On polynomials in spectral projections of spin operators

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Abstract
We show that the operator norm of an arbitrary bivariate polynomial, evaluated on certain spectral projections of spin operators, converges to the maximal value in the semiclassical limit. We contrast this limiting behavior with that of the polynomial when evaluated on random pairs of projections. The discrepancy is a consequence of a type of Slepian spectral concentration phenomenon, which we prove in some cases.

Keywords Spectral projections · Quantization · Slepian concentration problem

Mathematics Subject Classification 81S10 · 53D50 · 47B35 · 17B

1 Introduction

Let $J_1, J_2, J_3 : \mathbb{C}^n \to \mathbb{C}^n$ be the generators of a unitary irreducible representation of $SU(2)$, satisfying the commutation relations

$[J_1, J_2] = iJ_3, \ [J_2, J_3] = iJ_1, \ [J_3, J_1] = iJ_2.$

Their spectrum equals the set $\sigma_n = \{j, j - 1, \ldots, -j\}$, where $2j + 1 = n$ ([1]).

Let $A$ denote the complex unital algebra generated by two non-commuting variables $x, y$ which satisfy $x^2 = x, \ y^2 = y$.

Let $(G_k(n), \mu_{k,n})$ be the Grassmannian of $k$-dimensional subspaces of $\mathbb{C}^n$, equipped with the uniform probability measure$^1$. We identify $V \in G_k(n)$ with the orthogonal projection $P : \mathbb{C}^n \to V$.

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$^1$ That is, the unique probability measure invariant under the action of the unitary group on $G_k(n)$.

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In this paper, we will study the following questions.

**Questions** Fix $f \in A$. Let $\mathbb{1}_{(0, \infty)}$ be the indicator function of $(0, \infty) \subset \mathbb{R}$.

1. What is the behavior of $f \left( \mathbb{1}_{(0, \infty)}(J_1), \mathbb{1}_{(0, \infty)}(J_3) \right)$ in the semiclassical limit $n \to \infty$?
2. How does it compare with $f(P, Q)$, where $P, Q \in G_{\frac{n^2}{2}}(n)$ are random projections?
3. To what extent do the answers to questions 1, 2 change if we replace $\mathbb{1}_{(0, \infty)}(J_1), \mathbb{1}_{(0, \infty)}(J_3) \in G_{\frac{n^2}{2}}(n)$ with spectral projections of ranks $[\alpha n]$, where $0 < \alpha \leq \frac{1}{2}$?

One of the main points of the paper is that pairs of spectral projections of spin operators are quite different from random pairs of projections (as $n \to \infty$). This will be demonstrated rigorously (Corollary 2, Theorem 4), as well as through numerical simulations. The non-generic behavior is a consequence of a type of Slepian spectral concentration phenomenon, closely related to that of the prolate matrix and its variants ([13,31,35,38]). Namely, our simulations suggest that the principal angles between the ranges of the projections cluster near 0 and $\frac{\pi}{2}$, and we manage to prove this in some cases. The principal angles associated with a pair of projections largely determine the algebra that the projections generate. In the case of random pairs of projections, these angles are distributed relatively evenly in a sub-interval of $[0, \frac{\pi}{2}]$.

The algebra generated by two projections has been researched extensively, and its general structure is well understood ([2,15,24,27,32]). Notably, it has been considered in the contexts of quantum measurement theory ([18,19]). The polynomial $PQP$ is perhaps the most fundamental, since its spectrum encodes the principal angles associated with the projections $P, Q$, and additionally, it is the object of research in Slepian concentration problems. The commutator $PQ - QP$ is also of interest, since it expresses the extent to which the projections are incompatible. However, other polynomials can occur quite naturally in the context of quantum theory, hence (some of) our motivation to study general polynomials rather than just a few specific choices. We refer the reader to Examples 1 and 2, which were explained to us by L. Polterovich, for concrete scenarios involving polynomials other than $PQP$ and $PQ - QP$.

The present paper continues an earlier one ([30]), which also examined pairs of spectral projections arising from certain specific non-commuting quantum observables, and from spin operators in particular. However, previously we restricted our attention only to commutators (i.e., $f(x, y) = xy - yx$) and did not compare with random projections.

The analogues of our current, more general result on spin operators (namely Theorem 3) also hold for the rest of the cases considered in the previous paper (position and momentum operators, for instance). Ultimately, we suspect that these results are instances of a rather general phenomenon. We refer the reader to Sect. 6 for further details.
1.1 Main results

We consider the case \( \alpha = \frac{1}{2} \) first. Denote \( P_{1,n} = \mathbb{1}_{(0,\infty)}(J_1) \), \( P_{3,n} = \mathbb{1}_{(0,\infty)}(J_3) \). Let \( \Omega_n = G_{\lfloor \frac{n}{2} \rfloor}(n) \times G_{\lfloor \frac{n}{2} \rfloor}(n), v_n = \mu_{\lfloor \frac{n}{2} \rfloor,n} \times \mu_{\lfloor \frac{n}{2} \rfloor,n}, \) and

\[
I_n^f = \int_{\Omega_n} \| f(P,Q) \|_{\text{op}} \, dv_n(P,Q).
\]

The following result was essentially conjectured by D. Kazhdan.

**Theorem 1** Let \( 0 \neq f \in \mathcal{A} \). There exists a constant \( M_f > 0 \), depending only on \( f \), such that

\[
\lim_{n \to \infty} I_n^f = \lim_{n \to \infty} \| f(P_{3,n}, P_{1,n}) \|_{\text{op}} = M_f.
\]

\( M_f \) is a universal upper bound for \( \| f(P,Q) \|_{\text{op}} \), where \( P, Q \) are arbitrary orthogonal projections on a separable complex Hilbert space, and it is a tight bound, since

\[
\max_{(P,Q) \in \Omega_n} \| f(P,Q) \|_{\text{op}} = M_f
\]

for every \( n \geq 2 \).

As we shall see, \( M_f \) essentially appears in the literature on pairs of projections (along with a rather concise formula in terms of \( f \)).

Theorem 1 is illustrated for \( f(x,y) = xy - yx \) (where \( M_f = \frac{1}{2} \)) in Figs. 1, 2. The pair \( P_{1,n}, P_{3,n} \) is non-generic as \( n \to \infty \) (as follows from Theorem 4). Accordingly, the graphs in Figs. 1 and 2 are clearly dissimilar, even though the limits coincide. The seemingly different convergence rates are discussed in Remark 7.

Theorem 1 is in fact atypical, since the analogous limits do not necessarily coincide when the projections involved are of ranks \( \lfloor \alpha n \rfloor \), with \( \alpha < \frac{1}{2} \). Indeed, for \( 0 < \alpha \leq \frac{1}{2} \),
Fig. 2 $\|\langle P, Q \rangle\|_{op}$ for random projections of rank $\lfloor \frac{n}{2} \rfloor$, as a function of $n$.

Let $\Omega_{n, \alpha} = G_{[\alpha n]}(n) \times G_{[\alpha n]}(n)$, equipped with the product probability measure $\nu_{n, \alpha} = \mu_{[\alpha n], n} \times \mu_{[\alpha n], n}$. Denote

$$I_{n, \alpha}^f = \int_{\Omega_{n, \alpha}} \| f(P, Q) \|_{op} d\nu_{n, \alpha}(P, Q).$$

Let $\mathbb{C}[z, w]$ denote the commutative algebra of complex polynomials, and let $T : \mathcal{A} \to \mathbb{C}[z, w]/\mathcal{I}$. (1)

map $f(x, y)$ to $f(z, w)$, where $\mathcal{I}$ is the ideal generated by $z^2 - z$, $w^2 - w$. For the sake of conciseness, we now restrict our attention to polynomials $f \in \ker(T)$ (the general case is addressed in Theorem 10). A combination of results from [3, 9] readily implies the following result (which is illustrated for $f(x, y) = xy - yx$ in Figs. 3, 7).

**Theorem 2** Let $f \in \ker(T)$ and $\lambda_{\alpha} = 4\alpha(1 - \alpha)$. There exists a continuous $\psi_f : [0, 1] \to [0, \infty)$, determined by $f$ only, such that $\psi_f(0) = \psi_f(1) = 0$, and

$$I_{\alpha}^f = \lim_{n \to \infty} I_{n, \alpha}^f = \max_{[0, \lambda_{\alpha}]} \psi_f. \quad (2)$$

Note that $M_f = \max_{[0, 1]} \psi_f > 0$ (for $f \neq 0$).

$\psi_f$ essentially appears in [32]. Since it is continuous, so is the non-decreasing function $\alpha \mapsto I_{\alpha}^f$.

**Corollary 1** If $f \in \ker(T)$, then $\lim_{\alpha \to 0^+} I_{\alpha}^f = \psi_f(0) = 0$.

On the other hand,

**Theorem 3** Let $f \in \mathcal{A}$. Define intervals $(0, \alpha_n) \subset \mathbb{R}$ containing exactly $|\alpha_n|$ elements of $\sigma_n$. Denote $P_{1, \alpha, n} = 1_{(0, \alpha_n)}(J_1)$, $P_{3, \alpha, n} = 1_{(0, \alpha_n)}(J_3)$. Then,

$$\lim_{n \to \infty} \| f(P_{3, \alpha, n}, P_{1, \alpha, n}) \|_{op} = M_f. \quad (3)$$
In particular, when $\alpha < \frac{1}{2}$, we can rigorously say that pairs of spectral projections of spin operators are unlike random pairs of projections.

**Corollary 2** For every $\alpha < \frac{1}{2}$, there exists $f \in \ker(T)$ such that $I_{\alpha}^f < M_f$. For every $0 \neq f \in \ker(T)$, there exists $\delta > 0$ such that $I_{\alpha}^f < M_f$ for every $\alpha < \delta$. Thus, there is a discrepancy between (2), (3).

Corollary 2 is illustrated for $f(x, y) = xy - yx$, $\alpha = \frac{1}{20}$ in Figs. 3, 4. Note that $\psi_f(t) = \sqrt{t(1-t)}$, $M_f = \frac{1}{2}$ in this case (see Example 3).

**Remark 1** Figure 4 provides an example of a curious, unproven phenomenon. When $\alpha = \frac{1}{4k}$, our numerical simulations suggest that the graph of $n \mapsto \| [P_{1,\alpha,n}, P_{3,\alpha,n}] \|_{op}$ decomposes, modulo 4, to pieces of length $k$.

Finally, we consider the pairs $P_{3,\alpha,n}$, $P_{1,n}$. Let $N_n(s, t)$ denote the number of eigenvalues of $P_{3,\alpha,n}P_{1,n}P_{3,\alpha,n} \in \text{End}(\text{Im}(P_{3,\alpha,n}))$ lying in the interval $[s, t]$, where $0 \leq s < t \leq 1$.

**Theorem 4** For every $0 < t < \frac{1}{2}$,

\[
\lim_{n \to \infty} \frac{1}{an} N_n(0, t) = \lim_{n \to \infty} \frac{1}{an} N_n(1 - t, 1) = \frac{1}{2},
\]

and

\[
N_n(t, 1 - t) = \mathcal{O}(\log n).
\]

The behavior described in Theorem 4 is typical in the context of Slepian spectral concentration problems ([13, 31, 35, 38]), which involve pairs of (spectral) projections analogous to $P_{3,\alpha,n}$, $P_{1,\alpha,n}$, as follows. Let $\mathcal{M}_{I_1}$, $\mathcal{M}_{I_2}$ denote the operators of multiplication by the indicator functions of the finite intervals $I_1$, $I_2$. The first projection
Fig. 4 $\| [P_{1,\alpha,n}, P_{3,\alpha,n}] \|_{op}$ as a function of $n$ for $\alpha = \frac{1}{20}$

(say, $Q_1$) is the multiplication operator $M_{1_1}$, and the second ($Q_2$) is obtained from $M_{\ell_2}$ through conjugation by the Fourier transform (on $\mathbb{R}$, $S^1$, or $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$). Then, the problem is essentially to investigate the spectral properties and the eigenfunctions of $Q_2 Q_1 Q_2$. While numerous variants of this problem have been explored in great detail, we were not able find literature on the specific one addressed here.

Theorem 4 is illustrated for $P_{3,n}, P_{1,n}$ in Figs. 5 and 6. In particular, it provides further evidence that pairs of spectral projections of spin operators are unlike random pairs of projections\(^2\). Namely,

\[ \text{Corollary 3} \] Applying Theorem 8 (which is a reformulation of a result from \cite{12}) to some suitable $F \in C[0, 1]$, we see that for every $0 < \varepsilon < \frac{1}{2}$ it is possible to choose $r > 0$ such that the measure of the set

\[
\left\{ (P, Q) \in G_{[\alpha n]}(n) \times G_{[\frac{1}{2} n]}(n) \mid \#(\sigma(PQP) \cap [\varepsilon, 1 - \varepsilon]) \geq rn \right\}
\]

converges to 1 as $n \to \infty$. Here, $\sigma(A)$ is the spectrum of a linear operator $A$ and $\#S$ denotes the number of elements of a finite set $S$.

The conclusion of Theorem 3 also applies to the pairs $P_{3,\alpha,n}, P_{1,n}$ (while the corresponding random pairs satisfy a version of Theorem 2, as specified in Theorem 10). This can be proven using the arguments of Sect. 4, and it implies that $\lim_{n \to \infty} N_n(t, 1 - t) = \infty$. According to our numerical simulations, a version of Theorem 4 applies to the pairs $P_{3,\alpha,n}, P_{1,\alpha,n}$ as well. Unfortunately, it is not clear whether this can be proven using the arguments of Sect. 5.

\^{2} Nonetheless, in the case of $P_{1,n}, P_{3,n}$, the limits (2), (3) coincide for every $f \in \mathcal{A}$.\footnote{\[^2\] Nonetheless, in the case of $P_{1,n}, P_{3,n}$, the limits (2), (3) coincide for every $f \in \mathcal{A}$.}
Fig. 5  The sorted eigenvalues of $PQP \in \text{End} \left(\text{Im}(P)\right)$, where $P, Q \in G_{n}(2n)$ are random and $n = 1000$

Fig. 6  The sorted eigenvalues of $P_{3,n}P_{1,n}P_{3,n} \in \text{End} \left(\text{Im} \left(P_{3,n}\right)\right)$ for $n = 2000$. The number of values that are visibly between 0 and 1 is $O(\log n)$

Let us now offer a few remarks about the proofs of the main results. First of all, if $P, Q$ are two orthogonal projections on a separable complex Hilbert space and $f \in \ker(T)$, then

$$\|f(P, Q)\|_{\text{op}} = \max_{\sigma(PQP)} \psi_f$$

(the general case $f \in \mathcal{A}$ is only slightly more complicated). If $(P, Q) \in \Omega_{n,\alpha}$, then by the existing results from random matrix theory, as $n \to \infty$ the spectrum of $PQP$ becomes dense in $[0, \lambda_{\alpha}]$ and vanishes from $(\lambda_{\alpha} + \delta, 1]$ for every $\delta > 0$, with high probability. Thus, $\max_{\sigma(PQP)} \psi_f \approx \max_{[0,\lambda_{\alpha}]} \psi_f$ with high probability, which yields Theorem 2.

The proof of the deterministic case (Theorem 3) is based on a different approach, since our understanding of $\sigma(P_{3,\alpha,n}P_{1,\alpha,n}P_{3,\alpha,n})$ as $n \to \infty$ is quite limited. Instead, we refer to a previous work ([30]), in which certain matrix entries of $P_{1,\alpha,n}$ were
shown to converge to the Fourier coefficients of the indicator function $\mathbb{1}_{E_{\alpha}}$, where $E_{\alpha} = \{ \zeta \in \mathbb{T} \mid 0 < \Re \zeta < 2\alpha \}$. Here, $\Re \zeta$ is the real part of $\zeta$, and $\mathbb{T} \subset \mathbb{C}$ is the unit circle. This allows us to show that $P_{3, \alpha, n}, P_{1, \alpha, n}$, considered as finite rank projections on $L^2(\mathbb{T})$, converge strongly to the Cauchy–Szegö projection on the Hardy space and to the multiplication operator $M_{1_{E_{\alpha}}}$. From here, it is straightforward to deduce Theorem 3 by application of standard results about Toeplitz operators on the circle.

Notably, the proof of Theorem 3 does not rely on a specific realization of the spin operators $J_1, J_2, J_3$, e.g., as operators on spaces of spherical harmonics. The spectral projections are related by

$$P_{1, \alpha, n} = e^{-i \frac{\pi}{2} J_2} P_{3, \alpha, n} e^{i \frac{\pi}{2} J_2},$$

and the convergence of the relevant matrix entries of $P_{1, \alpha, n}$ is established (in [30]) using standard formulas involving the entries of $e^{-i \frac{\pi}{2} J_2}$. Nonetheless, the proof can be understood in more geometric terms, as we will now outline.

Assume that $n = 2l + 1$, where $l \in \mathbb{N}$. Following [33], we identify the sphere of area $2\pi n$ with $D_n = [0, \frac{n}{2} \pi) \times [0, 2\pi)$ via the spherical coordinates

$$(\theta, \phi) \mapsto \sqrt{\frac{n}{2}} \left(\cos \phi \sin \left(\frac{2\theta}{n}\right), \sin \phi \sin \left(\frac{2\theta}{n}\right), \cos \left(\frac{2\theta}{n}\right)\right).$$

Here, $D_n$ is viewed as a disk, such that $\theta$ is the radial coordinate and $\phi$ is the polar angle. Let $\mathbb{H}_n$ denote the space of spherical harmonics of degree $l = \frac{n-1}{2}$, considered as functions on $D_n$, and let $\rho_n$ be the representation of $su(2)$ (the Lie algebra of $SU(2)$) on $\mathbb{H}_n$ defined by the standard action of $J_1, J_2, J_3$. Let $\mathbb{H}_\infty$ denote the Hilbert space consisting of circular plane waves (certain nonsingular solutions of the Helmholtz equation in $\mathbb{R}^2$), and let $\mathfrak{e}(2)$ denote the Lie algebra of the group of orientation preserving Euclidean plane isometries. Then, $\mathbb{H}_\infty$ carries a representation $\rho_\infty$ of $\mathfrak{e}(2)$, and $(\mathbb{H}_n, \rho_n)$ converge to $(\mathbb{H}_\infty, \rho_\infty)$ in a precise sense as $n \to \infty$. Namely, this is an example of a contraction of Lie algebra representations (see [33] for the details). Finally, $\rho_\infty$ is unitarily equivalent to the representation of $\mathfrak{e}(2)$ on $L^2(\mathbb{T})$ generated by the operators

$$X_1 = i X, \quad X_2 = X, \quad X_3 = -\frac{d}{d\phi}.$$

Thus, $J_1, J_3$ correspond to $i X_1, i X_3$ as $n \to \infty$, and accordingly, the proof can be reduced to the settings of $L^2(\mathbb{T})$.

The proof of Theorem 4 proceeds as follows. Let $R_n = P_{3, \alpha, n} P_{1, n} P_{3, \alpha, n}$. The eigenvalues of $R_n$ lie within the interval $[0, 1]$, and we show, using explicit formulas for the entries of $R_n$, that $\text{trace}(R_n) - \text{trace}(R_n^2)$ is of logarithmic order in $n$. This is a rather well known argument in this context, and it implies the desired result.

We conclude this part of the introduction with examples featuring polynomials in two projections $P, Q$ which are more complicated than $P Q P$ and $P Q - Q P$. 
**Example 1** Consider a pair of incompatible quantum observables, represented by the projection-valued measures $\mathcal{P} = \{P_0, P_1\}$ and $\mathcal{Q} = \{Q_0, Q_1\}$. Here, $P_1 = \text{Id} - P_0$ and $Q_1 = \text{Id} - Q_0$. The sequential measurement ([10], [4]) of $\mathcal{P}$ first and $\mathcal{Q}$ second gives rise to another observable $\mathcal{R}$, whose measurements are subject to noise (due to the incompatibility of $\mathcal{P}, \mathcal{Q}$). This noise is characterized by the noise operator $N = \mathcal{R}_2 - \mathcal{R}_1^2$, where

$$\mathcal{R}_1 = \sum_{k,l} r_{k,l} P_k Q_l P_k, \quad \mathcal{R}_2 = \sum_{k,l} r_{k,l}^2 P_k Q_l P_k$$

and $r_{k,l} \in \mathbb{R}$ are (arbitrary, real) constants. In particular, $\|N\|_{\text{op}}$ is a measure of the “magnitude of the noise” of $\mathcal{R}$. $\mathcal{N}$ is a polynomial in the projections $P_0, Q_0$, so Theorem 3 provides information about $\|N\|_{\text{op}}$ when $P_0, Q_0$ are spectral projections of $J_1, J_3$. Further examples can be constructed this way by considering longer sequences of measurements of $\mathcal{P}, \mathcal{Q}$.

**Example 2** Let $\hat{H}$ be a quantum Hamiltonian with exactly two eigenvalues, i.e., admitting a spectral decomposition

$$\hat{H} = \hbar (\lambda P + \mu (\text{Id} - P)) = \hbar ((\lambda - \mu) P + \mu \text{Id}).$$

Fix an initial mixed state $\rho = (\text{rank } Q)^{-1} Q$, where $Q$ is an orthogonal projection. The evolution of $\rho$ under the Schrödinger flow corresponding to $\hat{H}$ is given by $\rho(t) = e^{-\frac{i}{\hbar} \hat{H} t} \rho e^{\frac{i}{\hbar} \hat{H} t}$. Using the Baker–Campbell–Hausdorff formula and the fact that $[P, [P, [P, Q]]] = [P, Q]$, we find that $\rho(t)$ is a non-trivial linear combination of $Q, [P, Q]$ and $[P, [P, Q]]$ (i.e., a polynomial in $P, Q$).

### 1.2 Examples

Every polynomial $f \in \mathcal{A}$ can be written uniquely as

$$f(x, y) = c_0 + f_1(xy)xy + f_2(yx)yx + f_3(xy)x + f_4(yx)y,$$

where $c_0 \in \mathbb{C}$ and $f_1, f_2, f_3, f_4$ are univariate polynomials. $\psi_f$ admits a rather concise formula (see Theorem 6) in terms of $f_1, f_2, f_3, f_4$. For instance, when $f(x, y) = f_1(xy)xy - f_1(yx)yx$ (see Example 4),

$$\psi_f(t) = |f_1(t)| \sqrt{t(1-t)}.$$

More specifically,

**Example 3** Let $f(x, y) = (xy)^{k+1} - (yx)^{k+1}$, where $k \geq 0$. Then,

$$\psi_f(t) = i^k \sqrt{t(1-t)}.$$

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Fig. 7 A simulation of randomly drawn $\| [P_{\alpha n}, Q_{\alpha n}] \|_{\text{op}}$ as a function of $\alpha$ for $n = 1000$

and

$$M_f = \max_{[0,1]} \psi_f(t) = \psi_f \left( \frac{2k + 1}{2k + 2} \right) = \frac{1}{\sqrt{2k + 2}} \left( 1 - \frac{1}{2k + 2} \right)^{k + \frac{1}{2}}.$$ 

Thus,

$$I_f^\alpha = \max_{[0,4\alpha(1 - \alpha)]} \psi_f = \begin{cases} M_f & \frac{1}{2} - \frac{1}{2\sqrt{2k+2}} \leq \alpha \\ (4\alpha(1 - \alpha))^{k + \frac{1}{2}} (1 - 2\alpha) & 0 < \alpha < \frac{1}{2} - \frac{1}{2\sqrt{2k+2}} \\ 0 & \frac{1}{2} - \frac{1}{2\sqrt{2k+2}} \leq \alpha \leq \frac{1}{2} \end{cases}.$$ 

We further specialize and consider $f(x, y) = xy - yx$. The previous example shows that $M_f = \frac{1}{2}$ and

$$I_f^\alpha = \begin{cases} \frac{1}{2} & \frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \alpha \leq \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2\sqrt{2}} \sqrt{\alpha(1 - \alpha)} & 0 < \alpha < \frac{1}{2} - \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \alpha \leq \frac{1}{2} \end{cases}.$$ 

This is illustrated in Fig. 7. The corresponding image (Fig. 8) for spectral projections of spin operators is (again) very dissimilar.

2 Preliminaries on two orthogonal projections

The proofs of Theorems 2 and 3 are independent, but both rely on the general theory of two projections, which we now present. Nearly all of the contents of this section can be found in the excellent guide to the theory [3]. A few minor modifications and lemmas were added for use in subsequent parts of the work.

Let $\mathcal{H}$ be a separable complex Hilbert space, possibly infinite dimensional. A pair of orthogonal projections $P : \mathcal{H} \to \mathcal{V}_P$, $Q : \mathcal{H} \to \mathcal{V}_Q$ give rise to a decomposition
of $\mathcal{H}$ as the orthogonal direct sum
\[ \mathcal{H} = V_{00} \oplus V_{01} \oplus V_{10} \oplus V_{11} \oplus V_{0} \oplus V_{1}, \tag{4} \]
where
\[ V_{00} = V_P \cap V_Q, \quad V_{01} = V_P \cap V_Q^\perp, \]
\[ V_{10} = V_P^\perp \cap V_Q, \quad V_{11} = V_P^\perp \cap V_Q^\perp, \]
\[ V_0 = (V_{00} \oplus V_{01})^\perp \cap V_P, \quad V_1 = (V_{10} \oplus V_{11})^\perp \cap V_P^\perp, \]
so that
\[ V_P = V_{00} \oplus V_{01} \oplus V_0, \quad V_P^\perp = V_{10} \oplus V_{11} \oplus V_1. \]

Of course, some of the summands in the decomposition (4) of $\mathcal{H}$ may be trivial.

**Remark 2** Let $V = V_0 \oplus V_1$. Then, $P, Q$ commute on $V^\perp = \bigoplus_{l, k \in \{0,1\}} V_{lk}$. Hence, unless stated otherwise, we assume throughout this section that $V \neq \{0\}$.

Given $\alpha_{lk} \in \mathbb{C}$, where $l, k \in \{0, 1\}$, we abbreviate
\[ (\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}) = \alpha_{00} I_{V_{00}} \oplus \alpha_{01} I_{V_{01}} \oplus \alpha_{10} I_{V_{10}} \oplus \alpha_{11} I_{V_{11}} : V^\perp \to V^\perp. \]

Clearly,
\[ P|_{V^\perp} = (1, 1, 0, 0), \quad Q|_{V^\perp} = (1, 0, 1, 0) \]
and $P|_V = \text{diag}(I, 0)$. A canonical form of the pair $P, Q$ is specified as follows.
Theorem 5 ([16]) \( V_0 \neq \{0\} \) if and only if \( V_1 \neq \{0\} \). If \( V_0, V_1 \neq \{0\} \), then

\[
P = (1, 1, 0, 0) \oplus P|_V = (1, 1, 0, 0) \oplus \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
Q = (1, 0, 1, 0) \oplus Q|_V = (1, 0, 1, 0) \oplus U^* \begin{pmatrix} I - H & \sqrt{H(I - H)} \\ \sqrt{H(I - H)} & H \end{pmatrix} U.
\]

Here, \( 0 \leq H \leq 1 \) with \( \ker H = \ker(I - H) = \{0\} \) and \( U = \text{diag}(I, R) \) with \( R : V_1 \to V_0 \) unitary.

In particular, we note that \( P, Q \) are non-commuting if and only if \( V_0 \neq \{0\} \).

2.1 Polynomials in two non-commuting idempotents

Recall that \( A \) denotes the complex unital algebra generated by non-commuting variables \( x, y \) which satisfy the relations \( x = x^2, y = y^2 \). A basis of \( A \) as a vector space is provided by the monomials

\[ 1, (xy)^{k+1}, (yx)^{k+1}, (xy)^k x, (yx)^k y, \]

where \( k \geq 0 \). Thus, any \( f \in A \) decomposes uniquely as

\[ f(x, y) = a_0 + f_1(x)xy + f_2(y)yx + f_3(xy)x + f_4(yx)y, \]

where \( f_1, f_2, f_3, f_4 \) are complex univariate polynomials. Let \( r(t) = \sqrt{t(1-t)} \), and for \( l, k \in \{0, 1\} \) define \( g_{lk} : [0, 1] \to \mathbb{C} \) by

\[
\begin{align*}
g_{00}(t) &= a_0 + f_3(t) + t(f_1(t) + f_2(t) + f_4(t)), \\
g_{01}(t) &= r(t)(f_1(t) + f_4(t)), \\
g_{10}(t) &= r(t)(f_2(t) + f_4(t)), \\
g_{11}(t) &= a_0 + (1 - t)f_4(t).
\end{align*}
\]

Also, denote

\[ \alpha_{00} = g_{00}(1), \ \alpha_{01} = g_{00}(0), \ \alpha_{10} = g_{11}(0), \ \alpha_{11} = g_{11}(1). \]

Lemma 1 Let \( T : A \to \mathbb{C}[z, w]/I \) be the “abelianization” map of (1). Then, \( f \in \ker(T) \) if and only if

\[ \alpha_{00} = \alpha_{10} = \alpha_{01} = \alpha_{11} = 0. \]

Proof Write

\[ f_l(t) = \sum_{k=0}^{k_l} a_{lk}^{(l)} t^k, \ l = 1, 2, 3, 4. \]
A straightforward computation shows that

\[(Tf)(z, w) = a_0 + f_3(0)z + f_4(0)w + \left[ f_1(1) + f_2(1) + f_3(1) - f_3(0) + f_4(1) - f_4(0) \right]zw.\]

Thus, we obtain the required. \(\square\)

Next, by induction,

\[(PQ)^{k+1} = IV_{00} \oplus U^* \left( (I - H)^{k+1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} (I - H)^{r(H)} \right) U,\]

\[(QP)^{k+1} = IV_{00} \oplus U^* \left( (I - H)^{k+1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} (I - H)^{r(H)} \right) U,\]

\[(PQ)^k P = IV_{00} \oplus U^* \left( (I - H)^k \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} (I - H)^{r(H)} \right) U,\]

\[(QP)^k Q = IV_{00} \oplus U^* \left( (I - H)^k \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} (I - H)^{r(H)} \right) U,\]

where \(k \geq 0\). By linearity, we obtain a precise expression for \(f(P, Q)\) as follows.

**Corollary 4** ([15]) For every complex Hilbert space \(\mathcal{H}\) and for any pair of non-commuting orthogonal projections \(P : \mathcal{H} \to V_P, \ Q : \mathcal{H} \to V_Q,\)

\[f(P, Q) = a_0 + f_1(PQ)PQ + f_2(QP)QP + f_3(PQ)P + f_4(QP)Q = (\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}) \oplus U^* \left( \begin{pmatrix} g_{00}(I - H) & g_{01}(I - H) \\ g_{10}(I - H) & g_{11}(I - H) \end{pmatrix} \right) U.\]

**Proof** Denote \(f_l(t) = \sum_{k=0}^{k_l} a_k^{(l)} t^k\), where \(l = 1, 2, 3, 4\) and \(k_l \neq 0\). We compute each of the summands \(f_1, f_2, f_3, f_4\) of \(f(P, Q)\) separately.

The first summand is

\[f_1(PQ)PQ = \sum_{k=0}^{k_1} a_k^{(1)} (PQ)^{k+1} = \left( f_1(1)IV_{00} \right) \oplus U^* \left( \begin{pmatrix} f_1(1) - H & f_1(1) - Hr(H) \\ 0 & 0 \end{pmatrix} \right) U.\]

The second summand is

\[f_2(QP)QP = \sum_{k=0}^{k_2} a_k^{(2)} (QP)^{k+1}\]
\[
= (f_2(1)I_{V_0}) \oplus U^* \begin{pmatrix}
  f_2(I - H)(I - H) & 0 \\
  f_2(I - H)r(H) & 0
\end{pmatrix} U.
\]

The third summand is

\[
f_3(PQ)P = f_3(PQP)P = a^{(3)}_0 P + \sum_{k=0}^{k_3-1} a^{(3)}_{k+1}(PQP)^{k+1},
\]

where

\[
\sum_{k=0}^{k_3-1} a^{(3)}_{k+1}(PQP)^{k+1}
\]

\[
= (f_3(1) - a^{(3)}_0 I_{V_0}) \oplus U^* \begin{pmatrix}
  f_3(I - H) & 0 \\
  0 & 0
\end{pmatrix} U,
\]

so

\[
f_3(PQ)P = (f_3(1), f_3(0), 0, 0) \oplus U^* \begin{pmatrix}
  f_3(1 - H) & 0 \\
  0 & 0
\end{pmatrix} U.
\]

Finally,

\[
f_4(QP)Q = f_4(QQP)Q = a^{(4)}_0 Q + \sum_{k=0}^{k_4-1} a^{(4)}_{k+1}(QPQ)^{k+1},
\]

so

\[
f_4(QP)Q = (f_4(1), 0, f_4(0), 0) \oplus U^* \begin{pmatrix}
  f_4(I - H)(I - H) & f_4(I - H)r(H) \\
  f_4(I - H)r(H) & f_4(I - H)H
\end{pmatrix} U.
\]

Putting everything together, we obtain the required. \(\square\)

**Remark 3** We can factor \(T\) as \(\Pi \circ T_0\), where \(T_0 : A \to \mathbb{C}[z, w]\) is the linear map \(f(x, y) \mapsto f(z, w)\) (unlike \(T\), the map \(T_0\) is not a morphism of algebras) and \(\Pi\) is the projection onto \(\mathbb{C}[z, w]/\mathcal{I}\). Note that \(f \in \ker(T_0)\) if and only if

\[
f(x, y) = f_1(xy)xy - f_1(yx)yx = [x, f_1(yx)y],
\]

where \(f_1\) is a complex univariate polynomial, and then

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\[ f(P, Q) = [P, f_1(QP)Q] = (0, 0, 0, 0) \oplus U^* \begin{pmatrix} 0 & f_1(I - H)r(H) \\ -f_1(I - H)r(H) & 0 \end{pmatrix} U. \]

2.2 The operator norm of polynomials in two projections

We define (following [32]) two functions \( g_0, g_1 \in C([0, 1], \mathbb{C}) \) by

\[ g_0 = \sum_{l,k=0}^{1} |g_{lk}|^2, \quad g_1 = g_{00}g_{11} - g_{01}g_{10}, \]

(6)

where \( g_{00}, g_{01}, g_{10}, g_{11} \) are the functions appearing in Corollary 4.

**Remark 4** In the literature, it appears that \( g_{00}, g_{01}, g_{10}, g_{11}, \) and \( g_0, g_1 \) are considered only as functions on \( \sigma(I - H) \). For us, it is very useful that they are determined by \( f \) as continuous functions on \([0, 1]\), so as to apply by restriction to all separable Hilbert spaces and orthogonal projections.

\( g_0, g_1 \) can be used to express \( \sigma(f(P, Q)) \) in terms of \( \sigma(I - H) \), since \( f(P, Q)|_V \) has a bounded inverse if and only if \( g_1(I - H) \) does. In particular, the polynomial \( f(P, Q)f(P, Q)^* \) can be used to determine \( \|f(P, Q)\|_{op} \).

**Theorem 6** ([3,32]) Let \( \Lambda = \{(l, k) \in \{0, 1\}^2 \mid V_{lk} \neq \{0\}\} \). The operator norm of \( f(P, Q) \) is given by

\[ \|f(P, Q)\|_{op} = \max \left\{ \max_{(l,k) \in \Lambda} |\alpha_{l,k}|, \max_{t \in \sigma(I - H)} \psi_f(t) \right\}, \]

where \( \psi_f \in C([0, 1], \mathbb{R}) \) is given by

\[ \psi_f(t) = \sqrt{\frac{g_0(t) + \sqrt{g_0^2(t) - 4|g_1(t)|^2}}{2}}. \]

Note that

\[ \psi_f(0) = \max(|\alpha_{10}|, |\alpha_{01}|), \quad \psi_f(1) = \max(|\alpha_{11}|, |\alpha_{00}|). \]

Assume \( f \in \ker(T) \), so that by Lemma 1, \( \alpha_{lk} = 0 \) for \( l, k = 0, 1 \), or equivalently \( \psi_f(0) = \psi_f(1) = 0 \). If \( P, Q \) commute, then \( f(P, Q) = 0 \) and \( \max_{\sigma(PQP)} \psi_f = 0 \). Otherwise if \( P, Q \) do not commute, \( \max_{\sigma(I - H)} \psi_f = \max_{\sigma(PQP)} \psi_f \). We use this to reformulate Theorem 6 as follows.

**Corollary 5** Let \( \Lambda = \{(l, k) \in \{0, 1\}^2 \mid V_{lk} \neq \{0\}\} \) as above, and \( f \in \mathcal{A} \).
1. Assume \( f \in \ker(T) \). Then, for every separable complex Hilbert space \( \mathcal{H} \) and orthogonal projections \( P, Q \) on \( \mathcal{H} \),

\[
\| f(P, Q) \|_{\text{op}} = \max_{\sigma(PQP)} \psi_f.
\]

2. Define \( \Psi_f : [0, 1] \cup \{0, 1\}^2 \to \mathbb{R} \) by \( \Psi_f \big|_{[0, 1]} = \psi_f \) and \( \Psi_f((l, k)) = |\alpha_{lk}| \) for \((l, k) \in \{0, 1\}^2\). Denote \( \sigma_{P, Q} = \sigma(I - H) \cup \Lambda \). Then, for every separable complex Hilbert space \( \mathcal{H} \) and orthogonal projections \( P, Q \) on \( \mathcal{H} \),

\[
\| f(P, Q) \|_{\text{op}} = \max_{\sigma_{P, Q}} \Psi_f.
\]

Note that \( \max_{[0, 1]} \psi_f = \max_{[0, 1] \cup \{0, 1\}^2} \Psi_f \), since \( \psi_f(1) = \max\{|\alpha_{00}|, |\alpha_{11}|\} \) and \( \psi_f(0) = \max\{|\alpha_{01}|, |\alpha_{10}|\} \).

The latter, together with Proposition 1, immediately leads to the following.

**Corollary 6** The constant

\[
M_f = \max_{[0, 1]} \Psi_f = \max_{[0, 1] \cup [0, 1]^2} \Psi_f
\]

is a universal, tight upper bound for \( \| f(P, Q) \|_{\text{op}} \), where \( P, Q \) are any orthogonal projections on an arbitrary complex Hilbert space \( \mathcal{H} \).

We conclude with the following example.

**Example 4** Let \( f \in \ker(T_0) \) where \( T_0 \) is as in Remark 3, and recall that \( r(t) = \sqrt{t(1 - t)} \). Then,

\[
f(P, Q) = f_1(PQ)PQ - f_1(QP)QP
\]

for some univariate polynomial \( f_1 \), therefore

\[
g_{00} = g_{11} = 0, \quad g_{01} = g_{10} = f_1(t)r(t).
\]

It follows that \( g_0(t) = 2|g_0(t)|^2 \), \( g_1(t) = g_{01}(t)^2 \), hence we find that

\[
\psi_f(t) = \sqrt{\frac{g_0(t) + \sqrt{g_0(t)^2 - 4|g_1(t)|^2}}{2}} = \sqrt{\frac{2|g_0(t)|^2 + \sqrt{4|g_0(t)|^4 - 4|g_{01}(t)|^4}}{2}} = |g_{01}(t)|.
\]

Thus,

\[
\| f(P, Q) \|_{\text{op}} = \max_{\sigma(I - H)} (|f_1|r) \leq \max_{[0, 1]} (|f_1|r) = M_f.
\]
2.3 The canonical form and angles between subspaces

Assume that dim $\mathcal{H} = n$. Recall the notation

$$
V_{00} = V_P \cap V_Q, \ V_{01} = V_P \cap V_Q^\perp, \\
V_{10} = V_P^\perp \cap V_Q, \ V_{11} = V_P^\perp \cap V_Q^\perp \\
V_0 = (V_{00} \oplus V_{01})^\perp \cap V_P, \ V_1 = (V_{10} \oplus V_{11})^\perp \cap V_P^\perp,
$$

and let $m_{lk} = \dim V_{lk}$, $l, k \in \{0, 1\}$ and $m = \dim V_0 = \dim V_1$.

Definition 1 Denote the eigenvalues of $H$ by $0 < \mu_1 \leq \ldots \leq \mu_m < 1$. The reduced principal angles $0 < \theta_1 \leq \ldots \leq \theta_m < \frac{\pi}{2}$ associated with the pair $(P, Q)$ are defined by

$$
\sin^2 \theta_l = \mu_l, \ l = 1, \ldots, m.
$$

The pair $(P, Q)$ is determined, up to unitary equivalence, by the numbers $m_{00}, m_{01}, m_{10}, m_{11}, m$ together with the reduced principal angles.

Definition 2 Let $m_P = \dim V_P \leq \dim V_Q = m_Q$ (so $m_P = m_{00} + m_{01} + m$). The principal angles $0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq \phi_{m_P} \leq \frac{\pi}{2}$ of the pair $(P, Q)$ are defined recursively. The angle $\phi_1$ is specified by

$$
\cos \phi_1 = \max \{ |\langle x, y \rangle| \mid x \in V_P, y \in V_Q, \|x\| = \|y\| = 1 \},
$$

and if $\cos \phi_1 = |\langle x_1, y_1 \rangle|$, then

$$
\cos \phi_2 = \max \{ |\langle x, y \rangle| \mid x \in V_P \cap \{x_1\}^\perp, \ y \in V_Q \cap \{y_1\}^\perp, \|x\| = \|y\| = 1 \}.
$$

Next, if we denote $V_{P,k} = \text{Span}\{x_1, \ldots, x_k\}$ and $V_{Q,k} = \text{Span}\{y_1, \ldots, y_k\}$, where $\cos \varphi_l = |\langle x_l, y_l \rangle|$ for $l = 1, \ldots, k$, then

$$
\cos \phi_{k+1} = \max \{ |\langle x, y \rangle| \mid x \in V_P \cap V_{P,k}^\perp, \ y \in V_Q \cap V_{Q,k}^\perp, \|x\| = \|y\| = 1 \}.
$$

The reduced principal angles are the principal angles lying in $(0, \frac{\pi}{2})$, i.e.,

$$
\phi_1 = \ldots = \phi_{m_{00}} = 0, \\
\phi_{m_{00}+1} = \theta_1, \ldots, \phi_{m_{00}+m} = \theta_m, \\
\phi_{m_{00}+m+1} = \ldots = \phi_{m_{00}+m+m_{01}} = \frac{\pi}{2}.
$$

The following elementary examples will be of immediate use.

Example 5 Let $n \geq 2$ and $\phi \in \left[0, \frac{\pi}{2}\right]$. For $0 < m_P \leq m_Q \leq n - m_P$ integers, there exists a pair of orthogonal projections $P : \mathbb{C}^n \rightarrow V_P, \ Q : \mathbb{C}^n \rightarrow V_Q$ with $m_P = \dim V_P, \ m_Q = \dim V_Q$ and smallest principal angle $\phi_1 = \phi$. 

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Indeed, if \( \{e_l \mid l = 1, \ldots, n\} \) is an orthonormal basis of \( \mathbb{C}^n \), and
\[
V_P = \text{Span} \{e_1, \ldots, e_{m_P}\},
\]
then
\[
V_Q = \text{Span} \{\cos \phi e_1 + \sin \phi e_n, e_{m_P+1}, \ldots, e_{m_P+m_Q-1}\}
\]
satisfies the required. In particular, if \( \phi \in (0, \frac{\pi}{2}) \), then \( \sin^2 \phi \in \sigma(H) \).

**Example 6** Let \( n \geq 2, \phi \in [0, \frac{\pi}{2}] \) and assume that \( 0 < m_P < m_Q < n \). Then, there exists a pair of orthogonal projections \( P : \mathbb{C}^n \to V_P, Q : \mathbb{C}^n \to V_Q \) with \( m_P = \dim V_P \) and \( m_Q = \dim V_Q \) such that \( \phi \in \{\phi_1, \ldots, \phi_{m_P}\} \).

Indeed, if \( m_P + m_Q < n \), we saw that it is possible to define \( P, Q \) such that \( \phi = \phi_1 \). If \( m_P + m_Q > n \), then \( m_{00} = \dim V_P \cap V_Q > 0 \), so \( \phi_1 = \ldots = \phi_{m_{00}} = 0 \). Thus, if \( \phi = 0 \), we are done. Otherwise, we may set \( m_{00} = m_P + m_Q - n \) (note that \( m_{00} < m_P \)), \( V_P = \text{Span}\{e_1, \ldots, e_{m_P}\} \) and
\[
V_Q = \text{Span}\{e_1, \ldots, e_{m_{00}}, \cos \phi e_{m_P} + \sin \phi e_n, e_{m_P+1}, \ldots, e_{n-1}\}.
\]

Then, \( P, Q \) are as required.

The previous examples essentially amount to the proof of the following, where \( M_f \) is the universal bound of Corollary 6.

**Proposition 1** Let \( \mathcal{H} \) be a separable complex Hilbert space with \( 2 \leq \dim \mathcal{H} \leq \infty \). Let \( 0, \text{Id} \neq P : \mathcal{H} \to V_P \) denote an orthogonal projection. Then, for every \( 1 \leq m_Q \leq \dim \mathcal{H} - 1 \) there exists an orthogonal projection \( Q : \mathcal{H} \to V_Q \) with rank \( Q = m_Q \) such that \( \|f(P, Q)\|_{op} = M_f \).

**Proof** We need to find an orthogonal projection \( Q : \mathcal{H} \to V_Q \) such that
\[
\|f(P, Q)\|_{op} = M_f.
\]

We assume without loss of generality that \( \dim \mathcal{H} = 2 \), so that \( \dim V_P = 1 \). Indeed, if we choose arbitrary nonzero \( v_1 \in V_P \) and \( v_2 \in \ker(P) \), and denote \( \mathcal{H}_0 = \text{Span}\{v_1, v_2\} \), and find an orthogonal projection \( Q : \mathcal{H}_0 \to \mathcal{H}_0 \) such that \( \|f(P|_{\mathcal{H}_0}, Q)\|_{op} = \max_{[0,1]} \psi_f \), then clearly we can extend \( Q \) to \( \mathcal{H}_0 \geq \mathcal{H} \) so that rank \( Q = m_Q \), which yields the required. Hence, we assume dim \( \mathcal{H} = 2 \).

Let \( t_{\text{max}} \in [0, 1] \) be such that \( \max_{[0,1]} \psi_f(t) = \psi_f(t_{\text{max}}) \). If \( 0 < t_{\text{max}} < 1 \), then there exists \( \phi \in (0, \frac{\pi}{2}) \) such that \( \sin^2 \phi = 1 - t_{\text{max}} \). Clearly, the previous examples imply that there exists \( Q : \mathcal{H} \to V_Q \) such that \( \phi \) is among the (reduced) principal angles associated with \( P, Q \), which means that \( t_{\text{max}} \in \sigma(I - H) \). Hence,
\[
\|f(P, Q)\|_{op} = \max_{[0,1]} \psi_f.
\]

Otherwise, \( t_{\text{max}} \in (0, 1) \), which means that
\[
\max_{[0,1]} \psi_f = \max_{[0,1]} \psi_f = \max_{\{\alpha_{00}, |\alpha_{01}|, |\alpha_{10}|, |\alpha_{11}|\}}.
\]
If the latter equals $|\alpha_{00}|$ or $|\alpha_{11}|$, we may set $Q = P$, so that $V_P \cap V_Q = V_P$ and $V_P^\perp \cap V_Q^\perp = V_P^\perp$. Then, $\|f(P, Q)\|_{\text{op}} = \max\{|\alpha_{00}|, |\alpha_{11}|\} = \max_{[0,1]} \psi_f$. Otherwise, we can set $Q = I - P$, so that $V_P \cap V_Q^\perp = V_P$, $V_P^\perp \cap V_Q = V_P^\perp$ to obtain the required.

\[\square\]

3 Proof of Theorem 2

The present section is dedicated to the proof of Theorem 10, which includes Theorem 2 as a special case. The proof consists of little more than a straightforward combination of results from [3] and [9]. Indeed, if $f \in \ker(T)$, then by Corollary 5, $\|f(P, Q)\|_{\text{op}} = \max_{\sigma(PQP)} \psi_f$ (the general case $f \in A$ is only slightly more complicated). The operator $PQP \in \text{End}(V_P)$ is known to provide a model of the so-called Jacobi ensemble of random matrices. In particular, the behavior of $\sigma(PQP)$ is well understood as $n \to \infty$.

Let $\Omega_{n,\alpha,\beta} = G_{an}(n) \times G_{bn}(n)$, equipped with the probability measure $\nu_{n,\alpha,\beta} = \mu_{an,n} \times \mu_{bn,n}$. Here, $a_n = \lfloor \alpha n \rfloor$ and $b_n = \lfloor \beta n \rfloor$, where $0 < \alpha \leq \beta$ satisfy $\alpha + \beta \leq 1$.

For $(P, Q) \in \Omega_{n,\alpha,\beta}$, we will use the notations of (4). Finally, let

$$I_{n,\alpha,\beta}^f = \int_{\Omega_{n,\alpha,\beta}} \|f(P, Q)\|_{\text{op}} d\nu_{n,\alpha,\beta}.$$

Our goal is to compute $I_{\alpha,\beta}^f = \lim_{n \to \infty} I_{n,\alpha,\beta}^f$ in terms of $\alpha, \beta$ and $f$, where according to Fubini’s theorem,

$$I_{n,\alpha,\beta}^f = \int_{G_{an}(n)} \left( \int_{G_{bn}(n)} \|f(P, Q)\|_{\text{op}} d\mu_{bn,n}(Q) \right) d\mu_{an,n}(P).$$

We note the following observation.

**Remark 5** $\|f(P, Q)\|_{\text{op}} \leq M_f$ for all $(P, Q) \in \Omega_{n,\alpha,\beta}$ by Corollary 6, hence $I_{n,\alpha,\beta}^f \leq M_f$. Also note that $\max_{Q \in G_{bn}(n)} \|f(P, Q)\|_{\text{op}} = M_f$ for every $P \in G_{an}(n)$, by Proposition 1.

The invariance of $\mu_{bn,n}$ implies that the inner integral in (7) is independent of the choice of $P$. To see this, let $E_n = \{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{C}^n$, and let $P_0$ denote the orthogonal projection on $\text{Span}\{e_1, \ldots, e_{an}\}$. Then, for every $P \in G_{an}(n)$ there exists a (non-unique) unitary operator $U_0$ such that

$$P = U_0^* P_0 U_0.$$

**Lemma 2** Let $Q \in G_{bn}(n)$. Then, $\|f(P, Q)\|_{\text{op}} = \|f(P_0, U_0 Q U_0^*)\|_{\text{op}}$.
Proof It will be convenient to introduce the evaluation homomorphisms
\[ \text{ev}(P, Q) : A \to \text{End}(\mathbb{C}^n), \]
specified by \( \text{ev}(P, Q)(f) = f(P, Q) \). Then,
\[ \text{ev}(P, Q)(\cdot) = U_0^* \text{ev}(P_0, U_0QU_0^*)(\cdot)U_0. \]
Indeed,
\[ \text{ev}(P, Q)(1) = I = U_0^* \text{ev}(P_0, U_0QU_0^*)(1)U_0, \]
\[ \text{ev}(P, Q)(x) = P = U_0^* P_0 U_0 = U_0^* \text{ev}(P_0, U_0QU_0^*)(x)U_0 \]
and
\[ \text{ev}(P, Q)(y) = Q = U_0^* (U_0QU_0^*) U_0 = U_0^* \text{ev}(P_0, U_0QU_0^*)(y)U_0. \]
This holds, similarly, for all the monomials in \( A \), and by linearity, extends to all of \( A \). Since \( U_0 \) is unitary, we obtain the required. \( \square \)

The invariance of \( \mu_{b_n,n} \) implies that
\[ \int_{G_{b_n}(n)} \| f(P_0, UQU^*) \|_{op} d\mu_{b_n,n}(Q) = \int_{G_{b_n}(n)} \| f(P_0, Q) \|_{op} d\mu_{b_n,n}(Q) \]
for every unitary operator \( U \) on \( \mathbb{C}^n \), hence we can conclude that
\[ I_{n,\alpha,\beta}^f = \int_{G_{b_n}(n)} \| f(P_0, Q) \|_{op} d\mu_{b_n,n}(Q). \] (8)

Let \( H_0 \) be the operator associated with the pair \( (P_0, Q) \) as in Theorem 5. According to Theorem 6, \( \| f(P_0, Q) \|_{op} \) can be expressed conveniently using \( \sigma(I - H_0) \subset \sigma(P_0QP_0) \) (assuming that \( P_0, Q \) do not commute). Thus, we are led to consider the joint distribution of the eigenvalues of \( P_0QP_0 \).

Theorem 7 ([12]) The joint eigenvalue distribution of \( P_0QP_0 \in \text{End}(V_{P_0}) \) in \([0, 1]^{a_n}\) is given by
\[ d\mathcal{J}(\lambda_1, \ldots, \lambda_{a_n}) = \frac{1}{C_{n,a,\beta}} \prod_{l=1}^{a_n} \lambda_l^{b_n-a_n} (1 - \lambda_l)^{n-(a_n+b_n)} \prod_{1 \leq l < k \leq a_n} (\lambda_l - \lambda_k)^2 d\lambda, \]
where \( C_{n,a,\beta} \) is a normalization constant (and \( d\lambda = d\lambda_1 \cdots d\lambda_{a_n} \)). It is the joint eigenvalue distribution of the (unitary) Jacobi ensemble with parameters \( N = a_n, N_1 = b_n, N_2 = n - b_n \) (we adopt the convention of [12]).
In what follows, we view $P_0Q P_0$ as an element of End($V_{P_0}$) (using the notation of (4)). Then,

$$P_0 Q P_0 = I_{V_{00}} \oplus 0_{V_{01}} \oplus (I - H_0).$$

Denote

$$G_{n,0} = \{ Q \in G_{b_n}(n) \mid \dim V_{00} = \dim V_{01} = 0 \}.$$  

Then, for $n$ sufficiently large$^3$,

$$\mu(b_{n,n}(G_{n,0}) = 1,$$ (9)

since by assumption $a_n + b_n \leq n$ and $b_n - a_n \geq 0$. The limiting behavior of $d\mathcal{J}_n$ that is relevant to us is as follows.

**Theorem 8** ([12]) If $d\mathcal{J}_n$ is the joint distribution of $(\lambda_1, \ldots, \lambda_{a_n}) \in [0, 1]^{a_n}$, then for $F \in C[0, 1]$,

$$\frac{1}{a_n} \sum_{l=1}^{a_n} F(\lambda_l) \xrightarrow{p} \int_0^1 F(t) d\mathcal{J}_\infty(t)$$

as $n \to \infty$, where

$$d\mathcal{J}_\infty(t) = \frac{1}{2\pi \alpha \beta (1 - t)} \sqrt{- (t - \lambda_-) (t - \lambda_+)} \mathbb{1}_{[\lambda_-, \lambda_+]} dt,$$

and $\lambda_\pm = (\sqrt{\beta (1 - \alpha)} \pm \sqrt{\alpha (1 - \beta)})^2$. Note that $\lambda_- < \lambda_+$, and $\lambda_- = 0$ if and only if $\alpha = \beta$, and $\lambda_+ = 1$ if and only if $\alpha + \beta = 1$.

The latter does not rule out the possibility, when $0 < \lambda_- \text{ or } \lambda_+ < 1$, that a subset of order $o(a_n)$ of eigenvalues of $P_0 Q P_0$ remains outside of $[\lambda_-, \lambda_+]$, and a priori it could be that

$$\mu(b_{n,n}(\{ Q \mid \sigma(P_0 Q P_0) \cap ([0, 1] \setminus (\lambda_- - \delta, \lambda_+ + \delta)) \neq \emptyset \}) > \varepsilon$$

for some $\varepsilon, \delta > 0$. Hence, we will use the following.

**Theorem 9** ([9]) For any compact set $K$ such that $K \cap [\lambda_-, \lambda_+] = \emptyset$, there exists $C > 0$ such that $\mu(b_{n,n}(\{ Q \mid \sigma(P_0 Q P_0) \cap K \neq \emptyset \}) < e^{-Cn}$.

We conclude as follows.

$^3$ Any $n$ works if $\alpha = \beta$, otherwise any $n$ for which $(\beta - \alpha)n \geq 1$ suffices.
Corollary 7 Let $\delta > 0$ and $t \in [\lambda_-, \lambda_+]$. Denote

$$A_{n,t,\delta} = \{Q \in G_{b_n}(n) \mid d(\sigma(I - H_0), t) < \delta\},$$

$$B_{n,\delta} = \{Q \in G_{b_n}(n) \mid \sigma(I - H_0) \subset (\lambda_- - \delta, \lambda_+ + \delta)\}.$$

Then, $\lim_{n \to \infty} \mu_{b_n,n}(B_{n,\delta}) = \lim_{n \to \infty} \mu_{b_n,n}(A_{n,t,\delta}) = 1$.

**Proof** As we noted previously (9), we can assume that $\mu_{b_n,n}(G_{n,0}) = 1$. Thus, almost surely $\sigma(P_0 Q P_0) = \sigma(I - H_0)$, so it is a basic consequence of the fact that the density of $d\mathcal{G}_\infty$ is non-vanishing on $[\lambda_-, \lambda_+]$ that $\mu_{b_n,n}(A_{n,t,\delta})$ converges to 1 as $n \to \infty$. Next, by Theorem 9,

$$\lim_{n \to \infty} \mu_{b_n,n}(\{Q \in G_{b_n}(n) \mid \sigma(P_0 Q P_0) \cap K\}) = 0,$$

where

$$K = [-\delta, \lambda_- - \delta] \cup [\lambda_+ + \delta, 1 + \delta],$$

hence $\lim_{n \to \infty} \mu_{b_n,n}(B_{n,\delta}) = 1$.



Theorem 10 Let $f \in \mathcal{A}$. Then, $I_{f_{\alpha,\beta}} = \lim_{n \to \infty} I_{f,\alpha,\beta}$ is well defined, and

1. If $\alpha = \beta = \frac{1}{2}$ then $I_{f,\alpha,\beta} = \max_{[0,1]} \psi_f = M_f$,
2. If $\alpha < \beta = 1 - \alpha$, then $I_{f,\alpha,\beta} = \max \{||f|_{10}, \max_{[\lambda_-,\lambda_+]} \psi_f\}$,
3. If $\alpha = \beta < 1 - \alpha$, then $I_{f,\alpha,\beta} = \max \{||f|_{11}, \max_{[\lambda_-\lambda_+]} \psi_f\}$,
4. If $\alpha < \beta < 1 - \alpha$, then $I_{f,\alpha,\beta} = \max \{||f|_{10}, ||f|_{11}, \max_{[\lambda_-\lambda_+]} \psi_f\}$.

We recall that $\lambda_- = 0 \Leftrightarrow \alpha = \beta$ and $\lambda_+ = 1 \Leftrightarrow \beta = 1 - \alpha$ (and that $\lambda_- < \lambda_+$).

**Proof** As before, we use the notation of (4), and assume that $\mu_{b_n,n}(G_{n,0}) = 1$. Fix $\varepsilon > 0$. Then,

$$I_{f,\alpha,\beta} = \int_{G_{b_n}(n)} ||f(P_0, Q)||_{op} d\mu_{b_n,n}(Q) = \int_{G_{b_n}(n)} \left(\max_{\sigma P_0, Q} \psi_f\right) d\mu_{b_n,n}(Q),$$

where $\psi_f$ and $\sigma P_0, Q$ are as in Corollary 5. If $\dim V_{10} = 0$ and $\dim V_{11} = 0$, then $\max_{\sigma P_0, Q} \psi_f = \max_{\sigma(I - H_0)} \psi_f$. The latter observation will guide our proof, i.e., we will first show that

$$\tilde{I}_{\alpha,\beta} = \lim_{n \to \infty} \int_{G_{b_n}(n)} \left(\max_{\sigma(I - H_0)} \psi_f\right) d\mu_{b_n,n}(Q) = \max_{[\lambda_-,\lambda_+]} \psi_f \quad (10)$$

Let $t_{\max} \in [\lambda_-, \lambda_+]$ be such that

$$M = \max_{[\lambda_-,\lambda_+]} \psi_f = \psi_f(t_{\max}).$$

---

4 Here, $d(K, t)$ denotes the distance between the set $K \subset \mathbb{R}$ and $t \in \mathbb{R}$.
By the continuity of $\psi_f$, there exists $\delta > 0$ such that if $t \in [0, 1]$ and $|t - t_{\max}| < \delta$ then

$$|\psi_f(t) - M| < \frac{\varepsilon}{2}, \quad (11)$$

and additionally,

$$\max_{[\lambda_- - \delta, \lambda_+ + \delta]} \psi_f - M < \frac{\varepsilon}{2}. \quad (12)$$

Note that the latter is trivial if $\lambda_- = 0, \lambda_+ = 1$ (since then the left-hand side equals 0). We will use (11) to show $\tilde{I}_{f, \alpha, \beta} \geq M$, and (12) to show $\tilde{I}_{f, \alpha, \beta} \leq M$.

Using Corollary 7, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\left( M - \frac{\varepsilon}{2} \right) \mu_{b_n, n} \left( G_{n, 0} \cap A_{n, t_{\max}, \delta} \right) > M - \varepsilon. \quad (13)$$

Thus, since $\max_{\sigma_{P_0, Q}} \psi_f \geq \max_{\sigma(I - H_0)} \psi_f \geq 0$, we get that for all $n > n_0$,

$$M_f \geq I_{f, n, \alpha, \beta} = \int_{G_{b_n}(n)} \left( \max_{\sigma_{P_0, Q}} \psi_f \right) d\mu_{b_n, n} \geq \int_{G_{n, 0} \cap A_{n, t_{\max}, \delta}} \left( \max_{\sigma(I - H_0)} \psi_f \right) d\mu_{b_n, n} \geq M - \varepsilon.$$

The last inequality holds first since $\max_{\sigma(I - H_0)} \psi_f > M - \frac{\varepsilon}{2}$ for every projection $Q \in G_{n, 0} \cap A_{n, t_{\max}, \delta}$ by (11), and then using (13).

If $\alpha = \beta = \frac{1}{2}$, then $\lambda_- = 0, \lambda_+ = 1$, so $M = M_f$ and $\lim_{n \to \infty} I_{n, \alpha, \beta} = M_f$ as required. Otherwise, by Corollary 7, for $\varepsilon_1 > 0$ which satisfies $M_f \varepsilon_1 < \frac{\varepsilon}{2}$, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$, it holds that

$$\mu_{b_n, n}(C_{n, \delta}) > 1 - \varepsilon_1, \quad (14)$$

where

$$C_{n, \delta} = G_{n, 0} \cap A_{n, t_{\max}, \delta} \cap B_{n, \delta},$$

$$B_{n, \delta} = \{ Q \in G_{b_n}(n) | \sigma(I - H_0) \subset (\lambda_- - \delta, \lambda_+ + \delta) \}.$$
Thus,
\[
\int_{G_{bn}(n)} \left( \max_{\sigma(I-H_0)} \psi_f \right) d\mu_{b_n,n} \\
= \int_{C_{n,\delta}} \left( \max_{\sigma(I-H_0)} \psi_f \right) d\mu_{b_n,n} + \int_{G_{n,0}(G_{bn}(n) \setminus C_{n,\delta})} \left( \max_{\sigma(I-H_0)} \psi_f \right) d\mu_{b_n,n} \\
\leq M + \frac{\varepsilon}{2} + \varepsilon_1 M_f < M + \varepsilon.
\]

Here, the first integral is not greater than \( M + \frac{\varepsilon}{2} \) since \( \max_{\sigma(I-H_0)} \psi_f \leq M + \frac{\varepsilon}{2} \) for every \( Q \in C_{n,\delta} \) using (12), and the second integral is bounded from above by \( \varepsilon_1 M_f \) since \( \max_{\sigma(I-H_0)} \psi_f \leq M_f = \max_{[0,1]} \psi_f \) and then using (14).

The above completes the proof of (10), and the proof for the case \( \alpha = \beta = \frac{1}{2} \).

We turn to address the remaining cases, namely when \( \alpha + \beta = 1 \), \( \alpha < \beta \), and when \( \alpha + \beta < 1 \), \( \alpha = \beta \) and finally when \( \alpha + \beta < 1 \), \( \alpha < \beta \). Note that for \( n \) large enough, if \( \alpha < \beta \) then \( \dim V_{10} > 0 \) for every \( Q \in G_{bn}(n) \), and if \( \alpha + \beta < 1 \) then \( \dim V_{11} > 0 \) for every \( Q \in G_{bn}(n) \).

Assume that \( \alpha + \beta = 1 \) and that \( \alpha < \beta \). Let \( M_- = \max(|\alpha_{10}|, M) \). If \( M_- = |\alpha_{10}| \), then for \( n \) large enough,
\[
\max_{\sigma_{n,0,Q}} \Psi_f = |\alpha_{10}|
\]
for every \( Q \in G_{bn}(n) \), hence clearly \( I_{n,\alpha,\beta}^f = |\alpha_{10}| \) for every \( n \) large enough. Otherwise, if \( |\alpha_{10}| < M \), we assume without loss of generality that \( M - |\alpha_{10}| > \varepsilon \). Then, by the above,
\[
M - \frac{\varepsilon}{2} \leq \max_{\sigma_{n,0,Q}} \Psi_f = \max_{\sigma(I-H_0)} \psi_f \leq M + \frac{\varepsilon}{2}
\]
for every \( Q \in C_{n,\delta} \), hence by the above, \( \lim_{n \to \infty} I_{n,\alpha,\beta}^f = M \).

Similarly, if \( \alpha = \beta < \frac{1}{2} \), then either \( M_+ = \max(|\alpha_{11}|, M) = |\alpha_{11}| \), in which case we note that for \( n \) large enough, \( \max_{\sigma_{n,0,Q}} \psi_f = |\alpha_{11}| \) for every \( Q \in G_{bn}(n) \) so \( I_{n,\alpha,\beta}^f = |\alpha_{11}| \) for every \( n \) large enough, or \( M_+ = M \), in which case as for the previous case, \( \lim_{n \to \infty} I_{n,\alpha,\beta}^f = M \).

Finally, if \( \alpha + \beta < 1 \), \( \alpha < \beta \), then either \( M_{-,+} = \max(|\alpha_{10}|, |\alpha_{11}|, M) = \max(|\alpha_{10}|, |\alpha_{11}|) \), in which case we note that for \( n \) large enough, \( \max_{\sigma_{n,0,Q}} \psi_f = M_{-,+} \) for all \( Q \in G_{bn}(n) \), hence \( I_{n,\alpha,\beta}^f = M_{-,+} \) for all \( n \) sufficiently large, or otherwise \( M_{-,+} = M \), in which case we repeat the reasoning above to obtain the required. \( \square \)

We obtain Theorem 2 immediately. Note that if \( \alpha = \beta \leq \frac{1}{2} \) then (see Theorem 8 for the formulas of \( \lambda_- \), \( \lambda_+ \)) we obtain
\[
\lambda_- = 0, \quad \lambda_+ = 4\alpha (1 - \alpha) = \lambda_{10}.
\]
Corollary 8 If \( f \in \ker(T) \), then by Lemma 1, \( |\alpha| \neq 0 \) for \( l, k \in \{0, 1\} \), hence the limits in the distinct cases addressed in Theorem 10 admit the same formula, namely
\[
\lim_{n \to \infty} I_{n, \alpha, \beta}^f = \max_{[\lambda_-, \lambda_+]} \psi_f.
\]
If \( \alpha = \beta \), then \( [\lambda_-, \lambda_+] = [0, \lambda_\alpha] \), giving Theorem 2.

4 Proof of Theorem 3

The present section contains the proof of Theorem 3. In the next subsection, we prove (for the sake of completeness) some basic lemmas about weak and strong convergence in Hilbert spaces. In Sect. 4.2, we specify the limits of “central” matrix coefficients of the projections, as computed in a previous work ((30)) by the author. The contents of Sects. 4.1 and 4.2 suffice to reduce (in Sects. 4.3 and 4.4) the evaluation of the limit to the case of two projections on \( L^2(T) \) (where \( T \subset \mathbb{C} \) is the unit circle), namely the Cauchy–Szegö projection on the Hardy space (denoted \( H_T \)), and the operator of multiplication by the indicator function of \( E_\alpha = \{ 0 < \Re \zeta < 2\alpha \} \subset T \) (denoted \( M_{\chi_{E_\alpha}} \)). Finally, in Sect. 4.5, we use a classical result on the spectrum of Toeplitz operators with bounded symbols (together with Theorem 6) to conclude the proof.

4.1 Preliminaries

Let \( B(H) \) denote the space of bounded operators on a complex Hilbert space \( H \). We will use the following elementary notions and facts.

Definition 3 Let \( \{A_n\}_{n \in \mathbb{N}} \subset B(H) \) and \( A \in B(H) \).

1. We say that \( \{A_n\}_{n \in \mathbb{N}} \) converges strongly to \( A \) if \( \lim_{n \to \infty} A_n v = A v \) for every \( v \in H \).

2. We say that \( \{A_n\}_{n \in \mathbb{N}} \) converges weakly to \( A \) if \( \lim_{n \to \infty} \langle A_n u, v \rangle = \langle A u, v \rangle \) for every \( u, v \in H \).

Lemma 3 Let \( E = \{e_k \mid k \in \mathbb{N}\} \) denote an orthonormal basis of \( H \). Let \( \{A_n\}_{n \in \mathbb{N}} \subset B(H) \) and \( A \in B(H) \). Then, \( \{A_n\}_{n \in \mathbb{N}} \) converges weakly to \( A \) if and only if \( \sup_n \|A_n\|_{\text{op}} < \infty \) and \( \lim_{n \to \infty} \langle A_n e_l, e_k \rangle = \langle A e_l, e_k \rangle \) for every \( l, k \in \mathbb{N} \).

Proof It is well known that weakly convergent sequences are bounded. Indeed, for \( v \in H \) let \( \psi_{n,v}(u) = \langle A_n v, u \rangle \), \( \psi_{n,v} \in \mathcal{H}^* \). Then, \( \{\|\psi_{n,v}\|_{\mathcal{H}^*}\}_{n \in \mathbb{N}} \) is bounded for every \( u \in \mathcal{H} \), hence \( \{\|A_n v\|_{\mathcal{H}}\}_{n \in \mathbb{N}} \) is bounded by the uniform boundedness principle. Since \( \{\|A_n v\|_{\mathcal{H}}\}_{n \in \mathbb{N}} \) is bounded for all \( v \in \mathcal{H} \), then again using uniform boundedness, we deduce that \( \{\|A_n\|_{\text{op}}\}_{n \in \mathbb{N}} \) is bounded.

Conversely, assume that \( \|A_n\|_{\text{op}} < M \) for all \( n \), let \( u, v \in \mathcal{H} \) and let \( \varepsilon > 0 \). For \( \delta > 0 \), we may take
\[
\tilde{u} = \sum_{k \leq k_0} \langle u, e_k \rangle e_k, \quad \tilde{v} = \sum_{k \leq k_0} \langle v, e_k \rangle e_k
\]
such that \( \| u - \tilde{u} \| < \delta \), \( \| v - \tilde{v} \| < \delta \). Then, for sufficiently large \( n \) so that \( \| (A_n - A)\tilde{u}, \tilde{v} \| < \delta \), we have that

\[
\begin{align*}
&\| (A_n u, v) - (Au, v) \| \\
&\quad = \| (A_n - A)(u - \tilde{u}), v \| + \| (A_n - A)\tilde{u}, v - \tilde{v} \| + \| (A_n - A)\tilde{u}, \tilde{v} \| \\
&\quad \leq \| A_n - A \|_{op} (\| v \| + \| \tilde{u} \|) \delta + \delta \leq (M + \| A \|_{op})(\| v \| + \| u \|)\delta + \delta < \varepsilon ,
\end{align*}
\]

where the latter holds for \( \delta \) sufficiently small. \( \square \)

**Lemma 4**

Let \( \mathcal{E} = \{ e_k \mid k \in \mathbb{N} \} \) denote an orthonormal basis of \( \mathcal{H} \). If \( \{ A_n \}_{n \in \mathbb{N}} \subset \mathcal{B}(\mathcal{H}) \) converges weakly to \( A \in \mathcal{B}(\mathcal{H}) \) and \( \lim_{n \to \infty} \| A_n e_k \| = \| Ae_k \| \) for every (fixed) \( k \in \mathbb{N} \), then \( A_n \) converges strongly to \( A \).

**Proof**

We note that

\[
\| A_n e_k - Ae_k \| = \| A_n e_k \| + \| Ae_k \| - (Ae_k, A_n e_k) - \langle A_n e_k, Ae_k \rangle,
\]

hence by weak convergence,

\[
\lim_{n \to \infty} \| A_n e_k - Ae_k \| = 0,
\]

that is, \( \lim_{n \to \infty} A_n e_k = Ae_k \) for every fixed \( k \in \mathbb{N} \).

Let \( v = \sum_{k \in \mathbb{N}} v_k e_k \in \mathcal{H} \). Let \( \varepsilon > 0 \). Note that \( \sup_n \| A_n \|_{op} < \infty \). Hence, there exists \( k_0 > 0 \) such that \( v_k = \sum_{k \geq k_0} v_k e_k \) satisfies \( \| (A - A_n) v_k \| < \frac{\varepsilon}{2} \). Let \( u_k = v - v_k \).

Then,

\[
\lim_{n \to \infty} A_n u_k = Au_k,
\]

so there exists \( n_0 \) such that for all \( n \geq n_0 \), it holds that \( \| (A_n - A) u_k \| < \frac{\varepsilon}{2} \). Thus, for all \( n \geq n_0 \),

\[
0 \leq \| (A_n - A) v \| = \| (A_n - A)(u_k + v_k) \| < \varepsilon,
\]

as required. \( \square \)

**Lemma 5**

Assume that \( \{ A_n \}_{n \in \mathbb{N}} \subset \mathcal{B}(\mathcal{H}) \) converges to \( A \in \mathcal{B}(\mathcal{H}) \) strongly. Then, \( \lim \inf \| A_n \|_{op} \geq \| A \|_{op} \) (this is also true if \( A_n \) converges to \( A \) weakly).

**Proof**

Let \( \varepsilon > 0 \). Assume that \( v \in \mathcal{H} \) satisfies \( \| Av \| > \| A \|_{op} - \frac{1}{2} \varepsilon \). Then, there exists \( n_0 \) such that for all \( n > n_0 \), it holds that \( \| A_n v \| > \| A \|_{op} - \varepsilon \), hence the required. \( \square \)

**Corollary 9**

If \( \{ A_n \}_{n \in \mathbb{N}} \subset \mathcal{B}(\mathcal{H}) \) converges to \( A \in \mathcal{B}(\mathcal{H}) \) strongly and \( \| A_n \|_{op} \leq \| A \|_{op} \) for all \( n \in \mathbb{N} \), then \( \lim_{n \to \infty} \| A_n \|_{op} = \| A \|_{op} \).

**Lemma 6**

Assume that \( \{ A_n \}_{n \in \mathbb{N}} \) and \( \{ B_n \}_{n \in \mathbb{N}} \) converge to \( A, B \) strongly. Then, \( A_n B_n \) converges strongly to \( AB \).
Proof Let $M_A > 0$ be such that $\|A_n\|_{\text{op}} < M_A$ for all $d$. Let $v \in H$. Then,

$$
\|(A_n B_n - AB)v\| = \|(A_n B_n - A_n B + A_n B - AB)v\|
\leq \|A_n (B_n - B)v\| + \|(A_n - A)Bv\|
\leq M_A \| (B_n - B)v\| + \|(A_n - A)Bv\|,
$$

hence by strong convergence of $A_n$, $B_n$, $\lim_{n \to \infty} \|(A_n B_n - AB)v\| = 0$. \hfill \Box

4.2 Matrices of spectral projections of spin operators

We consider the standard basis $B_{3,n} = \{e_j, e_{j-1}, ..., e_{-j}\}$ of eigenvectors of $J_3$, where as before $2j + 1 = n$. Recall that the spectrum of $J_1$, $J_2$, $J_3$ is

$$
\sigma_n = \{j, j-1, ..., -j\},
$$

and denote $P_{1,\alpha,n} = \mathbb{1}_{(0,\alpha_n)}(J_1)$ and $P_{3,\alpha,n} = \mathbb{1}_{(0,\alpha_n)}(J_3)$, where $\alpha_n > 0$ satisfies

$$
\# ((0, \alpha_n) \cap \sigma_n) = \lfloor \alpha n \rfloor
$$

for every $n \in \mathbb{N}$. Let $[A]_{B_{3,n}}$ denote the matrix of $A : \mathbb{C}^n \to \mathbb{C}^n$ relative to $B_{3,n}$. Clearly,

$$
[P_{3,\alpha,n}]_{B_{3,n}} = \begin{pmatrix} \tilde{I}_{\alpha,n} & 0 \\ 0 & 0 \end{pmatrix}, \quad (16)
$$

where

$$
\tilde{I}_{\alpha,n} = \begin{pmatrix} 0 & 0 \\ 0 & I_{\lfloor \alpha n \rfloor} \end{pmatrix}.
$$

Write $[P_{1,\alpha,n}]_{B_{3,n}} = (P_{1,\alpha,n,m',m})_{m'=j, j-1, ..., -j}$.

Theorem 11 ([30]) Fix $m', m \in \sigma_n$. Let $\hat{1}_{E_\alpha}(k)$ denote the $k$-th Fourier coefficient of the indicator function of $E_\alpha = \{0 < \Re \zeta < 2\alpha\} \subset \mathbb{T}$. Then,

$$
\lim_{k \to \infty} P_{1,\alpha,n+2k,m',m} = \hat{1}_{E_\alpha}(m - m').
$$

In particular,

Corollary 10 Fix $m \in \sigma_n$. Then, $\lim_{k \to \infty} \| P_{1,\alpha,n+2k} e_m \|^2 = \hat{1}_{E_\alpha}(0)$.

Proof Note that

$$
\langle P_{1,\alpha,n} e_m, P_{1,\alpha,n} e_m \rangle = \langle P_{1,\alpha,n}^* P_{1,\alpha,n} e_m, e_m \rangle
= \langle P_{1,\alpha,n} e_m, e_m \rangle = P_{1,\alpha,n,m,m}.
$$

Thus, by Theorem 11, we obtain the required. \hfill \Box
4.3 The $l^2(\mathbb{Z})$ settings

It will be convenient to work in $L^2(\mathbb{T})$ rather than $\mathbb{C}^n$. As an intermediate step, we consider $l^2(\mathbb{Z})$. Let $\widehat{B} = \{\hat{e}_k \mid k \in \mathbb{Z}\}$ denote the standard basis of $l^2(\mathbb{Z})$. Define an embedding $\Psi_n : \mathbb{C}^n \to l^2(\mathbb{Z})$ by

$$\Psi_n(e_m) = \begin{cases} \hat{e}_{m - \frac{1}{2}} & n \in 2\mathbb{N} \\ \hat{e}_{m - 1} & n \in 2\mathbb{N} - 1 \end{cases}, \quad m = j, j - 1, ..., -j, \quad n = 2j + 1.$$

Let $V_n = \Psi_n(\mathbb{C}^n)$, and let $\Pi_n : l^2(\mathbb{Z}) \to V_n \subset l^2(\mathbb{Z})$ denote the orthogonal projection on $V_n$. For an operator $A : \mathbb{C}^n \to \mathbb{C}^n$, we set

$$A^Z = \Psi_n \circ A \circ \Psi_n^{-1} \circ \Pi_n.$$

**Corollary 11** If the matrix of $A$ in $\mathcal{B}_{3,n}$ is $(a_{m',m})_{|m|,|m'| \leq j}$, then the matrix elements of $A^Z$ in the basis $\widehat{B}$ are as follows.

If $n \in 2\mathbb{N} - 1$, then

$$\langle A^Z \hat{e}_l, \hat{e}_k \rangle = \begin{cases} 0 & |l + 1| > j \text{ or } |k + 1| > j \\ a_{k+1,l+1} & |l + 1|, |k + 1| \leq j \end{cases}.$$

If $n \in 2\mathbb{N}$, then

$$\langle A^Z \hat{e}_l, \hat{e}_k \rangle = \begin{cases} 0 & \left|l + \frac{1}{2}\right| > j \text{ or } \left|k + \frac{1}{2}\right| > j \\ a_{k+\frac{1}{2},l+\frac{1}{2}} & \left|l + \frac{1}{2}\right|, \left|k + \frac{1}{2}\right| \leq j \end{cases}.$$

4.4 The $L^2(\mathbb{T})$ settings

Let $\mathcal{B} = \{\zeta^k \mid k \in \mathbb{Z}\}$ denote the standard orthonormal basis of $L^2(\mathbb{T})$, which we identify with $l^2(\mathbb{Z})$ in the obvious way (i.e., we identify $\hat{e}_k$ with $\zeta^k$). Let $P_{1,\alpha,n}^T, P_{3,\alpha,n}^T$ be the equivalents of $P_{1,\alpha,n}^Z, P_{3,\alpha,n}^Z$. Let $\Pi_T : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the orthogonal Cauchy–Szegö projection on the Hardy space $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$. We recall that

$$H^2(\mathbb{T}) = \left\{F \in L^2(\mathbb{T}) \mid \langle F, \zeta^k \rangle = 0 \forall k < 0\right\}.$$

Finally, for $\psi \in L^\infty(\mathbb{T})$ let $\mathcal{M}_\psi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the multiplication operator $G \mapsto \psi G$.

**Proposition 2** $P_{3,\alpha,n}^T, P_{1,\alpha,n}^T$ converge strongly to $\Pi_T, \mathcal{M}_{1_Ea}$, respectively.

**Proof** Clearly, $\|P_{3,\alpha,n}^T\|_{op} = \|P_{1,\alpha,n}^T\|_{op} = 1$ for all $n \in \mathbb{N}$. By (16), Theorem 11 and Corollary 11, we have for every $k, l \in \mathbb{Z}$
\[
\lim_{n \to \infty} \langle P_{3,\alpha,n}^T, \xi^l, \xi^k \rangle_{L^2(\mathbb{T})} = \langle \Pi_T \xi^l, \xi^k \rangle_{L^2(\mathbb{T})},
\]
\[
\lim_{n \to \infty} \langle P_{1,\alpha,n}^T, \xi^l, \xi^k \rangle_{L^2(\mathbb{T})} = \langle \mathcal{M}_{\alpha} \xi^l, \xi^k \rangle_{L^2(\mathbb{T})}.
\]

Thus, Lemma 3 gives us weak convergence. Then, as evident in (16),
\[
\lim_{n \to \infty} P_{3,\alpha,n}^T \xi^k = \Pi_T \xi^k
\]
for all \( k \in \mathbb{Z} \). Also,
\[
\left\| \mathcal{M}_{\alpha} \xi^k \right\|^2 = \langle \mathbb{1}_E \xi^k, \mathbb{1}_E \xi^k \rangle_{L^2(\mathbb{T})} = \langle \mathbb{1}_E, 1 \rangle_{L^2(\mathbb{T})} = \hat{1}_E(0).
\]

Thus, noting Corollary 10, we see that Lemma 4 applies to both \( \{ P_{1,\alpha,n}^T \}_{n \in \mathbb{N}} \) and \( \{ P_{3,\alpha,n}^T \}_{n \in \mathbb{N}} \).

**Corollary 12** \( f(P_{3,\alpha,n}^T, P_{1,\alpha,n}^T) \) converges strongly to \( f(\Pi_T, \mathcal{M}_{\alpha}) \).

**Proof** This follows for all monomials by induction using Lemma 6, then for all polynomials using linearity.

**4.5 The pair \( \Pi_T, \mathcal{M}_{\alpha} \)**

We now apply the general theory of pairs of orthogonal projections to the pair \( P = \Pi_T, Q = \mathcal{M}_{\alpha} \). In the notations of Theorem 5,
\[
PQP = (1, 0, 0, 0) \oplus \begin{pmatrix} I - H & 0 \\ 0 & 0 \end{pmatrix}.
\]

**Definition 4** The Toeplitz operator associated with the symbol \( \phi \in L^\infty(\mathbb{T}) \) is \( T_\phi = \Pi_T \mathcal{M}_\phi \Pi_T : H^2(\mathbb{T}) \to H^2(\mathbb{T}) \).

Thus,
\[
PQP = (1, 0, 0, 0) \oplus (I - H) \oplus 0 = T_{\mathbb{1}_E}.
\]

We note the following classical result.

**Theorem 12** (Hartman–Wintner, [11,17], 7.20) If \( \phi \in L^\infty(\mathbb{T}) \) is real-valued, then the spectrum of \( T_\phi \) is given by
\[
\sigma(T_\phi) = [\text{ess inf } \phi, \text{ess sup } \phi].
\]
Corollary 13 Hartman–Wintner’s theorem implies that
\[ \sigma(I - H) = \sigma(H) = [0, 1]. \]

Thus, by Theorem 6 and Corollary 6, \( \|f(P, Q)\|_{op} = M_f \). The polynomial
\[ f(P_{3,\alpha,n}, P_{1,\alpha,n}^T) \]
converges to \( f(P, Q) \) strongly, hence by Corollary 9
\[ \lim_{n \to \infty} \|f(P_{3,\alpha,n}, P_{1,\alpha,n}^T)\|_{op} = M_f, \]
and since \( \|f(P_{3,\alpha,n}, P_{1,\alpha,n})\|_{op} = \|f(P_{3,\alpha,n}^T, P_{1,\alpha,n}^T)\|_{op} \), this completes the proof of
Theorem 3.

Remark 6 If \( G \) is a holomorphic function on the unit disk \( \mathbb{D} \subset \mathbb{C} \) with zero radial
boundary values on an arc \( E \subset \mathbb{T} \), then \( G \equiv 0 \) (e.g., by the classical Luzin–Privalov
theorems). Thus, the same holds for an anti-holomorphic function. Hence (in the
notation of (4)), for the projections \( P, Q \), we have \( V_{lk} = \{0\} \) for every \( l, k \in \{0, 1\} \),
so that in fact
\[ PQP = T_{E,\alpha}^1 = I - H : H^2(\mathbb{T}) \to H^2(\mathbb{T}). \]

5 Proof of Theorem 4

Denote \( R_n = P_{3,\alpha,n} P_{1,n} P_{3,\alpha,n} \in \text{End} \left( \text{Im}(P_{3,\alpha,n}) \right) \). Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\lfloor \alpha n \rfloor} \) be
the eigenvalues of \( R_n \). For sequences of real numbers \( a_n, b_n \), we write \( a_n = \Theta(b_n) \)
if there exist positive constants \( C_1, C_2 \) and \( n_0 \in \mathbb{N} \) such that \( C_1 b_n < a_n < C_2 b_n \) for
every \( n > n_0 \). We will prove, by direct computation, that
\[ \text{trace}(R_n) - \text{trace} \left( R_n^2 \right) = \sum_{k=1}^{\lfloor \alpha n \rfloor} \lambda_k (1 - \lambda_k) = \Theta(\log n), \]
which readily implies Theorem 4. This method is quite standard in the context of
Slepian spectral concentration problems; specifically, our proof follows [13].

We denote the matrix of \( P_{1,n} \) with respect to the eigenbasis \( B_{3,n} \) of \( J_3 \) by
\[ \left[ P_{1,n} \right]_{B_{3,n}} = \left( p_{m',m}^j \right)_{m,m'=j-1,\ldots,-j}, \quad n = 2j + 1. \]

Proposition 3 Fix \( 0 < \gamma < 1 \). Let \( S_n \) be a sub-matrix of \( \left[ P_{1,n} \right]_{B_{3,n}} \) of the form
\[ S_n = \left( p_{m',m}^j \right)_{m,m' \geq \gamma j, m \geq m_0}, \]
where \( |m_0| \leq \gamma j \). Then, the Frobenius norm of \( S_n \) satisfies \( \|S_n\|_F = \Theta(\log n) \) (with
constants depending only on \( \gamma \)).
The proposition will be established through a series of lemmas, and our main tool is the formula ([30])

\[ p_{m',m}^j = \begin{cases} 
\frac{1}{2} e^{i\frac{\pi}{2} (m'-m)} \left( \delta_{m',m} - \frac{d_{m',m}^j(0)}{m' - m} \right) & \text{if } m' - m \in 2\mathbb{Z}, \\
-\frac{i}{2} e^{i\frac{\pi}{2} (m'-m)} \mathcal{H}_\mathbb{T}(d_{m',m}^j(0)) & \text{if } m' - m \in 2\mathbb{Z} + 1
\end{cases} \tag{17} \]

where \( \mathcal{H}_\mathbb{T} : L^2(\mathbb{T}) \to L^2(\mathbb{T}) \) is the periodic Hilbert transform and \( d_{m',m}^j : \mathbb{T} \to \mathbb{T} \) are the Wigner small-d functions, specified by

\[ d_{m',m}^j(\theta) = \langle e^{-i0J^2}e_m, e_{m'} \rangle. \]

The zeroth Fourier coefficient of \( d_{m',m}^j \) is specified by ([1], 3.78, [14,30])

\[ \widehat{d}_{m',m}^j(0) = \begin{cases} 
e^{-i\frac{\pi}{2} (m'-m)} d_{m',m}^j(0) \left( \frac{\pi}{2} \right) & \text{if } j \in \mathbb{N}, \\
0 & \text{if } j \notin \frac{1}{2} \mathbb{N} \setminus \mathbb{N}
\end{cases} \tag{18} \]

Moreover, when \( j \in \mathbb{N} \) ([36], 4.16(6)),

\[ d_{m,0}^j \left( \frac{\pi}{2} \right) = \begin{cases} 
\frac{(-1)^{j+m}}{2j} \sqrt{(j+m)(j-m)} & \text{if } j - m \in 2\mathbb{N} \\
0 & \text{if } j - m \notin 2\mathbb{N} + 1
\end{cases} \tag{19} \]

Throughout, we will make use of the symmetry relations ([36], 4.4)

\[ d_{m',m}^j(-\theta) = (-1)^{m'-m} d_{m',m}^j(\theta) = d_{m',-m}^j(\theta), \tag{20} \]

and

\[ d_{m',m}^j(\theta + 2\pi) = (-1)^{j-m} d_{m',-m}^j(\theta + \pi) = (-1)^2 d_{m',m}^j(\theta). \tag{21} \]

**Lemma 7** \[ \sum_{m' - m \in 2\mathbb{Z}\setminus\{0\}} (p_{m',m}^j)^2 = \mathcal{O}(1). \]

**Proof** Combining (17) and (18), we see that if \( j \notin \mathbb{N} \) then \( p_{m',m}^j = 0 \) for every \( m, m' \) such that \( m' - m \in 2\mathbb{Z} \setminus \{0\} \). Otherwise (assuming \( m' \neq m \)),

\[ p_{m',m}^j = -\frac{1}{2} \widehat{d}_{m',m}^j(0) = -\frac{1}{2} d_{m',0}^j \left( \frac{\pi}{2} \right) d_{m,0}^j \left( \frac{\pi}{2} \right). \]

If we fix \( m' \in \{j, j-1, \ldots, -j\} \), then

\[ \sum_{m' - m \in 2\mathbb{Z}\setminus\{0\}} (p_{m',m}^j)^2 \leq \frac{1}{4} \left( d_{m',0}^j \left( \frac{\pi}{2} \right) \right)^2 \sum_{m=-j}^j \left( d_{m,0}^j \left( \frac{\pi}{2} \right) \right)^2 = \frac{1}{4} \left( d_{m',0}^j \left( \frac{\pi}{2} \right) \right)^2. \]
where we used that \( d_{m',m}^j (\frac{\pi}{2}) \in \mathbb{R} \) are the elements of a unitary matrix, which also implies that

\[
\frac{1}{4} \sum_{m=-j}^{j} \left( d_{m',0}^j \left( \frac{\pi}{2} \right) \right)^2 = \frac{1}{4},
\]

hence the required. \( \Box \)

Next, by the symmetries (20), we may assume that \( m_0 \geq 1 \) without loss of generality.

**Proposition 4** When \( j \in \mathbb{N} \), \( \| S_n \|_F^2 = \Theta(\log n) \).

**Proof** The norms of the rows and columns of \( S_n \) are obviously bounded from above by 1, hence we can erase \( \mathcal{O}(1) \) of them without loss of generality. Thus, we will estimate the Frobenius norm of

\[
\tilde{S}_n = \left( p_{m',m}^j \right)_{m_0 > m' > -j+2, j-2 > m \geq m_0}
\]

and furthermore assume that \( m_0 \in 2\mathbb{N} + 1 \).

The previous lemma implies that it suffices to consider the entries of \( \tilde{S}_n \) corresponding to \( m' - m \in 2\mathbb{Z} + 1 \). Since \( j \in \mathbb{N} \), the functions \( d_{m',m}^j \) are \( 2\pi \)-periodic (21), so that

\[
\mathcal{H}_\mathbb{C}(d_{m',m}^j)(0) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} d_{m',m}^j(\theta) \cot\left( \frac{\theta}{2} \right) d\theta.
\]

Denote

\[
v_{j,m} = \sqrt{(j + m)(j - m + 1)} = v_{j, -(m-1)}.\]

By [1], 3.85,

\[
\cot\left( \frac{\theta}{2} \right) d_{m',m}^j(\theta) = \frac{1}{m - m'} \left( v_{j,m} d_{m',m-1}^j(\theta) + v_{j,m'} d_{m'-1,m}^j(\theta) \right).
\]

Using (18), we find that

\[
\left( p_{m',m}^j \right) = \frac{1}{2(m - m')} \left( v_{j,m} d_{m',0}^j \left( \frac{\pi}{2} \right) d_{m-1,0}^j \left( \frac{\pi}{2} \right) - v_{j,m'} d_{m'-1,0}^j \left( \frac{\pi}{2} \right) d_{m,0}^j \left( \frac{\pi}{2} \right) \right).
\]
Now, (19) implies, in particular, that either $a_{m,0}^j \left( \frac{\pi}{2} \right) = 0$ or $a_{m-1,0}^j \left( \frac{\pi}{2} \right) = 0$. Hence, we assume from now on that $j = 2l \in 2\mathbb{N}$ (the case $j \in 2\mathbb{N} + 1$ is essentially identical), so that

$$p_{m',m}^j = \begin{cases} -\frac{v_{j,m'}}{2(m-m')} d_{m-1,0}^j \left( \frac{\pi}{2} \right) a_{m,0}^j \left( \frac{\pi}{2} \right) m' \in 2\mathbb{Z} + 1, \\ \frac{v_{j,m'}}{2(m-m')} d_{m',0}^j \left( \frac{\pi}{2} \right) a_{m-1,0}^j \left( \frac{\pi}{2} \right) m' \in 2\mathbb{Z} \end{cases}.$$  

Write $m_0 = 2r + 1 \in 2\mathbb{N} + 1$. Then, we must estimate

$$\sum_{1 \leq m' \leq l-r-1, 1 \leq m \leq l-r-2} (p_{2r-2m'+1,2r+2m}^{2l})^2 + \sum_{1 \leq m' \leq l-r-1, 1 \leq m \leq l-r-1} (p_{2r-2m'+2,2r+2m-1}^{2l})^2.$$

We next show that the first sum is $\Theta(\log l)$. The same can be shown, in the same way, for the second sum, hence we omit the computation. We use (19) together with the estimate

$$\binom{2k}{k} = \frac{1}{\sqrt{\pi k}} 4^k \left( 1 + \mathcal{O}(1) \right)$$

(22)

to obtain

$$(p_{2r-2m'+1,2r+2m}^{2l})^2 = \frac{\Theta(1)}{(2m + 2m' - 1)^2} \frac{(2(l + r - m') + 1)(l - (r - m'))}{\sqrt{l^2 - (r - m')^2}(l^2 - (r + m)^2)}$$

$$= \frac{\Theta(1)}{(2m + 2m' - 1)^2} \frac{\sqrt{l^2 - (r - m')^2}}{\sqrt{l^2 - (r + m)^2}} \leq \frac{\Theta(1)}{(2m + 2m' - 1)^2} \frac{l}{\sqrt{l^2 - (r + m)^2}}.$$

Write $t = r + m, s = m' - r$. Then,

$$\sum_{1 \leq m' \leq l-r-1, 1 \leq m \leq l-r-2} (p_{2r-2m'+1,2r+2m}^{2l})^2 = \Theta(1) \sum_{1-r \leq s \leq l-1, 1 \leq t \leq l-2} \frac{1}{(2t + 2s - 1)^2} \frac{\sqrt{l^2 - s^2}}{\sqrt{l^2 - t^2}}.$$

Let $\varepsilon = \frac{1}{l} (1 - \gamma)$. If $s \geq 1 - r + \varepsilon l$, then $s + t \geq \varepsilon l$, so that

$$\sum_{s \geq 1 - r + \varepsilon l, 1+r \leq t \leq l-2} \frac{1}{(2t + 2s - 1)^2} \frac{\sqrt{l^2 - s^2}}{\sqrt{l^2 - t^2}} \leq \Theta(1) \sum_{s \geq 1 - r + \varepsilon l, 1+r \leq t \leq l-2} \frac{1}{l} \frac{1}{\sqrt{l^2 - t^2}}$$

$$\leq \Theta(1) \sum_{t=1}^{l-1} \frac{1}{\sqrt{l^2 - t^2}} = \Theta(1).$$
Next, assume \( s < 1 - r + \epsilon l \). Note that
\[
m_0 = 2r + 1 < (1 - 2\epsilon)2l,
\]
hence \( r < (1 - 2\epsilon)l - \frac{1}{2} \), so that \( 1 + r + \epsilon l < (1 - \epsilon)l + \frac{1}{2} \). If
\[
1 + r + \epsilon l \leq t \leq l - 1,
\]
then
\[
t + s \geq t + 1 - r > \epsilon l,
\]
and the same argument as above implies that
\[
\sum_{1-r \leq s < 1-r+\epsilon l \atop t \geq 1+r+\epsilon l} \frac{1}{(2t + 2s - 1)^2} \frac{\sqrt{l^2 - s^2}}{\sqrt{l^2 - t^2}} = O(1).
\]
Finally, assume \( t < 1 + r + \epsilon l \leq (1 - \epsilon)l + \frac{1}{2} \). Then, \( l^2 - t^2 = \Theta(l^2) \), so that
\[
\frac{\sqrt{l^2 - s^2}}{\sqrt{l^2 - t^2}} = \Theta(1).
\]
Thus,
\[
\sum_{1-r \leq s < 1-r+\epsilon l \atop 1+r \leq t < 1+r+\epsilon l} \frac{1}{(2t + 2s - 1)^2} \frac{\sqrt{l^2 - s^2}}{\sqrt{l^2 - t^2}} = \Theta(1) \sum_{1-r \leq s < 1-r+\epsilon l \atop 1+r \leq t < 1+r+\epsilon l} \frac{1}{(2t + 2s - 1)^2} = \Theta(\log l) = \Theta(\log n),
\]
as required. Namely, the original sum is \( \Theta(\log n) \).

\[
\text{Corollary 14} \quad \text{Assume } j \in \mathbb{N} \setminus \mathbb{N}. \text{ Then, } \|S_n\|_F^2 = \Theta(\log n).
\]

**Proof** If \( m' - m \in 2\mathbb{Z} \), then \( p_{m',m}^j = 0 \) by (17) and (18) (we know that \( m' \neq m \)). Hence, assume that \( m' - m \in 2\mathbb{Z} + 1 \). The functions \( d_{m',m}^j \) satisfy
\[
d_{m',m}^j(\theta + 2\pi) = -d_{m',m}^j(\theta).
\]
hence
\[
\mathcal{H}_T(d^j_{m',m})(0) = -\frac{1}{4\pi} \int_{-2\pi}^{2\pi} d^j_{m',m}(\theta) \cot \left( \frac{\theta}{4} \right) d\theta
\]
\[
= -\frac{1}{2\pi} \int_0^{2\pi} d^j_{m',m}(\theta) \frac{1}{\sin \left( \frac{\theta}{2} \right)} d\theta.
\]
As before, we can truncate \( O(1) \) rows and columns from \( S_n \) without changing our estimate. According to [1], 3.83,
\[
\frac{1}{\sin \left( \frac{\theta}{2} \right)} d^j_{m',m} = \sqrt{\frac{j + m}{j - m}} d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m-\frac{1}{2}}(\theta) + \sqrt{\frac{j - m}{j - m'}} d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m+\frac{1}{2}}(\theta),
\]
where \( \sqrt{\frac{j \pm m}{j - m}} = O(1) \) (since \( m' < m_0 \leq \nu j \)), and again using [1], 3.85,
\[
\cot \left( \frac{\theta}{2} \right) d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m+\frac{1}{2}}(\theta)
\]
\[
= \frac{1}{m - m'} \left( v_{j-\frac{1}{2},m} d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m-\frac{1}{2}}(\theta) + v_{j-\frac{1}{2},m'} d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m+\frac{1}{2}}(\theta) \right).
\]
Thus,
\[
\frac{1}{\sin \left( \frac{\theta}{2} \right)} d^j_{m',m}(\theta) = \left( \sqrt{\frac{j + m}{j - m}} + \sqrt{\frac{j - m}{j - m'}} \right) d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m-\frac{1}{2}}(\theta)
\]
\[
+ \sqrt{\frac{j - m}{j - m'}} \frac{v_{j-\frac{1}{2},m'} d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m+\frac{1}{2}}(\theta)}{v_{j-\frac{1}{2},m} d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m-\frac{1}{2}}(\theta)}.
\]
If \( m > \frac{1+\nu}{2} j \), then \( m - m' = \Theta(j) \). Thus,
\[
\frac{1}{\sin \left( \frac{\theta}{2} \right)} d^j_{m',m}(\theta) = O(1) \left( d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m-\frac{1}{2}}(0) + d^{j-\frac{1}{2}}_{m'-\frac{1}{2},m+\frac{1}{2}}(0) \right),
\]
which implies that
\[
p^j_{m',m} = O(1) \left( d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m-\frac{1}{2}}(0) + d^{j-\frac{1}{2}}_{m'-\frac{1}{2},m+\frac{1}{2}}(0) \right),
\]
where at most one of the summands is nonzero by (19), so
\[
(p^j_{m',m})^2 = O(1) \left( d^{j-\frac{1}{2}}_{m'+\frac{1}{2},m-\frac{1}{2}}(0)^2 + d^{j-\frac{1}{2}}_{m'-\frac{1}{2},m+\frac{1}{2}}(0)^2 \right).
\]
However, as in Lemma 7,
\[
\sum_{m', m} \left( \frac{d^{j - \frac{1}{2}}_{m' - \frac{1}{2}, m + \frac{1}{2}}(0)}{m' - \frac{1}{2}, m + \frac{1}{2}} \right)^2 = 1,
\]
where the sum is over all applicable \(m', m\). Hence, we deduce that
\[
\sum_{m > \frac{1}{1 + \gamma} j} (p^j_{m', m})^2 = \mathcal{O}(1).
\]

If \(m \leq \frac{1 + \gamma}{2} j\), then
\[
\sqrt{\frac{j - m}{j - m'}} = \Theta(1), \quad v_{j - \frac{1}{2}, m} = \sqrt{(j - m)(j + m)} = \Theta(j),
\]
so
\[
\frac{1}{\sin \left(\frac{\theta}{2}\right)} d^j_{m', m}(\theta)
= \Theta(1) \left( 1 + \frac{v_{j - \frac{1}{2}, m}}{m - m'} \right) d^{j - \frac{1}{2}}_{m' + \frac{1}{2}, m - \frac{1}{2}}(\theta) + \frac{v_{j - \frac{1}{2}, m'}}{m - m'} d^{j - \frac{1}{2}}_{m' - \frac{1}{2}, m + \frac{1}{2}}(\theta)
= \Theta(1) v_{j - \frac{1}{2}, m} d^{j - \frac{1}{2}}_{m' + \frac{1}{2}, m - \frac{1}{2}}(\theta) + v_{j - \frac{1}{2}, m'} d^{j - \frac{1}{2}}_{m' - \frac{1}{2}, m + \frac{1}{2}}(\theta)
= \Theta(1) \cot \left(\frac{\theta}{2}\right) d^{j - \frac{1}{2}}_{m' + \frac{1}{2}, m + \frac{1}{2}}(\theta),
\]
hence
\[
p^j_{m', m} = \Theta(1) p^{j - \frac{1}{2}}_{m' + \frac{1}{2}, m + \frac{1}{2}},
\]
and so the squared Frobenius norm of the sub-matrix of \(S_n\) corresponding to \(m \leq \frac{1 + \gamma}{2} j\) is \(\Theta(\log j)\) from the previous proposition.

**Corollary 15** Let \(P_{n,a,b} = P_{(aj, bj)}(J_3)\), where either \(a > -1\) or \(b < 1\). For \(s < t\), let \(N_n(s, t)\) denote the number of eigenvalues of
\[
R_{n,a,b} = P_{n,a,b} P_{1,n} P_{n,n} \in \text{End} \left( \text{Im}(P_{n,a,b}) \right)
\]
lying in the interval \([s, t]\). Then, for every \(0 < t < \frac{1}{2}\)
\[
\lim_{n \to \infty} \frac{1}{(b - a)n} N_n(0, t) = \lim_{n \to \infty} \frac{1}{(b - a)n} N_n(1 - t, 1) = \frac{1}{4},
\]
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and

\[ N_n(t, 1 - t) = \mathcal{O}(\log n). \]

**Proof** Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{k_n,a,b} \) denote the eigenvalues of \( R_{n,a,b} \). First of all, we note that

\[ \sum_{k=1}^{k_{n,a,b}} \lambda_k = \text{trace} R_{n,a,b} = \sum_{a_j < m < b_j} p_{m,m}^j, \]

where by (17), (18) and (19),

\[ p_{m,m}^j = \frac{1}{2} - \frac{d_{m,m}^j(0)}{2} \quad \text{for } j \in \mathbb{N}, \]

\[ p_{m,m}^j = \frac{1}{2} \quad \text{for } j \notin \mathbb{N}. \]

Thus, when \( j \notin \mathbb{N} \),

\[ \text{trace} R_{n,a,b} = \frac{b - a}{4} n + \mathcal{O}(1). \] (23)

The same holds for \( j \in \mathbb{N} \), since \( \sum_{m=-j}^{j} \left( d_{m,0}^j \left( \frac{3}{2} \right) \right)^2 = 1 \). Next, we wish to estimate

\[ \sum_{k=1}^{k_{n,a,b}} \lambda_k^2 = \text{trace} \left( R_{n,a,b}^2 \right) = \| R_{n,a,b} \|_F^2. \]

Instead of estimating the Frobenius norm of \( R_{n,a,b} \), we can estimate that of its “complement”

\[ S_{n,a,b} = (I - R_{n,a,b}) P_{1,n} P_{n,a,b}. \]

Indeed, denote \( v_m = [P_{1,n} e_m]_{B_3,n} \), where \( m = j, j - 1, \ldots, -j \). Then,

\[ \| v_m \|^2 = (P_{1,n} e_m, P_{1,n} e_m) = p_{m,m}^j, \]

so that

\[ \| S_{n,a,b} \|_F^2 + \| R_{n,a,b} \|_F^2 = \sum_{a_j < m < b_j} \| v_m \|^2 = \text{trace}(R_{n,a,b}). \]

Now, write \( \text{Im}(R_{n,a,b}) = \text{Span}\{ e_m \mid m_+ \geq m \geq m_- \} \), and let \( P_+ \), \( P_- \) denote the orthogonal projections on \( \text{Span}\{ e_m \mid m > m_+ \} \), \( \text{Span}\{ e_m \mid m < m_- \} \), respectively.
Then, \( S_{n,a,b} = S_{n,a,b}^+ + S_{n,a,b}^- \), where
\[
S_{n,a,b}^+ = P + P_{1,n} P_{n,a,b}, \quad S_{n,a,b}^- = P - P_{1,n} P_{n,a,b}
\]
have “off-diagonal” matrices as addressed in Proposition 3. Thus,
\[
\|S_{n,a,b}\|_F^2 = \|S_{n,a,b}^+\|_F^2 + \|S_{n,a,b}^-\|_F^2 = \Theta(\log n).
\]

We conclude that
\[
\sum_{k=1}^{k_{n,a,b}} \lambda_k (1 - \lambda_k) = \text{trace}(R_{n,a,b}) - \text{trace}\left(R_{n,a,b}^2\right) = \Theta(\log n).
\]

Now, if \( 0 < t < \lambda < 1 - s \), then \( \lambda (1 - \lambda) > ts \), hence
\[
\Theta(\log n) = \sum_{k=1}^{k_{n,a,b}} \lambda_k (1 - \lambda_k) \geq N_n(t, 1 - s) ts,
\]
so
\[
N_n(t, 1 - s) = \mathcal{O}(\log n).
\] (24)

Next, write
\[
x_{n,t} = \frac{1}{(b-a)n} N_n(0, t), \quad z_{n,t} = \frac{1}{(b-a)n} N_n(1 - t, 1).
\]

If \( t > s \), then \( x_{n,t} \geq x_{n,s} \) and \( z_{n,t} \geq z_{n,s} \). We know that
\[
N_n(0, t) + N_n(t, 1 - t) + N_n(1 - t, 1) = k_{n,a,b} + \mathcal{O}(1) = \frac{b-a}{2} n + \mathcal{O}(1),
\]
hence using (24),
\[
x_{n,t} + z_{n,t} = \frac{1}{2} + o(1).
\] (25)

Given \( \varepsilon > 0 \), choose \( 0 < \delta < t \) such that
\[
\frac{\delta^2}{1 - \delta^2} < \frac{\delta}{1 - \delta} < \varepsilon.
\]

Let
\[
\sigma_{n,t}^- = \{k \mid \lambda_k \leq t\}, \quad \sigma_{n,t}^+ = \{k \mid \lambda_k \geq 1 - t\}.
\]
Then, by (23) and (24),
\[
\frac{1}{4} + o(1) = \frac{1}{(b - a)n} \left( \sum_{k \in \sigma_{n,\delta}^-} \lambda_k + \sum_{k \in \sigma_{n,\delta}^+} \lambda_k \right) \geq (1 - \delta) z_{n,\delta},
\]
so
\[
z_{n,\delta} \leq \frac{1}{4(1 - \delta)} + o(1) = \frac{1}{4} + \frac{\delta}{4(1 - \delta)} + o(1) < \frac{1}{4} + \epsilon + o(1).
\]
Thus,
\[
z_{n,t} \leq z_{n,\delta} + \frac{1}{(b - a)n} N_n(1 - t, 1 - \delta) < \frac{1}{4} + \epsilon,
\]
where the last inequality is true for every \( n \) sufficiently large. Similarly,
\[
\frac{1}{(b - a)n} \left( \sum_{k \in \sigma_{n,\delta}^-} \lambda_k^2 + \sum_{k \in \sigma_{n,\delta}^+} \lambda_k^2 \right) \leq \delta^2 x_{n,\delta} + z_{n,\delta} = \frac{\delta^2}{2} + z_{n,\delta}(1 - \delta^2) + o(1),
\]
using (25), hence since the left-hand side equals \( \frac{1}{4} + o(1) \),
\[
z_{n,t} \geq z_{n,\delta} \geq \frac{1 - 2\delta^2}{4(1 - \delta^2)} + o(1) = \frac{1}{4} - \frac{1}{4} \frac{\delta^2}{1 - \delta^2} + o(1) > \frac{1}{4} - \epsilon,
\]
where the last inequality holds for every \( n \) large enough. Thus, \( \lim_{n \to \infty} z_{n,t} = \frac{1}{4} \).

Finally, \( x_{n,t} = \frac{1}{2} - z_{n,t} + o(1) = \frac{1}{4} + o(1) \) as well. \( \Box \)

6 Concluding remarks

6.1 Analogues of Theorem 3

The proof of Theorem 3 relies on the fact that in the semiclassical limit \( n \to \infty \), the spectral projections \( P_{1,\alpha,n} \) and \( P_{3,\alpha,n} \) converge to \( \mathcal{M}_{\mathbb{E}_\alpha} \) and \( \Pi_T \) in some appropriate sense. Analogous facts hold (in particular) for the rest of the pairs of spectral projections addressed in [30], hence the proof can be adapted so as to apply to them as well.

Notably, we considered pairs of spectral projections coming from position and momentum operators
\[
\hat{q} = \mathcal{M}_q, \quad \hat{p} = -i\hbar \frac{\partial}{\partial q}
\]
on \( L^2(\mathbb{R}) \). We also considered pairs of spectral projections corresponding to the operators \( \cos \hat{\theta} = M_{\cos \theta} \) and \( \cos \hat{L} \), where

\[
\hat{\theta} = M_{\theta}, \quad \hat{L} = -i \frac{2\pi}{n} \frac{\partial}{\partial \theta}
\]

are the analogues ([22, 26]) of \( \hat{q}, \hat{p} \) on \( L^2(\mathbb{T}) \cong L^2([0, 2\pi], \frac{d\theta}{2\pi}) \). Yet another example involved the generators \( g_1, g_2 \) of the finite Heisenberg groups \( H(\mathbb{Z}_n) \) ([29, 34, 37]), which act on \( F \in l^2(\mathbb{Z}_n) \) by

\[
g_1 F(k) = e^{\frac{2\pi k i}{n}} F(k), \quad g_2 F(k) = F(k + 1).
\]

Let \( E = \{ \zeta \in \mathbb{T} \mid \Re \zeta > 0 \} \). Then, the pair of projections

\[
\mathbb{1}_{(0, \infty)}(\hat{q}) = M_{1(0, \infty)}, \quad \mathbb{1}_{(0, \infty)}(\hat{p})
\]

is unitarily equivalent to \( M_{1_E}, \Pi_{T} \) independently of \( \hbar \) (since \( \mathbb{1}_{(0, \infty)}(\hat{p}) \) is just the projection on the Hardy space \( H^2(\mathbb{R}) \)). For the pair

\[
\mathbb{1}_{(0, \infty)}(\cos \hat{\theta}) = M_{\Pi_{T}}, \quad \mathbb{1}_{(0, \infty)}(\cos \hat{L}),
\]

a sequence of unitary operators \( U_n : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T}) \) is required in order to reduce the proof to the case of \( M_{1_E} \) and \( \Pi_{T} \). Similarly, for

\[
\Pi_1 = \mathbb{1}_{(0, \infty)}(\Re g_1), \quad \Pi_2 = \mathbb{1}_{(0, \infty)}(\Re g_2),
\]

where \( \Re A = \frac{1}{2}(A + A^*) \), we need a sequence of embeddings \( \hat{U}_n : l^2(\mathbb{Z}_n) \rightarrow L^2(\mathbb{T}) \). \( U_n, \hat{U}_n \) are defined by mapping elements of the standard bases of \( L^2(\mathbb{T}), l^2(\mathbb{Z}_n) \) to those of \( L^2(\mathbb{T}) \) in a certain suitable way, so as to obtain the convergence of the relevant spectral projections to the pair \( \Pi_{T}, M_{1_E} \). The arguments are essentially the same as those of Sects. 4.3 and 4.4. Finally, applying Corollary 13 to \( \Pi_{T}, M_{1_E} \), we obtain the following.

**Corollary 16** Analogues of Theorem 3 hold for the families of pairs of spectral projections detailed above, i.e.,

\[
\lim_{n \rightarrow \infty} \| f(\Pi_1, \Pi_2) \|_{\text{op}} = \lim_{n \rightarrow \infty} \| f \left( M_{1_E}, \mathbb{1}_{(0, \infty)}(\cos \hat{L}) \right) \|_{\text{op}} = \| f \left( M_{1(0, \infty)}, \mathbb{1}_{(0, \infty)}(\hat{p}) \right) \|_{\text{op}} = M_f
\]

for every \( f \in \mathcal{A} \).

\footnote{Here, \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \).}
6.2 Angles between subspaces and convergence rate

Let $H$ be a complex, finite-dimensional Hilbert space. Assume that $P : H \to V_P$ and $Q : H \to V_Q$ are two orthogonal projections with $\dim V_P \leq \dim V_Q$. Let us now consider $P Q P$ as an element of $\operatorname{End}(V_P)$. Then, the canonical form (Theorem 5) implies that

$$P Q P = \text{Id}_{V_{00}} \oplus 0_{V_{01}} \oplus (\text{Id} - H).$$

Thus (see Sect. 2.3), the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_{\dim V_P}$ of $P Q P$ are given by $\lambda_k = \cos^2 \phi_k$, where $\phi_1 \leq \ldots \leq \phi_{\dim V_P}$ are the principal angles between the subspaces $V_P, V_Q$.

**Corollary 17** If $f \in \ker(T)$, then using Corollary 5,

$$\|f(P, Q)\|_{\text{op}} = \max_{\sigma(P Q P)} \psi_f = \max_{\phi \in \Phi} \psi_f(\cos^2 \phi),$$

where $\Phi$ is the set of principal angles between $V_P, V_Q$.

Theorem 4 and Corollary 3 imply that there is a striking discrepancy between the spectrum of random $P Q P$ and that of $P_3, n P_1, n P_3, n$ (as illustrated in Figs. 5 and 6). According to [13], the same is true for the spectral projections $\Pi_1, \Pi_2$ specified in (26). These discrepancies imply that the rates of convergence in Theorem 3 (for $P_3, n, P_1, n$) and in Corollary 16 (for $\Pi_1, \Pi_2$) tend to be much slower than in Theorem 2 (for random $P, Q$).

To illustrate this, consider the case of commutators ($f(x, y) = xy - yx$), depicted in Figs. 1 and 2. Then, $\psi_f(t) = \sqrt{t(1 - t)}$ by Example 3, hence by Corollary 17,

$$\|[P, Q]\|_{\text{op}} = \max_{\phi \in \Phi} \sin \phi \cos \phi = \sin \phi_0 \cos \phi_0,$$

where $\phi_0 \in \Phi$ is the principal angle closest to $\pi / 4$.

**Remark 7** Let $\Phi_n$ be the set of principal angles between $\text{Im}(P_{1, n}), \text{Im}(P_{3, n})$. While Theorem 3 implies that $\Phi_n$ becomes dense in $[0, \pi / 4]$ as $n \to \infty$, this happens very slowly as per Theorem 4 (logarithmically, or slower). Thus, it appears that $\min_{\phi \in \Phi_n} |\phi - \pi / 4|$ tends to 0 slowly, so that $\|[P_{1, n}, P_{3, n}]\|_{\text{op}}$ tends to $1 / 2$ slowly.

By contrast, the set of principal angles $\Phi$ associated with a “typical” random pair $P, Q \in \mathcal{G}_{\frac{1}{2} n}^1$ is distributed much more evenly (as per Corollary 3). Thus, $\min_{\phi \in \Phi} |\phi - \phi_0|$ is typically quite small for every $\phi_0 \in [0, \pi / 2]$ and in particular for $\phi_0 = \pi / 4$, hence $\int_{\omega_{n}} \|[P, Q]\|_{\text{op}} d\nu_n$ converges quickly to $1 / 2$.

Similarly for general $f \in \ker(T)$.

The last remark is relevant for any sequence of pairs of projections which behaves as in Theorems 3 and 4 (e.g., $P_{3, \alpha, n}, P_{1, n}$, and likely also $P_{3, \alpha, n}, P_{1, \alpha, n}$).
6.3 A conjecture

Let us offer an informal, conjectured explanation for “maximality results” of the type of Theorem 3 and Corollary 16. The explanation is based on the notion of quantization and is inspired by findings from [25], [8]. In what follows, \( \mathcal{L}(\mathcal{H}) \) denotes the space of self-adjoint operators on a Hilbert space \( \mathcal{H} \).

Let \((M, \omega)\) denote a closed\(^6\), quantizable\(^7\) symplectic manifold. A Berezin–Toeplitz quantization ([7,20,28]) of \( M \) produces a sequence of finite-dimensional complex Hilbert spaces \( \mathcal{H}_n \), such that \( \lim_{n \to \infty} \dim \mathcal{H}_n = +\infty \), together with surjective linear maps \( T_n : \mathcal{C}^\infty(M) \to \mathcal{L}(\mathcal{H}_n) \). The maps are required to satisfy several desirable properties in the semiclassical limit \( n \to \infty \).

Example 7 Up to normalization, \( J_1, J_2 \) and \( J_3 \) are Berezin–Toeplitz operators associated with the Cartesian coordinate functions \( x_1, x_2, x_3 : S^2 \to \mathbb{R} \), where \( S^2 \subset \mathbb{R}^3 \) is the two-dimensional sphere.

Let \( F_1, F_2 \in \mathcal{C}^\infty(M) \) and assume that \( I_1, I_2 \subset \mathbb{R} \) are a pair of non-trivial intervals. We consider the spectral projections

\[
\Pi_{1,n} = \text{id}_{I_1} (T_n(F_1)) , \quad \Pi_{2,n} = \text{id}_{I_2} (T_n(F_2))
\]

as a pair of quantum observables that are “somehow related” (compare with [39]) to the domains

\[
D_1 = F_1^{-1}(I_1) , \quad D_2 = F_2^{-1}(I_2).
\]

Example 8 In this interpretation, \( P_{1,\alpha,n}, P_{3,\alpha,n} \) are “associated” with the domains \( \{0 < x_1 \leq 2\alpha\}, \{0 < x_3 \leq 2\alpha\} \subset S^2 \).

Recall that \( M_f \) denotes the universal, tight upper bound for \( \| f(P, Q) \|_{\text{op}} \), where \( P, Q \) are arbitrary orthogonal projections on a separable complex Hilbert space. Our various numerical simulations appear to support the following.

Conjecture 1 Fix \( f \in \ker(T) \). Assume that \( F_1, F_2 \) are Poisson non-commuting and that \( M \) is two dimensional.

1. If \( \partial D_1 \cap \partial D_2 \neq \emptyset \) is transversal, then

\[
\lim_{n \to \infty} \| f \left( \Pi_{1,n} , \Pi_{2,n} \right) \|_{\text{op}} = M_f .
\]

2. If the distance between \( \partial D_1, \partial D_2 \) is greater than some \( \varepsilon > 0 \), then

\[
\lim_{n \to \infty} \| f \left( \Pi_{1,n} , \Pi_{2,n} \right) \|_{\text{op}} = 0.
\]

---
\(^6\) That is, compact and without boundary.
\(^7\) That is, \( \frac{\omega}{2\pi} \) represents an integral de-Rham cohomology class.
The latter is essentially a conjecture about the principal angles between subspaces spanned by eigenstates of quantum observables, or equivalently, about the spectrum of $\Pi_{1,n}\Pi_{2,n}\Pi_{1,n}$ (see Corollary 17). When $\partial D_1 \cap \partial D_2 \neq \emptyset$ is transversal, we expect the eigenvalues of $\Pi_{1,n}\Pi_{2,n}\Pi_{1,n}$ to cluster near 0, 1, similarly to the situation specified in Theorem 4. If true, this would constitute an interesting formulation of the Slepian spectral concentration phenomenon.

We refer the reader to [30] (the final section in particular) for further details and simulations (mostly involving spin operators, but also some simulations for finite Heisenberg groups).

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

References

1. Biedenharn, L.C., Louck, J.D.: Angular momentum in quantum physics: theory and application, encyclopaedia of mathematics and its applications, 8. Addison-Wesley Publishing Company, Boston (1981)
2. Borac, S.: On the algebra generated by two projections. J. Math. Phys. 36, 863–874 (1995)
3. Böttcher, A., Spitkovsky, I.M.: A gentle guide to the basics of two projections theory. Linear Algebra Appl. 432, 1412–1459 (2010)
4. Busch, P., Cassinelli, G., Lahti, P.: On the quantum theory of sequential measurements. Found. Phys. 20(7), 757–775 (1990)
5. Busch, P., Heinonen, T., Lahti, P.: Noise and disturbance in quantum measurement. Phys. Lett. A 320(4), 261–270 (2004)
6. Busch, P., Heinonen, T., Lahti, P.: Heiseberg's uncertainty principle. Phys. Rep. 452, 155–176 (2007)
7. Charles, L.: Quantization of compact symplectic manifolds. J. Geom. Anal. 26, 2664–2710 (2016)
8. Charles, L., Polterovich, L.: Sharp correspondence principle and quantum measurements. Algebra i Analiz 29(1), 237–278 (2017)
9. Collins, B.: Product of random projections, Jacobi ensembles and universality problems arising from free probability. Probab. Theory Related Fields 133, 315–344 (2005)
10. Davies, E.B., Lewis, J.T.: An operational approach to quantum probability. Commun. Math. Phys. 17, 239–260 (1970)
11. Douglas, R.G.: Banach algebra techniques in operator theory. graduate texts in mathematics, 2nd edn. Springer-Verlag, New York (1998)
12. Dumitriu, I., Paquette, E.: Global fluctuations for linear statistics of $\beta$-Jacobi ensembles. Random Matrices Theory Appl. 1, 1250013 (2012)
13. Edelman, A., McCorquodale, P., Toledo, S.: The future fast Fourier transform? SIAM J. Sci. Comput. 20(3), 1094–1114 (1998)
14. Feng, X.M., Wang, P., Yang, W., Jin, G.R.: High-precision evaluation of Wigners d-matrix by exact diagonalization. Phys. Rev. E 92, 587 (2015)
15. Giles, R., Kummer, H.: A matrix representation of a pair of projections in a Hilbert space. Canad. Math. Bull 14(1), 35–44 (1971)
16. Halmos, P.: Two subspaces. Trans. Amer. Math. Soc. 144, 381–389 (1969)
17. Hartman, P., Wintner, A.: The spectra of Toeplitz matrices. Amer. J. Math. 76, 867–882 (1954)
18. Heinosaa, T., Kiukas, J., Reitzner, D.: Coexistence of effects from an algebra of two projections. J. Phys. A Math. Theor. 47, 225301 (2014)
19. Kiukas, J., Werner, R.F.: Maximal violation of Bell inequalities by position measurements. J. Math. Phys. 51, 072105 (2010)
20. Le Floch, Y.: A brief introduction to Berezin-Toeplitz operators on compact Kähler manifolds. Springer International Publishing, New York (2014)
21. Massar, S.: Uncertainty relations for positive-operator-valued measures. Phys. Rev. A 76(4), 042114 (2007). Erratum: Phys. Rev. A 78(5), 059901 (2008)
22. Mukunda, N.: Wigner distribution for angle coordinates in quantum mechanics. Am. J. Phys. 47, 182 (1979)
23. Ozawa, M.: Uncertainty relations for joint measurements of noncommuting observables. Phys. Lett. A 320(5–6), 367–374 (2004)
24. Pedersen, G.K.: Measure theory for $C^*$ algebras. II. Math. Scand 22, 63–74 (1968)
25. Polterovich, L.: Symplectic geometry of quantum noise. Commun. Math. Phys. 327, 481–519 (2014)
26. Przanowski, M.A., Tosiek, J.: Remarks on deformation quantization on the cylinder. Acta Physica Polonica B 31, 561–587 (2000)
27. Raeburn, I., Sinclair, A.M.: The $C^*$-algebra generated by two projections. Math. Scand. 65, 278–290 (1989)
28. Schlichenmaier, M.: Berezin-Toeplitz quantization for compact Kähler manifolds. A review of results. Adv. Math. Phys. (2010). https://doi.org/10.1155/2010/927280
29. Schwinger, J.: Unitary operator bases. Proc. Nat. Acad. Sci. USA 46, 570–579 (1960)
30. Shabtai, O.: Commutators of spectral projections of spin operators. J. Lie Theory 31(3), 599–624 (2020)
31. Slepian, D., Pollak, H.O.: Prolate spheroidal wave functions, Fourier analysis and uncertainty V: the discrete case. Bell System Tech. J. 57, 1371–1430 (1978)
32. Spitkovsky, I.M.: Once more on algebras generated by two projections. Linear Algebra Appl. 208(209), 377–395 (1994)
33. Subag, E.M., Baruch, E.M., Birman, J.L., Mann, A.: Strong contraction of the representations of the three dimensional Lie algebras. J. Phys. A 45(26), 265206 (2012)
34. Varadarajan, V.S., Weisbard, D.: Convergence of quantum systems on grids. J. Math. Anal. Appl. 336, 608–624 (2007)
35. Varah, J.M.: The prolate matrix. Linear Algebra Appl. 187, 269–278 (1993)
36. Varshalovich, D.A., Moskalev, A.N., Khernoskii, V.K.: Quantum theory of angular momentum. World Scientific, Singapore (1988)
37. Vourdas, A.: Quantum systems with finite Hilbert space. Rep. Prog. Phys. 67, 267–320 (2004)
38. Wang, L.L.: A review of prolate spheroidal wave functions from the perspective of spectral methods. J. Math. Study 50(2), 101–143 (2017)
39. Zelditch, S., Zhou, P.: Central limit theorem for spectral partial Bergman Kernels. Geom. Topol. 23(4), 1961–2004 (2019)

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