Uehling potential and lowest-order corrections on vacuum polarization to the cross sections of some QED processes

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Abstract Properties and different representations of the Uehling potential are investigated. Based on these properties and by using our formulas for the Fourier transform of the Uehling potential we have developed the new analytical, log-arithmically closed and physically transparent procedure which can be used to evaluate the lowest-order vacuum polarization correction to the cross sections of a number of QED processes, including the Mott electron scattering, bremsstrahlung, creation and/or annihilation of the \((e^-, e^+)\) pair in the field of a heavy Coulomb center, e.g., atomic nucleus.

1 Introduction

In this short communication we investigate radiative corrections to the Coulomb law. As is well known (see, e.g., [1–3]) these corrections can explicitly be described as a result of vacuum polarization around a point electric charge. In the lowest-order approximation the vacuum polarization, which always arises around an arbitrary (point) electrical charge, is described by the Uehling potential [4] (see, also, [1] and [2]). For atomic and Coulomb few- and many-body systems the Uehling potential [4] generates a small correction on vacuum polarization which must be added to the leading contribution from the original Coulomb potential. The sum of the original Coulomb and Uehling potentials is represented in the following integral form [1]

\[
V(r) = -\frac{Qe}{r} + U(r) = -\frac{Qe}{r} \left[ 1 + \frac{2\alpha}{3r} \right] \int_1^{\infty} d\varsigma \left( 1 + \frac{1}{2\varsigma^2} \right) \frac{\sqrt{\varsigma^2 - 1}}{\varsigma^2} \exp(-2m_e r \varsigma) \]

(1)

where \(-Qe\) is the electric charge of the central atomic nucleus, \(e\) is the charge of the electron \(e\), where \(e < 0\), \(\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}\) is the fine-structure constant, while \(m_e\) is the electron mass at rest and \(r\) is the electron-nuclear distance. In this formula and everywhere below in this study we apply the relativistic units, where \(\hbar = 1\) and \(c = 1\) [5]. In these units the Uehling potential \(U(r)\) from Eq. (1) is written in the form

\[
U(r) = -\frac{2Qe\alpha}{3\pi r} \int_1^{\infty} \left( 1 + \frac{1}{2\varsigma^2} \right) \frac{\sqrt{\varsigma^2 - 1}}{\varsigma^2} \exp(-2m_e r \varsigma) d\varsigma
\]

(2)

Below we shall assume that such a vacuum polarization is produced and described by the Uehling potential, Eq. (2), only. In other words, in the course of our analysis all higher-order corrections for vacuum polarization will be ignored. Applications of the Uehling potential to various atomic and muon-atomic systems can be found, e.g., in [6,7] and references mentioned in these papers. Here we will not discuss such applications, since they are quite different from the goals of this work.

The main goal of this study is to analyze the basic properties of Uehling potential, Eq. (2). We also derive a few different representations of the Uehling potential, Eq. (2). Then, by using our formula(s) for the Fourier transform of the Uehling potential we have developed the new transparent and covariant procedure which allows one to evaluate the lowest-order correction on vacuum polarization to the cross sections of a number of QED processes. As follows from the results of our study, the Uehling potential, Eq. (2), can be included in the set of fundamental and covariant QED rules which is used for analytical calculations of the \(S_{fi}\)-matrix elements from the very beginning.

2 Analytical formulas for the Uehling potential

In numerous books and textbooks on Quantum Electrodynamics (see, e.g., [1–3] and references therein) one usually...
finds the following statement (or one of its variations): the ‘closed analytical expression for the Uehling potential, which is correct for arbitrary electron-nuclear distances \( r \), does not exist’. A different form of the same statement is ‘analytical formula for the Uehling potential can be found only for very small \( r \) and/or for very large \( r \)’ (see, e.g., [3]). For the first time this statement appeared in some QED book(s) published in early 1950’s. In fact, quite a few of such books were published immediately after appearance of the fundamental papers by Feynman [9,10] and Dyson [11]. Since then this statement has not been verified, and now it simply wanders from one QED-book into another. In this Section we want to show that this statement is wrong and misleading.

Indeed, let us derive a few simple and explicit analytical formulas for the Uehling potential \( U(r) \) from Eq. (2). First, we introduce the new variable \( \varsigma = \cosh \chi \) in Eq. (2), where \( 0 \leq \chi < +\infty \), and obtain the following expression

\[
U(r) = -\frac{2Q\alpha}{3\pi r} \int_0^\infty \left( 1 - \frac{1}{\cosh^2 \chi} \right) \frac{1}{2\cosh^2 \chi} \exp(-2m_e r \cosh \chi) d\chi
\]

\[
= -\frac{2Q}{3\pi r} \int_0^\infty \left( 1 - \frac{1}{\cosh^2 \chi} \right) \frac{1}{2\cosh^2 \chi} \exp(-2m_e r \cosh \chi) d\chi
\]

\[
= -\frac{2Q}{3\pi r} \int_0^\infty \exp(-2m_e r \cosh \chi) d\chi - \frac{1}{2} \int_0^\infty \frac{\exp(-2m_e r \cosh \chi) d\chi}{\cosh^2 \chi}
\]

\[
= -\frac{1}{2} \int_0^\infty \frac{\exp(-2m_e r \cosh \chi) d\chi}{\cosh^2 \chi}
\]

where the first integral is the modified Bessel function \( K_0(2m_e r) \) of the second kind, which is also called the Macdonald function and/or Hankel function of purely imaginary argument [12,13] and [14], while the second and third integrals in Eq. (3) are the multiple integrals of the \( K_0(2m_e r) \) function. Briefly, we can write (see, e.g., Chapter 11 in [12])

\[
K_0(2m_e r) = K_0(2m_e r) = \sum_{k=0}^{\infty} \left[ \psi(k + 1) - \ln z + \ln 2 \right] \frac{z^{2k}}{2^{2k} (k!)^2}
\]

\[
= \sum_{k=0}^{\infty} \psi(k + 1) \frac{z^{2k}}{2^{2k} (k!)^2} - \ln \left( \frac{z}{2} \right) \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k} (k!)^2}
\]

\[
K_0(z) = \sum_{k=0}^{\infty} \left[ \psi(k + 1) - \ln z + \ln 2 \right] \frac{z^{2k}}{2^{2k} (k!)^2}
\]

\[
= \sum_{k=0}^{\infty} \psi(k + 1) \frac{z^{2k}}{2^{2k} (k!)^2} - \ln \left( \frac{z}{2} \right) \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k} (k!)^2}
\]

\[
= \sum_{k=0}^{\infty} \psi(k + 1) \frac{z^{2k}}{2^{2k} (k!)^2}
\]

\[
= \ln z - \sum_{n=0}^{\infty} \left[ 1 \frac{1}{x + n} - \ln \left( 1 + \frac{1}{x + n} \right) \right]
\]

is widely used to determine numerical values of the \( \psi(x) \) function.

Based on Eqs. (3)–(5) one can write the following explicit formula for the Uehling potential

\[
U(r) = -\frac{Q\alpha}{3\pi r} \left[ 2K_0(2m_e r) - Ki_2(2m_e r) - Ki_4(2m_e r) \right]
\]

\[
(8)
\]

This is the first analytical formula for the Uehling potential [15]. Advantage of this simple formula is obvious, since all coefficients in this formula are the numerical constants. However, by using the well known recursive relation [12] for the \( Ki_n(z) \) functions (or integrals)

\[
nKi_{n+1}(z) = -zKi_n(z) + (n - 1)Ki_{n-1}(z) + zKi_{n-2}(z)
\]

\[
(9)
\]

for \( n = 2, 3, 4, \ldots \), one can reduce the formula, Eq. (8), to a different form

\[
U(r) = -\frac{Q\alpha}{18\pi r} \left[ (12 + z^2)Ki_0(z) - zKi_1(z) \right.
\]

\[
- (10 + z^2)Ki_2(z)
\]

\[
= -\frac{Q\alpha}{18\pi r} \left[ (1 + q(z))Ki_0(z) - zKi_1(z) \right.
\]

\[
+ (1 - q(z))Ki_2(z)
\]

\[
(10)
\]

where \( z = 2m_e r \) and \( q(z) = 11 + z^2 \) is the quadratic function of \( z \). This is the second analytical formula known [15] for the Uehling potential. This formula contains only the two lowest multiple integrals of the \( K_0(2m_e r) \) function, but coefficients in this formula are the two quadratic functions of \( z \) and one linear function of \( z = 2m_e r \). By using more formulas known for the modified Bessel functions one can derive more analytical formulas for the Uehling potential \( U(r) \).
3 Fourier transform of the Uehling potential

Let us obtain the explicit formulas for the Fourier transform of the Uehling potential [16]. This problem was considered earlier in [17] and [18]. First, we derive the explicit formula for the three-dimensional Fourier transform [16] of an arbitrary Yukawa-type potential (∝ exp(−2mₑξr))

\[
\mathcal{F}\left[\frac{1}{r}\right] = \frac{4\pi}{q} \int_{0}^{\infty} \frac{dr}{r} \sin(qr) \exp(-2mₑξr) = \frac{\pi}{mₑ^2(ξ^2 + b^2)}
\]

(11)

where \( b = \frac{q}{2m_e} \) (or \( b = \frac{q^2}{4m_e^2} \)). Here we applied the formula (5) in Eq. (12) for which is

\[
\int_{0}^{+\infty} x^{\mu-1} \exp(-\beta x) \sin(\delta x) dx = \frac{\Gamma(\mu)}{(\beta^2 + \delta^2)^{\frac{\mu}{2}}} \sin\left(\mu \arctan \frac{\delta}{\beta}\right)
\]

For \( \mu = 1 \) one finds Eq. (12)

\[
\int_{0}^{+\infty} \exp(-\beta x) \sin(\delta x) dx = \frac{\delta}{\beta^2 + \delta^2}
\]

(13)

By using the formula, Eq. (11), we reduce the problem of analytical computation of the Fourier transform of the Uehling potential \( \tilde{U}(q) \) to the following one-dimensional integral

\[
\tilde{U}(q) = -\frac{2Qe a}{3m_e^2} \int_{1}^{+\infty} dt \left(1 + \frac{1}{2\pi^2}\right) \frac{\sqrt{t^2 - 1}}{t^2(t^2 + b^2)}
\]

(14)

where \( b^2 = \frac{q^2}{4m_e^2} \). The explicit form for this integral is derived by introducing the new variable \( v = \frac{\sqrt{t^2 - 1}}{t} \). Now, the integral from Eq. (14) is reduced to the following form

\[
\int_{1}^{+\infty} dt \left(1 + \frac{1}{2\pi^2}\right) \frac{\sqrt{t^2 - 1}}{t^2(t^2 + b^2)} = \frac{1}{2\pi^2} \int_{0}^{1} dv \frac{(3 - v^2)v^2}{d^2 - v^2}
\]

(15)

where \( d^2 = \frac{t^2 + 1}{b^2} = 1 + \frac{1}{b^2} = \frac{q^2 + 4m_e^2}{4m_e^2} \). After a few additional steps we obtain the following formula for the Fourier transform of the Uehling potential

\[
\tilde{U}(q) = -\frac{4Qe a}{3q^2} \left\{ \frac{5}{3} + \frac{4m_e^2}{q^2} + \left[1 - \frac{1}{2}\left(\frac{4m_e^2}{q^2}\right)\right] \ln\left(\frac{\sqrt{q^2 + 4m_e^2} + 2m_e}{\sqrt{q^2 + 4m_e^2} - 2m_e}\right) \right\}
\]

(16)

which exactly coincides with the result obtained in [17] (see also [18]). The sum of this Fourier transform of the Uehling potential with the original Coulomb potential \( W(q) \) is

\[
W(q) = -\frac{4\pi Qe}{q^2} + \tilde{U}(q)
\]

\[
= -\frac{4\pi Qe}{q^2} \left\{1 + \frac{\alpha}{3\pi} \left[-\frac{5}{3} + \frac{4m_e^2}{q^2} + \left[1 - \frac{1}{2}\left(\frac{4m_e^2}{q^2}\right)\right] \ln\left(1 + \frac{1}{b^2} \ln\left(\frac{\sqrt{b^2 + 1} + 1}{\sqrt{b^2 + 1} - 1}\right)\right)\right] \right\}
\]

(17)

where \( b^2 = \frac{q^2}{4m_e^2} \). By using the inverse variable \( a^2 = \frac{1}{b^2} = \frac{4m_e^2}{q^2} \) one can re-write this formula in the form

\[
W(q) = -\frac{4\pi Qe}{q^2} \left\{1 + \frac{\alpha}{3\pi} \left[-\frac{5}{3} + a^2 + (\frac{1}{2}a^2 - 1)\sqrt{a^2 + 1} \ln\left(\frac{\sqrt{a^2 + 1} + a}{\sqrt{a^2 + 1} - a}\right)\right] \right\}
\]

(18)

where \( a = \frac{2m_e}{q} \). The expression \( W(q) \) is the Fourier transform of the potential \( V(r) \) (Coulomb + Uehling) which is defined by Eq. (1). The vector-parameter \( \mathbf{q} \) is called the momentum transfer to the heavy nucleus.

Now, by using Eq. (18) we can determine the lowest-order correction on vacuum polarization for a number of QED processes which include electron/positron interaction with an external Coulomb field. Indeed, let us assume that we have calculated the cross section \( \sigma \) of some QED process which includes a direct interaction with an external Coulomb field, or proceeds in the Coulomb field of a heavy (immovable), electrically charged center. In this case the corresponding \( S \)-matrix element includes the factor \( \frac{1}{q^2} \) which describes interaction between electron and/or positron and an external Coulomb field. The factor \( \frac{1}{q^2} \) is the Fourier transform of the interaction potential. At the next step we want to evaluate the lowest-order vacuum polarization correction to this cross section. If such a correction is represented by the Uehling potential, then to achieve our final goal we need to make the following substitution in our formula for the cross section and/or for the \( S \)-matrix element:

\[
\frac{1}{|\mathbf{q}|^2} \Rightarrow \frac{1}{|\mathbf{q}|^2} \left\{1 + \frac{\alpha}{3\pi} \left[-\frac{5}{3} + a^2 + (\frac{1}{2}a^2 - 1)\sqrt{a^2 + 1} \ln\left(\frac{\sqrt{a^2 + 1} + a}{\sqrt{a^2 + 1} - a}\right)\right] \right\}
\]

(19)
In other words, the Fourier transform of the pure Coulomb potential is replaced by the Fourier transform of some ‘realistic’ potential which is the sum of the Coulomb and Uehling potentials.

In general, the differential and total cross sections as well as the corresponding 5–matrix elements of actual QED processes are represented in the form of series upon even powers of $\frac{1}{|q|^n}$, i.e., for the total cross sections we can write the formula

$$\sigma = A_0 + \frac{A_1}{|q|^2} + \frac{A_2}{|q|^4} + \frac{A_3}{|q|^6} + \frac{A_4}{|q|^8} + \ldots$$

To obtain the lowest order correction on vacuum polarization to this cross section we need to make different substitutions for each term in Eq. (20). However, all these substitutions are described by the following general formula

$$\frac{1}{|q|^{2n}} \Rightarrow \frac{1}{|q|^{2n}} \left( 1 + \frac{2n \alpha}{3 \pi} \left[ - \frac{5}{3} + a^2 \left( \frac{1}{2} a^2 - 1 \right) \right] \frac{\sqrt{a^2 + 1} + a}{\sqrt{a^2 + 1} - a} \right)$$

for $n = 1, 2, 3, \ldots$ and $a = \frac{2m_e}{q}$. The arising new formula for the total cross section will automatically include the lowest-order vacuum polarization correction. A number of useful formulas is considered in the next Section.

### 4 Vacuum polarization correction to the Mott scattering formula

Let us assume that we have calculated the cross section of some QED process which include interaction of the electron and/or positron with the external Coulomb field. Our goal is to evaluate the lowest-order correction on vacuum polarization to this cross section. First, consider derivation of the Mott scattering formula for the elastic scattering of the relativistic electron at the immovable Coulomb center. The procedure itself is well described in [1] (see also [19]). This allows us to omit all computational details which are not crucial for our current purposes. Below, the notations $p_i$ and $p_f$ stand for the three-dimensional momenta of the initial and final electron, respectively. The differential cross section of the elastic scattering of an electron at external Coulomb potential is [1]

$$\frac{d\sigma}{d\Omega_f} = \frac{Q^2 \alpha^2}{2 |q|^4} \left( E^2 + m_e^2 \right) \left[ \frac{1}{4 \sin^2 \frac{\theta}{2}} \left( 1 - 2 \sin^2 \frac{\theta}{2} \right) \right]$$

where $E$ is the total energy of the initial and/or final electron, while $q = p_f - p_i$ is the momentum transferred to the heavy Coulomb center, e.g., atomic nucleus, and $\theta$ is the scattering angle. In the case of elastic scattering we can write $q^2 = |p|^2 = |p_f|^2 = |p_i|^2$. Now, by using the formulas, Eqs. (19)–(21), one finds

$$\frac{d\sigma}{d\Omega_f} = \frac{Q^2 \alpha^2}{2 |q|^4} \left( E^2 + m_e^2 \right) \left[ 1 + \frac{4\alpha}{3\pi} \left[ - \frac{5}{3} + a^2 \right] + (\frac{1}{2} a^2 - 1) \sqrt{a^2 + 1} + a^2 \right]$$

$$+ \frac{Q^2 \alpha^2}{8 |q|^2} \left( \frac{1}{\sin^2 \frac{\theta}{2}} - 2 \right) \left[ 1 + \frac{2\alpha}{3\pi} \left( \frac{5}{3} + a^2 \right) - (\frac{1}{2} a^2 - 1) \sqrt{a^2 + 1} + a^2 \right]$$

where $a = \frac{m_e}{|q| \sin \frac{\theta}{2}}$. This formula contains the lowest-order correction on the vacuum polarization which is described by the Uehling potential, Eq. (2). The angular dependence of the differential cross section $\frac{d\sigma}{d\Omega_f}$ is quite complicated, since the parameter $a$ depends upon $\theta$. Therefore, the differential cross section $\frac{d\sigma}{d\Omega_f}$, Eq. (23), has a slightly different angular dependence (or $\theta$–dependence), than the original formula, Eq. (22). In principle, the actual difference in the cross sections Eqs. (22) and (23) can be observed and measured in modern experiments.

Analogous formulas for the differential and total cross sections, which explicitly include the lowest-order correction on the vacuum polarization (or V.P., for short), can be derived for the bremsstrahlung, creation and/or annihilation of the $(e^-, e^+)$—pair in the field of a heavy atomic nucleus and some other processes. In all these cases we have to make the same substitutions, Eqs. (19)–(21), in the original formulas for the cross sections which can be found in the corresponding Chapters from [1] (see also [2] and [3]). This simple procedure allows one to obtain simultaneously the cross sections of various QED processes themselves and the lowest-order V.P. corrections to them. Here we do not to discuss these obvious transformations of the formulas for the cross sections. Instead, we want to show how to make the Uehling potential an integral part of the complete and covariant procedure developed for analytical computations of the second-order $S$–matrix elements. In other words, we want to include the Uehling potential, Eq. (2), in the set of fundamental and covariant QED rules (Feynmann rules) from the very beginning.

For simplicity, consider the process of pair creation in the field of atomic nucleus (see, e.g., [1–3]). To simplify discussion and notations even further, below we shall apply the Feynmann rules in the form presented in [1]. First of all, we note that the Uehling potential, Eq. (2), is a scalar and static (i.e., time-independent) potential. Therefore, the second-order $S$–matrix element in coordinate space takes the form

$$q^2 = |p_f - p_i|^2 = 2 |p|^2 \sin^2 \frac{\theta}{2}$$
\[ S_{fi} = -e^2 \sqrt{\frac{4\pi m_e^2}{E_+ E_- \omega V^3}} \int \int d^4x d^4y \left\{ u(p_-, s_-) e^{ip_- x} \left[ (-i\epsilon)(e^{-i k x} + e^{-i k y}) \right] t S_F(x - y) \right. \\
\left. + (-i\gamma^0) \left( A_0^{Coul}(y) + A_0^{Uehl}(y) \right) \right. \\
\left. + (-i\gamma^0) \left( A_0^{Coul}(x) + A_0^{Uehl}(x) \right) \right. \\
\left. + (-i\epsilon) \left( e^{-iky} + e^{-ikx} \right) v(p_+ s_+) \right\} \]

where all notations are the same as in [1], \( \epsilon \) is the imaginary unit, while the symbols \( A_0^{Coul}(x) \) and \( A_0^{Uehl}(x) \) are the scalar component of the Coulomb and Uehling potentials, respectively (other components of these potentials equal zero identically). The rest mass of the electron is designated here by \( m_e \) (not \( m_0 \)). In our notation the sum of the \( A_0^{Coul}(x) \) and \( A_0^{Uehl}(x) \) components simply coincides with the three-dimensional potential \( V(r) \) defined by Eq. (1). By performing the Fourier integration upon the two sets of four-dimensional \( x \) and \( y \) coordinates and by reducing the arising delta-functions one finds the following formula for the second-order \( S \)-matrix element in momentum space

\[ S_{fi} = \frac{8\pi^2 e^3}{|q|^2} \delta(E_+ + E_- - \omega) \sqrt{\frac{4\pi m_e^2}{E_+ E_- \omega V^3}} \left\{ 1 + \frac{\alpha}{3\pi} \left[ \frac{5}{3} + a^2 + \left( \frac{1}{2} a^2 - 1 \right) \sqrt{a^2 + 1} \right] \frac{\ln(\sqrt{a^2 + 1} + a)}{\sqrt{a^2 + 1} - a} \right. \\
\left. \times \left[ (-i\epsilon) \frac{t}{p_- - k - m} (-i\gamma^0) + (-i\gamma^0) \frac{t}{p_- + k + m} \right] \right\} \]

where \( q = p_+ + p_- - k \) in this problem and \( a = \frac{2m_e}{q} \). The following steps of the regular procedure used to determine the \( |S_{fi}|^2 \) value and the corresponding cross sections do not change.

5 Conclusion

We have considered the basic properties of the Uehling potential and derived its different analytical representations. It is shown that the Uehling potential can be written in the closed analytical form(s). In fact, there are quite a few of such analytical formulas which differ from each other by the relations known for the Bessel and modified Bessel functions. The explicit formulas for the Fourier transform of the Uehling potential is also derived in the simple analytical forms, Eqs. (16)–(18). By using these properties of the Uehling potentials and formulas for its Fourier transform we have developed the new procedure which allows one to determine simultaneously cross sections of a number of QED processes and the lowest-order V.P. corrections to them. Furthermore, the Uehling potential, Eq. (2), can directly be included in the set of fundamental and covariant QED rules which is used for analytical calculations of the \( S_{fi} \)-matrix elements at the first stage of the process.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This paper is a transparent and logically closed investigation. All formulas needed for the goal of this manuscript are either derived in the text, or taken from the mentioned reliable sources. There is no additional ‘hidden’ data which must be deposited anywhere.]

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