Generalized Soliton Solutions to Generalized KdV Equation with Variable Coefficients by Exp-function Method

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Abstract. In this paper, the generalized KdV equation with variable coefficients is investigated by Exp-function method. The generalized soliton solutions and periodic solutions of this equation are obtained with the help of symbolic computation. It is shown that the Exp-function method provides a straightforward and powerful mathematical tool for solving nonlinear equations.

1. Introduction
The nonlinear equations of mathematical physics are major subjects in physical science. The investigation of exact traveling wave solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. The wave phenomena are observed in fluid dynamics, plasma, elastic media, optical fibers, etc. The importance of the exact solutions facilitates the verification of numerical solvers and aids in the stability analysis of solutions. Recently many new methods to nonlinear wave equations have been proposed, for example, variational iteration method [1], Adomian decomposition method [2], homogeneous balance method [3], F-expansion method [4], Tanh-method [5], inverse scattering method [6], Hirota’s bilinear method [7], Backlund transformation [8], painleve expansion [9], sine-cosine method [10], Jacobi elliptic expansion method [11] and so on. Many new approaches with advantages on the one hand and disadvantages on the other hand have been suggested to solve various nonlinear equations. Tian and Yin [12] studied KdV equation by using variational iteration method and obtained shock-peakon solution and shock-compacton solution. Tian and Yin [13] introduced a fifth-order K(m, n, 1) equation with nonlinear dispersion to obtain multi-compacton solutions by Adomian decomposition method. Using the homogeneous balance method, they derived a Backlund transformation, Lagrangian and some conservation laws. Finally the linear stability of all multi-compacton solutions is given.

Very recently, He et al. [14-16] proposed a straightforward and concise method, which was called Exp-function method and obtained generalized solitary solutions and periodic solutions. The method transforms the nonlinear equation to a simple algebraic computation. This method is easily extended to other kinds of nonlinear evolution equations.

To our knowledge, most of the aforementioned methods are related to the constant-coefficient models. Recently, the study of the variable-coefficient nonlinear equations has attracted much attention [17-18] because most of real nonlinear physical equations possess variable coefficients.

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In this paper, we study the generalized KdV equation with variable coefficients

\[ u_t + 2 \beta(t) u + \left[ \alpha(t) + \beta(t)x \right] u_x - 3 A \gamma(t) uu_x + \gamma(t) u_{xxx} = 0 \]  

(1)

by Exp-function method and educe the forms and the figures of the solutions. Where \( \alpha(t), \beta(t), \gamma(t) \) are arbitrary functions of \( t \). Eq.(1) includes many interesting equations, such as , when \( \gamma(t) = -K_0(t), A = -2, \beta(t) = h(t), \alpha(t) = -4K_1(t) \), Eq.(1) can be degenerated to the nonisospectral KdV equation with variable coefficients in [19-20]:

\[ u_t = K_0(t)(u_{xxx} + 6uu_x) + 4K_1(t)u_x - h(t)(2u + xu_x) \]  

(2)

when \( \gamma(t) = 1, A = -2, \alpha(t) = C_0 \), Eq.(1) can be degenerated to the KdV equation with variable coefficients discussed in[21-22]:

\[ u_{xxx} + 6uu_x + \left[ (C_0 + \beta(t)x)u_x \right] + \beta(t)u + u_t = 0 \]  

(3)

2. Generalized soliton solutions of generalized KdV equation with variable coefficients

We discuss the exp-function method application to generalized KdV equations with variable coefficients and study the generalized soliton solutions. Considering Eq.(1) and making the transformation

\[ u = U(\eta), \eta = kx + \int \tau(t) dt \]  

(4)

the Eq.(1) becomes an ordinary differential equation

\[ \tau(t) U' + 2 \beta(t) U + \left[ \alpha(t) + \beta(t)x \right] U' - 3 A \gamma(t) UU' + k^3 \gamma(t) U^{'''} = 0 \]  

(5)

where \( U' \) denotes the differential with respect to \( \eta \). According to the Exp-function method [14], we assume that the solution of Eq. (5) can be expressed in the form

\[ U(\eta) = \sum_{n=-q}^{c} a_n \exp(n \eta) \]  

\[ \sum_{m=-q}^{p} b_m \exp(m \eta) \]  

(6)

where \( c, d, p, q \) are positive integers which are unknown to be determined later. \( a_n \) and \( b_m \) are unknown constants. Eq.(6) can be rewritten in the form as follows:

\[ U(\eta) = \frac{a_c \exp(c \eta) + \cdots + a_d \exp(-d \eta)}{b_p \exp(p \eta) + \cdots + b_q \exp(-q \eta)} \]  

(7)

In order to determine values of \( c, p \), we balance the linear term of the highest order in Eq. (5) with the highest order nonlinear term \( U''' \) and \( UU' \), we have

\[ U''' = \frac{c_1 \exp([7p+c] \eta)}{c_2 \exp(8p \eta)} + \cdots \]  

(8)

\[ UU' = \frac{c_1 \exp([p+2c] \eta)}{c_4 \exp(3p \eta)} + \cdots = \frac{c_2 \exp([2(3p+c)] \eta)}{c_4 \exp(8p \eta)} + \cdots \]  

(9)
where \( c_i \) are unknown coefficients. Balancing the highest order of Exp-function in Eq.(8) and Eq.(9), we have

\[
7p + c = 2(3p + c)
\]

which leads to the result

\[
p = c
\]

Similarly, we determine values of \( d, q \). Balancing the lowest order in Eq. (5) of \( U^\alpha \) and \( UU' \),

\[
U^\alpha = \cdots + d_1 \exp[-(7q + d)\eta] \cdots + d_2 \exp[-(8q\eta)]
\]

\[
UU' = \cdots + d_3 \exp[-(q + 2d)\eta] + \cdots + d_4 \exp[-(3q\eta)] + \cdots + d_4 \exp[-(8q\eta)]
\]

where \( d_i \) are determined coefficients. Balancing the lowest order of Exp-function in Eq.(12) and Eq.(13), we have

\[
-(7q + d) = -2(3q + d)
\]

which leads to the result

\[
q = d
\]

By above homogeneous balance principle, we find that \( p = c, q = d \). We will discuss the generalized soliton solution of Eq.(1).

2.1. Case 1 \( p = c = 1, q = d = 1 \)

For simplicity, we set \( p = c = 1, q = d = 1 \) and find that the final solution does not depend upon the choice of values of \( c \) and \( d \). The solution (7) becomes

\[
U(\eta) = \frac{a_0 \exp(\eta) + a_n + a_\pm \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_\pm \exp(-\eta)}
\]

Substituting Eq.(16) into Eq.(5) and setting the coefficients of all powers of \( \exp(n\eta) \) to zero, we can obtain a series of over-determined algebraic equations for \( a_0, b_0, a_1, b_1, a_\pm, b_\pm \) and \( \alpha(t), \beta(t), \gamma(t) \), which are omitted to avoid the tediousness. With the aid of Maple, we obtain

\[
a_1 = \frac{-b_1C_1 - b_2k - b_3C_3k^3}{6C_3k}, \quad a_\pm = \frac{-b_1C_1 - b_2k - b_3C_3k^3}{6C_3k}, \quad b_0 = \frac{6a_0C_3k}{C_1 + k + C_3k^3}
\]

\[
a_0 = a_0, \quad b_1 = b_1, \quad b_\pm = b_\pm, \quad \alpha(t) = \alpha(t), \quad \beta(t) = C_2\alpha(t)
\]

\[
\gamma(t) = C_3\alpha(t), \quad \tau(t) = C_4\alpha(t)
\]

where \( a_0, b_1, b_\pm \) and \( C_i (i = 1,2,3) \) are constants. Substituting Eq.(17) into (16), we obtain the following generalized soliton solution of Eq.(1):
\[
\begin{align*}
\frac{-b_1 C_i - b_i k - b_1 C_i k^3}{6 C_i k} e^{i C_i \int \alpha(t) dt} + a_0 - \frac{b_1 C_i + b_i k + b_1 C_i k^3}{6 C_i k} e^{-i C_i \int \alpha(t) dt} \\
= \frac{b_i e^{i C_i \int \alpha(t) dt}}{C_1 + k + C_i k^3} + \frac{6 a_2 C_i k}{C_1 + k + C_i k^3} + b_2 e^{-i C_i \int \alpha(t) dt}
\end{align*}
\]

(18)

Setting \( b_{-1} = b_1 = k = C_1 = C_i = a_0 = 1 \), when (a) \( \alpha(t) = \tanh(t) \), (b) \( \alpha(t) = \sin(t) \) (c) \( \alpha(t) = \frac{1}{1 + t^2} \), the figures of (18) are shown as follows:

(a) Figure 1 Generalized soliton solution

When \( k \) and \( C_i \) are imaginary numbers, the obtained soliton solution can be converted into periodic solution. We let \( k = iK \), \( C_i = iC_i \). Using the transformation

\[
\begin{align*}
e^{i C_i \int \alpha(t) dt} &= \cos(Kx + C_i \int \alpha(t) dt) + i \sin(Kx + C_i \int \alpha(t) dt) \\
e^{-i C_i \int \alpha(t) dt} &= \cos(Kx + C_i \int \alpha(t) dt) - i \sin(Kx + C_i \int \alpha(t) dt)
\end{align*}
\]

then Eq.(18) becomes

\[
u = \frac{A \cos[Kx + C_i \int \alpha(t) dt] + a_i + iB \sin[Kx + C_i \int \alpha(t) dt]}{(b_1 + b_{-1}) \cos[Kx + C_i \int \alpha(t) dt] + \frac{6 a_0 C_i K}{c_i + K + c_i K^3} + i(b_1 - b_{-1}) \sin[Kx + C_i \int \alpha(t) dt]}
\]

(19)

Where

\[
A = \frac{-b_1 C_i - b_i K - b_1 C_i K^3}{6 c_i K} - \frac{b_{-1} C_i + b_{-1} K + b_{-1} C_i K^3}{6 c_i K}
\]

\[
B = \frac{-b_1 C_i - b_i K - b_1 C_i K^3}{6 c_i K} + \frac{b_{-1} C_i + b_{-1} K + b_{-1} C_i K^3}{6 c_i K}
\]

If we search for a periodic solution or compact-like solution, the imaginary part in Eq. (19) must be zero, which requires that

\[
b_1 = b_{-1}
\]

(20)

Substituting (20) into (19) yields two periodic solution
\[
\begin{align*}
    u &= \frac{-b_1 c_1 - b_1 K - b_1 c_3 K^3}{3 c_3 K} \cos \left[ Kx + c_1 \int \alpha(t) dt \right] + a_0 \\
    &= \frac{2b_1 e^{Kx + c_1 \int \alpha(t) dt} + 6a_0 c_3 K}{c_1 + K + c_3 K^3}
\end{align*}
\]

(21)

2.2. Case 2 \( p = c = 2 \), \( q = d = 2 \)

Setting \( p = c = 2 \), \( q = d = 2 \), the Eq.(7) becomes

\[
U(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{b_2 \exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)}
\]

(22)

For simplicity, we set \( b_{-1} = b_1 = 0 \), \( b_2 = 1 \), the Eq.(19) is simplified as follows

\[
U(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{\exp(2\eta) + b_0 + b_{-2} \exp(-2\eta)}
\]

(23)

By the same manipulation as above, we obtain

\[
\begin{align*}
    a_1 &= \frac{a_1^2 C_1 + a_1^2 k - 11a_1^2 C_3 k^3}{12 C_3 k^5} \quad a_{-2} = -\frac{a_0 (C_1 + k + C_3 k^3)}{96 C_3 k^9} \quad a_{-1} = \frac{a_1^3}{4k^4} \\
    a_2 &= -\frac{C_1 - k - C_3 k^3}{6C_3 k} \quad b_{-2} = \frac{a_1^2}{16k^5} \quad b_0 = -\frac{a_1^2}{2k^8} \quad a_1 = a_1
\end{align*}
\]

\[
\alpha(t) = \alpha(t) \quad \beta(t) = C_3 \alpha(t) \quad \gamma(t) = C_3 \alpha(t) \quad \tau(t) = C_3 \alpha(t)
\]

(24)

Substituting (24) into (23) yields the following solution

\[
\begin{align*}
    u &= \frac{-C_1 - k - C_3 k^3}{6C_3 k} e^{\frac{\frac{a_1^2}{2k^5} + \alpha_1^2}{96C_3 k^9}} \]
\]

(25)

2.3. Case 3 \( p = c = 2 \), \( q = d = 1 \)

We set \( p = c = 2 \), \( q = d = 1 \), \( b_{-1} = 1 \), the Eq.(7) is simplified as follows

\[
U(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_2 \exp(2\eta) + b_1 \exp(\eta) + b_0 + \exp(-\eta)}
\]

(26)

By the same manipulation as above, we obtain

\[
\begin{align*}
    a_1 &= \frac{-b_1 C_1 - b_1 k - b_1 C_3 k^3}{6C_3 k} \quad a_0 = \frac{-b_0 (C_1 + k + C_3 k^3)}{6C_3 k} \quad a_{-1} = \frac{-C_1 - k - C_3 k^3}{6C_3 k}
\end{align*}
\]
\[ b_0 = -\frac{6a_1 C_3 k}{C_1 + k + C_3 k^3}, \quad a_0 = a_0, \quad b_1 = b_1, \quad b_0 = b_0, \quad \alpha(t) = \alpha(t), \]

\[ \beta(t) = C_2 \alpha(t), \quad \gamma(t) = C_3 \alpha(t), \quad \tau(t) = C_4 \alpha(t), \quad (27) \]

Substituting (24) into (23) yields the following solution

\[
u = -\frac{2^{[k \sigma_c [\alpha(t)]]} - h C_1 + k + h C_3 k^3}{6 C_1 k} e^{\int [k \sigma_c [\alpha(t)]}] - \frac{h_0 (C_1 + k + C_3 k^3)}{6 C_1 k} e^{\int [k \sigma_c [\alpha(t)]}] - \frac{h_0 (C_1 + k + C_3 k^3)}{6 C_1 k} e^{\int [k \sigma_c [\alpha(t)]}] + \frac{h e^{\int [k \sigma_c [\alpha(t)]]} - h_0 + e^{-2^{[k \sigma_c [\alpha(t)]]}}}{6 C_1 k}
\]

(28)

We find that the (18) can be transformed to the (25) and (28) when the coefficients choose appropriately. So we can conclude that the generalized soliton solution of Eq.(1) does not depend on the choice of values of \( c \) and \( d \).

3. Conclusion

The generalized KdV equation with variable coefficients is investigated by Exp-function method. The generalized soliton solutions of this equation are obtained with the help of symbolic computation. It is shown that the Exp-function method provides a straightforward and powerful mathematical tool for solving nonlinear equations. The Exp-function method can be used to solve other nonlinear equation.

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