Dividing lines in unstable theories and subclasses of Baire 1 functions

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Abstract
We give a new characterization of SOP (the strict order property) in terms of the behaviour of formulas in any model of the theory as opposed to having to look at the behaviour of indiscernible sequences inside saturated ones. We refine a theorem of Shelah, namely a theory has OP (the order property) if and only if it has IP (the independence property) or SOP, in several ways by characterizing various notions in functional analytic style. We point out some connections between dividing lines in first order theories and subclasses of Baire 1 functions, and give new characterizations of some classes and new classes of first order theories.

Keywords Strict order property · Shelah’s theorem · Eberlein-Šmulian Theorem · Baire class 1 functions

Mathematics Subject Classification Primary 03C45 · 26A21; Secondary 46A50 · 03E15

1 Introduction

This paper aims to continue a new approach to Shelah stability theory (in classical logic), which was followed in [12, 13]. This approach is based on the fact that the study of the model-theoretic properties of formulas in ‘models’ instead of only these properties in ‘theories’ develops a sharper stability theory and establishes important links between model theory and other areas of mathematics, such as functional analysis. These links lead to new results, in both model theory and functional analysis, as well as better understanding of the known results.
Let us give the background and our own point of view. In the 70’s Saharon Shelah developed local (formula-by-formula) stability theory and combinatorial properties of formulas and used them to gain global properties of theories. The independence property and the strict order property of a formula for a ‘theory’ were introduced in 1971 in [19]. It is quite natural to try to develop local stability theory for formulas in ‘models’ instead of only theories. Such a theory was developed in [1, 15, 20] for the order property and recently in [12, 13] for the independence property. In [12], even a further step was taken and the strict order property was studied and a connection between a theorem of Shelah and an important theorem in functional analysis was discovered. What is interesting is that some model-theoretic notions appeared independently in topology and function theory, and moreover various characterizations yield, via routine translations, the characterization of NOP/NIP/NSOP in a model M or set A, and some important theorems in model theory have twins there.

Recall that in [19] Shelah introduced the strict order property as complementary to the independence property:

Shelah’s Theorem\(^1\): ([18, Theorem II.4.7]) A complete first order theory has the order property (OP) if and only if it has the independence property (IP) or the strict order property (SOP).

Later many classes of independent NSOP theories, such as simple and NSOP\(_n\), were found. In [12], it is shown that there is a correspondence between Shelah’s theorem above and the well known compactness theorem of Eberlein and Šmulian. In the current paper, we complete some results of [12] and give a new characterization of SOP for classical logic. In fact, the correspondence mentioned above is completed in this article. What is substantial is that there are connections between classification in model theory and classification of Baire class 1 functions which lead to a better understanding of both of these topics.

It is worth recalling more historical points. Stability in a model is not a new notion. In [9, 14, 15, 20] this notion was studied in the various contexts. (Although, the work of Krivine–Maurey [14] is about the stability of the formula \(\|x + y\|\) inside a fixed Banach space and not the models of its theory.) In [4] some variants of NOP/NIP/NSOP in a type were defined and a local version of Shelah’s theorem was proved. Recently, in [8, 12, 21] the connection between NIP and functional analysis was noticed. The notion “NIP of \(\phi(x, y)\) in a model” was introduced in [12]. We emphasize that our aim, approach and results in [12, 13] and the present paper are different from the previous works. In fact, the crucial idea in the paper is to study the model theoretic properties of theories by studying model theoretic properties of formulas in models.

This paper is organized as follows. In the next section we first review some basic notions from functional analysis and translate them into model theory. We then give a characterization of SOP that does not involve indiscernible sequences and allows us to relate the property to the behaviour of a class of Baire 1 functions (Proposition 2.10 and Remark 2.11 below). We also refine Shelah’s theorem (Theorem 2.6 below) using a criteria for formulas inside a model. We remark some equivalences on NIP in the terms of function spaces (Proposition 2.14 and Remark 2.15) and define the notion “NSOP in a model” (Definition 2.12). In Sect. 3, we point out connections between

\(^1\) In this article, when we refer to Shelah’s theorem, we mean this theorem.
some dividing lines in first order theories and subclasses of Baire class 1 functions (Remarks 3.2, 3.4 and Proposition 3.6).

2 Model theory and function spaces

We work in classical (\{0, 1\}-valued) model theory context, although similar results are valid in the continuous logic framework. Our model theory notation is standard, and a text such as [18] will be sufficient background for the model theory part of the paper. For the function theory part, read this paper with [12, 13] in your hand. We frequently switch from model theory to function theory and vice versa, so we provide some necessary functional analysis background.

First we recall some definitions and facts from functional analysis and then translate them into model theoretic language.

Function spaces

We give definitions of the function spaces with which we shall be concerned, with some of elementary relations between them.

Let \( X \) be a set and \( A \) a subset of \( \mathbb{R}^X \). The topology of pointwise convergence on \( A \) is that inherited from the usual product topology on \( \mathbb{R}^X \); namely the coarsest topology on \( A \) for which the map that sends each \( f \in A \) to \( f(x) \) is continuous for every \( x \in X \).

Let \( B \) be some collection of real valued functions on \( X \), containing \( A \). \( A \) is said to be relatively compact (or precompact) in \( B \) if the closure \( \text{cl}_B(A) \) of \( A \) in \( B \) is compact. In this case \( \text{cl}_B(A) \) is closed (and compact) in the space \( \mathbb{R}^X \), so in particular it implies that the closure of \( A \) in \( \mathbb{R}^X \) is contained in \( B \).

Recall that for a topological space \( X \), \( C(X) \) denotes the space of all bounded continuous functions on \( X \); it is a linear space under pointwise addition. We can equip the space \( C(X) \) with the uniform norm topology, the uniform metric defined by \( d(f, g) = \sup_{x \in X} |f(x) - g(x)| \). The weak topology on \( C(X) \) is the coarsest topology such that every bounded linear functional on \( C(X) \) is continuous. So, \( C(X) \) has three different topologies; namely the topology of pointwise convergence, the uniform topology, and the weak topology.

A well known fact in functional analysis states that for a compact space \( X \), the weak topology and the pointwise convergence topology on norm-bounded subsets of \( C(X) \) are the same. (See Proposition 462E in [3].)

For a complete metric space \( X \), a real-valued function \( f \) on \( X \) is said to be of the first Baire class or Baire 1, if it is the pointwise limit of a sequence \( (f_n) \) of continuous functions on \( X \). This means that for each \( \varepsilon > 0 \) and each \( x \in X \) there is a natural number \( k \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( n \geq k \). The set of Baire 1 functions on \( X \) is denoted by \( B_1(X) \).

A real-valued function \( f \) on a topological space \( X \) is upper (resp. lower) semi-continuous if and only if \( \{ x : f(x) \geq r \} \) (resp. \( \{ x : f(x) \leq r \} \)) is closed for every real number \( r \). A function \( f \) is called semi-continuous if it is either upper or lower semi-continuous. A known classical theorem, due to Baire, asserts that:
Fact 2.1 ([6, p. 274]) A real-valued function $f$ on a complete metric space $X$ is lower semi-continuous if and only if there is a sequence $(f_n)$ of continuous functions such that $f_1 \leq f_2 \leq \cdots$ and $(f_n)$ converges pointwise to $f$ (for short we write $f_n \nearrow f$).

A real-valued function $f$ on $X$ is called a difference of bounded semi-continuous functions (short DBSC) if there exist bounded semi-continuous functions $F_1$ and $F_2$ on $X$ with $f = F_1 - F_2$. The class of such functions is denoted by $DBSC(X)$. Since every lower (or upper) semi-continuous function is the limit of a monotone sequence of continuous functions, so $DBSC(X)$ is a proper subclass of $B_1(X)$ (see Remark 2.15 below). To summarize, $\mathbb{C}(X) \not\subseteq DBSC(X) \subseteq B_1(X) \not\subseteq \mathbb{R}^X$. (More details can be found in [2].)

We will see shortly, the order property corresponds to $\mathbb{C}(X)$ and the strict order property has connection to $DBSC(X)$.

In this paper, typically $A$ will be a subset of $\mathbb{C}(X)$, the set of bounded continuous functions on $X$. Moreover, it suffices to assume that $A$ is countable and $X$ is compact and Polish, i.e. a separable completely metrizable topological space. Note that the uniform closure of $A$ is contained in $\mathbb{C}(X)$ but in general the poinwise closure of $A$ is not contained in $\mathbb{C}(X)$, or even in $B_1(X)$.

Model theory translation

We fix an $L$-formula $\phi(x, y)$, a complete $L$-theory $T$ and a subset $A$ of the monster model of $T$. We let $\tilde{\phi}(y, x) = \phi(x, y)$. Let $X = S_\tilde{\phi}(A)$ be the space of complete $\tilde{\phi}$-types on $A$, namely the Stone space of ultrafilters on Boolean algebra generated by formulas $\phi(a, y)$ for $a \in A$. Each formula $\phi(a, y)$ for $a \in A$ defines a function $\phi(a, y) : X \to \{0, 1\}$, which takes $q \in X$ to 1 if $\phi(a, y) \in q$ and to 0 if $\phi(a, y) \notin q$. Note that $X$ is compact and these functions are continuous, and as $\phi$ is fixed we can identify this set of functions with $A$.\footnote{We should be more careful with this assertion; if the variables $y$ are just dummy variables that play no role, we can not recover $A$ from $X$. This result is true for the space of full types, but not necessarily for just the $\phi$-types. However, for the sake of simplicity we continue to write $A \subseteq \mathbb{C}(X)$.} So, $A$ is a subset of all bounded continuous functions on $X$, denoted by $A \subseteq \mathbb{C}(X)$. Just as we did above, one can define $B_1(X)$ and $DBSC(X)$.

To summarize, for an $L$-formula $\phi(x, y)$ and a subset $A$ of an $L$-structure $M$, we can assume that $A$ is a subset of $\mathbb{R}^X$ where $X = S_\tilde{\phi}(A)$ and $A$ has the topology of pointwise convergence as above. Moreover, every $f \in A$ is continuous, i.e. $A \subseteq \mathbb{C}(X)$.

The only additional thing we need to remark on is the following result (see [12, Corollary 2.10] and [16, Proposition 2.2]):

Fact 2.2 (Eberlein–Grothendieck Criterion) Let $(a_i)$ be a sequence in some model of $T$ and $\phi(x, y)$ a formula. Then the following are equivalent:

(i) There is no any sequence $(b_j)$ such that $\phi(a_i, b_j)$ holds iff $i < j$.
(ii) For any sequence $(b_j)$, $\lim_i \lim_j \phi(a_i, b_j) = \lim_j \lim_i \phi(a_i, b_j)$ when the limits on both sides exist.
(iii) Every function in the closure of $\{\phi(a_i, y) : S_\tilde{\phi}([a_i]) \to \{0, 1\} : i < \omega\}$ is continuous.
History The equivalence (ii) ⇔ (iii) in the general case, i.e. for real-valued functions on arbitrary compact spaces, is due to Grothendieck [5], which he says it is based on an idea of Eberlein. Pillay [15] proved the equivalence (i) ⇔ (iii) and pointed out that these conditions are equivalent to definability of coheirs (see [16]). In [9], Iovino also provided a proof of Fact 2.2 for real-valued formulas.

2.1 A new characterization of SOP

First, we recall some notions and facts. Let \( \phi(x, y) \) be formula and \( n \) a natural number. We say that a formula \( \psi(x_1, \ldots, x_n) \) is a \( \phi\)-\( n \)-formula if it is of the forms
\[
\exists y (\bigwedge_{i \in E} \phi(x_i, y) \land \bigwedge_{j \in F} \neg \phi(x_j, y)) \quad \text{or} \quad \forall y (\bigvee_{i \in E} \phi(x_i, y) \lor \bigvee_{j \in F} \neg \phi(x_j, y))
\]
where \( E, F \) are disjoint subsets of \( \{1, \ldots, n\} \). In this case, \( \psi(x_1, \ldots, x_n) \) has \( n \) free variables \( x_1, \ldots, x_n \) and a bounded variable \( y \).

If \( M \) is a model of a theory and \( \vec{a} = (a_1, \ldots, a_n) \in M^n \), the \( \phi\)-\( n \)-type of \( \vec{a} \), denoted by \( tp_{\phi, n}(\vec{a}) \), is the set of all \( \phi\)-\( n \)-formulas \( \psi(\vec{x}) \) such that \( \models \psi(\vec{a}) \).

Definition 2.3 ([18, Definition I.2.3]) Let \( T \) be a complete \( L \)-theory, \( \phi(x, y) \) an \( L \)-formula, \( N \) a number and \( (a_i) \) a sequence in some model. The sequence \( (a_i) \) is a \( \phi\)-\( N \)-indiscernible sequence (over the empty set) if for each \( i_1 < \cdots < i_N < \omega \), \( j_1 < \cdots < j_N < \omega \),
\[
\text{tp}_{\phi, N}(a_1 \ldots a_{i_N}) = \text{tp}_{\phi, N}(a_{j_1} \ldots a_{j_N}).
\]

Fact 2.4 Let \( T \) be a complete \( L \)-theory, \( \phi(x, y) \) an \( L \)-formula, \( M \) a model of \( T \), and \( N \) a natural number.

(i) If \( (a_i) \) is an infinite sequence in \( M \), there is an infinite subsequence \( (b_i) \) which is a \( \phi\)-\( N \)-indiscernible sequence.

(ii) If \( I \subset J \) are two (infinite) linear ordered sets and \( (a_i)_{i \in I} \) is an infinite \( \phi\)-\( N \)-indiscernible sequence in \( M \), there is a sequence \( (b_j)_{j \in J} \) (possibly in an elementary extension of \( M \)) which is a \( \phi\)-\( N \)-indiscernible sequence and \( (b_j)_{j \in J} \) has the same \( \phi\)-\( N \)-type as \( (a_i)_{i \in I} \).

Proof (i) follows from (infinite) Ramsey’s theorem (see Theorem I.2.4 of [18]) and (ii) follows from the compactness theorem.

Definition 2.5 (SOP for a theory)

(i) Let \( T \) be a complete \( L \)-theory, \( \mathcal{U} \) the monster model of \( T \), and \( \phi(x, y) \) an \( L \)-formula. We say that \( \phi(x, y) \) has the strict order theory (for the theory \( T \)) if there exists a sequence \( (a_i : i < \omega) \) such that for all \( i < \omega \),
\[
\phi(\mathcal{U}, a_i) \subsetneq \phi(\mathcal{U}, a_{i+1}).
\]

3 Although these notions seems very restrictive and unnatural, they are very useful for proving the main theorem of this section, i.e., Theorem 2.6 below. Note that the notion \( \phi\)-\( n \)-type is completely different from the notion \( \phi\)-type we defined earlier.
A complete theory $T$ has the strict order property if there is a formula $\phi(x, y)$ which has the strict order property (for $T$).

$SOP$ stands for the strict order property, and $NSOP$ for not the strict order property.

In Definition 2.12 below we give a localized version of $SOP$. (See also Remark 2.13 below.)

As we will see shortly, the following localized version of Shelah’s theorem leads to a new characterization of $SOP$ for a theory. In the following theorem, we will follow the argument in Theorem 4.7, chapter 2 [18].

**Theorem 2.6** (Localized Shelah’s theorem) Let $T$ be a complete $L$-theory and $\phi(x, y)$ an $L$-formula. Suppose that there are infinite sequences (not necessarily indiscernible) $(a_i), (b_j)$ in some model, a natural number $N$ and a set $E \subseteq \{1, \ldots, N\}$ such that

(i) for each $i_1 < \cdots < i_N < \omega$, $\psi(a_{i_1}, \ldots, a_{i_N})$ holds, where

$$\psi(x_1, \ldots, x_N) := \neg\left(\exists y\left(\bigwedge_{i \in E} \phi(x_i, y) \land \bigwedge_{i \in N \setminus E} \neg\phi(x_i, y)\right)\right),$$

and

(ii) $\phi(a_i, b_j)$ holds if and only if $i < j$.

Then the theory $T$ has $SOP$.

Before giving the proof let us remark:

**Remark 2.7** (i) Note that Theorem 2.6(i) identifies a weaker condition $\mathbb{P}_{\phi, \bar{a}}$ than $NIP$ such that $OP_{\phi, \bar{a}, \bar{b}} + \mathbb{P}_{\phi, \bar{a}}$ implies $SOP$, where $\bar{a} = (a_i), \bar{b} = (b_j)$ and $OP_{\phi, \bar{a}, \bar{b}}$ means that $\bar{a}, \bar{b}$ witness $\phi$ has the order property. We will see shortly, in fact, $SOP$ is equivalent to the existence of $\bar{a}, \bar{b}$ and $\phi$ such that $OP_{\phi, \bar{a}, \bar{b}} + \mathbb{P}_{\phi, \bar{a}}$ holds. (See Proposition 2.10 below.)

(ii) We will establish a connection between this presentation of $SOP$ and a well-known subclass of Baire 1 functions. (See Remark 2.11 below.)

**Proof of Theorem 2.6** By Fact 2.4, we can assume that $(a_i)$ is a $\phi$-$N$-indiscernible sequence. Now, we repeat the argument of Theorem 4.7, chapter 2 of [18]. By (i), there are the natural number $N$ and $\eta : N \to \{0, 1\}$ defined by $\eta(i) = 1$ if $i \in E$, and $= 0$ otherwise, such that $\bigwedge_{i \leq N} \phi(a_i, y)^{\eta(i)}$ is inconsistent. (Recall that for a formula $\phi$, we use the notation $\phi^0$ to mean $\neg\phi$ and $\phi^1$ to mean $\phi$.) Starting with that formula, we change one by one instances of $\neg\phi(a_i, y) \land \phi(a_{i+1}, y)$ to $\phi(a_i, y) \land \neg\phi(a_{i+1}, y)$. Finally, we arrive at a formula of the form $\bigwedge_{i < k} \phi(a_i, x) \land \bigwedge_{k \leq i \leq N} \neg\phi(a_i, x)$. By (ii), the tuple $b_k$ satisfies that formula. Therefore, there is some $i_0 \leq N, \eta_0 : N \to \{0, 1\}$ such that

$$\bigwedge_{i \neq i_0, i_0+1} \phi(a_i, y)^{\eta_0(i)} \land \neg\phi(a_{i_0}, y) \land \phi(a_{i_0+1}, y)$$

is inconsistent, but

$$\bigwedge_{i \neq i_0, i_0+1} \phi(a_i, y)^{\eta_0(i)} \land \phi(a_{i_0}, y) \land \neg\phi(a_{i_0+1}, y)$$
is consistent. Let us define \( \varphi(\bar{a}, y) = \bigwedge_{i \neq i_0, i_0+1} \phi(a_i, y)^{\eta_0(i)} \). By Fact 2.4, we may increase the sequence \((a_i : i < \omega)\) to a \(\varphi\)-\(\check{N}\)-indiscernible sequence \((a_i : i \in \mathbb{Q})\). Then for \(i_0 \leq i < i' \leq i_0 + 1\), the formula \(\varphi(\bar{a}, y) \land \phi(a_i, y) \land \neg \phi(a_{i'}, y)\) is consistent, but \(\varphi(\bar{a}, y) \land \neg \phi(a_i, y) \land \phi(a_{i'}, y)\) is inconsistent. Thus the formula \(\theta(x, y) = \varphi(\bar{a}, y) \land \phi(x, y)\) has the strict order property. \(\Box\)

Note that the formula \(\theta(x, y)\) above has parameters. However it is clear that if the formula \(\eta(x, y, \bar{a})\) has \(SOP\), where \(\bar{a}\) are parameters, then so does the formula \(\eta(x, y\bar{z})\).

Now we want to establish a connection between \(SOP\) and a class of functions. Recall that a real-valued function on a complete metric space is said to be of the first Baire class, or Baire 1, if it is the pointwise limit of a sequence of continuous functions. The following lemma provides a connection between \(SOP\) and a proper subclass of Baire 1 functions, namely \(DBSC\).

For easier reading, we note that the conditions (i), (ii) in Lemma 2.8 below are abstractions of the notion \emph{alternation number} in model theory. Of course, they are not equivalent to the notion \(NIP\) for a formula. (See the explanations after Proposition 2.10 below.) It seems that the direction (i) \(\Rightarrow\) (iii) of Lemma 2.8 is new to model theorists.

**Lemma 2.8** Let \((f_n)\) be a sequence of \(\{0, 1\}\)-valued functions on a set \(X\). Then the following are equivalent:

(i) There are a natural number \(N\) and a set \(E \subseteq \{1, \ldots, N\}\) such that for each \(i_1 < \cdots < i_N < \omega\),

\[
\bigcap_{j \in E} f_{ij}^{-1}(1) \cap \bigcap_{j \in N \setminus E} f_{ij}^{-1}(0) = \emptyset.
\]

(ii) There is a natural number \(M\) such that \(\sum_{1}^{\infty} |f_n(x) - f_{n+1}(x)| \leq M\) for all \(x \in X\).

Suppose moreover that \(X\) is a compact metric space and \(f_n\)’s are continuous, then (ii) above (or equivalently (i)) implies (iii) below:

(iii) \((f_n)\) converges pointwise to a function \(f\) which is \(DBSC\).

**Proof** (i) \(\Leftrightarrow\) (ii): Suppose that (i) holds. Note that (i) states that we have a special pattern that never exists; that is, \(\bigcap_{j \in E} f_{ij}^{-1}(1) \cap \bigcap_{j \in N \setminus E} f_{ij}^{-1}(0) = \emptyset\). Suppose, for a contradiction, that there is an element \(x \in X\) such that \(\sum_{1}^{\infty} |f_k(x) - f_{k+1}(x)| \geq 2N - 1\). Therefore, there are \(f_{k_1}, \ldots, f_{k_{2N}}\) such that \((f_k(x) = 1 \Leftrightarrow f_{k_i+1}(x) = 0)\) for all \(i < 2N\). Let \(i_1\) be in \(\{k_1, k_2\}\) such that the value of \(f_{i_1}(x)\) is the same as the pattern above, i.e. \(f_{i_1}(x) = 1\) iff \(1 \in E\). Let \(i_2\) be in \(\{k_3, k_4\}\) such that the value of \(f_{i_2}(x)\) is the same as the pattern above. We can choose \(i_3, \ldots, i_N\) similarly. Note that \(x \in \bigcap_{j \in E} f_{ij}^{-1}(1) \cap \bigcap_{j \in N \setminus E} f_{ij}^{-1}(0)\). This is the special pattern above, a contradiction. The other direction is even easier. Indeed, let \(N = M + 2\), and \(E = \{k : k \text{ is odd and } k \leq N\}\). Then \(\bigcap_{j \in E} f_{ij}^{-1}(1) \cap \bigcap_{j \in N \setminus E} f_{ij}^{-1}(0) = \emptyset\).

(ii) \(\Rightarrow\) (iii): Clearly, \((f_n)\) converges pointwise to a function \(f\). (We can define \(f_0(x) = 0\) for all \(x\).) Set \(F_1(x) = \sum_{0}^{\infty} (f_{n+1} - f_n)^+(x)\) and \(F_2(x) = \sum_{0}^{\infty} (f_{n+1} - \)
There are no formula \( \phi(123) \). (Recall that for a function \( h : X \to \mathbb{R} \), \( h^+(x) = \max(h(x), 0) \) and \( h^-(x) = \max(-h(x), 0) \).) Then \( f = F_1 - F_2 \) and \( F_1, F_2 \) are both lower semi-continuous. (Note that \( g_k = \sum_{j=0}^{k-1} (f_{n+1} - f_n)^+ \not\to F_1 \) and since the limit of an increasing sequence of continuous functions is lower semi-continuous (Fact 2.1), so \( F_1 \) is lower semi-continuous. Similarly for \( F_2 \).

\[ \square \]

**Remark 2.9**

(i) Note that Lemma 2.8(i) is an abstraction of the condition (i) of Theorem 2.6. Indeed, let \( f_n(y) = \phi(a_n, y) \) where \( a_n \) is a parameter in some model.

(ii) Let us do a model theoretic translation, we set \( f_n(y) = \phi(a_n, y) \) and \( X = S_\phi(A) \) where \( A = \{a_n : n < \omega \} \subseteq M \models T. \) Clearly, \( \phi(a_n, y) \) is continuous and since \( A \) is countable, so \( X \) is a metric space. This means that additional assumptions of (iii) in Lemma 2.8 hold.

(iii) We can expect a converse to (ii) \( \Rightarrow \) (iii) of the above lemma. Indeed, by Fact 2.1 above, if \( X \) is a compact metric space and \( f \) is the DBSC then there are a sequence \( (f_n) \) of (bounded) continuous functions and a natural number \( M \) such that \( (f_n) \) converges pointwise to \( f \) and \( \sum_1^\infty |f_n(x) - f_{n+1}(x)| \leq M \) for all \( x \).

(iv) Note that Lemma 2.8(ii) guarantees that the sequence \( (f_n) \) converges pointwise, but there are Baire 1 functions which are not DBSC (see Remark 2.15(i) below).

The following gives a new characterization of \( SOP \) (for a theory) and shows that the converse of Theorem 2.6 above is also true.

**Proposition 2.10** (Characterization of \( NSOP \)) Let \( T \) be a complete \( L \)-theory and \( U \) the monster model of \( T \). Then the following are equivalent:

(i) \( T \) is \( NSOP \).

(ii) There are no formula \( \phi(x, y) \) and sequences \( (a_i) \) and \( (b_{j}) \) in \( U \), a natural number \( N \) and a set \( E \subseteq \{1, \ldots, N\} \) such that two conditions (i) and (ii) in Theorem 2.6 hold, simultaneously.

(iii) There are no formula \( \phi(x, y) \) and indiscernible sequences \( (a_i) \) and \( (b_{j}) \) in \( U \), a natural number \( N \) and a set \( E \subseteq \{1, \ldots, N\} \) such that two conditions (i) and (ii) in Theorem 2.6 hold, simultaneously.

(iv) For any formula \( \phi(x, y) \) and any sequence (not necessarily indiscernible) \( (a_i : i < \omega) \), if there is a natural number \( N \) such that for any \( b \in U, \sum_1^\infty |\phi(a_i, b) - \phi(a_{i+1}, b)| \leq N \), then there is no infinite sequence \( (b_{j}) \) such that \( \phi(a_i, b_{j}) \) holds iff \( i < j \).

(v) For any formula \( \phi(x, y) \) and any indiscernible sequence \( (a_i : i < \omega) \), if there is a natural number \( N \) such that for any \( b \in U, \sum_1^\infty |\phi(a_i, b) - \phi(a_{i+1}, b)| \leq N \), then there is no infinite sequence \( (b_{j}) \) such that \( \phi(a_i, b_{j}) \) holds iff \( i < j \).

Moreover, if \( T \) is \( NIP \) then \( T \) is \( NSOP \) iff for any formula \( \phi(x, y) \) there is a natural number \( N \) such that for any sequence (not necessarily indiscernible) \( (a_i : i < \omega) \), if for any \( b \in U, \sum_1^\infty |\phi(a_i, b) - \phi(a_{i+1}, b)| \leq N \), then there is no infinite sequence \( (b_{j}) \) such that \( \phi(a_i, b_{j}) \) holds iff \( i < j \).

**Proof** (i) \( \Rightarrow \) (ii) is Theorem 2.6. (ii) \( \Rightarrow \) (iii) is evident, and (iii) \( \Rightarrow \) (ii) follows from Ramsey’s theorem and the compactness theorem. (i) \( \Rightarrow \) (iv) follows from Theorem 2.6 and Lemma 2.8. (iv) \( \Rightarrow \) (ii) follows from Lemma 2.8. (iv) \( \Rightarrow \) (v) is evident.
(ii) ⇒ (i): Suppose, in order to get a contradiction, that \( \phi(x, y) \) has \( \text{SOP} \) for the theory \( T \). This means that there is an indiscernible sequence \( (a_i) \) such that \( \exists y (\neg \phi(a_i, y) \land \phi(a_j, y)) \) iff \( i < j \). So, there is some sequence \( (b_j) \) such that \( \phi(a_i, b_j) \) holds iff \( i < j \), i.e., the condition (ii) in Theorem 2.6 holds. Let us define \( \psi(x_1, x_2) = \exists y (\phi(x_1, y) \land \neg \phi(x_2, y)) \). So, for \( i < j \), \( \psi(a_i, a_j) \) does not hold. Let \( N = \{1, 2\} \) and \( E = \{1\} \) and \( \psi(x_1, x_2) \) be as above. Then the condition (i) in Theorem 2.6 holds as well. This is a contradiction.

(v) ⇒ (i): By Lemma 2.8 and an argument similar to the direction (ii) ⇒ (i), the proof is completed.

Recall that for a formula \( \phi(x, y) \), an indiscernible sequence \( (a_i) \) and a parameter \( b \), the alternation of \( \phi(x, b) \) on \( (a_i) \) is bounded by a natural number \( n \), if there are at most \( n \) increasing indices \( i_1 < \cdots < i_n \) such that \( \models \phi(a_i, b) \iff \neg \phi(a_{i+1}, b) \) for all \( i < n \). A theory \( T \) has \( \text{NIP} \) if for any formula \( \phi(x, y) \) there is a natural number \( n_{\phi} \) such that for any indiscernible sequence \( (a_i) \) and any parameter \( b \), the alternation of \( \phi(x, b) \) on \( (a_i) \) is bounded by \( n_{\phi} \). Note that in \( \text{NIP} \) case, such numbers depend just on formulas.

Using this notion, Proposition 2.10(ii) above asserts that a theory \( T \) is \( \text{NSOP} \) if for any formula \( \phi(x, y) \) and any sequence \( \bar{a} = (a_i) \), if there is a natural number \( n_{\phi, \bar{a}} \) such that for any \( b \) the alternation of \( \phi(x, b) \) on \( \bar{a} \) is bounded by \( n_{\phi, \bar{a}} \), then there is no infinite sequence \( (b_j) \) such that \( \phi(a_i, b_j) \) holds iff \( i < j \). Note that the sequences are not necessarily indiscernible and such natural numbers \( n_{\phi, \bar{a}} \) depend on both the formulas and the sequences; not just on formulas. Thus, Lemma 2.8 above presents a ‘localized and wider’ notion of alternation number.

Remark 2.11 Recall that, for a set \( A \) of an \( L \)-structure \( M \) and an \( L \)-formula \( \phi(x, y) \), one can consider the continuous function \( \phi(a, y) : S_{\tilde{\phi}}(A) \to \{0, 1\} \) defined by \( \phi(a, q) = 1 \) if \( \phi(a, y) \in q \) and 0 if \( \phi(a, y) \notin q \). (Here \( \tilde{\phi} \) is the same formula as \( \phi \), but we have exchanged the role of variables and parameters, and \( S_{\tilde{\phi}}(A) \) is the space of complete \( \tilde{\phi} \)-types over \( A \).) If \( A \) is countable, \( S_{\tilde{\phi}}(A) \) is a compact Polish space. Recall that, using a crucial result due to Eberlein and Grothendieck (Fact 2.2), for a sequence (not necessarily indiscernible) \( (a_i) \) there is no infinite sequence \( (b_j) \) such that \( \phi(a_i, b_j) \iff i < j \) if and only if every function in the pointwise closure of \( \{\phi(a_i, y) : S_{\tilde{\phi}}([a_i]_{<\omega}) \to \{0, 1\}, i < \omega\} \) is continuous. By Lemma 2.8 and Proposition 2.10, \( \text{NSOP} \) corresponds to the class of functions which are difference of bounded semi-continuous functions (\( \text{DBSC} \)) on the type spaces. For a formula \( \phi(x, y) \) we set \( D(\phi) = \{f : \text{there exist } (a_i) \text{ and natural number } N \text{ such that } \phi(a_i, y) \text{ converges pointwise to } f \text{ and } \sum_{i=1}^{\infty} |\phi(a_i, q) - \phi(a_{i+1}, q)| \leq N \text{ for all } q \in S_{\tilde{\phi}}([a_i]_{<\omega})\} \). Similarly, we set \( C(\phi) = \{f : \text{there exists } (a_i) \text{ such that } \phi(a_i, y) \text{ converges pointwise to } f \text{ on } S_{\tilde{\phi}}([a_i]_{<\omega}) \text{ and } f \text{ is continuous}\} \). Using these definitions and the facts above, a complete theory \( T \) has \( \text{SOP} \) if and only if there is a formula \( \phi \) such that \( D(\phi) \setminus C(\phi) \neq \emptyset \). (See Fact 2.2 above.) Notice that the above characterization of \( \text{NSOP} \) is of the form “if ... then ...”. Indeed, by Fact 2.2 and Proposition 2.10, a theory \( T \) is \( \text{NSOP} \) if and only if
“for any formula $\phi(x, y)$ and any (infinite) sequence $(a_i : i < \omega)$, if for some natural number $N$, $\sum_{i=1}^{\infty} |\phi(a_i, q) - \phi(a_{i+1}, q)| \leq N$ for all $q \in S_\phi^{\infty}((a_i)_{i<\omega})$, then $\phi(a_i, y)$ converges to a continuous function.”

In [13], the notions $NIP$ and/or $NOP$ relative to a set or model were studied. We are now ready to introduce the analogous notion of $NSOP$ in a model or a set.

**Definition 2.12** ($NSOP$ in a model) Let $T$ be a complete $L$-theory, $\phi(x, y)$ an $L$-formula, and $M$ a model of $T$.

(i) A set $\{a_i : i < \kappa\}$ of $l(y)$-tuples from $M$ is said to be a $BSOP$-witness for $\phi(x, y)$ if the following conditions (1),(2) hold, simultaneously.

1. there are a natural number $N$ and a set $E \subseteq \{1, \ldots, N\}$ such that for each $i_1 < \cdots < i_N < \kappa$, $M \models \psi(a_{i_1}, \ldots, a_{i_N})$ where

$$\psi(x_1, \ldots, x_N) := \neg\left(\exists y \left( \bigwedge_{i \in E} \phi(x_i, y) \land \bigwedge_{i \in N \setminus E} \neg \phi(x_i, y) \right) \right),$$

and

2. for each natural number $n$ and $i_1 < \cdots < i_n < \kappa$,

$$M \models \exists y_1 \cdots y_n \left( \bigwedge_{k<j \leq n} \phi(a_{i_k}, y_j) \land \bigwedge_{j' \leq k', \leq n} \neg \phi(a_{i_{j'}}, y_{j'}) \right).$$

(ii) Let $A$ be a set of $l(x)$-tuples from $M$. Then $\phi(x, y)$ has $BSOP$-witness in $A$ if there is a countably infinite sequence $(a_i : i < \omega)$ of elements of $A$ which is a $BSOP$-witness for $\phi(x, y)$.

(iii) Let $A$ be a set of $l(x)$-tuples in $M$. We say that $\phi(x, y)$ has $NBSOP$-witness in $A$ if it does not have $BSOP$-witness in $A$.

(iv) $\phi(x, y)$ has $NBSOP$-witness in $M$ if it has $NBSOP$-witness in the set of $l(x)$-tuples from $M$.

**Remark 2.13**

(i) If $\phi$ has $BSOP$-witness in $A$, then a Boolean combination of instances of $\phi$ has $SOP$ for the theory $T$. Of course, if $\phi$ has $SOP$ for $T$, then it has $BSOP$-witness in some models of $T$.

(ii) $\phi$ has $NSOP$ for the theory $T$ iff it has $NBSOP$-witness in every model $M$ of $T$ iff it has $NBSOP$-witness in some model $M$ of $T$ in which all types over the empty set in countably many variables are realised.

(iii) If $\phi(x, y)$ has $BSOP$-witness in some model $M$ of $T$, then there are arbitrarily long $BSOP$-witness for $\phi$ (of course in different models).

We will shortly give examples that indicate why this notion is useful (see Examples 2.17 and 2.18 below).

**2.2 Remarks on $NIP$**

We already knew that a theory is $NIP$ iff for any formula $\phi(x, y)$ and any sequence $(a_i : i < \omega)$ in the monster model there is a subsequence $(a_{j_i} : i < \omega)$ such that for any
element $b$ (in the monster model) there is an eventual truth value of $(\phi(a_{ji}, b) : i < \omega)$. In the language of function theory, the subsequence $(\phi(a_{ji}, y) : i < \omega)$ converges to a (Baire 1) function $f$. In the following we will see that the criterion presented in Lemma 2.8 makes it possible to say more: the limit $f$ should be DBSC.

**Proposition 2.14** (Characterization of NIP) Let $T$ be a complete $L$-theory, $\phi(x, y)$ an $L$-formula and $\mathcal{U}$ the monster model of $T$. Then the following are equivalent:

(i) $\phi$ has NIP for $T$.

(ii) For any sequence (not necessarily indiscernible) $(a_i : i < \omega)$, there is a subsequence $(a_{ji} : i < \omega)$ such that for any $b \in \mathcal{U}$ there is an eventual truth value of $(\phi(a_{ji}, b) : i < \omega)$.

(iii) For any sequence (not necessarily indiscernible) $(a_i : i < \omega)$, there are a subsequence $(a_{ji} : i < \omega)$ and a natural number $N$ such that $\sum_{i=1}^{\infty} |\phi(a_{ji}, b) - \phi(a_{ji+1}, b)| \leq N$ for each $b \in \mathcal{U}$.

(iv) For any sequence (not necessarily indiscernible) $(a_i : i < \omega)$, there is a subsequence $(a_{ji} : i < \omega)$ such that the sequence $\phi(a_{ji}, y)$ converges to a function $f$ which is DBSC.

**Proof** The equivalence (i) $\iff$ (ii) is folklore. The direction (iii) $\Rightarrow$ (iv) follows from Lemma 2.8. The direction (iv) $\Rightarrow$ (ii) is evident.

(i) $\Rightarrow$ (iii): Suppose, for a contradiction, that there is a sequence $(a_i)$ such that (iii) fails. Let $n$ be an arbitrary natural number and $\varphi$ be an arbitrary formula. By Fact 2.4, we can assume that $(a_i)$ is $\varphi$-$n$-indiscernible. Then, by Lemma 2.8, there is a subsequence $(a_{ji}, \ldots, a_{jn})$ and $b \in \mathcal{U}$ such that $\phi(a_{ji}, b)$ holds if $i$ is even. As $n$ and $\varphi$ are arbitrary, the following set is a type

$$\left\{ \exists y \left( \bigwedge_{i=1}^{n} \phi(x_{2i}, y) \land \neg\phi(x_{2i+1}, y) \right) \land (x_1, \ldots, x_{2n+1}) \text{ is indiscernible} : n < \omega \right\}.$$ 

By the compactness theorem, there are an indiscernible sequence $(c_i)$ and an element $d$ such that $\phi(c_i, d)$ holds if and only if $i$ is even, a contradiction. □

Proposition 2.14(iv) above and the following remark show that the approach of the present paper would be useful.

**Remark 2.15** (i) Recall that for a compact metric space $X$ and a subset $A \subset X$, then the indicator function $1_A$ is Baire 1 if and only if $A$ is both $F_\sigma$ and $G_\delta$. (See Definition 24.1 and Theorem 24.10 of [10]. In this case, notice that the sets $A = \{ x : 1_A(x) = 1 \}$ and $A^c = \{ x : 1_A(x) = 0 \}$ are both $G_\delta$; that is, they are countable intersections of open sets). Notice that the class of functions which are difference of bounded semi-continuous functions is a proper subclass of Baire 1 functions. Furthermore, every $\{0, 1\}$-valued function is the DBSC if and only if there exist disjoint differences of closed sets $W_1, \ldots, W_m$ such that $f = \sum_{i=1}^{m} 1_{W_i}$ (see [2, Proposition 2.2]). This result makes clear why DBSC$(X)$ is a proper subclass of $B_1(X)$.

(ii) As Pierre Simon pointed out to us, it is known that for every sequence $(a_i)$ in the monster model of a NIP theory one can find a subsequence $(a_{ji})$ that their types
converges to a finitely satisfiable type and it is known that invariant types in \(NIP\) theories have definitions which are finite Boolean combinations of closed sets (see [7, Proposition 2.6]). In fact, by the above remark, that is equivalent to Proposition 2.14(iv).

The following statement clearly indicates why some people— not all them— say that the independence property and the strict order property are orthogonal. That is, Shelah’s theorem is of the form “\(p \land (p \rightarrow s) \equiv \text{stability}\).”

**Corollary 2.16** (Shelah’s Theorem, revisited) *Let \(T\) be a complete \(L\)-theory. Then the following are equivalent:

(1) \(T\) is stable.

(2) The following two properties hold:

(i) \((NIP)\): For any formula \(\phi(x, y)\) and any sequence (not necessarily indiscernible) \((a_i : i < \omega)\) and a natural number \(N\) such that the sequence \(\phi(a_{ji}, y)\) converges to a function \(f\) and for any \(b\) in the monster model, \(\sum_{i}^{\infty} |\phi(a_{ji}, b) - \phi(a_{ji+1}, b)| \leq N\), and

(ii) \((NSOP)\): For any formula \(\phi(x, y)\) and any sequence (not necessarily indiscernible) \((a_i : i < \omega)\), if the sequence \(\phi(a_i, y)\) converges to a function \(f\) and there is some natural number \(N\) such that for any \(b\) in the monster model, \(\sum_{i}^{\infty} |\phi(a_{i}, b) - \phi(a_{i+1}, b)| \leq N\), then \(f\) is continuous.

**Proof** By Proposition 2.14, \(NIP\) is equivalent to (i). By Proposition 2.10 and Fact 2.2 (or just Remark 2.11), \(NSOP\) is equivalent to (ii). Now, by the usual form of Shelah’s theorem the proof is completed. \(\square\)

We can give a proof of Shelah’s theorem above using a well-known theorem of functional analysis, namely the Eberlein–Šmulian Theorem (Fact 3.1 below). Also, one can provide a local version of Shelah’s theorem: A formula \(\phi\) is stable for the theory \(T\) iff the conditions (i),(ii) above hold for \(\phi\). We will compare shortly the above observations with the Eberlein–Šmulian Theorem.

### 2.3 Examples

To clarify the results, we build some examples. First, we give a model \(M\) and a formula \(\phi(x, y)\) such that \(\phi\) has \(OP\) and \(NIP\) in \(M\), and \(Th(M)\) has \(SOP\). This example is not interesting in itself but it is a step towards an example with interesting properties.

**Example 2.17** Let \(A = \{a_i : i < \omega\}\) and \(B = \{b_i : i < \omega\}\). We define a binary relation \(R(x, y)\) on \(D = (A \cup B) \times (A \cup B)\) as follows:

(1) \(R(a_i, a_j)\) holds iff \(i < j\),

(2) For each \(k < \omega\), we define:

(2–k) for any \(i \leq k\), \(R(a_{2k+i}, b_k)\) holds iff \(i\) is even, and for any \(j > 2^k + k\), \(\neg R(a_j, b_k)\) holds.

(3) For any other \((c, d) \in D\), \(R(c, d)\) does not hold.
(Note that (1) says that \( R(x, y) \) has the order in \( M = A \cup B \). It is easy to verify that the formula \( R(x, y) \) is \( NIP \) in \( M \).)

Moreover, \( Th(M) \) has \( SOP \). Indeed, notice that \( \neg R(a_i, \mathcal{U}) \subseteq \neg R(a_j, \mathcal{U}) \) for all odd numbers \( i < j \).

In the following, we give a model \( N \) and a formula \( \phi(x, y) \) such that \( \phi \) has \( NIP \) and \( OP \) in \( N \), and moreover \( Th(N) \) has \( IP \) and \( \phi \) has \( NSOP \) for \( Th(N) \).

**Example 2.18** Let \( A = \{a_i : i < \omega \} \) and for any infinite subsequence \( I \) of \( \omega \), let \( B_I = \{b^I_i : i < \omega \} \). We define a binary relation \( R(x, y) \) on \( D = (A \cup \bigcup I B_I) \times (A \cup \bigcup I B_I) \) as follows:

1. \( R(a_i, a_j) \) holds iff \( i < j \),
2. For each infinite subset \( I \) of \( \omega \) and each \( k < \omega \), the condition (2–k) in the above example holds for \( A = \{a_i : i \in I \} \) and \( B = B_I \).
3. For any other \((c, d) \in D\), \( R(c, d) \) does not hold.

Now, it is easy to verify that the formula \( R(x, y) \) is \( NIP \) in \( N = A \cup \bigcup I B_I \) but it has \( OP \) in \( N \). Also, (2) guarantees that the complete theory of this structure has \( IP \) (see Proposition 2.14(iii)). But, by Lemma 2.8, one can show that its theory does not have \( SOP \). In fact, the type of \( SOP \) (for any formula) is not consistent with \( Th(N) \). Indeed, notice that for any natural number \( N \), there is some natural number \( m \) such that there is no any subsequence \( c_1, \ldots, c_m \) of \( (a_i) \) such that for each \( i_1 < \cdots < i_N \leq m \), \( \models \psi(c_{i_1}, \ldots, c_{i_N}) \), where \( \psi(x_1, \ldots, x_N) \) is the formula in Theorem 2.6(i) with \( \phi(x, y) = R(x, y) \) (or any other formula).

This example confirms that there is a formula \( \phi \) \( NSOP \) for a theory and a sequence \((\phi(a_i, y) : i < \omega)\) pointwise converges to a non-continuous function. This statement contrasts with the theory of Random Graph (see Example 3.5 below).

### 3 Dividing lines in model theory and Baire class 1 functions

This part is mainly expository but is (in our view) very illuminating. We point out some parallels between model theoretic dividing lines for first order theories and subclasses of Baire 1 functions, and propose a new thesis. For this, we recall some notions and the following well-known theorem of functional analysis.

If \( X \) is a topological space then \( C(X) \) denotes the space of bounded continuous functions on \( X \). A subset \( A \subseteq C(X) \) is relatively weakly (pointwise) compact if it has compact closure in the weak (pointwise) topology on \( C(X) \). Notice that for a compact space \( X \), a subset \( A \) of \( C(X) \) is weakly compact if and only if it is norm-bounded and pointwise compact (cf. [3, Theorem 462E(ii)]).

**Fact 3.1** (Eberlein–Šmulian Theorem) *Let \( X \) be a compact Hausdorff space and \( A \) a norm-bounded subset of \( C(X) \). Then for the topology of pointwise convergence the following are equivalent:*

1. \( A \) is relatively compact in \( C(X) \).
2. The following two properties hold:
(i) \((RSC)\) A is relatively sequentially compact in \(\mathbb{R}^X\), and
(ii) \((SCP)\) A has the sequential completeness property.

Explanation. See [22] for a proof of the Eberlein–Šmulian theorem. Recall that, A is relatively sequentially compact in \(\mathbb{R}^X\) if every sequence of A has a convergent subsequence in \(\mathbb{R}^X\), and A has the sequential completeness property if the limit of every convergent sequence of A is continuous. (2) is precisely the condition \(B\) in the main theorem of [22]. Indeed, each sequence contains a subsequence converging to an element of \(C(X)\) if and only if (i) each sequence has a convergent subsequence in \(\mathbb{R}^X\), and (ii) the limit of every convergent sequence is continuous. Also, (1) is the condition \(A\) in [22].

Remark 3.2 Recall from [17] (or Proposition 2.14 above) that NIP implies 2.(i) in the Eberlein–Šmulian Theorem. By Proposition 4.6 of [12] (or Remark 3.4 below), if for every countable set A of the monster model and every formula \(\phi\) the condition 2.(ii) holds, then the theory is NSOP. Notice that, by Proposition 2.10 (or Remark 2.11) and Proposition 2.14, the converses do not hold. Recall from [12] that 2.(ii) is called the weak sequential completeness property (short SC P), and 2.(i) is called the relative sequential compactness (short RSC). Notice that relative compactness of A corresponds to stability, by a criterion due to Eberlein and Grothendieck (Fact 2.2). Now, we can complete the diagram presented in [12]:

\[
\begin{array}{ccc}
\text{Shelah} & \iff & \text{NIP} \quad \& \quad \text{NSOP} \\
\text{Eberlein–Grothendieck} & \downarrow & \uparrow \\
\text{Eberlein–Šmulian} & \iff & \text{RSC} \quad \& \quad \text{SCP}
\end{array}
\]

We will shortly prove that \(\text{NIP}\) is equivalent to \(\text{RSC}\) under compactness, and \(\text{SCP}\) and \(\text{NSOP}\) are not equivalent. We make a point that \(\text{NIP}\) together with many conditions correspond to stability, so of course there is no reason to expect that all notions agree.

A thesis

In the Eberlein–Šmulian Theorem, notice that (ii) is the weakest topological property such that (i) and (ii) imply relative compactness. This leads to the following definition.

Definition 3.3 Let \(T\) be a complete \(L\)-theory. We say that \(T\) has

(i) the relative sequential compactness property (short \(RSC\)) if

\((RSC)\) for any formula \(\phi(x, y)\) and any infinite sequence \((a_i : i < \omega)\), there is a subsequence \((a_{j_i} : i < \omega)\) such that for any parameter \(b\) there is an eventual truth value of \((\phi(a_{j_i}, b) : i < \omega)\).
Dividing lines in unstable theories and subclasses of…

(ii) the sequential completeness property (short SCP) if

\((SCP)\) for any formula \(\phi(x, y)\) and any infinite sequence \((a_i : i < \omega), \text{ if}\) for every \(b\) in the monster model there is an eventual truth value of the sequence \((\phi(a_i, b) : i < \omega), \text{ then}\) there is no infinite sequence \((b_j : j < \omega)\) such that \(\phi(a_i, b_j)\) holds iff \(i < j\).

**Remark 3.4** (i) Every stable theory has the SCP.

(ii) A theory is NSOP if it has the SCP.

(iii) A theory is NIP if and only if it has RSC.

(iv) A theory is stable if and only if it is NIP and has the SCP.

**Proof** (i): Immedaite.

(ii): Suppose that there are sequences \((a_i), (b_j)\) and formula \(\phi(x, y)\) such that the conditions of Theorem 2.6 hold. (Equivalently, the theory is SOP.) Then \(\phi(a_i, y)\) converges to a function \(f\) which is not continuous. (See Fact 2.2.) So, the SCP fails.

(iii): This is the equivalence (i) \(\Leftrightarrow\) (ii) of Proposition 2.14. (Note that in function theory the condition (iv) of Proposition 2.14 (equivalently NIP) strictly implies RSC, but their equivalence in model theory is due to compactness theorem.)

(iv): By (ii) above, SCP implies NSOP. So, NIP and SCP imply stability, by Shelah’s theorem. (One can give a proof using the Eberlein–Šmulian Theorem and Fact 2.2.) The converse is evident.

\(\square\)

**Example 3.5** (i) The theory of Random Graph has the SCP. Indeed, for any formula \(\phi(x, y)\), either \(\phi(x, y)\) is stable, or there is no infinite sequence \((a_i : i < \omega)\) such that for any \(b\) in the monster model there is an eventual truth value of the sequence \((\phi(a_i, b) : i < \omega)\). Recall that the theory of Random Graph has quantifier elimination. Therefore, one can easily check that atomic formulas are either stable, or satisfy the second alternative. By quantifier elimination, this holds for every formula.

(ii) The theory in Example 2.18 is NSOP but it does not have the SCP.

Note that in Lemma 2.8 we did not give a converse to (ii) \(\Rightarrow\) (iii). This suggests the following definition: A complete theory \(T\) has the DBSC if and only if for any formula \(\phi(x, y)\) and any infinite sequence \((a_i : i < \omega), \text{ if}\) the sequence \((\phi(a_i, y) : i < \omega)\) converges to a function which is DBSC, then there is no infinite sequence \((b_j : j < \omega)\) such that \(\phi(a_i, b_j)\) holds iff \(i < j\). Clearly, if a theory is DBSC then it is NSOP. Now we want to continue this process to create a hierarchy of theories. Let \(C\) be some subclass of Baire 1 functions, containing DBSC. We say that a theory \(T\) is (or has) \(C\) if

for any formula \(\phi(x, y)\) and any infinite sequence \((a_i : i < \omega), \text{ if}\) the sequence \((\phi(a_i, y) : i < \omega)\) converges to a function which is \(C, \text{ then}\) there is no infinite sequence \((b_j : j < \omega)\) such that \(\phi(a_i, b_j)\) holds iff \(i < j\).

Now we can give other Shelah-like theorems.

**Proposition 3.6** Let \(T\) be complete theory and \(C\) as above. Then \(T\) is stable if and only if it is both NIP and \(C\).
Proof. The proof is similar to the argument of Remark 3.4(iv). (Note that \( C \) implies DBSC and so NSOP.)

Notice that the SCP (DBSC) asserts that for any formula \( \phi(x, y) \), every Baire 1 (DBSC) function in the closure of \( \phi(a, y) \)'s is continuous. Set \( \text{Baire} 1(\phi) = \{ f : \text{there exists} \ (a_i) \text{ such that} \phi(a_i, y) \text{ converges pointwise to} \ f \} \), DBSC(\( \phi \)) = \{ f : \text{there exists} \ (a_i) \text{ such that} \phi(a_i, y) \text{ converges pointwise to} \ f \text{ and} \ f \text{ is DBSC} \} \) and \( C(\phi) = \{ f : \text{there exists} \ (a_i) \text{ such that} \phi(a_i, y) \text{ converges uniformly to} \ f \} \) as Remark 2.11 above. (Notice the difference between DBSC(\( \phi \)) and D(\( \phi \)) in Remark 2.11.) By these notations, we say that \( T \) has Baire 1 property (equivalently the SCP) iff for any formula \( \phi \), \( \text{Baire} 1(\phi) \setminus C(\phi) = \emptyset \). Similarly, we say that \( T \) is (or has) DBSC iff for any formula \( \phi \), DBSC(\( \phi \)) \setminus C(\( \phi \)) = \emptyset. We can do this process for each subclass \( C \supseteq \text{DBSC} \) of Baire 1 functions in the sense of [11]; a theory is (or has) C iff for any formula \( \phi \), \( C(\phi) \setminus C(\phi) = \emptyset \).

Notation: In the rest of this part, the symbol \( \mathfrak{P} \) (generated by frak{P}) denotes an arbitrary model theoretic property such that \( \mathfrak{P} \) implies NSOP (for example, NSOP\( _n \) or simplicity), and \( \mathcal{C} \) denotes an arbitrary subclass of Baire 1 functions on compact Polish spaces, containing DBSC. For a theory \( T \) and a formula \( \phi(x, y) \) we set \( C(\phi) = \{ f : \text{there exist} \ (a_i) \text{ such that} \phi(a_i, y) \text{ converges pointwise to} \ f \text{ and} \ f \in C \} \), and \( C(\phi) \) as Remark 2.11 above. For a model theoretic property \( \mathfrak{P} \), if there is a subclass \( \mathcal{C} \) of Baire 1 functions such that any theory \( T \) is \( \mathfrak{P} \) if and only if for any formula \( \phi \), \( C(\phi) \setminus C(\phi) = \emptyset \), then we write \( \mathfrak{P} = \mathfrak{P}_\mathcal{C} \). Similarly, for a subclass \( \mathcal{C} \), if there is a model theoretic property \( \mathfrak{P} \) such that any theory \( T \) has \( \mathfrak{P} \) if and only if for any formula \( \phi \), \( C(\phi) \setminus C(\phi) = \emptyset \), then we write \( \mathfrak{P} = \mathfrak{P}_\mathcal{C} \).

Recall that DBSC implies NSOP, and stability (or NOP) corresponds to the class of continuous functions (short Continuous). With these notations, \( \mathfrak{P}_{\text{DBSC}} \subseteq_\tau \mathfrak{P}_{\text{NSOP}} \), \( \mathfrak{P}_{\text{Continuous}} = \mathfrak{P}_{\text{NOP}} \) and \( \mathcal{C}_{\text{NSOP}} \subset_\tau \text{DBSC}, \mathcal{C}_{\text{NOP}} = \text{Continuous} \). Now, one can suggest the following diagram:

\[
\text{NOP} = \mathfrak{P}_{\text{Continuous}} \supseteq \cdots \supseteq \mathfrak{P}_{\text{Baire} 1} \supseteq \cdots \supseteq \mathfrak{P}_{\text{DBSC}} \subseteq_\tau \mathfrak{P}_{\text{NSOP}}
\]

\[
\text{Baire} 1 = \mathcal{C}_? \supseteq \cdots \supseteq \mathcal{C} \supseteq \cdots \supseteq \text{DBSC}, \supset \mathcal{C}_{\text{NSOP}} \supseteq \mathcal{C}_{\text{NOP}} = \text{Continuous}
\]

There are so many questions: for a model theoretic property \( \mathfrak{P} \), what is the right class \( \mathfrak{P}_\mathcal{C} \)? And converse, for a subclass \( \mathcal{C} \), what is the right model theoretic property \( \mathfrak{P}_\mathcal{C} \)?

Let us discuss possible answers. There are four possibilities. First: there are correspondences between some model theoretic classes and subclasses of Baire 1 functions. (See Question 3.7 below.) Second: some model theoretic dividing lines imply some subclasses of Baire 1 functions, or vice versa. Third: some model theoretic classes are divided by some subclasses of Baire 1 functions, or vice versa. Fourth: there are connections between subclasses of Baire 1 functions and classes in Keisler’s order. Everything that is the case is good.

Question 3.7 Are there any interesting relations between subclasses of Baire 1 functions and notions like NSOP\(_n\)?

Finally, we point out that the notion NSOP says that if any sequence of the form \( \phi(a_n, y) \) converges with a ‘special rate’, then the limit is continuous. One can expect

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other properties also have the same nature. If that is the case, the special rate for NSOP is stronger than the special rate for $\mathfrak{Q}$. The above points strongly inspire us to believe that model theoretic classification is correlated with a classification of Baire class 1 functions similar to the work of Kechris and Louveau in [11].

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