LVMB MANIFOLDS AND QUOTIENTS OF TORIC VARIETIES

L. BATTISTI

Abstract. In this article, we study a class of manifolds introduced by Bosio (see [3]) called LVMB manifolds. We provide an interpretation of his construction in terms of quotient of toric manifolds by complex Lie groups. Furthermore, LVMB manifolds extend a class of manifolds obtained by Meersseman in [9], called LVM manifolds, and we give a characterization of these manifolds using our toric description. Finally, we give an answer to a question asked by Cupit-Foutou and Zaffran in [5].

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0. Introduction

In [8], López de Medrano and Verjovsky introduce a family of complex compact manifolds, obtained as quotients of a dense open subset $U$ of $\mathbb{P}^n(\mathbb{C})$ by the action of a complex Lie group isomorphic to $\mathbb{C}$. This construction...
is extended to the case of an action of $\mathbb{C}^m$ (with $m$ a positive integer) by Meersseman in [9], and these manifolds are called LVM manifolds.

Then, Bosio extends in [3] the construction due to Meersseman by allowing other actions of $\mathbb{C}^m$ on certain open subsets of $\mathbb{P}^n(\mathbb{C})$, and these manifolds are called LVMB manifolds. In short, given a family $E_{m,n}$ of subsets of $\{0, \ldots, n\}$ of cardinal $2m + 1$ (we assume $n$ and $m$ are integers such that $2m \leq n$) and a family $\mathcal{L}$ of $n + 1$ linear forms on $\mathbb{C}^m$ satisfying some conditions, Bosio associates to $E_{m,n}$ an open subset $U$ of $\mathbb{P}^n(\mathbb{C})$ and to $\mathcal{L}$ an action of $\mathbb{C}^m$ on $\mathbb{P}^n(\mathbb{C})$, such that the quotient $U/\mathbb{C}^m$ is a compact complex manifold. We say that the pair $(E_{m,n}, \mathcal{L})$ is an LVMB datum, and that it is an LVM datum if the manifold we obtain is an LVM manifold. We will recall this construction in details later.

Our first goal in this article is to express Bosio’s construction in terms of toric geometry. We find a fan $\Delta$ in $\mathbb{R}^n$ such that the corresponding toric manifold is the open set $U$ and we see how the projection of this fan by a suitable $2m$-dimensional linear subspace of $\mathbb{R}^n$ helps to understand the action of $\mathbb{C}^m$ on $U$. Roughly speaking, the set $E_{m,n}$ will correspond to the fan $\Delta$, and the choice of the linear forms on $\mathbb{C}^m$ will give the subspace $\mathbb{R}^{2m}$. The converse will also work, i.e. with suitable conditions on a fan $\Delta$ and the choice of a suitable $2m$-dimensional subspace of $\mathbb{R}^n$, we will get an LVMB datum. We have:

\textbf{Theorem 2.2.} i) Let $(\mathcal{L}, E_{n,m})$ be an LVMB datum. Then there is a pair $(E, \Delta)$ where $E$ is a $2m$-dimensional linear subspace of $\mathbb{R}^n$ and $\Delta$ is a subfan of the fan $\Delta_{\mathbb{P}^n_{(\mathbb{C})}}$ of $\mathbb{P}^n(\mathbb{C})$, satisfying the following two properties:

\begin{enumerate}
  \item[a)] the projection map $\pi : \mathbb{R}^n \to \mathbb{R}^n/E \cong \mathbb{R}^{n-2m}$ is injective on $|\Delta|$, 
  \item[b)] the fan $\pi(\Delta)$ is complete in $\mathbb{R}^n/E$, i.e. $|\pi(\Delta)| = \mathbb{R}^n/E$.
\end{enumerate}

ii) Conversely, given a pair $(E, \Delta)$ having the two properties above, one obtains an LVMB datum.

To prove this theorem, we first need to slightly extend the notion of the manifold with corners associated to a fan. This is what we do in the first section.

In [3], Bosio gives a criterion for deciding whether an LVMB datum is an LVM datum or not. Our second goal is to translate this criterion in our new toric setting, it is given by the following:

\textbf{Theorem 3.10.} Let $(\mathcal{L}, E_{m,n})$ be an LVMB datum and $(E, \Delta)$ its associated pair given by theorem 2.2. Then, $(\mathcal{L}, E_{m,n})$ is an LVM datum if and only if the projection by $E$ of the fan $\Delta$ is polytopal.
Finally, we will use this characterization to show that if two LVMB manifolds $X$ and $Y$ are biholomorphic and if $X$ is an LVM manifold, then $Y$ is also an LVM manifold. This question was outlined and partly answered by Cupit-Foutou and Zaffran in [5] and we answer it with theorem 4.2:

**Theorem 4.2.** Let $(\mathcal{L}_1, \mathcal{E}_{m,n})$ and $(\mathcal{L}_2, \mathcal{E}'_{m',n'})$ be two LVMB data giving two biholomorphic LVMB manifolds, then $n = n'$, $m = m'$ and $(\mathcal{L}_1, \mathcal{E}_{m,n})$ is an LVM datum if and only if $(\mathcal{L}_2, \mathcal{E}'_{m,n})$ is an LVM datum.

This article is organized as follows: in the first part, we extend the notion of the manifold with corners of a fan and we give some basic properties of it. Here, the fan we consider is non-necessarily rational. We then use this new object in the second part, where we prove theorem 2.2 after recalling Bosio’s construction. We also study the case when two LVMB data give the same pair $(E, \Delta)$. This leads us to a correspondence statement between the set of all such pairs and the quotient of the set of LVMB data by a suitable equivalence relation. In the third part we detect LVM data among LVMB data by proving theorem 3.10 and in the fourth part we use this criterion to obtain theorem 4.2.

1. The manifold with corners of a fan

The main definition of this section (the manifold with corners of a fan and its topology) is a basic one in the theory of toric manifolds, and it can be found in [1] and [10] for instance. Here we extend this definition to a fan without the “rationality condition”, that is, the vectors generating the cones we work with are not necessarily located in some rational lattice of a vector space. We also give basic properties of the topology of the associated manifold with corners that will be of use later.

1.1. Definition of the manifold with corners associated to a fan

Let $N_\mathbb{R}$ be a real vector space of dimension $n$.

**Definition 1.1.** A subset $\sigma$ of $N_\mathbb{R}$ is called a **convex polyhedral cone (with apex at the origin)** if there exist a finite number of elements $v_1, ..., v_m$ such that

$$\sigma = \mathbb{R}_{\geq 0}v_1 + ... + \mathbb{R}_{\geq 0}v_m,$$

and that $\sigma \cap (-\sigma) = \{0\}$, where $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers.

The **dimension** of a cone is the dimension of the smallest linear subspace of $N_\mathbb{R}$ containing this cone. We denote by $L(\sigma)$ this vector space. We say that $\sigma$ is **simplicial** if the vectors generating this cone are linearly independent.

**Definition 1.2.** The cone in $N_\mathbb{R}^*$ (the dual of $N_\mathbb{R}$) **dual** to $\sigma$ is the set

$$\tilde{\sigma} := \{ \varphi \in N_\mathbb{R}^* \mid \varphi(x) \geq 0 \text{ for all } x \in \sigma \}.$$
It is a convex polyhedral cone. A subset $\tau$ of $\sigma$ is a face of $\sigma$ if there is a $\varphi_0 \in \delta$ such that

$$\tau = \{x \in \sigma \mid \varphi_0(x) = 0\}. $$

We denote this relation between $\tau$ and $\sigma$ by $\tau < \sigma$. The cone $\sigma$ is a face of itself, and we say that $\tau < \sigma$ is a proper face of $\sigma$ if $\tau \neq \sigma$.

**Remark 1.3.** In the following we always, for short, say “cone” for a convex polyhedral cone with apex at the origin, since we will only consider such sets.

**Definition 1.4.** A fan $\Delta$ of $N_R$ is a set of cones satisfying the following two properties:

- each face of a cone of $\Delta$ is a cone of $\Delta$,
- the intersection of two cones of $\Delta$ is a face of each of these cones.

The support of a fan $\Delta$ is the set

$$|\Delta| := \bigcup_{\sigma \in \Delta} \sigma. $$

Let $\Delta$ be a fan of $N_R$. For each cone $\sigma \in \Delta$, $L(\sigma)$ denotes the linear subspace of $N_R$ generated by $\sigma$ and we denote by $N^\sigma_R$ any complementary subspace of $L(\sigma)$ in $N_R$ (that is, $N_R = N^\sigma_R \oplus L(\sigma)$). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in $N_R$ and write the decomposition $x_n = y_n + z_n \in N^\sigma_R \oplus L(\sigma)$ for each $n \in \mathbb{N}$.

**Definition 1.5.** Given a fan $\Delta$, we call $N_\Delta$ the set of sequences $(x_n)_{n \in \mathbb{N}}$ of points in $N_R$ such that there exists a cone $\sigma((x_n)_{n \in \mathbb{N}}) \in \Delta$ satisfying:

- there is a point $y \in N^\sigma_R$ such that $\lim_{n \to \infty} y_n = y$ and
- for every $w \in L(\sigma)$, there is an integer $p$ such that $z_n \in w + \sigma$ for all $n \geq p$.

There are two remarks to be made here:

**Remark 1.6.** The set $N_\Delta$ is well-defined, that is, its definition does not depend on the choice of a complementary subspace $N^\sigma_R$ of $L(\sigma)$ for each $\sigma \in \Delta$. Given a cone $\sigma$, let $N_\sigma^\sigma$ and $N_\sigma^r$ be two complementary subspaces of $L(\sigma)$, and write the two decompositions $x_n = y_n + z_n \in N_\sigma^\sigma \oplus L(\sigma)$ and $x_n = y'_n + z'_n \in N_\sigma^r \oplus L(\sigma)$ for a sequence $(x_n)_{n \in \mathbb{N}}$ of $N_R$. It is straightforward to check that if the sequences $(y_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ satisfy the conditions of the previous definition, then $(y'_n)_{n \in \mathbb{N}}$ and $(z'_n)_{n \in \mathbb{N}}$ also do.

**Remark 1.7.** If $(x_n)_{n \in \mathbb{N}} \in N_\Delta$ is a bounded sequence of $N_R$, then necessarily $\sigma((x_n)_{n \in \mathbb{N}}) = \{0\}$.

The following lemma shows the uniqueness of $\sigma \in \Delta$ satisfying the conditions of the previous definition.

**Lemma 1.8.** Let $(x_n)_{n \in \mathbb{N}} \in N_\Delta$. Then, the cone $\sigma((x_n)_{n \in \mathbb{N}})$ is unique.
Proof: Assume that there exist two distinct cones \( \sigma, \sigma' \in \Delta \) both satisfying the conditions of definition \( \text{[1.5]} \). Their intersection \( \tau \) is a face of each of these cones and we can assume that it is different from at least one of them, say \( \tau \neq \sigma \), otherwise \( \tau = \sigma = \sigma' \).

Write the two decompositions \( x_n = y_n + z_n = y'_n + z'_n \) for all \( n \in \mathbb{N} \) corresponding to \( \sigma \) and \( \sigma' \) respectively. The condition on the sequence \( (x_n)_{n \in \mathbb{N}} \) for \( \sigma \) means that \( (y_n)_{n \in \mathbb{N}} \) converges, therefore it is bounded; thus there is a compact set \( K_\sigma \) and an integer \( p \) such that the sequence \( (x_n)_{n \in \mathbb{N}} \) has values in \( K_\sigma + \tau \); similarly there is a compact set \( K_{\sigma'} \) and an integer \( p' \) such that the elements of the sequence \( (x_n)_{n \in \mathbb{N}} \) all lie in \( K_{\sigma'} + \tau \) for \( n \geq p' \).

As a consequence, there is a compact set \( K \) such that all the elements of \( (x_n)_{n \in \mathbb{N}} \) are (for \( n \geq \max(p, p') \)) in the intersection of \( K + \tau \) and \( K + \tau \). Chose \( w \) in a complementary subspace \( V \) of \( L(\tau) \) in \( L(\sigma) \) such that \( w + \tau \) and \( \tau' \) have empty intersection. It is possible because \( \tau \) is a proper face of \( \sigma \), so \( \dim L(\tau) + 1 \leq \dim L(\sigma) \). Now choose \( \lambda > 0 \) large enough such that \( K + \lambda w + \tau \) and \( K + \tau' \) also have empty intersection. By definition of \( \mathcal{N}_\Delta \), there is an integer \( p'' \) such that the elements of \( (x_n)_{n \in \mathbb{N}} \) are also all in \( K + \lambda w + \tau \) and \( K + \tau' \) for \( n \geq p'' \). This is a contradiction. \( \blacksquare \)

Now, we define an equivalence relation on \( \mathcal{N}_\Delta \). Keeping the previous notations, one has the following definition:

**Definition 1.9.** Two sequences \( (x_n)_{n \in \mathbb{N}} \) and \( (x'_n)_{n \in \mathbb{N}} \) in \( \mathcal{N}_\Delta \) are equivalent when \( \sigma((x_n)_{n \in \mathbb{N}}) = \sigma((x'_n)_{n \in \mathbb{N}}) \) and \( y - y' \in L(\sigma) \). In this case, we shall write \( (x_n)_{n \in \mathbb{N}} \sim (x'_n)_{n \in \mathbb{N}} \) and denote by \( y + \infty \cdot \sigma \) the equivalence class of \( (x_n)_{n \in \mathbb{N}} \).

The manifold with corners of \( \Delta \) is the quotient of \( \mathcal{N}_\Delta \) by this equivalence relation. We denote this space by \( \mathcal{M}_c(\Delta) \).

To every \( x \in \mathcal{N}_\mathbb{R} \) and each cone \( \sigma \in \Delta \), we associate an element of \( \mathcal{M}_c(\Delta) \), denoted by \( x + \infty \cdot \sigma \). It is the equivalence class of any sequence \( (x_n)_{n \in \mathbb{N}} \in \mathcal{N}_\Delta \) such that \( y = p(x) \) where \( p(x) \) is the projection of \( x \) on a complementary subspace of \( L(\sigma) \).

We use the conventions \( x + \infty \cdot \{0\} = x \) and \( x + \infty \cdot \sigma = x' + \infty \cdot \sigma \) for two points \( x, x' \in \mathcal{N}_\mathbb{R} \) with \( x - x' \in L(\sigma) \).

**Remark 1.10.** The vector space \( \mathcal{N}_\mathbb{R} \) is a subset of the manifold with corners of \( \Delta \). For \( x \in \mathcal{N}_\mathbb{R} \), we simply consider the constant sequence \( (x_n)_{n \in \mathbb{N}} \) with \( x_n = x \). We still denote by \( \mathcal{N}_\mathbb{R} \) the image of \( \mathcal{N}_\mathbb{R} \) in \( \mathcal{M}_c(\Delta) = \mathcal{N}_\Delta/\sim \) and specify \( \mathcal{N}_\mathbb{R} \subset \mathcal{M}_c(\Delta) \) if necessary. Notice that any cone \( \sigma \), as a subset of \( \mathcal{N}_\mathbb{R} \), is then also a subset of \( \mathcal{M}_c(\Delta) \).

**Remark 1.11.** As one would expect, if \( \Delta \) is a rational fan and \( X_\Delta \) is the associated toric variety, the set \( \mathcal{M}_c(\Delta) \) is homeomorphic to the quotient \( X_\Delta/(\mathbb{S}^1)^n \), i.e. the manifold with corners of \( X_\Delta \). For this fact, we refer to [II], section 1.1.
1.2. **The topology of a manifold with corners.**

Now that the manifold with corners of a fan is defined, we endow this space with a topology and we then prove some properties of this topology that we will use later. These properties are very close to what we have in the case of toric manifolds: there is an action of $\mathbb{N}_\mathbb{R}$ on the manifold with corners of a fan, and it is compact if and only if the fan is finite and complete.

As before, let $\Delta$ be a fan in a vector space $\mathbb{N}_\mathbb{R}$.

**Definition 1.12.** We equip $\mathcal{M}_c(\Delta)$ with a topology in the following way: a neighbourhood basis of a point $y + \infty \cdot \sigma \in \mathcal{M}_c(\Delta)$ is the collection of sets

$$U_{\epsilon,w}(y + \infty \cdot \sigma) := \bigcup_{\tau < \sigma} (y + w + B_\epsilon + \sigma + \infty \cdot \tau),$$

for $\epsilon > 0$ and $w \in L(\sigma)$ ($B_\epsilon$ is the unit open ball around 0 in $\mathbb{N}_\mathbb{R}$).

Figure 1 below depicts a fan $\Delta = \{0, \tau, \tau_1, \tau_2, \sigma\}$ in $\mathbb{R}^2$, its associated manifold with corners and the neighbourhood of a point.

![Figure 1. A fan and its manifold with corners](image)

**Lemma 1.13.** The space $\mathcal{M}_c(\Delta)$ has a natural continuous action of $\mathbb{N}_\mathbb{R}$ which extends the action of $\mathbb{N}_\mathbb{R}$ on itself.
Proof: Let \( y + \infty \cdot \sigma \in \text{MC}(\Delta) \) (this point being \( y \) if \( \sigma = \{0\} \) or \( 0 + \infty \cdot \sigma \) if \( \sigma \) is \( n \)-dimensional), and \( x \in N_\mathbb{R} \). Set
\[
x.(y + \infty \cdot \sigma) := p(x) + y + \infty \cdot \sigma,
\]
where \( p(x) \) is the projection of \( x \) to the linear subspace \( N_\mathbb{R}^\sigma \). This action is clearly continuous.

\[ \blacksquare \]

Remark 1.14. Let \((x_n)_{n \in \mathbb{N}}\) be an element of \( N_\Delta \) and \( y + \infty \cdot \sigma \) its equivalence class in \( \text{MC}(\Delta) \). It is easy to see that if we consider \((x_n)_{n \in \mathbb{N}}\) as a sequence of points in \( \text{MC}(\Delta) \), then it converges to \( y + \infty \cdot \sigma \).

Lemma 1.15. Let \( \sigma \) be a cone of a fan \( \Delta \). Then the closure \( S \) of \( \sigma \subset \text{MC}(\Delta) \) in \( \text{MC}(\Delta) \) is compact.

Proof: We consider a sequence of \( S \), denoted by \((x_n)_{n \in \mathbb{N}}\). Call \( S := S \setminus \sigma \) and write \( S \) as the disjoint union \( S = \sigma \sqcup S \). We distinguish cases.

\[ \alpha \] First, assume that \((x_n)_{n \in \mathbb{N}}\) has infinitely many elements in \( \sigma \). After a possible extraction of subsequence, we can assume that the sequence \((x_n)_{n \in \mathbb{N}}\) has all its elements in this set. Then, either \((x_n)_{n \in \mathbb{N}}\) is bounded and we are done, or it is not bounded. In this case, after another extraction of subsequence if needed, we can assume that \((\|x_n\|)_{n \in \mathbb{N}}\) is strictly increasing, where \( \| \cdot \| \) is a norm on \( N_\mathbb{R} \). Denote by \( \tau_1, \ldots, \tau_r \) the vectors generating \( \sigma \), and write for all \( n \in \mathbb{N} \):
\[
x_n = x_{1,n} \tau_1 + \ldots + x_{r,n} \tau_r.
\]
Note that this decomposition is not necessarily unique (in fact it is only the case if \( \sigma \) is simplicial). Since \((x_n)_{n \in \mathbb{N}}\) is not bounded, one at least of the sequences \((x_{i,n})_{n \in \mathbb{N}}\) (for \( i \in \{1, \ldots, r\} \)) is also not bounded. Assume (to simplify writing) that the sequences \((x_{1,n})_{n \in \mathbb{N}}, \ldots, (x_{j,n})_{n \in \mathbb{N}}\) are bounded while the sequences \((x_{j+1,n})_{n \in \mathbb{N}}, \ldots, (x_{r,n})_{n \in \mathbb{N}}\) are not. Consider the cone generated by \( \tau_{j+1}, \ldots, \tau_r \). We call this cone \( \kappa \) if it is a proper face of \( \sigma \), otherwise we set \( \kappa := \sigma \). The sequence \((x_{1,n})_{n \in \mathbb{N}}\) is bounded so it admits a convergent subsequence, say \((x_{1,\varphi(n)})_{n \in \mathbb{N}}\). Similarly, the sequence \((x_{2,\varphi(n)})_{n \in \mathbb{N}}\) is bounded, so we can find a convergent subsequence for it, and so on. After extracting subsequences a finite number of times, we can therefore assume that \((x_{1,n})_{n \in \mathbb{N}}, \ldots, (x_{j,n})_{n \in \mathbb{N}}\) are convergent and that \((x_{j+1,n})_{n \in \mathbb{N}}, \ldots, (x_{r,n})_{n \in \mathbb{N}}\) are strictly increasing.

Set \( y_n := \pi(x_{1,n} \tau_1 + \ldots + x_{j,n} \tau_j) \) for all \( n \in \mathbb{N} \) where \( \pi \) is the projection on a complementary space \( N_\mathbb{R}^\sigma \) of \( L(\kappa) \) and denote by \( y \) the limit of this sequence. Notice that if \( \kappa = \sigma \), we have \( y_n = 0 \) for all \( n \in \mathbb{N} \). Write, for all \( n \in \mathbb{N} \), \( x_n = y_n + z_n \) with \( z_n = x_n - y_n \). One sees that for all \( n \in \mathbb{N} \), \( z_n \in L(\kappa) \) hence the sequence \((x_n)_{n \in \mathbb{N}}\) is an element of \( N_\Delta \). Indeed, the sequence \((y_n)_{n \in \mathbb{N}}\) is convergent, taking values in a complementary space \( N_\mathbb{R}^\sigma \) of \( L(\kappa) \) and the sequence \((z_n)_{n \in \mathbb{N}}\) satisfies, for all \( w \in L(\kappa) \), the existence of a rank \( p \) such that for all \( n \geq p \), \( z_n \in w + \kappa \). Lemma 1.8 tells us that \( \kappa \)
is uniquely defined, hence the fact that the decomposition above is not unique has no incidence. According to remark 1.14 we have proved that the sequence $(x_n)_{n \in \mathbb{N}}$ admits a convergent subsequence with limit $y + \infty \cdot \kappa \in S$.

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\( \beta \) Now, assume that $(x_n)_{n \in \mathbb{N}}$ has infinitely many terms in $S$. As above, after a possible subsequence extraction we may assume that all the values of $(x_n)_{n \in \mathbb{N}}$ are in $S$. By the previous part of the proof, we see that every element of $S$ is of the form $\pi(y) + \infty \cdot \kappa$ where $y \in \sigma$, $\kappa$ is a face of $\sigma$ and $\pi$ is the projection on a complementary space $N_\sigma^\kappa$ of $L(\kappa)$ (in particular, $y = 0$ if $\kappa = \sigma$). Since $\sigma$ only possesses a finite number of faces, after a subsequence extraction if needed, we may assume that there is a unique face $\kappa$ of $\sigma$ such that for all $n \in \mathbb{N}$, $x_n = y_n + \infty \cdot \kappa$ with $y_n \in \pi(\sigma)$. If $\kappa = \sigma$, the sequence is constant hence the result is proved. Now, assume that $\kappa$ is a proper face of $\sigma$. As earlier, we denote by $\tau_1, ..., \tau_r$ the generators of $\sigma$, the face $\kappa$ being generated over $\mathbb{R}_{>0}$ by $\tau_{j+1}, ..., \tau_r$. We now use the same reasoning as in the first part of the proof, this time applied to the sequence

\[ y_n = y_{1,n} \pi(\tau_1) + ... + y_{j,n} \pi(\tau_j) \]

(notice that since $\kappa$ is a face of $\sigma$, $\pi(\sigma)$ is a cone in $N_\sigma^\kappa$). As before, there is an integer $\ell \in \{1, ..., j + 1\}$ such that the sequences $(y_{1,n})_{n \in \mathbb{N}}, ..., (y_{\ell,n})_{n \in \mathbb{N}}$ are convergent (none of them is convergent if $\ell = j + 1$ by convention), $(y_{\ell+1,n})_{n \in \mathbb{N}}, ..., (y_{j,n})_{n \in \mathbb{N}}$ are strictly increasing and we see that $(x_n)_{n \in \mathbb{N}}$ converges in $\mathcal{M}(\Delta)$ to a point $y' + \infty \cdot \kappa'$ where $\kappa'$ is either the proper face of $\sigma$ generated by $\tau_{\ell}, ..., \tau_r$, either $\sigma$ itself (and $\kappa$ is a face of $\kappa'$ in each case).

This concludes the proof. \[ \square \]

We now prove the following proposition, which just extends what we already now in the toric case:

**Proposition 1.16.** The manifold with corners associated to a fan $\Delta$ in $\mathbb{R}^n$ is compact if and only if $\Delta$ is finite and complete.

**Proof:** If we assume that $\Delta$ is finite and complete, the result is a consequence of lemma 1.15. Assume now that $\mathcal{M}(\Delta)$ is compact. First, suppose $\Delta$ is not finite and take any sequence $(x_n)_{n \in \mathbb{N}}$ of points in $\mathcal{M}(\Delta)$ with $x_n \in N_{\mathbb{R}} + \infty \cdot \sigma_n$ and $\sigma_n \neq \sigma_p$ for $n \neq p$. The definition of the topology of $\mathcal{M}(\Delta)$ tells us that if a subsequence of $(x_n)_{n \in \mathbb{N}}$ were convergent, then a subsequence of $(\sigma_n)_{n \in \mathbb{N}}$ would become constant after some rank, which is a contradiction.

Now that we know $\Delta$ is finite, assume it is not complete. There is a vector $v \in N_{\mathbb{R}}$ such that $\mathbb{R}_{>0}v \subset \mathbb{R}^n \setminus |\Delta|$. We now claim that the sequence $(nv)_{n \in \mathbb{N}}$ has no convergent subsequence in $\mathcal{M}(\Delta)$. If it were the case, its limit would lie in $\mathcal{M}(\Delta) \setminus N_{\mathbb{R}}$, i.e. there would exist $\sigma \in \Delta$ and $y \in N_{\mathbb{R}}^\kappa$ such that \( \lim_{n \to +\infty} nv = y + \infty \cdot \sigma \). The definition of the topology tells us that for $\varepsilon > 0$, there exists an integer $p$ such that for $n \geq p$,
$nv \in \bigcup_{\tau < \sigma} (y + B_z + \sigma + \infty \cdot \tau)$, i.e. $nv \in y + B_z + \sigma$. This is impossible because for all $y \in N^{\sigma}_G$ and $\varepsilon > 0$, the set $(y + B_z + \sigma) \cap \mathbb{R}_{\geq 0}v$ is bounded. Indeed, it is a convex set (it is an intersection of convex sets) therefore connected, which means that it is an interval of $\mathbb{R}_{\geq 0}v$. If it were not bounded, this would mean that $\mathbb{R}_{\geq a}v \subset y + B_z + \sigma$ for some $a > 0$. Then, one would have $av \in (y + B_z + \sigma) \setminus \sigma$ (because $v \notin \sigma$) and we obtain a contradiction because every point of $(y + B_z + \sigma) \setminus \sigma$ is mapped outside $y + B_z + \sigma$ by homotheties of center 0 and large enough ratio. We thus have proved that $(nv)_{n \in \mathbb{N}}$ has no convergent subsequence and $\mathcal{M}(\Delta)$ is not compact if $\Delta$ is not complete. ■

2. Describing LVMB manifolds in terms of toric geometry

In this section we translate Bosio’s construction in terms of toric geometry. In Bosio’s construction, one needs an open subset $U$ of $\mathbb{P}^n(\mathbb{C})$ and an $m$-dimensional complex Lie group $G$ (with $2m \leq n$) such that $U/G$ is a compact complex manifold. First we recall this construction, then we see how we can associate a fan $\Delta$ to $U$, a linear subspace $E$ of $\mathbb{R}^n$ to $G$, and see how the manifold with corners of the fan $\Delta$ helps understanding the quotient map $U \to U/G$.

2.1. Bosio’s construction. The construction we explain here is due to Bosio and it generalizes a work of Meersseman (see [3] and [9] respectively).

Let $m, n$ be positive integers such that $2m \leq n$; let $\mathcal{L} := (\ell_0, ..., \ell_n)$ be $n + 1$ linear forms of $\mathbb{C}^m$ such that any subfamily of $2m + 1$ elements of $\mathcal{L}$ is an $\mathbb{R}$-affine basis of $(\mathbb{C}^m)^*$, where $(\mathbb{C}^m)^*$ is the dual space of $\mathbb{C}^m$. Call $\mathcal{E}_{m,n}$ a family of subsets of $\{0, ..., n\}$, each having $2m + 1$ elements. For every $P \in \mathcal{E}_{m,n}$, call $\mathcal{L}_P$ the corresponding subfamily of $\mathcal{L}$. We are interested in the two following conditions on $\mathcal{L}$ and $\mathcal{E}_{m,n}$:

- if for all $P \in \mathcal{E}_{m,n}$ and for all $i \in \{0, ..., n\}$, there exists $j \in P$ such that $(P \setminus \{j\}) \cup \{i\} \in \mathcal{E}_{m,n}$, we say that $(\mathcal{L}, \mathcal{E}_{m,n})$ satisfies the **SEP** (for substitute existence principle),
- if for all $P, Q \in \mathcal{E}_{m,n}$, the interiors of the convex envelopes of $\mathcal{L}_P$ and $\mathcal{L}_Q$ have non-empty intersection, we say that $(\mathcal{L}, \mathcal{E}_{m,n})$ satisfies the **imbrication condition**.

An **LVMB datum** is a pair $(\mathcal{L}, \mathcal{E}_{m,n})$ satisfying these two conditions. Following Bosio’s denomination, we say that an integer $i \in \{0, ..., n\}$ is **indispensable** if $i \in P$ for all $P \in \mathcal{E}_{m,n}$. If $i$ is indispensable, we also say that $\ell_i$ is indispensable.

Given a pair $(\mathcal{L}, \mathcal{E}_{m,n})$ (not necessarily an LVMB datum), we construct an open subset of $\mathbb{P}^n(\mathbb{C})$ and an action on this set by a Lie group $G$ and see when the quotient is a compact complex manifold:
First, call $U$ the (open) set of points $z = [z_0 : \ldots : z_n] \in \mathbb{P}^n(\mathbb{C})$ such that there exists $P_z \in \mathcal{E}_{m,n}$ satisfying: for all $i \in P_z$, $z_i \neq 0$.

Then, define an action of $G \cong \mathbb{C}^m$ on $U$ by:

$$
\mathbb{C}^m \times U \rightarrow U \quad (Z, [z_0 : \ldots : z_n]) \mapsto \exp(\ell_0(Z))z_0 : \ldots : \exp(\ell_n(Z))z_n.
$$

Bosio relates the properness and the cocompactness of this action on $U$ with the SEP and the imbrication condition, this is the following

**Lemma 2.1.** The action of $G$ on $U$ is proper if and only if $(\mathcal{L}, \mathcal{E}_{m,n})$ satisfies the imbrication condition.

If this action is proper, it is cocompact if and only if $(\mathcal{L}, \mathcal{E}_{m,n})$ satisfies the SEP.

Hence the quotient $X_{n,m} := U/G$ is a compact complex manifold if and only if $(\mathcal{L}, \mathcal{E}_{m,n})$ is an LVMB datum. In this case, the manifold $X_{n,m}$ is of complex dimension $n - m$ and is called an LVMB manifold.

### 2.2. The toric viewpoint.

As it is well-known, the complex projective space $\mathbb{P}^n(\mathbb{C})$ is a toric manifold given by the fan $\Delta_{\mathbb{P}^n}(\mathbb{C})$ in $\mathbb{R}^n$ which consists of the $n + 1$ cones generated by $n$ of the $n + 1$ vectors $e_1, \ldots, e_n, -e_1 - \ldots - e_n$ (where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$) and their faces. We say that a fan $\Delta$ is a subfan of $\Delta_{\mathbb{P}^n}(\mathbb{C})$ of $\mathbb{P}^n(\mathbb{C})$.

Our goal in this section is to prove the following

**Theorem 2.2.** i) Let $(\mathcal{L}, \mathcal{E}_{m,n})$ be an LVMB datum. Then there is a pair $(E, \Delta)$ where $E$ is a $2m$-dimensional linear subspace of $\mathbb{R}^n$ and $\Delta$ is a subfan of the fan $\Delta_{\mathbb{P}^n}(\mathbb{C})$ of $\mathbb{P}^n(\mathbb{C})$, satisfying the following two properties:

a) the projection map $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n/E \cong \mathbb{R}^{n-2m}$ is injective on $|\Delta|$, 

b) the fan $\pi(\Delta)$ is complete in $\mathbb{R}^n/E$, i.e. $|\pi(\Delta)| = \mathbb{R}^n/E$.

ii) Conversely, given a pair $(E, \Delta)$ having the two properties above, one obtains an LVMB datum.

#### 2.2.1. Preliminary results.

We prove the two parts of this theorem separately in the next two subsections. Before this, we need some preliminary statements: we see how the two conditions of the first part of this theorem are related with the SEP and the imbrication condition.

The following two lemmas are respectively translations of the first and the second part of lemma 2.1 in our toric setting, in the sense that they characterize the properness and the cocompactness of an action of a linear space $E$ on a manifold with corners with conditions on its fan:

**Lemma 2.3** (Properness). Let $\Delta$ be a fan in $\mathbb{R}^n$ and $E \cong \mathbb{R}^k$ be a linear subspace of $\mathbb{R}^n$. Then there is an action of $E$ on $\mathcal{M}(\Delta)$ and this action is
proper if and only if the restriction of the quotient map \( \pi : \mathbb{R}^n \to \mathbb{R}^n/E \) to the support \( |\Delta| \) of \( \Delta \) is injective.

**Proof:** First, assume that \( \pi \) is not injective on \( |\Delta| \), i.e. there exist two cones \( \sigma_1 \) and \( \sigma_2 \in \Delta \), and \( x \in \sigma_1, y \in \sigma_2 \) such that \( y - x \in E \). We know that the closures \( K_1 \) and \( K_2 \) in \( \text{Mc}(\Delta) \) of \( \sigma_1 \) and \( \sigma_2 \) respectively are compact subsets (lemma [1.15]). We then write, for \( \lambda \geq 1 \), the equality \( \lambda(y - x) + \lambda x = \lambda y \). Hence the set \( \{ t \in E \mid tK_1 \cap K_2 \neq \emptyset \} \) is not bounded, therefore the action of \( E \) on \( \text{Mc}(\Delta) \) is not proper.

Conversely, assume now that \( E \) is not acting properly on \( \text{Mc}(\Delta) \). This means that there are two compact subsets \( K_1 \) and \( K_2 \) of \( \text{Mc}(\Delta) \) such that the set \( E := \{ t \in E \mid tK_1 \cap K_2 \neq \emptyset \} \) is not bounded. We introduce the following notation, for \( \sigma \in \Delta \) and \( i = 1, 2 \):

\[
K_i^\sigma := K_i \cap (\mathbb{R}^n + \infty \cdot \sigma).
\]

Since the sets \( K_i \) are compact, there exist only a finite number of cones \( \sigma \in \Delta \) satisfying \( K_i^\sigma \neq \emptyset \). We call \( \Delta' \) the collection of these cones.

We then have \( K_i = \bigcup_{\sigma \in \Delta'} K_i^\sigma \). The action of \( E \) induces an action on each component \( \mathbb{R}^n + \infty \cdot \sigma \) for \( \sigma \in \Delta \), so we have the following decomposition:

\[
E = \bigcup_{\sigma \in \Delta'} \{ t \in E \mid tK_i^\sigma \cap K_j^\sigma \neq \emptyset \}.
\]

Since the set \( \Delta' \) is finite, there exists a cone \( \sigma \in \Delta' \) such that \( E^\sigma := \{ t \in E \mid tK_i^\sigma \cap K_j^\sigma \neq \emptyset \} \) is not bounded. Consequently there is a sequence \( (t_n)_{n \in \mathbb{N}} \) of \( E^\sigma \) with \( \|t_n\| \to +\infty \).

If \( E \cap L(\sigma) \neq \{0\} \), we are done. Indeed, in this case there exists \( x = \lambda_1 \tau_1 + \ldots + \lambda_k \tau_k \in E \) where the \( \tau_i \)'s are the generating rays of \( \sigma \).

We then write

\[
x = \sum_{\lambda_i \geq 0} \lambda_i \tau_i - \sum_{\lambda_j < 0} (-\lambda_j) \tau_j = x^+ - x^-,
\]

where the vectors \( x^+ \) and \( x^- \) are both in \( |\Delta| \), so the quotient map \( \pi \) is not injective on \( |\Delta| \).

Assume now that \( E \cap L(\sigma) = \{0\} \). We now study \( \mathbb{R}^n + \infty \cdot \sigma \cong \mathbb{R}^n/L(\sigma) = \mathbb{R}^n/\sigma \).

Let \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) be sequences of \( K_1^\sigma \) and \( K_2^\sigma \) respectively (with at least one being not bounded) satisfying \( t_n + x_n = y_n \). Up to extraction of subsequences, we may assume that they are convergent (\( K_1 \) and \( K_2 \) being compact sets), with respective limits \( x_0 + \infty \cdot \sigma_1 \) and \( y_0 + \infty \cdot \sigma_2 \) where \( \sigma < \sigma_i \) for \( i = 1, 2 \) (with eventually \( \sigma_i = \sigma \) for at most one index \( i \)). The definition of the topology of \( \text{Mc}(\Delta) \) then gives us that for any \( \varepsilon > 0 \), there exists some \( N \) such that for \( n \geq N \), \( x_n \in x_0 + B_{\varepsilon} + \sigma_1 + \infty \cdot \sigma \) and \( y_n \in y_0 + B_{\varepsilon} + \sigma_2 + \infty \cdot \sigma \).

We can then write \( x_n = x_0 + x_{e,n} + x_{e,n}^{\sigma_1} + \infty \cdot \sigma \) and \( y_n = y_0 + y_{e,n} + y_{e,n}^{\sigma_2} + \infty \cdot \sigma \) where \( x_{e,n}, y_{e,n} \in B_{\varepsilon} \) and \( x_{e,n}^{\sigma_1}, y_{e,n}^{\sigma_2} \in \sigma_i \).
The relation \( t_n + x_n = y_n \) means
\[
 t_n = p(t_n) = p(y_0 - x_0 + y_{\varepsilon,n} - x_{\varepsilon,n}) + p(y_{n}^2 - x_{n}^2),
\]
where \( p \) is the projection of \( \mathbb{R}^n = E \oplus F \oplus L(\sigma) \) on \( E \oplus F \) with respect to \( L(\sigma) \). Recall that \( \|t_n\| \to +\infty \). We now write
\[
 \frac{t_n}{\|t_n\|} = \frac{p(y_0 - x_0 + y_{\varepsilon,n} - x_{\varepsilon,n})}{\|t_n\|} + \frac{p(y_{n}^2 - x_{n}^2)}{\|t_n\|}.
\]

First notice that the sequence \( t_n/\|t_n\| \) is bounded, so we can assume it converges to some \( t \in \mathbb{R}^k \setminus \{0\} \). Also notice that the sequence \( (p(y_0 - x_0 + y_{\varepsilon,n} - x_{\varepsilon,n}))/\|t_n\| \) has limit 0, therefore the sequence \( (p(y_{n}^2 - x_{n}^2))/\|t_n\| \) also converges to \( t \). Since the Minkowski sum of closed cones is closed, we know that \( t \in p(\tilde{\sigma}_2 - \tilde{\sigma}_1) \). Therefore \( t = y - x \) with \( x \in p(\tilde{\sigma}_1) \) and \( y \in p(\tilde{\sigma}_2) \). Moreover, there exists \( z_1 = \lambda_1 \tau_1 + \ldots + \lambda_k \tau_k \in L(\sigma) \) such that \( x + z_1 \in \tilde{\sigma}_1 \). Then for all \( z'_1 = \mu_1 \tau_1 + \ldots + \mu_k \tau_k \) with \( \mu_i \geq \lambda_i \), we have \( x + z'_1 \in \tilde{\sigma}_1 \). Similarly, there exists \( z_2 \in L(\sigma) \) satisfying \( y + z_2 \in \tilde{\sigma}_2 \) and for all \( z'_2 = \mu'_1 \tau_1 + \ldots + \mu'_k \tau_k \) with \( \mu'_i \geq \lambda_i \), we have \( y + z'_2 \in \tilde{\sigma}_2 \). As a consequence there exists \( w \in \sigma \) such that \( y + w \in \tilde{\sigma}_2 \) and \( x + w \in \tilde{\sigma}_1 \). This gives us \( t = (y + w) - (x + w) \in \tilde{\sigma}_2 - \tilde{\sigma}_1 \), which implies the non-injectivity of \( \pi \) on \( |\Delta| \).

For the second lemma, we need to define the SEP condition for a set of cones:

**Definition 2.4.** Let \( \Delta \) be a set of cones in a vector space \( \mathbb{R}^n \) (\( \Delta \) not necessarily being a fan) and \( V := \{v_1, \ldots, v_p\} \) a set of generating rays of \( \Delta \) (that is, every cone of \( \Delta \) is positively generated by a subfamily of \( V \)). We say that \( \Delta \) satisfies the SEP condition if for every cone \( \sigma = \mathbb{R}_{\geq 0}v_{i_1} + \ldots + \mathbb{R}_{\geq 0}v_{i_k} \in \Delta \) and every \( i \in \{1, \ldots, p\} \), there exists \( j \in \{i_1, \ldots, i_k\} \) such that \( \sigma' := \mathbb{R}_{\geq 0}v_{i_1} + \ldots + \mathbb{R}_{\geq 0}v_{j} + \ldots + \mathbb{R}_{\geq 0}v_{i_k} + \mathbb{R}_{\geq 0}v_{i} \in \Delta \).

**Lemma 2.5** (Cocompactness). Let \( \Delta \) be a simplicial fan in \( \mathbb{R}^n \) (i.e. all its cones are simplicial) and \( E \cong \mathbb{R}^k \) be a linear subspace of \( \mathbb{R}^n \). Suppose that the quotient map \( \pi : \mathbb{R}^n \to \mathbb{R}^n/E \) is injective on \( |\Delta| \). Consider the set \( \Delta_{\text{max}} \) of all cones of \( \Delta \) of maximal dimension. Then \( \pi(\Delta) \) is complete if and only if the cones of \( \Delta_{\text{max}} \) are of dimension \( n - k \) and \( \Delta_{\text{max}} \) satisfies the SEP condition.

**Proof:** The assertion is clear.

**2.2.2. From LVMB data to toric data.**

In this subsection we prove part i) of theorem 2.2. We see how one recovers toric information from an LVMB datum.

Let \( (L, E_{m,n}) \) be an LVMB datum and \( U \) the corresponding open subset of \( \mathbb{P}^n(\mathbb{C}) \) (see section 2.1). We want to find a pair \( (E, \Delta) \) (where \( E \) is a 2\( m \)-dimensional subspace of \( \mathbb{R}^n \) and \( \Delta \) is a fan of \( \mathbb{R}^n \)) satisfying conditions a) and b) of theorem 2.2. Consider the fan \( \Delta_{\mathbb{P}^n(\mathbb{C})} \) in \( \mathbb{R}^n \) defining the toric manifold.
$\mathbb{P}^n(\mathbb{C})$: its rays are generated by the vectors $\{e_1, ..., e_n, -(e_1 + ... + e_n)\}$ where $(e_1, ..., e_n)$ is the canonical basis of $\mathbb{R}^n$. We call $e_0 := -(e_1 + ... + e_n)$ and to each $P \in \mathcal{E}_{m,n}$ we associate the simplicial cone $\sigma_P$ of dimension $n - 2m$ generated by the $n - 2m$ vectors $e_i$ satisfying $i \notin P$. A direct computation shows that the open set $U$ is the toric manifold given by the subfan of $\Delta_{\mathbb{P}^n(\mathbb{C})}$ consisting of the cones $\{\sigma_P, P \in \mathcal{E}\}$ and their faces. Call this fan $\Delta$.

Now that we have found the fan $\Delta$, we must detect the subspace $E \cong \mathbb{R}^{2m} \subset \mathbb{R}^n$. For this, we need a preliminary lemma. Since $\mathbb{C}^m$ acts on $\mathbb{P}^n(\mathbb{C})$ by $Z.[z_0: ...: z_n] = [\exp(\ell_0(Z))z_0: ...: \exp(\ell_n(Z))z_n]$, we see $\mathbb{C}^m$ as a closed subgroup of $(\mathbb{C}^*)^n$ by

$$Z \mapsto (\exp(\ell_1(Z) - \ell_0(Z)), ..., \exp(\ell_n(Z) - \ell_0(Z))).$$

**Lemma 2.6.** The intersection of the two subgroups $\mathbb{C}^m$ and $(\mathbb{S}^1)^n$ of $(\mathbb{C}^*)^n$ is trivial.

**Proof:** Choose $P \in \mathcal{E}_{m,n}$. An element $Z$ of $\mathbb{C}^m \cap (\mathbb{S}^1)^n$ must satisfy

$$\ell_k(Z) - \ell_0(Z) \in i\mathbb{R}, \text{ for all } k \in P;$$

therefore we have $\mathbb{R}(\ell_0(Z)) = \mathbb{R}(\ell_k(Z))$ for all $k \in P$. This means that $Z = 0$ since $\{\ell_k, k \in P\}$ is a $\mathbb{R}$-affine basis for $(\mathbb{C}^m)^*$. $\blacksquare$

Now we define $E \cong \mathbb{R}^{2m} \subset \mathbb{R}^n$ to be the image of $\mathbb{C}^m \subset (\mathbb{C}^*)^n$ by the ord map:

$$\text{ord} : \quad (\mathbb{C}^*)^n \to \mathbb{R}^n \quad (z_1, ..., z_n) \mapsto (- \log |z_1|, ..., - \log |z_n|),$$

which is injective on $\mathbb{C}^m$ by the previous lemma. A quick computation shows that a basis of this linear subspace of $\mathbb{R}^n$ is given by the following $2m$ vectors:

$$(2) \quad X_k = \begin{pmatrix} \mathbb{R}(\ell_{1,k} - \ell_{0,k}) \\ \vdots \\ \mathbb{R}(\ell_{n,k} - \ell_{0,k}) \end{pmatrix}, \quad Y_k = \begin{pmatrix} \Im(\ell_{1,k} - \ell_{0,k}) \\ \vdots \\ \Im(\ell_{n,k} - \ell_{0,k}) \end{pmatrix}, \quad k \in \{1, ..., m\},$$

where $\ell_{i,k}$ is the $k$-th component of $\ell_i$.

We now have a pair $(E, \Delta)$ so we have to prove that they satisfy the needed properties. For this, first notice that we have the following commutative diagram:

$$(3) \quad U = X_{\Delta} \xrightarrow{(\mathbb{S}^1)^n} \mathcal{M}_c(\Delta) \quad \mathbb{C}^m \xrightarrow{q((\mathbb{S}^1)^n) \cong (\mathbb{S}^1)^n} \mathcal{M}_c(\pi(\Delta)), \quad E = \text{ord}(\mathbb{C}^m) \cong \mathbb{R}^{2m}$$
where ord : \((\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n/\mathbb{S}^1)^n\) and \(q : (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n/\mathbb{C}^n\) are the quotient maps. Let us call \(\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n/E \cong \mathbb{R}^{n-2m}\) the projection map with respect to \(E\). The fact that the quotient \(\mathcal{M}c(\Delta)/E\) is homeomorphic to \(\mathcal{M}c(\pi(\Delta))\) is easily checked. Indeed, the homeomorphism is the map \(\tilde{\pi}\) sending the orbit of \(y + \infty \cdot \sigma\) to \(\pi(y) + \infty \cdot \pi(\sigma)\). Bijectivity is a consequence of the injectivity of \(\pi\) on \(|\Delta|\). To prove it is open, consider an open set \(U\) of \(\mathcal{M}c(\Delta)/E\) and a point \(\tilde{\pi}(x) = \tilde{x} \in \tilde{\pi}(U)\). Then, there are \(\varepsilon > 0\) and \(w\) such that \(U_{\varepsilon, w}(\tilde{x})/E\) (see definition \ref{def:0}) is a neighbourhood of \(x\) contained in \(U\), and for \(\delta > 0\) small enough, \(U_{\varepsilon, \pi(w)}(\pi(x))\) is a subset of \(\tilde{\pi}(U_{\varepsilon, w}(x))/E\). To prove continuity, we use a similar reasoning.

Since the group \((\mathbb{S}^1)^n\) is compact, the properness of the action of \(\mathbb{C}^m\) on \(U\) is equivalent to the properness of the action of \(E\) on \(\mathcal{M}c(\Delta)\), which in turn is equivalent to the fact that \(\pi\) is injective on \(|\Delta|\) according to lemma \ref{lem:0}. On the other hand, \(\mathcal{M}c(\pi(\Delta))\) is compact because \(X\) is, and the completeness of the fan \(\pi(\Delta)\) is now a consequence of proposition \ref{prop:0}. By lemma \ref{lem:3}, one now can see that conditions \(a\) and \(b\) of theorem \ref{th:2} are satisfied by the pair \((E, \Delta)\), hence part \(i\) of this theorem is proved.

2.2.3. From toric data to LVMB data.

We now prove part \(ii\) of theorem \ref{th:2}. Suppose we are given a subfan \(\Delta\) of \(\Delta_{\mathbb{P}^n(\mathbb{C})}\) and a linear subspace \(E \cong \mathbb{R}^{2m} \subset \mathbb{R}^n\) satisfying both conditions \(a\) and \(b\) of theorem \ref{th:2}. In order to recover an LVMB datum, we first choose \(U = X_\Delta\). For \(\sigma = \mathbb{R}_{\geq 0}e_{i_1} + \ldots + \mathbb{R}_{\geq 0}e_{i_{2m}} \in \Delta\) a cone of maximal dimension \(n - 2m\), define \(P_\sigma := \{0, \ldots, n\} \setminus \{i_1, \ldots, i_{2m}\}\) and \(\mathcal{E}_{m,n} := \{P_\sigma, \sigma \in \Delta\text{ of dimension } n - 2m\}\). It is clear that \(U\) is the open set corresponding to \(\mathcal{E}_{m,n}\) (as defined in section 2.1).

We now have to see how one recovers the set of linear forms \(\mathcal{L}\) and check if the pair \((\mathcal{L}, \mathcal{E}_{m,n})\) is an LVMB datum.

To do this, we first choose a basis for \(E \cong \mathbb{R}^{2m} \subset \mathbb{R}^n\) which we write as a matrix \(A = (a_{i,j}) \in \mathbb{M}_{n,2m}(\mathbb{R})\). Then we define \(n + 1\) vectors of \(\mathbb{C}^m\) by taking each of the \(n\) rows of \(A\) and sending them to \(\mathbb{C}^m\) via the map \((x_1, \ldots, x_{2m}) \mapsto (x_1, \ldots, x_n) + i(x_{n+1}, \ldots, x_{2m})\) (call these vectors \(\ell_1, \ldots, \ell_n\)), along with the zero vector of \(\mathbb{C}^m\). Call this vector \(\ell_0\) and set \(\mathcal{L} = \{\ell_0, \ldots, \ell_n\}\).

We now consider the pair \((\mathcal{L}, \mathcal{E}_{m,n})\). According to lemma \ref{lem:2}, we know that the action of \(\mathbb{C}^m\) is proper on \(X_\Delta\), meaning that the imbrication condition is satisfied. Since \(\pi(\Delta)\) is complete, lemma \ref{lem:3} tells us that the set of all \(n - 2m\)-dimensional cones of \(\Delta\) satisfies the SEP, hence \(\mathcal{E}_{m,n}\) also does. Finally, the two Bosio conditions are satisfied, i.e. the pair \((\mathcal{L}, \mathcal{E}_{m,n})\) is an LVMB datum.
2.3. A correspondence between Bosio and toric data.

Let \((L, E_{m,n})\) be an LVMB datum. For the proof of theorem 2.2, we associated to this datum a unique pair \((E, \Delta)\) where \(E\) is a \(2m\)-dimensional linear subspace of \(\mathbb{R}^n\) and \(\Delta\) is a fan of \(\mathbb{R}^n\).

As one can see from the proof of theorem 2.2, it is possible that two different LVMB data give the same pair \((E, \Delta)\). We discuss this fact now.

Let \((L, E_{m,n})\) and \((L', E'_{m',n'})\) be two LVMB data and let \((E, \Delta), (E', \Delta')\) be the two associated pairs given by theorem 2.2. If \((E, \Delta) = (E', \Delta')\), one has \(n = n'\) and \(m = m'\) because the dimensions of \(E\) and \(E'\) and their ambient spaces are equal respectively. Also, one sees that \(E_{m,n} = E'_{m',n'}\) because it is clear from the proof of theorem 2.2 that for a subfan \(\Delta\) of the fan of \(P^n(\mathbb{C})\) there is a unique corresponding set \(E_{m,n}\). On the other hand, the fact that \(E = E'\) does not imply \(L = L'\), but it tells us that there exists a real affine automorphism of \(\mathbb{R}^{2m} \cong (\mathbb{C}^m)^*\) sending each \(\ell_i \in L\) to an element \(\ell_i' \in L'\).

Call \(P\) the set of pairs \((E, \Delta)\) such that \(E\) and \(\Delta\) satisfy conditions \(a\) and \(b\) of theorem 2.2. Define an equivalence relation \(\approx\) on the set of LVMB data, setting \((L, E_{m,n}) \approx (L', E'_{m',n'})\) if and only if \(E_{m,n} = E'_{m',n'}\) and there exists a real affine automorphism of \(\mathbb{R}^{2m} \cong (\mathbb{C}^m)^*\) such that its restriction to \(L\) is a bijection with \(L'\). The discussion above leads to the following statement:

**Proposition 2.7.** There is a bijective correspondence

\[
\text{\{LVMB data\}} / \approx \leftrightarrow P.
\]

A natural question is now to ask when two LVMB data give the same manifolds (up to biholomorphism). We discuss this in section 4 below.

3. Detecting LVM data among LVMB data

In [9], Meersseman gave a method of construction of compact complex manifolds, called LVM manifolds. Bosio shows that these manifolds can be obtained by his construction and gives a criterion to detect when an LVMB datum leads to an LVM manifold. This is proposition 1.3 in [3] which we recall now.

Let \((L, E_{m,n})\) be an LVM datum and define \(O\) to be the set of points in \((\mathbb{C}^m)^*\) (the dual space of \(\mathbb{C}^m\)) which are not in the convex hull of any family of \(2m\) elements of \(L\).

**Proposition 3.1.** Given an LVMB datum \((L, E_{m,n})\), one obtains an LVM manifold if and only if there is a bounded connected component \(O\) of \(O\) such that \(E_{m,n}\) is the collection of subsets \(P\) of \(\{1, \ldots, n\}\) having \(2m + 1\) elements, with the property that the convex envelope of \(L_P\) contains \(O\).

**Definition 3.2.** We say that an LVMB datum is an LVM datum if it satisfies the previous condition.

Notice that proposition 3.1 can be rephrased as follows:
Proposition 3.3. An LVMB datum \((L, E_{m,n})\) is an LVM datum if and only if

\[ \bigcap_{P \in E_{m,n}} \hat{C}_P \neq \emptyset, \]

where \(\hat{C}_P\) is the interior of the convex envelope of \(L_P\).

In [5], Cupit-Foutou and Zaffran give another characterization of LVM data among LVMB data with a supplementary assumption, called “condition (K),” see proposition 3.2 in [5]. (We say that an LVMB datum \((L, E_{m,n})\) satisfies condition (K) if there exists an affine automorphism of \((\mathbb{C}^m)^*\) which maps each \(\ell_i \in L\) to a vector with integer coefficients.) Theorem 3.10 below can be seen as a generalization of their result. Before we can state it and give a proof, we need to recall some preliminary definitions and statements:

Definition 3.4. A fan \(\Delta\) is called strongly polytopal (for short, in the following, polytopal) if there exists a polytope \(P\) containing \(0\) in its interior, such that \(\Delta\) is the set of cones generated by the faces of \(P\).

Example 3.5. Every complete fan in \(\mathbb{R}^2\) is polytopal.

Shephard gives in [11] a criterion for a fan to be polytopal, we recall it now.

Definition 3.6. (See for instance [7] or [6].) Let \(X = (x_1, ..., x_r) \in \mathbb{R}^n\) be a family of vectors and \(A\) the matrix whose columns are the elements of \(X\) (hence, \(A\) is a matrix with \(r\) columns and \(n\) rows). Also assume that \(\dim \text{Aff } X = r\) where \(\text{Aff } X\) is the affine hull of \(X\). Let \((\alpha_1, ..., \alpha_{r-n})\) be a basis of the kernel of \(A\) and \(B\) the \((r-n) \times n\) matrix whose columns are the \(\alpha_i's\). The family \(X = (\pi_1, ..., \pi_r) \subset \mathbb{R}^{r-n}\) of vectors given by the rows of \(B\) is called a linear transform of \(X\).

Notice that we can not talk about the linear transform of a family \(X\), since the construction depends on the choice of the basis of the kernel of \(A\).

We have the following lemma (see for instance [6]):

Lemma 3.7. Let \(X = (x_1, ..., x_r)\) be a family of vectors as in the previous definition and \(\hat{X} = (\hat{x}_1, ..., \hat{x}_r)\) a linear transform of \(X\). Notice that \(X\) is a linear transform of \(\hat{X}\). Then \(\sum_{i=1} x_i = 0\) (resp. \(\sum_{i=1} \hat{x}_i = 0\)) if and only if the vectors \(x_i\) (resp. \(\hat{x}_i\)) all belong to a hyperplane \(H \subset \mathbb{R}^n\) (resp. \(\subset \mathbb{R}^{r-n}\)) which does not contain 0.

Definition 3.8. Let \(X = (x_1, ..., x_r)\) be a family of vectors in \(\mathbb{R}^n\) which positively span \(\mathbb{R}^n\) (i.e. \(\mathbb{R}_{\geq 0}x_1 + ... + \mathbb{R}_{\geq 0}x_r = \mathbb{R}^n\)). Choose a suitable family \(\lambda_i > 0\) such that \(\sum \lambda_i x_i = 0\) (this is always possible, see [11], p. 258). Choose a linear transform of the family \((\lambda_1 x_1, ..., \lambda_r x_r)\) such that the last coordinate is always equal to 1 and call such a family a Shephard transform of \(X\). Denote it by \(\hat{X} = (\hat{x}_1, ..., \hat{x}_r)\).
Now consider a complete fan $\Delta$ in $\mathbb{R}^n$ and assume that $X = (x_1, \ldots, x_r)$ is a family of vectors which generate the rays of $\Delta$. If $\sigma = \mathbb{R}_{\geq 0}x_{i_1} + \cdots + \mathbb{R}_{\geq 0}x_{i_p}$ is a cone of $\Delta$ of (maximal) dimension $n$, we denote by $\hat{C}(\sigma)$ the relative interior of the convex envelope of $\hat{X} \setminus \{\hat{x}_{i_1}, \ldots, \hat{x}_{i_p}\}$. Then we have the following:

**Theorem 3.9** (Shephard’s criterion). With the notations above, the fan $\Delta$ is polytopal if and only if

$$\bigcap_{\sigma \in \Delta_{\text{max}}} \hat{C}(\sigma) \neq \emptyset,$$

where $\Delta_{\text{max}}$ is the set of all cones of $\Delta$ of maximal dimension $n$.

For the previous definition and theorem we refer to [11]. We are now able to prove the following:

**Theorem 3.10.** Let $(L, E_{m,n})$ be an LVMB datum and $(E, \Delta)$ its associated pair given by theorem 2.2. Then, $(L, E_{m,n})$ is an LVM datum if and only if the projection by $E$ of the fan $\Delta$ is polytopal.

**Proof:** Form a basis of $\mathbb{R}^n$ with first $2m$ vectors being the vectors $X_k, Y_k$ defined in equation (2) which generate $E$ and, for last $n - 2m$ vectors, any of the canonical basis. Call $B'$ this basis, and $B$ the canonical basis of $\mathbb{R}^n$:

$$B' = (X_1, \ldots, Y_{2m}, e_{i_1}, \ldots, e_{i_{n-2m}}).$$

We have the direct sum decomposition $\mathbb{R}^n = E \oplus F$ where $F \cong \mathbb{R}^{n-2m}$. The projection onto $F$ with respect to $E$ is given by the matrix $P := J_{n,m}P^{-1}$ where $P$ is the invertible matrix whose columns are the vectors of $B'$ and

$$J_{n,m} = \begin{pmatrix} 0_{n-2m,2m} & I_{n-2m} \end{pmatrix}.$$  

(Here, $0_{n-2m,2m}$ is the zero matrix with $n - 2m$ rows and $2m$ columns, and $I_{n-2m}$ is the identity matrix of size $n - 2m$.) We now see that the vectors $e_1, \ldots, e_n$ and $e_0 = -(e_1 + \cdots + e_n)$ are sent by $P$ respectively to the $n$ columns of $P$ and the opposite of their sum.

We now separate cases, depending on the number of indispensable $\ell_i$’s (note that there can be between 0 and $2m$ indispensable $\ell_i$’s).

First, assume that there is no indispensable $\ell_i$. It is readily seen that the vectors $(P.e_0, P.e_1, \ldots, P.e_n)$ generate the rays of the (complete) projected fan $\pi(\Delta)$ of $F \cong \mathbb{R}^{n-2m}$. It is also straightforward to check that a Shephard transform of this family of vectors is given by the vectors $\tilde{\ell}_i := (\ell_i, 1) \in \mathbb{R}^{2m+1}$, for $i = 0, \ldots, n$. Now, the use of both Shephard and Bosio criteria (theorem 3.9 and proposition 3.1) gives the equivalence between the polytopality of $\pi(\Delta)$ and the fact that $(L, E_{m,n})$ is LVM.

Assume now that there is one indispensable $\ell_i$, say $\ell_0$. First, notice that it implies that the 1-dimensional cones of the fan $\Delta$ are generated by $e_1, \ldots, e_n$.
(and not \(e_0\)). Then, in order to characterize the polytopality of \(\pi(\Delta)\), one must compute a Shephard transform of \((\Pi.e_1, \ldots, \Pi.e_n)\), which we do in two steps. First, we project the vectors \(\hat{\ell}_i\) (for \(i = 1, \ldots, n\)) with respect to \(\mathbb{R}\cdot \hat{\ell}_0\). This procedure leads to a linear transform of \((\Pi.e_1, \ldots, \Pi.e_n)\). Recall that the vectors \(\hat{\ell}_i\) belong to the affine hyperplane \(H := \{ (x_1, \ldots, x_{2m+1}) \in \mathbb{R}^{2m+1} \mid x_{2m+1} = 1 \}\), so we can define \(H_0\) to be the hyperplane \(H - \hat{\ell}_0\), then one sees that \(H_0\) is a linear subspace of \(\mathbb{R}^{2m+1}\), and that \(\hat{\ell}_0 \notin H_0\). Call \(\pi_0 : \mathbb{R}^{2m+1} \to H_0\) the projection with respect to \(\mathbb{R}\cdot \hat{\ell}_0\). Endow \(H\) with a vector space structure by choosing \(\hat{\ell}_0\) as its origin, then the restriction of \(\pi_0\) between \(H\) and \(H_0\) is an isomorphism. Remark that if \(\ell_0\) is indispensable, it can not be an element of the convex envelope of \(\{\ell_1, \ldots, \ell_n\}\). Indeed, since \((\Pi.e_1, \ldots, \Pi.e_n)\) positively spans \(\mathbb{R}^{n-2m}\), one has \(\pi_0(\ell_0) \notin \text{conv}(\pi_0(\ell_1), \ldots, \pi_0(\ell_n))\) (see [11]). Hence, there exists a hyperplane \(H'_0\) of \(H_0\) containing \(2m\) elements of \(\{\pi_0(\ell_1), \ldots, \pi_0(\ell_n)\}\), separating \(\pi_0(\ell_0) = 0 \in H_0\) and \(\{\pi_0(\ell_1), \ldots, \pi_0(\ell_n)\}\). See figure 2 for a picture of the situation.

![Figure 2. The projection \(\pi_0\)](image-url)

The second step we have to do in order to obtain a Shephard transform of \((\Pi.e_1, \ldots, \Pi.e_n)\) is to multiply each vector of \((\pi_0(\ell_1), \ldots, \pi_0(\ell_n))\) by suitable scalars such that they all belong to \(H'_0\). Call \(L' := (\ell'_1, \ldots, \ell'_n)\) the vectors
obtained that way. Remark that any family of $2m$ vectors of this set forms an affine basis for $H'_0$. For $P = (0, i_1, ..., i_{2m}) \in \mathcal{E}_{m,n}$, call $P' := (i_1, ..., i_{2m})$ and denote by $\mathcal{E}'_{m,n}$ the set \{ $P' \mid P \in \mathcal{E}_{m,n}$ \}. It is clear that $\mathcal{E}'_{m,n}$ still satisfies Bosio’s SEP (with $\{1, ..., n\}$ instead of $\{0, ..., n\}$). We show that to a point in the convex envelope of a family of points $\mathcal{L}_P$ it is possible to associate a point in the convex envelope of $\mathcal{L}'_{P'}$ and reciprocally; this fact proves that the imbrication condition is also satisfied by $(\mathcal{L}', \mathcal{E}'_{m,n})$. Assume $x_0 \in \text{conv}(\mathcal{L}_{P'})$ for $P' \in \mathcal{E}'_{m,n}$, then the point $\frac{1}{2}x_0$ is an element of $\text{conv}(\mathcal{L}_P)$. Reciprocally if $x_0 \in \text{conv}(\mathcal{L}_P)$ (for $P \in \mathcal{E}_{m,n}$), there exists a unique $\lambda > 0$ (which does not depend on $P$) such that $\lambda x_0 \in H'_0$. Write $x_0 = \sum_{j=1}^{2m} \lambda_j \beta_j \ell_j'$ (where $\beta_j \ell_j' = \pi_0(\ell_i)$), and $\lambda x_0 = \sum_{j=1}^{2m} \alpha_j \lambda \ell_j'$ with $\sum_j \alpha_j = 1$. Since any subfamily of $\mathcal{L}'$ of $2m$ elements is an $\mathbb{R}$-affine basis of $H'_0$, one has $\lambda \lambda_j \beta_j = \alpha_j$, hence $\alpha_j \geq 0$ for all $j$, that is $\lambda x_0 \in \text{conv}(\mathcal{L}_{P'})$.

An important consequence of this remark is also that $\bigcap_{P' \in \mathcal{E}'_{m,n}} \text{conv}(\mathcal{L}_{P'}) \neq \emptyset$ if and only if $\bigcap_{P \in \mathcal{E}_{m,n}} \text{conv}(\mathcal{L}_P) \neq \emptyset$.

Now, the consecutive use of Shephard’s criterion and Bosio’s criterion applied to the Shephard transform that we have computed gives us that $\pi(\Delta)$ is polytopal if and only if $(\mathcal{L}, \mathcal{E}_{m,n})$ is LVM.

The same reasoning now applies if there are up to $2m-1$ indispensable $\ell_i$’s, say $\ell_0, ..., \ell_k$ by applying the previous method “recursively”: to compute a Shephard transform of $(\Pi.e_{k+1}, ..., \Pi.e_n)$, one can compute a Shephard transform of $(\Pi.e_1, ..., \Pi.e_n)$ just as above, then project it with respect to $\mathbb{R}\Pi.e_1$ and multiply by the proper scalars, which gives a Shephard transform of $(\Pi.e_2, ..., \Pi.e_n)$ and so on. Call $(\mathcal{L}^{(j)}, \mathcal{E}^{(j)}_{m,n})$ the families obtained at each step (for $j = 1, ..., k+1$) and $(\mathcal{L}^{(0)}, \mathcal{E}^{(0)}_{m,n}) := (\mathcal{L}, \mathcal{E}_{m,n})$. Notice (just as above) that for $j = 1, ..., k+1$, the intersection $\bigcap_{P \in \mathcal{E}^{(j)}_{m,n}} \text{conv}(\mathcal{L}^{(j)}_P)$ is non-empty if and only if $\bigcap_{P \in \mathcal{E}^{(j-1)}_{m,n}} \text{conv}(\mathcal{L}^{(j-1)}_P)$ is non-empty. Then, using Shephard and Bosio criteria again, we obtain the result.

Finally, assume there are $2m$ indispensable items, say $\{0, ..., 2m-1\}$ for simplicity. The SEP condition then forces that

$$\mathcal{E}_{m,n} = \{(0, ..., 2m-1, j) \mid j \in \{2m, ..., n\}\}.$$ 

The second Bosio condition (imbrication) implies that the hyperplane $H$ containing $\{0, ..., \ell_{2m-1}\}$ is a supporting hyperplane for the polytope obtained as the convex envelope of $\mathcal{L}$. As it can be easily seen, there is exactly one bounded connected component $O$ of $\mathcal{O}$ such that $\text{Aff}(\overline{O \cap H}) = H$, .
and this component lies in the intersection of all the convex envelopes of 
\{L_P, P \in E_{m,n}\}, hence the LVMB datum we consider is LVM by proposition \[3.1\]. On the other hand, the fact that there are \(2m\) indispensable elements says that the projected fan has exactly \(n + 1 - 2m\) generating rays and each cone of maximal dimension is generated by \(n - 2m\) vectors. Such a complete fan in \(\mathbb{R}^{n-2m}\) is always polytopal. \[\blacksquare\]

4. WHEN ARE TWO LVMB MANIFOLDS BIHOLOMORPHIC?

It is possible that two different LVMB data give rise to two biholomorphic manifolds, this is why we make a distinction between LVMB data and manifolds. In this section we prove that if an LVMB manifold \(X_{n,m}\) built with an LVMB datum \((L,E_{m,n})\) is biholomorphic to an LVM manifold, then the datum \((L,E_{m,n})\) itself is LVM. This problem was raised and partly answered by Cupit-Foutou and Zaffran in [5]. For the proof, we will use the criterion given by theorem \[3.10\].

We now study the case when two LVMB data \((L_1,E_{m_1,n_1})\) and \((L_2,E'_{m_2,n_2})\) give the same manifold (up to biholomorphism).

Let \(X\) be an LVMB manifold. Let \(A := \text{Aut}_\mathcal{O}(X)^o\) be the identity component of the group of holomorphic automorphisms of \(X\) which is a complex Lie group because \(X\) is compact (this is a result of Bochner and Montgomery, see [2]).

We first prove the following

**Lemma 4.1.** Let \((L_1,E_{m_1,n_1})\) and \((L_2,E'_{m_2,n_2})\) be two LVMB data such that the corresponding LVMB manifolds are biholomorphic. Then, \(m_1 = m_2\) and \(n_1 = n_2\).

**Proof:** Assume that \(X_1\) and \(X_2\) are two isomorphic LVMB manifolds obtained as quotient of two open sets \(U_1 \subset \mathbb{P}^{m_1}(\mathbb{C})\) and \(U_2 \subset \mathbb{P}^{m_2}(\mathbb{C})\) by two closed subgroups \(H_1 \cong \mathbb{C}^{m_1} \subset (\mathbb{C}^*)^{m_1}\) and \(H_2 \cong \mathbb{C}^{m_2} \subset (\mathbb{C}^*)^{m_2}\). Call \(\varphi\) a biholomorphism between \(X_1\) and \(X_2\). It induces an isomorphism 

\[\varphi^* : \text{Aut}_\mathcal{O}(X_2)^o =: A_2 \rightarrow A_1 := \text{Aut}_\mathcal{O}(X_1)^o,\]

given by \(\varphi^*(g) = \varphi^{-1} \circ g \circ \varphi\).

Then, \(G_i := (\mathbb{C}^*)^{m_i}/H_i\) is a subgroup of \(A_i\) and has an open dense orbit \(G_i x_i \subset X_i\) for \(i = 1, 2\) (see [3], proposition 2.4). Call \(\pi_i : (\mathbb{C}^*)^{m_i} \rightarrow (\mathbb{C}^*)^{m_i}/H_i\) the two quotient maps. Now the subgroup \(T_1 := \pi_1((\mathbb{S}^1)^{m_1}) \subset G_1\) (resp. \(T_2 = \pi_2((\mathbb{S}^1)^{m_2}) \subset G_2\) is contained in a maximal torus of \(A_1\) (resp. \(A_2\)), say \(\tilde{T}_1\) (resp. \(\tilde{T}_2\)). The two maximal tori \(\tilde{T}_1\) and \(\varphi^*(\tilde{T}_2)\) of \(A_1\) are conjugated (see, for instance, [4]) so, up to conjugacy, we can assume that \(T_1\) and \(\varphi^*(T_2)\) are contained in the same maximal torus \(\tilde{T}\) of \(A_1\). Now we
denote by $\tilde{T}^C$ the complexification of $T$, and both $G_1$ and $\varphi^*(G_2)$ are Lie subgroups of $\tilde{T}^C$ (because $T_i^C = G_i$ for $i = 1, 2$).

Suppose we have $\dim_C \tilde{T}^C > \dim_C G_1 = \dim_C X_1$, then the action of $A_1$ is not effective anymore (in this case indeed, there exists a non-trivial element $f \in \tilde{T}^C$ such that $f(x_1) = x_1$ and since $\tilde{T}^C$ is abelian (and contains $G_1$), $f$ is the identity map on all $G_1, x_1$, hence on all $X_1$ by analytic continuation), a contradiction. Hence we necessarily have $\dim_C \tilde{T}^C = \dim_C G_1 = \dim_C \varphi^*(G_2)$ and $G_1 = \varphi^*(G_2)$. In particular, these two Lie groups have isomorphic fundamental groups, $\mathbb{Z}^{n_1}$ and $\mathbb{Z}^{n_2}$ respectively, which leads to $n_1 = n_2$ and since $n_1 - m_2 = n_2 - m_2$, we also have $m_1 = m_2$. 

Now we prove the following

**Theorem 4.2.** Let $(\mathcal{L}_1, E_{m,n})$ and $(\mathcal{L}_2, E'_{m',n'})$ be two LVMB data giving two biholomorphic LVMB manifolds, then $n = n'$, $m = m'$ and $(\mathcal{L}_1, E_{m,n})$ is an LVMB datum if and only if $(\mathcal{L}_2, E'_{m,n})$ is an LVMB datum.

**Proof:** We keep the notations of the previous proof. Up to conjugacy, we can assume that $(\varphi^{-1})^*(G_1) = G_2$ and $\varphi(G_1, x_1) = G_2, x_2$, where $G_1, x_1$ is the open dense orbit of the action of $G_1$ on $X_1$. Call $(E, \Delta)$ and $(E', \Delta')$ the toric data associated respectively to these two LVMB data and let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n/E$ and $\pi' : \mathbb{R}^n \rightarrow \mathbb{R}^n/E'$ be the corresponding projections.

The maximal compact tori of $G_1$ and $G_2$ are isomorphic to $(S^1)^n$, hence the restriction of $(\varphi^{-1})^*$ to these maximal tori is given by a matrix $A \in \text{GL}(n, \mathbb{Z})$, and it extends to an isomorphism of $(\mathbb{C}^*)^n$ which we call $\tilde{\varphi}$.

As one can check by passing to the Lie algebras, the following diagram is commutative:

\[
\begin{array}{ccc}
(C^*)^n & \xrightarrow{\tilde{\varphi}} & (C^*)^n \\
\downarrow_{\iota_1} & & \downarrow_{\iota_2} \\
U_1 & \xrightarrow{p_1} & U_2 \\
\downarrow_{\varphi} & & \downarrow_{p_2} \\
X_1 & \xrightarrow{X_2,} &
\end{array}
\]

where $\iota_1$ and $\iota_2$ are the canonical injections.

By using the Riemann extension theorem, one sees that the application $p_2 \circ \iota_2 \circ \tilde{\varphi} = \varphi \circ p_1 \circ \iota_1$ extends on $U_1$, that is, we can extend $\tilde{\varphi}$ to a toric application between $U_1$ and $U_2$, so it induces a (bijective) linear map from $\mathbb{R}^n$ to itself (given by the matrix $A$), which maps the fan $\Delta$ of $\mathbb{R}^n$ defining the toric manifold $U_1$ to the fan $\Delta'$ of $\mathbb{R}^n$ defining $U_2$. (10, theorem 1.13). The commutativity of diagram (1) implies that this linear map induces a linear map from $\mathbb{R}^n/E \cong \mathbb{R}^{n-2m}$ to $\mathbb{R}^n/E' \cong \mathbb{R}^{n-2m}$ sending $\pi(\Delta)$ to $\pi'(\Delta')$. This means that these two projected fans are simultaneously polytopal and
the conclusion now comes from theorem 3.10.

5. A generalization of Bosio’s construction

5.1. Detecting all open subsets with a compact quotient. The following proposition tells us that Bosio’s construction is the most general in this context, in the sense that any open subset $U \subset \mathbb{P}^n(\mathbb{C})$ with an action of $\mathbb{C}^m$ such that the quotient is a complex compact manifold is in fact given by a subfan of $\Delta_{\mathbb{P}^n(\mathbb{C})}$.

**Proposition 5.1.** Let $G \cong \mathbb{C}^m \subset (\mathbb{C}^*)^n$ be a closed subgroup and $U \subset \mathbb{P}^n(\mathbb{C})$ be an open subset such that $U/G$ is a compact manifold. Then $U$ is stable by $(\mathbb{C}^*)^n$, therefore given by a subfan of the fan of $\mathbb{P}^n(\mathbb{C})$.

**Proof:** Let $\pi$ be the quotient map $\pi: U \to U/G$ and let $v_1, \ldots, v_n$ be a basis of the Lie algebra of $(\mathbb{C}^*)^n$ such that $v_1, \ldots, v_m$ is a basis of the Lie algebra of $G$. We define the vector fields $v_1^*, \ldots, v_n^*$ on $U/G$ by $v_i^*(x) := d\pi_y(v_i(y))$ for $y \in \pi^{-1}(x)$. Since $U$ is stable by $G$, it is enough to show that $U$ is invariant by the action of $v_{m+1}, \ldots, v_n$. Assume this is not the case, say for instance that $U$ is not stable under the action of $v_{m+1}$: there exists a point $z_0 \in U$, a holomorphic map $\gamma: \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ and $t_0 \in \mathbb{C}$ such that $\dot{\gamma}(0) = v_{m+1}(z_0)$, $\gamma(0) = z_0$ and $\gamma(t_0) \not\in U$. Call $\Omega := \gamma^{-1}(U)$. Since $U/G$ is compact, the vector field $v_{m+1}^*$ is complete and $\pi(\gamma|_\Omega)$ is an integral curve of $v_{m+1}^*$ passing through $\pi(z_0)$ hence it can be extended to $\mathbb{C}$, a contradiction because then $\gamma(t_0) \in U$. ■

5.2. A generalization. In light of the proof of proposition 5.1 where the property of $\mathbb{P}^n(\mathbb{C})$ used is the fact that it is a compact toric variety, we can study a “toric” open subset of any compact toric manifold:

**Theorem 5.2.** Let $\Delta$ be a finite rational fan in $\mathbb{R}^n$ and $E \cong \mathbb{R}^{2m}$ be a linear subspace of $\mathbb{R}^n$ such that:
- the projection map $\pi: \mathbb{R}^n \to \mathbb{R}^n/E \cong \mathbb{R}^{n-2m}$ is injective on $|\Delta|$,
- the fan $\pi(\Delta)$ is complete in $\mathbb{R}^n/E$, i.e. $|\pi(\Delta)| = \mathbb{R}^n/E$.

Define a closed subgroup $G \cong \mathbb{C}^m$ of $(\mathbb{C}^*)^n$ in the same way as in section 2.2.3. Then, the quotient $X_{\Delta}/G$ exists and it is a complex compact manifold.

**Proof:** Since $\Delta$ is finite we can construct a complete rational fan $\Delta'$ containing $\Delta$ as a subfan. The group $G$ acts properly and freely on $X_{\Delta}$ by lemma 2.3 hence we know that $X_{\Delta}/G$ is a complex manifold. Compactness is a consequence of the completeness of $\pi(\Delta)$. ■
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