HERMITIAN METRICS WITH NEGATIVE MEAN CURVATURE ON HOLOMORPHIC VECTOR BUNDLES

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ABSTRACT. In this paper, we study holomorphic vector bundles over compact Gauduchon manifolds. Using the continuity method, we prove the existence of $L^p$-approximate critical Hermitian structure. As its application, we show that a holomorphic vector bundle admits a Hermitian metric with negative mean curvature if and only if the maximum of slopes of all of its subsheaves is negative. Furthermore, we study the limiting behavior of the Hermitian-Yang-Mills flow. We prove the minimum of the smallest eigenvalue of the mean curvature is increasing along the Hermitian-Yang-Mills flow, while the maximum of the biggest eigenvalue is decreasing; we show that they converge to certain geometric invariants. Finally, we give some applications.

1. INTRODUCTION

Let $(M, \omega)$ be a compact Hermitian manifold of dimension $n$, and $(E, \overline{\partial}_E)$ be a holomorphic vector bundle. Given a Hermitian metric $H$ on $E$, for every holomorphic section $\zeta$ of $(E, \overline{\partial}_E)$, we have the following Bochner type formula:

\begin{equation}
\sqrt{-1} \Lambda_\omega \partial \overline{\partial} |\zeta|^2_H = |D_H \zeta|^2 - \langle \sqrt{-1} \Lambda_\omega F_H \zeta, \zeta \rangle_H,
\end{equation}

where $D_H$ is the Chern connection with respect to the Hermitian metric $H$, $F_H$ is the curvature form, $\Lambda_\omega$ denotes the contraction with $\omega$, and $\sqrt{-1} \Lambda_\omega F_H$ is called the mean curvature of $H$. By using the maximum principle and the above Bochner type formula, Kobayashi and Wu (1983) obtained the following vanishing theorem: if a holomorphic vector bundle admits a Hermitian metric with negative mean curvature, then there doesn’t exist nontrivial holomorphic sections. So, there is a natural question: how to determine whether a holomorphic vector bundle admits a Hermitian metric with negative or positive mean curvature?

We say the metric $\omega$ is Gauduchon if it satisfies $\partial \overline{\partial} \omega^{n-1} = 0$. Gauduchon (1977) proved that on a compact complex manifold, there is a unique Gauduchon metric $\omega$ up to a positive constant in the conformal class of every Hermitian metric $\hat{\omega}$. Assume $(M, \omega)$ is a...
compact Gauduchon manifold (i.e. $\partial \bar{\partial} \omega^{n-1} = 0$) and $\mathcal{F}$ is a coherent sheaf over $M$. The $\omega$-degree of $\mathcal{F}$ is given by
\begin{equation}
\deg_\omega(\mathcal{F}) := \deg_\omega(\det(\mathcal{F})) = \int_M c_1(\det(\mathcal{F}), H) \wedge \omega^{n-1},
\end{equation}
where $H$ is an arbitrary Hermitian metric on $\det \mathcal{F}$. This is a well-defined real number independent of $H$ since $\omega^{n-1}$ is $\partial \bar{\partial}$-closed. We define the $\omega$-slope of $\mathcal{F}$ as
\begin{equation}
\mu_\omega(\mathcal{F}) := \frac{\deg_\omega(\mathcal{F})}{\text{rank}(\mathcal{F})}.
\end{equation}
A holomorphic vector bundle $(E, \bar{\partial} E)$ is called $\omega$-stable (semi-stable) if for every proper saturated sub-sheaf $S \subset E$, there holds
\begin{equation}
\mu_\omega(S) < (\leq) \mu_\omega(E).
\end{equation}
We say $H$ is a Hermitian-Einstein metric on $(E, \bar{\partial} E)$ if it satisfies
\begin{equation}
\sqrt{-1} \Lambda_\omega F_H = \lambda \cdot \text{Id}_E,
\end{equation}
where $\lambda = \frac{2\pi}{\text{Vol}(M, \omega)} \mu_\omega(E)$. The classical Donaldson-Uhlenbeck-Yau theorem ([50, 20, 57]) states that, when $\omega$ is Kähler, the stability implies the existence of Hermitian-Einstein metric. According to [10, 42], we know that the Donaldson-Uhlenbeck-Yau theorem is also valid for compact Gauduchon manifolds. There are many other interesting and important works related ([2, 3, 4, 5, 6, 7, 14, 29, 32, 27, 31, 34, 42, 43, 35, 36, 37, 44, 45, 46, 49, 55, 56, 58], etc.). In [51], Nie and the third author proved that on a compact Gauduchon manifold $(M, \omega)$, every semistable holomorphic vector bundle $(E, \bar{\partial} E)$ admits an approximate Hermitian-Einstein structure, i.e. for any $\delta > 0$, there exists a Hermitian metric $H_\delta$ such that
\begin{equation}
\sup_M |\sqrt{-1} \Lambda_\omega F_{H_\delta} - \lambda \cdot \text{Id}_E|_{H_\delta} < \delta.
\end{equation}
This means that every semistable holomorphic vector bundle $(E, \bar{\partial} E)$ over the compact Gauduchon manifold $(M, \omega)$ must admit a Hermitian metric with negative mean curvature if $\mu_\omega(E) < 0$.

Let $S$ be a coherent subsheaf of the holomorphic vector bundle $(E, \bar{\partial} E)$, and $H$ be a Hermitian metric on $E$. Bruasse ([9]) obtained the following Chern-Weil formula:
\begin{equation}
\deg_\omega(S) = \int_{M \setminus \Sigma_{\text{alg}}} \frac{\sqrt{-1}}{2\pi} \text{tr} F_{H_S} \wedge \frac{\omega^{n-1}}{(n-1)!}
= \frac{1}{2\pi} \int_{M \setminus \Sigma_{\text{alg}}} (\sqrt{-1} \text{tr}(\pi_S^H \Lambda_\omega F_H) - |\bar{\partial}_S^{H} P_S^H|^2 \mathrm{d}\omega^n),
\end{equation}
where $\Sigma_{\text{alg}}$ is the singularity set of $S$, $H_S$ is the induced metric on $S|_{M \setminus \Sigma_{\text{alg}}}$ and $\pi_S^H$ is the orthogonal projection onto $S$ with respect to the metric $H$. We know that $\deg_\omega(S)$ is bounded from above. Bruasse ([9]) also proved that one can find a maximal subsheaf which realizes the supremum of the slopes, i.e. there exists a coherent subsheaf $\mathcal{F}$ such that
\begin{equation}
\mu_\omega(\mathcal{F}) = \mu_\omega(E, \omega) := \sup\{\mu_\omega(S) | S \text{ is a coherent sub-sheaf of } E \}.
\end{equation}
In the same way, we know that the infimum of the slopes of coherent quotient sheaves can be attained, i.e. there exists a coherent quotient sheaf $Q$ such that

$$\mu_\omega(Q) = \mu_L(E, \omega) := \inf \{ \mu_\omega(Q) \mid Q \text{ is a coherent quotient sheaf of } E \}.$$  

Furthermore, there is a unique filtration of $(E, \bar{\partial}E)$ by sub-sheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_i = E,$$

such that every quotient sheaf $Q_\alpha = \mathcal{E}_\alpha/\mathcal{E}_{\alpha-1}$ is torsion-free and $\omega$-semistable, and $\mu_\omega(Q_\alpha) > \mu_\omega(Q_{\alpha+1})$, which is called the Harder-Narasimhan filtration of $(E, \bar{\partial}E)$. If $\text{rank}(E) = r$, we have a nonincreasing $r$-tuple of numbers

$$\bar{\mu}_\omega(E) = (\mu_{1, \omega}, \ldots, \mu_{r, \omega})$$

from the HN-filtration by setting: $\mu_{i, \omega} = \mu_\omega(Q_\alpha)$, for $\text{rank}(\mathcal{E}_{\alpha-1}) + 1 \leq i \leq \text{rank}(\mathcal{E}_\alpha)$. We call $\bar{\mu}_\omega(E)$ the Harder-Narasimhan type of $(E, \bar{\partial}E)$. It is easy to see that

$$\mu_{1, \omega} = \mu_U(E, \omega) \quad \text{and} \quad \mu_{r, \omega} = \mu_L(E, \omega).$$

For each $\mathcal{E}_\alpha$ and the Hermitian metric $K$, we have the associated orthogonal projection $\pi^K_\alpha : E \to \mathcal{E}_\alpha$ with respect to metric $K$. It is well known that every $\pi^K_\alpha$ is an $L^2$-bounded Hermitian endomorphism. We define an $L^2$-bounded Hermitian endomorphism by

$$\Phi^H_{\omega}(E, K) = \sum_{\alpha=1}^l \mu_\omega(Q_\alpha)(\pi^K_\alpha - \pi^{-K}_{\alpha-1}),$$

which will be called the Harder-Narasimhan projection of $(E, \bar{\partial}E)$.

Let $(M, \omega)$ be a compact Hermitian manifold, $\omega$ be a Gauduchon metric in the conformal class of $\omega$. If the mean curvature $\sqrt{-1} \Lambda_\omega F_H < 0$, equivalently $\sqrt{-1} \Lambda_\omega F_H < 0$, by the formula (1.7) and the Gauss-Codazzi equation, we know that $\mu_\omega(S) < 0$ for any coherent subsheaf $S$, then

$$\mu_U(E, \omega) < 0.$$ 

An interesting question arises as follows. Does $\mu_U(E, \omega) < 0$ imply that the holomorphic vector bundle $(E, \bar{\partial}E)$ admits a Hermitian metric with negative mean curvature? To answer this question is also our motivation to study special metrics on un-semistable holomorphic vector bundles over compact Gauduchon manifolds.

We denote the $r$ eigenvalues of the mean curvature $\sqrt{-1} \Lambda_\omega F_H$ by $\lambda_1(H, \omega), \lambda_2(H, \omega), \cdots, \lambda_r(H, \omega)$, sorted in the descending order. Then each $\lambda_\alpha(H, \omega)$ is Lipschitz continuous. Set

$$\tilde{\lambda}(H, \omega) = (\lambda_1(H, \omega), \lambda_2(H, \omega), \cdots, \lambda_r(H, \omega)),$$

$$\lambda_L(H, \omega) = \lambda_r(H, \omega), \quad \lambda_U(H, \omega) = \lambda_1(H, \omega),$$

$$\hat{\lambda}_L(H, \omega) = \inf_M \lambda_L(H, \omega), \quad \hat{\lambda}_U(H, \omega) = \sup_M \lambda_U(H, \omega)$$

and

$$\lambda_{mL}(H, \omega) = \frac{1}{\text{Vol}(M, \omega)} \int_M \lambda_L(H, \omega) \frac{\omega^n}{n!}, \quad \lambda_{mU}(H, \omega) = \frac{1}{\text{Vol}(M, \omega)} \int_M \lambda_U(H, \omega) \frac{\omega^n}{n!}.$$
In this paper, we first study the following perturbed Hermitian-Einstein equation on $(E, \bar{\partial}_E)$:

\[
(1.19) \quad \sqrt{-1} \Lambda F_H - \lambda \cdot \text{Id}_E + \varepsilon \log(K^{-1}H) = 0,
\]

where $K$ is any fixed background metric. The above perturbed equation was firstly introduced by Uhlenbeck-Yau in [57], where they used the continuity method to prove the Donaldson-Uhlenbeck-Yau theorem. Due to the fact that the elliptic operators are Fredholm if the base manifold is compact, the equation (1.19) can be solved for any $\varepsilon \in (0, 1]$. Let $H_\varepsilon$ be a solution of perturbed equation (1.19), when $(M, \omega)$ is a compact Gauduchon manifold, Nie and the third author ([51, Proposition 3.1]) have the key observation:

\[
(1.20) \quad \int_M \left( \text{tr}(\sqrt{-1} \Lambda \omega F_K - \lambda \cdot \text{Id}_E)s_\varepsilon \right) + \langle \Phi(s_\varepsilon)(\bar{\partial}s_\varepsilon), \bar{\partial}s_\varepsilon \rangle_K \frac{\omega^n}{n!} = -\varepsilon \int_M \text{tr}(s_\varepsilon^2) \frac{\omega^n}{n!},
\]

where $s_\varepsilon = \log(K^{-1}H_\varepsilon)$ and

\[
(1.21) \quad \Phi(x, y) = \begin{cases} 
\frac{e^y - e^{-x}}{y - x}, & x \neq y; \\
1, & x = y.
\end{cases}
\]

By using the above identity (1.20) and Uhlenbeck-Yau’s result ([57]) that $L^2_1$ weakly holomorphic subbundles define saturated coherent subsheaves, following Simpson’s argument in [55], we can obtain the existence of $L^p$-approximate critical Hermitian structure on $(E, \bar{\partial}_E)$, i.e. we proved the following theorem.

**Theorem 1.1.** Let $(M, \omega)$ be a compact Gauduchon manifold of complex dimension $n$, $(E, \bar{\partial}_E)$ be a holomorphic vector bundle over $M$, $K$ be a fixed Hermitian metric on $E$ and $H_\varepsilon$ be a solution of perturbed equation (1.19). Then there exists a sequence $\varepsilon_i \to 0$ such that

\[
(1.22) \quad \lim_{i \to \infty} \left\| \sqrt{-1} \Lambda F_{H_{\varepsilon_i}} - \frac{2\pi}{\text{Vol}(M, \omega)} \Phi^H(N(E, K)) \right\|_{L^p(K)} = 0
\]

for any $0 < p < +\infty$. In particular,

\[
(1.23) \quad \lim_{i \to \infty} \lambda_{mL}(H_{\varepsilon_i}, \omega) = \frac{2\pi}{\text{Vol}(M, \omega)} \mu_{r, \omega} = \frac{2\pi}{\text{Vol}(M, \omega)} \mu_L(E, \omega),
\]

and

\[
(1.24) \quad \lim_{i \to \infty} \lambda_{mU}(H_{\varepsilon_i}, \omega) = \frac{2\pi}{\text{Vol}(M, \omega)} \mu_{1, \omega} = \frac{2\pi}{\text{Vol}(M, \omega)} \mu_U(E, \omega).
\]

The $L^p$-approximate critical Hermitian structure was firstly introduced by Daskalopoulos and Wentworth ([14]), and its existence plays a crucial role in proving the Atiyah-Bott-Bando-Siu conjecture ([14, 30, 54, 37]). It should be pointed out that, even in the Kähler case, our proof is new and very different from previous proofs, where they depend on the resolution of singularities theorem of Hironaka ([28]) and the Donaldson-Uhlenbeck-Yau theorem of reflexive sheaf by Bando and Siu ([8]).

Now we can answer the above question by applying Theorem 1.1. If $\mu_{1, \omega} < 0$, there exists a Hermitian metric $H_\varepsilon$ satisfying $\lambda_{mU}(H_{\varepsilon}, \omega) = -\delta < 0$. Let $\lambda$ be a smooth function such that $\lambda > \lambda_{mU}(H_{\varepsilon}, \omega) - \delta$ and $\int_M \lambda \frac{\omega^n}{n!} = \lambda_{mU}(H_{\varepsilon}, \omega) \text{Vol}(M, \omega)$, then $\lambda_{U}(H_{\varepsilon}, \omega) - \lambda + \delta > 0$. If $\mu_{1, \omega} > 0$, there
Let $(M, \omega)$ be a compact Hermitian manifold, and $(E, \bar{\mathcal{D}}_E)$ be a holomorphic vector bundle over $M$. Then, there exists a Hermitian metric $H$ such that the mean curvature $\sqrt{-1}\Lambda_{\omega} F_H$ is negative (or positive) if and only if $\mu_U(E, \omega) < 0$ ($\mu_L(E, \omega) > 0$), where $\omega$ is a Gauduchon metric conformal to $\bar{\omega}$.

Consider the Hermitian-Yang-Mills flow which was firstly introduced by Donaldson [20],

\begin{equation}
H^{-1}(t) \frac{\partial H(t)}{\partial t} = -2(\sqrt{-1} \Lambda_{\omega} F_{H(t)} - \lambda \cdot \mathrm{Id}_E),
\end{equation}

where $H(t)$ is a family of Hermitian metrics on $E$ and $\lambda$ is a constant. When $\omega$ is Gauduchon, we usually choose $\lambda = \frac{2\pi}{\text{Vol}(M, \omega)} \mu_{\omega}(E)$. The long time existence and uniqueness of the solution for the above heat equation (1.25) were proved by Donaldson [20] for the Kähler manifolds case, and by the third author [65] for the general Hermitian manifolds case. In the following part of this paper, we study the asymptotic behavior of (1.25). We deduce the monotonicity of $\hat{\lambda}_L(H(t), \omega)$ and $\hat{\lambda}_U(H(t), \omega)$ along the Hermitian-Yang-Mills flow.

Theorem 1.3. Let $(M, \omega)$ be a compact Hermitian manifold, $(E, \bar{\mathcal{D}}_E)$ be a holomorphic vector bundle over $M$. Assume $H(t)$ is a smooth solution of the Hermitian-Yang-Mills flow (1.25) on $(E, \bar{\mathcal{D}}_E)$. Then $\hat{\lambda}_L(H(t), \omega)$ is increasing and $\hat{\lambda}_U(H(t), \omega)$ is decreasing on $[0, \infty)$. If $\omega$ is Gauduchon, then $\lambda_{mL}(H(t), \omega)$ is increasing and $\lambda_{mU}(H(t), \omega)$ is decreasing.

Furthermore, we have

\begin{align}
\lim_{t \to \infty} \lambda_{mL}(H(t), \omega) &= \lim_{t \to \infty} \hat{\lambda}_L(H(t), \omega), \\
\lim_{t \to \infty} \lambda_{mU}(H(t), \omega) &= \lim_{t \to \infty} \hat{\lambda}_U(H(t), \omega).
\end{align}

Suppose $H_1(t)$ and $H_2(t)$ are two smooth solutions of the Hermitian-Yang-Mills flow (1.25). We can show that

\begin{equation}
\lim_{t \to \infty} \| \sqrt{-1} \Lambda_{\omega}(F_{H_1(t)} - F_{H_2(t)}) \|_{L^2(H_1(t))} = 0.
\end{equation}

The readers can see Theorem 4.10 for details. Then, by Theorem 1.1 and 1.3, we obtain the asymptotic behavior of the eigenvalue functions of the mean curvature along the Hermitian-Yang-Mills flow.

Theorem 1.4. Let $(M, \omega)$ be a compact Gauduchon manifold, $(E, \bar{\mathcal{D}}_E)$ be a holomorphic vector bundle over $M$. Assume $H(t)$ is a smooth solution of the Hermitian-Yang-Mills flow (1.25) on $(E, \bar{\mathcal{D}}_E)$. Then

\begin{equation}
\lim_{t \to \infty} \left\| \hat{\lambda}(H(t), \omega) - \frac{2\pi}{\text{Vol}(M, \omega)} \bar{\mu}_{\omega}(E) \right\|_{L^2} = 0.
\end{equation}
Specially, we have
\[
\lim_{t \to \infty} \lambda_m L(H(t), \omega) = \lim_{t \to \infty} \lambda L(H(t), \omega) = \frac{2\pi}{\text{Vol}(M, \omega)} \mu_{r, \omega},
\]
(1.30)
\[
\lim_{t \to \infty} \lambda_m U(H(t), \omega) = \lim_{t \to \infty} \lambda U(H(t), \omega) = \frac{2\pi}{\text{Vol}(M, \omega)} \mu_{1, \omega}.
\]
(1.31)

**Corollary 1.5.** Let \((M, \omega)\) be a compact Gauduchon manifold, \((E, \overline{\partial}_E)\) be a holomorphic vector bundle over \(M\). Then for any \(\delta > 0\), there exists a Hermitian metric \(H_\delta\) on \(E\), such that
\[
\frac{2\pi}{\text{Vol}(M, \omega)} \mu_L(E, \omega) - \delta \text{ Id}_E \leq \sqrt{-1} \Lambda_\omega F_{H_\delta} \leq \frac{2\pi}{\text{Vol}(M, \omega)} \mu_U(E, \omega) + \delta \text{ Id}_E.
\]
(1.32)

**Definition 1.6.** Let \((E, \overline{\partial}_E)\) be a holomorphic vector bundle over a compact complex manifold \(M\). The holomorphic vector bundle \(E\) is called HN-positive (HN-nonnegative) if there is a Gauduchon metric \(\omega\) on the base manifold \(M\) such that \(\mu_L(E, \omega) > 0\) (\(\mu_L(E, \omega) \geq 0\)). We say \(E\) is HN-negative (HN-nonpositive) if its dual bundle \(E^*\) is HN-positive (HN-nonnegative).

A compact complex manifold \(M\) is called HN-positive (HN-nonnegative, HN-negative, HN-nonpositive) if its tangent bundle \(T^{1,0}M\) is HN-positive (HN-nonnegative, HN-negative, HN-nonpositive).

**Remark 1.7.** If \((M, \omega)\) is a compact Gauduchon manifold and \((E, \overline{\partial}_E)\) is \(\omega\)-semistable, then \(\mu_U(E, \omega) = \mu_L(E, \omega) = \mu_\omega(E)\). So Corollary 1.5 implies the existence of approximate Hermitian-Einstein metrics on semistable holomorphic bundles over compact Gauduchon manifolds which was proved in [51]. By Corollary 1.5, it is easy to see that the following statements on \(E\) are equivalent:

1. \(E\) is HN-positive;
2. there is a Hermitian metric \(\omega\) on the base manifold \(M\) and a Hermitian metric \(H\) on \(E\) such that the mean curvature \(\sqrt{-1} \Lambda_\omega F_H\) is positive;
3. there is a Hermitian metric \(\omega\) on the base manifold \(M\) and a Hermitian metric \(H\) on \(E\) such that the mean curvature \(\sqrt{-1} \Lambda_\omega F_H\) is quasi-positive;
4. there is a Gauduchon metric \(\omega\) on the base manifold \(M\) and a Hermitian metric \(H\) on \(E\) such that \(\lambda_{mL}(H, \omega) > 0\).

A holomorphic vector bundle \((E, \overline{\partial}_E)\) is said to be ample if its tautological line bundle \(O_E(1)\) is ample over the projective bundle \(\mathbb{P}(E)\) of hyperplanes of \(E\). The notion of positivity is very important in both algebraic geometry and complex differential geometry. In [23], Griffiths introduced the following positivity: Given a Hermitian metric \(H\) on \(E\), an \(H\)-Hermitian \((1,1)\)-form \(\sqrt{-1} \Theta\) valued in End \(E\) is said to be Griffiths positive, if at every \(p \in M\), it holds that
\[
\langle \Theta(v, \overline{v})u, u \rangle_H > 0
\]
for any non-zero vector \(u \in E_p\) and any non-zero vector \(v \in T^{1,0}_p M\). We say \((E, \overline{\partial}_E)\) is Griffiths positive if \(E\) admits a Hermitian metric \(H\) such that \(\sqrt{-1} F_H\) is Griffiths positive.
Of course a Griffiths positive vector bundle is ample. However, it is still an open problem of Griffiths ([23]) that the ampleness implies Griffiths-positivity. In [23], Griffiths raised the question to determine which characteristic forms are positive on Griffiths positive vector bundles, and proved the second Chern form is positive for the rank two case. Recently, there are several interesting works on the above Griffiths’ question ([47, 24, 17, 52, 38, 21, 59, 18]). On semi-stable ample vector bundles over complex surfaces, Pingali proved that there must exist a Hermitian metric $H$ such that its second Chern form is positive. In [59], Xiao showed some Schur classes have strictly positive representations on ample bundles, but it is unclear whether these representations can be built from the Chern curvatures of some Hermitian metrics. As an application of Theorem 1.4, we get

**Theorem 1.8.** Let $(E, \bar{\partial}_E)$ be an ample holomorphic vector bundle over a compact complex manifold $M$. Then

1. For any Hermitian metric $\omega$ on $M$, there exists a Hermitian metric $H$ on $E$ such that $\sqrt{-1} \Lambda \omega F_H > 0$.
2. If dim$^C M = 2$ and rank$(E) = 2$, there must exist a Hermitian metric $H$ such that its second Chern form is positive, i.e. $c_2(E, H) > 0$.

**Remark 1.9.** A Hermitian metric $H$ on the bundle $E$ over a complex manifold $M$ is called RC-positive if for any non-zero local section $u$ of $E$, there exists a local section $v$ of $T^{1,0}M$ such that $\langle F_H(v, \bar{v})u, u \rangle_H > 0$. The concept of RC-positivity was introduced by Yang in [61], and it is very effective in studying vanishing theorems. From the definition, one can easily see that a Hermitian metric $H$ with positive mean curvature (i.e. $\sqrt{-1} \Lambda \omega F_H > 0$ with respect to some Hermitian metric $\omega$ on $M$) must be RC-positive. If $E$ is ample, by virtue of Theorem 1.8 there exists a Hermitian metric $H$ on $E$ with positive mean curvature. So, for every $1 \leq p \leq \text{rank}(E)$ (every $k \geq 1$), the induced metric $\wedge^p H (\otimes^k H)$ on the bundle $\wedge^p E (\otimes^k E)$ must have positive mean curvature, and then is also RC-positive. This confirms a conjecture proposed by Yang ([61, Conjecture 7.10]).

The holomorphic map is an important research object in complex geometry. There are many generalizations of the classical Schwarz Lemma and rigidity result on holomorphic maps via the work of Ahlfors, Chern, Lu, Yau and others ([1, 12, 41, 63, 53, 60, 48, 62, 66]). As another application of Theorem 1.4, we obtain the following integral inequality for holomorphic maps.

**Theorem 1.10.** Let $f$ be a holomorphic map from a compact Gauduchon manifold $(M, \omega)$ to a Hermitian manifold $(N, \nu)$. If $f$ is not constant, then there holds

\[
2\pi \mu_L(T^{1,0}M, \omega) \leq \int_M H B^\nu_f(x) \cdot f^*(\nu) \wedge \frac{\omega^{m-1}}{(m-1)!},
\]

where $m = \dim^C M$ and $H B^\nu_f(x)$ is the supremum of holomorphic bisectional curvature at $f(x) \in (N, \nu)$.

Therefore, we conclude the following rigidity result of holomorphic maps.

**Corollary 1.11.** Let $f$ be a holomorphic map from a compact complex manifold $M$ to a complex manifold $N$. If $M$ is HN-nonnegative (resp. HN-positive) and $N$ admits
a Hermitian metric with negative holomorphic bisectional curvature (resp. nonpositive holomorphic bisectional curvature), then \( f \) must be constant.

This paper is organized as follows. In Section 2, we give some estimates and preliminaries which will be used in the proof of Theorems 1.1, 1.3 and 1.4. In Section 3, we give a proof of Theorem 1.1. In Section 4, we consider the asymptotic behavior of the Hermitian-Yang-Mills flow, and complete the proof of Theorem 1.3 and 1.4. In Section 5, we prove Theorem 1.8 and Theorem 1.10.

2. Preliminary results

Let \((M, \omega)\) be a compact Hermitian manifold of complex dimension \( n \), and \((E, \bar{\partial}_E)\) be a rank \( r \) holomorphic vector bundle over \( M \). If \( H \) is a Hermitian metric on holomorphic vector bundle \( E \), we denote the Chern connection by \( D_H \) and the curvature form by \( F_H \). Suppose \( K \) is another Hermitian metric and set \( h = K^{-1}H \). We have the following identities

\[
\begin{align*}
\partial_H - \partial_K &= h^{-1} \partial_K h, \\
F_H - F_K &= \bar{\partial}_E (h^{-1} \partial_K h),
\end{align*}
\]

(2.1)

where \( \partial_H = D_H^{1,0} \) and \( \partial_K = D_K^{1,0} \) are the \((1,0)\)-parts of \( D_H \) and \( D_K \), respectively. The perturbed Hermitian-Einstein equation (1.19) can be written as

\[
(2.2) \quad \sqrt{-1} \Lambda_\omega \bar{\partial}_E (h^{-1} \partial_K h) + \sqrt{-1} \Lambda_\omega F_K - \lambda \cdot \text{Id}_E + \varepsilon \log h = 0.
\]

Now recall some results on the equation (1.19) which have been proved in [39].

**Lemma 2.1** ([39]). There exists a solution \( H_\varepsilon \) to the perturbed equation (1.19) for all \( \varepsilon > 0 \). And there holds that

\[
\begin{align*}
(i) & \quad -\frac{\sqrt{\pi}}{2} \Lambda_\omega \bar{\partial} (| \log h_\varepsilon |_K) + \varepsilon \cdot | \log h_\varepsilon |_K \leq | \sqrt{-1} \Lambda_\omega F_K - \lambda \cdot \text{Id}_E |_K | \log h_\varepsilon |_K; \\
(ii) & \quad m = \max_M | | \log h_\varepsilon |_K \leq \frac{1}{\varepsilon} \cdot \max_M | \sqrt{-1} \Lambda_\omega F_K - \lambda \cdot \text{Id}_E |_K; \\
(iii) & \quad m \leq C \cdot (| | \log h_\varepsilon |_L^2 + \max_M | \sqrt{-1} \Lambda_\omega F_K - \lambda \cdot \text{Id}_E |_K),
\end{align*}
\]

where \( h_\varepsilon = K^{-1}H_\varepsilon \), \( C \) is a constant depending only on \((M, \omega)\). Moreover, if the background Hermitian metric \( K \) satisfies \( \text{tr}(\sqrt{-1} \Lambda_\omega F_K - \lambda \cdot \text{Id}_E) = 0 \), then

\[
(2.3) \quad \text{tr} \log(h_\varepsilon) = 0
\]

and \( \text{tr} F_{H_\varepsilon} = \text{tr} F_K \).

If \( \omega \) is a Gauduchon metric, i.e. \( \bar{\partial} \omega^{n-1} = 0 \), by an appropriate conformal change, we can assume that \( K \) satisfies

\[
(2.4) \quad \text{tr}(\sqrt{-1} \Lambda_\omega F_K - \lambda \cdot \text{Id}_E) = 0,
\]

where \( \lambda = \frac{2\pi}{\text{Vol}(M, \omega)} \mu_\omega(E) \).

**Proposition 2.2** ([31], Proposition 3.1). Let \((E, \bar{\partial}_E)\) be a holomorphic vector bundle with a fixed Hermitian metric \( K \) over a compact Gauduchon manifold \((M, \omega)\). Assume \( H \) is a
Hermitian metric on $E$ and $s := \log(K^{-1}H)$. Then we have
\begin{equation}
(2.5) \quad \int_M \text{tr}((\sqrt{-1}\Lambda_w F_K)s)\frac{\omega^n}{n!} + \int_M \langle \Psi(s)(\bar{\partial}E s), \bar{\partial}E s \rangle_K \frac{\omega^n}{n!} = \int_M \text{tr}((\sqrt{-1}\Lambda_w F_H)s)\frac{\omega^n}{n!},
\end{equation}
where $\Psi$ is the function which is defined in (1.21).

The above identity (2.5) also works for compact manifolds with nonempty boundary case and some noncompact manifolds case, see Proposition 2.6 in [64].

2.1. Basic results of the Hermitian-Yang-Mills flow. Set $h(t) = K^{-1}H(t)$, then the Hermitian-Yang-Mills flow (1.25) can be written as
\begin{equation}
(2.6) \quad \frac{\partial h(t)}{\partial t} = -2\sqrt{-1}h(t)\Lambda_w(F_K + \bar{\partial}E(h^{-1}(t)\partial_K h(t))) + 2\lambda h(t).
\end{equation}
By (1.25), the evolution equation of $\sqrt{-1}\Lambda_w F_{H(t)}$ is
\begin{equation}
(2.7) \quad \frac{\partial \sqrt{-1}\Lambda_w F_{H(t)}}{\partial t} = -2\sqrt{-1}\Lambda_w \bar{\partial}E_{H(t)} \sqrt{-1}\Lambda_w F_{H(t)}.
\end{equation}

**Proposition 2.3** ([20,63]). Let $(M, \omega)$ be a compact Hermitian manifold and $(E, \bar{\partial}E)$ be a holomorphic vector bundle over $M$. The Hermitian-Yang-Mills flow (1.25) with initial data $H_0$ must have a unique solution $H(t)$ which exists for $0 \leq t < +\infty$.

When there is no confusion, we sometimes omit the parameter $t$ in the sequel for simplicity. Denote $\Phi(H) = \sqrt{-1}\Lambda_w F_H - \lambda \cdot \text{Id}_E$, then we have

**Proposition 2.4** ([65], Proposition 2.2). Let $(M, \omega)$ be a compact Hermitian manifold, $(E, \bar{\partial}E)$ be a holomorphic vector bundle over $M$ and $H(t)$ be a solution of Hermitian-Yang-Mills flow (1.25). Then
\begin{equation}
(2.8) \quad (2\sqrt{-1}\Lambda_w \bar{\partial}E - \frac{\partial}{\partial t}) \Phi(H(t)) = 0
\end{equation}
and
\begin{equation}
(2.9) \quad (2\sqrt{-1}\Lambda_w \bar{\partial}E - \frac{\partial}{\partial t})|\Phi(H(t))|^2_{H(t)} = 2|D_{H(t)} \Phi(H(t))|^2_{H(t)}.
\end{equation}
Moreover, if the initial Hermitian metric $H_0$ satisfies $\text{tr}(\sqrt{-1}\Lambda_w F_{H_0} - \lambda \cdot \text{Id}_E) = 0$, then
\begin{equation}
(2.10) \quad \text{tr} F_{H(t)} = \text{tr} F_{H_0}
\end{equation}
and $\det(H_0^{-1}H(t)) = 1$.

Since $\text{tr} \sqrt{-1}\Lambda_w F_H = \text{tr} \Phi(H) + r\lambda$ and $|\sqrt{-1}\Lambda_w F_H|^2_{H} = |\Phi(H)|^2_{H} - 2\lambda \text{tr} \Phi(H) + r\lambda^2$, both (2.8) and (2.9) still hold if we replace $\Phi(H)$ by $\sqrt{-1}\Lambda_w F_H$. The maximum principle yields
\begin{equation}
(2.11) \quad \sup_M |\Phi(H)|^2_{H}(t) \leq \sup_M |\Phi(H)|^2_{H}(0)
\end{equation}
and
\begin{equation}
(2.12) \quad \sup_M |\sqrt{-1}\Lambda_w F_H|^2_{H}(t) \leq \sup_M |\sqrt{-1}\Lambda_w F_H|^2_{H}(0).
\end{equation}
For simplicity, we use the same notation \( \partial \) to denote both the ordinary \( \partial \) operator and holomorphic structure \( \bar{\partial} \), and set \( \theta_H = \sqrt{-1} \Lambda_\omega F_H \). Like in [15, Chapter VII], we denote by \( \tau \) the type \((1,0)\) operator of zero order defined by \( \tau = [\Lambda_\omega, \partial_\omega] \). One has the fact that

\[
2.13 \quad [\sqrt{-1} \Lambda_\omega, \partial] = \partial^* + \tau^*, \quad [\sqrt{-1} \Lambda_\omega, \partial_\omega] = - (\bar{\partial}^* + \bar{\tau}^*),
\]

where \( \partial^* \) and \( \bar{\partial}^* \) are the formal adjoint operators of \( \partial_\omega \) and \( \bar{\partial}_\omega \) with respect to \( H \) respectively.

The following lemma is well-known.

**Lemma 2.5.** Assume that \((M, \omega)\) is a compact Gauduchon manifold. Let \( H(t) \) be a solution of Hermitian-Yang-Mills flow (1.25), then

\[
2.14 \quad \int_0^\infty \int_M |D_H \theta_H|^2_{H(t)} \frac{\omega^n}{n!} dt \leq \frac{1}{2} \int_M |\theta_H|^2_{H(0)} \frac{\omega^n}{n!}
\]

and

\[
2.15 \quad \lim_{t \to \infty} \int_M |D_H \theta_H|^2_{H(t)} \frac{\omega^n}{n!} = 0.
\]

For the convenience of the readers, we give a proof here.

**Proof.** As mentioned above, we see

\[
2.16 \quad \left( \frac{\partial}{\partial t} - 2 \sqrt{-1} \Lambda_\omega \partial \bar{\partial} \right) |\theta_H|^2_{H(t)} = -2 |D_H \theta_H|^2_{H(t)}.
\]

Then (2.14) follows directly.

In order to show (2.15), we need to estimate the growth of \( \int_M |D_H \theta_H|^2_{H(t)} \frac{\omega^n}{n!} \). Based on (2.17), one can find

\[
2.17 \quad \frac{d}{dt} \int_M |\bar{\partial} \theta_H|^2_{H(t)} \frac{\omega^n}{n!} = 2 \int_M \langle [\bar{\partial} \theta_H, \theta_H], \bar{\partial} \theta_H \rangle_{H(t)} \frac{\omega^n}{n!} - 4 \text{Re} \int_M \langle \bar{\partial} \sqrt{-1} \Lambda_\omega \bar{\partial} \theta_H, \bar{\partial} \theta_H \rangle_{H(t)} \frac{\omega^n}{n!}.
\]

For the last integral, we have

\[
2.18 \quad \int_M \langle \bar{\partial} \sqrt{-1} \Lambda_\omega \bar{\partial} \theta_H, \bar{\partial} \theta_H \rangle_{H(t)} \frac{\omega^n}{n!} = \int_M \langle \sqrt{-1} \Lambda_\omega \partial_\omega \theta_H, \partial_\omega \theta_H \rangle_{H(t)} \frac{\omega^n}{n!}
\]

and

\[
2.19 \quad \bar{\partial}^* \bar{\partial} \theta_H = - ([\sqrt{-1} \Lambda_\omega, \partial_\omega] + \bar{\tau}^*) \bar{\partial} \theta_H = - \sqrt{-1} \Lambda_\omega \partial_\omega \theta_H - \bar{\tau}^* \bar{\partial} \theta_H = \sqrt{-1} \Lambda_\omega \bar{\partial} \partial_\omega \theta_H - \bar{\tau}^* \bar{\partial} \theta_H.
\]
Then it follows that
\[
\text{Re} \int_M \langle \bar{\partial} \sqrt{-1} \Lambda \omega \bar{\partial} \theta_H, \bar{\partial} \theta_H \rangle_{H, \omega} \frac{\omega^n}{n!} \\
= \text{Re} \int_M \langle \sqrt{-1} \Lambda \omega \bar{\partial} \theta_H, \sqrt{-1} \Lambda \omega \bar{\partial} \theta_H - \bar{\tau}^* \bar{\partial} \theta_H \rangle_{H, \omega} \\
\geq \int_M |\sqrt{-1} \Lambda \omega \bar{\partial} \theta_H|_H^2 \frac{\omega^n}{n!} - \int_M |\sqrt{-1} \Lambda \omega \bar{\partial} \theta_H|_H |\bar{\tau}^* \bar{\partial} \theta_H|_H \frac{\omega^n}{n!} \\
\geq -\frac{1}{4} \int_M |\bar{\tau}^* \bar{\partial} \theta_H|_H^2 \frac{\omega^n}{n!}.
\] (2.20)

Therefore there holds
\[
\frac{d}{dt} \int_M |\bar{\partial} \theta_H|_{H, \omega}^2 \frac{\omega^n}{n!} \leq 2 \int_M \langle [\bar{\partial} \theta_H, \theta_H], \bar{\partial} \theta_H \rangle_{H, \omega} \frac{\omega^n}{n!} + \int_M |\bar{\tau}^* \omega|_H |\bar{\partial} \theta_H|_H \frac{\omega^n}{n!}
\leq C \int_M |\bar{\partial} \theta_H|_{H, \omega}^2 \frac{\omega^n}{n!},
\] (2.21)

where \(C\) is a constant depending only on \(\sup_M |\theta_H|_H(0)\) and \(\sup_M |d\omega|_\omega\). Since \(|D_H \theta_H|_H^2 = 2|\bar{\partial} \theta_H|_H^2\), we get
\[
\frac{d}{dt} \int_M |D_H \theta_H|_{H, \omega}^2 \frac{\omega^n}{n!} \leq C \int_M |D_H \theta_H|_{H, \omega}^2 \frac{\omega^n}{n!}.
\] (2.22)

This together with (2.14) gives (2.15). \(\square\)

### 2.2. Long-time behavior of weak super-solutions of the parabolic equation

In this subsection we introduce some results that are useful for studying the eigenvalues of the mean curvature along the Hermitian-Yang-Mills flow.

Let \(M\) be an \(m\)-dimensional oriented compact smooth manifold, and \(L\) a second order linear elliptic operator with smooth coefficients and no zero-order term on \(M\). Locally \(L\) can be expressed as
\[
L = a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + b^i(x) \frac{\partial}{\partial x^i}.
\] (2.23)

According to [11], there exists a Riemannian metric \(g\) together with a smooth vector field \(X\) such that globally
\[
L = \Delta_g + X.
\] (2.24)

For later use, we would like to introduce the following definition.

**Definition 2.6.** An \(m\)-form \(\Omega\) is said to be compatible with \(L\) if for any \(u \in C^2(M)\), we have
\[
\int_M Lu \Omega = 0.
\] (2.25)

In section 4, we particularly consider the case that \((M, \omega)\) is a compact Hermitian manifold and \(L = 2\sqrt{-1} \Lambda \omega \bar{\partial} \bar{\partial}\). When \(\omega\) is Gauduchon, the volume form \(\frac{\omega^n}{n!}\) is compatible with \(L\).
The following result can be easily deduced from the classical PDE theory.

**Lemma 2.7.** If $u$ is a lower semi-continuous function on $M \times [0,T]$ ($T > 0$) satisfying \((\frac{\partial}{\partial t} - L) u \geq 0\) in the viscosity sense. Then $\inf_M u(\cdot, t)$ is monotone increasing on $[0,T]$.

**Lemma 2.8** (parabolic Harnack inequality). If $u$ is a non-negative continuous solution of the equation \((\frac{\partial}{\partial t} - L) u = 0\) on $M \times [0,T]$ ($T > 0$), then for any $0 < t_1 < t_2 \leq T$, we have
\[
\inf_M u(\cdot, t_2) \geq \delta \sup_M u(\cdot, t_1),
\]
where $\delta$ is a constant depending only on $M$, $L$, $t_1$ and $t_2 - t_1$.

**Lemma 2.9** (convergence of the solution). If $u$ is a continuous solution of the equation \((\frac{\partial}{\partial t} - L) u = 0\) on $M \times [0,T]$ ($T > 1$), then for any $t \in [1,T]$, we have
\[
\sup_M u(\cdot, t) - \inf_M u(\cdot, t) \leq (1 - \delta)^{t-1} \left( \sup_M u(\cdot, 0) - \inf_M u(\cdot, 0) \right),
\]
where $\delta \in (0,1)$ is a constant depending only on $M$ and $L$. Consequently, if $u$ is a long-time solution, then $u$ converges uniformly to a constant when $t \to \infty$.

**Lemma 2.10** (limit of the solution). Assume that there is a non-trivial non-negative $m$-form $\Omega$ compatible with $L$. If $u$ is a continuous solution of the equation
\[
\begin{cases}
(\frac{\partial}{\partial t} - L) u = 0, & \text{on } M \times (0,T], \\
u = \varphi, & \text{on } M \times \{0\},
\end{cases}
\]
where $T > 0$ and $\varphi \in C(M)$. Then for any $t \in [0,T]$, we have
\[
\int_M u(\cdot, t) \Omega = \int_M \varphi \Omega.
\]
Furthermore, for any $0 \leq t_1 < t_2 \leq T$, we have
\[
\sup_M |u(\cdot, t_2) - c| \leq (1 - \delta) \sup_M |u(\cdot, t_1) - c|,
\]
where $c = \frac{\int_M \varphi \Omega}{\int_M \Omega}$ and $\delta \in (0,1)$ is a constant depending only on $M$, $L$ and $t_2 - t_1$. Consequently, if $u$ is a long-time solution, then $u$ converges uniformly to $c$ as $t \to \infty$.

Combining Lemma 2.7 and 2.10, we obtain

**Lemma 2.11.** Assume that there is a non-trivial non-negative $m$-form $\Omega$ compatible with $L$. If $u \in C(M \times [0,T])$ ($T > 0$) satisfies \((\frac{\partial}{\partial t} - L) u \geq 0\) in the viscosity sense, then
\[
\mu_m(t) \triangleq \frac{\int_M u(\cdot, t) \Omega}{\int_M \Omega}
\]
is monotone increasing on $[0,T]$. Furthermore, for any $0 \leq t_1 < t_2 \leq T$, we have
\[
\inf_M u(\cdot, t_2) - \inf_M u(\cdot, t_1) \geq \delta \left( \mu_m(t_1) - \inf_M u(\cdot, t_1) \right),
\]
where $\delta \in (0,1)$ is a constant depending only on $M$, $L$ and $t_2 - t_1$. Consequently, if $u$ is a long-time super-solution, then $\lim_{t \to \infty} \inf_M u(\cdot, t) = \lim_{t \to \infty} \mu_m(t)$. 

Hermitian metrics with negative mean curvature

Proof. For any \(0 \leq t_1 < t_2 \leq T\), let \(\varphi\) be the continuous solution of the equation

\[
\begin{cases}
\left(\frac{\partial}{\partial t} - L\right)\varphi = 0, & \text{on } M \times (t_1, t_2], \\
\varphi = u, & \text{on } M \times \{t_1\},
\end{cases}
\]

then \(\varphi\) is smooth on \(M \times (t_1, t_2]\). Clearly \(u - \varphi\) is a super-solution in the viscosity sense and equal to 0 on \(M \times \{t_1\}\). From Lemma 2.7 one can see \(u \geq \varphi\) on \(M \times [t_1, t_2]\).

Since \(\varphi\) is a solution, we know

\[
\int_M \varphi(\cdot, t_1) \Omega = \int_M \varphi(\cdot, t_2) \Omega
\]

and

\[
\left(\inf_M \varphi(\cdot, t_2) - \inf_M \varphi(\cdot, t_1)\right) \int_M \Omega \geq \delta \int_M \left(\varphi(\cdot, t_1) - \inf_M \varphi(\cdot, t_1)\right) \Omega,
\]

where \(\delta \in (0, 1)\) is a constant depending only on \(M, L, t_1\) and \(t_2 - t_1\). Note that \(u = \varphi\) on \(M \times \{t_1\}\) and \(u \geq \varphi\) on \(M \times \{t_2\}\). It follows that \(\mu_m(t_2) \geq \mu_m(t_1)\) and the weak Harnack inequality (2.32).

When \(u\) is a long-time super-solution in the viscosity sense and bounded from below on any finite time interval, for any \(t \geq 1\), we have

\[
\inf_M u(\cdot, t + 1) - \inf_M u(\cdot, t) \geq \delta_1 \left(\mu_m(t) - \inf_M u(\cdot, t)\right),
\]

where \(\delta_1\) is a constant depending only on \(M\) and \(L\). Since both \(\inf_M u(\cdot, t)\) and \(\mu_m(t)\) are monotone increasing, they converge to some finite numbers or \(\infty\) when \(t \to \infty\). The fact \(\mu_m(t) \geq \inf_M u(\cdot, t)\) tells us that if \(\lim \inf_M u(\cdot, t) = \infty\), then

\[
\lim_{t \to \infty} \mu_m(t) = \infty = \lim_{t \to \infty} \inf_M u(\cdot, t).
\]

Otherwise, \(\lim \inf_M u(\cdot, t) = c\) for some \(c \in \mathbb{R}\). Because

\[
0 \leq \mu_m(t) - \inf_M u(\cdot, t) \leq \delta_1^{-1} \left(\inf_M u(\cdot, t + 1) - \inf_M u(\cdot, t)\right),
\]

taking \(t \to \infty\), we have \(\lim_{t \to \infty} \left(\mu_m(t) - \inf_M u(\cdot, t)\right) = 0\). \(\square\)

Now we conclude the subsection with a remark on notions of weak super-solutions. Viscosity super-solution ([13]) is a very weak notion that admits the comparison principle, and it coincides with distribution super-solution in the continuous case ([40] [28]). In fact, the weak super-solutions considered in Section 4 also belong to a stronger notion introduced in [11] [19].

3. The existence of \(L^p\)-approximate critical Hermitian structure

In this section, we give a proof of Theorem 1.1. Let \((M, \omega)\) be a compact Gauduchon manifold of complex dimension \(n\) and \((E, \partial_E)\) a rank \(r\) holomorphic vector bundle associated with an Hermitian metric \(K\) over \(M\). Suppose \(H_\varepsilon\) is the solution of the perturbed
equation (1.19), i.e.

\[(3.1) \quad \sqrt{-1}A_{\omega}FH_{\epsilon} - \lambda \text{Id}_E + \epsilon \log(K^{-1}H_{\epsilon}) = 0.\]

By Lemma 2.1, we have

\[(3.2) \quad \| \epsilon \log(K^{-1}H_{\epsilon}) \|_{L^\infty} \leq \| \sqrt{-1}A_{\omega}FK - \lambda \text{Id}_E \|_{L^\infty},\]

In the sequel, we denote \(H_{\epsilon, i}\) by \(H_i\) and set \(h_i = K^{-1}H_i, s_i = \log h_i, l_i = \|s_i\|_{L^2}, u_i = \frac{s_i}{l_i}\) for simplicity. Then

\[(3.3) \quad \sqrt{-1}A_{\omega}FH_i - \lambda \text{Id}_E = -\epsilon s_i l_i = -u_i.\]

From (1.20), one can see

\[(3.4) \quad \int_M (\text{tr}(\sqrt{-1}A_{\omega}FK - \lambda \text{Id}_E)u_i) + l_i \langle \Psi(l_i u_i)(\bar{\partial}u_i), \bar{\partial}u_i \rangle K \frac{\omega^n}{n!} = -\epsilon l_i.\]

By (3.8) and following Simpson’s argument ([55, Lemma 5.4]), we have

\[(3.9) \quad \|u_i\|_{L^2} = 1, \quad \|u_i\|_{L^\infty} \leq \tilde{C} \quad \text{and} \quad \|D_Ku_i\|_{L^2} < \tilde{C},\]

i.e. \(u_i\) are uniformly bounded in \(L^\infty\) and \(L^2\). So one can choose a subsequence, which is also denoted by \(\{u_i\}\) for simplicity, such that \(u_i \rightharpoonup u_{\infty}\) weakly in \(L^2\). By Kondrachov compactness theorem ([22, Theorem 7.22]), we know that \(L^2\) is compactly embedded in \(L^q\) for any \(0 < q < \frac{2n}{n-1}\). This tells us that

\[(3.10) \quad \lim_{i \to \infty} \|u_i - u_{\infty}\|_{L^q} = 0\]

and

\[(3.11) \quad \lim_{i \to \infty} \|\sqrt{-1}A_{\omega}FH_i - \lambda \text{Id}_E + \delta u_{\infty}\|_{L^q} = 0,\]

for any \(0 < q < \frac{2n}{n-1}\). Hence \(\|u_{\infty}\|_{L^2} = 1.\)
Let $\mathcal{F} \subset E$ be a torsion-free subsheaf. Note that $(\pi_{\mathcal{F}}^H)^* h_i (\pi_{\mathcal{F}}^H)^* h_i^{-1} = h_i \pi_{\mathcal{F}}^H h_i^{-1}$ and $(h_i \pi_{\mathcal{F}}^H h_i^{-1})^* h_i^\frac{1}{2} = h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}}$. Thus $|h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}}|^2_K = \text{rank}(\mathcal{F})$. Then

$$2\pi \deg(\mathcal{F}) = \int_M (\text{tr}(\pi_{\mathcal{F}}^H \sqrt{-1} \Lambda \omega F_{H_i}) - |\bar{\partial} \pi_{\mathcal{F}}^H|^2) \frac{\omega^n}{n!}$$

$$\leq \int_M \text{tr}(\pi_{\mathcal{F}}^H \sqrt{-1} \Lambda \omega F_{H_i}) \frac{\omega^n}{n!}$$

$$= \int_M \text{tr}(h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}} (\sqrt{-1} \Lambda \omega F_{H_i}) h_i^{-\frac{1}{2}}) \frac{\omega^n}{n!}$$

$$= \int_M \text{tr}(h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}} (\lambda \text{Id}_E - \varepsilon_i \log h_i)) \frac{\omega^n}{n!}$$

$$= \lambda \cdot \text{rank}(\mathcal{F}) \cdot \text{Vol}(M, \omega) + \int M \text{tr}(h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}} \varepsilon_i l_i (u_\infty - u_i)) \frac{\omega^n}{n!}$$

$$- \int M \text{tr}(h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}} \varepsilon_i l_i u_\infty) \frac{\omega^n}{n!},$$

where we have used that $h_i^\frac{1}{2} (\sqrt{-1} \Lambda \omega F_{H_i}) h_i^{-\frac{1}{2}} = \sqrt{-1} \Lambda \omega F_{H_i}$ under the condition that $\sqrt{-1} \Lambda \omega F_{H_i} = \lambda \text{Id}_E - \varepsilon_i \log h_i$. Clearly there holds that when $i \to \infty$,

$$\int M \text{tr}(h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}} \varepsilon_i l_i (u_\infty - u_i)) \frac{\omega^n}{n!} \leq \varepsilon_i \cdot (\text{rank}(\mathcal{F}))^2 \int M |u_\infty - u_i| K \frac{\omega^n}{n!} \to 0.$$  

Again by (3.8) and following Simpson’s argument ([55 Lemma 5.5]), one can check that the eigenvalues of $u_\infty$ are constants and not all equal. Assume $\mu_1 < \mu_2 < \cdots < \mu_r$ are the distinct eigenvalues of $u_\infty$. Let $\{e_1, ..., e_r\}$ be an orthonormal basis of $E$ with respect to $H_i$ on the considered point such that

$$\pi_{\mathcal{F}}^H e_\alpha = \begin{cases} e_\alpha, & \alpha \leq \text{rank}(\mathcal{F}), \\ 0, & \alpha > \text{rank}(\mathcal{F}). \end{cases}$$

Then $\langle h_i^\frac{1}{2} e_\alpha, h_i^\frac{1}{2} e_\beta \rangle_K = \langle h_i e_\alpha, e_\beta \rangle_K = \delta_{\alpha, \beta}$. Set $\tilde{e}_\alpha = h_i^\frac{1}{2} e_\alpha$. Obviously $\{\tilde{e}_1, ..., \tilde{e}_r\}$ is an orthonormal basis of $E$ with respect to $K$. It is easy to find that

$$- \text{tr}(h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}} u_\infty) = \sum_{\alpha=1}^r \langle h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}} u_\infty (\tilde{e}_\alpha, \tilde{e}_\alpha) K \rangle$$

$$= \sum_{\alpha=1}^r \langle -u_\infty \tilde{e}_\alpha, \tilde{e}_\alpha \rangle_K \leq -\mu_1 \text{rank}(\mathcal{F}).$$

Thus

$$- \int M \text{tr}(h_i^\frac{1}{2} \pi_{\mathcal{F}}^H h_i^{-\frac{1}{2}} \varepsilon_i l_i u_\infty) \frac{\omega^n}{n!} \leq \varepsilon_i l_i \mu_1 \cdot \text{rank}(\mathcal{F}) \cdot \text{Vol}(M),$$

and then

$$\frac{2\pi \deg(\mathcal{F})}{\text{rank}(\mathcal{F})} \leq (\lambda - \delta \mu_1) \text{Vol}(M).$$
For $A < l$, define a smooth function $P_A : \mathbb{R} \to \mathbb{R}$ such that

\begin{equation}
P_A(x) = \begin{cases}
1, & x \leq \mu_A, \\
0, & x \geq \mu_A + 1.
\end{cases}
\end{equation}

Setting $\pi_A = P_A(u_\infty)$, by the argument as that in [55, p. 887], we have

\begin{itemize}
  \item[(i)] $\pi_A \in L^2_{1}$;
  \item[(ii)] $\pi_A^2 = \pi_\alpha = \pi_A^* K$;
  \item[(iii)] $(\text{Id} - \pi_A)\partial \pi_A = 0$.
\end{itemize}

According to the regularity statement for $L^2_{1}$-subbundles in [57], we know that $\pi_A$ defines a saturated subsheaf $E_A$ of $E$ (i.e. sub-sheaf with torsion-free quotient). Away from the singular set $\text{Sing}(E_A)$, $E_A$ is a holomorphic subbundle of $E$. We also set $E_0 = 0$ and $E_l = E$. In the following, write $r_A = \text{rank}(E_A)$ for simplicity.

**Lemma 3.1.** We have

\begin{equation}
- \text{tr}(h_i^{\frac{1}{2}} \pi_{E_A}^{H_i} h_i^{-\frac{1}{2}} u_\infty) \leq \sum_{B=1}^{A} (-\mu_B)(\text{rank}(E_B) - \text{rank}(E_{B-1})),
\end{equation}

where $\pi_{E_A}^{H_i}$ is the orthogonal projection onto $E_A$ with respect to $H_i$, then

\begin{equation} 2\pi \text{ deg}(E_A) \leq \text{Vol}(M) \sum_{B=1}^{A} (\lambda - \delta \mu_B)(\text{rank}(E_B) - \text{rank}(E_{B-1})).
\end{equation}

**Proof.** At $x \in M \setminus \text{Sing}(E_A)$, there is a basis $\{e_1, \cdots, e_{r_A}\}$ of $E_A|_x$. We choose $\{\hat{e}_1, \cdots, \hat{e}_{r_A}\}$ as an orthonormal basis of $E_A|_x$ with respect to $H_i$, and extend it to $\{\hat{e}_1, \cdots, \hat{e}_{r_A}, \cdots, \hat{e}_r\}$ as an orthonormal basis of $E|_x$ with respect to $H_i$. Set $\hat{e}_\alpha = h_i^{\frac{1}{2}} \hat{e}_\alpha$, so $\langle \hat{e}_\alpha, \hat{e}_\beta \rangle_K = \delta_{\alpha\beta}$, i.e. $\{\hat{e}_\alpha\}_{\alpha=1}^{r}$ is an orthonormal basis with respect to $K$. Define $\pi_l = \text{Id}_E$ and $\pi_0 = 0$. Then one has the fact that

\begin{equation}
u_\infty = \sum_{B=1}^{l} \mu_B(\pi_B - \pi_{B-1}),
\end{equation}

where $\pi_B$ defined as above is the orthogonal projection onto $E_B$ with respect to $K$. 
The straightforward calculation yields that

\[-\text{tr}(h_i^\frac{1}{2} \pi_{E_A}^H h_i^{-\frac{1}{2}} u_\infty) = -\sum_{\alpha=1}^r \langle h_i^\frac{1}{2} \pi_{E_A}^H h_i^{-\frac{1}{2}} u_\infty (\hat{\epsilon}_\alpha), \hat{\epsilon}_\alpha \rangle_K\]

\[= \sum_{\alpha=1}^r \langle -u_\infty (\hat{\epsilon}_\alpha), \hat{\epsilon}_\alpha \rangle_K\]

\[= \sum_{\alpha=1}^r \langle (\mu_A \Id_E - u_\infty) \hat{\epsilon}_\alpha, \hat{\epsilon}_\alpha \rangle_K - \mu_A \cdot r_A\]

\[(3.22)\]

\[\leq \sum_{\alpha=1}^r \sum_{B=1}^i \langle (\mu_A - \mu_B)(\pi_B - \pi_{B-1})(\hat{\epsilon}_\alpha), \hat{\epsilon}_\alpha \rangle_K - \mu_A \cdot r_A\]

\[\leq \sum_{\alpha=1}^r \sum_{B=1}^{A-1} \langle (\mu_A - \mu_B)(\pi_B - \pi_{B-1})(\hat{\epsilon}_\alpha), \hat{\epsilon}_\alpha \rangle_K - \mu_A \cdot r_A\]

\[= \sum_{B=1}^A (\mu_A - \mu_B)(\text{rank}(E_B) - \text{rank}(E_{B-1})),\]

where the first inequality comes from the facts that \(\mu_A - \mu_B \leq 0\) if \(B \geq A\) and \(\langle (\pi_B - \pi_{B-1})(\hat{\epsilon}_\alpha), \hat{\epsilon}_\alpha \rangle_K \geq 0\), which is due to \((\pi_B - \pi_{B-1})^2 = \pi_B^2 - \pi_B \circ \pi_{B-1} - \pi_{B-1} \circ \pi_B + \pi_{B-1}^2 = \pi_B - \pi_{B-1}\) and \((\pi_B - \pi_{B-1})^\ast K = \pi_B - \pi_{B-1}\), in the second equality from the bottom we have used \(\text{tr} \pi_B = \text{rank}(E_B)\). Then

\[2\pi \deg(E_A) = \int_M (\text{tr}(\pi_{E_A}^H \sqrt{-1} \Lambda_\omega F_H) - |\bar{\partial} \pi_{E_A}^H|^2) \frac{\omega^n}{n!}\]

\[\leq \int_M \text{tr}(h_i^{\frac{1}{2}} \pi_{E_A}^H h_i^{-\frac{1}{2}} \sqrt{-1} \Lambda_\omega F_H h_i^{-\frac{1}{2}}) \frac{\omega^n}{n!}\]

\[= \int_M \text{tr}(h_i^{\frac{1}{2}} \pi_{E_A}^H h_i^{-\frac{1}{2}} (\lambda \Id - \varepsilon l_i u_i)) \frac{\omega^n}{n!}\]

\[(3.23)\]

\[= \lambda \cdot \text{rank}(E_A) \cdot \text{Vol}(M, \omega) + \int_M \text{tr}(h_i^{\frac{1}{2}} \pi_{E_A}^H h_i^{-\frac{1}{2}} \varepsilon l_i (u_\infty - u_i)) \frac{\omega^n}{n!}\]

\[= \int_M \text{tr}(h_i^{\frac{1}{2}} \pi_{E_A}^H h_i^{-\frac{1}{2}} \varepsilon l_i u_\infty) \frac{\omega^n}{n!}\]

\[\leq \lambda \cdot \text{rank}(E_A) \cdot \text{Vol}(M, \omega) + \int_M \text{tr}(h_i^{\frac{1}{2}} \pi_{E_A}^H h_i^{-\frac{1}{2}} \varepsilon l_i (u_\infty - u_i)) \frac{\omega^n}{n!}\]

\[- \varepsilon l_i \text{Vol}(M, \omega) \left(\sum_{B=1}^A \mu_B(\text{rank}(E_B) - \text{rank}(E_{B-1}))\right)\]

Therefore, we achieve (3.20). \(\square\)
For simplicity, we write \( \lambda_A = \lambda - \delta \mu_A \). Then it follows that \( \lambda_1 > \lambda_2 > \cdots > \lambda_l \). For any torsion-free subsheaf \( \mathcal{F} \subset E \), by (3.17), we know that

\[
\frac{2 \pi \deg(\mathcal{F})}{\rank(\mathcal{F})} \leq \lambda_1 \Vol(M, \omega).
\]

Now consider the exact sequence

\[
0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow Q \longrightarrow 0.
\]

There holds that

\[
2 \pi \deg(Q) = \int_M (\tr((\Id - \pi^H) \sqrt{-1} \Lambda \omega F_{H_i}) + |\bar{\partial} \pi^H|^2) \frac{\omega^n}{n!} \\
\geq \int_M \tr(h_i^2 (\Id - \pi^H) h_i^{-2} \sqrt{-1} \Lambda \omega F_{H_i}) \frac{\omega^n}{n!} \\
= \int_M \tr(h_i^2 (\Id - \pi^H) h_i^{-2} (\sqrt{-1} \Lambda \omega F_{H_i} - (\lambda \Id - \varepsilon_i u_{\infty})) \frac{\omega^n}{n!} \\
+ \int_M \tr(h_i^2 (\Id - \pi^H) h_i^{-2} (\lambda \Id - \varepsilon_i u_{\infty})) \frac{\omega^n}{n!}.
\]

Take \( i \to \infty \), then

\[
2 \pi \deg(Q) \geq \lambda_1 \cdot \rank(Q) \cdot \Vol(M, \omega).
\]

Apply the same argument to the exact sequence

\[
0 \longrightarrow E_B \longrightarrow E \longrightarrow E/E_B \longrightarrow 0.
\]

Then

\[
2 \pi \deg(E/E_B) = 2 \pi (\deg(E) - \deg(E_B)) \\
= \int_M (\tr((\Id - \pi_{E_B}^H) \sqrt{-1} \Lambda \omega F_{H_i}) + |\bar{\partial} \pi_{E_B}^H|^2) \frac{\omega^n}{n!} \\
\geq \int_M \tr(h_i^2 (\Id - \pi_{E_B}^H) h_i^{-2} (\sqrt{-1} \Lambda \omega F_{H_i} - (\lambda \Id - \varepsilon_i u_{\infty})) \frac{\omega^n}{n!} \\
+ \int_M \tr(h_i^2 (\Id - \pi_{E_B}^H) h_i^{-2} (\lambda \Id - \varepsilon_i u_{\infty})) \frac{\omega^n}{n!}.
\]

After a similar computation as that in Lemma 3.1, one can see

**Lemma 3.2.**

\[
2 \pi \deg(E/E_B) \geq \sum_{A=B+1}^l \lambda_A (\rank(E_A) - \rank(E_{A-1})) \Vol(M, \omega).
\]

**Proof.** Note that \( \lambda_1 > \lambda_2 > \cdots > \lambda_l \) and

\[
0 \subset E_1 \subset E_2 \subset \cdots \subset E_l = E.
\]

At the point on the locally free part, let \( \{ \hat{e}_1, \cdots, \hat{e}_{r_B} \} \) be an orthonormal basis of \( E_B \) with respect to \( H_i \), and extend it to \( \{ \hat{e}_1, \cdots, \hat{e}_{r_B}, \cdots, \hat{e}_r \} \) as the orthonormal basis of \( E \) with respect to \( H_i \). Set \( \hat{e}_a = h_i^{1/2} \hat{e}_a \), so \( \langle \hat{e}_a, \hat{e}_\beta \rangle_K = \delta_{a\beta} \), i.e. \( \{ \hat{e}_a \}_{a=1}^r \) is an orthonormal basis with respect to \( K \). Recall \( u_{\infty} = \sum_{A=1}^l \mu_A (\pi_A - \pi_{A-1}) \), where \( \pi_A \) is the orthogonal
projection onto $E_A$ with respect to $K$, $\pi_0 = 0$ and $\pi_l = \text{Id}_E$. Denote $\tilde{u}_\infty = \lambda \text{Id}_E - \delta u_\infty$ and then

$$\tilde{u}_\infty = \sum_{A=1}^l \lambda_A (\pi_A - \pi_{A-1}).$$

Direct computing gives that

$$\text{tr}(h_i^{-\frac{1}{2}} (\text{Id}_E - \pi_{E_B}^H) h_i^{-\frac{1}{2}} \tilde{u}_\infty)$$

$$= \sum_{A=r_B+1}^r \langle \tilde{u}_\infty \hat{e}_\alpha, \hat{e}_\alpha \rangle_K$$

$$= \sum_{A=r_B+1}^r \langle (\tilde{u}_\infty - \lambda_B \text{Id}_E) \hat{e}_\alpha, \hat{e}_\alpha \rangle + \lambda_B (r - r_B)$$

$$= \sum_{A=r_B+1}^r \sum_{A=1}^l \langle (\lambda - \lambda_B) (\pi_A - \pi_{A-1}) \hat{e}_\alpha, \hat{e}_\alpha \rangle + \lambda_B (r - r_B)$$

$$\geq \sum_{A=r_B+1}^r \sum_{A=1}^l (\lambda - \lambda_B) \text{tr}(\pi_A - \pi_{A-1}) + \lambda_B (r - r_B)$$

$$= \sum_{A=r_B+1}^r \lambda_A (r_A - r_{A-1}).$$

So

$$\int_M \text{tr}(h_i^{-\frac{1}{2}} (\text{Id}_E - \pi_{E_B}^H) h_i^{-\frac{1}{2}} (\lambda \text{Id}_E - \delta u_\infty)) \frac{\omega^n}{n!} \geq \text{Vol}(M) \sum_{A=B+1}^l (\lambda - \delta \mu_A) (r_A - r_{A-1}).$$

Putting this into (3.29) and letting $i \to \infty$, we get the desired inequality (3.30). □

Recall that $\lambda_{mU}(H_i, \omega)$ is the average of the largest eigenvalue function $\lambda_U(H_i, \omega)$ of $\sqrt{-1} \Lambda \omega F_{H_i}$, $\lambda_{mL}(H_i, \omega)$ is the average of the smallest eigenvalue function $\lambda_L(H_i, \omega)$ of $\sqrt{-1} \Lambda \omega F_{H_i}$. By the Chern-Weil formula (1.7), it is easy to verify that

$$\lim_{i \to \infty} \lambda_{mU}(H_i, \omega) \text{Vol}(M, \omega) \geq \sup_{F \subset E} 2\pi \frac{\text{deg}(F)}{\text{rank}(F)},$$

where $F$ runs over all the subsheaves of $E$, and

$$\lim_{i \to \infty} \lambda_{mL}(H_i, \omega) \text{Vol}(M, \omega) \leq \inf_{Q \subset E} 2\pi \frac{\text{deg}(Q)}{\text{rank}(Q)},$$

where $Q$ runs over all the quotient sheaves of $E$. Furthermore, we have:
Lemma 3.3.

\( \lim_{i \to \infty} \lambda_{mU}(H_i, \omega) \leq \lambda_1 \) and

\( \lim_{i \to \infty} \lambda_{mL}(H_i, \omega) \geq \lambda_l. \)

Proof. Suppose \( e_1^i \) is an eigenvector of \( \sqrt{-1} \Lambda_{\omega} F_{H_i} \) with respect to \( \lambda_U(H_i, \omega) \), and \( |e_1^i|_K = 1 \). Of course one has

\[
\lambda_U(H_i, \omega) = \langle \sqrt{-1} \Lambda_{\omega} F_{H_i}(e_1^i), e_1^i \rangle_K
= \langle (\sqrt{-1} \Lambda_{\omega} F_{H_i} - (\lambda \text{Id}_E - \delta u_\infty)) e_1^i, e_1^i \rangle_K + \langle (\lambda \text{Id}_E - \delta u_\infty) e_1^i, e_1^i \rangle_K.
\]

This means that

\[
\lim_{i \to \infty} \lambda_{mU}(H_i, \omega) \text{Vol}(M, \omega)
= \lim_{i \to \infty} \int_M \lambda_U(H_i, \omega) \frac{\omega^n}{n!}
\leq \lim_{i \to \infty} \left( \| \sqrt{-1} \Lambda_{\omega} F_{H_i} - (\lambda \text{Id}_E - \delta u_\infty) \|_{L^1} + \int_M \langle (\lambda \text{Id}_E - \delta u_\infty) e_1^i, e_1^i \rangle_K \frac{\omega^n}{n!} \right)
= \int_M \langle (\lambda \text{Id}_E - \delta u_\infty) e_1^i, e_1^i \rangle_K \frac{\omega^n}{n!}
\leq \lambda_1 \text{Vol}(M, \omega).
\]

Immediately (3.38) can be proved in a similar way. \( \square \)

Define

\[
\nu = 2\pi \sum_{A=1}^{l-1} (\mu_{A+1} - \mu_A) \text{rank}(E_A) \left( \frac{\deg(E)}{\text{rank}(E)} - \frac{\deg(E_A)}{\text{rank}(E_A)} \right);
\]

then

\[
\nu = 2\pi (\mu_1 \deg(E) - \sum_{A=1}^{l-1} (\mu_{A+1} - \mu_A) \deg(E_A))
= 2\pi (\mu_1 \deg(E) + \sum_{A=1}^{l-1} \mu_A \deg(E_A) - \sum_{A=2}^{l} \mu_A \deg(E_{A-1}))
= 2\pi \sum_{A=1}^{l} \mu_A (\deg(E_A) - \deg(E_{A-1})).
\]

The fact \( \| u_\infty \|_{L^2} = 1 \) yields that

\[
\sum_{A=1}^{l} \mu_A^2 (\text{rank}(E_A) - \text{rank}(E_{A-1})) \text{Vol}(M) = 1.
\]
Recall $\mu_A = \frac{\lambda_A - \lambda}{\delta}$. Clearly it holds that

\begin{equation}
\sum_{A=1}^{l} (\lambda - \lambda_A)^2 (\text{rank}(E_A) - \text{rank}(E_{A-1})) = \frac{\delta^2}{\text{Vol}(M)}.
\end{equation}

By (3.8) and the same discussion in [55, Lemma 5.4] (51, (3.23)), we know

\begin{equation}
\delta + \int_{M} \left( \text{tr}(u_{\infty} \sqrt{-1} \Lambda \omega F_K) + \langle \zeta(u_{\infty}) \bar{\partial} u_{\infty}, \bar{\partial} u_{\infty} \rangle_K \right) \frac{\omega^n}{n!} \leq 0,
\end{equation}

where $\zeta \in C^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ satisfies $\zeta(x, y) < (x - y)^{-1}$ whenever $x > y$. Notice that

\begin{equation}
2\pi \deg(E_B) = \int_{M} \left( \text{tr}(\pi_B \cdot \sqrt{-1} \Lambda \omega F_K) - |\partial \pi_B|^2_K \right) \frac{\omega^n}{n!}.
\end{equation}

So by (3.45) and following the arguments in [35, p. 793-794], we obtain

\begin{equation}
\nu = \int_{M} \left( \text{tr}(u_{\infty} \sqrt{-1} \Lambda \omega F_K) + \sum_{A=1}^{l-1} (\mu_{A+1} - \mu_A)(dP_A)^2(u_{\infty})(\bar{\partial} u_{\infty}), \bar{\partial} u_{\infty} \right)_K \frac{\omega^n}{n!} \leq -\delta,
\end{equation}

where the function $dP_A : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

\[ dP_A(x, y) = \begin{cases} 
P_A(x) - P_A(y), & x \neq y; \\
P'_A(x), & x = y.
\end{cases} \]

Taking into account $\text{tr} u_{\infty} \equiv 0$, one has

\begin{equation}
\sum_{A=1}^{l} \mu_A(\text{rank}(E_A) - \text{rank}(E_{A-1})) = 0.
\end{equation}

Then

\begin{equation}
0 \geq \delta^2 + \delta \nu = \delta^2 + 2\pi \sum_{A=1}^{l} (\lambda - \lambda_A)(\deg(E_A) - \deg(E_{A-1}))
\end{equation}

\begin{equation}
= \sum_{A=1}^{l} (\lambda - \lambda_A)(2\pi(\deg(E_A) - \deg(E_{A-1})) - \lambda_A(r_A - r_{A-1}) \text{Vol}(M)).
\end{equation}

At the same time, we can conclude that

**Lemma 3.4.**

\begin{equation}
\sum_{A=1}^{l} (\lambda - \lambda_A)(2\pi(\deg(E_A) - \deg(E_{A-1})) - \lambda_A(r_A - r_{A-1}) \text{Vol}(M)) \geq 0.
\end{equation}
Proof. Computing straightforwardly gives that

\[
\sum_{A=1}^{l} (\lambda - \lambda_A)(2\pi(\deg(E_A) - \deg(E_{A-1})) - \Vol(M) \cdot \lambda_A(r_A - r_{A-1}))
\]

\[
= \sum_{A=1}^{l} (\lambda - \lambda_A) \left( 2\pi \deg(E_A) - \Vol(M) \cdot \sum_{B=1}^{A} \lambda_B(r_B - r_{B-1}) \right)
\]

\[
- \left( 2\pi \deg(E_{A-1}) - \Vol(M) \cdot \sum_{B=1}^{A-1} \lambda_B(r_B - r_{B-1}) \right)
\]

\[
= \sum_{A=1}^{l} (\lambda - \lambda_A) \left( 2\pi \deg(E_A) - \Vol(M) \cdot \sum_{B=1}^{A} \lambda_B(r_B - r_{B-1}) \right)
\]

\[
- \sum_{A=1}^{l-1} (\lambda - \lambda_{A+1}) \left( 2\pi \deg(E_A) - \Vol(M) \cdot \sum_{B=1}^{A} \lambda_B(r_B - r_{B-1}) \right)
\]

\[
= \sum_{A=1}^{l-1} (\lambda - \lambda_A - (\lambda - \lambda_{A+1})) \left( 2\pi \deg(E_A) - \Vol(M) \cdot \sum_{B=1}^{A} \lambda_B(r_B - r_{B-1}) \right)
\]

\[
+ (\lambda - \lambda_l) \left( 2\pi \deg(E) - \Vol(M) \cdot \sum_{A=1}^{l} \lambda_B(r_B - r_{B-1}) \right)
\]

\[
= \sum_{A=1}^{l-1} (\lambda_{A+1} - \lambda_A) \left( 2\pi \deg(E_A) - \Vol(M) \cdot \sum_{B=1}^{A} \lambda_B(r_B - r_{B-1}) \right)
\]

\[
\geq 0,
\]

where the inequality is based on (3.20) and \( \lambda_{A+1} < \lambda_A \), in the last equality we have used

\[
(3.52) \quad 2\pi \deg(E) = \Vol(M) \cdot \sum_{A=1}^{l} \lambda_A(r_A - r_{A-1}). \quad \Box
\]

Since \( \lambda_{A+1} < \lambda_A \), combining (3.49), (3.51), (3.20) and (3.52), one can find that

\[
(3.53) \quad 2\pi \deg(E_A) = \Vol(M) \cdot \sum_{B=1}^{A} \lambda_B(r_B - r_{B-1}),
\]

for \( 1 \leq A \leq l \). Consequently we have

\[
(3.54) \quad \frac{2\pi(\deg(E_A) - \deg(E_{A-1}))}{r_A - r_{A-1}} = \Vol(M) \cdot \lambda_A.
\]
By (3.35), (3.36), (3.37) and (3.38), we establish
\[
\sup_{F \subset E} 2\pi \left( \frac{\deg(F)}{\rank(F)} \right) \leq \lim_{i \to \infty} \lambda_{mU}(H_i, \omega) \Vol(M, \omega)
\]
\[(3.55) \leq \lambda_1 \Vol(M, \omega) = 2\pi \cdot \frac{\deg(E_1)}{\rank(E_1)}
\]
\[\leq \sup_{F \subset E} 2\pi \left( \frac{\deg(F)}{\rank(F)} \right)
\]
and
\[
\inf_{Q} 2\pi \left( \frac{\deg(Q)}{\rank(Q)} \right) \geq \lim_{i \to \infty} \lambda_{mL}(H_i, \omega) \Vol(M, \omega)
\]
\[(3.56) \geq \lambda_l \Vol(M, \omega) = 2\pi \cdot \frac{\deg(E) - \deg(E_l)}{\rank(E) - \rank(E_{l-1})}
\]
\[\geq \inf_{Q} 2\pi \left( \frac{\deg(Q)}{\rank(Q)} \right).
\]
So it follows that
\[
\lim_{i \to \infty} \lambda_{mU}(H_i, \omega) \Vol(M, \omega) = \lambda_1 \Vol(M, \omega) = 2\pi \cdot \frac{\deg(E_1)}{\rank(E_1)} = \max_{F \subset E} \frac{\deg(F)}{\rank(F)}
\]
\[(3.57)
\]
and
\[
\lim_{i \to \infty} \lambda_{mL}(H_i, \omega) \Vol(M, \omega) = \lambda_l \Vol(M, \omega) = 2\pi \cdot \frac{\deg(E/E_{l-1})}{\rank(E/E_{l-1})} = \min_{Q} \frac{\deg(Q)}{\rank(Q)}.
\]
\[(3.58)
\]
Assume \(F\) is a subsheaf of \(E\) with \(\rank(F) > r_{A-1}\) for some \(A \geq 2\). Clearly we have already known
\[
2\pi \deg(F) = \int_M (\tr(\pi_F^H \sqrt{-1} \Lambda_\omega F_H) - |\bar{\partial} \pi_F^H|^2) \frac{\omega^n}{n!}
\]
\[(3.59) \leq \int_M (\tr(h_i^\frac{1}{2} \pi_F^H h_i^{-\frac{1}{2}} (\sqrt{-1} \Lambda_\omega F_H - \bar{u}_\infty)) \frac{\omega^n}{n!}
\]
\[+ \int_M (\tr(h_i^\frac{1}{2} \pi_F^H h_i^{-\frac{1}{2}} \bar{u}_\infty) \frac{\omega^n}{n!}.
\]
Notice that \(F\) is a subbundle of \(E\) away from the singular set \(\text{Sing}(F)\). Suppose \(x \in M \setminus \text{Sing}(F)\). We choose \(\{\hat{e}_1, \ldots, \hat{e}_{\rank(F)}\}\) as the \(H_i\)-orthonormal basis of \(F|_x\), and extend it to \(\{\hat{e}_1, \ldots, \hat{e}_{\rank(F)}, \ldots, \hat{e}_r\}\) as the \(H_i\)-orthonormal basis of \(E|_x\). Set \(\hat{e}_\alpha = h_i^\frac{1}{2} \hat{e}_\alpha\), so \(\langle \hat{e}_\alpha, \hat{e}_\beta \rangle_K = \delta_{\alpha\beta}\), i.e. \(\{\hat{e}_\alpha\}_{\alpha=1}^r\) is an orthonormal basis with respect to \(K\). As before, we
also have

\[
\text{tr}(h_i^{1/2} \pi^H_i h_i^{-1/2} \tilde{u}_\infty) = \sum_{\alpha=1}^{\text{rank}(\mathcal{F})} \langle \tilde{u}_\infty(\hat{e}_\alpha), \hat{e}_\alpha \rangle_K
\]

\[
= \sum_{\alpha=1}^{\text{rank}(\mathcal{F})} \langle (\tilde{u}_\infty - \lambda_A \text{Id}_E)\hat{e}_\alpha, \hat{e}_\alpha \rangle_K + \lambda_A \cdot \text{rank}(\mathcal{F})
\]

\[
= \sum_{\alpha=1}^{\text{rank}(\mathcal{F})} \left( \sum_{B=1}^{A-1} (\lambda_B - \lambda_A) \langle (\pi_B - \pi_{B-1})(\hat{e}_\alpha), \hat{e}_\alpha \rangle_K + \lambda_A \cdot \text{rank}(\mathcal{F}) \right)
\]

(3.60)

\[
\leq \sum_{B=1}^{A-1} \left( \sum_{\alpha=1}^{r} (\lambda_B - \lambda_A) \langle (\pi_B - \pi_{B-1})(\hat{e}_\alpha), \hat{e}_\alpha \rangle_K + \lambda_A \cdot \text{rank}(\mathcal{F}) \right)
\]

\[
= \sum_{B=1}^{A-1} (\lambda_B - \lambda_A) \text{tr}(\pi_B - \pi_{B-1}) + \lambda_A \cdot \text{rank}(\mathcal{F})
\]

\[
= \sum_{B=1}^{A-1} \lambda_B (r_B - r_{B-1}) + \lambda_A \cdot (\text{rank}(\mathcal{F}) - r_{A-1}).
\]

Then

\[
2\pi \deg(\mathcal{F}) \leq (\sum_{B=1}^{A-1} \lambda_B (r_B - r_{B-1}) + \lambda_A \cdot (\text{rank}(\mathcal{F}) - r_{A-1})) \text{Vol}(M)
\]

(3.61)

\[
= 2\pi \sum_{B=1}^{A-1} (\deg(E_B) - \deg(E_{B-1})) + \lambda_A \cdot (\text{rank}(\mathcal{F}) - r_{A-1}) \text{Vol}(M)
\]

\[
= 2\pi \deg(E_{A-1}) + \lambda_A \cdot (\text{rank}(\mathcal{F}) - r_{A-1}) \text{Vol}(M).
\]

It follows that

\[
\frac{2\pi(\deg(\mathcal{F}) - \deg(E_{A-1}))}{\text{rank}(\mathcal{F}) - \text{rank}(E_{A-1})} \leq \frac{2\pi(\deg(E_A) - \deg(E_{A-1}))}{\text{rank}(E_A) - \text{rank}(E_{A-1})} < \lambda_{A-1} \text{Vol}(M).
\]

(3.62)

Next we are going to show that

\[
0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l = E
\]

is exactly the Harder-Narasimhan filtration of \((E, \bar{\partial}_E)\). Obviously (3.57) tells us that

(3.63)

\[
\frac{\deg(E_1)}{\text{rank}(E_1)} = \max_{\mathcal{F} \subset E} \left( \frac{\deg(\mathcal{F})}{\text{rank}(\mathcal{F})} \right).
\]

If \text{rank}(\mathcal{F}) > \text{rank}(E_1), from (3.62), we get

(3.64)

\[
\deg(\mathcal{F}) - \deg(E_1) < \frac{\text{rank}(\mathcal{F}) - \text{rank}(E_1)}{\text{rank}(E_1)} \deg(E_1),
\]
and then
\[
\frac{\deg(\mathcal{F})}{\text{rank}(\mathcal{F})} < \frac{\deg(E_1)}{\text{rank}(E_1)}.
\]
Consider
\[
\text{rank}(\mathcal{F}) < \text{rank}(E_1).
\]

(3.66) \quad 0 \subset E_B \subset \hat{\mathcal{F}} \subset E,

where \(\text{rank}(\hat{\mathcal{F}}) > \text{rank}(E_B)\) and \(B \geq 1\). Using (3.62) again, one can see
\[
\frac{\deg(\hat{\mathcal{F}}) - \deg(E_B)}{\text{rank}(\hat{\mathcal{F}}) - \text{rank}(E_B)} 
\leq \frac{\deg(E_{B+1}) - \deg(E_B)}{\text{rank}(E_{B+1}) - \text{rank}(E_B)},
\]
and if \(\text{rank}(\hat{\mathcal{F}}) > \text{rank}(E_{B+1})\), then
\[
\frac{\deg(\hat{\mathcal{F}}) - \deg(E_B)}{\text{rank}(\hat{\mathcal{F}}) - \text{rank}(E_B)} < \frac{\deg(E_{B+1}) - \deg(E_B)}{\text{rank}(E_{B+1}) - \text{rank}(E_B)}.
\]

Therefore, we have showed that
\[
0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l = E
\]
is the Harder-Narasimhan filtration of \((E, \tilde{\partial}_E)\).

Now we can prove Theorem 1.1. By the previous argument, there holds
\[
\lambda \text{Id}_E - \delta u_\infty \lambda = \lambda \sum_{A=1}^{l} (\pi_A - \pi_{A-1}) - \delta \sum_{A=1}^{l} \mu_A (\pi_A - \pi_{A-1}) = \sum_{A=1}^{l} \lambda_A (\pi_A - \pi_{A-1})
\]
\[
= \frac{2\pi}{\text{Vol}(M, \omega)} \sum_{A=1}^{l} \mu_{\omega} (E_A/E_{A-1}) (\pi_A - \pi_{A-1}) = \frac{2\pi}{\text{Vol}(M, \omega)} \Phi_{\omega}^H (E, K).
\]

Together with (3.11), we have for any \(0 < q < \frac{2n}{n-1}\),
\[
\lim_{i \to \infty} \left\| \sqrt{-1} \Lambda_{\omega} F_{H_i} - \frac{2\pi}{\text{Vol}(M, \omega)} \Phi_{\omega}^H (E, K) \right\|_{L^q (K)} = 0.
\]

Since \(\left\| \sqrt{-1} \Lambda_{\omega} F_{H_i} \right\|_{K}\) is uniformly bounded, (1.22) follows.

4. Long-Time Behavior of the Eigenvalues of the Mean Curvature

Let \((M, \omega)\) be an \(n\)-dimensional compact Hermitian manifold, \((E, \overline{\partial}_E)\) a rank \(r\) holomorphic vector bundle over \(M\) and \(H\) be a Hermitian metric on \(E\). As mentioned in
Introduction, \(\lambda_1(H, \omega), \lambda_2(H, \omega), \ldots, \lambda_r(H, \omega)\) are the eigenvalues of \(\theta_H = \sqrt{-1} \Lambda \omega F_H\), sorted in the descending order. For \(1 \leq k \leq r\), we denote

\[
\lambda_{L,k}(H, \omega) = \sum_{i=1}^{k} \lambda_{r-i+1}(H, \omega), \quad \lambda_{U,k}(H, \omega) = \sum_{i=1}^{k} \lambda_i(H, \omega),
\]

and

\[
\hat{\lambda}_{L,k}(H, \omega) = \inf_M \lambda_{L,k}(H, \omega), \quad \hat{\lambda}_{U,k}(H, \omega) = \sup_M \lambda_{U,k}(H, \omega).
\]

If additionally, \(\omega\) is Gauduchon, we set

\[
\lambda_{mL,k}(H, \omega) = \frac{1}{\Vol(M, \omega)} \int_M \lambda_{L,k}(H, \omega) \omega^n/n!.
\]

\[
\lambda_{mU,k}(H, \omega) = \frac{1}{\Vol(M, \omega)} \int_M \lambda_{U,k}(H, \omega) \omega^n/n!.
\]

4.1. Convergence of the eigenvalues. Let \(H(t)\) be a smooth solution of the Hermitian-Yang-Mills flow \([12, 25]\). We write for short

\[
\lambda_{L,k}(\cdot, t) = \lambda_{L,k}(H(t), \omega)(\cdot), \quad \lambda_{U,k}(\cdot, t) = \lambda_{U,k}(H(t), \omega)(\cdot),
\]

\[
\hat{\lambda}_{L,k}(t) = \hat{\lambda}_{L,k}(H(t), \omega), \quad \hat{\lambda}_{U,k}(t) = \hat{\lambda}_{U,k}(H(t), \omega),
\]

\[
\lambda_{mL,k}(t) = \lambda_{mL,k}(H(t), \omega), \quad \lambda_{mU,k}(t) = \lambda_{mU,k}(H(t), \omega).
\]

We have the following generalization of Theorem [1.3].

**Theorem 4.1.** Let \(k = 1, 2, \ldots, r\), then \(\hat{\lambda}_{L,k}(t)\) is monotone increasing and \(\hat{\lambda}_{U,k}(t)\) is monotone decreasing on \([0, \infty)\). If additionally \(\omega\) is Gauduchon, then \(\lambda_{mL,k}(t)\) is increasing and \(\lambda_{mU,k}(t)\) is decreasing. Furthermore, we have

\[
\lim_{t \to \infty} \lambda_{mL,k}(t) = \lim_{t \to \infty} \hat{\lambda}_{L,k}(t), \quad \lim_{t \to \infty} \lambda_{mU,k}(t) = \lim_{t \to \infty} \hat{\lambda}_{U,k}(t).
\]

By Lemma [2.7] and [2.11], the proof of Theorem 4.1 can be reduced to prove that

\[
<\partial_t - \sqrt{-1} \Lambda \omega \bar{\partial} \partial \partial> \lambda_{L,k} \geq 0, \quad <\partial_t - \sqrt{-1} \Lambda \omega \bar{\partial} \partial \partial> \lambda_{U,k} \leq 0,
\]

in the viscosity sense. Actually, the latter is a simple corollary of the following technical lemma.

**Lemma 4.2.** Let \(k = 1, 2, \ldots, r\). For any \((p_0, t_0) \in M \times [0, \infty)\), we can find an open neighborhood \(U\) of \(p_0\), and smooth functions \(f_{L,k}, f_{U,k}\) on \(U \times [0, \infty)\), such that

1) \(f_{L,k} \geq \lambda_{L,k}\) on \(U \times [0, \infty)\) and \(f_{L,k}(p_0, t_0) = \lambda_{L,k}(p_0, t_0)\).
2) \(<\partial_t - 2\sqrt{-1} \Lambda \omega \bar{\partial} \partial \partial> f_{L,k}(p_0, t_0) = 0\).
3) \(f_{U,k} \leq \lambda_{U,k}\) on \(U \times [0, \infty)\) and \(f_{U,k}(p_0, t_0) = \lambda_{U,k}(p_0, t_0)\).
4) \(<\partial_t - 2\sqrt{-1} \Lambda \omega \bar{\partial} \partial \partial> f_{U,k}(p_0, t_0) = 0\).

**Proof.** It suffices to show the construction of \(f_{L,k}\), the construction of \(f_{U,k}\) is similar.

Certainly one can find an orthonormal basis \(\{u_{\alpha}\}_{\alpha=1}^{r}\) of \((E_{p_0}, H(t_0))\) such that

\[
<\sqrt{-1} \Lambda \omega F_{H(t_0)} u_{\alpha}, u_{\beta}> H(t_0)(p_0) = \lambda_{\alpha}(H(t_0), \omega)(p_0) \delta_{\alpha\beta}.
\]
Choose a holomorphic frame field \( \{e_{\alpha}\}_{\alpha=1}^{r} \) on some open neighborhood \( U \) of \( p_0 \) such that

1) \( e_{\alpha}(p_0) = u_{\alpha} \) for \( \alpha = 1, 2, \ldots, r \);

2) \( dH_{\beta\alpha}|_{(p_0, t_0)} = 0 \), where \( H_{\beta\alpha} = \langle e_{\alpha}, e_{\beta} \rangle_{H(t)} \).

Set \( \theta_{\beta\alpha} = \langle \sqrt{-1}\Lambda_{\omega}F_{H}(e_{\alpha}), e_{\beta} \rangle_{H} \), then we have on \( U \times [0, \infty) \)

\[
\frac{\partial H_{\beta\alpha}}{\partial t} = -2(\theta_{\beta\alpha} - \lambda H_{\beta\alpha}), \tag{4.8}
\]

\[
\theta_{\beta\alpha} = \sqrt{-1}\Lambda_{\omega}(\bar{\partial}H_{\beta\alpha} - \bar{\partial}H_{\beta\gamma}H^{\gamma\delta}\partial H_{\delta\alpha}), \tag{4.9}
\]

where \( (H^{\alpha\beta}) \) is the inverse matrix of \( (H_{\beta\alpha}) \). Because \( dH_{\beta\alpha}|_{(p_0, t_0)} = 0 \), it is easy to check

\[
\left( \frac{\partial}{\partial t} - 2\sqrt{-1}\Lambda_{\omega}\bar{\partial}\right)H_{\beta\alpha}|_{(p_0, t_0)} = 2\lambda H_{\beta\alpha}|_{(p_0, t_0)}, \tag{4.10}
\]

\[
\left( \frac{\partial}{\partial t} - 2\sqrt{-1}\Lambda_{\omega}\bar{\partial}\right)\theta_{\beta\alpha}|_{(p_0, t_0)} = 2\lambda \theta_{\beta\alpha}|_{(p_0, t_0)}. \tag{4.11}
\]

Running the Gram-Schmidt process, we can construct smooth maps \( \tilde{e}_{1}, \ldots, \tilde{e}_{r} : U \times [0, \infty) \to E \)

\[
\tilde{e}_{1} = \frac{e_{1}}{|e_{1}|_{H}},
\]

\[
\tilde{e}_{2} = \frac{e_{2} - \langle e_{2}, \tilde{e}_{1} \rangle_{H}\tilde{e}_{1}}{|e_{2} - \langle e_{2}, \tilde{e}_{1} \rangle_{H}\tilde{e}_{1}|_{H}},
\]

\[
\vdots,
\]

\[
\tilde{e}_{r} = \frac{e_{r} - \langle e_{r}, \tilde{e}_{1} \rangle_{H}\tilde{e}_{1} - \cdots - \langle e_{r}, \tilde{e}_{r-1} \rangle_{H}\tilde{e}_{r-1}}{|e_{r} - \langle e_{r}, \tilde{e}_{1} \rangle_{H}\tilde{e}_{1} - \cdots - \langle e_{r}, \tilde{e}_{r-1} \rangle_{H}\tilde{e}_{r-1}|_{H}}.
\]

At any \( (p, t) \in M \times [0, \infty) \), \( \{\tilde{e}_{1}(p, t), \ldots, \tilde{e}_{r}(p, t)\} \) forms an orthonormal basis of \( (E_{p}, H(t)|_{p}) \).

If we write

\[
\tilde{e}_{\alpha} = a_{\alpha}^{\beta}e_{\beta}, \tag{4.13}
\]

then the coefficients \( a_{\alpha}^{\beta} \) satisfy

1) \( a_{\alpha}^{\gamma}H_{\gamma\delta}a_{\delta}^{\beta} = \delta_{\alpha\beta} \);

2) \( a_{\alpha}^{\beta}(p_0, t_0) = \delta_{\alpha\beta} \);

3) \( da_{\alpha}^{\beta}(p_0, t_0) = 0 \).

Define

\[
f_{L,k} = \sum_{\alpha=r-k+1}^{r} \langle \theta_{H}\tilde{e}_{\alpha}, \tilde{e}_{\alpha} \rangle_{H} = \sum_{\alpha=r-k+1}^{r} a_{\alpha}^{\beta}\theta_{\gamma\delta}a_{\delta}^{\alpha}. \tag{4.14}
\]

Obviously, \( f_{L,k}(p_0, t_0) = \lambda_{L,k}(p_0, t_0) \).

According to the definition of \( \lambda_{L,k} \) and the fact that \( \theta_{H} \in \text{Herm}(H) \), one has

\[
\lambda_{L,k}(p, t) = \inf \left\{ \sum_{\alpha=1}^{k} \langle \theta_{H(t)}v_{\alpha}, v_{\alpha} \rangle_{H(t)} \left| \{v_{\alpha}\}_{\alpha=1}^{k} \subset E_{p} \text{ is } H(t)-\text{orthonormal} \right. \right\}, \tag{4.15}
\]

consequently \( f_{L,k} \geq \lambda_{L,k} \) on \( U \times [0, \infty) \).
For any $\alpha = 1, 2, \cdots, r$, we denote
\begin{equation}
(4.16) \quad f_\alpha = \langle \theta_H \bar{e}_\alpha, \bar{e}_\alpha \rangle_H = \overline{a_\alpha} \theta_{\bar{\delta}} a_\alpha^\delta.
\end{equation}
By the properties that $e_\alpha$ and $a^\delta_\alpha$ have, a direct computation yields
\begin{equation}
(4.17) \quad \left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \bar{\partial} \partial \right) f_\alpha \bigg|_{(p_0, t_0)} = \left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \bar{\partial} \partial \right) a_\alpha \theta_{\bar{\delta}} a^\delta_\alpha \bigg|_{(p_0, t_0)} = \left( [ \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \bar{\partial} \partial ] \theta_{\bar{\alpha}} + \theta_{\bar{\alpha}} ( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \bar{\partial} \partial ) [ a^\alpha_\alpha ] \right) \bigg|_{(p_0, t_0)}.
\end{equation}
Similarly,
\begin{equation}
(4.18) \quad 0 = \left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \bar{\partial} \partial \right) a_\alpha H_{\bar{\delta}} a^\delta_\alpha \bigg|_{(p_0, t_0)} = \left( [ \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \bar{\partial} \partial ] H_{\bar{\alpha}} + \theta_{\bar{\alpha}} ( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \bar{\partial} \partial ) [ a^\alpha_\alpha ] \right) \bigg|_{(p_0, t_0)}.
\end{equation}

Together with (4.10) and (4.11), one can deduce
\begin{equation}
(4.19) \quad \left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \bar{\partial} \partial \right) f_\alpha \bigg|_{(p_0, t_0)} = 0.
\end{equation}
Since $f_{L,k}$ is a sum of several $f_\alpha$, we arrive at $\left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \bar{\partial} \partial \right) f_{L,k} \bigg|_{(p_0, t_0)} = 0$. \hfill \Box

There is the following evident fact.

**Lemma 4.3.** Let $u(\cdot, t) (t \in [0, \infty))$ be a family of uniformly bounded lower semi-continuous functions on $M$, and
\begin{equation}
(4.20) \quad \mu(t) = \inf_M u(\cdot, t), \quad \mu_m(t) = \frac{1}{\text{Vol}(M, \omega)} \int_M u(\cdot, t) \omega^n = \frac{1}{n!}.
\end{equation}
If $\lim_{t \to \infty} \mu(t) = \lim_{t \to \infty} \mu_m(t) = c$ for some $c$, then for any $p \in [1, \infty)$,
\begin{equation}
(4.21) \quad \lim_{t \to \infty} \| u(\cdot, t) - c \|_{L^p(M)} = 0.
\end{equation}
Combining Lemma 4.3 and Theorem 4.1, we conclude

**Corollary 4.4.** For $k = 1, 2, \cdots, r$, let
\begin{equation}
(4.22) \quad c_{L,k} = \lim_{t \to \infty} \lambda_{mL,k}(t), \quad c_{U,k} = \lim_{t \to \infty} \lambda_{mU,k}(t).
\end{equation}
Then for any $p \in [1, \infty)$,
\begin{equation}
(4.23) \quad \lim_{t \to \infty} \| \lambda_{L,k}(\cdot, t) - c_{L,k} \|_{L^p(M)} = 0, \quad \lim_{t \to \infty} \| \lambda_{U,k}(\cdot, t) - c_{U,k} \|_{L^p(M)} = 0.
\end{equation}
Consequently for any $i = 1, 2, \cdots, r$,
\begin{equation}
(4.24) \quad \lim_{t \to \infty} \| \lambda_i(H(t), \omega) - c_i \|_{L^p(M)} = 0,
\end{equation}
where $c_i = c_{L,i-1} - c_{L,i-1} - c_{U,i-1} - c_{U,i-1}$ and $c_{L,0} = c_{U,0} = 0$.
4.2. Limits of the eigenvalues.

**Theorem 4.5.** Let $H(t)$ be a smooth solution of the Hermitian-Yang-Mills flow (1.25). Then we have

$$\lim_{t \to \infty} \lambda_{mL,k}(H(t), \omega) = \frac{2\pi}{\Vol(M, \omega)} \sum_{i=1}^{k} \mu_{r+1-i,\omega}(E), \tag{4.25}$$

$$\lim_{t \to \infty} \lambda_{mU,k}(H(t), \omega) = \frac{2\pi}{\Vol(M, \omega)} \sum_{i=1}^{k} \mu_{i,\omega}(E). \tag{4.26}$$

Clearly Theorem 1.4 is derived from Corollary 4.4 and Theorem 4.5.

Once the following lemma is proved, the “≤” part of (4.25) and “≥” part of (4.26) come.

**Lemma 4.6.** Let $H$ be a smooth Hermitian metric on $E$, then for $i = 1, \cdots, r$, we have

$$\lambda_{mL,k}(H, \omega) \leq \frac{2\pi}{\Vol(M, \omega)} \sum_{i=1}^{k} \mu_{r+1-i,\omega}(E), \tag{4.27}$$

$$\lambda_{mU,k}(H, \omega) \geq \frac{2\pi}{\Vol(M, \omega)} \sum_{i=1}^{k} \mu_{i,\omega}(E). \tag{4.28}$$

**Proof.** We only give the proof of (4.28), the proof of (4.27) is similar.

Let $0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$ be the Harder-Narasimhan filtration of $E$, and set

$$r_j = \text{rank}(E_j), \tag{4.30}$$

for $j = 1, \cdots, l$. By (1.7) and the definition of $\bar{\mu}_\omega(E)$, one can easily verify that

$$\lambda_{mU,r_j}(H, \omega) \geq \frac{2\pi}{\Vol(M, \omega)} \deg_{\omega}(E_j) = \frac{2\pi}{\Vol(M, \omega)} \sum_{i=1}^{r_j} \mu_{i,\omega}(E). \tag{4.32}$$

We will proceed by induction on $k$.

1) For $k = 1$, we have

$$r_1 \lambda_{mU,1} \geq \lambda_{mU,r_1}(H, \omega) \geq \frac{2\pi}{\Vol(M, \omega)} \sum_{i=1}^{r_1} \mu_{i,\omega}(E) = \frac{2r_1\pi}{\Vol(M, \omega)} \mu_{1,\omega}(E). \tag{4.31}$$

So (4.28) holds for $k = 1$.

2) Assume that $1 \leq s \leq r-1$ and (4.28) holds for $k = s$. Namely

$$\lambda_{mU,s}(H, \omega) \geq \frac{2\pi}{\Vol(M, \omega)} \sum_{i=1}^{s} \mu_{i,\omega}(E). \tag{4.32}$$
If \( s + 1 = r_j \) for some \( j = 1, \ldots, l \), then of course (4.28) holds for \( k = s + 1 \). Otherwise we can find some \( j = 1, \ldots, l \) such that \( r_{j-1} < s + 1 < r_j \). Based on (4.32), one can see
\[
\lambda_{mU,s}(H, \omega) + \frac{r_j - s}{\text{Vol}(M, \omega)} \int_M \lambda_{s+1}(H, \omega) \frac{\omega^n}{n!} \\
\geq \lambda_{mU,r_j}(H, \omega) \\
\geq \frac{2\pi}{\text{Vol}(M, \omega)} \sum_{i=1}^{r_j} \mu_{i,\omega}(E) \\
= \frac{2\pi}{\text{Vol}(M, \omega)} \left( \sum_{i=1}^{s} \mu_{i,\omega}(E) + (r_j - s)\mu_{s+1,\omega}(E) \right).
\]

Making use of (4.32) again, we deduce
\[
(4.34) \quad (r_j - s)\lambda_{mU,s+1}(H, \omega) \geq \frac{2(r_j - s)\pi}{\text{Vol}(M, \omega)} \sum_{i=1}^{s+1} \mu_{i,\omega}(E).
\]

Hence (4.28) holds for \( k = s + 1 \).

This concludes the proof. \( \square \)

In order to show the “\( \geq \)” part of (4.25) and the “\( \leq \)” part of (4.26), we only need to apply the following two lemmas.

**Lemma 4.7.** For any \( \delta > 0 \), we can find a smooth solution \( H_\delta(t) \) of the Hermitian-Yang-Mills flow (1.25) on \( (E, \partial E) \) such that for any \( k = 1, \ldots, r \),
\[
\lim_{t \to \infty} \lambda_{mL,k}(H_\delta(t), \omega) \geq \frac{2\pi}{\text{Vol}(M, \omega)} \sum_{i=1}^{k} \mu_{r+1-i,\omega}(E) - k\delta,
\]
\[
\lim_{t \to \infty} \lambda_{mU,k}(H_\delta(t), \omega) \leq \frac{2\pi}{\text{Vol}(M, \omega)} \sum_{i=1}^{k} \mu_{i,\omega}(E) + k\delta.
\]

**Lemma 4.8.** Let \( H_1(t) \) and \( H_2(t) \) be two smooth solutions of the Hermitian-Yang-Mills flow (1.25), then
\[
\lim_{t \to \infty} \int_M |\tilde{\lambda}(H_1(t), \omega) - \tilde{\lambda}(H_2(t), \omega)|^2 \frac{\omega^n}{n!} = 0.
\]

Theorem 1.1 and Theorem 4.1 imply Lemma 4.7. Next we will work on the proof of Lemma 4.8.

Suppose \( H_i(t) \) \( (i = 1, 2) \) are two smooth solutions of the Hermitian-Yang-Mills flow (1.25) on \( E \). Set
\[
(4.38) \quad h(t) = H_1^{-1}(t)H_2(t), \quad A(t) = DH_2(t) - DH_1(t).
\]

Let \( \sigma(t) \in \text{Herm}^+(H_1(t)) \) be the unique element satisfying
\[
(4.39) \quad \sigma^*H_1(t)\sigma(t) = \sigma^2(t) = h(t).
\]
Lemma 4.9. Let $C = \sup_M \text{tr}(h + h^{-1})(0) - 2r$, then we have

\begin{equation}
\sup_M \text{tr}(h + h^{-1})(t) - 2r \leq C,
\end{equation}

\begin{equation}
\int_0^\infty \int_M \frac{|A|^2_{H_1,\omega}(t)}{n!} dt \leq \frac{C(2 + C)}{2} \text{Vol}(M, \omega).
\end{equation}

Furthermore, we have

\begin{equation}
\lim_{t \to \infty} \int_M |A|^2_{H_1,\omega}(t) \frac{\omega^n}{n!} = 0.
\end{equation}

Proof. The evolution equation of $h(t)$ is

\begin{equation}
\frac{\partial}{\partial t} + 2\sqrt{-1} \Lambda_{\omega} \bar{\partial} \partial H_1 \right) h(t) = 2([\theta_{H_1}, h] + \sqrt{-1} \Lambda_{\omega}(\bar{\partial} h \wedge h^{-1} \partial H_1, h)),
\end{equation}

consequently we have

\begin{equation}
\left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_{\omega} \bar{\partial} \right) \text{tr}(h + h^{-1})
\end{equation}

\begin{equation}
= 2\sqrt{-1} \Lambda_{\omega} \text{tr}(\bar{\partial} h \wedge h^{-1} \partial H_1, h - h^{-1} \partial H_1, h \wedge h^{-1} \bar{\partial} h h^{-1}).
\end{equation}

Notice that $A = h^{-1} \partial H_1, h$, $\bar{\partial} h h^{-1} = A^{*H_1}$ and $\sigma^2 = h$. One can find that

\begin{equation}
-\sqrt{-1} \Lambda_{\omega} \text{tr}(\bar{\partial} h \wedge h^{-1} \partial H_1, h) = -\sqrt{-1} \Lambda_{\omega} \text{tr}((\sigma A)^{H_1} \wedge (\sigma A))
\end{equation}

\begin{equation}
= |\sigma A|^2_{H_1,\omega},
\end{equation}

and

\begin{equation}
\sqrt{-1} \Lambda_{\omega} \text{tr}(h^{-1} \partial H_1, h \wedge h^{-1} \bar{\partial} h h^{-1}) = \sqrt{-1} \Lambda_{\omega} \text{tr}((A\sigma^{-1}) \wedge (A\sigma^{-1})^{*H_1})
\end{equation}

\begin{equation}
= |A\sigma^{-1}|^2_{H_1,\omega}.
\end{equation}

Thus

\begin{equation}
\left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_{\omega} \bar{\partial} \right) (\text{tr}(h + h^{-1}) - 2r) = -2 (|\sigma A|^2_{H_1,\omega} + |A\sigma^{-1}|^2_{H_1,\omega}).
\end{equation}

The parabolic maximal principle tells us that

\begin{equation}
\sup_M \text{tr}(h + h^{-1}) - 2r \leq \sup_M \text{tr}(h + h^{-1})(0) - 2r = C.
\end{equation}

This means

\begin{equation}
\frac{1}{2 + C} H_1 \leq H_2 \leq (2 + C) H_1,
\end{equation}

and

\begin{equation}
\frac{1}{2 + C} \text{Id}_E \leq h \leq (2 + C) \text{Id}_E, \quad \frac{1}{\sqrt{2 + C}} \text{Id}_E \leq \sigma \leq \sqrt{2 + C} \text{Id}_E,
\end{equation}

in Herm$^+(H_1)$. Then

\begin{equation}
\left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_{\omega} \bar{\partial} \right) (\text{tr}(h + h^{-1}) - 2r) \leq -\frac{4}{2 + C} |A|^2_{H_1,\omega},
\end{equation}

\begin{equation}
\int_0^\infty \int_M \frac{|A|^2_{H_1,\omega}(t)}{n!} dt \leq \frac{C(2 + C)}{2} \text{Vol}(M, \omega).
\end{equation}
which implies (4.41),
\[
\int_0^\infty \int_M |A|_{H_{1,\omega}}^2(t) \frac{\omega^n}{n!} dt \leq \frac{2 + C}{4} \int_0^\infty (\text{tr}(h + h^{-1}) - 2r)(0) \frac{\omega^n}{n!}
\leq \frac{C(2 + C)}{4} \text{Vol}(M, \omega).
\]

Write \(C_1 = 2 + C + \sup_M (|\theta_{H_1}|_{H_1} + |\theta_{H_2}|_{H_2})(0)\), then we always have
(4.53) \[2 + \text{tr}(h + h^{-1})(t) - 2r + |\theta_{H_1}|_{H_1}(t) + |\theta_{H_2}|_{H_2}(t) \leq C_1.\]

The evolution equation of \(A(t)\) is
(4.54) \[\frac{\partial A(t)}{\partial t} = -2\partial_{H_2} \theta_{H_2}(t) + 2\partial_{H_1} \theta_{H_1}(t),\]
so
(4.55) \[\left| \frac{\partial A(t)}{\partial t} \right|_{H_{1,\omega}} \leq 2|\partial_{H_2} \theta_{H_2}|_{H_{1,\omega}}(t) + 2|\partial_{H_1} \theta_{H_1}|_{H_{1,\omega}}(t) \leq 2C_1|\partial_{H_2} \theta_{H_2}|_{H_{2,\omega}}(t) + 2|\partial_{H_1} \theta_{H_1}|_{H_{1,\omega}}(t).\]

This yields that
(4.56) \[
\frac{\partial}{\partial t} |A|_{H_{1,\omega}}^2(t) = 2 \langle [A, \theta_{H_1}], A \rangle_{H_{1,\omega}} + 2 \text{Re} \langle \frac{\partial A}{\partial t}, A \rangle_{H_{1,\omega}} \\
\leq 4C_1 |A|_{H_{1,\omega}}^2 + 4C_1 |A|_{H_{1,\omega}}|\partial_{H_2} \theta_{H_2}|_{H_{2,\omega}} \\
+ 4 |A|_{H_{1,\omega}}|\partial_{H_1} \theta_{H_1}|_{H_{1,\omega}} \\
\leq 8C_1 |A|_{H_{1,\omega}}^2 + 2(C_1 |\partial_{H_2} \theta_{H_2}|_{H_{2,\omega}}^2 + |\partial_{H_1} \theta_{H_1}|_{H_{1,\omega}}^2).\]

Consider
(4.57) \[f(t) = \int_M |A|_{H_{1,\omega}}^2(t) \frac{\omega^n}{n!}\]
and
(4.58) \[a(t) = 2 \int_M (C_1 |\partial_{H_2} \theta_{H_2}|_{H_{2,\omega}}^2 + |\partial_{H_1} \theta_{H_1}|_{H_{1,\omega}}^2)(t) \frac{\omega^n}{n!}.\]

There holds that
(4.59) \[\lim_{t \to \infty} \int_{t-1}^t (f(s) + a(s)) \, ds = 0\]
and
(4.60) \[f'(t) \leq 8C_1 f(t) + a(t).\]

For any fixed \(t_0 \in [0, \infty)\), we have
(4.61) \[(e^{-8C_1(t-t_0)})f'(t) \leq a(t), \quad \text{for } t \geq t_0.\]
Then on \([t_0, +\infty)\),
(4.62) \[f(t) \leq e^{8C_1(t-t_0)} \left( f(t_0) + \int_{t_0}^t a(s) \, ds \right).\]
Hermitian metrics with negative mean curvature

Now assume that \( t \in [1, \infty) \). It follows that for any \( s \in [t - 1, t] \),

\[
(4.63) \quad f(t) \leq e^{8C_1} \left( f(s) + \int_{t-1}^t a(x) \, dx \right).
\]

Integrating the right hand side with respect to \( s \) on \([t - 1, t]\), one can get

\[
(4.64) \quad f(t) \leq e^{8C_1} \int_{t-1}^t (f(s) + a(s)) \, ds.
\]

Combined with (4.59), this gives the desired equality (4.42). \( \Box \)

**Theorem 4.10.**

\[
(4.65) \quad \lim_{t \to \infty} \int_M |\theta_{H_2} - \theta_{H_1}|^2_{H_1(t)} \frac{\omega^n}{n!} = 0.
\]

**Proof.** Obviously it holds that

\[
(4.66) \quad \theta_{H_2} - \theta_{H_1} = \sqrt{-1} \Lambda \bar{\partial} A = [\sqrt{-1} \Lambda, \bar{\partial}] A = (\partial^* H_1 + \tau^*) A.
\]

A simple calculation implies

\[
\int_M |\theta_{H_2} - \theta_{H_1}|^2_{H_1} \frac{\omega^n}{n!} = \int_M \langle (\partial^* H_1 + \tau^*) A, \theta_{H_2} - \theta_{H_1} \rangle_{H_1} \frac{\omega^n}{n!} = \int_M \langle A, (\partial H_1 + \tau)(\theta_{H_2} - \theta_{H_1}) \rangle_{H_1} \frac{\omega^n}{n!} = \int_M \langle A, \tau(\theta_{H_2} - \theta_{H_1}) \rangle_{H_1} \frac{\omega^n}{n!} = \int_M \langle A, [A, \theta_{H_2}] \rangle_{H_1} \frac{\omega^n}{n!} - \int_M \langle A, \partial H_1 \theta_{H_1} \rangle_{H_1} \frac{\omega^n}{n!}.
\]

Clearly there is a constant \( C \) depending only on \( \omega, H_1(0) \) and \( H_2(0) \) such that

\[
(4.68) \quad \text{tr}(h + h^{-1}) + |\tau|_{\omega} + |\theta_{H_1}|_{H_1} + |\theta_{H_2}|_{H_2} \leq C.
\]

Then we can find a constant \( C_1 \) depending only on \( C \), such that

\[
(4.69) \quad \int_M |\theta_{H_2} - \theta_{H_1}|^2_{H_1} \frac{\omega^n}{n!} \leq C_1 \int_M |A|_{H_1, \omega} (1 + |A|_{H_1, \omega} + |\partial H_1 \theta_{H_1}|_{H_1, \omega} + |\partial H_2 \theta_{H_2}|_{H_2, \omega}) \frac{\omega^n}{n!}.
\]

On account of (2.15) and (4.42), the right hand term converges to 0 when \( t \to \infty \), which concludes the proof. \( \Box \)

Point-wise by choosing an orthonormal basis with respect to \( H_1 \), we can treat \( \sigma, \theta_{H_2} \) and \( \theta_{H_1} \) as matrices. Since \( \sigma \theta_{H_2} \sigma^{-1} \) and \( \theta_{H_1} \) are Hermitian, one can find unitary matrices
$U_1$ and $U_2$, such that $U_1 \sigma \theta_{H_2} \sigma^{-1} U_1^{-1}$ and $U_2 \theta_{H_1} U_2^{-1}$ are real diagonal matrices. According to [8, Theorem VIII.3.9], we know
\[
|\vec{\lambda}(H_2) - \vec{\lambda}(H_1)|^2 \leq \text{cond}(U_1 \sigma) \text{cond}(U_2) |\theta_{H_2} - \theta_{H_1}|_{H_1}^2,
\]
where \text{cond}(B) is the condition number of a nonsingular matrix $B$. Because $U_1$ and $U_2$ are unitary, one has \text{cond}(U_1 \sigma) = \text{cond}(\sigma)$ and \text{cond}(U_2) = 1. Thus
\[
(4.70) \quad |\vec{\lambda}(H_2) - \vec{\lambda}(H_1)|^2 \leq \text{cond}(\sigma) |\theta_{H_2} - \theta_{H_1}|_{H_1}^2.
\]
Moreover, \text{cond}(\sigma) is equal to the quotient of the biggest and smallest eigenvalues of $\sigma$. We observe
\[
(4.71) \quad \text{cond}(\sigma) \leq \frac{1}{2} \text{tr}(h + h^{-1}) \leq \frac{1}{2} \sup_M \text{tr}(h + h^{-1})(0).
\]
Then Lemma 4.8 comes immediately from Theorem 4.10.

5. Some applications

5.1. A proof of Theorem 1.8. Let $(E, \bar{\partial}_E)$ be a rank $r$ holomorphic vector bundle over a compact complex manifold $M$. By [25] and [23], we have

(i) If $E$ is ample, then $\Lambda^k E$ is ample for $1 \leq k \leq r$.

(ii) If $E$ is ample, then any quotient bundle $Q$ of $E$ is ample.

(iii) If $E$ is Griffiths positive, then $E$ is ample.

(iv) If $E$ is ample, then when $k$ is sufficiently large, $S^k E$ is Griffiths positive.

Of course any quotient bundle of an ample bundle has positive first Chern class. However, it is not clear whether this property still holds in the quotient sheaf case. Fortunately we can confirm that there exists a Kähler current in the first Chern class.

Proposition 5.1. Let $\omega$ be a Kähler metric on $M$. Assume that $E$ is ample and $p = 1, \ldots, r - 1$. Then we can find $\delta_p > 0$, such that for any $p$-rank coherent quotient sheaf $Q$ of $E$, there exists a current $\theta \in c_1(Q)$ such that $\theta \geq \delta_p \omega$ in the sense of current.

Proof. Our idea of the proof originates from the proof of [16, Theorem 1.18]. Because $Q$ is a $p$-rank coherent quotient sheaf of $E$, $Q^*$ is a $p$-rank coherent subsheaf of $E^*$. Then there is the following injective sheaf morphism
\[
(5.1) \quad j : \det(Q^*) \to \Lambda^p E^*.
\]
Passing to symmetric powers, for $k = 1, 2, \ldots$, we have injective sheaf morphism
\[
(5.2) \quad j_k : (\det(Q^*))^k \to S^k(\Lambda^p E^*).
\]
Thanks to these sheaf morphisms, one can construct a singular Hermitian metric on $\det E$ whose “Chern curvature” is a Kähler current.

Since $\Lambda^p E$ is also ample, we can find $k_0 \geq 1, a > 0$ and Hermitian metric $\hat{H}$ on $S^{k_0}(\Lambda^p E)$ such that
\[
(5.3) \quad \sqrt{-1} F_{\hat{H}} \geq a \text{Id} \otimes \omega.
\]
in the sense of Griffiths. Suppose $\tilde{H}$ is the induced Hermitian metric on $S^{k_0}(\Lambda^pE^*) = (S^{k_0}(\Lambda^pE))^*$ by $\tilde{H}$, then
\[ \sqrt{-1}F_{\tilde{H}} \leq -a \text{Id} \otimes \omega \]
in the sense of Griffiths.

Let $h$ be a smooth Hermitian metric on $\text{det} Q$. We define $\varphi : M \to [-\infty, \infty)$ as
\[ e^{k_0 \varphi(z)} = \frac{|j_{k_0}(\xi)|^2_{\tilde{H}}}{|\xi|^2_{h^{-k_0}}}, \]
where $z \in M$ and $\xi$ is an arbitrary non-zero element of $(\text{det}(Q^*))^{k_0}_{|z}$. Replacing $\xi$ by nowhere vanishing local holomorphic sections of $(\text{det}(Q^*))^{k_0}$, we can get local expression of $\varphi$. Based on the local expression of $\varphi$, one can easily check that $\varphi \in L^1(M)$ and consequently $\sqrt{-1}F_h + \sqrt{-1}\partial\bar{\partial}\varphi$ is a well-defined $(1, 1)$-current. Furthermore, consider
\[ Z = \{z \in M \mid j_{|\text{det}(Q^*)|z} : \text{det}(Q^*)|z \to \Lambda^pE^*|z \text{ is not injective}\}, \]
then over $M \setminus Z$, $j_{k_0} : (\text{det}(Q^*))^{k_0} \to (S^{k_0}(\Lambda^pE))^*$ is a sub-bundle and $h^{-k_0}e^{k_0\varphi}$ is actually the induced Hermitian metric by $\tilde{H}$. If $u$ is a nowhere vanishing holomorphic section of $(\text{det}(Q^*))^{k_0}$ on some open subset of $M \setminus Z$, set $s = j_{k_0}(u)$ and $\tilde{s} = |s|^{-1}_{\tilde{H}}s$. Then by virtue of the Gauss-Codazzi equation for sub-bundles, we have
\[ -\sqrt{-1}k_0(F_h + \partial\bar{\partial}\varphi) = \sqrt{-1}F_{h^{-k_0}e^{k_0\varphi}} = \sqrt{-1}\langle F_{\tilde{H}}\tilde{s}, \tilde{s} \rangle_{\tilde{H}} - \sqrt{-1}\langle \tilde{\beta}\tilde{s}, \tilde{\beta}\tilde{s} \rangle_{\tilde{H}}, \]
where $\beta$ is the $(1, 0)$ component of the second fundamental form. One can directly verify that
\[ \sqrt{-1}(F_h + \partial\bar{\partial}\varphi) \geq \frac{a}{k_0} \omega \]
in the sense of current on the whole of $M$. The fact that $\sqrt{-1}(F_h + \partial\bar{\partial}\varphi) \in 2\pi c_1(\text{det} Q)$ finishes this proof. \hfill \square

As a simple corollary of Proposition 5.1, we infer

**Corollary 5.2.** If $E$ is ample, then we can find a Kähler metric $\omega_0$ on $M$, such that for any Gauduchon metric $\omega$ and any quotient sheaf $Q$ of $E$ of rank $p$ $(1 \leq p \leq r - 1)$, we have
\[ \mu_\omega(Q) \geq p \int_M \omega_0 \wedge \frac{\omega^{n-1}}{(n-1)!}. \]

**Proof of Theorem 1.8.** Let $(E, \bar{\partial}_E)$ be an ample holomorphic vector bundle with rank $r$ over a compact Hermitian manifold $(M, \omega)$, and $\hat{\omega}$ be a Gauduchon metric conformal to $\omega$. Corollary 5.2 implies
\[ \mu_L(E, \hat{\omega}) = \mu_r(\hat{\omega}) > 0. \]
By Corollary 1.5, it follows that there must exist a Hermitian metric $H$ on $E$ such that $\sqrt{-1}\Lambda_\omega F_H > 0$, and then
\[ \sqrt{-1}\Lambda_\omega F_H > 0. \]
From now on, we suppose that \( n = \dim \mathcal{M} = 2 \) and \( r = \rank(E) = 2 \). The ampleness of \((E, \bar{\partial}_E)\) tells us that \( c_1(E) > 0 \) and there exists a Hermitian metric \( H_0 \) such that 
\[
\sqrt{-1} \tr F_{H_0} > 0.
\]
Set the Kähler metric \( \omega = \sqrt{-1} \tr F_{H_0} \). Let \( H(t) \) be the solution of the Hermitian-Yang-Mills flow \((1.25)\) with initial data \( H_0 \). Noting that \( \tr(\sqrt{-1} \Lambda_\omega F_{H_0} - \Id_E) = 0 \), together with Proposition 2.4, one immediately has
\[
\sqrt{-1} \tr F_{H(t)} = \sqrt{-1} \tr F_{H_0} = \omega,
\]
for any \( 0 \leq t < \infty \). Applying Corollary 5.2 again yields
\[
\mu_L(E, \omega) > 0.
\]

To compute the second Chern form, we recall the following well-known formula
\[
4\pi^2(2c_2(E, H(t)) - \frac{1}{2} c_1^2(E, H(t))) = (|\sqrt{-1} F_{H(t)}^\perp|_{H(t), \omega}^2 - |\sqrt{-1} \Lambda_\omega F_{H(t)}^\perp|_{H(t)}^2) \frac{\omega^2}{2} ,
\]
where \( \sqrt{-1} F_{H(t)}^\perp = \sqrt{-1} F_{H(t)} - \frac{1}{2} \Id_E \otimes \omega \) and \( \sqrt{-1} \Lambda_\omega F_{H(t)}^\perp = \sqrt{-1} \Lambda_\omega F_{H(t)} - \Id_E \). Notice that \( 4\pi^2 c_1(E, H(t))^2 = \omega^2 \) and
\[
|\sqrt{-1} F_{H(t)}^\perp|_{H(t), \omega}^2 = |\sqrt{-1} F_{H(t)}^\perp|_{H(t)}^2 - (\sqrt{-1} \Lambda_\omega F_{H(t)}^\perp) \otimes \frac{\omega}{2} |_{H(t)}^2 + \frac{1}{2} |\sqrt{-1} \Lambda_\omega F_{H(t)}^\perp|_{H(t)}^2 .
\]
Hence
\[
8\pi^2 c_2(E, H(t)) \geq (2 - |\sqrt{-1} \Lambda_\omega F_{H(t)}^\perp|_{H(t)}^2) \frac{\omega^2}{4} .
\]
As above, we denote the 2 eigenvalues of the mean curvature \( \sqrt{-1} \Lambda_\omega F_{H(t)} \) by \( \lambda_1(H(t)) \), \( \lambda_2(H(t)) \), sorted in the descending order. It holds that
\[
\lambda_1(H(t)) + \lambda_2(H(t)) = 2
\]
and consequently
\[
2 - |\sqrt{-1} \Lambda_\omega F_{H(t)}^\perp|_{H(t)}^2 = 2 - \sum_{i=1}^2 (\lambda_i(H(t)) - 1)^2
\]
\[
= 4 - (\lambda_1(H(t)))^2 - (\lambda_2(H(t)))^2
\]
\[
= (\lambda_1(H(t)) + \lambda_2(H(t)))^2 - (\lambda_1(H(t)))^2 - (\lambda_2(H(t)))^2
\]
\[
= 2\lambda_1(H(t)) \lambda_2(H(t)).
\]

Owing to Theorem 1.4, we know that when \( t \) is large enough, \( \lambda_1(H(t)) \geq \lambda_2(H(t)) > 0 \), then
\[
c_2(E, H(t)) \geq \frac{\lambda_1(H(t)) \lambda_2(H(t))}{16\pi^2} \omega^2 > 0.
\]
This completes the proof of Theorem 1.8. \( \square \)
5.2. A integral inequality for holomorphic maps. Let \((M, \omega)\) be a Hermitian manifold of complex dimension \(m\). In the local complex coordinate \(\{z^\alpha\}_{\alpha=1}^m\), the Kähler form \(\omega\) and the curvature tensor \(F_\omega\) of the Chern connection \(D_\omega\) can be expressed as

\[
\omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta,
\]

\[
F_\omega \left( \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right) \frac{\partial}{\partial z^\gamma} = (F_\omega)^\gamma_{\alpha\bar{\beta}} \frac{\partial}{\partial z^\gamma},
\]

and

\[
(F_\omega)^\gamma_{\alpha\bar{\beta}} = -g^{\xi\bar{\eta}} \frac{\partial g_{\xi\eta}}{\partial z^\alpha} \bar{z}^\beta + g^{\eta\bar{\xi}} \frac{\partial g_{\eta\xi}}{\partial \bar{z}^\beta},
\]

where \((g^{\alpha\bar{\beta}})\) is the transpose of the inverse matrix of \((g_{\alpha\bar{\beta}})\). For any \(X, Y \in T_x^{1,0}(M) \setminus \{0\}\), \(x \in M\), the holomorphic bisectional curvature is defined by

\[
HB^\omega_x(X, Y) = \frac{\langle F_\omega(X, X)Y, Y \rangle_\omega}{|X|^2 |Y|^2},
\]

where \(\langle \cdot, \cdot \rangle_\omega\) is the Hermitian inner product induced by \(\omega\). The supremum of holomorphic bisectional curvature at \(x \in M\) is given by

\[
HB^\omega_x := \sup \{ HB^\omega_x(X, Y) \mid X, Y \in T_x^{1,0}(M) \setminus \{0\} \}.
\]

**Proposition 5.3.** Let \(f\) be a holomorphic map from a Gauduchon manifold \((M, \omega)\) to a Hermitian manifold \((N, \nu)\). Then for any Hermitian metric \(H\) on \(T^{1,0}M\), there holds

\[
\sqrt{-1} \Lambda_\omega \partial \bar{\partial} |f|^2_{H,\nu} \geq g^{\alpha\bar{\beta}} \left( \nabla_\omega \frac{\partial}{\partial z^\alpha} \right) (f) \left( \nabla_\omega \frac{\partial}{\partial \bar{z}^\beta} \right)_{H,\nu} + \lambda_L(H, \omega) |f|_{H,\nu}^2 - HB^\nu_f(\cdot) |f|_{H,\nu}^2 + |f|_{w,\nu}^2\nu_{ij},
\]

where \(\nabla\) is the connection on \(\Lambda^{1,0}M \otimes f^*(T^{1,0}N)\) induced by the Chern connection \(D_H\) on \(T^{1,0}M\) and the Chern connection \(D_\nu\) on \(T^{1,0}N\), \(HB_f(\cdot)\) is the supremum of holomorphic bisectional curvature at \(f(\cdot) \in (N, \nu)\).

**Proof.** Of course \(\partial f\) can be seen as a section of \(\Lambda^{1,0}M \otimes f^*(T^{1,0}N)\), where \(f^*(T^{1,0}N)\) is the pull-back bundle. Denote by \(|\partial f|_{H,\nu}\) the norm of \(\partial f\) with respect to Hermitian metrics \(H\) on \(T^{1,0}M\) and \(\nu\) on \(T^{1,0}N\). In the local complex coordinates \(\{z^\alpha\}_{\alpha=1}^m\) on \(M\) and \(\{w^i\}_{i=1}^n\) on \(N\), we set \(H(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}) = \delta_{\alpha\beta}\) and \(\nu(\frac{\partial}{\partial w^i}, \frac{\partial}{\partial \bar{w}^j}) = \nu_{ij}\). Let \(H^*\) be the Hermitian metric on \(\Lambda^{1,0}M\) induced by \(H\) and \(H^*(dz^\alpha, dz^\beta) = \delta_{\alpha\beta}\). Then one has the following local expressions

\[
|\partial f|_{H,\nu}^2 = \frac{\partial f_i}{\partial z^\alpha} dz^\alpha \otimes \frac{\partial}{\partial w^i} \quad \text{and} \quad |\partial f|^2_{w,\nu} = \frac{\partial f_i}{\partial z^\alpha} (\frac{\partial f_i}{\partial \bar{z}^\beta}) g_{\alpha\bar{\beta}} \nu_{ij}.
\]

Let \(g\) be the Hermitian metric whose associated \((1, 1)\)-form is \(\omega\). Write \(g(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}) = g_{\alpha\bar{\beta}}\). So

\[
\sqrt{-1} \Lambda_\omega \partial \bar{\partial} |f|^2_{H,\nu} = \frac{\sqrt{-1} \Lambda_\omega \partial \bar{\partial} |f|^2_{H,\nu}}{\omega^m/(m-1)!} = g^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} |f|^2_{H,\nu}.
\]
Moreover, the condition that \( f \) is holomorphic gives us
\[
\nabla \frac{\partial f}{\partial \bar{z}^\beta} = 0.
\]

Clearly there is
\[
\partial^2 f^2_{H,\nu} = \langle \nabla \frac{\partial f}{\partial \bar{z}^\beta} \nabla \frac{\partial f}{\partial \bar{z}^\beta} \rangle_{H,\nu} + \langle \nabla \frac{\partial \nabla f}{\partial \bar{z}^\beta} \nabla \frac{\partial f}{\partial \bar{z}^\beta} \rangle_{H,\nu}.
\]

A direct calculation yields that
\[
\nabla \frac{\partial f}{\partial \bar{z}^\beta} \nabla \frac{\partial f}{\partial \bar{z}^\beta} = \left( \nabla \frac{\partial f}{\partial \bar{z}^\beta} \nabla \frac{\partial f}{\partial \bar{z}^\beta} - \nabla \frac{\partial \nabla f}{\partial \bar{z}^\beta} \right) \partial f
\]
\[
= \partial^2 f^i \left( \nabla \frac{\partial f}{\partial \bar{z}^\beta} \nabla \frac{\partial f}{\partial \bar{z}^\beta} - \nabla \frac{\partial \nabla f}{\partial \bar{z}^\beta} \right) (dz^\gamma \otimes \partial \frac{\partial f}{\partial \bar{z}^\beta})
\]
\[
= \partial^2 f^i \left( \left( F_{H^*} \left( \frac{\partial}{\partial \bar{z}^\beta} \right) \right) dz^\gamma \otimes \partial \frac{\partial f}{\partial \bar{z}^\beta} + dz^\gamma \otimes \left( F_{\nu} \left( f_s \left( \frac{\partial}{\partial \bar{z}^\beta} \right) \right) \right) \partial \frac{\partial f}{\partial \bar{z}^\beta} \right),
\]

and then
\[
g_{\alpha \bar{\beta}} \langle \nabla \frac{\partial f}{\partial \bar{z}^\beta} \nabla \frac{\partial f}{\partial \bar{z}^\beta} \rangle_{H,\nu}
\]
\[
= \langle - \frac{\partial f^i}{\partial \bar{z}^\gamma} \left( \sqrt{-1} \Lambda_{\omega} F_{H^*} \left( dz^\gamma \right) \right) \otimes \partial \frac{\partial f}{\partial \bar{z}^\beta} \rangle_{H,\nu}
\]
\[
+ \langle g_{\alpha \bar{\beta}} \frac{\partial f^i}{\partial \bar{z}^\gamma} (dz^\gamma \otimes \left( F_{\nu} \left( f_s \left( \frac{\partial}{\partial \bar{z}^\beta} \right) \right) \right)) \partial \frac{\partial f}{\partial \bar{z}^\beta} \rangle_{H,\nu}
\]
\[
= - \langle \sqrt{-1} \Lambda_{\omega} F_{H^*} \left( dz^\gamma, dz^\xi \right) \frac{\partial f^i}{\partial \bar{z}^\gamma} \left( \frac{\partial f^i}{\partial \bar{z}^\xi} \right) \nu \rangle_{H^*}
\]
\[
+ \langle \frac{\partial f^i}{\partial \bar{z}^\gamma} \left( \frac{\partial f^i}{\partial \bar{z}^\xi} \right) \tilde{g}^{\gamma \xi} g_{\alpha \bar{\beta}} \langle F_{\nu} \left( f_s \left( \frac{\partial}{\partial \bar{z}^\beta} \right) \right), f_s \left( \frac{\partial}{\partial \bar{z}^\beta} \right) \rangle \frac{\partial \partial f}{\partial \bar{z}^\beta} \rangle_{H,\nu},
\]

where \( F_{H^*} \) and \( F_{\nu} \) are the curvatures of \( D_{H^*} \) and \( D_{\nu} \), respectively.

On the considered point \( x \in M \), one can choose the local complex coordinate \( \{ z^1, \cdots, z^m \} \) centered at \( x \) such that
\[
g_{\alpha \bar{\beta}}(x) = \delta_{\alpha \beta} \quad \text{and} \quad \tilde{g}_{\gamma \xi}(x) = a_{\gamma} \delta_{\xi \xi},
\]

where for every \( 1 \leq \gamma \leq m \), \( a_{\gamma} \) is a positive number. Notice that
\[
f_s \left( \frac{\partial}{\partial z^\alpha} \right) = \frac{\partial f^i}{\partial \bar{z}^\gamma} \frac{\partial}{\partial \bar{z}^\beta},
\]

At \( x \), we can write
\[
|\partial f|_{\omega,\nu}^2 = g_{\alpha \bar{\beta}} \langle \frac{\partial f^i}{\partial \bar{z}^\gamma} \frac{\partial}{\partial \bar{z}^\beta}, \frac{\partial f^j}{\partial \bar{z}^\gamma} \frac{\partial}{\partial \bar{z}^\beta} \rangle_{\nu} = \sum_{\alpha=1}^m |f_s \left( \frac{\partial}{\partial z^\alpha} \right)|^2_{\nu}.
Hermitian metrics with negative mean curvature

and

\[ |\partial f|^2_{H,\nu} = g^{\gamma\xi} \left( \frac{\partial f^i}{\partial z^\gamma} \frac{\partial}{\partial w^i}, \frac{\partial f^j}{\partial z^\xi} \frac{\partial}{\partial w^j} \right)_\nu \]

\[ = \sum_{\gamma=1}^m \frac{1}{a_\gamma} \left( \frac{\partial f^i}{\partial z^\gamma} \frac{\partial}{\partial w^i}, \frac{\partial f^j}{\partial z^\xi} \frac{\partial}{\partial w^j} \right)_\nu \]

\[ = \sum_{\gamma=1}^m \frac{1}{a_\gamma} |f^*(\frac{\partial}{\partial z^\gamma})|^2_\nu. \]

Furthermore, we compute

\[ \frac{\partial f^i}{\partial z^\gamma} \left( \frac{\partial f^j}{\partial z^\xi} \right) g^{\gamma\xi} g^{\alpha\beta} \left( F_\nu(f^*, \frac{\partial}{\partial z^\alpha}), f^*(\frac{\partial}{\partial z^\beta}) \right) \left( \frac{\partial}{\partial w^i}, \frac{\partial}{\partial w^j} \right)_\nu(x) \]

\[ = \sum_{\gamma=1}^m a^{-1}_\gamma \frac{\partial f^i}{\partial z^\gamma} \left( \sum_{\alpha=1}^m F_\nu(f^*, \frac{\partial}{\partial z^\alpha}), f^*(\frac{\partial}{\partial z^\beta}) \right) \left( \frac{\partial}{\partial w^i}, \frac{\partial}{\partial w^j} \right)_\nu(x) \]

\[ = \sum_{\gamma=1}^m \sum_{\alpha=1}^m \left( F_\nu(f^*, \frac{\partial}{\partial z^\alpha}), f^*(\frac{\partial}{\partial z^\beta}) \right) \left( \sqrt{a^{-1}_\gamma f^*(\frac{\partial}{\partial z^\alpha})}, \sqrt{a^{-1}_\gamma f^*(\frac{\partial}{\partial z^\beta})} \right)_\nu(x) \]

\[ \geq - HB^\nu_{f^*(x)} \sum_{\gamma=1}^m \sum_{\alpha=1}^m |f^*(\frac{\partial}{\partial z^\alpha})|^2_\nu \cdot |\sqrt{a^{-1}_\gamma f^*(\frac{\partial}{\partial z^\beta})}|^2_\nu(x) \]

\[ = - HB^\nu_{f^*(x)} |f^2|^2_{H,\nu}. \]

On the other hand, the assumption \( \sqrt{-1} \Lambda_\omega F_H \geq \lambda_L(H,\omega) \Id \) implies \( -\sqrt{-1} \Lambda_\omega F_H^* \geq \lambda_L(H,\omega) \Id \). Thus

\[ - \left( \sqrt{-1} \Lambda_\omega F_H^*(dz^\gamma), dz^\xi \right)_H^* \frac{\partial f^i}{\partial z^\gamma} \frac{\partial f^j}{\partial z^\xi} \nu_{ij} \geq \lambda_L(H,\omega) |\partial f|^2_{H,\nu}. \]

This together with (5.26), (5.28), (5.30) and (5.35) gives (5.24).  

\[ \square \]

**Theorem 5.4.** Let \( f \) be a holomorphic map from a compact Gauduchon manifold \((M,\omega)\) to a Hermitian manifold \((N,\nu)\). If \( f \) is not constant, then for any Hermitian metric \( H \) on \( T^{1,0} M \), there holds

\[ \int_M \lambda_L(H,\omega) \omega^m \frac{m!}{m!} \leq \int_M HB^\nu_{f^*(\nu)} \cdot f^*(\nu) \wedge \frac{\omega^{m-1}}{(m-1)!}, \]

where \( m = \dim^C M \).
Proof. By (5.24), we have

\[
\begin{align*}
\sqrt{-1} \Lambda_\omega \partial \bar{\partial} \log (|\partial f|_{H,\nu}^2 + \varepsilon) &= \sqrt{-1} \Lambda_\omega \partial \left( \frac{\bar{\partial} |\partial f|_{H,\nu}^2}{|\partial f|_{H,\nu}^2 + \varepsilon} \right) \\
&= \frac{\sqrt{-1} \Lambda_\omega \partial \bar{\partial} |\partial f|_{H,\nu}^2}{|\partial f|_{H,\nu}^2 + \varepsilon} + \frac{\sqrt{-1} \Lambda_\omega \bar{\partial} |\partial f|_{H,\nu}^2 \wedge \partial |\partial f|_{H,\nu}^2}{(|\partial f|_{H,\nu}^2 + \varepsilon)^2} \\
&= \frac{g^{\alpha\beta} \langle \nabla \frac{\partial}{\partial z^\alpha}, \partial f, \nabla \frac{\partial}{\partial z^\beta} \partial f \rangle_{H,\nu}}{|\partial f|_{H,\nu}^2 + \varepsilon} - \frac{g^{\alpha\beta} \frac{\partial}{\partial z^\alpha} |\partial f|_{H,\nu}^2 \cdot \frac{\partial}{\partial z^\beta} |\partial f|_{H,\nu}^2}{(|\partial f|_{H,\nu}^2 + \varepsilon)^2} \\
&\quad + \lambda_L(H,\omega) \frac{|\partial f|_{H,\nu}^2}{|\partial f|_{H,\nu}^2 + \varepsilon} - HB_{f(\cdot)}^\nu \frac{|\partial f|_{H,\nu}^2 |\partial f|_{H,\omega}^2}{|\partial f|_{H,\nu}^2 + \varepsilon}.
\end{align*}
\]

(5.38)

Choose the local complex coordinate \( \{z^1, \ldots, z^m\} \) such that \( g^{\alpha\beta} = \delta_{\alpha\beta} \) at the considered point. Then

\[
\begin{align*}
\frac{g^{\alpha\beta} \langle \nabla \frac{\partial}{\partial z^\alpha}, \partial f, \nabla \frac{\partial}{\partial z^\beta} \partial f \rangle_{H,\nu}}{|\partial f|_{H,\nu}^2 + \varepsilon} &= \frac{g^{\alpha\beta} \frac{\partial}{\partial z^\alpha} |\partial f|_{H,\nu}^2 \cdot \frac{\partial}{\partial z^\beta} |\partial f|_{H,\nu}^2}{(|\partial f|_{H,\nu}^2 + \varepsilon)^2} \\
&= \frac{1}{|\partial f|_{H,\nu}^2 + \varepsilon} \left( \sum_{\alpha=1}^m \left| \nabla \frac{\partial}{\partial z^\alpha} \partial f \right|^2_{H,\nu} - \sum_{\alpha=1}^m \left( \frac{\partial}{\partial z^\alpha} |\partial f|_{H,\nu}^2 \right)^2 \right) \\
&\geq \frac{\varepsilon}{(|\partial f|_{H,\nu}^2 + \varepsilon)^2} \left( \sum_{\alpha=1}^m \left| \nabla \frac{\partial}{\partial z^\alpha} \partial f \right|^2_{H,\nu} \right) \\
&= \frac{\varepsilon}{(|\partial f|_{H,\nu}^2 + \varepsilon)^2} g^{\alpha\beta} \langle \nabla \frac{\partial}{\partial z^\alpha} \partial f, \nabla \frac{\partial}{\partial z^\beta} \partial f \rangle_{H,\nu},
\end{align*}
\]

where the inequality is due to

\[
(\frac{\partial}{\partial z^\alpha} |\partial f|_{H,\nu}^2)^2 = (\langle \nabla \frac{\partial}{\partial z^\alpha} \partial f, \partial f \rangle_{H,\nu})^2 \leq |\nabla \frac{\partial}{\partial z^\alpha} \partial f|_{H,\nu}^2 \cdot |\partial f|_{H,\nu}^2.
\]

Hence

\[
\begin{align*}
\sqrt{-1} \Lambda_\omega \partial \bar{\partial} \log (|\partial f|_{H,\nu}^2 + \varepsilon) &\geq \frac{\varepsilon}{(|\partial f|_{H,\nu}^2 + \varepsilon)^2} g^{\alpha\beta} \langle \nabla \frac{\partial}{\partial z^\alpha} \partial f, \nabla \frac{\partial}{\partial z^\beta} \partial f \rangle_{H,\nu} \\
&\quad + \lambda_L(H,\omega) \frac{|\partial f|_{H,\nu}^2}{|\partial f|_{H,\nu}^2 + \varepsilon} - HB_{f(\cdot)}^\nu \frac{|\partial f|_{H,\nu}^2 |\partial f|_{H,\omega}^2}{|\partial f|_{H,\nu}^2 + \varepsilon} \\
&\geq \frac{|\partial f|_{H,\nu}^2}{|\partial f|_{H,\nu}^2 + \varepsilon} (\lambda_L(H,\omega) - HB_{f(\cdot)}^\nu |\partial f|_{H,\omega}^2).
\end{align*}
\]

(5.41)

Integrating (5.41) with respect to \( \frac{\omega^m}{m!} \) over \( M \), and noting that \( \omega \) is Gauduchon, one has

\[
\int_M \frac{|\partial f|_{H,\nu}^2}{|\partial f|_{H,\nu}^2 + \varepsilon} (\lambda_L(H,\omega) - HB_{f(\cdot)}^\nu |\partial f|_{H,\omega}^2) \frac{\omega^m}{m!} \leq 0.
\]

(5.42)
Hermitian metrics with negative mean curvature

If $f$ is not constant, $\tilde{\Sigma} := \{x \in M \mid \partial f(x) = 0\}$ is a proper subvariety of $M$. Applying Lebesgue’s dominated convergence theorem, we deduce

$$\int_M \left( \lambda_L(H, \omega) - HB_{f(\cdot)}^\nu |\partial f|_{\omega, \nu}^2 \right) \omega^m \frac{1}{m!}$$

$$= \lim_{\varepsilon \to 0} \int_M \frac{|\partial f|_{H, \nu}^2}{|\partial f|^2_{H, \nu} + \varepsilon} \left( \lambda_L(H, \omega) - HB_{f(\cdot)}^\nu |\partial f|_{\omega, \nu}^2 \right) \omega^m \frac{1}{m!}$$

$$\leq 0.$$  \hfill (5.43)

If $H$ is the Hermitian metric on $T^{1,0}(M)$ induced by $\omega$, the mean curvature $\sqrt{-1} \Lambda_{\omega} F_H$ is just the second Chern-Ricci curvature of $\omega$. In this special case, the inequality (5.37) was proved recently by Zhang (66).

**Proof of Theorem 1.10** Let $H(t)$ be a smooth solution of the Hermitian-Yang-Mills flow (1.25) on $T^{1,0}M$ over the Gauduchon manifold $(M, \omega)$. Combining (5.37) and (1.30), we derive

$$2\pi \mu_L(T^{1,0}M, \omega) = \lim_{t \to \infty} \int_M \lambda_L(H(t), \omega) \omega^m \frac{1}{m!} \leq \int_M HB_{f(\cdot)}^\nu \cdot f^* (\nu) \wedge \omega^{m-1} \frac{1}{(m-1)!}.$$

This completes the proof of Theorem 1.10. \hfill \Box

**References**

[1] L.V. Ahlfors, *An extension of Schwarz’s lemma*, Trans. Amer. Math. Soc. 43(1938), 359-364.

[2] L. Alvarez-Consul and O. Garcia-Prada, *Hitchin-Kobayashi correspondence, quivers, and vortices*, Commun. Math. Phys. 238(2003), 1-33.

[3] S. Bando and Y.T. Siu, *Stable sheaves and Einstein-Hermitian metrics*, in Geometry and Analysis on Complex Manifolds, World Scientific, 1994, 39-50.

[4] O. Biquard, *On parabolic bundles over a complex surface*, J. London Math. Soc. 53(1996), 302-316.

[5] I. Biswas, *Stable Higgs bundles on compact Gauduchon manifolds*, C. R. Math. Acad. Sci. Paris 349(2011), 71-74.

[6] I. Biswas and G. Schumacher, *Yang-Mills equation for stable Higgs sheaves*, Inter. J. Math. 20(2009), 541-556.

[7] S.B. Bradlow, *Vortices in holomorphic line bundles over closed Kähler manifolds*, Commun. Math. Phys. 135(1990), 1-17.

[8] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[9] L. Bruasse, *Harder-Narasimhan filtration on non Kähler manifolds*, Internat. J. Math. 12(2001), no. 5, 579-594.

[10] N.P. Buchdahl, *Hermitian-Einstein connections and stable vector bundles over compact complex surfaces*, Math. Ann. 280(1988), 625-648.

[11] E. Calabi, *An extension of E. Hopf’s maximum principle with an application to Riemannian geometry*, Duke Math. J. 25(1958), 45-56.

[12] S.S. Chern, *On holomorphic mappings of Hermitian manifolds of the same dimension*, Proc. Symp. Pure Math. 11, Amer. Math. Soc. (1968), 157-170.

[13] M. Crandall, H. Ishii and P.-L. Lions, *User’s guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) 27(1992), no. 1, 1-67.
[14] G. Daskalopoulos and R. Wentworth, *Convergence properties of the Yang-Mills flow on Kähler surfaces*, J. Reine Angew. Math. 575 (2004), 69-99.

[15] J.-P. Demailly, *Complex Analytic and Differential Geometry*, http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf.

[16] J.-P. Demailly, T. Peternell and M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom. 3 (1994), no. 2, 295-345.

[17] S. Diverio, *Segre forms and Kobayashi-Lübke inequality*, Math. Z. 283 (2016), no. 3-4, 1033-1047.

[18] S. Diverio and F. Fagioli, *Pointwise Universal Gysin formulae and Applications towards Griffiths’ conjecture*, arXiv: 2009.14587v2.

[19] J. Dodziuk, *Maximum principle for parabolic inequalities and the heat flow on open manifolds*, Indiana Univ. Math. J. 32 (1983), no. 5, 703-716.

[20] S.K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Adv. Math. 50 (1985), 1-26.

[21] S. Finski, *On characteristic forms of positive vector bundles, mixed discriminants and pushforward identities*, arXiv: 2009.13107v2.

[22] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition. Classics in Mathematics, Springer-Verlag, Berlin, 2001. xiv+517 pp. ISBN: 3-540-41160-7.

[23] D. Guler, *On Segre forms of positive vector bundles*, Canad. Math. Bull. 55 (2012), no. 1, 108-113.

[24] R. Hartshorne, *Ample vector bundles*, Inst. Hautes Études Sci. Publ. Math. 29 (1966), 63-94.

[25] H. Hironaka, *Flattening theorem in complex-analytic geometry*, Amer. J. Math., 97 (1975), no. 2, 503-547.

[26] J. Jost and K. Zuo, *Harmonic maps and \(SL(r, \mathbb{C})\)-representations of fundamental groups of quasiprojective manifolds*, J. Algebraic Geom. 5 (1996), 77-106.

[27] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, 15, Princeton University Press, Princeton, NJ, 1987.

[28] J.Y. Li and H.-H. Wu, *On holomorphic sections of certain hermitian vector bundles*, Math. Ann. 189 (1970), 1-4.

[29] J.Y. Li and M.S. Narasimhan, *Hermitian-Einstein metrics on parabolic stable bundles*, Acta Math. 181 (1998), 93-114.

[30] J.Y. Li and X. Zhang, *Existence of approximate Hermitian-Einstein structures on semi-stable bundles*, Calc. Var. Partial Differential Equations 52 (2015), 783-795.

[31] J.Y. Li, C.J. Zhang and X. Zhang, *Semi-stable Higgs sheaves and Bogomolov type inequality*, Calc. Var. Partial Differential Equations 56 (2017), no. 3, Paper No. 81, 33 pp.

[32] J.Y. Li, C.J. Zhang and X. Zhang, *The limit of the Hermitian-Yang-Mills flow on reflexive sheaves*, Adv. Math. 325 (2018), 165-214.

[33] P. Li, *Nonnegative Hermitian vector bundles and Chern numbers*, Math. Ann. 380 (2021), no. 1-2, 21-41.

[34] J. Li and S.T. Yau, *Hermitian-Yang-Mills connection on non-Kähler manifolds*, Mathematical aspects of string theory (San Diego, Calif., 1986), 560-573, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.
Hermitian metrics with negative mean curvature

[40] P.-L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness*, Comm. Partial Differential Equations 8(1983), no. 11, 1229-1276.

[41] Y.C. Lu, *Holomorphic mappings of complex manifolds*, J. Differential Geometry 2(1968), 299-312.

[42] M. Lübke and A. Teleman, *The universal Kobayashi-Hitchin correspondence on Hermitian manifolds*, Mem. Amer. Math. Soc. 183(2006), no. 863, vi+97 pp.

[43] M. Lübke and A. Teleman, *The Kobayashi-Hitchin correspondence*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[44] T. Mochizuki, *Kobayashi-Hitchin correspondence for tame harmonic bundles and an application*, Astérisque, 309(2006), viii+117 pp. ISBN: 978-2-85629-226-6.

[45] T. Mochizuki, *Kobayashi-Hitchin correspondence for tame harmonic bundles II*, Geom. Topol. 13(2009), 359-455.

[46] T. Mochizuki, *Kobayashi-Hitchin correspondence for analytically stable bundles*, Trans. Amer. Math. Soc. 373(2020), no. 1, 551-596.

[47] C. Mourougane, *Computations of Bott-Chern classes on \( \mathbb{P}(E) \)*, Duke Math. J. 124(2004), no. 2, 389-420.

[48] L. Ni, *Liouville theorems and a Schwarz lemma for holomorphic mappings between Kähler manifolds*, Comm. Pure Appl. Math., 74(2021), 1100-1126.

[49] L. Ni and H. Ren, *Hermitian-Einstein metrics for vector bundles on complete Kähler manifolds*, Trans. Amer. Math. Soc. 353(2001), 441-456.

[50] M.S. Narasimhan and C.S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. Math. 82(1965), 540-567.

[51] Y.C. Nie and X. Zhang, *Semistable Higgs bundles over compact Gauduchon manifolds*, J. Geom. Anal. 28(2018), no. 2, 627-642.

[52] V.P. Pingali, *Representability of Chern-Weil forms*, Math. Z. 288(2018), no. 1-2, 629-641.

[53] H.L. Royden, *The Ahlfors-Schwarz lemma in several complex variables*, Comment. Math. Helv. 55(1980), no. 4, 547-558.

[54] B.Sibley, *Asymptotics of the Yang-Mills flow for holomorphic vector bundles over Kähler manifolds: the canonical structure of the limit*, J. Reine Angew. Math., 706(2015), 123-191.

[55] C.T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. 1(1988), 867-918.

[56] C.T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. 75(1992), 5-95.

[57] K.K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. 39(1986), S257-S293.

[58] Y. Wang and X. Zhang, *Twisted holomorphic chains and vortex equations over non-compact Kähler manifolds*, J. Math. Anal. Appl. 373(2011), 179-202.

[59] J. Xiao, *On the positivity of high-degree schur classes of an ample vector bundle*, arXiv: 2007.12425v1.

[60] X.K. Yang, *RC-positivity, rational connectedness and Yau’s conjecture*, Camb. J. Math. 6 (2018), no.2, 183-212.

[61] X.K. Yang, *RC-positivity, vanishing theorems and rigidity of holomorphic maps*, J. Inst. Math. Jussieu 20(2021), no. 3, 1023-1038.

[62] S.-T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. 100(1978), no. 1, 197-203.

[63] C.J. Zhang, P. Zhang and X. Zhang, *Higgs bundles over non-compact Gauduchon manifolds*, Trans. Amer. Math. Soc., 374(2021), no.5, 3735-3759.

[64] X. Zhang, *Hermitian-Einstein metrics on holomorphic vector bundles over Hermitian manifolds*, J. Geom. Phys. 53(2005), 315-335.

[65] Y.S. Zhang, *Integral inequalities for holomorphic maps and applications*, Trans. Amer. Math. Soc. 374(2021), no. 4, 2341-2358.
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