BOHR–SOMMERFELD LAGRANGIANS OF MODULI SPACES OF HIGGS BUNDLES

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Abstract. Let $X$ be a compact connected Riemann surface of genus at least two. Let $M_H(r,d)$ denote the moduli space of semistable Higgs bundles on $X$ of rank $r$ and degree $d$. We prove that the compact complex Bohr–Sommerfeld Lagrangians of $M_H(r,d)$ are precisely the irreducible components of the nilpotent cone in $M_H(r,d)$. This generalizes to Higgs $G$–bundles and also to the parabolic Higgs bundles.

1. Introduction

Let $M_H(r,d)$ be the moduli space of semistable Higgs bundles of rank $r$ and degree $d$ on a compact connected Riemann surface $X$ of genus $g$ at least two. It is an irreducible normal complex projective variety of complex dimension $2(r^2(g-1)+1)$. This moduli space is equipped with an algebraic symplectic form. In fact, there is a canonical algebraic one-form $\omega$ on $M_H(r,d)$ such that $d\omega$ is the symplectic form. Let

$$ \mathcal{L} := \mathcal{O}_{M_H(r,d)} = M_H(r,d) \times \mathbb{C} $$

be the trivial holomorphic line bundle on $M_H(r,d)$. Consider the holomorphic connection

$$ D := d + \omega $$
on $\mathcal{L}$, where $d$ denotes the de Rham differential on functions on $M_H(r,d)$. We note that the curvature of $D$ is $d\omega$.

A compact Lagrangian on $M_H(r,d)$ is a reduced irreducible compact complex analytic subset

$$ \mathbb{L} \subset M_H(r,d) $$
of dimension $(\dim M_H(r,d))/2 = r^2(g-1)+1$ such that the restriction of $(\mathcal{L}, D)$ to $\mathbb{L}$ is a flat line bundle. A compact Bohr–Sommerfeld Lagrangian on $M_H(r,d)$ is a reduced irreducible compact complex analytic subset $\mathbb{L} \subset M_H(r,d)$ of dimension $r^2(g-1)+1$ such that $\mathcal{L}$ admits a nonzero flat section over $\mathbb{L}$. Clearly, a Bohr–Sommerfeld Lagrangian is a Lagrangian.

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A flat section of $\mathcal{L}$ over $\mathbb{L}$ is given by a holomorphic function on $\mathbb{L}$ because the form $\omega$ is holomorphic. On the other hand, $\mathbb{L}$ does not admit any nonconstant holomorphic function because it is compact and irreducible. A constant function is a flat section of $\mathcal{L}$ if and only if the pullback of the connection $D$ to $\mathcal{L}|_{\mathbb{L}}$ is the de Rham differential. Consequently, a reduced irreducible compact complex analytic subset

$$\iota : \mathbb{L} \rightarrow M_H(r,d)$$

is a Bohr–Sommerfeld Lagrangian if and only if $\iota^*\omega = 0$.

The Hitchin map

$$\mathcal{H} : M_H(r,d) \rightarrow \bigoplus_{i=1}^{r} H^0(X, K_X^{\otimes i})$$

sends any $(E, \theta)$ to $\sum_{i=1}^{r} \text{trace}(\theta^i)$. The fiber of $\mathcal{H}$ over 0 is the nilpotent cone.

We prove the following:

**Theorem 1.1.** *The compact Bohr–Sommerfeld Lagrangians in $M_H(r,d)$ are precisely the irreducible components of the nilpotent cone.*

Theorem 1.1 generalizes to other contexts; see Section 3.1.

## 2. One-form on the moduli space

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. The holomorphic cotangent bundle of $X$ will be denoted by $K_X$. Fix an integer $d$ and also fix a positive integer $r$. Let $M_H(r,d)$ denote the moduli space of semistable Higgs bundles on $X$ of rank $r$ and degree $d$. This $M_H(r,d)$ is an irreducible normal complex projective variety of complex dimension $2(r^2(g-1) + 1)$.

There is a natural algebraic 1-form on $M_H(r,d)$; we will recall its construction. Take any Higgs bundle $(E, \theta)$. Define

$$C^0 := \text{End}(E) = E \otimes E^\vee \quad \text{and} \quad C^1 := \text{End}(E) \otimes K_X.$$

Let $\text{ad}(\theta) : C^0 \rightarrow C^1$ be the homomorphism defined by

$$s \mapsto [s, \theta] = s \circ \theta - \theta \circ s,$$

where the composition is the usual composition of endomorphisms. Now consider the two-term complex

$$C^\bullet : C^0 \xrightarrow{\text{ad}(\theta)} C^1.$$  \hfill (2.1)

The infinitesimal deformations of the Higgs bundle $(E, \theta)$ are parametrized by the first hypercohomology $\mathbb{H}^1(C^\bullet)$ [Hi2], [BR], [Bo], [Ma].

The natural homomorphism of complexes

$$\begin{array}{ccc}
C^0 & \rightarrow & C^1 \\
\downarrow & & \downarrow \\
\text{End}(E) & \rightarrow & 0
\end{array}$$
induces a homomorphism
\[ q : H^1(C^\bullet) \longrightarrow H^1(X, C^0) = H^1(X, \text{End}(E)). \]  

(2.2)

We note that \( H^1(X, \text{End}(E)) \) parametrizes the infinitesimal deformations of the holomorphic vector bundle \( E \). Let

\[ SD : H^1(X, \text{End}(E)) \otimes H^0(X, \text{End}(E) \otimes K_X) \longrightarrow H^1(X, K_X) = \mathbb{C} \]

be the pairing given by Serre duality. Define

\[ \omega(E, \theta) : H^1(C^\bullet) \longrightarrow \mathbb{C}, \ \alpha \longmapsto SD(q(\alpha) \otimes \theta), \]

(2.3)

where \( q \) is the homomorphism in (2.2). The one-form \( \omega \) on \( M_H(r, d) \) defined by \((E, \theta) \longmapsto \omega(E, \theta)\) is algebraic, and \( d\omega \) is the algebraic symplectic form on \( M_H(r, d) \) [Hi1], [Hi2], [BR], [Bot], [Ma].

Let \( N^s(r, d) \) denote the moduli space of stable vector bundles on \( X \) of rank \( r \) and degree \( d \). The total space \( T^*N^s(r, d) \) of the (algebraic) cotangent bundle is a Zariski open subset of \( M_H(r, d) \). The restriction of \( \omega \) to \( T^*N^s(r, d) \) coincides with the Liouville one-form on \( T^*N^s(r, d) \).

For any \( \lambda \in \mathbb{C}^* \), let

\[ T_\lambda : M_H(r, d) \longrightarrow M_H(r, d), \ (E, \theta) \longmapsto (E, \lambda \cdot \theta) \]

(2.4)

be the automorphism. Clearly these together define an action of the multiplicative group \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) on \( M_H(r, d) \). Let

\[ \xi \in H^0(M_H(r, d), TM_H(r, d)) \]

be the vector field associated to this action of \( \mathbb{C}^* \).

**Lemma 2.1.** The one-form \( i_\xi d\omega \) obtained by contracting using the vector field \( \xi \) satisfies the equation

\[ i_\xi d\omega = \omega. \]

**Proof.** From the definitions of \( \omega \) and \( \xi \) it follows that

\[ \omega(\xi) = 0. \]  

(2.5)

Indeed, the action of \( \mathbb{C}^* \) does not change the underlying vector bundle, so \( q(\xi) = 0 \), where \( q \) is the homomorphism in (2.2). It also follows from the definitions of \( \omega \) and \( T_\lambda \) that \( T_\lambda^* \omega = \lambda \cdot \omega \). Therefore, we have

\[ L_\xi \omega = \omega. \]  

(2.6)

Consider the identity \( L_\xi \omega = i_\xi d\omega + d(\omega(\xi)) \). In view of (2.5) and (2.6), the lemma follows from it. \( \square \)
3. Bohr–Sommerfeld Lagrangians on the moduli space

Consider the symplectic form $d\omega$ on $M_H(r, d)$, where $\omega$ is constructed in (2.3). A \textit{compact complex Lagrangian} on $M_H(r, d)$ is a reduced irreducible compact complex analytic subset

$$\iota : \mathbb{L} \hookrightarrow M_H(r, d)$$

of dimension $r^2(g - 1) + 1$ such that $\iota^*d\omega = 0$. A \textit{compact Bohr–Sommerfeld Lagrangian} on $M_H(r, d)$ is a reduced irreducible compact complex analytic subset $\iota : \mathbb{L} \hookrightarrow M_H(r, d)$ of dimension $r^2(g - 1) + 1$ such that

$$\iota^*\omega = 0.$$

Since $\iota^*d\omega = d\iota^*\omega$, a Bohr–Sommerfeld Lagrangian is indeed a Lagrangian.

\textbf{Lemma 3.1.} Let $\iota : \mathbb{L} \hookrightarrow M_H(r, d)$ be a Bohr–Sommerfeld Lagrangian. For any smooth point $z \in \mathbb{L}$, the tangent vector

$$\xi(z) \in T_zM_H(r, d)$$

lies in the subspace $T_z\mathbb{L} \subset T_zM_H(r, d)$.

\textit{Proof.} Since $\mathbb{L}$ is a Bohr–Sommerfeld Lagrangian, we have

$$\omega(v) = 0, \quad \forall \ v \in T_z\mathbb{L}.$$  

Hence from Lemma 2.1 it follows that

$$d\omega(\xi(z), v) = 0, \quad \forall \ v \in T_z\mathbb{L}.$$  

Since $\mathbb{L}$ is Lagrangian for the symplectic form $d\omega$, this implies that $\xi(z) \in T_z\mathbb{L}$. \hfill \Box

Let

$$\mathcal{H} : M_H(r, d) \longrightarrow \mathcal{V} := \bigoplus_{i=1}^r H^0(X, K_X^{\otimes i}), \quad (E, \theta) \longmapsto \sum_{i=1}^r \text{trace}(\theta^i) \quad (3.1)$$

be the Hitchin map. Every irreducible component of every fiber of $\mathcal{H}$ is a compact Lagrangian $\text{Hi2}$. The fiber

$$\mathcal{N} := \mathcal{H}^{-1}(0) \subset M_H(r, d)$$

is known as the nilpotent cone.

\textbf{Proposition 3.2.} Any compact Bohr–Sommerfeld Lagrangian in $M_H(r, d)$ is contained in the nilpotent cone $\mathcal{N}$.

\textit{Proof.} For any $\lambda \in \mathbb{C}^*$, consider the linear automorphism of $\mathcal{V}$ in (3.1) defined by

$$(c_1, \cdots, c_i, \cdots , c_r) \longmapsto (\lambda \cdot c_1, \cdots, \lambda^i \cdot c_i, \cdots, \lambda^r \cdot c_r).$$

These together define an action of $\mathbb{C}^*$ on $\mathcal{V}$. The morphism $\mathcal{H}$ in (3.1) is clearly $\mathbb{C}^*$–equivariant (for the action on $M_H(r, d)$ in (2.4)).

Let

$$\iota : \mathbb{L} \hookrightarrow M_H(r, d)$$
be a compact Bohr–Sommerfeld Lagrangian. So $L$ does not admit any nonconstant holomorphic map to $V$. In particular, $H \circ \iota$ is a constant map. So

$$\iota(L) \subset H^{-1}(t_0)$$ (3.2)

for some point $t_0 \in V$.

From Lemma 3.1 it follows that $\iota(L)$ is preserved by the action of $\mathbb{C}^*$ on $M_H(r,d)$ in (2.4). Since $H$ is $\mathbb{C}^*$–equivariant, we conclude that $t_0$ in (3.2) is fixed by the action of $\mathbb{C}^*$ on $V$. This implies that $t_0 = 0$. 

**Theorem 3.3.** The compact Bohr–Sommerfeld Lagrangians in $M_H(r,d)$ are precisely the irreducible components of the nilpotent cone $N$.

**Proof.** In view of Proposition 3.2, it suffices to show each irreducible component $N$ is indeed a Bohr–Sommerfeld Lagrangian. Let

$$\iota : L \hookrightarrow N$$

be an irreducible component. We note that $L$ is compact because $H$ is a proper morphism [H1], [H2], [N1]. Associated to $L$ there is an integer $n \geq 2$ and pairs of integers $(r_1 d_1), \cdots, (r_n d_n)$ such that

(1) for any $(E, \theta) \in L$, there is filtration of subbundles

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E$$ (3.3)

such that $\text{rank}(E_i/E_{i-1}) = r_i$ and degree$(E_i/E_{i-1}) = d_i$ for all $1 \leq i \leq n$, and

(2) $\theta(E_i) \subset E_{i-1} \otimes K_X$ for all $1 \leq i \leq n$.

(See the proof of Theorem 5.3 in [BR, p. 228].)

Take a smooth point $z = (E, \theta) \in L$. We will describe the tangent space $T_z L$. Consider the filtration in (3.3). Let

$$\text{End}^p(E) \subset \text{End}(E)$$

be the subbundle defined by the condition that $\text{End}^p(E)(E_i) \subset E_i$ for all $1 \leq i \leq n$. Let

$$\text{End}^n(E) \subset \text{End}^p(E)$$

be the subbundle defined by the condition that $\text{End}^n(E)(E_i) \subset E_{i-1}$ for all $1 \leq i \leq n$. Let $D^*$ be the two-term complex

$$D^* : D^0 := \text{End}^p(E) \xrightarrow{\text{ad}(\theta)} D^1 := \text{End}^n(E) \otimes K_X,$$

where $\text{ad}(\theta)$ is the homomorphism in (2.1). Note that the condition $\theta(E_i) \subset E_{i-1} \otimes K_X$, $1 \leq i \leq n$, ensures that $\text{ad}(\theta)(\text{End}^p(E)) \subset \text{End}^n(E) \otimes K_X$. The inclusion of complexes

$$\begin{array}{c}
D^0 \xrightarrow{\text{ad}(\theta)} D^1 \\
\Downarrow \quad \Downarrow \\
\text{End}(E) \xrightarrow{\text{ad}(\theta)} \text{End}(E) \otimes K_X
\end{array}$$
induces a homomorphism of hypercohomologies
\[ \varphi : H^1(D^\bullet) \longrightarrow H^1(C^\bullet) \]
(see \ref{2.1}). We have
\[ T_zL = H^1(D^\bullet), \]
and the differential \[ dt : T_zL \rightarrow T_zM_H(r, d) \] coincides with the above homomorphism \( \varphi \).

We note that
- \([\text{End}^p(E), \text{End}^n(E)] \subset \text{End}^n(E)\), and
- the homomorphism
\[ \text{trace} : \text{End}(E) \longrightarrow \mathcal{O}_X \]
vanishes identically on \( \text{End}^n(E) \).

Therefore, for the homomorphism \( \omega(E, \theta) \) in \ref{2.3}, we have
\[ \omega(E, \theta) \circ \varphi = 0. \]
Hence \( i^*\omega = 0 \). In other words, \( L \) is a Bohr–Sommerfeld Lagrangian. \( \square \)

3.1. **Principal Higgs bundles and parabolic Higgs bundles.** Theorem \ref{3.3} generalizes to the moduli spaces of \( G \)-Higgs bundles of fixed topological type, where \( G \) is any reductive complex algebraic group. The proof goes through without any change. The only point to note is that if \( p \) is the Lie algebra of a parabolic subgroup \( P \) of \( G \), and \( B \) is any \( G \)-invariant nondegenerate symmetric bilinear form on the Lie algebra of \( G \), then the annihilator of \( p \), with respect to \( B \), is the Lie algebra of the unipotent radical of \( P \).

Theorem \ref{3.3} generalizes to the moduli spaces of parabolic vector bundles. In this case the assumption that \( g \geq 2 \) is not needed. It is enough to assume that there are stable parabolic Higgs bundles. Theorem \ref{3.3} also generalizes to parabolic analog of principal Higgs bundles; see \[ \text{BBN}, \text{Boa}, \text{PR}, \text{He} \] for parabolic analog of principal Higgs bundles.

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