Limits of Short-Time Quantum Annealing

Ali Hamed Moosavian,1 Seyed Sajad Kahani,1 and Salman Beigi1

1QuOne Lab, Phanous Research and Innovation Centre, Tehran, Iran

(Dated: March 16, 2021)

Quantum annealing is a general purpose optimization algorithm that is based on the quantum adiabatic theorem. Quantum annealing involves an evolving Hamiltonian that is local. Thus, we expect that short-time quantum annealing algorithms to be inherently local and limited as well. In this paper, we validate this intuition by proving some limitations of short-time quantum annealing algorithms. We show that the distribution of the measurement output of short-time (at most logarithmic) quantum annealing computations are concentrated and satisfy an isoperimetric inequality. To showcase explicit applications, we also study the MAXCUT problem and conclude that quantum annealing needs at least a run-time that scales logarithmically in the problem size to beat classical algorithms. To establish our results, we also prove a Lieb-Robinson bound that works for time-dependent Hamiltonians which might be of independent interest.

Keywords: Quantum annealing, Lieb-Robinson bound, Concentration of measure, MaxCut problem

I. INTRODUCTION

The Quantum Annealing (QA) algorithm was first introduced as a general purpose optimization algorithm for quantum computers in [1]. The QA algorithm is based on the adiabatic theorem in quantum mechanics [2]. The adiabatic theorem states that if we initialize a system in its ground state, and then evolve the Hamiltonian of the system slowly enough, as long as the spectral gap of the Hamiltonian does not close, the system remains close to the ground state of the system at all times [3].

Because of its simple conceptual structure, QA has been implemented on many different platforms and applied to a variety of different optimization problems, see e.g. [4–18].

In this manuscript, we focus on short-time QA algorithms. This setup is motivated by the current state of the art quantum technologies. On one hand, we still lack universal quantum computers capable of simulating general QA algorithms [19, 20], and on the other hand the analog QA devices lack fault-tolerance. Therefore, the total time of reliably performing a continuous computation on these analog devices is bounded by their decoherence time.

Quantum Approximate Optimization Algorithm (QAOA) is another general purpose quantum algorithm that can yield approximate solutions to many optimization problems, particularly combinatorial ones [21]. Based on QAOA’s local nature, recent developments have shown that low-depth QAOAs are unable to produce better approximate solutions to some optimization problems than the best known classical algorithms [22–26].

In particular, Bravyi et al [24] use the isoperimetric inequality of [27] that puts restrictions on the output distributions of all low-depth circuits, to show that low-depth (at most logarithmic) QAOA for MAXCUT does not outperform best classical algorithms on Ramanujan graphs. Also, Farhi et al [22] show that low-depth QAOA for the problems of MAXCUT and Maximum Independent Set, does not perform well on random regular bipartite graphs.

Thinking of the correspondence between short-time QA algorithms and low-depth quantum circuits, particularly low-depth QAOAs, it is natural to ask whether the above bounds still hold for QA algorithms or not. In this manuscript, we show that most of the aforementioned results on limitations of QAOAs still holds for short-time QA algorithms as well. Throughout this paper we use short-time to represent times that scale at most logarithmically with the system size. We first show that the isoperimetric inequality of [27] remains to hold for output distributions of short-time QA computations. Then, using this inequality as a tool (besides others), we generalize the results of [24] and [22] to the QA case.

QA and QAOA can be thought of two instances of optimal control problems. In QA we look for an optimal path in the space of Hamiltonians from a starting Hamiltonian to a target one, while in QAOA we look for optimal rotation angles of some local circuits. Applying Pontryagin’s maximum principle, some authors have argued that the optimal solution to the first control theory problem takes the so called bang-bang form in most cases [28, 29] and then reduces to the second problem. However, it was shortly noted that in some common problem instances the control Hamiltonian of the system becomes singular, a condition that was presumed to be extremely rare, and then the optimal control protocol is no longer of the bang-bang form [30]. This means that, in general, the optimal control problem for QA does not reduce to that of the QAOA, and needs thoughtful considerations. In fact, the observation of [30] shows that we cannot directly use the results on the obstructions of QAOAs, to prove the aforementioned limitations of short-time QA algorithms.

Another idea for relating results on low-depth quantum circuits (QAOAs) to QA algorithms is trotterization (see [31] and references therein), which approximates the unitary evolution of a local Hamiltonian with a quantum circuit. Although one can imagine that this approach works in some instances, the deducible error bound is not tight enough in general.

We use Lieb-Robinson bounds as a main tool for proving our results [32–35]. To this end, we first prove a Lieb-Robinson bound that works for time-dependent Hamiltonians and is amenable to QA algorithms. To the best of our knowledge, this is the first time-dependent Lieb-Robinson bound.

As another technical tool, we use the notion of $\gamma_2$-norm, also
called factorization norm, from matrix analysis to bound the derivative of certain functions on the space of operators. This enables us to bound certain errors induced by applying the Lieb-Robinson approximation.

Here is a summary of our results and the structure of the paper. In Section II we state our version of the Lieb-Robinson bound that is a fundamental tool in our work. Then in Section III we prove two different bounds on the output distributions of short-time QA computations. Our first bound is a generalization of the well-known Chebyshev inequality which says that the Hamming weight of the measurement outcome of a short-time QA computation is concentrated around its mean. The next bound, that is independent of the first one, is a generalization of the isoperimetric inequality of [27] for short-time QA algorithms. Then, in Section IV we take the approach of [24], and show that the $\mathbb{Z}_2$-symmetry of the MaxCut Hamiltonian limits the performance of QA algorithms on Ramanujan graphs. To prove this result we use our isoperimetric inequality. Next, in Section V, we prove a similar limitation on random regular bipartite graphs using the Lieb-Robinson bound. This result is an extension of the result of [22] for QA algorithms. Final remarks come in the last section, and some technical parts of the proofs are left in the appendix.

II. LIEB-ROBINSON BOUND

One of our main tools in this paper is the Lieb-Robinson bound [20]. Roughly speaking, it states that the evolution of a local observable in the Heisenberg picture, remains almost local in short times. We use the following version of the Lieb-Robinson bound that notably works for time-dependent Hamiltonians as well.

To state our theorem we need a definition. Given a graph $G$ and a subset of its vertices $A$, we denote the set of vertices on the $L$-boundary of $A$ by $\partial_L(A)$. That is,

$$\partial_L(A) = \{x \in A : \exists y \in A^c, \text{dist}_G(x,y) \leq L\}$$
(1)

$$\cup \{x \in A^c : \exists y \in S, \text{dist}_G(x,y) \leq L\},$$
(2)

where $\text{dist}_G(x,y)$ is the graph distance between vertices $x$ and $y$.

Theorem 1 (Lieb-Robinson Bound). Let $G$ be a graph that may contain loops and parallel edges with maximum degree $\Delta > 1$. Let $\hat{H}(t)$ be a time-dependent Hamiltonian defined on vertices of $G$ of the form

$$\hat{H}(t) = \sum_e u_e(t) \hat{h}_e,$$
(3)

where $\hat{h}_e$ is time-independent and acts non-trivially only on the two ends of edge $e$ in $G$. Let $U(t)$ be the unitary evolution associated with $\hat{H}(t)$. Suppose that $\|u_e(t)\| \leq g$ for any $e$ and $0 \leq t \leq T$. Let $\hat{O}_A$ be an observable acting on region $A$ of the graph. For $L > 1$ let

$$\hat{H}_A(t) := \sum_{e \subseteq A \cup \partial_L(A)} u_e(t) \hat{h}_e,$$
(4)

be the Hamiltonian consisting of terms in the region $A \cup \partial_L(A)$, and let $V_A(T)$ be the unitary evolution associated to $\hat{H}_A(t)$. Then we have

$$\|U(t)O_AU(T) - V_A(T)O_AV_A(T)\|
\leq \sqrt{\frac{2}{\pi}} |A| \cdot \|O_A\| \cdot e^{-L \left( \log L - \log T - \log(4g(\Delta-1)) \right) - \frac{1}{2} \log L}.$$  
(5)

Intuitively speaking, this theorem says that, as $O_A$ is local, it remains almost local under a short-time evolution of a local Hamiltonian. Indeed, since $H_A(t)$ is non-trivial only in the region $A \cup \partial_L(A)$, then $V_A(T)$ also acts non-trivially only on this region. Therefore, by this theorem $U(t)O_AV_A(T)$ is approximated by $V_A(T)O_AV_A(T)$ that acts only on $A \cup \partial_L(A)$.

Some remarks are in line regarding this theorem. First, as the proof of this theorem given in Appendix A shows, the above Lieb-Robinson bound can easily be generalized for local Hamiltonians with $k$-body interactions beyond two-body ones, by replacing the term $\log(4g(\Delta - 1))$ on the right hand side of Eq. (5) with $\log(2k g(\Delta - 1))$. Second, this theorem in the case where $H$ is time-independent is proven in [33, 34, 37]. Third, the type of time-dependent Hamiltonians Eq. (6) considered in this theorem are motivated with the applications in quantum annealing. Nevertheless, note that in principle any time-dependent Hamiltonian can be written in this form; simply consider the expansion of each local term of the Hamiltonian in the Pauli basis.

III. LIMITS ON THE OUTPUT DISTRIBUTION OF SHORT-TIME EVALUATIONS

In this section we prove our main results, namely, the output distribution of a short-time quantum annealing algorithm is constrained. Our first main result is a generalization of the well-known Chebyshev inequality in probability theory that is a concentration of measure inequality. Our second result is an isoperimetric inequality that is first proven for low-depth circuits in [27].

In the following we assume that $\hat{H}(t)$ is a time-dependent Hamiltonian defined on the vertices of a graph $G$ as in Eq. (3). We assume that starting with a product state

$$|\psi_0\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_n\rangle,$$
(6)

the system evolves under the Hamiltonian $\hat{H}(t)$. Let

$$|\psi_t\rangle = U(t)|\psi_0\rangle,$$
(7)

where as before $U(t)$ is the associated unitary evolution. We assume that at time $T$ we make a measurement in the computational basis to get a distribution $p(x)$ on $\{0, 1\}^n$. In the following we prove limitations on $p(x)$ when $T$ is bounded.

A. A Chebyshev bound

Theorem 2 (Quantum Chebyshev bound). Let $\hat{H}(t)$ be a Hamiltonian on a graph with maximum degree $\Delta > 1$ as in Eq. (3) with
\[ \|u_c(t)h_c\| \leq g \text{ for any edge } c. \] Let
\[ T \leq \frac{\kappa_1}{8g\Delta} \frac{\sqrt{2\log(n)}}{\log \Delta} \log n, \tag{8} \]
where \( 0 < \kappa_1 < 1 \) is an arbitrary constant. Let \( p(x) \) be the output distribution of measuring \( |\psi_T\rangle = U(T)|\psi_0\rangle \) in the computational basis. We assume \( |\psi_0\rangle \) to be a product state as in Eq. \( \ref{eq:state} \). Then for any \( c > 0 \) and \( \kappa_1/2 < \kappa_2 < 1/2 \) we have
\[ \Pr \left[ |w_H(x) - \left( \frac{n}{2} - \left\lceil \frac{m}{2} \right\rceil \right) \geq cn^{\frac{2}{2} + \kappa_2} \right] \leq \frac{3}{2e^2} n^{-(2\kappa_2 - \kappa_1)}, \tag{9} \]
where \( w_H(x) \) is the hamming weight of \( x \in \{0, 1\}^n \), i.e. number of non-zero coordinates of \( x \), and \( m = \sum_i \langle \psi_T | Z_i | \psi_T \rangle \) with \( Z \) being the Pauli-z operator.

This theorem says that, when \( T \) is bounded, with high probability the hamming weight of the measurement output is close to \( \frac{1}{2}(n - m) \) which is the average of \( w_H(x) \). That is, the Hamming weight of the measurement outcome is concentrated.

The proof idea of this theorem is similar to that of the standard Chebyshev inequality; we need to bound the variance of \( w_H(x) \). In the standard Chebyshev inequality the coordinates of \( x \) are independent while here they are correlated. Nevertheless, since \( T \) is bounded, by the Lieb-Robinson bound (Theorem \ref{thm:LR}) pairs of coordinates whose associated Hamming weights are far from each other are almost independent. This allows us to bound the variance of \( w_H(x) \). A detailed proof of this theorem is given in Appendix \ref{app:proof}.

**B. An isoperimetric inequality**

We now prove that the output distribution of a short-time quantum annealing algorithm satisfies an isoperimetric inequality. Such a result for low-depth circuits was first proved in \cite{22}.

**Theorem 3.** Let \( H(t) \) be a Hamiltonian on a graph with maximum degree \( \Delta > 1 \) as in Eq. \( \ref{eq:Ham} \) with \( \|u_c(t)h_c\| \leq g \) for any edge \( c \). Let \( |\psi_0\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_n\rangle \) be an arbitrary product state and let \( |\psi_T\rangle = U(T)|\psi_0\rangle \) be the state of the system at time \( T > 0 \). Let \( p(x) \) be the distribution on \( \{0, 1\}^n \) resulting from measuring the first \( n_0 \) qubits of \( |\psi_T\rangle \) in the computational basis, i.e.,
\[ p(x) = \sum_{y \in \{0, 1\}^{n-n_0}} |\langle x, y | \psi_T \rangle|^2. \tag{10} \]
Then for any \( F \subset \{0, 1\}^n \) with \( p(F) \leq \frac{1}{2} \) and any \( 0 \leq \theta \leq 1/2 \) the following results hold:

(i) Assume that
\[ T \leq \frac{\kappa_1}{4g\log(\Delta)\Delta} \frac{\sqrt{2\log(n)}}{\log \Delta} \log n_0, \tag{11} \]
for some constants \( \kappa_1, \kappa_2 > 0 \). Then we have
\[ p(\partial\ell(F)) \geq \frac{1}{8} (2n_0^{1+\kappa_1})^{-2\theta} p(F) - 2n_0^{-\kappa_2}. \tag{12} \]

(ii) Assume that we measure all the qubits, i.e. \( n_0 = n \). Moreover, assume that
\[ T \leq \frac{\kappa_1}{4g(\Delta - 1)} \log n, \tag{14} \]
and
\[ \ell = \frac{\kappa_1(1 + \kappa_2)}{2} \log(n) \sqrt{n} \tag{15} \]
for some constants \( \kappa_1, \kappa_2 > 0 \). Then we have
\[ p(\partial\ell(F)) \geq \frac{1}{8} p(F) - \frac{5}{4} n^{-\kappa_1(1+\kappa_2)\log(1+\kappa_2)-1}. \]

We follow similar steps as in the proof of the isoperimetric inequality for low-depth circuits of \cite{22} to prove our theorem. To this end, we need to use the Lieb-Robinson bound to replace some operators that are almost local, with local ones. Nevertheless, in the proof we need to consider certain functions of those operators. Therefore, we need a new ingredient in the proof in order to bound the operator derivative of those functions. Such a bound is proven in Appendix \ref{app:proof} via the so called \( \gamma_2 \)-norm. Then using the tools developed in Appendix \ref{app:proof} we prove Theorem \ref{thm:isoperimetric} in Appendix \ref{app:proof}.

**Corollary 4.** Consider the setup of Theorem \ref{thm:isoperimetric} with \( \theta = 0 \) and \( n_0 = n \). Assume that \( T \) and \( \ell \) satisfy Eq. \( \ref{eq:LR_bound} \) and Eq. \( \ref{eq:isoperimetric} \). Then for two arbitrary disjoint sets \( F_1, F_2 \subset \{0, 1\}^n \) with Hamming distance \( D = \text{dist}_H(F_1, F_2) \) and \( \mu := \min\{p(F_1), p(F_2)\} \), we have
\[ D < \frac{16\ell}{\mu} + \frac{10D}{\mu} n^{-\kappa_1(1+\kappa_2)\log(1+\kappa_2)-1} \tag{16} \]
\[ \leq \frac{16}{\mu} n^{-\kappa_1(1+\kappa_2)\log(1+\kappa_2)-2}. \tag{17} \]

The contra-positive version of this corollary says that if two far apart subsets of \( \{0, 1\}^n \) have high probability under the distribution produced by the annealing algorithm, then the runtime of the annealing algorithm is necessarily long.

**Proof.** For an integer \( 0 \leq d < D/\ell \) define \( K_d \subseteq \{0, 1\}^n \) by
\[ K_d = \{x : (d - 1)\ell < \text{dist}_H(x, F_1) \leq d \ell\}, \tag{18} \]
where \( \text{dist}_H(x, F_1) = \min_{y \in F_1} \text{dist}_H(x, y) \). Clearly, \( K_0 = F_1 \) and \( K_d \cap F_2 = \emptyset \) for all \( d < D/\ell \). Then, since \( K_d \)'s are disjoint, we have
\[ \sum_{1 \leq d < \frac{D}{\ell} + 1} p(K_d) + p(K_{d+1}) \leq 1 - (p(F_1) + p(F_2)) \leq 1 - 2\mu. \tag{19} \]
Therefore, there exists \( d_0 \) such that
\[ p(K_{d_0}) + p(K_{d_0+1}) \leq \frac{1 - 2\mu}{\frac{\mu}{D}}. \tag{20} \]
Now define
\[ \tilde{F}_1 := \bigcup_{j \leq d_0} K_d. \] (21)

Observe that by definition \( F_1 \subseteq \tilde{F}_1 \) and that \( \partial_t(\tilde{F}_1) \subseteq K_{d_0} \cup K_{d_0+1} \). Therefore, using Theorem 3 we obtain
\[
\frac{1 - 2\mu}{D} \geq p(K_{d_0}) + p(K_{d_0+1}) \geq p(\partial_t(\tilde{F}_1)) \geq \frac{1}{8} p(\tilde{F}_1) - \frac{5}{4} n - (\kappa_1(1+\kappa_2)\log(1+\kappa_2) - 1) \geq \frac{\mu}{8} - \frac{5}{4} n - (\kappa_1(1+\kappa_2)\log(1+\kappa_2) - 1),
\]
and
\[
\frac{D}{2\ell} - 2 \leq \frac{1 - 2\mu}{\mu/8} + \frac{5}{4} n - (\kappa_1(1+\kappa_2)\log(1+\kappa_2) - 1) \left( \frac{D}{\mu/8} - 2 \right) \leq \frac{8}{\mu} - 2 + \frac{5}{4} n - (\kappa_1(1+\kappa_2)\log(1+\kappa_2) - 1) \left( \frac{D}{\mu/8} - 2 \right). \] (27)

Multiplying both sides by \( 2\ell \) yields the desired inequality. \( \square \)

IV. MAXCUT ON RAMANUJAN GRAPHS

In this section and the following one, we present limits of short-time quantum annealing algorithms for the MaxCut problem. Here, inspired by [24] our strategy is to use the \( \mathbb{Z}_2 \)-symmetry of the MaxCut Hamiltonian, and show that short-time quantum annealing algorithm cannot outperform best classical algorithms for MaxCut on Ramanujan graphs [38].

We first describe the quantum annealing algorithm for MaxCut.

Given a graph \( G = (V, E) \), the MaxCut problem asks for a cut with maximum size in \( G \); what is the maximum number of edges between vertices in \( V_+ \) and \( V_- \) where \( V_+ \cup V_- = V \) form a partition of vertices? Labeling vertices in \( V_+ \) with \( +1 \) and vertices in \( V_- \) with \( -1 \), this problem is equivalent to minimizing the energy of the following Hamiltonian:
\[
C := -\frac{1}{2} \sum_{(i,j) \in E} (I - Z_i Z_j), \] (28)

Where \( Z_i \)'s are the Pauli-\( z \) operators. Indeed, letting \( \text{Cut}^* \) be the maximum cut of \( G \) we have
\[
\text{Cut}^* = -\min_{|\phi\rangle} \langle \phi | C | \phi \rangle. \] (29)

We may use quantum annealing to find the ground state and minimum energy of \( C \). To this end, a common choice for the driving Hamiltonian is the summation of Pauli-\( x \) operators:
\[
B := -\sum_{k \in V} X_k. \] (30)

Using a control parameter \( u : [0, T] \rightarrow [0, 1] \) with \( u(0) = 0 \) and \( u(T) = 1 \), the resulting time-dependent Hamiltonian is given by
\[
H(t) := (1 - u(t)) B + u(t) C \] (31)
At time \( t = 0 \), we start with the ground state of \( H(0) = B \) that is equal to
\[
|\psi_0\rangle = |+\rangle \otimes^n, \] (32)
where \( n = |V| \) is the number of vertices. In the following we show that if \( T \) is bounded, then
\[
\langle \psi_T | C | \psi_T \rangle, \] (33)
is bounded away from the ground energy, i.e., \( \text{Cut}^* \), for a certain family of graphs to be defined.

For a graph \( G = (V, E) \) on \( n \) vertices, its Cheeger constant is defined by [39]:
\[
h(G) = \min_{S \subseteq V, 0 < |S| \leq \frac{4}{\alpha}} \frac{|\partial^E(S)|}{|S|}, \] (34)
where \( \partial^E(S) \) is the set of edges with one end in \( S \) and one end outside of \( S \). Ramanujan graphs are certain families of graphs that have a large Cheeger constant; for a \( \Delta \)-regular Ramanujan graph \( G \) we have
\[
h(G) \geq \frac{1}{2} (\Delta - 2\sqrt{\Delta - 1}). \] (35)
It is known that [40–42] for any \( \Delta \geq 3 \) there exists an infinite family of \( \Delta \)-regular bipartite Ramanujan graphs. We note that for such a graph, as a bipartite graph, we have
\[
\text{Cut}^* = |E| = \frac{1}{2} \Delta n. \] (36)

**Theorem 5.** Let \( G \) be a \( \Delta \)-regular bipartite Ramanujan graph on \( n \) vertices. Consider the quantum annealing algorithm given by Eq. (31) and Eq. (32). Assume that \( T \) satisfies
\[
T \leq \frac{\kappa_1}{4\Delta} \log n, \] (37)
for some \( \kappa_1 > 0 \). Then, for any \( 0 < \alpha < 1/2, 0 < \epsilon < 1 \) and sufficiently large \( n \) we have
\[
\frac{-\langle \psi_T | C | \psi_T \rangle}{\text{Cut}^*} < 1 - \alpha (1 - \epsilon) + 2\alpha (1 - \epsilon) \frac{\sqrt{\Delta - 1}}{\Delta}. \] (38)

We note that for appropriate choices of \( \alpha, \epsilon \) and for sufficiently large (but constant) \( \Delta \geq 6 \), the right hand side of Eq. (38) can be made smaller than 0.87856. This means that the quantum annealing algorithm with \( T \leq O(\log n) \) does not improve on the classical algorithm of Goemans and Williamson for MaxCut [43].

As the following proof of the theorem shows, the smallest \( n \) for which the bound Eq. (35) holds can be explicitly estimated in terms of \( \alpha, \epsilon, \kappa_1 \) and \( \Delta \).

We remark that, comparing to Eq. (14), in Eq. (37) the parameter \( \Delta \) is replaced with \( \Delta + 1 \). The point is that in the MaxCut Hamiltonian Eq. (31), the number of terms acting on a vertex is \( \Delta + 1 \) due to the extra term coming from \( B \).
Proof: We follow similar steps as in [24] to prove this theorem. Let \( x \in \{0,1\}^n \) be the measurement outcome on \( |\psi_T\rangle \) in the computational basis. Let \( \text{Cut}(x) \) be the size of the cut associated to \( x \):

\[
\text{Cut}(x) = -\langle x | C | x \rangle.
\]

Then we have

\[
-\langle \psi_T | C | \psi_T \rangle = \mathbb{E}[\text{Cut}(x)].
\]

Here a crucial observation is that \( \text{Cut}(x) = \text{Cut}(\bar{x}) \) where \( x \in \{0,1\}^n \) is obtained from \( x \) by flipping each coordinate: \( \bar{x}_i = 1 - x_i \). On the other hand, the distribution of \( \bar{x} \) is the same as the distribution of \( x \). This is because \( X^{\otimes n} |\psi_T\rangle = |\psi_T\rangle \) which itself can be proven using the fact that the starting state of QA satisfies \( X^{\otimes n} |\psi_0\rangle = |\psi_0\rangle \) and that \( X^{\otimes n} \) commutes with \( H(t) \) for any \( t \).

Let \( x^* \) be the configuration associated to the bipartition of \( G \). Then we have \( \text{Cut}^* = \text{Cut}(x^*) = |E| = \Delta n/2 \). Let \( d = an \) and define

\[
F_1 = \{ x : \text{dist}_H(x, x^*) \leq d \},
\]

where as before, \( \text{dist}_H(x, x^*) \) denotes the hamming distance between \( x \) and \( x^* \). Similarly, define

\[
F_2 = \{ x : \text{dist}_H(x, \bar{x}^*) \leq d \} = \{ x : \bar{x} \in F_1 \}.
\]

Now using Corollary [4] and the fact that \( D := \text{dist}_H(F_1, F_2) \geq n - 2d \) we have

\[
p(F_1) = p(F_2) = \min\{p(F_1), p(F_2)\} \leq \frac{16\ell}{D} + 10n^{-\kappa_3},
\]

where the first equality follows from the aforementioned symmetry and \( \ell \) is given by Eq. [13] for some \( \kappa_2 > 0 \) and

\[
\kappa_3 = \kappa_1(1 + \kappa_2) \log(1 + \kappa_2) - 1.
\]

Suppose that \( x \) is not in \( F_1 \cup F_2 \). This means that either \( d < \text{dist}_H(x, x^*) \leq n/2 \) or \( d < \text{dist}_H(x, \bar{x}^*) \leq n/2 \). Then, e.g., in the first case, using the definition of the Cheeger constant we have

\[
\text{Cut}(x) \leq |E| - h(G)d < \text{Cut}(x^*).
\]

This means that, letting

\[
J = \{ x : \text{Cut}(x) \geq |E| - h(G)d \},
\]

we have \( J \subseteq F_1 \cup F_2 \) and

\[
p(J) \leq p(F_1) + p(F_2) \leq 2\left(\frac{16\ell}{D} + 10n^{-\kappa_3}\right).
\]

Therefore,

\[
p\left( |E| - \text{Cut}(x) > h(G)d \right) \geq 1 - \frac{32\ell}{D} - 20n^{-\kappa_3}.
\]

Next, using Markov's inequality we have

\[
\mathbb{E}[|E| - \text{Cut}(x)] \geq h(G)d \left( 1 - \frac{32\ell}{D} - 20n^{-\kappa_3} \right).
\]

Now, note that \( D \geq n - 2d = (1 - 2\alpha)n \) and that for sufficiently large \( n \) we have

\[
1 - \frac{32\ell}{D} - 20n^{-\kappa_3} \geq 1 - \epsilon.
\]

Thus, using \( |E| = \frac{n\Delta}{2} \) and \( h(G) \geq \frac{\Delta^2 - \Delta - 1}{2} \), we obtain

\[
\mathbb{E}[\text{Cut}(x)] \leq |E| - \alpha(1 - \epsilon)h(G)n.
\]

Thus, using \( |E| = \frac{n\Delta}{2} \) and \( h(G) \geq \frac{\Delta^2 - \Delta - 1}{2} \), we obtain

\[
\mathbb{E}[\text{Cut}(x)] \leq 1 - \alpha(1 - \epsilon) + 2\alpha(1 - \epsilon)\sqrt{\Delta - 1}\Delta.
\]

\[\square\]

V. MAXCUT ON RANDOM BIPARTITE GRAPHS

Inspired by the recent work of Farhi, Gamarnik and Gutmann [22], in this section we prove that short-time QA algorithm for MAXCUT does not perform well on random bipartite graphs. The idea is that such a short-time QA algorithm does not see the whole graph, so it cannot distinguish between a bipartite random graph and an arbitrary random graph. Thus, the output cut-size of these algorithms on random bipartite graphs is bounded by the MAXCUT of arbitrary random graphs.

Let \( G_\Delta(n) \) be a random \( \Delta \)-regular graph on \( n \) vertices and \( G_\Delta^R(n) \) be a bipartite random \( \Delta \)-regular graph on \( n \) vertices. Let \( L \) be a parameter that grows at most logarithmically with \( n \). Then, it is known [44] that there exists a constant \( \alpha_\Delta \) independent of \( n, L \) such that

\[
\mathbb{E}[G_\Delta(n)] \leq \alpha_\Delta \Delta^{2L+1},
\]

\[
\mathbb{E}[G_\Delta^R(n)] \leq \alpha_\Delta \Delta^{2L+1}.
\]

Given a graph \( G = (V, E) \) let \( R_L(G) \) be the set of edges in \( G \) that have a cycle in their \( L \)-neighborhood:

\[
R_L(G) = \{ e \in E | \exists \text{ a cycle in the } L\text{-neighborhood of } e \}.
\]

It is easy to verify that if \( e \) belong to \( R_L(G) \), then there is cycle of length at most \( 2L + 1 \) in the \( L \)-neighborhood of \( e \). On the other hand, for any such cycle, the number of edges in its \( L \)-neighborhood is at most:

\[
(2L+1) \left[ 1 + \Delta + \Delta(\Delta - 1) + \cdots + (\Delta - 1)^{L-1} \right] \leq 2(2L+1)\Delta^L
\]

Therefore, by the above bound on the expectation of the number of cycles of length at most \( 2L + 1 \) we have

\[
\mathbb{E}[G_\Delta(n)] \leq \alpha_\Delta n \Delta^{2L+1} \leq \alpha_\Delta n^e,
\]

\[
\mathbb{E}[G_\Delta^R(n)] \leq 2\alpha_\Delta (2L+1)\Delta^{3L+1} \leq \alpha_\Delta n^e.
\]
Theorem 6. Consider the QA algorithm using the Hamiltonian \([31]\) with the initial state \(|\psi_0\rangle = |+\rangle^{\otimes n}\), and time
\[ T \leq \frac{\kappa}{16 \log(\Delta) \Delta^{1+4/\kappa}} \log n, \] (59)
for some \(0 < \kappa < 1\). Then, for any \(\nu > 0\) and sufficiently large \(n\), the expected cut size from applying the algorithm on a random \(\Delta\)-regular bipartite graphs is at most
\[ \left( \frac{\Delta}{4} + O(\sqrt{\Delta}) + 2\nu \right) n. \] (60)

Note that, by the bound \([60]\), the performance of the QA algorithm on random \(\Delta\)-regular bipartite graphs is not much better than outputting a random cut.

Proof. Let
\[ L = \frac{\kappa}{4 \log \Delta} \log n. \] (61)
For an edge \(e = \{i, j\}\) in \(G\) define
\[ \mathcal{E}(e, L) = \langle \psi_0 | V_e(T)^\dagger C_e V_e(T) | \psi_0 \rangle, \]
where
\[ C_e = \frac{1}{2} (I - Z_i Z_j), \] (63)
and \(V_e(T)\) is the unitary evolution associated to the Hamiltonian that includes the terms of \(H(t)\) that correspond to edges/vertices at distance at most \(L\) from \(e\). Note that by Theorem \([1]\) we have
\[ \| \Upsilon \| \leq \epsilon(L) = \sqrt{\frac{2}{\pi}} e^{-L \left( \log L - \log T - \log(4\Delta) \right) - \frac{1}{4} \log L}, \] (64)
where
\[ \Upsilon_e = U(T)^\dagger C_e U(T) - V_e(T)^\dagger C_e V_e(T). \] (65)

On the other hand, observe that the \(L\)-neighborhood of any edge \(e\) not in \(R_L(G)\) is a tree, and that this is a fixed tree. Then, since \(V_e(T)\) depends only on the structure of the graph around \(e\), where \(|\psi_0\rangle\) is symmetric, for any \(e \notin R_L(G)\), the quantity
\[ \mathcal{E}(e, L) = \mathcal{E}_{\text{tree}}(L), \] (66)
is independent of \(e\). Thus, using the fact that \(\|C_e\| \leq 1\), we have
\[ -\langle \psi_T | C | \psi_T \rangle = -\langle \psi_0 | U(T)^\dagger C U(T) | \psi_0 \rangle - \sum_{e \notin R_L(G)} \langle \psi_0 | U(T)^\dagger C_e U(T) | \psi_0 \rangle + \sum_{e \in R_L(G)} \langle \psi_0 | U(T)^\dagger C_e U(T) | \psi_0 \rangle \leq |E| \cdot \left( \mathcal{E}_{\text{tree}}(L) + \epsilon(L) \right) + |R_L(G)| \] (67)
and
\[ \mathcal{E}_{\text{tree}}(L) \leq \mathcal{E}_{\text{tree}}(L) + \epsilon(L) + |R_L(G)|, \] (68)

Therefore,
\[ \mathbb{E}_{G_B(n)} \left[ -\langle \psi_T | C | \psi_T \rangle \right] \leq \frac{n \Delta}{2} \left( \mathcal{E}_{\text{tree}}(L) + \epsilon(L) \right) + 2 \alpha \Delta (2L + 1) \Delta^{3L + 1}. \] (69)

On the other hand, we also have
\[ -\langle \psi_T | C | \psi_T \rangle = \sum_{e \notin R_L(G)} \langle \psi_0 | U(T)^\dagger C_e U(T) | \psi_0 \rangle + \sum_{e \in R_L(G)} \langle \psi_0 | U(T)^\dagger C_e U(T) | \psi_0 \rangle \geq \left( |E| - |R_L(G)| \right) \left( \mathcal{E}_{\text{tree}}(L) - \epsilon(L) \right), \] (70)
and
\[ \mathbb{E}_{G_B(n)} \left[ -\langle \psi_T | C | \psi_T \rangle \right] \geq \left( \frac{n \Delta}{2} - 2 \alpha \Delta (2L + 1) \Delta^{3L + 1} \right) \left( \mathcal{E}_{\text{tree}}(L) - \epsilon(L) \right). \] (71)

It is known that for any \(\Delta\), there exists a constant \(\rho_\Delta\) such that
\[ \mathbb{E}_{G_B(n)} \left[ \text{size of MaxCut} \right] = \rho_\Delta n + o(n). \] (72)
Therefore, since \(-\langle \psi_T | C | \psi_T \rangle\) itself is an expectation of sizes of certain cuts, we have
\[ \mathbb{E}_{G_B(n)} \left[ -\langle \psi_T | C | \psi_T \rangle \right] \leq \rho_\Delta n + o(n). \] (73)

Next, using the definition of \(L\) and the bound on \(T\), we fine that for any \(\nu > 0\) and sufficiently large \(n\) we have
\[ \frac{\Delta}{2} \mathcal{E}_{\text{tree}} \leq \rho_\Delta + \nu, \] (74)
where \([43]\)
\[ \mathcal{E}_{\text{tree}} = \limsup_{n \to \infty} \mathcal{E}_{\text{tree}}(L). \] (75)
Using this in \([71]\) we find that for sufficiently large \(n\)
\[ \mathbb{E}_{G_B(n)} \left[ -\langle \psi_T | C | \psi_T \rangle \right] \leq (\rho_\Delta + 2\nu) n. \] (76)

It is known that \(\rho_3 \leq 1.4026 < \frac{1}{2}\), and that for large \(\Delta\) \([46]\)
\[ \rho_\Delta \leq \frac{\Delta}{4} + O(\sqrt{\Delta}). \] (77)

Thus, for sufficiently large \(n\), short time QA on a random bipartite graphs gives a cut of size of at most
\[ \left( \frac{\Delta}{4} + O(\sqrt{\Delta}) + 2\nu \right) n, \] (78)
which is far from the optimal value of \(\frac{\Delta}{\pi} n.\)

There is another result of \([22]\) that proves limitations of low-depth quantum circuits for the maximum independent set problem. That result can also be generalized to short-time QA algorithm similarly.
VI. CONCLUSIONS

In this paper we provided a framework for proving limitations on short-time quantum annealing algorithms. In particular, we showed that the distribution of the measurement outcome of a short-time QA computation is concentrated and satisfies an isoperimetric inequality. Using our framework, we generalized results about limitations of low-depth quantum circuits, to short-time QA algorithms. To this end, a main tool that we developed, is a Lieb-Robinson bound that works for time-dependent Hamiltonians and is amenable to QA algorithms. This tool is as an essential ingredient of all of our proofs. In proving Theorem 3 we need other tools besides the Lieb-Robinson bound. The point is that sometimes we need to consider functions of operators that (by the Lieb-Robinson bound) are known to be almost local. Then to approximate the output of those functions, we need to bound the derivative of those functions with respect to an operator. In the proof of Theorem 4 we took this approach when the underlying function is a variation of Chebyshev’s polynomial. The techniques used to bound the $\gamma_2$-norm of this polynomial may be useful elsewhere as well.

We believe that the tools of this paper can be applied to generalize other results about the limitation of low-depth circuits, see e.g., 23, 25, for short-time QA algorithm. We leave this for future research.

[1] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, International Journal of Heat and Mass Transfer 17, 401 (2000) arXiv:0003106 [quant-ph].
[2] T. Kato, Journal of the Physical Society of Japan 5, 435 (1950) https://doi.org/10.1143/JPSJ.5.435.
[3] M. Born and V. Fock, Zeitschrift fur Physik 51, 165 (1928).
[4] I. Hen and F. M. Spedalieri, Physical Review Applied 5, 034007 (2016) arXiv:1508.04212.
[5] S. Puri, C. K. Andersen, A. L. Grimsmo, and A. Blais, Nature Communications 8, 15785 (2017).
[6] W. Lechner, P. Hauke, and P. Zoller, Science Advances 1, e1500838 (2015).
[7] S. Jiang, K. A. Britt, A. J. McCaskey, T. S. Humble, and S. Kais, Scientific Reports 7, 17667 (2018).
[8] R. Y. Li, R. Di Felice, R. Rohs, and D. A. Lidar, NPJ quantum information 4, 1 (2018).
[9] L. Stella, G. E. Santoro, and E. Tosatti, Physical Review B 72, 014303 (2005).
[10] O. Titiloye and A. Crispin, Discrete Optimization 8, 376 (2011).
[11] A. Mott, J. Job, J.-R. Vlimant, D. Lidar, and M. Spiropulu, Nature 550, 375 (2017).
[12] K. L. Pudenz, T. Albash, and D. A. Lidar, Physical Review A 91, 042302 (2015).
[13] A. Perdomo-Ortiz, N. Dickson, M. Drew-Brook, G. Rose, and A. Aspuru-Guzik, Scientific reports 2, 571 (2012).
[14] K. L. Pudenz, T. Albash, and D. A. Lidar, Nature communications 5, 1 (2014).
[15] R. Martinòjak, G. E. Santoro, and E. Tosatti, Physical Review E 70, 057701 (2004).
[16] S. H. Adachi and M. P. Henderson (2015) arXiv:1510.06356.
[17] M. W. Johnson, M. H. Amin, S. Gildert, T. Lanting, F. Hamze, N. Dickson, R. Harris, A. J. Berkley, J. Johansson, P. Bunyk, et al., Nature 473, 194 (2011).
[18] S. Boixo, T. Albash, F. M. Spedalieri, N. Chancellor, and D. A. Lidar, Nature communications 4, 1 (2013).
[19] B. Foxen, C. Neill, A. Dunsworth, P. Roushan, B. Chiaro, A. Megrant, J. Kelly, Z. Chen, K. Satzinger, R. Barends, F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, S. Boixo, D. Buell, B. Burkett, Y. Chen, R. Collins, E. Farhi, A. Fowler, C. Gidney, M. Giustina, R. Graff, M. Harrigan, T. Huang, S. V. Isakov, E. Jeffrey, Z. Jiang, D. Kafri, K. Kechedzhi, P. Klimov, A. Korotkov, F. Kostritsa, D. Landsdell, E. Lucero, J. McClean, M. McEwen, X. Mi, M. Mohseni, J. Y. Mutus, O. Naaman, M. Neeley, M. Niu, A. Petukhov, C. Quintana, N. Rubin, D. Sank, V. Smelyanskiy, A. Vainschencher, T. C. White, Z. Yao, P. Yeh, A. Zalcman, H. Neven, and J. M. Martinis, Physical Review Letters 125, 120504 (2020).
[20] K. Wright, K. Beck, S. Deb Nath, J. Amini, Y. Nam, N. Grzesiak, J.-S. Chen, N. Pisenti, M. Chmielowski, C. Collins, et al., Nature communications 10, 1 (2019).
[21] E. Farhi, J. Goldstone, and S. Gutmann (2014) arXiv:1411.4028.
[22] E. Farhi, D. Gamarnik, and S. Gutmann (2020) arXiv:2005.08747.
[23] E. Farhi, D. Gamarnik, and S. Gutmann (2020) arXiv:2004.09002.
[24] S. Bravyi, A. Kliess, R. Koenig, and E. Tang, Physical Review Letters 125, 260505 (2020) arXiv:1910.08980.
[25] S. Bravyi, D. Gosset, and R. Movassagh, Nature Physics 17, 337 (2021) arXiv:1909.11485.
[26] S. Bravyi, A. Kliess, R. Koenig, and E. Tang, Hybrid quantum-classical algorithms for approximate graph coloring (2020) arXiv:2011.13420 [quant-ph].
[27] L. Eldar and A. W. Harrow, in 2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS) Vol. 2017-Octob (IEEE, 2017) pp. 427–438, arXiv:1510.02082.
[28] A. Bapat and S. Jordan, Quantum Information and Computation 19, 424 (2019).
[29] G. B. M. Mbeng, R. Fazio, and G. Santoro (2019) arXiv:1906.08948.
[30] L. T. Brady, K. Wright, K. Beck, S. Debnath, J. Amini, Y. Nam, N. Grzesiak, A. Kliess, R. Koenig, and E. Tang, Hybrid quantum-classical algorithms for approximate graph coloring (2020) arXiv:2003.08952.
[31] A. M. Childs, Y. Su, M. C. Tran, N. Wiebe, and S. Zhu, Physical Review X 11, 011020 (2021) arXiv:1912.08854.
[32] B. Nachtergaele, Y. Ogata, and R. Sims, Journal of Statistical Physics 124, 1 (2006).
[33] B. Nachtergaele and R. Sims (AMS, 2010) pp. 141–176, arXiv:1004.2086.
[34] S. Bravyi, M. B. Hastings, and F. Verstraete, Physical review letters 97, 050401 (2006).
[35] C.-F. Chen and A. Lucas (2019) arXiv:1905.03682.
[36] E. H. Lieb and D. W. Robinson, Communications in Mathematical Physics 28, 251 (1972).
[37] J. Haah, M. B. Hastings, R. Kothari, and G. H. Low, 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS) arXiv:1801.03922.
[38] A. Lubotzky, R. Phillips, and P. Sarnak, Combinatorica 8, 261 (1988).
[39] B. Mohar, Journal of Combinatorial Theory, Series B 47, 274 (1989).
[40] A. W. Marcus, D. A. Spielman, and N. Srivastava, in 2015 IEEE 56th Annual Symposium on Foundations of Computer Science Vol. 2015-Decem (IEEE, 2015) pp. 1358–1377.
[41] A. W. Marcus, D. A. Spielman, and N. Srivastava, SIAM Journal on Computing 47, 2488 (2018) arXiv:1505.08010.
[42] C. Hall, D. Puder, and W. F. Sawin.
Appendix A: Proof of the Lieb-Robinson Bound

In this appendix we give a proof of Theorem 1. As mentioned before, the proof ideas are essentially the same as those of the original Lieb-Robinson bound. Here, for simplicity of presentation of the proof we first give two lemmas.

**Lemma 1. (from [33])** Let \( X = X(t) \) and \( Y = Y(t) \) be time-dependent bounded operators with \( X \) being Hermitian. Let \( F = F(t) \) be given by

\[
\frac{d}{dt} F = i[X,F] + Y. \tag{A1}
\]

Then we have

\[
\frac{d}{dt}\|F(t)\| \leq \|Y(t)\|. \tag{A2}
\]

**Proof.** Writing the definition of the derivative we have

\[
\frac{d}{dt}\|F\| = \lim_{h \to 0} \frac{\|F(t + h)\| - \|F(t)\|}{h} = \lim_{h \to 0} \frac{\|F(t) + i[hX(t), F(t)] + hY(t)\| + O(h^2) - \|F(t)\|}{h} = \lim_{h \to 0} \frac{\|(I + i[hX,F] + iXFX + hY)\| - \|F\|}{h} \leq \lim_{h \to 0} \frac{\|e^{i\tau X} F e^{-i\tau X} - \|F\|}{h} + h\|XFX\| + \|Y\|, \tag{A3}
\]

where in the forth line we use \( e^{\pm i\tau X} = I \pm i\tau X + O(\tau^2) \), and in the last line we use the fact that \( X \) is Hermitian and \( e^{\pm i\tau X} \) is unitary.

Here is our second building block of the proof.

**Lemma 2.** Using the notation of Theorem 1 and assuming that \( L > 1 \) we have

\[
\|U(t)^\dagger H_A U(t) - V(t)^\dagger V(t)\| \leq 2\|A\| \sum_{e \sim A} \int_0^t |u_e(\tau)| \cdot |U^\dagger(\tau) h_e U(\tau) - V(\tau)^\dagger h_e V(\tau)| \, d\tau, \tag{A4}
\]

where \( \tilde{H}_A(t) := \sum_{e \subseteq A \cup \partial_L(A)} u_e(t) h_e \) is the Hamiltonian consisting of terms in the region \( A \cup \partial_L(A) \), and \( V(t) \) is the unitary evolution associated to \( \tilde{H}_A(t) \). Also, by \( e \sim A \) we mean \( e \) intersects \( A \). Moreover, if \( A = \{e_0\} \) for some edge \( e_0 \) and \( O_A = h_{e_0} \), then the sum in Eq. (A3) can be further restricted to \( e \neq e_0 \).

**Proof.** Let

\[
F(t) := U(t)^\dagger H_A U(t) - V(t)^\dagger V(t). \tag{A5}
\]
Then from Schrödinger’s equation we have

\[
\frac{d}{dt} \mathbf{F}(t) = i \mathbf{U}(t)^\dagger \left[ \mathbf{H}(t), \mathbf{O} \right] \mathbf{U}(t) - i \mathbf{V}(t)^\dagger \left[ \mathbf{H}(t), \mathbf{O} \right] \mathbf{V}(t)
\]

\[
= \sum_{e \sim A} i \mathbf{U}(t)^\dagger \left[ \mathbf{h}_e(t), \mathbf{O} \right] \mathbf{U}(t) - i \mathbf{V}(t)^\dagger \left[ \mathbf{h}_e(t), \mathbf{O} \right] \mathbf{V}(t)
\]

\[
= \sum_{e \sim A} i \left[ \mathbf{U}(t)^\dagger \mathbf{h}_e(t) \mathbf{U}(t), \mathbf{U}(t)^\dagger \mathbf{O} \mathbf{U}(t)^\dagger \right] - i \left[ \mathbf{V}(t)^\dagger \mathbf{h}_e(t) \mathbf{V}(t), \mathbf{V}(t)^\dagger \mathbf{O} \mathbf{V}(t) \right]
\]

\[
= \sum_{e \sim A} i \left[ \mathbf{U}(t)^\dagger \mathbf{h}_e(t) \mathbf{U}(t), \mathbf{F}(t) + \mathbf{V}(t)^\dagger \mathbf{O} \mathbf{V}(t) \right] - i \left[ \mathbf{V}(t)^\dagger \mathbf{h}_e(t) \mathbf{V}(t), \mathbf{V}(t)^\dagger \mathbf{O} \mathbf{V}(t) \right]
\]

\[
= i \left[ \sum_{e \sim A} \mathbf{U}(t)^\dagger \mathbf{h}_e(t) \mathbf{U}(t), \mathbf{F}(t) \right] + i \sum_{e \sim A} \left[ \mathbf{U}(t)^\dagger \mathbf{h}_e(t) \mathbf{U}(t) - \mathbf{V}(t)^\dagger \mathbf{h}_e(t) \mathbf{V}(t) \right] \mathbf{V}(t)^\dagger \mathbf{O} \mathbf{V}(t) \right] .
\]

Therefore, using \( \mathbf{F}(0) = 0 \) and Lemma 1 we can bound

\[
\| \mathbf{F}(t) \| = \left\| \int_0^t \frac{d}{d\tau} \mathbf{F}(\tau) d\tau \right\|
\]

\[
\leq \int_0^t \left\| \frac{d}{d\tau} \mathbf{F}(\tau) \right\| d\tau
\]

\[
\leq \int_0^t \sum_{e \sim A} \left\| \mathbf{U}(\tau)^\dagger \mathbf{h}_e(\tau) \mathbf{U}(\tau) - \mathbf{V}(\tau)^\dagger \mathbf{h}_e(\tau) \mathbf{V}(\tau), \mathbf{V}(\tau)^\dagger \mathbf{O} \mathbf{V}(\tau) \right\| d\tau
\]

\[
\leq 2\| \mathbf{O} \| \sum_{e \sim A} \int_0^t |u_e(\tau)| \cdot \left\| \mathbf{U}(\tau)^\dagger \mathbf{h}_e(\tau) \mathbf{U}(\tau) - \mathbf{V}(\tau)^\dagger \mathbf{h}_e(\tau) \mathbf{V}(\tau) \right\| d\tau ,
\]

where we use \( \| \mathbf{X}, \mathbf{Y} \| \leq 2 \| \mathbf{X} \| \| \mathbf{Y} \| \) in the last line. Note that if \( \mathbf{O}_A = \mathbf{h}_{e_0} \), in the second line of Eq. (A6) and thereafter, we can further restrict the sum to \( e \neq e_0 \).

\[ \square \]

We can now give the proof of Theorem 1. As before, let \( \mathbf{F}(t) := \mathbf{U}(t)^\dagger \mathbf{O} \mathbf{U}(t) - \mathbf{V}(t)^\dagger \mathbf{O} \mathbf{V}(t) \). By Lemma 2 we have

\[
\| \mathbf{F}(T) \| \leq 2\| \mathbf{O} \| \sum_{e_1 \sim A} \int_0^T |u_{e_1}(t_1)| \cdot \left\| \mathbf{U}(t_1)^\dagger \mathbf{h}_{e_1} \mathbf{U}(t_1) - \mathbf{V}(t_1)^\dagger \mathbf{h}_{e_1} \mathbf{V}(t_1) \right\| dt_1 .
\]

(A8)

Assuming that \( L > 2 \), we can apply Lemma 2 on each of the terms in the above sum, in which \( A = e_1 \) and \( \mathbf{O}_A \) is replaced by \( \mathbf{h}_{e_1} \). Therefore,

\[
\| \mathbf{F}(T) \| \leq 2\| \mathbf{O}_A \| \sum_{e_1 \sim A} \int_0^T \sum_{e_2 \sim A} \left\| u_{e_2}(t_2) \right\| \cdot \left\| \mathbf{U}(t_2)^\dagger \mathbf{h}_{e_2} \mathbf{U}(t_2) - \mathbf{V}(t_2)^\dagger \mathbf{h}_{e_2} \mathbf{V}(t_2) \right\| dt_2 dt_1
\]

\[
\leq 2\| \mathbf{O}_A \| 2g \sum_{e_1 \sim A} \sum_{e_2 \sim A} \int_0^T \int_0^{t_1} |u_{e_2}(t_2)| \cdot \left\| \mathbf{U}(t_2)^\dagger \mathbf{h}_{e_2} \mathbf{U}(t_2) - \mathbf{V}(t_2)^\dagger \mathbf{h}_{e_2} \mathbf{V}(t_2) \right\| dt_2 dt_1 .
\]

(A9)

We can repeat the same process for \( L \) times to obtain

\[
\frac{\| \mathbf{F}(T) \|}{2\| \mathbf{O}_A \| (2g)^{L-1}} \leq \sum_{\text{walk starting at } A} \int_0^T \int_0^{t_1} \cdots \int_0^{t_{L-1}} |u_{e_L}(t_L)| \cdot \left\| \mathbf{U}(t_L)^\dagger \mathbf{h}_{e_L} \mathbf{U}(t_L) - \mathbf{V}(t_L)^\dagger \mathbf{h}_{e_L} \mathbf{V}(t_L) \right\| dt_L \cdots dt_2 dt_1
\]

\[
\leq \sum_{\text{walk starting at } A} \int_0^T \int_0^{t_1} \cdots \int_0^{t_{L-1}} 2g dt_L \cdots dt_2 dt_1
\]

\[
= 2g \frac{T^L}{L!} N_L ,
\]

(A10)
where $N_L$ denotes the number of walks of length $L$ starting at $A$, i.e., $N_L$ is the number of sequences of edges $(e_1, \ldots, e_L)$ such that $e_1 \sim A, e_i \sim e_{i+1}$ and $e_i \neq e_{i+1}$ for $i = 1, \ldots, L - 1$. We note that as the maximum degree of the graph $G$ is $\Delta$, the number of choices for $e_1$ is at most $\Delta |A| \leq 2(\Delta - 1)|A|$. Moreover, the number of choices for $e_{i+1}$ given $e_i$ is at most $2(\Delta - 1)$. Thus $N_L \leq (2(\Delta - 1))^L |A|$.

Therefore, using the Stirling’s approximation we have

$$
\|F(T)\| \leq 2|A| \|O_A\| \frac{(4g(\Delta - 1)T)^L}{L!}
\leq 2|A| \cdot \|O_A\| \frac{(4g(\Delta - 1)T)^L}{L!} e^{-L \log L + L - \frac{1}{2} \log L}
\leq \frac{\sqrt{2\pi}}{\sqrt{2\pi}} |A| \cdot \|O_A\| e^{-L \left( \log L - \log T - \log (4g(\Delta - 1)) \right) - \frac{1}{2} \log L}.
$$

**Appendix B: Proof of Theorem**

Let $\zeta = \sum_i Z_i$ and use Markov’s inequality to obtain

$$
\Pr \left[ \left| w_H(x) - \left( \frac{n}{2} - \frac{m}{2} \right) \right| \geq cn^{\frac{1}{4} + \epsilon/2} \right] = \Pr \left[ \left| \langle x | \zeta | x \rangle - m \right| \geq 2cn^{\frac{1}{4} + \epsilon/2} \right]
\leq \frac{1}{2cn^{\frac{1}{4} + \epsilon/2}} \mathbb{E} \left[ \left( \langle x | \zeta | x \rangle - m \right)^2 \right].
$$

Thus to prove the theorem it suffices to bound $\mathbb{E} \left[ \left( \langle x | \zeta | x \rangle - m \right)^2 \right]$.

Let $m_i = \langle \psi_T | Z_i | \psi_T \rangle$. Then we have

$$
\mathbb{E} \left[ \left( \langle x | \zeta | x \rangle - m \right)^2 \right] = \mathbb{E} \left[ \left( \sum_i \langle x | Z_i | x \rangle - m_i \right)^2 \right]
= \sum_{i,j} \mathbb{E} \left[ \left( \langle x | Z_i | x \rangle - m_i \right) \cdot \left( \langle x | Z_j | x \rangle - m_j \right) \right]
= \sum_{i,j} \langle \psi_T | (Z_i - m_i)(Z_j - m_j) | \psi_T \rangle
+ \sum_{\text{dist}_G(i,j) > 2L} \langle \psi_T | (Z_i - m_i)(Z_j - m_j) | \psi_T \rangle,
$$

where in the third line we use the fact that $Z_i$ is diagonal in the computational basis, $\text{dist}_G(i,j)$ denotes the distance of vertices $i, j$ on the graph $G$, and $L$ is given by

$$
L = \frac{\kappa_1}{2 \log \Delta} \log n \geq 4\Delta^{\frac{2-\kappa_1}{\kappa_1}} T.
$$

We analyze terms in the above sum separately. If $\text{dist}_G(i,j) \leq 2L$, then by the Cauchy-Schwarz inequality we have

$$
\langle \psi_T | (Z_i - m_i)(Z_j - m_j) | \psi_T \rangle \leq \sqrt{\langle \psi_T | (Z_i - m_i)^2 | \psi_T \rangle \cdot \langle \psi_T | (Z_j - m_j)^2 | \psi_T \rangle}
= \sqrt{1 - m_i^2} \sqrt{1 - m_j^2}
\leq 1.
$$

Next, by the Lieb-Robinson bound (Theorem 1), for any $i$ there is an operator $\tilde{Z}_i$ that acts only on $\partial_L(\{i\})$ and

$$
\|U(T)^\dagger (Z_i - m_i) U(T) - \tilde{Z}_i \| \leq e^{-L \left( \log L - \log T - \log 4\Delta \right)} \leq e^{-L \left( \frac{2-\kappa_1}{\kappa_1} \log \Delta \right)} = n^{-(1-\kappa_1)},
$$
where in the last line we use the facts that $|\psi_0\rangle$ is a product state and that the supports of $\hat{Z}_i, \hat{Z}_j$ do not intersect. Now we have

$$\langle \psi_T | (Z_i - m_i) (Z_j - m_j) | \psi_T \rangle = \langle \psi_0 | (U(T) \dagger (Z_i - m_i) (Z_j - m_j) U(T) ) |\psi_0 \rangle - \langle \psi_0 | Y_i | \psi_0 \rangle | \psi_0 \rangle = \langle \psi_0 | Z_i | \psi_0 \rangle + \langle \psi_0 | Y_i | \psi_0 \rangle + \langle \psi_0 | Z_j | \psi_0 \rangle + \langle \psi_0 | Y_j | \psi_0 \rangle + \langle \psi_0 | Z_i Z_j | \psi_0 \rangle + \langle \psi_0 | Y_i Y_j | \psi_0 \rangle ,$$

where in the last line we use the facts that $|\psi_0\rangle$ is a product state and that the supports of $\hat{Z}_i, \hat{Z}_j$ do not intersect. Now we have

$$| \langle \psi_0 | \hat{Z}_i | \psi_0 \rangle | = | \langle \psi_T | (Z_i - m_i) | \psi_T \rangle - \langle \psi_0 | Y_i | \psi_0 \rangle | \leq \| Y_i \| \leq n^{-(1 - \kappa_1)} .$$

Using this, and similar inequalities, we find that

$$\langle \psi_T | (Z_i - m_i) (Z_j - m_j) | \psi_T \rangle \leq 2 \left( n^{-(1 - \kappa_1)} \right)^2 + 2n^{-2(1 - \kappa_1)} \leq 4n^{-2(1 - \kappa_1)} .$$

Putting these together yields

$$E \left[ (x | \zeta | x) - m \right]^2 \leq \sum_{i,j} 1 + \sum_{ \text{dist}_{G(i,j)} \leq 2L} 4n^{-2(1 - \kappa_1)} \leq 2n\Delta_{2L} + n^2 \left( 4n^{-2(1 - \kappa_1)} \right) \leq 6n^{1 + \kappa_1} ,$$

where in the second line we use

$$1 + \Delta \left( 1 + (\Delta - 1) + (\Delta - 1)^2 + \cdots (\Delta - 1)^{2L-1} \right) \leq 2\Delta_{2L} .$$

Using this in (B2) the desired inequality follows.

**Appendix C: Derivative of a matrix function**

In this appendix we compute a bound on the derivative of a certain polynomial function of matrices. This bound will be used in the proof of Theorem 3.

We first need to fix some notations. Given two matrices $A$ and $B$ of the same dimensions, we denote their entry-wise product by $A \circ B$. In other words, $A \circ B$ is a matrix whose $(i, j)$-th entry equals the product of $(i, j)$-th entries of $A$ and $B$, i.e., $(A \circ B)_{ij} = A_{ij} \cdot B_{ij}$. The matrix $A \circ B$ is called the Schur or Hadamard product of $A$ and $B$.

For a continuously differentiable function $f : (a, b) \to \mathbb{R}$ and a sequence $\lambda = (\lambda_1, \ldots, \lambda_d)$ of numbers in $(a, b)$ we let $D_{f, \lambda}$ be the $d \times d$ matrix whose $(i, j)$-th entry equals

$$(D_{f, \lambda})_{ij} = \begin{cases} \frac{f(\lambda_j) - f(\lambda_i)}{\lambda_j - \lambda_i} & \lambda_i \neq \lambda_j, \\ f'(\lambda_i) & \lambda_i = \lambda_j, \end{cases}$$

where $f'(\lambda_i)$ is the derivative of $f$ at $\lambda_i$. For a diagonal matrix $\Lambda = \text{diag}(\lambda)$ with diagonal elements in $\lambda = (\lambda_1, \ldots, \lambda_d)$ we let

$$D_{f, \Lambda} := D_{f, \lambda} .$$

Also, for a self-adjoint matrix $A$, that can be diagonalized as $A = U \Lambda U^\dagger$ where $U$ is unitary and $\Lambda = \text{diag}(\lambda)$ is a diagonal matrix containing eigenvalues of $A$, we let

$$D_{f, A} = UD_{f, \Lambda} U^\dagger .$$
Lemma 3. \cite{43} pp. 124] Let \(A, E\) be two Hermitian matrices and \(f\) be a real continuously differentiable function on an interval containing the eigenvalues of \(A + tE\) for all \(t \in (-1, 1)\). Then we have

\[ \frac{d}{dt} f(A + tE) \Big|_{t=0} = D_{f,A} \circ E. \] (C4)

By this lemma, to bound the derivative of a matrix function as above, we need to bound the norm of the Schur product as a super-operator on the space of matrices. This super-operator norm is given by the so called \(\gamma_2\)-norm.

Lemma 4. \cite{50} pp. 79] For a \(d \times d\) matrix \(M\) let

\[ \gamma_2(M) := \inf \max \{ \| v_i \|_2 : 1 \leq i \leq d \} \cup \{ \| w_j \|_2 : 1 \leq j \leq d \}, \] (C5)

where the infimum is taken over all vectors \(|v_i\rangle, |w_j\rangle\) satisfying \(M_{ij} = \langle v_i | w_j \rangle\) for all \(i, j\). Then for any \(d \times d\) matrix \(E\) we have

\[ \| M \circ E \| \leq \gamma_2(M) \| E \|. \] (C6)

It is not hard to verify that \(\gamma_2\)-norm as defined in Eq. (C5) satisfies the triangle inequality and is a norm. We also note that if \(M\) is a matrix and \(M'\) is another matrix, then we have \((M \circ M')_{ij} = \langle \tilde{v}_i | \tilde{w}_j \rangle\) where \(\tilde{v}_i = |v_i\rangle \otimes |v_i\rangle\) and \(\tilde{w}_j = |w_j\rangle \otimes |w_j\rangle\). Therefore,

\[ \gamma_2(M \circ M') \leq \gamma_2(M) \cdot \gamma_2(M'). \] (C7)

The following lemma is a simple consequence of the lemmas above.

Lemma 5. For any two Hermitian matrices \(A, E\) and polynomial \(P\), we have

\[ \| P(A + E) - P(A) \| \leq \| E \| \cdot \int_0^1 \gamma_2(D_{P,A+tE}) \, dt. \] (C8)

Proof: By Lemma 3 we have

\[ \| P(A + E) - P(A) \| = \int_0^1 \frac{d}{dt} P(A + tE) \, dt = \int_0^1 (D_{P,A+tE} \circ E) \, dt \] (C9)

Then the result follows using Lemma 5 and the triangle inequality.

In the proof of Theorem 8 we will use this lemma for a polynomial that is defined in terms of the Chebyshev’s polynomials. Recall that the Chebyshev’s polynomial \(T_n(x)\) is defined by

\[ T_n(\cos \theta) = \cos(n \theta). \] (C10)

This equation shows that \(|T_n(x)| \leq 1\) if \(|x| \leq 1\). The Chebyshev polynomials can also be defined by recursive equations: we have \(T_0(x) = 1, T_1(x) = x\) and

\[ T_{2n+1}(x) = 2T_{n+1}(x)T_n(x) - x, \]
\[ T_{2n}(x) = 2T_n^2(x) - 1. \] (C11) (C12)

Lemma 6. For any \(0 \leq \delta < 1\) and any sequence \(\lambda = (\lambda_1, \ldots, \lambda_d)\) with \(-1 \leq \lambda_i \leq 1 + \delta\) we have

\[ \gamma_2(D_{T_n,\lambda}) \leq (2n^2 - 1)T_n(1 + \delta), \quad \forall n \geq 1. \] (C13)

Proof: We prove the lemma by induction on \(n\). For \(n = 1\) it can be easily shown by setting \(T_1(x) = x\).

For an even number \(2n\) we can write:

\[ (D_{T_{2n}, \lambda})_{ij} = \frac{T_{2n}(\lambda_i) - T_{2n}(\lambda_j)}{\lambda_i - \lambda_j} \]
\[ = 2 \frac{T_n^2(\lambda_i) - T_n^2(\lambda_j)}{\lambda_i - \lambda_j} \]
\[ = 2 \frac{T_n(\lambda_i) - T_n(\lambda_j)}{\lambda_i - \lambda_j} (T_n(\lambda_i) + T_n(\lambda_j)) \]
\[ = 2 (D_{T_n, \lambda})_{ij} T_n(\lambda_i + T_n(\lambda_j)) \]
\[ = 2 (D_{T_n, \lambda})_{ij} \left( R_{T_n, \lambda} + R_{T_n, \lambda}^T \right)_{ij}, \] (C14) (C15) (C16) (C17) (C18)
where \((R_{T_n, \lambda})_{ij} = T_n(\lambda_i), \) and \(R_{T_n, \lambda}^T\) is its transpose. Now we can write:

\[
\gamma_2(D_{T_n, \lambda}) \leq 2\gamma_2(D_{T_n, \lambda})\gamma_2(R_{T_n, \lambda} + R_{T_n, \lambda}^T),
\]

\[
\leq 2 \left( (2n^2 - 1)T_n(1 + \delta) \right) \left( \gamma_2(R_{T_n, \lambda}) + \gamma_2(R_{T_n, \lambda}^T) \right),
\]

Using \(|T_n(x)| \leq T_n(1 + \delta)\) for \(x \in [-1, 1 + \delta]\), we can find that

\[
\gamma_2(R_{T_n, \lambda}), \gamma_2(R_{T_n, \lambda}^T) \leq T_n(1 + \delta).
\]

As a result

\[
\gamma_2(D_{T_n, \lambda}) \leq 4(2n^2 - 1)T_n^2(1 + \delta).
\]

Now recall that \(T_n^2(1 + \delta) = \frac{T_{2n(1+\delta)+1}^2}{2}\) and knowing that \(T_{2n}(1 + \delta) \geq 1\)

\[
\gamma_2(D_{T_n, \lambda}) \leq 2(2n^2 - 1)(T_{2n}(1 + \delta) + 1),
\]

\[
\leq 4(2n^2 - 1)T_{2n}(1 + \delta),
\]

\[
\leq (2(2n)^2 - 1)T_{2n}(1 + \delta).
\]

Similarly for an odd number \(2n + 1\), we can write:

\[
(D_{T_{2n+1}, \lambda})_{ij} = \frac{T_n(\lambda_i)T_{n+1}(\lambda_i) - T_n(\lambda_j)T_{n+1}(\lambda_j) - 1}{\lambda_i - \lambda_j}
\]

\[
= 2 \left( (D_{T_n, \lambda})_{ij} T_{n+1}(\lambda_i) + T_n(\lambda_j) \right) - 1
\]

\[
= 2 \left( (D_{T_n, \lambda})_{ij} + \left(R_{T_n, \lambda}^T \right) \left(D_{T_{n+1}, \lambda}\right)_{ij} \right) - 1.
\]

Now by bounding \(\gamma_2(R_{T_n, \lambda})\) and \(\gamma_2(R_{T_{n+1}, \lambda})\)

\[
\gamma_2(D_{T_{2n+1}, \lambda}) \leq 2\gamma_2(D_{T_n, \lambda})\gamma_2 \left( (R_{T_{n+1}, \lambda}) + 2\gamma_2(D_{T_{n+1}, \lambda}) \gamma_2(R_{T_{n+1}, \lambda}^T) + 1,\right.
\]

\[
\leq 2(2n^2 - 1)T_n(1 + \delta)\gamma_2(R_{T_{n+1}, \lambda}) +
\]

\[
2 \left( (2n + 1)^2 - 1 \right) T_{n+1}(1 + \delta) \gamma_2(R_{T_{n+1}, \lambda}^T) + 1,
\]

\[
\leq 2(2n^2 - 1)T_n(1 + \delta)T_{n+1}(1 + \delta) +
\]

\[
2 \left( (2n + 1)^2 - 1 \right) T_{n+1}(1 + \delta)T_n(1 + \delta) + 1,
\]

\[
\leq 2 \left( (2n + 1)^2 - 1 \right) T_{n+1}(1 + \delta)T_n(1 + \delta) + 1.
\]

Next using \(2T_n(1 + \delta)T_{n+1}(1 + \delta) = T_{2n+1}(1 + \delta) + 1 + \delta\) and the fact that \(T_{2n+1}(1 + \delta) \geq 1 + \delta\) we obtain

\[
\gamma_2(D_{T_{2n+1}, \lambda}) \leq \left( (2n + 1)^2 - 1 \right) \left( T_{2n+1}(1 + \delta) + 1 + \delta \right) + 1,
\]

\[
\leq (2(2n + 1)^2 - 1)T_{2n+1}(1 + \delta).
\]

We can now state the main lemma that will be used in the proof of Theorem\[E\]

**Lemma 7.** For any \(n\) define the polynomial \(C_n(x)\) by

\[
C_n(x) := 1 - \frac{T_n(f(x))}{T_n(f(0))},
\]

where

\[
f(x) = \frac{1 + \epsilon - 2x}{1 - \epsilon},
\]

and \(0 \leq \epsilon \leq 1/3\), then for any sequence \(\lambda = (\lambda_1, \ldots, \lambda_d)\) with \(0 \leq \lambda_i \leq 1\) we have

\[
\gamma_2(D_{C_n, \lambda}) \leq 6n^2 - 3
\]
Proof. One can easily verify that
\[
\mathcal{D}_{C_n, \lambda} = -\frac{1}{T_n(f(0))} \mathcal{D}_{T_n \circ f, \lambda}
= -\frac{1}{T_n(f(0))} \mathcal{D}_{T_n \circ f(\lambda)} \circ \mathcal{D}_{f, \lambda}
= -\frac{1}{T_n(f(0))} \mathcal{D}_{T_n \circ f(\lambda)} \circ (\frac{-2}{1 - \epsilon} \mathbf{J})
\]
Where \( \mathbf{J} \) is an all-one matrix.

Then we introduce \( \delta := \frac{2\epsilon}{1 - \epsilon} \) and we have \( f(0) = 1 + \delta \)
\[
\gamma_2(\mathcal{D}_{C_n, \lambda}) = -\frac{1}{T_n(1 + \delta)} \gamma_2(\mathcal{D}_{T_n \circ f(\lambda)}) (\frac{-2}{1 - \epsilon})
\leq \frac{1}{T_n(1 + \delta)} \gamma_2(\mathcal{D}_{T_n \circ f(\lambda)}) (\frac{2}{1 - \epsilon})
\]
Knowing \( f(\lambda_i) \leq 1 + \delta \) and \( 0 \leq \delta \leq 1 \), by applying Lemma 5
\[
\gamma_2(\mathcal{D}_{C_n, \lambda}) \leq \frac{3}{T_n(1 + \delta)} (2n^2 - 1) T_n(1 + \delta)
\gamma_2(\mathcal{D}_{C_n, \lambda}) \leq 6n^2 - 3
\]  

Appendix D: Proof of Theorem

Let \( N_1 \) be the set of qubits within distance \( L \) from qubits \( N_0 = \{1, \ldots, n_0\} \) in the graph, i.e.,
\[
N_1 = N_0 \cup \partial_L(N_0).
\]  
Observe that using \( B_{21} \) we have \( n_1 = |N_1| \leq \min\{2n_0 \Delta L, n\} \). For simplicity of presentation we assume that \( N_1 = \{1, \ldots, n_1\} \).
Define
\[
P_0 := |s_1\rangle \langle s_1| \otimes \cdots \otimes |s_{n_1}\rangle \langle s_{n_1}| \otimes I^{\otimes (n - n_1)},
\]  
and
\[
\Gamma_0 := \frac{1}{n_1} \sum_{i=1}^{n_1} |\bar{s}_i\rangle \langle \bar{s}_i|_i,
\]  
where \( |\bar{s}_i\rangle \) is the qubit state orthogonal to \( |s_i\rangle \) and \( |\bar{s}_i\rangle \langle \bar{s}_i|_i \) is the operator acting on the \( i \)-qubit by projecting on \( |\bar{s}_i\rangle \).

Let \( T_m(x) \) for \( m = \frac{1}{2} n_1 - \theta \) be the Chebyshev polynomial of degree \( m \) given by Eq. \( C_{10} \). Let
\[
f(x) = \frac{1 + \frac{1}{n_1} - 2x}{1 - \frac{1}{n_1}},
\]  
and define
\[
C_m(x) := 1 - \frac{T_m(f(x))}{T_m(f(0))},
\]  
Next, define
\[
K := UC_m(\Gamma_0)U^\dagger = C_m(\Gamma_0 U^\dagger),
\]  
where \( U = U(T) \). We note that \( C_m(0) = 0 \), and for \( 0 \leq x \leq 1 \) we have
\[
0 \leq C_m(x) \leq 1 + \frac{1}{T_m(f(0))} \leq 2.
\]
Moreover, as shown in [27] we have

\[ C_m(x) \geq 1 - \frac{1}{1 + 2^{\frac{n_2}{n_1}}} \geq \frac{n_2^2}{n_1} = \frac{1}{4} n_1^{-2\theta}, \quad \frac{1}{n_1} \leq x \leq 1. \] (D8)

Note that the spectrum of \( \Gamma_0 \), regardless of multiplicities, equals \( \{0, \frac{1}{n_1}, \frac{2}{n_1}, \ldots, 1\} \). Thus, by the above properties of \( C_m(x) \), the spectrum of \( K \) belongs to \( \{0\} \cup [n_1^{-2\theta}/4, 2] \). Moreover, the projection on the eigenspace of \( K \) associated to the 0-eigenvalue equals \( P = U \Gamma_0 U^\dagger \). Therefore,

\[ \frac{1}{4} n_1^{-2\theta} (I - P) \leq K \leq 2(I - P). \] (D9)

The next step is to approximate \( K \) with a sum of local operators. Based on Theorem 1 for \( 1 \leq i \leq n_1 \) let \( Q_i \) be an \( L \)-local operator such that

\[ \| Q_i - U [s_i] [s_i] U^\dagger \| \leq \epsilon(L), \] (D10)

where

\[ \epsilon(L) = \sqrt{2} e^{-L \left( \log L - \log T - \log(4g(D - 1)) \right)} - \frac{4}{L} \log L. \] (D11)

Indeed, \( Q_i \) acts on qubits within distance \( L \) from \( i \) in the graph. Then letting

\[ \tilde{\Gamma} := \frac{1}{n_1} \sum_{i=1}^{n_1} Q_i, \] (D12)

we have

\[ \| \tilde{\Gamma} - U \Gamma_0 U^\dagger \| \leq \epsilon(L). \] (D13)

We claim that \( \tilde{K} = C_m(\tilde{\Gamma}) \) is close to \( K = C_m(\Gamma_0 U^\dagger) \). To prove this we use Lemma 5 and Lemma 7. To apply the latter lemma we need to have a bound on the spectrum of \( (1 - r) U \Gamma_0 U^\dagger + r \tilde{\Gamma} \). Here, a crucial observation is that by construction of \( Q_i \) in the proof of Theorem 1 we have \( 0 \leq Q_i \leq I \) which gives \( 0 \leq \tilde{\Gamma} \leq I \). On the other hand, by definition \( 0 \leq \Gamma_0 \leq I \) which yields \( 0 \leq (1 - r) U \Gamma_0 U^\dagger + r \tilde{\Gamma} \leq I \) for \( 0 \leq r \leq 1 \). As a result, by Lemma 5 and Lemma 7 we have

\[ \| \tilde{K} - K \| \leq (6m^2 - 3) \epsilon(L) \leq 6m^2 \epsilon(L). \] (D14)

Now recall that \( \tilde{\Gamma} \) is a sum of \( L \)-local operators, and \( C_m(x) \) is a degree \( m \) polynomial. As a result \( \tilde{K} = C_m(\tilde{\Gamma}) \) is a sum of \( \ell' \)-local operators where

\[ \ell' = mL. \]

Next, as in [27] we partition \( \{0,1\}^n \) into four sets:

\[ F_1 := (F \setminus \partial_{\ell'}(F)) \times \{0,1\}^{n-n_0}, \] (D15)

\[ F_2 := (F \cap \partial_{\ell'}(F)) \times \{0,1\}^{n-n_0}, \] (D16)

\[ F_3 := (F^c \cap \partial_{\ell'}(F)) \times \{0,1\}^{n-n_0}, \] (D17)

\[ F_4 := (F^c \setminus \partial_{\ell'}(F)) \times \{0,1\}^{n-n_0}. \] (D18)

Here, \( \partial_{\ell'}(F) \) is defined with respect to the Hamming distance on \( \{0,1\}^{n_0} \). Then, expanding \( |\psi\rangle = |\psi_T\rangle \) in the computational basis, we can write

\[ |\psi\rangle = |\phi_1\rangle + |\phi_2\rangle + |\phi_3\rangle + |\phi_4\rangle, \] (D19)

where \( |\phi_j\rangle \) is a linear combination of \( |x\rangle \)'s with \( x \in F_j \). The point of this decomposition is that \( p(\partial_{\ell'}(F)) = ||\phi_2||^2 + ||\phi_3||^2 \). On the other hand, since \( K \) is \( \ell' \)-local we have

\[ \langle \phi_1 | K \phi_4 \rangle = \langle \phi_1 | K \phi_4 \rangle = \langle \phi_2 | K \phi_4 \rangle = 0. \] (D20)
Therefore, because $\Gamma_0 |\psi_0\rangle = 0$, we have

$$\langle \psi | K | \psi \rangle = \langle \psi | U C_m(\Gamma_0) U^\dagger | \psi \rangle = \langle \psi_0 | C_m(\Gamma_0) | \psi_0 \rangle = 0. \quad (D21)$$

Thus by Eq. (D13) we have

$$| \langle \psi | K | \psi \rangle | \leq 6m^2 \epsilon(L). \quad (D22)$$

Let $|\psi'\rangle := -|\phi_1\rangle - |\phi_2\rangle + |\phi_3\rangle + |\phi_4\rangle$. Then using Eq. (D20) we have

$$\langle \psi' | K | \psi' \rangle = \sum_{(i,j) \in \{(1,2)^4 \cup \{3,4\}^2}} \langle \phi_i | K | \phi_j \rangle - \sum_{(i,j) \in \{(1,2) \times \{3,4\}\}} \langle \phi_i | K | \phi_j \rangle \quad (D23)$$

$$= \| \phi_1 \|^2 + \| \phi_2 \|^2 + \| \phi_3 \|^2 + \| \phi_4 \|^2 + 2 \text{Re} \langle \phi_1 | K | \phi_2 \rangle + 2 \text{Re} \langle \phi_3 | K | \phi_4 \rangle - 2 \text{Re} \langle \phi_2 | K | \phi_3 \rangle. \quad (D24)$$

Considering the same expansion for $\langle \psi' | K | \psi' \rangle$ and using Eq. (D22), we obtain

$$\langle \psi' | K | \psi' \rangle \leq 4 \| \phi_2 \| \| \phi_3 \| + 6m^2 \epsilon(L) \leq 4 \| K \| \| \phi_2 \| \| \phi_3 \| + 6m^2 \epsilon(L) \leq 4 \left( \| \phi_2 \|^2 + \| \phi_3 \|^2 \right) + 6m^2 \epsilon(L) \leq 4 \left( \| \phi_2 \|^2 + \| \phi_3 \|^2 \right) + 6m^2 \epsilon(L), \quad (D25)$$

where in the third line we use $0 \leq K \leq I$ which gives $\| K \| = \| C_m(\Gamma) \| \leq 2$.

To this end we will use Eq. (D26), so we first prove an upper bound on $\langle \psi' | P | \psi' \rangle$. Define

$$R_0 := \sum_{x \in F^n} | x \rangle \langle x | \otimes I - \sum_{x \in F^n} | x \rangle \langle x | \otimes I. \quad (D26)$$

Note that $R_0 |\psi\rangle = |\psi'\rangle$ and that $R_0$ is an $n_0$-local operator acting on the first $n_0$ qubits. Then by Theorem 1 there is $R$ acting on qubits in $N_1$ such that

$$\| R - R_0 \| \leq n_0 \epsilon(L). \quad (D27)$$

Then we have

$$P_0 U^\dagger |\psi'\rangle = P_0 U^\dagger R_0 |\psi\rangle = P_0 U^\dagger R_0 U |\psi_0\rangle = P_0 R |\psi_0\rangle + |\delta\rangle, \quad (D28)$$

where $|\delta\rangle$ is a vector with $\| \delta \| \leq n_0 \epsilon(L)$. On the other hand, since $R$ acts only on the first $n_1$ qubits and $|\psi_0\rangle = |s_0\rangle \cdots |s_n\rangle$, the vector $R |\psi_0\rangle$ is a linear combination of vectors of the form $|x\rangle \otimes |s_{n+1}\rangle \cdots |s_n\rangle$ for $x \in \{0,1\}^{n_1}$. Therefore, applying the projection $P_0$ on $R |\psi_0\rangle$ we get a vector parallel to $|\psi_0\rangle$, i.e., $P_0 R |\psi_0\rangle = \alpha |\psi_0\rangle$ for some $\alpha \in \mathbb{C}$. This means that

$$P_0 U^\dagger |\psi'\rangle = \alpha |\psi_0\rangle + |\delta\rangle. \quad (D29)$$

To compute $\alpha$ we write

$$\alpha = \langle \psi_0 | P_0 U^\dagger |\psi'\rangle - \langle \psi_0 | \delta \rangle \quad (D30)$$

$$= \langle \psi_0 | P_0 U^\dagger RU |\psi_0\rangle - \langle \psi_0 | \delta \rangle \quad (D31)$$

$$= \langle \psi_0 | U^\dagger RU |\psi_0\rangle - \langle \psi_0 | \delta \rangle \quad (D32)$$

$$= \langle \psi | R |\psi\rangle - \langle \psi_0 | \delta \rangle \quad (D33)$$

$$= \langle \psi | \psi' \rangle - \langle \psi_0 | \delta \rangle \quad (D34)$$

$$= \| \phi_1 \|^2 - \| \phi_2 \|^2 + \| \phi_3 \|^2 - \| \phi_4 \|^2 - \langle \psi_0 | \delta \rangle \quad (D35)$$

$$= (1 - 2p(F)) - \langle \psi_0 | \delta \rangle. \quad (D36)$$
where in the last line we have used $p(F) = \| \phi_1 \|^2 + \| \phi_2 \|^2$. Putting these together we arrive at

$$P_0 U_\dagger |\psi'\rangle = \left((1 - 2p(F)) - \langle \psi_0 |\delta\rangle \right) |\psi_0\rangle + |\delta\rangle.$$ \hspace{1cm} (D39)

Therefore,

$$\langle \psi' | P |\psi'\rangle = \langle \psi' | U P_0 U_\dagger |\psi'\rangle$$

$$= \| P_0 U_\dagger |\psi'\rangle \|^2$$

$$\leq \left| \left(1 - 2p(F)\right) - \langle \psi_0 |\delta\rangle \right|^2 + \| |\delta\rangle \|^2 + 2 \left| \left(1 - 2p(F)\right) - \langle \psi_0 |\delta\rangle \right| \cdot \| |\delta\rangle \|$$

$$\leq \left(1 - 2p(F)\right)^2 + 8\| |\delta\rangle \|$$

$$\leq \left(1 - 2p(F)\right)^2 + 8n_0 \epsilon(L).$$ \hspace{1cm} (D44)

Next, using Eq. (D39) we find that

$$\langle \psi' | K |\psi'\rangle \geq \frac{1}{4} n_1^{-2g} \left(1 - \langle \psi' | P |\psi'\rangle \right)$$

$$\geq \frac{1}{4} n_1^{-2g} \left(1 - \left(1 - 2p(F)\right)^2 - 8n_0 \epsilon(L)\right)$$

$$\geq \frac{1}{2} n_1^{-2g} p(F) - 2n_1^{-2g} n_0 \epsilon(L),$$ \hspace{1cm} (D47)

where in the last line we used $p(F) \leq 1/2$ to conclude that $1 - \left(1 - 2p(F)\right)^2 \geq 2p(F)$. Then by Eq. (D14) we have

$$\left| \langle \psi' | K |\psi'\rangle \right| \geq \frac{1}{2} n_1^{-2g} p(F) - \left(2n_1^{-2g} n_0 + 6m^2\right) \epsilon(L).$$ \hspace{1cm} (D48)

Finally, comparing this inequality with Eq. (D25) yields

$$4p(\partial_{\epsilon}(F)) + 6m^2 \epsilon(L) \geq \frac{1}{2} n_1^{-2g} p(F) - \left(2n_1^{-2g} n_0 + 6m^2\right) \epsilon(L),$$

or equivalently

$$p(\partial_{\epsilon}(F)) \geq \frac{1}{8} n_1^{-2g} p(F) - \left(\frac{1}{2} n_1^{-2g} n_0 + 3m^2\right) \epsilon(L).$$ \hspace{1cm} (D50)

Now for the first part of the theorem let $L = \frac{6}{\log \Delta} \log n_0 = \beta 4g \Delta T$ with

$$\beta = \frac{\kappa_1}{4g \Delta \log(\Delta) T} \log n_0 \geq \Delta^{\frac{1+\kappa_1+\kappa_2}{\kappa_1}},$$ \hspace{1cm} (D51)

where the inequality follows from the assumption

$$T \leq \frac{\kappa_1}{4g \Delta \log(\Delta) T} \Delta^{\frac{1+\kappa_1+\kappa_2}{\kappa_1}} \log n_0.$$ \hspace{1cm} (D52)

Then we have $n_1 \leq 2n_0 \Delta L = 2n_0^{1+\kappa_1}$ and $m = \frac{1}{2} n_1^{-2g} \leq \frac{1}{2} n_0^{\frac{1}{2} \left(1+\kappa_1\right)}$. We also have

$$\epsilon(L) = \sqrt{\frac{g}{\pi}} e^{-L \left(\log L - \log(4g(\Delta - 1))\right)} \log L$$

$$\leq e^{-\log \beta \log n_0 \log \beta}$$

$$\leq n_0^{-\left(1+\kappa_1+\kappa_2\right)}.$$ \hspace{1cm} (D56)

Therefore,

$$\left(\frac{1}{2} n_1^{-2g} n_0 + 3m^2\right) \epsilon(L) \leq 2 n_0^{1+\kappa_1} n_0^{-\left(1+\kappa_1+\kappa_2\right)} = 2n_0^{-\kappa_2}.$$ \hspace{1cm} (D57)
On the other hand,

\[ \ell' = mL \leq \frac{1}{2}(2n_0^{1+\kappa_1})^{\frac{1}{2} - \frac{\kappa_1}{\log(\Delta)}} \log n_0 \leq \ell. \]  

We conclude that

\[ p(\partial_\ell(F)) \geq p(\partial_{\ell'}(F)) \geq \frac{1}{8}n_1^{-2\theta}p(F) - 2n_0^{-\kappa_2} \geq \frac{1}{8}(2n_0^{1+\kappa_1})^{-2\theta}p(F) - 2n_0^{-\kappa_2}. \]  

For the second part of the theorem, note that when \( n_0 = n \), then \( n_1 = n \). Using this in (D50) for \( \theta = 0 \), we obtain

\[ p(\partial_\ell) \geq \frac{1}{8}p(F) - \frac{5}{4}n\epsilon(L). \]  

Let \( L = \kappa_1(1 + \kappa_2)\log n \) and note that \( \ell' = mL = \frac{1}{2}\sqrt{n}L = \ell \). On the other hand, using the given bound on \( T \) we find that \( \epsilon(L) \leq n^{-\kappa_1(1+\kappa_2)\log(1+\kappa_2)} \). Putting this in the above inequality yields the desired result.
This figure "lieb-shape-1.png" is available in "png" format from:

http://arxiv.org/ps/2104.12808v1