THE ALGEBRA OF ADJACENCY PATTERNS: REES MATRIX SEMIGROUPS WITH REVERSION

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Abstract. We establish a surprisingly close relationship between universal Horn classes of directed graphs and varieties generated by so-called adjacency semigroups which are Rees matrix semigroups over the trivial group with the unary operation of reversion. In particular, the lattice of subvarieties of the variety generated by adjacency semigroups that are regular unary semigroups is essentially the same as the lattice of universal Horn classes of reflexive directed graphs. A number of examples follow, including a limit variety of regular unary semigroups and finite unary semigroups with NP-hard variety membership problems.

Introduction and overview

The aim of this paper is to establish and to explore a new link between graph theory and algebra. Since graphs form a universal language of discrete mathematics, the idea to relate graphs and algebras appears to be natural, and several useful links of this kind can be found in the literature. We mean, for instance, the graph algebras of McNulty and Shallon [20], the closely related flat graph algebras [25], and “almost trivial” algebras investigated in [15, 16] amongst other places. While each of the approaches just mentioned has proved to be useful and has yielded interesting applications, none of them seem to share two important features of the present contribution. The two features can be called naturalness and surjectivity.

Speaking about naturalness, we want to stress that the algebraic objects (adjacency semigroups) that we use here to interpret graphs have not been invented for this specific purpose. Indeed, adjacency semigroups belong to a well established class of unary semigroups\textsuperscript{1} that have been considered by many authors. We shall demonstrate how graph theory both sheds a new light on some previously known algebraic results and provides their extensions and generalizations. By surjectivity we mean that, on the level

\textsuperscript{1}Here and below the somewhat oxymoronic term “unary semigroup” abbreviates the precise but longer expression “semigroup endowed with an extra unary operation”.

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of appropriate classes of graphs and unary semigroups, the interpretation map introduced in this paper becomes “nearly” onto; moreover, the map induces a lattice isomorphism between the lattices of such classes provided one excludes just one element on the semigroup side. This implies that our approach allows one to interpret both graphs within unary semigroups and unary semigroups within graphs.

The paper is structured as follows. In Section 1 we recall some notions related to graphs and their classes and present a few results and examples from graph theory that are used in the sequel. Section 2 contains our construction and the formulations of our main results: Theorems 2.1 and 2.2. These theorems are proved in Sections 3 and 4 respectively while Section 5 collects some of their applications.

We assume the reader’s acquaintance with basic concepts of universal algebra and first-order logics such as ultraproducts or the HSP-theorem, see, e.g., [4]. As far as graphs and semigroups are concerned, we have tried to keep the presentation to a reasonable extent self-contained. We do occasionally mention some non-trivial facts of semigroup theory but only in order to place our considerations in a proper perspective. Thus, most of the material should be accessible to readers with very basic semigroup-theoretic background (such as some knowledge of Green’s relations and of the Rees matrix construction over the trivial group, cf. [12]).

1. Graphs and their classes

In this paper, graph is a structure \( G := (V; \sim) \), where \( V \) is a set and \( \sim \subseteq V \times V \) is a binary relation. In other words, we consider all graphs to be directed, and do not allow multiple edges (but do allow loops). Of course, \( V \) is often referred to as the set of vertices of the graph and \( \sim \) as the set of edges. As is usual, we write \( a \sim b \) in place of \((a,b) \in \sim\). Conventional undirected graphs are essentially the same as graphs whose edge relation is symmetric (satisfying \( x \sim y \rightarrow y \sim x \)), while a simple graph is a symmetric graph without loops. It is convenient for us to allow the empty graph \( \emptyset := (\emptyset; \emptyset) \).

All classes of graphs that come to consideration in this paper are universal Horn classes. We recall their definition and some basic properties. Of course, the majority of the statements below are true for arbitrary structures, but our interest is only in the graph case. See Gorbunov [9] for more details.

Universal Horn classes can be defined both syntactically (via specifying an appropriate sort of first order formulas) and semantically (via certain class operators). We first introduce the operator definition for which we recall notation for a few standard class operators. The operator for taking isomorphic copies is \( I \). We use \( S \) to denote the operator taking a class \( K \) to the class of all substructures of structures in \( K \); in the case when \( K \) is a class of graphs, substructures are just induced subgraphs of graphs in \( K \). Observe that the empty graph \( \emptyset \) is an induced subgraph of any graph and
thus belongs to any S-closed class of graphs. We denote by \( P \) the operator of taking direct products. For graphs, we allow the notion of an empty direct product, which we identify (as is the standard convention) with the 1-vertex looped graph \( \mathbf{1} := \langle \{0\}; \{(0,0)\} \rangle \). If we exclude the empty product, we obtain the operator \( P^+ \) of taking nonempty direct products. By \( \mathbb{P}_u \) we denote the operator of taking ultraproducts. Note that ultraproducts— unlike direct products—are automatically nonempty.

A class \( K \) of graphs is an universal Horn class if \( K \) is closed under each of the operators \( I, S, P^+, \text{ and } \mathbb{P}_u \). In the sequel, we write “uH class” in place of “universal Horn class”. It is well known that the least uH class containing a class \( L \) of graphs is the class \( \mathbb{ISP}^+ \mathbb{P}_u(L) \) of all isomorphic copies of induced subgraphs of nonempty direct products of ultraproducts of \( L \); this uH class is referred to as the uH class generated by \( L \).

If the operator \( P^+ \) in the above definition is extended to \( P \), then one obtains the definition of a quasivariety of graphs. The quasivariety generated by a given class \( L \) is known to be equal to \( \mathbb{ISP} \mathbb{P}_u(L) \). It is not hard to see that \( \mathbb{ISP} \mathbb{P}_u(L) = I(\mathbb{ISP}^+ \mathbb{P}_u(L) \cup \{1\}) \), showing that there is little or no difference between the uH class and the quasivariety generated by \( L \). However, as examples described later demonstrate, there are many well studied classes of graphs that are uH classes but not quasivarieties.

As mentioned, uH classes also admit a well known syntactic characterization. An atomic formula in the language of graphs is an expression of the form \( x \sim y \) or \( x \approx y \) (where \( x \) and \( y \) are possibly identical variables). A universal Horn sentence (abbreviated to “uH sentence”) in the language of graphs is a sentence of one of the following two forms (for some \( n \in \omega := \{0,1,2,\ldots\} \)):

\[
\forall x_1 \forall x_2 \ldots \left( \bigwedge_{1 \leq i \leq n} \Phi_i \rightarrow \Phi_0 \right) \quad \text{or} \quad \forall x_1 \forall x_2 \ldots \left( \bigvee_{0 \leq i \leq n} \neg \Phi_i \right)
\]

where the \( \Phi_i \) are atomic, and \( x_1, x_2, \ldots \) is a list of all variables appearing. In the case when \( n = 0 \), a uH sentence of the first kind is simply the universally quantified atomic expression \( \Phi_0 \). Sentences of the first kind are usually called quasi-identities. As is standard, we omit the universal quantifiers when describing uH sentences; also the expressions \( x \neq y \) and \( x \approx y \) abbreviate \( \neg x \approx y \) and \( \neg x \sim y \) respectively. Satisfaction of uH sentences by graphs is defined in the obvious way. We write \( G \models \Phi \) (\( K \models \Phi \)) to denote that the graph \( G \) (respectively, each graph in the class \( K \)) satisfies the uH sentence \( \Phi \).

The Birkhoff theorem identifying varieties of algebras with equationally defined classes has a natural analogue for uH classes, which is usually attributed to Mal’cev. Here we state it in the graph setting.

**Lemma 1.1.** A class \( K \) of graphs is a uH class if and only if it is the class of all models of some set of uH sentences.
In particular, the uH class \( \text{ISP}^+ \text{P}_u(L) \) generated by a class \( L \) is equal to the class of models of the uH sentences holding in \( L \).

Recall that we allow the empty graph \( \emptyset := \langle \emptyset; \emptyset \rangle \). Because there are no possible variable assignments into the empty set, \( \emptyset \) can fail no uH sentence and hence lies in every uH class. Thus, allowing \( \emptyset \) brings the advantage that the collection of all uH classes forms a lattice whose meet is intersection: \( A \wedge B := A \cap B \) and whose join is given by \( A \vee B := \text{ISP}^+ \text{P}_u(A \cup B) \). Furthermore, the inclusion of \( \emptyset \) allows every set of uH sentences to have a model (for example, the contradiction \( x \not\approx x \) axiomatizes the class \( \{ \emptyset \} \)). In the world of varieties of algebras, it is the one element algebra that plays these roles.

When \( \text{IP}_u(L) = \text{I}(L) \) (such as when \( L \) consists of finitely many finite graphs), we have \( \text{ISP}^+ \text{P}_u(L) = \text{ISP}^+(L) \), and there is a handy structural characterization of the uH class generated by \( L \).

**Lemma 1.2.** Let \( L \) be an ultraproduct closed class of graphs and let \( G \) be a graph. We have \( G \in \text{ISP}^+ \text{P}_u(L) \) if and only if there is at least one homomorphism from \( G \) into a member of \( L \) and the following two separation conditions hold:

1. for each pair of distinct vertices \( a, b \) of \( G \), there is \( H \in L \) and a homomorphism \( \phi : G \to H \) with \( \phi(a) \neq \phi(b) \);
2. for each pair of vertices \( a, b \) of \( G \) with \( a \not\approx b \) in \( G \), there is \( H \in L \) and a homomorphism \( \phi : G \to H \) with \( \phi(a) \not\approx \phi(b) \) in \( H \).

The 1-vertex looped graph \( 1 \) always satisfies the two separation conditions, yet it fails every uH sentence of the second kind; this is why the lemma asks additionally that there be at least one homomorphism from \( G \) into some member of \( L \). If \( G = 1 \) and no such homomorphism exists, then evidently, each member of \( L \) has nonlooped vertices, and so \( L \models x \not\approx x \), a law failing on \( 1 \). Hence \( 1 \notin \text{ISP}^+ \text{P}_u(L) \) by Lemma 1.1. Conversely, if there is such a homomorphism, then \( 1 \) is isomorphic to an induced subgraph of some member of \( L \) and hence \( 1 \in \text{ISP}^+ \text{P}_u(L) \). If the condition that there is at least one homomorphism from \( G \) into some member of \( L \) is dropped, then Lemma 1.2 instead characterizes membership in the quasivariety generated by \( L \).

We now list some familiar uH sentences.

- **reflexivity**: \( x \sim x \),
- **anti-reflexivity**: \( x \not\sim x \),
- **symmetry**: \( x \sim y \to y \sim x \),
- **anti-symmetry**: \( x \sim y \& y \sim x \to x \approx y \),
- **transitivity**: \( x \sim y \& y \sim z \to x \sim y \).

All except anti-reflexivity are quasi-identities.

These laws appear in many commonly investigated classes of graphs. We list a number of examples that are of interest later in the paper (mainly in its application part, see Section 5 below).
Example 1.3. Preorders.

This class is defined by reflexivity and transitivity and is a quasivariety. Some well known subclasses are:

- equivalence relations (obtained by adjoining the symmetry law);
- partial orders (obtained by adjoining the anti-symmetry law);
- anti-chains (the intersection of partial orders and equivalence relations);
- complete looped graphs, or equivalently, single block equivalence relations (axiomatized by $x \sim y$).

In fact it is easy to see that, along with the 1-vertex partial orders and the trivial class $\{0\}$, this exhausts the list of all uH classes of preorders, see Fig. 1 (the easy proof is sketched before Corollary 6.4 of [7], for example).

![Figure 1. The lattice of uH classes of preorders](image)

Example 1.4. Simple (that is, anti-reflexive and symmetric) graphs.

Sub-uH classes of simple graphs have been heavily investigated, and include some very interesting families. In order to describe some of these families, we need a sequence of graphs introduced by Nešetřil and Pultr [21]. For each integer $k \geq 2$, let $C_k$ denote the graph on the vertices $0, \ldots, k + 1$ obtained from the complete loopless graph on these vertices by deleting the edges (in both directions) connecting $0$ and $k + 1$, $0$ and $k$, and $1$ and $k + 1$. Fig. 2 shows the graphs $C_2$ and $C_3$; here and below we adopt the convention that an undirected edge between two vertices, say $a$ and $b$, represents two directed edges $a \sim b$ and $b \sim a$.

![Figure 2. Graphs $C_2$ and $C_3$](image)
Recall that a simple graph $G$ is said to be $n$-colorable is there exists a homomorphism from $G$ into the complete loopless graph on $n$ vertices.

**Example 1.5.** The 2-colorable graphs (equivalently, bipartite graphs).

This is the uH class $\text{ISP}^+(C_2)$ generated by the graph $C_2$ (Nešetřil and Pultr [21]) and has no finite axiomatization. Caicedo [5] showed that the lattice of sub-uH classes of $\text{ISP}^+(C_2)$ is a 6-element chain: besides $\text{ISP}^+(C_2)$, it contains the class of disjoint unions of complete bipartite graphs, which is axiomatized within simple graphs by the law

$$x_0 \sim x_1 \ & \ x_1 \sim x_2 \ & \ x_2 \sim x_3 \rightarrow x_0 \sim x_3;$$

the class of disjoint unions of paths of length at most 1 (axiomatized within simple graphs by $x \sim y \lor y \sim z$); the edgeless graphs (axiomatized by $x \sim y$), the 1-vertex edgeless graphs ($x \approx y$); and the trivial class $\{0\}$.

Every finite simple graph either lies in a sub-uH class of $\text{ISP}^+(C_2)$ or generates a uH class that: 1) is not finitely axiomatizable, 2) contains $\text{ISP}^+(C_2)$, and 3) has uncountably many sub-uH classes [10, Theorem 4.7], see also [17].

**Example 1.6.** The $k$-colorable graphs.

More generally, Nešetřil and Pultr [21] showed that for any $k \geq 2$, the class of all $k$-colorable graphs is the uH class generated by $C_k$. These classes have no finite basis for their uH sentences and for $k > 2$ have NP-complete finite membership problem, see [8].

**Example 1.7.** A generator for the class $G$ of all graphs.

The class of all graphs is generated as a uH class by a single finite graph. Indeed, it is trivial to see that for any graph $G$, there is a family of 3-vertex graphs such that the separation conditions of Lemma 1.2 hold. Since there are only finitely many non-isomorphic 3-vertex graphs, any graph containing these as induced subgraphs generates the uH class of all graphs. Alternatively, the reader can easily verify using Lemma 1.2 that the following graph $G_1$ generates the uH class of all graphs:

![Figure 3. Generator for the uH class of all graphs](image)

**Example 1.8.** A generator for the class $G_{\text{symm}}$ of all symmetric graphs.

Using Lemma 1.2 it is easy to prove that the class of symmetric graphs is generated as a uH class by the graph $S_1$ shown in Fig. 4.
Example 1.9. The class of simple graphs has no finite generator.

The class of all simple graphs is not generated by any finite graph, since a finite graph on \( n \) vertices is \( n \)-colorable, while for every positive integer \( n \) there is a simple graph that is not \( n \)-colorable (the complete simple graph on \( n + 1 \) vertices, for example). However the \( \mathcal{UH} \) class generated by the following 2-vertex graph \( S_2 \) contains all simple graphs (this is well known and follows easily using Lemma 1.2).

Example 1.10. A generator for the class \( \mathcal{G}_{\text{ref}} \) of all reflexive graphs.

The class of reflexive graphs is generated by the following graph \( R_1 \), while the class of reflexive and symmetric graphs is generated by the graph \( \text{RS}_1 \).

2. The Adjacency Semigroup of a Graph

Given a graph \( G = \langle V; \sim \rangle \), its adjacency semigroup \( A(G) \) is defined on the set \( (V \times V) \cup \{0\} \) and the multiplication rule is

\[
(x, y)(z, t) = \begin{cases} 
(x, t) & \text{if } y \sim z, \\
0 & \text{if } y \not\sim z;
\end{cases}
\]

\( a0 = 0a = 0 \) for all \( a \in A(G) \).

In terms of semigroup theory, \( A(G) \) is the Rees matrix semigroup over the trivial group using the adjacency matrix of the graph \( G \) as a sandwich matrix. We describe here the Rees matrix construction in a specific form that is used in the present paper.
Let $I, J$ be nonempty sets and $0 \notin I \cup J$. Let $P = (P_{i,j})$ be a $J \times I$ matrix (the sandwich matrix) over the set $\{0, 1\}$. The Rees matrix semigroup over the trivial group $M^0[P]$ is the semigroup on the set $(I \times J) \cup \{0\}$ with multiplication

$$a \cdot 0 = 0 \cdot a = 0 \text{ for all } a \in (I \times J) \cup \{0\}, \text{ and}$$

$$(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} 0 & \text{if } P_{j_1, i_2} = 0, \\ (i_1, j_2) & \text{if } P_{j_1, i_2} = 1. \end{cases}$$

The Rees-Sushkevich Theorem (see [12, Theorem 3.3.1]) states that, up to isomorphism, the completely 0-simple semigroups with trivial subgroups are precisely the Rees matrix semigroups over the trivial group and for which each row and each column of the sandwich matrix contains a nonzero element. If the matrix $P$ has no 0 entries, then the set $M[P] = M^0[P] \setminus \{0\}$ is a subsemigroup. Semigroups of the form $M[P]$ are called rectangular bands, and they are precisely the completely simple semigroups with trivial subgroups.

Back to adjacency semigroups, we always think of $A(G)$ as endowed with an additional unary operation $a \mapsto a'$ which we call reversion and define as follows:

$$(x, y)' = (y, x), \quad 0' = 0.$$

Notice that by this definition $(a')' = a$ for all $a \in A(G)$.

The main contribution in this paper is the fact that uH classes of graphs are in extremely close correspondence with unary semigroup varieties generated by adjacency semigroups, and our proof of this will involve a translation of uH sentences of graphs into unary semigroup identities. However, before we proceed with precise formulations and proofs of general results, the reader may find it useful to check that several of the basic uH sentences used in Section I correspond via the adjacency semigroup construction to rather natural semigroup-theoretic properties. Indeed, all the following are quite easy to verify:

- reflexivity of $G$ is equivalent to $A(G) \models xx'x \approx x$;
- anti-reflexivity of $G$ is equivalent to $A(G) \models xx'z \approx zxx' \approx xx'$ (these laws can be abbreviated to $xx' \approx 0$);
- symmetry of $G$ is equivalent to $A(G) \models (xy)' \approx y'x'$;
- $G$ is empty (satisfies $x \not\approx x$) if and only if $A(G) \models x \approx y$;
- $G$ has one vertex (satisfies $x \approx y$) if and only if $A(G) \models x \approx x'$; also, $G$ is the one vertex looped graph (satisfies $x \sim y$) if and only if $A(G)$ additionally satisfies $xx \approx x$.

Observe that the unary semigroup identities that appear in the above examples are in fact used to define the most widely studied types of semigroups endowed with an extra unary operation modelling various notions of an inverse in groups. For instance, a semigroup satisfying the identities

$$x'' \approx x \quad \text{(1)}$$


(which always holds true in adjacency semigroups) and
\[(xy)' \approx y'x'\]
(which is a semigroup counterpart of symmetry) is called \textit{involution semigroup} or \textit{*}-semigroup. If such a semigroup satisfies also
\[xx'x \approx x\]
(which corresponds to reflexivity), it is called a \textit{regular *-semigroup}. Semigroups satisfying (1) and (3) are called \textit{I-semigroups} in Howie [12]; note that an I-semigroup satisfies \(x'xx' \approx x'x'x' \approx x'\), so that \(x'\) is an inverse of \(x\). Semigroups satisfying (3) are often called \textit{regular unary semigroups}. There exists vast literature on all these types of unary semigroups; clearly, the present paper is not a proper place to survey this literature but we just want to stress once more that the range of the adjacency semigroup construction is no less natural than its domain.

When \(K\) is a class of graphs, we use the notation \(A(K)\) to denote the class of all adjacency semigroups of members of \(K\). As usual, the operator of taking homomorphic images is denoted by \(H\). We let \(A\) denote the variety \(\text{HSP}(A(G))\) generated by all adjacency semigroups of graphs, and let \(A_{\text{ref}}\) and \(A_{\text{symm}}\) denote the varieties \(\text{HSP}(A(G_{\text{ref}}))\) and \(\text{HSP}(A(G_{\text{symm}}))\) generated by all adjacency semigroups of reflexive graphs and of symmetric graphs respectively.

Our first main result is:

\textbf{Theorem 2.1.} Let \(K\) be any nonempty class of graphs and let \(G\) be a graph. The graph \(G\) belongs to the \(uH\)-class generated by \(K\) if and only if the adjacency semigroup \(A(G)\) belongs to the variety generated by the adjacency semigroups \(A(H)\) with \(H \in K\).

This immediately implies that the assignment \(G \mapsto A(G)\) induces an injective join-preserving map from the lattice of all \(uH\)-classes of graphs to the subvariety lattice of the variety \(A\). The latter fact can be essentially refined for the case of reflexive graphs. In order to describe this refinement, we need an extra definition.

Let \(I\) be a nonempty set. We endow the set \(B = I \times I\) with a unary semigroup structure whose multiplication is defined by
\[(i,j)(k,\ell) = (i,\ell)\]
and whose unary operation is defined by
\[(i,j)' = (j,i)\].

It is easy to check that \(B\) becomes a regular \(*\)-semigroup. We call regular \(*\)-semigroups constructed this way \textit{square bands}. Clearly, square bands satisfy
\[x^2 \approx x\] and \(xyz \approx xz\],

\[x'x \approx x'x'x' \approx x'\].
and in fact it can be shown that the class $SB$ of all square bands constitutes a variety of unary semigroups defined within the variety of all regular *-
semigroups by the identities (4).

Let $L(G_{\text{ref}})$ denote the lattice of sub-uH classes of $G_{\text{ref}}$ and let $L(A_{\text{ref}})$ denote the lattice of subvarieties of $A_{\text{ref}}$. Let $L^+$ denote the result of adjoining a new element $S$ to $L(G_{\text{ref}})$ between the class of single block equivalence relations and the class containing the empty graph. (The reader may wish to look at Fig. 1 to see the relative location of these two uH classes.) Meets and joins are extended to $L^+$ in the weakest way. So $L^+$ is a lattice in which $L(G_{\text{ref}})$ is a sublattice containing all but one element.

We are now in a position to formulate our second main result.

**Theorem 2.2.** Let $\iota$ be the map from $L^+$ to $L(A_{\text{ref}})$ defined by $S \mapsto SB$ and $K \mapsto \text{HSP}(A(K))$ for $K \in L(G_{\text{ref}})$. Then $\iota$ is a lattice isomorphism. Furthermore, a variety in $L(A_{\text{ref}})$ is finitely axiomatized (finitely generated as a variety) if and only if it is the image under $\iota$ of either $S$ or a finitely axiomatized (finitely generated, respectively) uH class of reflexive graphs.

We prove Theorems 2.1 and 2.2 in the next two sections.

### 3. Proof of Theorem 2.1

#### 3.1. Equations satisfied by adjacency semigroups.

The variety of semigroups generated by the class of Rees matrix semigroups over trivial groups is reasonably well understood: it is generated by a 5-element semigroup usually denoted by $A_2$ (see [19] for example). (In context of this paper $A_2$ can be thought as the semigroup reduct of the adjacency semigroup $A(S_2)$ where $S_2$ is the 2-vertex graph from Example 1.9.) This semigroup was shown to have a finite identity basis by Trahtman [23], who gave the following elegant description of the identities: an identity $u \approx v$ (where $u$ and $v$ are semigroup words) holds in $A_2$ if and only if $u$ and $v$ start with the same letter, end with the same letter and share the same set of two letter subwords. Thus the equational theory of this variety corresponds to pairs of words having the same “adjacency patterns”, in the sense that a two letter subword $xy$ records the fact that $x$ occurs next to (and before) $y$. This adjacency pattern can also be visualized as a graph on the set of letters, with an edge from $x$ to $y$ if $xy$ is a subword, and two distinct markers indicating the first and last letters respectively.

In this subsection we show that the equational theory of $A$ has the same kind of property with respect to a natural unary semigroup notion of adjacency. The interpretation is that each letter has two sides—left and right—and that the operation $'$ reverses these. A subword $xy$ corresponds to the right side of $x$ matching the left side of $y$, while $x'y$ or any subword $(x\ldots)'y$ corresponds to the left side of $x$ matching the left side of $y$. To make this more precise, we give an inductive definition. Under this definition, each letter $x$ in a word will have two associated vertices corresponding to the left
and right side. The graph will have an initial vertex, a final vertex as well as a set of (directed) edges corresponding to adjacencies.

Let \( u \) be a unary semigroup word, and \( X \) be the alphabet of letters appearing in \( u \). We construct a graph \( G[u] \) on the set

\[
\{ \ell_x \mid x \in X \} \cup \{ r_x \mid x \in X \}
\]

with two marked vertices. If \( u \) is a single letter (say \( x \)), then the edge set (or adjacency set) of \( G[u] \) is empty. The initial vertex of a single letter \( x \) is \( \ell_x \) and the final (or terminal) vertex is \( r_x \).

If \( u \) is not a single letter, then it is of the form \( v' \) or \( vw \) for some unary semigroup words \( v, w \). We deal with the two cases separately. If \( u \) is of the form \( v' \), where \( v \) has set of adjacencies \( S \), initial vertex \( p_a \) and final vertex \( q_b \) (where \( \{p,q\} \subseteq \{\ell,r\} \) and \( a, b \) are letters appearing in \( v \)), then the set of adjacencies of \( u \) is also \( S \), but the initial vertex of \( u \) is equal to the final vertex \( q_b \) of \( v \) and the final vertex of \( u \) is equal to the initial vertex \( p_a \) of \( v \).

Now say that \( u \) is of the form \( vw \) for some unary semigroup words \( v, w \), with adjacency set \( S_v \) and \( S_w \) respectively and with initial vertices \( p_{av} \), \( p_{aw} \) respectively and final vertices \( q_{bv} \) and \( q_{bw} \) respectively. Then the adjacency set of \( G[u] \) is \( S_v \cup S_w \cup \{(q_{bv}, p_{aw})\} \), the initial vertex is \( p_{av} \) and the final vertex is \( q_{bv} \). Note that the word \( u \) may be broken up into a product of two unary words in a number of different ways, however it is reasonably clear that this gives rise to the same adjacency set and initial and final vertices (this basically corresponds to the associativity of multiplication).

For example the word \( a'(baa')' \) decomposes as \( a' \cdot (baa')' \), and so has initial vertex equal to the initial vertex of \( a' \), which in turn is equal to the terminal vertex of \( a \), which is \( r_a \). Likewise, its terminal vertex should be the terminal vertex of \( (baa')' \), which is the initial vertex of \( baa' \), which is \( \ell_b \). Continuing, we see that the edge set of the corresponding graph has edges \( \{(\ell_a, \ell_a), (r_a, r_a), (r_a, \ell_a)\} \). This graph is the first graph depicted in Figure 7 (the initial and final vertices are indicated by a sourceless and targetless arrow respectively). The second is the graph of either of the words \( a(bc)' \) or \( (b(ac')')' \). The fact that \( G[a(bc)'] = G[(b(ac')')'] \) will be of particular importance in constructing a basis for the identities of \( A \).

Figure 7. Two examples of graphs of unary words
We can also construct a second kind of graph from a word \( w \), in which all loops are added to the graph of \( G[w] \) (that is, it is the reflexive closure of the edge set), we call this \( G_{\text{ref}}[w] \). For example, it is easy to see that \( G_{\text{ref}}[a'(ba')] = G_{\text{ref}}[(ba)'] \) (most of the work was done in the previous example). Lastly, we define the graph \( G_{\text{symm}}[w] \) corresponding to the symmetric closure of the edge set of \( G[w] \).

**Notation 3.1.** Let \( u \) be a unary semigroup word and let \( \theta \) be an assignment of the letters of \( u \) into nonzero elements of an adjacency semigroup \( A(H) \); say \( \theta(x) = (i_x, j_x) \) for each letter \( x \). Note that \( \theta(x') = (j_x, i_x) \), so we use the notation \( i_{x'} := j_x \) and \( j_{x'} = i_x \).

**Lemma 3.2.** Let \( u \) and \( \theta \) be as in Notation 3.1. If \( \lambda_a \) is the initial vertex of \( G[u] \) and \( \rho_b \) is the terminal vertex (so \( \lambda, \rho \in \{\ell, r\} \) and \( a \) and \( b \) are letters in \( u \)) and \( \theta(u) \neq 0 \), then \( \theta(u) = (i_{\bar{a}}, j_{\bar{b}}) \), where

\[
\bar{a} = \begin{cases} 
  a & \text{if } \lambda = \ell \\
  a' & \text{if } \lambda = r
\end{cases}
\quad \text{and} \quad
\bar{b} = \begin{cases} 
  b & \text{if } \rho = r \\
  b' & \text{if } \rho = \ell.
\end{cases}
\]

**Proof.** This follows by an induction following the inductive definition of the graph of \( u \). \( \square \)

**Lemma 3.3.** Let \( u \) and \( \theta \) be as in Notation 3.1. Then \( \theta(u) \neq 0 \) if and only if the map defined by \( \ell_x \mapsto i_x \) and \( r_x \mapsto j_x \) is a graph homomorphism from \( G[u] \) to \( H \).

**Proof.** Throughout the proof we use the notation of Lemma 3.2.

(Necessity.) Say \( \theta(u) \neq 0 \), and let \((\rho_b, \lambda_a)\) be an edge in \( G[u] \), where \( \rho, \lambda \in \{\ell, r\} \) and \( a \) and \( b \) are letters in \( u \). We use Lemma 3.2 to show that \((j_{\bar{b}}, i_{\bar{a}})\) is an edge of \( H \). Note that in the case where no applications of \( \ell' \) are used (so we are dealing in the nonunary case), the edge \((\rho_b, \lambda_a)\) will necessarily be \((r_b, \ell_a)\); and we would want \((j_{\bar{b}}, i_{\bar{a}})\) to be an edge of \( H \).

Now, since \((\rho_b, \lambda_a)\) is an edge in \( G[u] \), some subwords of \( u \)—say \( u_1 \) and \( u_2 \)—have \( u_1 u_2 \) a subword of \( u \), and \( \rho_b \) the terminal vertex of \( G[u_1] \), and \( \lambda_a \) the initial vertex of \( G[u_2] \). Applying Lemma 3.2 to both \( G[u_1] \) and \( G[u_2] \), we find that \( \theta(u_1) \) has right coordinate \( j_{\bar{b}} \), and \( \theta(u_2) \) has left coordinate \( i_{\bar{a}} \).

But \( u_1 u_2 \) is a subword, so \( \theta(u_1) \theta(u_2) \neq 0 \), whence \((j_{\bar{b}}, i_{\bar{a}})\) is an edge of \( H \), as required.

(Sufficiency.) This is easy. \( \square \)

Lemma 3.3 is easily adapted to the graph \( G_{\text{ref}}(u) \) or \( G_{\text{symm}}(u) \), where the graph \( H \) is assumed to be reflexive or symmetric, respectively.

**Proposition 3.4.** An identity \( u \approx v \) holds in \( A \) if and only if \( G[u] = G[v] \). An identity holds in \( A_{\text{ref}} \) if and only if \( G_{\text{ref}}[u] = G_{\text{ref}}[v] \). An identity holds in \( A_{\text{symm}} \) if and only if \( G_{\text{symm}}[u] = G_{\text{symm}}[v] \).

**Proof.** We prove only the first case; the other two cases are similar.

First we show sufficiency. Let us assume that \( G[u] = G[v] \), and consider an assignment \( \theta \) into an adjacency semigroup \( A(H) \). Now the vertex sets
are the same, so \( u \) and \( v \) have the same alphabet. So we may assume that \( \theta \) maps the alphabet to nonzero elements of \( A(H) \). By Lemma 3.3, we have \( \theta(u) \neq 0 \) if and only if \( \theta(v) \neq 0 \). By Lemma 3.2, we have \( \theta(u) = \theta(v) \) whenever both sides are nonzero. Hence \( \theta(u) = \theta(v) \) always.

Now for necessity. Say that \( G[u] \neq G[v] \). If the vertex sets are distinct, then \( u \approx v \) fails on \( A(\Lambda) \), which is isomorphic to the unary semigroup formed by the integers 0 and 1 with the usual multiplication and the identity map as the unary operation. Now say that \( G[u] \) and \( G[v] \) have the same vertices. Without loss of generality, we may assume that either \( G[v] \) contains an edge not in \( G[u] \), or that the two graphs are identical but have different initial vertices. Let \( A_u := A(G[u]) \) and consider the assignment into \( A_u \) that sends each variable \( x \) to \((\ell_x,r_x)\). Observe that the value of \( u \) is equal to \((\lambda_a,\rho_b)\) where \( \lambda_a \) is the initial vertex of \( G[u] \) and \( \rho_b \) is the final vertex, while the value of \( v \) is either 0 (if there is an adjacency not in \( G[v] \); we fail to get a graph homomorphism) or has different first coordinate (if \( G[v] \) has a different initial vertex). So \( u \approx v \) fails in \( A \).

3.2. A normal form. Proposition 3.4 gives a reasonable solution to the word problem in the \( A \)-free algebras. In this subsection we go a bit further and show that every unary semigroup word is equivalent in \( A \) to a unary semigroup word of a certain form. Because different forms may have the same adjacency graph, this by itself does not constitute a different solution to the word problem in \( A \)-free algebras, however it is useful in analyzing identities of \( A \).

Most of the work in this section revolves around the variety of algebras of type \( \langle 2,1 \rangle \) defined by the three laws:

\[
\Psi = \{x'' \approx x, \ x(yz)' \approx (yxz)'', \ (xy)'z \approx ((xz')y)' \}.
\]

as interpreted within the variety of unary semigroups. By examining the adjacency graphs, it is easy to see that these identities are all satisfied by \( A \) (see Fig. 7 for one of these). In fact \( \Psi \) defines a strictly larger variety than \( A \) (it contains all groups for example), but they are close enough for us to obtain useful information. For later reference we refer to the second and third laws in \( \Psi \) as the first associativity across reversion law (FAAR) and second associativity across reversion law (SAAR), respectively. We let \( B \) denote the unary semigroup variety defined by \( \Psi \).

Surprisingly, the laws in \( \Psi \) are sufficient to reduce every unary semigroup word to one in which the nesting height of the unary \( ' \) is at most 2. The proof of this is the main result of this subsection.

**Lemma 3.5.** \( \Psi \) implies \((a(bcd)'e)' \approx (b'e)'(c(ad)'')\), where \( c \) is possibly empty.

**Proof.** We prove the case where \( c \) is non-empty only. We have

\[
(a(bcd)'e)' \overset{\text{FAAR}}{\approx} (bc(ad)'')' \overset{\text{SAAR}}{\approx} ((b'e)'c(ad'')'' \approx (b'e)'c(ad)''.
\]

Let \( X := \{x_1, x_2, \ldots \} \). Let \( F(X) \) denote the free unary semigroup freely generated by \( X \) and \( F_\Psi(X) \) denote the \( B \)-free algebra freely generated
by $X$. We let $\psi$ denote the fully invariant congruence on $F(X)$ giving $F_\psi(X) = F(X)/\psi$. We find a subset $N \subseteq F(X)$ with $X \subseteq N$ and show that multiplying two words from $N$ in $F(X)$, or applying $'$ to a word in $N$ produces a word that is $\psi$-equivalent to a word in $N$. In other words, $N$ forms a transversal of $\psi$; equivalently, it shows that every word in $F(X)$ is $\psi$-equivalent to a word in $N$. In this way the members of $N$ are a kind of weak normal form for terms modulo $\Psi$ (we do not claim that distinct words in $N$ are not $\psi$-equivalent; for example, Proposition 3.3 shows that $\mathcal{A} \models x(x'y')' \equiv x(x'y')'$, but the two words are distinct elements of $N$).

We let $N$ consist of all (nonempty) words of the form

$$u_1(v_1)'u_2(v_2)\ldots u_n(v_n)'u_{n+1}$$

for some some $n \in \omega$, where for $i \leq n$,

- the $v_i$ are semigroup words in the alphabet

$$X \cup X' = \{x_1, x_1', x_2, x_2', \ldots\},$$

and all have length at least 2 as semigroup words;

- the $u_i$ are possibly empty semigroup words in the alphabet $X \cup X'$ and if $n = 0$, then $u_1$ is non-empty.

Notice that $X \subseteq N$ since the case $n = 0$ corresponds to semigroup words over $X \cup X'$. For a member $s$ of $N$, we refer to the number $n$ in this definition as the breadth of $s$.

The following lemma is trivial.

**Lemma 3.6.** If $s$ and $t$ be two words in $N$, then $s \cdot t$ is $\psi$-equivalent to a word in $N$.

**Lemma 3.7.** If $s$ is a word in $N$, then $s'$ is $\psi$-equivalent to a word in $N$.

**Proof.** We prove the lemma by induction on the breadth of $s$. If the breadth of $s$ is 0 then $s' = (u_1)'$ is either in $N$, or is of the form $x''$ for some variable $x$, in which case it reduces to $x \in N$ modulo $\Psi$. Now say that the result holds for breadth $k$ members of $N$, and say that the breadth of $s$ is $k + 1$. So $s$ can be written in the form $p(y_1 \cdots y_m)'u$ where $p$ is either empty or is a word from $N$ of breadth $k$, $u = u_{k+2}$ is a possibly empty semigroup word in the alphabet $X \cup X'$ and $y_1, \ldots, y_m$ is a possibly repeating sequence of variables from $X \cup X'$ with $v_{k+1} = y_1 \cdots y_m$ (so $m > 1$). Note that $p$ can be empty only if $k = 0$.

Let us write $w$ for $y_2 \cdots y_{m-1}$ (if $m = 2$, then $w$ is empty). If both $p$ and $u$ are empty, then $s' \in N$ already. If neither $p$ nor $u$ are empty, then by Lemma 3.3 $\Psi$ implies $s' \approx (py_m')'w(y_1'u)'$. The breadth of $py_m'$ is $k$, so the induction hypothesis and Lemma 3.6 complete the proof.

Now say that $p$ is empty and $u$ is not. We have $(y_1wy_m)'u' \approx (y_1'u)'wy_m$, and the latter word is contained in $N$ (modulo $x'' \approx x$).
Lastly, if \( u \) is empty and \( p \) is not, then we have \( s' \equiv (pwy_m)' \overset{\text{FAAR}}{\approx} w(py_m)' \), and the induction hypothesis applies to \( (py_m)' \) since \( py_m \) is of breadth \( k \). By Lemma 3.6 \( s' \) is \( \psi \)-equivalent to a member of \( N \). \( \square \)

As explained above, Lemmas 3.6 and 3.7 give us the following result.

**Proposition 3.8.** Every unary semigroup word reduces modulo \( \Psi \) to a word in \( N \).

A algorithm for making such a reduction is to iterate the method of proof of Lemma 3.6 and 3.7, however we will not need this here.

### 3.3. Subvarieties of \( \mathcal{A} \) and sub-\( uH \) classes of \( \mathcal{G} \).

In this subsection we complete the proof of Theorem 2.1. Recall that the theorem claims that, for any nonempty class \( K \) of graphs, any graph \( G \) belongs to the \( uH \)-class generated by \( K \) if and only if the adjacency semigroup \( A(G) \) belongs to the variety generated by the adjacency semigroups \( A(H) \) with \( H \in K \). For the “only if” statement we use a direct argument. For the “if” statement, we use a syntactic argument, translating \( uH \) sentences of \( K \) into identities of \( A(K) \).

**Lemma 3.9.** If \( G \in \text{ISP}^+(\mathcal{P}_u(K)) \), then \( A(G) \in \text{HSP}(A(K)) \).

**Proof.** First consider a nonempty family \( L = \{ H_i \mid i \in I \} \) of graphs from \( K \) and an ultraproduct \( H := \prod_U L \) (for some ultrafilter \( U \) on \( 2^I \)). It is easy to see that the ultraproduct of the family \( \{ A(H_i) \mid i \in I \} \) over the same ultrafilter \( U \) is isomorphic to \( A(H) \) (we leave this elementary proof to the reader). Hence, we have \( I(A\mathcal{P}_u(K)) = I\mathcal{P}_u(A(K)) \). Now we have \( G \in \text{ISP}^+(\mathcal{P}_u(K)) \). So it will suffice to prove that \( A(G) \in \text{HSP}(A(\mathcal{P}_u(K))) \), since \( \text{HSP}(A(\mathcal{P}_u(K))) = \text{HSP}^u(A(K)) = \text{HSP}(A(K)) \). We let \( P \) denote \( \mathcal{P}_u(K) \).

Now \( G \) is isomorphic to an induced subgraph of the direct product \( \prod_{i \in I} H_i \) with \( H_i \in P \). It does no harm to assume that this embedding is the inclusion map. Let \( \pi_i : G \to H_i \) denote the projection. Evidently the following properties hold:

(i) if \( u \) and \( v \) are distinct vertices of \( G \) then there is \( i \in I \) such that \( \pi_i(u) \neq \pi_i(v) \);

(ii) if \( (u,v) \) is not an edge of \( G \) then there is \( i \in I \) with \( (\pi_i(u), \pi_i(v)) \) not an edge of \( H_i \).

We aim to show that \( A(G) \) is a quotient of a subalgebra of \( \prod_{i \in I} A(H_i) \). We define a map \( \alpha : A(G) \to \prod_{i \in I} A(H_i) \) by letting \( \alpha(0) \) be the constant 0 and \( \alpha(u,v) \) be the map \( i \mapsto (\pi_i(u), \pi_i(v)) \). The map \( \alpha \) is unlikely to be a homomorphism. Let \( B \) be the subalgebra of \( \prod_{i \in I} A(H_i) \) generated by the image of \( A(G) \), and let \( J \) be the ideal of \( B \) consisting of all elements with a 0 coordinate.

**Claim 1.** Say \((u_1, v_1)\) and \((u_2, v_2)\) are (nonzero) elements of \( A(G) \). If \( v_1 \sim u_2 \) then \( \alpha(u_1, v_1)\alpha(u_2, v_2) = \alpha((u_1, v_1)(u_2, v_2)) \).
Proof. Now $\alpha((u_1, v_1)(u_2, v_2))[i] = \alpha(u_1, v_2)[i] = (\pi_i(u_1), \pi_i(v_2))$, because $v_1 \sim u_2$ in $G$. Also, for every $i \in I$ we have $\pi_i(v_1) \sim \pi_i(u_2)$, so that $\alpha(u_1, v_1)[i] \alpha(u_2, v_2)[i] = (\pi_i(u_1), \pi_i(v_1))(\pi_i(u_2), \pi_i(v_2)) = (\pi_i(u_1), \pi_i(v_2))$ as required. \hfill \Box

Claim 2. Say $(u_1, v_1)$ and $(u_2, v_2)$ are nonzero elements of $A(G)$. If $v_1 \not\sim u_2$ then $\alpha(u_1, v_1) \alpha(u_2, v_2) \in I$.

Proof. By the definition of $v_1 \not\sim u_2$ there is $i \in I$ with $\pi_i : G \to H_i$ with $\pi_i(v_1) \not\sim \pi_i(u_2)$. Then $(u_1, v_1)[i](u_2, v_2)[i] = 0$. \hfill \Box

Claims 1 and 2 show that $\alpha$ is a semigroup homomorphism from $A(G)$ onto $B/I$ (at least, if we adjust the co-domain of $\alpha$ to be $B/I$ and identify the constant 0 with $I$). Now we show that this map is injective. Say $(u_1, v_1) \neq (u_2, v_2)$ in $A(G)$. Without loss of generality, we may assume that $u_1 \neq u_2$. So there is a coordinate $i$ with $\pi_i(u_1) \neq \pi_i(u_2)$. Then $\alpha(u_1, v_1)$ differs from $\alpha(u_2, v_2)$ on the $i$-coordinate. So we have a semigroup isomorphism from $A(G)$ to $B/I$. Lastly, we observe that $\alpha$ trivially preserves the unary operation, so we have an isomorphism of unary semigroups as well. This completes the proof of Lemma 3.9. \hfill \Box

To prove the other half of Theorem 2.1 we take a syntactic approach by translating uH sentences into unary semigroup identities. To apply our technique, we first need to reduce arbitrary uH sentences to logically equivalent ones of a special form.

Our goal is to show that if $G \not\in \ISP + P_u(K)$ then $A(G) \not\in \HSP(A(K))$. We first consider some degenerate cases.

If $K = \{\emptyset\}$, then $A(K)$ is the class consisting of the one element unary semigroup and $\HSP(A(K)) \models x \approx y$. The statement $G \not\in \ISP + P_u(K)$ simply means that $|G| \geq 1$ and so $A(G) \not\models x \approx y$. So $A(G) \not\in \HSP(K)$.

Now we say that $K$ contains a nonempty graph. We can then further assume that the empty graph is not in $K$. If $G$ is the 1-vertex looped graph $1$, then the statement $G \not\in \ISP + P_u(K)$ simply means that $K$ consists of antireflexive graphs. In this case, $A(K) \models xx' \approx 0$, while $A(G) \not\models xx' \approx 0$. So again, $A(G) \not\in \HSP(A(K))$.

So now it remains to consider the case where $G$ is not the 1-vertex looped graph and $K$ does not contain the empty graph. Lemma 1.1 shows that there is some uH sentence $\Gamma$ holding in each member of $K$, but failing on $G$. We now show that $\Gamma$ can be chosen to be a quasi-identity.

If $\Gamma$ is a uH sentence of the second kind, say $\bigvee_{1 \leq i \leq n} \neg \Phi_i$, then choose some atomic formula $\Xi$ in variables not appearing in any of the $\Phi_i$ and that fails on $G$ under some assignment: for example, if $|G| \geq 2$, then a formula of the form $x \approx y$ suffices, while if $|G| = 1$, then $G$ is the one element edgeless graph and $x \sim y$ suffices. Now replace $\Gamma$ with the quasi-identity $\bigwedge_{1 \leq i \leq n} \Phi_i \to \Xi$. We need to show that this new quasi-identity holds in $K$ and fails in $G$. It certainly holds in $K$, since it is logically equivalent to $\bigvee \neg \Phi_i \lor \Xi$, while $\bigvee \neg \Phi_i$ is constantly true. On the other hand, since
$\forall_{1 \leq i \leq n} \neg \Phi_i$ does not hold on $G$, there is an assignment $\theta$ making $\&_{1 \leq i \leq n} \Phi_i$ true, and this assignment can be extended to the variables of $\Xi$ in such a way that $\Xi$ is false under $\theta$. In other words, we have a failing assignment for the new quasi-identity on $G$.

Next we need to show that the quasi-identity $\Gamma$ can be chosen to have a particular form. Let us call a quasi-identity reduced if the equality symbol $\approx$ does not appear in the premise. One may associate any quasi-identity with a reduced quasi-identity in the obvious way: if $x \approx y$ appears in the premise of the original, then we may replace all occurrences of $y$ in the quasi-identity with $x$ (including in the conclusion) and remove $x \approx y$ from the premise. If a quasi-identity fails on a graph $H$ under some assignment $\theta$, then the corresponding reduced quasi-identity also fails under $\theta$. Conversely, if the reduced quasi-identity fails on $H$ under some assignment $\theta$, then we may extend $\theta$ to a failing assignment of the original quasi-identity. This means that we may choose $\Gamma$ to be a reduced quasi-identity.

Let $\Phi := \&_{1 \leq i \leq n} u_i \sim v_i$ be a conjunction of adjacencies, where the

$$u_1, \ldots, u_n, v_1, \ldots, v_n \in \{a_1, \ldots, a_m\}$$

are not necessarily distinct variables. For each adjacency $u_i \sim v_i$ in $\Phi$, let $w_i$ denote the word $(u_i v_i)' s_i(u_i v_i)' s_i(u_i v_i)' s_i(u_i v_i)'$, where $s_i$ is a new variable.

Now let $\sigma : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ be some finite sequence of numbers from $\{1, \ldots, n\}$ with the property that for each pair $i, j \in \{1, \ldots, n\}$ there is $k < m$ with $\sigma_k = i$ and $\sigma_{k+1} = j$ and such that $\sigma(1) = \sigma(m) = 1$. Define a word $D_\Phi$ (depending on $\sigma$) as follows:

$$\left( \prod_{1 \leq i < m} w_{\sigma(i)} t_{\sigma(i), \sigma(i+1)} \right) w_{\sigma(m)},$$

where the $t_{i,j}$ are new variables. As an example, consider the conjunction $\Phi := x \sim y \& y \sim z$, where $n = 2$, $u_1 = x$, $v_1 = u_2 = y$ and $v_2 = z$. Using the sequence $\sigma = 1, 2, 2, 1, 1$ we get $D_\Phi$ equal to the following expression:

$$(xy)' s_1(x'y)' s_1(x'y)' s_1(x'y)' t_{1,2}(yz)' s_2(y'z)' s_2(y'z)' t_{2,2}$$

$$(yz)' s_2(y'z)' s_2(y'z)' t_{2,1}$$

$$(xy)' s_1(x'y)' s_1(x'y)' s_1(x'y)' t_{1,1}(xy)' s_1(xy)' s_1(xy)' s_1(xy)' s_1(xy)'.$$

**Lemma 3.10.** Let $H$ be a graph, $\Phi$ be a conjunction of adjacencies in variables $a_1, \ldots, a_m$ and $\theta$ be an assignment of the variables of $D_\Phi$ into $A(H)$, with $\theta(a_i) = (\ell_i, r_i)$ say. Let $\gamma$ be any member of $\{L, R\}^m$. If $\theta(D_\Phi) \neq 0$ then the map $\phi_\gamma$ from $a_1, \ldots, a_m$ into the vertices $V_H$ of $H$ defined by

$$\phi_\gamma(a_i) = \begin{cases} 
\ell_i & \text{if } \gamma(i) = L; \\
ri & \text{if } \gamma(i) = R 
\end{cases}$$

satisfies $\Phi$. 

Proof. Let \(a_i \sim a_j\) be one of the adjacencies in \(\Phi\). So all of \(a_i a_j, a_i' a_j, a_i a_j'\) and \(a_i' a_j'\) appear in \(D_\Phi\) and hence are given nonzero values by \(\theta\). We have \(\ell_i \sim \ell_j, \ell_i \sim r_j, r_i \sim \ell_j, r_i \sim r_j\) in \(H\). So regardless of the choice of \(\gamma\) we have \(\phi_\gamma(a_i) \sim \phi_\gamma(a_j)\) in \(H\).

\[\Box\]

Lemma 3.11. Let \(\Phi = \&_{1 \leq i \leq n} u_i \sim v_i\) be a nonempty conjunction in the variables \(a_1, \ldots, a_m\) and let \(\theta\) be an assignment of these variables into a graph \(H\) such that \(H \models \theta(\Phi)\). Define an assignment \(\theta^+\) of the variables of \(D_\Phi\) into \(A(H)\) by \(a_i \mapsto (\theta(a_i), \theta(a_i))\), \(\theta^+(t_{i,j}) := (\theta(v_i), \theta(u_j))\) and \(\theta^+(s_i) = \theta^+(t_{i,i})\).

We have \(\theta^+(D_\Phi) = (\theta(v_1), \theta(u_1))\).

Proof. This is a routine calculation. For each adjacency \(u_i \sim v_i\) in \(\Phi\) (here \(\{u_i, v_i\} \subseteq \{a_1, \ldots, a_m\}\)) we have

\[
\theta^+(u_i v_i')', \theta^+(u_i' v_i'), \theta^+(u_i' v_i')', \theta^+(u_i' v_i') \quad \text{all taking the same nonzero value } (\theta(v_i), \theta(u_i)).
\]

Then we also have \(\theta^+(u_i) = (\theta(v_i), \theta(u_i))\) which shows that

\[
\theta^+(D_\Phi) = [\theta(u_1), \theta(v_1)] \cdots [\theta(v_i), \theta(u_i)] \theta^+(t_{i,j}) [\theta(v_j), \theta(u_j)] \cdots [\theta(u_1), \theta(v_1)]
\]

\[
= [\theta(u_1), \theta(v_1)] \cdots [\theta(v_i), \theta(u_i)][\theta(v_j), \theta(u_j)][\theta(v_j), \theta(u_j)] \cdots [\theta(u_1), \theta(v_1)]
\]

\[
= [\theta(u_1), \theta(v_1)]
\]

(where the square brackets are used for clarity only). \(\Box\)

Lemma 3.12. Let \(H\) be a nonempty graph and \(\Phi \rightarrow u \approx v\) be a reduced quasi-identity where \(\Phi\) is nonempty and one of \(u\) or \(v\) does not appear in \(\Phi\) (say it is \(u\)). We have \(A(H) \models uD_\Phi \approx u'D_\Phi\) if and only if \(H \models \Phi \rightarrow u \approx v\).

Proof. First assume that \(H \models \Phi \rightarrow u \approx v\), where \(u\) does not appear in \(\Phi\). Both sides of the identity contain the subword \(D_\Phi\) so we may consider an assignment \(\theta\) sending \(D_\Phi\) to a nonzero value (if there are none, then we are done). By Lemma 3.10, we have an interpretation of \(\Phi\) in \(H\). But then, we can choose any value for \(\theta(u)\) and find that it is the same value as \(\theta(v)\).

In other words, \(H\) has only one vertex. Also, since \(D_\Phi\) takes a nonzero value on \(A(H)\), we find that the semigroup reduct of \(A(H)\) is not a null semigroup (that is, a semigroup in which all products are equal to 0). Hence \(A(H)\) is isomorphic to the unary semigroup formed by the integers 0 and 1 with the usual multiplication and the identity map as the unary operation. In this case we have \(u \approx u'\) satisfied and the identity holds.

Now say that \(H \not\models \Phi \rightarrow u \approx v\), and let \(\theta\) be a failing assignment. As \(\theta(u) \neq \theta(v)\) we can find a vertex \(a\) of \(H\) such that \(a \neq \theta(u_1)\). Extend the assignment \(\theta^+\) of Lemma 3.11 by \(u \mapsto (a, \theta(u_1))\). Evidently, \(\theta^+(uD_\Phi) = (a, \theta(u_1))(\theta(v_1), \theta(u_1)) = (a, \theta(u_1))\), but \(\theta^+(u'D_\Phi)\) is either equal to 0, or is non zero but has left coordinate different to \(\theta^+(uD_\Phi)\). \(\Box\)

Lemma 3.13. Let \(H\) be a nonempty graph and \(\&_{1 \leq i \leq n} u_i \sim v_i \rightarrow u \approx v\) be a reduced quasi-identity where \(\Phi\) is nonempty and both \(u\) and \(v\) appear in \(\Phi\).
If \( u = u_i \) for some \( i \) then we have \( \Lambda(H) \models u_i t_{i,1} D_{\Phi} \approx vt_{i,1} D_{\Phi} \) if and only if \( H \models \Phi \rightarrow u \approx v \). If \( u = v_i \) for some \( i \) then we have \( \Lambda(H) \models D_{\Phi} t_{1,i} u_i \approx D_{\Phi} t_{1,i} v \) if and only if \( H \models \Phi \rightarrow u \approx v \).

Proof. First assume that \( H \models \Phi \rightarrow u \approx v \). We consider only the case that \( u = u_i \); the other case follows by symmetry. As before, we can consider the case where there is an assignment \( \theta \) into \( H \) satisfying \( \Phi \). So we have \( \theta(u_i) = \theta(v) \) and hence \( \theta^+(u_i) = \theta^+(v) \), in which case both sides of the identity take the same value.

Now say that \( H \not\models \Phi \rightarrow u \approx v \) and let \( \theta \) be a failing assignment. Now, the left side of the identity contains the same adjacencies as \( D_{\Phi} \) and so takes a nonzero value in \( \Lambda(H) \) under the assignment \( \theta^+ \); moreover the left coordinate is \( \theta(u_i) \). However the right hand side either takes the value 0 (if \( \theta(v) \not\approx \theta(v_i) \)) or has left coordinate equal to \( \theta(v) \not= \theta(u_i) \). In either case, the identity fails.

Now we come to reduced quasi-identities in which the conclusion is an adjacency \( u \sim v \). We consider 9 cases according to whether or not \( u \) and \( v \) appear in \( \Phi \), and if so, whether they appear as the “source” or “target” of an adjacency. The nine identities \( \tau_1, \ldots, \tau_9 \) are defined in the following table. In this table, the first row corresponds to the situation where neither \( u \) nor \( v \) appear while the second corresponds to the situation where \( u \) does not appear, but \( v \) does appear as some \( u_j \) (in other words, as a “source”), and so on. If one of \( u \) or \( v \) appears as both a source and a target, then there will be choices as to which identity we can choose. The variables \( z \) and \( w \) are new variables not appearing in \( D_{\Phi} \).

| \( k \) | \( u \sim v \) | \( \tau_k \) |
|---|---|---|
| 1. | \( z \sim w \) | \( w D_{\Phi} z \approx (w D_{\Phi} z)^2 \) |
| 2. | \( z \sim u_j \) | \( u_j t_{j,1} D_{\Phi} z \approx (u_j t_{j,1} D_{\Phi} z)^2 \) |
| 3. | \( z \sim v_j \) | \( (u_j v_j) t_{j,1} D_{\Phi} z \approx ((u_j v_j) t_{j,1} D_{\Phi} z)^2 \) |
| 4. | \( u_i \sim w \) | \( w D_{\Phi} t_{1,i} (u_i v_i) \approx (w D_{\Phi} t_{1,i} (u_i v_i))^2 \) |
| 5. | \( v_i \sim w \) | \( w D_{\Phi} t_{1,i} v_i \approx (w D_{\Phi} t_{1,i} v_i)^2 \) |
| 6. | \( u_i \sim u_j \) | \( u_j t_{j,1} D_{\Phi} t_{1,i} (u_i v_i) \approx (u_j t_{j,1} D_{\Phi} t_{1,i} (u_i v_i))^2 \) |
| 7. | \( u_i \sim v_j \) | \( (u_j v_j) t_{j,1} D_{\Phi} t_{1,i} (u_i v_i) \approx ((u_j v_j) t_{j,1} D_{\Phi} t_{1,i} (u_i v_i))^2 \) |
| 8. | \( v_i \sim u_j \) | \( u_j t_{j,1} D_{\Phi} t_{1,i} v_i \approx (u_j t_{j,1} D_{\Phi} t_{1,i} v_i)^2 \) |
| 9. | \( v_i \sim v_j \) | \( (u_j v_j) t_{j,1} D_{\Phi} t_{1,i} v_i \approx ((u_j v_j) t_{j,1} D_{\Phi} t_{1,i} v_i)^2 \) |

**Lemma 3.14.** Let \( H \) be a graph and \( \Phi \rightarrow u \sim v \) be a quasi-identity where \( \Phi \) is nonempty. Consider the corresponding identity \( \tau_k \). We have \( \Lambda(H) \models \tau_k \) if and only if \( H \models \Phi \rightarrow u \approx v \).

**Proof.** We prove the case of \( \tau_1 \) and leave the remaining (very similar) cases to the reader. First assume that \( H \models \Phi \rightarrow u_i \sim v \). Consider some assignment \( \theta \) into \( \Lambda(H) \) that gives \( D_{\Phi} \) a nonzero value. As \( w \) appears on both sides, we may further assume that \( \theta(w) \) is nonzero. Observe that the graph of the right hand side of the identity is identical to that of the left side except...
for the addition of a single edge from \( e_{ui} \) to \( \ell_w \). Also, the initial and final vertices are the same. So to show that the two sides are equal, it suffices to show that \( \theta(u_i') \theta(w) \) is non zero.

Choose any map \( \gamma \) from the variables of \( \tau_4 \) to \( \{L, R\} \) with \( \gamma(u_i) = R \). By Lemma 3.10 we have \( H \models \phi_{\gamma}(\Phi) \). Using \( \Phi \rightarrow u_i \sim v \) it follows that for any vertex \( w \) we have \( \phi_{\gamma}(u_i) \sim w \). In other words, \( \theta(u_i') \theta(w) \) is nonzero as required.

Now say that \( \Phi \rightarrow u_i \sim v \) fails on \( H \) under some assignment \( \theta \). Extend \( \theta + \) to \( \theta(w) \rightarrow (\theta(v), \theta(u_i)) \). Under this assignment the left hand side of \( \tau_4 \) takes the value \( (\theta(v), \theta(u_i)) \), while the right hand side equals 0. \( \square \)

Lastly we need to consider the case where \( \Gamma \) has empty premise, that is, where \( \Gamma \) is a universally quantified atomic formula \( \tau \). In the language of graphs, there are essentially four different possibilities for \( \tau \) (up to a permutation of letter names): \( x \sim y \), \( x \sim x \), \( x \approx y \) and \( x \approx x \). The last of these is a tautology. The first three are nontautological and correspond to the uH-classes of complete looped graphs, reflexive graphs, and the one element graphs. For \( \Phi \) one of the three atomic formulas, we let \( \tau_{\Phi} \) denote the identities \( xx \approx x \), \( xx'x \approx x \), and \( x' \approx x \), respectively.

**Lemma 3.15.** Let \( H \) be a graph and \( \Phi \) be one of the three nontautological atomic formulas in the language of graphs. We have \( H \models \Phi \) if and only if \( \Lambda(H) \models \tau_{\Phi} \).

**Proof.** If \( \Phi \) is \( x \sim y \), then it is easy to see that \( H \models \Phi \) if and only if the underlying semigroup of \( \Lambda(H) \) satisfies \( xx \approx x \). The case of \( \Phi = x \sim x \) has been discussed already in Section 2. The case of \( x \approx y \) corresponds to the 1-vertex graphs, which is clearly equivalent to the property that \( \Lambda(H) \models x' \approx x \). \( \square \)

Now we can complete the proof of Theorem 2.1. We have a reduced quasi-identity \( \Gamma \) satisfied by \( K \) and failing on \( G \). By the appropriate choice out of Lemmas 3.12, 3.13, 3.14 or 3.15 we can construct an identity \( \tau \) such that \( \Lambda(K) \models \tau \) and \( \Lambda(G) \models \tau \). Hence \( \Lambda(G) \not\models \HSP(\Lambda(K)) \). \( \square \)

**4. Proof of Theorem 2.2**

In contrast to the proof of Theorem 2.1, this section requires some basic notions and facts from semigroup theory such as Green’s relations \( J, L, R, H \) and their manifestation on Rees matrix semigroups. For details, refer to the early chapters of any general semigroup theory text; Howie [12] for example.

The first step to proving Theorem 2.2 is the following.

**Lemma 4.1.** Let \( \mathcal{V} \) be a variety of unary semigroups satisfying

\[ (x'x) \approx x, \quad x'' \approx x, \quad (x'x)' \approx x'x, \quad (xy)' \approx y'(x'xy')'x'. \]

If \( A \in \mathcal{V} \) as a semigroup is a completely 0-simple semigroup with trivial subgroups, then \( A \) is of the form \( \Lambda(H) \) for some reflexive graph \( H \).
Proof. Since \( A \) is a completely 0-simple semigroup with trivial subgroups, the Green relation \( \mathcal{H} \) is trivial. Now every \( a \neq 0 \) in \( A \) is \( \mathcal{L} \)-related to \( a'a \) (since \( a(a'a) = a \)) and \( \mathcal{R} \)-related to \( aa' \). Also, these elements are fixed by \( ' \) by identity \( (x'x)' \approx x'x \) (and \( x'' \approx x \)). Next we observe that \( a \mathcal{L} b \) if and only if \( a' \mathcal{R} b' \). For this we can use identity \( (xy)' \approx y'(x'xyy')'x' \): if \( a \mathcal{L} b \) then \( xa = b \) for some \( x \), so \( b' = (xa)' = a'z \), for \( z = (x'xxa')x' \). So \( b' \mathcal{R} a' \). The other case follows by symmetry (or using \( (x')' \approx x \)).

This implies that each \( \mathcal{L} \)-class and each \( \mathcal{R} \)-class contain precisely one fixed point of \( ' \) (if \( a' = a \), \( b' = b \) and \( a \mathcal{L} b \), then \( a = a' \mathcal{R} b' = b \), so \( a \mathcal{H} b \)). Represent \( A \) as a Rees matrix semigroup (with matrix \( P \)) in which fixed points of \( ' \) correspond to diagonal elements (as \( xx' \) is an idempotent, \( P \) will have 1 down the diagonal). It is easily seen this is \( A(H) \) for the graph \( H \) with \( P \) as adjacency matrix. This graph is reflexive since the identity \( xx' \approx x \) holds. \( \square \)

In the case where \( H \) is a universal relation, the set \( A(H) \setminus \{0\} \) is a subuniverse, and the corresponding subalgebra of \( A(H) \) is a square band.

Lemma 4.2. Let \( \mathcal{V} \) be a variety of unary semigroups satisfying the identities \( \mathbf{5} \). If \( A \in \mathcal{V} \) as a semigroup is a completely simple semigroup with trivial subgroups, then \( A \) is a square band.

Proof. The proof is basically the same as for Lemma 4.1. \( \square \)

In order to get a small basis for the identities of \( A_{\text{ref}} \) the following lemma is useful.

Lemma 4.3. The following laws are consequences of

\[
\Psi_1 := \{ x \approx xx'x, (x'x)' \approx x'x, x'' \approx x, x(yz)' \approx (y(xz')')', (xy)'z \approx ((x'z')'y')':
\]

- \( (xy)' \approx y'(x'xyy')' \approx (x'xy)'x' \approx y'(x'xyy')'x' \)
- \( (xyz)' \approx (yz)'y(xy)' \)

Proof. For the first item we have \( \Psi_1 \) implies \( (xy)' \approx ((xx')'xy)' \approx (x'xy)'x' \). The other two cases of this item are very similar.

For the second item, first note that using item 1 and \( \Psi_1 \), we have \( (xyy')' \approx y(xy'y)' \approx y(xy)' \). Using this we obtain

\[
(xy)'z \approx (xy(y'y)'z)' \approx ((y'(xxy')')'z)' \approx (yz)'(xyy')' \approx (yz)'y(xy)' \]

The second item of Lemma 4.3 enables a refinement of Proposition 3.8.

Corollary 4.4. The identities \( \Psi_1 \) reduce every unary semigroup word to a member of \( N \) in which each subword of the form \( (v) \) has the property that \( v \) is a semigroup word of length 1 or 2 (over the alphabet \( X \cup X' \)).
We now let $\Sigma_{\text{ref}}$ denote the following set of unary semigroup identities:

\[(\Psi)\quad x'' \approx x, \quad x(yz)' \approx (y(xz)')', \quad (xy)'z \approx ((x'z)'y)'
\]

(6) \quad $xx'x \approx x$

(7) \quad $(x'x)' \approx xx'$

(8) \quad $x^3 \approx x^2$

(9) \quad $xyxx \approx zzyyxx \approx zzyx$

(10) \quad $x'yxx \approx (zx)'yxx$

(11) \quad $xyxz \approx (yx)xx$

Proposition 3.4 easily shows that all but identity (6) hold in $A$, while (6) obviously holds in the subvariety $A_{\text{ref}}$. Hence, to prove that $\Sigma_{\text{ref}}$ is a basis for $A_{\text{ref}}$, we need to show that every model of $\Sigma_{\text{ref}}$ lies in $A_{\text{ref}}$. Before we can do this, we need some further consequences of $\Sigma_{\text{ref}}$.

In the identities that occur in the next lemma we use $u$, where $u$ is either $x$ or $xyx$, to denote either $u$ or $u'$. We assume that the meaning of the operation $-$ is fixed within each identity: either it changes nothing or it adds $'$ to all its arguments.

Lemma 4.5. The following identities all follow from $\Sigma_{\text{ref}}$:

- $(\overline{u_1})'u_2xy \approx (\overline{u_1}u_2)xy$;
- $xyxu_2(u_1\overline{u})' \approx xyxu_2(u_1uxy)'$;
- $(u_1\overline{u})'u_2xy \approx (u_1uyx)u_2xy$;
- $xyxu_2(u_1\overline{u})' \approx xyxu_2(u_1uyx)'$.

where $u_1$ and $u_2$ are possibly empty unary semigroup words.

Proof. In each of the eight cases, if $u_1$ is empty, then the identity is equivalent modulo $x'' \approx x$ to one in $\Sigma_{\text{ref}}$ up to a change of letter names. So we assume that $u_1$ is non-empty. We can ensure that $u_2$ is non-empty by rewriting $u_2xy$ and $xyu_2$ as $(u_2xx')xy$ and $xy(x'xu_2)$ respectively (a process we reverse at the end of each deduction). For the first identity we have $\Psi$ implies

\[\overline{(u_1)'}u_2xy \approx (u_1u_2)xy,\]

and then we use (9) or (10) to replace $\overline{u}$ by $u_1u$. Reversing the application of SAAR, we obtain the corresponding right hand side.

The second identity is just a dual to the first so follows by symmetry. Similarly, the fourth will follow from the third by symmetry.

For the third identity, Lemma 4.3 can be applied to the left hand side to get $\overline{(u_1)'}u_2xy \approx (u_1u_2)xy$. Now, the subword $\overline{u_1}$ is either $x'x$ or $xx'$. We will write it as $t(x, x')$ (where $t(x, y)$ is one of the words $xy$ or $yx$). Using (9), we have $t(x, x')(u_1\overline{u})'u_2xy \approx t(xy, x')(u_1\overline{u})'u_2xy$. But the subword $t(xy, x')(u_1\overline{u})'$ is of the form required to apply the second identity in the lemma we are proving. Since this second identity has been established, we can use it to deduce $t(xy, x')(u_1u_2)xy$ and then reverse the procedure.
Recall that a unary polynomial $p(x)$ on an algebra $S$ is a function $S \to S$ defined for each $a \in S$ by $p(a) = t(a, a_1, \ldots, a_n)$ where $t(x, x_1, \ldots, x_n)$ is a term, and $a_1, \ldots, a_n$ are elements of $S$. We let $P_x$ denote the set of all unary polynomials on $S$. The syntactic congruence $\text{Syn}(\theta)$ of an equivalence $\theta$ on $S$ is defined to be

$$\text{Syn}(\theta) := \{(a, b) \mid p(a) \theta p(b) \text{ for all } p(x) \in P_x\}.$$ 

$\text{Syn}(\theta)$ is known to be the largest congruence of $S$ contained in $\theta$ (see [1] or [6]). It is very well known that for standard semigroups, one only needs to consider polynomials $p(x)$ built from the semigroup words $x, x_1x, xx_1, x_1xx_2$ (see [12] for example). In fact there is a similar—though more complicated—reduction for the variety defined by $\Sigma_{\text{ref}}$ (and more generally still $\Psi$). This can be gleaned fairly easily from Proposition 3.8 (see [6] for a general approach for establishing this), however we do not need an explicit formulation of it here, and so omit any proof.

We may now prove the key lemma, a variation of [11, Lemma 3.2].

**Lemma 4.6.** Every model of $\Sigma_{\text{ref}}$ (within the variety of unary semigroups) is a subdirect product of members of $A(G_\text{ref}) \cup \mathcal{B}$.

**Proof.** Let $S \models \Sigma_{\text{ref}}$. If $S$ is the one element semigroup we are done. Now assume that $|S| > 1$. We need to show that for every pair of distinct elements $a, b \in S$ there is a homomorphism from $S$ onto a square band or an adjacency semigroup $A(G)$ for some $G \in \mathcal{G}_{\text{ref}}$.

For each element $z \in S$, we let $I_z := \{u \in S \mid z \not\in S^1uS^1\}$, in other words, $I_z$ is the ideal consisting of all elements that do not divide $z$. Note that $I_z$ is closed under the reversion operation (since $u'$ divides $u$). Define equivalence relations $\rho_z$ and $\lambda_z$ on $S$:

$$\rho_z := \{(x, y) \in S \times S \mid (\forall t \in SzS) \quad xt \equiv yt \mod I_z\};$$

$$\lambda_z := \{(x, y) \in S \times S \mid (\forall t \in SzS) \quad tx \equiv ty \mod I_z\}.$$ 

So far the proof is identical to that of [11 Lemma 3.2]. In the semigroup setting, both $\rho_z$ and $\lambda_z$ are congruences, however this is no longer true in the unary semigroup setting. Instead, we replace $\rho_z$ and $\lambda_z$ by their syntactic congruences $\text{Syn}(\rho_z)$ and $\text{Syn}(\lambda_z)$.

Let $a$ and $b$ be distinct elements of $S$. Our goal is to show that one of the congruences $\text{Syn}(\rho_a)$, $\text{Syn}(\rho_b)$, $\text{Syn}(\lambda_a)$ and $\text{Syn}(\lambda_b)$ separate $a$ and $b$, and that $S/\text{Syn}(\rho_z)$ and $S/\text{Syn}(\lambda_z)$ are isomorphic to a square band or an adjacency semigroup of a reflexive graph. The first part is essentially identical to a corresponding part of the proof of [11 Lemma 3.2]. We include it for completeness only.
First suppose that \( a \notin SbS \). So \( b \in I_a \). Choose \( t = a't \in SaS \) so that \( a = at \equiv bt \) mod \( I_a \). Hence \((a,b) \notin \hat{\rho}_a\). Now suppose that \( SaS = SbS \), so that \( a \) and \( b \) lie in the same \( \mathcal{S} \)-class \( SaS \setminus I_a \) of \( S \). One of the following two equalities must fail: \( ab'b = b \) or \( aab' = a \) for otherwise \( a = a'ab'b = aab'b = ab'b = b \). Hence as neither \( a \) nor \( b \) is in \( I_a = I_b \), we have either \((a,b) \notin \hat{\rho} \) or \((a,b) \notin \lambda \).

Now it remains to prove that \( S/S\text{Syn}(\rho_z) \) and \( S/S\text{Syn}(\lambda_z) \) are adjacency semigroups or square bands. Lemmas \([1.1] \) \([4.2] \) and \([4.3] \) show that it suffices to prove that the underlying semigroup of \( S/S\text{Syn}(\rho_z) \) is completely 0-simple or completely simple. We look at the \( \text{Syn}(\rho_z) \) case only (the \( \text{Syn}(\lambda_z) \) case follows by symmetry). Now it does no harm to assume that \( I_z \) is empty or \( \{0\} \), since \( v, w \in I \) obviously implies that \((v,w) \in \text{Syn}(\rho_z) \). Hence \( K_z := SzS/(I_z \cap S\overline{z}S) \) is a 0-simple semigroup or a simple semigroup. Since \( S \) is periodic (by identity \([5] \) of \( \Sigma_{\text{ref}} \)), we have that \( K_z \) is completely 0-simple or completely simple. We need to prove that every element of \( S/I_z \) is \( \text{Syn}(\rho_z) \)-related to a member of \( S\overline{z}S/I_z \).

Let \( c \in S \). If \( c \in SzS \) or \( c \in I_z \) are done, so let us assume that \( c \notin S\overline{z}S \cup I_z \). So \( z = pqz \) for some \( p,q \in S^1 \). So \( z = pqz'p \). Put \( w = qz'p \). Note that \( w \in SzS \) and \( cwc \neq 0 \). Our goal is to show that \( c\text{Syn}(\rho_z)cwc \). Let \( s(x,y) \) be any unary semigroup word in some variables \( x, y, \ldots \) and let \( t \in SaS \). We need to prove that for any \( \vec{d} \) in \( S^1 \) we have \( s(c,\vec{d})t \equiv s(cwc,\vec{d})t \) modulo \( I_z \). Write \( t \) as \( ucwcv \), which is possible since both \( t \) and \( cwc \) are \( \mathcal{S} \)-related. (Note that modulo the identity \( xx'x \approx x \) we may assume both \( u \) and \( v \) are nonempty.) We want to obtain

\begin{equation}
(12) \quad s(c,\vec{d})ucwcv = s(cwc,\vec{d})ucwcv.
\end{equation}

Now using Corollary \([4.4] \) we may rewrite \( s(c,\vec{d}) \) as a word in which each application of \( ' \) covers either a single variable or a word of the form \( gh \) where \( g, h \) are either letters or \( ' \) applied to a letter. There may be many occurrences of \( c \) in this word. We show how to replace an arbitrary one of these by \( cwc \) and by repeating this to each of these occurrences we will achieve the desired equality \([12] \). Let us fix some occurrence of \( c \). So we may consider the expression \( s(c,\vec{d})ucwcv \) as being of one of the following forms: \( w_1cw_2cwc; w_1c'w_2cwc; w_1(c'z)w_2cwc; w_1(zc')w_2cwc; w_1(zc')w_2cwc \). In each case, we can make the required replacement using a single application of Lemma \([4.5] \). This gives equality \([12] \), which completes the proof.

As an immediate corollary we obtain the following result.

**Corollary 4.7.** The identities \( \Sigma_{\text{ref}} \) are an identity basis for \( A_{\text{ref}} \).

Let \( SL \) denote is the variety generated by the adjacency semigroup over the 1-vertex looped graph and let \( U \) denote the variety generated by adjacency semigroups over single block equivalence relations (equivalently, \( U \) is
the variety generated by the adjacency semigroup over the universal relation on a 2-element set). Recall that $SB$ denotes the variety of square bands.

**Lemma 4.8.** $SL \lor SB = U$.

*Proof.* The direct product of the semigroup $A(1)$ with an $I \times I$ square band has a unique maximal ideal and the corresponding Rees quotient is (isomorphic to) the adjacency semigroup over the universal relation on $I$. So $SL \lor SB \supseteq U$. However if $|I| \geq 2$, and $U_I$ denotes the universal relation on $I$, then the adjacency semigroup $A(U_I) \in U$ contains as subalgebras both $A(1)$ (a generator for $SL$) and the $I \times I$ square band (a generator for $SB$). So $SL \lor SB \subseteq U$. \hfill \Box

**Lemma 4.9.** Let $V$ be a subvariety of $A_{ref}$ containing the variety $SB$. Either $V = SB$ or $V \supseteq U$ and $V = HSP(A(K))$ for some class of (necessarily reflexive) graphs $K$.

*Proof.* Let $A$ be a nonfinitely generated $V$-free algebra. If $A \models xyz \approx x$ then $V$ is equal to $SB$. Now say that $xyz \approx x$ fails on $A$. Lemma 4.6 shows that $A$ is a subdirect product of some family $J$ of adjacency semigroups and square bands. Note that we have $V = HSP(J)$. Our goal is to replace all square bands in $J$ by adjacency semigroups over universal relations.

Since $xyz \approx x$ fails on $A$, at least one of the subdirect factors of $A$ is an adjacency semigroup that is not the one element algebra. Hence $V$ contains the semigroup $A(1)$. By Lemma 4.8 $V$ contains $U$. Now replace all square bands in $J$ by the adjacency semigroup of a universal relation of some set of size at least 2, and denote the corresponding class by $\bar{J}$; let $G_j$ denote the corresponding class of graphs. Then $V = HSP(J) = HSP(\bar{J}) = HSP(A(G_j))$. \hfill \Box

Now we may complete the proof of Theorem 2.2

*Proof.* Theorem 2.1 shows the map $\iota$ described in Theorem 2.1 is an order preserving injection from $L(G_{ref})$ to $L(A_{ref})$. Now we show that it is a surjection. That is, every subvariety of $A_{ref}$ other than $SB$ is the image under $\iota$ of some $uH$ class of reflexive graphs. Lemma 4.9 shows this is true if $SB \subseteq V$. However, if the square bands in $V$ are all trivial, then Lemma 4.6 shows that either $V$ is the trivial variety (and equal to $\iota(\{0\})$) or $\omega$-generated $V$-free algebra is a subdirect product of members of $A(G_{ref})$. Let $F$ be a set consisting of the subdirect factors and $G_F$ the corresponding graphs. Then $V = HSP(F) = \iota(HSP^+(G_F))$. To show that $\iota$ is a lattice isomorphism, it will suffice to show that $\iota$ preserves joins, since meets follow from the fact that $\iota$ is an order preserving bijection.
Let \( \bigvee_{i \in I} R_i \) be some join in \( L^+ \). First assume that \( S \) is not amongst the \( R_i \). Then

\[
\text{HSP}(A(\bigvee_{i \in I} R_i)) = \text{HSP}(A(\text{ISP}^+ \cup (\bigcup_{i \in I} R_i))) = \text{HSP}(\bigcup_{i \in I} A(R_i)) = \bigvee_{i \in I} \text{HSP}(A(R_i)).
\]

If \( S \) is amongst the \( R_i \) then either the join is a join of \( S \) with the trivial uH class \( \{0\} \) (and the join is obviously preserved by \( \iota \)), or using Lemma 4.8 we can replace \( S \) by the uH class of universal relations, and proceed as above. This completes the characterization of \( L(A_{\text{ref}}) \).

Next we must show that a class \( K \) of graphs generates a finitely axiomatizable uH class if and only if \( \text{HSP}(A(K)) \) is a finitely axiomatizable. The “only if” case is Corollary 5.3. Now say that \( K \) has a finite basis for its uH sentences. Following the methods of Subsection 3.3, we may construct a finite set \( \Xi \) of identities such that an adjacency semigroup \( A \) lies in \( \text{HSP}(A(K)) \) if and only if \( A \models \Xi \). We claim that \( \Sigma_{\text{ref}} \cup \Xi \) is an identity basis for \( \text{HSP}(A(K)) \). Indeed, if \( S \) is a unary semigroup satisfying \( \Sigma_{\text{ref}} \cup \Xi \), then by Lemma 4.6, \( S \) is a subdirect product of adjacency semigroups (or possibly square bands) satisfying \( \Xi \). So these adjacency semigroups lie in \( \text{HSP}(A(K)) \), whence so does \( S \).

The proof that \( \iota \) preserves the property of being finitely generated (and being nonfinitely generated) is very similar and left to the reader. \( \Box \)

5. Applications

The universal Horn theory of graphs is reasonably well developed, and the link to unary Rees matrix semigroups that we have just established provides numerous corollaries. We restrict ourselves to just a few ones which all are based on the examples of uH classes presented in Section 1.

We start with presenting finite generators for unary semigroup varieties that we have considered.

**Proposition 5.1.** The varieties \( A, A_{\text{symm}}, \) and \( A_{\text{ref}} \) are generated by \( A(G_1) \), \( A(S_1) \) and \( A(R_1) \) respectively.

**Proof.** This follows from Theorem 2.1 and Examples 1.7, 1.8 and 1.10. \( \Box \)

Observe that the generators are of fairly modest size, with 17, 17 and 10 elements respectively.

Recall that \( C_3 \) is a 5-vertex graph generating the uH class of all 3-colorable graphs (Example 1.6 see also Fig. 2).

**Proposition 5.2.** The finite membership problem for the variety generated by the 26-element unary semigroup \( A(C_3) \) is NP-hard.
Proof. By Theorem 2.1 \( A(G) \in \mathbb{HSP}_{\text{fin}}(A(C_3)) \) if and only if \( G \) is 3-colorable, a known \( \text{NP}\)-complete problem, see [8]. Of course, the construction of \( A(G) \) can be made in polynomial time, so this is a polynomial reduction. \( \square \)

A similar (but more complicated) example in the plain semigroup setting has been found in [14]. Observe that we do not claim that the finite membership problem for \( \mathbb{HSP}(A(C_3)) \) is \( \text{NP}\)-complete since it is not clear whether or not the problem is in \( \text{NP}\).

One can also show that the equational theory of \( A(C_3) \) is \( \text{co-NP}\)-complete. (It means that the problem whose instance is a unary semigroup identity \( u \approx v \) and whose question is whether or not \( u \approx v \) holds in \( A(C_3) \) is \( \text{co-NP}\)-complete.) This follows from the construction of identities modelling \( uH \) sentences in Subsection 3.3. The argument is an exact parallel to that associated with [14, Corollary 3.8] and we omit the details.

Proposition 5.3. If \( K \) is a class of graphs without a finite basis of \( uH \) sentences, then \( A(K) \) is without a finite basis of identities. If \( K \) is a class of graphs whose \( uH \) class has (infinitely many) uncountably many sub-\( uH \) classes, then the variety generated by \( A(K) \) has (infinitely many) uncountably many subvarieties.

Proof. This is an immediate consequence of Theorem 2.1. \( \square \)

In particular, recall the 2-vertex graph \( S_2 \) of Example 1.9, and let \( K_2 \) denote the 2-vertex complete simple graph.

Corollary 5.4. There are uncountably many varieties between the variety generated by \( A(S_2) \) and that generated by \( A(K_2) \). The statement is also true if \( S_2 \) is replaced by any simple graph that is not 2-colorable.

Proof. The first statement follows from Theorem 2.1 and statements in Example 1.9. The second statement follows similarly from statements in Example 1.4. \( \square \)

Note that the underlying semigroup of \( A(S_2) \) is simply the familiar semigroup \( A_2 \), see Subsection 3.1. The subvariety lattice of the semigroup variety generated by \( A_2 \) is reasonably well understood (see Lee and Volkov [19]). This variety contains all semigroup varieties generated by completely 0-simple semigroups with trivial subgroups but has only countably many subvarieties, all of which are finitely axiomatized (see Lee [18]).

Theorem 2.2 reduces the study of the subvarieties of \( A_{\text{ref}} \) to the study of \( uH \) classes of reflexive graphs. This class of graphs does not seem to have been as heavily investigated as the antireflexive graphs, but contains some interesting examples.

Recently Trotta [24] has disproved a claim made in [22] by showing that there are uncountably many \( uH \) classes of reflexive antisymmetric graphs. From this and Theorem 2.2 we immediately deduce:

Proposition 5.5. The unary semigroup variety \( A_{\text{ref}} \) has uncountably many subvarieties.
In contrast, it is easy to check that there are only 6 uH classes of reflexive symmetric graphs, see [3] for example. The lattice they form is shown in Fig. 8 on the left. Theorem 2.2 then implies that the subvariety lattice of

![Figure 8. The lattice of uH classes of reflexive symmetric graphs vs the lattice of varieties of strict regular semigroups](image)

the corresponding variety of unary semigroups contains 7 elements (it is one of the cases when the "extra" variety $SB$ of square bands comes into the play); the lattice is shown in Fig. 8 on the right. The variety is generated by the adjacency semigroup of the graph $RS_1$ of Example 1.10 and is nothing but the variety $CSR$ of so-called combinatorial strict regular *-semigroups which have been one of the central objects of study in [2]. The other join-indecomposable varieties in Fig. 8 are the trivial variety $T$, the variety $SL$ of semilattices with identity map as the unary operation, and the variety $BR$ of combinatorial strict inverse semigroups.

The main results of [2] consisted in providing a finite identity basis for $CSR$ and determining its subvariety lattice. We see that the latter result is an immediate consequence of Theorem 2.2. A finite identity basis for $CSR$ can be obtained by adding the involution identity (2) to the identity basis $\Sigma_{\text{ref}}$ of the variety $A_{\text{ref}}$, see Corollary 4.7. (The basis constructed this way is different from that given in [2].)

**Example 5.6.** The adjacency semigroup $A(2)$ of the two element chain 2 (as a partial order) generates a variety with a lattice of subvarieties isomorphic to the four element chain. The variety is a cover of the variety $BR$ of combinatorial strict inverse semigroups.

**Proof.** This follows from Example 1.3, Theorem 2.2 and the fact that the uH class of universal relations is not a sub-uH class of the partial orders (so $SB$ is not a subvariety of $HSP(A(2))$).

The underlying semigroup of $A(2)$ is again the semigroup $A_2$. Thus, Example 5.6 makes an interesting contrast to Corollary 5.4.

For our final application, consider the 3-vertex graph $P$ shown in Fig. 9.

It is known (see [3]) and easy to verify that the uH-class $ISP+P_0(P)$ is not finitely axiomatizable and the class of partial orders is the unique maximal
sub-uH class of $\mathbb{ISP}^+u(P)$. Recall that a variety $V$ is said to be a limit variety if $V$ has no finite identity basis while each of its proper subvarieties is finitely based. The existence of limit varieties is an easy consequence of Zorn’s lemma but concrete examples of such varieties are quite rare. We can use the just registered properties of the graph $P$ in order to produce a new example of a finitely generated limit variety of I-semigroups.

**Proposition 5.7.** The variety $\mathbb{HSP}(A(P))$ is a limit variety whose subvariety lattice is a 5-element chain.

**Proof.** This follows from Theorem 2.2 and Example 1.3. \qed

**Conclusion**

We have found a transparent translation of facets of universal Horn logic into the apparently much more restricted world of equational logic. A general translation of this sort has been established for uH classes of arbitrary structures (even partial structures) by the first author [13]. We think however that the special case considered in this paper is of interest because it deals with very natural objects on both universal Horn logic and equational logic sides.

We have shown that the unary semigroup variety $A_{\text{ref}}$ whose equational logic captures the universal Horn logic of the reflexive graphs is finitely axiomatizable. The question of whether or not the same is true for the variety $A$ corresponding to all graphs still remains open. A natural candidate for a finite identity basis of $A$ is the system consisting of the identities ($\Psi$) and (7)–(11), see Section 4.

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