Thermodynamical analogues in quantum information theory

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(printed October 24, 2018)

Abstract

The first step in quantum information theory is the identification of entanglement as a valuable resource. The next step is learning how to exploit this resource efficiently. We learn how to exploit entanglement efficiently by applying analogues of thermodynamical concepts. These concepts include reversibility, entropy, and the distinction between intensive and extensive quantities. We discuss some of these analogues and show how they lead to a measure of entanglement for pure states. We also ask whether these analogues are more than analogues, and note that, locally, entropy of entanglement is thermodynamical entropy.

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I. INTRODUCTION

We are familiar with the distinction between “pure” and “applied” research. In pure research, knowledge of the physical world is an end in itself, while in applied research, knowledge is only a means to an end. Usually, this distinction is a meaningful one; we can easily find examples of pure research that has no applications, and of applied research that does not increase our knowledge. But sometimes the distinction is not meaningful. For example, what about Carnot’s invention of the cycle that bears his name—was it pure research or applied research? In this example, the distinction simply doesn’t exist. It doesn’t exist, and not only because Carnot invented his cycle while considering a practical engineering problem. It doesn’t exist, because his “applied” research on the efficiency of heat engines was also essential “pure” research. Indeed, if Carnot had never considered a practical engineering problem, he would not have started thinking about limits to efficiency, and he would not have discovered the second law of thermodynamics.

A second example, quite analogous to the first, arises in quantum information theory. Research on quantum entanglement led to quantum information theory only after physicists found uses for entanglement and thought about how to use entanglement efficiently. Limits on the efficient use of entanglement are fundamental to quantum information theory; hence applied research on entanglement is also essential pure research in quantum information theory. The striking analogy between the roles of efficiency in thermodynamics and in quantum information theory is the subject of the next section. This analogy leads us to several other analogies between the two theories. Sect. III shows how analogues of heat engines, entropy, the thermodynamic limit and the second law of thermodynamics appear in quantum information theory. The analogies between thermodynamics and quantum information theory are so striking that we ask, in the concluding section, if they are more than analogies. Is quantum information theory, after all, a branch of thermodynamics?

II. ENTANGLEMENT AS A RESOURCE

One of the most salient facts about quantum information theory is the fact that it came about recently, and not 60 or more years ago. On the one hand, it could not have come before the birth of quantum mechanics in 1926 and the identification of entanglement by Schrödinger \[1\] and Einstein, Podolsky and Rosen (EPR) \[2\] in 1935. On the other hand, it could have, in principle, come soon after 1935. Why didn’t it? What happened instead?

In the decades following the EPR paper, most physicists ignored it. Bell was one of the few who did not; he agreed with EPR that “no reasonable theory” should allow such an unreasonable thing as quantum entanglement. Bell \[3\] published his famous inequalities almost 30 years after the EPR paper, and another five years passed until Clauser, Horne, Shimony and Holt (CHSH) \[4\] suggested testing Bell’s inequalities experimentally. The CHSH paper sparked intense interest in entanglement, and the many papers that followed it demonstrated again and again, in experiment and in theory, that quantum mechanics is unreasonable—but true. Thus entanglement became a wall for physicists to beat their heads against. How can the world be so unreasonable? Even physicists who were ready to stop beating their heads against the wall lacked a sense of direction. If we cannot understand
entanglement, at least we could try, for example, to measure it, to quantify it. But various proposed measures of entanglement seemed equally plausible [5].

What provided the direction was not the attempt to understand entanglement, but attempts to use it. Consider singlet pairs of spin-1/2 particles or photons shared by two remote observers, Alice and Bob; Alice has one particle in each pair, and Bob has the other. Alice and Bob can use these pairs in at least two ways. They can construct unbreakable codes, and they can teleport quantum states. Quantum cryptography began with a paper by Wiesner [6] in 1983; in 1991, Ekert [7] applied entanglement to quantum cryptography. Quantum teleportation appeared in a paper by Bennett et al. [8] in 1993. Let us consider each of these applications in turn.

The use of entanglement in quantum cryptography is straightforward. On each shared singlet pair, Alice and Bob measure polarization along identical axes. Since the pair is a singlet, Alice and Bob will always find the same (for photons) or opposite (for spin-1/2 particles) polarizations. Either way, Alice and Bob can thus construct identical sequences of binary data that only they know. Suppose that Alice wants to send a coded message to Bob. First, she translates her message into binary. For example, let the binary message be 10010111001101010010010, which contains 25 bits. Next, she and Bob generate a shared random binary sequence of the same length. Then Alice adds the two sequences, in binary, and transmits the sum to Bob using any (public or private) channel. Finally, Bob subtracts the random binary sequence from the sum and recovers Alice’s message. Only Bob can read Alice’s message, because only Bob knows what to subtract.

Teleportation, as everyone knows, is Captain Kirk’s way of getting around. He would turn into a column of glimmering light and disappear, only to reappear far away. To explain teleportation, we approximate Captain Kirk as a single spin-1/2 particle in an unknown state \( |K\rangle \):

\[
|K\rangle = a|\uparrow_K\rangle + b|\downarrow_K\rangle \quad .
\]  

(\( K \) stands for Kirk). Suppose Bob wants to transmit the state \( |K\rangle \). Alice and Bob join to this state a singlet pair, such that the overall state is

\[
\frac{1}{\sqrt{2}}(a|\uparrow_K\rangle + b|\downarrow_K\rangle)\left[|\uparrow_A\rangle|\downarrow_B\rangle - |\downarrow_A\rangle|\uparrow_B\rangle\right] \quad ,
\]  

where \( A \) and \( B \) stand for Alice and Bob, respectively. Next, Bob measures a nondegenerate operator with four eigenstates \( |\Psi(-)\rangle, |\Psi(+)\rangle, |\Phi(-)\rangle \) and \( |\Phi(+)\rangle \):

\[
|\Psi(+)\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_K\rangle|\downarrow_B\rangle + |\downarrow_K\rangle|\uparrow_B\rangle\right] \quad ,
\]

\[
|\Psi(-)\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_K\rangle|\downarrow_B\rangle - |\downarrow_K\rangle|\uparrow_B\rangle\right] \quad ,
\]

\[
|\Phi(+)\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_K\rangle|\uparrow_B\rangle + |\downarrow_K\rangle|\downarrow_B\rangle\right] \quad ,
\]

\[
|\Phi(-)\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_K\rangle|\uparrow_B\rangle - |\downarrow_K\rangle|\downarrow_B\rangle\right] \quad .
\]  

These states are called the Bell operator basis. Rewriting Eq. (2) in this basis, we obtain
filter and define local filtering according to a unitary operator $U$ state $| \downarrow \rangle$. So let Alice, say, run her spin through a selective filter that never absorbs the $\alpha$ with $| \uparrow \rangle$. Here $| \uparrow \rangle$ represents the state of the filter if it absorbs Alice’s spin, and $| \downarrow \rangle$ represents the initial state of the filter. Let $| 0 \rangle$ denote the initial state of the filter and define local filtering according to a unitary operator $U$ that sends

$$
| \uparrow_A \rangle | 0 \rangle \rightarrow x | \uparrow_A \rangle | 0 \rangle + y | \uparrow_A \rangle | 1 \rangle,
$$

$$
| \downarrow_A \rangle | 0 \rangle \rightarrow | \downarrow_A \rangle | 0 \rangle.
$$

Here $| 1 \rangle$ represents the state of the filter if it absorbs Alice’s spin, and $| x |^2 + | y |^2 = 1$. After Alice runs her spin through the filter, the combined state of the two spins and the filter is

$$
\left[ x \alpha | \uparrow_A \rangle | \uparrow_B \rangle + (1 - \alpha^2)^{1/2} | \downarrow_A \rangle | \downarrow_B \rangle \right] | 0 \rangle + y \alpha | \uparrow_A \rangle | \uparrow_B \rangle | 1 \rangle.
$$

Now Alice looks in the filter. The chance is $| \alpha y |^2$ that she finds her spin there. If she does not find her spin in the filter, however, she knows that the state of the two spins is given by the bracketed term in Eq. (7), up to normalization. In particular, if we choose $x = (1 - \alpha^2)^{1/2} / \alpha$, the state of the two spins will now be an ebit, suitable for coding and teleportation. So Alice has a chance $1 - | \alpha y |^2 = 2(1 - \alpha^2)$ of producing an exact ebit. Of course, she loses all the entanglement in $| \Psi_\alpha \rangle$ if the filter absorbs the $| \uparrow \rangle$ state, but with a little luck, Alice and Bob can turn pairs in the state $| \Psi_\alpha \rangle$ into a valuable resource. (It follows that pairs in the state $| \Psi_\alpha \rangle$ are a valuable resource, too.)

So we can extract singlets from other entangled pairs! Immediately, the question arises—just as it did for Carnot—whether local filtering is the most efficient method for extracting
The answer is that, in general, it is not. If Alice and Bob share many pairs in the state $|\Psi_{\alpha}\rangle$, they can do better than locally filter the pairs, one by one. But to do better, Alice and Bob must apply collective operations to their entangled pairs—they must operate on many pairs together, and not one by one. Suppose Alice and Bob share two pairs in the entangled state $|\Psi_{\alpha}\rangle$. The state of two pairs is

$$
\begin{align*}
\alpha|\uparrow_A\rangle|\uparrow_{A'}\rangle + (1 - \alpha^2)^{1/2} (|\downarrow_A\rangle|\downarrow_{A'}\rangle + |\downarrow_{A'}\rangle|\downarrow_A\rangle)
+ \sqrt{2}\alpha(1 - \alpha^2)^{1/2} |\downarrow_A\rangle|\downarrow_{A'}\rangle + \sqrt{2}\alpha(1 - \alpha^2)^{1/2} |\uparrow_A\rangle|\uparrow_{A'}\rangle
+ \sqrt{2}\alpha(1 - \alpha^2)^{1/2} |\downarrow_A\rangle|\downarrow_{A'}\rangle + \sqrt{2}\alpha(1 - \alpha^2)^{1/2} |\uparrow_A\rangle|\uparrow_{A'}\rangle
\end{align*}
$$

(8)

as the overall state. Now let Bob measure $\sigma_z^B + \sigma_z^{B'}$, or Alice measure $\sigma_z^A + \sigma_z^{A'}$. (The result is the same.) With probability $2\alpha^2(1 - \alpha^2)$ the result is zero, and the state of the spins after the measurement is the bracketed term in Eq. (8). Now the bracketed term is an ebit; we can define

$$
\begin{align*}
|\downarrow_A\rangle &\equiv |\uparrow_A\rangle|\downarrow_{A'}\rangle, \quad |\downarrow_{A'}\rangle &\equiv |\downarrow_A\rangle|\uparrow_{A'}\rangle \\
|\downarrow_B\rangle &\equiv - |\uparrow_B\rangle|\downarrow_{B'}\rangle, \quad |\uparrow_B\rangle &\equiv |\downarrow_B\rangle|\uparrow_{B'}\rangle
\end{align*}
$$

(10)

to write it explicitly as a singlet.

For two pairs, this method is not more efficient than local filtering. But it gets more efficient as Alice and Bob apply it collectively to more pairs at a time [9]. To apply the method to many pairs at a time, Alice and Bob first measure the $z$-component of total spin. Whatever result they get, their pairs are left in a superposition of biorthogonal states with coefficients of equal magnitude. For example, let Alice and Bob have three distinct pairs in the state $|\Psi_{\alpha}\rangle$. Suppose Alice measures the $z$-component of total spin to be $1/2$; then the state of their pairs is

$$
\begin{align*}
|\uparrow_A\rangle|\uparrow_B\rangle|\uparrow_{A'}\rangle|\uparrow_{B'}\rangle|\downarrow_{A''}\rangle|\downarrow_{B''}\rangle + |\downarrow_A\rangle|\downarrow_B\rangle|\uparrow_{A'}\rangle|\uparrow_{B'}\rangle|\downarrow_{A''}\rangle|\uparrow_{B''}\rangle
+ |\uparrow_A\rangle|\uparrow_B\rangle|\downarrow_{A'}\rangle|\downarrow_{B'}\rangle|\uparrow_{A''}\rangle|\uparrow_{B''}\rangle
\end{align*}
$$

(11)

up to normalization. Each time they measure the $z$-component of total spin on a set of pairs, they get a state of this form. The tensor product of such states is therefore also a state of this form. For example, two groups of three spins, each in the state of Eq. (11), yield a tensor product state having nine terms with equal coefficients. Alice and Bob can build such states, with various numbers of terms in the superposition, until they get a number of terms equal to, or nearly equal to, a power of 2. A superposition with number of terms equal to $2^n$ is unitarily equivalent to $n$ ebits. Thus Alice and Bob can apply local unitary operations to transform their superposition into ebits. For example, Alice and Bob could, by locally filtering a state having nine terms, reduce nine terms to eight terms. Eight terms are unitarily equivalent to three ebits. Bennett, Bernstein, Popescu and Schumacher (BBPS) [9] showed that Alice and Bob can obtain $n$ singlets from $k$ pairs of spins in the state $|\Psi_{\alpha}\rangle$, where the ratio $n/k$ approaches the limit.
\[
\lim_{n,k \to \infty} \frac{n}{k} = E(|\Psi_\alpha\rangle) \\
= -\alpha^2 \log_2 \alpha^2 - (1 - \alpha^2) \log_2 (1 - \alpha^2)
\]  

(12)

\(E(|\Psi_\alpha\rangle)\) is called the entropy of entanglement; it is the Shannon entropy of the squares of the coefficients of the Schmidt decomposition. It equals 1 if \(\alpha = 1/\sqrt{2}\) and equals 0 for a product state.

III. THERMODYNAMICAL ANALOGUES

Applications of entanglement naturally raise the question of efficiency. We have two methods to extract singlets from generic entanglement; we know they are not equally efficient. Are there more efficient methods? Is there a maximally efficient method? These questions are analogous to the questions that Carnot asked. We will now see also that the answers are analogous to his answers.

First, the answer to the thermodynamical question involves a principle—the principle that there is no way to build a perpetuum mobile, i.e. to build a machine that works for free, without changing its environment. This principle is the second law of thermodynamics. For entanglement, there is an analogous principle: local operations cannot increase the entanglement between remote systems. Measurements, local unitary operations, and additional unentangled systems cannot increase the entanglement between Alice’s and Bob’s systems; neither can classical communication (i.e. communication that does not involve entanglement) between Alice and Bob. We can accept this principle as an axiom, or we can prove it as a theorem of quantum mechanics. It is analogous to the second law also in that it is a statistical law. The method of local filtering, for example, can increase the entanglement between the systems held by Alice and Bob, but on average it decreases the entanglement.

Second, Carnot had the insight to focus on reversible transformations. Consider two reversible heat engines; suppose that both absorb heat \(Q_1\) at \(T_1\) and expel heat \(Q_2\) at \(T_2\), but one does work \(W\), and the other does work \(W' > W\), per cycle. The first engine, if run in reverse, is a refrigerator—absorbs heat \(Q_2\) at \(T_2\) and expels heat \(Q_1\) at \(T_1\)—and requires only work \(W\) per cycle. Thus the two engines together could provide \(W' - W\) in work per cycle without changing their environment. Such a conclusion contradicts the second law of thermodynamics, so both engines must do the same work: \(W = W'\).

Are the collective operations of BBPS reversible? Alice and Bob can turn shared singlets into shared pairs in the state \(|\Psi_\alpha\rangle\). Alice, say, can prepare pairs in the state \(|\Psi_\alpha\rangle\) in her laboratory, and then teleport one spin out of each entangled pair to Bob. However, Alice then uses up one singlet pair for every spin that she teleports to Bob, so Alice and Bob use up \(k\) shared singlets to produce \(k\) shared pairs in the state \(|\Psi_\alpha\rangle\), whereas from \(k\) pairs in the state \(|\Psi_\alpha\rangle\) they can recover only \(n < k\) singlet pairs. So this is not an efficient way to produce pairs in the state \(|\Psi_\alpha\rangle\). But Alice can teleport the pairs more efficiently using a method called quantum data compression. The idea behind quantum data compression is as follows. Alice has to teleport \(k\) spins, i.e. a state in a \(2^k\)-dimensional Hilbert space. But the effective dimension of the Hilbert space is much smaller than \(2^k\), because the \(k\) spins have a common bias. In the state \(|\Psi_\alpha\rangle\) with \(\alpha > 1\), \(|\uparrow\rangle\) is more likely than \(|\downarrow\rangle\), so a
sequence with every spin in the state $|\uparrow\rangle$ is much more likely than a sequence with every spin in the state $|\downarrow\rangle$; still more likely are sequences with most, but not all, spins in the state $|\uparrow\rangle$. In fact, the effective dimension of the Hilbert space approaches $2^n$, rather than $2^k$; that is, Alice can actually teleport the $k$ spins to Bob without using more than the $n$ singlets that she and Bob can obtain from $k$ pairs in the state $|\Psi_\alpha\rangle$. Hence the extraction of singlets from pairs in the state $|\Psi_\alpha\rangle$ is reversible. It is not reversible for finite $k$ and $n$, in general, but it is reversible as the number of systems approaches infinity, just as heat engines are reversible only in the thermodynamical limit.

Now suppose that Alice and Bob share $k$ pairs of systems in an entangled state $|\Psi_\alpha\rangle$, which they transform into $n$ singlets, using the method of BBPS. Did they use the most efficient method possible? That is, could Alice and Bob apply a more efficient method, using only local operations and classical communication, to obtain a greater number $n' > n$ of singlets from the same number $k$ of initial pairs? The answer is that they cannot. For if it were possible to transform $k$ of the initial pairs into $n'$ singlets by a different method, Alice and Bob could then reverse the BBPS operations on $n$ of the singlets and transform them into $k$ pairs in the entangled state $|\Psi_\alpha\rangle$. They could then obtain $n' - n$ entangled pairs only using local operations and classical communication, contradicting the general principle that local operations cannot increase entanglement. Hence $n' = n$; the BBPS method is maximally efficient (in the limit $k, n \to \infty$).

As a byproduct, this proof gives us a measure of the entanglement of the state $|\Psi_\alpha\rangle$. The $k$ systems in the state $|\Psi_\alpha\rangle$ have the same entanglement as $n$ singlet pairs. Thus the measure of entanglement for $k$ pairs in the state $|\Psi_\alpha\rangle$ must equal the measure of entanglement for $n$ singlets. At first, it might seem that we could assign an arbitrary measure of entanglement, such as $n$, $n^2$ and $e^n$, to $n$ singlets. But actually, the measure must be proportional to $n$, because the BBPS collective operations are reversible only when the number of systems becomes arbitrarily large. (The ratio $n/k$ nearly always tends to an irrational number, and if the number is irrational, we can never reversibly extract $n$ singlets from a finite number $k$ of systems.) Reversibility requires us to go to the limit of infinite $n$, and for infinite $n$ there is no way to define total entanglement. We can only define entanglement per system. Here too, we find a thermodynamical analogue: the thermodynamic limit requires us to define intensive quantities. Likewise, the measure of entanglement must be intensive, i.e. the measure of entanglement of $n$ singlets must be proportional to $n$. It follows that the measure of entanglement for pure states is unique (up to a constant factor). Since the measure of entanglement of $k$ systems $|\Psi_\alpha\rangle$ approaches the measure of entanglement of $n$ pairs in a singlet state, and since the measure is intensive, we have $kE(|\Psi_\alpha\rangle) = n$, where $E$ now denotes the measure, and the measure of entanglement of a singlet state is 1. Thus

$$E(|\Psi_\alpha\rangle) = \lim_{n,k \to \infty} \frac{n}{k}.$$ (13)

This limit indeed equals the entropy of entanglement of $|\Psi_\alpha\rangle$, Eq. (12); so the measure of entanglement of $|\Psi_\alpha\rangle$ must equal its entropy of entanglement, up to a conventional proportionality constant—measuring the entanglement of a singlet pair or ebit—that we set it to 1.
IV. BEYOND ANALOGUES

Thermodynamical analogues are powerful tools for quantum information theory. However, so far they remain mere analogues. Feynman [13] constructed an amusing analogue of the Carnot cycle to prove that gravitational potential energy near the surface of the earth is the product of weight and height. But his proof does not make gravity a branch of thermodynamics! So thermodynamical analogues in quantum information theory are not necessarily more than analogues. In particular, the previous section did not mention temperature, heat or heat baths in the context of entanglement. We did not need to ask how much work, if any, the BBPS method entails. So far, we have no reason to consider quantum information theory a branch of thermodynamics. I claim, however, that it is.

In a Carnot cycle, the entropy of the heat engine changes twice per cycle. The heat engine absorbs entropy at the high temperature and releases entropy at the low temperature. At first glance, the flow of entropy in the Carnot cycle seems not to fit the thermodynamical analogues: entropy changes in the Carnot cycle, while BBPS collective operations conserve the entropy of entanglement. A closer look, however, reveals yet another analogue. The heat engine indeed absorbs and releases entropy, but so does the environment, such that the total entropy is unchanged. Otherwise, the heat engine would not be reversible. Likewise, the BBPS operations would not be reversible if they did not conserve entropy of entanglement. We can therefore guess that the systems shared by Alice and Bob are the analogue, not of the heat engine, but of the heat engine plus environment.

How does this analogy work? Initially, say, Alice and Bob share \( k \) pairs of spins in an entangled state \( |\Psi_\alpha\rangle \). From these \( k \) pairs they extract, by the BBPS method, \( n \) ebits, with \( n < k \). Aside from the \( n \) ebits, then, there remain \( k - n \) spins; by conservation of entropy of entanglement, these \( k - n \) spins contain zero entanglement. Thus, the extraction of ebits suggests a process in which an ensemble of spins at thermodynamical equilibrium divides into two ensembles, with differing average entropy per spin. If we now visit, say, Alice’s laboratory, and forget about Bob, we cannot help but describe the process as heating and cooling of two subensembles: the \( k - n \) spins are in a pure state, hence they are “cold”, while the \( n \) ebits are in state of maximum entropy, hence they are “hot”. The inverse of the BBPS method, in which \( n \) ebits are combined with \( k - n \) pairs of spins (shared by Alice and Bob) in product states, to yield \( k \) equally entangled pairs, is a process in which two ensembles at different temperatures are reversibly brought to the same temperature.

We might conclude, then, that the BBPS method consumes work—precisely the amount of work required to separate the ensemble at thermodynamical equilibrium reversibly into “hot” and “cold” ensembles—and that the inverse of the BBPS method produces work—precisely the amount of work produced when the “hot” and “cold” ensembles come reversibly into thermodynamical equilibrium. However, this conclusion is premature. The reason is that our use of the word “reversibility” in the context of entanglement does not quite match its usage in thermodynamics. A local increase of entropy in Alice’s laboratory, or in Bob’s, may conserve the entanglement of the pairs they share, but it is not thermodynamically reversible. We must check whether the collective operations of the BBPS method have local thermodynamical effects that we have not taken into account [14]. We have not done so here, because we have treated collective operations abstractly, without considering their physical implementation. But we can already conclude that entropy of entanglement is more than
an analogue of thermodynamical entropy; locally, it is thermodynamical entropy.

It may seem paradoxical that Alice’s spins can have entropy and temperature (if we forget Bob) yet they belong to a pure state (if we remember Bob). But such is the nature of entanglement. Consider a closed system comprising a measuring device, a measured system, and an environment, in an initial pure state. After the measurement, which entangles the measuring device and the measured system, decoherence sets in. The process of decoherence does not change the fact that the closed system is in a pure state; it merely makes the fact irrelevant, for all practical purposes, because no one can keep track of it. Furthermore, even though the system as a whole is in a pure state, its entangled subsystems are not. The objective—not subjective—entropy of the subsystems derives from entanglement [15]. In just this sense, Alice’s and Bob’s spins have objective entropy.
REFERENCES

[1] E. Schrödinger, Proc. Camb. Phil. Soc. 31, 555 (1935).
[2] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47, 777 (1935).
[3] J. S. Bell, Physics 1, 195 (1964).
[4] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[5] For a review, see A. Shimony, in The Dilemma of Einstein, Podolsky and Rosen – 60 Years Later (Annals of the Israel Physical Society, 12), A. Mann and M. Revzen, eds., Institute of Physics Publishing (1996).
[6] S. Weisner, Sigact News 15, 78 (1983).
[7] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[8] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[9] C. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A53, 2046 (1996).
[10] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin and W. K. Wootters, Phys. Rev. Lett. 76, 722 (1996).
[11] S. Popescu and D. Rohrlich, Phys. Rev. A56, Rapid Comm. R3319 (1997); S. Popescu and D. Rohrlich, The joy of entanglement, in Introduction to Quantum Computation and Information; eds. H.-K. Lo, S. Popescu, and T. P. Spiller (Singapore: World Scientific), 1998, pp. 29-48.
[12] R. Jozsa and B. Schumacher, J. Mod. Optics 41, 2343 (1994); B. Schumacher, Phys. Rev. A51, 2738 (1995).
[13] R. P. Feynman, R. B. Leighton and M. Sands, The Feynman Lectures on Physics, I (New York: Addison-Wesley Pub. Co.), 1963, Sect. 4-2.
[14] D. Rohrlich, in preparation.
[15] Y. Aharonov, personal communication.