Expected lengths and distribution functions for Young diagrams in the hook

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Abstract

We consider $\beta$–Plancherel measures [5] on subsets of partitions – and their asymptotics. These subsets are the Young diagrams contained in a $(k, \ell)$–hook, and we calculate the asymptotics of the expected shape of these diagrams, relative to such measures. We also calculate the asymptotics of the distribution function of the lengths of the rows and the columns for these diagrams. This might be considered as the restriction to the $(k, \ell)$–hook of the fundamental work of Baik, Deift and Johansson [3]. The above asymptotics are given here by ratios of certain Selberg-type multi–integrals.

1 Introduction

This paper studies the asymptotics of certain ”$\beta$–Plancherel” measures on subsets of partitions. Let $Y_n = \{ \lambda \mid \lambda \vdash n \}$ denote the partitions of $n$, and let $f^\lambda$ denote the number of standard Young tableaux of shape $\lambda = (\lambda_1, \lambda_2, \ldots)$. For general references regarding partitions, Young diagrams and Young tableaux – see [13], [20]. Let $0 < \beta \in \mathbb{R}$. Baik and Rains [5] consider the following ”$\beta$–Plancherel” measure $M_n^\beta$ on $Y_n$:

$$M_n^\beta(\lambda) := \frac{(f^\lambda)^\beta}{\sum_{\mu \vdash n} (f^\mu)^\beta}. \quad (1)$$

Indeed, $M_n^2$ is the so called Plancherel measure on $Y_n$. We generalize to subsets $\Gamma_n \subseteq Y_n$, considering the subset of the partitions (i.e. diagrams) in the $(k, \ell)$ hook: Let $k, \ell \geq 0$ be integers and let $\Gamma_n = H(k, \ell; n)$ denote the following subset of $Y_n$:

$$H(k, \ell; n) = \{ \lambda \vdash n \mid \lambda_{k+1} \leq \ell \}.$$ 

These subsets arise in the representation theory of Lie groups, algebras and superalgebras, see for example [4]. The measures $\rho_{n}^{(\beta;k,\ell)}$ below are the $(k, \ell)$–hook restrictions of the above measures $M_n^\beta$. 

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Definition 1.1 Let $\lambda \in H(k, \ell; n)$ and $\beta > 0$, then

$$
\rho^{(\beta; k, \ell)}(\lambda) = \rho_n^{(\beta; k, \ell)}(\lambda) := \frac{(f^{\lambda})^\beta}{\sum_{\mu \in H(k, \ell; n)}(f^{\mu})^\beta}.
$$

1.1 Expected shape

Given $\Gamma_n \subseteq Y_n$, $n = 1, 2, \ldots$ and the probability measures $\rho = \{\rho_n\}_{n=1}^{\infty}$ on the $\Gamma_n$-s, one studies the asymptotics of the expected value (i.e. average length) of the first row $\lambda_1$, denoted $\lambda_{1,E}$, and similarly for the second row $\lambda_{2,E}$, etc. Similarly for the columns. Explicitly, when $\Gamma_n = H(k, \ell; n)$, expected values are given by the following definition.

Definition 1.2 If $\lambda \vdash n$, we write $\lambda = (\lambda_1, n, \lambda_2, n, \ldots)$. Also, $\lambda'$ is the conjugate partition of $\lambda$. Let $1 \leq p \leq k$ and $1 \leq q \leq \ell$. The expected value of the $p$-th row is $E(\lambda_p) = \lambda^{(\beta; k, \ell)}_{p,E}(n)$, where

$$
\lambda^{(\beta; k, \ell)}_{p,E}(n) = \frac{\sum_{\lambda \in H(k, \ell; n)} \lambda_{p,n} \cdot (f^{\lambda})^\beta}{\sum_{\lambda \in H(k, \ell; n)}(f^{\lambda})^\beta}
$$

and $\lambda^{(\beta; k, \ell)}_E(n) = (\lambda^{(\beta; k, \ell)}_{1,E}(n), \lambda^{(\beta; k, \ell)}_{2,E}(n), \ldots)$.

Similarly for the expected $q$-th column

$$
\lambda^{(\beta; k, \ell)}_{q,E}(n) = \frac{\sum_{\lambda \in H(k, \ell; n)} \lambda_{q,n} \cdot (f^{\lambda})^\beta}{\sum_{\lambda \in H(k, \ell; n)}(f^{\lambda})^\beta}
$$

and $\lambda^{(\beta; k, \ell)}_E(n) = (\lambda^{(\beta; k, \ell)}_{1,E}(n), \lambda^{(\beta; k, \ell)}_{2,E}(n), \ldots)$.

Of course, one can replace $H(k, \ell; n)$ in the above definition by other subsets $\Gamma_n \subseteq Y_n$.

The case $\Gamma_n = Y_n$ and $\beta = 2$ (Plancherel) has a long history. Let

$$
w(n) = \frac{\sum_{\lambda \vdash n} \lambda_{1,n} \cdot (f^{\lambda})^2}{\sum_{\lambda \vdash n}(f^{\lambda})^2} = \frac{\sum_{\lambda \vdash n} \lambda_{1,n} \cdot (f^{\lambda})^2}{n!}
$$

be the expected value of the first row – for the Plancherel measure $M_n^2$. Hammersley [10] showed that the limit $c = \lim_{n \to \infty} w(n)/\sqrt{n}$ exists. Vershik and Kerov [24] proved that $c = 2$ (independently, Logan and Shepp [12] proved that $c \leq 2$). Vershik and Kerov – and Logan and Shepp also determined the asymptotics of the expected shape $\lambda$ in this case.

Recently, in a major breakthrough paper, Baik, Deift and Johansson [3] determined the distribution function of the asymptotics of the first row, relating it to the Tracy-Widom distribution [22], see also [5] and [23]. The distribution function for the second row is given, by these same authors, in [4]. The distribution functions for the general rows are given in [8], [11] and [16]; see also [7] for the analogue results for colored permutations. The above results also establish deep connections with the theory of random matrices [15]. For detailed reviews of these results - see [11] and [21].
The main objective of the present paper is to compute the asymptotics of the above expected values (i.e. shapes) \( \lambda^{(\beta;k,\ell)}_E(n) \), as well as the corresponding distribution functions. The first term approximation is relatively simple, as we show that for each \( 1 \leq p \leq k \) and \( 1 \leq q \leq \ell \)
\[
\lambda^{(\beta;k,\ell)}_p(n), \quad \lambda^{(\beta;k,\ell)}_q(n) \sim \frac{n}{k + \ell},
\]
see Theorem 4.1. Second term approximations of \( \lambda^{(\beta;k,\ell)}_p(n) \) are introduced and studied in Sections 5, 6, and they have different values for different rows and for different columns. These second-term approximations are given as ratios of certain Selberg-type integrals, see Theorems 5.3 and 6.3 below.

1.2 Distribution functions

In Section 7 we introduce and study the asymptotics of the distribution functions \( \lambda^{(\beta;k,\ell)}_p(n, z) \), \( \lambda^{(\beta;k,\ell)}_q(n, z) \) for the lengths of the rows and the columns in \( H(k, \ell; n) \) – with respect to the above measures. We are able to calculate, asymptotically, a first-term approximation of these functions, but only a conjecture is given, about the second term approximations. That first-term approximation is
\[
\lambda^{(\beta;k,\ell)}_p(n, z), \quad \lambda^{(\beta;k,\ell)}_q(n, z) \sim \frac{n}{k + \ell} \cdot r(k, \ell, \beta)(z),
\]
where \( r(k, \ell, \beta)(z) \) is given by Equation (14), see Theorem 7.3.

1.3 Comparison with maximal shape

Given a subset of partitions \( \Gamma_n \subseteq Y_n \), one looks for \( \lambda \in \Gamma_n \) with maximal degree \( f^{\lambda} \). Call it maximal shape (with respect to \( \Gamma_n \)) and denote it by \( \lambda_{\text{max}} \). In Sections 8, 9, and 10 the expected shapes for \( \beta = 1, 2 \) are compared with the maximal shape. When \( \Gamma_n = Y_n \), the asymptotics of \( \lambda_{\text{max}} \) was calculated by Vershik and Kerov [24], [25], and by Logan and Shepp [12]. In particular, they proved that asymptotically, the expected shape for \( \beta = 2 \) and the maximal shape are the same, and that shape is given by the two axes and by the curve
\[
y = 1 + \left( \frac{2}{\pi} \right) \left[ x \cdot \sqrt{1 - x^2} - \arccos x \right]. \tag{2}
\]
A comparison with the case \( \Gamma_n = H(k, \ell; n) \) is intriguing.

When \( \Gamma_n \) is the \( k \)-strip \( \Gamma_n = H(k, 0; n) \), the asymptotics of \( \lambda \) with maximal \( f^{\lambda} \) was calculated in [2], and is given by the curve
\[
y = \left( \frac{2}{\pi} \right) \left[ x \cdot \arcsin x + \sqrt{1 - x^2} \right]. \tag{3}
\]
When \( \Gamma_n = H(k, \ell; n) \), the maximal \( \lambda \) was given in [18]. These results are reviewed in Section 8.2. Consider for example the 'strip' case \( \ell = 0 \), and denote the maximal \( \lambda \) by
\( \lambda^{(k,0)}_{\text{max}} \). Comparing it with \( \lambda^{(2,k,0)}_E \), the 'Plancherele' expected \( \lambda \) in \( H(k,0;n) \), we show that these asymptotic shapes are not equal – even in their first row. Nevertheless, numerically \( \lambda^{(k,0)}_{\text{max}} \) and \( \lambda^{(2,k,0)}_E \) are remarkably close, at least in the few special cases we check below, see Section 8.

Also, the asymptotics of \( \lambda_{\text{max}} \) for \( \Gamma_n = Y_n \) is not the limit case of \( \lambda^{(k,0)}_{\text{max}} \) as \( k \to \infty \), but the similarity between (2) and (3) is intriguing. It should be interesting to see if the ratios of the Selberg-type integrals, which give the expected shapes and the distribution functions for \( \Gamma_n = H(k,\ell;n) \), are in any way related to the Tracy–Widom distributions [22] [23], which give the (Plancherel and the 'involution') distribution functions in \( Y_n \).

1.4 RSK

In the case of \( \rho^{(1;k,\ell)}_n \) and \( \rho^{(2;k,\ell)}_n \), the RSK correspondence provides an interesting interpretation of the above asymptotics. The RSK (Robinson–Schensted–Knuth) correspondence \( \sigma \leftrightarrow (P_{\lambda}, Q_{\lambda}) \) corresponds \( \sigma \in S_n \) with a pair of standard Young tableaux of shape \( \lambda \) [20]. In the Plancherel case \( \beta = 2 \) it relates the above expected values of the first row to the statistics of the longest increasing (and decreasing) subsequences in permutation. For example, when \( \sigma \leftrightarrow (P_{\lambda}, Q_{\lambda}) \), \( \lambda_1 \) is the length of a longest increasing subsequence in \( \sigma \), while \( \lambda'_1 \) is the length of a longest decreasing subsequence in \( \sigma \). By C. Green’s theorem there are similar interpretations for \( \lambda_2, \lambda_3 \), etc. For a detailed account of the RSK see [20]. Thus the results in [3] etc. can also be stated in terms of longest increasing subsequences in permutations.

It is well known that \( \sigma \) is an involution if'f \( \sigma \leftrightarrow (P_{\lambda}, P_{\lambda}) \). The analogue Probability theory of longest increasing subsequences in involutions in \( S_n \) is done in [5].

Denote by \( S_{k,\ell;n} \subseteq S_n \) the subset of the permutations \( \sigma \in S_n \) such that under the RSK correspondence \( \sigma \leftrightarrow (P_{\lambda}, Q_{\lambda}) \), we have \( \lambda \in H(k,\ell;n) \). For example, \( S_{k,0;n} \) is the subset of those permutations in \( S_n \) where any descending subsequence has length \( \leq k \). Thus, \( \lambda^{(1;k,\ell)}_{1,E} \) is the expected value of the longest increasing subsequence in the involutions in \( S_{k,\ell;n} \).

2 Selberg type integrals

As mentioned above, the main results in this paper involve Selberg-type integrals, hence we briefly review these type of multi-integrals. In [19] A. Selberg proved the following formula:

\[
\int_0^1 \cdots \int_0^1 (u_1 \cdots u_n)^{x-1} \cdot [(1 - u_1) \cdots (1 - u_n)]^{y-1} \cdot \prod_{1 \leq i < j \leq n} |u_i - u_j|^{2z} du_1 \cdots du_n = \]

\[
= \prod_{k=1}^n \frac{\Gamma(1 + kz) \cdot \Gamma(x + (k - 1)z) \cdot \Gamma(y + (k - 1)z)}{\Gamma(1 + z) \cdot \Gamma(x + y + (n + k - 2)z)}. \]
Various integral formulas can be deduced from Selber's integrals, see for example [14] for the Macdonald–Mehta integrals. For example Mehta's integral formula (which was a conjecture for some time)

\[
\int_{\mathbb{R}^k} e^{-(1/2)\sum x_i^2} \prod_{1 \leq i < j \leq k} |x_i - x_j|^{2z} dx_1 \cdots dx_k = (\sqrt{2\pi})^k \prod_{j=1}^{k} \frac{\Gamma(1 + jz)}{\Gamma(1 + z)}
\]

can be deduced from Selberg’s formula, see [15] for details. We call these and related integrals "Selberg–type integrals". A connection between the RSK and these integrals, as well as with random matrices, appears in [17], [6]. Since the formulas from [17], [6] are needed later, we record it here, together with certain variations of these asymptotics and integrals that are also needed below. Let

\[\Omega_k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1 \geq x_2 \geq \cdots \geq x_k \text{ and } x_1 + \cdots + x_k = 0\},\]

and more generally,

\[\Omega_{(k, \ell)} = \{(x_1, \ldots, x_k, y_1, \ldots, y_\ell) \mid x_1 \geq \cdots \geq x_k; y_1 \geq \cdots \geq y_\ell; \sum x_i + \sum y_j = 0\}.
\]

**Theorem 2.1** (Theorem 2.10 in [17]) Let \(\gamma_k = (1/\sqrt{2\pi})^{k-1} \cdot k^{k^2/2}\) and \(D_k(x) = \prod_{1 \leq i < j \leq k} (x_i - x_j)\), then

\[
\sum_{\lambda \in H(k,0;n)} (f^\lambda)^\beta \sim \left[\gamma_k \cdot \left(\frac{1}{n}\right)^{(k-1)(k+2)/4} \cdot k^n\right]^\beta \cdot (\sqrt{n})^{k-1} \cdot I(k,0,\beta),
\]

where

\[
I(k,0,\beta) = \int_{\Omega_k} \left[D_k(x) \cdot e^{-\frac{k}{2}\sum x_i^2}\right]^\beta d(k-1)x.
\]

Note that by certain symmetry properties of the above integrand, \(\Omega_k\) is transformed in [17] into \(\mathbb{R}^k\).

Given \(\lambda = (\lambda_1, n, \lambda_2, n, \ldots) \vdash n\), write \(\lambda_{p,n} = n/k + c_{p,n} \cdot \sqrt{n}\) and denote \(c_{p,n} = c_{p,n}(\lambda)\). The same arguments in [17] prove the following theorem.

**Theorem 2.2**

\[
\sum_{\lambda \in H(k,0;n)} c_{p,n}(\lambda) \cdot (f^\lambda)^\beta \sim \left[\gamma_k \cdot \left(\frac{1}{n}\right)^{(k-1)(k+2)/4} \cdot k^n\right]^\beta \cdot (\sqrt{n})^{k-1} \cdot \Gamma^*(k,0,\beta),
\]

where

\[
\Gamma^*(k,0,\beta) = \int_{\Omega_k} x_p \cdot \left[D_k(x) \cdot e^{-\frac{k}{2}\sum x_i^2}\right]^\beta d(k-1)x.
\]

The \((k, \ell)\)–hook analogue of Theorem 2.1 is proved in [6].
Theorem 2.3 (See Theorem 7.18 in [6]) Let
\[ \gamma_{k,\ell} = \left(1/\sqrt{2\pi}\right)^{k+\ell-1} \cdot (k+\ell)^{(k^2+\ell^2)/2} \cdot (1/2)^{k\ell}, \]
then
\[ \sum_{\lambda \in H(k,\ell;n)} (f^\lambda)^\beta \sim \left[ \gamma_{k,\ell} \cdot \left(\frac{1}{n}\right)^{(k(k+1) + \ell(\ell+1) - 2)/4} \cdot (k + \ell)^n \right]^\beta \cdot (\sqrt{n})^{k+\ell-1} \cdot I(k,\ell,\beta), \]
where
\[ I(k,\ell,\beta) = \int_{\Omega_{k,\ell}} \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2} (\sum x_i^2 + \sum y_j^2)} \right]^\beta d^{(k+\ell-1)}(x; y). \]
The same arguments also prove

Theorem 2.4 Let \(1 \leq p \leq k\), let \(\lambda = (\lambda_1, n, \lambda_2, n, \ldots) \in H(k,\ell;n)\) and define \(c_{p,n}(\lambda)\) via:
\[ \lambda_{p,n} = \frac{n}{k+\ell} + c_{p,n}(\lambda) \cdot \sqrt{n}. \]
Similarly for \(1 \leq q \leq \ell\) and \(c'_{q,n}(\lambda) := c_{q,n}(\lambda').\)

Then
\[ \sum_{\lambda \in H(k,\ell;n)} c_{p,n}(\lambda) \cdot (f^\lambda)^\beta \sim \left[ \gamma_{k,\ell} \cdot \left(\frac{1}{n}\right)^{(k(k+1) + \ell(\ell+1) - 2)/4} \cdot (k + \ell)^n \right]^\beta \cdot (\sqrt{n})^{k+\ell-1} \cdot I^*(k,\ell,\beta), \]
where
\[ I^*(k,\ell,\beta) = \int_{\Omega_{k,\ell}} x_p \cdot \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2} (\sum x_i^2 + \sum y_j^2)} \right]^\beta d^{(k+\ell-1)}(x; y). \]
Similarly for the sum
\[ \sum_{\lambda \in H(k,\ell;n)} c'_{q,n}(\lambda) \cdot (f^\lambda)^\beta, \]
with the corresponding integral
\[ I^*(k,\ell,\beta) = \int_{\Omega_{k,\ell}} y_q \cdot \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2} (\sum x_i^2 + \sum y_j^2)} \right]^\beta d^{(k+\ell-1)}(x; y). \]

Generalizations of Theorem 2.4 – with more general functions of the \(c_{p,n}(\lambda), c'_{q,n}(\lambda)\) – are rather obvious, but will not be given here.

The same arguments of Section 7 of [6], applied to the asymptotics of both numerator and denominator, prove the following theorem – which is needed later.
Theorem 2.5 Let \( z > 0 \),
\[
H(k, \ell; n, z) = \{ \lambda \in H(k, \ell; n) \mid \lambda_{1,n} \leq \frac{n}{k+\ell} + z\sqrt{n} \}
\]
and let
\[
\Omega(k, \ell, z) = \{(x_1, \ldots, x_k; y_1, \ldots, y_\ell) \in \Omega(k, \ell) \mid x_1, y_1 \leq z \}.
\]
Then
\[
\sum_{\lambda \in \Omega(k, \ell, z)} (f^\lambda)^\beta \sim \int_{\Omega(k, \ell, z)} \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_i^2)} \right] d(k+\ell-1)(x; y).
\]

We call all the above "Selberg–type integrals", and remark that the above expected values and distribution functions are given below as ratio o f such integrals.

3 The main results

3.1 The expected values

We study the expected values of the row and of the column lengths in \( \Gamma_n = H(k, \ell; n) \) with respect to the measures \( \rho_n^{(\beta;k,\ell)} \) introduced in Definition \( \square \). The first term asymptotics is given by

Theorem 3.1 (See Theorem \[4,1\]) Let \( \beta > 0 \) and \( \Gamma_n = H(k, \ell; n) \). For each \( 1 \leq p \leq k \),
\[
\lambda_{p,E}^{(\beta;k,\ell)}(n) \sim \frac{n}{k+\ell}, \quad \text{namely} \quad \lim_{n \to \infty} \left( \frac{\lambda_{p,E}^{(\beta;k,\ell)}(n)}{\frac{n}{k+\ell}} \right) = 1.
\]
Similarly, for each \( 1 \leq q \leq \ell \),
\[
\lambda_{q,E}^{(\beta;k,\ell)}(n) \sim \frac{n}{k+\ell}.
\]

The second term approximations are given as follows. Define \( c_{p,E}^{(\beta;k,\ell)}(n) \) and \( c_{p,E}^{(\beta;k,\ell)} \) via
\[
\lambda_{p,E}^{(\beta;k,\ell)}(n) = \frac{n}{k+\ell} + c_{p,E}^{(\beta;k,\ell)}(n) \cdot \sqrt{n}, \quad \text{and} \quad c_{p,E}^{(\beta;k,\ell)} = \lim_{n \to \infty} c_{p,E}^{(\beta;k,\ell)}(n).
\]

Similarly for the columns. Then

Theorem 3.2 (see Theorem \[6,3\]) Let \( 1 \leq p \leq k \), then the limit \( c_{p,E}^{(\beta;k,\ell)} \) exists, and is given as follows:
\[
c_{p,E}^{(\beta;k,\ell)} = \frac{\int_{\Omega_k} x_p \cdot \left[ D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k+\ell}{2}(x_1^2 + \cdots + x_k^2)} \right]^\beta d(k-1)(x)}{\int_{\Omega_k} \left[ D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k+\ell}{2}(x_1^2 + \cdots + x_k^2)} \right]^\beta d(k-1)(x)} \quad (4)
\]

Similarly for the columns.
Equation (4) (or (9)) is a consequence of the seemingly more symmetric Equation (11).

In Sections 8, 9 and 10 the expected shapes for $\beta = 1, 2$ are compared with the maximal shape $\lambda_{\text{max}}$ in few special cases. As mentioned in Section 1.3, this shows a different behavior in the hook case $\Gamma_n = H(k, \ell; n)$ compared with the general case $\Gamma_n = Y_n$.

### 3.2 The distribution function for the first row

In Section 7 we study the distribution functions of the length on the rows and the columns. We calculate the first term approximations and conjecture the second term approximations. Given $0 < z \in \mathbb{R}$, denote

$$H(k, \ell; n, z) = \{\lambda \in H(k, \ell; n) \mid \lambda_{1,n}, \lambda'_{1,n} \leq \frac{n}{k+\ell} + z \sqrt{n}\}.$$ 

Let $1 \leq p \leq k, 1 \leq q \leq \ell$. The distribution of the length of the $p$-th row as a function of $z$ is defined as

$$\lambda_p^{(\beta; k, \ell)}(n, z) = \frac{\sum_{\lambda \in H(k, \ell; n, z)} \lambda_{p,n} \cdot (f^\lambda)^\beta}{\sum_{\lambda \in H(k, \ell; n)} (f^\lambda)^\beta},$$

and similarly for the columns. Recall $\Omega(k, \ell)$ from Section 2 and denote

$$\Omega(k, \ell; z) = \{(x_1, \ldots, x_k; y_1, \ldots, y_\ell) \in \Omega(k, \ell) \mid x_1, y_1 \leq z\}.$$ 

Then

**Theorem 3.3 (see Theorem 7.3)** Let $\beta, z > 0$ and denote

$$r(k, \ell, \beta, z) = \frac{\int_{\Omega(k, \ell, z)} [D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}]^\beta \cdot d^{(k+\ell-1)}(x; y)}{\int_{\Omega(k, \ell)} [D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}]^\beta \cdot d^{(k+\ell-1)}(x; y)}. \quad (5)$$

Then

$$\lambda_{p}^{(\beta; k, \ell)}(n, z), \lambda_{q}^{(\beta; k, \ell)}(n, z) \sim \frac{n}{k+\ell} \cdot r(k, \ell, \beta, z).$$

In Section 7.2 we make some conjectures about the second term approximations of $\lambda_p^{(\beta; k, \ell)}(n, z)$ and $\lambda_q^{(\beta; k, \ell)}(n, z)$, both in terms of ratios of Selberg-type integrals. In the last three sections (Sections 7, 8 and 9), we calculate some special cases and include also some computer calculations.
Part I

Expected shape, the $\beta$–Plancherel probability

4 First term approximation

Recall the notation $\sim$: Let $a_n, b_n$ be two sequences of, say, real numbers, and assume $b_n \neq 0$ if $n$ is large enough. Then $a_n \sim b_n$ if $\lim_{n \to \infty} a_n/b_n = 1$. Extend $\sim$ to vectors as follows: $(a_1, n, \ldots, a_r, n) \sim (b_1, n, \ldots, b_r, n)$ if $a_i, n \sim b_i, n$ for $i = 1, \ldots, r$.

Following the techniques and arguments in Section 7 of [6], (see also [17]) we show below the following first approximation of the expected shape $\lambda_E = \lambda_E^{(\beta,k,\ell)}$.

**Theorem 4.1** Let $\beta > 0$ and $\Gamma_n = H(k, \ell; n)$ and let $\lambda_{p,E}^{(\beta,k,\ell)}(n)$ and $\lambda_{q,E}^{(\beta,k,\ell)}(n)$ be given by Definition 1.2. For each $1 \leq p \leq k,$

$$\lambda_{p,E}^{(\beta,k,\ell)}(n) \sim \frac{n}{k + \ell}, \text{ namely } \lim_{n \to \infty} \left(\lambda_{p,E}^{(\beta,k,\ell)}(n) / \left(\frac{n}{k + \ell}\right)\right) = 1.$$

Similarly, for each $1 \leq q \leq k,$

$$\lambda_{q,E}^{(\beta,k,\ell)}(n) \sim \frac{n}{k + \ell}.$$

**Proof.** We sketch the proof for the expected row length in the case $\ell = 0$, thus showing that

$$\lambda_{p,E}^{(\beta,k,0)}(n) \sim \frac{n}{k}.$$

The main point is that since $\beta > 0$, both sums in the numerator and the denominator of definition 1.2 are dominated by the summands corresponding to the partitions $\lambda$, such that $\lambda_i = \frac{n}{k} + c_i \cdot \sqrt{n}$ and with $c_i$–s in a bounded interval. In other words, let $a > 0$ and denote

$$H_a(k, 0; n) = \{ \lambda \in H(k, 0; n) \mid \lambda_i = \frac{n}{k} + c_i \cdot \sqrt{n}, \text{ where } |c_i| \leq a, \ i = 1, 2, \ldots, k \}.$$

Then

$$\lim_{n \to \infty} \lambda_{p,E}^{(\beta,k,0)}(n) = \lim_{n \to \infty} \frac{\sum_{\lambda \in H(k, 0; n)} \lambda_p \cdot (f^{(k)})^\beta}{\sum_{\lambda \in H(k, 0; n)} (f^{(k)})^\beta} = \lim_{a \to \infty} \frac{\lim_{n \to \infty} \sum_{\lambda \in H_a(k, 0; n)} \lambda_p \cdot (f^{(k)})^\beta}{\lim_{n \to \infty} \sum_{\lambda \in H_a(k, 0; n)} (f^{(k)})^\beta}.$$

Writing $\lambda_p = \frac{n}{k} + c_p \cdot \sqrt{n}$, the expression in the brackets equals

$$\frac{\sum_{\lambda \in H_a(k, 0; n)} \left(\frac{n}{k}\right) (f^{(k)})^\beta}{\sum_{\lambda \in H_a(k, 0; n)} (f^{(k)})^\beta} + \frac{\sum_{\lambda \in H_a(k, 0; n)} c_p \cdot \sqrt{n} (f^{(k)})^\beta}{\sum_{\lambda \in H_a(k, 0; n)} (f^{(k)})^\beta}.$$

The first summand equals $\frac{n}{k}$ while the absolute value of the second summand is bounded by $a \sqrt{n}$ since all $|c_p(n)| \leq a$. Since $\frac{n}{k} + a \sqrt{n} \sim \frac{n}{k}$, it follows that

$$\lambda_{p,E}^{(\beta,k,0)}(n) \sim \frac{n}{k},$$

which completes the proof in the case $\ell = 0$. q.e.d.
5 Second term approximation, the 'strip' case ($\ell = 0$)

Because of Theorem 4.1, we look for a more subtle approximation of $\lambda_{p,E}^{(\beta,k,0)}(n)$, namely, we look for the expected deviation – of the form $c \cdot \sqrt{n}$ – from $\frac{n}{k + \ell}$. This leads us to introduce the asymptotic expected value $c_{p,E}^{(\beta,k,0)}$ below. We begin with the 'strip' case $\ell = 0$. The general $(k, \ell)$–hook case is given in the next section.

Definition 5.1 Let $\Gamma_n = H(k,0;n)$ and let $1 \leq p \leq k$, with $\lambda_{p,E}^{(\beta,k,0)}(n)$ given by Definition 4.2. Define $c_{p,E}^{(\beta,k,0)}(n)$ via the equation

$$\lambda_{p,E}^{(\beta,k,0)}(n) = \frac{n}{k} + c_{p,E}^{(\beta,k,0)}(n) \cdot \sqrt{n}, \quad \text{and} \quad c_{p,E}^{(\beta,k,0)} = \lim_{n \to \infty} c_{p,E}^{(\beta,k,0)}(n).$$

Thus, when $n$ goes to infinity,

$$\lambda_{p,E}^{(\beta,k,0)}(n) \sim \frac{n}{k} + c_{p,E}^{(\beta,k,0)} \cdot \sqrt{n}.$$

Remark 5.2 It is not obvious that the limit $c_{p,E}^{(\beta,k,0)} = \lim_{n \to \infty} c_{p,E}^{(\beta,k,0)}(n)$ exists. However, Theorem 5.3 asserts that in fact, this limit does exist.

Our aim is to calculate $c_{p,E}^{(\beta,k,0)}$, thus calculating "the second term" in the approximation of the expected value of the $p$–th row–length $\lambda_{p,E}^{(\beta,k,0)}$. Definitions 4.2 and 5.1 obviously imply the equation

$$c_{p,E}^{(\beta,k,0)}(n) = \frac{\sum_{\lambda \in H(k,0;n)} \lambda_{p,n} (f^\lambda)^\beta}{\sum_{\lambda \in H(k,0;n)} (f^\lambda)^\beta} \cdot \frac{1}{\sqrt{n}}. \quad (6)$$

If $k = 1$, Equation (6) implies that for any $\beta > 0$, $c_{1,E}^{(\beta,1,0)} = 0$.

Theorem 5.3 Let $k \geq 2$. The limit $c_{p,E}^{(\beta,k,0)} = \lim_{n \to \infty} c_{p,E}^{(\beta,k,0)}(n)$ exists, and is given by

$$c_{p,E}^{(\beta,k,0)} = \frac{\int_{\Omega_k} \sum_{i=1}^k x_p \cdot [D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(\sum_i x_i^2)}]^{\beta} d^{(k-1)} x}{\int_{\Omega_k} [D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(\sum_i x_i^2)}]^{\beta} d^{(k-1)} x}. \quad (7)$$

Proof. In Equation (7) write $\lambda_{i,n} = \frac{n}{k} + c_{i,n} \cdot \sqrt{n}$, and consider $\lambda$–s with $c_i$ bounded in some interval: $|c_i| \leq a$ for some $a > 0$. By an argument similar to the proof of Theorem 4.1, it follows that as $n \to \infty$, $c_{p,E}^{(\beta,k,0)}(n)$ is approximated by the ratio

$$\frac{\sum_{\lambda \in H_{a}(k,0;n)} \lambda_{p,n} (f^\lambda)^\beta}{\sum_{\lambda \in H_{a}(k,0;n)} (f^\lambda)^\beta}. \quad (8)$$

The proof now follows by applying Theorem 2.1 to the denominator, Theorem 2.2 to the numerator, then cancelling equal terms. q.e.d.
Remark 5.4 Note that the denominator of (7) is actually a "Selberg"–or a Macdonald–Mehta – integral, which can be evaluated for any $\beta$. For example, let $\beta = 2$. By comparing (F.2.10) with (F.4.5.2) of [17], deduce that

$$\int_{\Omega} \left[ D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(x_1^2 + \cdots + x_k^2)} \right]^2 d^{(k-1)}x =$$

$$= \left( \frac{\sqrt{2}}{2\pi} \right)^{k-1} \cdot \left( \frac{1}{\sqrt{2}} \right)^{k-1} \cdot \left( \frac{1}{\sqrt{k}} \right)^{k^2} \cdot 1! \cdot 2! \cdots (k-1)!.$$ 

Similarly, by comparing (F.2.10) with (F.4.5.1) of [17], deduce that when $\beta = 1$,

$$\int_{\Omega} \left[ D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(x_1^2 + \cdots + x_k^2)} \right]^2 d^{(k-1)}x =$$

$$= \left( \frac{1}{\sqrt{k}} \right)^{k(k+1)} \cdot \frac{1}{k!} \cdot (\sqrt{2})^{3k-1} \cdot \frac{1}{\sqrt{\pi}} \cdot \prod_{j=1}^{k} \Gamma \left( 1 + \frac{1}{2}j \right).$$

For explicit values, recall that $\Gamma \left( \frac{3}{2} \right) = \frac{\sqrt{\pi}}{2}$, that $\Gamma(1) = 1$, and that $\Gamma(z+1) = z\Gamma(z)$.

6 Second term approximation, the $(k, \ell)$–hook case

We turn now to the general $(k, \ell)$–hook case.

**Definition 6.1** Let $\beta > 0$ and let $1 \leq p \leq k$ and $1 \leq q \leq \ell$, with $\lambda_{p,E}(\beta; k, \ell)$ given by Definition 1.2. Define $c_{p,E}^{(\beta; k, \ell)}$ via the equation

$$\lambda_{p,E}^{(\beta; k, \ell)}(n) = \frac{n}{k + \ell} + c_{p,E}^{(\beta; k, \ell)}(n) \cdot \sqrt{n}, \quad \text{and} \quad c_{p,E}^{(\beta; k, \ell)} = \lim_{n \to \infty} c_{p,E}^{(\beta; k, \ell)}(n).$$

Similarly for the columns:

$$\lambda_{q,E}^{(\beta; k, \ell)}(n) = \frac{n}{k + \ell} + c_{q,E}^{(\beta; k, \ell)}(n) \cdot \sqrt{n}, \quad \text{and} \quad c_{q,E}^{(\beta; k, \ell)} = \lim_{n \to \infty} c_{q,E}^{(\beta; k, \ell)}(n).$$

The existence of the limits $c_{p,E}^{(\beta; k, \ell)}$ and $c_{q,E}^{(\beta; k, \ell)}$ is asserted by Theorem 6.3 below.

Definitions 1.2 and 6.1 obviously imply

**Remark 6.2**

$$c_{p,E}^{(\beta; k, \ell)}(n) = \left( \sum_{\lambda \in H(k, \ell; n)} \frac{\lambda_{p,n}(f^\lambda)^\beta}{\sum_{\lambda \in H(k, \ell; n)} (f^\lambda)^\beta} - \frac{n}{k + \ell} \right) \cdot \frac{1}{\sqrt{n}},$$

and

$$c_{q,E}^{(\beta; k, \ell)}(n) = \left( \sum_{\lambda \in H(k, \ell; n)} \frac{\lambda_{q,n}(f^\lambda)^\beta}{\sum_{\lambda \in H(k, \ell; n)} (f^\lambda)^\beta} - \frac{n}{k + \ell} \right) \cdot \frac{1}{\sqrt{n}}.$$
Theorem 6.3  Let $1 \leq p \leq k$, $1 \leq q \leq \ell$, then the limits $c_{p,E}^{(\beta;k,\ell)}$ and $c_{q,E}^{(\beta;k,\ell)}$ exist, and are given as follows.

\[
c_{p,E}^{(\beta;k,\ell)} = \frac{\int_{\Omega} x_p \cdot \left[ D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k+\ell}{2}(x_1^2 + \cdots + x_k^2)} \right]^{\beta} d^{(k-1)}(x)}{\int_{\Omega} \left[ D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k+\ell}{2}(x_1^2 + \cdots + x_k^2)} \right]^{\beta} d^{(k-1)}(x)}.
\]  

and

\[
c_{q,E}^{(\beta;k,\ell)} = \frac{\int_{\Omega} y_q \cdot \left[ D_\ell(y_1, \ldots, y_\ell) \cdot e^{-\frac{k+\ell}{2}(y_1^2 + \cdots + y_\ell^2)} \right]^{\beta} d^{(\ell-1)}(y)}{\int_{\Omega} \left[ D_\ell(y_1, \ldots, y_\ell) \cdot e^{-\frac{k+\ell}{2}(y_1^2 + \cdots + y_\ell^2)} \right]^{\beta} d^{(\ell-1)}(y)}.
\]

Thus, as $n$ goes to infinity,

\[
\lambda_{p,E}^{(\beta;k,\ell)}(n) \sim \frac{n}{k + \ell} + c_{p,E}^{(\beta;k,\ell)} \cdot \sqrt{n} \quad \text{and} \quad \lambda_{q,E}^{(\beta;k,\ell)}(n) \sim \frac{n}{k + \ell} + c_{q,E}^{(\beta;k,\ell)} \cdot \sqrt{n}.
\]

Proof. We prove for $c_{p,E}^{(\beta;k,\ell)}$ in two steps.

Step 1. We claim that the limit $c_{p,E}^{(\beta;k,\ell)}$ exists, and is given by

\[
c_{p,E}^{(\beta;k,\ell)} = \frac{\int_{\Omega} x_p \cdot \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)} \right]^{\beta} d^{(k+\ell-1)}(x; y)}{\int_{\Omega} \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)} \right]^{\beta} d^{(k+\ell-1)}(x; y)}.
\]

The proof of Equation (11) is essentially the same as that of Equation (7). Its starting point is Remark 6.2 (instead of Equation (6)). Here we write, for $1 \leq i \leq k$ and for $1 \leq j \leq \ell$,

\[
\lambda_{i,n} = \frac{n}{k + \ell} + c_{i,n} \cdot \sqrt{n} \quad \text{and} \quad \lambda^{\prime}_{j,n} = \frac{n}{k + \ell} + c_{j,n} \cdot \sqrt{n},
\]

and consider $\lambda$’s with $c_i$ and $c_{j,}^\prime$ bounded in some interval. Now follow a ‘hook’–generalization of the proof of Theorem 5.3, applying Theorems 2.3 and 2.4 and complete the proof of Equation (11).

Step 2. We now transform (11) into (9). Let $I_p^{(\beta;k,\ell)}$ denote the numerator of (11):

\[
I_p^{(\beta;k,\ell)} = \int_{\Omega} x_p \left[D_k(x) \cdot D_\ell(y) \cdot e^-{(k+\ell)/2)(\sum x_i^2 + \sum y_j^2)} \right]^{\beta} d^{(k+\ell-1)}(x; y).
\]

Setting $\sum x_i = u$ we have $\sum y_j = -u$, and $I_p^{(\beta;k,\ell)} = \int_{-\infty}^{\infty} K(u) L(-u) \, du$, where

\[
K(u) = \int_{M_k(u,u)} x_p \left[D_k(x) \cdot e^-{(k+\ell)/2)(\sum x_i^2)} \right]^{\beta} d^{(k-1)}x
\]
and
\[ L(-u) = \int_{M_u(x,-u)} D_\ell(y) \cdot e^{-(k+\ell)/2}(\sum y_j^2)^\beta d^{(k-1)}x. \]
Here \( M_k(x,u) = \{(x_1, \ldots, x_k) \mid x_1 \geq \cdots \geq x_k \text{ and } \sum x_i = u\} \) and similarly, \( M_\ell(x,-u) = \{(y_1, \ldots, y_\ell) \mid y_1 \geq \cdots \geq y_\ell \text{ and } \sum y_j = -u\}. \)

To evaluate \( I_p^{(\beta,k,\ell)} \), proceed as follows. In \( K(u) \) and \( L(-u) \) substitute \( x_i' = x_i - (u/k) \) and \( y_j' = y_j + (u/\ell) \). The Jacobians are \( 1, x_i = x_i' + (u/k); D_k(x') = D_k(x); D_\ell(y') = D_\ell(y); \sum x_i^2 = \sum x_i'^2 + (u^2/k) \) and \( \sum y_j^2 = \sum y_j'^2 + (u^2/\ell) \).
Replacing \( x_i' \) by \( x_i \) and \( y_j' \) by \( y_j \), it follows that
\[ I_p^{(\beta,k,\ell)} = J_1(x) \cdot J_3(y) \cdot A(u) + J_2(x) \cdot J_3(y) \cdot B(u). \]
Here
\[ A(u) = \int_{-\infty}^{\infty} e^{-\frac{(k+\ell)^2u^2}{2k}} du, \quad B(u) = \int_{-\infty}^{\infty} e^{-\frac{(k+\ell)^2u^2}{2k}} du, \]
\[ J_1(x) = \int_{M_k(x,0)} D_k(x) \cdot e^{-\frac{k+\ell}{k}(\sum x_i^2)} d^{(k-1)}x, \quad (J_1(x) = 1 \text{ if } k = 1), \]
\[ J_2(x) = \int_{M_k(x,0)} D_k(x) \cdot e^{-\frac{k+\ell}{\ell}(\sum x_i^2)} d^{(k-1)}x, \]
and
\[ J_3(y) = \int_{M_\ell(y,0)} D_\ell(y) \cdot e^{-\frac{k+\ell}{\ell}(\sum y_j^2)} d^{(\ell-1)}y, \quad (J_3(y) = 1 \text{ if } \ell = 1), \]
\( (M_\ell(x,0) = \{(x_1, \ldots, x_r) \mid x_1 \geq \cdots \geq x_r \text{ and } x_1 + \cdots + x_r = 0\}, \text{etc.}) \). Since, trivially, \( B(u) = 0 \), deduce that \( I_p^{(\beta,k,\ell)} = J_1(x) \cdot J_3(y) \cdot A(u) \).
Let \( I^{(\beta,k,\ell)} \) denote the denominator in Equation (11). By exactly the same arguments it follows that \( I^{(\beta,k,\ell)} = J_2(x) \cdot J_3(y) \cdot A(u) \).
By (11)
\[ c_{p,E}^{(\beta,k,\ell)} = \frac{I_p^{(\beta,k,\ell)}}{I^{(\beta,k,\ell)}} = \frac{J_1(x)}{J_2(x)}, \]
which is the right-hand-side of Equation (9). This completes the proof. \( \qquad \text{q.e.d.} \)

Remark 6.4 The denominator-integral in (11) is
\[ I^{(\beta,k,\ell)} = \int_{\Omega(k,\ell)} D_k(x_1, \ldots, x_k) \cdot D_\ell(y_1, \ldots, y_\ell) \cdot e^{-\frac{k+\ell}{2}(x_1^2 + \cdots + x_k^2 + y_1^2 + \cdots + y_\ell^2)} d^{(k+\ell-1)}(x; y), \]
and is calculated explicitly in [22, sec 7] (here \( \beta = 2z \)):
\[ I^*(k, \ell) = I(k, \ell, 2) = \frac{1}{k!} \sqrt{2\pi k+\ell-1} \cdot \sqrt{\frac{\beta}{2\pi}} \cdot \frac{1}{\int \frac{1}{2(k+\ell)}((k(k-1)+\ell(\ell-1))(\beta/2)+k+\ell)} \]
\[ \times \frac{\prod_{i=1}^k \Gamma(i\beta/2+1) \cdot \prod_{j=1}^\ell \Gamma(j\beta/2+1)}{\Gamma(\beta/2+1)}, \]
where \( \Gamma \) is the Gamma function \( (\Gamma(n+1) = n!) \).
Theorems 5.3 and 6.3 imply

**Corollary 6.5** Let \( 1 \leq p \leq k \) and \( 1 \leq q \leq \ell \). Then

\[
\frac{c_{p,E}(\beta;k,\ell)}{c_{p,E}(\beta;k,0)} = \sqrt{\frac{k}{k+\ell}} \cdot \frac{c_{p,E}(\beta;k,0)}{c_{p,E}(\beta;k,0)}, \quad \text{and similarly} \quad \frac{c_{q,E}(\beta;k,\ell)}{c_{q,E}(\beta;k,0)} = \sqrt{\frac{\ell}{k+\ell}} \cdot \frac{c_{q,E}(\beta;k,0)}{c_{q,E}(\beta;k,0)}.
\]

**Proof.** Let \( \alpha = \frac{k}{k+\ell} \) and in Equation (9) substitute \( x = \sqrt{\alpha} \cdot v \). By routine calculations, this substitution transforms the ratio of integrals into the ratio in (7) – multiplied by the factor \( \sqrt{\alpha} \), which completes the proof.

# 7 The distribution functions

## 7.1 First term approximation

**Definition 7.1** Let \( z > 0 \). Denote

\[
H(k,\ell;n,z) = \{ \lambda \in H(k,\ell;n) \mid \lambda_{1,n} \leq \frac{n}{k+\ell} + z\sqrt{n} \}.
\]

Let \( \beta > 0 \), \( 1 \leq p \leq k \) and \( 1 \leq q \leq \ell \). The distribution of the length of the \( p \)-th row as a function of \( z \) is defined as

\[
\lambda_p^{(\beta;k,\ell)}(n,z) = \frac{\sum_{\lambda \in H(k,\ell;n,z)} \lambda_{p,n} \cdot (f^\lambda)^\beta}{\sum_{\lambda \in H(k,\ell;n)} (f^\lambda)^\beta}.
\]

Similarly, the distribution of the length of the \( q \)-th column as a function of \( z \) is defined as

\[
\lambda_q^{(\beta;k,\ell)}(n,z) = \frac{\sum_{\lambda \in H(k,\ell;n,z)} \lambda_{q,n} \cdot (f^\lambda)^\beta}{\sum_{\lambda \in H(k,\ell;n)} (f^\lambda)^\beta}.
\]

Let \( \lambda \in H(k,\ell;n,z) \) with \( n \) large, then necessarily \( \lambda_1, \ldots, \lambda_k \geq \ell \) and \( \lambda_1', \ldots, \lambda_\ell' \geq k \) (in fact, Lemma 7.2 proves a much stronger property) so that \( \lambda_1, \ldots, \lambda_k + \lambda_1', \ldots, \lambda_\ell' = n + k\ell \). Write

\[
\lambda_p = \frac{n+k\ell}{k+\ell} + c_p \cdot \sqrt{n}, \quad p = 1, \ldots, k \quad \text{and} \quad \lambda_q' = \frac{n+k\ell}{k+\ell} + c_q' \cdot \sqrt{n}, \quad q = 1, \ldots, \ell,
\]

and notice that \( \sum c_p + \sum c_q' = 0 \).

**Lemma 7.2** With the above notations (and \( n \) large),

\[
-\frac{\ell + p - 1}{k - p + 1} : z \leq c_p \leq z, \quad p = 1, \ldots, k \quad \text{and} \quad -\frac{k + q - 1}{\ell - q + 1} : z \leq c_q' \leq z, \quad q = 1, \ldots, \ell.
\]

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Proof. Clearly, all \( c_p, c'_q \leq z \). Assume for example that \[
c_p < \frac{\ell + p - 1}{k - p + 1} \cdot z,
\]
hence also \( c_k, c_{k-1}, \ldots, c_p < \frac{-\ell + p - 1}{k - p + 1} \cdot z \). Thus \[
0 = c_1 + \cdots + c_k + c'_1 + \cdots + c'_\ell < -(k-p+1) \cdot \frac{\ell + p - 1}{k - p + 1} \cdot z + c_1 + \cdots + c_{p-1} + c'_1 + \cdots + c'_\ell
\]
\[
\leq -(\ell + p - 1) \cdot z + \ell + p - 1) \cdot z = 0,
\]
a contradiction. Similarly for \( c'_q \). This proves the lemma.

Recall that \( \Omega_{(k, \ell)} = \{(x_1, \ldots, x_k, y_1, \ldots, y_\ell) \mid x_1 \geq \cdots \geq x_k; y_1 \geq \cdots \geq y_\ell; \sum x_i + \sum y_j = 0\} \)
and \( \Omega_{(k, \ell), z} = \{(x_1, \ldots, x_k; y_1, \ldots, y_\ell) \in \Omega_{(k, \ell)} \mid x_1, y_1 \leq z\} \). For example, \( \Omega_{(2,0), z} = \{(x, -x) \mid 0 \leq x \leq z\} \).

**Theorem 7.3** Let \( \beta, z > 0 \) and denote
\[
r_{(k, \ell), \beta}(z) = \frac{\int_{\Omega_{(k, \ell), z}} \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)} \right]^\beta d^{(k+\ell-1)}(x; y)}{\int_{\Omega_{(k, \ell)}} \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)} \right]^\beta d^{(k+\ell-1)}(x; y)}.
\]

Let \( 1 \leq p \leq k, 1 \leq q \leq \ell \) and let \( n \to \infty \), then
\[
\lambda_p^{(\beta,k,\ell)}(n, z), \lambda_q^{(\beta,k,\ell)}(n, z) \sim \frac{n}{k+\ell} \cdot r_{(k, \ell), \beta}(z).
\]

In other words,
\[
\lim_{n \to \infty} \frac{k + \ell}{n} \cdot \lambda_p^{(\beta,k,\ell)}(n, z) = \lim_{n \to \infty} \frac{k + \ell}{n} \cdot \frac{\sum_{\lambda \in H(k,\ell;n,z)} \lambda_{p,n} \cdot (f^{\lambda})^\beta}{\sum_{\lambda \in H(k,\ell;n)} (f^{\lambda})^\beta} = \frac{\int_{\Omega_{(k, \ell), z}} \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)} \right]^\beta d^{(k+\ell-1)}(x; y)}{\int_{\Omega_{(k, \ell)}} \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)} \right]^\beta d^{(k+\ell-1)}(x; y)}.
\]

Similarly for
\[
\lim_{n \to \infty} \frac{k + \ell}{n} \cdot \lambda_q^{(\beta,k,\ell)}(n, z).
\]

**Proof** is similar to the proof of Theorem 4.1. Let \( \lambda_{p,n} = \frac{n}{k+\ell} + c_{p,n} \cdot \sqrt{n} \), then
\[
\frac{k + \ell}{n} \cdot \lambda_p^{(\beta,k,\ell)}(n, z) = \frac{\sum_{\lambda \in H(k,\ell;n,z)} (f^{\lambda})^\beta}{\sum_{\lambda \in H(k,\ell;n)} (f^{\lambda})^\beta} + \frac{k + \ell}{\sqrt{n}} \cdot \left( \frac{\sum_{\lambda \in H(k,\ell;n,z)} c_{p,n} \cdot (f^{\lambda})^\beta}{\sum_{\lambda \in H(k,\ell;n)} (f^{\lambda})^\beta} \right).
\]

By Theorem 2.5 the first summand approaches \( r_{(k, \ell), \beta}(z) \) as \( n \) goes to infinity, and by Lemma 7.2 \( |c_{p,n}| \leq b \cdot z \) for an appropriate constant \( b > 0 \), therefore the second summand obviously goes to zero as \( n \) goes to infinity. q.e.d.

Theorem 7.3 is a first–term approximation of the distribution function.
7.2 Conjectures about the second term approximation

Let

\[ s_{(k, \ell), \beta}(n, z) = \frac{\sum_{\lambda \in H(k, \ell, n, z)} (f^\lambda)^\beta}{\sum_{\lambda \in H(k, \ell, n)} (f^\lambda)^\beta}. \] \hspace{1cm} (16)

As in the proof of Theorem\textsuperscript{[3]} \lim_{n \to \infty} s_{(k, \ell), \beta}(n, z) = r_{(k, \ell), \beta}(z). \) Numerical evidence suggest the following (vague) conjecture.

**Conjecture 7.4** For all \( k, \ell \geq 0 \) and \( \beta, z > 0 \), as \( n \) goes to infinity the expression

\[ \frac{\sqrt{n}}{k + \ell} \cdot [s_{(k, \ell), \beta}(n, z) - r_{(k, \ell), \beta}(z)] = \] \hspace{1cm} (17)

oscillates in some symmetric bounded interval centered at zero. We denote that interval as \((-L(k, \ell, \beta, z), L(k, \ell, \beta, z))\), where (we conjecture that) \(-L(k, \ell, \beta, z)\) and \(L(k, \ell, \beta, z)\) are the respective infimum and supremum of the values in Equation \((17)\).

**Definition 7.5** Let \( \beta, z > 0 \), \( 1 \leq p \leq k \) and \( 1 \leq q \leq \ell \). Then \( \lambda_{p}^{(\beta, k, \ell)}(n, z) \) is given by Equation \((12)\). Similarly for the columns \( \lambda_{q}^{(\beta, k, \ell)}(n, z) \). Now define \( c_{p}^{(\beta, k, \ell)}(n, z) \) via the equation

\[ \lambda_{p}^{(\beta, k, \ell)}(n, z) = \frac{n}{k + \ell} \cdot r_{(k, \ell), \beta}(z) + c_{p}^{(\beta, k, \ell)}(n, z) \sqrt{n}. \]

Similarly for \( c_{q}^{(\beta, k, \ell)}(n, z) \). We would like to understand the behavior of \( c_{p}^{(\beta, k, \ell)}(n, z) \) and \( c_{q}^{(\beta, k, \ell)}(n, z) \) as \( n \) goes to infinity. \( \) We consider \( c_{p}^{(\beta, k, \ell)}(n, z) \).

Note that Definition\textsuperscript{[4]} Theorem\textsuperscript{[3]} and Definition\textsuperscript{[5]} imply

**Proposition 7.6**

\[ c_{p}^{(\beta, k, \ell)}(n, z) = \left( \frac{\sum_{\lambda \in H(k, \ell, n, z)} \lambda_{p, n} \lambda_{n, \beta}}{\sum_{\lambda \in H(k, \ell, n)} (f^\lambda)^\beta} \right) \cdot \frac{1}{\sqrt{n}}. \] \hspace{1cm} (18)

**Conjecture 7.7** Recall the interval \((-L(k, \ell, \beta, z), L(k, \ell, \beta, z))\) from Conjecture \textsuperscript{[4]} \hspace{1cm} As \( n \) goes to infinity, \( c_{p}^{(\beta, k, \ell)}(n, z) \) oscillates in the interval \((-L(k, \ell, \beta, z), L(k, \ell, \beta, z))\) + \( s_{p}^{(\ell, \beta)}(z) \), where

\[ s_{p}^{(\ell, \beta)}(z) = \frac{\int_{\Omega(k, \ell, z)} x_{p} \cdot \left[ D_{x}(x) \cdot D_{y}(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_{i}^{2} + \sum y_{j}^{2})} \right] d^{(k+\ell-1)}(x)}{\int_{\Omega(k, \ell)} \left[ D_{x}(x) \cdot D_{y}(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_{i}^{2} + \sum y_{j}^{2})} \right] d^{(k+\ell-1)}(x)}. \] \hspace{1cm} (19)
The proof is based on Conjecture 7.4 as follows. In (18) write \( \lambda_{p,n} = \frac{n}{k+\ell} + c_{p,n} \cdot \sqrt{n} \), then \( c_p^{(\beta; k, \ell)}(n, z) = A(n) + B(n) \), where

\[
A(n) = \frac{\sqrt{n}}{k+\ell} \cdot \left( \sum_{\lambda \in H(k, \ell; n)} \frac{(f_{\lambda})^\beta}{(f_{\lambda})^\beta} - r_{(k, \ell), \beta}(z) \right)
\]

and

\[
B(n) = \frac{\sum_{\lambda \in H(k, \ell; n)} c_{p,n} \cdot (f_{\lambda})^\beta}{\sum_{\lambda \in H(k, \ell; n)} (f_{\lambda})^\beta}.
\]

By arguments similar to those in previous proofs, \( \lim_{n \to \infty} B(n) = s_{(k, \ell), \beta}(z) \), and by Conjecture 7.4, \( A(n) \) oscillates in the interval \((-L(k, \ell, \beta, z), L(k, \ell, \beta, z))\).

Part II
Some special cases

8 \( \beta = 2 \), comparison of expected and maximal shapes

8.1 Expected shape

Recall the RSK bijection \( \sigma \leftrightarrow (P_{\lambda}, Q_{\lambda}) \) and the subsets

\[
S_{k,\ell;n} = \{ \sigma \in S_n \mid \text{under the RSK } \sigma \leftrightarrow (P_{\lambda}, Q_{\lambda}), \lambda \in H(k, \ell; n) \}
\]

from Section 1.4. For example if \( \ell = 0 \), it follows from well known properties of the RSK that \( S_{k,0;n} \) are the permutations in \( S_n \) with longest decreasing subsequence having length \( \leq k \). In general, if \( \sigma \in S_n \) is of shape \( \lambda = (\lambda_1, \lambda_2, \ldots) \), then \( \lambda_1 \) is the length of a maximal increasing subsequence in \( \sigma \). Thus, for example, \( \lambda_{1,E}^{(2; k, \ell)} \) (see Definition 1.2) is the expected length of the longest increasing subsequences in \( S_{k,\ell;n} \).

Consider first the case \( \ell = 0 \). Since the case \( k = 1 \) is trivial, assume \( k \geq 2 \) (and \( \ell = 0 \)). In that case, by Theorem 5.3, the expected shape in \( S_{k,0;n} \) is

\[
\lambda_{E}^{(2; k, 0)} \sim (\frac{2}{k} + c_1 \cdot \sqrt{n}, \frac{2}{k} + c_2 \cdot \sqrt{n}, \ldots, \frac{2}{k} + c_k \cdot \sqrt{n}),
\]

where for \( 1 \leq p \leq k \),

\[
c_p = c_{p,E}^{(2; k, 0)} = \frac{\int_{\Omega_k} x_p \cdot D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(x_1^2 + \cdots + x_k^2)}}{\int_{\Omega_k} D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(x_1^2 + \cdots + x_k^2)}} d(k-1)x.
\]

In general, the expected shape of the permutations in \( S_{k,\ell;n} \) is given by Definition 1.2 and the asymptotic shape as \( n \to \infty \) is given by Theorem 6.3 both with \( \beta = 2 \).
8.2 Maximal $f^\lambda$

Given a subset of partitions $\Gamma_n \subseteq Y_n$, one looks for $\lambda = \lambda_{max} \in \Gamma_n$ with maximal degree $f^\lambda$, see Section 13. In the case $\Gamma_n = H(k,0;n)$, the asymptotics of $\lambda_{max}$ is given in [2], which we briefly describe here. The analogue result from [18], for the case $\Gamma_n = H(k,\ell;n)$, is also described below.

Let $H_k(x)$ denote the $k$-th Hermit polynomial. It is defined via the equation

$$\frac{d^k}{dx^k} \left(e^{-x^2}\right) = (-1)^k H_k(x)e^{-x^2}. $$

Thus $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 4x(2x^2 - 3)$, $H_4(x) = 16x^4 - 48x^2 + 12$, etc. $H_k(x)$ is of degree $k$ and its roots are real and distinct, denoted $x_1^{(k)} < x_2^{(k)} < \ldots < x_k^{(k)}$.

Also, $x_1^{(k)} + x_2^{(k)} + \ldots + x_k^{(k)} = 0$. The following theorem is proved in [2]:

**Theorem 8.1** As $n \to \infty$, the maximum $\max \{ f^\lambda \mid \lambda \in H(k,0;n) \}$ occurs when

$$\lambda \sim \lambda_{max}^{(k,0)} = \left( \frac{n}{k} + x_1^{(k)} \sqrt{\frac{k}{n}}, \ldots, \frac{n}{k} + x_k^{(k)} \sqrt{\frac{k}{n}} \right).$$

The analogue result for $\Gamma_n = H(k,\ell;n)$ is given in [18]:

**Theorem 8.2** Let $\lambda = \lambda_{max}^{(k,\ell)} \in H(k,\ell;n)$ maximize $f^\lambda$ in $H(k,\ell;n)$ and write $\lambda_{max}^{(k,\ell)} = (\lambda_1, \ldots, \lambda_k, \ldots)$ and $\lambda_{max}^{(k,\ell)} = (\lambda_1', \ldots, \lambda_\ell, \ldots)$, and assume $n \to \infty$, then

$$(\lambda_1, \ldots, \lambda_k) \sim \left( \frac{n}{k + \ell} + x_1^{(k)} \sqrt{\frac{n}{k + \ell}}, \ldots, \frac{n}{k + \ell} + x_k^{(k)} \sqrt{\frac{n}{k + \ell}} \right)$$

and

$$(\lambda_1', \ldots, \lambda_\ell) \sim \left( \frac{n}{k + \ell} + x_1^{(\ell)} \sqrt{\frac{n}{k + \ell}}, \ldots, \frac{n}{k + \ell} + x_\ell^{(\ell)} \sqrt{\frac{n}{k + \ell}} \right).$$

Here $x_1^{(k)} < \ldots < x_k^{(k)}$ are the roots of the $k$-th Hermit polynomial and similarly for the $x_\ell^{(\ell)}$-s.

8.3 Examples of some $(k,0)$ cases

Let $\Gamma_n = H(k,0;n)$ and $\beta = 2$ ($\rightarrow_{RISK} S_{k,0;n}$) and compare the expected shape $\lambda_E^{(2,k,0)}$ with the maximizing shape $\lambda_{max}^{(k,0)}$. We begin with

**The case $k = 2$.**

Here the expected shape is given by

$$\lambda_E^{(2,2,0)} \sim \left( \frac{n}{2} + c_1^{(2,2,0)} \sqrt{n}, \frac{n}{2} + c_2^{(2,2,0)} \sqrt{n} \right),$$

where $c_1^{(2,2,0)}$ and $c_2^{(2,2,0)}$ are constants.
\( c_{2,E}^{(2,0)} = -c_{1,E}^{(2,0)} \), and by Equation (20), \( c_{1,E}^{(2,0)} = I_1 / I_2 \) is the ratio of the following integrals:

\[
I_1 = \int_0^\infty x \left[ 2x \cdot e^{-2x^2} \right]^2 \, dx = \frac{1}{8}, \quad \text{and} \quad I_2 = \int_0^\infty \left[ 2x \cdot e^{-2x^2} \right]^2 \, dx = \frac{\sqrt{\pi}}{8}.
\]

Thus \( c_{1,E}^{(2,0)} = \frac{1}{\sqrt{\pi}} \), hence

\[
\lambda_{E}^{(2,0)} \approx \left( \frac{n}{2} + \frac{1}{\sqrt{\pi}} \cdot \sqrt{n} \cdot \frac{n}{2} - \frac{1}{\sqrt{\pi}} \cdot \sqrt{n} \right) = \left( \frac{n}{2} + \frac{\sqrt{n}}{\sqrt{\pi}} \right)^2 (21)
\]

Compare \( \lambda_{E}^{(2,0)} \) with \( \lambda_{max}^{(2,0)} \). Here \( x_2^{(0)} = 1/\sqrt{2} \), \( x_1^{(2,0)} = -1/\sqrt{2} \), so

\[
\lambda_{max}^{(2,0)} = \left( \frac{n}{2} + \frac{1}{\sqrt{\pi}} \cdot \sqrt{n} \cdot \frac{n}{2} - \frac{1}{\sqrt{\pi}} \cdot \sqrt{n} \right) = \left( \frac{n}{2} + \frac{\sqrt{n}}{\sqrt{\pi}} \right).
\]

**Note:** working with Equation (6) and with 'Mathematica' we calculated \( c_{1,E}^{(2,0)}(n) \). For \( n = 100, 200, 300, 400, 500, 600, 700, 800, 900 \), the corresponding values of \( c_{1,E}^{(2,0)}(n) \) are: 0.517699, 0.530593, 0.536496, 0.54007, 0.542533, 0.544364, 0.545795, 0.546952 and 0.547915, agreeing with Theorem 5.3.

**The case \( k=3 \)**

Here \( \lambda_{E}^{(2,3,0)} \approx \left( \frac{4}{3} + \frac{1}{\sqrt{\pi}} \cdot \sqrt{n} \cdot \frac{4}{3} + \frac{1}{\sqrt{\pi}} \cdot \sqrt{n} \cdot \frac{4}{3} + \frac{1}{\sqrt{\pi}} \cdot \sqrt{n} \right) \), so we calculate the c-3. By Equation (20), \( c_{1,E}^{(2,3,0)} = J_1 / J_2 \) is the ratio of the following integrals:

\[
J_1 = \int_{\Omega_3} x_1 \left[ D_3(x)e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)} \right]^2 \, d(2) \, x \quad \text{and} \quad J_2 = \int_{\Omega_3} \left[ D_3(x)e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)} \right]^2 \, d(2) \, x.
\]

By Remark 5.4, with \( \beta = 2 \) and \( k=3 \), \( J_2 = \frac{\pi}{324 \sqrt{3}} \). We calculate the numerator \( J_1 \).

The domain \( \Omega_3 \) of integration is defined by: \( x_1 \geq x_2 \geq x_3 \) and \( x_3 = -(x_1 + x_2) \), so \( x_1 \geq x_2 \geq -(x_1 + x_2) \). When \( x_2 \leq 0 \), that condition is equivalent to \( x_2 \geq -x_1/2 \).

When \( x_2 \geq 0 \), that condition is equivalent to \( x_2 \leq x_1 \). It follows that

\[
J_1 = \int_0^\infty \left[ \int_{-1/2}^{x_1} \left( x_1 \left( x_1 - x_2 \right) \left( x_1 + x_2 \right) e^{-3(x_1^2 + x_2^2 + x_3^2)} \right)^2 \, dx_2 \right] \, dx_1.
\]

After some routine calculations ('Mathematica' was used here) we obtain \( J_1 = \frac{\sqrt{\pi}}{288 \sqrt{2}} \).

It follows that when \( k=3 \),

\[
\lim_{n \to \infty} c_{1,E}^{(2,3,0)}(n) = c_{1,E}^{(2,3,0)} = \frac{9 \sqrt{3}}{8 \sqrt{2 \pi}} = 0.777362...
\]

By similar calculations it follows that \( c_{2,E}^{(2,3,0)} = 0 \), hence \( c_{3,E}^{(2,3,0)} = -c_{1,E}^{(2,3,0)} \). Thus

\[
\lambda_{E}^{(2,3,0)} \approx \left( \frac{n}{3} + \frac{9 \sqrt{3}}{8 \sqrt{2 \pi} \sqrt{n}} \cdot \frac{n}{3} - \frac{9 \sqrt{3}}{8 \sqrt{2 \pi} \sqrt{n}} \right).
\]

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\[
(n \frac{2}{3} + 0.777362 \sqrt{n}, \ n \frac{2}{3} - 0.777362 \sqrt{n}).
\]

Compare now with \(\lambda_{\text{max}}^{(3,0)}\). Since \(H_3(x) = 4x(2x^2 - 3)\), \(x_3^{(3,0)} = \frac{\sqrt{3}}{\sqrt{2}}\), \(x_2^{(3,0)} = 0\) and \(x_1^{(3,0)} = -\frac{\sqrt{3}}{\sqrt{2}}\). Thus
\[
\lambda_{\text{max}}^{(3,0)} \sim \left(\frac{n}{3} + 0.707107 \sqrt{n}, \frac{n}{3} - 0.707107 \sqrt{n}\right).
\]

**Note:** For \(n = 200, 300, 400, 500, 700, 850\) and \(1100\), 'Mathematica' and Equation (6) give the following corresponding values of \(c_{1,E}^{(2;3,0)}(n)\): 0.719084, 0.729014, 0.735096, 0.739317, 0.74494, 0.747816 and 0.751261, in accordance with Theorem 5.3.

9. Examples of some \((k, \ell)-hook\) cases, \(\beta = 2\)

9.1 The \((1, 1)\) case

By Equation (9) \(c_{1,E}^{(2,1,1)} = 0\). [Alternatively, calculate \(c_{1,E}^{(2,1,1)}\) by Equation (11). Here \(\Omega_{1,1} = \{(x, y) | x + y = 0\}\), so \(y = -x\) and \(-\infty \leq x \leq \infty\). Thus \(c_{1,E}^{(2,1,1)} = I_1(\beta)/I_2(\beta)\) where \(I_1(\beta) = \int_{-\infty}^{\infty} x e^{-2x^2} dx = 0\).] Similarly \(c_{1,E}^{(2,1,1)} = 0\) by Equation (10). It follows that (for any \(\beta > 0\))
\[
\lambda_{E}^{(2,1,1)} \sim \left(\frac{n}{2}, \frac{1}{1^n}\right).
\]

9.2 The \((2, 1)\) case

By Corollary 6.5
\[
\lim_{n \to \infty} c_{1,E}^{(2,2,1)}(n) = c_{1,E}^{(2,2,1)} = \sqrt{\frac{2}{3}} \cdot c_{1,E}^{(2,2,0)} = \sqrt{\frac{2}{3}} \cdot \frac{1}{\sqrt{\pi}} = 0.460659.
\]

(23)

Since \(c_{2,E}^{(2,2,0)} = -c_{1,E}^{(2,2,0)}\), deduce that
\[
c_{2,E}^{(2,2,1)} = -\sqrt{\frac{2}{3}} \cdot \frac{1}{\sqrt{\pi}} = -0.460659.
\]

(24)

Since the sum of the coordinates in \(\lambda_{E}^{(2,2,1)}(n)\) is \(n\), it follows that \(c_{1,E}^{(2,2,1)} = 0\) (this can also be deduced directly from Theorem 6.3). Therefore
\[
\lambda_{E}^{(2,2,1)} \sim \left(\frac{n}{3} + 0.460659 \sqrt{n}, \frac{n}{3} - 0.460659 \sqrt{n}, 1^{n/3}\right).
\]

**Note:** for \(n = 100, 200, 300, 400, 700\) and \(1100\), Remark (6.2) and 'Mathematica' give the following values of \(c_{1,E}^{(2,2,1)}(n)\): 0.45423, 0.45571, 0.456475, 0.456962, 0.457777 and 0.458317, agreeing with Theorem 6.3.
10 Expected shape, $\beta = 1$

10.1 The general case

Since $\sigma \in S_n$ is an involution if the RSK yields $\sigma \leftrightarrow (P_\lambda, P_\lambda)$, therefore $\lambda^{(1;k,\ell)}_E(n)$ is the expected shape of the involutions in $S_{k,\ell,n}$. Note that when $\Gamma_n = Y_n$ (and $\beta = 1$) Baik and Rains [5] showed that the expected length of the first row (or of the longest increasing subsequence in involutions in $S_n$) is again $2\sqrt{n}$, i.e. the same as the first row of $\lambda_{\text{max}}$, see [5] (1.5). As we show below, this is not the case when $\Gamma_n = H(k, \ell; n)$.

Denote $\tilde{\lambda}^{(k,\ell)}(n) = \lambda^{(1;k,\ell)}_E(n)$. We summarize:

1. Let $1 \leq p \leq k, \ 1 \leq q \leq \ell$, then
   \[
   \tilde{\lambda}^{(k,\ell)}_{p,n} = \frac{\sum_{\lambda \in H(k,\ell,n)} \lambda_{p,n} \cdot f^\lambda}{\sum_{\lambda \in H(k,\ell,n)} f^\lambda} \quad \text{and} \quad \tilde{\lambda}^{(k,\ell)}_{q,n} = \frac{\sum_{\lambda \in H(k,\ell,n)} \lambda_{q,n} \cdot f^\lambda}{\sum_{\lambda \in H(k,\ell,n)} f^\lambda}.
   \]

2. Define $\tilde{c}^{(k,\ell)}_{p,n}$ via
   \[
   \tilde{\lambda}^{(k,\ell)}_{p,n} = \frac{n}{k + \ell} + \tilde{c}^{(k,\ell)}_{p,n} \sqrt{n}, \quad \text{and} \quad \tilde{c}^{(k,\ell)}_{p} = \lim_{n \to \infty} \tilde{c}^{(k,\ell)}_{p,n}.
   \]

   Similarly for $\tilde{c}^{(k,\ell)}_{q,n}$ and $\tilde{c}^{(k,\ell)}_{q}$. Thus, when $n \to \infty$,
   \[
   \tilde{\lambda}^{(k,\ell)}_{p,n} \sim \frac{n}{k + \ell} + \tilde{c}^{(k,\ell)}_{p,n} \cdot \sqrt{n} \quad \text{and} \quad \tilde{\lambda}^{(k,\ell)}_{q,n} \sim \frac{n}{k + \ell} + \tilde{c}^{(k,\ell)}_{q,n} \cdot \sqrt{n}.
   \]

3. (Theorem 6.3, $\beta = 1$) Let $1 \leq p \leq k, \ 1 \leq q \leq \ell$, then the limits $\tilde{c}^{(k,\ell)}_{p}$ and $\tilde{c}^{(k,\ell)}_{q}$ exist, and are given as follows.
   \[
   \tilde{c}^{(k,\ell)}_{p} = \int_{\Omega_k} x_p \cdot D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k+\ell}{2}(x^2_1 + \cdots + x^2_k)} d^{(k-1)}(x) \quad \text{and} \quad \tilde{c}^{(k,\ell)}_{q} = \int_{\Omega_k} y_q \cdot D_\ell(y_1, \ldots, y_\ell) \cdot e^{-\frac{k+\ell}{2}(y^2_1 + \cdots + y^2_\ell)} d^{(\ell-1)}(y).
   \]

   Similarly for the expected columns:
   \[
   \tilde{c}^{(k,\ell)}_{q} = \int_{\Omega_k} y_q \cdot D_\ell(y_1, \ldots, y_\ell) \cdot e^{-\frac{k+\ell}{2}(y^2_1 + \cdots + y^2_\ell)} d^{(\ell-1)}(y).
   \]

4. (Corollary 6.5, $\beta = 1$) We have
   \[
   \tilde{c}^{(k,\ell)}_{p} = \sqrt{\frac{k}{k + \ell}} \cdot \tilde{c}^{(k,0)}_{p}, \quad \text{and similarly} \quad \tilde{c}^{(k,\ell)}_{q} = \sqrt{\frac{\ell}{k + \ell}} \cdot \tilde{c}^{(0,\ell)}_{q}.
   \]
10.2 Examples for involutions in $S_{k,0;n}$

(i.e. $\beta = 1$ and $\ell = 0$).

When $k = 2$, we obtain $\int_0^\infty 2x \cdot e^{-2x^2} = \frac{1}{2}$ and $\int_0^\infty x^2 \cdot e^{-2x^2} = \frac{\sqrt{\pi}}{4\sqrt{2}}$

Thus $\tilde{c}_1^{(2,0)} = \frac{\sqrt{\pi}}{2\sqrt{2}}$ and

$\tilde{\chi}^{(2,0)} = (\tilde{\lambda}_1^{(2,0)}, \tilde{\lambda}_2^{(2,0)}) \sim \left( \frac{n}{2} + \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \sqrt{n}, \frac{n}{2} - \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \sqrt{n} \right) = \left( \frac{n}{2} + 0.626657\sqrt{n}, \frac{n}{2} - 0.626657\sqrt{n} \right)$. \hspace{1cm} (27)

Note: for $n = 200, 400, 600, 800, 1000, 1200$ and $1400$, Remark 6.2 and 'Mathematica' give the following values of $\tilde{c}_1^{(2,0)}(n)$: 0.592086, 0.602049, 0.606506, 0.609175, 0.611002, 0.612354 and 0.613406, agreeing with Equation (25).

When $k = 3$ we have

By Equation (25), $c_{1,E}^{(1,3,0)} = \tilde{J}_1/\tilde{J}_2$ is the ratio of the following integrals:

$\tilde{J}_1 = \int_{\Omega_3} x_1 D_3(x) e^{-\frac{3}{4}(x_1^2 + x_2^2 + x_3^2)} d^2 x$ and $\tilde{J}_2 = \int_{\Omega_3} D_3(x) e^{-\frac{3}{4}(x_1^2 + x_2^2 + x_3^2)} d^2 x$

By Remark 5.4 ($\beta = 1$, $k = 3$), $\tilde{J}_2 = \frac{\sqrt{\pi}}{2\pi}$. We calculate the numerator $\tilde{J}_1$. Similar to the evaluation of $J_1$ in Section 8.3 here

$\tilde{J}_1 = \int_0^\infty \left[ \int_{-\frac{x_1}{2}}^{x_1} \left( x_1(x_1 - x_2)(2x_1 + x_2)(x_1 + 2x_2) e^{-3(x_1^2 + x_2^2 + x_1x_2)} \right) dx_2 \right] dx_1 = 1/18.$

It follows that when $k = 3$, $\tilde{c}_1^{(3,0)} = c_{1,E}^{(1,3,0)} = \frac{3}{2\sqrt{\pi}} = 0.846284...$ Again, $\tilde{c}_2^{(3,0)} = 0$ so $\tilde{c}_3^{(3,0)} = -\tilde{c}_1^{(3,0)}$. Thus, when $k = 3$,

$\tilde{\chi}^{(3,0)} = \chi_E^{(1,3,0)} = \left( \frac{n}{3} + \frac{3}{2\sqrt{\pi}} \cdot \sqrt{n}, \frac{n}{3} - \frac{3}{2\sqrt{\pi}} \cdot \sqrt{n} \right) = \left( \frac{n}{3} + 0.846284 \sqrt{n}, \frac{n}{3} - 0.846284 \sqrt{n} \right)$. \hspace{1cm} (28)

Note: for $n = 200, 400, 600, 800, 1000$, and $1200$, Remark 6.2 and 'Mathematica' give the following values of $c_{1,E}^{(3,0)}(n) = c_{1,E}^{(1,3,0)}$:

0.789051, 0.80486, 0.812115, 0.816513, 0.819547 and 0.821802, as predicted by Equation (25).

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