Nonrelativistic Shifted-\(l\) Expansion Technique for Three- and Two-Dimensional Schrödinger Equation

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Abstract The shifted-\(l\) expansion techniques (SLET) has been developed to get eigenvalues of Schrödinger equation in three (3D) and two dimensions (2D). SLET simply consists of \(\frac{1}{I}\) as a perturbation parameter, where \(I = l - \beta\). \(\beta\) is a suitable shift, \(l\) is the angular-momentum quantum number for the 3D-case, \(l = |m|\) for the 2D-case, and \(m\) is the magnetic quantum number. Unlike the shifted large-\(N\) expansion theory (SLNT), SLET seems to be applicable to a wider number of problems of significant interest in physics.

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I. Introduction

A large number of problems of physics require solutions of Schrödinger, Klein-Gordon and Dirac wave equations. Exact solutions of these equations exist only for a handful of potentials. This fact has culminated into the development of many fascinating approximation techniques. However, highly accurate analytical solutions of these equations for potentials of arbitrary coupling constants are hard to find.

In this work we shall consider the Schrödinger equation for spherically (three-dimensional, 3D) and cylindrically (two-dimensional, 2D) symmetric potentials. Among the possible solutions of this wave equation in such symmetries exist, for example, the coupling constant perturbation theory, the variational method, the WKB approximation, the two-step procedure,\(^1\) the operator method,\(^2\) the \(1/N\) expansion technique,\(^3\) the rational fraction approach,\(^4\) the direct numerical integration,\(^5,7\) the supersymmetric quantum-mechanics-based methods,\(^8\) the Hill determinant method,\(^9\) the multiple-step recursion relation procedure,\(^10\) and the shifted large-\(N\) (or \(1/N\)) expansion technique (SLNT).\(^11-13\)

SLNT has surpassed the other approximations in its domain of applicability, effectively it puts no constraint on the potential or the quantum numbers involved. Moreover, its analytical expressions have yielded very accurate and fast converging results.\(^11-16\) SLNT has also shown that even if the results of physical interest are in two\(^{12}\) or three dimensions,\(^13,14\) it is advantageous to work in \(N\)-dimensions and use \(1/\vec{k}\) as a perturbation parameter. \(\vec{k} = N + 2l - a\), \(N\) being the number of spatial dimensions of interest, \(l\) the angular-quantum number, and \(a\) a suitable shift which has the meaning of additional degree of freedom and is responsible for speeding up the convergence of the resulting energy series. However, the scope of its applicability to relativistic problems, in particular, to atomic, molecular and quarkonium physics, has not been adequately explored. This is, most probably, because of the coupling nature of the Dirac equation that causes difficulties in inflating the dimensions of the mentioned equation.\(^14,15\) The Schrödinger equation for 3D hydrogenic state in a uniform electric field is yet another example of the many problems that cannot be solved using SLNT. Therefore, in spite of its success in the Schrödinger equation for spherically\(^{11}\) and cylindrically\(^{13}\) symmetric potentials of arbitrary coupling constants, the shifted \(1/N\) expansion has to be reformatted to widen its domain of applicability.

In this paper a new technique is introduced to solve the Schrödinger equation. What we shall call the shifted-\(l\) expansion technique (SLET) is a reformation of SLNT and can
be easily extended in a straightforward manner to solve the Dirac, the Klein–Gordon and any other wave equation that can be reduced to the form of the Schrödinger equation below, Eq. (1). SLET simply consists of $1/l$ as an expansion parameter, where $l = l - \beta$. $l$ is the angular-momentum quantum number for the spherically symmetric (3D) problems, $l = |m|$ for cylindrically symmetric (2D) problems, and $m$ is the magnetic quantum number. $\beta$ is a suitable shift which is mainly introduced to avoid the trivial case $l=0$. In addition, $\beta$ is chosen so that the next leading contribution to the energy eigenvalue series vanishes. This choice is physically motivated by requiring the agreement between the shifted-$l$ expansions and the exact analytical results for the harmonic oscillator and the Coulomb potentials both in 2D and 3D cases.\(^{11-13}\) We shall not adopt the classical limit, $l \to \infty$, to work out the leading term in the energy series, as Imbo and his co-workers\(^{[11]}\) have done. We shall fix $l$ and determine $r_0$ by minimizing the leading term of the energy eigenvalue series.\(^{[12-15]}\) Like SLNT, SLET is also a pseudoperturbative technique in the sense that it proposes a perturbation parameter that is not directly related to the coupling constant in the potential of interest.

In Sec. II we will discuss the method for spherically symmetric potentials of arbitrary coupling constants and give the resulting analytical expressions in such a way that allows the reader to use them without proceeding into their derivation. Inspired by the considerable attention that has been given to systems of reduced dimensions,\(^{12,17-21}\) we also discuss the method for cylindrically symmetric potentials of arbitrary coupling constants in Sec. III. In Sec. IV we shall show that the analytical results of SLET for the power-law and logarithmic potentials coincide with those of SLNT.\(^{[11]}\) In the same section we shall also demonstrate that the contribution of each term in the energy series, Eq. (6), of SLET is the same as the corresponding one of SLNT.\(^{[11]}\) In Sec. V we shall consider a 2D electron gas in the $x$-$y$ plane in the presence of a hydrogenic potential and a magnetic field in the $z$-direction. Therein, we compare our results with those of ZHU et al.\(^{[20]}\) Finally, we conclude and remark in Sec. VI.

II. SLET for the 3D Schrödinger Equation

To formulate SLET we start with the radial part of Schrödinger equation (in $\hbar = 2m = 1$ units)

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r)\right] \Psi(r) = E \Psi(r) ,$$  \hspace{1cm} (1)

where $V(r)$ is an arbitrary spherically symmetric potential. If the angular momentum quantum number $l$ is shifted through the relation $\tilde{l} = l - \beta$, equation (1) becomes

$$\left\{ -\frac{d^2}{dr^2} + \frac{[l^2 + (2\beta + 1)l + \beta(l + \beta)]}{r^2} + V(r)\right\} \Psi(r) = E \Psi(r) .$$  \hspace{1cm} (2)

It is convenient to shift the origin of coordinate by the definition\(^{[3,11]}\)

$$x = \frac{\tilde{l}^{1/2}(r - r_0)}{r_0} ,$$  \hspace{1cm} (3)

and to expand about $x = 0$ to obtain

$$\left( \frac{x}{\tilde{l}^{1/2}} + 1 \right)^{-2} = 1 - \frac{2}{\tilde{l}^{1/2}} + \frac{3x^2}{\tilde{l}} - \frac{4x^3}{\tilde{l}^{3/2}} + \cdots ,$$  \hspace{1cm} (4)

$$V(x(r)) = \frac{\tilde{p}^2}{Q} \left[ V(r_0) + V'(r_0) \frac{r_0 x}{\tilde{l}^{1/2}} + V''(r_0) \frac{r_0^2 x^2}{2\tilde{l}} + V'''(r_0) \frac{r_0^3 x^3}{6\tilde{l}^{3/2}} + \cdots \right] .$$  \hspace{1cm} (5)

We also propose the expansion of $E$ as\(^{[15]}\)

$$E = \frac{\tilde{p}^2}{Q} \left[ E_0 + \frac{E_1}{\tilde{l}} + \frac{E_2}{\til{l}^2} + \frac{E_3}{\til{l}^3} + \cdots \right] ,$$  \hspace{1cm} (6)

where $Q$ is a scale to be determined below. With the above expansions equation (2) becomes

$$\left\{ -\frac{d^2}{dx^2} + \left[ \tilde{l} + (2\beta + 1) + \frac{\beta(1 + \beta)}{\tilde{l}} \right] \left( 1 - \frac{2x}{\til{l}^{1/2}} + \frac{3x^2}{\til{l}} - \frac{4x^3}{\til{l}^{3/2}} + \cdots \right) + \frac{\tilde{p}^2}{Q} \times$$
\[
\left[ V(r_0) + V'(r_0) \frac{r_0x}{\hbar/2} + V''(r_0) \frac{r_0^2x^2}{2\hbar} + V'''(r_0) \frac{r_0^3x^3}{6\hbar^2} + \cdots \right] \Phi_n(x) = \xi_n \Phi_n(x),
\]

where
\[
\xi_n = \frac{r_0^3\mathcal{I}}{Q} \left[ E_0 + \frac{E_1}{\mathcal{I}} + \frac{E_2}{\mathcal{I}^2} + \frac{E_3}{\mathcal{I}^3} + \cdots \right].
\]

On the other hand, the Schrödinger equation for a one-dimensional anharmonic oscillator has been discussed in detail by Imbo et al., \[11\] which in turn implies that
\[
\xi_n = i\left(1 + \frac{r_0^2V(r_0)}{Q}\right) + \left[2\beta + 1 + \left(n_r + \frac{1}{2}\right)w\right] + \frac{1}{\mathcal{I}}\left[\beta(\beta + 1) + \alpha_1\right] + \frac{\alpha_2}{\mathcal{I}^2} + \cdots.
\]

Comparing Eq. (8) with Eq. (9) and equating terms of same order in \(\mathcal{I}\) one obtains
\[
E_0 = \frac{Q}{r_0^2} + V(r_0),
\]
\[
E_1 = \frac{Q}{r_0^2} \left[2\beta + 1 + \left(n_r + \frac{1}{2}\right)w\right],
\]
\[
E_2 = \frac{Q}{r_0^2} \left[\beta(\beta + 1) + \alpha_1\right],
\]
\[
E_3 = \frac{Q}{r_0^2} \alpha_2,
\]

where \(\alpha_1\) and \(\alpha_2\) are given in the Appendix. Here, \(r_0\) is chosen to minimize \(E_0,[13,14,18]\) i.e.
\[
\frac{dE_0}{dr_0} = 0 \text{ and } \frac{d^2E_0}{dr_0^2} > 0,
\]
which gives, with \(\mathcal{I} = \sqrt{Q},\)
\[
\mathcal{I} - \beta = \sqrt{\frac{r_0^3V'(r_0)}{2}},
\]
where \(\beta\) is fixed by the requirement that \(E_1\) vanishes, then
\[
\beta = -\frac{2 + (2n_r + 1)w}{4}
\]
with
\[
w = 2\sqrt{3 + \frac{r_0V''(r_0)}{V'(r_0)}}.
\]

Equation (15) is explicit in \(r_0\) and is the same equation as that given by Imbo et al.\[11\] to solve for \(r_0\).

Finally, the eigenvalues are calculated by
\[
E = E_0 + \frac{1}{r_0^2} \left[\beta(1 + \beta) + \alpha_1\right] + \frac{\alpha_2}{\mathcal{I}^2},
\]

where \(n_r\) is the radial quantum number, and \(\mathcal{I}\) is as much as \(\hbar/2\) of Imbo et al.\[11\]

### III. SLET for the 2D Schrödinger Equation

The radial Schrödinger equation for cylindrically symmetric potentials can be reduced (in \(\hbar = 2m = 1\) units) to the form
\[
\left[ -\frac{d^2}{d\rho^2} + \frac{4l^2 - 1}{4\rho^2} + V(\rho) \right] \Phi(\rho) = E\Phi(\rho),
\]
where \(\rho^2 = x^2 + y^2, l = |m|,\) and \(m\) is the magnetic quantum number. If \(l\) is shifted through the relation \(\mathcal{I} = l - \beta,\) equation (19) becomes
\[
\left[ -\frac{d^2}{d\rho^2} + \frac{(l + \beta - 1/2)(l + \beta + 1/2)}{\rho^2} + V(\rho) \right] \Phi(\rho) = E\Phi(\rho).
\]
Following the procedure described in Sec. II, Eqs (3) ~ (6), one obtains

\[
\left\{ -\frac{d^2}{dx^2} + \left[ I + 2\beta + \frac{(\beta^2 - 1/4)}{I} \right] \left( 1 - \frac{2x}{I^{1/2}} + \frac{3x^2}{I} - \frac{4x^3}{I^{3/2}} + \cdots \right) + \frac{\rho_0^2}{Q} \left[ V(\rho_0) + V'(\rho_0)\frac{\rho_0 x}{I^{1/2}} + V''(\rho_0)\frac{\rho_0^2 x^2}{2I} + \cdots \right] \right\} \chi_{n_\nu}(x) = \lambda_{n_\nu} \chi_{n_\nu}(x),
\]

where

\[
\lambda_{n_\nu} = \frac{\rho_0^2}{Q} \left[ E_0 + \frac{E_1}{I} + \frac{E_2}{I^2} + \frac{E_3}{I^3} + \cdots \right].
\]

Also \( \lambda_{n_\nu} \) can be written as\[11\]

\[
\lambda_{n_\nu} = I \left( 1 + \frac{\rho_0^2 V(\rho_0)}{Q} \right) + \left[ 2\beta + \left( n_\rho + \frac{1}{2} \right) w \right] + \frac{1}{I} \left[ \beta^2 - \frac{1}{4} \right] + \alpha_1 + \frac{\alpha_2}{I^2} + \cdots .
\]

Comparing Eq. (23) with Eq. (22) we obtain

\[ E_0 = \frac{Q}{\rho_0^2} + V(\rho_0), \]
\[ E_1 = \frac{Q}{\rho_0^2} \left[ 2\beta + \left( n_\rho + \frac{1}{2} \right) w \right], \]
\[ E_2 = \frac{Q}{\rho_0^2} \left[ \left( \beta^2 - \frac{1}{4} \right) + \alpha_1 \right], \]
\[ E_3 = \frac{Q}{\rho_0^2} \alpha_2, \]

where \( \rho_0 \) is given through the condition in Eq. (14) yielding

\[ I - \beta = \sqrt{\frac{\rho_0^2 V'(\rho_0)}{2}}, \]
\[ \beta = -\frac{1}{2} \left( n_\rho + \frac{1}{2} \right) w, \]
\[ w = 2 \sqrt{3 + \frac{\rho_0^2 V''(\rho_0)}{V'(\rho_0)}}, \]

and \( n_\rho \) is the radial quantum number, \( n_\rho = 0, 1, 2, \ldots \) Finally, the eigenvalues are

\[ E = E_0 + \frac{1}{\rho_0^2} \left[ \left( \beta^2 - \frac{1}{4} \right) + \alpha_1 \right] + \frac{\alpha_2}{I^2}, \]

where \( E_0 \) is given in Eq. (24).

IV. Applications to the 3D Case

For the sake of comparison with the results of the shifted large-\( N \) expansion technique SLNT\[11\] the power-law, \( V(r) = Ar^\nu \), and the logarithmic, \( V(r) = A\ln(r/b) \), potentials are considered. The above potentials have, however, been used in heavy quarkonium spectroscopy\[11,15-17\] Our results are given in such a way that the comparison with SLNT is made clear.

4.1. Power-Law Potential \( V(r) = Ar^\nu \)

For the power-law potential, \( V(r) = Ar^\nu \), the general formalism developed above leads to

\[ \tau_0 = \left[ \frac{2\rho_0^2}{A\nu} \right]^{1/(\nu+2)}, \]

where

\[ w = 2\sqrt{\nu + 2}, \]
\[ I = \frac{[(2n_\tau + 1)\sqrt{\nu + 2} + (2I + 1)]}{2}, \]
\[ E_0 = [2A\nu]^2/(\nu+2) \left( \frac{4l}{8\nu} \right) (\nu + 2) \left[ 2l \right]^{(\nu-2)/(\nu+2)} \]  
(35)

\[ E_2 = \left[ -2A\nu^2/(\nu+2) \right]^{2\nu+1}(\nu + 2) \left[ 2l \right]^{(\nu-2)/(\nu+2)} \]  
(36)

\[ \frac{E_3}{l^3} = 2l^{(\nu-2)/(\nu+2)} \left[ 2A\nu^2/(\nu+2) \right] \left[ \frac{2(\nu+1)(\nu-2)}{123(2l)^2\sqrt{\nu+2}} \right] \times \left[ (\nu+1)(\nu-2) + (7\nu^2 - 31\nu - 62)n_r + (5\nu^2 - 29\nu - 58)(3n_r^2 + 2n_r) \right]. \]  
(37)

It is worth mentioning that the exact energy eigenvalues for the harmonic oscillator, \( B^2r^2/4 \), and the Coulomb, \(-2/r\), potentials are obtained from the leading term, \( E_0 \), in Eq. (6) as \( E = B(2n_r + l + 3/2) \) and \( E = -1/(n_r + l + 1)^2 \), respectively, where higher-order terms vanish identically.

4.2. Logarithmic Potential \( V(r) = A \ln(r/b) \)

For the logarithmic potential, \( V(r) = A \ln(r/b) \), one may simply obtain the following results

\[ r_0 = l\sqrt{\frac{2}{A}}, \]  
(38)

where

\[ w = 2\sqrt{2}, \]  
(39)

\[ l = \left[ (2l + 1) + (2n_r + 1)\sqrt{2} \right] \]  
(40)

\[ E_0 = A \left[ \ln \left( \frac{l}{b\sqrt{A/2}} \right) + \frac{1}{2} \right], \]  
(41)

\[ \frac{E_2}{l^2} = \frac{A}{72l^2} (6n_r^2 + 6n_r + 1), \]  
(42)

and

\[ \frac{E_3}{l^3} = \frac{A}{864l^3}\sqrt{2} (58n_r^3 + 87n_r^2 + 31n_r + 1). \]  
(43)

For both of the above potentials, it should be pointed out that each term in Eq. (6) has the same contribution as that of the corresponding one in SLNT.\(^\text{[11]}\) Mathematically speaking, \( (E_0)_{\text{SLNT}} = (E_0)_{\text{SLNT}}, (E_2/l^2)_{\text{SLNT}} = (E_2/l^2)_{\text{SLNT}} \) and \( (E_3/l^3)_{\text{SLNT}} = (E_3/l^3)_{\text{SLNT}}. \)

As found previously,\(^\text{[11]}\) equations (22) \( \sim \) (24) along with Eq. (6) yield remarkably good results even for large values of \( n_r \). It can easily be checked that the rate of convergence of the three terms in Eq. (6) is approximately the same as \( n_r \to \infty \) as it is for \( n_r \to 0 \). So, if the results for small \( n_r \) are accurate, one expects roughly similar accuracy for all \( n_r \). For more details on the accuracy of the above predictions, Eqs (34) \( \sim \) (37) for the power law and Eqs (40) \( \sim \) (43) for the logarithmic potentials, the reader may refer to Imbo et al.\(^\text{[11]}\)

V. Application to the 2D Case

In this section we shall consider the potential that describes a 2D hydrogenic donor in the presence of a magnetic field of arbitrary strength applied perpendicular to the 2D plane. If we adopt the symmetric gauge \( A = (B/2)(-y, x, 0) \), we can express this potential as (in cylindrical polar coordinates)\(^\text{[12,20,21]}\)

\[ V(\rho) = -\frac{2}{\rho} + m\gamma + \frac{\gamma^2 \rho^2}{4}, \]  
(44)

where \( m \) is the magnetic quantum number and the magnetic field strength is given through the parameter \( \gamma, \gamma \sim B. \) The units of energy and length are the effective Rydberg and the effective Bohr radius, respectively. For more details on the potential in Eq. (44) and the parameters involved in it, one may refer to Refs [12], [17]–[21].
Considering Eq. (31) along with Eq. (24) and Eqs (28) ~ (30), we have obtained the well-known limiting values of the energies of such a system at the zero and high-magnetic-field limits\cite{23} as

\[ E_{\text{donor}} = -(n_p + |m| + 1)^{-2} \]  

(45)

and

\[ E_{\text{Landau}} = \gamma(2n_p + |m| + m + 1), \]  

(46)

respectively. Herein, we shall report that \( E_{\text{donor}} \) and \( E_{\text{Landau}} \) are obtained by the leading term \( E_0 \) of Eq. (31), where higher-order terms have vanished identically, i.e., \( E_2 = E_3 = 0 \).

For the donor state in an arbitrary magnetic field, we have numerically solved Eq. (28) through Eqs (29) ~ (30) and Eq. (24) to find these states by Eq. (31).

In Fig. 1, our results (curve of long dashes marked with solid circles) for the 1s state show excellent agreement with the other results in the weak-field regime. In the strong-field regime our results fall in between the results of the perturbation treatment\cite{24} (curve of small dashes) and the direct numerical integration\cite{23} (solid curve) on which ZHU et al.\cite{20} have located their predictions (solid circles). Our results are, however, unique in that they tend to approach the strong- and weak-field perturbation theory results. The perturbation theory coupling constants were appropriately defined in these regimes.\cite{24} Likewise, we believe, it should be the tendency of the results of any approximation technique.

Figure 2 shows the ground 1s and the 2p\textsubscript{−} states being weakly affected by the magnetic field since the Coulomb interaction dominates over the magnetic interaction for low-lying states. The higher excited states, on the other hand, are more weakly bound and in this case the parabolic quantum well, which is formed by the magnetic field, determines the energy spectrum. Our results (solid curves marked with solid circles) for the 1s, 2p\textsubscript{−} and 2p\textsubscript{+} agree with those of the series expansion calculations\cite{20} (solid curves). Whereas, the results for
higher excited states converge more rapidly to Landau levels (curve of small dashes) than those of ZHU et al., especially in the strong-field regime wherein the parabolic quantum well is the dominating interaction that determines the energy spectrum.

VI. Conclusions and Remarks

In this paper, the shifted-\( l \) expansion technique (SLET) has been developed to solve 3D and 2D Schrödinger equations. The development is a reformation of the shifted large-\( N \) expansion technique (SLNT) that widens its domain of applicability.

For the spherically symmetric (3D) power law and the logarithmic potentials, both SLET and SLNT have yielded the same analytical results. The observations
\[
\begin{align*}
(E_0)_{\text{SLET}} &= (E_0)_{\text{SLNT}}, \\
(E_2/k^2)_{\text{SLET}} &= (E_2/k^2)_{\text{SLNT}}, \\
(E_3/k^3)_{\text{SLET}} &= (E_3/k^3)_{\text{SLNT}}
\end{align*}
\]
are significant for the conclusion that SLET is indeed a reformation of SLNT. The accuracies and the speeds of convergence of both techniques are the same, at least for the power law and the logarithmic potentials.

For cylindrically symmetric (2D) potentials, we have considered a 2D Coulomb field in the presence of a magnetic field of arbitrary strength, Eq. (44). SLET results have appeared to be fast converging in the sense that the dominating contribution to the energy series, Eq. (6), is that of the leading term \( E_0 \). The results were compared with those of the perturbation, the direct numerical integration, and the series expansion methods according to whichever was available. Figures 1 and 2 show that SLET results seem more appropriate than the others. Figure 1 shows that they approach the perturbation theory results at weak- and strong-field limits, and figure 2 shows that they approach the Landau levels as one goes to higher-excited states where the magnetic interaction is more effective in determining the energy spectrum.

In the near absence of highly accurate analytical approximation methods to solve the Schrödinger equation, even for simple cases such as a 3D hydrogenic state in a uniform electric field (to be investigated in the near future), SLET provides highly accurate analytical expressions. Moreover, the difficulties associated with the application of SLNT to Dirac equation should vanish for SLET. To appear elsewhere, we have applied SLET to Dirac, and Klein–Gordon wave equations. Finally, we should like to point out that the energy states of excitons in a harmonic quantum dot and shallow donors and heavy hole excitons in quantum well in the presence of magnetic field can be correctly obtained by SLET and compared with others.

Appendix

Although some of the following definitions can be found in some other references we would like to repeat them so as to make this article self-contained. The definitions of \( \alpha_1 \) and \( \alpha_2 \) and the parameters involved are as follows:

\[
\begin{align*}
\alpha_1 &= [(1 + 2n_r)e_2 + 3(1 + 2n_r + 2n_r^2)e_4] - w^{-1}[e_1^2 + 6(1 + 2n_r)e_1e_3 + (11 + 30n_r + 30n_r^2)e_3^2], \\
\alpha_2 &= (1 + 2n_r)d_2 + 3(1 + 2n_r + 2n_r^2)d_4 + 5(3 + 8n_r + 6n_r^2 + 4n_r^3)d_6 - w^{-1}[(1 + 2n_r)e_2^2 + 12(1 + 2n_r + 2n_r^2)e_2e_4 + 2e_1d_1 + \\
&2(21 + 59n_r + 51n_r^2 + 34n_r^3)e_4^2 + 6(1 + 2n_r)e_1d_3 + \\
&30(1 + 2n_r + 2n_r^2)e_1d_5 + 6(1 + 2n_r)e_3d_1 + \\
&2(11 + 30n_r + 30n_r^2)e_3d_3 + 10(13 + 40n_r + 42n_r^2 + 28n_r^3)e_3d_5] + \\
&w^{-2}[4e_1^2e_2 + 36(1 + 2n_r)e_1e_2e_3 + 8(11 + 30n_r + 30n_r^2)e_2e_3^2 + \\
&24(1 + n_r)e_1^2e_4 + 8(31 + 78n_r + 78n_r^2)e_1e_3e_4 + \\
&12(57 + 189n_r + 225n_r^2 + 150n_r^3)e_3^2e_4] - \\
&w^{-3}[8e_1^3e_3 + 108(1 + 2n_r)e_1^2e_3^2 + 48(11 + 30n_r + 30n_r^2)e_1e_3^3 + \\
&30(31 + 109n_r + 141n_r^2 + 94n_r^3)e_3^3],
\end{align*}
\]
with
\[ e_j = \frac{\epsilon_j}{w^{j/2}} \quad \text{and} \quad d_i = \frac{\delta_i}{w^{i/2}}, \] (A3)

where \( j = 1, 2, 3, 4, i = 1, 2, 3, 4, 5, 6. \)

\( \epsilon_j \) and \( \delta_i \) for 3D case are given as
\[ \begin{align*}
\epsilon_1 &= -2(2\beta + 1), \\
\epsilon_2 &= 3(2\beta + 1), \\
\epsilon_3 &= -4 + \rho_0^5 V'''(r_0)/6Q, \\
\epsilon_4 &= 5 + \rho_0^5 V'''(r_0)/24Q, \\
\delta_1 &= -2\beta(1 + \beta), \\
\delta_2 &= 3\beta(1 + \beta), \\
\delta_3 &= -4(2\beta + 1), \\
\delta_4 &= 5(2\beta + 1), \\
\delta_5 &= -6 + \rho_0^7 V'''''(r_0)/120Q, \\
\delta_6 &= 7 + \rho_0^7 V'''''(r_0)/720Q.
\end{align*} \] (A4)

For the 2D case \( \epsilon_j \) and \( \delta_i \) are given by
\[ \begin{align*}
\epsilon_1 &= -4\beta, \\
\epsilon_2 &= 6\beta, \\
\epsilon_3 &= -4 + \rho_0^5 V'''(\rho_0)/6Q, \\
\epsilon_4 &= 5 + \rho_0^5 V'''(\rho_0)/24Q, \\
\delta_1 &= -2(\beta^2 - \frac{1}{4}), \\
\delta_2 &= 3(\beta^2 - \frac{1}{4}), \\
\delta_3 &= -8\beta, \\
\delta_4 &= 10\beta, \\
\delta_5 &= -6 + \rho_0^7 V'''''(\rho_0)/120Q, \\
\delta_6 &= 7 + \rho_0^7 V'''''(\rho_0)/720Q.
\end{align*} \] (A9)

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