Ermakov-Lewis invariant in Koopman-von Neumann mechanics

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Abstract In the paper Sci. Rep. 8, 8401 (2018), among other things, the Ermakov-Lewis invariant was constructed for the time dependent harmonic oscillator in Koopman-von Neumann mechanics. We point out that there is a simpler method that allows one to find this invariant.

Keywords Koopman-von Neumann mechanics · Ermakov-Lewis invariant · Time dependent harmonic oscillator

1 Introduction

In the interesting paper [1], the Ermakov-Lewis invariant is used to study the time-dependent harmonic oscillator (TDHO) in the framework of Koopman-von Neumann (KvN) mechanics [2,3]. To find the invariant, a system of coupled differential equations was obtained, which then was reduced to a single equation related to the Ermakov equation. Although this method is quite simple, our goal in this short note is to show that there is an even simpler way to find the Ermakov-Lewis invariant for TDHO in KvN mechanics.

Recent interest in KvN mechanics is motivated by experiments exploring the quantum-classical border. To formulate a consistent framework for a hybrid quantum-classical dynamics is a long-standing problem [4]. Its solution, in addition to practical interest, for example, in quantum chemistry, can clarify deep conceptual issues in quantum mechanics, such as the problem of measurement. An interesting result in this direction was obtained in [5]. It was found
that the Wigner function in the nonrelativistic limit turns into the Koopman-von Neuman wave function, which explains why the Wigner function is not positive-definite.

Time-dependent harmonic oscillators arise in many quantum mechanical systems [6,7]. At the same time, the existence of Ermakov-Lewis invariants in such systems has attracted much attention [7]. In our opinion, extension of these results to the case of KvN mechanics is of considerable interest.

2 KvN evolution equation for TDHO in new variables

The KvN evolution equation for TDHO wave-function has the form [1]:

$$i \frac{\partial}{\partial t} \psi(x, p; t) = \left[ \hat{p} \hat{\lambda}_x - k(t) \hat{x} \hat{\lambda}_p \right] \psi(x, p; t),$$

(1)

where $\hat{\lambda}_x$ and $\hat{\lambda}_p$ operators satisfy the following commutation rules

$$[\hat{x}, \hat{\lambda}_x] = [\hat{p}, \hat{\lambda}_p] = i.$$

(2)

Note that $m = 1$ and $\hbar = 1$ was assumed for simplicity. As Sudarshan remarked [8], any KvN-mechanical system can be considered as a hidden variable quantum system. Correspondingly, we will make a slight change in notations as follows:

$$x = q, \quad \lambda_x = P, \quad \lambda_p = -Q,$$

(3)

where $Q$ and $P$ are quantum variables that are hidden for classical observers. Thus eq.(1) takes the following form in new notations

$$i \frac{\partial}{\partial t} \psi(q, p; t) = \left[ \hat{p} \hat{P} + k(t) \hat{q} \hat{Q} \right] \psi(q, p; t).$$

(4)

Correspondingly, the KvN Hamiltonian is given by

$$\mathcal{H} = \hat{p} \hat{P} + k(t) \hat{q} \hat{Q}.$$  

(5)

Let us make the following canonical transformation

$$\hat{q} = \frac{1}{\sqrt{2}} (\hat{q}_1 - \hat{q}_2), \quad \hat{Q} = \frac{1}{\sqrt{2}} (\hat{q}_1 + \hat{q}_2),$$

$$\hat{p} = \frac{1}{\sqrt{2}} (\hat{p}_1 + \hat{p}_2), \quad \hat{P} = \frac{1}{\sqrt{2}} (\hat{p}_1 - \hat{p}_2).$$

(6)

The transformation is canonical in the sense that it doesn’t change the canonical form of the commutation relations:

$$[\hat{q}_i, \hat{q}_j] = 0, \quad [\hat{q}_i, \hat{p}_j] = i \delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0.$$

(7)

The KvN Hamiltonian when written in new variables $\hat{q}_1, \hat{q}_2, \hat{p}_1, \hat{p}_2$ splits into difference of two Schrödinger type quantum Hamiltonians:

$$\mathcal{H} = \left( \frac{\hat{p}_1^2}{2} + \frac{1}{2} k(t) \hat{q}_1^2 \right) - \left( \frac{\hat{p}_2^2}{2} + \frac{1}{2} k(t) \hat{q}_2^2 \right) = \mathcal{H}_1 - \mathcal{H}_2.$$  

(8)

In the next section we will use this splitting to find the Ermakov-Lewis invariant.
3 Ermakov-Lewis invariant for KvN TDHO

Let $I$ be the Ermakov-Lewis invariant for KvN TDHO. It must satisfy the following equation

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} - i[\hat{I}, \mathcal{H}] = 0. \quad (9)$$

Since $\mathcal{H} = \mathcal{H}_1(\hat{q}_1, \hat{p}_1) - \mathcal{H}_2(\hat{q}_2, \hat{p}_2)$, $\hat{I}$ will have the form $\hat{I} = \hat{I}_1 + \hat{I}_2$, where $\hat{I}_1$ and $\hat{I}_2$ are the usual quantum mechanical Ermakov-Lewis invariants associated respectively with $\mathcal{H}_1(\hat{q}_1, \hat{p}_1)$ and $\mathcal{H}_2(\hat{q}_2, \hat{p}_2)$. However, there is a subtlety associated with the minus sign in $\mathcal{H} = \mathcal{H}_1(\hat{q}_1, \hat{p}_1) - \mathcal{H}_2(\hat{q}_2, \hat{p}_2)$, which indicates that in the second Ermakov-Lewis invariant we should assume time-reversal. More formally, we have

$$\frac{\partial \hat{I}_1}{\partial t} - i[I_1, \mathcal{H}_1] = 0, \quad (10)$$

The individual terms in the brackets must vanish. However for $\frac{\partial \hat{I}_2}{\partial \tau} - i[I_2, \mathcal{H}_2] = 0$, the corresponding quantum-mechanical invariant is well known

$$I_1 = \frac{1}{2} \left[ \left( \frac{\dot{q}_1}{\rho} \right)^2 + (\dot{p}_1 - \rho \dot{q}_1)^2 \right], \quad (12)$$

where $\rho$ obeys the Ermakov equation

$$\ddot{\rho} + k(t)\rho = \rho^{-3}. \quad (13)$$

As for the equation containing $\hat{I}_2$, define $\tau = -t$ and the equation takes the form:

$$\frac{\partial I_2}{\partial \tau} - i[I_2, \mathcal{H}_2] = 0. \quad (14)$$

It is clear that the corresponding invariant is

$$I_2 = \frac{1}{2} \left[ \left( \frac{\dot{q}_2}{\rho} \right)^2 + (\dot{p}_2 - \rho \dot{q}_2)^2 \right], \quad (15)$$

where $\rho' = \frac{\partial \rho}{\partial \tau}$. Restoring the derivatives with respect to $t$, we get

$$I_2 = \frac{1}{2} \left[ \left( \frac{\dot{q}_2}{\rho} \right)^2 + (\dot{p}_2 + \rho \dot{q}_2)^2 \right]. \quad (16)$$

Thus the Ermakov-Lewis invariant $I$ in KvN mechanics takes the following form, after using eq.(12) and eq.(16) and re-writing the result in terms of $\dot{q}, \dot{Q}, \dot{p}$ and $\dot{P}$:

$$I = \frac{\dot{Q}^2}{2\rho^2} + \frac{\dot{q}^2}{2\rho^2} + \frac{1}{2} \left\{ (\dot{p} - \rho \dot{q})^2 + (\dot{p} - \rho \dot{Q})^2 \right\}. \quad (17)$$
Using the correspondence given by eq. (3), and re-arranging the above expression, we get the invariant in the form found in [1]:

\[
\hat{I} = \frac{1}{2} \left[ \dot{x}^2 \rho^2 + \left( \dot{\rho} \dot{x} - \rho \ddot{p} \right)^2 + \lambda^2 \rho^2 + \left( \dot{\rho} \dot{\lambda} + \rho \dot{\lambda} \right)^2 \right].
\]  
(18)

4 Conclusions

We have shown that one can find the Ermakov-Lewis invariant in the case of KvN mechanics using the well-known quantum-mechanical expression for this invariant [3] and some simple algebra. In this method, there is no need to consider any coupled differential equations. However, it is less general than the method considered in [1], which, in principle, can be applied to any potential, even if it does not allow \( \mathcal{H} \) to split in a way described in this note.

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