ON THE ASYMPTOTIC BEHAVIOUR OF TRAVELING WAVE SOLUTION FOR A DISCRETE DIFFUSIVE EPIDEMIC MODEL

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Abstract. A recent paper [Y.-Y. Chen, J.-S. Guo, F. Hamel, Traveling waves for a lattice dynamical system arising in a diffusive endemic model, Nonlinearity, 30 (2017), 2334-2359] presented a discrete diffusive Kermack-McKendrick epidemic model, and the authors proved the existence of traveling wave solutions connecting the disease-free equilibrium to the endemic equilibrium. However, the boundary asymptotic behavior of the traveling waves converge to the endemic equilibrium at $+\infty$ is still an open problem. In this paper, we investigate the above open problem and completely solve it by constructing suitable Lyapunov functional and employing Lebesgue dominated convergence theorem.

1. Introduction. Lattice dynamical systems are infinite systems of ordinary differential equations with a discrete spatial structure. Such systems arise from practical backgrounds, such as population biology [6, 11, 12], chemical reaction [5, 7] and material science [1, 2]. Recently, many researchers have paid attention to the traveling wave solutions of lattice dynamical systems. In [3], Chen et al. studied the traveling wave solutions for the following lattice dynamical system arising in a diffusive epidemic model:

\[
\begin{align*}
\frac{dS_n(t)}{dt} &= [S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] + \mu - \beta S_n(t)I_n(t) - \mu S_n(t), \\
\frac{dI_n(t)}{dt} &= d[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \beta S_n(t)I_n(t) - (\gamma + \mu)I_n(t),
\end{align*}
\]

for $n \in \mathbb{Z}$, where $S_n(t)$ and $I_n(t)$ denote the populations densities of susceptible and infectious individuals at niche $n$ and time $t$, respectively; $1$ and $d$ denote the random migration coefficients for susceptible and infectious individuals, respectively; $\mu$ is the input rate of the susceptible population, meanwhile, the death rates of susceptible and infectious individuals are also assumed to be $\mu$; $\beta$ is the transmission coefficient between susceptible and infectious individuals; $\gamma$ is the recovery rate of infectious individuals.

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Obviously, system (1) always has a disease-free equilibrium $E_0 = (S_0, 0) = (1, 0)$. Denote $\mathcal{R}_0 = \frac{\beta}{\mu + \gamma}$ as the basic reproduction number, which is the expected number of new infective individuals produced by a single infective individual in the susceptible individuals. When $\mathcal{R}_0 > 1$, system (1) also has a positive equilibrium which called endemic equilibrium as following:

$$E^* = (S^*, I^*) = \left( \frac{1}{\mathcal{R}_0}, \frac{\mu}{\beta}(\mathcal{R}_0 - 1) \right).$$

For the classical SIR model without migration described by ordinary differential equations (see [8]), it is well known that the global dynamics is completely determined by the basic reproduction number $\mathcal{R}_0$: that is, if the number is less than unity, then the disease-free equilibrium $E_0$ is globally asymptotically stable, while if the number is greater than unity, then a positive endemic equilibrium $E^*$ exists and it is globally asymptotically stable.

In [3], the authors showed that the existence of traveling wave solutions of system (1) connecting the disease-free equilibrium to endemic equilibrium, but the traveling wave solutions they obtained is so called weak traveling wave solutions (the definition will be given in the next section). In addition, the strong traveling wave solutions is obtained by assuming some priori conditions. In this paper, we will be devoted to study the strong traveling wave solutions of system (1) without that priori condition.

The rest of paper is organized as follows. In Section 2, we present preliminaries. In Section 3, we first construct suitable Lyapunov functional to deal with the lattice systems and employing Lebesgue dominated convergence theorem to show that the boundary asymptotic behaviour of traveling wave solutions at $+\infty$.

2. Preliminaries. In this section, we start with the some definitions of traveling wave solution.

**Definition 2.1.** A pair $(\phi, \psi)$ is said to be a traveling wave solution of system (1) if $(\phi, \psi)$ is a nontrivial and bounded solution of system (1) having the form $S_n(t) = \phi(n + ct)$ and $I_n(t) = \psi(n + ct)$. The function $(\phi, \psi)$ is called the wave profile and the number $c$ is called the wave speed.

Furthermore, we give the definition of weak and strong traveling wave solutions of system (1).

**Definition 2.2.** [15] A traveling wave solution is weak or persistent if there exist two positive constants $M_1$ and $M_2$ such that

$$\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (1, 0)$$

and

$$M_1 < \liminf_{\xi \to +\infty} \phi(\xi), \limsup_{\xi \to +\infty} \phi(\xi) < M_2, M_1 < \liminf_{\xi \to +\infty} \psi(\xi), \limsup_{\xi \to +\infty} \psi(\xi) < M_2.$$

A traveling wave solution is strong if it satisfies

$$\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (1, 0)$$

and

$$\lim_{\xi \to +\infty} (\phi(\xi), \psi(\xi)) = (S^*, I^*).$$
Letting $\xi = n + ct$, system (1) can be rewritten as the following wave form equations:

\[
\begin{align*}
    c\phi'(\xi) &= D[\phi](\xi) + \mu(1 - \phi(\xi)) - \beta \phi(\xi)\psi(\xi), \\
    c\psi'(\xi) &= dD[\psi](\xi) - (\mu + \gamma)\psi(\xi) + \beta \phi(\xi)\psi(\xi),
\end{align*}
\]

(2)
for all $\xi \in \mathbb{R}$, where

\[D[f](\xi) := f(\xi + 1) + f(\xi - 1) - 2f(\xi).\]

Consider the following linearized system of (2) at disease-free equilibrium $(1, 0)$:

\[
dD[\psi](\xi) - c\phi'(\xi) - (\mu + \gamma)\psi(\xi) + \beta \phi(\xi)\psi(\xi) = 0.
\]

(3)
Let $\psi(\xi) = e^{\lambda \xi}$ and substituting it into (3), we have

\[
\Delta(\lambda, c) := \beta^2 e^{\lambda} + e^{-\lambda} - 2 - c\lambda + \beta - \mu - \gamma = 0.
\]

(4)

By some calculations, we have

\[
\begin{align*}
    \Delta(0, c) &= \beta - (\mu + \gamma), \quad \lim_{c \to +\infty} \Delta(\lambda, c) = -\infty, \quad \lim_{\lambda \to +\infty} \Delta(\lambda, c) = +\infty, \\
    \frac{\partial \Delta(\lambda, c)}{\partial \lambda} &= d[e^\lambda - e^{-\lambda}] - c, \quad \frac{\partial \Delta(\lambda, c)}{\partial c} = -\lambda < 0, \\
    \frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2} &= d[e^\lambda + e^{-\lambda}] > 0, \quad \frac{\partial \Delta(\lambda, c)}{\partial \lambda}\bigg|_{(0, c)} = -c < 0,
\end{align*}
\]

for $\lambda > 0$ and $c > 0$. Therefore, we have the following lemma.

**Lemma 2.3.** Let $\mathbb{R}_0 > 1$. Then there exists $c^* > 0$ and $\lambda^* > 0$ such that

\[
\left. \frac{\partial \Delta(\lambda, c)}{\partial \lambda} \right|_{(\lambda^*, c^*)} = 0 \quad \text{and} \quad \Delta(\lambda^*, c^*) = 0.
\]

Furthermore,

(i): if $c = c^*$, then $\Delta(\lambda, c) = 0$ has only one positive real root $\lambda^*$;
(ii): if $0 < c < c^*$, then $\Delta(\lambda, c) > 0$ for all $\lambda \in (0, +\infty)$;
(iii): if $c > c^*$, then $\Delta(\lambda, c) = 0$ has two positive real roots $\lambda_1$, $\lambda_2$ with $\lambda_1 < \lambda^* < \lambda_2$.

One of the main results in [3] is the following theorem.

**Theorem 2.4.** [3] If $\mathbb{R}_0 > 1$, then for any $c \geq c^*$, there exists a bounded classical solution $(\phi(\xi), \psi(\xi))$ of system (2) such that

\[
0 < \phi(\xi) < 1 \quad \text{and} \quad \psi(\xi) > 0 \quad \text{in} \quad \mathbb{R}
\]

and

\[
\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (1, 0),
\]

(6)

together with

\[
0 < \liminf_{\xi \to +\infty} \phi(\xi) \leq S^* \leq \limsup_{\xi \to +\infty} \phi(\xi) < 1,
\]

(7)
and

\[
0 < \liminf_{\xi \to +\infty} \psi(\xi) \leq I^* \leq \limsup_{\xi \to +\infty} \psi(\xi) < +\infty.
\]

(8)

In fact, it is not sure whether the traveling wave solutions converge to the endemic equilibrium as $\xi \to +\infty$ in Theorem 2.4. Furthermore, if we assume that $\phi(+\infty)$ or $\psi(+\infty)$ exists, the authors in [3] showed that following lemma on the convergence of the traveling wave solutions as $\xi \to +\infty$. 
Lemma 2.5. [3] Let \((\phi, \psi)\) be a bounded classical solution of (2) satisfying (5)-(7), with speed \(c \geq c^*\). If \(\phi(+\infty)\) or \(\psi(+\infty)\) exists, then they both exist and
\[
\lim_{\xi \to +\infty} (\phi(\xi), \psi(\xi)) = (S^*, I^*).
\]

In Lemma 2.5, there is a priori condition assuming \(\phi(+\infty)\) or \(\psi(+\infty)\) exists. Without this condition, the boundary asymptotic behaviour of traveling wave solutions at \(+\infty\) is still an open problem. As explained in [3], the difficulties come from the fact that model (1) is a system and is non-local. In [4], Ducrot and Magal studied a diffusive epidemic model with age-structure and constant recruitment. They proved an existence and nonexistence result for traveling wave solutions, and constructed a suitable Lyapunov functional to discuss their convergence towards equilibria at \(\pm \infty\). This gives us a new sight to investigate the boundary asymptotic behaviour of traveling wave solutions (see also [10, 14] for the nonlocal dispersal epidemic model; [9, 13] for the reaction-diffusion epidemic model with time delay).

3. Main results. In this section, we show the convergence of the traveling wave solutions as \(\xi \to +\infty\). The key point is to construct a suitable Lyapunov function. Firstly, we give the following lemmas on the boundedness of traveling wave solution \((\phi(\xi), \psi(\xi))\) which proved in [3].

Lemma 3.1. [3, Lemma 3.6] The function \(\psi(\xi)\) is bounded in \(\mathbb{R}\).

From Lemma 3.1, without losing generality, we assume there exists a constant \(H > 0\) such that \(\psi(\xi) \leq H < +\infty\). Furthermore, it is easy to verify that \(\phi(\xi) = \frac{\mu}{\mu + \beta H}\) is a lower solution of system (2), that is \(\phi(\xi) \geq \frac{\mu}{\mu + \beta H}\) for all \(\xi \in \mathbb{R}\). The next lemma is to show that \(\psi\) can not approach zero when \(\xi \to +\infty\).

Lemma 3.2. [3, Lemma 3.8] Let \(0 < c_1 \leq c_2\) be given and \((\phi(\xi), \psi(\xi))\) be a solution of system (2) with speed \(c \in [c_1, c_2]\) satisfying \(0 < \phi(\xi) < 1\) and \(\psi(\xi) > 0\) in \(\mathbb{R}\). Then there exists some small enough constant \(\varepsilon_0 > 0\), such that \(\psi'(\xi) > 0\) provided that \(\psi(\xi) \leq \varepsilon_0\) for all \(\xi \in \mathbb{R}\).

Now we prove the convergence of traveling wave solution as \(\xi \to +\infty\), define
\[
\mathcal{D} := \left\{ (\phi(\cdot), \psi(\cdot)) \in (C^1(\mathbb{R}, (0, +\infty))) \times C^1(\mathbb{R}, (0, +\infty))) \mid \frac{\mu}{\mu + \beta H} < \phi(\xi) < 1, \ 0 < \psi(\xi) < H \ \text{for all} \ \xi \in \mathbb{R} \right\}.
\]

Let
\[
g(x) = x - 1 - \ln x
\]
for all \(x \in \mathbb{R}\). For each \((\phi, \psi) \in \mathcal{D}\), define the following Lyapunov functional:
\[
L(\phi, \psi)(\xi) = V_1(\phi, \psi)(\xi) + S^* V_2(\phi, \psi)(\xi) + d I^* V_3(\phi, \psi)(\xi),
\]
where
\[
V_1(\phi, \psi)(\xi) = c S^* g \left( \frac{\phi(\xi)}{S^*} \right) + c I^* g \left( \frac{\psi(\xi)}{I^*} \right),
\]
\[
V_2(\phi, \psi)(\xi) = \int_0^1 g \left( \frac{\phi(\xi - \theta)}{S^*} \right) \ d\theta - \int_{-1}^0 g \left( \frac{\phi(\xi - \theta)}{S^*} \right) \ d\theta,
\]
and
\[
V_3(\phi, \psi)(\xi) = \int_0^1 g \left( \frac{\psi(\xi - \theta)}{I^*} \right) \ d\theta - \int_{-1}^0 g \left( \frac{\psi(\xi - \theta)}{I^*} \right) \ d\theta.
\]
Then we have the following result.
Theorem 3.3. Let the solution $(\phi, \psi)$ be a positive solution of system (2) in the set $D$. Then there exists a constant $m > 0$ such that

$$-m \leq L(\phi, \psi)(\xi) \leq +\infty, \quad \forall \xi \in \mathbb{R}.$$ 

and the map $\xi \mapsto L(\phi, \psi)(\xi)$ is non-increasing in $\mathbb{R}$. Furthermore, if the map $\xi \mapsto L(\phi, \psi)(\xi)$ is a constant, then $\phi = S^*$ and $\psi = I^*$.

Proof. It is easy to check that $g(x) \geq 0$ for all $x \in \mathbb{R}$ since $g(x)$ has the global minimum value 0 only at $x = 1$. Moreover, $g(x)$ is strictly monotone decreasing for $x \in (0, 1)$ and is strictly monotone increasing for $x \in (1, +\infty)$, see Figure 1.

Thanks to the boundedness of traveling wave solutions, we have the Lyapunov functionals $V_1(\phi, \psi)(\xi)$ and $V_2(\phi, \psi)(\xi)$ are well defined and bounded from below. Recall that Lemma 3.2 means that $\lim_{\xi \to +\infty} \psi(\xi) \geq \varepsilon_0$ for some constant $\varepsilon_0 \in (0, I^*)$. But since $\lim_{\xi \to -\infty} \psi(\xi) = 0$, so we need to consider the process of approaching negative infinity for $V_3(\phi, \psi)(\xi)$. For the $\varepsilon_0$ in Lemma 3.2, we define $\xi^* = \min\{\xi \in \mathbb{R} | \psi(\xi) = \varepsilon_0\}$. By using Lemma 3.2, then $\psi(\xi)$ is increasing in $(-\infty, \xi^*)$. Obviously,

$$\psi(\tilde{\xi} - \theta_1) \leq \psi(\tilde{\xi} - \theta_2) \leq \psi(\xi^*) = \varepsilon_0,$$

for any $\tilde{\xi} \in (-\infty, \xi^* - 1]$, where $\theta_1 \in (0, 1)$ and $\theta_2 \in (-1, 0)$. Hence, by the properties of function $g$, we have $V_3(\phi, \psi)(\xi) \geq 0$ for $\xi \in (-\infty, \xi^*)$. Thus the Lyapunov functional $L(\phi, \psi)(\xi)$ is well defined and bounded from below, that is, there exists a constant $m > 0$ such that

$$-m \leq L(\phi, \psi)(\xi) \leq +\infty, \quad \forall \xi \in \mathbb{R}.$$ 

Next we show that the map $\xi \mapsto L(\phi, \psi)(\xi)$ is non-increasing. By some calculation, the derivative of $V_1(\phi, \psi)(\xi)$ along system (2) is as follows:

$$\frac{dV_1(\phi, \psi)(\xi)}{d\xi} = \left(1 - \frac{S^*}{\phi(\xi)}\right) \frac{d\phi(\xi)}{d\xi} + \left(1 - \frac{I^*}{\psi(\xi)}\right) \frac{d\psi(\xi)}{d\xi}$$

$$= \left(1 - \frac{S^*}{\phi(\xi)}\right) D[\phi](\xi) + \left(1 - \frac{I^*}{\psi(\xi)}\right) dD[\psi](\xi) + \Theta(\xi),$$
where
\[
\Theta(\xi) = \left(1 - \frac{S^*}{\phi(\xi)}\right)(\mu(1 - \phi(\xi)) - \beta\phi(\xi)\psi(\xi)) \\
+ \left(1 - \frac{I^*}{\psi(\xi)}\right)(\beta\phi(\xi)\psi(\xi) - (\mu + \gamma)\psi(\xi)).
\] (9)

Note that the endemic equilibrium \((S^*, I^*)\) of system (1) satisfying
\[
\begin{cases}
\mu = \beta S^* I^* + \mu S^*, \\
\beta S^* = \mu + \gamma.
\end{cases}
\] (10)

Substituting (10) into (9), we obtain
\[
\Theta(\xi) = \left(1 - \frac{S^*}{\phi(\xi)}\right)(\mu S^* - \mu \phi(\xi)) + \beta S^* I^* - \beta S^* I^* \frac{S^*}{\phi(\xi)} - \beta \phi(\xi) I^* + (\mu + \gamma) I^* \\
= (\mu S^* + \beta S^* I^*) \left(2 - \frac{S^*}{\phi(\xi)} - \frac{\phi(\xi)}{S^*}\right).
\]

For \(V_2(\phi, \psi)(\xi)\), one has that
\[
\frac{dV_2(\phi, \psi)(\xi)}{d\xi} = \frac{d}{d\xi} \left[ \int_0^1 g\left(\frac{\phi(\xi - \theta)}{S^*}\right) d\theta - \int_{-1}^0 g\left(\frac{\phi(\xi - \theta)}{S^*}\right) d\theta \right] \\
= \int_0^1 \frac{d}{d\xi} g\left(\frac{\phi(\xi - \theta)}{S^*}\right) d\theta - \int_{-1}^0 \frac{d}{d\xi} g\left(\frac{\phi(\xi - \theta)}{S^*}\right) d\theta \\
= - \int_0^1 \frac{d}{d\theta} g\left(\frac{\phi(\xi - \theta)}{S^*}\right) d\theta + \int_{-1}^0 \frac{d}{d\theta} g\left(\frac{\phi(\xi - \theta)}{S^*}\right) d\theta \\
= 2g\left(\frac{\phi(\xi)}{S^*}\right) - g\left(\frac{\phi(\xi - 1)}{S^*}\right) - g\left(\frac{\phi(\xi + 1)}{S^*}\right).
\]

Similarly,
\[
\frac{dV_3(\phi, \psi)(\xi)}{d\xi} = 2g\left(\frac{\psi(\xi)}{I^*}\right) - g\left(\frac{\psi(\xi - 1)}{I^*}\right) - g\left(\frac{\psi(\xi + 1)}{I^*}\right).
\]

By some calculations, it can be shown that
\[
\left(1 - \frac{S^*}{\phi(\xi)}\right) D[\phi](\xi) + S^* \frac{dV_2(\phi, \psi)(\xi)}{d\xi} \\
= S^* \left(2 - \frac{\phi(\xi - 1)}{\phi(\xi)} - \frac{\phi(\xi + 1)}{\phi(\xi)} - 2 \ln \frac{\phi(\xi - 1)}{S^*} + \ln \frac{\phi(\xi - 1)}{S^*} + \ln \frac{\phi(\xi + 1)}{S^*}\right) \\
= - S^* \left[g\left(\frac{\phi(\xi - 1)}{\phi(\xi)}\right) + g\left(\frac{\phi(\xi + 1)}{\phi(\xi)}\right)\right]
\]
and
\[
\left(1 - \frac{I^*}{\psi(\xi)}\right) D[\psi](\xi) + I^* \frac{dV_3(\phi, \psi)(\xi)}{d\xi} = - I^* \left[g\left(\frac{\psi(\xi - 1)}{\psi(\xi)}\right) + g\left(\frac{\psi(\xi + 1)}{\psi(\xi)}\right)\right].
\]
Thus
\[
\frac{dL(\phi, \psi)(\xi)}{d\xi} = \left(\mu S^* + \beta S^* I^*\right) \left(2 - \frac{S^*}{\phi(\xi)} - \frac{\phi(\xi)}{S^*}\right) - S^* \left[g \left(\frac{\phi(\xi - 1)}{\phi(\xi)}\right) + g \left(\frac{\phi(\xi + 1)}{\phi(\xi)}\right)\right] - dI^* \left[g \left(\frac{\psi(\xi - 1)}{\psi(\xi)}\right) + g \left(\frac{\psi(\xi + 1)}{\psi(\xi)}\right)\right].
\] (11)

Since the arithmetic mean of non-negative real numbers is greater than or equal to the geometric mean of the same list, then we have
\[
2 - \frac{S^*}{\phi(\xi)} - \frac{\phi(\xi)}{S^*} \leq 0.
\]

Recall that \(g(x) \geq 0\) for all \(x \in \mathbb{R}\). Hence, \(\frac{dL(\phi, \psi)(\xi)}{d\xi} \leq 0\) for all \(\xi \in \mathbb{R}\), that is, the map \(\xi \mapsto L(\phi, \psi)(\xi)\) is non-increasing in \(\mathbb{R}\). When \(L(\phi, \psi)(\xi)\) is a constant, we have \(\frac{dL(\phi, \psi)(\xi)}{d\xi} = 0\) for all \(\xi \in \mathbb{R}\). Since each term of (11) is less than or equal to 0, we have
\[
\phi(\xi) \equiv S^* \quad \text{and} \quad \psi(\xi) = \psi(\xi \pm 1) \quad \text{for all} \quad \xi \in \mathbb{R}.
\]

Recall that \((S^*, I^*)\) is a positive equilibrium of system (2), that means, if \(\phi(\xi) = S^*\), we must have that \(\psi(\xi) = I^*\). Insert \(\phi(\xi) \equiv S^*\) back into system (2) give us \(\psi(\xi) \equiv I^*\). The proof is completed. \(\square\)

**Theorem 3.4.** If \(R_0 > 1\), then for each \(c > c^*\), system (1) has a traveling wave solution \((\phi(\xi), \psi(\xi))\) satisfying the asymptotic boundary condition
\[
\lim_{\xi \to \pm \infty} (\phi(\xi), \psi(\xi)) = (S^*, I^*).
\]

**Proof.** Consider an arbitrary increasing sequence \(\{\xi_k\}_{k \geq 0}\) with \(\xi_k > 0\) such that \(\xi_k \to +\infty\) when \(k \to +\infty\) and denote
\[
\{\phi_k(\xi) = \phi(\xi + \xi_k)\}_{k \geq 0} \quad \text{and} \quad \{\psi_k(\xi) = \psi(\xi + \xi_k)\}_{k \geq 0}.
\]

Since both functions \(\phi\) and \(\psi\) are bounded, the system (2) give us that the functions \(\phi\) and \(\psi\) have bounded derivatives. Then by Arzela-Ascoli theorem, the functions \(\phi_k(\xi)\) and \(\psi_k(\xi)\) converge in \(C_{\text{loc}}^\infty(\mathbb{R})\) as \(k \to +\infty\), up to extraction of a subsequence, one may assume that the sequences \(\phi_k(\xi)\) and \(\psi_k(\xi)\) convergence towards some non-negative \(C^\infty\) functions \(\phi_\infty\) and \(\psi_\infty\). Furthermore, since \(L(\phi, \psi)(\xi)\) is non-increasing on \(\xi\) and bounded from below, then there exists a constant \(C_0\) and large \(k\) such that
\[
C_0 \leq L(\phi_k, \psi_k)(\xi) = L(\phi, \psi)(\xi + \xi_k) \leq L(\phi, \psi)(\xi).
\]

Therefore there exists some \(\delta \in \mathbb{R}\) such that \(\lim_{k \to \infty} L(\phi_k, \psi_k)(\xi) = \delta\) for any \(\xi \in \mathbb{R}\). By Lebesgue dominated convergence theorem, gives us
\[
\lim_{k \to +\infty} L(\phi_k, \psi_k)(\xi) = L(\phi_\infty, \psi_\infty)(\xi), \quad \xi \in \mathbb{R}.
\]

Thus
\[
L(\phi_\infty, \psi_\infty)(\xi) = \delta.
\]

Note that \(\frac{dL(\phi, \psi)(\xi)}{d\xi} = 0\) if and only if \(\phi(\xi) \equiv S^*\) and \(\psi(\xi) \equiv I^*\), it follows that
\[
(\phi_\infty, \psi_\infty) \equiv (S^*, I^*).
\]

By the arbitrariness of the sequence \(\{\xi_k\}\), we obtain
\[
\lim_{\xi \to +\infty} \phi(\xi) = S^* \quad \text{and} \quad \lim_{\xi \to +\infty} \psi(\xi) = I^*.
\]
This completes the proof.

Since the Lyapunov function $L(\phi, \psi)(\xi)$ is independent of $c$, we can directly obtain the following theorem for the case $c = c^*$ by [3, Remark 3.10] and Theorem 3.4.

**Theorem 3.5.** If $R_0 > 1$, then for each $c = c^*$, system (1) has a traveling wave solution $(\phi(\xi), \psi(\xi))$ satisfying the asymptotic boundary condition

$$\lim_{\xi \to +\infty} (\phi(\xi), \psi(\xi)) = (S^*, I^*).$$

Hence, we established the convergence of traveling wave solutions at $+\infty$ for $c > c^*$, which extends the result obtained by Chen et al. [3].

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