Newtonian and Post-Newtonian approximations of the $k = 0$
Friedmann Robertson Walker Cosmology.

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Abstract
In a previous paper [9], we derived a post-Newtonian approximation to cosmology which,
in contrast to former Newtonian and post-Newtonian cosmological theories, has a well-posed
initial value problem. In this paper, this new post-Newtonian theory is compared with the
fully general relativistic theory, in the context of the $k = 0$ Friedmann Robertson Walker
cosmologies. It is found that the post-Newtonian theory reproduces the results of its general
relativistic counterpart, whilst the Newtonian theory does not.

1 Introduction
The equations of general relativity are difficult to solve and often become tractable only in a
Newtonian context. Hence, it is desirable to use Newtonian theory rather than general relativ-
ity where possible [3]. By Newtonian, we mean that theory which is obtainable from general
relativity by taking the limit of weak gravitational fields and small velocities. Unfortunately,
when Newtonian theory is applied to cosmology, the boundary conditions become redundant,
and subsequently the Poisson equation no longer has a unique solution. Thus, the theory is no
longer well-posed and causality may be violated.

In a previous work [9], we explored the insufficiencies of the Newtonian theory in detail and
showed that in addition to the lack of well-posedness Newtonian theory is also incomplete. The
reasons for these insufficiencies are the following. In Newtonian theory, which is the expansion
of general relativity in terms of weak gravitational fields and small velocities up to order $c^{-2}$, all
that remains of the field equations is the Poisson equation, ie. the evolution equation and the
constraint equations have become one and the same. With no time evolution equation for the
Newtonian potential $\phi$, a unique solution for $\phi$ cannot be found, and so the theory is not well-
posed. Furthermore, the term $-\nabla \phi$, a term of order $c^{-4}$, does not feature in the field equations,
and hence the Bianchi identities cannot be fully obtained. Although completeness is obtainable
at order $c^{-4}$, it is necessary to go to order $c^{-6}$ and then to reformulate the field equations
as wavelike equations, in order to obtain a well-posed initial value problem. Throughout the
remainder of this paper we will call this approximation to general relativity “post-Newtonian
theory”.

In the following we will explore the Friedmann Robertson Walker (FRW) cosmology in the
context of the above Newtonian and post-Newtonian approximations. It has been shown that
the FRW models have a simple Newtonian interpretation [7, 1]. Howsoever, it will be outlined
here that the Newtonian theory cannot reproduce all the solutions of the fully general relativistic
theory: Variations in the equation of state do not cause changes in the solutions of the theory
since the pressure does not enter into the dynamics. This is reflected in the Raychaudhuri
equation being reproducible only for the case of vanishing pressure. Thus, the Newtonian theory
is only useful for the special case of dust. In the post-Newtonian theory, on the other hand, such
difficulties are overcome. The pressure does feature in the dynamics, and varying the equation of state produces correspondingly varying solutions, allowing for the full range of possibilities of its general relativistic counterpart. We therefore argue that the post-Newtonian theory should be used whenever we have non-zero pressure in the universe.

In Section 2 we write down what constitutes the general relativistic $k = 0$ FRW cosmology. In Section 3 we will introduce the Newtonian and post-Newtonian approximations and put the FRW metric into a form such that we may draw comparisons between these three theories. In Section 4 we explore the Newtonian approximation, and we will show that the theory may only be used in the special case of dust. In Section 5 we will find that the post-Newtonian theory is able to fully reproduce the results of the general relativistic case. We will discuss our results in Section 6.

2 The Fully General Relativistic $k = 0$ Friedmann Robertson Walker Cosmology

In the case of the flat FRW metric, the field equations are the Friedmann equation and the Raychaudhuri equation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8}{3}\pi G \rho,$$

and

$$3\frac{\ddot{R}}{R} = -4\pi G (\rho + 3pc^{-2}),$$

from which one may obtain the Bianchi identity

$$\dot{\rho} + 3(\rho + pc^{-2})\frac{\dot{R}}{R} = 0.$$ 

Assuming an equation of state of the form $p = wpc^2$ implies $\rho = \frac{3C}{8\pi G}R^{-3(1+w)}$, with $C$ a constant. Thus, from (1) it follows that

$$R(t) = \left(\frac{3}{2}(1 + w)C^{\frac{1}{2}}t\right)^{\frac{2}{3(1+w)}}.$$

Typical values for $w$ include:

- Matter ($w = 0$): $\rho \propto R^{-3}$, $R \propto t^{\frac{2}{3}}$,
- Radiation ($w = \frac{1}{3}$): $\rho \propto R^{-4}$, $R \propto t^{\frac{1}{3}}$,
- Stiff Matter ($w = 1$): $\rho \propto R^{-6}$, $R \propto t^{\frac{1}{3}}$. (3)
3 Newtonian and Post-Newtonian Approximations and the Friedmann Robertson Walker Metric

We now would like to study the FRW cosmology in the Newtonian and post-Newtonian approximations. Following a scheme similar to that of Weinberg’s [11], we adopt units in which the typical velocity has magnitude 1, i.e. $\beta \approx c^{-1}$, and assume a one parameter family of metrics $g_{\mu\nu}(x^i,c)$ for which there is a system of coordinates $(x^0,x^i)$ in which the components of the metric have the following asymptotic behaviour as $c \rightarrow \infty$:

\[ g_{00} = -1 - 2\phi c^{-2} - 2\alpha c^{-4} - 2\alpha' c^{-6} - 2\alpha'' c^{-8} \ldots , \]
\[ g_{0i} = \zeta_i c^{-3} + \zeta_i' c^{-5} + \zeta_i'' c^{-7} \ldots , \]
\[ g_{ij} = \delta_{ij} - 2\phi \delta_{ij} c^{-2} + \alpha_{ij} c^{-4} + \alpha_{ij}' c^{-6} + \alpha_{ij}'' c^{-8} \ldots . \]  
(4)

The usual Newtonian theory is obtained as the $O(c^{-2})$ limit of (4), while the complete Newtonian approximation is the $O(c^{-4})$ limit. Reformulating the field equations of the $O(c^{-6})$ limit as wavelike equations defines the post-Newtonian theory (see [9]).

We need to put the $k = 0$ FRW metric,

\[ ds^2 = -dx_0^2 + \Sigma_i R(t)^2 dx_i^2, \]

where $x_0 = ct$, into a form of which we can read off the potentials $\phi$, $\zeta_i$, $\alpha$ and $\alpha_{ij}$. To do so, we consider the following coordinate transformation

\[ x_0 = Tc + \tau c^{-1} + \tau' c^{-3}, \]
\[ x_i = R^{-1}X_i + \chi_i c^{-2} + \chi_i' c^{-4}, \]

with

\[ \tau = A(t) + A_{ij}(t)X_{ij}, \]
\[ \tau' = B_{ij}(t)X_{ij} + B_{ijkl}(t)X_{ijkl}, \]
\[ \chi_i = C_{ij}(t)X_j + C_{ijkl}(t)X_{ijkl}, \]
\[ \chi_i' = D_{ijkl}(t)X_{jkl} + D_{ijklmn}(t)X_{jklmn}, \]

where $X_{ij} \equiv X_i X_j$ and similar for $X_{ijk}$ etc. Throughout the remainder of the paper we assume: $\dot{A} \ll c^2$ to ensure convergence of our expansion in $c^{-2}$. $A(t)$, $A_{ij}(t)$, $B_{ij}(t)$, $B_{ijkl}(t)$, $C_{ij}(t)$, $C_{ijkl}(t)$, $D_{ijkl}(t)$ and $D_{ijklmn}(t)$ are arbitrary functions of time. In these coordinates the metric becomes

\[ ds^2 = c^2dT^2 \left[ -1 + c^{-2}(-2\dot{A} - 2\dot{A}_{ij}X_{ij} + \left( \frac{\dot{R}}{R} \right)^2 X_{ii}) ight. \]
\[ +c^{-4}(-2\dot{B}_{ij}X_{ij} - 2\dot{B}_{ijkl}X_{ijkl} + (\dot{A} + \dot{A}_{ij}X_{ij})^2 - 2\dot{R}\dot{C}_{ij}X_{ij} - 2\dot{R}\dot{C}_{ijkl}X_{ijkl} \]
\[ -2 \left( \frac{\dot{R}}{R} \right)^2 \dot{A}X_{ii} - 2 \left( \frac{\dot{R}}{R} \right)^2 \dot{A}_{ij}X_{kk}X_{ij} + O(c^{-6}) \right] \]
\[ +cdTdX_i \left[ c^{-1}(-4A_{ij}X_{ij} - 2\frac{\dot{R}}{R}X_i) \right. \]

3
\[ +c^{-3}(-4B_{ij}X_j - 8B_{ijkl}X_{jkl} - 4 \left( \frac{\dot{R}}{R} \right)^2 A_{ij}X_{kk}X_j + 2R\dot{C}_{ij}X_j + 2R\dot{C}_{ijkl}X_{jkl} \] \\
\[ -2\dot{R}C_{ij}X_j - 6\dot{R}C_{ijkl}X_{jkl} + 2\frac{\dot{R}}{R} A X_i + 2\frac{\dot{R}}{R} A_{jk}X_{ijk} \right) + O(c^{-5}) \] \\
\[ +dX_i dX_j \left[ \delta_{ij} + c^{-2}(-4A_{ik}A_{jl}X_{kl} + 2RC_{ij} + 6RC_{ijkl}X_{jkl} + 4\frac{\dot{R}}{R} A_{ik}X_{jk} \right) \] \\
\[ +c^{-4}(-8\dot{A}_{ik}A_{jl}X_{kl} - 8B_{ik}A_{jl}X_{kl} + 8A_{ik}A_{jl}\dot{A}_{mn}X_{klmn} - 16A_{jl}B_{ikmn}X_{klmn} + 6RD_{ijkl}X_{kl} \] \\
\[ +10RD_{ijklmn}X_{klmn} + R^2(C_{ij} + 3C_{ijkl}X_{kl})^2 + 4 \left( \frac{\dot{R}}{R} \right)^2 A_{ik}A_{jl}X_{km} + 4\dot{A} A_{il}C_{jk}X_{lk} \] \\
\[ -4RA_{in}\dot{C}_{jmkl}X_{mnkl} - 4RA_{il}\dot{C}_{jmkl}X_{lk} + 12\dot{R}A_{in}C_{jmkl}X_{mnkl} - 4\frac{\dot{R}}{R} \dot{A} A_{il}X_{jl} \] \\
\[ -4\frac{\dot{R}}{R} A_{im}\dot{A}_{kl}X_{jkl} + 4\frac{\dot{R}}{R} B_{ik}X_{jk} + 8\frac{\dot{R}}{R} B_{ijkl}X_{mkl} + O(c^{-6}) \right]. \tag{5} \]

Since in (4) there are no terms of order $c^{-1}$, we require that

\[ A_{ij} = -\frac{1}{2} \frac{\dot{R}}{R} \delta_{ij}. \tag{6} \]

In addition, the terms of order $c^{-2}$ in $g_{00}$ and $g_{ij}$ may be identified, and from the powers in $X_i$ one can read off

\[ C_{ij} = -\frac{\dot{A}}{R} \delta_{ij}, \]

and

\[ C_{ijkl} = \frac{1}{18R} \left( \frac{\ddot{R}}{R} - \left( \frac{\dot{R}}{R} \right)^2 \right) \delta_{(i} \delta_{kl)}. \]

4 The Homogeneous and Isotropic Newtonian Cosmological Theory

The Newtonian theory is the $c^{-2}$ cut-off of (4) and consists of the field (Poisson) equation

\[ \phi_{,ii} = 4\pi G \rho, \tag{7} \]

(\(\phi\) is the Newtonian potential and \(\rho\) is the density) and the continuity and Euler equations of fluid dynamics

\[ \dot{\rho} + \rho v_i, = 0, \tag{8} \]

\[ v_i + \phi, + \frac{1}{\rho} p, = 0, \tag{9} \]

where \(v_i\) is the velocity field and \(p\) is the pressure. Homogeneity implies that the density and pressure are merely functions of time and that the velocity field is the same relative to all
observers. It can be shown, \cite{1} and \cite{10}, that this amounts to \( v_i = V_{ij}(t)X_j \). By substituting the Poisson equation into the Euler equation we see that the Newtonian potential must be of the form \( \phi = a_{ij}(t)X_iX_j + a(t) \). Therefore, the Newtonian approximation of a homogeneous cosmology is

\[
a_{ii} = 4\pi G\rho,
\]

\[
\dot{\rho} + \rho V_{ii} = 0,
\]

\[
\dot{V}_{ij} + V_{ik}V_{kj} = a_{ij}.
\]

(10)

The FRW cosmology is the most general isotropic and homogeneous solution. Thus, we will only consider the case where \( \phi \) becomes isotropic, i.e. shear-free and rotation-free. Heckmann and Schücking in \cite{3} formulate the more general anisotropic Newtonian cosmology where there is shear and rotation. To this end we make the following decomposition

\[
V_{ij} = \frac{1}{3}\theta\delta_{ij} + \sigma_{ij} + w_{ij},
\]

(11)

where

\[
\theta = V_{ii},
\]

\[
\sigma_{ij} = \frac{1}{2}(V_{ij} + V_{ji}) - \frac{1}{3}\theta\delta_{ij},
\]

\[
w_{ij} = \epsilon_{jik}w_k = \frac{1}{2}(V_{ij} - V_{ji}),
\]

The trace part \( \theta \) is the expansion, the trace-free symmetric piece \( \sigma_{ij} \) is the shear and \( w_{ij} \), the anti-symmetric part, is the rotation. Using this decomposition in the Euler equation and setting \( \sigma = 0 \) and \( w = 0 \), we get

\[
\dot{\theta} = -\frac{1}{3}\theta^2 - 4\pi G\rho,
\]

\[
a_{ij} = \frac{1}{3}a_{kk}\delta_{ij},
\]

with continuity equation

\[
\dot{\rho} + \rho\theta = 0.
\]

(12)

Defining a function \( R'(t) \) such that \( \theta = 3\frac{\dot{R'}}{R'} \), then the solution of the continuity equation \( \cite{12} \) is

\[
\rho = C'R'^{-3},
\]

(13)

with \( C' \) a constant. Finally, using \( \cite{13} \) in the Euler equation, we see that the Newtonian, isotropic, homogeneous cosmology is given by

\[
a_{ii} = 4\pi G\rho,
\]

\[
\frac{\dot{R}'}{R'} = -\frac{4}{3}\pi G\rho,
\]

(15)

\[
\rho = C'R'^{-3}.
\]

(16)
Notice here the use of $R(t)$ as the general relativistic scale factor in (1) through to (3), and $R'(t)$ as the Newtonian scale factor in the Newtonian theory (14) through to (16). We are now in a position to compare these two theories. The general relativistic theory is well-posed. Equations (1) and (2) are consistent with the Bianchi identities. In the Newtonian theory there is only one field equation (14), and there is no completeness because (14) does not give (15) and (16), and nor is the theory well-posed. Also, notice that in the general relativistic theory pressure occurs in the dynamics of the theory, whereas in the Newtonian theory, pressure does not occur anywhere in the dynamics and is only defined through an equation of state.

Equation (15) has the same form as the Raychaudhuri equation (at least when $p = 0$). Using the Raychaudhuri equation it may be deduced that $R' \propto t^2$. Thus the equations (15) and (16) of the Newtonian theory predict the same results, at least for the case of matter, that the general relativistic equations (1) and (2) do.

How, if at all, do $R(t)$ and $R'(t)$ differ? To answer this question we use the remaining piece of information - (14), the Poisson equation. Since $\phi = a_{ij}(t)X_iX_j + a(t)$ is the term of order $c^{-2}$ in the $c^2dT^2$ piece of the FRW metric (5); a comparison of (4) and (5), and using (6) yields

\[ a_{ii} = -3\frac{\ddot{R}}{R}, \]

and

\[ a(t) = \dot{A}, \]

from which, with the aid of (14), we can deduce

\[ \frac{\ddot{R}}{R} = -\frac{4}{3}\pi G \rho. \]  

This is again the Raychaudhuri equation of general relativity for the case of vanishing pressure. (Since the Newtonian potential $\phi$ only appears as $\phi_{kk} = 2a_{kk}(t)$ in equations (16) to (18) we may set the piece $a(t) = 0$ without loss of generality; then $A$ is just a constant.)

Hence, the correct general relativistic scale factor $R(t)$ is equivalent to the Newtonian scale factor $R'(t)$. Dautcourt [2], shows how the validity of the Friedmann equation within Newtonian cosmology can be understood: Newtonian cosmology is applicable only when confined to a neighbourhood of the observer, corresponding to distances which are small compared to the Hubble distance.

Although reproducing results similar to the general relativistic theory, the Newtonian theory suffers in that varying the equation of state will have no effect on the outcome of the solutions for $\rho(t)$ and $R'(t)$. This being due to the fact that the pressure has not appeared anywhere in the dynamics. Thus we can only reproduce the results of the matter dominated case of general relativity.

5 The Homogeneous and Isotropic Post-Newtonian Cosmological Theory

The field equations for the post-Newtonian theory [3] are

\[ \phi_{kk} = 4\pi G \rho + \frac{1}{4c^2}(-\phi_{jk,jk} - 2\dot{\mathcal{A}}), \]  

\[ \zeta_{i,kk} = 16\pi G \rho v_i + \frac{1}{c^2} \left( \phi_{ij,j} - 2\mathcal{B}_i \right), \]

\[ \ddot{\phi}_{ij} - c^2 \phi_{ij, kk} = 2\mathcal{B}_{ij} + c^2 \left[ 16\pi G(\rho v_i v_j + \delta_{ij} p) - 2\mathcal{A}_{ij} \right], \]
where the $\alpha$ and $\alpha_{ij}$ of (4) are such that

$$\phi_{ij} = \alpha_{ij} - 2\delta_{ij}\alpha,$$

with

$$\mathfrak{A} \equiv 6\phi_i\phi_{,i} - 16\pi G(\rho v^2 + 4\rho\phi),$$

$$\mathfrak{B}_i \equiv 3\zeta_{,j}\phi_{,i} + 2\zeta_j\phi_{,ij} - 2\phi_j\zeta_{,i},$$

$$-16\pi G \left[ v_i p + \rho v_i v^2 - \frac{1}{2} \rho \zeta_{,i} \right],$$

$$\mathfrak{A}_{ij} \equiv 8\phi_{,ij} + 4\phi_{,i}\phi_{,j} - \delta_{ij}(6\phi_k\phi_{,k} + 32\pi G\rho\phi),$$

$$\mathfrak{B}_{ij} \equiv -\frac{1}{2} (\zeta\zeta_{,kj} + \zeta_{j,k} - \zeta_{,k} - \zeta_{,j} + 2\zeta_{k,ij} + \zeta_{i,j,k} + \zeta_{i,k,j} + \zeta_{j,k})$$

$$-2\phi_k (\phi_{ki,j} + \phi_{kj,i} - 2\phi_{ij,k}) - 16\phi_{,i}\phi_{,j} + \phi_{,i}\phi_{,k}$$

$$+\phi_{,j}\phi_{,k} - 2\phi_{,k}\phi_{ij} - \delta_{ij} \left[ \frac{1}{2} \zeta_{,m}\zeta_{,m} + \frac{1}{2} \zeta_{m,ij}\zeta_{,m} + \frac{1}{2}(\zeta_{,k})^2 \right]$$

$$-\zeta_{,m}\zeta_{m,k} - 4\phi_k\phi_{k,m} + 4\phi_k\phi_{m,k} - 12\phi_{,k}\phi_{,k}$$

$$-\phi (2\phi_{k,m} - 2\phi_{m,k} + 8\pi G[2pv_i v_j$$

$$+2\rho(2\phi + v^2) v_i v_j + \rho\phi_{,ij} + \delta_{ij} (2\rho\phi v^2 - \frac{1}{2} \rho p$$

$$+\frac{3}{4} \phi_k p_{,k} + \frac{1}{2} \rho \phi_{,k,k})].$$

Along with the harmonic gauge conditions

$$\dot{\phi} = -\frac{1}{4} \dot{\zeta}_{,i},$$

$$\dot{\zeta}_{i} = \phi_{,ij,j},$$

this system forms a closed set which is consistent because the Bianchi identities are obtainable from the field equations:

$$\dot{\phi} \left( 1 + \frac{v^2 - 4\phi}{c^2} \right) + (\rho v_{ij})_{,i} \left( 1 + \frac{v^2}{c^2} \right) + \frac{1}{c^2} \rho \left( 2v_{j} v_{j} + 2v_{j} v_{k} v_{k,j} + \frac{1}{2} \zeta_{j,j} \right)$$

$$-\frac{1}{2} \rho_{,j} \zeta_{j} + (v_{j} P)_{,j} + \frac{1}{16\pi G} \left[ 2\phi_{,i}\zeta_{,j,j} - 2\zeta_{i,j}\phi_{,j} - 3\zeta_{i,i,\phi_{,j,j}} \right]) = 0, \quad (23)$$

and

$$\rho(\dot{v}_{i} + v_{i,j} v_{j} + \phi_{,i}) + P_{,i} = \frac{1}{16\pi G c^2} \left[ -(\mathfrak{A} + \mathfrak{B}_{,j}) v_{i} + \mathfrak{B}_{i - \mathfrak{A}_{i,j,j} - 2\phi(\mathfrak{A}_{,i} + \phi_{j,k,i})} - \phi_{,i}(\mathfrak{A} + \phi_{j,k,i}) \right]. \quad (24)$$

Specifically, (23) and the $,i$ derivative of (19) imply $\partial/\partial t$ of the first constraint equation (18), while (24) and $,i$ of eq. (24) imply the time derivative of (19).

To make comparisons with the FRW cosmologies we need to consider the case where this theory is both homogeneous and isotropic. The following are the most general ansätze, providing isotropy and homogeneity, for tensors of rank $n$, expanded up to order $X^{n+2}$:
\[
\begin{align*}
\phi &= (2)\phi(t)X^2 + (1)\phi(t), \\
\zeta_i &= (2)\zeta(t)X^2X_i + (1)\zeta(t)X_i, \\
\phi_{ij} &= (5)\phi(t)X^4\delta_{ij} + (4)\phi(t)X^2X_{ij} + (3)\phi(t)X^2\delta_{ij} + (2)\phi(t)X_{ij} + (1)\phi(t)\delta_{ij}.
\end{align*}
\]

Again, we decompose the velocity field, \( V_{ij} \), and set \( w_{ij} = 0 \) and \( \sigma_{ij} = 0 \). Thus we have \( v_i = V_{ij}X_j = X^2\theta'X_i \). Pro tempore we do not make the substitution \( \theta = \frac{\theta'}{2} \), instead, we put \( \theta' \).

The field equations, (18) to (20), are then

\[3 (2)\phi = 2\pi G\rho + c^{-2} \left[ -\frac{3}{4} (3)\phi - \frac{3}{2} (2)\phi + 8\pi G\rho (1)\phi \right],\]

\[10 (5)\phi + 15 (4)\phi + 12 (2)\phi^2 - 8\pi G\rho \left( \frac{1}{9} \theta'^2 + 4 (2)\phi \right) = 0,\]

\[5 (2)\zeta - 8\pi G\rho \frac{1}{3} \theta' + c^{-2} \left[ -\frac{1}{3} (3)\zeta - 2 (2)\phi \theta' + 9 (2)\phi (1)\zeta + 8\pi G \left( -\frac{1}{3} \theta'\rho + \frac{1}{2} \theta' (1)\zeta \right) \right] = 0,\]

\[2 (5)\phi + 3 (4)\phi - 6 (2)\phi^2 \theta + 8\pi G\rho \left( \frac{1}{27} \theta'^3 - \frac{1}{2} (2)\zeta \right) = 0,\]

\[6 (3)\phi + 2 (2)\phi - 16 (2)\phi (1)\phi + 16\pi G(p + 2 (1)\rho) \]
\[+ c^{-2} \left[ -\frac{1}{2} (1)\phi^2 - 4 (2)\phi (1)\phi + 4 (1)\phi (2)\phi + 8 (1)\phi (3)\phi + 8\pi G \left( \rho (1)\phi + 3 (2)\phi \right) - \frac{1}{2} (2)\phi \right] = 0,\]

\[20 (5)\phi + 2 (4)\phi + 8 (2)\phi^2 + 32\pi G\rho (2)\phi + c^{-2} \left[ -\frac{1}{3} (3)\phi + 11 (1)\phi (2)\phi - 44 (2)\phi (3)\phi \right. \]
\[- 16 (2)\phi (2)\phi + 32 (1)\phi (5)\phi - 8 (1)\phi (4)\phi + 48 (2)\phi^2 (1)\phi + 8\pi G \left( \rho (3)\phi + 2 (1)\phi - \frac{1}{2} \theta'^2 \right) - \frac{1}{2} (2)\phi \left. \right] = 0,\]

\[14 (4)\phi - 16 (2)\phi^2 + 16\pi G\rho \frac{1}{9} \theta'^2 \]
\[+ c^{-2} \left[ -\frac{1}{2} (3)\phi + 2 (1)\phi (2)\phi - 64 (2)\phi^2 (1)\phi - 8 (2)\phi (3)\phi - 16 (2)\phi (2)\phi - 16 (1)\phi (5)\phi + 4 (1)\phi (4)\phi \right. \]
\[+ 8\pi G \left( 2 \frac{1}{9} \theta'^2 (p + 2 (1)\rho) + \rho (2)\phi \right) \left. \right] = 0,\]

\[(5)\phi = \frac{23}{2} (2)\phi^2 + 60 (2)\phi (5)\phi - 20 (2)\phi (4)\phi - 48 (2)\phi^3 - 8\pi G\rho \left( 2 (2)\phi \frac{1}{9} \theta'^2 + (5)\phi \right) = 0,\]
\[(4) \ddot{\phi} - 6 (2) \dot{\zeta}^2 + 64 (2) \phi^3 - 32 (2) \dot{\phi}(5) \ddot{\phi} - 12 (2) \phi(4) \ddot{\phi}

-8\pi G \rho \left(4 (2) \phi \frac{1}{9} \phi'^2 + 2 \frac{1}{81} \phi'^4 + (4) \ddot{\phi}\right) = 0. \tag{34}\]

The homogeneous harmonic gauge conditions \([21]\) and \([22]\) become

\[
\begin{align*}
(1) \dot{\phi} &= \frac{3}{4} (1) \zeta, \\
(2) \dot{\phi} &= \frac{5}{4} (2) \zeta, \\
(1) \dot{\zeta} &= 2 (3) \ddot{\phi} + 4 (2) \ddot{\phi}, \\
(2) \dot{\zeta} &= 4 (5) \ddot{\phi} + 6 (4) \ddot{\phi},
\end{align*}
\tag{35-38}\]

Using \([28]\) and the time derivative of \([26]\), the harmonic gauge condition \([36]\) reads

\[
\dot{\rho} + \theta' (\rho + p c^{-2}) + c^{-2} \left[ -\frac{27}{8\pi G} (2) \phi (1) \zeta - 4 \dot{\phi} (1) \phi + \frac{3}{2} \rho (1) \zeta \right] = 0, \tag{39}\]

where \([27]\) and \([29]\) have been used to express the \(c^{-2}\) parts of the equation in this relatively simple form. This is the continuity equation \([23]\) for the special case of homogeneity and isotropy.

Next, consider equations \([31]\) and \([32]\) with the time derivative of \([28]\). These equations, with the help of \([33]\) and \([34]\), provide us with the Euler equation, \([24]\), for the case of homogeneity and isotropy:

\[
\begin{align*}
\rho \left(\frac{1}{3} \dot{\theta}' + \frac{1}{9} \theta'^2 + 2 (2) \phi\right) + \frac{1}{16\pi G} e^{-2} \left[ 265 (2) \phi^2 (1) \phi + 132 (2) \phi (3) \ddot{\phi} + 120 (2) \phi (2) \ddot{\phi} \\
-18 (2) \phi (1) \zeta - 18 (2) \phi (1) \zeta - 30 (2) \zeta (2) \zeta + 80 (1) \phi (5) \ddot{\phi} + 120 (1) \phi (4) \ddot{\phi} + 54 (2) \phi (1) \phi \frac{1}{3} \theta'
\right.
\end{align*}
\]

\[
+16\pi G \left( -\dot{\rho} \left(\frac{1}{2} (1) \zeta + (1) \phi \frac{1}{3} \theta'\right) + \rho \left(\frac{1}{3} \theta' + \frac{1}{2} (2) \phi - \frac{7}{49} \theta'^2\right) + p \left(\frac{1}{3} \theta' - (3) \phi - 14 (1) \phi \frac{1}{9} \theta'^2 - 2 (2) \phi - 24 (2) \phi (1) \phi - 8 (1) \phi \frac{1}{3} \theta + \frac{1}{2} (1) \zeta \theta'\right) \right] = 0. \tag{40}\]

We may now read off the potentials \(\phi, \zeta, \) and \(\phi_{ij}\) from the FRW metric \([5]\). With the help of the harmonic gauge conditions \([33]\) to \([38]\) we are able to write down the functions \((1) \phi, (2) \phi\) etc. which appear in the ansätze \([25]\):

\[
\begin{align*}
(1) \phi &= \dot{A}, \\
(2) \phi &= -\frac{1}{2} \frac{\ddot{R}}{R}, \\
(1) \zeta &= -\frac{4}{3} \ddot{A}, \\
(2) \zeta &= \frac{2}{5} \left( \frac{\ddot{R}}{R} - \frac{\dot{R} \ddot{R}}{R^2} \right), \\
(1) \ddot{\phi} &= 4 \dot{\ddot{A}}^2, \\
(2) \ddot{\phi} &= -\frac{2}{5} \dot{A} + \frac{8}{15} \dot{A} \frac{\ddot{R}}{R} + \frac{2}{5} \ddot{A} \frac{\dot{R}}{R} + 2 \dot{A} \left( \frac{\ddot{R}}{R} \right)^2,
\end{align*}
\]
\( (3) \tilde{\phi} = \frac{2}{15} \ddot{A} - \frac{16}{15} \frac{2 \dot{A}}{R} - \frac{4}{5} \frac{\dot{A} R}{R} - 4A \left( \frac{\dot{R}}{R} \right)^2, \)

\( (4) \tilde{\phi} = \frac{1}{21} \ddot{R} + \frac{9}{35} \frac{\dot{R} \ddot{R}}{R R} - \frac{16}{21} \left( \frac{\dot{R}}{R} \right)^2 + \frac{29}{28} \left( \frac{\dot{R}}{R} \right)^4 - \frac{39}{35} \left( \frac{\dot{R}}{R} \right)^2, \)

\( (5) \tilde{\phi} = \frac{4}{35} \ddot{R} + \frac{87}{14} \left( \frac{\dot{R}}{R} \right)^4 + \frac{219}{210} \left( \frac{\dot{R}}{R} \right)^2 - \frac{32}{105} \frac{\ddot{R} R}{R} + \frac{92}{105} \left( \frac{\dot{R}}{R} \right)^2. \)

(41)

Using these potentials in (40) and adopting a power expansion for \( \theta' \)

\[ \theta' = \theta + e^{-2}\theta'', \quad \text{where} \quad \theta = 3\frac{\dot{R}}{R}, \]

one obtains, with the help of (30) and after considerable simplification

\[ 3\frac{\ddot{R}}{R} = -4\pi G (\rho + 3pc^{-2}) + e^{-2} \left[ -\dddot{A} + 4\ddot{A} \frac{\dot{R}}{R} - 3\dot{A} \frac{\ddot{R}}{R} \right]. \]

(42)

This is the Raychadhuri equation of general relativity (2), with the required inclusion of the pressure term, which was missing in the Newtonian case (17), and with \( e^{-2} \) corrections. Since \( \dot{A} \ll c^2 \), we can be sure that the corrections are of higher order and do not contribute to the theory at the \( c^0 \) level. We may now use this result, along with the now known values for the potentials (41), in (39) to derive the Friedmann equation

\[ \left( \frac{\dot{R}}{R} \right)^2 = 8\pi G \frac{3}{3} \rho + \Upsilon c^{-2}, \]

(43)

where \( \Upsilon \) is defined to be the solution of the differential equation

\[ \dddot{\Upsilon} + 2 \dot{R} \Upsilon + \frac{2}{3} \dddot{A} \frac{\dot{R}}{R} + \dddot{A} \left( -\frac{2}{3} \left( \frac{\dot{R}}{R} \right)^2 + 6 \dddot{R} \right) + \dddot{\dot{A}} \left( 5 \dot{\dddot{R}} - 3 \left( \frac{\dot{R}}{R} \right)^3 \right) + 2\theta'' \left( \frac{1}{3} \frac{\dddot{R}}{R} - 3 \left( \frac{\dot{R}}{R} \right)^2 \right) = 0. \]

The Bianchi identities (39) and (40), the field equations (26) to (34) and the harmonic gauge conditions (35) to (38) may be understood in the following manner: The time derivative of equation (28) along with the equations (31), (32), (33), (34) and with the aid of the harmonic gauge conditions give us the Bianchi identity (41), which leads to the Raychadhuri equation (42). The time derivative of equation (24) with equations (28), (27), (29) and the harmonic gauge condition (36) gives the Bianchi identity (39), which leads to the Friedmann equation (43). This leaves us with the field equation (38) which may also be combined with (24) to give the Raychadhuri equation.

It may also be shown that the following relationships between the field equations exist. The time derivative of (26) is (28). The time derivative of (27) with the aid of (34) is (29). Equations (31) and (32) may be combined in such a way as to be (24). Finally, Equations (33) and (34) may be combined in such a way as to be the second time derivative of (24).

Thus the post-Newtonian approximation may be completely defined for the FRW cosmology with the Raychadhuri equation (42), the Friedmann equation (43) and an extended Poisson equation (26), which after substitution of (41), becomes
\[ 3 \frac{\ddot{R}}{R} = -4\pi G \rho + c^{-2}(-\ddot{A} - 16\pi G \rho \dot{A}). \] (44)

The unknown \( \theta'' \) may be set to zero without loss of generality. Equation (43) may be equivalently represented by (39). Given an equation of state \( (pc^{-2} = w\rho, \ w \text{ an arbitrary constant}) \), the system (39), (42) and (44) gives equations for \( \dot{\rho}, \dot{R} \) and \( \ddot{A} \) respectively, and thus, forms a well-posed set, allowing the unknowns \( R(t), \rho(t) \) and \( A(t) \) to be determined.

It is the gravitational potential \( A(t) \) that incorporates the pressure into the theory. The pressure and \( A(t) \) are related through a third order differential equation one can obtain by combining (42) and (44). Varying the equation of state therefore results in varying solutions for the density \( \rho(t) \) and for \( R(t) \), which are analogous to those solutions predicted by the fully general relativistic theory for the FRW cosmologies, (3). Setting \( A = 0 \), for example, implies vanishing pressure, \( p = 0 \), which is the post-Newtonian approximation of FRW for dust. In this case the higher order corrections of the post-Newtonian theory disappear and one ends up with the Newtonian theory, which we have shown in Section 4 describes the case of dust.

6 Concluding Remarks

The FRW cosmology is a very good approximation to the large scale structure of the universe, at least to the present epoch [4]. Due to its obvious simplicity, Newtonian approximations, where they reproduce results which are similar to the fully general relativistic theory, are preferable. General relativity has a well-posed Cauchy problem in the case of perfect fluids with a barotropic equation of state [3]. Newtonian theory, on the other hand, is not well-posed [9]. We have seen that the Newtonian theory reproduces the results of it’s general relativistic counterpart only for the special case of dust, i.e. for a matter-dominated universe with an equation of state, \( p = \rho c^2 \). Although, any equation of state may be written down, it won’t make any difference since the pressure does not appear in the dynamics.

The post-Newtonian theory, which is consistent and well-posed, does provide field equations with pressure entering the dynamics through the potential \( A(t) \). We are thus able to vary the equation of state and, in doing so, will obtain various solutions for \( R(t) \) and the density, \( \rho \). Hence, the post-Newtonian theory seems to be a favourable approximation of the fully general relativistic theory.

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