CYCLES THAT ARE INCIDENCE EQUIVALENT TO ZERO

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Abstract. In this paper, we apply incidence divisors constructed through Archimedean height paring to prove that Griffiths’ conjecture ([8]) on incidence equivalence is correct.

1 Introduction

Let $X$ be a smooth projective variety of dimension $n$ over complex numbers. There is the Chow group $CH^r(X)$ which is the group of all codimensional $r$ algebraic cycles with integer coefficients modulo rational equivalence. Let $CH^r_{alg}(X)$ denote the subgroup of $CH^r(X)$ whose cycles are algebraically equivalent to zero. There is the Griffiths’ intermediate Jacobian $J^r$ (or $J_{n-r}$) which is a complex torus defined via Hodge structure of $X$ as follows. Let

$$H^{2r-1}(X; \mathbb{R})$$

be the cohomology group of the real manifold. Then the complexified cohomology group $H^{2r-1}(X; \mathbb{C})$ has a Hodge decomposition as a vector space

$$H^{2r-1}(X; \mathbb{C}) = \bigoplus_{i=0}^{2r-1} H^{i,2r-1-i}(X)$$  (1.1)

The decomposition can be re-grouped as

$$H^{2r-1}(X; \mathbb{C}) = F^rH^{2r-1}(X) \oplus F^cH^{2r-1}(X),$$  (1.2)

where $F^rH^{2r-1}(X)$ denotes the sum of summands $H^{p,q}$ in the Hodge decomposition (1.1) with $p > q$. Then it is clear the complex conjugate $F^cH^{2r-1}(X)$ is just the rest of other summands with $p < q$. Next we identify $H^{2n-2r+1}(X; \mathbb{Z})$ with a subgroup of $H^{2r-1}(X; \mathbb{C})$ via Poincaré duality and the projection in the vector space. Then the Griffiths’ intermediate Jacobian $J^r(X)$ (or $J_{n-r}(X)$) is defined to be

$$J^r(X) = \frac{F^rH^{2r-1}(X)}{H^{2n-2r+1}(X; \mathbb{Z})}.$$  (1.3)

Another expression of this is:

$$J^r(X) = \frac{(F^{n-r+1}H^{2n-2r+1}(X))^{\ast}}{H^{2n-2r+1}(X; \mathbb{Z})},$$  (1.4)

where the identification is made through the cup product between

$$H^{2r-1}(X), H^{2n-2r+1}(X).$$

The Jacobian $J^r(X)$ is equipped with a natural complex structure inherited from $F^rH^{2r-1}(X)$. There is the Abel-Jacobi map $AJ$ from $CH^r_{alg}(X)$ to $J^r(X)$ defined as follows. For any $B \in CH^r_{alg}(X)$, there is a real chain $\Gamma_B$ in $X$ such that $\partial \Gamma_B = B$. The map $AJ$,

$$B \mapsto \frac{\int_{\Gamma_B} (\cdot)}{H^{2n-2r+1}(X; \mathbb{Z})} \in \frac{(F^{n-r+1}H^{2n-2r+1}(X))^{\ast}}{H^{2n-2r+1}(X; \mathbb{Z})}$$  (1.5)
is defined to be the Abel-Jacobi map. This is a well-defined map by Hodge theory. Abel-Jacobi map is a regular homomorphism in the sense that for any smooth projective variety $T$ with a fixed point $t_0 \in T$ and a correspondence

$$Z \in CH^r(T \times X), \quad (1.6)$$

the map

$$T \xrightarrow{AJ} J^r(X) \quad \begin{array}{c} \text{t} \\ \rightarrow \end{array} AJ(Z(t) - Z(0)) \quad (1.7)$$

is a complex analytic map, where $Z(t) = (Pr_{TX})* (Z \cdot (\{t\} \times X))$ with projection $Pr_{TX}$ to $X$. In order to understand the kernel of $AJ$, Griffiths in [7] introduced another equivalence, called "incidence equivalence": let $T$ be a smooth projective variety parametrizing $n - q - 1$-algebraic cycles and

$$\Sigma \subset CH^{q+1}(T \times X)$$

be a correspondence. Now the cycle $B \in CH^{n-q}(X)$ is called incidence equivalent to zero if for all couples $(T, \Sigma)$ above, the divisor $\Sigma(B)$ is well-defined and rationally equivalent to zero on $T$. Let $CH_{inc}^{n-q}(X) \subset CH_{alg}^{n-q}(X)$ be the collection of all cycles in $CH_{alg}^{n-q}(X)$ that are incidence equivalent to zero. Then he further proved that for an algebraic cycle $B \in CH_{alg}^{n-q}(X)$,

$$AJ(B) = 0 \implies B \in CH_{inc}^{n-q}(X). \quad (1.8)$$

The converse was left as a conjecture ([8]): For $B \in CH_{alg}^{n-q}(X)$,

$$B \in CH_{inc}^{n-q}(X) \implies AJ(mB) = 0, \text{ for some positive integer } m. \quad (1.9)$$

The conjecture has an importance in the study of algebraic cycles. In the past 46 years, there had been a number of work on the conjecture. But they only yielded partial solutions. The most noticeable ones are the following theorems given by J. Murre ([11]), H. Saito ([12]).

**THEOREM 1.1.** (Murre) For $B \in CH_2^{alg}(X)$, the conjecture (1.9) is correct.

**THEOREM 1.2.** (Saito) If the conjecture (1.9) is true on all odd dimensional smooth projective varieties $X$ for cycles in $CH^{\dim(X)+1}_{dim(X)+1}(X)$, then the conjecture is true.

In this paper, we prove the full conjecture

**THEOREM 1.3.** Griffiths’ conjecture (1.9) is correct.

**Sketch of the proof.**

Our main technique is the Archimedean height pairing from Arakelov geometry.

Notations: (1) Throughout the paper all homology and cohomology in
interior coefficients are defined modulo torsion.

(2) For any cycle \( a \) (algebraic or non-algebraic), \([a]\) denotes its class.

(3) Let \( \text{int}(,)_V \) denote the intersection number of two cycles on a manifold \( V \).

The Intermediate Jacobians are complex analytic when they first appeared. Later H. Saito made more algebraic construction of them ([12]). In that paper he reduced the Griffiths’ conjecture. Using corollary 5.5 and proposition 4.4 in [12], Saito proved the Griffiths’ conjecture is correct under one assumption—the conjecture holds for arbitrary odd dimensional \( X \) with \( \text{dim}(X) = 2p + 1 \) and \( \text{dim}(B) = p \). Our paper is a proof of this assumption: Griffiths’ conjecture holds for any \( X, B \) with \( \text{dim}(X) = 2p + 1 \) and \( \text{dim}(B) = p \). Therefore with Saito’s result we prove the whole Griffiths’ conjecture.

Let \( X \) be a smooth projective variety of odd dimension \( n \). We use the notation \( T \) to represent a family of \( p \)-cycles in the following set-up: there is an algebraic cycle \( Z \) whose support is projected onto the smooth \( T \),

\[
Z \in \mathbb{Z}^{p+1}(T \times X),
\]

satisfying that \( Z \) intersects \( \{t\} \times X \) properly. We denote

\[
Z(t) = (\text{Pr}_X)_*(Z \cdot (\{t\} \times X))
\]

and \( \text{Pr}_X \) is the projection from \( T \times X \) to \( X \).

Let

\[
B \in \text{CH}^p_{\text{alg}}(X)
\]

where \( n = 2p + 1 \). If \( B = [Z(t_1) - Z(t_0)] \), for points \( t_0, t_1 \in T \). We say \( T \) is going through the cycle \( B \).

**Definition 1.4.** (see [11] )

(1) For any smooth parameter space \( T \) of \( p \)-cycles in \( X \), we let

\[
H^T_{2p+1}(X; \mathbb{Z})
\]

be the subgroup of \( H_{2p+1}(X; \mathbb{Z}) \) defined to be the image of the composition map

\[
\nu_T : H_1(T; \mathbb{Z}) \to H_1(J_p(X)) \to H_{2p+1}(X; \mathbb{Z}).
\]

The second map is obtained via the torus structure of the intermediate Jacobian. Let

\[
H^T_{2p+1}(X; \mathbb{A}) = H^T_{2p+1}(X; \mathbb{Z}) \otimes \mathbb{A}
\]

and

\[
H^T_{2p+1}(X; \mathbb{A})
\]

be its Poincaré dual, for \( \mathbb{A} = \mathbb{Q}, \mathbb{R}, \mathbb{C} \).

(2) Let \( H^a_{2p+1}(X; \mathbb{Z}) \) be the subgroup of \( H_{2p+1}(X; \mathbb{Z}) \) generated by all \( H^T_{2p+1}(X; \mathbb{Z}) \). Similarly let

\[
H^a_{2p+1}(X; \mathbb{A}) = H^a_{2p+1}(X; \mathbb{Z}) \otimes \mathbb{A}
\]
be its Poincaré dual, for \( \mathbb{A} = \mathbb{Q}, \mathbb{R}, \mathbb{C} \).

Our proof is based on a technique from Arithmetic geometry. So it bypassed classical ideas in [11]. Yet classical ideas are looming behind the entire proof for our new approach is an extension of Murre’s. Since our paper will not give a detail of this, it is certainly beneficiary to include a rather philosophical description of such a connection. In Arithmetic geometry (more specifically, Arakelov geometry), R. Hain ([10]) developed Beilinson-Bloch’s original idea ([2], [3]) to connect the Archimedean height pairing with intersection pairing via Poincaré line bundle. On the other hand our previous work ([13], [14]) connects the incidence equivalence with Archimedean height pairing. Combining both directions, we have a logic relation described in the following graph

\[
\begin{array}{c}
\text{Abel – Jacobi equivalence} \\
\downarrow \\
\text{Intersection pairing (Poincaré line bundle)} \\
\downarrow \\
\text{Archimedean height pairing} \\
\downarrow \\
\text{Incidence equivalence}
\end{array}
\]

(1.18)

(where the arrows or equal sign do not mean the exact equalities). The work of Caibăr and Clemens ([5]) yields similar results. This picture leads to an observation: the equality of the top and the bottom is the same as the non-degeneracy of the middle pairings (either one of them). Non-degeneracy of Archimedean height pairing does not have a rigorous definition unless it is replaced by the arithmetic, so called global height pairing over number fields. But non-degeneracy of intersection pairing \( \text{Int}_\alpha \) on the “algebraic parts, \( H^2_\alpha(X; \mathbb{Z}) \)” is a perfectly well-defined concept. Thus we concentrate on intersection pairing and claim:

**Claim 1.1.** _Non-degeneracy of the intersection pairing \( \text{Int}_\alpha \) restricted to the non-zero \( H^2_{2p+1}(X; \mathbb{Z}) \) is exactly the conjecture (1.9)._

This assertion is not only implied by the content of this paper, but it is also hinted in [4]. Furthermore, in the absence of Archimedean height pairing and the graph (1.18) the claim 1.1 was proved implicitly by Murre ([11]) via Hodge structure. This is the Murre’s key idea in [11] (see lemma 5.2, [11]), the classical idea mentioned above. In this paper, we changed the form of claim 1.1 as needed. More specifically based on the graph (1.18) we obtained a different version of claim 1.1 concerning only one cycle \( B \) that is incidence equivalent to zero. We show that in order to prove the conjecture (1.9), it suffices to show the non-degeneracy of restricted pairing \( \text{Int}_\alpha(1, 2) \) of \( \text{Int}_\alpha \) to subgroups

\[
H^2_{2p+1}(X; \mathbb{Z}) \times H^2_{2p+1}(X; \mathbb{Z})
\]

(1.19)

for some smooth curves \( T_1, T_2 \) depending on the cycle \( B \). This is much weaker than claim 1.1, but it is detailed to the cycle \( B \). So a rule of thumb is: if it is too hard
to prove the non-degeneracy of \( \text{Int} \) as a whole map\(^1\), we restricted it to a subgroup determined by the individual degenerate element \( B \). Finally we see that \( T_1, T_2 \) can be obtained by taking curves corresponding to sets that are “orthogonal” to the preimage of \( AJ(CH_{inc}^{p+1}(X)) \) under the map

\[
F^pH^{2p+1}(X) \to \frac{F^pH^{2p+1}(X)}{H_{2p+1}(X; \mathbb{Z})}.
\]

The detailed proof is divided into two steps.

First step.

In this step, section 3, 4, we prove the conjecture under an assumption. Let \( B \in CH_{inc}^{p+1}(X) \). Let \( T_1, T_2 \) be two smooth projective curves parameterizing \( p \)-cycles of \( X \) and \( Z_1, Z_2 \) be the correspondence for \( T_1, T_2 \) respectively. Also assume \( T_2 \) goes through \( B \), and \( Z_1(t), Z_2(t') \) are disjoint for generic \((t, t') \in T_1 \times T_2\).

Assumption 1.2. (see section 4). Assume that there are \( T_1, T_2 \) as above such that the intersection pairing on

\[
H^{2p+1}(X; \mathbb{Z}) \times H^{2p+1}(X; \mathbb{Z})
\]

is non-degenerate on left when restricted to the subgroup

\[
H^{T_1}_{2p+1}(X; \mathbb{Z}) \times H^{T_2}_{2p+1}(X; \mathbb{Z}),
\]

called algebraic part from \( T_i, i = 1, 2 \), and also

\[
\dim(H^{T_2}_{2p+1}(X; \mathbb{Q})) \leq \dim(H^{T_1}_{2p+1}(X; \mathbb{Q})).
\]

(1.20)

Then the Griffiths’ conjecture is true. By “on left”, we mean that for any fixed non-zero cycle \( \gamma'_1 \in H^{T_1}_{2p+1}(X; \mathbb{Z}) \), the intersection pairing with at least one cycle \( \gamma'_2 \in H^{T_2}_{2p+1}(X; \mathbb{Z}) \) is non zero. Let’s describe the result in more detail. Recall the intermediate Jacobian is defined to be

\[
J^{p+1} = \frac{(F^{p+1}H^{2p+1}(X))^*}{H^{2p+1}(X; \mathbb{Z})}.
\]

(1.21)

There is the Abel-Jacobi map \( AJ \) from \( CH_{alg}^{p+1}(X) \) to \( J^{p+1}(X) \). For any \( B \in CH_{alg}^{p+1}(X) \), there is a real chain \( \Gamma_B \) in \( X \) such that \( \partial \Gamma_B = B \). The map \( AJ \),

\[
B \mapsto \frac{\int_{\Gamma_B}()}{H^{2p+1}(X; \mathbb{Z})} \in \frac{(F^{p+1}H^{2p+1}(X))^*}{H^{2p+1}(X; \mathbb{Z})}
\]

(1.22)

is defined to be the Abel-Jacobi map. Denote the image of \( CH_{alg}^{p+1}(X) \) by

\[
J^{p+1}_a(X) \quad (or \ J^{p}_a(X)).
\]

The first step relies on a decomposition of the cohomology. This is lemma 3.4 in section 3 which states

\(^1\) This actually is wrong if one assumes \( AJ(CH_{inc}^{p+1}(X)) \) is not a torsion.
Claim 1.3. (see lemma 3.4). If the intersection pairing on $H^{2p+1}_{T_1}(X;\mathbb{Z}) \times H^{2p+1}_{T_2}(X;\mathbb{Z})$ is non-degenerate on left and

$$\dim(H^{T_2}_{2p+1}(X;\mathbb{Q})) \leq \dim(H^{T_1}_{2p+1}(X;\mathbb{Q})),$$

(1.23)

there is a decomposition

$$H^{2p+1}(X;\mathbb{Q}) = H^{T_2}_{2p+1}(X;\mathbb{Q}) \oplus W_Q$$

(1.24)

where $W_Q$ has the property that $\text{int}(h_1, h_2)_X = 0$ for $h_1 \in H^{T_2}_{2p+1}(X;\mathbb{Q}), h_2 \in W_Q$.

The Poincaré dual of this is

$$H_{2p+1}(X;\mathbb{Q}) = H^{T_1}_{2p+1}(X;\mathbb{Q}) \oplus W_Q$$

(1.25)

where $W_Q$ has the property that $\text{int}(h_1, h_2)_X = 0$ for $h_1 \in H^{T_2}_{2p+1}(X;\mathbb{Q}), h_2 \in W_Q$.

Then conjecture follows from this decomposition with a help of an easy claim below.

Claim 1.4. (see section 7, Appendix).

$$\int_{\Gamma_B} h = 0$$

(1.26)

where $\Gamma_B$ is the chain on $X$ that comes from $T_2$ and $h \in W_Q$.

Remark Claim 1.4 shows that the integrals (1.26) annihilates all cohomological elements $h \in W_C$. By the decomposition (1.24), the only non-annihilated integral cohomological elements are from $H^{T_1}_{2p+1}(X;\mathbb{Z})$. Those must be paired to rational numbers with $B$ if $B \in CH_{\text{inc}}^{p+1}(X)$. Therefore $AJ(B)$ is a torsion.

Second step

The conjecture now boils down to the proof of assumption 1.2, which is a finer version of claim 1.1. But it is obvious that the assumption does not hold if $H^{2p+1}_{T_1}(X;\mathbb{Z}) = 0$. It turns that the only situation where the assumption 1.2 fails is that $J^p_{B}(X) = 0$ (which implies $H^{2p+1}_{T_2}(X;\mathbb{Z}) = 0$). But for this situation we don’t need the assumption 1.2 because if $J^p_{B}(X) = 0$, the conjecture (1.9) is certainly correct. So in this step, section 5, 6, we assume $J^p_{B}(X) \neq 0$. With that we’ll prove assumption 1.2 for a special choice of $T_i, i = 1, 2$.

The main technique in this step is the incidence divisor constructed through Archimedean height pairing. Under the set-up in introduction we have the incidence divisor $D_{\Delta}(1, 2)$ on $T_1 \times T_2$, where the indexes 1, 2 are used to indicated the order in the product $T_1 \times T_2$. Set-theoretically, $D_{\Delta}(1, 2)$ can be naively defined to be

$$|D_{\Delta}(1, 2)| = \{ (t_1, t_2) \in T_1 \times T_2 : |Z_{t_1}| \cap |Z_{t_2}| \neq \emptyset \}.$$
See section 2 for a rigorous definition.

For the incidence divisor we prove that

**Claim 1.5.** *(see lemma 3.1).* For any \( \gamma_i \in H_1(T_i; \mathbb{Z}), i = 1, 2, \)

\[
\text{int}\left( \nu_{T_1}(\gamma_1), \nu_{T_2}(\gamma_2) \right)_X = \text{int}\left( \gamma_1 \times \gamma_2, \mathcal{D}_\Delta(1, 2) \right)_{T_1 \times T_2}
\] (1.28)

The assumption 1.2 assumes that the bilinear form (1.28) is non-degenerate on left.

We will show that there is a reversed map through another algebraic correspondence

\[
\mathcal{D}(2, 1) : H_1(T_2; \mathbb{Q}) \to H_1(T_1; \mathbb{Q})
\] (1.29)

such that

\[
\mathcal{D}(2, 1) \circ \mathcal{D}_\Delta(1, 2)
\] (1.30)

as a linear map on rational homology

\[
H_1(T_1; \mathbb{Q}) \to H_1(T_1; \mathbb{Q})
\] (1.31)

is non-degenerate.

This is the following assertion:

**Claim 1.6.** *(see section 6).* If \( J_p^{p+1}(X) \neq 0 \), we can choose \( T_1, T_2 \) such that

\[
\mathcal{D}(2, 1) \circ \mathcal{D}_\Delta(1, 2)
\] (1.32)

has a matrix representation

\[
\xi_0 I
\] (1.33)

where \( I \) is the identity matrix, and \( \xi_0 \) is a non-zero integer.

The claim 1.5 implies that on \( T_1 \times T_2, \mathcal{D}_\Delta(1, 2) \) is homologically non-degenerate on left. Then it follows that

**Claim 1.7.** *(see section 6).* The assumption 1.2 holds for such parameter spaces \( T_1, T_2 \).

Using step 1, we complete the proof.

We organize the rest of paper as follows. In section 2, we introduce our technique Archimedean height pairing, its relation with incidence equivalence, Poincaré line bundle, and Abel-Jacobi map. In section 3, 4, we complete the proof of the step 1. In section 5, 6, we prove the step 2. Appendix includes a corollary proved by Caibăr and Clemens.

**Acknowledgment** This paper is inspired by the work of Mirel Caibăr and C. Herbert Clemens ([4]). We would like to thank Herbert Clemens for the communication.
2 Incidence structure and Archimedean height pairing

In this section, we introduce the technique of the proof: Mazur’s incidence line bundle associated to Archimedean height pairing. This is a description of a geometric structure through Archimedean height pairing from Arakelov geometry.

We adapt everything in section 1 except the dimensions. Let \( \text{dim}(X) = n \) be any positive integer. We let \( p, q \) be any two nature numbers satisfying \( p + q = n - 1 \). Let \( \mathcal{C}_r(X) \) denote an irreducible component of Chow-variety of \( X \) of effective algebraic cycles of dimension \( r \geq 0 \).

**Definition 2.1.** (Archimedean height pairing, [1], [6]) Assume \( X \) is equipped with a Kähler metric. Let \( A \in \mathcal{C}_p(X), B \in \mathcal{C}_q(X) \). Assume \( |A| \cap |B| = \emptyset \). Define the Archimedean height pairing \( \langle A, B \rangle \) by the integral

\[
\int_A G_B.
\]

where \( G_B \) is a normalized Green’s form of \( B \). A normalized Green’s form of \( B \) is a smooth form on \( X \| \; |B| \) and \( L^1 \) on \( X \) that satisfies

1. \( \text{dd}^c < G_b > = \delta_B - < \omega_B > \) where \( < \cdot > \) is the notation for currents, \( \delta_B \) is the current of integration over \( B \), \( \omega_B \) is the harmonic, Poincaré dual to \( B \).
2. Harmonic projection of the current \( < G_B > \) is zero.

**Theorem 2.2.** ([13]). Let \( B \in \mathbb{Z}_q(X) \). Assume that there is a cycle \( A \in \mathcal{C}_p(X) \) such that \( |A| \cap |B| = \emptyset \). Then \( B \) determines a rational section \( s_B \) of some metrized line bundle \( \mathcal{L}_{[B]} \) such that the Archimedean height pairing, as a real function on \( \mathcal{C}_p \setminus \text{div}(s_B) \), is

\[
\langle A, B \rangle = \frac{1}{p!} \log ||s_B(A)||^2.
\] (2.1)

The line bundle \( \mathcal{L}_{[B]} \) is called “Mazur’s incidence line bundle”, and \( \text{div}(s_B) \) is called incidence divisor of \( B \), denoted by \( \mathcal{D}_B \).

By pulling back Mazur’s incidence line bundle, we obtain

**Corollary 2.3.** Let \( T_p \) be a smooth projective variety with a regular map

\[
\phi : T_p \to \mathcal{C}_p(X),
\] (2.2)

whose image does not lie in \( \text{div}(s_B) \). Then there is a rational section \( s'_B \) of some metrized line bundle \( \mathcal{L}'_{[B]} \) on \( T_p \) such that the Archimedean height pairing, as a real function on \( T_p \setminus \text{div}(s'_B) \), is

\[
\langle A, B \rangle = \log ||s'_B(A)||^2.
\] (2.3)

**Remark.** There is an easy, but non-trivial assertion: since \( T_p \) is smooth,

\[
\frac{\phi^*(\mathcal{L}_B)}{p!}
\]
is also a line bundle on $T_p$.

**Definition 2.4.**

1. The divisor $\text{div}(s_B')$ will be denoted by $\mathcal{D}_B$, called the incidence divisor of $B$.
2. If $\mathcal{D}_B$ is zero in the Chow group $CH(T_p)$ for all $T_p$, we say $B$ is incidence equivalent to zero.

Definition (2) coincides with Griffiths’ mentioned in introduction.

**Remark** In the context, the ambient space $X$ and the parameter space $T$ for the incidence divisor $\mathcal{D}_B$ will be omitted for simplicity.

Next we see the map $[B] \to \mathcal{L}[B]$ factors through intermediate Jacobian.

Using the result of ([10]), we obtained that

**Corollary 2.5.** *(Theorem 3.15, [14])*

Let

$$\mathcal{L} : CH^{n-q}_{\text{hom}}(X) \to \text{Pic}^0(C_p(X))$$

$$[B] \to \mathcal{L}_[B].$$

Then $\mathcal{L}$ is equal to the following composition of maps:

$$CH^{n-q}_{\text{hom}}(X) \xrightarrow{AJ} J_q(X) \xrightarrow{f} \text{Pic}^0(J_p(X)) \xrightarrow{p!} \text{Pic}^0(J_p(X)) \xrightarrow{AJ^*} \text{Pic}^0(C_p(X)).$$

where $f$ is induced from the Poincaré line bundle, $AJ^*$ is induced from the Abel-Jacobi map $AJ$ with a fixed base point in $C_p(X)$ and $p!$ is the map $\otimes p!$ on the line bundles.

### 3 Decomposition of cohomology

Let’s go back to the setting in the introduction.

Recall

$$Z_i \subset Z^{p+1}(T_i \times X)$$

be the correspondence as before. Let

$$Z^2 = Z_1 \times Z_2 \subset (T_1 \times X) \times (T_2 \times X).$$

be the Cartesian product. The projections from $T_2 \times X$ to $T_2$ and $X$ will be denoted by $p_1$ and $p_2$ respectively. Let $(p_1)_*, (p_2)^*$ denote the pull-back and push-forward through the intersection with $Z_2$.

Let $(Z_i)_L, Z_L^2$ be the operators that push the cycles on $T_i$ and $T_1 \times T_2$.
to

\[ X \text{ and } X \times X \]

respectively. Let \((Z_i)_R, I_i^2\) be operators that push the differential forms, currents or algebraic cycles from \(X\) and \(X \times X\) to \(T_i\) and \(T_1 \times T_2\) respectively. Without a confusion, we'll use the same notations \((I_i)_L, (I_i)_R, \ldots\) for the operations on their various equivalent classes.

**Lemma 3.1.** Let \(\gamma_i, i = 1, 2\) be two real cycles that represent two classes in \(H_1(T_i; \mathbb{Z})\) respectively. Then

\[
\int \left( (Z_1)_L(\gamma_1), (Z_2)_L(\gamma_2) \right)_X = \int \left( \gamma_1 \times \gamma_2, D_\Delta(1, 2) \right)_{T_1 \times T_2},
\]

(3.3)

where \(\Delta\) is the diagonal of \(X \times X\), \(D_\Delta(1, 2)\) is the incidence divisor of \(\Delta\) over the parameter space \(T_1 \times T_2\) defined in section 2 (corollary 2.3), and \(\gamma_1 \times \gamma_2\) is the tensor product in the Künneth decomposition.

**Definition 3.2.** Using the notation in the lemma 3.1, we define

\[
F(\gamma_1, \gamma_2) = \int \left( \gamma_1 \times \gamma_2, D_\Delta(1, 2) \right)_{T_1 \times T_2}.
\]

(3.4)

Then \(F\) is a bilinear form of integer values. The Poincaré dual of \(F\) on the cohomology is also denoted by \(F\).

**Proof.** of lemma 3.2: Using the “reduction to diagonal” for intersection pairing

\[
\int \left( (Z_1)_L(\gamma_1), (Z_2)_L(\gamma_2) \right)_X = \int \left( (Z_2^2)_L(\gamma_1 \times \gamma_2), [\Delta] \right)_{X \times X},
\]

(3.5)

we obtain that

\[
\int \left( (Z_1)_L(\gamma_1), (Z_2)_L(\gamma_2) \right)_X = \int \left( \gamma_1 \times \gamma_2, Z_R^2(\Delta) \right)_{T_1 \times T_2}.
\]

(3.6)

As it is known in the construction of the incidence divisor [13] that, when \(T_1 \times T_2\) is smooth, \(Z_R^2(\Delta)\) is the incidence divisor \(D_\Delta(1, 2)\).

This completes the proof.

**Definition 3.3.** As before we let

\(B \in CH_{alg}^{p+1}(X)\).

Next we define the topological linking number map \(l_B\).

\[
F^{p+1}H^{2p+1}(X) \xrightarrow{l_B} C \xrightarrow{\int_B} \int_{F^p} h
\]

(3.7)
where $\Gamma_B$ is a simplicial chain such that $\partial \Gamma_B = B$, and $\Lambda_C$ is the integer lattice of the complex plane. So $\frac{\Gamma}{\Lambda}$ is an elliptic curve. This map is well-defined by Hodge theory.

The map $B \to l_B$ is the Abel-Jacobi map, where $l_B$ is an element of $(F_{p+1}^p H_{2p+1}^2(X))^\ast$.

**Lemma 3.4.** Assume the assumption 1.2. There exists a subgroup $W_Q$ of $H_{2p+1}^2(X; \mathbb{Q})$ such that

$$H_{2p+1}^2(X; \mathbb{Q}) = H_{2p+1}^2(X; \mathbb{Q}) \oplus W_Q,$$

(or)

$$H_{2p+1}^2(X; \mathbb{Q}) = H_{2p+1}^2(X; \mathbb{Q}) \oplus W_Q$$

and

$$\text{int}(h_1, h_2)_X = 0 \quad (3.9)$$

for all

$$h_1 \in H_{2p+1}^2(X; \mathbb{Q}), h_2 \in W_Q.$$

**Proof.**

Let

$$W_Q = \{ h_1 \in H_{2p+1}^2(X; \mathbb{Q}) : \text{int}(h_1, h_2)_X = 0, \text{ for all } h_2 \in H_{2p+1}^2(X; \mathbb{Q}) \}.$$  

Using the linearity, we obtain

$$\text{cod}(W_Q) \leq \text{dim}(H_{2p+1}^2(X; \mathbb{Q})).$$  

(3.10)

Hence

$$\text{cod}(W_Q) \leq \text{dim}(H_{2p+1}^2(X; \mathbb{Q})).$$  

(3.11)

Therefore

$$\text{dim}(W_Q) \geq \text{dim}(H_{2p+1}^2(X; \mathbb{Q})) - \text{dim}(H_{2p+1}^2(X; \mathbb{Q})).$$  

(3.12)

Because $F(\gamma_1, \gamma_2)$ is non-degenerate on left, $W_Q$ can not have non-zero intersection with $H_{2p+1}^2(X; \mathbb{Q})$. Then using the dimension count for vector spaces, we must have

$$H_{2p+1}^2(X; \mathbb{Q}) = H_{2p+1}^2(X; \mathbb{Q}) \oplus W_Q.$$  

(3.13)

We complete the proof.

The following lemma is the equivalence of Griffiths' conjecture:
Lemma 3.5. Let $B \in CH_{\text{alg}}^{p+1}(X)$ and $T_1, T_2$ be families of cycles that satisfy lemma 3.4. Then there exists a positive integer $m$ such that the following two sets in $C_{\Lambda C}$ satisfy

$$\{l_m B(\alpha(H^{2p+1}(X;Z)))\} \subset \{l_B(\alpha(H^{2p+1}_{T_1}(X;Z)))\},$$

(3.14)

where $\alpha$ is the projection via the Hodge decomposition

$$H^{2p+1}(X;\mathbb{C}) \xrightarrow{\sim} F^{p+1}H^{2p+1}(X).$$

(3.15)

Proof. Let's work with cycles instead of cycle classes. So let $B \in Z_p(X)$ be algebraically equivalent to zero. Recall $T_2$ is a smooth irreducible, non-rational projective curve $T_2$ that parametrizes a family of $p$-cycles $Z(t) \in Z_p(Z), t \in T_2$ such that $B = Z(t_1) - Z(t_0)$ where $t_1 \neq t_0$ are in $T_2$.

Let $\phi^j \in H^{2p+1}(X;\mathbb{Z})$ be any one of the element in a basis. By lemma 3.4,

$$\phi^j = \phi_1^j + \phi_2^j$$

(3.16)

where $\phi_1^j \in H^{2p+1}_{T_1}(X;\mathbb{Q}), \phi_2^j \in \mathbb{W}_Q$. Thus there is an integer $m_j$ such that

$$m_j \phi_1^j \in H^{2p+1}_{T_1}(X;\mathbb{Z}).$$

Notice that

$$\int_G \phi_2^j = 0.$$

(3.17)

for any $G = \nu_{T_2}(\gamma) \in H^{2p+1}_{T_2}(X;\mathbb{Q})$ (where $\gamma \in H_1(T_2;\mathbb{Q})$). Hence

$$\int_\gamma (p_1)_*(p_2)^*(\phi_2^j) = \int_G \phi_2^j = 0.$$

(3.18)

Thus the intersection number on $T_2$

$$\text{int}(p_1)_*(p_2)^*(\phi_2^j), \gamma)_{T_2} = 0$$

(3.19)

for all $\gamma \in H_1(T_2;\mathbb{C})$. By the non-degeneracy of the intersection numbers on $T_2$,

$$(p_1)_*(p_2)^*(\phi_2^j)$$

(3.20)

is an exact form.

Then by the corollary 4 in [4] (see section 7, Appendix),

$$\int_{\Gamma_B} \phi_2^j = 0,$$

(3.21)

where $\Gamma_B$ is a chain obtained from a real path in $T_2$ connecting the point $t_1$ and the fixed $t_0$, in particular,

$$\partial \Gamma_B = Z(t_1) - Z(t_0).$$
This shows that \((p_1)_*(p_2)^*(\phi^j_2)\) is a zero form on \(T_2\). Hence
\[(p_1)_*(p_2)^*(\alpha(\phi^j_2))\]
is also a zero form. Then
\[l_{m_j}(\alpha(\phi^j)) = l_B(\alpha(m_j\phi^j_1)).\]
Since there are only finitely many such \(j\), we let \(m = \prod_j m_j\). Then the cycle \(mB\) satisfies lemma 3.4. We complete the proof.

4 Griffiths’ conjecture under the assumption.

In this section we prove theorem 1.3 under the assumption 1.2.

Proof. In this proof, the cycle \(B\) is only required to be incidence equivalent to zero until the last sentence. But in the last setence which is right after (4.6), it crucial for \(T_2\) to be through \(B\). Therefore we assume \(T_2\) goes through \(B\). We apply the long sequence (2.5) to the family of cycles, \(T_1\) to obtain the same long sequence with the composition map still denoted by \(L\):

\[CH_{alg}^{p+1}(X) \xrightarrow{AJ} J^{p+1}(X) \xrightarrow{J} Pic^0(J^{p+1}(X)) \xrightarrow{AJ^*} Pic^0(T_1) \]  
(4.1)

(where the last \(AJ\) has a fixed base point \(t_0 \in T_1\). The cycle \(B\) is irrelevant to this sequence. ). Next we analyze two line bundles, \(L_B\) over \(T_1\) and \(f \circ AJ(B)\) over

\[Pic^0(J^{p+1}(X)) \simeq J^{p+1}(X).\]

By the assumption \(L_B\) is a trivial bundle. But \(f \circ AJ(B)\) may not be trivial over the entire Jacobian \(J^{p+1}(X)\). However we would like to show that restricted to the sub Abelian variety

\[\frac{(F_{p+1}H_{T_1}^{2p+1}(X))}{H_{2p+1}^{T_1}(X; \mathbb{Z})} = J_{T_1}^{2p+1} \subset J^{p+1}(X)\]  
(4.2)

it is a trivial bundle. First we assure that

\[\frac{(F_{p+1}H_{T_1}^{2p+1}(X))}{H_{2p+1}^{T_1}(X; \mathbb{Z})} = J_{T_1}^{2p+1} \subset J^{p+1}(X)\]  
(4.3)

exists because \(H_{2p+1}^{T_1}(X; \mathbb{Z})\) induces a full lattice of

\[(F_{p+1}H_{T_1}^{2p+1}(X))^*.\]

The trivial bundle \(L_B\) corresponds to the trivial representation of the \(\pi_1(T_1)\), i.e. the \(U(1)\)-representation of the flat bundle is the trivial map

\[\eta_1 : \pi_1(T_1) \to \{1\} \subset U(1).\]  
(4.4)

On the other hand \(f \circ AJ(B)\) is also a flat bundle over the entire Jacobian \(J^{p+1}(X)\), which includes a representation of

\[\eta_2 : H_{2p+1}^{T_1}(X; \mathbb{Z}) \to U(1).\]  
(4.5)
By (4.1), \( L_B \) is the pull-back of \( f \circ AJ(B) \). This leads to the commutative diagram

\[
\begin{array}{c}
\pi_1(T_1) \rightarrow H_{2p+1}^1(X; \mathbb{Z}) \\
\eta_1 \downarrow \quad \eta_2 \downarrow \\
U(1) \quad U(1),
\end{array}
\]  

(4.6)

Since \( \eta_1 \) is trivial, \( \eta_2 \) must be trivial. Then the bundle

\[
(f \circ AJ(B))_{|J_{p+1}^1(X)}
\]

is trivial. By the definition of this bundle

\[
\int_{\Gamma_n} \alpha(h) \in \Lambda_C.
\]

(4.7)

where \( h \in H_{1p+1}^2(X; \mathbb{Z}) \). By lemma 3.5, exists a positive integer \( m \) such that

\[
l_mB(\alpha(\phi)) = 0
\]

in \( \frac{C}{\Lambda_C} \) for all \( \phi \in H^{2p+1}(X; \mathbb{Z}) \).

Notice the linear map

\[
F^{p+1}H^{2p+1}(X) \rightarrow \frac{C}{\Lambda_C} \\
\phi \rightarrow l_mB(\phi)
\]

(4.8)

is the Abel-Jacobi image of \( [mB] \). Thus \( AJ([mB]) = 0 \).

5 Non-vanishing intersection pairing

In this section, we would like to prove the following,

**Proposition 5.1.** Let \( G(\cdot, \cdot) \) be the restricted bilinear intersection form to

\[
H_a^{2p+1}(X; \mathbb{Z}) \times H_a^{2p+1}(X; \mathbb{Z}).
\]

If \( J_a^{p+1}(X) \) is non-zero, \( G(\cdot, \cdot) \) is non-zero.

**Remark.** At the first glance proposition 5.1 is not sufficient even though it is necessary for the claim 1.1. However the content of this paper will show that it is also a sufficient condition for the claim 1.1.

**Proof.** It is well-known (see proposition 1.2, [12]) that there are an Abelian variety \( A \) and a cycle

\[
Z \in CH^{p+1}(A \times X)
\]

such that the composition map \( \zeta \)

\[
A \xrightarrow{Z(a) - Z(0)} CH_{alg}^{p+1}(X) \xrightarrow{AJ} J_a^{p+1}(X)
\]

(5.2)
is an isogeny. Let $\theta \in A$ such that
\[ \zeta(\theta) \neq 0. \tag{5.3} \]
Let $T$ be a smooth, irreducible curve in $A$ that passes through $\theta, 0$. Let
\[ F_{p+1}H_{2p+1}^2(X) = H_{2p+1}^2(X; \mathbb{C}) \cap F_{p+1}H_{2p+1}^2(X). \tag{5.4} \]
Notice
\[ \frac{F_{p+1}H_{2p+1}^2(X)}{H_{2p+1}^2(X; \mathbb{C})} = (F_{p+1}H_{2p+1}^2(X))^* \cong J_{p+1}^p(X). \tag{5.5} \]
(The isomorphism is obtained through the projection in the linear space $F_{p+1}H_{2p+1}^2(X)$.) Under this isomorphism the Abel-Jacobi map can be expressed as
\[ B \rightarrow \int_{\Gamma_B} (\cdot) \tag{5.6} \]
where the map $\int_{\Gamma_B} (\cdot)$ is restricted to the subspace
\[ F_{p+1}H_{2p+1}^2(X). \]
Let $\omega$ be any integral form in
\[ H_{2p+1}^2(X; \mathbb{Z}). \]
We use the same notations $p_1, p_2$ to denote the projections from $T \times X$ to $T$ and $X$ respectively. Let $Z_T = Z \cdot (T \times X)$. Let $(p_1)_*, (p_2)^*$ denote the pull-back and push-forward through the intersection with $Z_T$. Suppose proposition 5.1 is false. Then for any $\gamma \in H_1(T; \mathbb{Z})$,
\[ \int_{\gamma} (p_1)_* \circ p_2^* (\alpha(\omega)) = 0, \tag{5.7} \]
Hence the cohomology of $(p_1)_* \circ p_2^* (\alpha(\omega))$ must be zero. Then by the corollary 7.1 in Appendix,
\[ \int_{\Gamma_\theta} \alpha(\omega) \tag{5.8} \]
is also zero where $\partial \Gamma_\theta = Z(\theta) - Z(0)$ and $\Gamma_\theta$ is reduced from a real 1-chain in $T$. Hence
\[ \int_{\Gamma_\theta} \alpha(\omega) \in \lambda_\mathbb{Z} \tag{5.9} \]
for any real chain $\Gamma_\theta$ in $X$ whose boundary is $Z(\theta) - Z(0)$. By the formula (5.5), $\zeta(\theta) = 0$. This is a contradiction. The lemma is proved.

Recall that there is a Poincaré line bundle $\mathcal{L}$ over
\[ J_{p+1}^p(X) \times Pic^0(J_{p+1}^p(X)) \tag{5.10} \]
Then the intersection pairing \( G(\cdot, \cdot) \) induces a section \( G_h(\cdot) \) of the line bundle \( \mathcal{L} \). It composes with a duality map to give a homomorphism \( f^- \):

\[
J^{p+1}(X) \xrightarrow{f^-} J^{p+1}(X) \xrightarrow{\sim} \text{Pic}^0(J^{p+1}(X))
\] (5.11)

This homomorphism \( f^{-1} \) is essentially the same map \( f \) in long sequence (2.5). To see this we observe in our setting \( p = q = \frac{n+1}{2} \).

\[
\text{Pic}^0(J_q(X)) \cong \text{Pic}^0(\text{Pic}^0(J^{p+1}(X))) \xrightarrow{\sim} J^{p+1}(X).
\] (5.12)

Proposition 5.1 implies that \( f, f^{-1} \) are non-zero maps.

### 6 Incidence correspondence

This is the step 2. Recall that there are an Abelian variety \( A \) and a cycle

\[
Z \in CH^{p+1}(A \times X)
\]

such that the composition map \( \varsigma \)

\[
A \xrightarrow{Z(a) - Z(0)} CH^{p+1}_{\text{alg}}(X) \xrightarrow{A^T} J^{p+1}_a(X)
\] (6.1)

is an isogeny.

**Theorem 6.1.** If \( J^{p+1}_a(X) \neq 0 \), there exist families \( T_1, T_2 \) of \( p \)-cycles as in the introduction such that the assumption 1.2 is correct.

**Proof.** Let \( B \in CH^{p+1}_{\text{inc}}(X) \). The goal of this section is to construct two smooth curves \( T_1, T_2 \) that parametrize cycles of dimension \( p \) in \( X \) such that

1. \( T_2 \) is through \( B \in CH^{p+1}_{\text{inc}}(X) \),
2. the correspondence induces the injective homomorphism on homology,

\[
H_1(T_1; \mathbb{Z}) \xrightarrow{\partial_1^{[1,2]}} H_1(T_2; \mathbb{Z})
\] (6.2)

In the following, we show how to obtain these curves via an isomorphism of a complex linear space. The property of these curves is that they must be “orthogonal” to \( CH^{p+1}_{\text{inc}}(X) \).

We start with a detailed construction (5.11) of the duality map \( f^- \) induced from Poincaré line bundle

\[
J^{p+1}(X) \xrightarrow{f^-} J^{p+1}(X) \cong \text{Pic}^0(J^{p+1}(X)).
\] (6.3)

\(^2\)For non “orthogonal sets”, \( T_1, T_2 \), the above claim is not correct.
We recall the construction of the intermediate Jacobian first. The real linear space $H^{2p+1}(X; \mathbb{R})$ has a complex structure such that
\[ H^{2p+1}(X; \mathbb{C}) = V \oplus \bar{V} \] (6.4)
for
\[ V = F^{p+1}H^{2p+1}(X), \]
and
\[ J^{p+1}(X) = \frac{V}{\Lambda}, \] (6.5)
where $\Lambda$ is the projection of $H^{2p+1}(X; \mathbb{Z})$ into $V$ in the decomposition (6.4). Then
\[ \text{Pic}^0(J^{p+1}(X)) = \frac{\bar{V}^*}{\Lambda^*}. \] (6.6)
where $\Lambda^* = \text{hom}(\Lambda, \mathbb{Z})$.

Fix an isomorphism
\[ \bar{V}^* \rightarrow V \] (6.7)
through a fixed basis. Composing $f$ with this isomorphism, we obtain a homomorphism $f^-$ in (5.11),
\[ J^{p+1}(X) \xrightarrow{f^-} J^{p+1}(X). \] (6.8)

The homomorphism $f^-$ is compatible with the similar linear map on linear space. It is constructed in the similar way. There is the sesquilinear intersection map $int_h(\cdot)_X$,
\[ V \times V \rightarrow \mathbb{C} \] (6.9)
The map induces the linear map
\[ V \rightarrow \bar{V}^* \] (6.10)
which similarly can be composed with (6.7) to have a linear homomorphism $f^+$,
\[ V \xrightarrow{f^+} V. \] (6.11)

Therefore we have the diagram
\[ \begin{array}{ccc}
V & \xrightarrow{f^+} & V \\
\downarrow & & \downarrow \\
J^{p+1}(X) & \xrightarrow{f^-} & J^{p+1}(X)
\end{array} \] (6.12)

We observe that $F^{p+1}H^{2p+1}_a(X)$ is a sub linear space of $V$. Hence with a suitable choice of the isomorphism (6.7), $f^+$ naturally induces a restricted morphism
\[ \begin{array}{ccc}
F^{p+1}H^{2p+1}_a(X) & \xrightarrow{f^+} & F^{p+1}H^{2p+1}_a(X) \\
\downarrow & & \downarrow \\
F^{p+1}H^{2p+1}(X) & \xrightarrow{f^+} & F^{p+1}H^{2p+1}(X)
\end{array} \] (6.13)
Then we restricted the diagram (6.12) to algebraic parts to obtain another diagram,

\[
\begin{array}{ccc}
F_{p+1}H^{2p+1}_a(X) & \xrightarrow{f_a} & F_{p+1}H^{2p+1}_a(X) \\
\downarrow \pi_1 & & \downarrow \pi_1 \\
J_{a+1}(X) & \xrightarrow{f_a} & J_{a+1}(X)
\end{array}
\]  
(6.14)

where \(F_{p+1}H^{2p+1}_a(X)\) is proved to be a sub-Hodge structure ([11]).

At last we obtain an isomorphism

\[
\begin{array}{ccc}
\mathbb{C}^m & \xrightarrow{G} & \mathbb{C}^m \\
\downarrow F_{p+1}H^{2p+1}_a \ker (f_a) & & \downarrow F_{p+1}H^{2p+1}_a \ker (f_a) \\
J_{p+1}(X) & \xrightarrow{Q_1(G)} & J_{p+1}(X)
\end{array}
\]  
(6.15)

We have a vector space decomposition

\[
F_{p+1}H^{2p+1}_a(X) = \frac{F_{p+1}H^{2p+1}_a(X)}{\ker (f_a)} \oplus \ker (f_a^+)
\]  
(6.16)

where \(\frac{F_{p+1}H^{2p+1}_a(X)}{\ker (f_a)}\) denotes a subspace of \(F_{p+1}H^{2p+1}_a(X)\). Then we obtain a linear map \(\beta\):

\[
\begin{array}{ccc}
\frac{F_{p+1}H^{2p+1}_a(X)}{\ker (f_a)} & \xrightarrow{\beta} & F_{p+1}H^{2p+1}_a(X) \\
\downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\
F_{p+1}H^{2p+1}_a(X) & \xrightarrow{Q_1(G)} & F_{p+1}H^{2p+1}_a(X)
\end{array}
\]  
(6.17)

whose composition with linear projection \(Pr\)

\[
\begin{array}{ccc}
F_{p+1}H^{2p+1}_a(X) & \xrightarrow{Pr} & \frac{F_{p+1}H^{2p+1}_a(X)}{\ker (f_a)} \\
\pi \downarrow & & \pi \downarrow \\
J_{a+1}(X) & \xrightarrow{f_a} & J_{a+1}(X)
\end{array}
\]  
(6.18)

is the identity. Let’s describe the process to obtain \(T_1, T_2\). This will be done through the linear spaces. We start with a diagram of maps.

\[
\begin{array}{ccc}
\frac{F_{p+1}H^{2p+1}_a(X)}{\ker (f_a)} & \xrightarrow{G} & \frac{F_{p+1}H^{2p+1}_a(X)}{\ker (f_a)} \\
\beta \downarrow & & \beta \downarrow \\
F_{p+1}H^{2p+1}_a(X) & \xrightarrow{Q_1(G)} & F_{p+1}H^{2p+1}_a(X) \\
\pi \downarrow & & \pi \downarrow \\
J_{a+1}(X) & \xrightarrow{f_a} & J_{a+1}(X) \\
\end{array}
\]  
(6.19)

where \(Q_1(G), Q_1'(G)\) will be described immediately in the following, and \(Q_1'(G)\) exists because of the existence of the inverse \(\beta\).
Next we use the property of the top row of linear spaces to obtain curves in the
bottom rows of Abelian varieties. First by proposition 5.1, we notice $G$ can’t be a
zero isomorphism \footnote{This does not mean the restricted intersection pairing is non-degenerate. So we need to continue.}. Thus $\det(G) \neq 0$. Let

$$P(\lambda)$$  \hspace{1cm} (6.20)

be the characteristic polynomial of $G$. Let $Q_1(G)$ be the homomorphism between $\mathbb{C}^n$ such that

$$Q_1(G) = Q_2(G) \cdot G^{-1}$$  \hspace{1cm} (6.21)

where $Q_2(G) = P(G) - \det(G)I$ is another homomorphism with identity matrix $I$. Since the intersection pairing holds in the field of rational numbers $\mathbb{Q}$. All homomorphisms above are the the base extension of homomorphisms over $\mathbb{Q}$, i.e. under the integral bases from integral cohomology, the matrices for these homomorphisms have rational entries. Since $P(G) = 0$,

$$Q_2(G) = -\det(G)I$$  \hspace{1cm} (6.22)

Let $m$ be an positive integer such that the product of $m$ with all coefficients of $P(\lambda)$, and $\det(G)$ are integers. Now we let $B' \subset J^p_{a+1}$ and incidence equivalent to zero, i.e. $B = Z^{-1}(B')$ is incidence to zero. So $f_a^-(B') = 0$. We start with the space $J^p_{a+1} \ni B'$ in the middle of the bottom row of (6.19). Choose a smooth curve $C \subset J^p_{a+1}$ such that it passes through $B'$ and $\pi^{-1}(C)$ meets $\ker(f_a^+)$ properly (Note $\pi^{-1}(C)$ is the “orthogonal” set mentioned in the introduction). Then the projection $\text{Pr}(\pi^{-1}(C))$ does not lie in $\ker(f_a^+)$ entirely. We also obtain the image

$$Q_1(G) \circ G(\text{Pr}(\pi^{-1}(C_f)))$$  \hspace{1cm} (6.23)

where $C_f$ is $f^{-1}(C)$. By the choice,

$$Q_1(G) \circ G(\text{Pr}(\pi^{-1}(C_f)))$$

is just the dilation of $\text{Pr}(\pi^{-1}(C_f))$, obtained by multiplying it with the non-zero integer

$$-m \cdot \det(G),$$

i.e.

$$Q_1(G) \circ G(\text{Pr}(\pi^{-1}(c_f))) = -m \cdot \det(G) \cdot \text{Pr}(\pi^{-1}(c_f))$$  \hspace{1cm} (6.24)

for any $c_f \in C_f$.

Now using the diagram (6.19), we define $T_1$ to be a smooth curve in

$$Z^{-1}(AJ^{-1}(C_f)),$$

and $T_2$ be to be a smooth curve in

$$Z^{-1}(AJ^{-1}(C)).$$
(Take a smooth resolution of $T_1$ if necessary).

Now we concentrate on the curves $T_1, T_2$. We should note $T_1$ can’t be rational curves because $H_{2p+1}^{T_1}(X; \mathbb{Q})$ are not zero. Secondly $T_2$ is obtained through the image of $T_1$ it is clear that

$$\dim(H_{2p+1}^{T_2}(X; \mathbb{Q})) \leq \dim(H_{2p+1}^{T_1}(X; \mathbb{Q})).$$

(6.25)

Let $D(2,1)$ be the divisor on $T_2 \times T_1$ whose algebraic correspondence makes the following diagram commute

$$
\begin{array}{ccc}
CH^1(T_2) & \xrightarrow{D(2,1)} & CH^1(T_1) \\
\downarrow & & \downarrow \\
J_a^{p+1}(X) & \xrightarrow{Q_1^*(G_1)} & J_a^{p+1}(X)
\end{array}
$$

(6.26)

where $Q_1^*(G_1)$ is the map induced from $Q_1(G_1)$ in the diagram (6.19). Because the group structure on the Picard group $Pic^0(J_a^{p+1}(X)) = J_a^{p+1}(X)$ is exactly the group structure of the Abelian group $J_a^{p+1}(X)$, the equation (6.24) and (6.16) imply that

$$
\left( mD(2,1) \circ D_\Delta(1,2) \right)([t] - [t_0]) + m \cdot det(G)([t] - [t_0])
$$

is the trivial line bundle on $T_1$. Since $m \cdot det(G)$ is an integer and $T_1$ is not a rational curve, we can use the §2, 5 in [9], in which the integer $m \cdot det(G)$ is called the valence of the correspondence

$$mD(2,1) \circ D_\Delta(1,2)$$

from $T_1$ to $T_1$. Then

$$mD(2,1) \circ D_\Delta(1,2)$$

is homologous to

$$\xi_1([t_1] \times X) + \xi_2[X \times \{t_2\}] - m \cdot det(G)\Delta_{T_1}$$

(6.28)

where $\xi_i, i = 1, 2$ are some integers and $\Delta_{T_1}$ is the diagonal of $T_1 \times T_1$.

The formula (6.28) says that the correspondence of

$$mD(2,1) \circ D_\Delta(1,2)$$

on homology

$$H_1(X; \mathbb{Z}) \to H_1(X; \mathbb{Z})$$

(6.29)

under a basis is just

$$- m \cdot det(G)I$$

(6.30)

where $I$ is the identity matrix. Hence it is injective. Therefore $D_\Delta(1,2)$ is also injective. This completes the proof.
7 Appendix

At last for its completeness we list the corollary 4 in [4] proved by Caibăr and Clemens. Consider the correspondence in the introduction

\[
\begin{array}{c}
\xymatrix{T & Z \ar@{-}[ld]_{p_1} \ar@{-}[rd]^{p_2} & X.}
\end{array}
\] (7.1)

Corollary 7.1. (Caibăr and Clemens).
Let \( D \in CH^1_{\text{hom}}(T) \) be a zero-cycle of degree zero. Then the Abel-Jacobi image

\[ AJ(D) \in J^{p+1}(X) \] (7.2)

pairs to zero with

\[
\ker \left( (p_1)_* (p_2)^* : \mathcal{H}^{2p+1}(X; \mathbb{R}) \rightarrow H^1(T; \mathbb{R}) \right),
\] (7.3)

where \( \mathcal{H}^{2p+1}(X; \mathbb{R}) \) is the set of harmonic forms for any Kähler metric on \( X \).

The proof is written for Weil’s Jacobian. But it is correct for Griffiths’ Jacobian too, and the proof is identical.

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