On the stability of call/put option’s prices in incomplete models under statistical estimations.

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Abstract

In exponential semi-martingale setting for risky asset we estimate the difference of prices of options when initial physical measure \( P \) and corresponding martingale measure \( Q \) change to \( \tilde{P} \) and \( \tilde{Q} \) respectively. Then, we estimate \( L_1 \)-distance of option’s prices for corresponding parametric models with known and estimated parameters. The results are applied to exponential Levy models with special choice of martingale measure as Esscher measure, minimal entropy measure and \( f^q \)-minimal martingale measure. We illustrate our results by considering GMY and CGMY models.

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1 Introduction

We consider the following semi-martingale model of risky asset \( S = (S_t)_{t \geq 0} \):

\[
S_t = S_0 \exp(X_t)
\]

where \( X = (X_t)_{t \geq 0} \) is a semi-martingale. Usually the law of this semi-martingale depend on unknown parameter, say \( \theta \in \Theta \), where \( \Theta \) is some space. For exemple, in Black-Scholes model we have:

\[
X_t = (\mu - \sigma^2/2)t + \sigma W_t
\]

where \( W = (W_t)_{t \geq 0} \) is a standard Wiener process, the parameter \( \theta = (\mu, \sigma) \) and \( \Theta = \mathbb{R} \times \mathbb{R}^+ \). In Geometric Variance Gamma model (cf. [3],[9]), as well known,

\[
X_t = \mu \tau_t + \sigma W_{\tau_t}
\]

where \( \mu \in \mathbb{R}, \sigma > 0, W = (W_t)_{t \geq 0} \) is again Wiener process and \( (\tau_t)_{t \geq 0} \) is, independent from \( W \), Gamma process with parameters \((1, \nu), \nu > 0 \). In this case \( \theta = (\mu, \sigma, \nu) \) and \( \Theta = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \).

In GMY model, as well known ( cf.[3],[9]) the process \( X \) has the same structure as in (1) but with \((\tau_t)_{t \geq 0}\) being Levy process with Levy measure

\[
\nu(dx) = \frac{C \exp(-Nx) \mathbb{1}_{\{x>0\}}}{x^{1+\alpha}} dx
\]

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where $\alpha < 2$, $C > 0$ and $N \geq 0$. Then, obviously, $\theta = (C,N,\alpha)$ and $\Theta = \mathbb{R}^{+,*} \times \mathbb{R}^{+,*} \times -\infty,2[. $

In CGMY model (cf. [9,2]) the process $X$ is simply a Levy process with the Levy measure

$$

\nu(dx) = \frac{C \exp(-Nx)\mathbb{I}_{\{x>0\}} + C \exp(-Mx)\mathbb{I}_{\{x<0\}}}{x^{1+\alpha}} dx

$$

(3)

where $C, M, N$ are positive constants and $\alpha < 2$. Then we have $\theta = (C,M,N,\alpha)$.

We will also mention Hyperbolic Levy process $X^\theta = (X^\theta_t)_{t \geq 0}$ which is often used in modelisation because of its flexibility to fit the form of one-dimensional distributions of log of returns (cf. [6], [25]). As well known, there exist several parametrisations of Hyperbolic Levy processes. Under one of them, say $\theta = (\alpha, \beta, \delta, \mu)$, the one dimensionnal densities of $X^\theta_1$ with respect to Lebesgue measure are given by:

$$
f(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta \sqrt{\alpha^2 - \beta^2})} \exp(-\alpha \sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu))
$$

(4)

where $\alpha > 0$, $0 \leq |\beta| < \alpha$, $\delta > 0$, $\mu \in \mathbb{R}$ and $K_1(\cdot)$ is a Bessel function of the third type of index 1. We know (cf. [25]) that Levy measure of this process is equal to:

$$

\nu(dx) = \frac{\exp(\beta x)}{|x|} \left( \exp^{-\alpha|x|} + \int_0^{+\infty} \frac{\exp(-\sqrt{2y + \alpha^2|x|})}{\pi^2y(J^2_1(\delta \sqrt{2y}) + Y^2_1(\delta \sqrt{2y}))} dy \right)

$$

(5)

where $J_1(\cdot)$ and $Y_1(\cdot)$ are Bessel functions of the first and second type of index 1.

The classical procedure of calculus of call/put option price $C_T$ of maturity time $T$ consists to take payoff function given by a continuous in the space $D([0,T])$ functional $g(\cdot)$, then to select in the set of equivalent martingale measures $\mathcal{M}(P)$, supposed non-empty, a ”good” one, say $Q$, and to put:

$$

C_T = \mathbb{E}_Q(g(S)).

$$

As we know, there exist many approaches to choose a ”good” martingale measure: it can be done using the minimisation of the risk in $L^2$-sense( see [7,24]), using the minimisation of Hellinger integrals (see [1,10]), it can be based on the minimisation of entropy (see [22,23,3]), one can take minimal $f^\theta$-martingale measures (see [14]) or use Esscher measures (see [16,22]) e.t.c.

We remark that since the law of $X^\theta$ depends on $\theta$, the price $C_T$ does it as well. To adjust the ”good” value of $\theta$ one perform then so called calibration which is equivalent, from statistical point of view, to find a minimal distance estimator or contrast estimator with very special contrast. About the properties of these estimators see for instance [1,20,18,26] and references there. One can use also another approach and consider maximum likelihood estimators or Bayesian estimators for the unknown parameters. The properties of these estimators were studied, for example, in [12,18], the conditions for weak convergence of these processes in terms of Hellinger processes can be found in [29,30,13]. When the density of the law of $X$ with respect to some majomeasuring measure cannot be expressed explicitly or when it is too complicated, one can use moment estimators ( see [12]). In practice often the combination of some statistical estimations and some calibration procedure also is used.

Let $\hat{\theta}$ be an estimator of unknown parameter $\theta$. Then, we replace $\theta$ in formulas for $C_T(\theta)$ by its estimator $\hat{\theta}$ and it becomes $C_T(\hat{\theta})$. So, it is important from point of view of stability of the procesure to measure the distance between estimated $C_T(\hat{\theta})$ and ”true” price $C_T(\theta)$. In this

2
paper we are interested to evaluate $L^1$ distance between these quantities, namely $E^\theta | \mathcal{C}_T(\hat{\theta}) - \mathcal{C}_T(\theta) |$ where the expectation is taken with respect to ”physical” measure $P_\theta$. We remark that
in the same manner one can obtain the estimation of $E^\theta [d(\mathcal{C}_T(\hat{\theta}), \mathcal{C}_T(\theta))]$ with different possible choice of the distance $d$. We notice the importance of use of consistent estimators of $\theta$ in this procesure. In fact, usually $\mathcal{C}_T(\theta) \neq \mathcal{C}_T(\theta')$ for $\theta \neq \theta'$. If the sequence of estimators is not consistent, then under some mild conditions one can extract a subsequence $(\hat{\theta}^n)$ converging $P$-a.s. to $\theta + \delta$ with $\delta \neq 0$. Then $E^\theta | \mathcal{C}_T(\hat{\theta}^n) - \mathcal{C}_T(\theta) |$ will converge to $| \mathcal{C}_T(\theta + \delta) - \mathcal{C}_T(\theta) |$ which is different from zero. It means that without arbitrage for initial model we can have asymptotic arbitrage consequences due to estimation procedure if $\mathcal{C}_T(\theta + \delta) \neq \mathcal{C}_T(\theta)$.

In this paper we consider only payoff functions $g$ verifying the condition (8). But similar results can be obtained in more general cases. The paper is organized in the following way. In §2 we give the results for binary model, i.e. for the parametric models with two values of parameter. The main result is presented in Theorem 1. In Corollary 1 the case of the processes with independent increments is considered. Then, in section 3 we give the results for general parametric model. The main results are presented in Theorem 2 and Corollaries 2,3. Finally, we apply the results for Levy processes, and we consider different possibilities to choose a martingale measure, namely as Esscher measure, Minimal entropy martingale measure and $f^\nu$- minimal martingale measure. It is shown that under conditions of Theorem 1 we obtain the estimation of the type (23). Then, the results are applied to Geometric Variance Gamma and CGMY models.

2 Results for binary statistical model

We suppose that we are given with a filtered canonical space of cadlag functions $(\Omega, \mathcal{F}, \mathbb{F})$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the right-continuous filtration such that $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $P$ and $\bar{P}$ be two equivalent probability measures on $(\Omega, \mathcal{F})$ and we denote by $P_t$ and $\bar{P}_t$ the restrictions of these measures on the $\sigma$-algebra $\mathcal{F}_t$, $t \geq 0$. In this setting the measures $P$ and $\bar{P}$ correspond to the laws of our semimartingale $X = (X_t)_{t \geq 0}$ under two fixed values of parameter. We suppose that $X$ has predictable representation property with respect to $P$ and the caracteristics of $X$ are $(B, C, \nu)$ and $(\bar{B}, \bar{C}, \bar{\nu})$ respectively. We remark that since the measures $P$ and $\bar{P}$ are equivalent, $C = \bar{C}$ (P-a.s.) and we have the representation property with respect to $\bar{P}$. For more details about caracteristics see [13].

We suppose that there are only two assets. For simplicity we assume that the interest rate $r$ of the bond $B = (B_t)_{t \geq 0}$ is equal to zero, i.e. $B_t = 1$, and that the risky asset $S = (S_t)_{t \geq 0}$ is given by:

$S_t = S_0 \exp(X_t)$

with $S_0 = 1$. To avoid technical difficulties we suppose that the processes $\mathbb{I}_{\{x>1\}} \exp(x) * \nu$ and $\mathbb{I}_{\{x>1\}} \exp(x) * \bar{\nu}$ have bounded variation on finite intervals. This supposition implies that $S = S_0 \exp(X)$ is a special semimartingale under $P$ and $\bar{P}$.

As usual we denote by $||P - \bar{P}||$ the variation distance between the measures $P$ and $\bar{P}$, i.e.

$||P - \bar{P}|| = 2 \sup_{A \in \mathcal{F}} |P(A) - \bar{P}(A)|$

We recall that

$||P - \bar{P}|| = \mathbb{E}_P |1 - \frac{d\bar{P}}{dP}|$.
Let $\mathcal{M}(P)$ and $\mathcal{M}(\tilde{P})$ be the sets of equivalent martingale measures which are supposed to be non-empty. Let $g$ be measurable functional in $D([0, T])$. We choose, then, using some procedure, two martingale measures: $Q$ and $\tilde{Q}$ to calculate call/put option prices: $C_T$ and $\tilde{C}_T$ of maturity time $T$:

$$C_T = \mathbb{E}_Q[g(S)], \quad \tilde{C}_T = \mathbb{E}_{\tilde{Q}}[g(S)].$$

We introduce also dual measures $Q'$ and $\tilde{Q}'$ (cf. [5]) by:

$$\frac{dQ'}{dQ} = S_T, \quad \frac{d\tilde{Q}'}{d\tilde{Q}} = S_T. \quad (6)$$

We notice that since $S$ is a martingale with respect to martingale measure $Q$, $S' = 1/S$ is also martingale but with respect to $Q'$. The same is true for $\tilde{S}' = 1/\tilde{S}$ with respect to $\tilde{Q}'$.

So, the measures involved in calculation can be represented by the following diagrams containing initial measure, martingale measure and dual measure:

$$P \to Q \to Q' \quad \text{and} \quad \tilde{P} \to \tilde{Q} \to \tilde{Q}' \quad (7)$$

**Lemma 2.1.** Let $g$ be measurable functional in $D([0, T])$ verifying:

$$|g(x)| \leq c|x_T| + d \quad (8)$$

where $c, d$ are positif constants. Then for call/put option’s price corresponding to $g$ we have:

$$|C_T - \tilde{C}_T| \leq c||Q'_T - \tilde{Q}'_T|| + d||Q_T - \tilde{Q}_T||$$

where $|| \cdot ||$ is a variation distance between the restriction of the corresponding measures on $\sigma$-algebra $\mathcal{F}_T$.

**Proof.**

We have:

$$|C_T - \tilde{C}_T| = |\mathbb{E}_Q[g(S)] - \mathbb{E}_{\tilde{Q}}[g(S)]| \leq \mathbb{E}_Q[(cS_T + d)|1 - \frac{d\tilde{Q}_T}{dQ_T}|]$$

But using (6) we obtain:

$$\mathbb{E}_Q(S_T|1 - \frac{d\tilde{Q}_T}{dQ_T}|) = ||Q'_T - \tilde{Q}'_T||$$

and by definition

$$\mathbb{E}_Q|1 - \frac{d\tilde{Q}_T}{dQ_T}| = ||Q_T - \tilde{Q}_T||$$

It is known (see [25], [27], [13]) that the behaviour of variation distance is closely related to the Hellinger distance and Hellinger processes. Let $h_{\alpha}(\frac{1}{2}, Q, \tilde{Q}) = (h_t(\frac{1}{2}, Q, \tilde{Q}))_{t \geq 0}$ be the Hellinger process of order $1/2$ for the measures $Q$ and $\tilde{Q}$.
Lemma 2.2. We have the following estimation for the variation distance via Hellinger processes: for $\epsilon > 0$:

$$||Q_T - \tilde{Q}_T|| \leq 4 [E_{Q} h_T(\frac{1}{2}, Q, \tilde{Q}))]^{1/2}$$  \tag{9}$$

$$||Q_T - \tilde{Q}_T|| \leq 3\sqrt{2}\epsilon + 2Q(h_T(\frac{1}{2}, Q, \tilde{Q})) \geq \epsilon)$$  \tag{10}$$

Proof See [13] p. 279.

To obtain the expressions for Hellinger processes we need the results on characteristics of the process $X$ with respect to mentioned above measures. First of all we remark that since the measure $Q$ is absolutely continuous with respect to $P$, $X$ is a semi-martingale with respect to this measure and Girsanov theorem permit us to find the characteristics of $X$ under $Q$ (see [13], p. 159):

$$\begin{align*}
B^Q &= B + \beta^Q \cdot C + l \cdot (Y^Q - 1) \star \nu \\
C^Q &= C \\
\nu^Q &= Y^Q \cdot \nu
\end{align*}$$

where $l(\cdot)$ is a truncation function and $\beta^Q$ and $Y^Q$ are predictable functions verifying the following integrability condition: for all $t \geq 0$ and $P$-a.s.

$$((\beta^Q)^2 \cdot C)_t + (l \cdot (Y^Q - 1) \star \nu)_t < \infty.$$  \tag{11}$$

Here and further $\cdot$ denotes a Lebesgue-Stieltjes integral and $\star$ means the integration with respect to a random measure (for the details see [13]). In the mentioned above situation we say that $(\beta^Q, Y^Q)$ are Girsanov parameters to pass from $P$ to $Q$.

The measures $Q'$ and $\tilde{Q}'$ are also absolutely continuous with respect to $P$. In the following lemma we give predictable characteristics of $X$ with respect to the measures $Q'$, $\tilde{Q}$ and $\tilde{Q}'$ via the characteristics of the measure $P$.

Lemma 2.3. a) The predictable characteristics of $X$ with respect to the measure $Q'$ via $P$ are given by:

$$\begin{align*}
B' &= B + (1 + \beta^Q) \cdot C + l \cdot (e^x Y^Q - 1) \star \nu \\
C' &= C \\
\nu' &= e^x Y^Q \cdot \nu
\end{align*}$$

where $l(\cdot)$ is a truncation function and $(\beta^Q, Y^Q)$ are Girsanov parameters to pass from $P$ to $Q$.

b) The predictable characteristics of $X$ with respect to the measure $\tilde{Q}$ via $P$ are given by:

$$\begin{align*}
B^\tilde{Q} &= B + (\beta + \beta^\tilde{Q}) \cdot C + l \cdot (Y^\tilde{Q} Y - 1) \star \nu \\
C^\tilde{Q} &= C \\
\nu^\tilde{Q} &= Y^\tilde{Q} Y \cdot \nu
\end{align*}$$

where $(\beta^\tilde{Q}, Y^\tilde{Q})$ and $(\beta, Y)$ are Girsanov parameters which permit us to pass from $\tilde{P}$ to $\tilde{Q}$ and from $P$ to $\tilde{P}$ respectively.

c) The predictable characteristics of $X$ with respect to the measure $\tilde{Q}'$ via $P$ are given by:

$$\begin{align*}
B^\tilde{Q}' &= B + (1 + \beta + \beta^\tilde{Q}) \cdot C + l \cdot (e^x Y^\tilde{Q} Y - 1) \star \nu \\
C^\tilde{Q}' &= C \\
\nu^\tilde{Q}' &= e^x Y^\tilde{Q} Y \cdot \nu
\end{align*}$$
Proof. To prove this Lemma we use (7). We denote by \( Z = (Z_t)_{t \geq 0}, \tilde{Z} = (\tilde{Z}_t)_{t \geq 0}, Z' = (Z'_t)_{t \geq 0}, \tilde{Z}' = (\tilde{Z}'_t)_{t \geq 0}, \) the processes such that for \( t \geq 0 \) and \( P \) - a.s.

\[
Z_t = \frac{dQ_t}{dP_t}, \quad \tilde{Z}_t = \frac{d\tilde{Q}_t}{dP_t}, \quad Z'_t = \frac{dQ'_t}{dP_t}, \quad \tilde{Z}'_t = \frac{d\tilde{Q}'_t}{dP_t},
\]

and \( Q_t, \tilde{Q}_t, Q'_t, \tilde{Q}'_t \) stand for the restrictions of the corresponding measures to the \( \sigma \)-algebra \( \mathcal{F}_t \). To prove a) we note that for all \( t \geq 0 \) we have:

\[
Z'_t = \int_0^t e^{\Delta X_s} Z_s d\tilde{X}_s + \int_0^t e^{\Delta X_s} dZ'_s.
\]

According to Girsanov theorem (see [13], p. 160) the Girsanov parameters \( (\beta^Q, Y^Q) \) are given by: for \( t \geq 0 \)

\[
\beta^Q_t = \frac{1}{Z'_t} \frac{d(Z'_c, X^c)_t}{dC_t} \tag{12}
\]

where \( Z'_c \) and \( X^c \) denote continuous martingale part of the corresponding processes. Using Itô formula for the function \( f(x, y) = e^x y \) we find that

\[
Z'_c = \int_0^t e^{X_{s-}} Z_s dX^c_s + \int_0^t e^{X_{s-}} dZ^c_s.
\]

Using the same formula as (12) for \( \beta^Q \) we obtain \( (P\text{-a.s.}) \) that \( \beta^Q_t = \beta^Q_t + 1 \).

Again according to Girsanov theorem

\[
Y^Q = M^P_\mu \left( \frac{Z'}{Z'_-} \mid \tilde{P} \right) \tag{13}
\]

where \( \tilde{P} = \mathcal{P} \times \mathcal{B}(\mathbb{R}^*) \) is \( \sigma \)-algebra of predictable sets in \( \tilde{\Omega} = \Omega \times [0, T] \times \mathbb{R}^* \) and for measurable non-negative functions \( W(\omega, t, x) \) on \( \tilde{\Omega} \)

\[
M^P_\mu(W)_T = \mathbb{E}_P[(W \ast \mu)_T]
\]

with \( \mathbb{E}_P \) being the expectation with respect to \( P \). Then

\[
M^P_\mu \left( \frac{Z'}{Z'_-} \mid \tilde{P} \right) = M^P_\mu \left( e^{\Delta X \frac{Z}{Z_-}} \mid \tilde{P} \right)
\]

and, since the function \( e^{\Delta X} \) is \( \tilde{P} \)-measurable, we obtain that the right-hand side of the previous equality is equal \( (P\text{-a.s.}) \) to:

\[
e^x M^P_\mu \left( \frac{Z}{Z_-} \mid \tilde{P} \right)
\]

and we have a).

For b), c) we first write the characteristics of \( X \) with respect to \( \tilde{P} \) via \( P \):

\[
\begin{align*}
\tilde{B} &= B + \beta \ast C + l \ast (Y - 1) \ast \nu \\
\tilde{C} &= C \\
\tilde{\nu} &= Y \ast \nu
\end{align*}
\]

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\tilde{\nu} &= Y \ast \nu
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\[
\begin{align*}
\tilde{B} &= B + \beta \ast C + l \ast (Y - 1) \ast \nu \\
\tilde{C} &= C \\
\tilde{\nu} &= Y \ast \nu
\end{align*}
\]
Now we take the Girsanov parameters \((\beta \tilde{Q}, Y \tilde{Q})\) to pass from \(\bar{P}\) to \(\tilde{Q}\):

\[
\begin{aligned}
B^{\tilde{Q}} &= \bar{B} + \beta^{\tilde{Q}} \cdot C + l \cdot (Y^{\tilde{Q}} - 1) \star \tilde{\nu} \\
C^{\tilde{Q}} &= C \\
\nu^{\tilde{Q}} &= Y^{\tilde{Q}} \cdot \tilde{\nu}
\end{aligned}
\]

Putting together these two decompositions we obtain b). Then, using the same procedure as in the proof of a), we obtain c).

Now we give the expressions for Hellinger processes. To avoid technical difficulties we suppose that \(X\) has no fixed points of discontinuity and that for \(\nu\) we have a desintegration formula. In fact, these suppositions are not too restrictive. In fact, from one hand, atom’s part can be also estimated, and, from another hand, a desintegration formula with respect to some predictable increasing process always exists (see [13], p. 77). We introduce the following integrability condition:

\[
\int_{\mathbb{R}^*} |e^x - 1|(d\nu^\tilde{Q} + d\nu^{\tilde{Q}}) < \infty. \tag{14}
\]

**Lemma 2.4.** Let \(X\) be a process without fixed points of discontinuity with respect to \(P\). We assume that there exists a kernel \(K(dx,t)\) such that we have a desintegration formula:

\[
d\nu = K(dx,t)dC_t \tag{15}
\]

where \(C\) is predictable variation of continuous martingale part of \(X\) if it is not zero, and some increasing predictable process if not. We suppose that (14) holds. Then the Hellinger processes of order 1/2 of the measures \(P\) and \(\bar{P}\), \(Q\) and \(\tilde{Q}\), \(Q'\) and \(\tilde{Q}'\) are given respectively by:

\[
\begin{align*}
\hat{h}(\frac{1}{2},P,\bar{P}) &= \frac{1}{8}(\beta^2) \cdot C + \frac{1}{2} \left( 1 - \sqrt{Y} \right)^2 \star \nu, \\
\hat{h}(\frac{1}{2},Q,\tilde{Q}) &= \frac{1}{8}(\beta^Q - \beta^{\tilde{Q}} - \beta)^2 \cdot C + \frac{1}{2} \left( \sqrt{Y^Q} - \sqrt{Y^{\tilde{Q}}} \cdot Y \right)^2 \star \nu, \\
\hat{h}(\frac{1}{2},Q',\tilde{Q}') &= \frac{1}{8}(\beta^Q - \beta^{\tilde{Q}} - \beta)^2 \cdot C + \frac{\exp(x)}{2} \left( \sqrt{Y^Q} - \sqrt{Y^{\tilde{Q}}} \cdot Y \right)^2 \star \nu.
\end{align*}
\]

In addition we have \((P \times \lambda_C)\)-a.s.

\[
\beta^Q - \beta^{\tilde{Q}} - \beta = (\exp(x) - 1) \left( Y^{\tilde{Q}} \cdot Y - Y^Q \right) \star K(dx,\cdot)
\]

where \(\lambda_C\) is a positive measure with the distribution function \(C\).

**Proof.** To obtain the expressions for the Hellinger processes we take in account that the compensator of \(X\) has no atoms, we use the characteristics given in Lemma 2 and the formula in [13], p. 221 (see also [19] for \(X\) being the processes with independent increments). Since \(e^X\) is a martingale with respect to the measures \(Q\) and \(\tilde{Q}\) we can write again using Ito formula and Girsanov theorem that (cf.[13]), p.556): \(P\)-a.s.

\[
\begin{align*}
B^Q + \frac{1}{2} C^Q + (e^x - 1 - l(x)) \star \nu^Q &= 0, \\
B^{\tilde{Q}} + \frac{1}{2} C^{\tilde{Q}} + (e^x - 1 - l(x)) \star \nu^{\tilde{Q}} &= 0.
\end{align*}
\]
Then $P$-a.s.
\[ B^Q - B^\tilde{Q} + (e^x - 1 - l(x))(Y^Q - Y^\tilde{Q} \cdot Y) \star \nu = 0. \]
Taking into account (15) and that $P$-a.s.
\[ B^Q - B^\tilde{Q} = (\beta^Q - \beta^\tilde{Q} - \beta) \cdot C + l \cdot (Y^Q - 1) \star \nu - l \cdot (Y^\tilde{Q} \cdot Y - 1) \star \nu \]
we obtain that
\[ (\beta^Q - \beta^\tilde{Q} - \beta) \cdot C + (e^x - 1) \cdot (Y^Q - Y^\tilde{Q} \cdot Y) K(dx, \cdot) \cdot C = 0 \]
and that $P \times \lambda_C$ -a.s.
\[ (\beta^Q - \beta^\tilde{Q} - \beta) + (e^x - 1)(Y^Q - Y^\tilde{Q} \cdot Y) K(dx, \cdot) = 0. \]

Let us introduce the processes $\rho(Q, \tilde{Q})$ and $\rho(P, \tilde{P})$ which are closely related with the Hellinger processes, namely with their integral part with respect to the compensator of the jump measure of $X$: for all $t \geq 0$
\[ \rho_t(Q, \tilde{Q}) = \int_0^t \int_{\mathbb{R}^*} \left( \sqrt{Y^Q} - \sqrt{Y^\tilde{Q}} \right)^2 d\nu, \quad (16) \]
\[ \rho_t(P, \tilde{P}) = \int_0^t \int_{\mathbb{R}^*} (1 - \sqrt{Y})^2 d\nu. \quad (17) \]
For a given non-negative constants $a, k$ we put
\[ A = 4a \sup_{0 \leq t \leq T} \int_{\mathbb{R}^*} |e^x - 1| e^{kx} K(dx, t) \quad (18) \]
and we suppose that this quantity is finite $P$-a.s. We introduce the functions
\[ p(x) = \frac{A|e^x - 1|}{4} + 1, \quad q(x) = \frac{A|e^x - 1|}{4} + e^x \]
We introduce also the processes $U = (U_t)_{t \geq 0}$ and $V = (V_t)_{t \geq 0}$ by:
\[ U_t = \int_0^t \int_{\mathbb{R}^*} p(x) d\rho_s(Q, \tilde{Q}) + \int_0^t \int_{\mathbb{R}^*} a e^{kx} p(x) d\rho_s(P, \tilde{P}) \quad (19) \]
\[ V_t = \int_0^t \int_{\mathbb{R}^*} q(x) d\rho_s(Q, \tilde{Q}) + \int_0^t \int_{\mathbb{R}^*} a e^{kx} q(x) d\rho_s(P, \tilde{P}) \quad (20) \]

**Lemma 2.5.** We suppose that $Y^Q$ and $Y^\tilde{Q}$ are bounded by $ae^{kx}$ where $a, k$ are non-negative constants satisfying $A < \infty$ ($P$-a.s.), and that (14) holds. Then we have:
\[ h_T(\frac{1}{2}, Q, \tilde{Q}) \leq U_T, \]
\[ h_T(\frac{1}{2}, Q', \tilde{Q}') \leq V_T \]
where the processes $U$ and $V$ are given by the formulas (19), (20).
Proof. We begin with the estimation of $h_T(\frac{1}{2}, Q, \tilde{Q})$. Using Lemma 3 and Lemma 4 we write:

$$h_T(\frac{1}{2}, Q, \tilde{Q}) = \frac{1}{8} \int_0^T \left( \int_{\mathbb{R}} (e^x - 1)((Y^Q - Y^Q \cdot Y)K(dx, \cdot)) \right)^2 dC_s + \frac{1}{2} \int_0^T \int_{\mathbb{R}^*} \left( \sqrt{Y^Q \cdot Y} - \sqrt{\tilde{Y}^Q} \right)^2 d\nu$$

For the first term on the right-hand side we have by Schwartz inequality:

$$\int_0^T \left( \int_{\mathbb{R}^*} (e^x - 1)((Y^Q - Y^Q \cdot Y)K(dx, \cdot)) \right)^2 dC_s \leq A \int_0^T \int_{\mathbb{R}^*} |e^x - 1|(\sqrt{Y^Q} - \sqrt{\tilde{Y}^Q} \cdot Y)^2 d\nu$$

where $A$ is given by (13). This leads to the following inequality:

$$h_T(\frac{1}{2}, Q, \tilde{Q}) \leq \frac{1}{8} \int_0^T \int_{\mathbb{R}^*} \left( e^x - 1 \right) \left( \sqrt{Y^Q} - \sqrt{\tilde{Y}^Q} \right)^2 d\nu, \quad (21)$$

Now we remark that

$$\left( \sqrt{Y^Q} - \sqrt{\tilde{Y}^Q} \cdot Y \right)^2 \leq 2(\sqrt{Y^Q} - \sqrt{\tilde{Y}^Q})^2 + 2Y^Q(1 - \sqrt{Y})^2$$

and that $Y^Q$ is bounded by $ae^{kx}$. Then from the inequality (21) we obtain the first result. The second result can be obtained in similar way.

Theorem 2.6. Suppose that $X$ is a process without fixed points of discontinuity under $P$. We assume that (14), (15) hold and that $Y^Q$ and $\tilde{Y}^Q$ are bounded by $ae^{kx}$ where $a,k$ are constants satisfying $\mathbb{A} < \infty$ $(P$-a.s.). Then for payoff function satisfying (3) we have:

$$|C_T - \tilde{C}_T| \leq 4c \left[ E_Q U_T \right]^{1/2} + 4d \left[ E_Q' V_T \right]^{1/2},$$

Moreover, for $\epsilon > 0$,

$$|C_T - \tilde{C}_T| \leq 3\sqrt{2}\epsilon(c + d) + 2c Q (U_T \geq \epsilon) + 2d Q' (V_T \geq \epsilon)$$

where the processes $U$ and $V$ given by the formulas (19), (20) and $Q$, $Q'$ are martingale and dual martingale measure for $P$.

Proof. We combine the Lemmas 2.2 and 2.5 to obtain the result.

Let us introduce the function

$$f(x) = \frac{A}{2} |e^x - 1| + \max(1, e^x)$$

and the process $R = (R_t)_{t \geq 0}$ such that

$$R_t = \int_0^t \int_{\mathbb{R}^*} f(x) \rho_s(Q, \tilde{Q}) + \int_0^t \int_{\mathbb{R}^*} ae^{kx} f(x) \rho_s(P, \tilde{P})$$

(22)
Corollary 2.7. Suppose that \( X \) is a process with independent increments under \( P \) and \( \tilde{P} \). Assume that the conditions of Theorem 2.6 are satisfied. If in addition under the measures \( Q, \tilde{Q} \) the process \( X \) remains the process with independent increments then for payoff function satisfying (\( \mathbb{S} \)) we have:

\[
|C_T - \tilde{C}_T| \leq 3\sqrt{2}(c + d) \sqrt{R_T}
\]

Proof. Use Theorem 1 and the fact that the processes \( \rho(Q, \tilde{Q}) \) and \( \rho(P, \tilde{P}) \) are deterministic.

3 Results for general statistical model

We suppose that \((\Omega, \mathcal{F}, \mathbb{P})\) is filtered space endowed by the equivalent measures \( P_\theta, \theta \in \Theta \), where \( \theta \) is unknown parameter. We suppose that for each \( \theta \in \Theta \), there exists a martingale measure \( Q_\theta \). We denote as before by \( C_T(\theta) \) the price of risky asset obtained under physical measure \( P_\theta \). Let \( \hat{\theta} \) be an estimator of \( \theta \) and let \( \hat{C}_T(\hat{\theta}) \) be the result of the replacement in \( C_T(\theta) \) of the unknown parameter \( \theta \) by its estimator.

We denote by \((\beta^\theta, Y^\theta)\) the Girsanov parameters to pass from \( P_\theta \) to \( Q_\theta \) and we introduce the processes \( U(\theta, \theta') \) and \( V(\theta, \theta') \) by the formulas (19), (20) with replacement \( P, Q \) by \( P_\theta, Q_\theta \), and \( \tilde{P}, Q_\theta \) by \( P_{\theta'}, Q_{\theta'} \) respectively. As before we assume that \( S_0 = B_0 = 1 \) and \( r = 0 \).

Theorem 3.1. Suppose that the conditions of Theorem 2.6 are satisfied for each pair of measures \( P_\theta \) and \( Q_\theta, \theta \neq \theta', \theta, \theta' \in \Theta \). Then for payoff function satisfying (\( \mathbb{S} \)) we have:

\[
\mathbb{E}^\theta |C_T(\hat{\theta}) - C_T(\theta)| \leq 2(c + d) P_\theta \left( |\hat{\theta} - \theta| > \epsilon \right) + 4c \sup_{|\theta - \theta'| \leq \epsilon} \left[ E_{Q_\theta} U_T(\theta, \theta') \right]^{1/2} + 4d \sup_{|\theta - \theta'| \leq \epsilon} \left[ E_{Q_\theta'} V_T(\theta, \theta') \right]^{1/2}.
\]

Moreover, for any \( \epsilon > 0 \) we have:

\[
\mathbb{E}^\theta |C_T(\hat{\theta}) - C_T(\theta)| \leq 2(c + d) P_\theta \left( |\hat{\theta} - \theta| > \epsilon \right) + 3\sqrt{2} \epsilon (c + d) + 2c \sup_{|\theta - \theta'| \leq \epsilon} Q_\theta \left( U_T(\theta, \theta') \geq \epsilon \right) + 2d \sup_{|\theta - \theta'| \leq \epsilon} Q_{\theta'} \left( V_T(\theta, \theta') \geq \epsilon \right)
\]

where \( Q_\theta \) is the martingale measure of "physical" measure \( P_\theta \) and \( Q'_{\theta} \) is the respective dual measure.

Proof. We remark that

\[
\mathbb{E}^\theta |C_T(\hat{\theta}) - C_T(\theta)| = \int_\Omega |C_T(\theta') - C_T(\theta)| dP_\theta(\theta')
\]

and that for any \( \epsilon > 0 \) the right-hand side can be majorated by:

\[
2 \sup_{\theta \in \Theta} C_T(\theta) \cdot P \left( |\hat{\theta} - \theta| > \epsilon \right) + \sup_{|\theta - \theta'| \leq \epsilon} |C_T(\theta) - C_T(\theta')|
\]

Due to (\( \mathbb{S} \)) and martingale properties of \( S \), we have \( C_T(\theta) \leq c + d \). Then we use the estimations of Theorem 2.6 to conclude.
Let also
\[ R_T(\theta, \theta') = \int_0^T \int_{\mathbb{R}^2} f_{\theta, \theta'}(x)dp_s(Q_{\theta}, Q_{\theta'}) + \int_0^T \int_{\mathbb{R}^2} a_{\theta, \theta'} e^{k_{\theta, \theta'}x} f_{\theta, \theta'}(x)dp_s(P, \tilde{P}) \]  
(23)
where \( f_{\theta, \theta'} \), \( A_{\theta, \theta'} \), \( a_{\theta, \theta'} \) and \( k_{\theta, \theta'} \) are the function and the constants corresponding to \( f, A, a \) and \( k \) of Theorem 2.6.

**Corollary 3.2.** Suppose that the process \( X \) is a process with independent increments under \( P, \theta \in \Theta \), as well as under corresponding martingale measures \( Q, \theta \in \Theta \). Suppose also that the conditions of Theorem 3.1 are satisfied. Then for payoff function satisfying [8] we have:
\[ \mathbb{E}_\theta |C_T(\tilde{\theta}) - C_T(\theta)| \leq 2(c + d) P_\theta (|\tilde{\theta} - \theta| > \epsilon) + 3\sqrt{2}(c + d) \left( \sup_{|\theta - \theta'| \leq \epsilon} R_T(\theta, \theta') \right)^{1/2} \]

**Corollary 3.3.** Suppose that we have a sequence of processes with independent increments involving the physical measures \( (P^n_\theta)_{n \geq 1}, \theta \in \Theta \), the corresponding martingale measures \( (Q^n_\theta)_{n \geq 1}, \theta \in \Theta \), and the respective sequence of the consistent estimators \( (\tilde{\theta}^n)_{n \geq 1} \). Suppose also that the conditions of Theorem 3.1 are satisfied. Let \( R^n_T(\theta, \theta') \) be defined by [23] with replacement of \( P, Q \) and \( P', Q' \) by \( P^n_\theta, Q^n_\theta \) and \( P^n_\theta, Q^n_\theta \) respectively.

If uniformly in the neighbourhood of \( \theta \) as \( n \to \infty \)
\[ R^n_T(\theta, \theta') \to 0 \]
then for payoff function satisfying [8] we have:
\[ \mathbb{E}_\theta^n |C_T(\tilde{\theta}^n) - C_T(\theta)| \to 0 \]
where \( \mathbb{E}_\theta^n \) is a mathematical expectation with respect to \( P^n_\theta \).

## 4 Applications to Levy processes

Suppose now that \( X \) is Levy process with parameters \((b, c, \nu)\) under the measure \( P \). We emphasize that here \( \nu \) is no more the compensator of the measure of jumps of \( X \) but a Levy measure, i.e. positive \( \sigma \)-finite measure on \( \mathbb{R} \) such that
\[ \int_{\mathbb{R}} (x^2 \wedge 1)d\nu < \infty. \]
We recall that the characteristic function of \( X_t \) for \( t \geq 0 \) and \( \lambda \in \mathbb{R} \) is given by:
\[ \phi_t(\lambda) = \exp(t\psi(\lambda)) \]
where \( \psi(\lambda) \) is a characteristic exponent of Levy process,
\[ \psi(\lambda) = ib\lambda - \frac{1}{2}\lambda^2c + \int_{\mathbb{R}} (\exp(i\lambda x) - 1 - i\lambda l(x))d\nu, \]
and \( l \) is the truncation function. Let now \( \tilde{P} \) be the measure corrsponding to the parameters \((\tilde{b}, \tilde{c}, \tilde{\nu})\). According to Corollary 2.7 of section 2 we have to find, for chosen martingale measures \( Q \) and \( \tilde{Q} \), the Girsanov parameters \((\beta^Q, Y^Q)\) and \((\beta^{\tilde{Q}}, Y^{\tilde{Q}})\) and write the expressions for the processes \( \rho(Q, \tilde{Q}) \) and \( \rho(P, \tilde{P}) \). We recall that as before \( S_t = \exp(X_t) \). Let \( r \) be positive constant, and, let us suppose that the value process of the bond is deterministic and given by \( B_t = \exp(rt) \).
4.1 Esscher measures

Esscher measures play very important role in actuary theory as well as in the option pricing theory and they were studied in [16], [22], [23]. Let

\[ D = \{ \lambda \in \mathbb{R} \mid EP e^{\lambda X_1} < \infty \} \]

where \( EP \) is the expectation with respect to the physical measure \( P \). Then for \( \lambda \in D \) we define Esscher measure \( P_{ES}^{\lambda} \) of the parameter \( \lambda \) and risk process \((X_t)_{t\geq0}\) by: for \( t \geq 0 \)

\[
\frac{dP_{ES}^{\lambda}}{dP_t} = \frac{e^{\lambda X_t}}{EP[e^{\lambda X_t}]}
\]

It is known that \((e^{-rt}S_t)_{t\geq0}\) is a martingale under \( Q = P^{ES} \) iff

\[
\psi(-i(1 + \lambda)) - \psi(-i\lambda) = r
\]

and the last equation is equivalent to:

\[
b + \left( \frac{1}{2} + \lambda \right)c + \int_{\mathbb{R}^*} ((e^x - 1) e^{\lambda x} - l(x))d\nu = r
\]

(24)

About existence and uniqueness of solution of (24) see [11] and [17].

Suppose again that \( X \) is Levy process with parameters \((b, c, \nu)\) under \( P \), and that it has the parameters \((\bar{b}, \bar{c}, \bar{\nu})\) under \( \bar{P} \). Suppose that the solution of (24) exists as well as the solution of the same equation with the replacement \((b, c, \nu)\) by \((\bar{b}, \bar{c}, \bar{\nu})\) denoted \( \lambda^* \) and \( \bar{\lambda}^* \) respectively. Then \( Q = P^{ES}(\lambda^*) \) and \( \bar{Q} = P^{ES}(\bar{\lambda}^*) \).

Now we show that the Girsanov parameters for \( Q \) and \( \bar{Q} \) are: \( \beta^Q = \lambda^* \ Y^Q = e^{\lambda^* x} \) and \( \beta^\bar{Q} = \bar{\lambda}^* \), \( Y^Q = e^{\bar{\lambda}^* x} \) respectively. We write

\[
Z_t = \frac{dQ_t}{dP_t} = \frac{e^{\lambda^* X_t}}{\phi(-i\lambda^* t)}
\]

(25)

From the formula (25) we see that

\[
\frac{Z_t}{Z_{t-}} = e^{\lambda^* \Delta X_t}
\]

and according to Girsanov theorem

\[
Y^Q = M_{\mu}^{P}\left(e^{\lambda^* \Delta X} \mid \bar{P}\right) = e^{\lambda^* x}.
\]

We use Ito formula to find \( Z^c \):

\[
Z^c_t = \int_0^t \frac{\lambda^* \exp(\lambda^* X_{s-})}{\phi(-i\lambda^* s)}dX^c_s
\]

and, hence,

\[
\beta^Q_t = \frac{1}{Z_{t-}} \frac{d\langle Z^c, X^c \rangle_t}{dC_t} = \lambda^*.
\]
Now, we have to write the expression of $\rho_T(Q, \tilde{Q})$ and $\rho_T(P, \tilde{P})$:

$$
\rho_T(Q, \tilde{Q}) = T \int_{\mathbb{R}^*} (\sqrt{e^{\lambda^* x}} - \sqrt{e^{\tilde{\lambda}^* x}})^2 d\nu
$$

$$
\rho_T(P, \tilde{P}) = T \int_{\mathbb{R}^*} (1 - \sqrt{Y})^2 d\nu
$$

where $Y = d\tilde{\nu}/d\nu$.

In the case when $\lambda^* \leq 0$ and $\tilde{\lambda}^* \leq 0$ we can find easily that the conditions of Theorem 2.6 are verified with $k = 0$ and $a = 1$. We remark that mean value theorem gives:

$$
(\sqrt{e^{\lambda^* x}} - \sqrt{e^{\tilde{\lambda}^* x}})^2 \leq |x|^2 (\lambda^* - \tilde{\lambda}^*)^2
$$

So, for payoff function satisfying (8) we obtain the estimation:

$$
|C_T - \tilde{C}_T| \leq T(\lambda^* - \tilde{\lambda}^*)^2 \int_{\mathbb{R}^*} f(x) x^2 d\nu + T \int_{\mathbb{R}^*} f(x)(\sqrt{d\nu} - \sqrt{d\tilde{\nu}})^2
$$

where $f(x) = \frac{A}{2} |e^{\lambda^* x} - 1| + \max(1, e^{\lambda^* x})$ and $A = 4aT \int_{\mathbb{R}^*} |e^{\lambda^* x} - 1| d\nu$. In the case when $\lambda^*$ and/or $\tilde{\lambda}^*$ are not negative we can obtain similar estimations.

### 4.2 Minimal entropy measures

Let $Q$ and $P$ be two equivalent probability measures then the relative entropy of $Q$ with respect to $P$ (or Kulback-Leibler information in $Q$ with respect to $P$) is:

$$
H(Q|P) = E_Q \left( \ln \frac{dQ}{dP} \right) = E_P \left( \frac{dQ}{dP} \ln \frac{dQ}{dP} \right)
$$

We are interested in minimal entropy martingale measure, i.e. the measure $P^{ME}$ such that $(e^{-rt}S_t)_{t \geq 0}$ is a $P^{ME}$-martingale, and that for all $Q$ martingale measures

$$
H(P^{ME}|P) \leq H(Q|P)
$$

It turns out (cf. [23]) that in the case of Levy processes $P^{ME}$ is nothing else as Esscher measure but for another risque process $(\hat{X}_t)_{t \geq 0}$, namely for the process appearing in the representation:

$$
S_t = S_0 \mathcal{E}(\hat{X})_t
$$

where $\mathcal{E}(\cdot)$ is Doleans-Dade exponential,

$$
\mathcal{E}(\hat{X})_t = \exp(\hat{X}_t - \frac{1}{2}(\hat{X})_t) \prod_{0 \leq s \leq t} (1 + \Delta \hat{X}_s)e^{-\Delta \hat{X}_s}
$$

Writing Ito formula for $f(x) = e^x$ we obtain that $S_t = S_t - d\hat{X}_t$ with

$$
\hat{X}_t = X_t + \frac{1}{2}(X^c)_t + \int_0^t \int_{\mathbb{R}^*} (e^{\tilde{x}} - 1 - x) d\mu(x)
$$
where $\mu$ is the measure of jumps of $X$. This permits us to find the characteristics of $\hat{X}$:

$$
\begin{cases}
\hat{B} = B + \frac{1}{2} (X^c) + (e^x - 1 - x) \nu \\
\hat{C} = C \\
\hat{\nu} = (e^x - 1) \cdot \nu
\end{cases}
$$

We see that if $X$ is a Levy process verifying $\int_{\mathbb{R}^*} |e^x - 1| d\nu < \infty$ where $\nu$ is a Levy measure of $X$, then $\hat{X}$ is also Levy process and the parameters of $\hat{X}$ are:

$$
\begin{cases}
\hat{b} = b + \frac{1}{2} c + (e^x - 1 - x) \nu \\
\hat{c} = c \\
\hat{\nu} = (e^x - 1) \cdot \nu
\end{cases}
$$

Now let $D = \{ \lambda \in \mathbb{R} \mid E_P e^{\lambda X_1} < \infty \}$ and let us introduce Esscher measure corresponding to the risque process $\hat{X}$ and $\lambda \in D$:

$$
\frac{dP_{tME}^M}{dP_t} = \frac{e^{\lambda X_t}}{E_P[e^{\lambda X_t}]}
$$

We remark that one can write easily the characteristic function of $\hat{X}$ and the expression for characteristic exponent:

$$
\hat{\psi}(\lambda) = i\lambda(b + \frac{1}{2} c + (e^x - 1 - x) \nu) - \frac{1}{2} \lambda^2 c + \int_{\mathbb{R}^*} (\exp(i\lambda x) - 1 - i\lambda l(x))(e^x - 1) d\nu,
$$

where $l$ is the truncation function.

As it was mentionned before, this measure is a martingale measure for $(e^{-rt}S_t)_{A \geq 0}$ iff

$$
\hat{\psi}(-i(1 + \lambda)) - \hat{\psi}(-i\lambda) = r
$$

and the last equation is equivalent to:

$$
b + (\frac{1}{2} + \lambda)c + \int_{\mathbb{R}^*} ((e^x - 1) e^{\lambda(e^x-1)} - l(x)) d\nu = r \tag{27}
$$

About existence and uniqueness of solution of (24) see [11] and [17].

Let us suppose that the solution $\lambda^*$ of the equation (27) exists as well the solution $\tilde{\lambda}^*$ of the similar equation with replacing $(b, c, \nu)$ by $(\tilde{b}, c, \tilde{\nu})$. We can show in the same way as before that Girsanov parameters of minimal entropy martingale measures are $(\lambda^*, e^{\lambda^*(e^x-1)})$ and $(\tilde{\lambda}^*, e^{\tilde{\lambda}^*(e^x-1)})$ respectively. Then, if $\lambda^*$ and $\tilde{\lambda}^*$ are negatives, for payoff function satisfying (8) we have:

$$
|C_T - \tilde{C}_T| \leq T(\lambda^* - \tilde{\lambda}^*)^2 \int_{\mathbb{R}^*} f(x)(e^x - 1)^2 d\nu + T \int_{\mathbb{R}^*} f(x)(\sqrt{d\nu} - \sqrt{d\tilde{\nu}})^2 \tag{28}
$$

where $f(x) = \frac{A}{2} |e^x - 1| + \max(1, e^x)$ and $A = 4aT \int_{\mathbb{R}^*} |e^x - 1| d\nu$. In the case when $\lambda^*$ and/or $\tilde{\lambda}^*$ are not negative we can obtain similar estimations.
Example 4.1. In Geometric Variance Gamma model the parameters \((b,c,\nu)\) are equal to \((0,0,\nu)\). The Levy measure of this model has the following form:

\[
\nu(dx) = \frac{C(\mathbb{I}_{\{x<0\}}e^{-M|x|} + \mathbb{I}_{\{x>0\}}e^{-N|x|})}{|x|}dx
\]

where \(C > 0\) and \(M,N \geq 0\).

We denote the left-hand side of (27) with given \(\nu\) by \(\hat{f}\). It is known (see [21]) that if \(0 \leq N \leq 1\), or \(N > 1\) and \(\hat{f}(0) \geq r\), then \(\lambda^* < 0\). If \(N > 1\) and \(\hat{f}(0) < r\), then \(\lambda^*\) does not exist. So, we have the estimation (28) when the solution of (27) exists.

Example 4.2. In Geometric CGMY model the parameters \((b,c,\nu)\) are equal to \((0,0,\nu)\). The Levy measure of this model has the following form:

\[
\nu(dx) = \frac{C(\mathbb{I}_{\{x<0\}}e^{-M|x|} + \mathbb{I}_{\{x>0\}}e^{-N|x|})}{|x|^{1+\alpha}}dx
\]

where \(\alpha < 2\), \(C > 0\) and \(M,N \geq 0\). We recall that the case of \(\alpha = 0\) corresponds to Geometric Variance Gamma model and it was already considered.

We denote again the left hand side of (27) by \(\hat{f}\). It is known (cf.[21]) that if \(M = N = 0\) and \(0 < \alpha < 2\) then \(X\) is symmetric stable process and if, in addition \(C > 0\), then \(\lambda^* < 0\). If \(0 \leq N \leq 1\) or if \(N > 1\) and \(\hat{f}(0) \geq r\) then again \(\lambda^* < 0\). If \(N > 1\) and \(\hat{f}(0) < r\) the equation (27) has no solution. So, we have the estimation (28) when the solution of (27) exists.

4.3 \(f^q\)-martingale measures

These measures take part of the measures minimising so called \(f\)-divergence between two probability measures. Let \(Q\) and \(P\) be two probability measures, \(Q << P\), and \(f\) be a convex function with the values in \(\mathbb{R}^{+,*}\). Then \(f\)-divergence (cf.[3]) of \(Q\) given \(P\), denoted \(f(Q|P)\) is given by:

\[
f(Q|P) = EP[f(\frac{dQ}{dP})]
\]

If \(f(x) = x \ln x\) we obtain as \(f(Q|P)\) the entropy or Kulback-Leibler information, if \(f(x) = |1-x|\) we obtain the variation distance, if \(f(x) = (1-x)^2\) we obtain variance squared distance, if \(f(x) = (1-\sqrt{x})^2\) we obtain Hellinger distance. We remark also that the minimisation of variance squared distance is equivalent to minimise \(EP[(\frac{dQ}{dP})^2]\), and that the minimising of Hellinger distance is equivalent to minimise \(-EP(\sqrt{\frac{dQ}{dP}})\).

In the papers [3], [4], [14] the authors consider \(f\)-divergences with

\[
f(x) = \begin{cases} 
-x^q, & \text{if } 0 < q < 1, \\
x^q, & \text{if } q < 0 \text{ or } q > 1.
\end{cases}
\]

It is not difficult to see that such \(f\) is a convex function. It was shown that in the case of Levy processes the Girsanov parameters \((\beta_q,Y_q)\) of the measure \(P^{(q)}\) minimising \(f\)-divergence given by the above expression, are deterministic. So, \(X\) is also Levy process under \(P^{(q)}\).

It can be also shown that if \(X\) is not monotone Levy process and if we allow as \(P^{(q)}\) not only equivalent, but also absolute continuous measures, then the Girsanov parameters \((\beta_q,Y_q)\) are unique minimizers of the function

\[
k(\beta,Y) = \frac{q(q-1)}{2}\beta^2c + \int_{\mathbb{R}^*} (Y^q - 1 - q(Y-1))d\nu
\]
under constraint
\[ b + c\beta + \int_{\mathbb{R}^*} (xY(x) - l(x))d\nu = 0 \]
on the set
\[ \mathcal{A} = \{ (\beta, Y) \mid \beta \in \mathbb{R}, Y \geq 0, \int_{\mathbb{R}^*} |xY(x) - l(x)|d\nu < \infty \} \]

Via an application of the Kuhn-Tucker theorem it can be shown that
\[ Y_q(x) = \begin{cases} 
(1 + (q-1)\beta_q(e^x - 1))^{\frac{1}{q-1}} & \text{if } 1 + (q-1)\beta_q(e^x - 1) \geq 0, \\
0 & \text{in opposite case},
\end{cases} \]

where \( \beta_q \) is the first Girsanov parameter which can be find from the constraint. We remark that if in addition
\[ \text{supp}(\nu) \subseteq \{ x : 1 + (q-1)\beta_q(e^x - 1) > 0 \} \]
then \( P^{(q)} \) is equivalent to \( P \). We will suppose that the last condition is satisfied.

Let \((\beta_q, Y_q)\) and \((\tilde{\beta}_q, \tilde{Y}_q)\) be Girsanov parameters of \( f^q \)-minimal martingale measures for \( P \) and \( \tilde{P} \) respectively. To evaluate \( \rho(P^{(q)}, \tilde{P}^{(q)}) \) we remark that
\[ (\sqrt{Y_q(x)} - \sqrt{\tilde{Y}_q(x)})^2 \leq C(e^x - 1)^2(\beta_q - \tilde{\beta}_q)^2 \]
with some constant \( C \). So, we have the estimations similar to \( (28) \).

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