On the $S$-matrix of Schrödinger operator with nonlocal $\delta$-interaction

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Abstract. Schrödinger operators with nonlocal $\delta$-interaction are studied with the use of the Lax-Phillips scattering theory methods. The condition of applicability of the Lax-Phillips approach in terms of non-cyclic functions is established. Two formulas for the $S$-matrix are obtained. The first one deals with the Krein-Naimark resolvent formula and the Weyl-Titchmarsh function, whereas the second one is based on modified reflection and transmission coefficients. The $S$-matrix $S(z)$ is analytical in the lower half-plane $\mathbb{C}^-$ when the Schrödinger operator with nonlocal $\delta$-interaction is positive self-adjoint. Otherwise, $S(z)$ is a meromorphic matrix-valued function in $\mathbb{C}^-$ and its properties are closely related to the properties of the corresponding Schrödinger operator. Examples of $S$-matrices are given.

Keywords: Lax-Phillips scattering scheme, scattering matrix, $S$-matrix, nonlocal $\delta$-interaction, non-cyclic function

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1 Introduction

Theory of non self-adjoint operators attracts a steady interests in various fields of mathematics and physics, see, e.g., [7] and the reference therein. This interest grew considerably due to the recent progress in theoretical physics of pseudo-Hermitian Hamiltonians [5].

In the present paper we study non-self-adjoint Schrödinger operators with nonlocal point interaction. Self-adjoint operators have been investigated by Nizhnik et al. [4, 5, 6, 10]. The case of non-self-adjoint operators with nonlocal point interaction is more complicated and it requires more detailed analysis. One of the simplest models of a non-local $\delta$-interaction is

$$-\frac{d^2}{dx^2} + a <\delta, \cdot> \delta(x) + <\delta, \cdot> q(x) + (\cdot, q)\delta(x) \quad a \in \mathbb{C},$$

(1.1)

where $\delta$ is the delta-function, $q \in L_2(\mathbb{R})$, and $(\cdot, \cdot)$ is the inner product (linear in the first argument) in $L_2(\mathbb{R})$. The expression (1.1) determines the following
operator acting in $L_2(\mathbb{R})$:
\[ H_{aq}f = -\frac{d^2 f}{dx^2} + f(0)q(x), \quad (1.2) \]
\[ \mathcal{D}(H_{aq}) = \left\{ f \in W^2_2(\mathbb{R}) : f_s(0) = 0, \quad f'_s(0) = af_r(0) + (f, q) \right\} \quad (1.3) \]

where $f_s(0) = f(0^+) - f(0^-)$ and $f_r(0) = \frac{f(0^+) + f(0^-)}{2}$. The operator $H_{aq}$ is self-adjoint if and only if $a \in \mathbb{R}$ and it can be interpreted as a Hamiltonian corresponding to the non-local $\delta$-interaction (1.1). Setting $q = 0$, we obtain an operator $H_a := H_{a0}$ generated by the ordinary $\delta$-interaction $-\frac{d^2}{dx^2} + a \delta$.

The spectral analysis of non-self-adjoint $H_{aq}$ ($a \in \mathbb{C} \setminus \mathbb{R}$) was carried out in [21]. One of interesting features is that non-real $a$ determines the measure of non-self-adjointness of $H_{aq}$, while the function $q$ is responsible for the appearance of exceptional points and eigenvalues on continuous spectrum [21, Example 5.3 and Sec. 6].

In the present paper, we investigate $H_{aq}$ by the scattering theory methods. For the case $a = 0$, the scattering matrix $S(\delta)$ of $H_{0q}$ was constructed in [4, Sec. 5] with the use of modified Jost solutions. In contrast to [4] we study the general case $a \in \mathbb{C}$ with the use of an operator-theoretical interpretation of the Lax-Phillips approach in scattering theory [23] that was consistently developed in [12, 16, 18, 19]. We prefer this approach because it involves a simple algorithm for an explicit calculation of the analytic continuation of the scattering matrix into the lower half-plane $\mathbb{C}^-$. The paper is organized as follows. We begin with presentation of necessary facts about the Lax-Phillips scattering theory. Further, in Sec. 3 we analyze for which operators $H_{aq}$ one can apply the Lax-Phillips approach. For technical reasons it is convenient to work with unitary equivalent copies $H_{aq}$ of the operators $H_{aq}$ acting in the Hilbert space $L_2(\mathbb{R}, \mathbb{C}^2)$, see (3.2), (3.3). The main result (Theorem 3.3) implies that $H_{aq}$ can be investigated in framework of the Lax-Phillips theory under the condition that $q$ is non-cyclic with respect to the backward shift operator. For such kind of positive self-adjoint operators $H_{aq}$, two formulas of the analytical continuation $S(z)$ of the scattering matrix into the lower half-plane $\mathbb{C}^-$ are obtained in Sec. 4. The first one (4.8) deals with the Krein-Naimark resolvent formula (3.7) and the Weyl-Titchmarsh function (3.9), whereas the second one (4.19) is based on the modified reflection $R_z^i$ and the transmission $T_z^i$ coefficients that is more familiar for non-stationary scattering theory.

We mention that the relationship between scattering matrices and the extension theory subjects like Krein-Naimark formula and Weyl-Titchmarsh function

\textsuperscript{1}The most beautiful and important aspect of the Lax-Phillips approach is that certain analyticity properties of the scattering operator arise naturally [24, p.211]
was established for various cases [2, 9, 11] and it provides additional possibilities for the study of scattering systems.

In Sec 5, the formula (4.8) is used for the definition of $S$-matrix $S(z)$ for each operator $H_{aq}$ (assuming, of course, that $q$ is non-cyclic). If $H_{aq}$ is positive self-adjoint, then the $S$-matrix is the direct consequence of proper arguments of the Lax-Phillips theory and it coincides with the analytical continuation of the Lax-Phillips scattering matrix into $\mathbb{C}_-$. Otherwise, $S(z)$ defined by (4.8) is a meromorphic matrix-valued function in $\mathbb{C}_-$ and it can be considered as a characteristic function of $H_{aq}$. Lemmas 5.1-5.5 and Corollary 5.6 justify such a point of view by showing a close relationship between properties of non-self-adjoint $H_{aq}$ and theirs $S$-matrices. Examples of $S$-matrices for various non-cyclic $q$ are given in Sec. 5.1.

Throughout the paper, $\mathcal{D}(H)$, $\mathcal{R}(H)$, and ker $H$ denote the domain, the range, and the null-space of a linear operator $H$, respectively, whereas $H \mid \mathcal{D}$ stands for the restriction of $H$ to the set $\mathcal{D}$ and $\bigvee_{t \in \mathbb{R}} X_t$ means the closure of linear span of sets $X_t$. The symbol $H^2(\mathbb{C}_+)$, where $\mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im} \ z > 0 \}$ is used for the Hardy space. The Sobolev space is denoted as $W^p_2(I)$ ($I \in \{ \mathbb{R}, \mathbb{R}_+ \}$, $p \in \{ 1, 2 \}$).

2 Elements of Lax-Phillips scattering theory

Here all necessary results about the Lax-Phillips scattering theory are presented. The monographs [23], [20, Chap. III] and the papers [16, 19] are recommended as complementary reading on the subject.

2.1 Applicability of the Lax-Phillips scattering approach

A continuous group of unitary operators $W(t)$ acting in a Hilbert space $\mathfrak{H}$ is a subject of the Lax-Phillips scattering theory [23] if there exist so-called incoming $D_-$ and outgoing $D_+$ subspaces of $\mathfrak{H}$ with properties:

(i) $W(t)D_+ \subset D_+$, $W(-t)D_- \subset D_-$, $t \geq 0$;
(ii) $\bigcap_{t > 0} W(t)D_+ = \bigcap_{t > 0} W(-t)D_- = \{ 0 \}$.

Conditions (i) – (ii) allow to construct incoming and outgoing spectral representations for the restrictions of $W(t)$ onto the subspaces

$$
M_- = \bigvee_{t \in \mathbb{R}} W(t)D_- \quad \text{and} \quad M_+ = \bigvee_{t \in \mathbb{R}} W(t)D_+,
$$

(2.1)

respectively and define the corresponding Lax-Phillips scattering matrix $S(\delta)$ ($\delta \in \mathbb{R}$) whose values are contraction operators [1], [20, Chap. 3]. Furthermore, the additional condition of orthogonality

(iii) $D_- \perp D_+$

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guarantees that $S(\delta)$ is the boundary value of a contracting operator-valued function $S(z)$ holomorphic in the lower half-plane $\mathbb{C}_-$. 

Usually, the Lax-Phillips scattering matrix is defined with the use of an operator-differential equation

\[
\frac{d^2}{dt^2}u = -Hu,
\]

where $H$ is a positive self-adjoint operator in a Hilbert space $\mathcal{H}$. Denote by $\mathcal{H}_H$ the completion of $\mathcal{D}(H)$ with respect to the norm $\|\cdot\|_H := (H\cdot, \cdot)$.

The Cauchy problem for (2.2) determines a continuous group of unitary operators $W(t)$ in the space

\[
\mathcal{W} = \mathcal{H}_H \oplus \mathcal{H} = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : u \in \mathcal{H}_H, \ v \in \mathcal{H} \right\}.
\]

If $H = -\Delta$ and $\mathcal{H} = L^2(\mathbb{R}^n)$, then (2.2) coincides with the wave equation $u_{tt} = \Delta u$ and the corresponding subspaces $D_\pm$ constructed in [23] possess the additional property

\[
J D_- = D_+,
\]

where $J$ is a self-adjoint and unitary operator in $\mathcal{W}$ (so-called time-reversal operator):

\[
J \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ -v \end{bmatrix}.
\]

Relation (2.3) is a characteristic property of dynamics governed by wave equations.

It is clear that, the existence of subspaces $D_\pm$ for $W(t)$ is determined by specific properties of $H$ in (2.2). Before explaining which properties of $H$ are needed, we recall that a symmetric operator $B$ is called simple if its restriction on any nontrivial reducing subspace is not a self-adjoint operator. The maximality of $B$ means that there are no symmetric extensions of $B$. The latter is equivalent to the fact that one of defect numbers of $B$ is equal to zero. In what follows, without loss of generality, we assume that $B$ has zero defect number in $\mathbb{C}_+$, i.e.,

\[
\dim \ker(B^* - \mu I) = 0, \quad \mu \in \mathbb{C}_-.
\]

**Theorem 2.1.** [13, 20] Let $H$ be a positive self-adjoint operator in a Hilbert space $\mathcal{H}$. The following are equivalent:

(i) the group $W(t)$ of solutions of the Cauchy problem of (2.2) has subspaces $D_\pm$ with properties (i) -- (iii) and (2.3);

(ii) there exists a simple maximal symmetric operator $B$ acting in a subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $H$ is an extension (with exit in the space $\mathcal{H}$) of the symmetric operator $B^2$. 

\footnote{i.e. $(Hf, f) > 0$ for nonzero $f \in \mathcal{D}(H)$}
2.2 The Lax-Phillips scattering matrix and its analytical continuation

By Theorem 2.1, the unitary group $W(t)$ can be investigated by the Lax-Phillips scattering methods if and only if $\mathcal{H}$ is an extension of a symmetric operator $B^2$ acting in a subspace $\mathcal{H}_0$ of $\mathcal{H}$. A simple maximal symmetric operator $B$ in Theorem 2.1 turns out to be a useful technical tool allowing one to exhibit principal parts of the Lax-Phillips theory in a simple form. In particular, the subspaces $D_{\pm}$ coincide with the closure of the sets:

\[ \left\{ \begin{bmatrix} u \\ iBu \end{bmatrix} \mid \forall u \in \mathcal{D}(B^2) \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} u \\ -iBu \end{bmatrix} \mid \forall u \in \mathcal{D}(B^2) \right\}, \quad (2.6) \]

respectively. Moreover, for all $t \geq 0$,

\[ W(t) \begin{bmatrix} u \\ iBu \end{bmatrix} = \begin{bmatrix} V(t)u \\ iBV(t)u \end{bmatrix}, \quad W(-t) \begin{bmatrix} u \\ -iBu \end{bmatrix} = \begin{bmatrix} V(t)u \\ -iBV(t)u \end{bmatrix}, \quad (2.7) \]

where $V(t) = e^{iBt}$ is a semigroup of isometric operators in $\mathcal{H}_0$.

The formulas (2.1), (2.6), and (2.7) allow one to construct the incoming/outgoing spectral representations for the restrictions of $W(t)$ onto $M_{\pm}$ in an explicit form \[14, \text{Sec. 2.1}\]. The latter leads to a simple method for the calculation of the Lax-Phillips scattering matrix $S(\cdot)$ \[12, 18\]. Actually, we need only a positive boundary triplet \[4, (\mathcal{H}, \Gamma_0, \Gamma_1)\] of $B^* B$ defined as follows: denote $\mathcal{H} = \ker(B^* B + I)$, then $\mathcal{D}(B^* B) = \mathcal{D}(B^* B)^{\perp} \mathcal{H}$ and each vector $f \in \mathcal{D}(B^* B)$ can be decomposed:

\[ f = u + h, \quad u \in \mathcal{D}(B^* B), \quad h \in \mathcal{H}. \quad (2.8) \]

The formula (2.8) allows to define the linear mappings $\Gamma_i : \mathcal{D}(B^* B) \rightarrow \mathcal{H}$

\[ \Gamma_0 f = \Gamma_0 (u + h) = h, \quad \Gamma_1 f = \Gamma_1 (u + h) = P_\mathcal{H}(B^* B + I)u, \quad (2.9) \]

where $P_\mathcal{H}$ is the orthogonal projector of $\mathcal{H}_0$ onto the subspace $\mathcal{H}$.

**Theorem 2.2** \[12, 18\]. If conditions of Theorem 2.1 hold, then the Lax-Phillips scattering matrix $S(\cdot)$ for the unitary group $W(t)$ of Cauchy problem solutions of (2.2) has the following analytical continuation into $\mathbb{C}_-$:

\[ S(z) = [I - 2(1 + iz)C(z)][I - 2(1 - iz)C(z)]^{-1}, \quad z \in \mathbb{C}_-, \quad (2.10) \]

where the operators $C(z) : \mathcal{H} \rightarrow \mathcal{H}$ are determined by the relation

\[ C(z)\Gamma_1 u = \Gamma_0 u, \quad u \in P_{\mathcal{H}_0}(H - z^2 I)^{-1} \ker(B^* + \tau I), \quad z \in \mathbb{C}_-. \quad (2.11) \]

An investigation of $C(z)$ carried out in \[18\] shows that the values of $S(z)$ are contraction operators in $\mathcal{H}$ and $S^*(z) = S(-\tau)$. 

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3in the space $\mathcal{H}$
4see \[15\] Chap 3 for definition of boundary triplets and positive boundary triplets
In what follows, the analytical continuation (2.10) of the Lax-Phillips scattering matrix will be called the $S$-matrix of the positive self-adjoint operator $H$ in (2.2). For this reason it is natural to ask: to what extend the $S$-matrix determines $H$?

We recall that a self-adjoint operator $H$ is called minimal if each subspace of $\mathcal{H} \ominus \mathcal{H}_0$ that reduces $H$ is trivial. Minimal self-adjoint extensions $H_1$ and $H_2$ of $B^2$ are called unitary equivalent if there exists an unitary operator $Z$ in $\mathcal{H}$ such that $ZH_1 = H_2Z$ and $Zf = f$ for all $f \in \mathcal{H}_0$.

It follows from [18] that the $S$-matrix determines a minimal positive self-adjoint extension $H$ of $B^2$ up to unitary equivalence.

**Remark 2.3.** Various approaches in non-stationary scattering theory are based on the comparing of two evolutions: “unperturbed” and “perturbed”. The subspaces $D_\pm$ characterize unperturbed evolution in the Lax-Phillips approach. Due to (2.6), the subspaces $D_\pm$ are described by the operator $B$. The operator $B^*B$ is a positive self-adjoint extension of $B^2$ in the space $\mathcal{H}_0$ and the group $W_0(t)$ of solutions of the Cauchy problem of (2.2) (with $B^*B$ instead of $H$) determines an unperturbed evolution. The corresponding wave operators $\Omega_\pm = s \lim_{t \to \pm \infty} W(-t)W_0(t)$ exist and are isometric in $\mathcal{H}_0$. The scattering operator $\Omega_+^*\Omega_-$ coincides with the Lax-Phillips scattering matrix $S(\delta)$ in the spectral representation of the unperturbed evolution $W_0(t)$ [18].

3 Properties of operators $H_{aq}$

3.1 Preliminaries

For technical reasons it is convenient to calculate the $S$-matrix for unitary equivalent copy of the operator $H_{aq}$ in the Hilbert space $L_2(\mathbb{R}, \mathbb{C}^2)$. To do that, for each function $f \in L_2(\mathbb{R})$, we define the operator

$$ Yf = \begin{bmatrix} f(x) \\ f(-x) \end{bmatrix} = f(x), \quad x > 0 $$

that maps isometrically $L_2(\mathbb{R})$ onto $L_2(\mathbb{R}_+, \mathbb{C}^2)$ and maps $W_2^2(\mathbb{R}\setminus\{0\})$ onto $W_2^2(\mathbb{R}_+, \mathbb{C}^2)$. For all $f = Yf, \ f \in W_2^2(\mathbb{R}\setminus\{0\})$ we denote $[f]_r = f_r(0)$ and $[f]_s = f_s(0)$. In other words,

$$ [f]_r = \frac{1}{2} \lim_{x \to +0} (f_1(x)+f_2(x)), \quad [f]_s = \lim_{x \to +0} (f_1(x)-f_2(x)), \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \quad (3.1) $$

It is easy to see that $YH_{aq} = H_{aq}Y$, where $H_{aq}$ is defined by (1.2), (1.3) and the operator

$$ H_{aq}f = -\frac{d^2f}{dx^2} + [f], q(x), \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = Yq \quad (3.2) $$

\textsuperscript{5}we will use the mathbf font for $\mathbb{C}^2$-valued functions of $L_2(\mathbb{R}_+, \mathbb{C}^2)$ in order to avoid confusion with functions from $L_2(\mathbb{R})$. In particular, $e^{-i\mu x} \equiv \begin{bmatrix} e^{-i\mu x} \\ e^{-i\mu x} \end{bmatrix}$. 

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acts in $L_2(\mathbb{R}^+, \mathbb{C}^2)$ with domain of definition

$$D(H_{aq}) = \{ f \in W^2_2(\mathbb{R}^+, \mathbb{C}^2) : \langle f, \gamma \rangle = 0, \quad \gamma = a[f]_r + (f, q)_+ \}, \quad (3.3)$$

where $(f, q)_+ = (Y f, Y q)_+ = (f, q)$ is the scalar product in $L_2(\mathbb{R}^+, \mathbb{C}^2)$.

When $a \to \infty$, the formulas (3.2) and (3.3) determine a positive self-adjoint operator in $L_2(\mathbb{R}^+, \mathbb{C}^2)$

$$H_\infty \equiv H_{\infty q} = -\frac{d^2}{dx^2}, \quad D(H_\infty) = \{ f \in W^2_2(\mathbb{R}^+, \mathbb{C}^2) : f(0) = 0 \}$$

that does not depend on the choice of $q$ and can be decomposed

$$H_\infty f = \begin{bmatrix} H_\infty f_1 \\ H_\infty f_2 \end{bmatrix}, \quad H_\infty = -\frac{d^2}{dx^2}, \quad D(H_\infty) = \{ f \in W^2_2(\mathbb{R}^+) : f(0) = 0 \}. \quad (3.4)$$

By analogy with [21, Sec. 5] (where the case of operators $H_{aq}$ has been studied) we consider $H_{aq}$ and $H_\infty$ as restrictions of the maximal operator

$$H_{\max} f = -\frac{d^2 f}{dx^2} + [f]_r q(x), \quad D(H_{\max}) = \{ f \in W^2_2(\mathbb{R}^+) : [f]_s = 0 \}. \quad (3.5)$$

onto the corresponding domain of definition.

The maximal operator $H_{\max}$ has a boundary triplet $(\mathbb{C}, \Gamma_0, \Gamma_1)$, where

$$\Gamma_0 f = [f]_r, \quad \Gamma_1 f = 2[f]_r - (f, q)_+, \quad f \in D(H_{\max}) \quad (3.4)$$

and the formulas (3.2) and (3.3) are rewritten:

$$H_{\max} f = H_{\max} f (D(H_{\max})), \quad D(H_{\max}) = \{ f \in D(H_{\max}) : a\Gamma_0 f = \Gamma_1 f \}. \quad (3.5)$$

In particular, $H_\infty$ is the restriction of $H_{\max}$ onto $\ker \Gamma_0$ and its resolvent is

$$(H_\infty - \zeta^2 I)^{-1} f = \frac{i}{2\zeta} \left[ A_\zeta(x) e^{-i\zeta x} + B_\zeta(x) e^{i\zeta x} \right], \quad f \in L_2(\mathbb{R}^+, \mathbb{C}^2), \quad (3.6)$$

where $\zeta \in \mathbb{C}_-$ and

$$A_\zeta(x) = \int_0^\infty e^{-i\zeta s} f(s) ds - \int_x^\infty e^{i\zeta s} f(s) ds, \quad B_\zeta(x) = -\int_x^\infty e^{-i\zeta s} f(s) ds.$$

**Lemma 3.1.** The Krein-Naimark resolvent formula

$$(H_{aq} - \zeta^2 I)^{-1} f = (H_\infty - \zeta^2 I)^{-1} f + \frac{\langle f, u - \zeta \rangle}{a - W(\zeta^2)} u_\zeta(x) \quad (3.7)$$

holds for $a \neq W(\zeta^2)$. Here,

$$u_\mu(x) = e^{-i\zeta x} - (H_\infty - \mu^2 I)^{-1} q, \quad \mu \in \{ z, -\zeta \} \subset \mathbb{C}_- \quad (3.8)$$

is an eigenfunction of $H_{\max}$ corresponding to the eigenvalue $\mu^2$ and

$$W(\zeta^2) = -2iz - 2(e^{-i\zeta x}, Re q)_+ + (H_\infty - \zeta^2 I)^{-1} q, q)_+, \quad z \in \mathbb{C}_-. \quad (3.9)$$
Proof. It follows from [21] that the subspace \( \ker(H_{\max} - \mu^2 I) \) is one dimensional and it is generated by the function \( u_\mu \) defined by (3.8). Setting \( \mu = z \) and using (3.4), we conclude that \( \Gamma_0 u_z = 1 \) and the Weyl-Titchmarsh function associated to the boundary triplet \((C, \Gamma_0, \Gamma_1)\) takes the form

\[
W(z^2) = \Gamma_1 u_z = -2iz - 2[v']_r - (e^{-izx} + v, q)_+,
\]

where \( v = (H_\infty - z^2 I)^{-1}q \). In view of (3.6), \( v'(0) = \int_0^\infty e^{-isz} q(s) ds \) and hence,

\[
2[v']_r + (e^{-izx}, q)_+ = 2(e^{-izx}, Re q)_+ \quad Re q = \begin{bmatrix} Re q_1 \\ Re q_2 \end{bmatrix}.
\]

Substituting this expression into the formula for \( W(z^2) \) we obtain (3.9).

In terms of the boundary triplet \((C, \Gamma_0, \Gamma_1)\), the Krein-Naimark resolvent formula has the form [26, Theorem 14.18, Proposition 14.14]

\[
(H_{aq} - z^2 I)^{-1}f = (H_\infty - z^2 I)^{-1}f + \frac{\Gamma_1 u}{a - W(z^2)} u_z(x),
\]

where \( u = (H_\infty - z^2 I)^{-1}f \). In view of (3.10), \( u'(0) = \int_0^\infty e^{-isz} f(s) ds \). Taking (3.11) into account,

\[
2[u']_r = \int_0^\infty e^{-isz} (f_1(s) + f_2(s)) dx = (f, e^{ix})_+.\]

Finally, using (3.1) and (3.8) with \( \mu = -z \), we obtain

\[
\Gamma_1 u = (f, e^{ix})_+ - (u, q)_+ = (f, e^{ix} - (H_\infty - z^2 I)^{-1}q)_+ = (f, u - z)_+
\]

that completes the proof.

3.2 Applicability of the Lax-Phillips approach for \( H_{aq} \)

Denote by

\[
B = i \frac{d}{dx}, \quad D(B) = \{ u \in W^1_2(\mathbb{R}_+) : u(0) = 0 \}
\]

the first derivative operator in \( L_2(\mathbb{R}_+) \). The same notation will be used for its analog acting in \( L_2(\mathbb{R}_+, C^2) \). The both operators are simple maximal symmetric with zero defect numbers in \( C_+ \), and theirs Cayley transforms

\[
T = (B - iI)(B + iI)^{-1}
\]

are forward shift operators in the corresponding spaces.

A function \( q \in L_2(\mathbb{R}_+, C^2) \) is called non-cyclic for the backward shift operator \( T^* \) if the subspace

\[
E_q = \bigvee_{n=0}^\infty T^{*n}q
\]

does not coincide with \( L_2(\mathbb{R}_+, C^2) \).
Considering $L_2(\mathbb{R}_+)$ as a subspace of $L_2(\mathbb{R})$ we conclude that the Fourier transform
\[
Ff(\delta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\delta s} f(s) ds
\]
maps isometrically $L_2(\mathbb{R}_+)$ onto the Hardy space $H^2(\mathbb{C}_+)$ and
\[
FBu = \delta Fu, \quad FTf = \frac{\delta - i}{\delta + i} Ff, \quad u \in D(B), \quad f \in L_2(\mathbb{R}_+).
\]

Let $\psi \in H^\infty(\mathbb{C}_+)$ be an inner function. Then
\[
\psi(B) = (F^{-1} \psi(\delta) F)
\]
is an isometric operator in $L_2(\mathbb{R}_+)$ which commutes with $B$ [14, Sec. 5].

**Lemma 3.2.** The following are equivalent:

(i) a function $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is non-cyclic for the backward shift operator $T^*$;

(ii) there exists an inner function $\psi \in H^\infty(\mathbb{C}_+)$ such that the subspace $H_0 = \psi(B) L_2(\mathbb{R}_+)$ of $L_2(\mathbb{R}_+)$ is orthogonal to at least one of the functions $q_i$.

**Proof.** (i) $\rightarrow$ (ii) Since $E_q = E_{q_1} \oplus E_{q_2}$, the function $q$ is non-cyclic if and only if at least one of the functions $q_i \in L_2(\mathbb{R}_+)$ is non-cyclic for the backward shift operator $T^*$ in $L_2(\mathbb{R}_+)$. Let $q \equiv q_i$ be non-cyclic. Then the non-zero subspace
\[
H_0 = L_2(\mathbb{R}_+) \oplus E_q
\]
is invariant with respect to $T$. This means that $F H_0$ is invariant with respect to the multiplication by $\frac{\delta - i}{\delta + i}$ in $H^2(\mathbb{C}_+)$. The Beurling theorem [22, p. 164] yields the existence of an inner function $\psi \in H^\infty(\mathbb{C}_+)$ such that $F H_0 = \psi(\delta) H_2(\mathbb{C}_+)$. Therefore
\[
H_0 = F^{-1} \psi(\delta) F L_2(\mathbb{R}_+) = \psi(B) L_2(\mathbb{R}_+).
\]

By the construction, $H_0$ is orthogonal to $q$ (since, $q$ belongs to $E_q$).

(ii) $\rightarrow$ (i) Let $H_0 = \psi(B) L_2(\mathbb{R}_+) \oplus E_q$ be orthogonal to $q$. Then
\[
(\psi(B)f, T^n q_+)_+ = (T^n \psi(B)f, q)_+ = (\psi(B) T^n f, q)_+ = 0 \quad \text{for all} \quad f \in L_2(\mathbb{R}_+).
\]

Therefore, $T^n q$ is orthogonal to $H_0$. This means that $E_q$ is orthogonal to $H_0$. Therefore, $E_q$ is a proper subspace of $L_2(\mathbb{R}_+)$ and $q$ is non-cyclic.

**Theorem 3.3.** If $q$ is non-cyclic for $T^*$, then there exists a simple maximal symmetric operator $B$ acting in a subspace $H_0$ of $L_2(\mathbb{R}_+, \mathbb{C}^2)$ such that the operators $H_{aq}$ are extensions of the symmetric operator $B^2$ for all $a \in \mathbb{C}$. 

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6here, $(\cdot, \cdot)_+$ is the scalar product in $L_2(\mathbb{R}_+)$. 

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Proof. If $q$ is non-cyclic, then at least one of $q_i$ is non-cyclic. Consider firstly the case where the both of functions $q_i$ are non-cyclic. Due to the proof of Lemma 3.2 for each $q_i$ there exists an inner function $\psi_i$ such that the subspace $\psi_i(B)L_2(\mathbb{R}^+)_{\mathbb{R}^+}$ is orthogonal to $q_i$. Denote

$$\mathcal{H}_0 = \begin{bmatrix} \psi_1(B)L_2(\mathbb{R}^+) \\ \psi_2(B)L_2(\mathbb{R}^+) \end{bmatrix} = \psi(B)L_2(\mathbb{R}^+, \mathbb{C}^2),$$

(3.13)

where

$$\psi(B) = \begin{bmatrix} \psi_1(B) & 0 \\ 0 & \psi_2(B) \end{bmatrix}$$

(3.14)

is an isometric operator in $L_2(\mathbb{R}^+, \mathbb{C}^2)$ that commutes with $B$. This allows to define a simple maximal symmetric operator in $\mathcal{H}_0$:

$$B = \psi(B)B\psi(B)^*, \quad D(B) = \psi(B)D(B).$$

(3.15)

Since $\psi(B)$ commutes with $B$, the formula (3.15) can be rewritten as

$$Bu = B_{\mathcal{H}_0}, \quad u \in D(B) = \psi(B)D(B) = D(B) \cap \mathcal{H}_0.$$  

(3.16)

(i.e., $B$ is a part of $B$ restricted on $\mathcal{H}_0$). In view of (3.10) and (3.16)

$$B^2 = -\frac{d^2}{dx^2}, \quad D(B^2) = \{u \in W_2^2(\mathbb{R}^+, \mathbb{C}^2) \cap \mathcal{H}_0 : u(0) = u'(0) = 0\}.$$  

(3.17)

By Lemma 3.2 and (3.16), the subspace $\mathcal{H}_0$ is orthogonal to $q$. Hence, in view of (3.2), (3.3), and (3.17), $D(H_{aq}) \supset D(B^2)$ and

$$H_{aq}u = -\frac{d^2u}{dx^2} = B^2u \quad \text{for all} \quad u \in D(B^2).$$

The case where only one $q_i$ is considered similarly. For example, if $q_1$ is non-cyclic whereas $q_2$ is cyclic (i.e., $E_{q_2} = L_2(\mathbb{R}^+)$), then $\mathcal{H}_0$ and $\psi(B)$ are determined as above with $\psi_2 = 0$. \hfill \Box

**Corollary 3.4.** Assume that $H = H_{aq}$ is a positive self-adjoint operator. If $q$ is non-cyclic for $T^*$, then the group $W(t)$ of Cauchy problem solutions of (2.2) has incoming/outgoing subspaces $D_\pm$ defined by (2.10), where $B$ is from (3.10).

**Proof.** It follows from Theorems 2.1 and 3.3. \hfill \Box

### 4 S-matrix for positive self-adjoint operator

In this section we suppose that $H_{aq}$ is a positive self-adjoint operator and the function $q$ is non-cyclic. By Theorem 3.3 $H_{aq}$ is an extension of the symmetric operator $B^2$ defined by (3.17) that acts in the subspace $\mathcal{H}_0 = \psi(B)L_2(\mathbb{R}^+, \mathbb{C}^2)$. In view of Corollary 3.4 and Theorem 2.2 the $S$-matrix of $H_{aq}$ exists and is given by (2.10). Our goal is to modify this general formula taking into account the specific choice of $B$ in (3.16).
4.1 Preliminaries

The following technical results are needed for the calculation of S-matrix.

**Lemma 4.1.** Let an isometric operator \( \psi(B) \) be defined by (3.11). Then
\[
\psi(B)^* e^{-i\mu x} = \overline{\psi(B)} e^{-i\mu x}, \quad \mu \in \mathbb{C}_-.
\]

*Proof.* It follows from (3.10) that \( B^* = i \frac{d}{dx}, \) \( \mathcal{D}(B^*) = W^1_2(\mathbb{R}_+) \). Therefore, \( \ker(B^* - \mu I) = \{ ce^{-i\mu x} : c \in \mathbb{C} \} \). This means that, for all \( u \in \mathcal{D}(B) \),
\[
((B-\overline{\mu}I)u, \psi(B)^* e^{-i\mu x})_+ = (\psi(B)(B-\overline{\mu}I)u, e^{-i\mu x})_+ = ((B-\overline{\mu}I)\psi(B)u, e^{-i\mu x})_+ = 0.
\]
Hence \( \psi(B)^* e^{-i\mu x} \) belongs to \( \ker(B^* - \mu I) \) and
\[
(\psi(B)^* e^{-i\mu x}, e^{-i\mu x})_+ = c(e^{-i\mu x}, e^{-i\mu x})_+ = \frac{c}{2\Im \mu}.
\]

Using (3.12) and taking into account that \( F_{\chi_{\mathbb{R}_+}}(x)e^{-i\mu x} = \frac{i}{\sqrt{2\pi}} \cdot \frac{1}{\delta - \mu} \), we verify that the inner product
\[
(\psi(B)^* e^{-i\mu x}, e^{-i\mu x})_+ = (e^{-i\mu x}, \psi(B)e^{-i\mu x})_+ = (F_{\chi_{\mathbb{R}_+}}(x)e^{-i\mu x}, \psi(\delta)F_{\chi_{\mathbb{R}_+}}(x)e^{-i\mu x})
\]
is equal to \( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-(\Im \mu) \overline{\psi(\delta)}}{(\Re \mu - \delta)^2 + (\Im \mu)^2} d\delta \). The Poisson formula, [24, p.147] and (4.1) lead to the conclusion that
\[
c = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{-(\Im \mu) \overline{\psi(\delta)}}{(\Re \mu - \delta)^2 + (\Im \mu)^2} d\delta = \overline{\psi(\Re \mu \pm i\Im \mu)} = \overline{\psi(\overline{\mu})}
\]
that completes the proof. \( \square \)

**Lemma 4.2.** Let \( B \) and \( \psi(B) \) be defined by (3.14) and (3.14), respectively. Then, for any \( \mu \in \mathbb{C}_- \),
\[
\ker(B^* - \mu^2 I) = \ker(B^* - \mu I) = \psi(B) \left\{ h_\mu = \begin{bmatrix} \alpha_\mu \\ \beta_\mu \end{bmatrix} e^{-i\mu x} : \alpha_\mu, \beta_\mu \in \mathbb{C} \right\}.
\]

*Proof.* The first identity follows from (2.5). It follows from (3.15) that
\[
B^* = \psi(B)B^* \psi(B)^*, \quad \mathcal{D}(B^*) = \mathcal{D}(B^*) = \psi(B)W^1_2(\mathbb{R}_+, \mathbb{C}^2).
\]
By virtue of (4.2) we conclude that \( \ker(B^* - \mu I) = \psi(B) \ker(B^* - \mu I) \). It follows from the proof of Lemma 4.1 that \( \ker(B^* - \mu I) \) coincides with the set of vectors \( \{ h_\mu \} \) defined above. \( \square \)

**Corollary 4.3.** Let \( \psi(B) \) be defined by (3.14). Then, for any \( \mu \in \mathbb{C}_- \),
\[
\psi(B)^* e^{-i\mu x} = \begin{bmatrix} \psi_1(\overline{\mu}) \\ \psi_2(\overline{\mu}) \end{bmatrix} e^{-i\mu x}, \quad \psi(B)^* u_\mu = \begin{bmatrix} c(\mu, q_1) \\ c(\mu, q_2) \end{bmatrix} e^{-i\mu x},
\]
where \( u_\mu \) is defined by (3.12) and
\[
c(\mu, q_j) = \psi_j(\overline{\mu}) + 2(\Im \mu)((H_{\infty} - \mu^2 I)^{-1} q_j, \psi_j(B)e^{-i\mu x})_+.
\]
Proof. The first relation in (4.3) follows from Lemma 4.1.

The function $u_\mu$ in the second relation is an eigenfunction of the operator $H_{\text{max}}$ (see Lemma 3.1). Since $(C, \Gamma_0, \Gamma_1)$ defined by (3.4) is a boundary triplet of $H_{\text{max}}$, its adjoint $H_{\text{max}}^*$ coincides with the symmetric operator $H_{\text{min}} = H_{\text{max}} |_{\ker \Gamma_0 \cap \ker \Gamma_1}$. Precisely,

$$H_{\text{min}} = -\frac{d^2}{dx^2} \quad D(H_{\text{min}}) = \{ f \in W^2_2(\mathbb{R}_+, \mathbb{C}^2) : [f]'_r = 0, 2[f'_r] = (f, q)_r \}.$$ 

Comparing this formula with (3.17) leads to the conclusion that $H_{\text{min}} \supseteq B^2$, i.e., $H_{\text{min}}$ is an extension of $B^2$ with the exit into the space $L_2(\mathbb{R}_+, \mathbb{C}^2)$. Then, for $f \in D(H_{\text{max}})$ and $u \in D(B^2)$,

$$(P_{\mathcal{D}_0} H_{\text{max}} f, u)_+ = (H_{\text{max}} f, u)_+ = (f, H_{\text{min}} u)_+ = (P_{\mathcal{D}_0} f, B^2 u)_+ = (B^{*2} P_{\mathcal{D}_0} f, u)_+,$$

where $P_{\mathcal{D}_0}$ is the orthogonal projection in $L_2(\mathbb{R}_+, \mathbb{C}^2)$ on the subspace $\mathcal{D}_0$ defined by (3.13). The obtained relation means that

$$P_{\mathcal{D}_0} H_{\text{max}} f = B^{*2} P_{\mathcal{D}_0} f, \quad \text{for all } f \in D(H_{\text{max}}) = W^2_2(\mathbb{R}_+, \mathbb{C}^2). \quad (4.5)$$

Setting $f = u_\mu$ in (4.5) and taking into account that $H_{\text{max}} u_\mu = \mu^2 u_\mu$, we obtain

$$P_{\mathcal{D}_0} H_{\text{max}} u_\mu = B^{*2} P_{\mathcal{D}_0} u_\mu = \mu^2 P_{\mathcal{D}_0} u_\mu.$$ 

This relation and (2.5) mean

$$P_{\mathcal{D}_0} u_\mu \in \ker(B^{*2} - \mu^2 I) = \ker(B^* - \mu I).$$

In view of Lemma 4.2 $P_{\mathcal{D}_0} u_\mu = \psi(B) h_\mu$ for some choice of $h_\mu = \left[ \begin{array}{c} \alpha_\mu \\ \beta_\mu \end{array} \right] e^{-i\mu x}$ or $\psi(B)^* h_\mu = \psi(B)^* u_\mu$ since $P_{\mathcal{D}_0} = \psi(B)^* \psi(B)^*$. Therefore $\psi(B)^* u_\mu = h_\mu$ that leads to the second relation in (4.3) with unspecified parameters $\alpha_\mu$, $\beta_\mu$.

Taking (3.8) into account and arguing by the analogy with the determination of $c$ in the proof of Lemma 4.1 we arrive at the conclusion that $\alpha_\mu = c(\mu, q_1)$ and $\beta_\mu = c(\mu, q_2)$, where $c(\mu, q_i)$ are defined in (4.4).

4.2 Positive boundary triplet

In view of Sec. 2.2 the S-matrix can not be constructed without finding the positive boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of $B^{*2}$. Since $B$ is the restriction of the first derivative operator $\mathcal{B}$ on $\mathcal{D}_0$, see (3.10), one can try to express $(\mathcal{H}, \Gamma_0, \Gamma_1)$ in terms of well-known positive boundary triplet $(\mathcal{H}', \Gamma_0', \Gamma_1')$ of $B^{*2}$.

Lemma 4.4. The following relations hold:

$$\mathcal{H} = \psi(B) \mathcal{H}', \quad \Gamma_0 \psi(B) = \psi(B) \Gamma_0' \quad \Gamma_1 \psi(B) = \psi(B) \Gamma_1'.$$

Proof. It follows from (4.2) that

$$B^{*2} = \psi(B) B^{*2} \psi(B)^*, \quad D(B^{*2}) = \psi(B) D(B^{*2}) = \psi(B) W^2_2(\mathbb{R}_+, \mathbb{C}^2) \quad (4.6)$$

By definition $\mathcal{H} = \ker(B^{*2} + I)$ and $\mathcal{H}' = \ker(B^{*2} + I)$. Using (4.6), we obtain

$$\mathcal{H} = \ker(B^{*2} + I) = \psi(B) \ker(B^{*2} + I) = \psi(B) \mathcal{H}'.$$
It follows from (3.15) and (4.2) that
\[ B^*B = \psi(B)B^*\psi(B)^*, \quad D(B^*B) = \psi(B)D(B^*B) \] (4.7)

For brevity, we denote \( V = \psi(B) \) and consider \( f \in D(B^{*2}) \). Then \( f = u + h \), where \( u \in D(B^*B) \) and \( h \in \mathcal{H}' \). By virtue of (4.6), (4.7), \( Vf \in D(B^{*2}) \) and \( Vf = Vu + Vh \), where \( Vu \in D(B^*B) \) and \( Vh \in \mathcal{H} \). In view of (2.9), \( \Gamma_0 Vf = Vh = V\Gamma_0 f \).

Since \( \mathcal{H} = V\mathcal{H}' \) and \( \mathcal{R}(B^2 + I) = VR(B^2 + I) \), the orthogonal projectors \( P_H \) and \( P_{H'} \) are related as follows: \( VP_H = P_HV \). Therefore,
\[ \Gamma_1 Vf = P_H(B^*B + I)Vu = P_H(VB^*BV^* + I)Vu = P_H V(B^*B + I)u = V\Gamma_1 f \]
that completes the proof. \( \square \)

**Corollary 4.5.** The positive boundary triplet \((\mathcal{H}, \Gamma_0, \Gamma_1)\) of \( B^{*2} \) consists of the space
\[ \mathcal{H} = \psi(B) \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{-x} : \alpha, \beta \in \mathbb{C} \right\} \]
and the mappings \( \Gamma_1 : \psi(B)W_2^2(\mathbb{R}_+, \mathbb{C}^2) \to \mathcal{H} \) that are defined as follows:
\[ \Gamma_0 \psi(B)f(x) = \psi(B)f(0)e^{-x}, \quad \Gamma_1 \psi(B)f(x) = 2\psi(B)[f'(0) + f(0)]e^{-x}. \]

**Proof.** It is well known (see, e.g., [12]) that the positive boundary triplet \((\mathcal{H}', \Gamma_0', \Gamma_1')\) of \( B^{*2} \) has the form: \( \mathcal{H}' = \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{-x} : \alpha, \beta \in \mathbb{C} \right\} \) and
\[ \Gamma'_0 f = f(0)e^{-x}, \quad \Gamma'_1 f = 2[f'(0) + f(0)]e^{-x}, \quad f \in W_2^2(\mathbb{R}_+, \mathbb{C}^2). \]
Applying Lemma 4.4 we complete the proof. \( \square \)

### 4.3 The S-matrix for positive self-adjoint \( \mathbf{H}_{aq} \)

**Theorem 4.6.** The S-matrix for positive self-adjoint operator \( \mathbf{H}_{aq} \) has the form
\[ S(z) = \begin{bmatrix} \Psi_1(z) & 0 \\ 0 & \Psi_2(z) \end{bmatrix} - \frac{2zi}{a - W(z)} \begin{bmatrix} \c(z, q_1)\c(-\bar{z}, q_1) & \c(z, q_1)\c(-\bar{z}, q_2) \\ \c(z, q_2)\c(-\bar{z}, q_1) & \c(z, q_2)\c(-\bar{z}, q_2) \end{bmatrix}, \] (4.8)
where \( \c(\mu, q_i) \) are determined by (4.4) and \( \Psi_j(z) \) are holomorphic continuations of the functions \( \psi_j(\cdot - \delta)/\psi_j(\delta) \) (\( \delta \in \mathbb{R} \)) into \( \mathbb{C}_- \) such that \( |\Psi_j(z)| < 1 \) and \( \Psi_j(z) = \Psi_j(-\bar{z}) \).

**Proof.** By Theorem 2.2 for the calculation of S-matrix, one need to find operators \( C(z) \) in (2.11). To do that we analyze vectors
\[ u \in P_{\delta_0}(\mathbf{H}_{aq} - z^2I)^{-1} \ker(B^* + \bar{z}I) \]
where and (4.12) in more detail. First of all we note that \( \ker(B^* + \tau I) = \psi(B) \{ h_{\tau} \} \) by Lemma 4.2. Consider the equation

\[
(H_{aq} - z^2 I)f = (\tau^2 - z^2)\psi(B)h_{\tau}, \quad z \in \mathbb{C} \setminus i\mathbb{R}.
\] (4.9)

Its solution \( f \in D(H_{aq}) \) is determined uniquely and

\[
u = P_{\beta_0}f = (\tau^2 - z^2)P_{\beta_0}(H_{aq} - z^2 I)^{-1}\psi(B)h_{\tau} \tag{4.10}
\]

belongs to \( D(B^{*2}) \) due to 4.5. In view of 4.10, \( u = \psi(B)v \), where \( v \in W_2^2(\mathbb{R}_+, \mathbb{C}^2) \) and \( B^{*2}\psi(B)v = \psi(B)B^{*2}v \). Moreover, since \( P_{\beta_0} = \psi(B)^*\psi(B) \), the relation (4.10) yields

\[
u = (\tau^2 - z^2)\psi(B)^*(H_{aq} - z^2 I)^{-1}\psi(B)h_{\tau}. \tag{4.11}
\]

Applying \( P_{\beta_0} \) to the both parts of (4.9) and using (4.5) we obtain

\[
(B^{*2} - z^2 I)\nu = \psi(B)(B^{*2} - z^2 I)v = (\tau^2 - z^2)\psi(B)h_{\tau}.
\]

Therefore, \( (B^{*2} - z^2 I)v = \left(-\frac{d^2}{dx^2} - z^2 I\right)v = (\tau^2 - z^2)h_{\tau} \). This means that

\[
u = h_{\tau} + h_z, \quad u = \psi(B)v = \psi(B)h_{\tau} + \psi(B)h_z, \tag{4.12}
\]

where \( h_z \in \ker(B^* - z I) \) is determined uniquely by the choice of \( h_{\tau} \). Applying operators \( \Gamma_i \) from Corollary 4.5 we obtain

\[
\Gamma_0\nu = \psi(B)\left[\frac{\alpha_{\tau}}{\beta_{\tau} + \beta_z}\right]e^{-z}, \quad \Gamma_1\nu = 2\psi(B)\left[\frac{1 + i\tau\alpha_{\tau} + (1 - iz)\alpha_z}{1 + i\tau\beta_{\tau} + (1 - iz)\beta_z}\right]e^{-z}.
\]

Since \( \dim \mathcal{H} = 2 \), the function \( C(z) \) in Theorem 2.2 is \( 2 \times 2 \)-matrix-valued. The substitution of \( \Gamma_0\nu \) into the characteristic relation (2.11) gives

\[
2C(z)\left[\frac{(1 + i\tau)\alpha_{\tau} + (1 - iz)\alpha_z}{1 + i\tau\beta_{\tau} + (1 - iz)\beta_z}\right] = \left[\frac{\alpha_{\tau} + \alpha_z}{\beta_{\tau} + \beta_z}\right],
\]

and, after elementary transformations,

\[
[I - 2(1 - iz)C(z)]^{-1}\left[\frac{\alpha_{\tau}}{\beta_{\tau}}\right] = \frac{1}{2iRe z}\left[\frac{(1 + i\tau)\alpha_{\tau} + (1 - iz)\alpha_z}{1 + i\tau\beta_{\tau} + (1 - iz)\beta_z}\right]. \tag{4.13}
\]

The substitution of (4.13) into (2.10) gives the \( S \)-matrix

\[
S(z)\left[\frac{\alpha_{\tau}}{\beta_{\tau}}\right] = -i\frac{Im z}{Re z}\left[\frac{\alpha_{\tau}}{\beta_{\tau}}\right] - \frac{z}{Re z}\left[\frac{\alpha_z}{\beta_z}\right], \quad z \in \mathbb{C} \setminus i\mathbb{R}. \tag{4.14}
\]

Here \( \alpha_z, \beta_z \) are functions of parameters \( \alpha_{\tau}, \beta_{\tau} \in \mathbb{C} \). Indeed, in view of (1.11) and (4.12) \( h_z = -h_{\tau} + (\tau^2 - z^2)\psi(B)^*(H_{aq} - z^2 I)^{-1}\psi(B)h_{\tau} \) and hence,

\[
\left[\frac{\alpha_z}{\beta_z}\right]e^{-iax} = (-I + (\tau^2 - z^2)\psi(B)^*(H_{aq} - z^2 I)^{-1}\psi(B))\left[\frac{\alpha_{\tau}}{\beta_{\tau}}\right]e^{iax}, \tag{4.15}
\]

\( ^7 \)The coefficient \( (\tau^2 - z^2) \) is used for the simplification of formulas below.
The $S$-matrix $S(z)$ depends on the choice of $H_{aq}$. If $H_{aq} = H_\infty$, then this operator is a positive self-adjoint extension of the symmetric operators $B^2$ and $B^2$. By Theorem 2.1 one can construct two pairs of subspaces $D_\pm$ that are determined by $B$ and $B$, respectively. Therefore, one can define two $S$-matrices $S_1(\cdot)$ and $S(\cdot)$ for $H_\infty$ corresponding to the cases where $H_\infty$ is considered as an extension of $B^2$ or an extension of $B^2$. The both of $S$-matrices are defined by (2.10) but, in the first case, $C(z) = 0$ and, therefore $S_1(z) = \sigma_0$. In view of [13, Proposition 3.1],

$$S(z) = \begin{bmatrix} \Psi_1(z) & 0 \\ 0 & \Psi_2(z) \end{bmatrix} S_1(z) = \begin{bmatrix} \Psi_1(z) & 0 \\ 0 & \Psi_2(z) \end{bmatrix}, \quad (4.16)$$

where $\Psi_j(z)$ are holomorphic functions in $\mathbb{C}_-$ such that $|\Psi_j(z)| < 1$ and $\Psi_j(z) = \Psi_j(-\bar{z})$. Moreover, the boundary values of $\Psi_j(z)$ on $\mathbb{R}$ coincide with $\psi_j(-\delta)/\psi_j(\delta)$. Due to (4.15), the coefficients $\alpha_z, \beta_z$ in (4.14) depend on the choice of $H_{aq}$. The resolvent formula (3.7) and (4.15) allow one to present $S_1(\cdot)$ as the sum of $\alpha_z H_{aq}$, $\beta_z H_{aq}$ and a function that is determined by the difference between $(H_{aq} - z^2 I)^{-1}$ and $(H_{aq} - z^2 I)^{-1}$ (see the second part in (5.7)). Such decomposition and (4.16) allows one to rewrite (4.14):

$$S(z) \begin{bmatrix} \alpha_z \beta_z \\ - & \beta_z \end{bmatrix} = \begin{bmatrix} \Psi_1(z) & 0 \\ 0 & \Psi_2(z) \end{bmatrix} - \frac{ze^{i\bar{z}}}{Re z} (\bar{z}^2 - z^2) \left( \frac{h_{-\bar{z}}, \psi(B)^* u_{-\bar{z}}}{a - W(z^2)} \psi(B)^* u_z \right). \quad (4.17)$$

In view of (4.3) with $\mu = -\bar{z}$

$$\frac{(\bar{z}^2 - z^2) (h_{-\bar{z}}, \psi(B)^* u_{-\bar{z}})}{Re z} = 2i \left\langle \begin{bmatrix} \alpha_z \\ \beta_z \end{bmatrix}, \begin{bmatrix} c(\bar{z}, q_1) \\ c(\bar{z}, q_2) \end{bmatrix} \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^2$. Substituting this expression into (4.17) and using (4.3) with $\mu = z$, we obtain

$$S(z) \begin{bmatrix} \alpha_z \\ \beta_z \end{bmatrix} = \begin{bmatrix} \Psi_1(z) & 0 \\ 0 & \Psi_2(z) \end{bmatrix} - \frac{2zi}{a - W(z^2)} \left\langle \begin{bmatrix} \alpha_z \\ \beta_z \end{bmatrix}, \begin{bmatrix} c(-\bar{z}, q_1) \\ c(-\bar{z}, q_2) \end{bmatrix} \right\rangle \begin{bmatrix} c(z, q_1) \\ c(z, q_2) \end{bmatrix}.$$  

A rudimentary linear algebra exercise leads to the conclusion this formula for $S(z)$ can be rewritten as (4.8) for $z \in \mathbb{C}_- \backslash i\mathbb{R}_-$. Since the $S$-matrix is holomorphic in the lower half-plane, the formula (4.8) remains true for $\mathbb{C}_-$.  

The expression (4.8) is based on the Krein-Naimark resolvent formula (3.7) and it allows one to establish various useful relationships between $S$-matrix and the operator $H_{aq}$. An alternative formula for $S$-matrix in terms of reflection and transmission coefficients is presented below.

By virtue of Lemma 4.1

$$P_{\delta_0} \begin{bmatrix} e^{i\bar{z}} \\ 0 \end{bmatrix} = \psi(B)^* \begin{bmatrix} e^{i\bar{z}} \\ 0 \end{bmatrix} = \psi(B) \begin{bmatrix} \overline{\psi_1(-z)} \\ 0 \end{bmatrix} e^{i\bar{z}} \quad (4.18)$$

and, similarly, $P_{\delta_0} \begin{bmatrix} \alpha_z \\ \beta_z \end{bmatrix} e^{-i\bar{z}} = \psi(B) \begin{bmatrix} \alpha_z \overline{\psi_1(\bar{z})} \\ \beta_z \overline{\psi_2(\bar{z})} \end{bmatrix} e^{-i\bar{z}}.$
Setting \( h \mapsto e^{ixz} \) in (4.9) and using (4.18) we obtain

\[
(H_{aq} - z^2 I)f = (\zeta - z^2)\psi(B)h \mapsto (\zeta - z^2) P_{\bar{z}} \begin{bmatrix}
\psi_1(-z) \\
0
\end{bmatrix}, \quad z \in \mathbb{C} \setminus i\mathbb{R}_{-}
\]

and, in view of (4.10), (4.12), its solution \( f \) satisfies the relation

\[
P_{\bar{z}} f = \psi(B) \begin{bmatrix}
\psi_1(-z) \\
0
\end{bmatrix} e^{ixz} + \psi(B) \begin{bmatrix}
\alpha_x \\
\beta_x
\end{bmatrix} e^{-ixz} = P_{\bar{z}} \begin{bmatrix}
\psi_1(z) \\
\beta_1\psi_2(z)
\end{bmatrix} e^{i\alpha x} + R^1 e^{-i\alpha x},
\]

where

\[
R^1 = \frac{\alpha_x}{\psi_1(z)}, \quad T^1 = \frac{\beta_x}{\psi_2(z)}
\]

are called the reflection and the transmission coefficients, respectively.

Similarly, assuming \( h \mapsto e^{ixz} \) and considering the solution \( f \) of

\[
(H_{aq} - z^2 I)f = (\zeta - z^2) P_{\bar{z}} \begin{bmatrix}
0 \\
e^{ixz}
\end{bmatrix},
\]

we obtain

\[
P_{\bar{z}} f = P_{\bar{z}} \begin{bmatrix}
T^2 e^{-i\alpha x} \\
e^{ixz} + R^2 e^{-i\alpha x}
\end{bmatrix}, \quad R^2 = \frac{\beta_x}{\psi_2(z)}, \quad T^2 = \frac{\alpha_x}{\psi_1(z)}
\]

The reflection \( R^j \) and the transmission \( T^j \) coefficients described above allow one to obtain an alternative formula for \( S \)-matrix.

**Theorem 4.7.** The \( S \)-matrix of a positive self-adjoint operator \( H_{aq} \) has the form

\[
S(z) = \frac{-z}{Re z} \begin{bmatrix}
\theta_{11}(z)R^1 + i\frac{L_m z}{2} \\
\theta_{21}(z)T^1
\end{bmatrix} \begin{bmatrix}
\theta_{12}(z)T^2 \\
\theta_{22}(z)R^2 + i\frac{L_m z}{2}
\end{bmatrix}, \quad \theta_{nm}(z) = \frac{\psi_n(z)}{\psi_m(-z)}.
\]

(4.19)

**Proof.** Setting in (4.14):

\[
\alpha_\mapsto = \psi_1(-z), \quad \beta_\mapsto = 0, \quad \alpha_z = \psi_1(z)R^1, \quad \beta_z = \psi_2(z)T^1
\]

and

\[
\alpha_\mapsto = 0, \quad \beta_\mapsto = \psi_2(-z), \quad \alpha_z = \psi_1(z)T^2, \quad \beta_z = \psi_2(z)R^2
\]

we obtain a system of four linear equations with respect to unknowns coefficients of the \( S \)-matrix \( S(z) = \begin{bmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{bmatrix} \). Its solution gives rise to (4.19) for all \( z \in \mathbb{C} \setminus i\mathbb{R}_{-} \). Since \( S(z) \) is holomorphic in \( \mathbb{C} \), the formula (4.19) holds for all \( z \in \mathbb{C} \). \( \square \)
4.3.1 Example of ordinary δ-interaction

In view of (3.2), the ordinary δ-interaction corresponds to \( q = 0 \). The operators \( H_a = H_{a0} = -\frac{d^2}{dx^2} \) have the domains:

\[
D(H_{a0}) = \{ f \in W_2^2(\mathbb{R}_+, \mathbb{C}^2) : [f]_s = 0, \quad [f']_r = a[f]_r \}.
\]

The function \( q = 0 \) is non-cyclic and one can set \( \psi_1 = \psi_2 = 1 \). Then \( P_{a0} = I \) and the reflection and the transmission coefficients are determined as follows:

\[
R_1 = R_2 = -a + i(\pi - z), \quad T_1 = T_2 = \frac{2ire^z}{a + 2iz}.
\]

Substituting the obtained expressions in (4.19) and taking into account that \( \theta_{nm}(z) = 1 \), we obtain a matrix-valued \( S \)-function

\[
S(z) = \frac{1}{a + 2iz} \begin{bmatrix} a & -2iz \\ -2iz & a \end{bmatrix},
\]

which is holomorphic on \( \mathbb{C}_- \) for positive self-adjoint operators \( H_a \) (the positivity of \( H_a \) is distinguished by the condition \( a \geq 0 \)).

The same formula (4.20) can be deduced from (4.8) if one take into account that \( \Psi_j = 1 \) since \( \psi_j = 1 \) and \( W(z^2) = -2iz, c(z, q_j) = 1 \) by virtue of (3.9) and (4.4), respectively.

5 Operators \( H_{aq} \) and their \( S \)-matrices

The example above leads to a natural assumption that the formulas (4.8), (4.19) allow to construct a function \( S(z) \) for each operator \( H_{aq} \) (assuming, of course, that \( q \) is non-cyclic). We will call it the \( S \)-matrix of \( H_{aq} \). If \( H_{aq} \) is positive self-adjoint, then the \( S \)-matrix is the consequence of proper arguments of the Lax-Phillips theory and it coincides with the analytical continuation of the Lax-Phillips scattering matrix into \( \mathbb{C}_- \). Otherwise, \( S(z) \) is defined directly by (4.8), (4.19) and it can be considered as a characteristic function of \( H_{aq} \). In this section, we describe properties of \( H_{aq} \) in terms of the corresponding \( S \)-matrix.

It follows from (4.8) that a \( S \)-matrix of \( H_{aq} \) is a meromorphic matrix-valued function on \( \mathbb{C}_- \). Its poles describe the point spectrum of \( H_{aq} \) in \( \mathbb{C} \setminus [0, \infty) \).

Lemma 5.1. If \( z \in \mathbb{C}_- \) is a pole of \( S(z) \), then \( z^2 \) belongs to the point spectrum of \( H_{aq} \).

Proof. By virtue of (4.3), if \( z \in \mathbb{C}_- \) is a pole for \( S(z) \) then \( a = W(z^2) \). This identity means that \( z^2 \in \sigma_p(H_{aq}) \) because \( H_{aq} \) is defined by (5.5) and \( W(z^2) \) is the Weyl-Titchmarsh function associated to the boundary triplet \( (\mathcal{C}, \Gamma_0, \Gamma_1) \) (see Sec. 5.1 and [26, Proposition 14.17]).

Remark 5.2. It may happen that the \( S \)-matrix ‘does not hear’ an eigenvalue \( z^2 \). This is the case where the corresponding eigenfunction \( u_z \) is orthogonal to \( \psi(B)L_2(\mathbb{R}_+, \mathbb{C}^2) \) and, as a result, the coefficients \( c(z, q_i) \) vanish, see Sec. 5.1.1.
Divide the half-plane \( \mathbb{C}_- \) into three parts

\[
\mathbb{C}_- = \{ z : \Re z < 0 \}; \quad \mathbb{C}_0 = \{ z : \Re z = 0 \}; \quad \mathbb{C}_+ = \{ z : \Re z > 0 \}.
\]

**Lemma 5.3.** If \( S(z) \) has a pole in \( \mathbb{C}_+ \), then \( S(z) \) has to be analytical on the opposite part \( \mathbb{C}_- \). If \( S(z) \) has a pole on the middle part \( \mathbb{C}_0 \), then \( S(z) \) is analytical on \( \mathbb{C}_- \cup \mathbb{C}_+ \) and \( \mathbf{H}_{aq} \) is a self-adjoint operator.

**Proof.** Let \( z \in \mathbb{C}_- \) be a pole for \( S(z) \). By virtue of (4.8), \( a = W(z^2) \), where \( \Im z^2 > 0 \) and \( \Im a > 0 \) since \( \Im W(z^2)/\Im z^2 > 0 \) [26, Sec. 14.5]. Similar arguments for a pole \( z \in \mathbb{C}_+ \) lead to the conclusion that \( \Im a < 0 \). The obtained contradiction means that the existence of a pole in \( \mathbb{C}_+ \) (\( \mathbb{C}_- \)) implies the absence of poles in \( \mathbb{C}_- \) (\( \mathbb{C}_+ \)).

If \( z \in \mathbb{C}_0 \) is a pole, then \( \mathbf{H}_{aq} \) has a negative eigenvalue and \( \mathbf{H}_{aq} \) has to be self-adjoint due to [21, Corollary 5.2].

An eigenvalue \( z^2 \in \mathbb{C} \setminus [0, \infty) \) of \( \mathbf{H}_{aq} \) is called an **exceptional point** if its geometrical multiplicity does not coincide with the algebraic one. The presence of an exceptional point means that \( \mathbf{H}_{aq} \) cannot be self-adjoint for any choice of inner product. It follows from Lemma 5.3 that an exceptional point \( z^2 \) is necessarily non-real and \( z \in \mathbb{C}_- \cup \mathbb{C}_+ \).

**Lemma 5.4.** A non-simple pole\(^8\) \( z \) of \( S(z) \) corresponds to an exceptional point \( z^2 \) of \( \mathbf{H}_{aq} \).

**Proof.** A non-simple pole \( z \) of \( S(z) \) means that the function \( (a - W(\lambda))^{-1} \) has a non-simple pole for \( \lambda = z^2 \). This yields that \( W'(z^2) = 0 \), where \( W'(\lambda) = dW/d\lambda \). In view of [21, Theorem 5.4], an eigenvalue \( z^2 \) of \( \mathbf{H}_{aq} \) is an exceptional point if and only if \( W'(z^2) = 0 \).

**Lemma 5.5.** Let \( \mathbf{S}_{H_{aq}}(z) \) be a \( S \)-matrix of \( \mathbf{H}_{aq} \). Then

\[
\mathbf{S}_{H_{aq}}(z) = \mathbf{S}_{H_{aq}}(-z) = \mathbf{S}_{H_{aq}}(-\overline{z}).
\]

**Proof.** Using (4.8) for the calculation of the adjoint, we get

\[
\mathbf{S}_{H_{aq}}(z) = \begin{bmatrix} \Psi_1(z) & 0 \\ 0 & \Psi_2(z) \end{bmatrix} + \frac{2\pi i}{\pi - W(z^2)} \begin{bmatrix} c(-\overline{z}, q_1)c(z, q_1) & c(-\overline{z}, q_1)c(z, q_2) \\ c(-\overline{z}, q_2)c(z, q_1) & c(-\overline{z}, q_2)c(z, q_2) \end{bmatrix}.
\]

In view of Theorem 4.6 \( \Psi_j(z) = \Psi_j(-\overline{z}) \). Moreover, \( W(z^2) = W((-\overline{z})^2) \). This well-known property of the Weyl-Titchmarsh functions [26, Chap. 14] can easily be derived from (3.3). Taking these facts into account and using (4.3) for the calculation of \( \mathbf{S}_{H_{aq}}(-\overline{z}) \), we arrive at the conclusion that \( \mathbf{S}_{H_{aq}}(z) = \mathbf{S}_{H_{aq}}(-\overline{z}) \). Now, to complete the proof it suffices to remark that \( \mathbf{H}_{aq} = \mathbf{H}_{aq}^* \) due to (3.3) and [26, Lemma 14.6].

---

\(^8\) a pole of order greater then one
Corollary 5.6. Let $S(z)$ be a $S$-matrix of $H_{aq}$. Then $H_{aq}$ is self-adjoint if and only if $S^*(z) = S(-\overline{z})$.

Proof. If $H_{aq}$ is self-adjoint, then $a \in \mathbb{R}$ and $S^*(z) = S(-\overline{z})$ due to Lemma 5.5. Conversely, as follows from the proof above, the relation $S^*(z) = S(-\overline{z})$ is possible only in the case of real $a$. This implies the self-adjointness of $H_{aq}$. \qed

5.1 Examples

5.1.1 Even function $q$ with finite support.

We consider the simplest example of even function with finite support

$q(x) = M\chi_{[-\rho,\rho]}(x), \quad M \in \mathbb{C}, \quad \rho > 0.$

In this case, $Yq = q = M \begin{bmatrix} \chi_{[0,\rho]}(x) \\ \chi_{[0,\rho]}(x) \end{bmatrix}$.

Denote $\psi(\delta) = e^{i\delta \rho}$. The function $\psi$ belongs to $H_{\infty}(\mathbb{C}^+)$ and the operator $\psi(B)$ in (3.12) acts in $L^2(\mathbb{R}^+)$ as follows:

$\psi(B)f = \begin{cases} f(x - \rho) & \text{for } x \geq \rho \\ 0 & \text{for } x < \rho \end{cases}$ (5.1)

Further, we extend the action of $\psi(B)$ onto $L^2(\mathbb{R}^+, \mathbb{C}^2)$ assuming in (3.14) that $\psi_1(B) = \psi_2(B) = \psi(B)$. It follows from (5.1) that $\psi(B)^*f = f(x + \rho)$. Hence,

$P_{\delta_0}f = \psi(B)\psi(B)^*f = \begin{cases} f(x) & \text{for } x \geq \rho \\ 0 & \text{for } x < \rho \end{cases}$ (5.2)

The formula (5.2) and Lemma 3.2 imply that $q$ is non-cyclic. Therefore, for $H_{aq}$ there exists a $S$-matrix defined by (4.8). Let us specify the counterparts of (4.8). First of all we note that $\Psi_1(\delta) = \Psi_2(\delta) = e^{-2i\delta \rho}$ as the holomorphic continuation of $e^{-i\delta \rho} = \psi(\delta)$ into $\mathbb{C}^-$. Further, in view of (3.6),

$(H_{\infty} - \mu^2 I)^{-1}q = \frac{M}{2\mu^2}[(e^{-i\mu \rho} + e^{i\mu m(x)} - 2)e^{-i\mu x} + (e^{-i\mu m(x)} - e^{-i\mu \rho})e^{i\mu x}],$

where $m(x) = \min\{x, \rho\}$ and $\mu \in \mathbb{C}^-$. This formula and (4.4) lead to the conclusion that

$c(\mu, q_1) = c(\mu, q_2) = e^{-i\mu \rho}\left(1 - \kappa_\mu \frac{M}{\mu^2}\right), \quad \kappa_\mu = 1 - \cos \mu \rho.$

Our next step is the calculation of $W(z^2)$ using formula (9.9) and the expression for $(H_{\infty} - \mu^2 I)^{-1}$, that gives

$W(z^2) = -2iz - \frac{4Re M}{iz} (1 - e^{-iz \rho}) + \frac{|M|^2}{iz^2} [(e^{-iz \rho} - 2)^2 - 2iz \rho - 1].$
Substituting the expressions obtained above into (4.8) we find the $S$-matrix for $H_{\omega q}$

$$S(z) = e^{-2iz\rho} \left( \sigma_0 - \frac{2i(z^2 - \kappa_z M)(z^2 - \kappa_z \bar{M})}{z^3(a - W(z^2))} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right).$$

Let us assume that $z_0 \in \mathbb{C}_-$ satisfies the relation $z_0^2 - \kappa_{z_0} M = 0$ and $W'(z_0^2) \neq 0$. Set $a = W(z_0^2)$. Then the operator $H_{\omega q}$ has the eigenvalue $z_0^2$ with eigenfunction $u_{z_0}$. It follows from (3.8) and the explicit expression for $(H_{\infty} - \mu^2 I)^{-1}$ that

$$u_{z_0} = \frac{1 - \cos z_0(\rho - x)}{z_0^2} q.$$

In view of (5.2), the eigenfunction $u_{z_0}$ is orthogonal to $\mathcal{H}_0$ and it has no impact on the $S$-matrix $S(z)$ (no pole for $z = z_0$).

### 5.1.2 Odd function $q$ with finite support.

Similarly to the previous case, we consider the odd function

$$q(x) = M \text{sign}(x)\chi_{[-\rho,\rho]}(x), \quad M \in \mathbb{C}, \quad \rho > 0.$$ 

In this case, $q = M \begin{bmatrix} \chi_{[0,\rho]}(x) \\ -\chi_{[0,\rho]}(x) \end{bmatrix}$ is non-cyclic and it is orthogonal to the same subspace $\mathcal{H}_0 = \psi(B)L_2(\mathbb{R}^+, \mathbb{C}^2)$ as above. Further,

$$c(\mu, q_1) = e^{-i\mu\rho} \left( 1 - \kappa_\mu \frac{M}{\mu^2} \right), \quad c(\mu, q_2) = e^{-i\mu\rho} \left( 1 + \kappa_\mu \frac{M}{\mu^2} \right)$$

and $W(z^2) = -2iz + \frac{|M|^2}{iz^2} [(e^{-iz}\rho - 2)^2 - 2i\rho - 1]$. Then (4.8) takes the form:

$$S(z) = e^{-2iz\rho} \left( \sigma_0 - \frac{2iz}{a - W(z^2)} \begin{bmatrix} 1 - \kappa_2 \frac{2\Re M}{z^2} + \kappa_2 \frac{|M|^2}{z^2} & 1 - \kappa_2 \frac{2\Im M}{z^2} - \kappa_2 \frac{|M|^2}{z^2} \\ 1 + \kappa_2 \frac{2\Re M}{z^2} - \kappa_2 \frac{|M|^2}{z^2} & 1 + \kappa_2 \frac{2\Im M}{z^2} + \kappa_2 \frac{|M|^2}{z^2} \end{bmatrix} \right).$$

It is easy to see that the entries of the last matrix can not vanish simultaneously. This means that $z \in \mathbb{C}_-$ is a pole of $S(z)$ if and only if $a = W(z^2)$. Therefore, in contrast to Sec. 5.1.1, the poles of $S(z)$ completely determine the point spectrum of $H_{\omega q}$ in $\mathbb{C} \setminus \mathbb{R}_+$.

### 5.1.3 Functions $q$ with infinite support.

The range of applicability of our results is not limited to operators $H_{\omega q}$, where $q = Yq$ has finite support. Due to Lemma [3.2] and Theorem [1.3] the $S$-matrix (4.8) can be constructed for an operator $H_{\omega q}$ when $q$ is non-cyclic with respect to the backward shift operator $T^*$ in $L_2(\mathbb{R}^+, \mathbb{C}^2)$. Various examples of non-cyclic functions can be found in [13, 17]. Consider, for instance, the function $q(x) = P_m(x)e^{-|x|}$, where $P_m$ is a polynomial of order $m$. Then

$$q = \begin{bmatrix} P_m(x) \\ P_m(-x) \end{bmatrix} e^{-x}, \quad x \geq 0.$$
Decompose the functions $P_m(\pm x)e^{-x} \in L_2(\mathbb{R}_+)$:

$$e^{-x}P_m(x) = \sum_{n=0}^{m} c_n q_n(2x), \quad e^{-x}P_m(-x) = \sum_{n=0}^{m} d_n q_n(2x), \quad (5.3)$$

with respect to the orthonormal basis of the Laguerre functions

$$q_n(x) = \frac{e^{x/2}}{n!} \frac{d^n}{dx^n} (x^n e^{-x}), \quad n = 0, 1 \ldots$$

Using the relation $T q_n(2x) = q_{n+1}(2x)$ [3, p. 363], where $T$ is defined by (3.11) and taking (5.3) into account we arrive at the conclusion that $q$ is orthogonal to the subspace $T^{m+1}L_2(\mathbb{R}_+) = \psi(\mathbb{B})L_2(\mathbb{R}_+)$, where $\psi(\delta) = (\frac{\delta - i}{\delta + i})^{m+1}$ belongs to $H^\infty(\mathbb{C}_+)$. Hence, $q$ is a non-cyclic function and for operators $H_{aq}$ there exist $S$-matrices defined by (4.8).

Let us calculate the $S$-matrix for the function $q(x) = Me^{-|x|}$. In this case, one can set $m = 0$, $\psi(\delta) = \frac{\delta - i}{\delta + i}$, and $\Psi_1(z) = \Psi_2(z) = (\frac{z + i}{z - i})^2$ as the holomorphic continuation of $\frac{\psi(-\delta)}{\psi(\delta)} = (\frac{\delta - i}{\delta + i})^2$ into $\mathbb{C}_-$. Further,

$$(H_\infty - z^2 I)^{-1}e^{-x} = \frac{e^{-ix} - e^{-x}}{1 + z^2}, \quad W(z) = -2iz - \frac{4Re M}{1 + iz} + \frac{|M|^2}{(1 + iz)^2}.$$

It follows from (4.4) and the Poisson formula [24, p.147] that

$$c(\mu, q_i) = \frac{\mu + i}{\mu - i} - \frac{M}{(\mu - i)^2} = \frac{\mu^2 + 1 - M}{(\mu - i)^2}.$$

After substitution of the expressions above into (4.8) and elementary transformations we find

$$S(z) = \left(\frac{z + i}{z - i}\right)^2 \left(\sigma_0 - \frac{2iz(1 - \frac{M}{z + i})(1 - \frac{\sqrt{\lambda}}{z + i})}{a - W(z^2)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right).$$

Let us assume for the simplicity that $M \in i\mathbb{R}$. Then

$$S(z) = \left(\frac{z + i}{z - i}\right)^2 \left(\sigma_0 - \frac{2iz(1 - \frac{|M|^2}{(z + i)^2})}{a - W(z^2)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) \quad (5.4)$$

and $W(\lambda) = -2i\sqrt{\lambda} + \frac{|M|^2}{(1 + i\sqrt{\lambda})^2}$, where $\lambda = z^2$ and $\sqrt{\lambda} = z$.

Since the first derivative of $W(\lambda)$ is

$$W'(\lambda) = -\frac{i}{\sqrt{\lambda}} \left(1 + \frac{|M|^2}{(1 + i\sqrt{\lambda})^3}\right),$$

it follows from (4.4) and the Poisson formula that

$$c(\mu, q_i) = \frac{\mu + i}{\mu - i} - \frac{M}{(\mu - i)^2} = \frac{\mu^2 + 1 - M}{(\mu - i)^2}.$$
the equation $W'(\lambda) = 0$ have the following roots $\lambda_j = z_j^2$, $j \in \{1, 2, 3\}$, where

$$z_1 = -\frac{\sqrt{3}}{2}|M|^\frac{3}{2} + i(1 - \frac{1}{2}|M|^\frac{3}{2}), \quad z_2 = -\overline{z_1}, \quad z_3 = i(|M|^\frac{3}{2} + 1).$$

Assume that $|M|^2 > 8$. Then $z_1, z_2 \in \mathbb{C}_\pm$. Denote $a = W(z_1^2)$. Then the $S$-matrix (5.4) has a non-simple pole for $z = z_1$ and, by Lemma 5.4, the operator $H_{aq}$ has an exceptional point $z_1^2$. (The choice of $z_2 = -\overline{z_1}$ instead of $z_1$ leads to the conclusion that the point $z_1^2$ is exceptional for the adjoint operator $H_{aq}^* = H_{aq}$.)

The obtained result shows that the existence of exceptional points for some operators of the set $\{H_{aq}\}_{a \in \mathbb{C}}$, where $q(x) = Me^{-x}$, $M \in i\mathbb{R}$ depends on the absolute value of the imaginary $M$. If $|M|^2 > 8$, then there exist two operators $H_{aq}$ and $H_{aq}$ with the exceptional points $z_1^2$ and $\overline{z_1^2}$, respectively. On the other hand, if $|M|$ is sufficiently small ($|M|^2 \leq 8$), then the collection of operators $\{H_{aq}\}_{a \in \mathbb{C}}$ has no exceptional points.

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References
[1] V. M. Adamyan, Nondegenerate unitary couplings of semiunitary operators, Funct. Anal. Appl. 7 (1973) 4, 255–267.
[2] V. M. Adamyan, B.S. Pavlov, Null-range potentials and M.G. Krein’s formula for generalized resolvents, J. Sov. Math. 42 (1988), 1537–1550.
[3] N.I. Akhiezer, I.M. Glazman, Theory of Linear Operators in Hilbert Spaces, Dover Publication Inc, New York, 1993.
[4] S. Albeverio, L. Nizhnik, Schrödinger operators with nonlocal point interactions, J. Math. Anal. Appl. 332 (2007), 884–895.
[5] S. Albeverio, L. Nizhnik, Schrödinger operators with nonlocal potentials, Methods Funct. Anal. Topology 19 (2013) 3, 199–210.
[6] S. Albeverio, R. Hryniv, L. Nizhnik, Inverse spectral problems for nonlocal Sturm-Liouville operators, Inverse problems 23 (2007), 523–536.
[7] F. Bagarello, J.-P. Gazeau, F. H. Szafraniec, M. Znojil, (eds.): Non-Selfadjoint Operators in Quantum Physics. Mathematical Aspects, J. Wiley & Sons, Hoboken, New Jersey, 2015.
[8] C. M. Bender, et al: PT-Symmetry in Quantum and Classical Physics, World Scientific, Singapore, 2019.
[9] J. Behrndt, M.M. Malamud, H. Neidhardt, Scattering matrices and Dirichlet-to-Neumann maps, J. Funct. Anal. 273 (2017), 1970–2025.

[10] J. Brasche, L. P. Nizhnik, One-dimensional Schrödinger operators with general point interactions, Methods Funct. Anal. Topology 19 (2013) 1, 4–15.

[11] K.D. Cherednichenko, A.V. Kiselev, L.O. Silva, Functional model for extensions of symmetric operators and applications to scattering theory, Networks and Heterogeneous Media 13 (2018) 2, 191–215.

[12] P. A. Cojuhari, S. Kuzhel, Lax-Phillips scattering theory for PT-symmetric ρ-perturbed operators, J. Math. Phys. 53 (2012), 073514 .

[13] R. G. Douglas, H. S. Shapiro, A. L. Shields, Cyclic vectors and invariant subspaces for the backward shift operator, Ann. Inst. Fourier 20 (1970), 37–76.

[14] M. Gawlik, A. Głowczyk, S. Kuzhel, On the Lax-Phillips scattering matrix of the abstract wave equation, Banach J. Math. Anal. 13 (2019) 2, 449–467.

[15] M. L. Gorbachuk, V. I. Gorbachuk, Boundary-Value Problems for Operator Differential Equations, Kluwer, Dordrecht, 1991.

[16] S. Kuzhel, On the determination of free evolution in the Lax-Phillips scattering scheme for second-order operator-differential equations, Math. Notes 68 (2000), 724-729.

[17] S. Kuzhel, Nonlocal perturbations of the radial wave equation. Lax-Phillips approach, Methods Funct. Anal. Topology 8 (2002) 2, 59-68.

[18] S. Kuzhel, On the inverse problem in the Lax-Phillips scattering theory method for a class of operator-differential equations, St. Petersburg Math. J. 13 (2002), 41-56.

[19] S. Kuzhel, On conditions of applicability of the Lax-Phillips scattering scheme to investigation of abstract wave equation, Ukrainian Math. J. 55 (2003), 621-630.

[20] A. Kuzhel, S. Kuzhel, Regular Extensions of Hermitian Operators, VSP, Utrecht, 1998.

[21] S. Kuzhel, M. Znojil, Non-self-adjoint Schrödinger operators with nonlocal one-point interactions, Banach J. Math. Anal. 11 (2017) 4, 923-944.

[22] P. Lax, Translation invariant spaces, Acta Math. 101 (1959), 163–178.

[23] P. Lax, R. Phillips, Scattering Theory, Revised Edition, Academic Press, London, 1989.

[24] N. K. Nikolski, Operators, Functions, and Systems: an Easy Reading, Volume I, AMS, USA, 2002.
[25] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Volume III: Scattering Theory*, Academic Press, New York[etc.], 1979.

[26] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Springer, Berlin, 2012.

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