RELATIVE OPERATOR ENTROPIES AND TSALLIS RELATIVE OPERATOR ENTROPIES IN JB-ALGEBRAS

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ABSTRACT. We initiate the study of relative operator entropies and Tsallis relative operator entropies in the setting of JB-algebras. We establish their basic properties and extend the operator inequalities on relative operator entropies and Tsallis relative operator entropies to this setting. In addition, we improve the lower and upper bounds of the relative operator $(\alpha, \beta)$-entropy in the setting of JB-algebras that were established in Hilbert space operators setting by Nikoufar [18, 20]. Though we employ the same notation as in the classical setting of Hilbert space operators, the inequalities in the setting of JB-algebras have different connotations and their proofs requires techniques in JB-algebras.

1. INTRODUCTION

Motivated by the study of quantum mechanics, Jordan, von Neumann, and Wigner investigated finite dimensional Jordan algebras in [13]. Later, von Neumann studied the infinite dimensional Jordan algebras [17]. In [21], Segal initiated the study of JC-algebras, and Effros and Størmer [6], Størmer [23, 24, 25] and Topping [26], among others, studied these algebras more thoroughly. The theory of JB-algebras was inaugurated by Alfsen, Shultz, and Størmer [2] and later considered by many others. As a motivation for this line of research, observables in a quantum system constitute a JB-algebra which is non-associative, therefore JB-algebras were considered as natural objects of study for quantum system. In mathematics, JB-algebras also have many powerful applications in many fields, such as analysis, geometry, operator theory, etc; more information on these can be found in [3, 28, 29].

Entropy, as a measure of uncertainty, is a fundamental notion in quantum information theory. A mathematical formulation of entropy was given by Segal [22]. In order to understand the basis of Segal’s notion, operator entropy $-A \log(A)$ for positive invertible operator $A$ was considered by Nakamura and Umegaki [16]. Later, the relative operator entropy was used by Umegaki [27] to study the measures of entropy and information. The concept of relative operator entropy $S(A|B)$ for strictly positive operators in noncommutative information theory was first introduced by Fujii and Kamei in [7, 8]. As an extension of relative operator entropy, generalized relative operator entropy $S_\alpha(A|B)$ was studied by Furuta in [10]. Meanwhile, Tsallis relative operator entropy $T_\lambda(A|B)$ was investigated by Yanagi, Kuriyama and Furutichi [32], which has the property $\lim_{\lambda \to 0} T_\lambda(A|B) = S(A|B)$. Furthermore,
the notion of relative operator \((\alpha, \beta)-\text{entropy}\) was introduced by Nikoufar [19] as follows:

\[
S_{\alpha, \beta}(A|B) = A^{\frac{\alpha}{2}} [(A^{-\frac{\alpha}{2}} B A^{-\frac{\alpha}{2}}) \log (A^{-\frac{\alpha}{2}} B A^{-\frac{\alpha}{2}})] A^{\frac{\alpha}{2}}
\]

(1.1)

for invertible positive operators \(A, B\) and any real numbers \(\alpha, \beta\). This has the properties that \(S_{\alpha, 1}(A|B) = S_{\alpha}(A|B)\) and \(S_{0, 1}(A|B) = S(A|B)\). More recently, relative operator entropies was used as a powerful tool to study quantum coherence [11], a key notion in quantum information processing.

In [30], we initiated the study of relative operator entropies in the settings of \(C^*\)-algebras, real \(C^*\)-algebras and JC-algebras. We extended operator inequalities on relative operator entropies to these settings, and improved the lower and upper bounds of the relative operator entropy which are new even for relative operator entropy defined on Hilbert space.

In another preprint [31], we studied operator means in the setting of JB-algebras, and obtained basic operator inequalities using nonassociative perspective function, which is defined as follows:

\[
P_{f, \Delta h}(A, B) = \left\{ h(B)^{\frac{1}{2}} f \left( \left\{ h(B)^{-\frac{1}{2}} A h(B)^{-\frac{1}{2}} \right\} \right) h(B)^{\frac{1}{2}} \right\},
\]

(1.2)

where \(f\) and \(h\) are real continuous function on a closed interval \([1, \infty)\) with \(h > 0\) and \(A, B\) are two elements in a unital JB-algebra with spectra contained in \([1, \infty)\). Our notion is a generalization of [4] for noncommutative associative case of Hilbert space operators. For commutative case, perspective function was inaugurated by Effros [5], in which an ingenious and simple proof of the celebrated Lieb’s concavity theorem [14, 15] was given.

In the present paper, we study relative operator entropies and Tsallis relative operator entropies in the setting of JB-algebras. We define in section 3 relative operator entropies and Tsallis relative operator entropies in the setting of JB-algebras and investigate their properties. In section 4, we extend the operator inequalities on relative operator entropies and Tsallis relative operator entropies to this setting; we also improve the lower and upper bounds of the relative operator \((\alpha, \beta)\)-entropy in the setting of JB-algebras that were established in Hilbert space operators setting by Nikoufar [18, 20] which refined the bounds for relative operator entropy obtained earlier by Fujii and Kamei [7, 8]. Though we employ the same notation as in the classical setting of Hilbert space operators, the properties and inequalities in the setting of JB-algebras we establish in this paper have different connotations and their proofs requires techniques in JB-algebras.

2. Preliminaries

For convenience of the reader, we give some background on JB-algebras and fix the notation in this section.

**Definition 2.1.** A Jordan algebra \(\mathcal{A}\) over real number is a vector space \(\mathcal{A}\) over \(\mathbb{R}\) equipped with a bilinear product \(\circ\) that satisfies the following identities:

\[
a \circ b = b \circ a, \quad (a^2 \circ b) \circ a = a^2 \circ (b \circ a).
\]

Any associative algebra \(\mathcal{A}\) has an underlying Jordan algebra structure with Jordan product given by

\[
a \circ b = (ab + ba)/2.
\]

Jordan subalgebra of such underlying Jordan algebras is called special.
As the important example in physics, $B(H)_{sa}$, the set of bounded self adjoint operators on a Hilbert space $H$, is a special Jordan algebra. Note that $B(H)_{sa}$ is not an associative algebra.

**Definition 2.2.** A concrete **JC-algebra** $\mathcal{A}$ is a norm-closed Jordan subalgebra of $B(H)_{sa}$.

**Definition 2.3.** A **JB-algebra** is a Jordan algebra $\mathcal{A}$ over $\mathbb{R}$ with a complete norm satisfying the following conditions for $A, B \in \mathcal{A}$:

\[ \|A \circ B\| \leq \|A\| \|B\|, \quad \|A^2\| = \|A\|^2, \text{ and } \|A^2\| \leq \|A^2 + B^2\| . \]

A JC-algebra is a JB-algebra, but the converse is not true. For example, the Albert algebra is a JB-algebra but not a JC-algebra, cf. [1, Theorem 4.6].

**Definition 2.4.** Let $\mathcal{A}$ be a unital JB-algebra. We say $A \in \mathcal{A}$ is **invertible** if there exists $B \in \mathcal{A}$, which is called **Jordan inverse** of $A$, such that

\[ A \circ B = I \quad \text{and} \quad A^2 \circ B = A. \]

The **spectrum** of $A$ is defined by

\[ \text{Sp}(A) := \{ \lambda \in \mathbb{R} \mid A - \lambda I \text{ is not invertible in } \mathcal{A} \}. \]

If $\text{Sp}(A) \subset [0, \infty)$, we say $A$ is **positive**, and write $A \geq 0$.

**Definition 2.5.** Let $\mathcal{A}$ be a unital JB-algebra and $A, B \in \mathcal{A}$. We define a map $U_A$ on $\mathcal{A}$ by

\[ U_A B := \{ABA\} := 2(A \circ B) \circ A - A^2 \circ B. \quad (2.1) \]

It follows from (2.1) that $U_A$ is linear, in particular,

\[ U_A(B - C) = \{ABA\} - \{ACA\}. \quad (2.2) \]

Note that $ABA$ is meaningless unless $\mathcal{A}$ is special, in which case $\{ABA\} = ABA$. The following proposition will be used repeatedly in this paper.

**Proposition 2.6.** [1, Lemma 1.23-1.25] Let $\mathcal{A}$ be a unital JB-algebra and $A, B$ be two elements in $\mathcal{A}$.

1. If $B$ is positive, then $U_A(B) = \{ABA\} \geq 0$.
2. If $A, B$ are invertible, then $\{ABA\}$ is invertible with inverse $\{A^{-1}B^{-1}A^{-1}\}$.
3. If $A$ is invertible, then $U_A$ has a bounded inverse $U_{A^{-1}}$.

For an element $A$ in $\mathcal{A}$ and a continuous function $f$ on the spectrum of $A$, $f(A)$ is defined by functional calculus in JB-algebras (see e.g. [1, Proposition 1.21]).

**Definition 2.7.** Let $f$ be a real valued continuous function $f$ on $\mathbb{R}$.

1. $f$ is said to be **operator monotone (increasing)** on a JB-algebra $\mathcal{A}$ if $0 \leq A \leq B$ implies $f(A) \leq f(B)$.
2. $f$ is **operator convex** if for any $\lambda \in [0, 1]$ and $A, B \geq 0$,

\[ f((1 - \lambda)A + \lambda B) \leq (1 - \lambda)f(A) + \lambda f(B). \]

We say that $f$ is **operator concave** if $-f$ is operator convex.
3. Relative operator entropy and Tsallis relative operator entropy

**Definition 3.1.** Let $A$, $B$ be two positive invertible elements in a unital JB-algebra $\mathcal{A}$. The relative operator entropy $S(A|B)$ is defined by

$$S(A|B) := \left\{ A^{\frac{1}{2}} \log \left( \left\{ A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right\} \right) A^{\frac{1}{2}} \right\}. \tag{3.1}$$

For any $\lambda \in (0, 1]$, the Tsallis relative operator entropy $T_\lambda(A|B)$ is defined by

$$T_\lambda(A|B) := \frac{A\#_\lambda B - A}{\lambda}, \tag{3.2}$$

where $A\#_\lambda B := \left\{ A^{\frac{1}{2}}\{ A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \}^\lambda A^{\frac{1}{2}} \right\}$ is the weighted geometric mean; for more information, see [31].

**Theorem 3.2.** Let $A$ and $B$ be positive invertible. Then

$$S(A|B) = \int_0^1 \frac{A_t B - A}{t} dt, \tag{3.3}$$

where $A_t B := ((1 - \lambda)A^{-1} + \lambda B^{-1})^{-1}$ is the weighted harmonic mean.

**Proof.** The following identity is established in the proof of [31, Proposition 3] for $x > 0$,

$$\log x = \int_0^1 \frac{(1 - t + tx^{-1})^{-1} - 1}{t} dt. \tag{3.4}$$

Applying functional calculus in JB-algebras (see [1, Proposition 1.21]) to (3.4) and by Proposition 2.6,

$$\log \left( \left\{ A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right\} \right) = \int_0^1 \frac{[(1 - t)I + t\{ A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} \}]^{-1} - I}{t} dt. \tag{3.5}$$

Therefore, by Proposition 2.6 again,

$$S(A|B) = \left\{ A^{\frac{1}{2}} \int_0^1 \frac{[(1 - t)I + t\{ A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} \}]^{-1} - I}{t} dt A^{\frac{1}{2}} \right\}$$

$$= \int_0^1 \frac{A^{-\frac{1}{2}}[(1 - t)I + t\{ A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} \}]A^{-\frac{1}{2}}}{t} dt - A$$

$$= \int_0^1 \frac{[(1 - t)A^{-1} + tB^{-1}]^{-1} - A}{t} dt$$

$$= \int_0^1 \frac{A_t B - A}{t} dt. \tag{3.6}$$

\[\square\]

**Proposition 3.3.** Let $A$ and $B$ be positive invertible. Then

$$S(A|B) = \left\{ B^{\frac{1}{2}} \left[ -\{ B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \} \circ \log(\{ B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \}) \right] B^{\frac{1}{2}} \right\}. \tag{3.7}$$
Proof. According to the proof of [31, Proposition 3], we have
\[ -x \log x = \int_0^1 \frac{(1-t)x^{-1} + t}{t} - x \, dt. \]  (3.8)

Denote
\[ E = - \left\{ B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right\} \circ \log \left( \{B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\} \right). \]

Utilizing functional calculus in JB-algebras for (3.8),
\[ E = \int_0^1 \left( (1-t)\left\{ B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right\}^{-1} + t \right)^{-1} - \{B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\} \, dt \]  (3.9)

Therefore,
\[ \left\{ B^{\frac{1}{2}}EB^{\frac{1}{2}} \right\} = \left\{ B^{\frac{1}{2}} \int_0^1 \frac{(1-t)\left\{ B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right\}^{-1} + t}{t} - \{B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\} \, dt \right\} B^{\frac{1}{2}} \]
\[ = \int_0^1 \left\{ B^{-\frac{1}{2}}\left( (1-t)\left\{ B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \right\}^{-1} + t \right)^{-1} - A \right\} \, dt \]
\[ = \int_0^1 \left\{ (1-t)A^{-1} + tB^{-1} \right\}^{-1} - A \, dt \]
\[ = \int_0^1 A_1B - A \, dt \]
\[ = S(A|B). \]

\[ \square \]

**Proposition 3.4.** Let \( A \) be a unital JB-algebra. The functions \( x \to -x \log x \) is operator concave on \((0, \infty)\).

**Proof.** Denote \( h_\alpha(x) = 1 - \alpha(\alpha + x)^{-1} - x(\alpha + 1)^{-1} \). By (4.5) in [31], \( -\alpha(\alpha + x)^{-1} \) is operator concave on \((0, \infty)\). Thus, \( h_\alpha(x) \) is also operator concave on \((0, \infty)\). By (4.9) in [31],
\[ -x \log x = \int_0^\infty x[(\alpha + x)^{-1} - (\alpha + 1)^{-1}] \, d\alpha \]
\[ = \int_0^\infty h_\alpha(x) \, d\alpha \]

By [31, Lemma 1], \( -x \log x \) is operator concave. \( \square \)

**Proposition 3.5.** The relative operator entropy \( S(A|B) \) defined in \( A \) has the following properties:

(i) \( S(\alpha A|\alpha B) = \alpha S(A|B) \) for any positive number \( \alpha \).

(ii) If \( B \leq C \), then \( S(A|B) \leq S(A|C) \).

(iii) \( S(A|B) \) is operator concave with respect to \( A, B \) individually.

(iv) \( S(\{CA\} | \{CB\}) = \{CS(A|B)C\} \) for any invertible \( C \) in \( A \).

**Proof.** For (i), it follows directly from the definition.
Proof of (ii). If \( B \leq C \), then
\[ \left\{ A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right\} \leq \left\{ A^{-\frac{1}{2}}CA^{-\frac{1}{2}} \right\}. \]
By [31, Proposition 5],
\[
\log \left( \{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\} \right) \leq \log \left( \{A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\} \right).
\]

This implies that \( S(A|B) \leq S(A|C) \).

(iii) For any \( 0 \leq t \leq 1 \), we denote
\[
E = S(A|(1-t)B_1 + tB_2).
\]
Since \( \log x \) is operator concave then
\[
E = \left\{ A^{\frac{1}{2}} \log \left( (1-t)\{A^{-\frac{1}{2}}B_1A^{-\frac{1}{2}}\} + t\{A^{-\frac{1}{2}}B_2A^{-\frac{1}{2}}\} \right) A^{\frac{1}{2}} \right\}
\geq (1-t) \left\{ A^{\frac{1}{2}} \log \left( \{A^{-\frac{1}{2}}B_1A^{-\frac{1}{2}}\} \right) A^{\frac{1}{2}} \right\}
+ t \left\{ A^{\frac{1}{2}} \log \left( \{A^{-\frac{1}{2}}B_2A^{-\frac{1}{2}}\} \right) A^{\frac{1}{2}} \right\}
= (1-t)S(A|B_1) + tS(A|B_2)
\]
On the other hand, we denote
\[
F = \left[ -\{B^{-\frac{1}{2}}((1-t)A_1 + tA_2)B^{-\frac{1}{2}}\} \circ \log(\{B^{-\frac{1}{2}}((1-t)A_1 + tA_2)B^{-\frac{1}{2}}\}) \right]
G = S(\{(1-t)A_1 + tA_2\}|B)
\]
By Proposition 3.4,
\[
F \geq (1-t) \left[ -\{B^{-\frac{1}{2}}A_1B^{-\frac{1}{2}}\} \circ \log(\{B^{-\frac{1}{2}}A_1B^{-\frac{1}{2}}\}) \right]
+ t \left[ -\{B^{-\frac{1}{2}}A_2B^{-\frac{1}{2}}\} \circ \log(\{B^{-\frac{1}{2}}A_2B^{-\frac{1}{2}}\}) \right]
\]
From Propsotion 3.3, one sees that
\[
G = \left\{ B^{\frac{1}{2}}FB^{\frac{1}{2}} \right\}
\geq (1-t) \left\{ B^{\frac{1}{2}} \left[ -\{B^{-\frac{1}{2}}A_1B^{-\frac{1}{2}}\} \circ \log(\{B^{-\frac{1}{2}}A_1B^{-\frac{1}{2}}\}) \right] B^{\frac{1}{2}} \right\}
+ t \left\{ B^{\frac{1}{2}} \left[ -\{B^{-\frac{1}{2}}A_2B^{-\frac{1}{2}}\} \circ \log(\{B^{-\frac{1}{2}}A_2B^{-\frac{1}{2}}\}) \right] B^{\frac{1}{2}} \right\}
= (1-t)S(A_1|B) + tS(A_2|B)
\]
Proof of (iv). By definition of harmonic mean in [31] and Proposition 2.6, for any invertible element \( C \) in \( \mathcal{A} \)
\[
\{CAC\}!t\{ABC\} = [(1-t)\{CAC\}^{-1} + t\{CBC\}]^{-1}
= \{C^{-1}[(1-t)A^{-1} + tB^{-1}]C^{-1}\}^{-1}
= \{C[(1-t)A^{-1} + tB^{-1}]^{-1}C\}
= \{C(A_1tB)C\}
\]
According to Theorem 3.2 and (3.10)
\[
S(\{CAC\}|\{CBC\}) = \int_0^1 \frac{\{C(A_1tB)C\} - \{CAC\}}{t} \, dt
= \left\{ C \int_0^1 \frac{\{(A_1tB - A)\}}{t} \, dt \right\} C
= \{CS(A|B)C\}.
\]
Proposition 3.6. Let $A$ be a JC-algebra. The relative operator entropy $S(A|B)$ is jointly operator concave.

Proof. A JC-algebra can be realized as self-adjoint operators on a Hilbert space. Since $f(x) = \log x$ and $h(x) = x$ are operator concave, then by $[4$, Corollary 2.6$]$, the relative operator entropy $S(A|B) = P_{f\triangle h}(B, A)$ is jointly operator concave for operators $A$ and $B$ on a Hilbert space.

Theorem 3.7. Let $A, B$ be two positive invertible elements in a unital JB-algebra $A$. For any $\lambda \in (0, 1)$,

$$T_\lambda(A|B) = \frac{\sin(\lambda \pi)}{\lambda \pi} \int_0^1 \left( \frac{t}{1-t} \right)\lambda A_t B - A dt.$$  \hspace{1cm} (3.11)

Proof. By $[31$, Theorem 2$]$, $A\#_\lambda B = \frac{\sin(\lambda \pi)}{\pi} \int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda} (A_t B) dt.$ (3.12)

Applying functional calculus in JB-algebras to the following identity of $\Gamma$-function,

$$\frac{\sin(\lambda \pi)}{\pi} \int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda} dt = 1,$$  \hspace{1cm} (3.13)

we have

$$\frac{\sin(\lambda \pi)}{\pi} \int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda} A dt = A.$$  \hspace{1cm} (3.14)

Combining (3.12) and (3.14),

$$A\#_\lambda B - A = \frac{\sin(\lambda \pi)}{\lambda \pi} \int_0^1 \left( \frac{t}{1-t} \right)\lambda A_t B - A dt.$$  \hspace{1cm} (3.15)

Proposition 3.8. The Tsallis relative operator entropy $T_\lambda(A|B)$ defined in $A$ has the following properties:

(i) $T_\lambda(\alpha A|B) = \alpha T_\lambda(A|B)$ for any positive number $\alpha$

(ii) If $B \leq D$, then $T_\lambda(A|B) \leq T_\lambda(A|D)$.

(iii) $T_\lambda(\{CAC\}||\{CBC\}) = \{CT_\lambda(A|B)C\}$ for any invertible $C$ in $A$.

(iv) $T_\lambda(A|B)$ is operator concave with respect to $A, B$ individually.

(v) $\lim_{\lambda \to 0} T_\lambda(A|B) = S(A|B).$

Proof. By $[31$, Proposition 6(i)$]$, $(\alpha A)\#_\lambda (\alpha B) = \alpha (A\#_\lambda B)$. Then (i) follows immediately.

For (ii), it follows from $[31$, Proposition 6(ii)$]$ that

$$A\#_\lambda B - A \leq A\#_\lambda D - A.$$  

Then, $T_\lambda(A|B) \leq T_\lambda(A|D)$.

(iii) According to $[31$, Proposition 6(iv)$]$, $T_\lambda(\{CAC\}||\{CBC\}) = \{CAC\}\#_\lambda \{CBC\} - \{CAC\}$

$$= \{C(A\#_\lambda B)C\} - \{CAC\} \hspace{1cm} (C)$$

$$= \{CT_\lambda(A|B)C\}.$$
Proof of (iv), it follows from the fact $A\#_{\lambda}B$ is operator concave with respect to $A, B$ individually (See e.g. [31, Proposition 6(iii)]).

(v) Denote $\ln_{x} = \frac{x^{\lambda-1}}{\lambda}$. By Dini’s theorem, $\ln_{x}$ uniformly converges to $\log x$ on any bounded closed interval $[a, b] \subset [0, \infty)$. It implies that

$$
\lim_{\lambda \to 0} \ln_{\lambda}\left(\left\{ A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right\}^{\lambda} \right) = \log \left(\left\{ A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right\} \right).
$$

Since $U_{\frac{1}{2}}$ is continuous, then

$$
\lim_{\lambda \to 0} T_{\lambda}(A|B) = \left\{ A^{\frac{1}{2}} \log \left(\left\{ A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right\} \right) A^{\frac{1}{2}} \right\} = S(A|B).
$$

Similar argument as in Proposition 3.6 gives

**Proposition 3.9.** Let $A$ be a JC-algebra. The Tsallis relative operator entropy $T_{\lambda}(A|B)$ is jointly operator concave.

4. Upper and lower bounds of generalized relative operator entropies and Tsallis relative operator entropies

**Definition 4.1.** Let $A, B$ be positive invertible elements in a unital JB-algebra $A$. The **relative operator** $(\alpha, \beta)$-entropy $S_{\alpha, \beta}(A|B)$ and **Tsallis relative operator** $(\lambda, \beta)$-entropy $T_{\lambda, \beta}(A|B)$ are defined respectively by

$$
S_{\alpha, \beta}(A|B) := \left\{ A^{\frac{\alpha}{\beta}} \left[ \left\{ A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}} \right\}^{\alpha} \circ \log \left(\left\{ A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}} \right\}\right) \right] A^{\frac{\alpha}{\beta}} \right\},
$$

$$
T_{\lambda, \beta}(A|B) := \left\{ A^{\frac{\alpha}{\beta}} \ln_{\lambda} \left(\left\{ A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}} \right\}\right) A^{\frac{\alpha}{\beta}} \right\}.
$$

Clearly, $S_{0,1}(A|B) = S(A|B)$ and $T_{\lambda,1}(A|B) = T_{\lambda}(A|B)$.

In this section, we study the bounds of $S_{\alpha, \beta}(A|B)$ and $T_{\lambda, \beta}(A|B)$ in the setting of JB-algebras. For real numbers $\alpha \geq 0$ and $\beta > 0$, we set

$$
I = 2 \left\{ A^{\frac{\alpha}{\beta}} \left( \left(1^{-2} \left\{ A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}} \right\}^{-1} \right) \circ \left\{ A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}} \right\}^{\alpha} \right) A^{\frac{\alpha}{\beta}} \right\}
$$

$$
II = 4A\#_{(\alpha, \beta)}B - 8 \left\{ A^{\frac{\alpha}{\beta}} \left[ \left\{ A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}} \right\}^{\alpha} \circ \left\{ A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}} \right\}^{\frac{1}{2}} + 1 \right] \right\} A^{\frac{\alpha}{\beta}}
$$

$$
III = A\#_{(\alpha + \frac{1}{2}, \beta)}B - A\#_{(\alpha - \frac{1}{2}, \beta)}B
$$

$$
V = \frac{1}{2} \left( A\#_{(\alpha + 1, \beta)}B - A\#_{(\alpha - 1, \beta)}B \right)
$$

where $A\#_{(\alpha, \beta)}B := \left\{ A^{\frac{\alpha}{\beta}} \left( \left\{ A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}} \right\} \right) A^{\frac{\alpha}{\beta}} \right\}$ is the operator $(\alpha, \beta)$-geometric mean. Then I, II, III, and V defined above are in $A$.

For convenience of the reader, we recall here Theorem 1 from [31]:

**Theorem 4.2.** Let $A$ be a unital JB-algebra. Let $r, q$ and $h$ be real valued continuous functions on a closed interval $I$ such that $h > 0$ and $r(x) \leq q(x)$. For elements $A$ and $B$ in $A$ such that the spectra of $B$ and $h(B)^{-1/2}Ah(B)^{-1/2}$ are contained in $I$,

$$
P_{r\triangle h}(A, B) \leq P_{q\triangle h}(A, B).
$$

We extend [30, Proposition 2.5] to the setting of JB-algebras.
Proposition 4.3. Let $A$ and $B$ be two positive invertible elements in a unital JB-algebras $A$. Then for $\alpha \geq 0$ and $\beta > 0$,

(i) $A^{\#(\alpha,\beta)}B - A^{\#(\alpha-1,\beta)}B \leq 1 \leq A^{\#(\alpha+1,\beta)}B - A^{\#(\alpha,\beta)}B$.

(ii) $A^{\#(\alpha,\beta)}B - A^{\#(\alpha-1,\beta)}B \leq V \leq A^{\#(\alpha+1,\beta)}B - A^{\#(\alpha,\beta)}B$.

(iii) $A^{\#(0,\beta)}B - A^{\#(-1,\beta)}B \leq A^{\#(\alpha,\beta)}B - A^{\#(\alpha-1,\beta)}B$.

Proof. Proof of (i). Let

$$r(x) = x^\alpha - x^{\alpha-1},$$
$$q(x) = 2 \left( 1 - \frac{2}{x+1} \right) x^\alpha,$$
$$k(x) = x^{\alpha+1} - x^\alpha.$$

Then,

$$r(x) \leq q(x) \leq k(x) \tag{4.8}$$

for $x > 0$ as shown in the proof of [30, Proposition 2.5].

Denoting $h(x) = x^\beta$, we have

$$P_{r\triangle h}(B, A) = A^{\#(\alpha,\beta)}B - A^{\#(\alpha-1,\beta)}B$$

$$P_{t\triangle h}(B, A) = 2 \left\{ 2^\beta \left( 1 - 2(1 + \{ A^{-\frac{\beta}{2}} B A^{-\frac{\beta}{2}} \}^{-1}) \right) \circ \{ A^{-\frac{\beta}{2}} B A^{-\frac{\beta}{2}} \}^\alpha \right\} - 1$$

$$P_{k\triangle h}(B, A) = A^{\#(\alpha+1,\beta)}B - A^{\#(\alpha,\beta)}B.$$

Applying Theorem 4.2 to the inequalities (4.8), we obtain desired inequalities.

For (ii), the inequalities

$$x^\alpha - x^{\alpha-1} \leq \frac{1}{2} (x^{\alpha+1} - x^{\alpha-1}) \leq x^{\alpha+1} - x^\alpha \tag{4.9}$$

hold for all $x > 0$. Using the perspective functions associated with the three functions in (4.9) with $h$ as in the proof of (i) and applying Theorem 4.2, (ii) follows.

(iii) Note that $1 - \frac{1}{x} \leq x^\alpha - x^{\alpha-1}$ for all $x > 0$. Applying Theorem 4.2 to this inequality with $h(x) = x^\beta$, we know (iii) is true. \hfill \Box

Let

$$r_\delta(x) = \left[ \ln \delta + 2 \left( 1 - \frac{2\delta}{x+\delta} \right) \right] x^\alpha$$

$$s_\delta(x) = \left[ \ln \delta + 4 - \frac{8\sqrt{\delta}}{\sqrt{x} + \sqrt{\delta}} \right] x^\alpha$$

$$q(x) = x^\alpha \ln(x)$$

$$j_\delta(x) = x^{\alpha+\frac{1}{2}} \frac{1}{\sqrt{\delta}} - x^{\alpha-\frac{1}{2}} \sqrt{\delta} + x^\alpha \ln \delta$$

$$k_\delta(x) = \frac{x^{\alpha+1}}{2\delta} - \frac{x^{\alpha-1}}{2} \delta + x^\alpha \ln \delta$$

Utilizing refined Hermite-Hadamard inequality (see e.g. [30, Lemma 2.7]), we obtained the following proposition, which is crucial for future results.

Proposition 4.4. Let $r_\delta(x), s_\delta(x), q(x), j_\delta(x), k_\delta(x)$ be defined as above.

(a) If $\alpha \geq 0$ and $x \geq \delta \geq 1$, then

$$r_\delta(x) \leq s_\delta(x) \leq q(x) \leq j_\delta(x) \leq k_\delta(x).$$
(b) If $\alpha \geq 0$ and $x \geq \delta \geq 1$, then
\[ s_1(x) \leq s_\delta(x) \quad \text{and} \quad j_\delta(x) \leq j_1(x). \]

(c) If $\alpha \geq 0$ and $x \leq \delta \leq 1$, then
\[ k_\delta(x) \leq j_\delta(x) \leq q(x) \leq s_\delta(x) \leq r_\delta(x) \]

(d) If $\alpha \geq 0$ and $x \leq \delta \leq 1$, then
\[ s_\delta(x) \leq s_1(x) \quad \text{and} \quad j_1(x) \leq j_\delta(x). \]

**Proof.** One could find them in the proof of Theorem 2.8, Theorem 2.11, Theorem 3.1 and Proposition 3.2 in [30]. \hfill $\square$

**Theorem 4.5.** Let $A$ and $B$ be two positive invertible elements in a unital JB-algebra $\mathcal{A}$, $\alpha \geq 0$ and $\beta > 0$.

(i) If $A^\delta \leq B$, then
\[ I \leq \Pi \leq S_{\alpha,\beta}(A|B) \leq \Pi \leq V \quad (4.11) \]

(ii) If $A^\delta \geq B$, then
\[ V \leq \Pi \leq S_{\alpha,\beta}(A|B) \leq \Pi \leq I \quad (4.12) \]

**Proof.** (i). Applying Theorem 4.2 to Proposition 4.4 (a) with $\delta = 1$ and with $h(t) = t^\beta$, we derive the inequality
\[ I \leq \Pi \leq S_{\alpha,\beta}(A|B) \leq \Pi \leq V. \]

Proof of (ii). Using the perspective functions associated with the five functions in Proposition 4.4 (c) with $\delta = 1$ and applying Theorem 4.2, (4.12) follows. \hfill $\square$

Theorem 4.5 above improves the upper and lower bounds of relative operator $(\alpha, \beta)$-entropy established by Nikoufar [20] and extends it to the setting of JB-algebras.

Combining Theorem 4.5 and Proposition 4.3, we have

**Corollary 4.6.** Let $A$ and $B$ be two positive invertible elements in a unital JB-algebra $\mathcal{A}$, $\alpha \geq 0$ and $\beta > 0$.

(i) If $A^\delta \leq B$, then
\[ A_{\#(\alpha,\beta)}B - A_{\#(\alpha-1,\beta)}B \leq I \leq \Pi \leq S_{\alpha,\beta}(A|B) \leq \Pi \leq V \leq A_{\#(\alpha+1,\beta)}B - A_{\#(\alpha,\beta)}B \]

(ii) If $A^\delta \geq B$, then
\[ A_{\#(\alpha,\beta)}B - A_{\#(\alpha-1,\beta)}B \leq V \leq \Pi \leq S_{\alpha,\beta}(A|B) \leq \Pi \leq I \leq A_{\#(\alpha+1,\beta)}B - A_{\#(\alpha,\beta)}B. \]

The following result refines the upper and lower bounds of relative operator entropy obtained by Nikoufar [18, 20], which improved the bounds established by Fujii and Kamei [7, 8], and extends it to the setting of JB-algebras.
Corollary 4.7. Let $A$ and $B$ be two positive invertible elements in a unital JB-algebra $\mathcal{A}$, $\alpha \geq 0$ and $\beta > 0$.

(i) If $A \leq B$, then
\[
A - \{AB^{-1}A\} \leq 2(A - 2\{A(A + B)^{-1}A\}) \leq 4A - 8\{A(A\#_{\frac{1}{2}}B + A)^{-1}A\} \leq S(A|B) \leq A\#_{\frac{1}{2}}B - A\#_{-\frac{1}{2}}B \leq \frac{1}{2}(B - \{AB^{-1}A\}) \leq B - A.
\]

(ii) If $A \geq B$, then
\[
A - \{AB^{-1}A\} \leq \frac{1}{2}(B - \{AB^{-1}A\}) \leq A\#_{\frac{1}{2}}B - A\#_{-\frac{1}{2}}B \leq S(A|B) \leq 4A - 8\{A(A\#_{\frac{1}{2}}B + A)^{-1}A\} \leq 2(A - 2\{A(A + B)^{-1}A\}) \leq B - A.
\]

Suppose that $A, B$ are two positive invertible elements. For any real number $\delta > 0$, $\alpha \geq 0$ and $\beta > 0$, we denote
\[
I' = (\ln \delta + 2)A\#(\alpha, \beta)B - 2\delta \left\{A^\frac{\alpha}{\beta}[(A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}}) + \delta]^{-1} \circ (A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}})^\alpha|A^\frac{\alpha}{\beta}\right\}
\]
\[
II' = (\ln \delta + 4)A\#(\alpha, \beta)B - 8\sqrt{\delta} \left\{A^\frac{\alpha}{\beta}[(A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}})^\frac{1}{2} + \sqrt{\delta}]^{-1} \circ (A^{-\frac{\alpha}{\beta}}BA^{-\frac{\alpha}{\beta}})^\alpha|A^\frac{\alpha}{\beta}\right\}
\]
\[
III' = \frac{1}{\sqrt{\delta}}A\#(\alpha + \frac{1}{2}, \beta)B - \sqrt{\delta}A\#(\alpha - \frac{1}{2}, \beta)B + \ln \delta A\#(\alpha, \beta)B
\]
\[
V' = \frac{1}{2}\left\{\frac{1}{\sqrt{\delta}}A\#(\alpha + 1, \beta)B - \delta A\#(\alpha - 1, \beta)B + \ln \delta A\#(\alpha, \beta)B\right\}.
\]

Theorem 4.8. Let $A$ and $B$ be two positive invertible elements in a unital JB-algebra $\mathcal{A}$, $\alpha \geq 0$ and $\beta > 0$.

(i) If $\delta \geq 1$ and $\delta A^\beta \leq B$, then
\[
I' \leq II' \leq S_{\alpha, \beta}(A|B) \leq III' \leq V'.
\]

(ii) If $\delta \geq 1$ and $\delta A^\beta \leq B$, then
\[
II \leq II' \quad \text{and} \quad III' \leq III.
\]

(iii) If $\delta \leq 1$ and $\delta A^\beta \geq B$, then
\[
V' \leq III' \leq S_{\alpha, \beta}(A|B) \leq II' \leq I'.
\]

(iv) If $\delta \leq 1$ and $\delta A^\beta \geq B$, then
\[
II' \leq II \quad \text{and} \quad III \leq III'.
\]
Proof. (i). Using perspective functions associated with the five functions in Proposition 4.4(a) with $h(t) = t^\delta$, and applying Theorem 4.2, (i) follows.

Proof of (ii). Applying Theorem 4.2 to Proposition 4.4(b) with $h$ as in the proof of (i), we obtain the inequalities

$$II \leq II' \quad \text{and} \quad III \leq III'.$$

(iii) follows by applying Theorem 4.2 to Proposition 4.4(c). Similar argument as in the proof of (ii) gives (iv). \hfill \Box

The following corollary improves Corollary 4.7 and also sharply refines the lower and upper bounds of the relative operator entropy established by Nikoufar in [18, 20], which refined the bounds obtained earlier by Fujii and Kamei [7, 8].

**Corollary 4.9.** Let $A$ and $B$ be two positive invertible elements in a unital JB-algebra $A$.

(i) If $\delta \geq 1$ and $\delta A \leq B$, then

$$A - \{AB^{-1}A\} \leq 2 \left[ A - 2\{A(A + B)^{-1}A\} \right]$$

$$\leq (\ln \delta)A + 2 \left[ A - 2\delta\{A\delta A + B\}^{-1}A \right]$$

$$\leq (\ln \delta + 4)A - 8\sqrt{\delta} \left\{ A(A^\#_\frac{1}{2}B + \sqrt{\delta}A)^{-1}A \right\}$$

$$\leq S(A|B)$$

$$\leq \left( \frac{1}{\sqrt{\delta}} A^\#_\frac{1}{2}B - \sqrt{\delta}A^\#_{-\frac{1}{2}}B \right) + (\ln \delta)A$$

$$\leq \frac{1}{2} \left( \frac{1}{\delta}B - \delta\{AB^{-1}A\} \right) + (\ln \delta)A$$

$$\leq \frac{1}{2} \left( B - \{AB^{-1}A\} \right)$$

$$\leq B - A.$$

(ii) If $\delta \leq 1$ and $\delta A \geq B$, then

$$A - \{AB^{-1}A\} \leq \frac{1}{2} \left( B - \{AB^{-1}A\} \right)$$

$$\leq \frac{1}{2} \left( \frac{1}{\delta}B - \delta\{AB^{-1}A\} \right) + (\ln \delta)A$$

$$\leq \left( \frac{1}{\sqrt{\delta}} A^\#_\frac{1}{2}B - \sqrt{\delta}A^\#_{-\frac{1}{2}}B \right) + (\ln \delta)A$$

$$\leq S(A|B)$$

$$\leq (\ln \delta + 4)A - 8\sqrt{\delta} \left\{ A(A^\#_\frac{1}{2}B + \sqrt{\delta}A)^{-1}A \right\}$$

$$\leq (\ln \delta)A + 2 \left[ A - 2\delta\{A\delta A + B\}^{-1}A \right]$$

$$\leq 2 \left[ A - 2\{A(A + B)^{-1}A\} \right]$$

$$\leq B - A.$$
Proof. Let $\alpha = 0$ and $\beta = 1$. By Proposition 2.6 and Macdonald Theorem,
\[ I = 2 \left[ A - 2\{A(A + B)^{-1} A\} \right] \]
\[ I' = (\ln \delta) A + 2 \left[ A - 2\{A(\delta A + B)^{-1} A\} \right] \]
\[ II' = (\ln \delta + 4) A - 8\sqrt{\delta} \left\{ A(A\#_{\frac{1}{2}} B + \sqrt{\delta} A)^{-1} A \right\} \]
\[ III' = \frac{1}{\sqrt{\delta}} A\#_{\frac{1}{2}} B - \sqrt{\delta} A\#_{-\frac{1}{2}} B \right\} + (\ln \delta) A \]
\[ V' = \frac{1}{2} \left( \frac{1}{\delta} B - \delta \{AB^{-1} A\} \right) + (\ln \delta) A \]
\[ V = \frac{1}{2} \left( B - \{AB^{-1} A\} \right) \]

Proof of (i). Combining Theorem 4.5(i), Corollary 4.7, and Theorem 4.8 (i), (ii), we obtain desired inequalities
\[ A - \{AB^{-1} A\} \leq I \leq II \leq II' \leq III' \leq V' \leq V \leq B - A. \]

Similar arguments as in the proof of (i) gives (ii). \qed

The following result is the ordering relation between Tsallis relative operator $(\lambda, \beta)$-entropy and relative operator $(0, \beta)$-entropy in the setting of JB-algebras.

**Proposition 4.10.** Let $A$ and $B$ be invertible positive elements in a unital JB-algebra $A$. For any $0 < \lambda \leq 1$ and $\beta > 0$ we have
\[ T_{-\lambda,\beta}(A|B) \leq S_{0,\beta}(A|B) \leq T_{\lambda,\beta}(A|B). \]

**Proof.** One sees that for any $0 < \lambda \leq 1$,
\[ \ln_{-\lambda} x \leq \log x \leq \ln_{\lambda} x \tag{4.13} \]
hold for all $x > 0$. Applying Theorem 4.2 to (4.13) with $h(t) = t^\beta$, we derive the inequalities
\[ T_{-\lambda,\beta}(A|B) \leq S_{0,\beta}(A|B) \leq T_{\lambda,\beta}(A|B). \]

**Proposition 4.11.** For any positive invertible elements $A$ and $B$ in a unital JB-algebra $A$, $0 < \lambda \leq 1$ and $\beta > 0$,
\[ A\#_{(0,\beta)} B - A\#_{(-1,\beta)} B \leq T_{\lambda,\beta}(A|B) \leq A\#_{(1,\beta)} B - A\#_{(0,\beta)} B. \tag{4.14} \]

Moreover, $T_{\lambda,\beta}(A|B) = 0$ if and only if $A^\beta = B$.

**Proof.** Note that for any $0 < \lambda \leq 1$ and $x > 0$
\[ 1 - x^{-1} \leq \ln_{\lambda} x \leq x - 1. \tag{4.15} \]
Using perspective function associated with the three functions above with $h(t) = t^\beta$ and applying Theorem 4.2 to (4.15), we obtain desired result.

By Macdonald’s theorem,
\[ A\#_{(0,\beta)} B - A\#_{(-1,\beta)} B = A^\beta - \{A^\beta B^{-1} A^\beta\} \]
\[ A\#_{(1,\beta)} B - A\#_{(0,\beta)} B = B - A^\beta \]
Suppose that $T_{\lambda,\beta}(A|B) = 0$, then
\[ A^\beta - \{A^\beta B^{-1}A^\beta\} \leq 0 \leq B - A^\beta. \]
Consequently,
\[ A^{-\beta} - B^{-1} \leq 0 \leq B - A^\beta. \]
According to [12, Lemma 3.5.3], $A^\beta \leq B \leq A^\beta$. \hfill \Box

**Remark 4.12.** If $\beta = 1$, then (4.14) becomes
\[ A - \{AB^{-1}A\} \leq T_{\lambda}(A|B) \leq -A + B. \]  
(4.16)
If in addition $A$ is special, then Proposition 4.11 reduces to [9, Proposition 3.4].

Denote
\[ IV = \frac{1}{2} \left( A^{\#(\lambda,\beta)}B - A^{\#(\lambda^{-1},\beta)}B + A^{\#(1,\beta)}B - A^{\#(0,\beta)}B \right). \]

The following result provides an improvement for the lower and upper bounds of Tsallis relative operator $(\lambda, \beta)$-entropy.

**Proposition 4.13.** Let $A$ and $B$ be two positive invertible elements in a unital JB-algebra $A$, $0 < \lambda \leq 1$ and $\beta > 0$.

(i) If $A^\beta \leq B$, then
\[ A^{\#(0,\beta)}B - A^{\#(-1,\beta)}B \leq A^{\#(\lambda,\beta)}B - A^{\#(\lambda^{-1},\beta)}B \leq T_{\lambda,\beta}(A|B) \leq IV \leq A^{\#(1,\beta)}B - A^{\#(0,\beta)}B. \]

(ii) If $B \leq A^\beta$, then
\[ A^{\#(0,\beta)}B - A^{\#(-1,\beta)}B \leq A^{\#(\lambda,\beta)}B - A^{\#(\lambda^{-1},\beta)}B \leq IV \leq T_{\lambda,\beta}(A|B) \leq A^{\#(1,\beta)}B - A^{\#(0,\beta)}B. \]

**Proof.** Proof of (i). For any $x \geq 1$ and $0 < \lambda \leq 1$, we have the following inequalities
\[ 1 - \frac{1}{x} \leq x^\lambda - x^{-\lambda^{-1}} \leq \ln x \leq \frac{1}{2}(x^\lambda - x^{-\lambda^{-1}} + x - 1) \leq x - 1. \]  
(4.17)
Applying Theorem 4.2 to (4.17) with $h(t) = t^\beta$, we obtain desired result.

For (ii), applying Hermite-Hadamard integral inequality to $f(t) = t^{\lambda^{-1}}$ on $[x, 1],$
\[ 1 - \frac{1}{x} \leq x^\lambda - x^{-\lambda^{-1}} \leq \frac{1}{2}(x^\lambda - x^{-\lambda^{-1}} + x - 1) \leq \ln x \leq x - 1 \]  
(4.18)
Using the perspective functions associated with the functions in (4.18) and applying Theorem 4.2, (ii) follows. \hfill \Box

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