Some Remarks on the Coupling Prescription of Teleparallel Gravity

R. A. Mosna∗ and J. G. Pereira†

Abstract

By using a nonholonomic moving frame version of the general covariance principle, an active version of the equivalence principle, an analysis of the gravitational coupling prescription of teleparallel gravity is made. It is shown that the coupling prescription determined by this principle is always equivalent with the corresponding prescription of general relativity, even in the presence of fermions. An application to the case of a Dirac spinor is made.

Keywords: teleparallel gravity, gravitational coupling, spinor fields, Dirac equation.

∗Instituto de Física Gleb Wataghin, Universidade Estadual de Campinas, 13083-970, Campinas SP, Brazil. Also at Departamento de Matemática Aplicada, Universidade Estadual de Campinas, Campinas SP, Brazil. E-mail: mosna@ifi.unicamp.br

†Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-900 São Paulo SP, Brazil. E-mail: jpereira@ift.unesp.br
1 Introduction: The principle of general covariance

The principle of equivalence rests on the equality of inertial and gravitational masses. It establishes the local equivalence between gravitational and inertial effects on all physical systems. An alternative version of this principle is the so-called principle of general covariance \([1]\). It states basically that a physical equation will hold in a gravitational field if it is generally covariant, that is, if it preserves its form under a general transformation \(x \rightarrow x'\) of the spacetime coordinates. Of course, in the absence of gravitation, it must agree with the corresponding law of special relativity. The first statement can be considered as the active part of the principle in the sense that, by making a special relativity equation covariant, it is possible to obtain its form in the presence of gravitation. The second statement can be interpreted as its passive part in the sense that the special relativity equation must be recovered in the absence of gravitation. It is important to notice that, as is well known, any physical equation can be made covariant through a transformation to an arbitrary coordinate system. What the general covariance principle states is that, due to its general covariance, this physical equation will be true in a gravitational field if it is true in the absence of gravitation \([2]\). In other words, to get a physical equation that holds in the presence of gravitation, the active and the passive parts of the principle must be true.

In order to make an equation generally covariant, new ingredients are necessary: A metric tensor and a connection, which are in principle inertial properties of the coordinate system under consideration. Then, by using the equivalence between inertial and gravitational effects, instead of inertial properties, these quantities can be assumed to represent a true gravitational field. In this way, equations valid in the presence of a gravitational field are obtained from the corresponding free equations. This is the reason why the general covariance principle can be considered as an active version of the passive equivalence principle. In fact, whereas the former says how, starting from a special relativity equation, to obtain the corresponding equation valid in the presence of gravitation, the latter deals with the reverse argument, that is, it says that in any locally inertial coordinate system, the equations valid in the presence of gravitation must reduce to the corresponding equations valid in special relativity.

The above description of the general covariance principle refers to its usual holonomic version. An alternative, more general version of the principle can be obtained in the context of nonholonomic moving frames. The basic difference between these two versions is that, instead of requiring that an equation be covariant under a general transformation of the spacetime coordinates, in the moving frame version the equation is required to preserve its form under a local Lorentz rotation of the frame. Of course, in spite of the different nature of the involved transformations, the physical content of both approaches are the same \([3]\).

It is important to emphasize that the principle of general covariance is not an invariance principle, but simply a statement about the effects of gravitation. However, when use is made of the
equivalence between inertial and gravitational effects, the principle is seen to naturally yield a gravitational coupling prescription. By using a moving frame version of this principle, the basic purpose of this paper will then be to determine the form of the coupling prescription of teleparallel gravity\textsuperscript{1} implied by the general covariance principle.

2 Moving frames and associated structures

We begin by introducing in this section the strictly necessary concepts associated with moving frames. Let $M$ be a 4-dimensional Lorentzian manifold representing our physical spacetime. We assume that $M$ admits a global orthonormal moving frame\textsuperscript{2} (or tetrad) $\{e_a\}_a=0$. Let $g$ be the metric on $M$ according to which the elements of $\{e_a\}$ are orthonormal vector fields, i.e., $g_x(e_a|_x, e_b|_x) = \eta_{ab}$ for each $x \in M$, with $(\eta_{ab}) = \text{diag}(1, -1, -1, -1)$. Let $\{x^\mu\}$ be local coordinates\textsuperscript{3} in an open set $U \subset M$. Denoting $\partial_\mu = \partial/\partial x^\mu$, one can always expand the coordinate basis $\{\partial_\mu\}$ in terms of $\{e_a\}$,

$$\partial_\mu = h^a_\mu e_a$$

for certain functions $h^a_\mu$ on $U$. This immediately yields $g_{\mu\nu} := g(\partial_\mu, \partial_\nu) = h^a_\mu h^b_\nu \eta_{ab}$. Let us then see how the global basis of vector fields $\{e_a\}$ gives rise to both a Riemannian and a teleparallel structure on $M$.

2.1 Riemannian structure

This is obtained by noting that the metric $g$ on $M$ defines a unique metric-compatible torsion-free connection, which we denote by $\overset{\circ}{\nabla}$. This is the so-called Levi-Civita connection on $(M, g)$. It has a possibly nonvanishing curvature

$$\overset{\circ}{\mathcal{R}}(X, Y)e_a = (\overset{\circ}{\nabla}_X \overset{\circ}{\nabla}_Y - \overset{\circ}{\nabla}_Y \overset{\circ}{\nabla}_X - \overset{\circ}{\nabla}_{[X,Y]})e_a,$$

and its torsion vanishes identically:

$$\overset{\circ}{\mathcal{T}}(X,Y) = \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X,Y] \equiv 0.$$

We see in this way that a tetrad can be used to define a Riemannian structure on $M$.

\textsuperscript{1}The name teleparallel gravity is normally used to denote the general three-parameter theory introduced in [4]. Here, however, we use it as a synonymous for the teleparallel equivalent of general relativity, which is the theory obtained for a specific choice of these parameters.

\textsuperscript{2}We note that a classical result [5] asserts that every noncompact spacetime on which spinors may be defined carries a global orthonormal moving frame.

\textsuperscript{3}We use the Greek alphabet $\mu, \nu, \rho, \cdots = 0, 1, 2, 3$ to denote holonomic spacetime indices, and the Latin alphabet $a, b, c, \cdots = 0, 1, 2, 3$ to denote anholonomic indices related to the tangent Minkowski spaces.
2.2 Teleparallel structure

The moving frame \( \{e_a\} \) gives rise also to a global notion of parallelism on \( M \). Given two vectors \( v \in T_x M \) and \( w \in T_y M \) (with possibly \( x \neq y \)), one can simply compare their components with respect to the global frame \( \{e_a\} \). This concept can be elegantly formalized by defining another connection on \( M \), according to which the basis vectors \( e_a \) are parallel. It is easily seen that there exists a unique connection \( \nabla \) on \( M \) satisfying \( \nabla e_a = 0 \). This is the so-called Weitzenböck connection associated with the moving frame \( \{e_a\} \) (we note, however, that each tetrad induces its own \( \nabla \)). Given a vector field \( X = X^a e_a \) on \( M \), we have \( \nabla_\mu X = (\partial_\mu X^a)e_a + X^a \nabla_\mu e_a = (\partial_\mu X^a)e_a \). Thus, the \( \{e_a\} \)-components of \( \nabla_\mu X \) are simply the ordinary derivatives of the \( \{e_a\} \)-components of \( X \):

\[
\nabla_\mu (X^a e_a) = (\partial_\mu X^a)e_a.
\]

The connection \( \nabla \) has null curvature \( \mathring{R} \), but in general a non-trivial torsion \( T \). In fact,

\[
\mathring{R}(X,Y)e_a = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})e_a \equiv 0,
\]

for \( \nabla_X e_a = 0, \forall X \). Also, the Weitzenböck connection yields

\[
\mathring{T}(e_a, e_b) = -[e_a, e_b] \equiv -f_{ab}^c e_c,
\]

with \( f_{ab}^c \) the coefficient of nonholonomy of the basis \( \{e_a\} \). Notice that \( f_{ab}^c \) can be expressed in terms of the tetrad components \( h_a^{\mu} \) by

\[
f_{ab}^c = (h_a^{\mu} \partial_\mu h_b^{\nu} - h_b^{\mu} \partial_\mu h_a^{\nu})h^c_\nu.
\]

Now, remember that a local basis \( \{e_a\} \) of vector fields can be expressed as a coordinate basis only if \( f_{ab}^c = 0 \). Therefore, the fact that \( \nabla \) has a non-null torsion is closely related to the non-holonomicity of the tetrad \( \{e_a\} \). It is also important to note that \( \nabla \) is compatible with the above defined metric \( g \), i.e. \( g_x(e_a|_x, e_b|_x) = \eta_{ab} \) for all \( x \in M \). In fact, it follows from Eq. (1) that \( \partial_\mu (g(X,Y)) = \partial_\mu (X^a Y^b \eta_{ab}) = g(\nabla_\mu X, Y) + g(X, \nabla_\mu Y) \).

2.3 Relation between the structures

As we have seen, each moving frame \( \{e_a\} \) gives rise to both a Levi-Civita (\( R \neq 0, T = 0 \)) and a Weitzenböck (\( R = 0, T \neq 0 \)) connection in \( M \). It is important to keep in mind that, strictly speaking, curvature and torsion are properties of a connection, and not of spacetime \([6]\). Notice, for example, that both the Levi-Civita and the Weitzenböck connections are defined on the very same metric spacetime. The difference between these connections defines the contorsion tensor \( \mathring{K}_{\mu \alpha}^\beta \) (we recall that the space of connections is an affine space \([7]\)). More precisely, defining the connection coefficients \( \nabla_\mu \partial_\alpha = \mathring{\Gamma}_{\mu \alpha}^\beta \partial_\beta \) and \( \nabla_\mu e_a = \mathring{\Gamma}_{\mu a}^b e_b \), with analogous expressions for \( \nabla \), we have

\[
\mathring{\Gamma}_{\mu \alpha}^\beta = \mathring{\Gamma}_{\mu \alpha}^\beta + \mathring{K}_{\mu \alpha}^\beta.
\]
with
\[ \overline{K}_{abc} = \frac{1}{2} \left( T_{cab} + T_{cbe} - T_{bac} \right) \tag{5} \]
and \( T(e_a, e_b) = T_{ab} e_c \). From Eq. (4) it follows two important properties. First, if we choose a local Lorentz frame at a certain point \( x \in M \) (free fall), we have that \( \overline{\Gamma}_{\mu \alpha \beta} |_x = 0 \), and consequently \( \overline{K}_{\mu \alpha \beta} |_x = \overline{K}_{\mu \alpha \beta} |_x \). Second, seen from the tetrad frame \( \{e_a\} \), we have that \( \overline{\Gamma}_{\mu ab} = 0 \), and thus \( \overline{\Gamma}_{\mu ab} = -\overline{K}_{\mu ab} \). Notice that this relation holds in this particular frame. This justifies the apparent equality between an affine and a tensor quantities in this expression.

### 3 Gravitational coupling prescriptions

In this section, we use the general covariance principle to study the gravitational coupling prescription of teleparallel gravity. We start by briefly reviewing the usual holonomic coordinate version of the principle, as well as the coupling prescription it implies for the specific case of general relativity.

#### 3.1 General covariance principle: holonomic coordinate formulation

Let us consider the Minkowski spacetime (special relativity) endowed with an inertial reference frame with global coordinates \( \{x^\mu\} \). The motion of a free particle is then described by
\[ \frac{d^2 x^\mu}{ds^2} = 0. \tag{6} \]
In terms of general curvilinear coordinates \( \{\bar{x}^\mu\} \), the corresponding equation of motion is given by
\[ \frac{d^2 \bar{x}^\mu}{ds^2} + \bar{\Gamma}_{\alpha \beta}^\mu \frac{d \bar{x}^\alpha}{ds} \frac{d \bar{x}^\beta}{ds} = 0, \tag{7} \]
where the coefficients \( \bar{\Gamma}_{\alpha \beta}^\mu \) are the Christoffel symbols associated with the transformation \( x^\mu \rightarrow \bar{x}^\mu \).

More explicitly,
\[ \bar{\Gamma}_{\alpha \beta}^\mu = \frac{1}{2} \eta^{\mu \nu} \left( \frac{\partial \bar{\eta}_{\alpha \nu}}{\partial \bar{x}^\beta} + \frac{\partial \bar{\eta}_{\beta \nu}}{\partial \bar{x}^\alpha} - \frac{\partial \bar{\eta}_{\alpha \beta}}{\partial \bar{x}^\nu} \right), \]
where \( \bar{\eta}_{\mu \nu} \) is the expression of the same flat metric, but now written in terms of the curvilinear coordinates \( \bar{x}^\mu \), that is, \( \eta = \eta_{\mu \nu} dx^\mu dx^\nu = \bar{\eta}_{\mu \nu} d\bar{x}^\mu d\bar{x}^\nu \).

The general covariance principle can now be invoked to formulate the physical and non-trivial hypothesis that the metric \( \bar{\eta}_{\mu \nu} \) can be replaced by a true gravitational field \( g_{\mu \nu} \), and consequently \( \bar{\Gamma}_{\alpha \beta}^\mu \) will represent a dynamical field, given now by
\[ \bar{\Gamma}_{\alpha \beta}^\mu = \frac{1}{2} g^{\mu \nu} \left( \frac{\partial g_{\alpha \nu}}{\partial y^\beta} + \frac{\partial g_{\beta \nu}}{\partial y^\alpha} - \frac{\partial g_{\alpha \beta}}{\partial y^\nu} \right), \]
with its own degrees of freedom. Of course, the metric \( g_{\mu \nu} \) reduces to \( \eta_{\mu \nu} \) only when we are in the gravitational vacuum and \( y^\mu \) are inertial coordinates. Since \( \bar{\Gamma}_{\alpha \beta}^\mu \) is the symmetric connection, we can say that the general covariance principle leads naturally to the coupling of General Relativity.
Notice that the minimal coupling prescription is already contained in the above analysis. To see that, consider a vector field with local expression \( X^\mu \partial_\mu \), where \( x^\mu \) are inertial coordinates in flat spacetime, as above. Consider the variation \( \partial_\alpha X^\mu \) of \( X^\mu \) in the direction given by the coordinate \( x^\alpha \). The corresponding expression in curvilinear coordinates \( \{ \bar{x}^\mu \} \) is easily seen to be

\[
D_\alpha X^\mu := \partial_\alpha X^\mu + \bar{\Gamma}^\mu_{\alpha\beta} X^\beta.
\]  

Under the hypothesis that in the presence of gravity the corresponding expression is that obtained by replacing \( \bar{\Gamma}^\mu_{\alpha\beta} \) by the dynamical field \( \Gamma^\mu_{\alpha\beta} \), we see immediately that

\[
\partial_\alpha X^\mu \rightarrow \overset{\circ}{D}_\alpha X^\mu := \partial_\alpha X^\mu + \overset{\circ}{\Gamma}^\mu_{\alpha\beta} X^\beta
\]  

(9)

couples gravity to the original field \( \partial_\alpha X^\mu \). This is the well known minimal coupling prescription of general relativity. The general covariance principle, therefore, says that gravitation is minimally coupled to matter through the Levi-Civita connection.

### 3.2 General covariance principle: nonholonomic formulation

We start again with the special relativity spacetime endowed with the Minkowskian metric \( \eta \). If \( \{ x^\mu \} \) are inertial Cartesian coordinates in flat spacetime, the basis of (coordinate) vector fields \( \{ \partial_\mu \} \) is then a global orthonormal coordinate basis for the flat spacetime. The frame \( \delta_a = \delta^\mu_a \partial_\mu \) can then be thought of as a trivial tetrad, with components \( \delta^\mu_a \) (Kronecker delta). Consider now a local (that is, point-dependent) Lorentz transformation \( \Lambda_a^b(x) \), yielding the new moving frame

\[
e_a = e^\mu_a \partial_\mu,
\]  

(10)

where

\[
e^\mu_a(x) = \Lambda_a^b(x) \delta^\mu_b.
\]  

(11)

Notice that, on account of the locality of the Lorentz transformation, the new moving frame \( e_a = e^\mu_a \partial_\mu \) is possibly anholonomous, with

\[
[e_a, e_b] = f_{abc} e_c.
\]  

(12)

For this kind of tetrads, defined on flat spacetime, it follows from Eq. (3) that

\[
\partial_a (\Lambda_b^d) \Lambda^c_d = \frac{1}{2} (f^c_{ab} + f^c_{ba} - f_{ba}^c),
\]  

(13)

where \( \partial_a = e^\mu_a \partial_\mu \) denotes the ordinary directional derivative along \( e_a \), and \( \Lambda_b^a \Lambda^c_b = \delta^c_a \).

The free particle worldline is a curve \( \gamma : \mathbb{R} \rightarrow M \), with \( \dot{\gamma} = (dx^\mu / ds) \partial_\mu \) the particle 4-velocity. In terms of the moving frame \( \{ e_a \} \), we have \( \dot{\gamma} = V^a e_a \), where \( (dx^\mu / ds) = e^\mu_a V_a \). Seen from the moving frame, a straightforward calculation shows that the free equation of motion (6) can be written in the form

\[
\frac{dV^c}{ds} + \frac{1}{2} (f^c_{ab} + f^c_{ba} - f_{ba}^c) V^a V^b = 0,
\]  

(14)
where use has been made of Eq. (13). It is important to emphasize that, although we are in the flat spacetime of special relativity, we are free to choose any tetrad \( \{ e_a \} \) as a moving frame. The fact that, for each \( x \in M \), the frame \( \{ e_a \vert _x \} \) can be arbitrarily rotated introduces the compensating term \( \frac{1}{2} (f^{ab}_{\ c} + f^{ba}_{\ c} - f^{ba}_{\ c}) \) in the free particle equation of motion.

Let us now assume that, in the presence of gravity, it is possible to define a global moving frame \( \{ e_a \} \) on \( M \). As we have seen in the preceding section, such moving frame gives rise to both a Riemannian and a teleparallel structures. The hypothesis to be made here is that—according to the general covariance principle—the coefficient of nonholonomy can be assumed to represent a true gravitational field. In the context of general relativity, as is well known, we make the identification

\[
\frac{1}{2} (f^{ab}_{\ c} + f^{ba}_{\ c} - f^{ba}_{\ c}) = \circ \Gamma_{abc},
\]

where \( \circ \Gamma_{abc} \) is the Ricci coefficient of rotation, the torsionless spin connection of general relativity. In this case, the equation of motion (14) becomes

\[
\frac{dV^c}{ds} + \circ \Gamma_{abc} V^a V^b = 0,
\]

which is the geodesic equation of general relativity. To obtain the equation of motion in the teleparallel case, we have to identify, in accordance with Eq. (2), the coefficient of anholonomy \( f^{ab}_{\ c} \) with minus the torsion tensor:

\[
f^{ab}_{\ c} = - w^{\ c}_{ab}.
\]

Accordingly, the equation of motion (14) becomes

\[
\frac{dV^c}{ds} - w^{\ c}_{ab} V^a V^b = 0,
\]

where Eq. (5) has been used. This is the force equation of teleparallel gravity [9]. Of course, it is equivalent to the geodesic equation (16) in the sense that both describe the same physical trajectory.

Now, the above procedure can be employed to obtain a coupling prescription for gravitation. Consider again a vector field \( X \) with local expression \( X^\mu \partial_\mu \), where \( x^\mu \) are inertial coordinates in flat spacetime. Then, the variation of the components of \( X = X^\mu \partial_\mu \) in the \( \alpha \)-direction is trivially given by \( \partial_\alpha X^\mu \). Still in the context of flat spacetime, let us again consider a more general tetrad \( e_a = e_a^\mu \partial_\mu \) as in Eqs. (10) and (11). Let us denote \( X = X^\mu \partial_\mu = X^a e_a \), where \( X^a = e^a_\mu X^\mu \). Now, the variation of the components \( X^a \) must take into account the intrinsic variation of \( \{ e_a \} \). A straightforward calculation shows that

\[
[\partial_\alpha X^\mu] \partial_\mu = \left[ \partial_\alpha X^c + (\partial_\alpha e^\mu_{\ b}) e^c_\mu X^b \right] e_c = \left[ \partial_\alpha X^c + e^a_\alpha e_a (\Lambda_b^d \Lambda^c_d X^b) \right] e_c.
\]

It follows from Eq. (13) that

\[
[\partial_\alpha X^\mu] \partial_\mu = \left[ \partial_\alpha X^c + e^a_\alpha \frac{1}{2} (f^{ab}_{\ c} + f^{ba}_{\ c} - f^{ba}_{\ c}) X^b \right] e_c.
\]
Now, by considering again the general covariance principle, we assume that the presence of gravity is obtained by

(i) replacing $e_a^\mu$ by a nontrivial tetrad field $h_a^\mu$, which gives rise to a Riemannian metric $g_{\mu\nu} = \eta_{ab} h_a^\mu h_b^\nu$, and

(ii) replacing the coefficient of anholonomy either by Eq. (15) or (17). In the first case we obtain

$$\partial_\alpha X^c \rightarrow \overset{\circ}{D}_\alpha X^c := \partial_\alpha X^c + \overset{\circ}{\Gamma}_{\alpha b c} X^b,$$

which is the usual minimal coupling prescription of general relativity. In the second case we obtain

$$\partial_\alpha X^c \rightarrow D_\alpha X^c := \partial_\alpha X^c - \overset{w}{K}_{\alpha b c} X^b,$$

which gives the teleparallel coupling prescription of the vector field to gravity [10].

The covariant derivative (20) can alternatively be written in the form

$$D_\alpha X^c = \partial_\alpha X^c - \frac{i}{2} w K_{\alpha ab} (S_{ab})^c_d X^d,$$

with [11]

$$(S_{ab})^c_d = i(\delta^c_a \delta^d_b - \delta^c_d \delta^a_b)$$

the vector representation of the Lorentz generators. For a field belonging to an arbitrary representation of the Lorentz group, it assumes the form

$$D_\alpha = \partial_\alpha - \frac{i}{2} w K_{\alpha ab} \Sigma_{ab},$$

with $\Sigma_{ab}$ denoting a general representation of the Lorentz generators. This covariant derivative defines the teleparallel coupling prescription of fields carrying an arbitrary representation of the Lorentz group.

4 Dirac spinor field

4.1 Dirac equation

We consider now the specific case of a Dirac spinor [12], and apply the general covariance principle at the level of the Lagrangian formulation. Once again, let \{x^\mu\} be inertial Cartesian coordinates in flat spacetime, so that \{\partial_\mu\} is an orthonormal coordinate basis for the flat spacetime. Let $\varphi$ represent a spin-1/2 field with respect to this frame. The Dirac equation in flat spacetime can then be obtained from the Lagrangian (we use units in which $\hbar = c = 1$)

$$\mathcal{L}_M = \frac{i}{2} \left( \bar{\varphi} \gamma^\alpha \delta_\alpha^\mu \partial_\mu \varphi - \partial_\mu \bar{\varphi} \gamma^\alpha \delta_\alpha^\mu \varphi \right) - m \bar{\varphi} \varphi,$$

(24)
where $\delta_a^\mu$ is the trivial tetrad (in the sense that the corresponding basis of vector fields is just $\{\partial_\mu\}$, as in section 3.2), $m$ is the particle’s mass, $\{\gamma^a\}$ are (constant) Dirac matrices in a given representation, and $\varphi = \varphi^\dagger \gamma^0$.

Let us consider now, as in section 3.2, a local Lorentz transformation $\Lambda_a^b(x)$, yielding the moving frame $e_a = e_a^\mu \partial_\mu$ defined by Eqs. (10) and (11). Substituting $\delta_a^\mu = \Lambda_b^a \epsilon_b^\mu$, Eq. (24) reads

$$L_M = \frac{i}{2} \left( \bar{\varphi} \gamma^a \Lambda_b^a \partial_\mu \varphi - \partial_\mu \bar{\varphi} \gamma^a \Lambda_b^a \varphi \right) - m\bar{\varphi} \varphi. \tag{25}$$

Taking into account the identity

$$\gamma^a \Lambda_b^a = L_{\gamma^b} L^{-1}, \tag{26}$$

with $L$ a matrix representing an element of the covering group $Spin_+(1,3)$ of the restricted Lorentz group, we get

$$L_M = \frac{i}{2} \left[ \bar{\varphi}_e \gamma^a \left( \partial_a - (\partial_a L)^{-1} \right) \varphi_e - \varphi_e \left( \partial_a + (\partial_a L)^{-1} \right) \gamma^a \varphi_e \right] - m\bar{\varphi}_e \varphi_e, \tag{27}$$

where $\varphi_e = L\varphi$ is the representative of the spinor field with respect to the moving frame $\{e_a\}$, and $(\partial_a L)^{-1}$ is a connection term that appears due to the anholonomicity of the tetrad frame. A straightforward calculation shows that (see Appendix)

$$(\partial_a L)^{-1} = -\frac{i}{8} \left( f_{cab} h_{e_a^\mu} + f_{cba} h_{e_d^\mu} - f_{bac} h_{e_d^\mu} \right) \sigma^{bc},$$

with $\frac{1}{2} h_{e_a^\mu} := \frac{i}{4} [\gamma^b, \gamma^c]$ the spinor representation of the Lorentz generators. Substituting into Eq. (27), it becomes

$$L_M = \frac{i}{2} \left[ \bar{\varphi}_e \gamma^a \left( \partial_a + \frac{i}{8} \left( f_{cab} + f_{cba} - f_{bac} \right) \sigma^{bc} \right) \varphi_e - \varphi_e \left( \partial_a - \frac{i}{8} \left( f_{cab} + f_{cba} - f_{bac} \right) \sigma^{bc} \right) \gamma^a \varphi_e \right] - m\bar{\varphi}_e \varphi_e.$$

Now, by considering again the general covariance principle as in section 3.2, we assume that the presence of gravitation in teleparallel gravity is obtained by replacing $e_a^\mu$ by a nontrivial tetrad field $h_d^\mu$, which gives rise to a Riemannian metric $g_{\mu\nu} = \eta_{ab} h_a^\mu h_b^\nu$, and the coefficient of anholonomy by minus the torsion tensor: $f_{ab}^c = -T_{ab}^c$. In addition, the spinor field $\varphi_e$, as seen from the moving frame $\{e_a\}$, is to be replaced by a spinor field $\psi$ as seen from the corresponding frame in the presence of gravitation. This yields the following matter Lagrangian, corresponding to the Dirac equation for teleparallel gravity

$$L_M = \frac{i}{2} \left[ \bar{\psi} \gamma^a \left( \partial_a - \frac{i}{4} \tilde{K}_{abc} \sigma^{bc} \right) \psi - \bar{\psi} \left( \partial_a + \frac{i}{4} \tilde{K}_{abc} \sigma^{bc} \right) \gamma^a \psi \right] - m\bar{\psi} \psi,$$

where Eq. (5) has been used. Equivalently, we can write

$$L_M = \frac{i}{2} \left( \bar{\psi} h_a^\mu \gamma^a D_\mu \psi - D_\mu \bar{\psi} h_a^\mu \gamma^a \psi \right) - m\bar{\psi} \psi, \tag{28}$$
with the teleparallel version of the Fock-Ivanenko derivative operator given by [14]

\[ D_\mu \psi = \partial_\mu \psi - \frac{i}{4} K_\mu^{\, bc} \sigma^{bc} \psi, \]  

(29a)

\[ D_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \frac{i}{4} \bar{\psi} K_\mu^{\, bc} \sigma^{bc}. \]  

(29b)

This gives the coupling prescription for spin-1/2 fields in the teleparallel formalism. Moreover, in terms of the underlying Riemannian structure, Eq. (29a) can be rewritten in the form

\[ \bar{\circ} D_\mu = \partial_\mu + \frac{i}{4} \Gamma_\mu^{ab} \sigma^{bc}, \]

which is the well-known minimal coupling prescription of general relativity, as defined by the usual Fock-Ivanenko derivative.

A straightforward calculation shows that the matter Lagrangian (28) gives rise to the equation of motion

\[ i\gamma^a h_\mu^{\, a} D_\mu \psi = m\psi, \]  

(30)

which is the Dirac equation in teleparallel gravity. It is interesting to note that Eq. (30) can also be obtained directly from the flat spacetime Dirac equation

\[ i\gamma^a \delta_\mu^{\, a} \partial_\mu \psi = m\psi \]

by substituting \( \partial_\mu \) and \( \delta_\mu^{\, a} \) with \( D_\mu \) and \( h_\mu^{\, a} \). This is a distinguished feature of the coupling considered here, since it is well known that the usual minimal coupling prescription in Riemann-Cartan spacetimes leads to different results when applied to the Lagrangian or to the field equations [15].

### 4.2 Irreducible decomposition for torsion

We decompose now the torsion tensor in irreducible components under the global Lorentz group,

\[ T_{abc} = \frac{1}{3} (T_a \eta_{bc} - T_b \eta_{ac}) + q_{abc} - \frac{1}{6} \varepsilon_{abcd} S^d, \]

where \( T_a = T_{ab} \) is the torsion trace, \( S^d := \varepsilon^{abcd} T_{abc} \) is the torsion pseudo-trace, also called axial torsion, and \( q_{abc} \) has null trace and null pseudo-trace. A straightforward calculation then yields

\[ -\frac{i}{4} K_{abc} \gamma^a \sigma^{bc} = \gamma^a \left( \frac{1}{2} T_a - \frac{i}{8} S_a \gamma^5 \right), \]

where \( \gamma^5 = \gamma_5 := i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \). After substituting in Eq. (30), we get (see also [16])

\[ \gamma^\mu(x) \left( i\partial_\mu + \frac{i}{2} T_\mu + \frac{1}{8} S_\mu \gamma^5 \right) \psi = m\psi, \]  

(31)

where \( \gamma^\mu \equiv \gamma^\mu(x) = h_\mu^{\, a} \gamma^a \). This is the Dirac equation in the teleparallel formalism in terms of irreducible components for torsion.
In contrast to the usual General Relativity formalism, where the gravitational coupling is given by the affine quantities $\Gamma_{\mu a}^b$, in the above equation all coupling terms have a tensorial or pseudo-tensorial nature. We observe that this is not a contradiction, since in the teleparallel formalism a global moving frame $\{e_a\}$ is fixed. This fixes, in a sense, the gauge corresponding to the affine transformations in $\Gamma_{\mu a}^b$. As a result, the Dirac equation exhibits a simpler final form, since the gravitational coupling is then realized through vector ($T_\mu$) and pseudo-vector ($S_\mu$) quantities. It is interesting to notice that, if $\psi$ is a state with a definite parity (say, $P\psi = +\psi$), then the axial torsion $S_\mu$ will couple to the state with opposite parity $\gamma^5\psi$.

5 Final remarks

The general covariance principle can be considered as an active version of the passive equivalence principle. In fact, whereas the former says how, starting with a special relativity equation, to obtain the corresponding equation valid in the presence of gravitation, the latter deals with the reverse argument, namely, that in a locally inertial coordinate system any equation of general relativity must reduce to the corresponding equation of special relativity. More specifically, what the general covariance principle states is that any physical equation can be made covariant through a transformation to an arbitrary coordinate system, and that, due to its general covariance, this physical equation will be true in a gravitational field. Of course, the passive equivalence principle must also hold. Now, in order to make an equation generally covariant, new ingredients are necessary: A metric tensor and a connection, which are in principle inertial properties of the coordinate system. Then, by using the equivalence between inertial and gravitational effects, instead of inertial properties, these quantities can be assumed to represent a true gravitational field. In this way, equations valid in the presence of a gravitational field can be obtained.

The above description of the general covariance principle refers to its usual holonomic version. An alternative, more general version of the principle can be obtained in the context of nonholonomic moving frames, whose application is mandatory, for example, in the presence of spinor fields. The basic difference between these two approaches is that, instead of requiring that an equation be covariant under a general coordinate transformation, in the moving frame version the equation is required to preserve its form under a local Lorentz rotation of the frame. Of course, in spite of the different nature of the involved transformations, the physical content of both approaches is the same.

An important property of the general covariance principle is that it naturally yields a coupling prescription of any field to gravitation. In other words, it yields the form of the spin connection appearing in the covariant derivative. By using the nonholonomic version of this principle, we have found that the spin connection of teleparallel gravity is given by minus the contorsion tensor, in which case the coupling prescription of teleparallel gravity becomes always equivalent to the
corresponding prescription of general relativity, even in the presence of spinor fields.

Finally, we remark that the form of the coupling prescription for spinor fields in teleparallel gravity has been a matter of recurrent interest [14, 17, 18] (see also [19, 20]). In [17], the authors argue that, when applied to teleparallel gravity, the minimal coupling associated with the Weitzenböck connection leads to some inconsistencies. As shown in [18], such problems do not arise if the adopted teleparallel coupling is equivalent to the corresponding prescription of General Relativity. The analysis presented here shows that such coupling follows naturally from the general covariance principle.

Acknowledgments

RAM thanks W. A. Rodrigues for useful discussions. The authors are grateful to FAPESP and CNPq for financial support.

Appendix

As we have already mentioned, \( L \) is an element of \( \text{Spin}_+(1, 3) \), the covering of the restricted Lorentz group. Thus, \((\partial_a L)L^{-1}\) belongs to the corresponding Lie algebra, which is generated by \( \frac{\sigma^{bc}}{2} = \frac{i}{4} [\gamma^b, \gamma^c] \). Therefore, we can write

\[
(\partial_a L)L^{-1} = \frac{1}{2} \omega_{abc} \frac{\sigma^{bc}}{2},
\]

for certain functions \( \omega_{abc} \) satisfying \( \omega_{abc} = -\omega_{acb} \). These functions are actually Lorentz-valued connections representing the gravitational vacuum, that is, a connection accounting for the frame anholonomy only.

Let us take now the identity (26). Taking the derivative on both sides,

\[
(\partial_a L)\gamma^e L^{-1} + L\gamma^e \partial_a L^{-1} = (\partial_a \Lambda^e_d) \gamma^d,
\]

which is equivalent to

\[
\Lambda^e_d \left[ (\partial_a L)L^{-1}, \gamma^d \right] = (\partial_a \Lambda^e_d) \gamma^d.
\]

Substituting (22), we get

\[
\frac{1}{2} \omega_{abc} \Lambda^e_d \left[ \frac{\sigma^{bc}}{2}, \gamma^d \right] = (\partial_a \Lambda^e_d) \gamma^d.
\]

Using the commutation relation \( \left[ \frac{\sigma^{bc}}{2}, \gamma^d \right] = i(\eta^d \gamma^b - \eta^b \gamma^d, \gamma^c) \), we get

\[
\omega_{ab}^c = -i(\partial_a \Lambda^d_b) \Lambda^e_d.
\]

It follows from (13) that

\[
\omega_{ab}^c = -\left( f^c_{ab} + f^c_{ba} - f^c_{ba} \right).
\]
which finally implies

\[(\partial_a L) L^{-1} = -\frac{i}{4} (f_{cab} + f_{eba} - f_{bac}) \sigma^{bc}/2.\]  

(33)

References

[1] Sciama, D. W. (1964). *The Physical Structure of General Relativity*, Rev. Mod. Phys. **36**, 463.

[2] Weinberg, S. (1972). *Gravitation and Cosmology* (Wiley, New York), p. 91.

[3] Calçada, M. and Pereira, J. G. (2002). *Gravitation and the Local Symmetry of Spacetime*, Int. J. Theor. Phys. **41**, 729 [gr-qc/0201059].

[4] Hayashi, K. and Shirafuji, T. (1979). *New General Relativity*, Phys. Rev. **D19**, 3524.

[5] Geroch, R. (1968). *Spinor Structure of Space-Times in General Relativity I*, J. Math. Phys. **9**, 1739.

[6] Aldrovandi, R. and Pereira, J. G. (1995). *An Introduction to Geometrical Physics* (World Scientific, Singapore).

[7] Kobayashi, S. and Nomizu, K. (1963). *Foundations of Differential Geometry*, Vol. I (Wiley, New York).

[8] Misner, C. W., Thorne, K. S. and Wheeler, J. A. (1973). *Gravitation* (Freeman, New York).

[9] de Andrade, V. C. and Pereira, J. G. (1997). *Gravitational Lorentz Force and the Description of the Gravitational Interaction*, Phys. Rev. **D56**, 4689 [gr-qc/9703059].

[10] de Andrade, V. C. and Pereira, J. G. (1999). *Torsion and the Electromagnetic Field*, Int. J. Mod. Phys. **D8**, 141 [gr-qc/9708051].

[11] See, for example, Ramond, P. (1989). *Field Theory: A Modern Primer*, 2nd edition (Addison-Wesley, Redwood).

[12] For a related discussion of the Dirac equation on Riemannian spacetime, see Chapman, T. C. and Leiter, D. J. (1976). *On the Generally Covariant Dirac Equation*, Am. J. Phys. **44**, 858.

[13] See, for example, Itzykson C. and Zuber, J. B. (1980). *Quantum Field Theory* (McGraw-Hill, New York).

[14] de Andrade, V. C., Guillen, L. C. T. and Pereira, J. G. (2001). *Teleparallel Spin Connection*, Phys. Rev. **D64**, 027502 [gr-qc/0104102].
[15] Saa, A. (1993). *On Minimal Coupling in Riemann-Cartan Space-Times*, Mod. Phys. Lett. **A8**, 2565.

[16] Zhang, C. M. and Beesham, A. (2001). *Rotation Intrinsic Spin Coupling–The Parallelism Description*, Mod. Phys. Lett. **A16**, 2319 [gr-qc/0111075].

[17] Obukhov, Y. N. and Pereira, J. G. (2003). *Metric-Affine Approach to Teleparallel Gravity*, Phys. Rev. **D67**, 044016 [gr-qc/0212080].

[18] Maluf, J. W. (2003). *Dirac Spinor Fields in the Teleparallel Gravity: Comment on “Metric-Affine Approach to Teleparallel Gravity”*, Phys. Rev. **D67**, 108501 [gr-qc/0304005].

[19] Mielke, E. W. (2003). *Consistent Coupling to Dirac Fields in Teleparallelism: Comment on “Metric-Affine Approach to Teleparallel Gravity”*, Phys. Rev. D **69**, 128501.

[20] Obukhov, Y. N. and Pereira, J. G. (2004). *Lessons of Spin and Torsion: Reply to “Consistent Coupling to Dirac Fields in Teleparallelism”*, Phys. Rev. D **69**, 128502 [gr-qc/0406015].