Reciprocity laws and $K$-theory

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Abstract

We associate to a full flag $F$ in an $n$-dimensional variety $X$ over a field $k$, a "symbol map" $\mu_F : K(F_X) \to \Sigma^n K(k)$. Here, $F_X$ is the field of rational functions on $X$, and $K(\cdot)$ is the $K$-theory spectrum. We prove a "reciprocity law" for these symbols: Given a partial flag, the sum of all symbols of full flags refining it is 0. Examining this result on the level of $K$-groups, we re-obtain various reciprocity laws. Namely, when $X$ is a smooth complete curve, we obtain degree of a principal divisor is zero, Weil reciprocity, Residue theorem, Contou-Carrère reciprocity. When $X$ is higher-dimensional, we obtain Parshin reciprocity.

1 Introduction

Several statements in number theory and algebraic geometry are "reciprocity laws". Let us consider, as an example, Weil reciprocity. Let $X$ be a complete smooth curve over an algebraically closed field $k$, and let us fix $f, g \in F_X^\times$, two non-zero rational functions on $X$. Given a point $p \in X$, one defines the tame symbol:

$$(f, g)_p := (-1)^{v_p(f)} v_p(g) \frac{f_{v_p(g)}}{g_{v_p(f)}}(p).$$

Here, $v_p$ is the valuation at $p$ (order of zero). Weil reciprocity states that $(f, g)_p = 1$ for all but finitely many $p \in X$, and that $\prod_{p \in X} (f, g)_p = 1$. More generally, one can describe the pattern as follows: There is a global object, exhausted by local pieces. One then associates an invariant to each local piece, as well as to the global object itself. The desired claim is then two-fold:

(i) ("Global is trivial") The global invariant is trivial.

(ii) ("Local to Global") The product of the local invariants equals the global invariant (usually this is an infinite product, and one should realize how to make sense of it).

In the above example, the global object is the curve, which is exhausted by the local pieces - the points of the curve. The invariant associated to a local piece is the tame symbol, while the global invariant is quite implicit.
It is well known, that one can define the tame symbol and prove Weil reciprocity using algebraic $K$-theory. Let us recall that Weil reciprocity admits a higher-dimensional analog, known as Parshin reciprocity, which will be recalled below. In this article, we use algebraic $K$-theory to prove Parshin reciprocity. In fact, our setup allows a unified treatment of several reciprocity laws in the geometric setting, such as the theorem that sum of degrees of a rational function is zero, Weil reciprocity, the residue theorem, Contou-Carrère reciprocity (all attached to a curve), and Parshin reciprocity (attached to a higher-dimensional variety).

Let us describe our setup in more detail. Fix an $n$-dimensional irreducible variety $X$ over an algebraically closed field $k$. These assumptions on $X$ and $k$ are made here merely to simplify matters, and will be relaxed below. By a full flag $F$ in $X$ we mean a chain of closed irreducible subvarieties $X = X_0 \supset X_1 \supset \ldots \supset X_n$, where the codimension of $X_i$ in $X$ is $i$. Given a full flag $F$, we define below a morphism of spectra ("symbol map")

$$\mu_F : K(F_X) \to \Sigma^n K(k).$$

Here $F_X$ is the field of rational functions on $X$, $K(\cdot)$ denotes the $K$-theory spectrum, and $\Sigma$ denotes suspension. By a partial flag $G$ in $X$ we mean a full flag with one level omitted. Then, fixing a partial flag $G$, we may consider the set $fl(G)$ of full flags which refine it. For the next theorem to be valid, in the case that the 0-dimensional level in $G$ is the omitted one, we should additionally assume that the 1-dimensional level in $G$ is proper over $k$. Theorem 2.1 then states:

**Theorem.**

$$\sum_{F \in fl(G)} \mu_F = 0.$$

**Remark.** The sum figuring in the theorem is infinite, however in appendix A we make sense of it (inspired by [3]).

In fact, it is more "correct" to additionally define a symbol map

$$\mu_G : K(F_X) \to \Sigma^n K(k),$$

associated to a partial flag $G$. The theorem then divides into two parts, that $\mu_G$ equals zero, and that the sum of all the morphisms $\mu_F$ for $F \in fl(G)$ equals $\mu_G$.

We notice how this setup instantiates the general pattern above. A fixed partial flag is the global object, exhausted by the local pieces which are the full flags refining the given partial flag. The symbol map is the associated invariant.

In order to derive the concrete reciprocity laws promised above from this abstract one, one considers its effect on $K$-groups, i.e., applies homotopy group functors.

Let us note that, in principle, the symbol map between spectra appears to contain more information than its "shadows" on $K$-groups. However, in this text we have only re-obtained known reciprocity laws from it.
There are several further directions which one may consider. For example, one may consider the "curve" Spec(Z). Could our setup be altered, so that Hilbert reciprocity law would fit in? For this, at least three phenomena should be addressed: The prime at infinity, ramification at the prime 2 and the sphere spectrum, which underlies all primes. A very relevant treatment of the case of Spec(Z) is in [3].

1.1 Relation to n-Tate spaces

There is a strong relation of our mechanism with the theory of n-Tate vector spaces. In fact, n-Tate vector spaces could be seen as the actual "geometric" objects whom our elements in K-theory classify, so that, in a sense, our approach "decategorifies" the actual picture.

The technical result underlying such a connection is the following one. Let \( \mathcal{C} \) be an exact category, and \( \text{Tate}(\mathcal{C}) \) the exact category of "pro-ind" objects in \( \mathcal{C} \), introduced by Beilinson [2].

**Theorem ([5]).**

\[
K(\text{Tate}(\mathcal{C})) \approx \Sigma K(\mathcal{C}).
\]

Thus we can say that the Tate construction acts as a delooping, when one passes to K-theory spectra.

In this paper we associate to a full flag \( \mathcal{F} \) in an \( n \)-dimensional variety \( X \) a symbol map

\[
\mu_{\mathcal{F}} : K(F_X) \to \Sigma^n K(k).
\]

Taking the above theorem into account, one might interpret it as a map

\[
\mu_{\mathcal{F}} : K(F_X) \to K(\text{Tate}^n(k)),
\]

where \( \text{Tate}^n(k) \) is the \( n \)-fold application of the \( \text{Tate}(\cdot) \) construction to the exact category \( \text{Vect}(k) \) of finite-dimensional vector spaces over \( k \). At this point, one might wonder if this map comes from a functor \( \text{Vect}(F_X) \to \text{Tate}^n(k) \). Indeed, such a functor can be constructed, and is essentially the adelic construction of [1].

We will address and develop the above interesting ideas elsewhere.

1.2 Organization

This text is organized as follows. Section 2 contains the formulation of the abstract reciprocity law (subsection 2.1) and the formulations of concrete reciprocity laws (subsection 2.2) which are obtained from the abstract reciprocity law by considering its effect on specific K-groups. Section 3 contains the construction of the abstract symbol map (subsection 3.1) and the proof of the abstract reciprocity law (subsection 3.2). Section 4 deals with calculation of the symbol map on specific K-groups. In appendix A, we describe how to make sense of an infinite sum of morphisms of spectra. In appendix B, we state some lemmas about K-theory, which are used in calculations.
1.3 Notations

1.3.1

We use [9] as a reference for $K$-theory of schemes. Given a Noetherian scheme $X$, $K(X)$ will denote the ”$K$-theory of $X$” spectrum. Given $Z \subset X$ a closed subset, $K(X \text{ on } Z)$ will denote the ”$K$-theory of $X$ with supports in $Z$” spectrum. By abuse of notation, for a commutative ring $A$ and an ideal $I \subset A$, we will also write $K(A) = K(X)$ and $K(A \text{ on } I) = K(X \text{ on } Z)$ where $X = \text{Spec}(A)$ and $Z \subset X$ is the closed subset of associated to the ideal $I$.

1.3.2

We use some notation for the scheme $X$ in this article:

- $n = \dim(X)$ will denote the Krull dimension of $X$.
- $|X|$ will denote the underlying topological space of $X$. $\leq$ will denote the usual partial order on $|X|$ (that of ”containment in the closure of”). $|X|^i$ will denote the subset of $|X|$ consisting of points of codimension $i$.
- $\gamma$ will denote the generic point of $|X|$ ($X$ will be assumed to be irreducible) - i.e., the only point in $|X|^0$, and $F = F_X = \mathcal{O}_{X,\gamma}$ will denote the local ring at that point.
- For $p \in |X|$, we will write $X_p := \text{Spec}(\mathcal{O}_{X,p})$. There is a canonical map $X_p \to X$. As usual, we will write $k(p)$ for the residue field of $\mathcal{O}_{X,p}$.
- If $X$ is affine and $p$ is a prime ideal in $\mathcal{O}(X)$, then $p \in |X|$ will denote the corresponding point.

2 Statements

2.1 Abstract reciprocity law

Let $X \to B$ be a morphism of schemes. We make the following assumptions:

1. $B$ is Noetherian, 0-dimensional (i.e., a finite disjoint union of $\text{Spec}$’s of local Artinian rings).
2. $X$ is Noetherian, of finite Krull dimension and irreducible.
3. For every $p \in |X|^n$ (recall $n := \dim(X)$), the composition $\text{Spec}(k(p)) \to X \to B$ is a finite morphism.

We give two examples of morphisms that satisfy the above assumptions. Let $k$ be a field and let $B = \text{Spec}(k)$.

1. Let $X \to B$ be an irreducible scheme of finite type over $B$. 

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2. Let $X = \text{Spec}(A)$, where $(A, \mathfrak{m})$ is a Noetherian local integral $k$-algebra, such that $A/\mathfrak{m}$ is finite over $k$. Let $X \to B$ be the corresponding structure map.

A convenient technical notion will be that of a collection $C$, by which we mean a family $C = (\mathcal{C}^i)_{0 \leq i \leq n}$, where $\mathcal{C}^i \subset |X|^i$. Our collections will always satisfy $\mathcal{C}^0 = \{\gamma\}$.

Given a collection $C$ we will construct in subsection 3.1 a map of spectra ("symbol map"):

$$\mu_C : K(F) \to \Sigma^n K(B).$$

We will, in this text, only consider and use collections attached to full and partial flags (to be now defined), for which we will state a reciprocity law. First, let $F : x_n < x_{n-1} < \ldots < x_0 = \gamma$ be a full flag of points in $|X|$ (thus, $\text{codim}(x_i) = i$). We define a collection $C(F)$, by setting $C(F)^i = \{x_i\}$. Second, let $G : x_n < x_{n-1} < \ldots < x_{d+1} < x_{d-1} < \ldots < x_0 = \gamma$ be a partial flag, with one level $0 < d \leq n$ omitted, where we require $\text{codim}(x_i) = i$. We define a collection $C(G)$, by setting $C(G)^i = \{x_i\}$ for $i \neq d$, and $C(G)^d = \{p \in |X|^d \mid x_{d+1} < p < x_{d-1}\}$. Note that we have the obvious notion of a full flag refining a partial flag (if $C(F) \subset C(G)$), which we will denote by $F > G$. We will sometimes write $\mu_F$ instead of $\mu_{C(F)}$.

We prove the following "reciprocity" laws (for the meaning of the infinite sum in this statement, consult Appendix A).

**Theorem 2.1.** Let $G$ be a partial flag with level $d$ omitted, where $0 < d \leq n$.

1. ("Global is trivial")

$$\mu_{C(G)} = 0,$$

where in case $d = n$ we should assume that $x_{n-1}$ is proper over $B$.

2. ("Local to Global")

$$\mu_{C(G)} = \sum_{F > G} \mu_{C(F)}.$$

2.2 Concrete reciprocity laws

In the following, we give examples of "concrete" reciprocity laws which one obtains by considering the effect of the "abstract" reciprocity law on various homotopy groups of the involved spectra.
2.2.1 The case \( \dim(X) = 1 \)

Let \( k \) be a field, \( B = \text{Spec}(k) \) and \( X \to B \) a regular, connected, proper curve over \( B \). We obtain, for every closed point \( p \in |X| \), a map \( \mu_p : K(F) \to \Sigma K(B) \). Here \( \mu_p = \mu_{C(F)} \), where \( F : p < \gamma \). Applying the functor \( \pi_i \), one has maps \( \mu_i^p : K_i(F) \to K_{i-1}(k) \).

2.2.1.1 Degree law

We have the map \( \mu_1^p : F^\times \cong K_1(F) \to K_0(k) \cong \mathbb{Z} \).

Claim 2.2. The integer \( \mu_1^p(f) \) is equal to the valuation \( v_p(f) \) of \( f \) at the point \( p \), multiplied by \([k(p) : k]\.\)

Applying the abstract reciprocity law, we recover the theorem about sum of degrees [6, II.3, Prop. 1]:

Corollary 2.3. For \( f \in F^\times \),

\[ \sum_{p \in |X|} [k(p) : k] \cdot v_p(f) = 0. \]

2.2.1.2 Weil reciprocity law

We have the map \( \mu_2^p : K_2(F) \to K_1(k) \). Precomposing with the product in \( K \)-theory \( K_1(F) \wedge K_1(F) \to K_2(F) \), we get a bilinear anti-symmetric form \( \mu_2^p : F^\times \wedge F^\times \to k^\times \) (we also call it \( \mu_2^p \), by abuse of notation).

Claim 2.4. \( \mu_2^p(f \wedge g) = N_{k(p)/k} \left( (-1)^{v_p(f) \cdot v_p(g)} \frac{F^v_p(g)}{g^v_p(f)}(p) \right) \).

Applying the abstract reciprocity law, we recover the Weil reciprocity law [6, III.4]:

Corollary 2.5. For \( f, g \in F^\times \),

\[ \prod_{p \in |X|} N_{k(p)/k} \left( (-1)^{v_p(f) \cdot v_p(g)} \frac{F^v_p(g)}{g^v_p(f)}(p) \right) = 1. \]

2.2.1.3 Residue law

Suppose now that \( k \) is algebraically closed. Set \( k_\epsilon := k[\epsilon]/(\epsilon^3) \), \( B_\epsilon = \text{Spec}(k_\epsilon) \) and \( X_\epsilon = k_\epsilon \otimes_k X \). Then, the local ring at the generic point of \( X_\epsilon \) is just \( F_\epsilon = k_\epsilon \otimes_k F \). By applying our construction to the morphism \( X_\epsilon \to B_\epsilon \) we get a map \( K(F_\epsilon) \to \Sigma K(k_\epsilon) \) for every closed point \( p \in |X_\epsilon| \). Applying the functor \( \pi_2 \) and using the product in \( K \)-theory as before, one has a pairing \( r_p : F_\epsilon^\times \wedge F_\epsilon^\times \to k_\epsilon^\times \).
Claim 2.6.

\[ r_p((1 - \epsilon f) \wedge (1 - \epsilon g)) = 1 - \epsilon^2 \text{Res}_p(f \cdot dg). \]

Here, \( \text{Res}_p \) is the usual residue [6] II.7.

Applying the abstract reciprocity law, we recover the residue theorem [6] II.7, Prop. 6:

Corollary 2.7. For \( f, g \in F \),

\[ \sum_{p \mid |X|} \text{Res}_p(f \cdot dg) = 0. \]

2.2.1.4 Contou-Carrère law

Let, more generally, \( k \) be a local Artinian ring. Set \( B = \text{Spec}(k) \) and \( X = \text{Spec}(k[[t]]) \). Applying the functor \( \pi_2 \) to the abstract reciprocity map \( K(k((t))) \rightarrow \Sigma K(k) \), one has a pairing \( k((t))^\times \wedge k((t))^\times \rightarrow k^\times \). One can check that it is the Contou-Carrère symbol [4], although we don’t make the explicit computations in this text. Then the abstract reciprocity law implies the Contou-Carrère reciprocity law.

2.2.2 The case \( \dim(X) > 1 \)

Let \( k \) be a field, let \( B = \text{Spec}(k) \) and let \( X \rightarrow B \) be an irreducible scheme of finite type over \( B \) (recall \( n := \dim(X) \)). For every full flag \( F \) one has a map \( \mu_F : K(F) \rightarrow \Sigma^n K(B) \). Applying the functor \( \pi_i \), one has maps \( \mu_F^i : K_i(F) \rightarrow K_{i-n}(k) \).

2.2.2.1 Parshin law

Let us assume that the flag \( F = x_n < x_{n-1} < \ldots < x_0 = \gamma \) is regular in the following sense: Considering \( X^i := X_{x_n}^i \) as an integral closed subscheme of \( X \), we demand \( \mathcal{O}_{X^i} \) to be regular (here, \( 1 \leq i \leq n \)).

We have the map \( \mu_F^{n+1} : K_{n+1}(F) \rightarrow K_1(k) \). Precomposing with the product in \( K \)-theory \( \bigwedge^{n+1} K_1(F) \rightarrow K_{n+1}(F) \), one has a multilinear anti-symmetric form \( \mu_F^{n+1} : \bigwedge^{n+1} F^\times \rightarrow k^\times \) (we also call it \( \mu_F^{n+1} \), by abuse of notation).

In order to write an explicit formula for the Parshin symbol, we introduce the following [5] Appendix A. For every \( 1 \leq i \leq n \), choose a uniformizer \( z_i \) in \( \mathcal{O}_i := O_{X^i,x_i} \) (we fix that choice in what follows). For \( f \in F^\times \), we will define a sequence of integers \( a_1, \ldots, a_n \) as follows. Note that the residue field of \( \mathcal{O}_{i-1} \) can be identified with the fraction field of \( \mathcal{O}_i \). We write \( f = z_1^{a_1} u_1 \), where \( u_1 \) is a unit in \( \mathcal{O}_1 \). Considering the residue class of \( u_1 \) as an element of the fraction field of \( \mathcal{O}_2 \), we proceed to write \( u_1 = z_2^{a_2} u_2 \) where \( u_2 \) is a unit in \( \mathcal{O}_2 \). We continue in this way to construct the sequence \( a_1, \ldots, a_n \). Note that, generally speaking, this sequence depends on the choice of uniformizers \( z_1, \ldots, z_n \).

Let \( f_1, \ldots, f_{n+1} \in F^\times \). Write \( a_{i1}, \ldots, a_{in} \) for the sequence of integers assigned to \( f_i \) as above. Construct the \((n + 1) \times n\) matrix \( A = (a_{ij}) \). Set \( A_i \) to
be the determinant of the $n \times n$ matrix that we get from $A$ by omitting the $i$-th row. Set $A^k_{ij}$ to be the determinant of the $(n-1) \times (n-1)$ matrix that we get from $A$ by deleting the $i$-th and $j$-th rows, and the $k$-th column. Set $B = \sum_k \sum_{i<j} a_{ik}a_{jk}A^k_{ij}$.

Claim 2.8.

$$
\mu^{n+1}_2(f_1, \ldots, f_{n+1}) = N_{k(x_n)/k}\left((-1)^B\left(\prod_{1 \leq i \leq n+1} f_i^{(-1)^{i+1}A_i}(x_n)\right)\right).
$$

By applying the abstract reciprocity law, we recover Parshin reciprocity law [8, Appendix A].

3 Construction of $\mu_C$ and proof of the abstract reciprocity law

3.1 Construction of $\mu_C$

Let us recall the codimension filtration in $K$-theory [9, (10.3.6)]. Write $S^d K(X)$ for the homotopy colimit of the spectra $K(X_{p \text{ on } Z})$, where $Z$ runs over closed subsets of $X$ of codimension $\geq d$. Also, write

$$Q^d K(X) := \bigvee_{p \in |X|^d} K(X_{p \text{ on } p}).$$

Then we have the following homotopy fiber sequence:

$$S^{d+1} K(X) \to S^d K(X) \xrightarrow{\partial_p} Q^d K(X) \xrightarrow{\partial_{d+1}} \Sigma S^{d+1} K(X).$$

Let us define $\Psi^d$ to be the composition:

$$\Psi^d : Q^d K(X) \xrightarrow{\partial_{d+1}} \Sigma S^{d+1} K(X) \xrightarrow{\partial_{d+1}} \Sigma Q^{d+1} K(X).$$

Also, given a collection $C = (C^i)_{0 \leq i \leq n}$, $(C^i \subset |X|^i)$, let us define a map

$$sel_{C^d} : Q^d K(X) \to Q^d K(X),$$

given by projecting on summands corresponding to $p \in C^d$.

Let us define a map

$$I : Q^n K(X) \to K(B);$$

In order to do this, we need to define maps $K(X_{p \text{ on } p}) \to K(B)$, and we do it just by pushing-forward along $X_p \to B$. Technically, as a model for $K(X_{p \text{ on } p})$ we take perfect complexes on $X_p$ which are acyclic outside of the closed point $p$ [9, Section 3], and as a model for $K(B)$ we take cohomologically
bounded complexes of quasi-coherent sheaves with coherent cohomology on $B$

Note that the important thing here is that coherent sheaves on $X_p$ which are supported on $p$ remain coherent after pushing-forward to $B$ (by our assumption that for $p \in |X|^n$, the map $\text{Spec}(X_p) \to \text{Spec}(B)$ is finite).

Finally, we define $\mu_C$ as follows:

$$\mu_C = I \circ \text{sel}_{\mathcal{C}^n} \circ \Psi^{n-1} \circ \ldots \circ \Psi^1 \circ \text{sel}_{\mathcal{C}^1} \circ \Psi^0.$$  

### 3.2 Proof of the reciprocity law

Let us show part (1) of theorem 2.1. First, consider $d \neq n$. Then, we notice that $\text{sel}_{\mathcal{C}(G)} \circ \Psi^d \circ \text{sel}_{\mathcal{C}(G)^{d-1}} = \Psi^d \circ \text{sel}_{\mathcal{C}(G)^{d-1}}$, so that our formula for $\mu_{\mathcal{C}(G)}$ contains $\Psi^d \circ \text{sel}_{\mathcal{C}(G)^{d-1}}$. But $\Psi^d \circ \Psi = 0$, since we are dealing with a composition of two consequent arrows in a long exact sequence.

Let us show the case $d = n$ of part (1). Write $Y = x^{n-1}$. We will deal first with the case $X = Y$, to simplify matters.

Note that $\mu_{\mathcal{C}(G)}$ equals the composition on the top horizontal line of the following commutative diagram:

$$
\begin{array}{c}
Q^n K(X) \\
\downarrow \partial_0 \\
\Sigma S^n K(X)
\end{array}
\quad
\begin{array}{c}
\downarrow i \\
\Sigma S^{n-1} K(X)
\end{array}

\begin{array}{c}
\Sigma Q^n K(X) \\
\downarrow I \\
\Sigma K(B)
\end{array}

\begin{array}{c}
\Sigma S^n K(X) \\
\downarrow \partial_0 \\
\Sigma Q^n K(X)
\end{array}

Here, $i$ is the natural arrow, and $\tilde{I}$ is the arrow induced by push-forward. The crucial moment here is that $X$ is proper; Thus pushing-forward preserves coherence, which in turn enables us to construct the map $\tilde{I}$ on $K$-spectra. Now, noticing that $i \circ \partial_0 = 0$ (as a composition of two consequent arrows in a long exact sequence), finishes the proof.

In general (not assuming $X = Y$), we want to do the same as in the case $X = Y$, but working with $(X on Y)$-versions. To proceed, one considers the commutative diagram:

$$
\begin{array}{c}
Q^n K(X) \\
\downarrow \partial_0 \\
\Sigma Q^n K(X)
\end{array}
\quad
\begin{array}{c}
\downarrow I \\
\Sigma K(B)
\end{array}

\begin{array}{c}
\Sigma Q^n K(X) \\
\downarrow \partial_0 \\
\Sigma Q^n K(X on Y)
\end{array}

\begin{array}{c}
\Sigma Q^n K(X on Y) \\
\downarrow \partial_{n-1}^n \\
\Sigma S^n K(X on Y)
\end{array}

\begin{array}{c}
Q^n K(X on Y) \\
\downarrow \partial_{n-1}^n \\
\Sigma Q^n K(X on Y)
\end{array}

and shows $I^Y \circ \partial_{n-1}^n = 0$ as before.

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1 we assume that $C^0 = \{\gamma\}$.

2 In this formula, as we compose, the target becomes more and more suspended; we do not write the obvious suspensions, by abuse of notation.

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Let us now show part (2) of theorem 2.1. We note that the map $sel_{C(G)}^d$ is the sum of the maps $sel_{C(F)}^d$ (where $F > G$). Thus, the desired follows using claims A.4 and A.5.

4 Calculation of local symbols

In this section, we calculate some symbol maps for local schemes. Using lemma 4.6, these calculations imply the claims of subsection 2.2.

Let us fix the following notations and assumptions for this section. Let $k$ be a field, and let $B = \text{Spec}(k)$. Also, let $A$ be a regular Noetherian local $k$-algebra, and set $X = \text{Spec}(A)$. Denote by $m$ the maximal ideal of $A$, and $k' = A/m$. We assume that $k'$ is finite over $k$. We denote by $F$ the fraction field of $A$.

4.1 The case \( \dim(X) = 1 \)

In this subsection, we additionally assume that $A$ is of Krull dimension 1. Let $v : F^\times \to \mathbb{Z}$ be the valuation, and let $[,] : A \to k'$ be the projection. Finally, choose a uniformizer $z \in A$ (i.e., $v(z) = 1$).

Consider the unique full flag $\mathcal{F} : p_m < p_0$ in $X$. We have the corresponding symbol map

$$
\mu = \mu_\mathcal{F} : K(F) \to \Sigma K(k).
$$

We write $\mu^i$ for the induced map $K_i(F) \to K_{i-1}(k)$.

4.1.1 Degree

Claim 4.1. The morphism $F^\times \cong K_1(F) \xrightarrow{\mu^1} K_0(k) \cong \mathbb{Z}$ is equal to $[k' : k] \cdot v$.

Proof. Since the composition $K_1(A) \to K_1(F) \to K_0(A_{\text{on } m})$ is zero (as part of a long exact sequence), it is enough to prove that

$$
F^\times \cong K_1(F) \to K_0(A_{\text{on } m}) \to K_0(k) \cong \mathbb{Z}
$$

maps $z$ to $[k' : k]$. By lemma B.3, the image of $z$ under the above map is equal to the alternating sum of dimensions (over $k$) of cohomologies of the complex

$$
\begin{array}{ccc}
A & \xrightarrow{z} & A \\
0 & \to & 0
\end{array}
$$

which is $[k' : k]$. \qed

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4.1.2 Tame Symbol

Claim 4.2. The morphism

\[ F^\times \wedge F^\times \cong K_1(F) \wedge K_1(F) \to K_2(F) \overset{\mu^2}{\to} K_1(k) \cong k^\times \]

is given by:

\[ f \wedge g \mapsto N_{k'/k} \left( (-1)^{v(f) v(g)} \begin{bmatrix} f^{v(g)} \\ g^{v(f)} \end{bmatrix} \right) . \]

Proof. We call the above morphism \( F^\times \wedge F^\times \to k^\times \), by abuse of notation, \( \mu^2 \).

By bilinearity and anti-symmetry of \( \mu^2 \), it is enough to verify:

(i) For \( f, g \in A^\times \), \( \mu^2(f \wedge g) = 0 \).

(ii) For \( f \in A^\times \), \( \mu^2(f \wedge z) = N_{k'/k}([f]) \).

(iii) \( \mu^2(z \wedge z) = N_{k'/k}(-1) \).

As for the first item, it follows since the following composition is zero (being a part of the localization long exact sequence):

\[ K_2(A) \to K_2(F) \to K_1(A \text{ on } k') . \]

For the second item, consider the commutative diagram:

\[
\begin{array}{c}
K_1(A) \wedge K_1(F) \\
\downarrow \\
K_1(F) \wedge K_1(F) \\
\downarrow \\
K_2(F) \\
\downarrow \\
K_1(A \text{ on } k') \\
\downarrow \\
K_1(k)
\end{array}
\]

We have the element \( f \wedge z \) in the upper-left group \( K_1(A) \wedge K_1(F) \), and we should walk it through down, and then all the way right. Using commutativity of the diagram, we can chase instead by the upper path, and using lemma [B.3], the result is represented by the automorphism of the following complex:

\[
\begin{array}{c}
A \\
\downarrow f \\
A
\end{array} \quad \begin{array}{c}
z \\
\downarrow f \\
z
\end{array} \quad \begin{array}{c}
A \\
\downarrow -1 \\
A
\end{array} \quad \begin{array}{c}
0
\end{array}
\]

Taking alternating determinant of cohomology, the above automorphism represents the element \( N_{k'/k}([f]) \in k^\times \cong K_1(k) \).

Let us handle the third item on our list. Denote by \( \{\cdot, \cdot\} : K_1(F) \wedge K_1(F) \to K_2(F) \) the multiplication in \( K \)-theory. Recall the Steinberg relation:

\[ \{x, 1-x\} = 0 \]
for $x, 1 - x \in F^\times \cong K_1(F)$. We then calculate:
\[ \{z, z\} = \{z, (1 - z^{-1})^{-1}\} \{z, 1 - z\} \{z, -1\} = \{z^{-1}, 1 - z^{-1}\} \{z, 1 - z\} \{z, -1\} = \{z, -1\} \]
(see [7, Theorem 2.6]). Hence, $\mu^2(z \wedge z) = \mu^2(-1 \wedge z) = N_{k'/k}(-1)$, by (ii) above.

### 4.1.3 Residue

In this subsection, $k$ is assumed to be algebraically closed. We consider a base change of our setup from $k$ to $k_\epsilon := k[e]/(e^3)$. So we have $A_\epsilon, F_\epsilon, X_\epsilon, B_\epsilon$, etc.

**Claim 4.3.** The morphism
\[ F_\epsilon^\times \wedge F_\epsilon^\times \cong K_1(F_\epsilon) \wedge K_1(F_\epsilon) \to K_2(F_\epsilon) \xrightarrow{\mu^2} K_1(k_\epsilon) \cong k_\epsilon^\times \]
sends $(1 + f_\epsilon)(1 + g_\epsilon)$ to $1 - \epsilon^2 \text{Res}(fdg)$.

**Proof.** Given $f \in F$, let us present a useful decomposition of $1 + f_\epsilon$. One has the Laurent decomposition $f = \sum_{-m \leq i \leq n} a_i z^i$ where $f \in A$, and $a_i \in k$. Thus we can write:
\[ 1 + f_\epsilon = \prod_{-m \leq i \leq n} (1 + a_i z^i) (1 + \tilde{f} z^{n+1} \epsilon) (1 + f_1 \epsilon^2), \]

for some $f_1 \in F$. We proceed to write $f_1$ as a Laurent series as well, getting a presentation:
\[ 1 + f_\epsilon = \prod_{-m \leq i \leq n} (1 + a_i z^i) (1 + \tilde{f} z^{n+1} \epsilon) \prod_{-m \leq j \leq n} (1 + b_j z^j \epsilon^2) (1 + \tilde{f}_1 z^{n+1} \epsilon^2), \]

for some $\tilde{f}_1 \in A, b_j \in k$ (here we perhaps enlarge $m, n$).

In order to calculate $\mu^2(1 + f_\epsilon, 1 + g_\epsilon)$ we use the above decomposition for $1 + f_\epsilon, 1 + g_\epsilon$, and we can enlarge $n$ to ensure that $n > 2m$. By the bilinearity of $\mu^2$ it is enough to prove the following:

(i) For $f, g \in A_\epsilon$, we have
\[ \mu^2(1 + f_\epsilon, 1 + g_\epsilon) = 1. \]

(ii) For $f \in A_\epsilon, a \in k_\epsilon, s > 0$ and $r > 2s$ we have
\[ \mu^2(1 + f z^r \epsilon, 1 + az^{-s} \epsilon) = 1. \]

(iii) For $a, b \in k_\epsilon, r, s \geq 0$, we have
\[ \mu^2(1 + az^{-r} \epsilon, 1 + bz^{-s} \epsilon) = 1. \]
(iv) For \( a, b \in k, r, s > 0, r \neq s \), we have
\[
\mu^2(1 + az^r \epsilon, 1 + bz^{-s} \epsilon) = 1.
\]

(v) For \( c, d \in k, r > 0 \), we have
\[
\mu^2(1 + cz^r \epsilon, 1 + dz^{-r} \epsilon) = 1 - rcd \epsilon^2.
\]

Item (i) is clear, since both elements are in \( A^\times \).

Let us show item (iii). First, we use lemma 4.7 and see that it is enough to show the claim in the case when \( A \) is complete. Then \( A \) is isomorphic to \( k[[z]] \).

We now consider the scheme \( Y = P^1_k \), and note that we can identify \( k[[z]] \) with the completion of \( \mathcal{O}_{Y,0} \). Hence, by the same lemma 4.7 it is enough to show the claim in the case when \( A = \mathcal{O}_{Y,0} \). Considering \( z \) (a local parameter in \( \mathcal{O}_{Y,0} \)) as a rational function on \( Y \), we consider the symbols \( \mu^2(1 + az^{-r} \epsilon, 1 + bz^{-s} \epsilon) \) for different points of \( |Y_\epsilon|^1 = |Y|^1 \). At all points except 0, these two elements will be units in the corresponding local rings, hence the symbol equals 1. Hence, by the reciprocity law (i.e., the product of the symbols over all points is 1), we get that also the symbol at the point 0 is equal to 1, as wanted.

Let us show item (v). Note that:
\[
\mu^2(1 + cz^r \epsilon, 1 + dz^{-r} \epsilon) = \mu^2(1 + cz^r \epsilon, z^r + de) \mu^2(1 + cz^r \epsilon, z^r),
\]
and hence it is enough to calculate \( \mu^2(1 + cz^r \epsilon, z^r + de) \). By lemmas B.3 and B.4, we should calculate the determinant of multiplication by \( 1 + cz^r \epsilon \) on the cohomology of:

\[
\begin{array}{c|cc}
A & z^r + de & A \\
\hline
-1 & 0
\end{array}
\]

The only non-zero cohomology is the 0-th one. It is a free \( k \)-module (with basis 1, \( z, \ldots, z^{r-1} \)). Multiplication by \( 1 + cz^r \epsilon \) is just multiplication by \( 1 - cde \), thus we are done.

The rest of the items are verified similarly to item (v).

\[\square\]

4.2 The case \( \dim(X) > 1 \)

In this subsection, we drop the assumption that \( A \) is 1-dimensional; We denote the Krull dimension of \( A \) by \( n \).

4.2.1 Parshin symbol

Fix a full flag \( \mathcal{F} : x_n < \ldots < x_0 \) in \( X \), corresponding to a chain of prime ideals \( 0 = p_0 \subset \ldots \subset p_n = m \). Consider \( X^\times := \overline{\mathcal{F}} \) as an integral closed subscheme of \( X \). We obtain a symbol:
\[ \mu = \mu_F : K(F) \to \Sigma^n K(k). \]

As in 2.2.2.1 consider the resulting map \( \mu^{n+1}_F : \bigwedge^{n+1} F^\times \to k^\times \). In 2.2.2.1 we essentially wrote a formula for this map (which we now want to verify), under the assumption that our flag is regular. In order to compute this map "recursively", we will use Quillen’s devissage (lemma B.5) - application of which will be possible due to regularity of \( F \).

Claim 4.4. The symbol \( \mu_F : K(F_X) \to \Sigma^n K(k) \) equals to the following composition:

\[
\begin{align*}
K(F_X) & \longrightarrow \Sigma K(X_x \text{ on } x_1) \xleftarrow{\sim} \Sigma K(F_{X_1}) \longrightarrow \Sigma^2 K(X_{x_2} \text{ on } x_2) \\
K(F_{X_2}) & \longrightarrow \ldots \longrightarrow \Sigma^n K(F_{X_n}) \longrightarrow \Sigma^n K(k),
\end{align*}
\]

where the arrows \( \sim \) stand for Quillen’s devissage.

In view of this claim, \( \mu^{n+1}_F \) equals to the composition

\[
\bigwedge^{n+1} F^\times \to K_{n+1}(F_X) \xrightarrow{\partial_0} K_n(F_{X_1}) \xrightarrow{\partial_1} \ldots \xrightarrow{\partial_{n-1}} K_1(F_{X_n}) \to K_1(k)
\]

where \( \partial_i \) is the composition of the boundary map and the inverse of the devissage.

The following lemma, will allow us, in principle, to calculate \( \mu^{n+1}_F(f_1, \ldots, f_{n+1}) \) for any \( f_1, \ldots, f_{n+1} \in F^\times \).

Lemma 4.5. Let \( R \) be a 1-dim. regular local Noetherian ring with maximal ideal \( n \), residue field \( \ell \) and fraction field \( L \). Let \( z \in R \) a uniformizer. Consider the map

\[ K(L) \to \Sigma K(R \text{ on } n) \xleftarrow{\sim} \Sigma K(\ell) \]

- composition of boundary with devissage. Using it we construct a map

\[ \nu^m : \bigwedge^m L^\times \to K_m(L) \to K_{m-1}(\ell). \]

The following hold:

(i) \( \nu^m(f_1, \ldots, f_m) = 0 \) for \( f_1, \ldots, f_m \in R^\times \).

(ii) \( \nu^m(f_1, \ldots, f_{m-2}, z, z) = \nu^m(f_1, \ldots, f_{m-2}, -1, z) \) for \( f_1, \ldots, f_{m-2} \in R^\times \).

(iii) \( \nu^m(f_1, \ldots, f_{m-1}, z) = [f_1] \wedge \ldots \wedge [f_{m-1}] \) for \( f_1, \ldots, f_{m-1} \in R^\times \) (we recall that \( [f] \) denotes the residue in \( \ell^\times \) of \( f \in R^\times \), considered as an element of \( K_1(\ell) \) in our case).
Proof. The first item is clear, since \( \nu^m(f_1, \ldots, f_m) \) is the value of the composition \( K_m(R) \to K_m(L) \to K_{m-1}(R \text{ on } n) \) on \( f_1 \land \ldots \land f_m \in K_m(R) \), and the composition is zero as part of a long exact sequence.

The second item follows from Steinberg relation (like in the proof of claim 4.2).

The third item follows from the commutativity of the following diagram:

\[
\begin{array}{ccc}
K_{m-1}(R) \land K_1(L) & \longrightarrow & K_{m-1}(R) \land K_0(R \text{ on } n) \\
\downarrow & & \downarrow \\
K_m(L) & \longrightarrow & K_{m-1}(R \text{ on } n)
\end{array}
\]

Here the left square commutes as the boundary morphism is a morphism of \( K(A) \)-modules, while the right square commutes as Quillen’s devissage morphism is a morphism of \( K(A) \)-modules.

Note that our element is the result of going right on the lower line, applied to \( f_1 \land \ldots \land f_{m-1} \land z \). However, this element comes from an element at the upper-left corner, which we can chase though the right on the upper line, and then to the lower-left corner through the right line.

\[ \square \]

4.3 Auxiliary lemmas

We state two lemmas which are used above, and whose proof is straightforward.

**Lemma 4.6.** Let \( X \to B \) be as in 2.1. Let 
\[
\mathcal{F} : x_n < x_{n-1} < \ldots < x_0 = \gamma
\]
be a full flag of points in \( |X| \). Writing \( p := x_n \), we consider also the setting \( X_p \to B \) and the obvious flag \( \mathcal{F}_p \) on \( X_p \) induced by \( \mathcal{F} \). We have two symbol maps;

\[
\mu_{\mathcal{F}} : K(F) \to \Sigma^n K(k)
\]

and

\[
\mu_{\mathcal{F}_p} : K(F) \to \Sigma^n K(k)
\]

(note that the function field of \( X_p \) is identified with \( F \)). These two symbol maps are equal.

**Lemma 4.7.** Let \( A \) be a 1-dim. regular local Noetherian \( k \)-algebra whose residue field is finite over \( k \), and let \( \hat{A} \) be its completion. Write, as usual, \( X = \text{Spec}(A) \) and \( B = \text{Spec}(k) \), and write \( \hat{X} = \text{Spec}(\hat{A}) \). Also, denote by \( F \) (resp. \( \hat{F} \)) the
fraction field of \( A \) (resp. \( \hat{A} \)). Associated to the unique full flag in \( X \) (resp. \( \hat{X} \)) we have the symbol \( K(F) \to \Sigma K(k) \) (resp. \( K(\hat{F}) \to \Sigma K(k) \)). Then the diagram

\[
\begin{array}{ccc}
K(F) & \longrightarrow & K(\hat{F}) \\
\downarrow & & \downarrow \\
\Sigma K(k) & & \Sigma K(k)
\end{array}
\]

commutes.

A Infinite sums of maps of spectra

We recall that a compact spectrum is a spectrum, maps from which commute with small direct sums. An example of a compact spectrum is \( \Sigma^k S \), a suspension of the sphere spectrum.

**Definition A.1.** Let \( f_i : \mathcal{S} \to \mathcal{T} \) \((i \in I)\) be a family of maps of spectra, and \( f : \mathcal{S} \to \mathcal{T} \) an additional map. We say that \( f \) is the sum of the \( f_i \) (written \( f = \sum_{i \in I} f_i \)) if for every compact spectrum \( C \), and every element \( e \in [C, S] \), almost all (i.e., all except of finitely many) of the maps \( f_i \circ e \) are equal to zero, and the sum of all these \( f_i \circ e \) is equal to \( f \circ e \).

We note that we don’t claim some uniqueness of the sum. In reality, this notion of ”summability and summation on probes” is derived from a more holistic notion:

**Definition A.2.** Let \( f_i : \mathcal{S} \to \mathcal{T} \) \((i \in I)\) be a family of maps of spectra, and \( f : \mathcal{S} \to \mathcal{T} \) an additional map. An evidence for \( f \) being the sum of the \( f_i \) is a map:

\[
g : \mathcal{S} \to \bigvee_{i \in I} \mathcal{T},
\]

so that when we compose \( g \) with the projection to the \( i \)-th summand we get \( f_i \), while when we compose \( g \) with the fold map, we get \( f \).

The following is evident:

**Claim A.3.** If we have an evidence for \( f \) being the sum of the \( f_i \), then \( f \) is the sum of the \( f_i \).

Let us also note the following two auxiliary claims (whose proof is straightforward):

**Claim A.4.** Let \( h : \mathcal{U} \to \mathcal{S}, g : \mathcal{T} \to \mathcal{V} \). If \( f \) is the sum of the \( f_i \) (we have evidence for \( f \) being the sum of the \( f_i \)) , then \( g \circ f \circ h \) is the sum of the \( g \circ f_i \circ h \) (we have evidence for \( g \circ f \circ h \) being the sum of the \( g \circ f_i \circ h \)).
Claim A.5. Let $S_i \ (i \in I)$ be a collection of spectra, and write $S = \bigvee_{i \in I} S_i$. Then we have evidence for $id$ being the sum of $pr_i \ (i \in I)$, where $id$ is the identity morphism of $S$, while $pr_i$ is the morphism of projection on the $i$-th summand. In particular, $id = \sum_{i \in I} pr_i$.

B $K$-theory calculation lemmas

We state some simple lemmas, which will be of use when calculating the concrete symbols. In what follows, $X$ is a Noetherian scheme, $U \subset X$ an open subscheme, and $Z$ the closed complement.

We denote by $\text{SP erf}(X)$ the category of (strictly) bounded complexes of $O_X$-modules, whose terms are locally free of finite rank. By $\text{SP erf}(X \text{ on } Z)$ we denote the full subcategory of $\text{SP erf}(X)$ consisting of complexes whose cohomologies are supported on $Z$.

Fact B.1. There is a natural map from (the geometric realization of) the core groupoid of $\text{SP erf}(X)$ to $K(X)$. In particular, every object in $\text{SP erf}(X)$ defines a point in $K(X)$. In addition, the automorphism group of any object of $\text{SP erf}(X)$ maps into $K_1(X)$; Especially, since $O(X)^\times$ maps into the automorphism group of the object $O_X \in \text{SP erf}(X)$, one has a map $O(X)^\times \to K_1(X)$. Thus, given an object or an automorphism in $\text{SP erf}(X)$, one can view it as an element of an appropriate $K$-group $K_i(X)$. We will abuse this without further notice.

Claim B.2. Let $X$ be local (i.e., the spectrum of a local ring). Then the above map $O(X)^\times \to K_1(X)$ is an isomorphism.

Lemma B.3. Let $f \in O(X)$ be such that $f|_U$ is invertible. Then the image of $f|_U \in O(U)^\times$ under the map $K_1(U) \to K_0(X \text{ on } Z)$ which is obtained from the localization sequence [9, Theorem 7.4]:

$$K(X \text{ on } Z) \to K(X) \to K(U)$$

is given by the complex:

$$
\begin{array}{ccc}
O_X & \xrightarrow{f} & O_X \\
-1 & & 0 \\
\end{array}
$$

Lemma B.4. Let $f \in O(X)^\times$, and $C \in \text{SP erf}(X \text{ on } Z)$. Then the image of $f \wedge C$ under the product map $K_1(X) \wedge K_0(X \text{ on } Z) \to K_1(X \text{ on } Z)$ is given by the automorphism $C \otimes O_X \xrightarrow{1 \otimes f} C \otimes O_X$.

Lemma B.5 (Quillen’s devissage). Suppose that $X$ and $Z$ are regular. Then the morphism $K(Z) \to K(X \text{ on } Z)$ (induced by push-forward) is an equivalence of spectra.
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