Quasi-long-range order in trapped systems

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We investigate the effects of a trapping space-dependent potential on the low-temperature quasi-
long-range order phase of two-dimensional particle systems with a relevant U(1) symmetry, such
as quantum atomic gases. We characterize the universal features of the trap-size dependence using
scaling arguments. The resulting scenario is supported by numerical Monte Carlo simulations of a
classical two-dimensional XY model with a space-dependent hopping parameter whose inhomogeneity
is analogous to that arising from the trapping potential in experiments of atomic gases.

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Statistical systems are generally nonhomogeneous in
nature, while homogeneous systems are often an ideal
limit of experimental conditions. Thus, in the study of
critical phenomena, an important issue is how critical be-
haviors develop in nonhomogeneous systems. Particularly
interesting physical systems are interacting particles con-
strained within a limited region of space by an external
potential. This is a common feature of the experimental
realizations of the Bose-Einstein condensation (BEC) in
diluted atomic vapors [1] and of optical lattices of cold
atoms [2], which have provided a great opportunity to in-
vestigate the interplay between quantum and statistical
behaviors in particle systems.

In the BEC scenario, a macroscopic number of bosonic
atoms accumulate in a single quantum state and are de-
scribed by a condensate wave function, which naturally
provides the complex order parameter ψ(x) of the phase transition and its relevant U(1) symmetry. These global
features characterize the XY universality class which de-
scribes the universal critical behavior of a large class of
systems, see, e.g., Ref. [3]. The critical behavior arising
from the formation of the condensate in a trapped Bose gas has been investigated experimentally [4], ob-
serving an increasing correlation length compatible with
the behavior expected at a continuous transition of ho-
mo geneous systems belonging the three-dimensional (3D)
XY universality class. Two-dimensional (2D) homogeneous gases of bosonic particles do not show a real BEC
with decreasing the temperature T. Nevertheless, they are expected to experience a finite-T Kosterlitz-Thouless
(KT) transition [5] separating the high-T phase with short-ranged correlations from a low-T phase character-
ized by a quasi-long-range order (QLRO), where the one-
bond correlation function decays algebraically at large
distance. Experimental evidences of such a transition in
trapped Bose atomic gases have been provided in Refs. [6,8].

However, the inhomogeneity due to the trapping po-
tential drastically changes the general features of the critical behavior at the transition separating the high-T and
low-T phases, and, in the case of 2D systems, of the
QLRO phase. For example, correlation functions of the
critical modes do not develop a diverging length scale in
a trap. The critical behavior of the unconfined homo-
genous system could be observed around the middle of
the trap only when the length scale ξ of the correlations
is much smaller than the length scale ξ† induced by the
trap size, and one looks at small-distance correlations rela-
tively to ξ. If ξ is large but not much smaller than the
trap size, the critical behavior gets somehow distorted by
the trap, although it gives rise to universal effects in the
large trap-size limit, controlled by the universality class
of the phase transition of the unconfined system [8,10].

The understanding of the trap effects is necessary for an
accurate determination of the critical parameters, see,
e.g., Refs. [6,11].

In this paper we consider 2D systems showing a low-T
QLRO phase in their phase diagram, after a KT transi-
tion. We investigate how the presence of the trap changes
the main features of the QLRO of the homogeneous sys-
tem and, therefore, how one may get evidence of the
QLRO phase from the behavior of the system in the pre-

cence of the trap. For this purpose, we resort to a scaling
analysis which allows us to take into account the trap
length scale when it becomes sufficiently large, exploit-
ing the universality of the scaling behavior.

The above considerations also apply to other physi-
cally interesting models, such as the Bose-Hubbard (BH)
model [12] at its finite-T superfluid transition, whose
Hamiltonian in the presence of confining potential reads

\[ \mathcal{H}_{BH} = -\frac{J}{2} \sum_{\langle ij \rangle} (\hat{b}_i^\dagger \hat{b}_j + \text{h.c.}) + \frac{U}{2} \sum_i n_i (n_i - 1) + \sum_i (\mu + v^2 r^2) n_i, \]  

(1)

where the sum runs over the bonds \( \langle ij \rangle \) of a d-dimensional
lattice, $b_i$ are bosonic operators, $n_i \equiv b_i^\dagger b_i$ is the particle density, $r$ the distance from the center of the trap. The trap size is defined as $l \equiv J^{1/2}/v$, see, e.g., [14]. The BH model is of experimental relevance because it describes cold bosonic atoms trapped in a limited space region of optical lattices [13]. Other transitions in the XY universality class are the 4He superfluid transition, insulator-superconductor transitions, like that of the attractive Hubbard model, etc.

In a standard scenario for a continuous transition, see, e.g., [3], the critical behavior of a $d$-dimensional system is characterized by two relevant parameters $u_t$ and $u_h$, which may be associated with $T$, i.e., $u_t \sim T/T_c - 1$ and the external field $h$ coupled to the order parameter, with renormalization-group (RG) dimension $y_t = 1/\nu$ and $y_h = (d + 2 - \eta)/2$. The presence of a trap of size $l$ generally induces a further length scale $\xi_l$, which must be taken into account to describe the critical correlations. Within the trap-size scaling (TSS) framework [4, 10], the scaling law of the singular part of the free energy density around the center of the trap can be written as

$$F_{\text{sing}} = l^{-\theta d} \mathcal{F}(u_t^{\theta y_t}, u_h^{\theta y_h}, x l^{-\theta}) \quad (2)$$

where $\theta$ is the trap exponent. At the critical point ($u_t = 0$), the length scale induced by the trap behaves as $\xi_l \sim l^\theta$, and the correlation function of the order parameter as

$$G(x, y) \equiv \langle \bar{\psi}(x)\psi(y) \rangle_c = l^{-\theta d} G(x l^{-\theta}, y l^{-\theta}) \quad (3)$$

Finite size effects, due to a finite volume $L^d$, can be taken into account by adding a further dependence on $L l^{-\theta}$ in the above scaling Ansatz [13].

The value of $\theta$ depends on the way the external confining field is coupled to the model variables. In the case relevant for the above-mentioned particle systems, the external trapping potential is coupled to the particle density. The corresponding perturbation can be inferred from the many-body Hamiltonian [16] in the presence of the external potential $V(x) = v^p x^p$, i.e.,

$$P_V = \int d^d x V(x)|\psi(x)|^2,$$

where $\psi(x)$ is the complex order-parameter field. The computation of the RG dimensions of the trap parameter leads to [3] $\theta = p\nu/(1 + p\nu)$. In order to apply it to the KT transition of 2D $U(1)$-symmetric systems, we formally set $\nu = \infty$, corresponding to the KT exponential behavior of the correlation length $\xi \sim \exp(\tau^{-1/2})$ where $\tau \equiv T/T_c - 1 \rightarrow +0$, thus obtaining $\theta = 1$ for any power $p$ of the potential. For comparison, we mention that $\theta = 0.57327(4)$ at the 3D BEC transition in a harmonic trap [3]. Moreover, in a Gaussian theory perturbed by $P_V$, since $\nu = 1/2$ we have $\theta = p/(2 + p)$, for any spatial dimension. It is worth mentioning that analogous TSS behaviors [10], with the same trap exponents, apply to the quantum $T = 0$ superfluid-Mott transition of the BH model [11] at fixed integer density, which belongs to the $(d + 1)$-dimensional XY universality class [12].

A standard representative model of the 2D XY universality class is the classical square-lattice XY model,

$$H = -J \sum_{\langle ij \rangle} \text{Re} \bar{\psi}_i \psi_j, \quad \psi_i \equiv e^{i\bar{\phi}_i} \in U(1), \quad (4)$$

which presents the same universal features of 2D systems whose phase diagram shows a KT transition between high-$T$ and low-$T$ QLRO phases. We may further exploit universality to investigate the effects of an inhomogeneity analogous to that of 2D trapped particle systems, where the external confining potential is generally coupled to the particle density, which can be associated with an energy-density operator in a corresponding effective model. A 2D XY model with an external space-dependent field coupled to the energy density is obtained by considering a space-dependent hopping parameter $U_{ij}, [17]$

$$H_U = -J \sum_{\langle ij \rangle} \text{Re} \bar{\psi}_i U_{ij} \psi_j, \quad (5)$$

$$U_{ij} = 1 + V(r_{ij}), \quad V(r) = v^p r^p, \quad (6)$$

where $p$ is an even positive integer, $r_{ij}$ is the distance from the origin of the midpoint of nearest-neighbor sites. We set $J = 1$. The inhomogeneity arising from the space dependence of $U_{ij}$ is analogous to that arising from a trapping potential in particle systems, such as the BH model [1].

Thus, $l = 1/v$ may be considered as the analog of the trap size. At large distance, since $V(r) \rightarrow \infty$, the spin variables get effectively frozen. When $p \rightarrow \infty$ the effect of the external potential $V$ is equivalent to confining a homogeneous system in a box of size $L = l = 1/v$, and the TSS becomes the standard finite-size scaling (FSS). At the critical KT temperature $T_c = 0.893(1)$ [18] of the homogeneous model [1], the TSS of the model [5] is expected to follow the scaling Ansatz [2] and [3], with $\eta = 1/4$ and $\theta = 1$ as computed above by RG arguments.

Now we turn to the QLRO phase, which is the main issue of this paper. The homogeneous model is critical in the whole low-$T$ region $T < T_c$, where the correlation function $\langle \bar{\psi}_x \psi_y \rangle$ decays as $1/|x - y|^{\eta(T)}$ with a $T$-dependent exponent $\eta(T)$: $\eta(T) = T/(2\pi) + O(T^2)$, increasing up to $\eta(T_c) = 1/4$ (some numerical estimates are reported in Refs. [19, 20]). This critical behavior is controlled by a line of Gaussian fixed points, essentially given by the spin-wave theory $H_{sw} = \int d^d x (\nabla \varphi)^2$, which is the leading nontrivial term for $T \rightarrow 0$. We may apply the same spin-wave approximation to infer the value of $\theta$ controlling the TSS in the QLRO phase. In the spin-wave limit we obtain

$$H_{sw} = \int d^d x \frac{1}{2}(v^p r^p)(\nabla \varphi)^2, \quad (7)$$

The trap exponent $\theta$ is related to the RG dimension $y_v$ of parameter $v$, $\theta = 1/y_v$, which can be obtained from
FIG. 1: Log-log plot of $m_0 \equiv \langle \psi_0 \rangle$ for the model (5), at $T = 0.4$, $0.5$, $0.8$, for $L/l \approx 2$ [21]. Statistical errors are hardly visible. The lines show fits to $a l^{-\zeta/2}$.

FIG. 2: $G(0, x)/m_0^2$ vs $x/l$ for the model (5), at $T = 0.5$, $0.8$, for several values of $l$, and $L/l \approx 2$ [21]. The sets of data at fixed $T$ are clearly converging to a nontrivial large-$l$ limit

The boundary conditions $\phi_0 = 1$ breaks the U(1) symmetry, thus allowing a nonzero local magnetization. According to the above scaling considerations, we expect that their asymptotic trap-size dependence is

$$m_0 \sim l^{-\eta(T)\theta/2},$$

$$G(\vec{x}, \vec{y}) \approx l^{-\eta(T)\theta} G(\vec{x}/l^\theta, \vec{y}/l^\theta),$$

where $\theta = 1$ and $\eta(T)$ is the $T$-dependent exponent of the homogeneous system. Since $\theta = 1$, a nontrivial simultaneous FSS and TSS limit can be achieved by keeping $L/l$ fixed, where scaling behaviors analogous to Eqs. (9) and (10) apply. [22]

Fig. 1 shows data of $m_0$ for some values of $T < T_c$. A power-law trap-size dependence is clearly supported by the data. Fits to $a l^{-\zeta/2}$, see Fig. 1, give $\zeta = 0.072(1), 0.093(1), 0.179(1)$ respectively for $T = 0.4, 0.5, 0.8$. Since according to Eq. (9) $\zeta = \eta\theta$ and $\theta = 1$, these results should be compared with the available estimates of $\eta(T)$ obtained in homogeneous systems, which are in good agreement [23].

Fig. 2 shows results for $G(0, \vec{x})$. In agreement with Eqs. (9) and (10), they show that $G(0, \vec{x})/m_0^2 = g(x/l)$ in the large-$l$ limit. At small distance $x \ll l$, $g(y) \sim y^{-\eta(T)}$, to recover the behavior of the homogeneous system.

The spatial dependence of the local magnetization does not show a simple scaling behavior. The numerical results appear consistent with

$$\langle \psi_2 \rangle \sim l^{-\eta(T)x/2} f(X)$$

where $X \equiv x/l$ and $T X \equiv T/(1 + X^2)$, which can be derived using arguments based on a local-temperature approximation, noting that $T X$ may be considered as an effective local (space-dependent) temperature.

We also consider a 2D XY model where the hopping parameter decreases moving far from the origin, i.e., replacing $U_{ij} = [1 + V(r_{ij})]^{-1}$, $V(r) = v^2 r^2$, [12] in Eq. (5). In this case the regions far from the origin are effectively in the high-$T$ phase. The lattice system is set as before, but we use open boundary conditions which are compatible with a diverging hopping parameter at large distance. Thus, $\langle \psi_2 \rangle = 0$ everywhere, including the origin. MC simulations show that the TSS is again characterized by the trap exponent $\theta = 1$. This is shown by Fig. 3 where we report $l^n(T) G(0, x)$ for several values of $l$, $L/l \approx 2$ [21], with $n(T)$ obtained from the data of Fig. 1. The sets of data for $T = 0.5$ and $T = 0.8$ are clearly converging to a nontrivial large-$l$ limit, at least up to $X = x/l$ corresponding to $T X \sim T/(1 + X^2) = T_c \approx 0.893 (X_c \approx 0.89, 0.34$ for $T = 0.5, 0.8$ respectively). [23]

In conclusion, we have characterized the trap-size dependence within the low-$T$ QLRO phase and at the KT finite-$T$ transition of 2D trapped systems. Using scaling
arguments, we have argued that it is described by the TSS Ansatz \[2\] and \[10]\) with the trap exponent \(\theta = 1\) in the whole QLRO phase, up to the KT transition. This scenario has been supported by numerical results for classical 2D XY models with space-dependent hopping parameters, which give rise to inhomogeneities analogous to that of trapped atomic gases in actual experiments. These results should be useful to get evidence of QLRO in trapped systems, and also determine the critical parameters, which give rise to inhomogeneities analogous to those of trapped atomic gases in actual experiments.

\[\theta = 0\]  

\[\text{FIG. 3: } p^{(T)}(G(0,x) \text{ vs } x/l, \text{ for the } U_{ij} \text{ given in Eq. (22), at } T = 0.5, 0.8, \text{ for several values of } l, \text{ and } L/l \approx 2.\]

For example, the KT critical point corresponds to a TSS with \(\eta = 1/4\), while values \(\eta < 1/4\) corresponds to the low-\(T\) QLRO phase.

\[1\] E.A. Cornell and C.E. Wieman, Rev. Mod. Phys. 74, 875 (2002); N. Ketterle, Rev. Mod. Phys. 74, 1131 (2002).

\[2\] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).

\[3\] A. Pelissetto and E. Vicari, Phys. Rep. 368, 549 (2002).

\[4\] T. Donner, S. Ritter, T. Bourdel, A. Öttl, M. Köhl, and T. Esslinger, Science 315, 1556 (2007).

\[5\] J.M. Kosterlitz and D.J. Thouless, J. Phys. C: Solid State 6, 1181 (1973).

\[6\] Z. Hadzibabic, P. Krüger, M. Cheneau, B. Battelier, and J. Dalibard, Nature 441, 1118 (2006).

\[7\] P. Krüger, Z. Hadzibabic, and J. Dalibard, Phys. Rev. Lett. 99, 040402 (2007).

\[8\] P. Cladé, C. Ryu, A. Ramanathan, K. Helmerson, and W.D. Phillips, Phys. Rev. Lett. 102, 170401 (2009).

\[9\] M. Campostrini and E. Vicari, Phys. Rev. Lett. 102, 240601 (2009).

\[10\] M. Campostrini and E. Vicari, Phys. Rev. A 81, 023606 (2010); J. Stat. Mech.: Theory Exp. P08020 (2010).

\[11\] L. Pollet, N.V. Prokof’ev, and B.V. Svistunov, Phys. Rev. Lett. 104, 245705 (2010).

\[12\] M.P.A. Fisher, P.B. Weichmann, G. Grinstein and D.S. Fisher, Phys. Rev. B 40, 546 (1989).

\[13\] D. Jaksch, C. Bruder, J.I. Cirac, C.W. Gardiner, and P. Zoller, Phys. Rev. Lett. 81, 3108 (1998).

\[14\] M. Campostrini and E. Vicari, arXiv:1010.0806

\[15\] S.L.A. de Queiroz, R.R. dos Santos, and R.B. Stinchcombe, Phys. Rev. E 81, 051122 (2010).

\[16\] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).

\[17\] As put forward in Ref. 26, a trap for atomic systems in an optical lattice, described by the BH model, may be realized through space-dependent hopping parameters.

\[18\] M. Hasenbusch and K. Pinn, J. Phys. A 30, 63 (1997).

\[19\] V. Alba, A. Pelissetto, and E. Vicari, J. Phys. A 41, 175001 (2008).

\[20\] B. Berche, J. Phys. A 36, 586 (2003).

\[21\] In our simulations, we consider square lattices with \(L/l\) corresponding to \(v = 1\), they should be obtained analogous results for \(c_v = 8\), i.e., \(L/l \approx 2.8\). We chose sufficiently large values of \(c_v\) to make finite-\(L\) effects negligible. The data shown in Figs. 1 and 2 were obtained using \(c_v = 4\), corresponding to \(L/l \approx 2\). We obtained analogous results for \(c_v = 8\), thus \(L/l \approx 2.8\). We chose sufficiently large values of \(c_v\) to make finite-\(L\) effects negligible.

\[22\] Estimates of \(\eta\) for the square-lattice nearest-neighbor XY model at \(T = 0.4, 0.5, 0.8\) are respectively \(\eta = 0.072(6), 0.098(7), 0.171(3)\) from Ref. 10, and \(\eta = 0.074(6), 0.100(8), 0.19(2)\) from Ref. 21.

\[23\] Finite-\(L\) effects turn out to be stronger in this case. The data shown in Fig. 3 were obtained using \(c_v = 4\) \(22\), corresponding to \(L/l \approx 2\). We also performed MC simulations with \(c_v = 16\), thus \(L/l \approx 4\), which showed a scaling behavior consistent with Eq. (10) as well.

\[24\] The behavior for \(X > X_c\) is not completely clear. Our MC data, in particular for \(L/l \approx 4\) up to \(L = 256\) (not shown here), suggest a vanishing large-\(L\) limit for \(X > X_c\).

\[25\] V.G. Rousseau, G.G. Batrouni, D.E. Sheehy, J. Moreno, and M. Jarrell, Phys. Rev. E 81, 051122 (2010).

\[26\] M. Hasenbusch, A. Pelissetto, and E. Vicari, J. Stat. Mech. P12002 (2005).