A SIMPLE PROOF FOR THE EXISTENCE OF ZARISKI DECOMPOSITIONS ON SURFACES

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In his fundamental paper [5], Zariski established the following result:

Theorem. Let $D$ be an effective $\mathbb{Q}$-divisor on a smooth projective surface $X$. Then there are uniquely determined effective (possibly zero) $\mathbb{Q}$-divisors $P$ and $N$ with

$$D = P + N$$

such that

(i) $P$ is nef,

(ii) $N$ is zero or has negative definite intersection matrix,

(iii) $P \cdot C = 0$ for every irreducible component $C$ of $N$.

The decomposition $D = P + N$ is called the Zariski decomposition of $D$, the divisors $P$ and $N$ are respectively the positive and negative parts of $D$. Zariski’s result has been used to study linear series on surfaces, and in the classification of surfaces (see [1, Chapt. 14] and [4, Sect. 2.3.E], as well as the references therein). We also mention that there is an extension to pseudo-effective divisors due to Fujita (see [2] and the nice account in [1]).

Given an effective divisor $D$, Zariski’s original proof employs a rather sophisticated procedure to construct the negative part $N$ out of those components $C$ of $D$ satisfying $D \cdot C \leq 0$. Our purpose here is to provide a quick and simple proof, based on the idea that the positive part $P$ can be constructed as a maximal nef subdivisor of $D$. This maximality condition is in the surface case equivalent to the defining condition of Nakayama’s $\nu$-decomposition of pseudo-effective $\mathbb{R}$-divisors (see the Remark below). It may be useful that this approach yields a practical algorithm for the computation of $P$.

Notation. For $\mathbb{Q}$-divisors $P$ and $Q$ we will write $P \preceq Q$, if $P$ is a subdivisor of $Q$, i.e., if the difference $Q - P$ is effective or zero. Similarly, we will use the partial ordering $\preceq$ in $\mathbb{Q}^r$ that is defined by $(x_1, \ldots, x_r) \preceq (y_1, \ldots, y_r)$, if $x_i \leq y_i$ for all $i$.

Proof of existence. Write $D = \sum_{i=1}^r a_i C_i$ with distinct irreducible curves $C_i$ and positive rational numbers $a_i$. Consider now all effective $\mathbb{Q}$-subdivisors $P$ of $D$, i.e., all divisors of the form $P = \sum_{i=1}^r x_i C_i$ with rational coefficients $x_i$ satisfying $0 \leq x_i \leq a_i$. A divisor $P$ of this kind is nef if and only if

$$\sum_{i=1}^r x_i C_i \cdot C_j \geq 0 \quad \text{for } j = 1, \ldots, r. \quad (1)$$

This system of linear inequalities for the rational numbers $x_i$ has a maximal solution (with respect to $\preceq$) in the rational cuboid

$$[0, a_1] \times \ldots \times [0, a_r] \subset \mathbb{Q}^r.$$
To see this, note first that the subset $K$ of the cuboid that is described by {11} is a rational convex polytope defined by finitely many rational halfspaces. It is therefore the convex envelope of finitely many rational points. We are done if $(a_1, \ldots, a_r) \in K$. In the alternative case consider for rational $t < 1$ the family of hyperplanes $H_t = \{(x_1, \ldots, x_r) \in \mathbb{Q}^r \mid \sum_i x_i = t \sum_i a_i\}$. There is then a maximal $t$ such that $H_t$ intersects $K$, the point of intersection being a vertex of $K$.

Let now $P = \sum_{i=1}^r b_i C_i$ be a divisor that is determined by a maximal solution, and put $N = D - P$. Then both $P$ and $N$ are effective, and $P$ is a maximal nef $\mathbb{Q}$-subdivisor of $D$. We will now show that (ii) and (iii) are satisfied as well.

As for (iii): Suppose $P \cdot C > 0$ for some component $C$ of $N$. As $C \leq N$, we have $b_i < a_i$, so that for sufficiently small rational numbers $\varepsilon > 0$, the divisor $P + \varepsilon C$ is a subdivisor of $D$. For curves $C'$ different from $C$ we clearly have $(P + \varepsilon C) \cdot C' \geq 0$. Moreover, $(P + \varepsilon C) \cdot C = P \cdot C + \varepsilon C^2 > 0$ for small $\varepsilon$. So $P + \varepsilon C$ is nef, contradicting the maximality of $P$.

As for (ii): Supposing that the divisor $N$ is non-zero, we need to show that its intersection matrix is negative definite. We will prove:

\[ (*) \] If $N$ is a divisor, whose intersection matrix $S$ is not negative definite, then there is an effective non-zero nef divisor $E$, whose components are among those of $N$.

Granting (*) for a moment, let us show how to complete the proof. Assume by way of contradiction that the intersection matrix of $N$ is not negative definite, and take $E$ as in (*). Consider then for rational $\varepsilon > 0$ the $\mathbb{Q}$-divisor

\[ P' =_{\text{def}} P + \varepsilon E. \]

Certainly $P'$ is effective and nef. As all components of $E$ are among the components of $N$, it is clear that $P'$ is a subdivisor of $D$ when $\varepsilon$ is small enough. But this is a contradiction, because $P'$ is strictly bigger than $P$.

It remains to prove (*). To this end we distinguish between two cases:

Case 1: $S$ is not negative semi-definite. In this case there is a divisor $B$ whose components are among those of $N$ such that $B^2 > 0$. Then, writing $B = B' - B''$ as a difference of effective divisors having no common components, we have $0 < B^2 = (B' - B'')^2 = B'^2 - 2B'B'' + B''^2$, and hence $B'^2 > 0$ or $B''^2 > 0$. Therefore, replacing $B$ by $B'$ or $B''$ respectively, we may assume that $B$ is effective. But then it follows from the Riemann-Roch theorem that the linear series $|\ell B|$ is large for $\ell \gg 0$. So we can write $\ell B = E_\ell + F_\ell$, where $|E_\ell|$ is the non-zero moving part of $|\ell B|$. Then $E = E_\ell$ is a nef divisor as required, so that the proof of (*) is complete in this case.

Case 2: $S$ is negative semi-definite. Let $C_1, \ldots, C_k$ be the components of $N$. We argue by induction on $k$. If $k = 1$, then $N^2 = C_1^2 = 0$, so $C_1$ is nef and we are done taking $E = C_1$. Suppose then $k > 1$. The hypotheses on $S$ imply that $S$ does not have full rank. Therefore there is a non-zero divisor $R$, whose components are among $C_1, \ldots, C_k$, having the property that $R \cdot C_i = 0$ for $i = 1, \ldots, k$. If one of the divisors $R$ or $-R$ is effective, then it is nef, and we are done, taking $E = R$ or $E = -R$ respectively. In the alternative case we write $R = R' - R''$, where $R'$ and $R''$ are effective non-zero divisors without common components. We have

\[ 0 = R^2 = R'^2 - 2R'R'' + R''^2. \]

As by hypothesis $R'^2 \leq 0$ and $R''^2 \leq 0$, we must have $R'^2 = 0$. The divisor $R'$ has fewer components than $R$, and its intersection matrix is still negative semi-definite,
but not negative definite. It now follows by induction that there is a divisor as claimed, consisting entirely of components of $R'$.

We now give the

\textit{Proof of uniqueness.} We claim first that in any decomposition $D = P + N$ satisfying the conditions of the theorem, the divisor $P$ is necessarily a maximal nef $\mathbb{Q}$-subdivisor of $D$. To see this, suppose that $P'$ is any nef divisor with $P \preceq P' \preceq D$. Then $P' = P + \sum_{i=1}^{k} q_i C_i$, where $C_1, \ldots, C_k$ are the components of $N$ and $q_1, \ldots, q_k$ are rational numbers with $q_i \geq 0$. We have

$$0 \preceq P' \cdot C_j = \sum_{i=1}^{k} q_i C_i \cdot C_j \quad \text{for } j = 1, \ldots, k,$$

and hence

$$\left( \sum_{i=1}^{k} q_i C_i \right)^2 = \sum_{i=1}^{k} q_i \sum_{j=1}^{k} q_i C_i \cdot C_j \geq 0.$$

As the intersection matrix of $C_1, \ldots, C_k$ is negative definite, we get $q_i = 0$ for all $i$. So $P' = P$.

To complete the proof it is now enough to show that a maximal effective nef $\mathbb{Q}$-subdivisor of $D$ is in fact unique. This in turn follows from:

(**) If $P' = \sum_{i=1}^{r} x'_i C_i$ and $P'' = \sum_{i=1}^{r} x''_i C_i$ are effective nef $\mathbb{Q}$-subdivisors of $D$, then so is $P = \sum_{i=1}^{r} x_i C_i$, where $x_i = \max(x'_i, x''_i)$.

As for (**): The divisor $P$ is of course an effective $\mathbb{Q}$-subdivisor of $D$, so it remains to show that it is nef, i.e., that the tuple $(x_1, \ldots, x_r)$ satisfies the inequalities (1). This, finally, is a consequence of the following elementary fact: Let $H \subset \mathbb{Q}^r$ be a halfspace, given by a linear inequality $\sum_{i=1}^{r} \alpha_i x_i \geq 0$, where the coefficients $\alpha_i$ are rational numbers with at most one of them negative. If two points $(x'_1, \ldots, x'_r)$ and $(x''_1, \ldots, x''_r)$ with $x'_i \geq 0$ and $x''_i \geq 0$ lie in $H$, then so does $(x_1, \ldots, x_r)$, where $x_i = \max(x'_i, x''_i)$.

\textbf{Remark.} As experts may recognize, the maximality condition that defines $P$ is in the surface case equivalent to the defining condition of Nakayama's $\nu$-decomposition (see [3 Sect. III.1]). As Nakayama pointed out, it is also possible to obtain a proof by using results on $\nu$-decompositions and $\sigma$-decompositions (in particular [3 Proposition III.1.14], [3 Lemma III.3.1], [3 Lemma III.3.3], and [3 Remark III.3.12 and the subsequent Remark (1)], when combined with arguments making use of [3 Lemma 7.3] and [5 Lemma 7.4].

When viewed from the point of view of $\nu$-decompositions, the essential content of the present note is to provide a quick, simple, and self-contained proof of the fact that in the surface case the $\nu$-decomposition of an effective $\mathbb{Q}$-divisor is a rational decomposition enjoying properties (ii) and (iii), and that it is the unique decomposition with these properties.

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