Throughout, the term lens space will mean a 3-dimensional lens space $L(m,q)$ other than $L(1,0)$ (the 3-sphere), $L(0,1)$ (the product $S^2 \times S^1$), and other than $L(2,1)$ (the real projective 3-space $\mathbb{R}P^3$). In addition, we always select $q$ so that $1 \leq q < m/2$.

Lens spaces are elliptic 3-manifolds. That is, they may be regarded as the quotient of the standard round 3-sphere $S^3$ by a finite subgroup of the group $\text{SO}(4)$ of orientation-preserving isometries of $S^3$. Then, they inherit a Riemannian metric of constant positive curvature. For this metric, the isometry group $\text{Isom}(L(m,q))$ is a Lie group of dimension either 2 or 4. These groups are given in Table 1.

S. Smale [33] proved that for the standard round 2-sphere $S^2$, the inclusion of the isometry group $O(3)$ into the diffeomorphism group $\text{Diff}(S^2)$ is a homotopy equivalence. He conjectured that the analogous result holds true for the 3-sphere, that is, that $O(4) \to \text{Diff}(S^3)$ is a homotopy equivalence. J. Cerf [6] proved that the inclusion induces a bijection on path components, and the full conjecture was proven by A. Hatcher [13]. This is a result of fundamental importance in the theory of 3-manifolds, because it shows that smooth structures on 3-manifolds are unique up to diffeomorphism, and that for many purposes, there is no essential difference between the group of homeomorphisms and the group of diffeomorphisms of a 3-manifold [7].

A natural extension of Smale’s conjecture is that if $M$ is any elliptic 3-manifold, then $\text{Isom}(M) \to \text{Diff}(M)$ is a homotopy equivalence. This has been proven for some cases [19, 20, 28], among them the lens spaces $L(4n,2n-1), n \geq 2$. Our main result extends this to all lens spaces:

**Theorem** (Smale Conjecture for Lens Spaces). For any lens space $L$, the inclusion $\text{Isom}(L) \to \text{Diff}(L)$ is a homotopy equivalence.

One consequence of this is the determination of the homeomorphism type of $\text{Diff}(L)$. Recall that a Fréchet space is a locally convex complete metrizable linear space (see for example [2, Proposition 6.4], or [25]). If $M$ is a closed smooth manifold, then with the $C^\infty$-topology, $\text{Diff}(M)$ is a second countable infinite-dimensional manifold locally modeled on the Fréchet
space of smooth vector fields on $M$ (see for example [1, section 1.2] for the local structure, and for the second countability see chapter 2 of [17], especially section 2.4). By the Anderson-Kadec Theorem [2, Corollary VI.5.2], every infinite-dimensional separable Fréchet space is homeomorphic to $\mathbb{R}^\infty$, the countable product of lines. A theorem of Henderson and Schori ([2, Theorem IX.7.3], originally announced in [15]) shows that if $Y$ is any locally convex space with $Y$ homeomorphic to $Y^\infty$, then manifolds locally modeled on $Y$ are homeomorphic whenever they have the same homotopy type. Therefore the Smale Conjecture for Lens Spaces gives immediately the homeomorphism type of $\text{Diff}(L)$:

**Corollary.** For any lens space $L$, $\text{Diff}(L)$ is homeomorphic to $\text{Isom}(L) \times \mathbb{R}^\infty$.

This combines with the known calculations of $\text{Isom}(L)$ in Table 1 to give a complete classification of $\text{Diff}(L)$ for lens spaces into four homeomorphism types. In the following corollary, we assume as usual that $L(m,q)$ is written with $1 \leq q < m/2$, and we write $P_n$ for the discrete space with $n$ points.

**Corollary.** For a lens space $L(m,q)$, the homeomorphism type of $\text{Diff}(L)$ is as follows:

1. For $m$ odd, $\text{Diff}(L(m,1)) \approx S^1 \times S^3 \times P_2$.
2. For $m$ even, $\text{Diff}(L(m,1)) \approx S^1 \times \text{SO}(3) \times P_2$.
3. For $q > 1$ and $q^2 \not\equiv \pm 1 \pmod{m}$, $\text{Diff}(L(m,q)) \approx S^1 \times S^1 \times P_2$.
4. For $q > 1$ and $q^2 \equiv \pm 1 \pmod{m}$, $\text{Diff}(L(m,q)) \approx S^1 \times S^1 \times P_4$.

Our homeomorphism classification contrasts with the fact that the isomorphism type of $\text{Diff}(L)$ determines $L$. In fact, every abstract group isomorphism between the diffeomorphism groups of two smooth manifolds without boundary is induced by a diffeomorphism between the manifolds [1, 8, 34].

The authors gratefully acknowledge support from many sources during the pursuance of this work, including the Australian Research Council, the Basic Science Research Center of Korea University, the U. S. National Science Foundation, the University of Oklahoma College of Arts and Sciences, and the University of Oklahoma Vice President for Research.
1. Outline of the proof

In this section, we will outline the proof of the Smale Conjecture for Lens Spaces.

Some initial reductions, detailed in section 2, reduce the Smale Conjecture for Lens Spaces to showing that the inclusion \( \text{diff}_f(L) \to \text{diff}(L) \) is an isomorphism on homotopy groups. Here, \( \text{diff}(L) \) is the connected component of the identity in \( \text{Diff}(L) \), and \( \text{diff}_f(L) \) is the connected component of the identity in the group of diffeomorphisms that are fiber-preserving with respect to a Seifert fibering of \( L \) induced from the Hopf fibering of its universal cover, \( S^3 \). To simplify the exposition, most of the paper is devoted just to proving that \( \text{diff}_f(L) \to \text{diff}(L) \) is surjective on homotopy groups, that is, that a map from \( S^d \) to \( \text{diff}(L) \) is homotopic to a map into \( \text{diff}_f(L) \). The injectivity is obtained in section 13 by a combination of tricks and minor adaptations of the main program.

Of course, a major difficulty in working with elliptic 3-manifolds is their lack of incompressible surfaces. In their place, we use another structure which has a certain degree of essentiality, called a sweepout. This means a structure on \( L \) as a quotient of \( P \times I \), where \( P \) is a torus, in which \( P \times \{0\} \) and \( P \times \{1\} \) are collapsed to core circles of the solid tori of a genus 1 Heegaard splitting of \( L \). For \( 0 < t < 1 \), \( P \times \{t\} \) becomes a Heegaard torus in \( L \), denoted by \( P_t \) and called a level. The sweepout is chosen so that each \( P_t \) is a union of fibers. Sweepouts are examined in section 5.

Start with a parameterized family of diffeomorphisms \( f: L \times S^d \to L \), and for \( u \in S^d \) denote by \( f_u \) the restriction of \( f \) to \( L \times \{u\} \). The procedure that deforms \( f \) to make each \( f_u \) fiber-preserving has three major steps.

Step 1 ("finding good levels") is to perturb \( f \) so that for each \( u \), there is some pair \((s,t)\) so that \( f_u(P_s) \) intersects \( P_t \) transversely, in a collection of circles each of which is either essential in both \( f_u(P_s) \) and \( P_t \) (a biessential intersection), or inessential in both (a discal intersection), and at least one intersection circle is biessential. This pair is said to intersect in good position, and if none of the intersections is discal, in very good position. These concepts are developed in section 4, after a preliminary examination of annuli in solid tori in section 3.

To accomplish Step 1, the methodology of Rubinstein and Scharlemann in [31] is adapted. This is reviewed in section 6. First, one perturbs \( f \) to be in "general position," as defined in section 8. The intersections of the \( f_u(P_s) \) and \( P_t \) are then sufficiently well-controlled to define a graphic in the square \( I^2 \). That is, the pairs \((s,t)\) for which \( f_u(P_s) \) and \( P_t \) do not intersect transversely form a graph imbedded in the square. The complementary regions of this graph in \( I^2 \) are labeled according to a procedure in [31], and in section 9 we show that the properties of general position salvage enough of the combinatorics of these labels developed in [31] to deduce that at least one of the complementary regions consists of pairs in good position.
Perhaps the hardest work of the paper, and certainly the part that takes us farthest from the usual confines of low-dimensional topology, is the verification that sufficient “general position” can be achieved. Since we use parameterized families, we must allow \( f_u(P_s) \) and \( P_t \) to have large numbers of tangencies, some of which may be of high order. It turns out that to make the combinatorics of [31] go through, we must achieve that at each parameter there are at most finitely many pairs \((s,t)\) where \( f_u(P_s) \) and \( P_t \) have multiple or high-order tangencies (at least, for pairs not extremely close to the boundary of the square). The need for this requirement is illustrated by examples in section 7, where we construct pairs of sweepouts with all tangencies of Morse type, but having no pair of levels intersecting in good position. To achieve the necessary degree of general position, we use results of a number of people, notably J. W. Bruce [4] and F. Sergeraert [32].

Step 2 (“from good to very good”) is to deform \( f \) to eliminate the discal intersections of \( f_u(P_s) \) and \( P_t \), for certain pairs in good position that have been found in Step 1, so that they intersect in very good position. This is an application of Hatcher’s parameterization methods [10]. One must be careful here, since an isotopy that eliminates a discal intersection can also eliminate a biessential intersection, and if all biessential intersections were eliminated by the procedure, the resulting pair would no longer be in very good position. Lemma 10.2 ensures that not all biessential intersections will be eliminated.

Step 3 (“from very good to fiber-preserving”) is to use the pairs in very good position to deform the family so that each \( f_u \) is fiber-preserving. This is carried out in sections 11 and 12. The basic idea is first to use the biessential intersections to deform the \( f_u \) so that \( f_u(P_s) \) actually equals \( P_t \) (for certain \((s,t)\) pairs that originally intersected in good position), then use known results about the diffeomorphism groups of surfaces and Haken 3-manifolds to make the \( f_u \) fiber-preserving on \( P_s \) and then on its complementary solid tori. This process is technically complicated for two reasons. First, although a biessential intersection is essential in both tori, it can be contractible in one of the complementary solid tori of \( P_t \), and \( f_u(P_s) \) can meet that complementary solid torus in annuli that are not parallel into \( P_t \). So one may be able to push the annuli out from only one side of \( P_t \). Secondly, the fitting together of these isotopies requires one to work with not just one level but many levels at a single parameter.

Two natural questions are whether Bonahon’s original method for determining the mapping class group \( \pi_0(\text{Diff}(L)) \) [3] can be adapted to the parameterized setting, and whether our methodology can be used to recover his results. Concerning the first question, we have had no success with this approach, as we see no way to perturb the family to the point where the method can be started at each parameter. For the second, the answer is yes. In fact, the key geometric step of [3] is the isotopy uniqueness of genus-one Heegaard surfaces in \( L \), which was already reproven in Rubinstein and Scharlemann’s original work [31, Corollary 6.3].
Table 1. Isometry groups of $L(m,q)$

| $m, q$ | $\text{isom}(L(m,q))$ | $\mathcal{I}(L(m,q))$ |
|--------|----------------------|---------------------|
| $m = 1$ | O(4) | $C_2$ |
| $m = 2$ | SO(3) $\times$ SO(3) | $C_2$ |
| $m > 2$, $m$ odd, $q = 1$ | O(2)$^*$$\times$ S$^3$ | $C_2$ |
| $m > 2$, $m$ even, $q = 1$ | O(2) $\times$ SO(3) | $C_2$ |
| $m > 2$, $1 < q < m/2$, $q^2 \not\equiv \pm 1 \text{ mod } m$ | Dih($S^1 \times S^1$) | $C_2$ |
| $m > 2$, $1 < q < m/2$, $q^2 \equiv -1 \text{ mod } m$ | $S^1 \times S^1$ | $C_4$ |
| $m > 2$, $1 < q < m/2$, $q^2 \equiv 1 \text{ mod } m$, gcd($m, q + 1$) gcd($m, q - 1$) = $m$ | O(2) $\times$ O(2) | $C_2 \times C_2$ |
| $m > 2$, $1 < q < m/2$, $q^2 \equiv 1 \text{ mod } m$, gcd($m, q + 1$) gcd($m, q - 1$) = $2m$ | O(2) $\times$ O(2) | $C_2 \times C_2$ |

2. Reductions

In this section, we carry out initial reductions. The Conjecture will be reduced to a purely topological problem of deforming parameterized families of diffeomorphisms to families of diffeomorphisms that preserve a certain Seifert fibering of $L$, which we call the Hopf fibering.

The paper [27] contains a calculation of the isometry groups of all elliptic 3-manifolds (calculations for lens spaces were also given in [18] and [23]). Among the elliptic 3-manifolds, the lens spaces have the most complicated isometry groups, given in Table 1. In the table, $\text{isom}(L(m,q))$ is the path component of the identity map in the isometry group $\text{Isom}(L(m,q))$, and $\mathcal{I}(L(m,q))$ is the group of path components of $\text{Isom}(L(m,q))$. The orthogonal groups are denoted by O(4), SO(3) and O(2), $C_k$ is the cyclic group of order $k$, and Dih($S^1 \times S^1$) is the semidirect product ($S^1 \times S^1$) $\circ$ $C_2$, where $C_2$ acts by complex conjugation in both factors. Also, O(2)$^*$ is the nontrivial central extension of O(2) by $C_2$, that is, the preimage of O(2) $\subset$ SO(3) under the 2-fold covering map $S^3 \to$ SO(3). If $H_1$ and $H_2$ are groups, each containing $-1$ as a central involution, then the quotient $(H_1 \times H_2)/((-1, -1))$ is denoted by $H_1 \times H_2$. In particular, SO(4) itself is $S^3 \times S^3$, and contains the subgroups O(2)$^* \times S^3$ and $S^1 \times S^1$. The latter is isomorphic to $S^1 \times S^1$.

From Table 1, one sees that $\text{isom}(L(m,1))$ is homeomorphic to $S^1 \times S^3$ for $m$ odd, and to $S^1 \times SO(3)$ for $m$ even, while for $q > 1$, isom($L(m,q)$) is homeomorphic to $S^1 \times S^1$. These observations were used in the second corollary stated in the introduction.

The following theorem from [27] is the “$\pi_0$-part” of the Smale Conjecture for elliptic 3-manifolds.
Theorem 2.1. Let $M$ be an elliptic 3-manifold. Then the inclusion of $\text{Isom}(M)$ into $\text{Diff}(M)$ is a bijection on path components.

Consequently, to prove the Smale Conjecture for a lens space $L$, it is sufficient to prove that the inclusion of the connected components of the identity map $\text{isom}(L) \to \text{diff}(L)$ is a homotopy equivalence.

Since $\text{Diff}(M)$ is an infinite-dimensional manifold locally modeled on $\mathbb{R}^\infty$, Corollary IX.7.1 of [2] (originally theorem 4 of [14]) shows that $\text{Diff}(M)$ admits an open imbedding into $\mathbb{R}^\infty$. Theorems II.6.2 and II.6.3 of [2] then show that $\text{Diff}(M)$ has the homotopy type of a CW-complex (as far as we know, this fact is due originally to Palais [30]). So $\text{diff}(M)$ has the homotopy type of a CW-complex, and the same is true for $\text{isom}(M)$, since it is a manifold. Therefore it is sufficient to prove that $\text{isom}(L) \to \text{diff}(L)$ is a weak homotopy equivalence.

Section 1.4 of [27] gives a certain way to imbed $\pi_1(L)$ into $\text{SO}(4)$ so that its action on $S^3$ is fiber-preserving for the fibers of the Hopf bundle structure of $S^3$. Consequently, this bundle structure descends to a Seifert fibering of $L$, which we call the Hopf fibering of $L$. If $q = 1$, this Hopf fibering is actually an $S^1$-bundle structure, while if $q > 1$, it has two exceptional fibers with invariants of the form $(k, q_1), (k, q_2)$ where $k = m / \gcd(q - 1, m)$ (see Table 4 of [27]). We will always use the Hopf fibering as the Seifert-fibered structure of $L$.

A diffeomorphism from $L$ to $L$ is called fiber-preserving if the image of each fiber is a fiber, and vertical if it preserves each fiber. By $\text{diff}_f(L)$ we denote the connected component of the identity map in the space of fiber-preserving diffeomorphisms. Theorem 2.1 of [27] shows that (since $m > 2$) every orientation-preserving isometry of $L$ preserves the Hopf fibering on $L$. In particular, $\text{isom}(L) \subset \text{diff}_f(L)$, so there are inclusions

$$\text{isom}(L) \to \text{diff}_f(L) \to \text{diff}(L).$$

Theorem 2.2. The inclusion $\text{isom}(L) \to \text{diff}_f(L)$ is a weak homotopy equivalence.

Proof. The argument is similar to the latter part of the proof of theorem 4.2 of [28], so we only give a sketch. There is a diagram

$$
\begin{array}{ccc}
S^1 & \longrightarrow & \text{isom}(L) \\
\downarrow & & \downarrow \\
\text{vert}(L) & \longrightarrow & \text{diff}_f(L) \\
\downarrow & & \downarrow \\
& & \text{diff}_{\text{orb}}(L_0)
\end{array}
$$

where $L_0$ is the quotient orbifold and $\text{diff}_{\text{orb}}(L_0)$ is the group of orbifold diffeomorphisms of $L_0$, and $\text{vert}(L)$ is the group of vertical diffeomorphisms. The first row is a fibration, in fact an $S^1$-bundle, and the second row is a fibration by theorem 8.3 of [22]. The vertical arrows are inclusions. When $q = 1$, $L_0$ is the 2-sphere and the right-hand vertical arrow is the inclusion of $\text{SO}(3)$ into $\text{diff}(S^2)$, which is a homotopy equivalence by [33]. When $q > 1$,
$L_0$ is a 2-sphere with two cone points, $\text{isom}(L_0)$ is homeomorphic to $S^1$, and $\text{diff}_{\text{orb}}(L_0)$ is essentially the connected component of the identity in the diffeomorphism group of the annulus. Again the right-hand vertical arrow is a weak homotopy equivalence. The left-hand vertical arrow is a weak homotopy equivalence in both cases, so the middle arrow is as well. □

Theorem 2.2 reduces the Smale Conjecture for Lens Spaces to proving that the inclusion $\text{diff}_f(L) \to \text{diff}(L)$ is a weak homotopy equivalence. For this it is sufficient to prove that for all $d \geq 1$, any map $f: (D^d, S^{d-1}) \to (\text{diff}(L), \text{diff}_f(L))$ is homotopic, through maps taking $S^{d-1}$ to $\text{diff}_f(L)$, to a map from $D^d$ into $\text{diff}_f(L)$. To simplify the exposition, we work until the final section with a map $f: S^d \to \text{diff}(L)$ and show that it is homotopic to a map into $\text{diff}_f(L)$. In the final section, we give a trick that enables the entire procedure to be adapted to maps $f: (D^d, S^{d-1}) \to (\text{diff}(L), \text{diff}_f(L))$, completing the proof.
3. Annuli in solid tori

Annuli in solid tori will appear frequently in our work. Incompressible annuli present little difficulty, but we will also need to examine compressible annuli, whose behavior is somewhat more complicated. In this section, we provide some basic definitions and lemmas.

A loop $\alpha$ in a solid torus $V$ is called a longitude if its homotopy class is a generator of the infinite cyclic group $\pi_1(V)$. If in addition there is a product structure $V = S^1 \times D^2$ for which $\alpha = S^1 \times \{0\}$, then $\alpha$ is called a core circle of $V$. A subset of a solid torus $V$ is called a core region when it contains a core circle of $V$. An imbedded circle in $\partial V$ which is essential in $\partial V$ and contractible in $V$ is called a meridian of $V$; a properly imbedded disk in $V$ whose boundary is a meridian is called a meridian disk of $V$.

Annuli in solid tori will always be assumed to be properly imbedded, which for us includes the property of being transverse to the boundary, unless they are actually contained in the boundary. The next three results are elementary topological facts, and we do not include proofs.

**Proposition 3.1.** Let $A$ be a boundary-parallel annulus in a solid torus $V$, which separates $V$ into $V_0$ and $V_1$, and for $i = 0, 1$, let $A_i = V_i \cap \partial V$. Then $A$ is parallel to $A_i$ if and only if $V_{1-i}$ is a core region.

**Proposition 3.2.** Let $A$ be a properly imbedded annulus in a solid torus $V$, which separates $V$ into $V_0$ and $V_1$, and let $A_i = V_i \cap \partial V$. The following are equivalent:

1. $A$ contains a longitude of $V$.
2. $A$ contains a core circle of $V$.
3. $A$ is parallel to both $A_0$ and $A_1$.
4. Both $V_0$ and $V_1$ contain longitudes of $V$.
5. Both $V_0$ and $V_1$ are core regions of $V$.

An annulus satisfying the conditions in proposition 3.2 is said to be longitudinal. A longitudinal annulus must be incompressible.

**Proposition 3.3.** Let $V$ be a solid torus and let $\cup A_i$ be a union of disjoint boundary-parallel annuli in $V$. Let $C$ be a core circle of $V$ that is disjoint from $\cup A_i$. For each $A_i$, let $V_i$ be the closure of the complementary component of $A_i$ that does not contain $C$, and let $B_i = V_i \cap \partial V$. Then $A_i$ is parallel to $B_i$. Furthermore, either

1. no $A_i$ is longitudinal, and exactly one component of $V - \cup A_i$ is a core region, or
2. every $A_i$ is longitudinal, and every component of $V - \cup A_i$ is a core region.

There are various kinds of compressible annuli in solid tori. For example, there are boundaries of regular neighborhoods of properly imbedded arcs,
possibly knotted. Also, there are annuli with one boundary circle a merid-
ian and the other a contractible circle in the boundary torus. When both
boundary circles are meridians, we call the annulus meridional. As shown
in figure 1, meridional annuli are not necessarily boundary-parallel.

Although meridional annuli need not be boundary-parallel, they behave
homologically as though they were, and as a consequence any family of
meridional annuli misses some longitude.

**Lemma 3.4.** Let $A_1, \ldots, A_n$ be disjoint meridional annuli in a solid torus
$V$. Then:

1. Each $A_i$ separates $V$ into two components, $V_{i,0}$ and $V_{i,1}$, for which
   $A_i$ is incompressible in $V_{i,0}$ and compressible in $V_{i,1}$.
2. $V_{i,1}$ contains a meridian disk of $V$.
3. $\pi_1(V_{i,0}) \to \pi_1(V)$ is the zero homomorphism.
4. The intersection of the $V_{i,1}$ is the unique component of the comple-
   ment of $\bigcup A_i$ that contains a longitude of $V$.

**Proof.** For each $i$, every loop in $V$ has even algebraic intersection with $A_i$,
since every loop in $\partial V$ does, so $A_i$ separates $V$. Since $A_i$ is not incompress-
ible, it must be compressible in one of its complementary components, $V_{i,1}$,
and since $V$ is irreducible, $A_i$ must be incompressible in the other comple-
mentary component, $V_{i,0}$.

Notice that $V_{i,1}$ must contain a meridian disk of $V$. Indeed, if $K$ is the
union of $A_i$ with a compressing disk in $V_{i,1}$, then two of the components of
the frontier of a regular neighborhood of $K$ in $V$ are meridian disks of $V_{i,1}$.
Consequently, $\pi_1(V_{i,0}) \to \pi_1(V)$ is the zero homomorphism. The Mayer-
Vietoris sequence shows that $H_1(A) \to H_1(V_{i,0})$ and $H_1(V_{i,1}) \to H_1(V)$ are
isomorphisms.

Let $V_1$ be the intersection of the $V_{i,1}$, and let $V_0$ be the union of the
$V_{i,0}$. The Mayer-Vietoris sequence shows that $V_1$ is connected, and that
$H_1(V_1) \to H_1(V)$ is an isomorphism, so $V_1$ contains a longitude of $V$. No
other complementary component of $\bigcup A_i$ contains a longitude, since each
such component lies in one of the $V_{i,0}$ and all of its loops must be contractible
in $V$. \qed
4. Heegaard Tori in Very Good Position

A Heegaard torus in a lens space $L$ is a torus that separates $L$ into two solid tori. In this section we will develop some properties of Heegaard tori. Also, we introduce the concepts of discal and biessential intersection circles, good position, and very good position, which will be used extensively in later sections.

When $P$ is a Heegaard torus bounding solid tori $V$ and $W$, and $Q$ is a Heegaard torus contained in the interior of $V$, $Q$ need not be parallel to $\partial V$. For example, start with a core circle in $V$, move a small portion of it to $\partial V$, then pass it across a meridian disk of $W$ and back into $V$. This moves the core circle to its band-connected sum in $V$ with an $(m,q)$-curve in $\partial V$. By varying the choice of band—for example, by twisting it or tying knots in it—and by iterating this construction, one can construct complicated knotted circles in $V$ which are isotopic in $L$ to a core circle of $V$. The boundary of a regular neighborhood of such a circle is a Heegaard torus of $L$. But here is one restriction on Heegaard tori:

**Proposition 4.1.** Let $P$ be a Heegaard torus in a lens space $L$, bounding solid tori $V$ and $W$. If a loop $\ell$ imbedded in $P$ is a core circle for a solid torus of some genus-1 Heegaard splitting of $L$, then $\ell$ is a longitude for either $V$ or $W$.

**Proof.** Since $L$ is not simply-connected, $\ell$ is not a meridian for either $V$ or $W$, consequently $\pi_1(\ell) \to \pi_1(V)$ and $\pi_1(\ell) \to \pi_1(W)$ are injective. So $P - \ell$ is an open annulus separating $L - \ell$, making $\pi_1(L - \ell)$ a free product with amalgamation $\mathbb{Z} \ast \mathbb{Z}$. Since $\ell$ is a core circle, $\pi_1(L - \ell)$ is infinite cyclic, so at least one of the inclusions of the amalgamating subgroup to the infinite cyclic factors is surjective.  

Let $F_1$ and $F_2$ be transversely intersecting imbedded surfaces in the interior of a 3-manifold $M$. A component of $F_1 \cap F_2$ is called discal when it is contractible in both $F_1$ and $F_2$, and biessential when it is essential in both. We say that $F_1$ and $F_2$ are in good position when every component of their intersection is either discal or biessential, and at least one is biessential, and we say that they are in very good position when they are in good position and every component of their intersection is biessential.

Later, we will go to considerable effort to obtain pairs Heegaard tori for lens spaces that intersect in very good position. Even then, the configuration can be complicated. Consider a Heegaard torus $P$ bounding solid tori $V$ and $W$, and another Heegaard torus $Q$ that meets $P$ in very good position. When the intersection circles are not meridians for either $V$ or $W$, the components of $Q \cap V$ and $Q \cap W$ are annuli that are incompressible in $V$ and $W$, and must be as described in proposition 3.3. But if the intersection circles are meridians for one of the solid tori, say $V$, then $Q \cap V$ consists of meridional annuli, and as shown in figure 2, they need not be boundary-parallel. To obtain that configuration, one starts with a torus $Q$ parallel to $P$ and outside
Figure 2. Heegard tori in very good position, with non-boundary-parallel meridional annuli.

$P$, and changes $Q$ only by an isotopy on a regular neighborhood of a meridian $c$ of $Q$. First, $c$ passes across a meridian in $P$, then shrinks down to a small circle which traces around a knot. Then, it expands out to another meridian in $P$ and pushes across. The resulting torus meets $P$ in four circles which are meridians for $V$, and meets $V$ in two annuli, both isotopic to the non-boundary-parallel annulus in figure 1. The next lemma gives a small but important restriction on meridional annuli of $Q \cap V$.

**Lemma 4.2.** Let $P$ be a Heegaard torus which separates a lens space into two solid tori $V$ and $W$. Let $Q$ be another Heegaard torus whose intersection with $V$ consists of a single meridional annulus $A$. Then $A$ is boundary-parallel in $V$.

**Proof.** From lemma 3.4, $A$ separates $V$ into two components $V_0$ and $V_1$, such that $A$ is compressible in $V_1$ and $V_1$ contains a longitude of $V$. Suppose that $A$ is not boundary-parallel in $V$.

Let $A_0 = V_0 \cap \partial V$. Of the two solid tori in $L$ bounded by $Q$, let $X$ be the one that contains $A_0$, and let $Y$ be the other one. Since $Q \cap V$ consists only of $A$, $Y$ contains $V_1$, and in particular contains a compressing disk for $A$ in $V_1$ and a longitude for $V$.

Suppose that $A_0$ were incompressible in $X$. Since $A_0$ is not parallel to $A$, it would be parallel to $\partial X - A$. So $V_0$ would contain a core circle of $X$. Since $\pi_1(V_0) \to \pi_1(V)$ is the zero homomorphism, this implies that $L$ is simply-connected, a contradiction. So $A_0$ is compressible in $X$. A compressing disk for $A_0$ in $X$ is part of a 2-sphere that meets $Y$ only in a compressing disk of $A$ in $V_1$. This 2-sphere has algebraic intersection $\pm 1$ with the longitude of $V$ in $V_1$, contradicting the irreducibility of $L$. \qed

Regarding $D^2$ as the unit disk in the plane, for $0 < r < 1$ let $rD^2$ denote $\{(x, y) \mid x^2 + y^2 \leq r^2\}$. A solid torus $X$ imbedded in a solid torus $V$ is called *concentric in $V$* if there is some product structure $V = D^2 \times S^1$ such that $X = rD^2 \times S^1$. Equivalently, $X$ is in the interior of $V$ and some (hence every) core circle of $X$ is a core circle of $V$. 
The next lemma shows how we will use Heegard tori that meet in very good position.

**Lemma 4.3.** Let $P$ be a Heegaard torus which separates a lens space into two solid tori $V$ and $W$. Let $Q$ be another Heegaard torus, that meets $P$ in very good position, and assume that the annuli of $Q \cap V$ are incompressible in $V$. Then at least one component $C$ of $V - (Q \cap V)$ satisfies both of the following:

1. $C$ is a core region for $V$.
2. Suppose that $Q$ is moved by isotopy to a torus $Q_1$ in $W$, by pushing the annuli of $Q \cap V$ one-by-one out of $V$ using isotopies that move them across regions of $V - C$, and let $X$ be the solid torus bounded by $Q_1$ that contains $V$. Then $V$ is concentric in $X$.
3. After all but one of the annuli have been pushed out of $V$, the image $Q_0$ of $Q$ is isotopic to $P$ relative to $Q_0 \cap P$.

**Proof.** Assume first that $Q \cap V$ has only one component $A$. Then $\partial A$ separates $P$ into two annuli, $A_1$ and $A_2$. Since $A$ is incompressible in $V$, it is parallel in $V$ to one of the $A_i$, say $A_1$. Let $A' = Q \cap W$.

If $A'$ is longitudinal, then $A'$ is parallel in $W$ to $A_2$. So pushing $A$ across $A_1$ moves $Q$ to a torus in $W$ parallel to $P$, and the lemma holds, with $C$ being the region between $A$ and $A_2$. An isotopy from $Q$ to $P$ can be carried out relative to $Q \cap P$, giving the last statement of the lemma. Suppose that $A'$ is not longitudinal. If $A'$ is incompressible, then it is boundary parallel in $W$. If $A'$ is parallel to $A_2$, then we are finished as before. If $A'$ is parallel to $A_1$, but not to $A_2$, then there is an isotopy moving $Q$ to a regular neighborhood of a core circle of $A_1$. By proposition 4.1, $A$ is longitudinal, so must also be parallel in $V$ to $A_2$. In this case, we take $C$ to be the region between $A$ and $A_1$.

Suppose now that $Q \cap V$ and hence also $Q \cap W$ consist of $n$ annuli, where $n > 1$. By isotopies pushing outermost annuli in $V$ across $P$, we obtain $Q_0$ with $Q_0 \cap V$ consisting of one annulus $A$. At least one of its complementary components, call it $C$, satisfies the lemma. Let $Z$ be the union of the regions across which the $n - 1$ annuli were pushed. Since $C$ is a core region, $C \cap (V - Z)$ is also a core region (since a core circle of $V$ in $C$ can be moved, by the reverse of the pushout isotopies, to a core circle of $V$ in $C \cap (V - Z)$). So $C \cap (V - Z)$ satisfies the conclusion of the lemma. □

Here is a first consequence of lemma 4.3.

**Corollary 4.4.** Let $P$ be a Heegaard torus which separates a lens space into two solid tori $V$ and $W$, and let $Q$ be another Heegaard torus separating it into $X$ and $Y$. Assume that $Q$ meets $P$ in very good position. If the circles of $P \cap Q$ are meridians (respectively, longitudes) in $X$ or in $Y$, then they are meridians (longitudes) in $V$ or in $W$. 
Proof. We may choose notation so that the annuli of $Q \cap V$ are incompressible in $V$. Use lemma 4.3 to move $Q$ out of $V$. After all but one annulus has been pushed out, the image $Q_0$ of $Q$ is isotopic to $P$ relative to $Q_0 \cap P$. That is, the original $Q$ is isotopic to $P$ by an isotopy relative to $Q_0 \cap P$. If the circles of $Q \cap P$ were originally meridians of $X$ or $Y$, then in particular those of $Q_0 \cap P$ are meridians of $X$ or $Y$ after the isotopy, that is, of $V$ or $W$. The case of longitudes is similar. \qed
5. Sweepouts, and levels in very good position

In this section we will define sweepouts and related structures. Also, we will prove an important technical lemma concerning pairs of sweepouts having levels that meet in very good position.

By a sweepout of a closed orientable 3-manifold, we mean smooth map \( \tau: P \times [0, 1] \rightarrow M \), where \( P \) is a closed orientable surface, such that

1. \( T_0 = \tau(P \times \{0\}) \) and \( T_1 = \tau(P \times \{1\}) \) are disjoint graphs with each vertex of valence 3.
2. Each \( T_i \) is a union of a collection of smoothly imbedded arcs and circles in \( M \).
3. \( \tau|_{P \times (0, 1)}: P \times (0, 1) \rightarrow M \) is a diffeomorphism onto \( M - (T_0 \cup T_1) \).
4. Near \( P \times \partial I \), \( \tau \) gives a mapping cylinder neighborhood of \( T_0 \cup T_1 \).

Associated to any \( t \) with \( 0 < t < 1 \), there is a Heegaard splitting \( M = V_t \cup W_t \), where \( V_t = \tau(P \times [0, t]) \) and \( W_t = \tau(P \times [t, 1]) \). For each \( t \), \( T_0 \) is a deformation retract of \( V_t \) and \( T_1 \) is a deformation retract of \( W_t \). We denote \( \tau(P \times \{t\}) \) by \( P_t \) and call it a level of \( \tau \). Also, for \( 0 < s < t < 1 \) we denote the closure of the region between \( P_s \) and \( P_t \) (that is, \( \tau(P \times [s, t]) \)) by \( R(s, t) \). Note that any genus-1 Heegaard splitting of \( L \) provides sweepouts with \( T_0 \) and \( T_1 \) as core circles of the two solid tori, and the Heegaard torus as one of the levels.

A sweepout \( \tau: P \times [0, 1] \rightarrow M \) induces a continuous projection function \( \pi: M \rightarrow [0, 1] \) by the rule \( \pi(\tau(x, t)) = t \). By composing this with a smooth bijection from \([0, 1]\) to \([0, 1]\) all of whose derivatives vanish at 0 and at 1, we may reparameterize \( \tau \) to ensure that \( \pi \) is a smooth map. We always assume that \( \tau \) has been selected to have this property.

The next lemma gives very strong restrictions on levels of two different sweepouts of a lens space that intersect in very good position. For its proof, recall that a spine for a connected surface \( P \) is a 1-dimensional cell complex in \( P \) whose complement consists of open disks.

**Lemma 5.1.** Let \( L \) be a lens space. Let \( \tau: T \times [0, 1] \rightarrow L \) be a sweepout as above, where \( T \) is a torus. Let \( \sigma: T \times [0, 1] \rightarrow L \) be another sweepout, with levels \( Q_s = \sigma(T \times \{s\}) \). Suppose that for \( t_1 < t_2 \), \( s_1 \neq s_2 \), and \( i = 1, 2 \), \( Q_{s_i} \) and \( P_{t_i} \) intersect in very good position, and that \( Q_{s_1} \) has no discal intersections with \( P_{t_2} \). If \( Q_{s_1} \) has nonempty intersection with \( P_{t_2} \), then either

1. every intersection circle of \( Q_{s_1} \) with \( P_{t_2} \) is biessential, and consequently \( Q_{s_1} \cap R(t_1, t_2) \) contains an annulus with one boundary circle essential in \( P_{t_1} \) and the other essential in \( P_{t_2} \), or
2. for \( i = 1, 2 \), \( Q_{s_i} \cap P_{t_i} \) consists of meridians of \( W_{t_i} \), and \( Q_{s_1} \cap R(t_1, t_2) \) contains a surface \( \Sigma \) which is a homology from a circle of \( Q_{s_1} \cap P_{t_1} \) to a union of circles in \( P_{t_2} \).

Figure 3 illustrates case (2) of lemma 5.1.

We mention that that to apply lemma 5.1 when \( t_1 > t_2 \), we interchange the roles of \( V_{t_i} \) and \( W_{t_i} \). The intersection circles in case (2) are then meridians of the \( V_{t_i} \) rather than the \( W_{t_i} \).
Proof of lemma 5.1. Assume for now that the circles of \( Q_{s_2} \cap P_{t_2} \) are not meridians of \( W_{t_2} \).

We first rule out the possibility that there exists a circle of \( Q_{s_1} \cap P_{t_2} \) that is inessential in \( Q_{s_1} \). If so, there would be a circle \( C \) of \( Q_{s_1} \cap P_{t_2} \), bounding a disk \( D \) in \( Q_{s_1} \) with interior disjoint from \( P_{t_2} \). Since \( Q_{s_1} \) and \( P_{t_2} \) have no discal intersections, \( C \) is essential in \( P_{t_2} \), so \( D \) is a meridian disk for \( V_{t_2} \) or \( W_{t_2} \). It cannot be a meridian of \( V_{t_2} \), contradicting the fact that \( Q_{s_1} \) and \( P_{t_2} \) meet in very good position. But \( D \) cannot be a meridian disk for \( W_{t_2} \), since \( D \) is disjoint from \( Q_{s_2} \) and the circles of \( Q_{s_2} \cap P_{t_2} \) are not meridians of \( W_{t_2} \).

We now rule out the possibility that there exists a circle of \( Q_{s_1} \cap P_{t_2} \) that is essential in \( Q_{s_1} \) and inessential in \( P_{t_2} \). There is at least one biessential intersection circle of \( Q_{s_1} \) with \( P_{t_1} \), hence also an annulus \( A \) in \( Q_{s_1} \) with one boundary circle essential in \( P_{t_1} \), the other essential in \( P_{t_2} \), and all intersections of the interior of \( A \) with \( P_{t_1} \cup P_{t_2} \) essential in \( A \). The interior of \( A \) must be disjoint from \( P_{t_1} \), since \( Q_{s_1} \) meets \( P_{t_1} \) in very good position. It must also be disjoint from \( P_{t_2} \), by the previous paragraph. So, since \( A \) has at least one boundary circle in \( P_{t_2} \), it is properly imbedded either in \( R(t_1, t_2) \) or in \( W_{t_2} \). It cannot be in \( R(t_1, t_2) \), since it has one boundary circle inessential in \( P_{t_2} \) and the other essential in \( P_{t_1} \cup P_{t_2} \).

So \( A \) is in \( W_{t_2} \), and since one boundary circle is inessential in \( P_{t_2} \), the other must be a meridian, contradicting the assumption that no circle of \( Q_{s_2} \cap P_{t_2} \) is a meridian of \( W_{t_2} \). Thus conclusion (1) holds when circles of \( Q_{s_2} \cap P_{t_2} \) are not meridians of \( W_{t_2} \).

Assume now that the circles of \( Q_{s_2} \cap P_{t_2} \) are meridians of \( W_{t_2} \). We will achieve conclusion (2).

Suppose first that some circle of \( Q_{s_1} \cap P_{t_2} \) is essential in \( Q_{s_1} \). Then there is an annulus \( A \) in \( Q_{s_1} \) with one boundary circle essential in \( P_{t_1} \), the other essential in \( P_{t_2} \), and all intersections of the interior of \( A \) with \( P_{t_1} \cup P_{t_2} \) inessential in \( A \). Since \( Q_{s_1} \) meets \( P_{t_1} \) in very good position, the interior of \( A \) must be disjoint from \( P_{t_1} \). So \( A \cap R(t_1, t_2) \) contains a planar surface \( \Sigma \) with one boundary component a circle of \( Q_{s_1} \cap P_{t_1} \) and the other
boundary components circles in $P_{t_2}$ which are meridians in $W_{t_2}$, giving the conclusion (2) of the lemma.

Suppose now that every circle of $Q_{s_1} \cap P_{t_2}$ is contractible in $Q_{s_1}$. We will show that this case is impossible. An intersection circle innermost on $Q_{s_1}$ bounds a disk $D$ in $Q_{s_1}$ which is a meridian disk for $W_{t_2}$, since $\partial D$ is essential in $P_{t_2}$ and disjoint from $Q_{s_2} \cap P_{t_2}$. Now, use lemma 4.3 to push $Q_{s_2} \cap V_{t_2}$ out of $V_{t_2}$ by an ambient isotopy of $L$. Suppose for contradiction that one of these pushouts, say, pushing an annulus $A_0$ in $Q_{s_2}$ across an annulus in $P_{t_2}$, also eliminates a circle of $Q_{s_1} \cap P_{t_1}$. Let $Z$ be the region of parallelism across which $A_0$ is pushed. Since $Z$ contains an essential loop of $Q_{s_1}$, and each circle of $Q_{s_1} \cap P_{t_2}$ is contractible in $Q_{s_1}$, $Z$ contains a spine of $Q_{s_1}$. This spine is isotopic in $Z$ into a neighborhood of a boundary circle of $A_0$. Since this boundary circle is a meridian of $W_{t_2}$, every circle in the spine is contractible in $L$. This contradicts the fact that $Q_{s_1}$ is a Heegaard torus. So the pushouts do not eliminate intersections of $Q_{s_1}$ with $P_{t_1}$, and after the pushouts are completed, the image of $Q_{s_1}$ still meets $P_{t_1}$.

During the pushouts, some of the intersection circles of $Q_{s_1}$ with $P_{t_2}$ may disappear, but not all of them, since the pushouts only move points into $W_{t_2}$. So after the pushouts, there is a circle of $Q_{s_1} \cap P_{t_2}$ that bounds a innermost disk in $Q_{s_1}$ (since all the original intersection circles of $Q_{s_1}$ with $P_{t_2}$ bound disks in $Q_{s_1}$, and the new intersection circles are a subset of the old ones). Since the boundary of this disk is a meridian of $W_{t_2}$, the disk it bounds in $Q_{s_1}$ must be a meridian disk of $W_{t_2}$. The image of $Q_{s_2}$ lies in $W_{t_2}$ and misses this meridian disk, contradicting the fact that $Q_{s_2}$ is a Heegaard torus. □
6. The Rubinstein-Scharlemann graphic

The purpose of this section is to present a number of definitions, and to sketch the proof of theorem 6.1 below, originally from [31]. It requires the hypothesis that two sweepouts meet in general position in a strong sense that we call Morse general position. In section 9, this proof will be adapted to the weaker concept of general position developed in section 8.

Consider a smooth function \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \). A critical point of \( f \) is stable when it is locally equivalent under smooth change of coordinates of the domain and range to \( f(x, y) = x^2 + y^2 \) or \( f(x, y) = x^2 - y^2 \). The first type is called a center, and the second a saddle. An unstable critical point is called a birth-death point if it is locally \( f(x, y) = x^2 + y^3 \).

Let \( \tau : P \times [0, 1] \to M \) be sweepouts as in section 5. As in that section, we denote \( \tau(P \times \{0, 1\}) \) by \( T \), \( \tau(P \times \{t\}) \) by \( P_t \), \( \tau(P \times \{[0, t]\}) \) by \( V_t \), and \( \tau(P \times \{t, 1\}) \) by \( W_t \). For a second sweepout \( \sigma : Q \times [0, 1] \to M \), we denote \( \sigma(Q \times \{0, 1\}) \) by \( S \), \( \sigma(Q \times \{s\}) \) by \( Q_s \), \( \sigma(Q \times [0, s]) \) by \( X_s \), and \( \sigma(Q \times [s, 1]) \) by \( Y_s \). We call \( Q_s \) a \( \sigma \)-level and \( P_t \) a \( \tau \)-level.

A tangency of \( Q_s \) and \( P_t \) at a point \( w \) is said to be of Morse type at \( w \) if in some local \( xyz \)-coordinates with origin at \( w \), \( P_t \) is the \( xy \)-plane and \( Q_s \) is the graph of a function which has a stable critical point or a birth-death point at the origin. Note that this condition is symmetric in \( Q_s \) and \( P_t \). We may refer to a tangency as stable or unstable, and as a center, saddle, or birth-death point.

A tangency of \( S \) with a \( \tau \)-level is said to be stable if there are local \( xyz \)-coordinates in which the \( \tau \)-levels are the planes \( \mathbb{R}^2 \times \{z\} \) and \( S \) is the graph of \( z = x^2 \) in the \( xz \)-plane. In particular, the tangency is isolated and cannot occur at a vertex of \( S \). There is an analogous definition of stable tangency of \( T \) with a \( \sigma \)-level.

We will say that \( \sigma \) and \( \tau \) are in Morse general position when the following hold:

1. \( S \) is disjoint from \( T \),
2. all tangencies of \( S \) with \( \tau \)-levels and of \( T \) with \( \sigma \)-levels are stable,
3. all tangencies of \( \sigma \)-levels with \( \tau \)-levels are of Morse type, and only finitely many are birth-death points,
4. each pair consisting of a \( \sigma \)-level and a \( \tau \)-level has at most two tangencies, and
5. there are only finitely many pairs consisting of a \( \sigma \)-level and a \( \tau \)-level with two tangencies, and for each of these pairs both tangencies are stable.

The following concept due to A. Casson and C. McA. Gordon [5] is a crucial ingredient in [31]. A Heegaard splitting \( M = V \cup_p W \) is called strongly irreducible when every compressing disk for \( V \) meets every compressing disk for \( W \).
Suppose that $P$ is a Heegaard surface in $M$, bounding a handlebody $V$. We define a precompression or precompressing disk for $P$ in $V$ to be an imbedded disk $D$ in $M$ such that

1. $\partial D$ is an essential loop in $P$,
2. $D$ meets $P$ transversely at $\partial D$, and $V$ contains a neighborhood of $\partial D$,
3. the interior of $D$ is transverse to $P$, and its intersections with $P$ are discal.

Provided that $M$ is irreducible, a precompression for $P$ in $V$ is isotopic relative to a neighborhood of $\partial D$ to a compressing disk for $P$ in $V$. In particular, if the Heegaard splitting is strongly irreducible, then the boundaries of a precompression for $P$ in $V$ and a precompression for $P$ in $M - V$ must intersect.

A sweepout is called strongly irreducible when the associated Heegaard splittings are strongly irreducible. We can now state the main technical result of [31].

**Theorem 6.1** (Rubinstein-Scharlemann). Let $M \neq S^3$ be a closed orientable 3-manifold, and let $\sigma, \tau : F \times [0,1] \to M$ be strongly irreducible sweepouts of $M$ which are in Morse general position. Then there exists $(s,t) \in (0,1) \times (0,1)$ such that $Q_s$ and $P_t$ meet in good position.

The closure in $I^2$ of the set $(s,t)$ for which $Q_s$ and $P_t$ have a tangency is a graph $\Gamma$. On $\partial I^2$, it can have valence-1 vertices corresponding to valence-3 vertices of $S$ or $T$, and valence-2 vertices corresponding to points of tangency of $S$ with a $\tau$-level or $T$ with a $\sigma$-level (see p. 1008 of [31], see also [24] for an exposition with examples). In the interior of $I^2$, it can have valence-4 vertices which correspond to a pair of levels which have two stable tangencies, and valence-2 vertices which correspond to pairs of levels having a birth-death tangency.

The components of the complement of $\Gamma$ in the interior of $I^2$ are called regions. Each region is either unlabeled or bears a label consisting of up to four letters. The labels are determined by the following conditions on $Q_s$ and $P_t$, which by transversality hold either for every $(s,t)$ or for no $(s,t)$ in a region.

1. If $Q_s$ contains a precompression for $P_t$ in $V_t$ (respectively, in $W_t$), the region receives the letter $A$ (respectively, $B$).
2. If $P_t$ contains a precompression for $Q_s$ in $X_s$ (respectively, in $Y_s$), the region receives the letter $X$ (respectively, $Y$).
3. If the region has neither an $A$-label nor a $B$-label, and $V_t$ (respectively, $W_t$), contains a spine of $Q_s$, the region receives the letter $b$ (respectively, $a$).
4. If the region has neither an $X$-label nor a $Y$-label, and $X_s$ (respectively, $Y_s$), contains a spine of $P_t$, the region receives the letter $y$ (respectively, $x$).
With these conventions, $Q_s$ and $P_t$ are in good position if and only if the region containing $(s, t)$ is unlabeled. To check this, assume first that they are in good position. Since all intersections are biessential or discal, neither surface can contain a precompressing disk for the other, and since there is a biessential intersection circle, the complement of one surface cannot contain a spine for the other. For the converse, an intersection circle which is not biessential or discal leads to a precompression as in (1) or (2), so assume that all intersections are discal. Then the complement of the intersection circles in $Q_s$ contains a spine, so the region has either an $a$- or $b$-label, and by the same reasoning applied to $P_t$ the region has either an $x$- or $y$-label. This verifies the assertion, as well as the following lemma.

**Lemma 6.2.** If the label of a region contains the letter $a$ or $b$, then it must also contain either $x$ or $y$. Similarly, if it contains $x$ or $y$, then it must also contain $a$ or $b$.

We call the data consisting of the graph $\Gamma \subset I^2$ and the labeling of a subset of its regions the *Rubinstein-Scharlemann graphic* associated to the sweepouts. Regions of the graphic are called *adjacent* if there is an edge of $\Gamma$ which is contained in both of their closures.

At this point, we begin to make use of the fact that the sweepouts are strongly irreducible. The labels will then have the following properties, where $A$ stands for either of $A$ and $a$, and $B$, $X$, and $Y$ are defined similarly.

1. **(RS1)** A label cannot contain both an $A$ and an $a$, or both an $X$ and a $Y$ (direct from the labeling rules and the definition of strong irreducibility).
2. **(RS2)** If the label of a region contains $A$, then the label of any adjacent region cannot contain $B$. Similarly for $X$ and $Y$ (Corollary 5.5 of [31]).
3. **(RS3)** If all four letters $A$, $B$, $X$, and $Y$ appear in the labels of the regions that meet at a valence-4 vertex of $\Gamma$, then two opposite regions must be unlabeled (Lemma 5.7 of [31]).

Property (RS2) warrants special comment, since it will play a major role in our later work. The analysis of labels of adjacent regions given in section 5 of [31] uses only the fact that for the points $(s, t)$ in an open edge of $\Gamma$, the corresponding $Q_s$ and $P_t$ have a single stable tangency. The open edges of the more general graphics we will use for the diffeomorphisms in parameterized families in general position will still have this property, so the labels of their graphics will still satisfy property (RS2). They will not satisfy property (RS3), indeed the $\Gamma$ for their graphics will have vertices of high valence, so property (RS3) will not even be meaningful.

We now analyze the labels of regions whose closures meet $\partial I^2$, as on p. 1012 of [31]. Consider first a region whose closure meets the side $s = 0$ (we consider $s$ to be the horizontal coordinate, so this is the left-hand side of the square). The region must contains points $(s, t)$ with $s$ arbitrarily close
to 0. These correspond to \(Q_s\) which are extremely close to \(S_0\). For almost all \(t\), \(S_0\) is transverse to \(P_t\), and for sufficiently small \(s\) any intersection of such a \(P_t\) with \(Q_s\) must be an essential circle of \(Q_s\) bounding a disk in \(P_t\) that lies in \(X_s\), in which case the region must have an \(X\)-label. If \(P_t\) is disjoint from \(Q_s\), then \(P_t\) lies in \(Y_s\) so the region has an \(x\)-label. That is, all such regions have an \(X\)-label. Similarly, the label of any region whose closure meets the edge \(t = 0\) (respectively, \(s = 1, t = 1\)) contains \(A\) (respectively, \(Y, B\)).

We will set up some of the remaining steps a bit differently from those of [31], so that their adaptation to our later arguments will be more transparent. We have seen that it is sufficient to prove that there exists an unlabeled region in the graphic defined by the sweepouts. To accomplish this, Rubinstein and Scharlemann use the shaded subset of the square shown in figure 4. It is a simplicial complex in which each of the four triangles is a 2-simplex. Henceforth we will refer to it as the Diagram.

Suppose for contradiction that every region in the Rubinstein-Scharlemann graphic is labeled. Let \(\Delta\) be a triangulation of \(I^2\) such that each vertex of \(\Gamma\) and each corner of \(I^2\) is a 0-simplex, and each edge of \(\Gamma\) is a union of 1-simplices. Let \(K\) be \(I^2\) with the structure of a regular 2-complex dual to \(\Delta\). We observe the following properties of \(K\):

(K1) Each 0-cell of \(K\) lies in the interior of a side of \(\partial I^2\) or in a region.

(K2) Each 1-cell of \(K\) either lies in \(\partial I^2\), or is disjoint from \(\Gamma\), or crosses one edge of \(\Gamma\) transversely in one point.

(K3) Each 2-cell of \(K\) either contains no vertex of \(\Gamma\), in which case all of its 0-cell faces that are not in \(\partial I^2\) lie in one region or in two adjacent regions, or contains one vertex of \(\Gamma\), in which case all of its 0-cell faces which do not lie in \(\partial I^2\) lie in the union of the regions whose closures contain that vertex.

We now construct a map from \(K\) to the Diagram. First, each 0-cell in \(\partial K\) is sent to one of the single-letter 0-simplices of the diagram: if it lies in the side \(s = 0\) (respectively, \(t = 0\), \(s = 1\), \(t = 1\)) then it is sent to the 0-simplex labeled \(X\) (respectively, \(sA, sY, sB\)). Similarly, any 1-cell in a side of \(\partial K\) is sent to the 0-simplex that is the image of its endpoints, and the four 1-cells in \(\partial K\) dual to the original corners are send to the 1-simplex whose
endpoints are the images of the endpoints of the 1-cell. Notice that $\partial K$
maps essentially onto the circle consisting of the four diagonal 1-simplices
of the Diagram.

We will now show that if there is no unlabeled region, this map extends to
$K$, a contradiction. Since an unlabeled region produces pairs $Q_s$ and $P_t$
that meet in good position, this will complete the proof sketch of theorem 6.1.

Now we consider cells of $K$ that do not lie entirely in $\partial K$. Each 0-cell in
the interior of $K$ lies in a region. By (RS1), the label of each 0-cell has a
form associated to one of the 0-simplices of the Diagram, and we send the
0-cell to that 0-simplex.

Consider a 1-cell of $K$ that does not lie in $\partial K$. Suppose it has one endpoint
in $\partial K$, say in the side $s = 0$ (the other cases are similar). The other endpoint
lies in a region whose closure meets the side $s = 0$, so its label contains $x$.
Therefore the images of the endpoints of the 1-cell both contain $x$, so lie
either in a 0-simplex or a 1-simplex of the Diagram. We extend the map to
the 1-cell by sending it into that 0- or 1-simplex. Suppose the 1-cell lies in
the interior of $K$. Its endpoints lie either in one region or in two adjacent
regions. If the former, or the latter and the labels of the regions are equal,
we send the 1-cell to the 0-simplex for that label. If the latter and the labels
of the regions are different, then property (RS2) shows that the labels span
a unique 1-simplex of the Diagram, in which case we send the 1-cell to that
1-simplex.

Assuming that the map has been extended to the 1-cells in this way,
consider a 2-cell of $K$. Suppose first that it has a face that lies in the side
$s = 0$ (the other cases are similar). Then each of its 0-cell faces lies in one
of the sides $s = 0$, $t = 0$, or $t = 1$, or in a region whose closure meets $s = 0$.
In the latter case, we have seen that the label of the region must contain $x$,
so it cannot contain $y$, and in particular it cannot be a single letter $y$. In
no case does the 0-cell map to the vertex $y$ of the Diagram, so the image of
the boundary of the 2-cell maps into the complement of that vertex in the
Diagram. Since that complement is contractible, the map extends over the
2-cell.

Suppose now that the 2-cell lies entirely in the interior of $K$. If it is dual
to a 0-simplex of $\Delta$ that lies in a region or in the interior of an edge of $\Gamma$,
then all its 0-cell faces lie in a region or in two adjacent regions. In this
case, all of its 1-dimensional faces map into some 1-simplex of the Diagram,
so the map on the faces extends to a map of the 2-cell into that 1-simplex.
Suppose the 2-cell is dual to a vertex of $\Gamma$. Its faces lie in the union of
regions whose closures contain the vertex. If the vertex has valence 2, then
all 0-cell faces lie in two adjacent regions (actually, in this case, the regions
must have the same label) and the map extends to the 2-cell as before. If
the vertex has valence 4, then by (RS3), the labels of the four regions whose
closures contain the vertex must all avoid at least one of the four letters.
This implies that the boundary of the 2-cell of $K$ maps into a contractible
subset of the Diagram. So again the map can be extended over the 2-cell, giving us the desired contradiction.

We emphasize that the map from $K$ to the Diagram carries each 1-cell of $K$ to a 0-simplex or a 1-simplex of the Diagram, principally due to property (RS2).
7. Graphics having no unlabeled region

One cannot hope to perturb a parameterized family of sweepouts to be in Morse general position. One must allow for the possibility of levels having tangencies of high order, and having more than two tangencies. We will see in section 8 that all such phenomena can be isolated at the vertices of the graph $\Gamma$ in the graphic. In particular, the $(s,t)$ that lie on the open edges of $\Gamma$ will still correspond to pairs of levels that have a single stable tangency, and therefore their associated graphics will still have property (RS2). Achieving this property for the edges of $\Gamma$ will require considerable effort, so before beginning the task, we will show that the hard work really is necessary. We will give here examples of pairs of sweepouts which have a graphic with no unlabeled region. It will be clear that what goes wrong is the existence of edges of $\Gamma$ that consist of pairs having multiple tangencies, and the corresponding failure of the graphic to have property (RS2).

This section is not part of the proof of the Smale Conjecture for Lens Spaces, and can be read independently from the rest of the paper (provided that one is familiar with Rubinstein-Scharlemann graphics and their labeling scheme).

We will first construct examples in $S^2 \times S^1$, then show how to further modify them to obtain examples in any $L(m,q)$.

The first step is to construct a pair of sweepouts of $S^2 \times S^1$, with the graphic shown on the left in figure 5. In figure 5, the edges of pairs for which the corresponding levels have a single center tangency are shown as dotted. The four corner regions are not labeled, since their labels are the same as the regions that are adjacent to them along an edge of centers.

After constructing the sweepouts that produce the first graphic, we will see how to move one of the sweepouts by isotopy to "collapse" the unlabeled region. Two edges of the first graphic are moved to coincide, producing the graphic on the right in figure 5. The three open edges that lie on the diagonal $y = x$ consist of pairs of levels which have two saddle tangencies. The two vertices where the edges labeled 1 and 4 cross the diagonal correspond to pairs having three saddle tangencies.

As it is rather difficult to visualize the sweepouts directly, we describe them by level pictures for various $P_t$. The $Q_s$ appear as level curves in each $P_t$. Here are some general conventions:

(i) A solid dot is a center tangency.
(ii) An open dot (i.e., a tiny circle) is a point in one of the singular circles $S_i$ of the $Q_s$-sweepout.
(iii) Double-thickness lines are intersections with a $Q_s$ that have more than one tangency.
(iv) In figures 6 and 7, dashed lines are biessential intersection circles (in figure 9, they have a different meaning).

In a picture of a $P_t$, the level curves $P_t \cap Q_s$ that contain saddles appear as curves with self-crossings, and we label the crossings with 1, 2, 3, or 4 to
indicate which edge of the graphic in figure 5 contains that \((s,t)\)-pair. For a fixed \(t\), \(s(n)\) will denote the \(s\)-level of saddle \(n\). That is, in the graphic the edge of \(\Gamma\) labeled \(n\) contains the point \((s(n),t)\).

Figure 6 shows some \(P_t\) with \(t \leq 1/2\), for a sweepout of \(S^2 \times S^1\) whose graphic is the one shown in the left of figure 5. Here are some notes on figure 6.

1. In (a)-(f), the circles \(x = \text{constant}\) are longitudes of \(V_t\), and the circles \(y = \text{constant}\) are meridians.
2. The point represented by the four corners is the point of \(P_t\) with largest \(s\)-level. In (a) it is a tangency of \(P_{t/2}\) with \(S_1\), and in (b)-(f) it is a center tangency of \(P_t\) with \(Q_{t+1/2}\).
3. The open dots in the interior of the squares are intersections of \(P_t\) with \(S_0\). In (a) it is a tangency of \(P_{t/2}\) with \(S_0\), in (b)-(e) they are transverse intersections. In (f), \(P_t\) is disjoint from \(S_0\).
4. In (b), saddle 1 has appeared. Circles of \(Q_s \cap P_t\) with \(s < s(1)\) are essential in \(Q_s\), and these \((s,t)\) lie in the region labeled \(X\) in the graphic. Circles of \(Q_s \cap P_t\) with \(s(1) < s < s(2)\) enclose the figure-8 in (b), which is \(P_t \cap Q_{s(1)}\). They are inessential in both \(Q_s\) and \(P_t\), and these \((s,t)\) lie in the region labeled \(bx\). The vertical dotted lines are biessential intersections corresponding to a pair in the unlabeled region. Finally, one crosses \(Q_{s(3)}\), and eventually reaches the center tangency.
5. The horizontal level curves shown in (f) are meridians of \(V_t\) that bound disks in the \(Q_s\) that contain them. This \((s,t)\) lies in the region labeled \(A\) in the graphic.

For \(t > 1/2\), the intersection pattern of \(P_t\) with the \(Q_s\) is isomorphic to the pattern for \(P_{1-t}\), by an isomorphism for which \(Q_s\) corresponds to \(Q_{1-s}\). As one starts \(t\) at \(1/2\) and moves upward through \(t\)-levels, saddle 4 appears
inside the component of $P_t - Q_{s(3)}$ that is an open disk, and expands until the level where $s(3) = s(4)$. The biessential intersection circles in (a)-(d) are again longitudes in $V_t$ and in $W_t$, and the horizontal intersection circles in (f) are meridians of $W_t$. These $(s,t)$ lie in the region labeled $B$ in the graphic. This completes the description of the sweepouts in Morse general position.

Figure 7 shows some $P_t$ for a sweepout of $S^2 \times S^1$ whose graphic is the one shown in the right of figure 5. This sweepout is obtained from the previous one by an isotopy that moves parts of the $Q_s$ levels down (to lower $t$-levels) near saddle 2 and up near saddle 3. Again, the portion that is shown fits together with a similar portion for $1/2 \leq t \leq 1$. As $t$ increases past $1/2$, saddle 4 appears in the component of $P_t - S_{s(2)}$ that contains the point which appears as the four corners.
Figure 6. Intersections of the $Q_s$ with fixed $P_t$ as $t$ decreases from $1/2$ to 0, for the sweepouts with an unlabeled region.

(a) $P_{1/2}$.
(b) $P_t$ where $s(1) < s(2) < s(3)$.
(c) $P_t$ where $s(1) = s(2)$.
(d) $P_t$ where $s(2) < s(1) < s(3)$.
(e) $P_t$ where $s(1) = s(3)$.
(f) $P_t$ where $s(3) < s(1)$, and after saddle 2 changes to a center.
Figure 7. Intersections of the $Q_s$ with fixed $P_t$ as $t$ decreases from $1/2$ to $0$, for the sweepouts with no unlabeled region.

(a) $P_{1/2}$.
(b) $P_t$ where $s(1) < s(2) = s(3)$.
(c) $P_t$ where $s(1) = s(2) = s(3)$.
(d) $P_t$ where $s(2) = s(3) < s(1)$.
(e) $P_t$ where $s(2) < s(3) < s(1)$.
(f) $P_t$ where $s(3) < s(1)$, and after saddle 2 changes to a center.
We will now explain how to modify this construction to obtain a pair of sweepouts with no unlabeled region for any $L(m, q)$. The graphic will be the same as the one on the right in figure 5, except that near the point $(1/2, 1/2)$, a small portion of the 2,3-edge will have a sequence of elaborations, called bowties, two of which are shown in figure 8. We remark that for any $t$ near $1/2$, one has $s(2) = s(3) = t$, since the 2,3-edge is the diagonal of the graphic. That is, $P_t \cap Q_t$ contains saddles 2 and 3 for $t$ near $1/2$.

Figure 9 shows various $P_t$ for a bowtie elaboration. Consider the lower-left bowtie in figure 8. Let $(t_0, t_0)$ be the point where its two saddle edges cross the diagonal 2,3-edge. Figure 9(a) shows $P_t \cap Q_t$ in $P_t$ for $t$ near $1/2$ but with $t$ below the level where the bowtie elaboration begins. This could be the lower endpoint of the portion of the 2,3-edge shown in figure 8. We have drawn the levels in $P_t$ a bit differently from the picture of $P_{1/2}$ in figure 7, but this picture is isotopic to figure 7(a). In figure 9(a), the closure $X$ of one of the components of $P_t - (P_t \cap Q_t)$ deformation retracts to a figure-8 $C \cup D$, where the circle $C$ passes through saddle 2 and is a longitude $L$ of $V_t$, and the circle $D$ passes through saddle 3 and is a meridian $M$ of $V_t$.

As one moves up in $t$-levels, one passes through two birth-death points, producing two center-saddle pairs. One birth-death point is at an $s$-level with $s < t$, and the other is at an $s$-level with $s > t$, so they lie in different components of $P_t - Q_t$. Figure 9(b) shows the two new center and saddle tangencies, along with $P_t \cap Q_t$, in such a level with $t < t_0$. Figure 9(c) shows $P_{t_0}$. The two new saddles are then in $Q_{t_0}$, along with saddles 2 and 3. In figure 9(d), we see $P_t \cap Q_t$ in a $P_t$ for $t > t_0$. The effect is to reposition the component $X$ so that $C$ still represents $L$, but $D$ represents $M + L$.

We remark that figure 9 is rather schematic. Since the bowtie elaborations lie very close to the diagonal, the new centers and saddles actually lie very close to $Q_t$. The elongations on $P_t \cap Q_t$ that reach toward the saddle points in figure 9(c) would actually follow along very close to the $P_t \cap Q_t$ of figure 9(a).

By a sequence of such bowtie elaborations, one can change $P_t \cap Q_t$ so that $C$ and $D$ represent any pair of generators of $H_1(P_t)$. In particular, we may move them so that $D$ represents $mL + qM$, the meridian of $W_t$ in $L(m, q)$. Then, $C$ is a longitude of $W_t$, and the portion in $W_{1/2}$ of the sweepouts of
Figure 9. Various $P_t$-levels illustrating a bowtie elaboration.
(a) $P_t$ for $t$ below the bowtie elaboration.
(b) $P_t$ above the two birth-death points, but below $t_0$.
(c) $P_{t_0}$.
(d) $P_t$ for $t > t_0$.

$S^2 \times S^1$ with no unlabeled region can be placed into this $W_t$ in $L(m, q)$, producing a pair of sweepouts whose graphic is obtained from the one on the right in figure 5 by bowtie elaborations. Since the bowtie elaborations produce no unlabeled regions, this graphic has no unlabeled region.
8. Graphics for parameterized families

In this section we prove that a parameterized family of sweepouts can be perturbed so that a suitable graphic exists at each parameter. As discussed in section 7, in a parameterized family one must allow for the possibility of levels having tangencies of high order, and having more than two tangencies.

Additional complications arise because one cannot avoid having parameters where the singular sets of the sweepouts intersect, or where the singular sets have high-order tangencies with levels. We sidestep these complications by working only with sweepout parameters that lie in an interval $[\epsilon, 1 - \epsilon]$. The graphic is only considered to exist on the square $[\epsilon, 1 - \epsilon] \times [\epsilon, 1 - \epsilon]$, which we call $I^2_\epsilon$. The number $\epsilon$ is chosen so that the labels of regions whose closure meets a side of $I^2_\epsilon$ will be known to include certain letters. Just as before, this will ensure that the map to the Diagram be essential on the boundary of the dual complex $K$.

These considerations, and the examples in section 7, motivate our definition of a general position family of diffeomorphisms. As usual, let $M$ be a closed orientable 3-manifold and $\tau : P \times [0, 1] \to M$ a sweepout with singular set $T = T_0 \cup T_1$ and level surfaces $P_t$ bounding handlebodies $V_t$ and $W_t$. Let $f : M \times W \to M$ be a parameterized family of diffeomorphisms, where $W$ is a compact manifold. For $u \in W$ we denote the restriction of $f$ to $M \times \{u\}$ by $f_u$. When a choice of parameter $u$ has been fixed, we denote $f_u(P_s)$ by $Q_s$, and $f_u(V_s)$ and $f_u(W_s)$ by $X_s$ and $Y_s$ respectively. When $Q_s$ meets $P_t$ transversely, a label is assigned to $(s,t)$ as in section 6.

A preliminary definition will be needed. We say that a positive number $\epsilon$ gives border label control for $f$ if the following hold at each parameter $u$:

1. If $t \leq 2\epsilon$, then there exists $r$ such that $Q_r$ meets $P_t$ transversely and contains a compressing disk of $V_t$.
2. If $t \geq 1 - 2\epsilon$, then there exists $r$ such that $Q_r$ meets $P_t$ transversely and contains a compressing disk of $W_t$.
3. If $s \leq 2\epsilon$, then there exists $r$ such that $P_r$ meets $Q_s$ transversely and contains a compressing disk of $X_s$.
4. If $s \geq 1 - 2\epsilon$, then there exists $r$ such that $P_r$ meets $Q_s$ transversely and contains a compressing disk of $Y_s$.

Throughout this section, a graph is a compact space which is a disjoint union of a CW-complex of dimension $\leq 1$ and circles. The circles, if any, are considered to be open edges of the graph.

We say that $f$ is in general position (with respect to the sweepout $\tau$) if there exists $\epsilon > 0$ such that $\epsilon$ gives border label control for $f$ and such that the following hold for each parameter $u \in W$.

(GP1) For each $(s,t)$ in $I^2_\epsilon$, $Q_s \cap P_t$ is a graph. At each point in an open edge of this graph, $Q_s$ meets $P_t$ transversely. At each vertex, they are tangent.
(GP2) The \((s,t) \in I^2_\epsilon\) for which \(Q_s\) has a tangency with \(P_t\) form a graph \(\Gamma_u\) in \(I^2_\epsilon\).

(GP3) If \((s,t)\) lies in an open edge of \(\Gamma_u\), then \(Q_s\) and \(P_t\) have a single stable tangency.

Here is the main result of this section.

**Theorem 8.1.** Let \(f: M \times W \to M\) be a parameterized family of diffeomorphisms. Then by an arbitrarily small deformation, \(f\) can be put into general position with respect to \(\tau\).

The proof of theorem 8.1 will constitute the remainder of this section. Since the argument is rather long, we will break it into subsections. Until subsection 8.7, \(M\) can be a closed manifold of arbitrary dimension \(m\).

8.1. **The Parameterized Extension Principle.** Let \(f: M \times W \to M\) be a parameterized family of diffeomorphisms. Here, \(W\) and \(M\) are smooth manifolds, with \(M\) closed and \(W\) compact, and \(f\) is continuous as a map from \(W\) to \(\text{Diff}(M)\), where \(\text{Diff}(M)\) has as always the \(C^\infty\)-topology.

We first recall that \(\text{Diff}(M)\) is locally convex in the following strong sense. Fix a Riemannian metric on \(M\) for which \(\partial M\) is totally geodesic, and let \(f': M \times W \to M\) be a parameterized family of smooth maps. Assume that \(f'\) is close enough to \(f\), in the compact-open topology on maps from \(W\) to \(\text{Diff}(M)\), so that for each \(u \in W\) and each \(x \in M\), there is a unique short vector \(v_{x,u}\) at \(f(x,u)\) such that \(\text{Exp}(v_{x,u}) = f'(x,u)\). Putting \(F_t(x,u) = \text{Exp}(tv_{x,u})\) defines a parameterized family \(F_t\) of homotopies from \(f_u\) to \(f'_u\). The diffeomorphisms form an open subset of the smooth maps from \(M\) to \(M\), so when \(f'\) is sufficiently close to \(f\), each \((F_t)_u\) and in particular \(f'\) will be a parameterized family of diffeomorphisms. Consequently, if a modification of a parameterized family \(f\) of diffeomorphisms can be achieved by taking a family which can be selected to be arbitrarily close in the \(C^\infty\)-topology, then it can be achieved by a deformation of \(f\) through families of diffeomorphisms.

This observation works just as well when \(M\) is an open manifold, provided that we use the spaces \(\text{Diff}_c(M)\) and \(C^\infty_c(M,M)\) of diffeomorphisms and maps with compact support (those which agree with the identity outside of a compact subset of \(M\)).

By very similar considerations, if \(f: M' \times W \to M\) is a parameterized family of imbeddings of a submanifold \(M'\) of \(M\) (possibly of codimension 0) into \(M\), then any map \(f': M' \times W \to M\) sufficiently close to \(f\) will also consist of imbeddings, and is homotopic to \(f\) through parameterized families of imbeddings.

We now state a powerful extension theorem for isotopies of submanifolds, due to R. Palais [30]. In the theorem, \(N\) is a not necessarily compact manifold, all spaces of maps have the strong \(C^r\)-topology, \(1 \leq r \leq \infty\), \(\text{Imb}(X,N)\) denotes the space of smooth imbeddings of the submanifold \(X\).
(with $X$ a closed manifold) of $N$ into $N$, and $\text{Diff}_c(N)$ denotes the space of diffeomorphisms of $N$ with compact support.

**Theorem 8.2** (Palais Extension Theorem). Let $N$ be a smooth manifold without boundary, and $X$ and $Y$ submanifolds of $N$, with $Y \subseteq X$. Then the restriction maps $\text{Imb}(X, N) \to \text{Imb}(Y, N)$ and $\text{Diff}_c(N) \to \text{Imb}(Y, N)$ are locally trivial fibrations.

In most of our applications of the Palais Extension Theorem, we will need considerable control. This control is present in Palais' setup, but not explicit in the statement of the Palais Extension Theorem, so we will rephrase Palais' method to prove the precise statement that will be needed.

**Theorem 8.3** (Parameterized Extension Principle). Let $M$ and $W$ be compact smooth manifolds, let $M_0$ be a submanifold of $M$ of positive codimension, and let $U$ be an open subset of the interior of $M$ with $M_0 \subset U$. Suppose that $F: M \times W \to M$ is a parameterized family of diffeomorphisms of $M$. If $g \in C^\infty(M_0 \times W, M)$ is sufficiently close to $F|_{M_0 \times W}$, then there is a deformation $G$ of $F$ such that $G|_{M_0 \times W} = g$, and $G = F$ on $(M - U) \times W$. By selecting $g$ sufficiently close to $F|_{M_0 \times W}$, $G$ may be selected arbitrarily close to $F$.

**Proof.** The key step in the proof of the Palais Extension Theorem is a method of extending imbeddings to diffeomorphisms, given as Lemmas c and d in [29]. Choose any Riemannian metric on $M$. Fixing an imbedding $i: M_0 \to U$ and an imbedding $j$ sufficiently close to $i$, a section of the tangent bundle of $M$ is defined over $i(M_0)$ by choosing at each $i(x)$ the unique short vector $w_x$ such that the exponential function at $i(x)$ sends $w_x$ to $j(x)$. Using a construction involving parallel translation along paths in the fibers of a tubular neighborhood of $i(M_0)$, the section over $i(M_0)$ is extended to a vector field $w$ on $M$, with compact support in $U$. The map $J: M \to M$ that carries each $p$ to $\text{Exp}_p(w_p)$ sends each $i(x)$ to $j(x)$. When $j$ is close to $i$, the vector field $w$ is close to the zero vector field, so $J$ is close to the identity in the $C^\infty$-topology. Since in the $C^\infty$-topology the diffeomorphisms form an open subset of the smooth maps, $J$ will be a diffeomorphism when $j$ is sufficiently close to $i$. If $g$ is sufficiently close to $F|_{M_0 \times W}$ so that each $g_u$ is an imbedding, then this process can be applied at each parameter $u$ to the imbeddings $i = F|_{M_0 \times \{u\}}$ and $j = g_u$. The tubular neighborhoods must be selected to vary continuously, so that the resulting $J_u$ vary continuously in $u$. The family $G$ is defined by $G_u = J_u \circ F_u$. $\square$

### 8.2. Weak transversality

Although individual maps may be put transverse to a submanifold of the range, it is not possible to perturb a parameterized family so that each individual member of the family is transverse. But a very nice result of J. W. Bruce, Theorem 1.1 of [4], allows one to simultaneously improve the members of a family.
**Theorem 8.4** (J. W. Bruce). Let \( A, B \) and \( U \) be smooth manifolds and \( C \subset B \) a submanifold. There is a residual family of mappings \( F \in C^\infty(A \times U, B) \) such that:

(a) For each \( u \in U \), the restriction \( F_u = F|_{A \times \{u\}} : A \to B \) is transverse to \( C \) except possibly on a discrete set of points.

(b) For each \( u \in U \), the set \( F_u^{-1}(C) \) is a smooth submanifold of codimension equal to the codimension of \( C \) in \( B \), except possibly at a discrete set of points. At each of these exceptional points \( F_u^{-1}(C) \) is locally diffeomorphic to the germ of an algebraic variety, with the exceptional point corresponding to an isolated singular point of the variety.

That is, \( F_u^{-1}(C) \) is smooth except at isolated points where it has topologically a nice cone-like structure. It is not assumed that any of the manifolds involved is compact.

Theorem 1.3 of [4] is a version of theorem 8.4 in which \( C \) is replaced by a bundle \( \phi : B \to D \). The statement is:

**Theorem 8.5** (J. W. Bruce). For a residual family of mappings \( F \in C^\infty(A \times U, B) \), the conclusions of theorem 8.4 hold for all submanifolds \( C = \phi^{-1}(d) \), \( d \in D \).

We should comment on the significance of the residual subset in these two theorems. The method of proof of these theorems is to define, in an appropriate jet space, a locally algebraic subset which contains the jets of all the maps that fail these weak transversality conditions. These subsets have increasing codimension as higher-order jets are taken. A variant of Thom transversality (lemma 1.6 of [4]) allows one to perturb a parameterized family of maps so that these jets are avoided and the conclusion holds. When \( A \) and \( W \) are compact, the image of \( A \times W \) will lie in the open complement of the locally algebraic sets of sufficiently high codimension. Consequently, any map sufficiently close to the perturbed map will also satisfy the conclusions of the theorems. In all of our applications, the spaces involved will be compact, and we tacitly assume that the result of any procedure holds on an open neighborhood of the perturbed map.

We now adapt the methodology of Bruce to prove a version of theorem 8.4 in which the submanifold \( C \) is replaced by the zero set of a nontrivial polynomial. We will prove it only for the case when \( A = I \), although a more general version should be possible.

**Proposition 8.6.** Let \( P : \mathbb{R}^n \to \mathbb{R} \) be a nonzero polynomial and put \( V = P^{-1}(0) \). Let \( W \) be compact. Then for all \( G \) in an open dense subset of \( C^\infty(I \times W, \mathbb{R}^n) \), each \( G_u^{-1}(V) \) is finite.

**Proof.** Let \( J^k_0(1, n) \) be the space of germs of degree-\( k \) polynomials from \((\mathbb{R}, 0)\) to \( \mathbb{R}^n \); an element of \( J^k_0(1, n) \) can be written as \((a_{1,0} + a_{1,1}t + \cdots + a_{1,k}t^k, \ldots, a_{n,0} + a_{n,1}t + \cdots + a_{n,k}t^k)\), so that \( J^k_0(1, n) \) can be identified with
Note that the jet space \( J^k(I, \mathbb{R}^n) \) can be regarded as \( I \times J^k_0(1, n) \), by identifying the jet of \( \alpha: I \to \mathbb{R}^n \) at \( t_0 \) with the jet of \( \alpha(t - t_0) \) at 0.

Define a polynomial map \( P_*: J^k_0(1, n) \to J^k_0(1, 1) \) by applying \( P \) to the \( n \)-tuple \((a_{1,0} + a_{1,1}t + \cdots + a_{1,k}t^k, \ldots , a_{n,0} + a_{n,1}t + \cdots + a_{n,k}t^k)\), and then taking only the terms up to degree \( k \). The preimage \( P_*^{-1}(0) \) is the set of \( \alpha \) for which \( P \circ \alpha(0) = 0 \) and the first \( k \) derivatives of \( P \circ \alpha \) at 0 vanish, that is, the set of germs of paths that lie in \( V \) up to \( k \)-th order.

**Lemma 8.7.** If \( P \) is nonconstant, then as a map from \( \mathbb{R}^{(k+1)n} \) to \( \mathbb{R}^{k+1} \), \( P_* \) has maximal rank.

**Proof.** We may select notation so that \( P(X, Y_1, \ldots , Y_m) = P_0(Y_1, \ldots , Y_m) + X^r P_r(Y_1, \ldots , Y_m) + \cdots + X^s P_s(Y_1, \ldots , Y_m) \) with \( P_r \) nonzero, and write elements of \( J^k_0(1, n) \) as \((a_0 + a_1t + \cdots + a_k t^k, b_0 + b_1 t + \cdots)\). The Jacobian of \( P_\alpha \) is a \((k + 1) \times (k + 1)n\) matrix, and we will show that its leftmost \((k + 1) \times (k + 1)\)-block is lower triangular with nonzero (as polynomials) diagonal entries.

Write \( P_\alpha(a_0 + a_1 t + \cdots + a_k t^k, b_0 + b_1 t + \cdots) \) as \( Q_0 + Q_1 t + \cdots + Q_k t^k \), where the \( Q_i \) are polynomials on \( \mathbb{R}^{(k+1)n} \). Note that \( Q_0 = P(a_0, b_0, \ldots) \). So, the \((1, 1)\)-entry of the Jacobian is \( \frac{\partial Q_0}{\partial a_0} = \frac{\partial}{\partial a_0} P(a_0, b_0, \ldots) \), while the \((1, i + 1)\)-entries, \( 1 \leq i \leq k \), are \( \frac{\partial Q_0}{\partial a_i} = 0 \).

For \( j \geq 1 \), we have \( Q_j = \frac{1}{j!} \frac{\partial^j P_*}{\partial t^j} \bigg|_{t=0} = \frac{\partial X}{\partial a_i} = t^i \), and \( \frac{\partial Y_i}{\partial a_i} = 0 \), so \( \frac{\partial Q_j}{\partial a_i} \) is

\[
\frac{1}{j!} \frac{\partial^j}{\partial t^j} \left( \frac{\partial P}{\partial X} \frac{\partial X}{\partial a_i} + \sum_{\ell=1}^m \frac{\partial P}{\partial Y_\ell} \frac{\partial Y_\ell}{\partial a_i} \right) \bigg|_{t=0} = \frac{1}{j!} \frac{\partial^j}{\partial t^j} \left( \frac{\partial P}{\partial X} t^i \right) \bigg|_{t=0}
\]

which vanishes for \( i > j \), and is \( \frac{\partial P}{\partial X} \bigg|_{t=0} = \frac{\partial}{\partial a_0} P(a_0, b_0, \ldots) \) for \( i = j \). \( \square \)

For each \( k \), put \( Z_k = P_*^{-1}(0) \). Lemma 8.7 shows that \( Z_k \) is a variety of codimension \( k + 1 \) in \( J^k_0(1, n) \). Observe that if \( \alpha: (\mathbb{R}, 0) \to \mathbb{R}^n \) is a germ of a smooth map, and 0 is a limit point of \( \alpha^{-1}(V) \), then all derivatives of \( P \circ \alpha \) vanish at 0. That is, the \( k \)-jet of \( \alpha \) at \( t = 0 \) is contained in \( Z_k \) for every \( k \).

By Lemma 1.6 of [4], there is a residual set of maps \( G \in C^\infty(I \times W, \mathbb{R}^n) \) such that the jet extensions \( j^k G: I \times W \to J^k(I, \mathbb{R}^n) \) defined by \( j^k G(t, u) = j^k G_u(t) \) are transverse to \( I \times Z_k \). For \( k + 1 \) larger than the dimension of \( I \times W \), this says that no point of \( G_u^{-1}(0) \) is a limit point, so each \( G_u^{-1}(0) \) is finite. \( \square \)

**8.3. Finite singularity type.** For our later work, we will need some ideas from singularity theory. Let \( f: (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0) \) be a germ of a smooth map. There is a concept of *finite singularity type* for \( f \), whose definition is readily available in the literature (for example, [4, p. 117]). The basic idea of the proof of theorem 8.4 (given as Theorem 1.1 in [4]) is to regard
the submanifold \( C \) locally as the preimage of 0 under a submersion \( s \), then to perturb \( f \) so that for each \( u \), the critical points of \( s \circ f_u \) are of finite singularity type. In fact, this is exactly the definition of what it means for \( f_u \) to be weakly transverse to \( C \). In particular, when \( C \) is a point, the submersion can be taken to be the identity, so we have:

**Proposition 8.8.** Let \( f : M \to \mathbb{R} \) be smooth. If \( f \) is weakly transverse to a point \( r \in \mathbb{R} \), then at each critical point in \( f^{-1}(r) \), the germ of \( f \) has finite singularity type.

Let \( f \) and \( g \) be germs of smooth maps from \((\mathbb{R}^m, a)\) to \((\mathbb{R}^p, f(a))\). They are said to be \( A \)-equivalent if there exist a germ \( \varphi_1 \) of a diffeomorphism of \((\mathbb{R}^m, a)\) and a germ \( \varphi_2 \) of a diffeomorphism of \((\mathbb{R}^p, f(a))\) such that \( g = \varphi_2 \circ f \circ \varphi_1 \). If \( \varphi_2 \) can be taken to be the identity, then \( f \) and \( g \) are called \( \mathbb{R} \)-equivalent (for right-equivalent). There is also a notion of contact equivalence, denoted by \( K \)-equivalence, whose definition is readily available, for example in [37].

It is implied by \( A \)-equivalence.

We use \( j^k f \) to denote the \( k \)-jet of \( f \); for fixed coordinate systems at points \( a \) and \( f(a) \) this is just the Taylor polynomial of \( f \) of degree \( k \). For \( G \) one of \( A, K, \) or \( \mathbb{R} \), one says that \( f \) is finitely \( G \)-determined if there exists a \( k \) so that any germ \( g \) with \( j^k g = j^k f \) must be \( G \)-equivalent to \( f \). In particular, if \( f \) is finitely \( G \)-determined, then for any fixed choice of coordinates at \( a \) and \( f(a) \), \( f \) is \( G \)-equivalent to a polynomial.

The elaborate theory of singularities of maps from \( \mathbb{R}^m \) to \( \mathbb{R}^p \) simplifies considerably when \( p = 1 \).

**Lemma 8.9.** Let \( f \) be the germ of a map from \((\mathbb{R}^m, 0)\) to \((\mathbb{R}, 0)\), with 0 is a critical point of \( f \). The following are equivalent.

(i) \( f \) has finite singularity type.

(ii) \( f \) is finitely \( A \)-determined.

(iii) \( f \) is finitely \( \mathbb{R} \)-determined.

(iv) \( f \) is finitely \( K \)-determined.

**Proof.** In all dimensions, \( f \) is finitely \( K \)-determined if and only if it is of finite singularity type (Corollary III.6.9 of [9], or alternatively the definition of finite singularity type on [4, p. 117] is exactly the condition given in Proposition (3.6)(a) of [26] for \( f \) to be finitely \( K \)-determined). Therefore (i) is equivalent to (iv). Trivially (ii) implies (iii), and (iii) implies (iv), and by Corollary 2.13 of [37], (iv) implies (ii). \( \square \)

### 8.4. Semialgebraic sets.

Recall (see for example Chapter I.2 of [9]) that the class of semialgebraic subsets of \( \mathbb{R}^m \) is defined to be the smallest Boolean algebra of subsets of \( \mathbb{R}^m \) that contains all sets of the form \( \{x \in \mathbb{R}^m \mid p(x) > 0\} \) with \( p \) a polynomial on \( \mathbb{R}^m \). The collection of semialgebraic subsets of \( \mathbb{R}^m \) is closed under finite unions, finite intersections, products, and complementation. The inverse image of a semialgebraic set under a polynomial
Lemma 8.10. Let $S$ be a semialgebraic subset of $\mathbb{R}^n$. If $S$ has empty interior, then $S$ is contained in the zero set of a nontrivial polynomial in $\mathbb{R}^n$.

Proof. Since the union of the zero sets of two polynomials is the zero set of their product, it suffices to consider a single semialgebraic set of the form $\left(\bigcap_{i=1}^r \{x \mid p_i(x) \geq 0\}\right) \cap \left(\bigcap_{j=1}^s \{x \mid q_j(x) > 0\}\right)$ where $p_i$ and $q_j$ are nontrivial polynomials. We will show that if $S$ is of this form and has empty interior, then $r \geq 1$ and $S$ is contained in the zero set of $\prod_{i=1}^r p_i$. Suppose that $x \in S$ but all $p_i(x) > 0$. Since all $q_j(x) > 0$ as well, there is an open neighborhood of $x$ on which all $p_i$ and all $q_j$ are positive. But then, $S$ has nonempty interior. □

8.5. The codimension of a real-valued function. It is, of course, fundamentally important that the Morse functions form an open dense subset of $C^\infty(M, \mathbb{R})$, the smooth maps from a closed connected manifold $M$ of dimension $m$ to $\mathbb{R}$, with the $C^\infty$-topology. But a great deal can also be said about the non-Morse functions. There is a “natural” stratification of $C^\infty(M, \mathbb{R})$ by subsets $F_i$, where stratification here means that the $F_i$ are disjoint subsets such that for every $n$ the union $\bigcup_{i=0}^n F_i$ is open. The functions in $F_n$ are those of “codimension” $n$, which we will define below. In particular, $F_0$ is exactly the open dense subset of Morse functions.

The union $\bigcup_{i=0}^\infty F_i$ is not all of $C^\infty(M, \mathbb{R})$. However, the residual set $C^\infty(M, \mathbb{R}) - \bigcup_{i=0}^\infty F_i$ is of “infinite codimension,” and any parameterized family of maps $F: M \times U \to \mathbb{R}$ can be perturbed so that each $F_u$ is of finite codimension. In fact, by applying theorem 8.5 to the trivial bundle $1_\mathbb{R}: \mathbb{R} \to \mathbb{R}$ and noting proposition 8.8, we may perturb any parameterized family so that each $F_u$ is of finite singularity type at each of its critical points. The definition of $f \in C^\infty(M)$ being of finite codimension, given below, is exactly equivalent to the algebraic condition given in (3.5) of [26] for $f$ to be finitely $A$-determined at each of its critical points (as noted in [26], this part of (3.5) was first due to Tougeron [35], [36]). By lemma 8.9, this is equivalent to $f$ having finite singularity type at each of its critical points. We summarize this as

Proposition 8.11. A map $f \in C^\infty(M, \mathbb{R})$ is of finite codimension if and only if it has finite singularity type at each of its critical points.

We now recall material from section 7 of [32]. Denote the smooth sections of a bundle $E$ over $M$ by $\Gamma(E)$. Until we reach theorem 8.14, we will denote $C^\infty(M, \mathbb{R})$ by $C(M)$. For a compact subset $K \subset \mathbb{R}$, define $\text{Diff}_K(\mathbb{R})$ to be the diffeomorphisms of $\mathbb{R}$ supported on $K$.

Fix an element $f \in C(M)$ and a compact subset $K \subset \mathbb{R}$ for which $f(M)$ lies in the interior of $K$. Define $\Phi: \text{Diff}(M) \times \text{Diff}_K(\mathbb{R}) \to C(M)$
by $\Phi(\varphi_1, \varphi_2) = \varphi_2 \circ f \circ \varphi_1$. The differential of $\Phi$ at \((1_M, 1_\mathbb{R})\) is defined by $D(\xi_1, \xi_2) = f_\ast \xi_1 + \xi_2 \circ f$. Here, $\xi_1 \in \Gamma(TM)$, which is regarded as the tangent space at $1_M$ of $\text{Diff}(M)$, $\xi_2 \in \Gamma_K(T \mathbb{R})$, similarly identified with the tangent space at $1_\mathbb{R}$ of $\text{Diff}_K(\mathbb{R})$, and $f_\ast \xi_1 + \xi_2 \circ f$ is regarded as an element of $\Gamma(f^\ast T \mathbb{R})$, which is identified with $C(M)$. The codimension $\text{cdim}(f)$ of $f$ is defined to be the real codimension of the image of $D$ in $C(M)$. As will be seen shortly, the codimension of $f$ tells the real codimension of the $\text{Diff}(M) \times \text{Diff}_K(\mathbb{R})$-orbit of $f$ in $C(M)$.

Suppose that $f$ has finite codimension $c$. In section 7.2 of [32], a method is given for computing $\text{cdim}(f)$ using the critical points of $f$. Fix a critical point $a$ of $f$, with critical value $f(a) = b$. Consider $D_a : \Gamma_a(TM) \times C_b(\mathbb{R}) \to C_a(M)$, where a subscript as in $\Gamma_a(TM)$ indicates the germs at $a$ of $\Gamma(TM)$, and so on. Notice that the codimension of the image of $D_a$ is finite, indeed it is at most $c$.

Let $A$ denote the ideal $f_\ast \Gamma_a(TM)$ of $C_a(M)$. This can be identified with the ideal in $C_a(M)$ generated by the partial derivatives of $f$. An argument using Nakayama’s lemma [32, p. 645] shows that $A$ has finite codimension in $C_a(M)$, and that some power of $f(x) - f(a)$ lies in $A$. Define $\text{cdim}(f, a)$ to be the dimension of $C_a(M)/A$, and $\dim(f, a, b)$ to be the smallest $k$ such that $(f(x) - f(a))^k \in A$.

Here is what these are measuring. The ideal $A$ tells what local deformations of $f$ at $a$ can be achieved by precomposing $f$ with a diffeomorphism of $M$ (near $1_M$), thus $\text{cdim}(f, a)$ measures the codimension of the $\text{Diff}(M)$-orbit of the germ of $f$ at $a$. The additional local deformations of $f$ at $a$ that can be achieved by postcomposing with a diffeomorphism of $\mathbb{R}$ (again, near $1_\mathbb{R}$) reduce the codimension by $k$, basically because Taylor’s theorem shows that the germ at $a$ of any $\xi_2(f(x))$ can be written in terms of the powers $(f(x) - f(a))^i, i < k$, plus a remainder of the form $K(x)(f(x) - f(a))^k$, which is an element of the ideal $A$. Thus $\text{cdim}(f, a) - \dim(f, a, b)$ is the codimension of the image of $D_a$. For a noncritical point or a stable critical point such as $f(x, y) = x^2 - y^2$ at $(0, 0)$, this local codimension is 0, but for unstable critical points it is positive.

Now, let $\dim(f, b)$ be the maximum of $\text{dim}(f, a, b)$, taken over the critical points $a$ such that $f(a) = b$ (put $\text{dim}(f, b) = 0$ if $b$ is not a critical value). The codimension of $f$ is then $\sum_{a \in M} \text{cdim}(f, a) - \sum_{b \in \mathbb{R}} \dim(f, b)$.

Here is what is happening at each of the finitely many critical values $b$ of $f$. Let $a_1, \ldots, a_\ell$ be the critical points of $f$ with $f(a_i) = b$. Let $f_i$ be the germ of $f - f(a_i)$ at $a_i$, and consider the element $(f_1, \ldots, f_\ell) \in C_{a_1}(M)/A_1 \oplus \cdots \oplus C_{a_\ell}(M)/A_\ell$. The integer $\text{dim}(f, b)$ is the smallest power of $(f_1, \ldots, f_\ell)$ that is trivial in $C_{a_1}(M)/A_1 \oplus \cdots \oplus C_{a_\ell}(M)/A_\ell$. The sum $\sum_{i} \text{cdim}(f, a_i)$ counts how much codimension of $f$ is produced by the inability to achieve local deformations of $f$ near the $a_i$ by precomposing with local diffeomorphisms at the $a_i$. This codimension is reduced by $\dim(f, b)$, because the germs of the additional deformations that can be achieved by postcomposition with diffeomorphisms of $\mathbb{R}$ near $b$ are the linear combinations of $(1, \ldots, 1)$,
8.6. The stratification of $C^\infty(M, \mathbb{R})$ by codimension. The functions whose codimension is finite and equal to $n$ form the stratum $\mathcal{F}_n$. In particular, $\mathcal{F}_0$ are the Morse functions, $\mathcal{F}_1$ are the functions either having all critical points stable and exactly two with the same critical value, or having distinct critical values and all critical points stable except one which is a birth-death point. Moving to higher strata occurs either from more critical points sharing a critical value, or from the appearance of more singularities of positive but still finite local codimension.

We use the natural notations $\mathcal{F}_{\geq n}$ for $\cup_{i \geq n} \mathcal{F}_i$, $\mathcal{F}_{> n}$ for $\cup_{i > n} \mathcal{F}_i$, and so on. In particular, $\mathcal{F}_{\geq 0}$ is the set of all elements of $C(M)$ of finite codimension, and $\mathcal{F}_{> 0}$ is the set of all elements of finite codimension that are not Morse functions.

The main results of [32] (in particular, Theorem 8.1.1 and Theorem 9.2.4) show that the Sergeraert stratification is locally trivial, in the following sense.

Theorem 8.12 (Sergeraert). Suppose that $f \in \mathcal{F}_n$. Then there is a neighborhood $V$ of $f$ in $C(M)$ of the form $U \times \mathbb{R}^n$, where

1. $U$ is a neighborhood of 1 in $\text{Diff}(M) \times \text{Diff}_K(\mathbb{R})$, and
2. there is a stratification $\mathbb{R}^n = \cup_{i=0}^n \mathcal{F}_i$, such that $\mathcal{F}_i \cap V = U \times \mathcal{F}_i$.

The inner workings of this result are as follows. Select elements $f_1, \ldots, f_n \in C(M)$ that represent a basis for the quotient of $C(M)$ by the image of the differential $D$ of $\Phi$ at $(1_M, 1_K)$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the function $g_x = f + \sum_{i=1}^n x_i f_i$ is an element of $C(M)$. If the $x_i$ are chosen in a sufficiently small ball around 0, which is again identified with $\mathbb{R}^n$, then these $g_x$ form a copy $E$ of $\mathbb{R}^n$ “transverse” to the image of $\Phi$. Then, $\mathcal{F}_i$ is defined to be the intersection $E \cap \mathcal{F}_i$. A number of subtle results on this local structure and its relation to the action of $\text{Diff}(M) \times \text{Diff}_K(\mathbb{R})$ are obtained in [32], but we will only need the local structure we have described here.

We remark that $\mathcal{F}_n$ is not necessarily just $\{0\} \in \mathbb{R}^n$, that is, the orbit of $f$ under $\text{Diff}(M) \times \text{Diff}_K(\mathbb{R})$ might not fill up the stratum $\mathcal{F}_n$ near $f$.

This result, due to H. Hendriks [16], has been interpreted as saying that the Sergeraert stratification of $C(M)$ is not locally trivial (a source of some confusion), or that it is “pathological” (which we find far too pejorative).

Denoting $\cup_{i \geq 1} \mathcal{F}_i$ by $\mathcal{F}_{\geq 1}$, we have the following key technical result.

Proposition 8.13. For some coordinates on $E$ as $\mathbb{R}^n$, there are a neighborhood $L$ of 0 in $\mathbb{R}^n$ and a nonzero polynomial $p$ on $\mathbb{R}^n$ such that $p(L \cap \mathcal{F}_{\geq 1}) = 0$.

Proof. We begin with a rough outline of the proof. By lemma 8.9, we may choose local coordinates at the critical points of $f$ for which $f$ is polynomial.
near each critical point. We will select the $f_i$ in the construction of the transverse slice $E = \mathbb{R}^n$ to be polynomial on these neighborhoods. Now $F_{\geq 1}$ consists exactly of the choices of parameters $x_i$ for which $f + \sum x_i f_i$ is not a Morse function, since they are the intersection of $E$ with $F_{\geq 1}$. We will show that they form a semialgebraic set. But $F_{\geq 1}$ has no interior, since otherwise (using theorem 8.12) the subset of Morse functions $F_0$ would not be dense in $C(M)$. So lemma 8.10 show that $F_{\geq 1}$ lies in the zero set of some nontrivial polynomial.

Now for the details. Recall that $m$ denotes the dimension of $M$. Consider a single critical value $b$, and let $a_1, \ldots, a_k$ be the critical points with $f(a_i) = b$. Fix coordinate neighborhoods $U_i$ of the $a_i$ with disjoint closures, so that $a_i$ is the origin 0 in $U_i$. By lemma 8.9, $f$ is finitely determined near each critical point, so on each $U_i$ there is a germ $\varphi_i$ of a diffeomorphism at 0 so that $f \circ \varphi_i$ is the germ of a polynomial. That is, by reducing the size of the $U_i$ and changing the local coordinates, we may assume that on each $U_i$, $f$ is a polynomial $p_i$. As explained in subsection 8.5, the contribution to the codimension of $f$ from the $a_i$ is the dimension of the quotient

$$Q_b = (\bigoplus_{i=1}^{\ell} C_{a_i}(U_i)/A_i)/B$$

where $B$ is the vector subspace spanned by $\{1, (p_1(x) - b, \ldots, p_{\ell}(x) - b), \ldots, ((p_1(x) - b)^{k-1}, \ldots, (p_{\ell}(x) - b)^{k-1})\}$. Choose $q_{i,j}$, $1 \leq j \leq n_i$, where $q_{i,j}$ is a polynomial on $U_i$, so that the germs of the $q_{i,j}$ form a basis for $Q_b$. Fix vector spaces $\Lambda_i \cong \mathbb{R}^{n_i} = \{(x_{i,1}, \ldots, x_{i,n_i})\}$; these will eventually be some of the coordinates on $E$.

In each $U_i$, select round open balls $V_i$ and $W_i$ centered at 0 so that $W_i \subset \overline{V_i} \subset V_i \subset \overline{U_i}$. We select them small enough so that the closures in $\mathbb{R}$ of their images under $f$ do not contain any critical value except for $b$. Fix a smooth function $\mu: M \to [0,1]$ which is 1 on $\overline{\bigcup_i V_i}$ and is 0 on $M - \bigcup_i V_i$, and put $f_{i,j} = \mu \cdot q_{i,j}$, a smooth function on all of $M$. Now choose a product $L = \prod_i L_i$, where each $L_i$ is a round open ball centered at 0 in $\Lambda_i$, small enough so that if each $(x_{i,1}, \ldots, x_{i,n_i}) \in L_i$, then each critical point of $f + \sum x_{i,j} f_{i,j}$ either lies in $\bigcup W_i$, or is one of the original critical points of $f$ lying outside of $\bigcup U_i$.

We repeat this process for each of the finitely many critical values of $f$, choosing additional $W_i$ and $L_i$ so small that all critical points of $f + \sum x_{i,j} f_{i,j}$ lie in $\bigcup W_i$. That is, these perturbations of $f$ are so small that each of the original critical points of $f$ breaks up into critical points that lie very near the original one and far from the others.

The sum of all $n_i$ is now $n$. We again use $\ell$ for the number of $U_i$, and write $\Lambda$ and $L$ for the direct sum of all the $\Lambda_i$ and the product of all the $L_i$ respectively. For $x \in L$, write $g_x = f + \sum x_{i,j} f_{i,j}$. It remains to show that the set of $x$ for which $g_x$ is not a Morse function — that is, has a critical point with zero Hessian or has two critical points with the same value — is contained in a union of finitely many semialgebraic sets.
Denote elements of $W_i$ by $\overline{u_i^*} = (u_{i,1}, \ldots, u_{i,m})$, and similarly for elements $\overline{v_i}$ of $L_i$. Define $G_i : W_i \times L_i \to \mathbb{R}$ by $G_i(\overline{u_i}, \overline{v_i}) = p_i(\overline{u_i}) + \sum_{j=1}^{n_i} x_i j g_{i,j}(\overline{v_i})$. Note that for $x = (\overline{x_1}, \ldots, \overline{x_L})$, $(G_1)_{\overline{x_i}}$ is exactly the restriction of $g_x$ to $W_i$.

We introduce one more notation that will be convenient. For $X \subseteq L_i$ define $E(X)$ to be the set of all $(\overline{x_1}, \ldots, \overline{x_L})$ in $L$ such that $\overline{x_i} \in X$. When $X$ is a semialgebraic subset of $L_i$, $E(X)$ is a semialgebraic subset of $L$. Similarly, if $X \times Y \subseteq L_i \times L_j$, we use $E(X \times Y)$ to denote its extension to a subset of $L$, that is, $E(X) \cap E(Y)$.

For each $i$, let $S_i$ be the set of all $(\overline{u_i, v_i})$ in $W_i \times L_i$ such that $\partial G_i / \partial u_{i,j}$ for $1 \leq j \leq n_i$ all vanish at $(\overline{u_i, v_i})$, that is, the pairs such that $\overline{v_i}$ is a critical point of $(G_i)_{\overline{u_i}}$. Since $S_i$ is the intersection of an algebraic set in $\mathbb{R}^m \times \mathbb{R}^n$ with $W_i \times L_i$, and the latter are round open balls, $S_i$ is semialgebraic. Let $H_i$ be the set of all $(\overline{u_i, v_i})$ in $W_i \times L_i$ such that the Hessian of $(G_i)_{\overline{u_i}}$ vanishes at $\overline{v_i}$, again a semialgebraic set. The intersection $H_i \cap S_i$ is the set of all $(\overline{u_i, v_i})$ such that $(G_i)_{\overline{v_i}}$ has an unstable critical point at $\overline{v_i}$. By the Tarski-Seidenberg Theorem, its projection to $L_i$ is a semialgebraic set, which we will denote by $A_i$. The union of the $E(A_i)$, $1 \leq i \leq \ell$, is precisely the set of $x$ in $L$ such that $g_x$ has an unstable critical point.

Now consider $G_i \times G_j : S_i \times S_j - \Delta_i \to \mathbb{R}^2$, where $\Delta_i$ is the diagonal in $S_i \times S_i$. Let $\widetilde{B}_i = (G_i \times G_j)^{-1}(\Delta_2)$, where $\Delta_2$ is the diagonal of $\mathbb{R}^2$. Now, let $\Delta_i'$ be the set of all $((\overline{u_i, v_i}), (\overline{u_j, v_i}))$ in $W_i \times L_i \times W_i \times L_i$ such that $\overline{v_i} = \overline{v_i'}$. Then the projection of $\widetilde{B}_i \cap \Delta_i'$ to its first two coordinates is the set of all $(\overline{u_i, v_i})$ in $W_i \times L_i$ such that $\overline{v_i}$ is a critical point of $(G_i)_{\overline{v_i}}$ and $(G_i)_{\overline{v_i}}$ has another critical point with the same value. The projection to the second coordinate alone is the set $B_i$ of $\overline{v_i}$ for which $(G_i)_{\overline{v_i}}$ has two critical points with the same value.

Finally, for $i \neq j$, consider $G_i \times G_j : S_i \times S_j \to \mathbb{R}^2$ and let $\widetilde{B}_{i,j}$ be the preimage of $\Delta_2$. Let $B_{i,j}$ be the projection of $\widetilde{B}_{i,j}$ to a subset of $L_i \times L_j$. The union of the $E(B_i)$ and the $E(B_{i,j})$ is precisely the set of all $x$ such that $g_x$ has two critical points with the same value. Since these are semialgebraic sets, the proof is complete. \(\square\)

Here is the main result of this subsection.

**Theorem 8.14.** Let $M$ and $W$ be compact smooth manifolds. Then for a residual set of smooth maps $F$ from $I \times W$ to $C^\infty(M, \mathbb{R})$, the following hold.

(i) $F(I \times W) \subset F_{\geq 0}$.

(ii) Each $F_u^{-1}(F_{> 0})$ is finite.

**Proof.** Start with a smooth map $G : I \times W \to C^\infty(M, \mathbb{R})$. Regarding it as a parameterized family of maps $M \times (I \times W) \to \mathbb{R}$, we apply Theorem 8.5 to perturb $G$ so that each $G_u$ is weakly transverse to the points of $\mathbb{R}$. By Proposition 8.11, this implies that $G(I \times W) \subset F_{\geq 0}$. Since $I \times W$ is compact, $G(I \times W) \subset F_{\leq n}$ for some $n$. 


For each \( f \in \mathcal{F}_{>0} \), choose a neighborhood \( V_f = U_f \times \mathbb{R}^n \) as in theorem 8.12. Using proposition 8.13, select a neighborhood \( L_f \) of 0 in \( \mathbb{R}^n \) and a nonzero polynomial \( p_f : L_f \to \mathbb{R} \) such that \( p_f(L \cap F_{\geq 1}) = 0 \).

Now, partition \( I \) into subintervals and triangulate \( W \) so that for each subinterval \( J \) and each simplex \( \Delta \) of maximal dimension in the triangulation, \( G(J \times \Delta) \) lies either in \( \mathcal{F}_0 \) or in some \( U_f \times L_f \). Fix a particular \( J \times \Delta \). If \( G(J \times \Delta) \) lies in \( \mathcal{F}_0 \), do nothing. If not, choose \( f \) so that \( G(J \times \Delta) \) lies in \( U_f \times L_f \). Let \( \pi : U_f \times L_f \to L_f \) be the projection, so that \( p_f \circ \pi(U_f \times F_{\geq 1}) = 0 \).

By proposition 8.6, we may perturb \( G|_{J \times \Delta} \) (changing only its \( L_f \)-coordinate in \( U_f \times L_f \)) so that for each \( u \in \Delta \), \( G_u|^{-1}_{J \times \Delta} \) is finite, and any map sufficiently close to \( G|_{J \times \Delta} \) on \( J \times \Delta \) will have this same property. As usual, of course, this is extended to a perturbation of \( G \).

This process can be repeated sequentially on the remaining \( J \times \Delta \). The perturbations must be so small that the property of having each \( G_u|^{-1}_{J \times \Delta} \) finite is not lost on previously considered sets. When all \( J \times \Delta \) have been considered, each \( G_u^{-1} (\mathcal{F}_{\geq 1}) \) is finite.

\[ \square \]

8.7. **Border label control.** We now return to the case when \( M \) is a closed 3-manifold, as in the introduction of section 8. In this subsection, we will obtain a deformation of \( f : M \times W \to M \) for which some \( \epsilon \) gives border label control.

We begin by ensuring that no \( f_u \) carries a component of the singular set \( T \) of \( \tau \) into \( T \). Consider two circles \( C_1 \) and \( C_2 \) imbedded in \( M \). By theorem 8.4, applied with \( A = C_1 \times W \), \( B = M \), and \( C = C_2 \), we may perturb \( f|_{C_1 \times W} \) so that for each \( u \in W \), \( f_u|_{C_1} \) meets \( C_2 \) in only finitely many points.

Recall that \( T \) consists of smooth circles and arcs in \( M \). Each arc is part of some smoothly imbedded circle, so \( T \) is contained in a union \( \bigcup_{i=1}^n C_i \) of imbedded circles in \( M \). By a sequence of perturbations as above, we may assume that at each \( u \), each \( f_u(C_i) \) meets each \( C_j \) in a finite set (including when \( i = j \)), so that \( f_u(T) \) meets \( T \) in a finite set.

The next potential problem is that at some \( u \), \( f_u(T_0) \) or \( f_u(T_1) \) might be contained in a single level \( P_t \). Recall that the notation \( R(s, t) \), introduced in section 5, means \( \tau^{-1}([s, t]) \). For some \( \delta > 0 \), every \( f_u(T_0) \) meets \( R(3\delta, 1 - 3\delta) \), since otherwise the compactness of \( W \) would lead to a parameter \( u \) for which \( f_u(T_0) \subset T \). Let \( \phi : R(\delta, 1 - \delta) \to [\delta, 1 - \delta] \) be the restriction of the map \( \pi(\tau(x, t)) = t \). This \( \phi \) makes \( R(\delta, 1 - \delta) \) a bundle with fibers that are level tori. As before, let \( C_1 \) be one of the circles whose union contains \( T \). Only the most superficial changes are needed to the proof of theorem 8.5 given in [4] so that it applies when \( \phi \) is a bundle map defined on a codimension-zero submanifold of \( B \) rather than on all of \( B \); the only difference is that the subsets of jets which are to be avoided are defined only at points of the subspace rather than at every point of \( B \). Using this slight generalization of theorem 8.5 (and as usual, the Parameterized Extension Principle), we perturb \( f \) so that each \( f_u|_{C_1} \) is weakly transverse to each \( P_t \) with \( \delta \leq t \leq 1 - \delta \). Since \( C_1 \) is 1-dimensional, weakly transverse implies that
$f_u(C_1)$ meets each such $P_t$ in only finitely many points. Repeating for the other $C_i$, we may assume that each $f_u(T_0)$ meets the $P_t$ with $\delta \leq t \leq 1 - \delta$ in only finitely many points. We also choose the perturbations small enough so that each $f_u(T_0)$ still meets $R(2\delta, 1 - 2\delta)$. So $f_u^{-1}(P_t) \cap T_0$ is nonempty and finite at least some $t$. In particular, $\pi(f_u(T_0))$ contains an open set, so by Sard’s Theorem applied to $\pi \circ f_u|_{T_0}$, for each $u$, there is an $r$ so that $f_u(T_0)$ meets $P_r$ transversely in a nonempty set (we select $r$ so that $P_r$ does not contain the image of a vertex of $T_0$). For a small enough $\epsilon$, a component of $X_s \cap P_r$ will be a compressing disk of $X_s$ whenever $s \leq 2\epsilon$, and by compactness of $W$, there is an $\epsilon$ such that for every $u$, there is a level $P_r$ such that some component of $X_s \cap P_r$ contains a compressing disk of $X_s$ whenever $s \leq 2\epsilon$.

Applying the same procedure to $T_1$, we may assume that for every $u$, there is a level $P_r$ such that some component of $Y_s \cap P_r$ is a compressing disk of $Y_s$ whenever $s \geq 1 - 2\epsilon$.

Let $h: M \times W \to M$ be defined by $h(x, u) = f_u^{-1}(x)$. Applying the previous procedure to $h$, making sure that all perturbations are small enough to preserve the conditions developed for $f$, and perhaps making $\epsilon$ smaller, we may assume that for each $u$, there is a level $Q_r$ such that $V_t \cap Q_r$ is a compressing disk of $V_t$ whenever $t \leq 2\epsilon$, and a similar $Q_r$ for $W_t$ with $t \geq 1 - 2\epsilon$. Thus the number $\epsilon$ gives border label control for $f$. Since border label control holds, with the same $\epsilon$, for any map sufficiently close to $f$, we may assume it is preserved by all future perturbations.

8.8. Building the graphics. It remains to deform $f$ to satisfy conditions (GP1), (GP2), and (GP3). As before, let $i: I \to \mathbb{R}$ be the inclusion, and consider the smooth map $i \circ \pi \circ f \circ (\tau \times 1_I): P \times I \times W \to \mathbb{R}$. Regard this as a family of maps from $I$ to $C^\infty(P, \mathbb{R})$, parameterized by $W$. Apply theorem 8.14 to obtain a family $k: P \times I \times W \to \mathbb{R}$. For each $I \times \{u\}$, there will be only finitely many values of $s$ in $I$ for which the restriction $k_{(s,u)}$ of $k$ to $P \times \{s\} \times \{u\}$ is not a Morse function. At these levels, the projection from $Q_s$ into the transverse direction to $P_t$ is an element of some $F_n$, so each tangency of $Q_s$ with $P_t$ looks like the graph of a critical point of finite multiplicity. This will ultimately ensure that condition (GP1) is attained when we complete our deformations of $f$.

We will use $k$ to obtain a deformation of the original $f$, by moving image points vertically with respect to the levels of the range. This would not make sense where the values of $k$ fall outside $(0, 1)$, so the motion will be tapered off so as not to change $f$ at points that map near $T$. It also would not be well-defined at points of $T \times W$, so we taper off the deformation so as not to change $f$ near $T \times W$. The fact that $f$ is unchanged near $T \times W$ and near points that map to $T$ will not matter, since border label control will allow us to ignore these regions in our later work.

Regard $P \times I \times W$ as a subspace of $P \times \mathbb{R} \times W$. For each $(x, r, u) \in P \times I \times W$, let $w'_{(x,r,u)}$ be $k(x, r, u) - i \circ \pi \circ f_u \circ \tau(x, r)$, regarded as a tangent vector to $\mathbb{R}$ at $i \circ \pi \circ f_u \circ \tau(x, r)$. 


We will taper off the $w'(x,r,u)$ so that for each fixed $u$ they will produce a vector field on $M$. Fix a number $\epsilon$ that gives border label control for $f$, and a smooth function $\mu: \mathbb{R} \to I$ which carries $(-\infty, \epsilon/4] \cup [1 - \epsilon/4, \infty)$ to 0 and carries $[\epsilon/2, 1 - \epsilon/2]$ to 1. Define $w(x,r,u)$ to be $\mu(r) \mu(i \circ \pi \circ f_u \circ \tau(x,r)) w'(x,r,u)$.

These vectors vanish whenever $r \notin [\epsilon/4, 1 - \epsilon/4]$ or $i \circ \pi \circ f_u \circ \tau(x,r,u) \notin [\epsilon/4, 1 - \epsilon/4]$, that is, whenever $\tau(x,r)$ or $f_u \circ \tau(x,r)$ is close to $T$. Using the map $i \circ \pi: M \to \mathbb{R}$, we pull these back to vectors in $M$ that are perpendicular to $P_t$; this makes sense near $T$ since the $w'(x,r,u)$ are zero at these points.

For each $u$, we obtain at each point $f_u \circ \tau(x,r) \in M$ a vector $v_{(x,r,u)}$ that points in the $I$-direction (i.e. is perpendicular to $P_t$) and maps to $w(x,r,u)$ under $(i \circ \pi)_u$.

If $k$ was a sufficiently small perturbation, the $v_{(x,r,u)}$ define a smooth map $j_u: M \to M$ by $j_u(\tau(x,r)) = \exp(v_{(x,r,u)})$. Put $g_u = j_u \circ f_u$. Since $\mu(r) = 1$ for $\epsilon/2 \leq r \leq 1 - \epsilon/2$, we have $i \circ \pi \circ g_u \circ \tau(x,r) = k(x,r,u)$ whenever both $\epsilon/2 \leq r \leq 1 - \epsilon/2$ and $\epsilon/2 \leq i \circ \pi \circ f_u \circ \tau(x,r) \leq 1 - \epsilon/2$. The latter condition says that $f_u \circ \tau(x,r)$ is in $P_s$ for some $\epsilon/2 \leq s \leq 1 - \epsilon/2$. Assuming that $k$ was close enough to $i \circ \pi \circ f \circ (\tau \times 1_W)$ so that each $\pi \circ g_u \circ \tau(x,r)$ is within $\epsilon/4$ of $\pi \circ f_u \circ \tau(x,r)$, the equality $i \circ \pi \circ g_u \circ \tau(x,r) = k(x,r,u)$ holds whenever $\tau(x,r)$ is in a $P_s$ and $g_u \circ \tau(x,r)$ is in a $P_t$ with $\epsilon \leq s, t \leq 1 - \epsilon$.

Carrying out this construction for a sequence of $k$ that converge to $i \circ \pi \circ f \circ (\tau \times 1_W)$, we obtain vector fields $v_{(x,r,u)}$ that converge to the zero vector field. For those sufficiently close to zero, $g$ will be a perturbation of $f$. Choosing $g$ sufficiently close to $f$, we may ensure that $\epsilon$ still gives border label control for $g$.

We will now analyze the graphic of $g_u$ on $I^2$. For $s, t \in [\epsilon, 1 - \epsilon]$, $\pi \circ g_u(x)$ equals $k_{(s,u)}(x)$ whenever $x \in P_s$ and $g_u(x) \in P_t$. Therefore the tangencies of $g_u(P_s)$ with $P_t$ are locally just the graphs of a critical point of $k_{(s,u)}: P \to \mathbb{R}$, so $g$ has property (GP1).

Let $s_1, \ldots, s_n$ be the values of $s$ in $[\epsilon, 1 - \epsilon]$ for which $k_{(s,u)}: P \to \mathbb{R}$ is not a Morse function. Each $k_{(s,u)}$ is still a function of finite codimension, so has finitely many critical points. Those with critical values in $[\epsilon, 1 - \epsilon]$ produce the points of the graphic of $g_u$ that lie in the vertical line $s = s_i$ as suggested in figure 10. We declare the $(s_i,t)$ at which $k_{(s_i,u)}$ has a critical point at $t$ to be vertices of $\Gamma_u$.

When $s$ is not one of the $s_i$, $k_{(s,u)}$ is a Morse function, so any tangency of $g_u(P_s)$ with $P_t$ is stable, and there is at most one such tangency. Since these tangencies are stable, all nearby tangencies are equivalent to them and hence also stable, so in the graphic for $g_u$ in $I^2$, the pairs $(s,t)$ corresponding to levels with a single stable tangency form ascending and descending arcs as suggested in figure 10. These arcs may enter or leave $I^2$, or may end at a point corresponding to one of the finitely many points of the graphic with $s$-coordinate equal to one of the $s_i$. We declare the intersection points of these arcs with $\partial I^2$ to be vertices of $\Gamma_u$. The conditions (GP2) and (GP3) have been achieved, completing the proof of theorem 8.1.
Finding good regions

In this section, we adapt the arguments of section 6 to general position families. The graphics associated to the $f_u$ of a general position family $f: M \times W \to M$ satisfy property (RS1) of section 6 (provided that the Heegaard splittings associated to the sweepout are strongly irreducible) and property (RS2) (since the open edges of the $\Gamma$ correspond to pairs of levels that have a single stable tangency, see the remark after the definition of (RS2) in section 6), but not property (RS3). Indeed, property (RS3) does not even make sense, since the vertices of $\Gamma$ can have high valence. Property (RS1) is what allows the map from the 0-cells of $K$ to the 0-simplices of the Diagram to be defined. Property (RS2) (plus conditions on regions near $\partial K$, which we will still have due to border label control) allows it to be extended to a cellular map from the 1-skeleton of $K$ to the 1-skeleton of the Diagram. What ensures that it still extends to the 2-cells is a topological fact about pairs of levels whose intersection contains a common spine, lemma 9.3. Because it involves surfaces that do not meet transversely, its proof is complicated and somewhat delicate. Since the proof does not introduce any ideas needed elsewhere, the reader may wish to skip it on a first reading, and go directly from the statement of lemma 9.3 to the last four paragraphs of the section.

We specialize to the case of a parameterized family $f: L \times W \to L$, where $L$ is a lens space and $W$ is a compact manifold. We retain the notations $P_t$, $Q_u$, $V_t$, $W_t$, $X$, and $Y_s$ of section 8. As usual, only $P_t$, $V_t$, $W_t$, and a number $\epsilon$ which gives border label control for $f$ are independent of the parameter $u$. As was mentioned above, properties (RS1) and (RS2) still hold for the labels of the regions of the graphic of each $f_u$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graphic.png}
\caption{A portion of the graphic of $g_u$.}
\end{figure}
Theorem 9.1. Suppose that $f : L \times W \to L$ is in general position with respect to $\tau$. Then for each $u$, there exists $(s,t)$ such that $Q_s$ meets $P_t$ in good position.

The proof of theorem 9.1 will constitute the remainder of this section.

We begin by examining the labels of parameters near the boundary of $I^2_{\epsilon}$; this will ultimately ensure that the boundary of $I^2_{\epsilon}$ maps to the Diagram in an essential way. Fix a parameter $u$, and suppose that $(s,t)$ is a point in the interior of $I^2_{\epsilon}$ such that $Q_s$ meets $P_t$ transversely. The next lemma is immediate from the definition of border label control and the labeling rules for regions. It does not require that we be working with lens spaces, so we state it as a lemma with weaker hypotheses.

Lemma 9.2. Suppose that $f : M \times W \to M$ is in general position with respect to $\tau$. Assume that $M \neq S^3$ and that the Heegaard splittings associated to $\tau$ are strongly irreducible. Suppose that $\epsilon$ gives border label control for $f$.

1. If $t \leq \epsilon$, then the label of $(s,t)$ contains $A$.
2. If $t \geq 1 - \epsilon$, then the label of $(s,t)$ contains $B$.
3. If $s \leq \epsilon$, then the label of $(s,t)$ contains $X$.
4. If $s \geq 1 - \epsilon$, then the label of $(s,t)$ contains $Y$.

We now prove a key geometric lemma that is particular to lens spaces.

Lemma 9.3. Let $f : L \times W \to L$ be a parameterized family of diffeomorphisms in general position, and let $(s,t) \in I^2_{\epsilon}$. If $Q_s \cap P_t$ contains a spine of $P_t$, then either $V_t$ or $W_t$ contains a core circle which is disjoint from $Q_s$.

Proof. We will move $Q_s$ by a sequence of isotopies. All isotopies will have the property that if $V_t - Q_s$ (or $W_t - Q_s$) did not contain a core circle of $V_t$ (or $W_t$) before the isotopy, then the same is true after the isotopy. We say this succinctly with the phrase that the isotopy does not create core circles. Typically some of the isotopies will not be smooth, so we work in the PL category. At the end of an initial “flattening” isotopy, $Q_s$ will intersect $P_t$ nontransversely in a 2-dimensional simplicial complex $X$ in $P_t$ whose frontier consists of points where $Q_s$ is PL imbedded but not smoothly imbedded. A sequence of simplifications called tunnel moves and bigon moves, plus isotopies that push disks across balls, will make $Q_s \cap P_t$ a single component $X_0$, which will then undergo a few additional improvements. After this has been completed, an Euler characteristic calculation will show that a core circle disjoint from the repositioned $Q_s$ exists in either $V_t$ or $W_t$, and consequently one existed for the original $Q_s$.

Since $f$ is in general position, $Q_s \cap P_t$ is a 1-complex satisfying the property (GP1) of section 8. Each isolated vertex of $Q_s \cap P_t$ is an isolated tangency of $Q_s \cap P_t$, so we can move $Q_s$ by a small isotopy near the vertex to eliminate it from the intersection. After this step, $Q_s \cap P_t$ is a graph $\Gamma$ which contains a spine of $Q_s \cap P_t$, such that each vertex of $\Gamma$ has positive valence.

By property (GP1), each vertex $x$ of $\Gamma$ is a point where $Q_s$ is tangent to $P_t$, and the edges of $\Gamma$ that emanate from $x$ are arcs where $Q_s$ intersects $P_t$. 
Along each arc, $Q_s$ crosses from $V_t$ into $W_t$ or vice versa, so there is an even number of these arcs. Near $x$, the tangent planes of $Q_s$ are nearly parallel to those of $P_t$, and there is an isotopy that moves a small disk neighborhood of $x$ in $Q_s$ until it coincides with a small disk neighborhood of $x$ in $P_t$. Perform such isotopies near each vertex of $\Gamma$. This enlarges $\Gamma$ in $Q_s \cap P_t$ to the union of $\Gamma$ with a union $E$ of disks, each disk containing one of the original vertices.

The closure of the portion of $\Gamma$ that is not in $E$ now consists of a collection of arcs and circles where $Q_s$ intersects $P_t$ transversely, except at the endpoints of the arcs, which lie in $E$. Consider one of these arcs, $\alpha$. At points of $\alpha$ near $E$, the tangent planes to $Q_s$ are nearly parallel to those of $P_t$, and starting from each end there is an isotopy that moves a small regular neighborhood of a portion of $\alpha$ in $Q_s$ onto a small regular neighborhood of the same portion of $\alpha$ in $P_t$. This flattening process can be continued along $\alpha$. If it is started from both ends of $\alpha$, it may be possible to flatten all of a regular neighborhood of $\alpha$ in $Q_s$ onto one in $P_t$. This occurs when the vectors in a field of normal vectors to $\alpha$ in $Q_s$ are being moved to normal vectors on the same side of $\alpha$ in $P_t$. If they are being moved to opposite sides, then we introduce a point where the configuration is as in figure 11, in which $P_t$ appears as the $xy$-plane, $\alpha$ appears as the points in $P_t$ with $x = -y$, and $Q_s$ appears as the four shaded half- or quarter-planes. These points will be called crossover points. Perform such isotopies in disjoint neighborhoods of all the arcs of $\Gamma - E$. For the components of $\Gamma$ that are circles of transverse intersection points, we flatten $Q_s$ near each circle to enlarge it to an annulus.

At the end of this initial process, $\Gamma$ has been been enlarged to a 2-complex $X$ in $Q_s \cap P_t$ that is a regular neighborhood of $\Gamma$, except at the crossover points where $\Gamma$ and $X$ look locally like the antidiagonal $x = -y$ of the $xy$-plane and the set of points with $xy \leq 0$. We will refer to $X$ as a pinched regular neighborhood of $\Gamma$.

Since $\Gamma$ originally contained a spine of $P_t$, $X$ contains two circles that meet transversely in one point that lies in the interior (in $P_t$) of $X$. Therefore $X$ contains a common spine of $P_t$ and $Q_s$. Let $X_0$ be the component of $X$ that contains a common spine of $Q_s$ and $P_t$. All components of $P_t - X_0$ and
Q_s − X_0 are open disks. Let X_1 = X − X_0, and for each i, denote Γ ∩ X_i by Γ_i.

The next step will be to move Q_s by isotopy to remove X_1 from Q_s ∩ P_1. These isotopies will be fixed near X_0. Some of them will have the effect of joining two components of V_1 − Q_s (or of W_1 − Q_s) into a single component of V_1 − Q_s (or of W_1 − Q_s) for the repositioned Q_s, so we must be very careful not to create core circles.

The frontier of X_1 in P_1 is a graph Fr(X_1) for which each vertex is a crossover point, and has valence 4 (as usual, our “graphs” can have open edges that are circles). Its edges are of two types: up edges, for which the component of Q_s − X that contains the edge lies in W_t, and down edges, for which it lies in V_t. At each disk of E, the up and down edges alternate as one moves around ∂E (see figure 12). For each of the arcs of Γ_1 − E, the flattening process creates an up edge on one side and a down edge on the other, but there is a fundamental difference in the way that the up and down edges appear in Q_s and P_1. As shown in figure 12, up edges (the solid ones) and down edges (the dotted ones) alternate as one moves around a crossover point, while in Q_s they occur in adjacent pairs. This is immediate upon examination of figure 11.

For our inductive procedure, we start with a pinched regular neighborhood X_1 ⊂ Q_s ∩ P_1 of a graph Γ_1 in Q_s ∩ P_1, all of whose vertices have positive even valence. Moreover, the edges of the frontier of X_1 are up or down according to whether the portion of Q_s − X that contains them lies in W_t or V_t. We call this an inductive configuration.

To ensure that our isotopy process will terminate, we use the complexity −χ(Γ_1) − χ(Fr(X_1)) + N, where N is the number of components of Γ_1. Since all vertices of Γ_1 and Fr(X_1) have valence at least 2, each of their components has nonpositive Euler characteristic, so the complexity is a non-negative integer. The remaining isotopies will reduce this complexity, so our procedure must terminate.

We may assume that the complexity is nonzero, since if N = 0 then X_1 is empty. Consider X_1 as a subset of the union of open disks Q_s − X_0. Since X_1 is a regular neighborhood of a graph with vertices of valence at least 2, it separates these disks, and we can find a closed disk D in Q_s with ∂D ⊂ X_1 and D ∩ X = ∂D. It lies either in V_t or W_t. Assume it is in W_t.
Figure 13. A portion of $P_t$ showing a tunnel arc in $X_1$, and the new $\Gamma_1$ and $X_1$ after the tunnel move.

(the case of $V_t$ is similar), in which case all of its edges are up edges. Since $\partial D \subset P_t - X_0$, $\partial D$ bounds a disk $D'$ in $P_t - X_0$. Since the interior of $D$ is disjoint from $P_t$, $D \cup D'$ bounds a 3-ball $\Sigma$ in $L$. Of course, $D'$ may contain portions of the component of $X_1$ that contains $\partial D'$, or other components of $X_1$. Let $X'_1$ be the component of $X_1$ that contains $\partial D'$; it is a pinched regular neighborhood of a component $\Gamma'_1$ of $\Gamma$.

Suppose that $X'_1$ contains some vertices of $\Gamma_1$ of valence more than 2. We will perform an isotopy of $Q_s$ that we call a tunnel move, illustrated in figure 13, that reduces the complexity of the inductive configuration. Near the vertex, select an arc in $X'_1$ that connects the edge of $\text{Fr}(X'_1)$ in $D'$ with another up edge of $\text{Fr}(X'_1)$ that lies near the vertex (this arc may lie in $D'$, in a portion of $X_1$ contained in $D'$). An isotopy of $Q_s$ is performed near this arc, that pulls an open regular neighborhood of the arc in $X'_1$ into $W_t$. This does not change the interior of $V_t - Q_s$ (it just adds the regular neighborhood of the arc to $V_t - Q_s$), but in $W_t$ it creates a tunnel that joins two different components of $W_t - Q_s$. One of these components was in $\Sigma$, so the isotopy cannot create core circles. After the tunnel move, we have a new inductive configuration. The Euler characteristic of $\Gamma_1$ has been increased by the addition of one vertex, while $\chi(\text{Fr}(X_1))$ and $N$ are unchanged, so the new inductive configuration is of lower complexity. The procedure continues by finding a new $D$ and $D'$ and repeating the process.

When a $D$ has been found for which no tunnel moves are possible, all vertices of $\Gamma'_1$ (if any) have valence 2. Suppose that $X'_1$ contains crossover points. It must contain an even number of them, since up and down edges alternate in $P_t$ around vertices of $\Gamma'_1$. Some portion of $X'_1$ is a disk $B$ whose frontier consists of two crossover points and two edges of $\text{Fr}(X'_1)$, each connecting the two crossover points. There is an isotopy of $Q_s$, supported in a neighborhood of $B$, that repositions $Q_s$ and replaces a neighborhood of $B$ in $X$ with a rectangle containing no crossover points. Figure 14 illustrates this isotopy. It cannot create core circles, indeed the interiors of $V_t - Q_s$ and $W_t - Q_s$ are unchanged during the isotopy. We call such an isotopy a bigon move.
Since bigon moves increase the Euler characteristic of $\text{Fr}(X_1)$, without changing $\Gamma_1$ or $N$, they reduce complexity. So we eventually arrive at the case when $X'_1$ is an annulus. Assume for now that the interior of $D'$ is disjoint from $X_1$. There is an isotopy of $Q_s$ that pushes $D$ across $\Sigma$, until it coincides with $D'$. This cannot create core circles, since its effect on the homeomorphism type of $W_t - Q_s$ is simply to remove the component $\Sigma - Q_s$. Perform a small isotopy that pulls $D'$ off into the interior of $W_t$, again creating no new core circles. An annulus component of $X_1$ has been eliminated, reducing the complexity. If the interior of $D'$ meets $X_1$, then $D' \cap X_1 = X'_1$, and a similar isotopy eliminates $X'_1$.

Suppose now that the interior of $D'$ contains components of $X_1$ other than perhaps $X'_1$. Let $X''_1$ be their union. It is a pinched regular neighborhood of a union $\Gamma''_1$ of components of $\Gamma_1$. If $\Gamma''_1$ has vertices of valence more than 2, then tunnel moves can be performed. These cannot create new core circles, since they do not change the interior of $V_t - Q_s$, and in $W_t - Q_s$ they only connect regions that are contained in $\Sigma$. If no tunnel move is possible, but there are crossover points, then a bigon move may be performed. So we may assume that every component of $X''_1$ is an annulus.

Let $S$ be a boundary circle of $X''_1$ innermost on $D'$, bounding a disk $D''$ in $D'$ whose interior is disjoint from $X$. Let $E''$ be the disk in $Q_s$ bounded by $S$, so that $D'' \cup E''$ bounds a 3-ball $\Sigma''$ in $L$.

We claim that if $(V_t - Q_s) \cup E''$ contains a core circle of $V_t$, then $V_t - Q_s$ contained a core circle of $V_t$ (and analogously for $W_t$). The closures of the components of $E'' - X_1$ are planar surfaces, each lying either in $V_t$ or $W_t$. Let $F$ be one of these, lying (say) in $V_t$. Its boundary circles lie in $P_t - X_0$, so bound disks in $P_t$. A regular neighborhood in $V_t$ of the union of $F$ and these disks is a punctured 3-cell $Z(F)$ meeting $P_t$ in disks. Suppose that $C$ is a core circle in $V_t$ that is disjoint from $Q_s - F$. We may assume that $C$ meets $\partial Z(F)$ transversely, so cuts through $Z(F)$ is a collection of arcs. Since $Z(F)$ is a punctured 3-cell, there is an isotopy of $C$ that pushes the arcs to the frontier of $Z(F)$ and across it, removing the intersections of $C$ with $F$ without creating new intersections (since the arcs need only be pushed slightly outside of $Z(F)$). Performing such isotopies for all components of $E'' - X_1$ in $V_t$ produces a core circle disjoint from $E''$, proving the claim.

By virtue of the claim, an isotopy that pushes $E''$ across $\Sigma''$ until it coincides with $D''$ does not create core circles. Then, a slight additional
isotopy pulls $D''$ and the component of $X_1$ that contained $\partial D''$ off of $P_t$, reducing the complexity.

Since we can always reduce a nonzero complexity by one of these isotopies, we may assume that $Q_s \cap P_t = X_0$. The frontier $\text{Fr}(X_0)$ in $P_t$ is the union of a graph $\Gamma_2$, each of whose components has vertices of valence 4 corresponding to crossover points, and a graph $\Gamma_3$ whose components are circles.

A component of $\Gamma_3$ must bound both a disk $D_Q$ in $Q_s - X_0$ and a disk $D_P$ in $P_t - X_0$. Since $Q_s \cap P_t = X_0$, the interiors of $D_P$ and $D_Q$ are disjoint, and $D_Q$ lies either in $V_t$ or in $W_t$. So we may push $D_Q$ across the 3-ball bounded by $D_Q \cup D_P$ and onto $D_P$, without creating core circles. Repeating this procedure to eliminate the other components of $\Gamma_3$, we achieve that the frontier of $Q_s \cap P_t$ equals the graph $\Gamma_2$.

Figure 15 shows a possible intersection of $Q_s$ with $P_t$ at this stage. The shaded region is $Q_s \cap P_t$; the portion of it that lies between $C_1$ and $C_2$ is a single octagon that passes around the back of the torus. The closure of $Q_s - (Q_s \cap P_t)$ consists of two meridian disks in $V_t$, bounded by the circles $C_1$ and $C_2$, and two boundary-parallel disks in $W_t$, bounded by the circles $C_3$ and $C_4$.

Suppose that there are now $2k_1$ meridian disks of $Q_s$ in $V_t$ and $2k_2$ in $W_t$ (their numbers must be even since $Q_s$ is zero in $H_2(L)$), and a total of $k_0$ boundary-parallel disks in $Q_s \cap V_t$ and $Q_s \cap W_t$. Since $\chi(Q_s) = 0$, we have $\chi(Q_s \cap P_t) = -k_0 - 2k_1 - 2k_2$. To prove the lemma, we must show that either $k_1$ or $k_2$ is 0.

Let $V$ be the number of vertices of $\Gamma_2$. Since all of its vertices have valence 4, $\Gamma_2$ has $2V$ edges. The remainder of $Q_s \cap P_t$ consists of 2-dimensional faces. Each of these faces has boundary consisting of an even number of edges, since up and down edges alternate around a face. If some of the faces are bigons, such as two of the faces in figure 15, they may be eliminated by bigon moves. These may create additional components of the frontier of $X_0$ that are circles, indeed this happens in the example of figure 15. These are eliminated as before by moving disks of $Q_s$ onto disks in $P_t$. After all bigons
have been eliminated, each face has at least four edges, so there are at most $V/2$ faces. So we have $\chi(Q_s \cap P_t) \leq V - 2V + V/2 = -V/2$.

Each boundary-parallel disk in $Q_s \cap V_t$ or $Q_s \cap W_t$ contributes at least two vertices to the graph, since at each crossover point, $X_0$ crosses over to the other side in $P_t$ of the boundary of the disk. This gives at least $2k_0$ vertices. The meridian disks on the two sides contribute at least $2k_1 \cdot 2k_2 \cdot m$ additional vertices, where $L = L(m, q)$, since the meridians of $V_t$ and $W_t$ have algebraic intersection $\pm m$ in $P_t$. Thus $V \geq 2k_0 + 4k_1k_2m$. We calculate:

$$-k_0 - 2k_1 - 2k_2 = \chi(Q_s \cap P_t) \leq -V/2 \leq -k_0 - 2k_1k_2m.$$ 

Since $m > 2$, this can hold only when either $k_1$ or $k_2$ is 0. □

Lemma 9.3 fails (at the last sentence of the proof) for the case of $L(2, 1)$. Indeed, there is a flattened Heegaard torus in $L(2, 1)$ which meets $P_{1/2}$ in four squares and has two meridian disks on each side. In a sketch somewhat like that of figure 15, the boundaries of these disks are two meridian circles and two $(2, 1)$-loops intersecting in a total of 8 points, and cutting the torus into 8 squares. There are two choices of four of these squares to form $Q_s \cap P_t$.

Now, we will complete the proof of theorem 9.1. As in section 6, assume for contradiction that all regions are labeled, and triangulate $I^2$. The map on the 1-skeleton is defined exactly as in section 6, using lemma 9.2 and the fact that the labels satisfy property (RS2). Using lemma 9.2, each 1-cell maps either to a 0-simplex or a 1-simplex of the Diagram, and exactly as before the boundary circle of $K$ maps to the Diagram in an essential way. The contradiction will be achieved once we show that the map extends over the 2-cells.

There is no change from before when the 2-cell meets $\partial K$ or lies in the interior of $K$ but does not contain a vertex of $\Gamma$, so we fix a 2-cell in the interior of $K$ that is dual to a vertex $v_0$ of $\Gamma$, located at a point $(s_0, t_0)$.

Suppose first that $Q_{s_0} \cap P_{t_0}$ contains a spine of $P_{t_0}$. By lemma 9.3, either $V_{t_0}$ or $W_{t_0}$ has a core circle $C$ which is disjoint from $Q_{s_0}$; we assume it lies in $V_{t_0}$, with the case when it lies in $W_{t_0}$ being similar. The letter $A$ cannot appear in the label of any region whose closure contains $v_0$, since $C$ is a core circle for all $P_t$ with $t$ near $t_0$, and $Q_s$ is disjoint from $C$ for all $s$ near $s_0$. By lemma 6.2, any letter $a$ that appears in the label of one of the regions whose closure contains $v_0$ must appear in a combination of either $ax$ or $ay$, so none of these regions has label $A$. Since each 1-cell maps to a 0- or 1-simplex of the Diagram, the map defined on the 1-cells of $K$ maps the boundary of the 2-cell dual to $v_0$ into the complement of the vertex $A$ of the Diagram. Since this complement is contractible, the map can be extended over the 2-cell.

Suppose now that $Q_{s_0} \cap P_{t_0}$ does not contain a spine of $P_{t_0}$. Then there is a loop $C_{(s_0, t_0)}$ essential in $P_{t_0}$ and disjoint from $Q_{s_0}$. For every $(s, t)$ near $(s_0, t_0)$, there is a loop $C_{(s, t)}$ essential in $P_t$ and disjoint from $Q_s$, with the property that $C_{(s, t)}$ is a meridian of $V_t$ (respectively $W_t$) if and only if $C_{(s_0, t_0)}$ is a meridian of $V_{t_0}$ (respectively $W_{t_0}$). In particular, any intersection circle
of $Q_s$ and $P_t$ which bounds a disk in $Q_s$ which is precompressing for $P_t$ in $V_t$ or in $W_t$ must be disjoint from $C(s,t)$. Since the meridian disks of $V_t$ and $W_t$ have nonzero algebraic intersection, the meridians for $V_t$ and $W_t$ cannot both be disjoint from $C(s,t)$. So for all $(s,t)$ in this neighborhood of $(s_0,t_0)$, either all disks in $Q_s$ that are precompressions for $P_t$ are precompressions in $V_t$, or all are precompressions in $W_t$. In the first case, the letter $B$ does not appear in the label of any of the regions whose closure contain $v_0$, while in the second case, the letter $A$ does not. In either case, the extension to the 2-cell can now be obtained just as in the previous paragraph. This completes the proof of theorem 9.1.
10. From good to very good

By virtue of theorem 9.1, we may perturb a parameterized family of diffeomorphisms of $M$ so that at each parameter $u$, some level $P_t$ and some image level $f_u(P_s)$ meet in good position. In this section, we use the methodology of A. Hatcher [10, 11] (see [12] for a more detailed version of [11], see also N. Ivanov [21]) to change the family so that we may assume that $P_t$ and $f_u(P_s)$ meet in very good position. In fact, we will achieve a rather stronger condition on discal intersections.

Following our usual notation, we fix a sweepout $\tau: P \times [0,1] \rightarrow M$ of a closed orientable 3-manifold $M$, and give $P_t$, $V_t$, and $W_t$ their usual meanings. Given a parameterized family of diffeomorphisms $f: M \times W \rightarrow M$, we give $f_u$, $Q_s$, $X_s$, and $Y_s$ their usual parameter-dependent meanings. From now on, we refer to the $P_t$ as levels and the $Q_s$ as image levels.

Throughout this section, we assume that for each $u \in W$, there is a pair $(s,t)$ such that $Q_s$ and $P_t$ are in good position. Before stating the main result, we will need to make some preliminary selections.

By transversality, being in good position is an open condition, so there exist a finite covering of $W$ by open sets $U_i$, $1 \leq i \leq n$, and pairs $(s_i,t_i)$, so that for each $u \in U_i$, $Q_{s_i}$ and $P_{t_i}$ meet in good position. Note that we may reselect the $U_i$ to be open $d$-balls whose closures are $d$-balls, by shrinking them and covering the shrunken ones with finitely many such $d$-balls contained in the original $\overline{U(t_i)}$. Moreover, by shrinking of the open cover, we can and always will assume that all transversality and good-position conditions that hold at parameters in $U_i$ actually hold on $\overline{U_i}$.

We want to select the sets and parameters so that at parameters in $U_i$, $Q_{s_i}$ is transverse to $P_{t_i}$ for all $t_i$. First note that for any $s$ sufficiently close to $s_i$, $Q_s$ is transverse to $P_{t_i}$ at all parameters of $U_i$ (here we are already using our condition that the transversality for the $Q_s$, holds for all parameters in $\overline{U_i}$). On $U_1$, $Q_{s_1}$ is already transverse to $P_{t_1}$. Sard’s Theorem ensures that at each $u \in U_2$, there is a value $s$ arbitrarily close to $s_2$ such that $Q_s$ is transverse to $P_{t_1}$ at all parameters in a neighborhood of $u$. Replace $U_2$ by finitely many open sets (with associated $s$-values), for which on each of these sets the associated $Q_s$ are transverse to $P_{t_1}$. The new $s$ are selected close enough to $s_2$ so that these $Q_s$ still meet $P_{t_2}$ in good position. Repeat this process for $U_3$, that is, replace $U_3$ by a collection of sets and associated values of $s$ for which the associated $Q_s$ are transverse to $P_{t_1}$ and still meet $P_{t_3}$ in good position. Proceeding through the remaining original $U_i$, we have a new collection, with many more sets $U_i$, but only the same $t_i$ values that we started with, and at each parameter in one of the new $U_i$, $Q_{s_i}$ is transverse to $P_{t_1}$. Now proceed to $P_{t_2}$. For the $U_i$ whose associated $t$-value is not $t_2$, we perform a similar process, and we also select the new $s$-values so close to $s_i$ that the new $Q_s$ are still transverse to $P_{t_i}$ and still meet their associated $P_{t_i}$ in good position. After finitely many repetitions, all $Q_{s_i}$ are transverse to each $P_{t_j}$. 
Since transversality is an open condition, we are free to replace $s_i$ by a very nearby value, while still retaining the good position of $Q_{s_i}$ and $P_{t_j}$ and the transverse intersection of $Q_{s_i}$ with all $P_{t_j}$, for all parameters in $U_i$, and similarly we may reselect any $t_j$. So (with the argument in the previous paragraph now completed) we can and always will assume that all $s_i$ are distinct, and all $t_i$ are distinct. We may then use $U(t_i)$ to denote the open set in $W$ associated to the pair $(s_i, t_i)$.

We can now state the main result of this section. With notation as above:

**Theorem 10.1.** Let $f : W \to \text{diff}(M)$ be a parameterized family, such that for each $u$ there exists $(s, t)$ such that $Q_s$ and $P_t$ meet in good position. Then $f$ may be changed by homotopy so that there exists a covering $\{U(t_i)\}$ as above, with the property that for all $u \in U(t_i)$, $Q_{s_i}$ and $P_{t_i}$ meet in very good position, and $Q_{s_i}$ has no discal intersection with any $P_{t_j}$. If these conditions already hold for all parameters in some closed subset $W_0$ of $W$, then the deformation of $f$ may be taken to be constant on some neighborhood of $W_0$.

Before starting the proof, we introduce a simplifying convention. Although strictly speaking, $Q_{s_i}$ is meaningful at every parameter, as is every $Q_s$, throughout the remainder of this section we speak of $Q_{s_i}$ only for parameters in $U(t_i)$. That is, unless explicitly stated otherwise, an assertion made about $Q_{s_i}$ means that the assertion holds at parameters in $U(t_i)$, but not necessarily at other parameters. Also, to refer to $Q_{s_i}$ at a single parameter $u$, we use the notation $Q_{s_i}(u)$. By our convention, $Q_{s_i}(u)$ is meaningful only when $u$ is a value in $U(t_i)$.

Now, to preview some of the complications that appear in the proof of theorem 10.1, consider the problem of removing, just for a single parameter $u \in U(t_i)$, a discal component $c$ of the intersection of $Q_{s_i}(u)$ with some $P_{t_j}$. Suppose that the disk $D'$ in $Q_{s_i}(u)$ bounded by $c$ is innermost among all disks in $Q_{s_i}(u)$ bounded by discal intersections of $Q_{s_i}(u)$ with the $P_{t_j}$. Note that $D'$ can contain a nondiscal intersection of $Q_{s_i}(u)$ with a $P_{t_k}$; such an intersection will be a meridian of either $V_{t_k}$ or $W_{t_k}$ (although $k$ cannot equal $i$, since $Q_{s_i}(u)$ and $P_{t_i}$ meet in good position). Let $D$ be the disk in $P_{t_j}$ bounded by $c$, so that $D \cup D'$ is the boundary of a 3-ball $E$. There is an isotopy of $f_u$ that moves $D'$ across $E$ to $D$, and on across $D$, eliminating $c$ and possibly other intersections of the $Q_{s_i}(u)$ with the $P_{t_k}$. We will refer to this as a basic isotopy.

It is possible for a basic isotopy to remove a biessential component of some $Q_{s_k}(u) \cap P_{t_k}$. Examples are a bit complicated to describe, but involve ideas similar to the construction in figure 2. Fortunately, the following lemma ensures that good position is not lost.

**Lemma 10.2.** After a basic isotopy as described above, each $Q_{s_k}(u) \cap P_{t_k}$ still has a biessential component.
Proof. Throughout the proof of the lemma, \(Q_s\) is understood to mean \(Q_s(u)\).

Suppose that a biessential component of some \(Q_{s_k} \cap P_{t_k}\) is contained in the ball \(E\), and hence is removed by the isotopy. Since a spine of \(Q_{s_k}\) cannot be contained in a 3-ball, there must be a circle of intersection of \(Q_{s_k}\) with \(D\) that is essential in \(Q_{s_k}\). This implies that \(k \neq j\). Now \(D'\) must have nonempty intersection with \(P_{t_k}\), since otherwise \(P_{t_k}\) would be contained in \(E\). An intersection circle innermost on \(D'\) cannot be inessential in \(P_{t_k}\), since \(c\) was an innermost discal intersection on \(Q_{s_i}\), so \(D'\) contains a meridian disk \(D'_0\) for either \(V_{t_k}\) or \(W_{t_k}\). Choose notation so that \(D\) is contained in \(V_{t_k}\) (that is, \(t_j < t_k\)).

Suppose first that \(D'_0 \subset V_{t_k}\). The basic isotopy pushing \(D'\) across \(E\) moves \(Q_{s_k} \cap E\) into a small neighborhood of \(D\), so that it is contained in \(V_{t_k}\). If there is no longer any biessential intersection of \(Q_{s_k}\) with \(P_{t_k}\), then \(Q_{s_k}\) is now on the \(V_{t_k}\)-side of \(P_{t_k}\). This is a contradiction, since \(Q_{s_k}\) is a Heegaard torus.

Suppose now that \(D'_0 \subset W_{t_k}\). Since the biessential circles of \(Q_{s_k} \cap P_{t_k}\) are disjoint from \(D'_0\), they are meridians for \(W_{t_k}\) and hence are essential in \(V_{t_k}\). Now, let \(A\) be innermost among the annuli on \(Q_{s_k}\) bounded by a biessential component \(C\) of \(Q_{s_k} \cap P_{t_k}\) and a circle of \(Q_{s_k} \cap D\). Since \(Q_{t_k}\) and \(P_{t_k}\) meet in good position, the intersection of the interior of \(A\) with \(P_{t_k}\) is discal. This implies that \(C\) is contractible in \(V_{t_k}\), a contradiction. \(\square\)

Proof of theorem 10.1. We will adapt the approach of Hatcher [10]. The principal difference for us is that in [10], there is only a single domain level, whereas we have the different \(Q_{s_i}\) on the sets \(U(t_i)\).

The first step is to construct a family \(h_{u,t}, 0 \leq t \leq 1\) of isotopies of the \(f_u = h_{u,0}\), which eliminates the discal intersections of every \(Q_{s_i}(u)\) with every \(P_{t_j}\). Let \(\mathcal{C}\) be the set of all discal intersection curves of \(Q_{s_i} \cap P_{t_j}\). Since \(Q_{s_i}\) is transverse to \(P_{t_j}\) at all \(u \in \overline{U(t_i)}\), the curves in \(\mathcal{C}\) fall into finitely many families which vary by isotopy as the parameter moves over (the connected set) \(\overline{U(t_i)}\). Thus we may regard \(\mathcal{C}\) as a disjoint union containing finitely many copies of each \(\overline{U(t_i)}\). It projects to \(W\), with the preimage of \(u\) consising of the discal intersection curves \(C_u\) of the \(Q_{s_i}(u)\) and \(P_{t_j}\) for which \(u \in U(t_i)\).

By assumption, no element of \(\mathcal{C}\) projects to any parameter \(u \in W_0\).

Each \(c \in C_u\) bounds unique disks \(D_c \subset P_{t_j}\) and \(D'_c \subset Q_{s_i}(u)\) for some \(i\) and \(j\). The inclusion relations among the \(D_c\) define a partial ordering \(\prec\) on \(C_u\), by the rule that \(c_1 \prec c_2\) when \(D_{c_1} \subset D_{c_2}\). Similarly, \(c_1 \prec c_2\) when \(D'_{c_1} \subset D'_{c_2}\).

If \(c\) is minimal for \(\prec\), then \(D'_c \cup D_c\) is an imbedded 2-sphere in \(M\) which bounds a 3-ball \(E_c\). By lemma 10.2, the basic isotopy that pushes \(D'_c\) across \(E_c\) to \(D_c\) and on to the other side of \(D_c\) retains the property that every \(Q_{s_k}(u) \cap P_{s_k}\) has a biessential intersection. This ensures that when all
discal intersections have been eliminated, each \( Q_{s_i}(u) \cap P_{t_k} \) will still have an intersection, so they will be in very good position.

Shrink the open cover \( \{U(t_i)\} \) to an open cover \( \{U(t_i)\}' \) for which each \( U(t_i)' \subset U(t_i) \). To construct the \( h_{u,t} \), Hatcher introduced an auxiliary function \( \Psi : C \to (0, 2) \) that gives the order in which the elements of \( C \) are to be eliminated, and allows the basic isotopies to be tapered off as one nears the frontier of \( U(t_i) \). Denoting by \( \psi_u \) the restriction of \( \Psi \) to \( C_u \), we will select \( \Psi \) so that the following conditions are satisfied:

1. \( \psi_u(c) < \psi_u(c') \) whenever \( c < Q c' \)
2. \( \psi_u(c) < 1 \) if \( c \subset Q_{s_i}(u) \) and \( u \in \overline{U(t_i)} \)
3. \( \psi_u(c) > 1 \) if \( c \subset Q_{s_i}(u) \) and \( u \in U(t_i) - U(t_i) \).

One way to construct such a \( \Psi \) is to choose a Riemannian metric on \( \tau(P \times (0, 1)) \) for which each \( P_t \) has area 1, and define \( \Psi_0(c) \) to be the area of \( f_{u}^{-1}(D_i) \) in \( P_{s_i} \). Then, choose continuous functions \( \alpha_t \) which are 0 on \( \overline{U(t_i)} \) and 1 on \( W - U(t_i) \), and define \( \Psi(c) = \Psi_0(c) + \alpha_t(u) \) for \( c \subset Q_{s_i}(u) \).

Roughly speaking, the idea of Hatcher’s construction is to have \( h_{u,t} \) perform the basic isotopy that eliminates \( c \) during a small time interval \( I_u(c) \) which starts at the number \( \psi_u(c) \). In order to retain control of this process, preliminary steps must be taken to ensure that basic isotopies that move points in intersecting 3-balls \( E_c \) do not occur at the same time.

Define \( G_0 \) to be the subset of \( W \times [0, 2] \) consisting of all \( (u, \psi_u(c)) \) with \( c \in C_u \). For a fixed isotopic family of \( c \in C \) with \( c \subset Q_{s_i}(u) \), the points \( (u, \psi_u(c)) \) form a \( d \)-dimensional sheet \( i(c) \) lying over \( \overline{U(t_i)} \), where \( d \) is the dimension of \( W \). If \( i(c_1) \) meets \( i(c_2) \), then by the first property of \( \Psi \), \( c_1 \) and \( c_2 \) cannot be \( <Q \)-related.

Thicken each \( i(c) \) to a plate \( I(c) \) intersecting each \( \{u\} \times [0, 2] \) in an interval \( I_u(c) = [\psi_u(c), \psi_u(c) + \epsilon] \), for some small positive \( \epsilon \). This interval will contain the \( t \)-support of the portion of \( h_{u,t} \) that eliminates \( c \), assuming that all other loops in \( C_u \) with smaller \( \psi_u \)-value have already been eliminated. By condition (1), \( c \) will be \( <Q \)-minimal at the times \( t \in I_u(c) \). Since \( C_u \) is empty for \( u \in W_0 \), the \( h_{u,t} \) will be constant for all \( u \in W_0 \).

Choose the \( \epsilon \) small enough so that \( I(c_1) \cap I(c_2) \) is nonempty only near the intersections of \( i(c_1) \) and \( i(c_2) \). This ensures that if basic isotopies eliminating \( c_1 \) and \( c_2 \) occur on overlapping time intervals, then \( c_1 \) and \( c_2 \) are \( <Q \)-unrelated. Also, choose \( \epsilon \) small enough so that \( I_u(c) \subset [0, 1] \) whenever \( u \in U(t_i)' \).

It may happen that for some \( c_1, c_2 \in C_u \) with \( \psi_u(c_1) < \psi_u(c_2) \), we have \( c_2 < P c_1 \). In this case the isotopy which eliminates \( c_1 \) will also eliminate \( c_2 \). So reduce \( G_0 \) by deleting all points \( (u, \psi_u(c_2)) \) such that \( \psi_u(c_1) < \psi_u(c_2) \) for some \( c_1 \) with \( c_2 < P c_1 \). Make a corresponding reduction of \( I(c_2) \) by deleting points \( t \in I_u(c_2) \) such that \( t > \psi_u(c_1) \) for some \( c_1 \) with \( c_2 < P c_1 \).

At values of \( t \) where the interiors of \( I(c_1) \) and \( I(c_2) \) still overlap, \( c_1 \) and \( c_2 \) are \( <Q \)-unrelated, and the reduction just made ensures that they are not \( <P \)-related. In Hatcher’s context, all intersections are discal, so the combined
effect of these is to eliminate the possibility of simultaneous isotopies on intersecting 3-balls $E_{c_1}$ and $E_{c_2}$. In our context, however, $E_{c_1}$ and $E_{c_2}$ can intersect on overlaps of $I(c_1)$ and $I(c_2)$ even when $c_1$ and $c_2$ are neither $<_P$-related nor $<_Q$-related. Figure 16 shows a simple example. The intersections of $P_{t_1}$ with $Q_{s_2}$, are not discal, nor are the intersections of $P_{t_2}$ with $Q_{s_1}$, but $Q_{s_2}$ has a discal intersection with $P_{t_2}$ inside $E(c_1)$. When this happens, however, $E_{c_1}$ and $E_{c_2}$ must be either disjoint or nested:

**Lemma 10.3.** Suppose that $c_1$ and $c_2$ are $<_Q$-minimal discal intersections, and are neither $<_P$-related nor $<_Q$-related. Then $\partial E_{c_1}$ and $\partial E_{c_2}$ are disjoint.

**Proof.** Since $c_1$ and $c_2$ are not $<_Q$-related, $D'(c_1)$ and $D'(c_2)$ are disjoint, and since they are not $<_P$-related, $D(c_1)$ and $D(c_2)$ are disjoint. An intersection circle of $D(c_1)$ and $D'(c_2)$ would be smaller than $c_2$ in the $<_Q$-ordering, and similarly an intersection circle of $D'(c_1)$ and $D(c_2)$ would be smaller than $c_1$ in the $<_Q$-ordering. \(\square\)

When $E_{c_1}$ and $E_{c_2}$ are nested, say, $E_{c_2}$ lies in $E_{c_1}$, a basic isotopy that removes $c_1$ will also remove $c_2$. So we make the further reduction in $G_0$ of deleting all $(u, \psi_u(c_2))$ for which there is a $c_1$ such that $i(c_1)$ meets $i(c_2)$, $\psi_u(c_1) < \psi_u(c_2)$, and $E_{c_2} \subset E_{c_1}$. Also, reduce $I(c_2)$ by removing any $t$ in $I_u(c_2)$ with $t > \psi_u(c_1)$.

For fixed $u \in W$, the basic isotopies are combined by proceeding upward in $W \times [0, 2]$ from $t = 0$ to $t = 1$, performing each basic isotopy involving $c$ on the interval $I_u(c)$. Condition (3) on the $\psi_u$ ensures that the basic isotopies involving $c \subset Q_{s_i}(u)$ taper off at parameters near the frontier of $U(t_i)$. On a reduced interval $I_u(c)$, which is an initial segment of $[\psi_u(c), \psi_u(c) + \epsilon]$, perform only the corresponding initial portion of the basic isotopy. On the
overlaps of the $I(c)$, perform the corresponding basic isotopies concurrently; the reductions of the $I(c)$ have ensured that these basic isotopies will have disjoint supports. Since $\epsilon$ was chosen small enough so that $I_u(c) \subset [0,1]$ whenever $u \in U(t_i)'$, the basic isotopies involving $Q_{s_i}$ will be completed at all $u$ in $U(t_i)'$. Since $\mathcal{C}_u$ is empty for $u \in W_0$, no isotopies take place at parameters in $W_0$.

The remaining concern is that the basic isotopies eliminating $c \subset Q_{s_i}(u)$ must be selected so that they fit together continuously in the parameter $u$ on $U(t_i)$. This can be achieved using the method in the last paragraph on p. 345 of [10] (which applies in the smooth category by virtue of [13], see also the more detailed version in [12]).
11. Setting up the last step

In this section, we present some technical lemmas that will be needed for the final stage of the proof.

The first two lemmas give certain uniqueness properties for the fiber of the Hopf fibration on $L$. Both are false for $\mathbb{R}P^3$, so require our convention that $L = L(m, q)$ with $m > 2$, and as usual we select $q$ so that $1 \leq q \leq m/2$. From now on, we endow $L$ with the Hopf fibering and assume that our sweepout of $L$ is selected so that each $P_t$ is a union of fibers. Consequently the exceptional fibers, if any, will be components of the singular set $S$.

**Lemma 11.1.** Let $P$ be a Heegaard torus in $L$ which is a union of fibers, bounding solid tori $V$ and $W$. Suppose that a loop in $P$ is a longitude for $V$ and for $W$. Then $q = 1$ and the loop is isotopic in $P$ to a fiber.

**Proof.** Let $a$ and $b$ be loops in $P$ which are respectively a longitude and a meridian of $V$, and with $a$ determined by the condition that $ma + qb$ is a meridian of $W$. Let $c$ be a loop in $P$ which is a longitude for both $V$ and $W$. Since $c$ is a longitude of $V$, it has (for one of its two orientations) the form $a + kb$ in $H_1(P)$ for some $k$. The intersection number of $c$ with $ma + qb$ is $q - km$, which must be $\pm 1$ since $c$ is a longitude of $W$. Since $1 \leq q \leq m/2$ and $m > 2$, this implies that $k = 0$ and $q = 1$. Since $k = 0$, $c$ is uniquely determined and $c = a$. Since $q = 1$, the Hopf fibering is nonsingular, so the fiber is a longitude of both $V$ and $W$ and hence is isotopic in $P$ to $c$. □

**Lemma 11.2.** Let $h: L \to L$ be a diffeomorphism isotopic to the identity, with $h(P_s) = P_t$. Then the image of a fiber of $P_s$ is isotopic in $P_t$ to a fiber.

**Proof.** Composing $f$ with a fiber-preserving diffeomorphism of $L$ that moves $P_s$ to $P_t$, we may assume that $s = t$. Write $P_s$, $V$, and $W$ for $P_t$, $V_t$, and $W_t$. Let $a$ and $b$ be loops in $P$ selected as in the proof of lemma 11.1, and write $h_*: H_1(P) \to H_1(P)$ for the induced isomorphism.

Suppose first that $h(V) = V$. Since the meridian disk of $V$ is unique up to isotopy, we have $h_*(b) = \pm b$. Since $h$ is isotopic to the identity on $L$ and $m > 2$, $h$ is orientation-preserving and induces the identity on $\pi_1(V)$. This implies that $h_*(b) = b$. Similar considerations for $W$ show that $h_*(ma + qb) = ma + qb$, so $h_*(a) = a$. Thus $h_*$ is the identity on $H_1(P)$ and the lemma follows for this case.

Suppose now that $h(V) = W$. Then $h$ is orientation-reversing on $P$. Since $h$ must take a meridian of $V$ to one of $W$, we have $h_*(b) = \epsilon_1(ma + qb)$ where $\epsilon_1 = \pm 1$. Writing $h_*(a) = ua + vb$, we find that $1 = a \cdot b = -h_*(a) \cdot h_*(b) = -\epsilon_1(qu - mv)$. The facts that $h$ is isotopic to the identity on $L$, $a$ generates $\pi_1(L)$, and $b$ is in $\pi_1(V)$ imply that $u \equiv 1 \pmod{m}$, so modulo $m$ we have $1 \equiv -\epsilon_1 q$. Since $1 \leq q \leq m/2$, this forces $q = 1$, $\epsilon_1 = -1$, and $h_*(b) = -ma - b$. Since $a$ has intersection number $-1$ with the meridian $-ma - b$ of $W$, it is also a longitude of $W$. Since $h$ is a homeomorphism interchanging $V$ and $W$, $h(a)$ is a longitude of $V$ and of $W$, and an application of lemma 11.1 completes the proof. □
We now give several lemmas which allow the deformation of diffeomorphisms and imbeddings to make them fiber-preserving or level-preserving. For \( Y \subset X \), \( \text{imb}(Y,X) \) means the connected component of the inclusion in the space of all imbeddings of \( Y \) in \( X \). When \( X \) is a fibered object, \( \text{Diff}_f(X) \) means the space of diffeomorphisms of \( X \) that take fibers to fibers, and \( \text{diff}_f(X) \) is the connected component of the identity \( \text{Diff}_f(X) \).

**Lemma 11.3.** Let \( X \) be either a solid torus or \( S^1 \times S^1 \times I \), with a fixed Seifert fibering. Then the inclusion \( \text{diff}_f(X) \to \text{diff}(X) \) is a homotopy equivalence.

**Proof.** Here is a very brief sketch of the proof; for detailed arguments of this kind, see the final section of [28]. All needed results on fibrations of spaces of diffeomorphisms appear in [22].

Results from surface theory imply that \( \text{diff}(S^1 \times S^1) \simeq S^1 \times S^1 \). Using [22], \( \text{diff}_f(X) \) is homotopy equivalent to \( S^1 \times S^1 \), and if \( T \) is a boundary component, the restriction map \( \text{diff}_f(X) \to \text{diff}(T) \) is a weak homotopy equivalence. Using [10], one can show that the fiber of \( \text{diff}(X) \to \text{diff}(T) \) is contractible, so in \( \text{diff}_f(X) \to \text{diff}(X) \to \text{diff}(T) \), the composition and the second map are homotopy equivalences, hence the first is as well. \( \square \)

Lemma 11.3 guarantees that if \( g: \Delta \to \text{diff}(X) \) is a continuous map from an \( n \)-simplex, \( n \geq 1 \), with \( g(\partial \Delta) \subset \text{diff}_f(X) \), then \( g \) is homotopic relative to \( \partial \Delta \) to a map with image in \( \text{diff}_f(X) \). Analogous observations hold for the next four lemmas as well.

When \( X \) is fibered or Seifert-fibered and \( Y \subset X \) is a union of fibers, we write \( \text{imb}_f(Y,X) \) for the connected component of the inclusion in the subspace of \( \text{imb}(Y,X) \) consisting of all imbeddings that take fibers to fibers. The next two lemmas were proven in [28], using results on fibrations of spaces of mappings from [22].

**Lemma 11.4.** Let \( T \) be a torus with a fixed \( S^1 \)-fibering, and let \( C_n \) be a union of \( n \) distinct fibers. Then \( \text{imb}_f(C_n,T) \to \text{imb}(C_n,T) \) is a weak homotopy equivalence.

**Lemma 11.5.** Let \( \Sigma \) be a compact 3-manifold with nonempty boundary and having a fixed Seifert fibering. Let \( F \) be a compact 2-manifold properly imbedded in \( \Sigma \), such that \( F \) is a union of fibers. Let \( \text{imb}_{\partial_f}(F,\Sigma) \) be the connected component of the inclusion in the space of (proper) imbeddings for which the image of \( \partial F \) is a union of fibers. Then \( \text{imb}_f(F,\Sigma) \to \text{imb}_{\partial_f}(F,\Sigma) \) is a weak homotopy equivalence.

The proof of the next lemma uses surface theory, somewhat along the lines of the proof of theorem 2.2, and we do not include the details.

**Lemma 11.6.** Let \( T \) be a torus with a fixed \( S^1 \)-fibering. Let \( \text{Diff}_h(T) \) be the subspace of \( \text{Diff}(T) \) consisting of the diffeomorphisms that take some fiber to a loop isotopic to a fiber. Then the inclusion \( \text{Diff}_f(T) \to \text{Diff}_h(T) \) is a weak homotopy equivalence.
For $e \in (0, 1)$ we let $eD^2$ denote the concentric disk of radius $e$ in the standard disk $D^2 \subset \mathbb{R}^2$. Let $X$ be either a solid torus $D^2 \times S^1$, or $T \times I$ where $T$ is a torus. Let $F = \bigcup F_i$ be a disjoint union of finitely many tori. Fix an inclusion of $F$ into $X$ such that each $F_i$ is of the form $\partial(e_i D^2 \times S^1)$, in the solid torus case, or of the form $T \times \{e_i\}$, in the $T^2 \times I$ case, for distinct numbers $e_i$ in $(0, 1)$. Let $\text{imb}_{\text{int}}(F, X)$ be the connected component of the inclusion in the space of all imbeddings of $F$ into the interior of $X$, and let $\text{imb}_{\text{conc}}(F, X)$ be the connected component of the inclusion in the set of imbeddings for which each $F_i$ is of the form $\partial(eD^2) \times S^1$ or $T \times \{e\}$ for some $e \in (0, 1)$. The next lemma is essentially the uniqueness of collars of a boundary component.

**Lemma 11.7.** Let $X$ be a Seifert-fibered solid torus or $S^1 \times S^1 \times I$. Then the inclusion $\text{imb}_{\text{conc}}(F, X) \to \text{imb}_{\text{int}}(F, X)$ is a homotopy equivalence.
12. Deforming to fiber-preserving families

**Theorem 12.1.** Let $L = L(m,q)$ with $m > 2$ and let $f: S^d \to \text{diff}(L)$. Then $f$ is homotopic to a map into $\text{diff}_f(L)$.

**Proof.** Applying theorems 8.1, 9.1, and 10.1, we may assume that $f$ satisfies the conclusion of theorem 10.1. That is, there are pairs $(s_i, t_i)$ and an open cover $\{U(t_i)\}$ of $S^d$ with the property that for every $u \in U(t_i)$, $Q_{s_i}(u)$ and $P_{t_i}$ meet in very good position, and $Q_{s_i}(u)$ meets every $P_{t_j}$ transversely, with no discal intersections. The $U(t_i)$ are selected to be connected, so the intersection $Q_{s_i}(u) \cap P_{t_j}$ is independent, up to isotopy in $P_{t_j}$, of the parameter $u$. We remind the reader of our convention that assertions about $Q_{s_i}$ implicitly mean “for every $u \in U(t_i)$.” We can and always will assume that each $U(t_i)$ is connected, and that conditions stated for parameters in $U(t_i)$ actually hold for all parameters in $U(t_i)$.

Since the $t_j$ are distinct, we may select notation so that $t_1 < t_2 < \cdots < t_m$. The corresponding $s_i$ typically are not in ascending order. Figure 17 shows a schematic picture of a block of three levels for which the image levels $Q_{s_1}$, $Q_{s_2}$, and $Q_{s_3}$ have $s_1 < s_3 < s_2$.

The basic idea of the proof is to make the $f_u$ fiber-preserving on the $P_{s_i}$, then use lemma 11.3 to make the $f_u$ fiber-preserving on the complementary $S^1 \times S^1 \times I$ or solid tori of the $P_{s_i}$-levels. We must be very careful that none of the isotopic adjustments to a $Q_{s_i}$ destroys any condition that must be preserved on the other $Q_{s_j}$.

Before listing the steps in the proof of theorem 12.1, a definition is needed. For each $i$, the intersection circles of $Q_{s_i} \cap P_{t_i}$ cannot be meridians in both $V_{t_i}$ and $W_{t_i}$, so $Q_{s_i}$ must satisfy exactly one of the following:

1. The circles of $Q_{s_i} \cap P_{t_i}$ are not longitudes or meridians for $V_{t_i}$, so the annuli of $Q_{s_i} \cap V_{t_i}$ are uniquely boundary parallel in $V_{t_i}$.
2. The circles of $Q_{s_i} \cap P_{t_i}$ are longitudes or meridians for $V_{t_i}$, but are not longitudes or meridians for $W_{t_i}$, so the annuli of $Q_{s_i} \cap W_{t_i}$ are uniquely boundary parallel in $W_{t_i}$.
3. The circles of $Q_{s_i} \cap P_{t_i}$ are longitudes both for $V_{t_i}$ and for $W_{t_i}$.

![Figure 17. A block of level tori with the $Q_{s_i}$ out of order.](image-url)
In the first case, we say that $Q_{s_i}$ and $P_{t_i}$ are $V$-cored, in the second that they are $W$-cored, and in the third that they are bilongitudinal. If they are either $V$-cored or $W$-cored, we say they are cored. Lemma 11.1 shows that the bilongitudinal case can occur only when $q = 1$, and then only when the intersection circles are isotopic in $P_{t_i}$ to fibers of the Hopf fibering.

We can now list the steps in the procedure. In this list, and in the ensuing details, “push $Q_{s_i}$” means perform a deformation of $f$ that moves $Q_{s_i}$ as stated, and preserves all other conditions needed. Making $Q_{s_i}$ “vertical” (at a parameter $u$) means making the restriction of $f_u$ to $P_{s_i}$ fiber-preserving. When we say that something is done “at all parameters of $U(t_i)$,” we mean that a deformation of $f$ will be performed, and that $U(t_i)$ is replaced by a smaller set, so that the result is achieved for all parameters in the new $U(t_i)$, while retaining all other needed properties (such as that $\{U(t_i)\}$ is an open covering of $S^d$).

1. Push the $Q_{s_i}$ that meet $P_{t_j}$ out of $V_{t_j}$, for all the $V$-cored $P_{t_j}$, at all parameters in $U(t_j)$. At the end of this step, each $Q_{s_i}$ that was $V$-cored is parallel to $P_{t_i}$.
2. Push the $Q_{s_i}$ that meet $P_{t_j}$ out of $W_{t_j}$, for all the $W$-cored $P_{t_j}$, at all parameters in $U(t_j)$. At the end of this step, each $Q_{s_i}$ that was $W$-cored is parallel to $P_{t_i}$.

These first two steps are performed using a method of Hatcher like that of the proof of section 10, although simpler. After they are completed, a triangulation of $S^d$ is fixed with mesh smaller than a Lebesgue number for the open cover by the $U(t_i)$. Each of the remaining steps is performed by inductive procedures that move up the skeleta of the triangulation, achieving the objective for $Q_{s_i}$ at all parameters that lie in a simplex completely contained in $U(t_i)$.

3. Push the $Q_{s_i}$ that originally were cored so that each one equals some level torus. These level tori may vary from parameter to parameter.
4. Push the $Q_{s_i}$ that originally were cored to be vertical.
5. Push the bilongitudinal $Q_{s_i}$ to be vertical.
6. Use lemma 11.3 to make $f_u$ fiber-preserving on the complementary $S^1 \times S^1 \times I$ or solid tori of the $P_{s_i}$-levels.

The underlying fact that allows all of this pushing to be carried out without undoing the results of the previous work is lemma 5.1. Its use involves the concepts of compatibility and blocks, which we will now define.

Recall that $R(t_i, t_j)$ means the closure of the region between $P_{t_i}$ and $P_{t_j}$. For a connected subset $Z$ of $S^d$, which in practice will be either a single parameter or a simplex of a triangulation, denote by $B_Z$ the set of $t_i$ such that $Z \subset U(t_i)$. Elements $t_i$ and $t_j$ of $B_Z$ are called $Z$-compatible when $Q_{s_i}(u) \cap P_{t_i}$ and $Q_{s_i}(u) \cap P_{t_k}$ are homotopic in $R(t_i, t_k)$ for every $t_k \in B_Z$ with $t_i < t_k \leq t_j$. Whether or not $t_i$ and $t_j$ are $u$-compatible typically varies
as \( u \) varies over \( U(t_i) \cap U(t_j) \), since \( B_u \) depends on the other \( U(t_k) \) that contain \( u \).

Because our family \( f \) satisfies the conclusion of theorem 10.1, lemma 5.1 has the following consequence: if \( t_i \) and \( t_j \) are \( u \)-compatible for any \( u \), then \( P_{t_i} \) and \( P_{t_j} \) are both \( V \)-cored, or both \( W \)-cored, or both bilongitudinal. The next proposition is also immediate from lemma 5.1.

**Proposition 12.2.** Suppose that \( t_i, t_j, t_k \in B_Z \). Then at parameters in \( Z \), \( Q_{s_k} \) can meet both \( P_{t_i} \) and \( P_{t_j} \) only if \( t_i \) and \( t_j \) are \( Z \)-compatible.

For a simplex \( \Delta \), write \( B_\Delta = \{ b_1, \ldots, b_m \} \) with each \( b_i < b_{i+1} \), and for each \( i \leq m \) define \( a_i \) to be the \( s_j \) for which \( b_i = t_j \). Decompose \( B_\Delta \) into maximal \( \Delta \)-compatible blocks \( C_1 = \{ b_{1}, b_{2}, \ldots, b_{t_1} \}, C_2 = \{ b_{t_{1}+1}, \ldots, b_{t_2} \}, \ldots, \)
\( C_r = \{ b_{t_{r−1}+1}, \ldots, b_{t_r} \} \), with \( t_r = m \). Since the blocks are maximal, proposition 12.2 shows that \( Q_{a_i} \) is disjoint from \( P_{b_j} \) if \( b_i \) and \( b_j \) are not in the same block. In steps 3-6, this disjointness will ensure that isotopies of these \( Q_{a_i} \) do not disturb the results of previous work.

Note that if \( b_i \) and \( b_j \) lie in the same block, then either both \( P_{b_i} \) and \( P_{b_j} \) are \( V \)-cored, or both are \( W \)-cored, or both are bilongitudinal. Thus we can speak of \( V \)-cored blocks, and so on.

When \( \delta \) is a face of \( \Delta \), \( B_\delta \subseteq B_\delta \). Therefore if \( b_i \) and \( b_j \) in \( B_\Delta \) are \( \delta \)-compatible, then they are \( \Delta \)-compatible. So for each block \( C \) of \( B_\delta \), \( C \cap B_\Delta \) is contained in a block of \( B_\Delta \). However, levels that are not compatible in \( B_\delta \) may become compatible in \( B_\Delta \), since the \( t_i \) for intervening levels in \( B_\delta \) may fail to be in \( B_\Delta \). Typically, the intersections of blocks of \( B_\delta \) with \( B_\Delta \) will combine into larger blocks in \( B_\Delta \).

We should emphasize that the blocks of \( B_Z \), and whether a level \( P_{t_i} \) is \( V \)-cored, \( W \)-cored, or bilongitudinal, are defined with respect to the original configuration, not the new positioning after the procedure begins. Indeed, after steps 1 and 2, many of the \( Q_{s_k} \) will be disjoint from their \( P_{t_j} \).

We now fill in the details of these procedures.

**Step 1:** Push the \( Q_{s_i} \) that meet \( P_{t_j} \) out of \( V_{t_j} \), for all the \( V \)-cored \( P_{t_j} \), at all parameters in \( U(t_j) \).

We perform this in order of increasing \( t_j \) for the \( V \)-cored image levels. Begin with \( t_1 \). If \( Q_{s_1} \) is \( W \)-cored or bilongitudinal, do nothing. Suppose it is \( V \)-cored. Then for each \( u \) in \( U(t_1) \), the \( Q_{s_1}(u) \) that meet \( P_{t_1} \) intersect \( V_{t_1} \) in a union of incompressible uniquely boundary-parallel annuli. Since any such \( Q_{s_i} \) are transverse to \( P_{t_1} \) at each point of \( U(t_j) \), the set of intersection annuli \( Q_{s_i} \cap V_{t_1} \) falls into finitely many isotopic families, with each family a copy of the connected set \( U(t_j) \). For each \( j \) with \( U(t_1) \cap U(t_j) \) nonempty, let \( A_{t_j} \) be the collection of the annuli \( Q_{s_j} \cap V_{t_1} \), over all parameters in \( U(t_j) \), and let \( A \) be the union of these \( A_{t_j} \). The nonempty intersection of \( U(t_1) \) and \( U(t_j) \) ensures that the loops of \( Q_{s_j} \cap P_{t_1} \) and \( Q_{s_k} \cap P_{t_1} \) are all in the same isotopy class in \( P_{t_1} \).

One might hope to push these families of annuli out of \( V_{t_1} \) one at a time, beginning with an outermost one, but an outermost family might not exist.
There could be a sequence $U(t_{j_1}), \ldots, U(t_{j_k})$ such that $U(t_{j_i}) \cap U(t_{j_{i+1}})$ is nonempty for each $i$, $U(t_{j_i}) \cap U(t_{j_{i+1}})$ is nonempty, and for some parameters $u_{j_i}$ in $U(t_{j_i})$, an annulus $Q_{s_{j_i}+1}(u_{j_i}) \cap V_{t_1}$ lies outside one of $Q_{s_{j_i}}(u_{j_i}) \cap V_{t_1}$ for each $i$, and an annulus of $Q_{s_{j_i}}(u_{j_i}) \cap V_{t_1}$ lies outside one of $Q_{s_{j_i}}(u_{j_i}) \cap V_{t_1}$. Since an outermost family might not exist, we will need to utilize the method of Hatcher as in the proof of theorem 10.1, but only a simple version of it.

Shrink the $U(t_i)$ slightly, obtaining a new open cover by sets $U(t_i)'$ with $U(t_i)' \subset U(t_i)$. We will use a function $\Psi : \mathcal{A} \to (0, 2)$, so that at each parameter $u$, the restriction $\psi_u$ of $\Psi$ to the annuli at that parameter has the property that $\psi_u(A_1) < \psi_u(A_2)$ whenever $A_1, A_2 \in \mathcal{A}_i$ and $A_1$ lies in the region of parallelism between $A_2$ and $\partial V_{t_1}$. Moreover, we will have $\psi_u(A) < 1$ whenever $A \in \mathcal{A}_i$ and $u \in U(t_i)'$, while $\psi_u(A) > 1$ for $u$ near the boundary of $U(t_i)$. We construct $\Psi$ by letting $\Psi_0(A)$ be the volume of the region of parallelism between $A$ and an annulus in $\partial V_{t_1}$ (assuming that the volume of $L$ has been normalized to 1 to ensure that $\Psi_0(A) < 1$), then adding on auxiliary values of $Hatcher$ as in the proof of theorem 10.1.

Form the union $G_0 \subset S^d \times (0, 2)$ of the $(u, \psi_u(A))$ as in the proof of theorem 10.1, and thicken each of its sheets as was done there, obtaining an interval for each parameter. These intervals tell the supports of the isotopies that push the annuli of $Q_{s_{j_i}} \cap V_{t_1}$ out of $V_{t_1}$. If two sheets of $\mathcal{A}$ cross in $S^d \times (0, 2)$, then the corresponding regions of parallelism have the same volume, so must be disjoint and the isotopies can be performed simultaneously without interference. At each individual parameter $u$, each annulus is outermost during the time it is being pushed out of $V_{t_1}$, but the times need to be different since there may be no outermost family.

After the process is completed, $Q_{s_{j_i}}$ will lie outside of $V_{t_1}$ at all parameters in $U(t_i)'$, whenever $U(t_{j_i})$ had nonempty intersection with $U(t_{j_{i+1}})$. Replacing each $U(t_{j_i})$ by $U(t_{j_i})'$, we have $Q_{s_{j_i}}$ pushed out of $V_{t_1}$ at all parameters in these $U(t_{j_i})$. Moreover, lemma 4.3(2) shows that $V_{t_1}$ is concentric in either $X_{s_{j_i}}$ or $Y_{s_{j_i}}$ at all parameters in $U(t_i)$.

Some of the $Q_{s_{j_i}}$ for which $U(t_{j_i})$ did not meet $U(t_{j_{i+1}})$ may be moved by the isotopies of the $Q_{s_{j_i}}$ at parameters in $U(t_{j_i}) \cap U(t_{j_{i+1}})$. The condition that these $Q_{s_{j_i}}$ meet $P_{t_{i+1}}$, transversely may be lost, but this will not matter, because these intersections never matter when $U(t_k)$ does not meet $U(t_{j_i})$.

Now consider $t_2$. Again, we do nothing if $Q_{s_{j_1}}$ is $W$-cored or bilongitudinal, so suppose that it is $V$-cored. Use the Hatcher process as before, to push annuli in the $Q_{s_{j_i}}$ out of $V_{t_2}$, when $Q_{s_{j_i}}$ meets $P_{t_2}$ and $U(t_{j_i})$ meets $U(t_{j_{i+1}})$. Notice that these $Q_{s_{j_i}}$ cannot meet $V_{t_1}$ at parameters in $U(t_{j_{i+1}})$. For if $t_{j_{i+1}}$ is not $u$-compatible with $t_{j_i}$ at some parameters in $U(t_{j_{i+1}})$, then (by lemma 5.1) $Q_{s_{j_i}}$ cannot meet both $P_{t_2}$ and $P_{t_{i+1}}$, while if it is $u$-compatible at some parameter in $U(t_{j_{i+1}})$, then it has already been pushed out of $V_{t_1}$. And $V_{t_1}$ cannot lie in any of the regions of parallelism for the pushouts from $V_{t_2}$, since the intersection circles of the $Q_{s_{j_i}}$ with $P_{t_2}$ are not longitudes in $V_{t_2}$.
After these pushouts are completed, if \( i = 1 \) or \( i = 2 \) and \( Q_{s_i} \) was \( V \)-cored, then \( V_{t_i} \) is concentric in either \( X_{s_i} \) or \( Y_{s_i} \) at all parameters in \( U(t_i) \).

We continue working up the increasing \( t_i \) in this way. At the end of this process, \( V_{t_i} \) is concentric in either \( X_{s_i} \) or \( Y_{s_i} \) for all \( i \) such that \( Q_{s_i} \) was \( V \)-cored, and at all parameters in \( U(t_i) \). For \( Q_{s_i} \) that were \( W \)-cored or bilongitudinal, the intersections \( Q_{s_i} \cap P_{t_i} \) have not been disturbed at parameters in \( U(t_i) \). We have not introduced any new intersections of \( Q_{s_i} \) with \( P_{t_j} \), so we still have the property that at any parameter \( u \) in \( U(t_i) \cap U(t_j) \), \( Q_{s_j} \) can meet \( P_{t_i} \) only if \( t_i \) and \( t_j \) were originally \( u \)-compatible.

**Step 2:** Push the \( Q_{s_i} \) that meet \( P_{t_j} \) out of \( W_{t_j} \) for all the \( Q_{s_j} \) that are \( W \)-cored, at all parameters in \( U(t_j) \).

The entire process is repeated with \( W \)-cored levels, except that we start with \( t_m \) and proceed in order of decreasing \( t_i \). Each \( W \)-cored \( Q_{s_i} \) is pushed out of \( W_{t_i} \), and at the end of the process \( W_{t_i} \) is concentric in either \( X_{s_i} \) or \( Y_{s_i} \) at all parameters in \( U(t_i) \), whenever \( Q_{s_i} \) was \( W \)-cored. No intersection of a \( Q_{s_j} \) with a \( V \)-cored or bilongitudinal level \( P_{t_i} \) is changed at any parameter in \( U(t_i) \).

For the remaining steps, we fix a triangulation of \( S^d \) with mesh smaller than a Lebesgue number for \( \{ U(t_i) \} \), which will ensure that \( B_\Delta \) is nonempty for every simplex \( \Delta \). We will no longer proceed up or down all \( t_i \)-levels, working on the sets \( U(t_i) \), but instead will work inductively up the skeleta of the triangulation. Recall that each \( B_\Delta \) is decomposed into blocks, according to the original intersections of the \( Q_{s_i} \) and \( P_{t_i} \) before steps 1 and 2 were performed.

**Step 3:** Push the \( Q_{s_i} \) that were originally cored so that each one equals some level torus.

We will proceed inductively up the skeleta of the triangulation, moving cored \( Q_{s_i} \) to level tori, without changing \( Q_{s_k} \cap P_{s_k} \) for the bilongitudinal \( Q_{s_k} \). We want to use the fact that \( V_{t_i} \) (or \( W_{t_i} \)) is concentric with \( X_{s_i} \) or \( Y_{s_i} \) to push \( Q_{s_i} \) onto a level torus, but when moving multiple levels at a given parameter, there is a consistency condition needed. As shown in figure 18, it might happen that \( V_{t_i} \) is concentric in \( X_{s_i} \) while \( V_{t_j} \) is concentric in \( Y_{s_j} \). Then, we might not be able to push \( Q_{s_i} \) and \( Q_{s_j} \) onto level tori without disrupting other levels. The following lemma rules out this bad configuration.

**Lemma 12.3.** Suppose, after steps 1 and 2 have been completed, that \( u \in U(t_i) \cap U(t_j) \), \( t_i < t_j \), and that \( Q_{s_i} \) is \( V \)-cored.

1. The region between \( Q_{s_i} \) and \( Q_{s_j} \) does not contain a core circle of \( V_{t_j} \).
2. Suppose that \( t_i \) and \( t_j \) are \( u \)-compatible, and \( V_{t_i} \) is concentric in \( Z_{s_i} \), where \( Z \) is \( X \) or \( Z \) is \( Y \). Then \( V_{t_j} \) is concentric in \( Z_{s_j} \).
3. If \( t_i \) and \( t_j \) are \( u \)-incompatible, then \( Q_{s_i} \) is parallel to \( P_{t_i} \) in \( R(t_i, t_j) \).

The analogous statement holds when \( Q_{s_j} \) is \( W \)-cored and \( W_{t_j} \) is concentric in \( Z_{s_j} \).
Proof. It suffices to consider the case when $Q_{s_i}$ is $V$-cored. In the situation at the start of Step 1 above, when annuli in the $Q_{s_k}$ were being pushed out of $V_{t_i}$, the intersection of $Q_{s_i} \cup Q_{s_j}$ with $V_{t_i}$ was a union $F$ of incompressible nonlongitudinal annuli. Since $Q_{s_i}$ met $P_{t_i}$, $F$ was nonempty. By proposition 3.3, exactly one complementary region of $F$ in $V_{t_i}$ contained a core circle $C$ of $V_{t_i}$. For at least one of $s_i$ and $s_j$, say for $s_k$, $Q_{s_k}$ met this complementary region.

Since the annuli of $F$ are nonlongitudinal, there is an imbedded circle $C'$ in $Q_{s_k}$ that is homotopic in the core region to a proper multiple of $C$. If $C$ were in the region $R = f_u(R(s_i, s_j))$ between $Q_{s_i}$ and $Q_{s_j}$, then the imbedded circle $C'$ in $\partial R$ would be a proper multiple in $\pi_1(R)$, which is impossible since $R$ is homeomorphic to $S^1 \times S^1 \times I$. This proves (1).

Assume that $t_i$ and $t_j$ are $u$-compatible and suppose that $V_{t_i} \subset X_{s_i}$ and $V_{t_j} \subset Y_{s_j}$. Then $C$ is contained in $X_{s_i} \cap Y_{s_j}$, forcing $s_i > s_j$ and $C$ in the region between $Q_{s_i}$ and $Q_{s_j}$, contradicting (1). The case of $V_{t_i} \subset Y_{s_i}$ and $V_{t_j} \subset X_{s_j}$ is similar, so (2) holds.

For (3), if $t_i$ and $t_j$ are not $u$-compatible, then $Q_{s_j}$ was disjoint from $P_{t_j}$ before the pushouts, so $i = k$ and $Q_{s_i}$ is disjoint from $P_{t_j}$. If $Q_{s_i}$ is not parallel to $P_{t_j}$ in $R(t_i, t_j)$, then $Q_{s_i}$ does not separate $P_{t_j}$ from $P_{t_i}$. This implies that before pushouts from $V_{t_i}$ it did not separate $Q_{s_j} \cap P_{t_j}$ from $P_{t_i}$, so after pushouts it could not separate $Q_{s_j}$ from $P_{t_i}$, contradicting (1). \(\square\)

It will be convenient to extend our previous notation $R(s,t)$ for the closure of the region between $P_s$ and $P_t$, by putting $R(0,t) = V_t$, $R(t,1) = W_t$, and $R(0,1) = L$.

We will now define target regions. The isotopies that we will use in the rest of our process will only change values within a single target region, ensuring that the necessary positioning of the $Q_{s_i}$ is retained. Let $\Delta$ be a simplex of the triangulation, and recall the decomposition of $B_\Delta = \{b_1, \ldots, b_m\}$.
into maximal $\Delta$-compatible blocks $C_1 = \{b_1, b_2, \ldots, b_{t_1}\}$, $C_2 = \{b_{t_1+1}, \ldots, b_{t_2}\}, \ldots, C_r = \{b_{t_{r-1}+1}, \ldots, b_{t_r}\}$. Define the target region of a block $C_n$ to be the submanifold $T_\Delta(C_n)$ of $L$ defined as follows. Put $\ell_0 = 0$, $b_0 = 0$, and $b_{r+1} = 1$.

1. If $C_n$ is $V$-cored, then $T_\Delta(C_n) = R(b_{t_n-1+1}, b_{t_n+1})$.
2. If $C_n$ is $W$-cored, then $T_\Delta(C_n) = R(b_{t_n-1}, b_{t_n})$.
3. If $C_n$ is bilongitudinal, then $T_\Delta(C_n) = R(b_{t_n-1}, b_{t_n+1})$.

We remark that $T_\Delta(C_n)$ is all of $L$ when $B_\Delta$ consists of a single bilongitudinal block, otherwise is of the form $V_t$ when $n = 1$ and $C_1$ is $W$-cored or bilongitudinal and of the form $W_t$ when $n = r$ and $C_n$ is $V$-cored or bilongitudinal, and in all other cases it is a region $R(s, t)$ diffeomorphic to $S^1 \times S^1 \times I$.

As noted in the next lemma, the interior of the target region of a block contains the $Q_{a_i}$ for the $b_i$ in the block, at this point of our argument.

**Lemma 12.4.** Target regions satisfy the following.

1. If $b_i \in C_n$ and $u \in \Delta$, then $Q_{a_i}(u)$ is in the interior of $T_\Delta(C_n)$.
2. If $\delta$ is a face of $\Delta$, and $C_1', \ldots, C_r'$ are the blocks of $B_\delta$, then for each $i$, there exists a $j$ such that $T_\delta(C_i') \subseteq T_\Delta(C_j)$.

**Proof.** Property (1) is a consequence of proposition 12.2 and the fact that Steps 1 and 2 do not create new intersections of the $Q_{s_i}(u)$ with the $P_{t_j}$.

For part (2), the proof is direct from the definitions, dividing into various subcases. $\square$

Target regions can overlap in the following ways: the target region for a $V$-cored block $C_n$ will overlap the target region of a succeeding $W$-cored block $C_{n+1}$, and the target region of a bilongitudinal block will overlap the target region of a preceding $V$-cored block or of a succeeding $W$-cored block (note that by lemma 11.1, successive blocks cannot both be bilongitudinal). The latter cause no difficulties, but the conjunctions of a $V$-cored block and a succeeding $W$-cored block will necessitate some care during the ensuing argument.

We can now begin the process that will complete Step 3. We will start at the parameters that are vertices of the triangulation and move the $Q_{a_i}$ for each $V$-cored or $W$-cored block to be level, that is, so that each $Q_{a_i}(u)$ equals some $P_t$. The isotopies will be fixed on each $P_{b_i}$ for which $Q_{a_i}$ is bilongitudinal, and these unchanged $Q_{a_i} \cap P_{b_i}$ will be used to work with the bilongitudinal levels in a later step. For each cored block, the isotopy that levels the $Q_{a_i}$ will move points only in the interior of the target region of the block. As we move to higher-dimensional simplices, the $Q_{a_i}$ will already be level at parameters on the boundary, and the deformation will be fixed at those parameters. Each deformation for the parameters in a simplex $\delta_0$ of dimension less than $d$ must be extended to a deformation of $f$. The extension will change an $f_u$ only when $u$ is in the open star of $\delta_0$, by a deformation that performs some initial portion of the deformation of $f_u$ at
a parameter $u_0$ of $\delta_0$— the parameter that is the $\delta_0$-coordinate of $u$ when the simplex that contains it is written as a join $\delta_0 * \delta_1$. We will see that because the target regions can overlap, the deformation of an $f_u$ might not preserve all target regions, but enough positioning of the image levels $Q_{a_i}$ will be retained to continue the inductive process.

Fix a vertex $\delta_0$ of the triangulation, and consider the first block $C_1$ of $B_{\delta_0}$. If it is bilongitudinal, we do nothing. Suppose that it is $V$-cored. All of the $Q_{a_1}, \ldots, Q_{a_{i+1}}$ lie in the interior of the target region $T_{\delta_0}(C_1)$. Lemma 12.3(2) shows that for either $Z = X$ or $Z = Y$, $V_b$ is concentric in $Z_{a_i}$ for $b_i \in C_1$. We claim that there is an isotopy, supported on $T_{\delta_0}(C_1)$, that moves each $Q_{a_i}$ to be level. If $C_1$ is the only block, then $T_{\delta_0}(C_1) = L$ and the isotopy exists by the definition of concentric. If there is a second block, then lemma 12.3(3) shows that the $Q_{a_i}$ for $b_i \in C_1$ are parallel to $P_{b_i}$ in $T_{\delta_0}(C_1) = R(b_1, b_{i+1})$, and again the isotopy exists. After performing the isotopy, we may assume that the $Q_{a_i}(u_0)$ are level.

To extend this deformation of $f_{b_0}$ to a deformation of the parameterized family $f$, we regard each simplex $\Delta$ of the closed star of $\delta_0$ in the triangulation as the join $\delta_0 * \delta_1$, where $\delta_1$ is the face of $\Delta$ spanned by the vertices of $\Delta$ other than $\delta_0$. Each point of $\Delta$ is uniquely of the form $u = s\delta_0 + (1 - s)u_1$ with $u_1 \in \delta_1$. Write the isotopy of $f_{b_0}$ as $j_t \circ f_{b_0}$, with $j_0$ the identity map of $L$. Then, at $u$ the isotopy at time $t$ is $j_t \circ f_u$ for $0 \leq t \leq s$ and $j_s \circ f_u$ for $s \leq t \leq 1$. For any two simplices containing $\delta_0$, this deformation agrees on their intersection, so it defines a deformation of $f$.

The target region $T_{\delta_0}(C_1)$ will overlap $T_{\delta_0}(C_2)$ if $C_2$ is bilongitudinal or $W$-cored. When $C_2$ is bilongitudinal, this does not affect any of our necessary positioning. If it is $W$-cored, then $Q_{a_{i+1}}$ with $b_{i+1} \in C_2$ may be moved into $T_{\delta_0}(C_1)$. At $\delta_0$, they will end up somewhere between the now-level $Q_{a_{i+1}}$ and $P_{b_{i+1}}$, and at other parameters in the star of $\delta_0$ they will lie somewhere in $R(b_1, b_{i+1})$. This will require only a bit of attention in the later argument.

In case $C_1$ was $W$-cored, we use lemmas 12.3(2) and 11.7, producing a deformation of $f_{\delta_0}$ supported on the interior of the solid torus $T_{\delta_0}(C_1) = V_{b_{i+1}}$, which does not meet any other target region. This is extended to a deformation of $f$ just as before.

We move on to consider $C_2$ in analogous fashion, doing nothing if $C_2$ is bilongitudinal, and moving the $Q_{a_{i+1}}$ to be level at the parameter $\delta_0$. If $C_1$ was $V$-cored and $C_2$ is $W$-cored, then instead of the initial target region $T_{\delta_0}(C_2)$ we must use the region between the now-level $Q_{a_{i+1}}(u)$ and $P_{b_{i+1}}$, but otherwise the argument is the same. Proceed in the same way through the remaining blocks $C_n$ of $B_{\delta_0}$, ending with all the cored $Q_{a_i}(u_0)$ moved to be level. This process for $u_0$ is repeated for each 0-simplex of the triangulation.

Now, consider a simplex $\delta$ of positive dimension. Inductively, we may assume that at each $u$ in $\partial \delta$, each cored $Q_{a_i}$ has been moved to a level torus, and $Q_{a_i} \cap P_{a_i}$ is unchanged for each bilongitudinal $Q_{a_i}$. Moreover, if $a_i$ is contained in a target block $T_{\delta}(C_1)$, then $Q_{a_i}$ lies in its target region,
or else lies in the union of the target regions for a $V$-cored block and a succeeding $W$-cored block. Note that we are using lemma 12.3(2) here.

We apply lemma 11.7 to each cored block of $B_\Delta$, sequentially up the cored blocks. We obtain a sequence of deformations of $f$ on $\delta$, constant at parameters in $\partial \delta$. There is no interference between different blocks, except when a $W$-cored block $C_{n+1}$ succeeds a $V$-cored block $C_n$. First, the $Q_{a_i}$ for the $V$-cored block are moved to be level. Then, at each parameter in $\delta$, the $Q_{a_i}(u)$ for the $W$-cored block lie between the now-level $Q_{a_i}(u)$ and $P_{b_i}$. We regard the union of these regions over the parameters of $\delta$ as a product $\delta \times S^1 \times S^1 \times I$, and apply lemma 11.7. Thus the isotopy that levels the $Q_{a_i}$ from the $W$-cored block need not move any of the $Q_{a_i}$ from the $V$-cored block. In other cases, the successive isotopies take place in disjoint regions. To extend this to a deformation of $f$, we adapt the join method from above (of course when $\delta_0$ is $d$-dimensional, no extension is necessary). Regard each simplex $\Delta$ of the closed star of $\delta_0$ in the triangulation as the join $\delta_0 \ast \delta_1$, where $\delta_1$ is the face of $\Delta$ spanned by the vertices of $\Delta$ not in $\delta_0$. Each point of $\Delta$ is uniquely of the form $u = su_0 + (1 - s)u_1$ with $u_i \in \delta_i$. Write the isotopy of $f_{u_0}$ as $j_t \circ f_{u_0}$, with $j_0$ the identity map of $L$. Then, at $u$ the isotopy at time $t$ is $j_t \circ f_u$ for $0 \leq t \leq s$ and $j_s \circ f_u$ for $s \leq t \leq 1$. For any two simplices containing $\delta_0$, this deformation agrees on their intersection, so it defines a deformation of $f$.

At the completion of this process, each cored $Q_{s_i}$ is level at all parameters in $\Delta$, whenever $\Delta \subset U(t_i)$. The bilongitudinal $Q_{s_i}$ may have been moved around some, but their intersections $Q_{s_i} \cap P_{t_i}$ will not be altered at parameters for which $t_i \in B_\Delta$ since these intersections will not lie in the interior of any target region for a cored level.

Step 4: Push all cored $Q_{s_i}$ to be vertical, that is, make each image of a fiber of $P_{s_i}$, a fiber in $L$.

Again we work our way up the simplices of the triangulation. Start at a 0-simplex $\delta_0$. Each cored $Q_{a_i}(\delta_0)$ for $b_i \in B_{\delta_0}$ is now level. By lemma 11.2, the image fibers in $Q_{a_i}(\delta_0)$ are isotopic in that level torus to fibers of $L$. Using lemma 11.6, there is an isotopy of $f_{\delta_0}$ that preserves the level tori and makes $Q_{a_i}(\delta_0)$ vertical. This isotopy can be chosen to fix all points in other $Q_{a_j}(\delta_0)$, and is extended to a deformation of $f$ by using the method of Step 3. We work our way up the skeleton; if $\delta \subset U(b_i)$, then for every $u$ in $\delta$, each $Q_{a_i}(u)$ is level torus, and at parameters $u \in \delta$, $Q_{a_i}(u)$ is vertical. Using lemma 11.6, we make the $Q_{a_i}(u)$ vertical at all $u \in \delta$, and extend to a deformation of $f$ as before. We repeat this for all levels of cored blocks.

Step 5: Push all bilongitudinal $Q_{s_i}$ to be vertical.

Now, we examine the bilongitudinal levels. For a bilongitudinal level $Q_{s_i}$ at a vertex $\delta_0$, corollary 4.4 shows that the intersection circles are longitudes for $X_{a_i}$ and $Y_{a_i}$. Lemma 11.1 then shows that the circles of $Q_{a_i} \cap P_{b_j}$ are isotopic in $Q_{a_i}$ and in $P_{b_j}$ to fibers. First, use lemma 11.4 to find an isotopy preserving levels, such that postcomposing $f_{\delta_0}$ by the isotopy makes the
intersection circles fibers of the $P_{b_j}$. Then, use lemma 11.4 applied to $f_{\delta_0}$ to find an isotopy preserving levels of the domain, such that precomposing $f_{\delta_0}$ by the isotopy makes the intersection circles the images of fibers of $P_{s_i}$. After this process has been complete for the bilongitudinal $Q_{a_i}$, the preimage (in their union $\cup Q_{a_i}$) of each region $R(b_j, b_{j+1})$ with $b_j$ or $b_{j+1}$ in a bilongitudinal block is a collection of fibered annuli which map into $R(b_j, b_{j+1})$ by imbeddings that are fiber-preserving on their boundaries. We use lemma 11.5 to find an isotopy that makes the $Q_{a_i}$ vertical. Again, we extend to a deformation of $f$ and work our way up the skeleta, to assume that $Q_{s_i}(u)$ is vertical whenever $u \in \Delta$ and $\Delta \subset U(t_i)$.

**Step 6: Make $f$ fiber-preserving on the complementary $S^1 \times S^1 \times I$ or solid tori of the $P_{s_i}$-levels**

We work our way up the skeleta one last time, using lemma 11.3 to make $f$ fiber-preserving on the complementary $S^1 \times S^1 \times I$ or solid tori of the $P_{a_i}$.
13. Parameters in $D^d$

Regard $D^d$ as the unit ball in $d$-dimensional Euclidean space, with boundary the unit sphere $S^{d-1}$. As mentioned in section 2, to prove that $\text{diff}_f(L) \to \text{diff}(L)$ is a homotopy equivalence, we actually need to work with a family of diffeomorphisms $f$ of $L$ parameterized by $D^d$, $d \geq 1$, for which $f(u)$ is fiber-preserving whenever $u$ lies in the boundary $S^{d-1}$. We must deform $f$ so that each $f(u)$ is fiber-preserving, by a deformation that keeps $f(u)$ fiber-preserving at all times when $u \in S^{d-1}$.

We now present a trick that allows us to gain good control of what happens on $S^{d-1}$. The Hopf fibering we are using on $L$ can be described as a Seifert fibering of $L$ over the round 2-sphere $S^2$, in such a way that each isometry of $L$ projects to an isometry of $S$ (details appear in [27], see also [28]). By conjugating $\pi_1(L)$ in $\text{SO}(4)$, we may assume that the singular fibers, when $q > 1$, are the preimages of the poles. We choose our sweepout so that the level tori are the preimages of latitude circles. Denote by $p_t$ the latitude circle that is the image of the level torus $P_t$.

There is an isotopy $J_t$ with $J_0$ the identity map of $L$ and each $J_t$ fiber-preserving, so that the images of the level tori $P_s$ under $J_1$ project to circles in the 2-sphere as indicated in figure 19. Denote the image of $J_1(P_s)$ in $S$ by $q_s$. Their key property is that when moved by any orthogonal rotation of $S$, each $p_t$ meets the image of some $q_s$ transversely in two or four points.

Using theorem 2.2, we may assume that $f_u$ is actually an isometry of $L$ for each $u \in S^{d-1}$. Denote the isometry that $f_u$ induces on $S$ by $\overline{f_u}$. Now, deform the entire family $f$ by precomposing each $f_u$ with $J_t$. At points in $S^{d-1}$, each $f_u \circ J_t$ is fiber-preserving, so this is an allowable deformation of $f$. At the end of the deformation, for each $u \in S^{d-1}$, $f_u \circ J_1(P_s)$ is a fibered torus $Q_s$ that projects to $\overline{f_u}(q_s)$. Since $\overline{f_u}$ is an isometry of $S$, it follows that for any latitude circle $p_t$, some $\overline{f_u}(q_s)$ meets $p_t$ transversely, in either two or four points. So $P_t$ and this $Q_s$ meet transversely in either two or four circles which are fibers of $L$. In particular, they are in very good position. We call such a pair $P_t$ and $Q_s$ at $u$ an instant pair.
Cover $S^{d-1}$ by finitely many open sets $Z'_i$ such that for each $i$, there is an $(x_i, y_i)$ such that $Q_{x_i}$ and $P_{y_i}$ are an instant pair at every point of $Z'_i$. We may assume that there are open sets $Z_i$ in $D^d$ such that $Z_i \cap S^{d-1} = Z'_i$ and $Q_{x_i}$ and $P_{y_i}$ meet in very good position at each point of $Z'_i$. For any sufficiently small deformation of $f$, $Q_{x_i}$ and $P_{y_i}$ will still meet in very good position at all points of $Z_i$. Let $V$ be a neighborhood of $S^{d-1}$ in $D^d$ such that $V$ is contained in the union of the $Z_i$.

Now, we apply to $D^d$ the entire process used for the case when the parameters lie in $S^d$, using appropriate fiber-preserving deformations at parameters in $S^{d-1}$. Here are the steps:

1. By theorem 8.1, there are arbitrarily small deformations of $f$ that put it in general position with respect to the sweepout. Select the deformation sufficiently small so that the $Q_{s_i}$ and $P_{t_i}$ still meet in very good position at every point of $Z_i$. Within $V$, we taper the deformation off to the identity, so that no change has taken place at parameters in $S^{d-1}$. At every parameter, either there is already a pair in very good position, or $f_u$ satisfies the conditions (GP1), (GP2), and (GP3) of a general position family.

2. Theorem 9.1 guarantees that at each of the parameters in $D^d - V$, there is a pair $Q_s$ and $P_t$ meeting in good position.

3. Applying theorem 10.1 to $D^d$, with $S^{d-1}$ in the role of $W_0$, we find a deformation of $f$, fixed on $S^{d-1}$, and a covering $U(t_i)$ of $D^d$ and associated values $s_i$ so that for every $u \in U(t_i)$, $Q_{s_i}$ and $P_{t_i}$ meet in very good position, and $Q_{s_i}$ has no discal intersection with any $P_{t_j}$.

4. In the pushout step of the proof of theorem 12.1, we may assume that all the $U(t_i)$ that meet $S^{d-1}$ are the open sets $Z_i$. At parameters $u$ in $S^{d-1}$, the annuli to be pushed out of each $V_{t_i}$ will be vertical annuli. So the pushouts may be performed using fiber-preserving isotopies at these parameters, because the necessary deformations can be taken as lifts of deformations of circles in the quotient sphere $S$, and [28] provides fiber-preserving lifts of any such deformations.

5. After the triangulation of $D^d$ is chosen, the deformation that move the $Q_{s_i}$ onto level tori can be performed using fiber-preserving isotopies at parameters in $S^{d-1}$, again because the necessary deformations cover deformations of circles in the quotient surface $S$. No further deformation will be needed on simplices in $S^{d-1}$, since the $f_u$ are already fiber-preserving there.

This completes the discussion of the case of parameters in $D^d$, and the proof of the Smale Conjecture for lens spaces.

References

1. A. Banyaga, The structure of classical diffeomorphism groups, Mathematics and its Applications, Vol. 400, Kluwer Academic Publishers Group, Dordrecht (1997).
2. C. Bessaga and A. Pelczynski, *Selected Topics in Infinite-dimensional Topology*, Polish Academy of Sciences Mathematical Monographs Vol. 58 (1975).
3. F. Bonahon, Difféotopies des espaces lenticulaires, *Topology* 22 (1983), 305-314.
4. J. W. Bruce, On transversality, *Proc. Edinburgh Math. Soc.* (2) 29 (1986), 115-123.
5. A. J. Casson and C. McA. Gordon, Reducing Heegaard splittings, *Topology Appl.* 27 (1987), 275-283.
6. J. Cerf, Sur les difféomorphismes de la sphère de dimension trois $(\Gamma_4 = 0)$, Springer-Verlag Lecture Notes in Mathematics No. 53, 1968.
7. J. Cerf, Groupes d'automorphismes et groupes de difféomorphismes des variétés compactes de dimension 3, *Bull. Soc. Math. France* 87 (1959), 319-329.
8. R. P. Filippkiewicz, Isomorphisms between diffeomorphism groups, *Ergodic Theory Dynamical Systems* 2 (1982), 159–171.
9. C. Gibson, K. Wirthmüller, A. du Plessis, and E. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Mathematics, Vol. 552, Springer-Verlag, Berlin-New York, 1976.
10. A. Hatcher, Homeomorphisms of sufficiently large $\mathbb{P}^2$-irreducible 3-manifolds, *Topology* 15 (1976), 343-347.
11. A. Hatcher, On the diffeomorphism group of $S^1 \times S^2$, *Proc. Amer. Math. Soc.* 83 (1981), 427–430.
12. A. Hatcher, On the diffeomorphism group of $S^1 \times S^2$, revised version posted at [http://www.math.cornell.edu/~hatcher/](http://www.math.cornell.edu/~hatcher/).
13. A. Hatcher, A proof of the Smale Conjecture, Diff$(S^3) \cong O(4)$, *Annals of Math.* 117 (1983), 553-607.
14. D. Henderson, Corrections and extensions of two papers about infinite-dimensional manifolds, *General Topology and Appl.* 1 (1971), 321–327.
15. D. Henderson, and R. Schori, Topological classification of infinite dimensional manifolds by homotopy type, *Bull. Amer. Math. Soc.* 76 (1970), 121-124.
16. H. Hendriks, La stratification naturelle de l’espace des fonctions différentiables réelles n’est pas le bonne, *C. R. Acad. Sci. Paris Sér. A-B* 274 (1972), A618-A620.
17. M. Hirsch, *Differential topology*, Graduate Texts in Mathematics No. 33, Springer-Verlag, New York-Heidelberg, 1976.
18. C. Hodgson and J. H. Rubinstein, Involutions and isotopies of lens spaces, *Knot theory and manifolds* (Vancouver, B.C., 1983), 60-96, Lecture Notes in Math., 1144, Springer, Berlin-New York, 1985.
19. N. Ivanov, Homotopy of spaces of automorphisms of some three-dimensional manifolds, *Soviet Math. Dokl.* 20 (1979), 47-50.
20. N. Ivanov, Homotopy of spaces of diffeomorphisms of some three-dimensional manifolds, *J. Soviet Math.* 26 (1984), 1646-1664.
21. N. Ivanov, Diffeomorphism groups of Waldhausen manifolds, *J. Soviet Math.* 12 (1979), 115–118.
22. J. Kalliongis and D. McCullough, Fiber-preserving imbeddings and diffeomorphisms, preprint available at [http://front.math.ucdavis.edu/math.GT/9802003](http://front.math.ucdavis.edu/math.GT/9802003)
23. J. Kalliongis and A. Miller, Geometric group actions on lens spaces, *Kyungpook Math. J.* 42 (2002), 313-344.
24. T. Kobayashi and S. Tsuyoshi, The Rubinstein-Scharlemann graphic of a 3-manifold as the discriminant set of a stable map, *Pacific J. Math.* 195 (2000), 101–156.
25. J. A. Leslie, On a differential structure for the group of diffeomorphisms, *Topology* 6 (1967), 263–271.
26. J. Mather, Stability of $C^{\infty}$ mappings. III. Finately determined mapgerms, *Inst. Hautes Études Sci. Publ. Math.* 35 (1968), 279-308.
27. D. McCullough, Isometries of elliptic 3-manifolds, *J. London Math. Soc.* (2) 65 (2002), 167-182.
28. D. McCullough and J. H. Rubinstein, The generalized Smale conjecture for 3-manifolds with genus 2 one-sided Heegaard splittings, preprint.
29. R. Palais, Local triviality of the restriction map for embeddings, Comment. Math. Helv. 34 (1960), 305-312.
30. R. Palais, Homotopy theory of infinite-dimensional manifolds, Topology 5 (1966), 1-16.
31. J. H. Rubinstein and M. Scharlemann, Comparing Heegaard splittings of non-Haken 3-manifolds, Topology 35 (1996), 1005-1026.
32. F. Sergeraert, Un théorème de fonctions implicites sur certains espaces de Fréchet et quelques applications, Ann. Sci. École Norm. Sup. (4) 5 (1972), 599-660.
33. S. Smale, Diffeomorphisms of the 2-sphere, Proc. Amer. Math. Soc. 10 (1959), 621-626.
34. F. Takens, Characterization of a differentiable structure by its group of diffeomorphisms, Bol. Soc. Brasil. Mat. 10 (1979), 17–25.
35. J.-C. Tougeron, Une généralisation du théorème des fonctions implicites, C. R. Acad. Sci. Paris Sér. A-B 262 (1966), A487–A489.
36. J.-C. Tougeron, Idéaux de fonctions différentiables I, Ann. Inst. Fourier (Grenoble) 18 (1968), 177–240.
37. C. T. C. Wall, Finite determinacy of smooth map-germs, Bull. London Math. Soc. 13 (1981), 481–539.

Department of Mathematics  
Korea University  
Seoul 136-701  
Korea  
E-mail address: shong@semi.korea.ac.kr  
URL: http://elie.korea.ac.kr/~shong/  

Department of Mathematics  
University of Oklahoma  
Norman, OK 73019  
U. S. A.  
E-mail address: dmccullough@math.ou.edu  
URL: www.math.ou.edu/~dmccullough/  

Department of Mathematics  
University of Melbourne  
Parkville, Victoria 3052  
Australia  
E-mail address: rubin@ms.unimelb.edu.au  
URL: www.ms.unimelb.edu.au/~rubin/