Some Generalizations in Supersymmetric Quantum Mechanics and the Supersymmetric $\varepsilon$-System Revisited

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Abstract

We discuss two distinct aspects in supersymmetric quantum mechanics. First, we introduce a new class of operators $A$ and $\bar{A}$ in terms of anticommutators between the momentum operator and $N + 1$ arbitrary superpotentials. We show that these operators reduce to the conventional ones which are the starting point in standard supersymmetric quantum mechanics. In this context, we argue furthermore that supersymmetry does not only connect Schrödinger-like operators, but also a more general class of differential operators. Second, we revisit the supersymmetric $\varepsilon$-system recently introduced in the literature by exploiting its intrinsic supersymmetry. Specifically, combining the Hamilton hierarchy method and the $\delta$-expansion method, we determine an energy for the first excited state of the bosonic Hamiltonian close to that calculated in earlier works.

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I. INTRODUCTION

Since its invention by Witten in connection with supersymmetry (SUSY) breaking issues in quantum field theory, supersymmetric quantum mechanics (SQM) has become an independent and a fruitful realm of research. Not only old problems such as the hydrogen atom were rephrased in terms of SUSY, but also more recently its conformal extensions (i.e., superconformal quantum mechanics) have shown to be relevant in quantum black holes and AdS/CFT correspondence (see e.g. and references herein).

As is well known, a supersymmetric system in quantum mechanics is described by a Hamiltonian which is expressed in terms of a set of supercharges (at least two). The superalgebra of these elements (Hamiltonian and supercharges, see Eqs. (1)-(2)) is realized by introducing two operators $A$ and $\bar{A}$ which in the standard approach depend merely on the momentum $p$ operator and on a function $W(x)$, called superpotential. It is possible to show that the Hamiltonians $H_-$ and $H_+$, diagonal entries of the supersymmetric Hamiltonian, constructed from these operators are Schrödinger-like operators, being both connected by supersymmetry. A footprint of this relationship is the isospectral energy structure which both Hamiltonians possess.

We discuss some generalizations of the standard supersymmetric quantum mechanics. By introducing a new class of operators $A$ and $\bar{A}$ which reduce to the old ones, we show that supersymmetry does not only link Schrödinger-like operators, but also a more general class of differential operators. The $N = 1$ case corresponds to supersymmetric systems described by Shrödinger-like operators and deformations of them, whereas the $N > 1$ case corresponds to supersymmetric systems with higher-derivative operators. A particular study of this last case was carried out long ago in [4], concerning the Witten index (a topological quantity which indicates whether or not supersymmetry is broken in usual theories).

In this work we also treat the supersymmetric $\varepsilon$-system recently introduced and studied (using the variational method) in [5]. Here we combine the Hamiltonian hierarchy method and the $\delta$-expansion method, for solving the Riccati equations, in order to find the first excited state energy of the bosonic Hamiltonian. Our result is close to that found in [5] and more recently in [9].

The paper is organized as follows. In Sec. II we review the “standard” supersymmetric quantum mechanics (SQM) and discuss some generalizations of it. In addition, we treat
briefly the Hamiltonian hierarchy method which will be used in the next section. In Sec. III we revisit the supersymmetric \( \varepsilon \)-system following another approach. As mentioned above, here we exploit its intrinsic supersymmetry to face the same energy eigenvalue problem than in [5]. Finally, Sec. IV contains our main results.

II. SUSY QUANTUM MECHANICS AND SOME GENERALIZATIONS

In this section we review the core of standard supersymmetric quantum mechanics (SQM) and discuss some generalizations of it. To this end, we follow closely [6, 10].

The simplest SQM is described in terms of two supercharges \( Q \) and \( \bar{Q} \) (its Hermitian adjoint), which obey the following algebra [1]

\[
H = \{Q, \bar{Q}\}, \quad Q^2 = 0 = \bar{Q}^2, \tag{1}
\]

where \( H \) denotes the Hamiltonian of the supersymmetric system. It is easy to show, using the above algebra, that the supercharges \( Q, \bar{Q} \) are constants of motion, that is

\[
[Q, H] = 0, \quad [\bar{Q}, H] = 0. \tag{2}
\]

A simple realization of the algebra (1) is achieved by choosing

\[
Q = \frac{1}{2} (\sigma^1 - i\sigma^2) A \quad \text{and} \quad \bar{Q} = \frac{1}{2} (\sigma^1 + i\sigma^2) \bar{A}, \tag{3}
\]

where \( \sigma^1 \) and \( \sigma^2 \) are the usual Pauli matrices and where \( A \) is an arbitrary differential operator (\( \bar{A} \) being its Hermitian adjoint). The supersymmetric Hamiltonian \( H \) in (1) takes the form

\[
H = \frac{1}{2} (\sigma^0 + \sigma^3) \bar{A}A + \frac{1}{2} (\sigma^0 - \sigma^3) AA\bar{A} = \begin{pmatrix} \bar{A}A & 0 \\ 0 & A\bar{A} \end{pmatrix}, \tag{4}
\]

a structure (diagonal) which allows us to identify two distinct but intimately connected by supersymmetry sectors in the state space of the system described by \( H \). Adopting the notation \( H_- = \bar{A}A \) and \( H_+ = A\bar{A} \) for the diagonal elements of \( H \) and writing the state function of the system as

\[
\Psi (x) = \begin{pmatrix} \psi^- \\ \psi^+ \end{pmatrix} = \begin{pmatrix} \psi^- \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \psi^+ \end{pmatrix}, \tag{5}
\]

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it is straightforward to observe that the two functions on the right hand side of (5) are independent one to another, and so belong to different sectors of the state space. Notice also that since $H$ has a diagonal structure, this operator does not “smear” the pureness of these kinds of functions. In other words, the operators $H_{\mp}$ do act on the respective component functions $\psi^\pm$ of $\Psi$:

$$H\Psi = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \begin{pmatrix} \psi^- \\ \psi^+ \end{pmatrix} = \begin{pmatrix} H_- \psi^- \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ H_+ \psi^+ \end{pmatrix}.$$  \hspace{1cm} (6)

In what follows a pure state $\Psi^- = \frac{1}{2} (\sigma^0 + \sigma^3) \Psi$ will be called bosonic and a pure state $\Psi^+ = \frac{1}{2} (\sigma^0 - \sigma^3) \Psi$ fermionic. We stress however that “bosonic” and “fermionic” are simply labels and have nothing to do with the geometrical concept of spin which does not exist in one-dimensional space.

Before proceeding with the construction of operators $A$ and $\bar{A}$, there are two direct consequences of the graded algebra (1, 2) which are worthwhile to mention. The positivity of energy of a supersymmetric system on the one hand, and the intertwining relationship between its bosonic and fermionic sectors by means of the supercharges on the other. The positive feature of the spectrum becomes evident if we compute, with the aid of the $H$-$Q$ anticommutator in (1), the expectation value of $H$ corresponding to an arbitrary state $|\Psi\rangle$,

$$E_\Psi = \langle \Psi | H | \Psi \rangle = |Q |\Psi\rangle|^2 + |\bar{Q} |\Psi\rangle|^2 \geq 0,$$ \hspace{1cm} (7)

while the bosonic-fermionic relationship is ascertained by regarding the $H$-$Q$ commutators (2) and the own structure of the supercharges. For example, let $\Psi^+ = \frac{1}{2} (\sigma^0 - \sigma^3) \Psi$ be a given fermionic eigenfunction of $H$ with eigenvalue $E_+$, i. e. $H\Psi^+ = E_+ \Psi^+$. Applying the supercharge $\bar{Q}$ on both sides of this eigenvalue equation, one obtains by using (2)

$$\bar{Q} \left( H\Psi^+ \right) = H \left( \bar{Q} \Psi^+ \right) \Rightarrow H \left( \bar{Q} \Psi^+ \right) = E_+ \left( \bar{Q} \Psi^+ \right),$$ \hspace{1cm} (8)

which indicates that the function $\bar{Q} \Psi^+$ is also an eigenfunction of $H$ with the same eigenvalue $E_+$ as $\Psi^+$. Moreover, taking into account the peculiar form of $\bar{Q}$ given in (3), one easily observes that the eigenfunction $\bar{Q} \Psi^+$ has the structure of a bosonic function $\Psi^-$. In fact,

$$\bar{Q} \Psi^+ = \begin{pmatrix} 0 & \bar{A} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \phi^+ \end{pmatrix} = \begin{pmatrix} \bar{A} \phi^+ \\ 0 \end{pmatrix} \sim \Psi^-.$$ \hspace{1cm} (9)
Hence if $\Psi^+$ is a normalized fermionic eigenfunction of $H$ with eigenvalue $E_+$, then its normalized bosonic partner $\Psi^-$ (also eigenfunction of $H$, with the same eigenvalue as $\Psi^+$) is given by $\Psi^- = (E_+)^{-1/2} \tilde{Q}\Psi^+$. A similar analysis can be done for $\Psi^-$. 

So far we have simply provided an overview of standard supersymmetric quantum mechanics, without nothing new. In what follows let us consider a variant of this by introducing a general class of operators $A$ and $\bar{A}$. This is possible because of the weak constraints imposed on such operators, namely, that one must be the Hermitian adjoint of the other ($\bar{A} = A$) and that $H_- = \bar{A}A$ and $H_+ = A\bar{A}$ must be positive semi-definite operators.

Since in the $x$-representation $A$ and $\bar{A}$ have to depend on the basic $x$ and $p = -id/dx$ operators, we propose a finite series in $p$ with $x$-dependent coefficients for each of them,

$$A(x, p) = \frac{1}{2} \sum_{n=0}^{N} a_n (x) \ast p^n \quad \quad \bar{A}(x, p) = \frac{1}{2} \sum_{n=0}^{N} a_n (x) \ast p^n,$$

where $X \ast Y$ denotes the anticommutator of $X$ and $Y$, i. e., $X \ast Y = \{X, Y\}$. Here $a_n (x)$ are in general $(N + 1)$ complex functions of $x$, which we call from now on generalized superpotentials. In this way, one needs to fix $(N + 1)$ complex superpotentials in order to specify completely $A$ and its adjoint $\bar{A}$. Note that for consistency the $N = 0$ case must be ruled out. The $N = 1$ and $N > 1$ cases contain “dynamic” in their structures and are rich in possibilities (possible choices for the functions $a_n (x)$).

We analyze now the simplest cases, i. e., $N = 1$ and $N = 2$. We show below that $N = 1$ leads to theories governed by Schrödinger-like equations and deformations of them, while $N = 2$ leads to theories with higher derivatives.

Taking $N = 1$ in (10), one obtains

$$A = \frac{1}{2} \left( a_0 \ast p^0 + a_1 \ast p^1 \right) = a_0 + \frac{1}{2} (pa_1) + a_1 p$$

$$\bar{A} = \frac{1}{2} \left( a_0 \ast p^0 + a_1 \ast p^1 \right) = a_0^* + \frac{1}{2} (pa_1^*) + a_1^* p.$$

Here we have made use of the (anti)commutator relations,

$$[X, Y] = - [X, \bar{Y}] \quad \quad \{X, Y\} = \{\bar{X}, \bar{Y}\},$$

along with the Hermiticity of the momentum operator: $\bar{p} = p = -id/dx$. Note that the use of the anticommutator property allows us to obtain from $A$ an adjoint $\bar{A}$ completely symmetric. This is the reason for which anticommutators were introduced in the definitions (10).
Using the explicit form of $A$ and $\bar{A}$ in (11,12), one sets up easily the $H_{\pm}$ “Hamiltonians”:

\[
H_{\pm} = |a_1|^2 p^2 + \left( a_0 a_1 + a_1^* a_0 + \frac{3}{2} a_1^* (pa_1) + \frac{1}{2} a_1 (pa_1^*) \right) p
\]
\[
+ \left( |a_0|^2 + \frac{1}{2} a_0^* (pa_1) + \frac{1}{2} a_0 (pa_1^*) + \frac{1}{4} (pa_1^*) (pa_1) + a_1^* (pa_0) + \frac{1}{2} a_1 (p^2 a_1) \right),
\]  
(14)

\[
H_{\pm} = |a_1|^2 p^2 + \left( a_0 a_1^* + a_1 a_0^* + \frac{3}{2} a_1 (pa_1^*) + \frac{1}{2} a_1^* (pa_1) \right) p
\]
\[
+ \left( |a_0|^2 + \frac{1}{2} a_0^* (pa_1) + \frac{1}{2} a_0 (pa_1^*) + \frac{1}{4} (pa_1^*) (pa_1) + a_1 (pa_0^*) + \frac{1}{2} a_1^* (p^2 a_1^*) \right).
\]  
(15)

These second-order linear differential operators become Schrödinger-like operators only if one chooses adequately the functions $a_0(x)$ and $a_1(x)$. The right choice at first sight is $a_0 = a_0^*$ (real function) and $a_1 = i/\sqrt{2m}$, since in this way the coefficient of $p^2$ turns out $1/(2m)$ and the $p$-linear terms vanish. Setting $2m \doteq 1$ and $a_0(x) \doteq W(x)$, we can write $H_{\pm}$ as

\[
H_{\pm} = p^2 + V_{\pm},
\]  
(16)

where

\[
V_{\pm} = W^2 \pm dW/dx
\]  
(17)

are known as Riccati’s equations. Here the operators $A$ and $\bar{A}$ in (11,12) become simple functions of $W$:

\[
A = W(x) + i p \quad \text{and} \quad \bar{A} = W(x) - i p.
\]  
(18)

These kinds of operators were considered long ago in [11] and actually constitute the starting point of “standard” supersymmetric quantum mechanics. Nevertheless, the liberty of choosing the functions $a_{0,1}(x)$ opens the door to regard some modifications (deformations) of the connected Schrödinger-like equations in SQM. For instance, taking $a_0 = a_0^* = W(x)$ and $a_1 = i\alpha(x)$, with $\alpha(x)$ real, in (14,15), one modifies the standard Schrödinger-like Hamiltonians $H_{\pm}$ given in (16). In this case, the modified Hamiltonians $\tilde{H}_{\pm}$ may be written as

\[
\tilde{H}_{\pm} = H_{\pm} + (\alpha^2 - 1) p^2 + \left[ \pm i (\alpha - 1) (pW) + \frac{1}{4} (p\alpha)^2 + \frac{1}{2} \alpha (p^2 \alpha) + 2\alpha (p\alpha) p \right],
\]  
(19)

where the last two terms modify the kinetic and potential parts of $H_{\pm}$. The key point in this analysis is that by construction the operators $\tilde{H}_-$ and $\tilde{H}_+$ must also be linked by supersymmetry.
Now we are going to consider briefly the \( N = 2 \) case. From \((10)\), setting \( N = 2 \), one can easily verify that
\[
A = \frac{1}{2} \left( a_0 \star p^0 + a_1 \star p^1 + a_2 \star p^2 \right)
\]
\[
= a_2 p^2 + (a_1 + (pa_2)) p + \left( a_0 + \frac{1}{2} (pa_1) + \frac{1}{2} (p^2 a_2) \right)
\] (20)
\[
\bar{A} = \frac{1}{2} \left( a_0 \star p^0 + a_1 \star p^1 + a_2 \star p^2 \right)
\]
\[
= a_2^* p^2 + (a_1^* + (pa_2^*)) p + \left( a_0^* + \frac{1}{2} (pa_1^*) + \frac{1}{2} (p^2 a_2^*) \right).
\] (21)

As these operators are second-order differential ones, \( H_- = \bar{A} A \) and \( H_+ = A \bar{A} \) will be in general fourth-order differential operators. Even though \( H_- \) and \( H_+ \) must still be intimately connected by supersymmetry, they turn out rather intricate without the imposition of additional conditions on the superpotentials \( a_i (x) \). A particular case of \((20, 21)\), which is obtained by putting \( a_0 = a_0^* = \varphi (x) \), \( a_1 (x) = i f (x) \), and \( a_2 = 1 \), was studied long ago in \[4\]. However, as can be easily seen, there are an infinity of possibilities which can be of interest from the physical or mathematical point of view.

Coming back to the \( N = 1 \) case and focusing, in particular, on formulas \((16-18)\), we show that it is always possible to write a one dimensional Hamiltonian \( H_- = p^2 + V (x) \) in the form \( H_- = \bar{A} A + c \), where the operators \( A \) and \( \bar{A} \) are given in \[18\] and \( c \) is an arbitrary constant. Here we follow the same line of reasoning as in \[6\]. Writing \( H_- = \bar{A} A + c \) in terms of the superpotential \( W \) with the help of \[18\] and comparing the result with the standard form \( H_- = p^2 + V (x) \), one arrives at
\[
W^2 - dW/dx = V (x) - c.
\] (22)

So the superpotential \( W \) which defines the operators \( A \) and \( \bar{A} \) must be a solution of the above equation. Obviously, this solution will depend on the form of the energy potential \( V (x) \) and the value of the constant \( c \). If now we fix the arbitrariness of \( c \) by equaling it to a given eigenvalue \( E \) of \( H \), \( \psi_E = E \psi_E \), one finds a solution \( W_E \) of \((22)\):
\[
W_E (x) = -\frac{1}{\psi_E} \frac{d\psi_E}{dx}.
\] (23)

Notice that \( W_E \) is implicitly a function of the eigenvalue \( E \) by means of its corresponding eigenfunction \( \psi_E \). Solving this differential equation one obtains a way of expressing the
eigenfunction $\psi_E$ in terms of its superpotential $W_E$:

$$\psi_E(x) = \psi_E(0) e^{-\int_0^x W_E(y)dy}. \quad (24)$$

Some comments concerning equations (22-24) are in order. First, note that (24) is true for any given eigenfunction $\psi_E$ with eigenvalue $E$ of $H_- = p^2 + V(x)$. However the simplest factorization of the Hamiltonian $H_-$, i.e. $H_- = \bar{A}A$, is achieved if and only if one chooses the ground state $\psi_{E_0}$ corresponding to $E_0 = 0$. This is always possible since in quantum mechanics one can fix the ground state energy $E_0$ to zero by subtracting $E_0$ from the Hamiltonian $H$. Second, the bosonic Hamiltonian $H_- = \bar{A}A + c$ has a partner $H_+ = A\bar{A} + c$ in such way that they are linked by the supercharges: $H_- \leftrightarrow H_+$. Finally, from (24) one realizes that the normalizability of the eigenfunction $\psi_E(x)$ depends on the behavior of the superpotential $W_E(x)$ when $x \to \pm\infty$.

From the above analysis, it is evident that one can always associate a set of Hamiltonians (constructed successively by following the procedure described in the previous paragraph) to a given Hamiltonian so that the eigenvalues and eigenfunctions of any two adjacent Hamiltonians are connected by supersymmetry. This hierarchy of Hamiltonians was studied for the first time in [6] and, as we shall see in the next section, becomes to be a powerful tool (in conjunction with the $\delta$-expansion [7, 8]) in finding approximate eigenvalues and eigenfunctions of a given Hamiltonian.

III. THE SUPERSYMMETRIC $\varepsilon$-SYSTEM REVISITED

In this section we study the supersymmetric $\varepsilon$-system (of order two in $x$) defined by the superpotential $W(x) = g|x|^2\varepsilon(x)$, with $g > 0$. This type of model was recently introduced and studied in detail in [5]. Indeed, by means of the variational technique, the authors in [5] computed the approximate energy eigenvalues of the first excited states of the partner Hamiltonians, establishing explicitly the SUSY relationship between them. In what follows we revisit the $\varepsilon$-system and exploit its supersymmetry in order to tackle the same energy eigenvalue problem. According to the method described in [8], we first modify the superpotential $W$ by introducing an extra $\delta$ parameter in terms of which we will carry out the perturbation expansion:

$$W(x) = g|x|^{1+\delta} \varepsilon(x), \quad (25)$$
where our original superpotential is obviously recovered by taking $\delta = 1$. Notice also that the absolute value of $x$ it is necessary to guarantee the right behavior of $W(x)$ at infinity: negative sign of $W(x)$ at minus infinity and positive sign at plus infinity.

Inserting (25) into (17), one obtains the partner potentials

$$V_{\pm} = g^2 x^{2(1+\delta)} \pm g (1 + \delta) |x|^\delta$$

and the corresponding Schrödinger equations

$$-\frac{d^2 \psi_n^\pm}{dx^2} + \left[ g^2 x^{2(1+\delta)} \pm g (1 + \delta) |x|^\delta \right] \psi_n^\pm = E_n^\pm \psi_n^\pm.$$  \hspace{1cm} (27)

Recall that here we are considering $2m = 1$.

Since the ground state wavefunction $\psi_0^-(x)$ of $H_- (= \bar{A}A)$ corresponds to a zero energy $E_0^- = 0$, i.e. $H_- \psi_0^- = 0$, this may be found by using the formula (24) along with (25) or by imposing the condition $A \psi_0^- = 0$. From (24), it follows easily that

$$\psi_0^- (x) = N \exp \left[ -\frac{g}{2 + \delta} |x|^{2+\delta} \right],$$

where $N$ is the normalization constant given by $N = \left( \frac{2g}{2+\delta} \right)^{1/[2(2+\delta)]} \sqrt{2^{1+1/[2+\delta]} \Gamma[1+1/[2+\delta]]}$.

In order to be able to gain an understanding of the method that will be employed later on, we compute the superpotential $W(x)$ assuming that this is unknown through the perturbation $\delta$-expansion. For this purpose, we first expand the potential $V_-(x)$ and the “unknown” superpotential $W(x)$ in powers of $\delta$, and then substitute these results in the corresponding Riccati equation. In other words, we are going to solve perturbatively the Riccati equation in the $\delta$ parameter.

The series expansion of the potential $V_-$ is

$$V_-(x) = g^2 x^2 - g + \sum_{n=1}^{\infty} \left[ \frac{g^2 x^2 \ln^n |x|^2 - g \ln^n |x|}{n!} - \frac{g \ln^{n-1} |x|}{(n - 1)!} \right] \delta^n,$$

$$= (g^2 x^2 - g) + (g^2 x^2 \ln |x|^2 - g \ln |x| - g) \delta^1 + \cdots$$

and assuming as mentioned before that the superpotential $W$ is unknown, we write it as a power series in $\delta$ with $x$-dependent coefficients

$$W(x) = \sum_{n=0}^{\infty} \omega_n (x) \delta^n = \omega_0 (x) + \omega_1 (x) \delta + \omega_2 (x) \delta^2 + \cdots.$$
As will be seen below, the unknown coefficients $\omega_n(x)$ will be determined by means of the Riccati equation.

Substituting the expansions (29) and (30) into the Riccati equation $V = W^2 - W'$ and comparing terms with the same power in $\delta$, one obtains in general an infinity set of coupled differential equations (except for one independent equation which results from $\delta = 0$). Up to order two in $\delta$, this process leads to

\begin{align}
\omega_0^2 - \omega_0' &= g^2 x^2 - g \\
2\omega_0\omega_1 - \omega_1' &= g^2 x^2 \ln |x|^2 - g \ln |x| - g \\
2\omega_0\omega_2 + \omega_1^2 - \omega_2' &= \frac{1}{2} \left[ g^2 x^2 \left( \ln |x|^2 \right)^2 - g \left( \ln |x| \right)^2 \right] - g \ln |x|.
\end{align}

The method for solving this system of differential equations is sequential, i.e., one first solves the independent equation (31) to find $\omega_0$, then with this function at hand solves (32) to find $\omega_1$, and so on. However, caution is needed here, for the differential equation (31) has a family of solutions:

\begin{equation}
\omega_0(x) = gx - \frac{2\sqrt{g} e^{g x^2}}{2\sqrt{g} e - i \sqrt{\pi} \text{erf}(i \sqrt{g} x)},
\end{equation}

where $c$ is an arbitrary constant and $\text{erf}(x) = 2/\sqrt{\pi} \int_0^x dy e^{-y^2}$ is the well-known error function. Therefore to choose the correct solution $\omega_0(x)$ we must contrast it with the corresponding one of the unperturbed model which results of taking $\delta = 0$, i.e., the linear harmonic oscillator (LHO). As the ground state wavefunction $\psi_0^{LHO}$ of the harmonic oscillator is $\psi_0^{LHO} \sim e^{-g x^2/2}$ (in our units), then the right $\omega_0$ solution is $\omega_0(x) = g x$.

Inserting the value of $\omega_0(x)$ into (32) and using the integration factor $e^{-g x^2}$ to simplify the integration as well as the initial condition $\omega_1(0) = 1$, it is straightforward to show that $\omega_1(x) = g x \ln |x|$. In a similar manner, using the results for $\omega_0(x)$ and $\omega_1(x)$, one solves (33) for $\omega_2$, finding that $\omega_2(x) = (g/2) x \left( \ln |x| \right)^2$.

In a nutshell, we have found perturbatively that the $\delta$-expansion of the superpotential $W(x)$ is given by

\begin{equation}
W(x) = \omega_0(x) + \omega_1(x) \delta + \omega_2(x) \delta^2 + \cdots,
\end{equation}

where, as previously shown,

\begin{equation}
\omega_0(x) = g x, \quad \omega_1(x) = g x \ln |x|, \quad \omega_2(x) = (g/2) x \left( \ln |x| \right)^2.
\end{equation}

Notice that this expansion coincides (as should be expected) with that obtained by using the exact form of $W(x)$ given in (25).
We now pass to compute the energy $E_{-1}$ of the first excited state of $H_-$, namely $\psi_{-1}(x)$, and to this end we shall use the following trick. Since the ground state wavefunction $\psi_0^+(x)$ of $H_+$ is connected by supersymmetry to $\psi_{-1}(x)$, $\psi_{-1}(x) \sim \tilde{A}\psi_0^+(x)$, and both have the same energy eigenvalue $E_{-1} = E_0^+$, we are going to work with the Hamiltonian $H_+$ rather than $H_-$, by refactoring it and then by solving approximately the corresponding Riccati equation. Let us see below how effectively this trick works.

Considering the “fermionic” Hamiltonian $H_+ = A\tilde{A} = -d^2/dx + V_+$ and following the procedure described in the final part of Sec. [11] we factor $H_+$ in the form

$$H_+ = A\tilde{A} = \bar{S}S + \mathcal{E},$$

(37)

where $\mathcal{E} = E_{-1} = E_0^+$ and

$$S = U(x) + ip, \quad \bar{S} = U(x) - ip,$$

(38)

where $p = -id/dx$. Note that the operators $S$ and $\bar{S}$ play the same role as $A$ and $\tilde{A}$ respectively, whereas $U(x)$ is a new superpotential to be determined later on and plays the same role as $W$.

Substituting the definitions of the operators $A$, $S$ and of their Hermitian adjoints into the second equality of (37), we find a relation between the superpotentials $W$, $U$ and the energy $\mathcal{E}$ of the first excited state of $H_-$:

$$W^2 + W' = U^2 - U' + \mathcal{E}.$$  

(39)

This relation is a Riccati-like equation and will be solved perturbatively in the $\delta$ parameter. Analogously to what was done in getting (35), we will assume a power series in the $\delta$ parameter for all elements involved in (39). That is,

$$W(x) = \sum_{n=0}^{\infty} \omega_n \delta^n = \omega_0(x) + \omega_1(x) \delta + \omega_2(x) \delta^2 + \cdots$$

(40)

$$U(x) = \sum_{n=0}^{\infty} u_n \delta^n = u_0(x) + u_1(x) \delta + u_2(x) \delta^2 + \cdots$$

(41)

$$\mathcal{E} = \sum_{n=0}^{\infty} \varepsilon_n \delta^n = \varepsilon_0 + \varepsilon_1 \delta + \varepsilon_2 \delta^2 + \cdots$$

(42)

Inserting these expressions into (39) and matching the coefficients of terms with the same power in $\delta$ at both sides of the equality, we get up to order two in $\delta$ a set of three differential
equations,
\[ \omega_0^2 + \omega'_0 = u_0^2 - u'_0 + \varepsilon_0 \]  \quad \text{(43)}
\[ 2\omega_0\omega_1 + \omega'_1 = 2u_0u_1 - u'_1 + \varepsilon_1 \]  \quad \text{(44)}
\[ 2\omega_0\omega_2 + \omega'_2 = 2u_0u_2 + u_2^2 - u'_2 + \varepsilon_2, \]  \quad \text{(45)}

where it should be noted that the functions \( u_i(x) \) and the quantities \( \varepsilon_i \) are unknowns, while the functions \( \omega_i \) are given in (36).

Solving (43) with \( \omega_0(x) = gx \), we find as a possible solution
\[ \varepsilon_0 = 2g \quad \text{and} \quad u_0(x) = gx. \]  \quad \text{(46)}

This solution is indeed the right one since it corresponds to the linear harmonic oscillator which results of taking \( \delta = 0 \). Note however that there is a family of solutions, for instance, \( \varepsilon_0 = 4g \) and \( u_0 = gx - 1/x \) constitute also a solution of (43).

The next step is to work out \( \varepsilon_1 \) and \( \omega_1(x) \) by integrating (44). Doing this one arrives at
\[ u_1(x) = e^{g x^2} \int_0^x e^{-gy^2} \left[ \varepsilon_1 - 2g^2y^2 \ln |y| - g (1 + \ln |y|) \right] dy, \]  \quad \text{(47)}
where we have made use of the initial condition \( u_1(0) = 0 \).

Using the boundary condition \( u_1(x \to \infty) \to 0 \), which comes from the requirement of finiteness of the wavefunction at infinity, we obtain directly from (47) the first contribution \( \varepsilon_1 \) to the energy \( \mathcal{E} \),
\[ \varepsilon_1 = \frac{g \left[I(0) + \partial_\alpha I(0)\right] + 2g^2 \partial_\alpha I(2)}{I(0)} = g \left[\psi(3/2) - \ln g\right], \]  \quad \text{(48)}
where \( \psi(x) \doteq d \ln \Gamma(x)/dx \), and
\[ I(\alpha, x) \doteq \int_0^x y^\alpha e^{-gy^2} dy = \frac{1}{2} g^{-\frac{1}{2}(1+\alpha)} \gamma \left(\frac{1+\alpha}{2}, g x^2\right), \]  \quad \text{(49)}

with \( \gamma(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t} dt \) (the incomplete gamma function). Note that for simplicity in (48) we have omitted the second argument \( x = \infty \) of the function \( I(\alpha, x) \).

On the other hand, the function \( u_1(x) \) in terms of \( \gamma \) and its first derivative with regard to \( \alpha \) is given by
\[ u_1(x) = gx \ln x + \frac{\sqrt{g}}{2} e^{g x^2} \left[ \psi(1/2) \gamma(1/2, g x^2) - \partial_\alpha \gamma(1/2, g x^2) \right]. \]  \quad \text{(50)}
With these results at hand the third differential equation (45) is tackled. Solving this equation for \( u_2(x) \), with the help of the integration factor \( e^{-gy^2} \) and the initial condition \( u_2(0) = 0 \), we find that

\[
\varepsilon_2 = \frac{g}{4\sqrt{\pi}} \left\{ \psi(1/2) \left[ \partial_\alpha i_{1/2} (\alpha, \alpha) - \psi(1/2) i_{1/2} (1/2, 1/2) \right] - \partial_{\alpha\beta} i_{1/2} (\alpha, \beta) + \sqrt{\pi} (1 + \psi(1/2) - \ln g)^2 + \sqrt{\pi} \right\},
\]

(53)

where \( \partial_{\alpha\beta} = \partial_\alpha \partial_\beta \) and the vertical bar \( | \) means evaluation, after performing the respective differentiations, at \( \alpha = \beta = 1/2 \). As before, for economy in notation, the third argument \( (x = \infty) \) of the function \( i_a (\alpha, \beta, x) \) has been dropped.

The majority of the integrals which appear in (51) were calculated by reducing them to the master integral (49) or to its \( \alpha \)-derivatives. For instance, an integral like \( \int_0^\infty \ln y e^{-gy^2} \) is simply the derivative of \( I(\alpha) \) with respect to \( \alpha \), i.e. \( \partial_\alpha I(\alpha) \), evaluated at \( \alpha = 0 \). By contrast, the remaining integrals in (53) are very complex due to the product of two incomplete \( \gamma \) functions involved in the definition of \( i_a (\alpha, \beta, x) \) so that they have been evaluated numerically. As a result, we have found that

\[
\varepsilon_2 = -0.17638 g + 0.48176 g \ln g + 0.25 g (\ln g)^2.
\]

(54)

Taking \( g = 1 \) and grouping all the contributions \( \varepsilon_i \), the \( \delta \) expansion for \( E \) becomes

\[
E = \varepsilon_0 + \varepsilon_1 \delta + \varepsilon_2 \delta^2,
\]

(55)

where \( \varepsilon_0 = 2 \), \( \varepsilon_1 = 0.03649 \) and \( \varepsilon_2 = -0.17638 \).
If we now focus on the supersymmetric $\varepsilon$-system defined by $W(x) = gx^2\varepsilon(x)$ and take so $\delta = 1$ in [55], we obtain the energy $E_1^- = 1.86011$ for the first excited state of the Hamiltonian $H_-$. This result can be improved by using the $[1,1]$ Padé approximant

$$\tilde{E} = \frac{\varepsilon_0\varepsilon_1 + \delta (\varepsilon^2_1 - \varepsilon_0\varepsilon_2)}{\varepsilon_1 - \delta\varepsilon_2}.$$  \hspace{1cm} (56)

For $\delta = 1$ an energy of $E_1^- = 2.00626$ for the first excited state of $H_-$ is obtained. This result is closer to its supersymmetric partner result $E_0^+ = 1.94605$ calculated in [9] by using a variant of the logarithmic perturbation theory, improved with the same $[1,1]$ Padé approximant. Both results must be compared with the very precise values $E_1^- = E_0^+ = 1.969507538$ obtained in [9] by a variational (Rayleigh-Ritz method) with the use of a seven parameter trial solution.

IV. CONCLUSIONS

The purpose of this paper is two-fold. First, we generalize the standard supersymmetric quantum mechanics by introducing a new class of operators $A$ and $\bar{A}$, and show that in the linear definition these operators reduce to the conventional ones proposed in [11]. Higher-order operator formulation is in progress, which for the second order reduces to those proposed in [4]. Second, we revisit the supersymmetric $\varepsilon$-system introduced in [5] and exploit its supersymmetry in order to determine the first excited state energy of the bosonic Hamiltonian $H_-$. Comparison with the results of [5] shows that the logarithmic approximation developed in [7, 8] does not give better results for the energy levels of the $\varepsilon$-system than the simpler linear logarithmic approximation used in [5].

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