Bose–Einstein Condensate in a Harmonic Trap with an Eccentric Dimple Potential

H. Uncu\textsuperscript{a}, D. Tarhan\textsuperscript{d}, E. Demiralp\textsuperscript{c,d}, and Ö. E. Müstecaplıoğlu\textsuperscript{e} \textsuperscript{*}

\textsuperscript{a}Department of Physics, Adnan Menderes University, Kepezli Mektep, Aydın, 34342 Turkey
\textsuperscript{b}Department of Physics, Harran University, Osmanbey Yerleşkesi, Şanlıurfa, Turkey
\textsuperscript{c}Department of Physics, Boğaziçi University, Bebek, İstanbul, 34342 Turkey
\textsuperscript{d}Boğaziçi University-TÜBITAK, Feza Gürsey Institute, Kandilli, İstanbul, 81220 Turkey
\textsuperscript{e}Department of Physics, Koç University, Rumelifeneri yolu, Sarıyer; İstanbul, 34450 Turkey

\textsuperscript{e-mail: omustecap@ku.edu.tr}

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1. INTRODUCTION

The phase-space density of a Bose–Einstein condensate (BEC) can be increased by modification of the shape of the potential \cite{11}. “Dimple”-type potentials are the most commonly used potentials for this purpose \cite{2–4}. A small dimple potential at the equilibrium point of the harmonic trapping potential is used to enhance the phase-space density by an arbitrary factor \cite{2}. A tight dimple potential was used for a recent demonstration of a caesium BEC \cite{3}. Quite recently, such potentials were proposed for efficient loading and fast evaporative cooling to produce large BECs \cite{5}. Attractive applications, such as controlling the interaction between dark solitons and sound \cite{6}, introducing defects such as atomic quantum dots in optical lattices \cite{7}, or quantum tweezers for atoms \cite{8} are offered by using tight dimple potentials for (quasi) one-dimensional BECs. Such systems can also be used for the spatially selective loading of optical lattices \cite{9}. In combination with the condensates on atom chips, tight and deep dimple potentials can lead to rich novel dynamics for potential applications in atomic lasers, atom interferometers, and in quantum computations (see \cite{10} and references therein).

In this paper, we continue the discussion of our recent paper \cite{11}. In that paper, we modeled the dimple-type potentials by Dirac δ functions and investigated the change of the chemical potential, critical temperature, and condensate fraction of a harmonic trap with respect to the various strengths of Dirac δ functions. In this paper, we investigate the behavior of the same physical quantities for a δ function, which can be located at positions other than the center of a harmonic trap. We find that, while a centrally positioned dimple potential is most effective in a large condensate formation at enhanced temperatures, there is a critical location for which the condensate fraction and the critical temperature can also be relatively enhanced. This might be useful in the spatial fragmentation of atomic condensates.

The paper is organized as follows. In Section 2, we briefly review the analytical solutions of the Schrödinger equation for a harmonic potential with a finite number of Dirac δ-decorated harmonic potentials and give the eigenvalue equation of the harmonic potential with a Dirac δ function. In Section 3, determining the eigenvalues numerically, we show the effect of the dimple potential on the condensate fraction and the transition temperature and investigate the change of these values with respect to the position of the Dirac δ function. Finally, we conclude in Section 4.

2. HARMONIC POTENTIAL DECORATED WITH DIRAC DELTA FUNCTIONS

We begin our discussion by reviewing the one-dimensional harmonic potential decorated with the Dirac δ functions \cite{11–14} for the sake of completeness. The potential is given as

\begin{equation}
V(x) = \frac{1}{2} m \omega^2 x^2 - \frac{\hbar^2}{2m} \sum_{i}^{P} \sigma_i \delta(x - x_i),
\end{equation}

where \(\omega\) is the frequency of the harmonic trap, \(P\) is a finite integer, and \(\sigma_i\) is the strength (depth) of the dimple potentials located at \(x_i\) with \(x_1 < x_2 < \ldots < x_P\) with
and introducing dimensionless quantities are parabolic cylinder functions and Wronskian \[ \[11, 14, 16\], we get the following eigenvalue equation:

\[ E = -\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) = E\Psi(x). \quad (2) \]

By defining \( E = \left(\xi + \frac{1}{2}\right) \hbar \omega \), with \( \xi \) as a real number, and introducing dimensionless quantities \( z = x/x_0 \) and \( z_i = x_i/x_0 \) with \( x_0 = \sqrt{\hbar/2m\omega} \) as the natural length scale of the harmonic trap, we can rewrite Eq. (2) as

\[ \frac{d^2\Psi(z)}{dz^2} + \left[ \xi + \frac{1}{2} \frac{z^2}{4} + \sum_i \Lambda_i \delta(z - z_i) \right] \Psi(z) = 0. \quad (3) \]

where \( \Lambda_i = x_0 \sigma_i \). By using the transfer matrix approach \[11, 14, 16\], we get the following eigenvalue equation:

\[ 1 - \frac{\Lambda_i D_\xi(z_i) D_\xi(-z_i)}{W} = 0, \quad (4) \]

for \( \xi \) and using \( E = \left(\xi + \frac{1}{2}\right) \hbar \omega \). Here, \( D_\xi(z) \) and \( D_\xi(-z) \) are parabolic cylinder functions and \( z_1 = x_1/x_0 \). The Wronskian \( W \) of \( D_\xi(z) \) and \( D_\xi(-z) \) is

\[ W = W[D_\xi(z), D_\xi(-z)] = \frac{2^{\xi+3/2} \pi}{\Gamma(-\xi/2) \Gamma(1-\xi/2)}. \quad (5) \]

For \( z_1 = 0 \), these results reduce to the results in \[11, 13, 14\].

3. BEC IN A ONE-DIMENSIONAL HARMONIC POTENTIAL WITH A DIRAC \( \delta \) FUNCTION

In this section, we calculate the condensate fraction, chemical potential, critical temperature, and density profile for different depths, sizes, and positions of a dimple potential modeled by a Dirac \( \delta \) function. In order to describe the depth and size of a dimple potential in a systematic way, we define a dimensionless variable in terms of the strength of the Dirac \( \delta \) functions as

\[ \Lambda = \sigma \sqrt{\frac{\hbar}{2m\omega}}. \quad (6) \]

We will present our results with respect to \( \Lambda \) and \( z_1 \) defined in the previous section.

In \[11\], we estimated the \( \sigma \) values approximately according to the parameters in \[3, 17\]. We find that, if \( 10^{4} \leq \sigma \leq 10^{10} \) \( \text{L/m} \), then \( 320 \leq \Lambda \leq 32000 \) for the experimental parameters \( m = 23 \text{amu} (^{23}\text{Na}) \), \( \omega = 2\pi \times 21 \text{Hz} \) \[17\], and for the experimental parameters \( m = 133 \text{amu} (^{133}\text{Cs}) \), \( \omega = 2\pi \times 14 \text{Hz} \) \[3\]. In \[11\], we have shown that, even for small \( \Lambda \) values, the condensate fraction and critical temperature change considerably. How sensitive such a change would occur depending on the location of the dimple trap was a question left unanswered in \[11\].

We begin our discussion by investigating the change of the critical temperature with respect to the position of the Dirac \( \delta \) function. The critical temperature \( (T_c) \) is obtained by taking the chemical potential equal to the ground-state energy \( (\mu = E_\varepsilon = E_0) \) and by solving

\[ N = \sum_{i=1}^{\infty} \frac{1}{\beta_i e^\beta_i - 1} \quad (7) \]

for \( \beta_i \), where \( \beta_i = 1/(k_B T_c) \). For finite \( N \) value, we define \( T_c^0 \) as the solution of Eq. (7) for \( \Lambda = 0 \) (only the harmonic trap).

In Eq. (7), \( \varepsilon_i \) is the eigenvalue for the harmonic potential decorated with a single eccentric dimple potential at \( z_1 \). The energies of the decorated states are found by solving Eq. (4) numerically. Then, these values are substituted into Eq. (7); and, finally, this equation is solved numerically to find \( T_c \). We obtain \( T_c \) for different \( z_1 \) values and \( \Lambda = 32 \), and show our results in Fig. 1. Since the harmonic potential is symmetric, negative \( z_1 \) values will give a the same values for a critical temperature with positive ones. As \( z_1 \) increases, the energy of the ground state increases, so that the critical temperature decreases as \( z_1 \) becomes larger. On the other hand, as the dimple trap becomes farther from the center of the harmonic trap, the critical temperature ceases to decrease and starts rising again, as seen in Fig. 1.
Fig. 1. Bose–Einstein condensate in a harmonic trap.

Fig. 2. The chemical potential vs. temperature $T/T_c^0$ for $N = 10^4$. The solid line shows $\mu$ only for the harmonic trap. The dotted line shows $\mu$ for $z_1 = 1$ and $\Lambda = 32$. The dashed line shows $\mu$ for $z_1 = 0$ and $\Lambda = 32$. The other parameters are the same as in Fig. 1.

Fig. 3. $N_{0}/N$ vs. $T/T_c^0$ for $N = 10^4$ and $\Lambda = 32$. The solid line is for $z_1 = 0$; the dashed line is for $z_1 = 1$. The other parameters are the same as in Fig. 1.

Fig. 4. $N_{0}/N$ vs. $T/T_c^0$ for $N = 10^4$ and $\Lambda = 32$. The other parameters are the same as in Fig. 1.

Fig. 5 Comparison of the density profiles of a BEC in a harmonic trap with a BEC in a harmonic trap decorated with a $\delta$ function ($\Lambda = 3.6, z_1 = 1$). The solid curve is the density profile of the BEC in the decorated potential. The dashed curve is the density profile of the 1D harmonic trap ($\Lambda = 0$). The parameter $z$ is a dimensionless length defined using Eq. (2). The other parameters are the same as in Fig. 1.

Fig. 1 around $z_1 = 3.5$. Finally, at very large separations between the dimple trap and the harmonic trap center, the critical temperature no longer changes with the location of the dimple trap and saturates at the value corresponding to the critical temperature for the single-harmonic trap per se. The increase of the critical temperature around $z_1 = 3.5$ can be explained as follows: As $z_1$ increases, it becomes closer to the node of the first exited-state wavefunction of the harmonic potential. At that value, the change of the energy eigenvalue of the first excited state vanishes and the difference between the first excited state and ground-state eigenvalues increase. Thus, the particles favor the ground state, which increases the critical temperature. In Fig. 1, we take $N = 10^4$ and use typical experimental parameters $m = 23$ amu ($^{23}$Na) and $\omega = 2\pi \times 21$ Hz [17].

For a gas of $N$ identical bosons, the chemical potential $\mu$ is obtained by solving

$$N = \sum_{\ell=0}^{\infty} \frac{1}{\beta(e_{\ell}-\mu)} = N_0 + \sum_{\ell=1}^{\infty} \frac{1}{\beta(e_{\ell}-\mu)} - 1$$

(8)
at a constant temperature and for a given \( N \), where \( \epsilon_i \) is the energy of state \( i \). We present the change of \( \mu \) as a function of \( T/T_c^0 \) in Fig. 2 for \( N = 10^4 \), \( \Lambda = 32 \), \( z_1 = 0 \), and \( \Lambda = 32 \), \( z_1 = 1 \).

By inserting the \( \mu \) values into the equation

\[
N_0 = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}, \tag{9}
\]

we find the average number of particle in the ground state. \( N_0/N \) versus \( T/T_c^0 \) for \( N = 10^4 \), \( \Lambda = 32 \) are shown in Fig. 3. In this figure, the solid line shows the condensate fraction for \( z_1 = 0 \) and the dashed line shows the condensate fraction for \( z_1 = 1 \). In [18], Ketterle and Druten mention that the phase transitions due to the discontinuity in an observable macroparameter occurs only in the thermodynamic limit, where \( N \to \infty \). However, we make our calculations for a realistic system with a finite number of particles in a confining potential. Thus, \( N_0/N \) is a finite nonzero quantity for \( T < T_c \) without having any discontinuity at \( T = T_c \).

We also investigate the behavior of the condensate fraction as a function of the position of Dirac \( \delta \) for \( \Lambda = 32 \), \( T = T_c^0 \), and present the results in Fig. 4.

Finally, we compare the density profiles of condensates for a harmonic trap and a harmonic trap decorated with a delta function (\( \Lambda = 3.2 \) and \( z_1 = 1 \)) in Fig. 5. Since the ground state wavefunctions can be calculated analytically for both cases, we find the density profiles by taking the absolute square of the ground-state wavefunctions. Comparing the graphics of the density profiles, we see that an off-center dimple potential maintains a higher density at the position of the Dirac \( \delta \) function, which may be utilized for the fragmentation of a BEC.

4. CONCLUSIONS

We have investigated the effect of the location of the tight dimple potential on the results reported recently in our paper [11]. We model the tight dimple potential with the Dirac \( \delta \) function. This allows for analytical expressions for the eigenfunctions of the system and the simple eigenvalue equations greatly simplify subsequent numerical treatment. We have calculated the critical temperature, chemical potential, and condensate fraction and presented the effects of the location of the dimple potential. We find that the dimple-type potentials are most effective when they are applied to the center. It is also advantageous to place the dimple potential at the nodes of the excited state, where our results revealed a relative enhancement of the critical temperature and the condensate fraction. Determining the density profiles of the BECs in the harmonic trap and in the decorated trap with the Dirac \( \delta \) function at this critical position, we argued that an eccentric dimple trap at such a critical location can be used for the spatial fragmentation of large enhanced BECs.

The presented results are obtained for the case of noninteracting and one-dimensional condensates for simplicity. In such a case, the stability of the condensate may become questionable and should be addressed separately in detail [19]. The treatment should be extended for the case of interacting condensates in larger (or quasi) dimensional traps in order to make the results more relevant to experimental investigations.

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