S-Matrix approach to Compton scattering at the tree level in a strong magnetic field

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Abstract

We have studied the Compton scattering (γe− → γe−) at the tree level in a homogeneous background of strong magnetic field (|eB| ≫ m^2, m is the mass of electron) through the S-matrix approach. For that purpose, using the Schwinger propagator for the electron, we have first calculated the square of the S-matrix element in the Landau gauge by summing over the final states of electron and photon and averaging over the initial states of the same. In the strong magnetic field, only the lowest Landau level for electron is considered. Finally we have computed the crosssection for Compton scattering as a function of initial photon energy for the different strengths of strong magnetic fields, where we have found that the crosssection in vacuum gets decreased due to the presence of strong magnetic field. However, for a fixed initial photon energy, the crosssection increases linearly with the magnetic field.

1 Introduction

The study of the Compton scattering in a strong magnetic field is of astrophysical importance because the strong magnetic field tremendously affects this matter-radiation scattering. This type of magnetic field is generally found in magnetized neutron stars, where the Compton scattering plays a key role in the formation of neutron star spectra. For neutron star applications, it thus becomes necessary to modify the scattering properties like the spinors and its orthogonality and completeness condition etc. in a strong magnetic field. A strong magnetic field is also expected to be produced in recent heavy-ion program in RHIC at BNL and in LHC at Geneva at very early
stages of collisions for the non-central events [1–5]. Depending on the centralities, the strength of the magnetic field may reach between \( m_\pi^2 \) (\( \simeq 10^{18} \) Gauss) at RHIC [6] to 15 \( m_\pi^2 \) at LHC [7]. At extreme cases, it may reach values of 50 \( m_\pi^2 \) at LHC. A very strong magnetic field (\( \sim 10^{23} \) Gauss) may have existed in the early universe during the electroweak phase transition due to the gradients in Higgs field [8].

The nonrelativistic treatment of Compton scattering in strong magnetic field is done by [9,10]. However, the relativistic effect becomes dominant as the strength of magnetic field exceeds 10^{12} Gauss. The strength of the magnetic field decides the transition between the Landau levels, so that for the weak magnetic field, a number of resonances corresponding to the transition between the Landau levels appear. The relativistic quantum electrodynamics (QED) treatment allows the scattering at resonances with higher harmonics and can consider the transition for electrons to their corresponding Landau levels. Such results from QED for Compton scattering have been offered in various papers [11–14] at various levels of analytic and numerical development. However, they consider the infinitely long lived intermediate states, so it will be far from resonance. In order to consider the resonant Compton scattering, one needs to consider the finite life time decay width for the electrons to do the cyclonic transitions to their Landau levels. In particular, the Ref. [15,16] has considered the spin-dependent influences at the cyclotron resonance and assumes the soft photon emitted due to transition of electrons from their corresponding Landau levels with very low energy and taken the finite decay width of landau levels.

In the presence of magnetic field, the momentum of electron (\( p \)) is factorized into components with respect to the direction of the magnetic field, where the transverse component gets quantized and longitudinal component (\( p_Z \)) remains unaffected. As a result, the dispersion relation for the electron is modified quantum mechanically into [17]
\[
E_n(p_Z) = \sqrt{p_Z^2 + m^2 + 2neB},
\]
where \( n = 0, 1, 2, \ldots \) are the quantum numbers specifying the Landau levels. In strong magnetic field, the electrons are rarely excited to the higher Landau levels, only the lowest Landau levels (LLL) (\( n = 0 \)) are populated. Thus the dynamics of electrons are effectively restricted to (1 + 1) dimensions.

We have studied the Compton scattering process through the S-matrix, which is based on the second-order QED perturbation theory and provides a ground for detailed investigation in a field of radiation transfer in case of strong external magnetic field. This paper is divided into
following sections. Using the notations of four momenta of electron and photon suitable for
the description in a strong magnetic field, we revisit the solution of Dirac equation in a strong
and homogeneous magnetic field in sections 2.1, 2.2 and 2.3 and then considered the Schwinger
propagator in section 2.4 for calculating the S-matrix. Using those notations, we calculate the
S-matrix element for the s and u-channel in sections 3.1 and 3.2, respectively. Then we wish
to calculate the S-matrix element squared, which involves the M-matrix element squared for s-,
u-channel and the interference between them. Thus we calculate the M-matrix element squared
for the s-channel, u-channel and the interference term, after summing over the final states and
averaging over the initial states in sections 4.1, 4.2 and 4.3, respectively. In section 5, we first
revisit the formula for calculating the crosssection by constructing the Lorentz invariant phase
space, flux factor, energy-momentum conserving Dirac-Delta function etc. in the presence of
strong magnetic field and finally, we evaluate the crosssections for the s and u channels in sections
5.1, 5.2, respectively because the matrix element for the interference term vanishes. Finally we
conclude in section 6.

2 Fermions in strong magnetic field

2.1 Notation

As mentioned earlier, the momentum of an electron in a magnetic field gets separated into the lon-
gitudinal and transverse components with respect to the direction of the magnetic field. Therefore,
using the following convention of the metric tensor

\[ g^{\mu\nu} = (1, -1, -1, -1), \; g^{\mu\perp\perp} = (0, -1, -1, 0) \; \text{and} \; g^{\mu\parallel\parallel} = (1, 0, 0, -1) , \]

(2)

it will be useful to define the four momentum for the electron as

\[ x^\mu = (X^0, X, Y, Z), \]
\[ p_\perp^\mu = (0, p^1, p^2, 0) = (0, p_X, p_Y, 0), \]
\[ p_\parallel^\mu = (p^0, 0, 0, p^3), = (E, 0, 0, p_Z), \]
\[ \tilde{p}_\parallel^\mu = (\tilde{p}^0, 0, 0, \tilde{p}^3) = (p_Z, 0, 0, E). \]

(3) (4) (5)

However, the usual definition for the photon four momentum is

\[ k^\mu = (k^0, k^1, k^2, k^3) = (\omega, k_X, k_Y, k_Z). \]

(6)
2.2 Spinors

The methods of Ritus eigenfunction [18] along with Schwinger Proper-time formalism [19] are commonly used to solve the Dirac equation of charged fermions in the presence of a constant magnetic field. There are different approaches which have been adopted in the literature [20–22] to obtain the spinor in a magnetic field. However, we have employed the procedure to solve the Dirac equation in a constant external field from Ref. [17].

For the sake of simplicity, we assume a static and homogeneous magnetic field along the $Z$-direction, $\vec{B} = B \hat{Z}$. Such a magnetic field can be obtained from a vector potential $A^\mu = (0, 0, BX, 0)$. The choice of vector potential is not unique as the same magnetic field can also be obtained in a symmetric gauge, $A^\mu = (0, -BY/2, BX/2, 0)$. Thus the positive energy Dirac spinors with the gauge $A^\mu = (0, -BY, 0, 0)$ are given by the shifted coordinate, $\xi = \sqrt{eB} \left( Y + \frac{pY}{eB} \right)$ [17, 23]

$$e^{-ip \cdot x/Y} U_s(Y, n, pY),$$

where $U_s$’s are

$$U_+ (Y, n, \vec{p}_Y) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{pz}{E_n + m} I_{n-1}(\xi) \\ -\frac{\sqrt{2neB}}{E_n + m} I_n(\xi) \end{pmatrix}, \quad U_- (Y, n, \vec{p}_Y) = \begin{pmatrix} 0 \\ I_n(\xi) \\ -\frac{\sqrt{2neB}}{E_n + m} I_{n-1}(\xi) \\ -\frac{pz}{E_n + m} I_n(\xi) \end{pmatrix}. \tag{7}$$

Similarly the negative energy Dirac spinors with $\tilde{\xi} = \sqrt{eB} \left( Y - \frac{pY}{eB} \right)$ are given by [17, 23]

$$V_- (Y, n, \vec{p}_Y) = N \begin{pmatrix} \frac{pz}{E_n + m} I_{n-1}(\tilde{\xi}) \\ \frac{\sqrt{2neB}}{E_n + m} I_n(\tilde{\xi}) \\ I_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix}; \quad V_+ (Y, n, \vec{p}_Y) = N \begin{pmatrix} \frac{\sqrt{2neB}}{E_n + m} I_{n-1}(\tilde{\xi}) \\ -\frac{pz}{E_n + m} I_n(\tilde{\xi}) \\ 0 \\ I_n(\tilde{\xi}) \end{pmatrix}, \tag{8}$$

where the normalization constant ($N$) is $N = \sqrt{E_n + m}$ and the symbol, $p_Y$ denotes the absence of the $Y$-component of momentum in the spinors. The energy eigenvalues are given by the above
Landau quantization \((1)\) and the energy eigenfunctions, \(I_n(\xi)\)'s are expressed in terms of Hermite polynomials, \(H_n(\xi)\) as

\[
I_n(\xi) = \sqrt{\frac{\sqrt{eB}n!}{2\pi}} e^{-\frac{\xi^2}{2}} H_n(\xi),
\]

having their completeness relation

\[
\sum_{\nu} I_{\nu}(\xi)I_{\nu}(\xi') = \sqrt{|eB|} \delta(\xi - \xi') = \delta(Y - Y').
\]

As mentioned earlier, in a strong magnetic field, only the lowest Landau level \((n=0)\) is populated so only \(U_-(Y, n, \vec{p}_Y)\) will be non-zero and the \(Y\)-dependence can also be extracted from the spinors for \(n = 0\) case. This will made our task easy to handle the \(Y\)-dependence contained in \(I_n(\xi)\) while solving the S-matrix. Thus, for positive energy solutions, the nonvanishing spinor is

\[
U_-(Y, n, \vec{p}_Y) = NI_0(\xi) \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p_z}{E_0 + m} \end{pmatrix}.
\]

Similarly, for negative energy solutions, the nonvanishing spinor is

\[
V_+(Y, n, \vec{p}_Y) = NI_0(\xi) \begin{pmatrix} 0 \\ \frac{p_z}{E_0 + m} \\ 0 \\ 1 \end{pmatrix}.
\]

### 2.3 Completeness Condition

We can now calculate the spin sums for the particles \((P_U)\) and anti-particles \((P_V)\) in the presence of external magnetic field as \([17, 23]\)

\[
P_U(Y, Y', n, \vec{p}_Y) = \sum_s U_s(Y, n, \vec{p}_Y) \overline{U_s}(Y', n, \vec{p}_Y) \\
= \frac{1}{2} \left[ \left\{ m(1 + \Sigma_z) + \not{p}_\parallel - \not{p}_\parallel \gamma_5 \right\} I_{n-1}(\xi)I_{n-1}(\xi') \\
+ \left\{ m(1 - \Sigma_z) + \not{p}_\parallel + \not{p}_\parallel \gamma_5 \right\} I_n(\xi)I_n(\xi') \\
- \sqrt{2neB}(\gamma_1 - i\gamma_2)I_n(\xi)I_{n-1}(\xi') - \sqrt{2neB}(\gamma_1 + i\gamma_2)I_{n-1}(\xi)I_n(\xi') \right].
\]

In the ultra-relativistic \((p^2 >> m^2)\) and strong magnetic field \((n = 0)\) limits, the spin sum reduces to

\[
P_U(Y, Y', n, \vec{p}_Y) = \frac{I_0(\xi)I_0(\xi')}{2} [\not{p}_\parallel + \not{p}_\parallel \gamma_5].
\]
Similarly, the spin-sum for the negative energy spinors can also be calculated as

$$P_V(Y, Y', n, \vec{p}_Y) \equiv \sum_s V_s(Y, n, \vec{p}_Y) V_s(Y', n, \vec{p}_Y)$$

$$= \frac{1}{2} \left\{ -m(1 + \Sigma_z) + \vec{p}_\parallel - \tilde{p}_\parallel \gamma_5 \right\} I_{n-1}(\tilde{\xi}) I_{n-1}(\tilde{\xi}')$$

$$+ \left\{ -m(1 - \Sigma_z) + \vec{p}_\parallel + \tilde{p}_\parallel \gamma_5 \right\} I_n(\tilde{\xi}) I_n(\tilde{\xi}')$$

$$+ \sqrt{2neB(\gamma_1 - i\gamma_2)} I_{n-1}(\tilde{\xi}) I_{n-1}(\tilde{\xi}') + \sqrt{2neB(\gamma_1 + i\gamma_2)} I_n(\tilde{\xi}) I_n(\tilde{\xi}') \right\}. \quad (15)$$

In ultra-relativistic and strong magnetic field limits, the spin sum reduces to

$$P_V(Y, Y', n = 0, \vec{p}_Y) = I_0(\tilde{\xi}) I_0(\tilde{\xi}') \frac{\theta(t - t')}{2} \langle 0 | \psi(x) \psi(x') | 0 \rangle - \frac{\theta(t' - t)}{2} \langle 0 | \psi(x') \psi(x) | 0 \rangle \right\}. \quad (16)$$

However, the $Y$-dependence will be later absorbed in the $S$-matrix element and results in the spin sum in the strong magnetic field as

$$P_U(n = 0, \vec{p}_Y) = \frac{1}{2} [\vec{p}_\parallel + \tilde{p}_\parallel \gamma_5]. \quad (17)$$

### 2.4 Electron propagator: Schwinger proper-time method

Usually the electron propagator is obtained by the vacuum expectation value of the time-ordered product of the Dirac field operators

$$S(x - x') = i \langle 0 | T\psi(x) \overline{\psi}(x') | 0 \rangle$$

$$= \theta(t - t') \langle 0 | \psi(x) \overline{\psi}(x') | 0 \rangle - \theta(t' - t) \langle 0 | \overline{\psi}(x') \psi(x) | 0 \rangle, \quad (18)$$

where, $\psi(x)$ and $\overline{\psi}(x')$ are the solutions obtained from the Dirac equation in the strong magnetic field from (32) and (33), respectively. However, the electron propagator satisfies the Dirac-equation in the Green’s function approach as

$$i\gamma^\mu (\partial_\mu + iqA_\mu - m) S(x, x') = -\delta^4(x - x'). \quad (19)$$

The solution of eq.(19) can be obtained from the Schwinger proper-time method [19]. However, the gauge transformation introduces the phase factor into the solution. Since the magnetic field breaks the translational invariance of space so we can not take its Fourier transform directly. To
take its Fourier transform, a phase factor \((\phi)\) is introduced in the solution, which is responsible for the breaking of the translational invariance, thus the propagator becomes

\[
S(x, x') = \phi(x, x') \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot (x-x')} iS(q).
\]  

(20)

The above phase factor is not gauge invariant and its form is given in path representation [24–27]

\[
\phi(x, x') = \exp \left( ie \int_x^{x'} A_\mu(x) dx_\mu \right),
\]

(21)

which becomes unity for single fermionic line. Since \(S(q)\) does not depend on the spatial coordinate due to gauge invariance, so \(S(q)\) is translationally invariant and it can be written in a discrete form as [24–27]

\[
S(q) = i \exp \left( \frac{-q^2_\perp}{eB} \right) (-1)^n \sum_{n=0}^{\infty} \frac{D_n}{q^2_\parallel - m^2 - 2neB},
\]

(22)

where

\[
D_n = (\gamma \cdot q_\parallel + m) \left[ (1 - i\gamma_1 \gamma_2) L_n \left( \frac{2q^2_\parallel}{eB} \right) - (1 + i\gamma_1 \gamma_2) L_{n-1} \left( \frac{2q^2_\parallel}{eB} \right) \right] + 4\gamma_+ q_\perp L_{n-1} \left( \frac{2q^2_\perp}{eB} \right).
\]

(23)

In strong magnetic field limit \((n = 0)\), the terms containing Laguerre polynomial, like \(L_{-1} \left( \frac{2q^2_\perp}{eB} \right)\) will become zero and \(L_0 \left( \frac{2q^2_\perp}{eB} \right)\) will become unity, so \(D_0\) reduces to

\[
D_0 = (\gamma \cdot q_\parallel + m)(1 - i\gamma_1 \gamma_2),
\]

(24)

which yields the electron propagator in the momentum space in a strong magnetic field [28, 29] in the extreme relativistic limit

\[
iS(q) = \left[ (1 + \gamma^0 \gamma_3 \gamma^5)(\gamma^0 q_0 - \gamma^3 q_3) \right] \exp \left( \frac{-q^2_\perp}{eB} \right)
\]

(25)

and is factorizable into the longitudinal and the transverse component as

\[
iS(q) = S_{\parallel}(q_\parallel) S_{\perp}(q_\perp),
\]

(26)

where,

\[
S_{\parallel}(q_\parallel) = \left[ (1 + \gamma^0 \gamma_3 \gamma^5)(\gamma^0 q_0 - \gamma^3 q_3) \right],
\]

(27)

\[
S_{\perp}(q_\perp) = \exp \left( \frac{-q^2_\perp}{eB} \right).
\]

(28)
3 S-Matrix Element for Compton Scattering ($\gamma e^- \rightarrow \gamma e^-$)

In Compton scattering, both $s$- and $u$-channel diagrams contribute, which are represented in Figure(s) 1 and 2, respectively. Thus the S-matrix element for Compton scattering due to both $s$ and $u$-channel diagrams is

$$S_{fi} = \langle f | (S^s + S^u) | i \rangle,$$

where the initial and final states are denoted by

$$| i \rangle = | e^-(p_Y), \gamma(k) \rangle,$$

$$| f \rangle = | e^-(P_Y), \gamma(K) \rangle,$$

respectively. The operators, $S^s$ and $S^u$ denote the transition operators for the $s$ and $u$-channel, respectively and we will now derive their forms from the corresponding Feynman diagrams.

3.1 S-Matrix element in $s$-channel

The $s$-channel diagram in Compton scattering at the lowest-order is depicted in Figure 1, where $\psi$ and $\bar{\psi}$ denote the incoming and outgoing electron, respectively whereas $A_\mu(k)$ and $A_\nu(K)$ represent
the incoming and outgoing photon, respectively. The incoming electron field operator is given by

\[
\psi(x) = N_1 \sum_s \sum_{n=0}^{\infty} \int \frac{dp_x dp_z}{(2\pi)^2} \left[ \hat{a}_s e^{-ip_x X_\gamma} U_s(Y, p_\gamma) + \hat{b}_s^\dagger e^{ip_x X_\gamma} V_s(Y, p_\gamma) \right],
\]

\[
\equiv \psi^+(x) + \psi^-(x), \tag{32}
\]

where \(\sum_s\) and \(\sum_n\) denote the summation over spins and the Landau levels and \(\hat{a}_s\) and \(\hat{b}_s^\dagger\) are the annihilation and creation operators for fermions and anti fermions, respectively. \(U_s(Y, p_\gamma)\) and \(V_s(Y, p_\gamma)\) represent the positive and negative energy spinors, respectively and \(N_1\) is the normalization constant.

Similarly the adjoint field operator is given by

\[
\bar{\psi}(x) = N_1 \sum_s \sum_{n=0}^{\infty} \int \frac{dp_x dp_z}{(2\pi)^2} \left[ \hat{a}_s^\dagger e^{-ip_x X_\gamma} \bar{U}_s(Y, p_\gamma) + \hat{b}_s e^{ip_x X_\gamma} \bar{V}_s(Y, p_\gamma) \right],
\]

\[
\equiv \bar{\psi}^+(x) + \bar{\psi}^-(x). \tag{33}
\]

For the real photon, the field operator expression can be written as

\[
A_\mu(x) = N_2 \sum_r \int \frac{d^2k}{(2\pi)^2} \left[ \hat{b}_r(k) e^{-ik \cdot x} \epsilon^r_\mu(k) + \hat{b}_r^\dagger(k) e^{ik \cdot x} \epsilon^r_\mu^s(k) \right],
\]

\[
\equiv A_\mu^+(x) + A_\mu^-(x), \tag{34-35}
\]

where the index, \(r\) labels the sum over the transverse polarization states and \(A_\mu^+(x)\) \((A_\mu^-(x))\) represents the positive (negative) energy part, respectively.

Therefore, the S-matrix element for the \(s\)-channel in Compton scattering process is given by

\[
S^s_{fi} = -e^2 \left\langle f \left| \int d^4x_1 d^4x_2 \bar{\Psi}^-(x_1) A_\alpha^- (x_1) \gamma^\alpha iS_F(x_1 - x_2) \gamma^\beta A^\beta_\beta (x_2) \Psi^+(x_2) \right| i \right\rangle. \tag{36}
\]

Since the electron and photon states are independent of each other, we can factorize the initial and final states as

\[
|i\rangle = |e^-(p_\gamma)\rangle \otimes |\gamma(k)\rangle, \tag{37}
\]

\[
|f\rangle = |e^-(P_\gamma)\rangle \otimes |\gamma(K)\rangle. \tag{38}
\]

So the S-matrix element for the \(s\)-channel becomes

\[
S^s_{fi} = \int d^4x_1 d^4x_2 \left\langle e^-(P_\gamma) | \bar{\Psi}^-(x_1) \gamma^\alpha iS_F(x_1 - x_2) \gamma^\beta \Psi^+(x_2) | e^-(p_\gamma) \right\rangle \left\langle \gamma(K) | A_\alpha^- (x_1) A^\beta_\beta (x_2) | \gamma(k) \right\rangle. \tag{39}
\]
By substituting the eigenvalue equations for the electron and photon field operators

\[ \psi^+ | e^-(p_Y) \rangle = U_s(p_Y, Y) e^{-ip \cdot Y} | 0 \rangle, \]
\[ \langle e^-(p_Y) | \bar{\Psi} = \langle 0 | U_s(p_Y, Y) e^{-ip \cdot Y}, \]
\[ A^+_\mu | \gamma(K) \rangle = e^{-iK \cdot \gamma \epsilon_\mu} | 0 \rangle, \]
\[ A^-_\mu | \gamma(K) \rangle = e^{iK \cdot \gamma \epsilon_\mu} | 0 \rangle, \]

the above matrix element becomes simplified as

\[ S^s_{fi} = -e^2 \int d^4x_1 d^4x_2 e^{i(p_Y + K) \cdot x_1} e^{-i(p_Y + k) \cdot x_2} U(P_Y, Y_1) f(K) iS_F(x_1 - x_2) U(p_Y, Y_2) f(k). \]  

Using the Fourier transform, the electron propagator in the coordinate space is converted into the momentum space, thus \( S^s_{fi} \) becomes

\[ S^s_{fi} = -e^2 \int d^4x_1 d^4x_2 \int \frac{d^4q}{(2\pi)^4} e^{i(p_Y + K - q) \cdot x_1} e^{-i(p_Y + k - q) \cdot x_2} U(P_Y, Y_1) \]
\[ \times f(K) e^{-iq \cdot (x_1 - x_2)} iS(q) U(p_Y, Y_2) f(k), \]

where \( q = p + k \) is the momentum of electron propagator in the s-channel diagram.

Rearranging the exponential terms and using the factorization of the electron propagator into longitudinal and transverse components from (26), the S-matrix element takes the form

\[ S^s_{fi} = -e^2 \int d^4x_1 d^4x_2 \int \frac{d^4q}{(2\pi)^4} e^{i(p_Y + K - q) \cdot x_1} e^{-i(p_Y + k - q) \cdot x_2} \]
\[ \times [U(P_Y, Y_1) f(K) S_{||}(q) U(p_Y, Y_2) f(k) S_{\perp}(q)], \]

Now, integrating over all the components except the X and Y components, we will get two-dimensional Dirac-delta function, so

\[ S^s_{fi} = -e^2(2\pi)^2 \delta_{X,Y} (P + K - p - k) \int dX_1 dX_2 dY_1 dY_2 dq_X dq_Y e^{i(K_Y - q_Y)Y_1} e^{-i(K_Y - q_Y)Y_2} \]
\[ \times e^{i(P_X + K_{X - q_X})X_1} e^{-i(p_X + K_{X - q_X})X_2} [U(P_Y, Y_1) f(K) S_{||}(q) U(p_Y, Y_2) f(k)] S_{\perp}(q). \]

Using the integration below over the X-variable

\[ \int dq_X \delta(P_X + K_{X - q_X}) \delta(p_X + K_{X - q_X}) \exp \left( \frac{-q_X^2}{eB} \right) \]
\[ = (2\pi)^2 \delta(P_X + K_{X - p_X - k_X}) \exp \left( \frac{-(P_X + K_{X})^2}{eB} \right), \]
the matrix element, \( S_{fi}^a \) becomes

\[
S_{fi}^a = - (2\pi)^4 e^2 \delta_\gamma^3 (P + K - p - k) \exp \left( \frac{-(P_X + K_X)^2}{eB} \right) \\
\times \int dY_1 dY_2 dq_Y e^{i(K_Y - q_Y)Y_1} e^{-i(k_Y - q_Y)Y_2} \\
\times \left[ \overline{U}(P_Y, Y_1) \gamma(K) S_\parallel(q||) U(p_Y, Y_2) \right] \exp \left( \frac{-q_Y^2}{eB} \right).
\]

Our next task is to solve the spatial (\( Y \)) integration. So factorizing the spinors into the spatial and momentum components (as done in eq. (11)) and defining the new variables by \( P_Y = K_Y - q_Y \) and \( P_Y' = k_Y - q_Y \); the matrix element \( S_{fi}^a \) is written as

\[
S_{fi}^a = - (2\pi)^4 e^2 \delta_\gamma^3 (P + K - p - k) \exp \left( \frac{-(P_X + K_X)^2}{eB} \right) \\
\times \int dY_1 dY_2 dq_Y I_0(\xi_1) I_0(\xi_2) e^{iP_Y Y_1} e^{iP_Y' Y_2} \\
\times \left[ \overline{U}(P_Y') \gamma(K) S_\parallel(q||) U(p_Y') \right] \exp \left( \frac{-q_Y^2}{eB} \right).
\]

In the strong magnetic field along the \( Z \)-direction, \( p_\perp \) (i.e. \( p_X \)) is much smaller than \(|eB|\), so \( \xi \) approximately becomes

\[
\xi = \sqrt{eB} \left( Y + \frac{p_X}{eB} \right) \approx \sqrt{eB} Y,
\]

so the integrations over \( Y_1 \) and \( Y_2 \) becomes decoupled in the form of standard integrals:

\[
\int dY_1 I_0(\sqrt{eB} Y_1) e^{iP_Y Y_1} = \sqrt{\frac{2\pi}{eB}} I_0 \left( \frac{P_Y}{\sqrt{eB}} \right),
\]

\[
\int dY_2 I_0(\sqrt{eB} Y_2) e^{iP_Y' Y_2} = \sqrt{\frac{2\pi}{eB}} I_0 \left( \frac{P_Y'}{\sqrt{eB}} \right).
\]

Thus, after performing the spatial integration, \( S_{fi}^a \) finally becomes

\[
S_{fi}^a = - (2\pi)^4 e^2 \delta_\gamma^3 (P + K - p - k) \exp \left( \frac{-(P_X + K_X)^2}{eB} \right) \\
\times \left[ \overline{U}(P_Y') \gamma(K) S_\parallel(q||) U(p_Y') \right] \int dq_Y \exp \left( \frac{-q_Y^2}{eB} \right) I_0 \left( \frac{P_Y}{\sqrt{eB}} \right) I_0 \left( \frac{P_Y'}{\sqrt{eB}} \right).
\]

Finally after the momentum integration over \( q_Y \) through the relation

\[
\int dq_Y \exp \left( \frac{-q_Y^2}{eB} \right) I_0 \left( \frac{P_Y}{\sqrt{eB}} \right) I_0 \left( \frac{P_Y'}{\sqrt{eB}} \right) = \sqrt{2\pi} \ \exp \left[ - \frac{3(K_Y^2 + k_Y^2)}{8eB} - \frac{K_Y k_Y}{4eB} \right],
\]

\[11\]
the S-matrix element for $s$-channel diagram results in a form

$$S'_{fi} = -16\sqrt{2}\pi^5 e^2 \exp \left[ -\frac{3(K_Y^2 + k_Y^2)}{8eB} + \frac{K_Y k_Y}{4eB} \right] \exp \left( \frac{-\left(p_X + K_X\right)^2}{eB} \right) \delta_\Psi^3 \left(P+K - p - k\right) \times \left[ U(P_Y)\gamma(k)S_{||}(q_{||})U(p_Y)\gamma(k) \right].$$

(52)

### 3.2 S-Matrix element in $u$-channel

The S-matrix element for the $u$-channel diagram in Figure 2 is given by

$$S'_{fi} = -16\sqrt{2}\pi^5 e^2 \exp \left[ -\frac{3(K_Y^2 + k_Y^2)}{8eB} + \frac{K_Y k_Y}{4eB} \right] \exp \left( \frac{-\left(p_X + K_X\right)^2}{eB} \right) \delta_\Psi^3 \left(P+K - p - k\right) \times \left[ U(P_Y)\gamma(k)S_{||}(q_{||})U(p_Y)\gamma(k) \right].$$

(53)

The further simplification needs the same procedure as done for the $s$-channel but the crossing-symmetry, *i.e.* replacing $K$ by $-k$ in the $s$-channel expression (52), solves it easily as

$$S'_{fi} = -16\sqrt{2}\pi^5 e^2 \exp \left[ -\frac{3(K_X^2 + k_X^2)}{8eB} + \frac{K_X k_X}{4eB} \right] \exp \left( \frac{-\left(p_X - K_X\right)^2}{eB} \right) \delta_\Psi^3 \left(P+K - p - k\right) \times \left[ U(P_Y)\gamma(k)S_{||}(q_{||})U(p_Y)\gamma(k) \right].$$

(54)
where \( q' (= p - K) \) is the momentum in the electron propagator for the \( u \)-channel diagram. Finally, the total S-matrix element for the Compton scattering is obtained from the \( s \)- (52) and \( u \)-channel (54) contributions and is expressed in terms of matrix elements for \( s \)- and \( u \)-channel, viz. \( M^s \) and \( M^u \), respectively

\[
S_{fi} = S_{fi}^s + S_{fi}^u = -16\sqrt{2}\pi^3 \delta^3_Y (P + K - p - k) \times \left[ eM^s + dM^u \right],
\]

where \( c \) and \( d \) are given by

\[
c = \exp \left[ -\frac{3}{8eB} \left( K_Y^2 + k_Y^2 \right) + \frac{K_Y k_Y}{4eB} \right] \exp \left( \frac{-\left(P_X + K_X\right)^2}{eB} \right),
\]

\[
d = \exp \left[ -\frac{3}{8eB} \left( K_Y^2 + k_Y^2 \right) + \frac{K_Y k_Y}{4eB} \right] \exp \left( \frac{-\left(P_X - k_X\right)^2}{eB} \right),
\]

respectively.

Now we need to square \( S_{fi} \), which involves the square of the three-dimensional Dirac delta function. So we first evaluate the square of the Dirac-delta functions, by taking the limits \( X \to \infty \), \( Y \to \infty \), and \( T \to \infty \),

\[
|\delta(P_X + K_X - k_X - p_X)|^2 = \frac{L_X}{2\pi} \delta(P_X + K_X - k_X - p_X),
\]

\[
|\delta(P_Z + K_Z - p_Z - k_Z)|^2 = \frac{L_Z}{2\pi} \delta(P_Z + K_Z - p_Z - k_Z),
\]

\[
|\delta(E' + \omega' - E - \omega)|^2 = \frac{T}{2\pi} \delta(E' + \omega' - E - \omega),
\]

respectively. The values \( L_X \) and \( L_Z \) cancel out due to the normalization constant. Thus, the square of the S-matrix element in large time \( T \) for Compton scattering takes the form

\[
\frac{|S_{fi}|^2}{T} = (2\pi)^6 \pi \delta^3_Y (P + K - p - k) |eM^s + dM^u|^2,
\]

where the matrix elements for the \( s \)- and \( u \)-channel are given by

\[
M^s = -e^2 \left[ \overline{U}(p_Y) \gamma(k) S_{\parallel}(q_{\parallel}) U(p_Y) \gamma(k) \right],
\]

\[
M^u = -e^2 \left[ \overline{U}(p_Y) \gamma(k) S_{\parallel}(q_{\parallel}) U(p_Y) \gamma(k) \right],
\]

respectively.
4 M-Matrix Element squared for Compton Scattering

Our aim is to calculate the square of the matrix element for \( s \)- and \( u \)-channel diagrams and the interference term.

4.1 M-Matrix Element squared for \( s \)-channel

The matrix element given by (60) can be rewritten in a compact form

\[
\mathcal{M}^s = A U^s(P_{\gamma}) \Gamma_1 U^s(p_{\gamma}), \tag{62}
\]

where \( s \) and \( s' \) denote the spins for the initial and final electrons, respectively and \( A \) and \( \Gamma_1 \) are given by

\[
A = -e^2 \left[ \frac{1}{q_0^2 - m^2 + i\epsilon} \right], \tag{63}
\]

\[
\Gamma_1 = \varepsilon_\mu^s(K) \gamma^\mu (1 + \gamma^0 \gamma^3 \gamma^5)(\gamma^0 q_0 - \gamma^3 q_3) \gamma^\nu \varepsilon_\nu^r(k), \tag{64}
\]

whereas \( r \) and \( r' \) represent the polarization for the initial and final photons, respectively. Thus the square of the matrix element becomes

\[
|\mathcal{M}^s|^2 = A^2 \left[ \bar{\mathcal{M}}^s(P_{\gamma}) \Gamma_1 U^s(p_{\gamma}) \right] \left[ \mathcal{M}^s(P_{\gamma}) \Gamma_1 U^s(p_{\gamma}) \right]^*, \tag{65}
\]

To calculate the unpolarized cross-section, we need to average over the quantum states of incoming particles and sum over the final states, therefore we replace the above matrix element squared

\[
|\mathcal{M}|^2 \to \frac{1}{(2s + 1)(2r + 1)} \sum_{\text{all states}} |\mathcal{M}|^2 \equiv |\bar{\mathcal{M}}|^2. \tag{66}
\]

Now we need to do the spin sum for both electrons and photons. First we will do the spin sum for electrons, therefore

\[
|\bar{\mathcal{M}}|^2 = \frac{A^2}{6} \sum_{r,r'} \sum_{s,s'} \left[ \bar{\mathcal{M}}^s(P_{\gamma}) \Gamma_1 U^s(p_{\gamma}) \right] \left[ \mathcal{M}^{s'}(p_{\gamma}) \Gamma_1 U^{s'}(P_{\gamma}) \right], \tag{67}
\]

where

\[
\Gamma_1 = \gamma^0 \Gamma_1^0 \gamma^0. \tag{68}
\]
It can be further rewritten in a short hand notation as

$$\overline{M^2} = \frac{A^2}{6} \sum_{r,s',r'} \text{Tr} \left[ Q \sum_{s'} U^s(P_\gamma) \overline{U}^{s'}(P_\gamma) \right] ,$$

where

$$Q = \Gamma_1 \sum_s U^s(p_\gamma) \overline{U}(p_\gamma) \Gamma_1 .$$

Now summing over the photon polarization states, $\overline{M^2}$ becomes

$$\overline{M^2} = A^2 \sum_{r,s,\mu,\nu} \varepsilon^\mu_{\mu'}(K) \varepsilon^{s'}_\nu(k) \text{Tr} \left\{ \left\{ \gamma^\mu(1 + \gamma^0 \gamma^3 \gamma^5)(\gamma^\nu q_0 - \gamma^\nu q_3)(\gamma^\lambda)(\tilde{p}_\parallel + \tilde{p}_\parallel \gamma^5) \right\} \times \left\{ (\gamma^k)(\gamma^0 q_0 - \gamma^3 q_3)(1 - \gamma^5 \gamma^3 \gamma^0)(\gamma^\lambda)(\tilde{p}_\parallel + \tilde{p}_\parallel \gamma^5) \right\} \right\} .$$

Using the spin-sum over the photon polarization states

$$\sum_r \varepsilon^r_{\mu} \varepsilon^r_{\nu} = -g_{\mu\nu} ,$$

$\overline{M^2}$ becomes

$$\overline{M^2} = A^2 \sum_{r,s,\mu,\nu} \text{Tr} \left\{ \left\{ \gamma^\mu(1 + \gamma^0 \gamma^3 \gamma^5)q_\parallel(\gamma^\nu)(\tilde{p}_\parallel + \tilde{p}_\parallel \gamma^5) \right\} \times \left\{ (\gamma^k)(1 - \gamma^5 \gamma^3 \gamma^0)\gamma_\mu(\tilde{p}_\parallel + \tilde{p}_\parallel \gamma^5) \right\} \right\} .$$

Applying the cyclic properties of traces and the following properties of the gamma matrices

$$\gamma_\mu \gamma^\mu = -2\gamma_\mu \text{ and } \gamma^5 \gamma^\nu + \gamma^\nu \gamma^5 = 0 ,$$

and multiplying all the terms carefully, $\overline{M^2}$ will have sixteen terms and after evaluating the sixteen terms \footnote{Detailed calculation is given in Appendix A.1}, the square of the matrix element for the $s$-channel becomes,

$$\overline{M^2} = A^2 \left[ 16(\tilde{p}_\parallel \cdot q_\parallel)P^3 q_0 - 16(\tilde{p}_\parallel \cdot q_\parallel)q_3^3 P^0 + 8q_3^2 P^0 q_3 P^3 - 8q_3^2 P^3 q_3 P^0 + 8(p_\parallel \cdot q_\parallel)\tilde{P}^3 q_0 \\
- 8(p_\parallel \cdot q_\parallel)\tilde{P}^0 q_3^3 - 4q_3^2 P^0 P^3 - 16(\tilde{p}_\parallel \cdot q_\parallel)\tilde{P}^3 q_0^3 - 8q_3^2 P^3 P^0 + 8q_3^2 P^0 P^3 - 16(\tilde{p}_\parallel \cdot q_\parallel)q_3^3 P^0 q_0 - 16(p_\parallel \cdot q_\parallel)P^3 q_3^3 - 8q_3^2 P^0 P^3 + 8q_3^2 P^3 P^3 \right] .$$

The above equation can be further written in Lorentz invariant form as

$$\overline{M^2} = \frac{A^2}{6} \left[ 32(\tilde{p}_\parallel \cdot q_\parallel)(\tilde{P}_\parallel \cdot q_\parallel) + 24(p_\parallel \cdot q_\parallel)(P_\parallel \cdot q_\parallel) + 8q_\parallel^2 (P_\parallel \cdot p_\parallel) \right] .$$

\footnote{Detailed calculation is given in Appendix A.1.1}
4.2 M-Matrix element squared for $u$-channel

The matrix element given by (61) can be rewritten in a compact form

$$\mathfrak{M}^u = A U^T (P_P)_{\gamma} \Gamma_1 U^*(p_P).$$ \hfill (77)

Like in the $s$-channel (73), the same procedure have been followed to obtain the squared matrix element for the $u$-channel after averaging over the spin states

$$|\mathfrak{M}^u|^2 = \frac{B^2}{24} \text{Tr} \left[ \gamma^\mu \left( 1 + \gamma^0 \gamma^3 \gamma^5 \gamma^\nu \right) \left( \gamma^\nu (\not p + \not p') \not p' \right) \gamma^\mu \right] \times \left[ (\gamma^\nu) (\gamma^0 q_0' - \gamma^3 q_3') (1 - \gamma^5 \gamma^3 \gamma^0) \gamma^\nu \right] \left( \not p + \not p' \not p' \right).$$ \hfill (78)

where $B$ is given by

$$B = -e^2 \left[ \frac{1}{q'^2 - m^2 + i\epsilon} \right].$$ \hfill (79)

The structure of $|\mathfrak{M}^u|^2$ is the same as $|\mathfrak{M}^s|^2$ which is given by (75), except the fact that $q$ is replaced by $q'$, thus $|\mathfrak{M}^u|^2$ takes the form

$$|\mathfrak{M}^u|^2 = \frac{B^2}{6} \left[ 16 (\not p \cdot q') P^3 q^0 - 16 (\not p \cdot q') q^3 P^0 + 8 q^0 P^2 P^0 P^3 - 8 q^2 P^0 P^3 + 8 (p \cdot q') P^3 P^0 - 8 q^0 P^2 P^3 - 8 q^0 P^2 P^3 + 16 (\not p \cdot q') P^3 q^0 - 8 q^0 P^2 P^3 + 8 q^0 P^2 P^3 \right].$$ \hfill (80)

The above equation can be further written in Lorentz-invariant form as

$$|\mathfrak{M}^u|^2 = \frac{B^2}{6} \left[ 32 (\not p \cdot q') (\not p \cdot q') + 24 (p \cdot q') (p \cdot q') + 8 q^2 (p \cdot p) \right].$$ \hfill (81)

4.3 M-Matrix element squared for interference term

In Compton scattering, there are two possibilities of the interaction of electron with photon. Since photons are the identical particles, both $s$ and $u$-channel diagrams need to be taken into account, where in $s$-channel, initially electron absorbs a photon and in $u$-channel, initially electron emits a photon. So initially we are assuming that there is finite probability of crossing between these two processes, known as the interference term $\mathfrak{M} = \mathfrak{M}^s + \mathfrak{M}^u$ in the total squared matrix element...
to make the total transition amplitude invariant. We will first evaluate the \( \mathcal{M}^{s\mu} \), which can be written by the eqs (60) and (61) as

\[
\mathcal{M}^{s\mu} = \frac{AB}{6} \sum_{r,r'} \sum_{s,s'} \left[ \mathcal{U}^r(P_Y) \Gamma_1 U^s(p_Y) \right] \left[ \mathcal{U}^s(p_Y) \Gamma_2 U^r(P_Y) \right],
\]

where

\[
\Gamma_1 = \varepsilon_{\mu}^r(K) \varepsilon_{\nu}^r(k) \gamma^\nu(\gamma^0 q_0 - \gamma^3 q_3)(1 - \gamma^5 \gamma^3 \gamma^0) \gamma^\mu,
\]

\[
\Gamma_2 = \varepsilon_{\nu}^r(K) \varepsilon_{\mu}^r(k) \gamma^\mu(\gamma^0 q_0' - \gamma^3 q_3')(1 - \gamma^5 \gamma^3 \gamma^0) \gamma^\nu.
\]

The sum over the spin states for the electrons can be written as

\[
\mathcal{M}^{s\mu} = \frac{AB}{6} \sum_{r,r'} \sum_{s,s'} \left[ \sum_{s'} U^r_s(p_Y) \frac{Q}{p} \left\{ \gamma^\mu(1 + \gamma^0 \gamma^3 \gamma^5)(\gamma^0 q_0 - \gamma^3 q_3)(\gamma^\nu)\right\} \right] \left[ \sum_{s} U^s(p_Y) \frac{Q}{p} \left\{ \gamma^\nu(1 - \gamma^5 \gamma^3 \gamma^0)(\gamma^\lambda)\right\} \right],
\]

where

\[
Q = \Gamma_1 \sum_s \left[ U^s(p_Y) \frac{Q}{p} \right] \Gamma_2.
\]

Now we sum over the spin states of photons,

\[
\mathcal{M}^{s\mu} = \frac{AB}{6} \sum_{r,r'} \varepsilon_{\mu}^r(K) \varepsilon_{\nu}^r(k) \varepsilon_{\lambda}(k) \varepsilon_{\lambda}(k) \frac{Q}{p} \left\{ \gamma^\mu(1 + \gamma^0 \gamma^3 \gamma^5)(\gamma^0 q_0 - \gamma^3 q_3)(\gamma^\nu)\right\} \left\{ \gamma^\nu(1 - \gamma^5 \gamma^3 \gamma^0)(\gamma^\lambda)\right\},
\]

which can be further simplified, using the completeness condition of photons, as

\[
\mathcal{M}^{s\mu} = \frac{AB}{24} \frac{Q}{p} \left\{ \gamma^\mu(1 + \gamma^0 \gamma^3 \gamma^5)\right\} \left\{ \gamma^\nu(1 - \gamma^5 \gamma^3 \gamma^0)\right\},
\]

and is split into 32 terms ((171) in Appendix A2). After evaluating the trace of each term\(^5\), we finally obtain the first interference term in Compton scattering in a strong magnetic field

\[
\mathcal{M}^{s\mu} = 0.
\]

\(^5\)which is calculated in Appendix A.2.1
For calculating the other interference term, $\mathcal{M}^{u\to s\star}$, we follow the same procedure as for $\mathcal{M}^{s\to u\star}$ and obtain

$$
\mathcal{M}^{u\to s\star} = \frac{AB}{24} \text{Tr} \left\{ \gamma^\mu (1 + \gamma^0 \gamma^3 \gamma^5) \gamma_\parallel (\gamma^\nu) (p_\parallel + \bar{p}_\parallel \gamma^5) \right\} 
\times \left\{ (\gamma_\mu) \gamma_\parallel (1 - \gamma^5 \gamma^3 \gamma^5) (\gamma_\nu) (\bar{P}_\parallel + \bar{P}_\parallel \gamma^5) \right\}.
$$

(90)

This can be further simplified similar to the first interference term. Thus the second interference term becomes

$$
\mathcal{M}^{u\to s\star} = 0.
$$

(91)

Thus the interference between the $s$- and $u$-channel in strong magnetic field limit vanishes.

## 5 Crosssection

Let us illustrate the usual procedure to compute the crosssection for the Compton scattering from the above transition amplitude in the presence of strong magnetic field. In general, the crosssection is given by

$$
d\sigma = \frac{|S_{fi}|^2}{TF} \, d\rho,
$$

(92)

where $F$ is the flux factor and $d\rho$ is the differential phase space. In case of magnetic field, the differential phase factor for the electron gets modified due to the gauge dependence. We have used the Landau gauge ($A^\mu = (0, -BY, 0, 0)$), in which the Hamiltonian does not commute with the $Y$-component of the momentum ($p_Y$), so $p_Y$ will not be a conserved quantity. So we will have the three dimensional Dirac-delta function and the $dp_Y$ component of the momentum is missing in the phase space of the electron. Since the photon is not affected by the magnetic field so the phase space factor of the photon will remain the same as in vacuum. We will calculate the crosssection in the lab frame, where the initial photon is along the direction of magnetic field and the electron is at rest initially. So we assign the initial ($p, k$) and final ($P, K$) four momentum of the electron and photon, respectively.

$$
p^\mu = (m, 0, 0, 0), \quad P^\mu = (E', P_X, P_Y, P_Z),
$$

$$
k^\mu = (\omega, 0, 0, \omega), \quad K^\mu = (\omega', K_X, K_Y, \omega' \cos \theta),
$$

(93)
where, $\theta$ is the angle between the initial and the final direction of photon. Hence the momentum appeared in the electron propagator in extreme relativistic limit for the $s$- and $u$-channel are given by

$$
q^0 = p^0 + K^0 = m + \omega, \quad q^3 = \omega, \\
q'^0 = p^0 - K^0 = m - \omega', \quad q'^3 = -\omega' \cos \theta,
$$

respectively.

Now the differential phase space for the electron is modified by the mass-shell condition in the strong magnetic field, $P_\parallel^2 = m^2$ [30, 31],

$$
\frac{dP_0 dX dP_Z}{(2\pi)^2} \delta(P_\parallel^2 - m^2) \Theta(P_0) = \frac{dP_X dP_Z dP_0}{(2\pi)^2} \frac{\delta(P_0 - E')}{2E'},
$$

where $E' = \sqrt{p_Z^2 + m^2}$. However, the photon phase factor is simply given by

$$
\frac{d^4 K}{(2\pi)^3} \delta(K^2) \Theta(K^0) = \frac{d^3 K dK^0}{(2\pi)^3} \frac{\delta(K^0 - \omega')}{2\omega'}.
$$

Now substituting the simplified photon and electron phase factors and using the property of Dirac-delta function, the crosssection for the Compton scattering is reduced into

$$
\sigma = \frac{\pi^2}{2} \frac{d^4 K}{(2\pi)^3} \delta(K^2) \Theta(K^0) \frac{d^3 K dK^0}{(2\pi)^3} \frac{\delta(K^0 - \omega')}{2\omega'} \frac{c\Omega^s + d\Omega^u}{E'},
$$

where the flux factor for the process $1 + 2 \rightarrow 3 + 4$ is given by

$$
F = |v_1 - v_2| \frac{2E_1 E_2}{E'},
$$

where $v_1$ and $v_2$ are the velocities and $E_1$ and $E_2$ are the energies of the target and projectile, respectively. In the lab-frame, the flux factor becomes

$$
F = 4v_2 m_1 p_2 = 4m_1 E_2.
$$

In our case, where the target is the electron and the projectile is the photon, hence the flux factor becomes

$$
F = 4m_\omega.
$$

Thus the crosssection becomes

$$
\sigma = \frac{\pi^2}{8} \frac{d^4 K}{(2\pi)^3} \delta(K^2) \Theta(K^0) \frac{d^3 K dK^0}{(2\pi)^3} \frac{\delta(K^0 - \omega')}{2\omega'} \frac{c\Omega^s + d\Omega^u}{m \omega},
$$
where,
\[
|c\overrightarrow{M} + d\overrightarrow{M}|^2 = c^2|\overrightarrow{M}^s|^2 + d^2|\overrightarrow{M}^u|^2 + cd\overrightarrow{M}^s\overrightarrow{M}^u + cd\overrightarrow{M}^u\overrightarrow{M}^s. \tag{102}
\]

Since, the matrix element due to interference has no contribution as given in (89) and (91). So finally
\[
|c\overrightarrow{M} + d\overrightarrow{M}|^2 = c^2|\overrightarrow{M}^s|^2 + d^2|\overrightarrow{M}^u|^2 \tag{103}
\]

Evaluating \(c^2\) and \(d^2\) in the lab frame
\[
c^2 = e^{-\left(\frac{3K^2}{4eB}\right)} \exp\left(\frac{-2q_X^2}{eB}\right), \tag{104}
\]
\[
d^2 = e^{-\left(\frac{3K^2}{4eB}\right)} \exp\left(\frac{2q_X^2}{eB}\right), \tag{105}
\]
the \(K_Y\)-momentum integration results in
\[
\int |c\overrightarrow{M} + d\overrightarrow{M}|^2 dK_Y = \sqrt{\frac{4eB}{3}} \left[ \exp\left(\frac{-2q_X^2}{eB}\right) |\overrightarrow{M}^s|^2 \right] \left[ \exp\left(\frac{-2q_X^2}{eB}\right) |\overrightarrow{M}^u|^2 \right]. \tag{106}
\]

After doing the \(P_X\)-momentum integration, the crosssection comes to be
\[
\sigma = \frac{eB\pi^2}{4\sqrt{3}} \int \frac{dP_Z}{mE'\omega'} \delta(E' + \omega' - E - \omega) \left[ \exp\left(\frac{-2q_X^2}{eB}\right) |\overrightarrow{M}^s|^2 \right] \left[ \exp\left(\frac{-2q_X^2}{eB}\right) |\overrightarrow{M}^u|^2 \right]. \tag{107}
\]
where the remaining \(P_Z\) integration will be done for the matrix elements in different channels as well as for the cross-terms one by one.

5.1 Crosssection due to the s-channel diagram

The crosssection expression for the s-channel diagram is given by
\[
\sigma^s(\gamma e^- \rightarrow \gamma e^-) = \frac{eB\pi^2}{4\sqrt{3}} \int \frac{dP_Z}{E'\omega'} \delta(E' + \omega' - E - \omega) \exp\left(\frac{-2q_X^2}{eB}\right) |\overrightarrow{M}^s|^2. \tag{108}
\]

Now we express \(|\overrightarrow{M}|^2\) in the lab frame by using the eqs. (93)-(94) as
\[
|\overrightarrow{M}|^2 = \frac{4A^2}{3} \left[ 4m(q^0)^2 + 3m(q^3)^2 - 7mq^0q^3 \right]. \tag{109}
\]
In the ultra-relativistic limit \((E' = P_Z)\), the real part of \(|\overrightarrow{M}|^2\) becomes
\[
|\overrightarrow{M}|^2 = \frac{4A^2}{3} [m^2\omega] P_Z = e^4 P_Z/3\omega, \tag{110}
\]
20
hence the crosssection due to the s-channel is

\[ \sigma^s(\gamma e^- \rightarrow \gamma e^-) = e^4 \frac{eB\pi^2}{4\sqrt{3}} \int \frac{dP_Z}{P_z} \delta(E' + \omega' - E - \omega) \frac{\exp(-\frac{q_X^2}{2eB}) P_z}{m\omega}. \] (111)

Using the kinematic relations from the energy-momentum conservation in strong magnetic field limit \((p_\perp \approx 0)\)

\[ \omega' = \frac{\omega}{1 + \frac{\omega(1 - \cos \theta)}{m}}, \quad P_Z = \frac{(m\omega + \omega^2)(1 - \cos \theta)}{m + \omega(1 - \cos \theta)}, \] (112)

the above crosssection becomes

\[ \sigma^s(\gamma e^- \rightarrow \gamma e^-) = e^4 \frac{eB\pi^2}{12\sqrt{3}m\omega^2} \exp\left(\frac{-2q_X^2}{eB}\right) \int_0^\pi \frac{(m + \omega)\sin \theta}{m + \omega(1 - \cos \theta)} d\theta \times \delta \left( \sqrt{\left[ \frac{(m\omega + \omega^2)(1 - \cos \theta)}{m + \omega(1 - \cos \theta)} \right]^2 + m^2 - \frac{(\omega^2)(1 - \cos \theta)}{m + \omega(1 - \cos \theta)} - m} \right). \] (113)

Using the property of Dirac-delta function

\[ \delta[f(x)] = \sum_{i=1}^n \frac{\delta(x - x_i)}{f'(x_i)}, \] (114)

\[ \sigma^s \] becomes \(^6\),

\[ \sigma^s(\gamma e^- \rightarrow \gamma e^-) = e^4 \frac{eB\pi^2(m + \omega)}{12\sqrt{3}m\omega^2} \exp\left(\frac{-2q_X^2}{eB}\right) \left[ \frac{1}{\omega^2} - \frac{m + 2\omega}{9m\omega^2 + 2\omega^3} \right], \] (115)

where the factor \(\exp\left(\frac{-2q_X^2}{eB}\right)\) can be approximated to unity in strong magnetic field limit. In the lab frame, considering the direction of incoming photon along the direction of magnetic field \((Z\text{-direction})\), i.e. \(q_X = p_X + k_X = m\), \(\sigma^s\) is further simplified into

\[ \sigma^s(\gamma e^- \rightarrow \gamma e^-) = e^4 \frac{eB\pi^2(m + \omega)}{12\sqrt{3}m\omega^2} \left[ \frac{1}{\omega^2} - \frac{m + 2\omega}{9m\omega^2 + 2\omega^3} \right]. \] (116)

### 5.2 Crosssection due to the u-channel diagram

The crosssection due to the u-channel is given by

\[ \sigma^u(\gamma e^- \rightarrow \gamma e^-) = \frac{eB\pi^4}{4\sqrt{3}} \int \frac{dP_Z}{E'\omega} \delta(E' + \omega' - E - \omega) \frac{\exp(-\frac{q_X^2}{2eB}) |M_u|^2}{m\omega}. \] (117)

\(^6\)which is calculated in Appendix A.2
Compton scattering in vacuum

For the sake of comparison, we have also calculated the cross-section (in units of mb) for the

Thus the total cross-section for the Compton scattering (in units of mb) is obtained as

Similar to the $s$-channel, $|M|^2$ has been translated into the lab frame by eqs. (93)- (94)

\[
|M|^2 = \frac{4B^2}{3} \left[ 4m(q^0)^2 + 3m(q^3)^2 - 7mq^0q^0 \right]
\]

\[
= \frac{4e^4}{3} \left[ \frac{m^3\omega^2(4 + 3\cos^2\theta - 7\cos\theta) + m^3\omega(7\cos\theta - 8)(m + \omega(1 - \cos\theta)) [m + \omega(1 - \cos\theta)]^2}{m^2\omega^2(1 - \cos^2\theta) - 2m^2\omega \cos\theta (m + \omega(1 - \cos\theta))^2} \right] P_Z.
\]

Substituting $dP_Z$, $\omega'$, $\delta(E' + \omega' - E - \omega)$ from Eq.(112), the above matrix element squared gives the
cross-section for the $u$-channel

\[
\sigma^u = \frac{e^4 eB\pi^2}{3\sqrt{3}m\omega} \exp \left( \frac{-2q_X^2}{eB} \right) \times \int_0^\pi \frac{(m + \omega) \sin\theta}{m + \omega(1 - \cos\theta)} d\theta
\]

\[
\times \frac{[m^3\omega^2(4 + 3\cos^2\theta - 7\cos\theta) + m^3\omega(7\cos\theta - 8)(m + \omega(1 - \cos\theta))] [m + \omega(1 - \cos\theta)]^2}{m^2\omega^2(1 - \cos^2\theta) - 2m^2\omega \cos\theta (m + \omega(1 - \cos\theta))^2} \]

\[
\times \delta \left[ \sqrt{\left[ \frac{(m\omega + \omega^2)(1 - \cos\theta)}{m + \omega(1 - \cos\theta)} \right]^2 + m^2 - \frac{(\omega^2)(1 - \cos\theta)}{m + \omega(1 - \cos\theta)} - m} \right].
\]

Using the same property of Dirac-delta function (114), the cross-section for the $u$-channel in the
strong magnetic field becomes

\[
\sigma^u = \frac{e^4 eB\pi^2}{12\sqrt{3}} \exp \left( \frac{-2q_X^2}{eB} \right) \left[ \frac{61m^2 + 78m\omega^2 + 32\omega^2}{m^2\omega[9m\omega^2 + 2\omega^3]} - \frac{1}{m\omega^3} \right].
\]

In the lab frame, we consider the initial direction of photon is along the direction of strong magnetic
field, so, $q'_X (= P_X - k_X)$ becomes zero, therefore the $\sigma^u$ becomes

\[
\sigma^u = \frac{e^4 eB\pi^2}{12\sqrt{3}} \left[ \frac{61m^2 + 78m\omega^2 + 32\omega^2}{m^2\omega[9m\omega^2 + 2\omega^3]} - \frac{1}{m\omega^3} \right].
\]

Thus the total cross-section for the Compton scattering (in units of mb) is obtained as

\[
\sigma = \sigma^s + \sigma^u = \frac{e^4 eB\pi^2}{12\sqrt{3}} \left[ \frac{(m + \omega)}{m\omega^2} \left( \frac{1}{\omega^2} - \frac{m + 2\omega}{9m\omega^2 + 2\omega^3} \right) + \frac{61m^2 + 78m\omega^2 + 32\omega^2}{m^2\omega[9m\omega^2 + 2\omega^3]} - \frac{1}{m\omega^3} \right].
\]

For the sake of comparison, we have also calculated the cross-section (in units of mb) for the
Compton scattering in vacuum [32,33] as

\[
\sigma_{\text{Vacuum}}(\gamma e^- \rightarrow \gamma e^-) = \frac{e^4}{8\pi m^2} \left[ \frac{1 + r}{(1 + 2r)^2} + \frac{2}{r^2} - \frac{2(1 + r) - r^2}{2r^3} \ln(1 + 2r) \right],
\]

with a dimensionless variable, $r = \frac{q}{m}$. To see the effect of strong magnetic field on the Compton scattering, we have plotted the
cross-section as a function of photon energy for the different strengths of magnetic fields in Figure
3, in addition to the crosssection in the vacuum only. We have found that the crosssection gets decreased in the presence of strong magnetic field. However, with the further increase of strong magnetic field, the crosssection increases.

![Figure 3: A plot between the crosssection and initial photon energy for different strengths of magnetic fields and vacuum (B=0) case.](image)

6 Conclusions

In neutron stars as well as in noncentral events of ultrarelativistic heavy ion collisions at Relativistic Heavy-ion Collider and Large Hadron Collider, a very strong magnetic field in the range $10^{16} - 10^{20}$ Gauss is expected to be present. Also a very strong magnetic field ($\sim 10^{23}$ Gauss) may have existed in the early universe. Thus Compton scattering in strong magnetic field has huge importance in astrophysics as well as in terrestrial laboratory, which therefore motivates us to revisit the Compton scattering in a strong magnetic field. For that purpose, using the Dirac spinor in strong magnetic field, we have first calculated the square of the S-matrix element and then have summed (averaged) over the final (initial) states. Thereafter we have illustrated the usual procedure to compute the crosssection in the lab-frame, where the photons are incident parallel to the magnetic field and the electron is at rest initially, by constructing the Lorentz invariant phase space, incident flux.
factor, energy-momentum conserving Dirac delta functions etc. in the strong magnetic field. We have noticed that the Compton scattering at the tree level gets suppressed due to the presence of strong magnetic field compared to the vacuum, when we increase the incident photon energy. But the crosssection increases linearly with the strength of magnetic field in strong magnetic field limit. In some related works on Compton scattering in strong magnetic field [15, 16], authors have considered the resonant Compton scattering, spin-dependent influences etc. They have also considered the finite width of the Landau levels, so that electrons can excite to the higher Landau levels. But we consider the magnetic field to be strong enough so that no electron can excite to the higher landau levels, it will remain only in its ground state.

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A Matrix Element for Compton scattering:($\gamma e^- \rightarrow \gamma e^-$)

This appendix contains detailed calculation of the squared matrix element for the $s$-channel, the $u$-channel and the interference between them.

A.1 Matrix Element for $s$-channel process

In this appendix we will show the calculation of $|M|^2$. Then for the unpolarized crosssection, we need to average over the quantum states of incoming particles and sum over the final states, therefore we replace the above matrix element squared

$$|M|^2 \rightarrow \frac{1}{(2s+1)(2r+1)} \sum_{\text{all states}} |M|^2 \equiv \overline{|M|^2}. \quad (125)$$

We will now calculate the average square of the matrix element $|\overline{M}|^2$ and then sum over the spin-states. Firstly we will do the spin-sum for the electron and then after some simplification of $|\overline{M}|^2$, we will do the spin sum for the photon polarization vectors. Now the averaged squared matrix element for $s$-channel process using(62) can be written as

$$\overline{|M|^2} = \frac{A^2}{6} \sum_{r,r'} \sum_{s,s'} \left[ \overline{U'}(P_Y)\Gamma_1 U^s(p_Y) \right] \left[ \overline{U'}(P_Y)\Gamma_1 U^{s'}(p_Y) \right]^* \quad (126)$$
Since, $\left[ \mathcal{U}^s \left( P_\gamma \right) \Gamma_1 U^s \left( p_\gamma \right) \right]^* \! = \! \left[ \mathcal{U}^{s'} \left( P_\gamma \right) \Gamma_1 U^s \left( p_\gamma \right) \right]^\dagger = \left[ U^{s'} \left( P_\gamma \right) \gamma^0 \Gamma_1 \gamma^0 U^s \left( p_\gamma \right) \right]^\dagger$.

Using $(\gamma^0)^2 = 1,$

$$\left[ \mathcal{U}^s \left( P_\gamma \right) \Gamma_1 U^s \left( p_\gamma \right) \right]^* \! = \! \left[ \mathcal{U}^s \left( p_\gamma \right) \Gamma_1 U^s \left( P_\gamma \right) \right] ,$$

Using Eq. (129), $|\mathcal{M}|^2$ becomes

$$|\mathcal{M}|^2 = \frac{A^2}{6} \sum_{r,r'} \sum_{s,s'} \left[ \sum_{s'} \mathcal{U}^s \left( P_\gamma \right) \Gamma_1 \sum_s \left( U^s \left( p_\gamma \right) \mathcal{U}^{s'} \left( p_\gamma \right) \Gamma_1 U^{s'} \left( P_\gamma \right) \right) \right] ,$$

where

$$\Gamma_1 = \gamma^0 \Gamma^\dagger \gamma^0 .$$

Now, we need to do spin sum for both electrons and photons. First we will do spin sum for electrons as follows

$$|\mathcal{M}|^2 = \frac{A^2}{6} \sum_{r,r'} \left[ \sum_{s'} \mathcal{U}^s \left( P_\gamma \right) \Gamma_1 \sum_s \left( U^s \left( p_\gamma \right) \mathcal{U}^{s'} \left( p_\gamma \right) \Gamma_1 U^{s'} \left( P_\gamma \right) \right) \right] .$$

Simplifying $|\mathcal{M}|^2,$

$$|\mathcal{M}|^2 = \frac{A^2}{6} \sum_{r,r'} \left[ \sum_{s'} \mathcal{U}^s \left( P_\gamma \right) Q \mathcal{U}^{s'} \left( p_\gamma \right) \Gamma_1 \right] ,$$

where,

$$Q = \Gamma_1 \sum_s U^s \left( p_\gamma \right) \mathcal{U}^{s'} \left( p_\gamma \right) \Gamma_1 \! = \! \gamma^0 \Gamma^\dagger \gamma^0 .$$

Since, $|\mathcal{M}|^2$ is a number, we can multiply all the elements explicitly as

$$|\mathcal{M}|^2 = \frac{A^2}{6} \sum_{r,r'} \sum_{s} \left[ \sum_{k,j} \mathcal{U}_k^s \left( P_\gamma \right) Q_{kj} U_j^{s'} \left( P_\gamma \right) \right] .$$
Now, taking spin sum over final state of electron spinor

\[
|\mathcal{M}|^2 = \frac{A^2}{6} \sum_{r,r'} \sum_{kj} Q_{kj} \sum_{s'} U^r_k (P_Y) U^r_s (P_Y),
\]

\[
= \frac{A^2}{6} \sum_{r,r'} \sum_{kj} \left[ \sum_{s'} QU^s_k (P_Y) \bar{U}^{s'}_j (P_Y) \right]_{jk},
\]

\[
= \frac{A^2}{6} \sum_{r,r'} \sum_{k} \left[ \sum_{s'} QU^s_k (P_Y) \bar{U}^{s'}_k (P_Y) \right]_{kk},
\]

\[
= \frac{A^2}{6} \sum_{r,r'} T r \left[ Q \sum_{s'} U^s_k (P_Y) \bar{U}^{s'}_k (P_Y) \right].
\]

(136)

Now, we need to simplify the value of \( Q \). To do this first simplify \( \Gamma(1) \)

\[
\Gamma_1 = \gamma^0 \Gamma^+_1 \gamma^0 = \gamma^0 [\epsilon^r_{\mu} (K) \gamma^\mu (1 + \gamma^0 \gamma^3 \gamma^5) (\gamma^0 q_0 - \gamma^3 q_3)] \gamma^0 \epsilon^r_{\nu}(k)^\dagger \gamma^0.
\]

(137)

Using the following identities of gamma matrices,

\[
(\gamma^0)^2 = I, \quad \gamma^0 (\gamma^\nu) \gamma^0 = \gamma^\nu, \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \gamma^{\mu \nu},
\]

we obtain,

\[
\Gamma_1 = \epsilon^r_{\mu} (K) \epsilon^r_{\nu}(k) \gamma^\nu (\gamma^0 q_0 - \gamma^3 q_3) (1 - \gamma^5 \gamma^3 \gamma^0) \gamma^\mu.
\]

(139)

Now, using the spin sum from (17) and substituting \( \Gamma_1 \) and \( \Gamma_1 \) from (64) and (139), respectively, into \( Q \) (134) and then substituting it in (69), we finally obtain (71). Taking the spin sum over the photon polarization index \( r \) and \( r' \) and using

\[
\sum_{r} \epsilon^r_{\mu} \epsilon^r_{\lambda} = -g_{\mu \lambda},
\]

(140)

Eq. (71) becomes

\[
|\mathcal{M}|^2 = \frac{A^2}{24} T r \left[ \gamma^\mu (1 + \gamma^0 \gamma^3 \gamma^5) \delta^r (\gamma^\nu (P_\parallel + \tilde{P}_\parallel \gamma^5)) \times [(\gamma^\nu) \delta^r (1 - \gamma^5 \gamma^3 \gamma^0) \gamma^\mu (P_\parallel + \tilde{P}_\parallel \gamma^5)] \right].
\]

Now, applying cyclic properties of traces and the following properties

\[
\gamma^\mu \gamma^\nu = -2 \delta^\mu \nu \quad and \quad \gamma^5 \gamma^\nu + \gamma^\nu \gamma^5 = 0,
\]

(141)
and multiplying it carefully, we get

$$\left| \mathcal{M} \right| = \frac{A^2}{6} \left[ \tilde{p}_{\parallel \alpha} \tilde{P}_{\parallel \nu} q_{\parallel \eta} q_{\parallel \theta} T r (\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\gamma) - \tilde{p}_{\parallel \alpha} \tilde{P}_{\parallel \nu} T r (\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\gamma) \right] = 0.$$ (142)

We obtain sixteen terms as shown below.

### A.1.1 Calculation of all sixteen Terms

**Term-1**

$$\tilde{p}_{\parallel \alpha} \tilde{P}_{\parallel \nu} q_{\parallel \eta} q_{\parallel \theta} T r (\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\gamma) = 8(\tilde{p}_{\parallel} q_{\parallel})(\tilde{P}_{\parallel} q_{\parallel}) - 4q_{\parallel}^2(\tilde{p}_{\parallel} \tilde{P}_{\parallel}).$$ (143)

**Term-2**

$$-\tilde{p}_{\parallel \alpha} \tilde{P}_{\parallel \nu} q_{\parallel \eta} q_{\parallel \theta} T r (\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\gamma) = 0.$$ (144)

**Term-3**

$$\tilde{p}_{\parallel \alpha} \tilde{P}_{\parallel \nu} q_{\parallel \eta} q_{\parallel \theta} T r (\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\gamma) = 0.$$ (145)

**Term-4**

$$\tilde{p}_{\parallel \alpha} \tilde{P}_{\parallel \nu} q_{\parallel \eta} q_{\parallel \theta} T r (\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\gamma) = -8(\tilde{p}_{\parallel} q_{\parallel})q^3 P^0 + 8(\tilde{p}_{\parallel} q_{\parallel})q^0 P^3 - 4q_{\parallel}^2 P^2 + 4q_{\parallel}^2 P^0 \tilde{P}.$$ (146)

**Term-5**

$$-p_{\parallel \parallel} \tilde{p}_{\parallel \nu} q_{\parallel \eta} q_{\parallel \theta} T r (\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\gamma) = 0.$$ (147)
Term-6
\[ p_{|\beta} P_{|\nu} q_{|\eta} q_{|\theta} \text{Tr}(\gamma^\nu \gamma^\beta \gamma^\theta \gamma^\nu) = 8(q_{|\beta} q_{|\nu})(q_{|\eta} q_{|\theta}) - 4q_{|\beta}^2 (P_{|\nu} q_{|\eta}). \]  
(148)

Term-7
\[-p_{|\beta} P_{|\nu} q_{|\eta} q_{|\theta} \text{Tr}(\gamma^\nu \gamma^\beta \gamma^\theta \gamma^3 \gamma^0 \gamma^\nu)\]
\[= 8(p_{|\beta} q_{|\nu})q^0 \tilde{P}^3 - 8(p_{|\beta} q_{|\nu})q^3 \tilde{P}^0 + 4q_{|\beta}^2 \tilde{P}^0 p^3 - 4q_{|\beta}^2 \tilde{P}^0 p^3. \]  
(149)

Term-8
\[-p_{|\beta} P_{|\nu} q_{|\eta} q_{|\theta} \text{Tr}(\gamma^\nu \gamma^\beta \gamma^\theta \gamma^5 \gamma^3 \gamma^0 \gamma^\rho) = 0. \]  
(150)

Term-9
\[\tilde{P}_{|\eta} \gamma q_{|\nu} q_{|\theta} \text{Tr}(\gamma^5 \gamma^0 \gamma^3 \gamma^0 \gamma^\rho \gamma^\eta \gamma^\theta \gamma^\epsilon) = 0. \]  
(151)

Term-10
\[-p_{|\lambda} P_{|\nu} q_{|\eta} q_{|\theta} \text{Tr}(\gamma^0 \gamma^3 \gamma^0 \gamma^\rho \gamma^\lambda \gamma^\theta \gamma^\epsilon)\]
\[= 8(\tilde{p}_{|\lambda} q_{|\nu} q^0 P^3 - 4q_{|\lambda}^2 P^0 P^3 + 4q_{|\lambda}^2 \tilde{P}^0 P^3 - 8(\tilde{p}_{|\lambda} q_{|\nu})q^3 P^0. \]  
(152)

Term-11
\[\tilde{p}_{|\lambda} \tilde{P}_{|\nu} q_{|\eta} q_{|\theta} \text{Tr}(\gamma^0 \gamma^3 \gamma^0 \gamma^\rho \gamma^\lambda \gamma^\theta \gamma^\epsilon)\]
\[= 16(\tilde{p}_{|\lambda} q_{|\nu} q^0 P^0 - 8q_{|\lambda}^2 \tilde{P}^0 P^0 - 16(\tilde{p}_{|\lambda} q_{|\nu})q^3 \tilde{P}^3 + 8q_{|\lambda}^2 \tilde{P}^3 \tilde{P}^0 - 8(\tilde{p}_{|\lambda} q_{|\nu})(\tilde{P}_{|\nu} q_{|\eta}) + 4(\tilde{q}_{|\lambda}^2)(\tilde{p}_{|\lambda} \tilde{P}_{|\nu}). \]  
(153)

Term-12
\[\tilde{p}_{|\lambda} P_{|\nu} q_{|\eta} q_{|\theta} \text{Tr}(\gamma^0 \gamma^3 \gamma^0 \gamma^\rho \gamma^\lambda \gamma^\theta \gamma^\epsilon) = 0. \]  
(154)

Term-13
\[p_{|\delta} \tilde{P}_{|\nu} q_{|\eta} q_{|\theta} \text{Tr}(\gamma^0 \gamma^3 \gamma^0 \gamma^\rho \gamma^\delta \gamma^\epsilon \gamma^\delta \gamma^\epsilon) = 0. \]  
(155)

Term-14
\[P_{|\nu} p_{|\delta} q_{|\eta} q_{|\theta} \text{Tr}(\gamma^5 \gamma^0 \gamma^3 \gamma^0 \gamma^\rho \gamma^\delta \gamma^\epsilon) = 0. \]  
(156)

Term-15
\[\tilde{p}_{|\delta} P_{|\nu} q_{|\eta} q_{|\theta} \text{Tr}(\gamma^0 \gamma^3 \gamma^5 \gamma^0 \gamma^\rho \gamma^\delta \gamma^\epsilon \gamma^3 \gamma^0 \gamma^\nu) = 0. \]  
(157)
\[ p_\parallel q_\parallel q_\theta \text{Tr}(\gamma^0 \gamma^3 \gamma^5 \gamma^7 \gamma^5 \gamma^3 \gamma^0 \gamma^\rho) \]

\[ = 16(p_\parallel q_\parallel q_\theta)q^0 p^0 - 8 q_\parallel^2 p^0 p^0 - 16(p_\parallel q_\parallel q_\theta)q^3 P^3 + 8 q_\parallel^2 p^3 P^3 - 8(p_\parallel q_\parallel q_\theta)(P_\parallel q_\parallel q_\theta) + 4 q_\parallel^2 (p_\parallel P_\parallel). \] (158)

Adding all the sixteen terms from (143)-(158), Eq. (142) becomes Eq. (75).

A.2 Interference-Term of Matrix element

Now we calculate the $M_s M_u^*$, for this we write the matrix elements for $s$-channel and $u$-channel process individually as,

\[ M_s = A \left[ U_s' (P_Y) \Gamma_1 U^s(p_Y) \right], \] (159)

\[ M_u = B \left[ U_u' (P_Y) \Gamma_2 U^s(p_Y) \right], \]

so first interference term $M_s M_u^*$ can be written as

\[ M_s M_u^* = AB \left[ U_s' (P_Y) \Gamma_1 U^s(p_Y) \right] \left[ U_u' (P_Y) \Gamma_2 U^s(p_Y) \right]^*. \] (160)

Since, $\left[ U_s' (P_Y) \Gamma_2 U^s(p_Y) \right]^*$ is a $1 \times 1$ matrix, so it is just a complex number, so we can write complex conjugate equal to its Hermitian conjugate, so

\[ \left[ U_s' (P_Y) \Gamma_2 U^s(p_Y) \right]^* = \left[ U_s' (P_Y) \Gamma_2 U^s(p_Y) \right]^\dagger = \left[ U_u^\dagger (P_Y) \gamma^0 \Gamma_2 \gamma^0 U^s(p_Y) \right]^\dagger. \] (161)

Then, first interference term $M_s M_u^*$ becomes

\[ M_s M_u^* = AB \left[ U_s' (P_Y) \Gamma_1 U^s(p_Y) \right] \left[ U_u^\dagger (p_Y) \Gamma_2 U_s' (P_Y) \right]. \] (162)

Now we need to do spin sum for both electrons and photons. First we will do the spin sum for electrons giving spinors as indices $s$ and $s'$ for the initial and final state of electrons, respectively and then for initial $(r)$ and final state $(r')$ for photon polarization and taking average over $M_s M_u^*$ as we have done in Eq. (125), we can write

\[ M_s M_u^* = \frac{AB}{6} \sum_{r,r'} \sum_{s,s'} \left[ U_s' (P_Y) \Gamma_1 U^s(p_Y) \right] \left[ U_r^s (p_Y) \Gamma_2 U_s' (P_Y) \right], \] (163)

where

\[ \Gamma_1 = \varepsilon^\nu_\mu (K) \varepsilon^\nu_r (k) \gamma^\nu (\gamma^0 q_0 - \gamma^3 q_3)(1 - \gamma^5 \gamma^3 \gamma^0) \gamma^\mu. \] (164)

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\[ \Gamma_2 = \varepsilon^r_\nu K \varepsilon^s_\mu (\gamma^\alpha q'_0 - \gamma^3 q'_3)(1 - \gamma^5 \gamma^3 \gamma^0) \gamma^\mu. \] (165)

Now, first we will do the spin sum for initial state of electron

\[
\overline{\mathfrak{M}}^{\mathfrak{M}_{\mu\nu}} = \frac{AB}{6} \sum_{r,r',s'} \left[ \sum_{s'} U^{s'}(p_Y) \Gamma_1 \sum_s \left[ U^s(p_Y) \overline{U}^s(p_Y) \right] \Gamma_2 U^{s'}(p_Y) \right],
\]

\[
= \frac{AB}{6} \sum_{r,r',s'} \left[ \sum_{s'} U^{s'}(p_Y) QU^{s'}(p_Y) \right],
\]

where,

\[ Q = \Gamma_1 \sum_s \left[ U^s(p_Y) \overline{U}^s(p_Y) \right] \Gamma_2. \] (167)

Since, \( \overline{\mathfrak{M}}^{\mathfrak{M}_{\mu\nu}} \) is a number we can multiply all the elements explicitly as

\[
\overline{\mathfrak{M}}^{\mathfrak{M}_{\mu\nu}} = \frac{AB}{6} \sum_{r,r',s'} \left[ \sum_{i,j} \sum_{s'} \overline{U}^{s'}_i(P_Y) Q U^{s'}_j(P_Y) \right],
\]

\[
= \frac{AB}{6} \sum_{r,r',s'} \left[ Q U^{s'}(P_Y) \overline{U}^{s'}(P_Y) \right],
\]

\[
= \frac{AB}{6} \sum_{r,r'} \sum_i \left[ \sum_{s'} QU^{s'}(P_Y) \overline{U}^{s'}(P_Y) \right],
\]

\[
= \frac{AB}{6} \sum_{r,r'} \sum_i \left[ \sum_{s'} QU^{s'}(P_Y) \overline{U}^{s'}(P_Y) \right],
\]

\[
= \frac{AB}{6} \sum_{r,r'} \sum_i \left[ \sum_{s'} QU^{s'}(P_Y) \overline{U}^{s'}(P_Y) \right] \gamma^i,
\]

\[
= \frac{AB}{6} \sum_{r,r'} \sum_i \left[ \sum_{s'} QU^{s'}(P_Y) \overline{U}^{s'}(P_Y) \right] \gamma^i,
\]

\[
= \frac{AB}{6} \sum_{r,r'} \sum_i \left[ \sum_{s'} QU^{s'}(P_Y) \overline{U}^{s'}(P_Y) \right] \gamma^i,
\]

\[
= \frac{AB}{6} \sum_{r,r'} \sum_i \left[ \sum_{s'} QU^{s'}(P_Y) \overline{U}^{s'}(P_Y) \right] \gamma^i,
\]

\[
= \frac{AB}{6} \sum_{r,r'} \sum_i \left[ \sum_{s'} QU^{s'}(P_Y) \overline{U}^{s'}(P_Y) \right] \gamma^i.
\] (168)

Now, using the spin sum from (17) and substituting \( Q \) from (167) in equation (85), \( \overline{\mathfrak{M}}^{\mathfrak{M}_{\mu\nu}} \) yields to (87).

Now, taking spin sum over all the polarization vectors using

\[
\sum_r \varepsilon^r_\mu \varepsilon^r_\nu = -g_{\mu\nu},
\] (169)

\( \overline{\mathfrak{M}}^{\mathfrak{M}_{\mu\nu}} \) in (87) becomes (88). Now, simplifying \( \overline{\mathfrak{M}}^{\mathfrak{M}_{\mu\nu}} \) and using the properties of gamma matrices, it becomes

\[
\overline{\mathfrak{M}}^{\mathfrak{M}_{\mu\nu}} = \text{CTr} \left[ \gamma^\mu \varepsilon^{\gamma}_\mu \gamma^\nu \varepsilon^{\gamma}_\nu \gamma^\alpha q'_0 + \gamma^\mu \gamma^0 \gamma^3 \gamma^5 \varepsilon^{\gamma}_\mu \gamma^\nu \varepsilon^{\gamma}_\nu \gamma^\alpha q'_3 + \gamma^\mu \gamma^0 \gamma^3 \gamma^5 \gamma^5 \varepsilon^{\gamma}_\mu \gamma^\nu \varepsilon^{\gamma}_\nu \gamma^\alpha q'_5 \right]
\]

\[
\times \left[ q'_\parallel \gamma^\mu \gamma^\nu \varepsilon^{\gamma}_\mu \gamma^\nu + q'_\parallel \gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \varepsilon^{\gamma}_\mu \gamma^\nu \gamma^\nu q'_5 - q'_\parallel \gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \varepsilon^{\gamma}_\mu \gamma^\nu \gamma^\nu q'_5 \right].
\] (170)
We obtain thirty-two terms as shown below.

\[ E = -2\tilde{p}_\parallel \gamma^\nu \tilde{g}_\parallel, \]

\[ F = -2\gamma^5 \gamma^3 \tilde{g}_\parallel \gamma^\nu \tilde{p}_\parallel \gamma^0 + 2\gamma^5 \gamma^0 \tilde{g}_\parallel \gamma^\nu \tilde{p}_\parallel \gamma^3 + 2\gamma^5 \gamma^0 \gamma^3 \tilde{p}_\parallel \gamma^\nu \tilde{g}_\parallel, \]

\[ G = -2\gamma^3 \tilde{g}_\parallel \gamma^\nu \tilde{p}_\parallel \gamma^0 + 2\gamma^0 \tilde{g}_\parallel \gamma^\nu \tilde{p}_\parallel \gamma^3 + 2\gamma^0 \gamma^3 \tilde{p}_\parallel \gamma^\nu \tilde{g}_\parallel, \]

\[ H = -2\gamma^5 \gamma^3 \tilde{g}_\parallel \gamma^\nu \tilde{p}_\parallel, \]

and substituting the value of E,F,G and H in Eq. (170) and multiplying among themselves, we get

\[ \text{Eq. (171)} \]

We obtain thirty-two terms as shown below.
A.2.1 Calculation of Thirty-two Terms

Term-1

\[-2\text{Tr}(\gamma_{\parallel} \gamma_{\parallel} \gamma_{\nu} \gamma_{\nu} P_{\parallel}) = -32(q_{\parallel} q'_{\parallel})(p_{\parallel} P_{\parallel}).\]  \hspace{1cm} (172)

Term-2

\[2\text{Tr}(\gamma_{\parallel} \gamma_{\parallel} \gamma_{\nu} P_{\parallel}) = 0.\]  \hspace{1cm} (173)

Term-3

\[2\text{Tr}(\gamma_{\parallel} \gamma_{\parallel} \gamma_{\nu} P_{\parallel}) = 0.\]  \hspace{1cm} (174)

Term-4

\[2\text{Tr}(\gamma_{\parallel} \gamma_{\parallel} \gamma_{\nu} P_{\parallel}) = 0.\]  \hspace{1cm} (175)

Term-5

\[-2\text{Tr}(\gamma_{\parallel} \gamma_{\parallel} \gamma_{\nu} P_{\parallel}) = 16(\tilde{P}_{\parallel} P_{\parallel})(\tilde{q}_{\parallel} q'_{\parallel}) - 16(q_{\parallel} q'_{\parallel}) \cdot (p_{\parallel} P_{\parallel}) \cdot \gamma_{\parallel} \gamma_{\nu} P_{\parallel} \]  
\[= -16(q_{\parallel} q'_{\parallel})(q^{3} P^{3} + q^{0} P^{3} + 16(q_{\parallel} P_{\parallel})(q^{3} P^{0} + q^{0} P^{3}) + 16(q_{\parallel} P_{\parallel})q^{3} P^{0} - 16(q_{\parallel} P_{\parallel})q^{0} P^{3} - 16(q_{\parallel} P_{\parallel})q^{0} P^{3}.\]  \hspace{1cm} (176)

Term-6

\[2\text{Tr}(\gamma_{\parallel} \gamma_{\parallel} \gamma_{\nu} P_{\parallel}) = -16(\tilde{P}_{\parallel} P_{\parallel})q^{0} q^{3} + 16(q_{\parallel} P_{\parallel})q^{0} P^{3} + 16(q_{\parallel} P_{\parallel})q^{0} - 16(\tilde{P}_{\parallel} P_{\parallel})q^{3} q^{0} \]  
\[= -16(q_{\parallel} P_{\parallel})q^{0} P^{3} + 16(q_{\parallel} P_{\parallel})q^{0} P^{0} + 16(q_{\parallel} P_{\parallel})q^{3} P^{0} - 16(q_{\parallel} P_{\parallel})q^{3} P^{0} \]  
\[+ 16(q_{\parallel} P_{\parallel})q^{3} P^{0} - 16(q_{\parallel} P_{\parallel})q^{3} P^{0} + 16(q_{\parallel} P_{\parallel})q^{0} P^{0} - 16(q_{\parallel} P_{\parallel})q^{3} P^{0}.\]  \hspace{1cm} (177)

Term-7
\[ 2\text{Tr}(\gamma^0 \gamma^5 \gamma^\nu \rho_{\parallel} g^\nu g^\rho P_{\parallel}) = -32(q_{\parallel} q'_{\parallel}) P^3 P^0 + 32(q_{\parallel} q'_{\parallel}) \tilde{P}^3 P^0 \] (178)

\[ = 32(p_{\parallel} \cdot P_{\parallel})(q_{\parallel} \cdot q'_{\parallel}). \] (179)

**Term-8**

\[-2\text{Tr}(\gamma^5 \gamma^\nu \rho_{\parallel} g^\nu g^\rho P_{\parallel}) = 0. \] (180)

**Term-9**

\[-2\text{Tr}(\gamma^5 \gamma^\nu \rho_{\parallel} g^\nu g^\rho P_{\parallel}) = 0. \] (181)

**Term-10**

\[-2\text{Tr}(\gamma^5 \gamma^3 \gamma^\nu \rho_{\parallel} g^\nu g^\rho P_{\parallel} \gamma^5) = 16(p_{\parallel} \cdot \tilde{P}_{\parallel}) q^3 q^0 - 16(q_{\parallel} q'_{\parallel}) q^3 \tilde{P}^0 + 16(q'_{\parallel} \cdot \tilde{P}_{\parallel}) q^3 p^0 - 16(p_{\parallel} \cdot \tilde{P}_{\parallel}) q^0 q^3 \]

\[+ 16(q_{\parallel} q'_{\parallel}) q^3 \tilde{P}^0 - 16(q_{\parallel} \cdot \tilde{P}_{\parallel}) q^3 p^0 + 16(q_{\parallel} q'_{\parallel}) \tilde{P}^0 p^0 + 16(q_{\parallel} q'_{\parallel}) p^3 q^0 \]

\[- 16(q_{\parallel} \cdot \tilde{P}_{\parallel}) q^0 p^3 + 16(q_{\parallel} q'_{\parallel}) \tilde{P}^0 p^0 - 16(q_{\parallel} q'_{\parallel}) q^0 \tilde{P}^3 + 16(q_{\parallel} q'_{\parallel}) q^0 \tilde{P}^3. \] (182)

**Term-11**

\[2\text{Tr}(\gamma^5 \gamma^0 \gamma^\nu \rho_{\parallel} g^\nu g^\rho P_{\parallel} \gamma^5) = -16(p_{\parallel} \cdot \tilde{P}_{\parallel}) q^0 q^3 + 16(q_{\parallel} p_{\parallel}) q^0 (a) \tilde{P}^3 - 16(q_{\parallel} q'_{\parallel}) q^0 p^3 + 16(p_{\parallel} q'_{\parallel}) q^3 q^0 \]

\[- 16(q_{\parallel} q'_{\parallel}) q^0 k^3 + 16(q_{\parallel} q'_{\parallel}) q^0 p^3 + 16(q_{\parallel} q'_{\parallel}) \tilde{P}^3 p^0 + 16(q_{\parallel} q'_{\parallel}) q^0 \tilde{P}^0 - 16(q_{\parallel} q'_{\parallel}) q^3 \tilde{P}^0 \]

\[+ 16(q_{\parallel} q'_{\parallel}) q^3 p^0 - 16(q_{\parallel} q'_{\parallel}) p^3 \tilde{P}^0 + 16(q_{\parallel} q'_{\parallel}) q^0 \tilde{P}^0 - 16(q_{\parallel} q'_{\parallel}) q^3 \tilde{P}^0. \] (183)

**Term-12**

\[2\text{Tr}(\gamma^5 \gamma^0 \gamma^3 \gamma^\nu \rho_{\parallel} g^\nu g^\rho P_{\parallel} \gamma^5) = -32(q_{\parallel} q'_{\parallel}) \tilde{P}^3 p^0 + 32(q_{\parallel} q'_{\parallel}) \tilde{P}^3 p^0 \]

\[= -32(p_{\parallel} \cdot P_{\parallel})(q_{\parallel} \cdot q'_{\parallel}). \] (184)

**Term-13**

\[-2\text{Tr}(\gamma^5 \gamma^0 \gamma^3 \gamma^\nu \rho_{\parallel} g^\nu g^\rho \tilde{P}_{\parallel} \gamma^5) = 0. \] (185)

**Term-14**

\[2\text{Tr}(\gamma^0 \gamma^\nu \rho_{\parallel} g^\nu g^\rho \tilde{P}_{\parallel} \gamma^5) = 0. \] (186)
Term-15

$$2 \text{Tr}(\gamma^0 \gamma^3 \tilde{p} \gamma^\nu q' \gamma^\nu \tilde{P} \gamma^5) = 0.$$  \hfill (187)

Term-16

$$-2 \text{Tr}(\gamma^5 \tilde{p} \gamma^\nu q' \gamma^\nu \tilde{P} \gamma^5) = -32(q_{||} \cdot q_{||})(\tilde{p}_{||} \cdot \tilde{P}_{||}) = 32(p_{||} \cdot P_{||})(q_{||} \cdot q_{||}).$$  \hfill (188)

Term-17

$$-2 \text{Tr}(p^\nu q' \gamma^\nu \gamma^5 \gamma^3 \gamma^0 \gamma_{||} P_{||}) = 0.$$  \hfill (189)

Term-18

$$-2 \text{Tr}(\gamma^5 \gamma^3 q' \gamma^\nu q' \gamma^0 q' \gamma^5 \gamma^3 \gamma^0 \gamma_{||} P_{||}) = 64 p^3 q^0 q^3 P^0 - 64 q^0 q^3 P^3 p^0 - 32(p_{||} \cdot P_{||}) q^0 q^0 + 32(q_{||} \cdot p_{||}) q^0 P^0$$
$$- 32(q_{||} \cdot P_{||}) q^0 p^0 + 32(p_{||} \cdot P_{||}) q^0 q^3 - 32(q_{||} \cdot P_{||}) q^3 p^3 + 32(q_{||} \cdot p_{||}) q^3 P^3$$
$$+ 16(q_{||} \cdot q_{||})(p_{||} \cdot P_{||} - 16(p_{||} \cdot q_{||})(P_{||} \cdot q_{||}) + 16(P_{||} \cdot q_{||})(p_{||} \cdot q_{||}).$$  \hfill (190)

Term-19

$$2 \text{Tr}(\gamma^5 \gamma^0 q' \gamma^\nu q' \gamma^3 q' \gamma^5 \gamma^3 \gamma^0 \gamma_{||} P_{||}) = 64 q^3 q^0 P^0 - 64 q^3 q^0 P^3 q^0 + 32 q^3 q^3 (p_{||} \cdot P_{||}) - 32(q_{||} \cdot p_{||}) q^3 P^3$$
$$+ 32(q_{||} \cdot P_{||}) q^3 p^3 - 32(p_{||} \cdot P_{||}) q^0 q^0 + 32(q_{||} \cdot P_{||}) p^0 q^0 - 32(q_{||} \cdot P_{||}) q^0 P^0$$
$$+ 16(q_{||} \cdot q_{||})(p_{||} \cdot P_{||} - 16(p_{||} \cdot q_{||})(P_{||} \cdot q_{||}) + 16(P_{||} \cdot q_{||})(p_{||} \cdot q_{||}).$$  \hfill (191)

Term-20

$$2 \text{Tr}(\gamma^5 \gamma^0 \gamma^3 q' \gamma^\nu q' \gamma^5 \gamma^3 \gamma^0 \gamma_{||} P_{||}) = -64 p^3 P^0 q^0 q^3 + 32(P_{||} \cdot q_{||}) p^3 q^3 + 128 q^3 P^0 p^0 q^3 - 32(q_{||} \cdot P_{||}) p^0 q^0$$
$$+ 32(q_{||} \cdot p_{||}) q^0 P^0 - 32(q_{||} \cdot q_{||}) q^0 P^0 - 32(P_{||} \cdot q_{||})(p_{||} \cdot q_{||}) + 32(p_{||} \cdot q_{||})(P_{||} \cdot q_{||})$$
$$+ 32(q_{||} \cdot q_{||})(p_{||} \cdot P_{||}) + 128 p^0 q^3 q^0 P^3 + 64 P^3 p^0 q^3 q^0 P^3 + 32(P_{||} \cdot q_{||}) p^0 q^0 q^0$$
$$+ 64 p^3 P^0 q^3 q^0 - 64 P^3 p^0 q^3 q^0 - 32(P_{||} \cdot q_{||}) q^3 p^3 + 32(p_{||} \cdot q_{||}) P^3 q^3$$
$$- 32(p_{||} \cdot q_{||}) P^0 q^3.$$  \hfill (192)
Term-21

\[-2\text{Tr}(\gamma^3 g_\parallel \gamma^\nu \tilde{p}_\parallel \gamma^0 g'_{\parallel} \gamma^5 \gamma^3 \gamma^0 \nu_\nu \tilde{P}_\parallel) = 0.\] (193)

Term-22

\[2\text{Tr}(\gamma^0 g_\parallel \gamma^\nu \tilde{p}_\parallel \gamma^3 g'_{\parallel} \gamma^5 \gamma^3 \gamma^0 \nu_\nu \tilde{P}_\parallel) = 0.\] (194)

Term-23

\[2\text{Tr}(\gamma^0 \gamma^3 \tilde{p}_\parallel \gamma^\nu g'_{\parallel} \gamma^5 \gamma^3 \gamma^0 \nu_\nu \tilde{P}_\parallel) = 0.\] (195)

Term-24

\[-2\text{Tr}(\gamma^5 \tilde{p}_\parallel \gamma^\nu g'_{\parallel} \gamma^0 \gamma^3 \gamma^0 \nu_\nu \tilde{P}_\parallel \gamma^5) = -32(\tilde{p}_\parallel \cdot P)_\parallel q^3 q^0 + 32(\tilde{p}_\parallel \cdot P)_\parallel q^3 q^0 = 32(\tilde{p}_\parallel \cdot P)_\parallel (\tilde{q}_\parallel \cdot q'_\parallel)\] (196)

Term-25

\[-2\text{Tr}(p_\parallel \gamma^\nu g'_{\parallel} \gamma^0 \gamma^3 \gamma^0 \nu_\nu \tilde{P}_\parallel) = -32(p_\parallel \cdot \tilde{P}_\parallel) q^3 q^0 + 32(p_\parallel \cdot \tilde{P}_\parallel) q^3 q^0 = 32(\tilde{P}_\parallel \cdot p_\parallel)(\tilde{q}_\parallel \cdot q'_\parallel)\] (197)

Term-26

\[-2\text{Tr}(\gamma^5 \gamma^3 g_\parallel \gamma^\nu \tilde{p}_\parallel \gamma^0 g'_{\parallel} \gamma^3 \gamma^0 \nu_\nu \tilde{P}_\parallel) = 0.\] (198)

Term-27

\[2\text{Tr}(\gamma^5 \gamma^0 g_\parallel \gamma^\nu \tilde{p}_\parallel \gamma^3 g'_{\parallel} \gamma^3 \gamma^0 \nu_\nu \tilde{P}_\parallel) = 0.\] (199)

Term-28

\[2\text{Tr}(\gamma^5 \gamma^0 \gamma^3 \tilde{p}_\parallel \gamma^\nu g'_{\parallel} \gamma^3 \gamma^0 \nu_\nu \tilde{P}_\parallel) = 0.\] (200)
Term-29

\[-2\text{Tr}(\gamma^3 \tilde{g} \gamma^\nu \gamma^0 \gamma^\nu \gamma^0 \gamma^\nu \widetilde{P}_\parallel^\nu) = 64\tilde{p}^3 q^0 q_0^3 \widetilde{P}_0^0 - 64q^0 q^3 \widetilde{P}_0^3 \tilde{p}_0^3 - 32(q\parallel \cdot \widetilde{P}_0^\parallel)q^0 q^0 + 32(q\parallel \cdot \widetilde{P}_0^\parallel)q^0 q_0^0 \]

\[-32(q\parallel \cdot \widetilde{P}_0^\parallel)q^0 q_0^0 + 32(\tilde{p}_0 \cdot \widetilde{P}_0^\parallel)q^0 q^3 - 32(q\parallel (a) \cdot \widetilde{P}_0^\parallel)q^3 \tilde{p}_0^3 + 32(q\parallel \cdot \widetilde{P}_0^\parallel)q^3 \tilde{p}_0^3 + 16(q\parallel \cdot q)_0^0(\tilde{p}_0 \cdot \widetilde{P}_0^\parallel) - 16(\tilde{p}_0 \cdot q)_0^0(\widetilde{P}_0^\parallel \cdot q)_0^0 + 16(\tilde{P}_0 \cdot q)_0^0(\widetilde{q}_0 \cdot q)^0. \tag{201} \]

Term-30

\[2\text{Tr}(\gamma^5 \gamma^0 \gamma^\nu \gamma^\nu \gamma^0 \gamma^\nu \gamma^\nu \widetilde{P}_\parallel^\nu) = 64q^0 q^0 \widetilde{P}_0^0 - 64q^0 q^0 \widetilde{P}_0^3 \tilde{p}_0^3 + 32q^0 q^3 (\tilde{p}_0 \cdot \widetilde{P}_0^\parallel) - 32(q\parallel \cdot \widetilde{P}_0^\parallel)q^3 \widetilde{P}_0^3 + 32(q\parallel \cdot \widetilde{P}_0^\parallel)q^3 \tilde{p}_0^3 + 32(q\parallel \cdot \widetilde{P}_0^\parallel)q^3 \tilde{p}_0^3 + 16(q\parallel q)_0^0(\tilde{p}_0 \cdot \widetilde{P}_0^\parallel) - 16(\tilde{p}_0 \cdot q)_0^0(\widetilde{P}_0^\parallel \cdot q)_0^0 + 16(\tilde{P}_0 \cdot q)_0^0(\widetilde{q}_0 \cdot q)_0^0. \tag{202} \]

Term-31

\[2\text{Tr}(\gamma^5 \gamma^0 \gamma^\nu \gamma^\nu \gamma^0 \gamma^\nu \gamma^\nu \widetilde{P}_\parallel^\nu) = -64\tilde{p}^3 \widetilde{P}_0^0 q^0 q_0^3 + 32(\widetilde{P}_0^\parallel \cdot q^0)_0^0 \tilde{p}^3 q^3 + 128q^0 \widetilde{P}_0^0 \tilde{p}^0 q_0^3 - 32(q\parallel \cdot \widetilde{P}_0^\parallel q^0 \tilde{p}^0 q_0^0 \]

\[+ 32(q\parallel \cdot q^0)_0^0(\tilde{p}_0 \cdot \widetilde{P}_0^\parallel) + 128q^0 q^3 \widetilde{P}_0^0 \tilde{p}^0 q_0^3 + 64 \widetilde{P}_0^3 \tilde{p}_0^3 q_0^3 q_0^0 + 32(\widetilde{P}_0^\parallel \cdot q^0) \tilde{p}_0^0 q_0^0 + 64 \tilde{p}^3 \widetilde{P}_0^0 q^0 q_0^0 - 64 \widetilde{P}_0^3 \tilde{p}_0^3 q^3 q_0^0 - 32(\widetilde{P}_0^\parallel \cdot q^0) \tilde{p}^3 q_0^3 - 32(\widetilde{P}_0^\parallel \cdot q^0) \tilde{p}^3 q_0^3. \tag{203} \]

Term-32

\[-2\text{Tr}(p^\parallel \gamma^\nu \gamma^\nu \gamma^0 \gamma^\nu \gamma^0 \gamma^\nu \widetilde{P}_\parallel^\nu) = 0. \tag{204} \]

Adding all the thirty two terms from (172)-(204), (171) becomes (89).

B Crosssection

In this appendix section, we will show the detailed calculation for the crosssection. So to calculate crosssection first we need to see differential phase factor. In magnetic field, differential phase factor is given by

\[d\rho = \frac{dP^0 dP_x dP_z}{(2\pi)^2} \delta(P^2 - m^2) \Theta(P^0) \frac{d^4 K}{(2\pi)^3} \delta(K^2) \Theta(K^0). \tag{205} \]
Now substituting the S-matrix element squared from (59) and the differential phase factor from (205) into (92), the crosssection for the Compton scattering in strong magnetic field becomes

\[
\sigma(\gamma e^- \rightarrow \gamma e^-) = \int \frac{dP_0 dP_x dP_z}{(2\pi)^2} \delta(P^2 - m_e^2) \Theta(P^0) \frac{d^3K}{(2\pi)^3} \delta(K^2) \Theta(K^0) \times (2\pi)^6 \pi \delta'(P + K - p - k) |c\mathcal{M}_s + d\mathcal{M}_u|^2.
\]

To solve the crosssection expression, we will first simplify the photon phase factor and then electron phase factor. Using the property of the Dirac-delta function

\[
\delta(K_0^2 - \omega'_{2}) \Theta(K^0) = \left[ \delta(K_0 + \omega') + \delta(K_0 - \omega') \right] \theta(K^0),
\]

the photon phase factor in Eq. (96) can be simplified as

\[
\frac{d^3K}{(2\pi)^3} \delta(K^2) \Theta(K^0) = \frac{d^3K_0 dK^0}{(2\pi)^3} \left[ \delta(K_0 + \omega') + \delta(K_0 - \omega') \right] \theta(K^0),
\]

\[
= \frac{d^3K dK^0}{(2\pi)^3} \frac{\delta(K_0 - \omega')}{2\omega'}.
\]

Using the mass-shell condition in strong magnetic field: \( P_+^2 = m^2 \), the differential phase factor for the electron in (95) can be simplified as

\[
\frac{dP_0 dP_x dP_z}{(2\pi)^2} \delta(P_+^2 - m^2) \Theta(P^0) = \frac{dP_0 dP_x dP_z}{(2\pi)^2} \frac{\delta(P_+^2 - p_+^2 - E^2 + p_0^2)}{2E'} \Theta(P^0),
\]

\[
= \frac{dP_0 dP_x dP_z}{(2\pi)^2} \frac{\delta(P_0^2 - E^2)}{2E'} \Theta(P^0).
\]

Now using equation (207), the electron phase factor is modified into

\[
\frac{dP_0 dP_x dP_z}{(2\pi)^2} \delta(P_+^2 - m^2) \Theta(P^0) = \frac{dP_0 dP_x dP_z}{(2\pi)^2} \frac{\delta(P_0^2 + E') + \delta(P_0^2 - E')}{2E'} \Theta(P^0),
\]

\[
= \frac{dP_0 dP_x dP_z}{(2\pi)^2} \frac{\delta(P_0^2 - E')}{2E'}.
\]

Now, substituting the simplified phase factors for photon and electron from (208) and (210), respectively, the crosssection for Compton scattering expression becomes

\[
\sigma(\gamma e^- \rightarrow \gamma e^-) = \int \frac{dP_0 dP_x dP_z}{(2\pi)^2} \frac{\delta(P_0^2 - E')}{2E'} \frac{d^3K dK^0}{(2\pi)^3} \delta(K_0 - \omega') \frac{(2\pi)^6 \pi \delta'(P + K - p - k) |c\mathcal{M}_s + d\mathcal{M}_u|^2}{F}.
\]

(211)
Using the property of the Dirac-delta function: \( \int f(x)\delta(x-x_0)dx = f(x_0) \), the Dirac-Delta functions \( \delta(K^0 - \omega') \) and \( \delta(P^0 - E') \) will be eliminated. Then substituting \( K^0 = \omega' \) and \( P^0 = E' \) in the remaining function, the differential phase factor of photon \( dK_X \ dK_Z \) will be eliminated by the energy-momentum conserving Dirac-delta function, hence the crosssection finally reduces to

\[
\sigma = \frac{\pi^2}{2} \int \frac{dP_X dP_Z \ dK_Y}{E'} \delta(E' + \omega' - E - \omega) \frac{[e\mathcal{M}^s + d\mathcal{M}^u]^2}{m\omega}.
\] (212)

**Flux factor:** For the process

\( 1 + 2 \rightarrow 3 + 4 \)

, the flux factor is given by

\[
F = |v_1 - v_2| \frac{2E_1 E_2}{E},
\] (213)

where, \( v_1 \) and \( v_2 \) are the velocities of the target and projectile, respectively and \( E_1 \) and \( E_2 \) are their corresponding energy. Assuming the lab frame where the initial momentum of target will be zero, then flux factor in lab-frame becomes,

\[
F = 4v_2 \sqrt{p_1^2 + m_1^2} \sqrt{p_2^2 + m_2^2} = 4v_2 m_1 p_2 = 4m_1 E_2.
\] (214)

For our case target is electron and projectile is photon, hence flux factor becomes

\[
F = 4m\omega.
\] (215)

Now, substituting the expression of flux factor from (215), the crosssection in (212) becomes

\[
\sigma(\gamma e^- \rightarrow \gamma e^-) = \frac{\pi^2}{8} \int \frac{dP_X dP_Z \ dK_Y}{E'} \delta(E' + \omega' - E - \omega) \frac{[e\mathcal{M}^s + d\mathcal{M}^u]^2}{m\omega} \exp \left(-\frac{2q^2 X eB}{eB}\right) \left|\mathcal{M}^s\right|^2 + \left|\mathcal{M}^u\right|^2 \exp \left(-\frac{2q^2 X eB}{eB}\right) m\omega.
\] (216)

Now using (104)-(105), (103), (106) and \( dP_X \) integration

\[
\int_0^{\sqrt{eB}} dP_X = \sqrt{eB},
\]

the crosssection gets simplified into

\[
\sigma = \frac{eB\pi^2}{4\sqrt{3}} \int \frac{dP_Z}{E' \omega'} \delta(E' + \omega' - E - \omega) \times \left(\exp \left(-\frac{2q^2 X eB}{eB}\right) \left|\mathcal{M}^s\right|^2 + \left|\mathcal{M}^u\right|^2 \exp \left(-\frac{2q^2 X eB}{eB}\right) m\omega\right).
\] (217)

Now, we will integrate over \( dP_Z \) for all the matrix elements due to different channels and cross-terms one by one.
FOR $s$-CHANNEL DIAGRAM:

The crosssection expression for the $s$-channel diagram is given by

$$
\sigma^s(\gamma e^- \rightarrow \gamma e^-) = \frac{eB\pi^2}{4\sqrt{3}} \int \frac{dP_Z}{E'\omega'} \delta(E' + \omega' - E - \omega) \exp\left(\frac{-2q_X^2}{eB}\right) \frac{|M_s|^2}{m\omega}.
$$

(218)

We will get eq.(113) from steps done in eqs.(109)- (112). Now our next task is to find the roots and integrate over the energy conserving Dirac-delta function, firstly we find the roots of the Dirac-delta function as follows

$$
\delta(E' + \omega' - E - \omega) = \delta(\sqrt{P_Z^2 + m^2 + \omega' - p_Z - \omega}),
$$

$$
= \delta(\sqrt{\left(\frac{(m\omega + \omega^2)(1 - \cos \theta)}{m + \omega(1 - \cos \theta)}\right)^2 + m^2 - \frac{(\omega^2)(1 - \cos \theta)}{m + \omega(1 - \cos \theta)} - m}).
$$

(219)

Now finding the roots of the equation inside the Dirac-delta function

$$
\left[\frac{(m\omega + \omega^2)(1 - \cos \theta)}{m + \omega(1 - \cos \theta)}\right]^2 + m^2 = \left[\frac{(\omega^2)(1 - \cos \theta)}{m + \omega(1 - \cos \theta)} + m\right]^2,
$$

(220)

simplifying this equation and finally, we get a quadratic equation,

$$
[2\omega^2 m^2 - \omega^2 m^2(1 - \cos \theta)](1 - \cos \theta) = 0,
$$

(221)

whose roots are given by

$$
\cos \theta = 1, \quad \cos \theta = -1.
$$

(222)

Now, putting $x = \cos \theta$ in eq.(113), then the crosssection for the $s$-channel becomes

$$
\sigma^s(\gamma e^- \rightarrow \gamma e^-)
$$

$$
= e^4 \frac{eB\pi^2(m + \omega)}{12\sqrt{3}m\omega^2} \exp\left(\frac{-2q_X^2}{eB}\right) \int_{-1}^{1} \frac{dx}{m + \omega(1 - x)} \delta\left[\sqrt{\left(\frac{(m\omega + \omega^2)(1 - x)}{m + \omega(1 - x)}\right)^2 + m^2 - \frac{(\omega^2)(1 - x)}{m + \omega(1 - x)} - m}\right],
$$

(223)

Since, we have two roots $x = 1$ and $x = -1$, so applying the property of the Dirac-delta function

$$
\delta[f(x)] = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{f'(x_i)},
$$

(224)
$\sigma^s$ becomes

$$
\sigma^s(\gamma e^- \rightarrow \gamma e^-) = e^4 eB \pi^2 (m + \omega) \exp \left( \frac{-2q_X^2}{eB} \right) \int_{-1}^{1} \frac{dx}{m + \omega(1 - x)} \left[ \frac{\delta(x - 1)}{|f'(x)|_{x=1}} + \frac{\delta(x + 1)}{|f'(x)|_{x=-1}} \right].
$$

(225)

where,

$$
f'(x) = \left[ \frac{4\omega^2 m^2 (1 - x) (-1 + \omega^2 (1 - x) - 2m^3 (-2) (1 - x) + 2m^3 \omega (-1)) [m + \omega (1 - x)]}{2 \sqrt{2 \omega^2 m^2 (1 - x)^2 + \omega^4 (1 - x)^2 + 2\omega^3 (1 - x)^2 + m^4 + 2m^3 \omega (1 - x)}} \right] \left( \frac{m + \omega (1 - x))^2}{\omega \sqrt{2 \omega^2 m^2 (1 - x)^2 + \omega^4 (1 - x)^2 + 2\omega^3 (1 - x)^2 + m^4 + 2m^3 \omega (1 - x)}} \right) - \omega^2 \left[ \frac{(-1)(m + \omega (1 - x)) + \omega i (1 - x)}{(m + \omega (1 - x))^2} \right].
$$

(226)

We obtain,

$$
|f'(x)|_{x=1} = \frac{\omega^2}{m},
$$

(227)

$$
|f'(x)|_{x=-1} = \frac{-9m^2 - 2\omega^3}{(m + 2\omega)^2} = \frac{9m^2 + 2\omega^3}{(m + 2\omega)^2}.
$$

Substituting the values of $|f'(x)|_{x=1}$ and $|f'(x)|_{x=-1}$ and integrating using the property of Dirac-delta function

$$
\int_{-a}^{a} f(x) \delta(x - a) dx = f(a) [2\Theta(a) - 1],
$$

(228)

$$
\sigma^s(\gamma e^- \rightarrow \gamma e^-) = e^4 eB \pi^2 (m + \omega) \exp \left( \frac{-2q_X^2}{eB} \right) \left[ \frac{1}{\omega^2} - \frac{m + 2\omega}{9m^2 + 2\omega^3} \right].
$$

(229)

The factor $\left( \frac{-2q_X^2}{eB} \right)$ can be eliminated by using strong magnetic field limit. In the lab frame, we consider the initial direction of photon along the magnetic field (Z) direction, so

$$
q_X = p_X + k_X = m.
$$

(230)

Also we have $eB \gg m^2$, so finally $\sigma^s(\gamma e^- \rightarrow \gamma e^-)$ becomes

$$
\sigma^s(\gamma e^- \rightarrow \gamma e^-) = e^4 eB \pi^2 (m + \omega) \left[ \frac{1}{\omega^2} - \frac{m + 2\omega}{9m^2 + 2\omega^3} \right].
$$

(231)

FOR $u$-CHANNEL DIAGRAM:
Putting $x = \cos \theta$ in (120), then $\sigma^u$ reduces to

$$
\sigma^u(\gamma e^- \rightarrow \gamma e^-) = e^4 \frac{e B \pi^2 (m + \omega)}{3 \sqrt{3} m \omega} \exp \left( \frac{-2 q_X^2}{e B} \right) \times \int \left[ \frac{[m^3 \omega^2 (4 + 3 x^2 - 7 x) + m^3 \omega (7 x - 8) \times (m + \omega (1 - x))] \times [m + \omega (1 - x)]^2}{[m^2 \omega^2 (1 - x^2) - 2 m^2 \omega x \times (m + \omega (1 - x))]^2} \right] P_Z \times \delta \left( \sqrt{\left[ \frac{(m \omega + \omega^2) (1 - x)}{m + \omega (1 - x)} \right] + m^2 - \frac{(\omega^2) (1 - x)}{m + \omega (1 - x)} - m} \right) \exp \left( \frac{-2 q_X^2}{e B} \right) \times \int \left[ \frac{[m^3 \omega^2 (4 + 3 x^2 - 7 x) + m^3 \omega (7 x - 8) \times (m + \omega (1 - x))] \times [m + \omega (1 - x)]^2}{[m^2 \omega^2 (1 - x^2) - 2 m^2 \omega x \times (m + \omega (1 - x))]^2} \right] P_Z \times \delta(x - 1) + \delta(x + 1) \right] \cdot \frac{|f'(x)|_{x=1}}{|f'(x)|_{x=-1}}. \quad (232)
$$

Substituting values of $|f'(x)|_{x=1}$ and $|f'(x)|_{x=-1}$ from (228) and integrating using property of Dirac-delta function as given in (228), finally the crosssection for the $u$-channel $\sigma^u$ becomes

$$
\sigma^u(\gamma e^- \rightarrow \gamma e^-) = e^4 \frac{e B \pi^2}{12 \sqrt{3}} \exp \left( \frac{-2 q_X^2}{e B} \right) \left[ \frac{61 m^2 + 78 m \omega^2 + 32 \omega^2}{m^2 \omega [9 m \omega^2 + 2 \omega^3]} - \frac{1}{m \omega^3} \right] \quad (233)
$$

The factor $\exp \left( \frac{-2 q_X^2}{e B} \right)$ becomes unity in strong magnetic field limit. In lab frame, we consider the initial direction of photon along the magnetic field (Z-direction), so $q'_X = P_X - k_X = P_X$. Also we have $P_\perp = 0$, so that we can write

$$
q'_X = 0. \quad (234)
$$

Finally $\sigma^u$ becomes

$$
\sigma^u(\gamma e^- \rightarrow \gamma e^-) = e^4 \frac{e B \pi^2}{12 \sqrt{3}} \left[ \frac{61 m^2 + 78 m \omega^2 + 32 \omega^2}{m^2 \omega [9 m \omega^2 + 2 \omega^3]} - \frac{1}{m \omega^3} \right] \quad (235)
$$