Abstract

We consider charged rotating black holes in $D = 2N + 1$ dimensions, $D \geq 5$. While these black holes generically possess $N$ independent angular momenta, associated with $N$ distinct planes of rotation, we here focus on black holes with equal-magnitude angular momenta. The angular dependence can then be treated explicitly, and a system of $5$ $D$-dependent ordinary differential equations is obtained. We solve these equations numerically for Einstein-Maxwell theory in $D = 5$, 7 and 9 dimensions. We discuss the global and horizon properties of these black holes, as well as their extremal limits.
1 Introduction

In $D = 4$ dimensions the Kerr-Newman solution presents the unique family of stationary asymptotically flat black holes of Einstein-Maxwell (EM) theory. It comprises the Kerr solution, representing rotating vacuum black holes, as well as the static Reissner-Nordström and Schwarzschild solutions.

The generalization of these black hole solutions to $D > 4$ dimensions was pioneered by Tangherlini [1] for static black holes, and by Myers and Perry [2] for rotating vacuum black holes. The corresponding $D > 4$ charged rotating black holes of EM theory could not yet be obtained in closed form [2, 3]. But in $D = 5$ dimensions rotating EM black holes have been constructed numerically [4].

In contrast to pure EM theory, exact higher dimensional charged rotating black holes are known in theories with more symmetries. The presence of a Chern-Simons (CS) term, for instance, leads to a class of odd-dimensional Einstein-Maxwell-Chern-Simons (EMCS) theories, comprising the bosonic sector of minimal $D = 5$ supergravity, whose stationary black hole solutions [5, 6, 7] possess surprising properties [8, 9, 10]. In particular, EMCS black holes (with horizons of spherical topology) are no longer uniquely characterized by their global charges [10]. The inclusion of additional fields, as required by supersymmetry or string theory, yields further exact solutions [11, 12, 13].

Stationary black holes in $D$ dimensions possess $N = [(D-1)/2]$ independent angular momenta $J_i$ associated with $N$ orthogonal planes of rotation [2]. $[(D-1)/2]$ denotes the integer part of $(D-1)/2$, corresponding to the rank of the rotation group $SO(D-1)$. The general black holes solutions then fall into two classes, in even-$D$ and odd-$D$ solutions [2].

Here we focus on charged rotating black holes in odd dimensions. We show, that when their $N$ angular momenta have all equal magnitude, the angular dependence can be treated explicitly for any odd dimension $D \geq 5$. The resulting system of coupled Einstein and matter field equations then simplifies considerably, yielding a system of $D$-dependent ordinary differential equations. We here solve these equations numerically for Einstein-Maxwell theory in $D = 5$, 7 and 9 dimensions.

In section 2 we recall the EM action, and present the stationary axially symmetric Ansätze for black hole solutions with $N$ equal-magnitude angular momenta in $D = 2N + 1$ dimensions, $N \geq 2$. We discuss the black hole properties in section 3, and present numerical results for EM black holes in section 4.

2 Metric and Gauge Potential

We consider the $D$-dimensional Einstein-Maxwell action with Lagrangian

$$L = \frac{1}{16\pi G_D} \sqrt{-g} \left( R - F_{\mu\nu} F^{\mu\nu} \right),$$

(1)
with curvature scalar \( R \), \( D \)-dimensional Newton constant \( G_D \), and field strength tensor \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), where \( A_\mu \) denotes the gauge potential.

Variation of the action with respect to the metric and the gauge potential leads to the Einstein equations

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2T_{\mu\nu},
\]

with stress-energy tensor

\[
T_{\mu\nu} = F_{\mu\rho} F^\rho_{\nu} - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma},
\]

and the gauge field equations,

\[
\nabla_\mu F^{\mu\nu} = 0.
\]

To obtain stationary black hole solutions, representing charged generalizations of the D-dimensional Myers-Perry solutions \([2]\), we consider black hole space-times with \( N \)-azimuthal symmetries, implying the existence of \( N + 1 \) commuting Killing vectors, \( \xi \equiv \partial_t \), and \( \eta(k) \equiv \partial_{\varphi_k} \), for \( k = 1, \ldots, N \). We parametrize the metric in isotropic coordinates, which are well suited for the numerical construction of rotating black holes \([3, 10, 14]\). (We consider only black holes with spherical horizon topology \([15]\).)

While the general EM black holes will then possess \( N \) independent angular momenta, we now restrict to black holes whose angular momenta have all equal magnitude. The metric and the gauge field parametrization then simplify considerably. In particular, for such equal-magnitude angular momenta black holes, the general Einstein and Maxwell equations reduce to a set of ordinary differential equations.

The metric for these equal-magnitude angular momenta black holes reads

\[
ds^2 = -f dt^2 + \frac{m}{f} \left[ dr^2 + r^2 \sum_{i=1}^{N-1} \left( \prod_{j=0}^{i-1} \cos^2 \theta_j \right) d\theta_i^2 \right] + \frac{n}{f} r^2 \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \left( \varepsilon_k d\varphi_k - \frac{\omega}{r} dt \right)^2 \\
+ \frac{m-n}{f} r^2 \left\{ \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \varepsilon_k d\varphi_k^2 \\
- \left[ \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \varepsilon_k d\varphi_k \right]^2 \right\},
\]

where \( \theta_0 \equiv 0, \theta_i \in [0, \pi/2] \) for \( i = 1, \ldots, N - 1 \), \( \theta_N \equiv \pi/2, \varphi_k \in [0, 2\pi] \) for \( k = 1, \ldots, N \), and \( \varepsilon_k = \pm 1 \) denotes the sense of rotation in the \( k \)-th orthogonal plane of rotation.

An adequate parametrization for the gauge potential is given by

\[
A_\mu dx^\mu = a_0 dt + a_\varphi \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \varepsilon_k d\varphi_k.
\]
Thus, independent of the odd dimension $D \geq 5$, this parametrization involves only four functions $f, m, n, \omega$ for the metric and two functions $a_0, a_0$ for the gauge field, which all depend only on the radial coordinate $r$.

To obtain asymptotically flat solutions, the metric functions should satisfy at infinity the boundary conditions
\begin{equation}
    f|_{r=\infty} = m|_{r=\infty} = n|_{r=\infty} = 1, \quad \omega|_{r=\infty} = 0,
\end{equation}
while for the gauge potential we choose a gauge, in which it vanishes at infinity
\begin{equation}
    a_0|_{r=\infty} = a_0|_{r=\infty} = 0.
\end{equation}

The horizon is located at $r_H$, and is characterized by the condition $f(r_H) = 0$. Requiring the horizon to be regular, the metric functions must satisfy the boundary conditions
\begin{equation}
    f|_{r=r_H} = m|_{r=r_H} = n|_{r=r_H} = 0, \quad \omega|_{r=r_H} = r_H \Omega,
\end{equation}
where $\Omega$ is (related to) the horizon angular velocity, defined in terms of the Killing vector
\begin{equation}
    \chi = \xi + \Omega \sum_{k=1}^N \varepsilon_k \eta(k),
\end{equation}
which is null at the horizon. Without loss of generality, $\Omega$ is assumed to be non-negative, any negative sign being included in $\varepsilon_k$. The gauge potential satisfies
\begin{equation}
    \chi^\mu A_\mu|_{r=r_H} = \Phi_H = (a_0 + \Omega a_\varphi)|_{r=r_H}, \quad \frac{da_\varphi}{dr}|_{r=r_H} = 0,
\end{equation}
with constant horizon electrostatic potential $\Phi_H$.

## 3 Black Hole Properties

The mass $M$ and the angular momenta $J^{(k)}$ of the black holes are obtained from the Komar expressions associated with the respective Killing vector fields
\begin{equation}
    M = \frac{-1}{16\pi G_D} \frac{D-2}{D-3} \int_{S^{D-2}} \alpha, \quad J^{(k)} = \frac{1}{16\pi G_D} \int_{S^{D-2}} \beta^{(k)},
\end{equation}
with $\alpha_{\mu_1...\mu_{D-2}} \equiv \epsilon_{\mu_1...\mu_{D-2}\rho\sigma} \nabla^\rho \xi^\sigma$, $\beta^{(k)}_{\mu_1...\mu_{D-2}} \equiv \epsilon_{\mu_1...\mu_{D-2}\rho\sigma} \nabla^\rho \eta^{(k)}$, and for equal-magnitude angular momenta $|J^{(k)}| = J$, $k = 1, \ldots, N$.

The electric charge is obtained from
\begin{equation}
    Q = \frac{-1}{8\pi G_D} \int_{S^{D-2}} \tilde{F},
\end{equation}
with $\tilde{F}_{\mu_1...\mu_{D-2}} \equiv \epsilon_{\mu_1...\mu_{D-2}\rho\sigma} F^{\rho\sigma}$.

The horizon mass $M_H$ and horizon angular momenta $J_{H(k)}$ are given by

$$M_H = \frac{-1}{16\pi G_D} \frac{D-2}{D-3} \int_{\mathcal{H}} \alpha , \quad J_{H(k)} = \frac{1}{16\pi G_D} \int_{\mathcal{H}} \beta_{(k)} ,$$

where $\mathcal{H}$ represents the surface of the horizon, and for equal-magnitude angular momenta $|J_{H(k)}| = J_H$, $k = 1, \ldots, N$.

Introducing further the area of the horizon $A_H$ and the surface gravity $\kappa$,

$$\kappa^2 = -\frac{1}{2} (\nabla_{\mu} \chi_{\nu}) (\nabla^\mu \chi^\nu) ,$$

the mass formulae [4, 8] for EM black holes with $N$ equal-magnitude angular momenta become

$$\frac{D-3}{D-2} M_H = \frac{\kappa A_H}{8\pi G_D} + N \Omega J_H ,$$

$$\frac{D-3}{D-2} M = \frac{\kappa A_H}{8\pi G_D} + N \Omega J + \frac{D-3}{D-2} \Phi_H Q .$$

The global charges and the magnetic moment $\mu_{\text{mag}}$, can be obtained from the asymptotic expansions of the metric and the gauge potential

$$f = 1 - \frac{\tilde{M}}{r^{D-3}} + \ldots , \quad \omega = \frac{\tilde{J}}{r^{D-2}} + \ldots , \quad a_0 = \frac{\tilde{Q}}{r^{D-3}} + \ldots , \quad a_\varphi = \frac{\tilde{\mu}_{\text{mag}}}{r^{D-3}} + \ldots ,$$

where

$$\tilde{M} = \frac{16\pi G_D}{(D-2)A(S^{D-2})} M , \quad \tilde{J} = \frac{8\pi G_D}{A(S^{D-2})} J ,$$

$$\tilde{Q} = \frac{4\pi G_D}{(D-3)A(S^{D-2})} Q , \quad \tilde{\mu}_{\text{mag}} = \frac{4\pi G_D}{(D-3)A(S^{D-2})} \mu_{\text{mag}} ,$$

and $A(S^{D-2})$ is the area of the unit $(D-2)$-sphere. The gyromagnetic ratio $g$ is defined via

$$\mu_{\text{mag}} = g \frac{Q J}{2M} .$$

4 Numerical results

In order to solve the coupled system of ODE’s, we take advantage of the existence of a first integral of that system,

$$r^{D-2} m^{(D-5)/2} \sqrt{f(D-3)/f} \left( \frac{da_0}{dr} + \frac{\omega \, da_\varphi}{r \, dr} \right) = -\frac{4\pi G_D}{A(S^{D-2})} Q ,$$

where

$$m = \frac{-1}{16\pi G_D} \frac{D-2}{D-3} \int_{\mathcal{H}} \alpha .$$
to eliminate \( a_0 \) from the equations, leaving a system of one first order equation (for \( n \)) and four second order equations.

For the numerical calculations we introduce the compactified radial coordinate \( \bar{r} = 1 - r_H/r \) \[14\], and we take units such that \( G_D = 1 \). We employ a collocation method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure \[16\]. Typical mesh sizes include \( 10^3 \) – \( 10^4 \) points. The solutions have a relative accuracy of \( 10^{-10} \). The estimates of the relative errors of the global charges and the magnetic moment are of order \( 10^{-6} \), giving rise to an estimate of the relative error of \( g \) of order \( 10^{-5} \).

Let us first address the domain of existence of rotating EM black holes with equal-magnitude angular momenta. We note, that unlike the case of a single non-vanishing angular momentum, where no extremal solutions exist in \( D > 5 \) dimensions \[2, 13\], extremal solutions do exist for odd \( D \) black holes with equal-magnitude angular momenta. We exhibit in Fig. 1 the scaled angular momentum \( J/M^{(D-2)/(D-3)} \) of the extremal EM black holes versus the scaled charge \( Q/M \) \[17\] for \( D = 5, 7 \) and 9 dimensions. Black holes exist only in the regions bounded by the \( J = 0 \)-axis and by the respective curves. The domain of existence is symmetric with respect to \( Q \rightarrow -Q \). Introducing the scaling factors \( \delta \) and \( \gamma \),

\[
\delta^2 = \frac{1}{2} \frac{D-2}{D-3}, \quad (2\gamma)^{D-3} = \frac{1}{32\pi}(D-3)(D-2)^{D-2} \left( \frac{D-1}{D-3} \right)^{(D-1)/2} A(S^{D-2}),
\]

we observe that the scaled domain of existence becomes almost independent of \( D \), as demonstrated in Fig. 1 (right). These extremal black holes have vanishing surface gravity, but finite horizon area.

Figure 1: Left: Scaled angular momentum \( J/M^{(D-2)/(D-3)} \) versus scaled charge \( Q/M \) for extremal black holes with equal-magnitude angular momenta in \( D = 5, 7 \) and 9 dimensions. Right: Scaled domain of existence.
We now turn to non-extremal black holes, and discuss their properties. We first consider sets of black hole solutions in $D = 5$, 7 and 9 dimensions, obtained by varying the horizon angular velocity $\Omega$, while keeping the isotropic horizon radius $r_H = 1$ and the charge $Q = 10$ fixed.

In Fig. 2 (left) we indicate the location of these sets of solutions within the respective domains of existence, by exhibiting their scaled angular momentum $J/M^{(D-2)/(D-3)}$ versus their scaled charge $Q/M$.

We exhibit the dependence of the mass $M$ of these solutions on the horizon angular velocity $\Omega$ in Fig. 2 (right) and compare with the corresponding $D$-dimensional Myers-Perry solutions, which have $Q = 0$. For each set of solutions we observe two branches, extending up to a maximal value of $\Omega$. The lower branch emerges from the static solution in the limit $\Omega = 0$, on the upper branch the mass diverges in the limit $\Omega \to 0$. The maximal value of $\Omega$ depends on the horizon radius $r_H$, the charge $Q$, and the dimension $D$. For a fixed value of the charge, its influence and thus the deviation from the Myers-Perry solution decreases with increasing dimension $D$, as expected from the scaling properties of the solutions [17].

Figure 2: Sets of non-extremal equal-magnitude angular momenta black hole solutions in $D = 5$, 7 and 9 dimensions, for fixed isotropic horizon radius $r_H = 1$, fixed charge $Q = 10$, and varying horizon angular velocity $\Omega$. Left: Scaled angular momentum $J/M^{(D-2)/(D-3)}$ versus scaled charge $Q/M$ within the respective domains of existence (represented by thin dotted lines). Right: Mass $M$ versus horizon angular velocity $\Omega$ (also for Myers-Perry black hole solutions with $Q = 0$).

The angular momentum $J$ and the gyromagnetic ratio $g$ are presented in Fig. 3 for the same sets of solutions. For small values of the charge $Q$ the gyromagnetic ratio has been obtained perturbatively, with perturbative value $g = D - 2$ [18]. We observe, that in the limit $\Omega \to 0$, i.e., for slow rotation the gyromagnetic ratio also tends to this perturbative value $g = D - 2$. On the lower branch, which emerges from a static solution, the gyromagnetic ratio then increases rapidly from its perturbative value. On
the upper branch the gyromagnetic ratio tends back to its perturbative value in the
limit $\Omega \to 0$, or whenever the mass and angular momentum become large so that the
perturbative regime for the charge is reached.

We emphasize that the deviation of the gyromagnetic ratio from the perturbative
value $g = D - 2$ is a physical effect. It is not due to numerical inaccuracy, since the error
estimate of $10^{-5}$ for $g$ is typically several orders of magnitude less than the observed
deviation of $g$ from the perturbative value $g = D - 2$.

![Figure 3](image1.png)

Figure 3: Same sets of solutions as in Fig. 2. Left: Angular momentum $J$ versus
horizon angular velocity $\Omega$ (also for Myers-Perry black hole solutions with $Q = 0$).
Right: Gyromagnetic ratio $g$ versus horizon angular velocity $\Omega$.

![Figure 4](image2.png)

Figure 4: Left: Gyromagnetic ratio $g$ versus horizon angular velocity $\Omega$ for fixed
isotropic horizon radius $r_H = 1$, fixed charge $Q = 10$ for $D = 5$, $Q = 100$ for $D = 7$,
$Q = 1000$ for $D = 9$. Right: Gyromagnetic ratio $g$ versus charge $Q$ for fixed isotropic
horizon radius $r_H = 1$, and fixed horizon angular velocity $\Omega = 0.1$ for $D = 5$, 7 and 9
dimensions.
As seen in Fig. 3, the deviation of the gyromagnetic ratio from its respective perturbative value decreases with increasing dimension $D$ for fixed charge. To obtain deviations of comparable size for $g$ in higher dimensions, one has to increase the value of the charge $Q$. This is illustrated in Fig. 4, where we compare the gyromagnetic ratio $g$ of the previous $D = 5$ data set with $Q = 10$ to the $D = 7$ and $D = 9$ data sets with $Q = 100$ and $Q = 1000$, respectively. We also exhibit the gyromagnetic ratio $g$ versus the charge $Q$, for fixed values of $r_H$ and $\Omega$ in Fig. 4. Obviously, for small values of $Q$ the gyromagnetic ratio $g$ tends to its respective perturbative value [18].

Horizon properties of the sets of black hole solutions (presented above in Figs. 2-3) are illustrated in Fig. 5, where we exhibit their area $A_H$ and their surface gravity $\kappa$. While the horizon area increases monotonically with increasing mass, the surface gravity tends to zero for large black holes.

![Figure 5](image-url)

Figure 5: Same sets of solutions as in Fig. 2 (including the Myers-Perry black hole solutions with $Q = 0$). Left: Area parameter $A_H$ versus horizon angular velocity $\Omega$. Right: Surface gravity $\kappa$ versus horizon angular velocity $\Omega$.

## 5 Conclusions

We have considered rotating black holes with equal-magnitude angular momenta in Einstein-Maxwell theory in odd dimensions. These black holes are asymptotically flat, and they possess a regular horizon of spherical topology. We have shown that, by employing suitable Ansätze for the metric and the gauge potential of these black holes, the coupled system of Einstein-Maxwell equations reduces to a set of five ordinary differential equations, which we have solved numerically.

We have studied the physical properties of these black holes, in particular their global charges and horizon properties. The numerical solutions satisfy the generalized
Smarr formula \[17\] with high accuracy. For generic values of the charge and angular momentum the gyromagnetic ratio of these black holes differs from \( g = D - 2 \). However, in the limit of vanishing electric charge or vanishing angular momentum, the gyromagnetic ratio does tend to the perturbative value \( g = D - 2 \) \[18\]. For a fixed value of the charge, its influence on the space-time decreases with increasing dimension \( D \), as expected from the scaling properties of the solutions \[17\].

Currently, we are generalizing these results to Einstein-Maxwell-Chern-Simons theory, and to Einstein-Maxwell-dilaton theory. EMCS black holes, in particular, exhibit already a number of surprising properties in five dimensions. These include non-uniqueness, rotational instability, or counterrotation \[10\]. Certainly further surprises are waiting here in higher dimensions.

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