A general formalism was introduced for reorganization of QCD evolution equations and derivation of hierarchical solution to DGLAP equation. The hierarchical solution separates two types of parton emissions: the flavour changing emissions and the diagonal ones (bremsstrahlung). It has been shown that both of the achieved processes are of the Markovian type, what gives a nice possibility of Monte Carlo implementation. Hierarchical algorithm has been implemented and crosschecked with standard Markovian algorithm implemented in program EvolFMC.

1 Introduction

General Form of QCD evolution equation is given by
\[ \partial_t D_k(t, x) = \sum_j P_{kj}(t, \cdot) \otimes D_j(t, \cdot)(x), \]  
where the convolution rule is defined by
\[ f_1(\cdot) \otimes f_2(\cdot)(x) = \int dz_1 dz_2 \delta(x - z_1 z_2) f_1(z_1) f_2(z_2). \]  
Parton momentum distribution function (PDF) \( D_k \) is a function of fraction of hadron longitudinal momentum - \( x \) and evolution variable \( t = \ln Q \) (where \( Q \) is virtuality). Splitting function \( P_{kj} \) which represents probability of parton emission, can be calculated perturbatively. General form of \( P_{kj} \) is
\[ P_{kj}(t, z) = -P_{kj}^\delta (t) \delta_{jk} \delta(1 - z) + P_{kj}^\Theta(t, z), \]
We impose the momentum sum rule: \( \sum_k \int_0^1 dx x D_k(t, x) = \text{const} \) what leads to relation between virtual and real part of kernel
\[ P_{kk}^\delta(t) = \sum_j \int_0^1 dz z P_{jk}^\Theta(t, z). \]

2 Reorganisation of evolution equation

Our aim is to separate two types of parton emissions: flavour changing emissions and the diagonal ones (bremsstrahlung). To do it we divide kernel into two parts the diagonal in flavour part and the rest
\[ P_{jk}^A(t, z) = -P_{jk}^\delta (t) \delta_{jk} \delta(1 - z) + \delta_{jk} P_{kk}^\Theta(t, z), \]
\[ P_{jk}^B(t, z) = P_{jk}(t, z) - P_{jk}^A(t, z) = -P_{jk}^\delta (t) \delta_{jk} \delta(1 - z) + (1 - \delta_{jk}) P_{jk}^\Theta(t, z). \]
We do it in such a way that for both parts we have relation analogous to equation (4). We will use the compact matrix notation, where $k$ and $x$ are treated as indices. But we need to remember that we do not have the properties of commutativity anymore. Now the evolution equation has the form:

$$\partial_t \mathbf{D}(t) = \mathbf{P}(t) \mathbf{D}(t) = (\mathbf{P}^A(t) + \mathbf{P}^B(t)) \mathbf{D}(t).$$

(6)

We introduce the diagonal transitions operator $\mathbf{G}_A(t, t_0)$ which is solution of the equation analogous to our evolution equation (1) but with only diagonal part of the kernel:

$$\partial_t \mathbf{G}_A(t, t_0) = \mathbf{P}^A(t) \mathbf{G}_A(t, t_0).$$

(7)

Solution to this equation can be written as time ordered exponent:

$$\mathbf{G}_A(t, t_0) = T \exp \left( \int_{t_0}^{t} dt' \mathbf{P}^A(t') \right).$$

(8)

Than we do a change of variables $\mathbf{P}^B(t) = \mathbf{G}_A^{-1}(t, t_0) \mathbf{P}^B(t) \mathbf{G}_A(t, t_0)$ and $\mathbf{D}(t) = \mathbf{G}_A^{-1}(t, t_0) \mathbf{D}(t)$. In this new variables evolution equation (1) takes form:

$$\partial_t \mathbf{D}(t) = \mathbf{P}^B(t) \mathbf{D}(t).$$

(9)

Now it is easy to see that solution of equation (9) is given by time ordered exponent. We iterate this exponent and translate it back to original variables:

$$\mathbf{D}(t) = \mathbf{G}_A(t, t_0) \mathbf{D}(t_0) +$$

$$+ \sum_{n=1}^{\infty} \left( \prod_{i=1}^{n} \int_{t_0}^{t} dt_i \Theta(t_i - t_{i-1}) \mathbf{G}_A(t_{i+1}, t_{i}) \mathbf{P}^B(t_i) \right) \mathbf{G}_A(t_1, t_0) \mathbf{D}(t_0).$$

(10)

For completeness we come back to standard notation which show all the indices explicitly. We also perform resummation of virtual part of flavour changing kernel $P^{\Phi B}$ what leads to introduction of flavour changing Sudakov formfactor $\Phi^B_k(t, t_0) = \int_{t_0}^{t} dt' \Phi^{\Phi B}(t').$

$$D_k(t, x) = \int_{0}^{1} dz' dx_0 e^{-\Phi^B(t, t_0)} G_{kk}(t, x, z') D_k(t_0, x_0) \delta(x - z' x_0)$$

$$+ \sum_{n=1}^{\infty} \sum_{k_n \neq \ldots \neq k_1 \neq k_0} \left( \prod_{i=1}^{n} \int_{t_0}^{t} dt_i \Theta(t_i - t_{i-1}) \int_{0}^{1} dz' dz_1 e^{-\Phi^B_{k_{i-1}}(t, t_{i-1})} G_{k_{i-1}k_{i-1}}(t, t_{i-1}, z_i')
\times P_{k_{i}k_{i-1}}^{\Phi}(t, z_i) \int_{0}^{1} dz_0 dz_0 D_{k_0}(t_0, x_0) \delta(x - x_0 \prod_{i=1}^{n} z_i \prod_{i=1}^{n+1} z_i').$$

(11)

Now to have the full solution we need to find the explicit form of function $G_{kk}^A(t, t_0)$ which represents diagonal transitions. By definition it is a solution of equation (9) so we can use analogous procedure to solve it. The solution in iterative form is given by:

$$G_{kk}^A(t, t_0, z') = e^{-\Phi^A(t, t_0)} \delta(1 - z') +$$

$$+ \sum_{n=1}^{\infty} \left[ \prod_{i=1}^{n} \int_{t_0}^{t} dt_i \Theta(t_i - t_{i-1}) \int_{0}^{1} dz_i' \int_{0}^{1} e^{-\Phi^A_{k_i}(t, t_{i-1})} \left[ \prod_{i=1}^{n} e^{-\Phi^A_{k_{i}}(t_{i-1})} P_{kk}^{\Phi}(t_i, z_i') \right] \delta(z' - \prod_{i=1}^{n} z_i') \right]$$

(12)

where the initial condition for $G_{kk}^A(t_0, t_0, z')$ as Dirac delta in $z' = 1$ was used.

In this way we obtained hierarchy of two processes. The external - flavour changing process and internal - diagonal process. For both processes one can define properly normalised transition
probability which depends only on previous step (emission). Probability in the flavour changing process is given by:

$$\omega(t_i, z_i z'_i x_{i-1}, k_i | t_{i-1}, x_{i-1}, k_{i-1}) = (1 - \delta_{k_i k_{i-1}}) e^{-\Phi^B_{k_i k_{i-1}}(z_i) z'_i G^A_{k_{i-1} k_i} (t_i, t_{i-1}, z'_i)},$$

with normalisation condition:

$$\int_{t_{i-1}}^{\infty} dt_i \sum_k \int_0^1 dz_i \int_0^1 dz'_i \omega(t_i, z_i z'_i x_{i-1}, k_i | t_{i-1}, x_{i-1}, k_{i-1}) \equiv 1. \tag{14}$$

For bremsstrahlung process we have:

$$p_k(t_i, Z_i z'_i | t_{i-1}, Z_{i-1}) = \Theta(t_i - t_{i-1}) P^{\Theta}_{kk}(t_i, Z_i) e^{-\Phi^A_k(t_i, t_{i-1})} \tag{15}$$

and normalisation condition:

$$\int_{t_{i-1}}^{\infty} dt_i \int_0^1 dZ_i \ p_k(t_i, Z_i z'_i | t_{i-1}, z_{i-1}) \equiv 1. \tag{16}$$

As we can see each of the processes on its own is of Markov type. This means one can use standard algorithm to solve each of the two processes separately. What connects this two levels are limits for evolution variable. So by means of presented reorganisation we obtained hierarchy of two Markov processes which has straightforward interpretation/implementation as a generalisation of well known Markovian algorithm.

## 3 Results

Presented algorithm has been used to solve LO DGLAP evolution equation with non-running $\alpha_S$. It takes into account three massless quarks and gluons. PDFs were calculated for scales reaching from $Q = 1\text{GeV}$ (initial conditions) up to scale of $Q = 1000\text{GeV}$. Results have been compared with standard Markovian algorithm implemented in program EvolFMC\[^1\][3]. Fig. 1 shows gluon momentum distribution $x D_G(t, x)$ and quark momentum distribution $x D_q(t, x)$ obtained from both algorithms and ratio for corresponding distributions for scales equal 1GeV, 10GeV, 100GeV and 1000GeV. Presented results were calculated for statistics equal $2 \times 10^8$ and as we can see agreement of both approaches is on the level of 0.5%.

## 4 Summary

The general formalism for reorganization of evolution equation has been used for derivation of hierarchical solution to DGLAP equation. Very good agreement with standard Markovian algorithm shows that it can compete with it. Generalisation for NLO and for running $\alpha_S$ case can be done and is only a matter of implementation. Also generalisation for different types of evolution equations is possible. Especially case of CCFM evolution seems to be worth considering because of possibility of simple changing of coupling constants argument for flavour changing and diagonal emissions.

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Figure 1: The left upper plot presents gluon distributions and right upper plot quark distributions obtained from hierarchical algorithm (dashed line) and standard Markovian algorithm-EvoFMC (straight line) for scales of 1 GeV, 10 GeV, 100 GeV, 1000 GeV. The lower plots represent corresponding ratio between these two distributions.

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