Critical properties of the $XXZ$ chain in 
external staggered magnetic field

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Abstract. We comment on the recent work of Alcaraz and Malvezzi [1995 J. Phys. A: Math. Gen. 19 1521] for the critical properties of the $S = 1/2$ $XXZ$ chain in staggered magnetic field. The method of determining the phase boundary from the finite-size numerical data is also discussed.

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Recently Alcaraz and Malvezzi (AM) [1] studied the ground-state phase diagram of the $S = 1/2$ XXZ spin chain in external homogeneous and staggered magnetic fields described by

$$H(\Delta, h, h_s) = -\frac{1}{2} \sum_{i=1}^{M} \{ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + 2[h + (-1)^i h_s] \sigma_i^z \}$$

where $\sigma_i^x, \sigma_i^y$ and $\sigma_i^z$ are Pauli matrices, $\Delta$ is the anisotropy parameter, and $h$ ($h_s$) is the uniform (staggered) magnetic field. They found that the ground-state phase diagram is composed of the antiferromagnetic (AF) phase, the massless (ML) phase and the ferromagnetic (FE) phase. Although we agree their schematic phase diagram of $H(\Delta, h, h_s)$ (figure 5 of [1]), we want to make comments on the nature of the ground-state phase transition and also on the method to determine the phase boundary from the finite-size numerical data.

First we discuss the nature of the ground-state phase transition. When $h \neq 0$, the uniform magnetic field breaks the spin reversal symmetry held in the $h = 0$ case, so that the nature of the phase transition may be different from that in the $h = 0$ case. Throughout this comment we restrict ourselves to the $h = 0$ case on which AM focused. Figure 1 shows the schematic phase diagram of $H(\Delta, h = 0, h_s)$, which is essentially the same as AM’s figure 3. They stated that the phase transition between the AF phase and the ML phase (path 1) is of the second order. We believe, however, it is of the infinite order, i.e., of the Kosterlitz-Thouless (KT) type. The operator coupled to the staggered magnetic field is irrelevant in the ML region and is relevant in the AF region. The excitation spectrum is either massless or massive depending on whether this operator is irrelevant or relevant. This mass-generating mechanism is the same as that of the sine-Gordon model, as AM themselves denoted. Thus the AF-ML transition of the path 1 is of the KT type, which is seen from the well-known properties of the sine-Gordon model [2].

We can also observe the AF-ML transition along the path 2. This transition is different from that of path 1, because it is due to the vanishing of the coefficient (which is proportional to the magnitude of the staggered field) of the relevant operator coupled to the staggered magnetic field. Thus this transition is of the second order and its critical exponents vary continuously.

Next we discuss on the method to determine the AF-ML phase boundary from the finite-size numerical data obtained by the numerical diagonalization of the Hamiltonian. AM used the $M \to \infty$ extrapolation of the sequences $(\Delta^{(M)}, h^{(M)}, h_s^{(M)})$ ($M = 2, 4, \cdots$) obtained by solving the so-called phenomenological renormalization group (PRG) equation

$$MG_M(\Delta, h, h_s) = (M - 2)G_{M-2}(\Delta, h, h_s)$$

where $G_M(\Delta, h, h_s)$ is the gap of the Hamiltonian (1) with $M$ sites. At the fixed point of the PRG equation (2), the gap $G_M$ behaves as

$$G_M \sim M^{-1}$$

in the lowest order of $M^{-1}$. If the transition is of the second order, the PRG method leads to the correct transition point because the system is massless and equation (3) holds only at the transition point. In case of the KT transition, on the other hand, care must be taken for the application of the PRG method. Since the present system is massless not only at the AF-ML transition line but also in the whole of the ML region, the PRG relation (2) is satisfied in the lowest order of $M^{-1}$ in the whole of the ML region. Where is the fixed point of the PRG equation? It is controlled by the lowest order correction to equation (3) which may comes from the operator coupled to the staggered magnetic field. Thus the fixed point of the PRG equation locates at the point where the staggered field vanishes. If this is the case, the transition point obtained through the PRG method is brought over from the AF-ML point to the $h_s = 0$ line. Then the simple application of the PRG method to the KT transition is dangerous. In the present problem, of course, there may be other corrections which make the situation more complicated.

Let us demonstrate that the PRG solution may lead to an incorrect critical point for the KT transition [3]. When $h = h_s = 0$, as is well-known, the Hamiltonian (1) is exactly solvable by the use of the Bethe ansatz method. Its ground-state is either the AF state or the ML state depending on whether $\Delta < -1$ or $-1 \leq \Delta < 1$. The excitation gap in the AF state behaves as [4,5]

$$G(\Delta) \simeq 8\pi \exp \left(-\frac{\pi^2}{2\sqrt{2(\Delta - 1)}}\right) \quad (\Delta \to -1 - 0)$$
which indicates that this AF-ML transition at $\Delta = -1$ is of the KT type. If we apply the PRG method to the finite-size numerical data of the excitation gap, we obtain $\Delta_c = -0.50706$ ($M = 10, 12$) and $\Delta_c = -0.47564$ ($M = 18, 20$). Therefore the critical value of $\Delta_c$ obtained from the PRG equation goes far off from the exact value $\Delta_c = -1$ as $M$ increases. Where is the fixed point of the PRG equation in this case? Since the mass in the AF state is generated by the operator coming from the Umklapp scattering between the Jordan-Wigner fermions originated from the $S^z_i S^z_{i+1}$ term in the spin Hamiltonian, the fixed point is the $XY$ point ($\Delta = 0$) where there is no $S^z_i S^z_{i+1}$ term resulting in the vanishment of the interaction between fermions. Thus the PRG solution is brought over from the true transition point $\Delta_c = -1$ to the $XY$ point. This example was also noticed by Bonner and Müller [6] and by Sólyom and Ziman [7].
References

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Figure caption

Figure 1: Schematic ground-state phase diagram of the Hamiltonian $H(\Delta, h = 0, h_s)$. The antiferromagnetic, massless and ferromagnetic phases are indicated by AF, ML and FE, respectively. The AF-ML transition along the path 1 is of the KT type and that along the path 2 is of the second order. The transition to the FE state is of the first order.