Solution of the dynamics of liquids in the large-dimensional limit

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We obtain analytic expressions for the time correlation functions of a liquid of spherical particles, exact in the limit of high dimensions \(d\). The derivation is long but straightforward: a dynamic virial expansion for which only the first two terms survive, followed by a change to generalized spherical coordinates in the dynamic variables leading to saddle-point evaluation of integrals for large \(d\). The problem is thus mapped onto a one-dimensional diffusion in a perturbed harmonic potential with colored noise. At high density, an ergodicity-breaking glass transition is found. In this regime, our results agree with thermodynamics, consistently with the general Random First Order Transition scenario. The glass transition density is higher than the best known lower bound for hard sphere packings in large \(d\). Because our calculation is, if not rigorous, elementary, an improvement in the bound for sphere packings in large dimensions is at hand.

Introduction – The physics of liquids and glasses belongs to the group of fields that are victims of the lack of a small parameter. Many approximations have been proposed over the years, but they suffer from the uncertainty about what is the limit in which they are supposed to become exact. This has been true both for equilibrium and for dynamic properties. From the point of view of dynamics, an extreme case is that of Mode-Coupling Theory (MCT) \cite{1–3}: it may be introduced by an (uncontrolled) resummation of an infinite subset of diagrams. The Mode-Coupling approximation yields Mode-Coupling dynamics: the phenomenology depends on the approximation itself \cite{4}, somewhat like a harmonic approximation is expected to predict oscillations.

An often used remedy to the absence of a small parameter \cite{5} is to promote the system to \(d\) dimensions, solve the large \(d\) limit, and (eventually) expand around. This strategy has been used with success for liquids \cite{6–8}, strongly coupled electrons \cite{9}, atomic physics \cite{10}, gauge field theory \cite{11}, and most recently the thermodynamics of amorphous systems \cite{12, 13}. In this paper we extend this procedure to the dynamics of liquids made of spherical particles. We restrict ourselves to equilibrium, although the extension to the glassy off-equilibrium “aging” regime is at hand. It has been a long-standing question whether MCT becomes exact in infinite dimension \cite{4, 14–18}, and the present computation gives an answer.

Statement of the main result – We shall consider a system of \(N\) identical particles, interacting via a spherical potential \(V(r)\) of typical interaction length \(\sigma\) in \(d\) dimensions, and obtain a solution for the equilibrium time correlations of the resulting liquid, that becomes exact in the limit \(d \to \infty\). We need to confine the particles in a finite volume \(\mathcal{V}\). It is very convenient to do this in such a way that the “box” does not break rotational and translational invariances, which are crucial in our developments. A practical way to do this is to consider particles living on points \(x_i\) \(i = 1, \ldots, N\) on the \(d\)-dimensional surface of a hypersphere with \(x_i \cdot x_i = \sum_{\mu} |x_i^\mu|^2 = R^2 \equiv \sigma^2 \Delta_{i\text{ii}}/(2d) \gg \sigma^2\), \(\mu = 1, \ldots, d + 1\). The thermodynamic limit \(R \to \infty\) with constant density \(\rho = N/\mathcal{V}\), in which the flat space is recovered, will be taken before \(d \to \infty\). Rotations and translations in \(d\) dimensions – with dimensions \(d(d-1)/2\) and \(d\), respectively – are encoded in the rotations in \(d + 1\) dimensions, with dimension \(d(d+1)/2\). We consider a Langevin dynamics

\[
\dot{\mathbf{x}}_i(t) + \gamma \mathbf{x}_i(t) = -\nabla_{\mathbf{x}} V(x_i(t)) - \nabla_{x_i} H + \mathbf{ξ}_i(t),
\]

where \(\mathbf{ξ}_i\) is a white noise with \(
\langle \xi_i^\mu(t) \xi_j^\nu(t') \rangle = 2 T \delta_{ij} \delta_{\mu\nu} \delta(t-t'),
\)

\(H = \sum_{i<j} V(x_i - x_j)\) and \(T = 1/\beta\) is the temperature. Here, and in what follows, \(\langle \cdot \rangle\) denotes average over noise \(\mathbf{ξ}_i\) and/or initial conditions. The \(\nu_i\) are Lagrange multipliers, imposing the spherical constraints \(x_i \cdot \hat{x}_i = 0\). For \(d \to \infty\), they do not fluctuate and their value \(\nu_i \sim \nu \equiv dTP/\sigma^2\) is proportional to the equilibrium reduced pressure \(p = \beta P/\rho\) \cite{19}. We shall in what follows treat the overdamped case \(m = 0\), but the inertial term may be reinstalled at any stage (and in that case the thermal bath may be disconnected setting \(\gamma = 0\), to recover a purely Newtonian dynamics).

We define the adimensional scaled correlation and response functions (see Fig.1 in the Appendix):

\[
C(t, t') \sim C_i(t, t') = \frac{2d}{\sigma^2} \tau_i(t) \cdot x_i(t'),
\]

\[
R(t, t') \sim R_i(t, t') = \frac{2d}{\sigma^2} \sum_{\mu} \frac{\Delta x_i^\mu(t)}{\Delta h_i^\mu(t')},
\]

\[
\Delta(t, t') \sim \frac{d}{\sigma^2} |x_i(t) - x_i(t')|^2 = \Delta_{i\text{ii}} - C(t, t'),
\]

where \(h_i\) is an external field conjugated to \(x_i\), and we have also introduced the single particle mean square displacement \(\Delta\) \cite{19}. As we shall see, these quantities have a finite and non-fluctuating limit for \(d \to \infty\). In the following and in the Appendix, we shall derive exact equations for the correlations (2). For simplicity, we will restrict to equilibrium where \(C(t, t') = C(t - t')\), and \(R(t) = -\beta \theta(t) \dot{C}(t)\); here \(\theta(t)\) is the Heaviside step function.

Our main result is that correlation functions are obtained in terms of a memory kernel \(M\) through the fol-
lowing equations:
\[ \dot{\rho}(t) = \frac{-T}{\Delta_{\text{liq}}} \rho(t) - \beta \int_0^t du \, M(t-u) \dot{\rho}(u), \]
\[ \dot{\Delta}(t) = \Delta - \beta \int_0^t du \, M(t-u) \dot{\Delta}(u), \] (3)
the second relation being valid for \( \Delta \ll \Delta_{\text{liq}}. \)

To give the expression of the memory kernel \( M \), let us define a scaled density \( \tilde{\rho} = \rho V_d(\sigma)/d \) where \( V_d(\sigma) \) is the volume of a \( d \)-dimensional sphere of radius \( \sigma \), a scaled \( \tilde{\gamma} = \sigma^2 \gamma/(2d^2) \), a scaled potential \( \tilde{V}(y) = V(\sigma(1+y/d)) \), and the inter-particle force \( F(y) = -\tilde{V}'(y) \). \( M \) is then obtained in terms of a one-dimensional effective dynamics with a colored noise \( \gamma \),
\[ \dot{\gamma}(t) = -u'(y(t)) - \beta \int_0^t du \, M(t-u) \dot{\gamma}(u) + \zeta(t), \] (4)
\[ \langle \zeta(t)\zeta(t') \rangle = 2\gamma T \delta(t-t') + M(t-t'), \]
from the self-consistent equation:
\[ M(t-t') = \frac{\tilde{\gamma}}{2} \int dy_0 \, e^{-\beta w(y_0)} \langle F(t)F(t') \rangle_{M,y_0}. \] (5)
The average in (5) is over the process (4) with \( y(t=0) = y_0 \). The total effective potential has the form (see Fig.4 in the Appendix):
\[ w(y_0) = \tilde{V}(y_0) - T y_0 + \frac{T y_0^2}{\Delta_{\text{liq}}}. \] (6)
The quadratic part of the potential plays the role of the confining “box”. In fact, it is negligible for finite times and large \( \Delta_{\text{liq}} \), and the probability distribution is exponential in \( y \) near \( y \sim 0 \) (expressing the growth of entropy as a function of distance along the \( d+1 \)-dimensional sphere), the region relevant for those times. Reassuringly, box details are irrelevant at short times.

Finally, Eq. (5) has a microscopic counterpart (cf. Appendix); in fact, the function \( M(t) \) is, in the large \( d \) limit, the autocorrelation of the inter-particle forces \( F_{ij}(t) = -\nabla V(x_i(t) - x_j(t)) \),
\[ M(t) = \frac{1}{2dN} \sum_{i \neq j} \langle F_{ij}(t) \cdot F_{ij}(0) \rangle, \] (7)
a result that provides a physical interpretation of \( M(t) \).

Relation with MCT – If \( M \) were a simple function of \( C, M = F(C) \), then Eq. (3) would be in the schematic MCT form. As is well known, schematic MCT is obtained as the exact dynamics of a system of spherical spins \( \sum_i s_i^2 = N \) with \( p \)-spin random interactions [20–22], for which
\[ M = F(C) = \frac{p}{2} C^{p-1}. \] (8)
However, as soon as one considers non-spherical variables, e.g. soft-spins with a potential \( V(s) = a(s^2 - 1)^2 \), one obtains an equation like (4) with this \( V(s) \) [21, 23, 24]:
\[ \dot{x}(t) = -V'(s) - \beta \int_0^t du \, M(t-u) \dot{x}(u) + \zeta(t). \] (9)
Here again, Eq. (8) holds and the system is closed by \( C(t-t') = \langle x(t)x(t') \rangle \). Within the liquid phase, this more general form of dynamic equation has essentially the same phenomenology as schematic MCT. Our system of equations belongs to this more general class, and they thus show exactly the same MCT phenomenology for what concerns universal quantities that are independent of details of the memory kernel (e.g. the dynamical scaling forms and the relations between critical exponents) [3, 25, 26].

However, important quantitative differences are observed with respect to applying the MCT approximation to the intermediate scattering function, which leads to the “standard” formulation of MCT for liquids [1, 2]. Standard MCT has the same qualitative structure as schematic MCT, but also provides quantitative results for the self and collective scattering functions in all dimensions, in particular in \( d = 3 \) [27, 28]; its \( d \to \infty \) limit was discussed in Refs. [15, 16]. Our result in \( d \to \infty \) is formulated in terms of \( \Delta(t,t') \), and most of the other natural observables are functionals of \( \Delta(t,t') \). For example, for \( q \sigma /d^{3/2} \ll 1 \), we have for the self intermediate scattering function (cf. Appendix):
\[ \phi_s^q(t,t') = \exp \left[ -\frac{q^2 \sigma^2}{2d^2} \Delta(t,t') \right], \] (10)
in contrast to the non-Gaussianity in \( q \) one finds within MCT close to the plateau [15, 16]. One could then write our equations in terms of \( \phi_s^q \). The result, however, is different from standard MCT and in particular our kernel \( M \) is not an analytic function of \( \phi_s^q \).

Because our equations fall in the same universality class as schematic MCT, but provide different quantitative results with respect to standard MCT in \( d \to \infty \), it remains a matter of taste if one wishes to call them the “standard” formulation of MCT for liquids [1, 2]. Standard MCT has the same qualitative structure as schematic MCT, but also provides quantitative results for the self and collective scattering functions in all dimensions, in particular in \( d = 3 \) [27, 28]; its \( d \to \infty \) limit was discussed in Refs. [15, 16]. Our result in \( d \to \infty \) is formulated in terms of \( \Delta(t,t') \), and most of the other natural observables are functionals of \( \Delta(t,t') \). For example, for \( q \sigma /d^{3/2} \ll 1 \), we have for the self intermediate scattering function (cf. Appendix):
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Sketch of the derivation – Let us outline the main steps in the derivation; more details are given in the Appendix. In order to construct the high-dimensional limit, a virial expansion is a reliable method. Following [29], we exploit the well-known analogy between trajectories and polymers. The dynamics are generated by a sum of trajectories in \( d \)-dimensional space, in our case, the surface of a \( d+1 \)-dimensional sphere. To each trajectory is associated an Onsager-Machlup probability weight which is the exponential of an action. The sum over all trajectories of this quantity is analogous to a partition function and is thus used to generate averages over the Langevin process (1). The action is expressed in terms of trajectories of the \( x_i \) and auxiliary “response” variables \( \dot{x}_i \) (Martin-
Siggia-Rose-De Dominicis-Janssen generating path integral) [20, 22]. The result reads, in the Itô convention:

\[ Z_N = \prod_{i=1}^{N} D\tilde{x}_i e^{-\sum_{i=1}^{N} \Phi[x_i, \tilde{x}_i] - \sum_{1 \leq i < j \leq N} W[x_i - x_j, \tilde{x}_i - \tilde{x}_j]} , \]

\[ \Phi[x, \tilde{x}] = \gamma \int dt \left( T \dd + \nu \cdot \dd + \nu \cdot x \right) , \]

\[ W[x_1 - x_2, \tilde{x}_1 - \tilde{x}_2] = i \int dt (\tilde{x}_1 - \tilde{x}_2) \cdot \nabla V(|x_1 - x_2|) . \]

Following standard liquid theory [19, 29], we introduce the density function for trajectories

\[ \rho[x, \tilde{x}] = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta[x_i - x_i] \delta[\tilde{x}_i - \tilde{x}_i] \right\rangle , \]  

(12)

where \( \delta[x] \) is the functional Dirac \( \delta \) (a product of deltas over all times) and we construct a virial (Mayer) expansion as a power series in \( \rho[x, \tilde{x}] \). One can show that all terms involving a product of more than two density fields are subleading for \( d \to \infty \) [8]: they are exponentially suppressed because of the requirement that the three trajectories overlap (see Fig.1 in the Appendix), which is exponentially unlikely in large \( d \). Truncating the virial expansion accordingly, we get (cf. Appendix):

\[ S \equiv \frac{\ln Z_N}{N} = -\int D\rho[x, \tilde{x}] (\Phi[x, \tilde{x}] + \ln \rho[x, \tilde{x}]) \]

\[ + \frac{N}{2} \int D\rho[x_1, \tilde{x}_1] D\rho[x_2, \tilde{x}_2] f[x_1 - x_2, \tilde{x}_1 - \tilde{x}_2] , \]

(13)

where \( D\rho[x, \tilde{x}] = D[x, \tilde{x}] \rho[x, \tilde{x}] \) and \( f[x_1 - x_2, \tilde{x}_1 - \tilde{x}_2] = e^{-W[x_1 - x_2, \tilde{x}_1 - \tilde{x}_2]} - 1 \). The physical \( \rho[x, \tilde{x}] \) is determined by \( \delta S/\delta \rho[x, \tilde{x}] = 0 \) and the normalization \( \int D\rho[x, \tilde{x}] = 1 \). The first term in Eq. (13) is an ideal gas contribution and the second accounts for interactions.

Following the thermodynamic treatment [30], we may now argue that due to rotational invariance on the hypersphere, \( \rho[x(t), \tilde{x}(t)] = \rho[C(t, t'), R(t, t'), D(t, t')] \) where \( R(t, t') \equiv (2d/\sigma^2)x(t) \cdot \dd(t') \) and \( D(t, t') \equiv (2d/\sigma^2)\dd(t) \cdot \dd(t') \). We can thus make a change of variables in the functional integration over \( x(t), \tilde{x}(t) \) to \( Q(t, t') \equiv \{C(t, t'), R(t, t'), D(t, t')\} \). The change of variables gives for density averages (cf. Appendix):

\[ \int D[x, \tilde{x}] \cdot \rho \to \int DQ \cdot e^{dstr \ln Q + d\Omega(Q)} , \]

(14)

where \( e^{dstr \ln Q} \) is the Jacobian of the transformation (cf. Appendix) and we defined \( \rho(Q) = e^{d\Omega(Q)} \). The appearance of the dimension in the exponent leads to a narrowing of fluctuations of correlations, and saddle-point evaluation becomes exact (cf. Appendix). In this way we can compute the ideal gas term in Eq. (13). For the interaction term, that involves two \( \rho \) functions, we need the variables corresponding to \( (x_1, \tilde{x}_1), (x_2, \tilde{x}_2) \) and also \( \omega = |x_1 - x_2|^2, \dot{\omega} = (x_1 - x_2) \cdot (\dd_1 - \dd_2) \). The Jacobian may be calculated with the same methods (cf. Appendix), and the crucial result is that at the saddlepoint level \( Q_1 = Q_2 = Q \) with \( Q \) determined by the same saddle-point as in Eq. (14), while the remaining integration over \( \omega(t), \dot{\omega}(t) \) is effectively one-dimensional. Changing variables with \( \omega(t) = \sigma^2(1 + y(t)/d) \) leads to a finite integration over \( y(t) \) that eventually gives Eq. (4). Hence, the typical distance between two trajectories turns out to be \( \sigma + O(1/d) \). This scaling physically tells us that a particle vibrate inside a cage with \( 1/d \) amplitude and interacts with \( O(d) \) neighbors. Finally, equilibrium and causality at the saddle-point level imply \( D(t, t') = 0 \) and the Fluctuation-Dissipation relation (FDT) \( \langle R(t, t') \rangle = \beta \theta(t-t') \delta_{t,t'} \). The saddle-point evaluation for the two-time variables is the mathematical justification of the above-mentioned fact that \( C, R, D \) do not fluctuate. These saddle-point equations give Eq. (5) and (3) (cf. Appendix).

In this paper we are treating an equilibrium situation. Within the liquid phase, this may be achieved by starting from any configuration in the distant past. A more practical way, however, is to assume equilibrium at a convenient time \( t_0 \) (e.g. \( t_0 = 0 \)). How does one deal with a non-Markovian equation of motion like (4)? The answer is simple: either one makes the memory kernel extend to the remote past, or, alternatively, one may assume equilibrium at \( t_0 \), in other words summing all the past histories passing through \( y(t_0) \) at \( t_0 \). It turns out [31] that this is implemented simply by cutting the memory at a lower limit \( t_0 \), as in Eq. (4) (cf. Appendix). This completes the derivation of our basic dynamical equations.

Dynamic transition and timescale separation – We now apply the standard MCT methodology to locate the density or temperature at which a dynamic transition occurs with freezing in a cage, corresponding to the development of a plateau in \( \Delta(t) \) [1–3, 22].

Consider the case when \( M(t) \) falls from \( M(0) \) to a plateau value \( M_{\text{EA}} \), and then, at much larger times, to zero. Concomitantly, \( \Delta(t) \) grows to a plateau value \( \Delta_{\text{EA}} \), and then continues to grow at a slower (diffusive) pace. Denote the fast part \( \delta M(t) = M(t) - M_{\text{EA}} \). In the limit in which the plateau times are much larger than the microscopic times, and much smaller than the final relaxation times, the noise breaks into a fast variable \( \delta \zeta(t) \) and a slow random variable \( \zeta \), as does the friction term. Their sum acts as an adiabatically slow field \( Y(t) \) at those times. We may thus split the equilibrium in two steps [32]: \( P(y|Y) \) and \( P_{\text{slow}}(Y) \). When \( t - t' \) is in the plateau region, we may write the expectations:

\[ \langle A(t)B(t') \rangle = \int dY P_{\text{slow}}(Y) \langle A \rangle_Y \langle B \rangle_Y , \]

(15)
where \( \langle \cdot \rangle_Y = \int dY \cdot P(y | Y) \), \( P_N(y) \propto e^{-\beta w_N(y)} \), and
\[
P(y | Y) = \frac{P_N(y) e^{-\frac{2 \pi \rho}{\sigma^2} (y - \frac{P_N(Y)}{2})^2}}{\int dy' P_N(y') e^{-\frac{2 \pi \rho}{\sigma^2} (y' - \frac{P_N(Y)}{2})^2}},
\]
\[
P_{\text{slow}}(Y) = \frac{\int dY P_N(y) e^{-\frac{2 \pi \rho}{\sigma^2} (y - \frac{P_N(Y)}{2})^2}}{\sqrt{2 \pi \rho \sigma^2}}.
\]

Obviously, \( P_N(y) = \int dY P(y | Y) P_{\text{slow}}(Y) \). We therefore obtain the self-consistent equation for \( M_{\text{EA}} \):
\[
M_{\text{EA}} = \frac{\hat{\zeta}_d}{2} \int dY P_{\text{slow}}(Y) \langle F \rangle_Y^2 \equiv \mathcal{M}(M_{\text{EA}}).
\]

From Eq. (3) we obtain
\[
\beta^2 M_{\text{EA}} = \frac{1}{\Delta_{\text{EA}}} - \frac{1}{\Delta_{\text{iq}}} \sim \frac{1}{\Delta_{\text{EA}}}.
\]

The dynamical transitional point, at which the plateau becomes infinite, happens when Eq. (17) first has a non-zero solution for \( M_{\text{EA}} = T^2/\Delta_{\text{EA}} \). This point happens as a bifurcation, and may be quickly obtained by solving Eq. (17) together with \( \mathcal{M}(M_{\text{EA}}) = 0 \) (cf. Appendix).

For hard spheres, the result is \( \hat{\zeta}_d = 4.807 \). This result is fully consistent with the one based on thermodynamics [12], and in fact Eq. (17) is exactly identical to the one that can be derived using the replica method (cf. Appendix), consistently with the general RFOT scenario [14, 33-38]. This is a particular instance of a general correspondence between thermodynamic and dynamical results that is verified by the infinite-d solution, and can be extended to critical MCT exponents [39] and to correlation functions [40-42]. In fact, expanding around \( \Delta_{\text{EA}} [1-3] \), one can show that \( \Delta_{\text{EA}} - \Delta(t) \sim t^{-a} \) upon approaching the plateau, while \( \Delta(t) - \Delta_{\text{EA}} \sim b \) upon leaving the plateau. The exponents \( a, b \) satisfy the famous relation [1-3, 39, 41]:
\[
\frac{\Gamma(1-a)^2}{\Gamma(1-2a)} = \frac{\Gamma(1+b)^2}{\Gamma(1+2b)} = \lambda.
\]

For hard spheres, we obtain \( \lambda = 0.707 \) which implies \( a = 0.324 \) and \( b = 0.629 \) [43]. Finally, from Eq. (10) one can show that the factorization property of MCT [1, 2] still holds in \( d \to \infty \), namely that close to the plateau,
\[
\phi^\beta_q(t) - \phi^\beta_{q,\text{EA}}(\Delta(t) - \Delta_{\text{EA}}) \sim -\frac{\gamma^2 T}{2d} \phi^\beta_{q,\text{EA}}(T^d D/\sigma^2) t
\]

in a function of \( q \) and a function of \( t \) (cf. Appendix).

**Diffusion, viscosity, Stokes-Einstein relation** – At long times, in the liquid phase \( \hat{\varphi} < \hat{\varphi}_d \), the motion is diffusive and \( \Delta(t) \sim (2d^2 D/\sigma^2) t \), where \( D \) is the diffusion coefficient. Plugging this form in Eq. (3), and recalling that \( M(t) \) decays to zero over a finite time, we obtain an exact result for \( D \):
\[
\frac{2d^2}{\sigma^2} D = \frac{T}{\gamma + \beta \int_0^\infty dt M(t)}.
\]

At low density \( M(t) = 0 \) and we recover the bare diffusion coefficient \( D = T/\gamma \). Upon increasing density, \( M(t) \) increases and \( D \) decreases. For \( \hat{\varphi} \to \hat{\varphi}_d \), where a finite plateau of \( M(t) \) emerges, \( \int_0^\infty dt M(t) \) diverges and the diffusion coefficient vanishes as \( D \sim (\hat{\varphi}_d - \hat{\varphi})^\gamma \) with the exponent \( \gamma = 1/(2a) + 1/(2b) = 2.34 \), which is consistent with the numerical results of [44].

Within linear response theory, the shear viscosity \( \eta_s \) of the liquid is given by [19, 45]
\[
\eta_s = \beta \int_0^\infty dt \langle \sigma_{\mu\nu}(t)\sigma_{\mu\nu}(0) \rangle,
\]
where \( \mu \neq \nu \) are two arbitrary components of the stress tensor \( \sigma_{\mu\nu} \). In the Appendix we show that \( \langle \sigma_{\mu\nu}(t)\sigma_{\mu\nu}(0) \rangle = d N M(t) \), and thus
\[
\eta_s = \beta d \int_0^\infty dt M(t).
\]

This relation shows that for \( \hat{\varphi} \to \hat{\varphi}_d \), \( \eta_s \sim 1/D \sim (\hat{\varphi}_d - \hat{\varphi})^{-\gamma} \), as it is found in MCT.

Combining Eqs. (20) and (22) we obtain a relation similar to the Stokes-Einstein relation (SER)
\[
D = \frac{T}{\gamma + \frac{2d}{\rho \sigma^2} \eta_s} \approx \frac{T \rho \sigma^2}{2d} \frac{1}{\eta_s},
\]
where the second expression holds close to \( \hat{\varphi}_d \). This relation is interesting because it predicts that the SER is not exactly satisfied in dense liquids: the quantity \( D \eta_s/T \), which is constant in SER, has instead a small variation proportional to \( \rho \). This is in agreement with results of [46, Fig.7b] that show a linear variation of \( D \eta_s/T \) with \( \rho \) in the dense regime for large enough dimension.

**Conclusions** – In this work we obtained an exact solution of the dynamics of liquids in the limit of infinite spatial dimension. The picture that emerges has a relevant “caging” lengthscale that is smaller than the particle radius by \( O(1/d) \) (it is about 1/5 for real colloids [47]). The physics of diffusion stems from interactions at that small scale, while all particle motion beyond that scale consists of uncorrelated steps of displacement, and memory of what happens at distances \( \gg 1/d \) is lost. Diffusion coefficients and viscosity are thus decided at distances much smaller than the particle radius. This strongly suggests that the wavevectors \( q \) that matter for this transition are not the ones associated with the first neighbor distance \( 1/\sigma \), but rather the much larger ones \( \sim d/\sigma \) corresponding to the cage size. Note that at the transition cages are correlated over large distances \( [33, 48-50] \). Also, an expansion in the number of collisions [7] seems difficult to reconcile with our results, because we expect multiple collisions within a cage.

Is the high-dimensional dynamics related to MCT? The answer is that the result is not the one obtained from the usual procedure for building up a MCT equation, which for example gives a different scaling of \( \hat{\varphi}_d \).
with dimension [15, 16]. Instead, the system we obtain is formally quite close to the slightly more general case of soft-spin mean-field dynamics, Eq. (9), because we have mapped the system into a one-dimensional dynamics in the presence of a colored noise and friction, that have to be determined self-consistently through Eqs. (4) and (5).

In the dense glassy regime we obtain predictions for the scaling of the cage radius, of the dynamical transition density $\hat{\varphi}$, and of the parameter $\lambda$, that differ from the ones of usual MCT [15, 16, 18]. Our results are fully consistent with those obtained from the thermodynamic approach [12, 13, 43], which proves the exactness of the RFOT scenario [33–38] for statics and dynamics in $d \to \infty$, as conjectured in [14]. Our results are also in agreement with numerical simulations of hard spheres in large spatial dimension [44, 46].

Interestingly, we find that an ergodic liquid phase of hard spheres exists for densities $\hat{\varphi} \leq \hat{\varphi}_d = 4.807$. This implies that hard sphere packings exist (at least) up to $\hat{\varphi}_d$, and they can be constructed easily through a sufficiently slow compression of the liquid [51, 52]. Note that the value of $\hat{\varphi}_d$ is larger than the best known lower bound for the existence of sphere packings, $\hat{\varphi} \geq 6/e$ [53], and that it took 20 years to improve the previous best lower bound $\hat{\varphi} \geq 2$ [54] by a small factor $3/e$. Our calculation is simple enough that there is hope to transform it into a rigorous proof, along the lines of [55]. This would result in an improved constructive lower bound for sphere packings.

Future extensions of this work include the investigation of the effect of dissipation [56–59], the study of out-of-equilibrium aging dynamics [60], and the study of non-perturbative processes in $1/d$ through an instantonic expansion. The thermodynamic partition function of quantum systems is formally very similar to Eq. (27) and could also be studied along these lines.

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# APPENDIX

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References
I. FORMULATION OF THE DYNAMICS

A. Definition of the model

We consider an assembly of $N$ spheres of mass $m$ interacting through a repulsive finite-ranged radial pair potential $V(r)$. The interaction Hamiltonian is then $H = \sum_{i < j} V(x_i - x_j)$ with $x_i(t)$ the positions of the particles. $\sigma$ is the diameter of the spheres, we note $V(\mu) = V(\sigma(1 + \mu / d))$ and the rescaled force $F(\mu) = -V'(\mu)$. In the following, we define the usual inverse temperature $\beta = 1 / k_B T$ and take $k_B = 1$. In order to exploit efficiently simplifications given by the translational and rotational symmetries of the system, we constrain each particle to live on the surface of a $d + 1$ dimensional hypersphere $S^d(R)$ of radius $R$. A particle is thus represented by a point $x_i \in \mathbb{R}^{d+1}$ with $x_i^2 = R^2$. The volume of this space is $V = \Sigma_{d+1}(R) = \Omega_{d+1} R^d$. $\Sigma$ and $\Omega$ are respectively the surface and the solid angle. We also denote by

$$V_d(R) = \frac{\Sigma_{d}(R)R}{d} = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} R^d$$

the volume of a $d$-dimensional ball of radius $R$ (i.e. the volume bounded by $S^d(R)$ is $V_{d+1}(R)$). Adding this extra dimension, rotation and translation invariances in usual $d$ dimensional space are transposed to rotational invariance only on the sphere $S^d(R)$. We recover in the limit $R \to \infty$ the original definition in a $d$ dimensional periodic cubic volume. However, note that in subsections IB and IC, we consider the original model in $d$ dimensional Euclidean space, so as to introduce the spherical model only when needed.

B. The dynamical action

A possible choice of the dynamics of the particles in the $d$-dimensional space is the following Langevin process:

$$m \ddot{x}_i + \gamma \dot{x}_i = -\sum_{j \neq i} \nabla V(x_i - x_j) + \xi_i$$

$\xi_i(t)$ being a centered white Gaussian noise with $\langle \xi_i^n(t) \xi_i^{n'}(t') \rangle = 2 \gamma T \delta_{ij} \delta_{\mu \nu} \delta(t - t')$. In the following, we will consider the overdamped\(^1\) case where $m = 0$. Given a set of initial conditions, the Martin-Siggia-Rose-De Dominicis-Janssen (MSRDDJ) \cite{20,22} generating path integral reads, with Itô convention:

$$Z_N = \int \prod_{i=1}^{N} Dx_i D\dot{x}_i e^{-\mathcal{A}[\{x_i, \dot{x}_i\}]}$$

where the action is:

$$\mathcal{A}[\{x_i, \dot{x}_i\}] = \sum_{i=1}^{N} \Phi[x_i, \dot{x}_i] + \sum_{i < j}^{1,N} W[x_i, \dot{x}_i, x_j, \dot{x}_j]$$

$$\Phi[x, \dot{x}] = \gamma \int dt \left( T \dot{x}^2 + i \dot{x} \cdot \dot{x} \right), \quad W[x, \dot{x}, y, \dot{y}] = i \int dt (\dot{x} - \dot{y}) \cdot \nabla V(x - y) = W[x - y, \dot{x} - \dot{y}]$$

The sum in $Z_N$ is done over all possible trajectories, with the measure given by $Dx_i = \prod_{n=1}^{M} \frac{dx_i^n}{(2\pi)^{d/2}}$ when discretizing the trajectory $x_i(t)$ in $M$ time steps. Time integrals are taken over an interval noted $[t_p, t_1]$ in section IV where the $\{x_i(t_p)\}$ are fixed (initial conditions, which would correspond to $n = 0$ in the latter discretization) and the $\{x_i(t_1)\}$ are summed over (which would correspond to $n = M$). In section IV we will need averages of the type $\langle x_i(t_0) x_i(t_1) \rangle$ with $t_0 \in [t_p, t_1]$, and it is not needed to consider times larger than $t_1$ in the action owing to causality, as they will give no contribution to the average by probability conservation (i.e. for the same reason as $Z_N = 1$).

The action $\mathcal{A}$ is rotation invariant (for both position and response fields using the same global rotation) and translation invariant along positions\(^2\). Because of the so-called kinetic term, it is not translation invariant along the response fields (which are already centered at the origin owing to the white noise), though the interaction term is.

\(^1\) Either Newtonian or Brownian dynamics can be treated by changing the kernel associated to $\Phi$ in the following sections.

\(^2\) The invariances for response fields follow if one studies the system modulo ‘rigid’ rotations and translations of the entire system.
FIG. 1. (a) The different dynamical regimes, described by equation (107): most of the dynamics is determined at the cage size scaling 1/d. We assume that the diffusive regime found already at this scale extends trivially at longer times. Concerning the slope at very short times, the mean-square displacement is $\propto t^2$ if inertia is not neglected; for purely overdamped motions it is $\propto t$. Finally, the particles feel the ‘box’ for distances of order $R$, hence a saturation at this scale. (b) The dynamic triangle diagram with three trajectories.

C. Derivation of the generating functional using a virial expansion

1. Dynamic virial

We define $\Xi = \sum_{N=0}^{+\infty} \frac{1}{N!} Z_N$ in order to use the Mayer expansion\(^3\) as in liquid theory [19]. This ‘grand canonical’ form is handy as a generating functional, but note that we assume there is no exchange of particles with a reservoir; $\ln \Xi$ and $\ln Z_N$ are related by a Legendre transform and virial expansions are more fruitful with the former. We have:

$$\Xi = \sum_{N=0}^{+\infty} \frac{1}{N!} \prod_{i=1}^N D[x_i, \hat{x}_i] z[x_i, \hat{x}_i] \prod_{i<j} (1 + f[x_i, \hat{x}_i, x_j, \hat{x}_j])$$

with a generalized fugacity $z = e^{-\Phi}$ and a Mayer function $f = e^{-W} - 1$. We Legendre transform $\ln \Xi$ with respect to $N \ln z$ since one has

$$\frac{\delta \ln \Xi}{\delta (N \ln z[x, \hat{x}])} = \left( \frac{1}{N} \sum_{i=1}^N \delta(x - x_i) \delta(\hat{x} - \hat{x}_i) \right) \equiv \rho[x, \hat{x}]$$

where the mean is generated by the functional $\Xi$. Next, the usual Mayer expansion can be carried out in this dynamical case, and inverting the Legendre transform, $\ln \Xi$ can be written as an ideal gas contribution and the sum of all connected 1-irreducible Mayer diagrams\(^4\) with $N \rho[x, \hat{x}]$ nodes and $f[x - y, \hat{x} - \hat{y}]$ bonds [19]:

$$\ln \Xi = -N \int D[x, \hat{x}] \rho[x, \hat{x}] (\Phi[x, \hat{x}] + \ln \rho[x, \hat{x}]) + \text{...}$$

In infinite dimension, the Mayer expansion reduces to its first term. However, this is strictly true if we assume that we are in a regime where the trajectories have the time to wander away only a finite fraction of the box. Because we expect (and confirm) that all interesting dynamics (namely, the $\beta$ relaxation with the formation of a plateau

\(^3\) This is why $W$ is cast in a symmetric form so that links in the product $\prod_{i<j}$ are not directed.

\(^4\) 1-irreducible means that they do not disconnect upon removal of a node.
in correlations and the onset of a relaxation towards equilibrium - \( \alpha \) relaxation -) occur on such scales, where the fluctuation around the initial position is of amplitude \( O(1/d) \) (a scaling consistent with the statics [12, 13, 30, 43]). We will show that one even gets the diffusive behaviour which is already decided at the 1/d scale (see figure 1(a)).

To justify this truncation, let us consider for example the second term of this expansion (triangle diagram):

\[
\Delta = \frac{N^3}{6} \int D[x, \hat{x}] D[y, \hat{y}] D[z, \hat{z}] \rho[x, \hat{x}] \rho[y, \hat{y}] \rho[z, \hat{z}] f[x - y, \hat{x} - \hat{y}] f[y - z, \hat{y} - \hat{z}] f[z - x, \hat{z} - \hat{x}]
\]  

(31)

For a finite-support potential \( V \), \( \nabla V(x - y) \propto \Theta(\sigma - |x - y|) \) (Heaviside’s step function\(^5\)), hence

\[
f[x - y, \hat{x} - \hat{y}] = \begin{cases} 
0 & \text{if } \forall t, |x(t) - y(t)| > \sigma \\
\tau[x - y, \hat{x} - \hat{y}] & \text{if } \exists t, |x(t) - y(t)| < \sigma = (\Theta(|x - y| - \sigma) - 1) \tau[x - y, \hat{x} - \hat{y}]
\end{cases}
\]  

(32)

Essentially, \( \tau \) is just a numerical value depending on the details of \( V \) (with \(|\tau| \leq 2\) and the physical content of \( f \) lies in its finite support, i.e.

\[
f[x - y, \hat{x} - \hat{y}] \simeq \Theta(|x - y| - \sigma) - 1 = -1 + \prod_{n=1}^{M} \Theta(|x^n - y^n| - \sigma)
\]  

(33)

For finite times, a trajectory stays in a bounded region of space, represented as a ball of diameter the typical size \( \sigma/d \) of the trajectory in figure 1(b). The Mayer functions require that each couple of trajectories get closer than \( \sigma \) at some time. To each set of three trajectories one can associate a corresponding static diagram with three overlapping balls (figure 1(b)). One can see this static diagram as the equivalence class of all dynamic diagrams with trajectories contained inside these balls. Actually there are lots of trajectories contained by these bounding balls that do not contribute because they do not get close enough. Then the sum over trajectories of the value of the integrand is at the truncation used here is valid largely above this bound.

\[
\mathcal{S} = \frac{\ln \Xi}{N} = -\int D[x, \hat{x}] \rho[x, \hat{x}] [\Phi[x, \hat{x}] + \ln \rho[x, \hat{x}]] + \frac{N}{2} \int D[x, \hat{x}] D[y, \hat{y}] \rho[x, \hat{x}] \rho[y, \hat{y}] f[x - y, \hat{x} - \hat{y}] 
\]  

(34)

with \( \delta \mathcal{S}/\delta \rho[x, \hat{x}] = 0 \) from the Legendre transform and the normalization \( \int D[x, \hat{x}] \rho[x, \hat{x}] = 1 \) We neglected purely additive constants irrelevant for the dynamics.

2. The Mayer expansion in infinite dimension

We discuss here the truncation of the Mayer expansion when \( d \to \infty \) in a more pedestrian way. As in IC1, we focus on the triangle diagram in (31) with the Mayer function given by (33).

Consider two typical trajectories \( x(t) \) and \( y(t) \) which we expect from the static case to dominate the dynamics at large \( d \), and let us focus on the positions \( x^1 \) and \( y^1 \) (any other couple would do). These typical trajectories scale as follows: the other positions of the trajectory \( \{x^n\}_{n \neq 1} \) (respectively \( \{y^n\}_{n \neq 1} \) fluctuate around \( x^1 \) (respectively \( y^1 \)) over a neighborhood of size \( 1/d \), see figures 1(a),(b). Besides, the typical distance between the two trajectories is the diameter \( \sigma \) (plus \( O(1/d) \) fluctuations). Then essentially \( \prod_{n=1}^{M} \Theta(|x^n - y^n| - \sigma) \simeq \Theta(|x^1 - y^1| - \sigma) \) except in the neighborhood \( |x^1 - y^1| \in [\sigma(1 - \frac{d}{a}), \sigma(1 + \frac{d}{a})] \) where these fluctuations of order \( 1/d \) matter (\( \mu \) is of order 1).

\(^5\) In the main text it is noted \( \theta \) but this notation will be used for Grassmann variables from subsection IE on.

\(^6\) The critical packing fraction later found in (147) is of order \( O(d/2^\theta) \) where it is known that the virial series does not converge anymore; nevertheless, Frisch & Percus [8] have shown that in high \( d \) the truncation used here is valid largely above this bound.

\(^7\) In the thermodynamic limit \( \ln \Xi/N = \ln Z_N/N \) (as if formally the chemical potential was zero in the static partition functions analogy). That is why for simplicity we used \( \ln Z_N/N \) in equation (13) of the main text.
However this neighborhood has to be accounted for since its volume \( V_d(\sigma)(e^{\mu} - e^{-\mu}) \), is comparable to the volume \( V_d(\sigma)e^{-\mu} \) of \([x^1 - y^1] \in [0, \sigma(1 - \frac{\rho}{\sigma})]\) where the contribution of \( f[x - y, \hat{x} - \hat{y}] \) is non zero (for \(|x^1 - y^1| > (1 + \frac{\rho}{\sigma})\), \( f[x - y, \hat{x} - \hat{y}] \) is zero).

Hence \( f[x - y, \hat{x} - \hat{y}] \cong f^*(x^1 - y^1) \) with \( f^*(x) \) being -1 for \(|x| < \sigma(1 - \frac{\rho}{\sigma})\), of order 1 for \(|x| > \sigma(1 - \frac{\rho}{\sigma})\), \( \sigma(1 + \frac{\rho}{\sigma})\) and 0 further. What counts for the truncation done in [6] is that \( f^*(x) \) is comparable to the volume \( V_d(\sigma)(e^{\mu} - e^{-\mu}) \) since \( \rho \cong \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \rho(x) dx \). The term in equation (31) reduces to

\[
\int \frac{dx^2}{(2\pi)^{\frac{d}{2}}} \cdots \frac{dx^M}{(2\pi)^{\frac{d}{2}}} \hat{\rho}(x^1, \ldots, x^M, \hat{x}) = \rho(x^1) = \text{constant} = \frac{(2\pi)^{\frac{d}{2}}}{V}
\]

since \( \rho(x^1) = \left\langle \frac{1}{N} \sum_{i=1}^N \delta(x^1 - x_i^1) \right\rangle \) must be a translation invariant function of \( x^1 \) i.e. a constant. \( V \) is the volume of the liquid. So the second term in the expansion is actually approximated for large dimension by

\[
\int dxdydz f_{HS}(x - y)f_{HS}(y - z)f_{HS}(z)
\]

where we defined the average particle density \( \rho = N/V \). We recognize the same term as in the usual static Mayer expansion of hard spheres and can conclude as in IC 1 from [6].

**D. Spherical setup**

From now on, we constrain the particles to live on the surface of a sphere of radius \( R \) embedded in \( d + 1 \) dimensional space, \( \mathbb{S}^d(R) \), see section IA. The field \( x(t) \) is promoted to \( x : \mathbb{R} \rightarrow \mathbb{R}^{d+1} \) and \( \forall t, \ (x(t))^2 = R^2 \) must be verified. Concerning the response field \( \hat{x}(t) \), we will rather constrain it to be orthogonal to the position field: \( \forall t, \ (x(t)) \cdot \hat{x}(t) = 0 \), thus living at each time in the hyperplane tangential to the sphere at \( x(t) \), cf. figure 2. This way we ensure that we will recover rotation and translation invariances for position fields and only rotation invariance for response fields once \( R \rightarrow \infty \) is taken9.

There are many possible ways to enforce these constraints, which are eventually equivalent:

1. One strategy is to use Lagrange multipliers via a term \( -\nu_i(t)x_i(t) \) in the right-hand side of (25), promoted to \( d + 1 \) dimensions, which ensures that the trajectory does not get out of the sphere due to interactions or thermal noise [22]. It is the one adopted in the main text for concision.

In \( d \rightarrow \infty \), the value of \( \nu \sim \frac{1}{\gamma} \) is non-fluctuating and is obtained by discretizing (25) as follows (in the Itô sense):

\[
\bar{x}_i(t + dt) = \bar{x}_i(t) - \frac{1}{\gamma} \nu_i(t) \bar{x}_i(t) dt - \frac{1}{\gamma} \nabla \bar{x}_i H dt + \frac{1}{\gamma} \eta_i,
\]

\[
\langle \eta_i^\mu \eta_j^\nu \rangle = 2\gamma T^\mu_\nu \delta_{ij} \delta_{\mu\nu}
\]

---

8 More generally bounded by a finite and independent of \( d \) quantity (for example here it is bounded by \( 2\sigma \) for large enough \( d \)).

9 Another way to see it is that the spherical constraint implies \( \dot{x}_i, x_i = 0 \) and only tangential fields \( \hat{x}_i \) are needed to exponentiate the Langevin equation in the MSRDDJ path integral.
At order $dt$, using that for large $d$ one has $B \cdot \eta_t \to 0$ (for any vector $B$ uncorrelated with $\eta_t$ in the Itô sense) and $\eta \cdot \eta_t \sim 2dT\gamma dt$ due to the central limit theorem, we impose the constraint

$$R^2 = x_i(t + dt) \cdot x_i(t + dt) = x_i(t) \cdot x_i(t) - \frac{2dt}{\gamma} x_i(t) \cdot \left[ \nu_i(t)x_i(t) + \nabla_x H \right] + \frac{2dTdt}{\gamma}$$  \hspace{1cm} (40)$$

and therefore

$$\nu_i(t) = -\frac{1}{R^2} x_i \cdot \nabla_x H + \frac{dT}{R^2}$$ \hspace{1cm} (41)$$

We have a general relation [19, Eq.(2.2.10)] for the reduced pressure $p$:

$$p \equiv \frac{\beta P}{\rho} = 1 - \frac{\beta}{dN} \left\langle \sum x_i \cdot \nabla_x H \right\rangle$$ \hspace{1cm} (42)$$

For $d \to \infty$ the fluctuations vanish because we average over $d$ dimensions, we thus have

$$\frac{1}{N} \sum x_i \cdot \nabla_x H \sim \frac{1}{N} \left( \sum x_i \cdot \nabla_x H \right) = dT(1 - p)$$ \hspace{1cm} (43)$$

and plugging this in Eq. (41) we obtain that all $\nu_i(t)$ are equal and constant in time, given by

$$\nu_i(t) \sim \nu = \frac{dT}{R^2} p$$ \hspace{1cm} (44)$$

From the static entropy in $d \to \infty$ [6, 8, 12, 29], $p = 1 + dA/2$ in the liquid phase, where $A$ is the rescaled packing fraction $A = 2d\varphi/d = \rho \nu d(\sigma)/d$ and the number density $\rho = N/V$.

This choice results in an additional term $\int dt \tilde{\nu} \hat{x} \cdot \hat{x}$ in the definition of $\Phi$ (27), and the summation of paths in (34) is over the sphere $S^d(R)$ for positions and the tangential hyperplane. In practice, we will integrate on the whole $d+1$ dimensional space and enforce the constraint through Dirac deltas. Note that, with $x$ (resp. $\hat{x}$) representing the position (resp. response) field at some time and $E = \text{Span}(x)$,

$$\int_{\mathbb{R}^{d+1}} dx \delta(x^2 - R^2) = \Omega_{d+1} \int_0^\infty d\rho \rho^d \delta(r^2 - R^2) = \Omega_{d+1} \frac{R^d}{2R} = \frac{V}{2R} = \frac{1}{2R} \int_V dx \int_{\mathbb{R}^{d+1}} d\hat{x} \delta(2x \cdot \hat{x})$$ \hspace{1cm} (45)$$

These choices rescale the path integral measures with respect to the ones on $\mathbb{R}^d$, which does not affect the dynamics. In the thermodynamic limit of infinite radius $R$ (with $\rho = N/V$ fixed), we recover the original $d$-dimensional space.

2. In the previous method, we would exponentiate the Dirac delta functions, giving additional terms in the exponent $\int dt \mu \hat{x} \cdot \hat{x}$ and $\int dt \tilde{\mu}(x^2 - R^2)$. We see that the Lagrange multiplier would just shift $\mu$. Hence, another technique is directly (and somewhat physically blindly) to promote the path integrals in (34) to $d+1$ dimensions and use Dirac deltas to constrain the $x$, $\hat{x}$ fields. This is what we are going to follow with fields $\nu$, $\tilde{\nu}$ but note that it is not exactly the Lagrange multiplier, even if it plays a similar role.

3. We might as well add a soft constraint $A \sum_i (x_i^2 - R^2)^2$ in the Hamiltonian. It would add a single-particle term $2A \int dt i\hat{x} \cdot \hat{x}(x^2 - R^2)$ to $\Phi$. Then we can write (still at a given time)

$$e^{-2Ai\hat{x} \cdot (x^2 - R^2)} \propto \int_{\mathbb{R}^{d+1}} d\tilde{\nu} \delta \left( \frac{\tilde{\nu}}{2A} - \hat{x} \cdot x \right) e^{-i\tilde{\nu}(x^2 - R^2)} \propto \int_{\mathbb{R}^{d+1}} d\nu d\tilde{\nu} e^{i\nu \hat{x} / A - 2i\nu x \cdot \hat{x} - i\tilde{\nu}(x^2 - R^2)}$$ \hspace{1cm} (46)$$

and in the hard limit $A \to \infty$ this is the same as enforcing the constraints through $\delta(x^2 - R^2)$, $\delta(2x \cdot \hat{x})$. 
E. Superspace notation

1. Translation of the dynamics into superfield language

As a compact way to write dynamical equations, we will use superspace notation [61, 62]. At any step, one can unfold this notation to recover the standard dynamical variables. \{θ₁, θᵢ\} are Grassmann variables\(^{10}\). Let us define \(\tilde{x} = i\dot{x}\) for convenience. We encode the position and response fields \([x, \tilde{x}]\) in a superfield\(^{11}\) \(x(a) = x(t) + \theta₁\thetaᵢ\tilde{x}(t)\), where arguments are denoted by \(a = (θ₁, θᵢ, t)\). The Mayer function and the kinetic part can be explicitly written:

\[
\begin{align*}
\Phi(x) & = \gamma \int da \frac{\partial x}{\partial θ₁} \cdot \left(T \frac{\partial x}{\partial θ₁} - θ₁ \frac{\partial x}{\partial t}\right) \\
\end{align*}
\]

\(x(a) = x(t)^2 + 2\tilde{θ}₁θᵢx(t) \cdot \tilde{x}(t)\) implies for the constraints that \(δ(x(t)^2 - R^2)δ(2x(t) \cdot \tilde{x}(t)) = δ(x(a)^2 - R^2)\).

The measure \(D[x, \tilde{x}]\) is replaced by \(Dx = D[x, \tilde{x}]\) where integration over \(\tilde{x}\) is on the ‘imaginary axis’ \(i\mathbb{R}^{d+1}\). Then the action can be written in the form:

\[
S = -\int Dx δ(x(a)^2 - R^2)ρ(x)(lnρ(x) + Φ(x)) + \frac{N}{2} \int Dx Dy δ(x(a)^2 - R^2)δ(y(b)^2 - R^2)ρ(x)ρ(y)f(x - y)
\]

still with \(\delta S/δρ(x) = 0\) and \(\int Dx δ(x(a)^2 - R^2)ρ(x) = 1\).

2. Definitions about superfields

Here we provide some useful generalization of usual field-theoretic tools and other identities in superfield language, used in the following sections. Proofs are not given here, but are easily obtained using the definitions and by direct computations.

1. **Superfields**: We will consider (one-component) superfields and define them as \(x(a) = x(t) + \tilde{θ}₁θᵢ\tilde{x}(t)\). We will also use operator (two components) superfields analogous to the replica case, such as \(q(a, b) = x(a) \cdot x(b)\). Similarly, an operator superfield \(r\) can be cast in the canonical expression with Grassmann variables and real scalar fields:

\[
r(a, b) = r₁(t, t') + \tilde{θ}₁θᵢ\tilde{r}₁(t, t') + \tilde{θ}₂θ₂\tilde{r}₂(t, t') + θ₁θᵢθ₂θ₂r₂(t, t') \tag{49}
\]

2. **Dirac deltas**: For superfields, they are simply functionally defined as \(δ(x(a)) = δ(x(t))δ(\tilde{x}(t))\). For operator superfields, they are simply functionally defined as a product of the functional deltas of their components appearing in notation (49). If the superfield is symmetric we need to introduce deltas only on the independent part, which is the case for \(q\).

3. **Path integral measure**: We clarify here the path integral measure for future needs. For a general superfield \(r\), the path integral measure is defined as \(Dr = D[\tilde{r}_1, \tilde{r}_2, r_1, r_2]\). For symmetric superfields such as \(q\), we will only sum on the symmetric components of it, and call it \(D^\ast q = D^\ast q₁(q₁, q₂)D[q₁, q₂]\) with \(D^\ast q₁ = Dq₁ \prod_{t > t'} δ(q₁(t, t') - q₁(t', t))\) in a discretized point of view. Therefore \(q(a, b)\) with ‘\(a > b\)’ (loosely speaking) will appear in the path integral, the previous Dirac deltas imposing them to be \(q(b, a)\), so that we can use all components and introduce \(q\) as a symmetric superfield in the path integral as it should be from its definition.

---

\(^{10}\) Let us emphasize here that this is only a compact notation without any physical content (for our present purpose), but we note by way of excuse that the computation is prohibitively complicated proceeding otherwise.

\(^{11}\) We do not need to resort to purely fermionic components (i.e. linear in \(θ₁\) and \(θᵢ\)) in the superfield \(x\) since we interpreted the Langevin equation in the Itô sense, implying the MSRDDDJ Jacobian is 1. Any other discretization scheme can be equivalently considered by adding two such ghosts.
4. **Product of superfields:** We define the product of two superfields \( r = pq \) as the generalization of the operator product

\[
r(a, b) = \int dc p(a, c)q(c, b)
\]  

(50)

5. **Identity:** The identity for superfields reads in components

\[
1(a, b) = \tilde{\theta}_1 \theta_1 \delta(t - t') + \tilde{\theta}_2 \theta_2 \delta(t - t')
\]  

(51)

6. **Inverse:** The definition of the inverse \( r^{-1} \) in the superfield sense reads, through the relation \( r^{-1} r = r r^{-1} = 1 \),

\[
r^{-1} = \beta + \tilde{\theta}_1 \theta_1 \beta + \tilde{\theta}_2 \theta_2 \gamma \beta + \tilde{\theta}_1 \theta_1 \tilde{\theta}_2 \theta_2 (\gamma \alpha + r^{-1}_1)
\]  

(52)

with \( \beta = (r_2 - r_1 r^{-1}_1 r_2)^{-1} \), \( \gamma = -r^{-1}_1 r_2 \) and \( \alpha = -\tilde{r}_1 r^{-1}_1 \).

7. **Determinant:** We define a ‘superdeterminant’ for superfields as

\[
s\det(r) = \det(r_1) \det(r_2 - \tilde{r}_1 r^{-1}_1 \tilde{r}_2)
\]  

(53)

8. **Trace:** The trace is defined by \( \text{str}(r) = \int da \, r(a, a) = \int dt \, (\tilde{r}_1 + \tilde{r}_2)(t, t) \).

9. **Integral representation of Dirac deltas for superfields:** an integral representation is expressed as

\[
\delta(q) = \int Dp \, e^{i \int da \, db \, p(a, b)q(b, a)} = \int Dp \, e^{i \text{str}(pq)} \quad \text{for a superfield operator}
\]

\[
\delta(x) = \int Dy \, e^{i \int da \, y(a) \cdot x(a)} \quad \text{similarly for a one-component superfield}
\]  

(54)

For a symmetric superfield such as \( q(a, b) = x(a) \cdot x(b) \), we only need to introduce the independent part of it, that is, taking the additional superfield \( p \) to be also symmetric, we only have to sum in the exponential over half of \( \int da \, db \, p(a, b)q(a, b) \). Rescaling \( p \) with the factor \( \frac{2}{\gamma} \), the formula above is unchanged\(^{12} \) provided that the measure is understood as \( D^p \). Since \( p \) is symmetric.

10. **Derivation with respect to a superfield:** Defining the derivative with respect to a superfield in a way similar to operators, using the convention for functional derivation:

\[
\frac{\delta r(a, b)}{\delta r(c, d)} = 1(a, c)1(b, d)
\]  

(55)

where \( r \) is a superfield, and \( 1 \) is the equivalent of the Dirac delta for superfields. Similarly for a symmetric superfield \( q \) we get

\[
\frac{\delta q(a, b)}{\delta q(c, d)} = 1(a, c)1(b, d) + 1(a, d)1(b, c)
\]  

(56)

which gives the factors 2. We will thus use the formulas

\[
\frac{\delta \ln s\det q}{\delta q} = 2q^{-1}, \quad \frac{\delta \text{str}(pq)}{\delta q} = 2p, \quad \frac{\delta \text{str}(q^{-1} p)}{\delta q} = -2q^{-1} pq^{-1}
\]  

(57)

11. **Gaussian integration on superfields:** A direct calculation with components shows that the Gaussian integral can be cast in the familiar form:

\[
\int Dx \, e^{-\frac{1}{2} \int da \, db \, q(a, b)x(a)x(b) + \int da \, h(a)x(a)} = \frac{e^{\frac{1}{2} \int da \, db \, q^{-1}(a, b)h(a)h(b)}}{\sqrt{s\det(q)}}
\]  

(58)

\( h \) and \( x \) being here scalar fields for the sake of clarity (i.e. \( d = 1 \)). \( q \) is assumed symmetric, i.e. \( q_1 \) and \( q_2 \) are symmetric operators and \( q_1 = q_2^T \). The Gaussian integrations require that \( q_1 \) (respectively \( q_2 \)) and \( q_2 - q_1 q^{-1}_1 q_2 \) (respectively \( q_1 - q_2 q^{-1}_2 q_1 \)) are positive definite.

\(^{12}\) All numerical constants, such as this one coming from the rescaling, will be omitted as we will eventually calculate all proportionality constants in another way.
12. **Gaussian moments**: Moments of a Gaussian random variable in notation SUSY enjoy similar properties than discrete Gaussian random variables. As an example, we can compute the following variance:

$$\sqrt{s\text{det}(q)} \int \text{D}x \, x(a) x(b) e^{-\frac{1}{2} \int da db q(a,b) x(a)x(b)} = q^{-1}(a,b)$$

(59)

13. **Other useful identities**: The following relations are useful for the derivation of the interaction term, for any superfields $A$ and $B$ such that the series converge:

$$\ln \text{s\text{det}}(A + B) = \ln \text{s\text{det}}A + \text{str} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (A^{-1}B)^n = \ln \text{s\text{det}}A + \text{str} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (BA^{-1})^n$$

(60)

These formulas are just generalizations of power series to superfields. The radius of convergence of the series used is 1.

14. **The projector $P$**: We define a superfield $P(a,b) = 1$ and $\lambda$ a scalar. We get the following important formulas:

$$\ln \text{s\text{det}}(\lambda P + B) = \ln \text{s\text{det}}(B) + \ln \left[1 + \lambda \text{str}(PB^{-1})\right]$$

$$\left(\lambda P + B\right)^{-1} = B^{-1} \left[1 - \frac{\lambda}{1 + \lambda \text{str}(PB^{-1})} PB^{-1}\right]$$

(61)

Both formulas require $|\text{str}(PB^{-1})| < 1/|\lambda|$.

II. **TRANSLATIONAL AND ROTATIONAL INVARIANCES**

We now take into account rotational and translational invariances and take the limit $d \to \infty$. In some cases the order of the two limits is irrelevant, but when relevant, we should take the $R \to \infty$ limit first. In other words, we should consider for example that $R/d$ is a large quantity.

A. ‘**Functional spherical coordinates**: invariances using the mean-square displacement

As emphasized in section I A, the aim of the introduction of $S^d(R)$ is to take into account both translation and rotation invariances on Euclidean $d$ dimensional space by only rotation invariance on a sphere of a $d + 1$ dimensional space, which is actually easier to handle in the viewpoint of the dynamics. Indeed, the MSRDDJ action $A$ in (27), now promoted to $d + 1$ dimensional fields, and the constraints in (48) are invariant by the same rotation $R$ for both fields $i.e.$ $(x, \dot{x}) \rightarrow (Rx, \dot{R}x)$, which is transposed to superfields as a global rotation $x \rightarrow Rx$.

Now considering expression (48), we define a superfield $q(a,b) = x(a) \cdot x(b)$. We assume that the $d + 1$ dimensional liquid is invariant by rotation $i.e.$ $\rho(x) = \rho(\{x(a) \cdot x(b)\}_{a,b}) = \rho(q)$. This way we will remove all irrelevant variables and be able to use a saddle point method.

Eventually, as regards the $R \to \infty$ limit, it is more convenient to consider the mean-square displacement (MSD) $D(a,b) = (x(a) - x(b))^2$ since it is a finite quantity as long as the difference $t - t'$ is finite, at equilibrium. Before the dynamical transition is met, $D$ is of order $R^2$ when $t - t' \to \infty$. We thus expect an artificial second plateau of order $R^2$ due to finite size effects, that will be removed when $R \to \infty$, giving back diffusion at long times (see figure 1(a)). One can check explicitly that in this limit the original $d$-dimensional MSD is recovered and that it is translation and rotation invariant for the position and rotation invariant for the response field, see figure 2.

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13 See subsection ID and footnote 2.
FIG. 2. Notations on the sphere for $x, \hat{x}$. The vector $x(t) - x(t')$ is along a chord and when $R \to \infty$, it lives in the same $d$-dimensional space as $\hat{x}$.

**B. Scalings in the infinite $d$ limit**

First, from now on we measure distances in units of the diameter $\sigma$, i.e. we take $\sigma = 1$.

To use the saddle point method, we need to specify how the quantities defined here scale with dimension. From the statics in [12], if we define $u = x - X$ where $X$ denotes the translational degree of freedom of a particle so that $u$ characterizes motion around an average position, $u$ scales as $1/d$. As $\Phi$ is translation invariant along $x$,

$$\Phi(x, \hat{x}) = \Phi(u, \hat{u}) \equiv \hat{x}$$

(62)

$$\dot{u} \sim 1/d$$

implies that, for the action to describe the Langevin process above ((25)), we need the two terms in the integral to scale identically in the limit $d \to \infty$. Thus $\hat{u} \sim 1/d$ as well. This implies that $D = \Delta/d$ where $\Delta$ is of order 1. Hence, for convenience, noticing that $D = 2R^2 P - 2q$, we define the rescaled quantities

$$\Delta_{\text{liq}} = 2dR^2, \quad Q = \Delta_{\text{liq}} P - \Delta = 2dq$$

(63)

$\Delta_{\text{liq}}$ corresponds to a typical MSD between particles (in the liquid phase) on the sphere $S^d(R)$. $Q(a, a) = \Delta_{\text{liq}}$ is the spherical constraint.

We then choose $\gamma = 2\hat{\gamma}d^2$ in order to write $\Phi = d\hat{\Phi}$ with $\hat{\gamma}$ and $\hat{\Phi}$ of order 1 so that $\Phi(x)$ scales like $\ln \rho(x)$, otherwise the infinite $d$ limit would not be well defined. Indeed, from the statics of the liquid phase [6, 8, 12, 29, 30], we will assume that $\rho$ is exponential in $d$, that is

$$\rho(q) = \Lambda(Q)e^{d\hat{\Omega}(Q)}$$

(64)

i.e. $\ln \rho(q) \sim d\Omega(Q)$ with $\Omega$ of order 1 and $\Lambda$ a subdominant factor (i.e. non-exponential in $d$).

Concerning the potential $V(r)$, $r$ will typically scale here as $r = \sigma(1 + \frac{\mu}{d})$ where $\mu$ is of order 1. Besides the two terms in the action $A$ in (27) must have the same scalings to describe the underlying Langevin process (25). As $A$ can be cast in a form similar to (34) using $\rho(x, \hat{x})$, thus similar to an ideal gas term plus an interaction term with ‘propagator’ $W$, the following derivation of the infinite $d$ limit will indicate that such interaction terms have the same scalings as the ideal gas term only if the ‘propagators’ $W$ (or $f$) scales as $\hat{\Phi}$ i.e. 1 here. Thus $V$, which scales like $W$ (cf. (47)), must be of order 1. For example, the soft harmonic spheres potential is $V_{\text{SS}}(r) = \kappa(1 - r)^2 \Theta(1 - r) = \kappa(\mu/d)^2 \Theta(-\mu)$. The amplitude must be $\kappa = \hat{\kappa}d^2$ with $\hat{\kappa}$ of order 1.
C. Ideal gas term

We write equation (48) as $S = S_{\text{IG}} + S_{\text{int}}$ and focus on the ideal gas term $S_{\text{IG}}$. Exploiting the invariances, we apply the program mentioned in II A:

\[
S_{\text{IG}} \propto \int \mathbf{x} \, \delta((x(a)^2 - R^2) \rho(x)(\ln \rho(x) + \Phi(x))
= \int \mathbf{x} \mathbf{D} \, \delta(q(a, b) - x(a) \cdot x(b)) \delta(q(a, a) - R^2) \rho(q)(\ln \rho(q) + \Phi(q))
= \int \mathbf{x} \mathbf{D} \, \mathbf{D}' \, \delta(q(a, a) - R^2) e^{\text{istra}(qq') - i \int \text{d}a \text{d}b q'(a, b) x(a) \cdot x(b) \rho(q)(\ln \rho(q) + \Phi(q))
= \int \mathbf{D} \, \mathbf{D}' \, \delta(q(a, a) - R^2) e^{\text{istra}(qq') - \frac{\delta x^2}{2} \ln \text{det}(2iq') \rho(q)(\ln \rho(q) + \Phi(q))
\]

The explicit expression of $\Phi(q)$ will be computed in III A. The integral over $q'$ is evaluated through a saddle point method for $d \to \infty$. Using the tools from subsection I E 2, the saddle-point equation is $2iq = (d + 1)q' - \frac{1}{s_{\text{sp}}}$. We introduce the rescaled variable $Q$ and neglect subdominant terms which will be calculated in subsection II C. For convenience with respect to the saddle-point equation in section III, we exponentiate the resulting spherical constraint $Q(a, a) - \Delta_{\text{liq}}$.

\[
S_{\text{IG}} \propto \int \mathbf{D}[Q, \nu] e^{\frac{1}{2} \ln \text{det} Q - \frac{1}{2} \int \text{d}a \nu(a)(Q(a, a) - \Delta_{\text{liq}}) \rho(Q)(\ln \rho(Q) + \Phi(Q))
\]

In the limit $d \to \infty$ we apply a saddle point method, thanks to the scalings provided in II B. We can get rid of proportionality constants using the normalization of the density:

\[
S_{\text{IG}} = \frac{\mathcal{C}(Q)}{\int \mathbf{x} \, \delta((x(a)^2 - R^2) \rho(x))} = -\frac{\int \mathbf{D}[Q, \nu] C(Q) e^{\frac{1}{2} \Gamma(Q, \nu)} \rho(Q)}{\int \mathbf{D}[Q, \nu] C(Q) e^{\frac{1}{2} \Gamma(Q, \nu)}
\]

where $C(Q)$ accounts for forgotten subdominant contributions to the integral, and

\[
\Gamma(Q, \nu) = \ln \text{det} Q - \int \text{d}a \nu(a) (Q(a, a) - \Delta_{\text{liq}}) + 2\Omega(Q)
\]

is the saddle point function. In $d \to \infty$ we maximize $\Gamma$, in particular

\[
\left. \frac{\delta \Gamma}{\delta Q} \right|_{\text{sp}} = 0 \quad \text{and} \quad \left. \frac{\delta \Gamma}{\delta \nu} \right|_{\text{sp}} = 0 \iff Q(a, a) = \Delta_{\text{liq}}
\]

giving the result

\[
S_{\text{IG}} \xrightarrow{d \to \infty} -d[\Omega(Q^{sp}) + \hat{\Phi}(Q^{sp})]
\]

This can be expressed explicitly using once again the normalization condition

\[
\int \mathbf{D}[Q, \nu] C(Q) e^{\frac{1}{2} \Gamma(Q, \nu)} = 1
\]

Evaluating it for $d \to \infty$ and taking the logarithm of the resulting equation at dominant order $O(d)$ provides $\Gamma(Q^{sp}, \nu^{sp}) = 0$ up to irrelevant additive constants\textsuperscript{14}, from which we get

\[
S_{\text{IG}} \xrightarrow{d \to \infty} \frac{d}{2} \ln \text{det} Q^{sp} - \frac{d}{2} \int \text{d}a \nu^{sp}(a) (Q^{sp}(a, a) - \Delta_{\text{liq}}) - d\hat{\Phi}(Q^{sp})
\]

\textsuperscript{14} These constants were hidden in $C(Q^{sp})$ and do not depend upon $Q^{sp}$, therefore having no influence on the dynamics.
D. Interaction term

1. Changes of variables to ‘functional spherical coordinates’

Now focusing on $S_{\text{int}}$, we introduce the superfields $q(a, b) = x(a) \cdot x(b), p(a, b) = y(a) \cdot y(b)$ (through a symmetric measure $D^p$) and the interaction superfield $\omega(a) = (x(a) - y(a))^2$, a one component variable since the Mayer function $f[\bar{x} - y, \bar{x} - y]$ in (47) needs only scalar products between its variables at equal times.

$$S_{\text{int}} \propto \int D[x, y] \delta(x(a)^2 - R^2) \delta(y(b)^2 - R^2) \rho(x) \rho(y) f(x - y)$$

$$= \int D[x, y, q, q', p, p', \omega, \omega'] \delta(q(a, a) - R^2) \delta(p(a, a) - R^2) e^{istr(qq' + pp')} \rho(q) \rho(p) f(\sqrt{\omega})$$

$$\times e^{-i \int dadb [q'(a,b)x(a)x(b)+p'(a,b)p(y(b)] - i \int da \omega'(a)[(x(a) - y(a))^2 - \omega(a)]}$$

$$= \int D[q, q', p, p', \omega, \omega'] \delta(q(a, a) - R^2) \delta(p(a, a) - R^2) \rho(q) \rho(p) f(\sqrt{\omega})$$

$$\times \exp \left(istr(qq' + pp') + i \int da \omega'(a) \omega(a) - \frac{d + 1}{2} \ln \det \left[ q' + \bar{\omega} - \omega \bar{p}' + \bar{\omega} p \right] \right)$$

(73)

The last line is obtained by Gaussian integration on superfields $x$ and $y$, introducing the superfield $\omega'(a) = \omega'(a)1 \omega(a, b)$, and det means the determinant of the $2 \times 2$ block matrix consisting of superfields. We now change variables to exploit the $x \leftrightarrow y$ symmetry in (73),

$$q_\pm = \frac{q \pm p}{2} \quad \text{and} \quad q'_\pm = \frac{q' \pm p'}{2}$$

(74)

whose Jacobian is subdominant. The $2 \times 2$ block matrix can be transformed through a (Weyl) rotation which does not affect the determinant:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \left( q' + \bar{\omega} - \omega \bar{p}' + \bar{\omega} p \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} q'_+ + 2\bar{\omega} & q'_- \\ q'_- & q'_+ \end{pmatrix}$$

(75)

In $d \to \infty$, owing to the $x \leftrightarrow y$ or $q \leftrightarrow p$ (respectively $q' \leftrightarrow p'$) symmetry in (73), the saddle-point value of $q_-$ (respectively $q'_-$) is zero. Dropping the $+$ index for the two other superfields$^{15}$, we get

$$S_{\text{int}} \propto \int D[q, q', \omega, \omega'] \delta(q(a, a) - R^2) e^{istr(qq' + i \int da \omega'(a) \omega(a) - \frac{d + 1}{2} \ln \det q' - \frac{d + 1}{2} \ln \det(q' + 2\bar{\omega})} \rho(q)^2 f(\sqrt{\omega})$$

(76)

To simplify the last term in the exponential, we make the following change of variables:

$$q = R^2 P - \frac{1}{2} D^s \quad \text{and} \quad q' = \frac{d + 1}{2i} \left( R^2 P - \frac{1}{2} D^s \right)^{-1}$$

(77)

The Jacobian is subdominant; all such subdominant terms will be calculated in subsection II D 2. The aim is to recover the same saddle point function $\Gamma$ at $O(d)$ in the exponential in $S_{\text{int}}$ as in the ideal gas term. Let us focus on the last term in the exponential, neglecting irrelevant constants:

$$\ln \det(q' + 2\bar{\omega}) = -\ln \det \left( R^2 P - \frac{D^s}{2} \right) + \ln \det \left( 1 - \frac{2i}{d + 1} \bar{\omega} D^s \right) + \ln \left( 1 + \frac{2R^2}{d + 1} \str \left[ P \left( \frac{\bar{\omega}^{-1} - D^s}{2i} \right)^{-1} \right] \right)$$

$$= -\ln \det \left( R^2 P - \frac{D^s}{2} \right) + \ln \det \left( 1 - \frac{2i}{d + 1} \bar{\omega} D^s \right) + \ln \str \left[ P \left( \frac{\bar{\omega}^{-1} - D^s}{2i} \right)^{-1} \right]$$

(78)

$^{15}$ The symmetry also implies that the saddle-point value of $q_+$ is the same than $q$ or equivalently $p$ (same for the primes).
We used (61) and expanded using the limit \( R \to \infty \) before \( d \to \infty \) in the last line as emphasized in the introduction to this section, which is valid if \( \text{str} \left[ P \left( \frac{\omega^{-1}}{2t} - \frac{D'}{d+1} \right) \right] \) is not zero.

We expect the same scalings for \( D' \) and \( D' \), so let us assume \( D' = \Delta'/d \) and \( \omega' = d\mu' \), hence \( \omega D' = \bar{\mu} \Delta' = O(d^0) \). Using (60), we can use expansions:

\[
\ln sdet \left( 1 - \frac{2i}{d+1} \omega D' \right) \bigg|_{d \to \infty} = - \frac{2i}{d+1} \text{str}(\mu \Delta') + O \left( \frac{1}{d^2} \right)
\]

\[
\ln \text{str} \left[ P \left( \frac{\omega^{-1}}{2t} - \frac{D'}{d+1} \right) \right] \bigg|_{d \to \infty} = \ln \text{str}(P \bar{\mu}) + \frac{2i}{d+1} \frac{\text{str}(P \mu \Delta' \bar{\mu})}{\text{str}(P \bar{\mu})} + O \left( \frac{1}{d^2} \right)
\]

Summarizing,

\[
S_{\text{int}} \propto \int D[D, D'] \delta(D(a, a)) e^{2\text{str} \left[ \left( R^2 P - \frac{D^2}{2} \right) \right] - \left( d+1 \right) \ln sdet \left( R^2 P - \frac{D^2}{2} \right) \rho(D)^2}
\]

\[
\times \int D[\omega, \mu] e^{i d \int d\mu'(a) \omega(a) + i \text{str}(\mu \Delta') - \frac{d+1}{2} \ln \text{str}(P \bar{\mu}) - i \text{str}(P \mu \Delta' \bar{\mu})/\text{str}(P \bar{\mu})} f(\sqrt{\omega})
\]

At order \( O(d) \) in the exponential in \( S_{\text{int}} \), we get (twice) the same terms that we had in (65). Therefore in the last line, all dependence in \( D' \) is of \( O(d^0) \). We now go back to the original variables \( q \) and \( q' \) by making again the change of variables (77). The terms in the exponent that must be kept for the saddle point in \( d \) becomes

\[
2i \text{str}(qq') - (d+1) \ln sdet(2iq') + 2 \ln \rho(q)
\]

which is exactly twice what we had for \( S_{\text{G}} \). Hence, \( 2i \omega = (d+1)q' \) \( \rho \) \( \text{str} \) \( D'(q) \) \( \text{str} \). Then the term \( \text{str}(\mu \Delta) = \int d\mu \cdot \mu(a) \Delta(a, a) = 0 \) because of the constraint. We also note that

\[
\omega(a) = (x(a) - y(a))^2 = (x(t) - y(t))^2 + 2 \theta_i \theta_j (x(t) - y(t)) \cdot (\dot{x}(t) - \dot{y}(t))
\]

As we expect \( (x(t) - y(t))^2 = \sigma^2(1 + O(1/d)) \), we define \( \omega(a) = 1 + 2 \mu(a)/d \). We set \( \lambda = \int d\mu' = \text{tr}(P \bar{\mu}) \) with a Dirac delta and exponentiate it with a conjugated \( \lambda' \) as usual. The interaction term now reads, once again exponentiating the constraint \( \delta(Q(a, a) - \Delta_{\text{int}}) \) through a superfield \( \nu \),

\[
S_{\text{int}} \propto \int D[Q, \mu] e^{i(\nu, \mu')} \mathcal{F}(Q), \quad \text{with } \mathcal{F} \text{ defined by:}
\]

\[
\mathcal{F}(Q) \propto \int D[\mu, \mu'] d\lambda d\lambda' \exp \left( i \lambda \lambda' + i d \lambda - \frac{d+1}{2} \ln \lambda + 2i \int d\mu' \cdot \mu(a) - \lambda'/2 \right)
\]

\[
\times \exp \left( -i \lambda \Delta_{\text{int}} + i \text{str}(P \mu Q \bar{\mu})/\lambda \right) f \left( \sqrt{1 + \frac{2\mu}{d}} \right)
\]

We used the simplification \( \text{str}(P \mu P \bar{\mu}) = \lambda^2 \). The integral over \( \lambda \) can be performed in the infinite \( d \) limit: the saddle-point equation gives \( \lambda_{\text{bp}} = 1/2i \). Performing the following steps:

1. In the Mayer function \( f \), expand the square root \( \sqrt{1 + 2\mu(a)/d} \) in the limit \( d \to \infty \)
2. Rescale \( \mu(a) - \lambda'/2 \to \mu(a) \), still calling the new superfield \( \mu \)
3. Rescale \( \lambda'/2 \to \lambda \), dropping the prime for convenience
4. Perform the Gaussian integration on \( \mu' \)

we obtain

\[
\mathcal{F}(Q) \propto \frac{e^{-\Delta_{\text{int}}/2}}{\sqrt{\text{det}Q}} \int d\lambda D\mu e^{\lambda - \frac{1}{4} \int d\lambda d\mu(a) \mu^{-1}(a, b) \mu(b)} f \left( \frac{1 + \mu}{d} \right)
\]

\[\text{for now we put all subexponential dependence in } \mathcal{F} ; \text{ explicit expressions will be given in II D2.}\]
2. Normalization

Note that all the nonexponential in $d$ dependences overlooked during the procedures of the different changes of variables does not depend upon the choice of the Mayer function $f$. Here we benefit from this to give the explicit expression of $S_{\text{int}}$.

In the MSRDDJ action $\mathcal{A}$ (27) we sum on times belonging to an interval $[t_p, t_1]$, where initial conditions are fixed at $t_p$. $t_1$ labels the final state, and if we sum on all positions at $t_1$, we have $Z_N = 1$ in (26). Let us pick $s \in [t_p, t_1]$ and define a test function $f_0[x, \hat{x}] = \Theta(1 - |x(s)|)$. Note that the choice of the test function is not completely arbitrary: as seen in I C, it should satisfy the properties of the true Mayer function $f$ that we used to derive $S$, as it must reject all trajectories that do not get close at some time. Making a choice that does not respect these properties would lead to absurd results. We obtain, using first the expression of $S_{\text{int}}$ in (34) and setting $y = u + x$ and $\hat{y} = \hat{u}$:

$$S_{\text{int}}[f] = \frac{N}{2} \int D[x, \hat{x}] D[u, \hat{u}] \rho(x, \hat{x}) \rho(u, \hat{u}) \Theta(1 - |u(s)|) = \frac{N}{2} \int dx^n_x \frac{d\nu^n_x}{(2\pi)^{\frac{d}{2}}} \frac{d\nu^n_{\hat{x}}}{(2\pi)^{\frac{d}{2}}} \rho(x^n_x + u^n_x) \Theta(1 - |u^n_x|)$$

$$= \frac{N}{2} \left( \frac{(2\pi)^{\frac{d}{2}}}{\nu} \right)^2 \frac{\nu}{(2\pi)^{\frac{d}{2}}} \int d\nu^n_x \Theta(1 - |u^n_x|) = \frac{\nu \nu_d(1)}{2}$$

(85)

As in I C, we discretized the trajectories and used that translation invariance and the normalization $\int D[x, \hat{x}] \rho(x, \hat{x}) = 1$ imply $\int D[x] \rho(x^1, \ldots, x^M, \hat{x}) = \text{constant} = (2\pi)^{\frac{d}{2}} / \nu$. $n_s$ labels the time $s$, i.e. $s = t_p + n_s(t_1 - t_p)/M$.

We define $C'(Q, \Delta_{\text{liq}})$ accounting for all the overlooked terms. We have, taking the saddle point over $Q$,

$$\frac{S_{\text{int}}[f]}{S_{\text{int}}[f_0]} = \frac{S_{\text{int}}[f]}{\nu \nu_d(1)/2}$$

$$= e^{-\Delta_{\text{liq}}/2} \int DQ, \nu, \mu) C'(Q, \nu, \Delta_{\text{liq}}) e^{i\nu(Q, \nu)e^{\lambda - \frac{\lambda}{d} \int d^d\mu(a)Q^{-1}(a, b)\mu(b)}} f \left(1 + \frac{\mu + \lambda}{d} \right)$$

$$= \frac{1}{C''(Q^{sp}, \Delta_{\text{liq}})} \int D\mu \lambda e^{\lambda - \frac{\lambda}{d} \int d^d\mu(a)Q^{-1}1_{sp}(a, b)\mu(b)}} f \left(1 + \frac{\mu + \lambda}{d} \right)$$

(86)

$C''$ is given by

$$C''(Q^{sp}, \Delta_{\text{liq}}) = \int D\mu \lambda e^{\lambda - \frac{\lambda}{d} \int d^d\mu(a)Q^{-1}1_{sp}(a, b)\mu(b)}} \theta \left( \frac{-\mu(s) + \lambda}{d} \right)$$

$$= \int D\mu e^{\lambda - \frac{\lambda}{d} \int d^d\mu(a)Q^{-1}1_{sp}(a, b)\mu(b)}} e^{\Delta_{\text{liq}}/2 \sqrt{\text{s\text{det}Q}^{sp}}}$$

(87)

where we introduced the superfield $g(a) = -\bar{\theta}_a \theta_b \delta(t - s)$ to integrate the Gaussian. We also used the constraint $Q_{sp}(a, a) = \Delta_{\text{liq}}$. Now we can conclude:

$$S_{\text{int}} \equiv \frac{\nu \nu_d(1)}{2} F(Q^{sp})$$

where

$$F(Q) = \frac{\nu \nu_d(1)}{2} \int D\mu \lambda e^{\lambda - \frac{\lambda}{d} \int d^d\mu(a)Q^{-1}(a, b)\mu(b)}} f \left(1 + \frac{\mu + \lambda}{d} \right)$$

(88)

E. Final result in the limit $d \to \infty$

Collecting the results from the last two subsections, and using (61), we obtain the final result in the infinite dimension limit:

$$S = \frac{d}{2} \text{ln}\text{det}Q - \frac{d}{2} \int da \nu(a) (Q(a, a) - \Delta_{\text{liq}}) - d\Phi(Q) + \frac{d^2}{2} F(Q)_{|_{sp}}$$

(89)
III. SADDLE-POINT EQUATION

A. Explicit form of \( \Phi(Q) \)

We now make explicit the \( Q \) dependence of \( \Phi \) using \( \hat{\Phi} \) justified in subsection II B. From (27), it is Gaussian in \( x \) and \( \tilde{x} \),

\[
\hat{\Phi}(x) = \frac{1}{d} \Phi(x) = \gamma_d \int dt \left( \tilde{x} \cdot \tilde{x} - T \tilde{x}\tilde{x} \right) = 2d \left[ \int dt dt' \tilde{x}(t)k(t,t')\tilde{x}(t') + 2 \int dt dt' \tilde{x}(t)\hat{k}(t,t')x(t') \right]
\]

(90)

with \( k(t,t') = -2\gamma T \delta(t - t') \) and \( \hat{k}(t,t') = \frac{\gamma}{2} \frac{\partial}{\partial t} \delta(t - t') \)

The kernel \( k \) is symmetric while \( \hat{k} \) is antisymmetric. Let us define a symmetric superfield:

\[
k(a,b) = k(t,t') - \bar{\theta}_1 \theta_1 \tilde{k}(t,t') + \bar{\theta}_2 \theta_2 \tilde{k}(t,t')
\]

(91)

One can check that \( \hat{\Phi}(Q) = \text{str}(kQ) \) gives back expression (90).

B. Saddle-point equation for the dynamic correlations

From subsection ID, the probability density of trajectories \( \rho \) is given by the saddle-point equation \( \delta S/\delta \rho(x) = \delta S/\delta \rho(Q) = 0 \). In the infinite \( d \) limit, \( S \) depends on \( \rho \), or equivalently on its logarithm \( \Omega \), only through its saddle-point value \( \Omega(Q^{\text{sp}}) \). From the relation \( \Gamma(Q^{\text{sp}}, \nu^{\text{sp}}) = 0 \) derived in II C thanks to the normalization of \( \rho \), \( \Omega(Q^{\text{sp}}) \) is explicitly determined by the saddle-point values \( Q^{\text{sp}} \) and \( \nu^{\text{sp}} \). Hence the saddle point condition is equivalent\(^{17} \) to \( \delta S/\delta Q_{\text{sp}} = 0 \) in (89), and is, as a consequence, equivalent to the saddle point condition used in the virial terms for \( d \to \infty \),

\[
\frac{\delta S}{\delta Q_{\text{sp}}} = 0 \iff -Q^{-1} - \nu + 2k + \frac{\hat{\varphi}}{2} \frac{\delta F}{\delta Q_{\text{sp}}} = 0
\]

(92)

The derivative of \( F \) is

\[
\frac{\delta F}{\delta Q(a,b)} = F(\Delta)Q^{-1}(a,b) - \int da'db' Q^{-1}(a,a') \int d\lambda e^{\lambda - \Delta u_{\text{liq}}/2} \int D\mu \mu(a') \mu(b') f \left( 1 + \frac{\mu + \lambda}{d} \right) Q^{-1}(b',b)
\]

(93)

with the Gaussian measure

\[
\int D\mu \cdot = \frac{1}{\sqrt{\text{sdet}Q}} \int D\mu \cdot e^{-\frac{1}{2} \int da db \mu(a) Q^{-1}(a,b) \mu(b)}
\]

Hence the saddle-point equation, \( \forall(a,b) \),

\[
\left( 1 - \frac{\hat{\varphi}(Q)}{2} \right) Q^{-1}(a,b) - 2k(a,b) + \frac{\hat{\varphi}}{2} Q^{-1} \int d\lambda e^{\lambda - \Delta u_{\text{liq}}/2} \int D\mu \mu f \left( 1 + \frac{\mu + \lambda}{d} \right) Q^{-1} \bigg|_{(a,b)} - \nu(a) 1(a,b) = 0
\]

(94)

Together with the spherical constraint \( Q^{\text{sp}} = \Delta_{\text{liq}} \) which shall provide \( \nu^{\text{sp}} \), this determines \( Q^{\text{sp}} \).

C. Simplification of the saddle-point equation

1. Exploiting Ward-Takahashi-like identities

Here we drop the labels ‘sp’ for convenience. Generically, derivatives of \( S \) are needed for example to compute the Mode-Coupling Theory (MCT) exponents. They can be simplified using Ward-Takahashi-like identities. From the

\(^{17} \) The condition \( \delta S/\delta \nu_{\text{sp}} = 0 \) is once again the spherical constraint \( Q^{\text{sp}}(a,a) = \Delta_{\text{liq}} \).
definition of the Mayer function \( f \), a quantity like \( \langle \bullet f \rangle_\mu \) is a difference between two averages, one with potential \( V \) and the other without. Let us focus on the non Gaussian part, with potential:

\[
\int \mathcal{D}\mu e^{-f d\nu (\mu(a) + \lambda)} \equiv \langle 1 \rangle_V
\]

(95)

Usually one shifts \( \mu \) by \( \epsilon \) in some well chosen average \( \langle O \rangle_V \) and demands the cancellation of all non-zero orders in \( \epsilon \), but a more compact way to get moments of \( Q\mu \) is to write, \( \forall (a, b) \),

\[
Q^{-1} (\mu \mu)_V Q^{-1} |_{(a, b)} = \frac{1}{\sqrt{\text{det} Q}} \int \mathcal{D}\mu e^{-\int d\nu (\mu(c) + \lambda)} \left[ \frac{\delta^2}{\delta \mu(a) \delta \mu(b)} + Q^{-1}(a, b) \right] e^{-\frac{1}{2} \int d\nu d\epsilon Q^{-1}(c, c') \mu(c)}
\]

\[
= Q^{-1}(a, b) + \int \mathcal{D}\mu \frac{\delta^2}{\delta \mu(a) \delta \mu(b)} e^{-\int d\nu (\mu(c) + \lambda)}
\]

(96)

\[
= Q^{-1}(a, b) + \langle F(\mu(a) + \lambda) F(\mu(b) + \lambda) \rangle_V + \langle F'(\mu(a) + \lambda) \rangle_V 1(a, b)
\]

where we integrated by parts twice. This method can be easily generalized to higher moments.

2. The value of \( F(Q) \) at the saddle point

The measure in (95) can be interpreted as an average over a Langevin process with potential \( V \). Provided we sum over all possible trajectories of \( \mu \), equation (95) is actually the normalization of probability \( \langle 1 \rangle_V = 1 \).

Similarly \( F(Q) \) can be interpreted as a difference between averages over two dynamical processes, one with potential \( V \) and the other free,

\[
F(Q) = \int d\lambda e^\lambda - \Delta_{\text{lin}}/2 \int \mathcal{D}\mu f \left[ 1 + \frac{\mu + \lambda}{d} \right] = \int d\lambda e^\lambda - \Delta_{\text{lin}}/2 \left[ \int \mathcal{D}\mu e^{-f d\nu (\mu(a) + \lambda)} - \int \mathcal{D}\mu 1 \right]
\]

(97)

Hence \( F \) is zero for all acceptable dynamical propagators (positive definite, as we expect \( Q \) is, at least at its saddle-point value dominating the dynamics) due to normalization.

3. Definition of the memory kernel \( M \)

From equations (94), (96) and (97), the saddle-point equation is simplified as:

\[
Q^{-1}(a, b) = 2k(a, b) - M(a, b) + (\nu(a) + \delta\nu(a)) 1(a, b)
\]

(98)

where

\[
M(a, b) = \frac{\hat{\omega}}{2} \int d\lambda e^\lambda - \Delta_{\text{lin}}/2 \left[ \langle F(\mu(a) + \lambda) F(\mu(b) + \lambda) \rangle_V \right]
\]

\[
\delta\nu(a) = -\frac{\hat{\omega}}{2} \int d\lambda e^\lambda - \Delta_{\text{lin}}/2 \left[ \langle F'(\mu(a) + \lambda) \rangle_V \right]
\]

(99)

In our study of the dynamics of the system, \( M \) will play the role of the analog of the MCT kernel.

IV. EQUATION FOR THE DYNAMIC CORRELATIONS

A. Equilibrium hypothesis

In this subsection and the next ones we will unfold the SUSY notation to get rid of it, coming back to the standard dynamical variables.
We focus on the equilibrium dynamics of the system, assuming that time translation invariance (TTI) as well as causality hold, and consequently fluctuation dissipation theorem (FDT): we assume we start in the remote past (time \(t_p\), formally sent to \(-\infty\)), so that the system is at equilibrium when a finite \(t_0\) is reached.

\(Q = 2dq\) (equivalently \(\Delta\)) takes its equilibrium form and so does \(M\):

\[
Q(a, b) = C(t - t') + \tilde{\theta}_1 t_1 R(t' - t) + \tilde{\theta}_2 t_2 R(t - t')
\]

\[
M(a, b) = M(t - t') + \tilde{\theta}_1 t_1 \tilde{M}(t' - t) + \tilde{\theta}_2 t_2 \tilde{M}(t - t')
\]

Along with TTI and (98), this implies that \(\nu(a) = \nu\) and \(\delta\nu(a) = \delta\nu\) are constant and real quantities. From this the inverse \(Q^{-1}\) reads:

\[
Q^{-1}(a, b) = \tilde{C}(t - t') + \tilde{\theta}_1 t_1 \tilde{R}(t' - t) + \tilde{\theta}_2 t_2 \tilde{R}(t - t') \quad \text{with} \quad \tilde{C} = -R^{-T} C R^{-1} \quad \text{and} \quad \tilde{R} = R^{-1}
\]

\(C\) and \(R\) satisfying fluctuation-dissipation theorem, one can check that \(\tilde{C}\) and \(\tilde{R}\) also do. Similarly, we can verify directly with (90) that \(k\) and 1 verify the latter relation. We conclude, from (98), that \(M\) also satisfies FDT (which was suggested by (99)).

**B. Mode-coupling form of the saddle-point equation and the effective stiffness**

We can cast the saddle-point equation into a mode-coupling form by multiplying (98) by \(Q\) on the right. The scalar component of the equation obtained reads at \((t', t)\):

\[
0 = -2(\dot{k}C + kR^T) - MC - MR^T + (\nu + \delta\nu) C
\]

\[
= \dot{\gamma} \dot{C}(t' - t) - 2\dot{\gamma} TR(t - t') + (\nu + \delta\nu) C(t - t') - \beta \int_{-\infty}^{t'} du (\partial_u M(t' - u)) C(u - t)
\]

\[
- \beta \int_{-\infty}^{t} du M(t' - u) \partial_u C(t - u)
\]

We can assume, for instance, that \(t > t'\), and use FDT:

\[
0 = \dot{\gamma} \dot{C}(t - t') + (\nu + \delta\nu) C(t - t') - \beta \int_{t'}^{t} du M(t' - u) \partial_u C(t - u) - \beta \int_{-\infty}^{t'} du \partial_u [M(t' - u) C(t - u)]
\]

Using the relaxation for long times and making the substitution \(v = t + t' - u\),

\[
\dot{\gamma} \dot{C}(t - t') = - (\nu + \delta\nu - \beta M(0)) C(t - t') - \beta \int_{t'}^{t} dv M(t - v) \dot{C}(v - t')
\]

\(\dot{C}(0) = -TR(0^+)\) represents the immediate response to a perturbation. At such very short times the potential is not relevant, particles follow a free dynamics. \(\dot{C}(0)\) can thus be obtained using (25) \(\gamma \dot{x}_i = -\nu_i x_i + \xi_i\) with \(\nu_i = dT/R^2\) since, at equilibrium, TTI and the equipartition theorem holds. One can solve this Ornstein-Uhlenbeck process and compute \(C(t) = \Delta_{\text{liq}} \exp(-dTt/\gamma R^2)\), giving \(-\dot{\gamma} \dot{C}(0) = T\), hence

\[
\nu + \delta\nu - \beta M(0) = \frac{T}{\Delta_{\text{liq}}}
\]

We conclude that the mode-coupling-like equation for \(C\) is, for \(t > t'\):

\[
\dot{\gamma} \dot{C}(t - t') = \left(-\frac{T}{\Delta_{\text{liq}}} C(t - t') - \beta \int_{t'}^{t} dv M(t - v) \dot{C}(v - t')
\]

or equivalently for the MSD \(\Delta = dD = \Delta_{\text{liq}} - C\) at \(t > t'\):

\[
\dot{\gamma} \Delta(t - t') = T - \frac{T}{\Delta_{\text{liq}}} \Delta(t - t') - \beta \int_{t'}^{t} dv M(t - v) \dot{\Delta}(v - t')
\]
C. Effective Langevin process

The aim is to compute $M$, the mode-coupling-like kernel, as a function of $Q^{op}$ to solve the saddle-point equation for $Q^{op}$, providing correlation and response of the system. To do this, we must calculate correlations of the force $F$ at two times $t_0$ and $t_1 > t_0$. To achieve this program, we will interpret, as mentioned in III.C.2, the average defining $M$ as two-point correlation functions of a Langevin dynamics with potential $V_{\text{sp}}$ at two times for $t > 0$. The initial condition in the remote past vanishes on a finite time scale $\hat{\tau}$ because we expect that $M$ using FDT for $t < t_0$ and $t > t_0$, as stressed in section I.B (see also figure 3). However one expects the Langevin equation (111) to describe a non-equilibrium difference above which correlations vanish, and (105), we get the generalized Langevin equation equivalent to (109):

$$M(t - t') = \frac{\sigma^2}{2} \int d\lambda e^{\lambda - \Delta_{\text{lin}}/2} \langle F(\mu(t) + \lambda) \rangle$$

We drop the notation $\sqrt{\text{det}Q}$ plays the role of normalization\(^{18}\). The corresponding Langevin process with potential $V$, depending on $\lambda$, is:

$$\tilde{\gamma} \dot{\mu}(t) = -\langle \nu + \delta \nu \rangle \mu(t) + \int_{t_p}^{t} dt' \tilde{M}(t - t') \mu(t') + F(\mu(t) + \lambda) + \zeta(t)$$

with $\langle \zeta(t) \rangle = 0$ and $\langle \zeta(t) \zeta(t') \rangle = 2\tilde{\gamma} T \delta(t - t') + M(t - t')$ (109)

Using FDT for $\hat{M}$, an integration by part with the fact that $t - t_p > t_0 - t_p > \tau_\alpha$ the relaxation time of the system, above which correlations vanish, and (105), we get the generalized Langevin equation equivalent to (109):

$$\tilde{\gamma} \dot{\mu}(t) = -\frac{T}{\Delta_{\text{lin}}} \mu(t) - \beta \int_{t_p}^{t} dt' M(t - t') \dot{\mu}(t') + F(\mu(t) + \lambda) + \zeta(t)$$

Note that so far in this section we have used equilibrium properties for all observables. It is either because we considered them at a time $t \geq t_0$ where equilibrium is reached, or in the case of convolution products with $M$ where the integral extends to the remote past, because we expect that $\tilde{M}$ (respectively $M$) vanishes quickly (respectively vanishes on a finite time scale $\tau_\alpha$). Then for finite times where they do not vanish, the system has equilibrated from the initial condition in the remote past $t_p$, and we can consider their equilibrium properties.

D. Memory kernel in equilibrium: resumming trajectories from the remote past

As emphasized before, dynamical two points functions at $(t_0, t_1)$ should be function of a single argument, the time difference $t_1 - t_0$, as we assume that equilibrium is reached. But equilibrium properties hold only if we consider times $t \geq t_0$. Therefore, we wish to use $t_0$ as our initial time, but in the understanding that at this time the system is at equilibrium. Similarly, we are not interested in what happens after $t_1$, as it should be irrelevant due to causality, as stressed in section 1.B (see also figure 3). However we expect the Langevin equation (111) to describe a non-Markovian process in which the memory kernel persists for a duration of the order of $\tau_\alpha$. How are we going to ignore the times $t < t_0$ if the kernel $M$ extends to the remote past?

An equation like (111) may be thought of as having originated in a system coupled linearly to a bath of harmonic oscillators, à la Zwanzig [31, 63]. Let us consider the Hamiltonian evolution of this system, described by coordinates denoted collectively by $\Gamma = \{ \mu, p_\alpha, q_\alpha \}$, according to the Hamiltonian\(^{19}\):

$$H_{\text{tot}}(\Gamma) = \frac{T}{2\Delta_{\text{lin}}} \mu^2 + \bar{V}(\mu + \lambda) + \sum_\alpha \left[ \frac{p_{\alpha}^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} \left( \frac{q_\alpha - c_\alpha}{m_\alpha \omega_\alpha^2} \right)^2 \right]$$

\(^{18}\) Note that $\text{det}Q$ = numerical constant at the saddle point level if the system is causal.

\(^{19}\) If we restore the inertia term in the dynamics, we would add to $\Gamma$ one more coordinate $p_\alpha$, momentum of the particle of position $\mu(t)$. Similarly, $H_F$ would have one more term $p_\alpha^2/2m_\alpha$. 
\( \{ c_\alpha, \omega_\alpha, m_\alpha \} \) are chosen suitably to reproduce dissipation and noise terms in (111). As before, assume we start in the remote past \( t_p \) at a point in phase space \( \Gamma_p \) and let us distinguish two times \( t_0 \) and \( t_1 \), such that \( t_1 > t_0 \gg t_p \). We can rewrite averages over the effective Langevin process as averages over the Markovian process \( \Gamma(t) \), by defining the transition probabilities, for fixed \( \Gamma_p \) and \( \Gamma_1 \),

\[
P_V(\Gamma_1, t_1|\Gamma_p, t_p) = \int_{\Gamma_p, t_p}^{\Gamma_1, t_1} D[\Gamma] \rho_V[\Gamma(t)]|\Gamma_p, t_p| \tag{113}
\]

where \( \rho_V \) is the transition probability density of the whole system \( \{ \text{effective particle} \} \cup \{ \text{bath} \} \). The marginal of \( \rho_V[\Gamma(t)]|\Gamma_p, t_p| \) over all trajectories of the bath degrees of freedom \( \{ p_\alpha, q_\alpha \} \) is the marginal of \( \rho_V[\mu, \tilde{\mu}|\mu_p, t_p] \) over trajectories of the response field \( \tilde{\mu} \) of the MSRDDJ probability density in (108), namely in compact SUSY notation with fixed initial conditions at \( t_p \):

\[
D[\mu] e^{-\frac{1}{2} \int_{t_p}^{t_1} f_p^1 \, da \mu(a) Q^{-1}(a,b) \mu(b) - \int_t^{t_1} da V(\mu(a) + \lambda)} = D[\mu, \tilde{\mu}] \rho_V[\mu, \tilde{\mu}|\mu_p, t_p] \tag{114}
\]

We can now write, using Markovianity:

\[
P_V(\Gamma_1, t_1|\Gamma_p, t_p) = \int d\Gamma_0 \, P_V(\Gamma_1, t_1|\Gamma_0, t_0) \, P_V(\Gamma_0, t_0|\Gamma_p, t_p) \tag{115}
\]

where \( \text{d} \Gamma = \frac{d\mu}{\sqrt{2\pi}} \prod_\alpha dp_\alpha dq_\alpha \). For \( t_0 - t_p \gg \tau_\alpha \) the system relaxes, \( i.e. \) we may assume that \( P_V(\Gamma_0, t_0|\Gamma_p, t_p) = P_V^\text{eq}(\Gamma_0) \), the equilibrium distribution in the phase space of the whole system:

\[
P_V^\text{eq}(\Gamma_0) = \frac{e^{-\beta H_{\text{tot}}(\Gamma_0)}}{Z_V} = P_V^\text{eq}(\mu_0) P_B^\text{eq}(\Gamma_0) \tag{116}
\]

with the marginals describing respectively the effective particle

\[
P_V^\text{eq}(\mu_0) = \frac{e^{-\beta H_{\text{eff}}(\mu_0)}}{Z_V} = \frac{e^{-\mu_0^2/2\Delta_{\text{eff}} - \beta V(\mu_0 + \lambda)}}{Z_V} \tag{117}
\]

and the bath

\[
P_B^\text{eq}(\Gamma_0) = \frac{e^{-\beta H_B(\Gamma_0)}}{Z_B} \quad \text{with} \quad Z_B = \int \prod_\alpha \, dp_\alpha dq_\alpha \, e^{-\beta H_B(\Gamma_0)} = \prod_\alpha \frac{2\pi}{\beta \omega_\alpha} \tag{118}
\]

We may now get rid of the bath degrees of freedom as in Zwanzig’s calculation, here integrating them out:

\[
P_V(\Gamma_1, t_1|\Gamma_p, t_p) = \int \frac{d\mu_0}{\sqrt{2\pi}} \, P_V^\text{eq}(\mu_0) \int \prod_\alpha \, dp_\alpha dq_\alpha \, e^{-\beta H_B(\Gamma_0)} \tag{119}
\]

Following Hänggi [31], the last term (underbraced) can be interpreted as the transition probability of a Langevin process, with full equilibrium at \( t_0 \) between the harmonic oscillators and the effective particle, given by \( H_B \), \( i.e. \) not neglecting the coupling to the particle. This generating functional reads, using again MSRDDJ:

\[
1 = \int d\Gamma_1 \, P_V(\Gamma_1, t_1|\Gamma_p, t_p) = \int \frac{d\mu_0}{\sqrt{2\pi}} \, P_V^\text{eq}(\mu_0) \int D[\mu, \tilde{\mu}] \rho_V[\mu, \tilde{\mu}|\mu_0, t_0] \tag{120}
\]

where now the effective Langevin process ruling the dynamics is:

\[
\hat{\gamma} \dot{\mu}(t) = -\frac{T}{\Delta_{\text{eff}}} \mu(t) - \beta \int_{t_0}^t dt' M(t-t') \dot{\mu}(t') + F(\mu(t) + \lambda) + \zeta(t) \tag{121}
\]

with \( \langle \zeta(t) \rangle = 0 \) and \( \langle \zeta(t) \zeta(t') \rangle = 2\delta T \delta(t-t') + M(t-t') \). \( \mu(t_0) = \mu_0 \) is picked with the equilibrium measure \( P_V^\text{eq}(\mu_0) \).

---

20 Constants are arbitrary here, they can be absorbed in the normalizations.
FIG. 3. The procedure can be more intuitively interpreted as a resummation before \( t_0 \) of all possible trajectories starting from a fixed initial position at \( t_0 \) (thanks to the relaxation towards equilibrium) and after \( t_1 \) (owing to causality), leaving us with finite times motions in the interval \([t_0, t_1]\).

E. Relaxation at long times

From now on we will be able to compute physically relevant observables, and with this in mind we will use standard MCT methodology [1, 2]. At long time difference \( t - t' \rightarrow \infty \), equation (106) gives

\[
0 = -\frac{T}{\Delta_{\text{liq}}} C(\infty) - \beta M(\infty)[C(\infty) - R^2] \Rightarrow C(\infty) = \frac{\Delta_{\text{liq}}^2 \beta^2 M(\infty)}{1 + \Delta_{\text{liq}}^2 \beta^2 M(\infty)} \Leftrightarrow \beta^2 M(\infty) = \frac{1}{\Delta_{\text{liq}}} \frac{C(\infty)}{\Delta_{\text{liq}} - C(\infty)}
\]

In the long time limit, decorrelation occurs if we assume ergodicity (which will be questioned at the dynamical transition, see VA), the average in (110) splits and by TTI we have to compute equilibrium averages:

\[
M(\infty) = \frac{\hat{\varphi}}{2} \int d\lambda e^{-\Delta_{\text{liq}}/2} \langle F(\mu_0 + \lambda) \rangle^2
\]

Using the procedure of the last subsection, \( \langle F(\mu_0 + \lambda) \rangle = \int \frac{d\mu_0}{\sqrt{2\pi}} P_{\text{eq}} V(\mu_0) F(\mu_0 + \lambda) \). Note that in equilibrium we can show through an integration by parts that, for any observable \( O(\mu_0) \),

\[
T \left\langle \frac{dO}{d\mu_0} \right\rangle = T \int \frac{d\mu_0}{\Delta_{\text{liq}}} \langle \mu_0 O \rangle - \langle F(\mu_0 + \lambda) O \rangle
\]

In particular by choosing \( O = 1 \) we obtain

\[
\langle F(\mu_0 + \lambda) \rangle = \frac{T}{\Delta_{\text{liq}}} \langle \mu_0 \rangle
\]

For simplicity we focus here on hard spheres. For this potential the normalization reads:

\[
Z_V = \int_{-\lambda}^{\infty} \frac{d\mu_0}{\sqrt{2\pi}} e^{-\mu_0^2/2\Delta_{\text{liq}}} = \sqrt{\Delta_{\text{liq}}} \Theta_0(-\lambda/\sqrt{\Delta_{\text{liq}}}) \quad \text{with} \quad \Theta_0(x) = \frac{1 + \text{erf}(-x/\sqrt{2})}{2}
\]

Given that

\[
\langle \mu_0 \rangle = \int_{-\lambda}^{\infty} \frac{d\mu_0}{\sqrt{2\pi}} e^{-\mu_0^2/2\Delta_{\text{liq}}} \mu_0 = \sqrt{\Delta_{\text{liq}}} \Theta_0(-\lambda/\sqrt{\Delta_{\text{liq}}}) \frac{e^{-\lambda^2/2\Delta_{\text{liq}}}}{2\pi \Theta_0(-\lambda/\sqrt{\Delta_{\text{liq}}})}
\]

after a short computation we find in the limit \( R \rightarrow \infty \):

\[
\beta^2 M(\infty) \sim \frac{\hat{\varphi}}{4\sqrt{\pi}} \frac{e^{-\Delta_{\text{liq}}/4}}{\sqrt{\Delta_{\text{liq}}}}
\]

which shows with (122) that both \( M(\infty) \) and \( C(\infty) \) go exponentially to zero for \( \Delta_{\text{liq}} \rightarrow \infty \), as one would expect.
F. Getting rid of the sphere $S^d(R)$: infinite radius limit

Applying the method in IV D, the equation for $M (110)$ reads

$$M(t - t') = \frac{\hat{\gamma}}{2} \int d\lambda e^{\lambda - \Delta_{\text{liq}}/2} \int \frac{d\mu_0}{\sqrt{2\pi}} P_V^\text{eq}(\mu_0) \langle F(\mu(t) + \lambda)F(\mu(t') + \lambda) \rangle$$

(129)

where the average is computed over the Langevin process (121). Fixing $\lambda$, we make the change of variables $21$ $h(t) = \mu(t) + \lambda$ centered at the wall. The normalization of $P_V^\text{eq}$ is unchanged and

$$P_V^\text{eq}(h_0) = e^{-(h_0 - \lambda)^2/2\Delta_{\text{liq}} - \beta \tilde{V}(h_0)} Z_V$$

(130)

We can simplify the expression of $M$ with a saddle point method$^{22}$ for $\Delta_{\text{liq}} \to \infty$, setting $\alpha = \lambda/\Delta_{\text{liq}}$:

$$M(t - t') = \frac{\hat{\gamma}}{2} \sqrt{\Delta_{\text{liq}}} \int \frac{dh_0}{\sqrt{2\pi}} e^{-\beta \tilde{V}(h_0) - \frac{h_0^2}{2\Delta_{\text{liq}}} - \frac{\Delta_{\text{liq}}}{2}(\alpha - 1)^2 + h_0\alpha} \langle F(h(t))F(h(t')) \rangle$$

$$\Rightarrow M(t - t') \sim \frac{\hat{\gamma}}{2} \int \frac{dh_0}{\sqrt{2\pi}} e^{-\beta w(h_0)} \langle F(h(t))F(h(t')) \rangle$$

(131)

with $w(h) = \tilde{V}(h) - Th + \frac{T}{2\Delta_{\text{liq}}} h^2$.

Since $\alpha = 1$ at the saddle point. We replaced the normalization by

$$Z_V = \int \frac{d\mu_0}{\sqrt{2\pi}} e^{-\mu_0^2/2\Delta_{\text{liq}} - \beta \tilde{V}(\mu_0 + \alpha \Delta_{\text{liq}})} \sim \sqrt{\Delta_{\text{liq}}}$$

(132)

since the potential goes to zero at long distances. The Langevin equation is affected by the change of variables only with an additional term $T\alpha \simeq T$:

$$\tilde{\gamma} \dot{h}(t) = T - \frac{T}{\Delta_{\text{liq}}} \dot{h}(t) - T \int_{t_0}^t dt' M(t - t') \dot{h}(t') + F(h(t)) + \zeta(t)$$

(133)

Our problem is mapped onto a one-dimensional diffusion with colored noise (as usual in mean field $^{22}$) and a harmonic effective potential $w(h)$ perturbed by the spheres’ repulsion (cf. figure 4). The underbraced terms -the harmonic potential well- are negligible for finite times, but necessary to confine the system: they represent the ‘box’.

FIG. 4. Effective potential landscape. If $V$ is hard, there is an infinite wall at $h = 0$ prohibiting any motion in the $h < 0$ half line. If the constraint is softer, as drawn here, motion is possible for $h < 0$ but is rather unlikely.

$^{21}$ Note that $h$ is called $y$ in the main text.

$^{22}$ From the Langevin equation, the bracketed term $\langle F(h(t))F(h(t')) \rangle$ depends upon $\alpha$ but it cannot be exponential in $\Delta_{\text{liq}}$ and as a consequence gives no contribution to the saddle-point equation.
G. The ‘Lagrange multiplier’

Plugging in (105) the value at equal times\(^{23}\) \(M(0)\), which can be computed at the equilibrium using TTI, we have

\[
\nu - \frac{T}{\Delta_{\text{liq}}} = -\delta \nu + \beta M(0) = \frac{\hat{\nu}}{2} \int_{-\infty}^{0} dh_{0} e^{h_{0} - \beta \tilde{V}(h_{0})} F'(h_{0}) + \beta \frac{\hat{\nu}}{2} \int_{-\infty}^{0} dh_{0} e^{h_{0} - \beta \tilde{V}(h_{0})} F(h_{0})^2 \tag{134}
\]

since \(\tilde{V}(h_{0}) = 0\) for \(h_{0} > 0\). We will focus in the following on hard spheres and assume a regularization of the potential which is of class \(C^1\), e.g. soft spheres\(^{24}\) \(\tilde{V}_{SS}(h) = \kappa \tau^2 \Theta(-h)\). We have, with integration by parts, due to the continuity of \(\tilde{V}\) and \(F\) in 0,

\[
\nu - \frac{T}{\Delta_{\text{liq}}} = \frac{\hat{\nu}}{2} \left\{ e^{h_{0} - \beta \tilde{V}(h_{0})} F(h_{0}) \right\}_{-\infty}^{0} - \int_{-\infty}^{0} dh_{0} e^{h_{0} - \beta \tilde{V}(h_{0})} F(h_{0}) \left( 1 + \beta F(h_{0}) \right) + \beta \int_{-\infty}^{0} dh_{0} e^{h_{0} - \beta \tilde{V}(h_{0})} F(h_{0})^2 \right\}
\]

The last integral being zero for hard spheres, we conclude that the Lagrange multiplier\(^{25}\) is (up to exponentially small corrections in \(\Delta_{\text{liq}} \rightarrow \infty\) due to \(M(0)\) and \(\delta \nu\)):

\[
\beta \nu = -\frac{\hat{\nu}}{2} + \frac{1}{\Delta_{\text{liq}}} \tag{136}
\]

V. PHYSICAL CONSEQUENCES OF THE DYNAMICAL EQUATIONS

A. Plateau and dynamical transition

1. Metastable glassy states: plateau value

We now look for a plateau in the dynamics: we assume a strong separation between a fast and a slow motion. We can split the correlations (and similarly the memory kernel) into a vibrational short-lived contribution and a slowly decaying function:

\[
C(t - t') = C^f(t - t') + C^s(t - t'), \quad M(t - t') = M^f(t - t') + M^s(t - t') \tag{137}
\]

each decaying on timescales \(\tau_f \ll \tau_s\), respectively. Let us look at intermediate times \(\tau_f \ll t - t' \ll \tau_s\). The slowly varying functions are approximately constant at this scale, equal to the plateau value noted \(C_{\text{EA}}\) (respectively \(M_{\text{EA}}\)). We have \(C_{\text{EA}} = \Delta_{\text{liq}} - \Delta_{\text{EA}}\) where \(\Delta_{\text{EA}}\) is the plateau of the MSD. Similarly to (122), we get from (106) the relation between \(M_{\text{EA}}\) and \(\Delta_{\text{EA}}\):

\[
\beta^2 M_{\text{EA}} = \frac{1}{\Delta_{\text{liq}}} \frac{C_{\text{EA}}}{\Delta_{\text{liq}} - C_{\text{EA}}} = \frac{1}{\Delta_{\text{EA}}} - \frac{1}{\Delta_{\text{liq}}} \tag{138}
\]

From (137), we can consider the Langevin noise as the sum of two independent centered Gaussian noises, a slowly varying one \(\zeta\) and a fast one \(\zeta^f\). In this limit, the Langevin equation (133) reads, using (138),

\[
\hat{\gamma} \dot{h}(t) = s - \frac{T}{\Delta_{\text{EA}}} h(t) - \beta \int_{t_0}^{t} dt' M^f(t - t') \dot{h}(t') + F(h(t)) + \zeta^f(t) \tag{139}
\]

with \(s \equiv \zeta + \beta M_{\text{EA}} h_0 + T\), \(\langle \zeta^f(t) \zeta^f(t') \rangle = 2 \hat{\gamma} T \delta(t - t') + M^f(t - t')\), \(\langle \zeta(t) \zeta(t') \rangle = M^s(t - t')\) and for \(\tau_f \ll t - t' \ll \tau_s\), \(\langle \zeta^2 \rangle \simeq M_{\text{EA}}\)

\(^{23}\) \(M(0)\) diverges in the hard-sphere limit, which is natural given its interpretation as a force-force correlation.

\(^{24}\) It is actually valid for a linear regularization \(\tilde{V}_{\text{lin}}(h) = -\kappa h \Theta(-h)\) too. Indeed we get \(\delta \nu = \frac{\hat{\nu}}{2\kappa}\) and \(\beta M(0) = \frac{\hat{\nu}}{2}(\kappa - T) + O(1/\kappa)\).

\(^{25}\) As noted in 1D, \(\nu\) is not the actual Lagrange multiplier, but is related to it.
Following [32, Sec. 4.], \( s \) acts as a quasistatic field: for times \( t - t' \ll \tau_s \), \( (\tilde{\zeta}, h_0) \) or equivalently \( s \) can be considered as quenched variables, picked with probability \( P_{\text{slow}}(s) \). For \( t - t' \gg \tau_f \), the process relaxes to an 'equilibrium' state selected by \( s \), which is the actual metastable glassy state, with probability\(^{26}\)

\[
P_h(h|s) = \frac{e^{-\beta H_1(h_s)}}{Z_1(s)} \quad \text{with} \quad H_1(h, s) = \frac{\tau_s}{2\Delta_{\text{EA}}} h^2 - s h + \tilde{V}(h)
\]

\[
P_{\text{slow}}(s) = \int \frac{d\theta_0}{\sqrt{2\pi}} d\zeta P_{\text{eq}}(h_0) e^{\frac{-\zeta^2}{2M_{\text{EA}}}} \delta(s - \zeta - \beta M_{\text{EA}} h_0 - T)
\]

\[= \int \frac{d\theta_0}{\sqrt{2\pi}} e^{-\theta_0 u(h_0)} e^{\frac{-\theta_0^2 M_{\text{EA}}}{2} (h_0 - \frac{T}{M_{\text{EA}}} (s - T))^2} \times e^{\Delta_{\text{liq}}/2 - \lambda h_0 (\alpha - 1) - \Delta_{\text{liq}}(\alpha - 1)^2/2} \times Z_V \]

with \( \alpha = \lambda / \Delta_{\text{liq}} \). Taking \( \Delta_{\text{liq}} \to \infty \) as in (131) provides the plateau value:

\[
\sqrt{M_{\text{EA}}} = \tilde{\varphi} \int ds e^{-(s - T)^2/2M_{\text{EA}}} Z_1(s) \langle F(h) \rangle_1^2
\]

As an example, we restrict ourselves to the hard-sphere case. Then, with an integration by parts,

\[
\langle F(h) \rangle_1 = \int \frac{d\theta}{\sqrt{2\pi}} P_h(h|s) F(h) = \frac{T}{Z_1(s) \sqrt{2\pi}}
\]

with \( Z_1(s) = \int \frac{d\theta}{\sqrt{2\pi}} e^{\beta s h - h^2/2\Delta_{\text{EA}} - \beta \tilde{V}(h)} = \sqrt{\Delta_{\text{EA}}} \Theta_0 \left(-s \beta \sqrt{\Delta_{\text{EA}}} \right) e^{\beta s^2/2\Delta_{\text{EA}}/2} \)

setting \( u = \sqrt{\Delta_{\text{EA}}} (\beta s - 1) \) finally gives

\[
\frac{1}{\sqrt{\Delta_{\text{EA}}}^2} = \tilde{\varphi} \int \frac{du}{4\pi} \frac{e^{-u^2/2 - (u + \sqrt{\Delta_{\text{EA}}})^2/2}}{\Theta_0 (-u - \sqrt{\Delta_{\text{EA}}})} \]

\[\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (143) \]

2. **Equivalence with the one-step replica-symmetry-breaking result**

In [12, 30], the equation for the plateau value was obtained, maximizing the hard spheres’ replicated entropy in \( d \to \infty \) in the glass phase where the IRSB ansatz is stable. It reads

\[
1 - \frac{m}{\hat{A}} = \tilde{\varphi} \frac{dG_m(\hat{A})}{d\hat{A}} \quad \text{with} \quad G_m(\hat{A}) = 1 - m \int \frac{dv}{\sqrt{2\pi}} e^{-v^2/2} \Theta_0 \left(v - \sqrt{2\hat{A}} \right)^{m-1}
\]

where \( 2\hat{A} \equiv \Delta_{\text{EA}} \). Here we are interested in the dynamical transition where the replica symmetric solution is valid (the liquid state is our 'paramagnetic' state), which can be recovered in the limit \( m \to 1 \) in Monasson’s scheme. Then

\[
G_m(\hat{A}) = (1 - m) \left[ 1 + \int \frac{dv}{\sqrt{2\pi}} e^{-v^2/2} \Theta_0 \left(v - \sqrt{2\hat{A}} \right) \right] + O((m - 1)^2)
\]

Deriving with respect to \( \hat{A} \), we get

\[
\frac{1}{\sqrt{2\hat{A}}} = \tilde{\varphi} \int \frac{dv}{4\pi} \frac{e^{-v^2/2 - (v - \sqrt{2\hat{A}})^2/2}}{\Theta_0 \left(v - \sqrt{2\hat{A}} \right)}
\]

which is the same as (143) with \( u = -v \).

\(^{26}\) One has to be careful here with the limit \( \Delta_{\text{liq}} \to \infty \): we cannot use directly equation (131), this is why we have to compute \( P_{\text{slow}} \) on the sphere and take the infinite radius limit. In principle we would also do it for \( P_h(h|s) \) but it is subdominant (as in footnote 22), so that we can compute it directly with (139), where \( \Delta_{\text{liq}} \to \infty \) has already been taken. Note that one recovers equation (16) of the main text by setting \( Y = s - T \).
3. Dynamical transition

Plotting the function $\hat{\phi}$ versus $\Delta_{EA}$ using (143) we get a minimum at the critical packing fraction

$$\varphi_d \approx \frac{4.80678 d}{2^7}$$

where the plateau is $\Delta_{EA}(\varphi_d) \approx 1.15336$. This packing fraction is larger than the best known lower bound for the existence of sphere packings $\varphi \geq \frac{6}{e}$ [53]. As mentioned in the abstract, this means that a hard sphere system may be prepared in equilibrium up to these densities in times that do not scale with the size of the box. In other words, packings as good as this are easy to obtain, and we conclude that this would be a constructive improvement on the best bound known in high $d$. It would require the present derivation to be turned into a rigorous proof, which seems feasible along the lines of [55] since our calculation, though a bit tedious, is quite elementary.

---

**FIG. 5. Plateau value and critical packing fraction**

B. Diffusion at long times

From (107) we obtain an expression for the diffusion coefficient for times larger than the relaxation time but still $\Delta \ll \Delta_{liq}$. In this regime, the mode-coupling-like equation for the MSD reduces to

$$\gamma \dot{D}(t - t') = 2dT - 2d^2 \beta \int_{t'}^t dv M(t - v) \dot{D}(v - t')$$

where $D = \Delta/d$ is the non-rescaled MSD, that is, the MSD of the original system of particles. Using Laplace transform we have

$$\tilde{D}(p) = \frac{1}{p^2 \gamma + 2d^2 \beta M(p)} \sim \frac{1}{p^2 \gamma + 2d^2 \beta M(0)}$$

By definition $\widetilde{M}(0) = \int_0^\infty dt M(t)$. A Tauberian theorem then gives the long-time diffusive behaviour of the MSD from the small $p$ behaviour of its Laplace transform [64]:

$$D(t) \sim 2dT \quad \text{with} \quad D = \frac{T}{\gamma + 2d^2 \beta \int_0^\infty M}$$

At low density $M \approx 0$ and we recover the usual diffusion coefficient $D = T/\gamma$ of the free dynamics. Upon increasing density, $M$ increases and the diffusion coefficient decreases. At the dynamical transition, $M$ displays a persistence of a plateau, the relaxation time $\tau_\alpha \propto \int_0^\infty M$ diverges and the diffusion coefficient vanishes. One usually defines an exponent $\gamma$ such that, for $\varphi \to \varphi_d$,

$$\tau_\alpha \sim \left(1 - \frac{\varphi}{\varphi_d}\right)^{-\gamma} \quad \text{and} \quad D \sim \left(1 - \frac{\varphi}{\varphi_d}\right)^{\gamma}$$

$\gamma$ is one of the so-called MCT exponents [1, 2].
C. Relation to the standard density formulation of MCT

1. Intermediate scattering functions

In its standard formulation, MCT provides equations for density correlators between time $t$ and the origin $\phi_q(t) = \langle \rho_q(t) \rho_q^*(0) \rangle / \langle \rho_q^2 \rangle$, where $\langle \bullet \rangle$ is a canonical average over initial conditions, and $\rho_q = \sum_{i=1}^{N} e^{iq \cdot x_i}$ is the Fourier transform of the particle density [1, 2]. This correlator thus reads

$$\phi_q(t) \propto \left\langle \sum_{ij} e^{iq \cdot (x_i(t) - x_j(0))} \right\rangle, \quad \phi_q(0) = 1 \tag{152}$$

We can define, as in IC 1 and similarly to standard liquid theory [19], the (non-averaged) local densities of trajectories so that the intermediate scattering functions can be written as

$$\phi_q(t) = \phi_q^s(t) + \phi_q^d(t) \propto \left\langle \int D[x, \hat{x}] \tilde{\rho}^{(1)}[x, \hat{x}] e^{iq \cdot x(t) - x(0)} \right\rangle = \left\langle \int D[x, \hat{x}] d\mathcal{R} \tilde{\rho}^{(1)}[\mathcal{R}x, \mathcal{R}\hat{x}] e^{iq \cdot \mathcal{R}[x(t) - x(0)]} \right\rangle$$

where the self ($i = j$) and distinct ($i \neq j$) parts are defined. Both parts can then be expressed as a function of $\Delta(t)$ through the saddle point evaluation of the integrals. In the following for simplicity we discuss only the self part.

2. The self part in infinite dimension

In $d \to \infty$, the self part is simply expressed in terms of the MSD. Using rotation invariance of $\tilde{\rho}^{(1)}$ to average over $d$-dimensional random rotations $\mathcal{R}$:

$$\phi_q^s(t) \propto \left\langle \int D[x, \hat{x}] \tilde{\rho}^{(1)}[x, \hat{x}] e^{iq \cdot x(t) - x(0)} \right\rangle = \left\langle \int D[x, \hat{x}] d\mathcal{R} \tilde{\rho}^{(1)}[\mathcal{R}x, \mathcal{R}\hat{x}] e^{iq \cdot \mathcal{R}[x(t) - x(0)]} \right\rangle$$

where $\theta$ denotes the angle between $q$ and $\mathcal{R}[x(t) - x(0)]$ and we used hyperspherical coordinates$^{27}$. Now we can proceed as in IIIC and express the dynamical variables $\rho_q$ as

$$\phi_q^s(t) \propto \left\langle \int D[Q, \nu] e^{\frac{1}{2} \ln \det \mathbf{Q} - \frac{1}{2} \int d\nu \langle \mathbf{Q} \cdot \mathbf{Q} \rangle - \Delta_{\text{in}}} \tilde{\rho}^{(1)}(Q) \right\rangle$$

The last integral can be evaluated through a saddle-point method in $d \to \infty$. Provided $|q|^{\Delta(t)} / d^{3/2} \ll 1$, the $q$-dependent term is irrelevant for the saddle-point evaluation$^{28}$ on $Q$. The saddle-point value of $\theta$ is imposed at the equator $\pi / 2$:

$$\int_{0}^{\pi} d\theta \sin^{d-2} \theta e^{i|q|/\Delta(t)/d} \cos \theta = \int_{0}^{\pi} d\epsilon e^{(d-2) ln \cos \epsilon - i|q|/\Delta(t)/d} \sin \epsilon \sim 1$$

$$\int_{\mathcal{R}} d\epsilon e^{-(d-2) \epsilon^2 / 2 - i|q|/\Delta(t)/d} \epsilon \sim e^{-q^2 \Delta(t)/2d^2}$$

$^{27}$ In $d$ dimensions, hyperspherical coordinates are defined by $x^1 = r \cos \theta_1$, $x^2 = r \sin \theta_1 \cos \theta_2$, ..., $x^{d} = r \sin \theta_1 \cdot \sin \theta_{d-2} \cos \theta_{d-1}$ with $\theta_{d-1} \in [0, 2\pi]$, $\theta_{i \neq d-1} \in [0, \pi]$. The measure is $\prod_{\mu=1}^{d-2} \sin^{d-\mu-1} \theta_{\mu-1} \sin \theta_{d-2} \sin \theta_{d-1} d\theta_{d-1} d\theta_{d-2} \cdots d\theta_1$. 

$^{28}$ If $q = O(d^{3/2})$, which, a priori, does not represent any physical distance (much less than the typical zone spanned by vibrations inside a cage $q = O(d)$), the saddle-point values of $\theta$ are different, although also simple; however the $q$-dependent term now contributes to the saddle-point equation on $\Delta$, changing its value and the formula (158) does not hold anymore.
The remaining integral over $Q$ and $\nu$ is dealt with as in II C and is normalized to 1, since the saddle-point is not affected by the last term in (156) as long as $q^2 \Delta(t)/d^2 \ll 1$. Together with the normalization $\phi_q^s(0) = 1$, we finally conclude for all wavevectors satisfying the latter condition,

$$\phi_q^s(t)\xrightarrow{d\to\infty} \exp\left(-\frac{q^2}{2d^2} \Delta(t)\right)$$

(158)

First, we note that the self correlator is Gaussian in $d \to \infty$, in contrast to what is found in [15, 16]. In these articles, the MCT equations for the plateau value (the so-called Debye-Waller factor or non-ergodic parameter, which reads here $\phi_{q,EA}^s = e^{-q^2 \Delta_{EA}/2d^2}$) are solved numerically for the hard spheres system up to $d = 800$, and its shape is found to be non-Gaussian.

This expression is also exact for any dimension both in the free-particle regime (lengths and time small compared to mean free path and collision time respectively) and hydrodynamic limit (lengths and time large compared to mean free path and collision time respectively) [19].

(158) implies, by substitution in (107), equations for the $\phi_q^s$, with noticeable qualitative differences with respect to MCT equations (such as a non-local memory kernel $M$).

3. The factorization property

A crucial outcome of MCT is the so-called factorization property [1, 2], which allows to get MCT scaling laws. It states that, in the $\beta$-relaxation window (i.e. close to the plateau), the difference between the value of the intermediate scattering functions and their value at the plateau can be factorized into a product of a function of the wavector only and a function of time only:

$$\delta\phi_q^s(t) \equiv \phi_q^s(t) - \phi_{q,EA}^s \simeq H(q)G(t)$$

(159)

This property is a stringent test of MCT in simulations [27]. In $d \to \infty$, the self intermediate scattering functions for all wavevectors are governed by a single quantity, the MSD. Close to the plateau, $\delta \Delta(t) = \Delta(t) - \Delta_{EA}$ is small and from equation (158),

$$\delta\phi_q^s(t) \simeq -\frac{q^2 \phi_{q,EA}^s}{2d^2} \delta \Delta(t)$$

(160)

We conclude that the factorization property holds in the infinite $d$ limit. Besides, equation (158) is more general since it provides all orders in $\delta \Delta$ and is valid even far from the plateau.

D. MCT exponents

Starting from equations (106),(107), one can compute the different MCT exponents related to the approach to the plateau $\Delta_{EA} - \Delta(t) \sim t^{-a}$ or the departure from it $\Delta(t) - \Delta_{EA} \sim t^b$, by expanding around the plateau value $\Delta_{EA}$ [1–3]. We do not report here the full dynamical computation [39–42]. One finds that the MCT exponents are controlled by the exponent parameter $\lambda$ through the relations [1–3, 39]:

$$\frac{\Gamma(1-a)^2}{\Gamma(1-2a)} = \frac{\Gamma(1+b)^2}{\Gamma(1+2b)} = \lambda, \quad \gamma = \frac{1}{2a} + \frac{1}{2b}$$

(161)

For hard spheres, we obtain $\lambda \simeq 0.70698$ which implies $a \simeq 0.324016$, $b \simeq 0.629148$ and $\gamma \simeq 2.33786$ [43]. We emphasize that the value of $\gamma$ is consistent with numerical results obtained in [44].

E. Connections with the microscopic model

1. Correlation, response and mean-square displacement

We wish to establish a connection between microscopic quantities and their counterpart at the saddle-point level in $d \to \infty$. Let us look at the MSD, but a similar reasoning can be done for correlations and responses (which are
related to it). Let us look at the microscopic MSD for a particle $i$

$$\left\langle (x_i(t) - x_i(t'))^2 \right\rangle_{Z_N} = \frac{1}{N} \left\langle \sum_{i=1}^{N} (x_i(t) - x_i(t'))^2 \right\rangle_{Z_N} = \frac{\delta (\ln Z_N[h]/N[h])}{\delta h(t,t')} \bigg|_{h=0}$$ (162)

since $Z_N[h = 0] = 1$, with $Z_N[h]$ (or $\Xi[h]$) obtained by adding to the dynamical action a coupling to a generating field $h(t,t')$,

$$\mathcal{A}([x_i, \dot{x}_i], h) = \mathcal{A}([x_i, \dot{x}_i]) - \sum_{i=1}^{N} \int dt dt' \left( x_i(t) - x_i(t') \right)^2 h(t,t')$$ (163)

This is a single-particle term, which amounts to shift the kinetic term in the following way:

$$\Phi[x, \dot{x}, h] = \Phi[x, \dot{x}] - \int dt dt' \left( x(t) - x(t') \right)^2 h(t,t')$$ (164)

Then we use the above-derived $d \to \infty$ limit of $\ln Z_N/N \sim N \ln \Xi/N = S$ with $S[h] = S_{IG}[h] + S_{int}$, where only the ideal gas term depends on $h$: $S_{IG} = - \int D[x, \dot{x}] \rho[x, \dot{x}] (\Phi[x, \dot{x}, h] + \ln \rho[x, \dot{x}])$. Consequently, going to generalized spherical coordinates,

$$\left\langle (x_i(t) - x_i(t'))^2 \right\rangle_{Z_N} = \frac{\delta S_{IG}[h]}{\delta h(t,t')} \bigg|_{h=0} = \frac{\delta (Q^{\nu \mu})}{\delta h(t,t')} \bigg|_{h=0} = \frac{\delta}{\delta h(t,t')} \int dudu' D(u, u') h(u, u') \bigg|_{h=0} = D(t,t')$$ (165)

We deduce that the adimensional rescaled correlation functions

$$\frac{2d}{\sigma^2 N} \sum_i x_i(t) \cdot x_i(t') \sim C(t,t'), \quad \frac{2d}{\sigma^2 N} \sum_{i, \mu} \frac{\delta x_i^\mu(t)}{\delta h^\mu(t')} \sim R(t,t'), \quad \frac{d}{\sigma^2 N} \sum_i (x_i(t) - x_i(t'))^2 \sim \Delta(t,t')$$ (166)

(with external fields $\tilde{h}_i$) are non-fluctuating, imposed by their saddle-point value.

2. Force-force correlation and its relation to the memory kernel

Here we establish a connection between a microscopic force-force correlation $\left\langle \sum_{i<j} F_{ij}(t) \cdot F_{ij}(t') \right\rangle_{Z_N}$, where $F_{ij} = -\nabla V(x_i - x_j)$, and the memory kernel $M$. To generate such terms we will use a random shift similar to the MK model [29], and once again resort to the SUSY notation for compactness. We will thus consider a shift of vector $A_{ij} g(a)$ for each pair of particles, where $A_{ij}$ are Gaussian centered random vectors in $d$ dimensions, of variance $\sum_{i,j,\mu} A_{ij}^\mu A_{ij}^{\nu}$. We will note $DA = \prod_{\mu=1}^{d} da_{\mu} e^{-\left(\sum_{\mu} a_{\mu}\right)^2/2 \Sigma_{A}}/\sqrt{2\pi \Sigma_{A}}$ their common measure. $g$ is a scalar time-dependent external field that will be sent to zero in the end, in order to recover the original model. Again for compactness, we will note averages over the $A_{ij}$ by an overbar, i.e. $\bar{A}_{ij} = 0$ and $\bar{A}_{ij} \bar{A}_{ik}^{\nu} = \sum_{\mu} \delta_{ik} \delta_{j\mu}$. The dynamical action becomes

$$\mathcal{A}([x_i, A_{ij}, g]) = \sum_{i=1}^{N} \Phi(x_i) + \sum_{i<j}^{1,N} \int da V(x_i(a) - x_j(a) + A_{ij} g(a))$$ (167)

We still note $F_{ij}^{\mu}(a) = -\nabla^{\mu} V(x_i(a) - x_j(a) + A_{ij} g(a))$, knowing that we recover the previously defined force by $F_{ij}^{\mu}(t) = F_{ij}^{\mu}(a)|_0$ where the 0 stands for $g$ and all Grassmann variables being sent to zero. Let us compute the second derivative of the generating dynamic functional $Z_N[g]$:

$$\frac{\delta Z_N[g]}{\delta g(a)} = \int \prod_{i=1}^{N} Dx_i e^{-\mathcal{A}([x_i, A_{ij}, g])} \sum_{i<j} A_{ij}^\mu F_{ij}^{\mu}(a)$$

$$\frac{\delta^2 Z_N[g]}{\delta g(a) \delta g(b)} = \int \prod_{i=1}^{N} Dx_i e^{-\mathcal{A}([x_i, A_{ij}, g])} \left[ \sum_{i<j} A_{ij}^\mu A_{ik}^{\nu} F_{ij}^{\mu}(a) F_{ij}^{\nu}(b) + \delta(a, b) \sum_{i<j} A_{ij}^\mu A_{ij}^{\nu} \nabla^{\nu} F_{ij}^{\mu}(a) \right]$$ (168)
where repeated Greek indices are summed over. Sending the external field to zero and averaging over the random shifts directly give $\delta Z_N \delta g(a) = 0$ and

$$
\frac{\delta^2 Z_N[\delta g]}{\delta g(a) \delta g(b)} = \Sigma_A^2 \left< \sum_{i<j} F_{ij}(t) \cdot F_{ij}(t') \right>_{Z_N} 
$$

(169)

which is the original force-force correlation looked for. As in [29], one can compute the average $Z_N[\delta g]$ introducing an averaged Mayer function

$$
\overline{\mathcal{F}_{ij} g} = \int \mathcal{D}A f(x_i(a) - x_j(a) + Ag(a)) = \int \mathcal{D}A \left[ e^{-\int da V(x_i(a) - x_j(a) + Ag(a))} - 1 \right] 
$$

(170)

For a non-zero $g(a)$ and large enough $\Sigma_A$, we still have the same crucial fact that $\overline{\mathcal{F}_{ij} g} = 1 + O(\nu_d/\nu)$ where $\Gamma$ is a typical length of a trajectory (for finite times), due to the requirement that two trajectories overlap to feel the effect of the potential. As a consequence, we can repeat the MK computation and obtain in the thermodynamic limit $Z_N[\delta g] = e^{NS[\delta g]}$ with the action

$$
S = -\int \mathcal{D}x \rho(x) (\ln \rho(x) + \Phi(x)) + \frac{N}{2} \int \mathcal{D}[x, y] \rho(x) \rho(y) \int \mathcal{D}A f(x - y + Ag(a)) 
$$

(171)

The difference here is that we cannot simplify further by translation invariance since the shift is time-dependent. Nevertheless, we can still compute derivatives of $Z_N[\delta g]$, reminding that due to probability conservation $Z_N[\delta g] = 1$:

$$
\frac{\delta Z_N}{\delta g(a)} \bigg|_{g=0} = N^{-1} \frac{\delta S}{\delta g(a)} \bigg|_{g=0} = 0
$$

$$
\frac{\delta^2 Z_N}{\delta g(a) \delta g(b)} \bigg|_{g=0} = N^{-2} \left[ \frac{\delta^2 S}{\delta g(a) \delta g(b)} \bigg|_{g=0} + N \frac{\delta S}{\delta g(a)} \bigg|_{g=b=0} \frac{\delta S}{\delta g(b)} \bigg|_{g=a=0} \right] = N^{-2} \frac{\delta^2 S}{\delta g(a) \delta g(b)} \bigg|_{g=0} 
$$

(172)

We will use, as in II D, the $d \to \infty$ limit to compute $S$. We know in this limit that the trajectory density $\rho(x)$ is determined, at leading order, by a saddle-point equation implying only the Jacobian of the change to generalized spherical coordinates, hence it is independent of the external field at this order. Thus we can overlook this dependence and compute derivatives of the Mayer function, leading to, after averaging:

$$
\frac{\delta^2 S}{\delta g(a) \delta g(b)} \bigg|_{g=0} = \Sigma_A^2 \frac{N}{2} \int \mathcal{D}[x, y] \rho(x) \rho(y) e^{-\int da V(x - y)} \left[ F(x - y)(a) \cdot F(x - y)(b) + \delta(a, b) \nabla \cdot F(x - y)(a) \right] 
$$

(173)

We are interested in the “boson-boson” part, so we can focus on the first term only, which reads

$$
F(x - y)(a) \cdot F(x - y)(b) \bigg|_{g=0} = \frac{(x - y)(t) \cdot (x - y)(t')}{|x - y|(t) |x - y|(t')} V'(|x - y|(t)) V'(|x - y|(t')) 
$$

(174)

The content of subsection II B is that the trajectory of $(x - y)\sim X$ plus a correction that is in $1/d$. For any finite time, the vector $X$ can be considered to be constant, with $|X| = \sigma$ and on average $(X^\nu)^2 \sim \sigma^2$. This is because particles do not move by $O(1)$ in a finite time. The correction term has zero average and $(x - y)\sim X$ plus a correction that is in $1/d$. As a consequence, we can repeat the MK computation and obtain in the thermodynamic limit $Z_N[\delta g] = e^{NS[\delta g]}$ with the action

$$
M(t, t') = \frac{\sigma^2}{Nd^3} \sum_{i<j} \left< F_{ij}(t) \cdot F_{ij}(t') \right>_{Z_N} 
$$

(176)

The normalization, giving a factor proportional to the packing fraction as in II D 2, follows from a similar analysis since, though the Mayer function $f(x - y)$ constraining the trajectories to be close is not there anymore (it is replaced by $1 + f$), $F(x - y)$ has the same role.
We make a final comment about two-body correlations that have the same structure as the interaction term. The truncated virial expansion in \( d \to \infty \) tells us that the two-body density of trajectories, which is the average of \( \hat{\rho}^{(2)} \) defined in (153), is simply given by\(^{30}\) [65, 66]

\[
\rho^{(2)}[x, y, \hat{x}, \hat{y}] = \left( \hat{\rho}^{(2)}[x, y, \hat{x}, \hat{y}] \right)_{Z_N} = N^2 \rho[x, \hat{x}] \rho[y, \hat{y}] (1 + f[x - y, \hat{x} - \hat{y}]) = N^2 \rho(x) \rho(y) e^{-f da V(x - y)(a)} \tag{177}
\]

We obtain the relation, for a function \( \mathcal{O} \),

\[
\frac{1}{N} \sum_{i \neq j} \mathcal{O}(x_{ij}(t)) \mathcal{O}(x_{ij}(0)) = \frac{1}{N} \int D[x, \hat{x}] D[y, \hat{y}] \rho^{(2)}[x, y, \hat{x}, \hat{y}] \mathcal{O}(x(t) - y(t)) \mathcal{O}(x(0) - y(0))
\sim N \int D[x, \hat{x}] D[y, \hat{y}] \rho[x, \hat{x}] \rho[y, \hat{y}] (1 + f[x - y, \hat{x} - \hat{y}]) \mathcal{O}(x(t) - y(t)) \mathcal{O}(x(0) - y(0)) \tag{178}
\]

To this kind of function we can apply the same reasoning as in subsection II D, with the replacement \( f \to (1 + f) \mathcal{O}(t) \mathcal{O}(0) \), if \( \mathcal{O} \) rejects trajectories that do not get close enough\(^{31}\), as was the Mayer function’s role, now replaced by \( 1 + f = e^{-W} \) (which is 1 for most trajectories), giving an average over the effective dynamics with potential. This argument gives more directly Eq. (176). However one has to be careful for correlations of a higher number of particles, as in the next subsection.

3. Stress-stress correlation and its relation to the memory kernel

Here we repeat the former procedure to obtain the link between the correlation of the off-diagonal (\( \mu \neq \nu \)) components of the stress tensor, which reads [19, 45]

\[
\sigma_{\mu \nu} = \sum_{i < j} (x_i - x_j)^\mu \nabla^\nu V(x_i - x_j) \tag{179}
\]

Once again, we use external random fields to generate the correlation function. This amounts to turn the dynamical action into

\[
A[\{x_i\}, \{A_{ij}, B_{ij}\}, g_1, g_2, h_1, h_2] = \sum_{i=1}^{N} \Phi(x_i) - \sum_{i < j}^{1, N} \int da \left[ A_{ij} g_1(a) + B_{ij} g_2(a) \right] \cdot (x_i - x_j)(a)
+ \frac{1}{2} \sum_{i < j}^{1, N} \int da \left[ V(x_i - x_j + A_{ij} h_1)(a) + V(x_i - x_j + B_{ij} h_2)(a) \right] \tag{180}
\]

where \( g_1, g_2, h_1, h_2 \) are external d-dimensional fields and \( \{A_{ij}, B_{ij}\} \) are one-dimensional independent identically distributed centered Gaussian random variables of variance \( \Sigma_A^2 \) and \( \Sigma_B^2 \), respectively. Note that when the external fields are zero, we recover the original model. Using the shorthand notation \( \partial_{1234} \equiv \delta^4 / \delta g_1^\mu(a) g_2^\nu(b) h_1^\sigma(a) h_2^\tau(b) \), we have, microscopically,

\[
\partial_{1234} Z_N \big|_{g_i=h_i=0} = \frac{1}{N} \prod_{i=1}^{N} D x_i e^{-A[\{x_i\}]} \sum_{i < j}^{1, N} \int_{k < l}^{m < n} \int_{p < q}^{m \neq n} \frac{1}{4} A_{ij}(x_i - x_j)^\mu(a) B_{kl}(x_k - x_l)^\nu(b) A_{mn}(a) F_{pq}(b) F_{pq}(b) \tag{181}
\]

\[
= \frac{\Sigma_A^2 \Sigma_B^2}{4} \left( \sum_{i < j}^{1, N} (x_i - x_j)^\mu(a) (x_k - x_l)^\nu(b) F_{ij}(a) F_{kl}(b) \right) = \frac{\Sigma_A^2 \Sigma_B^2}{4} \left( \sigma_{\mu \nu}(a) \sigma_{\mu \nu}(b) \right)_{Z_N}
\]

\(^{30}\) The \( N^2 \) factor only comes from the choice of a different normalization of \( \rho[x, \hat{x}] \) its definition (29), with respect to the one used in liquid theory.

\(^{31}\) This is important for e.g. the normalization as in II D2.
In \( \bar{Z}_N \) generates the connected correlation functions (cumulants), so that, using another shorthand notation \( \partial^n \) relative to any \( n^{th} \) derivative with respect to the fields involved in \( \partial_{1234} \):

\[
\partial_{1234} \ln \bar{Z}_N = \partial^4 \ln \bar{Z}_N = \frac{1}{\bar{Z}_N} \partial^4 \bar{Z}_N \left[ \begin{array}{l}
\frac{1}{\bar{Z}_N} \partial^4 \bar{Z}_N  \\
\frac{1}{\bar{Z}_N} \partial^3 \bar{Z}_N  \\
\frac{1}{\bar{Z}_N} \partial^2 \bar{Z}_N  \\
\frac{1}{\bar{Z}_N} \partial \bar{Z}_N  \\
\end{array} \right] + \frac{2}{\bar{Z}_N} \partial^3 \bar{Z}_N \partial \bar{Z}_N \partial^2 \bar{Z}_N
\]

(182)

By isotropy (or average over the disorder), when evaluated at zero external field, all terms containing a first derivative are zero. Furthermore, second derivative terms are also zero, either due to \( \overline{AB} = 0 \) for terms involving different times, or by isotropy for terms involving different indices\(^{32} \). We are thus left with only one term, which is \( \partial_{1234} \bar{Z}_N \) since \( \bar{Z}_N = 1 \). Once again as in [29], one can compute \( \ln \bar{Z}_N [g_1, g_2, h_1, h_2] \) introducing an averaged Mayer function

\[
\bar{f}_{ij} [g_1, g_2, h_1, h_2] = \int \mathcal{D}A \mathcal{D}B f(x_i(a) - x_j(a), g_1, g_2, h_1, h_2)
\]

\[
= \int \mathcal{D}A \mathcal{D}B \left[ e^{\int da \left[ (A(g_1(a) + Bg_2(a)) - V(x_i - x_j, a)) - \frac{1}{2} V(x_i - x_j, a) \right] - 1} \right]
\]

Note that, for finite times, \( e^{-\frac{1}{2} \int da V(x_i - x_j, a)} - 1 \) is zero except if \( h_1 \) is for some time approximately in the same direction as \( x_i - x_j \), and in that case it is of order \( O(\Gamma/L) \) where once again \( \Gamma \) is the typical length of a trajectory and \( L^d \sim \mathcal{V} \), for large enough \( \Sigma_A \). This implies, for fixed non-zero \( h_1 \) and small \( g_1 = \varepsilon \tilde{g}_1 / \Sigma_A \) with \( \tilde{g}_1 \) of order one,

\[
\int \mathcal{D}A \mathcal{D}B e^{\int da \left[ A g_1(a) - V(x_i - x_j, a) \right] - \frac{1}{2} \int da V(x_i - x_j, a)} \sim \int \mathcal{D}A \mathcal{D}B e^{\int da \left[ A g_1(a) - V(x_i - x_j, a) \right] + O(\Gamma/L)}
\]

\[
= e^{\Sigma_A^2 [\int da A g_1(a) - V(x_i - x_j, a)]^2/2 + O(\Gamma/L)}
\]

(184)

One must be careful with the order of limits, the reasoning is the following here:

1. we fix \( h_1 \) (non-zero)
2. we fix \( g_1 = \varepsilon \tilde{g}_1 / \Sigma_A \) with \( \tilde{g}_1 \) of order one
3. we take large \( \Sigma_A \) so that \( Ah_1 \) covers the whole line spanned by \( h_1 \), and \( \mathcal{D}A \sim da/L \)
4. we then take \( \varepsilon \to 0 \). In the end we will set all external fields to zero so defining them only in the neighborhood of 0 is enough\(^{33} \).

The same holds for the terms involving \( g_2 \) and \( h_2 \), with \( B \) instead of \( A \). This way, \( \bar{f}_{ij} = O(\varepsilon^2) + O(\Gamma/L) \) is small and we may repeat the same steps as in [29], so that \( \ln \bar{Z}_N = N S \), where

\[
S = - \int \mathcal{D}x [x] \rho(x) (\ln \rho(x) + \Phi(x)) + \frac{N}{2} \int \mathcal{D}[x, y] \rho(x) \rho(y) \int \mathcal{D}A \mathcal{D}B f(x - y, g_1, g_2, h_1, h_2)
\]

(185)

We only need to compute, to leading order\(^{34} \)

\[
\partial_{1234} S |_{g_1=h_1,h_2=0} = \frac{N}{2} \int \mathcal{D}[x, y] \rho(x) \rho(y) e^{-\int da V(x - y, a)} \frac{1}{4} A(x - y, a) A(x - y, b) \mathcal{F}(x - y, a) \mathcal{F}(x - y, b)
\]

\[
= \frac{N}{8} \Sigma_A^2 \nabla B \int \mathcal{D}[x, y] \rho(x) \rho(y) e^{-\int da V(x - y, a)} \{ (x - y, a) \mathcal{F}(x - y, a) \mathcal{F}(x - y, b)
\]

(186)

\(^{32} \) These are averages of an expression proportional to \( (x - y)^\mu(x - y)^\nu \) which is its own opposite when rotating by an angle \( \pi/2 \) in the \( (\mu, \nu) \) plane.

\(^{33} \) We can choose \( \varepsilon = 1/\Sigma_A \) if do not wish to introduce an additional parameter.

\(^{34} \) As in \( V \text{E2} \) for \( d \to \infty \) we can ignore the external fields dependence of the trajectory density \( \rho \).
Likewise (174), we have for $d \to \infty$

\[
(x - y)^\mu(a)(x - y)^\mu(b)F^\nu(x - y)(a)F^\nu(x - y)(b)|_0 = \frac{[(x - y)^\mu]^2(t) [(x - y)^\mu]^2(t')}{|x - y|(t)|x - y|(t')} V'(|x - y|(t)) V'(|x - y|(t')) \\
\sim \frac{(X^\nu)^2(X^\nu)^2}{|X|^2} V'(|x - y|(t)) V'(|x - y|(t')) \\
\sim F(h(t)) F(h(t'))
\]

(187)

We deduce, as in the last subsection,

\[
\left. \frac{\delta^4 \ln Z_N}{\delta g_1(a)g_2(b)h_1(a)h_2(b)} \right|_0 = \frac{\Sigma_A^2 \Sigma_B^2}{8} \frac{Nd^2}{8} \int dh_0 e^{-\beta w(h_0)} \left\langle F(h(t)) F(h(t')) \right\rangle = \frac{\Sigma_A^2 \Sigma_B^2}{4} dNM(t, t')
\]

(188)

and from (181) and (182)

\[
M(t, t') = \frac{1}{Nd} \left\langle \sigma_{\mu\nu}(t)\sigma_{\mu\nu}(t') \right\rangle_{Z_N}
\]

(189)

We conclude that the memory function coincides with the force-force and stress-stress correlations.
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