A tight Hermite-Hadamard’s inequality and a generic method for comparison between residuals of inequalities with convex functions

MILAN MERKLE AND ZORAN D. MITROVIĆ

Abstract. We present a tight parametrical Hermite-Hadamard type inequality with probability measure, which yields a considerably closer upper bound for the mean value of convex function than the classical one. Our inequality becomes equality not only with affine functions, but also with a family of V-shaped curves determined by the parameter. The residual (error) of this inequality is strictly smaller than in the classical Hermite-Hadamard inequality under any probability measure and with all non-affine convex functions. In the framework of Karamata’s theorem on the inequalities with convex functions, we propose a method of measuring a global performance of inequalities in terms of average residuals over functions of the type \( x \mapsto |x-u| \). Using average residuals enables comparing two or more inequalities as themselves, with same or different measures and without referring to a particular function. Our method is applicable to all Karamata’s type inequalities, with integrals or sums. A numerical experiment with three different measures indicates that the average residual in our inequality is about 4 times smaller than in classical right Hermite-Hadamard, and also is smaller than in Jensen’s inequality, with all three measures.

2010 Mathematics Subject Classification. 26A51, 60E15, 26D15.

Keywords. Jensen’s inequality, Lebesgue-Stieltjes integral, probability measure, average error.

1. INTRODUCTION

For a non-negative measure \( \mu \) on \([a,b]\), such that \( \mu[a,b]=1 \) (probability measure), let \( c = \int x \, d\mu(x) \). From long ago \[12, 13\], it is known that for a convex function \( f \) it holds

\[
(1) \quad f(c) \leq \int f(x) \, d\mu(x) \leq \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b).
\]

A case with \( \mu \) being the Lebesgue probability measure on \([a,b]\), with \( d\mu(x) = \frac{1}{b-a} \, dx \), is originally stated by C. Hermite and J. S. Hadamard independently in late 19th century (see \[18\] for more history):

\[
(2) \quad f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Hermite-Hadamard (HH) inequality from the beginning has been used in problems of approximations the integral in the middle, using left inequality (midpoint rule) or the right one (trapezoid rule). It is well known that the residual (error) in the right inequality is larger than in the left one, see \[2, 6\], and there is a voluminous literature on refinement of the right side of \( (2) \) like in \[11, 4, 5, 10, 15, 20\] and many more. Regardless of applications, it is always desirable to have an inequality with smaller residual. This paper offers two contributions to this topic. In Section 2 we present a new parametrical right bound in \[1\], which gives much smaller residual...
for all measures and all non-affine convex functions, with all values of the parameter and without any additional assumptions. In Section 3, we develop a method via Karamata’s theorem (know also as Levin-Stečkin) for estimating the residuals of inequalities for convex functions, and comparing residuals of different inequalities. To the best of our knowledge, this is the unique method capable to compare any two or more inequalities globally, without referring to a certain function. Numerical experiments in Section 4 confirm the theoretical results and also indicate that the residual in our tight inequality is smaller not only in comparison with the right bound in [1] but also with respect to the left bound.

In this paper we adopt the setup with countably additive probability measures on \( \mathbb{R} \) and Lebesgue-Stieltjes integrals (as the most general integral that incorporates Riemann and Riemann-Stieltjes one) over a compact interval \([a, b]\). To avoid repetitions, let us state several notions and conditions.

1.1 Notions and conditions. In the rest of this paper, by measure will be understood a countably additive probability measure \( \mu \) on Borel sigma algebra on \( \mathbb{R} \) such that \( \mu([a, b]) = 1 \), \( a < b \). Let \( X \) be the random variable associated to \( \mu \), with distribution function \( G \) defined as \( G(x) = \mu(-\infty, x] \). The integral of an integrable function \( f \) with respect to measure \( \mu \) is expressed as the Lebesgue-Stieltjes integral \( \int f(x) \, dG(x) \), or in more compact terms of expectation operator and random variables, as \( E(f(X)) \). Under "convex function on \([a, b]\)" we understand a function which is convex on some open interval \( I \) that contains \([a, b]\).

The left inequality in [1] is Jensen’s inequality originally proved by Jensen [8] and generalized by McShane [14]. A very simple proof can be find in [3]. The following theorem presents Jensen’s inequality on the compact interval, in our setup in 1.1.

1.2 Theorem. Let \( \mu \) and \( G \) be as in 1.1 and let \( f \) be a convex function on \([a, b]\). Then

\[
\int_{[a,b]} f(x) \, dG(x) \leq f\left(\int_{[a,b]} x \, dG(x)\right), \quad \text{or equivalently,} \quad f(\mathbb{E}X) \leq \mathbb{E}(f(X))
\]

The right inequality in [1] is generalized by A. Lupas [12] with an abstract linear functional. Here we give a formulation under our setup, and a short direct proof.

1.3 Theorem. Under the same conditions as in Theorem 1.2, for every convex function \( f \) it holds:

\[
\int_{[a,b]} f(x) \, dG(x) \leq \frac{b - \int_{[a,b]} x \, dG(x)}{b - a} f(a) + \frac{\int_{[a,b]} x \, dG(x) - a}{b - a} f(b),
\]

or equivalently,

\[
\mathbb{E}f(X) \leq \frac{b - \mathbb{E}X}{b - a} f(a) + \frac{\mathbb{E}X - a}{b - a} f(b)
\]

Moreover, if

\[
\mathbb{E}f(X) \leq \alpha f(a) + \beta f(b)
\]

holds for every convex function \( f \), then \( \beta = 1 - \alpha \) and \( \alpha a + \beta b = \mathbb{E}(X) \).
Proof. For a convex function \( f \) on \([a,b]\), it holds
\[
f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b),
\]
and after the integration on both sides, we get (4). The second part follows by plugging \( f = x \), \( f = -x \) and \( f = 1 \). \( \square \)

In what follows we use abbreviation \( J \) for the Jensen’s inequality (3) and \( H \) for the one in (4). A complete double-side inequality will be denoted as \( HH \). In the next section we present a \( H \)-type inequality with the right term being closer to \( E_f(X) \) than in (4). In Section 3 we use the Karamata’s theorem [9], see also [13, page 645] to define a method that can be used for comparison among inequalities with convex functions. In Section 4 we show the results of some numerical experiments and comparisons.

2. A NEW TIGHT \( H \)-TYPE INEQUALITY

From Theorem [1.3] it follows that the \( H \)-inequality can not be improved by changing the weights associated to \( f(a) \) and \( f(b) \). However, if we add an arbitrary point \( t \in (a,b) \) and re-calculate the weights, the sum on the right hand side becomes considerably closer to \( E_f(X) \) than in \( H \)-inequality, for all underlying measures and all non-affine convex functions.

2.1 Assumption. In order to avoid separations of cases, in this section we exclude measures concentrated on less than 3 points in \([a,b]\), i.e., we assume that there are no \( x_1, x_2 \in [a,b] \) such that \( \mu\{x_1\} + \mu\{x_2\} = 1 \) and \( \mu\{x_1\} \geq 0, \mu\{x_2\} \geq 0 \).

2.2 Theorem. Let \( f \) be a convex function on a compact interval \([a,b]\), \( a < b \), and let \( \mu \) be a probability measure with distribution function \( G \), under notations and conditions as in 1.1. Then the following holds:

(i) For any fixed \( t \in (a,b) \),
\[
\int_{[a,b]} f(x) \, dG(x) \leq \frac{f(a)}{t-a} \int_{[a,t]} (t-x) \, dG(x) + \frac{f(b)}{b-t} \int_{(t,b]} (b-x) \, dG(x) \nonumber + f(t) \left( \frac{1}{t-a} \int_{[a,t]} (x-a) \, dG(x) + \frac{1}{b-t} \int_{(t,b]} (b-x) \, dG(x) \right),
\]
(ii) Equivalently, for fixed \( t \) let \( \lambda = \lambda(t) = \frac{b-t}{b-a} \), so that \( t = \lambda a + (1-\lambda)b \). Then
\[
\int_{[a,b]} f(x) \, dG(x) \leq \frac{b-f \int_{[a,t]} x \, dG(x)}{b-a} f(a) + \frac{f(x) \int_{[a,t]} (x-a) \, dG(x) - a \int_{[a,b]} f(x) \, dG(x)}{b-a} f(b) \nonumber + (f(t) - \lambda f(a) - (1-\lambda)f(b)) \times \left( \frac{1}{t-a} \int_{[a,t]} (x-a) \, dG(x) + \frac{1}{b-t} \int_{(t,b]} (b-x) \, dG(x) \right)
\]

Proof. The convexity of \( f \) implies that, for arbitrary \( x \in [a,b] \),
where as a corollary of theorem 2.2.

\[ f(x) \leq \frac{t-x}{t-a} f(a) + \frac{x-a}{t-a} f(t), \quad a \leq x \leq t, \]

\[ f(x) \leq \frac{b-x}{b-t} f(t) + \frac{x-t}{b-t} f(b), \quad t \leq x \leq b. \]

Integrating (8) with respect to measure \( \mu \) on \( [a, t] \) and (9) on \( (t, b] \), and adding, we get

\[
\int_{[a, b]} f(x) \, d\mu(x) \leq \frac{E(t-X) \cdot I_{[a,t]}(X)}{t-a} f(a) + \frac{E(X-a) \cdot I_{[a,t]}(X)}{t-a} f(t) + \frac{E(b-X) \cdot I_{(t,b]}(X)}{b-t} f(t) + \frac{E(X-t) \cdot I_{(t,b]}(X)}{b-t} f(b),
\]

which is the inequality (6) in terms of random variables. To show the equivalence between (6) and (7), it suffices to verify that coefficients with \( f(a) \), \( f(b) \) and \( f(t) \) are equal in both formulae.

Unlike Jensen’s inequality, the right-hand side of HH inequality has not been much used so far in probability and statistics. Nevertheless, it might be of interest to formulate the new complete HH-type inequality in terms of random variables as a corollary of theorem 2.2.

2.3 Corollary. Let \( X \) be a random variable supported on \( [a, b] \), \( a < b \) and with distribution function \( G \). For any \( t \in (a, b) \) and a convex function \( f \), it holds

\[ f(E X) \leq E f(X) \leq p_a f(a) + p_t f(t) + p_b f(b), \]

where

\[ p_a = \frac{\int_{[a,t]} (t-x) \, dG(x)}{t-a}, \quad p_b = \frac{\int_{(t,b]} (x-t) \, dG(x)}{b-t} \]

\[ p_t = \frac{\int_{[a,t]} (x-a) \, dG(x)}{t-a} + \frac{\int_{(t,b]} (b-x) \, dG(x)}{b-t}, \]

and \( p_a + p_b + p_c = \int dG(x) = 1 \).

The inequality proved in Theorem 2.2 will be referred to as Tight Hermite-Hadamard (abbreviated TH) inequality. Let \( R_f \), \( R_H \) and \( R_{TH} \) be the corresponding residuals in \( J \), \( H \) and \( TH \) inequalities. Given the interval \([a, b]\), the size of residuals depends on the underlying measure \( \mu \), and on the function \( f \).

2.4 Lemma. For any convex function on \([a, b]\) and any measure that satisfies assumption 2.1, it holds

a) \( R_H(\mu, f) = 0 \) if and only if \( f \) is affine function. The same holds for \( R_f(\mu, f) \).

b) \( R_{TH}(\mu, f, t) < R_H(\mu, f) \) for all convex non-affine functions, for all \( t \in (a, b) \).

c) \( R_{TH}(\mu, f, t) = 0 \) if and only if

\[
f(x) = \left( \frac{t-x}{t-a} \alpha + \frac{x-a}{t-a} \tau \right) I_{[a,t]}(x) + \left( \frac{b-x}{b-t} \tau + \frac{x-t}{b-t} \beta \right) I_{(t,b]}(x),
\]

for some real numbers \( \alpha, \beta, \tau \).
Note that the functions defined by (11) are either affine or their graphs are V-shaped, with two lines that meet at the point \((t, \tau)\) and with endpoints \((a, \alpha)\) and \((b, \beta)\). Such functions are convex if and only if \(\tau \leq \min\{\alpha, \beta\}\).

**Proof.** We will prove in lemma 3.2 that the residuals \(R_J, R_H\) and \(RT_H\) are zero for all measures with an affine \(f\), so we need to prove "only if" part where applies.

a) Equality \(R_H(\mu, f) = 0\) is equivalent to \(E(f(X) = pf(a) + (1-p)f(b), p = \frac{b-x}{b-a}\). Suppose that \(f\) is not affine. This implies that the graph of \(f\) for \(x \in (a, b)\) lies under the chord that connects points \((a, f(a))\) and \((b, f(b))\), although the point \((E_X, E(f(X)))\) belongs to the chord. This is possible only if \(\mu\) is concentrated on the set \([a, b]\), which is excluded by assumption 2.1. For the proof of necessity for \(R_J = 0\), see [13, page 654].

b) From the representation (7) it follows that

\[
R_{TH}(\mu, f, t) = (f(t) - \lambda f(a) - (1 - \lambda)f(b))
\]

(12)

\[
\times \left( \frac{1}{t-a} \int_{[a, t]} (x-a) dG(x) + \frac{1}{b-t} \int_{(t, b)} (b-x) dG(x) \right)
\]

According to a), the first term is zero if and only if \(f\) is affine; otherwise it is negative. The second term is positive under the assumption 2.1 and the claim is proved.

c) The function \(f\) defined by (11) satisfies (8) and (9) with equalities. Tracing the proof of Theorem 2.2, the integration with respect to the given measure yields null residual. Moreover, the residual can be zero only if both (8) and (9) are equalities, and this is the case only if \(f\) is either affine or in the form (11). □

From Lemma 2.4 it follows that with any measure, inequality \(TH\) yields the better approximation to \(\int f(x) dG(x)\) than the inequality \(H\), for every non-affine convex function.

### 2.5 Optimal choice of parameter \(t\) for given \(f\).

Since inequality \(TH\) is valid for any \(t \in (a, b)\), it is natural to ask which \(t\) yields the smallest residual, or equivalently, the smallest (negative) difference \(R_{TH}(\mu, f, t) - R_H(\mu, f)\) for given \(f\) and \(\mu\). For given measure \(\mu\) with distribution function \(G\) and a convex function \(f\), this difference can be written as the function of \(\lambda\) using the relation \(t = \lambda a + (1 - \lambda)b\)

\[
D(\lambda) = R_{TH}(\mu, f, t) - R_H(\mu, f) = (f(t) - \lambda f(a) - (1 - \lambda)f(b))E(g(X)),
\]

where

\[
g(x) = \frac{\lambda(x-a)I_{[a, t]}(x) + (1 - \lambda)(b-x)I_{[t,b]}(x)}{\lambda(1 - \lambda)(b-a)}, \quad x \in \mathbb{R},
\]

and \(\lambda = \frac{b-t}{b-a}\). The graph of this function is the continuous triangular curve which connect points \((a, 0), (t, 1)\) and \((b, 0)\).

A value of \(t\) that minimizes \(D(\lambda(t))\) depends on the underlying measure. In the case of uniform distribution on \([a, b]\), we have \(G(x) = \frac{1}{b-a}\) and \(E(g(X)) = \frac{1}{2}\). The optimal value of \(\lambda\) is determined as the solution of \(D'(\lambda) = 0\), which yields \(t\) as a solution of \(f'(x) = \frac{\frac{b}{b-a} - f(a)}{b-a}\). If \(f\) does not have a derivative everywhere in \((a, b)\),
one can use methods relying on left and right derivatives. A discussion related to cases with non-uniform distribution is out of scope of this paper.

2.6 $TH$ inequality with purely discrete measures. Let $x_0 < x_1 < \ldots < x_n$, $n \geq 2$, and let $\mu(\{x_i\}) = p_i$, where $p_i\in(0, 1)$ and $\sum p_i = 1$, $p_i > 0$. The interval $[a, b]$ is here $[x_0, x_n]$. We can allow $t$ to be any point in the interval $(x_0, x_n)$; it can be one of points $t_i$ with positive probability, or not. Then with a discrete measure $\mu$ reads:

$$\sum_{i=0}^{n} p_i f(x_i) \leq \frac{x_n - \sum_{i=0}^{n} p_i x_i}{x_n - x_0} f(x_0) + \frac{\sum_{i=0}^{n} p_i x_i - x_0}{x_n - x_0} f(x_n)$$

$$+ \left( f(t) - \frac{x_n - t}{x_n - x_0} f(x_0) - \frac{t - x_0}{x_n - x_0} f(x_n) \right)$$

$$\times \left( \frac{1}{t - x_0} \sum_{x_i \leq t} p_i (x_i - x_0) + \frac{1}{x_n - t} \sum_{x_i > t} p_i (x_n - x_i) \right)$$

Although we will not discuss concrete examples, let us emphasize that all further results of this paper are also valid for discrete measures.

3. Quantifying the tightness via Karamata’s theorem

As an introduction to the topic of this section, let us note that all three inequalities that we considered so far are of the type

(15) \[ \int f(x) \, dG(x) \geq (\leq) \int f(x) \, dH(x) \]

where $G$ and $H$ are distribution functions of corresponding measures. Let $G$ be the distribution function that appears in the integral $\int f(x) \, dG(x)$ in inequalities $J$, $H$ and $TH$. The second measure is derived from $G$ as follows.

3.1 Second measure in inequalities $J$, $H$ and $TH$. Let $c := \int x \, dG(x)$. The second measures are discrete and derived from $G$ as follows.

(J) The second measure is the unit mass at $c$, with $H(x) = I_{[c, +\infty)}(x)$, and in these terms, the Jensen’s inequality can be written as $\int f(x) \, d(G(x) - H(x)) \geq 0$.

(H) The second measure is concentrated at points $a$ and $b$ with probabilities $\frac{b - c}{b - a}$ and $\frac{c - a}{b - a}$ respectively, so $H(x) = \frac{b - c}{b - a} I_{[a, b]}(x) + I_{[0, +\infty)}$. This inequality is of the form $\int f(x) \, d(G(x) - H(x)) \leq 0$.

(TH) The second measure is concentrated on the set $\{a, t, b\}$ with probabilities $p_a$, $p_t$, and $p_b$ in Corollary [2.3] The distribution function is

$$H(x) = p_a I_{[a, +\infty)}(x) + p_t I_{[t, +\infty)}(x) + p_b I_{[b, +\infty)}(x),$$

and the inequality is of the form $\int f(x) \, d(G(x) - H(x)) \leq 0$.

The next lemma gives some common properties of inequalities of type as in (15).

3.2 Lemma. Suppose that for measures $G$ and $H$ the inequality

(16) \[ \int f(x) \, dG(x) \geq \int f(x) \, dH(x) \]
holds with any convex function \( f \) on \([a, b]\). Then

\[
(17) \quad \int_{[a,b]} dG(x) = \int_{[a,b]} dH(x) \quad \text{and} \quad \int_{[a,b]} x \, dG(x) = \int_{[a,b]} x \, dH(x).
\]

Further, if \( f \) is an affine function, the inequality (15) turns to equality.

**Proof.** The first equality follows upon plugging \( f = 1 \) and \( f = -1 \) in (16). For the second equality take \( f(x) = x \) and \( f(x) = -x \). If \( f = \alpha x + \beta \), the statement above follows from (17) using the linearity of integral. \( \square \)

In the paper [9], Jovan Karamata in the year 1932 presented conditions for two given measures so that the inequality (16) holds with all convex functions. This result is often wrongly attributed to Levin and Stečkin [11]. In fact, [11] was originally written by Stečkin sixteen years after Karamata’s paper, as Supplement I in [22], with Karamata’s paper [9] in the list of references of [22]. In several recently published papers, (for example [21]), a related result is again rediscovered with the name Ohlin’s lemma, after the paper [19] of the year 1969 in the context of application in actuarial area.

#### 3.3 Theorem (Karamata [9]).

Given two measures with distribution functions \( G \) and \( H \) and assuming conditions (17), the inequality (16) holds for every convex function \( f \) if and only if for all \( u \in [a, b] \)

\[
(18) \quad \varphi(u) := \int_{[a,u]} (G(x) - H(x)) \, dx \geq 0 \quad \text{for all} \quad u \in [a, b]
\]

**Proof.** Let \( F(x) := G(x) - H(x) \). Then by conditions (17) we have that \( \int_{[a,b]} x \, dF(x) = 0 \), and

\[
\int_{[a,b]} |x - u| \, dF(x) = \int_{[a,u]} (u - x) \, dF(x) + \int_{[u,b]} (x - u) \, dF(x)
\]

\[
= 2uF(u) - 2 \int_{[a,u]} x \, dF(x),
\]

(21)
Further, an integration by parts yields
\[ \int_{[a,u]} x \, dF(x) = xF(x) \bigg|_{a}^{u} - \int_{[a,u]} F(x) \, dx = uF(u) - \int_{[a,u]} F(x) \, dx, \]
so, from (21) it follows
\[ \int_{[a,u]} F(x) \, dx = \int_{[a,b]} |x-u| \, dF(x), \]
which ends the proof.

3.5 Average residuals. In order to compare sharpness and tightness of two inequalities, we need to have a representative measure for the size of residuals of an inequality itself, with no particular function attached. In view of Lemma 3.4, a natural choice is the mean value of the Karamata’s function \( \varphi \). For inequality \( I \) which satisfies conditions of Karamata’s theorem, we define the average residual as

(22) \[ AR(I) = \frac{1}{b-a} \int_{a}^{b} \varphi(u) \, du \]

For comparing errors in two inequalities \( I \) and \( I_0 \) on the same interval, we define relative average residual of \( I \) with respect to \( I_0 \) as

(23) \[ RAR(I, I_0) = \frac{AR(I)}{AR(I_0)} = \frac{\int_{a}^{b} \varphi(u) \, du}{\int_{a}^{b} \varphi_0(u) \, du}. \]

For a concrete convex function, the size of residual depends on the second derivative (see explicit dependence formulae in [17] for some particular cases) or some other measures of convexity. Although the residual and relative residual here can be calculated directly, a representation of residuals in terms of Karamata’s function is meaningful to reveal to which extent the average residuals reflect particular ones. The next theorem gives the relationship between residual (with given function) and Karamata’s function.

3.6 Theorem. Let \( R(f, I) \) be the residual in inequality \( I \), with given measures \( G \) and \( H \) and with a twice differentiable convex function \( f \) on the interval \([a, b]\). Then,

(24) \[ R(f, I) = \frac{1}{2} \int_{a}^{b} f''(u) \varphi(u) \, du = \frac{1}{2} \varphi(\theta)(f'(b) - f'(a)), \]
for some \( \theta \in (a, b) \).

Proof. Let \( h(x) = \int_{a}^{b} f''(u)|x-u| \, du \). Performing the integration by parts on intervals \([a, u]\) and \([u, b]\) separately and adding, we find that \( h(x) = 2f(x) + g(x) \), where \( g(x) \) is affine. Therefore,

(25) \[ f(x) = \frac{1}{2} h(x) - \frac{1}{2} g(x). \]
Applying the inequality $I$ on both sides in (25), and using lemma 3.4 and second statement in lemma [3.2] we get

\[
R(f, I) = \int_{[a,b]} f(x) \, d(G(x) - H(x)) = \frac{1}{2} \int_{a}^{b} f''(u) \varphi(u) \, du.
\]

Since $\varphi$ is continuous and $f'' \geq 0$, the second equality in (24) follows from an integral mean value theorem.

For a given convex function $f$ in inequality $I$, we define a relative residual with respect to $I_0$ as

\[
RR(f, I, I_0) = \frac{R(f, I)}{R(f, I_0)} = \frac{\int_{a}^{b} f''(u) \varphi(u) \, du}{\int_{a}^{b} f''(u) \varphi_0(u) \, du}.
\]

If the function $f$ is not twice differentiable, the following theorem gives a possibility of approximate residuals in the form as above.

3.7 Theorem. \[TH\] If $f$ is convex on $[a, b]$, then for any $\varepsilon > 0$ there exists a convex $C^\infty$-function $\hat{f}$ such that $|f(x) - \hat{f}(x)| \leq \varepsilon$ for all $x \in [a, b]$. \(\Box\)

Let $\hat{f} = \hat{f}_\varepsilon$ be an approximation for $f$ as in the theorem above, with some $\varepsilon > 0$. It is not difficult to show that

\[
|R(f, I) - R(\hat{f}_\varepsilon, I)| \leq 2\varepsilon
\]

and

\[
|RR(f, I, I_0) - RR(\hat{f}_\varepsilon, I, I_0)| \leq \frac{2R(\hat{f}_\varepsilon, I) + R(\hat{f}_\varepsilon, I_0)}{R(f, I_0)R(f, \hat{f}_\varepsilon)} \varepsilon.
\]

Therefore, formulae (24) and (27) can be used with $\hat{f}$ in place of $f$, with small enough $\varepsilon$ to achieve an arbitrary small error of approximation.

4. Numerical evidence: Graphic contents and tables

In this section we compare the residuals of inequalities $J$, $H$ and $TH$, using the methodology presented in the section 3. The figures 1-3 are obtained by Maple calculation of Karamata’s function in an equivalent form adopted for measures with densities:

\[
\begin{align*}
(J) \quad \varphi(u) &= \int_{[a,u]} (u - x) \, dG(x) - (u - c)I_{[c,b]}(u) \quad (c = \int x \, dG(x)), \\
(H) \quad \varphi(u) &= \frac{u - a}{b - a} \int_{[a,b]} (b - x) \, dG(x) - \int_{[a,u]} (u - x) \, dG(x), \\
(TH) \quad \varphi(u) &= \left(\frac{u - a}{t - a} \int_{[a,t]} (t - x) \, dG(x) - \int_{[a,u]} (u - x) \, dG(x)\right) I_{[a,t]}(u) \\
&+ \left(\frac{u - t}{b - t} \int_{(t,b]} G(x) \, dx - \int_{(t,u]} G(x) \, dx\right) I_{(t,b]}(u) \quad (t = \frac{1}{2}).
\end{align*}
\]

Here $G$ is a main measure and $H$ is given in explicit form in terms of $G$, according to formulae in 3.1.

In all cases we set $a = 0$, $b = 1$, and we consider three distributions:

- Uniform distribution on $[0, 1]$, $G(x) = x, x \in [0, 1]$. 
• Beta $(2, 2)$ distribution, $G(x) = x^2(3 - 2x), x \in [0, 1]$.
• Exponential reduced to $[0, 1]$ $G(x) = \frac{1 - e^{-\lambda x}}{1 - e^{-\lambda x}}, \lambda = 1, x \in [0, 1]$

Figures 1-3 show graphs of Karamata’s functions $\varphi$ for J-inequality (with spike), H-inequality (the largest) and TH-inequality (lowest).

Figure 1. Uniform

Figure 2. Beta

Figure 3. Exponential

Since the domain is the interval $[0, 1]$, the average residual size is numerically equal to the area between the $x$-axis and the graph. It is obvious that the area under TH curve is the smallest, in each od three examples with different measures. This is confirmed in in the next table, where we present average residuals as in $\text{[22]}$.

| Inequality | Distribution (on $[0, 1]$) |
|------------|-----------------------------|
|            | Uniform | Exp (1) | Beta $(2, 2)$ |
| Jensen     | 42      | 25      | 40           |
| Classical H| 83      | 100     | 82           |
| Tight H    | 21      | 22      | 21           |

Table 1: The values of $AR \times 10^3$

Relative average residuals can be derived from Table 1. For example, if $\mathcal{I}$ is $\text{TH}$ with uniform distribution and $\mathcal{I}_0$ is $\text{H}$ with Exp (1) reduced to $[0, 1]$, then $RAR(\mathcal{I}, \mathcal{I}_0) = 0.21$.

We conclude that the theory in Sections 2 and 3, together with examples in this section, show an absolute superiority of the new tight approximation to $\int f(x) \, dG(x)$, compared to classical Hermite-Hadamard bonds, with $f$ being convex. The numerical evidences presented in the table above indicates that also it might be the case in comparison to Jensen’s lower bonds, which can be a topic of another research.

REFERENCES

[1] Allasia, G. Connections between Hermite-Hadamard inequalities and numerical integration of convex functions i. Bull. Allahabad Math. Soc. 30 (2015), 211–237.
[2] Bullen, P. Error estimates for some elementary quadrature rules. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602-633 (1978), 97–103.
[3] Durrett, R. Probability: Theory and examples. Cambridge University Press, 2010.
[4] Guessab, A., and Schmeisser, G. Sharp integral inequalities of the Hermite-Hadamard type. J. Approx. Theory 115 (2002), 260–288.
[5] Guessab, A., and Semisalov, B. A multivariate version of Hammer’s inequality and its consequences in numerical integration. Results Math. 73 (2018), Art. 33, 37 pp.
[6] Hammer, P. C. The midpoint method of numerical integration. Math. Mag 31 (1958), 97–103.
[7] Hardy, G. H., Littlewood, J. E., and Pólya, G. Some simple inequalities satisfied by convex function. Messenger Math. 58 (1929), 145–152.
Jensen, J. L. W. V. Sur les fonctions convexes et les inégalités entre les valeurs moyennes. Acta Math. 30 (1906), 175–193.

Karamata, J. Sur une inégalité relative aux fonctions convexes. Publ. Math. Univ. Belgrade 1 (1932), 145–148.

Koliha, J. J. Approximation of convex functions. Real Anal. Exchange 29 (2003), 465–471.

Levin, V. I., and Stečkin, S. B. Inequalities. Amer. Math. Soc. Transl 14 (1960), 1–22.

Lupas, A. A generalization of Hadamard inequalities for convex functions. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 544–576 (1976), 115–121.

Marshall, A. W., Olkin, I., and Arnold, B. C. Inequalities: theory of majorization and its applications. Springer, 2009. Springer series in Statistics, second edition.

McShane, E. J. Jensen’s inequality. Bull. Amer. Math. Soc. 8 (1937), 521–527.

Mercer, P. R. Hadamard’s inequality and trapezoid rules for the Riemann-Stieltjes integral. J. Math. Anal. Appl. 344 (2008), 921–926.

Merkle, M. Remarks on Ostrovs’ and Hadamard’s inequality. Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. 10 (1999), 113–117.

Merkle, M. Representation of the error term in Jensen’s and some related inequalities with applications. J. Math. Analysis Appl. 231 (1999), 76–90.

Mitrinović, D. S., and Lacković, I. Hermite and convexity. Aequations Math. 28 (1985), 229–232.

Ohlin, J. On a class of measures of dispersion with application to optimal reinsurance. ASTIN Bulletin 5 (1969), 249–266.

Olbryś, A., and Szostok, T. Inequalities of the Hermite-Hadamard type involving numerical differentiation formulas. Results Math. 67 (2015), 403–416.

Rajba, T. On the Ohlin lemma for Hermite-Hadamard-Fejer type inequalities. Math. Inequal. Appl 17 (2014), 557–571.

Stečkin, S. B. Supplement 1: Inequalities for convex functions. In: G. H. Hardy, J. E. Littlewood, G. Pólya: Inequalities (in Russian), translated from original by V. I. Levin, with supplements by V. I. Levin and S. B. Stečkin. Gosudarstvenoe izdatelstvo inostrannoi literaturi, Moskva, 1948, pp. 361–367.