Form-factors of exponential fields in the sine-Gordon model

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Abstract
An integral representation for form-factors of exponential fields $e^{ia\phi}$ in the sine-Gordon model is proposed.
In this letter we study the the sine-Gordon model defined by the Euclidean action

$$A_{SG} = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\nu \phi)^2 - 2\mu \cos (\beta \phi) \right\}. \quad (1)$$

Recently [1], [2], expectation values for exponential fields in this Quantum Field Theory (QFT) were proposed,

$$\langle e^{ia\phi} \rangle \equiv G_a = \left[ \frac{M \sqrt{\pi} \Gamma\left(\frac{1}{2} - 2\beta^2\right)}{2 \Gamma\left(\frac{\beta^2}{2 - 2\beta^2}\right)} \right]^{2a^2} \times \exp\left\{ \int_0^\infty \frac{dt}{t} \left[ \frac{\sinh^2(2\alpha \beta t)}{2 \sinh(\beta^2 t) \sinh(\beta t) \cosh\left((1 - \beta^2)t\right)} - 2a^2 e^{-2t} \right] \right\}, \quad (2)$$

where $M$ is the soliton mass. In writing (2) it was assumed that the exponential field is normalized in accordance with the short distance limiting form of the two-point function

$$\langle e^{ia\phi(x)} e^{-ia\phi(y)} \rangle \rightarrow |x - y|^{-2a^2} \quad \text{as} \quad |x - y| \rightarrow 0, \quad (3)$$

so that the field $e^{ia\phi(x)}$ has the dimension $[\text{mass}]^{2a^2}$. The result (2) is expected to hold in the domain

$$\beta^2 < 1, \quad (4)$$

where the discrete symmetry $\phi \rightarrow \phi + 2\pi n/\beta^{-1}$ ($n = \pm 1, \pm 2...$) of (1) is spontaneously broken (and the QFT is massive) and $\langle ... \rangle$ in (2) means the expectation value over one of the ground states $|0\rangle$ in which the field $\phi(x)$ is localized near 0. The expectation value (2) controls both short and long distance asymptotics of the two-point correlation function

$$G_{aa'}(|x - y|) = \langle e^{ia\phi(x)} e^{ia'\phi(y)} \rangle \quad (5)$$

with $|a + a'| < \beta/2$. Indeed, if this inequality is satisfied the short distance limit of (2) is dominated by OPE

$$e^{ia\phi(x)} e^{ia'\phi(y)} \rightarrow |x - y|^{4aa'} e^{i(a + a')\phi(y)} \quad \text{as} \quad |x - y| \rightarrow 0. \quad (6)$$

Therefore

$$G_{aa'}(r) \quad \rightarrow \quad \begin{cases} G_{a + a'} r^{4aa'} & \text{as} \quad r \rightarrow 0 \\ G_a \ G_{a'} & \text{as} \quad r \rightarrow \infty. \end{cases} \quad (7)$$

A systematic technique for analysis of the short distance expansion of (5) is the Conformal Perturbation Theory [3]. At the same time the most efficient way to study the large
r behavior is provided by the form-factor approach \[4], \[9]. Up to now, the whole form-factor sets (for general values of \(\beta^2\) from \[3]) have been obtained for fields \(\varphi, e^{\pm i \beta \varphi}\) and \(e^{\pm i \varphi}\) only \[4], \[3]. In this this paper we present form-factors of the exponential operators for arbitrary values of the parameters \(0 < \beta^2 < 1\) and \(a\).

The technique, providing an integral representation for form-factors in the sine-Gordon model, was developed in \[3]. For the exponential operators it is suggested that the matrix elements

\[
\langle 0 \mid e^{i a \varphi} \mid A_{\sigma_2 n} (\theta_2 n) ... A_{\sigma_1} (\theta_1) \rangle, \quad \sigma_k = \pm \left( \sum_{k=1}^{2n} \sigma_k = 0 \right),
\]

where \(\mid A_{\sigma_2 n} (\theta_2 n) ... A_{\sigma_1} (\theta_1) \rangle\) are multi-soliton states \[3], are given by the following bosonization procedure

\[
\langle 0 \mid e^{i a \varphi} \mid A_{\sigma_2 n} (\theta_2 n) ... A_{\sigma_1} (\theta_1) \rangle = \mathcal{G}_a \left( \langle \langle Z_{\sigma_2 n} (\theta_2 n) ... Z_{\sigma_1} (\theta_1) \rangle \rangle \right), \tag{8}
\]

with

\[
Z_+ (\theta) = \sqrt{i \frac{C_2}{4 i C_1}} e^{-\frac{a \theta}{\pi^2}} e^{i \phi (\theta)},
\]

\[
Z_- (\theta) = \sqrt{i \frac{C_2}{4 i C_1}} e^{-\frac{a \theta}{\pi^2}} \left\{ e^{\frac{i \pi}{2 \beta \tau}} \int_{C_+} \frac{d \gamma}{2 \pi} e^{(1 - \frac{2 a}{\pi^2}) (\gamma - \theta) - i \bar{\phi} (\gamma)} e^{i \phi (\theta)} - e^{-\frac{i \pi}{2 \beta \tau}} \int_{C_-} \frac{d \gamma}{2 \pi} e^{(1 - \frac{2 a}{\pi^2}) (\gamma - \theta) - i \bar{\phi} (\gamma)} e^{i \phi (\theta)} \right\}. \tag{9}
\]

The averaging \(\langle \langle ..., \rangle \rangle\) should be performed by Wick’s theorem using the prescriptions

\[
\langle \langle e^{i \phi (\theta_2)} e^{i \bar{\phi} (\theta_1)} \rangle \rangle = G (\theta_1 - \theta_2),
\]

\[
\langle \langle e^{i \phi (\theta_2)} e^{i \bar{\phi} (\theta_1)} \rangle \rangle = W (\theta_1 - \theta_2) = \frac{1}{G (\theta_1 - \theta_2 - \frac{i \pi}{2}) G (\theta_1 - \theta_2 + \frac{i \pi}{2})}, \tag{10}
\]

\[
\langle \langle e^{i \bar{\phi} (\theta_2)} e^{i \bar{\phi} (\theta_1)} \rangle \rangle = \bar{G} (\theta_1 - \theta_2) = \frac{1}{W (\theta_1 - \theta_2 - \frac{i \pi}{2}) W (\theta_1 - \theta_2 + \frac{i \pi}{2})}.
\]

The functions and the constants here read explicitly

\[
G (\theta) = i C_1 \sinh \left( \frac{\theta}{2} \right) \exp \left\{ \int_0^\infty \frac{dt}{t} \frac{\sinh^2 \left( t - \frac{i \theta}{\pi} \right)}{\sinh (2t) \cosh (t) \sinh (t \xi)} \right\},
\]

\[
W (\theta) = \frac{2}{\cosh (\theta)} \exp \left\{ -2 \int_0^\infty \frac{dt}{t} \frac{\sinh^2 \left( t - \frac{i \theta}{\pi} \right)}{\sinh (2t) \sinh (t \xi)} \right\}, \tag{11}
\]

\[
\bar{G} (\theta) = -\frac{C_2}{4} \xi \sinh \left( \frac{\theta + i \pi}{\xi} \right) \sinh (\theta),
\]

\[1\] The restriction on \(a\) is not completely clear at the moment.

\[2\] See also works \[10], \[11\] on an interesting algebraic interpretation of this procedure.

\[3\] Our convention on the normalization of the states is \(\langle A^\sigma (\theta) \mid A_{\sigma'} (\theta') \rangle = 2 \pi \delta_{\sigma, \sigma'} \delta (\theta - \theta').\]
\[ C_1 = \exp\left\{ -\int_0^\infty \frac{dt}{t} \frac{\sinh^2 \left( \frac{t}{2} \right) \sinh(t(\xi - 1))}{\sinh(2t) \cosh(t) \sinh(t\xi)} \right\} = G(-i\pi), \]

\[ C_2 = \exp\left\{ 4 \int_0^\infty \frac{dt}{t} \frac{\sin^2 \left( \frac{t}{2} \right) \sinh(t(\xi - 1))}{\sinh(2t) \sinh(t\xi)} \right\} = \frac{4}{\left[ W(i \frac{\pi}{2}) \xi \sin \left( \frac{\pi}{\xi} \right) \right]^2}, \]  

where

\[ \xi = \frac{\beta^2}{1 - \beta^2}. \]

In (9) we suggest that the integration contour \( C_+(C_-) \) goes from \( \gamma = -\infty \) to \( \gamma = +\infty \) and the pole \( \gamma = \theta + i\frac{\pi}{2} \) (\( \gamma = \theta - i\frac{\pi}{2} \)) lies below (above) the contour.

The integrals in (9) are worked out explicitly for the free fermion point \( \beta^2 = \frac{1}{2} \). In this case

\[ Z_+(\theta) = 2^{-1} e^{\frac{\pi i \theta}{2}} e^{i\sqrt{2\pi a} \theta} e^{i\phi(\theta)}, \]

\[ Z_-(\theta) = -e^{-\frac{\pi i \theta}{2}} e^{-\sqrt{2\pi a} \theta} \left\{ e^{-i\sqrt{2\pi a} \theta} e^{-i\phi(\theta+i\pi)} + e^{i\sqrt{2\pi a} \theta} e^{-i\phi(\theta-i\pi)} \right\} \]  

and

\[ \langle \langle e^{i\phi(\theta_2)} e^{i\phi(\theta_1)} \rangle \rangle = i \sinh \left( \frac{\theta_1 - \theta_2}{2} \right). \]  

With the formulas (13), (14) at hands it is easy to get the following simple expression

\[ \langle 0 | e^{i\alpha \varphi} | A_+(\theta_{2n})...A_+ (\theta_{n+1})A_- (\theta_n)...A_- (\theta_1) \rangle = G_a \left( -1 \right)^{\frac{n(n+1)}{2}} \left( \sinh (\sqrt{2\pi a}) \right)^n \times e^{\sqrt{2a} \sum_{k=1}^{n} (\theta_{n+k} - \theta_k)} \prod_{1 \leq k < j \leq n} \sinh \left( \frac{\theta_k - \theta_j}{2} \right) \sinh \left( \frac{\theta_{n+k} - \theta_{n+j}}{2} \right) \prod_{1 \leq k,j \leq n} \cosh \left( \frac{\theta_{n+k} - \theta_j}{2} \right), \]  

which is in agreement with the results of [12], [13].

It is instructive to consider two-point form-factors more closely. From (8) we have the integral representation

\[ \langle 0 | e^{i\alpha \varphi} | A_+ (\theta_2)A_\mp (\theta_1) \rangle = G_a \ F^a_{\mp} (\theta_1 - \theta_2), \]  

with

\[ F^a_{-+} (\theta) = -\frac{G(\theta)}{G(-i\pi)} \ e^{\frac{\theta + i\pi}{2\pi}} \left\{ e^{2i\pi a} I_a(\theta) + I_a(-2i\pi - \theta) \right\}, \]

\[ F^a_{+-} (\theta) = -\frac{G(\theta)}{G(-i\pi)} \ e^{-\frac{\theta + i\pi}{2\pi}} \left\{ e^{-2i\pi a} I_a(\theta) + I_a(-2i\pi - \theta) \right\}. \]
Here the argument of $I_a(-2i\pi-\theta)$ means the corresponding analytical continuation of the function $I_a(\theta)$. The latter is specified for real $\theta$ and

$$-\beta^{-1} + \frac{\beta}{2} < \Re a < \frac{\beta}{2}$$

by the integral

$$I_a(\theta) = \frac{1}{\left[W(i\frac{\pi}{2})\xi \sin \left(\frac{\pi}{2}\right)\right]^2} \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi} W(\gamma + \theta - i\pi) W(\gamma - \theta - i\pi) e^{(1 - \frac{2\beta}{\beta^2})\gamma}. \quad (18)$$

Eqs. (17) were checked against the known form-factors [4], [6]. In particular, if $a \rightarrow 0$

$$F_{\frac{l}{\beta}}^{\pm}(\theta) = \pm ia \ F(\theta) + O(a^2), \quad (19)$$

and $F(\theta)$ coincides with the two-particle form-factor of the sine-Gordon field $\varphi$. It can be evaluated by the method discussed in the Appendix 4 of [9] with the result,

$$F(\theta) = -\frac{G(\theta)}{G(-i\pi)} \frac{\pi}{\beta \cosh \left(\frac{\theta + i\pi}{2\xi}\right) \cosh \left(\frac{\theta}{2}\right)}. \quad (20)$$

Similarly, one can find

$$F_{\frac{\pi}{\beta}}^{\pm}(\theta) = \frac{G(\theta)}{G(-i\pi)} \frac{2i e^{\frac{\theta + i\pi}{2\xi}}}{\xi \sinh \left(\frac{\theta + i\pi}{2\xi}\right) \sinh \left(\frac{\theta}{2}\right)} \quad (21)$$

and

$$F_{\frac{\pi}{\beta}}^{\pm}(\theta) = \frac{G(\theta)}{G(-i\pi)} \cot \left(\frac{\pi\xi}{2}\right) \frac{4i \cosh \left(\frac{\theta}{2}\right) e^{\frac{\theta + i\pi}{2\xi}}}{\xi \sinh \left(\frac{\theta + i\pi}{2\xi}\right)}. \quad (22)$$

Eqs. (20), (21), (22) agree with known expressions from [4], [5]. Notice that the functions (17) admit simpler forms for any integer and half-integer values of $\frac{\alpha}{\beta}$. Under this condition, one can show that

$$F_{\frac{l}{\beta}}^{\pm}(\theta) = \frac{G(\theta)}{G(-i\pi)} \frac{4i e^{\frac{\theta + i\pi}{2\xi}}}{\xi \sinh \left(\frac{\theta + i\pi}{2\xi}\right)} \sum_{m=1}^{l} (-1)^{l-m} \cosh \left((m - \frac{1}{2})\theta\right) \times \prod_{k=m-l,k \neq 0}^{m+l-1} \cot \left(\frac{\pi \xi k}{2}\right), \quad (23)$$

$$F_{\frac{l+1}{\beta}}^{\pm}(\theta) = \frac{G(\theta)}{G(-i\pi)} \frac{2i e^{\frac{\theta + i\pi}{2\xi}}}{\xi \sinh \left(\frac{\theta + i\pi}{2\xi}\right)} \left\{ \prod_{k=1}^{l} \cot^2 \left(\frac{\pi \xi k}{2}\right) + \sum_{m=1}^{l} (-1)^{l-m} \cosh(m\theta) \prod_{k=m-l,k \neq 0}^{m+l} \cot \left(\frac{\pi \xi k}{2}\right) \right\}, \quad (24)$$
for \( l = 1, 2, ... \) and

\[
F_{\pm}^{-\frac{1}{2}\beta}(\theta) = F_{\pm}^{\frac{1}{2}\beta}(\theta), \\
F_{\pm}^{-(l+\frac{1}{2})\beta}(\theta) = F_{\pm}^{(l+\frac{1}{2})\beta}(\theta).
\]

Significant simplifications also appear at the so-called reflectionless points,

\[
\beta^2 = \frac{n}{n+1}, \quad n = 1, 2, ... ,
\]

for generic \( a \). Then,

\[
F_{\pm}^{a}(\theta) = \pm \frac{G'(\theta)}{G(-i\pi)} \frac{2i n e^{\frac{n(a+i\pi)}{2}}}{\sinh \left( n(\theta + i\pi) \right)} \sum_{p=1}^{n} e^{\pm \theta(p-\frac{1}{2}-\frac{i}{2})} \prod_{k=p-n}^{p-1} \frac{\sinh \left( \frac{\pi}{n} \left( \frac{a}{\beta} - \frac{k}{2} \right) \right) \cos \left( \frac{\pi k}{2n} \right)}{\cos^{2} \left( \frac{\pi(p-k-1)}{2n} \right)}.
\]

(25)

If \( 0 < \beta^2 < \frac{1}{2} \), the spectrum of the model (1) contains a number of soliton-antisoliton bound states ("breathers") \( B_n, n = 1, 2, ..., < 1/\xi \). The lightest of these bound states \( B_1 \) coincides with the particle associated with the field \( \varphi \) in perturbative treatment of the QFT (1). An existence of the \( B_1 \) bound state means that the operators (13) satisfy the requirement (14), (10)

\[
Z_+(\theta_2)Z_-(\theta_1) \rightarrow \frac{i \Gamma^l_{++} \Lambda(\theta_1) + i \Lambda(\theta_1)}{\theta_2 - \theta_1 - i\pi(1 - \xi)} \quad \text{as} \quad \theta_2 - \theta_1 - i\pi(1 - \xi) \rightarrow 0,
\]

(26)

where \( \Gamma^l_{++} = \sqrt{2 \cot \left( \frac{\pi \xi}{2} \right)} \) \( \mathbb{I} \) and \( \mathbb{I} \)

\[
\Lambda(\theta) = \frac{\lambda}{2 \sin(\pi \xi)} \left\{ e^{-i\pi \xi} e^{-i\omega(\theta + i\frac{\pi}{2})} - e^{-i\pi \xi} e^{i\omega(\theta - i\frac{\pi}{2})} \right\},
\]

(27)

with

\[
\omega(\theta) = \phi(\theta + i\frac{\pi \xi}{2}) - \phi(\theta - i\frac{\pi \xi}{2}),
\]

\[
\lambda = 2 \cos \frac{\pi \xi}{2} \sqrt{2 \sin \frac{\pi \xi}{2}} \exp \left\{- \int_{0}^{\pi \xi} \frac{dt}{2\pi} \frac{t}{\sin(t)} \right\}.
\]

Clearly \( B_1 \) form-factors of the exponential field admit the representation

\[
\langle 0 | e^{ia\varphi} | B_1(\theta_n) ... B_1(\theta_1) \rangle = G_{a} \langle \langle \Lambda(\theta_n) ... \Lambda(\theta_1) \rangle \rangle.
\]

(28)

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4 The soliton-antisoliton amplitude \( \mathbb{I} \) contains a simple pole in the physical strip corresponding to the \( B_1 \) particle with the residue \( S_{+-}^{(\frac{1}{2})2} \rightarrow \frac{i}{\theta - i\pi(1 - \xi)}, \quad 0 < \beta^2 < \frac{1}{2} \), \( \theta - i\pi(1 - \xi) \rightarrow 0 \).

5 The operator \( \Lambda(\theta) \) is closely related with a current of the "deformed Virasoro algebra" \( \mathbb{I} \), \( \mathbb{I} \), \( \mathbb{I} \).
The averaging \( \langle \langle \cdots \rangle \rangle \) here is performed by Wick’s theorem and
\[
\langle \langle e^{i\omega(\theta)} \rangle \rangle = 1,
\]
\[
\langle \langle e^{i\omega(\theta_2)} e^{i\omega(\theta_1)} \rangle \rangle = R(\theta_1 - \theta_2),
\tag{29}
\]
where the function \( R(\theta) \) for \(-2\pi + \pi \xi < 3m \xi < -\pi \xi\) is given by the integral
\[
R(\theta) = N \exp \left\{ 8 \int_0^\infty \frac{dt}{t} \frac{\sinh(t) \sinh(t\xi) \sinh(t(1+\xi)) \sinh^2 t(1 - \frac{i\theta}{\pi})}{\sinh^2(2t)} \right\},
\tag{30}
\]
\[
N = \exp \left\{ 4 \int_0^\infty \frac{dt}{t} \frac{\sinh(t\xi) \sinh(t(1+\xi))}{\sinh^2(2t)} \right\}.
\tag{31}
\]
Notice the useful relations
\[
R(\theta)R(\theta \pm i\pi) = \frac{\sinh(\theta)}{\sinh(\theta) \mp i\sin(\pi\xi)}.
\tag{32}
\]
Using (28)-(32), one can easily derive the first \( B_1 \) form-factors of the exponential fields
\[
\langle 0 \mid e^{ia\varphi} \mid B_1(\theta) \rangle = -i G_a \lambda [a],
\]
\[
\langle 0 \mid e^{ia\varphi} \mid B_1(\theta_2)B_1(\theta_1) \rangle = -G_a \lambda^2 [a]^2 R(\theta_1 - \theta_2),
\]
\[
\langle 0 \mid e^{ia\varphi} \mid B_1(\theta_3)B_1(\theta_2)B_1(\theta_1) \rangle = i G_a \lambda^3 [a] \prod_{1 \leq k < j \leq 3} R(\theta_k - \theta_j) \times
\left\{ [a]^2 + \frac{x_1x_2x_3}{(x_1+x_2)(x_2+x_3)(x_1+x_3)} \right\},
\]
\[
\langle 0 \mid e^{ia\varphi} \mid B_1(\theta_4)B_1(\theta_3)B_1(\theta_2)B_1(\theta_1) \rangle = G_a \lambda^4 [a]^2 \prod_{1 \leq k < j \leq 4} R(\theta_k - \theta_j) \times
\left\{ [a]^2 + \frac{(x_1+x_2+x_3+x_4)^2 x_1x_2x_3x_4 + (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)^2}{(x_1+x_2)(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)(x_3+x_4)} \right\},
\tag{33}
\]
here \( x_k = e^{\theta_k} (k = 1, 2, 3, 4) \) and \( [a] = \frac{\sin(\frac{\pi a}{\sin(\pi\xi)})}{\sin(\pi\xi)} \). The \( B_1 \) breather form-factors admit (for fixed \( a \)) a power series expansion in \( \beta \) with finite radius of convergence. Therefore it is natural to assume that the form-factors of exponential fields in the sinh-Gordon model can be obtained from the expression in (28) by the continuation \( \beta \rightarrow ib \). A simple check shows that the analytical continuation of (33) leads to the form-factors of the field \( e^{a\varphi} \) in the sinh-Gordon model proposed in [15].

Finally, we note that form-factors of the heavier breathers follow from (28) by the well known bootstrap procedure [13], [4], [8].

Acknowledgments

I am grateful A. Zamolodchikov and Al. Zamolodchikov for interesting discussions. This research is supported in part by NSF grant.
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