Cosmological Number Counts under Disformal Transformations

Basundhara Ghosh\textsuperscript{1}, Jérémie Francfort\textsuperscript{2} and Rajeev Kumar Jain\textsuperscript{1}

\textsuperscript{1}Department of Physics, Indian Institute of Science, C.V. Raman Road, Bangalore 560012, India
\textsuperscript{2}Département de Physique Théorique and Center for Astroparticle Physics, Université de Genève, 24 quai Ernest Ansermet, 1211 Genève 4, Switzerland

E-mail: basundharag@iisc.ac.in, jeremie.francfort@unige.ch, rkjain@iisc.ac.in

Abstract. We investigate whether true physical observables associated with the measurements of large scale structure in the universe are frame-independent. In particular, we study if cosmological observables such as the galaxy number counts and weak lensing observables are invariant under the disformal transformations. In a previous work, it was shown that this frame-invariance holds true for the case of conformal transformations. In this work, we find that although the cosmological number counts remain invariant under the disformal transformations, convergence and cosmic shear associated with the weak lensing potential are generally not invariant. Since the lightcone structure does not remain causal under disformal transformations, photon geodesics do not remain null anymore, and as a result, weak lensing observables are indeed affected. We also briefly comment on the disformal invariance of other cosmological observables.

Keywords: Jordan frame, Einstein frame, cosmological perturbation theory, cosmological number counts, lensing
Contents

1 Introduction 1

2 Einstein and Jordan frames - A dictionary for disformal transformations 3
   2.1 A new formalism 3
   2.2 Background variables 5
   2.3 First order perturbations 6

3 Cosmological number counts 7
   3.1 Density perturbations 8
   3.2 Volume perturbations 9

4 Conclusions 10

1 Introduction

Scalar-Tensor (ST) theories are considered viable alternatives to the general theory of relativity (GR) and are usually at play while describing the inflationary epoch in the early universe or the present accelerated expansion in the late universe. While GR still remains the minimal theory of gravity which is remarkably successful in explaining diverse cosmological observations, there has been a long quest to explore various modifications of GR as well as to formulate viable alternatives and test them with multiple observations. One of the well-known examples of a ST theory is the Brans-Dicke theory, in which the gravitational interaction is mediated by a scalar field in addition to the tensor field of GR [1] (See, for instance, Refs. [2–4] for reviews on modified gravity). Moreover, there usually exists a non-trivial coupling between the gravitational sector and the matter sector in all such theories, and they also often have more than two propagating degrees of freedom [5]. In comparison, GR strictly has two such physical degrees of freedom, i.e. two polarization modes of the graviton. In order to be consistent with the observational tests on smaller scales such as the scales of our solar system, a realistic modification of gravity should necessarily contain a physical mechanism – a screening mechanism to suppress the extra propagating scalar degrees of freedom on such scales [6–10].

In order to construct viable modified theories, one must ensure that the additional degrees of freedom arising in such modifications do not lead to a ghost behaviour or other unwanted instabilities, both at the background and the perturbative order. These requirements led to the construction of the Horndeski theory – the most general ST theory with a single scalar field leading to second-order equations of motion, thereby avoiding ghost degrees of freedom [11]. These theories have found numerous applications in cosmology, particularly in constructing viable models of inflation and dark energy [12]. Lately, healthy extensions beyond Horndeski theories have also been studied which are usually obtained by performing the so-called disformal transformation of the field space metric tensor which is essentially a generalization of the well-known conformal transformation (also sometimes called the Weyl transformation). A theory invariant under the conformal transformation is called conformally invariant which only involves rescaling of the metric tensor by a conformal factor. It is important to note that such a conformal transformation does not modify the causal structure.
of the spacetime. In general, the disformal transformation contains a scalar field \( \phi \) as well as its first order derivatives\(^1\) and can be written as

\[
g_{\mu\nu} = A(\phi, X) \tilde{g}_{\mu\nu} + B(\phi, X) \nabla_\mu \phi \nabla_\nu \phi, \tag{1.1}
\]

where \( A \) and \( B \) are some general functions, referred to as the conformal and disformal factors, respectively and \( X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \). Although \( \nabla_\mu \) indicates the covariant derivative, for a scalar field it becomes \( \partial_\mu \phi \). For \( B = 0 \), Eq. (1.1) reduces to the ordinary conformal transformation while \( A = 1 \) represents the pure disformal transformation. In [13], the concept and formulation of the disformal transformation was first introduced by Bekenstein in order to relate geometries of the same gravitational theory which suggests a richer set of possibilities for the transformed metric. Since some of these possibilities may be unphysical, the functions \( A \) and \( B \) are generally subject to some constraints.

Over the years, disformal transformations have been applied in different contexts in GR and cosmology. For the case of massless Klein-Gordon equation [14] and the vacuum Maxwell’s equations [15], the disformal transformations form a set of new symmetry transformations. Further, it has been shown that the gauge-invariant primordial cosmological perturbations are invariant under disformal transformation within the context of the Horndeski theory (in this context, see Refs. [16–19]). Moreover, they have been used to study black holes in the context of ST theory [20], to investigate allowed regions of the solution space and their symmetry [21], in the context of dark energy [22–24], dark matter [25, 26] and more importantly, in generating a wider class of healthy ST theories (without additional number of physical degrees of freedom) beyond Horndeski theories, so called the DHOST theories [27–34].

One may naively expect that true physical observables must be frame invariant. However, it may still be a tedious task to explicitly show this invariance in different physical frames. It is well known that conformal transformations leave physical observables invariant. In an earlier paper [35], we had shown that some specific observables associated with the distribution of large scale structure remain invariant under a purely conformal transformation. It was discussed that while the (unobservable) matter power spectrum is frame dependent, the observable number counts do not depend on the choice of the frame. In this paper, we study whether these observables such as the cosmological number counts, lensing potential, convergence and shear are invariant under the disformal transformations. Thus, the present work can be considered as a disformal extension of the previous work.

Since time and spatial coordinates are rescaled by the same factor under conformal transformations, they are usually referred to as the causality preserving transformations as the lightcone structure remains preserved. However, disformal transformations usually don’t preserve causality as the time and spatial coordinates are rescaled by different factors. In order to ensure causal behaviour for all particles, one therefore requires the condition \( B < 0 \) everywhere. In fact, demanding that the disformally transformed metric is healthy and well defined everywhere, there arises a set of conditions that must be imposed on it: (i) It must be causal (ii) It must preserve Lorentzian signature (iii) The inverse must exist and be non-singular and (iv) The volume element must be non-singular. All these conditions will ensure that the disformal frame is also a well-defined physical frame and the physical observables, if correctly calculated, must therefore be independent of the choice of any frame. However,

\(^1\)Note that the disformal transformation is not just a simple field redefinition because it involves derivatives of the scalar field.
one must keep in mind that disformal transformations do not preserve the causal structure of lightcones. As a result, the photons geodesics are not null anymore and therefore, the lensing observables are affected. In other words, disformal transformations introduce a new coupling of the scalar field to the stress energy tensor thereby leading to modifications of gravitational lensing [36, 37].

**Conventions and notations:** We work with the $(-, +, +, +)$ signature. Various quantities in the Jordan frame are indicated with a tilde. As stated in Eq. (1.1), the disformal transformation is given by

$$g_{\mu\nu} = A(\phi, X) \tilde{g}_{\mu\nu} + B(\phi, X) \nabla_\mu \phi \nabla_\nu \phi, \quad \text{with} \quad X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi.$$  

Note that, as $\phi$ is a scalar, the covariant derivative is nothing but a partial derivative, hence we do not put any tilde to avoid heavy notations. In a cosmological context, the FRLW metrics are

$$\tilde{g} = -dt^2 + \tilde{a}^2 dx^2,$$

$$g = -\alpha_T^2 dt^2 + \alpha_L^2 \tilde{a}^2 dx^2,$$

with

$$\alpha_T^2 = A - B \phi^2, \quad \alpha_L^2 = A.$$  

This implies that the measurements of length and time in both frames are related as

$$L = \alpha_L \tilde{L}, \quad T = \alpha_T \tilde{T}.$$  

Note that the change in notation we have employed here compared to [35] is slightly different in the sense that we have used upper-case $L$ and $T$ instead of the lower-case $\ell$.

This paper is organized as follows. In Section 2, we present a brief dictionary of conformal and disformal transformations and point out the scaling of various quantities, including the background variables and first order perturbations. In Section 3, we obtain the transformations of the density and volume perturbations, and prove that the cosmological number counts are indeed equal in the Jordan and Einstein frames. Finally, in Section 4 we summarise our results and provide an outlook for future directions.

2 Einstein and Jordan frames - A dictionary for disformal transformations

2.1 A new formalism

In this section, we present a formalism to compute quantities in different frames, if both the unit of length and the unit of time are different. For simplicity, we will only consider physical quantities which are product of length, time and masses (with a potential exponent). This is sufficient to describe most of the quantities relevant in cosmology. We assume that in each frame, there are three *basis units*: length, time and action. This means that, for example, a mass is of the form

$$m \propto \frac{AT}{L^2},$$

where $L$, $T$ and $A$ are respectively a length, a time and an action. As mentioned earlier, in the Einstein/Jordan frame there is a unit ruler $L/\tilde{L}$ and a unit time $T/\tilde{T}$. Any physical
quantity can be expressed as a product of powers of this quantity and a pure number, and in the Einstein frame, one finds

\[ X_p = X \times L^{n_L} T^{n_T} \Lambda^{n_\Lambda} . \]  

We assume that the action remains the same in both the frames, hence we only write \( \mathcal{A} \). At this stage, it is important to make our notations clear.

- \( X_p \) is a physical dimensionful quantity, and whose intrinsic value is independent of the unit system
- \( X \) is a dimensionless number whose value depends on the unit system
- \( L, T \) and \( A \) are given length, time and action which form the unit system
- \( n_L, n_T \) and \( n_\Lambda \) are rational numbers

In order to go from one frame to another, we define \( \alpha_L \) and \( \alpha_T \) such that

\[ L = \frac{1}{\alpha_L} \tilde{L}, \quad \text{and} \quad T = \frac{1}{\alpha_T} \tilde{T}. \]  

We can get the transformation rules for the dimensionless numbers by imposing the condition that any physical quantity should be invariant under a frame transformation. For the sake of the argument, let’s consider a length \( L_p \) given by

\[ L_p = L \times \mathcal{L} = L \frac{1}{\alpha_L} \tilde{L} \equiv \tilde{L} \times \tilde{L}, \]  

from which we deduce

\[ L = \alpha_L \tilde{L}, \]  

and a similar argument yields

\[ T = \alpha_T \tilde{T}, \]  

which is the same as Eq. (1.5).

From these two relations, it is straightforward to deduce (using that the unit of action \( \mathcal{A} \) is the same in both frames) that

\[ X_p = X \times \mathcal{L}^{n_L} \mathcal{T}^{n_T} \Lambda^{n_\Lambda} , \tag{2.6} \]

\[ X_p = \tilde{X} \times \mathcal{L}^{n_L} \mathcal{T}^{n_T} \Lambda^{n_\Lambda} , \tag{2.7} \]

\[ X = \tilde{X} \alpha_L^{n_L} \alpha_T^{n_T} . \tag{2.8} \]

These relations provide us a recipe to write the dictionary for any physical quantity i.e. write it in terms of length, time and action and read off the exponent of the length and time. In Table 1, we have constructed examples of some physical quantities of interest.

\[ \text{[In principle we could also allow real exponents, but is it not common to encounter, for example, a length to the power } \pi, \text{ hence we will only consider rational exponents.}] \]
| Quantity      | Dictionary                                      |
|---------------|------------------------------------------------|
| Length        | $L = \alpha L\tilde{L}$                       |
| Time          | $T = \alpha T\tilde{T}$                       |
| Mass          | $m = \frac{\alpha T}{\alpha L} m\tilde{m}$   |
| Velocity      | $v = \frac{\alpha T}{\alpha L} v\tilde{v}$   |
| Volume        | $V = \alpha L^3 V\tilde{V}$                   |
| Energy        | $E = \frac{1}{\alpha T} E\tilde{E}$           |
| Mass density  | $\rho = \frac{\alpha T}{\alpha L} \rho\tilde{\rho}$ |

Table 1. Dictionary for various quantities between the two frames

2.2 Background variables

In order to understand the correspondence between the two frames, we shall follow the same approach as adopted in [35]. We first proceed by obtaining the relations between the background variables in the Jordan and Einstein frames, followed by that of the perturbations as well in these two frames. In case of the disformal transformation, we can get the relation between the comoving Hubble parameters in the two frames as follows:

$$\tilde{H} = \frac{\alpha}{A^{1/2}} H - \frac{1}{2} \frac{A_0'}{A_0} \dot{\phi}_0$$

$$= \frac{\alpha_0 T}{\alpha_0 L} H - \frac{\alpha_0 L}{\alpha_0 L} \dot{\phi}_0$$

(2.9)

Here, the quantities with subscript 0 indicate the background quantities and an overprime denotes derivative with respect to $\phi_0$. From the energy-momentum tensors in the two frames, given by

$$T_{\mu\nu} = \rho u^\mu u_\nu + P (u^\mu u_\nu + \delta^\mu_\nu)$$

(2.10)

$$\tilde{T}_{\mu\nu} = \tilde{\rho} \tilde{u}^\mu \tilde{u}_\nu + \tilde{P} (\tilde{u}^\mu \tilde{u}_\nu + \delta^\mu_\nu)$$

(2.11)

we can obtain a generalised relation between the energy-momentum tensor components, i.e., the energy density and the pressure in the two frames in terms of the kinetic term $X$ as [38]

$$\rho = \xi_1 \tilde{\rho} + \xi_2 \tilde{P}$$

(2.12)

$$P = \frac{\tilde{P}}{\alpha A^{3/2}} = \frac{\tilde{P}}{\alpha T\alpha L^3}$$

(2.13)

where $\xi_1$ and $\xi_2$ are given by

$$\xi_1 = \frac{(1 - 2BX/A)\alpha}{A(A - A_X X + 2B_X X^2)}$$

(2.14)

$$\xi_2 = -\frac{3(1 - 2BX/A)^1/2 A_X X}{A^2 (A - A_X X + 2B_X X^2)}$$

(2.15)
where $A_X = \partial A / \partial X$. However, in order to preserve the Lorentzian signature of the metric, the dependence on $X$ is often done away with [27, 39]. Also for our purpose, the computation of number counts requires only the density and volume perturbations, so we can neglect the pressure term above. Then we find

$$\xi_1 = \alpha A^{-5/2} = \frac{\alpha T}{\alpha L}, \quad \text{and} \quad (2.16)$$

$$\xi_2 = 0 \quad (2.17)$$

These equations simply follow from the fact that $\alpha T = \sqrt{A - B\dot{\phi}^2} = \sqrt{A - 2EX}$, since $2X = -\nabla_\mu \phi \nabla^\mu \phi$.

In both the conformal and disformal cases, the conservation of the energy-momentum tensor $\nabla^\mu \tilde{T}_{\mu\nu} = 0$ gives (Eq. 2.18 of [35])

$$\dot{\rho} = - (\tilde{\rho} + \tilde{P}) 3 \tilde{H} \quad (2.18)$$

So, from Eq. (2.12) and the expression of $\xi_1$ above, we have,

$$\dot{\rho} = \dot{\xi}_1 \tilde{\rho} + \xi_1 \dot{\tilde{\rho}}$$

$$= -3H \rho \frac{\alpha T}{\alpha L} - \frac{2\alpha'_L \dot{\phi}}{\alpha L} \rho + \frac{\alpha'_T \dot{\phi}}{\alpha T} \rho \quad (2.19)$$

In the conformal case, $\xi_1 = A^{-2}$, $\xi_2 = 0$ and $\alpha = \sqrt{A}$.

### 2.3 First order perturbations

We now turn to the first order perturbed quantities for scalar perturbations and neglect the vector and tensor perturbations. The expression for a perturbed $\xi_1$ turns out to be

$$\xi_1 = \xi_{10} \left( 1 - \frac{5}{2} \frac{\dot{A}_0}{A_0} + \frac{\alpha'_0 \dot{\phi}}{\alpha_0} \right)$$

$$= \xi_{10} \left( 1 - \frac{5\alpha'_0 L \dot{\phi}}{\alpha_0 L} + \frac{\alpha'_T \dot{\phi}}{\alpha_0 T} \right) \quad (2.20)$$

where $\xi_{10}$ is the background quantity. Hence, the perturbations in both the frames can be related as,

$$\delta(x) = \frac{\rho(x) - \rho_0(t)}{\rho_0(t)} = \frac{\xi_1 \dot{\rho}(x) - \xi_{10} \dot{\rho}_0(t)}{\xi_{10} \dot{\rho}_0(t)}$$

$$= \bar{\delta}(x) - \left( \frac{5\alpha'_0 L \dot{\phi}}{\alpha_0 L} - \frac{\alpha'_T \dot{\phi}}{\alpha_0 T} \right)$$

$$= \bar{\delta}(x) - \left( \frac{5\delta L}{L} - \frac{\delta T}{T} \right) \quad (2.21)$$

As previously done in the conformal case, we can also check if the lensing potential, given below, is disformal frame-invariant [40].

$$\psi(n, z) = \int_0^{r(z)} \frac{dr}{r(z)r} \left[ \Phi(rn, t_{\text{now}} - r) + \Psi(rn, t_{\text{now}} - r) \right] \quad (2.22)$$
However, very fundamentally speaking, weak gravitational lensing and its observables are not expected to be disformally invariant because the causal lightcone structure is not preserved and hence null geodesics do not exist in disformal transformations [13, 41, 42]. This statement can be further supported mathematically by verifying that the sum of the two Bardeen potentials $\Phi + \Psi$, that the lensing potential is sensitive to, is not invariant under disformal transformation. According to [16], the gauge-invariant metric perturbations in the longitudinal gauge in terms of $\alpha_T$ and $\alpha_L$ are related by:

$$\Phi = \tilde{\Phi} - \frac{1}{\alpha_T^2} \left( (\alpha_L^2 - \alpha_T^2) \frac{\dot{a}}{a} + \frac{2\alpha_L \dot{\alpha}_L}{2} \right) \frac{\delta \phi}{\phi} \quad (2.23)$$

$$\Psi = \frac{1}{\alpha_T^2} \left( \alpha_L^2 \tilde{\Psi} + \frac{2\alpha_L \dot{\alpha}_L (2\alpha_T^2 - \alpha_L^2) + \alpha_T^2 (2\alpha_L \dot{\alpha}_L - 2\alpha_T \dot{\alpha}_T) \delta \phi}{2\alpha_T^2} \right) \quad (2.24)$$

Also in the conformal case, $\alpha_T = \alpha_L$, and we recover the conformal case relations [c.f. Eq. (2.27) of [35]]. However in general, we do not find the sum of two Bardeen potentials $\Phi + \Psi$ to be invariant in the disformal case.

This further means that other lensing observables such as the convergence $\kappa$ and shear $\gamma$ that are dependent on the lensing potential cannot be disformal frame-invariant.

$$\kappa = \frac{1}{2} \left( \partial_1 \partial_1 + \partial_2 \partial_2 \right) \psi = \frac{1}{2} \nabla^2 \psi$$

$$\gamma_1 = \frac{1}{2} \left( \partial_1 \partial_1 - \partial_2 \partial_2 \right) \psi; \quad \gamma_2 = \partial_1 \partial_2 \psi; \quad \gamma = \gamma_1 + i\gamma_2 \quad (2.25)$$

### 3 Cosmological number counts

The cosmological number counts are observables that are computed in terms of the number $N$ of galaxies observed in a patch of the sky in a particular direction $n$ per unit solid angle and per unit redshift bin and are quantified as the perturbation in the number density of galaxies as

$$\Delta(n, z) = \frac{N(n, z) - \langle N \rangle(z)}{\langle N \rangle(z)} \quad (3.1)$$

We follow the notation of [43], and define the number counts in the Jordan frame with a tilde

$$\tilde{\Delta}(n, z) = \tilde{\delta}_z(n, z) + \frac{\delta V(n, z)}{V(z)} \quad (3.2)$$

Here the first term $\tilde{\delta}_z$ is the redshift-space density perturbation, and the second term is the volume perturbation divided by the physical survey volume density per redshift bin, per solid angle. Our aim is to show that $\tilde{\Delta}(n, z)$ is invariant under a disformal transformation and is the same as the quantity in the Einstein frame, which we will denote without a tilde, $\Delta(n, z)$.

From the fundamental definition of the density perturbation in the redshift space, we have

$$\delta_z(n, z) = \frac{\rho(n, z) - \rho_0(z)}{\rho_0(z)} = \frac{N(n, z)(\alpha_T/\alpha_L^2)^{-1}}{V(n, z)} - \frac{N_0(z)(\alpha_T/\alpha_L^2)^{-1}}{V_0(z)}$$

$$\delta_z(n, z) = \Delta(n, z) - \frac{\delta V}{V_0} + 2 \left( \delta z \frac{1}{L_0} \frac{d L_0}{d z} - \frac{\delta L_0}{L_0} \right) - \left( \delta z \frac{1}{T_0} \frac{d T_0}{d z} - \frac{\delta T_0}{T_0} \right) \quad (3.3)$$
where the subscript 0 corresponds to background quantities. The second equality follows from the fact that \( \rho = mN/V \) and the mass \( m \) scales as \( \alpha T/\alpha L^2 \). The last equality is obtained in a manner similar to that explained in Appendix C of [35]. The quantity \( \delta z \) corresponds to the splitting of the observable redshift \( z \) into a background and a perturbative part, that is, \( z = z_0 + \delta z \) (see Figure 1 of [35]).

For a general function \( f \), we can assume that it has time and length dimensions \( n_T \) and \( n_L \) respectively, and the generalization of the formula becomes

\[
f = \tilde{f} L^n T^m.
\]

Again, as in [35], we define \( L_0(z), T_0(z), \delta L \) and \( \delta T \). Following the same steps we get

\[
\frac{\delta f(n, z)}{f_0(z)} = \frac{\delta \tilde{f}(n, z)}{f_0(z)} + \left( n_L \frac{\delta z}{L_0} dL_0 - n_L \frac{\delta L}{L_0} \right) + \left( n_T \frac{\delta z}{T_0} dT_0 - n_T \frac{\delta T}{T} \right),
\]

where \( \delta f \) is the first order perturbation in the function \( f \), as a function of direction \( n \) and redshift \( z \) and calculated in the Einstein frame and \( \delta \tilde{f} \) is the corresponding quantity in the Jordan frame.

### 3.1 Density perturbations

In this section, we aim to obtain a relation for the density perturbations between the two frames. Following the definition of the redshift-space density perturbation in [43], we have

\[
\delta z(n, z) = \frac{\delta \rho(n, z)}{\bar{\rho}(z_0)} - \frac{\delta \rho_0}{\rho_0(z_0)} \frac{\delta z(n, z)}{\rho_0(z_0)}
\]

(3.6)

where the first term corresponds to the relation established by Eq. (2.21). For the second term we first need to compute the expressions for \( \frac{d\rho_0}{dz_0} \) and \( \frac{d\rho_0}{dt} \). The latter has already been done in Eq. (2.19). In the Einstein frame we have,

\[
1 + z_0 = \frac{\alpha_0 T}{a}
\]

(3.7)

which implies that

\[
\frac{d\rho_0}{dt} = -\frac{\dot{a}}{a^2} \alpha_0 T + \frac{\alpha_0 T}{a} = -(1 + z_0) \left( H - \frac{\alpha_0 T \dot{\phi}_0}{\alpha_0 T} \right)
\]

(3.8)

We can now use this expression to obtain

\[
\frac{d\rho_0}{dz_0} = \frac{d\rho_0}{dt} \frac{dz_0}{dt} = \frac{-3H\rho_0 \alpha_0 T}{\alpha L} - \frac{2\alpha_0 \dot{\phi}_0 \rho_0}{\alpha_0 T} + \frac{\alpha_0 T \dot{\phi}_0}{\alpha_0 T} \rho_0
\]

\[
-(1 + z_0) \left( H - \frac{\alpha_0 T \dot{\phi}_0}{\alpha_0 T} \right)
\]

(3.9)

Now, from Eq. (3.6), and realising that the first term on the right hand side is essentially the quantity \( \delta(x) \) as defined in Eq. (2.21), we get an expression for the density contrast \( \tilde{\delta}z \) in the Jordan frame as

\[
\tilde{\delta}z(n, z) = \delta(x) - \left( \frac{d\rho_0}{dz_0} \right) \frac{\delta z(n, z)}{\rho_0(z_0)}
\]

(3.10)
which, using the previous equation, yields

$$
\tilde{\delta}(x) = \tilde{\delta}_z - \frac{3\mathcal{H} \alpha_T}{\alpha_L} + \frac{2\alpha_L \dot{\phi}_0}{\alpha_L} - \frac{\alpha_T \dot{\phi}_0}{\alpha_T} \delta z
$$

(3.11)

Upon using Eq. (2.21), we find

$$
\delta(x) = \tilde{\delta}_z - \frac{3\mathcal{H} \alpha_T}{\alpha_L} + \frac{2\alpha_L \dot{\phi}_0}{\alpha_L} - \frac{\alpha_T \dot{\phi}_0}{\alpha_T} \delta z - \left( \frac{5\delta L}{L} - \frac{\delta T}{T} \right)
$$

(3.12)

which leads to

$$
\delta_z = \tilde{\delta}_z - \left( \frac{5\delta L}{L} - \frac{\delta T}{T} \right) = \tilde{\delta}_z - 5\frac{\delta L}{L_0} + \frac{5}{L_0 \delta z_0} \delta z + \frac{\delta T}{T_0} - \frac{\delta T}{T_0} \delta z
$$

(3.13)

This equation is a generalised version of Eq. (3.9) of [35] since

$$
\delta(x) - \tilde{\delta}(x) = \delta_z - \tilde{\delta}_z
$$

The relation in Eq. (3.13) gives us the “unobservable” density contrast $\delta_z$ in the Einstein frame in terms of the same quantity in the Jordan frame, and as is evident, they are unequal. This renders the quantity $\delta_z$ frame-dependent. This also automatically implies that the “unobservable” matter power spectrum $P_{\delta_z}(k, z)$ which is nothing but a Fourier transform of the two-point correlation function of the density contrast, is also frame-dependent, as can be seen from the following definition of the matter power spectrum

$$
\langle \delta_z(k) \delta_z^*(k') \rangle = (2\pi)^3 P_{\delta_z}(k, z) \delta^3(k - k') \neq (2\pi)^3 P_{\tilde{\delta}_z}(k, z) \delta^3(k - k')
$$

(3.14)

where $\delta^3$ is the three-dimensional Kronecker delta function. Contrary to this result, in the next section, we will show that the “observable” matter power spectrum that includes both the density and volume perturbations is indeed frame-invariant.

### 3.2 Volume perturbations

In order to proceed with the transformation of the volume perturbations, we need to first understand what corrections are to be taken into consideration in the Einstein frame. An obvious correction comes into play in the prefactor for Eq. (14) of [43] that gives the expression for the volume element, which in the conformal case is (assuming the Jordan frame, and hence the tilde over $H$),

$$
\frac{a^4}{H} = \frac{a}{\mathcal{H}^3} = \frac{a^3 \alpha_{0L}}{1 + z_0} \frac{1}{\mathcal{H} - \frac{\alpha_{0L} A_0}{2A_0}} = \frac{\alpha_{0L}^3}{(1 + z_0)^4} \frac{1}{\mathcal{H} - \frac{\phi_0 A_0^2}{2A_0}}
$$

(3.15)

where the second and third equality come from the fact that $a = L_0/(1 + z_0) = \alpha_{0L}/(1 + z_0)$ (assuming that the length scale in the Jordan frame $L_0$ is unity) and the relation between
the Jordan and Einstein frame Hubble parameters (see Eq. (2.13) of [35]). Analogously, for the disformal case, we should have the prefactor as

\[
\frac{a^4}{\mathcal{H}} = \frac{a^3}{1 + z_0} \left( \mathcal{H} - \frac{\alpha'_{0T} \phi}{\alpha_{0T}} \right)^{-1}
\]

\[
= \frac{a^3_{0T}}{(1 + z_0)^2} \left( \mathcal{H} - \frac{\alpha'_{0T} \phi}{\alpha_{0T}} \right)^{-1}
\]

(3.16)

The volume transformations work the same way as the conformal case since the volume in the Einstein frame scales as \(a^3 L\). Following the steps underlined in [43] in order to obtain the volume perturbation, we obtain

\[
\frac{\delta V}{V_0} = \frac{\delta \tilde{V}}{V_0} + 3 \frac{\alpha'_{0T} \phi}{\alpha_L} = \frac{\delta \tilde{V}}{V_0} + 3 \frac{\delta L}{L_0} \delta z
\]

(3.17)

From Eqs. (3.3), (3.13) and (3.17) we finally have

\[
\Delta_E = \Delta(n, z) = \delta_z + \frac{\delta V}{V_0} - 2 \left( \delta z \frac{1}{L_0} \frac{dL_0}{dz_0} - \frac{\delta L}{L_0} \right) + \left( \delta z \frac{1}{T_0} \frac{dT_0}{dz_0} - \frac{\delta T}{T_0} \right)
\]

\[
= \left( \delta z - 5 \frac{\delta L}{L_0} + 5 \frac{dL_0}{L_0} \delta z + \frac{\delta T}{T_0} - \frac{dT_0}{T_0} \delta z \right) + \left( \frac{\delta \tilde{V}}{V_0} + 3 \frac{\delta L}{L_0} - 3 \frac{dL_0}{L_0} \delta z \right)
\]

\[
- 2 \left( \delta z \frac{1}{L_0} \frac{dL_0}{dz_0} - \frac{\delta L}{L_0} \right) + \left( \delta z \frac{1}{T_0} \frac{dT_0}{dz_0} - \frac{\delta T}{T_0} \right)
\]

\[
= \Delta(n, z) = \Delta_J
\]

(3.18)

where the subscripts \(E\) and \(J\) stand for Einstein and Jordan frames, respectively.

Therefore, we can show that in this case the matter power spectrum in both frames would indeed be invariant

\[
\langle \Delta(k) \Delta^*(k') \rangle = \langle \tilde{\Delta}(k) \tilde{\Delta}^*(k') \rangle
\]

Or, equivalently,

\[
(2\pi)^3 P_\Delta(k, z) \delta^3(k - k') = (2\pi)^3 P_{\Delta}(k, z) \delta^3(k - k')
\]

(3.19)

This establishes an interesting and important fact that while the “unobservable” power spectrum of the density contrast is frame-dependent, the “observable” matter power spectrum is indeed frame-invariant.

4 Conclusions

Disformal transformations are considered a generalisation of the conformal transformations which also involve derivative dependent terms of a scalar/vector field. While the conformal transformations usually involve a rescaling of the metric which also preserve causality, the disformal transformations are more general which affect the particle geodesics and also lead to a non-trivial coupling of the matter Lagrangian. These salient features provide very rich
phenomenology which has been explored in different cosmological contexts, in black hole physics and more interestingly, in constructing an even more general class of ST theories beyond Horndeski theories.

In this paper, we have investigated how physical observables associated with the galaxy surveys in the Jordan and Einstein frames are related by a disformal transformations constructed by scalar fields. In particular, we have successfully been able to show the frame-invariance of the cosmological number counts for the case of disformal transformations which is an interesting result. This establishes the property of number counts as a physical observable even further. The notable difference with respect to the conformal frame-invariance of cosmological observables is the fact that the lensing potential as a standalone observable does not turn out to be disformally invariant, because of the absence of null geodesics. As a result, various quantities such as the convergence and shear which are derived from the lensing potential are also not invariant under the disformal transformations. A summary of our results that we have obtained in this work, has been outlined in Table 2.

Although we have only considered the disformal factor $B$ as a function of $\phi$ only, and it might also seem non-trivial to carry out the same kind of proof for a disformal transformation including the $X$ dependence, we believe that our calculations provide a first step towards

| Quantity                  | Gauge dependent | Frame dependent | Background |
|---------------------------|-----------------|-----------------|------------|
| Density $\rho_0$          | Yes             | Yes             | $\rho_0 = \alpha_0 T_0 \rho_0$ |
| Pressure $P_0$            | Yes             | Yes             | $P_0 = \alpha_0 T_0 P_0$ |
| Redshift $z_0$            | Yes             | No              |            |
| Observed redshift $z$     | No              | No              |            |

| Perturbations             |                  |                 | Background |
|---------------------------|------------------|-----------------|------------|
| Density $\delta$          | Yes              | Yes             | $\delta = \tilde{\delta} - \left( \frac{5L}{T} \right)$ |
| Velocity $v$              | Yes              | No              |            |
| Bardeen potential $\Phi$  | No               | Yes             | Eq. (2.23) |
| Bardeen potential $\Psi$  | No               | Yes             | Eq. (2.24) |
| Lensing potential $\varphi$| No               | Yes             |            |
| Redshift density $\delta z$| No               | Yes             |            |
| Volume perturbation $\frac{\delta V}{V}$ | No | Yes | $\frac{\delta V}{V} = \frac{\delta V}{V_0} + 3 \frac{5L}{T_0} \delta z - 3 \frac{dl_0}{L_0 t_0 \delta z} \delta z$ |
| Number counts $\Delta(n, z)$ | No | No | |

**Table 2.** Gauge and frame dependence of various quantities.
a necessary generalisation, as we also observe that our results reduce to those of [35] under the pure conformal limit. However, an apparent appearance of Ostrogradsky ghosts for a $X$ dependent disformal transformation might happen, no real ghost in the theory should actually be present in the transformed frame due to the existence of hidden constraints [30]. Our work can, in fact, be further extended to more complex transformations, for example to those in [17–19, 42], employing much more rigorous and non-trivial calculations, which is beyond the scope of the current work.

Another direction in which one can explore the frame-invariance of cosmological number counts is by further generalisation of the Brans-Dicke ST theory itself, via the Horndeski Lagrangian [11]. An attempt to establish the invariance of the Horndeski Lagrangian under disformal transformations has been made in [27], where it was found that the $X$ dependence is detrimental to the frame invariance. However, it would be interesting to explore how the frame-invariance of number counts holds up beyond the ST theories. We leave these interesting directions for future work.

Acknowledgments

We would like to thank Ruth Durrer for useful discussions and comments on the draft. BG acknowledges partial financial support from the CV Raman Postdoctoral Fellowship and the DST-INSPIRE Faculty Fellowship DST/INSPIRE/04/2020/001534. JF acknowledges financial support from the Swiss National Science Foundation. RKJ wishes to acknowledge financial support from the new faculty seed start-up grant of the Indian Institute of Science, Bengaluru, India, Science and Engineering Research Board, Department of Science and Technology, Government of India, through the Core Research Grant CRG/2018/002200 and the Infosys Foundation, Bengaluru, India through the Infosys Young Investigator award.

References

[1] C. Brans and R. H. Dicke, Mach’s principle and a relativistic theory of gravitation, Phys. Rev. 124 (1961) 925–935.
[2] T. P. Sotiriou and V. Faraoni, $f(R)$ Theories Of Gravity, Rev. Mod. Phys. 82 (2010) 451–497, [arXiv:0805.1726].
[3] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Modified Gravity and Cosmology, Phys. Rept. 513 (2012) 1–189, [arXiv:1106.2476].
[4] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution, Phys. Rept. 692 (2017) 1–104, [arXiv:1705.11098].
[5] A. Iyonaga and T. Kobayashi, Distinguishing modified gravity with just two tensorial degrees of freedom from general relativity: Black holes, cosmology, and matter coupling, Phys. Rev. D 104 (2021), no. 12 124020, [arXiv:2109.10615].
[6] A. De Felice, R. Kase, and S. Tsujikawa, Vainshtein mechanism in second-order scalar-tensor theories, Phys. Rev. D 85 (2012) 044059, [arXiv:1111.5090].
[7] R. Kimura, T. Kobayashi, and K. Yamamoto, Vainshtein screening in a cosmological background in the most general second-order scalar-tensor theory, Phys. Rev. D 85 (2012) 024023, [arXiv:1111.6749].
[8] T. S. Koivisto, D. F. Mota, and M. Zumalacarregui, Screening Modifications of Gravity through Disformally Coupled Fields, Phys. Rev. Lett. 109 (2012) 241102, [arXiv:1205.3167].
[9] P. Brax, Screening mechanisms in modified gravity, Class. Quant. Grav. 30 (2013) 214005.
[10] P. Brax, S. Casas, H. Desmond, and B. Elder, *Testing Screened Modified Gravity*, Universe **8** (2021), no. 1 11, [arXiv:2201.10817].

[11] G. W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, Int. J. Theor. Phys. **10** (1974) 363–384.

[12] C. Deffayet and D. A. Steer, *A formal introduction to Horndeski and Galileon theories and their generalizations*, Class. Quant. Grav. **30** (2013) 214006, [arXiv:1307.2450].

[13] J. D. Bekenstein, *The Relation between physical and gravitational geometry*, Phys. Rev. D **48** (1993) 3641–3647, [gr-qc/9211017].

[14] F. T. Falciano and E. Goulart, *A new symmetry of the relativistic wave equation*, Class. Quant. Grav. **29** (2012) 085011, [arXiv:1112.1341].

[15] E. Goulart and F. T. Falciano, *Disformal invariance of Maxwell’s field equations*, Class. Quant. Grav. **30** (2013) 155020, [arXiv:1303.4350].

[16] M. Minamitsuji, *Disformal transformation of cosmological perturbations*, Phys. Lett. B **737** (2014) 139–150, [arXiv:1409.1566].

[17] S. Tsujikawa, *Disformal invariance of cosmological perturbations in a generalized class of Horndeski theories*, JCAP **04** (2015) 043, [arXiv:1412.6210].

[18] G. Domènech, A. Naruko, and M. Sasaki, *Cosmological disformal invariance*, JCAP **10** (2015) 067, [arXiv:1505.00174].

[19] H. Motohashi and J. White, *Disformal invariance of curvature perturbation*, JCAP **02** (2016) 065, [arXiv:1504.00846].

[20] E. Babichev, C. Charmousis, and A. Lehébel, *Asymptotically flat black holes in Horndeski theory and beyond*, JCAP **04** (2017) 027, [arXiv:1702.01938].

[21] J. Ben Achour, H. Liu, and S. Mukohyama, *Hairy black holes in DHOST theories: Exploring disformal transformation as a solution-generating method*, JCAP **02** (2020) 023, [arXiv:1910.11017].

[22] M. Zumalacarregui, T. S. Koivisto, D. F. Mota, and P. Ruiz-Lapuente, *Disformal Scalar Fields and the Dark Sector of the Universe*, JCAP **05** (2010) 038, [arXiv:1004.2684].

[23] J. Sakstein, *Disformal Theories of Gravity: From the Solar System to Cosmology*, JCAP **12** (2014) 012, [arXiv:1409.1734].

[24] P. Brax, C. Burrage, and C. Englert, *Disformal dark energy at colliders*, Phys. Rev. D **92** (2015), no. 4 044036, [arXiv:1506.04057].

[25] C. van de Bruck and J. Morrice, *Disformal couplings and the dark sector of the universe*, JCAP **04** (2015) 036, [arXiv:1501.03073].

[26] P. Brax, K. Kaneta, Y. Mambrini, and M. Pierre, *Disformal dark matter*, Phys. Rev. D **103** (2021), no. 1 015028, [arXiv:2011.11647].

[27] D. Bettoni and S. Liberati, *Disformal invariance of second order scalar-tensor theories: Framing the Horndeski action*, Phys. Rev. D **88** (2013) 084020, [arXiv:1306.6724].

[28] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, *Exploring gravitational theories beyond Horndeski*, JCAP **02** (2015) 018, [arXiv:1408.1952].

[29] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, *Healthy theories beyond Horndeski*, Phys. Rev. Lett. **114** (2015), no. 21 211101, [arXiv:1404.6495].

[30] M. Zumalacárregui and J. García-Bellido, *Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian*, Phys. Rev. D **89** (2014) 064046, [arXiv:1308.4685].

– 13 –
[31] D. Langlois and K. Noui, *Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability*, JCAP 02 (2016) 034, [arXiv:1510.06930].

[32] J. Ben Achour, D. Langlois, and K. Noui, *Degenerate higher order scalar-tensor theories beyond Horndeski and disformal transformations*, Phys. Rev. D 93 (2016), no. 12 124005, [arXiv:1602.08398].

[33] M. Crisostomi, M. Hull, K. Koyama, and G. Tasinato, *Horndeski: beyond, or not beyond?*, JCAP 03 (2016) 038, [arXiv:1601.04658].

[34] T. Kobayashi, *Horndeski theory and beyond: a review*, Rept. Prog. Phys. 82 (2019), no. 8 086901, [arXiv:1901.07183].

[35] J. Francfort, B. Ghosh, and R. Durrer, *Cosmological Number Counts in Einstein and Jordan frames*, JCAP 09 (2019) 071, [arXiv:1907.03606].

[36] J. D. Bekenstein and R. H. Sanders, *Gravitational lenses and unconventional gravity theories*, Astrophys. J. 429 (1994) 480, [astro-ph/9311062].

[37] M. Wyman, *Galilean-invariant scalar fields can strengthen gravitational lensing*, Phys. Rev. Lett. 106 (2011) 201102, [arXiv:1101.1295].

[38] T. Chiba, F. Chibana, and M. Yamaguchi, *Disformal invariance of cosmological observables*, JCAP 06 (2020) 003, [arXiv:2003.10633].

[39] M. Zumalacarregui, T. S. Koivisto, and D. F. Mota, *DBI Galileons in the Einstein Frame: Local Gravity and Cosmology*, Phys. Rev. D 87 (2013) 083010, [arXiv:1210.8016].

[40] M. Kilbinger, *Cosmology with cosmic shear observations: a review*, Rept. Prog. Phys. 78 (2015) 086901, [arXiv:1411.0115].

[41] G. Domènech, S. Mukohyama, R. Namba, and V. Papadopoulos, *Vector disformal transformation of generalized Proca theory*, Phys. Rev. D 98 (2018), no. 6 064037, [arXiv:1807.06048].

[42] A. L. Alinea and T. Kubota, *Transformation of primordial cosmological perturbations under the general extended disformal transformation*, Int. J. Mod. Phys. D 30 (2021), no. 08 2150057, [arXiv:2005.12747].

[43] C. Bonvin and R. Durrer, *What galaxy surveys really measure*, Phys. Rev. D 84 (2011) 063505, [arXiv:1105.5280].