SOME REMARKS ON THE ERDŐS DISTINCT SUBSET SUMS PROBLEM

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Abstract. Let \( \{a_1, \ldots, a_n\} \subset \mathbb{N} \) be a set of positive integers, \( a_n \) denoting the largest element, so that for any two of the \( 2^n \) subsets the sum of all elements is distinct. Erdős asked whether this implies \( a_n \geq c \cdot 2^n \) for some universal \( c > 0 \).

We prove, slightly extending a result of Elkies, that for any \( a_1, \ldots, a_n \in \mathbb{R}_{>0} \)

\[
\int_{\mathbb{R}} \left( \frac{\sin x}{x} \right)^2 \prod_{i=1}^{n} \cos (a_i x)^2 \, dx \geq \frac{\pi}{2^n}
\]

with equality if and only if all subset sums are 1-separated. This leads to a new proof of the currently best lower bound \( a_n \geq \sqrt{2/n} \cdot 2^n \). The main new insight is that having distinct subset sums and \( a_n \) small requires the random variable \( X = \pm a_1 \pm a_2 \pm \cdots \pm a_n \) to be close to Gaussian in a precise sense.

1. Introduction

A problem of Erdős [11] is as follows: if \( \{a_1, \ldots, a_n\} \subset \mathbb{N} \) is a set of positive integers, assumed to be ordered as \( a_1 < a_2 < \cdots < a_n \), such that for each of the \( 2^n \) subsets the sum of all elements is unique, does this force \( a_n \geq c \cdot 2^n \) for some universal \( c > 0 \)? The problem is quite old. Erdős [13] refers to it as “perhaps my first serious conjecture which goes back to 1931 or 32”. Since the sums over all subsets leads to \( 2^n - 1 \) distinct positive integers, one has \( \sum_{i=1}^{n} a_i \geq 2^n - 1 \) (sharp for the powers of 2) and \( a_n \gtrsim 2^n/n \). Currently, the best known bound is

\[
a_n \geq (c - o(1)) \frac{2^n}{\sqrt{n}}
\]

where different estimates for \( c \) have been given over the years

- \( c \geq 1/4 \) Erdős and Moser [11]
- \( \geq 2/3^{3/2} \) Alon and Spencer [2]
- \( \geq 1/\sqrt{\pi} \) Elkies [10]
- \( \geq 1/\sqrt{3} \) Bae [3], Guy [15]
- \( \geq \sqrt{3}/2\pi \) Aliev [11]
- \( \geq \sqrt{2}/\pi \) Dubroff, Fox and Xu [9].

The literature (see [11]) mentions an unpublished manuscript of Elkies and Gleason also showing \( c \geq \sqrt{2/\pi} \). Dubroff, Fox and Xu give two different proofs: one appeals to the Berry-Esseen Theorem, the other uses an isoperimetric principle of

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Harper [17]. In the other direction, we note that the powers of 2, with \(a_n = 2^{n-1}\), are not extremal: already in 1968, Conway and Guy [8] (answering another question by Erdős [12]) produced a candidate construction showing that \(a_n \leq 2^n - 2\) is possible (see Bohman [5]). The currently best construction is due to Bohman [6] showing \(a_n \leq 0.88908 \cdot 2^n - 2\), see also [4, 7, 18, 19, 20]. It is an interesting question whether relaxing the condition somewhat can give rise to interesting examples. More concretely, are there sets \(\{a_1, \ldots, a_n\} \subset \mathbb{N}\) such that the subset sums attain \((1-o(1)) \cdot 2^n\) distinct values and \(a_n = o(2^n)\)?

The main purpose of our paper is to give a new proof of \(c \geq \sqrt{2/\pi}\). Many arguments, starting with Erdős and Moser [11], have considered the random walk \(X = \pm a_1 \pm a_2 \pm \cdots \pm a_n\), where all signs are chosen independently and uniformly at random. If all subset sums are distinct, then all \(2^n\) possible outcomes of the random walk are equally likely and they are all at least distance 2 from each other. A well-known argument (see [3, 11, 15, 18]) exploits this by using

\[
n \cdot a_n^2 \geq \sum_{i=1}^{n} a_i^2 = \mathbb{E}(X^2) \geq \frac{2}{2^n} \sum_{k=1}^{2^n-1} (2k-1)^2 = 4^n - \frac{1}{3}
\]

which shows \(c \geq 1/\sqrt{3}\). This was further refined by Dubroff, Fox and Xu [9] who argued, using the Berry-Esseen theorem, that if \(a_n^2\) is relatively small compared to \(\sum_{i=1}^{n} a_i^2\) (the variance of the random walk), then the random walk is well-described by a Gaussian. Our argument will imply a somewhat converse result: unless the distribution of the random walk is close to a Gaussian (in a sense that will be made precise), the set cannot have distinct subset sums and \(a_n\) small. This leads to an interesting reformulation of the Erdős distinct subset sums problem as a problem in probability theory: whether it is possible for random walks with a large variance but relatively small largest stepsize to emulate a Gaussian distribution very well.

2. Results

2.1. Main Results. We start with a basic analytic characterization of what it means for a set of \(n\) positive real numbers to have the property that all subset sums are at least distance 1 from each other (if all numbers are integers, then this is the same as asking them to be distinct).

**Theorem 1.** Let \(a_1, \ldots, a_n > 0\) be positive, real numbers. Then

\[
\int_{\mathbb{R}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^{n} \cos (2\pi a_i x) dx \geq \frac{1}{2^{n+1}}.
\]

Equality occurs if and only if all subset sums are distance \(\geq 1\) from each other.

This result is very similar to the analytic approach of Elkies [10] based on Laurent series. If all \(a_i\) are integers, the product is \(2\pi\)-periodic which simplifies the integral and recovers the characterization used by Elkies.

**Corollary 1** (Elkies [10]). Let \(a_1, \ldots, a_n > 0\) be positive integers. Then

\[
\int_{0}^{1} \prod_{i=1}^{n} \cos (2\pi a_i x) dx \geq \frac{1}{2^n}
\]

with equality if and only if all subset sums are distinct.
All cosines in the product are aligned around \( x = 0 \). A natural approach is thus to bound the contribution coming from a small interval around the origin of length \( \sim 1/a_n \). If \( a_n \) is too small, that contribution is too large (see Lemma 1) and this was Elkies’ original approach to prove \( c \geq \sqrt{1/\pi} \) (see Lemma 1). The main novelty of our approach is to analyze the contribution coming from outside that interval. This leads to Corollary 2.

**Corollary 2.** We have

\[
a_n \geq (1 - o(1)) \cdot \sqrt{\frac{2^n}{\pi \sqrt{n}}}.\]

While Corollary 2 itself does not tell us anything new, the proof establishes a connection to probability theory which will be discussed in §2.2 and §2.3.

### 2.2. Proof of Corollary 2: Outline

We use Theorem 1. The first ingredient is a lower bound on how much the integrand contributes to the integral close to the origin where all the cosines are aligned.

**Lemma 1** (see Elkies [10]). Suppose that \( \{a_1, \ldots, a_n\} \) is a subset of the positive real numbers. Then

\[
\int_{|x| \leq \frac{1}{4\pi}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^{n} \cos (2\pi a_i x)^2 dx \geq (1 + o(1)) \cdot \frac{1}{2} \frac{1}{a_n} \frac{1}{\sqrt{\pi n}}.
\]

This Lemma in conjunction with Theorem 1 already shows \( c \geq \sqrt{1/\pi} \). The main new idea is to prove that contributions far away from the origin can also be analyzed and that they also contribute a substantial amount.

**Lemma 2.** Let \( c > 0 \). Suppose that \( \{a_1, \ldots, a_n\} \subset \mathbb{R}^+ \) has 1-separated subset sums and, for some \( \varepsilon > 0 \), we have \( a_n^2 \leq c \cdot n^{-2/3-\varepsilon} \sum_{i=1}^{n} a_i^2 \). Then, as \( n \to \infty \),

\[
\int_{|x| \geq \frac{1}{4\pi}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^{n} \cos (2\pi a_i x)^2 dx \geq (1 + o(1)) \frac{\sqrt{2} - 1}{2\sqrt{\pi}} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1/2}.
\]

We note that this lower bound can be bounded from below in terms of \( a_n \) using the trivial bound \( \sum_{i=1}^{n} a_i^2 \leq n \cdot a_n^2 \). Combining this with Theorem 1 and Lemma 1, we see that if all subset sums are 1-separated, then

\[
\frac{1}{2^n} \geq (1 + o(1)) \left( \frac{1}{2} \frac{1}{a_n} \frac{1}{\sqrt{\pi n}} + \frac{\sqrt{2} - 1}{2\sqrt{\pi}} \frac{1}{\sqrt{n} \cdot a_n} \right) = 1 + o(1) \frac{\sqrt{2/\pi} \cdot a_n}{\sqrt{\pi n} \cdot a_n}
\]

which shows \( c \geq \sqrt{2/\pi} \). Lemma 2 appears to be very technical but contains an interesting idea which will tell us something new. Lemma 2 can be written in a completely different way (Theorem 2) and this alternative formulation is also how we are going to prove Lemma 2.

### 2.3. Subset Sums and Gaussian Densities

Let \( A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^+ \) be a set of positive reals. As already indicated above, we consider the random variable \( X = \sum_{i=1}^{n} \varepsilon_i a_i \) where \( \varepsilon_i \in \{-1, 1\} \) independently and with equal likelihood (also known as Rademacher random variables). This random variable is distributed according to some probability measure \( \mu \) on \( \mathbb{R} \). Note that we can write

\[
X = \sum_{i=1}^{n} a_i + 2 \sum_{\varepsilon_i = 1}^{n} a_i.
\]
If the minimal distance between the sum of two different subsets of $A$ is 1, then the minimal distance between any two distinct values of $X$ is two. Moreover, by the subset sum condition, $X$ assumes $2^n$ distinct values which implies

$$
\mu = \frac{1}{2^n} \sum_{i=1}^{2^n} \delta_{x_i} \quad \text{where} \quad \min_{i \neq j} |x_i - x_j| = 2.
$$

Our main question of interest will now be whether $\mu$ is close to a Gaussian (and, if so, in what sense). Consider first a simple example: the set $\{1, 2, \ldots, 2^n - 1\}$. It is easy to see that all subset sums are distinct (the uniqueness of binary expansion) and, following the construction, we see that $\mu$ is supported on all $2^n$ odd numbers in $[-2^n, 2^n]$ roughly emulating a uniform distribution over that interval. A uniform distribution is not particularly close to a Gaussian overall. This will now be compared to a better construction: we take the first 22 terms induced by the Conway-Guy sequence [8] (where 22 was chosen so as to be ‘large’ while still computationally feasible). We end up with a set $\{a_1, \ldots, a_{22}\} \subset \mathbb{N}$ with distinct subset sums and $a_{22} = 1051905 \sim 0.51 \cdot 2^{21}$. The probability distribution of the associated random walk $\mu$ is shown in Figure 1. This is quite a bit closer to a Gaussian than uniform distribution would be. This is not a coincidence.

![Figure 1](image-url)

**Figure 1.** A histogram of the discrete measure $\mu$ derived from the first 22 terms from the Conway-Guy sequence.

We start by trying to understand which Gaussian we should compare the distribution $\mu$ to. A Gaussian is uniquely determined by mean and variance. Since $\mu$ is symmetric around the origin, the expectation is $\mathbb{E}X = 0$. Simultaneously, we have an explicit expression for the variance and

$$
\mathbb{E}(X^2) = \mathbb{E} \left( \sum_{i=1}^{n} \varepsilon_i a_i \right)^2 = \mathbb{E} \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j a_i a_j = \sum_{i=1}^{n} a_i^2.
$$

The probability density function of that Gaussian will be abbreviated as

$$
\gamma(x) = \frac{1}{\sqrt{2\pi}} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1/2} \exp \left( -\frac{x^2}{2} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1} \right).
$$
Note that $\gamma$ is a smooth function while $\mu$ is a singular measure. To facilitate a comparison between the two, we will introduce a smoothed version of $\mu$. Consider the normalized characteristic function $h(x) = (1/2) \cdot \chi_{[-1,1]}$. Since both $\mu$ and $h$ are probability measures, their convolution

$$(h * \mu)(x) = \frac{1}{2^n} \sum_{i=1}^{2^n} \chi_{[x_i-1,x_i+1]}(x)$$

is also a probability measure. We observe that $h * \mu$ is a sum of characteristic functions centered at the points $x_i$ at which $\mu$ is supported. Since $\mu$ is distributed over exponentially large scales, smoothing at scale 1 does not change any relevant characteristics. With this language in place, the second main result is as follows.

**Theorem 2.** Let $c > 0$. Suppose $\{a_1, \ldots, a_n\} \subset \mathbb{R} > 0$ has 1-separated subset sums and $a_n^2 \leq c \cdot n^{-1/2} \sum_{i=1}^{n} a_i^2$. Then, as $n \to \infty$, we have

$$\int_{\mathbb{R}} ((h * \mu)(x) - \gamma(x))^2 \, dx = \int_{|x| \geq \frac{1}{n^{1/2}}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^{n} \cos (2\pi a_i x)^2 \, dx + o(2^{-n}).$$

We emphasize that $h * \mu$ only assumes the values 0 and $2^{-n-1}$ (and the second value is assumed on $2^n$ intervals of length 2). This implies that

$$\int_{\mathbb{R}} (h * \mu)(x)^2 \, dx = \frac{1}{2^{n+1}} \quad \text{while} \quad \int_{\mathbb{R}} \gamma(x)^2 \, dx = \frac{1}{2\sqrt{n}} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1/2}.$$

We also remark that the probability density of the random walk behaving similarly to a Gaussian was already used by Dubroff, Fox and Xu who invoked the Berry-Esseen theorem. Under a slightly stronger assumption ($a_n^2 \leq c \cdot n^{-2/3 - \epsilon} \sum_{i=1}^{n} a_i^2$) the Berry-Esseen theorem guarantees that

$$\sup_{x \in \mathbb{R}} \left| \mu([-\infty, x]) - \int_{-\infty}^{x} \gamma(y) \, dy \right| = o(1)$$

which shows convergence of the cumulative distribution functions. Theorem 2 establishes that sets with distinct subset sums satisfy (using Theorem 1)

$$\int_{\mathbb{R}} ((h * \mu)(x) - \gamma(x))^2 \, dx \leq \frac{1 + o(1)}{2^n}$$

measuring proximity of the probability density functions in the $L^2$-sense.

### 2.4. Concluding Remarks.

Theorem 2 has a fascinating implication insofar as it allows us to reinterpret the Erdős distinct subset sums problem (the general version with real numbers being 1-separated) as a genuine problem in probability theory asking whether particularly excellent random walks exist. More precisely, are there positive real numbers $a_1, \ldots, a_n > 0$ such that the random unbiased random walk $X = \pm a_1 \pm a_2 \cdots \pm a_n$ has, simultaneously, (1) a large standard deviation, (2) a small largest element $a_n$ and (3) the ability to approximate the normal distribution very well in a concrete sense?

**Problem.** Fix $c > 0$. As $n \to \infty$, are there random walks $X = \pm a_1 \pm a_2 \cdots \pm a_n$ such that the largest step size is small compared to the variance

$$\text{largest stepsize} = a_n \leq c \cdot n^{-1/3} \sqrt{\mathbb{E}X}.$$
and, simultaneously, $X$ has a large variance and approximates a Gaussian well in the sense of

$$\frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{VX}} + \int_{\mathbb{R}} ((h * \mu)(x) - \gamma(x))^2 \, dx = \frac{1 + o(1)}{2^{n+1}} \gamma$$

If there exist $\{a_1, \ldots, a_n\} \subset \mathbb{N}$ with distinct subset sums and $a_n \lesssim n^{-1/3-\varepsilon} \cdot 2^n$, then such random walks do indeed exist: this follows from combining Theorem 1, Theorem 2 and the computation carried out after the proof of Lemma 1.

Note that, considering the constraint on $a_n$ being as small as possible and considering the structure of the first term, it does seem like one would like to have many of the $a_i$ to be roughly comparable to $a_n$. The Conway-Guy [8] sequence has this property: for each $\varepsilon > 0$ at least $n - c_\varepsilon \log n$ terms satisfy $a_i \geq (1 - \varepsilon) a_n$. We also observe that for sets of that type, where many of the $a_i$ are comparable in size to $a_n$, one can draw additional information from Theorem 1

$$\int_{\mathbb{R}} \left( \frac{\sin x}{x} \right)^2 n \prod_{i=1}^{\infty} \cos (a_i x)^2 \, dx = \frac{\pi}{2^n} .$$

The cosines are all aligned at $x = 0$, the contribution to the integral coming from close to the origin is really just a function of $\sum_{i=1}^{n} a_i^2$ (see the comment after the proof of Lemma 1) and fairly independent of the arithmetic structure. The next interesting point is $x = \pi/a_n$: if we have $a_i = (1 + o(1)) a_n$ for many $1 \leq i \leq n - 1$, then many of the cosines will still be aligned at $\pi/a_n$. The only way to avoid a large contribution is to have an $a_i \sim (1 + o(1)) a_n / 2$. So it is not inconceivable that Theorem 1 suggests a sort of multi-scale structure as being possibly favorable. The argument can then be continued for $x = k\pi/a_n$ for small $k \in \mathbb{Z}$. As $k$ gets larger, one would expect the cosines to decorrelate.

3. Proofs

3.1. Proof of Theorem 1.

Proof. As already mentioned, we will smooth $\mu$ by convolving with the normalized characteristic function $h(x) = (1/2) \cdot \chi_{[-1,1]}$. Since both $\mu$ and $h$ are probability measures, their convolution

$$(h * \mu)(x) = \frac{1}{2^n} \sum_{i=1}^{2^n} \frac{1}{2} \chi_{[x_i-1, x_i+1]}$$

is a probability measure. We observe that $h * \mu$ is a sum of characteristic functions and its $L^1$-norm is 1. Its $L^2$-norm is minimized if and only if these characteristic functions do not overlap which is equivalent to $\min_{i \neq j} |x_i - x_j| \geq 2$ and therefore, in turn, equivalent to all subset sums being at least distance 1 from
each other. Formally,
\[
\| h \ast \mu \|_{L^2}^2 = \frac{1}{4^n} \int_\mathbb{R} \left( \sum_{i=1}^{2^n} \frac{1}{2} \chi_{[x_i-1,x_i+1]} \right)^2 dx \\
= \frac{1}{4^n} \int_\mathbb{R} \sum_{i,j=1}^{2^n} \frac{1}{2} \chi_{[x_i-1,x_i+1]} \frac{1}{2} \chi_{[x_j-1,x_j+1]} dx \\
\geq \frac{1}{4^n} \int_\mathbb{R} \sum_{i=1}^{2^n} \frac{1}{2} \chi_{[x_i-1,x_i+1]} \frac{1}{2} \chi_{[x_i-1,x_i+1]} dx \\
= \frac{1}{4^n} 2^{n-1} = \frac{1}{2^{n+1}}.
\]
This is the only inequality in the entire argument and is attained if and only if all \(x_i\) are 2-separated. Using that the Fourier transform is unitary on \(L^2\) and sends convolution to products,
\[
\| \mu \ast h \|_{L^2}^2 = \| \hat{\mu} \ast \hat{h} \|_{L^2}^2 = \int_\mathbb{R} \hat{h}(\xi) \hat{\mu}(\xi)^2 d\xi.
\]
It remains to compute the Fourier transforms: the Fourier transform of the characteristic function \(h\) is completely explicit
\[
\hat{h}(\xi) = \frac{\sin (2\pi \xi)}{2\pi \xi}.
\]
The measure \(\mu\) can itself be defined as a convolution
\[
\mu = \left( \frac{\delta_{-a_1}}{2} + \frac{\delta_{a_1}}{2} \right) \ast \left( \frac{\delta_{-a_2}}{2} + \frac{\delta_{a_2}}{2} \right) \ast \cdots \ast \left( \frac{\delta_{-a_n}}{2} + \frac{\delta_{a_n}}{2} \right).
\]
Using again that the Fourier transform sends convolution to products and
\[
\left( \frac{\delta_{-a_i}}{2} + \frac{\delta_{a_i}}{2} \right)(\xi) = \frac{e^{2\pi i (-a_i)\xi}}{2} + \frac{e^{2\pi i a_i \xi}}{2} = \cos (2\pi a_i \xi)
\]
leads to
\[
\hat{\mu}(\xi) = \prod_{i=1}^n \cos (2\pi a_i \xi).
\]
Thus
\[
\int_\mathbb{R} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^n \cos (2\pi a_i x)^2 dx \geq \frac{1}{2^{n+1}}
\]
with equality if and only if all subset sums of \(\{a_1, \ldots, a_n\}\) are 1-separated. \(\square\)

### 3.2. Proof of Corollary 1.

**Proof.** If all \(a_i\) are integers, then the product is 1-periodic and, together with
\[
\sum_{k \in \mathbb{Z}} \left( \frac{\sin 2\pi (x - k)}{2\pi (x - k)} \right)^2 = \frac{1 + \cos (2\pi x)}{2},
\]
this implies
\[
2 \int_\mathbb{R} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^n \cos (2\pi a_i x)^2 dx = \int_0^1 (1 + \cos (2\pi x)) \prod_{i=1}^n \cos (2\pi a_i x)^2 dx.
\]
To further evaluate the integral, we switch back to exponentials and note that
\[
\cos (2\pi a_i x)^2 = \frac{1}{2} + \frac{e^{4\pi i a_i x} + e^{-4\pi i a_i x}}{4}
\]
leading to the integral
\[
\frac{1}{2^n} \int_0^1 \left(1 + \frac{e^{2\pi i x} + e^{-2\pi i x}}{2}\right) \prod_{i=1}^n \left(1 + \frac{e^{4\pi i a_i x} + e^{-4\pi i a_i x}}{2}\right).
\]
Selecting the constant 1 in all terms leads to a contribution of \(2^{-n}\). Any other choice of combinations from the big product leads to exponentials of the form \(\exp(4\pi i k x)\) where \(k \in \mathbb{Z} \setminus \{0\}\) whenever all subset sums are distinct. Thus every other contributions leads to 0 and
\[
\int_0^1 \prod_{i=1}^n \cos (2\pi a_i x)^2 dx \geq \frac{1}{2^n}
\]
if all subset sums are distinct. Conversely, if not all subset sums are distinct, then there is a corresponding choice of combinations in the product leading to a zero frequency: since all coefficients are nonnegative, we see that the integral will then be larger than \(2^{-n}\).

3.3. Proof of Lemma 1.

Proof, close to Elkies [10]. Note that, for example, \(a_n \geq 2^n/n\), already implies that the interval is very close to the origin where \(\sin (2\pi x)/(2\pi x) \sim 1\) and thus
\[
\int_{|x| \leq \frac{1}{\sqrt{n}}} \left(\frac{\sin 2\pi x}{2\pi x}\right)^2 \prod_{i=1}^n \cos (2\pi a_i x)^2 dx = (1 - o(1)) \int_{|x| \leq \frac{1}{\sqrt{n}}} \prod_{i=1}^n \cos (2\pi a_i x)^2 dx.
\]
On this interval, we have, for all \(1 \leq i \leq n - 1\) that \(\cos (2\pi a_i x) \geq \cos (2\pi a_n x)\) and
\[
\int_{|x| \leq \frac{1}{\sqrt{n}}} \prod_{i=1}^n \cos (2\pi a_i x)^2 dx \geq \int_{|x| \leq \frac{1}{\sqrt{n}}} \cos (2\pi a_n x)^{2n} dx
\]
A change of variables and evaluating the integral (see [10]) shows that
\[
\int_{|x| \leq \frac{1}{\sqrt{n}}} \cos (2\pi a_n x)^{2n} dx = \frac{1}{2\pi a_n} \int_{|x| \leq \frac{\pi}{2}} \cos (x)^{2n} dx
\]
\[
= \frac{1}{2\pi a_n} \pi \binom{2n}{n}
\]
\[
= (1 + o(1)) \frac{1}{2} \frac{1}{a_n} \frac{1}{\sqrt{n}}
\]
where evaluating \(\int \cos (x)^{2n} dx\) in terms of binomial coefficients is classical [16].

This implies a lower bound on \(a_n\) since
\[
\frac{1}{2^{n+1}} = \int_{\mathbb{R}} \left(\frac{\sin 2\pi x}{2\pi x}\right)^2 \prod_{i=1}^n \cos (2\pi a_i x)^2 dx
\]
\[
\geq \int_{|x| \leq \frac{1}{\sqrt{n}}} \left(\frac{\sin 2\pi x}{2\pi x}\right)^2 \prod_{i=1}^n \cos (2\pi a_i x)^2 dx \geq (1 + o(1)) \frac{1}{2} \frac{1}{a_n} \frac{1}{\sqrt{n}}
\]

showing that \( a_n \geq 2^n/\sqrt{\pi n} \) which is, in spirit, the original argument of Elkies. We note that, provided \( a_n \) is small, i.e. \( a_n = o(\sum_{i=1}^{n} a_i^2) \), one can Taylor expand the cosines and, for \( x \) small,

\[
\prod_{i=1}^{n} \cos (2\pi a_i x)^2 dx \sim \exp \left(-4\pi^2 x^2 \sum_{i=1}^{n} a_i^2\right)
\]

which then leads to the slightly refined estimate

\[
\int_{|x| \leq \frac{1}{\pi n}} \prod_{i=1}^{n} \cos (2\pi a_i x)^2 dx dx \geq \frac{1 + o(1)}{2\sqrt{\pi}} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1/2}.
\]

At this point, we do not know of any argument that excludes the possibility that \( n - o(n) \) of the \( a_i \) satisfy \( a_i = (1 + o(1))a_n \) and this refined estimate does not currently lead to any information different from that provided by the cruder estimate above. Indeed, the Conway-Guy sequence is an example of a set with distinct subset sums and this type of behavior, perhaps extremal configurations do behave like that.

3.4. Technical Lemma. The goal of this section is to establish an upper bound on the difference between \( \hat{\mu} \) and the approximating Gaussian measure close to the origin. Lemma 3 will then quickly imply Theorem 2.

**Lemma 3.** Let \( c > 0 \). Suppose \( \{a_1, \ldots, a_n\} \subset \mathbb{R}_{>0} \) has 1-separated subset sums and \( a_i^2 \leq c \cdot n^{-1/2} \sum_{i=1}^{n} a_i^2 \). Then, as \( n \to \infty \), we have

\[
\int_{|x| \leq \frac{1}{\pi n}} \left| \frac{\sin (2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos (2\pi a_i x) - \exp \left(-2\pi^2 x^2 \sum_{i=1}^{n} a_i^2\right) \right|^2 dx = o(2^{-n}).
\]

**Proof.** The first step is a Taylor expansion around \( x = 0 \)

\[
\frac{\sin (2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos (2\pi a_i x) = \frac{\sin (2\pi x)}{2\pi x} \exp \left( \sum_{i=1}^{n} \log (\cos (2\pi a_i x)) \right)
\]

\[
\leq \frac{\sin (2\pi x)}{2\pi x} \exp \left( \sum_{i=1}^{n} \log \left( 1 - 2\pi^2 a_i^2 x^2 + O(a_i^4 x^4) \right) \right)
\]

\[
= \frac{\sin (2\pi x)}{2\pi x} \exp \left( \sum_{i=1}^{n} -2\pi^2 a_i^2 x^2 + O(a_i^4 x^4) \right)
\]

\[
= e^{O(x^2 + na_i^4 x^4)} \exp \left(-2\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right).
\]

The goal is to bound

\[
X = \int_{|x| \leq \frac{1}{\pi n}} \left| \frac{\sin (2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos (2\pi a_i x) - \exp \left(-2\pi^2 x^2 \sum_{i=1}^{n} a_i^2\right) \right|^2 dx
\]

which, considering asymptotic expansion, can be bounded as

\[
X \leq \int_{|x| \leq \frac{1}{\pi n}} \left| e^{O(x^2 + na_i^4 x^4)} - 1 \right|^2 \exp \left(-4\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) dx.
\]

This bound by itself is a little bit too crude but is reasonably close to the origin; note that the integrand is the product of two functions the first of which is small.
for small values of \( x \) and the second of which is small for large \( x \). This suggests splitting the integral into the regions: for some \( 0 < \delta < 1/(4a_n) \) to be optimized later, we write \( I_1 = \{ x : |x| \leq \delta \} \) and let \( I_2 = \{ x : \delta \leq |x| \leq 1/(4a_n) \} \). Provided \( \delta^2 + na_n^2\delta^4 = O(1) \), we can estimate the integral over \( I_1 \) as

\[
Y = \int_{I_1} e^{O(x^2 + na_n^4x^4)} - 1 \left| \int_\mathbb{R} \exp \left( -4\pi^2 x^2 \sum_{i=1}^n a_i^2 \right) dx \right|^2 \leq O(\delta^2 + na_n^4\delta^4) \cdot \int_\mathbb{R} \exp \left( -4\pi^2 x^2 \sum_{i=1}^n a_i^2 \right) dx = O(\delta^2 + na_n^4\delta^4) \cdot \frac{1}{2\sqrt{\pi}} \left( \sum_{i=1}^n a_i^2 \right)^{-1/2}. 
\]

We use a different type of expansion for the second region: note that, for \( |x| < \pi/2 \),

\[
\log (\cos (x)) \leq -\frac{x^2}{2}
\]

and thus, for \( |x| < 1/(4a_n) \), \( \cos (2\pi a_i x) \leq \exp (-2\pi^2 x^2 a_i^2) \) from which we deduce

\[
\forall |x| \leq \frac{1}{4a_n} \quad 0 \leq \prod_{i=1}^n \cos (2\pi a_i x) \leq \exp \left( -2\pi^2 x^2 \sum_{i=1}^n a_i^2 \right).
\]

Therefore the contribution of the integrand to \( X \) over \( I_2 \), which is

\[
Z = \frac{\prod_{i=1}^n \cos (2\pi a_i x) \left\| \sin (2\pi x) \right\|^2}{2\pi x} dx,
\]

can be trivially bounded from above by

\[
Z \leq \int_{I_2} \exp \left( -2\pi^2 x^2 \sum_{i=1}^n a_i^2 \right) \left\| \sin (2\pi x) \right\|^2 dx \leq \frac{2}{\delta} \int_{\delta}^\infty x \exp \left( -4\pi^2 x^2 \sum_{i=1}^n a_i^2 \right) dx = \frac{2}{\delta} 8\pi^2 \sum_{i=1}^n a_i^2 \exp \left( -4\delta^2 \pi^2 \sum_{i=1}^n a_i^2 \right) \leq \frac{1}{\delta} \frac{1}{\sum_{i=1}^n a_i^2} \exp \left( -\delta^2 \sum_{i=1}^n a_i^2 \right).
\]

We want all error estimates to be \( o(2^{-n}) \) and achieve this by setting

\[
\delta = \alpha_n \left( \sum_{i=1}^n a_i^2 \right)^{-1/2}
\]

with \( \alpha_n \) an arbitrarily slowly growing sequence (think of \( \alpha_n = \log \log \log n \)). We start by checking whether our first asymptotic expansion is valid in this regime, i.e. whether \( \delta^2 + na_n^2\delta^4 = O(1) \). Moser’s estimate implies \( \sum_{i=1}^n a_i^2 \geq 4^n \) and thus

\[
\delta^2 + na_n^2\delta^4 = O(\alpha_n^2 4^{-n}) + O(na_n^2 \alpha_n^4 4^{-2n}) = o(2^{-n}).
\]

The next step is an estimate on \( Y \). Importing our upper bound on \( a_n \) shows

\[
Y \leq O(\delta^2 + na_n^4\delta^4) \left( \sum_{i=1}^n a_i^2 \right)^{-1/2} \leq na_n^4 \alpha_n^4 \left( \sum_{i=1}^n a_i^2 \right)^{-5/2} \leq n^{-\varepsilon} \left( \sum_{i=1}^n a_i^2 \right)^{-1/2}
\].
which is $\mathcal{O}(n^{-2}2^{-n}) = o(2^{-n})$. Finally, for the last error term,

$$Z \leq \frac{1}{\delta} \frac{1}{\sum_{i=1}^{n} a_i^2} \exp \left( -\delta^2 \sum_{i=1}^{n} a_i^2 \right) \leq 10 \frac{2^{-n}}{\alpha_n} e^{-\alpha_n^2} = o(2^{-n}).$$

This proves Lemma 3. □

3.5. Proof of Theorem 2.

Proof. Theorem 2 is a relatively easy consequence of Lemma 3. Recall that

$$\gamma(x) = \frac{1}{\sqrt{2\pi}} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1/2} \exp \left( -\frac{x^2}{2} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1} \right)$$

and $\hat{\gamma}(x) = \exp(-2\pi^2 x^2 \sum_{i=1}^{n} a_i^2)$. Using the Fourier transform we get that

$$X = \int_{\mathbb{R}} |(h \ast \mu)(x) - \gamma(x)|^2 dx = \int_{\mathbb{R}} |(h \ast \mu)(x) - \hat{\gamma}(x)|^2 dx$$

can be written as

$$X = \int_{\mathbb{R}} \left| \frac{\sin(2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos(2\pi a_i x) - \exp \left( -2\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) \right|^2 dx.$$

Lemma 3 implies that if $\{a_1, \ldots, a_n\} \subset \mathbb{R}_{>0}$ has 1-separated subset sums and $a_n^2 \leq c \cdot n^{-1/2} \sum_{i=1}^{n} a_i^2$, then

$$\int_{|x| \leq \frac{1}{4\pi n}} \left| \frac{\sin(2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos(2\pi a_i x) - \exp \left( -2\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) \right|^2 dx = o(2^{-n})$$

implying that, by splitting the integral into $\{|x| \leq 1/(4a_n)\}$ and $\{|x| \geq 1/(4a_n)\}$,

$$X = \int_{|x| \geq \frac{1}{4a_n}} \left| \frac{\sin(2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos(2\pi a_i x) - \exp \left( -2\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) \right|^2 dx + o(2^{-n}).$$

Using the upper bound on $a_n$

$$\int_{|x| \geq 1/(4a_n)} \exp \left( -4\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) dx \leq 8a_n \int_{1/(4a_n)}^{\infty} x \exp \left( -4\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) dx$$

$$\leq \frac{a_n}{\sum_{i=1}^{n} a_i^2} \exp \left( \frac{\pi^2}{4} \sum_{i=1}^{n} a_i^2 \right)$$

$$\leq \frac{1}{n^{1/4}} \frac{1}{(\sum_{i=1}^{n} a_i^2)^{1/2}} e^{-c\sqrt{n}}.$$

With Moser’s estimate $\sum_{i=1}^{n} a_i^2 \geq 4^n$ one deduces

$$\left\| \exp \left( -2\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) \chi_{|x| \geq \frac{1}{4a_n}} \right\|_{L^2(\mathbb{R})} \lesssim e^{-c\sqrt{n}} \frac{1}{2^n}.$$
Using the triangle inequality in $L^2$, we see that
\[
Z = \left\| \frac{\sin (2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos (2\pi a_i x) - \exp \left( -2\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) \chi_{|x| \geq \frac{1}{4an}} \right\|_{L^2(\mathbb{R})}
\]
\[
= \left\| \frac{\sin (2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos (2\pi a_i x) \chi_{|x| \geq \frac{1}{4an}} \right\|_{L^2(\mathbb{R})} + O \left( e^{-c\sqrt{n} \frac{1}{2^m}} \right).
\]
Squaring both sides and using Theorem 1, we deduce
\[
X = Z^2 + o(2^{-n}) = \int_{|x| \geq \frac{1}{4an}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^{n} \cos (2\pi a_i x) dx + o(2^{-n}).
\]

3.6. Proof of Lemma 2. Throughout this proof, we will abbreviate
\[
\gamma(x) = \frac{1}{\sqrt{2\pi}} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1/2} \exp \left( -\frac{x^2}{2} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1} \right)
\]
for the Gaussian approximating $\mu$. Before proving Lemma 2, we quickly recall the Berry-Esseen theorem which, in our setting, says that
\[
\sup_{x \in \mathbb{R}} \left| \mu([-\infty, x]) - \int_{-\infty}^{x} \gamma(x) dx \right| \leq \frac{\sum_{i=1}^{n} a_i^3}{(\sum_{i=1}^{n} a_i^2)^{3/2}}.
\]
Assuming that $a_n^2 = O \left( n^{-2/3} \sum_{i=1}^{n} a_i^2 \right)$, one can bound this by
\[
\frac{\sum_{i=1}^{n} a_i^3}{(\sum_{i=1}^{n} a_i^2)^{3/2}} \leq \frac{n \cdot a_n^3}{(\sum_{i=1}^{n} a_i^2)^{3/2}} \leq n^{-\frac{3}{2}} = o(1).
\]
The way we will use this information is that, for any interval $J \subset \mathbb{R}$
\[
\left| \mu(J) - \int_{J} \gamma(x) dx \right| = o(1).
\]
Note that this argument was also used by Dubroff, Fox and Xu [9] for $J$ an interval centered at the origin whose length is proportional to a small multiple of the standard deviation of the Gaussian. We will quickly summarize their short argument at an appropriate place in the proof of Lemma 2.

Proof of Lemma 2. We start the argument with a lower bound on
\[
X = \int_{\mathbb{R}} |(\mu * h)(x) - \gamma(x)|^2 dx.
\]
Taking a Fourier transform,
\[
X = \int_{\mathbb{R}} \left| \frac{\sin (2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos (2\pi a_i x) - \exp \left( -2\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) \right|^2 dx.
\]
We split the integral into two regions: $|x| \leq 1/(4a_n)$ and the remaining region. Lemma 3 implies that the integral over the first region is $o(2^{-n})$, it remains to
analyze the integral over the second region. Arguing exactly as in the proof of Theorem 2, we deduce that

\[
X = \int_{|x| \geq \frac{1}{4^n}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^n \cos (2\pi a_i x)^2 dx + o(2^{-n}).
\]

The next argument is completely independent of all the previous arguments: we will derive a lower bound on the same quantity via a completely different argument which will then imply Lemma 2. Recall that

\[
X = \int_{\mathbb{R}} |(\mu * h)(x) - \gamma(x)|^2 dx.
\]

\(\mu * h\) only assumes the values \(\{0, 2^{-n-1}\}\). Moreover, by the argument above,

\[
\sup_{J \subseteq \text{interval}} \left| \int_J (\mu * h)(x) dx - \int_J \gamma(x) dx \right| = o(1).
\]

This leads to an amusing setting: we know that \(\mu * h\) approximates the Gaussian in probability over intervals. Simultaneously, \(\mu * h\) can only assume two values one of which is 0: thus, the local density of the Gaussian predicts the density of intervals in the region where \(\mu * h\) assumes its nonzero value \(2^{-n-1}\). An example of what this could look like is shown in Fig. 2. We conclude with a simple proposition.

**Proposition.** Let \(\mu\) be the probability density function of a \(\mathcal{N}(0, \sigma^2)\) Gaussian. Let \((\nu_n)\) be a sequence of probability density functions such that

1. \(\nu_n \to \mu\) in probability: for every interval \(J \subset \mathbb{R}\) we have

   \[
   \lim_{n \to \infty} \int_J \nu_n(x) dx = \int_J \mu(x) dx
   \]

2. \(\nu_n(x)\) only assumes two values \(\{0, z_n\}\) for some \(z_n > 0\).

Then

\[
\lim inf_{n \to \infty} \int_{\mathbb{R}} (\mu(x) - \nu_n(x))^2 dx \geq \frac{\sqrt{2} - 1}{2\sqrt{\pi} \sigma}.
\]

**Proof of the Proposition.** The density of \(\mu\) is simply given by

\[
\mu(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left( -\frac{x^2}{2\sigma^2} \right).
\]

We note that both Properties combined require (by taking \(J\) to be a small interval centered around the origin) that

\[
\lim inf_{n \to \infty} z_n \geq \max_{x \in \mathbb{R}} \mu(x) = \frac{1}{\sqrt{2\pi} \sigma}.
\]

Let now \(J\) be a small interval centered around \(x_0 \in \mathbb{R}\), say \(J = (x_0 - \varepsilon, x_0 + \varepsilon)\). The two properties combined tell us what can be expected of \(\nu_n\): since

\[
\int_J \mu(x) dx = 2\varepsilon \mu(x_0) + O(\varepsilon^2)
\]

we have

\[
\lim_{n \to \infty} \int_J \nu_n(x) dx = \int_J \mu(x) dx = 2\varepsilon \mu(x_0) + O(\varepsilon^2).
\]
This allows us to deduce that the fraction \( \alpha \) of the interval \( J \) where \( \nu_n \) assumes the value \( z_n \) and the remaining fraction \( 1 - \alpha \) where it assumes the value 0 is determined by

\[
\alpha z_n = \mu(x_0) + \text{lower order terms}.
\]

This tells us that

\[
\int_{\mathbb{R}} (\mu(x) - \nu_n(x))^2 dx = (1 + o(1)) \int_{\mathbb{R}} \frac{\mu(x) - z_n}{z_n} (\mu(x) - z_n)^2 + \left(1 - \frac{\mu(x)}{z_n}\right) \mu(x)^2 dx.
\]

The integral algebraically simplifies to

\[
\int_{\mathbb{R}} \frac{\mu(x)}{z_n} (\mu(x) - z_n)^2 + \left(1 - \frac{\mu(x)}{z_n}\right) \mu(x)^2 dx = \int_{\mathbb{R}} \mu(x) (z_n - \mu(x)) dx.
\]

At this point, we recall that, up to lower order terms, \( z_n \geq \mu(0) \). Thus

\[
\int_{\mathbb{R}} \mu(x) (z_n - \mu(x)) dx \geq \int_{\mathbb{R}} \mu(x) \left( \frac{1}{\sqrt{2\pi\sigma}} - \mu(x) \right) dx = \frac{1}{\sqrt{2\pi\sigma}} - \frac{1}{2\sqrt{\pi\sigma}}
\]

which is the desired result. □

**Figure 2.** A step function assuming only two values approximating a Gaussian density.

At this point can we quickly note, in passing, the original argument of Dubroff, Fox and Xu [9]: the Gaussian attains its maximum density at the origin and therefore

\[
\gamma(0) = \frac{1}{\sqrt{2\pi}} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1/2} \leq (1 + o(1)) \cdot \| \mu \ast h \|_{L^\infty} = \frac{1 + o(1)}{2^{n+1}}
\]

from which one deduces

\[
\sqrt{n} \cdot a_n \geq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \geq (1 + o(1)) \sqrt{\frac{2}{\pi}} \cdot 2^n.
\]

We can now conclude by applying the Proposition. The variance \( \sigma \) of the mollified random walk is, up to lower order terms, given by the variance of the random walk which is \( \sum_{i=1}^{n} a_i^2 \). Thus, applying the Proposition, as \( n \) becomes large,

\[
X \geq (1 + o(1)) \frac{\sqrt{2} - 1}{2\sqrt{\pi} \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2}}.
\]

□
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