The Isaacson expansion in quantum cosmology

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Abstract

This paper is an application of the ideas of the Born-Oppenheimer (or slow/fast) approximation in molecular physics and of the Isaacson (or short-wave) approximation in classical gravity to the canonical quantization of a perturbed minisuperspace model of the kind examined by Halliwell and Hawking. Its aim is the clarification of the role of the semiclassical approximation and the backreaction in such a model. Approximate solutions of the quantum model are constructed which are not semiclassical, and semiclassical solutions in which the quantum perturbations are highly excited.

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I. INTRODUCTION

A. Semiclassical gravity and perturbed minisuperspace

There is a well-explored and consistent theory of quantized (free) fields propagating on a fixed classical spacetime [1]. In particular, one may construct the expectation value of their stress-energy tensor. Although it is formally infinite, there is a variety of regularization procedures which give the same and often sensible result. A simple example of such a result has been experimentally verified in the form of the Casimir effect. A more indirect link of quantum field theory in curved spacetime to physical reality is the possibility of explaining early universe density fluctuations as quantum fluctuations.

It is tempting to insert the quantum stress-energy tensor into the classical Einstein equations which govern the background spacetime. In the Heisenberg picture we could formally write

\[ G_{ab}[g] = \langle \psi | T_{ab}[g, \hat{q}] | \psi \rangle, \tag{1} \]

where \( g \) is the classical metric, \( \hat{q} \) the quantum field operators and \( | \psi \rangle \) their quantum state. To make the theory a consistent approximation, one must include among the quantum fields the linearized perturbations of the metric itself. Higher order effects from the quantum nature of the gravitational field should in contrast be suppressed by a factor of the background curvature scale over the Planck scale. It has been argued that this scheme, called semiclassical gravity, is a better approximation to the presumed quantum theory of gravity than classical general relativity if the curvature scale of the background spacetime is much larger than the Planck length. The question if it really is must be considered open in the absence of both experiments and a quantum theory of gravity. The effective equations of motion for the background metric, taking into account the backreaction of the quantum fields, contain higher than second time derivatives, which give rise to qualitatively different new solutions. It has been argued that these are artefacts of the approximation to full quantum gravity and should be excluded in a consistent way [2].
A tentative model for quantum gravity in which the derivation of semiclassical gravity as an approximation to a more general theory can be attempted is the canonical quantization of a minisuperspace model with generic but linearized perturbations. One such model has been given by Halliwell and Hawking [3], where the minisuperspace model is a closed Friedmann universe driven by a self-interacting real scalar field which is homogeneous on the surfaces of homogeneity.

The Halliwell-Hawking model does not seem to be totally consistent: It has been pointed out [4] that Halliwell and Hawking miss some components of the Einstein equation to a self-consistent order by restricting their ansatz for the background metric before variation. A related problem is the presence of linearization instabilities due to the symmetries of the spatial hypersurfaces [5,6]. In this paper we shall start from the quantized Halliwell-Hawking model as it stands in spite of these problems. Our reason for this neglect is simply that their resolution will require the imposition of a finite number of additional constraints on the wave function, in addition to the infinity that is already there. Although essential for consistency, these should not change the physical content of the theory, in which we are interested here.

Our aim is to reconsider the role of the semiclassical backreaction by a method which applies to the Halliwell-Hawking model, but which, in its physical meaning, should generalize to other perturbed minisuperspace models.

Halliwell [7,8] and other authors [9–11] have given a derivation of the equations of semiclassical gravity which is based on making a single-factor WKB ansatz for the wave function as in reference [3]. As we shall see directly, the backreaction term found this way is small by virtue of the WKB approximation itself, and the theory obtained therefore less general than semiclassical gravity. Semiclassical gravity by naive insertion of the quantum stress tensor into the Einstein equations would in contrast also allow for highly excited states of the quantum fields which contribute a large part or all of the total source term of the Einstein equations. It is this possibility which we examine here, based on the Wheeler-deWitt equation of reference [3]. For simplicity we consider only pure gravity, so that the Fried-
mann universe will be driven by gravitational waves only. Classically, this possibility has been examined a long time ago [12]. The approximation method we shall use is related both to the Born-Oppenheimer approximation in molecular physics [13] and to the short-wave approximation of Isaacson for gravitational waves [14]. In the remainder of this introduction we review the previous approach to the backreaction derivation and then the relevant features of the Born-Oppenheimer and Isaacson approximations.

B. The Halliwell-Hawking approach to the backreaction

Let us consider a model in canonical gravity where the Hamiltonian constraint is of the form

$$H(Q, P, q, p) = H_0(Q, P) + H_2(Q, P, q, p) = 0,$$  \hspace{1cm} (2)

where $H_2$ is homogeneous of order 2 in the $p$ and $q$. In reviewing the general way in which Halliwell and Hawking find an approximate solution to the corresponding Wheeler-deWitt equation, we can limit ourselves to a toy model which will display the features relevant for our argument. For simplicity of notation alone we assume therefore that the quantum operator version of $H_0$ is simply

$$H_0(Q, -i \frac{\partial}{\partial Q}) = -\frac{1}{2} \frac{\partial^2}{\partial Q^2} + V(Q).$$  \hspace{1cm} (3)

(We have assumed a positive definite kinetic energy term. In gravity it is indefinite.) With the ansatz $\Psi(Q, q) = e^{iS(Q)}\psi(Q, q)$ we then obtain

$$\frac{1}{2} S'^2 \psi - i S' \psi' - \frac{1}{2} \psi'' + V \psi + H_2 \psi = 0,$$  \hspace{1cm} (4)

where a prime denotes $\partial/\partial Q$ or $d/dQ$. Dividing by $\psi$ we can separate this equation into

$$\frac{1}{2} S'^2 - \frac{i}{2} S'' + V + E(Q) = 0,$$  \hspace{1cm} (5)

$$\left[ -i S' \frac{\partial}{\partial Q} - \frac{1}{2} \frac{\partial^2}{\partial Q^2} + H_2 - E(Q) \right] \psi = 0.$$  \hspace{1cm} (6)
Here $E(Q)$ is an arbitrary separation function. It has no direct physical significance, as its choice does not affect the product $e^{iS}\psi$, but an appropriate choice can help in the interpretation of the two separate equations of motion. We now assume that both $e^{iS}$ and the total wave function $e^{iS}\psi$ are of the WKB form in the variable(s) $Q$, that is we assume

$$|S''| \gg |S'|,$$  \hspace{1cm} (7)

$$|S'| \gg |(\ln \psi)'|.$$ \hspace{1cm} (8)

To leading order in this approximation we obtain the Hamilton-Jacobi equation for $S(Q)$,

$$\frac{1}{2}S'^{2} + V + E(Q) = 0.$$ \hspace{1cm} (9)

By the argument $S' = P = \partial L/\partial \dot{Q} = \dot{Q}$ (a dot denotes $\partial/\partial t$ or $d/dt$), we can identify $S'\partial/\partial Q$ with $\partial/\partial t$ in the WKB case. To leading order we then obtain a time-dependent Schrödinger equation \[\text{[15,3]}\]

$$\left[ -i \frac{\partial}{\partial t} + H_{2} - E(Q) \right] \psi = 0,$$ \hspace{1cm} (10)

where $\frac{\partial}{\partial t} \equiv S' \frac{\partial}{\partial Q}$.

One might be tempted to set $E(Q) = \langle \psi | H_{2} \psi \rangle$ in (9) in order to obtain the Hamilton-Jacobi form of the semiclassical backreaction \[\text{[9]}\] in the Einstein equations in this way, as was once proposed by Hartle \[\text{[16]}\]. But in (10) this leads to $\langle \psi | -i \partial/\partial t \psi \rangle = 0$, so that the other part of the semiclassical gravity scheme, the time-dependent Schrödinger equation describing the evolution of the quantum fields, is lost. If we want to obtain the correct time-dependent Schrödinger equation for the gravitons, i.e. one where the Hamiltonian is the usual Hamiltonian of linearized gravitational waves in the given background $Q(t)$, we must set $E(Q) = 0$. But then the background evolves without a backreaction from the perturbations. In order to obtain semiclassical gravity as it is commonly understood, one would have to set $Q(Q) = 0$ in (10), and equal to $\langle \psi | H_{2} \psi \rangle$ in (9), a question of having the cake and eating it.

A subtle argument for the presence of a backreaction in spite of this dilemma has been given by Halliwell \[\text{[7]}\]. He determines what the background spacetime really is, not from
equation (9), but by a Wigner function analysis of the total wave function. (The convention for $E(Q)$ is then irrelevant.) Following Halliwell, one may assume that the perturbations $q$ are for some reason unobservable in principle. Predictions should then be made, not from the wave function $\Psi(Q,q)$, but from the density matrix which is obtained by tracing over the perturbations, $\rho(Q,Q') = \int \Psi(Q,q)^* \Psi(Q',q) \, dq$. The Wigner function derived from this density matrix may then be positive definite and may be peaked around a classical trajectory in phase space $(Q,P)$. Halliwell showed that this trajectory is not given by $P(Q) = S'(Q)$, but rather by

$$P(Q) = S'(Q) + <\psi|-i\frac{\partial}{\partial Q}\psi>.$$  \hspace{1cm} (11)

Putting this into the Hamilton-Jacobi equation (9), using (10) and treating the second term in (11) as small, one obtains

$$\frac{1}{2}P^2 + V + <\psi|H_2\psi> = 0.$$  \hspace{1cm} (12)

(As predicted, the choice of $E(Q)$ does not matter.) As a way of finding the correct physics in spite of the appearances of the equations of motion (9,10), this is an impressive argument. It is limited in two ways. By the assumption $|S'| \gg |(\ln \psi)'|$ which went into its derivation, the backreaction term derived in this way is necessarily small and cannot qualitatively change the background. This limitation is also apparent in the fact that the equations one has to solve are still (9) and (10), and not those of semiclassical gravity.

Secondly, only those quantum perturbations contribute to the backreaction which are by assumption unobservable. There are good tentative reasons for this assumption. In a closed universe which has undergone inflation, for example, one would not be able to observe the quantum state of the perturbations completely because of the presence of particle horizons [8]. But why should one only ever be able to observe the gravitational backreaction of those quantum fields that are “out of sight”?

The limitation of the backreaction effect to a small perturbation of the background trajectory is even less satisfactory. One should expect to find a quantum equivalent of
the classical situation where the self-gravity of gravitational waves substantially curves the background spacetime on which they move. Examples of such spacetimes which have been examined are a nearly static spherically symmetric concentration of gravitational waves (a gravitational geon [17]), or a Friedmann universe closed by and driven by gravitational waves [12].

C. The classical Isaacson approximation

A key ingredient for both these cases was supplied by Isaacson [14] (see also [18]). He considered a one-parameter family of solutions of the vacuum Einstein equations of the form

\[ g_{ab}(x; \epsilon) = \gamma_{ab}(x) + h_{ab}(x; \epsilon) \]  

(13)

and assumed that the family of perturbations \( h_{ab}(x; \epsilon) \) is such that their amplitude scales like \( \epsilon \), but their “wavelength” like \( \epsilon^{-1} \). Formally, one assumes that there is a coordinate system in which

\[
\begin{align*}
    h_{ab}(x; \epsilon) &= O(\epsilon), \quad h_{ab,c}(x; \epsilon) = O(1), \quad h_{ab,cd}(x; \epsilon) = O(\epsilon^{-1}),
\end{align*}
\]

(14)

where a comma denotes a partial (coordinate) derivative. \( \gamma_{ab} \) and its derivatives are considered as \( O(1) \). Let the Ricci tensor of \( g_{ab} \) be expanded in powers of \( h_{ab} \):

\[
R_{ab}(g_{ab}, g_{ab,c}, g_{ab,cd}) = R_{ab}^0(\gamma) + R_{ab}^1(\gamma, h) + R_{ab}^2(\gamma, h) + \ldots = 0,
\]

(15)

where \( R_{ab}^1 \) is linear in \( h_{ab} \), \( R_{ab}^2 \) quadratic, and so on. Here \( R_{ab}^1 \) is a contraction (with \( \gamma^{ab} \)) of \( \nabla_a \nabla_b h_{cd} \), where \( \nabla_a \gamma_{bc} \equiv 0 \), and \( R_{ab}^2 \) is a contraction of \( \nabla_a \nabla_b h_{cd} h_{ef} \). Inserting the ansatz (13) into the vacuum Einstein equations, and separating powers of \( \epsilon \), one obtains to leading and next order

\[
R_{ab}^1(\gamma) = 0
\]

(16)

\[
R_{ab}^0(\gamma) = - < R_{ab}^2(\gamma, h) >_{\text{average}}
\]

(17)
(Strictly speaking, the connection terms in $R_{ab}^1$ and $R_{ab}^2$ are of a lower order than the partial derivative terms and do not appear in these equations). The brackets in (17) (and here only) refer to a suitable spacetime averaging. The right-hand side of (17) is an effective stress tensor of the perturbations which drives the background. Equation (16) is a linear wave equation for the perturbations propagating on the background. These equations have to be solved together in a self-consistent way, and as such are not part of a perturbation scheme in which each order is solved before and independently of the higher ones. The difference from conventional perturbation theory is also seen from the fact that the limit $\epsilon \to 0$ is singular.

D. The Born-Oppenheimer approximation

It is worth reviewing the Born-Oppenheimer approximation in a notation which is sufficiently abstract to be suggestive both of its original context of molecule physics and of the context it will be used in in this paper. Let us assume that we are dealing with a Hamiltonian of the form

$$H(Q, P, q, p) = H_0(Q, P) + H_2(Q, q, p).$$

(18)

The single but crucial difference to (2) is that $H_2$ depends only on $Q$, but not on the conjugate momentum (or momenta) $P$. We want to solve the time-independent Schroedinger equation

$$H(Q, -i \frac{\partial}{\partial Q}, q, -i \frac{\partial}{\partial q})\Psi(Q, q) = E\Psi(Q, q),$$

(19)

where $E$ is a given constant. (The case of canonical quantum gravity differs from this only by the presence of additional constraints on the wave function, and by the restriction $E = 0$.) We use the fact that $H_2$ does not contain any $Q$-derivatives to pose the eigenvalue problem

$$H_2(Q, q, -i \frac{\partial}{\partial q})f_\nu(Q, q) = E_\nu(Q)f_\nu(Q, q)$$

(20)

for the complete sets $f_\nu$ and $E_\nu$, for each value of the parameter $Q$. We choose the $f_\nu$ to be orthonormalized under the scalar product.
\[ <f_\nu|f_\mu> = \int f_\nu^* f_\mu dq. \] (21)

(They are of course already orthogonal for \( E_\nu \neq E_\mu \).) We now make the ansatz

\[ \Psi(Q,q) = \sum_\nu \Psi_\nu(Q) f_\nu(Q,q) \] (22)

and solve for the \( \Psi_\nu \). Let us again assume, for simplicity of notation, that \( H_0 \) is of the form (3). (It would be straightforward to write down what follows for \( H_0 \) being an arbitrary second order derivative operator in more than one variable. This general case includes canonical gravity.) We can simplify the result obtained from substituting (22) into (19) by splitting it into components with respect to the basis \( f_\nu \) using the scalar product (21). We obtain an infinite number (labelled by \( \nu \)) of coupled equations in the minisuperspace variable(s) \( Q \):

\[ (H_0 + E_\nu(Q) - E) \Psi_\nu(Q) = \sum_\mu \left( \frac{1}{2} <f_\nu|\frac{\partial^2}{\partial Q^2} f_\mu > \Psi_\mu + <f_\nu|\frac{\partial}{\partial Q} f_\mu \frac{\partial}{\partial Q} \Psi_\mu \right) \] (23)

In the molecular physics application, where \( Q \) are the nucleus positions and \( q \) the electron positions, it can be shown that the right-hand side is small by a factor of the square root of the ratio of the electron mass over the proton mass. The argument itself does not seem capable of extension to gravity, although it has been argued that the Planck mass must be involved in finding a small number in the problem. This is obviously inapplicable to the case of pure gravity, where the Planck mass is the only fundamental scale. In our particular example of a perturbed Friedmann universe the suppression factor will turn out to be the ratio of the length scales of the metric perturbations over the Hubble scale of the background metric, a very intuitive result given the usual language in which the \( q \) are called the fast variables and the \( Q \) the slow variables. Neglecting the right-hand side coupling terms in (23), and assuming \( E = 0 \), we can suggestively write it as

\[ (H_0 + <f_\nu|H_2 f_\nu>) \Psi_\nu(Q) = 0 \] (24)

Here we seem to have obtained the backreaction equation we were looking for in a direct manner and without making any arbitrary choices. However, we are not really solving two
equations self-consistently, in the manner of (16) and (17). Instead we have to do more, namely solve (20) for all values of \( Q \). There is no classical equivalent of this.

In section II we describe the Hamiltonian formulation of a perturbed minisuperspace model – that of Halliwell and Hawking without matter – which is is completely described by a Hamiltonian constraint of the form (18). We make the first steps towards the Born-Oppenheimer approximation by solving the fast-part eigenvalue problem (20) and then writing down the slow-part equation (23). In section III we find the Born-Oppenheimer approximation to that model, by finding the regime in which the right-hand side of (23) can be neglected. This turns out to be the Isaacson approximation. In section IV we recover semiclassical gravity by adding the WKB assumption. We compare our results with previous work. Section V resumes our results and raises some questions on the validity of our starting point (equation (26) below) and the possibility of generalizing our treatment to other perturbed minisuperspace models.

II. BORN-OPPENHEIMER TREATMENT OF A PERTURBED MINISUPERSPACE MODEL

Perhaps the simplest physically complete perturbative minisuperspace model is that of a Friedmann universe without matter, but with generic (small) perturbations of the metric. For a closed \( (K = 1) \) Friedmann universe this is a special case of the model of Halliwell and Hawking [3]. There is a single Hamiltonian constraint of the form (2) – where \( H_2 \) is the integral over all space of the quadratic Hamiltonian constraint, and \( H_0 \) is the minisuperspace Hamiltonian constraint – and an infinity of constraints (the linearized Hamiltonian and momentum constraints at each space point) which are homogenous of order 1 in the \( p \) and \( q \). (There should also be a finite number of additional constraints which are homogeneous of order 2, but as stated in the introduction, with Halliwell and Hawking we neglect these here.)

A canonical transformation that both solves the linearized Hamiltonian and momentum
constraints and brings the remaining part of the Hamiltonian constraint into the “Born-Oppenheimer” form (18) has been given by Wada [19]. The wavefunction then depends only on some of the new variables, and is subject only to a single Hamiltonian constraint. The remaining degrees of freedom in this particular model are the amplitudes of transverse traceless perturbations of the 3-metric, \( d_n \), where \( n \) labels the space dependence (the \( n \)th tensor harmonic on the 3-sphere), and the minisuperspace variable \( \tilde{\alpha} \), which is equal to \( \alpha \), the logarithm of the scale factor, up to terms quadratic in the perturbations.

Although physically interesting in its own right, this model will here also serve as a toy model for a more general perturbative minisuperspace ansatz. Therefore we use the previous notation \( Q \) for the minisuperspace variables and \( \vec{q} = \{q_3, q_4, \ldots q_n, \ldots \} \) for the perturbation variables. (\( n \) is an infinite discrete label which arises from splitting the perturbations into “Fourier” components with respect to their spatial dependence.) In the following \( Q \) is therefore the same as \( \tilde{\alpha} \) and \( q_n \) is the same as \( d_n \) of reference [19], where their precise definitions can be found. The definition of the perturbed spacetime metric in terms of these variables is given in [3]. For understanding the physics of this particular model it suffices to know that the background metric is (now using our notation)

\[
ds^2 = -N_0^2 dt^2 + e^{2Q} d\Omega_3^2,
\]

where \( d\Omega_3^2 \) is the round metric on the unit three-sphere, and where \( N_0 \) and \( Q \) are arbitrary functions of \( t \), and that \( q_n \) is the amplitude of a transverse traceless perturbation of the 3-metric of comoving wavelength \( \sim 1/n \). The total Hamiltonian of the perturbed system, in the remaining degrees of freedom given by Wada (but in our notation) is

\[
H = N_0 e^{-3Q} \left[ -\frac{1}{2} P^2 - e^{4Q} + \frac{1}{2} \sum_{n=3}^{\infty} \left( p_n^2 + (n^2 - 1) e^{4Q} q_n^2 \right) \right].
\]

It is constrained to vanish. We shall find a family of approximate solutions – in the Isaacson limit – of the corresponding Wheeler-deWitt equation

\[
\left[ \frac{1}{2} \frac{\partial^2}{\partial Q^2} - e^{4Q} + \frac{1}{2} \sum_{n=3}^{\infty} \left( -\frac{\partial^2}{\partial q^2} + (n^2 - 1) e^{4Q} q_n^2 \right) \right] \Psi(Q, \vec{q}) = 0.
\]
We have made an ad-hoc choice of factor ordering.

As a first step, as yet without an approximation, we split the Wheeler-de-Witt equation in an infinity of variables into an infinity of equations in only the minisuperspace variables, here only $Q$. This is precisely the ansatz that prepares the Born-Oppenheimer approximation in molecule physics. We make the ansatz

$$\Psi(Q, \vec{q}) = \sum_\nu \Phi_\nu(Q) \prod_{n=3}^{\infty} f_{\nu n} \left( (n^2 - 1)^{1/4} e^{Q q_n} \right).$$

(28)

Here $\vec{\nu} = \{ \nu_3, \nu_4, \nu_5, ..., \nu_n, ... \}$ is an infinite tupel of numbers which have range $\nu = 0, 1, 2, ....$ $f_\nu(x)$ is, for each value of $\nu$, a given function of its argument $x$, namely the normalized energy eigenfunction of the unit frequency harmonic oscillator, defined by $(1/2)(-d^2/dx^2 + x^2)f_\nu(x) = (\nu + 1/2)f_\nu(x)$ and $\int f_\nu^2 dx = 1$. In this definition we have already incorporated the solution of the eigenvalue equation (20) for $H_{\nu}$ given by (27). As the $f_\nu$ form a basis of square-integrable functions on the real line, our ansatz (28) is generic. We introduce the operators $a = (x + d/dx)/\sqrt{2}$, $a^\dagger = (x - d/dx)/\sqrt{2}$ and $N = a^\dagger a$, with the well-known properties $a f_\nu = \sqrt{\nu} f_{\nu-1}$, $a^{\dagger} f_\nu = \sqrt{\nu + 1} f_{\nu+1}$, $a f_0 = 0$, and $N f_\nu = \nu f_\nu$. Now we follow the Born-Oppenheimer procedure of separating the $\Phi_\nu$ by the orthonormality of the $f_\nu$. For this purpose we use the identity $x d/dx = (1/2)(a^2 - a^{\dagger 2} - 1)$ and its square $(x d/dx)^2 = (1/4)(a^4 + a^{\dagger 4} - 2a^2 + 2a^{\dagger 2} - 2N^2 - 2N - 1)$. The final result is

$$
\left( \frac{1}{2} \frac{d^2}{dQ^2} - e^Q - e^{2Q} \sum_n (\nu_n + \frac{1}{2})\sqrt{n^2 - 1} \right) \Phi_\nu = -\frac{1}{2} \sum_n \left[ \Phi_\nu |_{\nu_n+2} - \Phi_\nu |_{\nu_n-2} - \Phi_\nu \right] \\
-\frac{1}{8} \sum_n \left[ \Phi_\nu |_{\nu_n+4} + \Phi_\nu |_{\nu_n-4} - 2\Phi_\nu |_{\nu_n+2} + 2\Phi_\nu |_{\nu_n-2} - (2\nu_n + \nu_n + 1) \Phi_\nu \right] \\
-\frac{1}{8} \sum_n \sum_{m \neq n} \left[ \Phi_\nu |_{\nu_n+2,\nu_m+2} + \Phi_\nu |_{\nu_n-2,\nu_m-2} - 2\Phi_\nu |_{\nu_n-2,\nu_m+2} - 2\Phi_\nu |_{\nu_n+2,\nu_m-2} + 2\Phi_\nu |_{\nu_n+2} + 2\Phi_\nu |_{\nu_n-2} \right].
$$

(29)

Again, a prime denotes $\partial/\partial Q$ or $d/dQ$. To keep this expression readable, we have introduced a shorthand notation that implies certain factors arising from the Bose-Einstein statistics of the excitations of the perturbation modes $q_n$. It is most easily explained by example:
\[ \tilde{\Phi}_{\nu|\nu-4} = \sqrt{n_{\nu}}\sqrt{n_{\nu} - 1}\sqrt{n_{\nu} - 2}\sqrt{n_{\nu}} - 3\Phi_{\nu_3,\nu_4,...,\nu_{n-1},\nu_{n-4},...}, \quad (30) \]

\[ \tilde{\Phi}_{\nu|\nu+2} = \sqrt{n_{\nu}} + 2\sqrt{n_{\nu} + 1}\frac{d}{dQ}\Phi_{\nu_3,\nu_4,...,\nu_{n+2},...}, \quad \text{etc.} \quad (31) \]

The left-hand side of equation (29) contains an infinity of the form \( \sum_{n=3}^{\infty} \sqrt{n^2 - 1} \), and the right-hand side several infinities of the form \( \sum_{n=3}^{\infty} 1 \). We must eliminate these if the equation is to have more than formal meaning. A slightly more consistent procedure than to just cross out these terms is to normal-order all occurrences of \( a \) and \( a^\dagger \), i.e. to put \( a \) to the right of \( a^\dagger \) in all products. In \( H_2 \) we obtain \( N \) instead of \( N + 1/2 \). This is in analogy to standard regularization procedure for free quantum field theory in flat space. \( xd/dx \) becomes : \( xd/dx := (1/2)(a^2 - a^{12}) \). Replacing \( (xd/dx)^2 \) by : \( (xd/dx)^2 \) : does not get rid of all infinities. Instead we choose

\[ : (x d/dx :)^2 := (1/4)(a^4 + a^{14} - 2N(N - 1)). \quad (32) \]

In support of this choice one might argue that \( xd/dx \) should always be normal-ordered as a block as in (32), because it arises from the expression \( \partial f_{\nu}/\partial Q \). With that choice of normal ordering we obtain

\[ \left( \frac{1}{2}\frac{d^2}{dQ^2} - e^4 Q + e^{2Q} \sum_n \nu_n \sqrt{n^2 - 1} \right) \Phi_{\nu} = -\frac{1}{2} \sum_n \left[ \tilde{\Phi}''_{\nu|\nu+2} - \tilde{\Phi}''_{\nu|\nu-2} \right] - \frac{1}{8} \sum_n \left[ \tilde{\Phi}''_{\nu|\nu+4} + \tilde{\Phi}''_{\nu|\nu-4} - 2\nu_n (\nu_n - 1) \Phi_{\nu} \right] \]

\[ - \frac{1}{8} \sum_n \sum_{m \neq n} \left[ \tilde{\Phi}''_{\nu|\nu+2,\nu_m+2} + \tilde{\Phi}''_{\nu|\nu-2,\nu_m-2} - 2\Phi_{\nu|\nu-2,\nu_m+2} \right]. \quad (33) \]

In the following we use this equation as our starting point. We shall refer to the \( \nu_n \) as “occupation numbers” by analogy with the states of a set of harmonic oscillators. Furthermore we shall by the same analogy refer to the first line of the right-hand side of (33) as the “two-particle terms” and to the remainder as the “four-particle terms”.

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III. QUANTUM VERSION OF THE ISAACSON EXPANSION IS THE BORN-OPPENHEIMER APPROXIMATION

We now introduce the Isaacson expansion as a means of making the right-hand side of (33) small compared to the last term on the left-hand side, thus completing a Born-Oppenheimer approximation scheme. (To post-Born-Oppenheimer order we shall recover semiclassical gravity.) With Isaacson we consider a family of approximate solutions labeled by a small parameter $\epsilon$, such that the large-scale metric, in our case the Friedmann background, is independent of $\epsilon$. Both the amplitude and the wavelength of the perturbations are chosen to scale like $\epsilon$, so that their effective energy-momentum tensor, suitably averaged, is independent of $\epsilon$ and can act as the source for the background curvature. In our model, defined by the Hamiltonian (26), we identify as a guidance principle the perturbation wavelength with $1/n$ and the classical amplitude with the occupation number $\nu_n$. The total energy of the perturbations is then proportional to

$$\sum_{n=3}^{\infty} \nu_n \sqrt{n^2 - 1} \sim \int_{0}^{\infty} \nu(n) n \, dn.$$  \hspace{1cm} (34)

This is of course the term appearing on the left-hand side of (33) which we would like to retain as a backreaction term after the model of (24). To make this expression (approximately) independent of $\epsilon$, we may consider the one-parameter family of occupation numbers (classically, amplitudes)

$$\nu_n^{(\epsilon)} = \epsilon^2 \mu(\epsilon n; \epsilon).$$  \hspace{1cm} (35)

The explicit $\epsilon$-dependence in the second argument of the function $\mu(n; \epsilon)$ is necessary to make the perturbation energy precisely $\epsilon$-independent with this ansatz. Isaacson however uses the fact that for small $\epsilon$ this dependence becomes weak, so that for sufficiently small $\epsilon$ the function $\mu(n, 0)$ characterizes a whole family of approximate solutions $\nu_n^{(\epsilon)}$ by the scaling relation $\nu_n^{(\epsilon)} \simeq \epsilon^2 \mu(\epsilon n, 0)$.

The equivalent in the quantum theory as defined by (33) is a one-parameter family of wave functions $\Phi_n^{(\epsilon)}(Q)$. We define it as
\[
\Phi^{(\epsilon)}_{\vec{\mu}}(Q) = F_{\vec{\mu}}(Q; \epsilon) \text{ where } \mu_n = \epsilon^{-2} \nu_{\epsilon^{-1}n}. \quad (36)
\]

To keep both the value and the suffix of \(\mu_n\) integer one must restrict the range of \(\epsilon\) to \(\epsilon = N^{-1}\), with \(N\) any positive integer. A similar problem, that the family of approximate solutions cannot be continuous in \(\epsilon\), would arise also in the classical Isaacson approach to a closed universe. It therefore does not invalidate the existence of the approximation. In the case of open spatial hypersurfaces, the suffix \(n\) becomes continuous anyway, and one may then simply consider \(F(\vec{\mu})\) as a smooth functional on a continuous field. Alternatively one may set \(\epsilon = 1\) at the end of a formal expansion in \(\epsilon\). Then one no longer considers a one-parameter family of solution, but rather a single solution whose occupation numbers are nonvanishing for high frequencies only, and then small.

By substituting the ansatz (36) into the equations of motion for \(\Phi_{\vec{\nu}}\), and then changing variable from \(\vec{\nu}\) to \(\vec{\mu}\) and summation index from \(n\) to \(\epsilon n\), we obtain an equation of motion for \(F\) that contains explicit powers of \(\epsilon\) in its coefficients:

\[
\begin{align*}
&\left(\frac{1}{2} \frac{d^2}{dQ^2} - \epsilon^4 Q + \epsilon^2 Q \sum_n \nu_n \sqrt{n^2 - 1}\right) F_{\vec{\mu}}(Q; \epsilon) \\
&= -\frac{1}{2} \epsilon^2 \sum_n \left[ \sqrt{\mu_{\epsilon n} + 2\epsilon^{-2}} \sqrt{\mu_{\epsilon n} + \epsilon^{-2} F_{\vec{\mu}_{\epsilon n}+2\epsilon^{-2}}(Q; \epsilon)} - \sqrt{\mu_{\epsilon n}} \sqrt{\mu_{\epsilon n} - \epsilon^{-2} F_{\vec{\mu}_{\epsilon n}+2\epsilon^{-2}}(Q; \epsilon)} \right] \\
&- \frac{1}{8} \epsilon^4 \sum_n \left[ \sqrt{\mu_{\epsilon n} + 4\epsilon^{-2}} \sqrt{\mu_{\epsilon n} + 3\epsilon^{-2}} \sqrt{\mu_{\epsilon n} + 2\epsilon^{-2}} \sqrt{\mu_{\epsilon n} + \epsilon^{-2} F_{\vec{\mu}_{\epsilon n}+4\epsilon^{-2}}(Q; \epsilon)} \right] - \frac{1}{8} \epsilon^4 \sum_{n \neq m} \left[ \ldots \right].
\end{align*}
\]

(37)

We have not written out all terms. The remainder can easily be reconstructed from (33). The important point is that all “two-particle” terms of the right-hand side are multiplied by a factor of \(\epsilon^2\) and all “four-particle terms” by a factor of \(\epsilon^4\). These factors arise from the Bose statistics prefactors, which we have written out here for this reason.

We see from the presence of these explicit factors of \(\epsilon\) in (37) that it was necessary to have given \(F\) an explicit dependence on \(\epsilon\) (as well as the dependence of its formal argument \(\vec{\mu}\) on \(\epsilon\)). But by analogy with the work of Isaacson, we expect that solutions for different values of \(\epsilon\) are related by a simple scaling of their arguments, as \(\epsilon\) becomes small. Expecting the same physical phenomenon in the quantum theory, we have already incorporated that...
scaling behaviour into (36). We now express our expectation that the explicit ε-dependence
of $F$ disappears as $\epsilon \to 0$ by expanding it as

$$F_{\mu}(Q; \epsilon) = \chi_{\mu}(Q) \rho_{\mu}(\epsilon^2 Q) \sigma_{\mu}(\epsilon^4 Q) \ldots$$ \hspace{1cm} (38)

The final form of the equations of motion with our ansatz is then, after separating powers
of $\epsilon$ and then setting $\epsilon = 1$,

$$\left( \frac{1}{2} \frac{d^2}{dQ^2} + e^{2\epsilon Q} \sum_n \nu_n \sqrt{n^2 - 1} - e^{4\epsilon Q} \right) \chi_{\nu}(Q) = 0, \hspace{1cm} (39)$$

$$\chi_{\nu}'(Q) \frac{d}{dQ} \rho_{\nu}(Q) = -\frac{1}{2} \sum_n \left( \chi_{\nu+n}'(Q) \tilde{\rho}_{\nu+n}(Q) - \chi_{\nu+n-2}'(Q) \tilde{\rho}_{\nu+n-2}(Q) \right), \text{ etc.} \hspace{1cm} (40)$$

The leading order equation (39) is of course just (37) with $\epsilon$ set equal to zero. As such
it is the Born-Oppenheimer approximation to the quantum equations of motion. From the
analogy with the classical Isaacson expansion we expect that it describes the motion of the
background driven by the averaged perturbations. The next order (40) should describe “the
motion of the perturbations on that background”. In particular, if the solution to (39) is
of the WKB form, thus implying a classical background $Q(t)$, we should expect (40) to
describe a free quantum field theory on the curved classical spacetime given by (25). The
following order, a first-order linear differential equation for $\sigma$ with coefficients depending on
$\chi$ and $\rho$, should describe higher order corrections that we do not expect to understand in
this intuitive sense, and we have therefore not written it out. It may be worth clarifying
that, although the physical significance of $\epsilon$ here is the same as in (14), mathematically it
leads to a perturbation expansion as in (10,11) rather than a self-consistent field expansion
as in (16,17).

Before we examine the semiclasical gravity interpretation of (39), (40), we note here
that this set of equations has solutions which are definitely not semiclassical. We have
therefore found approximate solutions of the pure gravity Halliwell-Hawking model which
are not semiclassical, as promised in the abstract. In other words, it does not actually make
a difference to solving the perturbation equations if the background is behaving classically
or not, because we need not look for a time-dependent Schroedinger equation.
IV. WKB APPROXIMATION AND SEMICLASSICAL GRAVITY

We now compare our results with the corresponding quantum field theory of perturbations on a fixed background spacetime, with a given $Q(t)$ (for a given choice of $N_0(t)$) in the background metric $\text{(25)}$. As a first step we must formulate precisely what we mean by quantum field theory in a curved spacetime in the context of our model of a Friedmann universe with gravitational wave perturbations. The Friedmann universe $\text{(25)}$ is to be the classical background, and the metric perturbations are to be the quantum fields. As we want to make contact with quantum cosmology, we choose the Schrödinger picture.

Instead of a Wheeler-deWitt equation the perturbations obey the time-dependent Schrödinger equation

$$\left[ -i \frac{\partial}{\partial t} + \frac{1}{2} N_0(t) e^{-3Q(t)} \sum_{n=3}^{\infty} \left( -\frac{\partial^2}{\partial q^2} + (n^2 - 1) e^{4Q(t)} q_n^2 \right) \right] \psi(t, \vec{q}) = 0. \tag{41}$$

We again split the wave function into harmonic oscillator components

$$\psi(t, \vec{q}) = \sum_{\vec{\nu}} \Phi_{\vec{\nu}}(t) \prod_{n=3}^{\infty} f_{\nu_n} \left( (n^2 - 1)^{1/4} e^{Q(t)} q_n \right) \tag{42}$$

with the same notational conventions as before. We may then apply the same procedure of identifying the action of $\partial/\partial Q$ on the $f_{\nu_n}$ with the action of annihilation and creation operators. The only difference is that $\partial/\partial Q$ arises here from $-i \partial/\partial t$ acting on the $Q(t)$ inside the $f_{\nu_n}$. Therefore only first $Q$-derivatives appear, and the equations of motion for the coefficients are much simpler. We use normal ordering again as a regularization procedure. As there are no terms quartic in $a$ and $a^\dagger$, it is unambiguous. We obtain

$$\left( -i \frac{d}{dt} + N_0(t) e^{-Q(t)} \sum_{n} \nu_n \sqrt{n^2 - 1} \right) \Phi_{\vec{\nu}} = \frac{i}{2} \frac{dQ}{dt} \sum_{n} \left( \tilde{\Phi}_{\nu_n+2} - \tilde{\Phi}_{\nu_n-2} \right) \tag{43}$$

In these equations we recognize the well-known fact that “particles” are created in pairs (due to the symmetry of the background spacetime) and mainly in spacetime regions where the curvature scale is less than or comparable to their Compton wavelength. This is not the place to review definitions of the particle concept in curved spacetime. It will suffice to consider a simple situation in which $Q(t)$ is constant, then changes, then is constant again.
In the regions of constant $Q$ we have a well-defined notion of ground state and particles. The ground state is that in which all $\Phi_{\vec{\nu}}$ vanish apart from $\Phi_{\vec{0}}$. If we start with this state at an early time, then after the period of nonvanishing $\dot{Q}$, at late times, the other components $\Phi_{\vec{\nu}}$ will no longer vanish. The components with $\nu_n = 2, 4, \ldots$ for a given $n$ will be excited by being strongly coupled to the $\vec{\nu} = \vec{0}$ component, if $|dQ/dt| > N_0 e^{-Qn}$ during that period. But $e^Q/n$ is the physical wavelength of the perturbation labelled by $n$, and $N_0^{-1} dQ/dt$ the Hubble constant (with respect to proper time). Therefore the criterion that particles with comoving wavelength $1/n$ be produced abundantly is just that their physical wavelength be greater than the Hubble distance.

It is illuminating to separate off the trivial part of the time evolution of the $\Phi_{\vec{\nu}}$ (that is the part which is independent of their initial values):

$$\Phi_{\vec{\nu}}(t) = \rho_{\vec{\nu}}(t) \exp \left( -i \sum_n \nu_n \sqrt{n^2 - 1} \int_0^t N_0(t') e^{-Q(t')} dt' \right)$$  \hspace{1cm} (44)

($\rho_{\vec{\nu}}$ defined here is not the same as in equation (38), but we have chosen this notation consciously, as the two definitions will be seen to be related.) The resulting equation for the $\rho_{\vec{\nu}}$ is now real:

$$\frac{d}{dt} \rho_{\vec{\nu}}(t) = -\frac{1}{2} \frac{dQ}{dt} \sum_n \left( \tilde{\rho}_{\vec{\nu}n+2} - \tilde{\rho}_{\vec{\nu}n-2} \right)$$  \hspace{1cm} (45)

One way of looking at this equation is as an interaction picture of quantum field theory: The trivial part of the dynamics has been suppressed and only the interaction part is described explicitly. In flat space quantum field theory this is the nonlinear part of the equation of motion, but in curved space also the interaction with the background curvature, even for linear fields.

Although equation (44) is real, it still describes the entire evolution of the quantum fields $q$ on the background spacetime ($\mathbb{P}^2$) – the multiplication by the factor (44) is merely algebraic. But we have now formulated the time-dependent Schrödinger equation in a way which evokes Hartle’s original attempt at finding the backreaction in an unexpected manner. In particular, if we define $\psi(\vec{q}, t) = \sum_{\vec{\nu}} \rho_{\vec{\nu}} f_{\vec{\nu}}$, we find $<\psi| \partial/\partial t \psi > = 0$. The energy
of the perturbation has been hidden away in equation (44), and we can always reconstruct it trivially.

In order to establish the relation of (45) to the full Wheeler-deWitt equation, we can use $Q(t)$ locally as the independent variable instead of $t$, by replacing $d/dt$ by $(dQ/dt)d/dQ$. The equation for $\rho_{\vec{v}}$ we obtain is then very similar to equation (40):

$$
\frac{d}{dQ} \rho_{\vec{v}}(Q) = -\frac{1}{2} \sum_{n} \left( \tilde{\rho}_{|\nu_n+2} - \tilde{\rho}_{|\nu_n-2} \right)
$$

(46)

This resemblance is rather surprising, as we have not yet introduced a semiclassical approximation, only the Isaacson expansion. What is missing to recover (43) from (40)? The first requirement is that $\chi_{\vec{v}} \simeq \chi_{\vec{v}|\nu_n \pm 2}$ for all $n$, so that these factors cancel in (40), allowing us to go from there to (46). Clearly this is possible for a certain interval of $Q$ given appropriate initial conditions, because the equations (39) for $\chi_{\vec{v}}$ and $\chi_{\vec{v}|\nu_n \pm 2}$ have nearly the same coefficients: The difference is only the energy of two gravitons compared to the energy of all gravitons in the universe. Our first requirement is therefore that $\chi_{\vec{v}}(Q)$ be a smooth function not only of $Q$, but also of $\vec{v}$.

The second requirement, for going from (46) to (45), is not surprising: Clearly $dQ/dt$ is not defined for arbitrary $\chi_{\vec{v}}(Q)$, but only by the Banks [15]-Halliwell-Hawking [3] approximation. To obtain it, we must assume that the $\chi_{\vec{v}}$ are of WKB form. Finally, we obtain (43) from (45) by the trivial step (44).

What is the equation of motion for the semiclassical background? We substitute the WKB ansatz $\chi_{\vec{v}} = e^{-iS(Q)}$ into the minisuperspace equation (39) and in the WKB approximation $|S'|^2 \gg |S''|$ approximate $\chi''$ by $-S'^2 \chi$ and interpret $S'$ as the classical momentum $P$. From the Hamiltonian (26) we read off $\dot{Q} = -N_0 e^{-3Q} P$ and obtain finally the classical energy constraint

$$
\frac{1}{2} \left( \frac{\dot{Q}}{N_0} \right)^2 + e^{-2Q} = e^{-4Q} \sum_{n} \nu_n \sqrt{n^2 - 1}.
$$

(47)

As $e^{Q(t)}$ is the Friedmann scale factor, and $\dot{Q}/N_0$ the Hubble constant, this is just the Friedmann equation for a closed universe in the presence of a radiation fluid, whose density scales as (scale factor)$^{-4}$. 

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In rederiving semiclassical gravity we have not quite arrived at (1). From a naive interpretation of that equation one might expect that the backreaction term of a mixed state is just the sum of the backreaction terms of its pure-state components. This seems to be true in our model only if the components have approximately the same energy. In other words we cannot have the background driven by an effective perturbation stress tensor which is an average of the stress tensors of two different pure quantum states, unless these stress tensors are nearly the same. Presumably such a situation would not make experimental sense.

It should also be noted that we have found neither negative energy in the effective stress tensor of the quantum fields – it is just that of a radiation fluid – nor fourth-order effective background equations. This may be due to our choice of model to start from.

V. CONCLUSIONS

We have found a more general way of (approximately) solving the Wheeler-deWitt equation of the Halliwell-Hawking model than the one making a WKB ansatz for the background variables. Our approximation requires that the system splits into a slow and fast part, namely the Friedmann background metric and short-wave metric perturbations, with intermediate wavelengths in or close to their ground states. Physically, this assumption corresponds to the Isaacson approximation in classical gravity, mathematically to the Born-Oppenheimer approximation in quantum mechanics.

Under this condition the Wheeler-deWitt equation splits into a system of uncoupled minisuperspace Wheeler-deWitt equations for the Friedmann background and a system of coupled first order equations for the perturbations, with coefficients depending on the solutions of the background equations. The essential difference to the Halliwell-Hawking solution scheme is that one does not have to assume that the background is in a WKB state to obtain this split.

The WKB form of the background can be chosen as a second, and independent assumption, which allows to interpret the minisuperspace WdW equations as a Hamilton-Jacobi
equations for a classical background driven by a classical radiation fluid, and the system of perturbation equations as a time-dependent Schroedinger equation for QFT on that classical background. In other words, the WKB approximation works in the same way in our scheme as in the Halliwell-Hawking scheme, but the class of semiclassical solutions we obtain is more general: The Hamilton-Jacobi equation (the Friedmann equation in our model) contains the energy of a classical radiation field. This term, the semiclassical backreaction of the quantum metric perturbations, can be large, i.e. the quantum fields can be highly excited.

The most important remaining question is whether a Born-Oppenheimer expansion can be applied to any other perturbed minisuperspace models. We do not know why the canonical transformation described by Wada [19] manages to solve all the linearized constraints of the Halliwell-Hawking model restricted to pure gravity and at the same time to bring the Hamiltonian into a form that allows our Born-Oppenheimer treatment. The same kind of transformation, applied by Shirai and Wada [20] to the full model including scalar field matter, is not so successful: The linearized Hamiltonian constraint is not solved, and the zero-plus-second-order Hamiltonian is not of Born-Oppenheimer form in the remaining degrees of freedom. There is no obvious reason for this failure.

Wada’s treatment is hard to follow in geometrical terms, as he works from the Hamiltonian of Halliwell and Hawking, which is complicated for two different reasons. Firstly, Halliwell and Hawking split all perturbations into harmonics on $S^3$ from the start. Secondly, they truncate the action and then calculate the Hamiltonian. One would expect more insight from truncating the full Hamiltonian directly, a procedure which may be inequivalent if the background is not itself a solution. In truncating the full Hamiltonian directly, the source of terms of the form $f(Q)Ppq$ and $f(Q)PPqq$ in $H_2$ is clear: They arise from expanding the “supermetric times momentum times momentum” part of the full Hamiltonian constraint. Still, one may able to see for the general case of a homogeneous background if a canonical transformation cannot be found which gets rid of these terms. At least one should be able to formulate Wada’s transformation in geometrical terms and
explain why it does not generalize, if indeed it does not. In such a treatment it would be
necessary and probably illuminating to correct the inconsistencies of the Halliwell-Hawking
model we mentioned in the introduction.

After this work was completed, reference [21] was brought to the author’s attention.
There a general, exact solution of the quantum problem we have considered here is given. If
one makes the canonical transformation $\tilde{q}_n = (\exp Q)q_n$, $\tilde{Q} = Q + \frac{1}{2} \sum_n q_n^2$, the Hamiltonian
(26) transforms into a sum of simple harmonic oscillator Hamiltonians, the one representing
$\exp Q$ with a negative sign. (The second part of this transformation is not made explicit in
[21] ). One can therefore give the general exact solution of the corresponding Wheeler-deWitt
equation in terms of products of harmonic oscillator eigenfunctions.

The simple expression for the transformed Hamiltonian arises in the approximation in
which terms of order $\sum_n q_n^2$ are kept, but terms of order $(\sum_n q_n^2)^2$ are neglected. This is the
same approximation that was used in truncating the full Hamiltonian in the first place, i.e.,
to keep terms to quadratic order in the metric perturbations. It breaks down towards
the initial or the final singularity or both of any classical solution, where the perturbations
become large and must be treated nonlinearly.

The difference between $\exp Q$ and $\exp \tilde{Q}$ is of order $\sum_n q_n^2$ and therefore of physical
significance: Whereas the original Hamiltonian of the Friedmann universe of scale factor
$\exp Q$ driven by gravitons is the same as that of a Friedmann universe of the same scale
factor driven by a massless minimally coupled scalar field, the transformed Hamiltonian is
the same as that of a Friedmann universe with scale factor $\exp \tilde{Q}$ driven by a conformally
coupled scalar or an electromagnetic field. A minimally coupled and a conformally coupled
massless scalar field – or equivalently gravitational and electromagnetic radiation – driving
a Friedmann background do not give rise to the same spacetime, beyond leading order in
the Isaacson approximation. Only in the second case the effective stress-tensor is precisely
traceless. To interpret a classical or quantum solution of the transformed system one must
transform it back to variables $\exp Q$ and $q_n$ (when the closed form of the solution is lost). The
fact that the background spacetime with scale factor $\exp Q$ is more than just one of
several equivalent choices of variable is obscured by the simplicity of our model: In a more complete model the physical scale factor could be determined with the help of any additional test matter field introducing a fundamental scale. The present paper derives solutions of the quantum system directly in the physical variables. This scheme is directly linked to the derivation of semiclassical gravity, while the solution of [21] has the merit of being in closed form.

The “Born-Oppenheimer” solution scheme might have been introduced by starting with the canonical transformation $\tilde{q}_n = (\exp Q)q_n$, $\tilde{Q} = Q$, giving rise to terms $\tilde{p}\tilde{q}$ and $\tilde{p}\tilde{q}\tilde{p}\tilde{q}$ in the transformed Hamiltonian. These are just the 2- and 4-particle interactions we have described. Seen this way, the 4-particle interactions should be neglected as being of quartic order in the perturbations. This means that we cannot consistently derive the equation of motion for $\sigma$ and higher order corrections in $\epsilon$. The author takes this as an indication that the physical content of a quantized perturbed minisuperspace model does not go beyond semiclassical gravity – higher order perturbative approximations to quantum gravity require a better approximation to the full classical action.

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