THE WITTEN EQUATION AND ITS VIRTUAL FUNDAMENTAL CYCLE

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Abstract. We study a system of nonlinear elliptic PDEs associated with a quasi-homogeneous polynomial. These equations were proposed by Witten as the replacement in the Landau-Ginzburg/singularity setting for the Cauchy-Riemann equation.

We introduce a perturbation to the equation and construct a virtual cycle for the moduli space of its solutions. Then, we study the wall-crossing of the deformation of the perturbation and match it to classical Picard-Lefschetz theory. An extended virtual cycle is obtained for the original equation. Finally we prove that the extended virtual cycle satisfies a set of axioms similar to those of Gromov-Witten theory and r-spin theory.

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1. Introduction

This is the second installment of a series of papers devoted to the mathematical theory of Witten equation and its applications. The Witten equation is a system of nonlinear elliptic PDE associated to a quasihomogeneous polynomial $W$. It has the simple form

$$\bar{\partial}u_i + \frac{\partial W}{\partial u_i} = 0,$$

where $W$ is a quasi-homogeneous polynomial, and $u_i$ is interpreted as a section of an appropriate orbifold line bundle on a Riemann surface $\mathcal{C}$. Some simple examples are

- $(A_r)$-case: $\bar{\partial}u + \bar{u} = 0$
- $(D_n)$-case: $\bar{\partial}u_1 + nu_i^{-1} + u_2 = 0, \bar{\partial}u_2 + 2u_1 \bar{u}_2 = 0$
- $(E_7)$-case: $\bar{\partial}u_1 + 3u_2 + u_1^2 = 0, \bar{\partial}u_2 + 3u_1 u_2^2 = 0$.

To understand the importance of the Witten equation, we have to go back to the famous Landau-Ginzburg/Calabi-Yau correspondence [Mar], [VW], [Wi4]. Most of known examples of Calabi-Yau 3-folds are constructed as a hypersurface or a complete intersection of toric varieties. Its defining equation has a natural interpretation in terms of Landau-Ginzburg/singularities theory. To compute Gromov-Witten invariants is basically to solve the Cauchy-Riemann equation for maps from Riemann surfaces. It is generally difficult to solve the Cauchy-Riemann equation on a nonlinear space such as a Calabi-Yau manifold. If we compare the Cauchy-Riemann equation on a Calabi-Yau manifold to the anti-self dual equation in Donaldson theory, the Witten equation plays the role of the Seiberg-Witten equation. It takes a while to set up. But we expect that its invariants are much easier to compute. A program was launched by the authors six years ago to establish a mathematical theory of these equations and its various consequences in geometry. We believe that we have achieved some initial success, although a lot more remains to be done. In [FJR1], we formulated the algebraic-geometric foundations of the theory, its main properties in terms of axioms as well as the applications to mirror symmetry. In this article and its sequel [FJR2], we will establish its analytic foundations. A fourth article [FJR3] under preparation will establish the famous Witten conjecture for the integrable system of $ADE$-singularities.

A casual investigation of the Witten equation reveals that the Witten equation is much more subtle than its simple appearance would suggest. Suppose $u \in \Omega^{0}(\mathcal{L})$. A simple computation shows

$$\bar{\partial}u \in \Omega^{0,1}(\mathcal{L}), \frac{\partial W}{\partial u} \in \Omega^{0,1}_{\log}(\mathcal{L}_L^{\mathcal{L}_L^{-1}}),$$

where log means a $(0,1)$ form with possible singularities of order $\leq 1$. Namely, the Witten equation has singular coefficients! This is a fundamental phenomenon for the application of the Witten equation. A Calabi-Yau manifold has cohomological information which we expect to discover from the Witten equation. At first sight, it seems to be difficult to reproduce it from the Witten equation. Now we understand that the singularity of the Witten equation is the key to producing such cohomological data in our theory. Unfortunately, the appearance of singularities makes Witten equation very difficult to study. This is the main reason it took us such a long time to construct the theory. Another subtle issue is...
the fact that we need an isomorphism $L_i^{-1} \cong L_i$ for the two terms of the Witten equation living in the same space. The required isomorphism can be obtained by a choice of metric. A nontrivial fact is that such a metric can be constructed uniformly from a metric of the underlying Riemann surface. Then the question is: Which metric we should choose on the Riemann surface? We should mention that a different choice of metric often leads to a completely different looking theory including a different dimension for its moduli space. Apparently, there is no physical guidance for the correct metric we should choose. We have experimented with both smooth and cylindrical metric near marked points. Now we understand that both choices are important for the theory! In [FJR], we studied the theory of the smooth metric. In this article, our main choice is a cylindrical metric.

Our first important result is Theorem 5.4.3, which can be written in the simple form:

**Theorem 1.0.1.** If $W_0$ is strongly regular, then $\mathcal{M}_{g,k,W}(\kappa_1, \ldots, \kappa_h)$ is compact and has a virtual fundamental cycle $[\mathcal{M}_{g,k,W}(\kappa_1, \ldots, \kappa_h)]^\text{vir}$ of degree

$$2((\epsilon W - 3)(g - 1) + k - \sum_i \nu_i) - \sum_i N_i.$$ 

Here, $\nu_i$ is a certain degree shifting number defined in section two.

It turns out to be notionally convenient to map the above virtual cycle into $H_*([\mathcal{M}_{g,k,W}], \mathbb{Q})$ even though it is not a subspace of the latter in any way. This is the first step of our construction. We have not yet seen the cohomology data we hope for. Then, a crucial new phenomenon comes into play when we study how the above virtual cycle changes when we vary the perturbation. It turns out that the above virtual cycle does depend on the perturbation. It will change when $W_0$ fails to be strongly regular. The “wall crossing formula” proved in Theorem 6.1.6 shows the following quantum Picard-Lefschetz theorem:

**Theorem 1.0.2.** When $W_0$ varies, $[\mathcal{M}_{g,k,W}(\kappa_1, \ldots, \kappa_h)]^\text{vir}$ transforms in the same way as the so called Lefschetz thimble $S_{\gamma_i}$ attached to the critical point $k_i$.

The “wall crossing formula” of the above virtual cycle can be neatly packaged into the following formula. Let $S_{\gamma_i}$ be the set of dual Lefschetz thimbles. To simplify the notation, we assume that there is only one marked point with the orbifold decoration $\gamma$. Then, wall
crossing formula of $\{\bar{\mathcal{M}}_{g,1,W}(κ_j)\}^{vir}$ shows precisely that

$$\sum_j [\bar{\mathcal{M}}_{g,1,W}(κ_j)]^{vir} \otimes S_j$$

viewed as a class in $H_*(\bar{\mathcal{M}}_{g,1,W}(γ), \mathbb{Q}) \otimes H_{N_γ}(\mathbb{C}^N, W^∞_γ, \mathbb{Q})$ is independent of the perturbation. Now, we define

$$[\bar{\mathcal{M}}_{g,1,W}(γ)]^{vir} = \sum_j [\bar{\mathcal{M}}_{g,1,W}(κ_j)]^{vir} \otimes S_j.$$ 

The above definition can be generalized with multiple marked points in an obvious way. It is obvious that

$$[\bar{\mathcal{M}}_{g,k,W}(γ_1, \cdots, γ_k)]^{vir} \in H_*(\bar{\mathcal{M}}_{g,k,W}(γ_1, \cdots, γ_k), \mathbb{Q}) \otimes \prod_i H_{N_γ_i}(\mathbb{C}^N, W^∞_γ, \mathbb{Q})$$

of degree

$$2((c_W - 3)(g - 1) + k - \sum_i τ_γ).$$

**Corollary 1.0.3.** $[\bar{\mathcal{M}}_{g,k,W}(γ_1, \cdots, γ_k)]^{vir}$ is independent of the perturbation $W_0$.

$W_0$ is only part of the perturbation data. Eventually, we want to work on $\bar{\mathcal{M}}_{g,k,W}$. It is known from section 3 that $so : \bar{\mathcal{M}}_{g,k,W} \to \bar{\mathcal{M}}_{g,k,W}$ is a quasi-finite proper map by forgetting all the rigidifications. We can define

$$[\bar{\mathcal{M}}_{g,k,W}(γ_1, \cdots, γ_k)]^{vir} := \frac{1}{\deg(so)} [so]_* [\bar{\mathcal{M}}_{g,k,W}(γ_1, \cdots, γ_k)]^{vir}.$$ 

The independence of the above virtual cycle on rigidification implies that

$$[\bar{\mathcal{M}}_{g,k,W}(γ_1, \cdots, γ_k)]^{vir} \in H_*(\bar{\mathcal{M}}_{g,k,W}(γ_1, \cdots, γ_k), \mathbb{Q}) \otimes \prod_i H_{N_γ_i}(\mathbb{C}^N, W^∞_γ, \mathbb{Q})^G.$$ 

Similarly we can define the virtual cycle $[\bar{\mathcal{M}}_W(Γ)]^{vir}$ corresponding to the dual graph $Γ$.

Besides the quantum Picard-Lefschetz theory, another main aim of this paper is to show Theorem 4.1.3 in our paper [FJR1]. This theorem shows that the virtual cycles constructed in this paper satisfy a set of axioms similar to the standard Gromov-Witten theory. We state it below.

**Theorem 1.0.4.** The following axioms are satisfied for $[\bar{\mathcal{M}}_W(Γ)]^{vir}$:

1. **Dimension:** If $D_Γ$ is not a half-integer (i.e., if $D_Γ \notin \frac{1}{2} \mathbb{Z}$), then $[\bar{\mathcal{M}}_W(Γ)]^{vir} = 0$.

   Otherwise, the cycle $[\bar{\mathcal{M}}_W(Γ)]^{vir}$ has degree

   $$6g - 6 + 2k - 2D_Γ = 2\left(\hat{c} - 3\right)(1 - g) + k - \sum_{τ ∈ Γ(Γ)} τ_γ.$$ 

   So the cycle lies in $H_*(\bar{\mathcal{M}}_W(Γ), \mathbb{Q}) \otimes \prod_{τ ∈ Γ(Γ)} H_{N_τ}(\mathbb{C}^N, W^∞_γ, \mathbb{Q})$, where

   $$r := 6g - 6 + 2k - 2D - \sum_{τ ∈ Γ(Γ)} N_γ = 2\left(\hat{c} - 3\right)(1 - g) + k - \sum_{τ ∈ Γ(Γ)} τ(γ) - \sum_{τ ∈ Γ(Γ)} \frac{N_γ}{2}.$$
(2) **Symmetric group invariance:** There is a natural $S_k$-action on $\mathcal{W}_{g,k,W}$ obtained by permuting the tails. This action induces an action on homology. That is, for any $\sigma \in S_k$ we have:

$$\sigma_* : H_*(\mathcal{W}_{g,k,W}, \mathbb{Q}) \otimes \prod_i H_{N_i}(\mathbb{C}^N_{g_i}, W^\infty_{g_i}, \mathbb{Q})^G \to H_*(\mathcal{W}_{g,k,W}, \mathbb{Q}) \otimes \prod_i H_{N_i}(\mathbb{C}^N_{g_i}, W^\infty_{g_i}, \mathbb{Q})^G.$$  

For any decorated graph $\Gamma$, let $\sigma \Gamma$ denote the graph obtained by applying $\sigma$ to the tails of $\Gamma$.

We have

$$\sigma_* \left[ \mathcal{W}_W(\Gamma) \right]^{\text{vir}} = \left[ \mathcal{W}_W(\sigma \Gamma) \right]^{\text{vir}}. \quad (2)$$

(3) **Degenerating connected graphs:** Let $\Gamma$ be a connected, genus-$g$, stable, decorated $W$-graph.

The cycles $\left[ \mathcal{W}_W(\Gamma) \right]^{\text{vir}}$ and $\left[ \mathcal{W}_{g,k,W}(\gamma) \right]^{\text{vir}}$ are related by

$$\left[ \mathcal{W}_W(\Gamma) \right]^{\text{vir}} = \mathcal{I}^* \left[ \mathcal{W}_{g,k,W}(\gamma) \right]^{\text{vir}} \quad (3)$$

where $\mathcal{I} : \mathcal{W}_W(\Gamma) \to \mathcal{W}_{g,k,W}(\gamma)$ is the canonical inclusion map.

(4) **Disconnected graphs:** Let $\Gamma = \bigcup_i \Gamma_i$ be a stable, decorated $W$-graph which is the disjoint union of connected $W$-graphs $\Gamma_i$. The classes $\left[ \mathcal{W}_W(\Gamma) \right]^{\text{vir}}$ and $\left[ \mathcal{W}_W(\Gamma_i) \right]^{\text{vir}}$ are related by

$$\left[ \mathcal{W}_W(\Gamma) \right]^{\text{vir}} = \left[ \mathcal{W}_W(\Gamma_1) \right]^{\text{vir}} \times \cdots \times \left[ \mathcal{W}_W(\Gamma_d) \right]^{\text{vir}}. \quad (4)$$

(5) **Weak Concavity:** Suppose that all the decorations on tails and edges are Neveu-Schwarz, meaning that $\mathbb{C}^N_{g_i} = 0$. In this case we omit the $H_{N_i}(\mathbb{C}^N_{g_i}, W^\infty_{g_i}, \mathbb{Q})$ from our notation.

If, furthermore, the universal $W$-structure $(\mathcal{L}_1, \ldots, \mathcal{L}_N)$ on the universal curve $\pi : \mathcal{C} \to \mathcal{W}_W(\Gamma)$ is concave (i.e., $\pi_* \left( \bigoplus_{i=1}^t \mathcal{L}_i \right) = 0$), then the virtual cycle is given by capping the top Chern class of the orbifold vector bundle $-R^1\pi_* \left( \bigoplus_{i=1}^t \mathcal{L}_i \right)$ with the usual fundamental cycle of the moduli space:

$$\left[ \mathcal{W}_W(\Gamma) \right]^{\text{vir}} = c^\text{top} \left( -R^1\pi_* \left( \bigoplus_{i=1}^t \mathcal{L}_i \right) \right) \cap \left[ \mathcal{W}_W(\Gamma) \right]. \quad (5)$$

(6) **Index zero:** Suppose that $\dim \mathcal{W}(\Gamma) = 0$ and all the decorations on tails of $\Gamma$ and edges are Neveu-Schwarz.

If the pushforwards $\pi_* \left( \bigoplus \mathcal{L}_i \right)$ and $R^1\pi_* \left( \bigoplus \mathcal{L}_i \right)$ are both vector spaces of the same dimension, then the virtual cycle is just the degree $\deg(\mathcal{W})$ of the Witten map times the fundamental cycle:

$$\left[ \mathcal{W}_W(\Gamma) \right]^{\text{vir}} = \deg(\mathcal{W}) \left[ \mathcal{W}_W(\Gamma) \right]$$

where the $j$th term $\mathcal{W}_j : \pi_* \left( \bigoplus \mathcal{L}_i \right) \longrightarrow R^1\pi_* \mathcal{L}_j$ of the Witten map is given by $\mathcal{W}_j = \delta_j \mathcal{W}(x_1, \ldots, x_N).$
(7) **Composition law:** Given any genus-\(g\) decorated stable W-graph \(\Gamma\) with \(k\) tails, and given any edge \(e\) of \(\Gamma\), let \(\Gamma_{\text{cut}}\) denote the graph obtained by “cutting” the edge \(e\) and replacing it with two unjoined tails \(\tau_+\) and \(\tau_-\) decorated with \(\gamma_+\) and \(\gamma_-\), respectively.

In view of the gluing/cutting commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}(\Gamma_{\text{cut}}) & \xrightarrow{pr_1} & \mathcal{M}(\Gamma) \\
\downarrow{\varphi} & & \downarrow{\varphi_{\Gamma}} \\
\mathcal{M}(\Gamma) & \xrightarrow{\rho} & \mathcal{M}(\Gamma)
\end{array}
\]

the fiber product

\[
F := \mathcal{M}(\Gamma_{\text{cut}}) \times_{\mathcal{M}(\Gamma)} \mathcal{M}(\Gamma),
\]

has morphisms

\[
\mathcal{M}(\Gamma_{\text{cut}}) \xrightarrow{q} F \xrightarrow{pr_2} \mathcal{M}(\Gamma).
\]

We have

\[
\langle [\mathcal{M}(\Gamma_{\text{cut}})]^{vir} \rangle_\pm = \frac{1}{\deg(q)} q_{\ast} pr_2 \left( [\mathcal{M}(\Gamma)]^{vir} \right),
\]

where

\[
\langle , \rangle : H_s(\mathcal{M}(\Gamma_{\text{cut}}) \otimes \prod_{\tau \in T(\Gamma)} H_{N_\gamma}(\mathbb{C}_{\gamma_\ast}, W_{\gamma_\ast}, Q) \otimes H_{N_\gamma}(\mathbb{C}_{\gamma_\ast}, W_{\gamma_\ast}, Q) \otimes H_{N_\gamma}(\mathbb{C}_{\gamma_\ast}, W_{\gamma_\ast}, Q) \rightarrow H_{s}(\mathcal{M}(\Gamma_{\text{cut}}) \otimes \prod_{\tau \in T(\Gamma)} H_{N_\gamma}(\mathbb{C}_{\gamma_\ast}, W_{\gamma_\ast}, Q))
\]

(8) **Forgetting tails:**

(a) Let \(\Gamma\) have its \(i\)th tail decorated with \(J^{\ast}\), where \(J\) is the exponential grading element of \(G\). Further let \(\Gamma'\) be the decorated W-graph obtained from \(\Gamma\) by forgetting the \(i\)th tail and its decorations. Assume that \(\Gamma'\) is stable, and denote the forgetting tails morphism by

\[
\theta : \mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\Gamma').
\]

We have

\[
[\mathcal{M}(\Gamma)]^{vir} = \theta^\ast [\mathcal{M}(\Gamma')]^{vir}.
\]

(b) In the case of \(g = 0\) and \(k = 3\), then the space \(\mathcal{M}(\gamma_1, \gamma_2, J^{-1})\) is empty if \(\gamma_1 \gamma_2 \neq 1\) and \(\mathcal{M}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}) = \mathcal{M}(\gamma)\). We omit \(H_{N_\gamma}(\mathbb{C}_{\gamma_\ast}, W_{\gamma_\ast}, Q)^G = Q\) from the notation. In this case, the cycle

\[
[\mathcal{M}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1})]^{vir} \in H_s(\mathcal{M}(\mathcal{M}(\gamma), \mathbb{Q}) \otimes H_{N_\gamma}(\mathbb{C}_{\gamma_\ast}, W_{\gamma_\ast}, Q)^G \otimes H_{N_\gamma}(\mathbb{C}_{\gamma_\ast}, W_{\gamma_\ast}, Q)^G
\]
is the fundamental cycle of $\mathcal{B}G$ times the Casimir element. Here the Casimir element is defined as follows. Choose a basis $\{\alpha_i\}$ of $H_{N_1}(\mathbb{C}^N, W_c, Q)^G$, and a basis $\{\beta_j\}$ of $H_{N_1}(\mathbb{C}^N, W_c, Q)^G$. Let $\eta_{ij} = (\alpha_i, \beta_j)$ and $(\eta^{ij})$ be the inverse matrix of $(\eta_{ij})$. The Casimir element is defined as $\sum_{ij} \alpha_i \eta^{ij} \otimes \beta_j$.

(9) **Sums of Singularities:**

If $W_1 \in \mathbb{C}[z_1, \ldots, z_r]$ and $W_2 \in \mathbb{C}[z_{r+1}, \ldots, z_{r+r}]$ are two quasi-homogeneous polynomials with diagonal automorphism groups $G_1$ and $G_2$, and if we write $W = W_1 + W_2$ then the diagonal automorphism group of $W$ is $G = G_1 \times G_2$. Further, the state space $\mathcal{H}_W$ is naturally isomorphic to the tensor product

$$\mathcal{H}_W = \mathcal{H}_{W_1} \otimes \mathcal{H}_{W_2},$$

and the space $\mathcal{M}_{g,k,W}$ is naturally isomorphic to the fiber product

$$\mathcal{M}_{g,k,W} = \mathcal{M}_{g,k,W_1} \times_{\mathcal{M}_{g,k}} \mathcal{M}_{g,k,W_2}. $$

Indeed, since any $G$-decorated stable graph $\Gamma$ is equivalent to the choice of a $G_1$-decorated graph $\Gamma_1$ and $G_2$-decorated graph $\Gamma_2$ with the the same underlying graph $\overline{\Gamma}$, we have

$$\mathcal{M}_{W}(\Gamma) = \mathcal{M}_{W_1}(\Gamma_1) \times_{\mathcal{M}_W} \mathcal{M}_{W_2}(\Gamma_2). $$

The natural inclusion

$$\mathcal{M}_{g,k,W} = \mathcal{M}_{g,k,W_1} \times_{\mathcal{M}_{g,k}} \mathcal{M}_{g,k,W_2} \to \mathcal{M}_{g,k,W_1} \times \mathcal{M}_{g,k,W_2}$$

together with the isomorphism of middle homology induces a homomorphism

$$\Delta^* : \left( H_\ast(\mathcal{M}_{g,k,W_1}, Q) \otimes \prod_{i=1}^k H_{N_{0,i}}(\mathbb{C}^N, (W_1)_i^0, Q)^G \right)$$

$$\otimes \left( H_\ast(\mathcal{M}_{g,k,W_2}, Q) \otimes \prod_{i=1}^k H_{N_{0,i}}(\mathbb{C}^N, (W_2)_i^0, Q)^G \right)$$

$$\longrightarrow H_\ast(\mathcal{M}_{g,k,W_1+W_2}, Q) \otimes \prod_{i=1}^k H_{N_{0,i}}(\mathbb{C}^N, (W_1 + W_2)_i^0, Q)^{G_1 \times G_2}$$

The virtual cycle satisfies

$$\Delta^* \left( \mathcal{M}_{g,k,W_1} \right) \lor \mathcal{M}_{g,k,W_2} \lor \mathcal{M}_{g,k,W_1+W_2}$$

To complete our theory, we also need to establish the full concavity where we remove the NS-assumption on nodal points. The technique to deal with this case is quite different from our current set-up. We need to construct a theory with a smooth metric on nodes and prove a comparison theorem to our current set-up. We shall deal with it in a separate article [FJR2].

The paper is organized as follows. In Section 2 we will summarize the algebraic-geometric set up of [FJR1] and some of the easy consequences. The perturbed Witten equation and its nonlinear analysis on a smooth curve or orbicurve will be defined in Section 3. In Section 4 the perturbed Witten map and equation will be defined on the moduli space of rigidified $W$-curves. We will prove the Gromov compactness theorem for the space of $W$-sections. The virtual cycle will be constructed in Section 5. The main theorems will be proved in the last section.
2. Rigidified W-curves and their moduli

Recall that the variable $u_i$ of the Witten equation is supposed to be a section of certain orbifold line bundle $\mathcal{L}_i$ over the Riemann surface. The fact that $u_i$ satisfies the Witten equation forces a certain condition on $\mathcal{L}_i$. $\mathcal{L}_i$ satisfying the required condition is called a W-structure. The W-structure is the background data of the Witten equation. A detailed construction of the moduli of W-structures has been worked out in our previous paper [FJR1]. It is technically convenient to work over the moduli space of rigidified W-structures. In this section, we summarize the construction of [FJR1].

**Definition 2.0.5.** For any non-degenerate, quasi-homogeneous polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$, we define a W-structure on an orbicurve $\mathcal{C}$ to be the data of an $N$-tuple $(\mathcal{L}_1, \ldots, \mathcal{L}_N)$ of orbifold line bundles on $\mathcal{C}$ and isomorphisms $\varphi_j : W_j(\mathcal{L}_1, \ldots, \mathcal{L}_N) \to K_{\mathcal{C}, \mathcal{L}}$ for every $j \in \{1, \ldots, s\}$, where by $W_j(\mathcal{L}_1, \ldots, \mathcal{L}_N)$ we mean the $j$th monomial of $W$ in $\mathcal{L}_j$:

$$W_j(\mathcal{L}_1, \ldots, \mathcal{L}_N) = \mathcal{L}_1^{ob_{b_1}} \otimes \cdots \otimes \mathcal{L}_N^{ob_{b_j}}.$$ 

Note that for each point $p \in \mathcal{C}$, an orbifold line bundle $\mathcal{L}$ on $\mathcal{C}$ induces a representation $G_p \to \text{Aut}(\mathcal{L})$. Moreover, a W-structure on $\mathcal{C}$ will induce a representation $\rho_p : G_p \to G \subseteq U(1)^N$. For all our W-structures we require that this representation $\rho_p$ be faithful at every point.

The moduli space $\mathcal{M}_{g,k,W}$ of W-structures was constructed in [FJR1] along the lines of the Delign-Mumford moduli space. It is a smooth non-effective orbifold. There is an obvious finite map

$$\text{st} : \mathcal{M}_{g,k,W} \to \mathcal{M}_{g,k}.$$ 

It was shown in [FJR1] that

$$\mathcal{M}_{g,k,W} = \bigsqcup_{(\gamma_1, \ldots, \gamma_k)} \mathcal{M}_{g,k,W}(\gamma_1, \ldots, \gamma_k),$$

where $\mathcal{M}_{g,k,W}(\gamma_1, \ldots, \gamma_k)$ is the moduli space of W-structures with the orbifold structure at the $i$-th marked point decorated by $\gamma_i$.

It is convenient for us to work on the rigidified W-structure. Let $p$ be a marked point. A rigidification (at $p$) is a homomorphism

$$\psi : f_{p*}(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_N) \to [\mathbb{C}^N/G_{p*}].$$

A more geometric way to understand the rigidification is follows. Suppose the fiber of the W-structure at the marked point is $L_1 \oplus L_2 \cdots \oplus L_N/G_p$. The rigidification can be thought as a $G_p$-equivariant map $\psi : \oplus L_i \to \mathcal{L}$ commuting with the W-structure. For any element $g \in G_p$, $g\psi$ is considered to be an equivalent rigidification. Alternatively, $\psi$ is equivalent to a choice of basis $e_i \in L_i$ such that $W_j(e_1, \cdots, e_N) = dz/lz$ and $g(e_1), \cdots g(e_N)$ is considered to be an equivalent choice. In particular, if $L_i$ is fixed by $G_p$, or $L_i$ corresponds to the Ramond variable, there is a unique choice of basis $e_i$ with $W_j(e_1, \cdots, e_N) = dz/lz$. On the other hand, the choice of basis on the Neveu-Schwarz variables is only unique up to the action of $G_p$. It is clear that the group $G/G_p$ acts transitively on the set of rigidifications within a single orbit. Let $\mathcal{M}_{g,k}^{rig}(\Gamma)$ be the moduli space of equivalence classes of W-structures with a rigidification at $p$. $G/G_p$ acts on $\mathcal{M}_{g,k}^{rig}(\Gamma)$ by changing the rigidification. It is clear that

$$\mathcal{M}_{g,k}^{rig}(\Gamma)/(G/G_p) = \mathcal{M}_{g,k}(\Gamma).$$
We use $\overline{\mathcal{M}}_{g,k,W}^{\text{rig}}$ to denote the moduli space of rigidified $W$-structures at all the marked points.

Forgetting the rigidification gives a morphism

$$so : \overline{\mathcal{M}}_{g,k,W}^{\text{rig}} \longrightarrow \overline{\mathcal{M}}_{g,k,W},$$

which we call softening. The morphism $so$ is quasi-finite since one can always construct a rigidification of any unrigidified $W$-structure, and $G^k$ acts transitively, but usually not effectively, on the set of rigidifications. It easy to see that $so$ is proper and of Deligne-Mumford type. Furthermore, $so$ is representable, since the automorphisms of any rigidified $W$-curve are a subgroup of the automorphisms of the corresponding unrigidified $W$-curve.

Now we describe the gluing. To simplify notation, we ignore the orbifold structures at other marked points and denote the type of the marked point $p_+$, $p_-$ being glued by $\gamma_+$, $\gamma_-$. Because the resulting orbicurve must be balanced, we require that $\gamma_- = \gamma_+^{-1}$. Let

$$\psi_\pm : \tilde{\rho}_{p_\pm}(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_N) \longrightarrow [\mathbb{C}^N/G_{p_\pm}]$$

be the rigidifications. Moreover, the residues at $p_+$, $p_-$ are opposite to each other. The obvious identification will not preserve the rigidifications. Here, we fix once and for all an isomorphism

$$I : \mathbb{C}^N \longrightarrow \mathbb{C}^N$$

such that $W(I(x)) = -W(x)$. $I$ can be explicitly constructed as follows. Suppose that $q_i = n_i/d$ for common denominator $d$. Choose $\xi_i = -1$. Then,

$$I(x_1, \ldots, x_N) = (\xi_1^k x_1, \ldots, \xi_N^k x_N). \quad (13)$$

If $I'$ is another choice, then $I'^{-1}I' \in G_W$. Furthermore, $I^2 \in G_W$ as well. The identification by $I$ induces a $W$-structure on the nodal orbifold Riemann surface with a rigidification at the nodal point. Forgetting the rigidification at the node yields the lifted gluing morphisms

$$\tilde{\rho}_{\text{tree},y} : \overline{\mathcal{M}}_{g_1,k_1+1}(y) \times \overline{\mathcal{M}}_{g_2,k_2+1}(y^{-1}) \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,k_1+k_2}, \quad (14)$$

and

$$\tilde{\rho}_{\text{loop},y} : \overline{\mathcal{M}}_{g,k+2}(y, y^{-1}) \longrightarrow \overline{\mathcal{M}}_{g,k}, \quad (15)$$

where $\tilde{\rho}$ is defined by gluing the rigidifications at the extra tails and forgetting the rigidification at the node.

Next, we summarize some of the basic properties of $\overline{\mathcal{M}}_{g,k,W}^{\text{rig}}$. They are easy consequences of the existence of a universal family. We leave the proof to the interested readers. As a warm-up, we start with the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,k}$.

### 2.1. The structure of the Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$.

Let $\mathcal{M}_{g,k}$ be the moduli space of Riemann surfaces of genus $g$ and having $k$ marked points (assuming $k + 2g \geq 3$). The Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,k}$ of $\mathcal{M}_{g,k}$ is the set of all isomorphism classes of stable nodal Riemann surfaces in $\mathcal{M}_{g,k}$. The following is well-known and an easy consequence of the existence of a universal family.

The moduli space $\overline{\mathcal{M}}_{g,k}$ is a stratified space, indexed by a set Comb$(g, k)$. Each element in Comb$(g, k)$ is represented by a triple $(\Gamma, g, o)$ satisfying the following requirement:

1. $\Gamma$ is the dual graph of some nodal curve $C$ with each vertex assigned the genus $g_v$ of the component of $C$ corresponding to $v$.  

Proposition 2.1.3.

(3) Let \( k_v \) be the number of elements of the set \( T(\nu) \); then

\[
k_v + 2g_v \geq 3 \quad \text{and} \quad \sum_v g_v + \text{rank} \, H_1(\Gamma, \mathbb{Q}) = g
\]

From the definition, it is easy to see that the number of combinatorial types \((\Gamma, g_v, o)\) in \( \text{Comb}(g, k) \) is finite.

Each stable curve \( \mathcal{C} \in \overline{\mathcal{M}}_{g,k} \) has a dual graph \( \Gamma \) and corresponds to a combinatorial type \((\Gamma, g_v, o)\).

There is a partial order \( \succ \) on \( \text{Comb}(g, k) \) defined as follows. Let \((\Gamma_v, g_v, o_v) \in \text{Comb}(g, k)\). We consider \((\Gamma_v, g_{v,v}, o_v) \) for some vertices \( v = v_1, \cdots, v_s \) of \( \Gamma \). Now we replace the vertex \( v \) by the graph \( \Gamma_v \), and join the edges containing \( v \) to \( o_v(j) \); then we obtain a new graph \( \tilde{\Gamma} \). \( \tilde{\Gamma} \) is determined by \( g_v \) and \( g_{v,v} \) in a natural way. If \( o(j) \neq v_i \), \( i = 1, \cdots, a \), let \( \tilde{o}(j) = o(j) \); if \( o(j) = v_i \), then the \( j \)th marked point corresponds to some \( j' \in \{1, \cdots, k_{v_i}\} \), and we let \( \tilde{o}(j) = o_v(j') \). So we get an element \((\tilde{\Gamma}, \tilde{g}_v, \tilde{o}) \in \text{Comb}(g, k)\). If \((\Gamma, \tilde{g}_v, \tilde{o}) \) is obtained in this way from \((\Gamma, g_v, o)\), we define \((\Gamma, g_v, o) \succ (\tilde{\Gamma}, \tilde{g}_v, \tilde{o})\). There is a minimal element in \( \text{Comb}(g, k) \) which satisfies, for any vertex \( v \), \( g_v = 0 \) and \( k_v = 3 \).

Definition 2.1.1 (Cylindrical metric). We can view a disc neighborhood of a nodal or marked point as a half-infinite cylinder \( S^1 \times [T_0, \infty) \) for some \( T_0 > 0 \) plus the infinity point \( z \). We say that a metric is cylindrical near a neighborhood of a nodal or marked point if we take the flat metric \( dw \otimes d\bar{w} \) on the cylinder \( S^1 \times [T_0, \infty) \), where \( w = s + i\theta \) is the cylindrical coordinate.

Remark 2.1.2. Other uniform metrics near the marked points can also be chosen; for instance, one can choose the smooth metric or hyperbolic metric. See the discussion in [CL].

The moduli stack \( \overline{\mathcal{M}}_{g,k} \) has the following structure:

**Proposition 2.1.3.** Let \( \mathcal{M}_{g,k}(\Gamma) \) be the set of all stable curves having combinatorial type \((\Gamma, g_v, o)\). Then

- For each marked point \( z_v \), there is an orbifold disc bundle \( D_v \rightarrow \overline{\mathcal{M}}_{g,k} \) such that the fiber at \( \Sigma \) is a disc neighborhood of \( z_v \). Furthermore, there is a smooth family of metrics on \( D_v \) parameterized by \( \Sigma \in \overline{\mathcal{M}}_{g,k} \) such that in a uniformly small neighborhood of the zero point the metric is a cylindrical metric.
- \( \overline{\mathcal{M}}_{g,k} \) is a compact complex orbifold of (complex) dimension \( 3g - 3 + k \) which admits a stratification with finitely many strata, and each stratum is of the form \( \mathcal{M}_{g,k}(\Gamma) \). There is a minimal stratum containing only one point \((\mathcal{C}, z)\).
- There is a fiber bundle \( \mathcal{C}_{g,k}(\Gamma) \rightarrow \mathcal{M}_{g,k}(\Gamma) \) which has the following property. For each \( x = (\mathcal{C}_x, z_x) \in \mathcal{M}_{g,k}(\Gamma) \), there is a neighborhood of \( x \) in \( \mathcal{M}_{g,k}(\Gamma) \) of the form \( U_x = V_x/G_x \) where \( G_x = \text{Aut}(\mathcal{C}_x, z_x) \) such that the inverse image of \( U_x \) in \( \mathcal{M}_{g,k}(\Gamma) \) is diffeomorphic to \( V_x \times G_x \). There is a complex structure on each fiber such that the fiber of \( y = (\mathcal{C}_y, z_y) \) is identified with \((\mathcal{C}_x, z_x)\) itself, together with a Kähler metric \( \mu_y \) which is cylindrical in a neighborhood of the nodal points and varies smoothly in \( y \).
- \( \mathcal{M}_{g,k}(\Gamma) \) is contained in the compactification of \( \mathcal{M}_{g,k}(\Gamma) \) in \( \overline{\mathcal{M}}_{g,k} \) only if \((\Gamma, g_v, o) \succ (\Gamma', g_v', o')\).
- Different strata are patched together in a way which is described in the following local model of a neighborhood of a stable curve in \( \overline{\mathcal{M}}_{g,k} \). A neighborhood of
2.2. The structure of the moduli space \( \overline{\mathcal{M}}_{g,k}^{\text{rig}}(\gamma) \). The structure of \( \overline{\mathcal{M}}_{g,k}^{\text{rig}} \) can be described in the same way as that of \( \overline{\mathcal{M}}_{g,k} \). However, there are some new ingredients from the rigidified W-structures.

**Remark 2.2.1** (Dual graph with group element decoration). As shown in [FJR1], the dual graph describing a stratum of \( \overline{\mathcal{M}}_{g,k}^{\text{rig}} \) has an additional decoration \( \gamma \in G \) at each tail and a pair of decorations \( (\gamma, \gamma^{-1}) \) at each internal edge. We use the same \( \Gamma \) to denote the original dual graph together with the additional decoration and \( \overline{\mathcal{M}}^{\text{rig}}(\Gamma) \) to denote the space of rigidified W-structures whose combinatorial type is described by \( \Gamma \). Whether \( \Gamma \) contains the additional decoration should be clear from the context.

**Remark 2.2.2** (Canonical trivialization on cylindrical coordinate). Given a marked point \( z_j \), a rigidification defines a basis \( e_i \) of \( \mathcal{L}_i \) corresponding to a Ramond variable. Hence, it defines a canonical trivialization of \( \mathcal{L}_i \). We can extend this canonical trivialization over the cylindrical coordinate at \( z_j \). For \( \mathcal{L}_i \) corresponding to an NS-variable, \( \mathcal{L}_i \) has nontrivial orbifold structure at \( z_j \). A rigidification defines a canonical trivialization over the cylindrical coordinate in the orbifold sense.

**Remark 2.2.3** (Gluing rigidified W-structure). Suppose that \( \Sigma = \Sigma_1 \cup \Sigma_2 \) is a nodal curve obtained by gluing the marked point \( p \in \Sigma_1, q \in \Sigma_2 \). Given rigidified W-structures \( (\mathcal{L}_1^i, \psi_1^i), (\mathcal{L}_2^i, \psi_2^i) \) over \( \Sigma_1, \Sigma_2 \), there is a canonical way to glue them to get a rigidified W-structure on \( \Sigma \) with an additional rigidification at the node. The rigidified W-structure on \( \Sigma \) is obtained by forgetting the rigidification at the node. Alternatively, a rigidified W-structure of \( \Sigma \) is obtained by an isomorphism \( I : \mathcal{L}_1^i|_p \cong \mathcal{L}_2^i|_q \) preserving the W-structure while forgetting the rigidifications at \( p, q \).

**Proposition 2.2.4.** \( \overline{\mathcal{M}}_{g,k}^{\text{rig}} \) is a compact complex orbifold which is a finite cover of \( \overline{\mathcal{M}}_{g,k} \) and admits a stratification with finitely many strata, and each stratum is of the form \( \overline{\mathcal{M}}_{g,k}^{\text{rig}}(\Gamma) \).
• There is a trivialization of the restriction of $\mathcal{L}_i$ to each fiber of $D_l$. Furthermore, the trivialization can be chosen smoothly depending on $\Sigma$.

• There are two bundles (universal families). The first one is $\mathcal{C}(\Gamma) \to \overline{\mathcal{M}}^{\text{rig}}(\Gamma)$ which is the pull-back of the universal family of $\overline{\mathcal{M}}(\Gamma)$. Then there is a collection of (orbifold) universal line bundle, $\overline{\mathcal{M}}(\Gamma) \to \mathcal{C}(\Gamma)$ whose fiber is $\mathcal{L}_i$.

• $\overline{\mathcal{M}}^{\text{rig}}(\Gamma')$ is contained in the compactification of $\overline{\mathcal{M}}^{\text{rig}}(\Gamma)$ in $\overline{\mathcal{M}}^{\text{rig}}_{g,k}$ only if $\Gamma > \Gamma'$.

• Different strata are patched together in a way which is described in the following local model of a neighborhood of a nodal rigidified W-structure in $\overline{\mathcal{M}}^{\text{rig}}_{g,k}$. Let $x = (\mathcal{C}, z)$ be the underlying nodal stable Riemann surface. Recall that a neighborhood of $x = (\mathcal{C}, z)$ in $\overline{\mathcal{M}}_{g,k}$ is parameterized by

$$V_x \times (\oplus_i ([T_0, \infty] \times S^1)) \times \text{Aut}(\mathcal{C}, z),$$

where $\mathcal{C} = \pi_1(\mathcal{C}_x) = \pi_0(\mathcal{C}_x)$. An element $(\mathcal{C}, z)$ in a neighborhood of $(\mathcal{C}_x, z_x)$ is obtained by gluing cylindrical neighborhoods of corresponding marked points on each component. Now, the gluing of $\mathcal{L}_i$ is obtained as follows. Recall that a rigidified W-structure on $\mathcal{C}_x$ is obtained by identifying the fiber of $\mathcal{L}_i$ at the corresponding marked point and forgetting the corresponding rigidifications. Then, the canonical trivialization of $\mathcal{L}_i$ near the marked point extends the identification at the marked point to the cylindrical neighborhoods which glue to a rigidified W-structure on $(\mathcal{C}, z)$.

3. The Witten equation

We have laid down the algebraic-geometric foundations in the last several sections. Now we turn our attention to the analytic aspects of our theory. Even though we formulate our axioms in algebraic-geometric terms, the analytic aspect of the theory is at the heart of our construction. It remains a challenging problem to have a completely algebraic treatment of our theory.

Roughly speaking, the moduli space of W-structures $\overline{\mathcal{M}}_{g,k}^{\text{rig}}$ plays the role of Deligne-Mumford moduli space in ordinary Gromov-Witten theory. To construct the moduli space of stable maps, we use the solution of the Cauchy-Riemann equation $\overline{\partial}_J f = 0$ where $f$ is a map from a Riemann surface. In our theory, we replace $f$ by a section $s_l$ of $\mathcal{L}_i$ or equivalently $|\mathcal{L}|$. The replacement of the Cauchy-Riemann equation is the Witten equation, which we will describe in this section.

3.1. Perturbed Witten equation.

3.1.1. Cylindrical metric and Witten-equation. In this section, we fix a rigidified W-structure $(\mathcal{L}_1, \cdots, \mathcal{L}_N, \varphi_1, \cdots, \varphi_N, \psi_1, \cdots, \psi_N)$. Let $\gamma_1 \in G$ be the group element generating the orbifold structure at the marked point $z_{\ell}$. Recall that

$$W_l(|\mathcal{L}_1|, \ldots, |\mathcal{L}_N|) \equiv K_{l,0} \otimes \mathcal{O}(\sum_{l=1}^{k} \sum_{j=1}^{N} b_j \Theta_j^\nu z_{\ell})$$

Let $D = -\sum_{l=1}^{k} \sum_{j=1}^{N} b_j (\Theta_j^\nu - q_j)z_{\ell}$ be a divisor; then there is a canonical meromorphic section $s_0$ with divisor $D$. This section provides the identification

$$K_{\Sigma} \otimes \mathcal{O}(D) \cong K_{\Sigma}(D),$$

where $K_{\Sigma}$ is the canonical bundle of $\Sigma$.
where $K_2(D)$ is the sheaf of local possibly meromorphic sections of $K_2$ with zero or pole at $D$.

A rigidification $\varphi_i$ at $z_i$ actually gives a local trivialization of the orbifold line bundle $\Theta_i \otimes \mathcal{L}_i$, or equivalently determines a basis set $\eta_1, \ldots, \eta_N$. Taking a good coordinate system $\{z\}$ near $z_i$, we can assume that the holomorphic basis $e_i$ of $|\mathcal{L}_i|$ corresponding to $\eta_i$ satisfies the relation providing by $\varphi_i$.

$$W_i(e_1, \ldots, e_N) = \frac{dz}{z} \sum_{i=1}^N b_i \eta_i^{\gamma_i}$$

The relation between $e_i$ and $\eta_i$ will be fully discussed when we define the Witten equation on orbicurves.

Choose the cylindrical metric on the line bundle $K_2(D)$ such that $|dz| = 1$. The above identity together with the nondegeneracy of $W$ imply that the modulus $|e_i| = |e_i^{\gamma_i}|$ near marked points. In particular, if $s_i$ is Ramond variable, then $|e_i| = 1$. Let $u_j = \tilde{u}_j e_j$; then it is easy to see that

$$\overline{\partial W} \in K_{log} \otimes |\mathcal{L}_i|^{-1}.$$  \hspace{1cm} (16)

The bundle $|\mathcal{L}_i|^{-1}$ is isomorphic to $|\mathcal{L}_i|$ topologically. But there is no canonical isomorphism. However, we can choose an isomorphism compatible with the metric. It induces an isomorphism $I_1 : \Omega(\Sigma, |\mathcal{L}_i|^{-1} \otimes \Lambda^{0,1}) \to \Omega(\Sigma, |\mathcal{L}_i| \otimes \Lambda^{0,1})$ such that for a section $v = \tilde{v} e_j$, we have

$$I_1(\tilde{v} e_j) = \tilde{v} |e_j|^{\theta_j} e_j \otimes dz,$$

where $e_j$ is the holomorphic basis of $|\mathcal{L}_i|^{-1}$ such that $e_j \cdot e_j = 1$.

It is obvious that $I_1$ is a metric-preserving isomorphism between the corresponding two spaces and that it is independent of the choice of the local charts.

Since $I_1(\frac{dW}{dz} \otimes |\mathcal{L}_i|)$, the so-called Witten equation is defined below as the first order system for the sections $u_1, \ldots, u_N$:

$$\overline{\partial u}_j + I_1 \left( \frac{\partial W}{\partial u_j} \right) = 0, \quad \text{for all } j = 1, \ldots, N.$$

**Remark 3.1.1.** In this paper, we choose the cylindrical metric near the marked points; in [FJR] we used the smooth metric near marked points, i.e., let $|dz| = 1$ near marked points. Different choices give different equations near marked points. This will produce different theories. We will discuss the relations between the theories in future work.

Let $u_i = \tilde{u}_i e_i$, $\forall i$. Then near a marked point, the Witten equation can be written locally as

$$\overline{\partial u} + \sum_j \frac{\partial W_j(\tilde{u}_1, \ldots, \tilde{u}_N)}{\partial \tilde{u}_i} \overline{\partial u}_i = 0,$$

for $i = 1, \ldots, N$.  \hspace{1cm} (17)

**Note:** For simplicity, we often drop "---" from $\tilde{u}_i$ when discussing the local equation near the marked point. The reader can easily distinguish from the context when we are talking about the sections or the coordinate functions.

Near $z_i$, $W$ has the decomposition $W = W_1 + W_N$. Without loss of generality, we can assume the fixed point set $\mathbb{C}^N_{x_N} = \{(x_1, \ldots, x_N, 0, \ldots, 0)\}$. Then, the line bundles $\mathcal{L}_i$ for
$i \leq N_l$ have no orbifold structure at $z_i$ and $\Theta_i^{\gamma} = 0$. Now we drop the subscript $l$ without any confusion. (17) can be rewritten as

$$\frac{\partial u_i}{\partial z} + \frac{\partial W_i}{\partial u_i} + \sum_{W_j = W_j(u)} \frac{\partial W_{ij}(u_1, \ldots, W_{N_j})}{\partial u_i} z^{\sum_{i_j} b_i(\theta_{ij})} = 0,$$

(18)

for any $0 \leq i \leq N_i$, and

$$\frac{\partial u_i}{\partial z} + \left( \sum_{W_j = W_j(u)} \frac{\partial W_{ij}(u_1, \ldots, W_{N_j})}{\partial u_i} \right) z^{\sum_{i_j} b_i(\theta_{ij})} |z|^{-2\theta_i} = 0,$$

(19)

for any $i \geq N_i$.

**Example 3.1.2** ($D_n$-equation).

In this case, the quasi-homogeneous polynomial is $W(x, y) = x^n + xy^2$, so $W_1 = x^n$, $W_2 = xy^2$, and $b_{11} = n, b_{12} = 0; b_{21} = 1, b_{22} = 2$. The fractional degree is $(q_1, q_2) = (\frac{n-1}{2}, \frac{1}{2})$. $G$ is generated by $J^{-1} = (e^{2\pi i \frac{n-1}{2}}, e^{2\pi i \frac{1}{2}})$. There are some cases near a marked point:

Case 1. $\gamma = (1, 1)$. We have $\Theta_1^\gamma = \Theta_2^\gamma = 0$ and the equation becomes

$$\begin{cases}
\frac{\partial u_1}{\partial z} + nu_1^{n-1} + u_2^2 = 0 \\
\frac{\partial u_2}{\partial z} + 2u_1u_2 = 0
\end{cases}$$

(20)

Case 2. $\gamma = J^{-n}$ and $n$ is even. Then $\Theta_1^\gamma = 0$ and $\Theta_2^\gamma = \frac{1}{2} > 0$. In this case the Witten equation is

$$\begin{cases}
\frac{\partial u_1}{\partial z} + nu_1^{n-1} + u_2^2 = 0 \\
\frac{\partial u_2}{\partial z} + 2u_1u_2|z|^{-1} = 0
\end{cases}$$

(21)

Case 4. $\gamma \neq (1, 1), J^{-n}$, where $n$ is odd. This is the NS-case and $\Theta_1(\gamma) > 0, \Theta_2^\gamma > 0$, the equation is

$$\begin{cases}
\frac{\partial u_1}{\partial z} + (nu_1^{n-1} + u_2^2|z|^{-2\theta_1}) |z|^{-2\theta_1} = 0 \\
\frac{\partial u_2}{\partial z} + 2u_1u_2|z|^{-2\theta_2} = 0
\end{cases}$$

(22)

3.1.2. Perturbed Witten equation. When a Ramond variable is present, the Witten Lemma fails and there are nontrivial solutions to the Witten equation. Then, we have to study the moduli space of solutions to the Witten equation. The first step is to study the asymptotic behavior of the solution at the marked points. The moduli problem makes sense only if the solution has nice asymptotic behavior. From the analytic point of view, the Witten equation is highly degenerate in the same way that zero is a highly degenerate critical point of $W$. There is a sophisticated $L^2$-moduli space theory by Morgan-Mrowka-Ruberman [MMR] and Taubes [13] in the literature to deal with the situation being considered. However, we choose to perturb the equation, which simplifies the analysis immensely. An unexpected bonus of our approach is the appearance of vanishing cycles. Of course, we have to pay a price by studying the dependence of our invariants on the perturbation.

Our strategy is to modify the Witten equation for the Ramond variable without changing the equation for the NS variable. We only need to choose linear perturbations.

**Definition 3.1.3.** Let $W$ be a quasi-homogeneous polynomial. A linear polynomial $W_0 = \sum b_i x_i$ is called $W$-regular if $W + W_0$ has only nondegenerate critical points. Namely,
$W + W_0$ is a holomorphic Morse function. $W_0$ is called strongly $W$-regular if $W_0$ is regular and for any two different critical points $k' \neq k$, $\text{Im}(W + W_0)(k') \neq \text{Im}(W + W_0)(k')$.

**Theorem 3.1.4.** Given a quasi-homogeneous polynomial $W$. The set of $W$-regular $W_0$ forms a non-empty path connected open submanifold of all $W_0$. The subset of $W$-regular $W_0$ where two or more critical points have the same $\text{Im}(W + W_0)$ is a union of real hypersurfaces separating the set of $W$-regular $W_0$ into a system of chambers.

**Proof.** Let

$$X = \{(x_1, \ldots, x_N; b_1, \ldots, b_N); \frac{\partial W}{\partial x_1} + b_1 = 0, \det\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right) = 0\} \subset \mathbb{C}^N \times \mathbb{C}^N.$$

We claim that $X$ is an affine variety of complex dimension $N - 1$. It is clear that the set of regular $W_0$ is $\mathbb{C}^N - \pi_2(X)$, where $\pi_2 : \mathbb{C}^N \times \mathbb{C}^N \rightarrow \mathbb{C}^N$ is defined by $(x_1, \ldots, x_N, b_1, \ldots, b_N) \rightarrow (b_1, \ldots, b_N)$. Since $\pi_2$ is an algebraic map, $\pi_2(X)$ is an algebraic subset of dimension $\leq N - 1$. Therefore, $\mathbb{C}^N - \pi_2(X)$ is an open connected submanifold.

Let

$$F = (\frac{\partial W}{\partial x_1} + b_1, \ldots, \frac{\partial W}{\partial x_N} + b_N) : \mathbb{C}^N \times \mathbb{C}^N \rightarrow \mathbb{C}^N.$$

Let $\tilde{x}_i, \tilde{b}_j$ be the corresponding tangent vectors. Then,

$$DF_{\tilde{x}, \tilde{b}}(\tilde{x}_1, \ldots, \tilde{x}_N, \tilde{b}_1, \ldots, \tilde{b}_N) = \left(\sum_j \frac{\partial^2 W}{\partial x_i \partial x_j} \tilde{x}_j + \tilde{b}_1, \ldots, \sum_j \frac{\partial^2 W}{\partial x_N \partial x_j} \tilde{x}_j + \tilde{b}_N\right)$$

is obviously surjective. Hence, $F^{-1}(0)$ is a smooth affine variety. $X = F^{-1}(0) \cap \det(\frac{\partial^2 W}{\partial x_i \partial x_j})^{-1}(0)$ is a hypersurface of $F^{-1}(0)$ of dimension $N - 1$ unless $\det(\frac{\partial^2 W}{\partial x_i \partial x_j})$ vanishes on some component of $F^{-1}(0)$. On the other hand, $(x_1, \ldots, x_N)$ is a nondegenerate critical points iff $(x_1, \ldots, x_N)$ is a regular value of the projection map $\pi_1 : F^{-1}(0) \rightarrow \mathbb{C}'$. Hence, the set of nondegenerate critical points is dense and $\det(\frac{\partial^2 W}{\partial x_i \partial x_j})$ can not vanish on any open subset of $F^{-1}(0)$.

It is known that the cardinality of the set of nondegenerate critical points is precisely the dimension of the middle homology of the Milnor fiber and hence a constant number. The manifold of nondegenerate critical points $Y = F^{-1}(0) - \text{det}(\frac{\partial^2 W}{\partial x_i \partial x_j})^{-1}(0)$ is a disjoint union of components $Y = \bigcup_{1 \leq i \leq \mu} Y_i$. Furthermore, each component $Y_i$ is isomorphic to $\mathbb{C}^N - \pi_2(X)$. Let

$$f_i : \mathbb{C}^N - \pi_2(X) \rightarrow Y_i \rightarrow \text{Im}(W + W_0) \rightarrow \mathbb{R}.$$ 

Note that $D\text{Im}(W + W_0)(\tilde{x}_1, \ldots, \tilde{x}_N) = 0$ at critical points.

$$D\text{Im}(W + W_0)(\tilde{b}_1, \ldots, \tilde{b}_N) = \text{Im}(x_1 \tilde{b}_1 \ldots, x_N \tilde{b}_N).$$

Let $(x'_1, \ldots, x'_N, b_1, \ldots, b_N) \in Y_i$,

$$D(f_i - f_j)(\tilde{b}_1, \ldots, \tilde{b}_N) = \text{Im}(x'_1 - x'_j) \tilde{b}_1 \ldots, (x'_N - x'_j) \tilde{b}_N)$$

is surjective for $i \neq j$. Therefore, zero is a regular value of $f_i - f_j$ and $(f_i - f_j)^{-1}(0)$ is a smooth real hypersurface. We have finished the proof.

\[\square\]
Suppose that $\gamma$ parameterizes the orbifold structure at the marked point. The quasi-homogeneous polynomial can be decomposed into the sum of the polynomials $W_\gamma$ and $W_N$. Take a fixed $W_\gamma$-regular polynomial
\[ W_0 = W_0(x_1, \cdots, x_N) = \sum_{j=1}^{N_i} b_j x_j. \]

There exist line bundles $O_j(l_i)$, $j = 1, \cdots, N_i$, over the orbicurve $\tilde{\Sigma}$ such that

1. $O_j(l_i)$ has no orbifold structure near the marked point $z_i$;
2. the following isomorphisms hold:
\[ O_j(l_i) \otimes K_j \cong K_{\log}. \]

Hence, near the marked point $z_i$ we have the local isomorphisms
\[ |O_j(l_i)| \otimes |K_j| \cong K_{\log}. \]

If $j \leq N_i$, we define the section $\beta_j = \beta_r(z) dz/z \otimes e_j^* \in \Omega^0(|O_j|)$ and $\beta_r(z)$ is given by
\[ \beta_r(z) = \begin{cases} 1 & \text{if } |z| \leq e^{-T-1} \\ 0 & \text{if } |z| \geq e^{-T}. \end{cases} \]

Let
\[ W_{0,\beta}(u_1, \cdots, u_N) = \sum_{j=1}^{N_i} b_j \beta_j u_j. \]

Hence the global perturbed Witten equation is defined as
\[ \partial_u + l_1 \left( \frac{\partial W}{\partial u_i} + \frac{\partial W_{0,\beta}}{\partial u_i} \right) = 0, \quad \text{for all } i = 1, \cdots, N. \]

Locally near a marked point $z_i$, the perturbed Witten-equation has the following form:
\[ \frac{\partial}{\partial \bar{z}} + \frac{\partial W}{\partial u_i} + \beta_r \frac{\partial W_0}{\partial u_i} + \sum_{j: W_j \in W_N} \frac{\partial W_j(u_1, \cdots, u_N)}{\partial u_i} \gamma_{\nu, i}^{\nu, j} b_{ij}(\theta^j_\nu) = 0, \]

for any $i \leq N_i$, and
\[ \frac{\partial}{\partial \bar{z}} \left( \sum_{j: W_j \in W_N} \frac{\partial W_j(u_1, \cdots, u_N)}{\partial u_i} \gamma_{\nu, i}^{\nu, j} b_{ij}(\theta^j_\nu) \right) |z|^{-2\nu} = 0, \]

for any $i > N_i$. Here $u_i, \beta_r$ in the equations are understood not as sections but as the coordinate functions with respect to each basis of the line bundles.

If we set $v_i = u_i$, for all $i \leq N_i$ and $v_i = u_i e^{\theta^j_\nu}$, then the above two equations can be changed to a more symmetric form:
\[ \frac{\partial}{\partial \bar{z}} v_i + \frac{\partial (W + W_{0,\beta})}{\partial v_i} + \frac{\partial W_N(v_1, \cdots, v_N)}{\partial v_i} = 0, \quad \forall i \leq N_i \]

and
\[ \frac{\partial}{\partial \bar{z}} v_i + \frac{\partial W_N(v_1, \cdots, v_N)}{\partial v_i} = 0, \quad \forall i > N_i, \]

or simply written as one equation
\[ \frac{\partial}{\partial \bar{z}} v_i + \frac{\partial (W + W_{0,\beta})}{\partial v_i} = 0, \quad i = 1, \cdots, N. \]
Hence the equations (27), (28) and (29) only hold in any fan-shaped domain of the absolute value \(|z_i|\), i.e.,
\[
\begin{cases}
\frac{\partial v_i}{\partial s} + \sqrt{-1} \frac{\partial v_i}{\partial \theta} - 2 \frac{\partial (W + W_{0,\theta})}{\partial v_i} = 0, \quad \forall i \leq N_l \\
\frac{\partial v_i}{\partial s} + \sqrt{-1} \frac{\partial v_i}{\partial \theta} - 2 \frac{\partial W_N}{\partial v_i} = 0, \quad \forall i > N_l
\end{cases}
\]
(30) (31)

Set \(v_i = x_i + \sqrt{-1}y_i\). Let

\[H_{R0} := 2\text{Re}(W_N + W_{0,\theta}), \quad H_N := 2\text{Re}(W_N),\]

then the equation (31) becomes
\[
\frac{\partial X_i}{\partial s} + \sqrt{-1} \frac{\partial Y_i}{\partial s} + \sqrt{-1}(\frac{\partial X_i}{\partial \theta} + \sqrt{-1} \frac{\partial Y_i}{\partial \theta}) - \frac{\partial}{\partial x_i} (H_{R0} + H_N) = 0,
\]
i.e.,
\[
\begin{cases}
\frac{\partial X_i}{\partial s} - \frac{\partial Y_i}{\partial \theta} - \frac{\partial}{\partial x_i} (H_{R0} + H_N) = 0 \\
\frac{\partial Y_i}{\partial s} + \frac{\partial X_i}{\partial \theta} - \frac{\partial}{\partial y_i} (H_{R0} + H_N) = 0, \quad \forall i \leq N_l.
\end{cases}
\]
(32)

If we define \(v_R = (x_1, \cdots, x_{N_l}, y_1, \cdots, y_{N_l})\), and let

\[J_R = \begin{pmatrix} 0 & -I_{N_l} \\ I_{N_l} & 0 \end{pmatrix},\]

then the above equation is just

\[
\frac{\partial v_R}{\partial s} + J_R \frac{\partial v_R}{\partial \theta} - \nabla_R H_{R0} = \nabla_R H_N,
\]
(33)

where \(\nabla_R\) is the gradient operator with respect to the variables \((x_1, \cdots, x_{N_l}, y_1, \cdots, y_{N_l})\).

Similarly, if we set \(v_N = (x_{N_l+1}, \cdots, x_N, y_{N_l+1}, \cdots, y_N)\) and

\[J_N = \begin{pmatrix} 0 & -I_{N-N_l} \\ I_{N-N_l} & 0 \end{pmatrix},\]

then the equation (31) becomes

\[
\frac{\partial v_N}{\partial s} + J_N \frac{\partial v_N}{\partial \theta} = \nabla_N H_N,
\]
(34)

where \(\nabla_N\) is the gradient operator with respect to the variables \((x_{N_l+1}, \cdots, x_N, y_{N_l+1}, \cdots, y_N)\).

Thus the equations (30) and (31) are changed to

\[
\begin{cases}
\frac{\partial v_R}{\partial s} + J_R \frac{\partial v_R}{\partial \theta} - \nabla_R H_{R0}(v_R) = \nabla H_N(v_R, v_N) \\
\frac{\partial v_N}{\partial s} + J_N \frac{\partial v_N}{\partial \theta} = \nabla_N H_N(v_R, v_N)
\end{cases}
\]
(35)

Later we will show that the first equation of (35) is just the perturbation equation of the trajectory equation of a Hamiltonian system which has frequently appeared in symplectic geometry, and the second one is the perturbation equation of the Cauchy-Riemann equation of pseudo-holomorphic curves.
**Definition 3.1.5.** Sections \((u_1, \ldots, u_N)\) are said to be the solutions of the perturbed Witten equation (24); if they satisfy the following conditions:

1. for each \(i, u_j \in L^2_{1,\text{loc}}(\Sigma \setminus \{z_1, \ldots, z_k\}, |\mathcal{L}|), t_i \left( \frac{\partial W}{\partial u_j} + \frac{\partial u_j}{\partial t_i} \right) \in \{ L^2_{1,\text{loc}}(\Sigma \setminus \{z_1, \ldots, z_k\}, |\mathcal{L}|) \otimes \Lambda^{0,1} \};
2. \((u_1, \ldots, u_N)\) satisfy the perturbed Witten equation (24) almost everywhere;
3. near each marked point, the integral
   \[
   \sum_j \int_0^\infty \int_{S^1} \left| \frac{\partial u_j}{\partial s} \right|^2 ds < \infty.
   \]

3.1.3. Witten equations over orbicurves. In the last section, we have defined the Witten equation on a smooth Riemann surface. However, since our moduli theory will be constructed on orbicurves, actually we should define our Witten equation on orbifolds. In this part, we will define the Witten equation over orbifolds and will show that the resolution of these solutions of the Witten equation just satisfy the previously defined Witten equation on a Riemann surface and vice versa.

Because \(W(|\mathcal{L}|, \ldots, |\mathcal{L}|) \equiv K_{\text{orb}}\), we can choose a holomorphic basis \(\eta_i\) of the orbifold line bundle \(\mathcal{L}_i\) such that \(W_j(\eta_1, \ldots, \eta_N) = dz\) when away from the marked points and take the smooth metric in this part. If \(z_i\) is a marked point, we can assume that the orbifold structure near \(z_i\) is given by \((\tilde{\Delta}, \pi, G_{z_i} = \mathbb{Z}/m)\), where \(\tilde{\Delta}\) is a disc with coordinate \(z\), and \(\tilde{\Delta} \to \Delta\) is given by \(\pi(z) = z^m\), and \(G_{z_i}\) acts by \(z \to \exp(2\pi i/m)z\). The orbifold structure of \(\mathcal{L}_i\) is given by the uniformizing system \(\tilde{\Delta} \times \mathbb{C} \to [(\tilde{\Delta} \times \mathbb{C})/G_{z_i}]\) with the action of \(G_{z_i}\) given by \((z, w) \to (\exp(2\pi i/m)z, \exp(2\pi v_i/m)w)\) with \(m > v_i \geq 0\). Now the sheaf of sections of \(\mathcal{L}_i\) is generated as an \(O_{\tilde{\Delta}}\)-module by an element \(\eta_i\) such that for each \(j\) there is
\[
W_j(\eta_1, \ldots, \eta_N) = mdz/z.
\]

\(G_{z_i}\) acts on \(\eta_i\) by \(\eta_i \to \exp(-2\pi v_i/m)\eta_i\). The invariant sections are of the form \(u_i(z) = z^m \tilde{g}(z^m)\eta_i\). Let \(\tilde{u}_i\) be the coordinate function of \(u_i(z)\) with respect to \(\eta_i\); then \(\tilde{u}_i(z) = z^m \tilde{g}_i(z^m)\) which is equivariant. Let \(w = z^m\) be the coordinate in \(\tilde{\Delta}\). The sheaf of sections of \(|\mathcal{L}_i|\) is generated by the section \(e_i(w) = z^m \eta_i(z)\). So \(u_i = \tilde{g}_i(z^m)e_i(z^m)\) and the corresponding coordinate function is \(\tilde{g}_i(z^m)\) and actually we have the relation between two coordinate functions:
\[
\tilde{u}_i(z) = z^m \tilde{g}_i(z^m) = w^{\Theta_i} \tilde{g}_i(w).
\]

Now we choose the cylindrical metric in the small charts of all of the marked points on the resolved surface, i.e., let \(|dw/w| = 1\). This metric induces the metric on \(\Sigma\) such that \(|mdz/z| = 1\). By (36) this metric induces the metric on each orbifold line bundle such that \(|\eta_i| = 1\), \(\forall i\), in small charts of all marked points. Let \(\eta_i^\dagger\) be the dual basis of \(\eta_i\) on the dual line bundle \(\mathcal{L}_i^{-1}\). We define a metric preserving isomorphism:
\[
I_i : \Omega(\tilde{\Sigma}, \mathcal{L}_i^{-1} \otimes \Lambda^{0,1}) \to \Omega(\tilde{\Sigma}, \mathcal{L}_i \otimes \Lambda^{0,1})
\]
such that for \(v_i = v_i^\dagger\eta_i^\dagger\) there holds
\[
I_i(v_i \otimes d\tilde{z}) = v_i|\eta_i|^2 \eta_i \otimes d\tilde{z}.
\]

It is easy to see if \(u_i \in \Omega(\tilde{\Sigma}, \mathcal{L}_i)\), then
\[
I_i \left( \frac{\partial W}{\partial u_i} \right) \in \Omega(\tilde{\Sigma}, \mathcal{L}_i \otimes \Lambda^{0,1}).
\]
Now the Witten equation on orbifolds is defined as

\[ \bar{\partial} u_i + I \left( \frac{\partial W}{\partial u_i} \right) = 0, \forall i = 1, \ldots, N. \] (38)

We consider the local expression of the Witten equation in a neighborhood of the marked point \( z_i \). Assume that \( u_i = \tilde{u}_i \eta_i, i = 1, \ldots, N \), are solutions of equation (38) in this uniformizing system of \( z_i \); then the coordinate function \( \tilde{u}_i \) satisfies

\[ \frac{\partial \tilde{u}_i}{\partial \bar{z}} + \frac{\partial W(\tilde{u}_1, \ldots, \tilde{u}_N)}{\partial \tilde{u}_i} \frac{m}{z} = 0. \] (39)

Replacing \( \tilde{u}_i \) in (39) by the equality (37), we find out that

\[ \frac{\partial \tilde{g}_i}{\partial \bar{w}} + \sum_j \frac{\partial W_j(\tilde{g}_1, \ldots, \tilde{g}_N)}{\partial \tilde{g}_i} \frac{m}{w} = 0, \] (40)

where \( w = z^m \). This is just equation (17) when evaluated at \( z^m \).

Conversely, if one knows that the function \( \tilde{g}_i \) of \( |z| \) satisfies equation (40), then \( \tilde{u}_i(w) = w^{\xi_i} \tilde{g}_i(w) \) satisfies

\[ \frac{\partial \tilde{u}_i}{\partial \bar{w}} + \frac{\partial W(\tilde{u}_1, \ldots, \tilde{u}_N)}{\partial \tilde{u}_i} \frac{1}{w} = 0. \] (41)

Notice that equation (41) holds locally at Ramond point. On the other hand, the solutions of this Witten equation in the neighborhood of any orbifold point can be viewed as solutions satisfying some twisted periodic boundary condition. Equation (41) is changed to equation (39) when the potential \( W \) is replaced by \( mW \). This observation is beneficial in estimating the solutions of (39).

Similarly we can define the perturbed Witten equation if there exist any Ramond marked points. The definition is completely the same as before, since there is no orbifold structure for these Ramond line bundles.

### 3.2. Interior estimate of solutions.

When away from the marked points, the perturbed Witten equation can be written in the form

\[ \bar{\partial} u_j + \varphi_j \left( \frac{\partial W}{\partial u_j} + \beta_j \frac{\partial W_0}{\partial u_j} \right) = 0, \quad \text{for all } j = 1, \ldots, N. \] (42)

where \( \varphi_j \) is a smooth positive function. Here we understand that for \( i > N_i, \beta_i \equiv 0 \) and for \( i \leq N_i, \beta_i \) is defined as before, the polynomial \( W_0 \) can be chosen differently corresponding to different marked points, but they should satisfy the following uniform control:

\[ \left| \frac{\partial W_0}{\partial u_i} \right| \leq b \] (43)

To study the solutions, we also need

**Lemma 3.2.1 (FJR, Theorem 5.7).** Let \( W \in \mathbb{C}[x_1, \ldots, x_N] \) be a non-degenerate, quasihomogeneous polynomial with weights \( q_i := \text{wt}(x_i) < 1 \) for each variable \( x_i, i = 1, \ldots, N \). Then for any \( t \)-tuple \((u_1, \ldots, u_N) \in \mathbb{C}^N \) we have

\[ |u_i| \leq C \left( \sum_{j=1}^N \left| \frac{\partial W}{\partial x_j}(u_1, \ldots, u_N) \right| + 1 \right)^{\delta_i}, \]
where \( \delta_i = \frac{q_i}{\min_{j}(1-q_j)} \) and the constant \( C \) depends only on \( W \). If \( q_i \leq 1/2 \) for all \( i \in \{1, \ldots, N\} \), then \( \delta_i \leq 1 \) for all \( i \in \{1, \ldots, N\} \). If \( q_i < 1/2 \) for all \( i \in \{1, \ldots, N\} \), then \( \delta_i < 1 \) for all \( i \in \{1, \ldots, N\} \).

So by this lemma we have

\[
|u_i| \leq C \left( \sum_{i=1}^{N} \left| \frac{\partial W}{\partial x_i} (u_1, \ldots, u_N) \right| + 1 \right)^{\delta_0}, \quad \delta_0 = \max\{\delta_1, \ldots, \delta_N\}. \tag{44}
\]

In the rest of this paper, we always assume that \( q_i < 1/2 \) for any \( i \); therefore we have \( \delta_0 < 1 \).

**Theorem 3.2.2.** Let \((u_1, \ldots, u_N)\) be the solutions of the perturbed Witten equation (24); then for any \( m \in \mathbb{Z} \), there holds

\[
||u_j||_{C^m(B_R)} \leq C,
\]

where \( B_R \subset \Sigma \setminus \{z_1, \ldots, z_k\} \), and \( C \) is a constant depending only on the metric in \( B_{2R} \), the fractional degree \( q_i \), \( b \) and \( R \).

**Proof.** Multiplying (42) by \( \partial W/\partial u_i \) and taking the sum, one has

\[
\frac{\partial W}{\partial \bar{z}} + \left( \sum_i \psi_i \left| \partial W/\partial u_i \right|^2 + \sum_i \psi_i \beta_i \frac{\partial W}{\partial u_i} \frac{\partial W}{\partial u_i} \right) = 0 \tag{45}
\]

Take a small ball \( B_{2R}(z_0) \) and a test function \( \psi \) such that \( \psi \in C^\infty_0(B_{2R}(z)), \psi \equiv 1 \) on \( B_R(z) \), and \( \psi \equiv 0 \) outside \( B_{2R}(z) \), and \( |\nabla \psi| \leq \frac{C}{R} \).

Multiplying (45) by \( \psi \beta \sqrt{-1} \frac{\partial W}{\partial W} \) and integrating over \( B \), one has

\[
\int_B \frac{\partial W}{\partial \bar{z}} \psi^\beta \sqrt{-1} \frac{\partial W}{\partial u_i} \frac{\partial W}{\partial u_i} + \int_B \sum_j \phi_j \left| \frac{\partial W}{\partial u_j} \right|^2 |dzd\bar{z}| \psi^\beta \\
+ \int_B \sum_j \phi_j \beta_j \frac{\partial W}{\partial u_j} \frac{\partial W}{\partial u_j} \psi^\beta = 0 \tag{46}
\]

Now there holds

\[
\int_B \frac{\partial W}{\partial \bar{z}} \psi^\beta \sqrt{-1} \frac{\partial W}{\partial u_i} \frac{\partial W}{\partial u_i} = - \int \frac{\sqrt{-1}}{2} \psi^\beta d(Wdz) \\
= \int \frac{\sqrt{-1}}{2} \beta \psi^{\beta-1} d\psi \wedge Wdz = \int \partial_\beta \psi W \psi^{\beta-1}.
\]
By (46) and (43), one obtains for large $\beta$ that

$$
\int_B \psi^\beta \sum_j \varphi_j \left| \frac{\partial W}{\partial u_j} \right|^2 \leq \int_B \beta \psi^{\beta - 1} |\nabla \psi| |W|^{\beta - 1} + \int_B \sum_j \varphi_j \left| \frac{\partial W}{\partial u_j} \right|^2 \psi^\beta
$$

$$
\leq C \int_B \psi^{\beta - 1} |\nabla \psi| \cdot \left( \sum_i \left| \frac{\partial W}{\partial u_i} \right|^{1+\delta_0} \right) + C_2 \int_B |\tilde{\partial} \psi| + C_1 \max_j \varphi_j \int_B \sum_j \left| \frac{\partial W}{\partial u_j} \right|^{1+\delta_0} \psi^\beta + C_1 R^2
$$

Then take $\epsilon$ small enough such that

$$
\int_{B_\epsilon} \sum_j \left| \frac{\partial W}{\partial u_j} \right|^2 \leq C(q, b, \varphi, R) \quad (47)
$$

On the other hand, for the non-homogeneous $\bar{\partial}$-equation we have the interior estimate

$$
\|u_j\|_{L^2(B_\frac{\epsilon}{2})} \leq C \left( \sum_i \left| \frac{\partial W}{\partial u_i} \right|_{L^2(B_\epsilon)} + \sum_i \left| \frac{\partial W}{\partial u_i} \right|_{L^2(B_\epsilon)} + \|u_j\|_{L^2(B_\epsilon)} \right) \quad (48)
$$

By (44), there holds

$$
\|u_j\|_{L^2}^2 \leq C \left( \sum_j \left| \frac{\partial W}{\partial u_j} \right|_{L^2} + 1 \right)^{2\delta_0} \leq C_1 \sum_j \left| \frac{\partial W}{\partial u_j} \right|_{L^2}^2 + C_2,
$$

and hence by (47) we obtain

$$
\|u_j\|_{L^2(B_\frac{\epsilon}{2})} \leq C_1 \sum_j \left| \frac{\partial W}{\partial u_j} \right|_{L^2} + C_2 \leq C
$$

Applying the embedding theorem, there holds

$$
\|u_j\|_{L^p} \leq C \|u_j\|_{L^2} \leq C, \quad \forall 1 < p < \infty. \quad (49)
$$

Hence for any $p > 1$, we have

$$
\left| \frac{\partial W}{\partial u_j} \right|_{L^p}^p \leq C,
$$

where $C$ depends on $\|u_j\|_{L^2}$ and $\left| \frac{\partial W}{\partial u_j} \right|_{L^2}$. 


Now using the $L^p$-estimate of the $\bar{\partial}$-equation, (49) and (50), eventually we have
\[
\|u_i\|_{L^p(B_R)} \leq C \left( \sum_j \left\| \frac{\partial W}{\partial u_j} \right\|_{L^p(B_R)} + \left\| \frac{\partial W_0}{\partial u_j} \right\|_{L^p(B_R)} + \left\| u_j \right\|_{L^p(B_R)} \right) \leq C.
\]
Using embedding theorem again, we have
\[
\|u_j\|_{C^s(B_R)} \leq C.
\]
Furthermore, by the bootstrap argument, we have the following estimate for any $m \in \mathbb{Z}$:
\[
\|u_j\|_{C^m} \leq C.
\]

3.3. Asymptotic behavior.

Set the complex variable $\xi = s + it$; then the equations (31) and (30) can be written as
\[
\bar{\partial}_s v_i - 2i \frac{\partial (W + W_0 \beta)}{\partial v_i} = 0, \quad \forall i = 1, \cdots, N. \tag{51}
\]
Note that the function $v_i$ is only locally defined; hence the above relation only holds in a bounded domain $D \subset S^1 \times [0, \infty)$ such that $D$ is contractible.

**Theorem 3.3.1.** Let $B_R \subset S^1 \times [0, \infty)$ be a ball with radius $R$ such that $B_{2R}$ is contractible. Suppose that $v_1, \cdots, v_N$ satisfy (51) in $B_{2R}$; then for any $m > 0$, there is a constant $C$ depending only on $q, \beta, R, m$ such that
\[
\|v_j\|_{C^m(B_R)} \leq C,
\]
where the derivatives taken in the $C^m$ modulus correspond to the variables $s, \theta$.

**Proof.** Multiplying the two sides of (51) by $\frac{\partial W}{\partial v_i}$ and taking the sum over $i$ from 1 to $N$, we have
\[
\bar{\partial}_s W = 2 \sum_i \left| \frac{\partial W}{\partial v_i} \right|^2 + 2 \sum_i \frac{\partial W_0 \beta}{\partial v_i} \frac{\partial W}{\partial v_i}.
\]
Multiplying the above equality by $\psi \frac{\sqrt{-1}}{2} \partial_s d\xi \wedge d\bar{\xi}$ and integrating over $B_{2R}$, one has
\[
\int_{B_{2R}} \bar{\partial}_s W(\psi \frac{\sqrt{-1}}{2} \partial_s d\xi \wedge d\bar{\xi}) = 2 \int_{B_{2R}} \sum_i \left| \frac{\partial W}{\partial v_i} \right|^2 \psi |\partial_s d\xi d\bar{\xi}| + 2 \int_{B_{2R}} \sum_i \frac{\partial W_0 \beta}{\partial v_i} \frac{\partial W}{\partial v_i} |\psi \partial_s d\xi d\bar{\xi}|.
\]
Using Stokes’ theorem for the $\bar{\partial}$ operator, we have
\[
2 \int \sum_i \left| \frac{\partial W}{\partial v_i} \right|^2 \psi |\partial_s d\xi d\bar{\xi}| = \int_{B_{2R}} \bar{\partial}_s \psi \beta W \psi \beta^{-1} - 2 \int \sum_i \frac{\partial W_0 \beta}{\partial v_i} \frac{\partial W}{\partial v_i} |\psi \partial_s d\xi d\bar{\xi}|.
\]
Now we can proceed in the same way as in the proof of Theorem 3.2.2 to obtain the conclusion. \[\square\]

Define $\hat{v}_i = u_i e^{-\theta \gamma_i}$ for any $i$; then the $\hat{v}_i$ are well-defined functions in the cylinder $S^1 \times [0, \infty)$. We have the relations:
\[
\hat{v}_i = v_i, \quad \forall i \leq N_i;
\]
\[
|\hat{v}_i| = |v_i| = |u_i e^{-\theta \gamma_i}|, \quad \forall i > N_i. \tag{52}
\]
Hence \( \hat{v}_i \) for \( i > N_j \) satisfies

\[
(\partial_s + \sqrt{-1} \partial_{\varphi} + \Theta_i) \hat{v}_i = \frac{\partial W_N(v_1, \ldots, v_N)}{\partial v_i} e^{\Theta_i s + i \varphi}.
\] (54)

**Lemma 3.3.2.** Let \( u_1, \ldots, u_N \) be the solutions of the equations (26) and (25) defined in the cylinder \( S^1 \times [0, \infty) \). Then there holds

\[
\int_0^\infty \int_{S^1} \sum_{i > N_j} \left| \frac{\partial (W_y + W_{0, \varphi})}{\partial u_i} \right|^2 + \sum_{i > N_j} \left| \frac{\partial W_j(u_1, \ldots, u_N)}{\partial u_i} \right|^2 e^{-\sum b_j \Theta_i (s + i \varphi) + \Theta_i s} \, ds \, d\theta < \infty.
\] (55)

So what we need to prove is actually the following estimate:

\[
\int_0^\infty \int_{S^1} \sum_{i > N_j} \left| \frac{\partial (W_y + W_{0, \varphi})}{\partial v_i} \right|^2 + \sum_{i > N_j} \left| \frac{\partial W_N(v_1, \ldots, v_N)}{\partial v_i} \right|^2 \, ds \, d\theta < \infty.
\] (56)

or in equivalent form:

\[
\int_0^\infty \sum_{i = 1}^N \left| \frac{\partial (W + W_{0, \varphi})}{\partial v_i} \right|^2 \, d\theta < \infty.
\] (57)

To prove (57), we multiply the two sides of the equation (51) by \( \frac{\partial (W + W_{0, \varphi})}{\partial v_i} \) and take the sum of \( i \) from 1 to \( N \), then there holds

\[
\partial_s (W + W_{0, \varphi}) = 2 \sum_i \left| \frac{\partial (W + W_{0, \varphi})}{\partial v_i} \right|^2 + \sum_{j=1}^{N_j} b_j v_j \partial_s \beta_T.
\]

Integrating the above equality over \( S^1 \times [0, T_0] \) for \( T_0 > T \), where \( T \) appears in the definition of \( \beta_T \), and noting that \( (W + W_{0, \varphi})(v_1, \ldots, v_N) \) is a well-defined function on the cylinder, one has

\[
\int_0^{T_0} \partial_s \int_{S^1} W + \int_0^{T_0} \sum_{i = 1}^{N_j} b_j \partial_s v_j \beta_T = 2 \int_0^{T_0} \sum_{i = 1}^{N_j} \left| \frac{\partial (W + W_{0, \varphi})}{\partial v_i} \right|^2.
\]
Lemma 3.3.3.\] independent of Proof. 

Since solution has the form 

\[
\begin{align*}
\text{Now we have the integral estimate}
\end{align*}
\]

\[
\text{Then for any } T \in (0, \infty), \text{ there holds}
\]

\[
\int_T^\infty \int_{S^1} |v|^2 \leq \frac{1}{\Theta^2} \int_0^\infty \int_{S^1} |\partial_s v|^2
\]

and

\[
|v(s, \theta)| \leq \sqrt{\frac{2}{\Theta^2} e^{-\Theta s} \left( \int_0^\infty \int_{S^1} |\partial_s v|^2 \right)^{1/2} (1 - e^{-2T})^{-\frac{1}{2}}} \forall s \in [T, \infty)
\]

Proof. Since \(v(s, \theta)\) is a smooth function defined in \(S^1 \times [0, \infty)\), by Fourier analysis, any smooth solution of \(58\) has the following form:

\[
v(s, \theta) = \sum_{n=-\infty}^{\infty} C_n e^{-(\Theta+n) s} \cdot e^{-i n \theta}.
\]

Since \(v\) is assumed to be bounded, hence \(C_n = 0, \forall n < 0\). Thus the bounded smooth solution has the form

\[
v(s, \theta) = \sum_{n=0}^{\infty} C_n e^{-(\Theta+n) s} \cdot e^{-i n \theta}.
\]

Now we have the integral estimate

\[
\int_T^\infty \int_{S^1} |v|^2 = \sum_{n=0}^{\infty} |C_n|^2 \int_T^\infty e^{-2(\Theta+n) s} ds = \frac{1}{2} \sum_{n=0}^{\infty} |C_n|^2 (\Theta + n)^{-1} e^{-2(n+\Theta)T}
\]

\[
\leq \frac{1}{2} \sum_{n=0}^{\infty} |C_n|^2 (\Theta + n)^{-1} e^{-2(n+\Theta)T} \leq \frac{1}{\Theta^2} \int_T^\infty \int_{S^1} |\partial_s v|^2
\]

and the pointwise estimate

\[
|v(s, \theta)| \leq \sum_{n=0}^{\infty} |C_n| e^{-(\Theta+n) s} \leq e^{-\Theta s} \left( \sum_{n=0}^{\infty} |C_n|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} e^{-2nT} \right)^{1/2}
\]

\[
\leq \sqrt{\frac{2}{\Theta^2} e^{-\Theta s} \left( \int_0^\infty \int_{S^1} |\partial_s v|^2 \right)^{1/2} (1 - e^{-2T})^{-\frac{1}{2}}}
\]
Extend the function \( \frac{\partial W_N(v_1, \cdots, v_N)}{\partial v_i} e^{i\theta} \) symmetrically to \((-\infty, \infty)\), and view it as the free term of the following equation for \( v \) defined in \((-\infty, \infty)\):

\[
(\partial_s + \sqrt{-1} \partial_{\theta} + \Theta_i)v = - \frac{\partial W_N(v_1, \cdots, v_N)}{\partial v_i} e^{i\theta}.
\] (63)

This equation has a unique bounded solution in the whole interval \((-\infty, \infty)\) (see [D]). Since the “free term” is \( L^2 \) integrable by Lemma 3.3.2, we have the Fourier expansion

\[
- \frac{\partial W_N(v_1, \cdots, v_N)}{\partial v_i} e^{i\theta} = \sum_n \rho_n(s) e^{-in\theta}.
\]

The unique bounded solution

\[
\hat{v}_{i,0}(s, \theta) = - \sum_{n=0}^{\infty} e^{-(n+\Theta_i)s} \int_s^{\infty} e^{(n+\Theta_i)\tau} \rho_n(\tau) d\tau e^{-in\theta} + \sum_{n=-\infty}^{-1} e^{-(n+\Theta_i)s} \int_s^{-\infty} e^{-(n+\Theta_i)\tau} \rho_n(\tau) d\tau e^{in\theta}.
\]

**Lemma 3.3.4.** Let \( T > 1 \). For any \( i > N_i \), \( \hat{v}_{i,0} \) satisfies the integral estimate

\[
\|\hat{v}_{i,0}\|_{L^2(S^1 \times [T,\infty])} \leq C \left| \left| \frac{\partial W_N}{\partial v_i} \right|_{L^2(S^1 \times [T-1,\infty])} \right|,
\] (64)

where \( C \) is a constant.

**Proof.** By Lemma 3.22 of [D], we have

\[
\|\hat{v}_{i,0}\|_{L^2(S^1 \times [T,\infty])} \leq C \int_T^\infty \int_{S^1} \left| \frac{\partial W_N}{\partial v_i} \right|^2,
\]

where \( C \) depends only on \( \Theta_i^{\gamma} \). Hence by \( L^2 \) interior estimates of the Cauchy-Riemann operator, for any \( 1 < p < \infty \), there holds

\[
\|\hat{v}_{i,0}\|_{L^2(S^1 \times [T,\infty])} \leq C \left( \|\hat{v}_{i,0}\|_{L^2(S^1 \times [T-1,\infty])} + \left| \left| \frac{\partial W_N}{\partial v_i} \right|_{L^2(S^1 \times [T-1,\infty])} \right| \right)
\]

\[
\leq C \left| \left| \frac{\partial W_N}{\partial v_i} \right|_{L^2(S^1 \times [T-1,\infty])} \right|.
\]

\(\Box\)

**Theorem 3.3.5.** For any \( i > N_i \), \( \hat{v}_i(s, \theta) \) and its derivatives of any order tend to zero uniformly as \( s \to +\infty \).

**Proof.** The bounded function \( \hat{v}_i - \hat{v}_{i,0} \) satisfies the homogeneous equation

\[
(\partial_s + \sqrt{-1} \partial_{\theta} + \Theta_i)v = 0.
\]

On the other hand by Lemma 3.3.2 Lemma 3.3.4 and the definition of the solution \( u_i \), we have

\[
\|\partial_s(\hat{v}_i - \hat{v}_{i,0})\|_{L^2(S^1 \times [T,\infty])} \leq \|\partial_s \hat{v}_i\|_{L^2(S^1 \times [T,\infty])} + \|\partial_s \hat{v}_{i,0}\|_{L^2(S^1 \times [T,\infty])}
\]

\[
\leq \|\partial_s \hat{v}_i\|_{L^2(S^1 \times [T-1,\infty])} + C \left| \left| \frac{\partial W_N}{\partial v_i} \right|_{L^2(S^1 \times [T-1,\infty])} \right| \leq \infty
\] (65)

(66)
Hence by Lemma 3.3.3, we obtain
\[
\|\hat{v}_i\|_{L^2(S^1 \times [T, \infty))} \leq \|\hat{v}_{i,0}\|_{L^2(S^1 \times [T, \infty))} + \|\hat{v}_i - \hat{v}_{i,0}\|_{L^2(S^1 \times [T, \infty))}
\]
\[
\leq \frac{1}{\Theta} \|\partial_s(\hat{v}_i - \hat{v}_{i,0})\|_{L^2(S^1 \times [T, \infty))} + \|\hat{v}_{i,0}\|_{L^2(S^1 \times [T, \infty))}
\]
\[
\leq C\|\partial_s\hat{v}_i\|_{L^2(S^1 \times [T-1, \infty))} + C \left\| \frac{\partial W_N}{\partial \nu} \right\|_{L^2(S^1 \times [T-1, \infty))}.
\] (67)

By \(L^2\) estimates and the above inequality, there holds
\[
\|\hat{v}_i\|_{L^2(S^1 \times [T, \infty))} \leq C \left( \|\hat{v}_i\|_{L^2(S^1 \times [T-1, \infty))} + \left\| \frac{\partial W_N}{\partial \nu} \right\|_{L^2(S^1 \times [T-1, \infty))} \right).
\] (68)

Notice that the term on the right hand side tends to zero when \(T \to \infty\).

By the embedding theorem, any \(L^p\) norm of \(\hat{v}_i\) can be controlled by the \(L^2\) norm of \(\hat{v}_i\), therefore the \(L^p\) norm of \(\frac{\partial W_N}{\partial \nu}\) is also controlled. Using \(L^p\) estimates of the Cauchy-Riemann operator and furthermore by the embedding theorem and Schauder estimates, any \(C^m\) norm of \(\hat{v}_i\) can be controlled by the \(L^2\) norm of \(\hat{v}_i\). Therefore any \(C^m\) norm of \(\hat{v}_i\) in \([T, \infty)\) tends to zero as \(T \to \infty\).

\(\square\)

Now we turn to the study of the solution \(u_i = v_i = \hat{v}_i\) for \(i \leq N_1\).

**Lemma 3.3.6.** Suppose that \(u_i, i \leq N_1\) is the solution of (30) and (31); then \(\partial_s u_i(s, \theta), \partial_{\theta} u_i(s, \theta) \to 0\) uniformly for \(\theta \in S^1\) as \(s \to \infty\).

**Proof.** Take the derivative \(\partial_s\) of the equation (30); then we get
\[
\partial_s(\partial_s u_i) + \sqrt{-1} \partial_{\theta} \partial_s u_i - 2 \sum_{j \leq N_1} \frac{\partial^2(W_j + W_{0,j})}{\partial u_i \partial u_j} \partial_s u_j = \partial_s(2 \frac{\partial W_N(v_1, \cdots, v_N)}{\partial v_i})
\] (70)

For simplicity, we set \(X_{ij} = 2 \frac{\partial(W_j + W_{0,j})}{\partial u_i \partial u_j}\), which is a complex symmetric matrix, and set \(L_i(s, \theta) = \partial_s(2 \frac{\partial W_N}{\partial u_i})\). So (70) becomes
\[
\partial_s(\partial_s u_i) + \sqrt{-1} \partial_{\theta} \partial_s u_i - X_{ij}(\partial_s u_j) = L_i.
\] (71)

Here we also omit the summation sign if no confusion arises.

We have
\[
\Delta(\partial_s u_i) = (\partial_s + \sqrt{-1} \partial_{\theta})(X_{ij}(\partial_s u_j)) + (\partial_s - \sqrt{-1} \partial_{\theta})L_i
\]
\[
= X_{ij}(\partial_s u_j) + \sqrt{-1} \partial_{\theta} u_i(\partial_s u_j) + \overline{X}_{ij}(\partial_s u_j) + L_j + (\partial_s - \sqrt{-1} \partial_{\theta})L_i
\]
\[
= X_{ij}(\partial_s u_j) + \sqrt{-1} \partial_{\theta} u_i(\partial_s u_j) + X_{ij}(\partial_s u_j) + L_j + (\partial_s - \sqrt{-1} \partial_{\theta})L_i.
\] (72)

Denote by \(F_i\) the term on the right hand side. We know that \(F_i\) is a bounded term.

We have
\[
\Delta|\partial_s u_i|^2 = 4\Delta(\partial_s u_i \cdot \overline{\partial_s u_i}) + 4\Delta(\overline{\partial_s u_i} \cdot \partial_s u_i) + 4|\partial(\partial_s u_i)|^2 + 4|\overline{\partial(\partial_s u_i)}|^2
\]
\[
\geq 4(F_i \cdot \overline{\partial_s u_i} + F_i \cdot \partial_s u_i).
\] (73)
By the maximum principle, we have
\[
|\langle \partial_s u_i \rangle|^2(s, \theta) \leq C \left( \int_{s-1}^{s+1} \int_{S^1} |\partial_s u_i|^2 + \int_{s-1}^{s+1} \int_{S^1} |F_i|^2 |\partial_s u_i|^2 \right) \tag{74}
\]
\[
\leq C_1 \int_{s-1}^{s+1} \int_{S^1} |\partial_s u_i|^2. \tag{75}
\]
This shows that $\partial_s u_i$ converges to zero uniformly for $\theta \in S^1$. In the same way, one can prove that the conclusion is also true for $\partial_s u_i$. \qed

**Theorem 3.3.7.** Let $u_j, j \leq N_i$ be the solution of (3.3) and (3.1), then there holds $\|u_j(s, \theta) - \kappa_j\|_{C^0(S^1)} \to 0$ as $s \to 0$, where $\kappa = (\kappa_1, \cdots, \kappa_{N_i})$ is one of the critical points of $W_\gamma + W_0$.

**Proof.** Let $s_n$ be any sequence tending to infinity as $n \to \infty$. By Theorem 3.3.1 $\|u_i(s_n, \cdot)\|_{C^0(S^1)}$ is uniformly bounded for any $m \in \mathbb{Z}$. Hence by the Arzelà-Ascoli theorem there is a subsequence $s_{n_j}$ such that $u_i(s_{n_j}, \theta)$ converges uniformly in $C^{m-1}$ to a limit function $u^\omega_i(\theta)$. $u_i(s_{n_j}, \theta)$ satisfies the following relation:
\[
\partial_s u_i(s_{n_j}, \theta) + (\sqrt{1 - \partial_s u_i(s_{n_j}, \theta)} - 2 \frac{\partial(W_\gamma + W_0)}{\partial u_i}) = 2 \frac{\partial W^\gamma(v_1, \cdots, v_N)}{\partial v_i}. \tag{76}
\]
By Theorem 3.3.5 and Lemma 3.3.6 we know that
\[
\partial_s u_i(s_{n_j}) \to 0, \quad \left| \frac{\partial(W_\gamma + W_0)}{\partial v_i} \right| \to 0, \quad \text{as } n \to \infty.
\]
So let $k \to \infty$ in the above equation; we can obtain the limit equation
\[
\sqrt{1 - \partial_s u_i^\omega} - 2 \frac{\partial(W_\gamma + W_0)(u^\omega_1, \cdots, u^\omega_N)}{\partial u_i} = 0, \quad \forall i \leq N_i. \tag{77}
\]
Multiply $\frac{\partial(W_\gamma + W_0)}{\partial u_i}$ on the two sides of (76) and take the sum; then there holds
\[
\sum_{i \leq N_i} \sqrt{1 - \partial_s u_i^\omega} \frac{\partial(W_\gamma + W_0)}{\partial v_i} = \sum_{i \leq N_i} 2 \left| \frac{\partial(W_\gamma + W_0)}{\partial u_i} \right|^2. \tag{78}
\]
Integrating, we obtain
\[
\int_{S^1} \sum_{i \leq N_i} 2 \left| \frac{\partial(W_\gamma + W_0)}{\partial u_i} \right|^2 \, d\theta = 0,
\]
and so
\[
\sqrt{1 - \partial_s u_i^\omega} = \frac{\partial(W_\gamma + W_0)}{\partial u_i} = 0, \quad \forall i \leq N_i.
\]
Thus the solutions $u^\omega_i = \kappa_i, i \leq N_i$ are constants and $(\kappa_1, \cdots, \kappa_{N_i})$ is just one of the critical points of the polynomial $W_\gamma + W_0$. Notice that $W_\gamma + W_0$ is a holomorphic Morse function and has finitely many critical points, and so the values that $u^\omega_i$ can attain are also finitely many. Assume that $\kappa^1, \cdots, \kappa^m$ are different critical points. Define
\[
r = \min_{l \leq k} |\kappa^l - \kappa^i|.
\]
Now if we assume that the conclusion of this theorem does not hold, then there is a sequence $s_n \to \infty$ such that $u_i(s_n, \theta)$ does not converge. Hence there are at least two subsequences $s_{n_k}$ and $s_{n_k}'$ tending to infinity such that
\[
\|u(s_{n_k}, \theta) - \kappa_i\|_{C^0} \to 0, \quad \|u(s_{n_k}', \theta) - \kappa^j\|_{C^0} \to 0, \quad \text{as } k, k' \to \infty.
\]
where \( k' \neq k'' \), and we set \( u = (u_1, \ldots, u_N) \). In particular, for sufficiently large \( k, k' \), we have
\[
| \int_{S^1} u(s_{n_1}, \theta) d\theta - k' | \leq \frac{r}{4}, \quad | \int_{S^1} u(s_{n_2}, \theta) d\theta - k'' | \leq \frac{r}{4}.
\]
Now by continuity for sufficiently large \( k, k' \), there exists \( s_{k''} \in [s_{n_1}, s_{n_2}] \) such that
\[
\frac{r}{4} \leq | \int_{S^1} u(s_{k''}, \theta) d\theta - k'^{'} | \leq \frac{r}{3}.
\]
Again by the Arzelà-Ascoli theorem, there exists a subsequence of \( s_{k''} \), still denoted by \( s_{k''} \) such that \( u_i(s_{k''}, \theta) \) converges uniformly to some \( k'^{''} \) which should satisfy
\[
\frac{r}{4} \leq |k'^{''} - k'^{'}| \leq \frac{r}{3}.
\]
However, we know that there is no \( k'^{''} \) having distance from \( k'^{'} \) between \( \xi \) and \( \zeta \). This is a contradiction.

As one application of our analysis of asymptotic behavior, we have the following conclusion:

**Theorem 3.3.8 (Witten Lemma).** Suppose that \( z_k \) is the unique Ramond marked point on \( \Sigma \) and \((u_1, \ldots, u_N)\) are all the possible Ramond solutions near \( p \) such that their local coordinate functions \((\tilde{u}_1, \ldots, \tilde{u}_N)\) converge to a critical point \( \kappa \) of \( W_\gamma + W_0 \). Then we have
\[
\sum_i \| \tilde{\partial} u_i \|_{L^2(\Sigma)}^2 = \sum_i \| \frac{\partial(W + W_0)}{\partial u_i} \|_{L^2(\Sigma)}^2 = \pi(W_\gamma + W_0)(\kappa) - \frac{1}{2} \sum_i (b_i \tilde{u}_i \partial_\beta \kappa) ds d\theta.
\]
If there is no Ramond marked point, then the equation has only the zero solution.

**Proof.** Let \([z_1, \ldots, z_k]\) be \( k \) marked points and \( z_k \) be the only Ramond marked point. Our perturbed Witten equation is
\[
\tilde{\partial} u_i + I_1(\frac{\partial(W + W_0)}{\partial u_i}) = 0.
\]
Here the \( u_i \) are global sections on \( \Sigma \). Let \( u_i = \tilde{u}_i \epsilon_i \) in the disc \( D_r(z_i) \) (with radius \( r \) and centered at \( z_i \)) of the marked point \( z_i \). Then
\[
\sum_i \left( \tilde{\partial} u_i , I_1(\frac{\partial(W + W_0)}{\partial u_i}) \right)_{L^2(\Sigma)} = \sum_i \int_{\Sigma \setminus \{z_i\} \cup D_r(z_i)} (\tilde{\partial} u_i , I_1(\frac{\partial(W + W_0)}{\partial u_i})) +
\]
\[
\sum_{i=1}^{k-1} \sum_j \int_{D_r(z_i)} (\frac{\tilde{\partial} u_i}{\partial \Sigma} \epsilon_i \otimes \epsilon_j , \sum_j (\frac{\partial W_j(\tilde{u}_1, \ldots, \tilde{u}_N)}{\partial u_i} \Sigma^\_CART \beta_j(\theta_j'^{''}) + |e'_i|^2 \epsilon_i \otimes d\Sigma))
\]
\[
+ \sum_j \int_{D_r(z_i)} (\frac{\tilde{\partial} u_i}{\partial \Sigma} \epsilon_i \otimes \epsilon_j , \sum_j (\frac{\partial W_j(\tilde{u}_1, \ldots, \tilde{u}_N)}{\partial u_i} \Sigma^\_CART \beta_j(\theta_j'^{''}) + b_j \beta_{\kappa} - b_j \beta_{\kappa'} - \frac{1}{4} |e'_i|^2 \epsilon_i \otimes d\Sigma))
\]
\[
= - \lim_{r \to 0} \sum_{i=1}^{k-1} \pi \sum_j W_j(\tilde{u}_1(\xi), \ldots, \tilde{u}_N(\xi)) \Sigma^\_CART \beta_j(\theta_j'^{''})
\]
\[
- \lim_{r \to 0} \pi \sum_j W_j(\tilde{u}_1(\xi), \ldots, \tilde{u}_N(\xi)) \Sigma^\_CART \beta_j(\theta_j'^{''}) + \sum_j b_j \beta_{\kappa' \kappa} + \sum_j \frac{1}{2} \int_{\xi} \frac{d\Sigma \wedge d\Sigma}{z} b_j \partial_{\kappa'} \tilde{u}_i
\]
\[
= - \pi(W_\gamma + W_0)(\kappa) + \frac{1}{2} \sum_i (b_i \tilde{u}_i \partial_\beta \beta_{\kappa}) ds d\theta.
\]
We have used Stokes’ theorem in the second equality and Lemma 3.3.5, 3.3.7 in the last equality. Using the equation, we obtain

$$\sum_{i} ||\tilde{u}_i||_{L^2(\mathbb{C})}^2 = \sum_{i} ||\frac{\partial W}{\partial u_i}||_{L^2(\mathbb{C})}^2 = \pi(W_\gamma + W_0) - \frac{1}{2} \int_{T} \sum_{i} \sum_{j} (b_i \tilde{u}_i \partial_j \beta_T) ds d\theta.$$ 

If there is no Ramond marked point, we have the equality:

$$\sum_{i} ||\tilde{u}_i||_{L^2(\mathbb{C})}^2 = \sum_{i} ||\frac{\partial W}{\partial u_i}||_{L^2(\mathbb{C})}^2 = 0.$$ 

Since near each marked point the norm $|u_i|$ of the section $u_i$ equals $|\tilde{v}_i|$, which is an $L^2$-integrable function in view of the proof of Theorem 3.3.5, $u_i$ is a constant section near each marked point. Furthermore, $\frac{\partial W}{\partial u_i}$ equals $0$; then non-degeneracy of $W$ forces $u_i \equiv 0$ for each $i = 1, \cdots, N$. \hfill \Box

### 3.4. Exponential decay.

Let $u_i$, $i = 1, \cdots, N$, be the solutions of the perturbed Witten equation near a marked point $z_i$. Setting as before $v_i = u_i e^{\Theta(j)} = u_i e^{-\Theta(j)/\gamma}$ and $\hat{v}_i = u_i e^{-\Theta(j)/\gamma}$, we have

$$\hat{v}_i = v_i e^{\Theta(j)/\gamma}. \tag{79}$$

Note that $\hat{v}_i$ is a well-defined function on $S^1 \times [0, \infty)$ but $v_i$ is only a locally defined function. The locally defined section $v_i$ satisfies the equation

$$\partial_s v_i + \sqrt{-1} \partial_\theta v_i - 2 \frac{\partial (W + W_0)}{\partial v_i} = 0. \tag{80}$$

Suppose that the quasi-homogeneous polynomial $W$ has the form

$$W(u_1, \cdots, u_N) = \sum_j W_j = \sum_j c_j \prod_{i=1}^{N} u_i^{b_{ij}}.$$ 

Set $\Theta(j) := \sum_{k} b_{ij} \Theta_i^j \in \mathbb{Z}$. It is known that $W_j$ is Ramond at $z_j$ iff $\Theta(j) = 0$, and $W$ is Ramond at $z_j$ iff there exists one monomial $W_{j_0}$ such that $\Theta(j_0) = 0$.

Now an easy computation shows that $\hat{v}_i$ satisfies the equation

$$\partial_s \hat{v}_i + \sqrt{-1} \partial_\theta \hat{v}_i + \Theta_i^j \hat{v}_i = \frac{\partial \hat{W}(\hat{v}_1, \cdots, \hat{v}_N, \theta)}{\partial \hat{v}_j}, \tag{81}$$

where

$$\hat{W}(\hat{v}_1, \cdots, \hat{v}_N, \theta) := 2 \left( \sum_j W_j(\hat{v}_1, \cdots, \hat{v}_N) e^{-\Theta(j)/\gamma} + W_0(\hat{v}_1, \cdots, \hat{v}_N) \right). \tag{82}$$

Define $\hat{H} = 2 \text{Re} \hat{W}$, and let $\hat{v}_i = \hat{x}_i + \sqrt{-1} \hat{y}_i$. We want to change the complex system (81) into a real system. By (81), we have

$$\partial_s (\hat{x}_i + \sqrt{-1} \hat{y}_i) + \sqrt{-1} \partial_\theta (\hat{x}_i + \sqrt{-1} \hat{y}_i) + \Theta_i^j (\hat{x}_i + \sqrt{-1} \hat{y}_i) = \frac{1}{2}(\partial_{\hat{x}_i} + \sqrt{-1} \partial_{\hat{y}_i})(2 \hat{H}),$$

for $i = 1, \cdots, N$. So we obtain

$$\begin{align*}
\partial_s \hat{x}_i - \partial_\theta \hat{y}_i + \Theta_i^j \hat{x}_i &= \partial_{\hat{x}_i} \hat{H} \\
\partial_s \hat{y}_i + \partial_\theta \hat{x}_i + \Theta_i^j \hat{y}_i &= \partial_{\hat{y}_i} \hat{H},
\end{align*} \tag{83}$$

for $i = 1, \cdots, N$. 

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Define the following quantities:

\[ \hat{v}_R := (\hat{x}_1, \ldots, \hat{x}_{N_l}, \hat{y}_1, \ldots, \hat{y}_{N_l})^T, \quad \nabla_R = (\partial_{\hat{x}_1}, \ldots, \partial_{\hat{x}_{N_l}}, \partial_{\hat{y}_1}, \ldots, \partial_{\hat{y}_{N_l}})^T \]

\[ \hat{v}_N := (\hat{x}_{N_l+1}, \ldots, \hat{x}_{N_l+N_l}, \hat{y}_{N_l+1}, \ldots, \hat{y}_{N_l+N_l})^T, \quad \nabla_N = (\partial_{\hat{x}_{N_l+1}}, \ldots, \partial_{\hat{x}_{N_l+N_l}}, \partial_{\hat{y}_{N_l+1}}, \ldots, \partial_{\hat{y}_{N_l+N_l}})^T \]

\[ J_R := \begin{pmatrix} 0 & -I_R \\ I_R & 0 \end{pmatrix}, \quad \text{I}_R \text{ is the } N_l \times N_l \text{ identity matrix} \]

\[ J_N := \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}, \quad \text{I}_N \text{ is the } (N - N_l) \times (N - N_l) \text{ identity matrix} \]

\[ \Theta_N := \text{diag}(\Theta_{N_l+1}^\gamma, \ldots, \Theta_N^\gamma) \]

\[ A_N := \begin{pmatrix} \Theta_N & 0 \\ 0 & \Theta_N \end{pmatrix} \]

Then the system (83) can be written as follows:

\[
\begin{aligned}
\partial_{\hat{v}_R} + J_R \cdot \partial_{\hat{v}_R} &= \nabla_R \hat{H} \\
\partial_{\hat{v}_N} + J_N \cdot \partial_{\hat{v}_N} + A_N \cdot \hat{v}_N &= \nabla_N \hat{H}
\end{aligned}
\] (84)

Set

\[ \hat{v} := \begin{pmatrix} \hat{v}_R \\ \hat{v}_N \end{pmatrix}, \quad \nabla := \begin{pmatrix} \nabla_R \\ \nabla_N \end{pmatrix} \]

\[ J := \begin{pmatrix} J_R & 0 \\ 0 & J_N \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 0 \\ 0 & A_N \end{pmatrix} \]

Now we can write the equation (84) in the simple form

\[
\partial_{\hat{v}} \mathbf{v} + J \cdot \partial_{\hat{v}} \mathbf{v} + A \cdot \mathbf{v} = \nabla \hat{H}. \tag{85}
\]

Here \( J \) is an almost complex structure in \( \mathbb{R}^{2N} \), since \( J^2 = -I \).

Take the derivative \( \partial_s \) of the two sides of (83) and let \( \mathbf{w} = \partial_s \hat{v} \); we obtain the system for \( \mathbf{w} \):

\[
\partial_s \mathbf{w} + J \cdot \partial_{\hat{v}} \mathbf{w} + S \cdot \mathbf{w} = 0, \tag{86}
\]

where \( S := (A - \nabla^2 \hat{H}) \).

The following lemma gives a formulation of the asymptotic behavior of the coefficient matrix \( S = A - \nabla^2 \hat{H} \).

**Lemma 3.4.1.** The following conclusions hold when \( s \to \infty \):

1. \( S(s, \theta) \to S^\infty := \begin{pmatrix} -\nabla_R^2 2\text{Re}(W_y + W_0)_{(\kappa_1, \ldots, \kappa_N)} & 0 \\ 0 & \Theta_N - \nabla_N^2 2\text{Re}(W_N)_{(\kappa_1, \ldots, \kappa_N)} \end{pmatrix} \), where \( (\kappa_1, \ldots, \kappa_N) \) is some critical point of the polynomial \( W_y + W_0 \).
2. \( \sup_{s \in [0, \infty)} ||\partial_s S(s, \theta)|| \to 0 \), as \( s \to \infty \).
3. \( \sup_{s \in [0, \infty)} ||\partial_{\theta} S(s, \theta)|| < \infty \).

**Proof.** When \( s \to \infty \), we have \((\hat{v}_1, \ldots, \hat{v}_{N_l}, \hat{v}_{N_l+1}, \ldots, \hat{v}_{N}) \to (\kappa_1, \ldots, \kappa_N, 0, \ldots, 0)\). Note that the polynomial \( W_y + W_0 \) only contains the variables \( \hat{v}_1, \ldots, \hat{v}_{N_l} \); so we have two cases:

- for \( 1 \leq i, j \leq 2N_l \), there holds \( (\nabla^2 \hat{H})_{ij} = (\nabla_R^2 2\text{Re}(W_y + W_0 + W_N))_{ij} \);
- if \( i, j \) do not satisfy case 1, then \( (\nabla^2 \hat{H})_{ij} = (\nabla_N^2 2\text{Re}(W_N))_{ij} \).

Since there are at least two Neveu-Schwarz sections in each monomial of \( W_y \), so if \( 1 \leq i \leq 2N_l, N_l < j \) or \( 1 \leq j \leq 2N_l, N_l < i \), then \( (\nabla^2 \hat{H})_{ij} \to 0 \). Hence (1) is proved.

The proofs of (2) and (3) are easily obtained by Theorem 3.3.1, Theorem 3.3.5, and Lemma 3.3.6. \( \square \)
System (86) for \( w \) becomes the standard system that appears frequently in symplectic geometry, whose decaying behavior has been studied in detail. For example, we can cite Lemma 2.11 of D. Salamon’s lecture in [ET] as below (after minor changes in notations):

**Lemma 3.4.2.** Let \( J \) be a \( 2N \times 2N \) real matrix such that \( J^2 = -I \), and \( S(s,\theta) \) be a \( 2N \times 2N \) matrix function defined on \( S^1 \times [0, \infty) \) satisfying

\[
\lim_{s\to\infty} \sup_{\theta \in S^1} ||\partial_s S(s, \theta)|| = 0, \quad \sup_{s, \theta} ||\partial_\theta S(s, \theta)|| < \infty.
\]

Define the operator \( D = \partial_s + \partial_{\theta} + S \). If the unbounded self-adjoint operator \( J\partial_{\theta} + S : L^2(S^1, \mathbb{R}^{2N}) \to L^2(S^1, \mathbb{R}^{2N}) \) is invertible, then there exists a constant \( \delta > 0 \) such that the following holds. For every \( C^2 \)-function \( \xi : S^1 \times [0, \infty) \to \mathbb{R}^{2N} \) which satisfies \( D\xi = 0 \) and does not diverge to \( \infty \) as \( s \to \infty \) there exists a constant \( c > 0 \) such that for sufficiently large \( s \) and any \( \theta \in S^1 \),

\[
||\xi(s, \theta)|| < ce^{\delta s}.
\]

**Lemma 3.4.3.** Let \( W_0(u_1, \cdots, u_N) = \sum_{k=1}^N b_k u_k \) be any \( W_\gamma \)-regular polynomial at Ramond marked point \( z_i \). Then if \( \sum_{k=1}^N |b_k| \) is sufficiently small, there exist constants \( C, T, \delta > 0 \) such that for any \( (s, \theta) \in S^1 \times [T, \infty) \),

\[
|\partial_s \hat{v}_i| < Ce^{-\delta s}, \quad i = 1, \cdots, N.
\]

**Proof.** By Theorems 3.3.1 and (2), (3) of Lemma 3.4.1 we know that all the hypotheses in Lemma 3.4.2 except invertibility are satisfied. So we only need to show that the operator \( J\partial_{\theta} + S : L^2(S^1, \mathbb{R}^{2N}) \to L^2(S^1, \mathbb{R}^{2N}) \) is invertible when \( \sum_{k=1}^N |b_k| \) is sufficiently small. However, it is easy to see that if \( \sum_{k=1}^N |b_k| \) is sufficiently small, the absolute value of any critical point of \( W_\gamma + W_0 \) also becomes arbitrarily small. Hence we can take sufficiently small \( \kappa_1, \cdots, \kappa_N \) such that

\[
||S^\infty|| < 1, \quad and \ S^\infty \ is \ a \ non-degenerate \ matrix. \quad (87)
\]

Assume \( w \in L^2(S^1, \mathbb{R}^{2N}) \) is a solution of \( J\partial_{\theta} w + S^\infty w = 0 \). By Poincare inequality, there is

\[
||w - \int_{S^1} w||_{L^2} \leq ||\partial_\theta (w - \int_{S^1} w)||_{L^2}.
\]

Since the mean value \( \int_{S^1} w \) vanishes by the equation and the matrix \( S^\infty \) is non-degenerate, we obtain

\[
||w||_{L^2} \leq ||\partial_\theta w||_{L^2} = ||S^\infty w||_{L^2}.
\]

Since \( ||S^\infty|| < 1 \), we conclude that \( w = 0 \). Therefore the invertibility is proved. Applying Lemma 3.4.2 to our case, then we are done. \( \Box \)

By the above lemma, it is easy to obtain:

**Theorem 3.4.4.** Let \( W_0(u_1, \cdots, u_N) = \sum_{k=1}^N b_k u_k \) be any \( W_\gamma \)-regular polynomial at the Ramond marked point \( z_i \). Then if \( \sum_{k=1}^N |b_k| \) is sufficiently small, then there exist constants \( C, T, \delta > 0 \) such that for any \( (s, \theta) \in S^1 \times [T, \infty) \),

\[
|\hat{v}_i - \kappa_i| < Ce^{-\delta s}, \quad for \ i = 1, \cdots, N_i,
\]

\[
|\hat{v}_i| < Ce^{-\delta s}, \quad for \ i = N_i + 1, \cdots, N,
\]

where \( (\kappa_1, \cdots, \kappa_N) \) is some critical point of \( W_\gamma + W_0 \).
3.5. Liouville type theorem for $A_1$-equation.

If $W = u^2$, the Witten equation becomes

$$\bar{\partial}u + I_1(2u) = 0 \quad (89)$$

This is a linear equation and the existence of solutions is related to the spectrum problem. The previous interior estimate does not hold here since it holds for $W$ with all weights $q_i < 1/2$. Since $W = x^2$ is already a holomorphic Morse function, it is not useful to do linear perturbation again. To avoid the spectrum, we choose the following global perturbed equation

$$\bar{\partial}u + I_1((2 + \epsilon)u) = 0. \quad (90)$$

We have the following Liouville type theorem

**Theorem 3.5.1.** For generic perturbation parameter $\epsilon \in \mathbb{R}$, the perturbed $A_1$ equation (90) has only the zero solution.

**Proof.** Let $[1, \infty]$ be the cylindrical neighborhood of the marked point $p = \infty$. Choosing the cylindrical coordinate $\zeta = s + i\theta$, then (90) becomes

$$\bar{\partial}_s u - 2((2 + \epsilon)u) = 0. \quad (91)$$

Since the integral $\int_1^\infty \int_1^\infty |\partial_s u| d\theta ds < \infty$, we can use Lemma 3.4.2 to show that if $\epsilon \in \mathbb{R}$ is generic, then the solution converges to zero exponentially. Applying the Witten lemma to Equation (91), we are done. \hfill \square

4. Compactness

4.1. Sequence convergence of the moduli space $\overline{\mathcal{M}}_{g,k,W}(\gamma)$.

Traditionally, the compactness theorem of various moduli spaces is formulated in terms of possible geometric limits of its sequence. In Section two, we gave a description of the topology of $\overline{\mathcal{M}}_{g,k,W}(\gamma)$ in terms of its local neighborhood. Here, we take a different point of view in terms of converging sequences.

Take a sequence of rigidified $W$-curves $\mathcal{C}^n = (\mathcal{C}^n, p_1, \ldots, p_k, \mathcal{L}^n_1, \ldots, \mathcal{L}^n_{\ell_1}, \phi^n_1, \ldots, \phi^n_{\ell_1}, \psi^n_1, \ldots, \psi^n_{\ell_2})$ in $\overline{\mathcal{M}}_{g,k,W}(\gamma)$. Here we assume that the $W$-structure $(\mathcal{L}^n_1, \ldots, \mathcal{L}^n_{\ell_1}, \phi^n_1, \ldots, \phi^n_{\ell_1}, \psi^n_1, \ldots, \psi^n_{\ell_2})$ induces the data $\gamma = (\gamma_1, \ldots, \gamma_\ell)$, i.e., we assume the sequence of $W$-curves has type $\gamma$. The underlying curves $(\mathcal{C}^n, p_1, \ldots, p_k)$ have dual graph $\Gamma^n$. Since the combinatorial types of the dual graphs are finite in number, we can assume that $\Gamma^n = \Gamma$ for all $n$. Similarly, we can let $\psi^n_j = \psi_j$ at each marked point $p_i$, because the set of rigidifications at one marked point is finite and characterized by a group element in $G$. Hence it is sufficient to consider the sequence of $W$-curves $\mathcal{C}^n$ in $\overline{\mathcal{M}}_{g,k}(\Gamma)$, where $\Gamma$ is the $G$-decorated stable graph with each tail labelled by a group element in $G$. We can further assume that $\mathcal{C}^n \in \overline{\mathcal{M}}_{g,k}(\Gamma)$, where $\Gamma$ is also a fully $G$-decorated stable graph. In fact, each half-edge $\tau$ of $\Gamma$ corresponds to an orbifold point $p_\tau$ of the normalization of the underlying curve, thus has a corresponding choice of $\gamma_\tau \in G$. Since $G$ is a finite group, we can assume that all $W$-curves attain the same group element $\gamma_\tau \in G$. Notice that we don’t fix the rigidification at the nodal points. We start from the following obvious lemma:

**Lemma 4.1.1.** Suppose that the sequence of $W$-curves $\mathcal{C}^n \in \overline{\mathcal{M}}_{g,k}(\Gamma)$ converges to $\mathcal{C}$ in $\overline{\mathcal{M}}_{g,k}(\Gamma)$. Then $\text{st}^g(\mathcal{C}^n) \rightarrow \text{st}^g(\mathcal{C})$ in $\overline{\mathcal{M}}_{g,k}(\Gamma)$. 

In general when considering the convergence in $\overline{\mathcal{M}}_{g, k, W}^{\text{ig}}(\gamma)$ the limit curve of the sequence of underlying curves of $W$-curves $\mathcal{C}$ may change the combinatorial type. So to define the convergence in $\overline{\mathcal{M}}_{g, k, W}^{\text{ig}}(\gamma)$, we have to study the degeneracy behavior of the line bundles and the $W$-structures.

Consider a sequence of the rigidified $W$-curves $\mathcal{C}$ such that $\text{str}^k(\mathcal{C}) = \mathcal{C}^n \to \mathcal{C}$ degenerating only along a circle to a nodal point $z \in \mathcal{C}$. Hence by the description of Proposition 2.2.4 there is a sequence of gluing parameters $\zeta^n = (s^n, \theta^n) \in \{0, \infty\} \times S^1$ such that $\mathcal{C}^n = \mathcal{C}_0 \circ \mathcal{C}^n$ is obtained through gluing the corresponding cylinders $U_n = [\frac{1}{2}T^n, \frac{1}{2}T^n]$ ($s^n = 2T^n$) in two different components of $\mathcal{C}$ by a biholomorphic map.

The canonical bundle $K_{\mathcal{C}^n}$ when restricted to the cylinder $U_n$ can be trivialized. Fixing a trivialization, the $W$-structure gives the isomorphism $\varphi_j^n : W_i(\mathcal{L}_1^n, \cdots, \mathcal{L}_N^n) \to \mathbb{C}$.

Lemma 4.1.2. For each $n$, the line bundles $\mathcal{L}_i^n$ are flat and the structure group of the $N$-tuple $(\mathcal{L}_1^n, \cdots, \mathcal{L}_N^n)$ on $U_n$ is $G$.

Proof. Let $e^n(\alpha), i = 1, \cdots, N$ be the basis of $\mathcal{L}_i^n$ in a small chart $U_\alpha$ of $U_n$ such that $W_j(e^n(\alpha), \cdots, e^n(\alpha)) = 1$ for all $j$. Take an atlas $\{U_\alpha\}$ of $U_n$ such that all the line bundles can be trivialized in each chart and the chosen basis satisfies the above relation. Now it is easy to see that the transition function on each $U_\alpha \cap U_\beta$ must lie in the group $G$. Hence each $\mathcal{L}_i^n$ is a flat line bundle.

On $U_n$, each $\mathcal{L}_i^n$ is completely classified by its holonomy $p^n_i : \mathbb{Z} = \pi_1(U_n) \to U(1)$, and $p^n_i$ maps the generator to $e^{2\pi c_i^n/d}$ for some $0 \leq c_i^n < d$ such that $\gamma^n = (e^{2\pi c_1^n/d}, \cdots, e^{2\pi c_i^n/d}) \in G$. Such a flat line bundle can be explicitly constructed as follows. Let $(\overline{U}_n)^d$ be a degree-$d$ unramified cover of $U_n$. The line bundle is of the form $\mathcal{L}_i^n = (\overline{U}_n)^d \times_{e^{2\pi c_i^n/d}} \mathbb{C}$. After choosing a subsequence, we can assume that for all $n$, $\mathcal{L}_i^n$ corresponds to a fixed value $e^{2\pi c_i^n/d}$. When $n$ goes to infinity, we can associate a natural "limit". We change the coordinate of $U_n$ to $U_{n0} = [0, T^n] \times S^1$ and change $\overline{U}_n$ to $\overline{U}_{n0}$ accordingly. When $n$ goes to infinity, $U_{n0}$ converges to a punctual disc $U_0 - \{0\}$. $\mathcal{L}_i^n$ extends uniquely to a flat orbifold line bundle $\mathcal{L}_i = (\overline{U}_0)^d \times_{e^{2\pi c_i^n/d}} \mathbb{C}$ over $(\overline{U}_0, \mathbb{Z}_d)$. Now we change coordinates of $U_n$ to $U_{n1} = [-T^n, 0] \times S^1$. The same argument obtains a limiting orbifold line bundle $\mathcal{L}_i = (\overline{U}_1)^d \times_{e^{2\pi c_i^n/d}} \mathbb{C}$ over $(\overline{U}_1, \mathbb{Z}_d)$. Finally, since $\mathcal{L}_i^n, \mathcal{L}_i$ are really the limit of the same flat line bundle, there is a canonical identification $\mathcal{L}_{i0} \equiv \mathcal{L}_{i1}$. Therefore, we obtain a $W$-structure $\mathcal{L}_i$ on the nodal curve $\mathcal{C}$ locally represented by $U_0 \cap U_1$. Furthermore, $\varphi_j^n$ converges to an isomorphism $\varphi_j : W(\mathcal{L}_1, \cdots, \mathcal{L}_N) \to \mathbb{C}$.

When $\gcd(c_1, \cdots, c_d) = a > 1$, the orbifold structure $(\overline{U}_0, \mathbb{Z}_d)$ is redundant. The group action is not faithful; hence we can modify the uniformizing system of $\mathcal{L}_0$ by redefining the local group to be $\mathbb{Z}_d$.

Hence degeneracy of the underlying curves $\mathcal{C}$ along a circle induces a geometric limit $(\mathcal{L}_1, \cdots, \mathcal{L}_N, \varphi_1, \cdots, \varphi_N)$ on $\mathcal{C}$.

Theorem 4.1.3. $(\mathcal{L}_1, \cdots, \mathcal{L}_N, \varphi_1, \cdots, \varphi_N)$ converges to $(\mathcal{L}_1, \cdots, \mathcal{L}_N, \varphi_1, \cdots, \varphi_N)$ in $\overline{\mathcal{M}}_{g, k, W}^{\text{ig}}$.

Proof. For this purpose, we need to shows that the above sequence is in any neighborhood of $(\mathcal{L}_1, \cdots, \mathcal{L}_N, \varphi_1, \cdots, \varphi_N)$ for $i > 0$.

But this is automatic from our construction. Recall that the construction of a neighborhood of $(\mathcal{L}_1, \cdots, \mathcal{L}_N, \varphi_01, \cdots, \varphi_0N)$ in Proposition 2.2.4. Let $\mathcal{C} = (\mathcal{C}, p_1, \cdots, p_k, \mathcal{L}_1, \cdots, \mathcal{L}_N, \varphi_1, \cdots, \varphi_N)$,
ψ_1, · · · , ψ_k) be a rigidified W-curves having a orbifold nodal point z with local group \( G_z \cong \mathbb{Z}/d \), then we can construct the nearby rigidified W-curves. Let \( \xi = (s_k, \theta_k) \), \( s_k = 2\theta \) be the gluing parameter; then by Proposition 2.2.4 there is a nearby curve \( \mathcal{O}_{k, \xi} \). Take one component \( ([T_0, \infty) \times S^1)_k, k = 1, 2 \) with coordinates \((s_k, \theta_k)\). Let \( U_k = ((1/2T, 3/2T) \times S^1)_k \) and \( \overline{U}_k \) be the degree-\( d \) covering of \( U_k \). A local trivialization of \( \mathcal{L}_j \) is given by \( \overline{U}_k \times C \) with the group action \( e^{2\pi i/d} : (z_k, \omega_j) \rightarrow (e^{2\pi i/d} z_k, e^{(-1)^j 2\pi i c/d} \omega_j) \). Thus the orbifold line bundle \( \mathcal{L}_j \) when restricted on \( U_k \) is a flat line bundle with holonomy \( \rho_{k, j} : \pi_1(U_k) \rightarrow U(1), k = 1, 2, j = 1, \cdots, N \), where \( \rho_{k, j} \) sends the generator of \( \pi_1(U_k) \cong \mathbb{Z}/d \) to \( e^{(1)^j 2\pi i c/d} \omega_j \). As a rigidified W-structure on a nodal curve, the part of data is the identification of \( \mathcal{O}_{0, j} \) at nodal point compatible W-structure. Now we use this identification and the trivialization on \( U_k \) to glue the W-structures on \( U_1 \) and \( U_2 \) to get a W-structure on \( \mathcal{O}_{0, j} \). In our construction of \( \mathcal{O}_{0, j} \), it naturally carries a trivialization at the nodal point. The above gluing process gives precisely the \( \mathcal{L}_i^n = (\overline{U}_j)^d \times _{\rho_{0, j}, d} C \) on the cylinder. \( \square \)

4.2. Topology of the moduli spaces \( \overline{\mathcal{M}}_{g, k, W}^{\text{rig}}(\gamma, \kappa) \) and \( \overline{\mathcal{M}}_{g, k, W}^{\text{rig}}(\gamma, \kappa) \).

4.2.1. Perturbed Witten Map over \( \overline{\mathcal{M}}_{g, k, W}^{\text{rig}}(\gamma) \).

Fréchet stratified orbibundles over \( \overline{\mathcal{M}}_{g, k, W}^{\text{rig}}(\gamma) \). By the algebraic geometric construction in [FJRI], there exists the universal rigidified W-curve \( C_{g, W}^{\text{rig}} \rightarrow \overline{\mathcal{M}}_{g, k, W}^{\text{rig}}(\gamma) \). Also, by Proposition 2.2.4, we can obtain the required family of smooth metrics on \( C_{g, W}^{\text{rig}} \). We first construct the stratified Fréchet orbibundles \( B_0 \) and \( B_{0, 1} \) over \( \overline{\mathcal{M}}_{g, k, W}^{\text{rig}}(\gamma) \).

Let \( \mathcal{C} = (\mathcal{C}, p_1, \cdots, p_k, \mathcal{L}_1, \cdots, \mathcal{L}_N, \varphi_1, \cdots, \varphi_N, \psi_1, \cdots, \psi_k) =: (\mathcal{C}, \mathcal{L}, \Psi) \) be a rigidified W-curve representing an element \([\mathcal{C}]\) in \( \overline{\mathcal{M}}_{g, k, W}^{\text{rig}}(\gamma) \). \( \mathcal{C} \) can be decomposed into irreducible components: \( \mathcal{C} = \cup \mathcal{C}_r \). Denote by \( \pi_r : \mathcal{C}_r \rightarrow \mathcal{C} \) the projection map from the renormalized component \( \mathcal{C}_r \) to the nodal curve \( \mathcal{C} \). The automorphism group of \( \mathcal{C} \) is \( \text{Aut}(\mathcal{C}) \) which is the extension of \( \text{Aut}(\mathcal{C}_r) \) along \( \text{Aut}(\mathcal{L}, \Psi) \). We have the \( \text{Aut}(\mathcal{C}) \)-invariant metric of \( \mathcal{C} \) and the induced metrics on the line bundles \( \mathcal{L}_i \). Define

\[ C^0(\mathcal{C}, \mathcal{L}_i) := \{ (u, \zeta) \in \mathcal{C}_r, \mathcal{C}_{\mathcal{L}_i} | u_{xj}(p_r) = u_{xj}(p_{r'}) \text{ if } \pi_r(p_r) = \pi_r(p_{r'}) \} \]

\[ C^0(\mathcal{C}, \mathcal{L}_i \otimes \Lambda^{0, 1}) := \{ (u, \zeta) \in \mathcal{C}_r, \mathcal{C}_{\mathcal{L}_i \otimes \Lambda^{0, 1}} | \} \]

and let \( L^1(\mathcal{C}, \mathcal{L}_i) \) and \( L^1(\mathcal{C}, \mathcal{L}_i \otimes \Lambda^{0, 1}) \) be the p-integrable Sobolev spaces with respect to \( C^0(\mathcal{C}, \mathcal{L}_i) \) and \( C^0(\mathcal{C}, \mathcal{L}_i \otimes \Lambda^{0, 1}) \) respectively. We always take \( p > 2 \) to ensure the continuity of the functions in \( L^1(\mathcal{C}, \mathcal{L}_i) \times \cdots L^1(\mathcal{C}, \mathcal{L}_N) \) or \( C^0(\mathcal{C}, \mathcal{L}_i) \times \cdots C^0(\mathcal{C}, \mathcal{L}_N) \) and let \( B_{0, 1} \) represent \( L^0(\mathcal{C}, \mathcal{L}_i \otimes \Lambda^{0, 1}) \times \cdots L^0(\mathcal{C}, \mathcal{L}_N \otimes \Lambda^{0, 1}) \) or \( C^0(\mathcal{C}, \mathcal{L}_i \otimes \Lambda^{0, 1}) \times \cdots C^0(\mathcal{C}, \mathcal{L}_N \otimes \Lambda^{0, 1}) \). The automorphism group acts naturally on \( B_0 \) and \( B_{0, 1} \). For example, if \( (\tau, g) \in \text{Aut}(\mathcal{C}) \times \text{Aut}(\mathcal{L}, \Psi) \) is an element in \( \text{Aut}(\mathcal{C}) \) and \( u \in B_0 \), then \( (\tau, g) \cdot u(\zeta) = gu(\tau \cdot \zeta) \). Now \( B_0 \) (resp. \( B_{0, 1} \)) become the fiber at \([\mathcal{C}]\) of the Fréchet orbibundle \( B_0 \) (resp. \( B_{0, 1} \)). In fact, we can give a description of the uniformizing system of \( B_0 \) around \([\mathcal{C}]\). Let \( (U, \text{Aut}(\mathcal{C})) \) be a uniformizing system of \([\mathcal{C}] \in \overline{\mathcal{M}}_{g, k, W}^{\text{rig}} \) such that \([\mathcal{C}] \in U / \text{Aut}(\mathcal{C}) \); then the uniformizing system of \([\mathcal{C}] \in \overline{\mathcal{M}}_{g, k, W}^{\text{rig}}(\gamma) \) is given by \((U, \text{Aut}(\mathcal{C})) \). Hence the uniformizing system of \( B_0 \) around \([\mathcal{C}] \in (U \times B_0, \text{Aut}(\mathcal{C})) \). The action of \( g \in \text{Aut}(\mathcal{C}) \) on \( B_0|_U \) is an isometry from the Fréchet space \( B_0^{\mathbb{C}} \) to \( B_0^{\mathbb{C}} \).
Actually a local trivialization is given by the gluing map \textit{Glue} in Lemma 5.3.1. Similarly, one can get the description of the uniformizing system of $B^{0.1}$.

\textbf{Witten map.} Since the projection map $\pi : B^0 \rightarrow \mathcal{M}_{g,k,W}(\gamma)$ is an orbifold morphism, we have the pull-back Fréchet orbibundle $\pi^*B^{0.1}$ over $B^0$. The Witten map is defined as the section from $B^0$ to $\pi^*B^{0.1}$:

$$\tilde{WM}(\mathcal{C}, u) = \tilde{WM}_g(u_1, \cdots, u_N) = \left( \tilde{\partial}_\psi u_1 + \bar{\partial} \left( \frac{\partial W}{\partial u_1} \right), \cdots, \tilde{\partial}_\psi u_N + \bar{\partial} \left( \frac{\partial W}{\partial u_N} \right) \right).$$

Note that the Witten map is actually defined between uniformizing systems; hence it is required to be Aut($\mathcal{C}$)-equivariant. This equivariance can be easily seen from the equation.

\textbf{Remark 4.2.1.} It is not hard to see that Witten map descends to the moduli space of $W$-structures $\mathcal{M}_{g,k,W}(\gamma)$. However, the perturbed Witten map is only defined over $\mathcal{M}_{g,k,W}(\gamma)$. This is the main reason that we use $\mathcal{M}_{g,k,W}(\gamma)$. Alternatively, one can think that the perturbed Witten map is multi-valued over $\mathcal{M}_{g,k,W}(\gamma)$.

\textbf{Perturbed Witten map.} Since the nonlinear term appearing in the Witten map is degenerate, it is hard to get the asymptotic behavior of the solutions lying in the zero locus. We need to perturb the Witten map near the Ramond marked or Ramond nodal points. We will treat the two cases in different ways.

Let $\mathcal{C} = (\mathcal{C}, \mathcal{L}, \Psi)$ be a rigidified $W$-curve. Choosing a rigidification $\psi$ at a marked point $p$ means fixing a basis $e = (e_1, \cdots, e_N)$ of the line bundles $\mathcal{L}_1 \times \cdots \times \mathcal{L}_N$ near $p$ such that

$$W(e_1, \cdots, e_N) = dz/z.$$  

The group $G$ acts on $e$. We can fix a basis $e^0$, and call the corresponding rigidification $\psi^0$ the \textit{standard rigidification}. Then the set of rigidifications at $p$ is $g \cdot \psi^0$. Let $(\psi^0, b)$ be a pair of data, where $b = (b_1, \cdots, b_N)$ is the perturbation parameter. We define the group action of $G$ by $g \cdot (\psi^0, b) = (g \cdot \psi^0, g \cdot b)$.

For any rigidification $\psi = g \cdot \psi^0$, we take the pair $(\psi, g \cdot b)$. As the first step we will construct a perturbed map over $\mathcal{C}$ based on the data $(\psi, g \cdot b)$. If $p$ is a Neveu-Schwarz marked point, then we leave the Witten map unchanged. We only perturb the Witten map near a Ramond marked point $p$. Let $[0, \infty) \times S^1$ be the cylindrical neighborhood of $p$. Since $p$ is Ramond, the line bundles $\mathcal{L}_j$ are classified into Ramond line bundles and Neveu-Schwarz line bundles. As shown in Section 3 the group action of $\gamma$ at $p$ determines $W_\gamma$, the sum of partial monomials of $W$, and thus determines the choice of perturbed function $W_{0,\gamma}$. As done in Section 5 we choose a number $\bar{T}_0 > T_0$, where $T_0$ is the number related to the gluing parameter occurring in Proposition 2.2.4 and a section $\beta_j$ whose derivative has compact support in $[\bar{T}_0, \bar{T}_0 + 1] \times S^1$ of some line bundle such that $\beta_j \in \Omega(\mathcal{L}_j^{-1} \otimes K_{\log})$.

After choosing $\bar{T}_0$, we fix $\bar{T}_0$ in this paper. Hence the perturbed map on $\mathcal{C}$ is defined as

$$\tilde{WM}_{\bar{T}}(\psi, b)(u_1, \cdots, u_N) := \left( \tilde{\partial}_\psi u_1 + \bar{\partial} \left( \frac{\partial (W + \beta_1 W_{0,\gamma})}{\partial u_1} \right), \cdots, \tilde{\partial}_\psi u_N + \bar{\partial} \left( \frac{\partial (W + \beta_1 W_{0,\gamma})}{\partial u_N} \right) \right).$$

One can easily verify that

$$\tilde{WM}_{\bar{T}}(\psi, b) = \tilde{WM}_{\bar{T}}(\psi, b).$$  

(92)
We can perturb the Witten map near each Ramond marked point in this way and take \( \widetilde{WM}_\gamma \) as the global perturbed map whose restriction near each marked \( p \) is \( WM_\gamma^{(\Psi, \mathbf{b})} \).

**Lemma 4.2.2.** \( \widetilde{WM}_\gamma^{(\Psi, \mathbf{b})} \) is a section from \( B^0 \) to \( \pi^* B^1 \).

**Proof.** Assume that \( \mathcal{C} \) has several components \( \mathcal{C}_i \); then the group \( \text{Aut}_\mathcal{C}(\mathcal{L}, \Psi) \subset \prod \text{Aut}_\mathcal{C}_i(\mathcal{L}_i, \Psi_i) \). Each \( \text{Aut}_\mathcal{C}_i(\mathcal{L}_i, \Psi_i) \) acts trivially on the Ramond line bundles at each marked point on \( \mathcal{C}_i \). Hence \( \widetilde{WM}_\gamma^{(\Psi, \mathbf{b})} \) is \( \text{Aut}(\mathcal{C}) \)-equivariant. \( \square \)

Although we obtained the perturbed map \( \widetilde{WM}_\gamma^{(\Psi, \mathbf{b})} \), we still need to perturb this map at the Ramond nodal points. There is no natural rigidification at nodes. Here, we run into the same problem as defining perturbed Witten map over \( \mathcal{M}_{g, k, W}(\gamma) \). In this case, we simply treat it as a multi-valued perturbation or multi-section. We notice that Fukaya-Ono [FO] has used the same idea already in their construction of virtual fundamental cycles. The reader can find the definition and properties of multisection in subsection 5.2.

Now assume that \( p \) is the only Ramond nodal point connecting two components \( \mathcal{C}_i \), and \( \mathcal{C}_\mu \) of \( \mathcal{C} \). We want to define the new perturbed map over \( \mathcal{C} \). First we give the standard rigidification \( \psi^0 \) to the two half edges represented by \( p \) in the \( \mu \) and \( \nu \) components. Denote them by \( \psi^0_+ \) and \( \psi^0_- \) respectively. Take the standard pair \( (\psi^0_+, \mathbf{b}) \). If we take the map \( WM_\gamma^{(\Psi, \mathbf{b})} \) as the Witten map \( \widetilde{WM}_\gamma \), then we can define the new perturbed map over the component \( \mu \):

\[
WM_\mu^{(\psi^0, \mathbf{b})} := (WM_\gamma^{(\Psi, \mathbf{b})})^{(\psi^0_+)}_{\mu},
\]

where the latter operator has the same form as \( \Psi_\mu \). The new perturbed map on the \( \nu \)-component is not independent and it is defined as \( WM_\nu^{(\psi^0, \mathbf{b})} := (WM_\gamma^{(\Psi, \mathbf{b})})^{(\psi^0_-, \mathbf{b})}_{\nu} \). Here \( I = \text{diag}(\xi^1, \cdots, \xi^{s_\nu}) \) and \( \xi^d = -1 \). This is the requirement of the compatiblity from the gluing operation. The details will be discussed later in Remark 4.2.6. Thus we have the new perturbed map over the two components:

\[
WM_\mathcal{C}^{(\psi^0, \mathbf{b})} := WM_\mu^{(\psi^0, \mathbf{b})} \# WM_\nu^{(\psi^0, \mathbf{b})}
\]

Here we denote \( (\psi^0, \mathbf{b}) := (\psi^0_+, \psi^0_-, \mathbf{b}, -\mathbf{b}) \).

Now we consider two cases:

1. **Tree case** Let \( g = (g_1, g_2) \in \text{Aut}_\mathcal{C}(\Psi, \mathcal{L}) = \text{Aut}_{\mathcal{C}_\mu}(\Psi, \mathcal{L}_\mu) \times \text{Aut}_{\mathcal{C}_\nu}(\Psi, \mathcal{L}_\nu) \), then there exists a \( \delta \in \gamma_p > g \) such that \( g_1 = g_2 \delta \). We define \( g \cdot (\psi^0, \mathbf{b}) = (g \cdot \psi^0, \mathbf{b}) := (g_1 \cdot \psi^0_+, g_2 \cdot \psi^0_-, g_1 \cdot \mathbf{b}, -g_2 \cdot \mathbf{b}) \). So we can define \( WM_\mu^{(\psi^0, \mathbf{b})} \) and \( WM_\nu^{(\psi^0, \mathbf{b})} \). The latter map equals \( WM_\mu^{(\psi^0, \mathbf{b})} \) since \( \delta \) fixes the rigidification \( \psi^0 \) and it acts trivially on the perturbed term. Hence the compatibility condition for the two maps is satisfied and we obtain \( WM_\mathcal{C}^{(\psi^0, \mathbf{b})} \).

2. **Loop case** In this case, we have \( \text{Aut}_\mathcal{C}(\Psi, \mathcal{L}) = \gamma_p > \bigcup_{i \in I} < \gamma_i \). Let \( g \in \text{Aut}_\mathcal{C}(\mathcal{L}, \Psi) \); then the following map is well defined

\[
WM_\mathcal{C}^{(\psi^0, \mathbf{b})} = WM_\mu^{(\psi^0, \mathbf{b})} \# WM_\nu^{(\psi^0, \mathbf{b})}.
\]

Thus in either cases, for any \( g \in \text{Aut}_\mathcal{C}(\mathcal{L}, \Psi) \) the map \( WM_\mathcal{C}^{(\psi^0, \mathbf{b})} \) is well defined and equal to \( WM_\mathcal{C}^{(\psi^0, \mathbf{b})} \).

**Lemma 4.2.3.** If \( g \in \text{Aut}_\mathcal{C}(\mathcal{L}, \Psi) \), then for any \( \mathbf{u} \in B^0_\mu \), there is \( WM_\mathcal{C}^{(\psi^0, \mathbf{b})}(g \cdot \mathbf{u}) = g \cdot WM_\mathcal{C}^{(\psi^0, \mathbf{b})}(\mathbf{u}) \).
Proof. We only treat the tree case; the proof of the loop case is left to the reader. Assume that \( g = (g_1, g_2) \); then \( (g^{-1} \psi_1, b) = (g_1^{-1} \psi^0_1, g_2^{-1} \psi^0_2, b, -I(b)) \).

Let \( e = (e_1, \cdots, e_N) \) be the basis corresponding to the rigidification \( \psi^0_1 \); then the \( i \)-th component of \( WM_{\psi_1}^{(g_1, b)}(u) \) is

\[
\bar{\partial}_{\psi_1} u_i + \int \frac{\partial W}{\partial u_i} + \bar{b} \beta_T e_i \otimes \frac{dz}{z},
\]

(94)

Since \( g_1 \in G \), there is a number \( \lambda, \lambda^d = 1 \) such that \( g_1 = (\lambda^{k_1}, \cdots, \lambda^{k_N}) \). Hence it is easy to check that the \( i \)-th component of \( WM_{\psi_1}^{(g_1, b)}(\lambda^{k_1} u_1, \cdots, \lambda^{k_N} u_N) \) is \( \lambda^k WM_{\psi_1}^{(\psi^0_1, b)}(u) \). We have a conclusion similar to the \( v \) component. Hence

\[
WM_{\psi_1}^{(g, b)}(g \cdot u) = \left( WM_{\psi_1}^{(g, b)}(g_1 \cdot u_1), WM_{\psi_1}^{(g, b)}(g_2 \cdot u_2) \right)
\]

\[
= \left( g_1 \cdot WM_{\psi_1}^{(\psi^0_1, b)}(u_1), g_2 \cdot WM_{\psi_1}^{(\psi^0_2, b)}(u_2) \right) = g \cdot WM_{\psi_1}^{(\psi^0_1, b)}(u)
\]

\[\square\]

Definition 4.2.4. The perturbed Witten map \( WM_{\mathcal{C}}(u) \) over \( \mathcal{C} \) is the multisection \([WM_{\psi_1}^{(g, b)} : g \in \text{Aut}_\mathcal{C}(\mathcal{L}, \Psi)]\) from \( B^0_\mathcal{C} \) to \( B^1_\mathcal{C} \) which is \( \text{Aut}(\mathcal{C}) \)-equivariant by Lemma 4.2.3. In general, if \( \mathcal{C} \) has \( n \) Ramond nodal points, then we have the multiple index \((g^1, b) := (\psi^1_1(p_1), \psi^1_2(p_2), b, -I(b)), i = 1, \cdots, n \) and the group \( \text{Aut}_\mathcal{C}(\mathcal{L}, \Psi) \) acts naturally on it. The perturbed Witten map can be defined as

\[WM_{\mathcal{C}}(u) := [WM_{\psi_1}^{(g, b)} : g \in \text{Aut}_\mathcal{C}(\mathcal{L}, \Psi)]\]

which is also \( \text{Aut}(\mathcal{C}) \)-equivariant. Here \( WM_{\psi_1}^{(g, b)} \) is called a branch map of \( WM_{\mathcal{C}} \). The zero locus of \( WM_{\mathcal{C}} \) is defined as the union of the zero locus of all of its branch maps, which is also an \( \text{Aut}(\mathcal{C}) \)-invariant set.

Since the deformation domain in \( \mathcal{C} \) has empty intersection with the perturbation domain, the perturbed Witten map is naturally defined over \( \mathcal{M}_{g,k,W}(\Gamma; \gamma) \) for any combinatorial type \( \Gamma \). However, to construct a global perturbed Witten map over \( \mathcal{M}_{g,k,W}^{ig}(\gamma) \), we need to modify the existing perturbed Witten maps over the strata such that the compatibility condition holds for these multisections. We construct such a map by induction with respect to the order \( o \) of the dual graph.

As the first step, we consider the minimal stratum in \( \mathcal{M}_{g,k,W}^{ig}(\gamma) \). The dual graph \((\Gamma, (g_r), \alpha)\) is minimal if

\[g_r = 0, \ k_r = 3, \ \forall v,\]

In this case, the stratum \( \mathcal{M}_{g,k,W}(\Gamma) \) consists of only one point, and \( \mathcal{M}_{g,k,W}^{ig}(\Gamma) \) as the covering space consists of only finitely isolated points. Then the perturbed Witten map \( WM_{\mathcal{C}} \) is well-defined over \( \mathcal{M}_{g,k,W}^{ig}(\Gamma) \). The second step is to redefine the perturbed Witten map on the nearby curves. Define a smooth and monotone increasing function \( \sigma : \mathbb{R}^+ \times S^1 \to \mathbb{R} \) such that

\[\sigma(s, \theta) := \sigma(s) = \begin{cases} 0, & s \leq 8\tilde{T}_0 \\ 1, & s \geq 9\tilde{T}_0 \end{cases}\]

Without loss of generality, we show how to redefine the perturbed Witten map on nearby curves of \( \mathcal{C} \) which have only one Ramond nodal point \( p \) connecting two components \( \mathcal{C}_\mu \)
and $\mathcal{C}_v$. Then a neighborhood of $\mathcal{C}$ in $M_{g,k,w}(\gamma)$ is given by

$$V_{\mathcal{C}} \times ([T_0, \infty] \times S^1)_p / \text{Aut}(\mathcal{C}),$$

where $(V_{\mathcal{C}} \times ([T_0, \infty] \times S^1)_p, \text{Aut}(\mathcal{C}))$ is a uniformizing system of $[\mathcal{C}] \in M_{g,k}(\Gamma)$.

The rigidified $W$-curve $\mathcal{C}_{v,\epsilon,\zeta} = (p, \theta_p)$ is constructed by gluing the $W$-structures on the corresponding domains $[\mathcal{T}^1_p, \mathcal{T}^3_p]$ of two components of $C$. The redefined perturbed Witten map on $\mathcal{C}_{v,\epsilon}$ is defined as

$$WM_{v,\epsilon} := [WM_{\mathcal{C}_{v,\epsilon}}^{[g],h} : g \in G],$$

where

$$WM_{\mathcal{C}_{v,\epsilon}}^{[g],h}(u_1, \ldots, u_N) := \left\{ \tilde{\partial}_{\mathcal{C}_{v,\epsilon}} u_1 + I_1 \left( \frac{\partial(W + \sigma(\zeta)\beta_1 W_0,\gamma)}{\partial u_1} \right), \ldots, \tilde{\partial}_{\mathcal{C}_{v,\epsilon}} u_N + I_1 \left( \frac{\partial(W + \sigma(\zeta)\beta_N W_0,\gamma)}{\partial u_N} \right) \right\}.

(95)$$

It is easy to see that the redefined map $WM$ is $\text{Aut}(\mathcal{C})$-equivariant on $V_{\mathcal{C}} \times ([T_0, \infty] \times S^1)_p$.

So in this way we can redefine the perturbed Witten map on the nearby curves of $[\mathcal{C}] \in M_{g,k,w}(\Gamma)$. Note that the perturbation term $\sigma \beta_j W_0,\gamma$ will disappear if the gluing domain approaches $2T_0 \times S^1$; thus the redefined perturbed Witten map will be compatible with the original perturbed Witten map defined on $M_{g,k,w}(\Gamma')$ satisfying $\Gamma < \Gamma'$.

Now we do the induction assumption. Take the space $M_{g,k,w}(\Gamma')$ and assume that for any dual graph $\Gamma$ such that $\Gamma < \Gamma'$ we have already redefined the perturbed Witten map on the nearby curves of $M_{g,k,w}(\Gamma)$. We want to define the perturbed Witten map on the nearby curves of $M_{g,k,w}(\Gamma')$. Take a finite open covering $\mathcal{U}$ of $M_{g,k,w}(\Gamma)$ in $M_{g,k,w}(\Gamma')$ such that any point in the open set of this covering has the redefined perturbed Witten map. The perturbed Witten map was already constructed over any point in the compact complement $M_{g,k,w}(\Gamma') - \mathcal{U}$. We redefine the perturbed Witten map on nearby curves around those points as done in the first step. Now it is possible that for a nearby curve $[\mathcal{C}]$ there are two definitions of the perturbed Witten maps; one comes from the original definition in $M_{g,k,w}(\Gamma)$ and the other one from the new definition. But these two definitions are identical since the deformation domain and the resolution domain are separated. One can either deform the complex structure first and then redefine the perturbed Witten map or redefine the perturbed Witten map first and then deform the complex structure. Thus we construct the redefined perturbed Witten map on the nearby curves of $M_{g,k,w}(\Gamma')$. By induction, we can define the global perturbed Witten map at any point of $M_{g,k,w}(\gamma)$.

This global multisection $WM : B^0 \to \pi^* B^0,1$ is called the perturbed Witten map over $M_{g,k,w}(\gamma)$.

**Definition 4.2.5.** Let $WI_\mathcal{C}$ represent the branch $WM_{\mathcal{C}}^{[g],h}$ of the perturbed Witten map $WM_\mathcal{C}$; then the perturbed Witten equation over $\mathcal{C}$ is defined as

$$WI_\mathcal{C}(u_1, \ldots, u_N) = 0.

(96)$$

In the following, we also call the branch map $WI_\mathcal{C}$ as the perturbed Witten map.

This means that on each component $\mathcal{C}_v$, we have
\[ \bar{\partial}_g \bar{u}_{ix} + i \left( \frac{\partial(W + W_{0j})}{\partial u_{iy}} \right) = 0, \forall i = 1, \ldots, N, \]  

(97)

where \( u_{ix} \) is the \( v \)-component of the section \( u_i \) and \( W_{0j} \) represents the perturbed term.

In view of the definitions, we have

\[ WM^{-1}_g(0) = \text{Aut}_g \cdot W I^{-1}_g(0). \]  

(98)

Because of the group action, we have \( WM^{-1}_g(0) = \bigcup_{g \in G} WM^{-1}_g(0) / \text{Aut}(G) \subset B^0 \).

This set is rather complicated because transversality does not hold. Its topology is weak and can’t be characterized by the strong topology from the Banach bundle. But we can give it the Gromov-Hausdorff topology, and prove the Gromov compactness theorem. Finally, we can show that it carries an orientable Kuranishi structure if the perturbation is strongly regular. This method to construct the virtual cycle has already been used in the proof of the Arnold conjecture and in the construction of Gromov-Witten invariants in general symplectic manifolds (see [FO, LT, LT, R, Sb] and etc.).

**Remark 4.2.6.** Now we discuss the gluing of the Witten equations on two components connected by a nodal point. Let \( \mathcal{C}' \) be a nodal curve with one Ramond nodal point \( p \) connecting two components \( \mathcal{C}_r \) and \( \mathcal{C}_\mu \). Let \( (e_{ir}, \zeta_r) \) and \( (e_{i\mu}, \zeta_\mu) \) be the standard basis and coordinates of line bundles \( \mathcal{L}_r \) on \( \mathcal{C}_r \) and \( \mathcal{C}_\mu \), respectively, such that

\[
W(e_{1r}, \ldots, e_{Nr}) = \frac{dz_r}{z_r} = -\frac{dz_\mu}{z_\mu} = -W(e_{1\mu}, \ldots, e_{N\mu}).
\]

Then the perturbations we choose in the \( \mu \)-component and \( v \)-component are not independent. Now the relation between the two families of perturbation parameters is shown as the conclusion of the following facts.

As before, we take the cylindrical coordinates \( z_r = e^{i\zeta_r} \) and \( z_\mu = e^{i\zeta_\mu} \). Then we have the relation

\[
W(e_{1r}, \ldots, e_{Nr}) = -d\zeta_r, \quad \zeta_r \in [0, \infty)
\]

\[
W(e_{1\mu}, \ldots, e_{N\mu}) = -d\zeta_\mu, \quad \zeta_\mu \in [0, \infty),
\]

and

\[
d\zeta_r = -d\zeta_\mu, \quad \zeta_r + \zeta_\mu = \zeta_p,
\]

where \( \zeta_p \) is the gluing parameter if we want to do the gluing operation.

We have the following facts:

1. In the coordinate system \( (z_r, e_{i,r}) \), the perturbed polynomial

\[
W_{0r} = \sum_j b_{jr} \beta_{jr} u_{jr} \in \Omega(\frac{dz_r}{z_r}).
\]

A similar expression holds on the \( \mu \) component.

2. Assume that \( u_{ix} = \bar{u}_{ix}', e_{ix}' \); then the perturbed Witten equation has the form

\[
\bar{\partial}_g \bar{u}_{ix} - \frac{\partial W}{\partial u_{iy}} - 2 \partial_{i,y} = 0.
\]  

(99)

In \( (z_\mu, e_{i\mu}) \) coordinates, we have \( u_{i\mu} = \bar{u}_{i\mu}' e_{i\mu} \) and the corresponding equation

\[
\bar{\partial}_g \bar{u}_{i\mu} - \frac{\partial W}{\partial u_{i\mu}} - 2 \partial_{i,y} = 0.
\]  

(100)
(3) Let $\xi^e = -1$ and $\hat{e}_{i,J} = \xi^k e_{i,J}$, we have

$$W(\hat{e}_{i,J}, \cdots, \hat{e}_{N,J}) = d\xi^e.$$ 

When we do gluing, first we need to identify the coordinate $\zeta_\mu = \zeta_\mu - \zeta_\nu$ and so $d\zeta_\mu = d\zeta_\nu$. Secondly, we should identify the line bundles on two components by identifying the basis $e_{i,J}$ with $\hat{e}_{i,J}$.

(4) The local expression of the perturbed Witten equation in the coordinate system $(\zeta_\mu, \hat{e}_{i,J})$ is given as below. We have

$$u_{i,J} = \hat{u}_{i,J} \hat{e}_{i,J} = \tilde{u}_{i,J} e_{i,J},$$

and so

$$\tilde{u}_{i,J} = \xi^{-k} \hat{u}_{i,J}.$$ 

Substituting the above equality into Equation (100), one has

$$\tilde{\partial}_{\zeta_\nu} (\xi^k \tilde{u}_{i,J}) - \frac{2\partial W}{\partial \tilde{u}_{i,J}} (\xi^k \tilde{u}_{i,J}, \cdots, \xi^k \tilde{u}_{N,J}) - 2\tilde{b}_{i,J} = 0.$$ 

This is equivalent to

$$\tilde{\partial}_{\zeta_\nu} \tilde{u}_{i,J} - \frac{2\partial W}{\partial \tilde{u}_{i,J}} + 2\xi^k \tilde{b}_{i,J} = 0. \tag{101}$$

So in $(\zeta_\nu, \hat{e}_{i,J})$ coordinates, we have

$$\tilde{\partial}_{\zeta_\nu} \tilde{u}_{i,J} - \frac{2\partial W}{\partial \tilde{u}_{i,J}} = 0. \tag{102}$$

(5) After transformation, Equation (102) should be the same as Equation (99). So we obtain the relation between two parameter groups:

$$b_{i,J} = -\xi^k \tilde{b}_{i,J}. \tag{103}$$

and at the nodal point $p$ one has

$$e_{i,J} = \hat{e}_{i,J}, \tilde{u}_{i,J}(+\infty) = \hat{u}_{i,J}(+\infty).$$

(6) Let $I : \mathbb{C}^N \to \mathbb{C}^N$ be the map defined by the multiplication of the diagonal matrix $\text{diag}(\xi^k_1, \cdots, \xi^k_N)$. If $W + \xi^{k+\epsilon} \sum_j b_{j,J} \hat{u}_{j,J}$ has critical point $\hat{\alpha}^i$ and corresponding critical value $\hat{v}^i$, it is easy to show that $(I(\hat{\alpha}^i), -\hat{v}^i)$ is the critical point and critical value of $W + \sum_j b_{j,J} \hat{u}_{j,J}$.

(7) If the rigidifications over the two components are not standard but arbitrary, we can still glue the perturbed Witten equations. However the identification map $I$ is replaced by the composition of $I$ and the inverse of the corresponding rigidifications.

**Definition 4.2.7.** Sections $(u_1, \cdots, u_\nu)$ of $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_\nu$ on $\mathcal{C}$ are said to be solutions of the perturbed Witten equation (99) if they satisfy the following conditions:

1. for each $j, u_\nu \in L^2_{\text{loc}}(\mathcal{C}\setminus\{z_1, \cdots, z_\kappa\}, \mathcal{L}_j|_{\mathcal{C}_\nu}), I_1 \left( \frac{\partial W}{\partial u_\nu} + \frac{\partial W}{\partial u_\nu} \right) \in L^2_{\text{loc}}(\mathcal{C}\setminus\{z_1, \cdots, z_\kappa\}, \mathcal{L}_j \otimes \Lambda^{0,1}|_{\mathcal{C}_\nu})$, where $u_\nu$ is the component of $u_j$ and $\kappa$ is the number of marked points and nodal points on $\mathcal{C}_\nu$;
2. $(u_1, \cdots, u_\nu)$ satisfy the perturbed Witten-equation (99) on $\mathcal{C}_\nu$ almost everywhere;
3. near each marked or nodal point, the integral

$$\sum_j \int_0^\infty \int_{s_1} \frac{\partial u_\nu}{\partial \hat{s}} \hat{u}_\nu ds d\theta ds < \infty.$$
(4) If \( p \) is a Ramond nodal point of \( C \) between two components \( C_\nu \) and \( C_\mu \), then
\[
\lim_{s \to +\infty} (u_1(s, \theta), \ldots, u_s(s, \theta)) = \lim_{s \to -\infty} (u_{1, \mu}(s, \theta), \ldots, u_{s, \mu}(s, \theta)).
\]

Remark 4.2.8. Using the notations from Remark 4.2.6, we discuss the condition (4) in the above definition. The condition (4) is given in section form. Assume that on the \( \nu \)-component, \( u_{i, \nu} \) satisfies Equation (99) with \( u_{i, \nu}(+\infty) = \kappa'_i \), where \( \kappa'_i, \alpha_i \) are the \( j \)-th critical point and critical value of \( W + \sum b_i z u_i \). Now the condition (4) implies that the corresponding value in the \( \mu \)-coordinate should satisfy
\[
\bar{u}_{i, \mu}(+\infty) = \xi^i \kappa'_i,
\]
for a local solution \( \bar{u}_{i, \mu} \) of Equation (100).

On the component \( C_\nu \), the equation (97) has a corresponding equation defined on the resolved curve \( C \):
\[
\bar{\partial}_{C_\nu} g_{i, \nu} + J_i \left( \frac{\partial(W + W_{0, \rho})}{\partial g_{i, \nu}} \right) = 0, \; \forall i = 1, \ldots, N, \tag{104}
\]
where \( g_{i, \nu} \in \Omega(C_\nu, [Z]|_{C_\nu}) \). From subsection 3.3.3 we know that there is a one-one correspondence between the solutions of these two equations. In fact, let \( p \) be a marked or nodal point on the component \( C_\nu \), and take a uniformizing system \((\Delta \times \mathbb{C}, G_p)\) of the orbifold line bundle \( L_p \) near \( p \). If \((z, u_{i, \nu}(z))\) be the local solution of (97); then \( u_{i, \nu}(z) = g_{i, \nu}(z^m)(z^n)^{\theta_i} \). The properties of the solutions of Equation (104) have been fully studied in Section 3. Hence we can easily get the following theorems:

**Theorem 4.2.9.** The solutions of the perturbed Witten equation (96) satisfy the following conclusions:

1. **Interior estimate:** for any ball \( B_{2R} \) lying outside the cylindrical neighborhood of any marked or nodal points, there exists a constant \( C \) depending only on \( R, m \in \mathbb{Z} \), and the metric in \( B_{2R} \) such that
\[
\|u_i\|_{C^m(B_{2R})} \leq C.
\]

2. **Boundedness estimate on the cylinder:** Let \( p \) be a marked or nodal point with cylinder neighborhood \([0, \infty) \times S^1\). Then for any compact set \( K \subset [0, \infty) \times S^1 \), there is a constant depending only on \( m \in \mathbb{Z}, K \), and the perturbation parameters \( b \) in \( W_{0, \gamma_p} \), such that
\[
\|u_i\|_{C^m(K)} \leq C.
\]

Here the derivatives are taken with respect to the cylindrical coordinates \((s, \theta)\).

3. **Asymptotic behaviors:** Without loss of generality, we assume that for \( 1 \leq i \leq N_p \), the solutions \( u_{i, \nu} \)'s are sections of Ramond line bundles and for \( N_p + 1 \leq i \leq N \), the solutions \( u_{i, \nu} \)'s are sections of Neveu-Schwarz line bundles. Then we have:

- **Let** \( W_{0, \gamma_p}(u_1, \ldots, u_{N_p}) = \sum_{k=1}^{N_p} b_k u_k \) **be any** \( W_{\gamma_p} \)-regular polynomial at \( p \). **Then if** \( \sum_{k=1}^{N_p} |b_k| \) **is sufficiently small, there exist constants** \( C, T, \delta > 0 \) **such that for any** \((s, \theta) \in S^1 \times [T, \infty)\),
\[
|u_i - \kappa_i| < Ce^{-\delta s}, \text{for } i = 1, \ldots, N_p,
\]
where \((\kappa_1, \ldots, \kappa_{N_p})\) is some critical point of \( W_{\gamma_p} + W_{0, \gamma_p} \).
• If $N_\rho + 1 \leq i \leq t$, then there exist constants $C, T, \delta > 0$ such that for any $(s, \theta) \in S^1 \times [T, \infty)$,

$$
\|u_i\|_{C^1([T, \infty))} \leq Ce^{-\delta s}.
$$

(105)

(4) **Continuity:** Assume that $p$ is a nodal point connecting two components $\gamma_0$ and $\gamma_\rho$; then $u_i,\gamma(p) = 0 = u_i,\rho(p)$ naturally hold for Neveu-Schwarz sections $u_\rho, N_\rho + 1 \leq i \leq N$. For Ramond sections $u_\rho, 1 \leq i \leq N_\rho$, we have $u_\rho,\rho(p) = \lim_{s_\rho \to \infty} u_\rho,\rho(s_\rho) = \kappa_i = \lim_{s_\rho \to \infty} u_\rho,\rho(s_\rho) = u_\rho,\rho(p)$, which is required by the definition of solutions. Here $\kappa = (\kappa_1, \cdots, \kappa_N)$ is some critical point of $W + W_\rho$.

**Remark 4.2.10.** The above conclusions hold for any solutions of the perturbed Witten equations at any point $\gamma \in \mathcal{H}_{\tilde{g},\tilde{W}}(\gamma)$. One may worry about the equations on $\gamma$, which is the resolution curve of $\gamma$. In this case, the perturbed Witten map has a perturbation term appearing in the interior of the glued curve. Now the interior estimate on $\gamma$ is the combination of the interior estimate and the bounded estimate of $\gamma$.

> From the description of Theorem 4.2.9 we know that any solutions of the perturbed Witten equation actually lie in the space $C^\infty(\gamma, \mathcal{L}) \times \cdots \times C^\infty(\gamma, \mathcal{L}_N)$.

4.2.2. **Soliton space.** Let $\mathcal{E} = (\gamma, p_1, \cdots, p_k, \mathcal{L}_1, \cdots, \mathcal{L}_N, \varphi_1, \cdots, \varphi_N, \psi_1, \cdots, \psi_N)$ be a rigidified semistable $W$-curve with $k$ marked points. The $W$-structure induces a group element $\gamma_p \in G$ at $p_i$. The action of this local group determines whether the line bundles are Ramond or Neveu-Schwarz at $p_i$ and hence determines the choice of the polynomial $W_{\gamma_p}$ and the perturbed polynomial $W_{0,\gamma_p}$. Choose the coefficients such that each $W_{0,\gamma_p}$ is $W_{\gamma_p}$-regular and then fix $W_{0,\gamma_p}$. Under the rigidification, (i.e., choosing the local bases around the marked point), we obtain $\deg(W_{\gamma_p}) - 1$ critical points of the polynomial $W_{0,\gamma_p} + W_{\gamma_p}$ in $\mathbb{C}^N$, which are all non-degenerate. We use $\kappa_i$ to denote the critical points at the marked point $p_i$. Remember $\kappa_i$ can take $\deg(W_{\gamma_p}) - 1$ values. Denote $\kappa := (\kappa_1, \cdots, \kappa_N).

Now we study a special solution space related to a marked point $p$. The data attached to $p$ is $\gamma \in G$, the image of the generator of local group $G_p$ in $G$, and $\kappa_p$, some nondegenerate critical points of $W_{0,\gamma_p} + W_{\gamma_p}$. Let $N_\rho$ be the number of Ramond line bundles at $p$, we give an order to the set of these critical points $\kappa^1, \cdots, \kappa^{\deg(W_{\gamma_p}) - 1}$ in $\mathbb{C}^{N_\rho}$ space such that $\text{Re}((W_{\gamma_p} + W_{0,\gamma}(\kappa^i)) \geq \text{Re}((W_{\gamma_p} + W_{0,\gamma}(\kappa^j))$ if $i > j$.

Let $\mathcal{L}_1, \cdots, \mathcal{L}_N$ be flat orbifold line bundles defined on the infinitely long cylinder $\mathbb{R} \times S^1$, where the monodromy representation is given by

$$
\rho_p : 1 \in \mathbb{Z} \cong \pi_1(\mathbb{R} \times S^1) \to \gamma \in G \subset U(1)^N.
$$

At the point $-\infty$, the orbifold action on $\mathcal{L}_1, \cdots, \mathcal{L}_N$ is given by $\gamma^{-1}$ and at $\infty$ it is given by $\gamma$. We have the perturbed Witten equation (in cylindrical coordinates) defined on $\mathbb{R} \times S^1$:

$$
\frac{\partial u_i}{\partial \xi} - 2\frac{\partial (W + W_{0,\gamma})}{\partial u_i} = 0.
$$

(106)

By the asymptotic analysis in Section 3 we know that $u$ takes values $(k^+, 0)$ at $-\infty$ (notice the sign!), where $k^+$ or $k^-$ are two critical points of $W_\gamma + W_{0,\gamma}$. The solution of this equation is called the **soliton solution** of type $\gamma_p$ connecting $k^+$ and $k^-$ and is denoted by $(u_{\gamma_p}, k^+, \gamma_p)$.

We have the following Witten lemma.

**Lemma 4.2.11.** If $p$ is a Neveu-Schwarz point, then the related soliton equation (106) at $p$ has only the zero solution.
**Proof.** This is a special case of the Witten lemma we proved in section 3. □

By the above lemma, we only need to consider the soliton equation related to Ramond marked points. Set \( p \) as a Ramond marked point. Due to the proof of Lemma 3.3.2, we have

**Lemma 4.2.12.** Let \((u_{κ^+, κ^−}, γ)\) be a soliton; then we have

\[
(W_γ + W_{0, γ})(κ^−) - (W_γ + W_{0, γ})(κ^+) = 2 \int_{−∞}^{+∞} \int_S \left| \frac{∂(W + W_{0, γ})}{∂u_i} \right|^2.
\]

**Corollary 4.2.13.** There is no soliton solution connecting \( κ^− \) and \( κ^+ \), if \( \text{Im}(W_γ + W_{0, γ})(κ^−) ≠ \text{Im}(W_γ + W_{0, γ})(κ^+) \), and only the trivial solution if \( κ^− = κ^+ \).

According to this corollary, the soliton solution connecting two different critical points \( κ_i \) and \( κ_j \) exist, only if \( \text{Im}(W_γ + W_{0, γ})(κ') = \text{Im}(W_γ + W_{0, γ})(κ') \).

**Corollary 4.2.14.** Let \( p \) be a Ramond marked point on a \( W \)-curve. If the perturbed polynomial \( W_0, γ \) is strongly \( W \)-regular, then the related soliton equation has no nontrivial solution.

Now we consider a kind of special soliton solution which is independent of the angle \( θ \). Assume as before that the first \( N_p \) components of \( u \) are Ramond sections and the last components are Neveu-Schwarz sections. Since the Neveu-Schwarz line bundles are nontrivial bundles, there is no nontrivial section which is independent of \( θ \). Hence the equation (106) becomes

\[
\frac{∂u_i}{∂s} - 2 \frac{∂(W_γ + W_{0, γ})}{∂u_i} = 0, \quad s = 1, \cdots, N_p.
\]

(107)

This special solution is a called BPS soliton with respect to the superpotential \( W_γ + W_{0, γ} \) in Landau-Ginzburg theory in physics. By the above equation, we can easily get

\[
∂_s(W_γ + W_{0, γ})(s) = \left| \frac{∂(W_γ + W_{0, γ})}{∂u_i} \right|^2.
\]

This shows that the imaginary part of \( (W_γ + W_{0, γ})(s) \) is invariant under the evolution of the flow, and the real part increases monotonically.

**Stable manifolds and vanishing cycles.** So to count the number of solitons connecting two critical points, e.g., \( κ^1 \) and \( κ^2 \) we need to study the intersection behavior of the stable manifold at \( κ^2 \) and the unstable manifold at \( κ^1 \). Here we define the unstable manifold at \( κ^1 \) to be the following set in \( \mathbb{C}^{N_s} \):

\[
\mathbb{C}^u(κ^1) := \{(u_1, \cdots, u_{N_s}) ∈ \mathbb{C}^{N_s} | ((u_1, \cdots, u_{N_s})) : s → κ^1, \quad \text{as} \quad s → −∞\},
\]

where \( (u_1, \cdots, u_{N_s}) : s \) represents the flow line going through \( (u_1, \cdots, u_{N_s}) \) at time \( s = 0 \). Similarly, we can define the stable manifold of \( κ^2 \):

\[
\mathbb{C}^s(κ^2) := \{(u_1, \cdots, u_{N_s}) ∈ \mathbb{C}^{N_s} | ((u_1, \cdots, u_{N_s})) : s → κ^2, \quad \text{as} \quad s → +∞\}.
\]

It is wellknown that \( \mathbb{C}^u(κ^2) \) and \( \mathbb{C}^s(κ^1) \) are all real dimension \( n \). They are submanifolds of the \((2n − 1)\)-dimensional subspace

\[
\{(u_1, \cdots, u_{N_s}) ∈ \mathbb{C}^{N_s} | \text{Im}(W_γ + W_{0, γ})(u_1, \cdots, u_{N_s}) = \text{Im}(W_γ + W_{0, γ})(κ^1)\}.
\]

So for generic parameters \( b_1, \cdots, b_{N_s} \) such that \( \text{Im}(W_γ + W_{0, γ})(κ^1) = \text{Im}(W_γ + W_{0, γ})(κ^2) \), the geometric orbits connecting \( κ^1 \) and \( κ^2 \) are finitely many. Actually this can be given by the intersection numbers of two vanishing cycles which represent the two critical points.
respectively. In fact, take a point \( w = (W_γ + W_0γ)((u_1, \cdots, u_{Nγ}) \cdot (s_0)) \) on the segment connecting \( \text{Im}(W_γ + W_0γ)(κ^j) \) and \( \text{Im}(W_γ + W_0γ)(κ^i) \); then the \((n - 1)\)-dimensional intersection submanifolds \( Δ^1 := (W_γ + W_0γ)^{-1}(w) \cap C^i(κ^i) \) and \( Δ^2 := (W_γ + W_0γ)^{-1}(w) \cap C^i(κ^i) \) are just vanishing cycles representing \( κ^1 \) and \( κ^2 \). For example, when \( s_0 \to -∞, (W_γ + W_0γ)^{-1}(w) \cap C^i(κ^i) \) will shrink to the critical point \( κ^i \).

We have the well-known result from Picard-Lefschetz theory:

**Theorem 4.2.15.** The number of BPS solitons connecting \( κ^1 \) and \( κ^2 \) is given by the intersection number \( Δ^1 \cdot Δ^2 \).

The computation of the number of BPS soliton is an important part of classical singularity theory.

**Definition 4.2.16.** Let \( [κ^1, \cdots, κ^{\deg(Wγ)+1}] \) be the set of nondegenerate critical points of \( W_γ + W_0γ \). Let \( i_0 < j_1 < \cdots < j_p < i_1 \), define \( S_γ(κ^1, \cdots, κ^{i_p}) \) to be the space of broken soliton, \( \{u_{i_0j_1}, u_{j_1j_2}, \cdots, u_{i_pj_1}\} \), where \( u_{i_0j_1} \) is the soliton of type \( γ \) connecting \( κ^1 \) and \( κ^{i_1} \), and so on. We denote by \( S_γ(κ^1, κ^{i_1}) \) the space of all kinds of solitons of type \( γ \) (including broken solitons) from \( κ^1 \) to \( κ^{i_1} \).

Obviously \( S_γ(κ^1, κ^{i_1}) \) is not empty if and only if \( \text{Im}(W_γ + W_0γ)(κ^1)) = \text{Im}(W_γ + W_0γ)(κ^{i_1}) \).

The group \( C \) acts on \( S_γ(κ^1, κ^{i_1}) \) by \( (s_0, θ_0) \cdot (u_{i_0j_1}, u_{j_1j_2}, \cdots, u_{i_pj_1}) = (u_{i_0j_1}, u_{j_1j_2}, \cdots, u_{i_pj_1}) \) and the group \( \mathbb{C}^{p+1} \) also acts on \( S_γ(κ^1, κ^{i_1}) \) in an obvious way.

If the space \( S_γ(κ^1, κ^{i_1}) \) is not empty, the soliton \( \{u_{i_0j_1}, u_{j_1j_2}, \cdots, u_{i_pj_1}\} \) can be viewed as the broken trajectory connecting the periodic solutions \( κ^1 \) and \( κ^{i_1} \) in some sense. So imitating the construction of a trajectory space in symplectic geometry, one can consider the following equivalence relation:

\[
\{u_{i_0j_1}, u_{j_1j_2}, \cdots, u_{i_pj_1}\} \sim \{u'_{i_0j_1}, u'_{j_1j_2}, \cdots, u'_{i_pj_1}\}
\]

if there exist real constants \( s_1, \cdots, s_{p+1} \) such that

\[
(s_1, \cdots, s_{p+1}) \cdot \{u_{i_0j_1}, u_{j_1j_2}, \cdots, u_{i_pj_1}\} = \{u'_{i_0j_1}, u'_{j_1j_2}, \cdots, u'_{i_pj_1}\}.
\]

Then we define the moduli space of "geometrical" solitons as \( \hat{S}_γ(κ^1, κ^{i_1}) = S_γ(κ^1, κ^{i_1})/\sim \). There is an \( S^1 \)-action on the space \( \hat{S}_γ(κ_i, κ_{i+1}) \). The fixed points of this space are just the BPS solitons. However, in our setting, we will not care about the space \( \hat{S}_γ(κ_i, κ_{i+1}) \), but the quotient space \( \tilde{S}_γ(κ_i, κ_{i+1}) := \hat{S}_γ(κ_i, κ_{i+1})/S^1 \). BPS solitons become the singularities of the \( S^1 \)-action.

**4.2.3. Moduli space \( \tilde{M}_{g,k,W}(γ, χ) \).**

Define the section \( u := (u_1, \cdots, u_N) \in C^∞(C, \mathbb{L}_1 \times \cdots \times \mathbb{L}_N) \) by \( C^∞(C, \mathbb{L}_1) \times \cdots \times C^∞(C, \mathbb{L}_N) \).

**Definition 4.2.17.** Let \( C = (C, p_1, \cdots, p_k, \mathbb{L}_1, \cdots, \mathbb{L}_N, φ_1, \cdots, φ_s, ψ_1, \cdots, ψ_k) \) be a rigidified stable \( W \)-curve, and \( u \) be a solution of the perturbed Witten equation on \( C \). Then the tuple \( (C, u) \) is said to be a stable \( W \)-section.

**Definition 4.2.18.** The automorphism group of \((C, u)\) is defined as

\[
\text{Aut}(C, u) := \{τ ∈ \text{Aut}(C) | τ(u) = u\}
\]

Since \( \text{Aut}(C, u) \) is a subgroup of \( \text{Aut}(C) \), we have

**Lemma 4.2.19.** The automorphism group \( \text{Aut}(C, u) \) is a finite group if \( C \) is a \( W \)-stable curve.
Definition 4.2.20. We say that a stable section \((\mathcal{C}, u)\) is of type \((\gamma, \kappa)\) if at each marked point \(p_i\) of \(\mathcal{C}\) the generator of the local group \(G_{p_i}\) is \(\gamma_{p_i}\) and the section \(u\) which is viewed as the section in \(\mathbb{C}^N\) by the standard rigidification at \(p_i\) takes the given value \(\kappa_i\) for \(1 \leq i \leq k\).

Definition 4.2.21. The moduli space of \(\mathcal{M}_{g, k, W}^{rig}(\gamma, \kappa)\) is the space consisting of isomorphism classes of all sections \((\mathcal{C}, u)\) of type \((\gamma, \kappa)\) under the action of the automorphism group \(\text{Aut}((\mathcal{C}, u))\). The subspace \(\mathcal{M}_{g, k, W}^{rig}(\gamma, \kappa)\) of \(\mathcal{M}_{g, k, W}^{rig}(\gamma, \kappa)\) is the space consisting of the elements in \(\mathcal{M}_{g, k, W}^{rig}(\gamma, \kappa)\) having dual graph \(\Gamma\).

If the perturbed polynomial \(W_{\gamma, \kappa}\) at any Ramond marked or nodal point \(p_i\) is strongly \(W_{\gamma, \kappa}\)-regular, we call the moduli space \(\mathcal{M}_{g, k, W}^{rig}(\gamma, \kappa)\) strongly regular. However, when \(\mathcal{M}_{g, k, W}^{rig}(\gamma, \kappa)\) is not strongly regular, then it is not compact with respect to the Gromov convergence which will be defined later. The loss of compactness is due to the existence of solitons. So we need to add the corresponding limits to our moduli space.

Definition 4.2.22. Let \((\mathcal{C}, u)\) be a stable \(W\)-section of type \((\gamma, \kappa)\), where \(\gamma = (\gamma_1, \cdots, \gamma_k)\) and \(\kappa = (\kappa_1, \cdots, \kappa_k)\). Take a marked point \(p_0\) of \(\mathcal{C}\) such that \(\psi_0(u)(p_0) = \kappa_0\), where \(\psi_0\) is the rigidification at \(p_0\) and \(\kappa_0\) is the \(j_0\)th critical point of the perturbed polynomial \(W_{p_0} + W_{\gamma, \kappa}\). We simply write it as \(\mathbf{u}(p_0) = \kappa_0\) if no ambiguity can occur. Let \((\mathbf{u}_{j_0}, \gamma_{j_0})\) be a soliton in \(\mathcal{S}_{\gamma, \kappa}(\gamma_{j_0}, \kappa_{j_0})\). We define \(\mathcal{C}#_{p_0}(\mathbb{R} \times S^1)\) to be the connected sum of the \(W\)-curve \(\mathcal{C}\) and the \(W\)-curve \(\mathbb{R} \times S^1\) by identifying the marked point \(p_0\) in \(\mathcal{C}\) and the \(-\infty\) point of \(\mathbb{R} \times S^1\). Similarly, on the new \(W\)-curve, we can define the connected sum \(\mathbf{u}#_{p_0} \mathbf{u}_{j_0, j_1}\) in a natural way. We call the pair \((\mathcal{C}#_{p_0}(\mathbb{R} \times S^1), \mathbf{u}#_{p_0} \mathbf{u}_{j_0, j_1})\) a soliton \(W\)-section. If \((\mathcal{C}', \mathbf{u}')\) is another stable \(W\)-curve having a marked point \(p_i\) labelled by \(\gamma_i = \gamma_{j_0}\) and \(\mathbf{u}'(p_i) = \kappa_{j_0}\) = \(\mathbf{u}_{j_0, j_1}(+\infty)\), we can construct another pair \((\mathcal{C}#_{p_0}(\mathbb{R} \times S^1)\#_{p_i} \mathcal{C}', \mathbf{u}#_{p_0} \mathbf{u}_{j_0, j_1} \#_{p_i} \mathbf{u}')\). This pair is also called a soliton \(W\)-section. In the same way, one can continue to construct new soliton \(W\)-sections if the two glued soliton \(W\)-sections satisfy the compatibility conditions at the gluing point.

There is a natural group action on the soliton \(W\)-curve. For example, assume that \((\mathcal{C}_1, \mathbf{u}_1) \in \mathcal{M}_{g, k, W}^{rig}(\gamma_1, \kappa_1)\), \((\mathcal{C}_2, \mathbf{u}_2) \in \mathcal{M}_{g, k, W}^{rig}(\gamma_2, \kappa_2)\) and \((\mathbf{u}_1, \gamma_1, \kappa_1) \in \mathcal{S}_{\gamma, \kappa}(\gamma_{j_0}, \kappa_{j_0})\) satisfy the compatibility conditions at marked points \(p_0\) on \(\mathcal{C}_1\) and \(p_i\) on \(\mathcal{C}_2\). Then we can get a soliton \(W\)-section \((\mathcal{C}_1#_{p_0}(\mathbb{R} \times S^1)\#_{p_i} \mathcal{C}_2, \mathbf{u}_1#_{p_0} \mathbf{u}_{j_0, j_1} \#_{p_i} \mathbf{u}_2)\). Let \(g_i \in \text{Aut}(\mathcal{C}, \mathbf{u}_i), i = 1, 2\) and \((s_0, \theta_0) \in \mathbb{C}\); the action of \((g_1, (s_0, \theta_0), g_2)\) of the above soliton \(W\)-section is defined in an obvious way and we can define its automorphism group. Obviously the \(W\)-curve \(\mathcal{C}_1#_{p_0}(\mathbb{R} \times S^1)\#_{p_i} \mathcal{C}_2\) has genus \(g_1 + g_2 + k_1 + k_2 - 2\) marked points and may not be a stable \(W\)-curve. The \(k_1 + k_2 - 2\) marked points are labelled by the set \(\overline{\gamma} = (\gamma_1, \gamma_2) - \{\gamma_{p_0}, \gamma_{p_i}\}\) and \(\overline{\kappa} = (\kappa_1, \kappa_2) - \{\kappa_{p_0}, \kappa_{p_i}\}\). We say this soliton \(W\)-section is of type \((\overline{\gamma}, \overline{\kappa})\).

Definition 4.2.23. The moduli space of \(\mathcal{M}_{g, k, W}^{rig}(\gamma, \kappa)\) is the space consisting of the isomorphism classes of all soliton \(W\)-sections of type \((\gamma, \kappa)\) under the action of its automorphism group.

**Combinatorial types of soliton \(W\)-sections.** We have the relation \(\mathcal{M}_{g, k, W}^{rig}(\gamma) = \bigsqcup_{\Gamma} \mathcal{M}_{g, k, W}^{rig}(\gamma, \kappa)\), where the summation is taken over all possible \(G\)-decorated dual graphs. Note that each half-edge \(\tau\) of a \(G\)-decorated dual graph is decorated with a group element \(\gamma_{\tau} \in G\). The number of these \(\Gamma\) is finite. Each \(\mathcal{M}_{g, k, W}^{rig}(\Gamma)\) is a multiple covering of \(\mathcal{M}_{g, k, W}(\Gamma)\) under the
stable map \( s^{rig} \) and the degree of \( s^{rig} \) is finite because of different rigidified \( W \)-structures.

Define \( \text{Comb}(g, k, W; \gamma) \) to be the combinatorial types of \( W \)-curves in \( \mathcal{M}^{rig}_{g,k,W}(\gamma) \). Then this set is a finite set. The partial order \( > \) in \( \text{Comb}(g, k) \), the set of combinatorial types in \( \mathcal{M}^{rig}_{g,k} \), induces a more refined partial order \( > \) (including the information of rigidified \( W \)-structures) in \( \mathcal{M}^{rig}_{g,k,W}(\gamma) \) such that it becomes a stratified space.

Now we consider the combinatorial types in \( \mathcal{M}^{rig}_{g,k,W}(\gamma, \kappa) \). There are new combinatorial types in \( \mathcal{M}^{rig}_{g,k,W}(\gamma, \kappa) \) compared to those in \( \mathcal{M}^{rig}_{g,k,W}(\gamma) \) because of the existence of soliton components. We always assume that there exists only one group element \( \hat{\gamma} \in G \) such that the perturbed polynomial \( W_{\hat{\gamma}} + W_{\gamma,0} \) has only two critical points \( k^\pm \) with the property \( \text{Im}(W_{\hat{\gamma}} + W_{\gamma,0}(k^\pm)) = \text{Im}(W_{\gamma} + W_{\gamma,0}(k^\pm)) \).

If a dual graph \( \Gamma \in \text{Comb}(g, k, W; \gamma) \) has a half-edge \( \tau \) decorated with \( \hat{\gamma} \), then we replace the edge \( \hat{\gamma} \) by the dual graph of a soliton of type \( \hat{\gamma} \). We denote the new graph by \( \Gamma(\hat{\gamma}) \) and call it a soliton graph. Define \( \Gamma > \Gamma(\hat{\gamma}) \).

Furthermore, if \( \Gamma \) has multiple edges decorated by \( \hat{\gamma} \), then we can replace each edge by the dual graph of a soliton of type \( \hat{\gamma} \) to get a new graph \( \Gamma^\prime \). We define the order \( \Gamma > \Gamma^\prime \).

Since the number of half-edges \( \tau \) possibly decorated by \( \gamma \) is finite, the number of soliton graphs is finite. Let \( \text{Comb}(g, k, W; \gamma, \kappa) \) be the set of combinatorial types of dual graphs in \( \mathcal{M}^{rig}_{g,k,W}(\gamma, \kappa) \). We have a partial order relation \( > \) in \( \text{Comb}(g, k, W; \gamma, \kappa) \). This set gives a stratification to the moduli space \( \mathcal{M}^{rig}_{g,k,W}(\gamma, \kappa) \). We will use the partial order to glue our Kuranishi neighborhoods to obtain a Kuranishi structure.

Now we begin our construction of the topology in our moduli space. We first define the topology in \( \mathcal{M}^{rig}_{g,k,W}(\gamma, \kappa) \) when it is strongly regular.

Let \( (\mathcal{C}^n, u^n) \) be a sequence of isomorphism classes in \( \mathcal{M}^{rig}_{g,k,W}(\gamma, \kappa) \). Suppose that \( \mathcal{C}^n \in \mathcal{M}^{rig}_{g,k,W}(\gamma) \) converges to \( \mathcal{C} \in \mathcal{M}^{rig}_{g,k,W}(\Gamma) \). This means that there exists a sequence of \( \mathcal{C}^n \in \mathcal{M}^{rig}_{g,k,W}(\Gamma) \) such that \( \mathcal{C}^n \to \mathcal{C} \) in \( \mathcal{M}^{rig}_{g,k,W}(\Gamma) \) and there is a sequence of gluing parameters \( \zeta^n = (\zeta^n_1, \ldots, \zeta^n_{#E(\Gamma)}) \) such that \( \mathcal{C}^n = (\mathcal{C}^n_1, \ldots, \mathcal{C}^n_{#E(\Gamma)}) \) converges to \( \mathcal{C} \) as \( n \to \infty \).

**Definition 4.2.24 (Neck region).** Suppose that \( \mathcal{C}^n \in \mathcal{M}^{rig}_{g,k,W}(\gamma) \) converges to \( \mathcal{C} \in \mathcal{M}^{rig}_{g,k,W}(\Gamma) \). Let \( z \) be any nodal point of the underlying curve \( \mathcal{C} \) of \( \mathcal{C} \) connecting two components \( \mathcal{C}_\nu \) and \( \mathcal{C}_\mu \). Given \( \hat{T} \geq 10 T_0 \) (where \( T_0 \) comes from the definition of cut-off function \( \beta \)), the neck region of the underlying curve \( \mathcal{C} \) of \( \mathcal{C} \) at the nodal point \( z \) is defined as
\[
N_{z,\nu}(\hat{T}) := ([\hat{T}, T^\nu_\mu] \times S^1)_\nu \cup ([\hat{T}, T^\nu_\mu] \times S^1)_\mu
\]
Here \( s^n_\nu = (s^\nu, \theta_\nu) \) and \( s^n_\mu = 2T^\nu_\mu \).

Note that \( \mathcal{C}^n \) is obtained from \( \mathcal{C}^n_1 \) by gluing the corresponding domains \( [\frac{1}{2}T^\nu, \frac{3}{2}T^\nu] \) in two components connected at \( z \). In view of the definition, we can identify the regions \( \mathcal{C}^n - \cup_{E(\Gamma)} N_{z,\nu}(\hat{T}) \) in \( \mathcal{C}^n \) with \( \mathcal{C}^n_1 - \cup_{E(\Gamma)} N_{z,\nu}(\hat{T}) \) in \( \mathcal{C}^n_1 \) and with \( \mathcal{C}^n - \cup_{E(\Gamma)} N_{z,\nu}(\hat{T}) \) in \( \mathcal{C} \). So the section \( u^1 \) can be viewed as defined on \( \mathcal{C}^n_1 \) and by the pull-back of the deformation map we also can assume that the \( u^n \) are defined on the region \( \mathcal{C} - \cup_{E(\Gamma)} N_{z,\nu}(\hat{T}) \) in \( \mathcal{C} \).

**Definition 4.2.25 (Gromov convergence).** We say that \( (\mathcal{C}^n, u^n) \to (\mathcal{C}, u) \) in \( \mathcal{M}^{rig}_{g,k,W}(\gamma, \kappa) \) as \( n \to \infty \) if

1. For each \( \hat{T} \geq 10 T_0 \), \( u^n \) converges in the \( C^\infty \) topology to \( u \) on \( \mathcal{C} - \cup_{E(\Gamma)} N_{z,\nu}(\hat{T}) \).
2. \( \lim_{\hat{T} \to \infty} \lim_{n \to \infty} \text{Diam}(u^n(\cup_{z} N_{z,\nu}(\hat{T}))) = 0 \).
special marked or nodal points. Hence if there are more than two Ramond marked or
critical values as in the above definition of convergence, we can still define the Gromov
Theorem 4.3.1.

4.3. convergence in this case.

mials $W$
any of the nodal points of $(\gamma, \xi)$ is strongly regular, then $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi) = \mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$ and we can prove
below that $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$ is compact with respect to Gromov convergence. Now we consider
a special case. We assume that the perturbation polynomials $W_{\gamma_{0}} + W_{0, \gamma_{0}}$ are all strongly
$W_{\gamma_{0}}$-regular except at the marked (or nodal point) $p_{0}$, the perturbation polynomial $W_{\gamma_{0}} + W_{0, \gamma_{0}}$
has only two critical points $\kappa^{i}, \kappa^{j}$ such that $\text{Im}(W_{\gamma_{0}} + W_{0, \gamma_{0}})(\kappa^{i}) = \text{Im}(W_{\gamma_{0}} + W_{0, \gamma_{0}})(\kappa^{j})$. We want to define the convergence of $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$ in this case. Without loss
of generality, we can assume that $p_{0}$ is a Ramond nodal point.

**Definition 4.2.26.** Take $(\mathcal{C}^{n}, u^{n})$, $(\mathcal{C}, u)$ in $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$. If for sufficiently large $n$, $p_{0}$ is not
any of the nodal points of $(\mathcal{C}^{n}, u^{n})$, then we define that $(\mathcal{C}^{n}, u^{n}) \to (\mathcal{C}, u)$ in $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$
as in the Definition 4.2.25. Now assume that $(\mathcal{C}^{n}, u^{n}) = (\mathcal{C}^{n-1} - \mathcal{C}^{n-2}, u^{n-1} - u^{n-2})$, and $(\mathcal{C}, u) = (\mathcal{C}^{1} - \mathcal{C}^{0}, u^{1} - u^{0}, u^{0} - u^{2})$, where
$u_{1,j}(\kappa^{i}) = \kappa^{j} = u^{1}(p_{0}), u_{1,j}(\infty) = \kappa^{j} = u^{2}(p_{0}).$ We say that $(\mathcal{C}^{n}, u^{n}) \to (\mathcal{C}, u)$ in $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$ if the following conditions hold:

1. $(\mathcal{C}^{n-2}, u^{n-2}) \to (\mathcal{C}^{2}, u^{2})$ as in Definition 4.2.25,
2. Let $N(\hat{T})$ be the neck region on $\mathcal{C}^{1} - \mathcal{C}^{0} = (\mathcal{C}^{1} - \mathcal{C}^{0}) \times (\mathcal{T} \times S^{1})$ for any $\hat{T} \geq 10\hat{T}$. Then $(\mathcal{C}^{n-1} - (\hat{T}, \infty) \times S^{1}), u^{n-1}$ converges to $(\mathcal{C}^{1} - (10\hat{T}, \infty) \times S^{1}), u^{1}$ in the sense of Definition 4.2.25
3. There is a sequence of positive numbers $s_{n} \to \infty$ such that $\tilde{u}^{n} := u^{n}(\cdot + s_{n}, \cdot)$ converges in $\mathcal{C}^{n, \text{loc}}$ to $u_{1,j}^{1,2}$ on $\mathcal{T} \times S^{1}$.
4. $\lim_{\hat{T} \to \infty} \lim_{n \to \infty} \text{Diam}(u^{n}(\hat{T}, -\hat{T} + s_{n}) \times S^{1})) = 0$

**Remark 4.2.27.** Since the polynomial $W_{\gamma}$ is only determined by the group action of $\gamma \in G$, the perturbation polynomial $W_{\gamma} + W_{0, \gamma}$ can be chosen to depend only on $\gamma$ but not on the special marked or nodal points. Hence if there are more than two Ramond marked or nodal points labelled by the same group action and satisfying the same requirement of critical values as in the above definition of convergence, we can still define the Gromov convergence in this case.

4.3. Gromov compactness theorem.

**Theorem 4.3.1.** [Gromov Compactness theorem] Assume that the perturbation polynomials $W_{\gamma} + W_{0, \gamma}$ are all strongly $W_{\gamma}$-regular except possibly for only one group element $\gamma_{0} \in G$, the perturbation polynomial $W_{\gamma_{0}} + W_{0, \gamma_{0}}$ has only two critical points $\kappa^{i}, \kappa^{j}$ such that $\text{Im}(W_{\gamma_{0}} + W_{0, \gamma_{0}})(\kappa^{i}) = \text{Im}(W_{\gamma_{0}} + W_{0, \gamma_{0}})(\kappa^{j})$. Then the moduli space $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$ is a compact Hausdorff space for any $\gamma, \xi$. In particular, if $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$ is strongly regular, then $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$ is a compact Hausdorff space.

**Proof.** Let $(\mathcal{C}^{n}, u^{n}) \in \mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$; we need only find a convergent subsequence. Since
$\mathcal{C} \in \mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$ and $\mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$ is compact, we can assume that $\mathcal{C}$ has the same combinatorial type for each $n$ and converges to $\mathcal{C} \in \mathcal{M}_{g,k,W}^{\gamma}(\gamma, \xi)$. By Theorem 4.2.9 for each $\hat{T}$, $u^{n}$ has a subsequence (we always use the same notation if we take subsequences) such that it converges to a $C^{\infty}$ section $u$ on $\mathcal{C} - \bigcup_{\gamma \in G} N_{\gamma}(\hat{T})$. When restricting to each component of $\mathcal{C}$, $u$ is a solution of the perturbed Witten equation.

Now we consider the convergence. There are several cases:
(1) \( p \) is a nodal point of \( \mathcal{C} \) and all \( \mathcal{C}^0 \) connecting two components \( \mathcal{C}_v \) and \( \mathcal{C}_\mu \) such that the group element \( \gamma_p \neq \gamma_0 \). We assume that \( u^r_\mathcal{C} \) converges to \( u_r \) in the interior part of \( \mathcal{C}_v \).

Now we claim in this case that for any \( \varepsilon > 0 \), there exist a constant \( \hat{T} \) and a critical point \( \kappa_M \) such that for any sufficiently large \( n \) and \( (s, \theta) \in [\hat{T}, \infty) \times S^1 \), the following holds:

\[
|u^r_\mathcal{C}(s, \theta) - \kappa_M| < \varepsilon,
\]

and

\[
|u^r_\mu(s, \theta) - \kappa_M| < \varepsilon.
\]

If the conclusion is not true, then there exists a positive number \( \varepsilon_0 \) and a sequence \( s_n \to \infty, \theta_n \in S^1 \) such that for any critical points \( \kappa \) of \( W_{\gamma_p} + W_{0, \gamma_p} \) the following holds

\[
|u^r_\mathcal{C}(s_n, \theta_n) - \kappa| \geq \varepsilon_0.
\]

Let \( \tilde{u}^r_\mathcal{C}(s, \theta) = u^r_\mathcal{C}(s_n + s, \theta_n) \); then \( \tilde{u}^r_\mathcal{C} \) is the solution of the perturbed Witten equation defined on \([-s_n, \infty)\) satisfying the condition \( |\tilde{u}^r_\mathcal{C}(0, \theta_n) - \kappa| \geq \varepsilon_0 \).

By Theorem 4.2.9 \( \tilde{u}^r_\mathcal{C} \) will \( C_{\text{local}} \) converge to a solution \( u \) such that for some \( \theta_0 \in S^1 \),

\[
|u(0, \theta_0) - \kappa| \geq \varepsilon_0,
\]

for any critical point \( \kappa \). Hence \( u \) is a nontrivial solution of the perturbed Witten equation connecting two different critical points. However, we know this is not true because \( W_{0, \gamma_p} \) is strongly \( W_{\gamma_p} \)-regular.

Now by (108) and (109), we know that

\[
\text{and}
\]

\[
|u^r_\mathcal{C}(+\infty) - \kappa_M| = |u^r_\mu(+\infty) - \kappa_M|,
\]

and

\[
(u^r_\mathcal{C}(+\infty) = \kappa_M = u^r_\mu(+\infty) = \kappa_M
\]

Hence we have proved in this case that \( (\mathcal{C}^0, u^0) \) converges to \( (\mathcal{C}, u) \).

(2) \( p \) is a marked point such that \( \gamma_p \neq \gamma_0 \). Proof of compactness in this case is the same as in (1) except we need only consider one component.

(3) \( p \) is a nodal point of \( \mathcal{C} \) and all \( \mathcal{C}^0 \) connecting two components \( \mathcal{C}_v \) and \( \mathcal{C}_\mu \) satisfy the relation \( \gamma_p = \gamma_0 \).

Assume that \( u^r_\mathcal{C}(\infty) = \kappa = u^r_\mu(\infty) \).

There are also some cases:

(a) \( \kappa \neq \kappa^1, \kappa^2 \). We know that in the interior part of each component, \( u^r_\mathcal{C} \) and \( u^r_\mu \) converge to \( u \) and \( u_\mu \) respectively. In this case we can show in the same way as (1) that \( u_r(\infty) = u_\mu(\infty) \), and \( (\mathcal{C}^0, u^0) \to (\mathcal{C}, u) \) in \( \overline{H}_{\kappa, W}(\gamma, \kappa) \).

(b) \( \kappa = \kappa^1 \) (or \( \kappa^2 \)).

If the sequence \( (\mathcal{C}^0, u^0) \) satisfies for any \( \varepsilon > 0 \) that there exist a constant \( \hat{T} \) and a critical point \( \gamma_M \) such that for any sufficiently large \( n \) and \( (s, \theta) \in [\hat{T}, \infty) \times S^1 \), the following holds:

\[
|u^r_\mathcal{C}(s, \theta) - \kappa_M| < \varepsilon,
\]

and

\[
|u^r_\mu(s, \theta) - \kappa_M| < \varepsilon.
\]

Then we know that \( \kappa_M = \kappa_M = \kappa, u_r(\infty) = u_\mu(\infty) = \kappa \) and \( (\mathcal{C}^0, u^0) \) converges to \( (\mathcal{C}, u) \) in \( \overline{H}_{\kappa, W}(\gamma, \kappa) \).
If it is not the case described as above, then the argument of (1) shows that there exists a sequence $s_n \to \infty$ such that the reparametrized sequence $\tilde{u}_j(s, \theta) := u_j^0(s_n + s, \theta_n) \in C^{0, \infty}_{\text{local}}$ converges to a nontrivial solution $u_{j_1, j_2}$ on $\mathbb{R} \times S^1$ connecting the uniquely possible two critical points $x^1_i$ and $x^2_i$. In fact, it is easy to see that $u_j(+\infty) = x^1_i$ and $u_j(-\infty) = x^2_i$, because there is no other kind of trajectory connecting two critical points. Hence we obtain a soliton $W$-section $(\mathbb{C}, \#_{p_0}(\mathbb{R} \times S^1)^\nu, u_{j_1, j_2}, \#_{p_0} u_j)$. Now it is easy to show that $(\mathbb{C}, u^0) \to (\mathbb{C}, \#_{p_0}(\mathbb{R} \times S^1)^\nu, u_{j_1, j_2}, \#_{p_0} u_j)$ in Gromov convergence.

In summary, we have shown that $(\mathbb{C}, u^0) \to (\mathbb{C}, \#_{p_0} u_j)$ in $\mathcal{M}_{g, k, W}(\gamma)$. If $\gamma_{p_0} \neq \gamma_0$, then one can show as before that $(\mathbb{C}, u^0)$ will converge to a stable $W$-section $(\mathbb{C}, \#_{p_0} u_j, \#_{p_0} u_j)$. If $\gamma_{p_0} = \gamma_0$, then $(\mathbb{C}, u^0)$ may converge to a stable $W$-section or a soliton $W$-section in Gromov convergence.

In summary, we have shown that $\mathcal{M}_{g, k, W}(\gamma, \kappa)$ is a compact space in the Gromov topology. This is also a Hausdorff space, since the limit $W$-section $(\mathbb{C}, u)$ is uniquely determined by the approximating sequence.

5. Construction of the virtual cycle

The aim of this section is to construct a Kuranishi structure of the moduli space $\mathcal{M}_{g, k, W}(\gamma, \kappa)$, the space of soliton $W$-section. We know that if this moduli space is a strongly perturbed space, then there is no soliton $W$-section and $\mathcal{M}_{g, k, W}(\gamma, \kappa) = \mathcal{M}_{g, k, W}(\gamma, \kappa)$. As the first step, we construct the Fredholm theory of the perturbed Witten map and do some preparations.

5.1. Fredholm theory and implicit function theorem.

Note that we have defined for any $\mathcal{C} = (\mathcal{C}, p_1, \cdots, p_k, \mathcal{L}_1, \cdots, \mathcal{L}_n, \varphi_1, \cdots, \varphi_s, \psi_1, \cdots, \psi_s) \in \mathcal{M}_{g, k, W}(\gamma)$ the corresponding Witten map:

$$W_{I_1}(C) : C^{0, \infty}(\mathcal{C}, \mathcal{L}_1) \times \cdots \times C^{0, \infty}(\mathcal{C}, \mathcal{L}_n) \to C^{0, \infty}(\mathcal{C}, \mathcal{L}_1 \otimes \Lambda^{0,1}) \times \cdots \times C^{0, \infty}(\mathcal{C}, \mathcal{L}_n \otimes \Lambda^{0,1}),$$

which has the following form:

$$W_{I_1}(u) = (\tilde{\partial}_\varphi u_1 + \tilde{I}_1 \left( \frac{\partial (W + W_{0, \beta})}{\partial u_1} \right), \cdots, \tilde{\partial}_\psi u_N + \tilde{I}_1 \left( \frac{\partial (W + W_{0, \beta})}{\partial u_N} \right)).$$

Here the perturbation term $W_{0, \beta}$ has the form $\sigma(\zeta) \beta_i W_{0, \gamma}$ which is determined by the combinatorial type of $\mathcal{C}$ and the group element $\gamma$ and the cut-off section $\beta_i$.

Define

$$C_0^{0, \infty}(\mathcal{C}, \mathcal{L}_1 \times \cdots \times \mathcal{L}_n) := \left\{ u = (u_1, \cdots, u_N) \in C^{0, \infty}(\mathcal{C}, \mathcal{L}_1) \times \cdots \times C^{0, \infty}(\mathcal{C}, \mathcal{L}_n) | u_{j,i}(p_j) = u_{j,i}(p_{j+1}) = 0 \right\},$$

and

$$L_i^{0, \infty}(\mathcal{C}, \mathcal{L}_1 \times \cdots \times \mathcal{L}_n) = \oplus_i L_i^{0, \infty}(\mathcal{C}, \mathcal{L}_1 \times \cdots \times \mathcal{L}_n) \quad (112)$$

$$L^{0, \infty}(\mathcal{C}, \mathcal{L}_1 \otimes \Lambda^{0,1}) = \oplus_i L^{0, \infty}(\mathcal{C}, \mathcal{L}_1 \otimes \Lambda^{0,1}, i = 1, \cdots, N). \quad (113)$$

The metric used to define the $L^p$ norm is the cylindrical metric near each marked or nodal point. We will always set $p > 2$ in our discussion.
We have the linearized operator $D_{\xi,u} WI$ of $WI_\xi$ at $u$:

$$D_{\xi,u} WI(\phi) := D_{\xi,u} WI(\phi_1, \ldots, \phi_N) := 
\left( \partial_\xi \phi_1 + \sum_j I_1 \left( \frac{\partial^2 (W + W_{0,\phi})}{\partial u_1 \partial u_j} \right) \phi_j, \ldots, \partial_\xi \phi_N + \sum_j I_1 \left( \frac{\partial^2 (W + W_{0,\phi})}{\partial u_N \partial u_j} \right) \phi_j \right).$$  \hfill (114)

$D_{\xi,u} WI$ is a map from $C^\infty_0 (\mathcal{C}, \mathbb{L}_1 \times \cdots \times \mathbb{L}_N)$ to $(\oplus_i \mathcal{C}_i \otimes \mathbb{L}^{0,1}_i) \times \cdots \times (\oplus_i \mathcal{C}_i \otimes \mathbb{L}^{0,1}_i)$ or from $L^p_\xi (\mathcal{C}, \mathbb{L}_1 \times \cdots \times \mathbb{L}_N)$ to $L^p (\mathcal{C}, \mathbb{L}_1 \otimes \Lambda^{0,1}) \times \cdots \times L^p (\mathcal{C}, \mathbb{L}_N \otimes \Lambda^{0,1})$.

When restricted to the cylinder neighborhood, the linearized operator has the local form

$$D_{\xi,u} WI(\phi) := D_{\xi,u} WI(\phi_1, \ldots, \phi_N) := 
\left( \partial_\xi \phi_1 - 2 \sum_j \frac{\partial^2 (W + W_{0,\phi})}{\partial u_1 \partial u_j} \phi_j, \ldots, \partial_\xi \phi_N - 2 \sum_j \frac{\partial^2 (W + W_{0,\phi})}{\partial u_N \partial u_j} \phi_j \right),$$  \hfill (115)

where $u, \phi_i$'s are local sections or can be understood as functions satisfying the twisted periodic conditions.

**Theorem 5.1.1.** Let $(\xi, u) \in \mathcal{H}_{\xi,u}(\gamma, \kappa)$ and assume that $\mathcal{C}$ is connected. Then its linearized operator $D_{\xi,u} WI: L^p_\xi (\mathcal{C}, \mathbb{L}_1 \times \cdots \times \mathbb{L}_N) \to L^p (\mathcal{C}, \mathbb{L}_1 \otimes \Lambda^{0,1}) \times \cdots \times L^p (\mathcal{C}, \mathbb{L}_N \otimes \Lambda^{0,1})$ is a real linear Fredholm operator of index $2\hat{c}_{\mathcal{W}}(1 - \gamma) - \sum_{r=1}^k 2\gamma_r - \sum_{r=1}^k N_{\gamma_r}$, where $\hat{c}_{\mathcal{W}} = \sum_r (1 - 2q_r), \gamma_r = \sum_r (\gamma_r - q_r)$ and $N_{\gamma_r} = \dim \mathcal{C}^{\gamma_r}_{\gamma_r}$ (if $\mathcal{C}^{\gamma_r}_{\gamma_r} = \{0\}$, we set $N_{\gamma_r} = 0$).

**Proof.** Let $p_2$ be a marked point. Define the limit matrix of $A(\tau)$ as: $A_{ij}(\tau) = \lim_{r \to +\infty} -\frac{\partial^2 (W + W_{0,\phi})}{\partial u_i \partial u_j}$.

We define a new real linear operator $D$ such that in the cylindrical neighborhood $([1, \infty) \times S^1)_p$, $D$ has the form

$$D\phi := \left( \partial \partial_\xi \phi_1 + \sum_j A_{1j} \partial \phi_j, \ldots, \partial \partial_\xi \phi_N + \sum_j A_{Nj} \partial \phi_j \right).$$

Note the facts that by Theorem 4.2.9 $u$ decays exponentially to zero, So $D$ is a compact perturbation of $D_{\xi,u} WI$. To show that $D$ is a Fredholm operator and compute its index, we need some preparations.

**Index of $\overline{\partial}^\infty$ in a half tube.** Let $([0, \infty) \times S^1) \times \mathbb{C}$ be a bundle pair, i.e., $\mathbb{C}$ is a trivial line bundle and $F$ is the Lagrangian subbundle given by the fiber $F_{\nu} = \mathbb{C}^p_{\nu} \times \mathbb{R}$. Notice that we take the coordinate $\xi = s + i\theta$ in the half tube $[0, \infty) \times S^1$. Define the weighted Sobolev spaces of sections with weight $\delta > 0$ and $p > 2$ as follows:

$$L^p_{1,\nu} := \{ \phi \mid \int (|\nabla \phi|^p + |\phi|^p) e^{\delta \nu} \, ds \, d\theta < \infty, \phi(0, \theta) \in F_{\nu} \}$$

$$L^p := \{ \phi \mid \int |\phi|^p e^{\delta \nu} \, ds \, d\theta < \infty \}$$

We have the standard Cauchy-Riemann operator: $\overline{\partial}^\infty := \overline{\partial} : L^p_{1,\nu} \to L^p$.  

**Lemma 5.1.2.** $\overline{\partial}^\infty$ is a Fredholm operator. Let $k_{\delta,p} = \left[ \frac{\delta}{p} \right]$. Then

\[
\begin{aligned}
\text{index } \overline{\partial}^\infty &= 0 & \text{if } |k| &\leq k_{\delta,p} \\
\text{index } \overline{\partial}^\infty &= -2k + 2k_{\delta,p} + 1 & \text{if } k > k_{\delta,p} \\
\text{index } \overline{\partial}^\infty &= -2k - 2k_{\delta,p} - 1 & \text{if } k < -k_{\delta,p}
\end{aligned}
\]
Step 1: We lift this problem to the $d$-sheeted covering $([-\infty, \infty) \times S^1)$ and get the bundle pair $(\hat{\mathcal{C}}, \hat{\mathcal{F}})$. Meanwhile, we have the lifting operator $\hat{\partial}$. It is easy to see that $\text{index} \hat{\partial} = \text{index} \partial$. Hence in the following we always assume that $d = 1$.

Step 2: Prove $\text{coker} \partial = \{ \psi d\bar{\zeta} | \partial \psi = 0, \psi(0, \theta) \in i\mathcal{F}, \psi \in L^p \}$ and is a smooth function.

In fact, taking any $\phi \in L^p$, $\psi \in L^p \subset L^2$, we have

$$\langle \psi d\bar{\zeta}, \partial \phi \rangle = \text{Re} \left( \int_0^\infty \int_{S^1} \bar{\psi} (\partial_s \phi + i\bar{\partial}_\psi \phi) ds d\theta \right)$$

$$= \text{Re} \left( \int_0^\infty \int_{S^1} (\partial_s \bar{\psi} - i\bar{\partial}_\psi \psi) ds d\theta \right) + \text{Re} \left( \int_{S^1} \bar{\psi}(0, \theta) \phi(0, \theta) \right).$$

Therefore we have the conclusion.

Step 3: If $k \leq k_{\delta, p}$, then $\partial \phi$ is injective. If $k > k_{\delta, p}$, then $\dim \ker \partial = 2k - 2k_{\delta, p} - 1$.

The solution of $\partial \phi = 0$ has the Laurent series

$$\phi(s, \theta) = \sum a_n e^{i(s+\theta)}.$$  

The coefficient is given by

$$a_n = \frac{1}{2\pi} \int_{S^1} \phi(0, \theta) e^{-in\theta} d\theta.$$  

By the equality $\phi(0, \theta) = \phi(0, \theta) e^{-2ik\theta}$ we have

$$\bar{a_n} = \frac{1}{2\pi} \int_{S^1} \phi(0, \theta) e^{in\theta} d\theta$$

$$= \frac{1}{2\pi} \int_{S^1} \phi(0, \theta) e^{-i(2\pi/n)\theta}$$

$$= a_{2k-n}.$$  

Now by integrability we know that $a_n = 0, \forall n \geq -k_{\delta, p}$. So $\phi \equiv 0$ if $k \geq -k_{\delta, p}$, and this shows that $\partial$ is an injection. If $k < -k_{\delta, p}$, we have

$$\phi(s, \theta) = a_{2k+k_{\delta, p}+1} e^{i(2k+k_{\delta, p}+1)(s+\theta)} + \cdots + a_{-k_{\delta, p}} e^{-i(k_{\delta, p}+1)(s+\theta)},$$

and so $\ker \partial = -2k - 2k_{\delta, p} - 1$.

Step 4. If $k \leq k_{\delta, p}$, then $\partial$ is surjective. If $k > k_{\delta, p}$, then $\dim \ker \partial = 2k - 2k_{\delta, p} - 1$.

Let $\psi \in \text{coker} \partial$; then $\partial \bar{\psi}(0, \theta) = 0$ and $\bar{\psi}(0, \theta) \in e^{-2ik\theta}$. Now using the result of Step 3, we have the conclusion.

$\square$

**Index of $\partial^D$ in a disc.** We can also consider the bundle pair $(D \times \mathbb{C}, F)$ on a flat disc $D$, where the Lagrangian subbundle $F := e^{i\theta/\delta}$. We have the Fredholm map

$$\partial^D := \partial : L^p(D, \mathbb{C}) \to L^p \left( D, \Lambda^{0,1} \right).$$

The following lemma is proved in the appendix of [MS]:

**Lemma 5.1.3.** $\text{index} \partial^D = 1 + 2k$.  

Index of $\partial_{\text{orb}}$ in a disc with origin an orbifold point. Assume $(D \times \mathbb{C}, \mathbb{Z}_d)$ to be a chart of orbifold line bundle $\mathbb{C}$ at the origin, where the group action is given by 

$$e^{2\pi i/d}(z, w) = (e^{2\pi i/d}z, e^{2\pi i/d}w).$$

Since, near the boundary $\partial D$, the orbifold structure gives a line bundle structure, we can associate a Lagrangian subbundle $F$ to $\partial D$ defined as $F_{\text{orb}} := e^{i\theta|/d} \mathbb{R}$. We can define the Sobolev spaces of the orbifold sections: $L^1_{\text{orb}}(D, \mathbb{C})$ and $L^p_{\text{orb}}$. We have the Fredholm operator $\partial_{\text{orb}} := \partial : L^1_{\text{orb}}(D, \mathbb{C}) \to L^p_{\text{orb}}$.

**Lemma 5.1.4.**

$$\begin{cases}
\text{index } \partial_{\text{orb}} = 1 & \text{if } k = 0 \\
\text{index } \partial_{\text{orb}} = 2k - 1 & \text{if } k \neq 0.
\end{cases}$$

**Proof.** If $k = 0$, then the orbifold line bundle is a trivial non-orbifold bundle, and the index can be obtained from Lemma 5.1.3. If $k \neq 0$, then the orbifold line bundle is nontrivial and the group action forces all continuous sections to vanish at the origin. Hence the first and the last coefficients in a Taylor expansion similar to what we computed in the case of the half-infinite long tube will disappear. So, compared to the case of the disc, the index will drop by real dimension 2. □

Firstly we assume that $\mathcal{C}$ has only one component, and let $\mathcal{C}_b = \mathcal{C} \setminus \bigcup_{r=1}^{k} ([1, +\infty) \times S^1)_{p_r}$, where the $p_r$ are marked points. Let $S^1_+ = ([1, +\infty) \times S^1)_{p_0}$ be the oriented circle with the induced orientation from $([1, +\infty) \times S^1)_{p_r}$. So $\partial \mathcal{C}_b = \cup_r S^1_0$, where $-$ means the anti-orientation. Notice that the orbifold structure of $\mathcal{L}_1$ near the marked point $p_\tau$ induces a monodromy representation of the line bundle $\mathcal{L}^1_1$:

$$\rho_{\tau i} : \pi_1(([0, \infty) \times S^1)_{p_\tau}) \to G \subset U(1)$$

which is given by $\rho_{\tau i}(1) = \Theta^\tau$. Let $\mathbb{R}^1_+ = -e^{2\pi \sqrt{-1} \theta_{\mathcal{L}_1} \gamma_{\mathcal{L}_1}} \mathbb{R}$ be the Lagrangian subbundle on $S^1_+$. Then we obtain the bundle pairs $([\mathcal{C}_b \times \mathcal{L}_i, \mathbb{R}^1_+] \times \cdots \times \mathbb{R}^1_1)$ and $((1, +\infty) \times S^1)_{p_r} \times \mathcal{L}_i \times \mathbb{R}^1_1)$. Define

$$C^0(\mathcal{C}_b, \mathcal{L}_i) := \{ s \in C^0(\mathcal{C}_b, \mathcal{L}_i) | s \in \mathbb{R}^1_1 \}$$

Let $L^p_{1,b}(\mathcal{C}_b, \mathcal{L}_i)$ be the closure of $C^0(\mathcal{C}_b, \mathcal{L}_i)$ under the $L^p$ norm and $L^p(\mathcal{C}_b, \mathcal{L}_i \times L^{0,1})$. Then it is well-known that $\bar{\partial}_i(\mathcal{C}_b) : L^p_{1,b}(\mathcal{C}_b, \mathcal{L}_i) \to L^p(\mathcal{C}_b, \mathcal{L}_i \times L^{0,1})$ is a self-dual Fredholm operator in this boundary value problem.

Similarly we can define another boundary value problem on $[1, +\infty) 	imes S^1$ such that $\bar{\partial}_i(\mathcal{C}_b) : L^p_{1,b}([1, +\infty) \times S^1)_{p_r} \to L^p((1, +\infty) \times S^1)_{p_r} \times \mathcal{L}_i \times L^{0,1}$ is also a self-dual Fredholm operator.

Since $D$ is a real linear Cauchy-Riemann type operator, we can similarly define the following maps:

$$D(\mathcal{C}_b) := D : L^p_{1,b}(\mathcal{C}_b, \mathcal{L}_1 \times \cdots \times \mathcal{L}_N) \to L^p(\mathcal{C}_b, \mathcal{L}_i \times L^{0,1})$$

and

$$D^\tau(\mathcal{C}_b) : L^p_{1,b}([1, +\infty) \times S^1)_{p_r} \to L^p((1, +\infty) \times S^1)_{p_r} \times \mathcal{L}_i \times L^{0,1}).$$

We have the index gluing formula which can be proved in exactly the same way as in the appendix of [MS].
**Lemma 5.1.5.** Assume that $D(\mathcal{E}_b), D^\alpha_i(\tau), \tau = 1, \cdots, k$ are defined as above; then we have

\[
\text{index}(D) = \text{index } D(\mathcal{E}_b) + \sum_{\tau=1}^k D^\alpha_i(\tau).
\]

Since $D(\mathcal{E}_b)$ is a compact perturbation of $\bar{\partial}_1(\mathcal{E}_b) \oplus \cdots \oplus \bar{\partial}_i(\mathcal{E}_b)$, we have

\[
\text{index } D(\mathcal{E}_b) = \sum_i \text{index } \bar{\partial}_i(\mathcal{E}_b).
\]  \hspace{1cm} (116)

Notice that the operator

\[
D^\alpha_i(\phi) = \left\{ \bar{\partial}_1 + \sum_j A_{ij} \phi_j, \cdots, \bar{\partial}_N + \sum_j A_{Nj} \phi_j \right\}.
\]

is also a Fredholm operator and

\[
\text{index } D(\mathcal{E}_b) = \sum_i \text{index } \bar{\partial}_i(\mathcal{E}_b).
\]  \hspace{1cm} (117)

where the last equality is from Lemma 5.1.2.

$D_N(\tau)$ is a Fredholm operator in the usual Sobolev space of sections, since each Neveu-Schwarz bundle $\mathcal{L}_i$ is a nontrivial line bundle. Actually we can define a desingularization

\[
\eta(\Theta_i^\gamma) : \phi(s, e^{\theta}) \in C^\infty(((1, \infty) \times S^1)_{p_i}, \mathcal{L}_i) \to e^{-\gamma_i s^\theta} \phi(s, e^{\theta}) \in C^\infty(((1, \infty) \times S^1)_{p_i}, \mathcal{L}_i).
\]

This map provides the isomorphisms of Banach spaces from $\mathcal{L}_i^p(((1, \infty) \times S^1)_{p_i}, \mathcal{L}_i)$ to $\mathcal{L}_i^p(((1, \infty) \times S^1)_{p_i}, \mathcal{L}_i)$ and $L^p(((1, \infty) \times S^1)_{p_i}, \mathcal{L}_i) \to L^p(((1, \infty) \times S^1)_{p_i}, \mathcal{L}_i)$. Now the operator $\eta(\Theta_i^\gamma) \circ D_N(\tau) \circ \eta(\Theta_i^\gamma)^{-1} = \bar{\partial} \cdot + \text{diag}(\Theta_i^\gamma \cdots + A_{N}^\gamma)$. Hence we know that $D_N(\tau)$ is also a Fredholm operator and

\[
\text{index } D_N(\tau) = \text{index } \left( \eta(\Theta_i^\gamma) \circ D_N(\tau) \circ \eta(\Theta_i^\gamma)^{-1} \right).
\]

For the same reason, the sections in the kernel and cokernel of $\eta(\Theta_i^\gamma) \circ D_N(\tau) \circ \eta(\Theta_i^\gamma)^{-1}$ are also exponentially decaying; we can compute the index in a weighted Sobolev space and then drop the 0-term. By Lemma 5.1.2 and Lemma 5.1.4 we have

\[
\text{index } D_N(\tau) = \sum_{i, \theta_i \neq 0} \text{index } \bar{\partial}_i^\tau(\tau) = \sum_{i, \theta_i \neq 0} \text{index } \bar{\partial}_i^\tau(\tau), \hspace{1cm} (118)
\]
Notice that in Lemma 5.1.2 we take the coordinate \( \xi = s + i\theta \), but in Lemma 5.1.4 we take the coordinate \( z = e^{-\xi} \). For a different choice of coordinates, there is a minus sign difference in the corresponding Lagrangian subbundles. Therefore we can obtain

\[
\text{index } D^{\alpha}(\tau) = \text{index } D_{R}(\tau) + \text{index } D_{N}(\tau) = \sum_{i: \Theta_{i}^{\gamma} \neq 0} \text{index } \tilde{D}_{i}^{\alpha}(\tau).
\]  

(119)

By Lemma 5.1.5 (116) and (119), there holds

\[
\text{index } D = \sum_{i} \text{index } \tilde{D}_{i}(\mathcal{E}_{b}) + \sum_{\tau = 1}^{k} \sum_{i: \Theta_{i}^{\gamma} \neq 0} \text{index } \tilde{D}_{i}^{\alpha}(\tau).
\]  

(120)

Furthermore, by Lemma 5.1.4 and the index gluing formula, we have

\[
\text{index } D = \sum_{i} \text{index } \tilde{D}_{i}(\mathcal{E}_{b}) + \sum_{\tau = 1}^{k} \sum_{i} \text{index } \tilde{D}_{i}^{\alpha}(\tau) - \sum_{\tau = 1}^{k} \sum_{i: \Theta_{i}^{\gamma} \neq 0} 1
\]  

\[= \sum_{i} \text{index } \tilde{D}_{i} - \sum_{\tau = 1}^{k} \sum_{i: \Theta_{i}^{\gamma} \neq 0} 1.
\]  

(121)

Therefore now the index computation is changed to the index computation of \( \tilde{D}_{i} \) on the closed orbifold with orbifold line bundle \( \mathcal{L}_{i} \).

The following lemma is from Proposition 4.2.2. of (CR1).

**Lemma 5.1.6.**

\[
\text{index } \tilde{D}_{i} = \text{index}_{\mathcal{E}} \tilde{D}_{i} = 2C_{1}(\mathcal{L}_{i})(\text{cl}(\mathcal{E})) + 2(1 - g),
\]  

(122)

where \( \text{cl}(\mathcal{E}) \) means the closed orbifold.

Combining this Lemma, (121) and the following degree equality which was obtained before:

\[
C_{1}(\mathcal{L}_{i})(\text{cl}(\mathcal{E})) = \text{deg}(\mathcal{L}_{i}) = q_{i}(2g - 2 + k) - \sum_{\tau = 1}^{k} \Theta_{i}^{\gamma_{i}},
\]

we have

\[
\text{index } D = 2\sum_{i} q_{i}(2g - 2 + k) - \sum_{\tau = 1}^{k} \Theta_{i}^{\gamma_{i}} + 2\sum_{i} (1 - g) - \sum_{\tau = 1}^{k} \sum_{\Theta_{i}^{\gamma_{i}} \neq 0} 1
\]

\[= 2(1 - g)(\sum_{i} (1 - 2q_{i})) - 2\sum_{i} (\sum_{\tau = 1}^{k} \Theta_{i}^{\gamma_{i}} - q_{i}) - \sum_{\tau = 1}^{k} N_{\gamma_{i}}
\]

\[= 2\check{c}_{W}(1 - g) - \sum_{\tau} 2\ell(\gamma_{\tau}) - \sum_{\tau = 1}^{k} N_{\gamma_{i}},
\]  

(123)

where \( \check{c}_{W} = \sum (1 - 2q_{i}) \) and \( \ell(\gamma_{\tau}) = \sum (\Theta_{i}^{\gamma_{i}} - q_{i}) \).

If \( \mathcal{E} \) has more than one connected component, i.e., nodal points appear, than we use the relation

\[
\ell(\gamma_{\tau}) + \ell(\gamma_{\tau}^{-1}) + N_{\gamma_{i}} = \check{c}_{W}
\]

to obtain the general result. We have finished the proof of Theorem 5.1.1. \( \square \)

It is easy to get the following conclusion:
Corollary 5.1.7. Let \((u_{j_1,j_2}, \gamma) \in S_\gamma(k^j, k^{j_2})\). Then the linearized operator \(D_{u_{j_1,j_2}}(WI)\) is a real linear Fredholm operator of index 0 on \(\mathbb{R} \times S^1\).

In the following, we introduce the implicit function theorem in our required form, which is called the "Kuranishi model" in [CR2]. Here we have done a small modification.

Suppose \(F\) is a \(C^1\) Fredholm map from a neighborhood of 0 in the Banach space \(X\) to a neighborhood of 0 in the Banach space \(Y\). Then \(DF(0)\) has a finite-dimensional kernel and cokernel. Write \(F(x) = F(0) + DF(0)x + G(x)\).

Assume that \(E\) is a finite-dimensional subspace such that \(Y = E + \text{im} DF(0)\), and let \(V = \{x \in X | DF(0) \cdot x \in E\}\). Then \(V\) is a finite-dimensional subspace in \(X\), and \(\dim V - \dim E = \text{index} DF(0)\). There exist subspaces \(V'\) and \(E'\) in \(X\) and \(Y\) respectively such that \(X = V \oplus V'\) and \(Y = E \oplus E'\). Let \(P_E : V \rightarrow \mathbb{V}'\) and \(P_E : X \rightarrow V'\) be the projection map. The operator \(DF(0) : V' \rightarrow E'\) is invertible. We denote its inverse by \(Q\).

Lemma 5.1.8. Consider the ball \(U_{2r}\) in \(X\) of radius \(2r\) such that \(x \in U_{2r}\) satisfies the condition \(\forall x_1, x_2 \in U_{2r},\)

\[\|P_E \cdot G(x_1) - P_E \cdot G(x_2)\| \leq C(\|x_1\| + \|x_2\|)\|x_1 - x_2\|\]

Let \(0 < r < \frac{1}{2\|P_E \cdot F(0)\|}\), then for any \(v \in V \cap U_r\), there is a unique \(v'(v) \in V' \cap U_r\) such that \(F(v + v'(v)) \in E\), and \(\Psi : v \mapsto v + v'(v)\) is continuous and one to one. On the other hand, for any \(x \in U_{r/\|P_E \cdot F(0)\|}\) such that \(F(x) \in E\), there is a unique \(v \in V \cap U_r\) such that \(x = v + v'(v)\). In particular, let \(s : V \cap U_{r/\|P_E \cdot F(0)\|} \rightarrow E\) be defined by \(s(v) = F(v + v'(v))\); then \(s\) is continuous and \(s^{-1}(0)\) is homeomorphic to the zero set \(F^{-1}(0) \cap U_{r/\|P_E \cdot F(0)\|}\).

(R, E, s, \Psi) is called the Kuranishi model of \(F\) at 0.

Proof. The proof is similar to the proof of Lemma 3.2.1 in [CR2], if we put \(B_r(x) = -Q \cdot P(F(0) + G(v + x))\) there.

\(\square\)

Remark 5.1.9. In the construction of the local charts, we will see that \(F(x)\) represents the nonlinear Witten map and \(F(0)\) represents the approximating map.

5.2. Multisection and Kuranishi theory.

This section provides the machinery, multisection and Kuranishi theory, to produce the virtual cycles. All of the contexts we summarize below can be found in [FO]. We cite the context here only for the convenience of the reader.

Multisection. For a space \(Z\), let \(\mathcal{S}^k(Z) := Z^k / \mathcal{S}_k\) be the \(k\)-th symmetric power of \(Z\), where \(\mathcal{S}_k\) is the permutation group of order \(k\). If \(Z\) is an orbifold, then \(\mathcal{S}^k(Z)\) is also an orbifold.

Definition 5.2.1. Let \((V \times \mathbb{R}^k, \Gamma, pr)\) be a local model of smooth orbibundle of rank \(k\) over \((V, \Gamma)\) and \(l\) be a positive integer. An \(l\)-multisection of \((V \times \mathbb{R}^k, \Gamma, pr)\) is a continuous map \(s : V \rightarrow \mathcal{S}^l(\mathbb{R}^k)\) which is \(\Gamma\)-equivariant.

There is a canonical map \(tm_l : \mathcal{S}^l(Z) \rightarrow \mathcal{S}^{l'}(Z)\) for each \(l, l'\) defined by \(tm_l[x_1, \ldots, x_l] = [x_1, \ldots, x_1, \ldots, x_l, \ldots, x_l]\), where we write \(x_i \) \(l\) times.

It is easy to see if \(s\) is an \(l\)-multisection, then \(tm_l \circ s\) is an \(l'\)-multisection and the restriction of \(s\) is still an \(l\)-multisection.
Definition 5.2.2. Let \( pr : E \to X \) be an orbibundle. A multisection is an isomorphism class of the following objects \(((V_i \times \mathbb{R}^k, \Gamma_i, pr)), \{s_i\})\) such that

1. \(((V_i \times \mathbb{R}^k, \Gamma_i, pr))\) is a family of charts of \( E \) such that \( \cup_i V_i/\Gamma_i = X \).
2. \( s_i \) is an \( l \) -multisection of \((V_i \times \mathbb{R}^k, \Gamma_i, pr)\).
3. In the overlap \( V_i/\Gamma_i \setminus V_j/\Gamma_j \), the section \( tm_l \circ s_i \) equals \( tm_l \circ s_j \) under the transition map.

We say two multisections

\[ \left((V'_i \times \mathbb{R}^k, \Gamma'_i, pr)), \{s'_i\}\right) \]

and

\[ \left((V_i \times \mathbb{R}^k, \Gamma_i, pr)), \{s_i\}\right) \]

are equivalent if the sections \( tm_l \circ s'_i \) and \( tm_l \circ s_j \) satisfy the compatibility conditions on the overlaps.

For a locally liftable multisection \(((V_i \times \mathbb{R}^k, \Gamma_i, pr)), \{s_i\})\), \( s_{ij} \) is called a branch of \( s_i = [s_j, 1 \leq j \leq l_i] \). The sum of an \( l \) -multisection \( s^{(1)} \) and \( l' \) -multisection \( s^{(2)} \) is an \( l'l' \) -multisection \( s^{(1)} + s^{(2)} \) and its branches consist of all the possible sums of branches of \( s^{(1)} \) and \( s^{(2)} \). We can also naturally define the multiplication of a function with a multisection. For an open set \( \Omega \subset X \), we can define the space \( C^0_m(\Omega, E) \) of continuous liftable multisections. It is a \( C^0(\Omega) \) -module. Similarly, one can define the \( C^k \) -differentiable space \( C^m_k(\Omega, E) \).

Using the concept of multisection, Fukaya and Ono have constructed the transversality approximation theorems for orbifold sections (see Theorem 3.11 and Lemma 3.14 in [FO]).

Kuranishi theory. Let \( X \) be a compact, metrizable topological space.

Definition 5.2.3. Let \( V \) be an open subset of \( X \). A Kuranishi or virtual neighborhood of \( V \) is a system \((U, E, G, s, \Psi)\) where

1. \( \tilde{U} = U/G \) is an orbifold, and \( E \to U \) is a \( G \) -equivariant bundle.
2. \( s \) is a \( G \) -equivariant continuous section of \( E \).
3. \( \Psi \) is a homeomorphism from \( s^{-1}(0) \) to \( V \) in \( X \).

We call \( E \) the obstruction bundle and \( s \) the Kuranishi map. We say \((U, E, G, s, \Psi)\) is a Kuranishi neighborhood of a point \( p \in X \) if \( p \) has a neighborhood \( V \) carrying a Kuranishi neighborhood.

Definition 5.2.4. Let \((U_i, E_i, G_i, s_i, \Psi_i)\) be a Kuranishi neighborhood of \( V_i \) and \( f_{21} : V_1 \to V_2 \) be an open embedding. A morphism

\[ \{\phi\} : (U_1, E_1, G_1, s_1, \Psi_1) \to (U_2, E_2, G_2, s_2, \Psi_2) \]

covering \( f \) is a family of open embeddings

\[ \phi_{21} : U_1 \to U_2, \hat{\phi}_{21} : E_1 \to E_2, \lambda_{21} : G_1 \to G_2, \Phi_{21} : \phi_{21}^*TU_2/TU_1 \to \phi_{21}^*E_2/E_1 \]

called injections) such that

1. \( \phi_{21}, \hat{\phi}_{21} \) are \( \lambda_{21} \) -equivariant and commute with bundle projection.
2. \( \lambda_{21} \) induces an isomorphism from \( \ker(G_1) \) to \( \ker(G_2) \), where \( \ker(G) \) is the subgroup acting trivially.
3. \( \Psi_2 \phi_{21} = \Psi_1 \)
4. If \( g \phi_{21}(U_i) \cap \phi_{21}(U_i) \neq \emptyset \) for some \( g \in G_2 \), then \( g \) is in the image of \( \lambda_{21} \).
5. \( G_2 \) acts on the set \( \{\phi_{21}\} \) transitively, where \( g(\phi_{21}, \phi_{21}, \lambda_{21}) = (g\phi_{21}, g\phi_{21}, g\lambda_{21}g^{-1}) \).
6. \( \Phi_{21} \) is an \( G \) -equivariant bundle isomorphism.
We can also define morphisms between two spaces $X$ and $Y$ carrying Kuranishi structures. However the morphism we prefer is a special map from $X$ which carries a Kuranishi structure to $Y$ which is an orbifold. The following definition is given in Definition 6.6 of [FO].

Definition 5.2.6. Let $X$ be a space carrying a Kuranishi structure $\mathcal{Y}$ and $Y$ be an orbifold. $f : X \to Y$ is called a strongly continuous map if $f$ is a family of continuous maps $\{f_U\}$ for $V \in \mathcal{Y}$ such that $f_{V_2} \circ \phi_{V_2, V_1} = f_{V_1}$. $f$ is said to have maximal rank if for each $V \in \mathcal{Y}$, $f_V$ is smooth and $\forall p \in V$ we have rank $\phi_{f(p)} = \min\{\dim V, \dim Y\}$.

Remark 5.2.7. The concept of Kuranishi structure was introduced in symplectic geometry by Fukaya and Ono ([FO]) to define the Gromov-Witten invariants and prove the Arnold conjecture in general symplectic manifolds. The reason is that the moduli space of stable maps is not an orbifold in general but it can still carry a Kuranishi structure. The existence of this structure is sufficient for defining the virtual cycle.

Fukaya-Ono used the concept of germ to define the Kuranishi structure; then they proved that there is a good coordinate system, i.e., a finite covering of Kuranishi neighborhoods. Here we define the Kuranishi structure in a different but equivalent style. In the definition of morphism, we included the map $\Phi$ as our data and required $\Phi : \Phi^* T U_2 / T U_1 \to \Phi^* E_2 / E_1$ to be an isomorphism. In definition 5.2.5, $\Phi$ is required to satisfy the injection composition rule. If we assume that $U_3 \subset U_2 \subset U_1$, this is equivalent to the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Phi^* T U_2 / T U_3 & \longrightarrow & \Phi^* T U_1 / T U_3 & \longrightarrow & 0 \\
\downarrow \Phi_{13} & & \downarrow \Phi_{13} & & \downarrow \Phi_{12} \\
0 & \longrightarrow & E_2 / E_3 & \longrightarrow & E_1 / E_3 & \longrightarrow & 0 
\end{array}
$$

This is just the definition of “tangent bundle” in [FO]. In our definition, we require the existence of tangent bundle to be a part of the definition of Kuranishi structure.

To treat the orientation problem of the moduli problem, we follow Fukaya-Ono’s construction of the “bundle system”.

Definition 5.2.8. Assume that the space $X$ has a Kuranishi structure $\{(U, E, G, s, \Psi)\}$. We change these data into a new composition $\{(U, F_1, F_2, G, \Psi^F)\}$ (forget the information about $E, s, \Psi$) such that they satisfy the following conditions:

1. For each open set $V_q$ carrying a Kuranishi neighborhood, $F_{1,q}$ and $F_{2,q}$ are orbifold bundles over $U_q / G$.
2. For any embedding $\phi_{pq} : U_q \to U_p$ covering $f_{pq} : V_q \to V_p$, there are embeddings of orbibundles $\Psi_{i,pq}^F : F_{i,q} \to \phi_{pq}^* F_{i,p}$, $i = 1, 2$, and an isomorphism

$$
\phi_{pq}^* : \phi_{pq}^* F_{1,p} \to \phi_{pq}^* F_{2,p}.
$$
(3) Let $V_r \subset V_q \cap V_p$ be three open sets in $X$ all of which carry the Kuranishi neighborhoods, then we have the commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \phi_{r,F_1}^* F_{1,q} & \longrightarrow & \phi_{r,F_1}^* F_{1,q} & \longrightarrow & 0 \\
\psi_r & \downarrow & \psi_r & \downarrow & \psi_r & \downarrow & \\
0 & \longrightarrow & \phi_{r,F_2}^* F_{2,q} & \longrightarrow & \phi_{r,F_2}^* F_{2,q} & \longrightarrow & 0
\end{array}
\]

$(F_1, F_2, \Psi^F)$ is called a bundle system related to the Kuranishi structure $\{(U, E, G, s, \Psi)\}$ of $X$.

We say two bundle systems $(F_1, F_2, \Psi^F)$ and $(F'_1, F'_2, \Psi'^F)$ are isomorphic if there exist isomorphisms between $F_1$ and $F'_1$, $F_2$ and $F'_2$ in each chart of $X$ such that they commute with the morphisms $\Psi^F$’s and $\Psi'^F$’s.

**Example 5.2.9.** $(TU, E, \Phi)$ is an intrinsic bundle system related to the Kuranishi structure $\{(U, E, G, s, \Psi)\}$

**Definition 5.2.10.** A bundle system is said to be orientable if for any point $p \in X$, $F_{1,p}$ and $F_{2,p}$ are orientable as orbifold bundles on a small Kuranishi neighborhood $U_p$ of $p$ and if $U_q \subset U_p$, then

\[
\Psi_{pq} : \phi_{pq}^* F_{1,q} \longrightarrow \phi_{pq}^* F_{2,q}
\]

is orientation preserving.

Fukaya-Ono [FO] then defined the K-theory of the bundle system related to a Kuranishi structure of a space $X$. They defined the concepts of oriented bundle system, complex bundle system, stably oriented and stably almost complex. They showed that a bundle system is stably oriented iff it is oriented. Hence if one proves that a bundle system is stably almost complex, then it is stably orientable and so an orientable system.

Once the definitions of Kuranishi structure and orientation are constructed, Fukaya-Ono have the following theorem:

**Theorem 5.2.11.** Let $X$ be a topological space carrying an $n$-dimensional orientable Kuranishi structure $\{(U, E, G, s, \Psi)\}$. Then the section $s$ can be perturbed to a multisection $\tilde{s}$ which is transverse to the zero section and the zero set $\tilde{s}^{-1}(0)$ is an oriented cycle. The cobordism class $[\tilde{s}^{-1}(0)]$ of this cycle is independent of the choice of the multisection $\tilde{s}$. Furthermore, if $Y$ is an orbifold and there is a strongly continuous map $f : X \rightarrow Y$, then $f_*[\tilde{s}^{-1}(0)]$ is a homology class in $H_n(Y; \mathbb{Q})$.

**Kuranishi structure with boundary.**

Let $X$ be a compact metric space. We can also define an $n$-dimensional Kuranishi structure with boundary on $X$. We only need a minor change in Definition 5.2.4. We modify the definition of Kuranishi neighborhood $(U, E, G, s, \Psi)$ and don’t change the other conditions. $U$ is required to be a $G$-invariant open neighborhood of 0 in $\mathbb{R}^n$ or $\mathbb{R}^n \times [0, \infty)$ and $G$ is required to be a finite group with a linear representation to $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$ respectively. A point $p$ is said to be a boundary point of $X$ if there is a chart $(U_{p}^{n-1} \times [0, \varepsilon_p), E_p, G_p, s_p, \Psi_p)$, where $U_{p}^{n-1}$ is a small neighborhood of 0 in $\mathbb{R}^{n-1}$, such that $p \in U_{p}^{n-1}/G$. Let $\partial X$ be the boundary of $X$, which consists of all the boundary points of $X$. Obviously $\partial X$ is a space with $(n-1)$-dimensional Kuranishi structure.

Similarly, we can define the notions of bundle systems, orientation, etc. on $X$ carrying a Kuranishi structure with boundary.
5.3. Construction of the local chart of an inner point in $\mathcal{M}_{g,k,W}(\gamma, \kappa)$.

If a soliton W-section $(\mathcal{C}_\sigma, u_\sigma)$ in $\mathcal{M}_{g,k,W}(\gamma, \kappa)$ has no BPS soliton component, then we show that it is actually an interior point of the moduli space. In this part, we will construct the local chart of an interior point.

A non-BPS W-section $(\mathcal{C}_\sigma, u_\sigma)$ can still have a soliton component, for example, we denote one of them by $(u_{j_1, j_2}, \gamma_p)$, where $p$ is a marked or nodal point of $\mathcal{C}$. Since this soliton is not a BPS soliton, $u_{j_1, j_2}$ has at least one regular value and this guarantees that the automorphism group $\text{Aut}(\mathcal{C}_\sigma, u_\sigma)$ is a finite group.

For simplicity, we define

$$D_\sigma(WI) := D_{u_\sigma}((WI)_{\mathcal{C}_\sigma}); \text{Aut}(\sigma) := \text{Aut}(\mathcal{C}_\sigma, u_\sigma).$$

Using the standard method one can easily find a finite dimensional space $E_\sigma \subset L^p(\mathcal{C}, N, (\mathcal{L} \otimes \Lambda^{0,1}))$ satisfying the following properties:

1. $E_\sigma + \text{im} D_\sigma(WI) = L^p(\mathcal{C}, N, (\mathcal{L} \otimes \Lambda^{0,1})).$
2. $E_\sigma$ is complex linear and $\text{Aut}(\sigma)$-invariant.
3. There exists a compact set $K_{obst}$ away from the marked or nodal points containing the support of all elements in $E_\sigma$.
4. $E_\sigma$ is a finite-dimensional space consisting of smooth sections.

Let $E'_\sigma \oplus E_\sigma = L^p(\mathcal{C}, N, (\mathcal{L} \otimes \Lambda^{0,1}))$, and let $V_{\text{map}, \sigma} \oplus V'_\sigma = L^p(\mathcal{C}, N, \mathcal{L})$, where $V_{\text{map}, \sigma} = (D_\sigma(WI))^{-1}(E_\sigma)$. Thus $D_\sigma(WI) : V'_\sigma \rightarrow E'_\sigma$ is a bounded invertible operator. Its inverse is denoted by $Q_{\sigma}^{-1}$.

Let $\mathcal{C}_\sigma = (\mathcal{C}_\sigma, p_1, \ldots, p_\ell, \mathcal{L}_1, \ldots, \mathcal{L}_N, \varphi_1, \ldots, \varphi_\ell, \psi_1, \ldots, \psi_k)$. Then a neighborhood of $[\mathcal{C}_\sigma, p_1, \ldots, p_\ell]$ in $\mathcal{M}_{g,k}$ can be parametrized by the following neighborhood of $0 \times 0$:

$$V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma},$$

where $V_{\text{deform}, \sigma}$ is a small neighborhood of $0$ in $\prod E_i$ is stable $\mathbb{C}^{3g-3+k_c}$ and

$$V_{\text{resol}, \sigma} = \oplus \mathcal{C}_\sigma \text{ is nodal point}((T_0, \infty) \times S^1).$$

$V_{\text{deform}, \sigma}$ and $V_{\text{resol}, \sigma}$ are the deformation domain and resolution domain respectively. Now a uniformizing system of $\mathcal{C}_\sigma$ in $\mathcal{M}_{g,k,W}(\gamma)$ is given by

$$V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma}$$

$$\text{Aut} \mathcal{C}.$$

So for each data $(\gamma, \ell) \in V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma}$, we can obtain a nearby curve $\mathcal{C}_{\gamma, \ell} \in \mathcal{M}_{g,k,W}(\gamma)$. They are also $\text{Aut}(\sigma)$-invariant, i.e., for any $g \in \text{Aut}(\sigma)$ we have $g \cdot \mathcal{C}_{\gamma, \ell} = \mathcal{C}_{g(\gamma, \ell)}$.

Now we begin the construction of the Kuranishi neighborhood of $[\mathcal{C}_\sigma, u_\sigma]$ in $\mathcal{M}_{g,k,W}(\gamma)$.

The approximation solutions. Note that there are some parameters used in the neighborhood of a nodal point. They are fixed numbers $T_0, T_0, \hat{T}$ which have the relation $T_0 < 10 \hat{T} < \hat{T}$. Recall the meaning of these parameters: $T_0$ gives the range of the gluing parameter $\zeta = (s, \theta) \in ([T_0, \infty) \times S^1)$. The supports of the derivatives of the cut-off function $\beta, \sigma$ which are used to define the perturbed Witten map is in $[\hat{T}, \hat{T} + 1]$ and in $[8T_0, 9T_0]$ respectively. $\hat{T}$ appears in the definition of the neck region $N_{\mathcal{C}}(\hat{T}) = ([\hat{T}, \hat{T}^2] \times S^1) \cup ([\hat{T}, \hat{T}^2] \times S^1)$, where $\zeta = (s^2, \theta^2)$ is the gluing parameter and $s^2 = 2T^2$.

Now we define some sets near a nodal point $z$. Let $1/2 < \delta < 1$. Take the gluing parameter $\zeta = (s, 2T, \theta)$ and $T > 4\hat{T}$. 
N_z(δ, T_z) = \{(1 - δ)T_z, T_z\} \times S^1
N_z(δ, T_z) = N_z(δ, T_z) \cup N_y(δ, T_z)
N_z(δ, T_z, \infty) = \{(1 - δ)T_z, \infty\} \cup \{(1 - δ)T_z, \infty\} \times S^1
N_z(T_z, \infty) = \{T_z, \infty\} \times S^1

Notice that \((\mathbb{T}, \frac{2\pi}{T_z}) \times S^1\) \cup \((\mathbb{T}, \frac{2\pi}{T_z}) \times S^1\) is the gluing domain. We can take \(T_z\) to be sufficiently large such that \(\mathcal{K}_{obst}\) is disjoint from the gluing domain but contains the definition domain of \(\beta, \sigma\).

Take a section \(\phi_{\mu} \in \mathcal{C}^0(\mathcal{E}_\sigma, \mathcal{L}_1) \times \cdots \times \mathcal{C}^0(\mathcal{E}_\mu, \mathcal{L}_3)\); we want to define an approximating section \(\phi_{app, y, \xi}\) on the nearby curve \(\mathcal{C}_y, \xi\). We will only modify the solution \(\phi_{\mu}\) near each nodal point \(z\). So without loss of generality, we assume that \(\mathcal{E}\) has only one nodal point \(z\) connecting the components \(\mathcal{E}_y\) and \(\mathcal{E}_\mu\). Assume that \(\phi_{\mu}(\infty) = \chi = \phi_{\mu}(\infty)\) (under the same rigidification, i.e., viewed in \(\mathcal{C}_\mathcal{N}\)).

Let \(\beta_\delta\) be a smooth function satisfying
\[
\beta_\delta(s, \theta) = \begin{cases} 
0 & , if s \leq 0 \\
1 & , if s \geq \delta.
\end{cases}
\]

and \(|\nabla \beta_\delta| \leq C/\delta\).

Let \((s_r, \theta_r)\) and \((s_\mu, \theta_\mu)\) be the cylindrical coordinates of the two components respectively on the gluing domain of \(\mathcal{E}_{y, \xi}\) they satisfy the relation \((s_r, \theta_r) = (s_\mu, \theta_\mu)\).

Define a section \(\phi_{app, y, \xi}\) on \(\mathcal{C}_y, \xi\). When outside the gluing domain in \(\mathcal{C}_y, \xi\), \(\phi_{app, y, \xi} \equiv \phi\).

On the gluing domain, we define
\[
\phi_{app, y, \xi}(s_\mu, \theta_\mu) := \phi_{\mu}(s_\mu, \theta_\mu) + (1 - \beta_\delta(T_z - s_\mu/T_z))(\phi_{\mu}(2T_z - s_\mu, \theta_\mu) - \chi)
\]
in the \(s\) coordinate or in the \(y\) coordinate in the following form:
\[
\phi_{app, y, \xi}(s_r, \theta_r) := \phi_{\mu}(s_r, \theta_r) + (1 - \beta_\delta(T_z - s_r/T_z))(\phi_{\mu}(2T_z - s_r, \theta_r) - \chi).
\]

One can show this section is well defined since on \(T_z \times S^1\) there holds \(\phi_{app, y, \xi}(s_r, \theta_r) = \phi_{\mu}(T_z, \theta_r) + \phi_{\mu}(T_z, \theta_r - \theta_r) - \chi = \phi_{app, y, \xi}(s_\mu, \theta_\mu)\).

Define
\[
\text{Glue}_{y, \xi}(\phi) := \phi_{app, y, \xi}.
\]

\(\text{Glue}_{y, \xi}\) is a map from \(\mathcal{C}^0(\mathcal{E}_\sigma)\) to \(\mathcal{C}^0(\mathcal{E}_{y, \xi})\) or from \(L^p(\mathcal{E}_\sigma)\) to \(L^p(\mathcal{E}_{y, \xi})\).

Let \(\mathbf{u}_\sigma\) be a solution of the perturbed Witten equation \((\mathcal{W})_{\mathcal{E}_\sigma}(\mathbf{u}) = 0\); we can obtain the approximate solution \(\mathbf{u}_{app, y, \xi} = \text{Glue}_{y, \xi}(\mathbf{u})\).

If we consider the map \(\text{Glue}_{y, \xi}\) as a map between Sobolev spaces \(L^p\), we have the following lemma.

**Lemma 5.3.1.** Assume that \(\phi \in V_{map, \sigma}\). For any \(\varepsilon, 0 < \varepsilon < 1\), the following holds for sufficiently large \(\xi\) and sufficiently small \(y\):
\[
(1 + \varepsilon)||\phi||_{L^p(\mathcal{E}_\sigma)} \geq ||\text{Glue}_{y, \xi}(\phi)||_{L^p(\mathcal{E}_{y, \xi})} \geq (1 - \varepsilon)||\phi||_{L^p(\mathcal{E}_\sigma)}.
\]

**Proof.** It suffices to prove it on one component near a nodal point \(z\). Since \(\phi \in V_{map, \sigma}\), it satisfies the equation
\[
D_\sigma(\mathcal{W})(\phi) = 0 \mod \mathcal{E}_\sigma.
\]
Note that when $\zeta$ is sufficiently large, the gluing domain does not intersect $K_{\text{obst}}$. So in $N_\xi(\delta, T, \infty)$, we have

$$D_\sigma(WI)(\phi) = \left( \tilde{\partial}_\xi \phi_1 + \sum_j I_1 \left( \frac{\partial^2(W + W_{0,\beta})}{\partial u_1 \partial u_j} \phi_j \right), \ldots, \tilde{\partial}_\xi \phi_N + \sum_j I_1 \left( \frac{\partial^2(W + W_{0,\beta})}{\partial u_N \partial u_j} \phi_j \right) \right) = 0.$$  

(127)

Note that all the sections here are smooth orbifold sections. Let $e_i$ be the basis of $E_i$ and let $\tilde{u}_i, \phi_i$ be the coordinate functions of $u_i, \phi_i$ with respect to this basis. If we take the transformation $\tilde{u}_i = (\tilde{u}_1 e^{e_1 \Theta T} + b_1 \Theta T, \ldots, \tilde{u}_N e^{e_1 \Theta T} + b_N \Theta T)$ and $\hat{\phi}_i = (\hat{\phi}_1 e^{e_1 \Theta T} + b_1 \Theta T, \ldots, \hat{\phi}_N e^{e_1 \Theta T} + b_N \Theta T)$, then the equation (127) becomes the equation on the resolved $W$-curve $|E_\xi|$:

$$\left( \tilde{\partial}_\xi \hat{\phi}_1 + \sum_j I_1 \left( \frac{\partial^2(W + W_{0,\beta})}{\partial u_1 \partial u_j} \phi_j \right), \ldots, \tilde{\partial}_\xi \hat{\phi}_N + \sum_j I_1 \left( \frac{\partial^2(W + W_{0,\beta})}{\partial u_N \partial u_j} \phi_j \right) \right) = 0.$$  

(128)

By the analysis in Section 6.4, we know that $\hat{\phi} e^{e_1 \Theta T}$ and any of its derivatives are of exponential decay. This just means that $\phi_i$ and its derivatives are of exponential decay. Thus for sufficiently large $\zeta$, we have

$$\|\phi\|_{L^1_\omega(N_\xi(\delta, T, \infty))} \leq \frac{\epsilon}{6} \|\phi\|_{L^1_\omega(E_i)}.$$  

(129)

Since $\phi_{\text{app}, \xi, \zeta} = \phi$ in $\mathcal{C}_\xi \setminus N_\xi(\delta, T, \zeta)$, for sufficiently small deformation parameter $\gamma$, we have

$$(1 - \frac{\epsilon}{2})\|\phi\|_{L^1_\omega(\mathcal{C}_\xi \setminus N_\xi(\delta, T, \infty))} \leq \|\phi_{\text{app}, \xi, \zeta}\|_{L^1_\omega(\mathcal{C}_\xi \setminus N_\xi(\delta, T, \infty))} \leq (1 + \frac{\epsilon}{2})\|\phi\|_{L^1_\omega(\mathcal{C}_\xi \setminus N_\xi(\delta, T, \infty)))}.$$  

(130)

On the other hand, we have

$$\|\phi_{\text{app}, \xi, \zeta}\|_{L^1_\omega(N_\xi(\delta, T, \zeta))} = \|\phi_{\text{app}, \xi, \zeta}\|_{L^\rho} + \|\nabla \phi_{\text{app}, \xi, \zeta}\|_{L^\rho} \leq 2\|\phi\|_{L^\rho(N_\xi(\delta, T, \infty))} + 2\|\nabla \phi\|_{L^\rho(N_\xi(\delta, T, \infty))} + 2\|\gamma \phi\|_{L^\rho(N_\xi(\delta, T, \infty))} \leq 3\|\phi\|_{L^1_\omega(N_\xi(\delta, T, \zeta))}.$$  

By (129), we obtain for sufficiently large $\zeta$:

$$\|\phi_{\text{app}, \xi, \zeta}\|_{L^1_\omega(N_\xi(\delta, T, \zeta))} \leq \frac{\epsilon}{2} \|\phi\|_{L^1_\omega(\mathcal{C}_\xi)}.$$  

(131)

Combining the results (130) and (131), we get the conclusion.

\textbf{Obstruction bundle on $\mathcal{C}_\xi$.} The deformation map from $\mathcal{C}_\xi$ to $\mathcal{C}_\xi$ provides a bundle isomorphism when restricted to the domain $K_{\text{obst}}$:

$$\theta_\xi : \times_\iota(\mathcal{L}_{\xi, \iota} \otimes \Lambda^{0,1}|_{K_{\text{obst}}}) \rightarrow \times_\iota(\mathcal{L}_{\xi, \iota} \otimes \Lambda^{0,1}|_{K_{\text{obst}}})), \quad$$

which induces the isomorphism of sections:

$$\theta_\xi : C^\infty(K_{\text{obst}}, \times_\iota(\mathcal{L}_{\xi, \iota} \otimes \Lambda^{0,1}|_{K_{\text{obst}}})) \rightarrow C^\infty(K_{\text{obst}}, \times_\iota(\mathcal{L}_{\xi, \iota} \otimes \Lambda^{0,1}|_{K_{\text{obst}}})).$$

Since the support of each section in $E_\xi$ is contained in $K_{\text{obst}}$, we define $E_{\gamma, \xi, \zeta} : = \theta_\xi(E_\xi)$. Set $D_{\gamma, \xi}(WI) := D_{\text{app}, \xi, \zeta}((WI)_{\gamma, \xi})$. Our aim is to find the solution of the following equations:

$$D_{\gamma, \xi}(WI)(\phi) = 0 \mod E_{\gamma, \xi}.$$  

(132)
Lemma 5.3.2. Let $\phi \in V_{map, \sigma}$; then for sufficiently large $\zeta$ we have
\[
\|D_{\gamma, \zeta}(WI) \circ \text{Glue}_{\gamma, \zeta}(\phi) - \theta_{\gamma, \zeta} \circ D_{\sigma}(WI)(\phi)\|_{L^p(\mathcal{C}(\gamma, \zeta))} \leq C(\|y\| + \frac{1}{T_z} + e^{-\delta_0 T_z})\|\phi\|_{L^p_z(\mathcal{C}(\gamma, \zeta))},
\]
where $C$ is a constant depending on $u_{\sigma, \delta}$ and the decay exponent $\delta_0$ which is from Theorem 1.2.7.

Proof. We discuss the integral on $\mathcal{C}(\gamma, \zeta) \setminus N(\delta, T_z)$ and on $N(\delta, T_z)$ respectively. For simplicity, we write the operator as
\[
D_{\gamma, \zeta}(WI)(\phi) := \bar{\partial}_y \phi + A(u_{\text{app,}\eta, \zeta}, y) \cdot \phi,
\]
where $A(u_{\text{app,}\eta, \zeta}, y)$ is the corresponding matrix depending on $u_{\text{app,}\eta, \zeta}$ and the deformation parameter $\eta$, since the metric-preserving isomorphism $\bar{I}_1$ also depends on $\eta$ which is induced by the $W^s$-spin structure. This shows that the operator only depends on the deformation parameter if the resolution parameter is sufficiently large (because the function $\sigma \equiv 1$ for large $\zeta$).

Let $\phi_\eta$ and $\phi_\mu$ represent the $\eta$ and $\mu$ component of $\phi$. On $N(\delta, T_z)$ we have
\[
\|D_{\gamma, \zeta}(WI) \circ \text{Glue}_{\gamma, \zeta}(\phi) - \theta_{\gamma, \zeta} \circ D_{\sigma}(WI)(\phi)\|_{L^p(N(\delta, T_z))}
= \|\bar{\partial}_y (\phi_\eta + (1 - \beta_\sigma)\phi_\mu) + A(u_{\text{app,}\eta, \zeta}, y) \cdot (\phi_\eta + (1 - \beta_\sigma)\phi_\mu)\|_{L^p}
\leq \|\bar{\partial}_y - \bar{\partial}_{\gamma} \| \phi_\eta + (1 - \beta_\sigma)\| \phi_\mu\| + \|\bar{\partial}_y (1 - \beta_\sigma)\phi_\eta\|
+ \|\bar{\partial}_y \phi_\mu - A(u, \sigma)(\phi_\eta + (1 - \beta_\sigma)\phi_\mu)\|
\leq C(\|y\| + \frac{1}{T_z} + e^{-\delta_0 T_z})\|\phi\|_{L^p_z}.
\]
(134)
Here we have used the definition of $u_{\text{app,}\eta, \zeta}$ and the property of exponential decay of $u$ on $[T_z, \infty) \times S^1$ when $T_z$ is large enough.

On $\mathcal{C}(\gamma, \zeta) \setminus N(\delta, T_z)$, we have
\[
\|D_{\gamma, \zeta}(WI) \circ \text{Glue}_{\gamma, \zeta}(\phi) - \theta_{\gamma, \zeta} \circ D_{\sigma}(WI)(\phi)\|_{L^p(\mathcal{C}(\gamma, \zeta) \setminus N(\delta, T_z))}
\leq \|\|D_{\gamma, \zeta}(WI) - D_{\sigma}(WI)\| + \|I - \theta_{\gamma, \zeta}\|D_{\sigma}(WI)\phi\|
\leq C(\|y\| + \frac{1}{T_z})\|\phi\|_{L^p_z}.
\]
(135)
Combining the above two inequalities, we obtain the result. $\square$

Existence of right inverse and its uniform upperbound. Let $V_{\gamma, \zeta} = (D_{\gamma, \zeta}(WI))^{-1}(E_{\gamma, \zeta})$. Define $E_{\gamma, \zeta}'$ to be the complementary subspace of $E_{\gamma, \zeta}$ and $V_{\gamma, \zeta}'$ to be the complementary subspace of $V_{\gamma, \zeta}$. To solve the equation (132) is equivalent to proving the existence of the right inverse of $D_{\gamma, \zeta}(WI)$. Define a map
\[
I_{\gamma, \zeta}: L^p(\mathcal{C}(\gamma, \zeta), \chi(\mathcal{L}_{\gamma, \zeta} \otimes \Lambda^{0, 1}(\mathcal{C}(\gamma, \zeta)))) \to L^p(\mathcal{C}(\gamma, \zeta), \chi(\mathcal{L}_{\gamma, \zeta} \otimes \Lambda^{0, 1}(\mathcal{C}(\gamma, \zeta))))
\]
as
\[
I_{\gamma, \zeta}(\phi)(z) := \begin{cases} 
\theta_{\gamma, \zeta}^{-1} \circ \phi(z), & \text{if } z \in \mathcal{C}(\gamma, \zeta) \setminus N(\delta, T_z) \\
0, & \text{if } z \in N(\delta, T_z). 
\end{cases}
\]
We claim that the composition map $Q_{\text{app,}\eta, \zeta} := \text{Glue}_{\gamma, \zeta} \circ Q_{\sigma} \circ I_{\gamma, \zeta} : E_{\gamma, \zeta}' \to L^p_{\gamma, \zeta}$ is an approximating right inverse of $D_{\gamma, \zeta}(WI) : V_{\gamma, \zeta}' \to E_{\gamma, \zeta}'$. This is known from the following lemma.
Lemma 5.3.4. If the gluing parameter $\xi$ is sufficiently large and the deformation parameter $y$ is sufficiently small, then

$$\|D_{y,\xi}(WI) \circ Q_{app,\xi,y}(\phi) - \phi\|_{L^p(\mathcal{E}_{\xi,y})} \leq \frac{1}{2}\|\phi\|_{L^p(\mathcal{E}_{\xi,y})}. \quad (136)$$

Proof. Note that $\phi = \theta_{\bar{y},\xi} \circ D_{\sigma}(WI) \circ Q_{\sigma} \circ I_{y,\xi}(\phi)$; we only need to prove that for $\psi = Q_{\sigma} \circ I_{y,\xi}(\phi)$ the following inequality holds:

$$\|D_{y,\xi}(WI) \circ \text{Glue}_{y,\xi}(\psi) - \theta_{\bar{y},\xi} \circ D_{\sigma}(WI)(\psi)\|_{L^p} \leq C(|y| + \frac{1}{T_{\xi}} + e^{-\delta T_{\xi}})\|\psi\|_{L^p_y}. \quad (137)$$

Now the proof is the same as the proof of Lemma 5.3.2 while observing that $D_{\sigma}(WI)(\psi) = 0$

on $([T_{\xi}, (1 + \delta)T_{\xi}] \times S^1)$. \hfill $\Box$

This lemma implies that the right inverse $Q_{y,\xi}$ exists and $Q_{y,\xi} = Q'_{app,\xi} \circ (D_{y,\xi}(WI) \circ Q'_{app,\xi})^{-1}$.

Now it is easy to obtain the following lemma.

Lemma 5.3.5. For sufficiently large $\xi$ and sufficiently small $y$, $Q_{y,\xi}$ has uniform upper bound:

$$\|Q_{y,\xi}\| \leq \tilde{C}_1.$$  

Kuranishi model on $\mathcal{C}_{y,\xi}$. Define the map $F_{y,\xi}(\phi) := WI_{\xi,\phi} \circ (\mathbb{u}_{app,\xi} + \phi) : L^p_{\xi}(\mathcal{E}_{y,\xi} \times \mathbb{L}, \mathcal{L}, d\mathcal{L}) \rightarrow L^p_{\xi}(\mathcal{E}_{y,\xi} \times \mathbb{L} \otimes \Lambda^{0,1})$. We want to apply Lemma 5.1.8 to the nonlinear operator $F_{y,\xi}$ and hope to get a Kuranishi model centered at $\mathbb{u}_{app,\xi,\xi}$.

We have

$$F_{y,\xi}(0) = WI_{\xi}(\mathbb{u}_{app,\xi}) \circ (\mathbb{u}_{app,\xi}) = D_{\mathbb{u}_{app,\xi}}(WI_{\xi}(\mathbb{u}_{app,\xi})) = D_{y,\xi}(WI_{\xi}).$$

Let $G_{y,\xi}(\phi) := F_{y,\xi}(\phi) - F_{y,\xi}(0) - DF_{y,\xi}(0)\phi$.

Define the projection maps $P_{E,\xi} : L^p_{\xi}(\mathcal{E}_{y,\xi}) \rightarrow E_{y,\xi}$ and $P_{V,\xi} : L^p_{\xi}(\mathcal{E}_{y,\xi}) \rightarrow V_{y,\xi}$. We need two lemmas when applying Lemma 5.1.8.

Lemma 5.3.5. For sufficiently large $\xi$ and sufficiently small $y$, we have

$$\|P_{E,\xi} \circ F_{y,\xi}(0)\|_{L^p} \leq C(|y| + \frac{1}{T_{\xi}} + e^{-\delta T_{\xi}})\|\mathbb{u}\|_{L^p_y}. \quad (138)$$

Proof. We have

$$\|P_{E,\xi} \circ F_{y,\xi}(0)\|_{L^p} \leq \|WI_{\xi}(\mathbb{u}_{app,\xi}) - WI_{\xi}(\mathbb{u})\|_{L^p}$$

$$= \|\tilde{\bar{\theta}}_{\mathbb{u}_{app,\xi},\xi} + B(\mathbb{u}_{app,\xi},y) - \tilde{\bar{\theta}}_{\mathbb{u}} - B(\mathbb{u},\sigma)\|,$n

where $B(\mathbb{u},\sigma) = (\tilde{I}_{1}(\frac{\partial W + W_{1}}{\delta_{\xi}}), \cdots , \tilde{I}_{1}(\frac{\partial W + W_{\delta}}{\delta_{\xi}}))$ is a $t$-dimensional vector.

Then using the decay property of $\mathbb{u}$ on $[T_{\xi}, \infty) \times S^1$, we obtain

$$\|P_{E,\xi} \circ F_{y,\xi}(0)\|_{L^p} \leq C(|y| + \frac{1}{T_{\xi}} + e^{-\delta T_{\xi}})\|\mathbb{u}\|_{L^p_y}. \quad \Box$$

Lemma 5.3.6. For sufficiently large $\xi$ and sufficiently small $y$, we have

$$\|P_{E,\xi} \circ G_{y,\xi}(\phi_1) - P_{E,\xi} \circ G_{y,\xi}(\phi_2)\|_{L^p} \leq \tilde{C}_2(\|\phi_1\|_{L^p_y} + \|\phi_2\|_{L^p_y})\|\phi_1 - \phi_2\|_{L^p_y}, \quad (139)$$

where $\tilde{C}_2$ depends only on $\mathbb{u}$.  

\hfill \Box
Proof. This is a direct computation for which, among other things, the Sobolev embedding theorem and the interpolation formula of $L^p$ spaces are used. \hfill \Box

Now the hypothesis of Lemma 5.1.8 is satisfied by Lemma 5.3.4, 5.3.5, 5.3.6. Therefore we have

**Lemma 5.3.7.** Take $y$ small enough and $\zeta$ large enough. Let $\tilde{C}_1, \tilde{C}_2$ be the constants from Lemma 5.3.2 and Lemma 5.3.6 respectively. Choose $0 < r < \frac{1}{\tilde{C}_1} \zeta_1$ and let $U_{y,\zeta}(r)$ be a ball centered at the origin with radius $r$ in $L^p(\mathcal{C}_{y,\zeta})$. Then for any $\phi \in V_{y,\zeta} \cap U_{y,\zeta}(r)$, there is a unique $v'(\phi) \in V'_{y,\zeta} \cap U_{y,\zeta}(r)$ such that $F_{y,\zeta}(\phi + v'(\phi)) \in E_{y,\zeta}$. On the other hand, for any $\tilde{\phi} \in E_{y,\zeta}$, there is a unique $\phi \in V_{y,\zeta}$ such that $\tilde{\phi} = \phi + v'(\phi)$. Define $\Psi_{y,\zeta} : \phi \mapsto \tilde{\phi} + v'(\phi)$, and let $s_{y,\zeta} : V_{y,\zeta} \cap \mathcal{W}^{-1}(U_{y,\zeta}(r)) \to E_{y,\zeta}$ be defined by $s_{y,\zeta}(\phi) := F(\tilde{\phi} + v'(\phi))$. Then $(U_{y,\zeta}(r), E_{y,\zeta}, s_{y,\zeta}, \Psi_{y,\zeta})$ forms a Kuranishi model, where $s_{y,\zeta}, \Psi_{y,\zeta}$ are continuous and $\Psi_{y,\zeta}$ is a one to one map.

**Kuranishi neighborhood at $(\mathcal{C}_{y,\zeta}, u_{\ast})$.** Now we know that starting from the point $\sigma = (\mathcal{C}_{y,\zeta}, u_{\ast})$ in the interior of $\mathcal{W}_{\mathcal{C}_{y,\zeta}}(\mathbf{y}, \mathbf{x})$ we can construct the Kuranishi model $(U_{y,\zeta}(r), E_{y,\zeta}, s_{y,\zeta}, \Psi_{y,\zeta})$ on any nearby curve $\mathcal{C}_{y,\zeta}$. To construct the Kuranishi neighborhood, we only need to construct a family of isomorphisms $\eta_{y,\zeta} : V_{\text{map},\sigma} \to V_{y,\zeta}$ with uniformly bounded norms. For $\phi \in V_{\text{map},\sigma}$, we define

$$\eta_{y,\zeta}(\phi) = \text{Glue}_{y,\zeta}(\phi) - Q_{y,\zeta} \circ D_{\sigma}(\mathcal{W}) \circ \text{Glue}_{y,\zeta}(\phi).$$

**Lemma 5.3.8.** For sufficiently large $\zeta$ and sufficiently small $y$, $\eta_{y,\zeta}$ is an isomorphism.

Proof. By Lemmas 5.3.2, 5.3.6 there exists a uniform constant $C$ independent of $y, \zeta$ such that

$$\|\eta_{y,\zeta}(\phi)\|_{L^p_{y,\zeta}} \leq C\|\phi\|_{L^p_{y,\zeta}}.$$

On the other hand, we have

$$\|\eta_{y,\zeta}(\phi)\|_{L^p_{y,\zeta}} \geq \|\text{Glue}_{y,\zeta}(\phi)\|_{L^p_{y,\zeta}} - \|Q_{y,\zeta} \circ D_{\sigma}(\mathcal{W}) \circ \text{Glue}_{y,\zeta}(\phi) - \theta_{y,\zeta} \circ D_{\sigma}(\mathcal{W})(\phi)\| \geq (1 - \varepsilon)\|\phi\|_{L^p_{y,\zeta}} - C\varepsilon\|\phi\|_{L^p_{y,\zeta}},$$

where we have used Lemma 5.3.2 and 5.3.4. Hence if the parameters $\zeta, y$ satisfying our requirement, $\eta_{y,\zeta}$ is an isomorphism. \hfill \Box

For convenience, we identify the gluing parameter space $V_{\text{resol},\sigma}$ with a small neighborhood of $\Gamma_{\mathcal{C}}$ is nodal point $\mathcal{C}_y$ by the map $e^{-z}$.

Before formulating the main result, we have to consider the action of automorphisms of curves. Because of the existence of unstable soliton components, the automorphism group is of positive dimension. It is complex 1-dimensional for each unstable component. For example, if $(\mathbb{R} \times S^1, u_{p, j_i})$ is a soliton component (non-BPS soliton by assumption), then the field $\xi_{p,j_i} + t_{p,j_i} \frac{\partial}{\partial t}$ generates the automorphism group, the transition group. $d\xi_{p,j_i}, d\xi_{p,j_i} \frac{\partial}{\partial t}$ generates a complex dimension 1 subspace in $V_{\text{map},\sigma}$.

To eliminate the action of the transition group in the unstable component, we use a normalization technique used in [FO].

We add one Neveu-Schwarz marked point $z_{\ast}$ with the trivial orbifold structure (i.e., the group action is given by $\exp 2\pi i t$ in the unstable component $(\mathbb{R} \times S^1)$, such that if there exists a map in $\text{Aut}(\sigma)$ that maps an unstable component $\mathcal{C}_{y,\zeta}$ to $\mathcal{C}_{y,\zeta}'$, then this map will send the extra marked point $z_{\ast}$ to $z_{\ast}'$. Let $z_{\ast}'' = (z_{\ast}', \ldots, z_{\ast}'')$ be the set of extra marked points on $\mathcal{C}_{y,\zeta}$ chosen in this way. We can also assume that these marked points are chosen such that $u_{\ast}$ is an immersion near these points.
For each new marked point $z'_j$, take an $(2N - 2)$-dimensional disk $D_{z'_j}$ in $\mathbb{C}^N$ which is transversal to $\text{im}(u)$ at $u(z'_j)$. We assume that $D_{g z'_j} = D_{z'_j}$ when $z'_j$ and $g \cdot z'_j$ are marked points when $g \in \text{Aut}(\sigma)$.

Define the parameter space:

$$V^r_\sigma = V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma} \times V_{\text{map}, \sigma}$$

**Theorem 5.3.9.** Let $(\xi_\sigma, u_\sigma)$ be a non-BPS soliton $W$-section in $\mathcal{M}_{g,k,W}(\gamma, \chi)$. Let the set $V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma}$ be small enough such that for any $(y, \zeta) \in V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma}(0, 0) \equiv \sigma)$ the operators $\theta_{y, \zeta}, \eta_{y, \zeta}, s_{y, \zeta}, \Psi_{y, \zeta}$ are well defined on $\mathcal{C}_{y, \zeta}$. Define the set

$$Z'_\sigma = \{ (y, \zeta) \in V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma} \mid \Phi(F_{y, \zeta}(\phi)) \in E_{y, \zeta} \times V_{\text{resol}, \sigma}(0, 0) \}.$$

Define the map $\Psi_{\sigma} : V^r_\sigma \to Z'_\sigma$ by

$$\Psi_{\sigma}(y, \zeta, \phi) := \Psi_{y, \zeta}(\eta_{y, \zeta}(\phi)),$$

and the map $\tilde{s}_{\sigma} : V^r_\sigma \to E_{\sigma}$ by

$$\tilde{s}_{\sigma}(y, \zeta, \phi) := \theta_{y, \zeta}^{-1} \circ s_{y, \zeta} \circ \eta_{y, \zeta}(\phi).$$

Then for sufficiently small $r > 0$, we can obtain a $\text{Aut}(\sigma)$-invariant open neighborhood $U^r_\sigma$ of $0$ which is contained in $V^r_\sigma$ such that when restricting to the domain $U^r_\sigma$ the following conclusions hold:

1. $\Psi_{\sigma}$ is an $\text{Aut}(\sigma)$-equivariant continuous map, which is one to one and onto its image, and for fixed $(y, \zeta)$, $\Psi_{\sigma}(y, \zeta, \cdot)$ is a homeomorphism.
2. The map $s_{\sigma} : U^r_\sigma \cap V_{\text{map}, \sigma} \to \text{Aut}(\sigma)$ defined by its lifting map $\tilde{s}_{\sigma}$ is continuous.
3. Define a closed set in $V^r_\sigma$:

$$V^r_{\text{map}, \sigma} := \{ (y, \zeta, \phi) \in V^r_\sigma \mid \Psi_{\sigma}(y, \zeta, \phi) \in \text{Aut}(\sigma) \},$$

and $V^r_{\text{map}, \sigma}$ is continuous.

Then there is an $\text{Aut}(\sigma)$ action on $V^r_{\text{map}, \sigma}$, and if we let $U_{\sigma} = U^r_\sigma \cap V^r_{\text{map}, \sigma}$, $\Psi_{\sigma}$ induces a homeomorphism between $\tilde{s}_{\sigma}^{-1}(0) \subset U_{\sigma}/\text{Aut}(\sigma)$ and a neighborhood of $\sigma$ in a branch containing $\sigma$ in $\mathcal{M}_{g,k,W}(\gamma, \chi)$.

The data $(U_{\sigma}, E_{\sigma}, s_{\sigma}, \Psi_{\sigma})$ forms a Kuranishi neighborhood of $(\xi_\sigma, u_\sigma)$ in $\mathcal{M}_{g,k,W}(\gamma, \chi)$ of real dimension $6g - 6 + 2k - 2D = \sum_{i=1}^k N_\gamma$, where $D = c_\gamma(g - 1) + \sum_i (\gamma_i)$.

**Remark 5.3.10.** The closed set $V^r_{\text{map}, \sigma} \subset V^r_\sigma$ is actually a fiber bundle over $V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma}$ and each fiber is homeomorphic to the fiber at zero.

**Proof.** Since in our construction of $\mathcal{C}_{y, \zeta}$ and $E_{y, \zeta}$, they can be required to be $\text{Aut}(\sigma)$-invariant, so the operators $\theta_{y, \zeta}, \eta_{y, \zeta}, s_{y, \zeta}, \Psi_{y, \zeta}$ can also be required to be $\text{Aut}(\sigma)$-equivariant maps. Thus $\tilde{s}_{\sigma}, \Psi_{\sigma}$ are $\text{Aut}(\sigma)$-equivariant continuous maps. Continuity comes from the implicit function theorem, Lemma 5.1.8. Furthermore, $V^r_{\text{map}, \sigma}$ is an $\text{Aut}(\sigma)$-invariant closed set. The dimension is given by Theorem 5.1.1.

We need only to prove that $\Psi_{\sigma}$ does not induce a homeomorphism between $\tilde{s}_{\sigma}^{-1}(0) \cap U_{\sigma}/\text{Aut}(\sigma)$ and a neighborhood of $\sigma = (\xi_\sigma, u_\sigma) \in \mathcal{M}_{g,k,W}(\gamma, \chi)$.

First we prove the injectivity of $\Psi_{\sigma}$ when restricting to $\tilde{s}_{\sigma}^{-1}(0) \cap U_{\sigma}/\text{Aut}(\sigma)$.

Denote $\sigma' = (\xi_\sigma', (z_\sigma', z'_\sigma)) \in \mathcal{M}_{g,k+1}$. We remark that $V_{\text{deform}, \sigma'} = V_{\text{deform}, \sigma}, V_{\text{resol}, \sigma} = V_{\text{resol}, \sigma}$ and there is an open embedding

$$\frac{V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma}}{\text{Aut}(\xi_\sigma, (z_\sigma, z'_\sigma))} \to \mathcal{M}_{g,k+1}.$$
Let $\gamma_0 = \oplus_s (s, \theta_s)$ be a transition moving any point $(s, \theta)$ on the component $(S \times S^1)$ to $(s + s_r, \theta + \theta_r)$. We can define an action $\gamma_0 \cdot (\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0))) = (\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0)))$, i.e., fixing the marked points $z_{\hat{\gamma}}$ in stable components but moving the extra marked points $z'_0$ by $\gamma_0$.

Similarly, we can define

$$\gamma_0 \cdot (\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0))) = (\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0))).$$

Here one has to do some small modification such that the transition fixes the gluing domain and only moves the interior points of the unstable components.

If $\gamma_0$ is small enough, then the action of $\gamma_0$ will induce an action on $V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma}$. This action is described in [FO], and we follow their description. If $\gamma_0$ is small enough, the surface $\mathcal{C}_{\gamma, \xi}(\gamma_0)$ is still in the neighborhood of $(\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0))) \in \overline{M}_{g,k+l}$, hence there exists $(\gamma_0, z_{\hat{\gamma}}) \in V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma} = V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma}$ such that $(\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0)))$ is equivalent to $(\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0)))$. This map is unique modulo the finite group $\text{Aut}(\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0)))$. We define $\gamma_0 \cdot (\gamma, \xi) := (\gamma_0, \xi)$.

Now assume that there are $(y', \xi', \phi', (y, \xi, \phi)) \in \mathcal{C}_{\gamma, \xi}(0) \cap U_{\gamma}$ such that $(\mathcal{C}_{\gamma, \xi}(y, \xi, \phi))$ is equivalent to $(\mathcal{C}_{\gamma, \xi}(y', \xi', \phi'))$, where $u_{s'}, \phi' = u_{s', y', \xi', \phi}$ and $u_{s', y', \xi', \phi} = u_{s', y, \xi, \phi}$ are assumed to be solutions of the perturbed Witten equation $W_{\gamma, \xi}(u) = 0$. So there exists a biholomorphic map $\gamma : C_{\gamma, \xi} \to C_{\gamma', \xi'}$ such that $u_{s', \phi} = \gamma \cdot u_{s', \xi', \phi'}$. We want to prove that $\gamma \in \text{Aut}(\sigma)$. Now we have $(\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0))) \in \overline{M}_{g,k+l}$ and $\gamma \cdot (\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0))) = (\mathcal{C}_{\gamma', \xi'}(z_{\gamma', \xi'}(\gamma_0)))$. So $(\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0))) = (\mathcal{C}_{\gamma', \xi'}(z_{\gamma', \xi'}(\gamma_0)))$. We can take $\gamma_1 \in \text{Aut}(\sigma)$ such that $\gamma_1 \cdot (\gamma_0(z_{\gamma, \xi}(\gamma_0)))$ is close to $\gamma_i$ for $i = 1, \cdots, l$. Then there exists a unique $\gamma_0 = \oplus_s (s, \theta_s)$ such that $(\mathcal{C}_{\gamma_1, \xi}(z_{\gamma, \xi}(\gamma_0))) = (\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0)))$ modulo the group $\text{Aut}(\mathcal{C}_{\gamma, \xi}(z_{\gamma, \xi}(\gamma_0)))$. Therefore $\gamma_1 \gamma = \gamma_0$ and $\gamma_1 \cdot (\gamma_0(y, \xi, \zeta)) = (y', \xi')$. We claim that $\gamma_0 = 1$ under our normalization choice. In fact, from the relation

$$u_{s', \xi'} = \gamma_1 \cdot \gamma_0(u_{s', \xi', \phi})$$

we have at the extra marked point $z'_j$

$$u_{s', z'_j} - u_{s'}(\gamma_1 \cdot \gamma_0 \cdot z'_j) = \gamma_1 \cdot \gamma_0(\gamma_1 \cdot \gamma_0(y, \xi, \zeta)) - \gamma_1 \cdot \gamma_0(\gamma_1 \cdot \gamma_0(y, \xi, \zeta)) = \gamma_1 \cdot \gamma_0(y, \xi, \zeta) - \gamma_1 \cdot \gamma_0(y, \xi, \zeta) = 0.$$
In this case, we have (without loss of generality, we consider one nodal point and the two-component case)
\[
\lim_{\hat{T} \to \infty} \text{Diam}(u^w_N,\hat{T})) = 0.
\]
Here \( N_{n,\hat{T}} = ([\hat{T}, T^1_n] \times S^1)_\tau \cup ([\hat{T}, T^2_n] \times S^1)_\tau \) and \( T^1_n \) is a gluing parameter.

On the neck region we have two equations:
\[
\bar{\partial} u_{\text{app},z} - 2 \frac{\partial (W + W^s)(u_{\text{app},z})}{\partial u_t} = 0
\]
\[
\bar{\partial}(u_{\text{app},z} + \phi^\sigma) - 2 \frac{\partial (W + W^s)(u_{\text{app},z} + \phi^\sigma)}{\partial u_t} = 0.
\]

When taking the difference of the two equations, we obtain the equation of \( \phi^\sigma \):
\[
\bar{\partial}\phi^\sigma + A \cdot \phi^\sigma = 0,
\]
(140)
where \( A \) is the coefficient matrix which is differentiable and all of its derivatives are bounded (Notice that the \( C^m \) estimates of any solutions are uniformly bounded). Since the diameter of the image \( u^w(N_{n,\hat{T}}) \) is sufficiently small if \( \hat{T} \) and \( n \) are large enough, the \( C^0 \)

norm of \( \phi^{\sigma} \) is uniformly small on the neck region. So the matrix \( A(u_{\text{app},z} + \phi^\sigma) = A(u^w + \phi^\sigma) \)

will tend to \( A(u^\infty) \), which is a nondegenerate matrix. Thus, for large \( \hat{T} \) and \( n \), the absolute value of the eigenvalues of the matrix \( A(u_{\text{app},z} + \phi^\sigma) \) has uniformly positive lower bound. This means \( \phi^\sigma \) have uniformly exponential decay, hence \( \| \phi^{\sigma} \|_{L^2(N_{n,\hat{T}})} < r \) for large \( n \) and \( \hat{T} \). We have to fix \( \phi^{\sigma} \) on an unstable component of \( \mathcal{C} \). Since \( u^w \) is close to \( u \), there is another unique point \( z_j' \) such that \( u^w(z_j') - u_{\text{app},z}(z_j) \) is tangential to \( D_{z_j} \). So we can redefine \( \phi^{\sigma}(z) = u^w(z - z_j' + z_j) - u_{\text{app},z}(z) \) on the unstable part.

### 5.4. Interior gluing and orientation.

In this part, we will glue the Kuranishi neighborhoods of the interior points of \( \mathcal{M}_{g,k,W}(\gamma, \kappa) \)

into an open global Kuranishi structure. If our moduli space is strongly perturbed, i.e., there is no soliton \( W \)-section, then this gives a Kuranishi structure to the moduli space and hence obtains a virtual cycle by Fukaya-Ono theory if it is orientable. If the moduli space contains a soliton \( W \)-section, then we need to define the Kuranishi neighborhoods near the boundary and then glue the interior Kuranishi structure with the boundary neighborhoods to form a global structure. This will be discussed in latter section.

Since there is no canonical way to construct the Kuranishi structure, we will choose the obstruction bundle step by step and then patch up these charts by induction. Though the Kuranishi structure is dependent on the choice of such spaces, the virtual cycle is well-defined. We will use the partial order \( \succ \) in \( \mathcal{M}_{g,k,W}(\gamma, \kappa) \) defined before to do our induction.

Let \( \mathcal{M}_{g,k,W}(\Gamma; \gamma, \kappa) \) be a minimal (and nonempty) stratum in \( \mathcal{M}_{g,k,W}(\gamma, \kappa) \) with respect to the partial order \( \succ \). Notice that this minimal stratum is not unique.

Unlike the stratum in \( \mathcal{M}_{g,k,W}(\gamma, \kappa) \) is not in general an orbifold. So we can’t construct a universal orbifold obstruction bundle on it. We have to construct it in a more direct way.

Let \( (U_{\tau}, E_{\tau}, \text{Aut}(\sigma), s_{\tau}, \Psi_{\tau}) \) be the Kuranishi neighborhood of \( \sigma \in \mathcal{M}_{g,k,W}(\gamma, \kappa) \) constructed in Theorem \( 5.3.9 \) such that \( (s_{\tau})^{-1}(0) \) is homeomorphic to a neighborhood of \( \sigma \in \mathcal{M}_{g,k,W}(\Gamma; \gamma, \kappa) \). Let \( \Psi_{\tau} : s_{\tau}^{-1}(0) \cap U_{\tau} \to \mathcal{M}_{g,k,W}(\Gamma; \gamma, \kappa) \) be the restriction of \( \Psi_{\tau} \) to \( s_{\tau}^{-1}(0) \cap U_{\tau} \). \( \Psi_{\tau} \) is a homeomorphism by Theorem \( 5.3.9 \). Since \( \mathcal{M}_{g,k,W}(\Gamma; \gamma, \kappa) \) is compact, there are finitely many points \( \tau \) in \( \text{im} \Psi_{\tau} \cap \mathcal{M}_{g,k,W}(\Gamma; \gamma, \kappa) \)
covers $\overline{M}_{g,k,W}(\Gamma; \gamma, \chi)$. Assume $\Omega_\tau \subset \hat{\Omega}_\tau$ are closed subsets and their interior, also cover $\overline{M}_{g,k,W}(\Gamma; \gamma, \chi)$.

Fix a representative $(\xi_\tau, u_\tau) \in \tau$, then we have the bundle $E_\tau \subset L^p(\xi_\tau; \mathfrak{X}(L_i \otimes \Lambda^{0,1}(\xi_\tau)))$. For a point $(\xi, u)$ near $\overline{M}_{g,k,W}(\Gamma; \gamma, \chi)$ in the big Banach bundle, we want to embed $E_\tau$ in a more canonical way in $L^p(\xi; \mathfrak{X}(L_i \otimes \Lambda^{0,1}(\xi)))$.

**Definition 5.4.1.** Let $(\xi, u)$ be a point consisting of a map $u \in L^1(\xi, L_i \times \cdots \times L_i)$ and a $W$-curve $\xi \in \overline{M}_{g,k,W}(\gamma)$. $(\xi, u)$ is said to be closed to $\sigma$ if there exist a representative $(\xi_\sigma, u_\sigma) \in \sigma$, $(y, \zeta) \in V_{\text{deform}, \sigma} \times V_{\text{resolv}, \sigma}$ and a biholomorphic map $\theta: \xi \rightarrow \xi_\sigma$; such that $\theta$ preserves marked points and is an isomorphism of $W$-structures, and $u \theta^{-1}$ is close to $u_{y, \zeta}$ in the smooth topology on each irreducible components of $\xi_{y, \zeta}$, where $(\xi_{y, \zeta}, u_{y, \zeta}) = \Psi_\sigma(y, \zeta, 0)$. This actually gives a topology on the space of "perturbed $W$-sections".

Let $(\xi, u)$ be close to $\sigma \in \Omega_\tau$. By taking $\Omega_\tau$ sufficiently small, we can obtain $(y, \zeta) \in V_{\text{deform}, \tau} \times V_{\text{resolv}, \tau}$ and an automorphism $\theta: \xi \rightarrow \xi_{y, \zeta}$ such that

$$\sup_p |u_{y, \zeta} \circ \theta(p) - u(p)| < \epsilon$$

(141)

So if $\epsilon$ is small, we can use $\theta$ to define

$$\text{emb}_{y, \zeta; \theta}: E_\tau \rightarrow C^\infty(\xi, \mathfrak{X}(L_i \otimes \Lambda^{0,1}(\xi)))$$

(142)

Therefore the perturbed Witten equation we should discuss is

$$(W\xi)_\epsilon \equiv 0 \mod \oplus_{\tau \in \Omega_\tau} \text{emb}_{y, \zeta; \theta}(E_\tau).$$

(143)

**Remark 5.4.2.** Note that if $\beta$ is the cut-off section on $\xi_{y, \zeta}$ which is used to define the perturbed Witten map, then we use $\theta' \beta$ as the corresponding quantity to define the perturbed Witten map on $\xi$.

This means if $(\xi, u)$ is a $W$-section, then the obstruction bundle on it is $\oplus_{\tau \in \Omega_\tau} \text{emb}_{y, \zeta; \theta}(E_\tau)$.

However, there is an ambiguity in choosing the embedding $\text{emb}_{y, \zeta; \theta}$, because of the possible existence of unstable (soliton) components of $\xi$. To get rid of the ambiguity, we choose a similar normalization condition as in (3) of Theorem 5.3.9. We choose the minimal extra marked points as done in the proof of Theorem 5.3.9. We also let $u_\tau$ (where $\tau$ is some $\tau_i$) be an immersion near these extra marked points.

For each new marked point $p$, take an embedded $(2N - 2)$-dimensional disk $D_p$ in $\mathbb{C}^N$, which is transversal to $\text{im}(u_\tau)$ at $u_\tau(p)$. We assume that $D_{\phi(p)} = D_p$ when $p$ and $\phi(p)$ are marked points, and $\phi \in \text{Aut}(\tau)$. We choose those $\theta$ such that

$$u \circ \theta^{-1}(p) \in D_p$$

(144)

Now there are only finitely many $\theta$ satisfying (144), which is $\text{Aut}(\tau)$-invariant and the action of $\text{Aut}(\tau)$ is described in the appendix of [FO].

Let $\text{emb}_{y, \zeta}: E_\tau \rightarrow C^\infty(\xi, \mathfrak{X}(L_i \otimes \Lambda^{0,1}(\xi)))$ be the average of the map $\text{emb}_{y, \zeta; \theta}$ defined by: for $s \in \tau_i$,

$$\text{emb}_{y, \zeta}(s) = \sum_{\theta} \frac{\text{emb}_{y, \zeta; \theta}(s)}{|\text{Aut}(\tau_i)|}$$

So the equation (143) can be modified as

$$(W\xi)_\epsilon \equiv 0 \mod E_\epsilon$$

(145)

where $E_\epsilon := \oplus_{\tau_i} \text{emb}_{y, \zeta}(E_\tau)$. Now the definition of the equation (145) is independent of the choice of $\sigma$. 


Using Theorem 5.3.9, we can construct for any point \( \sigma \in \mathcal{M}_{g,k,W}(\Gamma; \gamma, \kappa) \) a Kuranishi neighborhood \((U_\sigma, E_\sigma, s_\sigma, \Psi_\sigma)\) with respect to the equation (145).

We next construct the coordinate change. We choose \( \{U_\sigma\} \) satisfying the following condition:

if \( \rho \in \text{im} \Psi_\sigma \) and if \( \rho \in \Omega_\tau, \) then \( \sigma \in \Omega_\tau \)

(146) is true since \( \Omega_\tau \) is closed. (146) implies if \( \rho \in \text{im} \Psi_\sigma \cap \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \), then \( E_\rho \subset E_\sigma \). Hence if \( (\xi, \mu) \) is a solution of (145) for \( \rho \) with the condition (144), then it solves (145) for \( \sigma \). Thus we find the required embeddings \( \phi_{\sigma \rho} : U_\rho \to U_\sigma \) and \( \phi_{\sigma \rho} : E_\rho \to E_\sigma \).

So \( \{\phi_{\sigma \rho}\} : (U_\rho, E_\rho, \text{Aut}(\rho), s_\rho, \Psi_\rho) \to (U_\sigma, E_\sigma, \text{Aut}(\sigma), s_\sigma, \Psi_\sigma) \) becomes the morphism defined in Definition 5.2.5. It is easy to check that those data construct a Kuranishi structure in a neighborhood of those strata \( \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \). By induction, we can assume we have constructed a Kuranishi structure in a neighborhood of those strata \( \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \) for \( \Gamma' < \Gamma \). Now we want to construct the Kuranishi structure near \( \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \). We already have finitely many \( \tau_i \) contained in some \( \overline{\mathcal{M}}_{g,k,W}(\Gamma'; \gamma, \kappa) \) with \( \Gamma' < \Gamma \) and maps \( \Psi_{\tau_i} : s_{\tau_i}^{-1}(0) \cap U_{\tau_i} \to \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \) such that \( \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \) minus the union of images of \( \Psi_{\tau_i} \) is compact. We then choose finitely many \( \tau_i' \) on \( \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \) such that

\[ \bigcup_i \text{im} \Psi_{\tau_i} \cup \bigcup_i \text{im} \Psi_{\tau_i'} \supset \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa). \]

Choose closed subsets \( \Omega_\tau, \subset \text{im} \Psi_{\tau}, \Omega_\tau' \subset \text{im} \Psi_{\tau}' \) such that their interiors cover \( \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \). For each \( \sigma \in \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \), we put

\[ E_\sigma = \bigoplus_{\tau \in \partial_\tau} E_{\tau}, \bigoplus_{\tau \in \partial_\tau} E_{\tau}' \]

(147)

We use this space to define a similar equation to (145), and also requires that condition (143) holds. So by the same argument used to study the first stratum, we can extend the Kuranishi structure to a neighborhood of \( \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \).

Finally we prove that the Kuranishi structure in the interior of \( \overline{\mathcal{M}}_{g,k,W}(\Gamma; \gamma, \kappa) \) is orientable. To prove that it is orientable, Fukaya-Ono showed it is enough to show the tangent bundle is stably almost complex. The key point in their proof is that the symbol of the linearization of the Cauchy-Riemann equation is complex linear and the moduli space \( \mathcal{M}_{g,k} \) is a complex orbifold which has a complex tangent bundle. Then they use a family of operators connecting the Cauchy-Riemann operator and its complex linear part and change the orientation problem of the Cauchy-Riemann operator to the orientation problem of its complex linear part. In this way, they proved that the Kuranishi structure is orientable. It is the same situation in our case. The linearization operator is only real linear, not complex linear, but the symbol of the first order term is complex linear. We can prove that the interior of our moduli space is orientable using their proof almost word for word. The only difference is that we don’t connect our operator directly to \( \overline{\partial} \) but to some complex linear operator having 0-order term, since the Sobolev spaces we use are based on a noncompact surface. The detailed proof can be seen in section 16 of [FO].

Now if the moduli space \( \overline{\mathcal{M}}_{g,k,W}(\gamma, \kappa) \) is strongly regular; we have \( \overline{\mathcal{M}}_{g,k,W}(\gamma, \kappa) = \overline{\mathcal{M}}_{g,k,W}(\gamma, \kappa) \). By abstract Kuranishi theory, we have the following conclusion.

**Theorem 5.4.3.** Suppose that the moduli space \( \overline{\mathcal{M}}_{g,k,W}(\gamma, \kappa) \) is strongly regular, then \( \overline{\mathcal{M}}_{g,k,W}(\gamma, \kappa) \) is a compact Hausdorff space carrying an orientable Kuranishi structure \( \{U_\sigma, E_\sigma, \text{Aut}(\sigma), s_\sigma, \Psi_\sigma\} \) with (real) dimension \( 6g - 6 + 2k - 2D - \sum_{i=1}^k N_i \), where \( D = \)
\[ \hat{c}_W(g - 1) + \sum_i \epsilon(\gamma_i), \text{ where } \hat{c}_W = \sum_i (1 - 2q_i) \text{ and } \epsilon(\gamma_i) = \sum_i (\Theta_{i}^{\gamma_i} - q_i). \]

becomes the virtual fundamental cycle of its Kuranishi structure.

5.5. Neighborhood around BPS soliton section.

We always assume that there is only one group element \( \bar{\gamma} \in G \) such that the perturbed polynomial \( W_{\bar{\gamma}} + W_{0,\bar{\gamma}} \) has only two critical points \( \kappa^\pm \) such that \( \text{Im}(W_{\bar{\gamma}} + W_{0,\bar{\gamma}})(\kappa^+) = \text{Im}(W_{\bar{\gamma}} + W_{0,\bar{\gamma}})(\kappa^-) \). We consider a BPS soliton \( W \)-section \( (\mathcal{S}, \mathbf{u}) \) in \( \mathcal{M}_{g,k,W}(\gamma, \mathbf{b}, W) \). We say a nodal point \( p \) of \( (\mathcal{S}, \mathbf{u}) \) is decorated by \( (\bar{\gamma}, \kappa) \) if there exists a rigidification \( \psi_p \) such that \( \psi_p(\mathbf{u}(p)) = \kappa \). Note that the combinatorial type of this kind of soliton section is finite. Several cases may occur for a soliton \( W \)-section in \( \mathcal{M}_{g,k,W}(\gamma, \mathbf{b}, W) \):

1. There is only one marked point decorated by \( (\bar{\gamma}, \kappa) \) and there is no nodal point decorated by \( (\bar{\gamma}, \kappa) \).
2. There is no marked point decorated by \( (\bar{\gamma}, \kappa) \) and there is only one nodal point decorated by \( (\bar{\gamma}, \kappa) \).
3. There is no marked point decorated by \( (\bar{\gamma}, \kappa) \) and there are several nodal points decorated by \( (\bar{\gamma}, \kappa) \).
4. There is one marked point decorated by \( (\bar{\gamma}, \kappa) \) and there are several nodal points decorated by \( (\bar{\gamma}, \kappa) \).

To construct the neighborhoods of BPS soliton sections of the above cases, we need to know more details about a BPS soliton \( \mathbf{u}_{\bar{\gamma}+} \in \mathcal{S}_{\bar{\gamma}}(\kappa^+, \kappa^-) \).

Kernel and cokernel of \( D_{\mathbf{u}_{\bar{\gamma}+}}(WI) \). The linearized operator

\[ D_{\mathbf{u}_{\bar{\gamma}+}}(WI) : L^p_1(\mathbb{R} \times S^1, \mathcal{L}_1) \times \cdots \times L^p_1(\mathbb{R} \times S^1, \mathcal{L}_N) \rightarrow L^p(\mathbb{R} \times S^1, \mathcal{L}_1 \otimes \Lambda^{0,1}) \times \cdots \times L^p(\mathbb{R} \times S^1, \mathcal{L}_N \otimes \Lambda^{0,1}) \]

is split into the direct sum of two operators:

\[ D^R_{\mathbf{u}_{\bar{\gamma}+}}(WI) = D^R_{\mathbf{u}_{\bar{\gamma}+}}(WI) \oplus D^N_{\mathbf{u}_{\bar{\gamma}+}}(WI) \]

where

\[ D^R_{\mathbf{u}_{\bar{\gamma}+}}(WI) : L^p_1(\mathbb{R} \times S^1, \mathcal{C}^N) \rightarrow L^p(\mathbb{R} \times S^1, \mathcal{C}^N \otimes \Lambda^{0,1}) \]

has the form

\[ D^R_{\mathbf{u}_{\bar{\gamma}+}}(WI)(\phi_1, \ldots, \phi_N) = (\bar{\partial}_{\phi} \phi_1 - 2 \sum_{j=1}^N \frac{\partial^2(W_{\bar{\gamma}} + W_{0,\bar{\gamma}})}{\partial u_i \partial u_j} \phi_j, \ldots, \bar{\partial}_{\phi} \phi_N - 2 \sum_{j=1}^N \frac{\partial^2(W_{\bar{\gamma}} + W_{0,\bar{\gamma}})}{\partial u_i \partial u_j} \phi_j). \]

and

\[ D^N_{\mathbf{u}_{\bar{\gamma}+}}(WI) : L^p_1(\mathbb{R} \times S^1, \mathcal{L}_1 \otimes \Lambda^{0,1}) \times \cdots \times L^p_1(\mathbb{R} \times S^1, \mathcal{L}_N \otimes \Lambda^{0,1}) \rightarrow L^p(\mathbb{R} \times S^1, \mathcal{L}_1 \otimes \Lambda^{0,1}) \]

has the form

\[ D^N_{\mathbf{u}_{\bar{\gamma}+}}(WI)(\phi_{N+1}, \ldots, \phi_N) = (\bar{\partial}_{\phi} \phi_1 - 2 \sum_{j=N+1}^N \frac{\partial^2 W_N}{\partial u_{N+1} \partial u_j} \phi_j, \ldots, \bar{\partial}_{\phi} \phi_N - 2 \sum_{j=N+1}^N \frac{\partial^2 W_N}{\partial u_{N+1} \partial u_j} \phi_j). \]

Note that we have the decomposition \( W = W_N + \hat{W} \) according to the action of \( \bar{\gamma} \).

**Proposition 5.5.1.** Suppose \( \mathbf{u}_{\bar{\gamma}+} \) is a BPS soliton. If the perturbation parameters \( b_i \) in \( W_{0,\bar{\gamma}} \) are sufficiently small, then the \( C^0 \) norm of \( \mathbf{u}_{\bar{\gamma}+} \) is also sufficiently small.

**Proof.** Since \( (WI)(\mathbf{u}_{\bar{\gamma}+}) = 0 \), we have

\[ \text{Re}(W_{\bar{\gamma}} + W_{0,\bar{\gamma}})(\kappa^-) - \text{Re}(W_{\bar{\gamma}} + W_{0,\bar{\gamma}})(\kappa^+) = \int_{-\infty}^{\infty} \sum_i \left| \frac{\partial(W_{\bar{\gamma}} + W_{0,\bar{\gamma}})}{\partial u_i} \right|^2. \]
Hence if the perturbation is small enough, the integral of the right hand side is small enough, which implies that the pointwise norm $|\tilde{\mathcal{H}}_{W_0,\gamma}|$ is small enough. This is because the $C^1$ norm of each section $u_i$ can be uniformly bounded by a constant which is independent of the perturbation parameters $b_i$ in $W_{0,\gamma}$ when all $b_i$ are less than 1 (cf. Theorem 5.5.9). Therefore $|\tilde{\mathcal{H}}_{W_0,\gamma}|$ is sufficiently small. By nondegeneracy of $W$, the absolute value of each section $u_i$ should be sufficiently small. □

**Lemma 5.5.2.** If the perturbation parameters $b_i$ in $W_{0,\gamma}$ are sufficiently small, then the linearized operator $D_{u^+}^R(W)$ is an isomorphism.

*Proof.* If we set $\phi^N := (\phi_N, \ldots, \phi_N)^T$ and $A^N(s) := (-2\tilde{\mathcal{H}}_{W_0,\gamma})_{1 \leq i, j \leq N}$, then

$$D_{u^+}^R(WI)(\phi^N) = \xi_i \phi^N + A^N(s) \cdot \phi^N.$$ 

Here $\xi = s + i\theta$ and the matrix $A^N$ depends on $u_{+}$. Define matrices

$$\Theta_N = \text{diag}(\Theta_{1-N}, \ldots, \Theta_{N}),$$

$$\eta(\Theta_N) = \text{diag}(e^{-i\theta}\Theta_{1-N}, \ldots, e^{-i\theta}\Theta_{N}).$$

Then the multiplication by $\eta(\Theta_N)$ from $L^p(\mathbb{R} \times S^1, \mathbb{C}^N)$ to $L^p(\mathbb{R} \times S^1, \mathbb{C}^{N-1})$ and from $L^p(\mathbb{R} \times S^1, \mathbb{C}^N \otimes \Lambda^1) \to L^p(\mathbb{R} \times S^1, \mathbb{C}^{N-1} \otimes \Lambda^0)$ are isomorphisms. The operator $\eta(\Theta_N) \circ D_{u^+}^R(WI) \circ \eta(\Theta_N)^{-1}$ is an isomorphism by Proposition 5.5.1 if the perturbation parameters $b_i$ are sufficiently small. Thus we know that $D_{u^+}^R(WI)$ is an isomorphism. □

Now we study the transversality of $D_{u^+}^R(WI)$. Define

$$H_{\pm} := \{ (u_1, \ldots, u_N) : \text{Im}(W_{\gamma} + W_{0,\gamma}(u_1, \ldots, u_N)) = \text{Im}(W_{\gamma} + W_{0,\gamma}(\kappa^+)) \}$$

This is a real codimension 1 submanifold in $\mathbb{C}^N$.

**Lemma 5.5.3.** Suppose the BPS soliton $u_{+}$ to be a Morse-Smale flow on the manifold $H_{\pm}$. If the perturbation parameters $b_i$ in $W_{0,\gamma}$ are sufficiently small, then the kernel $V_{\pm}$ and cokernel $E_{\pm}$ of the linearized operator $D_{u^+}^R(WI) : L^p(\mathbb{R} \times S^1, \mathbb{C}^N) \to L^p(\mathbb{R} \times S^1, \mathbb{C}^N \otimes \Lambda^0)$ are 1-dimensional and generated by $u_{\pm}$ and $\tilde{u}_{\pm}$, respectively.

*Proof.* Set $\phi^R := (\phi_1, \ldots, \phi_N)^T$ and $A^R(s) := (-2\tilde{\mathcal{H}}_{W_0,\gamma})_{1 \leq i, j \leq N}$; then

$$D_{u^+}^R(WI)(\phi^R) = \xi_i \phi^R + A^R(s) \cdot \phi^R.$$ 

By Proposition 5.5.1 the $C^1$ norm of $A^R(s)$ is sufficiently small if the $b_i$ in $W_{0,\gamma}$ are sufficiently small. Now it is a well-known fact in symplectic geometry that if the $C^1$ norm of $A^R(s)$ is small enough, then the kernel and cokernel of $D_{u^+}^R(WI)$ are the same as the kernel and cokernel of the operator

$$D_{u^+}^{R,\mathbb{R}}(W) : L^p(\mathbb{R}, \mathbb{C}^N) \to L^p(\mathbb{R}, \mathbb{C}^N \otimes \Lambda^0).$$

(see the proof on Page 1038 of [FO] or [ZZ] [Oî]).

Since all the BPS solitons connecting $\kappa^+$ and $\kappa^-$ should lie in the hypersurface $H_{\pm}$, we have $\ker D_{u^+}^{R,\mathbb{R}}(W) \subset L^p(\mathbb{R}, \mathbb{C}^N \otimes \Lambda^0)$. Since we assume that $u_{-}$ is a Morse-Smale flow on $H_{\pm},$ the $\ker D_{u^+}^{R,\mathbb{R}}(W)$ is just the 1-dimensional space generated by the field $\frac{\partial H_{\pm}}{\partial \mathbf{x}}$. On
the other hand, the dual operator \((D^+_{\mathcal{A}_-}(WI))^* = -\partial_s + A^R(s)\). It is easy to see that \(i\frac{\partial_s}{\partial t}\) satisfies \((D^+_{\mathcal{A}_-}(WI))^*(u) = 0\). Therefore it generates the 1-dimensional cokernel. \(\square\)

Let \(H^\gamma_{\text{para}} \subset \mathbb{C}^{N_1}\) be the set in the parameter space of \(b_1\) such that there exist two critical points \(\kappa^+\) and \(\kappa^-\) of \(W_\gamma + W_{0,\gamma}\) satisfying \(\text{Im}(W_\gamma + W_{0,\gamma})(\kappa^-) = \text{Im}(W_\gamma + W_{0,\gamma})(\kappa^+)\). By Theorem 3.1.4 \(H^\gamma_{\text{para}}\) is a union of finitely many real hypersurfaces.

**Theorem 5.5.4.** For generic points on \(H^\gamma_{\text{para}}\), the kernel and cokernel spaces of \(D_{\mathcal{A}_-}(WI)\) are just the 1-dimensional spaces \(V_\gamma\) and \(E_{\gamma,-}\). We can choose a Kuranishi neighborhood of \((R \times S^1, u_{\gamma,-})\) as \(((pt), E_{\gamma,-}, s \equiv 0)\).

**Proof.** For generic points on \(H^\gamma_{\text{para}}\), the function \(-2\text{Re}(W_\gamma + W_{0,\gamma})\) is a Morse function on \(H_{\gamma,-}\) such that \(u_{\gamma,-}\) is a Morse-Smale flow on \(H_{\gamma,-}\). By Lemmas 5.5.2 and 5.5.3 we finish the proof. \(\square\)

**Remark 5.5.5.** The cokernel space \(E_{\gamma,-}\) can be generated by a compactly supported element in \(L^p(R, u_{\gamma,-}^* T\mathbb{C}^{N_1})\), since we can multiply the section \(i\frac{\partial_s}{\partial t}\) by a cut-off function with sufficiently large compact support.

Now we begin the construction of the neighborhoods of BPS soliton sections case by case.

Case (1). Suppose that \(\mu = (\mu', \kappa^+)\). Take \((\zeta_1, u_1) \in \mathcal{M}^\mu_{g,k,W}(\gamma, \mu, \kappa^-)\) and a BPS soliton \(u_{\gamma,-} \in S_\gamma(\kappa^+, \kappa^-)\). Then \((\zeta_1 \# (R \times S^1), u_1 \# u_{\gamma,-}) \in \mathcal{M}^\mu_{g,k,W}(\gamma, \mu', \kappa^-)\). We define a map:

\[
\text{Glue}_{u_{\gamma,-}}^m: \mathcal{M}^\mu_{g,k,W}(\gamma, \mu', \kappa^-) \to \mathcal{M}^\mu_{g,k,W}(\gamma, \mu', \kappa^-)
\]

by

\[
\text{Glue}_{u_{\gamma,-}}^m(u_1) := u_1 \# u_{\gamma,-}.
\]

Here we want to say words about the gluing. Note that \(u_{\gamma,-}\) is defined in \(\mathbb{C}^N\) space. To do the gluing, we firstly map \(u_1\) to \(\mathbb{C}^N\) by the rigidification and then do gluing in \(\mathbb{C}^N\) space. The obtained element inherits the same rigidification.

By Theorem 5.4.3 we can give an oriented Kuranishi structure to each strongly regular moduli space \(\mathcal{M}^\mu_{g,k,W}(\gamma, \mu, \kappa^-)\). We repeat the procedures in the last section to construct the neighborhood of \(\text{Glue}_{u_{\gamma,-}}^m(u_1)\). Let \((U_\sigma, E_\sigma, s_\sigma)\) be a Kuranishi neighborhood of \(u_1\) in \(\mathcal{M}^\mu_{g,k,W}(\gamma, \mu', \kappa^-)\). By Theorem 5.5.4 we can take the Kuranishi neighborhood of \((R \times S^1, u_{\gamma,-})\) to be \(((pt), E_{\gamma,-}, s \equiv 0)\). Now we can use our gluing construction again:

1. Assume that \(U_\sigma = V_{\text{deform},\sigma} \times V_{\text{resol},\sigma} \times V_{\text{map},\sigma}\). Let \(\beta \in V_{\text{deform},\sigma} \times V_{\text{resol},\sigma}\) and \(\zeta \in D_\sigma(0) \in \mathbb{C}\). We can get the nearby curve \(\zeta_{\beta,\zeta}\).
2. Take the obstruction bundle \(E_{\beta,\zeta} = E_\sigma \oplus E_{\gamma,-}\).
3. Use the implicit function theorem to construct the Kuranishi map \(s_{\sigma,\zeta}\) on \(U_\sigma \times D_\sigma(0)\). Because of the \(S^1\) action, the “Kuranishi” neighborhood of the boundary point \((\zeta_{\sigma,\zeta} \# (R \times S^1), u_{\sigma,\zeta} \# u_{\gamma,-})\) is \(((U_\sigma \times [0, \varepsilon_\sigma]), E_\sigma \oplus E_{\gamma,-} \oplus s_{\sigma,\zeta}, \varepsilon_\sigma)\), where the length \(\varepsilon_\sigma\) of the cone may depend on the point \(\sigma\). We understand that the section \(s_{\sigma,\zeta}\) depends only on the first coordinate of \(\zeta\).

Now we have the following lemma.

**Lemma 5.5.6.** Let \((U_\sigma, E_\sigma, s_\sigma)\) be a Kuranishi neighborhood of \((\zeta_1, u_1) \in \mathcal{M}^\mu_{g,k,W}(\gamma, \mu', \kappa^-)\), then \((U_\sigma \times [0, \varepsilon_\sigma], E_\sigma \oplus E_{\gamma,-} \oplus s_{\sigma,\zeta}, \varepsilon_\sigma)\) is a Kuranishi neighborhood of \(\text{Glue}_{u_{\gamma,-}}^m(u_1)\) in \(\mathcal{M}^\mu_{g,k,W}(\gamma, \mu', \kappa^-)\). There exists a sequence of smooth multisections \(s_{\sigma,\zeta}\) is transversal to the zero section such
that it converges to \( s_{r,\ell} \) in the \( C^0 \) topology. For sufficiently large \( n \), the zero set \( s_{r,0,n}^{-1}(0) \) is diffeomorphic to \( s_{r,0,n}^{-1}(0) \times [0,e] \).

Proof. The approximating sequence of multisections \( s_{r,\ell,n} \) is given by Lemma 17.4 in [FO]. \[ \square \]

Case (2). There are still two types of gluing operations called tree gluing and loop gluing in this case.

**Tree gluing.** Suppose that \( g = g_1 + g_2, k = k_1 + k_2, \gamma = (\gamma_1, \gamma_2) \) and \( \kappa = (\kappa_1, \kappa_2) \).

Let \( (G_1, u_1) \in \mathcal{M}_{g_1,k+1,W}(\gamma_1, \gamma_1; \kappa; \kappa') \), \( (G_2, u_2) \in \mathcal{M}_{g_2,k+1,W}(\gamma_2, \gamma_2; \kappa; \kappa') \) and let \( u_{-1} \in S_{\gamma}(\kappa, \kappa') \) be a BPS soliton. Then \( (G_1, u_1, \#(\mathbb{R} \times S^1)) \) is a BPS soliton \( W \)-section in \( \mathcal{M}_{g,k,W}(\gamma, \kappa) \). We define the map from \( \mathcal{M}_{g_1,k+1,W}(\gamma_1, \gamma_1; \kappa; \kappa') \times \mathcal{M}_{g_2,k+1,W}(\gamma_2, \gamma_2; \kappa; \kappa') \) to \( \mathcal{M}_{g,k,W}(\gamma, \kappa) \) to be \( \text{Glue}_{u_1, \#(u_2)}(u_1, u_2) := u_1 \# u_{-1} \# u_2 \).

Let \( A \) be a space; we define the cone based on \( A \) with length \( \varepsilon \) to be \( C_\epsilon(A) \).

Let \( (U_{r,1}, E_{r,1}, s_{r,1}) \) be a Kuranishi neighborhood of \( (G_1, u_1) \) for \( i = 1, 2 \). Then \( (U_{r,1} \times U_{r,2}, E_{r,1} \oplus E_{r,2}, s_{r,1} \oplus s_{r,2}) \) is a neighborhood of \( u_1 \times u_2 \). Let \( U_{r,1} = V_{\text{def}, 1} \times V_{\text{res}, 1} \times V_{\text{map}, 1} \).

We can begin our gluing procedures:

1. The parameter space for the curves is \( V_{\text{def}, 1} \times V_{\text{res}, 1} \times V_{\text{def}, 2} \times V_{\text{res}, 2} \times D_\varepsilon(0) \times D_\varepsilon(0) \). We take a point \( (\beta_1, \beta_2, \xi_1, \xi_2) \), then we can get a nearby curve \( G_{\beta_1, \beta_2, \xi_1, \xi_2} \in \mathcal{M}_{g,k,W}(\gamma, \kappa) \), where \( \beta_1, \beta_2 \in V_{\text{def}, 1} \times V_{\text{res}, 1} \) and \( (\xi_1, \xi_2) \in D_\varepsilon(0) \times D_\varepsilon(0) \).

2. Set the obstruction bundle on \( G_{\beta_1, \beta_2, \xi_1, \xi_2} \) to be \( E_{\sigma,1} \oplus E_{\sigma,2} \oplus E_{\sigma,1} \).

3. From \( u_1, u_{-1}, u_2 \) we get the approximating solution \( \text{uapp} \beta_1, \beta_2, \xi_1, \xi_2 \). Use the implicit function theorem to obtain the Kuranishi map \( s_{r,1,\sigma,1,\xi_1,\xi_2} \) on \( U_{r,1} \times U_{r,2} \times D_\varepsilon(0) \times D_\varepsilon(0) \). Here the radius of these domains may shrink. Since \( u_{-1} \) is \( S^1 \)-invariant, the approximating solution is also \( S^1 \)-invariant. Hence the moduli space we need should be modulo the \( S^1 \) action. The \( S^1 \) group gives a diagonal action to the neighborhood \( D_\varepsilon(0) \times D_\varepsilon(0) \) while keeping the other parameter space fixed. We take the neighborhood \( C_\varepsilon(S^3) \subset D_\varepsilon(0) \times D_\varepsilon(0) \). We have \( C_\varepsilon(S^3)/S^1 = C_\varepsilon(S^2) \).

Hence we obtain a neighborhood of the point \( u_1 \# u_{-1} \# u_2 \) in \( \mathcal{M}_{g,k,W}(\gamma, \kappa) \):

\[ (U_{r,1} \times U_{r,2} \times C_\varepsilon(S^2), E_{\sigma,1} \oplus E_{\sigma,2} \oplus E_{\sigma,1} \oplus s_{r,1,\sigma,1,\xi_1,\xi_2}), \]

where \( e_{\sigma,1} \) depends on the point \( \sigma_1 \times \sigma_2 \), and \( [\xi_1, \xi_2] \) is the element in the quotient space \( C_\varepsilon(S^2)/C_\varepsilon(S^3)/S^1 \subset C \times C \).

Using the same method as Lemma 5.5.6, we can chose a uniform \( \varepsilon > 0 \) for the Kuranishi structure on \( \mathcal{M}_{g,1,k+1,W}(\gamma_1, \gamma_1; \kappa; \kappa') \times \mathcal{M}_{g,2,k+1,W}(\gamma_2, \gamma_2; \kappa; \kappa') \).

Using this method, we have:

**Lemma 5.5.7 (Tree).** Let \( (U_{r,1} \times U_{r,2}, E_{r,1} \oplus E_{r,2}, s_{r,1} \oplus s_{r,2}) \) be a Kuranishi neighborhood of \( (u_1, u_2) \in \mathcal{M}_{g,k,W}(\gamma_1, \gamma_2; \kappa; \kappa') \). Then \( (U_{r,1} \times U_{r,2} \times C_\varepsilon(S^2), E_{r,1} \oplus E_{r,2} \oplus E_{r,1} \oplus s_{r,1,\sigma,1,\xi_1,\xi_2}) \) is a Kuranishi neighborhood of \( \text{Glue}_{u_1, \#(u_2)}(u_1, u_2) \) in \( \mathcal{M}_{g,k,W}(\gamma, \kappa, \kappa') \).

There exists a sequence of smooth multisections \( s_{r,1,\sigma,1,\xi_1,\xi_2} \) which is transversal to the zero section such that it converges to \( s_{r,1,\sigma,1,\xi_1,\xi_2} \) in the \( C^0 \) topology. For sufficiently large \( n \), the zero set \( s_{r,1,\sigma,1,\xi_1,\xi_2,\nu}^{-1}(0) \) is diffeomorphic to \( s_{r,1,\sigma,1,\xi_1,\xi_2,\nu}^{-1}(0) \times C_\varepsilon(S^2) \).

Proof. Since the cone \( C_\varepsilon(S^2) \) is homeomorphic to a closed small neighborhood \( N_\varepsilon \) of \( \mathbb{R}^3 \), we can use the approximation theorem on the section \( s_{r,1,\sigma,1,\xi_1,\xi_2} \) over \( U_{r,1} \times U_{r,2} \times N_\varepsilon \). \[ \square \]
Loop gluing. Let \(( \mathcal{C}_1, u_1) \in \mathcal{M}_{g,k}^{\text{ig},2}(\mathbf{y}, \bar{y}; X, K, \kappa)\) and \((\mathbb{R} \times S^1, u_\infty) \in S^\circ(K, \kappa)\). Then we can glue \(( \mathcal{C}_1, u_1)\) and \((\mathbb{R} \times S^1, u_\infty)\) between the two marked points on \( \mathcal{C}_1\) decorated by \((\bar{y}, \kappa)\) and \((\bar{y}^{-1}, \kappa)\) to obtain an element in \(\mathcal{M}_{g,k}^{\text{ig},2}(\mathbf{y}, X)\). Denote this element by \(\text{Glue}^n_{u_\infty}(u_1)\). In the same way, we have

\section*{Lemma 5.5.8 (Loop).} Let \((U_{c_1}, E_{c_1}, s_{c_1})\) be a Kuranishi neighborhood of \(\mathcal{M}_{g,k}^{\text{ig},2}(\mathbf{y}, \bar{y}; X, K, \kappa)\). Then \((U_{c_1} \times C_\varepsilon(S^2), E_{c_1} \oplus E_{-\varepsilon}, s_{c_1}, \{0, \infty\})\) is a Kuranishi neighborhood of \(\text{Glue}^n_{u_\infty}(u_1)\) in \(\mathcal{M}_{g,k}^{\text{ig},2}(\mathbf{y}, X)\). There exists a sequence of smooth multisectons \(s_{c_1} \in \{0, \infty\}\) is transversal to the zero section such that it converges to \(s_{c_1, j_{123}}\) in the \(C^0\) topology. For sufficiently large \(n\), the zero set \(s_{c_1, j_{123}}^{-1}(0)\) is diffeomorphic to \(s_{c_1, j_{123}}^{-1}(0) \times C_\varepsilon(S^2)\).

**Case (3)** The gluing operation will become more complicated because of the possible tree gluing and the possible loop gluing. Here we only consider the simplest gluing which does not contain any loop gluing.

Let \(u_{i_{-n}} \in \mathcal{M}_{g,k}^{\text{ig},2}(\mathbf{y}, \bar{y}; X, K, \kappa)\). Assume that \(g = g_1 + \cdots + g_{n+1}, k = k_1 + \cdots + k_{n+1}\) and the index group \((\mathbf{y}, X)\) is divided into \(l+1\) parts: \((\mathbf{y}_1, X_1), \ldots, (\mathbf{y}_{l+1}, X_{l+1})\).

Take \(( \mathcal{C}_1, u_1) \in \mathcal{M}_{g_1,k_1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa), (\mathcal{C}_2, u_2) \in \mathcal{M}_{g_2,k_2}^{\text{ig},2}(\mathbf{y}_2, \bar{y}_2; X_2, K, \kappa), \ldots, (\mathcal{C}_l, u_l) \in \mathcal{M}_{g_l,k_l}^{\text{ig},2}(\mathbf{y}_l, \bar{y}_l; X_l, K, \kappa)\).

We can define the gluing map from \(\mathcal{M}_{g_1,k_1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa)\) to \(\mathcal{M}_{g,l+k+1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa)\) as follows:

\[
\text{Glue}^n_{u_{i_{-n}}}(u_1, \ldots, u_l) := (\mathcal{C}_1 \# \mathcal{M}_{g,l} \# \ldots \# \mathcal{C}_l)\times \mathcal{M}_{g_1,k_1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa).
\]

Similarly, we have the following conclusion.

**Lemma 5.5.9.** Let \((U_{c_1} \times \cdots \times U_{c_1}) \in \mathcal{M}_{g_1,k_1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa)\) be a Kuranishi neighborhood of \(\mathcal{M}_{g,1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa)\). Then \((U_{c_1} \times \cdots \times U_{c_1}) \in \mathcal{M}_{g,1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa)\) is a Kuranishi neighborhood of \(\text{Glue}^n_{u_{i_{-n}}}(u_1, \ldots, u_l)\) in \(\mathcal{M}_{g,l+k+1}^{\text{ig},2}(\mathbf{y}, X)\). There exists a sequence of smooth multisectons \(s_{c_1} \in \{0, \infty\}\) which are transversal to the zero section such that it converges to \(s_{c_1, j_{123}}\) in the \(C^0\) topology. For sufficiently large \(n\), the zero set \(s_{c_1, j_{123}}^{-1}(0)\) is diffeomorphic to \(s_{c_1, j_{123}}^{-1}(0) \times C_\varepsilon(S^2)\).

**Case (4)**. Like in Case (3), we only give the description of the simplest treegluing. Take the same notations as in Case (3) except that we require

\[
u_{i+1} \in \mathcal{M}_{g_1,k_1}^{\text{ig},2}(\mathbf{y}_{i+1}, \bar{y}_{i+1}; X_{i+1}, K, \kappa).
\]

Let \(u_{i_{-n}} \in \mathcal{M}_{g_1,k_1}^{\text{ig},2}(\mathbf{y}_{i+1}, \bar{y}_{i+1}; X_{i+1}, K, \kappa)\). Then we define the gluing operation:

\[
\text{Glue}^n_{u_{i_{-n}}}(u_1, \ldots, u_{i+1}) := \text{Glue}^n_{u_{i_{-n}}}(\text{Glue}^n_{u_{i_{-n}}}(u_1, \ldots, u_{i+1})).
\]

**Lemma 5.5.10.** Let \((U_{c_1} \times \cdots \times U_{c_1}) \in \mathcal{M}_{g_1,k_1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa)\) be a Kuranishi neighborhood of \(\mathcal{M}_{g,1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa)\). Then \((U_{c_1} \times \cdots \times U_{c_1}) \in \mathcal{M}_{g,1}^{\text{ig},2}(\mathbf{y}_1, \bar{y}_1; X_1, K, \kappa)\) is a Kuranishi neighborhood of \(\text{Glue}^n_{u_{i_{-n}}}(u_1, \ldots, u_{i+1}) \in \mathcal{M}_{g,l+k+1}^{\text{ig},2}(\mathbf{y}, X)\). There exists a
sequence of smooth multisections \( s_{r_1, \ldots, r_n, l_1, \ldots, l_n} \) which are transversal to the zero section such that it converges to \( s_{r_1, \ldots, r_n, l_1, \ldots, l_n} \) in the \( C^0 \) topology. For sufficiently large \( n \), the zero sequence \( s_{r_1, \ldots, r_n, l_1, \ldots, l_n}^{(0)} \) is diffeomorphic to \( s_{r_1, \ldots, r_n, [0,0], \ldots, [0,0], 0, 0}^{(0)} \) \( \times C_i(S^2) \times C_i(S^2) \times [0, \epsilon] \).

**Definition 5.5.11.** If the BPS soliton section \( (\mathcal{E}, u) \in \mathcal{M}_{g, k, W}(\gamma, \kappa) \) satisfies Case 2 or Case 3, we call it a cone point in \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \).

By Lemma 5.5.7, 5.5.8, and Lemma 5.5.9, we know that the cone point also carries a Kuranishi neighborhood and hence is actually an interior point of the moduli space \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \).

**Theorem 5.5.12.** Suppose that the moduli space \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \) is regular but not strongly regular. Then we have the following conclusions:

1. If \( (\gamma, \kappa) \) does not contain the pair \( (\gamma, \kappa) \), then \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \) is a compact Hausdorff space carrying an orientable Kuranishi structure. Its (real) dimension is \( 6g - 6 + 2k - 2D - \sum_{i=1}^k N_{\gamma_i} \), where \( D = \hat{c}_W(g - 1) + \sum_\tau (\gamma_\tau) \) and \( \gamma(\gamma_\tau) = \sum_i (\Theta^i_1 - q_i) \).
2. If \( (\gamma, \kappa) \) contains the pair \( (\gamma, \kappa) \), then \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \) is a compact Hausdorff space carrying an orientable Kuranishi structure with boundary. Its (real) dimension is \( 6g - 6 + 2k - 2D - \sum_{i=1}^k N_{\gamma_i} \), where \( D \) is given as above. The boundary points consist of those BPS soliton W-sections satisfying Case (1) and (4). Their neighborhoods in \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \) are characterized by Lemma 5.5.6 and 5.5.10. The boundary of \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \) together with its Kuranishi structure is diffeomorphic to that of \( \mathcal{M}_{g, k, W}(\gamma, \kappa, \kappa) \# (S^1(k^+, \kappa^-)/\mathbb{R})^\mathbb{S}^1 \), where \( (S^1(k^+, \kappa^-)/\mathbb{R})^\mathbb{S}^1 \) is the \( S^1 \)-invariant set of \( (S^1(k^+, \kappa^-))/\mathbb{R} \), i.e., the geometrical set of BPS solitons connecting \( \kappa^+ \) and \( \kappa^- \).

**Proof.** Proof of (1). \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \) is a compact Hausdorff space, which is proved by Theorem 5.5.1. It is a stratified space stratified by the partial order \( \succ \). Note that the decoration \( (\gamma, \kappa) \) may still occur at some nodal points. The minimum strata are composed of soliton sections. To construct the Kuranishi structure of \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \), we first construct the Kuranishi neighborhoods of those soliton W-sections. If the soliton section is a non-BPS soliton, then we have already constructed its neighborhood. If the soliton is a BPS soliton section, then its neighborhood has been constructed by Lemmas 5.5.7, 5.5.8, and 5.5.9. Subsequently, we extend them to the interior part of the moduli space. We use the gluing method described in the last section to get a global Kuranishi structure. Notice that those cone points are also interior points of the moduli space. The dimension is given by Theorem 5.1.1.

One can argue as in Section 5.4 that this the Kuranishi structure is also stably almost complex and hence is orientable.

Proof of (2). In this case, the moduli space \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \) is a compact Hausdorff space with boundary \( \mathcal{M}_{g, k, W}(\gamma', \gamma^{-1}, \kappa') \# (S^1(k^+, \kappa^-)/\mathbb{R})^\mathbb{S}^1 \). Boundary points consist of those BPS soliton sections satisfying Case (1) and (4). To construct the Kuranishi structure with boundary over \( \mathcal{M}_{g, k, W}(\gamma, \kappa) \), we begin our construction of neighborhoods of the cone points and the boundary points. Neighborhoods of cone points are constructed by Lemma 5.5.8, 5.5.7, and 5.5.9. By (1), we can choose an oriented Kuranishi structure.
\[\{(U_{\sigma}, E_{\sigma}, s_{\sigma})\} \text{ of } \mathcal{M}_{g,k,W}(\gamma, \kappa') \text{ such that the induced orientation of Kuranishi structure }\]
\[\{(U_{\sigma} \times \{0, \varepsilon_{W}\}, E_{\sigma} \oplus E_{\sigma}, s_{\sigma})\} \text{ is compatible with the orientation of the neighborhoods of the interior points. After construction of neighborhoods of these special points, we can use the same gluing method shown in Section 5.4 to build an oriented Kuranishi structure of the whole moduli space.}\]

6. Proof of the axioms

6.1. Virtual cycle and quantum Picard-Lefschetz theory.

In this part, we will define the virtual cycle \([\mathcal{M}^\text{vir}_{g,k,W}(\gamma, \kappa')] \text{ in } H_*(\mathcal{M}_{g,k,W}(\gamma))\) which is given by its oriented Kuranishi structure. The Kuranishi structure may depend on many choices, but we will show that the virtual cycle depends only on \(g, k, W\) and the Lefschetz thimbles and is independent of the other choices used to construct the Kuranishi structure. The interesting thing is that we can see another version of Picard-Lefschetz theory at the level of moduli space.

Let \(\mathcal{M}^\text{vir}_{g,k,W}(\gamma, \kappa')\) be a strongly regular moduli space, then by Theorem 5.4.3, \(\mathcal{M}^\text{vir}_{g,k,W}(\gamma, \kappa')\) carries an oriented Kuranishi structure \((U_{\sigma}, E_{\sigma}, s_{\sigma})\) [even almost stably complex]. By Theorem 5.2.11, the section \(s_{\sigma}\) can be perturbed to a transversal smooth multisection \(\{s^\tau_{\sigma}\}\) such that \(U_{\sigma}s^\tau_{\sigma}(0)\) forms a cycle which is independent of the choice of the perturbed multisection \(\{s^\tau_{\sigma}\}\).

We define a map \(\Pi : \mathcal{M}^\text{vir}_{g,k,W}(\gamma, \kappa') \to \mathcal{M}_{g,k,W}(\gamma)\) by

\[\{(\xi, u)\} \to \{i(\xi)\}.\]

Note that \(\Pi\) is a system of map germs \(\Pi_{\sigma} : U_{\sigma} \to \mathcal{M}^\text{vir}_{g,k,W}(\gamma)\).

It is easy to see that this map is strongly continuous. By Theorem 5.4.3 and Theorem 5.2.11 we have the following definition.

**Definition 6.1.1.** We define the homology class \([\mathcal{M}^\text{vir}_{g,k,W}(\gamma, \kappa')] \text{ in } H_*(\mathcal{M}_{g,k,W}(\gamma)).\)

Its (real) dimension is \(6g - 6 + 2k - 2D - \sum_{i=1}^k N_{\gamma_i} = 2(\ell_W - 3)(1 - g) + k - \sum_{i=1}^k (\ell_{\gamma_i}) - \sum_{i=1}^k N_{\gamma_i}.\)

Furthermore, if \(\Gamma\) is a decorated stable \(W\)-graph, then we have the homology class \([\mathcal{M}^\text{vir}_{g,k,W}(\Gamma; \gamma, \kappa')] \text{ in } H_*(\mathcal{M}_{g,k,W}(\Gamma; \gamma)).\)

Its (real) dimension is \(6g - 6 + 2k - 2D1 - \sum_{i=1}^k N_{\gamma_i} - 2\#E(\Gamma) = 2(\ell_W - 3)(1 - g) + k - \sum_{i \in T(\Gamma)} (\ell_{\gamma_i}) - \#E(\Gamma) - \sum_{i=1}^k N_{\gamma_i},\)

where \(E(\Gamma)\) is the set of edges of \(\Gamma\).

For each \(\gamma \in G\), we have a perturbed polynomial \(W_\gamma + W_{0,\gamma}\), where \(W_{0,\gamma}\) is assumed to be \(W_\gamma\)-regular. Recall that \(W_{0,\gamma}(u) = \sum_{j=1}^{N_{\gamma}} \beta_j u_j\), where \(N_{\gamma}\) is the number of Ramond sections with respect to the action of \(\gamma\). Now we consider a path of the perturbation \(\lambda : [-1, 1] \to W'_{0,\gamma}(u) = \sum_{j=1}^{N_{\gamma}} (\lambda)u_j\) such that the path of the perturbation is still \(W_\gamma\)-regular.

Such a path is generic in the path space of the perturbation parameter space. Assume that the \(i\)-th critical point of the path is \(\kappa'(\lambda) = (\kappa'_1(\lambda), \cdots, \kappa'_{N_{\gamma}}(\lambda))\), for \(i = 1, \cdots, \mu_{\gamma}\), where \(\mu_{\gamma}\) is the multiplicity of \(W_\gamma + W_{0,\gamma}\). These critical points are all nondegenerate critical points.

We know from Section 3 that there are real hypersurfaces in the perturbed coefficient space \(\mathbb{C}^{N_{\gamma}}\) separating \(\mathbb{C}^{N_{\gamma}}\) into a system of chambers. When the path \((b_1(\lambda), \cdots, b_{N_{\gamma}}(\lambda))\) crosses one hypersurface, e.g., at \(\lambda = 0\), then correspondingly there exist two critical points \(\kappa'\) and \(\kappa''\) such that

\[\text{Im}(W_\gamma + W_{0,\gamma}(\kappa')) = \text{Im}(W_\gamma + W_{0,\gamma}(\kappa'')).\] (148)
Since for different $\gamma \in G$, we can take different perturbations, in the following discussion, we always fix the perturbations about all group elements $\gamma$ except for one group element $\gamma$. For $\gamma$ we choose a path of perturbation such that all of its critical points are fixed except for one depending on $\lambda$. We denote it by $\kappa^\gamma(\lambda)$. We assume that the perturbation crosses only one chamber at $\lambda = 0$. Namely, if $\lambda \neq 0$, the perturbation is always strongly regular, and at $\lambda = 0$, there exists only one critical point $\kappa^\gamma$ satisfying (148). The existence of this perturbation path is generic in the path space. We call this path of perturbation generic crossing path.

Now our Kuranishi structure depends on the metric we chose near the marked points (we choose the cylindrical metric), cut-off functions $\beta_j$, $\sigma$ used to define the perturbed Witten map, the obstruction bundle $E_\sigma$ and the partition of unity. There are two natural ways to choose the metrics near marked points which correspond to “Smooth theory” and “Cylinder theory” we mentioned in the introduction. The different choice of the metrics will significantly change the Witten map and the Witten equation. In physics, people think these two theories are equivalent in some sense. This is just the conjecture we proposed in the introduction. Here we fix the cylindrical metric, and consider the influence of other choices.

Let $\gamma = (\gamma', \tilde{\gamma})$ and $\kappa^\pm = (\kappa', \kappa^\pm)$. We fix $\kappa = (\kappa', \kappa')$ and choose a generic crossing path of perturbation with the crossing point at $\lambda = 0$ and satisfying $\kappa^\gamma(\lambda) < \kappa^\gamma$, i.e., $\text{Re}(W_{\gamma} + W_{0,\gamma}) < \text{Re}(W_{\gamma} + W_{0,\gamma}) (\kappa^\gamma)$.. Then we get a family of perturbed Witten equations $(W_\gamma(U_\gamma) = 0$, and this family induces a family of moduli spaces $\mathcal{M}_{g,k,W}(\gamma', \tilde{\gamma}; \kappa', \kappa^\gamma; \lambda)$ for $\lambda \in [-1 + \epsilon, 1 - \epsilon]$.

Since $\mathcal{M}_{g,k,W}(\gamma', \tilde{\gamma}; \kappa', \kappa^\gamma; \lambda)$ are strongly regular moduli spaces, they have oriented (stable almost complex) Kuranishi structures $\mathcal{Y}_k = \{(U_\gamma(\pm), E_\sigma(\pm), s_\gamma(\pm))\}$. The Kuranishi structure can be trivially extended to the spaces $\mathcal{M}_{g,k,W}(\gamma', \tilde{\gamma}; \kappa', \kappa^\gamma; -1 + \epsilon) \times [-1, -1 + \epsilon]$ and $\mathcal{M}_{g,k,W}(\gamma', \tilde{\gamma}; \kappa', \kappa^\gamma; 1 - \epsilon) \times [1 - \epsilon, 1]$.

Define

$$\mathcal{M}_{g,k,W}(\kappa^\gamma) = \bigcup_{\lambda \in [-1, 1]} \mathcal{M}_{g,k,W}(\gamma', \tilde{\gamma}; \kappa', \kappa^\gamma; \lambda).$$

We can define the Gromov convergence on $\mathcal{M}_{g,k,W}(\kappa^\gamma)$ in the same way as in Section 7. The sole difference is that the sequence $(\mathcal{C}^\gamma, \mathbf{u}^\gamma)$ may depend on the extra parameter $\lambda$. Similarly we can prove that $\mathcal{M}_{g,k,W}(\kappa^\gamma)$ is a compact Hausdorff space.

Our aim is to examine the change of the virtual cycle $\mathcal{M}_{g,k,W}(\gamma', \tilde{\gamma}; \kappa', \kappa^\gamma; \lambda)$ when $\lambda$ changes from positive to negative.

**Theorem 6.1.2.** $\mathcal{M}_{g,k,W}(\kappa^\gamma)$ is a compact Hausdorff space carrying an orientable Kuranishi structure with boundaries. The boundaries appear at $\lambda = \pm 1$ and $\lambda = 0$. When $\lambda = \pm 1$, the boundaries correspond to $\mathcal{M}_{g,k,W}(\gamma', \tilde{\gamma}; \kappa', \kappa^\pm; \pm 1)$. When $\lambda = 0$, the boundary is $\mathcal{M}_{g,k,W}(\gamma', \tilde{\gamma}; \kappa^\pm) \# S^\gamma(\kappa^\pm, \kappa^\gamma)$, where $S^\gamma(\kappa^\pm, \kappa^\gamma)$ is the space of BPS solitons flowing from the critical point $\kappa^\pm$ to $\kappa^\gamma$.

**Proof.** The proof of $\mathcal{M}_{g,k,W}(\kappa^\gamma)$ to be a compact Hausdorff space under Gromov convergence is similar to the proof of Theorem 4.3.1. The next thing is to construct an oriented Kuranishi structure over $\mathcal{M}_{g,k,W}(\kappa^\gamma)$.

We already have the Kuranishi structure at the boundary points corresponding to $\lambda = \pm 1$. When $\lambda = 0$, the perturbation of $W_{\gamma}$ is regular but not strongly regular. The BPS
satisfies Case 3 of Section 5.5. We can still construct the neighborhoods near those BPS soliton sections. We still use the notations from Section 5.5.

By Theorem 5.5.12, we can give an oriented Kuranishi structure to the regular but not strongly regular moduli space \( \mathcal{M}_{g,k,W}^{reg}(γ, γ', κ^+) \).

The BPS soliton sections in \( \mathcal{M}_{g,k,W}^{1}(κ^-) \) can also be divided into four cases according to Section 5.5. We only construct the Kuranishi neighborhoods for those BPS soliton sections satisfying Case 1, and the other cases can be treated in the same way.

Case 1. Let \((ξ_1, u_1) \in \mathcal{M}_{g,k,W}(γ, γ', κ^-)\) be a non-soliton section and let \(u_{+\delta} \) be a BPS soliton. Then \((ξ_1#(\mathbb{R} \times S^1), u_{1#u_{+\delta}}) \in \mathcal{M}_{g,k,W}^{1}(κ^-)\) is a BPS soliton section satisfying Case 1 of Section 5.5.

Let \((U_δ, E_δ, s_δ)\) be a Kuranishi neighborhood of \(u_1 \) in \( \mathcal{M}_{g,k,W}(γ, γ', κ^-)\). By Theorem 5.5.14, we can take the Kuranishi neighborhood of \((\mathbb{R} \times S^1, u_{+\delta})\) to be \((\{pt\}, E_+, s \equiv 0)\).

Now we can use our gluing construction again:

1. Assume that \(U_δ = V_{\text{deform},\sigma} \times V_{\text{resol},\sigma} \times V_{\text{map},\sigma}\). Let \(β \in V_{\text{deform},\sigma} \times V_{\text{resol},\sigma}\) and \(ζ \in D_ε(0) \in \mathbb{C}\). We can get the nearby curve \(ξ_{β,ζ}\) and the approximating solution \(u_{\text{app},β,ζ} \equiv u_{\text{app},β,ζ}\) which is defined as before since the boundary values at marked points are fixed.

2. Take the obstruction bundle \(E_{β,ζ}(E_δ \oplus E_+)\), where \(θ_{β,ζ}\) is defined as before. Notice that the obstruction bundle is independent of \(λ\). Now the equation we study is

\[
D_{β,ζ}(W(λ)) = 0 \mod E_{β,ζ}.
\]

However, we find that the linearized operator \(D_{β,ζ}(W(λ))\) is independent of the perturbation term, i.e., independent of \(λ\). Thus all the Lemmas from 5.3.1 to 5.3.4 hold without any change.

3. Now we want to apply the implicit function theorem to the operator \(W(λ)\) on \(ξ_{γ,ξ}\).

We define

\[
F_{β,ζ,λ}(φ) := W_{β,ζ,λ}(u_{\text{app},γ,ζ} + φ).
\]

We have

\[
F_{β,ζ,λ}(0) = W_{β,ζ,λ}(u_{\text{app},γ,ζ}), \quad DF_{β,ζ,λ}(0)(δφ) = D_{u=0}(W(λ))_{β,ζ} = D_{β,ζ}(W(0)).
\]

Actually, we only need to modify Lemmas 5.3.5 and 5.3.6. After a simple computation, we find that we can obtain Lemmas 5.3.5 and 5.3.6 with the constants there independent of \(δ\) near 0. Then we can construct our Kuranishi neighborhood \((V_{\text{deform},\sigma} \times V_{\text{resol},\sigma} \times D_ε(0) \times V_{\text{map},\sigma}, E_δ \oplus E_+, \tilde{δ}_ε)\) by the same technique. Here the set \(V_{\text{map},\sigma}\) is obtained by using the implicit function theorem and hence should depend on \(λ\). However, because of uniform estimates, we can take the same \(V_{\text{map},\sigma}\).

Since \(S^1\) only acts on \(D_ε(0)\), when modulo the \(S^1\) action, we can obtain the “Kuranishi” neighborhood \(((U_δ \times \{0, ε_δ\}, E_δ \oplus E_+, \tilde{δ}_ε)\), where \(U_δ = V_{\text{deform},\sigma} \times V_{\text{resol},\sigma} \times V_{\text{map},\sigma}, \tilde{δ}(β, φ, 0) = s_δ(φ)\), and the length \(ε_δ\) of the cone may depend on the point \(σ\).

Note that the Kuranishi structure \((U_δ, E_δ, s_δ)\) and orientation on \(\mathcal{M}_{g,k,W}(γ, γ', κ^-)\) are well defined; we get a kuranishi structure \(((U_δ \times \{0, ε_δ\}, E_δ \oplus E_+, s_{δ,ξ})\) with natural orientation of a neighborhood of \(\text{Im}(\text{Glue}_{u_{δ}}) \in \mathcal{M}_{g,k,W}(γ, γ', κ^-)\).

Case 3. If \((ξ_1, u_1) \in \mathcal{M}_{g,k,W}(γ, γ', κ^-)\) is a BPS soliton section, then \((ξ_1#(\mathbb{R} \times S^1), u_{1#u_{+\delta}})\) satisfies Case 3 of Section 5.5. We can still construct the neighborhoods near those points.
Now we can take a good coordinate system of $\mathcal{M}_{g,k,W}(\gamma, \varepsilon', \kappa')$ (see Lemma 6.3 of [FO] for the existence) which is a finite covering of Kuranishi neighborhoods. Therefore we can take the length $\varepsilon_\sigma$ to be the minimal length $\varepsilon$.

Thus one collar of the boundary of $\mathcal{M}_{g,k,W}(\kappa')$ at $\lambda = 0$ is $\mathcal{M}_{g,k,W}(\gamma', \kappa') \times [0, \varepsilon]$.

The cone points of $\mathcal{M}_{g,k,W}(\kappa')$ at $\lambda = 0$ can also occur; one can construct their neighborhoods as done in Case 1 which is also characterized by Lemma 5.5.10. Now we constructed the Kuranishi neighborhoods of the boundary points and the cone points occurring at $\lambda = 0$. Using the gluing argument as before, we can extend the Kuranishi structure covering the compact set of boundary points and cone points to the whole space $\mathcal{M}_{g,k,W}(\kappa')$ and construct an oriented Kuranishi structure (See the proof of Theorem 17.11 of [FO] for the extension reason).

We still take the above special perturbation path. Define

$\mathcal{M}_{g,k,W}^1(\kappa') = \bigcup_{\lambda \in [-1,1]} [\lambda] \times \mathcal{M}_{g,k,W}(\gamma', \gamma, \kappa')$. 

**Theorem 6.1.3.** $\mathcal{M}_{g,k,W}^1(\kappa')$ is a compact Hausdorff space carrying an orientable Kuranishi structure with boundaries. The boundaries appear only at $\lambda = \pm 1$. When $\lambda = \pm 1$, the boundaries correspond to $\mathcal{M}_{g,k,W}(\gamma', \gamma, \kappa')$. 

**Proof.** The proof is analogous to the proof of Theorem 6.1.2 except there is no soliton $W$-section satisfying Case 1 and 3 of Section 5.5. Therefore when $\lambda = 0$, only the cone points appear. We need to construct the local charts of the ordinary interior points as well as the local charts of cone points. There is a subtle difference in the present construction when compared to the previous construction of local charts: the critical point $\kappa'(\lambda)$ is movable. We should modify our previous argument.

For example, we consider $\lambda \in (-\varepsilon, \varepsilon)$ and $u_\varepsilon \in \mathcal{M}_{g,k,W}(\gamma', \gamma, \kappa')$ a non-soliton section. The approximating solution $u'_{y,\gamma,\kappa}$ on $\mathcal{M}_{g,k,W}$ is defined as follows: near the nodal points we define $u'_{y,\gamma,\kappa}$ as before; near the marked point labelled by $\gamma$, we let

$u'_{y,\gamma,\kappa} = u_\varepsilon - \kappa' + \kappa'(\lambda)$.

The linearized operator at the point $(y, \gamma, \kappa, \lambda)$ is

$D_{y,\gamma,\kappa,\lambda}(W)(\phi) := D_{y,\gamma,\kappa,\lambda}(W)(\phi, \phi) = \partial_{y,\gamma,\kappa,\lambda} \phi + A(u'_{y,\gamma,\kappa,\lambda}, y) \phi$.

The parameter $\lambda$ appears in the nonlinear term. Then we modify Lemmas 5.3.1, 5.3.2 by a trick: we replace the symbol $y$ representing the deformation parameter by $y, \lambda$ and $|y|$ by $|y| + |\lambda|$. Then all those lemmas hold if $y, \gamma, \lambda, \kappa$ are sufficiently small. Using the implicit function theorem, we can find a Kuranishi chart for $u_\varepsilon$.

Similarly, after a small modification, we can construct the Kuranishi neighborhoods of cone points as done in Lemma 5.5.10.

Then we have constructed the local chart of each interior point (including cone points).

We can extend the given Kuranishi structure of a collar of the boundaries of $\mathcal{M}_{g,k,W}(\kappa')$ at $\lambda = \pm 1$ to the whole space as done before to obtain an oriented Kuranishi structure. 

**Classical Picard-Lefschetz theory.** We will give a simple description of vanishing cycles, Lefschetz thimbles and related transformation groups. It has already become a classical theory. The interested reader can see [AE] and etc.

Now we assume that the perturbation polynomial $W_{0,\tilde{\gamma}}$ is strongly $\tilde{\gamma}$-regular and sufficiently small such that there are exactly $\mu_0$ critical points of $W_{0,\tilde{\gamma}} + W_{0,\tilde{\gamma}}$ inside a small ball $B$ centered at 0. Let $\alpha'$ be the critical value of $W_{\tilde{\gamma}} + W_{0,\tilde{\gamma}}$ which lies inside a small
neighborhood \( T \subset \mathbb{C} \) corresponding to \( B \). Furthermore, we can require the order of these critical values to satisfy \( \text{Im}(\alpha_i) < \text{Im}(\alpha_j) \) if \( i < j \). Let \( \alpha_\ast \in \partial T \) be a regular value. We take \( T \) small enough and define \( Y = (W_\gamma + W_{0,\bar{\gamma}})^{-1}(T) \cap B \) and \( Y_\ast = (W_\gamma + W_{0,\bar{\gamma}})^{-1}(\alpha_\ast) \).

Take a system of paths \( l_i(\tau) : [0, 1] \to \mathbb{C} \) connecting \( \alpha_\ast \) and \( \alpha_l \) such that

1. the paths \( l_i \) have no self-intersections;
2. the paths \( l_i \) and \( l_j \) intersect only for \( \tau = 0 \), at the point \( \alpha_\ast = l_i(0) = l_j(0) \);
3. the paths \( l_i \) are ordered by the requirement that \( \arg l_i' < \arg l_j' \) for \( i < j \).

For each path \( l_i \), there exists a corresponding simple loop \( \beta_i \) which goes along \( l_i \) from \( \alpha_\ast \) to the critical value \( \alpha_l \), then goes anticlockwise around \( \alpha_l \) and finally returns to \( \alpha_\ast \) along \( l_i \). The system of these paths \( l_i \)'s is called distinguished if the corresponding system of simple loops \( \beta_i \) generates the group \( \pi_1(T', \alpha_\ast) \), where \( T' = T - \{ \alpha_1, \cdots, \alpha_l \} \).

Each path \( l_i \) induces a unique vanishing cycle \( \Delta_i \in H_{N_{\ast,1}}(Y_\ast) \) or a Lefschetz thimble \( S_i \in H_{N_{\ast}}(Y, Y_\ast) \) up to orientation. In singularity theory, the set of these cycles or thimbles forms a basis of the homology group \( H_{N_{\ast,1}}(Y_\ast) \cong \mathbb{Z}^l \), or the relative homology group \( H_{N_{\ast}}(Y, Y_\ast) \), which is called a distinguished basis of vanishing cycles or thimbles respectively. Let \( D_\ast \) represent the set of all the distinguished bases of vanishing cycles (or thimbles).

Assume that the boundary of the relative cycle \( S_i \) is just \( \Delta_i \); then when taking compatible orientations we have the connecting isomorphism:

\[
\partial_\ast : H_{N_{\ast}}(Y, Y_\ast) \to H_{N_{\ast,1}}(Y_\ast)
\]

such that \( \partial_\ast(S_i) = \Delta_i \).

Each simple loop \( \beta_i \) induces the monodromy operators

\[
h_{\beta_i} : H_{N_{\ast,1}}(Y_\ast) \to H_{N_{\ast,1}}(Y_\ast)
\]

and

\[
h_{S_i} : H_{N_{\ast}}(Y, Y_\ast) \to H_{N_{\ast}}(Y, Y_\ast),
\]

which have action given by Picard-Lefschetz theory as follows:

\[
h_{\beta_i}(\Delta_j) = \Delta_j + R_{\beta_i} \Delta_i, \forall j \quad (149)
\]

and

\[
h_{S_i}(S_j) = S_j + R_{S_i} S_j, \forall j \quad (150)
\]

where \( R_{\beta_i} = (-1)^{N_{\ast}(Y_{\ast,1})/2} \Delta_{\beta_i} \circ \Delta_i \) and \( \Delta_j \circ \Delta_i \) is the intersection number.

The map \( \beta_i \to h_{\beta_i}(h_{S_i}) \) induces a homomorphism \( \pi_1(T', \alpha_\ast) \to \text{Aut} H_{N_{\ast,1}}(Y_\ast)(\text{Aut} H_{N_{\ast}}(Y, Y_\ast)) \). The image of the homomorphism is called the (relative) monodromy group of the singularity \( W_\gamma \). It can be proved (shown in [AE]) that the (relative) monodromy group only depends on the type of the singular point of \( W_\gamma \). The set \( \{ h_{\Delta_i}, i = 1, \cdots, \mu_j \} \) \( (h_{S_i}, i = 1, \cdots, \mu_j) \) generates the (relative) monodromy group.

There are several groups acting on the set \( D_\ast \)(cf. [F]). Let \( (\delta_1, \cdots, \delta_{\mu_j}) \) be a distinguished basis of vanishing cycles or thimbles. We have

1. action of \( (\mathbb{Z}/2\mathbb{Z})^\mu \) (change of orientation)

\[
O_{\beta_\gamma}(\delta_1, \cdots, \delta_{\mu_j}) = (\delta_1, \cdots, \delta_{j-1}, -\delta_j, \delta_{j+1}, \cdots, \delta_{\mu_j}).
\]

2. action of monodromy groups

\[
h_t(\delta_1, \cdots, \delta_{\mu_j}) = (h_t(\delta_1), \cdots, h_t(\delta_{\mu_j})).
\]

3. action of braid group \( Br(\mu_\gamma) \). Let \( b_{\gamma_1}, \cdots, b_{\gamma_{\mu_j-1}} \) be the standard generators of \( Br(\mu_\gamma) \), then

\[
b_{\gamma_1}(\delta_1, \cdots, \delta_{\mu_j}) = (\delta_1, \cdots, \delta_{j-1}, h_{\gamma_1}(\delta_{j+1}), \delta_j, \delta_{j+2}, \cdots, \delta_{\mu_j}).
\]

(151)
(4) action of the symmetric group $\text{Sym}_{\mu_i}$.

$$\sigma(\delta_1, \cdots, \delta_{\mu_i}) = (\delta_{\sigma(1)}, \cdots, \delta_{\sigma(\mu_i)}), \quad \sigma \in \text{Sym}_{\mu_i}.$$  

(5) Gabrielov transformations

$$G_i(j)(\delta_1, \cdots, \delta_{\mu_i}) = (\delta_1, \cdots, \delta_{j-1}, h_i(\delta_j), \delta_{j+1}, \cdots, \delta_{\mu_i}).$$

One can also discuss the Dynkin diagrams corresponding to a distinguished basis. The above group actions also act on the Dynkin diagrams. The following proposition was proved by Gabrielov (cf. [13]).

**Proposition 6.1.4.** Any element in $D_\lambda$ can be obtained from a fixed element by iterations of transformations (1) and (3).

**Quantum Picard-Lefschetz theory.** By a cobordism argument, the classical Picard-Lefschetz theory can be seen at the level of the moduli space.

By Theorem 6.1.3 we have

**Corollary 6.1.5.** The virtual cycle $\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma; \mathcal{X}(\lambda))$ in $H_*(\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma))$ is independent of the various choices we made to construct the Kuranishi structure, which include the cut-off functions $\beta$, $\varpi$, the obstruction bundle $E_{\sigma}$ and the partition of unity. Assume that $\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma; \mathcal{X}(\lambda))$ is a family of moduli spaces depending on a perturbation path. If for each $\lambda$, the moduli space $\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma; \mathcal{X}(\lambda))$ is strongly regular, then each $\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma; \mathcal{X}(\lambda))$ defines the same cohomology class in $H_*(\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma))$.

Now assume $\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma'; \tilde{\gamma}; \tilde{\mathcal{X}}', \kappa')$ to be a fixed strongly regular moduli space. Here $\kappa'$ is one of the critical points of $W_{\tilde{\gamma}} + W_{b_{N_i}}$. Then we obtain $\mu_{\tilde{\gamma}}$ virtual cycles $\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma'; \tilde{\gamma}; \tilde{\mathcal{X}}', \kappa')^{\text{vir}}, i = 1, \cdots, \mu_{\tilde{\gamma}}$ in $H_*(\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma))$. Let $V_{\tilde{\gamma}} \subset H_*(\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma))$ be the vector space generated by these virtual cycles.

Let $a'$ be the corresponding critical value of $\kappa'$. We can further require that the order of the critical values satisfies $\text{Im}(a') < \text{Im}(a^j)$ if $i < j$. This order determines a basis of $V_{\tilde{\gamma}}$.

The perturbation parameter $(b_1, \cdots, b_{N_i})$ lies inside one chamber of $\mathbb{C}^{N_i}$. By Corollary 6.1.5, the virtual cycles do not change if we only move the point $(b_1, \cdots, b_{N_i})$ inside the same chamber.

However, when the point $(b_1, \cdots, b_{N_i})$ crosses the wall between two chambers, the basis of $V_{\tilde{\gamma}}$ is transformed to another basis. The change is given by the following wall crossing formula.

**Theorem 6.1.6 (Wall crossing formula).** Let $(b_1(\lambda), \cdots, b_{N_i}(\lambda)), \lambda \in [-1, 1]$ be a generic crossing path in $\mathbb{C}^{N_i}$. Let $[\kappa'(\pm), \cdots, \kappa'(\pm), \kappa^{i+1}(\pm), \cdots, \kappa^{i_{\text{new}}}(\pm)]$ be the set of ordered critical points at $\lambda = \pm 1$. We can assume that $\kappa'(\pm) = \kappa'$ is fixed for $j \neq i$, $\kappa'(\pm) = \kappa'(\lambda = \pm 1)$ and $\text{Im}(\sigma'(\lambda = 0)) = \text{Im}(\sigma'(\lambda + 1))$.

If the perturbation satisfies $\text{Re} a'(\lambda) < \text{Re} a'^{i+1}$, we have the left-transformation:

$$[\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma'; \tilde{\gamma}; \tilde{\mathcal{X}}', \kappa'(\pm))]^{\text{vir}} = [\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma'; \tilde{\gamma}; \tilde{\mathcal{X}}'; \kappa'(\pm))]^{\text{vir}}, \quad \forall j \neq i, i + 1$$

$$[\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma'; \tilde{\gamma}; \tilde{\mathcal{X}}', \kappa'(\pm))]^{\text{vir}} = [\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma'; \tilde{\gamma}; \tilde{\mathcal{X}}', \kappa'(\pm))]^{\text{vir}} +$$

$$R_{i+1} : [\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma'; \tilde{\gamma}; \tilde{\mathcal{X}}'; \kappa'(\pm))]^{\text{vir}}$$

$$[\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma'; \tilde{\gamma}; \tilde{\mathcal{X}}', \kappa'^{(i+1)}(\pm))]^{\text{vir}} = [\mathcal{M}_{\text{rig}}^{g_k,W}(\gamma'; \tilde{\gamma}; \tilde{\mathcal{X}}', \kappa'^{(i+1)}(\pm))]^{\text{vir}},$$

(152), (153), (154)
where $R_{i,i+1}$ is the intersection number defined as above.

If the perturbation satisfies $\text{Re}(\lambda) > \text{Re}(\lambda + 1)$, we have the right-transformation:

\begin{align}
&\text{Hom}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi', k'(+)\text{vir}) = \text{Hom}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi', k'(-))\text{vir}, \quad \forall j \neq i, i + 1 \quad (155) \\
&\text{Hom}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi', k'(+)\text{vir}) = \text{Hom}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi', k'(-))\text{vir}, \quad (156) \\
&\text{Hom}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi', k'(+)\text{vir}) = \text{Hom}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi', k'(-))\text{vir} + \\
&R_{i,i+1} : \text{Hom}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi', k'(+)\text{vir}) (157)
\end{align}

**Proof.**

It suffices to prove the left transformation formula.

At first, we shall prove (152). Define the moduli space

\[
\mathcal{M}^{1}_{g,k,W}(\kappa') = \bigcup_{i \in \mathbb{Z}, j \in \mathbb{N}} \mathcal{M}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)
\]

By Theorem 6.13, this moduli space is a compact Hausdorff space carrying an orientable Kuranishi structure with boundaries. Its boundaries consist of two parts: $\mathcal{M}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)$ and $\mathcal{M}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)$ which are strongly regular moduli spaces. Let $\{(U_{\sigma}(\pm), E_{\sigma}(\pm), s_{\sigma}(\pm))\}$ be a good coordinate system of Kuranishi neighborhoods of $\mathcal{M}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)$ and $\mathcal{M}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)$ respectively. Then by Theorem 6.4 of [FO], the section $\{s_{\sigma}(\pm)\}$ can be approximated by a sequence of multisections $\{\tilde{s}_{\sigma,n}\}$. By Lemma 17.4 of [FO], we can extend the multisections $\{\tilde{s}_{\sigma,n}\}$ to the multisections $\{\tilde{s}_{\sigma,n}\}$ over $\mathcal{M}^{1}_{g,k,W}(\kappa')$ such that the restriction $\tilde{s}_{\sigma,n}\big|_{v = 1} = \tilde{s}_{\sigma,n}$. By the proof of Theorem 4.9 of [FO], the zero set $(\tilde{s}_{\sigma,n})^{-1}(0)$ is a singular chain satisfying

\[
\partial(\tilde{s}_{\sigma,n})^{-1}(0) = (\tilde{s}_{\sigma,n})^{-1}(0) - (\tilde{s}_{\sigma,n})^{-1}(0).
\]

Define the map $\Pi : \mathcal{M}^{1}_{g,k,W}(\kappa') \to \mathcal{M}_{g,k,W}(\gamma')$ by

\[
[\{C, u\}] \to [\{C_{0}\}],
\]

where $C_{0}$ is the $W$ curve obtained from $C$ by shrinking the soliton components.

Since the map $\Pi : \mathcal{M}^{1}_{g,k,W}(\kappa') \to \mathcal{M}_{g,k,W}(\gamma')$ is strongly continuous, we have

\[
\partial \Pi_{i}((\tilde{s}_{\sigma,n})^{-1}(0)) = \Pi_{i}((\tilde{s}_{\sigma,n})^{-1}(0)) - \Pi_{i}((\tilde{s}_{\sigma,n})^{-1}(0)).
\]

Therefore, we obtain

\[
[\mathcal{M}^{1}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)]\text{vir} = [\mathcal{M}^{1}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)]\text{vir}.
\]

In particular, the cobordism argument can be applied to the moduli space

\[
\bigcup_{i \in \mathbb{Z}, j \in \mathbb{N}} \mathcal{M}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)
\]

to obtain the following relation:

\[
[\mathcal{M}^{1}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)]\text{vir} = [\mathcal{M}^{1}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)]\text{vir}.
\]

So we have finished the proof of (152).

The proof of (154) is almost the same as the proof of (152). The sole difference is that the decorated index $k'$ is moving when $\lambda$ changes from $+1$ to $-1$. In particular, one can also prove

\[
[\mathcal{M}^{1}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)]\text{vir} = [\mathcal{M}^{1}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)]\text{vir}.
\]

To prove equality (153), we consider the following moduli space:

\[
\mathcal{M}^{1}_{g,k,W}(\kappa^{i+1}) = \bigcup_{i \in \mathbb{Z}, j \in \mathbb{N}} \mathcal{M}^{1}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0)\times \mathcal{M}^{1}_{g,k,W}(\gamma', \bar{\gamma}; \chi', \chi'; 0).
\]
By Theorem 6.1.2, this moduli space can carry an oriented Kuranishi structure with boundaries. The boundaries consist of three parts: $\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', \nu)$, $\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', \nu^i)$, and $\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', \nu^j)$. Theorem 6.1.6 implies that the generic crossing path of perturbation provides the action of the braid group on $\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', \nu)$.

If the perturbation path of the parameter $b(\lambda)$ is generic, then the solutions of $\mathfrak{S}(k'(0), k'^{+})$ are Morse-Smale flows and there are exactly $R_{j+1}$ elements, where $R_{j+1}$ is the intersection number of the vanishing cycles representing the critical points $k'(0)$ and $k'^{+}$.

If $u_{\pm} \in S^{\mathfrak{S}}(k'(0), k'^{+})$ and $u_1 \in \overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'; \lambda = 0)$, then their neighborhoods in $\overline{M}_{g,k,W}(k'^{+})$ are exactly the same as those described by Lemmas 5.5.6 and 5.5.10. Now using a similar cobordism argument, one can show that $\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'; \lambda = 1)$ is cobordant to $\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'; \lambda = 0)$. Notice that

$$\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'; \lambda = 1) = \overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'(+)$$

and

$$\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'; \lambda = 1) = \overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'^{+}(-))$$

by relation (159), and thus we obtain (153).

**Correspondence.** With the given order of the critical points $\{k^1, \ldots, k^p\}$ or the critical values $\{\alpha_1, \ldots, \alpha_p\}$, we have the corresponding order of the system of paths $\{l_1, \ldots, l_p\}$. This system of paths induces a distinguished basis of vanishing cycles $\{\Delta_1, \ldots, \Delta_p\}$ in $H_{N_1}(Y, \nu)$ or thimbles $\{S_1, \ldots, S_p\}$ in $H_{N_1}(Y, \nu)$. For simplicity, in the following we only discuss the correspondence between virtual cycles in $V_\gamma$ and thimbles in $H_{N_1}(Y, \nu)$. The result is the same for vanishing cycles.

We define a linear map

$$\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', \nu) : H_{N_1}(Y, \nu) \to V_\gamma \subset \overline{M}_{g,k,W}(\gamma)$$

by setting the map between bases:

$$\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', S_i) := \overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'^{+})^{vir}.$$ 

In particular, we define

$$\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', -S_i) := -\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'^{+})^{vir}.$$ 

Theorem 6.1.6 implies that the generic crossing path of perturbation provides the action of the braid group on $V_\gamma$:

$$br_j^i \cdot ([\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'^{+})]^{vir}, \ldots, [\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k^p)]^{vir})$$

$$= ([\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'^{+})]^{vir}, \ldots, [\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k^i)]^{vir},$$

$$h_j([\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k'^{+})]^{vir}, [\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k^i)]^{vir}, \ldots, [\overline{M}_{g,k,W}(\gamma', \tilde{\gamma}; \chi', k^p)]^{vir}).$$

Here $br^i_j$ means the action of the braid group which is given by left-transformation. Hence, we have
Proposition 6.1.7. The map \( \overline{\mathcal{H}}_{g,k,W}^{ig}(\gamma', \gamma; \mathbf{x}, \mathbf{x}') : H_{N}(Y, Y_{t}) \to V_{k} \subset H_{*}(\overline{\mathcal{H}}_{g,k,W}^{ig}(\gamma)) \) is a homomorphism of \( Br(\mu_{g}) \)-modules. Furthermore, if \( \{ S_{i} \} \) is another distinguished basis of \( H_{N}(Y, Y_{t}) \) and \( S = \sum_{i} x_{i} S_{i} ' \), then
\[
\overline{\mathcal{H}}_{g,k,W}^{ig}(\gamma', \gamma; \mathbf{x}, \mathbf{x}') \in \sum_{i} x_{i} \overline{\mathcal{H}}_{g,k,W}^{ig}(\gamma', \gamma; S_{i}').
\]

Proof. The second conclusion is obtained by Proposition 6.1.4. \( \square \)

6.2. Axioms for the virtual cycle on the space of rigidified \( W \)-curves.

Theorem 6.2.1. Let \( \Gamma \) be a decorated stable \( W \)-graph (not necessarily connected) with each tail \( \tau \in T(\Gamma) \) decorated by an element \( \gamma_{\tau} \in G \). Denote by \( k := |T(\Gamma)| \) the number of tails of \( \Gamma \). There exists a virtual cycle
\[
\left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma) \right]^{vir} \in H_{*}(\overline{\mathcal{M}}_{W}^{ig}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau \in T(\Gamma)} H_{N}(C_{N}^{g}, W_{N}^{0}, \mathbb{Q})
\]
satisfying the axioms below. When \( \Gamma \) has a single vertex of genus \( g \), \( k \) tails, and no edges (i.e., \( \Gamma \) is a corolla) we denote the virtual cycle by \( \left[ \overline{\mathcal{M}}_{g,k,W}^{ig}(\gamma) \right]^{vir} \), where \( \gamma := (\gamma_{1}, \ldots, \gamma_{k}) \). The following axioms are satisfied:

1. **Dimension:** If \( D_{\Gamma} \) is not a half-integer (i.e., if \( D_{\Gamma} \notin \frac{1}{2} \mathbb{Z} \)), then \( \left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma) \right]^{vir} = 0 \). Otherwise, the cycle \( \left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma) \right]^{vir} \) has degree
\[
6g - 6 + 2k - 2D - 2E(\Gamma) = 2 \left( (\hat{c} - 3)(1 - g) + k - \sum_{\tau \in T(\Gamma)} t_{\tau} - #E(\Gamma) \right).
\]
So the cycle lies in \( H_{*}(\overline{\mathcal{M}}_{W}^{ig}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau \in T(\Gamma)} H_{N}(C_{N}^{g}, W_{N}^{0}, \mathbb{Q}) \), where
\[
r := 6g - 6 + 2k - 2D - 2E(\Gamma) - \sum_{\tau \in T(\Gamma)} N_{\gamma_{\tau}} = 2 \left( (\hat{c} - 3)(1 - g) + k - \sum_{\tau \in T(\Gamma)} t(\gamma_{\tau}) - #E(\Gamma) - \sum_{\tau \in T(\Gamma)} \frac{N_{\gamma_{\tau}}}{2} \right).
\]

2. **Symmetric group invariance** There is a natural \( S_{k} \)-action on \( \overline{\mathcal{M}}_{g,k,W}^{ig} \) obtained by permuting the tails. This action also induces the permutation of relative homology and cohomology groups. So for any \( \sigma \in S_{k} \) the induced map
\[
\sigma_{*} : H_{*}(\overline{\mathcal{M}}_{g,k,W}^{ig}, \mathbb{Q}) \otimes \prod_{i} H_{N}(C_{N}^{g}, W_{N}^{0}, \mathbb{Q}) \to H_{*}(\overline{\mathcal{M}}_{g,k,W}^{ig}, \mathbb{Q}) \otimes \prod_{i} H_{N}(C_{N}^{g}, W_{N}^{0}, \mathbb{Q}).
\]
satisfies
\[
< \sigma_{*} \left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma) \right]^{vir}, \sigma_{*}(\alpha) > = < \left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma) \right]^{vir}, \alpha >,
\]
for any \( \alpha \in \prod_{i} H_{N}(C_{N}^{g}, W_{N}^{0}, \mathbb{Q}) \).

3. **Disconnected graphs:** Let \( \Gamma = \bigcup_{i} \Gamma_{i} \) be a stable, decorated \( W \)-graph which is the disjoint union of connected \( W \)-graphs \( \Gamma_{i} \). The classes \( \left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma) \right]^{vir} \) and
\[
\left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma_{i}) \right]^{vir}
\]
are related by
\[
\left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma) \right]^{vir} = \left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma_{1}) \right]^{vir} \times \cdots \times \left[ \overline{\mathcal{M}}_{W}^{ig}(\Gamma_{d}) \right]^{vir}.
\]
(163)
(4) **Degenerating connected graphs:**

Let \( \Gamma \) be a connected, genus-\( g \), stable, decorated W-graph with a single edge \( e \) and let \( \bar{1} : \mathcal{M}_{g,k,W}(\Gamma) \to \mathcal{M}_{g,k,W}(\gamma) \) denote the canonical inclusion map. Then there holds

\[
[\mathcal{M}_{g,k,W}(\gamma)]^{\text{vir}} \cap \mathcal{M}_{g,W}(\Gamma) = [\mathcal{M}_{g,W}(\Gamma)]^{\text{vir}}.
\]

or in cohomology form

\[
[\mathcal{M}_{g,W}(\Gamma)]^{\text{vir}} = \bar{1}[\mathcal{M}_{g,k,W}(\gamma)]^{\text{vir}}.
\]

(5) **Weak Concavity** Suppose that all the decorations on tails and edges are Neveu-Schwarz, meaning that \( \mathbb{C}^N = \{0\} \). In this case we omit the \( H_{N_i}(\mathbb{C}^N, W_\gamma, \mathbb{Q}) \) from our notation.

If, furthermore, the universal W-structure \( (\mathcal{L}_1, \ldots, \mathcal{L}_N) \) on the universal curve \( \pi : \mathcal{G} \to \mathcal{M}_{g,W}(\Gamma) \) is concave (i.e., \( \pi_* \left( \bigoplus_{i=1}^t \mathcal{L}_i \right) = 0 \)), then the virtual cycle is given by capping the top Chern class of the orbifold vector bundle \( -R^1\pi_* \left( \bigoplus_{i=1}^t \mathcal{L}_i \right) \) with the usual fundamental cycle of the moduli space:

\[
[\mathcal{M}_{g,W}(\Gamma)]^{\text{vir}} = c_{\text{top}} \left( -R^1\pi_* \bigoplus_{i=1}^t \mathcal{L}_i \right) \cap [\mathcal{M}_{g,W}(\Gamma)]^{\text{vir}}.
\]

(6) **Index zero** Suppose that \( \dim \mathcal{M}_W(\Gamma) = 0 \) and all the decorations on tails of \( \Gamma \) and edges are Neveu-Schwarz.

If the pushforwards \( \pi_* \left( \bigoplus \mathcal{L}_i \right) \) and \( R^1\pi_* \left( \bigoplus \mathcal{L}_i \right) \) are both vector spaces of the same dimension, then the virtual cycle is just the degree \( \text{deg}(\mathcal{W}) \) of the Witten map times the fundamental cycle:

\[
[\mathcal{M}_W(\Gamma)]^{\text{vir}} = \text{deg}(\mathcal{W})[\mathcal{M}_W(\Gamma)]^{\text{vir}}
\]

where the \( j \)th term \( \mathcal{W}_j = \pi_* \left( \bigoplus \mathcal{L}_i \right) \rightarrow R^1\pi_* \mathcal{L}_j \) of the Witten map is given by \( \mathcal{W} = \bar{\partial}_\gamma \mathcal{W}(x_1, \ldots, x_N) \).

(7) **Forgetting tails:**

(a) Let \( \Gamma \) have its \( i \)th tail decorated with \( J^{-1} \), where \( J \) is the exponential grading element of \( \mathcal{G} \). Further, let \( \Gamma' \) be the decorated W-graph obtained from \( \Gamma \) by forgetting the \( i \)th tail and its decorations. Assume that \( \Gamma' \) is stable, and denote the forgetting tails morphism by

\[
\phi^{\text{vir}} : \mathcal{M}_{g,W}(\Gamma) \to \mathcal{M}_{g,W}(\Gamma')
\]

We have

\[
[\mathcal{M}_{g,k,W}(\Gamma)]^{\text{vir}} = (\phi^{\text{vir}})_* ([\mathcal{M}_{g,k-1,W}(\Gamma')]^{\text{vir}}) \tag{165}
\]

(b) \( (1) \) \( [\mathcal{M}_{0,3,W}(\gamma_1, \gamma_2, J^{-1})]^{\text{vir}} = 0 \) if \( \gamma_1 \neq \gamma_2^{-1} \).

(2) If \( S_i \) and \( S_i^\perp \) are arbitrary two basis in \( H_{N_i}(\mathbb{C}^N, (W_\gamma)^{\text{vir}}, \mathbb{Q}) \) and \( H_{N_i}(\mathbb{C}^N, (W_\gamma)^{\text{vir}}, \mathbb{Q}) \) respectively and if \( (\eta^{ij}) \) is the inverse matrix of \( (S_i, S_j^\perp) \), then

\[
[\mathcal{M}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1})]^{\text{vir}} = \sum_{i,j} \frac{|G|}{|<\gamma>|} \times \frac{|G|}{|<J^{-1}>|} \eta^{ij} S_i \otimes S_j^\perp.
\]

\[\tag{This axiom was called convexity in [JKV1] because the original form of the construction outlined by Witten in the \( A_{n-1} \) case involved the Serre dual of \( \mathcal{L} \), which is convex precisely when our \( \mathcal{L} \) is concave.}\]
Here $\sum_{i,j} \eta^{ij} S_i \otimes S_j$ is called the Casimir element of the intersection pairing. Furthermore, for any $\alpha \in H^N(C^N_{\gamma}, (W_\gamma)^\infty, \mathbb{Q})$ and $\beta \in H^N(C^N_{\gamma}, (W_\gamma)^\infty, \mathbb{Q})$, there holds

$$\langle M_{0,0,0,0}(\gamma, \gamma, 0, 0), \mu \rangle = \frac{|G|}{2(|\gamma|)} \times \frac{|G|}{2(|\gamma|)} < \alpha, \beta >,$$

where $< \cdot, \cdot >$ is the pairing induced by the intersection pairing.

(8) **Composition law:** Given any genus-$g$ decorated stable $W$-graph $\Gamma$ with $k$ tails, and given any edge $e$ of $\Gamma$, let $\hat{\Gamma}$ denote the graph obtained by "cutting" the edge $e$ and replacing it with two unjoined tails $\tau_+$ and $\tau_-$. Further, denote by

$$\hat{\rho} : \overline{\mathcal{M}}_W(\hat{\Gamma}) \to \overline{\mathcal{M}}_W(\Gamma)$$

the gluing loops or gluing trees morphism corresponding to gluing $\tau_+$ to $\tau_-$, as described in Section 2 Equations (14) and (15).

The virtual cycle $\overline{\mathcal{M}}_W(\hat{\Gamma})$ lies in $H_1(\mathcal{M}_{\hat{\Gamma}}^W, \mathbb{Q}) \otimes \prod_{(\gamma_1, \gamma_2) \in \Gamma} H_{N_{\gamma_1}}(C_{\gamma_1}, W_{\gamma_1}, \mathbb{Q}) \otimes H_{N_{\gamma_2}}(C_{\gamma_2}, W_{\gamma_2}, \mathbb{Q})$.

The homology $H_{N_{\gamma_1}}(C_{\gamma_1}, W_{\gamma_1}, \mathbb{Q})$, is naturally isomorphic to $H_{N_{\gamma_2}}(C_{\gamma_2}, W_{\gamma_2}, \mathbb{Q})$, so the intersection pairing $(\cdot, \cdot) : H_{N_{\gamma_1}}(C_{\gamma_1}, W_{\gamma_1}, \mathbb{Q}) \otimes H_{N_{\gamma_2}}(C_{\gamma_2}, W_{\gamma_2}, \mathbb{Q}) \to \mathbb{C}$ can be applied to contract these and give a map $(\cdot)_{\gamma_1, \gamma_2} : H_1(\mathcal{M}_{\hat{\Gamma}}^W, \mathbb{Q}) \otimes \prod_{(\gamma_1, \gamma_2) \in \Gamma} H_{N_{\gamma_1}}(C_{\gamma_1}, W_{\gamma_1}, \mathbb{Q}) \otimes H_{N_{\gamma_2}}(C_{\gamma_2}, W_{\gamma_2}, \mathbb{Q}) \to H_1(\mathcal{M}_{\hat{\Gamma}}^W, \mathbb{Q}) \otimes \prod_{(\gamma_1, \gamma_2) \in \Gamma} H_{N_{\gamma_1}}(C_{\gamma_1}, W_{\gamma_1}, \mathbb{Q})$.

We have

$$\overline{\mathcal{M}}_W(\hat{\Gamma}) = \overline{\mathcal{M}}_{W_1}(\Gamma_1) \times_{\mathcal{M}_{\hat{\Gamma}}} \overline{\mathcal{M}}_{W_2}(\Gamma_2).$$

(9) **Sums of Singularities:**

If $W_1 \in C[z_1, \ldots, z_{N_1}]$ and $W_2 \in C[z_{N_1+1}, \ldots, z_{N_1+N_2}]$ are two quasi-homogeneous polynomials with diagonal automorphism groups $G_1$ and $G_2$, then the diagonal automorphism group of $W = W_1 + W_2$ is $G = G_1 \times G_2$. For any $\gamma_1 + G_1$ and $\gamma_2 + G_2$, it is known [AGV] that the following isomorphism holds:

$$H_{mid}(C_{\gamma_1+G_1}, W_{\gamma_1, \gamma_2}) \cong H_{mid}(C_{\gamma_1}, W_{\gamma_1, \gamma_2}) \otimes H_{mid}(C_{\gamma_2}, W_{\gamma_1, \gamma_2}).$$

Further, since any $G$-decorated stable graph $\Gamma$ is equivalent to the choice of a $G_1$-decorated graph $\Gamma_1$ and a $G_2$-decorated graph $\Gamma_2$ which are based on the underlying graph $\Gamma$, we have the fiber bundle

$$\overline{\mathcal{M}}_W(\Gamma) = \overline{\mathcal{M}}_{W_1}(\Gamma_1) \times_{\mathcal{M}_{\Gamma}} \overline{\mathcal{M}}_{W_2}(\Gamma_2).$$

The natural inclusion

$$\overline{\mathcal{M}}_{g,k,W}(\Gamma) = \overline{\mathcal{M}}_{g,k,W_1}(\Gamma_1) \times_{\mathcal{M}_{\Gamma}} \overline{\mathcal{M}}_{g,k,W_2}(\Gamma_2) \Delta \overline{\mathcal{M}}_{g,k,W_1}(\Gamma_1) \times_{\mathcal{M}_{\Gamma}} \overline{\mathcal{M}}_{g,k,W_2}(\Gamma_2)$$

together with the isomorphism of middle homology induces a homomorphism

$$\Delta^* : \left( H_{\ast}(\overline{\mathcal{M}}_{g,k,W_1}(\Gamma_1), \mathbb{Q}) \otimes \prod_{i=1}^k H_{N_{\gamma_i+G_1}}(C_{\gamma_i+G_1}, W_{\gamma_i+G_1}) \right) \otimes \left( H_{\ast}(\overline{\mathcal{M}}_{g,k,W_2}(\Gamma_2), \mathbb{Q}) \otimes \prod_{j=1}^k H_{N_{\gamma_j+G_2}}(C_{\gamma_j+G_2}, W_{\gamma_j+G_2}) \right) \to H_{\ast}(\overline{\mathcal{M}}_{g,k,W}(\Gamma), \mathbb{Q}) \otimes \prod_{i=1}^k H_{N_{\gamma_i+G_1+G_2}}(C_{\gamma_i+G_1+G_2}, W_{\gamma_i+G_1+G_2}).$$
such that the virtual cycle satisfies

$$
\Delta^* \left( \overline{M}_{g,k,W_1}^{\text{vir}}(\Gamma_1) \right)^{\text{vir}} \otimes \left( \overline{M}_{g,k,W_2}^{\text{vir}}(\Gamma_2) \right)^{\text{vir}} = \left( \overline{M}_{g,k,W_1+W_2}^{\text{vir}}(\Gamma) \right)^{\text{vir}}
$$

(168)

Remark 6.2.2. By a Liouville type theorem for $A_1$-singularity, we see that the $A_1$-theory is uniquely determined by these axioms and it agrees with the theory of 2-spin curves as described in [IKVI] §4.5.

6.3. **Proof of Theorem 6.2.1 and Theorem 1.0.4**

In this section, we will first define the virtual cycle, prove the axioms given by Theorem 6.2.1 and finally prove Theorem 1.0.4.

**Definition of virtual cycle** $\overline{M}_{g,k,W}^{\text{vir}}(\Gamma)$. Fix $\gamma = \{\gamma_1, \cdots, \gamma_k\}$ and choose the moduli space $\overline{M}_{g,k,W}(\gamma, \kappa)$ to be strongly regular. For each $\gamma \in G$, choose the basis $\{S_\gamma^j(\gamma), j = 1, \cdots, \mu_\gamma\}$ in $H_{N_\gamma}(\mathbb{C}_\gamma, (W_\gamma + W_{0,\gamma})^{\pm \infty})$ corresponding to the critical points of $W_\gamma + W_{0,\gamma}$ and the dual basis $\{S_{\gamma}^j(\gamma), j = 1, \cdots, \mu_\gamma\}$ in $H_{N_\gamma}(\mathbb{C}_\gamma, (W_\gamma + W_{0,\gamma})^{\pm \infty})$. Then each combination $\{S_{\gamma_1}^j(\gamma_1), \cdots, S_{\gamma_k}^j(\gamma_k)\}$ corresponds to the combination of $k$ critical points, $\gamma_{j_1 \cdots j_k} := (\kappa_{\gamma_1}^j(\gamma_1), \cdots, \kappa_{\gamma_k}^j(\gamma_k))$. We obtain the virtual cycle $\overline{M}_{g,k,W}(\Gamma; \gamma, \kappa_{j_1 \cdots j_k})^{\text{vir}} =: \left[\overline{M}_{g,k,W}(\Gamma; \gamma, S_{\gamma_1}^j(\gamma_1), \cdots, S_{\gamma_k}^j(\gamma_k))^{\text{vir}}\right]$. Now we fix a strongly regular parameter $(b_0)$; the Gauss-Manin connection provides the isomorphisms

$$GM(b_0) : H_{N_\gamma}(\mathbb{C}_\gamma, (W_\gamma + W_{0,\gamma})^{\pm \infty}), Q) \to H_{N_\gamma}(\mathbb{C}_\gamma, (W_\gamma)^{\pm \infty}), Q)$$

Using the isomorphisms we can identify $H_{N_\gamma}(\mathbb{C}_\gamma, (W_\gamma + W_{0,\gamma})^{\pm \infty})$ with $H_{N_\gamma}(\mathbb{C}_\gamma, (W_\gamma)^{\pm \infty})$.

Define

$$\overline{M}_{g,k,W}^{\text{vir}}(\Gamma) := \sum_{j_1 \cdots j_k} \left[\overline{M}_{g,k,W}(\Gamma; \gamma, \kappa_{j_1 \cdots j_k})\right]^{\text{vir}} \otimes \prod_{i=1}^k S_{\gamma_i}(\gamma_i)
$$

(169)

$$\in H_{\gamma}(\overline{M}_{g,k,W}(\Gamma)) \otimes \prod_{\tau \in \mathcal{T}(\Gamma)} H_{N_{\gamma_{\tau}}}(\mathbb{C}_{\gamma_{\tau}}, W_{\gamma_{\tau}}^{\pm \infty}, Q)
$$

(170)

By Proposition 6.1.7, we have

**Proposition 6.3.1.** The virtual cycle $\overline{M}_{g,W}^{\text{vir}}(\Gamma)$ is independent of the choice of the basis $\{S_{\gamma_i}(\gamma_i)\}$ of $H_{N_\gamma}(\mathbb{C}_\gamma, (W_\gamma)^{\pm \infty})$ at each marked point $p_i$.

Since the parallel transport induced by the Gauss-Manin connection preserves the inner product of the homology bundle, the above proposition justifies the definition of the virtual cycle $\overline{M}_{g,W}^{\text{vir}}(\Gamma)$.

**Proof of Dimension axiom.** The dimension of the virtual cycle $\overline{M}_{g,k,W}(\Gamma; \gamma, \kappa_{j_1 \cdots j_k})^{\text{vir}}$ was already calculated in Theorem 5.1.1. The real dimension of the tensor product of the Lefschetz thimbles $\prod_{i=1}^k S_{\gamma_i}(\gamma_i)$ is $\sum_{i=1}^k N_{\gamma_i}$. So we obtain the dimension of the virtual cycle $\overline{M}_{g,W}^{\text{vir}}(\Gamma)$ Notice that its real dimension can be an odd number. This is a new result compared to the previous research in $A_r$-spin curves.
Proof of symmetric group invariance. Since the construction of the virtual cycle \( \mathcal{M}_W^{\text{ir}}(\Gamma; \gamma, \kappa) \) is independent of the order of the tails, hence if we denote by \( \sigma \Gamma \) the graph obtained by applying \( \sigma \) to the tails of \( \Gamma \) then we have

\[
\sigma \ast \left[ \mathcal{M}_W^{\text{ir}}(\Gamma) \right]^{\text{vir}} = \left[ \mathcal{M}_W^{\text{ir}}(\sigma \Gamma) \right]^{\text{vir}} = \left[ \mathcal{M}_W^{\text{ir}}(\Gamma) \right]^{\text{vir}}
\]

as homology classes.

Proof of Disconnected graphs. This is obvious.

Proof of Degenerating connected graphs. It suffices to prove the following conclusion:

\[
\left[ \mathcal{M}_{g,k,W}^{\text{ir}}(\gamma, \kappa) \right]^{\text{vir}} \cap \mathcal{M}_W^{\text{ir}}(\Gamma) = \left[ \mathcal{M}_W^{\text{ir}}(\Gamma; \kappa) \right]^{\text{vir}}.
\]

Let \( \{(V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma} \times V_{\text{map}, \sigma}, E_{\sigma}, \text{Aut}(\sigma), s_{\sigma})\} \) be an oriented Kuranishi structure of \( \mathcal{M}_{g,k,W}^{\text{ir}}(\gamma, \kappa) \). The restriction \( \{(V_{\text{deform}, \sigma} \times \{0\} \times V_{\text{map}, \sigma}, E_{\sigma}, \text{Aut}(\sigma), s_{\sigma})\} \) provides a Kuranishi structure of \( \mathcal{M}_W^{\text{ir}}(\Gamma; \kappa) \). We can take a smooth sequence of multisections \( s^n_\sigma \) such that \( s^n_\sigma \) approximates \( s_\sigma \), and the restriction \( s^n_{0,[\Gamma]} \) approximates \( s_{0,[\Gamma]} \). Hence

\[
\left[ \mathcal{M}_W^{\text{ir}}(\Gamma; \kappa) \right]^{\text{vir}} = \Pi \left( \bigcup_{\sigma} \left( (s^n_{0,[\Gamma]})^{-1}(0) / \text{Aut}(\sigma) \right) \right)
\]

\[
= \Pi \left( \bigcup_{\sigma} \left( \left( (s^n_\sigma)_{0}^{-1}(0) \right) \cap (V_{\text{deform}, \sigma} \times \{0\} \times V_{\text{map}, \sigma}) / \text{Aut}(\sigma) \right) \right)
\]

\[
= \left[ \mathcal{M}_{g,k,W}^{\text{ir}}(\gamma, \kappa) \right]^{\text{vir}} \cap \mathcal{M}_W^{\text{ir}}(\Gamma; \kappa),
\]

where \( \Pi : \mathcal{M}_W^{\text{ir}}(\Gamma; \kappa) \to \mathcal{M}_W^{\text{ir}}(\Gamma) \) is the forgetful map introduced before.

Proof of Weak Concavity axiom. By hypothesis all the marked or nodal points are Neveu-Schwarz points, then the perturbed Witten equation becomes the Witten equation which has only the zero solution by Theorem \( \text{[3.3.8]} \) and the virtual cycle \( \mathcal{M}_W^{\text{ir}}(\gamma, 0) \) is well defined without introducing any perturbation. The linearized operator of the Witten map becomes

\[
D(\bar{W}M) = \bar{\partial}.
\]

Now by assumption, \( \bar{\partial} \) has no holomorphic section. For any \( \sigma \in \mathcal{M}_W^{\text{ir}}(\gamma, 0) \), the Kuranishi neighborhood of \( \sigma \) can be taken as \( \{(V_{\text{deform}, \sigma} \times V_{\text{resol}, \sigma} \times \{0\}, R^1 \pi_*(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_i))\} \). Thus the virtual cycle \( \mathcal{M}_{g,k,W}^{\text{ir}}(\gamma; 0) \) is just the Poincaré dual to the Euler class (top Chern class) of the bundle \( R^1 \pi_*(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_i) \) on \( \mathcal{M}_W^{\text{ir}}(\gamma) \).

Proof of index zero axiom. Since all the decorations are Neveu-Schwarz the perturbed Witten equation becomes Witten equation. By the assumption, \( \mathcal{M}_W^{\text{ir}}(\Gamma) \) is a set of points. The Kuranishi structure is a set of maps from kernel to cokernel at each rigidified \( W \)-structure. It is obvious that the virtual cycle is given by the degree of the maps.

Proof of forgetting tails axiom. (a) Without loss of generality, we assume that \( i = k \) and

\[
\gamma_k = J^{-1} = (e^{2\pi i q_i}, \ldots, e^{2\pi i q_n}),
\]

\[
\gamma_k^* = J = (e^{2\pi i(1-q_i)}, \ldots, e^{2\pi i(1-q_n)}).
\]

Since \( c_{j-1}^N = \{0\} \), it suffices to prove

\[
\langle \theta_k^{\text{ir}}(\bar{\mathcal{M}}_{g,k,W}^{\text{ir}}(\Gamma; \gamma', \gamma_k, \kappa', 0)), \mathcal{M}_W^{\text{ir}}(\Gamma'; \gamma, \kappa) \rangle = \frac{|G|}{|J|^{k-1}} \left[ \mathcal{M}_{g,k-1,W}^{\text{ir}}(\Gamma'; \gamma', \kappa') \right]^{\text{vir}}
\]

(171)
Consider the semi-stable $W$-curve $\mathfrak{C}_j := (\mathbb{C}P^1, z_1, z_2, J, J^{-1}, \mathcal{L}_1, \cdots, \mathcal{L}_l, \varphi_1, \cdots, \varphi_l)$.

By the degree formula, we have
\[
\deg(|\mathcal{L}_j|) = -1.
\]
The Witten map on $\mathfrak{C}_j$ has no perturbation term and the index
\[
\text{index}(WI_{\mathfrak{C}_j}) = 0.
\]
Let $\mathcal{C}$ be a rigidified $W$-curve having $k$ marked points with the $k$-th marked point decorated by $\gamma_k$. We want to study the Witten equations nearby this Neveu-Schwarz point $(z_k, \gamma_k)$.

If we take the cylindrical metric near $z_k$ (i.e., let $\frac{d}{dz} = 1$), then its neighborhood is viewed as an infinitely long cylinder and $z_k$ is viewed as the infinite point. Let $u_j = \tilde{u}_je_j$; then the Witten equation is
\[
\frac{\partial \tilde{u}_i}{\partial \tilde{z}} + \frac{\partial W(\tilde{u}_1, \cdots, \tilde{u}_i)}{\partial \tilde{u}_i} \frac{1}{\tilde{z}} = 0. \tag{172}
\]
Let $z = e^w$; then we have
\[
\frac{\partial \tilde{u}_i}{\partial w} + \frac{\partial W(\tilde{u}_1, \cdots, \tilde{u}_i)}{\partial \tilde{u}_i} = 0, \quad \forall w \in [-\infty, 0] \times S^1. \tag{173}
\]
If we take the smooth metric, (i.e., let $|dz| = 1$), then the Witten equation over the orbicurve has the following form:
\[
\frac{\partial \tilde{u}_i}{\partial \tilde{z}} + \sum_j \frac{\partial W(\tilde{u}_1, \cdots, \tilde{u}_i)}{\partial \tilde{u}_i} \frac{1}{\tilde{z}|\tilde{z}|^{\gamma_k}} = 0. \tag{174}
\]
We can resolve this equation near $z_k$, and the corresponding function $\hat{u}$ satisfies the equation
\[
\frac{\partial \hat{u}_i}{\partial \tilde{z}} + \frac{\partial W(\hat{u}_1, \cdots, \hat{u}_i)}{\partial \hat{u}_i} = 0. \tag{175}
\]
Consider the moduli space $\mathcal{M}^{ig}_{g,k,W}(\Gamma; \gamma, \mathcal{X})$. Assume that the $k$-th marked point is decorated by $(\gamma_k, e_j, \cdots)$. Here we also assume that we take the cylindrical metric near each marked point. On the other hand, we can set the metric near $z_k$ to be the smooth metric and then the Witten equation has the form (173). After resolution, the solution $\hat{u}$ corresponds 1-1 to the solution $\tilde{u}$ of (175). In particular, the marked point $z_k$ becomes an ordinary point if we only consider the equation.

Let $\mathcal{M}^{ig}_{g,k,W}(\Gamma; \gamma_k, \mathcal{X}, sm)$ be the moduli space consisting of the isomorphism classes of $W$-sections ($\mathfrak{C}, u$), where $u$ is the solution of the perturbed Witten equation. Here the symbol ”$sm$” means that near the $k$-th marked point, the metric is set to be the smooth metric. As before we can give the Gromov topology to this moduli space. Moreover, we will show below that it can carry an oriented Kuranishi structure which is induced by the Kuranishi structure of $\mathcal{M}^{ig}_{g,k-1,W}(\Gamma; \gamma', \mathcal{X}')$.

The following diagram is commutative:
\[
\begin{array}{c}
\mathcal{M}^{ig}_{g,k,W}(\Gamma; \gamma, \gamma_k, \mathcal{X}, sm) \xrightarrow{\theta'^{ig}_k} \mathcal{M}^{ig}_{g,k-1,W}(\Gamma; \gamma', \mathcal{X}') \\
\downarrow \quad \downarrow \\
\mathcal{M}^{ig}_{g,k,W}(\Gamma; \gamma) \xrightarrow{\theta'^{ig}_k} \mathcal{M}^{ig}_{g,k-1,W}(\Gamma; \gamma')
\end{array}
\]

Here $\theta'^{ig}_k$ is the forgetful map. This map exists if and only if the marked point $z_k$ is decorated by the group element $J^{-1}$.  

Let $$\{(U_{\sigma}, \Gamma_{\sigma}, E_{\sigma}, s_{\sigma})\}$$ be a Kuranishi structure of $$\overline{\mathcal{M}}_{g,k-1,W}(\Gamma'; \gamma', \kappa')$$. We can choose the obstruction bundle $$E_{\sigma}$$ such that its generating functions have compact support disjoint from the $$k-1$$ marked points and the special point $$z_k$$. For a point $$\sigma = (\mathcal{C}_{\sigma}, u_{\sigma})$$, we can take a neighborhood $$\{(U_{\sigma}, \Gamma_{\sigma}, E_{\sigma}, s_{\sigma})\} \in \overline{\mathcal{M}}_{g,k-1,W}(\Gamma'; \gamma')$$ has a neighborhood $$(\tilde{U}_{\sigma}, \Lambda_{\sigma})$$. Then the embedding $$U_{\sigma} \hookrightarrow \tilde{U}_{\sigma}$$ is a $$\Lambda_{\sigma}$$ equivariant. There is a universal family of rigidified $$W$$-curves $$U_{\sigma} \rightarrow \tilde{U}_{\sigma}$$ on which $$\Lambda_{\sigma}$$ acts. Let $$(\mathcal{C}, u)$$ be a point in $$\overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma', \kappa, \kappa', sm)$$ such that $$\theta_{k}^{\text{rig}}([\mathcal{C}]) = [\mathcal{C}_{\sigma}]$$; then there is a neighborhood $$\tilde{U}_{\sigma}/\Lambda_{\sigma}$$ of $$[\mathcal{C}]$$ such that $$\theta_{k}^{\text{rig}}(\tilde{U}_{\sigma}/\Lambda_{\sigma}) = U_{\sigma}/\Lambda_{\sigma}$$. Now $$(U_{\sigma} \times \tilde{U}_{\sigma})/\Gamma$$ becomes a neighborhood of $$(\mathcal{C}, u)$$. The obstruction bundle $$E_{\sigma}$$ and the section $$s_{\sigma}$$ induce the obstruction bundle $$\tilde{E}_{\sigma}$$ and the section $$\tilde{s}_{\sigma}$$. It is straightforward to prove that $$\{(U_{\sigma} \times \tilde{U}_{\sigma}), \Gamma_{\sigma}, \tilde{E}_{\sigma}, s_{\sigma}\}$$ is an oriented Kuranishi structure of $$\overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma', \kappa, \kappa', sm)$$. Furthermore if $$\{s_{\sigma}^{\text{vir}}\}$$ is an approximation sequence of the continuous section $$\{s_{\sigma}\}$$ such that the multi-section $$s_{\sigma}^{\text{vir}}$$ is a transversal section, then $$\{(s_{\sigma}^{\text{vir}})^{-1}(0)\}$$ defines the virtual cycle $$\overline{\mathcal{M}}_{g,k,1,W}(\Gamma; \gamma', \kappa')^{\text{vir}}$$. Now the induced multisection $$\{\tilde{s}_{\sigma}^{\text{vir}}\}$$ satisfies the relation $$(\tilde{s}_{\sigma}^{\text{vir}})^{-1}(0) = (s_{\sigma}^{\text{vir}})^{-1}(0) \times \overline{\mathcal{M}}_{g,k,1,W}(\Gamma; \gamma') = \overline{\mathcal{M}}_{g,k,1,W}(\Gamma; \gamma')^{\text{vir}}$$.

Therefore we have

$$\overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma, \kappa, sm)^{\text{vir}} = (\theta_{k}^{\text{rig}})^{\text{vir}}(\overline{\mathcal{M}}_{g,k-1,W}(\Gamma'; \gamma', \kappa'))^{\text{vir}}$$

Now to prove this axiom we only need to show the following identity:

$$\overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma', \kappa, \kappa', sm)^{\text{vir}} = \overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma, \kappa)^{\text{vir}}. \quad (176)$$

Actually as the first step one can show after a simple computation that the two moduli spaces have the same virtual dimension. The difference between the two moduli spaces originates from the choice of different metric around $$z_k$$. When using the cylindrical metric, $$z_k$$ is the infinite point with an infinitely long cylinder neighborhood. When using the smooth metric, $$z_k$$ is just an ordinary point with a disc neighborhood. So to obtain the neighborhood of $$z_k$$ from the infinitely long cylinder one has to cap a disc at the infinitely far end. The inverse process is just the degeneration course of a $$W$$-curve along a circle centered at $$z_k$$.

Let $$(\mathcal{C}, u) \in \overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma, \kappa)$$, and let $$\mathcal{C}_J$$ be a semi-stable $$W$$-curve shown in (2). To construct the Witten map over $$\mathcal{C}_J$$, we can choose the cylindrical metric near $$z_1$$ which is decorated by $$J$$ and choose the smooth metric around $$z_2$$. So $$z_2$$ is actually an ordinary point and one can use the Witten lemma to show that the Witten equation over $$\mathcal{C}_J$$ has only the zero solution. When identifying the point $$z_k$$ on $$\mathcal{C}$$ with $$z_1$$ on $$\mathcal{C}_J$$, $$\mathcal{C}_J$$ becomes a nodal $$W$$-curve. Notice that this curve is not in the space $$\overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma')$$ because the $$\mathcal{C}_J$$ component is not stable. However, one can also do the gluing operation. The gluing parameter $$\zeta_k$$ is a complex number. However we only take a small real interval $$[0, \varepsilon]$$ as our gluing parameter.

When $$\zeta_k = 0$$, we do nothing. When $$\zeta_k \neq 0$$, we obtain a $$W$$-curve $$\mathcal{C}(\zeta_k)$$ from $$\mathcal{C}$$. Therefore from the moduli space $$\overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma')$$ of rigidified $$W$$-curves, we obtain another moduli space $$\overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma, \zeta_k)$$ by the gluing operation. These two spaces are totally equivalent as orbifolds. Let $$\overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma', \gamma_k, \kappa', \zeta_k, sm)$$ be the moduli space of $$W$$-sections over $$\overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma, \zeta_k)$. Define

$$\mathcal{M} = \bigcup_{\zeta \in [0, \varepsilon]} \overline{\mathcal{M}}_{g,k,w}(\Gamma; \gamma', \gamma_k, \kappa', \zeta, sm).$$
Using the decay estimate of the Witten equation, one can prove that this space is compact in the Gromov topology. One can construct an oriented Kuranishi structure over \( \overline{\mathcal{M}} \) from the Kuranishi structure of \( \mathcal{M}^\text{rig}_{g,k,W}(\Gamma; \gamma, \kappa) \) and \( \mathcal{M}^\text{rig}_{g,k,W}(\Gamma; \gamma', \kappa', \sm) \).

Let \( (U_\sigma, E_\sigma, s_\sigma) \) be a chart of \( (\mathcal{M}, \mathcal{U}) \in \mathcal{M}^\text{rig}_{g,k,W}(\Gamma; \gamma, \kappa) \). For any gluing parameter \( s_\sigma \in (0, e) \), we can obtain the \( W \)-curve \( \mathcal{C}(s_\sigma) \). By an index computation and the fact that the generators of the bundle \( E_\sigma \) have compact support away from \( s_\sigma \), we can view \( E_\sigma \) as the obstruction bundle over \( \mathcal{C}(s_\sigma) \).

Now by the implicit function theorem, there exists a section \( s_\sigma : U_\sigma \times [0, e/3] \rightarrow E_\sigma \). On the other hand, we let \( \mathcal{M}^\text{rig}_{g,k,W}(\Gamma; \gamma', \kappa', \mathcal{E}_s, sm) \) and extend the Kuranishi structure to \( \cup_{\mathcal{E}_s \in (2e/3, e]} \mathcal{M}^\text{rig}_{g,k,W}(\Gamma; \gamma', \kappa', \mathcal{E}_s, sm) \). By the extension theorem, we can construct an oriented Kuranishi structure \( \{(U_\sigma \times [0, e], E_\sigma, s_\sigma, \mathcal{E}_s)\} \) over \( \mathcal{M} \). So by a cobordism argument we have \( (\mathcal{E}_s) \) and then the final conclusion.

(b) \( \mathcal{M}_{0,3,W}(\gamma_1, \gamma_2, J^{-1}) \) is empty if \( \gamma_1 \neq \gamma_2^{-1} \) for degree reasons. So we only need to consider the moduli space \( \mathcal{M}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}) \) and the virtual cycle \( \mathcal{M}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}) \) for any \( \gamma \in G \). The moduli space \( \mathcal{M}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}) \) is 0-dimensional by the dimension formula. There is only one point in \( \mathcal{M}_{0,3} \) and \( \mathcal{M}_{0,3,W} \) respectively. For the space of rigidified \( W \)-curves, we have the decomposition

\[
\mathcal{M}^\text{rig}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}, k_1^1, k_2^1) = \bigcup_{\phi_1, \phi_2 \in \phi} \mathcal{M}^\text{rig}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}, k_1^2, k_2^2) \bigcup_{\phi_1, \phi_2 \in \phi} \mathcal{M}^\text{rig}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}, k_1^2, k_2^2, 0, \psi_1, \psi_2, \psi_3),
\]

where \( k_1, k_2 \) are critical points of the perturbed polynomial \( W_{\gamma} + W_{\delta \gamma} \) and \( W_{\gamma^{-1}} + W_{\delta \gamma^{-1}} \).

Recall that to define the perturbed Witten equation we first fix a standard rigidification and then choose the perturbation term with respect to this rigidification. After that the perturbation parameters with respect to the other rigidification is naturally defined, since the perturbed Witten equation is a globally defined equation and the solutions are independent of the choice of the rigidification. Assume that \( \psi^+ \) and \( \psi^- \) are the standard rigidifications attached to \( \gamma \) and \( \gamma^{-1} \). We have

**Lemma 3.6.2.** If the perturbation is strongly regular, then

\[
\mathcal{M}^\text{rig}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}, k_1^1, k_2^2) = \begin{cases} 1 & \text{if } I^{-1}(k_1^1) = k_2^2 \\ 0 & \text{otherwise} \end{cases}
\]

where \( I = \text{diag}(\epsilon^1, \ldots, \epsilon^3) \) and \( \epsilon^d = -1 \).

**Proof.** Take the rigidified \( W \)-curve \( \mathcal{C} = (\mathbb{R} \times S^1, p_1, p_2, p_3, \mathcal{L}, \psi^+, \psi^-, \psi_3) \) where we can set \( p_1, p_2 \) to be infinite far points at \( -\infty \) and \( +\infty \) and \( p_3 = (0, 0) \). The rigidification \( \psi^+ \) actually gives the basis \( e^u = (e_1^u, \ldots, e_3^u) \) on open half cylinders \( U_i, i = 1, 2 \), where we require that \( U_1 \cup U_2 = \mathbb{R} \times S^1 \) and \( U_1 \cap U_2 = U_3 \). Now by the disingularization operation and a cobordism argument similar to the proof in (a), we can suppose \( p_3 \) is an ordinary smooth point (not an infinitely far point) with the trivial orbifold structure without changing the virtual cycle. Consider the perturbed Witten equation over \( \mathcal{C} \):

\[
\bar{\partial}_u - \frac{\partial(W + W_{0,\beta})}{\partial u_i} = 0,
\]

where \( W_{0,\beta} = \sum b_i^u \beta_i u_i \). Let \( (\zeta^+, \theta^+) \) and \( (\zeta^-, \theta^-) \) be the local coordinates on \( U_1 \) and \( U_2 \) respectively. On the overlap \( U_3 \), we can do the following transformation:

- Coordinate transformation: \( (\zeta^+, \theta^+) = - (\zeta^-, \theta^-) \);
- Corresponding transformation of local basis: \( I(e^+) = e^- \).
If $u$ is a solution of the equation, and $u = \tilde{u}^+ e^x$, then we have the transformation: $I^{-1}(\tilde{u}^+) = \tilde{u}^-$. In particular, we have the relation:

$$I^{-1}(\tilde{u}^+(-\infty)) = \tilde{u}^-(\infty).$$ (180)

and the equation in the uniform coordinates and basis $(\zeta^-, \theta^)$, $e^-$ has the form

$$\bar{\partial} \tilde{u}^- - \frac{\partial (W + W_{0,\gamma})}{\partial \tilde{u}^-} = 0.$$ (181)

Note that the perturbation terms near the two ends have a cut-off function. Actually, we can consider a family of perturbed equations continuously changing to the following perturbed Witten equation without changing the cycle:

$$\bar{\partial} \tilde{u}^- - \frac{\partial (W + W_{0,\gamma})}{\partial \tilde{u}^-} = 0.$$ (182)

Therefore, to compute the virtual cycle is equivalent to considering the existence of the solutions of Equation (182). Multiplying Equation (182) by $\frac{\partial (W + W_{0,\gamma})}{\partial \tilde{u}^-}$ and integrating it over $\mathbb{R} \times S^1$, we have

$$(W_{\gamma^+} + W_{0,\gamma^+})(\tilde{u}^-)_{t=\infty} = \int_0^\infty \int_{S^1} \left| \frac{\partial (W + W_{0,\gamma})}{\partial \tilde{u}^-} \right|^2 ds d\theta.$$ (183)

Since the perturbation is strongly regular, by the above integral equality Equation (182) has solutions if and only if $\tilde{u}^+(-\infty) = \tilde{u}^-(\infty)$. By (180), this is just

$$I^{-1}(\gamma^+) = \gamma^-.$$

\[\square\]

**Corollary 6.3.3.** If the perturbation is strongly regular, then

$$[\mathcal{M}^{ig}_{0.3,\psi}(\gamma, \gamma^-, J^-, \kappa^1, \kappa^2, 0)]^{vir} = \begin{cases} \frac{|G|}{|\gamma^+|} \times \frac{|G|}{|\gamma^-|} & \text{if } I^{-1}(\gamma^+) = \gamma^- \\ 0 & \text{otherwise} \end{cases}$$ (184)

**Proof.** For any pair rigidification $(\psi_1, \psi_2)$, there exist $g_1, g_2 \in G$ such that $\psi_1 = g \cdot \psi^+$ and $\psi_2 = g \cdot \psi^-$. If $I^{-1}(\gamma^+) = \gamma^-$, we can choose $I = g_1 g_2^{-1} I$ satisfying the condition $I^{-1}(g_1 \kappa^1) = (g_2 \kappa^2)$ such that the corresponding equation has a unique solution. Thus the virtual cycle is just the number of different $I$, which is $\frac{|G|}{|\gamma^+|} \times \frac{|G|}{|\gamma^-|}$. \[\square\]

We begin the proof of (b). Let $S_j, j = 1, \cdots, \mu_j$ be a basis of $H_{N_j}(\mathbb{C}_\gamma^N, (W_\gamma)^{\infty}, \mathbb{Q})$ which is identified with $H_{N_j}(\mathbb{C}_\gamma^N, (W_\gamma + W_{0,\gamma})^{\infty}, \mathbb{Q})$ and let $\{S_j\}$ be the dual basis in $H_{N_j}(\mathbb{C}_\gamma^N, (W_\gamma)^{\infty}, \mathbb{Q}) \cong H_{N_j}(\mathbb{C}_\gamma^N, (W_\gamma + W_{0,\gamma})^{\infty}, \mathbb{Q}) \cong H_{N_j}(\mathbb{C}_\gamma^N, (W_{\gamma^+} + W_{0,\gamma^+})^{\infty}, \mathbb{Q})$. We have

$$[\mathcal{M}^{ig}_{0.3,\psi}(\gamma, \gamma^-, J^{-1})]^{vir} = \sum_{i,j} [\mathcal{M}^{ig}_{0.3,\psi}(\gamma, \gamma^-, J^{-1}, \kappa^1, \kappa^2, 0)]^{vir} \otimes S_i \otimes S_j.$$ (185)

The map $I$ is a 1-1 map from the set of critical points of $W_\gamma + W_{0,\gamma}$ to that of $W_\gamma + W_{0,\gamma^+}$ (cf. Remark 4.2.6) which induces a map

$$I : H_{N_j}(\mathbb{C}_\gamma^N, (W_\gamma + W_{0,\gamma})^{\infty}, \mathbb{Q}) \to H_{N_j}(\mathbb{C}_\gamma^N, (W_{\gamma^+} + W_{0,\gamma^+})^{\infty}, \mathbb{Q})$$

such that $I(S_i) = S_i$. Thus by Corollary 6.3.3, we have

$$[\mathcal{M}^{ig}_{0.3,\psi}(\gamma, \gamma^-, J^{-1})]^{vir} = \sum_i \frac{|G|}{|\gamma^+|} \times \frac{|G|}{<\gamma> \cdot |<J^{-1}>|} S_i \otimes S_i.$$
In general if $S_i$ and $S_j^-$ are two arbitrary bases in $H^0(C_{γ}^i, (W_{γ})^\infty, Q)$ and $H^0(C_{γ}^j, (W_{γ})^\infty, Q)$ respectively and if we denote by $(η^{ij})$ the inverse matrix of $(<S_i, S_j^->)$, then

$$\{[M^*_{0,3,W}(γ, γ_j^-)]^\mathrm{vir} = \sum_{i,j} \frac{|G|}{|<γ_j>|} \times \frac{|G|}{|<γ_j^->|} η^{ij} S_i ⊗ S_j^-.$$

Let $α ∈ H^0(C_{γ}^{N}, (W_{γ})^\infty, Q)$ and $β ∈ H^0(C_{γ}^{N}, (W_{γ})^\infty, Q)$; then

$$\{[M^*_{0,3,W}(γ, γ_j^-)]^\mathrm{vir}(α, β, e_{j-1}) = \sum_{i,j} \frac{|G|}{|<γ_j>|} \times \frac{|G|}{|<γ_j^->|} α(S_i)β(S_j^-)$$

$$= \frac{|G|}{|<γ_j>|} \times \frac{|G|}{|<γ_j^->|} <α, β>.$$

Proof of Composition law. We can choose two bases $S_i^+, i = 1, \cdots, μ_γ$, in $H^0(C_{γ}^{N}, (W_{γ})^\infty, Q) ⊂ H^0(C_{γ}^{N}, (W_{γ}+W_{0,γ})^\infty, Q)$ such that $S_i^+$ corresponds to the critical point $κ_i$ of $W_{γ}+W_{0,γ}$ and $I(κ_i)$ of $W_{γ}+W_{0,γ}$ respectively. We can write the virtual cycle as

$$\{[M^*_{g,k,W}(Γ)]^\mathrm{vir} = \sum_{i,j} \{[M^*_{g,k,W}(Γ, S_i^+, S_j^-)]^\mathrm{vir} ⊗ S_j^- ⊗ S_j^+.$$

Hence we have

$$\left\{\{M^*_{g,k,W}(Γ)\}^\mathrm{vir}\right\}_\pm = \sum_i \{[M^*_{g,k,W}(Γ, κ_i^+, I(κ_i^+))]^\mathrm{vir}\}$$

and then

$$\bar{ρ}_* \{[M^*_{g,k,W}(Γ)]^\mathrm{vir}\}_\pm = \frac{|G|}{|<γ_j>|} \times \frac{|G|}{|<γ_j^->|} \{[M^*_{g,k,W}(Γ)]^\mathrm{vir}\},$$

since we have $|G|/|<γ_j>|$ ways to obtain a rigidified $W$-curve in $M^*_{g,k,W}(Γ)$ (without rigidification at the nodal point).

Proof of Sums of singularity axiom. Let $γ_i = (γ_{1,i}, γ_{2,i}, \cdots, γ_{i,k})$ for $i = 1, 2$ and $γ = (γ_{1,1}, γ_{2,1}, \cdots, (γ_{1,k}, γ_{2,k}, \cdots, γ_{k,k}))$. Assume that $H^0(C_{y_{1,i}}, W_{1,y_{1,i}}^\infty, Q)$ is generated by a basis $S_i^- j_i = 1, \cdots, μ_γ$, for each $i$ and $H^0(C_{γ_i}, W_{2,y_{2,i}}^\infty, Q)$ is generated by a basis $S_{j_i}^- l_i = 1, \cdots, μ_γ$. Then the group $H^0(C_{γ_1+γ_2}, W_{1,y_{1,1}+y_{2,1}}^\infty, Q)$ is generated by the basis $(S_j^- ⊗ S_l^- j_i = 1, \cdots, μ_γ, l_i = 1, \cdots, μ_γ)$.

Now we consider the case that $Γ$ is a decorated stable $W$-graph with each tail decorated by the Lefschetz thimble $\{S_j^- j_i = 1, \cdots, k\}$. We have the moduli spaces $\overline{M}_{g,W}(Γ; γ, S_j^-) = \overline{M}_{g,W}(Γ; γ, S_j^- j_i, S_j^- l_i, \cdots, S_j^-), \overline{M}_{g,W}(Γ; γ, S_j^- l_i, S_j^- j_i, \cdots, S_j^-)$, and $\overline{M}_{g,W}(Γ; γ, S_j^- l_i, S_j^- j_i, \cdots, S_j^-)$. These are compact Hausdorff spaces when given the Gromov topology and have the relation

$$\overline{M}_{g,W}(Γ; γ, (S_j^-, S_j^-)) = \overline{M}_{g,W}(Γ; γ, S_j^- j_i, S_j^- l_i, \cdots, S_j^-), \overline{M}_{g,W}(Γ; γ, S_j^- l_i, S_j^- j_i, \cdots, S_j^-).$$

By Corollary 6.15 the moduli space $\overline{M}_{g,W}(Γ; γ, (S_j^-, S_j^-))$ carries an orientable Kuranishi structure. Consider its local charts. For any $σ ∈ \overline{M}_{g,W}(Γ)$, the local chart is given by $(V_{deformed} × V_{ext} × V_{map,ε}, E_σ, Aut(σ), s_σ, Ψ_σ)$. Let $\{s_σ\}$ be a family of multisections approximating $s_σ$; then for $n$ large enough $\overline{M}_{g,W}(Γ)^{vir}$ is the pushdown of $[s_σ]^{-1}(0))$. Since $W = W_1 + W_2$, we have the splitting $V_{map,ε} = V_{map,ε_1} × V_{map,ε_2}$, $Aut(σ) = Aut(σ_1) ×
\[ \text{Aut}(\sigma_2) \text{ and } E_\sigma = E_{\sigma_1} \otimes E_{\sigma_2}, \text{ where } (V_{\text{deform,}\sigma} \times V_{\text{resol,}\sigma} \times V_{\text{map,}\sigma}, E_\sigma, \text{Aut}(\sigma), s_\sigma, \Psi_\sigma) \]

for \( i = 1, 2 \) is the local chart of \( \mathcal{M}_{\text{rig}}(\Gamma_1) \). Thus

\[ [(\hat{\delta}_\sigma^0)^{-1}(0)] = [(\hat{\delta}_{\sigma_1}^0)^{-1}(0)] \times_{\mathcal{M}_1} [(\hat{\delta}_{\sigma_2}^0)^{-1}(0)], \]

i.e.,

\[ [(\hat{\mathcal{M}}_{\text{rig}}^g(\Gamma, \gamma, (S_\hat{f}, \hat{S}_\gamma))]^{\text{vir}} = [(\hat{\mathcal{M}}_{\text{rig}}^g(\Gamma_1; \gamma_1, S_\hat{f}_1))]^{\text{vir}} \times_{\mathcal{M}_1} [(\hat{\mathcal{M}}_{\text{rig}}^g(\Gamma_2; \gamma_2, \hat{S}_\gamma))]^{\text{vir}}. \quad (188) \]

On the other hand, we have the decomposition of the dual spaces

\[ H_{N_{\gamma_1}}(\mathbb{C}_{\gamma_1 \gamma_2}^N, W_{\gamma_1}^\infty, Q) = H_{N_{\gamma_2}}(\mathbb{C}_{\gamma_1 \gamma_2}^N, W_{\gamma_2}^\infty, Q) \otimes H_{N_{\gamma_1}}(\mathbb{C}_{\gamma_1 \gamma_2}^N, W_{\gamma_1}^\infty, Q). \]

This decomposition combined with the identity (188) induces the conclusion.

**Proof of Theorem 1.0.4.** The primary complication is the numerical factors. Recall following lemma (Lemma 4.2.3 of [FJR1]).

**Lemma 6.3.4.** For a diagram of schemes or DM stacks,

\[
\begin{array}{ccc}
W & \xrightarrow{i} & Y \\
p \downarrow & & q \\
Z & \xrightarrow{j} & X
\end{array}
\]

where \( i \) and \( j \) are regular imbeddings of the same codimension, \( p \) and \( q \) are finite morphisms, and there exists a finite surjective morphism \( f : W \twoheadrightarrow Z \times_Y W \) with \( p = pr_1 \circ f \) and \( i = pr_2 \circ f \), or such that there exists a finite surjective morphism \( f : Z \times_Y W \twoheadrightarrow W \) with \( pr_1 = p \circ f \) and \( pr_2 = i \circ f \), then for any \( c \in H_*(M_2, Q) \) we have

\[
\frac{p.\hat{\gamma}^*(c)}{\text{deg}(p)} = \frac{j.\hat{\gamma}.q_*(c)}{\text{deg}(q)}. \quad (190)
\]

We will use this lemma several times. All the diagrams in the proof satisfy the condition of Lemma.

Recall that

\[ \mathcal{M}_{\text{rig}}(\Gamma)^{\text{vir}} = \frac{1}{\text{deg so}_\Gamma} (\text{so}_\Gamma)_! \mathcal{M}_{\text{rig}}(\Gamma)^{\text{vir}}. \]

The Dimension and Symmetric group invariance axioms follows trivially from the corresponding axioms for the rigidified space.

Let's prove the Degenerating connected graph axioms.

Consider diagram

\[
\begin{array}{ccc}
\mathcal{M}_{\text{rig}}^g(\Gamma) & \xrightarrow{j} & \mathcal{M}_{g,k}(\gamma) \\
\downarrow \text{so}_\Gamma & & \downarrow \text{so} \\
\mathcal{M}_{\text{rig}}(\Gamma) & \xrightarrow{j} & \mathcal{M}_{g,k}(\gamma)
\end{array}
\]

It implies that

\[
\frac{1}{\text{deg (so)_\Gamma}} j^*_\gamma c = \frac{1}{\text{deg (so)}} j^*_\gamma s_\gamma c
\]

for any homology class \( c \in H_*(\mathcal{M}_{g,k}(\gamma), Q) \).
Let \( c = [\mathcal{M}_{g,k,W}^{\text{rig}}(\gamma)]^{\text{vir}} \). We obtain
\[
\tilde{j}^* \text{so}_* [\mathcal{M}_{g,k,W}^{\text{rig}}(\gamma)]^{\text{vir}} = \frac{\text{deg} (\text{so})}{\text{deg} (\text{so}_*)} \cdot \tilde{j}^* [\mathcal{M}_{g,k,W}^{\text{rig}}(\gamma)]^{\text{vir}}.
\]
Using the formula \([\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} = \tilde{j}^* [\mathcal{M}_{g,k,W}^{\text{rig}}(\gamma)]^{\text{vir}}\) from the corresponding axiom for the rigidified space, we obtain
\[
\tilde{j}^* [\mathcal{M}_{g,k,W}^{\text{rig}}(\gamma)]^{\text{vir}} = \frac{1}{\text{deg} (\text{so})} \tilde{j}^* \text{so}_* [\mathcal{M}_{g,k,W}^{\text{rig}}(\gamma)]^{\text{vir}} = \frac{1}{\text{deg} (\text{so}_*)} (\text{so}_*) \cdot \tilde{j}^* [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} = [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}}.
\]

The disconnected graph axiom, weak concavity axiom and index zero axiom follow trivially from the corresponding axioms of rigidified space.

Slightly more attention is required for the Composition axiom. It is easy to check that
\[
\text{so}_*^{\text{cut}} [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} = \frac{1}{\text{deg} (\text{so}_*)} \text{so}_*^{\text{cut}} [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} = [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}}.
\]

Hence,
\[
\frac{1}{\text{deg} (\text{so}_*)} (\text{so}_*) \cdot (\text{so} \circ \tilde{p})^* [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} = \frac{1}{\text{deg} (\text{so}_*)} (\text{so}_*) \cdot (\text{so} \circ \tilde{p})^* [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}}
\]
\[
= \frac{1}{\text{deg} (\text{so}_*)} (\text{so}_*) \cdot <[\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} >_{\alpha} = <[\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} >_{\alpha}.
\]

Recall that \( F = \mathcal{M}_W(W_{\text{cut}}) \times_{\mathcal{M}_1} \mathcal{M}_W(W) \). There is a natural map
\[
\tau : \mathcal{M}_W(W_{\text{cut}}) \rightarrow F
\]
such that
\[
\text{so}_* \circ \tilde{p} = p r_2 \circ \tau, \text{so}_* = q \circ \tau.
\]
Hence,
\[
\frac{1}{\text{deg} (\text{so}_*)} (\text{so}_*) \cdot (\text{so} \circ \tilde{p})^* [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} = \frac{1}{\text{deg} (\text{so}_*)} q \tau \cdot (\text{so} \circ \tilde{p})^* [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}}
\]
\[
= \frac{\text{deg} (\tau)}{\text{deg} (\text{so}_*)} q \cdot (pr_2^*) [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} = \frac{\text{deg} (\tau)}{\text{deg} (\text{so}_*)} q \cdot (pr_2^*) [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}}.
\]

Let’s consider the forgetting tail axiom. We have the fiber diagram
\[
\begin{array}{ccc}
\mathcal{M}_W^{\text{rig}}(\Gamma) & \xrightarrow{\theta} & \mathcal{M}_W^{\text{rig}}(\Gamma') \\
\downarrow \text{so}_* & & \downarrow \text{so}_* \\
\mathcal{M}_W(W) & \xrightarrow{\theta} & \mathcal{M}_W(W)
\end{array}
\]
Hence,
\[
\theta^* [\mathcal{M}_W^{\text{rig}}(\Gamma')]^{\text{vir}} = \frac{1}{\text{deg} (\text{so}_*)} \theta^* (\text{so}_*) \cdot [\mathcal{M}_W^{\text{rig}}(\Gamma')]^{\text{vir}} = \frac{1}{\text{deg} (\text{so}_*)} (\text{so}_*) \cdot \theta^* [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}}
\]
\[
= \frac{1}{\text{deg} (\text{so}_*)} (\text{so}_*) \cdot [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}} = [\mathcal{M}_W^{\text{rig}}(\Gamma)]^{\text{vir}}.
\]

For the second part of the forgetting tail axiom, we observe \( \frac{1}{|G| \cdot |\gamma|} \sum_{\phi_1, \phi_2} \sum_{i, j} \alpha_i \eta_i^j \phi_1 \phi_2 \beta_j \) corresponds to the Casimir element of pairing of invariant homology
\[
<, > : H_{N_1}(\mathbb{C}_\gamma^N, W_\gamma^{-\infty}, Q) \otimes H_{N_1}(\mathbb{C}_\gamma^{-1}, W_\gamma^{\infty}, Q) \rightarrow Q.
\]
Therefore, \( \overline{\mathcal{M}}_{g,W}^{vir}(\gamma, \gamma^{-1}, J^{-1}) \) is times the Casimir element of the pairing on invariant homology. On the other hand, the degree of so is \( \frac{|G|}{|G_{\gamma}|} \). Therefore, \( \left[ \overline{\mathcal{M}}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}) \right]^{vir} = \frac{|G|}{|G_{\gamma}|} \) times the Casimir element of invariant homology. The proof of the Sum of singularity axioms follows from the fact that all the relevant maps are fiber products.

Let \( \tilde{W} = W + Z \) where \( Z \) has no common monomials with \( W \). There is a natural inclusion map of components

\[
i : \overline{\mathcal{M}}_{g,k,\tilde{W}} \to \overline{\mathcal{M}}_{g,k,W}(\gamma).
\]

Furthermore, there is a natural isomorphism

\[
H_{N_{\gamma}}(\mathbb{C}_{\gamma}^N, (\tilde{W})_{\gamma}^\infty, \mathbb{Q}) \cong H_{N_{\gamma}}(\mathbb{C}_{\gamma}^N, (W)_{\gamma}^\infty, \mathbb{Q})
\]

We can restrict the moduli space to the components \( \overline{\mathcal{M}}_{g,k,W}(\gamma) \). For the Witten equation, we consider a family of quasihomogeneous polynomials \( \tilde{W}_t = W + tZ \). Then, a simple cobordism argument yields

**Theorem 6.3.5.** \( \left[ \overline{\mathcal{M}}_{g,k,W_C}(\gamma) \right]^{vir} = \overline{\mathcal{M}}_{g,k,\tilde{W}}(\gamma) \cap \left[ \overline{\mathcal{M}}_{g,k,W}(\gamma) \right]^{vir}. \)

\[\square\]

**References**

ADKMV. Aganagic, Mina; Dijkgraaf, Robbert; Klemm, Albrecht; Mariño, Marcos; Vafa, Cumrun. Topological strings and integrable hierarchies. Comm. Math. Phys. 261 (2006), no. 2, 451–516.

AGV. V. Arnold, A. Gusein-Zade and A. Varchenko. Singularities of differential maps, vol I, II, Monographs in Mathematics.

AJ. D. Abramovich and T. Jarvis. Moduli space of twisted spin curves. Proceedings of the American Mathematical Society, (2003) 131(3), 685–699.

Ar. V. I. Arnold(Ed.). Dynamical systems VI (Singularity Theory I) and VIII (Singularity Theory II). Encyclopaedia of Mathematical Sciences, Volume 6 and Volume 39, Springer-Verlag.

ABK. M. Aganagic, V. Bouchard and A. Klemm. Topological Strings and (Almost) Modular Forms, arXiv:hep-th/0607100

AKO. Auroux, Denis; Katzarkov, Ludmil; Orlov, Dmitri. Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves. Invent. Math. 166 (2006), no. 3, 537–582.

Ch1. A. Chiodo. The Witten top Chern class via K-theory, J. Algebraic Geom., 15 no 4, (2006) 681–707.

Ch2. A construction of Witten’s top Chern class via K-theory, J. Algebraic Geom., 15 no 4, (2006) 681–707.

Ch3. Towards an enumerative geometry of the moduli space of twisted curves and \( v \)-th roots. arXiv:math/0607324v3 [math.AG]

CL. Chen, Bohui and Li, An-Min. Symplectic virtual localization of Gromov-Witten invariants, arXiv:math/0610370.

CR1. W. Chen and Y. Ruan. A new cohomology theory for orbifold. Comm. Math. Phys. 248 (2004), no. 1, 1–31.

CR2. Orbifold Gromov-Witten theory. Orbifolds in mathematics and physics (Madison, WI, 2001), 25–85, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

CV. S. Cecotti, C. Vafa, On Classification of N=2 Supersymmetric Theories, Commun.Math.Phys. 158 (1993) 569-644

CW. Ya-Zhe Chen and Lan-Cheng Wu. Second order elliptic equations and elliptic systems. Translated from the 1991 Chinese original by Bei Hu. Translations of Mathematical Monographs, 174. American Mathematical Society, Providence, RI, 1998.

CZ. Conley, C. and Zeidler, E., Morse type index theory for flows and periodic solutions of Hamiltonian equations. Comm. Pure Appl. Math., Vol. 37, 207-253, 1984
[1] Dijkgraaf, Robbert, *Topological field theory and 2D quantum gravity*. Two-dimensional quantum gravity and random surfaces (Jerusalem, 1990/1991), 191–238, Jerusalem Winter School Theoret. Phys., 8, World Sci. Publ., River Edge, NJ, 1992.

[2] Dijkgraaf, Robbert, *Intersection theory, integrable hierarchies and topological field theory*. New symmetry principles in quantum field theory (Cargese, 1991), 95–158, NATO Adv. Sci. Inst. Ser. B Phys., 295, Plenum, New York, 1992.

[3] Donaldson, D. S., *Floer homology groups in Yang-Mills theory*, Cambridge Tracts in Mathematics 147, Cambridge university press, 2002.

[4] Ebeling, W., *The monodromy groups of isolated singularities of complete intersection*, Lecture Notes in Mathematics 1293, Springer-Verlag.

[5] Faber, C., Shadrin, S., and Zvonkine, D., *Tautological relations and the r-spin Witten conjecture*, arXiv:math/0612510.

[6] Fan, H., Jarvis, T., and Ruan, Y., *Geometry and analysis of spin equations*, arXiv:math/0409434.

[7] FJR, The Witten equation, mirror symmetry and quantum singularity theory, arXiv:math.0712.4021.

[8] FJR, Concavity for the Virtual Fundamental Cycle of the Witten Equation, in preparation.

[9] FJR, Generalized Witten conjecture for ADE singularity, in preparation.

[10] FJR, Landau-Ginzburg quantum cohomology for the complete intersections of toric varieties, in preparation.

[11] Fukaya, Kenji; Ono, Kaoru, *Arnold conjecture and Gromov-Witten invariant*. Topology 38 (1999), no. 5, 933–1048.

[12] Fulton, William, *Intersection theory*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol 2. Springer-Verlag, Berlin, 1998.

[13] Givental, A., *Symplectic geometry of Frobenius structures*. Frobenius manifolds, 91–112, Aspects Math., E36, Vieweg, Wiesbaden, 2004.

[14] Givental, A., *A_n^{−1} singularities and nKdV hierarchies*. Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. Mosc. Math. J. 3 (2003), no. 2, 475–505, 743.

[15] Givental, A., *A mirror theorem for toric complete intersections*. Topological field theory, primitive forms and related topics (Kyoto, 1996), 141–175, Progr. Math., 160, Birkhäuser Boston, Boston, MA, 2005.

[16] Hashimoto, private communication.

[17] Hertling, C., *Frobenius manifolds and moduli spaces for singularities*, Cambridge University Press.

[18] Horn, R. and Johnson, C., *Matrix-analysis*, Cambridge University Press, 1990.

[19] Hardy, G.H., Littlewood, J.E., and Polya, G., *Inequalities*, Cambridge Mathematical Library.

[20] Intriligator, K. and Vafa, C., *Landau-Ginzburg orbifolds*, Nuclear Phys. B 339 (1990), no. 1, 95–120.

[21] Jarvis, T., *Geometry of the moduli of higher spin curves*, Inter. J. Math. 11 (2000), 637–663.

[22] Jarvis, T., Kimura, T., and Vaintrob, A., *Moduli spaces of higher spin curves and integrable hierarchies*, Compositio Math. 126 (2001), 157–212.

[23] Jarvis, T., Kimura, T., and Vaintrob, A., *Gravitational descendants and the moduli space of higher spin curves*. Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), Contemp. Math., 276, Amer. Math. Soc., Providence, RI, 2001, 167–177.

[24] Kaufmann, R., *Singularities with symmetries, orbifold Frobenius algebras and mirror symmetry*. math.AG/0312417.

[25] Kaufmann, R., *Orbifolding Frobenius algebras*. Internat. J. Math. 14 (2003), no. 6, 573–617.

[26] Kaufmann, R., *Orbifold Frobenius algebras, cobordisms and monodromies*. Orbifolds in mathematics and physics (Madison, WI, 2001), 155–161, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[27] Konstevich, M., *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. 164 (1992), 1–23.

[28] Lee, Y.-P., *Witten's conjecture and the Virasoro conjecture for genus up to two*. Gromov-Witten theory of spin curves and orbifolds, 31–42, Contemp. Math., 403, Amer. Math. Soc., Providence, RI, 2006.

[29] Liang, B.; Liu, K.; and Yau, S., *Mirror principle. I*. Asian J. Math. 1 (1997), no. 4, 729–763.
LiT. Li, Jun; Tian, Gang, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. J. Amer. Math. Soc. 11 (1998), no. 1, 119–174.

LT. Liu, Gang; Tian, Gang, Floer homology and Arnold conjecture. J. Differential Geom. 49 (1998), no. 1, 1–74.

Ma. Y. Manin, Three constructions of Frobenius manifolds: a comparative study, arXiv:math/9801006

Mar. E. Martinec, Criticality, Catastrophes, and Compactifications, in Physics and Mathematics of strings, ed L. Brink, D. Friedan, and A. M. Polyakov.

Mo. Mochizuki, Takuro The virtual class of the moduli stack of stable r-spin curves. Comm. Math. Phys. 264 (2006), no. 1, 1–40.

MMR. J. Morgan, T. Mrowka and D. Ruberman, The $L^2$-moduli space and a vanishing theorem for Donaldson polynomial invariants, International Press, Cambridge, (1994)

MS. McDuff, Dusa; Salamon, Dietmar J-holomorphic curves and symplectic topology. American Mathematical Society Colloquium Publications, 52. American Mathematical Society, Providence, RI, 2004.

Mu. D. Mumford, Towards an enumerative geometry of the moduli space of curves, in “Arithmetic and Geometry,” eds. M. Artin and J. Tate, Part II, Progress in Math., Vol.36. Birkhäuser, Basel (1983), 271–328.

NY. M. Noumi and Y. Yamada: Notes on the flat structures associated with simple and simply elliptic singularities, in “Integrable Systems and Algebraic Geometry” (eds. M.-H.Saito, Y.Shimizu, K.Ueno), 372–383, World Scientific, 1998.

On. Ono, K., On the Arnold conjecture for weakly monotone symplectic manifolds, Inven. Math., Vol.119, 519-537, 1995

Po. A. Polishchuk, Witten’s top Chern class on the moduli space of higher spin curves. math.AG/0208112.

PV. Polishchuk, Alexander; Vaintrob, Arkady Algebraic construction of Witten’s top Chern class. Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), 229–249, Contemp. Math., 276. Amer. Math. Soc., Providence, RI, 2001.

R. Ruan, Yongbin Virtual neighborhoods and pseudo-holomorphic curves, Turkish J. Math. 23 (1999), no. 1, 161–231.

Sa1. C. Sabbah, Monodromy at infinity and Fourier transform. Publ. Rims. 33 (1997), 643-685.

Sa2. C. Sabbah, Frobenius manifolds: isomonodromic deformations and infinitesimal period mappings. Expo. Math. 16(1998), 1-58.

Sa3. C. Sabbah, Hypergeometric period for a tame polynomial. C.R. Acad. Sci. Paris Ser. I Math. 328(1999), 603-608.

Sb. Siebert, Bernd Symplectic Gromov-Witten invariants. New trends in algebraic geometry (Warwick, 1996), 375–424, London Math. Soc. Lecture Note Ser., 264. Cambridge Univ. Press, Cambridge, 1999.

Si. P. Seidel, Fukaya categories and Picard-Lefschetz theory, book, to appear

SS. R. Seeley and I. M. Singer, Extending $\bar{\partial}$ to singular Riemann surfaces, J. Geom. Phys. 5 (1989), 121–136.

Ta. C. Taubes, $L^2$-moduli spaces on open 4-manifolds, International Press, Cambridge, (1993)

Va. Varchenko, A. N. Local residue and the intersection form in vanishing cohomology. Izv. Akad. Nauk SSSR Ser. Mat. 49 (1985), no. 1, 32–54, 237.

VG. Varchenko, A. N.; Givental , A. B. The period mapping and the intersection form. (Russian) Funktsional. Anal. i Prilozhen. 16 (1982), no. 2, 7–20, 96.

VW. C. Vafa and N. Warner, Catastrophes and the classification of conformal field theories, Phys. Lett. 218B (1989)51

Wa1. Wall, C. T. C. A note on symmetry of singularities. Bull. London Math. Soc. 12 (1980), no. 3, 169–175.

Wa2. A second note on symmetry of singularities. Bull. London Math. Soc. 12 (1980), no.5, 347–354.

Wi1. E. Witten, Two-dimensional gravity and intersection theory on the moduli space, Surveys in Diff. Geom. (1991), 243–310.

Wi2. Algebraic geometry associated with matrix models of two-dimensional gravity, Topological models in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX (1993), 235–269.

Wi3. private communication.

Wi4. Phases of $N = 2$ Theories In Two Dimensions, Nucl.Phys. B403 (1993) 159-222
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