Quantum Error Correcting Subsystems are Unitarily Recoverable Subsystems

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We show that every correctable subsystem for an arbitrary noise operation can be recovered by a unitary operation, where the notion of recovery is more relaxed than the notion of correction insofar as it does not protect the subsystem from subsequent iterations of the noise. We also demonstrate that in the case of unital noise operations one can identify a subset of all correctable subsystems —those that can be corrected by a single unitary operation— as the noiseless subsystems for the composition of the noise operation with its dual. Using the recently developed structure theory for noiseless subsystems, the identification of such unitarily correctable subsystems is reduced to an algebraic exercise.

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A basic principle in quantum computation, communication and cryptography is the requirement that if quantum information encoded in a physical system is preserved during some storage or transmission process, then the information is represented at every time on a subsystem of the overall system. This heuristic concept has been formalized as part of the “subsystem principle” for realizing quantum information\(^1\)\(^2\), and provides a central motivation for quantum error correction techniques. Recently a framework for studying correctable subsystems for arbitrary quantum operations was introduced in \(^3\)\(^4\) under the moniker “operator quantum error correction”. The approach includes standard quantum error correcting (subspace) codes\(^5\)\(^6\)\(^7\)\(^8\)\(^9\), and decoherence-free/noiseless subspaces and subsystems\(^10\)\(^11\)\(^12\)\(^13\)\(^14\)\(^15\)\(^16\). Although it does not yield new codes, this approach enables more efficient correction procedures\(^17\)\(^18\)\(^19\).

Intuitively, the subsystem principle implies that if a subsystem is correctable for a given noise operation, then at the output stage of the transmission there should be an identifiable copy of the subsystem within the output Hilbert space. Our first main result in this paper formalizes this notion and thereby contributes to the rigorous formulation of the subsystem principle for quantum information. Specifically, we show that every correctable subsystem for an arbitrary noise operation only suffers a change in representation due to that operation. A unitary operation is therefore sufficient to recover from the noise, in the sense of restoring the quantum information to the subsystem in which it was encoded, but due to the fact that complementary subsystems are not necessarily preserved by the noise, the system is not necessarily protected from subsequent iteration of the noise operation. We therefore say that the subsystem is only recovered, but not corrected, by a unitary operation.

Quantum error-correcting codes are typically designed with particular noise models in mind. Given the myriad different physical implementations for quantum information processing and communication, and the diverse nature of the noise that may afflict these, techniques for identifying codes for arbitrary noise models are clearly of interest. To date, this has only been achieved for passive quantum error correction. It is accomplished by identifying the decoherence-free/noiseless subsystems for the noise model. A recently developed structure theory for passive error correction\(^2\)\(^20\) reduces the identification of such codes to an exercise in matrix algebra. (The algorithm is polynomial in the dimension of the system Hilbert space and can be implemented with available computer software.) Our second main result demonstrates that in certain cases finding active quantum error correcting codes reduces to the problem of finding passive codes. Specifically, in the case of a noise operation that is unital but otherwise arbitrary, we identify a subset of the actively correctable subsystems, those that can be corrected by a single unitary operation, as precisely the noiseless subsystems for the composition of the operation with its dual. Hence the identification of such actively correctable subsystems may be accomplished via an application of the passive theory\(^2\)\(^20\).

Let us first discuss nomenclature. Given a quantum system \(S\) represented on a (finite-dimensional) Hilbert space \(\mathcal{H}^S \equiv \mathcal{H}\), we say a quantum system \(B\) is a subsystem of \(S\) if there is a representation of \(B\) on a Hilbert space \(\mathcal{H}^B\) such that

\[
\mathcal{H}^S = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K},
\]

where \(A\) is also a subsystem of \(S\) and \(\mathcal{K} = (\mathcal{H}^A \otimes \mathcal{H}^B)^\perp\). We are interested in cases \(\dim \mathcal{H}^B > 1\), so at least one qubit can be encoded. We adopt the convention that \(\rho\), \(\sigma\) and \(\tau\) denote density operators and a superscript such as \(\sigma^S\) refers to the subsystem on which the operator is
Of course, subsystem that leads to the following notion. Associated with a Hilbert space of different dimension. For convenience we shall assume the noise operation maps \( \mathcal{H} \) to \( \mathcal{H}' \), but our results are easily adapted to the general setting simply by replacing \( \mathcal{H} \) with \( \mathcal{H}' \) at the output stage. Such maps describe time evolution of open quantum systems (in the Schrödinger picture), and can always be represented in the operator-sum representation 
\[
\mathcal{E}(\sigma) = \sum_a E_a \sigma E_a^\dagger
\]
by a set of “Kraus” operators \( \{E_a\} \) on \( \mathcal{H} \). The composition of two superoperators will be denoted by \( \mathcal{E} \circ \mathcal{F} \). The dual or adjoint map \( \mathcal{E}^\dagger \) is the superoperator on \( \mathcal{L}(\mathcal{H}) \) defined in the usual way by the equation 
\[
\text{Tr}(\mathcal{E}(\sigma) \tau) = \text{Tr}(\sigma \mathcal{E}(\tau)).
\]
A unitary operation \( \mathcal{U} \) satisfies \( \mathcal{U}^\dagger \circ \mathcal{U} = \mathcal{U} \circ \mathcal{U}^\dagger = \text{id} \), where \( \text{id} \) is the identity superoperator. This is equivalent to the existence of a unitary \( \mathcal{U} \in \mathcal{L}(\mathcal{H}) \) that implements the operation via the equation \( \mathcal{U}(\sigma) = \mathcal{U} \sigma \mathcal{U}^\dagger \). Given a decomposition of \( \mathcal{H} \) as in Eq. (1), \( \mathcal{P}_{AB} \) denotes the superoperator defined by 
\[
\mathcal{P}_{AB}(\sigma) = \mathcal{P}_{AB} \sigma \mathcal{P}_{AB}^\dagger
\]
where \( \mathcal{P}_{AB} \) is the projector onto the subspace \( \mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B \), and let \( \text{id}_B \) be the identity superoperator on \( \mathcal{L}(\mathcal{H}^B) \). If we are given maps \( \mathcal{E}_A \), \( \mathcal{E}_B \) on the subsystems, as a notational convenience we write \( \mathcal{E}_A \otimes \mathcal{E}_B \) both as a map on \( \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B) \) and for the natural extension of the map to \( \mathcal{L}(\mathcal{H}) \).

The definition from [3, 4] of a correctable subsystem for a quantum operation \( \mathcal{E} \) is as follows.

**Definition:** A recovery operation. In both cases, quantum information stored in subsystem \( B \) can be brought back to \( B \) after a single iteration of the noise operation \( \mathcal{E} \), but only if one implements a correction operation will \( B \) be recoverable from a subsequent iteration of the noise operation \( \mathcal{E} \). The reason is that if one only implements a recovery operation, then the state is left in \( CB \), while quantum information in \( B \) is only guaranteed to be recoverable if the initial state is in \( AB \).

Furthermore, when we constrain the nature of the recovery and correction operations, the two notions no longer coincide. Specifically, we can introduce the following two notions.

**Definition:** A unitarily correctable subsystem for an operation \( \mathcal{E} \) if it satisfies the definition of a correctable subsystem and the correction operation \( \mathcal{R} \) can be chosen to be a unitary \( \mathcal{U} \), so that 
\[
\mathcal{U} \circ \mathcal{E} \circ \mathcal{P}_{AB} = \mathcal{F}_A \otimes \text{id}_B.
\]

**Definition:** A unitarily recoverable subsystem for an operation \( \mathcal{E} \) if it satisfies the definition of a recoverable subsystem and the recovery operation \( \mathcal{R} \) can be chosen to be a unitary \( \mathcal{U} \), so that 
\[
\mathcal{U} \circ \mathcal{E} \circ \mathcal{P}_{AB} = \mathcal{F}_{C|A} \otimes \text{id}_B.
\]

Note that by operating with \( \mathcal{U}^\dagger \) from the left on Eqs. (7) and (8), we obtain an expression for the action of the noise operation on \( AB \). Heuristically, being
unitarily correctable or recoverable ensures that although quantum information stored in the subsystem $B$ may be moved to another subsystem, it remains coherent. In the general case of possibly distinct input and output Hilbert spaces, we note that the relevant notion to consider is that of isometric recovery.

Although a unitarily correctable subsystem is unitarily recoverable, the converse is not true unless $C$ satisfies $\dim C = \dim A$ because only in this case can one find a unitary operation $R'$ that maps all states of subsystem $C$ to states of $A$. Thus all we can say in this case is that a unitarily correctable subsystem is equivalent to a unitarily recoverable subsystem for which $\dim C$ is at most $\dim A$.

One might expect that not every correctable subsystem is unitarily correctable, and this is indeed the case. However, every correctable subsystem is unitarily recoverable. The converse of this implication is also true, and together these facts constitute our first main result.

**Theorem 1** Let $E$ be a quantum operation. The following conditions are equivalent:

(i) $B$ is a correctable subsystem for $E$.

(ii) $B$ is a unitarily recoverable subsystem for $E$.

The proof is presented in Appendix A. Note that it is a constructive proof in the sense that if an error model $E$ and a correctable subsystem $B$ are known, then the unitary operation which yields Eq. (2) may be explicitly obtained. Moreover, as discussed above, a correction operation can be obtained simply by composing $U$ with an additional operation $R'$ that maps states of $C$ to states of $A$ such that $R = R' \circ U$ satisfies Eq. (3).

There are two special cases of this theorem that are worthy of note. First, consider the case where $B$ is a correctable subsystem for $E$ and $H^B$ is one-dimensional, so that $H^A \otimes H^B$ is a correctable subspace for $E$. In this case, $F_{C|A}$ is simply a density operator $\omega_C$, and Eq. (3) becomes $\forall \sigma^B : U \circ E(\sigma^B) = \omega_C \otimes \sigma^B$. Theorem 1 therefore establishes that a correctable subspace for $E$ is mapped by $E$ to a subspace, modulo an overall unitary. This special case was proven by Nayak and Sen [22]. That the subspace gets mapped to a subsystem may be viewed from our general perspective as simply a representation change of the subsystem $B$.

The second special case of the theorem established previously is the case $H = H^A \otimes H^B$, or equivalently $K = \{0\}$ in Eq. (1). In other words, $B$ is associated with a factor space of $H$ rather than a factor space of a subspace of $H$. In this case, any subsystem $C$ in Eq. (3) must have the same dimension as the subsystem $A$. Consequently, there is no distinction between unitarily recoverable and unitarily correctable in this case. Dropping the projection operation $P_{AB}$, which is the identity operator in this case, Eq. (3) becomes $U \circ E = F_A \otimes id_B$ or equivalently, $E = U^\dagger \circ (F_A \otimes id_B)$. Theorem 1 therefore establishes that if a factor space $B$ is a correctable subsystem for $E$, then $E$ simply maps $B$ unitarily to another factor space. This case was proven by Nielsen and Poulin [23].

Theorem 1 allows us to immediately make a nice connection with the testable conditions for operator quantum error correction. Taking the adjoint of Eq. (5), we have

$$P_{AB} \circ E^\dagger \circ U = F_{C|A}^\dagger \otimes id_B. \quad (9)$$

Composing Eqs. (7) and (9), noting that $U^\dagger \circ U = id$ and defining the map $G_A : B(H_A) \rightarrow B(H_A)$ by $G_A = F_{C|A}^\dagger \circ F_{C|A}$, we obtain

Testable condition for $B$ to be a correctable subsystem for $E$ : There exists a positive superoperator $G_A$ on $A$ such that

$$P_{AB} \circ E^\dagger \circ E \circ P_{AB} = G_A \otimes id_B. \quad (10)$$

This is simply the superoperator form of the standard testable condition for operator quantum error correction [3, 4, 9]. This is seen by noting that two completely positive maps are equal if and only if the Kraus operators of one are related by a unitary remixing of the Kraus operators of the other [21]. If the Kraus operators of $E$ are denoted by $\{E_a\}$ and those of $G_A$ are denoted $\{G_{lc}\}$, then the Kraus operators of $P_{AB} \circ E^\dagger \circ E \circ P_{AB}$ are $\{P_{AB} E^\dagger_a E_a P_{AB}\}_{a,b}$ and those of $G_A \otimes id_B$ are $\{G_c \otimes I_B\}_c$, so that Eq. (10) is equivalent to

$$P_{AB} E^\dagger_a E_a P_{AB} = F_{ab} \otimes I_B \quad (11)$$

where $F_a \equiv \sum_c u_{ac} G_c$ is a unitary remixing of the $G_c$. This is the testable condition discovered in [3, 4]. It follows from the results of [22] that not only can one derive Eq. (10) from the fact that $B$ is a correctable subsystem for $E$, as we have done, but the opposite implication holds as well. (It would be interesting to have a version of this direction of the implication that is native to the superoperator formalism.)

If $A$ is one-dimensional, $B$ is a subspace of $S$ and the positive superoperator $G_A$ is simply a positive scalar $\gamma^2$, so that we have $P_B \circ E^\dagger \circ E \circ P_B = \gamma^2 P_B$. This is the superoperator form of the testable condition for correctable subspaces. The condition was presented in this form in [24].

Finding correctable subsystems (or subspaces) for arbitrary noise models in full generality appears to be an intractable problem. Nevertheless, it has been possible to do so in several special cases, and this has been one of the successes of quantum error correction. The case of passive error correction is one such instance. As an application of Theorem 1 together with further analysis, we next show how recently developed techniques for passive error correction [2, 20] can be extended to a special case of active error correction. The notion of a noiseless
Lemma, which applies to arbitrary positivity-preserving unital operations presented in Appendix B. The following the definition of Eq. (3). (The correction in this case sim-
and thus constitutes a correction operation that satisfies
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by Corollary 3 (provided in Appendix B), we have
one sees that the leading \( P_{AB} \) can be dropped if \( \text{supp}(E^\dagger \circ E(P_{AB})) \subseteq \text{supp}(P_{AB}) \), so it remains only to show that the latter condition is satisfied if \( B \) is unitarily correctable.
Since \( B \) is a unitarily correctable subsystem for \( E \), there exists a unitary operation \( U \) such that \( U \circ E \circ P_{AB} = \mathcal{F}_A \otimes \text{id}_B \), from which one finds that \( E(P_{AB}) = U^\dagger (\mathcal{F}_A(I_A) \otimes I_B) \). It follows that \( \text{rank}(E(P_{AB})) = \text{rank}(\mathcal{F}_A(I_A)) \text{rank}(I_B) \leq \text{rank}(I_A) \text{rank}(I_B) = \text{rank}(P_{AB}) \), where we have made use of the fact that \( P_{AB} = I_A \otimes I_B \). But \( E \) is unital, so the leading \( \mathcal{F}_C \) (provided in Appendix B), we have
one concludes that there must be equality between the two quantities, that is, \( \text{rank}(E(P_{AB})) = \text{rank}(P_{AB}) \). By Lemma 3 it follows that \( \text{supp}(E^\dagger \circ E(P_{AB})) \subseteq \text{supp}(P_{AB}) \).
We now prove \((ii) \Rightarrow (i)\). From the fact that \( B \) is a noiseless subsystem, we infer that \( \text{supp}(E^\dagger \circ E(P_{AB})) \subseteq \text{supp}(P_{AB}) \). By Lemma 4 it follows that \( \text{rank}(E(P_{AB})) = \text{rank}(P_{AB}) \). Now, given that \( B \) is a correctable subsystem for \( E \) (as established in the discussion above Theorem 2), by Theorem 4 it is also a unitarily recoverable subsystem for \( E \), so that \( \mathcal{F}_C \) in this condition.

Theorem 2 Let \( E \) be a unital quantum operation. The following are equivalent:

(i) \( B \) is a unitarily correctable subsystem for \( E \).
(ii) \( B \) is a noiseless subsystem for \( E^\dagger \circ E \).

The proof relies on a number of ancillary results for unital operations presented in Appendix B. The following lemma, which applies to arbitrary positivity-preserving superoperators, does not rely on these results and hence we prove it here.

Lemma 3 Let \( O \) be a positivity-preserving superoperator. Let \( P \) be a projector and let \( P(\cdot) = P(\cdot)P \) be the associated projective superoperator. Then

\[
P \circ O \circ P = O \circ P
\]

if and only if

\[
\text{supp}(O(P)) \subseteq \text{supp}(P),
\]

where \( \text{supp}(\sigma) \) denotes the support of the operator \( \sigma \).

Proof. The superoperator identity Eq. (14) implies \( PO(P)P = O(P) \) which is equivalent to Eq. (15). Conversely, suppose Eq. (16) holds and let \( \rho \) be an arbitrary density operator. Noting that \( P - \mathcal{P}(\rho) \) is a positive operator, it follows from the fact that \( O \) is linear and positivity-preserving that \( O(P) - O \circ \mathcal{P}(\rho) \) is also positive and consequently \( \text{supp}(O \circ \mathcal{P}(\rho)) \subseteq \text{supp}(O(P)) \). Given Eq. (15), we have \( \text{supp}(O \circ \mathcal{P}(\rho)) \subseteq \text{supp}(P) \). Finally, given that \( \rho \) is arbitrary, Eq. (14) follows. QED
and define a unital channel \( \mathcal{V} \circ \mathcal{U} \circ \mathcal{E} \circ \mathcal{P} \mathcal{A}_{B} = \mathcal{V} \circ \mathcal{F}_{C(A) \otimes \mathcal{I}_{B}} = \mathcal{F}_{A} \otimes \mathcal{I}_{B} \), which by Eq. (4) implies that the unitary operation \( \mathcal{V} \circ \mathcal{U} \) is a correction operation for \( \mathcal{E} \), and thus that \( B \) is unitarily correctable.

By combining Theorem 2 with the recently developed structure theory for noiseless subsystems [22, 20], we obtain a method for finding the unitarily correctable subsystems for any unital operation \( \mathcal{E} \). Specifically, the noiseless subsystems of \( \mathcal{E} \circ \mathcal{E} \circ \mathcal{E} \) are obtained from the fixed point set \( \text{Fix}(\mathcal{E} \circ \mathcal{E} \circ \mathcal{E}) = \{ \sigma : (\mathcal{E} \circ \mathcal{E} \circ \mathcal{E})(\sigma) = \sigma \} \), in the following way. This set is a \( \mathcal{A} \)-algebra \( \mathcal{A} \) and the representation theory for such algebras induces a Hilbert space decomposition \( \mathcal{H} = \bigoplus_{k} (\mathcal{H}^{A_{k}} \otimes \mathcal{H}^{B_{k}}) \) in which \( B \) is a noiseless subsystem for \( \mathcal{E} \circ \mathcal{E} \circ \mathcal{E} \) if and only if \( \mathcal{H}^{B} \subseteq \mathcal{H}^{B_{k}} \) for some \( k \) (See [22, 20] for further discussions and analysis.) The fixed point set of an operation \( \mathcal{E} \) is simply the eigenvalue-1 operator eigenspace of the superoperator \( \mathcal{E} \), which is straightforward to determine.

For an arbitrary unital noise operation \( \mathcal{E} \), not every correctable subsystem is unitarily correctable, and consequently the noiseless subsystems of \( \mathcal{E} \circ \mathcal{E} \circ \mathcal{E} \) do not in general capture all correctable codes for a typical unital channel \( \mathcal{E} \).

The generic two-qubit “binary unitary channels” provide a class of operations that illustrate this point [27]. As an example, let \( U \) be a unitary on \( \mathcal{H} = \mathbb{C}^{4} \otimes \mathbb{C}^{2} \) with distinct eigenvalues \( \lambda_{j} = \exp(i\theta_{j}) \) ordered so that \( 0 < \theta_{1} < \theta_{2} < \theta_{3} < \theta_{4} < 2\pi \). Let \( |\psi_{j}\rangle, j = 1, 2, 3, 4 \), be corresponding eigenstates. Fix a probability \( 0 < p < 1 \), and define a unit channel \( \mathcal{E} \) by \( \mathcal{E}(\sigma) = p \mathcal{U} + (1-p) \mathcal{U} \mathcal{U}^{\dagger} \). Then \( \mathcal{E} \) has two Kraus operators given by, up to normalization, \( \mathcal{E} = \{ I, U \} \). The noiseless subsystems (actually subspaces in this case) for \( \mathcal{E} \circ \mathcal{E} \) come from the so-called noise commutant \( \{ I \mathcal{U}, I \mathcal{U}^{\dagger}, U \mathcal{U}, U \mathcal{U}^{\dagger} \} = \{ U, U^{\dagger} \} \), which coincides with the fixed point set. But since \( U \) has distinct eigenvalues, this commutant is isomorphic as a \( \mathcal{A} \)-algebra to the algebra \( \mathcal{C} \oplus \mathcal{C} \oplus \mathcal{C} \oplus \mathcal{C} \), and hence can only be used to encode classical information.

On the other hand, this channel has correctable qubit codes. For example, let \( \lambda \) be the point of intersection of the line segments \( [\lambda_{1}, \lambda_{3}] \) and \( [\lambda_{2}, \lambda_{4}] \). Let \( s \) and \( t \) be fixed probabilities such that \( \lambda = s \lambda_{1} + (1-s) \lambda_{3} = t \lambda_{2} + (1-t) \lambda_{4} \), and define states \( |\psi \rangle, |\phi \rangle \) by

\[
|\psi \rangle = \sqrt{s}|\psi_{1}\rangle + \sqrt{1-s}|\psi_{3}\rangle,
|\phi \rangle = \sqrt{t}|\psi_{2}\rangle + \sqrt{1-t}|\psi_{4}\rangle.
\]

The two-dimensional subspace spanned by \( |\psi \rangle \) and \( |\phi \rangle \) is a correctable code for \( \mathcal{E} \). In particular, one can compute that \( \mathcal{P} \mathcal{U} \mathcal{P} = \lambda \mathcal{P} \) where \( \mathcal{P} = |\psi \rangle \langle \psi | + |\phi \rangle \langle \phi |, \) and thus the error correction condition from [2] is satisfied for \( \mathcal{E} \) on the subspace \( \mathcal{E} = \text{span} \{ |\psi \rangle, |\phi \rangle \} \).

There are many unitary noise operations, however, for which the composition of this map with its dual does have noiseless subsystems. As a simple example, consider the swap operation \( |\psi \rangle \otimes |\phi \rangle \mapsto |\phi \rangle \otimes |\psi \rangle \) on a composite quantum system \( \mathcal{H} = \mathcal{H}^{A} \otimes \mathcal{H}^{A} \) made up of a subsystem \( \mathcal{A} \) and a replication \( \mathcal{R}^{A} = \mathcal{H}^{A} \). It is clear that both the subsystem \( \mathcal{H}^{A} \) and its copy can be returned to their initial locations by simply applying the swap operation again (which is equal to its dual).

Of course, one could note that the swap operation itself has a noiseless subsystem of the same size: namely the symmetric space \( |\psi \rangle \otimes |\psi \rangle \). But it is easy to find examples of operations with no noiseless subsystem, for which the composition map has a non-trivial noiseless subsystem.

To this end, consider a two-qubit system exposed to decoupled phase flips. The associated error model satisfies \( \mathcal{E}(\rho) = p Z_{1} \rho Z_{1} + (1-p) Z_{2} \rho Z_{2} \) for some fixed probability \( 0 < p < 1 \) and \( Z_{1} = Z \otimes \mathbb{I}_{2}, Z_{2} = \mathbb{I}_{2} \otimes Z \). In this case there is no noiseless subsystem (or subspace) for \( \mathcal{E} \). This follows from the fact that the noise commutant \( \{ Z_{1}, Z_{2} \} \) is isomorphic to the algebra \( \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \). Thus, only classical information can be safely sent through the channel unscathed.

However, the operators supported on the subspace spanned by \( |0_{L}\rangle = |00\rangle \) and \( |1_{L}\rangle = |11\rangle \) form a noiseless subspace for \( \mathcal{E} \circ \mathcal{E} \). Indeed, the set of operators \( \sigma = a|00\rangle \langle 00| + b|11\rangle \langle 11| + c|01\rangle \langle 01| + d|10\rangle \langle 10| \) form a subalgebra of the commutant \( \text{Fix}(\mathcal{E} \circ \mathcal{E}) = \{ Z_{1} Z_{2}, Z_{1}^{3} Z_{2}, Z_{1}^{3} Z_{2}^{3} Z_{2}^{3} \} = \{ Z_{1}, Z_{2} \} \). The unitary correction operation guaranteed by Theorem 2 in this case happens to be the controlled phase flip operation \( \mathcal{U} = |00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| - |11\rangle \langle 11| \).

Conclusion. — We showed that every correctable subsystem for an arbitrary quantum operation is a unitarily recoverable subsystem for the operation. Thus, the effect of the operation on the subsystem can be reversed by a single unitary operation, up to a change in the representation of the subsystem. Our proof was constructive in nature, showing explicitly how the representation and unitary may be obtained, and hence a correction operation, if the operation and correctable subsystem are known. We also suggested that this result contributes to the rigorous formulation of the subsystem principle for quantum operations.

We showed that the unitarily correctable subsystems for unital quantum operations are precisely the noiseless subsystems for the operation followed by its dual. We indicated how such subsystems can be practically computed and discussed some simple examples. The possibility of extending this result to the case of nonunital quantum operations remains a problem for future research.

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APPENDIX A: PROOF OF THEOREM

Proof of Theorem. The implication (ii) \( \Rightarrow (i) \) follows immediately from the fact that a unitarily recoverable subsystem is a special case of a recoverable subsystem, and as discussed below Eq. \( \mathcal{E} \) being recoverable implies being correctable.

For (i) \( \Rightarrow (ii) \), let \( \mathcal{E} = \{ E_a \} \) be a Kraus operator representation for \( \mathcal{E} \) and assume \( B \) is correctable for \( \mathcal{E} \).

We proceed as follows. First we shall construct an operation \( \mathcal{G} = \{ G_a \} \) such that the operators \( \{ G_aP_{AB} \} \) have mutually orthogonal ranges, that is \( P_{AB}G_a^\dagger G_bP_{AB} = 0 \) for all \( a \neq b \), and \( \mathcal{G}(I_A \otimes \sigma^B) = \mathcal{E}(I_A \otimes \sigma^B) \) for all \( \sigma^B \). Then we shall find a subsystem \( C \) such that \( C \) can be identified with a subspace of \( \mathcal{H} \), an operation \( \mathcal{F}_{C|A} : \mathcal{L}(\mathcal{H}^A) \rightarrow \mathcal{L}(\mathcal{H}^C) \), and a unitary operation \( \mathcal{V} \) on \( \mathcal{L}(\mathcal{H}) \), such that

\[
\mathcal{G}(\sigma^A \otimes \sigma^B) = \mathcal{V}(\mathcal{F}_{C|A}(\sigma^A) \otimes \sigma^B) \quad \forall \sigma^A \forall \sigma^B. \quad (16)
\]

Then we will have

\[
\mathcal{V}^\dagger \circ \mathcal{E}(I_A \otimes \sigma^B) = \mathcal{F}_{C|A}(I_A) \otimes \sigma^B \quad \forall \sigma^B, \quad (17)
\]

and we can use a customary positivity-cum-linearity argument (c.f. Lemma 2.3) to show that Eq. \( \mathcal{E} \) holds for all \( \sigma^A \).

We begin by noting that the testable conditions from \( \mathcal{E} \) give us operators \( F_{ab} \) on \( \mathcal{H}^A \) such that

\[
P_{AB}E_a^\dagger E_bP_{AB} = F_{ab} \otimes I_B \quad \forall a, b. \quad (18)
\]

Observe that the operator block matrix \( F = (F_{ab}) \) is positive since \((I_m \otimes P_{AB})E^\dagger E(I_m \otimes P_{AB}) = F \otimes I_B \), where the row matrix \( E = [E_{a_1} E_{a_2} \cdots] \), the number of \( E_{a_i} \) is \( m \), and \( I_m \) is the identity operator on \( m \)-dimensional Hilbert space.

Next let \( U \) be a unitary such that \( UFU^\dagger = D \) is diagonal, and let \( U = (U_{ab}) \) and \( D = (D_{ab}) \) be the associated block decompositions. Then

\[
\sum_{c,d} U_{ac}F_{cd}U_{bd}^\dagger = \delta_{ab}D_{aa} \quad \forall a, b, \quad (19)
\]

\[
\sum_{c} U_{ca}^\dagger U_{cb} = \delta_{ab}I_A \quad \forall a, b. \quad (20)
\]

Define a superoperator \( \mathcal{G} = \{ G_a \} \) where for all \( a, \)

\[
G_a = \sum_b E_b(U_{ab}^\dagger \otimes I_B)P_{AB} + E_aP_{AB}^+. \quad (21)
\]

Let \( X_{ab} = E_b(U_{ab}^\dagger \otimes I_B)P_{AB} \). Then by Eqs. \( \mathcal{E} \), one can verify for all \( a, b, \)

\[
P_{AB}G_a^\dagger G_bP_{AB} = \sum_{c,d} X_{ac}^\dagger X_{bd} \quad \forall a, b
\]

\[
= \sum_{c,d} U_{ac}F_{cd}U_{bd}^\dagger \otimes I_B
\]

\[
= D_{ab} \otimes I_B. \quad (22)
\]
and $D_{ab} = 0$ for all $a \neq b$. Moreover, Eq. (20) yields for all $\sigma^B$:

$$\mathcal{G}(I_A \otimes \sigma^B) = \sum_a G_a(I_A \otimes \sigma^B)G_a^\dagger = \sum_{a,b,c} X_{ab}(I_A \otimes \sigma^B)X_{ac}^\dagger$$

$$= \sum_{b,c} E_{bc}\left(\sum_a U_{ab}^\dagger U_{ac}\right)\otimes \sigma^B E_{bc}^\dagger = \sum_{b} E_b(I_A \otimes \sigma^B)E_b^\dagger = \mathcal{G}(I_A \otimes \sigma^B).$$

To simplify notation, let $\dim B = n$ and fix an orthonormal basis $\{|\psi_k\rangle\}_{k=1}^n$ for $B$. Let $m$ be the cardinality of the set $\{a : D_{aa} \neq 0\}$, and for all $a$ let $r_a = \text{rank } D_{aa} = \text{rank } D_{aa}$.

By the polar decomposition applied to each $G_aP_{AB}$, and the fact that these operators have mutually orthogonal ranges, there are partial isometries $V_a$ with mutually orthogonal ranges for distinct $a$ such that

$$G_aP_{AB} = V_a \sqrt{P_{AB}G_a^2G_aP_{AB}} = V_a(\sqrt{D_{aa}} \otimes I_B). \quad (21)$$

It follows that we can find a decomposition of each $V_a$ of the form

$$V_a = \sum_{k=1}^n \sum_{l=1}^{r_a} |\psi_k^{(a)}\rangle |\phi_k^{(a)}\rangle \langle \phi_k^{(a)}| \langle \psi_k^{(a)}|,$$  

where $\{|\psi_k^{(a)}\rangle\}_{k=1}^{r_a}$ is an orthonormal basis for range $\sqrt{D_{aa}} = \text{range } D_{aa}$ for each $a$. The vectors $|\psi_k^{(a)}\rangle |\phi_k^{(a)}\rangle = V_a |\psi_k^{(a)}\rangle |\phi_k^{(a)}\rangle$ are given by the tensor product on the final space $V_aV_a^H$ induced by the tensor product on the initial space $V_a^HV_a^H$ and the isometric action of $V_a : V_aV_a^H \rightarrow V_aV_a^H$.

Now let $\{|w_a\rangle\}_{a=1}^m$ be an orthonormal set of vectors in an $m$-dimensional Hilbert space. The vectors $\{|\phi_k^{(a)}\rangle |w_a\rangle \langle \psi_k|\}$ are orthonormal since the $|w_a\rangle$ are, and because, for a fixed $a$, the vectors $\{|\phi_k^{(a)}\rangle \langle \psi_k|\}_{k,l}$ form an orthonormal set. Moreover, the orthogonality of the ranges of the operators $G_aP_{AB}$ ensures the dimension of $\mathcal{H}$ is at least $\sum_{a=1}^m r_a$.

Next define $V$ to be any unitary extension of the following partial isometry to all of $\mathcal{H}$:

$$V : |\phi_k^{(a)}\rangle |w_a\rangle \langle \psi_k| \mapsto |\psi_k^{(a)}\rangle |\phi_k^{(a)}\rangle.$$  

The image vectors $\{|\psi_k^{(a)}\rangle |\phi_k^{(a)}\rangle\}_{a,k,l}$ form an orthonormal set since the $V_a$ are partial isometries with mutually orthogonal ranges, and because these vectors are orthonormal for a fixed $a$. Further define $\mathcal{F}_{CIA} = \{\sqrt{D_{aa}} \otimes |w_a\rangle\}$ as a channel on $\mathcal{L}(\mathcal{H}^A)$, and so for all $\sigma^A$,

$$\mathcal{F}_{CIA}(\sigma^A) = \sum_a (\sqrt{D_{aa}} \sigma^A \sqrt{D_{aa}}) \otimes |w_a\rangle \langle w_a|.$$  

Finally, it can be verified by direct computation that

$$\mathcal{G}(\sigma^A \otimes \sigma^B) = \mathcal{V}(\mathcal{F}_{CIA}(\sigma^A) \otimes \sigma^B) \quad \forall \sigma^A \forall \sigma^B,$$  

and this completes the proof. QED

**APPENDIX B: THEOREM 2**

In this appendix we establish results used in the proof of Theorem.

We begin by recalling a result for classical maps. A vector of probabilities $p$ of dimension $n$ is said to be majorised by a vector $q$, denoted $p \prec q$, if for each $k$ in the range $1$ to $n$, $\sum_{j=1}^k p_j \leq \sum_{j=1}^k q_j$, with equality for $k = n$, where the $\downarrow$ indicates that the probabilities are to be taken in nonincreasing order. A bistochastic map is a matrix $\Lambda$ satisfying $\sum_k \Lambda_{kk} = 1$ and $\sum_k \Lambda_{kk'} = 1$. The Hardy-Littlewood-Polya theorem [34, 30] states that the output of a bistochastic map is majorised by the input, that is, if $p = \Lambda q$ for bistochastic $\Lambda$, then $p \prec q$.

A similar result holds for unital operations. It is a theorem due to Uhlmann [31, 32]. We include a short proof for completeness.

**Lemma 5** If $\rho = \mathcal{E}(\sigma)$ for a unital channel $\mathcal{E}$, then the ordered spectrum $r$ of $\rho$ is majorised by the ordered spectrum $s$ of $\sigma$; that is,

$$\rho = \mathcal{E}(\sigma) \text{ with } \mathcal{E} \text{ unital implies } r \prec s.$$  

**Proof.** Let $p_k$ and $|e_k\rangle$ (respectively, $q_k$ and $|f_k\rangle$) denote the $k$th eigenvalue and normalized eigenvector of $\rho$ (respectively, $\sigma$). Clearly, $q_k = \sum_k D_{kk'} p_{k'}$ where $D_{kk'} \equiv \text{Tr} (|f_k\rangle \langle f_k| \mathcal{E}(\sigma) |e_k\rangle \langle e_k|)$. From the fact that $\mathcal{E}$ is trace-preserving one infers that $\sum_k D_{kk'} = 1$ while from the fact that $\mathcal{E}$ is unital one infers that $\sum_k D_{kk'} = 1$. It follows that $D = (D_{kk'})$ is bistochastic. Thus, by the Hardy-Littlewood-Polya theorem, we have $r \prec s$. QED

Heuristically, this says that unital channels can only increase the impurity or “mixedness” of quantum states.

**Corollary 6** For a unital operation $\mathcal{E}$ acting on a projector $P$, we have $\text{rank}(\mathcal{E}(P)) \geq \text{rank}(P)$.

This is a simple consequence of the majorisation relation given by Eq. (20).

**Corollary 7** Let $\mathcal{E}$ be a unital operation and $P$ a projection. If $\text{rank}(\mathcal{E}(P)) = \text{rank}(P)$, that is, if the inequality of Corollary 4 is saturated, then $\mathcal{E}(P)$ is a projection.
Proof. A projection $P$ has a uniform spectrum, and the only spectrum that is majorised by the uniform spectrum while having the same number of non-zero elements is the uniform spectrum. Given that $\mathcal{E}$ is trace-preserving, it follows that $\mathcal{E}(P)$ is a projection. QED

Lemma 8  For a unital operation $\mathcal{E}$ acting on a projector $P$, we have $\mathcal{E}(P) = P$ if and only if $\mathcal{E}^\dagger(P) = P$.

Proof. By the fixed point theorem for unital operations 25, $P$ is in the fixed point set of $\mathcal{E}$ if and only if it commutes with all Kraus operators of $\mathcal{E}$. The lemma then follows from the fact that the Kraus operators of $\mathcal{E}^\dagger$ are the adjoints of those of $\mathcal{E}$ and that both maps are unital. QED

We are now in a position to prove Lemma 4.

Proof of Lemma 4. For (ii) $\Rightarrow$ (i), we assume that $\text{rank}(\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB})) = \text{rank}(P_{AB})$. It follows from the fact that $\mathcal{E}$ is unital and Corollary 7 that $\mathcal{E}(P_{AB})$ is a projection. A projection of the same rank as $P_{AB}$ is obtained by a unitary operation $\mathcal{U}$ acting on $P_{AB}$, so that $\mathcal{E}(P_{AB}) = \mathcal{U}(P_{AB})$. We infer that $\mathcal{U}^\dagger \circ \mathcal{E}(P_{AB}) = P_{AB}$. By Lemma 8 it follows that $\mathcal{E}^\dagger \circ \mathcal{U}(P_{AB}) = P_{AB}$. Consequently we have $\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB}) = \mathcal{E}^\dagger \circ \mathcal{U} \circ \mathcal{U}^\dagger \circ \mathcal{E}(P_{AB}) = P_{AB}$, and a fortiori condition (i) holds.

For (i) $\Rightarrow$ (ii), we assume that $\text{supp}(\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB})) \subseteq \text{supp}(P_{AB})$. It follows that $\text{rank}(\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB})) \leq \text{rank}(P_{AB})$. Thus, given that $\mathcal{E}^\dagger \circ \mathcal{E}$ is a unital operation, by Corollary 6 we deduce that in fact $\text{rank}(\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB})) = \text{rank}(P_{AB})$. By Corollary 6 it follows that $\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB})$ is a projection. The only projection with the same rank and support as $P_{AB}$ is $P_{AB}$, therefore $\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB}) = P_{AB}$. Given that $\mathcal{E}^\dagger$ is unital, it follows from Corollary 6 that $\text{rank}(\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB})) \geq \text{rank}(\mathcal{E}(P_{AB}))$. Given that $\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB}) = P_{AB}$, we deduce that $\text{rank}(P_{AB}) \geq \text{rank}(\mathcal{E}(P_{AB}))$. However, from Corollary 6 and the unitailness of $\mathcal{E}$, we infer the opposite inequality $\text{rank}(\mathcal{E}(P_{AB})) \geq \text{rank}(P_{AB})$, so that we must have equality, $\text{rank}(\mathcal{E}(P_{AB})) = \text{rank}(P_{AB})$. QED

Notice from this proof that the two conditions of Lemma 4 are equivalent to the seemingly stronger statement $\mathcal{E}^\dagger \circ \mathcal{E}(P_{AB}) = P_{AB}$.