Heavy flavour contributions to the
depth inelastic scattering sum rules

J. Blümlein and W.L. van Neerven

DESY-Zeuthen, Platanenallee 6, D-15738 Zeuthen, Germany

Abstract

We have calculated the first and second order corrections to several deep inelastic sum rules which are due to heavy flavour contributions. A comparison is made with the existing perturbation series which has been computed up to third order for massless quarks only. In general it turns out that the effects of heavy quarks are very small except when $Q \sim m$ or $Q \gg m$. Here $Q$ and $m$ denote the virtual mass of the vector boson and the mass of the heavy quark, respectively. For $Q \gg m$ the radiative corrections reveal large logarithms of the type $\ln Q^2/m^2$ which have to be absorbed in the running coupling constant so that the number of light flavours $n_f$ is enhanced by one unit. However this has to happen at much larger values of $Q$ i.e. $Q \sim 6.5 m$ than one usually assumes for the flavour thresholds which appear in the running coupling constant. An alternative description for the heavy flavour dependence of the running coupling constant in the context of the MOM-scheme is discussed.
The study of QCD sum rules, as represented by the first moments of the deep inelastic structure functions, has lead to a deeper insight of the behaviour of the perturbation series. This became possible after new techniques were invented to evaluate the Feynman integrals up to four-loop order. Examples of these techniques are infrared rearrangement [1], integration by parts [2], and the $R^*$-operation [3]. Also important was the appearance of new algebraic manipulation programs like FORM [4] which enables us to evaluate the complicated traces of the huge amount of Feynman graphs characteristic of higher order loop calculations. At this moment the sum rules computed up to third order in $\alpha_s$ are represented by the first Bjørken (polarized) sum rule [3], the second Bjørken (unpolarized) sum rule [5] and the Gross-Llewellyn Smith sum rule [6].

The perturbation series for these sum rules show a similar behaviour as is observed for other quantities which are calculated up to third order like e.g. the $Z$-boson and $\tau$-lepton decay widths (for a review of the literature see [8]). Quantities computed up to a very high order in perturbation theory provide us with a very good tool to understand methods used in improved perturbation theory. Examples are the principle of minimal sensitivity (PMS [9]) and the effective charge approach (ECH [10]). These methods were applied [8] to the above sum rules to obtain an estimate of the unknown order $\alpha_s^4$ contribution. Another way to get the latter term is to use Pade-approximants as carried out in [11] (for an estimate using renormalons see also [12]). One of the remarkable results of these methods is that all estimates agree very well with each other. Apart from the theoretical interest there is also a practical one. Quantities which can be calculated up to a very high order in perturbation theory provide us with an excellent tool to measure the running coupling constant $\alpha_s$. Notice that in many cases the perturbation series is only known up to next-to-leading order (NLO) which means that, apart from some resummation of dominant terms, we have no control on the higher order corrections. An example of the determination of $\alpha_s$ is given in [13] where it is extracted via the polarized Bjørken sum rule from the data obtained for the longitudinal structure function $g_1(x, Q^2)$.

The order $\alpha_s^3$ corrections to the sum rules mentioned above have been carried out in [14] (the unpolarized Bjørken sum rule) and [15] (the polarized Bjørken sum rule and the Gross-Llewellyn Smith sum rule). In these calculations only massless quarks were considered but mass effects coming from the contribution of heavy quarks were omitted. The latter are important because apart from additional corrections the mass effects indicate when a heavy quark has to be treated as a massless or as a massive quark. This also indicates which number of light flavours $n_f$ has to be chosen in the perturbation series in particular for the running coupling constant at a given value of $Q^2$. Here $Q$ denotes the virtual mass of the intermediate vector boson in deep inelastic lepton hadron scattering. Before presenting the heavy flavour contributions we first give the definitions of the three aforementioned sum rules and the corresponding perturbation series corrected up to third order in $\alpha_s$. The polarized [5] and unpolarized Bjørken [3] sum rules are defined by

$$\Delta g_1(Q^2) \equiv \int_0^1 dx \left[ g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2) \right] = \frac{1}{6} \frac{G_A}{G_V} A_{P1}(Q^2),$$

and

$$\Delta F_1(Q^2) \equiv \int_0^1 dx \left[ F_1^{ep}(x, Q^2) - F_1^{en}(x, Q^2) \right] = K(n_f) A_{F1}(n_f, Q^2)$$

$$K(3) = 1 + \sin^2 \theta_c \quad (SU_F(3)) \quad K(4) = 1 \quad (SU_F(4)),$$

respectively, whereas the Gross-Llewellyn Smith sum rule [4] is given by

$$\Delta F_3(Q^2) \equiv \int_0^1 dx \left[ F_3^{ep}(x, Q^2) + F_3^{en}(x, Q^2) \right] = K(n_f) A_{F3}(n_f, Q^2)$$
\[
A(Q^2) = A_s^0(Q^2) + A_s^1(Q^2) + A_s^2(Q^2)
\]

Table 1: The coefficients in the \( \overline{\text{MS}} \)-scheme of the perturbation series (4) corresponding to the three sum rules in Eqs. (1)-(3).

| \( A_s^0(Q^2) \) | \( A_s^1(Q^2) \) | \( A_s^2(Q^2) \) |
|------------------|------------------|------------------|
| \( a_1 \) | -1 | 2/3 | -1 |
| \( a_2 \) | -\( \frac{55}{12} \) | \( \frac{25}{6} \) | \( -\frac{55}{12} \) |
| \( b_2 \) | \( \frac{1}{3} \) | \( \frac{8}{27} \) | \( \frac{1}{3} \) |
| \( a_3 \) | \( -\frac{13841}{216} \) + \( \frac{44}{9} \zeta(3) - \frac{55}{2} \zeta(5) \) | \( -\frac{4075}{108} + \frac{622}{27} \zeta(3) - \frac{680}{27} \zeta(5) \) | \( -\frac{13841}{216} \) + \( \frac{44}{9} \zeta(3) + \frac{55}{2} \zeta(5) \) |
| \( b_3 \) | \( \frac{10339}{1296} + \frac{61}{54} \zeta(3) - \frac{5}{3} \zeta(5) \) | \( \frac{3565}{648} + \frac{59}{27} \zeta(3) + \frac{10}{3} \zeta(5) \) | \( \frac{10009}{1296} + \frac{91}{54} \zeta(3) - \frac{5}{3} \zeta(5) \) |
| \( c_3 \) | \( -\frac{115}{648} \zeta(3) \) | \( -\frac{155}{972} \) | \( \frac{115}{648} \) |
| \( a_4^{\text{PMS}}(3) \) | -130 | -133 | -130 |

Here \( \theta_c \) denotes the Cabibbo angle and for the constant \( K(n_f) \) we have quoted the values given by the flavour group \( SU_F(n_f) \) for \( n_f = 3, 4 \), where \( n_f \) represents the number of light flavours. The perturbation series of the above sum rules in the case of massless quarks can be written up to third order in \( \alpha_s \) as

\[
A_s(r, Q^2) = 1 + \frac{\alpha_s(n_f, \mu^2)}{\pi} a_1 + \left( \frac{\alpha_s(n_f, \mu^2)}{\pi} \right)^2 \left[ -a_1 \beta_0(n_f) \ln \left( \frac{Q^2}{\mu^2} \right) + a_2 + b_2 n_f \right]
\]

\[
+ \left( \frac{\alpha_s(n_f, \mu^2)}{\pi} \right)^3 \left[ a_1 \beta_1^2(n_f) \ln^2 \left( \frac{Q^2}{\mu^2} \right) - \left\{ a_1 \beta_1(n_f) + 2 \beta_0(n_f) \left( a_2 + b_2 n_f \right) \right\} \right]
\]

\[
\times \ln \left( \frac{Q^2}{\mu^2} \right) + a_3 + b_3 n_f + c_3 n_f^2 \quad \text{with} \quad r = g_1, F_1, F_3,
\]

where \( \beta_0 \) and \( \beta_1 \) stand for the first and second order contributions to the \( \beta \)-function which are given by

\[
\beta_0(n_f) = \frac{11}{4} - \frac{1}{6} n_f \quad \beta_1(n_f) = \frac{51}{8} - \frac{19}{24} n_f.
\]

The other coefficients \( a_i, b_i, c_i \), which are computed in the \( \overline{\text{MS}} \)-scheme in [14] and [15], are given in table 1. As has been mentioned above, the order \( \alpha_s^4 \) contribution to Eq. (1) is not known. However, there exist some estimates. Here we will adopt the results obtained from PMS given in [16]. They will be denoted by

\[
\delta A_s^{\text{PMS}}(Q^2) = \left( \frac{\alpha_s(n_f, \mu^2)}{\pi} \right)^4 \left[ a_4^{\text{PMS}}(n_f) \right],
\]

where the coefficient \( a_4^{\text{PMS}}(3) \) is given in table 1. It turns out that the other estimates originating
Figure 1: Forward Compton scattering graphs for heavy flavour production: $W + d \rightarrow g + c$. The down quark and the charm quark are indicated by a dashed line and a solid line, respectively.

from ECH [8] and the Pade-technique [11] are very close to the PMS value. Besides the sum rules above we also have the Adler sum rule [16] given by

$$
\Delta F_2(Q^2) \equiv \int_0^1 \frac{dx}{x} \left[ F_{2}^{ep}(x,Q^2) - F_{2}^{np}(x,Q^2) \right] = K(n_f)
$$

$$
K(3) = 2 + 2 \sin^2 \theta_c \quad (SU_F(3)) \quad K(4) = 2 \quad (SU_F(4)), \quad (7)
$$

which holds in all orders of perturbation theory. Furthermore it does not receive higher twist contributions or mass corrections. The latter we have checked in our computations presented below. The coefficients in table 1 are only determined for massless quarks (see [14, 15]). In the subsequent part of the paper we will show how the perturbation series is modified by including mass corrections due to heavy flavour contributions.

In our calculations we assume that in addition to the gluon the proton only contains three light flavours given by the quarks $u, d, s$, including their anti-particles. The heavy quarks only show up in the final state. Since the sum rules presented above only involve non-singlet contributions the perturbation series for heavy flavour contributions in the case of neutral current interactions starts in order $\alpha_s^2$. However, for the charged current interaction we get already contributions on the Born level. Starting with the latter interaction $\Delta F_1$ in Eq. (2) and $\Delta F_3$ in Eq. (3) are in lowest order given by the flavour excitation process

$$
d (\bar{d}) + W \rightarrow c (\bar{c}). \quad (8)
$$

In the process above we have only considered charm production because the other heavy quarks are heavily suppressed by the mixing angles occurring in the Kobayashi-Maskawa matrix. Moreover the relevant values of $Q^2$ are so small that they are far below the thresholds of bottom and top production. Further notice that the integrals over the strange quark and the anti-strange quark densities cancel against each other in the computation of $\Delta F_1$ and $\Delta F_3$ so that the process in Eq. (8) does not contribute to the sum rules when $d$ is replaced by $s$. In NLO one gets
Figure 2: Forward Compton scattering graphs for heavy flavour production: $V + q \rightarrow q' + H + \bar{H}$. The light quarks $q$ and the heavy quarks $H$ are indicated by a dashed line and a solid line, respectively.

contributions from the virtual corrections to reaction (3) and the gluon bremsstrahlungs process (see Fig. 1). The latter is given by

$$d(\bar{d}) + W \rightarrow c(\bar{c}) + g.$$  \hspace{1cm} (9)

The coefficient functions corresponding to the structure functions $F_i$ ($i = 1, 2, 3, L$) have been computed for processes (8) and (9) in Ref. [17] for $m_d = 0$ and $m_c \neq 0$ (see also Ref. [18]). Notice that due to the non-vanishing mass of the charm quark one gets already a contribution to $F_L$ on the Born level from reaction (8). If the quark in the initial state becomes massive the above processes also contribute to the neutral current interaction where $W$ is replaced by $Z$ or $\gamma$. The computation of the coefficient functions for the neutral current interaction where the masses of the initial and final state quarks are equal has been treated in Ref. [19]. Integration of the coefficient functions over the scaling variable $x$ provides us with the result for the unpolarized Bjørken sum rule

$$A_{F_i}^{F_1,(1)}(Q^2, m^2) = \left[ \frac{1}{1 + \xi} + C_F \frac{\alpha_s(\mu^2)}{4\pi} \left\{ -\frac{1}{1 + \xi} - \frac{2}{\xi} - \left( 6 \frac{1}{1 + \xi} - \frac{2}{\xi^2} \right) \ln (1 + \xi) \right\} \right] \sin^2 \theta_c,$$  \hspace{1cm} (10)

and the Gross-Llewellyn Smith sum rule

$$A_{F_i}^{F_3,(1)}(Q^2, m^2) = \left[ -\frac{1}{3(1 + \xi)} + C_F \frac{\alpha_s(\mu^2)}{4\pi} \left\{ \frac{1}{1 + \xi} + \frac{2}{1 + \xi} \ln(1 + \xi) \right\} \right] \sin^2 \theta_c \quad \xi = \frac{Q^2}{m^2},$$  \hspace{1cm} (11)

respectively, where $C_F$ denotes the colour factor which in QCD reads $C_F = 4/3$. Furthermore, we have also checked that the corrections to the Adler sum rule in Eq. (7) are zero as expected. The above expressions have to be added to the light quark contribution $A'(3, Q^2)$ in Eq. (4) to obtain the $O(\alpha_s)$ mass corrections to the sum rules in Eqs. (2), (3) with $K = K(4)$. In the

\footnote{Notice that $A_2$ in table I of [17] should be $K_A$ and not $K_A/2$.}


perturbation series above the following limits are of interest. When $Q^2$ is much larger than the mass of the charm quark, i.e. $\xi \to \infty$, then the corrections in Eqs. (10) and (11) tend to zero. This means that the sum rules are presented in a four light flavour scheme. However, when the mass of the charm quark becomes much larger than $Q^2$, i.e. $\xi \to 0$, then the heavy quark mass corrections are non-vanishing. After adding them to the massless quark result $A^t(3, Q^2)$ in Eq. (3) one can extract an overall factor $K$ which turns out to be $K(3)$ which is the value in the three light flavour scheme. This is expected because for infinite mass the heavy flavour disappears from the theory. Unfortunately this does not happen for the Adler sum rule in Eq. (7) because it is insensitive to the mass of the heavy flavours. The next process which shows up in neutral current as well as in charged current interactions is given by gluon splitting into a heavy quark anti-quark pair (see Fig. 2).

$$q + V \to q' + Q + \bar{Q} \quad \text{with} \quad V = \gamma, Z, W.$$  \quad (12)

The coefficient function for $g_1$, which is the same as for $F_3$, has been calculated in [20]. Notice that the heavy quark loop contribution in Fig. 2 to the gluon self-energy $\Pi(p^2, m^2)$, where $p$ denotes the gluon momentum, has been renormalized in such a way that $\Pi(0, m^2) = 0$. This implies that heavy quarks are decoupled from the running coupling constant. The result for the polarized Björken sum rule and the Gross-Llewellyn Smith sum rule becomes equal to

$$A_{H}^{g_1, (2)}(Q^2, m^2) = A_{H}^{F_3, (2)}(Q^2, m^2) = \left( \frac{\alpha_s(n_f, \mu^2)}{4\pi} \right)^2 C_F T_f \left[ \left( \frac{1}{105} \xi^2 + \frac{16}{45} \chi \right) \ln \xi \right. \right.$$

$$\left. + \frac{1}{\lambda^4} \left( \frac{2}{105} \xi + \frac{2783}{315} + \frac{6740}{63} \xi + \frac{137552}{315} \xi^2 + \frac{62528}{105} \xi^3 \right) \right.$$

$$\left. + \frac{1}{\lambda^5} \left( \frac{142}{315} + \frac{494}{63} \xi \right) \right.$$

$$\left. + \frac{1}{\lambda^6} \left( \frac{23024}{63} \xi^2 + \frac{298432}{315} \xi^3 + \frac{102656}{105} \xi^4 \right) \right.$$

$$\left. \ln \left( \frac{\lambda + 1}{\lambda - 1} \right) - \frac{20}{3} \xi^2 \ln^2 \left( \frac{\lambda + 1}{\lambda - 1} \right) \right]. \quad (13)$$

where $T_f$ stands for the colour factor which in QCD is given by $T_f = 1/2$ (for $C_F$ see below Eq. (17)). The coefficient function for $F_1$ can be derived from the ones obtained for the structure functions $F_2$ and $F_L$ which are presented in [24]. They are obtained using the same renormalization condition for the heavy quark loop contribution to the gluon self-energy as given above. The result for the unpolarized Björken sum rule is

$$A_{H}^{F_1, (2)}(Q^2, m^2) = \left( \frac{\alpha_s(n_f, \mu^2)}{4\pi} \right)^2 C_F T_f \left[ -\frac{2}{105} \xi^2 \ln \xi + \frac{1}{\lambda^4} \left( -\frac{4}{105} \xi + \frac{2162}{315} \xi \right) \right.$$

$$\left. + \frac{1}{\lambda^5} \left( \frac{315}{21} \xi^2 + \frac{13696}{21} \xi^3 \right) \right.$$

$$\left. + \frac{1}{\lambda^6} \left( \frac{2}{105} \xi^2 + \frac{4}{21} \xi^3 - \frac{1000}{21} \xi^2 - \frac{21}{21} \xi - \frac{6752}{21} \xi^2 - \frac{99712}{105} \xi^3 \right) \right.$$

$$\left. - \frac{22016}{21} \xi^4 \right) \ln \left( \frac{x + 1}{x - 1} \right) - \frac{8}{\xi^2} \ln^2 \left( \frac{\lambda + 1}{\lambda - 1} \right) \right], \quad \text{with} \quad \lambda = \sqrt{1 + \frac{4}{\xi}}. \quad (14)$$

Like for the flavour excitation mechanism (Eqs. (8), (9)) we have checked that the gluon splitting process does not contribute to the Adler sum rule (1). We are also interested in the asymptotic expansions of the expressions above. In the case the quark mass gets much larger than the
The leading terms in the expressions above which are given by the constant and the logarithm of the virtuality of the intermediate vector bosons we get

\[
A^{r,(2)}_H(Q^2, m^2) \big|_{m^2 \gg Q^2} = \left( \frac{\alpha_s(n_f, \mu^2)}{4\pi} \right)^2 C_F T_F \left[ \frac{16}{45} \xi + \frac{1}{105} \xi^2 \right] \ln \xi - \frac{232}{225} \xi - \frac{1933}{44100} \xi^2
- \frac{1}{62370} \xi^4 + \frac{2}{945945} \xi^5 \right] \quad r = g_1, F_3
\]  

(15)

and

\[
A^{F,(2)}_H(Q^2, m^2) \big|_{m^2 \gg Q^2} = \left( \frac{\alpha_s(n_f, \mu^2)}{4\pi} \right)^2 C_F T_F \left[ -\frac{2}{105} \xi^2 \ln \xi - \frac{16}{45} \xi + \frac{1093}{22050} \xi^2 + \frac{8}{4725} \xi^3
- \frac{1}{10395} \xi^4 + \frac{8}{945945} \xi^5 \right].
\]  

(16)

The expressions show that for infinite mass ($\xi \to 0$) the corresponding heavy flavour decouples from the radiative correction which is a consequence of the renormalization condition for the gluon self-energy in the graphs of Fig. \[.\] When the virtuality $Q$ of the vector bosons is much larger than the mass of the heavy quark, which implies that the latter behaves like a light flavour, we obtain

\[
A^{r,(2)}_H(Q^2, m^2) \big|_{Q^2 \gg m^2} = \left( \frac{\alpha_s(n_f, \mu^2)}{4\pi} \right)^2 C_F T_F \left[ -\frac{20}{3} \xi^2 \ln^2 \xi - \left( 4 + \frac{64}{3} \xi + \frac{226}{9} \xi^2 + \frac{32}{5} \xi^3 \right)
- \frac{56}{15} \xi + \frac{1}{63} \xi^4 \right] \ln \xi + 8 + \frac{272}{9} \xi - \frac{1775}{54} \xi^2 - \frac{536}{75} \xi^3 + \frac{118}{225} \xi^4 + \frac{7136}{6615} \xi^5 + \cdots
\]  

(17)

with $r = g_1, F_3$

\[
A^{F,(2)}_H(Q^2, m^2) \big|_{Q^2 \gg m^2} = \left( \frac{\alpha_s(n_f, \mu^2)}{4\pi} \right)^2 C_F T_F \left[ -\frac{8}{3} \xi^2 \ln \xi - \left( \frac{8}{3} + \frac{64}{3} \xi + \frac{28}{15} \xi^2 + \frac{128}{15} \xi^3 \right)
- \frac{16}{3} \xi + \frac{128}{21} \xi^5 \right] \ln \xi + \frac{64}{9} \xi + \frac{320}{9} \xi - \frac{35}{12} \xi^2 + \frac{1984}{225} \xi^3 + \frac{4}{9} \xi^4 + \frac{1376}{735} \xi^5 + \cdots
\]  

(18)

The leading terms in the expressions above which are given by the constant and the logarithm $\ln \xi$ can be predicted by the renormalization group. The general form up to order $\alpha_s^3$ becomes

\[
A^{r,\text{asymp}}_H(Q^2, m^2) = \left( \frac{\alpha_s(n_f, \mu^2)}{\pi} \right)^2 \left\{ \frac{1}{6} a_1 \ln \left( \frac{Q^2}{m^2} \right) + b_2 \right\} + \left( \frac{\alpha_s(n_f, \mu^2)}{\pi} \right)^3 \left\{ a_1 \left( -\frac{8}{9} \right)
+ \frac{1}{18} n_f \ln^2 \left( \frac{Q^2}{m^2} \right) + a_1 \left( \frac{11}{12} - \frac{1}{18} n_f \right) \ln \left( \frac{Q^2}{m^2} \right) \ln \left( \frac{\mu^2}{m^2} \right) + \frac{19}{24} a_1
+ \frac{1}{3} \left( a_2 + b_2 n_f \right) - b_2 \left( \frac{11}{2} - \frac{1}{3} (n_f + 1) \right) \ln \left( \frac{Q^2}{m^2} \right)
+ b_2 \left( \frac{11}{2} - \frac{1}{3} n_f \right) \ln \left( \frac{\mu^2}{m^2} \right) + b_3 + c_3 \left( 2n_f + 1 \right) \right\},
\]  

(19)
Table 2: The contributions to the sum rules originating from the light quarks $A(n_f)$, Eq. (4), the charm excitation $A_c^{(1)}$, Eqs. (10), (11), and gluon splitting into heavy quarks $A^{(2)}_H$ ($H = c, b$), Eqs. (13), (14). For a comparison we also presented the asymptotic expressions $A^{\text{asymp.(2)}}_H$, Eq. (19), in order $\alpha_s^2$.
where the coefficients $a_i, b_i, c_i$ originate from the light quark contributions presented in table [I]. Notice that in the equation above we have already substituted the values of coefficients $\beta_0, \beta_1$, Eq. (3), appearing in the series expansion of the $\beta$-function. The large logarithms of the type $\ln Q^2/m^2$, which originate from all heavy quark loop insertions like in Fig. 3, can be absorbed into the running coupling constant. This is equivalent to a redefinition given by

$$\alpha_s(n_f, \mu^2) = \alpha_s(n_f + 1, \mu^2) \left[ 1 + \frac{\alpha_s(n_f + 1, \mu^2)}{\pi} (\beta_0(n_f + 1) - \beta_0(n_f)) \ln \left( \frac{\mu^2}{m^2} \right) \right]$$

\[ \left. + \left( \frac{\alpha_s(n_f + 1, \mu^2)}{\pi} \right)^2 \left\{ (\beta_0(n_f + 1) - \beta_0(n_f))^2 \ln^2 \left( \frac{\mu^2}{m^2} \right) \right\} \right] \ . \tag{20} \]

If we add the asymptotic expression (19) for the heavy quark loop contributions to the perturbation series for the light quarks in Eq. (10) we obtain after substitution of $\alpha_s(n_f, \mu^2)$ the following result

$$A_H^{\text{asymp}}(Q^2, m^2) + A'(n_f, Q^2) = A'(n_f + 1, Q^2) \ . \tag{21}$$

Therefore for $Q^2 \gg m^2$ we obtain the expression of the perturbation series for massless flavours again but wherein now the number of light flavours is enhanced by one unit. The question is at which $Q^2$ this will happen. Here we will give the answer for the charm quark because the deep inelastic sum rules are studied in the region $2 < Q^2 < 100 \text{(GeV/c)}^2$.

In our analysis we will use the three-loop corrected running coupling constant which satisfies the matching conditions (22)

$$\alpha_s \left( n_f, \Lambda_{n_f}, \mu^2 \right) = \alpha_s \left( n_f + 1, \Lambda_{n_f+1}, \mu^2 \right) \quad \text{at} \quad \mu = m_{n_f} . \tag{22}$$

If we choose $\Lambda_3 = 397 \text{ MeV/c (MS-scheme)}$ we get $\alpha_s(3, \mu_0^2) = 0.375$ for $\mu_0^2 = 2.5 \text{ (GeV/c)}^2$. These values were obtained from a comparison of the polarized Bjørken sum rule with the data carried out in [13]. Following the matching conditions in Eq. (22) one gets $\alpha_s(5, M_Z^2) = 0.122$ (here $\Lambda_5 = 259 \text{ MeV/c}$) which lies a little bit above the world average of $\alpha_s(5, M_Z^2) = 0.119$. Nevertheless we will use $\alpha_s(3, \mu_0^2)$ as a starting point. The values for the heavy flavour masses are chosen to be $m_c = 1.5 \text{ GeV/c}^2$, $m_b = 4.5 \text{ GeV/c}^2$ and $m_t = 173.8 \text{ GeV/c}^2$. Further the number of light flavours in $A'$, Eq. (4), and the running coupling constant is taken to be $n_f = 3$ irrespective of the value of $Q^2$. In table 2 we have presented for $Q^2 = 2.5, 10, 100 \text{ GeV/c}^2$ the light and heavy flavour contributions to the perturbation series denoted by $A(3)$ and $A_H^{(3)} (H = c, b)$, respectively. In the case of $A(3)$, Eq. (4), we only consider the exact perturbation series corrected up to order $\alpha_s^2$ and omitted the fourth order estimate $\delta A_{\text{PMS}}^{\text{asymp}}(3)$, Eq. (5), which is listed separately in table 3. The charm contribution, represented by the order $\alpha_s$ corrected quantity $A_c^{(1)}$, is given by Eqs. (10), (11) which does not appear in the case of the polarized Bjørken sum rule. The remaining heavy flavour contributions show up in order $\alpha_s^2$ and they are represented in the table by $A_c^{(2)}, A_b^{(2)}$ (see Eqs. (13), (14)). The top quark contribution is so small that it is neglected. Besides the exact results for $A_H^{(2)}$ we have also made a comparison with the asymptotic expression given by the order $\alpha_s^2$ contribution $A_H^{\text{asymp},(2)}$ (see Eq. (19)) derived in the limit $Q^2 \gg m^2$. From table 2 we infer that the heavy flavour contributions are...
rather small even when compared with the estimate $\delta A_{\text{PMS}}(3)$. Only in the case of $\Delta F_1(Q^2)$ at $Q^2 = 2.5$ GeV/c$^2$ the charm component is of the same size as the order $\alpha_s^2$ estimate and it amounts to 0.017 which is about 2% of the light quark contribution given by $A_{F_1}(3) = 0.847$. This effect can be wholly attributed to the charm excitation mechanism (see Eqs. (8), (9)) represented by $A_c^{(1)}$, which also dominates $\Delta F_3(Q^2)$. At larger values of $Q^2$, $A_c^{(1)}$ decreases and it becomes of the same order of magnitude as $A_c^{(2)}$ which is due to the gluon splitting mechanism. Notice that the bottom quark contribution is always smaller than the charm quark component. The behaviour of the heavy quark contributions follows from their asymptotic behaviour at small and at large $Q^2$, see e.g. Eqs. (13)-(18). At increasing $Q^2$ the charm excitation contribution $A_c^{(1)}$ is decreasing whereas the gluon splitting part $A_c^{(2)}$ becomes larger. At $Q^2 = 100$ GeV/c$^2$, which is about $Q = 6.5 \, m_c$, the latter gets closer to its asymptotic expression $A_{c_{\text{asymp}}}^{(2)}$. However, there is still a discrepancy between the exact and asymptotic expressions which in the case of the sum rules $\Delta g_1(Q^2)$ and $\Delta F_3(Q^2)$ amounts to 15%. For $\Delta F_1(Q^2)$ this is much worse and the difference between the exact and asymptotic expression is 28 \% w.r.t. the exact one. In the case of the bottom quark one needs much larger values before $A_b^{(2)} \sim A_{b_{\text{asymp}}}^{(2)}$ which occurs for $Q^2 > 1000$ GeV/c$^2$. In order to get $A_{b_{\text{asymp}}}^{(2)} = A_b^{(2)}$ within 1 \% one needs the value $Q > 25 \, m_b$. Hence we can conclude that the large logarithmic terms given by $\ln(Q^2/m^2)$ start to dominate the heavy flavour contribution for $Q > 6.5 \, m_c$ which means that from this value onwards the heavy flavour behaves like a light quark. Therefore only for $Q > 6.5 \, m_c$ the large logarithms have to be resummed as is explained below Eq. (29) which will lead to a $n_f + 1$ flavour description. This means that the matching condition $\mu = m_{n_f}$ has to be changed into $\mu = 6.5 \, m_{n_f}$. Using this new matching condition we get, starting from $\alpha_s(3, 2.5) = 0.375$ as our experimental input value, the result $\alpha_s(5, M_Z^2) = 0.114$. The latter is very close to the value obtained in fixed target deep inelastic scattering experiments given by $\alpha_s(5, M_Z^2) = 0.113$ [23]. However, from the analysis above we think that the matching conditions as presented in Eq. (22) are rather artificial. There is no specific scale where nature suddenly jumps from an $n_f$-flavour to an $n_f+1$-flavour scheme. Moreover the relevant scale in deep inelastic scattering is $q^2 = -Q^2$ which is spacelike rather than $p^2 \geq 4 \cdot m_{n_f}^2$ which is timelike. Here $p$ denotes the gluon momentum in the graphs of Fig. 2. Therefore, in principle all heavy flavour channels may contribute for spacelike processes which proceeds via the coefficient functions rather than through the running coupling constant. The decoupling of the heavy flavours from the perturbation series is then ruled by the Appelquist–Carazzone theorem [24]. In order to get more continuity between the large and small $Q^2$ regions we substitute on the l.h.s. of Eq. (21) at $n_f = 3$ the coupling constant by

$$\alpha_s(\mu^2, 3) = \alpha_s^{\text{MOM}}(\mu^2) \left[ 1 + \frac{\alpha_s^{\text{MOM}}(\mu^2)}{4\pi} U_1 + \left( \frac{\alpha_s^{\text{MOM}}(\mu^2)}{4\pi} \right)^2 \left( U_2 + U_1^2 \right) + \cdots \right], \quad (23)$$

with

$$U_i = T_f \sum_{n_f=4}^6 \left[ \Pi_i \left( \mu^2 \frac{m_{n_f}^2}{m_{n_f}^2} \right) - \Pi_i \left( \mu_0^2 \frac{m_{n_f}^2}{m_{n_f}^2} \right) \right], \quad \text{with} \quad m_4 = m_c, m_5 = m_b, m_6 = m_t. \quad (24)$$

Here we are starting from a low input scale $\mu_0$ so that one is sure that at this value the perturbation series is described by a three-flavour number scheme. The functions $\Pi_i$ are the order $\alpha_i^2$ contributions to the gluon self-energy which can be attributed to the heavy quark loop only. They are presented in [25] up to order $\alpha_s^2$. For $\mu^2 \gg m_{n_f}^2$ and $m_{n_f}^2 \gg \mu_0^2$ the expression in Eq. (23) tends to its asymptotic result in Eq. (24) provided one considers in the sum of Eq. (24) one
flavour only. Finally one can resum the self-energies so that our new running coupling constant becomes
\[ \alpha_{s}^{\text{MOM}}(\mu^2) = \frac{\alpha_s(\mu^2, 3)}{1 + \frac{\alpha_s(\mu^2, 3)}{4\pi} U_1 + \frac{\alpha_s(\mu^2, 3)}{4\pi} \left( \frac{U_2}{U_1} \right) \ln \left( 1 + \frac{\alpha_s(\mu^2, 3)}{4\pi} U_1 \right) } . \] (25)

The coupling constant above was proposed in the context of the momentum subtraction (MOM) scheme in [24] (for earlier work on this subject see also [27]) where it also includes the resummation of the light quark contributions which we did not perform in this paper. The procedure outlined above guarantees that one gets a smooth transition between the low and large \( Q^2 \) regions. When e.g. \( \mu^2 = Q^2 \) and \( Q^2 \gg m_{n_f}^2 \) the large logarithms \( \ln(Q^2/m_{n_f}^2) \) occurring in the functions \( U_i \) (24) cancel against the corresponding terms in the perturbations series of the heavy quark component \( A_H(Q^2, m_{n_f}^2) \) of which the asymptotic expression is given by Eq. (19). In this way one gets effectively a \((n_f + 1)\)-flavour description. If \( Q^2 \ll m_{n_f}^2 \) then \( U_i \sim 0 \) and no large corrections appear in \( A_H(Q^2, m_{n_f}^2) \) (decoupling of heavy quarks !) so that one gets the \( n_f \)-flavour representation for the whole perturbation series. Following our approach and choosing \( \mu_0^2 = 2.5 \text{ GeV}/c^2 \) in Eq. (24) we get \( \alpha_s(5, M_Z^2) = 0.117 \) which is closer to the LEP measurement.

Finally we checked that the numbers in the table hardly change in passing from the \( n_f = 3 \) \( \overline{\text{MS}} \)-scheme to the MOM-scheme. The maximal deviation is observed for \( \Delta g_1(Q^2) \) where at \( Q^2 = 100 \text{ GeV}/c^2 \) the latter scheme leads to a decrease of \( A_{g1}(3) \) by 0.002 so that for this value of \( Q^2 \) the resummation effect is very small. Summarizing our findings we have calculated the heavy quark contribution to several deep inelastic sum rules. The corrections to the three-loop corrected perturbation series, computed for light quarks, are very small. Only at low \( Q^2 \) the correction in the case of the unpolarized Bjørken sum rule \( \Delta F_1(Q^2) \) is noticeable where it is of the same order of magnitude as the order \( \alpha_s^4 \)-estimate. The quark component of the sum rule attains its asymptotic value at much larger scales as given by the usual matching conditions. Matching at larger scales i.e. \( \mu = 6.5 \text{ } m_{n_f} \) leads to a smaller value of the running coupling constant at the \( Z \)-boson mass. The unnatural matching conditions which are characteristic of the \( \overline{\text{MS}} \)-scheme can be replaced by expressing the perturbation series in the MOM-scheme.

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References

[1] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, Nucl. Phys. B174 (1980) 345.
[2] K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159.
[3] K.G. Chetyrkin and F.V. Tkachov, Phys. Lett. B114 (1982) 340;
   K.G. Chetyrkin and V.A. Smirnov, Phys. Lett. B144 (1984) 419.
[4] A. Vermaseren, "Symbolic Manipulation with FORM", Published by CAN, Amsterdam, The Netherlands, ISBN 90-74116-01-9.
[5] J.D. Bjørken, Phys. Rev. 148 (1966) 1467, Phys. Rev. D1 (1970) 1376.
[6] J.D. Bjørken, Phys. Rev. 163 (1967) 1767.
[7] J.J. Gross and C.H. Llewellyn Smith, Nucl. Phys. B14 (1969) 337.
[8] A.L. Kataev and V.V. Starshenko, Mod. Phys. Lett. A10 (1995) 235.
[9] P.M. Stevenson, Phys. Rev. D23 (1981) 2916.
[10] G. Grunberg, Phys. Lett. B221 (1980) 70, Phys. Rev. D29 (1984) 2315.
[11] M.A. Samuel, J. Ellis, M. Karliner, Phys. Rev. Lett. 74 (1995) 4380.
[12] J. Ellis, E. Gardi, M. Karliner and M.A. Samuel, Phys. Lett. B366 (1996) 268.
[13] J. Ellis and M. Karliner, Phys. Lett. B341 (1995) 397.
[14] S.A. Larin, F.V. Tkachiev, J.A.M. Vermaseren, Phys. Rev. Lett. 66 (1991) 862.
[15] S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B259 (1991) 345.
[16] S.L. Adler, Phys. Rev. 143 (1966) 1144.
[17] Th. Gottschalk, Phys. Rev. D23 (1981) 56.
[18] M. Glück, S. Kretzer and E. Reya, Phys. Lett. B380 (1996) 171.
[19] O.V. Teryaev and O.L. Veretin, preprint hep-ph/9602362.
[20] M. Buza, Y. Matiounine, J. Smith, W.L. van Neerven, Nucl. Phys. B485 (1997) 420.
[21] M. Buza, Y. Matiounine, J. Smith, R. Migneron and W.L. van Neerven, Nucl. Phys. B472 (1996) 611.
[22] W.J. Marciano, Phys. Rev. D29 (1984) 580.
[23] M. Virchaux and A. Milsztajn, Phys. Lett. B274 (1992) 221.
[24] T. Appelquist and J. Carazzone, Phys. Rev. D11 (1975) 2856.
[25] F. Jegerlehner and O.V. Tarasov, DESY 98/093, hep-ph/9809485.
[26] D.V. Shirkov, Teor. Mat. Fiz. 93 (1992) 466, Nucl. Phys. B371 (1992) 467.
[27] T. Yoshino and K. Hagiwara, Z. Phys. C24 (1984) 185.