Asymptotic Density of Open p-brane States with Zero-modes included

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ABSTRACT

We obtain the asymptotic density of open $p$-brane states with zero-modes included. The resulting logarithmic correction to the $p$-brane entropy has a coefficient $-\frac{p+2}{2p}$, and is independent of the dimension of the embedding spacetime. Such logarithmic corrections to the entropy, with precisely this coefficient, appear in two other contexts also: a gas of massless particles in $p$-dimensional space, and a Schwarzschild black hole in $(p+2)$-dimensional anti de Sitter spacetime.
1. The asymptotic density of states $\rho(N)$ at level $N$, $N \gg 1$, for $p$-branes compactified on $(S^1)^p \times \mathbf{R}^{D-p}$ has been calculated within the semiclassical quantisation scheme [1, 2, 3, 4]. (For various applications of this result see the review article in [4] and [2, 5].) The corresponding $p$-brane entropy, given by $\ln \rho(N)$, then has a logarithmic correction with a particular coefficient $X$, which depends on the dimension $D$ of the embedding spacetime.

The correct counting of the asymptotic density of states must also include the zero-mode states. They have been included for the open string case ($p = 1$) in [6, 7]. As a consequence, the logarithmic correction coefficient $X$ becomes independent of $D$ and is given by $X = -\frac{3}{2}$. Such logarithmic corrections to the entropy, with precisely this coefficient, have appeared in other contexts also: in $(1+1)$-dimensional conformal field theories [6, 7], and in the entropies for $(2+1)$ and $(3+1)$ dimensional black holes calculated using the spin network formalism [8, 9].

In this paper, using the results of [4], we obtain the asymptotic density of states for open $p$-branes. We then include the zero-modes following the methods of [6, 7]. We find that the logarithmic correction coefficient $X$ becomes independent of the dimension $D$ of the embedding spacetime, and is given by $X = -\frac{p+2}{2p}$.

Logarithmic corrections to entropy also arise for statistical mechanical systems due to statistical fluctuations [10]. Using the results of [10], we find that logarithmic corrections to the entropy, with precisely the same coefficient as that obtained in the open $p$-brane case, namely $-\frac{p+2}{2p}$, appear in two other contexts also: a gas of massless particles in $p$-dimensional space, and a Schwarzschild black hole in $(p+2)$-dimensional anti de Sitter spacetime [10].

This paper is organised as follows. In section 2, we briefly present the results of [4] and, using them, obtain the asymptotic density of states for open $p$-branes. In section 3, we include the zero-modes. In section 4, we show that such logarithmic corrections with precisely the same coefficient appear in other contexts also. In section 5, we conclude by mentioning a few issues for further study.

2. The asymptotic density of states $\rho(N)$ at level $N$, $N \gg 1$, for $p$-branes compactified on $(S^1)^p \times \mathbf{R}^{D-p}$ can be calculated within the semiclassical quantisation scheme, and is of the form

$$\rho(N) \simeq C N^B e^{AN^s} \quad (1)$$
where $\delta$, $A$, $B$, and $C$ are constants. $\delta$ was obtained in [1, 2] and the correct expressions for the remaining constants $A$, $B$, and $C$ in [4]. The corresponding $p$-brane entropy $S(N)$ is given by

$$S(N) = \ln \rho(N) \simeq S_0 + X \ln S_0 + (\text{const}) \tag{2}$$

where the leading term $S_0 = AN^\delta$ and $X = \frac{B}{\delta}$ is the coefficient of the logarithmic correction to the entropy.

We now present briefly the results of [4]. See [4] for details. In the semiclassical quantisation, the total number operator $\mathcal{N}$ can be written in the proper time formalism as

$$\mathcal{N} = \sum_{i=1}^{d} \sum_{n \neq 0} \omega_n \mathcal{N}_n^i, \quad \omega_n = \sqrt{\sum_{j=1}^{p} n_j^2}, \tag{3}$$

where $d = (D - p - 1)$, $n = (n_1, n_2, \cdots, n_p) \in \mathbb{Z}^p$, $0 = (0, 0, \cdots, 0)$, and $\mathcal{N}_n^i$ are number operators [11]. For the sake of simplicity, we have set the $p$-brane tension to unity and taken all the circles in $(S^1)^p$ to be of unit radius.

Let $\rho(N)$ be the number of independent eigenstates of the total number operator $\mathcal{N}$ with eigenvalue $N$. Its generating function $F(z)$ is given by

$$F(z) = \sum_{N=0}^{\infty} \rho(N) e^{-zN} = Tr e^{-z\mathcal{N}} = \prod_{n \neq 0} (1 - e^{-z\omega_n})^{-d}. \tag{4}$$

Inverting the above relation then gives $\rho(N)$ in terms of $F(z)$:

$$\rho(N) = -\frac{1}{2\pi i} \oint dz \ e^{Nz} F(z) \tag{5}$$

where the integration contour is a small circle around the origin. Using Meinardus theorem [12] and the properties of Epstein zeta function, an asymptotic expression for $F(z)$ can be obtained in the limit $Re(z) \to 0$ which is sufficient to obtain $\rho(N)$ in the limit $N \gg 1$. The asymptotic expression for $F(z)$ is of the form [4]

$$F(z) \simeq c \ z^b e^{az-p} \tag{6}$$

where $a$, $b$, and $c$ are constants. The contour integral in (5), and thus $\rho(N)$, can then be evaluated by saddle point method. For $F(z)$ of the form given
in equation (6), the saddle point is located at 
\[ z = z_0 = \left( \frac{N}{ap} \right)^{-\frac{1}{p+1}} \]
in the limit \( N \gg 1 \), is given by
\[
\rho(N) \simeq \frac{c (ap)^{\frac{2b+1}{p+1}}}{\sqrt{2\pi(p+1)}} N^{-\frac{2b+p+2}{2(p+1)}} e^{AN^\frac{p}{p+1}}, \quad A = \frac{p + 1}{p} (ap)^{\frac{1}{p+1}}. \tag{7}
\]
In our case, \( a = \frac{2d \Gamma(p) \zeta(p+1)}{\Gamma(\frac{p}{2})} \) and \( b = d \) where \( d = (D - p - 1) \) and \( \zeta \) is the Riemann zeta function; the constant \( c \) is given explicitly in [4] and is not required for our purposes. See [4] for further details.

We now obtain the asymptotic density of states for an open \( p \)-brane using the above results. In this case modes \( n \) and \( -n \) together contribute to one standing wave (SW) mode of the open \( p \)-brane and, hence, should be counted only once. Thus, the corresponding total number operator is given by
\[
N = \sum_{i=1}^{d} \sum_{SW; n \neq 0} \omega_n N^i. \tag{8}
\]
The corresponding generating function \( F_0(z) \) is given by
\[
F_0(z) = \sum_{N=0}^{\infty} \rho_o(N) e^{-zN} = \prod_{SW; n \neq 0} \left( 1 - e^{-z\omega_n} \right)^{-d} = \prod_{n \neq 0} \left( 1 - e^{-z\omega_n} \right)^{-\frac{d}{2}} \tag{9}
\]
where the last equality follows since \( \omega_n = \omega_{-n} \) and the product in the last expression includes both \( n \) and \( -n \). Note that the generating function \( F_0(z) \) is identical to that in equation (4) with \( d \) there replaced by \( \frac{d}{2} \). It therefore follows that the density of states \( \rho_o(N) \) for an open \( p \)-brane in the limit \( N \gg 1 \) is given by (7), but now with \( d \) replaced by \( \frac{d}{2} \). Explicitly, in the limit \( N \gg 1 \),
\[
\rho_o(N) \simeq \frac{c (ap)^{\frac{D-p}{2(p+1)}}}{\sqrt{2\pi(p+1)}} N^{-\frac{D-1}{2(p+1)}} e^{AN^\frac{p}{p+1}}, \quad A = \frac{p + 1}{p} (ap)^{\frac{1}{p+1}}. \tag{10}
\]
where \( a = \frac{(D-p-1) \Gamma(p) \zeta(p+1)}{\Gamma(\frac{p}{2})} \), \( c \) is as given in [4] but with \( d = (D - p - 1) \) there replaced by \( \frac{d}{2} = \frac{D-p-1}{2} \), and we have used \( b = \frac{d}{2} = \frac{D-p-1}{2} \) in obtaining (10). Note that \( \rho(N) \) is of the form given in equation (1) with \( \delta = \frac{p}{p+1} \) and \( B = -\frac{D+1}{2(p+1)} \), and that string theory result [3] is obtained upon setting \( p = 1 \).
3. We now include the zero-modes and obtain the resulting asymptotic density of open $p$-brane states. The complete Hamiltonian for a $p$-brane in the proper time formalism is given by

$$ H = p^2 + N $$

(11)

where $p^2$ is the transverse momentum square operator and $N$ is the total number operator given in (8), and they both commute with each other [4].

The correct counting of the total number of states $\rho(N)$ must include the zero-mode states also, namely those corresponding to the transverse momentum. For open strings ($p = 1$), $\rho(N)$ with zero-modes included has been calculated in [6, 7] in the limit $N \gg 1$. The resulting $\rho(N)$ is still given by equation (1) with $\delta = \frac{1}{2}$ as before, but now with $B = -\frac{3}{4}$ independent of the dimension $D$ of the embedding spacetime. See [6, 7] for further discussions.

The total number of states $\rho(N)$ for open $p$-branes, with zero-modes included, can be calculated for other values of $p$ also in the limit $N \gg 1$. $\rho(N)$ is still given by equation (5), but now the relevant generating function $F(z)$, with zero-modes included, is given by

$$ F(z) = Tr e^{-zH} = \left( \int d^{D-p-1} p \ e^{-zp^2} \right) Tr e^{-zN} = \left( \int d^{D-p-1} p \ e^{-zp^2} \right) F_0(z) $$

(12)

where the second equality follows since the operators $p^2$ and $N$ commute with each other and $F_0(z)$ is the generating function given in equation (9). The momentum integral can be evaluated easily and results in an extra $z$-dependent factor given by

$$ \int d^{D-p-1} p \ e^{-zp^2} = c_0 \ z^{-\frac{D-p-1}{2}} $$

where $c_0$ is a constant. The contour integral in (5), and thus $\rho(N)$, can now be evaluated by saddle point method. The saddle point is at $z = z_0 = \left( \frac{N}{ap} \right)^{-\frac{1}{p+1}}$ as before, and the total number of open $p$-brane states $\rho_o(N)$, in the limit $N \gg 1$ and with zero-modes included, is now given by

$$ \rho_o(N) \simeq \frac{cc_0 \ (ap)^{\frac{1}{2(p+1)}}}{\sqrt{2\pi(p+1)}} \ N^{-\frac{p+2}{2(p+1)}} \ e^{AN\frac{p}{p+1}} $$

(13)
where the constants $a$ and $c$ are as in equation (10). Clearly, $\rho(N)$ is of the form given in equation (1) with $\delta = \frac{p}{p+1}$ as before, but now with $B = -\frac{p+2}{2(p+1)}$ independent of the dimension $D$ of the embedding spacetime.

The open $p$-brane entropy $S(N)$ with zero-modes included is thus given by

$$S(N) = \ln \rho(N) \simeq S_0 - \frac{p+2}{2p} \ln S_0 + (\text{const})$$

where $S_0 = AN^{\frac{p}{p+1}}$. The coefficient of the logarithmic correction to the open $p$-brane entropy with zero-modes included is now given by $-\frac{p+2}{2p}$ and is independent of the dimension $D$ of the embedding spacetime. Note that without zero-modes included it is given by $-\frac{p+1}{2p}$, as can be seen from equations (2) and (10), and depends on $D$.

4. For open strings ($p = 1$) with zero-modes included, the coefficient of the logarithmic correction to the entropy given above becomes $-\frac{3}{2}$, which agrees with the results of [6, 7]. Such logarithmic corrections, with precisely this coefficient, $-\frac{3}{2}$, have appeared in other contexts also: in $(1+1)$-dimensional conformal field theories [6, 7], and in the entropies for $(2+1)$ and $(3+1)$ dimensional black holes calculated using the spin network formalism [8, 9].

Similarly, it turns out that logarithmic corrections, with precisely the coefficient given in equation (14), namely $-\frac{p+2}{2p}$, also appear in two other contexts. Recently, logarithmic corrections to entropies of statistical mechanical systems, arising due to statistical fluctuations, have been obtained in [10]. One calculates the density of states $\rho(E)$ as an inverse Laplace transformation of the partition function in the canonical ensemble. Statistical fluctuations can then be incorporated naturally. Then, $\rho(E)\Delta$ is the number of states with energy in the range $E \pm \frac{\Delta}{2}$ where $\Delta$ depends on the precision with which the system is prepared and, in particular, is independent of $E$. The entropy $S(E)$ is therefore given by $S(E) = \ln \rho(E) + (\text{const})$.

The result of [10] is that for a system at temperature $T$, with specific heat $C$ (which must be positive for this formalism to be applicable), one obtains for its entropy

$$S = S_0 - \frac{1}{2} \ln(C T^2) + (\text{const})$$

where $S_0$ is the leading term. See [10] for details.
Now, consider a gas of massless particles in $p$-dimensional space. Then

\[ S_0 \propto T^p , \quad E \propto T^{p+1} , \quad C \propto T^p . \]

Equation (15), therefore, gives

\[ S = S_0 - \frac{p+2}{2p} \ln S_0 + (\text{const}) . \] (16)

For a Schwarzschild black hole, the specific heat is negative and, hence, the above formalism is inapplicable [10]. However, a Schwarzschild black hole of sufficiently large mass in a $d$-dimensional anti de Sitter spacetime (AdS$_d$) has positive specific heat. Consider an AdS$_{p+2}$ Schwarzschild black hole of mass $M$. For sufficiently large $M$, one has [13]

\[ S_0 \propto r_+^p , \quad E = M \propto r_+^{p+1} , \quad T \propto r_+ , \]

where $r_+$ is the horizon. It then follows that $C \propto r_+^p$. Equation (15), therefore, gives

\[ S = S_0 - \frac{p+2}{2p} \ln S_0 + (\text{const}) . \] (17)

See [10]) for details. For AdS$_3$, see also [8, 9]. From equations (16) and (17), we see that logarithmic corrections to the entropy of a gas of massless particles in $p$-dimensional space, and to that of an AdS$_{p+2}$ Schwarzschild black hole, both have a coefficient $-\frac{p+2}{2p}$ which is precisely the same as that obtained in the open $p$-brane case with zero-modes included.

5. To summarise, we have obtained the asymptotic density of open $p$-brane states with zero-modes included. The corresponding open $p$-brane entropy has a logarithmic correction, with a coefficient $-\frac{p+2}{2p}$. Such logarithmic corrections, with precisely the same coefficient, also appear for a $p$-dimensional gas and for an AdS$_{p+2}$ Schwarzschild black hole where the corrections arise due to statistical fluctuations.

The relation of a $p$-dimensional gas to AdS$_{p+2}$ Schwarzschild black hole, for $p = 1, 2, 3,$ and 5, can be understood in the context of AdS/CFT duality [14] as that of a boundary conformal field theory at high temperature [13]. In light of the present results, one may explore the relations between quantum/semi classical $p$-branes, $p$-dimensional gas, and AdS$_{p+2}$ spacetime in more detail. In particular, it will be interesting to know if the values of
$p$ are restricted for quantum/semi classical $p$-branes, or if a duality exists between $p$-dimensional gas and AdS$_{p+2}$ spacetime for any value of $p$.

As mentioned earlier, the coefficient $X = -\frac{3}{2}$, corresponding to $p = 1$, also appears for the entropy of a $(3 + 1)$ dimensional black hole calculated using the spin network formalism [8]. In this formalism, one considers punctures, each carrying a spin $J_{\text{puncture}}$, and counts the number of spin singlet states, namely those states with $J(\text{total}) = 0$. The leading term (and a part of the logarithmic correction) in the entropy corresponds to the number of states with $J_z(\text{total}) = 0$. However, such states include states with $J(\text{total}) \neq 0$ also. A correct counting, that counts states with $J(\text{total}) = 0$ only, then leads to the coefficient $-\frac{3}{2}$ for the logarithmic correction to the entropy [8]. It will be interesting to find if a similar interpretation exists for the logarithmic correction coefficient $X = -\frac{p+2}{2p}$ for other values of $p$ also.

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