Non-Hermitian avalanche effect: Non-perturbative effect induced by local non-Hermitian perturbation on a $\mathbb{Z}_2$ topological order

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Abstract – In this paper, based on a non-Hermitian toric-code model we found the fragileness of intrinsic topological orders under non-Hermitian perturbations. The effect of non-Hermitian avalanche for a designed toric-code model is uncovered: for the designed toric-code model with special external fields, a tiny non-Hermitian perturbation (local imaginary state-selective dissipation) leads to anomalous topological degeneracy and breaks down the bulk-degeneracy correspondence (a correspondence between bulk quasi-particles and topologically protected degenerate ground states). To support our theoretical predictions, we calculated the ground-state degeneracy and fidelity susceptibility of the ground states based on the non-Hermitian toric-code model on $2 \times 2 \times 2$ lattice and on $2 \times 3 \times 3$ lattice. The numerical results are consistent with theoretical prediction.

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Introduction. – Recently, there has been a lot of activities in the research on non-Hermitian topological systems [1–42], including non-Hermitian topological insulators, non-Hermitian topological superconductors, and non-Hermitian topological semi-metals. After considering the non-Hermitian extensions of the usual topological band systems, quantum exotic effects are uncovered, such as the fractional topological invariant and defective edge states [6,18], non-Hermitian skin effect [15,22,25,36,39,40,42], and the breakdown of bulk-boundary correspondence [11,15–17,31–33,35,39]. In addition to the research on non-Hermitian topological band systems, the non-Hermitian extensions of intrinsic topological orders that are many-body topological systems with long-range entanglement are studied [43–45]. In ref. [43], the non-Hermitian strings and the breakdown of the correspondence between bulk quasi-particles and topologically protected degenerate ground states are discovered. In ref. [44], a continuous quantum phase transition without gap closing was explored that occurs in non-Hermitian topological orders together with the breakdown of the Lieb-Robinson bound.

Therefore, one must give careful reconsideration of the non-Hermitian extensions of topological stability for intrinsic topological orders. It is well known that for the topological ordered states, due to the existence of energy gap, the ground states are robust. The degeneracy of the ground states depends on the topology of the system and is also robust against any small and local perturbations. Topological phase transition between topological ordered states and trivial states may occur when the perturbations become large enough and are beyond certain thresholds. In this paper, we point out that there exists the non-Hermitian avalanche effect for a non-Hermitian topological order: the effect of the changing of ground state’s degeneracy induced by a local non-Hermitian perturbation (even in the thermodynamic limit).

Topological stability of the (Hermitian) topological order. – Firstly, we show the topological stability of the (Hermitian) $\mathbb{Z}_2$ topological order.

For a (Hermitian) $\mathbb{Z}_2$ topological order, there are four types of topological sectors (ground state and three types of quasi-particles): $1$ (vacuum), $e$ (e-particle or $\mathbb{Z}_2$ charge), $m$ (m-particle or $\mathbb{Z}_2$ vortex), $f$ (fermion). The e-particle and m-particle are all bosons with mutual $\pi$ statistics between them. The fermion can be regarded as a bound state of an e-particle and an m-particle. All these quasi-particles have finite energy gaps $\Delta^I (I = e, m, f)$. The number of the topological sector of quasi-particles can be denotes as...
$N = 4$. When we consider the system on a torus, the ground states have topological degeneracy. Each degenerate ground state $|0⟩_i$ corresponds to the one by adding a virtual quasi-particle. We can use the basis of sectors of (virtual) quasi-particles to characterize the ground states, i.e., $ψ = \langle 0, |e⟩, |m⟩, |f⟩)$. This is named bulk-degeneracy correspondence (BDC). We denote the BDC by

$$\mathcal{N} = \mathcal{D},$$

where $\mathcal{D}$ denotes the number of ground-state degeneracy. For a system with infinite size, the four ground states $|0⟩, |e⟩, |m⟩, |f⟩$ become degenerate with exact zero energy splitting. When one considers the perturbations on the systems, the degeneracy of the four ground states does not change. In ref. [46], it is pointed out that the topological-order classes are stable against any small stochastic local transformations and there exists a phenomenon of emergence of unitarity.

We may use the Kitaev’s toric-code model as an example to illustrate the topological stability of the Hermitian $Z_2$ topological order. The toric-code model is an exactly solvable spin model, of which the Hamiltonian is

$$\hat{H}_{TC} = -g \left( \sum_{s} A_{s} + \sum_{p} B_{p} \right),$$

where $A_{s} = \prod_{i \in s} \sigma_{i},^{z}$ and $B_{p} = \prod_{i \in p} \sigma_{i},^{y}$, the subscripts $s$ and $p$ represent the vertices and plaquettes of a square lattice, respectively. In this paper, we set $g \equiv 1$. For the toric-code model, the ground states are defined as $A_{s}|ψ_{g}⟩ = |ψ_{g}⟩, B_{p}|ψ_{g}⟩ = |ψ_{g}⟩$ for all $A_{s}$ and $B_{p}$.

The quantum states of $Z_2$ topological order are characterized by different configurations of strings $\hat{W}(C) = \prod_{i \in C} \sigma_{i},^{α_{i}}$, where $\sigma_{i},^{α_{i}}$ is the $α_{i}$-type Pauli matrix on site $i$. $\hat{W}(C)$ is over all the sites on the string along a path $C$. i.e., $|Φ⟩ = \sum_{i} a_{c}^{i} \hat{W}(C)|0⟩$, where $a_{c}^{i}$ denotes the spin-polarized states with all spin down ({$\downarrow, \downarrow, \ldots, \downarrow$}), $\hat{W}(C)$ denotes the possible string operators, and $a_{c}^{i}$ is the weight of the string operator. The different configurations of open strings correspond to different excited states of different quasi-particles.

The ground state for $\hat{H}_{TC}$ is a $Z_2$ topological order [47–51]. The ground states have topological degeneracy, i.e., different topologically degenerate ground states are classified by different topological closed operation strings $\hat{W}_{o}(C^{\text{close, topo}})$ ($a = v, c, f$). The operator $\hat{W}_{o}(C^{\text{close, topo}})$ takes on binary values 0, 1 and denotes whether the loops $C^{\text{close, topo}}$ belong to the even or odd winding number sectors along the $x$/$y$-direction. So, we can use the basis of even-odd parity of the winding number of electric field lines around the torus $\{(m_{x}, m_{y}) | (0, 0), (0, 1), (1, 0), (1, 1)\}$ to describe the four ground states. As a result, the bulk-degeneracy correspondence is valid, i.e., $\mathcal{N} = \mathcal{D}$.

The local perturbations on the $Z_2$ topological order just locally and slightly deform the string configurations but can never change the degeneracy of ground states. The Kitaev’s toric-code model, the dissipation effect had been studied in ref. [52]. The results show that small dissipations cannot change the ground states. As a result, the degenerate ground states make up a protected code subspace and can be regarded as topological qubits to do possible topological quantum computation [48].

**Topological poisoning effect for the non-Hermitian topological order.** We then illustrate the string poisoning effect for the $Z_2$ topological order from the non-Hermitian local perturbations. Here, the non-Hermitian toric-code model is defined by adding non-Hermitian external fields,

$$\hat{H}_{NTC} = \hat{H}_{TC} + \hat{H}',$$

where

$$\hat{H}' = \sum_{i} h_{i} \sigma_{i} = \sum_{i} h_{i}^{x} \sigma_{i}^{x} + \sum_{i} h_{i}^{y} \sigma_{i}^{y} + \sum_{i} h_{i}^{z} \sigma_{i}^{z}.$$  

Now, we introduce $h_{i} \neq h_{i}^{x}$ for $i$-th spin, therefore the Hamiltonian satisfies $\hat{H}_{NTC} \neq \hat{H}_{TC}$.

To characterize the quantum properties of the non-Hermitian $Z_2$ topological order, the (non-Hermitian) dynamic strings were defined as [43] $D_{a}(CN) = \prod_{i \in C}^{p} e^{i \phi_{a,i}} \sigma_{i}^{α_{i}}$, where $\phi_{a,i}$ is the phase of quantum state at step $i$ for $a$-type ($a = v, c, f$) excitation, and $\sigma_{i}^{α_{i}}(ν_{i} = x, y, z)$ is an $ν_{i}$ type of Pauli matrix on site $i$. For the case of $D_{a}(CN) \neq D_{a}(CN)$, a dynamical string becomes non-Hermitian.

In this paper, we consider the non-Hermitian model with local non-Hermitian external field on single lattice site $i_{0}$, i.e., $h_{i_{0}} \neq h_{i_{0}}^{x}$, $h_{i_{0}} \neq h_{i_{0}}^{y}$, and $h_{i_{0}} \neq h_{i_{0}}^{z}$. Now, arbitrary dynamic (open or closed) strings passing through site $i_{0}$ become non-Hermitian,

$$D_{a}(CN \rightarrow i_{0}) \neq D_{a}(CN \rightarrow i_{0}),$$

where $CN \rightarrow i_{0}$ means the string crossing site $i_{0}$. We call it topological poisoning effect under local non-Hermitian perturbations. This effect from non-Hermitian perturbations for $Z_2$ topological order are quite different from those from Hermitian perturbations. Due to the topological poisoning effect, a local non-Hermitian perturbation may causes
Non-Hermitian avalanche effect induced by local non-Hermitian perturbation etc.

![Diagram](Fig. 2: The schematic diagram of the designed toric-code model. The external fields are applied only on three paths.)

highly non-local influence. See the illustration in fig. 1. The red dashed strings are all non-Hermitian dynamic strings poisoned by the local non-Hermitian perturbation at site $i_0$.

Local non-Hermitian perturbation on a designed toric-code model. — In this section, we introduce a designed toric-code model, of which the Hamiltonian is expressed as

$$\hat{H}_{TC} = \hat{H}_{TC}^{\prime} + \hat{H}''$$

where

$$\hat{H}'' = h_x \sum_{i \in L_1} \sigma_i^x + h_z \sum_{i \in L_2} \sigma_i^z + h'_z \sum_{i \in L_3} \sigma_i^z.$$  

Here, $h_x$, $h_z$ and $h'_z$ are real parameters, and $h'_z$ is a small real parameter. The dominating external fields are applied only on two crossing lines ($L_1$ and $L_2$). In addition, the auxiliary external fields are applied on $L_3$. See the illustration in fig. 2.

To characterize the ground states of a $Z_2$ topological order, we define topologically closed string operators. For the (Hermitian) toric-code model $\hat{H}_{TC}$ around a torus, there are four types of topologically closed string operators: $W_x(C_X)$, $W_y(C_Y)$, $W_f(C_X)$ and $W_f(C_Y)$, where $C_X$ and $C_Y$ denote a topologically closed loop in the $x$-direction and in the $y$-direction around the torus, respectively. Due to the commutation and anti-commutation relations between these topologically closed string operators, i.e., $\{W_x(C_X), W_y(C_Y)\} = 0$, $\{W_x(C_X), W_y(C_Y)\} = 0$, $\{W_f(C_X), W_f(C_X)\} = 0$, and $\{W_f(C_Y), W_f(C_Y)\} = 0$, we can represent $W_x(C_X)$, $W_y(C_Y)$, $W_f(C_X)$ and $W_f(C_Y)$ by pseudo-spin operators $\tau_i^x$, $\tau_i^y$, $\tau_i^z$ and $\tau_i^z$, respectively. Therefore, the ground states become the eigenstates of $\tau_i^x \otimes \tau_j^x$, i.e., $|m_1, m_2\rangle = |m_1\rangle \otimes |m_2\rangle$. For $m_1 = 0$ ($l = 1, 2$), we have $\tau_i^x |m_1\rangle = |m_1\rangle$, and for $m_1 = 1$, we have $\tau_i^x |m_1\rangle = -|m_1\rangle$. A ground state can be represented as a linear combination of the four degenerate ground states, $|\psi_0\rangle = \sum_{m_1, m_2 = 0, 1} \alpha_{m_1, m_2} |m_1, m_2\rangle$, where $\alpha_{m_1, m_2}$ is the weight of $|m_1, m_2\rangle$.

It is known that the degenerate ground states of Hermitian topological order have the same energy in the thermodynamic limit. For a finite system, there are small energy splitting between these (nearly) ground states due to tunneling processes, and the degeneracy of the ground states is (partially) removed [48,50]. However, the energy splitting exponentially decreases with the increase of system size, and it becomes 0 in the thermodynamic limit. Strictly speaking, the degeneracy of the ground states is what it is in the thermodynamic limit. Therefore the topological degeneracy of ground states for Hermitian topological order is robust against any local perturbations in the thermodynamic limit.

In this paper, we focus on the case of $g \gg h_x, h_z, h'_z$ and treat the external field term $\hat{H}'' = h_x \sum_{i \in L_1} \sigma_i^x + h_z \sum_{i \in L_2} \sigma_i^z + h'_z \sum_{i \in L_3} \sigma_i^z$, where $|m_1, m_2\rangle$ and $|m_1', m_2'\rangle$ ($m_1, m_2, m_1', m_2' = 0, 1$) are the four ground states of $\hat{H}_{TC}$, and $h_x, h_z, h'_z$ is the weight of the tunneling effect between $|m_1, m_2\rangle$ and $|m_1', m_2'\rangle$.

Under the perturbation $\hat{H}'' = h_x \sum_{i \in L_1} \sigma_i^x + h_z \sum_{i \in L_2} \sigma_i^z + h'_z \sum_{i \in L_3} \sigma_i^z$, there are small energy splitting between four (nearly) ground states due to tunneling effect for the designed toric-code model $\hat{H}_{TC}^{\prime}$. To characterize the effective low-energy physics of (nearly) degenerate ground states under the perturbation, we can use a four-level system to describe the (nearly) degenerate ground states [43,50]. The effective pseudo-spin Hamiltonian of (nearly) the degenerate ground states can be expressed as

$$\hat{H}_{eff} = \sum_{m_1, m_2, m_1', m_2'} \chi_{m_1, m_2, m_1', m_2'} |m_1, m_2\rangle \langle m_1', m_2'|,$$

where $\chi_{m_1, m_2, m_1', m_2'}$ and $\chi_{m_1, m_2, m_1', m_2'}$ are the four ground states of $\hat{H}_{TC}^{\prime}$, and $\chi_{m_1, m_2, m_1', m_2'}$ is the effective pseudo-spin operators. Under the perturbation $\hat{H}'' = h_x \sum_{i \in L_1} \sigma_i^x + h_z \sum_{i \in L_2} \sigma_i^z + h'_z \sum_{i \in L_3} \sigma_i^z$, the quasi-particles begin to hop. An e-particle (or $Z_2$ charge) is defined as $A_1 = -1$, and a m-particle (or $Z_2$ vortex) is defined as $B_0 = -1$. Therefore, the terms $h_x \sum_{i \in L_1} \sigma_i^x$ and $h_z \sum_{i \in L_2} \sigma_i^z$ can drive the hopping of m-particles along $L_1$ string and $L_3$ string without affecting e-particles, respectively. And the term $h_z \sum_{i \in L_2} \sigma_i^z$ drives the hopping of e-particles along $L_2$ string without affecting m-particles. As a result, after considering $\hat{H}''$, three quantum tunneling processes occur: 1) virtual $Z_2$-vortex propagating along $L_1$; 2) virtual $Z_2$-charge propagating along $L_2$; 3) virtual $Z_2$-vortex propagating along $L_3$ around the torus. With the help of the high-order perturbative theory, the four-level quantum system of the four nearly degenerate ground states on a $2 \times L_x \times L_y$ lattice (with $2 \times L_x \times L_y$ spins) is obtained as

$$\hat{H}_{eff}^{L_x \times L_y} = \Delta (\tau_2^z \otimes 1) + \epsilon (\tau_1^x \otimes \tau_2^z) + \kappa (1 \otimes \tau_2^x),$$

where $\Delta = (\alpha_1 h_x)^{L_x} / (\alpha_1 h_x)^{L_y}$, $\epsilon = (\alpha_2 h_x)^{L_y}$, and $\kappa = (\alpha_3 h'_z)^{L_y}$ with $\alpha_1 = (\alpha_1 h_x)^{L_y}$, $\alpha_2 = \alpha_3 = (\alpha_3 h'_z)^{L_y}$. We can see that $\alpha_1$, $\alpha_2$, and $\alpha_3$ are real parameters, and $\alpha_1 = \alpha_2 = \alpha_3$ when $L_x = L_y$. The eigenvalues of $\hat{H}_{eff}^{L_x \times L_y}$ can be obtained as $\pm \kappa \pm \sqrt{\Delta^2 + \epsilon^2}$.  

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Now, we consider a particular local non-Hermitian perturbation on the designed toric-code model,
\[ \hat{H}'_{\text{NTC}} = \hat{H}'_{\text{TC}} + \hat{H}' = \hat{H}_{\text{TC}} + \hat{H}' + \hat{H}'', \]
where \( \hat{H}' = (\lambda_{\text{Re}} + i\lambda_{\text{Im}})\sigma_z^x \). It is obvious that \( \hat{H}'_{\text{NTC}} \) does not have parity-time symmetry. However, the most important difference is topological poisoning effect under local non-Hermitian perturbations, \( D_{\text{N}}(C_N \mapsto i\delta) \).

When considering above extra non-Hermitian term, the effective Hamiltonian of the degenerate ground states \( \hat{H}_{\text{eff}}^{L_z + L_y} \) on designed toric-code model may change:

1) When the site \( i_0 \) is on the vertical dynamic string \( \mathcal{L}_2 \), \( \hat{H}_{\text{eff}}^{L_z + L_y} \) becomes
\[ \hat{H}_{\text{eff}}^{L_z + L_y} = \Delta'(\tau_1^x \otimes 1) + \varepsilon'(\tau_2^z \otimes \tau_2^z) + \kappa(1 \otimes \tau_2^z), \]
where \( \Delta = (\alpha_1 h_x)^{L_z} \), \( \varepsilon' = (\alpha_2 h_z)^{L_y-1}(h_x + \lambda_{\text{Re}} + i\lambda_{\text{Im}}) \), and \( \kappa = (\alpha_3 h_z)^{L_y} \). The eigenvalues of \( \hat{H}_{\text{eff}}^{L_z + L_y} \) can be obtained as \( \pm \kappa \pm \sqrt{\Delta^2 + \varepsilon'^2} \).

2) When the site \( i_0 \) is on the transverse dynamic string \( \mathcal{L}_1 \), \( \hat{H}_{\text{eff}}^{L_z + L_y} \) becomes
\[ \hat{H}_{\text{eff}}^{L_z + L_y} = \Delta'(\tau_1^x \otimes 1) + \varepsilon'(\tau_1^z \otimes \tau_2^z) + \kappa(1 \otimes \tau_2^z), \]
where \( \Delta' = (\alpha_4 h_y)^{L_z}(h_x + \lambda_{\text{Re}} + i\lambda_{\text{Im}}) \), \( \varepsilon' = (\alpha_5 h_z)^{L_z-1}(h_z + \lambda_{\text{Re}} + i\lambda_{\text{Im}}) \), and \( \kappa = (\alpha_6 h_z)^{L_y} \). The eigenvalues of \( \hat{H}_{\text{eff}}^{L_z + L_y} \) can be obtained as \( \pm \kappa \pm \sqrt{(\Delta')^2 + (\varepsilon')^2} \).

3) When the site \( i_0 \) is on the crossing between \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), \( \hat{H}_{\text{eff}}^{L_z + L_y} \) does not change. The eigenvalues of \( \hat{H}_{\text{eff}}^{L_z + L_y} \) can be obtained as \( \pm \kappa \pm \sqrt{\Delta^2 + \varepsilon^2} \).

In this paper, we focus on the case 1) and \( \hat{H}' = (-h_z + ih_z)\sigma_z^x \). Therefore, The eigenvalues of \( \hat{H}_{\text{eff}}^{L_z + L_y} \) can be obtained as \( E_{\pm} = \pm \kappa \pm \sqrt{\Delta^2 + \varepsilon^2} \).

**Spontaneous PT-symmetry breaking for the degenerate ground states.** An interesting property of the designed toric-code model is spontaneous PT-symmetry breaking for the degenerate ground states by tuning the strength of the local non-Hermitian term.

When the site \( i_0 \) is on the vertical dynamic string \( \mathcal{L}_2 \) and \( \hat{H}' = (-h_z + ih_z)\sigma_z^x \), we have \( \varepsilon' = i\varepsilon \). Then \( \hat{H}_{\text{eff}}^{L_z + L_y} \) becomes
\[ \hat{H}_{\text{eff}}^{L_z + L_y} = \Delta(\tau_1^x \otimes 1) + i\varepsilon(\tau_1^z \otimes \tau_2^z) + \kappa(1 \otimes \tau_2^z), \]
where \( \Delta = (\alpha_1 h_z)^{L_z} \), \( \varepsilon = (\alpha_2 h_z)^{L_y} \), and \( \kappa = (\alpha_3 h_z)^{L_y} \). The effective Hamiltonian \( \hat{H}_{\text{eff}}^{L_z + L_y} \) has PT-symmetry. The eigenvalues of \( \hat{H}_{\text{eff}}^{L_z + L_y} \) can be obtained as \( E_{1/2} = -\kappa \pm \sqrt{\Delta^2 + \varepsilon^2}, E_{3/4} = \kappa \pm \sqrt{\Delta^2 - \varepsilon^2} \). For the non-Hermitian effective Hamiltonian \( \hat{H}_{\text{eff}}^{L_z + L_y} \), we define right/left eigenstates as \( \hat{H}_{\text{eff}}^{L_z + L_y}\ket{\psi_i} = E_i\ket{\psi_i} \) and \( \hat{H}_{\text{eff}}^{L_z + L_y}\ket{\psi_i} = E_i\ket{\psi_i} \) (i = 1, 2, 3, 4), respectively. The corresponding right eigenstates \( \ket{\psi_i^R} \) of \( \hat{H}_{\text{eff}}^{L_z + L_y} \) can be written as
\[ \ket{\psi_i^R_{1/2}} = \frac{1}{N_{1/2}} \left( \begin{array}{c} i\varepsilon \pm \sqrt{\Delta^2 + \varepsilon^2} \\ \pm \Delta^2 + i(\varepsilon - i\sqrt{\Delta^2 - \varepsilon^2})(\varepsilon - i\kappa) \\ \sqrt{\Delta^2 - \varepsilon^2} \pm \kappa \end{array} \right), \]
(14)
\[ \ket{\psi_i^R_{3/4}} = \frac{1}{N_{3/4}} \left( \begin{array}{c} i\varepsilon \mp \sqrt{\Delta^2 - \varepsilon^2} \\ \pm \Delta^2 + i(\varepsilon + i\sqrt{\Delta^2 - \varepsilon^2})(\varepsilon - i\kappa) \\ \sqrt{\Delta^2 - \varepsilon^2} \pm \kappa \end{array} \right), \]
(15)
where \( N_i (i = 1, 2, 3, 4) \) are normalization constants.

For the case of \( |\Delta| > |\varepsilon| \), the system belongs to a phase with PT-symmetry, of which \( E_i(i = 1, 2, 3, 4) \) are real and the eigenvectors are eigenstates of the symmetry operator, i.e., \( PT\ket{\psi_i^R} = \ket{\psi_i^R} \). For the case of \( |\Delta| < |\varepsilon| \), \( E_i(i = 1, 2, 3, 4) \) are complex, and \( PT\ket{\psi_i^R} \neq \ket{\psi_i^R} \). A PT-symmetry-breaking transition occurs at the exceptional points \( |\Delta| = |\varepsilon| \), which leads to the following relation: \( h_x = h_z \), when \( L_x = L_y \). It is clear that \( \ket{\psi_i^R_1} \) and \( \ket{\psi_i^R_2} \) form a pair of PT-symmetry, and \( \ket{\psi_i^R_3} \) and \( \ket{\psi_i^R_4} \) form another pair of PT-symmetry.

From the result, one can see there exist exceptional points (EPs) at \( |\Delta| = |\varepsilon| \). In the limit of \( \Delta \to 0, \varepsilon \to 0 \) according to the condition of quantum phase transition \( (|\Delta| = |\varepsilon|) \), an arbitrary small local perturbation (a local complex external field) causes the quantum phase transition for the ground states. We call it non-Hermitian avalanche effect. In the following, we show the physics consequences of the non-Hermitian avalanche effect.

**Breakdown of bulk-degeneracy correspondence for Z2 topological order.** To characterize the non-Hermitian avalanche effect, we define the ground-state degeneracy \( D \) under local non-Hermitian perturbation. Because the effective Hamiltonian \( \hat{H}_{\text{eff}}^{L_z + L_y} \) is non-Hermitian, the orthogonality between two of the four right eigenstates may not be satisfied. Thus we need to estimate the similarity of any two of the four right eigenstates \( \ket{\psi_i^R} \).

Firstly, we define the right eigenstates with self-normalization constants of \( \hat{H}_{\text{eff}}^{L_z + L_y} \) as \( \ket{\tilde{\psi}_i^R} = \frac{\ket{\psi_i^R}}{\sqrt{\langle \psi_i^R | \psi_i^R \rangle}} \)
which satisfy the condition of \(|\langle \tilde{\psi}_i^R | \tilde{\psi}_j^R \rangle| = 1\). To check whether two right eigenstates (for example, near exceptional point) of \(\mathcal{H}_{eff,L_x+L_y}\) are the same, we define the similarity of \(\tilde{\psi}_i^R\) and \(\tilde{\psi}_j^R\) as \(O_{ij} = |\langle \tilde{\psi}_i^R | \tilde{\psi}_j^R \rangle|\). If \(O_{ij} = |\langle \tilde{\psi}_i^R | \tilde{\psi}_j^R \rangle| = 1\), we say that \(\tilde{\psi}_i^R\) and \(\tilde{\psi}_j^R\) are the same, i.e., \(\tilde{\psi}_i^R = \tilde{\psi}_j^R\).

Using the above definition, we calculate the similarity of any two of these four nearly degenerate eigenstates as follows:

\[
\begin{align*}
O_{12} &= |\langle \tilde{\psi}_2^R | \tilde{\psi}_1^R \rangle|, \\
O_{13} &= |\langle \tilde{\psi}_3^R | \tilde{\psi}_1^R \rangle|, \\
O_{14} &= |\langle \tilde{\psi}_4^R | \tilde{\psi}_1^R \rangle|, \\
O_{23} &= |\langle \tilde{\psi}_3^R | \tilde{\psi}_2^R \rangle|, \\
O_{24} &= |\langle \tilde{\psi}_4^R | \tilde{\psi}_2^R \rangle|, \\
O_{34} &= |\langle \tilde{\psi}_4^R | \tilde{\psi}_3^R \rangle|.
\end{align*}
\] (16)

By inserting eqs. (14), (15) into eqs. (16), (17), we obtain the overlap as

\[
\begin{align*}
O_{12} &= \frac{\varepsilon}{\Delta}, \\
O_{13} &= 0, \\
O_{14} &= 0, \\
O_{23} &= 0, \\
O_{24} &= 0,
\end{align*}
\] (18)

in the region of \(\mathcal{PT}\)-unbroken phase \(|\Delta| \geq |\varepsilon|\), and

\[
\begin{align*}
O_{12} &= \frac{\Delta}{\varepsilon}, \\
O_{13} &= 0, \\
O_{14} &= 0, \\
O_{23} &= 0, \\
O_{24} &= 0,
\end{align*}
\] (19)

in the region of \(\mathcal{PT}\)-broken phase \(|\Delta| < |\varepsilon|\).

Then, we define the degeneracy of the ground states in this case as \(\mathcal{D} = 4 - 2O_{12} - O_{14}\) that changes under the non-Hermitian perturbations. As a result, the bulk-degeneracy correspondence is broken, i.e.,

\[
N \neq \mathcal{D}.
\] (22)

where \(N = 4\) and \(\mathcal{D} = 4 - 12O_{12} - O_{14}\).

For the Hermitian toric-code model with Hermitian external field on a torus in the thermodynamic limit, the energy splitting between these (nearly) ground states approaches 0, i.e., these four states have the same energy. And these four states with the same energy are orthogonal to each other, therefore the degeneracy of the ground states of the Hermitian toric-code model on a torus is 4 (in the thermodynamic limit). However, this situation can be changed for the non-Hermitian toric-code model \(\tilde{\mathcal{H}}_{NTC}\). In the thermodynamic limit, we have \(\Delta \rightarrow 0\), \(\varepsilon \rightarrow 0\), and \(\kappa \rightarrow 0\) in eq. (13). As a result, the energy splitting between these (nearly) ground states approaches 0. However, the similarity of any two of these four states is related to relative magnitude between \(\varepsilon\) and \(\Delta\) even though \(\varepsilon, \Delta \rightarrow 0\). When \(O_{12} \neq 0, O_{14} \neq 0\), i.e., these four states with the same energy are not orthogonal to each other, we have \(\mathcal{D} = 4 - 12O_{12} - O_{14} \neq 4\). In particular, \(O_{12} = O_{14} \rightarrow 1\) when \(|\Delta| \rightarrow |\varepsilon|\), then \(\mathcal{D} \rightarrow 2\). To make it clear, we write down the right eigenstates of \(\tilde{\mathcal{H}}_{eff,L_x+L_y}\) at EPs \(|\Delta| = |\varepsilon|\) and find that \(\tilde{\psi}_1^R = \tilde{\psi}_2^R\) and \(\tilde{\psi}_3^R = \tilde{\psi}_4^R\). Therefore, in the thermodynamic limit the degeneracy of ground states may be different from 4, then the bulk-degeneracy correspondence is broken which is strikingly different from the Hermitian case.

**Fidelity susceptibility of the ground state.** – To confirm the existence of the quantum phase transition from the non-Hermitian avalanche effect, we calculate the fidelity susceptibility of ground states.

Fidelity susceptibility of ground states can be used to characterize the occurrence of the quantum phase transitions. In this section, we study fidelity susceptibility of a given ground state \(\tilde{\psi}_i^R\) \((i = 1, 2, 3, 4)\) for the non-Hermitian toric-code model. The fidelity of the ground state in terms of \(\varepsilon\) can be defined as \(F(\varepsilon, \delta) = |\langle \tilde{\psi}_i^R | \tilde{\psi}_i^R(\varepsilon + \delta) \rangle|\). The fidelity susceptibility of the ground state in terms of \(\varepsilon\) can be defined as \(\chi(\varepsilon, \delta) = \lim_{\delta \rightarrow 0} \frac{dF(\varepsilon, \delta)}{d\delta}\). The behavior of \(\tilde{\psi}_i^R\) of the effective model \(\tilde{\mathcal{H}}_{eff,L_x+L_y}\) is \(\Delta(\tau_k^x \otimes 1) + i\varepsilon(\tau_k^y \otimes \tau_k^z)\) \((\kappa \rightarrow 0)\) is the same as that of \(\tilde{H} = \Delta\tau_k^x + i\varepsilon\tau_k^y\). As a result, the fidelity and fidelity susceptibility of each ground state are obtained as

\[
F(\varepsilon, \delta) = \begin{cases} 
1 - \frac{\delta^2}{8(\Delta^2 - \varepsilon^2)}, & |\Delta| \geq |\varepsilon|, \\
1 - \frac{\Delta \delta^2}{8\varepsilon^2(\varepsilon^2 - \Delta^2)}, & |\Delta| < |\varepsilon|.
\end{cases}
\] (23)

and

\[
\chi(\varepsilon, \delta) = \begin{cases} 
\frac{1}{4(\Delta^2 + \varepsilon^2)}, & |\Delta| \geq |\varepsilon|, \\
\frac{\Delta^2}{4\varepsilon^2(\varepsilon^2 - \Delta^2)}, & |\Delta| < |\varepsilon|.
\end{cases}
\] (24)

**Numerical calculations of the non-Hermitian toric-code model.** – To support our theoretical predictions, we do numerical calculations based on the non-Hermitian toric-code model \(\tilde{\mathcal{H}}_{NTC}\) on the \(2 \times 2 \times 2\) lattice and on the \(2 \times 3 \times 3\) lattice.

In fig. 3, we plot the numerical results from the exact diagonalization technique of the non-Hermitian toric-code model \(\tilde{\mathcal{H}}_{NTC}\) on the \(2 \times 2 \times 2\) lattice with periodic boundary conditions. Figure 3 shows the global phase diagram of \(\mathcal{PT}\)-symmetry-breaking transition for topologically degenerate ground states. The phase boundary are all exceptional points characterized by the relation \(h_x = h_z\).

In fig. 4, we plot the numerical results from the exact diagonalization technique of the non-Hermitian toric-code model \(\tilde{\mathcal{H}}_{NTC}\) on the \(2 \times 3 \times 3\) lattice with periodic boundary conditions. Figure 4 shows the real part and imaginary part of energies \(E_{\phi}(i = 1, 2, 3, 4)\) for the four nearly degenerate ground states for the non-Hermitian toric-code model with \(h_x = 0.1\) and \(h_y = 0.1\) on the \(2 \times 3 \times 3\) lattice, respectively. The numerical results indicate that exceptional points occur when \(h_x = h_z\), which is consistent with theoretical prediction. In addition, we calculate the overlap \(O_{ij}\) of any two of these nearly degenerate eigenstates defined as above. We theoretically predict that the
Fig. 3: Phase diagram for spontaneous $\mathcal{PT}$-symmetry breaking for the topologically degenerate ground states on $2 \times 2$ lattice: in white regions, $\mathcal{PT}$-symmetry is broken; in the gray regions, $\mathcal{PT}$-symmetry is not broken. The phase boundaries are exceptional points.

Fig. 4: (a), (b): the real part and imaginary part of energies for the four degenerate ground states for the case of $h = 0.1$ and $h' = 0.1$ via $h_z$ based on the non-Hermitian toric-code model on $2 \times 3 \times 3$ lattice.

Fig. 5: The non-Hermitian degeneracy of the ground states for the case of $h_z = 0.1$ and $h' = 0.1$ via $h_z$ based on the non-Hermitian toric-code model on $2 \times 2 \times 2$ lattice and those on $2 \times 3 \times 3$ lattice.

In fig. 5, we present the numerical results from the exact diagonalization technique of the non-Hermitian toric-code model $\hat{H}_{NTC}$ on $2 \times 2 \times 2$ and $2 \times 3 \times 3$ lattices with periodic boundary conditions. We plot the non-Hermitian degeneracy as a function of $h_z$ for the case of $h_z = 0.1$ and $h' = 0.1$, which is consistent with the theoretical prediction. The results indicate the degeneracy of ground states may be different from 4. Now, the bulk-degeneracy correspondence is broken, i.e., $\mathcal{N} \neq \mathcal{D}$, where $\mathcal{N} = 4$.

In fig. 6, we show the fidelity susceptibility of the ground state from the exact diagonalization technique of the non-Hermitian toric-code model $\hat{H}_{NTC}$ on $2 \times 2 \times 2$ and $2 \times 3 \times 3$ lattices with periodic boundary conditions. The results show that the quantum $\mathcal{PT}$ phase transition occurs at EPs.

The overlaps are

$$O_{12} = O_{34} = \left| \frac{\Delta}{\Delta} \right| \sim \begin{cases} \frac{h_z^2}{h_z^2} & (2 \times 2 \times 2 \text{ lattice}), \\ \frac{h_z^3}{h_z^3} & (2 \times 3 \times 3 \text{ lattice}) \end{cases}$$

and $O_{13} = O_{14} = O_{23} = O_{24} = 0$ in $\mathcal{PT}$-unbroken phase;

$$O_{12} = O_{34} = \left| \frac{\Delta}{\Delta} \right| \sim \begin{cases} \frac{h_z^2}{h_z^2} & (2 \times 2 \times 2 \text{ lattice}), \\ \frac{h_z^3}{h_z^3} & (2 \times 3 \times 3 \text{ lattice}) \end{cases}$$

and $O_{13} = O_{14} = O_{23} = O_{24} = 0$ in $\mathcal{PT}$-broken phase.

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Conclusion. – In this paper, we studied topological stability of intrinsic topological orders under non-Hermitian perturbations by taking the non-Hermitian toric-code model as an example. The effect of non-Hermitian avalanche for a designed toric-code model is uncovered: for the designed toric-code model with special external fields, a tiny non-Hermitian perturbation (local imaginary state-selective dissipation) leads to anomalous topological degeneracy and the breakdown of bulk-degeneracy correspondence. In addition, we do numerical calculations and the numerical results for a non-Hermitian toric-code model on $2 \times 2 \times 2$ and $2 \times 3 \times 3$ lattices with periodic boundary conditions to support our theoretical predictions.

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