On Uniqueness of T–duality with Spectators

Ladislav Hlavatý, Filip Petrásek
Faculty of Nuclear Sciences and Physical Engineering,
Czech Technical University in Prague,
Břehová 7, 115 19 Prague 1, Czech Republic

Abstract

We investigate the dependence of non-Abelian T–duality on various identification of the isometry group of target space with its orbits, i.e. with respect to the location of the group unit on manifolds invariant under the isometry group. We show that T–duals constructed by isometry groups of dimension less than the dimension of the (pseudo)-Riemannian manifold may depend not only on the initial metric but also on the choice of manifolds defining positions of group units on each of the sub-manifold invariant under the isometry group. We investigate whether this dependence can be compensated by coordinate transformation.

1 Introduction

Both Abelian [1] and non-Abelian [2] T–duality assume that an "adapted" coordinate system consisting of group coordinates and spectators in the target space of sigma model is given. However, such system is not unique.

Let us give very simple example for the Euclidean metric $\eta = \text{diag}(1,1)$ in $\mathbb{R}^2 \ni (x,y)$. It is invariant with respect to one-dimensional isometry group $y \mapsto y + \alpha$ that can be used for dualization. Of course, the adapted coordinates $(s,g)$ in this case are $x = s, y = g$ but also

$$x = s, \ y = g + \xi(s)$$

where $\xi(s)$ is arbitrary. The Euclidean metric in this coordinates acquires the form

$$F_\xi(s,g) = \begin{pmatrix} 1 + \xi'(s)^2 & \xi'(s) \\ \xi'(s) & 1 \end{pmatrix}$$

and the dual tensor is

$$\hat{F}_\xi(s,\hat{g}) = \begin{pmatrix} 1 & -\xi'(s) \\ \xi'(s) & 1 \end{pmatrix}.$$  

Evidently, there is no coordinate transformation in the dual space that would transform $\hat{F}_\xi(s,\hat{g})$ to (symmetric) $\hat{F}_{\text{const}}(s,\hat{g}) = \eta$.

Therefore, it is interesting and generally important to investigate how the choice of adapted coordinates, i.e. the choice of function $\xi(s)$ influences dualization in more complicated cases.

T–dualities can be understood in the framework of the Poisson–Lie T–duality introduced in 1995 by C. Klimčík and P. Ševera using the concept of the Drinfel’d
double \[3, 4\]. An important point for construction of dual metrics and torsion potentials in this framework is whether the group of isometries that induces the duality acts transitively and freely on the (pseudo)-Riemannian manifold, in other words, if the manifold is diffeomorphic to the isometry group. If not, it is necessary to foliate the manifold by orbits of the isometry group and perform dualization by the Drinfel’d double method on each of them separately. The orbits are then numerated by one or more parameters called spectators since they do not participate in the dualization.

Examples in Refs. \[5\] and \[6\] choose a specific introduction of spectator \(t\) into space-time metric invariant under the isometry group containing dilations and shifts of the space and derive corresponding duals. We are going to show that introduction of spectators is ambiguous and this ambiguity may influence dualization. The ambiguity follows from identifications of the Lie group of isometries with its orbits and as such it is not unique, namely, it depends on the location of the group unit in the orbit. This ambiguity then appears in the dual metric and torsion potential as well.

The main question that we want to answer in this paper is whether these ambiguities are essential for the dual model or if they can be understood as coordinate effect. In other words, we search for transformations of coordinates in the dual space that transform between components of the dual metrics and torsion potentials obtained from various assignments of the group unit to the invariant manifolds numerated by the spectators.

In Ref. \[2\] where the dualization is performed by gauging the non-Abelian symmetry of initial action, similar ambiguity appears as choice of gauge. It is claimed there that "Obviously different gauge choices will not give different dual theories, they will give the same theory differing by a coordinate transformation." Goal of this paper is to confirm this conjecture in the framework of the Drinfel’d double method where the dual tensor field is obtained explicitly, and mainly, give the procedure for finding the coordinate transformations.

The paper is structured as follows. In Sec. \[2\] we give elements of dualization in the Drinfel’d double framework and present the appearance of ambiguities. In Sec. \[3\] we look for the coordinate transformations that should eliminate the ambiguities. All steps are illustrated by dualization of three-dimensional flat metric expressed in non-adapted coordinates.

### 2 T–dualities of \(\sigma\)–models with Spectators

We are going to deal with non-Abelian T–dualities of non-linear sigma models given by the action

\[
S_F[\phi] = \int_\Omega \mathcal{L} \, d\tau d\sigma = \int_\Omega \tilde{\mathcal{L}} \, d\xi_+ d\xi_-
\]

where \(\xi_\pm = \tau \pm \sigma \in \Omega \subset \mathbb{R}^2\),

\[
\mathcal{L} = \frac{1}{2} \mathcal{G}_{\mu\nu}(\phi) \left( \partial_{\tau} \phi^{\mu} \partial_{\tau} \phi^{\nu} - \partial_{\tau} \phi^{\mu} \partial_{\sigma} \phi^{\nu} \right) + \mathcal{B}_{\mu\nu}(\phi) \partial_{\tau} \phi^{\mu} \partial_{\sigma} \phi^{\nu},
\]

where 

\[\mathcal{G}_{\mu\nu}(\phi) = \frac{1}{2} \left( \partial_{\tau} G_{\mu\nu}(\phi) - \partial_{\sigma} G_{\mu\nu}(\phi) \right), \quad \mathcal{B}_{\mu\nu}(\phi) = \frac{1}{2} \left( \partial_{\tau} B_{\mu\nu}(\phi) - \partial_{\sigma} B_{\mu\nu}(\phi) \right). \]
and $G_{\mu\nu}$, $B_{\mu\nu}$ are components of the symmetric and antisymmetric part of a tensor field $F$ on a manifold $M$ so that

$$\mathcal{L} = \partial_\phi \phi^\mu F_{\mu\nu}(\phi) \partial_\phi \phi^\nu = \frac{1}{2} \mathcal{L}, \quad F_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}.\quad (6)$$

The functions $\phi^\mu : \Omega \to \mathbb{R}$, $\mu = 1, 2, \ldots, \dim M$ are obtained by the composition $\phi^{\mu} = x^{\mu} \circ \phi$ of a map $\phi : \Omega \to M$ and components of a coordinate map $x$ on an open set of $M$.

Equations of motion that follow from this action are

$$(F_{\mu\nu} + F_{\nu\mu}) \partial_+ \phi^\nu + (F_{\mu\nu,\beta} + F_{\beta\mu,\nu} - F_{\beta\nu,\mu}) \partial_- \phi^\beta \partial_+ \phi^\nu = 0.\quad (7)$$

If $G$ is a non-trivial Lie group of isometries of the tensor field $F$, there is an algebra generated by independent Killing vector fields $K_a$, $a = 1, \ldots, \dim G$ satisfying

$$(\mathcal{L}_{K_a} F)_{\mu\nu} = 0,\quad (8)$$

which is the condition for dualizability of $F$ where $\mathcal{L}$ denotes the Lie derivative.

Here we shall focus on the case, when $\dim G < \dim M$ so that we can not identify $G \approx M$ (atomic duality). Nevertheless, we shall assume that the group of isometries acts freely and transitively on sub-manifolds of $M$ invariant under the isometry group so that we can identify them with orbits of $G$. Let us summarize main points of construction of dual models by the Drinfel’d double method in this case.

### 2.1 Invariant sub-manifolds and adapted coordinates

Before the dualization procedure is started, we have to construct the invariant sub-manifolds $\Sigma$ of $M$. They are implicitly given by functions $\Phi(x^{\mu})$ satisfying linear partial differential equations

$$K_a \Phi = 0, \quad a = 1, \ldots, \dim G.\quad (9)$$

As the number of equations is less than the number of independent variables, we get $S = \dim M - \dim G$ independent solutions that define the invariant sub-manifolds $\Sigma(s)$ as

$$\Phi^\delta(x^{\mu}) = s^\delta, \quad \delta = 1, \ldots, S.\quad (10)$$

Assuming free action of the isometry group, we can identify each of the invariant sub-manifolds with the isometry group and Killing vectors with left-invariant fields of the group. The latter identification provides us with transformation to special – adapted – coordinates on $M$

$$x^{\mu} = \{s^\delta, g^\alpha\}, \quad \delta = 1, \ldots, S, \quad a = 1, \ldots, \dim G,\quad (11)$$

part of which numerate the sub-manifolds and the other parametrize group elements by

$$g = e^{g^1 T_1} e^{g^2 T_2} \ldots e^{g^{\dim G} T_{\dim G}}\quad (12)$$
where \( T_a \) form the basis of the Lie algebra of the group \( G \).

The left-invariant vector fields \( V(x') \) are extended to \( M \) so that

\[
V^\delta(x') = 0, \quad \delta = 1, \ldots, S
\]

and equations that determine transformations to the adapted coordinates are then

\[
K^\mu_a(x) = \frac{\partial x^\mu}{\partial x'^\nu} V_a^\nu(x'), \quad a = 1, \ldots, \text{dim } G
\]

where \( V_a \) are independent left-invariant fields that commute in the same way as the corresponding Killing vectors. Number of equations (14) is less than number of sought functions \( X^\mu(x') = x^\mu \), therefore solution will depend on "integration functions" of spectators \( \xi^\mu(s) \)

\[
x^\mu = X^\mu_\xi(x') = X^\mu_\xi(s, g).
\]

As Killing vectors are transformed to left-invariant vector fields of the group \( G \), coordinate transformations (15) can be understood as right action of the group \( G \) in the invariant sub-manifold \( \Sigma(s) \) and

\[
x^\mu = (u_\xi(s) \triangleleft g)^\mu
\]

where \( u_\xi(s) \) are points of sub-manifolds \( \Sigma(s) \) that correspond to the unit element \( u \in G \). It means that functions \( \xi^\mu \) must satisfy condition

\[
\Phi^\delta(\xi^\mu) = s^\delta, \quad \delta = 1, \ldots, S.
\]

For application of dualization procedure, components of the tensor field \( F \) must be expressed in the adapted coordinates (11) as

\[
F_{\kappa\lambda}(s, g) = \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\nu}{\partial x'\lambda} F_{\mu\nu}(x).
\]

It is clear that components of the tensor field \( F \) in adapted coordinates will depend on functions \( \xi^\mu \) since the transformation matrix \( \frac{\partial x^\mu}{\partial x'^\nu} \) does. The equations (8) take then the form

\[
(\mathcal{L}_{V_\xi} F)_{\mu\nu} = 0,
\]

which is the condition for non-Abelian dualizability of the \( \sigma \)-model [3].

### 2.1.1 Example

The simplest example of non-Abelian T–duality with one spectator can be given for three-dimensional metrics invariant under two-dimensional non-Abelian group. Let us have pseudo-Riemannian flat metric \( \eta = \text{diag}(-1, 1, 1) \) in \( \mathbb{R}^3 \ni (t, y, z) \) invariant under subgroup \( G_2 \) of inhomogeneous Lorentz group ISO(1,2) generated by the Killing vector fields [7]

\[
K_1 = -z \partial_t - t \partial_z, \quad K_2 = -y \partial_t - (t + z) \partial_y + y \partial_z
\]
commuting as
\[ [K_1, K_2] = -K_2. \] (21)
The sub-manifolds of \( \mathbb{R}^3 \) invariant under this subgroup are given by condition
\[ \Phi(t, y, z) = t^2 - y^2 - z^2 = \text{const} \] (22)
where the value of the constant depend on the sub-manifold and can be taken as spectator \( s \). The Killing vector fields are transformed to left-invariant vector fields
\[ V_1 = \partial_1 + g_2 \partial_2, \quad V_2 = \partial_2 \] (23)
of the group
\[ G_2 = \{ g \in \mathbb{R}^2, \; g \cdot g' := (g_1 + g_1', e^{g_1'} g_2 + g_2') \} \] (24)
by transformation \((t, y, z) \mapsto (s, g_1, g_2)\)
\[ t = \frac{1}{2} \left( e^{-g_1} \left( g_2^2 + 1 \right) \xi_1(s) + e^{g_1} \xi_3(s) - 2 g_2 \xi_2(s) \right), \]
\[ y = \xi_2(s) + e^{-g_1} g_2 \xi_1(s), \]
\[ z = -\frac{1}{2} \left( e^{-g_1} \left( g_2^2 - 1 \right) \xi_1(s) + e^{g_1} \xi_3(s) - 2 g_2 \xi_2(s) \right). \] (25)
The transformation depends on functions \( \xi_j(s) \) that fix the positions
\[ u_\xi(s) = \left( \frac{\xi_1(s) + \xi_3(s)}{2}, \frac{\xi_2(s) - \xi_3(s)}{2} \right) \] (26)
of the unit of \( G_2 \) in the invariant manifolds \( \Sigma(s) \). They are arbitrary up to condition
\[ \xi_1(s) \xi_3(s) - \xi_2(s)^2 = s \] (27)
that follows from the requirement \( t^2 - y^2 - z^2 = s \). The transformation (25) is transition from the space-time coordinates of \( \mathbb{R}^3 \) to the group coordinates of \( G_2 \) and spectator \( s \) numerating the invariant sub-manifolds. It is easy, even though tedious, to check that this transformation is right action \([16]\) of \( G_2 \) on \( \mathbb{R}^3 \). Moreover, for \( \xi_1(s) \neq 0 \), the Killing vectors are independent and the curve (26) in \( \mathbb{R}^3 \) is transversal to the invariant sub-manifolds of \( \mathbb{R}^3 \).

Components \( F_{\kappa\lambda}(s, g_1, g_2) \) of the flat metric \( \eta \) in the adapted coordinates are rather complicated and depend heavily on functions \( \xi_j(s) \) and their derivatives \( \xi_j'(s) \) as
\[ F_\xi(s, g) = \begin{pmatrix}
\frac{(\xi_2^2 + s)\xi_1^2 - \xi_1(2\xi_2\xi_2^2 + 1)\xi_1^2 + \xi_2^2 s^2}{\xi_1(2\xi_2\xi_2^2 + 1) - 2(\xi_2^2 + s)\xi_1^2} & \frac{\xi_1(2\xi_2\xi_2^2 + 1) - 2(\xi_2^2 + s)\xi_1^2}{\xi_1(2\xi_2\xi_2^2 + 1) - 2(\xi_2^2 + s)\xi_1^2} & e^{-g_1} \left( \xi_2 \xi_1' - \xi_1 \xi_2' \right) \\
\frac{\xi_2^2 + s}{e^{-g_1} (\xi_2 \xi_1' - \xi_1 \xi_2')} & \frac{\xi_2^2 + s}{e^{-g_1} (\xi_2 \xi_1' - \xi_1 \xi_2')} & e^{-g_1} \xi_1 \xi_2 \\
\frac{\xi_2^2 + s}{e^{-g_1} (\xi_2 \xi_1' - \xi_1 \xi_2')} & \frac{\xi_2^2 + s}{e^{-g_1} (\xi_2 \xi_1' - \xi_1 \xi_2')} & e^{-g_1} \xi_1 \xi_2
\end{pmatrix}. \] (28)
Here we have put
\[ \xi_3(s) = \frac{s + \xi_2(s)^2}{\xi_1(s)} \] (29)
to satisfy (27).
2.2 Dualization

Non-Abelian T–duality is a special case of the Poisson–Lie T–duality \[3, 4\] formulated in the framework of the Drinfel’d double – a Lie group whose Lie algebra \( \mathfrak{d} \) admits a decomposition \( \mathfrak{d} = \mathfrak{g} + \mathfrak{g}^{\hat{}} \) into a pair of sub-algebras maximally isotropic with respect to a symmetric ad-invariant non-degenerate bi-linear form \( \langle \ldots, \ldots \rangle_d \).

Poisson–Lie T–duality can be applied to models with tensor fields satisfying condition

\[
(\Omega_{V^a} F)_{\mu\nu} = F_{\mu\nu} V^\kappa_b \hat{f}_{a}^{bc} V^\lambda_c F_{\lambda\nu}
\]

(30)

where \( V^\kappa_b \) are components of the left-invariant vector fields \( V^b \) generating a Lie group \( G \) and \( \hat{f}_{a}^{bc} \) are structure coefficients of a "dual" Lie group \( \hat{G} \) of the same dimension. In case of non-Abelian T–duality \( \hat{f}_{a}^{bc} = 0 \) so that the dual group is Abelian. Self-consistency of the condition (30) restricts the structure coefficients in such way that \( G \) and \( \hat{G} \) can be interpreted as subgroups defining the Drinfel’d double \( D \equiv (G|\hat{G}) \).

Components of tensor field \( F \) satisfying the condition (19) can be written as

\[
(F_\xi)_{\mu\nu}(s, g) = e^j_\mu(g)(E_\xi(s))_{jk}e^k_\nu(g)
\]

(31)

where the matrix \( E_\xi(s) = F_\xi(s, 0) \),

\[
e^j_\mu(g) = \begin{pmatrix} 1_S & 0 \\ 0 & e^a_\mu(g) \end{pmatrix}
\]

(32)

and \( e^a_\mu(g) \) are components of right-invariant forms \( (dg)g^{-1} \) in adapted coordinates.

Components of the dual tensor \( \hat{F} \) obtained by the non-Abelian T–duality transformation are \[3\]

\[
\hat{F}_\xi(s, \hat{g}) = (\hat{E}_\xi(s)^{-1} + \hat{\Pi}(\hat{g}))^{-1}
\]

(33)

where \( \hat{E}_\xi(s) \) is

\[
\hat{E}_\xi(s) = (A + E_\xi(s) \cdot B)^{-1}(B + E_\xi(s) \cdot A),
\]

(34)

\[
A = \begin{pmatrix} 1_S & 0 \\ 0 & O_G \end{pmatrix}, \quad B = \begin{pmatrix} O_S & 0 \\ 0 & 1_G \end{pmatrix}
\]

(35)

Matrices \( 1_G, 1_S \) and \( O_G, O_S \) are unit and zero matrices of \( \dim G \) and \( (\dim M - \dim G) \),

\[
\hat{\Pi}(\hat{g}) = \begin{pmatrix} O_S & 0 \\ 0 & -f_{cd}^{\hat{b}}\hat{g}_b \end{pmatrix}
\]

(36)

where \( f_{cd}^{\hat{b}} \) are structure coefficients of the Lie algebra of the group \( G \) and \( \hat{g}_b \) are coordinates of the Abelian group \( \hat{G} \). As both factors in the definition of \( \hat{E}_\xi(s) \) (34) must have non-vanishing determinants, we get from (34), (35) conditions

\[
\det (A + E_\xi(s) \cdot B) \neq 0, \quad \det (B + E_\xi(s) \cdot A) \neq 0
\]

(37)

that further restrict the functions \( \xi^\mu \).

Formula (33) for the non-Abelian T–duality of \( \sigma \)–models with spectators follow from the Poisson–Lie T–duality formulated in the framework of the Drinfel’d double \( [3, 4, 6] \). By this formula we get dual tensors whose components will depend on functions \( \xi^\mu \) or \( X_\xi \) mapping for fixed \( s \) from the group \( G \) to the invariant manifold \( \Sigma(s) \).
2.2.1 Dual tensor field – example continues

From the expression \((31)\) we can get the matrix \(E(s) = E_\xi(s)\) by setting \(g_1 = 0\) in \((28)\). Formula \((34)\) then yields

\[
\hat{E}_\xi(s) = \begin{pmatrix}
-\frac{1}{2s} & \frac{\xi'_1(s)}{\xi_1(s)} - \frac{1}{2s} & -\frac{\xi_2(s) - 2s \xi'_2(s)}{2s \xi_1(s)} \\
\frac{1}{2s} & \frac{\xi'_2(s)}{\xi_1(s)} - \frac{1}{2s} & -\frac{\xi_2(s) - 2s \xi'_2(s)}{2s \xi_1(s)} \\
\frac{s \xi_2(s) - 2s \xi'_2(s)}{2s \xi_1(s)} & \frac{s \xi_1(s)}{s \xi_1(s)} & \frac{s \xi_1(s)}{s \xi_1(s)} + s \\
\end{pmatrix}
\]

and applying the formula \((33)\) with

\[
\hat{\Pi}(\hat{g}) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \hat{g}_2 \\
0 & -\hat{g}_2 & 0 \\
\end{pmatrix},
\]

we get the dual tensor field in adapted coordinates \((s, \hat{g}_1, \hat{g}_2)\) with non-vanishing torsion and scalar curvature

\[
\hat{R} = \frac{4 \xi_1(s)^2 \left(11 \hat{g}_2^2 - 3s \xi_1(s)^2\right)}{(\hat{g}_2^2 + s \xi_1(s)^2)^2}.
\]

Unfortunately, \(\hat{F}_\xi(s, \hat{g})\) is too extensive to display it here.

3 Change of Functions \(\xi^\mu\) versus Coordinate Transformations

It is clear from the above that components of both tensor fields \(F\) and \(\hat{F}\) in adapted coordinates may depend on functions \(\xi^\mu(s)\). The question is whether we can get rid off this dependence by a coordinate transformation in both initial and dual tensors. More precisely, we search for transformation of group coordinates \(g^a = \gamma^a(s, \hat{g})\) and \(\hat{g}^a = \hat{\gamma}^a(s, \hat{g})\) that would be equivalent to the change of functions \(\xi^\mu(s) \mapsto \xi^\mu(s)\), i.e.

\[
\gamma^a : e(g)E_\xi(s)e(g)^T \mapsto e(\hat{g})E_\xi(s)e(\hat{g})^T
\]

and

\[
\hat{\gamma}^a : (\hat{E}_\xi(s)^{-1} + \hat{\Pi}(\hat{g}))^{-1} \mapsto (\hat{E}_\xi(s)^{-1} + \hat{\Pi}(\hat{g}))^{-1}.
\]

It means that we look for functions \(\Gamma(s, \hat{g}) = (s^\delta, \gamma^a(s, \hat{g}))\) and \(\hat{\Gamma}(s, \hat{g}) = (s^\delta, \hat{\gamma}^a(s, \hat{g}))\) such that

\[
(F_\xi)_{\mu\nu}(s, \hat{g}) = \frac{\partial \Gamma_{\kappa\lambda}}{\partial x^\mu} \frac{\partial \Gamma^\lambda_{\nu}}{\partial x^\nu} (F_\xi(\Gamma))_{\kappa\lambda}
\]

and

\[
(\hat{F}_\xi)_{\mu\nu}(s, \hat{g}) = \frac{\partial \hat{\Gamma}_{\kappa\lambda}}{\partial x^\mu} \frac{\partial \hat{\Gamma}^\lambda_{\nu}}{\partial x^\nu} (\hat{F}_\xi(\hat{\Gamma}))_{\kappa\lambda}.
\]

The answer to the question above is partially positive in the sense that both functions \(\gamma^a\) and \(\hat{\gamma}^a\) are given implicitly in general, and many examples show that instead of \((41)\) we always get (see the example in the Introduction)

\[
(\hat{F}_\xi)_{\mu\nu} = \frac{\partial \hat{\Gamma}_{\kappa\lambda}}{\partial x^\mu} \frac{\partial \hat{\Gamma}^\lambda_{\nu}}{\partial x^\nu} (\hat{F}_\xi(\Gamma))_{\kappa\lambda} + \Omega, \quad d\Omega = 0.
\]
How to find functions $\Gamma$ and $\hat{\Gamma}$?

### 3.1 Transformation of coordinates on the initial manifold

Let us start with the case where the manifold $M$ has structure of Lie group but dualization is performed only with respect to a subgroup $G \subset M$. The right action of the subgroup $G$ on the location of the group unit $u_\xi(s)$ in the sub-manifold $\Sigma(s) \subset M$ is then realized by group multiplication in $M$ as

$$x = u_\xi(s) \cdot g, \quad g \in G.$$  \hfill (46)

From this we get the transformation (15) of manifold coordinates $x^\mu$ to the group parameters and spectators $x'^\mu$. Changing the location of the unit element on the orbit $\Sigma(s)$ to $u_\tilde{\xi}(s)$, the same point $x \in \Sigma(s)$ is obtained by a different group element

$$x = u_\tilde{\xi}(s) \cdot \tilde{g}, \quad \tilde{g} \in G.$$  \hfill (47)

Comparing these two equations, we get corresponding change of subgroup elements induced by the change of functions $\xi(s)$

$$g = u_\xi(s)^{-1} \cdot u_\tilde{\xi}(s) \cdot \tilde{g}$$  \hfill (48)

and consequently the transformation of the group parameters $g^a = \gamma^a(s, \tilde{g})$ corresponding to the change of functions $\xi \rightarrow \tilde{\xi}$.

If the manifold has not structure of the Lie group and the right action of the group $G$ is given by a more general map (15), it is not difficult to deduce that transformation of the group $G$ coordinates $g^a = \gamma^a(s, \tilde{g})$ will be obtained from requirement

$$X_\xi^\mu(s^\delta, g^a) = x^\mu = X_{\tilde{\xi}}^\mu(s^\delta, \tilde{g}^b).$$  \hfill (49)

However, in this case differently from (48), transformation $g^a = \gamma^a(s, \tilde{g})$ is given by the equation (19) only implicitly and must be calculated for each $X^\mu$ separately. E.g., the transformation of group coordinates following from (25) is

$$g_1 = \tilde{g}_1 + \log \frac{\xi_1(s)}{\tilde{\xi}_1(s)}, \quad g_2 = \tilde{g}_2 + e^{\tilde{g}_1} \frac{\xi_2(s) - \tilde{\xi}_2(s)}{\xi_1(s)}.$$  \hfill (50)

### 3.2 Transformation of coordinates on the dual manifold

More interesting and more important is transformation of components of the dual tensor. Its form $\tilde{g}^a = \tilde{\gamma}^a(s, \tilde{g})$ can be obtained from dual decomposition of elements of the Drinfel’d double

$$l = g \cdot \hat{h} = \tilde{g} \cdot \hat{h}, \quad g, h \in G, \quad \hat{h}, \tilde{g} \in \hat{G}.$$  \hfill (51)

This formula can be used for solution of the equations of motion of the $\sigma$–model in the dual tensor field [8] but here we will use it for finding the coordinate transformation in the dual space.
To find the dual decomposition for given \( g, \hat{h} \) is rather complicated problem in general but its solution simplifies substantially in case of (non)-Abelian T–duality where the dual group \( \hat{G} \) is Abelian. In this case we can use a representation \( r \) of an element of the semi-Abelian Drinfel’d double in terms of block matrices \((\dim G + 1) \times (\dim G + 1)\) with

\[
\begin{align*}
  r(g) &= \begin{pmatrix} Adg & 0 \\ 0 & 1 \end{pmatrix}, \\
  r(\hat{h}) &= \begin{pmatrix} 1_G & 0 \\ \vec{v}(\hat{h}) & 1 \end{pmatrix}
\end{align*}
\]

where \( \vec{v}(\hat{h}) = (\hat{h}^1, \ldots, \hat{h}^{\dim G}) \), \( h^j \) being parameters of the (Abelian) group element

\[
\hat{h} = e^{\hat{h}^1 T_1} e^{\hat{h}^2 T_2} \cdots e^{\hat{h}^{\dim G} T_{\dim G}}.
\]

From the equation (51) we then get

\[
\begin{align*}
  r(l) &= r(g \hat{h}) = \begin{pmatrix} Adg & 0 \\ \vec{v}(\hat{h}) & 1 \end{pmatrix} = r(\hat{g} h) = \begin{pmatrix} Ad h \vec{v}(\tilde{h}) & 0 \\ (Ad h) \vec{v}(\tilde{h}) & 1 \end{pmatrix}.
\end{align*}
\]

If the adjoint representation of the Lie algebra \( g \) is faithful then the representation \( r \) of the Drinfel’d double is faithful as well and the relation (54) immediately gives \( g = h \). If not, we can use formula

\[
e^A e^B = e^{(Ad A) B} e^A
\]

(55)

to permute the elements of \( G \) and \( \hat{G} \) in (51) and again we get \( g = h \). From the decomposition (51) we then get

\[
\hat{h} = g^{-1} \cdot \tilde{g} \cdot g.
\]

(56)

Similarly for \( \tilde{g}, \bar{g} \) we get

\[
\hat{h} = \tilde{g}^{-1} \cdot \bar{g} \cdot \tilde{g}, \quad \tilde{g} \in G, \quad \hat{h}, \bar{g} \in \hat{G}.
\]

(57)

Comparing (56) and (57), we get transition \( \tilde{g} \mapsto \hat{g} \)

\[
\hat{g} = g \cdot \tilde{g}^{-1} \cdot \bar{g} \cdot \tilde{g} \cdot g^{-1}
\]

(58)

corresponding to \( \xi \mapsto \tilde{\xi} \).

For manifolds with group structure one can see from (48) that

\[
\tilde{g} \cdot g^{-1} = u_\xi(s)^{-1} \cdot u_\xi(s)
\]

(59)

so that the relation between \( \tilde{g} \) and \( \hat{g} \) depends only on \( \xi^\mu(s) \) and \( \tilde{\xi}^{\mu}(s) \). By expressing (58) in group parameters, we get the function \( \hat{\gamma}^\mu(s, \tilde{g}) \) that transform dual tensor fields \( \tilde{F}_{\tilde{\xi}} \) to \( \tilde{F}_\xi \) by (55). For general case where relation between \( g \) and \( \hat{g} \) is given implicitly by (59), function \( \hat{\gamma}^\mu(s, \tilde{g}) \) again depends only on \( \xi^\mu(s) \) and \( \tilde{\xi}^{\mu}(s) \) since from (16) we get

\[
u_\xi(s) = u_\tilde{\xi}(s) \cdot (\tilde{g} \cdot g^{-1})
\]

(60)
so that solution of this equation for $\hat{g} \cdot g^{-1}$ must depend only on $\xi^\mu(s)$ and $\tilde{\xi}^\mu(s)$. E.g., the transformation of the dual group coordinates in the example given above is
\[
\hat{g}_1 = g_1 + g_2 \frac{\tilde{\xi}_2(s) - \xi_2(s)}{\tilde{\xi}_1(s)}, \quad \hat{g}_2 = g_2 \frac{\xi_1(s)}{\tilde{\xi}_1(s)}
\]
and
\[
\Omega = ds \wedge d\tilde{g}_1 \left( \frac{\xi_1(s)}{\tilde{\xi}_1(s)} - \frac{\tilde{\xi}_1(s)}{\xi_1(s)} \right) + ds \wedge d\tilde{g}_2 \frac{\xi_1(s)(\tilde{\xi}_2(s) - \xi_2(s)) + \xi_2(s)(\tilde{\xi}_1(s) - \xi_1(s))}{\xi_1(s)\tilde{\xi}_1(s)}.\tag{62}
\]

4 Conclusion

We have shown that both initial and dual tensors $F$ and $\hat{F}$ defining $\sigma$–models expressed in adapted coordinates may depend on functions of spectators $\xi^\mu$ that are arbitrary up to conditions (17) and (37). Corresponding choices of sub-manifolds (curves, surfaces, etc.) representing the choices of location of the isometry group unit in the invariant sub-manifolds can be to large extent compensated by coordinate transformation. We have given the formulas for the corresponding coordinate transformations.

Let us note that when deriving transition $\hat{g}^a \rightarrow \tilde{g}^a$ transforming components of dual tensor field $\hat{F}$ in various adapted coordinates, we have substantially used the fact that the Drinfel’d double is semi-Abelian. Therefore, it is not clear whether this transformation can be found for general Poisson–Lie T–duality or Poisson–Lie T–plurality with spectators and, especially, how to find $\hat{\gamma}^a$ in these general cases.

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