SINGULARITIES OF THE THETA DIVISOR AT POINTS OF ORDER TWO

SAMUEL GRUSHEVSKY AND RICCARDO SALVATI MANNI

ABSTRACT. In this note we study the geometry of principally polarized abelian varieties (ppavs) with a vanishing theta-null (i.e. with a singular point of order two and even multiplicity lying on the theta divisor) — denote \( \theta_{\text{null}} \) the locus of such ppavs. We describe the locus \( \theta_{g-1}^{\text{null}} \subset \theta_{\text{null}} \) where this singularity is not an ordinary double point. By using theta function methods we first show \( \theta_{g-1}^{\text{null}} \subseteq \theta_{\text{null}} \) (this was shown in [4], see below for a discussion). We then show that \( \theta_{g-1}^{\text{null}} \) is contained in the intersection \( \theta_{\text{null}} \cap N'_0 \) of the two irreducible components of the Andreotti-Mayer \( N_0 = \theta_{\text{null}} + 2N'_0 \), and describe by using the geometry of the universal scheme of singularities of the theta divisor which components of this intersection are in \( \theta_{g-1}^{\text{null}} \).

Some of the intermediate results obtained along the way of our proof were concurrently obtained independently by C. Ciliberto and G. van der Geer in [3] and by R. de Jong in [5], version 2.

INTRODUCTION

The theta divisor \( \Theta \) of a generic principally polarized abelian variety (ppav) is smooth. In [1] and [2] it was shown that ppavs \((A, \Theta)\) with a singular theta divisor form a divisor \( N_0 \) in the moduli space of ppavs \( A_g \).

In [11] and [4] it was shown that \( N_0 \) has two irreducible components \( \theta_{\text{null}} \) and \( N'_0 \), where \( \theta_{\text{null}} \) is the locus of those ppavs for which the theta divisor has a singularity at a point of order two, and \( N'_0 \) is the closure of the locus of those ppavs for which the theta divisor has a singularity at a point not of order two.

Moreover it has been proven that for a generic ppav \((A, \Theta) \in \theta_{\text{null}} \) the theta divisor \( \Theta \) has a unique singular point, which is a double point. Similarly the generic element of \( N'_0 \) is a principally polarized abelian variety for which the theta divisor \( \Theta \) has two distinct singular points \( x \) and \(-x\), which are double points — this is due to the symmetry of
the theta divisor. So we can write

\[ N_0 = \theta_{\text{null}} + 2N'_0. \]

Moreover, in [4] the author claims that in the case of \( \theta_{\text{null}} \), the singular point is generically an ordinary double point (i.e. the tangent cone to the theta divisor at such a point has maximal rank, i.e. rank \( g \)). He refers to [12] for a proof. It seems that the reference is not really appropriate, since [12] treats the restriction of \( \theta_{\text{null}} \) to \( \mathcal{M}_g \), the moduli space of curves of genus \( g \). But in this case, for \( g \geq 4 \) we know that the double points cannot be ordinary.

Upon reading a preliminary version of this note, O. Debarre has explained to us that this can be fixed by a little more work, using the results in his work — see remark 6 for that proof. The different method we use, however, yields a further insight into the solution of some interesting problems about the double points on the theta divisor, and thus is hopefully of independent interest.

Our interest in the divisor \( \theta_{\text{null}} \) comes from our recent paper [8] where we gave, in genus 4, a characterization of the intersection

\[ \theta_{\text{null}} \cap N'_0 \]

in terms of the rank of the tangent cone of the singular point of the theta divisor. Essentially we proved that the above locus is characterized by the fact that the singular point of order two is a double point, but not an ordinary double point. The existence of ppavs of dimension 4 for which the theta divisor has an ordinary double point was proved in [2].

We also study the intersection (2) and its relation with the locus \( \theta_{\text{null}}^{g-1} \) — the sublocus of \( \theta_{\text{null}} \) parametrizing ppavs \((A, \Theta)\) with a singular point of order two which is a double point, but not an ordinary double point.

Incidentally with our proof, we solve a question raised in [5], about the vanishing locus of a modular form defined on the universal theta divisor, which is constructed in that work. We were recently informed by R. de Jong that he has also obtained independently a proof of proposition 5 in the new version of his work.

At the last stages of editing the draft of this text we were informed by C. Ciliberto and G. van der Geer of their preprint [3] where along the way of their discussion of the dimension of Andreotti-Mayer loci they also discuss related questions about the singularities of theta divisors for points on \( \theta_{\text{null}} \). Their main interest is in the higher Andreotti-Mayer loci, and our results on \( \theta_{\text{null}}^{g-1} \) do not have a parallel in their work.
1. Notations and definitions

In this section we recall notations, definitions, as well as some results from [8]. We denote $\mathcal{H}_g$ the Siegel upper half-space, i.e. the set of symmetric complex $g \times g$ matrices $\tau$ with positive definite imaginary part. Each such $\tau$ defines a complex principally polarized abelian variety (ppav for short) $\mathbb{C}^g/\tau \mathbb{Z}^g + \mathbb{Z}^g$. If $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(g, \mathbb{Z})$ is a symplectic matrix in a $g \times g$ block form, then its action on $\tau \in \mathcal{H}_g$ is defined by $\sigma \cdot \tau := (a\tau + b)(c\tau + d)^{-1}$, and the moduli space of ppavs is the quotient $A_g = \mathcal{H}_g/\text{Sp}(g, \mathbb{Z})$. A period matrix $\tau$ is called decomposable if there exists $\sigma \in \text{Sp}(g, \mathbb{Z})$ such that $\sigma \cdot \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$, $\tau_i \in \mathcal{H}_{g_i}$, $g_1 + g_2 = g$, $g_i > 0$; otherwise we say that $\tau$ is indecomposable.

For $\varepsilon, \delta \in (\mathbb{Z}/2\mathbb{Z})^g$, thought of as vectors of zeros and ones, $\tau \in \mathcal{H}_g$ and $z \in \mathbb{C}^g$, the theta function with characteristic $[\varepsilon, \delta]$ is

$$\theta_{[\varepsilon, \delta]}(\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp \pi i \left[ t(m + \frac{\varepsilon}{2})\tau(m + \frac{\varepsilon}{2}) + 2 \cdot t(m + \frac{\varepsilon}{2})(z + \frac{\delta}{2}) \right].$$

Sometimes we shall write $\theta(\tau, z)$ for the theta function with characteristic $[0, 0]$.

A characteristic $[\varepsilon, \delta]$ is called even or odd depending on whether $\theta_{[\varepsilon, \delta]}(\tau, z)$ is even or odd as a function of $z$, which corresponds to the scalar product $\varepsilon \cdot \delta \in \mathbb{Z}/2\mathbb{Z}$ being zero or one, respectively. A theta constant is the evaluation at $z = 0$ of a theta function. All odd theta constants of course vanish identically in $\tau$.

Observe that

$$\theta_{[0, 0]}(\tau, z + \frac{\varepsilon}{2} + \frac{\delta}{2}) = \exp \pi i \left( -\frac{t\varepsilon}{2} \frac{\varepsilon}{2} - \frac{t\varepsilon}{2}(z + \frac{\delta}{2}) \right) \theta_{[\varepsilon, \delta]}(\tau, z),$$

i.e. theta functions with characteristics are, up to some non-zero factor, the Riemann theta function (the one with characteristic $[0, 0]$) shifted by points of order two.

A map $f : \mathcal{H}_g \to \mathbb{C}$ is called a (scalar) modular form of weight $k$ with respect to a finite index subgroup $\Gamma \subset \text{Sp}(g, \mathbb{Z})$ if

$$f(\sigma \cdot \tau) = \det(c\tau + d)^k f(\tau) \quad \forall \tau \in \mathcal{H}_g, \forall \sigma \in \Gamma,$$

and if additionally $f$ is holomorphic at all cusps of $\mathcal{H}_g/\Gamma$. 
We define
\[ \Gamma_g(n) := \left\{ \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(g, \mathbb{Z}) \mid \sigma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\} \]
\[ \Gamma_g(n, 2n) := \left\{ \sigma \in \Gamma_g(n) \mid \text{diag}(a'b) \equiv \text{diag}(c'd) \equiv 0 \mod 2n \right\}. \]
These are finite index normal subgroups of Sp\((g, \mathbb{Z})\).

Under the action of \(\sigma \in \text{Sp}(g, \mathbb{Z})\) the theta functions transform as follows:
\[
\theta \left[ \sigma \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\sigma \cdot \tau, c\tau + d)^{-1}z
= \phi(\varepsilon, \delta, \tau, z) \det(c\tau + d)^{1/2} \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\tau, z),
\]
where
\[ \sigma \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} := \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} \text{diag}(c'd) \\ \text{diag}(a'b) \end{pmatrix}, \]
considered in \((\mathbb{Z}/2\mathbb{Z})^g\), and \(\phi(\varepsilon, \delta, \tau, z)\) is some complicated explicit function. For more details, we refer to [9] and [6].

Thus theta constants with characteristics are modular forms of weight \(1/2\) with respect to \(\Gamma_g(4, 8)\), i.e. we have
\[
\theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\sigma \cdot \tau, 0) = \det(c\tau + d)^{1/2} \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\tau, 0) \quad \forall \sigma \in \Gamma_g(4, 8).
\]

The theta constants are known to define an embedding of the level moduli space \(A_g(4, 8) := H_g/\Gamma_g(4, 8)\) (see [9], chapter V):
\[
\text{Th} : A_g(4, 8) \to \mathbb{P}^{2g-1(2g+1)-1}
\]
\[
\tau \mapsto \left\{ \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\tau) \right\}^{[\varepsilon, \delta] \text{ even}},
\]
which extends to the Satake compactification \(\overline{A_g(4, 8)}\).

We call the theta-null divisor \( \theta_{\text{null}} \subset A_g \) the zero locus of the product of all even theta constants. We define a stratification of \( \theta_{\text{null}} \) as follows. For \( h = 0, \ldots, g \) we let
\[
\theta_{\text{null}}^h = \left\{ \tau \in H_g : \exists [\varepsilon, \delta] \text{ even}, \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\tau) = 0; \quad \text{rk} \left( \frac{\partial^2 \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\tau, z)}{\partial z_j \partial z_k} \right)_{z=0} \leq h \right\},
\]
i.e. the locus of points on \( \theta_{\text{null}} \) where the rank of the tangent cone to the theta divisor at the corresponding point \( \frac{\partial^2 \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right]}{\partial z_j \partial z_k} \) of order two is at most \( h \).
The partial derivatives of the theta function are not modular forms. However, since theta functions satisfy the heat equation

\[ \frac{\partial^2 \theta[\varepsilon \delta]}{\partial z_j \partial z_k}(\tau, z) = \pi i (1 + \delta_{j,k}) \frac{\partial \theta[\varepsilon \delta]}{\partial \tau_{jk}}(\tau, z), \]

(where \( \delta_{j,k} \) is Kronecker's symbol) and the partial \( \tau_{jk} \) derivative of a section \( \theta[\varepsilon \delta](\tau, 0) \) of a line bundle on \( \mathcal{A}_g(4,8) \), when restricted to the zero locus of this theta constant, is a section of the same bundle, on the locus \( \{ \theta[\varepsilon \delta](\tau, 0) = 0 \} \) the second derivative

\[ \frac{\partial^2 \theta[\varepsilon \delta]}{\partial z_j \partial z_k}(\tau, z) \bigg|_{z=0} \]

is a modular form for \( \Gamma_g(4,8) \).

Using the heat equation, the Hessian of the theta function with respect to \( z \) can be rewritten using the first derivatives with respect to \( \tau_{jk} \). Hence if a point \( x = \tau \frac{\varepsilon}{2} + \frac{\delta}{2} \) of order two is a singular point on the theta divisor, which is simply to say \( \theta[\varepsilon \delta](\tau, x) = 0 = \theta[\varepsilon \delta](\tau, 0) \) (the first derivatives at zero of an even function are all zero), the rank of the quadric defining the tangent cone at \( x \) is the rank of the matrix obtained by applying the \( g \times g \)-matrix-valued differential operator

\[ D := \begin{pmatrix} \frac{\partial}{\partial \tau_{11}} & \frac{1}{2} \frac{\partial}{\partial \tau_{12}} & \cdots & \frac{1}{2} \frac{\partial}{\partial \tau_{1g}} \\ \frac{1}{2} \frac{\partial}{\partial \tau_{12}} & \frac{\partial}{\partial \tau_{22}} & \cdots & \frac{1}{2} \frac{\partial}{\partial \tau_{2g}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2} \frac{\partial}{\partial \tau_{1g}} & \cdots & \cdots & \frac{\partial}{\partial \tau_{gg}} \end{pmatrix} \]

to \( \theta[\varepsilon \delta](\tau, 0) \).

In \( \mathcal{A}_g(4,8) \), the locus \( \theta_{null}^h \) is given by the conditions

\[ \exists [\varepsilon, \delta] \text{ even}; 0 = \theta[\varepsilon \delta](\tau); \text{ rk} D \theta[\varepsilon \delta](\tau) \leq h. \]

The divisor \( \theta_{null} \subset \mathcal{A}_g(4,8) \) is reducible. Its irreducible components are the divisors of individual theta constants with characteristics (cf. [7] page 88 for \( g \geq 3 \); it is easily verified also for \( g = 1, 2 \)). These components are all conjugate under the action of \( \text{Sp}(g, \mathbb{Z})/\Gamma(4,8) \), and
it follows that $\theta_{\text{null}}$ and $\theta_{\text{null}}^h$ are well-defined on $A_g$ and not only on $A_g(4,8)$.

2. Partial toroidal compactification

In this section we recall from [11] the partial toroidal compactification of the moduli space of ppavs, and the description of the intersection of a subvariety of $A_g$ with the boundary $\partial A_g$. We will be mainly interested in the intersection of the divisor $\theta_{\text{null}}$ with the boundary.

The partial compactification that we consider is

$$A_g^1 := A_g \cup \partial A_g.$$  

This is the coarse moduli space of ppavs $(A, \Theta)$ of dimension $g$ and their rank 1 degenerations, obtained as the blowup of $A_g \sqcup A_{g-1}$ along $A_{g-1}$.

We denote $p : A_g^1 \to A_g \sqcup A_{g-1}$ the projection map. An element of $\partial A_g$ is a pair $(G, D)$, where $G$ is a complete $g$-dimensional variety that is a limit of $g$-dimensional abelian varieties, and $D$ is an ample divisor that is the limit of the respective theta divisors. Obviously an element of $A_{g-1}$ is a pair $(B, \Xi)$ where $B$ is a ppav of dimension $g-1$ and $\Xi$ is its theta divisor.

The restriction of the map

$$p|_{\partial A_g} : \partial A_g \to A_{g-1}$$

has $B/\text{Aut}(B, \Xi)$ as fiber over $(B, \Xi)$. This means that the fiber over a general $(B, \Xi) \in A_{g-1}$ is the Kummer variety $B/\pm 1$.

We know from [11]

Theorem 1.

$$\theta_{\text{null}} \cap \partial A_g = \left( \bigcup_{(G,D)} 2_B(\Xi) \right) \cup p^{-1}(\theta_{\text{null}}, g-1).$$

where we denoted $2_B(\Xi) := \{2x|x \in \Xi\}$.

3. Double points on general elements of $\theta_{\text{null}}$ are ordinary

As we stated in the introduction we shall prove that a generic ppav in $\theta_{\text{null}}$ has an ordinary double point as the only singular point of the theta divisor. We need a technical lemma

Lemma 2. Let $F(x_1, \ldots, x_n) = 0$ be the equation of a hypersurface $X \subset \mathbb{C}^n$. Let $F_1, \ldots, F_n$ be the partial derivatives of $F$, and let $G : \mathbb{C}^n \to \mathbb{C}^n$
be the induced map
\[ G(x_1, \ldots, x_n) := (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n)). \]
Then \(dF\) ramifies at \(x \in X\) if and only if the matrix
\[
\begin{pmatrix}
\frac{\partial^2 F(x)}{\partial X_1 \partial X_1} & \frac{\partial^2 F(x)}{\partial X_1 \partial X_2} & \cdots & \frac{\partial^2 F(x)}{\partial X_1 \partial X_n} & \frac{\partial F(x)}{\partial X_1} \\
\frac{\partial^2 F(x)}{\partial X_2 \partial X_1} & \frac{\partial^2 F(x)}{\partial X_2 \partial X_2} & \cdots & \frac{\partial^2 F(x)}{\partial X_2 \partial X_n} & \frac{\partial F(x)}{\partial X_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial^2 F(x)}{\partial X_n \partial X_1} & \frac{\partial^2 F(x)}{\partial X_n \partial X_2} & \cdots & \frac{\partial^2 F(x)}{\partial X_n \partial X_n} & \frac{\partial F(x)}{\partial X_n}
\end{pmatrix}
\]
does not have maximal rank.

Proof. The map \(G\) ramifies if and only if there exists a vector \(v \in T_x(X)\) of the tangent space mapping to \(\lambda dF(x)\), where \(dF\) is the column vectors whose entries are the \(F_i\). Denoting by \(H\) the hessian matrix of \(F\), this becomes
\[ H(x)v = \lambda dF(x). \]
Obviously, since \(v \in T_x(X)\), we have the scalar product \(v \cdot dF(x) = 0\). Thus the two assertions are equivalent to
\[
\begin{pmatrix}
H(x) & dF(x) \\
{}^t dF(x) & 0
\end{pmatrix}
\begin{pmatrix}
v \\
\lambda
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
But this is true if and only if the matrix (7) does not have maximal rank. \(\square\)

Now we are able to prove the following

**Theorem 3.**
\[ \theta_{null}^{g-1} \subseteq \theta_{null} \]

Proof. Let us consider the intersection of (the closure in \(\overline{A}_g\)) of these loci with the boundary \(\partial A_g\). We shall restrict ourselves to considering in (6) the component
\[
\left( \bigcup_{(\mathcal{G}, \mathcal{D})} 2_B(\Xi) \right)
\]
of \(\theta_{null} \cap \partial A_g\). Let \(\theta(\tau', z)\) be the standard theta function of genus \(g - 1\). By using the heat equation we can easily check that \(\theta_{null}^{h} \cap \partial A_g\) restricted to the above component is described by the following analytic conditions
\[
\theta(\tau', z/2) = 0
\]
The fact that the determinant of (9) does not vanish identically follows immediately from the previous lemma, since it is a well known fact that the Gauss map of the theta divisor of an abelian variety does not ramify everywhere. But this is what we need to finish the proof of the theorem. Indeed, if the determinant of the above matrix is non-zero, it means that its rank is equal to $g$, and thus the corresponding boundary point lies in $(\theta_{\text{null}} \setminus \theta_{\text{null}}^{-1}) \cap \partial A_g$.

We now note that in [5] the determinant of matrix (9) restricted to $\theta(\tau', z) = 0$ was studied. It is denoted $\eta(\tau', z)$ there (though explained in a slightly different way — see the remark below for a discussion), and it is shown there in particular that it does not vanish identically for $\tau'$ in the Jacobian locus. □

**Remark 4.** From our approach to the proof of theorem 3 we see the geometric significance of the matrix (9), which is of independent interest, and allows us to answer a question posed in [5] about the vanishing of the modular form $\eta(\tau', z)$ defined on the theta divisor. Indeed, let us write the matrix in (9) as

$$B = \begin{pmatrix} \frac{\partial^2 \theta(\tau', z/2)}{\partial z_1} & \frac{\partial^2 \theta(\tau', z/2)}{\partial z_2} & \ldots & \frac{\partial^2 \theta(\tau', z/2)}{\partial z_{g-1}} & \frac{\partial \theta(\tau', z/2)}{\partial z_1} \\ \frac{\partial^2 \theta(\tau', z/2)}{\partial z_2} & \frac{\partial^2 \theta(\tau', z/2)}{\partial z_1} & \ldots & \frac{\partial^2 \theta(\tau', z/2)}{\partial z_{g-2}} & \frac{\partial \theta(\tau', z/2)}{\partial z_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 \theta(\tau', z/2)}{\partial z_{g-1}} & \frac{\partial^2 \theta(\tau', z/2)}{\partial z_{g-2}} & \ldots & \frac{\partial^2 \theta(\tau', z/2)}{\partial z_{g-2}} & \frac{\partial \theta(\tau', z/2)}{\partial z_{g-1}} \\ \frac{\partial \theta(\tau', z/2)}{\partial z_1} & \frac{\partial \theta(\tau', z/2)}{\partial z_2} & \ldots & \frac{\partial \theta(\tau', z/2)}{\partial z_{g-1}} & 0 \end{pmatrix} \leq h.
$$

As an immediate consequence of the proof of the previous theorem, we solve a problem raised in [5] (R. de Jong has informed us that he has also obtained a proof of this independently, included in the updated version of [5]).

**Proposition 5.** The function $\eta(\tau', z)$ vanishes at the point $(\tau_0, x_0)$ if and only if $x_0$ is a ramification point for the Gauss map $G_{\tau_0}$ of the theta divisor of the abelian variety with period matrix $\tau_0$. 

In [5] it is proved that $\eta(\tau', z)$ is a theta function of order $g$ with respect to $z$ and weight $(g + 4)/2$ with respect to $\tau$ (note that in our notations there is a shift of $g$ by $-1$ compared to [5]). In [8] we proved that the function

$$F(\tau) = \det \begin{pmatrix} \frac{\partial \theta(\tau, 0)}{\partial \tau_1} & \frac{\partial \theta(\tau, 0)}{\partial \tau_2} & \vdots & \frac{\partial \theta(\tau, 0)}{\partial \tau_g} \\ \frac{\partial \theta(\tau, 0)}{2\partial \tau_1} & \frac{\partial \theta(\tau, 0)}{2\partial \tau_2} & \vdots & \frac{\partial \theta(\tau, 0)}{2\partial \tau_g} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \theta(\tau, 0)}{2\partial \tau_1} & \frac{\partial \theta(\tau, 0)}{2\partial \tau_2} & \vdots & \frac{\partial \theta(\tau, 0)}{2\partial \tau_g} \end{pmatrix}$$

(10)

defines a modular form of weight $(g + 4)/2$ with respect to $\Gamma_g(4, 8)$ along $\theta(\tau, 0) = 0$. Obviously the determinant of the matrix (9) gives the intersection of $F(\tau)$ with a suitable boundary component of the partial toroidal compactification of $A_g(4, 8)$.

We end this section by observing that in the above discussion the intersection $\theta_{\text{null}}^h \cap \partial A_g$ is also described explicitly. We hope that this can be helpful for obtaining an estimate of the dimension of $\theta_{\text{null}}^h$.

**Remark 6.** O. Debarre has explained to us the following way to easily fix the proof of the theorem 3 using the results from his work [4], without referring to the moduli space of curves. Such a proof proceeds by induction in $g$. Indeed, just before “Quatrième pas” (p. 701 in [4]), it is shown that the morphism $S$ is smooth and that $S \to N_0$ is birational at a general point of $\partial \theta_{\text{null}}$. Thus the same must holds over a general point of $\theta_{\text{null}}$; moreover, the differential is injective if and only if the singularity is an ordinary double point, and thus the result is proven.

4. **The locus $\theta_{\text{null}}^{g-1}$**

We will now consider the intersection $\theta_{\text{null}} \cap N'_0$ and its eventual relation with $\theta_{\text{null}}^{g-1}$.

The intersection of the two components of $N_0 = \theta_{\text{null}} + 2N'_0$ is studied in the last section of [4]: it is proven that their intersection is not reduced for $g \geq 4$, reducible for $g \geq 5$ and irreducible for $g = 4$. In a recent paper [8] we proved that in genus 4 scheme-theoretically

$$\theta_{\text{null}} \cap N'_0 = 2\theta_{\text{null}}^3$$

(for genus 4 the locus $N'_0$ is the Jacobian locus). This was done by proving an inclusion and checking that the components of the two varieties have the same degree in the space $A_4(4, 8)$ (more precisely, in the projective space $\mathbb{P}^{135}$ containing $Th(A_4(4, 8))$).
It is natural to ask what the situation is for higher \( g \). We thus recall some more notations and results from [4] and [11]. Following Mumford, we denote by \( S := \text{Sing}_{\text{vert}} \Theta \) the locus of singular points of theta divisors of ppavs. This is a subvariety of the universal family 
\[
\pi : \mathcal{X}_g \to \mathcal{A}_g.
\]
Each component of \( S \) has codimension \( g + 1 \), cf. [11] and [3], and it is locally defined within \( \mathcal{X}_g \) by \( g + 1 \) equations
\[
\begin{align*}
\theta(\tau, z) &= 0, \\
\frac{\partial \theta}{\partial z_i}(\theta, z) &= 0 \quad \forall i = 1, \ldots, g.
\end{align*}
\] (11)

Thus set-theoretically \( S \) decomposes into \( S_{\text{null}} \) — the locus where \( z \) is an even point of order two on the ppav lying on the theta divisor — and the remaining component(s) \( S' \). The following lemma ties in the locus \( \theta_{g-1}^{\text{null}} \) with this description of the geometry.

**Proposition 7.** Set-theoretically we have \( \theta_{g-1}^{\text{null}} = \pi(S_{\text{null}} \cap \text{Sing } S) \).

**Proof.** Since the locus \( S \) is given as a subvariety of \( \mathcal{X}_g \) by the \( g + 1 \) equations (11), \( \text{Sing } S \) is the locus where the \( g + 1 \) gradients of these equations, with respect to all the local coordinates on \( \mathcal{X}_g \), i.e. with respect to all \( \tau_{ij} \) and \( z_i \), are linearly dependent, i.e. \( \text{Sing } S \) is the locus where the \( \left( \frac{g(g+1)}{2} + g \right) \times (g + 1) \) matrix
\[
\begin{pmatrix}
\frac{\partial \theta}{\partial \tau_{11}} & \cdots & \frac{\partial \theta}{\partial \tau_{g g}} & \frac{\partial \theta}{\partial z_1} & \cdots & \frac{\partial \theta}{\partial z_g} \\
\frac{\partial^2 \theta}{\partial z_1 \partial \tau_{11}} & \cdots & \frac{\partial^2 \theta}{\partial z_1 \partial \tau_{g g}} & \frac{\partial^2 \theta}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 \theta}{\partial z_1 \partial z_g} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\frac{\partial^2 \theta}{\partial z_{g} \partial \tau_{11}} & \cdots & \frac{\partial^2 \theta}{\partial z_{g} \partial \tau_{g g}} & \frac{\partial^2 \theta}{\partial z_{g} \partial z_1} & \cdots & \frac{\partial^2 \theta}{\partial z_{g} \partial z_g}
\end{pmatrix}
\]
evaluated at \((\tau, z)\) has rank at most \( g \).

If \( z \) is an even point of order two, then all \( \frac{\partial \theta}{\partial z_i} \) and, by using the heat equation, all \( \frac{\partial^2 \theta}{\partial z_i \partial \tau_{ij}} \) derivatives at \((\tau, z)\) are equal to zero, and thus the above matrix becomes
\[
\begin{pmatrix}
\frac{\partial \theta}{\partial \tau_{11}} & \cdots & \frac{\partial \theta}{\partial \tau_{g g}} & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{\partial^2 \theta}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 \theta}{\partial z_1 \partial z_g} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{\partial^2 \theta}{\partial z_{g} \partial z_1} & \cdots & \frac{\partial^2 \theta}{\partial z_{g} \partial z_g}
\end{pmatrix}
\]
The rank of this matrix is non-maximal if either all the derivatives \( \frac{\partial \theta}{\partial \tau_{ij}} = 0 \) for all \( 1 \leq i \leq j \) (and then by the heat equation in fact the matrix is completely zero), or if the \( g \times g \) Hessian matrix in the lower right is degenerate, i.e. exactly if \( \tau \in \theta_{\text{null}}^{g-1} \). \( \square \)
In the last section of [4] Debarre shows that \( \theta_{\text{null}} \cap N'_0 \) is reducible for \( g \geq 5 \). In fact he shows that there is a component \( R_g \) of \( \theta_{\text{null}} \cap N'_0 \) (the boundary of which in the partial toroidal compactification is described explicitly) such that \( R_g \subset \pi(S_{\text{null}} \cap S') \), and that there are other components \( D_g \), which do not lie in \( \pi(S_{\text{null}} \cap S') \).

For a generic ppav in \( R_g \) the theta divisor has only one singular point (of order two) that is the limit of singular points \( x \) and \(-x\) of theta divisors of ppav in \( N'_0 \). For generic points of the other component(s) \( D_g \) of \( \theta_{\text{null}} \cap N'_0 \) the theta divisor has a singular point of order two, and also two other singular points \( x \) and \(-x\) — this is exactly to say that \( D_g \) does not lie in \( \pi(S_{\text{null}} \cap S') \).

We now tie in the locus \( \theta_{\text{null}}^{g-1} \) with this picture.

**Proposition 8.** The locus \( \theta_{\text{null}}^{g-1} \), as a set, is equal to \( \pi(S_{\text{null}} \cap S') \).

**Proof.** Indeed, we formally compute (on the level of sets, i.e. with reduced scheme structure)

\[
\theta_{\text{null}}^{g-1} = \pi(S_{\text{null}} \cap \text{Sing } S) = \pi(S_{\text{null}} \cap \text{Sing}(S_{\text{null}} \cup S')) = \pi((S_{\text{null}} \cap S') \cup (\text{Sing } S_{\text{null}}))
\]

so what it remains to prove is that \( \pi(\text{Sing } S_{\text{null}}) \subset \pi(S_{\text{null}} \cap S') \). Note, however, that since the locus \( \theta_{\text{null}}^{g-1} \subset \mathcal{A}_g \) is given locally by two equations — some theta constant and its Hessian being zero — it is purely codimension two in \( \mathcal{A}_g \) (we proved above that it is not codimension one, since it is not equal to the irreducible divisor \( \theta_{\text{null}} \), in which it is contained). Therefore it suffices to show that \( \pi(\text{Sing } S_{\text{null}}) \) has codimension higher than two — then it cannot be an irreducible component of \( \theta_{\text{null}}^{g-1} \), and thus must be contained in \( \pi(S_{\text{null}} \cap S') \). Thus the following lemma finishes the proof of the proposition.

**Lemma 9.** The codimension of \( \pi(\text{Sing } S_{\text{null}}) \) in \( \mathcal{A}_g \) is higher than two.

**Proof.** By definition \( S_{\text{null}} \) is locally given in the universal family \( \mathcal{X}_g \) by the following \( g + 1 \) equations:

\[
\theta(\tau, z) = 0, \quad z = (\tau \varepsilon + \delta)/2
\]

where \( [\varepsilon, \delta] \) is an even theta characteristic. Thus the locus \( \text{Sing } S_{\text{null}} \) is where the \( g + 1 \) gradient vectors (with respect to all local coordinates on \( \mathcal{X}_g \), i.e. with respect to both \( \tau \) and \( z \)) of these defining equations
are linearly dependent. These gradients form the matrix

\[
\begin{pmatrix}
\frac{\partial \theta}{\partial \tau_1} & \cdots & \frac{\partial \theta}{\partial \tau_g} & \frac{\partial \theta}{\partial z_1} & \cdots & \frac{\partial \theta}{\partial z_g} \\
-\varepsilon_1/2 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & -\varepsilon_g/2 & 0 & \cdots & 1
\end{pmatrix}
\]

When we evaluate the derivatives of the theta function at the even point \((\tau \varepsilon + \delta)/2\) of order two, the derivatives \(\frac{\partial \theta}{\partial \tau_{ij}}\) all vanish. Thus the matrix only has non-maximal rank if all the derivatives \(\frac{\partial \theta}{\partial \tau_{ij}}\) are zero for all \(1 \leq i \leq j \leq g\), i.e. if the corresponding theta constants \(\theta \left[ \frac{\varepsilon}{\delta} \right]\) vanishes at \(\tau\) to order at least two. A naive dimension count would predict a high codimension for such a condition; here is an easy observation showing that this codimension is greater than two.

Indeed, by the heat equation the theta constants vanishes to order at least two only if the theta function \(\theta \left[ \frac{\varepsilon}{\delta} \right](\tau, z)\) vanishes at \(z = 0\) to order at least 4 (the third \(z\)-derivatives at zero are all zero by parity). In [12] Teixidor i Bigas studies the locus of curves having such a theta characteristic — this is \(\mathcal{M}_g^3\) in her notations — and shows that \(\text{codim}_{\mathcal{M}_g} \mathcal{M}_g^3 > 2\) (her results are actually better than this, but this is all we need). But if a locus in \(\mathcal{A}_g\) has non-empty intersection with \(\mathcal{M}_g \subset \mathcal{A}_g\), and the codimension of its intersection with \(\mathcal{M}_g\) within \(\mathcal{M}_g\) is at least \(n\), then the codimension of the locus itself in \(\mathcal{A}_g\) is at least \(n\), cf. [10]. Rigorously we can only use this for smooth varieties, while \(\mathcal{A}_g\) is an orbifold, but we can pass to the finite covering \(\mathcal{A}_g(4, 8)\) that is smooth. Alternatively, we can use the result from [3] that if the theta divisor has a point of multiplicity greater than two, then the period matrix belongs to the Andreotti-Mayer locus \(\mathcal{N}_1\) that has codimension at least 3 in \(\mathcal{A}_g\). \qed

**Corollary 10.** As sets (i.e. with reduced scheme structure) we have the inclusion \(\theta_{\text{null}}^g \subset (\theta_{\text{null}} \cap N'_0)\).

**Proof.** Indeed, by definition we have \(\theta_{\text{null}} = \pi(S_{\text{null}})\), and \(N'_0 = \pi(S')\), and thus \(\theta_{\text{null}}^{g-1} = \pi(S_{\text{null}} \cap S') \subset (\pi(S_{\text{null}}) \cap \pi(S')) = \theta_{\text{null}} \cap N'_0\). \qed

Furthermore, we can now describe more precisely the locus \(R_g\) introduced by Debarre.

**Proposition 11.** We have the equality of sets \(R_g = \theta_{\text{null}}^{g-1} = \pi(S_{\text{null}} \cap S')\).
Proof. Recall that \( R_g \) is defined in [4] by first studying the boundary of the locus \( \theta_{\text{null}} \cap N_0' \) in the partial toroidal compactification \( \overline{A}_g \), choosing a certain explicitly defined component \( \partial R_g \) of this boundary, and then arguing that \( \partial R_g \) must be the boundary of some locus \( R_g \subset A_g \). It then follows that \( R_g \subset \pi(S_{\text{null}} \cap S') \), and it is shown in [4] that no other component of \( \theta_{\text{null}} \cap N_0' \) is contained in \( \pi(S_{\text{null}} \cap S') \).

Since we know the equality \( \theta_{\text{null}}^{g-1} = \pi(S_{\text{null}} \cap S') \), it follows that \( \pi(S_{\text{null}} \cap S') \) is purely codimension two, and since it is contained in \( \theta_{\text{null}} \cap N_0' \), which is also purely codimension two, \( \pi(S_{\text{null}} \cap S') \) must be the union of a number of irreducible components of \( \theta_{\text{null}} \cap N_0' \). But then since this union cannot contain any of the components of \( D_g \), we must have \( \pi(S_{\text{null}} \cap S') = R_g \). \( \square \)

The equality proved above is set-theoretic. Since \( R_g \) and \( \theta_{\text{null}}^{g-1} \) have the same intersection with the boundary \( \partial A_g \), we also get

**Corollary 12.** Up to embedded subvarieties \( \theta_{\text{null}}^{g-1} = R_g \) as schemes.

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Mathematics Department, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544, USA. Research is supported in part by National Science Foundation under the grant DMS-05-55867.

E-mail address: sam@math.princeton.edu

Dipartimento di Matematica, Università “La Sapienza”, Piazzale A. Moro 2, Roma, I 00185, Italy

E-mail address: salvati@mat.uniroma1.it