Collapse and Revival of ‘Schrödinger Cat’ States

C. E. A. Jarvis, D. A. Rodrigues, B. L. Györffy, T. P. Spiller, A. J. Short, and J. F. Annett

1 H H Wills Physics Laboratory, University of Bristol, Bristol BS8 1TL, United Kingdom
2 School of Physics and Astronomy, University of Nottingham, Nottingham, NG7 2RD, United Kingdom
3 Hewlett Packard Laboratories, Filton Road, Bristol, BS34 8QZ, United Kingdom
4 DAMPT, Center of Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom

We study the dynamics of the Jaynes-Cummings Model for an array of $N_q$ two level systems (or qubits) interacting with a quantized single mode electromagnetic cavity (or quantum bus). For an initial cavity coherent state $|\alpha\rangle$ and the qubit system in a specified ‘basin of attraction’ in its Hilbert space, we demonstrate the oscillation of a superposition of two macroscopic quantum states between the qubit system and the field mode. From the perspective of either the qubit or the field system, there is collapse and revival of a ‘Schrödinger Cat’ state.

Quantum superposition of macroscopically different states of matter is of fundamental conceptual interest in many fields of physics such as Measurement Theory [1], Quantum Optics [2], Macroscopic Quantum Tunnelling [3] and Quantum Computation [4]. Many of the relevant experiments revolve around the preparation of and measurement on a specific class of states called ‘Schrödinger Cat’ states [5]. These are quantum superpositions of states which correspond to two (or more) different values of a macroscopic variable, such as the magnetization or the electric field in the cases of a large spin cluster, or large photon number or phase in a cavity mode. In this letter we report the surprising discovery that in a Jaynes-Cummings model (JCM), which describes an array of two-level systems (qubits) interacting with a single cavity mode, the time evolution can be such that sometimes the radiation field and sometimes the qubit subsystem is in a ‘Schrödinger Cat’ state. In other words, a superposition of macroscopically different states can shift from one set of physical variables to another, as a function of time.

The quantum dynamics of two-level systems (qubits), coupled to a single mode of an electromagnetic cavity, arise in many different physically interesting systems. These include Rydberg atoms [6], NMR studies of atomic nuclei [6, 7], Cooper Pair Boxes [8], Cavity Quantum Electrodynamics [9], trapped ions [10] and Quantum Computing [11]. A very general and simple Hamiltonian that captures the relevant physics in all these fields is the JCM [11] (for one qubit) and its generalization for multi-qubit systems by Tavis and Cummings [12]. Thus, our results are pertinent to a broad range of physical systems.

One of the most interesting and surprising predictions of the JCM is the ‘collapse and revival’ of Rabi oscillations of the occupation probabilities for various qubit states as the system evolves, from an initial state which is a product of a coherent state $|\alpha\rangle$ for the radiation field, and a generic qubit state $|\psi_{N_q}\rangle$ [2], where $N_q$ is the number of qubits. These remarkable dynamics occur only because both the matter and the cavity field are treated fully quantum mechanically. Indeed, our aim here is to study the ‘collapse and revival’ of ‘Schrödinger cat’-like states in a multi-qubit subsystem, as well as in the cavity field.

For clarity let us specify the multi-qubit JCM Hamiltonian, where each qubit labelled $i$ has ground (excited) state $|g_i\rangle$ ($|e_i\rangle$) with energy $\epsilon_{g,i}$ ($\epsilon_{e,i}$). Up to a constant, the Hamiltonian has the form

$$\hat{H} = \hbar \omega \hat{a}^\dagger \hat{a} + \frac{\hbar}{2} \sum_{i=1}^{N_q} \Omega_i \hat{\sigma}_z^i + \hbar \sum_{i=1}^{N_q} \lambda_i (\hat{a} \hat{\sigma}_z^i + \hat{a}^\dagger \hat{\sigma}_z^i)$$

where $\hat{\sigma}_z^i = |e_i\rangle\langle e_i| - |g_i\rangle\langle g_i|$, $\hat{\sigma}_z^i = |g_i\rangle\langle g_i| - |e_i\rangle\langle e_i|$, $\lambda_i (\hat{a})$ is the creation (annihilation) operator for a photon with frequency $\omega$. The cavity-qubit $i$ coupling constant is $\lambda_i$ and $\hbar \Omega_i = \epsilon_{e,i} - \epsilon_{g,i}$. Here we consider only the cases of resonance, so $\omega = \Omega_i$ for all $i$, and uniform coupling, so $\lambda_i = \lambda$.

The celebrated ‘collapse and revival’ can be observed in the one-qubit case [2]. It follows from an initial system state of $|\Psi_1(0)\rangle = |\psi_1\rangle |\alpha\rangle$, where $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$, $\alpha = \sqrt{\Omega \tau} e^{-i\theta}$, $\tau$ is the average number of photons in the field and $|\psi_1\rangle = (C_g |g\rangle + C_e |e\rangle)$. The Rabi oscillations (in the probability of the qubit being in its initial state) demonstrate collapse, on a time scale of $t_c \approx \frac{\hbar}{\epsilon_{g,e}}$, and then revival at $t_r \approx 2\sqrt{\frac{\pi}{\lambda \Omega}}$.

This is illustrated in Fig. 1 for $C_g = 1$, $C_e = 0$ by plotting $\sum_{n=0}^{\infty} \langle g,n|\Psi_1(t)\rangle^2$, where $\langle g,n|\Psi_1(t)\rangle$ corresponds to the qubit in its ground state with $n$ photons in the cavity.

A second notable feature of this time evolution, discovered by Gea-Banacloche [13], is that at $t = \frac{1}{2} t_r$, $|\Psi_1(\frac{1}{2} t_r)\rangle$ again factorises into a qubit part $|\psi\rangle^\dagger_{\text{att}}$ and a cavity part $|\Phi(\frac{1}{2} t_r)\rangle$. Moreover, remarkably, the former is given by

$$|\psi\rangle_{\text{att}}^\pm = \frac{1}{\sqrt{2}} (e^{-i\theta} |e\rangle \pm i |g\rangle)$$

where $\theta$ is the phase of the initial coherent state, for all initial conditions such that $|C_g|^2 + |C_e|^2 = 1$. Under same conditions $|\psi\rangle_{\text{att}}^-$ is attained at $t = \frac{3}{2} t_r$. Because of this strikingly non-linear behavior, following Phoenix and Knight [14], we shall refer to the states Eq. 2 as ‘attractors’. The probability that the qubit is in...
the state $|\psi\rangle^+, \text{ given by } \sum_{n=0}^\infty \langle \psi_{\text{att}}^+, n | \Psi_1 (t) \rangle |^2$ is also shown in Fig. 1 together with the von Neumann entropy $S^q(t) = -\text{Tr} (\rho^q(t) \ln \rho^q(t))$ associated with the qubit density matrix $\rho^q(t) = \text{Tr}_F (|\Psi_1(t)\rangle \langle \Psi_1(t)|)$, reduced by tracing over the initial state $|\psi\rangle$. Clearly, at $t = \frac{t_r}{2}$ the entropy $S^q(t)$ approaches zero, and the analytical solution shows that in the limit $\bar{n} \to \infty$ the entropy goes to zero, indicating that the radiation field and the qubit are not entangled.

Prompted by these results, we have investigated the $N_q > 1$ qubit evolution, starting in a state $|\Psi_{N_q}(0)\rangle = |\psi_{N_q}\rangle (\alpha)$. Analytically, in the large $\bar{n}$ limit, we have found that the states

$$|\psi_{N_q}\rangle^\pm = \frac{1}{\sqrt{2^{N_q}}} \left( e^{-i\theta} |e\rangle \pm i|g\rangle \right)^\otimes N_q$$

(3)

can also be regarded as ‘attractors’ in a similar, dynamical, sense as outlined above. The only difference is that in the $N_q > 1$ case, $|\psi_{N_q}\rangle^\pm$ only occurs at $t = t_r/2N_q$ for a restricted range of initial conditions, which we shall term the ‘basin of attraction’. As before, at half way to revival of the initial state, namely $t = t_r/2N_q$, the radiation field and the qubit system are not entangled. However, now the question of entanglement between the qubits arises. Compared to the one-qubit problem studied by Gea-Banacloche [13], this is a new feature of the multi-qubit case. Clearly, as $|\psi_{N_q}\rangle^\pm$ is a simple product of individual qubit states when the qubit subsystem is in this state, the qubits are not entangled with each other. This is surprising because, as we shall show later, this attractor state can be reached from initial states with arbitrary entanglement between qubits.

Before examining how such interesting dynamics can occur, it is useful to note that the above product state is a spin coherent state for a finite $N_q$-qubit system. Following Radcliffe [16] we define such states as

$$|\beta, N_q\rangle = \frac{1}{\sqrt{2^{N_q}}} \sum_{m=N_q/2}^{N_q/2} \sqrt{\binom{N_q}{m}} \beta^{N_q-m} |N_q, m\rangle$$

(4)

where the states $|N_q, m\rangle$ are the fully symmetrized $N_q$ qubit states, for $N_e$ qubits excited and $N_q$ in the ground state, with $m = \frac{N_q-N_e}{2}$ and $\beta$ a complex number that characterizes the the state. The normalization and combinatoric factors are $N = \left(1 + |\beta|^2\right)^{N_q/2}$ and

$$C_{N_q} = \frac{N_q!}{(\frac{N_q}{2}-m)! (\frac{N_q}{2}+m)!}.$$ 

As can be readily shown, the ‘attractor’ states identified in Eq. 3 are given by $|\beta = \pm i e^{i\theta}, N_q\rangle$. Below we explore the implications of this observation for the dynamics of qubit states in the ‘basin of attraction’.

Let us now investigate the ‘basin of attraction’ for the simplest multi-qubit system, with $N_q = 2$. In this case the time evolution described by $|\Psi_2(t)\rangle$ is readily found [17]. For the most general, normalized, initial state

$$|\psi_2\rangle = C_{ee} |ee\rangle + C_{eg} |eg\rangle + C_{ge} |ge\rangle + C_{gg} |gg\rangle$$

(5)

the exact analytical solution will be given elsewhere [18]. Here we consider only the sector determined by the restrictions: $a = e^{i\theta} C_{ee} = e^{-i\theta} C_{gg}$ and $\sqrt{1 - |a|^2} = C_{eg} = C_{ge}$. As will be illustrated presently, these define the ‘basin of attraction’ for the ‘attractor’ $|\psi_2\rangle^\pm$. Namely, for any complex number $a$ satisfying the condition $0 \leq |a| \leq 1/\sqrt{2}$ in

$$|\psi_2\rangle = a (e^{-i\theta} |ee\rangle + e^{i\theta} |gg\rangle) + \sqrt{1 - |a|^2} (|eg\rangle + |ge\rangle) ,$$

(6)
we found the ‘basin of attraction’ for all the attractor states identified in Eq. 4. We have established that, for a given \( N_q \), only initial states parametrized as follows:

\[
|\psi(t)\rangle_a = \sum_{m=-N_q/2}^{N_q/2} A(N_q, a) e^{-i(N_q/2 - m)\theta} \sqrt{\frac{N_q}{2} + m} \langle \frac{N_q}{2} - m | N_q, m \rangle
\]

where \( 0 \leq |a| \leq \frac{\sqrt{2N_q}}{N_q} \), will evolve into the spin coherent state \( |\beta = e^{-i\theta}N_q \rangle \) at the time \( t = t_r/2N_q \).

A remarkable feature of this ‘basin of attraction’ is that it includes ‘Schrödinger cat’ states of the qubit system. Indeed it can be readily shown, by carrying out the sum over \( m \) in Eq. (7) and identifying two different spin coherent states, that

\[
A(N_q, a) = \begin{cases} 
\sqrt{1 - |a|^2} & \text{if } k \text{ is even} \\
\frac{1}{\sqrt{2N_q}} - |a|^2 & \text{if } k \text{ is odd}
\end{cases}
\]

Therefore the ‘basin of attraction’ consists of linear superpositions of two quite different spin-coherent states, Eq. 4 \( |e^{-i\theta}N_q \rangle \) and \( |-e^{-i\theta}N_q \rangle \). For \( N_q \leq 3 \) these two states are hard to distinguish, but for large \( N_q \) they can be regarded as macroscopically different and hence their superposition is very similar to the finite \( N_q \)-qubit ‘Schrödinger cat’ state with equal coefficients. Indeed for \( a = 0 \) and \( a = \frac{\sqrt{2N_q}}{N_q} \), \( |\psi_N\rangle_a = |\beta, N_q \rangle \equiv |\beta, N_q \rangle_{\text{Sch}} \) for \( \beta = e^{-i\theta} \).

Thus, the dynamics governed by the JCM Hamiltonian can transfer an initial state which is a product of a coherent state for the radiation field, \( |\alpha\rangle \), and a highly quantum mechanical ‘Schrödinger cat’-like state for the qubits, at the time \( t^* = \frac{1}{2N_q} \), into another product state where the qubit component is a rather classical qubit (spin) coherent state, \( |\beta\rangle \), with no entanglement between the qubits. Given this intriguing time evolution, it is interesting to examine the state of the radiation field at \( t^* \). Again in the limit of large average photon number \( \overline{N} \) in the cavity we find

\[
\Phi(N_q) \left( \frac{t_r}{2N_q} \right) = \left[ \left( a - \frac{1}{2N_q - 1} - |a|^2 \right) e^{i\pi|\alpha|^2/2} |\alpha\rangle - \left( a + \frac{1}{2N_q - 1} - |a|^2 \right) e^{-i\pi|\alpha|^2/2} |\alpha\rangle \right] \sqrt{2N_q - 2}
\]

Evidently, this is a ‘Schrödinger cat’-like state for the radiation field. Hence, we may conclude that the initial
quantum information which was encoded in the qubit ‘Schrödinger cat’ state moves to and resides in the radiation field at $t^*$. As might now be expected, the time evolution after $t^*$ returns this information to the qubits, as the initial state revives. This remarkable time evolution is illustrated in Fig. 4.

In summary, we have studied the dynamics governed by the JCM for an array of qubits interacting with a single mode of quantized radiation field, a quantum bus. We have shown that a product state, comprising a coherent state for the photons and a macroscopic ‘Schrödinger cat’-like state for the qubits, transforms into a product of a ‘Schrödinger cat’-like state for the photons and a spin-coherent state for the qubits. Furthermore, as time goes on this transformation is reversed and then the process starts all over again. This suggests that some universal quantum information is being passed back and forth between the two subsystems, each of which manifests it through its own physics. Further study of this intriguing ‘collapse and revival’ phenomenon should include an investigation of the effects of decoherence and dissipation, of which initial studies have been made by Meunier et al. [22], and design of experiments to observe it.

The work of C.E.A.J. was supported by UK HP/EPSCC case studentship, and D.A.R. was supported by EPSRC-GB grant no EP/D06417/1. A.J.S. acknowledges support from a Royal Society University Research Fellowship and the EC QAP project. We thank the ESF network AQDJJ for partial support.

---

**References**

[1] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, 1983).
[2] C. C. Gerry and P. L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005).
[3] S. Takagi, *Macroscopic Quantum Tunneling* (Cambridge University Press, Cambridge, 2002).
[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2001).
[5] S. Haroche and J. M. Raimond, *Cavity Quantum Electrodynamics* (Academic Press, London, 1994), p.123.
[6] C. P. Slichter, *Principles of Magnetic Resonance* (Springer-Verlag, Berlin, 1978).
[7] L. P. Pryadko and G. Quiroz, Phys. Rev. A. **77**, 012330 (2008).
[8] A. Wallraff, D. I. Schuster, A. Blaies, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin and R. J. Schoelkopf, Nature **431**, 162 (2004).
[9] P. R. Berman, ed., *Cavity Quantum Electrodynamics* (Academic Press, London, 1994).
[10] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[11] E. T. Jaynes and F. W. Cummings, Proc IEEE **51**, 89 (1963).
[12] M. Tavis and F. W. Cummings, Phys. Rev. **170**, 279 (1968).
[13] J. Gea-Banacloche, Phys. Rev. Lett. **65**, 3385 (1990).
[14] S. J. D Phoenix and P. L. Knight, Phys. Rev. A **44**, 6023 (1991).
[15] J. Gea-Banacloche, Phys. Rev. A **44**, 5913 (1991).
[16] J. M. Radcliffe, J. Phys. A: Gen. Phys. **4**, 313 (1971).
[17] S. M. Chumakov, A. B. Klimov and J. J. Sanchez-Mondragon, Optics Commun. **118**, 529 (1995).
[18] C. E. A. Jarvis, D. A. Rodrigues, B. L. Györfy, T. P. Spiller, A. J. Short and J. F. Annett, In preparation (2008).
[19] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[20] W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, Phys. Rev. A **64**, 030302(R) (2001).
[21] D. A. Rodrigues, B. L. Györfy and T. P. Spiller, J. Phys.: Condens. Matter **16**, 4477 (2004).
[22] T. Meunier, A. Le Difon, C. Ruef, P. Degiovanni and J.-M. Raimond, Phys. Rev. A **74**, 033802 (2006).
[23] C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer, Berlin, 1991).
[24] $Q_\beta(\beta, t) \propto \langle \beta, N_q | \rho(t) | \beta, N_q \rangle$. 

---

**FIG. 4**: (color online) Diagrams of the $Q$ function [23] (left) and spin $Q$ function [24] (right) at three different times. (a) the time $t = 0$, where the cavity is in a coherent state and the qubits are in a spin Schrödinger cat state. (b) the time $t = t_r/2N_q$ which is the time of the first attractor. (c) the time $t = t_f/N_q$, when the field states are again overlapping and in a coherent state. They show the ‘Schrödinger Cat’ state moving from the qubits to the radiation field and back again in the limit $\pi \to \infty$, $N_q = 40$, $\theta = 0$. The $Q$ function for the field has been scaled to a unit circle.