I. INTRODUCTION

The determination of the signal template of the radiating companion of a black hole is of great importance for the forthcoming interferometric gravitational observatories. An inspiralling neutron star or a several solar-mass black hole orbiting a massive black hole, as well as a particle constituent contributing to the radiation of the debris trapped by the black hole are notable examples of these companions. Either is the case, one wants to describe the evolution of the orbit of a test particle in the neighborhood of a black hole under the influence of gravitational radiation backreaction. In a highly idealized picture, the orbit of the particle is a Carter geodesic [1] characterized by four constants of the motion, the energy, the total angular momentum, the rest-mass of the particle and the separation constant. In the spirit of perturbation theory, the radiation backreaction effects may be taken into account by evolving these constants of the motion, thus picturing the trajectory of the particle by a sequence of geodesic orbits.

The history of the orbit may be divided in separate epoches. For example, the first thing that happens to a particle trapped by a nonrotating Schwarzschild black hole on an arbitrary but distant orbit is that it loses eccentricity due to radiation losses, and the radius shrinks until the close region of the last stable circular orbit is approached. There one has to replace the adiabatic picture with one valid in the strong field regime.

The description of a generic orbit in the strong field region of a spinning Kerr black hole is difficult (as has been pointed out by Thorne [2]) due to the special nature of the separation constant which lacks a straightforward geometrical interpretation. Recently, progress in treating the separation constant has been reported by Ori [3].

There is a considerable ease in the description of special orbits such as equatorial or quasicircular, and in fact, several papers have obtained the signal templates for such special orbits. In order that any of these special orbits can be credited as significant contributor to the gravitational signal, one has to make sure that the Carter constant evolves to the envisaged special values.

Several authors have recently published results on radiative losses by systems of spinning masses. Kidder, Will and Wiseman [4] formulate the problem by using the Lagrangian (proposed by Barker and O’Connell [5]) for bodies with finite masses and spins. They and Kidder [6] compute the spin-orbit and spin-spin contributions to the momentary radiative loss of energy, linear and angular momenta. In the test particle limit their results have subsequently been used by Shibata [7], in his study of equatorial orbits, and by Ryan [8,9]. By computing the backreaction forces, Ryan obtains the power and angular momentum losses for circular and generic orbits.

In our work, we have taken up the evolution of the orbit in the far field region of the black hole. The trajectory of the idealized, nonradiating particle is described by the Lense-Thirring picture [10,11]. This picture emerges in the test particle limit of the scenario described in [6]. The spin $S$ of the black hole is fixed to the $z$ axis. The orbit is viewed approximately as an ellipse precessing in a plane which rotates about $S$. The motion is characterized by three constants: the energy $E$, the magnitude $L$ and spin projection $L_z$ of the orbital angular momentum, and by two
slowly changing angle parameters \(\Psi_0\) and \(\Phi_0\). The Lense-Thirring approximation is linear in the spin terms, thus the (additive) Schwarzschild effects may be neglected in this treatment.

In an earlier paper \([12]\), we have computed the radiation losses from the mass quadrupole tensor in the Lense-Thirring approximation, by the orbit smoothing method of Landau and Lifshitz \([11]\). However, one may be rightly concerned about the degradation of accuracy of this computation, as a price of the relative simplicity introduced by the smoothing method.

In the present paper, we drop the orbit smoothing, and we include the velocity quadrupole tensor \(J_ik\) in the description of the radiative losses. In the test particle limit, this (formally) corresponds to the 1PN approximation \([8]\). The rest of the paper is organized as follows. In Sec. 2 we describe the orbital motion in the Lense-Thirring approximation. In terms of a set of new angular variables (related to Euler rotations), the equations of motion take a particularly simple form: all time derivatives depend on the radial variable \(r\) alone.

In Sec. 3, we explore the advantages of having a pure radial equation of motion. We find the turning points at \(\dot{r} = 0\) which allow us to parameterize the radial motion by trigonometric functions. Integrations over the time can be replaced by integrations over the parameter, which are straightforward in principle, but can be messy in practice. We evade such complications by finding suitable parametrizations. The first parameter \(\xi\) we introduce is a generalization of the eccentric anomaly of the Keplerian orbits. It is employed for computing the half period defined as the time spent by the particle between consecutive turning points. For the purpose of averaging the radiative losses, and for determining the change in the new angular variables, a generalized true anomaly parameter \(\chi\) is properly defined. In terms of this parameter, when integration by the residue theorem is performed, the only pole of the integrands will be in the origin.

In Sec. 4, we compute the rates of change of the constants of motion due to radiation backreaction, to first order in \(S\), by employing the radiative multipole tensors of Kidder, Will and Wiseman \([4]\), which originate in the Blanchet-Damour-Iyer formalism \([13, 14]\). As we are interested in the cumulative effects of the secular motion, we compute averaged losses over the period of radial motion. We give the losses in an unambiguous manner in terms of the constants of motion \(E, L\) and \(L_z\).

As has been pointed out by Ryan \([9]\), the definition of the orbit parameters \(a\) and \(e\) is subject to ambiguities. In Sec. 5, we define the semimajor axis and eccentricity. With advantages explained in the main text, our definitions differ both from Ryan’s and the ones used in the Lense-Thirring paper. We rewrite all losses in terms of our orbital parameters \(i, a\) and \(e\).

In the concluding remarks, we complete the characterization of the changes caused by gravitational radiation by giving the change in the \(\Psi_0, \Phi_0\) angle parameters over one orbit. We then compare our results with those of other authors.

We wish to make some comments on the expansion procedure we have adopted. Earlier works employ either the expansion parameter \(\epsilon \approx \nu^2 \approx m/r\) or the inverse \(1/c^2\) of the speed of light. Since we eliminate \(\nu^2\) and \(\dot{r}^2\) by use of the first integrals of the motion, we choose to retain \(G\) and \(c\) for bookkeeping of the orders in our formalism. The correction terms in the Lagrangian, in the equations of motion and radiation losses will carry an extra \(1/c^2\) factor. In our computation, we keep only the terms containing the spin \(S\) from among these correction terms, thus we may as well choose \(S\) as our expansion parameter.

Our treatment of radiation backreaction effects can be generalized beyond the Lense-Thirring approximation. In a follow-up paper \([15]\) we will apply our averaging method and parameterization for computing the backreaction effects on a bound system of two finite masses.

**II. THE ORBIT IN THE LENSE-THIRRING APPROXIMATION**

We consider the orbit of a small particle of mass \(\mu\) about a body with mass \(M\) and angular momentum vector \(S\). In the Lense-Thirring approximation \([10]\), the Lagrangian of the system has the form \([11]\)

\[
\mathcal{L} = \frac{\mu \dot{r}^2}{2} + \frac{G \mu M}{r} + \delta \mathcal{L},
\]

where \(r = |\mathbf{r}|\), a dot denotes \(d/dt\) and the term describing (perturbative) rotation effects is

\[
\delta \mathcal{L} = \frac{2G\mu}{c^2 r^3} \mathbf{S} \cdot (\dot{\mathbf{r}} \times \mathbf{r}).
\]

The Cartesian coordinates of the test particle are \(\mathbf{r} = \{x, y, z\}\), with origin chosen at the black hole. Hence the orbital momentum
\( L = r \times p \)  \hspace{1cm} (2.3)

and Runge-Lenz vector

\[
A = \frac{p}{\mu} \times L - \frac{G\mu Mr}{r}
\]

satisfy the equations of motion, respectively

\[
\dot{L} = 2 \frac{G}{c^2 r^3} S \times L
\]

\hspace{1cm} (2.5)

\[
\dot{A} = 2 \frac{G}{c^2 r^3} S \times A + \frac{6G}{c^2 r^5 \mu} (S \cdot L) r \times L.
\]

\hspace{1cm} (2.6)

Here \( p = \frac{\partial L}{\partial \dot{r}} \) is the momentum of the orbiting mass. Note that the angular momentum \( L \) undergoes a pure rotation about the spin with angular velocity:

\[
\Omega_S = 2 \frac{G}{c^2 r^3} S.
\]

\hspace{1cm} (2.7)

From (2.3) and (2.4), the vector \( r \) can be written in terms of the basis \( \{ A, L \times A \} \) in the plane with normal \( L \):

\[
r = \frac{L^2 - G\mu^2 Mr}{\mu A^2} A + \frac{r \cdot p}{\mu A^2} L \times A,
\]

\hspace{1cm} (2.8)

where \( A = |A| \). Inserting this expression of \( r \) in (2.6), the direction \( A/A \) of the Runge-Lenz vector is found to rotate about a linear combination of the vectors \( S \) and \( L \):

\[
\left( \frac{A}{A^2} \right) \times \left[ \frac{2G}{c^2 r^3} S - \frac{6G(L^2 - G\mu^2 Mr)}{c^2 \mu A^2 A^2 r^5} (S \cdot L) L \right] \times \frac{A}{A^2}.
\]

\hspace{1cm} (2.9)

Thus the direction vector rotates about \( S \) with the angular velocity \( \Omega_S \) (as does \( L \)) and about \( L \) with the angular velocity:

\[
\Omega_L = -\frac{6GL(L^2 - G\mu^2 Mr)}{c^2 \mu A^2 A^2 r^5} (S \cdot L).
\]

\hspace{1cm} (2.10)

Although the orbital momentum and the Runge-Lenz vector are not conserved (unlike in the zeroth order in \( S \)), here also there are constants of motion. Because of the fact that the potential is axisymmetric and stationary, the orbital momentum component \( L_z \) and the energy \( E = \dot{r} \partial L / \partial \dot{r} - L \) are conserved. Since the motion of \( L \) is a pure rotation, its magnitude \( L \) is also constant:

\[
\dot{E} = \dot{L}_z = \dot{L} = 0.
\]

\hspace{1cm} (2.11)

The constants of the motion are

\[
E = \frac{\mu}{2}[r^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)] - \frac{G\mu M}{r}
\]

\hspace{1cm} (2.12)

\[
L_z = \mu r^2 \sin^2 \theta \dot{\phi} - 2\frac{G\mu S}{c^2 r} \sin \theta
\]

\hspace{1cm} (2.13)

\[
L^2 = \mu^2 r^4 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - 4 \frac{G\mu^2 S}{c^2} r \sin^2 \theta \dot{\phi}.
\]

\hspace{1cm} (2.14)

(Here we are using polar coordinates, \( x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta \)). As a consequence, the angle \( \iota \) subtended by \( L \) and \( S \) is also conserved:

\[
\cos \iota = \frac{L_z}{L}.
\]

\hspace{1cm} (2.15)
The first integrals of the equations of motion follow from (2.12)-(2.14) by simple algebra (we need not recourse to the Hamilton-Jacobi formalism). Thus the equation of the radial motion takes the first-order form, decoupled from the angular degrees of freedom:

\[ \dot{r}^2 = \frac{L^2}{\mu r^2} + 2 \frac{GM}{r} + 2 \frac{E}{\mu} - 4 \frac{GLz}{c^2 \mu r^3} S, \]  

(2.16)

the equations for the angular variables \( \theta, \varphi \), however, are coupled among themselves and with the \( r \) motion:

\[ \dot{\theta}^2 = \frac{L^2}{\mu^2 r^2} \left( 1 - \frac{\cos^2 \iota}{\sin^2 \theta} \right), \quad \dot{\varphi} = \frac{Lz}{\mu r^2 \sin^2 \theta} + \frac{2GS}{c^2 r^3}. \]  

(2.17)

There is an alternative description of the motion in the Lense-Thirring approximation which bears a more intimate geometrical relation to the picture provided by perturbation theory. We shall introduce this formalism by performing time-dependent Euler rotations about the origin so as to place the Kepler ellipse in its momentary orientation:

\[ \mathbf{r} = R_z(\Phi)R_x(\iota)R_z(\Psi)\mathbf{r}_0. \]  

(2.18)

Here

\[ \mathbf{r}_0 = \begin{pmatrix} r_0 \\ 0 \\ 0 \end{pmatrix} \]  

(2.19)

is the initial position of the particle on the plane perpendicular to \( \mathbf{L} \) (although \( \mathbf{L} \) is different from \( \mathbf{L}_N = \mu \mathbf{r} \times \mathbf{v} \), the condition \( \mathbf{r} \perp \mathbf{L} \) is fulfilled). A rotation about the Cartesian \( x \) and \( z \) axes is given, respectively, by

\[ R_x(\iota) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \iota & -\sin \iota \\ 0 & \sin \iota & \cos \iota \end{pmatrix}, \quad R_z(\Phi) = \begin{pmatrix} \cos \Phi & -\sin \Phi & 0 \\ \sin \Phi & \cos \Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]  

(2.20)

The coordinates of the particle are written in terms of the Euler angles \( \Phi, \iota \) and \( \Psi \) as

\[ x = r(\cos \Phi \cos \Psi - \cos \iota \sin \Phi \sin \Psi), \]
\[ y = r(\sin \Phi \cos \Psi + \cos \iota \cos \Phi \sin \Psi), \]
\[ z = r \sin \iota \sin \Psi. \]  

(2.21)

Hence we get the relations

\[ \dot{\theta} = -\frac{\sin \iota}{\sin \theta} \cos \Psi \dot{\Psi}, \quad \dot{\varphi} = \dot{\Phi} + \frac{\cos \iota}{\sin^2 \theta} \dot{\Psi}, \quad \sin^2 \theta = 1 - \sin^2 \iota \sin^2 \Psi, \]  

(2.22)

which allow us to express the time derivatives of the new variables:

\[ \dot{\Psi} = \frac{L}{\mu r^2} \]  

(2.23)
\[ \dot{\Phi} = \frac{2GS}{c^2 r^3}. \]  

(2.24)

Note that \( \dot{\Phi} = \Omega_S \) and \( \dot{\Psi} \) carries the interpretation of angular velocity of rotation about \( \mathbf{S} \) and in the plane perpendicular to \( \mathbf{L} \) respectively. An important feature of the new coordinates \( \Psi, \Phi \) is that their derivatives appear linearly in the equations of motion (2.23) and (2.24). This is not the case with the polar coordinates (2.17) because the root of \( \dot{\theta}^2 \) cannot be extracted. The geometrical interpretation of the angle variable \( \Psi \) assures that it is monotonously changing. We choose \( \Psi \) to increase with the motion evolving.

There is a freedom in the precise way one introduces the three Euler-angle variables in place of the polar angles. Even though the equations of motion are made simple by use of the angular variables (2.19-2.21), the description of the momentary plane of orbit is tied to a different choice of the Euler angles \( (\Psi', \iota', \Phi') \), where the initial \( z \) axis is chosen along \( \mathbf{L}_N \) rather than along \( \mathbf{L} \). These angular variables appear (with the appropriate change in notation) in the Lense-Thirring paper. The angle \( \iota' \) is not a constant, and the resulting equations are less simple. The primed angles will not be employed in our computations, and we relegate their discussion to the Appendix.
III. PARAMETRIZATION OF THE ORBIT

We now proceed with the solution of the equations of motion. First consider the radial equation (2.16). For the turning points $\dot{r} = 0$, we get the cubic equation

$$2E\mu r^3 + 2GM\mu^2 r^2 - L^2 r - 4G\mu L_z S = 0.$$  \hfill (3.1)

In the no-spin limit, $S = 0$, the solutions are (the unphysical) $r = 0$ and

$$r_{\pm} = \frac{-GM\mu \mp A_0}{2E},$$  \hfill (3.2)

where $A_0$ is the length of the Runge-Lenz vector to zeroth order:

$$A_0 = \sqrt{G^2M^2\mu^2 + 2EL^2}.$$  \hfill (3.3)

When a small spin term is present, the roots of (3.1) will have a slightly different form from (3.2): $0 + \epsilon$ and $r_+ + \epsilon_\pm$, where $\epsilon$ and $\epsilon_\pm$ are assumed to be small. Note that the constants $A_0$ are still defined by (3.3), although it is not identical with the length $A$ of the Runge-Lenz vector (which is not constant). Writing the cubic equation as $2E\mu(r - r_+ - \epsilon_+)(r - r_- - \epsilon_-)(r - \epsilon) = 0$, we get

$$2E\mu[r^2 - r(r_+ + r_-) + r_+ r_-] + 2E\mu(-r^2(\epsilon_+ + \epsilon_- + \epsilon) + r(\epsilon_+ r_- + \epsilon_- r_+ + \epsilon(r_+ + r_-)) - \epsilon r_+] = 0.$$  \hfill (3.4)

The first term is the zeroth-order part of the original equation, written in terms of the roots, thus it equals $(4G/c^2)\mu L_z S$. Inserting the chosen form of the roots we get

$$\epsilon = -\frac{4G\mu L_z S}{c^2 L^2}, \quad \epsilon_\pm = \frac{2G\mu L_z S}{c^2 L^2} \left(1 \mp \frac{GM\mu}{A_0}\right).$$  \hfill (3.5)

The orbit is supposed to lie at large distance, thus the first root will not occur.

Using the above results, a parameter $\xi$ generalizing the eccentric anomaly of the Kepler orbits is readily introduced:

$$r = -\frac{G\mu M}{2E} + \frac{2G\mu L_z S}{c^2 L^2} + \left(\frac{A_0}{2E} + \frac{2G^2\mu^2 ML_z S}{c^4 A_0 L^2}\right) \cos \xi,$$  \hfill (3.6)

such that $r_{\min}$ is at $\xi = \pi$ and $\xi = 0$, respectively. From here:

$$\frac{dr}{d\xi} = -\left(\frac{A_0}{2E} + \frac{2G^2\mu^2 ML_z S}{c^4 A_0 L^2}\right) \sin \xi.$$  \hfill (3.7)

For equatorial orbits, this parametrization has been employed by Shibata [7]. Expressing $1/\dot{r}$ from (2.16) and linearizing in $S$, the expression for $dt/d\xi$ is found:

$$\frac{dt}{d\xi} = \frac{1}{\dot{r}} \frac{dr}{d\xi} = \frac{\mu^2(GM\mu - A_0 \cos \xi)}{(-2GM\mu - \frac{A_0}{2E})^2} + \frac{2GM\mu^3 L_z S \cos \xi}{(-2GM\mu - \frac{A_0}{2E})^2}.$$  \hfill (3.8)

Integration of (3.8) from 0 to $2\pi$ gives the orbital period:

$$T = 2\pi \frac{GM\mu^3}{(-2GM\mu - \frac{A_0}{2E})^2}.$$  \hfill (3.9)

However the parametrization (3.4) is inconvenient for carrying out the averaging of the losses, because it yields a complicated array of poles of the integrands. Hence in the sequel we will, in place of (3.6), find a parametrization $r = r(\chi)$ satisfying the following two criteria:

(a) $r(0) = r_{\min}$ and $r(\pi) = r_{\max}$

$$\frac{dr}{d(\cos \chi)} = -(\gamma_0 + S\gamma_1) r^2,$$  \hfill (3.10)

(b) $\frac{dr}{d(\cos \chi)} = -(\gamma_0 + S\gamma_1) r^2.$  \hfill (3.11)
where \( \gamma_0, \gamma_1 \) are constants. That is, we want to keep the properties of the true anomaly parametrization of the Kepler orbit. Property (b) generalizes Kepler’s second law for the area \( [14] \). The unique parametrization satisfying both (a) and (b) is:

\[
\frac{L^2}{\mu (GM\mu + A_0 \cos \chi)} + \frac{4GLzS}{A_0 L^2 c^2} \frac{A_0(2G^2M^2\mu^3 + EL^2) + GM\mu(2G^2M^2\mu^3 + 3EL^2) \cos \chi}{(GM\mu + A_0 \cos \chi)^2}.
\]  

(3.12)

Hence

\[
\frac{dr}{d\chi} = \frac{\mu A_0}{L^2} - \frac{4G\mu^2LzS}{A_0 L^2 c^2} (2G^2 M^2 \mu^3 + 3EL^2) \right] r^2 \sin \chi
\]  

(3.13)

and to first order in \( S \):

\[
\frac{dt}{d\chi} = \frac{1}{r} \frac{dr}{d\chi} = \frac{\mu r^2}{L} \left[ 1 - \frac{2G\mu^2LzS}{c^2 L^4} (3GM\mu + A_0 \cos \chi) \right].
\]  

(3.14)

We will need also the relation of the type \( \Psi = \Psi(\chi) \). This can be obtained from integration of (2.23):

\[
\Psi = \Psi_0 + \chi - \frac{2G\mu^2LzS}{c^2 L^4} (3GM\mu + A_0 \sin \chi).
\]  

(3.15)

The constant of integration \( \Psi_0 \) measures the angle subtended by the semiminor axis \( (\chi = 0) \) and the node line at \( \Psi = 0 \). In a perturbative picture, the orientation of the momentary ellipse of the orbit is reinterpreted at each revolution, and the value of the integration constant is subject to a corresponding shift. Our parametrization \( \frac{L^2}{\mu (GM\mu + A_0 \cos \chi)} \) makes especially simple the integration over one period of all expressions \( F = F(r, \Psi) \) containing in the denominator no \( \Psi \) dependence and no other \( r \) dependence than \( r^{2+n} \), with \( n \) a positive integer. Note that (2.10), (2.23) and (2.24) have this property. Further, as will be shown in the next section, the instantaneous radiative losses of the constants of motion share this property. In all of these cases one has:

\[
\int_0^T F(r(t), \Psi(t)) dt = \int_0^{2\pi} F(r(\chi), \Psi(\chi)) \frac{dt}{d\chi} d\chi,
\]  

(3.16)

where the \( r \) dependence of the denominator of the integrand is especially simple: \( r^n \). The average over one period is

\[
< F > = \frac{1}{T} \int_0^T F(r(t), \Psi(t)) dt,
\]  

(3.17)

with \( T \) given by Eq. (1.9).

The integrals may conveniently be evaluated by use of the residue theorem. One introduces the complex variable \( \zeta = e^{i\chi} \), and the integral is taken over the unit circle in the \( \zeta \) plane. The value of the integral is given by the sum of the residues of the poles inside the circle. In all relevant cases our parametrization assures that the only pole is at \( \zeta = 0 \).

Simple examples are the integrations of (2.24), (2.23) and (2.10):

\[
\Delta\Phi = 2\pi S \frac{2G^2 M \mu^3}{c^2 L^3},
\]  

(3.18)

\[
\Delta\Psi = 2\pi - 2\pi S \frac{6G^2 M \mu^3 L_z}{c^2 L^4},
\]  

(3.19)

\[
\Delta\Omega_L = -2\pi S \frac{6G^2 M \mu^3 L_z}{c^2 L^4}.
\]  

(3.20)

Here (3.18) is the precession angle of the orbital angular momentum \( L \) about \( S \). The interpretation of the variable \( \Psi \) as the polar angle in the plane perpendicular to \( L \) is in accordance with the identical first order terms in (3.19) and (3.20). The ellipse is found to counterrotate (for \( L_z > 0 \)). The periastron shift is a combination of (3.18) and (3.20). Both \( \Delta\Psi \) and \( \Delta\Phi \) will be unchanged in the averaged-motion approximation of Landau and Lifshitz [11].

Our angles are related to the polar angle \( \varphi \) for equatorial orbits \( (\iota = 0) \) by \( \varphi = \Phi + \Psi \). This relation helps comparison with Shibata’s expression (2.19) for the periastron shift \( \Delta\varphi \).

Ryan [9] has introduced a parametrization \( r = r(\psi) \) which, when rewritten in terms of the constants of motion \( E, L \) and \( L_z \) (employing his definitions for the orbital parameters), will read:
\[ r = \frac{L^2}{\mu (GM\mu + A_0 \cos \psi_R)} + \frac{4GL_2S}{A_0L^2 c^2} \frac{A_0 (2G^2 M^2 \mu_0^3 + EL^2)}{GM\mu (2G^2 M^2 \mu^3 + 3EL^2) \cos \psi_R - (\mu A_0^2/2) \sin^2 \psi_R}. \]

(3.21)

This differs from our parametrization \([3.12]\) by the last \(\sin^2 \psi_R\) term, which shows that Ryan’s parameter \(\psi_R\) and our \(\chi\) reach the turning points simultaneously. In fact both \([3.10]\) and \([3.12]\) can be supplemented by arbitrary terms involving a \(\sin\) squared factor, and they still will satisfy (a). However none of these parameters, and among those Ryan’s, will satisfy the generalized second Kepler law (b).

**IV. INSTANTANEOUS AND AVERAGED LOSSES**

The mass and current quadrupole tensors of a test particle with coordinates \(r_i = (x, y, z)\) are

\[ I_{ij} = \mu(r_i r_j)^{\text{STF}} \]

(4.1)

\[ J_{ij} = -\mu \left[ r_i \left( \frac{dr}{dt} \right) \right]^{\text{STF}} + \frac{3\mu}{2M} (r_i S_j)^{\text{STF}}, \]

where STF means symmetrization and trace removal in the free indices. The rates of the energy and the angular momentum losses in the test particle 1PN approximation are, respectively \([4]\),

\[ \frac{dE}{dt} = -\frac{G}{5c^3} \left( \frac{d^3 I_{ij}}{dt^3} + \frac{16}{45c^2} \frac{d^3 J_{ij}}{dt^3} \right) \]

(4.3)

\[ \frac{dL_i}{dt} = -\frac{G}{c^5} \epsilon_{ijk} \left( \frac{2}{5} \frac{d^2 I_{jk}}{dt^2} \right) + \frac{32}{45c^2} \frac{d^2 J_{jk}}{dt^2} \frac{d^3 J_{kk}}{dt^3}. \]

(4.4)

Evaluating the right-hand sides by use of \((2.16), (2.23)\) and \((2.24)\) and dropping higher-order terms in \(S\), the total power is

\[ \frac{dE}{dt} = -\frac{8G^3M^2}{15c^3 \mu r^2} (2E\mu r^2 + 2GM\mu^2 r + 11L^2) + \frac{8SG^3ML\cos t}{15c^3 \mu r^2} (20E\mu r^2 - 12GM\mu^2 r + 27L^2). \]

(4.5)

This is independent of the angles \(\Psi\) and \(\Phi\). The loss in \(L\) is evaluated by \(2LdL = \delta L^2 = \delta(L_i L_i) = 2L_i \delta L_i\) and it has the form

\[ \frac{dL_i}{dt} = \left( \frac{8G^2LM}{5 \mu c^3 r^5} (2E\mu r^2 - 3L^2) + \frac{8G^2S\cos t}{15c^3 \mu r^6} (12GEM\mu^2 r^2 + 3G^2\mu^3 M^2 r - 11MGL^2 \mu + 18EL^2 r) \right). \]

(4.6)

The momentary loss in \(L_z\) is

\[ \frac{dL_z}{dt} = \left( \frac{8G^2LM \cos t}{5 \mu c^3 r^5} (2E\mu r^2 - 3L^2) \right) - \frac{4G^2S}{15 \mu c^3 r^6} \left\{ \frac{1}{2} (6G^2M^2 \mu^4 r^2 + 36E \mu^2 L^2 + 6GL^2 M^2 \mu^2 r + 18L^4 + 24\mu^3 E_r^3 GM) \sin^2 \psi \\ - 39GL^2 M^2 \mu^2 r + 36L^4 - 36E \mu^2 L^2 \right\} \sin^2 t \right] \right] \sin^2 t + 3r^2 \mu \sin 2\psi \sin^2 t \left( 2GM\mu^2 r - 3L^2 + 6E\mu r^2 \right) \\
-6G^2M^2\mu^4 r^2 + 22GL^2M^2\mu^2 r - 24\mu^3 E_r^3 GM - 36\mu E_r^2 L^2. \]

(4.7)

After parameterizing by \(\chi\) and averaging over the period \(T\) as described in the previous section, we obtain

\[ \left\langle \frac{dE}{dt} \right\rangle = \frac{4 (E\mu)^{3/2} G^2 M}{15 c^3 L^2 \sqrt{2}} (148E^2 L^4 + 732EG^2 L^2 M^2 \mu^3 + 425G^4 M^4 \mu^6) \\
+ \frac{2S\cos t G^2 (E\mu)^{3/2}}{5 c^3 L^10 \sqrt{2}} (520E^3 L^6 + 10740 G^2 E^2 L^4 M^2 \mu^3 + 24990 G^4 E L^2 M^4 \mu^6 + 12579 G^8 M^6 \mu^9). \]
These relations can be inverted:

\[ \frac{dL}{dt} = -\frac{16 G^2 (-E \mu)^{3/2} M}{5 c^5 L^4 \sqrt{2}} (14 E L^2 + 15 G^2 M^2 \mu^3) \]

\[ + \frac{4 G^2 (-E \mu)^{3/2} S \cos \iota}{15 c^7 L^9 \sqrt{2}} (1188 E^2 L^4 + 6756 G^2 E L^2 M^2 \mu^3 + 5345 G^4 M^4 \mu^6) \]

\[ \frac{dL_z}{dt} = -\frac{16 G^2 (-E \mu)^{3/2} M \cos \iota}{5 c^5 L^4 \sqrt{2}} (14 E L^2 + 15 G^2 M^2 \mu^3) \]

\[ + \frac{4 G^2 S (-E \mu)^{3/2}}{15 c^7 L^9 \sqrt{2}} (1188 E^2 L^4 + 6756 G^2 E L^2 M^2 \mu^3 + 5345 G^4 M^4 \mu^6) \]

\[ - \frac{2 G^2 S (-E \mu)^{3/2} \sin^2 \iota}{5 c^7 L^9 \sqrt{2}} (1172 E^2 L^4 + 5892 G^2 E L^2 M^2 \mu^3 + 4325 G^4 M^4 \mu^6) \]

\[ + \frac{8 G^2 S (-E \mu)^{3/2} \cos(2 \Psi_0) \sin^2 \iota}{5 c^7 L^9 \sqrt{2}} (52 E^2 L^4 + 92 G^2 E L^2 M^2 \mu^3 + 33 G^4 M^4 \mu^6) . \]

Among these losses averaged over one period of the radial motion, the only quantity depending on the initial angle \( \Psi_0 \) is \( \langle dL_z/dt \rangle \). This is to be interpreted such that the rate of loss of the angular momentum component \( L_z \) varies with the position of the periastron. In (4.8), the terms with \( \cos(2 \Psi_0) \) average to zero when the precession time scale is short compared to the radiation reaction time scale. (For a more extensive discussion of this subject, cf. [9].)

### V. AVERAGED LOSSES IN TERMS OF ORBIT PARAMETERS

The major axis \( a \) and the eccentricity \( e \) of the Kepler motion are constants and together with \( \iota \) they determine the full set of the constants of motion \( E, L \) and \( L_z \). In the Lense-Thirring picture of a perturbed Kepler motion \( E, L \) and \( L_z \) contain first-order terms, and the corresponding constants \( a \) and \( e \) should also contain first-order terms. There is an ambiguity in how to introduce these orbit parameters, as noted by Ryan [1]. He chooses these parameters by referring to the quasicircular orbits in the Kerr metric. This method is passing when wishing to approximate Carter orbits. However, there appears to be no reason for invoking phenomena tied to black holes when describing arbitrary weakly bound spinning bodies. In fact, we want to keep our treatment general enough to cover all axially symmetric gravitational fields with first order contributions from the spin-orbit interaction term in the Lagrangian (2.2). Thus we choose as the definition of the semimajor axis \( a \) and eccentricity \( e \):

\[ r_{\text{max}} \to r_{\text{min}} = a(1 \pm e) , \]

which implies:

\[ E = -\frac{G M \mu}{2a} \left( 1 + \frac{2GS \cos \iota}{c^2 a^{7/3} \sqrt{GM(1 - e^2)}} \right) \]

\[ L^2 = GM \mu^2 a(1 - e^2) \left( 1 - \frac{2GS \cos \iota}{c^2 a^{7/3} \sqrt{GM(1 - e^2)}} \left( 3 + e^2 \right) \right) . \]

These relations can be inverted:

\[ a = -\frac{2E L z S}{G E^2 M^3 \mu^3} \left( 1 - 4E L z S \right) \]

\[ 1 - e^2 = -\frac{2E L^2}{G^2 M^2 \mu^3} \left[ 1 + \frac{8L z S}{c^2 M L^3} (G^2 M^2 \mu^3 + E L^2) \right] . \]

The averaged losses in terms of \( a \) and \( e \) then read:

\[ \langle dE/dt \rangle = -(37 e^4 + 292 e^2 + 96) \frac{G^4 M^3 \mu^2}{15 (1 - e^2)^{7/2} a^5 c^9} \]

\[ + (491 e^6 + 5694 e^4 + 6584 e^2 + 1168) \frac{G^9/2 S M^5/2 \mu^2 \cos \iota}{30 c^7 a^{13/2} (1 - e^2)^5} . \]
\[
\left< \frac{dL}{dt} \right> = -(7e^2 + 8) \frac{4G^{7/2}M^{5/2} \mu^2}{5a^{7/2}(1-e^2)^2c^5} + (549e^4 + 1428e^2 + 488) \frac{G^4SM^2 \mu^2 \cos \iota}{15c^7a^5(1-e^2)^{7/2}}
\]
\[
\left< \frac{dL_z}{dt} \right> = -(7e^2 + 8) \frac{4G^{7/2}M^{5/2} \mu^2 \cos \iota}{5c^5a^{7/2}(1-e^2)^2} - (285e^4 + 1512e^2 + 488) \frac{G^4SM^2 \mu^2}{30c^2a^5(1-e^2)^{7/2}} + (461e^4 + 1450e^2 + 488) \frac{G^4SM^2 \mu^2 \cos^2 \iota}{10c^7a^5(1-e^2)^{7/2}} + (13e^2 + 20)2e^2 \frac{G^4SM^2 \mu^2 \sin \iota \cos (2\Psi_0)}{5c^7a^5(1-e^2)^{7/2}}.
\]

(5.7)

(5.8)

Differentiating (2.15) one has:

\[
\left< \frac{du}{dt} \right> = \frac{1}{L \sin \iota} \left[ \cos \iota \left< \frac{dL}{dt} \right> - \left< \frac{dL_z}{dt} \right> \right].
\]

(5.9)

Those terms in the brackets which are zero-order in \( S \), cancel.

The averaged losses of the ellipse parameters can be obtained by differentiating (5.10) and (5.11) and from (5.9):

\[
\left< \frac{de}{dt} \right> = -\frac{G^3M^2 \mu e}{15(1-e^2)^{5/2}a^3c^5} \left( 121e^2 + 304 \right) + \frac{G^{7/2}M^{3/2} \mu \epsilon}{(1-e^2)^4a^{11/2}c^3} S \cos \iota \left( \frac{1313}{30}e^4 + \frac{932}{5}e^2 + \frac{1172}{5} \right)
\]

(5.10)

\[
\left< \frac{da}{dt} \right> = -\frac{64G^3M^2 \mu}{5a^3c^5(1-e^2)^{7/2}} \left( \frac{37}{96}e^4 + \frac{73}{24}e^2 + 1 \right) + \frac{64G^{7/2}M^{3/2} \mu S \cos \iota}{5a^{9/2}c^7(1-e^2)^5} \left( \frac{121}{64}e^6 + \frac{585}{32}e^4 + \frac{124}{3}e^2 + \frac{133}{12} \right)
\]

(5.11)

\[
\left< \frac{dl}{dt} \right> = S \sin \iota \frac{G^{7/2}M^{3/2} \mu}{a^{11/2}c^7(1-e^2)^4} \left[ -\cos (2\Psi_0) \left( \frac{26}{5}e^4 + \frac{8}{5}e^2 \right) + \frac{19}{2}e^4 + \frac{252}{5}e^2 + \frac{244}{15} \right].
\]

(5.12)

Let us take an example of a binary system containing a massive black hole with mass 100\( M_\odot \) and spin \( S \approx 0.6 \mu \), and where the companion’s orbit has the semimajor axis \( 10^8 \) km, \( i.e. \), one-hundredth of the Hulse-Taylor system. Then the increase in the angle \( \iota \) per loss of log \( a \) (a dimensionless quantity) is \( \frac{dl}{d \log a} = \left( \frac{G/M_c}{c^3} \right)^{1/2} S \sin \iota \approx 10^{-3} \sin \iota \) for a circular orbit.

The enhancement factor for an eccentric orbit is

\[
f(e) = \frac{285e^4 + 1512e^2 + 488 - 6(26e^4 + 40e^2) \cos (2\Psi_0)}{4(37e^4 + 292e^2 + 96)(1-e^2)^{1/2}}.
\]

(5.13)

FIG. 1. The enhancement factor.
The enhancement factor is monotonously increasing (Fig.1) from $f(0) = 122/121$ to make the change in $t$ comparable to that in $\log a$ only when the eccentricity approaches the unit value to one part in $10^6$.

VI. CONCLUDING REMARKS

The Kepler orbits are completely characterized by the energy $E$, the angular momentum vector $L$, and the Runge-Lenz vector $A$ at the initial time $t_0$. These are six independent constants of the motion. In the Lense-Thirring picture, the constants of the motion alone do not suffice for the full description of the orbit. We complete the characterization by specifying the Euler angles $\Psi_0$ and $\Phi_0$ of the periastron. Their evolution over one period, under the influence of the spin (disregarding radiation backreaction) is given in (3.18) and (3.19). Taking into account the gravitational radiation backreaction, the shifts (3.19) and (3.18) will vary from one period to another, as the ‘constants’ $L$ and $L_z$ evolve. One may interpret these as the radiation-induced changes in the parameters $\Psi_0$ and $\Phi_0$:

$$\delta \Psi_0 := \Delta \Psi(T) - \Delta \Psi(0) = 2\pi S \frac{18G^2 M \mu^3}{c^2 L^4} \delta L_z$$

$$\delta \Phi_0 := \Delta \Phi(T) - \Delta \Phi(0) = -2\pi S \frac{6G^2 M \mu^3}{c^2 L^4} \delta L$$

where $\delta L = T \langle dL/dt \rangle$ and we have used that in zeroth order $\delta L_z = \delta L \cos \iota$. The total shift in the Euler angle $\Psi_0$ of the periastron point is given by the sum of (3.19) and (6.1), and likewise for $\Phi_0$.

We continue with some comments on the radiative losses in the constants of motion. The terms independent of $S$ agree with the results of Peters and Mathews [17,18]. The terms linear in the spin $S$ in our averaged losses agree with Ryan’s corresponding results, when the parameters $a$, $e$ and $t$ are converted to his parameters. In the present paper, the energy and angular momentum losses have been directly computed. This is to be contrasted with Ryan’s method where the backreaction forces have been used. The perfect agreement achieved underscores the reliability of these computations.

When $\iota = 0$, the averaged losses in $L$ are equal to the averaged losses in $L_z$ up to first order in $S$. Hence the equatorial plane of the orbit is stable with respect to radiation losses.

As we have shown in the previous section, the inclination of the orbit cannot change significantly in the adiabatic era. Hence we conclude that it is insufficient to restrict the computation of signal templates from a black hole – test particle system to equatorial Carter orbits.

In the limiting case of circular orbits, $e \to 0$, there is no distinguished point on the ellipse. The dependence of $\langle dL_z/dt \rangle$ on $\Psi_0$ vanishes. As expected from the axial symmetry, the radiative losses, for arbitrary eccentricity, do not depend on the initial angle $\Phi_0$.

In the present paper, we have developed a toolchest for the computation of radiation losses, which include a new parametrization of the orbit, new angular variables and the application of the residue theorem for obtaining time averages. Currently we are employing these procedures with the inclusion of finite mass effects. [15]

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APPENDIX A: ALTERNATIVE ANGULAR VARIABLES

Introduce three angle variables $\Psi'$, $\iota'$ and $\Phi'$, where $\Psi'$ is the angle of rotation about $L_N$, bringing the position of the particle to the node; $\iota'$ is the angle subtended by $L_N$ and $S$ and $\Phi'$ is the angle of precession of $L_N$ about $S$. The angle $\Psi'$ is the argument of the latitude, $\iota'$ the inclination and $\Phi'$ the longitude of the node [14]. Unlike $\iota$, the angle $\iota'$ is not constant. Let us express $\iota'$ in terms of $\iota$ and $\Psi$. Computing the $z$ component of $L_N$ in two different ways we get the two sides of the equation

$$\{ \mu^2 r^4 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \} \frac{\dot{\Psi}}{\dot{\Psi}} \cos \iota' = \mu^2 \sin^2 \theta \dot{\phi}.$$  \hspace{1cm} (A1)

With the time derivatives of polar angles (2.17) after linearization in $S$, the previous equation gives
\[
\cos \iota' = \cos \iota \left(1 + \frac{2G\mu S}{c^2 L_z r} \sin^2 \iota' \cos^2 \Psi \right). \tag{A2}
\]

Here we are using that the primed angles are equal, to zeroth order in \(S\), with the corresponding unprimed ones. Taking the time derivative of \(\iota'\) we get
\[
\dot{\iota'} = \frac{2GS \sin \iota \cos \Psi}{c^2 L_z \mu r} (\mu r \dot{r} \cos \Psi + 2L \sin \Psi). \tag{A3}
\]

With the new time-dependent Euler rotations the coordinates of the particle become
\[
x = r(\cos \Phi' \cos \Psi' - \cos \iota' \sin \Phi' \sin \Psi')
\]
\[
y = r(\sin \Phi' \cos \Psi' + \cos \iota' \cos \Phi' \sin \Psi')
\]
\[
z = r \sin \iota' \sin \Psi'
\tag{A4}
\]

and the time derivatives of the polar angles are
\[
\dot{\theta} = -\frac{\sin \iota' \cos \Psi'}{\sin \theta} \Psi' - \frac{\cos \iota' \sin \Psi'}{\sin \theta} \iota', \quad \sin^2 \theta = 1 - \sin^2 \iota' \sin^2 \Psi',
\tag{A5}
\]
\[
\dot{\phi} = \dot{\Phi}' + \frac{\cos \iota' \sin \theta}{\sin \theta} \dot{\Psi}' - \frac{\sin \iota' \sin 2\Psi'}{2 \sin^2 \theta} \iota'.
\tag{A6}
\]

Inserting these equations in (2.17), we get the changes of the primed Euler angles in terms of the unprimed ones
\[
\dot{\Psi}' = \frac{L}{\mu r^2} + \frac{GS \cos \iota}{c^2 L_z r^3} (2L - \mu r \dot{r} \sin 2\Psi)
\tag{A7}
\]
\[
\dot{\Phi}' = \frac{2GS \sin \Psi}{c^2 L_z r^3} (\mu r \dot{r} \cos \Psi + 2L \sin \Psi).
\tag{A8}
\]

Integrating these equations, we get the new angle variables and their shifts over one period
\[
\Delta \iota' = 0, \quad \Delta \Psi' = 2\pi - 2\pi \frac{6G^2 M^3 L_z}{c^2 L^4}, \quad \Delta \Phi' = 2\pi \frac{2G^2 M^3}{c^2 L^4}.
\tag{A9}
\]

Note that these changes in the primed variables are equal to the shifts (3.13) and (3.14) of the unprimed variables.

[1] B.Carter, Phys. Rev. 174, 1559 (1968)
[2] K.S.Thorne, private communication
[3] A.Ori, Phys. Rev. 174, 1559 (1968)
[4] L.Kidder, C.Will and A.Wiseman, Phys. Rev. D47, 4183 (1993)
[5] B.M.Barker and R.F.O’Connell, Gen.Rel.Gravitation 11, 149 (1979)
[6] The computations of [4] have been extended to the 5/2PN approximation by L.Kidder, Phys. Rev. D52, 821 (1995)
[7] M.Shibata, Phys. Rev. D50, 6297 (1994)
[8] F.Ryan, Phys. Rev. D52, R3159 (1995)
[9] F.Ryan, Phys. Rev. D53, 3064 (1996)
[10] H. Thirring and S. Lense, Phys. Zeitschr. 19, 156 (1918), English translation: Gen.Rel.Gravitation, 16 727 (1984)
[11] L.D.Landau and E.M.Lifitz, The Classical Theory of Fields, Pergamon, Oxford, 1975
[12] L. À. Gergely and Z.Perjés, in Proceedings of the Virgo Conference, Eds. I.Ciufolini and F.Fidecaro, World Scientific, Singapore, 1997
[13] L.Blanchet and T. Damour, Ann.Inst.Henri Poincaré, A50, 377 (1989)
[14] T. Damour and B. Iyer, Ann.Inst.Henri Poincaré, 54, 115 (1991)
[15] L. À. Gergely, Z.Perjés and M.Vasúth, in preparation.
[16] The parametrization of the Kepler orbit by the angle \(\chi\) is close in spirit to Ptolemy’s idea of the equant. Both the equant and the orbit parameter are means for setting up a rule for determining the angular velocity of the orbiting body.
[17] P.C. Peters and S.Mathews, Phys. Rev. 131, 435 (1963)
[18] P.C. Peters, Phys. Rev. 136, B1224 (1964)