Spectral Factorization of Rank-Deficient Polynomial Matrix-Functions

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Abstract. A spectral factorization theorem is proved for polynomial rank-deficient matrix-functions. The theorem is used to construct paraunitary matrix-functions with first rows given.

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Wiener’s spectral factorization theorem [12], [4] for polynomial matrix-functions asserts that if

\[ S(z) = \sum_{n=-N}^{N} C_n z^n \]

is an \( m \times m \) matrix-function (\( C_n \in \mathbb{C}^{m \times m} \) are matrix coefficients) which is positive definite for a.a. \( z \in \mathbb{T} \), \( \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \), then it admits a factorization

\[ S(z) = S^+(z)S^-(z) = \sum_{n=0}^{N} A_n z^n \cdot \sum_{n=0}^{N} A_n^* z^{-n}, \quad z \in \mathbb{C}\{0\}, \]

where \( S^+ \) is an \( m \times m \) polynomial matrix-function which is nonsingular inside \( \mathbb{T} \), \( \det S^+(z) \neq 0 \) when \( |z| < 1 \), and \( S^- \) is its adjoint, \( A_n^* = A_n^T \), \( n = 0,1,\ldots,N \). (Respectively, \( S^- \) is analytic and nonsingular outside \( \mathbb{T} \).) \( S^+ \) is unique up to a constant right unitary multiplier.

The factorization (2) is also known under the name of matrix-valued Fejér-Riesz theorem and its simple proof is provided in [2]. Various practical applications of this theorem in system analysis [6] and wavelet design [1] are widely recognized.

In the present paper we consider rank-deficient matrix polynomials and prove the corresponding spectral factorization theorem for them:

**Theorem 1.** Let \( S(z) \) be an \( m \times m \) (trigonometric) polynomial matrix-function (1) of order \( N \) (\( C_N = C_N^* \neq 0 \)) which is nonnegative definite and of rank \( k \leq m \) for a.a. \( z \in \mathbb{T} \). Then there exists a unique (up to a \( k \times k \) unitary matrix right multiplier) \( m \times k \) matrix-polynomial \( S^+(z) = \sum_{n=0}^{N} A_n z^n \), \( A_n \in \mathbb{C}^{m \times k} \) of order \( N \) (\( A_N \neq 0 \)), which is of full rank \( k \) for each \( z \) inside \( \mathbb{T} \), such that (2) holds.

**Remark.** If we require of \( S^+ \) to be just a rational matrix-function analytic inside \( \mathbb{T} \) and drop the uniqueness from the condition, then the theorem can be obtained in a standard algebraic manner (see [9]). Hence, as we will see below, the proof of the theorem provides a simple proof of the same theorem for the full rank case, \( k = m \), as well. This proof is even more elementary as compared with the one given in [2] since it avoids an application of the Hardy space theory.
Prior to proving the theorem, we make some simple observations on adjoint functions and prove Lemma 1 on paraunitary matrix-functions. We do not claim that this lemma is new, but include its proof for the sake of completeness.

If $f$ is an analytic $m \times k$ matrix-function in $\mathbb{C}\{z_1, z_2, \ldots, z_n\}$, then its adjoint $f^*(z) = \overline{f(1/z)^T}$ is an analytic $k \times m$ matrix-function in $\mathbb{C}\{z_1^*, z_2^*, \ldots, z_n^*\}$, $z^* := 1/z$, $\infty^* = 0$. Obviously, if $f$ is analytic inside $\mathbb{T}$, then $f^*$ is analytic outside $\mathbb{T}$ (including infinity). Namely, if $f_{ij}(e^{\theta}) \in L_1^+(\mathbb{T})$, ($f_{ij}$ is the $ij$th entry of $f$), then $f_{ji}^*(e^{\theta*}) \in L_1^-(\mathbb{T})$, where $L_1^+(\mathbb{T}) \ (L_1^-(\mathbb{T}))$ is the set of integrable functions defined on $\mathbb{T}$ which have Fourier coefficients with negative (positive) indices equal to zero.

Since $f$ is uniquely determined by its values on $\mathbb{T}$, and $f^*(z) = \overline{f(z)^T} = (f(z))^*$ for $|z| = 1$, usual relations for adjoint matrix-functions, like $(fg)^*(z) = g^*(z)f^*(z)$ and $(f^{-1})^*(z) = (f^*)^{-1}(z)$, etc., are valid.

Note that if $f$ is a rational $m \times m$ matrix-function, $f \in \mathcal{R}_m$, then

$$[f(e^{\theta})f^*(e^{\theta})]_{ii} \in L_\infty(\mathbb{T}) \implies f_{ij} \text{ are free of poles on } \mathbb{T}, j = 1, 2, \ldots, m,$$

($L_\infty(\mathbb{T})$ stands for the set of bounded functions) since $[f(z)f^*(z)]_{ii} = \sum_{j=1}^m |f_{ij}(z)|^2$ when $|z| = 1$.

$U \in \mathcal{R}_m$ is called paraunitary if

$$U(z)U^*(z) = I_m \quad \text{in the domain of } U \text{ and } U^*,$$

where $I_m$ stands for the $m$-dimensional unit matrix. Note that $U(z)$ is a usual unitary matrix for each $z \in \mathbb{T}$, since $U^*(z) = \overline{U(z)^T} = (U(z))^*$ when $|z| = 1$ and, consequently,

$$\overline{U(z)^T} = U^{-1}(z), \quad z \in \mathbb{T}.$$  \hfill (5)

**Lemma 1.** If $U \in \mathcal{R}_m$ is paraunitary and analytic inside $\mathbb{T}$ (its entries are free of poles inside $\mathbb{T}$), and $U^{-1} \in \mathcal{R}_m$ is analytic inside $\mathbb{T}$ as well, then $U$ is a constant unitary matrix.

**Proof.** The equation (4) implies that $U_{ij}(z)$, $1 \leq i, j \leq m$, are free of poles on $\mathbb{T}$ (see (3)). Since $U_{ij}(e^{\theta}) \in L_1^+(\mathbb{T})$ and $L_1^+(\mathbb{T}) \ni U_{ji}^{-1}(e^{\theta}) = U_{ji}(e^{\theta*})$ (see (5)), we have $U_{ij}(e^{\theta}) \in L_1^+(\mathbb{T}) \cap L_1^-(\mathbb{T})$. Thus $U_{ij}(z)$ is constant for a.a. $z \in \mathbb{T}$, and hence everywhere in the complex plane. \hfill $\square$

**Proof of Theorem 1.** Since $S$ is nonnegative definite on the unit circle, we have $S^*(z) = S(z)$, $z \in \mathbb{C}\{0\}$.

Observe that every polynomial matrix-function always has a constant rank in its domain except for a finite number of points. Without loss of generality, we can assume that the $k \times k$ left-upper submatrix of $S$, denoted by $S_{00}$, has the full rank $k$ (a.e.) so that $S$ has the block matrix form

$$S(z) = \begin{pmatrix} S_{00}(z) & S_{01}(z) \\ S_{10}(z) & S_{11}(z) \end{pmatrix},$$

where $S_{01}$, $S_{10} = S_{01}^*$ and $S_{11}$ are matrix-functions of dimensions $k \times (m-k)$, $(m-k) \times k$, and $(m-k) \times (m-k)$, respectively. Since every $k+1$ rows (columns) of $S(z)$ are
linearly dependent, we have
\begin{equation}
S_{10}(z)S_{00}^{-1}(z)S_{01}(z) = S_{11}(z) \quad \text{(a.e.)}
\end{equation}
Let
\begin{equation}
S_{00}(z) = S_{00}^{+}(z)S_{00}^{-}(z) = S_{00}^{+}(z)(S_{00}^{+})^{*}(z)
\end{equation}
be the polynomial spectral factorization of $S_{00}$ which exists by virtue of the matrix-valued Fejér-Riesz theorem. Define
\begin{equation}
\sigma_{10}(z) := S_{10}(z)(S_{00}^{-}(z))^{-1}
\end{equation}
and let $S_{0}$ have the block matrix form
\begin{equation}
S_{0}(z) = \begin{pmatrix} S_{00}^{+}(z) \\ \sigma_{10}(z) \end{pmatrix}.
\end{equation}
Then $S_{0}^{*}(z) = [S_{00}^{+}(z) \quad (S_{00}^{+}(z))^{-1}S_{01}(z)]$ and, taking (6) into account, one can directly check that
\begin{equation}
S(z) = S_{0}(z)S_{0}^{*}(z).
\end{equation}
Since $S_{00}^{+}$ is a polynomial matrix-function, $S_{0}$ is a rational matrix-function, however it might not be analytic inside $\mathbb{T}$. If $s_{ij}$ is the $ij$th entry of $S_{0}$ with a pole at $a$ inside $\mathbb{T}$, then we can multiply $S_{0}$ by the unitary matrix-function $U(z) = \text{diag}[1, \ldots, u(z), \ldots, 1]$, where $u(z) = (z - a)/(1 - \overline{a}z)$ is the $jj$th entry of $U(z)$, so that the $ij$th entry of the product $S_{0}(z)U(z)$ will not have a pole at $a$ any longer keeping the factorization (8):
\begin{equation}
(S_{0}U)(z)(S_{0}U)^{*}(z) = S_{0}(z)S_{0}^{*}(z) = S(z).
\end{equation}
In the same way one can remove every pole of the entries of $S_{0}$ at points inside $\mathbb{T}$. Thus $S$ can be represented as a product
\begin{equation}
S(z) = S_{0}^{+}(z)S_{0}^{-}(z),
\end{equation}
where $S_{0}^{+}$ is a rational matrix-function which is analytic inside $\mathbb{T}$, and $S_{0}^{-}(z)$ is its adjoint. Note that $S_{0}^{+}(z)$ remains of full rank $k$ for each $z \in \mathbb{T}$ except possibly a finite number of points.

Now, it might happen so that $S_{0}^{+}$ is not of full rank $k$ inside $\mathbb{T}$ everywhere. If $|a| < 1$ and rank $S_{0}^{+}(a) < k$, then there exists a unitary matrix $U$ such that the product $S_{0}^{+}(a)U$ has all 0’s in the first column. Hence $a$ is a zero of every entry of the first column of the matrix-function $S_{0}^{+}(z)U$ and the product $S_{1}^{+}(z) := S_{0}^{+}(z)U \text{diag}[u(z), 1, \ldots, 1]$, where $u(z) = (1 - \overline{a}z)/(z - a)$, remains analytic inside $\mathbb{T}$. While the factorization (9) remains true replacing $S_{0}^{+}$ and $S_{0}^{-}$ by $S_{1}^{+}$ and $S_{1}^{-}$, respectively, the minors of $S_{1}^{+}$ will have less zeros inside $\mathbb{T}$ than the minors of $S_{0}^{+}$. Thus, continuing this process if necessary, we get the factorization
\begin{equation}
S(z) = S^{+}(z)S^{-}(z),
\end{equation}
where $S^{+}$ is a rational matrix-function which is analytic and of full rank $k$ inside $\mathbb{T}$.

Now let us show that $S^{+}$ is in fact a polynomial matrix-function of order $N$. It suffices to show that $z^{N}S^{-}(z)$ is analytic inside $\mathbb{T}$. Indeed, since $S^{+}$ does not have poles on $\mathbb{T}$ (see (10) and (3)), $z^{N}S^{-}(z)$ should be an analytic (on the whole $\mathbb{C}$) rational matrix-function in this case, and therefore a polynomial.
It follows form (10) that
\[(11) \quad z^N S^-(z) = ((S^+(z))^T S^+(z))^{-1} \cdot (S^+(z))^T \cdot z^NS(z)\]
and $z^N S^-(z)$ is analytic inside $\mathbb{T}$ since each of the three factors on the right-hand side of (11) is such.

To complete the proof of the theorem, it remains to show that the factorization (2) is unique, i.e. if
\[S(z) = S_1^+(z)S_1^-(z)\]
where $S_1^+$ is a $m \times k$ polynomial matrix-function which has the full rank $k$ inside $\mathbb{T}$, then
\[S_1^+(z) = S^+(z)U\]
for some $k \times k$ (constant) unitary matrix $U$.

Since $S^+(z)$ is of the full rank $k$ for each $z \in \mathbb{C}$ except for some finite number of singular points, there exists a matrix-function $U(z)$ such that
\[(12) \quad S_1^+(z) = S^+(z)U(z)\]
Thus $U(z)$ can be determined by the equation
\[U(z) = ((S^+(z))^T S^+(z))^{-1} (S^+(z))^T(z)S_1^+(z)\]
as a rational function in $\mathbb{C}$. Note that $U(z)$ is analytic inside $\mathbb{T}$, and since $S^+$ and $S_1^+$ participate symmetrically in the theorem, $U^{-1}(z)$ is analytic inside $\mathbb{T}$ as well.

Due to Lemma 1, it remains to show that $U \in \mathcal{R}^{k \times k}$ is a paraunitary matrix-function. From the equation (12), one can determine $U(z)$ as
\[U(z) = (S^-(z)S^+(z))^{-1} S^-(z)S_1^+(z)\]
and, consequently,
\[U^*(z) = S_1^-(z)S^+(z) (S^-(z)S^+(z))^{-1}.\]
Hence
\[U(z)U^*(z) = (S^-(z)S^+(z))^{-1} S^-(z)S_1^+(z) \cdot S_1^-(z)S^+(z) (S^-(z)S^+(z))^{-1} = \]
\[= (S^-(z)S^+(z))^{-1} S^-(z)S^+(z)S^-(z)S^+(z) (S^-(z)S^+(z))^{-1} = I_k.\]
The proof of the theorem is complete.

**Remark.** As one can observe, the above proof of the existence of $S^+$ is constructive. There are several classical algorithms to perform the factorization (7) numerically in the full rank case (a new efficient algorithm of such type is proposed in [5]). Further using the steps described in the proof, one can compute $S^+$ numerically.

Our next theorem illustrates one of the applications of Theorem 1 in some areas of signal processing. Namely, $m \times m$ paraunitary matrix-functions
\[(13) \quad U(z) = \sum_{n=0}^N \rho_n z^n = \left[u_{ij}(z)\right]_{i,j=1}^m, \quad \rho_n \in \mathbb{C}^{m \times m},\]
defined by (4) play an important role in the theory of wavelets and multirate filter banks [8] where they are known under different names, for example, lossless systems [11], perfect reconstruction $m$-filters [7], paraunitary $m$-channel filters [10], and so on.
The positive integers \( m \) and \( N \) are called the size and the length of \( U \), respectively. Sometimes, the first row of a matrix-function \( U \) is called the low-pass filter, and the remaining rows are called the high-pass filters. Theorem 2 allows us to find the set of matching high-pass filters to each low-pass filter. First we give a simple proof of the following lemma which provides additional information about structures of paraunitary matrix-polynomials.

**Lemma 2.** (cf. [8, Lemma 4.13]) Let (13) be a paraunitary matrix-polynomial of length \( N \) (\( \rho_N \neq 0 \)). Then

\[
\text{det} U(z) = c \cdot z^k, \text{ where } |c| = 1, \text{ and } k \geq N.
\]

**Proof.** Since \( \text{det} U(z) \cdot \text{det} U^*(z) = 1 \) and \( \text{det} U(z) \) is a polynomial, it follows that \( \text{det} U(z) = cz^k \) for some nonnegative integer \( k \). We have

\[
\sum_{n=0}^{N} \rho_n^* z^{-n} = U^*(z) = U^{-1}(z) = \frac{1}{\text{det} U(z)} \left( \text{Cof} U(z) \right)^T = cz^{-k} \left( \text{Cof} U(z) \right)^T.
\]

Therefore \( k \geq N \), since \( \text{Cof} U(z) \) is a polynomial matrix-function and \( \rho_N^* \) is not the zero matrix. \( \Box \)

**Remark.** The positive integer \( k \) in (14) is called the degree of \( U \). Generically, a paraunitary matrix-polynomial \( U \) of length \( N \) has the same degree \( N \), although in some specific cases the degree is more than \( N \).

The following theorem was first established in [3] by a different method, however the presented approach gives a new insight to the problem.

**Theorem 2.** For any polynomial vector-function

\[
U_1(z) = [u_{11}(z), u_{12}(z), \ldots, u_{1m}(z)],
\]

\[
u_{1j}(z) = \sum_{n=0}^{N} \alpha_{jn} z^n, \quad j = 1, 2, \ldots, m, \text{ of length } N \left( \sum_{j=1}^{m} |\alpha_{jn}| > 0 \right) \text{ which is of unit norm on } \mathbb{T}
\]

\[
\|U_1(z)\|_{\mathbb{C}^m}^2 = \sum_{j=1}^{m} \|u_{1j}(z)\|^2 = 1, \quad z \in \mathbb{T},
\]

there exists a unique (up to a constant left multiplier of the block matrix form \( \begin{pmatrix} 0 & 0 \\ 1 & U \end{pmatrix} \)), where \( U \) is a \( (m-1) \times (m-1) \) unitary matrix) paraunitary matrix-function \( \hat{U}(z) \) (of size \( m \) and length \( N \)), with determinant \( cz^N \), \( |c| = 1 \), whose first row is equal to (16).

**Lemma 3.** Let \( \mathbf{v} = (v_1, v_2, \ldots, v_m)^T \in \mathbb{C}^m \) be a vector of unit norm, \( \|\mathbf{v}\|^2 = \mathbf{v}^* \mathbf{v} = \sum_{j=1}^{m} |v_j|^2 = 1 \). Then \( I_m - \mathbf{v} \mathbf{v}^* \) is a nonnegative definite matrix,

\[
I_m - \mathbf{v} \mathbf{v}^* \geq 0,
\]

and

\[
\text{rank} \left( I_m - \mathbf{v} \mathbf{v}^* \right) = m - 1.
\]
Proof. For each column vector $x \in \mathbb{C}^m$, we have
\[
x^*(I_m - vv^*)x = \|x\|^2 - |x^*v|^2 \geq \|x\|^2 - \|x^*\|^2\|v\|^2 = \|x\|^2 - \|x^*\|^2 = 0.
\]
Hence (18) holds and $x^*(I_m - vv^*)x = 0$ if and only if $x = \alpha v$ for some $\alpha \in \mathbb{C}$. Thus (19) holds as well. 

Proof of Theorem 2. Due to Lemma 3 and the property (17), the matrix-function
\[(20) \quad S(z) = I_m - U_1^T(z)(U_1^T)^*(z)\]
is positive definite and of rank $m - 1$ for each $z \in \mathbb{T}$. (Note that the order of $S$ is less than or equal to $N$.) Hence, by virtue of Theorem 1, there exists an $m \times (m - 1)$ matrix-function $S^+(z)$ of full rank $m - 1$, for each $z$ inside $\mathbb{T}$, such that (2) holds. Consequently,
\[
\begin{bmatrix}
U_1^T(z) & S^+(z)
\end{bmatrix}
\begin{bmatrix}
(U_1^T)^*(z) \\
S^-(z)
\end{bmatrix} = I_m
\]
and
\[
U(z) = 
\begin{bmatrix}
U_1(z) \\
(S^+)^T(z)
\end{bmatrix}
\]
is the paraunitary matrix-function we wanted to find. Indeed, clearly $U(z)$ is of size $m$ and length $N$, and we show that
\[(21) \quad \det U(z) = c \cdot z^N, \quad |c| = 1.
\]

Due to Lemma 2, $\det U(z) = cz^k, |c| = 1$, for some positive integer $k \geq N$. Hence (see (15))
\[(22) \quad \sum_{n=0}^{N} \alpha_{jn}z^{-n} = u_{1j}^*(z) = c \cdot z^{-k} \cdot \text{cof}(u_{1j}(z)), \quad j = 1, 2, \ldots, m.
\]

Since $S^+(0)$ is of rank $m - 1$, then $\text{cof}(u_{1j}(0)) \neq 0$ for at least one $j \in \{1, 2, \ldots, m\}$ so that the first coefficient of the polynomial $\text{cof}(u_{1j}(z))$ differs from 0 for at least one $j$. Thus it follows from (22) that $k \leq N$ and hence $k = N$, which yields (21). The desired $U(z)$ is found and let us show its uniqueness.

Assume now that $U(z)$ is any $m \times m$ paraunitary polynomial matrix-function, with the first row (16), which satisfies (21), and let $U_{m-1}(z)$ be the $(m - 1) \times m$ matrix-polynomial which is formed by deleting the first row in $U(z)$. It is obvious that $U_{m-1}^T(z)$ is an $m \times (m - 1)$ polynomial spectral factor of (20) so that, by virtue of Theorem 1, we get $U_{m-1}^T(z) = S^+(z)U \iff U(z) = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} U_1(z) \\ (S^+)^T(z) \end{pmatrix}$ immediately after we establish that $U_{m-1}(z)$ is of full rank $m - 1$ for each $z$ inside $\mathbb{T}$. But $\text{rank} U_{m-1}(z) = m - 1$ for any $z \neq 0$ since (21) implies that $\text{rank} U(z) = m, z \neq 0,$ and $\sum_{n=0}^{N} \alpha_{jn}z^{-n} = u_{1j}^*(z) = cz^{-N} \text{cof}(u_{1j}(z))$ (see (22)), $\alpha_{jN} \neq 0$, implies that $\text{cof}(u_{1j}(0)) \neq 0$, which means that $\text{rank} U_{m-1}(0) = m - 1$. 

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