Minimum Distance Estimators for Dynamic Games

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Abstract

We develop a minimum distance estimator for dynamic games of incomplete information. We take a two-step approach, following Hotz and Miller (1993), based on the pseudo-model that does not solve the dynamic equilibrium in order to circumvent the potential indeterminacy issues associated with multiple equilibria. The class of games estimable by our methodology includes the familiar discrete unordered action games as well as games where players’ actions are monotone (discrete, continuous or mixed) in their private values. We also provide conditions for the existence of pure strategy Markov perfect equilibria in monotone action games under increasing differences condition.

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1 Introduction

We propose a new estimator for a class of dynamic games of incomplete information that builds on the Markov discrete decision framework reviewed in Rust (1994). Our estimator adds to a growing list of methodologies to analyze empirical games discussed in the surveys of Ackerberg, Benkard, Berry and Pakes (2005) and Aguirregabiria and Mira (2010). Two well-known obstacles to structural estimation of dynamic games arise from multiple equilibria and the computational of value functions that represent future expected returns. More specifically, for each structural parameter, the model may have non unique equilibria predicting different distributions of actions and, even when there are no issues of equilibrium selection, it is numerically demanding to evaluate the value functions that are defined as fixed points of some nonlinear functional equations. We take a two-step approach that does not solve out the full dynamic optimization problem and is designed to circumvent these issues.

We begin with an assumption that pure strategy Markov perfect equilibria exist and data are generated from a single equilibrium. Most two-step estimators in the literature, following Hotz and Miller (1993)’s work in a single agent discrete choice problem, consider the pseudo-model where the intractable value functions are replaced by easy to compute policy value functions that can be constructed using beliefs observed from the data. Each player’s pseudo-decision problem can then be interpreted as playing a single stage game against nature. When the pseudo-decision problem has a unique solution almost surely, each player’s best response is a pure strategy so that any candidate structural parameter is mapped into an implied distribution function that defines a complete pseudo-model (as opposed to incomplete models, for instance, see Tamer (2003)). Conditions for the existence of Markov perfect equilibria, as well as the uniqueness of the solution to pseudo-decision problems, have been established for games where players actions are modeled to be (unordered) discrete, where players’ private values enter the payoff functions additively; see Aguirregabiria and Mira (2007, hereafter AM), Bajari, Chernozhukov, Hong and Nekipelov (2009), and Pesendorfer and Schmidt-Dengler (2008, hereafter PSD).\footnote{Bajari et al. (2009) also consider a one-step estimator.} In an independent work, Schrimpf (2011) also recently proposes an estimator for continuous action games. Whilst the aforementioned papers make use of the pseudo-decision problem and focus on games with a single type of actions, Bajari, Benkard and Levin (2007, hereafter BBL) take a different approach, using forward simulation, that can handle models with both discrete and/or continuous decisions. BBL’s methodology is versatile, particularly it has been applied to model games where players’ actions are monotone in their private values; for some examples, see Gowrisankaran, Lucarelli, Schmidt-Dengler and Town (2010), Ryan (2010) and Santos (2010).

The main contribution of this paper is to provide an alternative estimator for a large class of
games that includes the models considered in BBL and their subsequent applications. A distinctive feature of BBL’s methodology is the use of inequality restrictions to construct objective functions. Since there exists little guidance on how to select inequalities, we show that some popular classes of inequalities can lead to objective functions that do not have unique (minimizing) solutions as the sample size tend to infinity, even when the underlying model is actually point-identified. Our estimator is obtained by minimizing the distance between distributions of actions observed from the data and predicted by the pseudo-model. We provide a set of conditions to ensure our estimator is consistent and asymptotically normal.

We also contribute in providing important foundations for the modeling of games where players play monotone strategies. Existence of pure strategy Markov equilibria is often assumed in dynamic games where players employ monotone strategies with respect to their private information, for examples, see BBL, (ordered-discrete action) Gowrisankaran et al. (2010), and (continuous action) Schrimpf (2011). We provide primitive conditions based on increasing differences that ensure monotone pure strategy Markov equilibria exist for dynamic games when the action variable can be discrete, continuous, or a mixture of both. We also show that the same conditions are sufficient for each player’s best response to the pseudo-decision problem be a pure strategy almost surely. Therefore the pseudo-model can bypass the issues associated with multiple equilibria for this class of games.

BBL define their estimator using a system of moment inequality restrictions implied by the equilibrium condition. Their estimator satisfies a necessary condition of an equilibrium that the implied expected return from the optimal strategy is at least as large as the returns from employing alternative strategies, where each alternative strategy is represented by an inequality. To give an intuition of why inequality selection may have a non-trivial implication, suppose the parameter of interest is uniquely identified by the inequality restrictions implied by the equilibrium. However, the equilibrium imposes that inequalities must hold for all alternative strategies. If we restrict our attention to certain classes of inequalities, for examples additive or multiplicative perturbations, these inequalities may not be able to identify the parameter of interest in the sense that there are other elements in the parameter space that also satisfy these less restrictive sets of inequality restrictions. Our comment is closely related to the general issue of consistent estimation in conditional moment models. Particularly, in a familiar instrumental variable framework, Domínguez and Lobato (2004) provide explicit examples when there is a unique value in the parameter space that satisfies a conditional moment (equality) restriction but the uniqueness is lost when the conditional moment is converted into a finite number of unconditional moments. Domínguez and Lobato (2004) and Khan and Tamer (2009) also show how to construct objective functions that preserve the identifying information content of conditional moment models commonly used in economics, with equality and
inequality restrictions respectively. However, their techniques are not applicable to BBL’s estimation methodology. We show that the loss of identifying information associated with BBL’s inequality selection problem can occur even without any conditioning variable.

Our estimator is motivated by a characterization of a Markov perfect equilibrium, as fixed points of an operator that maps beliefs into distributions of best responses. Thus, our construction of the pseudo-model can be seen as a generalization of AM and PSD who provide analogous characterizations for unordered discrete games that also play central roles in their estimation methodologies. We show the game they consider is included in our general setup. We define a class of minimum distance estimators from the characterization of the equilibrium. Our estimation methodology proceeds in two stages. In the first stage we use the distributions of actions from the data as the nonparametric beliefs to simulate the distributions of the pseudo-model implied best responses. We then compare the simulated distributions with the nonparametric distributions in the second stage by minimizing some $L^2$-distance.

We prove our equilibrium existence results by closely following the arguments in Athey (2001), which show pure strategy equilibria exist for static games of incomplete information under single crossing conditions. Athey’s results are amenable to the dynamic games we consider once we restrict ourselves to players playing stationary Markov strategies. Existence of Markov equilibria in other related games can be found in AM and PSD for a class of unordered discrete action games, and Doraszelski and Satterthwaite (2010) for games with entry/exit decisions with investment decisions.

Throughout the paper we treat the transition law of the observed states nonparametrically since the transition law is a model primitive that we often have little information on. We also maintain a common assumption in this literature that the observable states take finitely many values. Therefore the estimation problem is a semiparametric one when the action variable is continuous. The effective rate of convergence of the nonparametric estimator in our methodology is determined by a one-dimensional object, which is consistent with the nature of a simultaneous-move game where each player forms an expectation conditioning only on her action. Therefore our proposed estimator does not suffer from the nonparametric curse of dimensionality with respect to the number of players. This is in contrast to extending the forward simulation method of BBL (Step 3, p.1343) to estimate a semiparametric model, where future states are drawn conditionally on the actions of all players.\footnote{BBL only consider a fully parametric estimation framework.}

We note that it is also possible to extend our estimation procedure to allow for continuous states, as illustrated by Srisuma and Linton (2012) when action is discrete, although this may be of limited practical interest when the action is also continuous.

The rest of the paper proceeds as follows. Section 2 introduces the class of games estimable by our two-step approach. We provide the details of our methodology in Section 3. A general large
sample theory is given in Section 4. Section 5 reports results from Monte Carlo studies, where we also consider the performance of BBL estimators when the objective functions used cannot identify the parameter of interest in the limit. Section 6 concludes.

Appendix A contains three parts, A.1 - A.3. In A.1, we give two examples where the inequality restrictions imposed by the equilibrium is satisfied by a unique element in the parameter space, but the uniqueness is lost when some well-known subclasses of all inequalities are considered. In A.2, we show a simple class of inequalities can be used to construct objective functions that preserve the identifying information from the equilibrium in discrete action games where players’ best response is characterized by some cut-off rules; by choosing alternative strategies based perturbing the cut-off values only in the first period. The suggested inequalities are applicable for unordered and ordered action games. A.3 provides some additional discussion. Appendix B contains proofs of the Theorems.

2 Markovian Games

This section introduces the class of estimable games for our methodology. We begin by describing the elements of the general model and define the equilibrium concept. We then consider the players’ decision problems and show that when players’ best responses to any Markovian beliefs are pure strategies almost surely, then the equilibrium can be characterized by a fixed point of an operator that maps beliefs into distributions of best responses. We end the section by providing examples of Markovian games that have been used in the literature. Particularly, we shall study in detail the games where payoffs satisfy increasing differences condition.

2.1 Model

We consider a dynamic game with $I$ players, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$, over an infinite time horizon. The elements of the game in each period are as follows:

**Actions.** We denote the action variable for player $i$ by $a_{it} \in A_i$. Let $a_t = (a_{1t}, \ldots, a_{It}) \in A = A_1 \times \cdots \times A_I$. We will also occasionally abuse the notation and write $a_t = (a_{it}, a_{-it})$ where $a_{-it} = (a_{1t}, \ldots, a_{i-1t}, a_{i+1t}, \ldots, a_{It}) \in A_{-i} = A \setminus A_i$.

**States.** Player $i$’s information set is represented by the state variables $s_{it} \in S_i$, where $s_{it} = (x_{it}, \varepsilon_{it})$ such that $x_{it} \in X_i$ is common knowledge to all players and $\varepsilon_{it} \in \mathcal{E}_i$ denotes private information only observed by player $i$. For notational simplicity we set $x_{it} = x_t$ for all $i$, this is without any loss of generality as we can define $x_t = (x_{1t}, \ldots, x_{It}) \in X$. We shall use $s_i$ and $(x, \varepsilon_i)$ interchangeably. We define $(s_i, s_{-it}, \varepsilon_t, \varepsilon_{-it}, \mathcal{E})$ analogously to $(a_t, a_{-it}, A)$, and denote the support of $s_i$ by $S = X \times \mathcal{E}$. 


**State Transition.** Future states are uncertain. Players’ actions and states today affect future states. The evolution of the states is summarized by a Markov transition law $P(s_{t+1}|s_t, a_t)$.

**Per Period Payoff Functions.** Each player has a payoff function, $u_i: A \times S_i \rightarrow \mathbb{R}$, that is time separable. The payoff function for player $i$ can depend generally on $(a_t, x_t, \varepsilon_{it})$ but not directly on $\varepsilon_{i, \tau}$.

**Discounting Factor.** Future period’s payoffs are discounted at the rate $\beta_i \in (0, 1)$ for each player.

Every period all players observe their state variables, then they choose their actions simultaneously. We consider a Markovian framework where players’ behavior is stationary across time and players are assumed to play pure strategies. More specifically, for some $\alpha_i: S_i \rightarrow A_i$, $a_{it} = \alpha_i(s_{it})$ for all $i, t$, so that whenever $s_{it} = s_{i\tau}$ then $\alpha_i(s_{it}) = \alpha_i(s_{i\tau})$ for any $\tau$. Next, we introduce three modeling assumptions that are assumed to hold throughout the paper.

**Assumption M1 (Conditional independence).** The transitional distribution of the states has the following factorization: $P(x_{t+1}, \varepsilon_{t+1}|x_t, \varepsilon_t, a_t) = Q(\varepsilon_{t+1}) G(x_{t+1}|x_t, a_t)$, where $Q$ is the cumulative distribution function of $\varepsilon_i$ and $G$ denotes the transition law of $x_{t+1}$ conditioning on $a_t$ and $x_t$.

**Assumption M2 (Independent private values).** The private information is independently distributed across players. I.e. $Q(\varepsilon) = \prod_{i=1}^I Q_i(\varepsilon_i)$ where $Q_i$ denotes the cumulative distribution function of $\varepsilon_{it}$.

**Assumption M3 (Discrete public values).** The support of $x_t$ is finite so that $X = \{x^1, \ldots, x^J\}$ for some $J < \infty$.

Assumptions M1 and M2 generalize Rust’s (1987) conditional independence framework to dynamic games. They are the key restrictions commonly imposed on the class of games in this literature. M1 implies that $\varepsilon_i$ is independent of $x_t$ and all variables in the past, and $\varepsilon_i$ is only correlated to $x_{t+1}$ through the choice variables $a_t$. It is conceptually straightforward to relax the former condition and allow for $\varepsilon_i$ to be conditionally independent of the past given $x_t$, although this is rarely done in practice. M2 rules out games with correlated private values. M3 is a simplifying assumption that has an important practical implication, however it is not necessary for a general estimation methodology; for examples, see Bajari et al. (2009) and Srisuma and Linton (2012).

Under M1 and M2, player $i$’s beliefs, which we denote by $\sigma_i$, is a stationary distribution of $a_t = (\alpha_1(s_{i1}), \ldots, \alpha_I(s_{iI}))$ conditional on $x_t$ for some pure Markov strategies $(\alpha_1, \ldots, \alpha_I)$. Then following Maskin and Tirole (2001), we define the equilibrium concept as follows.
Similar arguments can be applied recursively for any future periods.

2.2 Players’ Decision Problems

In order to characterize the players’ optimal behaviors, we consider the following decision problem faced by player $i$ for a given $\sigma_i$: for all $s_t$,

$$
\max_{a_t \in A_i} \{ E_{\sigma_i}[u_t(a_t, a_{-t}, s_t) | s_{it}] = s_t, a_{it} = a_i \} + \beta V_i(s_{it+1}; \sigma_i | s_{it} = s_t, a_{it} = a_i] \},
$$

(1)

where $V_i(s_t; \sigma_i) = \sum_{t=t}^{\infty} \beta^{t-t} E_{\sigma_i}[u_t(a_t, s_{it+1}) | s_{it} = s_t]$.

The subscript $\sigma_i$ on the expectation operator makes explicit that present and future actions are integrated out with respect to the beliefs $\sigma_i$; in particular, player $i$ forms an expectation for all players’ future actions including herself, and today’s actions of opposing players. $V_i$ is a policy value function since the expected discounted return needs not be an optimal value from an optimization problem since $\sigma_i$ can be any beliefs, not necessarily equilibrium beliefs. Note that the transition law for future states are completely determined by the primitives and the beliefs.\(^3\) Thus, we can interpret each player’s decision problem in (1) as a single stage game against nature determined by Markov beliefs. Clearly, any strategy profile that solves the decision problems for all $i$, and is

\(^3\)First, note that the use of Markovian beliefs imply that $I(s_{t+1}, a_{t+1}) = I(s_{t+1})$ and $I(s_{t+1}, a_{it+1}) = I(s_{it+1})$, where $I(\cdot)$ denotes the information set of $\cdot$. For some random vectors $X$ and $Y$, let $f_{X,Y}$ and $f_{X|Y}$ denote the joint density of $(X,Y)$ and $X$ given $Y$ respectively (components of $X$ and $Y$ can be either continuous, discrete or a mixture). Then, for a one-step-ahead transition, by M1,

$$
f_{s_{t+1}|s_{it},a_{it}} = f_{x_{t+1}|x_{t},a_{it}} \cdot f_{x_{t+1}|x_{t},a_{it}},
$$

where $f_{x_{t+1}}$ and $f_{x_{t+1}|x_{t},a_{it}}$ can be deduced from the model primitives. For two-period-ahead, note that $f_{s_{t+2}|s_{it},a_{it}} = \int f_{s_{t+2},s_{t+1}|s_{it},a_{it}} ds_{t+1}$, using the same line of arguments as above:

$$
f_{s_{t+2},s_{t+1}|s_{it},a_{it}} = f_{s_{t+2}|s_{t+1},a_{it}} \cdot f_{s_{t+1}|s_{it},a_{it}} = \int f_{x_{t+2}|x_{t+1},a_{it}} \cdot f_{x_{t+1}|x_{t},a_{it}}.
$$

Similar arguments can be applied recursively for any future periods.
consistent with the beliefs satisfies conditions in Definition 1 and is an equilibrium strategy. To avoid multiple predictions of best responses, the class of games estimable by our methodology requires (1) to have a unique solution almost surely. In this subsection, we show that Markov equilibrium can be represented by a fixed point of a particular mapping when the solution to the decision problem exists and is unique.

First we simplify the objective function of the decision problem by incorporating our modeling assumptions. It shall be convenient to write \( V_i \) recursively as

\[
V_i(s_t; \sigma_i) = E_{\sigma_i}[u_i(a_t, s_{it}) | s_{it} = s_i] + \beta_i E_{\sigma_i}[V_i(s_{it+1}; \sigma_i) | s_{it} = s_i].
\]

The ex-ante value function can be obtained by taking conditional expectation of \( V_i \) with respect to \( x_t \),

\[
E_{\sigma_i}[V_i(s_{it}; \sigma_i) | x_t] = E_{\sigma_i}[u_i(a_t, s_{it}) | x_t] + \beta_i E_{\sigma_i}[V_i(s_{it+1}; \sigma_i) | x_t].
\]

Under M1, by the law of iterated expectation, \( E_{\sigma_i}[V_i(s_{it+1}; \sigma_i) | x_t] = E_{\sigma_i}[E_{\sigma_i}[V_i(s_{it+1}; \sigma_i) | x_{t+1}] | x_t] \) so that the ex-ante value can be written as a solution to the following linear equation

\[
m_i(\sigma_i) = r_i(\sigma_i) + L_{i, \sigma_i} m_i(\sigma_i),
\]

where \( m_i(\sigma_i) = E_{\sigma_i}[V_i(s_{it+1}; \sigma_i) | x_t = \cdot], r_i(\sigma_i) = E_{\sigma_i}[u_i(a_t, s_{it}) | x_t = \cdot], \) and \( L_{i, \sigma_i} \) is a conditional expectation operator so that \( L_{i, \sigma_i} \phi = \beta_i E_{\sigma_i} [\phi(x_{t+1}) | x_t = \cdot] \) for any \( \phi : X \to \mathbb{R} \). Note that \( m_i(\sigma_i) \) exists and is unique under great generality since \( L_{i, \sigma_i} \) is typically a contraction map.\(^4\) Also, under M1, the choice specific expected future return under beliefs \( \sigma_i \) satisfies \( E_{\sigma_i}[V_i(s_{it+1}; \sigma_i) | s_{it}, a_{it}] = E_{\sigma_i}[E_{\sigma_i}[V_i(s_{it+1}; \sigma_i) | x_{t+1}] | x_t, a_{it}] \), this can be represented by \( g_i(\sigma_i) \) so that

\[
g_i(\sigma_i) = H_{i, \sigma_i} m_i(\sigma_i),
\]

where \( H_{i, \sigma_i} \) is a conditional expectation operator so that \( H_{i, \sigma_i} \phi = E_{\sigma_i}[\phi(x_{t+1}) | x_t = \cdot, a_{it} = \cdot] \) for any \( \phi : X \to \mathbb{R} \). Since, under M1 and M2, \( a_{it} \) and \( \epsilon_{it} \) has no additional information on \( a_{-it} \) given \( x_t \), then the objective function in (1), which we henceforth denote by \( \Lambda_i \), can be written as:

\[
\Lambda_i(a_i, s_i; \sigma) = E_{\sigma_i}[u_i(a_i, a_{-it}, x_t, \epsilon_i) | x_t = x] + \beta_i g_i(a_i, x; \sigma).
\]

\(^4\)Let \( X \) be some compact subset of \( \mathbb{R}^L \) and \( B \) be a space of bounded real-valued functions defined on \( X \). Consider a Banach space \( (B, \| \cdot \|) \) equipped with the sup-norm, i.e. \( \| \phi \| = \sup_{x \in X} |\phi(x)| \) for any \( \phi \in B \). For any \( x \in X \), \( L_{i, \sigma_i} \phi(x) = \beta_i E_{\sigma_i}[\phi(x_{t+1}) | x_t = x] \), then it follows that \( |L_{i, \sigma_i} \phi(x)| \leq \beta_i \sup_{x \in X} |\phi(x)| \). In other words \( \|L_{i, \sigma_i} \phi\| \leq \beta_i \| \phi \| \), hence the operator norm \( \|L_{i, \sigma_i}\| \) is bounded above by \( \beta_i \). Since \( \beta_i \in (0, 1) \), \( L_{i, \sigma_i} \) is a contraction. Therefore the inverse of \( I - L_{i, \sigma_i} \) exists. Furthermore, it is a linear bounded operator and admits a Neumann series representation

\[
\sum_{\tau=0}^{\infty} L_{i, \sigma_i}^\tau \quad (\text{see Kreyszig (1989)}).
\]
The corresponding set of best responses is defined as

\[ BR_i(s_i; \sigma_i) = \{ a_i \in A_i : \Lambda_i(a_i, s_i; \sigma_i) \geq \Lambda_i(a'_i, s_i; \sigma_i) \ \text{for all } a'_i \in A_i \}. \]

A pure strategy best response is a particular selection from the best response that satisfies \( \alpha_i(\cdot; \sigma_i) \in BR_i(\cdot; \sigma_i) \), i.e. for all \( s_i \)

\[ \Lambda_i(\alpha_i(s_i; \sigma_i), s_i; \sigma_i) \geq \Lambda_i(a'_i, s_i; \sigma_i) \ \text{for all } a'_i \in A_i. \] (2)

Since we assume that \( BR_i(s_i; \sigma_i) \) is a singleton for all \( s_i, \sigma_i \), there is no need for a selection from the best response set. Thus, there is a single-valued map \( \Psi_i \) such that

\[ F_i = \Psi_i(\sigma_i), \text{ where } F_i(a_i|x; \sigma_i) = \Pr[a_i(s_{it}; \sigma_i) \leq a_i|x_t = x] \ \text{for all } a_i, x. \] (3)

Under independence (Assumption M2), information on all marginal distribution of actions provide equivalent information for the joint distribution of actions. So that any equilibrium beliefs must satisfy condition (2) and the beliefs are consistent with the actions according to (3), where each \( \sigma_i \) can be represented by \( \prod_{l=1}^I F_l = \prod_{l=1}^I \Psi_l(\sigma_l) \) for all \( i \). We can therefore summarize the necessary condition that the equilibrium beliefs must satisfy by a fixed point of a map \( \Psi \) that takes any vector \( F = (F_1, \ldots, F_I) \) into \( \Psi(F) = (\Psi_1(\prod_{l=1}^I F_l), \ldots, \Psi_I(\prod_{l=1}^I F_l)) \), i.e. the condition:

\[ F = \Psi(F). \] (4)

The fixed point of \( \Psi \) fully characterizes the equilibrium since any \( F \) that satisfies equation (4) can be extended to construct a Markov perfect equilibrium, as \( \alpha_i(s_i; \prod_{l=1}^I F_l) = \arg \max_{a_i \in A_i} \Lambda_i(a_i, s_i; \prod_{l=1}^I F_l) \) is the best response that is consistent with the beliefs by construction.

Equation (4) shall form the basis of our minimum distance estimator, where in Section 3 we look to minimize the distance between the distribution of actions from the data and the implied distribution generated by the empirical version of \( \Psi(F) \). The characterization of an equilibrium as a fixed point to equation (4) is very similar to the approach taken by AM (Representation Lemma) and PSD (Proposition 1), where they consider a particular class of unordered discrete choice game (see Assumption D below).

### 2.3 Games Under Increasing Differences

In many economic applications it is natural to model players’ best responses to be monotone in their private values. The action space can be finite, for example, in investment models where firms

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5Equation (4) can also be useful for proving existence of a Markov perfect equilibrium when \( \Psi \) is known to satisfy regularity conditions to ensure that a fixed point exists, as well as providing an alternative numerical calculation of equilibrium probabilities; see Pesendorfer and Schmidt-Dengler (2008) for further discussions.
purchase or rent goods in discrete units, or the action variable can have a continuous contribution (with or without discrete component), as in the traditional investment and pricing models. The source of the monotonicity can often be derived from an intuitive restriction imposed on the interim payoff differences when player \(i\) chooses action \(a_i\) over \(a'_i\), which we denote by \(\Delta u_i(a_i, a'_i, a_{-i}, x, \varepsilon_i) = u_i(a_i, a_{-i}, x, \varepsilon_i) - u_i(a'_i, a_{-i}, x, \varepsilon_i)\), that increases with \(\varepsilon_i\). Increasing differences have seen numerous applications in economics; see the monograph by Topkis (1998) for examples. We consider games that satisfy the following conditions.

**Assumption S1** (Increasing differences). For any \(a_i > a'_i\) and \(\varepsilon_i > \varepsilon'_i\), \(\Delta u_i(a_i, a'_i, a_{-i}, x, \varepsilon_i) > \Delta u_i(a_i, a'_i, a_{-i}, x, \varepsilon'_i)\) for all \(i, a_{-i}, x\).

**Assumption S2.** \(E_i = [\varepsilon_i, \bar{\varepsilon}_i] \subseteq \mathbb{R}\), and the distribution of \(\varepsilon_{it}\) is absolutely continuous with respect to the Lebesgue measure with a bounded density on \(E_i\) for all \(i\).

Assumptions S1 and S2 are a version of the conditions used in Athey (2001) to study the equilibrium properties in static games. Importantly, increasing differences of \(u_i\) in \((a_i, \varepsilon_i)\) implies the incremental return satisfies single crossing condition in \((a_i, \varepsilon_i)\) (see Definition 1 in Athey). Our increasing differences condition is strict and holds uniformly over \((a_{-i}, x)\), which, although generally not necessary for pure strategy equilibria to exist, will be convenient for modeling games that players’ employ pure strategies almost surely. When \(u_i\) is differentiable in \((a_i, \varepsilon_i)\), the increasing differences condition has a simple characterization: \(\frac{\partial^2}{\partial a_i \partial \varepsilon_i} u_i(a_i, a_{-i}, x, \varepsilon_i) > 0\) for all \(a_{-i}, x\). We also comment that compactness of \(E_i\) is assumed here only for the purpose of establishing existence of equilibria. In an econometric application, \(E_i\) can have full support on \(\mathbb{R}\). Next, we show that existence theorems for equilibria in static games under single crossing condition of Athey (2001) can be applied to our dynamic games.

For the first case, we restrict the support of the action variable to be discrete and impose an integrability condition.

**Assumption S3.** \(A_i\) is finite for all \(i\) and \(\int |u_i(a_i, a_{-i}, x, \varepsilon_i)| dQ_i(\varepsilon_i) < \infty\) for all \(i, a_i, a_{-i}, x\).

Under Assumptions M1, M2, M3, S2 and S3: all expected returns, particularly \(\Lambda_i\), exist, and \(BR_i(s_i; \sigma_i)\) is non-empty by finiteness of \(A_i\) for all \(s_i, \sigma_i\). Let \(\Delta \Lambda_i(a_i, a'_i, s_i; \sigma_i) = \Lambda_i(a_i, s_i; \sigma_i) - \Lambda_i(a'_i, s_i; \sigma_i)\). Then we have the following results.

**Lemma 1** (Increasing differences in expected returns). Under M1, M2, M3, S1, S2 and S3, for any \(a_i > a'_i\) and \(\varepsilon_i > \varepsilon'_i\), \(\Delta \Lambda_i(a_i, a'_i, x, \varepsilon_i; \sigma_i) > \Delta \Lambda_i(a_i, a'_i, x, \varepsilon'_i; \sigma_i)\) for all \(i, x, \sigma_i\).
Proof of Lemma 1. Under M1 and M2 $g_i(\sigma_i)$ does not depend on $\varepsilon_i$. Therefore we have
\[
\Delta \Lambda_i(a_i, a'_{-i}, x, \varepsilon_i; \sigma_i) - \Delta \Lambda_i(a_i, a'_{-i}, x, \varepsilon'_i; \sigma_i) = E_{\sigma_i}[\Delta u_i(a_i, a'_{-i}, x_t, \varepsilon_i) - \Delta u_i(a_i, a'_{-i}, x_t, \varepsilon'_i)|x_t = x] > 0,
\]
where the inequality follows from Assumption S1.

Lemma 2 (Pure strategy best response). Under M1, M2, M3, S1, S2 and S3, $BR_i(s_{it}; \sigma_i)$ is a singleton set almost surely for all $i, \sigma_i$.

Proof of Lemma 2. For any $\sigma_i$, let $\alpha_i(\cdot; \sigma_i)$ and $\alpha'_i(\cdot; \sigma_i)$ denote distinct selections from $BR_i(\cdot; \sigma_i)$ so that for some $x$ there exists $\varepsilon_i > \varepsilon'_i$ such that (without any loss of generality) $\alpha_i(x, \varepsilon'_i; \sigma_i) > \alpha'_i(x, \varepsilon_i; \sigma_i)$. By definition of a best response: $\Delta \Lambda_i(\alpha_i(x, \varepsilon'_i; \sigma_i), \alpha'_i(x, \varepsilon_i; \sigma_i), x, \varepsilon'_i; \sigma_i) \geq 0 \geq \Delta \Lambda_i(\alpha_i(x, \varepsilon'_i; \sigma_i), \alpha'_i(x, \varepsilon_i; \sigma_i), x, \varepsilon_i; \sigma_i)$. However, this contradicts the strict increasing difference condition in the expected returns (Lemma 1).

Notice that finiteness of $\Lambda_i$ does not play any role in proving Lemmas 1 and 2 beyond ensuring $\Lambda_i$ exists and $BR_i$ is non-empty. An implication of Lemma 1 is that every selection from $BR_i(\cdot; \sigma_i)$ is nondecreasing in $\varepsilon_i$ for all $i, x, \sigma_i$ (by Monotone selection theorem of Milgrom and Shannon (1994, Theorem 4)). Together with Lemma 2, they ensure that, for any given beliefs, each player’s best response is a monotone pure strategy almost surely. Existence of an equilibrium then follows immediately from results developed in Athey (2001).

Proposition 1. Assume M1, M2, M3, S1, S2 and S3. Then a pure strategy Markov perfect equilibrium exists where each player’s equilibrium strategy $\alpha_i(x, \varepsilon_i)$ is nondecreasing in $\varepsilon_i$ for all $i, x$.

Proof of Proposition 1. Under S2 and S3, the regularity assumption A1 in Athey is satisfied with $\Lambda_i$ as player’s $i$ objective function. Lemmas 1 and 2 imply that each player’s best response to any Markov beliefs is a monotone pure strategy almost surely. Therefore $\Lambda_i$ satisfies the Single Crossing Condition for games of incomplete information in Definition 3 of Athey. The proof then follows from Theorem 1 in Athey.

Although we consider a dynamic game, by restricting the equilibrium concept to players using stationary Markov beliefs under the conditional independence and private values framework, the arguments used for static games in Athey are directly applicable.\(^6\) Athey also shows that finiteness

\(^6\)The objective function (see first display on p. 865) of the decision problem studied in Athey appears in a slightly different form to ours where, instead of using distribution of actions, she uses the strategy functions of opposing players as beliefs. However, the two approaches are analogous since any conditional distribution, $\sigma_i$, of $a_t$ given $x_t$ uniquely determines monotone strategies $\alpha_i(s_t) = (\alpha_i(s_{it}), \alpha_{-i}(s_{-it}))$ for all $x$ up to null sets on $\varepsilon_t$.\(^6\)
of $A_i$ can be replaced by compactness when the payoff function is continuous in the players’ actions. To apply her result in a dynamic setting, we also need to impose some continuity condition on the transition law of the states. Let $a_i^* = \inf A_i$ and $\bar{a}_i = \sup A_i$, and let $G_i(x_{t+1}|x_t, a_{it})$ denote the transition law of $x_{t+1}$ conditioning on $a_{it}$ and $x_t$.

Assumption S4. For all $i$:

(i) $A_i = [a_i, \bar{a}_i]$;

(ii) $u_i(a_i, a_{-i}, x, \varepsilon_i)$ is continuous in $(a_i, a_{-i}, \varepsilon_i)$ for all $x$;

(iii) $G_i(x'|x, a_i)$ is continuous in $a_i$ for all $x, x'$.

Assumptions M1, M2, M3, S2 and S4 ensure the regularity condition in Athey (A1) is satisfied and $\Lambda_i$ exists, and is continuous in $a_i$, hence $BR_i(s_i; \sigma_i)$ is non-empty for all $s_i, \sigma_i$ since $A_i$ is compact (Weierstrass theorem). Each player’s best response for any given beliefs is also a monotone pure strategy almost surely (by replacing S3 with S4 in Lemmas 1 and 2). Then we have the following proposition.

Proposition 2. Assume M1, M2, M3, S1, S2 and S4. Then a pure strategy Markov perfect equilibrium exists where each player’s equilibrium strategy $\alpha_i(x, \varepsilon_i)$ is nondecreasing in $\varepsilon_i$ for all $i, x$.

Proof of Proposition 2. Under S2 and S4, assumption A1 in Athey is satisfied with $\Lambda_i$ as player’s $i$ objective function. It is easy to see conditions (i) to (iii) in Theorem 2 of Athey are satisfied by our assumptions; in particular, for any finite $A'_1 \times \cdots \times A'_I \subset A_1 \times \cdots \times A_I$, monotone pure (Markov) strategy exists by Proposition 1. The proof then follows from Theorem 2 in Athey (2001).

For modeling purposes, note that strict increasing differences do not imply that $\alpha_i(x, \varepsilon_i)$ is strictly increasing in $\varepsilon_i$. A sufficient condition for strict monotonicity is given by Edlin and Shannon (1998), which in our case requires: (i) $\Lambda_i(a_i, x, \varepsilon_i; \sigma_i)$ is continuously differentiable in $a_i, \varepsilon_i$, and (ii) the best response satisfies the first order condition. Thus an intermediate case exists between purely continuous and discrete action games. For instance when there are corner solutions, then the distribution of the action variable has both continuous and discrete components. Proposition 2 (and Theorem 2 in Athey) accommodates mass points as long as the payoff function remains continuous on the action space. However, the continuity requirement does exclude some interesting games. For example, although continuity in payoffs over opponents’ mass points may be reasonable in Cournot oligopoly games, it rules out Bertrand-type pricing problems. A recent empirical study whose payoff structure satisfies the continuity requirement of Assumption S4 is the dynamic milk quota trading case in Hong and Shum (2009). There, an economic agent can have positive (negative) trade demand (supply),
which is modeled continuously, or she can produce using existing quota (mass point at zero). For further discussions of other games with discontinuities and the existence of their equilibria see Athey (Section 4).

In this subsection we have shown that games under increasing differences have a pure strategy equilibrium under weak primitive modeling conditions. Furthermore, Lemmas 1 and 2 show that players’ decision problems also have unique solutions. The consequence of the Lemmas are particularly important for inference since analogous conditions ensure that the parameterized pseudo-decision problem gives a unique prediction of an optimal behavior almost surely. However, without further restrictions, games under increasing differences may also have multiple equilibria. In this paper, we only consider the estimation problem for games that either have a unique equilibrium, or the observed data have been generated from a single equilibrium.

2.4 Other Dynamic Models

Note that a single agent Markov decision problem is a special case of a game when \( I = 1 \), where the player’s beliefs simplifies to the Markov distribution of her own action given the states. Indeed, a class of popular games which is included in our general framework is built on the discrete decision problem studied in Rust (1987). These discrete games have been extensively studied in this literature, see the surveys of Ackerberg, Benkard, Berry and Pakes (2005) and Aguirregabiria and Mira (2010); they impose the following assumptions to model games with unordered discrete actions.

**Assumption D (Discrete choice game).** For all \( i \):

(i) \( A_i = \{0, \ldots, K_i\} \);

(ii) \( \mathcal{E}_i = \mathbb{R}^{K_i+1} \) so that \( \varepsilon_{it} = (\varepsilon_{it}(0), \ldots, \varepsilon_{it}(K_i)) \);

(iii) the distribution of \( \varepsilon_{it} \) is absolutely continuous with respect to the Lebesgue measure whose density is bounded on \( \mathcal{E}_i \);

(iv) \( u_i(a_i, a_{-i}, x, \varepsilon_i) = \pi_i(a_i, a_{-i}, x) + \sum_{k=0}^{K_i} \varepsilon_i(k) 1[a_i = k] \) for all \( a_{-i}, x \).

Under M1, M2, M3 and D, it is easy to see that event where \( \Lambda_i(a_i, s_{it}; \sigma) = \Lambda_i(a'_i, s_{it}; \sigma) \) has probability zero so that each player’s best response for any given beliefs is a pure strategy almost surely; for further details see AM and PSD, who characterize the equilibrium by the choice probabilities analogous to our equation (4). Specifically, note that a vector of choice probabilities,

\[ 1^3 \] Recently, Mason and Valentinyi (2007) propose some sufficient conditions for a unique equilibrium under increasing differences; specifically, by employing a stronger version of increasing differences, and imposing a Lipschitz condition on the incremental return with respect to other players’ actions.
is just a linear transformation of a vector of conditional distributions,

\[
\begin{pmatrix}
P_i(0|x) \\
\vdots \\
P_i(K|x)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
-1 & \ddots & \ddots & 0 \\
\vdots & \ddots & -1 & 0 \\
0 & \cdots & -1 & 1
\end{pmatrix}
\begin{pmatrix}
P_i(0|x) \\
\vdots \\
P_i(K|x)
\end{pmatrix},
\]  
where the transformation matrix has 1’s on the main diagonal, -1’s on the subdiagonal and 0 everywhere else.

The general model discussed in this section can also be adapted to accommodate games where players have more than one decision variable. This feature is useful for many oligopoly games, for instance where the economic agents endogenously choose whether to participate in the market before deciding on the price or investment decisions. One can model such decision problems where players make sequential choices by combining the primitives from the games with a single action variable discussed previously: for a detailed discussion see Arcidiacono and Miller (2008), BBL and Srisuma (2010).

\section{Estimation Methodology}

In order to consider an estimation problem, we now parameterize \( \{u_i\}_{i=1}^I \) by a finite dimensional parameter \( \theta \in \Theta \subset \mathbb{R}^p \), and update the notation for the payoff functions with \( \{u_i, \theta\}_{i=1}^I \). We take \( \{\beta_i\}_{i=1}^I \) as known. We do not impose any particular distribution on \( G \) as this is nonparametrically identified under weak regularity conditions. To keep the notation as simple as possible, we shall assume the observed data are collected from games played over two periods across \( N \) markets. Specifically, we omit the time subscript and let \( \{(a_n, x_n, x'_n, e_n)\}_{n=1}^N \) denote a random sample generated from a particular equilibrium when \( \theta = \theta_0 \), where \( x'_n \) is the only variable observed from the second period. We state this as an assumption that we maintain for the remainder of the paper.

**Assumption E.** The data are generated by a Markov perfect equilibrium \( (\alpha, \sigma) = (\alpha_1, \ldots, \alpha_I, \sigma_1, \ldots, \sigma_I) \) for some \( \theta = \theta_0 \in \Theta \).

The econometrician only observes \( \{(a_n, x_n, x'_n)\}_{n=1}^N \). The goal is to estimate \( \theta_0 \). Assumption E implies that \( a_{in} = \alpha_i(x_n, e_{in}) \) for all \( i, n \). We shall simply denote the conditional distribution of the equilibrium actions for each player by \( F_i \) and let \( F = (F_1, \ldots, F_I) \), so that \( \sigma_i = \Pi_{i=1}^I F_i \) is the same for all \( i \). For any \( \theta \in \Theta \), we can then define the pseudo-decision problems where players use \( \sigma \) to construct the policy values. When each pseudo-decision problem has unique solution then there is a map, analogous to the previous section, that takes \( \theta \) into \( F_{i, \theta} \), the implied best response distribution.
of actions given $\sigma_i$. By construction, equilibrium condition requires that $F_{i,\theta_0} = F_i$ for all $i$, which is the condition that motivates our minimum distance estimator. Therefore our estimation strategy requires the construction of the distribution of best response mapping analogous to that found in Section 2.2. Section 3.1 gives the outline of our minimum distance estimator.

We provide details regarding practical implementation in Section 3.2. The section ends with a brief discussion. In what follows, since we only consider the policy value functions and associated pseudo-decision problems generated from $\sigma$, henceforth we suppress the dependence on the beliefs.

### 3.1 Minimum Distance Approach

To formally define $F_{i,\theta}$, we need to construct the pseudo-decision problem. As in Section 2.2, we begin by incorporating Assumptions M1 - M3 to simplify the nature of future expected returns under $\sigma$. The (policy) value function, here written recursively, for any $i$, is given by

$$V_{i,\theta}(s_{in}) = E[u_{i,\theta}(a_{in}, s_{in}) | s_{in}] + \beta_i E[V_{i,\theta} (s'_{in}) | s_{in}].$$

Under M1 and M3, by the law of iterated expectation, the ex-ante value, $E[V_{i,\theta} (s_{in}) | x_n]$, can be written as the solution a matrix equation:

$$m_{i,\theta} = r_{i,\theta} + L_i m_{i,\theta},$$

where $m_{i,\theta}$ and $r_{i,\theta}$ are $J$-dimensional vectors whose $j$-th entries are $m_{i,\theta}(x^j) = E[V_{i,\theta} (s_{in}) | x_n = x^j]$ and $r_{i,\theta}(x^j) = E[u_{i,\theta}(a_{in}, s_{in}) | x_n = x^j]$ respectively, and $L_i$ is a $J$ by $J$ matrix whose $(j,k)$-th entry is $\beta_i \times \Pr [x'_{n} = x^k | x_n = x^j]$. Since $L_i$ is the product between $\beta_i$ and a stochastic matrix, $I - L_i$ is invertible, ensuring the existence and uniqueness of $m_{i,\theta}$ for all $(i, \theta)$.

Under M1, by the law of iterated expectation, the choice specific expected future return, $E[V_{i,\theta} (s'_{in}) | x_n, a_{in}]$, is a linear transform of the ex-ante value,

$$g_{i,\theta} = H_i m_{i,\theta},$$

where, for all $(a_i, x)$, $g_{i,\theta}(a_i, x) = E[V_{i,\theta} (s'_{in}) | x_n = x, a_{in} = a_i]$, and $H_i \phi(a_i, x) = \sum_{x' \in X} \phi(x') G_i(x'|x, a_i)$ for any $\phi : X \to \mathbb{R}$ where $G_i$ is the transition law of $x'_n$ conditioning on $(x_n, a_{in})$. Then, under M1 and M2, the parameterized objective function for the pseudo-decision problem is given by

$$\Lambda_{i,\theta}(a_i, x, \varepsilon) = E[u_{i,\theta}(a_i, a_{-in}, x_n, \varepsilon) | x_n = x] + \beta_i g_{i,\theta}(a_i, x).$$

For $u_{i,\theta}$ that satisfies the modeling assumptions analogous to those in Sections 2.3 and 2.4, $\Lambda_{i,\theta}(\cdot, x_n, \varepsilon_{in})$ has unique maximizer on $A_i$ almost surely. We denote its corresponding best response function by

---

8This is a special case of footnote 2. Existence of $(I - L_i)^{-1}$ can also be seen to follow directly from the dominant diagonal theorem since the sum of the (nonnegative) elements in each row of $L_i$ is $\beta_i < 1$ (Taussky (1949)).
Then, the pseudo-model implied distribution function can be written as an outcome of the following map (cf. equation (3)):

\[ F_{i,\theta} = \Psi_{i,\theta}(\prod_{l=1}^{I} F_{l}), \quad \text{where} \quad F_{i,\theta}(a_{i}|x) = \int \mathbf{1}[\alpha_{i,\theta}(x,\varepsilon_{i}) \leq a_{i}] \, dQ_{i}(\varepsilon_{i}) \quad \text{for all} \quad (a_{i}, x). \tag{10} \]

By construction, the equilibrium condition implies that \( F_{i,\theta} = F_{i} \) when \( \theta = \theta_{0} \). We shall consider the limiting objective function that measures an \( L^{2} \)-distance between \( F_{i,\theta}(\cdot|x) \) and \( F_{i}(\cdot|x) \) over the support of \( A_{i} \) for all \( i \) and \( x \):

\[ M(\theta) = \sum_{i \in I} \sum_{x \in X} \int_{A_{i}} (F_{i,\theta}(a_{i}|x) - F_{i}(a_{i}|x))^{2} \mu_{i,x}(da_{i}), \]

for some measures \( \left\{ \mu_{i,x} \right\}_{i=1,x=X} \). The issues of identification and the choice of measures are discussed in Section 4. For now, we suppose \( M(\theta) \) has a unique minimum at zero when \( \theta = \theta_{0} \).

### 3.2 Implementation

In practice \( \Psi_{i,\theta} \) and \( F \) are infeasible, we replace them by their empirical counterparts. Our estimator minimizes the sample analog of \( M(\theta) \). The estimation procedure therefore proceeds in two stages. The first stage estimates the pseudo-model implied distributions. The second stage chooses \( \theta \) to minimize their distance with the distribution of actions from the data. For the convenience of the reader, we tabulate various elements and their possible estimators from equations (8) and (10) that

\[ \text{For the discrete action games considered in Section 3.4 (under Assumption D), there is no need to solve the pseudo-decision problem at all since the choice probabilities (hence distribution functions) have a one-to-one relationship with the normalized expected returns (Hotz and Miller (1993)). In particular, when the unobserved states vectors are also i.i.d. extreme values then } F_{i,\theta}(a_{i}|x) - F_{i,\theta}(a_{i} - 1|x) = \int \mathbf{1}[\alpha_{i,\theta}(x,\varepsilon_{i}) = a_{i}] \, dQ_{i}(\varepsilon_{i}) \text{ has a closed-form in the expected returns (for instance, see AM).} \]
are used to define $F_{i,\theta}$ in Table A.

| Variable | Definition | Possible estimator |
|----------|------------|--------------------|
| $p_X(x)$ | $Pr[x_n = x]$ | $\hat{p}_X(x) = \frac{1}{N} \sum_{n=1}^{N} 1[x_n = x]$ |
| $p_{X',X}(x', x)$ | $Pr[x'_n = x', x_n = x]$ | $\hat{p}_{X',X}(x', x) = \frac{1}{N} \sum_{n=1}^{N} 1[x'_n = x, x_n = x]$ |
| $G_i(x'|a_i)$ | $Pr[x'_n = x'|x_n = x, a_{in} = a_i]$ | $\hat{G}_i$ depends on $a_{in}$ |
| $F_i(a_{in}|x)$ | $Pr[a_{in} \leq a_i|x_n = x]$ | $\hat{F}_i(a_{in}|x) = \frac{1}{N} \sum_{n=1}^{N} 1[a_{in} \leq a_i, x_n = x]/\hat{p}_X(x)$ |

The elements from the linear equations can be found in (6) and (7). We shall also let $E_n[\psi(w_n)|x_n = x]$ denote an empirical version of $E[\psi(w_n)|x_n = x]$ for any function $\psi$ of $w_n$, which can be any vectors from the sample. In particular, since $x_n$ is a discrete random variable, a possible candidate of $E_n[\psi(w_n)|x_n = x]$ is simply $\frac{1}{N} \sum_{n=1}^{N} \psi(w_n) 1[x_n = x]/\hat{p}_X(x)$.

**First Stage Distribution of Best Responses**

A feasible estimator for $F_{i,\theta}$ can be obtained by estimating $\Lambda_{i,\theta}$ and simulating $\varepsilon_{in}$ as follows:

**Step 1.** Estimate the elements of $\Lambda_{i,\theta}$. From (8), let

$$\hat{\Lambda}_{i,\theta}(a_i, x, \varepsilon_i) = E_n[u_{i,\theta}(a_{in}, a_{-in}, x_n, \varepsilon_{in})|x_n = x] + \beta_i \hat{g}_{i,\theta}(a_i, x) \text{ for all }(a_i, x, \varepsilon_i).$$

Using equations (6) and (7), $g_{i,\theta}$ satisfies

$$g_{i,\theta} = \mathcal{H}_i(I - \mathcal{L}_i)^{-1} r_{i,\theta}. \quad (11)$$

Therefore $\hat{g}_{i,\theta}$ can be obtained from $(\hat{r}_{i,\theta}, \hat{\mathcal{L}_i}, \hat{\mathcal{H}_i})$, estimators for $(r_{i,\theta}, \mathcal{L}_i, \mathcal{H}_i)$, which we now consider.

**Estimation of $r_{i,\theta}$**

The estimation of $r_{i,\theta}$ is complicated by the fact that we do not observe $\{\varepsilon_{in}\}_{n=1}^{N}$. Estimable games in this literature impose modeling assumptions that allow $r_{i,\theta}$ to be nonparametrically identified for all $\theta$. For examples, unordered discrete action games (under Assumption D) make use of the well-known Hotz and Miller’s (1993) inversion theorem to identify and estimate $r_{i,\theta}$, and for games with
monotone actions, the identification and estimation of \( r_{i,\theta} \) rely on the quantile invariance between \( a_{in} \) and \( \varepsilon_{in} \). To illustrate, we consider the purely continuous and discrete action cases under monotonicity.

**Example 1.** Suppose \( \alpha_i(x, \varepsilon_i) \) is strictly increasing in \( \varepsilon_i \) almost everywhere on \( E_i \) for all \( i, x \). Then the inverse of \( \alpha_i \) exists and we denote it by \( \rho_i \), which is defined by the relation: \( \rho_i (\alpha_i(x, \varepsilon_i), x) = \varepsilon_i \) for all \( i, x, \varepsilon_i \). It follows that \( F_i(a_i|x) = Q_i(\rho_i(a_i, x)) \). Thus \( \varepsilon_{in} = Q^{-1}_i(F_i(a_{in}|x_n)) \) and we can generate the private value \( \widehat{\varepsilon}_{in} \) by \( Q^{-1}_i(\widehat{F}_i(a_{in}|x_n)) \). Then, one candidate for \( \widehat{r}_{i,\theta}(x) \) is \( E_n[u_{i,\theta} (a_n, x_n, \widehat{\varepsilon}_{in}) | x_n = x] \).

**Example 2.** Suppose \( \alpha_i(x, \varepsilon_i) \) is weakly increasing in \( \varepsilon_i \) almost everywhere on \( E_i \) for all \( i, x \). Let \( \{a^k_i\}_{k=1}^{K_i} \) be an increasing sequence of possible actions for some \( K_i < \infty \). Although the inverse of \( \alpha_i \) does not exist, by monotonicity, we have \( E_i = \bigcup_{k=1}^{K_i} C_k(x) \), where \( C_k(x) = [Q_i^{-1}(F_i(a^k_i|x)), Q_i^{-1}(F_i(a^{k-1}_i|x))] \) for \( k > 1 \). Therefore the cut-off values where the optimal action jumps to higher actions are identified. In particular,

\[
r_{i,\theta}(x) = \sum_{k=1}^{K_i} \Pr[a_{in} = a^k_i|x_n = x] \int_{C_k(x)} E[u_{i,\theta}(a^k_i, a_{-in}, x_n, \varepsilon_i) | x_n = x] dQ_i(\varepsilon_i).
\]

Then, for instance, we can estimate \( r_{i,\theta}(x) \) by replacing \( \Pr[a_{in} = a^k_i|x_n = x] \) with \( \frac{1}{N} \sum_{n=1}^{N} 1[a_{in} = a^k_i, x_n = x] / \hat{p}_X(x) \) and \( \int_{C_k(x)} E[u_{i,\theta}(a^k_i, a_{-in}, x_n, \varepsilon_i) | x_n = x] dQ_i(\varepsilon_i) \) by \( \int_{C_k(x)} E_n[u_{i,\theta}(a^k_i, a_{-in}, x_n, \varepsilon_i) | x_n = x] dQ_i(\varepsilon_i) \) with \( \hat{C}_k(x) = [Q_i^{-1}(\hat{F}_i(a^{k-1}_i|x)), Q_i^{-1}(\hat{F}_i(a^k_i|x))] \).

The mixed-continuous case can also be straightforwardly dealt with, using a combination of the two techniques above, since we can write

\[
r_{i,\theta}(x) = \Pr[a_{in} \in A_{i}^C | x_n = x] E[u_{i,\theta}(a_{in}, a_{-in}, x_n, \varepsilon_i) | x_n = x, a_{in} \in A_{i}^C] + \Pr[a_{in} \in A_{i}^D | x_n = x] E[u_{i,\theta}(a_{in}, a_{-in}, x_n, \varepsilon_i) | x_n = x, a_{in} \in A_{i}^D],
\]

where \( A_{i}^D \) denotes the support of \( A_{i} \) that \( a_{in} \) has positive mass points and \( A_{i}^C \) is the complement set of \( A_{i}^D \) with respect to \( A_{i} \).

**Estimation of \( L_i \)**

\( L_i \) can be represented by a \( J \) by \( J \) matrix of conditional probabilities. A simple estimator for \( L_i \) is the frequency estimator whose \((j, k)\) - th element satisfies:

\[
\hat{L}_i(j, k) = \begin{cases} 
\beta_i \hat{p}_{X', X}(x^k, x^j) / \hat{p}_{X}(x^j) & \text{if } \hat{p}_{X}(x^j) > 0 \\
0 & \text{otherwise}
\end{cases}
\]

An appealing feature of the frequency estimator is that \((I - \hat{L}_i)^{-1}\) necessarily exists as discussed previously.

**Estimation of \( H_i \)**

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$\mathcal{H}_i$ is a conditional expectation operator defined by $G_i$, the transition law of $x'_n$ conditioning on $a_{in}$ and $x_n$. The nature of the nonparametric estimator of $G_i$ depends whether $a_{in}$ is continuous or discrete, or mixed. For an estimator $\hat{G}_i$ of $G_i$, $\hat{H}_i$ is defined as $\hat{H}_i(\phi(a_i, x)) = \sum_{x' \in X} \phi(x') \hat{G}_i(x'| x, a_i)$ for any $a_i, x$ and any function $\phi : X \to \mathbb{R}$.

**Example 1 (continued).** There are many nonparametric estimators that can be used to estimate a conditional expectation. One candidate is a Nadaraya-Watson type estimator, where

$$\hat{G}_i(x'| x, a_i) = \frac{1}{N} \sum_{n=1}^{N} 1[x'_n = x', x_n = x]K_h(a_{in} - a_i) / \frac{1}{N} \sum_{n=1}^{N} 1[x_n = x]K_h(a_{in} - a_i)$$

and $K_h(\cdot)$ is a user-chosen kernel and $h$ is the bandwidth.

**Example 2 (continued).** Since all variables are discrete, we can simply use the frequency estimator $\hat{G}_i(x'| x, a_i) = \sum_{n=1}^{N} 1[x'_n = x', x_n = x, a_{in} = a_i] / \sum_{n=1}^{N} 1[x_n = x, a_{in} = a_i]$ whenever $\sum_{n=1}^{N} 1[x_n = x, a_{in} = a_i] > 0$ and define $\hat{G}_i(x'| x, a_i)$ to be zero otherwise.

For the mixed-continuous case, a candidate for $\hat{G}_i(x'| x, a_i)$ can be constructed in the same way as one of the two examples above, depending on whether $a_i$ lies in the support of $A_i$ that has positive mass or not.

**Estimation of $\hat{g}_{i, \theta}$**

This is simply the sample analog of equation (11), i.e. $\hat{g}_{i, \theta} = \hat{H}_i(I - \hat{L}_i)^{-1} \hat{r}_{i, \theta}$, which can be obtained following equations (6) and (7). First, for any $\hat{r}_{i, \theta}$, $\hat{m}_{i, \theta}$ can be estimated by a matrix multiplication: $\hat{m}_{i, \theta} = (I - \hat{L}_i)^{-1} \hat{r}_{i, \theta}$. Then, for any $a_i, x$, $\hat{g}_{i, \theta}(a_i, x) = \sum_{x' \in X} \hat{m}_{i, \theta}(x') \hat{G}_i(x'| x, a_i)$.

Note that $\hat{L}_i$ and $\hat{H}_i$ do not depend on $\theta$.

**Step 2.** Estimate $F_{i, \theta}$. Having obtained the pseudo-objective function $\hat{\Lambda}_{i, \theta}$, the implied best response and distributions are

$$\hat{\alpha}_{i, \theta}(x, \varepsilon_i) = \arg \max_{a_i \in A_i} \{\hat{\Lambda}_{i, \theta}(a_i, x, \varepsilon_i)\}, \text{ and}$$

$$\hat{F}_{i, \theta}(a_i | x) = \int 1[\hat{\alpha}_{i, \theta}(x, \varepsilon_i) \leq a_i] \ dQ_{i}(\varepsilon_i)$$

respectively. As shown in Section 2, the issue of existence and uniqueness of solutions to $\hat{\Lambda}_{i, \theta}(a_i, x, \varepsilon_i)$ depends crucially on the modeling of $u_{i, \theta}$. It is easy to see that we also have existence and uniqueness in finite sample when conditions in Sections 2.3 and 2.4 to hold for $u_i = u_{i, \theta}$ for all $\theta$ with the examples given above.

Note that $\hat{F}_{i, \theta}(a_i | x)$ is a random distribution function of $\hat{\alpha}_{i, \theta}(s_{in})$ conditioning on the event that $x_n = x$. In particular, $\hat{F}_{i, \theta}$ is generally different to $\hat{F}_i$ even when $\theta = \theta_0$ since the randomness of the former comes from the construction of the pseudo-model whilst the latter is driven purely by
the data. Although we know the distribution of $\varepsilon_{in}$, $\tilde{F}_{i,\theta}$ generally does not have a closed-form and is generally infeasible; special cases do exist for unordered discrete action games, see AM and PSD. We denote a feasible estimator for $F_{i,\theta}$ by $\tilde{F}_{i,\theta}$, which can be obtained by simulation. For instance, in our numerical studies, we use

$$\tilde{F}_{i,\theta}(a_i|x) = \frac{1}{R} \sum_{r=1}^{R} \mathbf{1} \left[ \tilde{\alpha}_{i,\theta}(x, \varepsilon_{r}^i) \leq a_i \right],$$

where $\{\varepsilon_{r}^i\}_{r=1}^{R}$ denote a random sample drawn from the known distribution of $\varepsilon_{in}$.

## Second Stage Optimization

Given the estimators $(\tilde{F}_{i,\theta}, \tilde{F}_{i})$ for $(F_{i,\theta}, F_{i})$, a class of $L^2$–distance functions can be constructed from (potentially random) measures $\{\mu_{i,x}\}_{i \in I, x \in X}$ defined on the support of $A_i$:

$$\tilde{M}_N(\theta) = \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( \tilde{F}_{i,\theta}(a_i|x) - \tilde{F}_{i}(a_i|x) \right)^2 \mu_{i,x}(da_i).$$

When $A_i$ is finite it is natural to choose each $\mu_{i,x}$ to be a count measure, where $\tilde{M}_N$ can then be written as $\sum_{i \in I} \sum_{x \in X} \sum_{a_i \in A_i} (\tilde{F}_{i,\theta}(a_i|x) - \tilde{F}_{i}(a_i|x))^2 \mu_{i,x}(\{a_i\})$. Our minimum distance estimator minimizes $\tilde{M}_N(\theta)$. The statistical properties of the estimator depend on the choice of $\{\mu_{i,x}\}_{i \in I, x \in X}$, we discuss the selection of these measures in Section 4.

**A Remark on Semiparametric Estimation.** Our methodology naturally generalizes to the case when $x_n$ is a continuous random variable (or vector), where equation (6) becomes a linear integral equation of type II that has a well-posed solution (see Srisuma and Linton (2012)). In this case, regardless whether $a_{in}$ is continuous or discrete, the estimation problem is a semiparametric one since $\mathcal{L}_{i}$ becomes an operator on an infinite dimensional space. Under Assumption M3, if $a_{in}$ has a continuous component then estimating $\mathcal{H}_i$ also leads to a semiparametric problem. However, the dimensionality of an infinite dimensional parameter is always one since each player forms an expectation based only on her action alone in the pseudo-decision problem. This is in contrast to the forward simulation method of BBL, where estimating value functions requires future states to be sequentially drawn from the estimator of $G$ (not $G_i$) which is a distribution conditioning on actions of all players. In our case, the nonparametric dimensionality problem is determined by the total number of continuous variables present in $a_{in}$ and $x_n$.

### 3.3 A Discussion

Having gone through our two-step procedure in detail, we can now put in perspective its practical advantages in relation to its full solution counterpart. In particular, an analogous estimator can
be defined by a two stage procedure similar to the one described above, where Step 1, in the first stage, now requires the equilibrium beliefs to be computed for each $\theta$. Even if we have unlimited computational resources, multiple equilibria give rise to multiple beliefs leading to more than one model implied distributions of actions. Without the indeterminacy issue, solving for the equilibrium numerically is also non-trivial, it typically involves fixed point iterations of some non-linear functional equation, for example see Pakes and McGuire (1994). The additional numerical cost required from solving for the equilibrium of dynamic games repeatedly is generally considered infeasible.

We use the insight from Hotz and Miller (1993) and its extension to dynamic games (AM and PSD), where we only consider the beliefs observed from the data that leads to the pseudo-model. As described in the previous section, there are no multiplicity issues associated with the pseudo-decision problem for the two main classes of games where players’ actions are modeled to be monotone in the unobserved states or to be unordered discrete. Given the beliefs, the implied value functions and objective functions for the pseudo-decision problem are also easy to compute for each $\theta$. Particularly, in Step 1, of the first stage, all the elements we require to estimate the continuation value function, $g_{i,\theta}$, either have explicit functional forms or are nonparametrically identified, hence they are easy to program (for instance see Table A).

We also comment on the prospect of solving equation (6), which we can think of as inverting the estimate of the matrix $I - L_i$. Although not all estimators of $L_i$ lead to a non-singular estimator of $I - L_i$, a simple frequency estimator does. Importantly, since we estimate $L_i$ nonparametrically, suppose $I - \hat{L}_i$ is invertible, this inversion only has to be done once. In addition, similar to Hotz et al. (1994) and BBL, we can also take advantage of the linear structure of the (policy) value equation. Specifically when the parameterization of $\theta$ in $u_{i,\theta}$ is linear, so that $u_{i,\theta} = \theta^T u_{i,0}$ for some $p$-dimensional vector $u_{i,0}$, then $r_{i,\theta}$ can be written as $\theta^T r_{i,0}$ where $r_{i,0}$ is a $p$-dimensional vector such that the $r_{i,0}(x) = E[u_{i,0}(a_n, s_{in}) | x_n = x]$ for all $x$. In a matrix notation $r_{i,\theta} = R_i \theta$, where $R_i$ is a $J \times p$ matrix whose $j$-th row is $r_{i,0}(x^j)$. Then $m_{i,\theta}$ equals $(I - L_i)^{-1} R_i \theta$. And, for the choice specific expected future return, $g_{i,\theta}$, in equation (11) simplifies to $H_i (I - L_i)^{-1} R_i \theta$, where $H_i (I - L_i)^{-1} R_i \theta$ does not depend on $\theta$.

In practice, the researcher has the freedom to choose any estimators for $r_{i,\theta}, L_i$ and $H_i$. Therefore it is also straightforward to carry out our methodology in a fully parametric framework by parameterize $L_i$ and $H_i$. Particularly, under the Markovian framework, $L_i$ and $H_i$ can be estimated independently of the dynamic parameters; they can then be used in to transform the estimator of $r_{i,\theta}$ as discussed in Step 1, and all of the above subsequent steps remain valid.
4 Inference

Before we proceed to the asymptotic theorems, it is important to first consider whether minimum distance approach suggested in the previous section provides a sensible method to uncover \( \theta_0 \) from the data. Particularly, similar to other two-step estimators in the literature, the extent of what we can learn about \( \theta_0 \) is restricted to the pseudo-best response functions \( \{\alpha_{i,\theta}\}_{\theta \in \Theta} \) defined in (9). Therefore it is appropriate to speak of identification in terms of the pseudo-model generated by the data.

**Definition 2.** The set \( \Theta_0 = \{\alpha_{i,\theta}(x, \varepsilon_{in}) = \alpha_i(x, \varepsilon_{in}) \text{ a.s. for all } (i, x)\} \) is called the identified set.

**Definition 3.** \( \theta_0 \) is said to be identified if \( \Theta_0 \) is a singleton set.

In Section 4.1 we show that, for the class of games discussed previously, \( \{F_{i,\theta}\}_{\theta \in \Theta} \) contains the same identifying information on the identified set in the sense that the following two conditions are equivalent:

\[
\alpha_{i,\theta}(x, \varepsilon_{in}) = \alpha_i(x, \varepsilon_{in}) \text{ a.s. for all } (i, x) \iff \theta \in \Theta_0, \tag{14}
\]

\[
F_{i,\theta}(a_{in}|x) = F_i(a_{in}|x) \text{ a.s. for all } (i, x) \iff \theta \in \Theta_0. \tag{15}
\]

Section 4.2 then takes the identified set to be a singleton, and provides conditions for our minimum distance estimator to be consistent and asymptotically normal.

4.1 Equivalence of Identification Conditions

We consider the parameterized versions of games discussed in Section 2.3. Specifically let Assumptions S1’, S3’ and S4’ be identical to Assumptions S1, S3 and S4 everywhere except that \( u_i \) is replaced by \( u_{i,\theta} \) and all conditions imposed on the former are assumed to hold for the latter for all \( \theta \). In what follows, we denote the probability measure of \( \varepsilon_{in} \) by \( Q_i \). We begin with games that have finite actions.

**Proposition 3.** Assume M1, M2, M3, S1’, S2 and S3’. Then Conditions (14) and (15) are equivalent.

**Proof of Proposition 3.** Suppose for each \( i, A_i = \{a_i^1, \ldots, a_i^K\} \), then Condition (15) only has to be checked on \( A_i \).

Suppose (14) holds. The implication is immediate for \( \theta \in \Theta_0 \). Let \( D_{i,x,\theta} = \{\alpha_{i,\theta}(x, \varepsilon_{in}) \neq \alpha_i(x, \varepsilon_{in})\} \). When \( \theta \notin \Theta_0 \) there exists some \( i, x \), such that \( Q_i(D_{i,x,\theta}) > 0 \). Let \( D_{i,x,\theta}(k) \) denote
$D_{i,x,\theta} \cap \{\alpha_i(x, \varepsilon_{in}) = a_i^k\}$, and let $k^* = \min\{k : Q_i(D_{i,x,\theta}(k)) > 0\}$. By Assumption S2 and the monotonicity of $\alpha_{i,\theta}(x, \cdot)$ and $\alpha_i(x, \cdot)$: $F_{i,\theta}(a_i|x) = F_i(a_i|x)$ for all $a_i < a_i^{k*}$ and $Q_i(\{\alpha_{i,\theta}(x, \varepsilon_{in}) = a_i^{k*}\}) \neq Q_i(\{\alpha_i(x, \varepsilon_{in}) = a_i^{k*}\})$, therefore $F_{i,\theta}(a_i^{k*}|x) \neq F_i(a_i^{k*}|x)$.

Suppose (15) holds. If $\theta \in \Theta_0$ then $Q_i(\{\alpha_{i,\theta}(x, \varepsilon_{in}) = a_i^k\}) = Q_i(\{\alpha_i(x, \varepsilon_{in}) = a_i^k\})$ for all $k$, hence it follows from Assumption S2 and the monotonicity of $\alpha_{i,\theta}(x, \cdot)$ and $\alpha_i(x, \cdot)$ that $Q_i(D_{i,x,\theta}) = 0$ for all $i, x$. If $\theta \notin \Theta_0$, let $k^* = \min\{k : F_{i,\theta}(a_i^k|x) - F_{i,\theta}(a_i^{k-1}|x) = F_i(a_i^k|x) - F_i(a_i^{k-1}|x)\}$ where we define $F_{i,\theta}(a_i^0|x) = F_i(a_i^0|x) = 0$. By Assumption S2 and the monotonicity of $\alpha_{i,\theta}(x, \cdot)$ and $\alpha_i(x, \cdot)$, it follows that $\{\alpha_{i,\theta}(x, \varepsilon_{in}) \leq a_i\}$ and $\{\alpha_i(x, \varepsilon_{in}) \leq a_i\}$ may differ only on a $Q_i$–null set for $a_i < a_i^{k*}$ therefore $Q_i(\{\alpha_{i,\theta}(x, \varepsilon_{in}) = a_i^{k*}\}) = Q_i(\{\alpha_i(x, \varepsilon_{in}) = a_i^{k*}\}) > 0$.

An equivalence result is also available when the distribution of $a_{in}$ is continuous, i.e. the best response is strictly monotone in $\varepsilon_i$.

**Proposition 4.** Assume M1, M2, M3, S1', S2, S4' and for all $i, x, \theta$, $\alpha_{i,\theta}(x, \varepsilon_i)$ is strictly increasing in $\varepsilon_i$. Then Conditions (14) and (15) are equivalent.

**Proof of Proposition 4.** The inverse of $\alpha_{i,\theta}(x, \cdot)$ exists and is unique for all $i, x, \theta$ by strict monotonicity. We denote the inverse by $\rho_{i,\theta}(\cdot, x)$, so that $\rho_{i,\theta}(\alpha_{i,\theta}(x, \varepsilon_i), x) = \varepsilon_i$ for all $i, \theta, x, \varepsilon_i$. Then for any $a_i, x$,

$$F_{i,\theta}(a_i|x) = \Pr[\alpha_{i,\theta}(x, \varepsilon_{in}) \leq a_i|x_n = x] = \Pr[\varepsilon_{in} \leq \rho_{i,\theta}(a_i, x)|x_n = x] = Q_i(\rho_{i,\theta}(a_i, x)).$$

Since $Q_i$ is a bijection map, as it is strictly increasing (Assumption S2), the one-to-one correspondence between $\alpha_{i,\theta}$ and $\rho_{i,\theta}$ for all $\theta$ completes the equivalence claim.

We have an analogous result when the distribution of $a_{in}$ has finite mass points as well as a continuous contribution. For notational simplicity we consider games where each action variable has a single mass point at the lower boundary of the support.

**Proposition 5.** Assume M1, M2, M3, S1', S2, S4' and for all $i, x, \theta$, there exists $\varepsilon_{i,x,\theta} \in \mathcal{E}_i$ such that $\alpha_{i,\theta}(x, \varepsilon_i) = a_i$ for all $\varepsilon_i \leq \varepsilon_{i,x,\theta}$ and $\alpha_{i,\theta}(x, \varepsilon_i)$ is strictly increasing in $\varepsilon_i$ for $\varepsilon_i > \varepsilon_{i,x,\theta}$, furthermore $\varepsilon_{i,x,\theta} = \varepsilon_{i,x} > \varepsilon_i$ whenever $\theta \in \Theta_0$. Then Conditions (14) and (15) are equivalent.

**Proof of Proposition 5.** We only consider $\theta$ such that $\varepsilon_{i,x,\theta} > \varepsilon_i$. As seen previously, we shall repeatedly make use of Assumption S2 and the monotonicity of $\alpha_{i,\theta}(x, \cdot)$ and $\alpha_i(x, \cdot)$.

\[^{10}\text{For any sets } A, B, A \Delta B = (A \cup B) \setminus (A \cap B) \text{ denotes the symmetric difference between } A \text{ and } B.\]
Suppose (14) holds. The implication is immediate for \( \theta \in \Theta_0 \). If \( \theta \notin \Theta_0 \) then for some \( i, x \), either (i) \( \xi_{i,x,\theta}^0 \neq \xi_{i,x} \) so that \( \alpha_{i,\theta}(x, \varepsilon_i) \) and \( \alpha_i(x, \varepsilon_i) \) do not agree when \( \varepsilon_i \in (\min\{\xi_{i,x,\theta}^0, \xi_{i,x}\}, \max\{\xi_{i,x,\theta}^0, \xi_{i,x}\}) \), in which case \( F_{i,\theta}(a_i|x) \neq F_i(a_i|x) \), otherwise (ii) \( \xi_{i,x,\theta}^0 = \xi_{i,x} \) then strict monotonicity implies \( \alpha_{i,\theta}(x, \cdot) \) and \( \alpha_i(x, \cdot) \) must have different inverses, hence a different implied distribution functions.

Suppose (15) holds. The implication is now obvious for \( \theta \in \Theta_0 \). If \( \theta \notin \Theta_0 \), then either (i) \( F_{i,\theta}(a_i|x) \neq F_i(a_i|x) \) in which case \( Q_i(\{\alpha_{i,\theta}(x, \varepsilon_i) = a_i\}) \neq Q_i(\{\alpha_i(x, \varepsilon_i) = a_i\}) \), otherwise (ii) the one-to-one correspondence between the best response and their implied distribution functions implies that \( \{\alpha_{i,\theta}(x, \varepsilon_i) \neq \alpha_i(x, \varepsilon_i)\} \) has a positive measure.

When \( \theta_0 \) is identified, equivalence between Conditions (14) and (15) means that minimum distance criterion function can be constructed so that it has a unique minimum only at \( \theta_0 \). For instance, it is sufficient that for all \( i, x \), any \( E \subset A_i \) that has positive probability measure with respect to the distribution of \( a_{in} \) also has a positive measure on \( \mu_{i,x} \). The equivalence of information content on the identified set between the pseudo-best response function and implied distribution is not restricted to games with monotone strategies. Conditions (14) and (15) are also equivalent for the discrete choice games studied in AM and PSD. Since (15) can be stated in terms of the choice probabilities (see (5)), the equivalence condition follows from the one-to-one relationship between the choice probabilities and optimal decision rule using Hotz and Miller (1993)'s well-known inversion result (see also Lemma 1 of Pesendorfer and Schmidt-Dengler (2003)).

### 4.2 Asymptotic Theorems

We state the regularity conditions for our Theorems in terms of the distribution functions and their estimators. These conditions are more informative than the usual high level conditions as they allow us to highlight key features of the minimum distance estimator. At the same time, they are also flexible enough to cover all the games considered in this paper, and admit any estimators for \( F_{i,\theta} \) and \( F_i \) as long as the conditions below are satisfied. Indeed, our Theorems 1 and 2 are also applicable to any estimation problem based on minimizing the distance of conditional distribution functions outside the context of dynamic games.

Specific to our application, for some estimators \( (\widehat{F}_{i,\theta}, \widehat{F}_i) \) of \( (F_{i,\theta}, F_i) \), recall that the objective function is

\[
\widehat{M}_N(\theta) = \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( \frac{1}{\mu_{i,x}} \right)^2 \mu_{i,x}(da_i),
\]

where \( \widehat{F}_{i,\theta} \) is a feasible estimator for \( F_{i,\theta} \). However, \( \widehat{F}_{i,\theta} \) may generally not be smooth in \( \theta \) due to simulation (see (13)). We denote a smooth version of \( \widehat{M}_N \) by \( \widetilde{M}_N \), where \( \widehat{F}_{i,\theta} \) is replaced by \( \widehat{F}_{i,\theta} \), an
infeasible estimator of \( F_{i, \theta} \), and denote its limiting function by \( M \), so that
\[
M_N (\theta) = \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( \hat{F}_{i, \theta} (a_i | x) - F_i (a_i | x) \right)^2 \mu_{i,x} (da_i),
\]
\[
M (\theta) = \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( F_{i, \theta} (a_i | x) - F_i (a_i | x) \right)^2 \mu_{i,x} (da_i).
\]

The minimum distance estimator is defined to be any sequence \( \hat{\theta} \) that satisfies
\[
\hat{M}_N (\hat{\theta}) \leq \inf_{\theta \in \Theta} M_N (\theta) + o_p (N^{-1}).
\]

**Assumption A1.**

(i) \( \Theta \) is a compact subset of \( \mathbb{R}^p \);
(ii) for all \( i, a_i, x, F_{i, \theta} (a_i | x) \) and \( F_i (a_i | x) \) exist, and \( F_{i, \theta} (a_i | x) = F_i (a_i | x) \) if and only if \( \theta = \theta_0 \);
(iii) for all \( i, a_i, x, F_{i, \theta} (a_i | x) \) is continuous on \( \Theta \);
(iv) for all \( i, x, \mu_{i,x} \) is a non-random finite measure on \( A_i \) that dominates the distribution of \( a_{in} \);
(v) for all \( i, x, \sup_{(\theta, a_i) \in \Theta \times A_i} \left| \hat{F}_{i, \theta} (a_i | x) - \hat{F}_{i, \theta} (a_i | x) \right| = o_p (1) \);
(vi) for all \( i, x, \sup_{(\theta, a_i) \in \Theta \times A_i} \left| \hat{F}_{i, \theta} (a_i | x) - F_{i, \theta} (a_i | x) \right| = o_p (1) \);
(vii) for all \( i, x, \sup_{a_i \in A_i} \left| \hat{F}_{i, \theta} (a_i | x) - F_i (a_i | x) \right| = o_p (1) \).

A1(ii) is the point-identification assumption of the pseudo-model. A1(iv) ensures that the measures used to define the objective function do not lose any identifying information on \( \theta_0 \). In application, \( A_i \) is generally compact hence finiteness of the measures is a mild assumption. Note that the integral representation of \( M_N, M \) and \( M \) encompasses games with discrete, continuous or mixed discrete-continuous actions. When \( A_i \) is finite \( \mu_{i,x} \) is a count measure, it is sufficient to choose measures that put positive weights on each point of \( A_i \). For purely continuous action game the domination condition is satisfied by choosing any measure dominated by the Lebesgue measure, for instance the uniform measure. For an intermediate case with \( a_{in} \) has a mixture of discrete-continuous distribution, then \( \mu_{i,x} \) is simply a combination of the count and continuous measures. We can also allow the measures to be random. Specifically we can also use any random measure \( \hat{\mu}_{i,x} \) as long as it converges (-weakly) to \( \mu_{i,x} \) that satisfy the finiteness and dominant conditions; one such candidate is the empirical measure, which puts equal mass on each observed data points \( \{a_{in}, x_n\} \) and zero measure outside it. A1(i) to A1(iv) imply that \( M (\theta) \) has a well-separated minimum over a compact set at \( \theta_0 \). The remaining conditions require our estimators for the distribution functions to

\[\text{The proofs in Appendix B can be lengthened leading to the same asymptotic results for random measures } \{\hat{\mu}_{i,x}\}_{i \in I, x \in X}, \text{ where } \hat{\mu}_{i,x} \text{ converges weakly to } \mu_{i,x} \text{ for all } (i, x); \text{ using repeated applications of continuous mapping theorem (see Ranga Rao (1962)).}\]
be uniformly consistent, which can generally be verified using empirical process theory (see van der Vaart and Wellner (1996)). Note that A1(v) is not relevant if \( \widehat{F}_{i,\theta} \) is feasible. An important special case is when \( \widehat{F}_{i,\theta} \) is the naive Monte Carlo integration estimator. Suppose \( \widehat{F}_{i,\theta} \) is defined as in (13), then
\[
\widehat{F}_{i,\theta}(a_i|x) - \widehat{F}_{i,\theta}(a_i|x) = \frac{1}{R} \sum_{r=1}^{R} \mathbf{1}_{\left\{ \widehat{\alpha}_{i,\theta}(x,\varepsilon'_i) \leq a_i \right\}} - \int \mathbf{1}_{\left\{ \widehat{\alpha}_{i,\theta}(x,\varepsilon_i) \leq a_i \right\}} dQ_i(\varepsilon_i),
\]
so that A1(v) is expected to hold as \( R \to \infty \) by an application of Glivenko-Cantelli theorem. A1(vi) requires a standard equicontinuity condition and uniform consistent estimation of the parameters in the first stage. A1(vii) follows from the classical uniform law of large numbers.

**Theorem 1 (Consistency).** Under Assumption A1: \( \widehat{\theta} \xrightarrow{p} \theta_0 \).

To show asymptotic normality we require additional assumptions. In what follows, let \( \rightsquigarrow \) denote weak convergence and \( l^\infty(A_i) \) denotes the space of all bounded functions on \( A_i \).

**Assumption A2.**
(i) \( \theta_0 \) lies in the interior of \( \Theta \);
(ii) for all \( i, a_i, x, \widehat{F}_{i,\theta}(a_i|x) \) and \( \widehat{F}_{i,\theta}(a_i|x) \) are twice continuously differentiable in \( \theta \) in a neighborhood of \( \theta_0 \), and \( \int_{A_i} \frac{\partial}{\partial \theta} F_{i,\theta}(a_i|x) \mu_{i,x} \mu_{i,x}(da_i) \) and \( \int_{A_i} \frac{\partial^2}{\partial \theta \partial \theta} F_{i,\theta}(a_i|x) \mu_{i,x}(da_i) \) exist for all \( l, l' \) for \( \theta \) in a neighborhood of \( \theta_0 \);
(iii) \( \sum_{i \in I} \sum_{x \in X} \int_{A_i} \frac{\partial}{\partial \theta} F_{i,\theta}(a_i|x) \frac{\partial}{\partial \theta} F_{i,\theta}(a_i|x) \mu_{i,x}(da_i) \) is positive definite at \( \theta = \theta_0 \);
(iv) for all \( i, l, x, \sup_{a_i \in A_i} \left| \frac{\partial}{\partial \theta} \widehat{F}_{i,\theta}(a_i|x) - \frac{\partial}{\partial \theta} F_{i,\theta}(a_i|x) \mu_{i,x}(da_i) \right| = o_p(1) \) as \( \| \theta - \theta_0 \| \to 0 \);
(v) for all \( i, l, l', x, \sup_{a_i \in A_i} \left| \frac{\partial^2}{\partial \theta \partial \theta} \widehat{F}_{i,\theta}(a_i|x) - \frac{\partial^2}{\partial \theta \partial \theta} F_{i,\theta}(a_i|x) \right| = o_p(1) \) as \( \| \theta - \theta_0 \| \to 0 \);
(vi) for all \( i, x, \sqrt{N} \left( \widehat{F}_{i,\theta}(\cdot|x) - F_{i,\theta}(\cdot|x) \right) \rightsquigarrow \mathbb{V}_{i,x} \) where \( \mathbb{V}_{i,x} \) is a tight Gaussian process that belongs to \( l^\infty(A_i) \);
(vii) for all \( i, x, \sqrt{N} \left( \widehat{F}_{i,\theta_0}(\cdot|x) - F_{i,\theta_0}(\cdot|x) \right) \rightsquigarrow \mathbb{W}_{i,x} \) where \( \mathbb{W}_{i,x} \) is a tight Gaussian process that belongs to \( l^\infty(A_i) \);
(viii) for all \( i, x, \sqrt{N} \left( \widehat{F}_{i,\theta_0}(\cdot|x) - \widehat{F}_{i,\theta_0}(\cdot|x) \right) \rightsquigarrow \mathbb{T}_{i,x} \) where \( \mathbb{T}_{i,x} \) is a tight Gaussian process that belongs to \( l^\infty(A_i) \).

Conditions A2(i) to A2(v) are standard regularity and smoothness assumptions. Since \( F_{i,\theta}(a_i|x) \) is twice continuously differentiable in \( \theta \) (near \( \theta_0 \)), sufficient conditions for A2(iv) and A2(v) are uniform consistency of the first and second derivatives of \( \widehat{F}_{i,\theta} \) to \( F_{i,\theta} \) respectively (cf. A1(vi)). A2(vi) imposes a rate for the simulation error. If \( \widehat{F}_{i,\theta} \) is defined by (13), then \( \sqrt{R} \left( \widehat{F}_{i,\theta} - F_{i,\theta} \right) \) is an empirical process (see equation (16)) that is expected to satisfy the Donsker’s theorem. The remaining conditions
assume uniform central limit theorems hold on \( A_i \). When \( A_i \) is finite, uniform limit theorem reduces to multivariate central limit theorem where tightness condition is trivially satisfied, otherwise these can be verified using empirical process theory (cf. A1(v) to A1(vii)). Specifically A2(vii) captures the effects from using a first step estimator, which typically can be verified by showing the linearization of \( \sqrt{N} \left( \hat{F}_{i,\theta_0} - F_{i,\theta_0} \right) \) satisfies the Donsker’s theorem. When the limiting distributions in A2(vii) and A2(viii) are jointly Gaussian, which is expected to hold in most applications, A2(ix) immediately follows from continuous mapping theorem.

**Theorem 2 (Asymptotic Normality).** Under Assumptions A1 and A2:

\[
\sqrt{N} \left( \hat{\theta} - \theta_0 \right) = \left( \frac{\partial^2}{\partial \theta \partial \theta'} M(\theta_0) \right)^{-1} \sqrt{N} \frac{\partial}{\partial \theta} M_N(\theta_0) + o_p(1),
\]

and \( \sqrt{N} \left( \hat{\theta} - \theta_0 \right) \overset{d}{\rightarrow} N(0, \mathcal{W}^{-1} \mathcal{V} \mathcal{W}^{-1}) \), where

\[
\mathcal{V} = \lim_{N \to \infty} \text{var} \left( 2 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( \frac{\partial}{\partial \theta} F_{i,\theta_0} (a_i|x) \left( \hat{F}_{i,\theta_0} (a_i|x) - \hat{F}_i (a_i|x) \right) \right) \mu_{i,x} (da_i) \right), \quad (17)
\]

\[
\mathcal{W} = 2 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \frac{\partial}{\partial \theta} F_{i,\theta_0} (a_i|x) \frac{\partial}{\partial \theta} F_{i,\theta_0} (a_i|x) \mu_{i,x} (da_i). \quad (18)
\]

The asymptotic distribution of our estimator shows no effect of using the feasible estimator \( \tilde{F}_{i,\theta} \) instead of \( \hat{F}_{i,\theta} \). In order to perform inference, a feasible estimator for the asymptotic variance is required. Bootstrapping is a natural candidate to estimate the standard error in this setting.\(^\text{12}\) In a closely related framework, Kasahara and Shimotsu (2008a) develop a bootstrap procedure for a parametric discrete decision model that can be applied to discrete action games (under Assumption D). Recently, Cheng and Huang (2010) provide some general conditions to validate the use of the bootstrap as an inferential tool for a general class of semiparametric M-estimators when the objective function is not smooth. We show in the next section that bootstrapping performs well with our minimum distance estimator.

**A Remark on Semiparametric Estimation.** Theorems 1 and 2 are applicable to both parametric and semiparametric problems. In the context of dynamic games, the first stage estimators

\(^{12}\)Recently Ackerberg, Chen and Hahn (2010) propose a way to simplify semiparametric inference when unknown functions are estimated by the method of sieves. They consider, as a specific example, a class of discrete action games, where they focus on estimating finite conditional moment models and also require the objective function to be smooth. Therefore, despite our theorems admitting sieves estimators, their results are generally not applicable to our estimator, and also other notable estimators in this literature (e.g. the iterative estimator of Aguirregabiria and Mira (2007), and the inequality estimator of BBL).
(finite and/or infinite dimensional) are defined implicitly in our objective function $\hat{M}_N$ through $\hat{F}_{i,\theta}$. The uniform consistency and functional central limit theory requirements in A1 and A2 are standard for a minimum distance estimator. These uniformity conditions can be verified using modern empirical process theory under weak conditions. Particularly, for the simulation estimator defined in (13), Andrews (1994, “type IV class”) and Chen, Linton and van Keilegom (2003, Theorem 3.2) provide conditions for the Donsker’s theorem to hold in a parametric and semiparametric setting respectively.\textsuperscript{13}

**Possible Extensions**

In this paper we have focused on consistent estimation method for a large class of dynamic games. However, there are two important aspects of our estimators we have not discussed. These are the issues of efficiency and finite sample bias.

Our minimum distance estimator is not efficient. For example when $A_i$ is finite we can create large vectors of the conditional distribution of actions across all players, action choices and observable states, then our objective function is a special case of the asymptotic least squares estimators analogous to the setup in PSD with a diagonal weighting matrix. In principle, we can provide a more efficient estimator by consider a more general metric to match the distribution functions and construct the efficient weights (that will rely on a consistent preliminary estimator). However, the efficient weights will generally require the estimates of $\frac{\partial}{\partial \theta} F_{i,\theta} (a_i|x)$ for all $a_i, x$, which rely on further numerical approximations when the feasible estimator of $F_{i,\theta}$ is not smooth (for recent results on statistical properties of estimators with numerical derivatives see Hong, Mahajan and Nekipelov (2010)). The issue of efficient estimation for this class of games is a challenging and interesting problem in both theory and practice, especially in a semiparametric model.

Another important concern for two-step estimators is the bias in small sample. In a single agent discrete choice setting, Aguirregabiria and Mira (2002) propose iteration methods that appears to improve the finite sample performance of their estimators. Kasahara and Shimotsu (2008a) give a theoretical explanation of Aguirregabiria and Mira’s findings, the idea is that a fixed point constraint of the pseudo-model implied choice probabilities provides an iteration operator that can be used to reduce the bias in the first stage estimation. Although such iteration procedure may not converge, especially in a game setting (Pesendorfer and Schmidt-Dengler (2010)), recently Kasahara and Shimotsu (2012) provide an alternative iteration method that leads to a consistent estimator even when the fixed point constraint is not a contraction (hence it need not ensure global convergence). The frameworks that the aforementioned papers consider are games under Assumption D in Section 2.4.

\textsuperscript{13}Srisuma (2010a) gives a set of primitive conditions where Assumptions A1 and A2 are satisfied for a single agent problem that coincides with purely continuous action game in Section 3.3 when $I = 1$. 

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Since equation (10) also represents a fixed point constraint, it will be interesting to study if analogous iterative schemes can be developed for other class of games such as those considered in this paper.

5 Numerical Examples

We apply our methodology described in Section 4 to estimate two simulated dynamic models with continuous actions. We construct our minimum distance estimators based on the estimators proposed in Table A and Example 1. First, a semiparametric dynamic price setting problem for a single agent firm. Second, in a parametric framework, we use our estimator and BBL’s to estimate a repeated Cournot duopoly game. Since it is generally difficult to solve a dynamic optimization problem, the models below are kept simple in order to generate the data. It is easy to check that both examples below satisfy conditions M1, M2, M3, S1’, S2 and S4’, so that players employ monotone optimal strategies.

**Design 1 (Markov decision problem).** At every period, each firm faces the following demand function

\[ D_\theta (a_n, x_n, \varepsilon_n) = \overline{D} - \theta_1 a_n + \theta_2 (x_n + \varepsilon_n), \]

where \( a_n \) denotes the price, \( x_n \) is the demand shifter (e.g. some observable measure of the consumer’s satisfaction), and \( \varepsilon_n \) is the firm’s private demand shock. \( \overline{D} \) can be interpreted as a constant market size, and \( (\theta_1, \theta_2) \) denote the parameters that represent the market elasticities that lie in \( \mathbb{R}^+ \times \mathbb{R}^+ \). The firm’s profit function is

\[ u_\theta (a_n, x_n, \varepsilon_n) = D_\theta (a_n, x_n, \varepsilon_n) (a_n - c), \]

where \( c \) denotes a constant marginal cost. The price setting decision affects the demand for the next period through \( x'_n \) that follows a Markov process. Specifically \( x_n \) takes value either 1 or \(-1\), and its transitional distribution is summarized by \( \Pr [x'_n = -1|x_n, a_n] = \frac{a_n - \bar{a}}{\bar{a} - \underline{a}} \), where \( \bar{a} \) and \( \underline{a} \) denote the minimum and maximum possible prices respectively. The evolution of private shocks are completely random and transitory, and \( \varepsilon_n \) is distributed uniformly on \([-1, 1]\). The firm chooses price \( a_n \) to maximize its discounted expected profit where future payoff is discounted by \( \beta = 0.9 \). The values of \( (\overline{D}, c) \) are assigned to be \((3, 1)\) and the data is generated using the optimal decision when \( \theta = (1, 0.5) \).

We generate 500 replications of the controlled Markov processes with sample size \( N \in \{20, 100, 200\} \), where each decision series spans 5 time periods. This leads to three sets of experiments with the total sample size, \( NT \), of 100, 500 and 1000.

We have two estimators, denoted by \( \hat{\theta}^{UM} \) and \( \hat{\theta}^{EM} \), which minimize the objective functions constructed using the uniform and empirical measures respectively. For the nonparametric estimator
of the transition law, $G(x'|x,a)$, we use a truncated 4-th order kernel based on the density of a standard normal random variable (see Rao (1983)). For each replication, we experiment with 3 different bandwidths $\{h_0 = 1.06s(NT)^{-\zeta} : \zeta = \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\}$; the order of the bandwidth is chosen to be consistent with a derivative of one-dimensional kernel estimator for a density or regression derivative (for example see Hansen (2008)). We simulate the pseudo-distribution function using $N \log (N)$ random draws. The number of bootstrap draws is 99.

We report the bias, median of the bias, standard deviation, coverage probability of 95% confidence interval based on a standard normal approximation, and the bootstrapped standard errors and coverage probabilities from the bootstrapped distributions. Tables 1 and 2 give the results for $\theta_1$ and $\theta_2$ respectively, where the bootstrapped values are given in italics.

Table 1: Monte Carlo results (Markov decision process). The bandwidth used in the nonparametric estimation is $h_0 = 1.06s(NT)^{-\zeta}$, where $s$ is the standard deviation of $\{a_{nt}\}_{n=1,t=1}^{NT}$.

| NT  | $\zeta$ | Bias | Mbias | Std  | 95%  | Bias | Mbias | Std  | 95%  |
|-----|--------|------|-------|------|------|------|-------|------|------|
| 100 | 1/6    | -0.0101 | 0.0118 | 0.1716 | 0.9560 | 0.0023 | 0.0205 | 0.1722 | 0.9640 |
|     | -      | 0.2594 | 0.9960 | -     | -     | 0.2370 | 0.9960 |
| 1/7 | 0.0014 | 0.0248 | 0.1543 | 0.9600 | 0.0128 | 0.0305 | 0.1515 | 0.9640 |
|     | -      | 0.2370 | 0.9900 | -     | -     | 0.2326 | 0.9840 |
| 1/8 | 0.0067 | 0.0296 | 0.1569 | 0.9680 | 0.0233 | 0.0409 | 0.1406 | 0.9580 |
|     | -      | 0.2227 | 0.9740 | -     | -     | 0.2129 | 0.9740 |
| 500 | 1/6    | 0.0009 | 0.0024 | 0.0695 | 0.9360 | 0.0025 | 0.0040 | 0.0693 | 0.9360 |
|     | -      | 0.0784 | 0.9760 | -     | -     | 0.0784 | 0.9800 |
| 1/7 | 0.0044 | 0.0086 | 0.0620 | 0.9480 | 0.0065 | 0.0091 | 0.0617 | 0.9440 |
|     | -      | 0.0706 | 0.9720 | -     | -     | 0.0708 | 0.9720 |
| 1/8 | 0.0105 | 0.0155 | 0.0574 | 0.9380 | 0.0121 | 0.0164 | 0.0583 | 0.9340 |
|     | -      | 0.0797 | 0.9660 | -     | -     | 0.0649 | 0.9700 |
| 1000| 1/6    | -0.0023 | 0.0003 | 0.0511 | 0.9380 | -0.0025 | -0.0007 | 0.0507 | 0.9400 |
|     | -      | 0.0552 | 0.9540 | -     | -     | 0.0552 | 0.9540 |
| 1/7 | 0.0021 | 0.0045 | 0.0475 | 0.9500 | 0.0028 | 0.0044 | 0.0474 | 0.9540 |
|     | -      | 0.0500 | 0.9600 | -     | -     | 0.0500 | 0.9640 |
| 1/8 | 0.0073 | 0.0086 | 0.0457 | 0.9460 | 0.0075 | 0.0081 | 0.0450 | 0.9460 |
|     | -      | 0.0463 | 0.9500 | -     | -     | 0.0462 | 0.9460 |

14 When $\Lambda_{i,\theta}$ is smooth, by implicit function theorem, $\alpha_{i,\theta}$ is a smooth functional of $E[\frac{\partial}{\partial a} u_i,\theta(\cdot,a_{-in},x_n,\cdot)|x_n=\cdot]$ and $\frac{\partial}{\partial a} g_{i,\theta} \alpha_{i,\theta} (s_i), s_i = 0$. Since $E_u[\frac{\partial}{\partial a} u_i,\theta(\cdot,a_{-in},x_n,\cdot)|x_n=\cdot]$ converges at the parametric rate, the rate of convergence of $\hat{\alpha}_{i,\theta}$ is determined by $\frac{\partial}{\partial a} g_{i,\theta}$. 

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Table 2: Monte Carlo results (Markov decision process). The bandwidth used in the nonparametric estimation is $h = 1.06s(NT)^{-\zeta}$, where $s$ is the standard deviation of $\{o_{nt}\}_{n=1,t=1}^{N,T}$.

| $NT$ | $\zeta$ | Bias | Mbias | Std | 95% | Bias | Mbias | Std | 95% |
|------|--------|------|-------|-----|-----|------|-------|-----|-----|
| 100  | 1/6    | 0.1140 | 0.0546 | 0.2633 | 0.9440 | 0.1079 | 0.0493 | 0.2460 | 0.9380 |
|      |        | -    | -     | 0.3245 | 0.9940 | -    | -     | 0.3118 | 0.9940 |
| 1/7  | 0.0986 | 0.0580 | 0.2213 | 0.9320 | 0.0985 | 0.0575 | 0.2257 | 0.9420 |
|      |        | -    | -     | 0.3126 | 0.9920 | -    | -     | 0.3040 | 0.9940 |
| 1/8  | 0.1030 | 0.0583 | 0.2267 | 0.9380 | 0.0987 | 0.0524 | 0.2110 | 0.9360 |
|      |        | -    | -     | 0.2969 | 0.9920 | -    | -     | 0.2885 | 0.9940 |
| 500  | 1/6    | 0.0381 | 0.0370 | 0.0878 | 0.9220 | 0.0386 | 0.0371 | 0.0889 | 0.9160 |
|      |        | -    | -     | 0.1091 | 0.9740 | -    | -     | 0.1086 | 0.9740 |
| 1/7  | 0.0383 | 0.0307 | 0.0860 | 0.9200 | 0.0373 | 0.0317 | 0.0867 | 0.9180 |
|      |        | -    | -     | 0.1078 | 0.9580 | -    | -     | 0.1026 | 0.9700 |
| 1/8  | 0.0374 | 0.0308 | 0.0839 | 0.9060 | 0.0367 | 0.0312 | 0.0854 | 0.9140 |
|      |        | -    | -     | 0.3005 | 0.9460 | -    | -     | 0.0964 | 0.9460 |
| 1000 | 1/6    | 0.0338 | 0.0319 | 0.0699 | 0.9260 | 0.0332 | 0.0310 | 0.0691 | 0.9240 |
|      |        | -    | -     | 0.0753 | 0.9560 | -    | -     | 0.0753 | 0.9560 |
| 1/7  | 0.0317 | 0.0275 | 0.0668 | 0.9320 | 0.0308 | 0.0255 | 0.0670 | 0.9260 |
|      |        | -    | -     | 0.0704 | 0.9420 | -    | -     | 0.0706 | 0.9440 |
| 1/8  | 0.0316 | 0.0256 | 0.0648 | 0.9240 | 0.0310 | 0.0232 | 0.0650 | 0.9220 |
|      |        | -    | -     | 0.0662 | 0.9300 | -    | -     | 0.0665 | 0.9300 |

We have the following general observations for our estimators across all bandwidths and measures: (i) the median of the bias is similar to the mean; (ii) the estimators are consistent, as $N$ increases the bias and standard deviation converge to zero; (iii) the performance of the bootstrapped standard errors steadily approaches the true with increasing sample size and appear to be consistent; (iv) the coverage probabilities improves with sample size, although the results for $\theta_1$ are closer to the nominal value than those for $\theta_2$, the bootstrapped confidence intervals appear to perform reasonably well, and even favorably in some cases, relative to the normal approximations with the infeasible variance at larger sample sizes. Therefore bootstrap appears to offer one reasonable mode to perform inference for our estimator.

**Design 2 (Cournot game).** We use a variant of a repeated Cournot duopoly competition studied
in PSD. We specify a linear inverse demand function:

\[ D_\theta(a_n) = x_n (\bar{D} - \theta_1 (a_{1n} + a_{2n})) , \]

where \( a_{in} \) denotes the quantity supplied by player \( i \), \( x_n \) is the demand shifter that rotates the slope of the demand curve, and \( \bar{D} \) represents the market size similar to Example 1. The parameter space for \((\theta_1, \theta_2)\) is \( \mathbb{R}^+ \times \mathbb{R}^+ \). Each firm also has a stochastic variable cost, so that the profit function for each period is

\[ u_{i,\theta}(a_i, a_j, x, \varepsilon_{in}) = a_{in} (D(a_n) - \theta_2 \varepsilon_{in}) \text{ for } i, j = 1, 2 \text{ and } i \neq j , \]

where \( \varepsilon_{in} \) is the private demand shock, which has a normal distribution with mean 0 and variance 1. \( \varepsilon_{in} \) is normally distributed independently across players, time periods and other variables. The observable state is the stochastic demand coefficient \( x_n \) that has 0.5 probability of taking values 2 or 4, independently of previous actions and states. Thus an equilibrium exist, particularly the symmetric strategy profile where each player maximizes her expected static profit (a non-cooperative Nash equilibrium) in every period is an equilibrium. We add a dynamic dimension to our estimation problem by misspecifying the model (see below). Our data is generated from the symmetric equilibrium from the static duopoly game, where \( \bar{D} \) is normalized to 1, we use \( \theta_0 = (0.2, 0.2) \) and the discounting factor is 0.9. For each simulation, we generate \( N \in \{100, 500, 1000\} \) independent draws from the equilibrium. The experiment is repeated 500 times for each \( N \).

For our estimators, as done previously, we use two estimators constructed from the objective functions with uniform measures and empirical measures. We allow for a particular misspecification such that our agent maximizes the following objective function (cf. (8)) in the pseudo-optimization stage:

\[ \tilde{\Lambda}_{i,\theta}(a_i, s_i) = E[u_{i,\theta}(a_i, a_{-in}, x_n, \varepsilon_i) | x_n = x] + \beta_i \tilde{g}_i(a_i, x) , \]

where \( \tilde{g}_i \) is a linear function of \( a_i \) that has a random slope, varying randomly with each player and state. The slope of \( \tilde{g}_i \) converges to zero at the parametric rate, it is determined by a random draw from a normal distribution with mean zero and variance \( \frac{1}{N} \). We simulate the pseudo-distribution function using \( N \log (N) \) random draws.

We also consider two versions of BBL estimators; one is based on choosing alternative strategy by an additive perturbation and the other by multiplicative perturbation. For additive perturbations, each inequality is represented by an alternative strategy \( \tilde{\alpha}_1(\cdot; \eta_1) \) for some \( \eta_1 \in \mathbb{R} \) such that \( \tilde{\alpha}_1(s_i; \eta_1) = \alpha_{0\theta}(s_i) + \eta_1 \) for all \( s_i \in S_i \), where \( \alpha_{0\theta} \) is the (symmetric) optimal strategy estimable from the data. We draw \( \eta_1 \) from a normal distribution with mean 0 and variance 0.5. For multiplicative perturbation, each inequality is represented by an alternative strategy \( \tilde{\alpha}_2(\cdot; \eta_2) \) for some \( \eta_2 \in \mathbb{R} \) such
that $\tilde{\alpha}(s_i; \eta_2) = \eta_2 \alpha_{\theta_0}(s_i)$ for all $s_i \in S_i$. We draw $\eta_2$ from a normal distribution with mean 1 and variance 0.5. The BBL type objective functions are constructed based on using $N_I \in \{300, 600\}$ randomly drawn inequalities and the number of simulations used to compute the expected returns is $2000$. BBL estimators correctly ignores the dynamics and estimates the repeated static game.

We show in the second example of Section A.1 in Appendix A that the parameters in the Cournot game is identified. However, with BBL’s approach, we also show that the class of additive perturbations preserves the identifying information of $\theta_{01}$ but not $\theta_{02}$, in the sense that the expected returns from employing the optimal strategies that generate the data (with $\theta = \theta_0$) is always at least as large as the returns from additively perturbed strategies for all $\theta' = (\theta_{01}, \theta_{02}')$ with any value of $\theta_{02}'$. On the other hand, the inequalities based on multiplicative perturbations can preserve the identifying information of both $\theta_{01}$ and $\theta_{02}$.

We report the bias, median of the bias, standard deviation, interquartile range scaled by 1.349 (which approximately equals to the standard deviation for a normal variable), coverage probability of 95% confidence interval based on a standard normal approximation, and mean square error. Tables 3 and 4 give the results for our estimators, with and without the misspecified the dynamics, and BBL’s estimators, constructed using additive and multiplicative perturbations, of $\theta_1$ and $\theta_2$ respectively.

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The number of inequalities and simulations we use represent the upper bound values that BBL use in their simulation studies, which conform to their asymptotic theorems. Specifically, see Assumption S2 (iii) on p. 1348, $N_I$ is allowed to grow to infinity at any rate, whilst the number of simulations is required to go to infinity at a faster rate than $\sqrt{N}$.
Table 3: Monte Carlo results (Cournot game). UM and EM are our minimum distance estimators for the static games obtained from using uniform and empirical measures respectively; UM-M and EM-M are their misspecified counterparts. AP-L and AP-H are BBL’s estimators obtained from using additive perturbations with 300 and 600 inequalities respectively. MP-L and MP-H are BBL’s estimators obtained from using multiplicative perturbations with 300 and 600 inequalities respectively.

| N    | $\hat{\theta}_1$ | Bias | Mbias | Std  | Iqr  | 95%  | Mse  |
|------|------------------|------|-------|------|------|------|------|
| 100  | UM               | 0.0000 | -0.0001 | 0.0014 | 0.0013 | 0.9460 | 0.0000 |
|      | UM-M             | 0.0001 | -0.0001 | 0.0026 | 0.0027 | 0.9460 | 0.0000 |
|      | EM               | -0.0004 | -0.0004 | 0.0014 | 0.0013 | 0.9400 | 0.0000 |
|      | EM-M             | -0.0003 | -0.0005 | 0.0026 | 0.0028 | 0.9580 | 0.0000 |
|      | AP-L             | -0.0000 | -0.0000 | 0.0016 | 0.0017 | 0.9560 | 0.0000 |
|      | AP-H             | -0.0001 | -0.0001 | 0.0017 | 0.0015 | 0.9540 | 0.0000 |
|      | MP-L             | 0.0002 | 0.0000 | 0.0020 | 0.0019 | 0.9440 | 0.0000 |
|      | MP-H             | 0.0001 | -0.0000 | 0.0021 | 0.0020 | 0.9400 | 0.0000 |
| 500  | UM               | 0.0000 | 0.0000 | 0.0007 | 0.0007 | 0.9580 | 0.0000 |
|      | UM-M             | 0.0000 | -0.0000 | 0.0012 | 0.0012 | 0.9580 | 0.0000 |
|      | EM               | -0.0001 | -0.0000 | 0.0007 | 0.0007 | 0.9540 | 0.0000 |
|      | EM-M             | -0.0000 | -0.0001 | 0.0012 | 0.0012 | 0.9560 | 0.0000 |
|      | AP-L             | 0.0000 | 0.0000 | 0.0008 | 0.0008 | 0.9380 | 0.0000 |
|      | AP-H             | -0.0000 | -0.0000 | 0.0007 | 0.0008 | 0.9580 | 0.0000 |
|      | MP-L             | 0.0000 | 0.0000 | 0.0010 | 0.0009 | 0.9580 | 0.0000 |
|      | MP-H             | -0.0000 | 0.0000 | 0.0010 | 0.0009 | 0.9460 | 0.0000 |
| 1000 | UM               | -0.0000 | 0.0000 | 0.0004 | 0.0004 | 0.9460 | 0.0000 |
|      | UM-M             | -0.0000 | 0.0000 | 0.0008 | 0.0008 | 0.9420 | 0.0000 |
|      | EM               | -0.0001 | -0.0000 | 0.0004 | 0.0004 | 0.9480 | 0.0000 |
|      | EM-M             | -0.0001 | -0.0000 | 0.0008 | 0.0008 | 0.9480 | 0.0000 |
|      | AP-L             | -0.0000 | 0.0000 | 0.0006 | 0.0005 | 0.9380 | 0.0000 |
|      | AP-H             | 0.0001 | 0.0000 | 0.0005 | 0.0005 | 0.9440 | 0.0000 |
|      | MP-L             | 0.0000 | 0.0000 | 0.0007 | 0.0005 | 0.9720 | 0.0000 |
|      | MP-H             | 0.0000 | 0.0000 | 0.0007 | 0.0007 | 0.9460 | 0.0000 |
Table 4: Monte Carlo results (Cournot game). UM and EM are our minimum distance estimators for the static games obtained from using uniform and empirical measures respectively; UM-M and EM-M are their misspecified counterparts. AP-L and AP-H are BBL’s estimators obtained from using additive perturbations with 300 and 600 inequalities respectively. MP-L and MP-H are BBL’s estimators obtained from using multiplicative perturbations with 300 and 600 inequalities respectively.

| N   | θ̂₂  | Bias | Mbias | Std  | Iqr  | 95%  | Mse   |
|-----|------|------|-------|------|------|------|-------|
| 100 | UM   | -0.0009 | -0.0011 | 0.0119 | 0.0131 | 0.9580 | 0.0001 |
|     | UM-M | 0.0054 | 0.0053 | 0.0142 | 0.0132 | 0.9500 | 0.0002 |
|     | EM   | 0.0033 | 0.0029 | 0.0140 | 0.0139 | 0.9360 | 0.0002 |
|     | EM-M | 0.0205 | 0.0164 | 0.0255 | 0.0206 | 0.8960 | 0.0011 |
|     | AP-L | -0.0174 | -0.0421 | 0.2613 | 0.2153 | 0.9300 | 0.0686 |
|     | AP-H | 0.0008 | -0.0062 | 0.1520 | 0.1390 | 0.9480 | 0.0231 |
|     | MP-L | 0.0268 | 0.0047 | 0.2623 | 0.2009 | 0.9400 | 0.0695 |
|     | MP-H | 0.0217 | 0.0064 | 0.2753 | 0.2623 | 0.9500 | 0.0762 |
| 500 | UM   | -0.0002 | -0.0003 | 0.0052 | 0.0050 | 0.9580 | 0.0000 |
|     | UM-M | 0.0005 | 0.0006 | 0.0055 | 0.0053 | 0.9500 | 0.0000 |
|     | EM   | 0.0006 | 0.0002 | 0.0059 | 0.0055 | 0.9520 | 0.0000 |
|     | EM-M | 0.0036 | 0.0036 | 0.0070 | 0.0069 | 0.9320 | 0.0001 |
|     | AP-L | -0.0241 | -0.0828 | 0.2012 | 0.1380 | 0.9380 | 0.0411 |
|     | AP-H | -0.0150 | -0.0248 | 0.1388 | 0.1097 | 0.9340 | 0.0195 |
|     | MP-L | -0.0010 | 0.0039 | 0.0945 | 0.0117 | 0.9260 | 0.0089 |
|     | MP-H | 0.0046 | 0.0043 | 0.1191 | 0.0841 | 0.9400 | 0.0142 |
| 1000| UM   | 0.0001 | -0.0000 | 0.0037 | 0.0038 | 0.9460 | 0.0000 |
|     | UM-M | 0.0004 | 0.0006 | 0.0039 | 0.0039 | 0.9600 | 0.0000 |
|     | EM   | 0.0006 | 0.0004 | 0.0042 | 0.0046 | 0.9560 | 0.0000 |
|     | EM-M | 0.0019 | 0.0022 | 0.0047 | 0.0045 | 0.9380 | 0.0000 |
|     | AP-L | -0.0288 | -0.0943 | 0.1833 | 0.1024 | 0.9400 | 0.0344 |
|     | AP-H | -0.0168 | -0.0295 | 0.1284 | 0.1141 | 0.9360 | 0.0168 |
|     | MP-L | 0.0021 | 0.0000 | 0.0643 | 0.0046 | 0.9280 | 0.0041 |
|     | MP-H | -0.0054 | 0.0005 | 0.0820 | 0.0455 | 0.9080 | 0.0068 |
For $\theta_1$, as expected, all estimators appear to be consistent and, from looking at the coverage probabilities and comparing the standard deviation with the scaled interquartile range, well approximated by a normal distribution. For $\theta_2$, as before, our estimators appear to be consistent and asymptotically normal. BBL’s estimators of $\theta_2$ show several interesting characteristics. The first general observation is that estimators obtained from using multiplicative perturbations perform better, as expected, at least for larger sample sizes; they also appear to be consistent, but seem to be less well approximated by a normal distribution compared to our estimators. For the estimators based on additive perturbations, the bias appears to increase with sample size, which can be explained by looking at the mathematical details of our examples in Appendix A, since the loss of identification only materializes in the limit. However, its standard deviation is decreasing with sample size, although it does so at an increasingly slower rate compared to the multiplicative ones. It is also unclear from our small scale studies, what is the role the number of inequalities have on the statistical properties of BBL’s estimators, for instance we see that more inequalities lead to an improvement in the mean square error for additive perturbations but not for multiplicative perturbations.

6 Conclusion

Discrete Markov decision processes studied in Rust (1987,1994) provide a useful framework to model dynamic games of incomplete information. In this paper we propose a two-step methodology, in a similar spirit to Hotz and Miller (1993) that uses the pseudo-model to estimate the same class of games estimable by BBL’s methodology. We give precise conditions that extend the scope of the pseudo-model, traditionally used to model games where players’ actions are unordered-discrete, to games where players’ actions are monotone discrete, continuous or mixed. We also show that Markov perfect equilibria exist for the latter class of games. Our estimator is defined to minimize the distance between the distribution of actions implied by the data and the pseudo-model that is motivated by a characterization of the equilibrium. Since the distribution functions are defined on the familiar Euclidean space, given an identified (pseudo-)model, there are natural objective functions that can be used to construct a consistent estimator. In contrast, BBL’s method requires selection of alternative strategies, where such candidates may be less obvious especially when actions are continuously distributed. We illustrate the importance of choosing objective functions for consistent estimation in finite sample with a Monte Carlo study, and provide the theoretical explanations in Appendix A.

There are several directions for future research. We focus on consistent estimation and have not provided an efficient estimator in this paper. Our methodology also appears to be amenable to adopt the iterative scheme along the line of Aguirregabiria and Mira (2002,2007) and Kasahara and
Shimotsu (2012) that may reduce the small sample bias of the first step estimator. Lastly, although we do not contribute to the development of ways to deal with unobserved heterogeneity and the related issues regarding multiple equilibria. We believe the recent progress made in the studies of dynamic discrete choice models, for example, the nonparametric finite mixture results of Kasahara and Shimotsu (2008b) or methods that take advantage of finite dependence structure in Arcidiacono and Miller (2008), can be adapted and extended to estimate the dynamic games considered in this paper.
Appendix A  Consistent Estimation with BBL’s Methodology

This appendix illustrates a potential problem with the inequality approach of BBL. We provide two examples in A.1, each showing a scenario where: the inequality restrictions imposed by the equilibrium is satisfied by a unique element in the parameter space, the uniqueness can be lost when a strict subclass of inequalities are considered. The first example does not have any conditioning variables to emphasize that the source of information loss here differs from the instrumental variable model in Domínguez and Lobato (2004). The second example corresponds to Design 2 of the simulation study in Section 5. In A.2, we provide a class of inequalities that retains the identifying information of the (identified) parameters some discrete action games. We conclude with a brief discussion in A.3.

A.1 Mathematical Examples

Single Agent Problem

Consider a simple optimization problem where an economic agent maximizes the following payoff function

$$u_\theta (a, \varepsilon) = -a^2 + 2\theta a \varepsilon.$$  

Here $a$ and $\varepsilon$ denote the action and state variables respectively, and $\theta$ belongs to $\Theta$, some positive subset of $\mathbb{R}$. The model is generated from some distribution of $\varepsilon_n$ that is absolutely continuous with respect to the Lebesgue measure and has finite second moment. Notice that the current setup satisfies conditions in Section 2.3, as a special case of a single agent static decision problem ($\beta = 0$ and $I = 1$). Since the payoff function is concave, the optimal strategy follows from the first order condition:

$$\alpha_\theta (\varepsilon_n) = \theta \varepsilon_n \text{ a.s. for all } \theta \in \Theta.$$  

It is clear the distribution of $\alpha_\theta (\varepsilon_n)$ is identified. Let $\theta_0$ denote the true parameter and suppose we observe a random sample $\{a_n\}_{n=1}^N$ where $a_n = \alpha_{\theta_0} (\varepsilon_n)$ for each $n$.

The inequality approach of BBL defines an estimator for $\theta_0$ that satisfies the following system of moment inequalities in the limit

$$E [u_\theta (\alpha_{\theta_0} (\varepsilon_n), \varepsilon_n)] \geq E [u_\theta (\tilde{\alpha} (\varepsilon_n), \varepsilon_n)] \text{ for all } \tilde{\alpha} \in \mathfrak{A}_0,$$  \hspace{1cm} (SA1)

where $\mathfrak{A}_0$ is some user-chosen class of functions (of alternative strategies). We first consider a popular class of strategies based on additive perturbations and show that it cannot be used to identify $\theta_0$. Formally, let $S$ be some subset of $\mathbb{R}$, then define $\mathfrak{A}_0(S) = \{\tilde{\alpha} (\cdot; \eta) \text{ for } \eta \in S : \tilde{\alpha} (\varepsilon; \eta) = \alpha_{\theta_0} (\varepsilon) + \eta$
for all $\varepsilon \in \mathcal{E}$.\footnote{In an application of the BBL methodology, the user puts a distribution on $\eta$ that has support $\mathcal{S}$. A random sequence from this distribution is then drawn to construct the objective function; for instance, if $\mathcal{S} = \mathbb{R}$ then $\eta$ can be drawn from a normal distribution.} It follows from some simple algebra that, for any $\eta$,\[ E [u_\theta (\alpha_{\theta_0} (\varepsilon_n), \varepsilon_n)] - E [u_\theta (\tilde{\alpha} (\varepsilon_n), \varepsilon_n)] = \eta^2 + 2\eta (\theta_0 - \theta) E [\varepsilon_n]. \]

When $\varepsilon_n$ has mean zero, $\mathcal{A}_0 (\mathcal{S})$ has no identifying information for $\theta_0$ in the sense that, for all $\theta \in \Theta$\[ E [u_\theta (\alpha_{\theta_0} (\varepsilon_n), \varepsilon_n)] \geq E [u_\theta (\tilde{\alpha} (\varepsilon_n), \varepsilon_n)] \quad \text{for all } \tilde{\alpha} \in \mathcal{A}_0 (\mathcal{S}), \]
even if $\mathcal{S} = \mathbb{R}$. Therefore $\mathcal{A}_0 (\mathcal{S})$ cannot be used to consistently estimate $\theta_0$.

However, the set of inequalities that considers all alternative strategies can actually identify $\theta_0$. To see this, we begin by calculating the difference between the expected returns from $\alpha_{\theta_0}$ and a generic alternative strategy $\tilde{\alpha}$,\[ E [u_\theta (\alpha_{\theta_0} (\varepsilon_n), \varepsilon_n)] - E [u_\theta (\tilde{\alpha} (\varepsilon_n), \varepsilon_n)] = - (\theta - \theta_0)^2 E [\varepsilon_n^2] + E [(\varepsilon_n - \tilde{\alpha} (\varepsilon_n))^2]. \]

If we consider an inequality based on multiplicative perturbation, say $\mathcal{A}_1 (\mathcal{S}) = \{ \tilde{\alpha} (\cdot; \eta) \text{ for } \eta \in \mathcal{S} : \tilde{\alpha} (\varepsilon; \eta) = \eta \alpha_{\theta_0} (\varepsilon) \text{ for all } \varepsilon \in \mathcal{E} \}$, then by choosing $\tilde{\alpha}$ from $\mathcal{A}_1 (\mathcal{S})$, the difference above simplifies to $((\theta - \theta_0)^2 - (\theta - \theta_0)^2) E [\varepsilon_n^2]$. It is easy to see that, whenever $\theta \neq \theta_0$ the inequality in (SA1) will be violated for some range of values of $\eta$ sufficiently close to 1; more precisely, if $\theta > \theta_0$ then violation occurs for $\eta \in (1/\theta_0, 1)$, otherwise take $\eta \in (\theta/\theta_0, 1)$. Therefore the class of multiplicative perturbations has sufficient identifying power for $\theta_0$ in the sense that, when $\mathcal{S}$ contains any open ball centred at 1, then \[ E [u_\theta (\alpha_{\theta_0} (\varepsilon_n), \varepsilon_n)] \geq E [u_\theta (\tilde{\alpha} (\varepsilon_n), \varepsilon_n)] \quad \text{for all } \tilde{\alpha} \in \mathcal{A}_1 (\mathcal{S}) \text{ iff } \theta = \theta_0. \]

**Cournot Game**

Consider the setup of Design 2 in Section 5. Here we give a slightly more informal argument for why inequalities based on additive perturbation lose some identifying information on the data generating parameter whilst multiplicative perturbations can preserve it.

Consider player 1. For any given $a_1, x, \varepsilon_1$, $u_{1, \theta} (a_1, a_2, x, \varepsilon_1)$ is concave in $a_1$ since $\theta_1 > 0$. Taking the first derivative gives \[ \frac{\partial}{\partial a_1} u_{1, \theta} (a_1, a_2, x, \varepsilon_1) = x - \theta_2 \varepsilon_1 - \theta_1 x (2a_1 + a_2). \]

Since $a_1$ and $a_2$ enter the first derivative linearly and separately, the expected (symmetric) optimal action, which we denote by $\gamma_\theta$, can be obtained by finding the zero to the solving the following first
order condition
\[
\gamma_{\theta} (x_n) = \arg \text{zero}_{a \in A} E \left[ \frac{\partial}{\partial a_1} u_{1,\theta} (a_1, a_2, x_n, \varepsilon_n) \mid x_n \right]_{a_1 = a_2 = a}.
\]

Given that \( \varepsilon_{1n} \) is a random variable with mean 0 and variance 1, it then follows that \( \gamma_{\theta} (x_n) = \frac{1}{3\theta_1} \).

Therefore, for any \( x, \varepsilon_1 \), player 1’s optimal choice, \( \alpha_{\theta} (x, \varepsilon_1) \), can be characterized by the zero of \( \frac{\partial}{\partial a_1} u_{1,\theta} (a_1, \gamma_{\theta} (x), x, \varepsilon_1) \) that is equaled to \( \frac{1}{3\theta_1} - \frac{\theta_{01}}{2\theta_2 \theta_1} \). It is clear the distribution of \( \alpha_{\theta} (x, \varepsilon_n) \) is identified.

Suppose the data is generated from a random sample of \( \{a_{1n}, a_{2n}, x\}_{n=1}^{N} \), where \( a_{in} = \alpha_{\theta_0} (x_n, \varepsilon_{in}) \) for \( i = 1, 2 \) and every \( n \), for some \( \theta_0 = (\theta_{01}, \theta_{02}) \in \mathbb{R}^+ \times \mathbb{R}^+ \). To study whether additive perturbations can be used to construct objective functions that identify \( \theta_0 \), we consider \( u_{1,\theta} (a_1 + \eta, \gamma_{\theta_0} (x), x, \varepsilon_1) \) for some \( \eta \). Through some tedious algebra, it can be shown that
\[
u_{1,\theta} (a_1 + \eta, \gamma_{\theta_0} (x), x, \varepsilon_1) = u_{1,\theta} (a_1, \gamma_{\theta_0} (x), x, \varepsilon_1) + \eta (x - \theta_1 x \gamma_{\theta_0} (x) - \theta_2 \varepsilon) - \theta_1 x (2a_1 \eta + \eta^2).
\]

Comparing the expected returns from using the optimal strategy and a perturbed one,
\[
E \left[ u_{1,\theta} (\alpha_{\theta} (s_{1n}), \gamma_{\theta_0} (x_n), s_{1n}) \mid x_n = x \right] = E \left[ u_{1,\theta} (\alpha_{\theta_0} (s_{1n}) + \eta, \gamma_{\theta_0} (x_n), s_{1n}) \mid x_n = x \right]
\]
\[
= -\eta x + \theta_1 x (2\gamma_{\theta_0} (x) \eta + \eta^2)
\]
\[
= -\eta x \left( 1 - \frac{\theta_1}{\theta_{01}} \right) + \theta_1 x \eta^2.
\]

Clearly \( \theta' = (\theta_{01}, \theta_{02}') \) satisfies the necessary condition implied by the equilibrium for all values of \( \theta_{02}' \). Therefore the objective functions constructed using additive perturbations cannot identify \( \theta_{02} \) in the limit. Next, we consider the multiplicative perturbation. For the calculations, it shall be convenient to write the multiplicative factor as \( (1 + \eta) \). Then, it can be shown that
\[
u_{1,\theta} (a_1 (1 + \eta), \gamma_{\theta_0} (x), s_1) = u_{1,\theta} (a_1, \gamma_{\theta_0} (x), s_1) + \eta a_1 (x - \theta_1 x \gamma_{\theta_0} (x) - \theta_2 \varepsilon) - \theta_1 x (2\eta + \eta^2) a_1^2.
\]

Taking conditional expectation and compare the expected returns,
\[
E \left[ u_{1,\theta} (\alpha_{\theta_0} (s_{1n}), \gamma_{\theta_0} (x_n), s_{1n}) \mid x_n = x \right] = E \left[ u_{1,\theta} (\alpha_{\theta_0} (s_{1n}) + \eta, \gamma_{\theta_0} (x_n), s_{1n}) \mid x_n = x \right]
\]
\[
= -\eta \frac{x}{3} \delta_1 + \frac{\theta_{02}}{2x} \left( \delta_1 \theta_{02} + \delta_2 \right) + \theta_1 x \eta^2 \left( \frac{1}{3\theta_{01}} + \frac{\theta_{02}}{2\theta_{01} x} \right),
\]

where \( \delta_1 = 1 - \frac{\theta_{01}}{\theta_{02}} \) and \( \delta_2 = \theta_2 - \theta_{02} \). For any \( \delta_1, \delta_2 \neq 0 \), with a small enough \( |\eta| \), the squared (second) term above is of smaller order and the first term will be strictly negative for some state \( x \) with either \( \eta > 0 \) or \( \eta < 0 \). Therefore we expect \( \{ \alpha (\cdot; \eta) \mid \eta \in \mathbb{S} : \alpha (s_i; \eta) = \eta \alpha_{\theta_0} (s_i) \text{ for all } s_i \in S_i \} \) to be able to preserve the identifying information of \( \theta_0 \) when \( \mathbb{S} \) contains an open ball centered at 1.
A.2 Perturbations for Discrete Action Games

We first consider a binary action game that satisfy Assumptions M1, M2, M3 and D’ in Section 2, where D’ is the parameterized version of D that replaces \( u_i \) everywhere with \( u_{i, \theta} \). To keep the calculation of the expected returns tractable, we only use the class of alternative strategies where players only deviate from the equilibrium action in the first stage; BBL (see p.1348) also suggest this amongst other ways to construct inequalities. In particular, we can therefore adopt the framework of the pseudo-model constructed in Section 3.1. Suppose the data \( \{(a_{in}, a_{-in}, x_n, \epsilon_n)\}_{n=1}^{N} \) are generated from a pure strategy Markov equilibrium when \( \theta = \theta_0 \). In the limit, the pseudo-objective function (see equation (8)) is

\[
\Lambda_{i, \theta}(a_i, x, \epsilon_i) = E[u_{i, \theta}(a_i, a_{-in}, x_n, \epsilon_i) | x_n = x] + \beta_i g_{i, \theta}(a_i, x) = v_{i, \theta}(a_i, x) + \epsilon_i(a_i),
\]

where \( v_{i, \theta}(a_i, x) = E[\pi_{i, \theta}(a_i, a_{-in}, x_n) | x_n = x] + \beta_i g_{i, \theta}(a_i, x) \); PSD calls \( v_{i, \theta} \) the continuation value net of the pay-off shocks. Since we only focus on identification \( v_{i, \theta} \) is taken as known, conditions for consistent estimation of \( v_{i, \theta} \) and other details can be found in AM and PSD. It shall also be convenient to define the differences between the choice specific continuation values and private values.

Let \( \Delta \Lambda_{i, \theta}(a_i, a_i', s_i) = \Lambda_{i, \theta}(a_i, s_i) - \Lambda_{i, \theta}(a_i', s_i) \), and also let \( \Delta v_{i, \theta}(x) = v_{i, \theta}(1, x) - v_{i, \theta}(0, x) \) and \( \omega_{in} = \epsilon_{in}(0) - \epsilon_{in}(1) \). Note that under Assumption D(iii) \( \omega_{in} \) is absolutely continuous with respect to the Lebesgue measure with support on \( \mathbb{R} \). The pseudo-best response is characterized by a cut-off rule:

\[
\alpha_{i, \theta}(s_{in}) = 1[\Delta v_{i, \theta}(x_n) > \omega_{in}] \text{ a.s. for all } \theta \in \Theta \text{ and } i = 1, \ldots, I.
\]

Then \( \Delta v_{i, \theta_0}(x) \) is identified from \( Q_{\omega_i}^{-1}(P_i(1|x)) \), where \( P_i(1|x) \) denotes the underlying equilibrium choice probability of choosing action 1 and \( Q_{\omega_i}^{-1} \) is the inverse of the distribution function of \( \omega_{in} \).

We assume \( \theta_0 \) is identified (see Definition 3 in Section 4.1). And we claim that a class of alternative strategies that consists of perturbing the cut-off values has sufficient identifying power for \( \theta_0 \). More formally let \( \mathfrak{A}_i(U) (S) = \{\tilde{\alpha}_i(\cdot; \eta) \text{ for } \eta \in S : \tilde{\alpha}_i(s_i; \eta) = 1[\Delta v_{i, \theta_0}(x) + \eta > \omega_i] \text{ for all } s_i \in S_i \} \), then \( \mathfrak{A}_i(U) (S) \) has sufficient identifying power for \( \theta_0 \) in the sense that,

\[
E[\Lambda_{i, \theta}(\alpha_{i, \theta_0}(s_{in}), s_{in}) | x_n = x] \geq E[\Lambda_{i, \theta}(\tilde{\alpha}_i(s_{in}), s_{in}) | x_n = x] \text{ for all } i, x \text{ and } \tilde{\alpha}_i \in \mathfrak{A}_i(U) (S) \text{ iff } \theta = \theta_0,
\]

for some appropriate \( S \). To see this, we first show whenever \( \theta \neq \theta_0 \), we can find some \( i, s_i \) and \( \eta \) such that \( \Delta \Lambda_{i, \theta}(\alpha_{i, \theta_0}(s_i), \tilde{\alpha}_i(s_i; \eta), s_i) < 0 \).

Since \( \theta_0 \) is identified, for any \( \theta \neq \theta_0 \) there exists some \( i, x \) and \( \xi \neq 0 \) such that \( \Delta v_{i, \theta}(x) = \).
\( \Delta v_{i, \theta_0} (x) + \xi \). Suppose \( \xi > 0 \), then any \( \eta \in (0, \xi) \) implies

\[
\Delta \Lambda_{i, \theta} (\alpha_{i, \theta_0} (s_i), \tilde{\alpha}_i (s_i; \eta), s_i) = \begin{cases} 
-(\Delta v_{i, \theta} (x) - \omega_i) < 0 & \text{for } \omega_i \in (\Delta v_{i, \theta_0} (x), \Delta v_{i, \theta_0} (x) + \eta) \\
0 & \text{otherwise}
\end{cases}
\]

By an analogous argument, when \( \xi < 0 \), choosing any \( \eta \in (\xi, 0) \) implies that \( \Delta \Lambda_{i, \theta} (\alpha_{i, \theta_0} (s_i), \tilde{\alpha}_i (s_i; \eta), s_i) \) takes strictly negative values for all \( \omega_i \in (\Delta v_{i, \theta_0} (x) + \eta, \Delta v_{i, \theta_0} (x)) \), and 0 otherwise. Since \( \omega_{in} \) has a continuous distribution on \( \mathbb{R} \), \( E [\Delta \Lambda_{i, \theta} (\alpha_{i, \theta_0} (s_{in}), \tilde{\alpha}_i (s_{in}; \eta), s_{in}) | x_n = x] < 0 \) for all \( \eta \) on either set \((-\xi, 0)\) or \((0, \xi)\) with small enough \( \xi > 0 \). Therefore the class of perturbations at the cut-off value has sufficient identifying power for \( \theta_0 \) if \( S \) contains any open ball that is centred at 0.

Although we do not provide any formal details, due to non-trivial additional notational complexity, an analogous idea can be used for multinomial action games. Suppose \( K_i = K \) for all \( i \). Then the optimality condition for the \((K + 1)\)-choice problem can be characterized, for each player and state, by \( K \) inequality constraints that partition \( \mathbb{R}^K \), the support of the normalized private values. The role of a cut-off value is then replaced by a locus point in \( \mathbb{R}^K \), which is uniquely identified by the inversion result of Hotz and Miller (1993) subject to the choice of a normalization action. Then analogous alternative strategies can be constructed by perturbation the locus point by a \( K \)-dimensional variables whose support includes a ball in \( \mathbb{R}^K \) that contains the origin.

The intuition used in the unordered binary action game can also be applied to the class of discrete monotone action games. Specifically, we now assume \( M1, M2, M3, S1', S2 \) and \( S3' \) and let the data \( \{(a_{in}, a_{-in}, x_n)\}_{n=1}^N \) be generated from a pure strategy Markov equilibrium when \( \theta = \theta_0 \). Recall that \( \alpha_{i, \theta} (x, \cdot) \) is a nondecreasing function on \( \mathcal{E}_i \) (by the arguments of Lemmas 1 and 2). For notational simplicity suppose that \( A_i = \{0, 1\} \) for all \( i \). Then the pseudo-best response is uniquely characterized by a cut-off rule:

\[
\alpha_{i, \theta} (s_{in}) = 1 \{C_{i, \theta} (x_n) \geq \varepsilon_{in}\} \text{ a.s. for all } \theta \in \Theta \text{ and } i = 1, \ldots, I,
\]

for some \( C_{i, \theta} \) such that \( \varepsilon_i \leq C_{i, \theta} (x) \leq \varepsilon_i \) for all \( i, x, \theta \). In particular, when \( \Pr [\alpha_{i, \theta} (s_{in}) = 1 | x_n = x] = 0 \), set \( C_{i, \theta} (x) = \varepsilon_i \), and when \( \Pr [\alpha_{i, \theta} (s_{in}) = 1 | x_n = x] = 1 \), set \( C_{i, \theta} (x) = \varepsilon_i \). As seen previously, \( C_{i, \theta_0} (x) \) is identified by \( Q_i^{-1} (P_i (1|x)) \) where \( Q_i^{-1} \) denotes the inverse of the distribution function of \( \varepsilon_{in} \). If \( \theta_0 \) is identified then the following class of alternative strategies \( \mathcal{R}_i^Q = \{\tilde{\alpha}_i (:; \eta) \in \mathcal{S} : \tilde{\alpha}_i (s_i; \eta) = 1 \{C_{i, \theta_0} (x) + \eta > \varepsilon_i\} \text{ for all } s_i \in S_i\} \), has sufficient identifying power for \( \theta_0 \) in the sense described in equation (SA2). When there are more than two actions, suppose \( K_i = K \) for all \( i \), then the data generating best response is generally characterized by \( K - 1 \) boundary points on \( \mathcal{E}_i \) for each player and state. These boundary points can be identified from \( F_i \) and \( Q_i \). Since \( \mathcal{E}_i \subseteq \mathbb{R} \), a simple way to apply the same technique used in binary action games above is to choose the set of alternative strategies that perturb only one of boundary points at a time and leave all other boundary points the same as those identified by the data.
A.3 A Discussion

The inequality moment restrictions imposed by the equilibrium condition considered in BBL is indexed by a class of functions of alternative strategies. Our examples in A.1 illustrate a general point that some alternative strategies may have no identifying information for a subset of the parameter of interest (or the entire parameter space in some cases). In contrast to the examples in Domínguez and Lobato (2004), objective functions constructed from certain classes of alternative strategies do not only lack global identification, i.e. does not have a unique optimum, they cannot even distinguish between different parameters locally. We only provide an example when the inequality approach suggested by BBL can fail for a point-identified model (most known applications of their methodology proceed under this assumption). Although BBL also suggest a set estimator for partially identified models. It is intuitively clear their set estimation approach is exposed to the same criticism above, in which case some classes of inequalities may only be able to identify a strict superset of the identified set.

We consider dynamic games in A.2. We focus on alternative strategies where each player only deviates in the first stage since it provides a more tractable starting point to study identification. It enables us to show that when the parameter is identified in binary action games, inequalities generated from perturbing the cut-off values preserves the identifying information. We also explain how such technique can be applied to multinomial choice games as well as discrete games where players play monotone strategies. However, it is clearly impractical to extend the suggested perturbation method for discrete action games to a continuous action one.

Finally, all of our analytical arguments above only apply to the limiting case where equilibrium and alternative strategies are perfectly known and there are no simulation errors. As the Monte Carlo study in Section 5 shows, it is always possible to obtain an estimate in finite sample even when the objective function cannot identify the parameter of interest in the limit. Our main message is the choice of alternative strategies, which can be viewed as tuning parameters, is very important since it affects not only efficiency but also consistency. It remains an interesting issue to find some sufficiency theory for choosing inequalities in a continuous action game.

Appendix B Proofs of Theorems

Since the first stage estimators are defined implicitly in our objective function \( \hat{M}_N (\theta) \), it suffices to show Assumptions A1 and A2 imply some familiar conditions from large sample theorems for parametric estimators. For Theorem 1, we make use of a well-known consistency result for extremum estimators; for instance, see Theorem 2.1 of Newey and McFadden (1994). For Theorem 2, we show
A1 and A2 are sufficient for the conditions of Theorem 7.1 of Newey and McFadden (1994), who provide a high level condition for the asymptotic normality of an extremum estimator that maximizes a non-smooth objective function.

**Proof of Theorem 1.** Under A1(i) \( \Theta \) is compact. A1(ii) - A1(iv) ensure that \( M(\theta) \) has a well-separated minimum at \( \theta_0 \). Next, we show the sample objective function converges uniformly in probability to its limit. By triangle inequality,

\[
\left| \hat{M}_N(\theta) - M(\theta) \right| \leq 4 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left| \hat{F}_{i,\theta}(a_i|x) - \hat{F}_{i,\theta}(a_i|x) \right| \mu_{i,x}(da_i)
+ 4 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left| \hat{F}_{i,\theta}(a_i|x) - F_{i,\theta}(a_i|x) \right| \mu_{i,x}(da_i)
+ 4 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left| \hat{F}_{i}(a_i|x) - F_{i}(a_i|x) \right| \mu_{i,x}(da_i)
\]

asymptotically since distribution functions are bounded above by 1, and \( \hat{F}_{i,\theta}, \hat{F}_{i,\theta} \) are uniformly consistent under A1(v) - A1(vii). Under A1(iv) the measures are finite, hence \( \sup_{\theta \in \Theta} \left| \hat{M}_N(\theta) - M(\theta) \right| = o_p(1) \) by A1(v) - A1(vii). Consistency then follows by a standard argument.

**Proof of Theorem 2.** Conditions (i) - (iii) of Newey and McFadden (1994, Theorem 7.1) are trivially satisfied by the definition of our estimator and conditions A2(i) and A2(ii). It remains to show that there exists a sequence \( C_N \) that has an asymptotic normality of the root-\( N \) rate, which satisfies the following (stochastic differentiability) condition,

\[
\sup_{|\theta - \theta_0| < \delta_N} \left| \frac{\mathcal{D}_N(\theta)}{1 + \sqrt{N} \|\theta - \theta_0\|} \right| = o_p(1)
\]

for any positive sequence \( \delta_N = o(1) \), where

\[
\mathcal{D}_N(\theta) = \sqrt{N} \frac{\hat{M}_N(\theta) - \hat{M}_N(\theta_0) - (M(\theta) - M(\theta_0)) - (\theta - \theta_0)^\top C_N}{\|\theta - \theta_0\|}
\]

We shall show

\[
\hat{M}_N(\theta) - \hat{M}_N(\theta_0) - \left( \hat{M}(\theta) - \hat{M}(\theta_0) \right) - (\theta - \theta_0)^\top C_N = o_p \left( \|\theta - \theta_0\|^2 + \frac{\|\theta - \theta_0\|}{\sqrt{N}} + \frac{1}{N} \right) \tag{SA3}
\]

holds uniformly for \( \|\theta - \theta_0\| \leq \delta_N \). The additional \( o_p(N^{-1}) \) term, added in (SA3), does not affect Newey and McFadden’s results as it is the rate that our estimator (approximately) minimizes the objective function, which coincides with condition (i) of their theorem.

For \( \theta \) in a neighborhood of \( \theta_0 \), we write \( \hat{M}_N(\theta) - \hat{M}_N(\theta_0) - (M(\theta) - M(\theta_0)) \) as a sum, \( E_1(\theta) + E_2(\theta) \), where

\[
E_1(\theta) = M_N(\theta) - M_N(\theta_0) - (M(\theta) - M(\theta_0)),
E_2(\theta) = \hat{M}_N(\theta) - \hat{M}_N(\theta_0) - (M_N(\theta) - M_N(\theta_0)).
\]
Under A2(ii) $M_N$ and $M$ are twice continuously differentiable in a neighborhood of $\theta_0$. By Taylor’s theorem

$$E_1 (\theta) = (\theta - \theta_0)^\top \frac{\partial}{\partial \theta} M_N (\theta_0) + \frac{1}{2} (\theta - \theta_0)^\top \frac{\partial^2}{\partial \theta \partial \theta^\top} (M_N (\overline{\theta}) - M(\overline{\theta})) (\theta - \theta_0)$$

for some mean value functions $\overline{\theta}, \overline{\theta}$ that depend on $(i, a_i, x)$. Note that $\frac{\partial}{\partial \theta} M (\theta)$ vanishes when $\theta = \theta_0$ under A1(ii). For $\frac{\partial}{\partial \theta} M_N (\theta_0)$, we have

$$\frac{\partial}{\partial \theta} M_N (\theta_0) = 2 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \frac{\partial}{\partial \theta} \hat{F}_{i, \theta_0} (a_i | x) \left( \hat{F}_{i, \theta_0} (a_i | x) - \hat{F}_i (a_i | x) \right) \mu_{i,x} (da_i)$$

$$= 2 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \frac{\partial}{\partial \theta} F_{i, \theta_0} (a_i | x) \left( \hat{F}_{i, \theta_0} (a_i | x) - \hat{F}_i (a_i | x) \right) \mu_{i,x} (da_i) + o_p \left( \frac{1}{\sqrt{N}} \right),$$

where the second equality follows from finiteness of $\{ \mu_{i,x} \}_{i \in I, x \in X}$. A2(ii), A2(iv) and A2(ix). Importantly, by A2(ix) and the continuous mapping theorem $\sqrt{N} \frac{\partial}{\partial \theta} M_N (\theta_0) \Rightarrow N (0, \mathcal{V})$, where $\mathcal{V}$ is defined in (17). For the Hessians of $M_N$ and $M$:

$$\frac{\partial^2}{\partial \theta \partial \theta^\top} M_N (\theta) = \sum_{i \in I} \sum_{x \in X} \int_{A_i} \frac{\partial^2}{\partial \theta \partial \theta^\top} \hat{F}_{i, \theta} (a_i | x) \left( \hat{F}_{i, \theta} (a_i | x) - \hat{F}_i (a_i | x) \right) \mu_{i,x} (da_i)$$

$$+ \sum_{i \in I} \sum_{x \in X} \int_{A_i} \frac{\partial}{\partial \theta} \hat{F}_{i, \theta} (a_i | x) \frac{\partial}{\partial \theta} \hat{F}_{i, \theta} (a_i | x) \mu_{i,x} (da_i),$$

$$\frac{\partial^2}{\partial \theta \partial \theta^\top} M (\theta) = \sum_{i \in I} \sum_{x \in X} \int_{A_i} \frac{\partial^2}{\partial \theta \partial \theta^\top} F_{i, \theta} (a_i | x) (F_{i, \theta} (a_i | x) - F_i (a_i | x)) \mu_{i,x} (da_i)$$

$$+ \sum_{i \in I} \sum_{x \in X} \int_{A_i} \frac{\partial}{\partial \theta} F_{i, \theta} (a_i | x) \frac{\partial}{\partial \theta} F_{i, \theta} (a_i | x) \mu_{i,x} (da_i).$$

By repeated applications of the triangle inequality, and making use of A2(ii), A2(iv) and A2(v), it is straightforward to show that $\left| \frac{\partial^2}{\partial \theta \partial \theta^\top} (M_N (\overline{\theta}) - M(\overline{\theta})) \right| = o_p (1)$ for all $(l, l')$ as $\| \theta - \theta_0 \| \to 0$. Therefore we have

$$E_1 (\theta) = (\theta - \theta_0)^\top \frac{\partial}{\partial \theta} M_N (\theta_0) + o_p (\| \theta - \theta_0 \|^2).$$

Let $\xi (\theta) = \widehat{M}_N (\theta) - M_N (\theta)$, so that $E_2 (\theta) = \xi (\theta) - \xi (\theta_0)$. From the definitions of $\widehat{M}_N$ and $M_N$,

$$\xi (\theta) = \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( \hat{F}_{i, \theta} (a_i | x) - \hat{F}_i (a_i | x) \right) \left( \hat{F}_{i, \theta} (a_i | x) + \hat{F}_i (a_i | x) - 2 \hat{F}_i (a_i | x) \right) \mu_{i,x} (da_i).$$

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By repeatedly adding nulls, we can write

\[ \xi (\theta) = \xi_1 (\theta) + \xi_2 (\theta) + \xi_3 (\theta) + \xi_4 (\theta), \]

where

\[ \xi_1 (\theta) = \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( \hat{F}_{i,\theta} (a_i|x) - \hat{F}_{i,\theta} (a_i|x) \right)^2 \mu_{i,x} \, (da_i), \]

\[ \xi_2 (\theta) = 2 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( \hat{F}_{i,\theta} (a_i|x) - \hat{F}_{i,\theta} (a_i|x) \right) \left( \hat{F}_{i,\theta} (a_i|x) - \hat{F}_{i,\theta} (a_i|x) \right) \mu_{i,x} \, (da_i), \]

\[ \xi_3 (\theta) = 2 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( \hat{F}_{i,\theta} (a_i|x) - \hat{F}_{i,\theta} (a_i|x) \right) \left( \hat{F}_{i,\theta} (a_i|x) - \hat{F}_{i,\theta} (a_i|x) \right) \mu_{i,x} \, (da_i), \]

\[ \xi_4 (\theta) = -2 \sum_{i \in I} \sum_{x \in X} \int_{A_i} \left( \hat{F}_{i,\theta} (a_i|x) - \hat{F}_{i,\theta} (a_i|x) \right) \left( \hat{F}_{i,\theta} (a_i|x) - \hat{F}_{i,\theta} (a_i|x) \right) \mu_{i,x} \, (da_i). \]

In sum $\xi (\theta)$ is $o_p(N^{-1/2} \| \theta - \theta_0 \| + N^{-1})$ since: $\xi_1 (\theta)$ is $o_p(N^{-1})$ by A2(vi); $\xi_2 (\theta)$ is $o_p(N^{-1/2} \| \theta - \theta_0 \|)$, use a mean value expansion in $\theta$ then apply A2(ii), A2(iv) and A2(vi); $\xi_3 (\theta)$ is $o_p(N^{-1})$ by A2(iv) and A2(vii); $\xi_4 (\theta)$ is $o_p(N^{-1})$ by A2(vi) and A2(viii). Therefore $E_2 (\theta) = o_p(N^{-1/2} \| \theta - \theta_0 \| + N^{-1})$. Thus, condition (SA3) is satisfied uniformly for $\| \theta - \theta_0 \| \leq \delta_N$ with $C_N = \frac{\partial}{\partial \theta} M_N (\theta_0)$. Since $\frac{\partial^2}{\partial \theta^2} M (\theta_0)$ equals $W$ (defined in equation (18)), the desired limiting distribution of $\hat{\theta}$ follows from applying Theorem 7.1 of Newey and McFadden (1994).
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