Solving a percolation inverse problem with a divide-and-concur algorithm

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We present a percolation inverse problem for diode networks: Given information about which pairs of nodes allow current to percolate from one to the other, can one construct a diode network consistent with the observed currents? We implement a divide-and-concur iterative projection method for solving the problem and demonstrate the supremacy of our method over an exhaustive approach for nontrivial instances of the problem. We find that the problem is most difficult when some but not all of the percolation data are hidden, and that the most difficult networks to reconstruct generally are those for which the currents are most sensitive to the addition or removal of a single diode.

I. INTRODUCTION

Percolation problems have long been a topic of interest in a variety of disciplines, including the chemistry of polymers [1, 2], the ecology of habitat fragmentation [3], and the dynamics of commuter traffic networks [4]. The archetypal percolation problem is the flow of liquid through a porous medium. One can model the medium as a network in which the interstices are nodes and two interstices share an edge if they are near one another. Each edge can be “open” or “closed” depending on whether liquid can flow from one node to the other. The task is to ascertain whether the liquid has a path from one side of the porous medium to the other [5].

One can argue that for an infinite medium, if the edges are opened randomly with probability $p$ then there is a critical $p$ below which there will never be a path and above which there will always be a path. A significant amount of research has focused on analytical [6] and numerical [7] techniques for evaluating the critical $p$ and exploring the behavior of networks near the critical point [8].

In this paper we take a different perspective on the percolation problem: Imagine an electrical circuit consisting solely of nodes, wires, and ideal diodes. Each wire can have

- zero diodes, so that current can flow freely in either direction,
- one diode, so that current can flow in one direction but not the other, or
- two diodes back-to-back, so that current cannot flow in either direction.

Suppose that we know which nodes share a wire but we have no knowledge of the location or number of diodes. However, we can apply a voltage across a pair of nodes $(i, j)$ and see whether current percolates from $i$ to $j$; that is, we can measure whether there exists a path for current to flow from $i$ to $j$ (passing through other nodes along the way if necessary), as in Figure 1. The task is to reconstruct a placement of diodes that is consistent with these observations. Many different networks can yield the same percolation data, so we do not seek to reconstruct the network that generated the data but merely a network that is consistent with the data.

The diodes introduce a directional aspect absent from the typical percolation problem: Current might flow one way but not the other. Others [9, 10] have studied percolation in networks with diodes, but our problem is different in several key ways. The most salient is that we are not taking a network and asking if current flows; we are tackling the inverse problem: Given the currents, we want to reconstruct the network that generated the data but merely a network that is consistent with the data.

There are some cases in which one can construct a solution to our problem by hand. In Appendix A we show that this is possible if the network is completely connected, meaning that every pair of nodes shares a
Let \( x \) denote the projection of \( x \) to \( A \). For the details of implementing this projection, see Appendix B.

Set \( A \) implements the “divide” half of the method: We break down the difficult task of finding a solution that is consistent with all of the observations into the much easier task of finding, for each observation, a solution that merely satisfies that single observation irrespective of any others.

What remains is to make the many solutions “concur.” For this we define set \( B \) as the set of \( x \) such that for each \( e \), all \( x_{ep} \) are equal. Since \( A \) already enforces the discreteness of the \( x \) variables, \( B \) need not be discrete. Projecting to \( B \) is therefore a simple average:

\[
x_{ep}^B = \frac{1}{n(n-1)} \sum_{p'} x_{ep'}.
\]

Any \( x \in A \cap B \) is a solution to our problem.

We initialize \( x \) with random values between 0 and 1. Each iteration takes

\[
x \rightarrow (1 - \beta/2) x + (\beta/2) R_B(R_A(x))
\]

where \( R_A(x) = 2x^A - x \) is the reflection of \( x \) across \( x^A \), and similar for \( R_B \). One can check that if \( x \) is a fixed point of (1), then \( x^A \in A \cap B \). The parameter \( \beta \) controls the size of the iteration steps. Taking \( \beta \to 0 \) gives a reliable, continuous trajectory but makes the search process slow. Larger \( \beta \) (1) can make the search quicker but sometimes leads the algorithm to get stuck in limit cycles. We find that \( \beta = 0.1 \) works well, so that is the setting we use for all the results presented in this report.

### III. RESULTS

#### A. Scaling with \( n \)

Before we can apply the algorithm we must choose our networks. We use bipartite networks with an even number of nodes \( n \). Two nodes \( i \) and \( j \) share a wire if \( i \) is even and \( j \) is odd, or vice versa. For \( n \geq 4 \) this ensures the network is not completely connected. Such a network has \( n^2/4 \) wires, and thus \( n^2/2 \) places to put a diode.

Figure 1 is an instance of our problem with \( n = 6 \) nodes, 10 diodes, and half of the data hidden. Figure 2 illustrates the behavior of our algorithm as it tries to solve such an instance. The upper panel shows the evolution of the concur estimate \( x^B \), with each row corresponding to \( x_{ep}^B \) for a single directed edge \( e \). (Since \( x_{ep}^B \) is...
the same for all \( p \), we can drop the \( p \) from the subscript.) White corresponds to \( x_{e1} \) closer to 1, meaning that the algorithm thinks there should be a diode on edge \( e \), while black indicates no diode. Gray shades indicate the algorithm is uncertain, or is “changing its mind” and adding or removing that diode. The lower panel plots the error, \( ||R_B(R_A(x)) - x|| \), for the same trial. The error quantifies how much the algorithm’s copies disagree with each other. Every so often the algorithm adds or removes a diode, and typically the error spikes when it does so. Suddenly, after about 1200 iterations the error drops by many orders of magnitude, reflecting the “aha moment” when the algorithm has found a solution. In practice, we let the algorithm stop and declare success once the error falls below \( 10^{-3} \).

The most obvious parameter of our problem is the number of nodes, and it is important to know how our method scales with \( n \). For each \( n \) from 4 to 10 we randomly generate 100 networks, with the number of diodes chosen to maximize the difficulty of the problem for our algorithm (see Section III C), then hide half of the current data. To provide a comparison for our algorithm, we also implement an exhaustive “brute force” solution finder, which simply lists all of the possible solutions and checks them one by one until finding one that works.

Table I gives the average time required for our projection method and the exhaustive approach. The computational expense grows quite quickly for the projection algorithm: From \( n = 4 \) to \( n = 10 \) the time required increases by five orders of magnitude. However, the exhaustive approach scales much worse: The required time grows by eight orders of magnitude just from \( n = 4 \) to \( n = 8 \). We have not attempted using the exhaustive checker for \( n = 10 \).

One can understand the growth of the search time by considering what each method actually does. In principle, each iteration of our algorithm may involve checking every possible path for current from any one node to any other. We only consider paths of length less than \( n \), because any path that visits a node more than once is irrelevant as far as current is concerned, but the number of such paths can still grow with \( n! \). Meanwhile, the exhaustive approach has to contend with \( n^2/2 \) undetermined diodes, yielding \( 2^{n^2/2} \) possible solutions. Factorial growth may be fast, but \( 2^{n^2/2} \) grows faster. For large networks the exhaustive approach is simply infeasible.

| \( n \) | projection | exhaustive |
|---|---|---|
| 4 | 0.000040(4) | 0.000018(1) |
| 6 | 0.008(1) | 0.058(6) |
| 8 | 0.15(2) | 3.0(2) \( \times 10^3 \) |
| 10 | 8(1) | |

TABLE I. Average time, in seconds, to solve an instance of the problem for several values of \( n \), comparing our projection method to an exhaustive “brute force” approach.

FIG. 3. Average number of iterations required to find a solution as a function of the number of hidden data \( h \) for several values of \( n \).

B. Hiding data

We have already argued that the problem is trivial when all or none of the data are hidden. We will now explore how exactly the algorithm responds to the two trivial extremes and all the cases in between.

For a given \( n \) the number of hidden data \( h \) can be any integer from 0 to \( n(n-1) \). For each \( h \) we generate 100 instances, once again with the number of diodes for each \( n \) chosen to maximize the difficulty, and record the average number of iterations required to find a solution.

Figure 3 plots the results. Beginning from \( h = 0 \) the required number of iterations increases as more data are hidden. The logarithmic scale disguises the magnitude of this effect somewhat; the number of iterations when half of the data are hidden is actually about twice as many as when none of the data are hidden. However, as \( h \) continues increasing there comes a point at which the algorithm begins to find solutions much more quickly, particularly when \( h \) reaches \( n(n-1) \).

Every piece of data is a constraint that limits the space of possible solutions; when very few of the data are hidden the algorithm seems to take advantage of the many constraints, finding solutions in fewer and fewer steps as \( h \) approaches 0. A human does much the same thing in constructing a solution by hand for \( h = 0 \) (see Appendix A).

On the other hand, when most of the data are hidden, the problem is underconstrained and there are many possible solutions. It is natural that the algorithm finds solutions quite easily in this regime—especially when \( h = n(n-1) \), as that is the case in which any network is a solution.

C. Number of diodes

Next we explore how the number of diodes affects the difficulty of the problem. For our networks, the number of diodes \( d \) can be any integer from 0 to \( n^2/2 \). For each \( d \) we have the algorithm attempt to solve 100 networks with \( d \) randomly placed diodes. In each instance we hide exactly half of the data and record the average number of iterations required to reach a solution.
Figure 4 plots the results. For each $n$ the required number of iterations is generally small when $d$ is near 0 or $n^2/2$, with a peak somewhere in between. The peak shifts farther to the right as $n$ increases.

In order to understand this behavior, consider the influence of the diodes on the current percolations of a network. The top panel of Figure 5 shows the current probability—the likelihood that current can percolate from one randomly chosen node to another—as a function of $d$. There is a range of small $d$ for which current almost always flows, and this range extends farther to the right as $n$ increases. As $d \to n^2/2$, the current probability for any $n$ approaches 0 linearly.

Both limits make sense intuitively: Current is very likely to flow when the number of diodes is small. Having more nodes allows many more paths, so as $n$ increases it takes many more diodes to block all the paths from one node to another. On the other hand, when $d$ is almost maximal the number of ordered pairs for which current percolates is simply the number of directed edges that have not been blocked by a diode, $n^2/2 - d$, which does indeed approach 0 linearly as $d \to n^2/2$.

The middle panel of Figure 5 gives what we call the current sensitivity, which is the magnitude of the slope of the upper panel. The slope is essentially the typical number of percolations that are affected by adding or removing a single diode, so a steeper slope corresponds to a more sensitive network. The current sensitivity curves are qualitatively similar to the curves in Figure 4: small on either side, with a peak that shifts to the right as $n$ increases.

To make the similarity more obvious, in the lower panel of Figure 5 we re-plot the data from Figure 4 with a few modifications: First, we plot with a linear rather than logarithmic vertical axis. Second, we subtract off the number of iterations it takes for a completely trivial instance with all of the data hidden. We are essentially removing the part of the search at the far right of Figure 2 when the algorithm has found an answer and is simply converging to reach our desired stopping error ($10^{-3}$).

This turns out to be about 65 iterations, regardless of $n$. Finally, since the larger networks take significantly more iterations, we divide the number of iterations by the square of the number of possible diodes, $n^4/4$. Our original motivation for doing so is simply that it makes all of the curves roughly the same magnitude and therefore easy to plot on a linear scale, but in fact it turns out to be a very good rescaling—the plots of this modified iteration number are remarkably similar to the plots of current sensitivity, not only in shape but also in magnitude.

We are hesitant to make any strong claims about whether the similarity of the magnitudes is meaningful, but we are confident that the similar shapes are no coincidence. The networks that require more iterations to solve tend to be those with greater current sensitivity. In other words, the hardest networks to solve are those for which the current percolations are most affected by adding or removing a single diode. Since the algorithm’s task is to match the percolation data by adding and removing diodes, it makes sense that the task is more difficult when the currents are more sensitive.
D. Number of wires

Finally, we explore how the number of wires affects the difficulty of the problem. For a network with \( n \) nodes the number of wires \( w \) can be any integer from 0 to \( n(n-1)/2 \). Here we will give results for networks of size \( n = 6 \); the behavior for other values of \( n \) is analogous. For each \( w \), the number of diodes \( d \) can be any integer from 0 to \( 2w \), so for each \( d \) we generate 1000 networks, hide half of the data, and record the number of iterations needed to find a solution.

There are many ways in which one can remove wires. One option is to prune at random. Another approach is to disconnect the network as quickly as possible: Pick a node and remove all the wires that connect to it before touching any other wires. Let us call this the recursive method, because once the chosen node is disconnected one can repeat the process with the remaining \( n-1 \) nodes. Alternatively, one might delay disconnection as long as possible. One way to do this is to pick two nodes and remove the wire they share. Then pick a third node and remove the wires it shares with the first two. Then pick a fourth node and remove the wires it shares with the first three, and so on. We call this the star method, because once we have picked all but one of the nodes we are left with a star graph (in which the unpicked node is the center and the \( n-1 \) remaining wires connect the center to the other nodes). Removing the last wires from the star completes the pruning.

How much does the pruning order matter? To find out, in Figure 6 we plot the average iteration count as a function of \( w \) for each of the three methods. As in sections III A and III B, we are choosing \( d \) to maximize difficulty: For each \( w \) we select the \( d \) that requires the most iterations to solve. The difficulty almost always grows monotonically with \( w \), with little difference between the methods. The disparity between the recursive and star methods is greatest at \( w = 5 \), the moment at which the star method has produced a star graph. An astute observer may notice that the iteration count for the recursive method is higher at \( w = 10 \) than \( w = 11 \). This coincides with the moment the network of \( n = 6 \) nodes becomes disconnected, leaving a single isolated node and the other 5 nodes completely connected by 10 wires. The same phenomenon occurs between \( w = 7 \) and \( w = 6 \), the moment when the next node becomes isolated and the remaining 4 are completely connected.

If all one cares about is the most difficult diode configuration for a given \( w \), then the order in which one removes wires does not seem to matter much. But what about the pattern from section III C in which the iteration count was proportional to the current sensitivity: Is this true for all \( w \)?

Figure 7 plots the current probability, current sensitivity, and average number of iterations (rescaled in the same way as in section III C) for the recursive pruning method, with lighter color corresponding to fewer wires. Thanks to recursion we only need to plot the results from \( w = 15 \) down to \( w = 11 \). After that it reduces to the analogous problem for \( n = 5 \). These results echo those of section III C in that the rescaled iteration count is roughly proportional to the current sensitivity, with the possible exception of the lightest curve, corresponding to \( w = 11 \).

Figure 8 plots the current probability, current sensitivity, and average number of iterations for the star pruning method. This time we must take \( w \) from 15 all the way down to 5 before the network becomes disconnected. Here we see a significant discrepancy between the iteration count and the current sensitivity, particularly for the networks with the fewest wires and relatively few diodes.

Evidently the pattern from section III C is not universal. It holds for well-connected networks with many paths from one node to another, such as the bipartite net-
FIG. 8. Current probability, current sensitivity, and rescaled iteration count for the star pruning method. The colors indicate the number of wires, with the darker curves corresponding to larger $w$.

works used in most of this paper, but it falls for more sparse networks, especially those with “peninsular” nodes that only have one wire connecting them to the rest of the network. Indeed, the $w = 11$ curve in Figure [7] the first in which we noticed a discrepancy between iteration count and current sensitivity, corresponds to the network with one node connected to the rest of the network by just a single wire. For the star method, the peninsulas appear at $w = 8, 7, 6, 5$—precisely the networks for which the discrepancy becomes most obvious in Figure [8].

IV. CONCLUSION

We have demonstrated a divide-and-concur iterative projection method for solving an inverse percolation problem in diode networks. Though the computational expense of our method grows rapidly with the size of the network, it grows much slower than that of the exhaustive approach, making our method far more practical for networks with more than a few nodes.

We find that the projection algorithm generally requires more iterations to find a solution when one hides some of the current data, but only up to a point: If nearly all of the data are hidden the lack of constraints makes the problem very easy to solve. We have also used our algorithm to explore how the difficulty of the problem depends on the number of diodes in the network. Our results suggest that the most difficult networks are those for which the currents are most sensitive to the addition or removal of a single diode, although this pattern breaks down for highly pruned networks in which many nodes only have a single wire.

V. ACKNOWLEDGMENTS

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Appendix A: Constructing a solution by hand

Suppose we know whether current can percolate from $i$ to $j$ for every $(i, j)$. For any $(i, j)$ that share a wire and for which current does not flow from $i \rightarrow j$, we must place a diode on wire $ij$ that will prevent current from $i \rightarrow j$. We have to place these diodes because otherwise current would flow from $i \rightarrow j$, in violation of our data.

Claim A.1. This placement of diodes solves the problem.

Proof. Suppose current can percolate from $i$ to $j$ in the actual circuit used to generate the data. We can break down the percolation path into each of its steps: $i \rightarrow k$, ..., $l \rightarrow j$. Then current also percolates from $i$ to $k$, so we must have left $i \rightarrow k$ unblocked in our solution circuit. The same holds for every other step in the path. Concatenating these steps together shows that current can percolate from $i$ to $j$ in our solution circuit.

In the converse direction, suppose current can percolate from $i$ to $j$ in our solution circuit. Again, we can break down the percolation path into each of its steps: $i \rightarrow k$, ..., $l \rightarrow j$. We would have blocked $i \rightarrow k$ in our solution circuit if the data had told us current does not flow from $i$ to $k$, so the fact that we did not implies that current can percolate from $i$ to $k$ in the actual circuit. The same holds for every other step in the path. Any complete dataset generated from an actual circuit has current transitivity: If we observe current from $a \rightarrow b$ and $b \rightarrow c$ then we must also observe current from $a \rightarrow c$. Thus, by the transitive property we conclude that current can percolate from $i$ to $j$.

Claim A.2. This placement of diodes solves the problem: The solution may not have the same percolations as the circuit that generated the data, but it is consistent with all the available data.

Proof. Suppose the data indicate that current can percolate from $i$ to $j$. Then we did not place a diode blocking $i \rightarrow j$ in our solution circuit, so there is certainly a path from $i$ to $j$ in our solution circuit. The contrapositive of this implication is that if there is no percolation from $i$...
to \( j \) in our circuit, then there is no percolation from \( i \) to \( j \) in the data table: The data table either says no percolation, or it contains no information on \((i,j)\). Either way, we are consistent with the data.

Now suppose there is percolation from \( i \) to \( j \) in our solution circuit. We can break down the percolation path into steps: \( i \rightarrow k, \ldots, l \rightarrow j \). We would not have left \( i \rightarrow k \) unblocked in our solution circuit unless the data told us current flows from \( i \) to \( k \), so the data must indicate that current percolates from \( i \) to \( k \). The same holds for every other step in the path. By the transitive property discussed above, current can percolate from \( i \) to \( j \) in the circuit that generated the data. So the data table either says so, or it contains no information on \((i,j)\); once again, we are consistent with the data.

The hypothesis of complete connectivity is necessary: If the network is not completely connected, there may be a pair of nodes \((i,j)\) that do not share a wire, but do have percolation according to the data. The direct path \( i \rightarrow j \) does not exist in this case, so we cannot guarantee that our solution allows percolation from \( i \) to \( j \).

There are certainly other ways to construct solutions, but these are among the simplest. In restricting ourselves to cases in which these constructions fail, we are in essence asking our algorithm to be more clever than we have been in making these constructions. At some point it becomes more time consuming to come up with better constructions than to simply implement our algorithm.

**Appendix B: Details of the \( A \) projection**

Strictly speaking, the projection of \( x \) to set \( A \) should be the point \( x^A \in A \) such that the (Euclidean) distance

\[
d(x, x^A) = \sum_{e \in p} (x_{e \mid x} - x^A_{e \mid x})^2 \tag{B1}
\]

is as small as possible. We define \( A \) as the set of \( x \) such that for all \( p = (i,j) \), \( x_{e \mid x} \) gives a solution consistent with the observation (if any) for percolation from \( i \) to \( j \).

Our “projection” to \( A \) is in fact only a quasi-projection: It always gives a point in set \( A \), but not necessarily the distance-minimizing point. First, for all \( e \) and \( p \), we set

\[
x^A_{e \mid x} = \begin{cases} 0 & x_{e \mid x} < 0.5 \\ 1 & x_{e \mid x} \geq 0.5 \end{cases}
\]

Then, for each \( p = (i,j) \):

- If there are no data for \( p \), move on to the next \( p \).

- If the data indicate current does percolate from \( i \) to \( j \), check if there is at least one path from \( i \) to \( j \) with \( x^A_{e \mid x} = 0 \) along every edge \( e \) of the path. If there is no such path, rank the paths by

\[
\sum_{e \in \text{path}} \min(x_{e \mid x} - 0.5, 0)
\]

and for the path with the lowest sum set \( x^A_{e \mid x} = 0 \) for every \( e \) on the path.

- If the data indicate current does not percolate from \( i \) to \( j \), then for every path from \( i \) to \( j \), check if at least one edge \( e \) along the path has \( x^A_{e \mid x} = 1 \). For any path for which this is not so, choose the \( e \) on the path with the largest \( x_{e \mid x} \) and set \( x^A_{e \mid x} = 1 \).

This last point, the method for blocking current, is not a strictly distance-minimizing method: Imagine a scenario in which two paths need to be blocked (they both have \( x_{e \mid x} < 0.5 \) on all of their edges \( e \)). The extra distance for changing \( x^A_{e \mid x} \) from 0 to 1 is

\[
(1 - x_{e \mid x})^2 - x^2_{e \mid x} = 1 - 2x_{e \mid x}.
\]

For each path individually, the best move is to block the edge with the largest \( x_{e \mid x} \). Let \( e_1 \) be that edge for path 1, and similar for path 2. For definiteness, suppose \( x_{e_{1p}} = x_{e_{2p}} = 0.1 \), so that the added distance for blocking each path is 0.8. But if the two paths share an edge \( e_0 \neq e_1, e_2 \) with \( x_{e_{0p}} = 0 \), then blocking \( e_0 \) is actually the best move overall, adding a distance of only 1 compared to \( 2 \cdot 0.8 = 1.6 \).

Finally, we stress that alternative implementations that may be better in some scenarios:

- One can take advantage of the percolation data for ordered pairs that happen to share an edge: For every directed edge \( e = i \rightarrow j \), if we observe no percolation from \( i \) to \( j \), permanently set \( x_{e \mid x} = 1 \) for all \( p \). Similarly, if we do observe percolation then by the transitive property discussed in Appendix A we can safely eliminate the diode from edge \( e \) and permanently set \( x_{e \mid x} = 0 \) for all \( p \). By not using these shortcuts in our implementation we are effectively asking the projection algorithm to be clever enough to discover them on its own.

- Another speed-up is to create copies of \( x_e \) only for the pairs \( p \) for which the percolation is known. When \( h > 0 \) this mean fewer copies, which makes each iteration faster. However, it can also make the algorithm more likely to get trapped in limit cycles, especially if \( h \) is very close to \( n(n-1) \).

- If speed is no concern but the exactness of the projection is, then create copies of \( x_e \) for every path, not just every edge. This creates an absurd number of copies when \( n \) is large, but it makes our approach of blocking path by path an exact projection.

There are surely others as well. After briefly testing the alternatives described above, we have seen that the choice of implementation can affect the behavior seen in Section III.B. For instance, using the elimination scheme makes the algorithm faster for when none or very few of the data are hidden. The implementation can also affect the times observed in Section III.A, though for a given \( d \) and \( h \) the algorithm would still become slower with increasing \( n \) in
much the same way. We believe that the main results of Section III C persist regardless of implementation—
networks with more sensitive currents are generally more difficult to solve.

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