Dissipative Kerr solitons at the edge state of the Su-Schrieffer–Heeger model

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Abstract. We investigate analytically and numerically dynamics of dissipative Kerr solitons (DKS) at the edge state of the Su-Schrieffer–Heeger model. We demonstrate that four-wave mixing processes can lead to the formation of DKSs in the edge state of the resonator chain which subsequently initiates photon transfer to the bulk states. We discuss how the edge state soliton can be stabilized by limiting its width within the band gap. Our results contribute to advanced dispersion engineering via mode hybridization in chains of resonators — one of promising ways to achieve broadband frequency combs generation on chip.

1. Introduction
Photonics often serves as a primary platform for observation of effects found in various areas Physics ranging from materials in the solid-state to Bose–Einstein condensate, etc. Topological states, initially found in solids, have triggered in photonics a remarkable interest due to fundamental aspects as well as to their potential applications [1]. However, up to now the majority of studies have considered linear and single (or few) mode(-s) operation regime of photonic devices. In this work we theoretically analyze the nonlinear dynamics of the edge state in the Su-Schrieffer–Heeger model [2], constructed of dimerized optical high-Q microcavities with Kerr nonlinearity. On the single resonator level, the four-wave mixing processes lead to the generation of coherent and broadband frequency combs [3] which are related to the formation of dissipative Kerr solitons (DKS) [3].

2. Homogeneously coupled resonators
In the simplest (and for now trivial, i.e. not topological) case — a chain of linearly coupled resonators with periodic boundary conditions — we assume the field envelope $A_\ell(\phi)$ of an uncoupled $\ell$-th resonator to be described by the Lugiato-Lefever equation (LLE) [3], and linear coupling to the neighbor resonators expressed by $j$. In the meanfield approximation, this system is described by the set of coupled LLEs [4] studied in details in Ref. [5]. Here we briefly recap the main features arising due to the nonlinear coupling between frequency modes of a single resonator $\omega_\mu$ ($\mu$ is integer) and spatial supermodes $k$. Introducing the collective excitation amplitude $\psi_{\mu k}$ which is connected to the field envelope amplitudes $A_\ell(\phi)$ via the Fourier transform

$$\psi_{\mu k} = \frac{1}{2\pi \sqrt{N}} \int \sum_{\ell=1}^{N} A_\ell e^{i(\ell k/N + \mu \phi)} d\phi.$$
we obtain the governing equation for $\psi_{\mu k}$:

$$\frac{\partial \psi_{\mu k}}{\partial \tau} = -(1 + i \zeta_0) \psi_{\mu k} - i \left[d_2 \mu^2 - 2j \cos \frac{2\pi k}{N}\right] \psi_{\mu k} + \frac{i}{N} \sum_{k_1,k_2,k_3} \psi_{\mu_1 k_1} \psi_{\mu_2 k_2} \psi_{\mu_3 k_3} \delta_{\mu_1 + \mu_2 - \mu_3 - \mu} \delta_{k_1 + k_2 - k_3 = k} + \delta_{k,k_0} \delta_{\mu,0} \tilde{f}_{k_0},$$

where $\mu$ is the comb index, $k$ is the spatial supermode index, $\zeta_0$ stands for normalized laser detuning, $d_2$ is normalized group velocity dispersion ($d_2 > 0$ corresponds to anomalous dispersion case), $j$ is normalized intraresonator coupling, $\tilde{f}_{k_0}$ is normalized pump to given supermode $k_0$. The term in square brackets depicts two-dimensional hybridized dispersion surface with different regions [e.g. anomalous-anomalous (anomalous-normal) in the vicinity $k = 0$ ($k_0 = N/2$)], the nonlinear term incorporates conservation laws in both frequency $\mu$ and spatial $k$ dimensions. Therefore, the four-wave mixing processes occur in $\mu$-$k$ space, leading to the effective two-dimensional dynamics on the system.

3. Staggered coupling

Now, we can perform the first step towards the SSH model. We need to open a gap in the supermode dispersion which is realized by dimerazing the chain. By creating unit cells made of two resonators, we introduce intra-cell $J_{\text{intra}}$ and inter-cell couplings $J_{\text{inter}}$ as shown in Fig. 1(a) keeping the periodic boundary conditions. Now, we can introduce collective excitation amplitudes $\partial \psi_{\mu k}$ and $\phi_{\mu k}$ for both sublattices

$$\psi_{\mu k} = \frac{1}{2\sqrt{N}} \int \sum_{\ell=1,3,5...} A_\ell e^{i(\ell k/N + \mu \varphi)} d\varphi, \quad \phi_{\mu k} = \frac{1}{2\sqrt{N}} \int \sum_{\ell=2,6,...} A_\ell e^{i(\ell k/N + \mu \varphi)} d\varphi,$$

where $N$ is the number of unit cells. In contrast to the previous case, now the dynamics is described by a set of coupled equations on $\psi_{\mu k}$, $\phi_{\mu k}$:

$$\frac{\partial \psi_{\mu k}}{\partial \tau} = -(1 + i \zeta_0) \psi_{\mu k} - i d_2 \mu^2 \psi_{\mu k} + i \left[J_{\text{intra}} + J_{\text{inter}} e^{-2\pi i k/N}\right] \phi_{\mu k} + \frac{i}{N} \sum_{k_1,k_2,k_3} \psi_{\mu_1 k_1} \psi_{\mu_2 k_2} \psi_{\mu_3 k_3} \delta_{\mu_1 + \mu_2 - \mu_3 - \mu} \delta_{k_1 + k_2 - k_3 = k} + \delta_{k,k_0} \delta_{\mu,0} \tilde{f}_{k_0},$$

$$\frac{\partial \phi_{\mu k}}{\partial \tau} = -(1 + i \zeta_0) \phi_{\mu k} - i d_2 \mu^2 \phi_{\mu k} + i \left[J_{\text{intra}} + J_{\text{inter}} e^{2\pi i k/N}\right] \psi_{\mu k} + \frac{i}{N} \sum_{k_1,k_2,k_3} \phi_{\mu_1 k_1} \phi_{\mu_2 k_2} \phi_{\mu_3 k_3} \delta_{\mu_1 + \mu_2 - \mu_3 - \mu} \delta_{k_1 + k_2 - k_3 = k} + \delta_{k,k_0} \delta_{\mu,0} \tilde{f}_{k_0}^2.$$
with corresponding single-mode band structure in Fig. 1(b) and spatial profile of the edge states in Fig. 1(c). Taking into account the frequency modes \( \omega_\mu \) and the group velocity dispersion, we obtain the hybridized dispersion to be of the form presented in Fig. 1(d). Since the edge parabolas are always at the center of the band gap, the four-wave mixing processes along \( k \) axes for fixed \( \mu \) are suppressed and one can expect the nonlinear dynamics to be similar to single resonator case, leading to soliton solution to be just a line under the edge parabolas [6]. However, once a soliton state is achieved, the dispersionless line under the edge parabola [black line in Fig. 1(d)] crosses the lower band and the phase matching condition between the edge and bulk modes can be satisfied, leading to presence of 2D dynamics. We investigate this effect numerically.

We consider a chain of 10 identical resonators with physical parameters accessible in experiments: internal linewidth \( \kappa_0/2\pi = 50 \text{ MHz} \), group velocity dispersion \( D_2/2\pi = 40 \text{ MHz} \), coupling coefficients \( J_{\text{intra}}/2\pi = 5 \text{ GHz} \), \( J_{\text{intra}}/2\pi = 1 \text{ GHz} \). We numerically solve the set of linearly coupled LLEs similar to [4] by scanning the resonance from blue to red detuned sides. For the pump power in the bus waveguide 800 mW we observe rich nonlinear dynamics presented in Fig. 2. In the transmission trace (Fig. 2(a)) one can see dynamics qualitatively similar to single resonator dynamics. Indeed, most of the optical power is concentrated in the first resonator, and the transmission traces represent typical regions: Turing rolls, chaotic MI, soliton steps. However, the soliton step has irregular deviations of power due to generation of dispersive waves whose presence can be seen in other resonators. The corresponding spatio-temporal diagrams indeed show that the solitons in the first resonator experience a random walk [Fig. 2(b)], which is accompanied by the generation of light in other resonators [Fig. 2(c,d)].

In order to understand the four-wave mixing processes with regard to the linear band structure, it is useful to reconstruct the nonlinear dispersion relation (NDR) [6]. NDR for a given resonator reveals presence of collective linear and nonlinear structures [5]. We investigate closely the soliton regime corresponding to the step in Fig. 2(a) for 1\textsuperscript{st}, 2\textsuperscript{nd} and 10\textsuperscript{th} resonators [Fig. 2(e-g)]. We observe ladder of tilted lines in the first resonator (Fig. 2(c)), which correspond to the breathing of the solitons and their random walk. In 2\textsuperscript{nd} and 10\textsuperscript{th} resonators we observe
Figure 2. Scan of the edge state. (a) Transmission trace of the edge state. (b-d) Spatio-temporal dynamics with increased detuning in 1st, 2nd and 10th resonators. (e-g) Nonlinear dispersion relation reconstruction of a soliton state.

the predicted effect: the lines cross the lower bulk parabolas, achieving efficient phase matching, giving rise to photon transfer to the corresponding supermodes. Moreover, since the parabolas are symmetric, and there is a huge number of modes in the bulk which can be crossed, the soliton line can be tilted in both directions, leading to soliton random walk at the spatio-temporal diagram [Fig. 2(b)].

5. Conclusion
We have instigated nonlinear dynamics and dissipative Kerr soliton formation at the edge state of the Su-Schrieffer–Heeger model. The edge state dynamics in this system is similar to the single resonator case, but the presence of the bulk modes can perturb the solitonic state. If the gap width is such that the soliton spectrum does not overlap with the bulk modes, the soliton stability will not be perturbed. Alternatively, this can be achieved by choosing a resonator with a larger group velocity dispersion. If the spectral overlap is possible, the soliton provides a photon transfer to the bulk parabolas whose dynamics has two-dimensional nature. Our results reveal insights in nonlinear interactions in the simplest topological model in the nonlinear and multimode case, that can be further used and generalized for more complex models.

References
[1] Lu L, Joannopoulos J D and Soljačić M 2014 Nature photonics 8 821–829
[2] Asbóth J K, Oroszlány L and Pályi A 2016 The Su-Schrieffer-Heeger (SSH) Model (Cham: Springer International Publishing) pp 1–22
[3] Kippenberg T J, Gaeta A L, Lipson M and Gorodetsky M L 2018 Science 361
[4] Tikan A, Riemensberger J, Komagata K, Hönl S, Churaev M, Skehan C, Guo H, Wang R N, Liu J, Seidler P et al. 2021 Nature Physics 1–7
[5] Tusnin A K, Tikan A M, Komagata K and Kippenberg T J 2021 arXiv preprint arXiv:2104.11731
[6] Leisman K P, Zhou D, Banks J W, Kovačić G and Cai D 2019 Phys. Rev. E 100(2) 022215