ON PARK’S EXOTIC SMOOTH FOUR-MANIFOLDS

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Abstract. In a recent paper, Park constructs certain exotic simply-connected four-manifolds with small Euler characteristics. Our aim here is to prove that the four-manifolds in his constructions are minimal.

1. Introduction

Since the seminal works of Donaldson [1] and Freedman [6], it has been known that closed, simply-connected four-manifolds can support exotic smooth structures. In fact, for many homeomorphism classes, gauge theory tools (Donaldson invariants and Seiberg-Witten invariants) have been very successful at proving the existence of infinitely many smooth structures, see for example [2], [8], [7], [5]. However, exotic examples with small Euler characteristics are much more difficult to find. For a long time, the smallest known example was the Barlow surface [10], which has Euler characteristic 11 and which is homeomorphic, but not diffeomorphic, to \(\mathbb{CP}^2 \# 8\mathbb{CP}^2\). Recently, in a remarkable paper, Park [14] constructs a symplectic manifold \(P\) with Euler characteristic 10 using the rational blow-down operation of Fintushel and Stern [4], and proves that it is homeomorphic, but not diffeomorphic to \(\mathbb{CP}^2 \# 7\mathbb{CP}^2\).

In this note, we compute the Seiberg-Witten invariants of \(P\) and prove the following:

**Theorem 1.1.** Park’s example \(P\) does not contain any smoothly embedded two-spheres with self-intersection number \(-1\); equivalently, it is not the blow-up of another smooth four-manifold.

In a similar manner, Park also constructs a symplectic four-manifold \(Q\) which is homeomorphic, but not diffeomorphic, to \(\mathbb{CP}^2 \# 8\mathbb{CP}^2\). We prove here the following:

**Theorem 1.2.** The manifold \(Q\) contains no smoothly embedded two-sphere with self-intersection number \(-1\), and in particular \(Q\) is not diffeomorphic to \(P \# 7\mathbb{CP}^2\).

Note that \(Q\) and the Barlow surface have the same Seiberg-Witten invariants, and we do not know whether or not they are diffeomorphic.

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2. Seiberg-Witten theory

We will deal in this paper with Seiberg-Witten theory for four-manifolds $X$ with $b_2^+(X) = 1$ (and $b_1(X) = 0$). For the reader’s convenience, we recall the basic aspects of this theory, and refer the reader to [11], [13] for more in-depth discussions.

The Seiberg-Witten equations can be written down on any four-manifold equipped with a Spin$^c$ structure and a Riemannian metric. We identify here Spin$^c$ structures over $X$ with characteristic classes for the intersection form of $X$, by taking the first Chern class of the Spin$^c$ structure. This induces a one-to-one correspondence in the case where $H^2(X; \mathbb{Z})$ has no two-torsion. Taking a suitable signed count of solutions, one obtains a smooth invariant of $X$ when $b_2^+(X) > 1$. In the case where $b_2^+(X) = 1$, the invariant depends on the choice of the Riemannian metric through the cohomology class of its induced self-dual two-form (compare also [1]).

Formally, then, for a fixed two two-dimensional cohomology class $H \in H^2(X; \mathbb{R})$ with $H^2 > 0$ and characteristic vector $K \in H^2(X; \mathbb{Z})$ with $K.H \neq 0$, the Seiberg-Witten invariant $SW_{X,H}(K)$ is an integer which is well-defined provided that $H.K \neq 0$. This integer vanishes whenever

$$K^2 < 2\chi(X) + 3\sigma(X).$$

For fixed $H$, then, the $H$-basic classes are those characteristic cohomology classes $K$ for which $SW_{X,H}(K) \neq 0$. The quantity $K^2 - 2\chi(X) - 3\sigma(X)$ is four times the formal dimension of the moduli space of solutions to the Seiberg-Witten equations over $X$ in the Spin$^c$ structure whose first Chern class is $K$. The Seiberg-Witten invariant vanishes when this formal dimension is negative; when it is positive, one cuts down the moduli space by a suitable two-dimensional cohomology class to obtain an integer-valued invariant.

More precisely, a Riemannian metric on $X$ induces a Seiberg-Witten moduli space. The signed count of the solutions in this moduli space depends only on the cohomology class of the induced self-dual two-form $\omega_g$, which in the above case was denoted by $H$. The dependence on $H$ is captured by the wall-crossing formula [11], [12]: if $X$ is a four-manifold with $b_1(X) = 0$, and $H$ and $H'$ are two cohomology classes with positive square and $H.H' > 0$, then

$$SW_{X,H}(K) = SW_{X,H'}(K) + \begin{cases} 0 & \text{if } K.H \text{ and } K.H' \text{ have the same sign} \\ \pm 1 & \text{otherwise.} \end{cases}$$

It follows readily from the compactness result for the moduli space of solutions to the Seiberg-Witten equations that for any $H$, there are only finitely many $H$-basic classes.

It is interesting to note that the wall-crossing formula together with the dimension formula (which states that $SW_{X,H}(K) = 0$ when $K^2 - 2\chi(X) - 3\sigma(X) < 0$), ensures that if $X$ is a four-manifold with $b_2^+(X) = 1$ but $b_2(X) \leq 9$, there is only one chamber.

2.1. Rational blow-downs. In [4], Fintushel and Stern introduce a useful operation on smooth four-manifolds, and calculate how the Seiberg-Witten invariants transform
under this operation. Specifically, let $C_p$ be the four-manifold which is a regular neighborhood of a chain of two-spheres $\{S_0, ..., S_p\}$ where $S_0$ has self-intersection number $-4-p$, and $S_i$ has self-intersection number $-2$ for all $i > 0$. The boundary of this chain (the lens space $L((p + 1)^2, p)$) also bounds a four-manifold $B$ with $H^2(B; \mathbb{Q}) = 0$. If $X$ is a smooth, oriented four-manifold with $b_+^2(X) > 1$ which contains $C_p$, then we can trade $C_p$ for the rational ball $B$ to obtain a new four-manifold $X'$. Clearly, $H^2(X')$ is identified with the orthogonal complement to $[S_i]_{i=0}^p$ in $H^2(X)$.

For each Spin$^c$ structure over $L((p + 1)^2, p + 1)$ which extends over $B$, there is an extension (as a characteristic vector $K_0$) over $C_p$ with the property that $K_0^2 - p - 1 = 0$.

Fintushel and Stern show that for any characteristic vector $K$ for the intersection form of $X'$,

$$SW_{X'}(K) = SW_X(\tilde{K}),$$

where $\tilde{K}$ is obtained from $K$, by extending over the boundary by the corresponding characteristic vector $K_0$ as above.

In the case where $b_+^2(X) = 1$, the relation is expressed by choosing a chamber for $X$ (and induced chamber for $X'$) whose metric form $H$ is orthogonal to each sphere in the configuration $C_p$.

### 3. The Four-Manifold $P$

We review Park’s construction of $P$ briefly. Start with a rational elliptic surface with an $\tilde{E}_6$ singularity (a configuration of $-2$ spheres arranged in a star-like pattern, with a central node and three legs of length two). There is a model of the rational elliptic surface with the property that there are four nodal curves in a complement of this singularity. Blowing up the nodal curves, one obtains four spheres of square $-4$. A section of the rational elliptic surface meets all four of these spheres, and also one of the leaves in the $\tilde{E}_6$ singularity. Adding the section and the four $-4$-spheres, one obtains a sphere $R_0$ with self-intersection number $-9$ and then inside the $\tilde{E}_6$ singularity, this can be extended to a chain of embedded spheres with self-intersection $-2 \{R_i\}_{i=1}^5$. Park’s example $P$ is obtained by performing a rational blow-down, in the sense of Fintushel and Stern [4], on the chain of spheres $\{R_i\}_{i=0}^5$. Since the spheres are all symplectic, a result of Symington [15] guarantees that $P$ is symplectic.

Theorem 3.1 follows from the following refinement:

**Theorem 3.1.** Let $K$ denote the canonical class of $P$. Then, the Seiberg-Witten basic classes of $P$ are $\{\pm K\}$.

It follows at once that $X$ is minimal. Specifically, if one could write $X \cong Y \# \mathbb{CP}^2$, then according to the blow-up formula [3], the basic classes of $X$ come in pairs of the form $K_0 \pm E$ where $K_0$ runs over the basic classes of $Y$, and $E$ denotes the exceptional curve in $\mathbb{CP}^2$. But this is impossible since $K^2 = 2$. 
We find it convenient to describe the manifold $P$ in a concrete model. Specifically, consider the four-manifold $X = S^2 \times S^2 \# 12\mathbb{CP}^2$, with the basis of two-spheres $A, B, \{E_i\}_{i=1}^{12}$. Here, $A$ and $B$ are supported in the $S^2 \times S^2$ factor, so that $A = \{a\} \times S^2$ and $B = S^2 \times \{b\}$, while $E_i$ is the “exceptional sphere” (sphere of square $-1$) in the $i$th $\mathbb{CP}^2$ summand. Alternatively, this manifold can be thought of as the blowup of rational elliptic surface with an $\tilde{E}_6$ singularity, and a complementary singularity consisting of three $-1$-spheres arranged in a triangular pattern, which is then blown up four times, to give a tree-like configuration of spheres with a central sphere of of square $-4$, and three legs consisting of a chain of a $-1$ sphere and another $-4$ sphere. See Figure 1 for an illustration.

More precisely, consider the elliptic surface singularity which can be described by three $-1$-framed unknots, each of which links the other two in one point apiece. Denote the corresponding two-dimensional homology classes by $A$, $B$, and $C$. It is well-known, c.f. [9] that this singularity can be perturbed into four nodal curves. By blowing up the four double-points, we obtain four disjoint $-1$-spheres. In fact, the homology class of the fiber is represented by the homology class of the fiber $A + B + C$. Thus, the four $-4$ spheres can written in the basis of homology as

$$\{A + B + C - 2E_i\}_{i=1}^{4},$$

where $E_i$ are the newly-introduced exceptional spheres.

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**Figure 1.** We have illustrated here a basis of two-spheres for $\mathbb{CP}^2 \# 12\mathbb{CP}^2$.
Armed with this principle, the chain of spheres in $X$ which are to be rationally blown down can be written homologically as:

\[
R_0 = 10A + 8B - 6E_1 - 4E_2 - 4E_3 - 4E_4 - 4E_5 - 4E_6 - 3E_7 - 4E_8 - 4E_9 - 2E_{10} - 2E_{11} - 2E_{12}
\]

\[
R_1 = B - E_1 - E_4
\]

\[
R_2 = A - E_2 - E_3
\]

\[
R_3 = E_3 - E_6
\]

\[
R_4 = E_6 - E_9
\]

\[
R_5 = E_4 - E_7
\]

Note that we are using here $E_7$ as our section, which is to be added to the four $-4$-spheres coming from the complement of the $\tilde{E}_6$ singularity. The four exceptional spheres in the complementary singularity are represented by the spheres $A - E_1, E_{10}, E_{11}, E_{12}$.

Let $P$ denote the Park manifold obtained by rationally blowing down the configuration $R_0, ..., R_5$ in $X$. Spin\(^c\) structures over $P$ (labelled by characteristic vectors $K$) correspond to characteristic vectors (labelled by characteristic vectors $\tilde{K}$) over $X$ whose evaluations on the configuration $\{R_i\}$ take one of the following seven forms:

\[
\begin{align*}
(7, & 0, 0, 0, 0, 0) \\
(-1, & 0, -2, 0, 0, 0) \\
(5, & 0, 0, 0, 0, -2) \\
(-3, & -2, 0, 0, 0, 0) \\
(3, & 0, 0, 0, -2, 0) \\
(-7, & 0, 0, 0, 0, 0) \\
(1, & 0, 0, -2, 0, 0)
\end{align*}
\]

(1)

According to the rational blow-down formula \[4\],

\[SW_P(K) = SW_{X,H}(\tilde{K}),\]

where here $H \in H^2(X; \mathbb{R})$ is any real two-dimensional cohomology class with $H^2 > 0$ and $H.H' > 0$ and which is orthogonal to all the $\{R_i\}$. Moreover, according to the wall-crossing formula, combined with the fact that $S^2 \times S^2$ has positive scalar curvature and hence trivial invariants in a suitable chamber (c.f. \[16\]), it follows that

\[
SW_{X,H}(\tilde{K}) = \begin{cases} 
0 & \text{if } \tilde{K}^2 + 4 < 0 \text{ or } \tilde{K}.H \text{ and } \tilde{K}.H' \text{ have the same sign} \\
\pm 1 & \text{otherwise},
\end{cases}
\]

where here $H' = \text{PD}(A + B)$. (The first condition for vanishing is the dimension formula for the moduli space, while the second condition comes from the wall-crossing formula.)

Explicitly, then, we see that the basic classes $\tilde{K}$ for $P$ are precisely those for which the extension $\tilde{K}$ (by one of the vectors from the list in Equation (1)) satisfies: $\tilde{K}^2 + 4 \geq 0$ and also $\text{sgn}(\tilde{K}.H) \neq \text{sgn}(\tilde{K}.H')$, where here $H$ is any (real) cohomology class with
\(H^2 > 0\) and \(H \cdot H' > 0\) and which is orthogonal to all the \(\{R_i\}_{i=0}^5\). For example, we could use the vector

\[
H = (105, 92, -67, -51, -41, -38, -36, -41, -36, -41, -18, -18)
\]

(written here with respect to the basis Poincaré dual to \(\{A, B, E_1, ..., E_{12}\}\)). In order to make this a finite computation, we proceed as follows.

Suppose that \(Z\) is a smooth four-manifold with \(b^+ (Z) > 1\), and we have homology classes \(C = \{C_i\}_{i=1}^n\) with negative self-intersection number \(C_i \cdot C_i = -p_i \leq 0\). A cohomology class \(K \in H^2 (X; \mathbb{Q})\) is called \(C\)-adjunctive if for each \(i \langle K, [C_i] \rangle\) is integral, and indeed the following two conditions are satisfied:

\[
\begin{align*}
|\langle K, [C_i] \rangle| & \leq p_i \\
\langle K, [C_i] \rangle & \equiv p_i \pmod{2}.
\end{align*}
\]

Clearly, the set of \(C\)-adjunctive cohomology classes has size \(\prod_{i=1}^n (p_i + 1)\).

**Lemma 3.2.** Let \(S = \{S_i\}_{i=1}^n\) be a collection of embedded spheres in \(X\) whose homology classes are orthogonal to the the \(\{R_i\}_{i=0}^5\). Let \(C = \{C_i\}_{i=1}^8\) denote their induced homology classes in \(H_2 (P)\). If every \(C\)-adjunctive basic class for \(P\) is zero-dimensional, then in fact every basic class for \(P\) is \(C\)-adjunctive.

**Proof.** If \(P\) has a basic class \(L_0\) which is not \(C\)-adjunctive, then by the rational blow-down formula, \(X\) has a basic class \(L_1\) and a smoothly embedded sphere \(S_i\) for which \(|\langle L_1, [S_i] \rangle| > -S_i \cdot S_i\), where we can use any metric whose period point \(H'\) is perpendicular to the configuration \(\{R_i\}_{i=0}^5\). By fixing \(H'\) to be also perpendicular to \(S_i\), and using the adjunction formula for spheres of negative square \[\text{we get another basic class } L_2 = L \pm 2\text{PD}[S_i] \text{ of } X.\] Applying the blowdown formula once more we get a basic class \(L_3\) of \(P\) where the dimension of \(L_3\) is bigger then the dimension of \(L_0\). Since \(P\) has only finitely many basic classes this process has to stop, which means that the final \(L_{3k}\) class is \(C\)-adjunctive. However it is also positive dimensional, thus proving the lemma. \(\square\)
Our next goal, then is to find a collection of embedded spheres \( \{S_i\}_{i=1}^8 \) in \( X \) which, together with the \( \{R_i\}_{i=0}^5 \) form a basis for \( H^2(X; \mathbb{Q}) \). To this end, we use the spheres:

\[
\begin{align*}
S_1 &= E_5 - E_8 \\
S_2 &= E_{12} - E_{10} \\
S_3 &= E_{11} - E_{12} \\
S_4 &= A - E_1 - E_{11} \\
S_5 &= A + B - E_1 - E_2 - E_5 - E_8 \\
S_6 &= -E_5 + E_{10} + E_{11} \\
S_7 &= 2E_7 + 2E_4 - 2A + E_{11} \\
S_8 &= E_6 + E_9 + E_3 - E_2 - 2E_5.
\end{align*}
\]

The spheres \( \{S_i\}_{i=1}^5 \) have square \(-2\), while \( S_6, S_7, \) and \( S_8 \) have squares \(-3\), \(-9\), and \(-8\) respectively. It is easy to see that these classes are all orthogonal to the homology classes generated by the spheres \( \{R_i\}_{i=0}^5 \).

It is easy to see, now, that there are 612360 \( \{S_i\} \)-adjunctive vectors in \( H^2(X; \mathbb{Q}) \) with integral evaluations on each of the \( S_i \), and whose extension over the blow-down configuration is one of the seven choices enumerated in Equation (1). Of these, 12498 correspond to characteristic cohomology classes in \( H^2(X; \mathbb{Z}) \). Of these, 8960 have length \( \geq -4 \) (i.e. satisfying \( K^2 - (2\chi + 3\sigma) \geq 0 \)). Finally, only two of these have the property that evaluation of \( H \) and \( H' \) have opposite sign. Indeed, these classes are the canonical class \( K \) and also \(-K\). Since these classes have dimension zero, it follows from Lemma 3.2 that these are the only two basic classes for \( P \).

4. THE FOUR-MANIFOLD \( Q \)

The manifold \( Q \) is constructed as follows. Start with a rational surface with an \( \tilde{E}_6 \) singularity as before, except now blow up only three of the nodes. In a manner similar to the previous construction, one finds now a sphere of self-intersection number \(-7\) (gotten by resolving a section and the three \(-4\) spheres). This is then completed by a chain of three \(-2\) spheres in the \( \tilde{E}_6 \) singularity. Forming the rational blow-down, one obtains a second manifold \( Q \) which is homeomorphic to \( \mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2} \).

For \( Q \), we prove the following:

**Theorem 4.1.** Let \( K \) denote the canonical class of \( Q \). Then, the Seiberg-Witten basic classes of \( Q \) are \( \{\pm K\} \).

The second construction starts again with a rational surface. For this surface, we can take the previous one, only blow down the curve \( E_{12} \).

Again, we use the section \( E_7 \); now the three \(-4\) spheres which are to be added are represented by \( E_1 - E_4 - E_7 - E_{10}, B - E_2 - E_5 - E_8 - E_{11}, \) and \( A - E_1 - E_{10} - E_{11} \).
Thus, our configuration which is to be rationally blown down consists of:

\[ R_0 = 7A + 6B - 4E_1 - 3E_2 - 3E_3 - 3E_4 - 3E_5 - 3E_6 - 2E_7 - 3E_8 - 3E_9 - 2E_{10} - 2E_{11} \]
\[ R_1 = E_4 - E_7 \]
\[ R_2 = B - E_1 - E_4 \]
\[ R_3 = A - E_2 - E_3. \]

The vector \( H = (229, 226, -143, -113, -113, -86, -87, -87, -86, -87, -58, -58) \) has positive square, and is orthogonal to all the \( \{R_i\}_{i=0}^3 \).

A rational basis the cohomology of \( (S^2 \times S^2)\#11\mathbb{CP}^2 \) is gotten by completing \( R_0, R_1, R_2, \) and \( R_3 \) with the following set of spheres with negative square:

\[ S_1 = E_{10} - E_{11} \]
\[ S_2 = E_5 - E_6 \]
\[ S_3 = E_8 - E_9 \]
\[ S_4 = E_5 - E_8 \]
\[ S_5 = E_2 - E_3 \]
\[ S_6 = A - E_1 - E_{10} \]
\[ S_7 = A + B - E_1 - E_2 - E_5 - E_8 \]
\[ S_8 = 2A - 2E_4 - 2E_7 - E_{11} \]
\[ S_9 = 2A + 2B - E_1 - E_2 - E_3 - E_4 - E_7 - 2E_5 - E_6 - E_{10}. \]

For this case, the Spin\(^c\) structures over \( L(25, 4) \) which extend over the rational ball can be uniquely extended over the configuration of spheres in one of the five possible ways:

\[
\begin{align*}
& \quad (5, \quad 0, \quad 0, \quad 0) \\
& \quad (-1, \quad -2, \quad 0, \quad 0) \\
& \quad (3, \quad 0, \quad 0, \quad -2) \\
& \quad (-5, \quad 0, \quad 0, \quad 0) \\
& \quad (1, \quad 0, \quad -2, \quad 0).
\end{align*}
\]

Again, there are 437400 \( \{S_i\}\)-adjunctive vectors in \( H^2 \) with rational coefficients, which have integral evaluations on each sphere and which extend over the configuration of spheres \( \{R_i\}_{i=0}^3 \) as above. Of these, 17496 correspond to (integral) characteristic cohomology classes. Of these, 3754 have square \( \geq -3 \). Finally, of these, exactly two \( (K' \) and \( -K') \) have the evaluations with opposite sign against \( H \) and \( H' \), hence correspond to basic classes for \( Q \). Arguing as in Lemma 3.2, we see that these are the only two basic classes for \( Q \).
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