Euclidean Space with Quantum OAM based on Tensor Analysis

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Abstract: Orbital Angular Momentum (OAM) is the intrinsic property of electromagnetic waves, which can be applied in wireless transmission. When OAM is applied with the traditional modulation and coding, the Euclidean space model with the OAM dimension is required. Due to the introduction of quantum OAM, the OAM can be used to enlarge the traditional Euclidean space, and need higher-order tensor space to analyze traditional multidimensional Euclidean space. In this paper, the high-dimension Euclidean space is established based on quantum OAM in high-order tensor model, which can be used for the optimization of the joint coding and modulation. The performances of bit error rate confirm the capability of high-dimension Euclidean space model in the corresponding signal processing.

Keywords: orbital angular momentum, Euclidean space, trellis coding modulation, OAM, TCM.

Classification: Wireless communication technologies

References

[1] David L. Andrews, Mohamed Babiker, The angular momentum of light, Cambridge University Press, New York, 2013.
[2] Zhang C, et al., “Vortex electron generated by microwave photon with orbital angular momentum in a magnetic field”, AIP Advances, 2020, 10(10):105230. DOI:10.1063/5.0019899
[3] Katoh M, et al., “Helical phase structure of radiation from an electron in circular motion”, Scientific Reports, 2017, 7(1):6130. DOI:10.1038/s41598-017-06442-2
[4] Chen Y, et al., “Mapping twisted light into and out of a photonic chip”, Physical Review Letters, 2018, 121. DOI:10.1103/PhysRevLett.121.233602
[5] Mcdonnell M.J, et al., “High-efficiency detection of a single quantum of angular momentum by suppression of optical pumping”, Physical Review Letters, 2004, 93(15):153601. DOI:10.1103/PhysRevLett.93.153601
[6] Allen L., et al., “Obital angular momentum of light and the transformation of Laguerre-Gaussian laser modes”, Physical Review A. 1992, 45(11):8185-8189. DOI:10.1103/PhysRevA.45.8185
[7] Chao Zhang, Jin Jiang, “Angular momentum spectrum of electromagnetic wave,” IEICE Trans. Fundamentals, vol. E103-A, no.4, pp.715-717. DOI: 10.1587/transfun.2019EAL2112
[8] Cohen-Tannoudji, Dupont-Roc, and Grynberg, Introduction to quantum electrodynamics, New York: Wiley, 1989.
1 Introduction

Orbital Angular Momentum (OAM) is an intrinsic physical property of Electromagnetic (EM) waves. In quantum states, the value of OAM mode number is discrete, which obeys the rule of quantum electrodynamics [1]. The quantum OAM owns the OAM term with mode $l$. The beam of quantum state radiates EM wave by cyclotron oscillating electron in magnetic field [2]. While the electron in magnetic field is cyclotron oscillating, it interacts with EM wave and radiates EM wave vortex quantum with OAM [3]. Generally, this process can be realized by cyclotron or gyrotron in engineering [2]. In principle, an infinite range of OAM modes can be obtained [1] in wireless transmission independent with the electric field strength. Moreover, the OAM detection methods and sensors are developed in some published papers. e.g. The OAM waveguide photonic chip is developed to detect OAM by measuring the interference between distorted light and reference light, and projecting the OAM mode before and after the chip [4]. While suppressing excitation on a closely separated unwanted transition using an intense pump laser, relying on resonant excitation of a wanted transition using a weak probe laser beam to detect OAM state of a single atom [5]. The orthogonality in OAM brings the additional spectrum efficiency, which was first discussed in the region with optical photon [6]. The quantum OAM is a new dimension in wireless transmission, which can establish its own OAM spectrum [7]. The traditional Euclidean space is determined by the electric field intensity in the traditional domain (frequency domain, space domain, code domain, etc.), while the quantum OAM transmission with independent and orthogonal dimensions can define a larger Euclidean space. However, the traditional definition of Euclidean space is limited to multi-dimensional first-order tensor, namely vector space. In this paper, we derive the definition of multi-dimensional Euclidean space in tensor analysis with the quantum OAM dimension. In addition, the multi-dimensional second-order tensor space description is established.

2 Euclidean Space with OAM in Tensor analysis

According to quantum electro-dynamics [8], the EM wave obeys the quantum mechanics rules which is composed of EM photons. Therefore, the Euclidean space can be established on OAM, which is the physical property of the EM photon. Statistically, the terminology of the Degrees of Freedom (DoF) is defined as the amount of independent information entering the parameter estimation. In wireless transmissions, the DoF is the ratio of the capacity $C(P)$ to the logarithm $\log(P)$ as shown in Eq. (1), where $P$ indicates the transmission power. If the channel matrix $H$ is given,
the DoF is equal to the rank of $H$ [9].

$$\xi = \lim_{P \to \infty} \frac{C(P)}{\log(P)} = \text{rank}(H).$$

(1)

The orthogonal basis of DoF is orthogonal to each other [10].

Assume that the detection in different dimensions is regarded as the Cartesian product of $\mathbb{R}$, the elements of which have component representation $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and give the usual vector addition and multiplication with the number of real fields $\mathbb{R}$. $\mathbb{R}^n$ constitutes a linear space, and its elements are represented by columns vector $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$. The entire linear function $\mathbb{R}^n \to \mathbb{R}^n$ constitutes a dual space $(\mathbb{R}^n)^*$, and its elements are represented by a row vector $f = [f_1, \cdots, f_n] \in (\mathbb{R}^n)^*$. The electric field strength $E$ gives the space expressed as a tensor as

$$E_i = [e_1 | e_2 | e_3]$$

(2)

where $i = 1, 2, 3$. $e_1, e_2$ and $e_3$ denote the electric field strengths detected by antennas on the $x$-axis, $y$-axis and $z$-axis in the three-dimensional space, respectively. Each electric field strength basis vector has components expressed as

$$e_i = [e_i^1 | e_i^2 | e_i^3]^T$$

(3)

where $j = 1, 2, \cdots, n$. The $e_i^j$ denotes the projection of different physical dimensions of electric field strength detection by antennas on different physical dimensions, e.g., $e_i^1$, $e_i^2$ and $e_i^3$ denote the time domain, the space domain and the frequency domain signal detected on the $i$-axis of antennas, respectively.

Similarly, as an independent and orthogonal dimension, OAM is a physical quantity that has the similar status as $E$, i.e., the three dimensions of OAM can be detected by the OAM sensor in the time domain, the space domain, the frequency domain, the code domain, etc. As indicated in Eq. (3), each OAM basis vector has components expressed as

$$l_i = [l_i^1 | l_i^2 | l_i^3]^T$$

(4)

where, the $l_i^j$ denotes the projection of different physical dimensions of OAM detected by OAM sensors. Assume that the $E$ and $l$ can be both detected simultaneously in the system, then, the two independent orthogonal dimensions $E$ and $l$ can be calculated by the tensor product as

$$L_{ij} = l_i \otimes e_j = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

(5)

where $i, j = 1, 2, 3$. $L_{ij}$ denotes the second-order tensor, i.e. $l_{ij}$ denotes the OAM in the $i$-axis and electric field strength in the $j$-axis. Similarly, each element of the tensor contains the signal of the time domain, the space domain, the frequency domain, the code domain, etc. $L_{ij}$ is composed of $n + 3$ orthogonal dimensions and two independent dimensions $E$ and $l$. Therefore, $L_{ij}$ is also called $n + 3$ dimensional second-order tensor.
According to the definition of Euclidean distance, the Euclidean distance between two second-order tensors can be defined as

\[ d_{L_{ij}} = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} (l_{ij} - \bar{l}_{ij})^2} \]  

(6)

Usually, the time domain contains \( n_1 \) and \( n_2 \) orthogonal components in two independent dimensions of \( L_{ij} \), respectively. Moreover, the frequency domain contains \( n_3 \) and \( n_4 \) orthogonal components in two independent dimensions of \( L_{ij} \), respectively. Assume that there are \( M \) OAM sensors. According to the definition of DoF, the DoF of the second-order tensor \( L_{ij} \) can be calculated as

\[ \xi_1 = 9n_1n_2n_3n_4M \]  

(7)

For OAM quantum transmission, the OAM is a new dimension, and the new dimension is independent and totally separated from the traditional domains. This OAM can be used to enlarge the traditional Euclidean space. According to Eq. (7), second-order tensor \( L_{ij} \) consists of \( \xi \) DoF. Assume that each DoF is represented by the variable \( s \), the Euclidean space can be defined as \( S_1 = \{ s_1, s_2, \cdots, s_{\xi_1} \} \).

3 Discussion and Analysis

3.1 Trellis with OAM Orthogonal Space

In the following, we give an example of trellis coding and modulation with proposed Euclidean Space. The Euclidean free distance can be defined as

\[ d_{\text{free}} = \sqrt{\min \sum |a_n - b_n|^2}; \{a_n\}, \{b_n\} \in \mathbb{C}; \{a_n\} \neq \{b_n\} \]  

(8)

where \( a_n \) and \( b_n \) are the transmitted and received bit sequences, respectively. \( \mathbb{C} \) is a complex number set, which denotes the largest number set.

Assume that 2 OAM modes multiplexing without interference are taken into account, each OAM mode transmits QPSK constellation signals. Furthermore, the Trellis Coding and Modulation (TCM) is utilized in 8 constellation points of the 2 OAM modes, the QPSK with 2 OAM modes signal is divided into 4 subsets \( \{S_0, S_1, S_2, S_3\} \). The constellation decomposition is shown in Fig. 1. The minimum Euclidean distance between constellation points in the first set segmentation is \( \sqrt{2} \). In the second layer, the minimum Euclidean distance between the constellation points in the same OAM mode set is 2. In addition, the minimum Euclidean distance between the constellation points in the different OAM modes is \( \sqrt{2} \).

We denote 2 shift registers for 2/3 convolutional coding as \( (3,2,2) \) convolutional coding, as shown in Fig. 2(a), the first bit \( b_1 \) and the second bit \( b_2 \) output \( \{d_0, d_1\} \) and \( \{d_2\} \) through \( (3,2,2) \) convolutional coding, respectively. The set \( \{00,01,10,11\} \) formed by \( \{d_0, d_1\} \) is used to select the subset \( \{S_0, S_1, S_2, S_3\} \), and the set \( \{0,1\} \) formed by \( \{d_2\} \) is used to select the constellation points in the subset \( \{S_0, S_1, S_2, S_3\} \). The spectral efficiency within the unit bandwidth is only 2 bit/s/Hz, which has the same spectral efficiency as QPSK. If TCM is not utilized, two orthogonal OAM modes are used with QPSK modulation (2 bit/s/Hz), 4 bit/s/Hz.
Fig. 1. (3,2,2) convolution coding 4 subset QPSK TCM constellation decomposition with OAM orthogonality.

Fig. 2. (3,2,2) TCM convolution encoder mapping and optimal path grid mapping with OAM orthogonality.

are expected in total. Due to the 2/3 convolutional coding, the spectral efficiency is only 2 bit/s/Hz, which has the same spectral efficiency as QPSK (2 bit/s/Hz).

Since the constraint length of convolutional code is 2, the number of coding states is 4, and the number of input bits of TCM is 2. There are four transfer paths starting from each state. The Euclidean distance between the two parallel transfer paths is the same. The state transition diagram is shown in Fig. 2(b). The path from the zero state to itself is from “00” to “10”, then to “01”, and then back to “00”. The free distance of subset segmentation can be calculated as $d(S_0, S_1)^2 + d(S_0, S_2)^2 + d(S_0, S_2)^2 = 8$. Therefore, the minimum free distance of (3,2,2) convolutional encoding 4-state QPSK TCM with four OAM modes is $d_{\text{free}} = 2\sqrt{2}$.

### 3.2 BER Performance with Euclidean Distance

In general, the error probability of the decoder can be approximately calculated as

$$P(e) \approx N_{\text{free}}Q\left(\frac{d_{\text{free}}}{\sqrt{2n_0}}\right)$$

(9)
where $N_{\text{free}}$ represents average number of nearest neighbors with the decoding path [11], $n_0$ is the power spectral density of the noise. For 4-state 8PSK/TCM, convolution code with the ratio of 2/3 is employed, which has the spectral efficiency with 2 bit/s/Hz. Among the 3 input bits, 2 bit is used for (3,2,2) convolution coding to select the subset, 1 bit is used for (3,2,2) convolution coding to select constellation points from the subset. On the basis of analysis in Sect. 3.1, $N_{\text{free}} = 1$, normalized the average signal power $E_s = 1$, $d_{\text{free}} = 2 \sqrt{E_s} = \sqrt{8E_b}$, where $E_b = E_s/2$ is the average transmitted energy per bit. Therefore, the BER of four-state TCM with 8PSK is given by $P_{b,4\text{state}} = Q\left(2 \sqrt{E_b/n_0}\right)$. The OAM applications modeled with Euclidean spaces is simulated with 2 bit/s/Hz. The advances of the TCM compared to QPSK without any coding and 4-state 8PSK/TCM without OAM mode, the BER decreases with using OAM modes, as shown in Fig. 3. The simulation results show that the quantum OAM constitutes the independent orthogonal channel. In addition, the resource of OAM orthogonal dimensional are used in exchange for the improvement of BER performance.

![Figure 3. BER performance comparison.](image)

### 4 Conclusion

In this paper, the Euclidean space is established based on physical property of OAM in tensor. With the dimension of OAM, traditional Euclidean space have been greatly improved. Besides, the paper consolidates the performance evaluation with Euclidean distances, such as the coding and modulation with OAM, which confirms the capability of the high-dimension Euclidean space model in the corresponding signal processing with quantum OAM.

### Acknowledgments

Research is sponsored by the Science and Technology Key Project of Guangdong Province under Grant 2019B010157001, in part by the National Natural Science Foundation of China (NSFC) with project number 61731011.

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