Classical theory of the Hall-effect
in an inhomogeneous magnetic field

Kasper Juel Eriksen and Per Hedegård

Ørsted Laboratory, Niels Bohr Institute,
Universitetsparken 5, DK-2100 Copenhagen, Denmark

(May 6, 2019)

Abstract

Inspired by recent experiments by Geim et al. we discuss the classical theory of the Hall effect of a 2 dimensional electron gas in an inhomogeneous magnetic field. The field modulation is in the form of flux tubes created by a superconductor overlayer. We find that an approach, where the vortices are treated as individual scatterers contributing to the collision term in the Boltzmann equation will not work — it leads to a vanishing Hall constant at \( T = 0 \). If the field is treated as a smooth contribution to the driving term in the Boltzman equation, the classical Hall constant emerges, in agreement with experiments when the Fermi wavelength is short in comparison with all other lengths in the problem.

73.50.J, 73.61.Ey, 73.50.Bk
I. INTRODUCTION

We will in this paper discuss the Hall effect in the case, where the applied magnetic field is spatially varying. The theory will be based on the Boltzmann equation, and we shall only consider the 2D case. Our motivation comes from recent experiments by Andrei Geim et al.\textsuperscript{1}, who put a superconductor over a 2 dimensional electron gas, and then measured the Hall voltage. The superconductor will only allow the magnetic field to penetrate in Abrikosov vortices, thereby modulating the field. For high fields the vortices are strongly overlapping and the field is only slowly varying, so that the usual Hall coefficient is to be expected. This is indeed what is seen experimentally and it is the result of our calculations. For low fields (below 100 G), where the vortices start to become spatially separated, the Hall coefficient depends on the 2D electron density and therefore on the de Broglie wavelength of the electrons at the Fermi surface. When the de Broglie wavelength is comparable to or greater than the diameter of a vortex, the Hall effect is reduced by an almost field independent fraction in low fields. The fraction is about 80\% in the electron gas with the smallest density experimentally obtainable. Since the effect depends on the electron density one might expect that it is a quantum mechanical effect. The only way to put quantum mechanics into the Boltzmann equation is through the scattering cross sections. Khaetskii\textsuperscript{2} has proposed treating the spatially separated vortices as asymmetric scatterers. At de Broglie wavelengths much shorter than the vortex diameter the electron will be scattered asymmetrically and in accordance with a classical picture. At the opposite limit, first treated by Aharonov and Bohm\textsuperscript{3}, where the wavelength is much larger than a vortex diameter the scattering is symmetric. Khaetskii and earlier Kuptsov and Moiseev\textsuperscript{4} showed that the degree of asymmetry gradually disappears as the diameter of the vortex is reduced in comparison with the electron wavelength. Khaetskii’s idea is that this reduced asymmetric scattering can account for the reduced Hall effect. His calculations show that this is indeed possible in a classical gas. We have extended his calculations to a degenerate gas obeying Fermi-Dirac statistics. Here the Pauli principle, and its restrictions on the scattering, will reduce the
calculated Hall coefficient to about $k_B T/\epsilon_F$, in strong disagreement with the experiment.

We have subsequently considered the case where the magnetic field is a slowly modulated field. This amounts to treating the $B$-field as a driving force on the left hand side of the Boltzmann equation. Here we find the classical Hall effect corresponding to a homogeneous field at all field strengths and electron densities. This is to be expected since we recover the experimental results when the electron wavelength is shorter than a vortex and we henceforth can talk about the magnetic field at the electron’s position with some confidence, while the procedure fails when the electron wavelength is longer than the field modulations.

II. VORTICES AS SCATTERING CENTERS

First we will describe the magnetic flux tubes as independent scattering centers, much like usual impurities. This of course is supposed to apply only at the low field limit, where the tubes are sufficiently far apart. The characteristic feature of scattering off flux tubes is that the scattering probability is asymmetric: There is an enhanced probability of electrons being scattered to the left. We will model this by an asymmetric scattering probability $w(k, \psi)$, where $\psi$ is the scattering angle and $k$ is the length of the momentum vector. The Boltzmann equation, linearized in the external electric field, has the familiar form

$$-e\vec{v} \cdot \vec{E} \frac{\partial f^0}{\partial \epsilon} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}},$$

where $f^0$ is the equilibrium distribution function. The collision term consists of two parts, a flux tube part and a usual impurity scattering part, which we will treat in the relaxation time approximation. Denoting the distribution function as $f(k, \phi)$, $\phi$ being the angle between $\vec{k}$ and the external field $\vec{E}$, we have

$$\left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = -\rho \int_0^{2\pi} \frac{d\psi}{2\pi} w(k, \psi)(f(k, \phi)(1 - f(k, \phi + \psi)) - f(k, \phi - \psi)(1 - f(k, \phi)))$$

$$- \frac{f(k, \phi) - f^0}{\tau}.$$

Here $\rho$ is the density of fluxtubes, i.e. $\rho = (BA)/\phi_0/2/A = eB/h$. The important difference to the work by Khaetskii\textsuperscript{3}, is the inclusion of the Pauli principle.
We will solve the equation by Fourier transforming in the angle $\phi$. Introducing

$$f_n(k) = \int_{0}^{2\pi} \frac{d\phi}{2\pi} e^{i n \phi} f(k, \phi), \quad w_n(k) = \int_{0}^{2\pi} \frac{d\phi}{2\pi} e^{i n \phi} w(k, \phi), \quad (3)$$

the Boltzmann equation (1) has the form

$$-e v E \frac{\partial f_0}{\partial \epsilon} (\delta_{n,1} + \delta_{n,-1})/2 = -(f_n(k) - \delta_{n,0} f^0)/\tau \quad (4)$$

$$-\rho \left( (w_0(k) - w_n(k)) f_n(k) + \sum_m (w_m(k) - w_{-m}(k)) f_{n-m}(k) f_m(k) \right).$$

The functions $w_n(k)$ satisfy $w_n(k)^* = w_{-n}(k)$. Accordingly there are both real and imaginary contributions to the effective relaxation time, due to the flux tubes. The imaginary parts have, as pointed out by Khaetskii, a simple interpretation, namely as an effective homogeneous magnetic field. Indeed in the Fourier transformed Boltzmann equation with a homogeneous magnetic field, the magnetic field term has the form:

$$e(\vec{v} \times \vec{B}) \frac{\partial f(\vec{k})}{\partial \vec{p}} \rightarrow -i n \omega_e f_n(k). \quad (5)$$

Upon linearization of the last term in (4) we get $f_0 = f^0$ and of the other terms only $f_1$ and $f_{-1}$ are non-vanishing and they become

$$f_1(k) = f_{-1}(k)^* = \frac{1}{2} \frac{ev \frac{\partial f_0}{\partial \epsilon} \tau(\epsilon)}{1 + i \rho (2 f_0(k) - 1) \text{Im}(w_1(k)) \tau(\epsilon)} E, \quad (6)$$

where

$$\tau(\epsilon)^{-1} = \rho (1 - \text{Re}(w_1(k))) + \tau^{-1}. \quad (7)$$

It is now straightforward to work out the conductivities. We get

$$\sigma_{xx} = \frac{n e^2}{m} \left\langle \frac{\tau(\epsilon)}{1 + ((2f_0 - 1) \alpha(\epsilon))^2} \epsilon \left( -\frac{\partial f_0}{\partial \epsilon} \right) \right\rangle, \quad (8)$$

and

$$\sigma_{xy} = -\frac{n e^2}{m} \left\langle \frac{\tau(\epsilon) \alpha(\epsilon)}{1 + ((2f_0 - 1) \alpha(\epsilon))^2 (2f_0 - 1)} \epsilon \left( -\frac{\partial f_0}{\partial \epsilon} \right) \right\rangle, \quad (9)$$

where the bracket $\langle \cdot \rangle$ is defined by
\[ \langle A \rangle = \frac{\int d\epsilon A(\epsilon)}{\int d\epsilon f_0(\epsilon)}, \]  
and \( \alpha(\epsilon) = \rho \tau(\epsilon) \text{Im}(w_1(\epsilon)) \). Khaetskii\(^2\) has shown that in the classical limit \( \alpha(\epsilon_F) = \omega_c \tau \).

The important difference between the result in equation (9) and Khaetskii’s result is the factor \( 2f_0(\epsilon) - 1 \), which is zero at the Fermi level. This means that the Hall voltage will disappear at \( T = 0 \). The factor comes from the proper implementation of the Pauli principle.

In the low-\( T \) \( (T \ll \epsilon_F) \) limit, where we approximate \( \alpha \) and \( \tau \) by their value at the Fermi level, we simply get

\[ \sigma_{xx} = \frac{n e^2 \tau(\epsilon_F)}{m} \frac{\text{Arctan}(\alpha(\epsilon_F))}{\alpha(\epsilon_F)}. \]  

By neglecting the term \( ((2f_0(\epsilon) - 1)\alpha(\epsilon))^2 \) in the denominator in (8) we get

\[ \sigma_{xy} \propto \frac{k_B T}{\epsilon_F} \frac{n e^2 \tau(\epsilon_F)}{m} \alpha(\epsilon_F). \]  

In the experiment by Andrei Geim et al.\(^1\) the temperature was 1.3 K. If we use an effective mass of \( 0.07m_e \), \( k_B T/\epsilon_F \) is less than 0.1. The electron mobilities were in the range of 40-100 \( \frac{m^2}{\text{Vs}} \) and the magnetic field was swept from 0 G to 200 G. Consequently \( \alpha \) is in the range of 0-2. In Figure 1 we have plotted the Hall resistance (normalized to the classical value \( B_{\text{eff}}/ne \)) as a function of the magnetic field. From here it is seen that the Hall effect is reduced by a factor of about \( k_B T/\epsilon_F \). To illustrate the crossover to the non-degenerate electron gas case treated by Khaetskii we have plotted the same quantity as a function of temperature in Figure 2.

The conclusion is that asymmetric scattering does not give rise to a Hall effect in a degenerate electron gas. This result is not in agreement with experiments. To explain this result we take as a simple model the scattering to the left through the same angle \( \phi_0 \) at each scattering event.

\[ w(\phi) \propto \delta(\phi - \phi_0) \]  

The effect of the \(-e\vec{E}\) field is to make more electrons go in it’s direction \( (\theta = 0) \). The effect of the scattering and therefore of the magnetic field is to oppose this effect by scattering electrons out of the \( \theta = 0 \) direction. In particular (considering only magnetic scattering)
From (2) we have with our model scattering (13) that
\[
\left( \frac{\partial f}{\partial t} \right)_{\text{coll-mag}}(k, 0) = f(k, -\phi_0)(1 - f(k, 0)) - f(k, 0)(1 - f(k, \phi_0))
\] (15)

We want to determine the angle dependence of \( f(k, \theta) \). If \( f(k, \theta) \) is small (\( k > \kappa_{\text{Fermi}} \)) it is the angle dependence of the \( f(k, \theta) \) outside the parentheses that dominates and we therefore drop the parentheses and get from (14)
\[
f(k, -\phi_0) - f(k, 0) < 0
\] (16)
\[
f(k, -\phi_0) < f(k, 0).
\] (17)

Consequently the electrons have a tendency to move to the left. This classical picture is due to the fact that the parentheses we have neglected are exactly the contribution from the Pauli principle. If, on the other hand, \( f(k, \theta) \) is close to 1 (\( k < \kappa_{\text{Fermi}} \)) the Pauli contribution dominates and we consequently drop the prefactor to the parentheses.
\[
(1 - f(k, 0)) - (1 - f(k, \phi_0)) < 0
\] (18)
\[
f(k, \phi_0) < f(k, 0)
\] (19)

The electrons now tend to move to the right. The reason is that in order to scatter in a dense Fermi gas it is essential that there are few electrons a scattering angle away. The electrons above and below the Fermi surface thus move in opposite directions (the \( 1 - 2f_0(\epsilon) \) factor) and the net asymmetry is very small (it arises solely from the difference in speed below and above the Fermi surface) — the Hall effect has disappeared.

III. MAGNETIC FIELD AS A DRIVING FORCE

From the above section we conclude that it is not correct to treat the vortices as scattering centers, that discontinuously changes a electron’s position in phasespace. In this section we
are going to discuss a complementary approach, where we treat the inhomogeneous field as a driving force, that changes a electron’s position continously in phase space. The approach is totally classical and is supposed only to apply in a dense electron gas, where the electron de Broglie wavelength is short in comparison with the length over which the magnetic field varies. We will assume that the magnetic field is random, correlated over lengths comparable to the effective London length.

To the usual linear order in the electric field and in the deviation $g(\vec{r}, \vec{k})$ from equilibrium $f^0(\vec{r}, k)$, the Boltzmann equation is

$$\vec{v} \cdot \frac{\partial g}{\partial \vec{r}}(\vec{r}, \vec{k}) - e\vec{E}(\vec{r}) \cdot \vec{v} \frac{\partial f^0}{\partial \epsilon}(k) - e(\vec{v} \times \vec{B}(\vec{r})) \cdot \frac{\partial f}{\partial \vec{p}}(\vec{r}, \vec{k}) = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}(\vec{r}, \vec{k}).$$

The collision contribution we will treat as scattering against fixed impurities. Accordingly in polar coordinates in the $\vec{k}$-space

$$\left( \frac{\partial f}{\partial t} \right)_{\text{coll}}(\vec{r}, \theta) = \rho \int_0^{2\pi} \frac{d\phi}{2\pi} w(\phi)(f(\vec{r}, \theta - \phi) - f(\vec{r}, \theta)),$$

where $\rho$ is the density of scatterers. We have suppressed the $k$ dependence.

We will write the magnetic field as $B(\vec{r}) = B^0 + \delta B(\vec{r})$, where $B^0$ is the average magnetic field. The Boltzmann equation can now be written

$$v \cos \theta \frac{\partial g}{\partial x}(\vec{r}, \vec{k}) + v \sin \theta \frac{\partial g}{\partial y}(\vec{r}, \vec{k}) + \omega_c \frac{\partial g}{\partial \theta}(\vec{r}, \vec{k})$$

$$+ \frac{e\delta B}{m} \frac{\partial g}{\partial \theta}(\vec{r}, \vec{k}) - ev \frac{\partial f^0}{\partial \epsilon} \cos \theta E_x(\vec{r}) - ev \frac{\partial f^0}{\partial \epsilon} \sin \theta E_y(\vec{r}) = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}(\vec{r}, \vec{k})$$

with $\omega_c = eB^0/m$.

In (22) we introduce the hermitian operator

$$D = i \frac{\partial}{\partial \theta} + i r_c \left( \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right),$$

where $r_c = v/\omega_c$ is the classical cyclotron radius in the magnetic field $B^0$. We now want to simulate the effect of the collisions by a relaxation time approximation, so we add and subtract a relaxation time contribution and arrive at
\[(D + \frac{i}{\omega_c \tau})g = \chi \]
\[\equiv \frac{i}{\omega_c} ev \frac{\partial f^0}{\partial \epsilon} (E_x \cos \theta + E_y \sin \theta) - \frac{i \delta B}{B^0} \frac{\partial g}{\partial \theta} + \frac{i g}{\omega_c \tau} + \frac{i}{\omega_c} \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}. \]  

(25)

Here \(\tau\) can be chosen arbitrarily, but later the usual value of \(\tau\) will emerge as a natural choice. The eigenfunctions of \(D\) with eigenvalue \(n\) (\(n\) integer) are

\[\psi_{\vec{q},n} = \frac{1}{\sqrt{2\pi A}} \exp \left( i \vec{q} \cdot \vec{r} - in\theta - ir_c q_x \sin \theta + ir_c q_y \cos \theta \right), \]

(26)

where \(A\) is the area of the electron gas. Therefore \(g\) can be written

\[g = \int d\vec{r}'' \int d\theta'' G(\vec{r}, \vec{r}'', \theta, \theta'') \chi(\vec{r}'', \theta'') \]

(27)

with the Green function

\[G(\vec{r}, \vec{r}'', \theta, \theta'') = \sum_{\vec{q},n} \frac{\psi_{\vec{q},n}(\vec{r}, \theta) \psi^*_{\vec{q},n}(\vec{r}'', \theta'')}{n + \frac{i}{\omega_c \tau}} \]
\[= \frac{1}{2\pi A} \sum_{\vec{q},n} \frac{e^{-in(\theta-\theta')}}{n + \frac{i}{\omega_c \tau}} \exp \left( i \vec{q} \cdot (\vec{r} - \vec{P}_\theta(\theta - \theta', \vec{r}'')) \right) \]
\[= \frac{1}{2\pi} \sum_n \frac{e^{-in(\theta-\theta')}}{n + \frac{i}{\omega_c \tau}} \delta(\vec{r} - \vec{P}_\theta(\theta - \theta', \vec{r}')), \]

(28)

where

\[\vec{P}_\theta(\phi) = \vec{r} + r_c \begin{pmatrix} \sin \theta \\ - \cos \theta \end{pmatrix} + r_c \begin{pmatrix} - \sin (\theta - \phi) \\ + \cos (\theta - \phi) \end{pmatrix} \]

(29)

is the classical cyclotron orbit in the homogeneous field \(B^0\) parametrised by the momentum coordinates (angles). We assume \(\omega_c \tau\) is positive and get with \(\phi = \theta - \theta'\), using Poisson’s summation formula

\[\frac{1}{2\pi} \sum_n \frac{e^{-i\phi}}{n + \frac{i}{\omega_c \tau}} = \frac{i \exp (-\frac{\phi}{\omega_c \tau})}{\exp (-\frac{2\pi}{\omega_c \tau}) - 1} \]

(30)

where \(|\phi|\) is the value of \(\phi\) in \([0, 2\pi]\). We, finally, have

\[g(\vec{r}, \theta) = \frac{i}{\exp(-2\pi/\omega_c \tau) - 1} \int_0^{2\pi} d\phi \exp (-\phi/\omega_c \tau) \chi(\vec{P}_\theta(\phi), \theta - \phi) \]

(31)
The physical interpretation of this formula is that you assume that the electrons move along their classical trajectory in a homogeneous magnetic field $B^0$. The correction to the local electron density is obtained by adding field corrections from the neighbouring points according to the number of electrons arriving from neighbouring points to your fieldpoint. The prefactor arises because we only integrate around the classical circular orbit once — we could drop it and instead integrate to infinity.

For our later choice of relaxation time the mean free path is very long (at least $2\mu m$ and normally more than $10\mu m$) so in fact we make a field average along the classical trajectory.

Now the cyclotron radius $r_c$ is of the order $2\mu m$ and therefore much bigger than the magnetic correlation length $\xi$, which is of the order $0.1\mu m$, so the system is strongly selfaveraging.

We now average and get

$$
\langle g \rangle = \frac{i}{e^{-2\pi/\omega_c \tau} - 1} \int_0^{2\pi} d\phi e^{-\phi/\omega_c \tau} \left< \chi(\vec{P}(\phi), \theta - \phi) \right>
$$

$$
= \frac{i}{e^{-2\pi/\omega_c \tau} - 1} \int_0^{2\pi} d\phi e^{-\phi/\omega_c \tau} \left( i \frac{\partial f^0}{\partial \epsilon} \frac{E_x^0}{\omega_c} (\langle E_x > \cos(\theta - \phi) + \langle E_y > \sin(\theta - \phi))
\right.

$$

$$
- \frac{i}{B^0} \frac{\partial g}{\partial \theta} (\theta - \phi) + i \frac{\langle g \rangle}{\omega_c} (\theta - \phi) + \frac{i}{\omega_c} \left< \frac{\partial f}{\partial t} \right>_{coll} (\theta - \phi) >
$$

$$
= g^0(\vec{r}, \theta, \tau) + \frac{i}{e^{-2\pi/\omega_c \tau} - 1} \int_0^{2\pi} d\phi e^{-\phi/\omega_c \tau} \left( - \frac{i}{B^0} \frac{\partial g}{\partial \theta} (\theta - \phi)
\right.

$$

$$
+ \frac{i}{\omega_c} \frac{\langle g \rangle}{\omega_c} (\theta - \phi) + \frac{i\rho}{\omega_c} \int_0^{2\pi} d\phi' w(\phi') (\langle g(\theta - \phi - \phi') > - \langle g(\theta - \phi) > )
$$

with

$$
g^0(\vec{r}, \theta, \tau) = \tau ev \frac{\partial f^0_k}{\partial \epsilon} \cos \theta \left( \frac{E_x^0}{1 + (\omega_c \tau)^2} - \frac{\omega_c \tau E_y^0}{1 + (\omega_c \tau)^2} \right)
$$

$$
+ \tau ev \frac{\partial f^0_k}{\partial \epsilon} \sin \theta \left( \frac{\omega_c \tau E_x^0}{1 + (\omega_c \tau)^2} + \frac{E_y^0}{1 + (\omega_c \tau)^2} \right).
$$

Notice that $g^0$ is the distribution function in a homogeneous magnetic field in the relaxation time approximation (with relaxation time $\tau$). We now decompose the angle part of the $\vec{k}$-space in Fourier components

$$
g(\theta) = \sum_n g_n e^{-in\theta}
$$

$$
w(\phi) = \sum_n w_n e^{-in\phi}
$$
Integrating out $\phi$ and $\phi'$ we get

\[
\langle g(\theta) \rangle = g^0(\vec{r}, \theta, \tau) + \sum_n \frac{ie^{-in\theta}}{\omega_c \tau - in} n \langle \delta B g_n \rangle
\]

\[
+ \sum_n \frac{\langle g_n \rangle e^{-in\theta}}{\omega_c \tau - in} \left( \frac{\rho(w_n - w_0)}{\omega_c} + \frac{1}{\omega_c \tau} \right) \tag{35}
\]

The $n = 0$ Fourier component is trivial ($\langle g_0 \rangle = \langle g_0 \rangle$) and for $n \neq 0$ we have

\[
\langle g_n \rangle = g^0_n(\tau) + \frac{1}{\omega_c \tau - in} \left[ in \langle \delta B g_n \rangle + \frac{\rho(w_n - 1)}{\omega_c} + \frac{1}{\omega_c \tau} \langle g_n \rangle \right] - i < \delta B g_1(\vec{r}) > \tag{36}
\]

where we, in the last step, have used $\langle \delta B \rangle = 0$.

The current is determined by $g_1$:

\[
j_x + ij_y = \int_0^\infty dk \frac{k}{2\pi^2} \int_0^{2\pi} d\theta g(k, \theta)(v \cos \theta + iv \sin \theta) \tag{37}
\]

\[
= \int_0^\infty \frac{kdk}{\pi} \frac{g_1}{B^0} \tag{38}
\]

If we choose

\[
\frac{1}{\tau} = \frac{1}{\tau_1} \equiv \rho(1 - w_1) = \rho \int_0^{2\pi} d\theta (1 - \cos \theta) w(\theta) \tag{39}
\]

we get

\[
\langle g_1 \rangle = g^0_1(\tau_1) + \frac{i}{\omega_c \tau_1 - i} \langle \delta B(\vec{r}) \delta g_1(\vec{r}) \rangle. \tag{40}
\]

In the appendix we have calculated the leading term in $\langle \frac{\delta B(\vec{r})}{B^0} \delta g_1(\vec{r}) \rangle$ to be

\[
\frac{-i}{e^{(-2\pi / \omega_c \tau_1)}} \int_0^{2\pi} d\phi e^{-i/\omega_c \tau_1} e^{i\phi} \frac{\delta B(\vec{r})}{B^0} \frac{\delta (\vec{P}_\phi(\phi))}{B^0} > \tag{41}
\]

\[
\sim \langle g_1 \rangle \frac{\xi}{r_c} \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(\vec{r})}{B^0} \tag{42}
\]

\[
= \langle g_1 \rangle > 0.06 \sqrt{10^{15} m^{-2}} \tag{43}
\]

We use here that for randomly distributed gaussian vortices each carrying half a flux quantum,

\[
\frac{\delta B(\vec{r})}{B^0} \frac{\delta B(\vec{r})}{B^0} = \frac{\hbar}{2\pi e^2}, \xi = 0.1 \mu m \text{ and that } r_c = \frac{522 G}{B^0} \sqrt{\frac{n m^{-2}}{10^{15}}} \mu m. \text{ In the experiment by Geim}
\]
et al. $\sqrt{\frac{10^{15}m^{-2}}{n}}$ varies between 0.5 and 1.7. Therefore, the average electron distribution is nearly as in the homogeneous case. If we use the approximation (41) in (40) and assume that $\tau_1$ can be treated as a constant in the energy integration in (38) we get that

$$\rho_{xy} \frac{B}{B_0/ne} = 1 + \frac{1}{1 - e^{-2\pi/\omega_c \tau_1}} \int_0^{2\pi} d\phi e^{-2\pi/\omega_c \tau_1} \sin \phi < \frac{\delta B(\vec{r}, \phi) \delta B(\vec{P}_\theta(\phi))}{B_0^2} >$$

(44)

$$\rho_{xx} \frac{m}{ne^2 \tau_1} = \frac{\omega_c \tau_1}{1 - e^{-2\pi/\omega_c \tau_1}} \int_0^{2\pi} d\phi e^{-2\pi/\omega_c \tau_1} \cos \phi < \frac{\delta B(\vec{r}, \phi) \delta B(\vec{P}_\theta(\phi))}{B_0^2} >$$

(45)

The integral in (44) is of the same order of magnitude as the prefactor in (43), implying the magnetoresistance is a few percent greater than it would have been in a homogeneous magnetic field. Because the $\phi$-integration in (44) is restricted by the correlation function to an interval from 0 to $\xi/\tau_c$, the integral is about $\left(\frac{\xi}{\tau_c}\right)^2 \approx 400$ times smaller than the prefactor in (43). The approximation (41) is therefore not the dominant contribution to the deviations in the Hall effect from the homogeneous case. Consequently, the deviation from the homogeneous Hall effect is at most a few promille. This is in perfect agreement with the fact that the experiment by Geim et al. showed no deviations from the homogeneous result in a dense electron gas.

**IV. CONCLUSION**

In the experiment by Geim et al., a reduced Hall effect in the Abrikosov vortex modulated field is only observed at electron densities below $4 \cdot 10^{15}m^{-2}$ and in a magnetic field of less than 100 Gauss. That is when the external magnetic field varies appreciably within a de Broglie wavelength of the electrons at the Fermi surface. In this regime it is expected that the Boltzmann equation description breaks down, but outside this regime our treatment of the vortices simply as a modulated magnetic field in the Boltzmann equation agrees with the experiment. To explain the reduced Hall effect one has to incorporate some kind of quantum mechanics. We have shown that it is not feasible to describe the vortices as scatterers and hide all the quantum mechanics in the calculation of the scattering cross sections.
A full quantum treatment should certainly include multiple coherent scattering by the vortices, because single scattering is contained in the present Boltzmann calculation. It is also clear that the Hall constant is reduced, since in the limit of very thin vortices (or what amounts to the same, a very dilute electron gas) the Hall constant will vanish. In this limit the time symmetry breaking will vanish, because one can without any change in the physics reverse the direction of the field by placing an infinitely thin Dirac vortex carrying one flux quantum \( h/e \) at each of the external vortices that carries half a flux quantum; the Dirac vortices having a field in the opposite direction of the external field.

We acknowledge discussions with Mads Brandbyge, Erland Brun Hansen, Ayoe Hoff, Dung-Hai Lee, Poul Erik Lindelof, Mads Nielsen and Rafael Taboryski.

**APPENDIX A:**

In this appendix we are going to calculate the correlation function

\[
< \frac{\delta B(\vec{r})}{B_0} \delta g_1(\vec{r}) > = \frac{1}{2\pi} e^{i\theta} \int_0^{2\pi} e^{i\theta} B_0 g(\vec{r}, \theta) d\theta.
\]

To do this we assume the higher order correlation functions factorize the second order correlation function out and we henceforth have the gaussian result

\[
< \delta B(\vec{r}) \Phi(B) > = \int d\vec{y} < \delta B(\vec{r}) \delta B(\vec{y}) > < \frac{\delta \Phi}{\delta B(\vec{y})} >.
\]

Using this in (A1) we get that

\[
< \frac{\delta B(\vec{r})}{B_0} \delta g_1(\vec{r}) > = \frac{1}{B_0} \int_0^{2\pi} e^{i\theta} \int d\vec{y} < \delta B(\vec{r}) \delta B(\vec{y}) > < \frac{\delta g(\vec{r}, \theta)}{\delta B(\vec{y})} >.
\]

Now we have from (B1) that

\[
\frac{\delta g(\vec{r}, \theta)}{\delta B(\vec{y})} = \exp \left( -\frac{2\pi}{\omega_c \tau} \right) \int_0^{2\pi} d\phi \exp (-\phi/\omega_c \tau) \times
\]

\[
\left\{ \frac{e \nu}{\omega_c} \frac{\partial f_0}{\partial \epsilon} \left( \frac{\delta E_x(\vec{P}_\theta(\phi))}{\delta B(\vec{y})} \cos (\theta - \phi) + \frac{\delta E_y(\vec{P}_\theta(\phi))}{\delta B(\vec{y})} \sin (\theta - \phi) \right) \right.
\]

\[
- i \frac{\delta (\vec{y} - \vec{P}_\theta(\phi))}{B_0} \frac{\partial g(\vec{P}_\theta(\phi), \theta - \phi)}{\partial \theta} - i \frac{\delta B(\vec{P}_\theta(\phi))}{B_0} \frac{\partial}{\partial \theta} \left( \frac{\delta g(\vec{P}_\theta(\phi), \theta - \phi)}{\delta B(\vec{y})} \right)
\]

\[
+ \frac{\delta}{\delta B(\vec{y})} \left( i \frac{g}{\omega_c \tau} + i \frac{\partial f}{\partial \epsilon} \right) \right\}. \tag{A4}
\]
When the last term is inserted in (A3) it is seen as before that if we choose \( \tau = \tau_1 \) the term cancels. When we use this expression below we will assume that this kind of cancellation can be done, and erase this term. (A4) is an equation to iteratively determine \( \frac{\delta g(\vec{r}, \theta)}{\delta B(y)} \) with the third term as the driving term. We henceforth expand \( \frac{\delta B(\vec{r})}{B^0} \delta g_1(\vec{r}) \) in this term. The first order contribution is

\[
< \frac{\delta B(\vec{r})}{B^0} \delta g_1(\vec{r}) > = \frac{1}{\exp(-2\pi/\omega c \tau_1) - 1} \int_0^{2\pi} d\phi e^{(-\phi/\omega c \tau_1)} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta} \int d\vec{y} \times \nonumber
\]

\[
< \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(y)}{B^0} > \delta(\vec{y} - \vec{P}_\theta(\phi)) \frac{\partial < g >}{\partial \theta} (\theta - \phi) \nonumber
\]

\[
= \frac{1}{\exp(-2\pi/\omega c \tau_1) - 1} \int_0^{2\pi} d\phi \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta} e^{(-\phi/\omega c \tau_1)} \times \nonumber
\]

\[
< \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(\vec{P}_\theta(\phi))}{B^0} > \frac{\partial < g >}{\partial \theta} (\theta - \phi). \quad \text{(A5)}
\]

Now \( < \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(\vec{P}_\theta(\phi))}{B^0} > \) only depends on the distance between \( \vec{r} \) and \( \vec{P}_\theta(\phi) \). Consequently \( < \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(\vec{P}_\theta(\phi))}{B^0} > \) is independent of \( \theta \) and we can move the \( \theta \)-integral through with the result that

\[
< \frac{\delta B(\vec{r})}{B^0} \delta g_1(\vec{r}) > = \frac{-i < g_1 >}{\exp(-2\pi/\omega c \tau_1) - 1} \int_0^{2\pi} d\phi e^{(-\phi/\omega c \tau_1)} e^{i\phi} < \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(\vec{P}_\theta(\phi))}{B^0} >. \quad \text{(A6)}
\]

\[
< \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(\vec{P}_\theta(\phi))}{B^0} > \text{ is only large within a correlation length } \xi \text{ and it’s size is estimated as } < \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(\vec{r})}{B^0} >. \quad \text{Accordingly, as an order of magnitude estimate we have}
\]

\[
< \frac{\delta B(\vec{r})}{B^0} \delta g_1(\vec{r}) > \sim < g_1 > \frac{\xi}{r_c} < \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(\vec{r})}{B^0} >. \quad \text{(A7)}
\]

This is in our case much less than \( < g_1 > \). To get the second order contribution we have to iterate (A4) once more, putting the driving term back into the first two terms and the fourth term on the right hand side of (A4). We will first take the fourth term and here we get

\[
< \frac{\delta B(\vec{r})}{B^0} \delta g_1(\vec{r}) > \text{4th term} = \frac{1}{\exp(-2\pi/\omega c \tau_1) - 1} \int_0^{2\pi} d\phi e^{(-\phi/\omega c \tau_1)} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta} \int d\vec{y} \times \nonumber
\]

\[
< \frac{\delta B(\vec{r})}{B^0} \frac{\delta B(y)}{B^0} > < \frac{\delta B(\vec{P}_\theta(\phi))}{B^0} \frac{\partial}{\partial \theta} \left( \frac{1}{\exp(-2\pi/\omega c \tau_1) - 1} \int_0^{2\pi} d\phi e^{(-\phi/\omega c \tau_1)} \times \right. \nonumber
\]

\[
\delta(\vec{y} - \vec{P}_\theta(\phi + \phi')) \frac{\partial g}{\partial \theta} (\vec{P}_\theta(\phi + \phi'), \theta - \phi - \phi') \left. \right) >. \quad \text{(A8)}
\]
\[
\begin{aligned}
&= \frac{1}{(\exp (-2\pi/\omega_0r \tau_1) - 1)^2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \exp\left(-\frac{-(\phi+\phi')}{\omega_0r \tau_1}\right) \left< \frac{\delta B(\vec{r})}{B_0} \frac{\delta B(\vec{r}_0(\phi + \phi'))}{B_0} \right> \times \\
&\int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta} \left< \frac{\delta B(\vec{r}_0(\phi))}{B_0} \frac{\partial^2 g}{\partial \theta^2} \left(\vec{B}_0(\phi + \phi'), \theta - \phi - \phi'\right) \right>.
\end{aligned}
\]

(A9)

The last integral is, apart from the differentiations, the same as the original integral, just now spatially separated. Therefore it is not greater than \(< g_1 > \frac{\xi}{r_c} < \frac{\delta B(\vec{r})}{B_0} \frac{\delta B(\vec{r})}{B_0} > - \text{ the order of magnitude from before.} \) Again the correlation function \(< \frac{\delta B(\vec{r})}{B_0} \frac{\delta B(\vec{r}_0(\phi'))}{B_0} > \) is only appreciable within a distance of \(\xi\). Since both \(\phi\) and \(\phi'\) are positive, the two remaining integrals are restricted to a region of size \(\xi r_c\). Implying that we have the following order of magnitude estimate:

\[
< \frac{\delta B(\vec{r})}{B_0} \delta g_1(\vec{r}) >^{4\text{th term}} \sim \frac{\xi}{r_c} \left( \frac{\xi}{r_c} < \frac{\delta B(\vec{r})}{B_0} \frac{\delta B(\vec{r})}{B_0} > \right)^2 < g_1 >.
\]

(A10)

If we use that the parenthesis is about 0.1 we get that this term is \(10^{2\mu m} \sim 10^{2\mu m} \), i.e. 200 times smaller than the first order contribution. To take care of the first two terms in (A4) we use that

\[
\frac{\delta E_x(\vec{P}(\theta))}{\delta B(\vec{y})} = \int d\vec{z} \frac{\delta E_x(\vec{P}(\phi))}{\delta g_0(\vec{z})} \frac{\delta g_0(\vec{z})}{\delta B(\vec{y})}.
\]

(A11)

The last term is treated as above. We find that the first iterate is 0. As explained in the main text the first order contribution to \(< \frac{\delta B(\vec{r})}{B_0} \delta g_1(\vec{r}) >\) mainly influences the magnetoresistance. Consequently, higher order terms contribute significantly to the Hall effect. The most important is the first term that arises when you go beyond the gaussian approximation\(\footnote{In a magnetic field consisting of fluxtubes placed at random}

\[
< \delta B(\vec{r}) \delta g_1(\vec{r}) > = \sum_{n=1}^{\infty} \frac{N}{\pi} \int d\vec{y}_1 \cdots d\vec{y}_n < \delta B(\vec{r}) \delta B(\vec{y}_1) \cdots \delta B(\vec{y}_n) >_{1\text{-flux}} < \frac{\partial g_1(\vec{r})}{\partial B(\vec{y}_1) \cdots \partial B(\vec{y}_n)} >,
\]

where \(N\) is the number of fluxes. In this appendix we have calculated the first term in the sum. The higher order terms may be calculated in exactly the same manner.
The order of magnitude of the relative deviation in the Hall effect from the homogeneous case, due to (AT2), is \(0.06 \sqrt{\frac{10^{15} m^2}{n}}\). Implying that in a dense electron gas the deviation in the Hall effect from the homogeneous case is about one promille.
REFERENCES

1 A. K. Geim, S. J. Bending and I. V. Grigorieva, Phys. Rev. Lett. 69, 2252 (1992).

2 A. V. Khaetskii, J. Phys. C. 3, 5515 (1991).

3 Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959).

4 D. A. Kuptsov and M. Yu. Moiseev, J. Phys. I France 1, 1165 (1991).
FIGURES

FIG. 1. The theoretical Hall resistivity coming from treating the magnetic fluxtubes as scatterers in the Boltzmann equation, normalized to the classical homogeneous result $\frac{B}{ne}$ as a function of $\alpha = \mu B$ at a temperature of 0.1, 0.5, 1.0 and 5.0 $\epsilon_F$. In the experiments by Andrei Geim et al., who used mobilities $\mu$ in the range of 40-100 $\frac{m^2}{Vs}$, they found in a dense gas the homogeneous result $\frac{B}{ne}$, at a temperature of less than 0.1 $\epsilon_F$.

FIG. 2. The hall resistivity normalized to the homogeneous value $\frac{B}{ne}$ as a function of the temperature for $\alpha = \mu B = 0.5$. 
