The Stochastic Gierer–Meinhardt System

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Abstract
The Gierer–Meinhardt system occurs in morphogenesis, where the development of an organism from a single cell is modelled. One of the steps in the development is the formation of spatial patterns of the cell structure, starting from an almost homogeneous cell distribution. Turing proposed different activator–inhibitor systems with varying diffusion rates in his pioneering work, which could trigger the emergence of such cell structures. Mathematically, one describes these activator–inhibitor systems as coupled systems of reaction-diffusion equations with different diffusion coefficients and highly nonlinear interaction. One famous example of these systems is the Gierer–Meinhardt system. These systems usually are not of monotone type, such that one has to apply other techniques. The purpose of this article is to study the stochastic reaction-diffusion Gierer–Meinhardt system with homogeneous Neumann boundary conditions on a one or two-dimensional bounded spatial domain. To be more precise, we perturb the original Gierer–Meinhardt system by an infinite-dimensional Wiener process and show under which conditions on the Wiener process and the initial conditions, a solution exists. In dimension one, we even show the pathwise uniqueness. In dimension two, uniqueness is still an open question.

Keywords Gierer–Meinhardt system · Pattern formation · Coupled system · Activator–inhibitor system · Stochastic partial differential equations · Stochastic systems · Wiener process · Biomathematics

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1 Introduction

Pattern formation is a phenomenon based on the interaction of different components, possibly under the influence of their surroundings. Alan Turing, a cryptographer and a pioneer in computer science, developed algorithms to describe complex patterns using simple inputs and random fluctuation. In his seminal paper [42] in 1952, he proposed that the interaction between two biochemical substances with different diffusion rates have the capacity to generate biological patterns. In his mathematical framework, there is one activating protein (activator) that activates both itself and an inhibitory protein (inhibitor), which only inhibits the activator. He detected that a stable homogeneous pattern could become unstable if the inhibitor diffuses more rapidly than the activator. The interplay between the concentrations of these substances forms a pattern whose spatiotemporal evolution is governed by coupled reaction-diffusion systems (activator–inhibitor model). This phenomenon is called diffusion-driven instability or Turing instability. Thus, the most fundamental phenomenon in pattern-forming activator–inhibitor systems is that a slight deviation from spatial homogeneity has vital positive feedback leading to increase further. The presence of nonlinearities in the local dynamics, for example, due to the inhibitor concentration, saturates the Turing instability into a stable and spatially inhomogeneous pattern. Pattern formation via diffusion-driven instabilities plays an essential role in, e.g., biology, chemistry, physics, ecology, and population dynamics.

One well-established activator–inhibitor system was suggested by Gierer and Meinhardt in 1972 to model the (re)generation phenomena in a hydra (see [15], and [16]). This model, called the Gierer–Meinhardt model, describes the interaction of two biochemical substances, a slowly diffusing activator $U$ and a rapidly diffusing inhibitor $V$. In particular, $U$ activates its production and the production of $V$. The inhibitor $V$ represses the production of $U$ and diffuses more rapidly than $U$.

1.1 The Deterministic Gierer–Meinhardt Model

For a domain, $\mathcal{O} \subset \mathbb{R}^d, d = 1, 2$, the model introduced by Gierer and Meinhardt reads as

$$\begin{align*}
\dot{u}(t) &= r_u \Delta u(t) + \kappa_u \frac{u^2(t)}{v(t)} - \mu_u u(t), \quad t > 0, \\
u(0) &= u_0,
\end{align*}$$

and

$$\begin{align*}
\dot{v}(t) &= r_v \Delta v(t) + \kappa_v u^2(t) - \mu_v v(t), \quad t > 0, \\
v(0) &= v_0,
\end{align*}$$

subjected to Neumann boundary conditions. Here, the unknowns $u$ and $v$ stand for the concentrations $U$ and $V$, with a source distribution $\kappa_u$ and $\kappa_v$ respectively. Here $\kappa_u$ and $\kappa_v$ are positive constants. Furthermore, the constants $r_u > 0$ and $r_v > 0$ are the diffusion coefficients, and the constants $\mu_u > 0$ and $\mu_v > 0$ are given decay rates.
This system explains that starting from a uniform condition (i.e., a homogeneous distribution with no pattern), they could spontaneously self-organise their concentrations into a repetitive spatial pattern. The parameters in the system can be tuned in such a way, that some interesting phenomena such as Turing instability and peak steady states occur.

The Gierer–Meinhardt system has been studied extensively by many authors, both in the biological and physical communities; more recently also in mathematics, to elucidate its role in pattern formation. There are several works worth mentioning on the deterministic Gierer–Meinhardt system, e.g., Masuda and Takahashi [31] showed the existence and boundedness of solutions. Gonpot et al. [18] studied the roles of diffusion and Turing Instability in the formation of spot and stripe patterns investigated by performing a nonlinear bifurcation analysis. Kelkel and Surulescu [25] proved the existence of a local weak solution for general initial conditions and parameters upon using an iterative approach. Chen et al. [8] have carried out Bifurcation analysis, including theoretical and numerical analysis. Kavallaris and Suzuki [24] focused on the derivation of blow-up results for the shadow Gierer–Meinhardt system. Recently, Wang et al. [17] investigated the stability of the equilibrium and the Hopf bifurcation of the Gierer–Meinhardt system of the Depletion type. In addition, we refer for a more general background on the model to the book of Britton [7], Meinhardt [32], the books of Murray [33, 34], the book of Ghergu and Rădulescu [14], Perthame [36], and of Wei and Winter [44].

1.2 Why Study Stochasticity?

The deterministic model, i.e., the macroscopic system of equations, is derived from the microscopic behavior studying the limit behavior. From the microscopic perspective, one interprets the movements of the molecules as a result of microscopic irregular movement. Taking the limit and passing from the microscopic to the macroscopic equation, one neglects the fluctuations around the mean value. Biological systems are frequently subject to noisy environments, inputs, and signalling. These stochastic perturbations are crucial when considering the ability of such models to reproduce results consistently. Murray [34] addressed the possible impact of the noise, where he mentioned that to study the effect of stochasticity would be illuminating.

For a more realistic model, it is necessary to consider features of the natural environment that are non-reproducible and, hence, modelled by random spatiotemporal forcing. An appropriate mathematical approach to establish more realistic models is the incorporation of stochastic processes. The randomness leads to a variate of new phenomena and may have a highly non-trivial impact on the behavior of the solution. In Karig et al. [23], the authors explore whether the stochastic extension leads to a broader range of parameters with Turing patterns by a genetically engineered synthetic bacterial population in which the signalling molecules form a stochastic activator–inhibitor system. Biancalani et al. [5] studied the impact of noise on Turing pattern on several examples and showed that with random noise, the range of parameter where Turing patterns may appear, are enlarged. Kolinichenko and Ryashko [27], respectively, Bashkirtseva et al. [3] addresses multistability and noise-induced
transitions between different states. In Kolinichenko et al. [28], scenarios of noise-induced pattern generation and stochastic transformations are studied using numerical simulations and modality analysis. In summary, the stochastic term gives rise to a new type of behavior and, therefore, often leads to a more practical description of natural systems than their deterministic counterpart.

1.3 The Stochastic Model

Modelling randomness is done by adding a random forcing term to the system (1.1) or the system (1.2) leading to a stochastic partial differential equation (SPDE). The internal noise results in a multiplicative term; since the white noise is an approximation of a continuously fluctuating noise with finite memory being much shorter than the dynamical timescales, the integration with respect to the noise is modelled by the Stratonovich integral.

Let \( \mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}) \) be a filtered probability space. Let \( W_j, j = 1, 2, \) be two independent Wiener processes defined on \( L^2(\mathcal{O}) \) over the probability space \( \mathfrak{A} \) with covariances \( Q_1 \) and \( Q_2 \) respectively, which will be explained in detail later. We are now interested in the existence and uniqueness of the solution of the following stochastic Gierer–Meinhardt system

\[
du(t) = \left[ ru \Delta u(t) + \kappa_u \frac{u^2(t)}{v(t)} - \mu_u u(t) \right] dt + \sigma_u u(t) \circ dW_1(t), \quad t > 0, \tag{1.3}
\]

and

\[
v(t) = \left[ rv \Delta v(t) + \kappa_v \frac{v^2(t)}{u(t)} - \mu_v v(t) \right] dt + \sigma_v v(t) \circ dW_2(t), \quad t > 0, \tag{1.4}
\]

again subjected to Neumann boundary conditions (or if \( \mathcal{O} \) is a torus to periodic boundary conditions) and initial conditions \( u(0) = u_0 \) and \( v(0) = v_0 \). Besides, \( \sigma_u, \sigma_v > 0 \) and \( \mu_u > 0, \mu_v > 0 \). In dimension one we can show existence and uniqueness of the solution, in dimension two we can only show uniqueness.

1.4 Novelty of the Paper

The main difficulty in the treatment of (1.1) and (1.2) is the lack of a variational structure. Jiang [21] have shown the global existence of solutions to the deterministic Gierer–Meinhardt system. However, the method cannot be transferred to the stochastic case due to the randomness’s peculiarity. Due to the lack of variational structure, the standard methods to show the existence and uniqueness of solutions to stochastic partial differential equations can not be applied. A way out is to use a stochastic Schauder–Tychonoff type Theorem. In dimension one, we were able to show pathwise uniqueness, see Section 5. By standard arguments based on the Yamada-Watanabe Theorem gives for 1D, we can therefore show the existence and uniqueness of a strong solution.
There are very few works known to the authors investigating the Gierer–Meinhardt system with full stochasticity. Li and Xu [30] consider the shadow Gierer–Meinhardt System with random initial data; Winter et al. [46] investigate the stochastic shadow Gierer–Meinhardt system. Kelkel and Surulescu [26] prove locally in time a pathwise unique mild solution via an iterative method, which is done for positive saturation constants (that is biologically expedient); whereas our paper shows the existence of a martingale solution (global-in-time) without the positive saturation restriction (i.e. for nonnegative constants).

The article is structured as follows. In Sect. 2, we list the hypotheses and the main result is presented, i.e., the existence of a martingale solution to the stochastic counterpart of the system (1.3) and (1.4). In Sect. 3 the actual proof is stated, in Sect. 4, we prove several technical propositions necessary for Sect. 3. The pathwise uniqueness in dimension one has been proved in Sect. 5. Several technical lemmata and supplementary material is published in the online resource [19]. In particular, in Section D.2 of the online resource [19], we present a stochastic version of a Schauder–Tychonoff type Theorem which is used to prove the main result.

Let us introduce few notations of functional spaces which will be used throughout the paper.

**Notation 1.1** For a Banach space $E$ and $0 \leq c < d < \infty$, let $C^\delta_b([c, d]; E)$ denote a set of all continuous and bounded functions $u : [c, d] \to E$ such that

$$
\|u\|_{C^\delta_b([c, d]; E)} := \sup_{c \leq t \leq d} |u(t)|_E + \sup_{c \leq s, t \leq d, t \neq s} \frac{|u(t) - u(s)|_E}{|t - s|^\delta},
$$

is finite. The space $C^\delta_b([c, d]; E)$ endowed with the norm $\|\cdot\|_{C^\delta_b([c, d]; E)}$ is a Banach space.

**Notation 1.2** For $1 < p < \infty$, let $W^1_p(O)$ be the Sobolev space defined by (compare [6, p. 263])

$$
W^1_p(O) := \left\{ u \in L^p(O) \mid \exists g_1, \cdots, g_d \in L^p(O) \text{ such that } \int_O u(x) \frac{\partial \phi(x)}{\partial x_i} \, dx = -\int_O g_i(x) \phi(x) \, dx \quad \forall \phi \in C^\infty_c(O), \forall i = 1, \ldots, d \right\}
$$

equipped with norm

$$
|u|_{W^1_p} := |u|_{L^p} + \sum_{j=1}^d \left| \frac{\partial u}{\partial x_j} \right|_{L^p}, \quad u \in W^1_p(O).
$$

Given an integer $m \geq 2$ and a real number $1 \leq p < \infty$, we define by induction the space

$$
W^m_p(O) := \left\{ u \in W^{m-1}_p(O) \mid Du \in W^{m-1}_p(O) \right\}
$$
equipped with norm
\[ |u|_{W^m_p} := |u|_{L^p} + \sum_{|\alpha| \leq m} |D^\alpha u|_{L^p}, \quad u \in W^m_p(O). \]

Let \( H^m_2(O) := W^m_2(O) \), and for \( \rho \in (0, 1) \), let \( H^\rho_2(O) \) be the real interpolation space given by \( H^\rho_2(O) := (L^2(O), H^1_2(O))_{\rho,2} \). In addition, let \( H^{-1}_2(O) \) be the dual space of \( H^2_2(O) \) and for \( \rho \in (0, 1) \), let \( H^{-\rho}_2(O) \) be the real interpolation space given by \( H^{-\rho}_2(O) := (L^2(O), H^{-1}_2(O))_{1-\rho,2} \). Note, by Theorem 3.7.1 [4], \( H^{-\rho}_2(O) \) is dual to \( H^\rho_2(O) \), \( \rho \in (0, 1) \). Furthermore, we have \( (H^{-\rho}_2(O), H^\rho_2(O))_{1,2} = L^2(O) \) and \( (H^\rho_2(O), H^\theta_2(O))_{\rho,2} = H^\theta_2(O) \) for \( \theta = \alpha(1 - \rho) + \beta \rho, \rho \in (0, 1) \) and \( |\alpha|, |\beta| \leq 1 \).

2 Hypotheses and the Main Result

Let us denote \( \mathcal{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) be a complete probability space with the filtration \( \mathbb{F} = \{ \mathcal{F}_t : t \in [0, T] \} \) satisfying the usual conditions i.e., \( \mathbb{P} \) is complete on \( (\Omega, \mathcal{F}) \), for each \( t \geq 0, \mathcal{F}_t \) contains all \( (\mathcal{F}, \mathbb{P}) \)-null sets, and the filtration \( \mathbb{F} \) is right-continuous.

Let in case \( d = 1 \) the domain \( O \) be an interval and in case \( d = 2 \), let \( O \subset \mathbb{R}^2 \) be a bounded domain with \( C^\infty \) boundary. Let \( W_1 \) and \( W_2 \) be two independent Wiener processes in \( \mathcal{H} := L^2(O) \) defined over the probability space \( \mathcal{A} \). Before introducing the hypothesis on the noise, let us introduce the unbounded operator \( A \) given by the Laplace operator \(-\Delta\) in \( L^2(O) \) with Neumann boundary conditions. In particular, let

\[
\begin{cases}
D(A) = \{ u \in H^2_2(O) : \frac{\partial}{\partial n} u(x) = 0, \ x \in \partial O \}, \\
Au = -\Delta, \quad u \in D(A).
\end{cases}
\]

Here, \( n \) denotes the outward normal vector of the boundary \( \partial O \).

**Hypothesis 2.1** We assume that \( W_1 \) and \( W_2 \) are two independent Wiener processes defined on \( \mathcal{H} := L^2(O) \). In particular, we have

\[ W_j(t) := \sum_{k \in \mathbb{N}} \beta^i_k(t) (1 + \lambda_k)^{-\gamma_j/2} e_k, \quad t \geq 0, \ j = 1, 2, \]

where \( \gamma_j > d, j = 1, 2, \{ \beta^i_k : k \in \mathbb{N} \}, j = 1, 2, \) are two independent families of mutually independent real-valued Brownian motions over \( \mathcal{A}, \{ e_k : k \in \mathbb{N} \} \) are the eigenfunctions of \( A \) and \( \{ \lambda_k : k \in \mathbb{N} \} \) with \( 0 = \lambda_1 < \lambda_2 < \cdots \) are the corresponding eigenvalues.

In the case of one dimension, we have \( \lambda_k = 4\pi^2 k^2, k \in \mathbb{N}, \) and if \( O \) is a square domain (i.e. \( O = \{(x, y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\} \)), then we have

\[
\lambda_{i,m} = \left( \frac{l\pi}{a} \right)^2 + \left( \frac{m\pi}{b} \right)^2, \quad i, m \in \mathbb{N}.
\]
For later on, let us define the covariances $Q_j$ by $Q_j := (\text{Id} + A)^{-\gamma_j}$, $j = 1, 2$. Next, we define the notion of solution to the system (1.3)–(1.4). The notion of solution will depend on a parameter $\rho$, later in Hypothesis 2.4 we will see that $\rho$ belongs to the interval $(1, \frac{6}{5})$ if $d = 2$ and $[1, \frac{6}{5})$ if $d = 1$.

**Definition 2.2** We say a pair of progressively measurable processes $(u, v)$ a strong solution to the system (1.3)–(1.4) on $[0, T]$, $T > 0$, for initial data $(u_0, v_0)$ if for $\rho > 0$ we have $\mathbb{P}$-a.s.

\[
 u \in C_b([0, T]; H^{1-\rho}_2(\mathcal{O})) \cap L^2([0, T]; H^{2-\rho}_2(\mathcal{O})),
\]

and

\[
 v \in C_b([0, T]; L^2(\mathcal{O})) \cap L^2([0, T]; H^1_2(\mathcal{O})),
\]

$u$ and $v$ are $(\mathcal{F}_t)_{t \geq 0}$-adapted, and satisfies for all $t \in [0, T]$, $\mathbb{P}$-a.s.

\[
 u(t) = u_0 + \int_0^t \left[ r_u \Delta u(s) + \kappa_u \frac{u^2(s)}{v(s)} - \mu_u u(s) \right] ds + \sigma_u \int_0^t u(s) \circ dW_1(s),
\]

and

\[
 v(t) = v_0 + \int_0^t \left[ r_v \Delta v(s) + \kappa_v u^2(s) - \mu_v v(s) \right] ds + \sigma_v \int_0^t v(s) \circ dW_2(s). \tag{2.1}
\]

Note that $r_u, r_v, \kappa_u, \kappa_v, \mu_u, \mu_v, \sigma_u, \sigma_v > 0$. As mentioned before, in the proof of the main result, we are using compactness arguments, which causes the loss of the original probability space. This means the solution will only be a weak solution in the probabilistic sense.

**Definition 2.3** A martingale solution to the problem (1.3)–(1.4) is a system

\[
 (\Omega, \mathcal{F}, \mathbb{P}, (W_1, W_2), (u, v))
\]

such that

- $\mathfrak{A} := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ satisfying the usual condition;
- $W_1$ and $W_2$ be two independent Wiener processes in $L^2(\mathcal{O})$, defined over the probability space $\mathfrak{A}$ with covariances $Q_1$ and $Q_2$;
- $u : [0, T] \times \Omega \to H^{1-\rho}_2(\mathcal{O})$ for $\rho > 0$ and $v : [0, T] \times \Omega \to L^2(\mathcal{O})$ are two $\mathbb{F}$-progressively measurable process such that the couple $(u, v)$ is a solution to the system (1.3)–(1.4) over the probability space $\mathfrak{A}$.

The range of $\rho$ will be specified in the next hypothesis. In this section, we list all the hypotheses on the initial data and on the Wiener process.
Hypothesis 2.4 We assume that the initial data \((u_0, v_0)\) satisfies

(i) \(u_0 \geq 0\) and \(v_0 > 0\) on \(\mathcal{O}\);
(ii) there exists a number with

\[ \rho \in \begin{cases} (1, \frac{6}{5}) & \text{if } d = 2, \\ [1, \frac{6}{5}) & \text{if } d = 1, \end{cases} \]

such that \(\mathbb{E}|u_0|^{2}_{H^{1-\rho}_2} < \infty\);
(iii) suppose that \(\mathbb{E}|v_0|_{L^2} < \infty\);
(iv) \(\mathbb{E}|\xi_0|^{p}_{L^p} < \infty\) for \(p = 31/7\) and \(\mathbb{E}(\int_{\mathcal{O}} \ln(|\xi_0(x)|) \, dx) < \infty\), where \(\xi_0 = v_0^{-1}\).

Remark 2.5 The constant \(31/7\) and \(\rho\) are mainly given by Lemma 4.10.

We assume that the noise is of linear type. With these hypotheses, we are now ready to state the main result of our paper.

Theorem 2.6 Under the Hypotheses 2.1 and 2.4, there exists a martingale solution to the system (1.3)–(1.4) in the sense of Definition 2.3, satisfying the following properties

(i) The solutions \(u(t, x) \geq 0\) and \(v(t, x) > 0\), \(\mathbb{P} \otimes \text{Leb a.s.}\);
(ii) We have \(\mathbb{P} \text{-}a.s.\ u \in C_b(0, T; H^{1-\rho}_2(\mathcal{O}))\) and \(v \in C_b(0, T; L^2(\mathcal{O}))\);
(iii) There exist constants \(C_1, C_2 > 0\) such that

\[ \mathbb{E} \sup_{t \in [0, T]} |u(t)|^{2}_{H^{1-\rho}_2} + 2\mathbb{E} \int_0^T |\nabla u(t)|^{2}_{H^{1-\rho}_2} \, dt \leq C_1 (1 + \mathbb{E}|u_0|^{2}_{H^{1-\rho}_2}), \]

and

\[ \mathbb{E} \sup_{t \in [0, T]} |v(t)|_{L^2} + 2\mathbb{E} \left( \int_0^T |\nabla v(t)|^{2}_{L^2} \, dt \right)^{1/2} \leq C_2 (1 + \mathbb{E}|v_0|_{L^2}). \]

(iv) There exists a constant \(C_3 > 0\) such that we have for \(\xi = v^{-1}\) and \(p = 31/7\)

\[ \mathbb{E} \sup_{t \in [0, T]} |\xi(t)|^{p}_{L^p} + p(p + 1)\mathbb{E} \int_0^T \int_{\mathcal{O}} \xi^{p+2}(t, x) (\nabla v)^2(t, x) \, dx \, dt \leq C_3 (1 + \mathbb{E}|\xi_0|^{p}_{L^p}). \]

The proof of this main result involves many steps and several tedious calculations. For the convenience of the reader, we assign a different section for the proof, which will constitute many lemmas and propositions. For the proof, we will use the stochastic version of the Schauder–Tychonoff type Theorem, which is stated and proved in Sect. D.2 of the online resource. In Sect. 3 the actual proof of the main theorem is stated, and in Sect. 4, the propositions used in the proof are postponed. The statement and the proof of the pathwise uniqueness in one dimension is obtained in Sect. 5.
3 Proof of the Main Result

3.1 Technical Preliminaries

As mentioned before, the proof is an application of the stochastic Schauder–Tychonoff fixed point theorem and consists of several steps.

- In the first step, we specify the underlying Banach spaces;
- In the second step, we construct the operator $T$;
- In the third step, we formulate the main claims and show that the assumptions of the stochastic Schauder–Tychonoff Theorem are satisfied;
- In the fourth step, we conclude the proof of the main result.

To keep the proof itself simple, the proof will use many technical propositions, which are the content of Sect. 4. Before starting with the actual proof, we will set up the notation we will use.

First, we are considering a linear noise, that is (see [1, Example 2.1.2]). The enumeration is chosen in increasing order by counting the multiplicity. The following is the estimate of the asymptotic behaviour of the eigenvalues: there exist two numbers $c, C > 0$ such that we have (e.g. see [2, Sect. 1.2.1] and references therein)

\[
ck^{\frac{d}{2}} \leq \lambda_k \leq Ck^{\frac{d}{2}}, \quad k \in \mathbb{N}.
\]  

(3.1)

In addition, there exists some constant $c > 0$ such that (see [1, p. 7])

\[
\sup_{x \in O} |e_k(x)| \leq c \lambda_k^{\frac{d-1}{2}}, \quad k \in \mathbb{N}.
\]  

(3.2)

A drawback of the Stratonovich stochastic integral is that it is not a martingale, and therefore, the Burkholder–Davis–Gundy inequality is not at its disposal. Hence, we changed within the proof from the Stratonovich stochastic integral to the Itô stochastic integral. For a detailed discussion, we refer to the book of Duan and Wang [11] or to the original work of Stratonovich [38, 39]. The step from the Stratonovich integral to the Itô integral can be done by adding a correction term, i.e., by adding drift to the Itô form. In this way, we will end up with a similar equation. In particular, we will consider the following system

\[
\begin{align*}
du(t) &= \left[ ru \Delta u(t) + \kappa_u \frac{u^2(t)}{v(t)} - \mu_u u(t) \right] dt + \sigma_u u(t) dW_1(t), \quad t > 0, \\
dv(t) &= \left[ rv \Delta v(t) + \kappa_v u^2(t) - \mu_v v(t) \right] dt + \sigma_v v(t) dW_2(t), \quad t > 0, 
\end{align*}
\]  

(3.3)

where the scalar $\mu_u$ and $\mu_v$ are replaced by the operators $\Upsilon_u := \mu_u \Id - \sigma_u (\Id + A)^{-\gamma_1}$ and $\Upsilon_v := \mu_v \Id - \sigma_v (\Id + A)^{-\gamma_2}$, respectively. Observe, since $\gamma_1, \gamma_2 \geq 0$, we know that $\Upsilon_u$ and $\Upsilon_v$ are bounded operator on $L^p(O)$, $1 \leq p < \infty$. Since $\mu_u \geq 0$ and $u$ is non-negative, $\mu_u u$ is non-negative. Since $\sigma_u > 0$, we are only concern about the term $(\Id + A)^{-\gamma_1} u$. Let $u$ be non-negative. If $z := (\Id + A)^{-\gamma_1} u$, then $(\Id + A)^{\gamma_1} z = u \geq 0$
in $\mathcal{O}$ and $\frac{\partial z}{\partial \nu} = 0$ on $\partial \mathcal{O}$. By the maximum principle, we obtain $z \geq 0$ in $\mathcal{O}$, i.e. $\Upsilon_{u} \geq 0$ in $\mathcal{O}$. Similarly, it can be shown if $v$ is positive, the $\Upsilon_{v} v > 0$ in $\mathcal{O}$.

Now, we can start with the actual proof of Theorem 2.6.

**Proof of Theorem 2.6**  **Step I** Definition of the functional spaces

Our main aim is to construct an integral operator, denoted by $\mathcal{T}$, whose fixed point is the solution of the coupled system (3.3). In this step, we define the required functional spaces for the operator $\mathcal{T}$ to act upon. Let the probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be given and let $W_1$ and $W_2$ be two independent cylindrical Wiener processes in $\mathcal{H} = L^2(\mathcal{O})$ defined over $\mathfrak{A}$. Let $W = (W_1, W_2)$, $H = \mathcal{H} \times \mathcal{H}$ and let $Q$ be a covariance operator defined by

$$Q = \begin{pmatrix} (\text{Id} + A)^{-\gamma_1} & 0 \\ 0 & (\text{Id} + A)^{-\gamma_2} \end{pmatrix}.$$ 

Let us define for $\rho \in (1, \frac{6}{5})$ the space

$$\mathcal{M}_{\mathfrak{A}} := \mathcal{M}_{\mathfrak{A}}(0, T) := \{ (\chi, \eta) : \Omega \times [0, T] \times \mathcal{O} \to (\mathbb{R}^+_0 \times \mathbb{R}^+_0) : (\chi, \eta) \text{ is } \mathbb{F}\text{-progressively measurable and}$$

$$\mathbb{E} \sup_{s \in [0, T]} |\chi(s)|^2_{H^1_2} + \mathbb{E} \sup_{s \in [0, T]} |\eta(s)|_{L^2} < \infty \} ,$$

equipped with the semi-norm

$$| (\chi, \eta) \rangle_{\mathcal{M}_{\mathfrak{A}}} := \left( \mathbb{E} \sup_{s \in [0, T]} |\chi(s)|^2_{H^1_2} \right)^{1/2} + \left( \mathbb{E} \sup_{s \in [0, T]} |\eta(s)|_{L^2} \right) , \text{ for } (\chi, \eta) \in \mathcal{M}_{\mathfrak{A}}.$$ 

Let $K_1$, $K_2$ and $K_3$ be fixed constant. Let us fix the set

$$\mathcal{U}_{\mathfrak{A}}(K_1, K_2, K_3) := \left\{ (\chi, \eta) \in \mathcal{M}_{\mathfrak{A}} : (\chi(t, x) \geq 0 \text{ and } \eta(t, x) > 0 \text{ Leb } \otimes \mathbb{P} \text{ a.s.} \forall (t, x) \in [0, T] \times \mathcal{O} : \mathbb{E} L_1(\chi, \eta) \leq K_1 \right.$$ 

$$\mathbb{E} L_2(\chi, \eta) \leq K_2, \text{ and } \sup_{0 \leq t \leq T} \mathbb{E} L_3(\chi(t), \eta(t)) \leq K_3 \right\} ,$$

where the Lyapunov functionals $L_1(\chi, \eta)$, $L_2(\chi, \eta)$, and $L_3(\chi, \eta)$ are defined by

$$L_1(\chi, \eta) := \sup_{s \in [0, T]} |\chi(s)|^2_{L^2} + \left( \int_0^T |\nabla \chi(s)|^2_{L^2} \, ds \right) + \sup_{s \in [0, T]} |\xi(s)|_p^p,$$

$$L_2(\chi, \eta) := \left( \int_0^T \int_\mathcal{O} \chi^2(s, x) \xi(s, x) \, dx \, ds \right)^2 + \left( \int_0^T \int_\mathcal{O} \xi^2(s, x) \chi^2(s, x) \, dx \, ds \right),$$

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and
\[ \mathcal{L}_3(\chi, \eta)(t) := |\xi(t)|_{L^p}^p + |\xi(t)|_{L^1} + \left( \int_{\mathcal{O}} \ln(\xi(t, x)) \, dx \right)^2. \]

Here, the process \( \xi \) is given by \( \xi := v^{-1} \), where \( v \) depends on \( \chi \) and solves
\[
\begin{align*}
dv(t) &= \left[ r_v \Delta v(t) + \kappa_v \chi^2(t) - \Upsilon_v v(t) \right] \, dt + \sigma_v v(t) \, dW_2(t), \quad t > 0, \\
v(0) &= v_0.
\end{align*}
\]

**Step II Construction of the operator** \( \mathcal{T} \) In this step we construct an integral operator whose fixed point is the solution of the system (1.3)–(1.4). Let us fix \( K_1, K_2, K_3 \) to be some positive real numbers. First, let us define the following operator:
\[
\mathcal{T} : \mathcal{U}_\mathcal{Q}(K_1, K_2, K_3) \rightarrow \mathcal{M}_\mathcal{Q}(0, T); \\
(\chi, \eta) \mapsto (u, v),
\]
where the pair \((u, v)\) solves the following system
\[
\begin{align*}
dv(t) &= \left[ r_v \Delta v(t) + \kappa_v \chi^2(t) - \Upsilon_v v(t) \right] \, dt + \sigma_v v(t) \, dW_2(t), \quad t > 0, \\
v(0) &= v_0, \\
(3.4)
\end{align*}
\]
and
\[
\begin{align*}
du(t) &= \left[ r_u \Delta u(t) + \kappa_u \frac{\chi^2(t)}{v(t)} - \Upsilon_u u(t) \right] \, dt + \sigma_u u(t) \, dW_1(t), \quad t > 0, \\
u(0) &= u_0. \\
(3.5)
\end{align*}
\]

**Remark 3.1** Here, \( \eta \) will not be used in the definition of the operator. However, we keep \( \eta \), since then the setting fits to the setting of the Schauder–Tychonoff-type Theorem.

The operator \( \mathcal{T} \) is well-defined. In fact, from Theorem 4.1, stated in the next section, we infer that for given \((\chi, \eta) \in \mathcal{U}_\mathcal{Q}(K_1, K_2, K_3)\), the existence of a non-negative unique solution \( v \) to (3.4) such that we have
\[
\mathbb{E} \sup_{0 \leq s \leq T} |v(s)|_{L^2} + \mathbb{E} \left( \int_0^T |\nabla v(s)|_{L^2}^2 \, ds \right)^{\frac{1}{2}} < \infty.
\]

From Theorem 4.3 for \( \bar{\rho} > \frac{d}{2} \) we infer that there exists a unique solution \( u \) to (3.5) such that
\[
\mathbb{E} \sup_{0 \leq s \leq T} |u(s)|_{H^{1-\bar{\rho}}_{d, \bar{\rho}}} + \mathbb{E} \int_0^T |u(s)|_{H^{1-\bar{\rho}}_{d, \bar{\rho}}}^2 \, ds < \infty.
\]
Step III Verification of the assumptions of the Schauder–Tychonoff type Theorem

In this step, we verify the assumptions of the Schauder–Tychonoff type Theorem D.2 of the online resource [19] by proving the three claims: Claims 3.2, 3.3, and 3.4.

Claim 3.2 There exist $K_1, K_2 > 0$, and $K_3 > 0$ such that $\mathcal{T}$ maps $U_{2\mathbb{R}}(K_1, K_2, K_3)$ into $U_{2\mathbb{R}}(K_1, K_2, K_3)$.

Proof of Claim 3.2 First, we will start to show the existence of numbers $K_2, K_3 > 0$ (independent of $K_1$) such that, given $(\chi, \eta) \in U_{2\mathbb{R}}(K_1, K_2, K_3)$, then

\[
E \mathcal{L}_2(u, v) \leq K_2, \quad \text{and} \quad \sup_{0 \leq t \leq T} E \mathcal{L}_3(u(t), v(t)) \leq K_3. \tag{3.6}
\]

In fact, by Proposition 4.2 we know that there exist constants $c_1 > 0$ and $\rho_1 > 0$ such that (3.6) is satisfied if

\[
K_2 \geq c_1 e^{\rho_1 T} \left( 1 + E|\xi_0|_{L^2}^2 \right), \tag{3.7}
\]

and

\[
K_3 \geq c_1 e^{\rho_1 T} \left( 1 + E|\xi_0|_{L^p}^p + E|\xi_0|_{L^1} + E \left( \int_\Omega \ln(|\xi_0(x)|) \, dx \right)^2 \right). \tag{3.8}
\]

It remains to find a number $K_1 > 0$ such that if $(\chi, \eta) \in U_{2\mathbb{R}}(K_1, K_2, K_3)$ then

\[
E |\mathcal{L}_1(\chi, \eta)| \leq K_1. \tag{3.9}
\]

In Proposition 4.4 it is shown that there exist constants $c_2 > 0$ and $\rho_2 > 0$ such that we have for any $T > 0$

\[
E \sup_{s \in [0, T]} |u(s)|_{L^2}^2 + E \left( \int_0^T |\nabla u(s)|_{L^2}^2 \, ds \right) \leq c_2 e^{\rho_2 T} \left( E|u_0|_{L^2}^2 + 2 \kappa u_0 \int_0^t \int_\Omega u(s, x) \chi^2(s, x) \xi(s, x) \, dx \, ds \right). \tag{3.10}
\]

By Corollary 4.6 we know that for any $\varepsilon_1, \varepsilon_2 > 0$ we have

\[
E \int_0^t \int_\Omega \chi^2(s, x) \xi(s, x) u(s, x) \, dx \, ds \
\leq c_1 E|\xi_0|_{L^1} + c_2 E|u_0|_{L^2}^2 + c_3 E|v_0|_{L^1} + c_4 E \int_0^t |\xi(s)|_{L^2}^2 \, ds \
+ c_5 \left( E \sup_{0 \leq s \leq T} |\eta(s)|_{L^2}^2 + E \int_0^T |\eta(s)|_{H^1}^2 \, ds \right)^\frac{1}{4} \
+ \varepsilon \left( E \sup_{0 \leq s \leq T} |u(s)|_{L^2}^2 + E \int_0^T |u(s)|_{L^2}^2 \, ds \right).
\]
Collecting together both estimates, choosing \( \varepsilon \in (0, \frac{1}{2}] \) and diving by \((1 - \varepsilon)\) gives for some \( c_1, \ldots, c_5 > 0 \)

\[
\begin{align*}
\mathbb{E} \sup_{s \in [0,T]} |u(s)|^2_{L_2} + \mathbb{E} \left( \int_0^T |\nabla u(s)|^2_{L_2} \, ds \right) \\
&\leq c_1 \mathbb{E} |\xi_0|_{L^1} + c_2 \mathbb{E} |u_0|^2_{L_2} + c_3 \mathbb{E} |v_0|_{L^1} + c_4 \mathbb{E} \int_0^t |\xi(s)|^2_{L_2} \\
&\quad + c_5 \left( \mathbb{E} \sup_{0 \leq s \leq T} |\eta(s)|^2_{L_2} + \mathbb{E} \int_0^T |\eta(s)|^2_{L_2} \, ds \right)^{\frac{1}{4}}.
\end{align*}
\]

(3.11)

Note that due to the definition of \( L_1(u, v) \) we have

\[
\mathbb{E} \sup_{0 \leq s \leq T} |u(s)|^2_{L_2} + \mathbb{E} \left( \int_0^T |\nabla u(s)|^2_{L_2} \, ds \right) \leq K_1.
\]

Substituting above in (3.11) and using the estimate in Proposition 4.2-(a) we obtain for some new constants \( c_5, \delta_5 > 0 \)

\[
\begin{align*}
\mathbb{E} \sup_{s \in [0,T]} |u(s)|^2_{L_2} + \mathbb{E} \left( \int_0^T |\nabla u(s)|^2_{L_2} \, ds \right) + \mathbb{E} \sup_{0 \leq s \leq T} |\xi(s)|^p_{L^p} \\
&\leq c_1 \mathbb{E} |\xi_0|_{L^1} + c_2 \mathbb{E} |u_0|^2_{L_2} + c_3 \mathbb{E} |v_0|_{L^1} + c_4 K_1^{\frac{1}{4}} + c_5 e^{\delta_5 T} \mathbb{E} |\xi_0|^p_{L^p}.
\end{align*}
\]

Since we are aiming to prove

\[
\mathbb{E} \sup_{0 \leq s \leq T} |\eta(s)|^2_{L_2} + \mathbb{E} \int_0^T |\eta(s)|^2_{L_2} \, ds + \mathbb{E} \sup_{0 \leq s \leq T} |\xi(s)|^p_{L^p} \leq K_1
\]

we chose \( K_1 > 0 \) according to

\[
K_1 \geq c_1 \mathbb{E} |\xi_0|_{L^1} + c_2 \mathbb{E} |u_0|^2_{L_2} + c_3 \mathbb{E} |v_0|_{L^1} + c_4 K_1^{\frac{1}{4}} + c_5 e^{\delta_5 T} \mathbb{E} |\xi_0|^p_{L^p}.
\]

With this choice, inequality (3.9) is satisfied. In this way, we have found three constants \( K_1, K_2 > 0 \) and \( K_3 > 0 \) such that, if \((\chi, \eta) \in \mathcal{U}_\mathcal{A}(K_1, K_2, K_3)\), then \((u, v) := \mathcal{T}[(\chi, \eta)] \in \mathcal{U}_\mathcal{A}(K_1, K_2, K_3)\).

**Claim 3.3** For any \( K_1, K_2 > 0 \), and \( K_3 > 0 \), the map

\[ \mathcal{T} : \mathcal{U}_\mathcal{A}(K_1, K_2, K_3) \rightarrow \mathcal{M}_\mathcal{A} \]

is continuous.
Proof of Claim 3.3} The continuity follows by a combination of the Proposition 4.7, Proposition 4.8, and Proposition 4.9. First, it follows by Proposition 4.7 that there exists some \( \gamma \in (0, 1] \) and \( \delta_1, \delta_2 > 0 \) such that we have

\[
\mathbb{E} \sup_{0 \leq s \leq T} \left| u_1(s) - u_2(s) \right|_2^{H_2^{1-\rho}} \leq C(K_1, K_2, T) \times \left[ \mathbb{E} \left[ \sup_{s \in [0, T]} \left| \chi_1(s) - \chi_2(s) \right|_2^{H_2^{1-\rho}} \right] \right]^{\delta_1} + \left[ \mathbb{E} \left[ \sup_{s \in [0, T]} \left| \xi_1(s) - \xi_2(s) \right|_2^{H_2^{1-\rho}} \right] \right]^{\delta_2}.
\]

Next, by Proposition 4.8 we have for \( m > q = 1, \rho \geq \frac{2mq}{m-q} \) and \( r = \frac{1}{1-\gamma} \)

\[
\mathbb{E} \sup_{s \in [0, T]} \left| \xi_1(s) - \xi_2(s) \right|_2^{H_2^{1-\rho}} \leq C(K_2, K_3) \left( \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| v_1(s) - v_2(s) \right|_{L^{m'}} \right] \right)^{\frac{1}{2}}.
\]

Finally, Proposition 4.9 gives for \( \rho > 2d(1 - \frac{1}{m}) \)

\[
\mathbb{E} \sup_{0 \leq s \leq T} \left| v_1(s) - v_2(s) \right|_{L^2} \leq C(K_3, T) \times \left[ \mathbb{E} \left[ \sup_{s \in [0, T]} \left| \chi_1(s) - \chi_2(s) \right|_2^{H_2^{1-\rho}} \right] \right]^{\gamma}.
\]

Collecting altogether and taking into account our choice of \( p \) gives the assertion. \( \square \)

Claim 3.4 For any \( K_1, K_2 > 0, \) and \( K_3 > 0, \) the map

\[ \mathcal{T} : \mathcal{U}_{2\lambda}(K_1, K_2, K_3) \to \mathcal{M}_{2\lambda} \]

is compact.

Proof of Claim 3.4} First, let us define the convolution operator \( \mathcal{C} \) by

\[
(\mathcal{C} f)(t) := \int_0^t e^{-(t-s)(A-\gamma_{1} I)} f(s) \, ds,
\]

and the convolution operators \( \mathcal{S}_j, j = 1, 2, \) by

\[
(\mathcal{S}_j f)(t) := \int_0^t e^{-(t-s)(A-\gamma_{1} I)} f(s) \, dW_j(s), \quad j = 1, 2.
\]

We can write the solution \( (u, v) \) for \( t > 0 \) as

\[
u(t) = e^{-t(A-\gamma_{1} I)} u_0 + \int_0^t e^{-(t-s)(A-\gamma_{1} I)} \chi^2(s) \xi(s) \, ds + \int_0^t e^{-(t-s)(A-\gamma_{1} I)} u(s) \, dW_1(s) = e^{-t(A-\gamma_{1} I)} u_0 + \left( \mathcal{C} \chi^2 \xi \right)(t) + \left( \mathcal{S}_1 u \right)(t),
\]

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and

\[ v(t) = e^{-t(A - \gamma_1 I)} v_0 + \int_0^t e^{-(t-s)(A - \gamma_1 I)} \chi^2(s) \, ds + \int_0^t e^{-(t-s)(A - \gamma_1 I)} v(s) \, dW_2(s) \]

\[ = e^{-t(A - \gamma_1 I)} v_0 + (\mathcal{E}[\chi^2])(t) + (\mathbb{S}_2[v])(t). \]

Since \( v \) depends only on \( \chi \), we study this process at first. Let us first examine the perturbation by \( \chi^2 \). Since \( (\chi, \eta) \in \mathcal{U}_\mathcal{Q}(K_1, K_2, K_3) \), by Proposition B.1 of the online resource [19], for \( p = \frac{5}{2} \) there exists a \( C > 0 \) such that

\[
\mathbb{E}\|\chi\|_{L^p(0,T;L^4)}^2 \leq C \mathbb{E}\left( \|\chi\|_{L^\infty(0,T;H_2^{1-\rho})}^2 + \|\chi\|_{L^p(0,T;H_2^{2-\rho})}^2 \right).
\]

In particular, we know for \( f = \chi^2 \), that there exists a constant \( C(K_1, K_2, K_3) > 0 \) such that

\[
\mathbb{E}\|f\|_{L^\frac{5}{2}(0,T;L^2)}^2 \leq C(K_1, K_2, K_3).
\]

It follows by Lemma A.3 of the online resource [19] with \( \alpha = 1, q = \frac{5}{2}, X = L^2(\mathcal{O}), \gamma = 0, \delta > 0, \text{ and } \beta > 0 \) with \( \beta + \frac{\delta}{2} < 1 - \frac{1}{q} \), that we have

\[
\mathcal{E}[\chi^2] \in C^\beta_b(0,T;H_2^{\frac{5}{2}}(\mathcal{O})),
\]

and

\[
\mathbb{E}\|\mathcal{E}[\chi^2]\|_{C^\beta_b(0,T;H_2^{\frac{5}{2}})} = \mathbb{E}\|\mathcal{E}[f]\|_{C^\beta_b(0,T;H_2^{\frac{5}{2}})} \leq C \mathbb{E}\|f\|_{L^q(0,T;L^2)} \leq C(K_1, K_2, K_3).
\]

Since we do not have the integrability of \( v \) for \( \rho > 2 \), we cannot rely on the factorisation method and have to use another argument. Observe, firstly, we know by Theorem 4.1 that for all \( (\chi, \eta) \in \mathcal{U}_\mathcal{Q}(K_1, K_2, K_3) \), \( \mathbb{E}\sup_{0 \leq s \leq T} |v(s)|_{L^2} \leq C(K_1, K_2, K_3) \). By Theorem 6.3 [13] we have to show that

(a) for every number \( t \in \mathbb{Q} \cup [0,T] \) for any \( \varepsilon > 0 \) there exists a compact set \( \Gamma_{t}^\varepsilon \subset L^2(\mathcal{O}) \) such that for any \( (\chi, \eta) \in \mathcal{U}_\mathcal{Q}(K_1, K_2, K_3) \) \( \mathbb{P}(v(t) \in \Gamma_{t}^\varepsilon) \geq 1 - \varepsilon \), where \( v \) denotes the solution of (3.4);

(b) for any \( T > 0 \) and any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that we have \( \mathbb{P}(\omega(v, \delta) \geq \varepsilon \leq \varepsilon \) for all \( (\chi, \eta) \in \mathcal{U}_\mathcal{Q}(K_1, K_2, K_3) \) (and \( v \) denotes the solution of (3.4)), where \( \omega(v, \delta) = \sup_{|t-s| \leq \delta} |v(t) - v(s)|. \)

Fix \( \delta > 0 \) and \( \Gamma_{t}^\varepsilon := \{w \in H_2^{\frac{5}{2}}(\mathcal{O}) : |w| \leq t^{-\delta} |v_0|/\varepsilon \}. \) Due to the compact embedding \( H_2^{\frac{5}{2}}(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \) and since straightforward calculations gives that \( e^{-t(A - \gamma_1 I)} v_0 \in \Gamma_{t}^\varepsilon \), we know \( t \mapsto e^{-t(A - \gamma_1 I)} v_0 \) satisfies (a). Secondly, straightforward calculations gives \( \sup_{t \in (0,T]} \mathbb{E}|\mathbb{S}_2[v](t)|_{H_2^{\frac{5}{2}}} < \infty \). The Chebycheff inequality gives (a). Finally, we
have to estimate the modulo of continuity, i.e. to verify condition (b). However, splitting the difference we arrive at the following sum

\[ \mathbb{E} \sup_{t \leq s \leq t+h} \int_t^s e^{-(s-r)(A-\mathbb{Y}_s)} v(r) dW_2(r) + \mathbb{E} \sup_{t \leq s \leq t+h} (I - e^{-s(A-\mathbb{Y}_s)}) \mathcal{G} v(t), \]

straightforward estimates and using estimates (4) of [20] gives the desired result. Next, we will investigate the regularity of \( u \). First, we will show that there exists a constant such that there exists some \( l = \frac{5}{2} \) such that for all \( (\chi, \eta) \in \mathcal{U}_2(K_1, K_2, K_3) \) and \( (u, v) = \mathcal{T}(\chi, \eta) \) and for \( r = 2 \)

\[ \mathbb{E} \left( \int_0^T |\chi^2(s)\xi(s)|_{L^4}^r ds \right)^{\frac{1}{r}} \leq C. \]

Here, let us note that we get for \( q = 4 \) and \( q' = \frac{4}{3} \) conjugate

\[ \mathbb{E} \left( \int_0^T |\chi^2(s)\xi(s)|_{L^4}^r ds \right)^{\frac{1}{r}} \leq \mathbb{E} \left( \int_0^T |\chi(s)|_{L^{2q'}}^r |\xi(s)|_{L^q}^r ds \right)^{\frac{1}{r}} \]

\[ \leq \mathbb{E} \sup_{0 \leq s \leq T} |\xi(s)|_{L^q}^r \left( \int_0^T |\chi(s)|_{L^{2q'}}^r ds \right)^{\frac{1}{r}} \]

\[ \leq \left\{ \mathbb{E} \sup_{0 \leq s \leq T} |\xi(s)|_{L^q}^4 \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left( \int_0^T |\chi(s)|_{L^{2q'}}^2 ds \right) \right\}^{\frac{2}{2q'-1}} \]

Due to Proposition 4.2-(a) we know that the first term is bounded. By Proposition B.1 of the online resource [19] and the fact that \( \frac{4}{3} - (1 - \rho) \leq \frac{2}{3} + \frac{d}{q'} \) we have

\[ \|\chi\|_{L^{2^*}(0,T;L^{2^*})} \leq C \left( \|\chi\|_{L^\infty(0,T;H_2^{1-\rho})} + \|\chi\|_{L^r(0,T;H_2^{2-\rho})} \right). \]  (3.12)

Now, the second term is bounded due to the definition of \( \mathcal{U}_2(K_1, K_2, K_3) \), (3.12). In particular, we have

\[ \mathbb{E} \|\chi\|_{H_1}^{2(1-\rho)} \leq K_1. \]

If \( l \) is chosen sufficiently large, i.e. \( \frac{2}{2^*-1} \leq \frac{1}{2} \). Therefore, we know that for any \( \delta > 0 \) there exists a constant \( C(K_1, K_2, K_3) > 0 \) such that

\[ \mathbb{E} \|\chi^2\xi\|_{L^r(0,T;H_2^{1-\delta})}^r \leq C(K_1, K_2, K_3), \quad (\chi, \xi) \in \mathcal{U}_2(K_1, K_2, K_3). \]

Now, by Lemma A.3 of the online resource [19] with \( \alpha = 1, q = r, \) and \( \gamma = 0 \) gives that for any \( \beta + \delta < 1 - \frac{1}{r}, 0 < \delta < \frac{\rho-1}{4} \)

\[ \mathcal{C}[\chi^2\xi] \in C^\beta_b(0, T; H_2^{1-\rho}(\Omega)). \]
In particular, there exists a constant $C > 0$ such that for $(\chi, \eta) \in \mathcal{U}_{\mathcal{Q}}(K_1, K_2, K_3)$ and we have

$$
\mathbb{E}\|\mathcal{C}[\chi^2 \xi]\|^\frac{1}{2}_{C_b^0(0,T;H_2^{\delta-1})} \leq C.
$$

(3.13)

Secondly, we know since $(u, v) \in \mathcal{U}_{\mathcal{Q}}(K_1, K_2, K_3)$ that

$$
\mathbb{E} \sup_{0 \leq s \leq T} |u(s)|^2_{H_2^{1-\rho}} + \mathbb{E} \int_0^T |\nabla u(s)|^2_{H_2^{1-\rho}} ds \leq K_1.
$$

Then, Corollary A.7 of the online resource [19] gives for $\beta + \frac{1}{p} = \frac{1}{2}$, $X = H_2^{1-\rho}(\Omega)$, $\delta = 0$, $\nu = \frac{1}{4}$

$$
\mathcal{S}_1[u] \in C_b^0(0, T; H_2^{1-\rho}(\Omega)).
$$

In particular, there exists a constant $C > 0$ such that for all $(\chi, \eta) \in \mathcal{U}_{\mathcal{Q}}(K_1, K_2, K_3)$

$$
\mathbb{E}\|\mathcal{S}_1[v]\|^\frac{1}{2}_{C_b^0(0,T;L^2)} \leq C.
$$

(3.14)

Collecting together (3.13), and (3.14), there exist $\beta, \delta > 0$ such that

$$
u \in C_b^0(0, T; H_2^{1-\rho}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).
$$

In particular, there exists a $C > 0$ such that for all $(\chi, \eta) \in \mathcal{U}_{\mathcal{Q}}(K_1, K_2, K_3)$

$$
\mathbb{E}\left(\|u\|_{C_b^0(0,T;H_2^{\delta-1})} + \|u\|_{L^\infty(0,T;L^2)}\right)^\frac{1}{t} \leq C.
$$

The Chebycheff inequality and the Prohorov Theorem gives compactness of the operator $\mathcal{T}$.

**Step IV Conclusion**

By Theorem D.2 of the online resource [19], there exists a probability space $\tilde{\mathcal{A}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a Wiener process $\tilde{\mathcal{W}} = (\tilde{W}_1, \tilde{W}_2)$ defined on the probability space $\tilde{\mathcal{A}}$ and an element $(\tilde{u}, \tilde{v}) \in \mathcal{U}_{\mathcal{Q}}(K_1, K_2, K_3)$ such that $\tilde{\mathbb{P}}$-a.s.

$$
\mathcal{T}_{\tilde{\mathcal{A}}}[(\tilde{u}, \tilde{v})](t) = (\tilde{u}(t), \tilde{v}(t)) \quad \forall t \in [0, T].
$$

By the construction of the operator $\mathcal{T}$, the pair $(\tilde{u}, \tilde{v})$ solves the system (2.1)–(2.2). In addition, the solution belongs to $\mathcal{U}_{\mathcal{Q}}(K_1, K_2, K_3)$, hence the estimate in item (iii) is satisfied.
4 Auxiliary Results and The Technical Propositions

In this particular section, we will prove various results and energy bounds which are crucial for the proof of the main result in Sect. 3. We assume that the Hypotheses 2.4 and 2.1 are satisfied. We split this section into four main subsections. In Sect. 4.1, we discuss the properties of Eq. (3.4) and prove several uniform bounds for $v$. Next, in Sect. 4.2, we analyse the properties of Eq. (3.5) and show various energy bounds for $u$. Next, in Sect. 4.3, we state the necessary results to prove the continuity of the map $T$. Finally, in Sect. A of the online resource [19], we state the necessary results to prove the compactness of the mapping $T$.

4.1 Properties of the System (3.4)

In the first part of this section, we will focus on Eq. (3.4). Let us recall the system for the convenience of the reader. We are interested in the solution of the following system for $x \in \mathcal{O}$, $t > 0$:

$$
\begin{cases}
    dv(t) = \left[ r_v \Delta v(t) + \kappa_v \chi^2(t) - \Upsilon v(t) \right] dt + \sigma_v v(t) dW_2(t), \\
v(0) = v_0.
\end{cases}
$$

(4.1)

At first, we will show the existence of a unique solution to the system (4.1) and then we will prove the solution is positive, suppose $(\chi, \eta) \in U_A(K_1, K_2, K_3)$.

**Theorem 4.1** Suppose $K_1, K_2 > 0$, $K_3 > 0$, and $T > 0$ are fixed. For any $v_0 \in L^2(\Omega; L^2(\mathcal{O}))$ and any $(\chi, \eta) \in U_A(K_1, K_2, K_3)$, the system (4.1) has a unique solution $v : \mathcal{O} \times [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ such that

$$
\mathbb{E} \sup_{0 \leq s \leq T} |v(s)|_{L^2} + \mathbb{E} \left( \int_0^T |\nabla v(s)|_{L^2}^2 \, ds \right)^{\frac{1}{2}} < \infty.
$$

Moreover, this solution is positive provided that $v_0 > 0$, $\mathbb{P} \times \text{Leb-a.s.}$

**Proof** Since $(\chi, \eta) \in U_A(K_1, K_2, K_3)$, $\chi$ is adapted. Existence of a unique solution follows by Theorem A.8 in the online resource. Next, by Proposition B.1 of the online resource [19], it follows that for $\frac{d}{2} \leq \frac{1}{2} + \frac{d}{l_2}$ there exists a constant $C > 0$ and $l_2 = 4$, such

$$
\mathbb{E} \| \chi \|^2_{L^1(0,T;L^{l_2})} \leq C \mathbb{E} \left( \| \chi \|^2_{L^\infty(0,T;L^2)} + \| \chi \|^2_{L^2(0,T;H^1_2)} \right).
$$

Since $(\chi, \eta) \in U_A(K_1, K_2, K_3)$, we know that there exists a constant $C_1 > 0$ such

$$
\mathbb{E} \| \chi \|^2_{L^1(0,T;L^{l_2})} \leq C_1.
$$

(4.2)

Observe we have for $l_2 = 4$,

$$
\mathbb{E} \| \chi \|^2_{L^2(0,T;L^{l_2/2})} \leq C \mathbb{E} \| \chi \|^2_{L^2(0,T;L^{l_2})} \leq C \mathbb{E} \| \chi \|^2_{L^4(0,T;L^2)} \leq C C_1.
$$
Let
\[
\mathcal{C}(\eta)(t) := \int_0^t e^{-(t-s)\Delta} \eta(s) \, ds,
\]
be the deterministic convolution. Then, we have by Lemma A.3 of the online resource [19] for any \( p \geq \frac{\nu}{2}, \gamma \in \mathbb{R}, \) and \( \delta \in (0, 2) \)
\[
\|\mathcal{C}(\eta)\|_{L^2(0,T; H^\delta_{p,\gamma})} \leq \|\eta\|_{L^2(0,T; H^{-\gamma}_{p,\gamma})}.
\]

Now, with \( \eta = \chi^2 \) and for any \( p > 2 \) we have
\[
E\|\mathcal{C}(\chi^2)\|_{L^2(0,T; H^1_p)} \leq C E\|\chi\|_{L^4(0,T; L^{2/2})}^2 < CK_1.
\]

Let \( \alpha = 1, q = 2, \) and \( \gamma = 0. \) Now, for any \( 2 \leq p < 6 \) we have
\[
\mathcal{C}[\chi^2] \in C^\beta_p(0,T; L^p(\mathcal{O})).
\]

In particular, there exists a constant \( C(K_1, K_2, K_3) > 0 \) such that for \( (\chi, \eta) \in U_A(K_1, K_2, K_3) \) and \( (u, v) = \mathcal{T}(\chi, \eta) \) we have
\[
E\|\mathcal{C}[\chi^2]\|_{C^\beta_p(0,T; L^2)} \leq C(K_1, K_2, K_3).
\] (4.3)

It remains to tackle \( \Upsilon v v \), but since \( \Upsilon v : L^2(\mathcal{O}) \to L^2(\mathcal{O}) \) and \( \Upsilon v : H^1_2(\mathcal{O}) \to H^1_2(\mathcal{O}) \) is bounded and linear, the term can be tackled by standard calculations. To give an estimate of the stochastic convolution term in \( L^2(0,T; H^1_{p,\gamma}(\mathcal{O})) \) we use the fact that \( \sum_{j=1}^\infty \lambda_j^{-2\gamma_2} |e_j|_{L^\infty}^2 \) is bounded, i.e. \( \gamma_2 > \frac{d}{2} \). To give an estimate of the stochastic convolution term we use estimate (4) of the main Theorem in [20]. A fix point argument leads the existence and uniqueness of a solution. For more details see Theorem A.8 of the online resource [19], where we have shown in Theorem C.1 the positivity of the solution of \( u(t) = u_0 + \int_0^t e^{A(t-s)} w(s) dW_2(s) \) along the proof of Theorem 2.3 in [41]. The positivity of \( v \) follows by comparison results, taking into account that \( \chi^2 \) is non-negative. \( \square \)

### 4.1.1 Uniform Bounds on \( v \) and \( \xi \)

In fact, uniform bounds on \( v \) are difficult to achieve, however, we can show several uniform bounds on \( \xi \), where \( \xi = v^{-1} \). Applying the Itô formula to \( \phi_1(v) = v^{-1} \) (see [10]) we infer that
\[
d\xi(t) = -\xi^2(t) \left( r_v \Delta v(t) + \kappa_v \chi^2(t) + \sigma_v (\text{Id} + A)^{-\gamma_2} v(t) \right) dt + \mu_v \xi(t) \, dt
\]
\[
+ \frac{1}{2} \text{Tr} \left[ D^2 \phi_1(v(t)) \left[ \sigma_v v(t) \sqrt{Q_2} \right] \left[ \sigma_v v(t) \sqrt{Q_2} \right]^* \right] - \sigma_v \xi(t) \, dW_2(t).
\]
Thus, we have the following equation in \( \xi \):

\[
\begin{aligned}
  d\xi(t) &= -\xi^2(t)\left[r_v \Delta v(t) + \kappa_v \chi^2(t) + \sigma_v (\text{Id} + A)^{-\gamma_2} v(t)\right]dt \\
  &+ \left[\mu_v + \frac{1}{2} \sigma_v^2 S(\gamma_2)\right]\xi(t)dt - \sigma_v \xi(t)dW_2(t), \\
  \xi(0) &= v_0^{-1},
\end{aligned}
\] (4.4)

where for \( \gamma > 0 \), \( S(\gamma) \) denotes the following operator

\[
S(\gamma) f := \sum_{k=1}^{\infty} (1 + \lambda_k)^{-\gamma} (e_k, f)e_k,
\]

which is bounded due to (3.1). Here, it should be noted that this formula holds if \( v \) is positive.

**Proposition 4.2** For any \( K_1, K_2 > 0 \) and \( K_3 > 0 \) there exist

(a) constants \( C_1, \delta_1 > 0 \) and \( \tilde{C}_1 > 0 \) such that for any \( T > 0 \), for any initial conditions \((u_0, v_0)\) satisfying Hypothesis 2.4, for any \((\chi, \eta) \in U_2(K_1, K_2, K_3)\) and \((u, v) = \mathcal{F}(\chi, \eta)\) we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} |\xi(s)|_{L_p}^p + 2p(p + 1)r_v \mathbb{E} \int_0^T \int_O \xi^{p+2}(s, x)|\nabla v(s, x)|^2 dx ds \\
+ \tilde{C}_1 \mathbb{E} \int_0^T \int_O \xi^{p+1}(s, x)\chi^2(s, x) dx ds \leq C_1 e^{\delta_1 T} \mathbb{E} |\xi_0|_{L_p}^p;
\] (4.5)

(b) constants \( C_2, \delta_2 > 0 \) such that for any \( T > 0 \), for any initial conditions \((u_0, v_0)\) satisfying Hypothesis 2.4, for any \((\chi, \eta) \in U_2(K_1, K_2, K_3)\) and \((u, v) = \mathcal{F}(\chi, \eta)\) we have

\[
\sup_{0 \leq s \leq T} \mathbb{E} |\xi(s)|_{L^1} + 4r_v \mathbb{E} \int_0^T \int_O \xi^3(s, x)|\nabla v(s, x)|^2 dx ds \\
+ \kappa_v \mathbb{E} \int_0^T \int_O \chi^2(s, x)\xi^2(s, x) dx ds \leq C_2 e^{\delta_2 T} \left( \mathbb{E} |\xi_0|_{L^1} + \mathbb{E} |\xi_0|_{L^2} \right);
\] (4.6)

(c) constants \( C_3, \tilde{C}_3 > 0 \) such that for any \( T > 0 \), for any initial conditions \((u_0, v_0)\) satisfying Hypothesis 2.4, for all \((\chi, \eta) \in U_2(K_1, K_2, K_3)\) and \((u, v) = \mathcal{F}(\chi, \eta)\) we have

\[
\mathbb{E} \left( \int_O |\ln \xi(t, x)| dx \right) + r_v \mathbb{E} \left( \int_0^T \int_O \xi^2(s, x)|\nabla v(s, x)|^2 dx ds \right) \\
+ \kappa_v \mathbb{E} \left( \int_0^T \int_O \chi^2(s, x)\xi(s, x) dx ds \right) \\
\leq \mathbb{E} |v_0|_{L^1} + C_3 \mathbb{E} \left( \int_O |\ln \xi_0(x)| dx \right) + \tilde{C}_3 T.
\] (4.7)
Proof of Part (a) For $k \in \mathbb{N}$ and $p \geq 2$, let us define the stopping time $	au_k := \inf\{t > 0 \mid |\xi(t)|_{L^p}^p \geq k\}$. Applying the Itô formula (see [29]) to the function $\phi_2(z) = |z|^p$, for $p \geq 2$, and taking into account that
\[
\text{Tr} \left[ D^2 \phi(\xi(t)) \left[ \sigma_v \xi(t) \sqrt{Q_2} \right] \left[ \sigma_v \xi(t) \sqrt{Q_2}^* \right] \right] 
= p(p-1)\sigma_v^2 \xi^p(t) \sum_{k=1}^{\infty} (1 + \lambda_k)^{-\gamma_2} |\epsilon_k|^2,
\]
for ($\phi = \phi_2 \circ \phi_1$) we infer for $t \in [0, T \wedge \tau_k]$
\[
d|\xi(t)|_{L^p}^p = \left[ -p \int_\Omega \left( r_v \xi^{p+1}(t,x) \Delta \nu(t,x) 
+ \kappa_v \xi^{p+1}(t,x) \chi^2(t,x) - \mu_v \xi^p(t,x) \right) dx 
+ \frac{p(p-1)}{2} \sigma_v^2 S(\gamma_2) \int_\Omega \xi^p(t,x) dx 
- p \int_\Omega \sigma_v \xi^{p+1}(t,x) (\text{Id} + A)^{-\gamma_2} \nu(t,x) dx \right] dt 
- p\sigma_v \int_\Omega \xi^p(t,x) dW_2(t,x). \tag{4.8}
\]
Using the positivity of $\nu$, the maximum principle and $\sigma_v > 0$, we conclude that the term $-p \int_\Omega \sigma_v \xi^{p+1}(t,x) (\text{Id} + A)^{-\gamma_2} \nu(t,x) dx$ can be dropped from the right hand side. Then, applying integration by parts to the first term in right hand side of (4.8) we get for $t \in [0, T \wedge \tau_k]$
\[
d|\xi(t)|_{L^p}^p \leq \left[ -p(p+1)r_v \int_\Omega \xi^p(t,x) \nabla \xi(t,x) \cdot \nabla \nu(t,x) dx 
+ r_v p \int_{\partial \Omega} \bar{n} \cdot \nabla \nu(t,x) \xi(t,x) dx 
- p \int_\Omega \left[ \kappa_v \xi^{p+1}(t,x) \chi^2(t,x) \right] dx 
+ p \left( \mu_v + \sigma_v^2 S(\gamma_2) \frac{(p-1)}{2} \right) \int_\Omega \xi^p(t,x) dx \right] dt 
- p\sigma_v \int_\Omega \xi^p(t,x) dW_2(t,x) \tag{4.9} 
\leq \left[ -p(p+1)r_v \int_\Omega \xi^{p+2}(t,x)(\nabla \nu(t,x))^2 dx 
- p \int_\Omega \left[ \kappa_v \xi^{p+1}(t,x) \chi^2(t,x) \right] dx 
+ p \left( \mu_v + \sigma_v^2 S(\gamma_2) \frac{(p-1)}{2} \right) \int_\Omega \xi^p(t,x) dx \right] dt 
- p\sigma_v \int_\Omega \xi^p(t,x) dW_2(t,x).
Here, the positive term $p \int_{\partial \Omega} \vec{n} \cdot \nabla v \xi \, dx$ vanishes due to the Neumann boundary conditions.

In the integral form, taking supremum over $s \in [0, t \wedge \tau_k]$ and taking expectation we infer that

$$
\mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_k} |\xi(s)|_{L^p}^p + p(p + 1)r_v \mathbb{E} \int_0^{t \wedge \tau_k} \int_{\Omega} \xi^{p+2}(s, x) |\nabla v(s, x)|^2 \, dx \, ds \\
+ p \kappa_v \mathbb{E} \int_0^{t \wedge \tau_k} \int_{\Omega} \xi^{p+1}(s, x) \chi^2(s, x) \, dx \, ds \\
\leq \mathbb{E} |\xi_0|_{L^p}^p + C(\mu_v, S(\gamma_2), \sigma_v, p) \mathbb{E} \int_0^{t \wedge \tau_k} |\xi(s)|_{L^p}^p \, ds \\
+ \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_k} \left| p \sigma_v \int_0^{s \wedge \tau_k} \int_{\Omega} \xi^p(r, x) \, dW_2(r, x) \right|. 
$$

Using the Burkholder–Davis–Gundy inequality, the Cauchy–Schwarz inequality, and the Young inequality we infer

$$
\mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_k} \left| p \sigma_v \int_0^{s \wedge \tau_k} \int_{\Omega} \xi^p(r, x) \, dW_2(r, x) \right| \\
\leq p^2 \sigma_v^2 C \mathbb{E} \left( \int_0^{t \wedge \tau_k} \left| \int_{\Omega} \xi^p(s, x) \, dx \right|^2 \, ds \right)^{1/2} \\
\leq p^2 \sigma_v^2 C \mathbb{E} \left( \sup_{s \in [0, t \wedge \tau_k]} |\xi(s)|_{L^p}^p \int_0^{t \wedge \tau_k} \int_{\Omega} \xi^p(s, x) \, dx \, ds \right)^{1/2} \\
\leq C \mathbb{E} \left( \left\{ \sup_{s \in [0, t \wedge \tau_k]} |\xi(s)|_{L^p}^p \right\}^{1/2} \left\{ p \sigma_v \int_0^{t \wedge \tau_k} |\xi(s)|_{L^p}^p \, ds \right\}^{1/2} \right) \\
\leq \frac{\epsilon}{2} \mathbb{E} \sup_{s \in [0, t \wedge \tau_k]} |\xi(s)|_{L^p}^p + \frac{p \sigma_v}{2\epsilon} \mathbb{E} \int_0^{t \wedge \tau_k} |\xi(s)|_{L^p}^p \, ds. 
$$

Using this in (4.10), choosing $\epsilon = \frac{1}{2}$ and finally rearranging we obtain,

$$
\mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_k} |\xi(s)|_{L^p}^p + 2p(p + 1)r_v \mathbb{E} \int_0^{t \wedge \tau_k} \int_{\Omega} \xi^{p+2}(s, x) |\nabla v(s, x)|^2 \, dx \, ds \\
+ 2p \kappa_v \mathbb{E} \int_0^{t \wedge \tau_k} \int_{\Omega} \xi^{p+1}(s, x) \chi^2(s, x) \, dx \, ds \\
\leq \mathbb{E} |\xi_0|_{L^p}^p + C(\mu_v, S(\gamma_2), \sigma_v, p) \mathbb{E} \int_0^{t \wedge \tau_k} |\xi(s)|_{L^p}^p \, ds. 
$$

Dropping the second and third terms from the left hand side and applying the Gronwall lemma we infer that, there exists a constant $C > 0$ depending on $\mu_v, S(\gamma_2), \sigma_v, p$ such that
\[ E \sup_{0 \leq s \leq T \wedge \tau_k} |\xi(s)|_{L^p}^p \leq E|\xi_0|_{L^p}^p e^{(T \wedge \tau_k)C(\mu_v, S(\gamma_2), \sigma_v, p)}. \]

By the Chebyshev inequality, we observe that as \( k \to \infty \), we have \( \mathbb{P}-\text{a.s.} \ T \wedge \tau_k \to T. \) Therefore, taking the limit \( k \to \infty \) we infer

\[ E \sup_{0 \leq s \leq T} |\xi(s)|_{L^p}^p \leq E|\xi_0|_{L^p}^p e^{TC(\mu_v, S(\gamma_2), \sigma_v, p)}. \]  (4.13)

Substituting (4.13) in (4.12), we infer that there exist constants \( \delta_1, C_1 > 0 \) such that we get the assertion in part (a).

**Proof of Part (b)** Now we will prove (4.7). For clarity, we write the calculation without defining a stopping time \( \tau_k := \inf \{ t > 0 \mid |\xi(t)|_{L^1} \geq k \} \). Tracing the proof it can be shown that taking into account the stopping time and being precise will not change the result. Applying the Itô formula to the function \( \phi : L^1(\mathcal{O}) \to \mathbb{R} \) defined by \( \phi(z) = \int_{\mathcal{O}} z(x) \, dx \), we infer that

\[
d|\xi(t)|_{L^1} = -rv \int_{\mathcal{O}} \xi^2(t, x) \Delta v(t, x) \, dx \, dt - \kappa_v \int_{\mathcal{O}} \xi^2(t, x) \chi^2(t, x) \, dx \, dt
\]
\[
+ \mu_v \int_{\mathcal{O}} \xi(t, x) \, dx \, dt + \sigma_v \int_{\mathcal{O}} \xi(t, x) \, dW_2(t, x)
\]
\[
- p \int_{\mathcal{O}} \sigma_v \xi(t, x) \, (\text{Id} + A)^{-\gamma_2} v(t, x) \, dx \, dt
\]
\[
\leq -2rv \int_{\mathcal{O}} \xi^3(t, x) |\nabla v(t, x)| \, dx \, dt - \kappa_v \int_{\mathcal{O}} \xi^2(t, x) \chi^2(t, x) \, dx \, dt
\]
\[
+ \mu_v S(\gamma_2) \int_{\mathcal{O}} \xi(t, x) \, dx + \sigma_v \int_{\mathcal{O}} \xi(t, x) \, dW_2(t, x),
\]

where the last term in the right hand side of the first equality is dropped using the similar arguments as in part (a) and used integration by parts to get

\[
-r_v \int_{\mathcal{O}} \xi^2(t, x) \Delta v(t, x) \, dx = r_v \int_{\mathcal{O}} \nabla (\xi^2(t, x)) \cdot \nabla v(t, x) \, dx
\]
\[
= -2rv \int_{\mathcal{O}} \xi^3(t, x) \cdot |\nabla v(t, x)|^2 \, dx.
\]

Finally, taking integration from 0 to \( t \), then taking expectation, and rearranging we obtain,

\[
E|\xi(s)|_{L^1} + 2rvE \int_0^t \int_{\mathcal{O}} \xi^3(s, x) |\nabla v(s, x)|^2 \, dx \, ds + \kappa_v E \int_0^t \int_{\mathcal{O}} \chi^2(s, x) \xi^2(s, x) \, dx \, ds
\]
\[
\leq E|\xi_0|_{L^1} + C(\mu_v, S(\gamma_2)) E \left( \int_0^t |\xi(s)|_{L^2}^2 \, ds \right)^{1/2}.
\]
Dropping the second, third and fourth term from the left hand side and by applying part (a) and the Gronwall lemma we infer that there exists a constant $\delta_2 > 0$ depending on $\mu_v$ and a constant $C_2 > 0$ such that we get the assertion in part (b).

**Proof of Part (c)** Again, for clarity, we write the calculation without defining a stopping time $\tau_k := \inf\{t > 0 \mid \int \ln(\xi(t, x)) \, dx \geq k\}$. Tracing the proof it can be shown that taking into account the stopping time and being precise will not change the result. We note that applying the Itô formula for $\Phi(z) = -\ln(z)$

$$
\ln(\xi(t)) - \ln(\xi(0)) = \int_0^t \xi^{-1}(s)\left[d\xi(s)\right] ds - \frac{\sigma_v^2}{2} \int_0^t \text{Tr}[D^2\Phi(v(t))(v(t)\sqrt{Q_2})(v(t)\sqrt{Q_2})^*] \, ds.
$$

(4.14)

From the previous energy estimates we have

$$
d\xi(t, x) = -\xi^2(t, x)\left[r_v \Delta v(t, x) + \kappa_v \chi^2(t, x) + \sigma_v \left(\text{Id} + A\right)^{-\gamma^2} v(t, x)\right] dt + \mu_v \xi(t, x) dt
\quad - \frac{1}{2} S(\gamma_2) \sigma_v^2 \xi(t, x) dt - \sigma_v \xi(t, x) dW_2(t, x).
$$

By integration by parts we infer that

$$
- r_v \int \xi(t, x) \Delta v(t, x) \, dx = r_v \int \nabla \xi(t, x) \cdot \nabla v(t, x) \, dx
\quad = - r_v \int \xi^2(t, x) |\nabla v(t, x)|^2 \, dx.
$$

Using this in (4.15), rearranging and rewriting in integral form we get

$$
\int \ln \xi(t, x) \, dx + r_v \int_0^t \int \xi^2(s, x) |\nabla v(s, x)|^2 \, dx \, ds
\quad + \kappa_v \int_0^t \int \xi(s, x) \chi^2(s, x) \, dx \, ds \quad \leq \int \ln \xi(0) \, dx + \int_0^t \int \sigma_v \, dW_2(t, x) + T\left(\mu_v - \frac{1}{2} S(\gamma_2) \sigma_v^2\right).
$$

(4.16)
Observe, firstly that we have for any \( n \in \mathbb{N} \) and \( x > 0 \)

\[
- \ln(1/x) \leq 1_{(1, \infty)}(x) \ln(x) \leq (x - 1) \leq x.
\]

Straight forward calculations using Burkholder–Davis–Gundy inequality and taking expectation we obtain

\[
\mathbb{E}\left( \int_{0}^{t} |\ln \xi(s, x) dx| \right) + r_{v} \mathbb{E}\left( \int_{0}^{t} \int_{\Omega} \xi^{2}(s, x)|\nabla v(s, x)|^{2} dx ds \right) \\
+ \kappa_{v} \mathbb{E}\left( \int_{0}^{t} \int_{\Omega} \chi^{2}(s, x) \xi(s, x) dx ds \right) \\
\leq \mathbb{E}|v_{0}|_{L^{1}} + \mathbb{E}\left( \int_{0}^{t} |\ln \xi_{0}(x)| dx \right) + TC(\mu_{v}, S(\gamma_{2}), \sigma_{v}).
\]

(4.17)

4.2 Properties of the System (3.5)

Given the couple \((\chi, \eta) \in \mathcal{U}_{A}(K_{1}, K_{2}, K_{3})\) we investigate in this section the existence and uniqueness of a solution to the auxiliary system

\[
\begin{aligned}
du(t) &= \left[ r_{u} \Delta u(t) + \kappa_{u} \frac{\chi^{2}(t)}{v(t)} - \gamma_{u} u(t) \right] dt + \sigma_{u} u(t) dW_{1}(t), \\
u(0) &= u_{0}.
\end{aligned}
\]

(4.18)

Note that \( v \) is a solution to system (3.4). Secondly, we investigate the regularity of \( u \).

**Theorem 4.3** Suppose \( K_{1}, K_{2} > 0 \) and \( K_{3} > 0 \) are fixed. Then for any \((\chi, \eta) \in \mathcal{U}_{A}(K_{1}, K_{2}, K_{3})\) and \( v \) being a solution to (4.1) there exists a unique solution \( u \) to system (4.18) such that for any \( \tilde{\rho} > \frac{d}{2} \)

\[
u \in C([0, T]; H_{2}^{-\tilde{\rho}}(\bar{\Omega}) \cap L^{2}(0, T; H_{2}^{1-\tilde{\rho}}(\bar{\Omega}))) \quad \mathbb{P} - a.s.
\]

Moreover, the solution to the system (4.18) is non-negative provided \( u_{0}(x) \geq 0 \) for all \( x \in \Omega \) and \( v(t, x) > 0 \), for \((t, x) \in [0, T] \times \Omega \).

**Proof** Since \([0, T] \ni t \mapsto \frac{\chi^{2}(t)}{v(t)} := G(t) \) belongs to \( L^{1}(\Omega \times [0, T]; L^{1}(\Omega)) \), we know by standard arguments that a solution exists. In particular, we will apply Theorem A.8 of the online resource to prove the existence and uniqueness of the system (4.18). Similarly, we may use the setting of the book of Da Prato and Zabczyk [10,Sect. 6.3.3] or use Theorem 4.5 [43]. Let us provide the basic steps of the proof for our case.

Since \((\chi, \eta) \in \mathcal{U}_{A}(K_{1}, K_{2}, K_{3})\), it follows by Proposition 4.2-(c) that \( \chi^{2}\xi = \chi^{2}/v \in L^{1}((0, T) \times \Omega; L^{1}(\Omega)) \). So, setting \( f(t) = G(t) \) for all \( t \in [0, T] \) we obtain the Hypotheses are valid.

The non-negativity of the solution can be proved following Sect. 2.6 in [1] or [2]. □
4.2.1 Uniform Bounds on $u$

In this subsection, we show several uniform bounds on $u$ under some assumptions.

**Proposition 4.4** For any $K_1, K_2 > 0$ and $K_3 > 0$, there exists constants $C_4, \delta_4 > 0$ and $\mathcal{C}_4 > 0$ such that for any $T > 0$, for any initial conditions $(u_0, v_0)$ satisfying Hypothesis 2.4, for all $(\chi, \eta) \in \mathcal{U}_2(K_1, K_2, K_3)$, $v$ being a solution to (4.1), and $u$ being a solution to system (4.18)

$$
\mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_{L^2}^2 + 4 \kappa_u \mathbb{E} \int_0^t |\nabla u(s)|_{L^2}^2 \, ds + \mathcal{C}_4 \mathbb{E} \int_0^t |u(s)|_{L^2}^2 \, ds
\leq C_4 e^{\delta_4 T} \left[ \mathbb{E}|u_0|^2_{L^2} + C \mathbb{E} \int_0^t \int_{\mathcal{O}} u(s, x) \chi^2(s, x) \xi(s, x) \, dx \, ds \right].
$$

(4.19)

**Proof** For clarity, we write the calculation without defining a stopping time $\tau_k := \inf\{t > 0 \mid |u(t)|_{L^2}^2 > k\}$ as done in Proposition 4.2-(a). Tracing the proof it can be seen easily that taking into account the stopping time and being precise will not change the result. Let us put $\psi_1(u) = |u|^p_{L^p}$. For $z, g, g_1, g_2 \in L^2(\mathcal{O})$ we infer that

$$
D\psi_1(z)[g] = p \int_{\mathcal{O}} |z(x)|^{p-1} g(x) \, dx, \quad D^2\psi_1(z)[g_1, g_2]
= p(p-1) \int_{\mathcal{O}} |z(x)|^{p-2} g_1(x)g_2(x) \, dx.
$$

Applying the Itô formula to the function $\psi_1(u)$ for $p = 2$ we obtain,

$$
d|u(t)|_{L^2}^2 = 2r_u \int_{\mathcal{O}} u(t, x) \cdot \Delta u(t, x) \, dx + 2\kappa_u \int_{\mathcal{O}} u(t, x) \frac{\chi^2(t, x)}{\nu(t, x)} \, dx
- 2\mu_u |u(t)|_{L^2}^2 + \sigma_u \int_{\mathcal{O}} u(t)[(\text{Id} + A)^{-\gamma_1}u(t)] \, dx
+ \frac{1}{2} S(\gamma_1) \sigma^2_u |u(t)|_{L^2}^2 \, dt
+ 2\sigma_u |u(t), u(t)\, dW_1(t)|_{L^2},
$$

(4.20)

where we calculated the trace term as before. Using integration by parts we get,

$$
2r_u \int_{\mathcal{O}} u(t, x) \cdot \Delta u(t, x) \, dx = -2r_u \int_{\mathcal{O}} (\nabla u(t, x))^2 \, dx.
$$

We may bound the term

$$
\sigma_u \int_{\mathcal{O}} u(t)[(\text{Id} + A)^{-\gamma_1}u(t)] \, dx \leq \sigma_u |u(t)|_{L^2} |(\text{Id} + A)^{-\gamma_1}u(t)|_{L^2}
\leq \sigma_u \|(\text{Id} + A)^{-\gamma_1}\|_{L^2(L^2)} |u(t)|_{L^2}^2
\leq C(\sigma_u) |u(t)|_{L^2}^2.
$$
Using the above estimates, we rewrite (4.20) in the integral form as follows:

\[ |u(t)|^2_{L^2} + 2r_u \int_0^t |\nabla u(s)|^2_{L^2} \, ds + 2\mu_u \int_0^t |u(s)|^2_{L^2} \, ds \]

\[ \leq |u_0|^2_{L^2} + C \int_0^t \int_D u(s, x) \chi^2(s, x) \xi(s, x) \, dx \, ds \]

\[ + C(S(\gamma_1), \sigma_u) \int_0^t |u(s)|^2_{L^2} \, ds + 2\sigma_u \int_0^t \int_D u^2(s, x) \, dW_1(s, x) \]  

where \( \xi = v^{-1} \) and the constant \( C > 0 \) depends on \( \kappa_u \). Now taking supremum over \( s \in [0, t] \) and then taking expectation we infer,

\[ \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|^2_{L^2} + 2r_u \mathbb{E} \int_0^t |\nabla u(s)|^2_{L^2} \, ds + 2\mu_u \mathbb{E} \int_0^t |u(s)|^2_{L^2} \, ds \]

\[ \leq \mathbb{E} |u_0|^2_{L^2} + C \mathbb{E} \int_0^t \int_D u(s, x) \chi^2(s, x) \xi(s, x) \, dx \, ds \]

\[ + C(S(\gamma_1), \sigma_u) \mathbb{E} \int_0^t |u(s)|^2_{L^2} \, ds + \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^t \int_D u^2(s, x) \, dW_1(s, x) \right| . \]  

Applying the Burkholder–Davis–Gundy inequality and following similar steps as in (4.11) we infer,

\[ \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^t \int_D u^2(s, x) \, dW_1(s, x) \right| \leq \frac{\varepsilon}{2} \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|^2_{L^2} + \frac{\sigma_u}{2\varepsilon} \mathbb{E} \int_0^t |u(s)|^2_{L^2} \, ds, \]  

where we have applied the Young inequality for \( \varepsilon > 0 \). Choosing \( \varepsilon = \frac{1}{2} \) in (4.23) and putting in (4.22) and rearranging we obtain

\[ \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|^2_{L^2} + 4r_u \mathbb{E} \int_0^t |\nabla u(s)|^2_{L^2} \, ds + 2\mu_u \mathbb{E} \int_0^t |u(s)|^2_{L^2} \, ds \]

\[ \leq 2\mathbb{E} |u_0|^2_{L^2} + C \mathbb{E} \int_0^t \int_D u(s, x) \chi^2(s, x) \xi(s, x) \, dx \, ds \]

\[ + C(S(\gamma_1), \sigma_u) \mathbb{E} \int_0^t |u(s)|^2_{L^2} \, ds. \]

Dropping the second and third term from the left hand side and by the application of Gronwall’s lemma, there exist constants \( \delta_4 \) and \( C_4 \) such that we get the assertion. \( \square \)

**Proposition 4.5** For any \( K_1, K_2 > 0 \) and \( K_3 > 0 \), there exist constants \( C_5, \delta_5 > 0 \) such that for any \( \varepsilon > 0 \), for any \( T > 0 \), for any initial conditions \((u_0, v_0)\) satisfying Hypothesis 2.4, for any \((\chi, \eta) \in U_\delta(K_1, K_2, K_3)\), \( v \) being a solution to (4.1), and \( u \) being a solution to system (4.18) we have for any \( t \in [0, T] \).
\[
\mathbb{E}\left( \ln(\xi(t)) \cdot u(t) \right) + \mu_x \mathbb{E} \int_0^t \left( \ln(\xi(s)) \cdot u(s) \right) ds \\
+ r_u \mathbb{E} \int_0^t \int_{\mathcal{O}} \xi^2(s, x)u(s, x) \cdot |\nabla v(s, x)|^2 \, dx \, ds \\
+ \kappa_v \mathbb{E} \int_0^t \int_{\mathcal{O}} \chi^2(s, x)\xi(s, x)u(s, x) \, dx \, ds \\
\leq \mathbb{E}\left( \ln(\xi_0) \cdot u_0 \right) + \epsilon \mathbb{E} \int_0^t |\nabla u(s)|^2_{L^2} \, ds + \frac{(r_u + r_v)}{4\epsilon} \mathbb{E} \int_0^t |\xi(s) \nabla v(s)|^2_{L^2} \, ds \\
+ \kappa_u \mathbb{E} \int_0^t \ln(\xi(s)) \chi^2(s, x)\xi(s, x)u(s, x) \, dx \, ds + \mu_v \mathbb{E} \int_0^t \int_{\mathcal{O}} \xi(s, x) u(s, x) \, dx \, ds \\
+ C(S(\gamma_1), \sigma_u) \mathbb{E} \int_0^t \left| \ln(\xi(s)) \cdot u(s) \right| \, ds \\
+ \sigma_u \mathbb{E} \int_0^t \int_{\mathcal{O}} \ln(\xi(s, x)) \left( \text{Id} + A \right)^{-\gamma_1} u(s, x) \, dx \, ds.
\]

**Proof** We will apply the Itô formula to \( \phi(t) = \ln(\xi(t)) \cdot u(t) \). For clarity, we write the calculation without defining a stopping time \( \tau_k := \inf\{t > 0 \mid (\ln(\xi(t)) \cdot u(t)) \geq k\} \) as done in Proposition 4.2-(a). Tracing the proof it can be seen easily that taking into account the stopping time and being precise will not change the result.

Before applying the Itô formula, let us note that using integration by parts and the Hölder inequality leads to

\[
\int_{\mathcal{O}} \ln(\xi(s, x)) \Delta u(s, x) \, dx = - \int_{\mathcal{O}} (\nabla \ln(\xi(s, x))) \cdot (\nabla u(s, x)) \, dx \\
= \int_{\mathcal{O}} \xi(s, x)(\nabla v(s, x)) \cdot (\nabla u(s, x)) \, dx \\
\leq \left( \int_{\mathcal{O}} \xi^2(s, x)|\nabla v(s, x)|^2 \, dx \right)^{1/2} \left( \int_{\mathcal{O}} |\nabla u(s, x)|^2 \, dx \right)^{1/2}.
\]

By the Young inequality, we get for any \( \epsilon > 0 \)

\[
r_u \int_{\mathcal{O}} \ln(\xi(s, x)) \Delta u(s, x) \, dx \leq \frac{r_u^2}{4\epsilon} \int_{\mathcal{O}} \xi^2(s, x)|\nabla v(s, x)|^2 \, dx + \epsilon \int_{\mathcal{O}} |\nabla u(s, x)|^2 \, dx.
\]

Inserting this into the Itô formula and applying integration by parts leads to

\[
- \int_{\mathcal{O}} \xi(s, x) \Delta v(s, x) u(s, x) \, dx = \int_{\mathcal{O}} \nabla(\xi(s, x)u(s, x)) \cdot (\nabla v(s, x)) \, dx \\
= \int_{\mathcal{O}} \xi(s, x)(\nabla u(s, x)) \cdot (\nabla v(s, x)) \, dx \\
+ \int_{\mathcal{O}} \nabla \xi(s, x) \cdot \nabla v(s, x) u(s, x) \, dx
\]

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\[ \mathbb{E}(\ln(\xi(s)) \cdot u(t)) - \mathbb{E}(\ln(\xi(0)) \cdot u(0)) \]
\[ \leq -r_v \mathbb{E} \int_0^t \int_{\mathcal{O}} \xi(s, x) \left[ r_v \Delta v(s, x) + \kappa_u \chi^2(s, x) \right] u(s, x) \, dx \, ds \]
\[ + \mu_v \mathbb{E} \int_0^t \int_{\mathcal{O}} \xi(s, x) u(s, x) \, dx \, ds - \frac{1}{2} S(\gamma_1) \sigma_v^2 \mathbb{E} \int_0^t \int_{\mathcal{O}} u(s, x) \, dx \, ds \]  
\[ + \mathbb{E} \int_0^t \int_{\mathcal{O}} \ln(\xi(s)) \left[ r_u \Delta u(s, x) + \kappa_u \chi^2(s, x) \xi(s, x) \right] \, dx \, ds \]
\[ - \mu_u u(s, x) + \sigma_u (\text{Id} + A)^{-\gamma_1} u(s, x) \]  
\[ \text{dx} \, ds. \]  

Using the previous estimates we get,
\[ \mathbb{E}(\ln(\xi(t)) \cdot u(t)) - \mathbb{E}(\ln(\xi(0)) \cdot u_0) \]
\[ \leq -r_v \mathbb{E} \int_0^t \int_{\mathcal{O}} \xi^2(s, x) u(s, x) |\nabla v(s, x)|^2 \, dx \, ds \]
\[ -\kappa_u \mathbb{E} \int_0^t \int_{\mathcal{O}} \chi^2(s, x) \xi(s, x) u(s, x) \, dx \, ds \]
\[ + \mu_v \mathbb{E} \int_0^t \int_{\mathcal{O}} \xi(s, x) u(s, x) \, dx \, ds \]
\[ + \frac{1}{4} r_v \mathbb{E} \int_0^t \int_{\mathcal{O}} \xi^2(s, x) |\nabla v(s, x)|^2 \, dx \, ds \]
\[ + \mathbb{E} \int_0^t \int_{\mathcal{O}} |\nabla u(s)|^2 \, dx \, ds + \kappa_u \mathbb{E} \int_0^t \int_{\mathcal{O}} \ln(\xi(s, x)) \chi^2(s, x) \xi(s, x) \, dx \, ds \]
\[ - \mu_u \mathbb{E} \int_0^t \int_{\mathcal{O}} \ln(\xi(s, x)) u(s, x) \, dx \, ds \]
\[ + \sigma_u \mathbb{E} \int_0^t \int_{\mathcal{O}} \ln(\xi(s, x)) (\text{Id} + A)^{-\gamma_1} u(s, x) \, dx \, ds. \]
Rearranging the terms we finally get

\[
\mathbb{E}\left( \ln(\xi(t)) \cdot u(t) \right) - \mathbb{E}\left( \ln(\xi_0) \cdot u_0 \right) \\
+ \mu_u \mathbb{E} \int_0^t \ln(\xi(s)) \cdot u(s) \, ds \\
+ r_v \mathbb{E} \int_0^t \int_O \xi^2(s, x) u(s, x) |\nabla v(s, x)|^2 \, dx \, ds \\
+ \kappa_u \mathbb{E} \int_0^t \int_O \chi^2(s, x) \xi(s, x) u(s, x) \, dx \, ds \\
\leq \mu_v \mathbb{E} \int_0^t \left( \frac{1}{2} \chi_1 \xi(s, x) u(s, x) \right) \, dx \, ds + \frac{1}{2} S(\gamma_1) \sigma_u^2 \mathbb{E} \int_0^t \ln(\xi(s)) \cdot u(s) \, ds \\
+ \kappa_v \mathbb{E} \int_0^t \ln(\xi(s)) \chi^2(s, x) \xi(s, x) \, dx \, ds \\
+ \frac{(r_u + r_v)^2}{4\epsilon} \mathbb{E} \int_0^t \int_O \xi^2(s, x) |\nabla v(s, x)|^2 \, dx \, ds \\
+ \epsilon \mathbb{E} \int_0^t |\nabla u(s)|^2_{L^2} \, ds + \sigma_u \mathbb{E} \int_0^t \int_O \ln(\xi(s, x)) \left( \text{Id} + A \right)^{-\frac{1}{2}} u(s, x) \, dx \, ds.
\]

Thus, we get the assertion. \(\square\)

**Corollary 4.6** For any \(K_1, K_2 > 0\) and \(K_3 > 0\),

(a) and any \(n \in \mathbb{N}\) there exist a constant \(C_4 > 0\) such that for any \(\epsilon > 0\), for any \(T > 0\), for any initial conditions \((u_0, v_0)\) satisfying Hypothesis 2.4, for any \((\chi, \eta) \in \mathcal{U}_Q(K_1, K_2, K_3)\), \(v\) being a solution to (4.1), and \(u\) being a solution to system (4.18) we have

\[
- \mathbb{E} \int_O \ln(\xi(t, x)) \, u(t, x) \, dx \\
\leq C_4 \frac{n}{4\epsilon} \left\{ \mathbb{E} \sup_{0 \leq s \leq T} |\eta(s)|^2_{L^2} + \mathbb{E} \int_0^T |\eta(s)|^2_{H^1} \, ds \right\}^{\frac{1}{n}} + \epsilon \mathbb{E} |u(t)|^2_{L^2}.
\]

(b) there exist constants \(c_1, \ldots, c_5 > 0\) such that for any \(\epsilon > 0\), for any \(T > 0\), for any initial conditions \((u_0, v_0)\) satisfying Hypothesis 2.4, for any \((\chi, \eta) \in \mathcal{U}_Q(K_1, K_2, K_3)\), \(v\) being a solution to (4.1), and \(u\) being a solution to system (4.18) we have

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Let us start to show item (a). Before starting, observe, firstly that we have for any \( n \in \mathbb{N} \) and \( x > 0 \)

\[- \ln(1/x) \leq n \ln(x^{-\frac{1}{n}}) \leq n(1_{(1,\infty)}(x) \ln(x^{-\frac{1}{n}}) \leq nx^{-\frac{1}{n}}\]

and, secondly,

\[
-\mathbb{E} \int_0^T \ln(\xi(t,x)u(t,x)) \, dx \leq n \mathbb{E} \int_0^1 \mathbb{E}[\xi(t,x)u(t,x)] \, dx \leq n \left\{ \mathbb{E} \int_0^1 v^{\frac{1}{n}}(t,x) \, dx \right\}^{\frac{1}{n}} \left\{ \mathbb{E}[\xi(t,x)]^2 \right\}^{\frac{1}{2}} \leq \frac{n}{4\varepsilon} \mathbb{E} \int_0^1 v^{\frac{2}{n}}(t,x) \, dx + \varepsilon \mathbb{E}[\xi(t,x)]^2 \leq \frac{n}{4\varepsilon} \left\{ \mathbb{E} \int_0^1 v(t,x) \, dx \right\}^{\frac{1}{n}} + \varepsilon \mathbb{E}[\xi(t,x)]^2_2. \tag{4.28}
\]

By Theorem 4.1, we know that there exists constants \( C_1, C_2 > 0 \) such that

\[
\mathbb{E}[v(t)]_{L^1} \leq C_1 \mathbb{E}[v(t)]_{L^2} \leq C_2 \left( \mathbb{E} \sup_{0 \leq s \leq T} |\eta(s)|^2_{L^2} + \mathbb{E} \int_0^T |\eta(s)|^2_{H^1} \, ds \right). \tag{4.29}
\]

The assertion follows by the Jensen inequality.

Now we will show (b). By Proposition 4.5 we know that for any \( \varepsilon_1, \varepsilon_2 > 0 \) we have

\[
\mathbb{E} \left( \ln(\xi(t)) \cdot u(t) \right) + \mu_u \mathbb{E} \int_0^t (\ln(\xi(s)) \cdot u(s)) \, ds + \mathbb{E} \int_0^t \int_0^1 \chi^2(s,x) \xi(s,x)u(s,x) \, dx \, ds 
\leq \mathbb{E} \int_0^1 \ln(\xi_0(x))u_0(x) \, dx + \varepsilon_1 \mathbb{E} \int_0^t |\nabla u(s)|^2_{L^2} \, ds 
+ \frac{(r_u + r_v)^2}{4\varepsilon_1} \mathbb{E} \int_0^t |\xi(s)\nabla v(s)|^2_{L^2} \, ds 
+ \kappa_u \mathbb{E} \int_0^t \ln(\xi(s,x)) \chi^2(s,x) \xi(s,x) \, dx \, ds|.
\]
\[ C(S(\gamma_1), \sigma_u) \mathbb{E} \int_0^t \int_{\Omega} (|\ln(\xi(s, x))| \cdot u(s, x) \, dx) \, ds \\
+ \mu_v \mathbb{E} \int_0^t \int_{\Omega} \xi(s, x) u(s, x) \, dx \, ds + \frac{\sigma_u^2}{4\varepsilon_2} \mathbb{E} \int_0^t |\xi(s)|^2_{L^2} \, ds \\
+ \varepsilon_2 \mathbb{E} \int_0^t |u(s)|^2_{L^2} \, ds. \tag{4.30} \]

First, note due to estimate (4.28) and (4.29) the negative part of
\[ \mathbb{E}(\ln(\xi(t)) \cdot u(t)) \]
and the negative part of
\[ \mathbb{E} \int_0^t \int_{\Omega} (\ln(\xi(s, x)) \cdot u(s, x) \, dx \, ds \]
can be estimated by
\[ \left( \frac{n}{4\varepsilon_0} + \frac{n}{4\tilde{\varepsilon}_0} \right) \mathbb{E} \sup_{0 \leq s \leq T} |\eta(s)|^2_{L^2} \\
+ \mathbb{E} \int_0^T |\eta(s)|^2_{H^1} \, ds \right)^{\frac{1}{2}} \varepsilon_0 \int_0^T \mathbb{E}|u(s)|^2_{L^2} \, ds + \tilde{\varepsilon}_0 \mathbb{E}|u(t)|^2_{L^2}, \tag{4.31} \]
where \( \varepsilon_0 \) and \( \tilde{\varepsilon}_0 \) are arbitrary. By Proposition 4.2-(c) the term \( \mathbb{E} \int_0^t |\xi(s)\nabla v(s)|^2_{L^2} \, ds \) can be bounded by
\[ \mathbb{E}|v_0|_{L^1} + C_1 \mathbb{E}\left( \int_{\Omega} |\ln \xi_0(x)| \, dx \right) + C_2 T. \]

By Proposition 4.2-(c) the term \( \mathbb{E} \int_0^t \xi^2(s, x) \chi^2(s, x) \, dx \, ds \) can be bounded by \( C e^{\delta_2 T} \mathbb{E}|\xi_0|_{L^1} \). Since there exists a constant \( C > 0 \) such that \( -C \leq \ln(x) - x^2 \leq 0 \), the term \( \mathbb{E} \int_0^t \ln(\xi(s, x)) \chi^2(s, x) \xi(s, x) \, dx \, ds \) can be estimated by \( \mathbb{E} \int_0^t \xi^2(s, x) \chi^2(s, x) \, dx \, ds \). In this way there exists constants \( C_3, \delta_3 > 0 \) such that
\[ \frac{(r_u + r_v)^2}{4\varepsilon_1} \mathbb{E} \int_0^t |\xi(s)\nabla v(s)|^2_{L^2} \, ds \\
+ \kappa_u \mathbb{E} \int_0^t \ln(\xi(s, x)) \chi^2(s, x) \xi(s, x) \, dx \, ds \leq C_3 e^{\delta_3 T} \mathbb{E}|\xi_0|_{L^1}. \]

Let us consider the term \( \mathbb{E} \int_0^t \int_{\Omega} \xi(s, x) u(s, x) \, dx \, ds \) and the positive part of
\( \int_0^t \int_{\Omega} \ln(\xi(s, x)) u(s, x) \, dx \, ds \). Firstly, since \( \ln(x) \leq x \) for \( x \geq 1 \), the positive part given by \( \mathbb{E} \int_0^t \int_{\Omega} 1_{[1, \infty)}(\xi(s, x)) \ln(\xi(s, x)) u(s, x) \, dx \, ds \) can be estimated by
\[ \mathbb{E} \int_0^t \int_{\mathcal{O}} \xi(s, x) u(s, x) \, dx \, ds. \]

Applying the Young inequality we know that we have for any \( \varepsilon_3 > 0 \)
\[ \mathbb{E} \int_0^t \int_{\mathcal{O}} \xi(s, x) u(s, x) \, dx \, ds \leq \varepsilon_3 \int_0^t \| u(s) \|_{L^2}^2 \, ds + \frac{1}{4\varepsilon_3} \int_0^t \| \xi(s) \|_{L^2}^2 \, ds. \]

Note, we have for \( x \geq 1, \ln(x) \leq 2 \frac{\ln(x^{\frac{1}{2}})}{x^{\frac{1}{2}}} \leq (x - 1)^{\frac{1}{2}} \leq x^{\frac{1}{2}} + 1. \) The Cauchy
Schwarz and Young inequality gives for the initial condition
\[ \mathbb{E} \int_{\mathcal{O}} 1_{[1, \infty]}(\xi_0(x)) \ln(\xi_0(x)) u_0(x) \, dx \leq C_1 \mathbb{E} |\xi_0|_{L^1} + C_2 \mathbb{E} |u_0|_{L^2}^2. \]

Collecting altogether we get by estimate (4.30) and estimates (4.31), we can con-
clude that there exist constants \( c_0, \ldots, c_{10}, \delta_6 > 0 \) such that we have for all
\( \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \)
\[ \mathbb{E} \int_0^t \int_{\mathcal{O}} \chi^2(s, x) \xi(s, x) u(s, x) \, dx \, ds \leq c_1 \mathbb{E} |\xi_0|_{L^1} + c_2 \mathbb{E} |u_0|_{L^2}^2 \]
\[ + c_3 \mathbb{E} |v_0|_{L^1} + c_4 \mathbb{E} \left( \int_{\mathcal{O}} |\ln \xi_0(x)| \, dx \right) + c_5 T + c_6 e^{\delta_6 T} \mathbb{E} |\xi|_{L^1} \]
\[ + c_7 (\varepsilon_1 + \varepsilon_2 + \varepsilon_0) \sup_{0 \leq s \leq T} |u(s)|_{L^2}^2 + c_8 (\varepsilon_0 + \varepsilon_3) \mathbb{E} \int_0^T |u(s)|_{L^2}^2 \, ds \]
\[ + c_9 \left( \frac{n}{4 \varepsilon_0} + \frac{n}{4 \varepsilon_0} \right) \left\{ \mathbb{E} \sup_{0 \leq s \leq T} |\eta(s)|_{L^2}^2 + c_4 \mathbb{E} \int_0^T |\eta(s)|_{H^1_2}^2 \, ds \right\}^{\frac{1}{2}} \]
\[ + c_{10} \left( \frac{n^2}{4 \varepsilon_2} + \frac{1}{4 \varepsilon_3} \right) \int_0^t |\xi(s)|_{L^2}^2 \, ds. \]

Summarizing, for all \( \varepsilon > 0 \) gives constants
\[ \left( \mathbb{E} \int_0^t \int_{\mathcal{O}} \chi^2(s, x) \xi(s, x) u(s, x) \, dx \, ds \right) \]
\[ \leq c_1 \mathbb{E} |\xi_0|_{L^1} + c_2 \mathbb{E} |u_0|_{L^2}^2 + c_3 \mathbb{E} |v_0|_{L^1} \]
\[ + C_3(n) \left( \mathbb{E} \sup_{0 \leq s \leq T} |\eta(s)|_{L^2}^2 + \mathbb{E} \int_0^T |\eta(s)|_{H^1_2}^2 \, ds \right)^{\frac{1}{2}} \]
\[ + C_4 \mathbb{E} \int_0^t |\xi(s)|_{L^2}^2 \, ds + C_5 \left( 1 + \mathbb{E} |\xi_0|^p_{L^p} \right) \]
\[ + \varepsilon \left( \mathbb{E} \sup_{0 \leq s \leq T} |u(s)|_{L^2}^2 + \mathbb{E} \int_0^T |u(s)|_{L^2}^2 \, ds \right), \]

which leads by taking \( n = 4 \) the assertion. \( \Box \)
4.3 Proof of Continuity of the Operator $\mathcal{T}$

In this section we state all results, that we are using to show that the operator

$$\mathcal{T} : U_{\mathcal{R}}(K_1, K_2, K_3) \longrightarrow \mathcal{M}_{\mathcal{R}}(0, T);$$

$$(\chi, \eta) \mapsto (u, v),$$

is continuous. The following proposition shows the continuity of the operator $\mathcal{T}$ with respect to $u$ in terms of $\chi$ and $\eta$.

**Proposition 4.7** Let $K_i > 0$ for $i = 1, 2, 3$. For all $(\chi_1, \eta_1), (\chi_2, \eta_2) \in U_{\mathcal{R}}(K_1, K_2, K_3)$ and $(u_1, v_1), (u_2, v_2) \in U_{\mathcal{R}}(K_1, K_2, K_3)$ such that $(u_1, v_1) = \mathcal{T}[(\chi_1, \eta_1)]$ and $(u_2, v_2) = \mathcal{T}[(\chi_2, \eta_2)]$, there exists a constant $C = C(K_1, K_2, K_3) > 0$, some numbers $\delta_1, \delta_2 > 0$ and $\gamma \in (0, 1)$ such that

$$\mathbb{E} \sup_{0 \leq s \leq T} |u_1(s) - u_2(s)|^2_{H_2^{1-\rho}}$$

$$\leq C(K_1, K_2, T) \times \left[ \mathbb{E} \left[ \sup_{s \in [0, T]} |\chi_1(s) - \chi_2(s)|^2_{H_2^{1-\rho}} \right]^\delta_1 \right.$$  

$$+ \left[ \mathbb{E} \left[ \sup_{s \in [0, T]} |\xi_1(s) - \xi_2(s)|^\gamma_{L^1} \right]^\delta_2 \right],$$

where $\xi_i = v_i^{-1}$ for $i = 1, 2$.

Before stating the proof, we present the following proposition where the continuity of $\xi$ on $v$ is stated. Since the proof is short, we present the proof after stating the proposition.

**Proposition 4.8** Let $\xi_i = v_i^{-1}$ for $i = 1, 2$. Then, for any $1 \leq q < 2$ there exists a constant $C > 0$ and numbers $p = \frac{2mq}{m-q}, m > q$, and $r = \frac{2}{1-\gamma}$ such that

$$\left( \mathbb{E} \left[ \sup_{0 \leq s \leq T} |\xi_1(s) - \xi_2(s)|^\gamma_{L^1} \right] \right)^\frac{1}{\gamma} \leq \left( \mathbb{E} \left[ \sup_{0 \leq s \leq T} |v_1(s) - v_2(s)|_{L^m} \right] \right)^\frac{1}{m} \left( \mathbb{E} |\xi_1(s)|_{L^p} \right)^\frac{1}{p} \left( \mathbb{E} |\xi_2(s)|_{L^p} \right)^\frac{1}{p}.$$

**Proof of Proposition 4.8** This follows by

$$\xi_1(s) - \xi_2(s) = \frac{v_2(s) - v_1(s)}{v_1(s)v_2(s)} = \left( \frac{v_2(s) - v_1(s)}{v_1(s)} \right) \frac{\xi_1(s)}{\xi_2(s)},$$

and the Hölder inequality. \hfill \Box

Finally, we present the following proposition, stating the continuity of $v$ on $\chi$. Again, since the proof is short, we present the proof after stating the proposition.

**Proposition 4.9** Let $K_i > 0$ for $i = 1, 2, 3$. Let us assume that $\rho \in [1, 2)$ and $2 \geq d(1 - \frac{1}{\rho})$. For all $(\chi_1, \eta_1), (\chi_2, \eta_2) \in U_{\mathcal{R}}(K_1, K_2, K_3)$ and $(u_1, v_1), (u_2, v_2) \in U_{\mathcal{R}}(K_1, K_2, K_3)$ such that $(u_1, v_1) = \mathcal{T}[(\chi_1, \eta_1)]$ and $(u_2, v_2) = \mathcal{T}[(\chi_2, \eta_2)]$, there
exists a constant $C = C(K_1, K_2, K_3) > 0$, some numbers $\delta_1, \delta_2 > 0$ and $\gamma \in (0, 1)$ such that

$$\mathbb{E} \sup_{0 \leq s \leq T} |v_1(s) - v_2(s)|_{L^q} \leq C(K_3, T) \times \left[ \mathbb{E} \sup_{s \in [0, T]} |\chi_1(s) - \chi_2(s)|_{H^{\gamma-1}}^2 \right]^\gamma.$$

**Proof of Proposition 4.9** We start with focusing on the non-linear part. The Minkowski inequality yields

$$\left| \int_0^t e^{-\sigma v(t-s)} \Delta (\chi_1^2(s) - \chi_2^2(s)) \, ds \right|_{L^q} \leq \int_0^t \left| e^{-\sigma v(t-s)} \Delta (\chi_1^2(s) - \chi_2^2(s)) \right|_{L^q} \, ds.$$

The smoothing property for the Laplace operator $\Delta$ and the Hölder inequality give for $\frac{q}{2} < 1 - \frac{1}{r}$

$$\left| \int_0^t e^{-\sigma v(t-s)} \Delta (\chi_1^2(s) - \chi_2^2(s)) \, ds \right|_{L^q} \leq \int_0^t (t - s)^{-\frac{q}{2}} \left| \chi_1^2(s) - \chi_2^2(s) \right|_{H^{\gamma}} \, ds \leq C(t) \left\| \chi_1^2 - \chi_2^2 \right\|_{L^r(0, t; H^{\gamma}_{q-\delta})}.$$

The embedding $L^1(\mathcal{O}) \hookrightarrow H^{-\delta}_{q}(\mathcal{O})$ gives for $\delta > d \left( 1 - \frac{1}{q} \right)$

$$\left| \int_0^t e^{-\sigma v(t-s)} \Delta (\chi_1^2(s) - \chi_2^2(s)) \, ds \right|_{L^q} \leq C(t) \left\| \chi_1^2 - \chi_2^2 \right\|_{L^r(0, t; L^1)}.$$

The Hölder inequality gives

$$\left| \int_0^t e^{-\sigma v(t-s)} \Delta (\chi_1^2(s) - \chi_2^2(s)) \, ds \right|_{L^q} \leq C(t) \left( \int_0^t |\chi_1(s) - \chi_2(s)|_{L^2}^{r} \, ds \right)^\frac{1}{r} \left( \int_0^t |\chi_1(s) + \chi_2(s)|_{L^2}^{r} \, ds \right)^{\frac{1}{r}}.$$

We know by real interpolation (see paragraph 1.2) that there exists a constant $C > 0$ such that $|w|_{L^2} \leq C |w|_{H^{\gamma}_{q-\delta}}^{\theta} |w|_{H^{\gamma}_{q-\delta}}^{1-\theta}$ for $\theta = 1/r$. Applying first interpolation, and, subsequently, the Hölder inequality gives

$$\left| \int_0^t e^{-\sigma v(t-s)} \Delta (\chi_1^2(s) - \chi_2^2(s)) \, ds \right|_{L^q} \leq C(t) \sup_{0 \leq s \leq t} |\chi_1(s) - \chi_2(s)|_{H^{\gamma}_{q-\delta}}^{\theta} \sup_{0 \leq s \leq t} |\chi_1(s) + \chi_2(s)|_{H^{\gamma}_{q-\delta}}^{1-\theta} \times \left( \int_0^t |\chi_1(s) - \chi_2(s)|_{H^{\gamma}_{q-\delta}}^{(1-\theta)r} \, ds \right)^{\frac{1}{r}} \left( \int_0^t |\chi_1(s) + \chi_2(s)|_{H^{\gamma}_{q-\delta}}^{(1-\theta)r} \, ds \right)^{\frac{1}{r}}.$$

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Taking expectation and applying the Cauchy–Schwarz inequality we get for $r \leq \frac{1}{1-\theta}$

$$
\mathbb{E} \left| \int_0^t e^{-\sigma_v(t-s)\Delta} (\chi_1^2(s) - \chi_2^2(s)) \, ds \right|_{L^q} 
\leq C(t) \left\{ \mathbb{E} \sup_{0 \leq s \leq t} |\chi_1(s) + \chi_2(s)|^2_{H_2^{1-\rho}} \right\}^{\theta \frac{q}{2}} \left\{ \mathbb{E} \sup_{0 \leq s \leq t} |\chi_1(s) - \chi_2(s)|^2_{H_2^{1-\rho}} \right\}^{\frac{q}{2}} 
\times \left\{ \mathbb{E} \int_0^t |\chi_1(s) + \chi_2(s)|^2_{H_2^1} \right\}^{\frac{1}{q}}.
$$

The rest follows by the definition of $\mathcal{U}_Q(K_1, K_2, K_3)$ and estimating the terms on the RHS. \hfill \square

Before embarking onto the proof of Proposition 4.7, we prove the following important inequality. This Lemma contains the main step in the proof of Proposition 4.7.

**Lemma 4.10** For all $(\chi_1, \eta_1), (\chi_2, \eta_2) \in \mathcal{U}_Q(K_1, K_2, K_3)$ and $(u_1, v_1), (u_2, v_2) \in \mathcal{U}_Q(K_1, K_2, K_3)$ such that $(u_1, v_1) = \mathcal{T}[(\chi_1, \eta_1)]$ and $(u_2, v_2) = \mathcal{T}[(\chi_1, \eta_1)]$, there exists a constant $C = C(K_1, K_2) > 0$, some numbers $\delta_1, \delta_2 > 0$ and $\gamma \in (0, 1)$ such that

$$
\mathbb{E} \left( \int_0^T \left| \chi_1^2(s)\xi_1(s) - \chi_2^2(s)\xi_2(s) \right|_{L^1} \, ds \right) 
\leq C(K_1, K_2, T) \times \left[ \mathbb{E} \sup_{0 \leq s \leq T} |\chi_1(s) - \chi_2(s)|^2_{H_2^{1-\rho}} \right]^{\delta_1} \tag{4.32} 
+ \left\{ \mathbb{E} \sup_{0 \leq s \leq T} |\xi_1(s) - \xi_2(s)|^\gamma_{L^1} \right\}^{\delta_2},
$$

where $\xi_i := v_i^{-1}$ for $i = 1, 2$.

**Proof of Lemma 4.10** Firstly, let us note that

$$
|\chi_1^2(s)\xi_1(s) - \chi_2^2(s)\xi_2(s)| \leq |\chi_1(s) - \chi_2(s)||\chi_1(s) + \chi_2(s)||\xi_1(s) - \xi_2(s)| \tag{4.33}
+ \chi_2(s)||\xi_1(s)| + |\chi_2^2(s)||\xi_1(s) - \xi_2(s)|.
$$

Secondly, let us note that for any $n \in \mathbb{N}$ we have

$$
|a - b| \leq \left| \sum_{k=1}^n a^{\frac{k-1}{n}} (a^{\frac{1}{n}} - b^{\frac{1}{n}}) b^{\frac{n-k}{n}} \right| \leq C(n - 1)(a - b)^\frac{1}{n} (a^{\frac{n-1}{n}} + b^{\frac{n-1}{n}}).
$$
Using the above inequality, we infer that

\[
\int_0^T \left| \chi_1^2(s) \xi_1(s) - \chi_2^2(s) \xi_2(s) \right|_{L^1} ds
\]

\[
\leq \int_0^T \left| \chi_1^2(s) \xi_1(s) - \chi_2^2(s) \xi_2(s) \right|^{\frac{1}{n}} \cdot \Psi(s) \left| L^1 \right| ds,
\]

\[
\leq \int_0^T \left| \left( (\chi_1^2(s) - \chi_2^2(s)) \xi_1(s) + \chi_2^2(s)(\xi_1(s) - \xi_2(s)) \right) \right|^{\frac{1}{n}} \cdot \Psi(s) \left| L^1 \right| ds,
\]

where

\[
\Psi(s) := (n - 1) \left[ (\chi_1^2 \xi_1)^{(n-1)/n}(s) + (\chi_2^2 \xi_2)^{(n-1)/n}(s) \right].
\] (4.34)

Now, let us fix some numbers \( p_1, q_1, n \geq 1 \) with \( n \in \mathbb{N}, n \geq 2, \frac{1}{p_1} + \frac{1}{q_1} \leq \frac{1}{2}, \) and

\[
\frac{q_1(n - 1)}{q_1n - q_1 - 1} \leq 2.
\] (4.35)

We split the right hand side of the above inequality into two summands, i.e., \( S_1 \) and \( S_2.\) In particular, let

\[
S_1 := \int_0^T \left| \left( (\chi_1^2(s) - \chi_2^2(s)) \xi_1(s) \right) \right|^{\frac{1}{n}} \cdot \Psi(s) \left| L^1 \right| ds,
\]

\[
S_2 := \int_0^T \left| \left( \chi_2^2(s)(\xi_1(s) - \xi_2(s)) \right) \right|^{\frac{1}{n}} \cdot \Psi(s) \left| L^1 \right| ds.
\]

By (4.33), using the Hölder inequality we obtain

\[
S_1 := \int_0^T \left| \chi_1(s) - \chi_2(s) \right|_{L^2}^{\frac{1}{n}} \left| \chi_1(s) + \chi_2(s) \right|_{L^p_1}^{\frac{1}{n}} \left| \xi_1(s) \right|_{L^{q_1} n}^{\frac{1}{n}} \left| \Psi(s) \right|_{L^{\frac{n}{n-1}}} \left( 4.36 \right)
\]

\[
\leq \int_0^T \left| \chi_1(s) - \chi_2(s) \right|_{L^2}^{\frac{1}{n}} \left| \chi_1(s) + \chi_2(s) \right|_{L^p_1}^{\frac{1}{n}} \left| \xi_1(s) \right|_{L^{q_1} n}^{\frac{1}{n}} \left| \Psi(s) \right|_{L^{\frac{n}{n-1}}} ds.
\]

We have

\[
\mathbb{E}[S_1] \leq \mathbb{E} \left( \int_0^T \left| \chi_1(s) - \chi_2(s) \right|_{L^2}^{\frac{1}{n}} \left| \chi_1(s) + \chi_2(s) \right|_{L^p_1}^{\frac{1}{n}} \left| \xi_1(s) \right|_{L^{q_1} n}^{\frac{1}{n}} \left| \Psi(s) \right|_{L^{\frac{n}{n-1}}} ds \right).
\]

Note that we have by interpolation between \( H_2^{1-\theta}(O) \) and \( H_2^1(O) \) for \( \theta = 1/\rho \)

\[
\left| \chi_1(s) - \chi_2(s) \right|_{L^2} \leq \left| \chi_1(s) - \chi_2(s) \right|_{H_2^{1-\theta}}^{\theta} \left| \chi_1(s) - \chi_2(s) \right|_{H_2^1}^{1-\theta}.
\]
In this way, we obtain by again applying the Hölder inequality

\[
\mathbb{E}[S_1] \leq \mathbb{E}\left[ \sup_{s \in [0, T]} |\chi_1(s) - \chi_2(s)|^2_{H_2^{1-\rho}} \right]^{\frac{\theta}{2n}} \left[ \int_0^T |\chi_1(s) - \chi_2(s)|^2_{H_2^{1-\rho}} \, ds \right]^{\frac{1-\theta}{2n}} \\
\times \left[ \int_0^T |\chi_1(s) + \chi_2(s)|^2_{L^p} \, ds \right]^{\frac{1}{2n}} \left[ \sup_{s \in [0, T]} |\xi_1(s)|^q_{L^q} \right]^{\frac{1}{q_n}} \\
\times \left[ \int_0^T |\Psi(s)|^q_{L^{q_1}} \, ds \right]^{\frac{2n-2+\theta}{2nq_n}} \left[ \int_0^T |\Psi(s)|^q_{L^{q_1}} \, ds \right]^{\frac{2n-2+\theta}{2nq_n}} \left[ \sup_{s \in [0, T]} |\xi_1(s)|^q_{L^q} \right]^{\frac{1}{q_n}}.
\]

Then, applying again the Hölder inequality

\[
\mathbb{E}S_1 \leq \left[ \mathbb{E}\left( \int_0^T \left( |\chi_1(s)|_{H_2^{2-\rho}} + |\chi_2(s)|_{H_2^{2-\rho}} \right)^2 \, ds \right) \right]^{\frac{1-\theta}{2n}} \\
\times \left[ \mathbb{E}\left( \int_0^T |\chi_1(s) + \chi_2(s)|_{L^p} \, ds \right) \right]^{\frac{1}{2n}} \\
\times \left[ \sup_{s \in [0, T]} |\xi_1(s)|^q_{L^q} \right]^{\frac{1}{q_n}} \left[ \mathbb{E}\left( \int_0^T |\Psi(s)|^q_{L^{q_1}} \, ds \right) \right]^{\frac{2n-2+\theta}{2nq_n}} \left[ \mathbb{E}\left( \int_0^T |\Psi(s)|^q_{L^{q_1}} \, ds \right) \right]^{\frac{2n-2+\theta}{2nq_n}} \left[ \sup_{s \in [0, T]} |\xi_1(s)|^q_{L^q} \right]^{\frac{1}{q_n}}.
\]

From the definition of \( \Psi \) in (4.34), and, since

\[
|\Psi(s)|_{L^{\frac{n}{n-1}}} \leq C \left( |(\chi_1^2 \xi_1)(t, x)|_{L^{\frac{n}{n-1}}} + |(\chi_2^2 \xi_2)(t, x)|_{L^{\frac{n}{n-1}}} \right) \\
\leq C \left[ \left( \int_\mathcal{O} (\chi_1^2 \xi_1)(t, x) \, dx \right)^{\frac{n-1}{n}} + \left( \int_\mathcal{O} (\chi_2^2 \xi_2)(t, x) \, dx \right)^{\frac{n-1}{n}} \right],
\]

we get

\[
\int_0^T |\Psi(s)|_{L^{\frac{n}{n-1}}} \, ds \\
\leq C \left[ \int_0^T \left( \int_\mathcal{O} (\chi_1^2 \xi_1)(s, x) \, dx \right)^{\frac{n-1}{n}} \frac{2n}{2n-2+\theta} \, ds \\
+ \int_0^T \left( \int_\mathcal{O} (\chi_2^2 \xi_2)(s, x) \, dx \right)^{\frac{n-1}{n}} \frac{2n}{2n-2+\theta} \, ds \right] \\
\leq C(\mathcal{O}, T) \left[ \left( \int_0^T \int_\mathcal{O} (\chi_1^2 \xi_1)(s, x) \, dx \, ds \right)^{\frac{2(n-1)}{2n-2+\theta}} \\
+ \left( \int_0^T \int_\mathcal{O} (\chi_2^2 \xi_2)(s, x) \, dx \, ds \right)^{\frac{2(n-1)}{2n-2+\theta}} \right],
\]
and

\[
\mathbb{E}\left|\Psi(s)\right|^\frac{2n-2+\theta}{2n} \left(0, T; L^\frac{n}{q_1n} \mathbb{I}_{n}^{-\frac{q_1}{q_1n} - \frac{1}{q_{1n}} - 1}\right) \leq C \left[\mathbb{E}\left(\int_0^T \int_0^1 (\chi_1^2 \xi_1(s, x)) dx ds\right)^\frac{q_1(n-1)}{q_{1n}n - q_1 - 1} + \mathbb{E}\left(\int_0^T \int_0^1 (\chi_2^2 \xi_2(s, x)) dx ds\right)^\frac{q_1(n-1)}{q_{1n}n - q_1 - 1}\right].
\]

Due to (4.35) we know

\[
\frac{q_1(n-1)}{q_{1n}n - q_1 - 1} \leq 2.
\]

Using the previous energy inequalities results in (4.37) we infer that

\[
\mathbb{E}[S_1] \leq \left[\mathbb{E}\sup_{s \in [0, T]} |\chi_1(s) - \chi_2(s)|^2 \xi_{H_2}^{1-\rho}\right]^\frac{1}{m\rho} \times K_1 \times K_2. \tag{4.38}
\]

In the next step, we deal with S_2, where S_2 is given by

\[
\mathbb{E}[S_2] := \mathbb{E}\int_0^T |\chi_2^2(s) |\xi_1(s) - \xi_2(s)| \frac{1}{\rho} \Psi(s) | L^1 d s.
\]

Let us choose \( n \in \mathbb{N} \) with \( \gamma \in (0, 1), q_2 = 1, p_2 > 2, m_0 \in (1, \infty) \), and \( r < 2 \), such that

\[
\frac{1}{n} + \frac{\theta}{\gamma n} + \frac{1 - \theta}{m_0 n} + \frac{n}{2(n - 1)} \leq 1, \quad \frac{1}{p_2} = \frac{\theta}{q_2} + \frac{1 - \theta}{m_0}, \quad \text{and} \quad \frac{1}{r} + \frac{1}{p_2} \leq 1.
\]

A short calculations gives that for \( \gamma = \frac{9}{10}, n = 5, p_2 = 5/2, \) and \( r = 5/3 \), there exists a number \( \theta \in (0, 1) \) and a number \( m_0 \geq 1 (\theta = 9/40, m_0 = 31/7) \) satisfying the conditions above. An application of the H"older inequality gives

\[
\mathbb{E}[S_2] \leq \mathbb{E}\int_0^T |\chi_2^2(s) |L^m\xi_1(s) - \xi_2(s)| \frac{1}{\rho} \Psi(s) | L_{m_0}^n d s
\]

\[
\leq \mathbb{E}\int_0^T |\chi_2(s) |L^m\xi_1(s) - \xi_2(s)| \frac{1}{\rho} \Psi(s) | L_{m_0}^n d s.
\]

Interpolation gives for \( \frac{1}{p_2} = \frac{\theta}{q_2} + \frac{1 - \theta}{m_0} \)

\[
|\xi_1(s) - \xi_2(s)|_{L^{p_2}} \leq |\xi_1(s) - \xi_2(s)|^{\theta}\xi_2(s) - |\xi_2(s)|_{L^{m_0}}^{1-\theta}.
\]
Hence, applying the Hölder inequality

\[
\mathbb{E}[S_2] \leq \mathbb{E} \left\{ \left( \int_0^T |\chi_2(s)|^2_{L^2} \, ds \right)^{\frac{1}{n}} \left[ \sup_{0 \leq s \leq T} |\xi_1(s) - \xi_2(s)|_{L^q}^{\gamma} \right]^{\frac{\alpha}{n}} \times \left( \sup_{0 \leq s \leq T} |\xi_1(s) - \xi_2(s)|_{L^q}^{m_0} \right)^{\frac{(1-\theta)n}{m_0}} \left( \int_0^T |\Psi(s)|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}} \, ds \right)^{\frac{n-1}{n}} \right\}.
\]

Note that we have

\[
\left( \int_0^T |\Psi(s)|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}} \, ds \right) \leq \left( \int_0^T \int_{\Omega} (\chi_2^2(s,x)\xi_1(s,x) + \chi_2^2(s,x)\xi_2(s,x)) \, dx \, ds \right)^{\frac{n}{n-1}}.
\]

Again applying the Hölder inequality gives

\[
\mathbb{E}[S_2] \leq \left( \mathbb{E} \int_0^T |\chi_2(s)|^2_{L^2} \, ds \right)^{\frac{1}{n}} \left( \mathbb{E} \sup_{0 \leq s \leq T} |\xi_1(s) - \xi_2(s)|_{L^q}^{\gamma} \right)^{\frac{\alpha}{n}} \times \left( \mathbb{E} \sup_{0 \leq s \leq T} \left( |\xi_1(s)|_{L^q}^{m_0} + |\xi_2(s)|_{L^q}^{m_0} \right)^{\frac{(1-\theta)n}{m_0}} \right) \times \left( \mathbb{E} \left( \int_0^T \int_{\Omega} (\chi_2^2(s,x)\xi_1(s,x) + \chi_2^2(s,x)\xi_2(s,x)) \, dx \, ds \right)^2 \right)^{\frac{n}{2(n-1)}}.
\]

In particular, for our choice of \( q \) we get

\[
\mathbb{E}[S_2] \leq K_1^{\frac{1}{n}} \times K_2^{\frac{(1-\theta)}{m_0}} \times \left( \mathbb{E} \sup_{0 \leq s \leq T} |\xi_1(s) - \xi_2(s)|_{L^q}^{\gamma} \right)^{\frac{\alpha}{n}} \times \left( \mathbb{E} \sup_{0 \leq s \leq T} \left( |\xi_1(s)|_{L^q}^{m_0} + |\xi_2(s)|_{L^q}^{m_0} \right)^{\frac{(1-\theta)n}{m_0}} \right). \tag{4.40}
\]

Using the estimates (4.38) and (4.40) in (4.36), there exists \( \delta_1, \delta_2 > 0 \) such that the inequality (4.32) holds. \( \square \)

Now we are ready to prove Proposition 4.7.

**Proof of Proposition 4.7** By analyticity of the semigroup we infer that

\[
\mathbb{E} \sup_{0 \leq s \leq t} |u_1(s) - u_2(s)|_{H^{1-\rho}_2}^2 \, dt \leq \mathbb{E} \sup_{0 \leq s \leq t} \int_0^t \left| \chi_1^2(s)\xi_1(s) - \chi_2^2(s)\xi_2(s) \right|_{H^{1-\rho}_2} \, ds \tag{4.41}
\]

\[
+ \mathbb{E} \sup_{0 \leq s \leq t} \left| \sigma_u \int_0^t (u_1(s) - u_2(s)) \, dW_1(s) \right|_{H^{1-\rho}_2},
\]

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where we used the results from previous sections. Using the Burkholder–Davis–Gundy inequality and the Young inequality we infer,

\[
E \sup_{0 \leq s \leq t} \left| \sigma_u \int_0^t (u_1(s) - u_2(s)) \, dW_1(s) \right|_{H_2^{1-\rho}} \\
\leq \epsilon E \sup_{0 \leq s \leq t} \left| u_1(s) - u_2(s) \right|^2_{H_2^{1-\rho}} + \frac{C(\sigma_u, S(\gamma_1))}{4\epsilon} \int_0^t E \sup_{0 \leq r \leq s} \left| u_1(r) - u_2(r) \right|^2_{H_2^{1-\rho}} \, dr.
\]

(4.42)

We consider the non-linear term. By Sobolev embedding theorem, we have \( L^1(\mathcal{O}) \hookrightarrow H_2^{1-\rho}(\mathcal{O}) \). Using the previous technical result, i.e., Lemma 4.10, we infer that there exists a constant \( C > 0 \) and \( \delta_1, \delta_2 > 0 \) such that

\[
E \int_0^t \left| \chi_1^2(s) \xi_1(s) - \chi_2^2(s) \xi_2(s) \right|_{H_2^{1-\rho}} \, ds \\
\leq E \int_0^t \left| \chi_1^2(s) \xi_1(s) - \chi_2^2(s) \xi_2(s) \right|_{L^1} \, ds \\
\leq C(K_1, K_2, K_3, T) \left[ \left\{ E \left[ \sup_{s \in [0,t]} \left| \chi_1(s) - \chi_2(s) \right|^2_{H_2^{1-\rho}} \right] \right\}^{\delta_1} \right. \\
+ \left. \left\{ E \left[ \sup_{s \in [0,t]} \left| \xi_1(s) - \xi_2(s) \right|^2_{L^1} \right] \right\}^{\delta_2} \right].
\]

(4.43)

Substituting the first and second term by the estimates given in (4.42) and (4.43) in the inequality (4.41), we obtain

\[
E \sup_{0 \leq s \leq t} \left| u_1(s) - u_2(s) \right|^2_{H_2^{1-\rho}} \, dt \\
\leq \epsilon E \sup_{0 \leq s \leq t} \left| u_1(s) - u_2(s) \right|^2_{H_2^{1-\rho}} \\
+ \frac{C\sigma_u}{4\epsilon} \int_0^t E \sup_{0 \leq r \leq s} \left| u_1(r) - u_2(r) \right|^2_{H_2^{1-\rho}} \, dr \\
+ C(K_1, K_2, T) \left[ \left\{ E \left[ \sup_{s \in [0,t]} \left| \chi_1(s) - \chi_2(s) \right|^2_{H_2^{1-\rho}} \right] \right\}^{\delta_1} \right. \\
+ \left. \left\{ E \left[ \sup_{s \in [0,t]} \left| \xi_1(s) - \xi_2(s) \right|^2_{L^1} \right] \right\}^{\delta_2} \right].
\]

Suitably choosing \( \epsilon \) and using the Gronwall lemma, we get the assertion. □

5 Pathwise Uniqueness in One Dimension

Since in the proof of the existence of solutions compactness arguments are used, the underlying probability space gets lost. In this section we will prove pathwise uniqueness for the Gierer–Meinhardt system, i.e. system (1.3)–(1.4), in dimension
one. Usually, by the Yamada-Watanabe theory we know that pathwise uniqueness for the system (1.3)–(1.4) implies joint uniqueness in law and strong existence for the equation the system (1.3)–(1.4), see [9, 12, 22, 35, 37, 40, 47]. By standard arguments, the existence of a strong solution follows.

Definition 5.1 Equations (1.3)–(1.4) are said to be pathwise unique if, whenever \((\Omega, F, (F_t)_{t \in [0, T]}, \mathbb{P}, (u_i, v_i), W_1, W_2), i = 1, 2,\) are solutions to (1.3)–(1.4) such that

\[
\mathbb{P} (u_1(0) = u_2(0), v_1(0) = v_2(0)) = 1,
\]

then

\[
\mathbb{P} (u_1(t) = u_2(t), v_1(t) = v_2(t)) = 1,
\]

for every \(0 < t \leq T\).

Under certain assumptions, the pathwise uniqueness of the stochastic Gierer–Meinhardt system can be shown.

Theorem 5.2 Let \(d = 1\). Let be given a filtered probability space \(\mathfrak{A} = (\Omega, F, \mathbb{P}, \mathbb{P})\) with the filtration \(\mathbb{F} = \{F_t : t \in [0, T]\}\) satisfying the usual conditions. Let \(W_1\) and \(W_2\) be two independent Wiener processes in \(\mathcal{H} := L^2(\mathcal{O})\), defined over the probability space \(\mathfrak{A}\), with covariances \(Q_1\) and \(Q_2\) and two couples of solutions \((u_1, v_1)\) and \((u_2, v_2)\) to equation (1.3)–(1.4) over \(\mathfrak{A}\), on \([0, T]\) such that \((u_1, v_1)\) and \((u_2, v_2)\) are continuous on \(L^2(\mathcal{O})\). If the solutions \((u_1, v_1)\) and \((u_2, v_2)\) are belonging \(\mathbb{P}\)-a.s. to \(C_b([0, T]; L^2(\mathcal{O})) \times C_b([0, T]; L^2(\mathcal{O}))\), then \((u_1, v_1)\) and \((u_2, v_2)\) are indistinguishable in \(L^2(\mathcal{O})\).

Since \((u_1, v_1)\) and \((u_2, v_2)\) are solutions to the system (1.3)–(1.4), \(\mathbb{P} (u_1(0) = u_2(0), v_1(0) = v_2(0)) = 1\), then we can write

\[
du_i(t) = \left[ ru_i \Delta u_i(t) + \kappa_{u_i} \frac{u_i^2(t)}{v_i(t)} - \mu_{u_i} u_i(t) \right] dt + \sigma_{u_i} u_i(t) \circ dW_1(t), \quad t > 0, i = 1, 2, \tag{5.1}
\]

and

\[
dv_i(t) = \left[ rv_i \Delta v_i(t) + \kappa_{v_i} u_i^2(t) - \mu_{v_i} v_i(t) \right] dt + \sigma_{v_i} v_i(t) \circ dW_2(t), \quad t > 0, i = 1, 2. \tag{5.2}
\]

Proof In the first step we will introduce a family of stopping times \(\{\tau_m : m \in \mathbb{N}\}\) and show that on the time interval \([0, \tau_m]\) the solutions \(u_1\) and \(u_2\) are indistinguishable. In the last step, we will show that \(\mathbb{P} (\tau_m < T) \to 0\) for \(m \to \infty\). From this, it follows that \(u_1\) and \(u_2\) are indistinguishable on the time interval \([0, T]\).
Step 1 Introducing the stopping times Let us define the stopping times for \( m \in \mathbb{N} \)

\[
\begin{align*}
\tau_m^1(\xi) & := \inf_{t > 0} \left\{ \sup_{0 \leq s \leq \min(t, \tau_m)} |\xi(s)|_{L^8} \geq m \right\}, \\
\tau_m^2(u) & := \inf_{t > 0} \left\{ \left( \int_0^t |u(s)|_{H^1}^2 \, ds \right) + \sup_{0 \leq s \leq \min(t, \tau_m)} |u(s)|_{L^2}^2 \geq m \right\}.
\end{align*}
\]

Let us introduce the stopping times \( \{\bar{\tau}_m^1 : m \in \mathbb{N}\} \) and \( \{\bar{\tau}_m^2 : m \in \mathbb{N}\} \) given by

\[
\bar{\tau}_m^j := \min(\tau_m^1(\xi_j), \tau_m^2(u_j)) \wedge T, \quad j = 1, 2,
\]

where \( \xi_j := v_j^{-1} \). The aim is to show that the couple \((u_1, v_1)\) and \((u_2, v_2)\) are indistinguishable on the time interval \([0, \tau_m]\), with \( \tau_m = \inf(\bar{\tau}_m^1, \bar{\tau}_m^2) \).

Fix \( m \in \mathbb{N} \). To get uniqueness on \([0, \tau_m]\) we first stop the original solution processes at time \( \tau_m \) and extend the processes \( u_1 \) and \( u_2 \) by other processes to the whole interval \([0, T]\). For this purpose, let \( y_j \) be a solution to

\[
y_j(t) = e^{-(t-\tau_m)(r_\Delta-\mu_u)} u_j(\tau_m) + \int_{\tau_m}^t e^{-(t-s)(r_\Delta-\mu_u)} y_j(s) \circ dW_1(s), \quad t \geq \tau_m, \quad j = 1, 2,
\]

and let \( z_j \) be a solution to

\[
z_j(t) = e^{-(t-\tau_m)(r_\Delta-\mu_v)} v_j(\tau_m) + \int_{\tau_m}^t e^{-(t-s)(r_\Delta-\mu_v)} z_j(s) \circ dW_2(s), \quad t \geq \tau_m, \quad j = 1, 2.
\]

Since \( u_1 \) and \( u_2 \), and \( v_1 \) and \( v_2 \) are continuous in \( L^2(\mathcal{O}) \), \( u_1(\tau_m), u_2(\tau_m), v_1(\tau_m) \) and \( v_2(\tau_m) \) are well defined and belong \( \mathbb{P}\)-a.s. to \( L^2(\mathcal{O}) \). Since, in addition, \( (e^{-t(r_\Delta-\mu_u)})_{t \in \mathbb{R}} \) and \( (e^{-t(r_\Delta-\mu_v)})_{t \in \mathbb{R}} \) are analytic semigroups on \( L^2(\mathcal{O}) \), the existence of unique solutions \( y_j \) and \( z_j \) in \( L^2(\mathcal{O}) \), \( j = 1, 2 \), can be shown by standard methods.

Now, let us define two processes \( \tilde{u}_{1,m} \) and \( \tilde{u}_{2,m} \) which are equal to \( u_1 \) and \( u_2 \) on the time interval \([0, \tau_m]\) and follow the linear equations \( y_1 \) and \( y_2 \) respectively afterwards. In the same way, let us define two processes \( \tilde{v}_{1,m} \) and \( \tilde{v}_{2,m} \) which are equal to \( v_1 \) and \( v_2 \) on the time interval \([0, \tau_m]\) and follow the linear equations \( z_1 \) and \( z_2 \) afterwards. In particular, let

\[
\tilde{u}_{j,m}(t) = \begin{cases} u_j(t) & \text{for } 0 \leq t < \tau_m, \\ y_j(t) & \text{for } \tau_m \leq t \leq T; \end{cases}
\]

and

\[
\tilde{v}_{j,m}(t) = \begin{cases} v_j(t) & \text{for } 0 \leq t < \tau_m, \\ z_j(t) & \text{for } \tau_m \leq t \leq T. \end{cases}
\]
Note, that the couples \((\bar{u}_{1,m}, \bar{v}_{1,m})\) and \((\bar{u}_{2,m}, \bar{v}_{2,m})\) solve the following system corresponding to (1.3)–(1.4) for the time interval \(t \in [0, \tau_m]\):

\[
d\bar{u}(t) = \left[ r_u \Delta \bar{u}(t) + \kappa_u \bar{u}^2(t) - \mu_u \bar{u}(t) \right] dt + \sigma_u \bar{u}(t) \circ dW_1(t),
\]

and

\[
d\bar{v}(t) = \left[ r_v \Delta \bar{v}(t) + \kappa_v \bar{v}^2(t) - \mu_v \bar{v}(t) \right] dt + \sigma_v \bar{v}(t) \circ dW_2(t).
\]

In addition, let us note that the process \(\xi_{j,m} := v_{j,m}^{-1}\) solves (4.4).

**Step II** The aim in this step is to show that the \(\mathbb{E}|\bar{u}_{1,m}(t) - \bar{u}_{2,m}(t)|^2_{L^2} = 0\). The first step is to show that for all \(m \in \mathbb{N}\), there exists a \(C(m) > 0\) such that

\[
\mathbb{E}|\bar{u}_{1,m}(t) - \bar{u}_{2,m}(t)|^2_{L^2} \leq C(m) \int_0^t \mathbb{E}|\bar{u}_{1,m}(s) - \bar{u}_{2,m}(s)|^2_{L^2} ds.
\]

Let us define the subsets of \(\Omega\)

\[
\begin{cases}
\Omega_1^m := \{ \omega \in \Omega : \tau_1^m(\xi_{1,m}) \leq t, \tau_1^m(\xi_{2,m}) \leq t \}, \\
\Omega_2^m := \{ \omega \in \Omega : \tau_2^m(\bar{u}_{1,m}) \leq t, \tau_2^m(\bar{u}_{2,m}) \leq t \}.
\end{cases}
\]

In addition, for a stochastic process \(\zeta\), let us define the convolution process \(\mathcal{C}\) by

\[
\mathcal{C}(\zeta)(t) := \int_0^t e^{-(t-s)r_A} \zeta(s) \, ds, \quad t \in [0, T].
\]

For simplicity we omit the index \(m\) in the following paragraph and write \(u_j\) and \(v_j\) instead of \(\bar{u}_{j,m}(t)\) and \(\bar{v}_{j,m}(t)\) for \(j = 1, 2\). Since the nonlinear term is the most critical term, we start by analysing the nonlinear term. Let us fix \(\gamma \in \left(\frac{1}{2}, \frac{14}{15}\right)\). Analysing the convolution term with respect to the nonlinear term by splitting it into two parts. By applying the smoothing property of the semigroup and the embedding \(L^1(\mathcal{O}) \hookrightarrow H_2^{1-\gamma}(\mathcal{O})\) we obtain for \(p = 2\)

\[
\begin{align*}
&|\mathcal{C}(u_1^2\xi_1 1_{\Omega_1} 1_{\Omega_2})(t) - \mathcal{C}(u_2^2\xi_2 1_{\Omega_1} 1_{\Omega_2})(t)|^p_{L_2} \\
&\quad \leq \left( \int_0^t (t-s)^{-\gamma/2} |u_1^2(s)\xi_1(s) - u_2^2(s)\xi_2(s)|_{H_2^{1-\gamma} 1_{\Omega_1} 1_{\Omega_2}} ds \right)^p \\
&\quad \leq \left( \int_0^t (t-s)^{-\gamma/2} |u_1(s) - u_2(s)|_{L^2} |(u_1(s) + u_2(s))\xi_1(s)|_{H_2^{1-\gamma} 1_{\Omega_1} 1_{\Omega_2}} ds \right)^p \\
&\quad \quad + \left( \int_0^t (t-s)^{-\gamma/2} (|u_1^2(s)|_{L^4} + |u_2^2(s)|_{L^4}) |\xi_1(s) - \xi_2(s)|_{L^4} 1_{\Omega_1} 1_{\Omega_2} ds \right)^p \\
&\quad =: S_1^p(t) + S_2^p(t).
\end{align*}
\]

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First we consider the term $S_1^p$. Due to the specific choice of $\gamma$, we apply the Hölder inequality with $k' = \frac{15}{4}$ and $k = \frac{15}{11}$ to obtain

$$S_1^p(t) \leq C(t) \left( \int_0^t |u_1(s) - u_2(s)|_{L^2}^k (|u_1(s)|_{L^5} + |u_2(s)|_{L^5})^k \right)^{p/k} \| \mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2} \|_L^0 ds. $$

Again applying the Hölder inequality we get for $q = \frac{22}{15}$

$$S_1^p(t) \leq C(t) \left( \int_0^t |u_1(s) - u_2(s)|_{L^2}^{kq} ds \right)^{\frac{p}{kq}} \times \left( \int_0^t (|u_1(s)|_{L^5} + |u_2(s)|_{L^5})^{kq} \mathbb{1}_{\Omega_2} ds \right)^{\frac{p}{kq}} \sup_{0 \leq s \leq t} \| \mathbb{1}_{\Omega_1} |\xi_1(s)|_{L^\frac{10}{3}}^p. $$

By our choice $p = 2 = \frac{15}{11} \times \frac{22}{15} = k \times q$. Using the embedding $H^1_2(\mathcal{O}) \hookrightarrow L^5(\mathcal{O})$, then using the definition of subsets $\Omega_1, \Omega_2$ and the definition of stopping times we infer that

$$S_1^p(t) \leq C \left( \int_0^t |u_1(s) - u_2(s)|_{L^2}^p ds \right) \left( \int_0^t (|u_1(s)|_{L^5}^p + |u_2(s)|_{L^5}^p) \mathbb{1}_{\Omega_2} ds \right) \sup_{0 \leq s \leq t} \| \mathbb{1}_{\Omega_1} |\xi_1(s)|_{L^\frac{10}{3}}^p. $$(5.6)

Next, we investigate the second summand. From Proposition B.1 of the online resource [19] we observe that

$$\| \cdot \|_{L^4(0,T;L^8)} \leq \| \cdot \|_{L^2(0,T;H^1_2)} + \| \cdot \|_{L^\infty(0,T;L^2)}. $$

Using this result, then applying again the Hölder inequality and using the definition of subsets $\Omega_1, \Omega_2$ and the definition of stopping times we infer that

$$S_2^p(t) \leq \left( \frac{1}{t} \right)^{-\gamma/2} \left( \int_0^t (|u_1(s)|_{L^4}^4 + |u_2(s)|_{L^4}^4) |\xi_1(s) - \xi_2(s)|_{L^3}^4 \mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2} ds \right)^p$$

$$\leq C(t) \left( \int_0^t \mathbb{1}_{\Omega_2} (|u_1(s)|_{L^4}^4 + |u_2(s)|_{L^4}^4) ds \right)^p \sup_{0 \leq s \leq t} \| \mathbb{1}_{\Omega_1} |\xi_1(s) - \xi_2(s)|_{L^\frac{4}{3}}^p. $$

$$\leq C(t, m) \sup_{0 \leq s \leq t} \| \mathbb{1}_{\Omega_1} |\xi_1(s) - \xi_2(s)|_{L^\frac{4}{3}}^p. $$(5.7)
Using the above two estimates (5.6) and (5.7) we get

\[ S_1^p(t) + S_2^p(t) \leq C(t, m) \sup_{0 \leq s \leq t} \mathbb{1}_{\Omega_1} |\xi_1(s) - \xi_2(s)|_{L^\frac{4}{3}}^p + C(t, m) \int_0^t |u_1(s) - u_2(s)|_{L^2}^p \, ds. \]

Now, to tackle the stochastic convolution part, for a stochastic process \( \tilde{\zeta} \), we define the operator

\[ \mathcal{G}(\tilde{\zeta})(t) = \int_0^t e^{-(t-s)ru} A \tilde{\zeta}(s) \, dW_1(s). \]

Here it is important that the noise coefficient depends linearly on \( u \), and we make use of the Corollary A.7 in the online resource [19] to get the desired estimate. Recollecting all the terms we arrive at

\[
|\mathcal{E}(u_1^2 \mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2})(t) - \mathcal{E}(u_2^2 \mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2})(t)|_{L^2}^p \\
\leq C(t, m) \sup_{0 \leq s \leq t} \mathbb{1}_{\Omega_1} |\xi_1(s) - \xi_2(s)|_{L^\frac{4}{3}}^p + C(t, m) \int_0^t |u_1(s) - u_2(s)|_{L^2}^p \, ds.
\]

(5.8)

It remains to estimate the entity \( \mathbb{E} \sup_{0 \leq s \leq t} |\xi_1(s) - \xi_2(s)|_{L^\frac{4}{3}}^p \). Here, we obtain by the Hölder inequality

\[ |\xi_1(s) - \xi_2(s)|_{L^\frac{4}{3}}^p \leq |\xi_1(s)\xi_2(s)|_{L^4} |v_1(s) - v_2(s)|_{L^2}^p. \]

Since \( |\xi_1(s)\xi_2(s)|_{L^4} \leq |\xi_1(s)|_{L^8}^p |\xi_2(s)|_{L^8}^p \), we use the definition of the stopping times and the definition of \( \Omega_1 \) and \( \Omega_2 \) to handle the term \( |\xi_1(s)\xi_2(s)|_{L^4}^p \). In this way we estimate it by a constant depending on \( m \).

We only have to estimate \( \sup_{0 \leq s \leq t} |v_1(s) - v_2(s)|_{L^2}^p \). For this, we first calculate the nonlinear term which is the difficult term to handle. Here, using the smoothing property of the semigroup, the embedding \( L^1(\mathcal{O}) \hookrightarrow H_2^{-\gamma}(\mathcal{O}) \), the Hölder inequality and the definition of stopping times we infer that

\[
|\mathcal{E}(u_1^2 \mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2})(t) - \mathcal{E}(u_2^2 \mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2})(t)|_{L^2}^2 \\
\leq \left( \int_0^t (t-s)^{-\frac{\gamma}{4}} |u_1^2(s) - u_2^2(s)|_{H_2^{-\gamma}} \, ds \right)^2 \\
\leq \left( \int_0^t \mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2} (t-s)^{-\frac{\gamma}{4}} |u_1(s) - u_2(s)|_{L^2} (|u_1(s)|_{L^2} + |u_2(s)|_{L^2}) \, ds \right)^2 \\
\leq C(t) \sup_{0 \leq s \leq t} \mathbb{1}_{\Omega_2} (|u_1(s)|_{L^2} + |u_2(s)|_{L^2})^2 \int_0^t |u_1(s) - u_2(s)|_{L^2}^2 \, ds \\
\leq C(t, m) \int_0^t |u_1(s) - u_2(s)|_{L^2}^2 \, ds.
\]

(5.9)
Using the above estimates in (5.8) and taking expectation we get
\[
\mathbb{E}|v_1(t) - v_2(t)|^2_{L^2} \\
\leq |\mathcal{E}(u_1^2 \mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2})(t) - \mathcal{E}(u_2^2 \mathbb{1}_{\Omega_1} \mathbb{1}_{\Omega_2})(t)|^2_{L^2} \\
+ \left| \mathbb{E} \int_0^t (v_1(s) - v_2(s), (v_1(s) - v_2(s)) dW_2(s)) \right| \\
\leq C(t, m) \int_0^t \mathbb{E}|u_1(s) - u_2(s)|^2_{L^2} \, ds + C \int_0^t \mathbb{E}|v_1(s) - v_2(s)|^2_{L^2} \, ds.
\]

Finally taking into account the linear part, we can show that for any \( \varepsilon > 0 \) and \( m \in \mathbb{N} \) there exists a constant \( C(m, T, \varepsilon) > 0 \) such that
\[
\mathbb{E}|u_1(t) - u_2(t)|^p_{L^2} \leq C(T, m, \varepsilon) \int_0^t \mathbb{E}|u_1(s) - u_2(s)|^2_{L^2} \, ds.
\]

For \( p = 2 \), applying the Gronwall Lemma, we get
\[
\mathbb{E}|u_1(t) - u_2(t)|^p_{L^2} \leq 0.
\]

Similarly, by estimate (5.9) and taking into account the linear part and stochastic convolution, we obtain
\[
\mathbb{E}|v_1(t) - v_2(t)|^p_{L^2} \leq 0.
\]

**Step III** We show that \( \mathbb{P}(\tau_m < T) \to 0 \) as \( m \to \infty \). Observe, that we have
\[
\{\tau_m \leq T\} \subset \{|\hat{\xi}_1|_{L^\infty([0,T];L^8)} \text{ or } |\hat{\xi}_2|_{L^\infty([0,T];L^8)} \geq m\} \\
\cup \{|u_1|_{L^\infty([0,T];L^2) \cap L^2([0,T];H^1)} \geq m \text{ or } |u_2|_{L^\infty([0,T];L^2) \cap L^2([0,T];H^1)} \geq m\}.
\]

Therefore,
\[
\mathbb{P}(\tau_m < T) \leq \mathbb{P}\left(|u_1|_{L^\infty([0,T];L^2) \cap L^2([0,T];H^1)} \geq m\right) + \mathbb{P}\left(|u_1|_{L^\infty([0,T];L^2) \cap L^2([0,T];H^1)} \geq m\right) \\
+ \mathbb{P}\left(|\hat{\xi}_1|_{L^\infty([0,T];L^8)} \geq m\right) + \mathbb{P}\left(|\hat{\xi}_2|_{L^\infty([0,T];L^8)} \geq m\right).
\]

Since \( u_1 \) and \( u_2 \) are continuous in \( L^2(\mathcal{O}) \) and \( \hat{\xi}_1 \) and \( \hat{\xi}_2 \) are continuous in \( L^8(\mathcal{O}) \), and \( \mathbb{P}\)-a.s.
\[
\sup_{0 \leq s \leq T} |u_1(s)|_{L^\infty([0,T];L^2) \cap L^2([0,T];H^1)} < \infty,
\]
\[
\sup_{0 \leq s \leq T} |u_2(s)|_{L^\infty([0,T];L^2) \cap L^2([0,T];H^1)} < \infty,
\]
\[
\sup_{0 \leq s \leq T} |\hat{\xi}_1(s)|_{L^\infty([0,T];L^8)} < \infty, \text{ and } \sup_{0 \leq s \leq T} |\hat{\xi}_2(s)|_{L^\infty([0,T];L^8)} < \infty,
\]

it follows that as \( m \to \infty \), \( \mathbb{P}(\tau_m \leq T) \to 0 \). Hence, both processes \((u_1, v_1)\) and \((u_2, v_2)\) are undistinguishable on \([0, T]\). \(\square\)
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