MODEL THEORY OF PROBABILITY SPACES
WITH AN AUTOMORPHISM

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ABSTRACT. The class of generic structures among those consisting of the measure algebra of a probability space equipped with an automorphism is axiomatizable by positive sentences interpreted using an approximate semantics. The separable generic structures of this kind are exactly the ones isomorphic to the measure algebra of a standard Lebesgue space equipped with an aperiodic measure-preserving automorphism. The corresponding theory is complete and has quantifier elimination; moreover it is stable with built-in canonical bases. We give an intrinsic characterization of its independence relation.

1. INTRODUCTION

One motivation for the work presented in this paper is to understand generic structures in the setting of measure algebras equipped with an automorphism. In recent years, existentially closed (or “generic”) structures have (again) attracted a lot of attention in model theory. Several important examples in simple and stable theories, like algebraically closed fields, differentially closed fields, random graphs and algebraically closed fields with a generic automorphism, can be seen as existentially closed structures. Expanding a stable structure by adding generic predicates, automorphisms or substructures is a common way to construct examples of simple theories. It is a natural idea to test these tools outside the first order context, in particular to apply them to familiar structures coming from analysis and probability. The study of some generic expansions of Hilbert spaces is carried in [5]. In this paper we examine measure algebras of probability spaces extended by a generic automorphism.

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A second motivation for this paper comes from the more general objective of understanding the model theory of probability spaces equipped with an automorphism. Said differently, it is the connections between model theory and ergodic theory that are being explored here, from a very general point of view. It is not surprising that some of this general theory might be needed to study generic structures. However, it turns out that the connection between these two projects is much closer than could be expected. Indeed, a generic structure in this setting turns out to be essentially the same as a probability space equipped with an aperiodic automorphism (i.e., one for which the set of points with a finite period has measure zero). Moreover, the theory of an arbitrary probability space with an automorphism can be reduced to the theory of its aperiodic part in a simple and direct way.

The model theoretic background for this paper comes largely from [10], suitably generalized for metric structures. On a probability space there is a canonical pseudometric, obtained by taking the distance between sets to be the measure of their symmetric difference. The quotient structure which turns this distance into a metric is exactly the measure algebra of the original probability space: identify two sets if they differ by a set of measure zero. Any automorphism of the probability space induces one on the measure algebra. A full discussion of these structures from a point of view very close to what we take here can be found in Chapter 7 of [19]. In general, an automorphism of a measure algebra need not arise from a point map of the underlying probability space. However, if the measure algebra is separable (as a metric space; equivalently, if it is countably generated as a measure algebra) then it is isomorphic to the measure algebra of a Lebesgue space and all of its automorphisms arise from point maps.

The study of aperiodic maps on Lebesgue spaces is one of the fundamental parts of ergodic theory. The basics of the theory of Lebesgue spaces and of aperiodic maps on such spaces is due to Rokhlin. In section two we introduce the basic definitions and tools that we need from analysis. Among them is Rokhlin’s theorem on aperiodic maps (stated here as Theorem 2.5), which is a fundamental tool for this paper.
Stability in probability spaces was studied by Ben-Yaacov in [2]. In section three we present some of the results from [2], but we translate them to the language of positive formulas (see [10]). In particular we show that the approximate positive theory of atomless probability spaces is well behaved from the model theoretic point of view: it has quantifier elimination, is separably categorical and is superstability with respect to the topology generated by the distance metric in the space of types. We also prove the definability of this distance metric and introduce the notion of built-in canonical bases. For this section we expect the reader to be familiar with the notions from [10], in a modified form suitable to the metric structures that are studied here.

In section four we construct an axiomatization (denoted by $T_A$) that isolates the aperiodic expansions among the atomless probability spaces expanded by an automorphism. We prove that the class of models of these axioms agree with the existentially closed models of the approximate theory of probability spaces expanded by an automorphism. In this section we also show that $T_A$ has quantifier elimination.

The fifth section is dedicated to showing the stability of the theory $T_A$ and giving a natural characterization of non-dividing. We also prove that $T_A$ has built-in canonical bases.

Entropy is an additive rank defined in ergodic theory. In section six we review its properties and state some of its model theoretic consequences.

In section seven we use the properties of entropy to show that the types in $T_A$ of finite tuples over a set of parameters $B$ are either non-principal or they belong to the definable closure of $B$. Using an argument like one that has been applied to ACFA, as well as the properties of entropy, we generalize a theorem of ergodic theory and show that the types of transformally independent elements are orthogonal to the types of transformally definable elements (see Definition 6.6).

Finally in the last section we discuss the model theory of an arbitrary (not necessarily atomless) probability space extended by a general (not necessarily aperiodic) automorphism. We show the approximate positive theory of such a structure is easily reduced to the theory of its aperiodic part.
2. Probability spaces, Lebesgue spaces and aperiodic maps

In this section we present the basic information about probability spaces including Lebesgue spaces, their measure algebras of events, and aperiodic maps.

A probability space is a triple \((X, \mathcal{B}, P)\), where \(X\) is a space, \(\mathcal{B}\) is a \(\sigma\)-algebra of subsets of \(X\) and \(P\) is a measure on \(\mathcal{B}\) such that \(P(X) = 1\). We say that a probability space \((X, \mathcal{B}, P)\) is atomless if for any \(B \in \mathcal{B}\) such that \(P(B) > 0\), there are \(B_1, B_2 \in \mathcal{B}\) such that \(B = B_1 \cup B_2\), \(B_1\) and \(B_2\) are disjoint and \(P(B_1) > 0\), \(P(B_2) > 0\).

We say that \(A_1, A_2 \in \mathcal{B}\) determine the same event, if the symmetric difference of the sets, denoted by \(A_1 \triangle A_2\), has measure zero. We denote the class of \(A\) under this equivalence relation by \(a\). Throughout this section lower case letters will stand for events and capital letters either for elements of the \(\sigma\)-algebra or for sets of events.

The operations of complement, union and intersection are well defined for events. We call events\((X, \mathcal{B}, P)\) a measure algebra and the pair \((\text{events}(X, \mathcal{B}, P), P)\) a measured algebra.

Whenever \(C \subset \mathcal{B}\) is a \(\sigma\)-subalgebra, we denote by \(\text{events}(C, P)\) the measure algebra of events coming from \(C\) with respect to \(P\). Whenever \(P\) is clear from context, we just write \(\text{events}(C)\). Conversely, whenever \(C \subset \text{events}(X, \mathcal{B}, P)\), we denote by \(\langle C \rangle\) the \(\sigma\)-subalgebra of \(\mathcal{B}\) generated by the elements \(\{A \in \mathcal{B} : a \in C\}\).

There are two approaches to understand isomorphisms on probability spaces. On the one hand, we have point maps between the spaces, on the other, measure preserving maps between measured algebras.

2.1. Definition. Let \((X_1, \mathcal{B}_1, m_1)\), \((X_2, \mathcal{B}_2, m_2)\) be probability spaces and let \(\hat{\mathcal{B}}_1 = \text{events}(X_1, \mathcal{B}_1, m_1)\), \(\hat{\mathcal{B}}_2 = \text{events}(X_2, \mathcal{B}_2, m_2)\) be their measure algebras. By an isomorphism of the measured algebras we mean a bijection \(\Phi: \hat{\mathcal{B}}_2 \to \hat{\mathcal{B}}_1\) which preserves complements, countable unions and intersections and satisfies \(m_1(\Phi(\hat{B})) = m_2(\hat{B})\) for all \(\hat{B} \in \hat{\mathcal{B}}_2\). The probability spaces are said to be conjugate if their measured algebras are isomorphic.

2.2. Definition. Let \((X_1, \mathcal{B}_1, m_1)\), \((X_2, \mathcal{B}_2, m_2)\) be probability spaces and let \(\hat{\mathcal{B}}_1 = \text{events}(X_1, \mathcal{B}_1, m_1)\), \(\hat{\mathcal{B}}_2 = \text{events}(X_2, \mathcal{B}_2, m_2)\) be their measure algebras. Let \(M_1 \in \mathcal{B}_1\), \(M_2 \in \mathcal{B}_2\) with \(m_1(M_1) = 1 = m_2(M_2)\). An invertible measure preserving
transformation $\phi: M_1 \to M_2$ is called an \textit{isomorphism} between $(X_1, B_1, m_1)$ and $(X_2, B_2, m_2)$. If $(X_1, B_1, m_1) = (X_2, B_2, m_2)$, we call $\phi$ an \textit{automorphism}. The induced map $\phi: \hat{B}_1 \to \hat{B}_2$ is called an \textit{induced isomorphism} of the measured algebras.

To bridge the gap between these two approaches we need to know how point maps are related to maps of measured algebras. Clearly any two isomorphic probability spaces are conjugate; however, the converse does not hold in general. The next definition concerns a well-known special class of probability spaces which are well behaved from this point of view.

2.3. \textbf{Definition.} A probability space $(X, B, m)$ is a \textit{Lebesgue space} if it is isomorphic to a probability space which is the disjoint union of a countable (or finite) set of points $\{y_1, y_2, \ldots\}$, each of positive measure, and the space $([0, s], L([0, s]), l)$, where $L([0, s])$ is the Lebesgue $\sigma$-algebra of $[0, s]$ and $l$ is Lebesgue measure. Here $s = 1 - \sum_{i=1}^{\infty} p_i$, where $p_i > 0$ is the measure of $\{y_i\}$.

The theory of Lebesgue spaces was developed by Rokhlin. On these spaces the notion of isomorphism and conjugacy coincide (see Theorem 2.2 in [21]). Thus, as long as we work on Lebesgue spaces, we can switch between point maps and maps on the measured algebra of events.

For the rest of this section we fix $(X, B, m)$ an atomless Lebesgue space. It is shown in [9] that for any $A, B \in B$ such that $m(A) = m(B)$, there is an automorphism $\eta$ of the space such that $\mu(\eta(A) \triangle B) = 0$.

Let $G$ be the group of measure preserving automorphisms on $(X, B, m)$, where we identify two maps if they agree on a set of measure one. There is a natural representation of $G$ in $B(L^2(X, B, m))$ (the space of bounded linear operators on $L^2(X, B, \mu)$); it sends $\tau \in G$ to the unitary operator $U_\tau$ defined for all $f \in L^2(X, B, \mu)$ by $U_\tau(f) = f \circ \tau$. The norm topology on $B(L^2(X, B, \mu))$ pulls back to a group topology on $G$, which is called in [9] the \textit{uniform topology} on $G$. For $\tau, \eta \in G$, let $\rho(\tau, \eta) = m(\{x \in X : \tau(x) \neq \eta(x)\})$. It is shown in [9] that $\rho$ is a metric for the uniform topology.

For the rest of this section we will study aperiodic maps and their properties. A good source for this material is the book of Halmos [4] on ergodic theory.
2.4. **Definition.** Let \((Y, C, \mu)\) be an atomless probability space and let \(\tau\) be an automorphism of \((Y, C, \mu)\). We say that \(\tau\) is *aperiodic* if for every \(n \in \mathbb{N}^+\), the set \(\{x \in X : \tau^n(x) = x\}\) has measure zero.

One of the key tools in studying aperiodic automorphisms is the following theorem by Rokhlin:

2.5. **Theorem.** *(Rokhlin’s Lemma [9], [13])* Let \((Y, C, \mu)\) be an atomless probability space and \(\tau\) an aperiodic automorphism of this space. Then for every positive integer \(n\) and \(\epsilon > 0\), there exist a measurable set \(E\) such that the sets \(E, \tau(E), \ldots, \tau^{n-1}(E)\) are disjoint and \(\mu(\bigcup_{i<n} \tau^i(E)) > 1 - \epsilon\).

2.6. **Definition.** We call a map \(\eta \in G\) a *cycle* of period \(k\) if there is a set \(E \in B\) such that \(E, \ldots, \eta^{k-1}(E)\) forms a partition of \(X\) and \(\eta^k = \text{id}\).

2.7. **Observation.** Let \(\tau \in G\) be aperiodic. By Rokhlin’s Lemma, for every \(N > 0\) there is a cycle \(\eta \in G\) of period \(N\) such that \(\rho(\tau, \eta) \leq 2/N\) (see [9, pp. 75]).

2.8. **Remark.** Any two cycles \(\eta_1, \eta_2 \in G\) of period \(k\) are conjugate in \(G\). Let \(E \in B\) be such that \(E, \ldots, \eta_1^{k-1}(E)\) forms a partition of \(X\) and let \(F \in B\) be such that \(F, \ldots, \eta_2^{k-1}(F)\) forms a partition of \(X\). Since \((X, B, \mu)\) is a Lebesgue space, there is a measure preserving invertible map \(\gamma\) such that \(\gamma(E) = F\). Extend \(\gamma\) by defining for \(x = \eta_1^i(y) \in \eta_1^i(E), \gamma(x) = \eta_2^i(\gamma(y))\). Then \(\gamma \eta_1 = \eta_2 \gamma\).

2.9. **Proposition.** Let \(\tau_1, \tau_2 \in G\) be aperiodic. Then for every \(\epsilon > 0\), there is a conjugate \(\tau_2'\) of \(\tau_2\) such that \(\rho(\tau_1, \tau_2') \leq \epsilon\).

*Proof.** Let \(N > 0\) be such that \(4/N < \epsilon\). By 2.7 we can find \(\eta_1, \eta_2 \in G\) cycles of period \(N\) such that \(\rho(\tau_1, \eta_i) < 2/N\) for \(i = 1, 2\). Using 2.8 we get \(\gamma \in G\) such that \(\eta_1 = \gamma^{-1} \eta_2 \gamma\). Let \(\tau_2' = \gamma^{-1} \tau_2 \gamma\). Then \(\rho(\tau_1, \tau_2') \leq \rho(\tau_1, \eta_1) + \rho(\eta_1, \tau_2') = \rho(\tau_1, \eta_1) + \rho(\eta_2, \tau_2) < 4/N\). \(\square\)

3. **The model theory of probability spaces**

We develop the model theory of probability spaces inside structures of the form \(M = (\text{events}(X, B, P), \emptyset, X, -1, \cap, \cup, P)\), where \((X, B, P)\) is an atomless probability space, \(\emptyset\) is the event corresponding to \(\emptyset\), \(X\) is the event corresponding to \(X\), \(\cup\),
stand for the union and intersection of events, \( \cap ^{-1} \) for the complement of events and \( P \) for the probability of events. We also include in \( M \) a second sort for the ordered field of real numbers and constants for all rationals. For \( a, b \in M \), let \( \rho(a, b) = P(a \triangle b) \). The distance \( \rho \) is a metric on the space of events, it is definable from \( P \) and makes \( M \) a complete metric space. We will use the tools from [10], modified for metric structures, to understand the model theory of probability spaces (and later their expansion by generic automorphisms). We call a structure \( M \) as above a probability structure.

We define positive formulas inductively. If \( q \in \mathbb{Q} \) and \( x_1, \ldots, x_n \) are variables in the sort of events and \( t(y_1, \ldots, y_n) \) is a polynomial with coefficients in \( \mathbb{Q} \), then \( t(P(x_1), \ldots, P(x_n)) \geq q \) and \( t(P(x_1), \ldots, P(x_n)) \leq q \) are positive formulas. If \( \varphi, \psi \) are positive formulas, so are \( \varphi \lor \psi \) and \( \varphi \land \psi \). Finally, if \( \varphi \) is a formula, so are \( \exists x \varphi \) and \( \forall x \varphi \), where \( x \) is a variable in the sort of events.

From an abstract point of view, the structures considered here consist of a complete metric space \((M, \rho)\) equipped with operations making \( M \) a Boolean algebra on which \( P(a) = \rho(a, 0) \) defines \( P \) to be a probability measure and it is translation invariant under the operation of symmetric difference. In Chapter 7 of [19] there is a full discussion of the fact that these structures are exactly the measured algebras of probability spaces.

Strictly speaking [10] is formulated in the setting of normed space structures, and the probability structures considered here are not of that type. However, the aspects of [10] on which we rely here are routinely seen to apply to probability structures, and we will cite results from [10] (such as the existence of highly saturated and homogeneous models) without additional comment. Note that since a probability structure is based on a bounded metric space (of diameter 1) there is no need to bound quantifiers; the key aspects of [10] that are essential to what we do here are the use of positive formulas (only) and the use of an approximate semantics.

In this section we denote by lower case letters the events and by capital letters elements in the \( \sigma \)-algebra and sets of events.

We need the following special case of the Radon-Nikodym theorem:

3.1. **Theorem.** (Radon-Nikodym) Let \((X, \mathcal{B}, P)\) be an atomless probability space, let \( \mathcal{C} \subseteq \mathcal{B} \) be a \( \sigma \)-subalgebra and let \( A \in \mathcal{B} \). Let \( a \) be the event corresponding to \( A \). Then
there is a unique $g_a \in L^1(X, \mathcal{C}, P)$ such that for any $B \in \mathcal{C}$, $\int_B g_a dP = \int_B \chi_A dP$. Such an element $g_a$ is called the conditional probability of $a$ with respect to $\mathcal{C}$ and it is denoted by $P(a|\mathcal{C})$.

3.2. Definition. Let $\kappa$ be a regular cardinal larger than $2^{\aleph_0}$. We say that a metric structure $M$ in a language $L$ is a $\kappa$-universal domain if it is $\kappa$-strongly homogeneous and $\kappa$-saturated for all reducts of the language (see [11]). We call a subset $C \subset M$ small if $|C| < \kappa$.

3.3. Definition. Let $M$ be a metric structure which is a $\kappa$-universal domain for its positive theory. Let $C \subset M$ be small and $n \in \mathbb{N}^+$. By an $n$-type over $C$ we mean the collection of positive formulas with parameters in $C$ realized by some tuple $\bar{a} \in M^n$.

The collection of $n$-types over a small set $C$ is independent of the choice of $M$; it only depends on $Th_A(M, c)_{c \in C}$.

We denote by $T$ the approximate theory [10] of an atomless probability structure.

3.4. Definition. Let $M$ be a metric structure which is a $\kappa$-universal domain for its theory. Let $C \subset M$ be small. The definable closure of $C$, denoted by $dcl(C)$, is the collection of elements in $M$ that are fixed under automorphisms of $M$ fixing $C$ pointwise.

The following two lemmas are proved in [2]:

3.5. Lemma. Let $M \models T$ be a $\kappa$-universal domain, let $C \subset M$ be small and let $a_1, \ldots, a_n; b_1, \ldots, b_n \in M$. Then $tp(a_1, \ldots, a_n/C) = tp(b_1, \ldots, b_n/C)$ iff $P(a_i^1 \land \cdots \land a_i^n | (C)) = P(b_i^1 \land \cdots \land b_i^n | (C))$ for all $i_j \in \{1, -1\}, j \leq n$.

In particular, any probability structure has quantifier elimination (see [11] pp. 86–88).

3.6. Lemma. Let $M \models T$ be a $\kappa$-universal domain and let $C \subset M$ be small. Then $dcl(C) = events((C))$.

The next lemma gives another basic fact about the model theory of probability spaces.
3.7. Lemma. The positive theory $T$ is separably categorical.

Proof. Let $N$ be a separable complete model of $T$. Then $N$ is a probability structure coming from an atomless Lebesgue space $(X, \mathcal{B}, m)$, where $\mathcal{B}$ is countably generated. Hence $N$ is isomorphic to the probability structure coming from the standard interval $([0, 1], \mathcal{B}, m)$. \hfill \Box

Let $M$ be a metric structure which is a $\kappa$-universal domain for its theory and denote by $\rho$ the metric of $M$. Then the collection of $n$-types over small sets also form a metric space $[10]$. When $C \subset M$ is small and $p, q$ are $n$-types over $C$, we define the $d$-metric by $d(p, q) = \inf \{ \max_{i \leq n} \rho(a_i, b_i) : (a_1, \ldots, a_n) \models p, (b_1, \ldots, b_n) \models q \}.$

In general, in a metric space structure the distance between types over finite sets need not be definable. The next lemma shows that these expressions on probability structures are actually definable (a fact known to analysts). We sketch a proof; see also Lemma 6.3 in [18]:

3.8. Lemma. Let $M \models T$ be a $\kappa$-universal domain and let $C \subset M$ be small. Let $\bar{a} = (a_1, \ldots, a_n) \in M^n, \bar{b} = (b_1, \ldots, b_n) \in M^n$ be partitions of the probability structure. Then $d(tp(\bar{a}/C), tp(\bar{b}/C)) = \max_{i \leq n} \| P(a_i|C) - P(b_i|C) \|_1$, where $\| \|_1$ is the $L_1$-norm.

Proof. We prove the result for $C = \emptyset$. It is sufficient to show that $d(tp(\bar{a}), tp(\bar{b})) \leq \max_{i \leq n} |P(a_i) - P(b_i)|$, since the other direction is obvious. Since $T$ is separably categorical and $C = \emptyset$, we can work in the probability structure of the standard Lebesgue space $([0, 1], \mathcal{B}, P)$. Let $(A_1, \ldots, A_n) \in \mathcal{B}^n, (B_1, \ldots, B_n) \in \mathcal{B}^n$ be partitions of $[0, 1]$. Reordering the partitions if necessary, we may assume there is $m \leq n$ such that $P(A_i) \geq P(B_i)$ iff $i \leq m$. Let $A'_i$ be the closed interval $[\sum_{j < i} P(A_j), \sum_{j \leq i} P(A_j)]$ for $i \leq n$. Write $a_i, b_i, a'_i$ for the events determined by $A_i, B_i$ and $A'_i$ for $i \leq n$. Then $tp(a_1, \ldots, a_n) = tp(a'_1, \ldots, a'_n).$ For $i \leq m$, let $B'_i$ be the closed interval $[\sum_{j < i} P(A_j), \sum_{j \leq i} P(A_j) + P(B_i)]$. Clearly $P(A'_i \Delta B'_i) = P(A_i) - P(B_i)$ for $i \leq m$. For $i > m$, let $B'_i$ be the set $[\sum_{j < i} P(A_j), \sum_{j \leq i} P(A_j)] \cup D'_i$, where $\{ D'_i : m < i \leq n \}$ are measurable sets, disjoint from each other and disjoint from $B'_1, \ldots, B'_m, A'_{m+1}, \ldots, A'_n$, such that $P(B_i) = P(A_i) + P(D'_i)$ for $m < i \leq n$. Let $b'_i$ be the event determined by $B'_i$. Then $tp(b_1, \ldots, b_n) = tp(b'_1, \ldots, b'_n)$ and $P(a'_i \Delta b'_i) = |P(a_i) - P(b_i)|$ for $i \leq n$. \hfill \Box
3.9. **Definition.** Let $M$ be a metric structure which is a $\kappa$-universal domain for its theory. Let $C \subset M$ be small and let $(I, <)$ be a countable infinite linear order. Let $I = (\bar{a}_i : i \in I)$ be a sequence of $n$-tuples. We say that $I$ is *indiscernible over $C$* if for any $m \in \mathbb{N}$ and elements $i_1 < i_2 < \cdots < i_{2m}$ of $I$, $tp(\bar{a}_{i_1}, \ldots , \bar{a}_{i_{2m}}/C) = tp(\bar{a}_{i_{m+1}}, \ldots , \bar{a}_{i_{2m}}/C)$.

Let $M \models T$ be a $\kappa$-universal domain, let $C \subset M$ be small and let $I = (a_i : i \in \omega)$ be an indiscernible sequence of $1$-tuples over $C$. Let $I = (\chi_{a_i} : i \in \omega)$ be the corresponding sequence of characteristic functions. Then the sequence $I'$ is spreadable \[3.13\] pp.168. Furthermore, for all $k_1 < \cdots < k_n \in \omega$ and $i_1, \ldots , i_n \in \{-1, 1\}$, we have $\mathbb{P}(a_{i_1}^{k_1} \land \cdots \land a_{i_n}^{k_n} | \langle C \rangle) = \mathbb{P}(a_{i_1}^{k_1} \land \cdots \land a_{i_n}^{k_n} | \langle C \rangle)$.

3.10. **Definition.** Let $M$ be a metric structure which is a $\kappa$-universal domain for its theory. Let $\bar{d}_0 \in M^n$ and let $\bar{a} \in M^n$. We say that $tp(\bar{a}/C \cup \bar{d}_0)$ does not divide over $C$ if for any indiscernible sequence $I = (\bar{d}_i : i < \omega)$ over $C$, there is $\bar{a}' \in M^n$ such that $tp(\bar{a}', \bar{d}_i/C) = tp(\bar{a}, \bar{d}_i/C)$ for all $i < \omega$. Let $D \subset M$ be small. We say that $tp(\bar{a}/C \cup D)$ does not divide over $C$ if for all finite $\bar{d} \subset D$, $tp(\bar{a}/C \cup \bar{d})$ does not divide over $C$. Whenever $tp(\bar{a}/C \cup D)$ does not divide over $C$ we say that $\bar{a}$ is independent from $D$ over $C$ and we write $\bar{a} \perp_C D$. We say that $tp(\bar{a}/C)$ is stationary if whenever $\bar{b} \in M^n$ and $D \supset C$ is small, $tp(\bar{a}/C) = tp(\bar{b}/C)$ and $\bar{a}, \bar{b} \perp_C D$ implies that $tp(\bar{a}/D) = tp(\bar{b}/D)$.

Let $\bar{a}_0 \in M^n$ and let $I = (\bar{a}_i : i \in \omega)$ be an indiscernible sequence over $C$. We say that $I$ is a Morley sequence in $tp(\bar{a}_0/C)$ if for every $n \in \mathbb{N}$, $tp(\bar{a}_{n+1}/C \cup \bar{a}_0 \cup \cdots \cup \bar{a}_n)$ does not divide over $C$.

We refer the reader to \[11, 12, 1, 3\] for the properties of non-dividing (non-forking) and stable structures.

In \[2\] Theorem 2.10 there is a natural characterization of non-dividing in probability structures:

3.11. **Proposition.** Let $M \models T$ be a $\kappa$-universal domain and let $C \subset M$ be small. Let $a_1, \ldots , a_n; b_1, \ldots , b_m \in M$. Then $tp(a_1, \ldots , a_n/\{b_1, \ldots , b_m\})$ does not divide over $C$ if and only if $\mathbb{P}(a_1^{i_1} \land \cdots \land a_n^{i_n} | \langle C \cup \{b_1, \ldots , b_m\} \rangle) = \mathbb{P}(a_1^{i_1} \land \cdots \land a_n^{i_n} | \langle C \rangle)$ for all $i_j \in \{-1, 1\}$, $j \leq n$. 
3.12. **Proposition.** Let $M \models T$ be a $\kappa$-universal domain. Then

1. $M$ is stable and types over sets are stationary.
2. $M$ is $\omega$-stable with respect to the $d$-metric.

**Proof.** (1) It is proved in [2, section 2.3]. (2) Let $C \subset M$ be countable. We may assume that $C$ is closed under finite intersections, unions and complements. Let $\text{Step}(C)$ be the set of step functions in $L^1(X, \langle C \rangle, m)$ with coefficients in $\mathbb{Q}$ and let $F = \{ \text{tp}(a/C) : P(a/\langle C \rangle) \in \text{Step}(C) \}$. Then $F$ is a countable set of types. By Lemma 3.8, $F$ is a dense subset of the space of 1-types over $C$ with respect to the $d$-metric. Then by [11], $M$ (equivalently $T$) is $\omega$-stable with respect to the $d$-metric. \(\square\)

3.13. **Remark.** In particular, $M$ is superstable with respect to the $d$-metric: for any $\bar{a} \in M^n$, $C \subset M$ small and $\epsilon > 0$, there is $C_0 \subset C$ finite and $\bar{a}' \in M^n$ such that $P(a_i \triangle a_i') < \epsilon$ for $i = 1, \ldots, n$ and $\bar{a}' \downarrow_{C_0} C$. The proof follows along the same lines as the proof of part (2) in the previous proposition.

Let $\epsilon > 0$ and let $B \subset C \subset M$ be small sets. We say that $\text{tp}(a/C)$ $\epsilon$-divides over $B$ if $d(\text{tp}(a/C), \text{tp}(a'/BC)) \geq \epsilon$, where $\text{tp}(a'/C)$ is the (unique) non-dividing extension of $\text{tp}(a/B)$. Let $SU_\epsilon(\text{tp}(a/B))$ be the foundation rank of $\epsilon$-dividing of the type $\text{tp}(a/B)$. Then for any $\epsilon > 0$, $a \in M$ and $B \subset M$ small, $SU_\epsilon(\text{tp}(a/B))$ is finite. This property translates the condition of being superstable of finite $SU$-rank into the current metric setting.

We now recall some definitions from [5]. These definitions apply to general metric structures (not just probability structures).

3.14. **Definition.** Let $M$ be a metric structure which is a $\kappa$-universal domain for its theory. Let $I, J \subset (M^n)^\omega$ be countable indiscernible sequences. We say $I$ and $J$ are **colinear** if the concatenation $IJ$ is an indiscernible sequence. We say that $I$ and $J$ are **parallel** if there is another infinite indiscernible sequence $K \subset (M^n)^\omega$ such that $I$ is colinear to $K$ and $J$ is colinear to $K$.

Let $M$ be a metric space structure which is a $\kappa$-universal domain for its theory and assume that $Th_A(M)$ is stable. We will show that parallelism is an equivalence relation. Assume that $I_1$ is parallel to $I_2$ and that $I_2$ is parallel to $I_3$. So there are
\[ K_1 = (a_i : i < \omega) \text{ and } K_2 = (b_i : i < \omega) \] indiscernible sequences such that all concatenations \( I_1 K_1, I_2 K_1, I_2 K_2, I_3 K_2 \) are indiscernible. Let \( tp(a/I_2 I_1 K_1 K_2) \) be the unique non-dividing extension of \( tp(a/I_2) \). Then by stability, \( tp(a/I_2 I_1 I_3 K_1 K_2) \) is also the non-dividing extension of \( tp(b/I_2) \). Let \( L \) be a Morley sequence in \( tp(a/I_1 I_2 I_3 K_1 K_2) \). Then \( I_1 L \) and \( I_3 L \) are colinear, so \( I_1 \) is parallel to \( I_3 \).

3.15. Definition. Let \( M \) be a metric structure which is a \( \kappa \)-universal domain for its theory. Let \( C \subset M \) be small, let \( \bar{a}_0 \in M^n \) and assume that \( tp(\bar{a}_0/A) \) is stationary. We say that \( tp(\bar{a}_0/A) \) has a built-in canonical base if there is a small set \( B \subset M \) such that for some (equivalently, any) Morley sequence \( I = (\bar{a}_i : i < \omega) \) over \( A \), the parallelism class of \( I \) is interdefinable with \( B \). That is, for every automorphism \( \Psi \) of \( M \), \( B \) is fixed (pointwise) by \( \Psi \) if and only if the parallelism class of \( I \) is fixed (setwise) by \( \Psi \). We call \( B \) a built-in canonical base for \( tp(\bar{a}_0/A) \). We say that \( M \) has built-in canonical bases if for all \( C \subset M \) small, \( n \in \mathbb{N} \) and \( \bar{a}_0 \in M^n \) such that \( tp(\bar{a}_0/A) \) is stationary, there is a built-in canonical base for \( tp(\bar{a}_0/A) \).

3.16. Remark. Let \( M \models T \) be a \( \kappa \)-universal domain, let \( C \subset M \) be small and let \( a_1, \ldots, a_n \in M \). Let \( \mathcal{D} \) be the (smallest) measure algebra making \( \mathbb{P}(a_i^1 \wedge \cdots \wedge a_i^n | \langle C \rangle) \) measurable for all \( i_j \in \{-1, 1\}, j \leq n \). Then \( \text{events}(\mathcal{D}) \) is a built-in canonical base for \( tp(a_1, \ldots, a_n/C) \). An exposition of this fact from a slightly different perspective can be found in [2] or [4].

4. Aperiodic algebras and quantifier elimination

We start by fixing the notation for the rest of the paper. Recall that we denote by \( L \) be the language of probability structures and by \( T \) the approximate theory of atomless probability structures. Write \( L_\tau \) for the language \( L \) expanded by a unary function with symbol \( \tau \) and let \( T_\tau \) be the theory \( T \cup \text{“}\tau \text{ is an automorphism”} \).

Let \( (X, \mathcal{B}, \mu) \) be an atomless Lebesgue space and let \( M \) be the probability structure of \( (X, \mathcal{B}, \mu) \). Let \( G \) be the group of automorphisms of this space, where we identify two maps if they agree on a set of measure one. Let \( \tau, \rho \in G \). Note that the map sending \( A \in \mathcal{B} \) to \( \rho^{-1}(A) \) is a measure preserving automorphism that induces an isomorphism between the structures \( (M, \tau), (M, \rho^{-1} \tau \rho) \). Let \( \tau_1, \tau_2 \in G \) be aperiodic. An application of proposition [23] with the values for \( \epsilon \) ranging over the sequence \( \{1/n : n \in \mathbb{N}^+\} \), shows that there are countable ultrapowers of \( (M, \tau_1) \)
and $(M, \tau_2)$ which are isomorphic. Thus any two aperiodic transformations in a Lebesgue space have the same approximate elementary theory. The aim of this section is to study the approximate theory of probability structures expanded by an aperiodic point automorphism. Since the elements of a probability structure are events, we need to define a notion of aperiodicity with respect to the measure algebra of events.

4.1. Definition. Let $(X, \mathcal{B}, m)$ be a probability space, let $\hat{\mathcal{B}}$ be the corresponding measure algebra of events and let $\tau$ be an automorphism of the measure algebra $\hat{\mathcal{B}}$. The map $\tau$ is called aperiodic if for all $n \in \mathbb{N}^+$ and $\epsilon > 0$ there is $b \in \mathcal{B}$ such that $m(b \cap \tau^n(b)) \leq \epsilon$ and $|m(b) - 1/2| \leq \epsilon$.

Note that the previous definition can be expressed by positive formulas in the language $L_\tau$. Denote by $T_A$ the theory $T_\tau \cup \text{"is aperiodic"}$. $T_A$ is $T_\tau$ plus a set of approximations, for every $n \in \mathbb{N}$, of the existence of a set $b$ of measure 1/2 which is disjoint from $T^n(b)$.

4.2. Lemma. Let $(X, \mathcal{B}, m)$ be a probability space and let $\hat{\mathcal{B}}$ be the corresponding measure algebra of events. Let $\tau$ be an automorphism of $(X, \mathcal{B}, m)$. Then $\tau$ is aperiodic iff the induced automorphism $\tau$ on the measured algebra $(\hat{\mathcal{B}}, m)$ is aperiodic.

Proof. If $\tau$ is an aperiodic automorphism of $(X, \mathcal{B}, m)$, then Rokhlin’s Lemma the induced automorphism $\tau$ on the measured algebra of events is also aperiodic.

Now assume that the induced automorphism on the measured algebra of events is aperiodic. By a way of contradiction, assume that $\tau$ is not aperiodic automorphism of $(X, \mathcal{B}, \mu)$. Then for some $n > 0$, $m(\{x : \tau^n(x) = x\}) \geq \epsilon$ for some $\epsilon > 0$. Since $\tau$ is an automorphism, $\{x : \tau^n(x) = x\} \in \mathcal{B}$. Let $a$ be the event corresponding to the set $\{x : \tau^n(x) = x\}$ and let $\delta > 0$ be such that $9\delta < \epsilon$. Since $\tau$ is aperiodic with respect to the measure algebra of events, there is an event $b$ such that $m(b \cap \tau^n(b)) \leq \delta$ and $|m(b) - 1/2| \leq \delta$. Then $m(a) = m(a \cap b) + m(a \cap \tau^n(b) \cap b^c) + m(a \cap b^c \cap \tau^n(b^c))$. Observe that $m(b^c \cap \tau^n(b^c)) = 1 - m(b) - m(\tau^n(b)) + m(b \cap \tau^n(b)) \leq 1 - 2(1/2 - \delta) + \delta = 3\delta$. Since $m(a) > 9\delta$, we must have that $m(a \cap b) > 3\delta$ or that $m(a \cap \tau^n(b)) > 3\delta$. Assume that $m(a \cap b) > 3\delta$. Then $a \cap b \subset a$ and thus $\tau^n(a \cap b) = a \cap b$. But $m(b \cap \tau^n(b)) \leq \delta$ which is a contradiction. A similar contradiction can be found when $m(a \cap \tau^n(b)) > 3\delta$. Thus $\tau$ has to be aperiodic.
The previous Lemma shows that the two notions of aperiodicity coincide for probability structures expanded by point automorphisms and that the approximate positive theory of atomless Lebesgue space expanded by an aperiodic automorphism is axiomatized by $T_A$.

4.3. Lemma. $T_A$ is complete.

Proof. Let $(M_1, \tau_1)$ and $(M_2, \tau_2)$ be two models of $T_A$. Then there are separable complete models $(M'_i, \tau'_i) \models T_A$ which are elementarily equivalent to $(M_i, \tau_i)$ for $i = 1, 2$ respectively. By separable categoricity of $T$, for $i = 1, 2$, $M'_i$ is isomorphic to the probability structure associated to a Lebesgue space and $\tau'_i$ can be assumed to be a point automorphism of the underlying probability space. By the previous lemma, $\tau'_1$ and $\tau'_2$ are aperiodic automorphisms as point set maps and thus by Proposition 2.9 we have $(M'_1, \tau'_1) \equiv_A (M'_2, \tau'_2)$. □

4.4. Definition. Let $M$ be a metric structure. We say that $\text{Th}_A(M)$ is model complete if for all $N_0 \subset N_1$ models of $\text{Th}_A(M)$, $N_0 \preceq_A N_1$.

Our next aim is to show that $T_A$ is model complete. The techniques used for the proof are similar to the ones used in proving the completeness of the theory $T_A$, but now we need to include parameters.

4.5. Lemma. Let $(M, S)$ be a separable complete model of $T_\sigma$, where $S$ is an $(n+1)$-shift. Let $f_1, \ldots, f_m \in M$. Let $(N, \sigma) \supset (M, S)$ be a separable complete extension which is a model of $T_\sigma$ and in which $\sigma$ is an $(n+1)$-shift. Then there is $\eta : M \to N$ measure preserving such that $\eta_S = \sigma \eta$ and $\eta(f_k) = f_k$ for $k \leq m$.

Proof. We may assume that $\{f_1, \ldots, f_m\}$ are the atoms of an algebra and that $S(\{f_1, \ldots, f_m\}) = \{f_1, \ldots, f_m\}$. Since $N$ is separable, we may assume that there is a Lebesgue space $(X, \mathcal{B}, \mu)$ such that $N = \text{events}(\mathcal{B})$ and that $\sigma$ is induced by a point map (also denoted by $\sigma$) such that $\sigma^{n+1} = \text{id}$. We may also suppose that $M = \text{events}(\mathcal{C})$ for some complete subalgebra $\mathcal{C}$ of $\mathcal{B}$ and that $S$ is induced by a point map $S$ such that $S^{n+1} = \text{id}$.

Let $A \in \mathcal{C}$ be such that $(A, \ldots, S^n(A))$ is a partition (up to measure zero) of $X$. Let $F_1, \ldots, F_m \in \mathcal{C}$ be disjoint sets such that $\text{event}(F_i) = f_i$ for $i \leq m$. Consider
η a measure preserving bijection between \((A,C \upharpoonright_A,m)\) and \((A,B \upharpoonright_A,m)\) such that 
\[ \eta(A \cap F_k) = A \cap F_k \text{ for } k \leq m. \]
Since \(F_1, \ldots, F_m\) are the atoms of an algebra, \(\eta\) is well defined for all \(x \in A\). Extend \(\eta\) by defining, for \(x \in A \cap F_k\), 
\[ \eta(S^i(x)) = \sigma^i(\eta(x)) \]
for \(i \leq n\). Then \(\eta(S^i(A \cap F_k)) = \sigma^i(A \cap F_k)\) for \(i \leq n, k \leq m\) and thus \(\eta(f_k) = f_k\) for \(k \leq m\). By the definition of \(\eta\) we also have \(\eta S = \sigma \eta\).

4.6. **Proposition.** \(T_A\) is model complete.

**Proof.** Let \((M,\tau) \models T_A\) be separable complete and let \((N,\tau_1) \models T_\tau\) be a separable complete complete extension of \((M,\tau)\). Note that \((N,\tau_1) \models T_A\). Let \(\bar{f} = (f_1, \ldots, f_m) \in M^m\).

By compactness we can find a separable complete elementary extension \((M_1,\tau)\) of \((M,\tau)\) such that for every \(n > 0\) there is \(a \in M_1\) such that \((a, \ldots, \tau^n(a))\) forms a partition of \(M_1\). We can amalgamate \((M_1,\tau)\) and \((N_1,\tau_1)\) over \((M,\tau)\) (see the discussion of relative independent joinings over a common factor in Chapter 6). Call this structure \((N_1,\tau_1)\). To prove the proposition it suffices to show that any formula \(\varphi(\bar{x},\bar{f})\) true in \((N_1,\tau_1)\) and any approximation \(\varphi'(\bar{x},\bar{f})\) of \(\varphi(\bar{x},\bar{f})\), there is a realization of \(\varphi'(\bar{x},\bar{f})\) in \((M_1,\tau)\).

Since \((N_1,\tau_1)\) is separable, we may assume that there is an atomless Lebesgue space \((X,B,\mu)\) such that \(N_1 = \text{events}(X,B,\mu)\) and \(\tau_1\) is an automorphism of \((X,B,\mu)\). Furthermore, we may assume that there is a complete \(\sigma\)-subalgebra \(C\) of \(B\) such that \(M_1 = \text{events}(C)\) and that \(\tau\) is an isomorphism of \((X,C,\mu)\).

For every \(n > 0\), we will construct a measure preserving isomorphism \(\eta_n: (X,C,\mu) \to (X,B,\mu)\) such that \(\eta_n(f_k) = f_k\) for \(k \leq m\) and \(\rho_{M_1}(\eta_n^{-1} \tau_1 \eta_n, \tau) \leq 2/(n+1)\).

We start by finding approximations of \(\tau\) and \(\tau_1\) by \((n+1)\)-shifts. Let \(A \in C\) with event \(a\). For \(x \in \cup_{i < n} \tau_1^i(A)\), let \(\sigma_1(x) = \tau_1(x)\) and for \(x \in \tau_1^n(A)\), let \(\sigma_1(x) = \tau_1^{-n}(x)\). Then \(\sigma_1^{n+1} = id, (a, \ldots, \sigma_1^n(a))\) is a partition of \(M_1\) and \(\rho(\tau_1, \sigma_1) \leq 1/(n+1)\). For \(x \in \cup_{i < n} \tau_i(A)\), let \(\sigma(x) = \tau(x)\) and for \(x \in \tau^n(A)\), let \(\sigma(x) = \tau^{-n}(x)\). Then \(\sigma^{n+1} = id\) and \(\rho(\tau, \sigma) \leq 1/(n+1)\).

By Lemma 4.3 there is a measure preserving isomorphism \(\eta_n: (X,C,\mu) \to (X,B,\mu)\) such that \(\eta_n \sigma = \sigma \eta_n\) and \(\eta_n(f_k) = f_k\) for \(k \leq m\). Note that \(\rho(\eta_n^{-1} \tau_1 \eta_n, \tau) \leq 2/(n+1)\).

Since \(n\) was arbitrary, we can find countable ultrapowers \((M_2,\tau)\) and \((N_2,\tau)\) of \((M_1,\tau)\) and \((N_1,\tau)\) respectively such that \((M_2,\tau,\bar{f}) \cong (N_2,\tau,\bar{f})\) and hence \((M_1,\tau,\bar{f}) \equiv_A (N_1,\tau,\bar{f})\).
4.7. **Definition.** We say that \((M, \tau) \models T\) is *existentially closed* if whenever \((N, \tau) \supset (M, \tau), \bar{m} \in M^n\) and \(\varphi(\bar{x}, \bar{m})\) is a quantifier free formula such that \((N, \tau) \models \exists \bar{x} \varphi(\bar{x}, \bar{m})\), then for any approximation \(\varphi'(\bar{x}, \bar{y})\) of \(\varphi(\bar{x}, \bar{y})\), \((M, \tau) \models \exists \bar{x} \varphi'(\bar{x}, \bar{m})\).

4.8. **Lemma.** The models of \(T_A\) are precisely the existentially closed models of \(T_\tau\).

**Proof.** Since any model of \(T_\tau\) can be extended to a model of \(T_A\), an existentially closed model of \(T_\tau\) is a model of \(T_A\). The other direction follows from the previous proposition. \(\square\)

4.9. **Remark.** The authors initially studied this subject to answer a question of Itay Ben-Yaacov about the axiomatizability of probability spaces expanded by generic automorphisms. Indeed, \(T_A\) is an axiomatization for this class. Another axiomatization comes from a suggestion of Anand Pillay. It is given by the following axioms indexed by \(\epsilon \in \mathbb{Q}^+\) and the arities of \(\bar{x}\) and \(\bar{a}\):

\[(ECN) \forall \bar{x} \forall \bar{y} \forall \bar{a} \left( d(tp(\tau(\bar{x})/\tau(\bar{a})), tp(\tau(\bar{y}))) \geq \epsilon \lor \\
\quad d(tp(\tau(\bar{x}'/\bar{a})), tp(\tau(\bar{y}))) \geq \epsilon \lor \\
\quad d(tp(\tau(\bar{y}'/\bar{a})), tp(\tau(\bar{y}))) \geq \epsilon \lor \\
\quad \exists \bar{c} \; d(tp(\bar{c}, \tau(\bar{c})/\bar{a}, \tau(\bar{a})), tp(\bar{x}, \bar{y} \tau(\bar{a}))) \leq 2\epsilon \right).\]

where \(d\) is the distance between types and \(tp(\bar{x}'/\bar{a}, \tau(\bar{d})), tp(\bar{y}'/\bar{a}, \tau(\bar{d}))\) are the unique non-dividing extensions of \(tp(\bar{x}/\bar{d})\) and \(tp(\bar{y}/\tau(\bar{d}))\) respectively.

This axiomatization says that any possible extension of an automorphism \(\tau\) is approximately realized already in \(\tau\). It is an exercise to the reader to show that this scheme axiomatizes the existentially closed models of \(T_\tau\).

The advantage of this approach is that it can also be used to show the existence of a model companion for other structures, for example Hilbert spaces expanded by an automorphism.

4.10. **Observation.** The theory ACFA in characteristic 0 can be seen as the limit, as the characteristic \(p\) goes to infinity, of the theories of algebraically closed fields expanded by adding the Frobenius automorphism. The theory of aperiodic automorphisms on atomless Lebesgue spaces is the limit, as \(n\) goes to infinity, of the theory of a probability space formed by \(n\) points of equal weight with a cycle of period \(n\).
Let \((M, \tau)\) be a \(\kappa\)-universal domain of \(T_A\). We denote the definable closure in the structure \(M\) by \(\text{dcl}\) and the definable closure in the structure \((M, \tau)\) by \(\text{dcl}_\tau\).

It is easy to give, as in first order theories (see [7]), a characterization for \(\text{dcl}_\tau\):

4.11. Lemma. Let \((M, \tau)\) be a \(\kappa\)-universal domain of \(T_A\) and let \(\bar{a} \subset M\). Then \(\text{dcl}_\tau(\bar{a}) = \text{dcl}(\{\tau^i(\bar{a}) : i \in \mathbb{Z}\})\).

In ergodic theory, *joinings* give different ways of amalgamating two probability structures with automorphisms into a common extension. In particular, the *relative independent joining over a common factor* (described in Section 6.1 of [8]) corresponds to the model-theoretic *free amalgamation*. Since \(T_A\) is model complete and has the amalgamation property, it should have quantifier elimination. That is the content of the next theorem:

4.12. Theorem. \(T_A\) has elimination of quantifiers.

Proof. Let \((M, \tau) \models T_A\) and let \(\bar{a}, \bar{b} \in M^n\) such that \(qftp_\tau(\bar{a}) = qftp_\tau(\bar{b})\). Then for every \(k < \omega\), \(tp(\tau^{-k}(\bar{a}), \ldots, \tau^k(\bar{a})) = tp(\tau^{-k}(\bar{b}), \ldots, \tau^k(\bar{b}))\). Let \(f\) be an \(L\)-isomorphism taking \((\tau^i(\bar{a}) : i \in \mathbb{Z})\) to \((\tau^i(\bar{b}) : i \in \mathbb{Z})\). The map \(f\) has a unique extension from \(\text{dcl}_\tau(\bar{a}) = \text{dcl}(\tau^i(\bar{a}) : i \in \mathbb{Z})\) to \(\text{dcl}_\tau(\bar{b})\), which is an \(L\)-isomorphism. By stationarity of types in atomless probability spaces, the sets \(\text{dcl}_\tau(\bar{a})\) and \(\text{dcl}_\tau(\bar{b})\) are amalgamation bases in \(T_\tau\). Since \((M, \tau)\) is existentially closed by a back and forth argument \(f\) is an \(L_\tau\)-isomorphism and \(tp_\tau(\bar{a}) = tp_\tau(\bar{b})\). \(\square\)

5. Independence and Stability

In this section we introduce an abstract notion of independence and show that it agrees with non-dividing. This idea follows the approach used in [7] to characterize non-dividing inside a first order stable structure expanded by a generic automorphism. We reserve the use of the word independence for independence of events in the sense of probability structures. Fix \((M, \tau) \models T_A\) a \(\kappa\)-universal domain.

5.1. Definition. Let \(\bar{a} \in M^n\) and let \(C \subset B \subset M\) be small. We say that \(\bar{a}\) is \(\tau\)-independent from \(B\) over \(C\) and write \(\bar{a} \independent C B\) if \(\text{dcl}_\tau(\bar{a})\) is independent from \(\text{dcl}_\tau(B)\) over \(\text{dcl}_\tau(C)\).

The next lemma shows that types in \(T_A\) are stationary with respect to \(\tau\)-independence. The main tool for this proof is quantifier elimination for \(T_A\).
5.2. Proposition. Let $\vec{a}, \vec{b} \in M^n$ and let $C \subseteq D \subseteq M$. Suppose that $tp_\tau(\vec{a}/C) = tp_\tau(\vec{b}/C)$ and that $\vec{a} \upharpoonright_C D$ and $\vec{b} \upharpoonright_C D$. Then $tp_\tau(\vec{a}/D) = tp_\tau(\vec{b}/D)$.

Proof. Let $\vec{a}, \vec{b}, C, D$ be as above. Then for every $k < \omega$,

$$tp(\tau^{-k}(\vec{a}), \ldots, \tau^k(\vec{a}))/\dcl_\tau(C)) = tp(\tau^{-k}(\vec{b}), \ldots, \tau^k(\vec{b}))/\dcl_\tau(C)).$$

By stationarity of types in probability spaces, we get

$$tp(\tau^{-k}(\vec{a}), \ldots, \tau^k(\vec{a}))/\dcl_\tau(D)) = tp(\tau^{-k}(\vec{b}), \ldots, \tau^k(\vec{b}))/\dcl_\tau(D)).$$

Since this equality holds for all $k < \omega$, by quantifier elimination of $T_A$, $tp_\tau(\vec{a}/D) = tp_\tau(\vec{b}/D)$. □

5.3. Corollary. The theory $T_A$ is stable and $\tau$-independence agrees with non-dividing.

Proof. By the properties of independence in $M$, it is clear that $\tau$-independence satisfies: symmetry, transitivity, extension, local character and finite character (see [20] for the definition of these properties). By the previous proposition it also satisfies stationarity. Since non-dividing can be characterized by these properties, $\tau$-independence agrees with non-dividing (see [20], [3]) and $T_A$ is stable. □

5.4. Remark. Since $T$ has built-in canonical bases (see Definition 3.16, Remark 3.16), $T_A$ will also have built-in canonical bases. For any $c_1, \ldots, c_m \in M$ and $A \subseteq M$, denote by $Cb(c_1, \ldots, c_m/A)$ a built-in canonical base for $tp(c_1, \ldots, c_m/A)$.

Let $\vec{a} = (a_1, a_2, \ldots, a_n) \in M^n$ and let $A \subseteq M$ be such that $A = dcl_\tau(A)$. Then a built-in canonical base for $tp_\tau(\vec{a}/A)$ is $\{Cb(c_1, \ldots, c_m/A) : c_1, \ldots, c_m \in dcl_\tau(a_1, \ldots, a_n), m < \omega\}$. This is reminiscent of what happens in ACFA, see [15], [9].

Roughly speaking, stability as developed in [11], [12], [13] corresponds to the study of universal domains that have a bound on the size of the space of types. This analysis is carried out through the density character of uniform structures. Independence is studied through the notion of non-forking and stability turns out to be equivalent to definability of types (see section 3 in [12]). The analysis of independence developed in [3] is based on the notion of non-dividing (defined by Shelah). A structure is stable when it has definability of types (see section 2 in [3]) and inside a stable structure, non-dividing can be characterized by its properties. Hence both points of view coincide and furthermore, non-forking in [11], [12], [13] corresponds to non-dividing from [3].
The previous corollary shows that $T_A$ is stable. Now we will explicitly count types. We will show that $T_A$ is $\omega$-stable with respect to the minimal uniform structure, which was introduced in [11]. We recall the definition:

5.5. Definition. Let $(M, \tau) \models T_A$ be a $\kappa$-universal domain and let $B \subset M$ be small. Given $L_\tau$ formulas $\phi_1(\bar{x}, \bar{y}) < \phi'_1(\bar{x}, \bar{y}), \ldots, \phi_k(\bar{x}, \bar{y}) < \phi'_k(\bar{x}, \bar{y})$, define $U[\phi_1, \phi'_1, \ldots, \phi_k, \phi'_k]$ as the set of all pairs of $\tau$-types $(p, q)$ with parameters in $B$ such that $\phi_i(\bar{x}, \bar{b}) \in p$ implies $\phi'_i(\bar{x}, \bar{b}) \in q$ and $\phi_i(\bar{x}, \bar{b}) \in q$ implies $\phi'_i(\bar{x}, \bar{b}) \in p$, for $i = 1, \ldots, k$ and $\bar{b} \subset B$.

The family $U[\phi_1, \phi'_1, \ldots, \phi_k, \phi'_k]$ forms a uniform structure on $S_n(A)$ and it is called the minimal uniform structure.

Before we show that $T_A$ is $\omega$-stable with respect to the minimal uniform structure, we need to introduce some new definitions.

5.6. Definition. Let $(M, \tau) \models T_A$, let $A \subset M$ and let $b \in M$. We say that $b$ is $m$-step independent over $A$ if $tp(b/A \cup \{\tau^j(b) : j \geq 1\})$ does not divide over $A \cup \{\tau^j(b) : 1 \leq j \leq m\}$.

5.7. Definition. Let $(M, \tau) \models T_A$, let $A \subset M$ be a subalgebra and let $b \in M$. We say that $b$ is $m$-step simple over $A$ if for any $c \in dcl(\tau^i(b) : 0 \leq i \leq m)$, $P(c\langle A\rangle)$ is a finite sum of rational multiples of characteristic functions of elements of $A$.

5.8. Theorem. $T_A$ is $\omega$-stable with respect to the minimal uniform structure.

Proof. Let $(M, \tau) \models T_A$ be a $\kappa$-universal domain and let $A_0 \subset M$ be countable. We need to prove that there is a countable dense subset of the space of $\tau$-types over $A_0$ with respect to the minimal uniform structure. Let $A_1 = \cup_{i \in \mathbb{Z}} \tau^i(A_0)$, so $A_1$ is countable. Finally let $A$ be the boolean algebra generated by $A_1$. Note that $A$ is countable. To show $\omega$-stability, it is enough to find a countable dense subset of the space of $\tau$-types over $A$ with respect to the minimal uniform structure. Let $\mathcal{F}$ be the set of all $\tau$-types $tp_{\tau}(b/A)$ where there is $m$ such that $b$ is $m$-step simple over $A$ and $m$-step independent over $A$. The set $\mathcal{F}$ is countable.

Claim: $\mathcal{F}$ is dense in the space of types over $A$ with respect to the minimal uniform structure.
Let \( \varphi(x, y) < \varphi'(x, y) \) be \( L_\sigma \)-formulas. Then, by quantifier elimination, there are \( m < \omega \) and \( \psi(x_1, \ldots, x_m, y_1, \ldots, y_m) < \psi'(x_1, \ldots, x_m, y_1, \ldots, y_m) \) quantifier free \( L \)-formulas such that
\[
(M, \tau) \models \varphi(x, y) \implies \psi(x, \ldots, \tau^m(x), y_1, \ldots, y_\ell) \quad \text{and} \quad
(M, \tau) \models \psi'(x, \ldots, \tau^m(x), y_1, \ldots, y_\ell) \implies \varphi'(x, y).
\]

By the perturbation lemma, there is \( \epsilon > 0 \) such that whenever
\[
(\bar{c}, \bar{d}) \models \psi(\bar{x}, \ldots, \tau^m(\bar{x}), \bar{y}, \ldots, \tau^m(\bar{y}))
\]
and
\[
d(tp(\bar{c}, \ldots, \tau^m(\bar{c}), \bar{d}, \ldots, \tau^m(\bar{d})), tp(\bar{c}', \ldots, \tau^m(\bar{c}'), \bar{d}', \ldots, \tau^m(\bar{d}'))) < \epsilon,
\]
then \( (\bar{c}', \bar{d}') \models \psi'(\bar{x}, \ldots, \tau^m(\bar{x}), \bar{y}, \ldots, \tau^m(\bar{y})). \)

Let \( b \in M \) and let \( b' \in \mathcal{F} \) be such that \( d(tp(b, \ldots, \tau^m(b)/A), tp(b', \ldots, \tau^m(b)/A)) < \epsilon \). Then for all \( \bar{a} \subset A \), if \( b \models \varphi(x, \bar{a}) \) then \( b' \models \varphi'(x, \bar{a}) \) and if \( b' \models \varphi(x, \bar{a}) \) then \( b \models \varphi'(x, \bar{a}). \)

\section{Ranks}

In this section we will follow \cite{21} and review the definition and the main properties of entropy. Let \( (X, \mathcal{B}, P) \) be a probability space.

\subsection{Definition}

let \( \mathcal{A} \) be a finite subalgebra of \( \mathcal{B} \) with atoms \( \{A_1, \ldots, A_k\} \). Let \( \mathcal{C} \) be a sub-\( \sigma \)-algebra of \( \mathcal{B} \). Then the \textit{entropy of} \( \mathcal{A} \) \textit{given} \( \mathcal{C} \) is
\[
H(\mathcal{A}/\mathcal{C}) = -\int \sum_{i \leq k} \mathbb{P}(A_i|\mathcal{C}) \ln(\mathbb{P}(A_i|\mathcal{C})) dP
\]

We write \( H(\mathcal{A}) \) for \( H(\mathcal{A}/\{\emptyset, X\}) \). If \( \mathcal{A} \) and \( \mathcal{C} \) are \( \sigma \)-algebras, we denote by \( \mathcal{A} \lor \mathcal{C} \) the \( \sigma \)-algebra generated by \( \mathcal{A} \) and \( \mathcal{C} \).

\subsection{Fact}

Let \( \mathcal{A}, \mathcal{C} \) be finite subalgebras of \( \mathcal{B} \) and let \( \mathcal{D} \) be a sub-\( \sigma \)-algebra of \( \mathcal{B} \). Then:
\begin{enumerate}
\item \( H(\mathcal{A} \lor \mathcal{C}/\mathcal{D}) = H(\mathcal{A}/\mathcal{D}) + H(\mathcal{C}/\mathcal{A} \lor \mathcal{D}). \) (Additivity)
\item \( \mathcal{A} \subset \mathcal{C} \implies H(\mathcal{A}/\mathcal{D}) \leq H(\mathcal{C}/\mathcal{D}). \)
\item \( H(\mathcal{A}/\mathcal{D}) \geq H(\mathcal{A}/\mathcal{C} \lor \mathcal{D}), \)
\item If \( \tau \) is a measure preserving automorphism, \( H(\tau^{-1}\mathcal{A}/\tau^{-1}\mathcal{D}) = H(\mathcal{A}/\mathcal{D}) \)
\item \( H(\mathcal{A}/\mathcal{C} \lor \mathcal{D}) = H(\mathcal{A}/\mathcal{D}) \) iff \( \mathcal{A} \) is independent from \( \mathcal{C} \) over \( \mathcal{D} \).
\end{enumerate}
6.3. Definition. Let \( \tau: X \to X \) be a measure preserving transformation of the probability space \((X, \mathcal{B}, m)\). If \( \mathcal{A} \) is a finite subset of \( \mathcal{B} \), then

\[
h(\tau, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \tau^{-i} \mathcal{A})
\]

is called the entropy of \( \tau \) with respect to \( \mathcal{A} \).

The value \( h(\tau) = \sup \{ h(\tau, \mathcal{A}) : \mathcal{A} \text{ is a finite subset of } \mathcal{B} \} \) is called the entropy of \( \tau \).

6.4. Fact. Let \( \mathcal{A} \) be a finite subalgebra of \( \mathcal{B} \) and \( \tau \) a measure preserving automorphism of \((X, \mathcal{B}, m)\). Then:

1. \( h(\tau, \mathcal{A}) \leq H(\mathcal{A}) \).
2. For \( n > 0 \), \( h(\tau, \mathcal{A}) = h(\tau, \bigvee_{i<n} \tau^{-i} \mathcal{A}) \).
3. \( h(\tau, \mathcal{A}) = 0 \) iff \( \mathcal{A} \subset \bigvee_{i=1}^{\infty} \tau^{-i} \mathcal{A} \).
4. \( h(\tau, \mathcal{A}) = \lim_{n \to \infty} H(\mathcal{A}/\bigvee_{i=1}^{n} \tau^{-i}(\mathcal{A})) \).

The proofs can be found in Section 4.5 of [21].

6.5. Definition. Let \((X, \mathcal{B}, m)\) be an atomless probability space and let \( \tau \) be an aperiodic measure preserving automorphism of this space. Let \( M \) be the probability structure associated to \((X, \mathcal{B}, m)\). Let \( \bar{a} = (a_1, \ldots, a_n) \in M^n \) and let \( D \subset M \). Let \( A_1, \ldots, A_n \in \mathcal{B} \) with events \( a_1, \ldots, a_n \) respectively and let \( \mathcal{A} \) be the algebra generated by \( A_1, \ldots, A_n \). Define \( H(\bar{a}/D) \), the entropy of \( \bar{a} \) with respect to \( D \), to be \( H(\mathcal{A}/\langle D \rangle) \). Define the entropy of \( \tau \) with respect to \( \bar{a} \) to be \( h(\tau, \mathcal{A}) \) and denote it by \( h(\tau, \bar{a}) \). Similarly, for \( C = \{a_1, \ldots, a_n\} \), let \( H(C/D) = H(\mathcal{A}/\langle D \rangle) \) and call it the entropy of \( C \) with respect to \( D \). Finally let the entropy of \( \tau \) with respect to \( C \) be \( h(\tau, \mathcal{A}) \) and denote it by \( h(\tau, C) \).

The properties listed in 6.2 and 6.4 still hold when measurable sets are replaced by events.
6.6. Definition. Let \((M, \tau) \models T_A\), let \(\bar{a} \in M^n\). We say that \(\bar{a}\) is transformally independent if \(\{\tau^i(\bar{a}) : i \in \mathbb{Z}\}\) is an independent sequence in \(M\). We say that \(\bar{a}\) is transformally definable if \(\bar{a} \in \text{dcl}(\tau^{-1}(\bar{a}) : i > 0)\).

6.7. Remark. Let \((M, \tau)\) be a model of \(T_A\), let \(\bar{a} \in M^n\). By the additivity of entropy, \(\bar{a}\) is transformally independent if and only if \(h(\tau, \bar{a}) = H(\bar{a})\). By property (3) in Fact 6.4, \(\bar{a}\) is transformally definable if and only if \(h(\tau, \bar{a}) = 0\). We call the algebra formed by the transformally definable elements the Pinsker algebra (compare with \([21]\)).

6.8. Remark. It is shown in Section 4.9 of \([21]\) that for every \(r \in \mathbb{R}^+\), there is a separable model \((M_r, \tau_r) \models T_A\) such that \(h(\tau_r) = r\). It follows that \(h(\tau) = \infty\) whenever \((M, \tau) \models T_A\) is \(\aleph_1\)-saturated.

6.9. Observation. We can use entropy to characterize the generic elements of the groups \((M, \triangle)\), where \(\triangle\) is the symmetric difference. When we work in \(M\) (just the probability structure), it is shown in \([2]\) that the generics of the group are the events of measure \(1/2\). Note that if \(a \in M\), then \(H(a) \leq \ln(2)\) and that the (maximal) value \(\ln(2)\) is attained only when \(P(a) = 1/2\). So the generics are the elements with maximal entropy.

Similarly, if we work in the structure \((M, \tau)\), it is easy to see that the generic elements of the group \((M, \tau, \triangle)\) are the transformally independent events of measure \(1/2\). Let \(a \in M\). Then \(h(a, \tau) \leq \ln(2)\) and equality holds iff \(a\) is an event of measure \(1/2\) which is transformally independent. So the generics of \((M, \tau, \triangle)\) are the elements \(a \in M\) such that the entropy of \(\tau\) with respect to \(a\) is maximal.

7. Orthogonality and omitting types

We start with a definition from \([21]\).

7.1. Definition. Let \((M, \tau) \models T_A\) and let \(A \subset M\) be such that \(\tau(A) = A\). We say that \(\tau\) has completely positive entropy on \(A\) if for all finite \(B \subset A\), \(h(\tau, B) > 0\).

Let \(M\) be a probability space structure and let \(\tau\) be a measure preserving aperiodic automorphism of the underlying probability space. Let \(A \subset M\) be countable. It is shown in \([10]\) that \(\tau\) has completely positive entropy on \(A\) if
and only if there is $A' \subset A$ such that $\tau^{-1}(A') \subset A'$, $\text{dcl}(\cup_{n \geq 0} \tau^n(A')) \supset A$ and $\cap_{n \geq 1} \text{dcl}(\tau^{-n}(A')) = \{\emptyset, X\}$.

It is well known (Theorem 4.37 in [21], Corollary 6.15 in [16]) that if $\tau$ has completely positive entropy on $A$, then $A$ is independent from the Pinsker $\sigma$-algebra of $\tau$. A similar result is known about ACFA([15]), namely, that types of transformally algebraic elements are orthogonal to types of transformally algebraic elements. The combination of these two ideas suggests that types of subsets of the Pinsker $\sigma$-algebra should be orthogonal to types of subsets where $\tau$ has completely positive entropy. This result will be the first aim of this section. We need the following notation from [16].

7.2. Notation. Let $(M, \tau) \models T_A$ and let $A \subset M$. Denote by $A^-$ the set $\text{dcl}(\cup_{i \geq 1} \tau^{-i}(A))$.

7.3. Theorem. Let $(M, \tau) \models T_A$ be a $\kappa$-universal domain. Let $A \subset M$ be a small set such that $\tau(A) = A$ and suppose that $\tau$ has completely positive entropy on $A$.

Let $B \subset M$ be a small subset of the Pinsker $\sigma$-algebra. Let $\bar{a}$ be an enumeration of $A$ and $\bar{b}$ an enumeration of $B$. Then $\text{tp}_\tau(\bar{a})$ is orthogonal to $\text{tp}_\tau(\bar{b})$.

Proof. We assume that $A$ and $B$ are countable. Then $A$ is $\tau$-interdefinable with a set $A'$ such that $\tau^{-1}(A') \subset A'$ and $\cap_{n \geq 1} \text{dcl}(\tau^{-n}(A')) = \{\emptyset, X\}$. Let $F \subset M$ be a small set which is $\tau$-independent from $A'$ over $\emptyset$ and $\tau$-independent from $B$ over $\emptyset$. Let $F_\tau = \text{dcl}_\tau(F)$ and $\bar{a}'$ an enumeration of $A'$. We need to show that $\bar{a}', \ldots, \tau^n(\bar{a}') \downarrow F_\tau, \bar{b}, \ldots, \tau^m(\bar{b})$ for any $n, m < \omega$. Replacing $A'$ by $A' \cup \cdots \cup \tau^n(A')$ and $B$ by $B \cup \cdots \cup \tau^m(B)$, it is enough to prove that $\bar{a}' \downarrow F_\tau, \bar{b}$.

By the finite character of non-dividing, we only need to show that for any finite subsets $C$ of $A'$ and $D$ of $B$, we have $C \downarrow_{F_\tau} D$.

By the additivity property of entropy, we can expand $H(C \cup \cdots \cup \tau^{-l}(C) \cup D \cup \cdots \cup \tau^{-l}(D)/F_\tau)$ in two different ways:

$$H(C \cup \cdots \cup \tau^{-l}(C)/F_\tau) + H(D \cup \cdots \cup \tau^{-l}(D)/F_\tau \cup C \cup \cdots \cup \tau^{-l}(C)) =$$

$$= H(D \cup \cdots \cup \tau^{-l}(D)/F_\tau) + H(C \cup \cdots \cup \tau^{-l}(C)/F_\tau \cup D \cup \cdots \cup \tau^{-l}(D))$$

Dividing by $l + 1$, taking limits as $l$ goes to infinity and using Fact [34] we get

$$H(C/F_\tau \cup C^-) + H(D/F_\tau \cup D^- \cup C^- \cup C) = H(D/F_\tau \cup D^-) + H(C/F_\tau \cup C^- \cup D \cup D^-)$$
Since $D$ is a subset of the Pinsker $\sigma$-algebra, this implies $H(C/F_\tau \cup C^-) = H(C/F_\tau \cup C^- \cup D \cup D^-)$. Thus $C \downarrow_{F_\tau \cup D^-} D \cup D^-$. The same argument holds if we replace $C$ by $\tau^{-i}(C)$, so we get $\tau^{-i}(C) \downarrow_{F_\tau \cup (\tau^{-i}(C))^-} D \cup D^-$ for all $i \geq 0$. By transitivity of independence, this implies $C \downarrow_{F_\tau} \tau^{-i}(C) \downarrow_{F_\tau \cup (\tau^{-i}(C))^-} D \cup D^-$ for any $i \geq 0$. Since $\cap_{i \geq 1} dcl(\tau^{-i}(A')) = \{\emptyset, X\}$, we get $C \downarrow_{F_\tau} D \cup D^-$ as we wanted. □

Since there are $2^{\aleph_0}$ many non-isomorphic Bernoulli shifts (see Section 4.9 in [21]), all of which induce aperiodic maps on separable complete probability structures, there are many non principal types over $\emptyset$. In the rest of this section we prove the stronger result that only algebraic types are principal. We start with some definitions.

7.4. Definition. Let $(M, \tau) \models T_A$ and let $B \subset M$ be such that $\tau(B) = B$. If $A \subset M$ is a finite set, then

$$h(\tau, A/B) = \lim_{n \to \infty} \frac{1}{n} H(\bigcup_{i=0}^{n-1} \tau^{-i}A/B)$$

is called the entropy of $\tau$ with respect to $A$ over $B$.

We also define $h(\tau/B) = \sup\{h(\tau, A/B) : A \text{ a finite subset of } M\}$. This is called the entropy of $\tau$ with respect to $B$.

7.5. Observation. Let $(M, \tau) \models T_A$ and let $B \subset M$ be such that $\tau(B) = B$.

1. Let $A$ be a finite subalgebra of $M$. Then $h(\tau, A/B) = H(A/B \cup A^-)$.

2. Let $C$ be a subalgebra of $M$ such that $dcl(C) = M$. Then $h(\tau/B) = \sup\{h(\tau, A/B) : A \text{ a finite subset of } C\}$.

Proof. When $B = \emptyset$, (1) is proved in Section 4.5 in [21] and (2) is proved in Section 4.6 in [21]. The proofs in [21] easily generalize to give what is stated here. □

7.6. Proposition. Let $(M, \tau) \models T_A$ be a $\kappa$-universal domain and let $B \subset M$ be countable such that $\tau(B) = B$. Then there is $(M_1, \tau_1) \subset (M, \tau)$ such that $(M_1, \tau_1) \models T_A$, $B \subset M_1$ and for any finite algebra $D \subset M_1$, $h(\tau, D/B) = 0$.

Proof. Let $(M_0, \tau_0)$ be the model of $T_A$ induced by an irrational rotation on the unit circle. Then $h(\tau, D) = 0$ for any finite subalgebra $D \subset M_0$ (see Section 4.7 in [21]). Since $(M, \tau)$ is $\kappa$-saturated, we may assume that $(M_0, \tau_0)$ is a substructure
of $(M, \tau)$ which is independent from $B$. Let $M_1 = \dcl(M_0, B)$ and let $\tau_1$ be the restriction of $\tau$ to $M_1$.

Let $B' = \dcl(B)$. Then $(B', \tau \restriction_{B'})$ is separable and can be seen as a probability structure with an automorphism. The structure $(M_1, \tau_1)$ is isomorphic to the product of the structures $(M_0, \tau_0)$ and $(B', \tau \restriction_{B'})$. Since $\tau_0$ is an aperiodic map, we get $(M_1, \tau_1) \models T_A$.

It remains to show that for any finite algebra $D \subset M_1$, $h(\tau, D/B) = 0$. By it is enough to prove that for any finite algebras $B_0 \subset B$ and $A_0 \subset M_0$, $h(\tau, B_0 \cup A_0)/B) = 0$. Now, since the entropy of $\tau_0$ is zero, the following inequalities hold: $h(\tau, B_0 \cup A_0/B) = H(B_0 \cup A_0/B \cup A_0^-) \leq H(A_0/A_0^-) = 0$. □

7.7. Proposition. Let $(M, \tau) \models T_A$ be a $\kappa$-universal domain and let $B \subset M$ be countable such that $\tau(B) = B$. Then there is $(M_1, \tau_1) \subset (M, \tau)$ such that $(M_1, \tau_1) \models T_A$, $B \subset M_1$ and for any finite algebra $D \subset M$, $h(\tau, D/B) = 0$ iff $D \subset \dcl(B)$.

Proof. Let $(M_0, \tau_0) \models T_A$ be the structure induced by a Bernoulli shift generated by a partition into two elements $\{a_1, a_2\}$ of probability 1/2 each. Then for any finite set $D \subset M_0$, $h(\tau, D) = 0$ iff $D \subset \{\emptyset, X\}$ (see Section 4.9 in [21] or [18]).

Since $(M, \tau)$ is $\kappa$-saturated, we may assume $(M_0, \tau_0)$ is a substructure of $(M, \tau)$ which is independent from $B$. Let $B' = \dcl(B)$. Then $(B', \tau \restriction_{B'})$ is separable. Let $M_1 = \dcl(M_0, B)$ and let $\tau_1$ be the restriction of $\tau$ to $M_1$. Then $(M_1, \tau_1)$ is isomorphic to the product of $(M_0, \tau_0)$ and $(B', \tau \restriction_{B'})$. Since $\tau_0$ is an aperiodic map, then $(M_1, \tau_1) \models T_A$.

Let $D \subset M_1$ be a finite subalgebra and assume that $h(\tau, D/B) = 0$. We want to show that $D \subset \dcl(B)$.

Let $A = \{a_1, a_2\}$. Let $A^- = \dcl(\cup_{i \geq 1} \tau^{-i}(A))$ and $D^- = \dcl(\cup_{i \geq 1} \tau^{-i}(D))$. Then by the additivity property of entropy, we can show that $H(A/A^- \cup B \cup D \cup D^-) + H(D/B \cup D^-) = H(D/D^- \cup B \cup A \cup A^-) + H(A/B \cup A^-)$. Since $h(\tau, D/B) = 0$, we get $H(A/A^- \cup B \cup D \cup D^-) = H(A/B \cup A^-)$. This proves $A \downarrow_{B \cup A^-} D$. Exchanging $A$ for $\tau^m(A)$, we get $\tau^m(A) \downarrow_{B \cup (\tau^m(A))} D$ for any $m \in \mathbb{Z}$. By transitivity of independence we obtain $\tau^m(A), \ldots, \tau^{-m}(A) \downarrow_{B \cup (\tau^{-m}(A))} D$ for any $m \in \mathbb{N}$. Since $A$ is transformally independent and $\tau$-independent from $B$, we obtain $\tau^m(A), \ldots, \tau^{-m}(A) \downarrow_B D$ for any $m \in \mathbb{N}$. By the finite character of
independence, this implies \( M_0 \downarrow_B D \). Since \( D \subseteq \mathrm{dcl}(M_0, B) \), we must also have \( D \subseteq \mathrm{dcl}(B) \).

7.8. **Proposition.** Let \((M, \tau) \models T_A\) be a \( \kappa \)-universal domain and let \( B \subseteq M \) be countable such that \( \tau(B) = B \). Let \( \bar{a} \in M^n \). Then \( \mathrm{tp}(\bar{a}/B) \) is principal iff \( \bar{a} \in \mathrm{dcl}(B)^n \).

**Proof.** Let \( B \) and \( \bar{a} \) be as above and assume that \( \bar{a} \notin \mathrm{dcl}(B)^n \). If \( h(\tau, \bar{a}/B) > 0 \), then we can omit \( \mathrm{tp}(\bar{a}/B) \) by Proposition 7.6. If \( h(\tau, \bar{a}/B) = 0 \), then we can omit \( \mathrm{tp}(\bar{a}/B) \) by Proposition 7.7. \( \square \)

8. **General automorphisms**

Let \((X, \mathcal{B}, m)\) be a probability space, let \( \tau \) be an automorphism of this space and let \( M \) be the probability structure associated to \((X, \mathcal{B}, m)\). The aim of this section is to discuss \( \mathrm{Th}_A(M, \tau) \).

Let \( Y \) be the union of the atoms from \((X, \mathcal{B}, m)\) and let \( \mathcal{B}_Y \) be the \( \sigma \)-algebra induced by \( \mathcal{B} \) on \( Y \). Then \( \tau \) is an automorphism of \( Y \) that permutes each set of atoms having the same (positive) measure. First we characterize \((\text{events}(Y', \mathcal{B}', m'), \tau') \equiv_A \text{events}(Y, \mathcal{B}, m), \tau) \). Then for every real number \( r \in (0, 1] \), the number of atoms in \( \mathcal{B}_Y \) with measure \( r \) (denoted by \( A_r \)) agrees with the number of atoms of \( \mathcal{B}' \) with measure \( r \) (denoted by \( A'_r \)) and the action of \( \tau \) on \( A_r \) is isomorphic to the action of \( \tau' \) on \( A'_r \). So for every \( r \in (0, 1] \), \( (A_r, \tau) \cong (A'_r, \tau') \). This implies \((\text{events}(Y', \mathcal{B}', m'), \tau') \cong (\text{events}(Y, \mathcal{B}, m), \tau) \). Thus, to characterize \((\text{events}(Y, \mathcal{B}, m), \tau) \), we only need to describe the permutation that \( \tau \) induces on the set of atoms of measure \( r \) for each \( r \in (0, 1] \). The structure \((\text{events}(Y, \mathcal{B}, m), \tau) \) behaves like a finite structure in first order theories.

Let \( Z \) be the atomless part of \( X \). We can decompose \( Z \) into a disjoint union \( \bigcup_{i \in \mathbb{N}} Z_i \), where \( Z_0 \) is the set of aperiodic elements of \( Z \) and for \( i > 0 \), \( Z_i = \{ x \in X : \tau^i(x) = x \} \setminus (Z_1 \cup \cdots \cup Z_{i-1}) \). The automorphism \( \tau \) acts on each of the sets \( Z_i \). Let \( \mathcal{B}_{Z_i} \) be the \( \sigma \)-algebra induced by \( \mathcal{B} \) on \( Z_i \). Let \( M_i \) be the probability structure associated to \((Z_i, \mathcal{B}_{Z_i}, m)\). To study the atomless part of \( \mathrm{Th}_A(M, \tau) \), it suffices to understand \( \mathrm{Th}_A(M_i, \tau) \) for \( i \in \mathbb{N} \). The behavior of the aperiodic part \( \mathrm{Th}_A(M_0, \tau) \), is described by \( T_A \) after rescaling \( m(Z_0) \) to be 1.
8.1. Lemma. Let \(([0,1],\mathcal{B},m)\) be the standard Lebesgue space, let \(n \in \mathbb{N}\) and let \(\tau\) be an automorphism such that \(\tau^{n+1} = \text{id}\) and \(m(\{ x : \tau^j(x) = x \}) = 0\) for all \(j < n + 1\). Then there is a set \(A \in \mathcal{B}\) such that \((A,\ldots,\tau^n(A))\) forms a partition of \([0,1]\) up to measure zero.

Proof. See Lemma 1 in [9, pp. 70].

We can now show that \(Th_A(M_i, \tau)\) is separably categorical for each \(i \geq 1\).

8.2. Proposition. Let \(([0,1],\mathcal{B},m)\) be the standard Lebesgue space, let \(i \in \mathbb{N}^+\) and let \(\tau, \eta\) be an automorphisms such that \(\tau^i = \text{id}\), \(\eta^i = \text{id}\) and \(m(\{ x : \tau^j(x) = x \}) = 0\), \(m(\{ x : \eta^j(x) = x \}) = 0\) for all \(j < i\). Let \(N\) be the probability structure associated to \(([0,1],\mathcal{B},m)\). Then \((N,\tau) \cong (N,\eta)\).

Proof. By the previous lemma, there are \(a \in N\) and \(b \in N\) such that \((a,\tau(a),\ldots,\tau^{i-1}(a))\) forms a partition of \(N\) and \((b,\eta(b),\ldots,\eta^{i-1}(a))\) forms a partition of \(N\). Let \(A \in \mathcal{B}\) with event \(a\) and let \(B \in \mathcal{B}\) with event \(b\). There is a measure preserving automorphism \(\gamma : A \rightarrow B\). We can extend \(\gamma\) by defining for \(x = \tau^i(y) \in \tau^i(A)\), \(\gamma(x) = \eta^i(\gamma(y))\). Then \(\gamma\tau = \eta\gamma\) on a set of measure one. This proves that \((N,\tau) \cong (N,\eta)\).

8.3. Lemma. Let \(([0,1],\mathcal{B},m)\) be the standard Lebesgue space, let \(n \in \mathbb{N}^+\) and let \(\tau\) be an automorphism such that \(\tau^{n+1} = \text{id}\) and \(m(\{ x : \tau^j(x) = x \}) = 0\) for all \(j < n + 1\). Let \(B_1,\ldots,B_m \in \mathcal{B}\). Then there is a set \(A \in \mathcal{B}\) of measure \(1/(n+1)\) such that \((A,\ldots,\tau^n(A))\) forms a partition of \([0,1]\) (up to measure zero) which is independent from \(\{B_1,\ldots,B_m\}\).

Proof. Replacing \(B_1,\ldots,B_m\) by the atoms in the algebra generated by \(\{\tau^i(B_j) : 0 \leq i \leq n, 1 \leq j \leq m\}\), we may assume that \(\tau\) acts on the set \(\{B_1,\ldots,B_m\}\). If \(\tau(B_i) = B_i\), by Lemma 8.3.1 we can find \(A_i \in \mathcal{B}\) such that \(A_i \subset B_i\), and \((A_i,\ldots,\eta^n(A_i))\) forms a partition of \(B_i\). If \(\tau\) acts transitively on \(B_{i_1},\ldots,B_{i_k}\), just repeat the argument from [9, pp 70] starting with \(B_{i_1}\) instead of \(E_1\). From the sets \(\{A_i : 1 \leq i \leq m\}\) we can construct \(A\).

Now we prove that \(Th_A(M_i, \tau)\) has quantifier elimination for each \(i \geq 1\).
8.4. **Lemma.** Let $([0,1], B, m)$ be the standard atomless Lebesgue space and let $M$ be the probability structure associated to it. Let $\eta_1, \eta_2$ be cycles of period $n + 1$ on $M$. Let $b, d \in M^m$ be such that $qftp(b, \ldots, \eta_1^n(b)) = qftp(d, \ldots, \eta_2^n(d))$. Then there is an automorphism $\gamma$ of $M$ such that $\gamma \eta_1 = \eta_2 \gamma$ and $\gamma(b) = (d)$. 

**Proof.** By the previous lemma we can find $a \in M$ such that $(a, \eta_1(a), \ldots, \eta_1^n(a))$ is a partition of $M$ which is independent from $\{b, \ldots, \eta_2^n(b)\}$. Similarly there is $c \in M$ such that $(c, \eta_2(c), \ldots, \eta_2^n(c))$ is a partition of $M$ which is independent from $\{d, \ldots, \eta_2^n(d)\}$.

In particular we obtain that $qftp(b, \ldots, \eta_1^n(b), a, \ldots, \eta_1^n(a)) = qftp(d, \ldots, \eta_2^n(d), c, \ldots, \eta_2^n(c))$. Let $b' = (b_1', \ldots, b_k')$ be the atoms of the algebra generated by $\{b, \ldots, \eta_2^n(b)\}$ and let $d' = (d_1', \ldots, d_k')$ be the atoms of the algebra generated by $\{d, \ldots, \eta_2^n(d)\}$. We may choose enumerations such that $qftp(b', a, \ldots, \eta_1^n(a)) = qftp(d', c, \ldots, \eta_1^n(c))$. It suffices to prove the lemma for $b'$ and $d'$.

Let $A, B_1', \ldots, B_k', C, D_1', \ldots, D_k'$ be sets in $B$ giving rise to the collection of events $a, b_1', \ldots, b_k', c_1, \ldots, d_k'$. Then $m(A \cap B_i') = m(C \cap D_i')$ for $i \leq k$. Let $\gamma: A \rightarrow C$ be an isomorphism sending $A \cap B_i$ to $C \cap D_i$ (up to measure zero) for $i \leq k$. Let $x \in \eta_1^n(A)$. Then there is $y \in A$ such that $\eta_1^n(y) = x$. Extend $\gamma$ by defining $\gamma(y) = \eta_2^n(\gamma(y))$. Note that $\eta_1(b') = d'$ for $i \leq k$ and $\gamma \eta_1 = \eta_2 \gamma$.  

By Lemma 8.4 $Th_\mathcal{A}(M_i, \tau)$ has quantifier elimination and thus it is model complete. Note that $Th_\mathcal{A}(M, \tau)$ need not have quantifier elimination. We first need to split $(M, \tau)$ into its atomic part and the $(M_i, \tau)$ in order to get quantifier elimination.

Since $T$ is stable, so are the theories $Th_\mathcal{A}(M_i, \tau)$ for $i > 0$ and $Th_\mathcal{A}(M, \tau)$.

Let $(M, \tau) \preceq_\mathcal{A} (N, \tau)$ and let $\{z_i : i \in \mathbb{N}\}$ be the events corresponding to the measurable sets $\{Z_i : i \in \mathbb{N}\}$ described above. Let $i \in \mathbb{N}^+$ and assume that $m(z_i) > 0$. Let $a \in N$ be such that $a \subseteq z_i$ and $m(a) > 0$. Then there is $b \subseteq a$ such that $m(b) > 0$, $\tau^i(b) = b$ and $\tau^j(b) \neq b$ for all $j < i$. This shows the decomposition $\{z_i : i \in \mathbb{N}\}$ is preserved in elementary superstructures.

Let $(X', B', m')$ be a probability space, let $\tau'$ be a measure preserving automorphism of this space and let $M'$ be the probability structure associated to the probability space. Follow the notation above and denote by $Y'$ the union of the atoms from $M'$ and $z_i'$ the event associated to the set formed by the elements...
with period $i$. Then $(M', \tau') \equiv_A (M, \tau)$ iff $(\langle Y', E_Y', m', \tau' \rangle) \equiv (\langle Y, E_Y, m, \tau \rangle)$ and $m(z'_i) = m(z_i)$ for $i \in \mathbb{N}$. We conclude that the theory $Th_A(M, \tau)$ can be described in terms of $Th_A(M_Y, \tau)$ and the sequence $(m(M_i) : i < \omega)$.

In conclusion, we give some open questions related to the subject of this paper.

As background to the first questions, note that for probability structures (i.e., without automorphism), Lemma 3.8 gives a very nice, explicit formula for the $d$ distance between types over a set of parameters $C$. This is a basis for a full analysis of stability of these structures. For example, see Remark 3.13.

8.5. Question. Can we characterize the $d$-metric in type spaces of $T_A$? That is, can we find a description of $d$ in the spirit of Lemma 3.8?

8.6. Question. What is the density character of the space of $\tau$-types over a given set of parameters $C$, with respect to the $d$-metric? (See [11, 12] for a proof that $T_A$ is stable with respect to the $d$-metric, since it is stable with respect to the minimal uniform structure on types. This gives a partial answer to this question.)

8.7. Question. Is $T_A$ superstable with respect to the $d$-metric? (See Remark 3.13.)

In [4] the model theory of the Banach lattices $L_p(\mu)$ is studied, for each $p \in [1, \infty)$. Each of these theories interprets the theory $T$ of probability structures (for any positive $L_p$-function $f$ of norm 1, consider the set of components of $f$ equipped with the $p$th power of the norm as probability measure). So, in a certain sense the results in [4] extend those in [2] and in section 3 of this paper. Furthermore, automorphisms of $L_p(\mu)$-spaces are well understood from the functional analysis point of view.

8.8. Question. Is there a model companion for the positive theory of the Banach lattices $L_p(\mu)$ expanded by an automorphism?

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Probability Spaces with an Automorphism

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