Free Field Dynamics in the Generalized $AdS$ (Super)Space

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Abstract

Pure gauge representation for general vacuum background fields (Cartan forms) in the generalized $AdS$ superspace identified with $OSp(L, M)$ is found. This allows us to formulate dynamics of free massless fields in the generalized $AdS$ space-time and to find their (generalized) conformal and higher spin field transformation laws. Generic solution of the field equations is also constructed explicitly. The results are obtained with the aid of the star product realization of ortosymplectic superalgebras.

1 Introduction

In the recent papers [1, 2] it was shown that infinite multiplets of massless higher spins in 4d flat Minkowski space-time admit description in terms of ten-dimensional space-time $\mathcal{M}_4$ with real symmetric bispinor matrix coordinates $X^{\alpha\beta} = X^{\beta\alpha}$, ($\alpha, \beta = 1 \ldots 4$). A single scalar field $c(X)$ in $\mathcal{M}_4$ describes massless fields of all integer spins in 4d Minkowski space-time upon imposing field equations found in [1]. Half-integer spin massless fields are described analogously by a spinor field $c_\alpha(X)$. That massless fields of all spins have to admit some formulation in $\mathcal{M}_4$ was argued by Fronsdal in the pioneering paper [3] where it was also stressed that such infinite sets of massless fields have to form representations of the extension of the 4d conformal group $su(2, 2)$ to $sp(8|R)$. Then in [4] it was found that world-line particle models based on $sp(8)$ give rise to massless higher spin excitations of all spins. The explicit realization of the $sp(8)$ symmetry by local transformations was given in [1] as well as the generalization of the proposed $sp(8)$ covariant dynamical equations to $\mathcal{M}_M$ with arbitrary even $M$.

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Properties of the $Sp(2M)$ invariant space-time $\mathcal{M}_M$ were analyzed in [2]. It was shown that the classical solutions of the field equations define a causal structure and give rise to correct quantization in a positive definite Hilbert space. Usual $d$-dimensional Minkowski space-time appears as a subspace of the generalized space-time. The analysis of [1, 2] was performed for the case of flat space-time although the formalism as a whole works in any (generalized) conformally flat background. In particular it is interesting to extend this analysis to the generalized anti-de Sitter space-time which was argued in [1] to be the group manifold $Sp(M)$ ($M$ is even) having $Sp(M) \times Sp(M) \subset Sp(2M)$ as the group of motions realized by left and right group actions on itself. Since the analysis of $Sp(2M)$ invariant higher spin systems is most naturally performed in terms of star product algebras, for its extension to the generalized $AdS$ space-time it is necessary to built star-product realizations of left invariant Cartan forms (i.e., flat connections) on $Sp(M)$. This is the primary goal of this paper. Obtained results will allow us to present explicit formulae for symmetries and solutions of the massless field equations in the generalized $AdS$ space-time. The analogous construction will also be given for the supersymmetric case associated with $OSp(L, M)$.

Let us note that since the star product formalism we apply leads to compact expressions for $OSp(L, M)$ Cartan superforms, apart from the higher spin problem, the results obtained in this paper may have applications to other problems where left-invariant forms of $OSp(L, M)$ appear. For example, in [5] it was shown how $OSp(1, M)$ Cartan forms can be used to construct twistor-like actions for super-particles and possible applications to super-branes were discussed while in [6] a toy model of $M$ theory based on $osp(1,64)$ was suggested.

### 1.1 Generalized conformal symmetry

The generators $L_{mn}, P_m, K_m, D$ of the conformal algebra $o(d, 2)$ satisfy the following commutation relations

\[
[L_{ab}, L_{cd}] = \eta_{ac}L_{bd} - \eta_{bc}L_{ad} + \eta_{ad}L_{cb} - \eta_{bd}L_{ca},
\]

\[
[L_{ab}, P_c] = \eta_{ac}P_b - \eta_{bc}P_a,
\]

\[
[L_{ab}, K_c] = \eta_{ac}K_b - \eta_{bc}K_a,
\]

\[
[L_{ab}, D] = [P_a, P_b] = [K_a, K_b] = 0,
\]

\[
[P_a, K_b] = 2(\eta_{ab}D + L_{ab}),
\]

\[
[P_a, D] = P_a, \quad [K_a, D] = -K_a,
\] (1.1)

$m, n = 0, \ldots d - 1, \eta_{mn} = diag(1, -1 \ldots -1)$. The conformal algebra can be realized by the vector fields

\[
L_{ab} = \eta_{ac}x^c \frac{\partial}{\partial x^b} - \eta_{bc}x^c \frac{\partial}{\partial x^a},
\]

\[
P_a = \frac{\partial}{\partial x^a}, \quad D = x^a \frac{\partial}{\partial x^a},
\]

\[
K_a = 2\eta_{ac}x^c x^b \frac{\partial}{\partial x^b} - \eta_{bc}x^b x^c \frac{\partial}{\partial x^a}.
\] (1.2)
The Poincare subalgebra is spanned by \( L_{mn} \) and \( P_m \). \( K_m \) and \( D \) are the generators of special conformal transformations and dilatations, respectively. To embed the \( AdS_d \) algebra \( o(d - 1, 2) \) into the \( d \)-dimensional conformal algebra \( o(d, 2) \) one identifies the \( AdS_d \) translations with the mixture of translations and special conformal transformations in the conformal algebra

\[
P_{AdS_d}^a = P^a - \lambda^2 K^a. \tag{1.3}
\]

The generators \( P_{AdS_d}^a \) and \( L_{ab} \) form the \( AdS_d \) subalgebra \( o(d - 1, 2) \subset o(d, 2) \). This embedding breaks down the manifest \( o(1, 1) \) dilatation covariance because it mixes the operators \( P^a \) and \( K^a \), which have different dimensions. \( \lambda \) is the dimensionful Wigner-Inönu contraction parameter to be identified with the inverse \( AdS_d \) radius.

The \( sp(2M) \) algebra admits analogous description in terms of the generators \( L_{\alpha\beta}, P_{\alpha\beta}, K_{\alpha\beta} \) and \( D \), where indices \( \alpha, \beta \) . . . range from 1 to \( M \) and \( L_{\alpha\beta} \) is traceless. The commutation relations are

\[
[K_{\alpha\beta}, K_{\gamma\delta}] = 0, \quad [P_{\alpha\beta}, P_{\gamma\delta}] = 0, \tag{1.4}
\]

\[
[D, P_{\alpha\beta}] = -P_{\alpha\beta}, \quad [D, K_{\alpha\beta}] = K_{\alpha\beta}, \quad [D, L_{\alpha\beta}] = 0, \tag{1.5}
\]

\[
[L_{\alpha\beta}, P_{\gamma\delta}] = -\delta_{\gamma\delta} P_{\alpha\beta} - \delta_{\delta\gamma} P_{\alpha\beta} + \frac{2}{M} \delta_{\alpha\beta} P_{\gamma\delta}, \tag{1.6}
\]

\[
[L_{\alpha\beta}, K_{\gamma\delta}] = \delta_{\alpha\gamma} K_{\beta\delta} + \delta_{\alpha\delta} K_{\gamma\beta} - \frac{2}{M} \delta_{\alpha\beta} K_{\gamma\delta}, \tag{1.7}
\]

\[
[P_{\alpha\beta}, K_{\gamma\delta}] = L_{\alpha\beta} \delta_{\gamma\delta} + L_{\beta\gamma} \delta_{\alpha\delta} + L_{\alpha\delta} \delta_{\gamma\beta} + L_{\gamma\delta} \delta_{\alpha\beta} + \frac{4}{M} D (\delta_{\alpha\delta} \delta_{\beta\gamma} + \delta_{\beta\delta} \delta_{\alpha\gamma}), \tag{1.8}
\]

\[
[L_{\alpha\beta}, L_{\gamma\delta}] = \delta_{\alpha\delta} L_{\gamma\beta} - \delta_{\gamma\beta} L_{\alpha\delta}. \tag{1.9}
\]

Note that the generalized Lorentz subalgebra generated by \( L_{\alpha\beta} \) is \( sl_M \). Analogously to the usual conformal algebra, generalized translations generated by \( P_{\alpha\beta} \) form Abelian subalgebra of \( sp(2M) \). Generalized special conformal transformations generate a dual Abelian subalgebra.

The commutation relations (1.4)-(1.9) can be realized by the vector fields

\[
P_{\alpha\beta} = \frac{\partial}{\partial X_{\alpha\beta}}, \quad K_{\alpha\beta} = 4X^{\alpha\gamma}X^{\beta\eta} \frac{\partial}{\partial X^{\gamma\eta}}, \tag{1.10}
\]

\[
L_{\alpha\beta} = 2X^{\beta\gamma} \frac{\partial}{\partial X^{\alpha\gamma}} - \frac{2}{M} \delta_{\alpha\beta} X^{\gamma\gamma} \frac{\partial}{\partial X^{\beta\gamma}}, \quad D = X^{\beta\gamma} \frac{\partial}{\partial X^{\beta\gamma}}, \tag{1.11}
\]

where \( X^{\alpha\beta} = X^{\beta\alpha} \) are coordinates of \( M_M \).

The simplest way to see that the commutation relations (1.4)-(1.9) are indeed of \( sp(2M) \) is to use its oscillator realization [7]. Actually, let \( \hat{a}_\alpha \) and \( \hat{b}^\beta \) be oscillators with the commutation relations

\[
[\hat{a}_\alpha, \hat{b}^\beta] = \delta_\alpha^\beta, \quad [\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{b}^\alpha, \hat{b}^\beta] = 0. \tag{1.12}
\]

The generators of \( sp(2M) \) are spanned by the bilinears

\[
\hat{T}_\alpha^\beta = \frac{1}{2} \{\hat{a}_\alpha, \hat{b}^\beta\}, \quad \hat{P}_{\alpha\beta} = \hat{a}_\alpha \hat{a}_\beta, \quad \hat{K}^\alpha_\beta = \hat{b}^\alpha \hat{b}^\beta. \tag{1.13}
\]
Instead of working in terms of operators it is convenient to use the star-product operation in the algebra of polynomials of commuting variables \(a_\alpha\) and \(b^\alpha\)

\[
(f \star g)(a,b) = \frac{1}{\pi^{2M}} \int f(a + u, b + t) g(a + s, b + v) e^{(s a \nu^\alpha - u a \nu^\alpha)} dM u dM t dM s dM v .
\]

(1.14)

The star-product defined this way, often called Moyal product, describes the product of symmetrized (i.e Weyl ordered) polynomials of oscillators in terms of symbols of operators. The integral is normalized in such a way that

\[
\frac{1}{\pi^{2M}} \int e^{(s a \nu^\alpha - u a \nu^\alpha)} dM u dM t dM s dM v = 1 ,
\]

(1.15)

so that 1 is the unit element of the algebra. Eq.(1.14) defines the associative algebra with the defining relations

\[
[a_{\alpha}, b^{\beta}]_{\star} = \delta_{\alpha}^{\beta} , \quad [a_{\alpha}, a_{\beta}]_{\star} = 0 , \quad [b^{\alpha}, b^{\beta}]_{\star} = 0 \quad (1.16)
\]

\(([a, b]_{\star} = a \star b - b \star a)\). The star product realization of the generators of \(sp(2M)\) is

\[
T_{\alpha}^{\beta} = a_{\alpha} b^{\beta} , \quad P_{\alpha \beta} = a_{\alpha} a_{\beta} , \quad K^{\alpha \beta} = b^{\alpha} b^{\beta} ,
\]

(1.17)

where the \(gl(M)\) generator \(T_{\alpha}^{\beta}\) decomposes into the \(sl(M)\) “Lorentz” and \(o(1,1)\) “dilatation” generators

\[
L_{\alpha}^{\beta} = a_{\alpha} b^{\beta} - \frac{1}{M} \delta_{\alpha}^{\beta} a_{\gamma} b^{\gamma} , \quad D = \frac{1}{2} a_{\alpha} b^{\alpha} .
\]

(1.18)

The bilinears of oscillators fulfil the commutation relations (1.4)-(1.9).

The embedding of the generalized \(AdS\) subalgebra into the conformal algebra \(sp(2M)\) is achieved by identification of the (generalized) \(AdS\) translations with the mixture of translations and special conformal transformations \(P_{\alpha \beta}^{AdS} = P_{\alpha \beta} + \lambda^{2} \eta_{\alpha \beta \gamma \delta} K_{\gamma \delta}^{\gamma \delta}\) with some bilinear form \(\eta_{\alpha \beta \gamma \delta}\). (Note that keeping the same number of translation generators we keep dimension of the generalized space-time intact).

As argued in [1], \(\eta_{\alpha \beta \gamma \delta}\) has to have the factorized form: \(\eta_{\alpha \beta \gamma \delta} = V_{\alpha \gamma} V_{\beta \delta}\), where \(V_{\alpha \beta}\) is some nondegenerate antisymmetric form (thus requiring \(M\) to be even). In what follows the form \(V_{\alpha \beta}\) will be used to raise and lower indices according to the rule

\[
A_{\alpha} = V_{\beta \alpha} A^{\beta} , \quad A^{\alpha} = V_{\alpha \beta} A_{\beta} , \quad V_{\alpha \beta} V^{\alpha \gamma} = \delta_{\beta \gamma} .
\]

(1.19)

Thus, the generalized \(AdS\) translations have the form

\[
P_{\alpha \beta}^{AdS} = P_{\alpha \beta} + \lambda^{2} V_{\alpha \gamma} V_{\beta \delta} K_{\gamma \delta}^{\gamma \delta} = P_{\alpha \beta} + \lambda^{2} K_{\alpha \beta} .
\]

(1.20)

The commutation relations of \(P_{\alpha \beta}^{AdS}\) have the form

\[
[P_{\alpha \beta}^{AdS} , P_{\gamma \delta}^{AdS}] = 2 \lambda^{2} (V_{\beta \gamma} L_{\alpha \delta}^{AdS} + V_{\beta \delta} L_{\alpha \gamma}^{AdS} + V_{\alpha \gamma} L_{\beta \delta}^{AdS} + V_{\alpha \delta} L_{\beta \gamma}^{AdS}) ,
\]

(1.21)

where \(L_{\alpha \beta}^{AdS} = L_{\beta \alpha}^{AdS}\) are generators of the \(sp(M)\) subalgebra of \(gl_{M}\) which leaves invariant the symplectic form \(V_{\alpha \beta}\). The full generalized \(AdS\) subalgebra is \(sp(M) \oplus sp(M) \subset sp(2M)\). Its Lorentz subalgebra \(sp^{I}(M)\) identifies with the diagonal \(sp(M)\) while \(AdS\) translations span \(sp(M) \oplus sp(M)/sp^{I}(M)\). Note that the generalized \(dS\) algebra obtained from (1.20) by virtue of the sign change \(\lambda^{2} \rightarrow -\lambda^{2}\) is \(Sp(M, C)^{R}\).
1.2 Fock space and $Sp(2M)$ covariant equations

The $sp(2M)$ invariant equations of all massless fields in three and four dimensions are naturally described [8, 1] in terms of sections of the Fock fiber bundle over $M_M$. In other words, consider functions on $M_M$ taking values in the Fock module $F$:

$$|\Phi(b|X)\rangle = C(b|X) \ast |0\rangle \langle 0|,$$  \hspace{1cm} (1.22)

where $C(b|X)$ is some “generating function”

$$C(b|X) = \sum_{m=0}^{\infty} \frac{1}{m!} c_\beta \cdots c_m(X) b^{\beta_1} \cdots b^{\beta_m}$$  \hspace{1cm} (1.23)

and $|0\rangle \langle 0|$ is the Fock vacuum defined by the relations

$$a_\alpha \ast |0\rangle \langle 0| = 0, \quad |0\rangle \langle 0| \ast b^\alpha = 0.$$  \hspace{1cm} (1.24)

$|0\rangle \langle 0|$ can be realized as an element of the star-product algebra

$$|0\rangle \langle 0| = e^{-2a_\alpha b^\alpha}.$$  \hspace{1cm} (1.25)

Note that the Fock vacuum is the space-time constant projector

$$d|0\rangle \langle 0| = 0, \quad |0\rangle \langle 0| \ast |0\rangle \langle 0| = |0\rangle \langle 0|,$$  \hspace{1cm} (1.26)

where $d$ is de Rahm differential

$$d = dX^{\alpha \beta} \frac{\partial}{\partial X^{\alpha \beta}}, \quad d^2 = 0.$$  \hspace{1cm} (1.27)

As shown in [1] the relevant flat space $Sp(2M)$ covariant equation can be formulated in the form

$$d|\Phi(b|X)\rangle - w_0 \ast |\Phi(b|X)\rangle = 0,$$  \hspace{1cm} (1.28)

where

$$w_0 = \frac{1}{2} dX^{\alpha \beta} a_\alpha a_\beta.$$  \hspace{1cm} (1.29)

That the equation (1.28) does indeed describe all conformal field equations in $d = 3$ and $d = 4$ was shown in [8] and [1] for the cases of $M = 2$ and $M = 4$, respectively. In this paper we will consider the general case of any even $M$. It is worth to mention that the cases of $M = 8$, $M = 16$, and $M = 32$ were argued in [2] to correspond to conformal systems in $d = 6$, $d = 10$ and $d = 11$, respectively.

The Fock fiber bundle realization of the higher spin equations guarantees generalized conformal symmetry of the system along with its infinite-dimensional higher spin extension. Actually, let $w_0$ be some 1-form, taking values in the higher spin algebra identified with the star product algebra (i.e., the algebra of regular functions of oscillators acting on the Fock module $F$)

$$w_0(X) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} w_0^{\alpha_1 \cdots \alpha_n} (X) a_{\alpha_1} \cdots a_{\alpha_n} b^{\beta_1} \cdots b^{\beta_m},$$  \hspace{1cm} (1.30)
which satisfies the zero-curvature condition
\[ dw_0 = w_0 \star \wedge w_0. \quad (1.31) \]

The equations (1.28), (1.31) are invariant under the gauge transformations
\[ \delta w_0 = d\epsilon - [w_0, \epsilon]_\star, \quad (1.32) \]
\[ \delta |\Phi(b|X)\rangle = \epsilon \star |\Phi(b|X)\rangle, \quad (1.33) \]
where \( \epsilon(a, b|X) \) is an arbitrary infinitesimal gauge parameter. Any fixed vacuum solution \( w_0 \) of the equation (1.31) breaks the local higher spin symmetry to its stability subalgebra with the infinitesimal parameters \( \epsilon_0(a, b|X) \) satisfying the equation
\[ d\epsilon_0 - [w_0, \epsilon_0]_\star = 0. \quad (1.34) \]

Consistency of this equation is guaranteed by (1.31). As a result, (1.34) admits locally a pure gauge solution
\[ w_0(X) = -g^{-1}(X) \star dg(X), \quad (1.35) \]
where \( g(a, b|X) \) is some invertible element of the star-product algebra. The global symmetry parameters satisfying (1.34) then have the form
\[ \epsilon_0(X) = g^{-1}(X) \star \xi \star g(X), \quad (1.36) \]
where an arbitrary \( X \)-independent element \( \xi = \xi(a, b) \) of the star-product algebra describes parameters of the global higher spin symmetry which acts on the solutions of the equation (1.28) (for any given \( w_0 \)). In particular, the \( sp(2M) \) subalgebra spanned by bilinears of oscillators is thus shown to be a symmetry of the equation (1.28).

Analogously one solves the equation (1.28) in the form
\[ |\Phi(b|X)\rangle = g^{-1} \star |\Phi(b|X_0)\rangle, \quad (1.37) \]
where \( |\Phi(b|X_0)\rangle \) plays a role of initial data. The meaning of this formula is that the Fock module \( |\Phi(b|X_0)\rangle \) parametrises all combinations of the derivatives of the dynamical fields at \( X = X_0 \) which are allowed to be nonzero by the field equations. The formula (1.37) plays a role of the covariantized Taylor expansion reconstructing generic solution in terms of its derivatives at \( X = X_0 \). Note that the Fock module \( F \) is not unitary because it decomposes into an infinite sum of finite-dimensional (tensor) representations of the generalized noncompact Lorentz algebra \( sl_M(\mathbb{R}) \). Nevertheless, the fact that initial data of the problem are formulated in terms of the Fock module \( F \) is closely related to the fact (see e.g. [3, 4]) that the collection of unitary massless representations corresponding to this dynamical system in \( d = 4 \) is described by the unitary Fock module \( U \) known as singleton representation of \( sp(8) \). (It is also well known that unitary representations of the 4d conformal algebra associated with massless fields admit Fock realization in terms of appropriate
oscillators [10].) As shown in [8, 1], the modules $U$ and $F$ are related by some Bogolyubov transform.

The formulae (1.35), (1.37) will play the key role in our analysis. They allow one to solve the equations of motion explicitly provided that the gauge function $g(X)$ is found that corresponds to a chosen zero-curvature connection $w_0$. This program for the flat connection (1.29) was accomplished in [1]. In this paper we will find a family of such gauge functions $g(X)$ that all nonvanishing components of $w_0$ take values in the $AdS$ subalgebra $osp(L|M) \oplus osp(L|M)$ of $osp(2L|2M)$.

## 2 $Sp(M)$ and star product

As argued in [1], the generalized $AdS$ space is identified with $Sp(M)$. Let us note that the generalized conformal group $Sp(2M)$ does not act globally on $Sp(M)$ analogously to the usual conformal group acting in the Minkowski space-time by Möbius transformations which have singularities. Recall that usual Minkowski space-time is the big cell of the compactified Minkowski space. Analogously, the generalized Minkowski space-time is the big cell in the compactified generalized space-time $\mathcal{M}_M$.

The universal covering space of $Sp(M)$ can be thought of as a sort of deformation of the generalized Minkowski space-time being the big cell of $\mathcal{M}_M$.

The group $Sp(M)$ is realized by the $M \times M$ matrices $U_{\alpha\beta}$ satisfying

$$U_{\alpha\beta}U_{\gamma\delta}V^{\alpha\gamma} = V^{\beta\delta},$$

where $V^{\alpha\beta}$ is some non-degenerate antisymmetric form $V^{\alpha\beta} = -V^{\beta\alpha}$ ($M$ is even). The manifold $Sp(M)$ is $\frac{M(M+1)}{2}$ dimensional. It can be described by local coordinates $X^{\alpha\beta} = X^{\beta\alpha}$. The simplest parametrization is

$$U_{\alpha\beta} = (\exp(\lambda X))_{\alpha\beta},$$

where $\lambda$ is inverse "radius" of $Sp(M)$ introduced to compensate the space dimensionality of $X^{\alpha\beta}$. Note that a particular value of $\lambda \neq 0$ is irrelevant unless there are some other dimensionful parameters in the theory (e.g., the gravitational constant). The exponential in (2.2) is the matrix exponential of

$$X_{\alpha\beta} = V_{\gamma\alpha}X^{\gamma\beta}.$$ 

It is elementary to see that the parametrization (2.2) solves the group equation (2.1). The exponential parametrization (2.2) provides the universal covering space [11] of $Sp(M)$ (metaplectic group $Mp(M)$) topologically equivalent to $R^{\frac{M(M+1)}{2}}$, the big cell of $\mathcal{M}_M$.

$Sp(M)$ is invariant under the action of $Sp(M) \times Sp(M)$ generated by left and right actions of $Sp(M)$ on itself. Using the oscillator realization of $sp(M) \oplus sp(M) \subset sp(2M)$ we can set

$$w_0(X) = \omega_{\alpha\beta}(X)a^\alpha b^\beta + h_{\alpha\beta}(X)(a^\alpha a^\beta + \lambda^2 b^\alpha b^\beta),$$

(2.4)
where the “Lorentz connection” $\omega_{\alpha\beta}(X)$ and the “frame” $h_{\alpha\beta}(X)$ have the form

$$\omega_{\alpha\beta} = -\frac{1}{2} (d(U^{-1})_\alpha^\gamma U_{\gamma\beta} + dU_\alpha^\gamma (U^{-1})_{\gamma\beta}) , \quad (2.5)$$

$$h_{\alpha\beta} = \frac{1}{4\lambda} \left( dU_\alpha^\gamma (U^{-1})_{\gamma\beta} - d(U^{-1})_\alpha^\gamma U_{\gamma\beta} \right) , \quad (2.6)$$

which guarantees that $w_0$ satisfies (1.31). In the exponential parametrization (2.2) one gets

$$\omega^{\alpha\beta} = \lambda X^{\mu\nu} \left( \int_0^1 \exp(\lambda X^t)^{\mu\beta} \exp(\lambda X^{t\alpha}) dt - \int_{-1}^0 \exp(\lambda X^{t\alpha}) \exp(\lambda X^t)^{\mu\beta} dt \right) , \quad (2.7)$$

$$h^{\alpha\beta} = \frac{1}{4} dX^{\mu\nu} \int_{-1}^1 \exp(\lambda X^t)^{\mu\beta} \exp(\lambda X^{t\alpha}) dt , \quad (2.8)$$

where we used the identity $\delta e^A = \int_0^1 e^{At} \delta e^A e^{-t} dt$ valid for an arbitrary matrix $A$.

As expected, in the flat limit $\lambda \to 0$ one recovers (1.29).

Let us now present the star product pure gauge form (1.35) of the connection (2.4)-(2.6). The final result is

$$g = \frac{1}{\det \left\| \text{ch} \frac{\lambda X}{2} \right\|} \exp \left( -\frac{1}{\lambda} \left( th \frac{\lambda X}{2} \right) ^{\alpha\beta} (a_\alpha a_\beta + \lambda^2 b_\alpha b_\beta) \right) , \quad (2.9)$$

$$g^{-1} = \frac{1}{\det \left\| \text{ch} \frac{\lambda X}{2} \right\|} \exp \left( \frac{1}{\lambda} \left( th \frac{\lambda X}{2} \right) ^{\alpha\beta} (a_\alpha a_\beta + \lambda^2 b_\alpha b_\beta) \right) . \quad (2.10)$$

This formula is derived as follows. Let $sp(M)$ be realized in terms of bilinears of oscillators $\alpha_\alpha$ satisfying the commutation relations

$$[\alpha_\alpha, \alpha_\beta]_* = -2V_{\alpha\beta} , \quad (2.11)$$

with the star-product

$$(f * g)(\alpha) = \frac{1}{(2\pi)^M} \int f(\alpha + u)g(\alpha + v)e^{u_\alpha v_\alpha} d^M u d^M v . \quad (2.12)$$

Consider star-product algebra elements $g_1$ and $g_2$ of the form

$$g_1 = r_1 e^{\frac{1}{4} f_1^{\alpha\beta} a_\alpha a_\beta} , \quad g_2 = r_2 e^{\frac{1}{4} f_2^{\alpha\beta} a_\alpha a_\beta} . \quad (2.13)$$

with some $\alpha$-independent $r_1, r_2, f_1^{\alpha\beta}$ and $f_2^{\alpha\beta}$. Elementary evaluation of the Gaussian integrals shows that

$$g_{1,2} = g_1 * g_2 = r_{1,2} e^{\frac{1}{4} (f_1^{\alpha\beta} a_\alpha a_\beta)_{g_{1,2}}} . \quad (2.14)$$

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where
\[ r_{1,2} = \frac{r_1 r_2}{\sqrt{\text{det} \| f_1 f_2 + 1 \|}} \]  
(2.15)

and
\[ f_1 \circ f_2 = \left( \frac{1}{1 + f_2 f_1} (1 + f_2) - \frac{1}{1 + f_2 f_1} (1 - f_1) \right) \]  
(2.16)

(with the usual matrix multiplication on the right hand side: \( AB \rightarrow A_{\gamma}^{\alpha} B_\beta^\gamma \), \( \frac{1}{A} B \rightarrow (A^{-1})_\alpha^{\gamma} B_\gamma^\beta \)). Let us look for a map
\[ g(U) = r(U)e^{\frac{i}{2}f^{\alpha\beta}(U)\epsilon_{\alpha\beta}} \]  
(2.17)
of \( Sp(M) \) into the star-product algebra, such that
\[ g(U_1) * g(U_2) = g(U_1 U_2) = r(U_1 U_2)e^{\frac{i}{2}f^{\alpha\beta}(U_1 U_2)\epsilon_{\alpha\beta}} . \]  
(2.18)

Equivalently, one can use the inverse map \( U(f) \) requiring
\[ U(f_1)U(f_2) = U(f_1 \circ f_2) . \]  
(2.19)

As shown in appendix, the multiplication law (2.16) requires
\[ U^{\alpha\beta}(f) = \left( \frac{1 + f}{1 - f} \right)^{\alpha\beta} . \]  
(2.20)
The inverse formula is analogous
\[ f^{\alpha\beta}(U) = \left( \frac{U - 1}{1 + U} \right)^{\alpha\beta} . \]  
(2.21)
The normalization factor is
\[ r(U) = \frac{2^M}{\sqrt{\text{det} \| U + 1 \|}} . \]  
(2.22)

To derive (2.9) it remains to use (2.2) and to observe that the two \( sp(M) \) subalgebras of \( sp(2M) \) are generated by the two mutually commuting sets of oscillators
\[ \alpha_\alpha^{\pm} = \frac{a_\alpha}{\sqrt{\lambda}} \pm \sqrt{\lambda} V_{\beta\alpha} b^\beta = \frac{1}{\sqrt{\lambda}} (a_\alpha \pm \lambda b_\alpha) , \]  
(2.23)
satisfying the commutation relations
\[ [\alpha_\alpha^{\pm}, \alpha_\beta^{\pm}]_* = \pm 2V_{\alpha\beta} . \]  
(2.24)
The map (2.20) has a number of interesting properties. In particular,
\[ U^{-1}(f) = U(-f) , \]  
(2.25)
\[ U(f) = -U^{-1}(-f^{-1}) . \]  
(2.26)
The property (2.25) is a consequence of the elementary fact (see e.g. [12]) that the star product (2.12) admits an antiautomorphism \( \rho(g(\alpha)) = g(i\alpha) \), i.e. \( \rho(g_1) * \rho(g_2) = \rho(g_2 * g_1) \). From (2.17) it follows that \( \rho(U(f)) = U(-f) \). The natural group antiautomorphism is \( \rho(U) = U^{-1} \). The formula (2.25) identifies the antiautomorphism \( \rho \) in the star product algebra with that of the group \( Sp(M) \).

The formula (2.26) is more interesting. It does not have a global interpretation within \( Sp(M) \) being singular at degenerate \( f^{\alpha\beta} \) (in particular for \( f^{\alpha\beta} = 0 \) and, therefore, \( U = I \)). However, these maps are expected to have global meaning in \( M \) where one can define inversion by analogy with the flat case considered in [2]

\[
I(f) = -f^{-1}, \quad I(U) = -U^{-1}.
\]

The formula (2.26) implies that these two definitions are consistent with each other. Note that inversion defined this way maps unit element of \( Sp(M) \) to the central element \( -I \) which does not belong to the connected component of unity \( PSp(M) \subset Sp(M) \).

### 3 Arbitrary coordinates

The gauge function (2.9) corresponds to the exponential realization of \( Sp(M) \), thus yielding global coordinates which cover the metaplectic group \( Mp(M) \). Our formalism allows one to write down explicit form of vacuum gauge connections (Cartan forms) in arbitrary coordinates, however. Indeed, let us consider a gauge function of the form

\[
g = \sqrt{\det \left| 1 - \lambda^2 f^2(X) \right|} \exp \left( -f(X)^{\alpha\beta} (a_\alpha a_\beta + \lambda^2 b_\alpha b_\beta) \right),
\]

\[
g^{-1} = \sqrt{\det \left| 1 - \lambda^2 f^2(X) \right|} \exp \left( f(X)^{\alpha\beta} (a_\alpha a_\beta + \lambda^2 b_\alpha b_\beta) \right),
\]

(3.1)

where \( f^{\alpha\beta}(X) = f^{\beta\alpha}(X) \) is an arbitrary function of matrix coordinates \( X^{\alpha\beta} \). The zero-curvature connection (1.35) can be written in the form

\[
w_0 = -g(-f) \left( \frac{\partial f^{\gamma\lambda}_1}{\partial X^{\alpha\beta}} \frac{\partial}{\partial f^{\gamma\lambda}_1} g(f_1) \right) \bigg|_{f_1 = f}.
\]

(3.2)

Direct computation leads to the expressions for the “Lorentz connection” and “frame”

\[
h^{\alpha\beta} = dX^{\rho\sigma} \left( \frac{1}{1 - \lambda^2 f^2} \right)^{\alpha\gamma} \left( \frac{\partial f^{\gamma\lambda}_1}{\partial X^{\rho\sigma}} - \lambda^2 f^{\mu}_\gamma \frac{\partial f^{\lambda\mu}}{\partial X^{\rho\sigma}} f^{\nu}_\lambda \right) \left( \frac{1}{1 - \lambda^2 f^2} \right)^{\beta},
\]

(3.3)

\[
\omega^{\alpha\beta} = 2\lambda^2 dX^{\rho\sigma} \left( \frac{1}{1 - \lambda^2 f^2} \right)^{\alpha\gamma} \left( \frac{\partial f^{\gamma\mu}_1}{\partial X^{\rho\sigma}} f^{\lambda}_\mu - f^{\mu}_\gamma \frac{\partial f^{\lambda\mu}}{\partial X^{\rho\sigma}} \right) \left( \frac{1}{1 - \lambda^2 f^2} \right)^{\beta}.
\]

(3.4)

Note that from these formulae it follows that

\[
\omega^{\alpha\beta} = 2h^{\alpha\gamma} f^{\gamma}_\beta - 2f^{\alpha\gamma} h^{\gamma}_\beta + \lambda^2 f^{\alpha\gamma} \omega^{\gamma\lambda} f^{\lambda}_\beta.
\]

(3.5)
An arbitrary function \( f^{\alpha\beta}(X) \) parametrizes various coordinate choices in \( Sp(M) \). A relationship with the coordinates of the exponential parametrization obviously is

\[
f(\tilde{X}) = \frac{1}{\chi} th\frac{\lambda X}{2}
\]

which implies locally

\[
sh\frac{\lambda X}{2} = \frac{\lambda f(\tilde{X})}{\sqrt{1 - \lambda^2 f^2(\tilde{X})}}, \quad ch\frac{\lambda X}{2} = \frac{1}{\sqrt{1 - \lambda^2 f^2(\tilde{X})}}.
\]

The formulae (3.3) and (3.4) thus provide a representation for Cartan forms in arbitrary coordinates associated with one or another function \( f^{\alpha\beta}(X) \). Consider now a few particular examples. Let \( f^{\alpha\beta}(X) \) be of the form

\[
f^{\alpha\beta}(X) = \phi(\det \|X\|) X^{\alpha\beta}.
\]

The corresponding connections are

\[
h^{\alpha\beta} = \phi \cdot \left( \frac{1}{1 - \lambda^2 \phi^2 X^2} \right)^{\alpha\gamma} (dX^\gamma - \lambda^2 \phi^2 X^\gamma dX^\mu X^\nu X^\lambda) \left( \frac{1}{1 - \lambda^2 \phi^2 X^2} \right)^\beta + \tilde{\phi} \cdot dX^{\rho\sigma} (X^{-1})_{\rho\sigma} \left( \frac{X}{1 - \lambda^2 \phi^2 X^2} \right)^{\alpha\beta},
\]

where

\[
\tilde{\phi} = \frac{\partial \phi}{\partial \ln \det \|X\|}.
\]

Another useful example results from

\[
f^{\alpha\beta}_\pm(X) = \left( \frac{X}{1 \pm \sqrt{1 - \lambda^2 X^2}} \right)_{\alpha\beta}.
\]

The corresponding gauge function is

\[
g^\pm = \det \left\| \frac{1 + \lambda X \pm \sqrt{1 - \lambda^2 X^2}}{\lambda X} \right\| \exp \left( - \left( \frac{X}{1 \pm \sqrt{1 - \lambda^2 X^2}} \right)^{\alpha\beta} (a_\alpha a_\beta + \lambda^2 b_\alpha b_\beta) \right).
\]

In these “stereographic” coordinates the “frame” gets the following simple form

\[
h^{\alpha\beta} = \frac{1}{2} \left( \frac{1}{\sqrt{1 - \lambda^2 X^2}} \right)^{\alpha\gamma} dX^\gamma \lambda \left( \frac{1}{\sqrt{1 - \lambda^2 X^2}} \right)^\beta.
\]

Let us now compare this formula with those obtained in [8, 9] to describe massless fields in \( AdS_3(M = 2) \) and \( AdS_4(M = 4) \).

Let us first consider the case \( M = 2 \). Using for example \( g^+ \), from (3.13) one obtains

\[
g = \frac{2\sqrt{z}}{1 + \sqrt{z}} \exp \left( - \left( \frac{1}{1 + \sqrt{z}} x^{\alpha\beta} (a_\alpha a_\beta + \lambda^2 b_\alpha b_\beta) \right) \right),
\]
\( g^{-1} = \frac{2\sqrt{z}}{1 + \sqrt{z}} \exp \left( \frac{1}{1 + \sqrt{z}} x^{\alpha \beta} (a_\alpha a_\beta + \lambda^2 b_\alpha b_\beta) \right), \) \hspace{1cm} (3.16)

where \( z = 1 + \frac{1}{2} \lambda^2 x_\alpha x_\beta x^{\alpha \beta} \). The "frame" and the "Lorentz connection" are

\[ h_{\alpha \beta} = \frac{1}{2z} dx_{\alpha \beta}, \quad \omega_{\alpha \beta} = \frac{1}{2z} (dx_{\alpha \gamma} x_{\beta \gamma} + dx_{\beta \gamma} x_{\alpha \gamma}). \] \hspace{1cm} (3.17)

To derive this result, which reproduces that of [8], we used a simple fact that, when \( M = 2 \), any antisymmetric matrix is proportional to \( V_{\alpha \beta} \) and, therefore, any polynomial of matrix coordinates \( P(x)_{\alpha \beta} \) decomposes into a combination of its symmetric part \( P_S(x_{\mu \nu} x^{\mu \nu} x_{\alpha \beta}) \) and antisymmetric part \( P_A(x_{\mu \nu} x^{\mu \nu} V_{\alpha \beta}) \). From (3.17) it follows that the metric tensor is

\[ g_{mn} = \frac{1}{2} h_{\alpha \beta, n} h_{\alpha \beta, m} = \frac{1}{4} \frac{\eta_{mn}}{(1 + \lambda^2 x_n x^m)^2}, \] \hspace{1cm} (3.18)

where

\[ x_n = \sigma_n^{\alpha \beta} x_{\alpha \beta}, \quad x_{\alpha \beta} = \frac{1}{2} \sigma_{\alpha \beta}^n x_n, \] \hspace{1cm} (3.19)

and \( \sigma_n^{\alpha \beta} \) is a set of basis symmetric real matrices normalized to satisfy

\[ \sigma_n^{\alpha \beta} \sigma_{m \alpha \beta} = 2 \eta_{mn}, \] \hspace{1cm} (3.20)

where \( \eta_{mn} \) is the flat Minkowski metric.

To consider the 4d case we embed \( AdS_4 \) space-time into \( M_4 \) as follows

\[ X^{\alpha \beta} = \begin{pmatrix} 0 & x^{\alpha \beta} \\ x^{\alpha \beta} & 0 \end{pmatrix}, \] \hspace{1cm} (3.21)

where \( \alpha, \beta = 1, 2 \), \( \bar{\alpha}, \bar{\beta} = 3, 4 \), and \( x_{\alpha \beta}^{\bar{\alpha} \bar{\beta}} \) are local \( AdS_4 \) coordinates which can be expressed via the vector coordinates \( x^n (n = 0 \ldots 3) \) with the aid of Pauli matrices \( \sigma_n^{\alpha \beta} = (I, \sigma_1^{\alpha \beta} \ldots \sigma_3^{\alpha \beta}) \) as

\[ x^n = \sigma_n^{\alpha \beta} x^{\alpha \beta}, \quad x_{\alpha \beta}^{\bar{\alpha} \bar{\beta}} = \frac{1}{2} x_n \sigma_{n \alpha \beta}^{n \alpha \beta}, \quad \sigma_n^{\alpha \beta} \sigma_m^{\alpha \beta} = 2 \eta_{mn}. \] \hspace{1cm} (3.22)

The gauge function and gravitational fields resulting from (3.13) and (3.14) are

\[ g = \left( \frac{2\sqrt{z}}{1 + \sqrt{z}} \right)^2 \exp \left( \frac{1}{1 + \sqrt{z}} x^{\bar{\alpha} \bar{\beta}} (a_\bar{\alpha} a_\bar{\beta} + \lambda^2 b_\bar{\alpha} b_\bar{\beta}) \right), \] \hspace{1cm} (3.23)

\[ h_{\bar{\alpha} \bar{\beta}} = \frac{1}{2z} dx_{\bar{\alpha} \bar{\beta}}, \] \hspace{1cm} (3.24)

\[ \omega_{\bar{\alpha} \bar{\beta}} = \frac{1}{2z} (dx_{\bar{\alpha}}^\bar{\gamma} x_{\bar{\beta} \gamma} + dx_{\bar{\beta}}^\bar{\gamma} x_{\bar{\alpha} \gamma}), \quad \bar{\omega}_{\bar{\alpha} \bar{\beta}} = \frac{1}{2z} (dx_{\bar{\alpha}}^\gamma x_{\bar{\beta} \gamma} + dx_{\bar{\beta}}^\gamma x_{\bar{\alpha} \gamma}), \] \hspace{1cm} (3.25)

where \( z = 1 + \frac{1}{2} \lambda^2 x_\alpha x_\beta x^{\alpha \beta} = 1 + \lambda^2 x_n x^n \). These \( AdS_4 \) gravitational fields coincide with those found in [9].
4 Symmetries

Having fixed some vacuum solution $w_0$ of (1.31), the local higher spin symmetry is broken down to the global one with the parameter $\epsilon_0(a, b|X)$ satisfying (1.34). Once the vacuum solution $w_0$ is fixed in the pure gauge form (1.35) with some gauge function $g$, it is easy to find the gauge parameter $\epsilon_0(a, b|X)$ of the leftover global symmetry. Indeed let the generating parameter $\xi(a, b; \mu, \eta)$ in (1.36) be of the form

$$\xi = \xi_0 \exp(a_\alpha \mu^\alpha - b^\alpha \eta_\alpha), \quad (4.1)$$

where $\xi_0$ is an infinitesimal constant while $\mu^\alpha$ and $\eta_\alpha$ are constant parameters. An arbitrary symmetry with star-product polynomial parameters can be obtained via differentiation of $\xi$ with respect to $\mu^\alpha$ and $\eta_\alpha$. Substitution of (2.9) into (1.36) gives

$$\epsilon_0(a, b; \mu, \eta|X) = g^{-1} \ast \xi \ast g = \xi_0 \exp(a_\alpha \hat{\mu}^\alpha - b^\alpha \hat{\eta}_\alpha), \quad (4.2)$$

where

$$\hat{\mu}^\alpha = ch(\lambda X)^{\alpha\beta} \mu_\beta - \frac{sh(\lambda X)^{\alpha\beta}}{\lambda} \eta_\beta, \quad \hat{\eta}^\alpha = ch(\lambda X)^{\alpha\beta} \eta_\beta - \lambda \cdot sh(\lambda X)^{\alpha\beta} \mu_\beta. \quad (4.3)$$

According to (3.7), in the arbitrary coordinates associated with the function $f_{\alpha\beta}(X)$ of section 3, we have

$$\hat{\mu}_\alpha = \left(1 + \lambda^2 f^2(X)\right)^{\alpha\beta} \frac{2 f(X)}{1 - \lambda^2 f^2(X)} \eta_\beta, \quad (4.4)$$

$$\hat{\eta}_\alpha = \left(1 + \lambda^2 f^2(X)\right)^{\alpha\beta} \frac{2 f(X)}{1 - \lambda^2 f^2(X)} \mu_\beta. \quad (4.5)$$

The global symmetry transformation of the higher spin generating function

$$\delta |\Phi(b|X)\rangle \equiv \epsilon_0 \ast |\Phi(b|X)\rangle = \xi_0 \exp(-\hat{\eta}_\alpha b^\alpha + \frac{1}{2} \hat{\eta}^\alpha \hat{\mu}_\alpha) \cdot C(b + \hat{\mu}|X) \ast |0\rangle\langle 0| \quad (4.6)$$

implies

$$\delta C(b|X) = \xi_0 C(b + \hat{\mu}|X) \exp\left(\frac{1}{2} \hat{\eta}^\alpha \hat{\mu}_\alpha - b^\alpha \hat{\eta}_\alpha\right). \quad (4.7)$$

The dynamical fields are associated with the scalar $c(X) = C(0|X)$ and scalar $c_\alpha(X) = \frac{\partial}{\partial b^\alpha} C(b|X)\big|_{b=0}$ in the expansion (1.23). (All other fields in $C(b|X)$ are expressed via derivatives of the dynamical fields [1].) Their transformation laws are

$$\delta c(X) = \xi_0 C(\hat{\mu}|X) \exp\left(\frac{1}{2} \hat{\eta}^\alpha \hat{\mu}_\alpha\right), \quad (4.8)$$

$$\delta c_\alpha(X) = \xi_0 \left(\frac{\partial}{\partial b^\alpha} C(b + \hat{\mu}|X)\big|_{b=0} - \hat{\eta}_\alpha C(\hat{\mu}|X)\right) \exp\left(\frac{1}{2} \hat{\eta}^\alpha \hat{\mu}_\alpha\right). \quad (4.9)$$

Differentiating over the parameters $\mu^\alpha$ and $\eta_\alpha$ and setting them then equal to zero one obtains explicit expressions for the higher spin symmetry transformations associated with any symmetry parameters $\epsilon_0(a, b|X)$ polynomial in the oscillators $a$ and $b$. In particular, the transformation law with the parameters bilinear in the oscillators reproduces the $Sp(2M)$ generalized conformal transformations in the generalized $AdS$ space-time $Sp(M)$. 

13
5 Light-like solutions

Once the gauge function $g$ is known one solves the system of free field equation (1.28) for all massless fields via (1.37). Let us consider basis light-like solutions generated by the initial data of the form

$$C(b|0) = C_0 \exp(\kappa_\alpha b^\alpha),$$

where $C_0$ is an arbitrary constant and $\kappa^\alpha$ is some space-time constant spinor. According to (1.22) the Fock representation of the initial data has the form

$$|\Phi(b|0)\rangle = C_0 \exp(\kappa_\alpha b^\alpha) \star |0\rangle \langle 0|.$$  

So the dynamical problem is solved by

$$|\Phi(b|X)\rangle = g^{-1}(X) \star |\Phi(b|0)\rangle = \frac{C_0}{\det \| ch(\lambda X) \|} e^{\frac{1}{2} (\lambda h(\lambda X))^{\alpha \beta} (a_\alpha a_\beta + \lambda^2 b_\alpha b_\beta) \star \epsilon^{\kappa_\alpha b^\alpha} \star e^{-2a_\alpha b^\alpha}}.$$  

Elementary evaluation of Gaussian integrals gives the following result

$$C(b|X) = \frac{C_0}{\sqrt{\det \| ch(\lambda X) \|}} \exp \left( t_{\alpha \beta}(\lambda^2 b_\alpha b_\beta + \kappa_\alpha \kappa_\beta) + p_\alpha \kappa_\alpha b^\alpha \right),$$

where we use notations

$$t_{\alpha \beta} = \left( \frac{\lambda h(\lambda X)}{2 \lambda} \right)_{\alpha \beta}, \quad p_\alpha = (ch^{-1}(\lambda X))_\alpha \beta,$$

equivalent by virtue of (3.7) to

$$t_{\alpha \beta} = \left( \frac{f(X)}{1 + \lambda^2 f^2(X)} \right)_{\alpha \beta}, \quad p_\alpha = \left( \frac{1 - \lambda^2 f^2(X)}{1 + \lambda^2 f^2(X)} \right)_\alpha \beta.$$

Let us stress that, according to [1, 2], for the particular case of $M = 4$ the obtained expressions describe solutions of massless equations for all spins in $AdS_4$, constructed in [9]. Using (3.23), these solutions take the form

$$C(b|x) = z \exp \left( \frac{x^{\alpha \beta}}{2} (\kappa_\alpha \kappa_\beta + \lambda^2 b_\alpha b_\beta) + \sqrt{z} \kappa_\alpha b^\alpha \right).$$

For the case of $M = 2$ we get solutions of the $AdS_3$ massless equations discussed in [8] of the form

$$C(b|x) = \sqrt{z} \exp \left( \frac{x^{\alpha \beta}}{2} (\kappa_\alpha \kappa_\beta + \lambda^2 b_\alpha b_\beta) + \sqrt{z} \kappa_\alpha b^\alpha \right).$$

Here we make use of the gauge function (3.15).

For the dynamical fields we obtain

$$c(X) = C_0 \sqrt{\det \| 1 - \lambda^2 f^2(X) \| \| 1 + \lambda^2 f^2(X) \|} \exp(t_{\alpha \beta} \kappa_\alpha \kappa_\beta).$$
\[ c_\alpha(X) = C_0 \sqrt{\det \left| \frac{1 - \lambda^2 f^2(X)}{1 + \lambda^2 f^2(X)} \right|} p_\alpha \kappa \kappa \exp \left( t^{\alpha \beta} \kappa \kappa \right). \quad (5.10) \]

Substitution of \( \epsilon_0 \) into (1.33) gives the global higher spin symmetry transformation of the solution (5.3)

\[ \delta C(b|X) = C_0 \xi_0 \sqrt{\det \left| \frac{1 - \lambda^2 f^2(X)}{1 + \lambda^2 f^2(X)} \right|} \exp \left( t^{\alpha \beta} \lambda^2 (b_\alpha + \hat{\mu}_\alpha)(b_\beta + \hat{\mu}_\beta) + \right. \]

\[ \left. t^{\alpha \beta} \kappa \kappa + p_\beta \kappa \kappa (b_\beta + \hat{\mu}_\beta) - \hat{\eta}_\alpha (b^\alpha + \frac{1}{2} \hat{\mu}^\alpha) \right). \quad (5.11) \]

The flat limit \( \lambda \to 0 \) gives

\[ \delta C(b|X) = C_0 \xi_0 \exp \left( \frac{1}{2} X^{\alpha \beta} (\kappa \kappa - 2 \kappa \eta + \eta_\alpha \eta_\beta) + \right. \]

\[ \left. + b^\alpha (\kappa - \eta_\alpha) + \kappa \eta_\alpha + \frac{1}{2} \mu_\alpha \eta^\alpha \right). \quad (5.12) \]

For the dynamical fields we get the plane wave solutions

\[ c_{\text{plane}}(X) = C_0 e^{\frac{1}{2} X^{\alpha \beta} \kappa \kappa}, \quad c_{\alpha \text{plane}}(X) = C_0 \kappa \alpha e^{\frac{1}{2} X^{\alpha \beta} \kappa \kappa}, \quad (5.13) \]

with the twistorial “wave vector” \( K_{\alpha \beta} = \frac{1}{2} \kappa \kappa \).

The solution deformed to the \( \text{AdS} \) case is not strictly speaking plane wave. However it is “conformally plane wave” in the sense that it has still leftover generalized conformal invariances identified with such global symmetry transformations that

\[ \delta_{\epsilon_0} |\Phi(b|X)\rangle = 0. \quad (5.14) \]

It is easy to see using (4.2) that this condition is solved by any parameter of the form

\[ \xi = f(a, b) \ast (\rho^\alpha a_\alpha), \quad (5.15) \]

where \( \rho \) is an arbitrary parameter such that \( \rho^\alpha \kappa \alpha = 0 \) and \( f(a, b) \) is an arbitrary function. Indeed, according to (1.37)

\[ \delta_{\epsilon_0} |\Phi(b|X)\rangle = g^{-1} \ast \xi \ast C(b|0) \ast |0\rangle \langle 0| = g^{-1} \ast f \ast (\rho^\alpha a_\alpha) \ast e^{\kappa \alpha b_\alpha} \ast |0\rangle \langle 0| = 0. \quad (5.16) \]

## 6 Superextension

The star-product formalism we use admits a straightforward generalization to the supersymmetric case associated with \( OSp(L|2M) \) where \( L \) is an arbitrary integer. To describe \( OSp(L|2M) \) superalgebra let us introduce the Clifford elements \( \psi_i \) (\( i = 1 \ldots L \)) satisfying the anticommutation relations

\[ \{ \psi_i, \psi_j \}_*= \eta_{ij}, \quad (6.1) \]
where $\eta_{ij} = \eta_{ji}$ is some non-degenerate symmetric form. The Clifford star-product in (6.1) is defined (see, e.g., [12]) according to

$$(f \ast g)(\psi) = \frac{1}{2\pi} \int f(\psi + \phi)g(\psi + \chi)e^{-2\chi^i\phi_i}d^L\phi d^L\chi,$$

where $\chi_i$ and $\phi_i$ are anticommuting variables. The supercharges

$$Q_{i\alpha} = a_{\alpha}\psi_i, \quad S_i^\alpha = b^\alpha\psi_i$$

satisfy

$$\{Q_{i\alpha}, Q_{i'\beta}\} = \eta_{ij}P_{\alpha\beta}, \quad \{S_i^\alpha, S_j^\beta\} = \eta_{ij}K^{\alpha\beta}.$$ (6.4)

Let the Grassmann odd coordinates $\theta_{i\alpha}$ be associated with the $Q$-supergenerators. It is convenient to require the differential $d\theta^{i\alpha}$ to anticommute to $dX^{\alpha\beta}$ and $\theta_{i\alpha}$.

It is easy to see [1] that the gauge function

$$g = e^{-X^{\alpha\beta}a_{\alpha}a_{\beta} - \theta^{i\alpha}a_{\alpha}\psi_i}$$ (6.5)

reproduces the flat superspace vacuum 1-form

$$w_0 = \left(dX^{\alpha\beta} + \frac{1}{2}d\theta^{i\alpha}\theta_i^{\beta}\right)P_{\alpha\beta} + d\theta^{i\alpha}Q_{i\alpha}.$$ (6.6)

The left Fock module $|\Phi(b, \psi^+|X, \theta\rangle\rangle$ satisfies the $osp(L|2M)$ supersymmetric equation

$$(d - w_0) \ast |\Phi(b, \psi^+|X, \theta\rangle\rangle = 0,$$ (6.7)

where the supersymmetric Fock vacuum $|0\rangle\langle0|$ in addition to (1.26) is annihilated by the $\frac{1}{2}L$ (in case of even $L$) or $\frac{1}{2}(L-1)$ (in case of odd $L$) annihilation Clifford elements $\psi^-$ and, when $L$ is odd, it is an eigenvector of the central element $\Psi^L = \psi_1 \cdots \psi_L$

$$\Psi^L \ast |0\rangle\langle0| = \pm |0\rangle\langle0|.$$ (6.8)

Let us now consider free field dynamics in the generalized $AdS$ superspace. The corresponding supersymmetry algebra is $osp(L, M) \oplus osp(L, M)$ while the superspace is $osp(L, M)$. To describe background fields (i.e., Cartan forms) in such a space we follow the same procedure as for $Sp(M)$.

The $OSp(L|M)$ supergroup is realized by $(M + L) \times (M + L)$ matrices $U_A^B$, where $A = (\alpha, i)(\alpha = 1 \cdots M, i = 1 \cdots L)$, satisfying the group condition

$$U_A^B U_C^D \Omega^{AC} = \Omega^{BD},$$ (6.9)

where $\Omega^{AB} = (-1)^{\pi_A\pi_B}\Omega^{BA}$ and

$$\pi_A = \begin{cases} 1, & A = i \\ 0, & A = \alpha \end{cases}.$$ (6.10)

It can be described by the local supercoordinates $X^{AB} = (-1)^{\pi_A\pi_B}X^{BA}$ with the aid of the exponential parametrization

$$U_A^B = \exp(\lambda X)_A^B.$$ (6.11)
Let us introduce the super-oscillators $a_A, b^A$ satisfying the (anti)commutation relations
\[ a_A \star b^B - (-1)^{\pi_A \pi_B} b^B \star a_A = \delta_A^B , \tag{6.10} \]
with respect to the star-product
\[ (f \star g)(a, b|X) = \frac{1}{2\pi^L} \int f(a+u, b+t)g(a+s, b+v) e^{2i(u^A s_A - v^A u_A)} du dv , \tag{6.11} \]
where the statistics of the integration variables is defined according to
\[ u_A u_B = (-1)^{\pi_A \pi_B} u_B u_A . \]
The integration measure is chosen so that 1 is the unit element of the star-product algebra (6.11).

Using the oscillator realization of $osp(L|M) \oplus osp(L|M) \subset osp(2L|2M)$ we can set
\[ w_0 = \omega^{AB} a_B b_A + h^{AB} (a_B a_A + \lambda^2 b_B b_A) . \tag{6.12} \]
The analysis analogous to that of section 2 shows that the gauge function
\[ g = \sqrt{s det\|1 - \lambda^2 f^2(X)\|} \exp\left( - f^{AB}(X) (a_B a_A + \lambda^2 b_B b_A) \right) \tag{6.13} \]
provides the "Lorentz connection" $\omega_{AB}$ and the "frame" $h_{AB}$ of the form
\[ \omega_{AB} = \frac{1}{2} (dU_A^C(U^{-1})_{CB} + d(U^{-1})_A^C U_{CB}) , \tag{6.14} \]
\[ h_{AB} = \frac{1}{4\lambda} (dU_A^C(U^{-1})_{CB} - d(U^{-1})_A^C U_{CB}) , \tag{6.15} \]
where
\[ U_A^B = \left( \frac{1 + \lambda f(X)}{1 - \lambda f(X)} \right)_A^B . \tag{6.16} \]
The relationship between $h_{AB}$ and $\omega_{AB}$ is analogous to (3.5)
\[ \omega_{AB} = 2h_A^C f_{CB} - 2f_A^C h_{CB} + \lambda^2 f_A^C \omega_{CD} f_{DB} . \tag{6.17} \]

Here is the list of the gauge functions and corresponding Cartan forms in different coordinates.
1. The exponential parametrization (6.9)
\[ g = \frac{1}{s det\|c h^{\lambda X} \|} \exp\left( - \left( th \frac{\lambda X}{2} \right)^{AB} (a_B a_A + \lambda^2 b_B b_A) \right) , \tag{6.18} \]
\[ \omega_{AB} = \frac{\lambda}{2} \left( \int_0^1 \exp(\lambda X t)_A^C dX_C^D \exp(\lambda X t)_{DB} dt - \int_{-1}^0 \exp(\lambda X t)_A^C dX_C^D \exp(\lambda X t)_{DB} dt \right) \tag{6.19} \]
\[ h_{AB} = \frac{1}{4} \int_{-1}^1 \exp(\lambda X t)_A^C dX_C^D \exp(\lambda X t)_{DB} dt \tag{6.20} \]
2. \( f^{AB} = \phi(s\text{det}||X||)X^{AB} \)

\[
g = \sqrt{s\text{det}||1 - \lambda^2 \phi^2 \cdot X^2||} \exp \left( -\phi X^{AB}(a_B a_A + \lambda^2 b_B b_A) \right), \quad (6.21)
\]

\[
h_{AB} = \phi \cdot \left( \frac{1}{1 - \lambda^2 \phi^2 X^2} \right)_A C (dX^D_C - \lambda^2 \phi^2 X^M_C dX^N_M)(X^D - \lambda^2 \phi^2 X^2)_{DB} - \tilde{\phi} \cdot \left( \frac{X}{1 - \lambda^2 \phi^2 X^2} \right)_{AB} (X^{-1}) (X)_{DB}, \quad (6.22)
\]

where

\[
\tilde{\phi} = \frac{\partial \phi}{\partial \ln s\text{det}||X||}. \quad (6.23)
\]

3. The “stereographic” coordinates \( f^{AB}(X) = \left( \frac{X}{1 + \sqrt{1 - \lambda^2 X^2}} \right)^{AB} \)

\[
g^\pm = \sqrt{s\text{det}||1 - \lambda^2 f^2(X)||} \exp \left( -\left( \frac{X}{1 + \sqrt{1 - \lambda^2 X^2}} \right)^{AB} (a_B a_A + \lambda^2 b_B b_A) \right), \quad (6.24)
\]

\[
h_{AB} = \frac{1}{2} \left( \frac{1}{\sqrt{1 - \lambda^2 X^2}} \right)_A C (dX^D_C - X^M_C dX^N_M)(X^D - \lambda^2 \phi^2 X^2)_{DB}. \quad (6.25)
\]

In the supersymmetric case, the global higher spin symmetry transformation law for the generating function \( C(b|X) \) with respect to infinitesimal parameter

\[
\xi = \xi_0 \exp(\mu^A a_A - \lambda^A \eta_A)
\]

is analogous to (4.7)

\[
\delta C(b|X) = C(b + \hat{\lambda} |X) \exp \left( -\left( b^A + \frac{1}{2} \hat{\mu}^A \right) \hat{\eta}_A \right), \quad (6.26)
\]

where

\[
\hat{\mu}^A = ch(\lambda X)^{AB} \mu_B - \frac{sh(\lambda X)^{AB}}{\lambda} \eta_B, \quad \hat{\eta}^A = ch(\lambda X)^{AB} \eta_B - \lambda \cdot sh(\lambda X)^{AB} \mu_B. \quad (6.27)
\]

It is straightforward to extend the rest of the analysis to the dynamics in the generalized superspace. Also, having found left invariant forms it is elementary to write down world-line particle actions (see e.g., [4, 5, 1] for more details and references). The form of the world-line particle Lagrangian suggested in [1] is

\[
L = \dot{X}^{AB} w_{0AB}(a,b|X) + a_A \dot{b}^A, \quad (6.28)
\]

where \( dX^{AB} w_{0AB}(a,b|X) = w_0(a,b|X) \) is the vacuum 1-form satisfying the zero-curvature equation (1.31) and dot denotes the derivative with respect to the world line parameter. Applying the Stokes theorem and using (1.31) the particle action
(6.28) can be rewritten in the string form as an integral over a two-dimensional surface bounded by a particle trajectory and parameterized by $\sigma^I$

$$S = \int_{S^2} (w_0(a, b|X) \star w_0(a, b|X) + da_A \wedge db^A +$$

$$+(da_A \partial_{\partial a_A} + db^A \partial_{\partial b^A}) \wedge w_0(a, b|X)), \quad (6.29)$$

where the pullback is defined as usual

$$w_0(a, b|X) = d\sigma^I \partial_{\partial a_A} w_{0AB}(a, b|X), \quad da_A = d\sigma^I \partial_{\partial a_A}, \quad db^A = d\sigma^I \partial_{\partial b^A}. \quad (6.30)$$

The problem of calculating Cartan superforms in $osp(1|2M)$ superspace was considered in [5] where a particular parametrization was found with bosonic Cartan forms being at most bilinear in fermionic coordinates. Note that the star-product algebra formalism simplifies some of the computational problems being reduced to evaluation of elementary Gaussian integrals.

### 7 Conclusions

It is demonstrated how the star-product algebra formalism can be applied to the calculation of the vacuum of fields of the generalized $AdS$ space associated with $sp(M) \oplus sp(M)$ subalgebra of the recently proposed in [2] generalized conformal symmetry $Sp(2M)$. The method is universal working equally well for the supersymmetric case of $OSp(L, M)$ with any $M$ and $L$. The formalism of star-product algebra is shown to be very efficient for solving free field equations in non-trivial (generalized conformally flat) geometries in $\mathcal{M}_M$ and calculating Cartan forms in arbitrary coordinates. Hopefully it may have applications to formulations of world line (super)particle dynamics as well as (super)string actions in $M$ theory backgrounds.

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### A Appendix

Let us prove that the formula

$$g(U) = \frac{2^M}{\sqrt{\det \|U + 1\|}} \exp \left( \frac{1}{2} \left( \frac{U - 1}{U + 1} \right)^{\alpha_\beta} \alpha_\alpha \alpha_\beta \right) \quad (A.1)$$
respects the group multiplication law of $Sp(M)$, i.e. that the formula (2.20) solves the equation (2.19). Let us look for $U(f)$ in the form

$$U(f) = \sum_{n=0}^{\infty} a_n f^n,$$  \hspace{1cm} (A.2)

where $a_n$ are some coefficients. Hence $U(f_1)U(f_2) = \sum_{m,n=0}^{\infty} a_m a_n f_1^m f_2^n$. Since this expression contains all $f_1$ on the left side, and $f_2$ on the right, we have to find such a function $U(f)$ that $U(f_1 \circ f_2)$ contains $f_1$ and $f_2$ in the correct order. We have

$$U(f_1 \circ f_2) = \sum_{m=0}^{\infty} a_m \{ \sum_{n=0}^{\infty} (-1)^n ((f_2 f_1)^n (1 + f_2) - (f_1 f_2)^n (1 - f_1)) \}^m. \hspace{1cm} (A.3)$$

All terms of wrong order must vanish. The analysis of a few first terms of $U(f_1 \circ f_2)$ gives a hint that the coefficients are: $a_n = \{ a_0, a, a, a, ... \}$, i.e.

$$U(f) = a_0 - a + \frac{a}{1 - f}. \hspace{1cm} (A.4)$$

The substitution of $U(f)$ into the equation $U^2(f) = U(f \circ f)$ fixes $a_0 = 1, a = 2$ so that

$$U(f) = \frac{1 + f}{1 - f}. \hspace{1cm} (A.5)$$

To prove that the obtained solution satisfies the equation (2.19) one has to check the identity

$$(1 + f_1 \circ f_2) \frac{1 - f_2}{1 + f_2} = (1 - f_1 \circ f_2) \frac{1 + f_1}{1 - f_1} \hspace{1cm} (A.6)$$

equivalent to the relation

$$\{ 1 + \sum_{n=0}^{\infty} (-1)^n ((f_2 f_1)^n (1 + f_2) - (f_1 f_2)^n (1 - f_1)) \} (1 + 2 \sum_{m=1}^{\infty} f_2^m) =$$

$$\{ 1 - \sum_{n=0}^{\infty} (-1)^n ((f_2 f_1)^n (1 + f_2) - (f_1 f_2)^n (1 - f_1)) \} (1 + 2 \sum_{m=1}^{\infty} f_1^m), \hspace{1cm} (A.7)$$

which is elementary to check.

The normalization factor solves the equation

$$\frac{r(U_1) r(U_2)}{\sqrt{\det \|f_1 f_2 + 1\|}} = r(U_1 U_2), \hspace{1cm} (A.8)$$

which is obviously true after the substitution (2.20)

$$\frac{2^N}{\sqrt{\det \|U_1 + 1\|}} \cdot \frac{2^N}{\sqrt{\det \|U_2 + 1\|}} \cdot \frac{1}{\sqrt{\det \|\frac{U_1 - 1}{U_1 + 1} \frac{U_2 - 1}{U_2 + 1} + 1\|}} = \frac{2^N}{\sqrt{\det \|U_1 U_2 + 1\|}}. \hspace{1cm} (A.9)$$

This completes the proof of eq. (2.20). The proof for the supersymmetric case is analogous.
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