The diameter of the directed configuration model

Xing Shi Cai* and Guillem Perarnau**

* Duke Kunshan University, China. Email: xingshi.cai@dukekunshan.edu.cn
** IMTech, Universitat Politècnica de Catalunya, and Centre de Recerca Matemàtica, Spain.
   Email: guillem.perarnau@upc.edu

April 27, 2021

Abstract

We show that the diameter of the directed configuration model with \(n\) vertices rescaled by \(\log n\) converges in probability to a constant. Our assumptions are the convergence of the in- and out-degree of a uniform random vertex in distribution, first and second moment. Our result extends previous results on the diameter of the model and applies to many other random directed graphs.

1 Introduction and notations

1.1 The directed configuration model

The configuration model \(\mathcal{G}_n\) is a uniform random multigraph on \([n] := \{1, 2, \ldots, n\}\) vertices conditioned on its degree sequence being fixed. It was introduced by Bollobás [6] and has since become one of the most well-studied random graph models, see, e.g., [30] for an overview.

The directed version of this model, introduced by Copper and Frieze [15], is defined analogously. Let \(\mathbf{d}_n = ((d^-_1, d^+_1), \ldots, (d^-_n, d^+_n))\) be a bi-degree sequence with \(m_n := \sum_{i \in [n]} d^+_i = \sum_{i \in [n]} d^-_i\). The directed configuration model \(\mathcal{G}_n\) is the random directed multigraph on \([n]\) obtained by first giving \(d^-_i\) in half-edges (called heads) and \(d^+_i\) out half-edges (called tails) to node \(i\), and then choosing a uniform pairing of heads and tails. In this paper, we mainly consider the diameter of \(\mathcal{G}_n\), i.e., the longest distance between two connected nodes in \(\mathcal{G}_n\).

Many real-world complex networks are by nature directed. Thus, the directed configuration model has been studied in many applied domains, such as neural networks [2], finance [3] and social networks [23].
Let $U$ be uniform random variable on $[n]$. Let $D_n = (d_i^-, d_i^+)$. We denote by $D_n^-$ and $D_n^+$ the marginals of $D_n$ in each component. Let $n_{k,\ell}$ be the number of $(k, \ell)$ in $\tilde{d}_n$. Let $\Delta_n = \max_{i \in [n]} \{d_i^-, \ldots, d_i^+\}$ be the maximum degree of $\tilde{d}_n$. Consider a sequence of degree sequences $(\tilde{d}_n)_{n \geq 1}$. We assume the following:

**Condition 1.1.** There exists a discrete probability distribution $D = (D^-, D^+)$ on $\mathbb{Z}_{\geq 0}^2$ with $\lambda_{k,\ell} := \mathbb{P} \{D = (k, \ell)\}$ such that

(i) $D_n$ converges to $D$ in distribution:

$$\lim_{n \to \infty} \frac{n_{k,\ell}}{n} = \lambda_{k,\ell}, \quad (k, \ell \in \mathbb{Z}_{\geq 0});$$

(ii) $D_n$ converges to $D$ in expectation and the expectation is finite:

$$\lim_{n \to \infty} \mathbb{E}[D_n^-] = \lim_{n \to \infty} \mathbb{E}[D_n^+] = \mathbb{E}[D^-] = \mathbb{E}[D^+] =: \lambda \in (0, \infty);$$

(iii) $D_n$ converges to $D$ in second moment and the second moments are finite:

(a) $\lim_{n \to \infty} \mathbb{E}[D_n^- D_n^+] = \mathbb{E}[D^- D^+] < \infty$;
(b) $\lim_{n \to \infty} \mathbb{E}[(D_n^+)^2] = \mathbb{E}[(D^+)^2] < \infty$;
(c) $\lim_{n \to \infty} \mathbb{E}[(D_n^-)^2] = \mathbb{E}[(D^-)^2] < \infty$.

**Remark 1.2.** A simple digraph (directed graph) has no self-loops and no parallel edges of the same direction between two vertices. Let $\tilde{G}_n^s$ be a uniform random simple digraph with degree sequence $\tilde{d}_n$. Conditioned on being simple, $\tilde{G}_n$ is distributed as $\tilde{G}_n^s$ [15, Section 2.1]. Moreover, under Condition 1.1, the probability that $\tilde{G}_n$ is a simple digraph is bounded away from 0, see [5, 21]. Thus results that hold whp (with high probability) for $\tilde{G}_n$ also hold whp for $\tilde{G}_n^s$.

**Remark 1.3.** Note that removing nodes of degree $(0, 0)$ does not change the diameter of a digraph. Nonetheless, Condition 1.1 allows $n_{0,0} > 0$ and $\lambda_{0,0} > 0$.

An important parameter of $D$ which governs the limit behaviour of $\tilde{G}_n$ is

$$\nu := \frac{\mathbb{E}[D^- D^+]}{\lambda}. \quad (1.1)$$

Note that by conditions (ii) and (iii), $\nu < \infty$.

Cooper and Frieze [15] proved that the phase transition for the existence of a giant strongly connected component is at $\nu = 1$. Their result holds under assumptions stronger than Condition 1.1, including $\Delta_n \leq n^{1/12}/\log n$. The condition on the maximum degree was relaxed by Graf to $\Delta_n = o(n^{1/4})$ [18]. In recent paper of the two authors [11], the condition to have a giant component has been further weakened to $\Delta_n = o(n^{1/2})$. Throughout the paper we assume that $\nu > 0$ and $\nu \neq 1$.

Before stating our main result, we need to introduce two additional parameters. Let $f(z, w)$ be the bivariate generating function of $D$. Let $s_-$ and $s_+$ be the survival probabilities of the branching processes with offspring distributions having generating functions $\frac{1}{\lambda} \frac{\partial f}{\partial w}(z, 1)$ and $\frac{1}{\lambda} \frac{\partial f}{\partial z}(1, w)$ respectively. Then, we define

$$\hat{\nu}_- := \frac{1}{\lambda} \frac{\partial^2 f}{\partial z \partial w} (1 - s_-, 1), \quad \hat{\nu}_+ := \frac{1}{\lambda} \frac{\partial^2 f}{\partial z \partial w} (1, 1 - s_+), \quad (1.2)$$
which satisfy \( \hat{\nu}_-, \hat{\nu}_+ \in [0, 1) \).

Let the diameter of \( \vec{G}_n \) be

\[
\text{diam}(\vec{G}_n) := \max_{i,j \in [n]} \{ \text{dist}(i,j) : \text{dist}(i,j) < \infty \}.
\]

Our main result is the following:

**Theorem 1.4.** Suppose that \((\vec{d}_n)_{n \geq 1}\) satisfies Condition 1.1.

(i) **The supercritical case:** If \( \nu > 1 \), then

\[
\frac{\text{diam}(\vec{G}_n)}{\log n} \to \frac{1}{\log(1/\hat{\nu}_+)} + \frac{1}{\log(1/\hat{\nu}_-)} + \frac{1}{\log \nu},
\]

in probability, where we use the convention that \( 1/\log(1/0) = 0 \).

(ii) **The subcritical case:** If \( 0 < \nu < 1 \), then

\[
\frac{\text{diam}(\vec{G}_n)}{\log n} \to \frac{1}{\log(1/\nu)},
\]

in probability.

In the supercritical case (1.3), there are three terms that contribute to the diameter. The first term is given by vertices whose out-neighbourhoods neither expand nor die for many steps \( (\text{thin out-neighbourhoods}) \), and the second one is the analogue for in-neighbourhoods \( (\text{thin in-neighbourhoods}) \). Due to the symmetry in (1.1), the typical expansion rate of the in- and out-neighbourhoods of a vertex is the same. However, conditioned on the rare event of “having a thin neighbourhood”, the expansion rate is different (see Section 9 for some particular examples). The last term in (1.3) corresponds to the typical distance between a thin in- and out-neighbourhood. The case \( \mathbb{P}\{D^+ = 0\} = 0 \) is of particular interest (and similarly for \( \mathbb{P}\{D^- = 0\} = 0 \)). If additionally \( \mathbb{P}\{D^+ = 1\} = 0 \), then almost all vertices in \( \vec{d}_n \) have out-degree at least 2 and there are no thin out-neighbourhoods, so \( \hat{\nu}_+ = 0 \) and the first term in (1.3) disappears. Otherwise \( \mathbb{P}\{D^+ = 1\} > 0 \), one can check that \( \hat{\nu}_+ = \frac{1}{2} \mathbb{P}\{D^+ = 1\} \) and the thin out-neighbourhoods are directed paths.

In the subcritical case, we have \( \hat{\nu}_+ = \hat{\nu}_- = \nu \). In other words, thin in- and out-neighbourhoods of length \( \log_{1/\hat{\nu}_\pm} n \) still exist whp, but instead of expanding to large size they die before intersecting each other. Thus there is only one term in (1.4), which comes from both long in- and out-neighbourhoods.

The proof of Theorem 1.4 is based on the analysis of a BFS (Breadth First Search) edge-exploration process of the out-neighbourhoods of a given tail in \( \vec{G}_n \) (and similarly for the in-neighbourhoods of heads) and its coupling with the corresponding branching process. Convergence in Condition 1.1 is usually required in this setting and is necessary to ensure that we can couple the exploration process with a Galton-Watson tree with offspring obtained from \( D \). It would be interesting to see if one could drop the condition \( \nu < \infty \), as in the case of the undirected configuration model [17] (see Subsection 1.2).

We make no assumption on the rate of convergence in Condition 1.1, thus, we cannot determine the second order term of \( \text{diam}(\vec{G}_n) \). Under explicit convergence rate assumptions, it might be possible to find the second order term, as in [14, 28].
1.2 Previous results on distances in configuration models

We first discuss the previous results obtained for the undirected configuration model $G_n$ with degree sequence $d_n = (d_1, \ldots, d_n)$. Bollobás and Fernandez de la Vega [7] determined the asymptotic diameter of random regular graphs; that is, the case where $d_n$ contains only a constant. Fernholz and Ramachandran [17] obtained an asymptotic expression for the diameter of $G_n$.

To state the result in [17], some notation is needed: Let $D_n$ be chosen uniformly at random from $d_n$. Let $D$ be a discrete random variable on $\mathbb{Z}_{\geq 0}$ with distribution $\lambda_k := \mathbb{P}\{D = k\}$. Let $n_k$ be the number of $k$ in $d_n$. Let $D^\ast$ be a random variable on $\mathbb{Z}_{\geq 0}$ with distribution $\lambda^\ast_k := \mathbb{P}\{D^\ast = k\} = (k + 1)\lambda_{k+1}/\mathbb{E}[D]$; $D^\ast$ is the size-biased distribution of $D$. Let $D$ be the conjugate of $D^\ast$ (see Subsection 3.1.2) and let $\nu = \mathbb{E}[D]$.

Remark 1.5. The size-biased distribution of $D$ is sometimes defined as $D^s$ with $\mathbb{P}\{D^s = k\} = k\lambda_k/\mathbb{E}[D]$. Note that $D^s = D^\ast + 1$. We use $D^\ast$ for the sake of convenience.

Theorem 1.6. Assume that $D_n \rightarrow D$ in distribution, first and second moment, $\mathbb{E}[D] < \infty$ and $\lambda_1 > 0$, and that $\nu := \mathbb{E}[D(D - 1)]/\mathbb{E}[D] > 1$. Then

$$\frac{\text{diam}(G_n)}{\log n} \rightarrow \frac{2}{\log(1/\nu)} + \frac{1}{\log \nu},$$

in probability.

The case $\lambda_1 = 0$ is discussed in [31, Theorem 7.16]. Under the extra conditions $n_1 = 0$ when $\lambda_1 = 0$ and $n_2 = 0$ when $\lambda_2 = 0$, Theorem 1.6 extends to

$$\frac{\text{diam}(G_n)}{\log n} \rightarrow 2 \cdot \mathbb{1}[\lambda_1 > 0] + \mathbb{1}[\lambda_1 = 0, \lambda_2 > 0] \frac{1}{\log(1/\lambda_1^\ast)} + \frac{1}{\log \nu},$$

(1.5)

in probability. Theorem 1.4 (i) can be seen as the directed analogue of (1.5). The main difference is that, in the directed case, the first term in (1.5) splits into two (corresponding to thin in- and out- neighbourhoods which behave differently), and that there is no exceptional behaviour in the case $\lambda_1 = 0, \lambda_2 > 0$. The proof of Theorem 1.4 draws similarities with the proof of Theorem 1.6, based on the analysis of a BFS exploration process. However, the proof of Theorem 1.6 restricts the exploration to the 2-core of $G_n$ and thus, it heavily relies on previous understanding of the size and degree distribution of the 2-core, which are not known in the directed setting.

Refinements of Theorem 1.6 have been obtained for particular degree sequences. For example, Riordan and Wormald [28] proved that for every $c > 1$ there exists $\eta_c > 0$ such that the binomial random graph $G(n, p)$ satisfies $\text{diam}(G(n, c/n)) = \eta_c \log n + O(1)$ whp. (In this case $D$ is distributed as a Poisson with mean $c$.) We will use some of the ideas introduced in [28], to analyse the exploration process on $G_n$ without taking into consideration its core.

For degree sequences with $\mathbb{E}[D^2] = \infty$ and $\lambda_1 = \lambda_2 = 0$, Theorem 1.6 implies the weak upper bound on the diameter $o(\log n)$. In the particular case of power-law distributions with exponent $\tau \in (2, 3)$ and provided that the minimum degree is at least 3, van der Hofstad showed that the diameter is of order $\log \log n$ [31, Theorem 7.17].

Recently, there has been some progress on the diameter of the supercritical directed configuration model. Caputo and Quattropani [14] determined the asymptotic behaviour of the diameter of $G_n$ provided that $2 \leq d_i^- , d_i^+ = O(1)$. One of the motivations to study the diameter of directed
random graphs is its close connection to the properties of a random walk on it. For instance, in [14]
the authors used their results on neighbourhood expansion to determine the extremal values for the
stationary distribution of a random walk in $\tilde{G}_n$. We refer the interested reader to [12, 13] for fur-
ther discussion on the extremal stationary values and to [9, 13] for typical values of the stationary
distribution and their relation to the mixing time of the random walk. Finally, typical distances
in $\tilde{G}_n$ have been recently studied by van der Hoorn and Olvera-Cravioto [32]. We are not aware of
any result in the subcritical regime.

A related model is the $d$-out random digraph. In this model, each node is given a set of $d$
out-edges that connect independently to other vertices. This model is of particular interest since
it provides a way to study random Deterministic Finite Automata. Penrose studied the emergence
of a linear order strongly connected component [26] and its diameter was determined by Addario-
Berry, Balle and the second author [1]. In [10], the first author and Devroye studied the diameter
outside the giant strongly connected component and other properties of the $d$-out model.

Theorem 1.4 implies the results on the asymptotic behaviour of the diameter previously obtained
in [1, 14].

1.3 Organization of the paper

The paper is organized as follows. In Section 2 we obtain results on the number of edges incident to
small sets of nodes. In Section 3 we study rare events in branching processes. Section 4 introduces
the in- and out-size-biased distributions. We present an edge-BFS-exploration process in Section 5
and couple it with the corresponding branching process. In Section 6 we find thin in- and out-
neighbourhoods with large depth that will give rise to the first two terms in (1.3) and the term
in (1.4). Typical distances between large sets of edges are studied in Section 7, giving rise to the
last term in (1.3). Theorem 1.4 is proved in Section 8. Finally, in Section 9, we present some
applications.

2 Small sets of nodes

At various stages of our proof, we will need the fact that Condition 1.1 implies that any small set
of nodes is incident to a small number of half-edges. We state this formally in Lemma 2.2.

Let $X$ and $Y$ be random variables. We say that $X$ is stochastically dominated by $Y$ if
$\mathbb{P}\{X \geq z\} \leq \mathbb{P}\{Y \geq z\}$ for all $z \in \mathbb{R}$, and we denote it by $X \lesssim_{st} Y$. First we need the following
simple statement whose simple proof we omit.

Lemma 2.1. Let $X_n \geq 0$ and $Y_n \geq 0$ be two sequences of random variables such that $X_n \overset{d}{\to} X$ and
$Y_n \overset{d}{\to} Y$. Assume that $X_n \leq_{st} Y_n$ and $\mathbb{E}[Y_n] \to \mathbb{E}[Y]$. Then $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

For $S \subseteq [n]$, let

$$d_S(i, j) := \sum_{v \in S} (d^-_v)^i (d^+_v)^j.$$ 

Lemma 2.2. Assume Condition 1.1. Let $s_n = o(n)$ be a sequence of numbers. Then uniformly for
all $S$ with $|S| \leq s_n$,

$$d_S(1, 1) = o(n),$$ (2.1)
and
\[ d_S(1, 0) = o(\sqrt{s_n n}), \quad d_S(0, 1) = o(\sqrt{s_n n}). \] (2.2)

**Proof.** We first show (2.1) by a coupling argument. Recall that \( D_n = (d_U^-, d_U^+) \) where \( U \) is a uniform random variable on \([n]\). Define a random variable \( D_{n,*} \) on \( \mathbb{Z}_{\geq 0}^2 \) by \( D_{n,*} = D_n \) if \( U \notin S \) and \( D_{n,*} = (0, 0) \) otherwise. Then
\[ \mathbb{P} \{ D_{n,*} \neq D_n \} \leq \frac{s_n}{n} = o(1). \]

By **Condition 1.1**, the above implies that \( D_{n,*} \xrightarrow{d} D \) and \( D_{n,*}^- D_{n,*}^+ \xrightarrow{d} D^- D^+ \). Note that our coupling ensures \( D_{n,*}^- D_{n,*}^+ \leq_{st} D_n^- D_n^+ \) which converges in mean and in distribution to \( D^- D^+ \). Thus it follows from **Lemma 2.1** that \( \mathbb{E}[D_{n,*}^- D_{n,*}^+] \to \mathbb{E}[D^- D^+] \) by taking \( X_n = D_{n,*}^- D_{n,*}^+ \) and \( Y_n = D_n^- D_n^+ \). Therefore,
\[ d_S(1, 1) = \sum_{v \in S} d_v^- d_v^+ = \sum_{v \in [n]} d_v^- d_v^+ - \sum_{v \notin S} d_v^- d_v^+ = n \mathbb{E}[D_n^- D_n^+] - n \mathbb{E}[D_{n,*}^- D_{n,*}^+] = o(n). \]

For (2.2), first note that **Lemma 2.1** together with **Condition 1.1** also implies that \( \mathbb{E}[(D_{n,*}^-)^2] \to \mathbb{E}[(D^-)^2] \). Thus
\[ \sum_{v \in S} (d_v^-)^2 = \sum_{v \in [n]} (d_v^-)^2 - \sum_{v \notin S} (d_v^-)^2 = n \mathbb{E}[(D_n^-)^2] - n \mathbb{E}[(D_{n,*}^-)^2] = o(n). \]

It follows from Cauchy-Schwarz inequality that
\[ d_S(1, 0) = \sum_{v \in S} d_v^- \leq \sqrt{\left( \sum_{v \in S} 1 \right) \left( \sum_{v \in S} (d_v^-)^2 \right)} = o(\sqrt{s_n n}). \]
The same argument works for \( d_S(0, 1) \).

For \( \mathcal{I} \subseteq [n] \), define
\[ \nu_\mathcal{I} := \frac{d_{\mathcal{I}}(1, 1)}{m_n} = \frac{\sum_{v \in \mathcal{I}} d_v^- d_v^+}{m_n}. \] (2.3)

**Corollary 2.3.** Under the hypothesis of **Lemma 2.2** and uniformly for all \( \mathcal{I} \subseteq [n] \) with \( |\mathcal{I}| \geq n - s_n \), we have \( \nu_\mathcal{I} = (1 + o(1))\nu \).

**Proof.** The corollary follows from (2.1) with \( S = [n] \setminus \mathcal{I} \).

**Corollary 2.4.** Under **Condition 1.1**, we have \( \Delta_n = o(\sqrt{n}) \).

**Proof.** Let \( \mathcal{S} \) be the set containing only a node with maximum out/in degree and apply (2.2).

### 3 Branching processes

Let \( \xi \) be a random variable on \( \mathbb{Z}_{\geq 0} \) and let \( (\xi_{t,i})_{i \geq 1, t \geq 0} \) be iid (independent and identically distributed) copies of \( \xi \). The branching process, also known as the Galton-Watson tree, \((X_t)_{t \geq 0}\) with offspring distribution \( \xi \) is defined by
\[ X_t = \begin{cases} 1 & (t = 0) \\ \sum_{i=1}^{X_{t-1}} \xi_{t-1,i} & (t \geq 1) \end{cases}. \]

Let \( h \) be the probability generating function of \( \xi \), i.e., \( h(z) = \sum_{i \geq 0} \mathbb{P} \{ \xi = i \} z^i \). Then
\[ \nu_\xi := h'(1) = \mathbb{E}[\xi]. \]
3.1 Supercritical branching process

In this subsection we will assume that $\nu_\xi \in (1, \infty)$, usually referred to as $(X_t)_{t \geq 0}$ being supercritical.

3.1.1 Convergence of branching processes

The sequence $\nu_\xi^{-t}X_t$ is a martingale which, provided that $\nu_\xi < \infty$, converges almost surely to a random variable $W$. However, stronger conditions are needed to show that $W$ is non-degenerated. The following is due to Kesten and Stigum (see [4, pp. 24–29] for a proof):

**Theorem 3.1.** Let $(X_t)_{t \geq 0}$ be the branching process defined above. Assume that $\mathbb{P}\{\xi = i\} \neq 1$ for all $i \in \mathbb{N}$. Let $W = \lim_{t \to \infty} \nu_\xi^{-t}X_t$ be the rescaled limiting random variable.

- (i) If $\mathbb{E}[\xi \log_+ (\xi)] < \infty$, then $\mathbb{E}[W] = 1$ and $W$ is absolutely continuous on $(0, \infty)$.
- (ii) If $\mathbb{E}[\xi \log_+ (\xi)] = \infty$, then $W = 0$ almost surely.

In the case that $\mathbb{E}[\xi \log_+ (\xi)] < \infty$, since almost sure convergence implies convergence in probability, it follows that for every fixed $0 < c_1 < c_2$,

$$\inf_t \mathbb{P}\left\{ c_1 \nu_\xi^t \leq X_t \leq c_2 \nu_\xi^t \right\} > 0,$$

where the infimum is taken over all $t$ such that $[c_1 \nu_\xi^t, c_2 \nu_\xi^t] \cap \mathbb{N}$ is non-empty.

If $\mathbb{E}[\xi \log_+ (\xi)] = \infty$, the growth of $X_t$ is not necessarily of order $\nu_\xi^t$ any more. However, there always exists a normalization sequence $\{m_{\xi,t}\}_{t \geq 0}$ with $\lim_{t \to \infty} (m_{\xi,t})^{1/t} = \nu_\xi$ and $m_{\xi,t} = O(\nu_\xi^t)$, such that $X_t/m_{\xi,t}$ converges almost surely to a non-degenerate limit (see [4, pp. 30] for a proof and further reference). In words, the exponential growth rate of $X_t$ is still $\nu_\xi$, but there might be subexponential fluctuations that slow it down. Again, almost sure convergence implies that for every $0 < c_1 < c_2$,

$$\inf_t \mathbb{P}\left\{ c_1 m_{\xi,t} \leq X_t \leq c_2 m_{\xi,t} \right\} > 0,$$

(3.1)

where the infimum is taken over all $t$ such that $[c_1 m_{\xi,t}, c_2 m_{\xi,t}] \cap \mathbb{N}$ is non-empty.

If $\mathbb{P}\{\xi = \nu_\xi\} = 1$ for some $\nu_\xi \geq 2$, then Theorem 3.1 does no longer apply. Nonetheless, we can define $m_{\xi,t} = \nu_\xi^t$ and (3.1) still holds for any $0 < c_1 \leq 1 \leq c_2$.

3.1.2 Duality and conditioned branching process

The survival probability of $(X_t)_{t \geq 0}$ is defined by

$$s := \mathbb{P}\{X_t > 0, \text{ for all } t \geq 1\}.$$

It is well-known that $s > 0$ if and only if $\nu_\xi > 1$, see, e.g., [30, Theorem 3.7].

For $s < 1$, the conjugate probability distribution of $\xi$, $\hat{\xi}$ is defined by

$$\mathbb{P}\{\hat{\xi} = \ell\} := (1 - s)^{\ell-1} \mathbb{P}\{\xi = \ell\}.$$

(3.2)

The definition of $\hat{\xi}$ can be extended to encompass the case $s = 1$ by taking the limit of (3.2) as $s \uparrow 1$. In other words, when $s = 1$,

$$\mathbb{P}\{\hat{\xi} = 1\} = \mathbb{P}\{\xi = 1\}, \quad \mathbb{P}\{\hat{\xi} = 0\} = 1 - \mathbb{P}\{\xi = 1\}.$$

(3.3)
By the assumption $\nu_\xi > 1$, we have $\mathbb{P}\{\xi = 1\} < 1$ and $s > 0$. Therefore,

$$\hat{\nu}_\xi := \mathbb{E}[\hat{\xi}] = h'(1-s) \in [0,1).$$

(3.4)

By the assumption that $\nu_\xi > 1$, we have $s > 0$. Thus $\hat{\nu}_\xi < 1$ and $\hat{\nu}_\xi = 0$ if and only if $\mathbb{P}\{\xi \leq 1\} = 0$.

**Lemma 3.2.** Let $\xi_n$ be a sequence of random variables such that $\xi_n \to \xi$ in distribution and in expectation, where $\mathbb{P}\{\xi = 1\} < 1$. Then $\hat{\xi}_n \to \hat{\xi}$ in distribution and in expectation.

**Proof.** Let $s_n$ and $s$ be the survival probabilities of the branching processes with offspring distributions $\xi_n$ and $\xi$ respectively. Let $\rho_n = 1 - s_n$ and $\rho = 1 - s$. Let $\phi_n(x)$ and $\phi(x)$ be the probability generating functions of $\xi_n$ and $\xi$ respectively. As $\xi_n \xrightarrow{d} \xi$,

$$\max_{x \in [0,1]} |\phi_n(x) - \phi(x)| \to 0.$$  

(3.5)

By taking subsequence of subsequences, we can assume that $\lim_{n \to \infty} \rho_n \to \rho' \in [0,1]$. If $\rho' < 1$, by (3.5), $\phi(\rho') = \rho'$. As $\mathbb{P}\{\xi = 1\} < 1$, $\phi(x)$ is strictly convex and $\rho(x) = x$ has at most one solution on $[0,1)$. Thus $\rho = \rho' < 1$ if $\rho' < 1$. If $\rho' = 1$, we have $\mathbb{E}[\xi] = \lim_{n \to \infty} \mathbb{E}[\xi_n] = 1$. Thus $\rho = 1$. So in both cases, $\rho_n \to \rho$.

Thus by the definition of conjugate distribution (3.2), we have $\hat{\xi}_n \to \hat{\xi}$ in distribution. To see that the convergence is also in expectation, note that $\hat{\xi}_n \xrightarrow{st} \xi_n$. Thus we can apply Lemma 2.1 with $X_n = \hat{\xi}_n$ and $Y_n = \xi_n$.

**Remark 3.3.** It follows from Lemma 4.5 in [8] that $s_n \to s$ if $\xi_n \xrightarrow{d} \xi$. However, the extra assumption that $\mathbb{E}[\xi_n] \to \mathbb{E}[\xi] < \infty$ in Lemma 3.2 makes our proof slightly simpler.

An important property of supercritical branching processes is duality [31, Theorem 3.7]:

**Theorem 3.4.** Let $(X_t)_{t \geq 0}$ be as in Theorem 3.1 and let $s$ be its survival probability. If $s < 1$, then the branching process $(X_t)_{t \geq 0}$ conditioned on extinction is distributed as a branching process with offspring distribution $\hat{\xi}$.

The next key result allows us to estimate the probability of certain rare events in branching processes. It generalizes a result of Riordan and Wormald [28, Lemma 2.1], who proved it for Poisson distributed offsprings.

**Theorem 3.5.** Let $(X_t)_{t \geq 0}$ be a branching process with offspring distribution $\xi$ with $\nu_\xi \in (1,\infty)$. Let $t_\xi(\omega) := \inf\{t \geq 0 : m_{\xi,t'} \geq \omega \text{ for all } t' \geq t\}$.

(i) If $\mathbb{P}\{\xi \leq 1\} > 0$, then there exist constants $c,C > 0$ depending on $\xi$ such that

$$c\hat{\nu}_\xi^t \leq \mathbb{P}\{\cap_{r=1}^t [0 < X_r < \omega]\} \leq C\hat{\nu}_\xi^{t-t_\xi(\omega)}, \quad (t \geq 1, \omega \geq t).$$

(ii) If $\mathbb{P}\{\xi \leq 1\} = 0$, then $\hat{\nu}_\xi = 0$ and there exists $c > 0$ depending on $\xi$ such that

$$\mathbb{P}\{\cap_{r=1}^t [0 < X_r < \omega]\} \leq \exp\{-c2^{t-t_\xi(\omega)}\}, \quad (t \geq 1, \omega \geq 1).$$
Remark 3.6. If $\mathbb{E}[\xi \log_+(\xi)] < \infty$, by Theorem 3.1, we may choose $m_{\xi,t} = \nu_{\xi}^t$ and $t_{\xi}(\omega) = \lfloor \log_{\nu_{\xi}} \omega \rfloor$. In fact, the result of Seneta [29] implies that $t_{\xi}(\omega) = (1+o(1)) \log_{\nu_{\xi}} \omega$ if $\mathbb{E}[\xi \log_+(\xi)] = \infty$.

Proof of the upper bound in (i). It suffices to provide an upper bound for the probability of the event $[0 < X_t < \omega]$. Since the proof follows the same ideas as the proof of Lemma 2.1 in [28], we omit the details that are identical to the aforementioned lemma.

Define $r_t := \mathbb{P}\{X_t < \omega \mid X_t > 0\}$ and write $t(\omega) = t_{\xi}(\omega)$. Equation (3.1) and the definition of $t(\omega)$ imply that

$$\mathbb{P}\{X_{t(\omega)} \geq \omega \mid X_{t(\omega)} > 0\} \geq \mathbb{P}\{X_{t(\omega)} \geq \omega\}$$

$$\geq \mathbb{P}\{X_{t(\omega)} \geq m_{\xi,t(\omega)}\}$$

$$\geq \mathbb{P}\{m_{\xi,t(\omega)} \leq X_{t(\omega)} \leq 2m_{\xi,t(\omega)}\} > c_0,$$

for some $c_0 > 0$, which implies that $r_{t(\omega)} < 1 - c_0$.

Let $s$ be the survival probability of $X_t$ and let $S(t) \subseteq X_t$ be the set of children of the initial particle that have progeny in $X_t$. Thus, as in [28], to show that $r_t$ has exponential decrease with basis $\hat{\nu}_{\xi}$ for $t \geq t(\omega)$, it suffices to show that

$$\mathbb{P}\{|S(t)| = 1 \mid |S(t)| \geq 1\} = \hat{\nu}_{\xi}(1 + O(q(s)\hat{\nu}_{\xi}^t)), \quad (3.6)$$

for some function $q$. This directly implies the upper bound in (i) since $\mathbb{P}\{0 < X_t < \omega\} \leq r_t$. We prove (3.6) in the following.

We first consider the case $s < 1$. Recall that $s > 0$ and let $s_t = \mathbb{P}\{X_t > 0\}$. We can assume that $t$ is large enough with respect to $s$, as we can set $C$ large enough with respect to $\hat{\nu}_{\xi}$ so the bound holds trivially for small values of $t$. Using Markov inequality and Theorem 3.4,

$$s \leq s_t = \mathbb{P}\{(X_r)_{r \geq 0} \text{ survives}\} + \mathbb{P}\{X_t > 0, (X_r)_{r \geq 0} \text{ extinguishes}\}$$

$$= s + (1 - s)\mathbb{P}\{X_t > 0 \mid (X_r)_{r \geq 0} \text{ extinguishes}\} \quad (3.7)$$

$$\leq s + (1 - s)\hat{\nu}_{\xi}^t.$$

Conditioning on $X_1$, the events $[x \in S(t)]$ for $x \in X_1$ happen independently with probability $s_{t-1}$. Thus, the random variable $|S(t)|$ has the distribution of a $s_{t-1}$-thinned version of $\xi$ and

$$\mathbb{P}\{|S(t)| = 1 \mid |S(t)| \geq 1\} = \frac{\mathbb{P}\{|S(t)| = 1\}}{\mathbb{P}\{|S(t)| \geq 1\}} = \frac{s_{t-1}h'(1 - s_{t-1})}{1 - h(1 - s_{t-1})}, \quad (3.8)$$

We use Taylor expansion to approximate $h(1 - s_{t-1})$ and $h'(1 - s_{t-1})$ around $1 - s$. First note that for every $m \geq 1$, the $m$-th derivative of $h$ is bounded at $1 - s$

$$h^{(m)}(1 - s) = \sum_{\ell \geq 0} (\ell)_m (1 - s)^{\ell - m} \mathbb{P}\{\xi = \ell\} \leq \sum_{\ell \geq 0} (\ell)_m (1 - s)^{\ell - m} = m! s^{-m-1},$$

where $(\ell)_m := \ell(\ell - 1) \ldots (\ell - m + 1)$. Using Taylor’s theorem, we have, uniformly for all $t \geq 1$

$$h(1 - s_{t-1}) = h(1 - s) + h'(1 - s)(s - s_{t-1}) + O((s - s_{t-1})^2) = 1 - s + O(\hat{\nu}_{\xi}^t),$$

$$h'(1 - s_{t-1}) = h'(1 - s) + h''(1 - s)(s - s_{t-1}) + O((s - s_{t-1})^2) = \hat{\nu}_{\xi} + O(\hat{\nu}_{\xi}^t),$$

where we use $h(1 - s) = 1 - s$, $h'(1 - s) = \hat{\nu}_{\xi}$ and (3.7). Using these estimates and (3.8), we obtain (3.6).

In the case $s = 1$, the event $[|S(t)| \geq 1]$ holds almost surely and (3.6) still holds since

$$\mathbb{P}\{|S(t)| = 1 \mid |S(t)| \geq 1\} = \mathbb{P}\{|S(t)| = 1\} = \mathbb{P}\{\xi = 1\} = \hat{\nu}_{\xi}. \quad \square$$
Proof of (ii). Let \( r_t \) and \( c_0 \) be as in the proof of the upper bound in (i). In this case, the event \( |S(t)| \geq 2 \) holds almost surely. Thus
\[
r_t \leq \mathbb{P}\{|S(t)| \geq 2\} r_{t-1}^2 = r_{t-1}^2,
\]
and \( r_t \leq (r_t(\omega))^{2^{t-\omega}} < (1 - c_0)^{2^{t-\omega}} \).

\[\square\]

Proof of the lower bound in (i). Let \( (X^*_r)_{r \geq 0} \subseteq (X_r)_{r \geq 0} \) be the subprocess of the elements that have some surviving progeny. For the lower bound, consider the following events:
\[
E_1 = [X^*_t = 1], \quad E_2 = [\cap_{r=1}^{t} [0 < X_r < \omega]].
\]

Instead of lower bounding the probability of \( E_2 \), we will show a lower bound for the probability of \( E := E_1 \cap E_2 \). Write
\[
\mathbb{P}\{E\} = \mathbb{P}\{E_1\} \mathbb{P}\{E_2 \mid E_1\}. \tag{3.9}
\]

We start by computing \( \mathbb{P}\{E_1\} \). Conditioning on that an element of \( X_t \) belongs to \( X^*_t \) is equivalent to conditioning on that the progeny of at least one of its children survives. So, conditional on \( X_t \) surviving, \( X^*_t \) is a branching process with offspring distribution \( \xi^* \), the \( s \)-thinned version of \( \xi \) conditioning on being at least 1. In other words, for \( \ell \geq 1 \)
\[
\mathbb{P}\{\xi^* = \ell\} = \sum_{m \geq \ell} \mathbb{P}\{\xi = m\} \binom{m}{\ell} s^\ell (1-s)^{m-\ell} = \frac{s^{\ell-1} h(\ell)(1-s)}{\ell!}.
\]

(The term in the sum is the probability that an element in \( X_t \) has \( m \) children and exactly \( \ell \) of them survive.) Thus, letting \( \hat{\nu} = \hat{\nu}_s \),
\[
\mathbb{P}\{\xi^* = 1\} = h'(1-s) = \hat{\nu}.
\]

We conclude that
\[
\mathbb{P}\{E_1\} = \mathbb{P}\{(X_r)_{r \geq 0} \text{ survives} \mid [\cap_{r=1}^{t} [X^*_r = 1]]\} = s \hat{\nu}^t. \tag{3.10}
\]

If \( \mathbb{P}\{\xi = 0\} = 0 \), then \( X^*_t = 1 \) implies that \( X_r = 1 \) for all \( r \leq t \). Therefore, \( \mathbb{P}\{E_2 \mid E_1\} = 1 \) and we are done. Thus, we assume that \( \mathbb{P}\{\xi = 0\} > 0 \), and so \( s < 1 \).

Conditioning on \( E_1 \), the tree can be seen as the main branch of length \( t \) (the part that has surviving progeny), with \( t \) independent branching processes with offspring distribution of the root being \( \xi - 1 \) conditioned on having exactly one surviving child, and \( \hat{\xi} \) for all other elements, attached to each node on the path. Let \( \bar{\xi} \) denote the number of children of the root of such a process. Then
\[
\mathbb{P}\{\bar{\xi} = m\} = \frac{(m+1)\mathbb{P}\{\xi = m+1\} s(1-s)^m}{\sum_{m=1}^{\infty} m\mathbb{P}\{\xi = m\} s(1-s)^{m-1}} = \hat{\nu}^{-1}(m+1)\mathbb{P}\{\xi = m+1\} (1-s)^m.
\]

Thus
\[
\mathbb{E}[\bar{\xi}] = \sum_{m \geq 0} m\mathbb{P}\{\bar{\xi} = m\} = \hat{\nu}^{-1}(1-s) h''(1-s) < \infty,
\]

10
where the last inequality follows from $1 - s < 0$ and that the radius of convergence of $h(z)$ is at least 1. By a similar computation,

$$\mathbb{E}[\hat{\xi}(\hat{\xi} - 1)] = \hat{\nu}^{-1}(1 - s)^2 h^{(3)}(1 - s) < \infty,$$

which implies $\text{Var}(\hat{\xi}) < \infty$.

By Theorem I.12.3 in [4], the probability generating function of $\hat{\xi}$ is

$$\hat{h}(z) := \mathbb{E}[z^{\hat{\xi}}] = \frac{h(z(1 - s))}{1 - s}.$$

Therefore, $\hat{h}''(1) = (1 - s)h''(1 - s) < \infty$. In other words, $\text{Var}(\hat{\xi}) < \infty$.

Thus, by Wald’s equation [16, Theorem 4.1.5], uniformly for $0 \leq r \leq t$,

$$\mathbb{E}[X_r | E_1] = 1 + \mathbb{E}[\hat{\xi}] \sum_{j=1}^{r} \hat{\nu}^j = O(1).$$

Moreover, by Wald’s second equation [16, Theorem 4.1.6] and the moment formula in [4, pp. 4], uniformly for $1 \leq r \leq t$,

$$\text{Var}(X_r | E_1) = \sum_{j=1}^{r} \left( \mathbb{E}[\hat{\xi}] \text{Var}(\hat{\xi}) \hat{\nu}^{j-2}(\hat{\nu}^{j-1} - 1) \frac{1}{\hat{\nu} - 1} + \hat{\nu}^{2(j-1)} \text{Var}(\hat{\xi}) \right) = O(1).$$

Therefore, it follows from Chebyshev’s inequality that

$$\mathbb{P}\{E_2^c | E_1\} \leq \sum_{j=0}^{t} \mathbb{P}\{X_j \geq \omega | E_1\} = O\left( \sum_{r=1}^{t} \frac{\text{Var}(X_r | E_1)}{\omega^2} \right) = O\left( \frac{t}{\omega^2} \right) = O(t^{-1}). \quad (3.11)$$

The lemma follows immediately by putting (3.10) and (3.11) into (3.9).

\section*{3.2 Subcritical branching process}

If $\nu_{\xi} \in (0, 1)$, then $s = 0$ and $\hat{\nu}_{\xi} = \nu_{\xi}$. Thus, the following theorem shows that the depth of thin supercritical branching processes is close to the depth of subcritical processes. This has already been observed in [17].

**Theorem 3.7.** Let $(X_t)_{t \geq 0}$ be a branching process with offspring distribution $\xi$ with $\nu_{\xi} \in (0, 1)$. Then

$$\lim_{t \to \infty} \mathbb{P}\{X_t > 0\}^{1/t} = \nu_{\xi}. \quad (3.12)$$

Moreover, letting $Y_t = \sum_{i=0}^{t} X_i$, for all $\omega(t)$ such that $\omega(t)/t = \infty$ as $t \to \infty$, we have

$$\lim_{t \to \infty} \mathbb{P}\{Y_t \leq \omega(t) \cap [X_t > 0]\}^{1/t} = \nu_{\xi}. \quad (3.13)$$

**Proof.** First note that $\mathbb{E}[X_t] = \nu_{\xi}^t$, by Markov’s inequality $\mathbb{P}\{X_t > 0\} \leq \nu_{\xi}^t$. Thus it suffices to prove a lower bound in all cases discussed below.

We first prove (3.12). If $\mathbb{E}[\xi \log \xi] < \infty$, then it follows from Theorem I.11.1 in [4] that

$$\lim_{t \to \infty} \frac{\mathbb{P}\{X_t > 0\}}{\nu_{\xi}^t} = c > 0, \quad (3.14)$$
for some constant $c$. From this (3.12) follows immediately.

If $\mathbb{E}[\xi \log \xi] = \infty$, fix a large $M > 0$, consider the distribution $\bar{\xi} = \min\{\xi, M\}$ and let $(\bar{X}_t)_{t \geq 0}$ be the branching process with offspring distribution $\bar{\xi}$. Since $\bar{\xi}$ is bounded, it follows from (3.14) that

$$\liminf_{t \to 0} \mathbb{P}\{X_t > 0\}^{1/t} \geq \liminf_{t \to 0} \mathbb{P}\{\bar{X}_t > 0\}^{1/t} = \mathbb{E}[\bar{\xi}] \to \nu_{\bar{\xi}}.$$ 

as $M \to \infty$. This proves the lower bound in (3.12).

For (3.13), first assume that $\mathbb{E}[\xi^2] < \infty$. In this case, Theorem 1 of [25] states that

$$\frac{(Y_t | X_t > 0)}{t} \to c_{\xi} := 1 + \frac{\mathbb{E}[\xi(\xi - 1)]}{\nu_{\xi}(1 - \nu_{\bar{\xi}})},$$

in probability. Therefore

$$\mathbb{P}\{Y_t < \omega(t) \cap [X_t > 0]\} = \mathbb{P}\{Y_t < \omega(t) | X_t > 0\} \mathbb{P}\{X_t > 0\} = (1 + o(1))(1 + o(1))\nu_{\bar{\xi}}^t, \quad (3.15)$$

from which (3.13) follows immediately.

In the case that $\mathbb{E}[\xi^2] = \infty$, we again use a truncation argument. Let $\xi$ and $(\bar{X}_t)_{t \geq 0}$ be as above. Let $Y_t = \sum_{i=1}^t \bar{X}_i$. Let $A_s$ be the event that the first $s$ nodes in BFS order in $(X_t)_{t \geq 0}$ have degree at most $M$. Then

$$\mathbb{P}\{Y_t < \omega(t) \cap [X_t > 0]\} \geq \mathbb{P}\{Y_t < 2c_{\xi}^t \cap [X_t > 0] | A_{2c_{\xi}^t}\} \mathbb{P}\{A_{2c_{\xi}^t}\} \quad (3.16)$$

By Markov inequality, $\mathbb{P}\{\xi \geq M\} \leq \nu_{\bar{\xi}}/M \leq 1/M$. Indeed, it is well-known that $\mathbb{E}[\xi] < \infty$ actually implies that $\mathbb{P}\{\xi \geq M\} = o(1/M)$, where the asymptotics is as $M \to \infty$. Since $\bar{\xi} \leq M$,

$$c_{\xi} = O(\mathbb{E}[\xi^2]) = O(M\mathbb{E}[\bar{\xi}]) = O(M).$$

Therefore

$$\mathbb{P}\{A_{2c_{\xi}^t}\} \geq (1 - \mathbb{P}\{\xi \geq M\})^{2c_{\xi}^t} \geq (1 - o(1/M))^{O(M)} = (1 - \delta)^t, \quad (3.17)$$

for all $\delta > 0$, provided that $M$ is large enough. Also by choosing $M$ large enough, we can get $\mathbb{E}[\bar{\xi}] > (1 - \delta/2)\nu_{\bar{\xi}}$ for all $\delta > 0$. As $\mathbb{E}[\bar{\xi}^2] < \infty$, it follows from (3.15) that

$$\mathbb{P}\{Y_t < 2c_{\xi}^t \cap [X_t > 0] | A_{2c_{\xi}^t}\} = \mathbb{P}\{Y_t < 2c_{\xi}^t \cap [X_t > 0]\}
= (1 + o(1))(1 + o(1))\mathbb{E}[\bar{\xi}]^t \geq (1 - \delta)(1 - \delta)^t, \quad (3.18)$$

for all $t$ large enough. Putting (3.17) and (3.18) into (3.16), we have

$$\liminf_{t \to \infty} \mathbb{P}\{Y_t < \omega(t) \cap [X_t > 0]\}^{1/t} \geq (1 - \delta)^2\nu_{\bar{\xi}}.$$

Since $\delta$ is arbitrary, we are done with (3.13).

\section{Size-biased distributions}

We will apply the results on branching processes obtained in Section 3 to distributions arising from $D$. Define the \emph{in- and out-size biased} distributions of $D_n$, denoted by $(D_n)_{\text{in}}$ and $(D_n)_{\text{out}}$ respectively, by

$$\mathbb{P}\{(D_n)_{\text{in}} = (k - 1, \ell)\} = \frac{k\ell n_{k,\ell}}{m_n}, \quad \mathbb{P}\{(D_n)_{\text{out}} = (k, \ell - 1)\} = \frac{\ell n_{k,\ell}}{m_n}.$$
In other words, if we choose a head $e^-$ uniformly at random (say $v$ is its incident node) and look at the number of heads/tails incident to $v$ different from $e^-$, what we get is a random pair of integers distributed as $(D_{in})_{in}$. Similarly, we get $(D_{out})_{out}$ choosing a tail uniformly at random.

We also define the in- and out-size biased distributions of $D$, denoted by $D_{in}$ and $D_{out}$ respectively, by
\[
\mathbb{P} \{ D_{in} = (k-1, \ell) \} = \frac{k \lambda k, \ell}{\lambda}, \quad \mathbb{P} \{ D_{out} = (k, \ell - 1) \} = \frac{\ell \lambda k, \ell}{\lambda}.
\]
Then, by (i) of Condition 1.1, $(D_{in})_{in} \rightarrow D_{in}$ and $(D_{out})_{out} \rightarrow D_{out}$ in distribution, and by (iii) of Condition 1.1,
\[
\lim_{n \to \infty} \mathbb{E} [(D_{in})^+] = \lim_{n \to \infty} \mathbb{E} [(D_{out})^-] = \mathbb{E} [D_{in}^+] = \mathbb{E} [D_{out}^-] = \frac{\mathbb{E} [D^+] D^-}{\lambda} = \nu. \tag{4.1}
\]

Alternatively, one can define the size-biased distributions of $D$ using generating functions. The bivariate probability generating function of $D$ is
\[
f(z, w) := \sum_{k, \ell} z^k w^\ell \lambda k, \ell.
\]
The distributions $D_{in}$ and $D_{out}$ have bivariate probability generating functions respectively
\[
f_{in}(z, w) = \frac{1}{\lambda} \partial f(1, 1), \quad f_{out}(z, w) = \frac{1}{\lambda} \partial f(1, 1).
\]
Note that $\frac{\partial f(1, 1)}{\partial z} = \frac{\partial f(1, 1)}{\partial w} = \lambda$, so $f_{in}(1, 1) = f_{out}(1, 1) = 1$. (This shows that $D_{in}$ and $D_{out}$ are indeed probability distributions.) Similarly, the probability generating functions of $D_{in}^+$ and $D_{out}^-$ are $f_{in}(1, w)$ and $f_{out}(z, 1)$ respectively.

We define
\[
g(z, w) := \frac{\partial^2 f}{\lambda \partial z \partial w}.
\]
Then
\[
g(1, 1) = \frac{\partial f_{in}}{\partial w} = \frac{\partial f_{out}}{\partial z} = \mathbb{E} [D_{in}^+] = \mathbb{E} [D_{out}^-] = \nu.
\]

Since $\nu > 1$, there exists a unique solution $\rho_{in} \in [0, 1)$ of $f_{in}(1, w) = w$. Similarly, there is a unique solution $\rho_{out} \in [0, 1)$ of $f_{out}(z, 1) = z$.

The following is classical from branching process theory (see, e.g., Corollary 4.2 in [8] or Theorem 3.1 in [31]):

**Theorem 4.1.** Assume Condition 1.1 and $\nu > 1$. Let $\rho_{in}, \rho_{out} \in [0, 1)$ be the unique roots of $f_{in}(1, w) = w$ and $f_{out}(z, 1) = z$, respectively. Then $s_+ := 1 - \rho_{in}$ and $s_- := 1 - \rho_{out}$ are the survival probabilities of the branching processes with distribution $D_{in}^+$ and $D_{out}^-$ respectively.

Recall the definitions $\hat{\nu}_+ = g(1, 1 - s_+)$ and $\hat{\nu}_- = g(1 - s_-, 1)$ given in (1.2). Let $\hat{D}_{in}^+$ and $\hat{D}_{out}^-$ be the conjugate distributions of $D_{in}^+$ and $D_{out}^-$ respectively, defined as in (3.2). It is easy to check that $\mathbb{E} [\hat{D}_{in}^+] = \hat{\nu}_+$ and $\mathbb{E} [\hat{D}_{out}^-] = \hat{\nu}_-$. Note that (3.4) implies that $\hat{\nu}_+, \hat{\nu}_- \in [0, 1)$.

**Remark 4.2.** While $\mathbb{E} [D_{in}^+] = \mathbb{E} [D_{out}^-] = \nu$, in general, $\hat{\nu}_+$ and $\hat{\nu}_-$ are different. As an example, fix an integer $\lambda \geq 2$, let $D^+$ be constant $\lambda$ and let $D^-$ have a Poisson distribution with expectation $\lambda$. Then $\nu = \lambda$, $D_{in}^+ = D^+$ and $D_{out}^- = D^-$. Thus, $s_+ = 1$ and $s_- < 1$, so $\hat{\nu}_+ = 0$ and $\hat{\nu}_- > 0$. 

13
5 Exploring the graph

We will explore $\bar{G}_n$ following a Breadth First Search (BFS) order. This technique to explore vertex-neighbourhoods of a vertex is standard in the study of random graphs, see, e.g., [30, Chapter 4]. However, in this paper it will be more convenient to study the edge-neighbourhoods of a half-edge. In this section we describe only the out-neighbourhoods of tails since the study of in-neighbourhoods of heads is identical with only the exploration direction reversed.

5.1 The exploration process conditioning on a partial pairing

For a set of nodes $I$, let $E^\pm(I)$ denote the set of heads and tails incident to the nodes in $I$. When $I = \{v\}$, we also use $E^\pm(v) = E^\pm(I)$. Let $E^\pm := E^\pm([n])$ denote the set of all heads and tails respectively. For $X \subseteq E^\pm$, let $V(X)$ denote the set of vertices incident to $X$. When $X = \{e\}$, we also use $v(e)$ to denote the only element of $V(e)$.

Let $H$ be a partial pairing of $E^\pm$. Let $P^\pm(H) \subseteq E^\pm$ be the set of heads and tails that have been paired in $H$ and write $V(H) = V(P^\pm(H))$ for the set of nodes incident to $P^\pm(H)$. Let $F^\pm(H) := E^\pm(V(H)) \setminus P^\pm(H)$ be the unpaired heads and tails that are incident to $V(H)$. Let $E_H$ denote the event that $H$ is part of the final half-edge pairing in $\bar{G}_n$. We will explore the graph conditioning on $E_H$.

The exploration starts from an arbitrary tail $e^+ \in E^+ \setminus E^+(V(H))$. Let $v_0 = v(e^+)$. In this process, we create random pairings of half-edges one by one and keep each half-edge in exactly one of the four states — active, paired, fatal or undiscovered.

More precisely, let $A^\pm_i$, $P^\pm_i$, $F^\pm_i$ and $U^\pm_i$ denote the set of heads and tails in the four states respectively after the $i$-th pairing of half-edges. Initially we have

$$A^+_0 = \{e^+\}, \quad A^-_0 = E^-(v_0), \quad P^+_0 = P^\pm(H), \quad F^+_0 = F^\pm(H), \quad U^+_0 = E^\pm \setminus (A^+_0 \cup P^\pm_0 \cup F^\pm_0).$$

Then we set $i = 1$ and run the following procedure:

(i) Let $e^+_i$ be the tail which became active earliest in $A^+_i$. (If multiple such tails exist, choose an arbitrary one among them. Note that $e^+_1 = e^+$.)

(ii) Pair $e^+_i$ with a head $e^-_i$ chosen uniformly at random from $E^- \setminus P^-_{i-1}$, i.e., from all unpaired heads. Let $P^\pm_i = P^\pm_{i-1} \cup \{e^+_i\}$.

(iii) If $e^-_i \in F^-_{i-1}$, then terminate; if $e^-_i \in A^-_{i-1}$, then $A^\pm_i = A^\pm_{i-1} \setminus \{e^+_i\}$; and if $e^-_i \in U^-_{i-1}$, then $A^\pm_i = (A^\pm_{i-1} \cup E^\pm(v_i)) \setminus \{e^+_i\}$ where $v_i = v(e^-_i)$.

(iv) If $A^+_i = \emptyset$, then terminate; otherwise set $F^\pm_i = F^\pm_{i-1}, U^\pm_i = E^\pm \setminus (A^\pm_i \cup P^\pm_i \cup F^\pm_i), i = i + 1$ and go to (i).

(Note that $F^\pm_i$ does not change in the process. We keep the subscript only to make the notations consistent.) In words, the exploration process exposes edge by edge of $\bar{G}_n$ in a BFS order and stops either when it hits $V(H)$ or when all tails that can be reached from $e^+$ have been paired.

In parallel to the exploration process, we construct a sequence of rooted trees $T_{e^+}(i)$, whose nodes represent tails in $E^+$. Let $T_{e^+}(0)$ be a tree with a single node corresponding to $e^+$. We construct $T_{e^+}(i)$ as follows: if $e^-_i \in U^-_{i-1}$, then construct $T_{e^+}(i)$ from $T_{e^+}(i-1)$ by adding $|E^+(v_i)|$ child nodes to the node representing $e^+_i$, each one representing a tail in $E^+(v_i)$; otherwise, let
\( T_{e^+}(i) = T_{e^+}(i - 1) \). See Figure 2 for an example of the exploration process and the corresponding tree.

Given half-edges \( e_1, e_2 \), we define the distance from \( e_1 \) to \( e_2 \), denoted by \( \text{dist}(e_1, e_2) \), to be the length of the shortest path from \( v(e_1) \) to \( v(e_2) \) which starts with the edge containing \( e_1 \) if \( e_1 \) is a tail, and which ends with the edge containing \( e_2 \) if \( e_2 \) is head. For example, in Figure 2, \( \text{dist}(e_1^+, e_5^+) = 2 \) and \( \text{dist}(e_2^-, e_2^-) = 1 \).

**Observation 5.1.** Note that the tree \( T_{e^+}(i) \) preserves distances of tails in \( \vec{G}_n \): if a node corresponding to a tail is at distance \( t \) from the root, then the tail is at distance \( t \) from \( e^+ \) in \( \vec{G}_n \). Therefore, the number of nodes in the \( t \)-th level of the tree is the number of tails in \( E^+ \) at distance \( t \) from \( e^+ \).

While \( T_{e^+}(i) \) is an unlabelled tree, its set of nodes corresponds to the set of tails \( P_i^+ \cup A_i^+ \). Therefore, we can assign a label *paired* or *active* to each node.

![Figure 2: An ongoing exploration process and its associated tree](image)

We split the exploration process into epochs. At the \( t \)-th epoch, we pair all the tails at distance \( t \) from \( e^+ \). Let \( i_t \) be the last step of epoch \( t \). Then, \( T_{e^+}(i_t) \) has the following properties: (i) has depth \( t \); (ii) all nodes in the \( t \)-th level are active; and (iii) all nodes in the \( j \)-th level for \( j < t \) are paired.

We call a rooted tree \( T \) *incomplete* if it satisfies (i)-(iii) (in the sense that the subtrees rooted at the last level have not been decided yet). We let \( p(T) \) be the number of *paired* nodes in \( T \).

### 5.2 Coupling the exploration and branching processes

Let \( Q_n := (D_n)_{in}^+ \) be the distribution obtained by taking the marginal on the second component of the in-size biased distribution of \( D_n \); i.e., for all \( \ell \geq 0 \),

\[
P\{Q_n = \ell\} = q_{n,\ell} := \frac{\sum_{k \geq 1} k n_{k,\ell}}{m_n}.
\]

Recall that in Section 4 it has been shown that \( Q_n \to D_{in}^+ \) in distribution and in expectation. In particular, by (4.1) \( \mathbb{E}[Q_n] \to \mathbb{E}[D_{in}^+] = \nu \). Let \( \hat{Q}_n \) be the conjugate of \( Q_n \). It follows from Lemma 3.2 that \( \hat{Q}_n \xrightarrow{d} \hat{D}_{in}^+ \) and \( \mathbb{E}[\hat{Q}_n] \to \mathbb{E}[\hat{D}_{in}^+] = \hat{\nu}_+ \). Hence, \( \mathbb{E}[Q_n] = (1 + o(1))\nu \) and \( \mathbb{E}[\hat{Q}_n] = (1 + o(1))\hat{\nu}_+ \).
In order to transfer the results from branching processes to the graph exploration process, we need to introduce two new probability distributions $Q^\downarrow_n$ and $Q^\uparrow_n$ by slightly perturbing $Q_n$.

For $\beta \in (0, 1/10)$ consider the probability distribution $Q^\downarrow_n = Q^\downarrow_n(\beta)$ defined by

$$
\mathbb{P}\left\{ Q^\downarrow_n = \ell \right\} = q^\downarrow_{n, \ell} := \begin{cases}
    c^\downarrow q_{n, \ell} & \text{if } q_{n, \ell} \geq n^{-2\beta} \text{ and } \ell \leq n^\beta \\
    0 & \text{otherwise}
\end{cases}
$$

where $c^\downarrow$ is a normalizing constant. It is easy to check that Condition 1.1 implies that $c^\downarrow = 1 + o(n^{-\beta})$ and

$$
\sum_{\ell > n^\beta} \ell q_{n, \ell} = o(1), \quad \sum_{\ell, q_{n, \ell} < n^{-2\beta}} \ell q_{n, \ell} = o(1).
$$

Therefore, $\mathbb{E}[Q^\downarrow_n] \to \nu$.

Similarly, the probability distribution $Q^\uparrow_n = Q^\uparrow_n(\beta)$ is defined by

$$
\mathbb{P}\left\{ Q^\uparrow_n = \ell \right\} = q^\uparrow_{n, \ell} := \begin{cases}
    c^\uparrow q_{n, \ell} & \ell \geq 1 \\
    c^\uparrow q_{n, 0} + n^{-1/2+2\beta} & \ell = 0
\end{cases}
$$

where $c^\uparrow = 1 + O(n^{-1/2+2\beta})$ is a normalizing constant. Again, by Condition 1.1, $\mathbb{E}[Q^\uparrow_n] \to \nu$.

**Observation 5.2.** Let $\hat{Q}^\downarrow_n$ be the conjugate distribution of $Q^\downarrow_n$. It follows from Lemma 3.2 that $\hat{Q}^\downarrow_n \overset{d}{\to} D^\downarrow_{\text{in}}$ and $\mathbb{E}[\hat{Q}^\downarrow_n] \to \mathbb{E}[D^\downarrow_{\text{in}}] = \hat{\nu}_+$. So we can apply Theorem 3.5 to a branching process with offspring distribution $Q^\downarrow_n$ by taking $\hat{\nu}_\xi = (1 + o(1))\hat{\nu}_+$. The same applies to $Q^\uparrow_n$ and $\hat{Q}^\uparrow_n$.

Let $GW_\xi$ be a Galton-Watson tree with offspring distribution $\xi$. For an incomplete rooted tree $T$ of depth $t$, we use the notation $GW_\xi \cong T$ to denote that $T$ is a root subtree of $GW_\xi$ and all paired nodes of $T$ have the same degree in $GW_\xi$.

**Lemma 5.3.** Let $\beta \in (0, 1/10)$ and let $H$ be a partial pairing with $|\mathcal{X}(H)| \leq n^{1-6\beta}$. For every incomplete tree $T$ with $p(T) \leq n^\beta$, we have

$$(1 + o(1))\mathbb{P}\left\{ GW_{Q^\downarrow_n(\beta)} \cong T \right\} \leq \mathbb{P}\left\{ T_{\mathcal{P}^+}(p(T)) = T \mid E_H \right\} \leq (1 + o(1))\mathbb{P}\left\{ GW_{Q^\uparrow_n(\beta)} \cong T \right\} ,$$

where the implicit functions $o(1)$ are uniform over all such $T$ and $H$.

**Proof.** We start with the upper bound. First, by Lemma 2.2, $|\mathcal{X}(H)| \leq n^{1-6\beta}$ implies that

$$|\mathcal{P}^-(H)| \leq d_{\mathcal{X}(H)}(1, 0) = o(n^{1-3\beta}).$$

Let $X_i = |\mathcal{A}^+_i| - |\mathcal{A}^+_{i-1}| + 1$ be the number of tails that became active during the $i$-th step of the process. Let $E_i$ be the number of children of the $i$-th node in $T$ in the BFS order. Let $E_i = E_H \cap [X_j = \ell,j]$. Let $q_{n, \ell}(i) := \mathbb{P}\{ X_i = \ell \mid E_{i-1} \}$. Then for all $\ell \geq 1$ and $i \leq p(T)$,

$$q_{n, \ell}(i) \leq \frac{\sum_{k \geq 1} k n_{k, \ell}}{m_n - (i - 1) - |\mathcal{P}^-(H)|} = (1 + o(n^{-3\beta})) q_{n, \ell} = (1 + o(n^{-3\beta})) q^\downarrow_{n, \ell}.$$

Recall that by Corollary 2.4 we have $\Delta_n = o(\sqrt{n})$. Since at most $p(T)\Delta_n = o(n^{1/2+\beta})$ heads are active, we also have

$$q_{n, 0}(i) \leq \frac{p(T)\Delta_n + \sum_{k \geq 1} k n_{k, 0}}{m_n - (i - 1) - |\mathcal{P}^-(H)|} = (1 + o(n^{-3\beta})) q_{n, 0} \leq (1 + o(n^{-\beta})) q^\uparrow_{n, 0},$$
where the last step uses that \( q_{n,0}^+ \geq n^{-1/2+2\beta} \). It follows that

\[
\mathbb{P} \left\{ T_e^+ (p(T)) = T \mid E_H \right\} = \prod_{i=1}^{p(T)} q_{n,\ell}(i) \leq \prod_{i=1}^{p(T)} (1 + o(n^{-\beta})) q_{n,\ell}^+ = (1 + o(1)) \mathbb{P} \left\{ GW_{Q_n^+} (\beta) \geq T \right\}.
\]

We now prove the lower bound. We first show that \( q_{n,\ell}(i) \geq (1 + o(n^{-\beta})) q_{n,\ell}^+ \). We may assume that \( \ell \leq n^\beta \) and \( q_{n,\ell} \geq n^{-2\beta} \), as otherwise \( q_{n,\ell}^+ = 0 \) and the claim holds trivially. Since there are at most \( p(T) \Delta_n + d_{y(H)} (1, 0) = o(n^{1-3\beta}) \) heads in \( \mathcal{A}_{i-1} \cup \mathcal{P}_{i-1} \cup \mathcal{F}_{i-1} \) and at most \( m_n \) heads that have not been paired yet, we have for \( \ell \geq 0, \)

\[
q_{n,\ell}(i) \geq \sum_{k \geq 1} k n k,\ell - p(T) \Delta_n - d_{y(H)} (1, 0) = q_{n,\ell} - o(n^{-3\beta}) = (1 + o(n^{-\beta})) q_{n,\ell}^+ = (1 + o(n^{-\beta})) q_{n,\ell}^+,
\]

where the last step uses that \( q_{n,\ell} = (1 + o(n^{-\beta})) q_{n,\ell}^+ \). The rest of the argument is analogous to the upper bound. \( \square \)

6 Thin neighbourhoods

Given \( e_1^\pm, e_2^\pm \in E^{\pm} \), recall that \( \text{dist}(e_1^\pm, e_2^\pm) \) is the distance between the nodes incident to them in \( \mathcal{G}_n \). Let \( \mathcal{N}_t^\pm (e^\pm) \) be the set of heads/tails at distance \( t \) from a head/tail \( e^\pm \) in \( \mathcal{G}_n \). We will only use tail-neighbourhoods of tails or head-neighbourhoods of heads. Thus, we talk about the edge-neighbourhood of a tail/head and denote it by \( \mathcal{N}_t(e^\pm) \).

Throughout the rest of the paper, we fix \( \omega := \log^6 n \).

6.1 Supercritical case

In this subsection, we assume that \( \nu > 1 \). Let \( t_{\omega}(e^\pm) \) be the first time that the edge-neighbourhood of \( e^\pm \) has size at least \( \omega \), i.e.,

\[
t_{\omega}(e^\pm) := \inf \left\{ t \geq 1 : |\mathcal{N}_t(e^\pm)| \geq \omega \right\}.
\]

We call \( t_{\omega}(e^\pm) \) the expansion time of \( e^\pm \). If such expansion never happens, let \( t_{\omega}(e^\pm) = \infty \).

When \( \hat{\nu}_\pm > 0 \) and \( \nu > 1 \), we are interested in expansions which happen at around time

\[
t^\pm := \frac{\log(n)}{\log(1/\hat{\nu}_\pm)}.
\]

More specifically, we will show that \( \text{whp} \) there is no expansion after time \( (1 + \delta) t^\pm \) and there exist out/in explorations expanding after time \( (1 - \delta) t^\pm \), producing atypically thin neighbourhoods.

When \( \hat{\nu}_\pm = 0 \), we will show that \( \text{whp} \), the expansion of out/in-neighbourhood happens before time \( \delta \log(n) \) for all \( \delta > 0 \).

The proof relies on Lemma 5.3, which allows approximating the probability of finding a thin neighbourhood with the probability of the corresponding event in a branching process.
Proposition 6.1. Assume that $\nu > 1$. Let $\gamma > 0$ and $e^\pm \in \mathcal{E}^\pm$. Let
\[
A(e^\pm, t) = [\cap_{r=1}^t [0 < |\mathcal{V}_r(e^\pm)| < \omega]]
\]
Then uniformly for every partial pairing $H$ with $|\mathcal{V}(H)| \leq n^{1-\gamma}$ and every $t = \Theta(\log n)$, we have
\[
\Pr \{ A(e^\pm, t) \mid E_H \} = \begin{cases} 
\hat{\nu}_\pm^{(1+o(1))t} & (\hat{\nu}_\pm > 0) \\
O(\zeta^t) & (\hat{\nu}_\pm = 0)
\end{cases} \quad (6.2)
\]
for all $\zeta > 0$.

Proof. We only prove the upper bounds for $A(e^\pm, t)$ in (6.2). The lower bound follows from similar arguments.

Let $\mathcal{T}_{t,\omega}$ be the class of incomplete trees of depth $t$ where each level has less than $\omega$ nodes. For $T \in \mathcal{T}_{t,\omega}$, we have $t-1 \leq p(T) \leq |T| \leq (\omega-1)t = o(n^{\gamma/6})$. Let $X_t^\nu$ be the size of the $r$-th generation of a branching process (Galton-Watson tree) with offspring distribution $H$. Then uniformly for every partial pairing $H$ with $|\mathcal{V}(H)| \leq n^{1-\gamma}$ and every $t = \Theta(\log n)$, we have
\[
\Pr \{ A(e^\pm, t) \mid E_H \} = \begin{cases} 
\hat{\nu}_\pm^{(1+o(1))t} & (\hat{\nu}_\pm > 0) \\
O(\zeta^t) & (\hat{\nu}_\pm = 0)
\end{cases} \quad (6.2)
\]
for all $\zeta > 0$.

The following lemma shows that whp no expansion happens later than $(1+\delta)t^\pm$, for $\delta > 0$.

Lemma 6.2. Assume that $\nu > 1$. Let $\delta \in (0, 1)$ and let
\[
B_1(e^\pm) = B_1(e^\pm; \delta) = \begin{cases} 
A(e^\pm, (1+\delta)t^\pm) & (\hat{\nu}_\pm > 0) \\
A(e^\pm, \delta \log(n)) & (\hat{\nu}_\pm = 0)
\end{cases}
\]
Let $B_1 = \cup_{e^\pm \in \mathcal{E}} B_1(e^\pm)$. Then $\Pr \{ B_1 \} = o(1)$.

Proof. By taking $H$ to be an empty partial pairing, it follows from Proposition 6.1 that when $\hat{\nu}_\pm > 0$
\[
\Pr \{ A(e^\pm, (1+\delta)t^\pm) \} \leq \hat{\nu}_\pm^{(1-\delta/2)(1+\delta)t^\pm} \leq \hat{\nu}_\pm^{(1+\delta/4)t^\pm} \leq n^{-(1+\delta/4)},
\]
and when $\hat{\nu}_\pm = 0$
\[
\Pr \{ A(e^\pm, \delta \log(n)) \} = O(\zeta^{\delta \log(n)}) \leq n^{-(1+\delta/4)},
\]
by choosing $\zeta$ small enough with respect to $\delta$. The lemma follows from a union bound over all half-edges. \qed
Lemma 6.3. Assume that $\nu > 1$. Let $\delta \in (0, 1)$ and $e^\pm \in \mathcal{E}^\pm$.

(i) If $\hat{\nu}_\pm > 0$, let
\[
B_2(e^\pm) = B_2(e^\pm; \delta) = \left[ \left\lfloor \frac{t_\omega(e^\pm)}{t^\pm} \right\rfloor - 1 < \delta \right] \cap \left[ |N_{t_\omega(e^\pm)}(e^\pm)| < \omega^2 \right].
\]

Then uniformly for every partial paring $H$ with $|\mathcal{V}(H)| \leq n^{1-\delta/5}$, we have
\[
 n^{-1+\delta/2} \leq \mathbb{P} \left\{ B_2(e^\pm) \mid E_H \right\} \leq n^{-1+3\delta/2}.
\]

(ii) If $\hat{\nu}_\pm = 0$, let
\[
B_2(e^\pm) = B_2(e^\pm; \delta) = \left[ t_\omega(e^\pm) < \delta \log(n) \right] \cap \left[ |N_{t_\omega(e^\pm)}(e^\pm)| < \omega^2 \right].
\]

Then uniformly for every partial paring $H$ with $|\mathcal{V}(H)| \leq n^{1-\delta/5}$, we have
\[
\mathbb{P} \left\{ B_2(e^\pm) \mid E_H \right\} = 1 - o(1).
\]

Proof of (i). We only bound the probability of $B_2(e^+)$; the proof for $B_2(e^-)$ is analogous. Let $t_1 = \left\lfloor (1-\delta)t^+ \right\rfloor$ and $t_2 = \left\lfloor (1+\delta)t^+ \right\rfloor$. Let
\[
A_1 = A(e^+, t_1), \quad A_2 = A(e^+, t_2), \quad A_3 = \left[ N_{t_2}(e^+) = \emptyset \right],
\]
\[
A_4 = \left[ t_\omega(e^+) < \infty \right] \cap \left[ |N_{t_\omega(e^+)}(e^+)| \geq \omega^2 \right].
\]

Using Proposition 6.1 with $\gamma = \delta/5$, we have
\[
\mathbb{P} \left\{ B_2(e^+) \mid E_H \right\} \leq \mathbb{P} \left\{ A_1 \mid E_H \right\} \leq n^{-1+3\delta/2}.
\]

We now prove the lower bound. If $A_1$ happens, then there are three cases in which $B_2(e^+)$ does not happen: (i) the neighbourhood of $e^+$ survives but does not expand by time $t_2$; (ii) the neighbourhood dies by time $t_2$; (iii) the neighbourhood expands by time $t_2$, but expands by too much. Thus
\[
\mathbb{P} \left\{ B_2(e^+) \mid E_H \right\} \geq \mathbb{P} \left\{ A_1 \mid E_H \right\} - \mathbb{P} \left\{ A_2 \mid E_H \right\} - \mathbb{P} \left\{ A_1 \cap A_3 \mid E_H \right\} - \mathbb{P} \left\{ A_1 \cap A_2 \cap A_4 \mid E_H \right\} \quad (6.3)
\]
\[
= \mathbb{P} \left\{ A_1 \mid E_H \right\} \left( 1 - \mathbb{P} \left\{ A_3 \mid A_1 \cap E_H \right\} - \mathbb{P} \left\{ A_2 \cap A_4 \mid A_1 \cap E_H \right\} \right) - \mathbb{P} \left\{ A_2 \mid E_H \right\}.
\]

It follows directly from Proposition 6.1 with $\gamma = \delta/5$ that
\[
\mathbb{P} \left\{ A_1 \mid E_H \right\} \geq n^{-1+3\delta/4}, \quad \mathbb{P} \left\{ A_2 \mid E_H \right\} \leq n^{-1-3\delta/4}. \quad (6.4)
\]

Let $X^+_t$ be the size of the $t$-th generation of a branching process (Galton-Watson tree) with offspring $Q^+_0$. Consider the analogue of the events $A_i$ for $X^+_t$:
\[
A^*_1 = \left[ \bigcap_{i=1}^{t_1} [0 < X^+_i < \omega] \right]
\]
\[
A^*_2 = \left[ \bigcap_{i=1}^{t_2} [0 < X^+_i < \omega] \right]
\]
\[
A^*_3 = \left[ X^+_{t_2} = 0 \right]
\]
\[
A^*_4 = [t_\omega < \infty, X^+_{t_\omega} \geq \omega^2]
\]
where \( t_\omega \) is the smallest \( t \) such that \( X^+_t \geq \omega \), or \( t_\omega = \infty \) if it does not exist. One can transfer the probability of any event for branching processes to the corresponding event in the graph exploration process conditional on \( E_H \) in a similar way as in the proof of Proposition 6.1. Thus, it suffices to upper bound the remaining probabilities in (6.3) for the branching process analogues.

Since the survival probability of \( X^+_t \) tends to \( s_+ \), we have

\[
P \left\{ A_{n}^+ \mid A^+_t \right\} \leq P \left\{ X^+_{t_2-t_1+1} = 0 \right\} \leq 1 - s_+ + o(1). \tag{6.5}
\]

By Markov inequality, for any \( t \geq t_1 \),

\[
P \left\{ X^+_{t+1} \geq \omega^2 \left| A^+_1 \cap [X^+_t < \omega] \right. \right\} \leq \frac{E \left[ X^+_{t+1} \left| A^+_1 \cap [X^+_t < \omega] \right. \right]}{\omega^2} \leq \frac{(1 + o(1))\nu_\omega}{\omega^2} = O(\omega^{-1}).
\]

The event \( A^+_1 \cap (A^+_2)^c \cap A^+_t \) implies that there exists \( t \in [t_1, t_2] \) such that \( X^+_t < \omega \) and \( X^+_{t+1} \geq \omega^2 \). Therefore,

\[
P \{ (A^+_2)^c \cap A^+_1 \mid A^+_t \} \leq \sum_{t=t_1}^{t_2-1} P \left\{ X^+_{t+1} \geq \omega^2 \left| A^+_1 \cap [X^+_t < \omega] \right. \right\} = O(\omega^{-1} \log n) = o(1). \tag{6.6}
\]

Then part (i) of the lemma follows by transferring the probabilities to the original events conditional on \( E_H \) and putting (6.4), (6.5) and (6.6) into (6.3).

Proof of (ii). Again we only bound the probability of \( B_2(e^+) \). Let \( t_2 = \lfloor \delta \log n \rfloor \). Let

\[
A_1 = A(e^+, t_2), \quad A_2 = [\mathcal{N}_t(e^+) = \emptyset], \quad A_3 = [t_\omega(e^+) < \infty] \cap [\mathcal{N}_{t_\omega(e^+)}(e^+) \geq \omega^2].
\]

When \( B_2(e^+) \) does not happen, there are three (non-exclusive) cases: (i) the neighbourhood of \( e^+ \) survives till \( t_2 \) but does not expand; (ii) the neighbourhood of \( e^+ \) dies by time \( t_2 \); (iii) the neighbourhood of \( e^+ \) expands too much. Thus

\[
P \left\{ B_2(e^+)^c \mid E_H \right\} \leq P \{ A_1 \mid E_H \} + P \{ A_2 \mid E_H \} + P \{ A_3 \mid A^+_1 \cap A^+_2 \cap E_H \}. \tag{6.7}
\]

Note that it follows from Proposition 6.1 that \( P \{ A_1 \mid E_H \} = o(1) \).

Let \( X^+_t \) and \( t_\omega \) be as in the proof of (i), where the conjugate is defined as in (3.3). Consider the branching process analogue of the events \( A_i \) for \( X^+_t \):

\[
A^+_1 = \left[ \cap_{t=1}^{t_2} \left( 0 < X^+_t < \omega \right) \right], \quad A^+_2 = \left[ X^+_{t_2} = 0 \right], \quad A^+_3 = \left[ t_\omega < \infty, X^+_{t_\omega} \geq \omega^2 \right].
\]

By the same argument as in (i), it suffices to compute the remaining probabilities in (6.7) for the branching process analogues.

Note that \( \nu_+ = 0 \) implies \( s_+ = 1 \). Thus

\[
P \{ A^+_3 \} \leq P \left\{ (X^+_t)_{t \geq 0} \text{ extinguishes} \right\} \to 1 - s_+ = 0. \tag{6.8}
\]

When \( (A^+_1)^c \cap (A^+_2)^c \) happens, there must be the expansion before time \( \delta \log n \). Thus by an argument similar to that of (6.6), we also have

\[
P \left\{ A^+_3 \mid (A^+_1)^c \cap (A^+_2)^c \right\} = O(\omega^{-1} \log n) = o(1). \tag{6.9}
\]

Then part (ii) of the lemma follows by transferring the probabilities to the original events conditional on \( E_H \) and putting (6.8), (6.9) into (6.7).
Next we show that whp there exist thin out-neighbourhoods and thin in-neighbourhoods of expected height which do not intersect.

**Lemma 6.4.** Assume that \( \nu > 1 \). Let \( B_2(e^\pm) \) be as in Lemma 6.3. Define

(i) if \( \hat{\nu}_+ > 0 \) and \( \hat{\nu}_- > 0 \), for every \( e^+ \in \mathcal{E}^+ \) and \( e^- \in \mathcal{E}^- \)

\[
B_3(e^+, e^-) = B_3(e^+, e^-; \delta) = B_2(e^+; \delta) \cap B_2(e^-; \delta) \cap \left[ \text{dist}(e^+, e^-) \geq t_\omega(e^+) + t_\omega(e^-) \right],
\]

and

\[
B_3 = \bigcup_{e^+ \in \mathcal{E}^+} \bigcup_{e^- \in \mathcal{E}^-} B_3(e^+, e^-).
\]

(ii) if \( \hat{\nu}_+ > 0 \) and \( \hat{\nu}_- = 0 \), let

\[
B_3 = \bigcup_{e^+ \in \mathcal{E}^+} B_2(e^+)
\]

(iii) if \( \hat{\nu}_+ = 0 \) and \( \hat{\nu}_- > 0 \), let

\[
B_3 = \bigcup_{e^- \in \mathcal{E}^-} B_3(e^-)
\]

Then \( \mathbb{P}\{B_3\} = 1 - o(1) \).

**Proof.** We will only prove (i); case (ii) and (iii) are proved analogously. Our proof is algorithmic and divided into two phases. Firstly we have the out-phase, where we run the exploration process described in Subsection 5.1 repeatedly until the desired thin out-neighbourhood appears. Secondly we have the in-phase, where we run the exploration process in reversed direction repeatedly until we find a thin in-neighbourhood, disjoint from the previous one. Although the probability of success in each trial is small, we can show that the probability of eventual success goes to 1. Without loss of generality, we can assume that the half-edges are ordered in some arbitrary way.

We provide more details. Let \( H_0^+ \) be the empty partial pairing. At the \( \ell \)-th trial for \( \ell \leq n^{1-\delta/4} \), choose \( e^+ \) to be the smallest unpaired tail and run the exploration process in Subsection 5.1 from \( e^+ \), epoch by epoch, and conditioning on \( E_{H_{\ell-1}^+} \). Recall that the process terminates when we hit \( H_{\ell-1}^- \) or when all tails that can be reached from \( e^+ \) have been paired. We add two extra termination conditions that are checked at the end of each epoch \( t \): (i) \( t \geq (1 + \delta)t^+ \) and (ii) \( |N_t(e^+)\| \geq \omega \) (or equivalently, \( |A_t^+| \geq \omega \)). If the process terminates with condition (ii), \( t \geq (1 - \delta)t^+ \) and \( |N_t(e^+)\| < \omega^2 \), then \( B_2(e^+) \) holds, we declare the \( \ell \)-th trial (and the out-phase) a success and proceed to the in-phase. Otherwise, we declare the trial a failure, obtain \( H_{\ell-1}^+ \) from \( H_{\ell-1}^- \) by adding the new pairs, set \( \ell \) to \( \ell + 1 \) and restart the exploration process from the smallest unpaired tail. If \( \ell > n^{1-\delta/4} \), then we declare out-phase a failure and terminate.

If the out-phase succeeded, let \( H_0^- \) be the partial pairing obtained after the successful trial. We start the in-phase and run at most \( n^{1-\delta/4} \) trials of the exploration process in reverse direction, where in the \( \ell \)-th trial we condition on \( E_{H_{\ell-1}^-} \).

If both phases succeed, then we have found \( e^\pm \) with \( B_2(e^\pm) \), and by the definition of \( H_{\ell}^- \), these neighbourhoods are disjoint.

Let us compute the probability that the out-phase fails. Let \( F_{\ell}^+ \) denote the event that the \( \ell \)-th trial in the out-phase failed, which implies the event \( B_2(e^+)^c \). In each trial of the exploration
process and regardless of whether it is a success or failure, at most \(O(\log(n)) = O(\log^7 n)\) half-edges are paired (although the number of half-edges that have been activated might be larger). As there are at most \(n^{1-\delta/4}\) trials, the event \(\cap_{j=1}^{\ell} E_j^+\) implies \(|\mathcal{V}(H_\ell^+)| = O(n^{1-\delta/4} \log^7 n) < n^{1-\delta/5}\). It follows from Lemma 6.3 that regardless of \(\hat{\nu}_+ > 0\) or \(\hat{\nu}_+ = 0\)

\[
P \left\{ F_\ell^+ \left| \cap_{j=1}^{\ell} E_j^+ \right. \right\} \leq 1 - n^{-1+\delta/2}.
\]

(In the case \(\hat{\nu}_+ = 0\), this is actually \(o(1)\).) Therefore

\[
P \left\{ \bigcap_{t=1}^{[n^{1-\delta/4}]} F_t^+ \right\} = \prod_{t=1}^{[n^{1-\delta/4}]} P \left\{ F_t^+ \left| \cap_{j=1}^{t-1} F_j^+ \right. \right\} \leq \left(1 - n^{-1+\delta/2}\right)^{n^{1-\delta/4}-1} \to 0.
\]

Thus, the probability that the out-phase fails is \(o(1)\). By a similar argument, the probability that the in-phase fails is also \(o(1)\), concluding the proof.

\section{Subcritical case}

When \(\nu \in (0, 1)\), i.e., in the subcritical case, we have \(s_\pm = 0\) and \(\hat{\nu}_\pm = \nu\). Thus \(t^+ = t^- = \log_{1/\nu} n\). By considering analogous events in branching process, we can show that whp the neighbourhoods of all half-edges die before time \((1 + \delta)t_\pm\) and that there exist half-edges \(e_\pm\) whose neighbourhood is of height \((1 - \delta)t_\pm\).

\textbf{Lemma 6.5.} Assume that \(0 < \nu < 1\). Define the event \(A'(e_\pm, t) := |N_t(e_\pm)| > 0\).

(i) Let \(B_4 = B_4(\delta) = \cup_{e_\pm \in e_\pm} A'(e_\pm, (1 + \delta)t_\pm)\). Then \(P \{ B_4 \} = o(1)\).

(ii) Let \(B_5 = B_5(\delta) = \cup_{e_\pm \in e_\pm} A'(e_\pm, (1 - \delta)t_\pm)\). Then \(P \{ B_5 \} = 1 - o(1)\).

\textbf{Proof.} The proof of (i) is analogous to that of Lemma 6.2. Let \((X_t^\pm)_{t \geq 0}\) be a branching process with offspring distribution \(Q_t^\pm\). Then it follows from Theorem 3.7 and Lemma 5.3 that

\[
P \left\{ A'(e_\pm, (1 + \delta)t_\pm) \right\} \leq (1 + o(1))P \left\{ X_{(1+\delta)t_\pm}^\pm > 0 \right\} \leq n^{-1-\delta/2}.
\]

Thus \(P \{ B_4 \} = o(1)\) follows from a union bound over all half-edges.

The proof of (ii) is analogous to that of Lemma 6.4. We start with the smallest tail and explore its neighbourhood. If its neighbourhood either dies or reaches total size \(\omega\) before time \((1 - \delta)t_\pm\), then we call it a failure and restart the exploration from the smallest unpaired tail. Otherwise we call it a success and terminate. The probability of success in one trial is at least \(n^{-1+\delta/2}\) by the same argument as in (i). Thus by repeating the process \(n^{1-\delta/4}\) times, whp we eventually succeed.

\section{Distance between two sets of edges in the supercritical regime}

In this section, we show that whp, the distance between two modestly large sets of edges is about \(\log_\nu(n)\), given that \(\nu > 1\).

Let \(H\) be a partial pairing of half-edges as defined in Subsection 5.1. Recall that \(\mathcal{P}_\pm(H)\) is the set of paired heads and tails in \(H\), \(\mathcal{V}(H)\) is the set of nodes incident to half-edges in \(\mathcal{P}_\pm(H)\), and \(E_H\) is the event that \(H\) is a subset of the pairing \(\mathcal{G}_n\).

We consider a triplet \((H, X^+, X^-)\) satisfying the following condition:
**Condition 7.1.** $H$ is a partial pairing of $\mathcal{E}^\pm$ and $\mathcal{X}^\pm \subseteq \mathcal{E}^\pm(H) \setminus \mathcal{P}^\pm(H)$.

For $I \subseteq [n]$, let $\text{dist}(\mathcal{X}^+, \mathcal{X}^-, I)$ be the minimal length of paths from $\mathcal{X}^+$ to $\mathcal{X}^-$ using only nodes in $I$. The main result in this section is the following:

**Proposition 7.2.** Uniformly over all choices of $\varepsilon, \gamma > 0$ and $(H, \mathcal{X}^+, \mathcal{X}^-)$ satisfying **Condition 7.1**, $|\mathcal{V}(H)| \leq n^{1-\gamma}$ and $|\mathcal{X}^+|, |\mathcal{X}^-| \geq \omega$ we have

$$
P \{ \text{dist}(\mathcal{X}^+, \mathcal{X}^-, [n]) > (1 + \varepsilon) \log_n n \mid E_H \} = o(n^{-100}),$$

and, assuming in addition that $|\mathcal{X}^+|, |\mathcal{X}^-| \leq \omega^2$

$$
P \{ \text{dist}(\mathcal{X}^+, \mathcal{X}^-, [n] \setminus \mathcal{V}(H)) < (1 - \varepsilon) \log_n n \mid E_H \} = o(n^{-\varepsilon/2}).$$

### 7.1 Lower bound by path counting

We prove the lower bound (7.2) in Proposition 7.2 by a technique called *path counting* which was introduced by Janson in [21], see also [31].

Given $I \subseteq [n] \setminus \mathcal{V}(H)$, a *simple path* of length $k$ from $\mathcal{X}^+$ to $\mathcal{X}^-$ using nodes in $I$ is a sequence

$$
\Pi = \{ e^+, (v_1, e^-_1, e^+_1), \ldots, (v_{k-1}, e^-_{k-1}, e^+_{k-1}), e^- \},
$$

where $e^+ \in \mathcal{X}^+$, $e^- \in \mathcal{X}^-$, $v_i \in I$ are distinct nodes, and $e^-_i \in \mathcal{E}^-(v_i)$ and $e^+_i \in \mathcal{E}^+(v_i)$. Let $P_k(\mathcal{X}^+, \mathcal{X}^-, I)$ be the number of simple directed paths of length $k$ from some $\mathcal{X}^+$ to some $\mathcal{X}^-$ only using nodes in $I$. The following proposition is an adaptation of [31, Proposition 7.4] for the directed configuration model and provides an upper bound for the expected number of paths of certain length from $\mathcal{X}^+$ to $\mathcal{X}^-$ conditioning on $E_H$ using nodes outside $\mathcal{V}(H)$:

**Lemma 7.3.** Let $I \subseteq [n] \setminus \mathcal{V}(H)$. Let $r$ be the number of nodes $i \in I$ with $d^+_i d^-_i \geq 1$. Let $s = |\mathcal{P}^-((H))|$, i.e., the number of paired heads in $H$. For any $H$ any $k \in [|I|+1]$, we have

$$
n_{k,H}(\mathcal{X}^+, \mathcal{X}^-, I) := \mathbb{E} \left[ P_k(\mathcal{X}^+, \mathcal{X}^-, I) \mid E_H \right] \leq \frac{\nu^k I}{m_n - k - s + 1} \prod_{i=0}^{k-2} \frac{1 - \frac{i}{s}}{1 - \frac{s}{m_n}},$$

where $\nu_I$ is defined as in (2.3). (We use the convention that an empty product equals 1.)

**Proof.** It suffices to consider only the case that $\mathcal{X}^+ = \{ e^+ \}$, $\mathcal{X}^- = \{ e^- \}$ and show that

$$
n_{k,H}(e^+, e^-, I) \leq \frac{\nu^k I}{m_n - k - s + 1} \prod_{i=0}^{k-2} \frac{1 - \frac{i}{s}}{1 - \frac{s}{m_n}}.
$$

Then (7.3) follows by adding up the previous bound for all $e^+ \in \mathcal{X}^+$ and $e^- \in \mathcal{X}^-$. Conditioning on $E_H$, the exact probability for a given $\Pi$ to exist is

$$
\prod_{i=1}^{k} \frac{1}{m_n - i - s + 1}.
$$
If we fix \(v_1, \ldots, v_{k-1}\), then the number of simple path using them in the given order is exactly 
\[
\prod_{i=1}^{k-1} d^-(v_i)d^+(v_i).
\]
Let \(I^k\) be the set of sequences of distinct nodes in \(I\) of length \(k\). Thus
\[
n_{k,H}(e^+, e^-, I) = \frac{1}{m_n-k-s+1} \sum_{(v_1, \ldots, v_{k-1}) \in I^k_{\ell}} \prod_{i=1}^{k-1} d^-(v_i)d^+(v_i).
\]

Let \(\mathcal{R} = \{i \in I : d^+_i d^-_i \geq 1\}\). Note that \(\nu = \nu_\ell\). Define \(\mathcal{R}^k\) as before. We use the following inequality of Maclaurin ([19, Theorem 52]), for \(r = |\mathcal{R}|\), \(1 \leq k-1 \leq r\) and \((a_i)_{i \in \mathcal{R}}\) with \(a_i \geq 0\), we have
\[
\frac{(r-k+1)!}{r!} \sum_{(\pi_1, \ldots, \pi_{k-1}) \in \mathcal{R}^k_{\ell}} k! \leq \left(\frac{1}{r} \sum_{i \in \mathcal{R}} a_i\right)^{k-1}.
\]
Applying the above inequality with \(a_{\pi_i} = d^-(v_i)d^+(v_i)\), we have
\[
n_{k,H}(e^+, e^-, I) \leq \frac{1}{m_n-k-s+1} \prod_{i=1}^{k-1} m_n - i - s + 1 \frac{r!}{(r-k+1)!} \left(\sum_{i \in \mathcal{R}} d^-(v_i)d^+(v_i)\right)^{k-1} \leq \frac{\nu_{\ell}^{k-1}}{m_n-k-s+1} \prod_{i=0}^{k-2} \frac{1 - i \nu_\ell}{1 - s \nu_\ell}.
\]

**Proof of lower bound in Proposition 7.2.** Let \(k = \left\lfloor (1 - \varepsilon) \log_\nu n \right\rfloor\) and \(I = [n] \setminus \mathcal{V}(H)\). Since \(|\mathcal{V}(H)| \leq n^{1-\gamma}\), by Lemma 2.2 we have \(s = |\mathcal{P}^-(H)| = o(n^{1-\gamma/2})\). Since \(|I| = n - o(n)\), it follows from Corollary 2.3 that \(\nu_\varepsilon = (1 + o(1))\nu\). Let \(r\) be as in Lemma 7.3. Then it follows from Lemma 7.3 that
\[
\mathbb{P} \{ \text{dist}(\mathcal{X}^+, \mathcal{X}^-, I) \leq k \mid E_H \} \leq \sum_{\ell=1}^k \mathbb{E} \left[ P_{E_H} (\mathcal{X}^+, \mathcal{X}^-, I) \mid E_H \right] \leq \sum_{\ell=1}^k \frac{|\mathcal{X}^+||\mathcal{X}^-|\nu_{\ell}^{k-1}}{m_n - \ell - s + 1} \prod_{i=0}^{k-1} \frac{1 - i \nu_{\ell}}{1 - s \nu_{\ell}} \leq O\left(n^{-\varepsilon} \log_\nu n\right) = o(n^{-\varepsilon/2}).
\]

### 7.2 Upper bound by bounded expansion

We prove the upper bound (7.1) in Proposition 7.2 by showing that a large set of half-edges typically expands with rate at least \(\nu\), even after conditioning on \(E_H\) for a small partial pairing \(H\). To this end, it suffices to consider only nodes with bounded degree, since deleting vertices can only increase the diameter.

Given \(\rho > 0\) that will be fixed later, choose \(K\) large enough so that
\[
\left(1 - \frac{\rho}{4}\right) \nu \leq \mathbb{E} \left[ D^+_{\text{in}} \mathbb{I}([D^+_{\text{in}} < K] \cap [D^-_{\text{in}} < K]) \right] \leq \nu. \tag{7.4}
\]

24
Note that such a $K$ exists as $\mathbb{E} \left[ D_{in}^+ \right] = \nu < \infty$ and $D_{in}^+$ takes non-negative values. Let $\mathcal{L}^\pm$ be the set of heads and tails incident to vertices with at least $K$ heads or at least $K$ tails.

Given $(H, \mathcal{X}^+, \mathcal{X}^-)$ satisfying Condition 7.1, we want to explore the edge out-neighbourhoods of $\mathcal{X}^+$ and the edge in-neighbourhoods of $\mathcal{X}^-$ using only nodes with small in- and out-degree, and conditioning on $E_H$. Let $\mathcal{N}_0^*(\mathcal{X}^+) = \mathcal{X}^+$ and for $k \geq 1$ define recursively

$$\mathcal{N}_k^*(\mathcal{X}^+) = \{e^+ \in \mathcal{E}^+ \setminus (\mathcal{L}^+ \cup \mathcal{N}_{\ast k}^*(\mathcal{X}^+) \cup \mathcal{F}^+(H)) \mid \exists f^+ \in \mathcal{N}_{k-1}^*(\mathcal{X}^+), f^- \in \mathcal{E}^-(v(e^+)), f^+f^- \text{ is a pair} \}$$

In words, $\mathcal{N}_k^*(\mathcal{X}^+)$ is the set of tails that can be reached from $\mathcal{X}^+$ by a path of length $k$ using only vertices of low in- and out-degree. Note that the distance in $\mathcal{E}_n$ from $\mathcal{X}^+$ to the elements of $\mathcal{N}_k^*(\mathcal{X}^+)$ is at most $k$. We define $\mathcal{N}_k^*(\mathcal{X}^-)$ in a similar way.

**Lemma 7.4.** Uniformly over all choices of $\gamma > 0$, $(H, \mathcal{X}^+, \mathcal{X}^-)$ satisfying Condition 7.1, $|\mathcal{V}(H)| \leq n^{1-\gamma}$ and $|\mathcal{X}^+|, |\mathcal{X}^-| \geq \omega$ we have:

(i) for all $k \leq \log_{1+\rho} (n^{1-\gamma}/|\mathcal{X}^+|)$

$$\mathbb{P} \left( \left| (1-\rho)\nu^k |\mathcal{X}^+| \leq |\mathcal{N}_k^*(\mathcal{X}^+) | \leq (1+\rho)\nu^k |\mathcal{X}^+| \mid E_H \right) = 1 - o(n^{-100}) \right); \quad (7.5)$$

(ii) for all $k \leq \log_{1+\rho} (n^{1-\gamma}/|\mathcal{X}^-|)$

$$\mathbb{P} \left( \left| (1-\rho)\nu^k |\mathcal{X}^-| \leq |\mathcal{N}_k^*(\mathcal{X}^-) | \leq (1+\rho)\nu^k |\mathcal{X}^-| \mid E_H \right) = 1 - o(n^{-100}) \right). \quad (7.6)$$

**Proof.** We will show (7.5), then (7.6) also holds by swapping $\mathcal{X}^+$ and $\mathcal{X}^-$ and reversing the direction of the edges.

For $k \geq 1$, let $r_k = |\mathcal{N}_k^*(\mathcal{X}^+) |$ and let $E_k$ denote the event

$$(1-\rho)\nu r_{k-1} \leq r_k \leq (1+\rho)\nu r_{k-1}. \quad (7.7)$$

We show that uniformly for all $k \leq \log_{1+\rho} (n^{1-\gamma}/|\mathcal{X}^+|),$

$$\mathbb{P} \left\{ E_k \mid E_H \cap \left[ \cap_{i=1}^{k-1} E_j \right] \right\} = 1 - o(n^{-1000}). \quad (7.8)$$

from which (7.5) follows directly.

We may assume that $|\mathcal{X}^+| \leq n^{1-\gamma}$, as otherwise $\log_{1+\rho} (n^{1-\gamma}/|\mathcal{X}^+|) < 0$ and there is nothing to prove. The event $\cap_{i=1}^{k-1} E_i$ implies that $r_{k-1} \leq n^{1-\gamma}$ and $\sum_{i=0}^{k-1} r_i + |\mathcal{V}(H)| = O(n^{1-\gamma}).$

Consider the exploration process defined in Subsection 5.1 and introduce the following three modifications:

- initially, we let $\mathcal{A}_0^+ = \mathcal{X}^+$ and $\mathcal{A}_0^- = \mathcal{E}^-(\mathcal{V}(\mathcal{X}^+));$
- in (iii), if $e_i^- \in \mathcal{L}^-$, then we let $\mathcal{A}_i^\pm = \mathcal{A}_{i-1}^\pm \setminus \{e_i^\pm\};$
- in (iii), if $e_i^- \in \mathcal{F}^-(H)$, we do not terminate the process and let $\mathcal{A}_i^\pm = \mathcal{A}_{i-1}^\pm \setminus \{e_i^\pm\}.$
Now the process generates a collection of rooted trees \( \{ T_e+(i) \}_{e \in \mathcal{X}^+} \). As in Observation 5.1, the union of the \( k \)-th level of each tree is equal to \( \mathcal{N}^+_k(\mathcal{X}^+) \) so it suffices to study the process.

Recall that \( i_k \) is the last time we pair a tail at distance \( k \) from \( \mathcal{X}^+ \), so the \( k \)-th epoch of the process goes from time \( i_{k-1} + 1 \) to \( i_k + r_{k-1} \), and \( r_k \) is precisely the number of tails that have been activated during this epoch. Let \( X_i = |A_i| - |A_{i-1}| + 1 \). The only way we activate new tails at the \( i \)-th step is if \( e_i^- \in U_{i-1}^- \setminus \mathcal{L}^- \), in which case we activate \( |E^+(v_i)| \) tails. Thus,

\[
X_i = |E^+(v_i)| 1(e_i^- \in U_{i-1}^- \setminus \mathcal{L}^-)
\]

and

\[
r_k = \sum_{i=i_k-1+1}^{i_k} X_i.
\]

All the heads that were paired or fatal at the beginning of the process, are incident to a vertex in \( \mathcal{V}(H) \). All heads that have been paired before the \( k \)-th epoch are incident to a vertex incident to \( \mathcal{N}^+_k(\mathcal{X}^+) \) and there are at most \( \sum_{i=0}^{k-1} r_i = O(n^1) \) such vertices. By applying Lemma 2.2 to the set of vertices incident to \( \mathcal{E}^- \setminus U_{i-1}^- \) we have \( |\mathcal{E}^- \setminus U_{i-1}^-| = o(n^{1-\gamma/2}) \).

Let \( H_{i-1} \) denote a history of the process before the \( i \)-th match that is compatible with \( E_H \cap \bigcap_{j=1}^{i-1} E_j \). Then, for all \( \ell \in \{0, \ldots, K-1\} \),

\[
\mathbb{P} \{ X_i = \ell | H_{i-1} \} = \frac{\sum_{e^- \in U_{i-1}^- \setminus \mathcal{L}^-} 1(\bar{d}(v(e^-)) = \ell)}{|m_n| - |\mathcal{P}^{-}(H)| - (i-1)}
\]

\[
\geq \frac{\sum_{e^- \in \mathcal{E}^- \setminus U_{i-1}^-} 1(\bar{d}(v(e^-)) = \ell)}{|m_n|} \geq \mathbb{P} \{ (D_n^+)_{in} = \ell \} \cap (D_n^-_{in} < K) \} - n^{-\gamma/4} =: b_{n,\ell}.
\]

Let \( \bar{X}_i \) and \( \hat{X}_i \) be two independent random variables with distributions

\[
\mathbb{P} \{ \bar{X}_i = \ell \} = \begin{cases} 1 - \sum_{j=1}^{K} (b_{n,j} \lor 0), & \text{if } \ell = 0 \\ b_{n,\ell} \lor 0, & \text{if } 1 \leq \ell < K, \\ 0 & \text{if } \ell \geq K \end{cases}
\]

and

\[
\mathbb{P} \{ \hat{X}_i = \ell \} = \begin{cases} b_{n,\ell} \lor 0, & \text{if } 0 \leq \ell < K - 1 \\ 1 - \sum_{j=0}^{K-1} (b_{n,j} \land 0), & \text{if } \ell = K - 1, \\ 0 & \text{if } \ell \geq K \end{cases}.
\]

Then \( \bar{X}_i \leq \text{st} (X_i | H_{i-1}) \leq \text{st} \hat{X}_i \). Moreover,

\[
\mathbb{E} \left[ X_i \right] = (1 + o(1)) \mathbb{E} \left[ D_{in}^+(D_n^+_{in} < K \cap [D_n^+ < K]) \right] - K n^{-\gamma/4} \geq \left( 1 - \frac{\rho}{2} \right) \nu \]

where the last step follows from our choice of \( K \) that satisfies (7.4). Similar computations show that \( \mathbb{E} \left[ \hat{X}_i \right] \leq (1 + \rho/2) \nu \).
Note that both \( \hat{X}_i \) and \( \bar{X}_i \) are bounded random variables so we can applying Hoeffding’s inequality [20]. It follows that
\[
\mathbb{P} \left\{ r_k < (1 - \rho)\nu r_{k-1} \mid \mathcal{H}_{i_{k-1}} \right\} \leq \mathbb{P} \left\{ \sum_{i=i_{k-1}+1}^{i_{k-1}+r_{k-1}} (\bar{X}_i - \mathbb{E} \bar{X}_i) > (\rho/2)\nu r_{k-1} \mid \mathcal{H}_{i_{k-1}} \right\} \\
\leq \exp \left( -\frac{\rho^2 \nu r_{k-1}}{8K^2} \right) = o \left( n^{-1000} \right).
\]
where we used that \( r_{k-1} \geq r_0 \geq \omega \) by the conditioning \( \mathcal{H}_{i_{k-1}} \). Similarly, using \( \hat{X}_i \), we obtain
\[
\mathbb{P} \left\{ r_k > (1 + \rho)\nu r_{k-1} \mid \mathcal{H}_{i_{k-1}} \right\} = o \left( n^{-1000} \right).
\]
This proves \((7.7)\) and \((7.5)\). The proof of \((7.6)\) is analogous.

**Proof of upper bound in Proposition 7.2.** Let \( \beta, \rho > 0. \) Let \( k^+ := \left\lceil \log_{(1-\rho)\nu} (n^{1/2+\beta}/|X^+|) \right\rceil \) and \( k^- := \left\lceil \log_{(1-\rho)\nu} (n^{1/2+\beta}/|X^-|) \right\rceil \). By choosing \( \beta, \rho \) small enough with respect to \( \epsilon \), we have \( k^+ + k^- + 1 \leq (1+\epsilon) \log_{\nu} n \). It follows from Lemma 7.4 that with probability \( 1 - o(n^{-100}) \),
\[
|\mathcal{N}^*_k(X^+)| \geq ((1 - \rho)\nu)^{k^+} |X^+| \geq n^{1/2+\beta},
\]
and similarly \( |\mathcal{N}^*_k(X^-)| \geq n^{1/2+\beta}. \)

If a tail in \( \mathcal{N}^*_k(X) \) has been paired with a head in \( \mathcal{N}^*_k(X) \), then \( \text{dist}(X^+, X^-) \leq k^+ + k^- \). Otherwise, the probability that no tail in \( \mathcal{N}^*_k(X) \) is paired to a head in \( \mathcal{N}^*_k(X) \) is at most
\[
\left( 1 - \frac{n^{1/2+\beta}}{m_n} \right)^{n^{1/2+\beta}} = o(n^{-100}).
\]
Therefore, that probability of \( \text{dist}(X^+, X^-) > k^+ + k^- + 1 \) is \( o(n^{-100}) \). □

### 7.3 Distance between thin neighbourhoods

We will use Proposition 7.2 to show that the distance between thin neighbourhoods are about \( \log_{\nu} n \).

**Lemma 7.5.** Assume that \( \tilde{\nu}_\pm > 0. \) Let \( t^+, t^- \) be as in \((6.1)\) and let \( B_3 \) be as in Lemma 6.4. Let
\[
k_n = t^+ + t^- + \log_{\nu} n.
\]
Let
\[
B_4(e^+, e^-) = B_4(e^+, e^-; \epsilon) = B_3(e^+, e^-; \epsilon/6) \cap \left\{ \left[ \text{dist}(e^+, e^-) / k_n - 1 \right] > \epsilon \right\},
\]
and let
\[
B_4 = \bigcup_{e^+ \in E^+ e^- \in E^-} B_4(e^+, e^-)
\]
Then \( \mathbb{P} \{ B_4 \} = o(1). \)
Proof. Fix \( e^+ \in \mathcal{E}^+ \) and \( e^- \in \mathcal{E}^- \). It suffices to show that \( \mathbb{P}\{B_4(e^+,e^-)\} = o(n^{-2}) \).

We have

\[
\mathbb{P}\{B_4(e^+,e^-)\} = \mathbb{P}\left\{B_3\left(e^+,e^-; \frac{\varepsilon}{6}\right) \mid \mathcal{E}^+ \right\} \mathbb{P}\left\{\left|\frac{\text{dist}(e^+,e^-)}{k_n} - 1\right| > \varepsilon \mid B_3\left(e^+,e^-; \frac{\varepsilon}{6}\right)\right\}. \tag{7.8}
\]

By applying (i) of Lemma 6.3 twice where in the second time we condition on \( E_{H_0} \), where \( H_0 \) is the partial pairing resulting from the exploration of the out-neighborhoods of \( e^+ \) and satisfies \( |V(H_0)| = O(\log^7 n) \), we obtain

\[
\mathbb{P}\{B_3(e^+,e^-; \varepsilon/6)\} \leq n^{-2+\varepsilon/2} \tag{7.9}
\]

Note that by the choice of \( H \), the two neighborhoods are disjoint.

Let \( H \) be the partial pairing of the edges exposed during the previous exploration process, conditional on \( B_3(e^+,e^-) \), so \( |V(H)| = O(\log^7 n) \). Let \( \mathcal{X}^+ = \mathcal{N}_{t_{\omega}(e^+)}(e^+) \) and \( \mathcal{X}^- = \mathcal{N}_{t_{\omega}(e^-)}(e^-) \). Note that \((H,\mathcal{X}^+,\mathcal{X}^-)\) satisfies Condition 7.1 and \(|\mathcal{X}^+|,|\mathcal{X}^-| \in [\omega,\omega^2] \). Since \( t_{\omega}(e^+) \leq (1+\varepsilon/6)t^z \), by applying (7.1) in Proposition 7.2,

\[
\mathbb{P}\{\text{dist}(e^+,e^-) > (1+\varepsilon)k_n \mid E_H\} = \mathbb{P}\{\text{dist}(\mathcal{X}^+,\mathcal{X}^-) > (1+\varepsilon)\log_\nu n \mid E_H\} = o(n^{-100}).
\]

Note that there is no simple path from \( \mathcal{X}^+ \) to \( \mathcal{X}^- \) that uses vertices in \( H \). Since \( t_{\omega}(e^+) \geq (1-\varepsilon/6)t^z \), if follows from (7.2) in Proposition 7.2 that

\[
\mathbb{P}\{\text{dist}(e^+,e^-) < (1-\varepsilon)k_n \mid E_H\} \leq \mathbb{P}\{\text{dist}(\mathcal{X}^+,\mathcal{X}^-,[n] \setminus V(H)) < (1-\varepsilon)\log_\nu n \mid E_H\} = o(n^{-\varepsilon/2}).
\]

As this is true for any \( H \), we have

\[
\mathbb{P}\left\{\left|\frac{\text{dist}(e^+,e^-)}{k_n} - 1\right| > \varepsilon \right\} = o(n^{-\varepsilon/2}) \tag{7.10}
\]

and the lemma follows by putting (7.9) and (7.10) in (7.8).

Next lemma holds in the case where in- or out- thin neighborhoods do not exist.

**Lemma 7.6.** Let \( t^+, t^- \) be as in (6.1), \( k_n = t^+ + t^- + \log_\nu n \) and let \( B_2 \) be as in Lemma 6.3. Then

(i) if \( \hat{\nu}_+ > 0 \) and \( \hat{\nu}_- = 0 \) (so \( t^- = 0 \)), for every \( \mathcal{X}^- \subseteq \mathcal{E}^- \) with \( |\mathcal{X}^-| \in [\omega,\omega^2] \) define

\[
B_4(e^+,\mathcal{X}^-) = B_4(e^+,\mathcal{X}^-;\varepsilon) = B_2(e^+;\varepsilon/3) \cap \left[\left|\frac{\text{dist}(e^+,\mathcal{X}^-)}{k_n} - 1\right| > \varepsilon\right],
\]

and \( B_4(\mathcal{X}^-) = \bigcup_{e^+ \in \mathcal{E}^+} B_4(e^+,\mathcal{X}^-) \). Then \( \mathbb{P}\{B_4(\mathcal{X}^-)\} = o(1) \).

(ii) if \( \hat{\nu}_+ = 0 \) and \( \hat{\nu}_- > 0 \) (so \( t^+ = 0 \)), for every \( \mathcal{X}^+ \subseteq \mathcal{E}^+ \) with \( |\mathcal{X}^+| \in [\omega,\omega^2] \) define

\[
B_4(e^-,\mathcal{X}^+) = B_4(e^-,\mathcal{X}^+;\varepsilon) = B_2(e^-;\varepsilon/3) \cap \left[\left|\frac{\text{dist}(\mathcal{X}^+,e^-)}{k_n} - 1\right| > \varepsilon\right],
\]

and \( B_4(\mathcal{X}^+) = \bigcup_{e^- \in \mathcal{E}^-} B_4(e^-,\mathcal{X}^+) \). Then \( \mathbb{P}\{B_4(\mathcal{X}^+)\} = o(1) \).

**Sketch of the proof.** We only sketch the proof of (i) as both proofs are analogous and similar to the proof of Lemma 7.5. We apply Lemma 6.3 (i) only once to upper bound the probability of \( B_2(e^-;\varepsilon/3) \) by \( n^{-1+\varepsilon/2} \), so such tails are rare but possible. Then we let \( \mathcal{X}^+ = \mathcal{N}_{t_{\omega}(e^+)}(e^+) \) and we use Proposition 7.2 to connect \( \mathcal{X}^+ \) and \( \mathcal{X}^- \) whp with \( H \) being the partial pairing resulting from the exploration of \( \mathcal{N}_{t_{\omega}(e^+)}(e^+) \).
8 Diameter
With all the preparation at hand, the proof of Theorem 1.4 is readily available.

8.1 Supercritical: Lower bound
We split into cases depending on \(\tilde{\nu}_+\) and \(\tilde{\nu}_-\). If \(\tilde{\nu}_+, \tilde{\nu}_- > 0\), by Lemma 6.4 (i), whp there exist a tail \(e^+\) and a head \(e^-\) satisfying \(B_2(e^+, e^-; \varepsilon/6)\). By Lemma 7.5 whp there is no such pair also satisfying \(\text{dist}(e^+, e^-) \not\in ((1 - \varepsilon)k_n, \infty)\), where \(k_n = t^+ + t^- + \log_\omega n\).

If \(\tilde{\nu}_+ > 0\) and \(\tilde{\nu}_- = 0\), then fix an arbitrary set of heads \(X^-\) with \(|X^-| \in [\omega, \omega^2]\). By Lemma 6.4 (ii), there exists a tail \(e^+\) satisfying \(B_2(e^+)\), but by Lemma 7.6 (i), no such tail satisfies \(\text{dist}(e^+, X^-) \not\in ((1 - \varepsilon)k_n, \infty)\). The proof is analogous if \(\tilde{\nu}_+ = 0\) and \(\tilde{\nu}_- > 0\).

If \(\tilde{\nu}_+ = \tilde{\nu}_- = 0\) (so \(t^+ = t^- = 0\)), then we fix two arbitrary sets \(X^+ \subseteq \mathcal{E}^+\) and \(X^- \subseteq \mathcal{E}^-\) with \(|X^+|, |X^-| \in [\omega, \omega^2]\). By Proposition 7.2 with \(H\) the empty pairing, we obtain \(\text{dist}(X^+, X^-) \in (1 \pm \varepsilon)\log_\omega n = (1 \pm \varepsilon)k_n\).

In each case we obtain the existence of \(e^+ \in \mathcal{E}^+\) and \(e^- \in \mathcal{E}^-\) at distance at least

\[
(1 - \varepsilon) \left( \frac{1}{\log(1/\tilde{\nu}_+)} + \frac{1}{\log(1/\tilde{\nu}_-)} + \frac{1}{\log \nu} \right) \log n.
\]

Let \(v^+\) be the node incident to the head paired with \(e^+\). Let \(v^-\) be the node incident to the tail paired with \(e^-\). Then \(\text{dist}(v^+, v^-) = \text{dist}(e^+, e^-) - 2\), concluding the proof of the lower bound in (i) of Theorem 1.4.

8.2 Supercritical: Upper bound
Assume first that \(\tilde{\nu}_+ > 0\) and \(\tilde{\nu}_- > 0\). By Lemma 6.2, for every pair of half-edges \(e^+ \in \mathcal{E}^+\) and \(e^- \in \mathcal{E}^-\), whp \(B^*_t\) holds; that is, either \(C_1\): there are no edges at distance more than \((1 + \varepsilon)t^+\) from \(e^+\), or \(C_2\): there are no edges at distance more than \((1 + \varepsilon)t^-\) to \(e^-\), or \(C_3\):

\[
t_\omega(e^+) < (1 + \varepsilon)t^+, \quad t_\omega(e^-) < (1 + \varepsilon)t^-,
\]

and,

\[
|X^+| > \omega, \quad |X^-| > \omega,
\]

where \(X^+ = N_{t_\omega(e^+)}(e^+)\) and \(X^- = N_{t_\omega(e^-)}(e^-)\).

If \(C_1 \cup C_2\) holds, then either

\[
\text{dist}(e^+, e^-) < (1 + \varepsilon)(t^+ + t^-),
\]

or there is no path from \(e^+\) to \(e^-\) and \(\text{dist}(e^+, e^-) = \infty\).

Suppose that \(C_3\) holds. If a tail in \(N_{t_\omega(e^+)}(e^+)\) has been paired with a head in \(N_{t_\omega(e^-)}(e^-)\), then \(\text{dist}(e^+, e^-) < (1 + \varepsilon)(t^+ + t^-)\) and we are done. Otherwise, let \(H\) be the partial pairing induced by \(N_{t_\omega(e^+)}(e^+)\) and \(N_{t_\omega(e^-)}(e^-)\). Since \((H, X^+, X^-)\) satisfies Condition 7.1 and \(|X^+|, |X^-| > \omega\), it follows from Proposition 7.2 that with probability \(o(n^{-100})\), \(\text{dist}(X^+, X^-) > (1 + \varepsilon)\log_\omega n\). Thus

\[
\mathbb{P} \left\{ (1 + \varepsilon)(t^+ + t^- + \log_\omega n) < \text{dist}(e^+, e^-) < \infty \right\} \leq \mathbb{P} \left\{ \text{dist}(X^+, X^-) > (1 + \varepsilon)\log_\omega n \mid C_3 \right\} = o(n^{-100}).
\]

The upper bound for the diameter follows from applying a union bound over all \(e^\pm \in \mathcal{E}^\pm\).

In the case \(\tilde{\nu}_\pm = 0\), the argument still works by replacing \((1 + \varepsilon)t^\pm\) by \(\varepsilon \log n\).
8.3 Subcritical

Part (ii) of Theorem 1.4 follows immediately from Lemma 6.5. By (i) of Lemma 6.5, whp for any pair of nodes \((u, v)\) in \(\vec{G}_n\), either \(\text{dist}(u, v) < (1 + \delta) \log_{1/\nu} n\) or \(\text{dist}(u, v) = \infty\). By (ii) of Lemma 6.5, whp there exist a tail \(e^+\) and a head \(e^-\) such that \(\text{dist}(e^+, e^-) \in ((1 - \delta) \log_{1/\nu} n, \infty)\). Let \(v^+\) be the node incident to the head paired with \(e^+\). Let \(v^-\) be the node incident to the tail paired with \(e^-\). Then \(\text{dist}(v^+, v^-) = \text{dist}(e^+, e^-) - 2\).

9 Applications

In this section, we give some applications of our results in the supercritical regime without delving into too much details.

9.1 Typical distance

Let \(U_1, U_2 \in [n]\) be two vertices chosen uniformly at random. Then \(\text{dist}(U_1, U_2)\) is called the typical distance of \(\vec{G}_n\). A distributional result of \(\text{dist}(U_1, U_2)\) is given by van der Hoorn and Olvera-Cravioto [32]. Here we give a weaker result under weaker assumptions:

**Theorem 9.1.** Assume Condition 1.1 and \(\nu > 1\). Let \(U_1, U_2 \in [n]\) be two vertices chosen uniformly at random in \(\vec{G}_n\). Then for all \(\varepsilon > 0\),

\[
P\left\{ \left| \frac{\text{dist}(U_1, U_2)}{\log_{\nu} n} - 1 \right| < \varepsilon \middle| \text{dist}(U_1, U_2) < \infty \right\} \to 1.
\]

The proof of the theorem is an easy application of Proposition 7.2 and we leave it to the reader.

9.2 Other random graphs

In many random digraphs, the degree sequence is not fixed but random. However, many such models, including \(d\)-out regular digraphs and binomial random digraph described below, can be studied via the directed configuration model using the following simple lemma whose proof we omit:

**Lemma 9.2.** Assume \(D_n\) is a random directed multi/simple graph of \(n\) vertices which is uniformly random conditioning on its degree sequence. Let \(D_n\) be the in- and out-degree of a uniform random vertex in \(D_n\) and assume that \(D_n\) satisfies Condition 1.1 with some distribution \(D\) on \(\mathbb{Z}_{\geq 0}^2\). Let \(\vec{G}_n\) be the directed configuration model satisfying Condition 1.1 with the same \(D\). If \(\vec{G}_n\) has a property \(P_n\) whp, then \(D_n\) has property \(P_n\) whp.

9.2.1 Regular digraphs

The \(d\)-out model \(D_{n,d\text{-out}}\) is a directed multigraph on \([n]\) in which each of vertex is given \(d\) out-edges whose end vertices are chosen independently and uniformly at random from all vertices. In this model, \(D_n\) converges in distribution, first moment, and second moment to \((D^+, D^-)\), where \(D^+ \equiv d\) and \(D^- \equiv \text{Poi}(d)\). When \(d \geq 2\), whp there are thin in-neighbourhoods but no thin out-neighbourhood in \(D_{n,d\text{-out}}\). Therefore, we recover the following result in [1] by applying Theorem 1.4, and Lemma 9.2:
Theorem 9.3. Assume that \( d \geq 2 \). Let \( \lambda_d \) be the unique solution of \( de^{-d} = \lambda_d e^{-\lambda_d} \) on \((0,1)\). Then

\[
\frac{\text{diam} \left( D_{n,d-\text{out}} \right)}{\log n} \to \frac{1}{\log(1/\lambda_d)} + \frac{1}{\log d},
\]

in probability.

A related model is the \( d \)-in/out model \( D_{n,d} \), i.e., the uniform random directed multigraph on \([n]\) in which each vertex has both in- and out-degree \( d \). When \( d \geq 2 \), we have neither thin out-neighbourhood nor thin in-neighbourhood and the diameter is of the same order as the typical distance.

Theorem 9.4. Assume that \( d \geq 2 \). Then

\[
\frac{\text{diam} \left( D_{n,d} \right)}{\log n} \to \frac{1}{\log d},
\]

in probability.

9.2.2 Binomial random digraph

A binomial random digraph \( D_{n,p} \) is a simple digraph on \([n]\) in which a directed edge is added between each ordered pair of vertices independently with probability \( p \), as described in [22, 24].

A slightly different model \( D_{n,p}^* \) recently introduced by Ralaivaosaona, Rasendrahasina and Wagner [27] is constructed by adding an undirected edge between each pair of vertices independently with probability \( 2p \) and choosing the direction of the edge with a fair coin toss.

In both models, assuming that \( np \to \nu > 1 \), \( D_n \) converges in distribution to a pair of independent Poisson random variables with expectation \( \nu \). Thus, we have the following result by applying Theorem 1.4, and Lemma 9.2:

Theorem 9.5. Assume that \( np \to \nu > 1 \). Let \( \hat{\nu} \) be the unique solution of \( \hat{\nu} e^{-\hat{\nu}} = \nu e^{-\nu} \) on \((0,1)\). Then

\[
\frac{\text{diam} \left( D_{n,p} \right)}{\log n} \to \frac{2}{\log(1/\hat{\nu})} + \frac{1}{\log \nu},
\]

in probability, and

\[
\frac{\text{diam} \left( D_{n,p}^* \right)}{\log n} \to \frac{2}{\log(1/\hat{\nu})} + \frac{1}{\log \nu},
\]

in probability.

Acknowledgements

We thank Svante Janson for helpful discussions on branching processes in the subcritical regime.

References

[1] L. Addario-Berry, B. Balle, and G. Perarnau. Diameter and stationary distribution of random \( r \)-out digraphs. The Electronic Journal of Combinatorics, page P3.28, 2020. doi:10/ghd74q.
[2] H. Amini. Bootstrap percolation in living neural networks. *J Stat Phys*, 141(3):459–475, 2010. doi: 10/c53hx4.

[3] H. Amini and A. Minca. Mathematical modeling of systemic risk. In *Advances in Network Analysis and Its Applications*, Mathematics in Industry, pages 3–26. Springer, Berlin, Heidelberg, 2013. doi: 10/dm56.

[4] K. B. Athreya and P. E. Ney. *Branching Processes*. Grundlehren Der Mathematischen Wissenschaften. Springer-Verlag, Berlin Heidelberg, 1972. doi: 10/dft4.

[5] J. Blanchet and A. Stauffer. Characterizing optimal sampling of binary contingency tables via the configuration model. *Random Structures & Algorithms*, 42(2):159–184, 2013. doi: 10/f4mtxh.

[6] B. Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European Journal of Combinatorics*, 1(4):311–316, 1980. doi: 10/ggktcq.

[7] B. Bollobás and W. Fernandez de la Vega. The diameter of random regular graphs. *Combinatorica*, 2(2):125–134, 1982. doi: 10/dr7rjg.

[8] C. Bordenave. *Lecture Notes on Random Graphs and Probabilistic Combinatorial Optimization*. Online, 2016. URL https://web.archive.org/web/20210420015408/http://www.i2m.univ-amu.fr/perso/charles.bordenave/_media/coursrg.pdf.

[9] C. Bordenave, P. Caputo, and J. Salez. Random walk on sparse random digraphs. *Probab. Theory Relat. Fields*, 170(3):933–960, 2018. doi: 10/gc8nxk.

[10] X. S. Cai and L. Devroye. The graph structure of a deterministic automaton chosen at random. *Random Structures & Algorithms*, 51(3):428–458, 2017. doi: 10/gbtqgb.

[11] X. S. Cai and G. Perarnau. The giant component of the directed configuration model revisited. *arXiv:2004.04998 [cs, math]*, 2020. URL http://arxiv.org/abs/2004.04998.

[12] X. S. Cai and G. Perarnau. Minimum stationary values of sparse random directed graphs. *arXiv:2010.07246 [cs, math]*, 2020. URL http://arxiv.org/abs/2010.07246.

[13] X. S. Cai, P. Caputo, G. Perarnau, and M. Quattropani. Rankings in directed configuration models with heavy tailed in-degrees. *arXiv:2104.08389 [cs, math]*, 2021. URL http://arxiv.org/abs/2104.08389.

[14] P. Caputo and M. Quattropani. Stationary distribution and cover time of sparse directed configuration models. *Probab. Theory Relat. Fields*, 178(3):1011–1066, 2020. doi: 10/ghd74v.

[15] C. Cooper and A. Frieze. The size of the largest strongly connected component of a random digraph with a given degree sequence. *Combinatorics, Probability and Computing*, 13(3):319–337, 2004. doi: 10/cn8q5j.

[16] R. Durrett. *Probability: Theory and Examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010. doi: 10/gfj8hd.
[17] D. Fernholz and V. Ramachandran. The diameter of sparse random graphs. Random Structures & Algorithms, 31(4):482–516, 2007. doi: 10/bzwvjr.

[18] A. Graf. On the Strongly Connected Components of Random Directed Graphs with given Degree Sequences. PhD thesis, University of Waterloo, 2016. URL http://hdl.handle.net/10012/10681.

[19] G. H. Hardy, J. E. Littlewood, and G. Pólya. Inequalities. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.

[20] W. Hoeffding. Probability inequalities for sums of bounded random variables. Journal of the American Statistical Association, 58(301):13–30, 1963. doi: 10/gf8mc6.

[21] S. Janson. The probability that a random multigraph is simple. Combinatorics, Probability and Computing, 18(1-2):205–225, 2009. doi: 10/bg4m2c.

[22] R. M. Karp. The transitive closure of a random digraph. Random Structures & Algorithms, 1(1):73–93, 1990. doi: 10/bd47bm.

[23] H. Li. Attack vulnerability of online social networks. In 2018 37th Chinese Control Conference (CCC), pages 1051–1056, 2018. doi: 10/ggh2kg.

[24] T. Łuczak and T. G. Seierstad. The critical behavior of random digraphs. Random Structures & Algorithms, 35(3):271–293, 2009. doi: 10/cq6mrk.

[25] A. G. Pakes. Some limit theorems for the total progeny of a branching process. Advances in Applied Probability, 3(1):176–192, 1971. doi: 10/dkshmc.

[26] M. D. Penrose. The strong giant in a random digraph. Journal of Applied Probability, 53(1):57–70, 2016. doi: 10/ggkpjf.

[27] D. Ralaivaosaona, V. Rasendrahasina, and S. Wagner. On the Probability That a Random Digraph Is Acyclic. In 31st International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2020), volume 159 of Leibniz International Proceedings in Informatics (LIPIcs), pages 25:1–25:18, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. doi: 10/gjmt6b.

[28] O. Riordan and N. Wormald. The diameter of sparse random graphs. Combin. Probab. Comput., 19(5-6):835–926, 2010. doi: 10/dgp6hh.

[29] E. Seneta. On Recent Theorems Concerning the Supercritical Galton-Watson Process. The Annals of Mathematical Statistics, 39(6):2098–2102, 1968. doi: 10/cjmg8f.

[30] R. van der Hofstad. Random Graphs and Complex Networks, volume 1 of Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, England, 2016. doi: 10/ggv8q7.

[31] R. van der Hofstad. Random Graphs and Complex Networks, volume 2. Online, 2020. URL https://web.archive.org/web/20201213052702/https://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf.
[32] P. van der Hoorn and M. Olvera-Cravioto. Typical distances in the directed configuration model. *Ann. Appl. Probab.*, 28(3):1739–1792, 2018. doi: 10/ggh2ch.