A VARIATION OF CONTINUITY IN \( n \)-NORMED SPACES

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ABSTRACT. The \( s \)-th forward difference sequence that tends to zero, inspired by the consecutive terms of a sequence approaching zero, is examined in this study. Functions that take sequences satisfying this condition to sequences satisfying the same condition are called \( s \)-ward continuous. Inclusion theorems that are related to this kind of uniform continuity and continuity are also considered. Additionally, the concept of \( s \)-ward compactness of a subset of \( X \) via \( s \)-quasi-Cauchy sequences are investigated. One finds out that the uniform limit of any sequence of \( s \)-ward continuous function is \( s \)-ward continuous and the set of \( s \)-ward continuous functions is a closed subset of the set of continuous functions.

1. Introduction and Preliminaries

Although some evaluations was first made about the axioms of an abstract \( n \)-dimensional metric, the main developments regarding the definition of the 2-metric, 2-normed spaces and their topological properties were described by Gähler [22] then the results of these concepts were extended to the most generalized case \( n \)-metric and \( n \)-normed spaces where \( n \) is a natural number by Gähler[23]. Shortly after the concept of \( n \)-normed space is introduced, the concept of 2-inner product space is also defined in [3]. Afterwards many authors have done lots of impressive improvements in \( n \)-normed spaces or in 2-inner product spaces ([2, 20, 19, 13, 15, 11, 16]).

Firstly we recall the notion of \( n \)-normed space:

**Definition 1.1.** An \( n \)-norm on a real vector space \( X \) of dimension \( d \), where \( 2 \leq n \leq d \) is a real valued function \( \| \cdot, \ldots, \cdot \| \) on \( X^n \) which satisfies the properties:

1. \( \| \zeta_1, \zeta_2, \ldots, \zeta_n \| = 0 \) if and only if \( \zeta_1, \zeta_2, \ldots, \zeta_n \) are linearly dependent,
2. \( \| \zeta_1, \zeta_2, \ldots, \zeta_n \| = \| \zeta_{k_1}, \ldots, \zeta_{k_n} \| \) for every permutation \((k_1, \ldots, k_n)\) of \((1, \ldots, n)\),
3. \( \| \zeta_1, \zeta_2, \ldots, \delta \zeta_n \| = \| \zeta_1, \zeta_2, \ldots, \zeta_n \| \) for any real number \( \delta \),
4. \( \| m + n, \zeta_1, \ldots, \zeta_{n-1} \| \leq \| m, \zeta_1, \ldots, \zeta_{n-1} \| + \| n, \zeta_1, \ldots, \zeta_{n-1} \|. \)

A set \( X \) is an \( n \)-normed space with an \( n \)-norm \( \| \cdot, \ldots, \cdot \| \).

In [12],

\[
\| \zeta_1, \ldots, \zeta_n \|_p = \left[ \frac{1}{n!} \sum_{t_1} \ldots \sum_{t_n} \det(\zeta_{it_k})^p \right]^{1/p}.
\]

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is given as an example of an \( n \)-norm on \( l^p \times \ldots \times l^p \) space for \( 1 \leq p < \infty \). Also if \( p = \infty \), the \( n \)-norm on \( l^\infty \times \ldots \times l^\infty \) is given as \([1]\)

\[
\|\zeta_1, \ldots, \zeta_n\|_\infty = \sup_{t_1} \ldots \sup_{t_n} \det(x_{it_k}).
\]

**Definition 1.2.** A sequence \((x_k)\) converges to an \( \zeta \in X \) in an \( n \)-normed space \( X \) if for each \( \epsilon > 0 \), there exists a positive integer \( \tilde{k} \) such that for every \( k \geq \tilde{k} \)

\[
\|x_k - \zeta, \mu_1, \ldots, \mu_{n-1}\| < \epsilon, \quad \forall \mu_1, \ldots, \mu_{n-1} \in X.
\]

**Definition 1.3.** A sequence \((x_k)\) is a Cauchy sequence if for each \( \epsilon > 0 \), there exists a positive integer \( t_0 \) such that for every \( k, m \geq t_0 \)

\[
\|x_k - x_m, \mu_1, \ldots, \mu_{n-1}\| < \epsilon, \quad \forall \mu_1, \ldots, \mu_{n-1} \in X.
\]

If each Cauchy sequence in \( X \) converges to an element of \( X \), we call \( X \) is complete and if \( X \) is complete, then it is called an \( n \)-Banach space.

In recent times, the notion of quasi-Cauchy sequences is given in \([4]\). The distance between consecutive terms of a sequence tending to zero is expressed by Burton and Coleman with the idea of quasi-Cauchy sequence. Then using this idea, different types of continuities were defined for real functions in \([6, 7]\) as ward continuity, statistically ward continuity, lacunary ward continuity and etc. They were also studied in 2-normed space in \([24, 9, 10]\).

The aim of this research is to give a generalization of the notions of a quasi-Cauchy sequence and ward continuity of a function to the notions of an \( s \)-quasi-Cauchy sequence and \( s \)-ward continuity of a function in an \( n \)-normed space for any fixed positive integer \( s \). Also interesting theorems related to ordinary continuity, uniform continuity, compactness and \( s \)-ward continuity are proved. This paper contains not only an extension of results of \([24]\) to an \( n \)-normed space, but also includes new results in 2-normed spaces as a special case for \( n = 2 \).

2. Main results

In this paper \( X, \mathbb{R} \) and \( s \) will denote a first countable \( n \)-normed space with an \( n \)-norm \( \|., \ldots, .\| \), the set of all real numbers and a fixed positive integer, respectively. Now we give the notion of \( s \)-quasi Cauchyness of a sequence in \( X \):

**Definition 2.1.** A sequence \((x_k)\) of points in \( X \) is said to be \( s \)-quasi-Cauchy if for all \( \mu_1, \mu_2, \ldots, \mu_{n-1} \in X \) it satisfies

\[
\lim_{k \to \infty} \|\Delta_s x_k, \mu_1, \mu_2, \ldots, \mu_{n-1}\| = 0
\]

where \( \Delta_s x_k = x_{k+s} - x_k \) for each positive integer \( k \).

If one chooses \( s = 1 \), the sequences returns to the ordinary quasi-Cauchy sequences and also using the equality

\[
x_{k+s} - x_k = x_{k+s} - x_{k+s-1} + x_{k+s-1} - x_{k+s-2} - x_{k+2} + x_{k+2} - x_{k+1} + x_{k+1} - x_k,
\]

we see that any quasi-Cauchy sequence is \( s \)-quasi-Cauchy, however the converse is not true.

Any Cauchy sequence is \( s \)-quasi-Cauchy, so is any convergent sequence. A sequence of partial sums of a convergent series is \( s \)-quasi-Cauchy. One notes that the
set $\Delta_s(X)$, the set of $s$-quasi-Cauchy sequences in $X$, is a vector space. If $(x_k), (y_k)$ are $s$-quasi-Cauchy sequences in $X$ so

\begin{align}
\lim_{k \to \infty} ||\Delta_s x_k, \mu_1, \mu_2, ..., \mu_{n-1}|| &= 0 \quad \text{and} \\
\lim_{k \to \infty} ||\Delta_s y_k, \mu_1, \mu_2, ..., \mu_{n-1}|| &= 0.
\end{align}

for all $\mu_1, \mu_2, ..., \mu_{n-1} \in X$. Therefore

\begin{equation}
\lim_{k \to \infty} ||\Delta_s (x_k + y_k), \mu_1, \mu_2, ..., \mu_{n-1}|| \leq \lim_{k \to \infty} ||\Delta_s x_k, \mu_1, \mu_2, ..., \mu_{n-1}|| + \lim_{k \to \infty} ||\Delta_s y_k, \mu_1, \mu_2, ..., \mu_{n-1}|| = 0.
\end{equation}

So the sum of two $s$-quasi-Cauchy sequences is again $s$-quasi-Cauchy, it is clear that $(ax_k)$ is an $s$-quasi-Cauchy sequence in $X$ for any constant $a \in \mathbb{R}$.

**Definition 2.2.** A subset $A$ of $X$ is called $s$-ward compact if any sequence in the set $A$ has an $s$-quasi-Cauchy subsequence.

If a set $A$ is an $s$-ward compact subset of $X$, then any subset of $A$ is $s$-ward compact. Moreover any ward compact subset of $X$ is $s$-ward compact. Union of finite number of $s$-ward compact subset of $X$ is $s$-ward compact. Any sequentially compact subset of $X$ is $s$-ward compact.

For each real number $\alpha > 0$, an $\alpha$-ball with center $a$ in $X$ is defined as

\begin{equation} B_\alpha(a, x_1, ..., x_{n-1}) = \{ x \in X : ||a - x, x_1 - x, ..., x_{n-1} - x|| < \alpha \}
\end{equation}

for $x_1, ..., x_{n-1} \in X$. The family of all sets $W_i(a) = B_\alpha(a, x_{i_1}, ..., x_{i_{n-1}})$ where $i = 1, 2, ..., \text{is an open basis in} \ a$. Let $\beta_{n-1}$ be the collection of linearly independent sets $B$ with $n - 1$ elements. For $B \in \beta_{n-1}$, the mapping

\begin{equation} p_B(x) = ||x, x_1, ..., x_{n-1}||, \quad \text{for} \ x \in X, \ x_1, ..., x_{n-1} \in B
\end{equation}

defines a seminorm on $X$ and the collection $\{p_B : B \in \beta_{n-1}\}$ of seminorms makes $X$ a locally convex topological vector space. For each $x \in X$, different from zero, there exists $x_1, ..., x_{n-1} \in B$ such that $x, x_1, x_2, ..., x_{n-1}$ are linearly independent so $||x, x_1, ..., x_{n-1}|| \neq 0$, which makes $X$ a Hausdorff space. A neighborhood of origin for this topology is in a form of a finite intersection

\begin{equation} \bigcap_{i=1}^{n} \{ x \in X : ||x, x_{i_1} - x, ..., x_{i_{n-1}} - x|| < \epsilon \}
\end{equation}

where $\epsilon > 0$.

Now the following theorem characterizes totally boundedness not only valid for $n$-normed spaces but also valid for the 2-normed spaces. It extends the results for quasi-Cauchy sequences given in [24] for 2-normed valued sequences to $n$-normed valued $s$-quasi-Cauchy sequences in which $s = 1$ gives earlier results given for 2-normed spaces. It should be noted that Theorem 3 in [8] can not be obtained just by putting $n = 1$ in the $n$-normed space to get in a normed space, which is awkward, whereas the following theorem is interesting as a point of studying a new space.

**Lemma 2.1.** If a subset of $X$ is totally bounded then every sequence in $A$ contains an $s$-quasi-Cauchy subsequence.
Proof. Let $A$ be totally bounded. Let $(x_n)$ be any sequence in $A$. Since $A$ is covered by a finite number of balls of $X$ of diameter less than 1. One of these sets, which we denote by $A_1$, must contain $x_n$ for infinitely many values of $n$. Choose a positive integer $n_1$ such that $x_{n_1} \in A_1$. Since $A_1$ is totally bounded, it is covered by a finite number of balls of diameter less than $1/2$. One of these balls, which we denote by $A_2$, must contain $x_{n_2}$ for infinitely many $n$. Let $n_2$ be a positive integer such that $n_2 > n_1$ and $x_{n_2} \in A_2$. Since $A_2 \subset A_1$, it follows that $x_{n_2} \in A_1$. Continuing in this way, a ball $A_k$ of $A_{k-1}$ of diameter less than $1/k$ and a term $x_{n_k} \in A_k$ of the sequence $(x_n)$, where $n_k > n_{k-1}$ for any positive integer $k$. Since $x_{n_k}, x_{n_{k+1}}, ..., x_{n_k+s}, ...$ lie in $A_k$ and diameter $(A_k)$ less than $1/k$, then $x_{n_k}$ is an $s$-quasi-Cauchy subsequence of $(x_n)$.

Theorem 2.2. A subset of $X$ is totally bounded if and only if it is $s$-ward compact for any positive integer $s$.

Proof. If $A$ is totally bounded, then every sequence of $A$ has an $s$-quasi-Cauchy subsequence by Lemma 2.1. So the set $A$ is $s$-ward compact for any fixed positive integer $s$. For the converse, think $A$ is not a totally bounded set. Choose any $x_1 \in A$ and $\alpha > 0$. Since $A$ is not totally bounded, the neighborhood of a point $x_1$ in $A$ which is defined by $B_\alpha(x_1, \mu_1^1, ..., \mu_{n-1}^1) = \{ y \in A : ||x_1 - y, \mu_1^1 - y, ..., \mu_{n-1}^1 - y|| < \alpha \}$ is not equal to $A$. There is an $x_2 \in A$ such that $x_2 \notin B_\alpha(x_1, \mu_1^1, ..., \mu_{n-1}^1)$, that is, $||x_2 - x_1, \mu_1^1 - x_1, ..., \mu_{n-1}^1 - x_1|| \geq \alpha$. Since $A$ is not totally bounded $B_\alpha(x_2, \mu_1^1, ..., \mu_{n-1}^1) \cup B_\alpha(x_2, \mu_1^{i^2}, ..., \mu_{n-1}^{i^2}) \neq A$ where $B_\alpha(x_2, \mu_1^{i^2}, ..., \mu_{n-1}^{i^2})$ is the neighborhood of a point $x_2$ in $A$. Continuing the procedure, a sequence $(x_k)$ of points in $A$ can be obtained as $x_{k+s} - x_k, \mu_1^1 - x_k, ..., \mu_{n-1}^1 - x_k|| \geq \alpha$ and all nonzero $\mu_1^1, ..., \mu_{n-1}^1 \in A$ where $i = 1, ..., k + s - 1$. So the sequence $(x_k)$ has not any $s$-quasi-Cauchy subsequence. Therefore $A$ is not $s$-ward compact.

Definition 2.3. A function $f$ is called $s$-ward continuous on a subset $A$ of $X$ if

$$\lim_{k \to \infty} ||\Delta_s x_k, \mu_1, \mu_2, ..., \mu_{n-1}|| = 0$$

is satisfied for all $\mu_1, \mu_2, ..., \mu_{n-1} \in X$, then

$$\lim_{k \to \infty} ||\Delta_s f(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})|| = 0.$$ 

In the following we give that any $s$-ward continuous function is continuous.

Theorem 2.3. Any $s$-ward continuous function on a subset $A$ of $X$ is continuous on $A$.

Proof. Let the function $f$ be $s$-ward continuous on $A \subset X$ and any sequence $(x_k)$ in $A$ converge to $\zeta$, that is

$$\lim_{k \to \infty} ||x_k - \zeta, \mu_1, \mu_2, ..., \mu_{n-1}|| = 0$$

for all $\mu_1, \mu_2, ..., \mu_{n-1} \in X$. Let us write a new sequence using some terms of the sequence $(x_k)$ as

$$(t_m) = (x_1, ..., x_1, \zeta, ..., \zeta, x_2, ..., x_2, \zeta, ..., \zeta, x_n, ..., x_n, \zeta, ..., \zeta, ...).$$

where same terms repeated $s$-times. Every convergent sequences is Cauchy and moreover any Cauchy sequence is $s$-quasi-Cauchy, then it follows that

$$\lim_{m \to \infty} ||\Delta_s t_m, \mu_1, \mu_2, ..., \mu_{n-1}|| = \lim_{m \to \infty} ||t_m - t_m, \mu_1, \mu_2, ..., \mu_{n-1}|| = 0.$$
in which either
\[ \lim_{m \to \infty} \| t_{m+s} - \zeta, \mu_1, \mu_2, \ldots, \mu_{n-1} \| = 0 \]
or
\[ \lim_{m \to \infty} \| \zeta - t_m, \mu_1, \mu_2, \ldots, \mu_{n-1} \| = 0 \]
for every \( \mu_1, \mu_2, \ldots, \mu_{n-1} \). This result implies \((t_m)\) is an \(s\)-quasi Cauchy sequence. Since the function \(f\) is assumed to be \(s\)-ward continuous, using this assumption we get
\[ \lim_{m \to \infty} \| \Delta_s f(t_m), f(\mu_1), f(\mu_2), \ldots, f(\mu_{n-1}) \| = 0 \]
in which either
\[ \lim_{m \to \infty} \| f(t_{m+s}) - f(t_m), f(\mu_1), f(\mu_2), \ldots, f(\mu_{n-1}) \| = 0 \]
or
\[ \lim_{m \to \infty} \| f(\zeta) - f(t_m), f(\mu_1), f(\mu_2), \ldots, f(\mu_{n-1}) \| = 0 \]
So \((f(x_k))\) converges to \(f(\zeta)\).

As the sum of two \(s\)-ward continuous function on \(A\) is \(s\)-ward continuous and \(cf\) is \(s\)-ward continuous for any constant real number \(c\), the set of \(s\)-ward continuous functions on \(A\) is a vector subspace of vector space of all continuous function on \(A\).

**Theorem 2.4.** Every \(s\)-ward continuous function on \(A \subset X\) is \(s\)-ward continuous on \(A\).

**Proof.** Assume that \((x_k)\) is a quasi-Cauchy sequence in \(A\) and \(f\) is any \(s\)-ward continuous function on \(A\). If \(s = 1\), the result is obvious. Let \(s > 1\) and a sequence
\[ (t_m) = (x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, \ldots, x_n, x_n, \ldots, x_n, \ldots) \]
be \(s\)-quasi-Cauchy, i.e.
\[ \lim_{m \to \infty} \| \Delta_s t_m, \mu_1, \mu_2, \ldots, \mu_{n-1} \| = 0. \]
We have
\[ \lim_{m \to \infty} \| \Delta_s f(t_m), f(\mu_1), f(\mu_2), \ldots, f(\mu_{n-1}) \| = 0 \]
by using the \(s\)-ward continuity of the function \(f\). Therefore
\[ \lim_{m \to \infty} \| \Delta f(t_m), f(\mu_1), f(\mu_2), \ldots, f(\mu_{n-1}) \| = 0 \]
So \(s\)-ward continuity of the function \(f\) implies that the \(s\)-ward continuity of \(f\) on \(A \subset X\).

**Theorem 2.5.** The image of an \(s\)-ward compact subset of \(X\) by an \(s\)-ward continuous function is \(s\)-ward compact.

**Proof.** Assume that \(f\) is an \(s\)-ward continuous function and \(A\) is an \(s\)-ward compact subset of \(X\). Choose a sequence \(t\) as \(t = (t_k) \in f(A)\) and say \((t_k) = f(x_k)\) where \(x_k \in A\). \(A\) is \(s\)-ward compact so there is a subsequence \((x_m)\) of \((x_k)\) with
\[ \lim_{m \to \infty} \| \Delta_s x_m, \mu_1, \mu_2, \ldots, \mu_{n-1} \| = 0 \]
for all \(\mu_1, \mu_2, \ldots, \mu_{n-1} \in X\). Using the \(s\)-ward continuity of \(f\) we have
\[ \lim_{m \to \infty} \| \Delta_s f(x_m), f(\mu_1), f(\mu_2), \ldots, f(\mu_{n-1}) \| = 0, \]
so there is an s-quasi-Cauchy subsequence \((f(x_m))\) of \(t\). The result implies that the subset \(f(A) \subset X\) is s-ward compact. \(\Box\)

s-ward continuous image of any compact subset of \(X\) is compact. It is easily evaluated from Theorem 2.8.

**Theorem 2.6.** If \(f\) is uniformly continuous on \(A \subset X\), then it is s-ward continuous on \(A\).

**Proof.** Let \(f\) be a uniformly continuous function on \(A\), and the sequence \((x_k)\) be an s-quasi-Cauchy sequence in \(A\). Our aim is to prove the sequence \((f(x_k))\) is also an s-quasi-Cauchy sequence in \(A\). Take any \(\varepsilon > 0\). There exists a \(\delta > 0\) such that if
\[
\|x - y, \mu_1, \mu_2, ..., \mu_{n-1}\| < \delta \quad \text{then} \quad \|f(x) - f(y), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\| < \varepsilon.
\]

There exists an \(\bar{k} = \bar{k}(\delta)\) for this \(\delta > 0\) such that
\[
\|\Delta_s f_k(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\| < \delta
\]
for every \(\mu_1, \mu_2, ..., \mu_{n-1} \in X\) whenever \(k > \bar{k}\). Uniform continuity of \(f\) on \(A\) for every \(k > \bar{k}\) implies
\[
\|\Delta_s f_k(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\| < \varepsilon
\]
for every \(f(\mu_1), f(\mu_2), ..., f(\mu_{n-1}) \in X\). The sequence \((f(x_k))\) is s-quasi-Cauchy so the function \(f\) is s-ward continuous. \(\Box\)

**Theorem 2.7.** Uniform limit of a sequence of s-ward continuous function is s-ward continuous.

**Proof.** Let \((f_t)\) be a sequence of s-ward continuous functions and it be uniformly convergent sequence to a function \(f\). Pick an s-quasi-Cauchy sequence \((x_k)\) in \(A\) and choose any \(\varepsilon > 0\). There is an integer \(N \in \mathbb{Z}^+\) such that
\[
\|f_t(x) - f(x), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\| < \frac{\varepsilon}{3}
\]
for every \(x \in A\), for all \(f(\mu_1), f(\mu_2), ..., f(\mu_{n-1}) \in X\) whenever \(t \geq N\). Using the s-ward continuity of \(f_N\), there is a positive integer \(N_1(\varepsilon) > N\) such that
\[
\|\Delta_s f_t(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\| < \frac{\varepsilon}{3}
\]
for every \(t \geq N_1\). Now for \(t \geq N_1\) we have
\[
\|\Delta_s f(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\|
\leq \|f(x_{k+s}) - f(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\|
\leq \|f(x_{k+s}) - f_N(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\|
+\|\Delta_s f_N(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\|
+\|f_N(x_k) - f(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
So the function \(f\) is s-ward continuous on \(A\). \(\Box\)

**Theorem 2.8.** The collection of the s-ward continuous functions on \(A \subset X\) is a closed subset of the collection of every continuous functions on \(A\).
Proof. Let $E$ be a collection of all s-ward continuous functions on $A \subset X$ and $\bar{E}$ is the closure of $E$. $\bar{E}$ is defined as for every $x \in X$ there exists $x_k \in E$ with $\lim_{k \to \infty} x_k = x$ and $E$ is closed if $E = \bar{E}$. It is obvious that $E \subseteq \bar{E}$. Let $f$ be any element of the set of all closure points of $E$ which means there exists a sequence of points $f_t$ in $E$ as

$$\lim_{t \to \infty} ||f_t - f, f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})|| = 0$$

for all $f(\mu_1), f(\mu_2), ..., f(\mu_{n-1}) \in X$ and also $f_t$ is a s-ward continuous function. Choose the sequence $(x_k)$ as any s-quasi-Cauchy sequence. Since $(f_t)$ converges to $f$, for every $\varepsilon > 0$ and $x \in E$, there is any $N_0$ such that for every $t \geq N_0$,

$$||f(x) - f_t(x), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})|| < \frac{\varepsilon}{3}.$$ 

As $f_N$ is p-ward continuous, $N_1 > N_0$ exists such that for all $t \geq N_1$,

$$||\Delta_s f_N(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})|| < \frac{\varepsilon}{3}.$$ 

Hence for all $t \geq N_1$,

$$||\Delta_s f(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})||$$

$$= ||f(x_{k+s}) - f(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})||$$

$$\leq ||f(x_{k+s}) - f_N(x_{k+s}), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})||$$

$$+ ||f(x_k) - f_N(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})||$$

$$+ ||\Delta_s f_N(x_k), f(\mu_1), f(\mu_2), ..., f(\mu_{n-1})|| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$ 

Since the function $f$ is s-ward continuous in $E$ then $E = \bar{E}$, using Theorem 2.3 it ends the proof. 

\[\square\]

3. Conclusion

The notion of an $n$-normed space was given by thinking if there is a problem where $n$-norm topology works however norm topology doesn’t. As an application of the notion of n-norm, we can examine that if a term in the definition of n-norm shows the change of shape then the n-norm stands for the associated volume of this surface. Suppose that for any particular output one needs $n$-inputs but with one main input and other (n-1)-inputs as dummy inputs required which accomplish the operation, so this concept may be used as an application in many areas of science. The generalization of the notions of quasi-Cauchy sequences and s-ward continuous functions to the notions of $n$-quasi-Cauchy sequences and s-ward continuous functions in $n$-normed spaces are investigated in this paper. Also we find out some interesting inclusion theorems related to the concepts of ordinary continuity, uniform continuity, s-ward continuity, and s-ward compactness. We prove that the uniform limit of a sequence of s-ward continuous function is s-ward continuous and the set of s-ward continuous functions is a closed subset of the set of continuous functions. We recommend research $n$-quasi-Cauchy sequences of points and fuzzy functions in an $n$-normed fuzzy space as a further study. However, due to the different structure of the methods of proof will not be similar to the one in this study (see [13], [21]). Also we recommend investigate $s$-quasi-Cauchy sequences of double sequences in $n$-normed spaces as another further study (see [17], [18]).
Declarations

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