Weak Insertion of a Perfectly Continuous Function

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Abstract: A sufficient condition in terms of lower cut sets are given for the insertion of a perfectly continuous function between two comparable real-valued functions on such topological spaces that \(A\)-sets are open.

1. INTRODUCTION

A generalized class of closed sets was considered by Maki in 1986 [10]. He investigated the sets that can be represented as union of closed sets and called them \(V\)-sets. Complements of \(V\)-sets, i.e., sets that are intersection of open sets are called \(\Lambda\)-sets [10].

Recall that a real-valued function \(f\) defined on a topological space \(X\) is called \(A\)-continuous [15] if the preimage of every open subset of \(\mathbb{R}\) belongs to \(A\), where \(A\) is a collection of subset of \(X\). Most of the definitions of function used throughout this paper are consequences of the definition of \(A\)-continuity. However, for unknown concepts the reader may refer to [2, 5].

Hence, a real-valued function \(f\) defined on a topological space \(X\) is called perfectly continuous [14] (resp. contra-continuous [3]) if the preimage of every open subset of \(\mathbb{R}\) is a clopen (i.e., open and closed) (resp. closed) subset of \(X\).

We have a function is perfectly continuous if and only if it is continuous and contra-continuous.

Results of Katetov [6, 7] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [1], are used in order to give a necessary and sufficient conditions for the insertion of a perfectly continuous function between two comparable realvalued functions on the topological spaces that \(A\)-sets are open [10].

If \(g\) and \(f\) are real-valued functions defined on a space \(X\), we write \(g \leq f\) in case \(g(x) \leq f(x)\) for all \(x\) in \(X\).

The following definitions are modifications of conditions considered in [8].

A property \(P\) defined relative to a real-valued function on a topological space is a \(pc\)-property provided that any constant function has property \(P\) and provided that the sum of a function with property \(P\) and any perfectly continuous function also has property \(P\). If \(P_1\) and \(P_2\) are \(pc\)-property, the following terminology is used: A space \(X\) has the weak \(pc\)-insertion property for \((P_1,P_2)\) if and only if for any functions \(g\) and \(f\) on \(X\) such that \(g \leq f\), \(g\) has property \(P_1\) and \(f\) has property \(P_2\), then there exists a perfectly continuous function \(h\) such that \(g \leq h \leq f\).

In this paper, is given a sufficient condition for the weak \(pc\)-insertion property. Also, several insertion theorems are obtained as corollaries of these results. In addition, the insertion and strong insertion of a contracontinuous function between two comparable contra-precontinuous (contrasemi-continuous) functions have also recently considered by the author in [11, 12].

2. THE MAIN RESULT

Before giving a sufficient condition for insertability of a perfectly continuous function, the necessary definitions and terminology are stated.

Let \((X,\tau)\) be a topological space, the family of all open, closed and clopen will be denoted by \(O(X,\tau)\), \(C(X,\tau)\) and \(Clo(X,\tau)\), respectively.

Definition 2.1. Let \(A\) be a subset of a topological space \((X,\tau)\). We define the subsets \(A^A\) and \(A^V\) as follows:
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\[ A^\Lambda = \cap \{ O : O \supseteq A, O \subseteq O(X,\tau) \} \] and \[ A^V = \cup \{ F : F \subseteq A, F \in C(X,\tau) \} \].

In [4, 9, 13], \( A^\Lambda \) is called the *kernel* of \( A \).

**Definition 2.2.** Let \( A \) be a subset of a topological space \((X,\tau)\). Respectively, we define the *closure*, *interior*, *clo-closure* and *clo-interior* of a set \( A \), denoted by \( Cl(A), Int(A), clo(Cl(A)) \) and \( clo(Int(A)) \) as follows:

\[
Cl(A) = \cap \{ F : F \supseteq A, F \in C(X,\tau) \}, \quad Int(A) = \cup \{ O : O \subseteq A, O \in O(X,\tau) \}, \quad clo(Cl(A)) = \cap \{ F : F \supseteq A, F \in Clo(X,\tau) \} \text{ and } clo(Int(A)) = \cup \{ O : O \subseteq A, O \in Clo(X,\tau) \}.
\]

If \((X,\tau)\) be a topological space whose \( \Lambda \)-sets are open, then respectively, we have \( A^V, clo(Cl(A)) \) are closed, clopen and \( A^\Lambda, clo(Int(A)) \) are open, clopen.

The following first two definitions are modifications of conditions considered in [6, 7].

**Definition 2.3.** If \( \rho \) is a binary relation in a set \( S \) then \( \rho^- \) is defined as follows: \( x \rho^- y \) if and only if \( y \rho \) implies \( x \rho \) and \( u \rho x \) implies \( u \rho y \) for any \( u \) and \( v \) in \( S \).

**Definition 2.4.** A binary relation \( \rho \) in the power set \( P(X) \) of a topological space \( X \) is called a *strong binary relation* in \( P(X) \) in case \( \rho \) satisfies each of the following conditions:

- If \( A, \rho B_i \) for any \( i \in \{1,...,m\} \) and for any \( j \in \{1,...,n\} \), then there exists a set \( C \in P(X) \) such that \( A, \rho C \) and \( C, \rho B_j \) for any \( i \in \{1,...,m\} \) and any \( j \in \{1,...,n\} \).
- If \( A \subseteq B \), then \( A \rho^- B \).
- If \( A \rho B \), then \( clo(Cl(A)) \subseteq B \) and \( A \subseteq clo(Int(B)) \).

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [1] as follows:

**Definition 2.5.** If \( f \) is a real-valued function defined on a space \( X \) and if \( \{ x \in X : f(x) < \ell \} \subseteq A(f,\ell) \subseteq \{ x \in X : f(x) \leq \ell \} \) for a real number \( \ell \), then \( A(f,\ell) \) is called a *lower indefinite cut set* in the domain of \( f \) at the level \( \ell \).

We now give the following main result:

**Theorem 2.1.** Let \( g \) and \( f \) be real-valued functions on a topological space \( X \), in which \( \Lambda \)-sets are open, with \( g \leq f \). If there exists a strong binary relation \( \rho \) on the power set of \( X \) and if there exist lower indefinite cut sets \( A(f,t) \) and \( A(g,t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f,t_1) \rho A(g,t_2) \), then there exists a perfectly continuous function \( h \) defined on \( X \) such that \( g \leq h \leq f \). **Proof.** Let \( g \) and \( f \) be real-valued functions defined on \( X \) such that \( g \leq f \). By hypothesis there exists a strong binary relation \( \rho \) on the power set of \( X \) and there exist lower indefinite cut sets \( A(f,t) \) and \( A(g,t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f,t_1) \rho A(g,t_2) \).

Define functions \( F \) and \( G \) mapping the rational numbers \( Q \) into the power set of \( X \) by \( F(t) = A(f,t) \) and \( G(t) = A(g,t) \). If \( t_1 \) and \( t_2 \) are any elements of \( Q \) with \( t_1 < t_2 \), then \( F(t_1) \rho^- F(t_2), G(t_1) \rho^- G(t_2), \) and \( F(t_1) \rho G(t_2) \). By Lemmas 1 and 2 of [7] it follows that there exists a function \( H \) mapping \( Q \) into the power set of \( X \) such that if \( t_1 \) and \( t_2 \) are any rational numbers with \( t_1 < t_2 \), then \( F(t_1) \rho H(t_2), H(t_1) \rho H(t_2) \) and \( H(t_1) \rho G(t_2) \).

For any \( x \in X \), let \( h(x) = \inf \{ t \in Q : x \in H(t) \} \).

We first verify that \( g \leq h \leq f \). If \( x \) is in \( H(t) \) then \( x \) is in \( G(t) \) for any \( t > t \); since \( x \) is in \( G(t) = A(g,t) \) implies that \( g(x) \leq t \), it follows that \( g(x) \leq t \). Hence \( g \leq h \). If \( x \) is not in \( H(t) \), then \( x \) is not in \( F(t) \) for any \( t > t \); since \( x \) is not in \( F(t) = A(f,t) \) implies that \( f(x) > t \), it follows that \( f(x) > t \). Hence \( h \leq f \).

Also, for any rational numbers \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \), we have \( h^{-1}(t_1,t_2) = clo(Int(H(t_2))) \setminus clo(Cl(H(t_1))) \). Hence \( h^{-1}(t_1,t_2) \) is a clopen subset of \( X \), i. e., \( h \) is a perfectly continuous function on \( X \).

The above proof used the technique of proof of Theorem 1 of [6].

3. Applications

The abbreviations \( c, pc \) and \( cc \) are used for continuous, perfectly continuous and contra-continuous, respectively.
Before stating the consequences of theorems 2.1, we suppose that $X$ is a topological space that $\Lambda$–sets are open.

**Corollary 3.1.** If for each pair of disjoint closed (resp. open) sets $F_1,F_2$ of $X$, there exist clopen sets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then $X$ has the weak $pc$–insertion property for $(c,c)$ (resp. $(cc,cc)$).

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$ are $c$ (resp. $cc$), and $g \leq f$. If a binary relation $\rho$ is defined by $A \rho B$ in case $Cl(A) \subseteq Int(B)$ (resp. $A^\alpha \subseteq B^\beta$), then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $Q$ with $t_1 < t_2$, then
\[ A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2) ; \]

since $\{x \in X : f(x) \leq t_1\}$ is a closed (resp. open) set and since $\{x \in X : g(x) < t_2\}$ is an open (resp. closed) set, it follows that $Cl(A(f,t_1)) \subseteq Int(A(g,t_2))$ (resp. $A(f,t_1)^\alpha \subseteq A(g,t_2)^\beta$). Hence $t_1 < t_2$ implies that $A(f,t_1) \rho A(g,t_2)$. The proof follows from Theorem 2.1.

**Corollary 3.2.** If for each pair of disjoint closed (resp. open) sets $F_1,F_2$, there exist clopen sets $G_1$ and $G_2$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then every continuous (resp. contra-continuous) function is perfectly continuous.

**Proof.** Let $f$ be a real-valued continuous (resp. contra-continuous) function defined on the $X$. By setting $g = f$, then by Corollary 3.1, there exists a perfectly continuous function $h$ such that $g = h = f$.

**Corollary 3.3.** If for each pair of disjoint closed (resp. open) sets $F_1,F_2$ of $X$, there exist clopen sets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then $X$ has the $pc$–insertion property for $(c,c)$ (resp. $(cc,cc)$).

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$ are $c$ (resp. $cc$), and $g < f$. Set $h = (f + g)/2$, thus $g < h < f$, and by Corollary 3.2, since $g$ and $f$ are perfectly continuous functions hence $h$ is a perfectly continuous function.

**Corollary 3.4.** If for each pair of disjoint subsets $F_1,F_2$ of $X$, such that $F_1$ is closed and $F_2$ is open, there exist clopen subsets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then $X$ have the weak $pc$–insertion property for $(c,cc)$ and $(cc,c)$.

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $g$ is $c$ (resp. $cc$) and $f$ is $cc$ (resp. $c$), with $g \leq f$. If a binary relation $\rho$ is defined by $A \rho B$ in case $A^\alpha \subseteq Int(B)$ (resp. $Cl(A) \subseteq B^\beta$), then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $Q$ with $t_1 < t_2$, then
\[ A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2) ; \]

since $\{x \in X : f(x) \leq t_1\}$ is an open (resp. closed) set and since $\{x \in X : g(x) < t_2\}$ is an open (resp. closed) set, it follows that $A(f,t_1)^\alpha \subseteq Int(A(g,t_2))$ (resp. $Cl(A(f,t_1)) \subseteq A(g,t_2)^\beta$). Hence $t_1 < t_2$ implies that
\[ A(f,t_1) \rho A(g,t_2) . \] The proof follows from Theorem 2.1.

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