SMALL-TIME LOCAL STABILIZATION OF THE TWO DIMENSIONAL INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

SHENGQUAN XIANG

Abstract. We provide explicit time-varying feedback laws that locally stabilize the two-dimensional internal controlled incompressible Navier–Stokes equations in arbitrarily small time. We also obtain quantitative rapid stabilization via stationary feedback laws, as well as quantitative null controllability with explicit controls having $e^{C/T}$ costs.

1. Introduction

Let $\Omega$ be a bounded connected open set in $\mathbb{R}^2$ with smooth boundary. Let the controlled domain $\omega \subset \Omega$ be a nonempty open subset. We are interested in the stabilization and null controllability of the two dimensional incompressible Navier-Stokes system with internal control,

\[
\begin{aligned}
 y_t - \Delta y + (y \cdot \nabla) y + \nabla p &= 1_{\omega} f & \text{in } \Omega, \\
 \text{div } y &= 0 & \text{in } \Omega, \\
 y &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]  

(1.1)

where, the state $y(t, \cdot)$ and the control term $f(t, \cdot)$ are in $\mathcal{H}$. We adapt the standard fluid mechanics framework,

\[
\mathcal{H} := \{ y \in L^2(\Omega)^2 : \text{div } y = 0 \text{ in } \Omega, \ y \cdot n = 0 \text{ on } \partial \Omega \},
\]

\[
\mathcal{V}_\sigma := \{ y \in H^1_0(\Omega)^2 : \text{div } y = 0 \text{ in } \Omega \} \quad \text{and} \quad \mathcal{V} := \{ y \in H^1_0(\Omega)^2 \},
\]

with $\mathcal{V}_\sigma \hookrightarrow \mathcal{V} \hookrightarrow L^2(\Omega)^2 \hookrightarrow \mathcal{V}' \hookrightarrow \mathcal{V}'_\sigma$. We have taken the viscosity coefficient as 1 to simplify the presentation.

When dealing with stabilization problems the control term $f$ is regarded as a feedback control governed by some “feedback application” that depends on current states and time, $U(t; y)$:

\[
f(t, x) := U(t; y(t, x)),
\]

(1.2)

where the application $U$ is the so called time-varying feedback law,

\[
\begin{aligned}
 U & : \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H} \\
 (t; y) & \mapsto U(t; y).
\end{aligned}
\]  

(1.3)

The closed-loop system associated to the preceding feedback law $U$ is the evolution system (1.1)–(1.3). A stationary feedback law is such an application only depends on $\mathcal{H}$. A $T$-periodic feedback law is a time-varying feedback law that is periodic with respect to time, i.e. $U(T + t; y) = U(t; y)$. A proper feedback law $U$, roughly speaking, is some time-varying feedback law such that, for every $s \in \mathbb{R}$, and for every $y_0 \in \mathcal{H}$ as initial state at time $s$, i.e. $y(s, x) = y_0(x)$, the closed-loop system (1.1)–(1.3) admits a unique solution. For the closed-loop system with proper feedback law we can define the “flow”, $\Phi(t, s; y_0)$, as the state at time $t$ of the solution of (1.1)–(1.3) with initial state $y(s, x) = y_0(x)$.

2010 Mathematics Subject Classification. 35Q30, 35S15, 93D15.

Keywords. Cost, finite time stabilization, null controllability, quantitative, spectral estimate.
The local controllability and stabilization of Navier–Stokes equations have been extensively studied in the literature. Based on global Carleman estimates introduced by Fursikov–Imanuvilov [24], a nearly complete local exact controllability result is obtained in [21], other works include but not limited to [12, 23, 20, 29, 32]. Eventually one can even control the system locally via some reduced control terms [13]. The global controllability of Navier–Stokes equations with controlling on boundary (namely Lions’ problem), which, different from the cases on Riemannian manifolds [1, 11], is far away from been answered due to boundary layer difficulties, by far the best results are given by [14, 15].

The study on local exponential stabilization around 0 and other trajectories of Navier–Stokes equations both with internal controls or with boundary controls is fruitful, notably based on Riccati type methods by optimal control theory. For example, [2, 5] for local exponential stabilization with finite dimensional internal control (feedback laws); exponential stabilization by boundary feedback laws [3, 22, 30]; stabilization around trajectories or unstable steady states [4, 7, 31], etc. To the best of our knowledge, result on quantitative rapid stabilization or even finite time stabilization of Navier–Stokes equations is extremely limited, we refer to [19] for a detailed review on these questions.

Recently, the author has introduced a method to stabilize the multi-dimensional heat equations in finite time [35], which is based on quantitative rapid stabilization relying on spectral estimates and Lyapunov functionals, as well as piecewise feedback laws. Methodologically speaking, the technical spectral estimate is achieved via local Carleman estimates on elliptic operators up to boundaries (as always fulfilling Hörmander’s pseudoconvex condition), starting from the seminal paper [26] these results can be regarded as standard, at least compared to wave type operators; the Lyapunov functions [18] aim at finding artfully chosen energy and multiplier to characterize the variation of the energy from a global point of view without knowing any microlocal information, which have been extensively developed in the study of hyperbolic systems of conservation laws as well as scattering theory [6, 25, 28, 36]; the piecewise (in time) feedback law is introduced in [16] to stabilize the one dimensional heat equation in finite time together with the backstepping method, instead of using stationary feedback laws. This method shares several advantages:

- The designed feedback laws are simple and explicit to be compared with some other stabilization techniques, for instance the powerful Riccati method requires on solving some algebraic nonlinear Riccati equation;
- The quantitative rapid stabilization combined with the piecewise continuous feedback law argument leads to null controllability without applying Lions’ fundamental H.U.M. [28]. Moreover, this constructive approach also provides explicit (and probably optimal) controlling costs;
- The feedback laws are stable under perturbation. As a direct consequence, the same feedback law can be used to stabilize (rapidly or even in finite time) nonlinear models with satisfying costs.

Inspired by [35] we prove the following theorems concerning quantitative rapid stabilization, local null controllability with cost estimates, and finite time stabilization for the two dimensional incompressible internal controlled Navier–Stokes equations, the proofs of which are presented in Section 3, Section 4, and Section 5 respectively.

**Theorem 1.1 (Quantitative rapid stabilization).** There exists an effectively computable constant $C_2 > 0$ such that for any $\lambda > 0$ we can construct an explicit stationary feedback law $F_\lambda : \mathcal{H} \to \mathcal{H}$, such that the closed-loop system (1.1)–(1.2) with the feedback law $U(t; y) := F_\lambda y$ is locally
expontially stable:
\[ \| \Phi(t, s; y_0) \|_H + \| \mathcal{F}_t \Phi(t, s; y_0) \|_H \leq 2 C_2 e^{C_2 \sqrt{\lambda} - \frac{2}{3} (t-s)} \| y_0 \|_H, \quad \forall s \in \mathbb{R}, \forall t \in [s, +\infty), \]
for any \( \| y_0 \|_H \leq C_2^{-1} e^{-C_2 \sqrt{\lambda}}. \)

**Theorem 1.2** (Quantitative null controllability with cost estimates). There exists an effectively computable constant \( C_3 > 0 \) such that, for any \( T \in (0, 1) \), and for any \( \| y_0 \|_H \leq e^{-C_3 T} \) we can find an explicit control \( f \mid _{[0,T]} \) satisfying
\[ \| f(t, x) \|_{L^\infty(0,T; H)} \leq e^{C_3 T} \| y_0 \|_H, \]
such that the unique solution of the controlled system (1.1) with initial state \( y(0, x) = y_0(x) \) and the control \( f \mid _{[0,T]} \) verifies \( y(T, x) = 0. \)

**Theorem 1.3** (Small-time local stabilization with explicit feedback laws). For any \( T > 0 \), we find an effectively computable constant \( \Lambda_T \) and construct an explicit \( T \)-periodic proper feedback law \( U \) satisfying
\[ \| U(t; y) \|_H \leq \min \{ 1, 2 \| y \|_H^{1/2} \}, \quad \forall y \in \mathcal{H}, \quad \forall t \in \mathbb{R}, \]
that stabilizes system (1.1)-(1.3) in finite time:

(i) (2T stabilization) \( \Phi(2T + t, x; y_0) = 0, \quad \forall t \in \mathbb{R}, \quad \forall \| y_0 \|_H \leq \Lambda_T. \)

(ii) (Uniform stability) For every \( \delta > 0 \) there exists an effectively computable \( \eta > 0 \) such that
\[ (\| y_0 \|_H \leq \eta) \Rightarrow (\| \Phi(t, t'; y_0) \|_H \leq \delta, \quad \forall t' \in \mathbb{R}, \quad \forall t \in (t', +\infty)). \]

2. Preliminary

2.1. Functional framework. We refer to the book by Chemin [9] for the functional analysis framework and well-posedness results concerning Navier-Stokes equations, and the book by Coron [10] for introduction on the related control theory. In the context if there is no confusing sometimes we simply denote \( L^2(\Omega)^2 \) by \( L^2(\Omega) \) or \( L^2 \).

1. (1) Leray projection and spectral decomposition.

According to Helmholtz decomposition, for any \( u \in L^2(\Omega)^2 \) there exist unique \( v \in \mathcal{H} \) and \( \nabla p \in L^2(\Omega)^2 \) such that \( u = v + \nabla p \), which defines the (orthogonal) Leray projection \( \mathbb{P} \) on \( L^2(\Omega)^2 \):
\[ \begin{cases} \mathbb{P} : L^2(\Omega)^2 \to \mathcal{H} \\ u \mapsto \mathbb{P} u := u - \nabla p. \end{cases} \]

Notice that for any \( f \in \mathcal{H} \),
\[ \| \mathbb{P}(1_\omega f) \|_H \leq \| 1_\omega f \|_{L^2(\Omega)^2} \leq \| f \|_{L^2(\Omega)^2} = \| f \|_H, \]
which allows us to estimate the control term via \( \| f \|_{L^2} \) (or equivalently \( \| f \|_H \)).

Let \( \{ e_i \}_{i=1}^{\infty} \subset \mathcal{V}_\sigma \) be the orthonormal basis of \( \mathcal{H} \) given by the eigenvectors of the the Stokes operator
\[ \begin{cases} -\Delta e_i + \nabla p_i = \tau_i e_i & \text{in } \Omega, \\ \text{div } e_i = 0 & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial \Omega, \end{cases} \]
with \( 0 < \tau_1 \leq \tau_2 \leq \tau_3 \leq \ldots \leq \tau_n \leq \ldots \) and \( \lim_{i \to \infty} \tau_i = +\infty. \) Let \( \mathcal{H}_N \) be the low frequency subspace of \( \mathcal{H} \), and \( \mathbb{P}_N \) be its orthogonal projection,
\[ \mathcal{H}_N := \text{Vect} \{ e_i \}_{i=1}^{N}. \]

In terms of the above eigenvectors Leray projection can be extended to \( \mathcal{V}' \),
\[ \begin{cases} \mathbb{P} : \mathcal{V}' \to \mathcal{V}' \\ u \mapsto \mathbb{P} u := u - \nabla p, \end{cases} \]
where \( p \in L^2_{\text{loc}}(\Omega) \), and \( \nabla p \) belongs to \( \mathcal{V}_\sigma^0 \) as polar space of \( \mathcal{V}_\sigma \),

\[
\mathcal{V}_\sigma^0 := \{ f \in \mathcal{V} : \langle f, v \rangle_{\mathcal{V}' \times \mathcal{V}} = 0, \ \forall \ v \in \mathcal{V}_\sigma \}.
\]

More precisely,

\[
\mathbb{P} u := \sum_{i=1}^{\infty} \langle u, e_i \rangle_{\mathcal{V}' \times \mathcal{V}} e_i \in \mathcal{V}' \quad \text{for } u \in \mathcal{V}',
\]

\[
\mathbb{P}_N u := \sum_{i=1}^{N} \langle u, e_i \rangle_{\mathcal{V}' \times \mathcal{V}} e_i \in \mathcal{V}_\sigma \quad \text{for } u \in \mathcal{V}',
\]

\[
\mathbb{P} u := \sum_{i=1}^{\infty} \langle u, e_i \rangle_{L^2(\Omega)^2} e_i \in \mathcal{H} \quad \text{for } u \in L^2(\Omega)^2,
\]

with

\[
\langle u, v \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle u, v \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} = \langle \mathbb{P} u, v \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma}, \ \forall \ u \in \mathcal{V}', \ \forall \ v \in \mathcal{V}_\sigma.
\]

Furthermore, the related \( \mathcal{H} \)-norm, \( \mathcal{V} \)-norm, and \( \mathcal{V}_\sigma \)-norm can be characterized by

\[
||u||^2_{\mathcal{H}} = \sum_{i=1}^{\infty} \left| \langle u, e_i \rangle_{L^2(\Omega)^2} \right|^2 \quad \text{for } u \in \mathcal{H},
\]

\[
||u||^2_{\mathcal{V}} = \sum_{i=1}^{\infty} \left| \langle u, e_i \rangle_{L^2(\Omega)^2} \right|^2 \tau_i \quad \text{for } u \in \mathcal{V},
\]

\[
||u||^2_{\mathcal{V}_\sigma} = ||\mathbb{P} u||^2_{\mathcal{V}_\sigma} = \sum_{i=1}^{\infty} \left| \langle u, e_i \rangle_{\mathcal{V} \times \mathcal{V}'} \right|^2 \tau_i^{-1} \quad \text{for } u \in \mathcal{V}'.
\]

(2) Spectral estimates.

For any \( \lambda > 0 \), we denote by \( N(\lambda) \) the number of the eigenvalues that are smaller than or equal to \( \lambda \), i.e. \( \tau_{N(\lambda)} \leq \lambda < \tau_{N(\lambda)+1} \), and define the following symmetric matrix \( J_{N(\lambda)} \),

\[
J_{N(\lambda)} := \left( \langle e_i, e_j \rangle_{L^2(\omega)^2} \right)_{i,j=1}^{N(\lambda)}.
\]

**Proposition 2.1** (Spectral estimates). There exists an effectively computable constant \( C_1 \geq 1 \) only depends on \( (\Omega, \omega) \) that is independent of \( \lambda > 0 \) such that, for any \( \lambda > 0 \) and for any \( E_{N(\lambda)} = (a_1, a_2, ..., a_{N(\lambda)}) \in \mathbb{R}^{N(\lambda)} \) the following inequality holds,

\[
E_{N(\lambda)}^T J_{N(\lambda)} E_{N(\lambda)} \geq C_1^{-1} e^{-C_1 \sqrt{\lambda}} ||E_{N(\lambda)}||_2^2.
\]

**Proof.** This is a Lebeau–Robbiano type spectral inequality [26]. By letting \( N \) be presenting \( N_\lambda \), we get

\[
E_N^T J_N E_N = \sum_{1 \leq i,j \leq N} a_i \langle e_i, e_j \rangle_{L^2(\omega)^2} a_j = \left| \sum_{i=1}^{N} a_i e_i \right|_{L^2(\omega)^2}^2 \geq C_1^{-1} e^{-C_1 \sqrt{\lambda}} ||E_N||_2^2,
\]

where we have recalled the recent paper [8, Theorem 3.1],

\[
C_1 e^{C_1 \sqrt{\lambda}} \int_{\omega} \left( \sum_{\tau_i \leq \lambda} a_i e_i(x) \right)^2 dx \geq \sum_{\tau_i \leq \lambda} a_i^2.
\]

\( \Box \)
(3) Nonlinear terms.

Next, we define the following bilinear map $Q$ as well as trilinear functional $B$,

$$
\begin{align*}
Q : \mathcal{V} \times \mathcal{V} & \to \mathcal{V}' \\
(u, v) & \mapsto -\text{div} \ (u \otimes v),
\end{align*}
$$

$$
B(u, v, w) := \left\langle Q(u, v), w \right\rangle_{\mathcal{V'} \times \mathcal{V}} = \int_\Omega [(u \cdot \nabla)v] \cdot w \, dx, \ \forall \ u, v, w \in \mathcal{V}.
$$

**Proposition 2.2** (Nonlinearity estimates). There exists a constant $c_0$ such that for any $u, v$ and $w$ in $\mathcal{V}$, we have the following estimates,

$$
|B(u, v, w) + B(u, w, v)| = 0 \text{ if } u \in \mathcal{V}_\sigma,
$$

$$
|B(u, v, w)| \leq c_0 ||u||^\frac{1}{2}_{L^2} ||v||^\frac{1}{2}_{L^2} ||\nabla u||^\frac{1}{2}_{L^2} ||\nabla v||^\frac{1}{2}_{L^2} ||\nabla w||_{L^2}.
$$

2.2. **Open loop controlled (inhomogenous) Navier-Stokes systems.** The open loop controlled equation is indeed an inhomogenous equation with a force term located in the controlled domain. A general inhomogenous equation (without any restriction on force terms) is presented by,

$$
\begin{align*}
\begin{cases}
y_t - \Delta y + (y \cdot \nabla) y + \nabla p = f(t, x), & (t, x) \in (t_1, t_2) \times \Omega, \\
\text{div} \ y(t, x) = 0, & (t, x) \in (t_1, t_2) \times \Omega,
\end{cases}
\end{align*}
$$

(2.2)

$$
\begin{align*}
y(t_1, x) = y_0(x), & \quad x \in \Omega,
\end{align*}
$$

where $t_2$ can be taken as $+\infty$. We are interested in the solutions under Leray’s weak solution sense [27]: for any $y_0 \in \mathcal{H}$ and any $f \in L^2_{loc}(t_1, t_2; \mathcal{V}')$, the solution of equation (2.2) is some $y \in C([t_1, t_2]; \mathcal{H}) \cap L^2_{loc}(t_1, t_2; \mathcal{V}_\sigma)$ such that, for any test function $\phi$ in $C^1([t_1, t_2]; \mathcal{V}_\sigma)$, the vector field $y$ satisfies the following condition:

$$
\begin{align*}
(y(t), \phi(t))_{\mathcal{H}} &= (y_0, \phi(0))_{\mathcal{H}} + \int_{t_1}^t \left\langle \Delta \phi(s) + \partial_t \phi(s), y(s) \right\rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma} \, ds \\
&+ \int_{t_1}^t \left\langle f(s), \phi(s) \right\rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma} \, ds,
\end{align*}
$$

(2.3)

for every $t \in [t_1, t_2]$.

**Theorem 2.3** (Leray theorem on well-posedness and stability of the solutions). For any $y_0 \in \mathcal{H}$ and any $f \in L^2_{loc}(t_1, t_2; L^2(\Omega)^2)$, the Cauchy problem (2.2) admits a unique solution. This unique solution is also in $H^1_{loc}(t_1, t_2; \mathcal{V})$. Moreover, there exists some constant $C_0$ independent of $t_1$ and $t_2$ such that this unique solution satisfies,

$$
\begin{align*}
\frac{1}{2}||y(t, x)||^2_{\mathcal{H}} + \int_{t_1}^t ||\nabla y(s, x)||^2_{L^2} ds &= \frac{1}{2}||y_0||^2_{\mathcal{H}} + \int_{t_1}^t \left\langle f(s), y(s) \right\rangle_{\mathcal{V}'_\sigma \times \mathcal{V}_\sigma} \, ds,
\end{align*}
$$

(2.4)

$$
\begin{align*}
||y(t, x)||^2_{\mathcal{H}} + \int_{t_1}^t ||\nabla y(s, x)||^2_{L^2} ds &\leq ||y_0||^2_{\mathcal{H}} + C_0 \int_{t_1}^t ||f(s)||^2_{L^2} ds,
\end{align*}
$$

(2.5)

for any $t \in [t_1, t_2]$.

Furthermore, the Leray solutions are stable in the following sense. Let $y$ (resp. $z$) be the Leray solution associated with $y_0$ (resp. $z_0$) in $\mathcal{H}$ and $f$ (resp. $g$) in the space $L^2_{loc}(t_1, +\infty; L^2(\Omega)^2)$, then
for \( w := y - z \) and for any \( t \in (t_1, +\infty) \) we have
\[
||w(t)||^2_H + \int_{t_1}^{t} ||\nabla w(s, x)||^2_{L^2} ds \leq \left( ||w_0||^2_H + C_0 \int_{t_1}^{t} ||(f - g)(s)||^2_{L^2} ds \right) \exp(CE^2(t)),
\]
\[
E(t) := \min \left\{ \||y_0||^2_H + C_0 \int_{t_1}^{t} ||f(s)||^2_{L^2} ds, ||z_0||^2_H + C_0 \int_{t_1}^{t} ||g(s)||^2_{L^2} ds \right\}.
\]
Actually Theorem 2.3 holds for \( f \) in \( L^2_{\infty}(t_1, +\infty; \mathcal{V}') \), for which the related inequalities are governed by the \( L^2(\mathcal{V}') \)-norm of \( f \) and the constant \( C_0 \) can be taken as 1.

2.3. Time-varying feedback laws, closed-loop systems, and finite time stabilization. In this section we recall the precise definition of time-varying feedback laws as well as the related closed-loop solutions.

**Definition 2.4** (Closed-loop systems). Let \( s_1 \in \mathbb{R} \) and \( s_2 \in \mathbb{R} \) be given such that \( s_1 < s_2 \). Let the time-varying feedback law on interval \([s_1, s_2]\) be an application
\[
(U : \mathbb{R}^2 \times H \rightarrow H)
\]
\[
(t; y) \mapsto U(t; y).
\]
Let \( t_1 \in [s_1, s_2] \), \( t_2 \in (t_1, s_2] \), and \( y_0 \in H \). A solution on \([t_1, t_2]\) to the Cauchy problem associated to the closed-loop system (1.1)-(1.2) with (2.6) for initial data \( y_0 \) at time \( t_1 \) is some \( y : [t_1, t_2] \rightarrow H \) such that
\[
t \in (t_1, t_2) \mapsto f(t, x) := 1_w U(t; y(t)) \in L^2(t_1, t_2; L^2(\Omega)^2),
\]
\[
y \text{ is a Leray solution of (2.2) with initial data } y_0 \text{ at time } t_1 \text{ and the above force term } f(t, x).
\]

**Definition 2.5** (Proper feedback laws). Let \( s_1 \in \mathbb{R} \) and \( s_2 \in \mathbb{R} \) be given such that \( s_1 < s_2 \). A proper feedback law on \([s_1, s_2]\) is an application \( U \) of type (2.6) such that, for every \( t_1 \in [s_1, s_2] \), for every \( t_2 \in (t_1, s_2] \), and for every \( y_0 \in H \), there exists a unique solution on \([t_1, t_2]\) to the Cauchy problem associated to the closed-loop system (1.1)-(1.2) with (2.6) for initial data \( y_0 \) at time \( t_1 \) according to Definition 2.4.

A proper feedback law is an application \( U \) of type (1.3) such that, for every \( s_1 \in \mathbb{R} \) and for every \( s_2 \in \mathbb{R} \) satisfying \( s_1 < s_2 \), the feedback law restricted to \([s_1, s_2] \times H\) is a proper feedback law on \([s_1, s_2]\).

For a proper feedback law, one can define the flow \( \Phi : \Delta \times H \rightarrow H \) associated to this feedback law, with \( \Delta := \{(t, s); t > s\} \): \( \Phi(t, s; y_0) \) is the value at time \( t \) of the solution \( y \) to the closed-loop system (1.1)-(1.3) which is equal to \( y_0 \) at time \( s \).

**Definition 2.6** (Finite time local stabilization of Navier-Stokes equations). Let \( T > 0 \). A \( T \)-periodic proper feedback law \( U \) locally stabilizes the two dimensional Navier-Stokes equation in finite time, if for some \( \varepsilon > 0 \) the flow \( \Phi \) of the closed-loop system (1.1)-(1.3) verifies,
\[
(i) \quad (2T \text{ stabilization}) \quad \Phi(2T + t, t; y_0) = 0, \quad \forall t \in \mathbb{R}, \forall ||y_0||_H \leq \varepsilon,
\]
\[
(ii) \quad (\text{Uniform stability}) \quad \text{For every } \delta > 0, \text{ there exists } \eta > 0 \text{ such that}
\]
\[
( ||y_0||_H \leq \eta) \Rightarrow ( ||\Phi(t, t'; y_0)||_H \leq \delta, \forall t' \in \mathbb{R}, \forall t \in (t', +\infty) )
\]

2.4. Well-posedness of closed-loop systems. Finally we present well-posedness results concerning closed-loop systems with stationary Lipschitz feedback laws. Concerning linear feedback laws one has the following well-posedness result.
Theorem 2.7. Let $T > 0$. Let given vector functions $\{\varphi_i\}_{i=1}^n \in \mathcal{H}$ and bounded linear operators $\{l_i\}_{i=1}^n : \mathcal{H} \to \mathbb{R}$. For any $y_0 \in \mathcal{H}$ the Cauchy problem
\[
\begin{aligned}
y_t - \Delta y + (y \cdot \nabla) y + \nabla p &= 1_\omega \left( \sum_{i=1}^n l_i(y) \varphi_i \right), & (t, x) \in (0, T) \times \Omega, \\
div y &= 0, & (t, x) \in (0, T) \times \Omega, \\
y(t, x) &= 0, & (t, x) \in (0, T) \times \partial \Omega, \\
y(0, x) &= y_0(x), & x \in \Omega,
\end{aligned}
\]

admits a unique solution.

As will consider finite time stabilization problems we also introduce “cutoff” type feedback laws. For any $r \in (0, 1/2]$ we find some smooth cutoff function $\chi_r \in C^\infty(\mathbb{R}^+; [0, 1])$ satisfying
\[
\chi_r(x) = \begin{cases} 
1 & \text{if } x \in [0, r], \\
0 & \text{if } x \in [2r, +\infty),
\end{cases}
\]
and further define the following related Lipschitz operator $\mathcal{K}_r : \mathcal{H} \to \mathcal{H}$,
\[
\mathcal{K}_r(y) := y \cdot \chi_r (||y||_\mathcal{H}), \ \forall \ y \in \mathcal{H},
\]

satisfying, for some constant $L_r$ depending on $r$,
\[
||\mathcal{K}_r(y)||_\mathcal{H} \leq \min\{1, ||y||_\mathcal{H}\},
\]

\[
||\mathcal{K}_r(y) - \mathcal{K}_r(z)||_\mathcal{H} \leq L_r ||y - z||_\mathcal{H}, \ \forall \ y, z \in \mathcal{H}.
\]

Theorem 2.8. Let $T > 0$. Let $r \in (0, 1/2]$. Let given vector functions $\{\varphi_i\}_{i=1}^n \in \mathcal{H}$ and bounded linear operators $\{l_i\}_{i=1}^n : \mathcal{H} \to \mathbb{R}$. For any $y_0 \in \mathcal{H}$ the Cauchy problem
\[
\begin{aligned}
y_t - \Delta y + (y \cdot \nabla) y + \nabla p &= 1_\omega \mathcal{K}_r \left( \sum_{i=1}^n l_i(y) \varphi_i \right), & (t, x) \in (0, T) \times \Omega, \\
div y &= 0, & (t, x) \in (0, T) \times \Omega, \\
y(t, x) &= 0, & (t, x) \in (0, T) \times \partial \Omega, \\
y(0, x) &= y_0(x), & x \in \Omega,
\end{aligned}
\]

admits a unique solution.

Both the closed-loop systems with linear feedback laws and with Lipschitz nonlinear feedback laws are well-posed correspond to Theorem 2.7 and Theorem 2.8, the detailed proofs of which we omit. Indeed, local (in time) existence and uniqueness of solutions are based on Leray’s theorem 2.3 and Banach fixed point theorem: let the Lipschitz constant of the feedback law be $L$ and let $||y_0||_\mathcal{H} = M$, for some $T$ small enough we consider the Banach space
\[
\mathcal{X}_T := C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}_\sigma),
\]

\[
\mathcal{X}_T(2M) := \{ y \in \mathcal{X}_T : ||y||_{\mathcal{X}_T}^2 = ||y||_{C([0,T];\mathcal{H})}^2 + ||\nabla y||_{L^2(0,T;L^2)}^2 \leq 4M^2 \},
\]

and find the fixed point of the following application,
\[
\begin{aligned}
\mathcal{S} : & \mathcal{X}_T(2M) \to \mathcal{X}_T, \\
y & \mapsto \mathcal{S}(y),
\end{aligned}
\]
where $\mathcal{S}(y)$ is the solution of Cauchy problem (2.2) with the initial state $y_0$ and force (control) term $f = 1_\omega \mathcal{K}_r \left( \sum_{i=1}^n l_i(y) \varphi_i \right)$. Moreover, since $y = \mathcal{S}(y)$ is the solution of the Cauchy problem (2.2) with control $f = 1_\omega \mathcal{K}_r \left( \sum_{i=1}^n l_i(y) \varphi_i \right)$, thanks to Theorem 2.3, this solution also belongs to the space $H^1(0, T; \mathcal{V}_\sigma')$. In the end, some a priori energy estimates lead to global (in time) solutions. We also emphasize that the Lipschitz condition is crucial in order to guarantee the uniqueness. Otherwise one may need to use other compactness arguments to prove existence of solutions, see for example [17].
2.5. **On the choice of constants.** In this section we conclude the values of the constants that will be used later on.

- For any given $\lambda > 0$, we define

$$\gamma_\lambda := C_1 e^{C_1 \sqrt{\lambda}} \quad \text{and} \quad \mu_\lambda := \frac{\gamma_\lambda^2}{\lambda^2} = C_1^2 e^{2C_1 \sqrt{\lambda}} > 1.$$  

- By recalling the definition of $C_1$ in Proposition 2.1, we further select some $C_2 \in [3C_1, +\infty)$ such that for all $\lambda > 0$,

$$\frac{C_1 e^{C_1 \sqrt{\lambda}}}{8(1 + \lambda C_1)} \leq C_2 e^{C_1 \sqrt{\lambda}}.$$  

and define

$$r_\lambda := \left( C_2 e^{C_2 \sqrt{\lambda}} \right)^{-1}.$$  

- Then we choose some constant $Q > 0$ satisfying

$$C_1 e^{C_1 Q m}, C_2 e^{C_2 Q m} \leq e^{Q m}, \quad \forall \ m \geq 1,$$

and select

$$C_3 := \frac{Q^2}{32}.$$  

3. **Quantitative rapid stabilization**

Inspired by the recent work [35] on the stabilization of the heat equations, we directly define the following stationary feedback law

$$F_\lambda y := -\gamma_\lambda \mathbb{P}_N(\lambda) y, \quad \forall y \in \mathcal{H},$$

and consider the following closed-loop system,

$$\begin{cases}
    y_t = \Delta y - (y \cdot \nabla)y - \nabla p - \gamma_\lambda \omega \mathbb{P}_N y & \text{in } \Omega, \\
    \text{div } y = 0 & \text{in } \Omega, \\
    y = 0 & \text{on } \partial \Omega,
\end{cases}$$

where, and from now on, we simply denote $N(\lambda)$ by $N$. Furthermore, the low frequency system satisfies

$$\frac{d}{dt}(\mathbb{P}_N y) = \mathbb{P}_N(\Delta y) - \mathbb{P}_N((y \cdot \nabla)y) - \gamma_\lambda \mathbb{P}_N \left( \omega \mathbb{P}_N y \right).$$

Since $y$ lives in $\mathcal{H}$, we decompose

$$y(t, x) = \mathbb{P} y(t, x) = \sum_{i=1}^{\infty} y_i(t) e_i,$$

and

$$\mathbb{P}(\omega e_i) = \sum_{j=1}^{\infty} \langle \omega e_i, e_j \rangle_{L^2(\Omega)} e_j = \sum_{j=1}^{\infty} \langle e_i, e_j \rangle_{L^2(\Omega)} e_j,$$

and

$$\mathbb{P}_N((y \cdot \nabla)y) = \sum_{i=1}^{N} \langle (y \cdot \nabla)y, e_i \rangle_{V' \times V} e_i,$$

and

$$\mathbb{P}_N \left( \omega \mathbb{P}_N y \right) = \sum_{i=1}^{N} \sum_{j=1}^{N} \langle y_i(t), e_j \rangle_{L^2(\omega)^2} e_j.$$
By defining

\[
X_N(t) := \begin{pmatrix}
y_1(t) \\
y_2(t) \\
\vdots \\
y_N(t)
\end{pmatrix},
Y_N(t) := \begin{pmatrix}
-\langle (y \cdot \nabla) y, e_1 \rangle_{V' \times V(t)} \\
-\langle (y \cdot \nabla) y, e_2 \rangle_{V' \times V(t)} \\
\vdots \\
-\langle (y \cdot \nabla) y, e_N \rangle_{V' \times V(t)}
\end{pmatrix},
A_N := \begin{pmatrix}
-\tau_1 \\
-\tau_2 \\
\vdots \\
-\tau_N
\end{pmatrix},
\]

we know that the finite dimensional system \(X_N(t)\) satisfies

\[
\dot{X}_N(t) = A_N X_N(t) - \gamma \lambda J N X_N(t) + Y_N(t).
\]

Let us consider the following Lyapunov functional on \(\mathcal{H}\),

\[
V(y) := \mu_\lambda \left( \mathbb{P}_N y, \mathbb{P}_N y \right)_{L^2(\Omega)} + \left( \mathbb{P}_N^+y, \mathbb{P}_N^+ y \right)_{L^2(\Omega)^2} = \mu_\lambda ||X_N||^2 + \left( \mathbb{P}_N^+ y, \frac{d}{dt} y \right)_{V'_\sigma \times V'_\sigma},
\]

for every \(y \in \mathcal{H}\), where \(||X_N||^2\) denotes \(\sum_{i=1}^N y_i^2\).

Concerning the variation of the Lyapunov functional, at least for \(y(t)\) regular enough, for example \(C^1([0, T]; V'_\sigma) \cap C^0([0, T]; V_\sigma)\), one has

\[
\frac{d}{dt} V(y(t)) = \mu_\lambda \frac{d}{dt} ||X_N||^2 + \frac{d}{dt} \left( \mathbb{P}_N^+ y, \mathbb{P}_N^+ y \right)_{L^2(\Omega)^2} = 2\mu_\lambda X_N^T X_N + 2 \left( \mathbb{P}_N^+ y, \frac{d}{dt} y \right)_{V'_\sigma \times V'_\sigma},
\]

and the value of \(-Y_N^T X_N\) is given by

\[
\left\langle \mathbb{P}_N \left( (y \cdot \nabla) y \right), \mathbb{P}_N y \right\rangle_{V'_\sigma \times V'_\sigma} = \left\langle \mathbb{P} \left( (y \cdot \nabla) y \right), \mathbb{P}_N y \right\rangle_{V'_\sigma \times V'_\sigma} = \left\langle (y \cdot \nabla) y, \mathbb{P}_N y \right\rangle_{V' \times V} = \mathcal{B}(y, y, \mathbb{P}_N y).
\]

Hence, on the one hand, thanks to Proposition 2.1, Proposition 2.2, and the choice of \(\gamma_\lambda\) and \(\mu_\lambda\), we have

\[
\mu_\lambda \frac{d}{dt} ||X_N||^2 = \mu_\lambda X_N^T (A_N^T + A_N - 2\gamma \lambda J N) X_N + \mu_\lambda (Y_N^T X_N + X_N^T Y_N)
\leq 2\mu_\lambda X_N^T (A_N - \gamma \lambda J N) X_N + 2\mu_\lambda X_N^T Y_N,
\]

\[
\leq -2\mu_\lambda \gamma \lambda \left( C_1 e^{C_1 \sqrt{\lambda}} \right)^{-1} ||X_N||^2 - 2\mu_\lambda ||\nabla \mathbb{P}_N y||^2_{L^2(\Omega)} + 2\mu_\lambda ||\mathcal{B}(y, y, \mathbb{P}_N y)||,
\]

\[
\leq -\mu_\lambda \gamma \lambda \left( ||X_N||^2 - 2\mu_\lambda ||\nabla \mathbb{P}_N y||^2_{L^2(\Omega)} + 2\mu_\lambda \gamma \lambda ||\mathcal{B}(y, y, \mathbb{P}_N y)|| + 2\mu_\lambda \gamma \lambda ||\nabla \mathbb{P}_N y|| ||\nabla \mathbb{P}_N y||_{L^2(\Omega)} + 2\mu_\lambda \gamma \lambda ||\nabla \mathbb{P}_N y|| ||\nabla \mathbb{P}_N y||_{L^2(\Omega)} - 2\mu_\lambda \gamma \lambda ||\nabla \mathbb{P}_N y|| ||\nabla \mathbb{P}_N y||_{L^2(\Omega)}
\]

On the other hand,

\[
\frac{d}{dt} \left( \mathbb{P}_N^+ y, \mathbb{P}_N^+ y \right)_{L^2(\Omega)^2} = 2 \left\langle \mathbb{P}_N^+ y, \Delta y - (y \cdot \nabla) y - \gamma \lambda 1_\omega \mathbb{P}_N y - \nabla p \right\rangle_{V'_\sigma \times V'_\sigma},
\]

\[
= -2 \left\langle \mathbb{P}_N^+ y, \nabla \mathbb{P}_N y, 1_\omega \mathbb{P}_N y \right\rangle_{L^2(\Omega)^2} - 2 \left\langle (y \cdot \nabla) y, \mathbb{P}_N^+ y \right\rangle_{V'_\sigma \times V'_\sigma},
\]

\[
= -2 \sum_{i=N+1}^\infty \tau_i y_i^2 - 2\gamma \lambda \left( \mathbb{P}_N^+ y, 1_\omega \mathbb{P}_N y \right)_{L^2(\Omega)^2} - 2 \mathcal{B}(y, y, \mathbb{P}_N y),
\]

\[
\leq -2 \sum_{i=N+1}^\infty \tau_i y_i^2 + 2\gamma \lambda ||\mathbb{P}_N^+ y|| ||1_\omega \mathbb{P}_N y|| ||\nabla \mathbb{P}_N y||_{L^2(\Omega)} + 2\mathcal{B}(y, y, \mathbb{P}_N y),
\]

\[
\leq -\frac{\gamma}{2} ||\mathbb{P}_N^+ y||^2_{L^2(\Omega)} - \frac{1}{2} ||\nabla \mathbb{P}_N y||^2_{L^2(\Omega)} + \lambda ||\mathbb{P}_N^+ y||^2_{L^2(\Omega)} + \frac{\gamma^2}{\lambda} ||X_N||^2 + 2c_0 ||y|| ||\nabla \mathbb{P}_N y|| ||\nabla \mathbb{P}_N y||_{L^2(\Omega)} + 2\mathcal{B}(y, y, \mathbb{P}_N y),
\]

\[
\leq -\frac{\gamma}{2} ||\mathbb{P}_N^+ y||^2_{L^2(\Omega)} - \frac{1}{2} ||\nabla \mathbb{P}_N y||^2_{L^2(\Omega)} + \mu_\lambda ||X_N||^2 + 2c_0 ||y|| ||\nabla \mathbb{P}_N y|| ||\nabla \mathbb{P}_N y||_{L^2(\Omega)}.
\]
Combining the preceding three inequalities, we further get
\[
\frac{d}{dt} V(y(t)) \leq -2\mu_\lambda \|X_N\|^2 - 2\mu_\lambda \|\nabla P_N y\|^2_{L^2(\Omega)} + 2\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|^2_{L^2(\Omega)}
\]
\[
- \frac{1}{2} \lambda \|P_N y\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla P_N y\|^2_{L^2(\Omega)} + \mu_\lambda \|X_N\|^2 + 2c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|^2_{L^2(\Omega)},
\]
\[
\leq -\mu_\lambda \|X_N\|^2 - \frac{1}{2} \|P_N y\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla y\|^2_{L^2(\Omega)} + 4\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|^2_{L^2(\Omega)},
\]
\[
\leq \left( -\frac{\lambda}{2} \right) V(y(t)) - \frac{1}{2}\|\nabla y\|^2_{L^2(\Omega)} + 4\mu_\lambda c_0 \|y\|_{L^2(\Omega)} \|\nabla y\|^2_{L^2(\Omega)},
\]
\[
\leq \left( -\frac{\lambda}{2} \right) V(y(t)) - \|\nabla y\|^2_{L^2(\Omega)} \left( \frac{1}{2} - 4\mu_\lambda c_0 V^\frac{1}{2}(y(t)) \right).
\]
Eventually, according to Theorem 2.3 and Theorem 2.7 the solution \( y \) lives indeed in the space \( H^1(0, T; \mathcal{V}) \cap C^0([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}_0) \), which is included in \( H^1(0, T; \mathcal{V}_0') \cap L^2(0, T; \mathcal{V}_0) \). Hence the preceding inequality holds in the distribution sense in \( L^1(0, T) \). Inspired by the same formula, at first by ignoring the first term in the right hand side we know that the value of \( \max \{V(y(t)), (8\mu_\lambda c_0)^{-2}\} \) decreases. Therefore, if \( V(y(0)) \leq (8\mu_\lambda c_0)^{-2} \) then the value of \( V(y(t)) \) is always smaller than \( (8\mu_\lambda c_0)^{-2} \). As a consequence in the preceding inequality one can next ignore the second term involving \( \nabla y \), which results in that the Lyapunov functional \( V(y(t)) \) decay exponentially with decay rate \( \lambda/2 \). More precisely, by the choice of \( r_\lambda \) for any initial data \( y_0 \in \mathcal{H} \) satisfying \( \|y_0\|_{L^2(\Omega)} \leq r_\lambda \) we have
\[
V(y_0) \leq \mu_\lambda \|y_0\|^2_{L^2(\Omega)} \leq \mu_\lambda r_\lambda^2 \leq (8\mu_\lambda c_0)^{-2},
\]
thus
\[
V(y(t)) \leq e^{-\frac{\lambda}{2}t} V(y(0)), \quad \forall \ t \geq 0.
\]

Consequently
\[
\|y(t)\|^2_{L^2(\Omega)} \leq V(y(t)) \leq e^{-\frac{\lambda}{2}t} V(y(0)) \leq e^{-\frac{\lambda}{2}t} \mu_\lambda \|y(0)\|^2_{L^2(\Omega)} \leq C_1^2 e^{2C_1 \sqrt{\lambda} e^{-\frac{\lambda}{2}t}} \|y(0)\|^2_{L^2(\Omega)}.
\]
Therefore, for any \( \|y_0\|_{L^2(\Omega)} \leq r_\lambda \) the solution decays exponentially,
\[
\|y(t)\|_{L^2(\Omega)} \leq C_1 e^{C_1 \sqrt{\lambda} e^{-\frac{\lambda}{2}t}} \|y(0)\|_{L^2(\Omega)}, \forall \ t \in [0, +\infty),
\]
\[
\|\mathcal{F}_\lambda y(t)\|_{L^2(\Omega)} \leq \gamma_\lambda \|y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2 \sqrt{\lambda} e^{-\frac{\lambda}{2}t}} \|y(0)\|_{L^2(\Omega)}, \forall \ t \in [0, +\infty).
\]

Therefore, we have proved the following theorem that quantifies Theorem 1.1.

**Theorem 3.1 (Local stabilization with linear feedback laws).** For any \( \lambda > 0 \), for any \( \|y_0\|_\mathcal{H} \leq r_\lambda \), and for any \( s \in \mathbb{R} \) the Cauchy problem

\[
\begin{aligned}
\begin{cases}
y_t - \Delta y + (y \cdot \nabla)y + \nabla p &= -\gamma_\lambda l \omega \mathcal{F}_\lambda y, & (t, x) \in [s, +\infty) \times \Omega, \\
div y &= 0, & (t, x) \in [s, +\infty) \times \Omega, \\
y(t, x) &= 0, & (t, x) \in [s, +\infty) \times \partial \Omega, \\
y(s, x) &= y_0(x), & x \in \Omega,
\end{cases}
\end{aligned}
\]

(3.7)

has a unique solution in \( C^0([s, +\infty); \mathcal{H}) \cap L^2_{loc}(s, +\infty; \mathcal{V}_0) \). Moreover, this unique solution verifies

\[
\|y(t)\|_{L^2(\Omega)} \leq C_1 e^{C_1 \sqrt{\lambda} e^{-\frac{\lambda}{2}(t-s)}} \|y_0\|_{L^2(\Omega)}, \forall \ t \in [s, +\infty),
\]

(3.8)
\[
\|\mathcal{F}_\lambda y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2 \sqrt{\lambda} e^{-\frac{\lambda}{2}(t-s)}} \|y_0\|_{L^2(\Omega)}, \forall \ t \in [s, +\infty).
\]

(3.9)

Actually similar result also holds for nonlinear feedback laws \( \mathcal{K}_\lambda (\mathcal{F}_\lambda y) \) provided by equations (2.7)--(2.8) and (3.1). From the preceding theorem we observe that for initial state \( \|y_0\|_\mathcal{H} \leq r_\lambda^2 \), the unique solution \( y(t) \) of the Cauchy problem (3.7) satisfies

\[
\|\mathcal{F}_\lambda y(t)\|_{L^2(\Omega)} \leq C_2 e^{C_2 \sqrt{\lambda} e^{-\frac{\lambda}{2}(t-s)}} \|y_0\|_{L^2(\Omega)} \leq C_2 e^{C_2 \sqrt{\lambda} \lambda^2} = r_\lambda, \forall \ t \in [s, +\infty).
\]

(3.10)
Now, we replace the linear feedback law $F_\lambda$ by $K_{r\lambda}(F_\lambda y)$ (see equation (2.8)), which satisfies
\begin{equation}
||K_{r\lambda}(F_\lambda y)||_{L^2(\Omega)} \leq \min\{1, \sqrt{2}||y||_{L^2(\Omega)}\}.
\end{equation}
Indeed, if $||F_\lambda y||_{L^2(\Omega)} \leq 2r_\lambda$, then since the operator norm $||F_\lambda|| \leq \gamma_\lambda \leq r_\lambda^{-1}$, we have,
\begin{equation}
||K_{r\lambda}(F_\lambda y)||_{L^2(\Omega)} \leq ||F_\lambda y||_{L^2(\Omega)} \leq 2r_\lambda||F_\lambda y||_{L^2(\Omega)} \leq \sqrt{2}||y||_{L^2(\Omega)}.
\end{equation}
If $||F_\lambda y||_{L^2(\Omega)} > 2r_\lambda$, then by the definition of $K_{r\lambda}$ we know that $K_{r\lambda}(F_\lambda y) = 0$, which completes the proof of the condition (3.11).

Finally, we show that for $||y_0||_H \leq r_\lambda^2$ the solution of the closed-loop system with feedback law $K_{r\lambda}(F_\lambda y)$ also decays exponentially. Indeed, it suffices to prove that the solution $y$ verifies
\begin{equation}
K_{r\lambda}(F_\lambda y(t)) = F_\lambda y(t), \forall t \in [s, +\infty),
\end{equation}
which, by recalling the definition of $K_{r\lambda}$ in (2.7)--(2.8), is true according to (3.10),
\begin{equation}
||F_\lambda y(t)||_{L^2(\Omega)} \leq C_2 e^{C_2 \sqrt{\lambda}}||y_0||_{L^2(\Omega)} \leq C_2 e^{C_2 \sqrt{\lambda}} = r_\lambda, \forall t \in [s, +\infty).
\end{equation}

**Theorem 3.2** (Local stabilization with nonlinear Lipschitz feedback laws). For any $\lambda > 0$, for any $||y_0||_H \leq r_\lambda^2$, and for any $s \in \mathbb{R}$ the Cauchy problem
\begin{equation}
\begin{aligned}
y_t - \Delta y + (y \cdot \nabla)y + \nabla p &= -\gamma_\lambda 1_\omega K_{r\lambda}(F_\lambda y), \quad (t, x) \in [s, +\infty) \times \Omega, \\
\text{div } y &= 0, \quad (t, x) \in [s, +\infty) \times \Omega, \\
y(t, x) &= 0, \quad (t, x) \in [s, +\infty) \times \partial \Omega, \\
y(s, x) &= y_0(x), \quad x \in \Omega,
\end{aligned}
\end{equation}
has a unique solution in $C^0([s, +\infty); H) \cap L^2_{\text{loc}}(s, +\infty; V_0)$. Moreover, this unique solution verifies
\begin{equation}
\begin{aligned}
||y(t)||_{L^2(\Omega)} &\leq C_1 e^{C_1 \sqrt{\lambda}} e^{-\frac{1}{4}(t-s)}||y_0||_{L^2(\Omega)}, \forall t \in [s, +\infty), \\
||F_\lambda y(t)||_{L^2(\Omega)} &\leq C_2 e^{C_2 \sqrt{\lambda}} e^{-\frac{1}{4}(t-s)}||y_0||_{L^2(\Omega)}, \forall t \in [s, +\infty).
\end{aligned}
\end{equation}

4. **Quantitative null controllability with cost estimates**

In this section we construct feedback laws (controls) that yields the solution decays to zero in finite time.

**Theorem 4.1.** There exists $C_3 > 0$ such that, for any $T \in (0, 1)$, for any $y_0 \in H$ satisfying $||y_0||_H \leq e^{-\frac{C_3}{T}}$ we construct an explicit control $f(t, x)$ for the controlled system (1.1) such that the unique solution with initial data $y(0, x) = y_0(x)$ verifies $y(T, x) = 0$. Moreover,
\begin{equation}
||f(t, x)||_{L^\infty(0, T; H)} \leq e^{\frac{C_3}{T}}||y_0||_H.
\end{equation}

**Proof.** For the ease of presentation, we only consider the case when $1/T = 2^{n_0}$ with $n_0 \in \mathbb{N}^*$. The other cases can be treated via time transition, i.e. if $T \in (2^{-m-1}, 2^{-m})$ then we simply let the feedback law $U(t; y) := 0$ on the time interval $[2^{-m-1}, T]$. More precisely, we consider the following partition of $[0, T]$ and piecewise feedback laws,
\begin{equation}
T_n := 2^{-n_0} \left(1 - \frac{1}{2^n}\right), \quad I_n := [T_n, T_{n+1}], \quad \lambda_n := Q^2 2^{2(n_0+n)} \text{ for any } n \geq 0,
\end{equation}
\begin{equation}
\text{for any } n \geq 0 \text{ we consider the control (feedback law) as } F_{\lambda_n} \text{ on interval } I_n.
\end{equation}

**Control design.**
Step 1. Let the constant $R_T > 0$ be sufficiently small to be fixed later on. First, for $\|y_0\|_\mathcal{H} \leq R_T$, on the interval $I_0$ we consider the closed-loop system (1.1)–(1.2) with feedback law $U := \mathcal{F}_{\lambda_0}$ and initial data $y(0, x) = y_0(x)$. Assuming that $R_T \leq r_{\lambda_0}$, then according to Theorem 3.1 the closed-loop system has a unique solution $\bar{y}|_{I_0}$ that decays exponentially with decay rate $\lambda_0/4$.

Step 2. Next, we consider the closed-loop system with feedback law $\mathcal{F}_{\lambda_1}$ and $y(T_1, x) := \bar{y}(T_1, x)$ on $I_1$. Again we assume $\|y(T_1)\| \leq r_{\lambda_1}$ to find a unique solution $\bar{y}|_{I_1}$ that is exponentially stable.

Step 3. By continuing this procedure on $I_n$ and by always assuming $\|y(T_n)\| \leq r_{\lambda_n}$, we find a stable solution $\bar{y}|_{I_n}$.

Step 4. We denote this constructed solution $\bar{y}|_{[0, T)} \in C^0([0, T); \mathcal{H})$ by $y|_{[0, T)}$.

Step 5. For some sufficiently small $R_T$ we prove that $\|y(T_n)\|$ is indeed smaller than $r_{\lambda_n}$ for every $n \in \mathbb{N}$, and show that the solution tends to zero as $y(T) := \lim_{t \to T^-} y(t) = 0$.

Step 6. Eventually, thanks to Step 5, $y|_{[0, T)}$ is the unique solution of the Cauchy problem (2.2) with the control term $f$ given by $f|_{I_n} := \mathcal{F}_{\lambda_n} y|_{I_n}, \forall n \geq 0$, which satisfies $y(T) = 0$.

Step 7. We calculate precise cost estimates.

First we assume that for every $I_n$ the value $\|y(T_n)\|_{L^2}$ is smaller than $r_{\lambda_n}$, which, together with Theorem 3.1, implies that the solution $y|_{I_n}$ verifies

\begin{align}
\|y(t)\|_{L^2(\Omega)} &\leq C_1 e^{C_1 Q 2^n + n} e^{-\frac{Q^2}{2} 2^{(n_0 + n)}(t - T_n)} \|y(T_n)\|_{L^2(\Omega)}, \forall t \in I_n, \\
\|\mathcal{F}_{\lambda_n} y(t)\|_{L^2(\Omega)} &\leq C_2 e^{C_2 Q 2^n + n} e^{-\frac{Q^2}{2} 2^{(n_0 + n)}(t - T_n)} \|y(T_n)\|_{L^2(\Omega)}, \forall t \in I_n.
\end{align}

Consequently, for every $n \geq 1$ the value of $y(T_n)$ is dominated by

\begin{align}
\|y(T_n)\|_{L^2(\Omega)} &\leq \left(\prod_{k=0}^{n-1} C_1 e^{C_1 \sqrt{n_k}} e^{-\frac{Q^2}{2} 2^{n_0 + k}}\right) \|y_0\|_{L^2(\Omega)}, \\
&= \left(\prod_{k=0}^{n-1} C_1 e^{C_1 Q 2^{n_0 + k}} e^{-\frac{Q^2}{2} 2^{n_0 + k}}\right) \|y_0\|_{L^2(\Omega)}, \\
&\leq \left(\prod_{k=0}^{n-1} e^{\frac{Q^2}{64} 2^{n_0 + k}} e^{-\frac{Q^2}{2} 2^{n_0 + k}}\right) \|y_0\|_{L^2(\Omega)}, \\
&= \left(\prod_{k=0}^{n-1} e^{-\frac{7Q^2}{64} 2^{n_0 + k}}\right) \|y_0\|_{L^2(\Omega)}, \\
&= \exp \left(\frac{7Q^2}{64} 2^{n_0} (2^n - 1)\right) \|y_0\|_{L^2(\Omega)}.
\end{align}

Observe that the above inequality also holds for $n = 0$. Furthermore, for any $n \geq 1$ and for any $t \in I_n$ the control term is bounded by

\begin{align}
\|\mathcal{F}_{\lambda_n} y(t)\|_{L^2(\Omega)} &\leq C_2 e^{C_2 Q 2^{n_0 + n}} \|y(T_n)\|_{L^2(\Omega)} \leq \exp \left(\frac{-5Q^2}{64} 2^{n_0 + n - 1}\right) \|y_0\|_{L^2(\Omega)},
\end{align}

Clearly, the right hand side of the inequalities (4.5) and (4.6) tend to 0 as $n$ tends to $\infty$. Therefore, it suffices to prove the assumption $\|y(T_n)\|_{L^2} \leq r_{\lambda_n}$ to close the “bootstrap” and to conclude the null controllability. By recalling the definitions of $\lambda_n$, $r_{\lambda_n}$, and $Q$ we know that

\[ e^{-\frac{Q^2}{64} 2^{n_0 + n}} \leq (C_2 e^{C_2 Q 2^{n_0 + n}})^{-1} = (C_2 e^{C_2 \sqrt{\lambda_n}})^{-1} = r_{\lambda_n}, \forall n \in \mathbb{N}. \]
Hence, it suffices to find some $R_T > 0$ such that
\begin{equation}
R_T \exp \left( -\frac{7Q^2}{64} 2^{n_0} (2^n - 1) \right) \leq e^{-\frac{Q^2}{64}2^{n_0+n}} \leq r_{\lambda_n}, \quad \forall \ n \in \mathbb{N}.
\end{equation}
Thus one can take
\begin{equation}
R_T := e^{-\frac{Q^2}{64}2^{n_0}} = e^{-\frac{Q^2}{64}} = e^{-C_3} \text{ where } C_3 = \frac{Q^2}{32}.
\end{equation}
It only remains to estimate the controlling cost. Thanks to (4.6) we know that
\[ ||f(t)||_{L^2(\Omega)} \leq ||y_0||_{L^2(\Omega)}, \quad \forall \ t \in [T_1, T]. \]
As for $t \in [0, T_1)$ and the control $f|_{I_0}(t)$, we have
\[ ||F_{\lambda_n} y(t)||_{L^2(\Omega)} \leq C_2 e^{-C_3} ||y_0||_{L^2(\Omega)} \leq e^{-\frac{Q^2}{64}2^{n_0}} ||y_0||_{L^2(\Omega)} \leq e^{-\frac{Q^2}{64}2^{n_0}} ||y_0||_{L^2(\Omega)}.
\]
In conclusion, for any $||y_0||_{\mathcal{H}} \leq e^{-\frac{Q^2}{64}}$, the constructed solution $y(t,x)$ and control $f(t,x)$ satisfy
\[ ||y(t,\cdot)||_{L^2(\Omega)} \text{ and } ||f(t,\cdot)||_{L^2(\Omega)} \rightarrow 0^+, \quad \text{as } t \rightarrow T^-, \]
\[ ||y(t,\cdot)||_{L^2(\Omega)} \text{ and } ||f(t,\cdot)||_{L^2(\Omega)} \leq e^{-\frac{Q^2}{64}2^{n_0}} ||y_0||_{L^2(\Omega)}, \quad \forall \ t \in [0, T]. \]
\[ \square \]
\begin{remark}
If we replace the linear feedback laws \( \{F_{\lambda_n} y\}_{n=1}^{\infty} \) by \( \{K_{\lambda_n} (F_{\lambda_n} y)\}_{n=1}^{\infty} \) on interval $I_n$, then similar result holds. Indeed, according to Theorem 3.2 it suffices to find some initial state such that for every $n \in \mathbb{N}$ the value of $||y(T_n)||$ is smaller than $r_{\lambda_n}^2$. More precisely, instead of taking some $R_T > 0$ that satisfies (4.7), one only needs to find $\tilde{R}_T := e^{-\frac{Q^2}{64}} = e^{-\frac{Q^2}{64}}$ satisfying
\[ \tilde{R}_T \exp \left( -\frac{7Q^2}{64} 2^{n_0} (2^n - 1) \right) \leq e^{-\frac{Q^2}{64}2^{n_0+n}} \leq r_{\lambda_n}^2, \quad \forall \ n \in \mathbb{N}, \]
to guarantee that for every $n \in \mathbb{N}$ we have $||y(T_n)||_{L^2} \leq r_{\lambda_n}^2$.
\end{remark}

5. Small-time local stabilization

As in the preceding section, we only focus on the case when $T = 1/2^{n_0}$ with $n_0$ be integer. We also adapt the same construction of $T_n$ and $\lambda_n$ given by (4.1) in Section 4.

\begin{theorem}[Small-time local stabilization of Navier-Stokes equations]
Let $T = 1/2^{n_0}$ with $n_0 \in \mathbb{N}^\ast$. The following $T$-periodic feedback law $U(t,y) : \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H}$ satisfying (3.11),
\begin{equation}
U|_{[0,T) \times \mathcal{H}} (t; y) := K_{\lambda_n} (F_{\lambda_n} y), \quad \forall \ y \in \mathcal{H}, \forall \ t \in I_n, \forall \ n \in \mathbb{N},
\end{equation}
is a proper feedback law for system (1.1)–(1.2). Moreover, for some effectively computable constant $\Lambda_T$ this feedback law stabilizes system (1.1)–(1.2) in finite time:
\begin{itemize}
\item[(i)] (2$T$ stabilization) $\Phi(2T + t, y_0) = 0$, \quad $\forall \ t \in \mathbb{R}$, \quad $||y_0||_{\mathcal{H}} \leq \Lambda_T$.
\item[(ii)] (Uniform stability) For every $\delta > 0$, there exists an effectively computable $\eta > 0$ such that
\[ (||y_0||_{\mathcal{H}} \leq \eta) \Rightarrow (||\Phi(t, t'; y_0)||_{\mathcal{H}} \leq \delta, \quad \forall \ t' \in \mathbb{R}, \forall \ t \in (t', +\infty)) . \]
\end{itemize}
\end{theorem}
\begin{proof}[Proof of Theorem 5.1] We mimic the prove of the finite time stabilization of the heat equations [16, 35], as relatively standard, see also [19, 34, 33] for similar results. The proof is followed by five steps:
\begin{enumerate}
\item **Step 1.** The feedback law $U$ is a proper feedback law, i.e. for any $y_0 \in \mathcal{H}$ and for any initial time $s \in \mathbb{R}$ there exists a unique global (in time) solution.
\item **Step 2.** Null controllability: $\Phi(T, 0; y_0) = 0$ for any $y_0$ satisfying $||y_0||_{\mathcal{H}} \leq \tilde{R}_T = e^{-\frac{Q^2}{64}}$. Moreover,
\begin{equation}
||\Phi(t, 0; y_0)||_{\mathcal{H}} \leq e^{\frac{C_3}{T}} ||y_0||_{\mathcal{H}}, \quad \forall \ ||y_0||_{\mathcal{H}} \leq e^{-\frac{Q^2}{64}}, \forall \ t \in [0, T].
\end{equation}
\end{enumerate}
Step 3. For any $\tilde{\eta} > 0$, there exists some $\varepsilon(\tilde{\eta}) \in (0, \tilde{\eta})$ such that
\begin{equation}
||\Phi(t, s; y_0)||_{L^2(\Omega)} \leq \tilde{\eta}, \quad \forall \ ||y_0||_{L^2(\Omega)} \leq \varepsilon(\tilde{\eta}), \quad \forall \ s \in [0, T], \quad \forall \ t \in [s, T].
\end{equation}

Step 4. $2T$ stabilization: $\Phi(2T, s; y_0) = 0$, for any $s \in [0, T)$, for any $y_0$ satisfying $||y_0||_H \leq \varepsilon \left( e^{-\frac{2C_1}{T}} \right) =: \Lambda_T$.

Step 5. Uniform stability as direct consequence of Step 2–4.

Step 1. It suffices to prove that for any $s \in [0, T)$ the closed-loop system has a unique solution on $[s, T]$. Indeed, thanks to Theorem 2.8 there exists a unique solution on $I_n$ for any $I_n$ that intersects with $[s, T)$. Hence we find a unique solution $y$ in $C^0([s, T]; H) \cap L^2_{loc}(s, T; V_a)$. Observe that the control (provided by the related feedback law) is smaller than 1, i.e. $||f(t, x)||_{L^2(s, T; H)} \leq \sqrt{T}$. Theorem 2.3 implies that the solution $y$ is indeed in $C^0([s, T]; H) \cap L^2(s, T; V_a)$. Finally, thanks to Theorem 2.3 again, the unique solution $y$ never blow up,
\[||y(t, x)||^2_H + ||\nabla y(t, x)||^2_{L^2(s, T; L^2)} \leq ||y_0||^2_H + C_0(t - s), \quad \forall \ t \in (s, +\infty).\]

Step 2. This step is a consequence of Theorem 4.1 and Remark 4.2.

Step 3. Thanks to the fact that $||f(t, x)||_H \leq 1$ and Theorem 2.3, there exists $\tilde{T} \in (0, T)$ such that
\[||\Phi(t, s; y_0)||_H \leq \tilde{\eta}, \quad \forall \ ||y_0||_H \leq \tilde{\eta}/2, \quad \forall \ s \in [\tilde{T}, T), \quad \forall \ t \in [s, T].\]

Observe that the feedback law $U$ on $[0, \tilde{T})$ is composed by finitely many stationary feedback laws on intervals $I_n$, while, thanks to Theorem 3.2, on each of these intervals $I_n$ the system is locally exponentially stable. Consequently, there exists some $\varepsilon = \varepsilon(\tilde{\eta}) \in (0, \tilde{\eta}/2)$ such that
\[||\Phi(t, s; y_0)||_{L^2(\Omega)} \leq \tilde{\eta}/2, \quad \forall \ ||y_0||_{L^2(\Omega)} \leq \varepsilon, \quad \forall \ s \in [0, \tilde{T}), \quad \forall \ t \in [s, \tilde{T}].\]

Step 4 is a trivial combination of Step 2 and Step 3 by taking $\varepsilon \left( e^{-\frac{2C_1}{\tilde{T}}} \right)$.

Step 5 follows directly from Step 2–4.

Acknowledgments. The author would like to thank Jean-Michel Coron for having attracted his attention to this problem and for fruitful discussions. He also thanks Emmanuel Trélat for valuable discussions on this problem.

References

[1] Andrey A. Agrachev and Andrey V. Sarychev. Navier-Stokes equations: controllability by means of low modes forcing. J. Math. Fluid Mech., 7(1):108-152, 2005.
[2] Mehdi Badra and Takéo Takahashi. Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers: application to the Navier-Stokes system. SIAM J. Control Optim., 49(2):420–463, 2011.
[3] Viorel Barbu, Irena Lasiecka, and Roberto Triggiani. Tangential boundary stabilization of Navier-Stokes equations. Mem. Amer. Math. Soc., 181(852):x+128, 2006.
[4] Viorel Barbu, Sérgio S. Rodrigues, and Armen Shirikyan. Internal exponential stabilization to a nonstationary 3D Navier-Stokes equations. SIAM J. Control Optim., 49(4):1454–1478, 2011.
[5] Viorel Barbu and Roberto Triggiani. Internal stabilization of Navier-Stokes equations with finite-dimensional controllers. Indiana Univ. Math. J., 53(5):1443–1494, 2004.
[6] Tobias Breiten, Karl Kunisch, and Laurent Pfeiffer. Feedback stabilization of the two-dimensional Navier-Stokes equations by value function approximation. Appl. Math. Optim., 80(3):599–641, 2019.
[7] Felipe W. Chaves-Silva and Gilles Lebeau. Spectral inequality and optimal cost of controllability for the Stokes system. ESAIM Control Optim. Calc. Var., 22(4):1137–1162, 2016.
[8] Jean-Yves Chemin. Perfect incompressible fluids, volume 14 of Oxford Lecture Series in Mathematics and its Applications. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.
[9] Jean-Michel Coron. Control and nonlinearity, volume 136 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2007.
[11] Jean-Michel Coron and Andrei V. Fursikov. Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. *Russian J. Math. Phys.*, 4(4):429–448, 1996.

[12] Jean-Michel Coron and Sergio Guerrero. Local null controllability of the two-dimensional Navier-Stokes system in the torus with a control force having a vanishing component. *J. Math. Pures Appl. (9)*, 92(5):528–545, 2009.

[13] Jean-Michel Coron and Pierre Lissy. Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components. *Invent. Math.*, 198(3):833–880, 2014.

[14] Jean-Michel Coron, Frédéric Marbach, and Franck Sueur. Small-time global exact controllability of the Navier-Stokes equation with Navier slip-with-friction boundary conditions. *J. Eur. Math. Soc. (JEMS)*, 22(5):1625–1673, 2020.

[15] Jean-Michel Coron, Frédéric Marbach, Franck Sueur, and Ping Zhang. Controllability of the Navier-Stokes equation in a rectangle with a little help of a distributed phantom force. *Ann. PDE*, 5(2):Paper No. 17, 49, 2019.

[16] Jean-Michel Coron and Hoai-Minh Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. *Arch. Ration. Mech. Anal.*, 225(3):993–1023, 2017.

[17] Jean-Michel Coron, Ivonne Rivas, and Shengquan Xiang. Local exponential stabilization for a class of Korteweg–de Vries equations by means of time-varying feedback laws. *Anal. PDE*, 10(5):1089–1122, 2017.

[18] Jean-Michel Coron and Emmanuel Trélat. Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM J. Control Optim.*, 43(2):549–569, 2004.

[19] Jean-Michel Coron and Shengquan Xiang. Small-time global stabilization of the viscous Burgers equation with three scalar controls. *Preprint*, hal-01723188, 2017.

[20] Jean-Michel Coron, Ivonne Rivas, and Shengquan Xiang. Local exponential stabilization for a class of Korteweg–de Vries equations by means of time-varying feedback laws. *Anal. PDE*, 10(5):1089–1122, 2017.

[21] Jean-Michel Coron and Emmanuel Trélat. Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM J. Control Optim.*, 43(2):549–569, 2004.

[22] Jean-Michel Coron and Shengquan Xiang. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. *Arch. Ration. Mech. Anal.*, 225(3):993–1023, 2017.