phi) \) of the group \( \Phi \) of the deck transformations of the covering map \( \tau : \hat{W} \to W \), and \( A^H(W) \) is isomorphic to the quotient group \( N(\Phi)/\Phi \). Let us denote this isomorphism by \( \Phi : A^H(W) \to N(\Phi)/\Phi \).

The effectiveness of the Cartan geometry \( \eta = (P(B,H),\omega) \) on \( B \), where \( P = \hat{W} \), implies the existence of the Lie group isomorphism \( \sigma : A^H(\hat{W}) \to A(B,\eta) \) (see Remark 2.1). Observe that \( \sigma(\Phi) = \Psi \) and \( \sigma(N(\Phi)) = N(\Psi) \), hence there exists the inducted group isomorphism \( \tilde{\sigma} : N(\Phi)/\Phi \to N(\Psi)/\Psi \). Thus, the composition of the Lie group
isomorphisms
\[ \tilde{\sigma} \circ \Theta \circ \varepsilon : A_B(M, F) \to N(\Psi)/\Psi \]
is the required Lie group isomorphism.

8.3 Proof of Corollary 1.4

As \((M, F)\) is the simple foliation of a simply connected manifold \(M\), its global holonomy group \(\Psi\) is trivial. Consequently the the normalizer \(N(\Psi)\) in the Lie group \(Aut(B, \eta)\) is coincided with the Lie group \(Aut(B, \eta)\).

9 Examples

9.1 Suspended foliations

Suspension foliation was introduced by A. Haefliger. Let \(Q\) and \(T\) be smooth connected manifolds. Denote by \(\rho : \pi_1(Q, x) \to Diff(T)\) homomorphism group. Let \(G := \pi_1(Q, x)\) and \(\Phi := \rho(G)\). Consider a universal covering map \(\hat{\rho} : \hat{Q} \to Q\). A right action of the group \(G\) on product of manifolds \(\hat{Q} \times T\) is defined as follows:
\[ \Theta : \hat{Q} \times T \times G \to \hat{Q} \times T : (x, t, g) \mapsto (x \cdot g, \rho(g^{-1})(t)), \]
where the covering transformation \(\hat{Q} \to \hat{Q} : x \mapsto x \cdot g\) is induced by an element \(g \in G\).

The quotient manifold \(M := (\hat{Q} \times T)/G\) with the canonical projection \(f_0 : \hat{Q} \times T \to M = (\hat{Q} \times T)/G\)
are determined.

Let \(\Theta_g := \Theta|_{\hat{Q} \times \{t\} \times \{g\}}\). Since \(\Theta_g(\hat{Q} \times \{t\}) = \hat{Q} \times \rho(g^{-1})(t) \ \forall t \in T\), then the action of the discrete group \(G\) on \((\hat{Q} \times T)\) preserves the trivial foliation \(F := \{\hat{Q} \times \{t\} | t \in T\}\) of the product \(\hat{Q} \times T\). Thus the projection \(f_0 : \hat{Q} \times T \to M\) induced on the \(M\) of the smooth foliation \(F\). The pair \((M, F)\) is called a suspended foliation and is denoted by \(Sus(T, Q, \rho)\).

We accentuate that \((M, F)\) is covered by the trivial fibration \(\hat{Q} \times T \to T\).

9.2 Examples of the calculation of the basic automorphisms groups

Definition 9.1. Let \(\xi = (P(N, H), \omega)\) is arbitrary Cartan geometry of the type \((G, H)\), of the effectiveness of which is not assumed. The group
\[ \text{Gauge}(\xi) := \{ \Gamma \in A(\xi) | p \circ \Gamma = p \} \]
is called of the gauge transformation group of the Cartan geometry \(\xi\).

Example 9.1. Let \(G\) be a Lie group and \(H\) be a closed subgroup of \(G\). Denote by \(\mathfrak{g}\) and \(\mathfrak{h}\) the Lie algebras of Lie groups \(G\) and \(H\) relatively. Suppose that the kernel of the pair of Lie groups \((G, H)\) is equal to the intersection \(K = Z(G) \cap Z(H)\) of the centers of the groups \(G\) and \(H\). Denote by \(\omega_G\) the Maurer-Cartan \(\mathfrak{g}\)-valued 1-form on the Lie group \(G\). Then \(\xi^0 = (G(G/H, H), \omega_G)\) is the Cartan geometry, and its transverse curvature is zero. Consider any smooth manifold \(L\). Denote by \(M\) the product of
manifolds $M = L \times (G/H)$, and $F = \{L \times \{x\} \mid x \in G/H\}$. Then $(M, F)$ is the trivial transverse homogeneous foliation with the transverse Cartan geometry $\xi^0$. Because the foliation $(M, F)$ is trivial, the group $A_B(M, F)$ is coincided with the automorphisms group $Aut(\xi^0)$ of the Cartan geometry $\xi^0$ in the category $\mathcal{C}$.

Any left action $L_a$, $g \in G$, of the Lie group $G$ satisfies the conditions: $L_a^*\omega_G = \omega_G$ and $L_g^* \circ L_a = L_a \circ L_g \forall a \in G$. Therefore, $L_g \in Aut(\xi^0)$ and dim$(Aut(\xi^0)) = \operatorname{dim}(G) = \operatorname{dim}(\mathfrak{g})$. By assumption, the kernel of the pair $(G, H)$ equal to $K = Z(G) \cap Z(H)$, hence $Gauge(\xi^0) = \{L_b \mid b \in K\}$. Thus, the basic automorphisms group $A_B(M, F)$ is equal to the quotient $Aut(\xi^0)/Gauge(\xi^0) \cong G/K$, and dim$(A_B(M, F)) = \operatorname{dim}(\mathfrak{g}) - \operatorname{dim}(\mathfrak{k})$, where $\mathfrak{k}$ is the algebra Lie of the kernel of $K$.

Example 9.1 shows that the estimation (1) of the dimension group $Aut_B(M, F)$ in Theorem 1.1 is exact.

**Example 9.2.** Let

$$G := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R}^1 \right\}, \quad H := \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}^1 \right\}.$$ 

Then $G$ is an Abelian Lie group and $H$ is a connected closed subgroup of the group Lie $G$. Hence $H = Z(G) \cap Z(H)$. Let $\xi^0 = (G(G/H, H), \omega_G)$ is canonical Cartan geometry with projection $p : G \to G/H$. Let us use notation from Example 9.1.

Consider the proper foliation $(G, F)$, where $F = \{gH \mid g \in G\}$, is Cartan foliation with transverse Cartan geometry $\xi^0$. Because Lie groups $G$ is simply connected, we may apply Corollary 1.5. Therefore the basic automorphisms group $A_B(M, F)$ is isomorphic to the Lie group $Aut(G/H, \xi^0)$, that is isomorphic to the quotient group $G/H$. Thus $A_B(M, F) \cong G/H$.

Since $K = H$, then dim$(A_B(M, F)) = \operatorname{dim}(G) - \operatorname{dim}(K) = \operatorname{dim}(\mathfrak{g}) - \operatorname{dim}(\mathfrak{k})$ and the estimation (1) of Theorem 1.1 is exact.

**Example 9.3.** Let $G$ be the similar group of the Euclidean space $\mathbb{E}^q$, $q \geq 1$. Then $G = CO(q) \rtimes \mathbb{R}^q$ is the semidirect product of the conformal group $CO(q)$ and the group $\mathbb{R}^q$. Let $H = CO(q)$ and $p : G \to G/H = \mathbb{E}^q$ be the canonical principal $H$-bundle. Let $\mathfrak{g}$ be the Lie algebra of the Lie group $G$, and $\omega_G$ be the Maurer-Cartan $\mathfrak{g}$-valued 1-form on $G$. Then $\xi = (G(\mathbb{E}^q, H), \omega_G)$ is an effective Cartan geometry. Foliations with this transverse geometry $(\mathbb{E}^q, \xi)$ are called transversally similar foliations [17].

Let $Q$ be a smooth $p$-dimensional manifold whose fundamental group $\pi_1(Q, x)$ contains an element $\alpha$ of infinite order. For an arbitrary natural number $q \geq 1$, denote by $\mathbb{E}^q$ a $q$-dimensional Euclidean space. Define a homomorphism $\rho : \pi_1(Q, x) \to Diff(\mathbb{E}^q)$ by setting $\rho(\alpha) = \psi$, where $\psi$ is the homothety transformation of the Euclidean space $\mathbb{E}^q$ with the coefficient $\lambda \neq 1$, i. e. $\psi(x) = \lambda x \forall x \in \mathbb{E}^q$, and $\rho(\beta) = i_{d_\psi}$ for any element $\beta \in \pi_1(Q, x)$ such that $\beta \neq \alpha^k$ with some integer $k$. Then $(M, F) = \text{Sus}(\mathbb{E}^q, Q, \rho)$ is a proper transversally similar foliation with a unique closed leaf diffeomorphic to $Q$.

According to Theorem 1.5 $A_B(M, F) \cong N(\Psi)/\Psi$. The foliation $(M, F)$ is covered by the fibration $\hat{Q} \times \mathbb{E}^q \to \mathbb{E}^q$, hence $\Psi := \rho(\pi_1(Q, x)) \cong \mathbb{Z}$ is the global holonomy group of $(M, F)$ and $K = H$ is the kernel of the pair $(G, H)$. Thus the assumption of which was made in Example 9.1 is realized.

In our case $\Psi = (\psi)$ and $N(\Psi) = CO(q) = \mathbb{R}^+ \cdot O(q)$, therefore $A_B(M, F) \cong U(1) \times O(q)$, where $U(1) \cong \mathbb{R}^+ / \Psi$ is the compact 1-dimensional Abelian group.

If $q = 1$, then $O(q) = \mathbb{Z}_2$ and $A_B(M, F) \cong U(1) \times \mathbb{Z}_2$.

Thus $\operatorname{dim}(A_B(M, F)) = \operatorname{dim}(\mathfrak{h})$ and the estimate (2) in Theorem 1.1 is exact.
Example 9.4. Let $\mathbb{E}^2 = (\mathbb{R}^2, g)$ be an Euclidean plane with an Euclidean metric $g$. Let $\psi$ be the rotation of the Euclidean plane $\mathbb{E}^2$ around the point $0 \in \mathbb{E}^2$ by the angle $\delta = 2\pi r$. Denote by $\mathcal{I}(\mathbb{E}^2)$ the full isometry group of $\mathbb{E}^2$. It is well known that $\mathcal{I}(\mathbb{E}^2) \cong O(2) \ltimes \mathbb{R}^2$.

Let $\rho : \pi_1(S^1, b) \cong \mathbb{Z} \to \mathcal{I}(\mathbb{E}^2)$ be defined by the equality $\rho(1) := \psi$, $1 \in \mathbb{Z}$. Then we have a suspended Riemannian foliation $(M, F) := \text{Sus}(\mathbb{E}^2, S^1, \rho)$. This foliation has a unique closed leaf which is compact.

By analogy with the proof of Theorem 4.3 we get that there exists a group isomorphism between $A_B(M, F)$ and the quotient group $N(\Psi)/\Psi$, where $\Psi = \langle \psi \rangle$ and $N(\Psi)$ is the normalizer of $\Psi$ in the Lie group $O(2) \ltimes \mathbb{R}^2$. As $N(\Psi) = O(2)$, so $A_B(M, F) = O(2)/\Psi$. Hence $A_B(M, F)$ admits a Lie group structure if and only if $\Psi$ is a closed subgroup of $O(2)$ or, equivalent, when $\delta = 2\pi r$ for some rational number $r$.

If $\delta = 2\pi r$, where $r$ is a nonzero rational number, then $A_B(M, F) \cong O(2)$.

Example 9.5. Consider the standard 2-dimension torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ and call the pair of vectors $e_1 = \left( \frac{1}{0} \right)$, $e_2 = \left( \frac{0}{1} \right)$, the standard basis of the tangent vector space $T_x \mathbb{T}^2$ with $x \in \mathbb{T}^2$. Let $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$ be the quotient map, which is the universal covering of the torus. Denote by $f_A$ the Anosov automorphism of the torus $\mathbb{T}^2$ determined by the matrix $A \in SL(2, \mathbb{Z})$, while by $E$ the identity $2 \times 2$ matrix.

Let $g$ be the flat Lorentzian metric on the torus $\mathbb{T}^2$ given in the standard basis by the matrix $G = \lambda \left( \begin{array}{cc} 2 & m \\ m & 2 \end{array} \right)$, where $\lambda$ is any non zero real number and $m \in \mathbb{Z}$, $|m| > 2$.

Introduce notations $\mathcal{I}(\mathbb{T}^2, g)$ for the full isometry group of this Lorentzian torus $(\mathbb{T}^2, g)$ and $\mathcal{I}_0(\mathbb{T}^2, g)$ for the stationary subgroup of the group $\mathcal{I}(\mathbb{T}^2, g)$ at point $0 = \Omega(0)$, $0 = (0, 0) \in \mathbb{R}^2$. It is known (20, Example 3) that $\mathcal{I}(\mathbb{T}^2, g) = \mathcal{I}_0(\mathbb{T}^2, g) \ltimes \mathbb{Z} \times \mathbb{Z}$, where $\Phi_0 := \mathcal{I}_0(\mathbb{T}^2, g)$ is generated by $f_A$, $\tilde{f}_A$ and $-E$, $A = \left( \begin{array}{cc} m & 1 \\ -1 & 0 \end{array} \right)$ and $\tilde{A} = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$, hence $\mathcal{I}(\mathbb{T}^2, g) \cong (\mathbb{Z} \times \mathbb{Z}) \ltimes \mathbb{T}^2$. Put $\mathbb{E}^2 = (\mathbb{T}^2, \Omega^* g)$.

Let $Q$ be a smooth $p$-dimensional manifold whose fundamental group $\pi_1(Q, x)$ contains an element $\alpha$ of infinite order and $T = \mathbb{T}^2$. Define the group homomorphism $\rho : \pi_1(Q) \to \mathcal{I}(\mathbb{T}^2, g)$ by the equality $\pi_1(\alpha) := f_A$ and $\rho(\beta) = \text{id}_{\mathbb{T}^2}$ for any element $\beta \in \pi_1(Q, x)$ such that $\beta \neq \alpha^k$ with some integer $k \in \mathbb{Z}$. Then the suspended foliation $(M, F) := \text{Sus}(\mathbb{T}^2, Q, \rho)$ is Lorentzian, and its global holonomy group $\Psi$ is the group of all transformations lying over the group $\Phi := \langle f_A \rangle$ relatively the universal covering map $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$.

Elements of affine group $\text{Aff}(A^2)$ will be denoted by $< C, c >, C \in GL(2, \mathbb{R})$, $c \in \mathbb{R}^2$, in compliance with $\text{Aff}(A^2) = GL(2, \mathbb{R}) \ltimes \mathbb{R}^2$. The composition of transformations $< C, c >$ and $< D, d >$ from $\text{Aff}(A^2)$ has the following form

$\langle C, c \rangle \langle D, d \rangle : = \langle CD, Cc + d \rangle.$

The direct check using the Buzemann’s theorem shows that $\Psi = \Psi^0 \ltimes (\mathbb{Z} \times \mathbb{Z})$, where the group $\Psi^0$ is generated by matrix $A$, i.e. $\Psi^0 \cong \Phi$. Let $\Gamma := \mathbb{Z} \times \mathbb{Z} \subset \mathcal{I}(\mathbb{E}^2)$.

Consider any $< C, c > \in N(\Psi)$, then for every $< E, a > \in \Gamma$ there are

$< D, d >, < K, b > \in \Psi$ such that

$< C, c > < E, a > = < D, d > < C, c >, \quad < C, c > < K, d > = < E, a > < C, c >. \quad (5)$

Hence $D = E$, $K = E$ and $< C, c > \in N(\Gamma)$. Consequently $N(\Psi) \subset N(\Gamma)$ and, due to $\Gamma$ is the deck transformation group of $\Omega$, by the statement (1) of Proposition 8.1 the following map

$\alpha : N(\Psi) \to \mathcal{I}(\mathbb{T}^2, g) : \hat{h} \mapsto h,$
where $\hat{h}$ lies over $h$ relatively $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$, is defined and it is a group homomorphism.

The relations (6) imply also the inclusion $\langle C, 0 \rangle \in N(\Gamma)$. Therefore $f_C \in \Phi_0 := \mathcal{J}_0(\mathbb{T}^2, g)$ and $C \in \Psi_0$, where $\Psi_0$ is the subgroup of $N(\Psi)$ generated by matrix $A$, $\tilde{A}$ and $-E$, i.e. $\Psi_0 \cong \Phi_0$. Thus, the stationary subgroup $N(\Psi)_0$ at $0 \in \mathbb{R}^2$ of the normalizer $N(\Psi)$ is equal to $\Psi_0$.

Since $\alpha(\Psi) = \Phi$, the homomorphism $\alpha$ has the property $\alpha(N(\Psi)) = N(\Phi)$.

Now let us compute the normalizer $N(\Phi)$ of $\Phi$ in the group $\mathcal{J}(\mathbb{T}^2, g)$. Take any $< D, d >$ from $N(\Phi)$. Then there is $k \in \mathbb{Z} - \{0\}$ such that

$$< D, d > < A, 0 > = < A^k, 0 > < D, d >,$$

consequently $A^k d = d$, i.e. 1 is the eigenvalue of $A^k$. Since $A^k$ is an Anosov automorphism, then its eigenvalues are irrational. Thus, it is necessary $d = 0$, hence $N(\Phi) \subset \Phi_0$.

Observe that $A\tilde{A} = \tilde{A}A^{-1}$ and $A(-E) = (-E)A$, therefore, $N(\Phi) = \Phi_0$. Thus, $N(\Phi)/\Phi = \Phi_0/\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Applying this fact and the statement (3) of Proposition 8.1 we get the following chain of the group isomorphisms $N(\Psi)/\Psi \cong (N(\Psi)/\Gamma)/(\Psi/\Gamma) \cong N(\Phi)/\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Thus, using Theorem 1.5 we have $A_B(M, F) \cong N(\Psi)/\Psi$, therefore

$$A_B(M, F) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

**Remark 9.1.** It is well known (see, for example, [14], Lemma 3.3) that the set of periodic orbits of a Anosov automorphism of the torus $\mathbb{T}^2$ is countable. Therefore the foliation $(M, F)$ constructed in Example 9.5 has a countable set of closed leaves and according to the item (b) of Theorem 1.5 its basic automorphism group $A_B(M, F)$ is a discrete Lie group. Our result $A_B(M, F) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ illustrates this assertion.

### Список литературы

1. Bagaev A.V., Zhukova N.I., The isometry group of Riemannian orbifolds, *Siberian Mathematical Journal*, 48:4 (2007), 579–592.
2. Belko I.V., Affine transformations of a transversal projectable connection on a foliated manifold, *Mathematics of the USSR-Sbornik*, 45:2 (1988), 191–203.
3. Busemann H., The geometry of geodesics, Academic Press, New York, 2011.
4. Conlon L., Transversally parallelizable foliations of codimension 2, *Trans. Amer. Math. Soc.*, 194 (1974), 79–102.
5. Hector G., Macias-Virgos E., Diffeological groups, *Research and Exposition in Math.*, 25 (2002), 247–260.
6. Kobayashi S., Nomizu K. Foundations of differential geometry, I, Interscience Publ., New York-London, 1963.
7. Molino P., Riemannian foliations, *Progress in Math.*, Birkhauser, Boston, 1988.
8. Macias-Virgos E., Sanmartin E., Manifolds of maps in Riemannian foliations, *Geometrica Dedicata*, 79 (2000), 143–156.
[14] Nitecki Z. Differentiable Dynamics: an Introduction to the Orbit Structure of Diffeomorphisms, The MIT Press, Cambridge, Mass., 1971.

[15] Sharpe R.W. Differential Geometry: Cartan’s Generalization of Klein’s Erlangen Program. Graduate Texts in Mathematics, Vol. 166, Springer-Verlag, New York, 1997.

[16] Tamura I. Topology of foliations. Translations of math. monograph., AMS: Publishing House, New York, 1992.

[17] Zhukova N.I., Minimal sets of Cartan foliations, Proc. of the Steklov Inst. of Math., 256 (2007), 105–135.

[18] Zhukova N.I., Complete foliations with transverse rigid geometries and their basic automorphisms, Bulletin of Peoples’ Friendship University of Russia. Ser. Math. Information Sci. Phys., 2, (2009), 14–35.

[19] Zhukova N.I., Global attractors of complete conformal foliations, Sbornik: Mathematics, 203:3 (2012), 380–405.

[20] Zhukova N.I., Rogozhina E.A., Classification of compact Lorentzian 2-orbifolds with noncompact isometry group, Siberian Mathematical Journal, 53:6 (2012), 1037–1050.