Composability
and
Generalized Entropy

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Abstract

We address in this paper how tightly the composability nature of systems:

\[ S_{A+B} = \Omega(S_A, S_B) \]

constrains definition of generalized entropies and investigate explicitly
the composability in some ansatz of the entropy form.
1 Introduction

For the recent decade generalization of Boltzmann-Gibbs statistics has gradually attracted attention and it has already been recognized that many interesting phenomena in various fields really need modification of their entropies for their precise descriptions [1, 2].

Even though the entropy is modified, it is expected to possess several fundamental properties in order to express macroscopic aspects of physical systems. Many efforts to investigate the properties, including the concavity and the H-theorem, have been made so far and shed light on understanding of the generalized thermodynamics itself [1, 2, 3].

We shall analyze in this paper the composability [3, 4] which is a candidate of natural property of the generalized entropy. Let us consider two systems A and B of the same material structure with different numbers of states. Their entropies are denoted by

\[ S_A = S_A(P_A^i), \]
\[ S_B = S_B(P_B^j). \]

(1)

(2)

We can always regard the two systems as one composite system \( A + B \). So the total entropy of the system \( A + B \) is requested to exist with some form:

\[ S_{A+B} = S_{A+B}(P_{ij}), \]

(3)

where \( P_{ij} \) is probability related to the composite system and the index \( i(j) \) corresponds to degree of freedom of the system \( A(B) \). Now let us keep focusing on cases in which the interaction between the systems \( A \) and \( B \) is so small that it can be neglected. Due to this the probability of the composite system may take the value as

\[ P_{ij} = P_A^i P_B^j. \]

(4)

Then the definition of the composability can be given as following.

\[ S_{A+B}(P_A^i P_B^j) = \Omega(S_A, S_B), \]

(5)

where \( \Omega \) is a function with two arguments \( S_A \) and \( S_B \). The function \( \Omega \) should not depend on the number of states of the systems. This definition means
that the total entropy is deterministic macroscopically and can be built of just the two macroscopic quantities, $S_A$ and $S_B$.

We point out that the composability constrains the form of the generalized entropy quite tightly. In Section 2, we discuss an ansatz:

$$S(P_i) = C + \sum_i \phi(P_i)$$

as a modification of the entropy definition and force the ansatz to respect the composability. Consequently uniqueness of the Tsallis entropy, up to constant and factor, is proven under some assumptions.

In Section 3, we extend the ansatz of the entropy into more complicated one and find a species of the generalized entropy holding the composability.

## 2 Composability and Tsallis Entropy

It is a quite interesting problem to clarify how strongly the composability constrains the form of the generalized entropy definition itself. In this section, we concentrate our attention on an ansatz of the generalized entropy definition as follows.

$$S(P_i) = C + \sum_i \phi(P_i),$$

where $C$ is a constant, $\phi$ is an unfixed smooth function independent of the number of states of the system, $P_i$ is probability corresponding to Event $i$. Taking the complete sum of all the probabilities,

$$\sum_i P_i = 1$$

should hold by definition.

Here it should be noted that the function $\phi$ satisfies

$$\phi(0) = 0.$$  \hspace{1cm} (8)

To prove this, let us consider a system with $N+1$ states. By taking $P_{N+1} = 0$, the system is straightforwardly reduced into a system with $N$ states. Then
the entropies \(S_{N+1}(P_{N+1} = 0)\) and \(S_N\) naturally coincide with each other:

\[ S_{N+1}(P_{N+1} = 0) = S_N. \]

From this eqn (8) is found easily.

Due to the definition (6) the entropies of each systems are written down explicitly as follows.

\[ S_{A+B}(P_{ij}) = C + \sum_{i=1}^{N} \sum_{j=1}^{M} \phi(P_{ij}), \] (9)
\[ S_A(P_i^A) = C + \sum_{i=1}^{N} \phi(P_i^A), \] (10)
\[ S_B(P_j^B) = C + \sum_{j=1}^{M} \phi(P_j^B), \] (11)

where \(N(M)\) is the number of states of the system \(A(B)\).

Now we address a problem what kind of constraints the comosability imposes on the functions \(\Omega\) and \(\phi\).

Firstly let us analyze the case with \(N = 2\) and \(M = 2\). The probabilities for the system \(A(B)\) are denoted by \(a\) and \(1 - a\) (\(b\) and \(1 - b\)) where \(a(b)\) is a parameter with \(0 \leq a \leq 1\) \((0 \leq b \leq 1)\). Then the following expression for the combined entropy is obtained from the definition (9).

\[ \Omega(S_A, S_B) = C + \phi(ab) + \phi(a(1 - b)) + \phi((1 - a)b) + \phi((1 - a)(1 - b)). \] (12)

Also the entropies for \(A\) and \(B\) read as follows.

\[ S_A = C + \phi(a) + \phi(1 - a), \] (13)
\[ S_B = C + \phi(b) + \phi(1 - b). \] (14)

It is noticed here that the set of eqns (12), (13) and (14) is just a parametric representation of the function \(\Omega(S_A, S_B)\) by two free parameters \(a\) and \(b\). Thus this fact tells us that the function \(\Omega\) is determined by giving the function \(\phi\), as naturally expected. Therefore we do not care about \(\Omega\)-hunt any more and can devote ourselves later to search for the function \(\phi\).
Next let us consider the case with \((N, M) = (N, 2)\) and the case with \((N, M) = (N, 3)\) simultaneously. For the system \(A\) with \(N\) states, let us make the probabilities denoted by \(P_i\) with
\[
\sum_{i=1}^{N} P_i = 1.
\]
And for the system \(B\) with 2 states, let \(x\) and \(1 - x\) denote the probabilities. Then the entropies are expressed using these variables as follows.
\[
S_A = C + \sum_{i=1}^{N} \phi(P_i),
\]
\[
S_B = C + \phi(x) + \phi(1 - x).
\]
Meanwhile for another remaining system \(B'\) with 3 states let us make \(y, z\) and \(1 - y - z\) denote the probabilities. Then the entropy is written as
\[
S_{B'} = C + \phi(y) + \phi(z) + \phi(1 - y - z).
\]
We try to adjust the two entropies \(S_B\) and \(S_{B'}\) to take the same value:
\[
S_B = S_{B'}.
\]
It is always possible to realize the situation by substituting a suitable function \(x = x(y, z)\)
into the entropy \(S_B\). Then the condition \((18)\) is reexpressed as
\[
\phi(x(y, z)) + \phi(1 - x(y, z)) = \phi(y) + \phi(z) + \phi(1 - y - z).
\]
Eqn \((20)\) makes the function \(x(y, z)\) fixed after determination of the function \(\phi(x)\). Due to adoption of the above function \(x(y, z)\) the relation:
\[
\Omega(S_A, S_B) = \Omega(S_A, S_{B'})
\]
trivially holds and is rewritten explicitly as follows.
\[
\sum_{i=1}^{N} \phi(xP_i) + \sum_{i=1}^{N} \phi((1 - x)P_i) = \sum_{i=1}^{N} \phi(yP_i) + \sum_{i=1}^{N} \phi(zP_i) + \sum_{i=1}^{N} \phi((1 - y - z)P_i).
\]
Eqn (22) must be satisfied for arbitrary probability variables, \( y, z \) and \( P_i \). Therefore we are able to set

\[
P_i = \frac{1}{N}
\]

and

\[
y = z.
\]

Then a compact version of the master equation (22) is induced as follows.

\[
\phi \left( \frac{x}{N} \right) + \phi \left( \frac{1-x}{N} \right)
= 2\phi \left( \frac{y}{N} \right) + \phi \left( \frac{1-2y}{N} \right),
\]

for \( N = 2, 3, \cdots \). It is noticed that eqn (20) with \( y = z \) corresponds to the \( N = 1 \) case of eqn (23). Thus in later discussion we shall deal with the equation as a special case of eqn (23) for positive integer \( N \). Just for convenience we use later not \( x = x(y, y) \) but \( y = y(x) \), the inverse function of \( x = x(y, y) \).

Next we shall prove that the index \( N \) in eqn (23) can be extended into a continuous variable. For this purpose let us take the following distribution for \( P_i \) of the system \( A \).

\[
P_1 = p, \\
P_i = \frac{1-p}{N-1} \quad (i = 2, \cdots, N).
\]

Substituting this into eqn (22) yields a new relation:

\[
\phi (xp) + \phi ((1-x)p) \\
+(N-1)\phi \left( x \frac{1-p}{N-1} \right) + (N-1)\phi \left( (1-x) \frac{1-p}{N-1} \right)
= 2\phi (yp) + \phi ((1-2y)p) \\
+2(N-1)\phi \left( y \frac{1-p}{N-1} \right) + (N-1)\phi \left( (1-2y) \frac{1-p}{N-1} \right).
\]

By differentiating with respect to \( p \) iteratively in eqn (26) and taking \( p = 1/N \), the following relations are obtained.

\[
x^n\phi^{(n)} \left( \frac{x}{N} \right) + (1-x)^n\phi^{(n)} \left( \frac{1-x}{N} \right)
= 2y^n\phi^{(n)} \left( \frac{y}{N} \right) + (1-2y)^n\phi^{(n)} \left( \frac{1-2y}{N} \right),
\]
where \( n = 0, 2, 3, 4, \ldots \) and \( \phi^{(n)}(x) \) is the \( n \)-th derivative of the function \( \phi(x) \) with respect to \( x \). Unfortunately the relation (27) corresponding to \( n = 1 \) does not appear due to a coincidence that the derived equation for \( n = 1 \) becomes trivial when \( p = 1/N \) is taken. However we can remedy the lack using another source provided that \( \phi'(0) \) exists. Let us replace \( N \) with \( N + 1 \) in eqn (26) and substituting

\[
P_i = \frac{1-p}{N}, \quad (i = 1, 2, \cdots, N) \tag{28}
\]

\[
P_{N+1} = p, \tag{29}
\]

into the equation. It turns out that the following equation holds.

\[
\phi(xp) + \phi((1-x)p) \\
+ N\phi \left( x\frac{1-p}{N} \right) + N\phi \left( (1-x)\frac{1-p}{N} \right) \\
= 2\phi(yp) + \phi((1-2y)p) \\
+ 2N\phi \left( y\frac{1-p}{N} \right) + N\phi \left( (1-2y)\frac{1-p}{N} \right). \tag{30}
\]

Assuming the existence of \( \phi'(0) \), we can set \( p = 0 \) after the differentiation with respect to \( p \) in eqn (30). After some manipulation the missing equation (27) for \( n = 1 \) is really derived:

\[
x\phi' \left( \frac{x}{N} \right) + (1-x)\phi' \left( \frac{1-x}{N} \right) \\
= 2y\phi' \left( \frac{y}{N} \right) + (1-2y)\phi' \left( \frac{1-2y}{N} \right). 
\]

After all it have been proven that

\[
A_n = x^n \phi^{(n)} \left( \frac{x}{N} \right) + (1-x)^n \phi^{(n)} \left( \frac{1-x}{N} \right) \\
-2y^n \phi^{(n)} \left( \frac{y}{N} \right) - (1-2y)^n \phi^{(n)} \left( \frac{1-2y}{N} \right) \\
= 0, \tag{31}
\]

where \( n \) takes all non-negative integer values. Next let us introduce a deviation parameter \( \Delta N \) which takes real values and add \( \Delta N \) to \( N \). Then we
can show that a function defined as
\[
\Psi(N + \Delta N) = \phi\left(\frac{x}{N + \Delta N}\right) + \phi\left(\frac{1 - x}{N + \Delta N}\right) - 2\phi\left(\frac{y}{N + \Delta N}\right) - \phi\left(\frac{1 - 2y}{N + \Delta N}\right)
\] (32)
vanishes for arbitrary values of the continuous parameter \(\Delta N\) as follows. The function \(\Psi\) may be Taylor-expanded as
\[
\Psi(N + \Delta N) = \sum_{n=0}^{\infty} \frac{1}{n!} \Psi_n(N) \Delta N^n.
\] (33)
After some manipulation it is shown that all the \(\Psi_n\) are equal to linear combinations of \(A_n\) like that
\[
\Psi_0 = A_0,
\] (34)
\[
\Psi_1 = -\frac{1}{N^2} A_1,
\] (35)
\[
\Psi_2 = \frac{2}{N^3} A_1 + \frac{1}{N^4} A_2,
\] (36)
\[
\Psi_n = \sum_{k=1}^{n} C_k^{(n)}(N) A_k. \quad (n \geq 3)
\] (37)
Thus from eqn (B1)
\[
\Psi_n = 0
\] (38)
must be satisfied for \(n = 0, 1, 2, \cdots\) and \(\Psi(N + \Delta N)\) vanishes. This means that eqn (B3) must hold for arbitrary real number \(N\). By virtue of this fact, our task has now been rather simplified and is to find the two functions \(\phi(x)\) and \(y(x)\) that satisfy eqn (23) for arbitrary real parameter, \(N\).

From eqn (B3), we are able to prove under some conditions that the generalized entropy (6) possessing the composability is equal to the Tsallis entropy, up to constant and factor. Actually we give two independent proofs in the following.

Firstly we assume a rather general ansatz for the function \(\phi(x)\) as follows.
\[
\phi(x) = \sum_{\nu=\nu_{\text{min}}}^{\infty} \sum_{n=0}^{\infty} \phi_n(\nu) x^{\nu+n},
\] (39)
where \( \nu \) are real numbers and \( \nu_1 - \nu_2 \neq 0 \) (mod integer) if \( \nu_1 \) and \( \nu_2 \) appear in the sum. Also the coefficient \( \phi_0(\nu) \) does not vanish. In eqn (39), there exists the minimum positive value for \( \nu; \nu_{\text{min}} \geq 1 \) because we must ensure that \( \phi(0) = 0 \) and the existence of \( \phi'(0) \). Substituting eqn (39) into eqn (23) and taking the expansion on \( 1/N \) for large \( N \) yield

\[
\sum_{\nu = \nu_{\text{min}}}^{\infty} \sum_{n=0}^{\infty} \left( \frac{1}{N} \right)^{\nu+n} \phi_n(\nu) \left[ x^{\nu+n} + (1 - x)^{\nu+n} - 2y^{\nu+n} - (1 - 2y)^{\nu+n} \right] = 0. \tag{40}
\]

By comparing each of the coefficients in the expansion on \( 1/N \), it is straightforwardly obtained that

\[
\phi_n(\nu) \left[ x^{\nu+n} + (1 - x)^{\nu+n} - 2y^{\nu+n} - (1 - 2y)^{\nu+n} \right] = 0. \tag{41}
\]

If we have two or more non-zero coefficients \( \phi_{n_i}(\nu_i) \), the function \( y = y(x) \) must satisfy the following equations for all the \( i \)'s.

\[
x^{\nu_i+n_i} + (1 - x)^{\nu_i+n_i} = 2y^{\nu_i+n_i} + (1 - 2y)^{\nu_i+n_i}. \tag{42}
\]

However, this is clearly impossible and we fails to find any solution \( y = y(x) \) like that. Thus we conclude that there exists only one non-vanishing coefficient \( \phi_{0}(\nu_{\text{min}}) \neq 0. \) Consequently from eqn (39) the function \( \phi(x) \) is uniquely given by

\[
\phi(x) = Bx^q \tag{43}
\]

where \( B = \phi_0(q) \) is some constant, \( q = \nu_{\text{min}} \geq 1 \) and the function \( y = y(x) \) is determined by the following equation;

\[
x^q + (1 - x)^q = 2y^q + (1 - 2y)^q \tag{44}
\]

due to eqn (42). Substituting \( (43) \) into definition of the generalized entropy (1) and introducing constants \( S_o \) and \( D \) like

\[
C = S_o - \frac{D}{1-q}, \tag{45}
\]
\[
B = \frac{D}{1-q}, \tag{46}
\]
we get the main result of this section;

$$S(P_i) = S_o + DS_q$$

where

$$S_q = -\frac{1 - \sum_i P_i^q}{1 - q}$$

is just the Tsallis entropy [4] and if we are able to take $q \to 1$ beyond the assumption that $\phi'(0)$ exists, Boltzmann-Gibbs entropy:

$$S_{q=1} = -\sum_i P_i \ln P_i$$

is reproduced. One can check easily that the entropy (47) actually has the composability and its additive law of the entropy is modified as

$$S_{A+B} = S_A + S_B - S_o + \frac{1-q}{D}(S_A - S_o)(S_B - S_o).$$

We expect that even if the ansatz (39) for the function $\phi$ is extended to a more complicated form, eqn (23) will prevent the result derived above from changing.

Besides the above proof we can show another one in order to argue the uniqueness. Let us introduce two independent variables $X$ and $Y$ as follows.

$$X = \phi(x) + \phi(1-x) - 2\phi\left(\frac{1}{2}\right),$$

$$Y = 2\phi(y) + \phi(1-2y) - 2\phi\left(\frac{1}{2}\right),$$

where $x = 1/2$ ($y = 1/2$) corresponds to $X = 0$ ($Y = 0$). Let us consider again the function $\Psi$, eqn (32). The function depends on only three independent variables, $(N, X, Y)$ or $(N, x, y)$. Let us assume later that the function $\Psi$ may be Maclaurin-expanded in terms of $X$ and $Y$ as

$$\Psi = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{1}{k!l!} \tilde{\Psi}_{kl}(N) X^k Y^l.$$  

It is worthwhile to note a simple fact that the function $\Psi$ is obtained by adding a function of $x$:

$$\phi\left(\frac{x}{N}\right) + \phi\left(\frac{1-x}{N}\right)$$
and a function of $y$:

$$-2\phi\left(\frac{y}{N}\right) - \phi\left(\frac{1-2y}{N}\right),$$

that is, takes the separate-variables form with respect to the variables $x$ and $y$. Moreover eqn (23) must hold when

$$\phi(x) + \phi(1-x) - 2\phi(y) - \phi(1-2y) = X(x) - Y(y) = 0$$

is satisfied. Because of these two conditions the coefficients $\tilde{\Psi}_{kl}$ are drastically constrained and eqn (53) can be replaced into the following equation.

$$\phi\left(\frac{x}{N}\right) + \phi\left(\frac{1-x}{N}\right) - 2\phi\left(\frac{y}{N}\right) - \phi\left(\frac{1-2y}{N}\right) = \sum_{k=0}^{\infty} \tilde{\Psi}_k(N)(X^k - Y^k),$$

(54)

where $\tilde{\Psi}_k$ are members of the coefficients $\tilde{\Psi}_{kl}$ that survive even after imposing the above two conditions. Therefore we acquire the two independent relations as

$$\phi\left(\frac{x}{N}\right) + \phi\left(\frac{1-x}{N}\right) = \sum_{k=0}^{\infty} \tilde{\Psi}_k(N)X^k + \delta(N),$$

(55)

$$2\phi\left(\frac{y}{N}\right) + \phi\left(\frac{1-2y}{N}\right) = \sum_{k=0}^{\infty} \tilde{\Psi}_k(N)Y^k + \delta(N),$$

(56)

where $\delta(N)$ is a undetermined function of $N$ needed when we separate the variables $x$ and $y$ in eqn (54). The function $\delta$ can be always absorbed by the coefficient $\tilde{\Psi}_0$, so we fix $\delta = 0$ later. The coefficients $\tilde{\Psi}_k$ are calculated by repeatedly differentiating the left-hand-side of eqn (55) on $X$ and setting $X = 0$, that is, $x = 1/2$ as follows.

$$\tilde{\Psi}_0 = 2\phi\left(\frac{1}{2N}\right),$$

(57)

$$\tilde{\Psi}_1 = \frac{\phi^{(2)}\left(\frac{1}{2N}\right)}{N^2\phi^{(2)}\left(\frac{1}{2}\right)}.$$  

(58)

On the other hand, the same coefficients must be obtained from repeatedly differentiation of eqn (56) with respect to $Y$ and setting $Y = 0$, or $y = 1/2$. 

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In fact substituting $y = 1/2$ into eqn (56) reproduces eqn (57). Meanwhile a non-trivial constraint is generated for the $\bar{\Psi}_1$ calculation. The form of $\bar{\Psi}_1$ obtained this time is as follows.

$$\bar{\Psi}_1 = \frac{1}{N} \phi' \left( \frac{1}{2N} \right) - \phi'(0).$$

Equating eqn (58) and eqn (59), and replacing $1/(2N)$ into $x$, it can be proven that the function $\phi(x)$ must satisfy the following differential equation.

$$x^2 \frac{d^2 \phi}{dx^2} - \frac{1}{2} \phi' \left( \frac{1}{2} \right) - \phi'(0) \frac{d\phi}{dx} = -\frac{1}{2} \phi'(0) \phi(2) \left( \frac{1}{2} \right) \phi' \left( \frac{1}{2} \right) - \phi'(0)x.$$  (60)

By integrating the equation, it is shown that the function $\phi(x)$ take the form of

$$\phi(x) = \phi'(0)x + C_1 + C_2 x^q,$$

where $C_1$ and $C_2$ are integrated constants and

$$q = 1 + \frac{1}{2} \phi(2) \left( \frac{1}{2} \right) \phi' \left( \frac{1}{2} \right) - \phi'(0).$$

Substituting eqn (61) into eqn (23), the same result of the first analysis is reproduced:

$$\phi(x) \propto x^q. \quad (q \geq 1)$$  (62)

Consequently it has been proven, assuming the ansatz of the function $\phi$ (58) or of the function $\Psi$ (53) besides the existence of $\phi'(0)$, that the composability requests the Tsallis entropy uniquely, up to the constant $S_o$ and the factor $D$, in eqn (5). This means that the composability can play a quite significant role in the generalized entropy search.

We comment that dos Santos also proved uniqueness of the Tsallis entropy, assuming the Tsallis pseudo-additivity:

$$S_{A+B} = S_A + S_B + (1 - q) S_A S_B$$  (63)

and other several conditions [8]. It should be stressed here that the pseudo-additivity (X) is not assumed a priori in our analysis.
3 Extension

In this section we try to extend the analysis in Section 2 for more complex definition of the generalized entropies. Instead of the ansatz in eqn (6), we adopt here the following form.

\[ S = S \left( \sum_i P_i^{q_1}, \sum_i P_i^{q_2}, \ldots, \sum_i P_i^{q_K} \right), \]  
(64)

where \( S \) is given as a Laurent series:

\[ S(X_1, X_2, \ldots, X_K) = \sum_{i_1 = -\infty}^{\infty} \sum_{i_2 = -\infty}^{\infty} \cdots \sum_{i_K = -\infty}^{\infty} S_{i_1,i_2,\ldots,i_K} X_1^{i_1} X_2^{i_2} \cdots X_K^{i_K}. \]  
(65)

Here the power exponents \( q_m (\neq 1) \) take different values each other and the coefficients \( S_{i_1,i_2,\ldots,i_K} \) are independent of the number of states of the systems. Also \( K \) is assumed finite positive integer.

We can repeat straightforwardly the analysis of the composability using the cases with \( (N, 2) \) and \( (N, 3) \) in Section 2. Probabilities for the system \( A \) with \( N \) states, \( B \) with 2 states and \( B' \) with 3 states are denoted in the same way of Section 2. Then it can be pointed out that the composability calls for the existence of a function \( y = y(x) \) which satisfies that

\[ S(f_1(x)Q_1, f_2(x)Q_2, \ldots, f_K(x)Q_K) = S(g_1(y)Q_1, g_2(y)Q_2, \ldots, g_K(y)Q_K), \]  
(66)

where

\[ Q_k = \sum_{i=1}^{N} P_i^{q_k}, \]  
(67)

\[ f_k(x) = x^{q_k} + (1 - x)^{q_k}, \]  
(68)

\[ g_k(y) = 2y^{q_k} + (1 - 2y)^{q_k}. \]  
(69)

If \( N \geq K \) is taken, it is noticed from counting the degree of freedom that all the \( Q_k \) are independent parameters each other. This fact may sound
trivial but shows us a significant result. Using eqn (65), the relation (66) is rewritten as

$$\sum S_{i_1,i_2,\ldots,i_K} \left[ \prod_{k=1}^{K} f_k(x)^{i_k} - \prod_{k=1}^{K} g_k(y)^{i_k} \right] Q_1^{i_1} Q_2^{i_2} \cdots Q_K^{i_K} = 0.$$  (70)

Thus

$$S_{i_1,i_2,\ldots,i_K} \left[ \prod_{k=1}^{K} f_k(x)^{i_k} - \prod_{k=1}^{K} g_k(y)^{i_k} \right] = 0$$  (71)

must hold simultaneously for all the indices. In order not to overfix the function \(y(x)\) due to two or more constraints in eqn (71), the coefficients \(S_{i_1,i_2,\ldots,i_K}\) vanish except

$$S_{jl_1,jl_2,\ldots,jl_K} \quad (j:\text{integer})$$  (72)

where \(l_k \ (k = 1 \sim K)\) are integers chosen arbitrarily. Then the function \(y\) is defined by the following equation.

$$\prod_{k=1}^{K} f_k(x)^{l_k} = \prod_{k=1}^{K} g_k(y(x))^{l_k}.$$  (73)

Introducing a function \(F(X)\) as

$$F(X) = \sum_{j=-\infty}^{\infty} S_{jl_1,jl_2,\ldots,jl_K} X^j,$$  (74)

the entropy now reads

$$S = F \left( \prod_{k=1}^{K} \left( \sum_i \left( P_i^A \right)^{l_k} \right) \right).$$  (75)

It can be checked straightforwardly using

$$\sum_{ij} \left( P_i^A P_j^B \right)^q = \left( \sum_i \left( P_i^A \right)^q \right) \left( \sum_j \left( P_j^B \right)^q \right).$$  (76)
that the result (75) really possesses the composability. Actually a parametric expression of the composability is explicitly given as

$$S_{A+B} = F(ab),$$  
(77)

$$S_A = F(a),$$  
(78)

$$S_B = F(b),$$  
(79)

where

$$a = \prod_{k=1}^{K} \left( \sum_i (P_{Ai})^{q_k} \right)^{l_k}$$  
(80)

and

$$b = \prod_{k=1}^{K} \left( \sum_j (P_{Bj})^{q_k} \right)^{l_k}$$  
(81)

are its free parameters.

Note that the form of eqn (75) involves a modified Tsallis entropy:

$$\tilde{S}_q = -\frac{1}{1-q} \frac{1 - \sum_i P_i^q}{\sum_i P_i^q}$$  
(82)

which was proposed by Rajagopal and Abe [7] and is known to be equipped with the composability nature.

In conclusion, it has been clarified that if one imposes the composability on the generalized entropy $S$ (64), the form of the entropy is restricted to eqn (75).

Finally we would like to comment that the composability we argue here is just a conjecture so far and it may be possible to propose attractive non-composable entropies which express significant aspects of some exotic systems. Actually Anteneodo and Plastino [9] give quite an interesting entropy form which possesses a lot of plausible physical properties but is clearly non-composable.

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