A Comment on the Zamolodchikov c-Function and the Black String Entropy

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Abstract

Using the spectral representation approach to the Zamolodchikov’s c-function and the Maldacena conjecture for the D1-branes, we compute the entropy of type IIB strings. An agreement, up to a numerical constant which cannot be determined using this approach, with the Bekenstein-Hawking entropy is found.
In two dimensional field theories, the two-point function of the energy-momentum tensor is a useful concept in studying the relation between the energy and the entropy. For conformal theories, they completely determine the entropy \[1\]. This fact, together with the observation of Brown and Henneaux \[2\] that the asymptotic group of \(AdS_3\) yields a two dimensional conformal theory, was used by Strominger to derive the Bekenstein-Hawking entropy for black holes in \(AdS_3\) \[3\].

In this note we study the SYM theory in 1+1 dimensions with gauge group \(SU(N)\) and sixteen supercharges which is a non-conformal theory. This theory can be thought of as the theory living on a collection of \(N\) D1-branes in the low energy “decoupling” limit. In the extreme UV the theory is free and conformal with central charge

\[
c_{UV} \sim N^2. \tag{1}
\]

Perturbation theory in SYM can be trusted in the UV as long as the effective coupling constant is small, that is

\[
1 \gg g_{\text{eff}}^2 = g_{YM}^2 N x^2 \quad \Rightarrow \quad x \ll \frac{1}{g_{YM} \sqrt{N}}, \tag{2}
\]

where \(x\) is the scale being probed by the two point function. In the deep IR region, the physical energy scale determined by the coupling constant becomes irrelevant and the theory flows to a conformal theory. In \[4, 5\] this theory was shown to be a conformal \(\sigma\)-model with the target space \((R^8)^N/S_N\) whose central charge is

\[
c_{IR} \sim N. \tag{3}
\]

Note that \(c_{UV} > c_{IR}\) as expected from the Zamolodchikov’s c-theorem \[7\]. The first correction to the orbifold CFT is given by the twist operator \(\frac{1}{g_{YM}} V_{ij}\) where \(i, j\) label the fields on which the twist operator is acting \[8\]. There are various ways to show that the perturbation theory with respect to the twist operators breaks down at \(x < \frac{\sqrt{N}}{g_{YM}}\) \[8, 9, 10\]. A simple argument which rests on the c-theorem is the following. Perturbation theory around the conformal point will break down when the difference between the Zamolodchikov c-function and \(c_{IR}\) is of the order of \(c_{IR}\). That is, when \(\langle T_{zz}(x) T_{zz}(0) \rangle\) is of the order of \(\langle T_{zz}(x) T_{zz}(0) \rangle\). Conservation of the energy-momentum tensor implies that (see e.g. \[11\])

\[
T_{zz} = -\frac{\pi \sum V_{ij}}{g_{YM}}. \tag{4}
\]

Therefore, perturbation theory around the conformal point can be trusted when

\[
\frac{c_{IR}}{x^4} \gg \frac{N^2}{g_{YM}^2 x^6}, \quad \Rightarrow \quad x \gg \frac{\sqrt{N}}{g_{YM}}, \tag{5}
\]
where we have used the fact that the weight of the twist operators is \((3/2, 3/2)\) \[3\].

In the large \(N\) limit there is a large region,

\[
\frac{1}{\sqrt{N g_{YM}}} \ll x \ll \frac{\sqrt{N}}{g_{YM}},
\]

(6)

for which neither the perturbative SYM nor the orbifold CFT description can be trusted. In \[4\] it was shown that this region is best described by the type IIB string theory on a background associated with the near horizon geometry of D1/F1 strings. Exactly at the points in UV and the IR regions where the perturbative field theory descriptions break down, the curvature (in string units) is small so that the supergravity approximation becomes reliable. The transitions between the perturbative conformal theories (at the UV and IR) and the supergravity description were studied in \[12, 13, 14\] to find a match from both sides up to a numerical coefficient which cannot be determined using current methods. This agreement (for the entropy \[12, 14\] and for the Wilson line \[13, 14\]) supports the Maldacena conjecture for this non-conformal theory but it does not give us much information about the way in which the supergravity description interpolates between the UV and the IR perturbative field theories descriptions.

In this article, we elaborate on the interpolation of the Zamolodchikov’s \(c\)-function between \(c_{IR}\) and \(c_{UV}\) and its relation to the entropy of the near-extremal D1/F1 string. To make contact with entropy it is useful to use the Kallen-Lehmann spectral representation of the correlator of two energy-momentum tensors \[15\],

\[
\langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle = \frac{\pi}{3} \int_0^\infty d\mu c(\mu) \int \frac{d^2 p}{(2\pi)^2} e^{ipx} \frac{(g_{\mu\nu}p^2 - p_\mu p_\nu)(g_{\rho\sigma}p^2 - p_\rho p_\sigma)}{p^2 + \mu^2}.
\]

(7)

The fact that the Zamolodchikov’s \(c\)-function is monotonically decreasing along the RG flow follows from \(c(\mu) \geq 0\) which must hold for any unitary theory \[13\]. In two dimensions, covariant quantity with four indices subject to the constraint following from the conservation of energy momentum-tensor is characterized by a single invariant. Thus, there is only one possible function of the intermediate mass scale, \(c(\mu)\), which is known as the spectral density. The quantity

\[
c_{eff}(\Lambda) = \int_0^\Lambda d\mu c(\mu),
\]

(8)

interpolates between \(c_{IR} = c_{eff}(0)\) and \(c_{UV} = c_{eff}(\infty)\). Since \(c(\mu)d\mu\) measures the density of degrees of freedom which couple to the energy-momentum tensor, and since all fields couple to the energy-momentum tensor, the spectral representation of the correlator of two energy-momentum tensors measures the density of degrees of freedom.
Using the methods developed in [16, 17] to compute field theory correlation functions via the bulk propagation of supergravity modes, we can calculate the two point function of the energy-momentum tensor. Suppressing numerical factors and Lorentz indices, we find

\[ \langle T(x)T(0) \rangle = \frac{N^{3/2}}{g_{YM} x^5}. \]  

(9)

Before substituting this into eq.(7) and discussing the spectral density, it is worth while to make a few comment about this result. Eq.(9) is obtained by repeating the procedure of [16, 17] for the minimally coupled scalar in the near horizon geometry of the D1-brane. In non-conformal theories it is harder to identify the correspondence between the supergravity modes and the field theory operators since the symmetry group is smaller (see however [18]). General covariance indicates that the energy momentum tensor must correspond to the metric fluctuation \( h_{\mu\nu} \). It is therefore more appropriate to analyze the field equations for metric fluctuations in this background. We expect nonetheless for the generic components of the metric fluctuations to behave essentially like a minimal scalar. The reason is that \( h_{\mu}^{\mu} \) mixes with a linear combination of a minimal and fixed scalar\(^1\) [19]. In the supergravity region the fixed scalar contribution is suppressed and we are left with eq.(9) for \( \langle T_{zz}(x)T_{zz}(0) \rangle \). All other components are determined in two-dimensions by the conservation of the energy-momentum tensor.

There are corrections to (9) suppressed by \( \frac{1}{g_{YM} x \sqrt{N}} \) and \( \frac{2g_{YM}}{\sqrt{N}} \) which can be thought of respectively as curvature and quantum corrections from the point of view of type IIB string theory in the near horizon geometry of the D1-brane. These corrections are very small and can be ignored in the region given by eq.(6) where supergravity approximation can be trusted. Conversely, the point in \( x \)-space where these corrections become significant mark the transition point to the UV and IR conformal fixed points. At these transition points, between supergravity and perturbative SYM and between supergravity and the orbifold CFT, eq.(9) agrees (up to a numerical factor) with the conformal results

\[ \langle T(x)T(0) \rangle = \frac{c}{x^4}, \]  

(10)

for the central charge appropriate for the UV and IR fixed points given in eqs.(1,3). In order to fix the numerical coefficient of eq.(9) unambiguously, it may be necessary to understand the supergravity-perturbative SYM crossover in detail so that the normalization in the supergravity region can be matched to the normalization in the perturbative SYM region. For our purpose, however, there is no need to fix the numerical constant since the relation be-

\(^1\)Strictly speaking, these fields are scalars in the 9 dimensional supergravity obtained by dimensionally reducing along the spatial direction of the D1-brane.
between the spectral density and the entropy can be determined only up to a numerical factor as we discuss below.

Combining eq. (9) with eq. (7) we find that $c(\mu) = N^{3/2}/g_{YM}$. Therefore, for a given temperature $\hat{T}$, the number of light degrees of freedom with $p^2 < \hat{T}^2$ is

$$N_{eff}(\hat{T}) \sim c_{eff}(\hat{T}) = \frac{N^{3/2}\hat{T}}{g_{YM}},$$

(11)

Here we encounter a fundamental ambiguity: the relative numerical coefficient between $c_{eff}(\hat{T})$ and $N_{eff}(\hat{T})$ cannot be determined for non-conformal theories (as opposed to conformal theories where $N_{eff} = c_{eff}$). In fact, one can construct two theories with the same $c_{eff}$ whose $N_{eff}$ agrees only up to a numerical factor of order one.

The contribution to the free energy is

$$F \sim N_{eff}(\hat{T})LT^2.$$

(12)

Hence the energy density and entropy density are

$$s = \frac{S}{L} = \frac{N^{3/2}\hat{T}^2}{g_{YM}},$$

$$\epsilon = \frac{E}{L} = \frac{N^{3/2}\hat{T}^3}{g_{YM}}.$$  

(13)

We wish to compare this with the black hole thermodynamics. In the Einstein frame the near horizon metric of $N$ near-extremal D1-branes is,

$$\frac{ds^2}{\alpha'} = \frac{U^{9/2}}{g_{YM}^{5/2}N^{3/4}} \left(-\left(1 - \frac{U_0^6}{U^6}\right)dt^2 + dx^2\right) + \frac{N^{1/4}}{\sqrt{g_{YM}U^{3/2}(1 - \frac{U_0^6}{U^6})}}dU^2 + \frac{N^{1/4}\sqrt{U}}{\sqrt{g_{YM}}}d\Omega_6^2,$$

(14)

where $U_0^6 = g_{YM}^4\epsilon$. This yields for the entropy density [3, 21],

$$s = g_{YM}^{-1/3}\sqrt{N}\epsilon^{2/3},$$

(15)

which is in agreement, up to a numerical factor, with eq. (13).

Note that unlike in the near horizon geometry of D5+D1 branes, the field theory describing string theory in the near horizon geometry of D1-branes is known in details: it is the SYM in two dimensions. However, our calculation did not rely on the detailed properties of the SYM action at all. What we did instead was to use the supergravity dual to compute a field theory quantity, the two-point function of the energy-momentum tensor. Then,

\footnote{For an attempt to go off criticality see [2].}
using rather general field theory arguments which are valid for any unitary field theory in two dimensions, we compute the entropy to find an agreement with the Bekenstein-Hawking formula. This agreement serves as a check of Maldacena conjecture for D1-branes [9].

It is interesting to note that in the supergravity region the entropy energy relation is similar to that of a gas of a free massless scalar field in (2+1) dimensions. In the extreme UV and IR, on the other hand, this theory behaves like a free gas in (1+1) dimensions (with a different number of field, since $c_{UV} > c_{IR}$). Amusingly, this general behavior is mimicked by the following simple statistical mechanical model. Consider $\hat{N}$ free fields in (2+1) dimensions which propagate on a “semi-lattice.” By “semi-lattice” we mean a chain of $\hat{N}$ continuous strings with lattice spacing $a$. So the size of the system is $L_1L_2$ where $L_1$ is an IR cutoff which we can take to infinity and $L_2 = \hat{N}a$. The dispersion relation for this system is

$$\omega^2 = k_1^2 + \frac{4}{a^2}\sin^2(k_2a/2), \quad k_2 = n\pi/L_2, \quad n = 0...\hat{N}. \quad (16)$$

In the IR where the thermal wave-length is larger than $L_2$, the contributions from the extra dimension are negligible and we have the entropy of a free massless gas in two dimensions. In the UV where the thermal wave-length is smaller than $a$, $\omega$ can grow only due to $k_1$ as $k_2$ is restricted to a single Brillouin zone. Therefore, the entropy is that of a free massless gas in two dimensions where the number of massless fields is $NN'$ as can be read from eq.(16). To have an agreement with eq.(1) we set $\hat{N} = N$. In the intermediate region the system behaves like a gas of $N$ free particles in (2+1) dimensions. To match with the number of degrees of freedom of 1+1 dimensional SYM in the deep UV and IR, we set $a = \sqrt{\frac{N}{g_{YM}}}$.  

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*The fields are not exactly massless as can be seen from eq.(16). Rather their masses are bounded by $1/a$ which is much smaller then the temperature so their contribution to the entropy at large temperature is similar to the massless fields contributions.*
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