Splitter Theorems for Graph Immersions

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Abstract

We establish splitter theorems for graph immersions for two families of graphs, $k$-edge-connected graphs, with $k$ even, and 3-edge-connected, internally 4-edge-connected graphs. As a corollary, we prove that every 3-edge-connected, internally 4-edge-connected graph on at least seven vertices that immerses $K_5$ also has $K_{3,3}$ as an immersion.

Keywords: reduction theorem, splitter theorem, graph immersion, edge-connectivity

1 Introduction

Throughout the paper, we use standard definitions and notation for graphs as in [17]. Let $G$ be a graph with a certain connectivity. One natural question is that whether there is a way to “reduce” $G$ while preserving the same connectivity, and possibly also the presence of a particular graph “contained” in $G$. Broadly speaking, in answering such questions, two types of theorems arise. In the first type, chain theorems, one tries to “reduce” the graph down to some basic starting point, which is typically a particular small graph, or a small family of graphs. The other type of theorems are splitter theorems. Here, there is the extra information that another graph $H$ is properly “contained” in $G$, and both have a certain connectivity. The idea is then to “reduce” $G$ to a graph “one step smaller”, while preserving the connectivity, and the “containment” of $H$.

The best known such results are the ones in the world where the connectivity concerned is vertex-connectivity, the “reduction” is an edge-contraction
or edge-deletion, and the “containment” relation is that of minor. In this realm, the first chain result is due to Tutte, who showed if a graph $G$ is 2-connected, for every edge $e \in E(G)$, either $G \setminus e$ or $G/e$ is 2-connected. The next result, also due to Tutte, is a classical result of a reduction theorem of chain variety. Here, a wheel is a graph formed by connecting a single vertex to all vertices of a cycle.

**Theorem 1.1 (Tutte [15])** If $G$ is a simple 3-connected graph, then there exists $e \in E(G)$ such that either $G \setminus e$ or $G/e$ is simple and 3-connected, unless $G$ is a wheel.

Another classical result of chain type is that every simple 3-connected graph other than $K_4$ has an edge whose contraction results in a 3-connected graph, see [8]. There is a wide body of literature sharpening these results and extending them to other connectivity, see, for instance [9, 1].

Reduction theorems of splitter variety for graph minors started with a result for 2-connected graphs, independently discovered by Brylawski [2], and Seymour [13]. This result asserts that if $G, H$ are 2-connected graphs, and $H$ is a proper minor of $G$, then there is an edge $e \in E(G)$ such that $G \setminus e$ or $G/e$ is 2-connected, and has $H$ as a minor. The more famous splitter theorem is Seymour’s Splitter Theorem for 3-connected graphs, which asserts:

**Theorem 1.2 (Seymour [14])** Let $G, H$ be 3-connected simple graphs, where $H$ is a proper minor of $G$, and $|E(H)| \geq 4$. Also, suppose if $H$ is a wheel, then $G$ has no larger wheel minor. Then $G$ has an edge $e$ such that either $G \setminus e$ or $G/e$ is simple and 3-connected, and contains $H$ as a minor.

There is an extremely wide body of literature extending these results to other connectivity and the realm of matroids, and binary matroids, see, for example, [12, 3, 4].

In this paper, however, we are not concerned with vertex-connectivity and minors, but rather the world of edge-connectivity and a less-explored type of containment, immersion. A pair of distinct edges $xy, yz$ with a common neighbour is said to split off at $y$ if we delete these edges and add a new edge $xz$. We say a graph $G$ immerses $H$, or has an $H$-immersion, and write $G \succ im H$, if a subgraph of $G$ can be transformed to a graph isomorphic to $H$ through a series of splitting pairs of edges.\footnote{This is sometimes called weak immersion.}
a vertex $v \in V(G)$ of even degree is completely split if $d(v)/2$ consecutive splits are performed at $v$, and then the resulting isolated vertex $v$ is deleted.

In the world of edge-connectivity, and immersions, there is a chain theorem due to Lovász ([10], Problem 6.53, see also [5]).

**Theorem 1.3 (Lovász [10])** Suppose $G$ is 2k-edge-connected. Then by repeatedly applying complete split, and edge-deletion it can be reduced to a graph on two vertices, with 2k parallel edges between them.

This theorem was later generalized by a significant theorem of Mader that is key in our proofs, and will be stated in section [2]. The goal of this paper is to establish two splitter theorems for immersions, the first of which is an analogue of the aforementioned result of Lovász.

**Theorem 1.4** Suppose $G \not\cong H$ are 2k-edge-connected loopless graphs, and $G \succ_{im} H$. Then there exists an operation taking $G$ to $G'$ so that $G'$ is 2k-edge-connected and $G' \succ_{im} H$, where an operation is either

- deleting an edge,
- splitting at a vertex of degree $\geq 2k + 2$,
- completely splitting a 2k-vertex,

each followed by iteratively deleting any loops, and suppressing vertices of degree 2.

In comparison with graph minors, the literature on splitter theorems for graph immersions is extremely sparse. Indeed, we only know of two significant papers concerning this, namely [5, 6], where Ding and Kanno have proved a handful of splitter theorems for immersion for cubic graphs, and 4-regular graphs. In particular, they have shown the following (see [6], Theorem 9):

**Theorem 1.5 (Ding, Kanno[6])** Suppose $G \not\cong H$ are 4-edge-connected 4-regular loopless graphs, and $G \succ_{im} H$. Then there exists a vertex whose complete split takes $G$ to $G'$ so that $G'$ is 4-edge-connected 4-regular, and $G' \succ_{im} H$. 


Our second theorem, stated below, is similar to the first one, but is for a different type of connectivity, and generalizes Theorem 1.5. Here, $G$ is said to be *internally $k$-edge-connected* if every edge-cut containing less than $k$ edges is the set of edges incident with a single vertex. Also, in the statement of the theorem, $Q_3$ denotes the graph of the cube, and $K^3_2$ denotes the graph on two vertices with three parallel edges between them.

**Theorem 1.6** Let $G \not\cong H$ be 3-edge-connected, and internally 4-edge connected loopless graphs, with $G \cong im H$. Further, assume $|V(H)| \geq 2$, and $(G,H) \not\cong (Q_3,K_4), (Q_3,K^3_2)$. Then there exists an operation taking $G$ to $G'$ such that $G'$ is 3-edge connected, internally 4-edge connected, and $G' \cong im H$, where an operation is either

- deleting an edge,
- splitting at a vertex of degree $\geq 4$,

each followed by iteratively deleting any loops, and suppressing vertices of degree 2.

In the world of graph minors, an immediate simple consequence of Seymour’s Splitter Theorem, first observed by Wagner[16], is that every 3-connected graph on at least six vertices containing $K_5$ as a minor, has a $K_{3,3}$-minor. This fact is then used to obtain a precise structural description of graphs with no $K_{3,3}$-minor. In parallel to this, and as an application of Theorem 1.6, we will establish the following analogue of this result for immersions. The result will be a step towards understanding graphs with no $K_5$-immersion.

**Corollary 1.7** Suppose $G$ is 3-edge-connected, and internally 4-edge-connected, where $G \cong im K_5$. Then

1. if $|V(G)| \geq 6$ then $G \cong K_{3,3}$, or $G \cong K_{2,2,2}$.
2. If $|V(G)| \geq 7$ then $G \cong K_{3,3}$.

The rest of the paper is organized as follows: In section 2, we state the preliminary definitions and key tools, and prove Theorem 1.4. Section 3 is dedicated to the family of 3-edge-connected, internally 4-edge-connected graphs, and includes the proof of Theorem 1.6 and Corollary 1.7.
2 k-edge-connected graphs, k even

We will assume the graphs are undirected and finite, which may have loops or parallel edges. For \(X \subset V(G)\), we use \(\delta_G(X)\) to denote the edge-cut consisting of all edges of \(G\) with exactly one endpoint in \(X\), the number of which is called the size of this edge-cut, and is denoted by \(d_G(X)\). When \(G\) is connected we refer to both \(X\) and \(X^c(= V(G) \setminus X)\) as sides of the edge-cut \(\delta(X)\). An edge-cut is called trivial if at least one side of the cut consists of only one vertex. Note that graph is \(k\)-edge-connected (internally \(k\)-edge-connected) if every edge-cut (non-trivial edge-cut) has size \(\ge k\). For distinct vertices \(x,y \in V(G)\), we let \(\lambda_G(x,y)\) denote the maximum size of a collection of pairwise edge-disjoint paths between \(x\) and \(y\). Whenever the graph concerned is clear from the context, we may drop the subscript \(G\). In Section 1 the notion of graph immersions was introduced. Equivalently, we could say \(H\) is immersed in \(G\) if there is a one-to-one mapping \(\phi : V(H) \to V(G)\) and a \(\phi(u) - \phi(v)\)-path \(P_{uv}\) in \(G\) corresponding to every edge \(uv\) in \(H\) so that \(P_{uv}\) paths are pairwise edge-disjoint. In this case, a vertex in \(\phi(V(H))\) is called a terminal of \(H\)-immersion.\footnote{It is worth mentioning that if \(G \succ_{im} H\) and the collection of paths \(P_{uv}\) are internally disjoint from \(\phi(V(H))\), it is standard in the literature to say \(G\) strongly immerses \(H\). It would be then in contrast with the notion of weak immersion, where \(P_{uv}\) paths are not necessarily internally disjoint from \(\phi(V(H))\). However, we are only studying weak immersion and, for the sake of simplicity, refer to it as immersion.}

We proceed by listing a couple facts and theorems which will feature in our proofs. The first is the observation below.

**Observation 2.1** Suppose \(G\) is a graph, and \(X \neq Y\) are distinct nonempty subsets of \(V(G)\). Then, by counting the edges contributing to the edge-cuts, we have

\[
d(X \cap Y) + d(X \cup Y) + 2e(X^c \cap Y, X \cap Y^c) = d(X) + d(Y).
\]

Observe it also implies the following inequality

\[
d(X \cap Y) + d(X \cup Y) \le d(X) + d(Y).
\]

Another frequently used fact is the classical theorem of Menger. A proof may be found, for instance, in \[17\].
Theorem 2.2 (Menger) Let $G$ be a graph, and $x, y$ distinct vertices of $G$. Then $\lambda_G(x, y)$ equals the minimum size of an edge-cut of $G$ separating $x$ from $y$.

The next theorem, due to Mader, is an extremely powerful tool when working with immersions, and is also a key ingredient in our proofs.

Theorem 2.3 (Mader [11], see also Frank [7]) Suppose for $s \in V(G)$ we have $d(s) \neq 3$, and $s$ is not incident with a cut-edge. Then there is a split at $s$ such that in the resulting graph $G'$, for any $x, y \in V(G')$ other than $s$, we have $\lambda_G(x, y) = \lambda_{G'}(x, y)$.

Throughout the rest of this section, we assume $k$ is even. The main result of this section is Theorem 1.4 which is restated below for convenience.

Theorem 2.4 Suppose $G \not\cong H$ are $k$-edge-connected loopless graphs, and $G \succ_{im} H$. Then there exists an operation taking $G$ to $G'$ so that $G'$ is $k$-edge-connected and $G' \succ_{im} H$, where an operation is either

- deleting an edge,
- splitting at a vertex of degree $\geq k + 2$,
- completely splitting a $k$-vertex,

each followed by iteratively deleting any loops, and suppressing vertices of degree 2.

Note that in order to have a splitter theorem for the family of $k$-edge-connected graphs, we do need to embrace completely splitting a $k$-vertex as one of our operations, since as soon as we do a split at a $k$-vertex, the graph will have a trivial $(k - 2)$-edge-cut.

Theorem 2.4 will be proved through a series of lemmas. We will begin by introducing a few definitions.

Definition. A graph $G$ is called nearly $k$-edge-connected if either $G$ is $k$-edge-connected, or there exists a single vertex, $u$, called the special vertex, of even degree $< k$ so that every nonempty edge-cut in $G$ apart from $\delta(u)$ has size at least $k$. 

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Definition. Suppose \( G \) is (nearly) \( k \)-edge-connected, and \( H \) a \( k \)-edge-connected graph, with \( G \succ im H \). We define a good operation to be either a split, a complete split for either a vertex of degree \( k \) or the special vertex, or a deletion of an edge from \( G \) which preserves (nearly) \( k \)-edge-connectivity of \( G \) (for the same special vertex), and an immersion of \( H \) in the resulting graph.

Note that the theorem can now be restated as follows: If we can step down from \( G \) towards \( H \) doing each of the three operations, then there is a good operation.

Throughout the rest of this section, we will assume that \( H \) is \( k \)-edge-connected, with \( k \) even, and \( G \succ im H \).

Lemma 2.5 Suppose \( G \) is (nearly) \( k \)-edge-connected, and there exists \( X \subset V(G) \) such that \( d(X) = k \), and every \( x \in X \) is of degree \( k + 1 \). Then there exists an edge lying in \( X \) such that \( G \setminus e \) is (nearly) \( k \)-edge-connected.

Proof. Choose \( X' \subseteq X \) such that \( d(X') = k \), and subject to this \( X' \) is minimal. Since every \( x \in X' \) has degree \( k + 1 \), we have \( |X'| \neq 1 \), and \( X' \) does not contain the special vertex (if existent at all). Also \( X' \) must be connected, so there exists an edge \( e \in X' \). We will show that \( G \setminus e \) is nearly \( k \)-edge-connected. For a contradiction, suppose \( e \) is in a \( k \)-edge cut, \( \delta(Y) \).

Note that \( d(X'^c) = d(X') = k \) implies that \( X'^c \) contains at least one vertex rather than the special vertex. We may assume (by possibly replacing \( Y \) by \( Y^c \)) it is in \( Y^c \). As \( G \) is (nearly) \( k \)-edge-connected, \( d(X' \cap Y), d(X'^c \cap Y^c) \geq k \). However, it follows from

\[
k + k \leq d(X' \cap Y) + d(X'^c \cap Y^c) \leq d(X') + d(Y) = k + k
\]

that \( d(X' \cap Y) = k \), which contradicts minimality of \( X' \). \( \square \)

Notation. If \( G \) is a graph with \( X \subset V(G) \), we will denote the graph obtained from identifying \( X \) to a single node by \( G.X \).

Observation 2.6 Suppose \( G \) is a graph with \( X \subset V(G) \) such that there exists an immersion of \( H \) with all terminals in \( X \). Then \( G.X^c \) contains \( H \) as an immersion.

The following lemma enables us to handle \( k \)-edge-cuts in (nearly) \( k \)-edge-connected graphs:
Lemma 2.7 If $G$ is a (nearly) $k$-edge-connected graph with a nontrivial edge-cut $\delta(X)$ of size $k$ such that some immersion of $H$ has no terminal in $X$, then there exists a good operation.

Proof. If the special vertex is in $X$, we will split it using Mader’s Theorem (Theorem 2.3). Else, if there exists a vertex $x \in X$ of degree $\neq k+1$, Mader’s Theorem may be applied to either split (if $d(x) \geq k+2$) or completely split $x$ (if $d(x) = k$). Call the graph resulting from applying Mader’s Theorem $G'$.

To see that the operation is good, observe that there remain $k$ edge-disjoint paths between any pair of nonspecial vertices, one in $X$, and the other in $X^c$ (as it was the case in $G$). Therefore $G'$ immerses $G.X$, thus, immerses $H$.

Now suppose every vertex in $X$ is of degree $k+1$. Applying Lemma 2.5 we can delete an edge from $X$ preserving (nearly) $k$-edge-connectivity. Also, the same argument as above shows that the resulting graph still has $H$ as an immersion, thus the deletion is indeed a good operation.

The next two lemmas which concern a broader family of graphs, will later be helpful dealing with $(k+1)$-edge-cuts in (nearly) $k$-edge-connected graphs.

Lemma 2.8 Let $G$ be an internally $k$-edge-connected graph in which every vertex of degree $< k$ is of even degree. If $d(x)$ is odd, then there exists $y \in V(G) \setminus x$ such that $\lambda(x,y) \geq k+1$.

Proof. We prove the statement by induction on $|V(G)|$. Note that by parity, there must exist another vertex of odd degree, $y$, in $G$. If every cut separating $x$ from $y$ is of size $\geq k+1$, by Menger’s Theorem (Theorem 2.2) we are done. Otherwise, there exists a $k$-edge-cut $\delta(Y)$, with $y \in Y$, separating $x$ from $y$.

Note that degree properties imply that $|Y| \geq 2$, so the graph $G' = G.Y$, which satisfies the lemma’s hypothesis, has fewer vertices than $G$. Also $x$ is of odd degree in $G'$ as well, thus, by induction hypothesis there exists $y' \in V(G') \setminus x$ such that $\lambda_{G'}(x,y') \geq k+1$. It follows, however, that $\lambda_{G}(x,y') \geq k+1$ as well, since $\lambda_{G}(x,y) = k$ implies that $G \succeq_{im} G'$.

Lemma 2.9 Let $G$ be an internally $k$-edge-connected graph in which every vertex of degree $< k$ is of even degree. If $\delta(X)$ is a $(k+1)$-edge-cut in $G$, there exist $x \in X, y \in X^c$ such that $\lambda(x,y) \geq k+1$.
Proof. Let \( G_1 = G.X, G_2 = G.X^c \), with \( s, t \) being the nodes replacing \( X, X^c \), respectively. Note that both \( G_1, G_2 \) satisfy Lemma 2.8's hypothesis. Also, \( s \) is a vertex of odd degree in \( G_1 \), so, by Lemma 2.8, there exists \( y \in X^c \) such that \( \lambda_{G_1}(s,y) \geq k + 1 \), thus \( G \succ im G_2 \). It can be similarly argued that there exists \( x \in X \) such that \( \lambda_{G_2}(x,t) \geq k + 1 \), which together with \( G \succ im G_2 \) shows that \( \lambda_G(x,y) \geq k + 1 \). □

Having the lemma above in hand, we can now efficiently handle \((k+1)\)-edge-cuts:

**Lemma 2.10** If \( G \) is a (nearly) \( k \)-edge-connected graph with a nontrivial \((k+1)\)-edge-cut \( \delta(X) \) such that some immersion of \( H \) has no terminal in \( X \), then there exists a good operation.

**Proof.** If the special vertex is in \( X \), we will split it using Mader’s Theorem. Else, if there exists a vertex in \( X \) of degree \( \neq k + 1 \), we will apply Mader’s Theorem to either split or completely split it. We claim this operation is good. Let \( G' \) be the resulting (nearly) \( k \)-edge-connected graph. First, note that \( \delta(X) \) remains a \((k+1)\)-edge-cut in \( G' \), since doing a split changes the size of an edge-cut by an even number, and, by the edge-connectivity of \( G' \), \( d_{G'}(X) \geq k \). We may now apply Lemma 2.9 to choose \( x \in X, y \in X^c \) with \( \lambda(x,y) \geq k + 1 \). Thus \( G' \) immerses \( G.X \), and therefore, immerses \( H \).

Now, suppose every vertex in \( X \) is of degree \( k + 1 \), and take an edge \( e \in X \), which has to exist because \( |X| \geq 2 \), and \( X \) must be connected. If \( G \setminus e \) is nearly \( k \)-edge-connected, then the same argument as above shows that deletion of \( e \) is a good operation. So, we may now assume that \( e \) is in a \( k \)-edge-cut, \( \delta(Y) \).

**Remark.** Let \( Z \subset V(G), Z = Z_1 \cup Z_2, Z_1 \cap Z_2 = \emptyset, \) and denote the number of edges from \( Z_1 \) to \( Z_2 \) by \( e(Z_1, Z_2) \). Then we have

\[
\quad d(Z) = d(Z_1) + d(Z_2) - 2e(Z_1, Z_2). \quad (\star)
\]

Using (\( \star \)), by possibly replacing \( Y \) with \( Y^c \), we may assume that \( d(X \cap Y^c) \) is even and \( d(X \cap Y) \) is odd. Also, by (\( \star \)), we conclude that \( d(X^c \cap Y) \) is odd, and so \( X^c \cap Y \) is nonempty. Thus, both \( X \cap Y^c \) and \( X^c \cap Y \) contain a non special vertex. We also have

\[
2k + 1 = d(X) + d(Y) \geq d(X \cap Y^c) + d(X^c \cap Y) \geq 2k,
\]
so by parity \(d(X \cap Y^c) = k\). Therefore, \(\delta(X \cap Y^c)\) is a nontrivial (as every vertex in \(X\) is of degree \(k + 1\)) \(k\)-edge-cut with no terminal of \(H\) in \(X \cap Y^c\). Applying Lemma 2.7, we may conclude that a good operation exists. \(\square\)

The next three lemmas concern the three operations allowed in stepping from \(G\) towards \(H\), which are \(k\)-edge-connected graphs with \(G \succ_{\text{im}} H\), and show that in each case we can take a step maintaining \(k\)-edge-connectivity.

**Lemma 2.11** If \(G \succ_{\text{im}} H\) are \(k\)-edge-connected graphs, and there is a complete split of a \(k\)-vertex \(u\) of \(G\) preserving an \(H\)-immersion, then there is a good operation.

**Proof.** Consider the complete split of \(u\) as \(\frac{k}{2}\) many splits at \(u\), and choose a sequence of splits which, while preserving an \(H\)-immersion, results in the fewest number of loops. If \(u\) could be completely split without ever creating a too small of an edge-cut rather than \(\delta(u)\) along the way, we are done. Otherwise, we will stop doing these splits the first time the resulting graph \(G'\) is about to lose nearly \(k\)-edge-connectivity (with \(u\) being the special vertex).

In \(G'\), therefore, there exists a subset \(X \neq \{u\}, \{u\}^c \) of \(V(G)\) for which \(d_{G'}(X) \geq k\), doing the next split, however, makes it a \(< k\)-edge-cut, so \(d_{G'}(X) = k\) or \(k + 1\). Moreover, since completely splitting \(u\) results in \(d(X) < k\), and preserves an immersion of \(H\), we may conclude that there is an immersion of \(H\) with all terminals on one side of \(\delta(X)\), say \(X^c\).

If \(\delta(X)\) is a nontrivial cut, Lemma 2.7 or 2.10 applied to \(G'\) guarantee the existence of a good operation, which may or may not be a split at \(u\). If it is not a split at \(u\), undoing the splits that took \(G\) to \(G'\) recovers \(k\)-edge-connectivity.

Now suppose \(\delta(X)\) is a trivial cut with \(X = \{v\}\). Therefore the next split at \(u\) would create a loop at \(v\). Note that there cannot be a vertex \(w \in N_{G'}(u) \setminus v\), because if there was one, then we could have split off \(vuw\) instead. It is because splitting \(vuw\) creates no loop while maintaining an immersion of \(H\), as splitting \(wv\) in the graph obtained results in the same graph as splitting off \(vuw\) would.

Therefore \(N_{G'}(u) = \{v\}\), implying that in \(G'\), \(d(v) = d(\{u, v\}) + d(u)\). This, however, contradicts \(d(X) = k\), or \(k + 1\), as \(d(X) = d(v) = d(\{u, v\}) + d(u) \geq k + d(u) \geq k + 2\), where the inequalities hold because \(G'\) is nearly \(k\)-edge-connected, and \(u\) is of even degree. This completes the proof. \(\square\)
Lemma 2.12 If \( G \succ_{im} H \) are \( k \)-edge-connected graphs, and there is an edge \( e \) such that \( G \setminus e \) has an \( H \)-immersion, then a good operation exists.

Proof. Suppose \( e \) is in a \( k \)-edge-cut. If it is incident with a \( k \)-vertex \( u \), then, by previous lemma, \( u \) could be completely split off maintaining an \( H \)-immersion and \( k \)-edge-connectivity. Otherwise, \( e \) is in a nontrivial \( k \)-edge-cut, with all terminals of \( H \) on one side of the cut, thus we can use Lemma 2.7 to find a good operation. \( \square \)

Lemma 2.13 If \( G \succ_{im} H \) are \( k \)-edge connected graphs, and there is a split at a vertex \( v \) preserving an \( H \)-immersion, then a good operation exists.

Proof. Suppose splitting at \( v \) makes an edge-cut \( \delta(X) \) too small, then \( d(X) = k \) or \( k + 1 \). Also, all terminals of \( H \) are on one side of the cut, say \( X^c \). If \( \delta(X) \) is a nontrivial edge-cut Lemma 2.7 or 2.10 may be applied. If \( |X| = 1 \), with \( d(X) = k \), we apply Lemma 2.11 to completely split the vertex in \( X \), and if \( d(X) = k + 1 \) we will apply Lemma 2.12 to delete an edge incident to it. \( \square \)

The proof of Theorem 2.4 is now immediate:

Proof of Theorem 2.4. Apply Lemmas 2.11, 2.12, and 2.13 \( \square \)

3 3-edge-connected, internally 4-edge-connected graphs

In this section we establish Theorem 1.6. Later, as an application, we will see that if a 3-edge-connected, internally 4-edge-connected graph other than \( K_{2, 2, 2} \) immerses \( K_5 \), it also has a \( K_{3,3} \)-immersion. First, we move towards proving Theorem 1.6 which, for convenience is restated here.

Theorem 3.1 Let \( G \not\cong H \) be 3-edge-connected, and internally 4-edge connected loopless graphs, with \( G \succ_{im} H \). Further, assume \( |V(H)| \geq 2 \), and \( (G, H) \not\cong (Q_3, K_4), (Q_3, K^3_3) \). Then there exists an operation taking \( G \) to \( G' \) such that \( G' \) is 3-edge connected, internally 4-edge connected, and \( G' \succ_{im} H \), where an operation is either

- deleting an edge,
- splitting at a vertex of degree \( \geq 4 \),

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each followed by iteratively deleting any loops, and suppressing vertices of degree 2.

As in the proof of Theorem 2.4, we will consider each operation separately, and the proof of the theorem will then be immediate. First, we will adjust our notion of a good operation as follows:

**Definition.** Suppose $G, H$ are 3-edge-connected, and internally 4-edge connected loopless graphs, with $G \succ_{im} H$. We define a good operation to be either a split at a vertex of degree $\geq 4$, or a deletion of an edge from $G$ which preserves 3-edge-connectivity, internal 4-edge-connectivity, and an immersion of $H$ in the resulting graph.

**Lemma 3.2** Suppose $G, H$ are as in Theorem 3.1, and there is an edge $e$ such that $G \setminus e$ has an $H$-immersion. Then if $(G, H) \not\succ (Q_3, K_4), (Q_3, K_3^3)$, a good operation exists.

**Proof.** Since deletion of $e$ is followed by suppression of any resulting vertices of degree two, $G \setminus e$ is clearly 3-edge-connected. If deletion of $e$ does not preserve internal 4-edge-connectivity, then $e$ must be contributing to some 4-edge-cut, $\delta(X)$, in which each side has either at least three vertices, or has two vertices which are not both of degree 3. We call such a cut an interesting cut.

Note that $H$ too is internally 4-edge-connected, thus all, but possibly one, of the terminals of an immersion of $H$ lie on one side of this cut, say $X$. Let $X'$ be the maximal subset of $V(G)$ containing $X$, such that $\delta(X')$ is interesting. Suppose there is an edge $uv$ in $X'^c$ not contributing to an interesting edge-cut, then deleting $uv$ is a good operation. It is because $G \setminus uv$ is 3-edge-connected, internally 4-edge-connected. Also $G \setminus uv$ has an $H$-immersion, because it immerses $(G \setminus e).X^c$.

We may now assume that $uv$ is in some interesting edge-cut $\delta(Y)$. Note that maximality of $X'$ implies that $X' \cap Y, X' \cap Y^c \neq \emptyset$. Also, we claim that there cannot be edges contributing to both $\delta(X'), \delta(Y)$. To prove the claim, suppose, to the contrary, that there are edges between, say, $X' \cap Y, X'^c \cap Y^c$, i.e. $e \neq 0$ in Figure 1. Then it follows from

$$8 = d(X') + d(Y) = d(X'^c \cap Y) + d(X' \cap Y^c) + 2e \geq 3 + 3 + 2e,$$

that if $e \neq 0$, it equals to 1, and, moreover, $d(X'^c \cap Y) = d(X' \cap Y^c) = 3$.  

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Using a similar argument, one can see that, if in addition to \( e \neq 0 \), there were also edges between \( X' \cap Y_c, X'^c \cap Y \), then \( d(X'^c \cap Y_c) = 3 \). Thus, both \( X'^c \cap Y_c, X'^c \cap Y \) would consist of a single vertex of degree 3, contradicting \( \delta(X') \) being interesting. Therefore the number of edges contributing to both \( \delta(X'), \delta(Y) \) equals \( e \).

We will now show that \( e \neq 0 \) results in a contradiction. Note that from \( d(X'^c \cap Y) = 3 \) we may conclude, without loss of generality, that \( b \geq 2 \). Now, by alternatively looking at the cuts \( \delta(X'^c \cap Y), \delta(X'), \delta(X' \cap Y^c), \delta(Y) \), we see that if \( b \geq 2 \), then \( c \leq 1 \), so \( a \geq 2 \), thus \( d \leq 1 \). Therefore, \( d(X' \cap Y) = c + d + e \leq 3 \), so \( X' \cap Y \) consists of a single vertex of degree three. This, however, together with the earlier conclusion of \( X' \cap Y^c \) consisting of a single vertex of degree three contradicts \( \delta(X') \) being interesting. Therefore \( e = 0 \), so there are no edges contributing to both \( \delta(X') \) and \( \delta(Y) \).

Now, we show that \( a = b = c = d = 2 \). For a contradiction, we will assume that, say \( a > 2 \), and, similar to the argument above, alternatively look at the cuts \( \delta(X'), \delta(X' \cap Y), \delta(Y) \). It then follows that \( c \leq 1 \), so \( d \geq 2 \), thus \( b \leq 2 \). So, in order for \( d(X'^c \cap Y) = b + c \geq 3 \), we must have \( b = 2, c = 1 \). Also, we have \( d(Y) = 4 = b + d \), so \( d = 2 \), thus \( d(X' \cap Y) = c + d = 3 \). Hence, each \( X'^c \cap Y \) and \( X' \cap Y \) consist of a single vertex of degree three, which contradicts \( \delta(Y) \) being interesting.

Therefore, \( a = b = c = d = 2 \), and thus \( \delta(X'^c \cap Y), \delta(X'^c \cap Y^c) \) are 4-edge-cuts. However, by maximality of \( X' \), they cannot be interesting cuts. Thus each of \( X'^c \cap Y, X'^c \cap Y^c \) consists of only one vertex, or two vertices both of degree 3.
We are now ready to prove that a good operation exists unless \((G, H) \cong (Q_3, K_4)\) or \((G, H) \cong (Q_3, K^3_2)\). Consider different possibilities for \(X'^c \cap Y, X'^c \cap Y^c\):

- Both sets consist of one vertex, see Fig. 2(a). Here, a good operation is to split off \(wuv\). Note that the resulting graph immerses \(H\), as it immerses \((G \setminus e).X^c\).

- Only one set consists of one vertex. Then it is easy to verify that \(X'^c\) should be as in Fig. 2(b). Here, deleting \(vw\) is a good operation.

![Figure 2: At least one of \(X'^c \cap Y, X'^c \cap Y^c\) consists of only one vertex](image)

- Both sets have two vertices in them, see Fig. 3. Here the operation

![Figure 3: Both \(X'^c \cap Y, X'^c \cap Y^c\) consist of two vertices](image)

will be deleting \(uw\) or \(vz\), from which we claim at least one is a good operation unless \(G \cong Q_3\). Suppose that deleting both \(uw\) and \(vz\) destroy internal 4-edge-connectivity, thus both these edges contribute to some interesting cuts.
As before, it can be argued that the cuts look like as in Fig. 4 with respect to each other. Now, ignoring \( \{u, v, w, z\} \) in Figures 3, and we can see that there exists a 2-edge cut separating \( \{n_u, n_w\} \) from \( \{n_v, n_z\} \), and another one separating \( \{n_u, n_v\} \) from \( \{n_w, n_z\} \), implying that \( n_u, n_v, n_w, n_z \) form a square, thus \( G \cong Q_3 \). It has now only remained to notice that \( K_4, K_2 \) are the only internally 4-edge-connected graph that \( Q_3 \) immerses.

\[ \square \]

Our next task is to deal with splits in \( G \) that preserve an \( H \)-immersion, which will be done in Lemma 3.4. The following statement, which holds for a broader family of graphs than the ones we work with, features in the proof of Lemma 3.4.

**Lemma 3.3** Suppose \( H \) is a 3-edge-connected graph, and \( Y \) is a minimal subset of \( V(H) \) such that \( \delta(Y) \) is a nontrivial 3-edge-cut in \( H \). Then for every edge \( e \) in \( H[Y] \), \( H \setminus e \) is internally 3-edge-connected.

**Proof.** For a contradiction, suppose an edge \( e = yz \) in \( H[Y] \) contributes to some nontrivial 3-edge-cut \( \delta(Z) \), where \( z \in Z \). We will look into how \( Y, Z \) look like with respect to one another. Note both \( Y \cap Z \) and \( Y \cap Z^c \) are nonempty, as \( z \in Y \cap Z, y \in Y \cap Z^c \). Also, both \( Y^c \cap Z \) and \( Y^c \cap Z^c \) are nonempty. It is because, if, say \( Y^c \cap Z = \emptyset \), then \( \delta(Y \cap Z) \) would be a nontrivial 3-edge-cut, which contradicts the choice of \( Y \), as \( Y \cap Z \) \( \subseteq \) \( Y \).

Now, since \( H \) is 3-edge-connected, we have \( d(Y \cap Z), d(Y^c \cap Z^c) \geq 3 \). It now follows from \( d(Y \cap Z) + d(Y^c \cap Z^c) + 2e(Y^c \cap Z, Y \cap Z^c) = d(Y) + d(Z) \).
that \( d(Y \cap Z) = d(Y^c \cap Z^c) = 3 \) and \( e(Y^c \cap Z, Y \cap Z^c) = 0 \). Similarly, we obtain \( d(Y \cap Z^c) = d(Y^c \cap Z) = 3 \) and \( e(Y \cap Z, Y^c \cap Z^c) = 0 \). Now, since \( d(Y) = 3 = e(Y \cap Z, Y^c \cap Z) + e(Y \cap Z^c, Y^c \cap Z) \), we have, say, \( e(Y \cap Z, Y^c \cap Z) \leq 1 \). Similarly, it follows from \( d(Z) = 3 \) that we have, say, \( e(Y \cap Z, Y \cap Z^c) \leq 1 \). Hence, \( d(Y \cap Z) \leq 2 \), a contradiction. \( \square \)

**Lemma 3.4** Suppose \( G, H \) are as in Theorem 3.1, and there is a split at a vertex \( v \) preserving an \( H \)-immersion. Then if \( (G, H) \not\cong (Q_3, K_4), (Q_3, K_3^2) \), a good operation exists.

**Proof.** Let \( uvw \) be the 2-edge-path that is to be split. Note if \( d(v) = 3 \), then deleting the edge incident to \( v \) other than \( vu, vw \) preserves the \( H \)-immersion. Hence, by Lemma 3.2 we are done. Also, observe that if a split is done at a vertex of degree at least four, the resulting graph is 3-edge-connected. Therefore, we only need to look into the case where splitting off \( uvw \) destroys internal 4-edge-connectivity. So, it must be the case that \( uv, vw \) contribute to some 4- or 5-edge-cut \( \delta(X) = \{uv, wv, x_1y_1, x_2y_2, x_3y_3 : u, w, x_i \in X\} \), where \( |X|, |X^c| \geq 2 \). We now split the analysis into cases depending on \( d(X) \).

**3.4.1 If** \( d(X) = 4 \), **a good operation exists.**

Since \( H \) is 3-edge-connected, all terminals of \( H \) lie on one side of the cut. Also, since \( G \) is 3-edge-connected, each side of the cut contains an edge completely lying in it, i.e. \( E(G[X]), E(G[X^c]) \neq \emptyset \).

First, suppose all terminals of \( H \) are in \( X \). Observe that if we can modify \( X^c \) in a way that it preserves the connectivity of \( y_1, y_2 \) in \( G[X^c] \), an \( H \)-immersion is present in the resulting graph. We propose to delete an edge \( e \in E(G[X^c]) \), and claim that deleting \( e \) preserves the \( H \)-immersion. It suffices to show \( e \) is not a cut-edge in \( G[X^c] \) separating \( y_1, y_2 \). For a contradiction, suppose \( e = \delta(Y) \) separates \( y_1, y_2 \) in \( G[X^c] \), where \( y_1 \in Y \). We may also assume, without loss of generality, that \( v \in Y \). Then \( \delta_G(Y^c) \) would be a 2-edge-cut in \( G \), a contradiction. Therefore, we can delete \( e \) using Lemma 3.2.

Next, suppose all terminals of \( H \) are in \( X^c \). Similar to the previous case, if we modify \( X \) in a way that preserves the connectivity of \( x_1, x_2 \) in \( G[X] \), an \( H \)-immersion is sure to exist in the resulting graph. Again, we propose to delete an edge \( e \in E(G[X]) \), and claim that deleting \( e \) preserves
the $H$-immersion. It suffices to show $e$ is not a cut-edge in $G[X]$ separating $x_1, x_2$. For a contradiction, suppose $e = \delta(Y)$ separates $x_1, x_2$ in $G[X]$, where $x_1 \in Y$. Note 3-edge-connectivity of $G$ implies that $\delta(Y)$ separates $u, w$ as well. We may assume, without loss of generality, that $u \in Y, w \in Y^c$. Then $d_G(Y) = d_G(Y^c) = 3$, thus it follows from internal 4-edge-connectivity of $G$ that $|Y| = |Y^c| = 1$ and $Y = \{u = x_1\}, Y^c = \{w = x_2\}$. Therefore, $X$ consists of two vertices $u, w$ of degree three, and thus deleting $uw$ preserves the $H$-immersion.

3.4.2 If $d(X) = 5$, a good operation exists.

By the internal edge-connectivity of $H$, all terminals of $H$, but possibly one, lie on one side of the cut. First, suppose that most terminals of $H$ are in $X$. Observe that if $X^c$ is modified in a way that preserves the presence of three edge-disjoint paths form a vertex in it to $X$ not using $uv, vw$, the presence of $H$-immersion is guaranteed. Next, suppose that most terminals of $H$ are in $X^c$. In this case, if we manage to modify $X$ in a way that preserves the presence of three edge-disjoint paths form a vertex in it to $X^c$ covering $\delta(X) \setminus \{uv, vw\}$, the presence of $H$-immersion is guaranteed. We claim such modifications are possible.

Let $G'$ be the graph resulting from splitting off $uvw$, followed by suppressing $v$ in case $d_G(v) = 4$. We denote the edge created by splitting $uvw$ by $e'$. Note, by 3.4.1, we may assume $G'$ is 3-edge-connected.

Take an arbitrary nontrivial 3-edge-cut $\delta_{G'}(Y)$ in $G'$. Observe that $\delta_G(Y)$ must have been a 5-edge-cut in $G$, which both edges of the split 2-path $uvw$ contributed to. So, in particular, $e'$ lies either completely in $Y$ or in $Y^c$. Also, there must be an edge other than $e'$ in $G'[Y]$. It is because 3-edge-connectivity of $G$ implies $6 \leq \sum_{v \in Y} d_G(v) = d_G(Y) + 2e_G(G[Y]) = 5 + 2e_G(G[Y])$. Thus $e_G(G[Y]) > 0$, and so there is an edge $\neq e'$ in $G'[Y]$.

Now, let $Z$ denote the side of $\delta(X)$ containing most terminals of $H$ (so $Z = X$ or $X^c$). We will show that there is an edge lying in $Z^c$ which we could delete, while preserving an $H$-immersion. Since $\delta_{G'}(Z)$ is a nontrivial 3-edge-cut, we may choose a minimal $Y \subseteq Z^c$ such that $\delta_{G'}(Y)$ is a nontrivial 3-edge-cut.

It is argued above that there exists an edge $e \neq e'$ in $G'[Y]$. We claim deletion of $e$ preserves the $H$-immersion. It is because it follows from Lemma 3.3 that $G' \setminus e$ is internally 3-edge-connected. Now, 3-edge-connectivity of
$G'$ and $d_{G'}(Y) = 3$ imply that $G'[Y] \setminus e$ has a vertex of degree at least three. Therefore, there exists in $G'[Y] \setminus e$ three edge-disjoint paths from such a vertex to $Z$. Observe that since these set of paths cover $\delta(Z)$, deletion of $e$ from $G$ preserves the presence of $H$-immersion. We now can use Lemma 3.2 to delete $e$ from $G$.

Proof of Theorem 3.1 is now immediate.

**Proof of Theorem 3.1.** Apply Lemmas 3.2 and 3.4.

Having established Theorem 3.1, we will now take advantage of it to prove Corollary 1.7. The idea is to examine 3-edge-connected, internally 4-edge-connected graphs “one step bigger”, or perhaps “a few steps bigger”, than $K_5$, and see if they immerse $K_{3,3}$. One subtlety here is that we are working with multigraphs, thus even graphs “much bigger than” $K_5$ may happen to be on five vertices, and thus not possess $K_{3,3}$-immersions. Therefore, we need some tool to limit the graphs necessary to examine. Given that $K_5$ itself is 4-edge-connected, Lemma 3.5 serves very well in doing so. First, however, we need the following definition.

**Definition.** We define a **good sequence** from $G$ to $H$ to be a sequence of graphs

$$G = G_t, G_{t-1}, \ldots, G_2, G_1, G_0 \cong H$$

in which each $G_i$ is 3-edge-connected, and internally 4-edge-connected, and $G_i$ is resulting from applying an operation $o_{i+1}$ (as defined in the statement of theorem 3.1) to $G_{i+1}$.

**Lemma 3.5** Let $G$ be 3-edge-connected, internally 4-edge-connected, and $H$ be 4-edge-connected. Suppose there is a good sequence from $G$ to $H$, and choose a good sequence from $G$ to $H$

$$G = G_t, G_{t-1}, \ldots, G_2, G_1, G_0 \cong H$$

such that $\min\{k : |V(G_k)| > |V(H)|\}$ is as small as possible. Then either

(a) $G_1$ is as in Fig. 3(a), with $v_1 \neq v_2$, $v_3 \neq v_4$, and the last operation, $o_1$, is to split off $v_1w_2$, and $v_3w_4$.

(b) $G_1$ is as in Fig. 3(b), with $v_1 \neq v_2$, $v_3 \neq v_4$, and $o_1$ is deleting $uw$.

(c) $G_1$ is as in Fig. 3(c) and $o_1$ is to delete $uw_1$. 

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Proof. Let $G_k$ be the graph in the sequence which attains the $\min\{k : |V(G_k)| > |V(H)|\}$, thus $V(G_{k-1}) = V(H) = \{v_1, v_2, \ldots, v_{|H|}\}$. First, consider the case where $o_k$ is a split. Since this split reduces the number of vertices, it must be a split at a vertex $u$ of degree 4, see Fig. 5(a). Let $v_1v_2, v_3v_4$ be the edges resulting from splitting $v$. We claim that $G_{k-1} = H$, since if there was $k' < k$ so that $o_{k'}$ was

- splitting a 2-edge-path where both edges are present in $G_k$, or deleting an edge present in $G_k$, then it could have been done before $o_k$.
- splitting a $v_1v_2v_i$ path, then we could have split $uv_2v_i$ instead.
- splitting a 2-edge-path, with both edges $v_1v_2$, and $v_3v_4$, with, say, $v_2 = v_3$, resulting from $o_k$, then we could have deleted one of $uv_2$ edges instead.
- deleting one of the edges, say $v_1v_2$, created by $o_k$, then we could have deleted $uv_1$. (It also implies that $v_1 \neq v_2$, and $v_3 \neq v_4$.)

Note that in all the cases above the alternative operation would result in another good sequence, with smaller $\min\{k : |V(G_k)| > |V(H)|\}$, contradicting our choice of the good sequence. Therefore the claim is proved, thus $k = 1$, and (a) occurs.

Now, consider the case where $o_k$ is a deletion of an edge $uw$. Since this deletion reduces the number of vertices, at least one of its endpoints is of degree 3. If both $u$ and $w$ are of degree 3 (see Fig. 5(b)), the same argument as above shows that $k = 1$, and thus (b) happens.
Otherwise, only $u$ is of degree 3, and $o_k$ is deleting $uv_1$, see Fig. 5(c). As before, it could be argued that there cannot be a $k' < k$ with $o_{k'}$ being splitting a 2-edge-path with both edges present in $G_k$, or deleting an edge present in $G_k$. Also, $o_{k'}$ cannot be splitting $v_2v_3v_i$, since we could have split $uv_3v_i$ before $o_k$, obtaining a good sequence with smaller $\min\{k : |V(G_k)| > |V(H)|\}$. However, it could be that $o_{k'}$ is deleting $v_2v_3$. Thus, if $v_2v_3$ is not to be deleted, we have $k = 1$, and (c) happens; else, $k = 2$, and $o_{k-1}$ would be deleting $v_2v_3$, i.e. (d) occurs. So, suppose case (a) of the previous lemma occurs. Again, it can easily be verified that if the two edges created by $o_1$ share an endpoint, then $G_1 \succ im K_{3,3}$, thus $G \succ im K_{3,3}$. Otherwise, $K_{3,3}$ is not immersed in $G_1$, as $G_1$ would be the octahedron, which, being planar, doesn’t have $K_{3,3}$ as a subgraph. On the other hand, it has six vertices, all of degree 4, so an immersion of $K_{3,3}$ cannot be found doing splits either.

Therefore, if $G \cong$ octahedron, $G \not\succ im K_{3,3}$. However, if $G$ properly immerses octahedron, then it immerses $K_{3,3}$ as well. To see that, note that the 6-vertex graphs from which octahedron is obtained after deletion of an edge or splitting a 2-edge path, all immerse $K_{3,3}$. On the other hand, if

Corollary 3.6 Suppose $G$ is 3-edge-connected, and internally 4-edge-connected, where $G \succ im K_5$. Then

1. if $|V(G)| \geq 6$ then $G \succ im K_{3,3}$, or $G \cong$ octahedron, where octahedron is the graph in Fig. 6.

2. If $|V(G)| \geq 7$ then $G \succ K_{3,3}$.

Proof. Observe part (2) is an immediate consequence of part (1). We will then prove (1). Suppose $G \succ im K_5$, and $|V(G)| > 5$. By Theorem 3.1 a good sequence from $G$ to $K_5$ exists. Thus, we can choose a good sequence

$G = G_t, G_{t-1}, \ldots, G_2, G_1, G_0 \cong K_5$

such that $\min\{k : |V(G_k)| > 5\}$ is as small as possible, and apply the previous lemma. It could be easily verified that if cases (b), or (c) of the previous lemma occur, then $G_1 \succ im K_{3,3}$, and if case (d) happens, $G_2 \succ im K_{3,3}$, thus $G \succ im K_{3,3}$.

So, suppose case (a) of the previous lemma occurs. Again, it can easily be verified that if the two edges created by $o_1$ share an endpoint, then $G_1 \succ im K_{3,3}$, thus $G \succ im K_{3,3}$. Otherwise, $K_{3,3}$ is not immersed in $G_1$, as $G_1$ would be the octahedron, which, being planar, doesn’t have $K_{3,3}$ as a subgraph. On the other hand, it has six vertices, all of degree 4, so an immersion of $K_{3,3}$ cannot be found doing splits either.

Therefore, if $G \cong$ octahedron, $G \not\succ im K_{3,3}$. However, if $G$ properly immerses octahedron, then it immerses $K_{3,3}$ as well. To see that, note that the 6-vertex graphs from which octahedron is obtained after deletion of an edge or splitting a 2-edge path, all immerse $K_{3,3}$. On the other hand, if
$|V(G)| > 6$, we may again use Lemma 3.5, since octahedron itself is 4-edge-connected.

To reduce the number of graphs we examine, it now helps to notice that we only need to consider the case where a 4-vertex 7 gets split to create edges $\{23, 15\}$, or $\{23, 14\}$. It is because in all other cases, the graph obtained by splitting 2-paths 163, 264 would be one of the graphs we already looked at, all of which immerse $K_{3,3}$.

If vertex 7 is split to create $\{23, 15\}$, an immersion of $K_{3,3}$ may be found after splitting 2-path 173. Also, if vertex 7 is split to create $\{23, 14\}$, then $K_{3,3}$ lies as a subgraph in $G$. \hfill $\square$

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