Statistical mechanics of confined binary system: Comparison of three and two dimensions

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ABSTRACT
We have used a toy model to study the behaviour of confined binary systems in 2D and compare it with previously known result in 3D. In the case of 2D, in which canonical distribution exists only above a critical temperature, we evaluate the exact form of partition function for this system and compare the exact partition function with the mean field partition function for the case of two-particle system. In contrast of its 3D counterpart, there is no phase transition here. If this system, however, studied in microcanonical ensemble, it shows two different phases of kinetic energy dominated with positive specific heat and potential energy dominated with negative specific heat in presence of short distance cutoff. In absence of short distance cutoff, surprisingly, the negative specific heat region will be replaced by region of large specific heat. This feature is completely new and there is no such a case in 3D.

Key words: gravitation, binaries: general

1 INTRODUCTION
The statistical behaviour of \(N\) particles interacting through Newtonian gravitational forces is very different from the statistical behaviour of other many body systems such as neutral gases and plasmas. The central feature of gravitating system, in contrast to normal many body systems, is the unshielded, long range nature of gravitational force. Because of this feature one of the fundamental concepts of statistical mechanics, the extensive nature of energy, breaks down. This, in turn, leads to different physical descriptions for the gravitating systems in the microcanonical and canonical distribution.

The statistical behaviour also strongly depends on the spatial dimension. For instance, in 3D, the available phase volume for the system diverges and one is forced to use short distance cutoff. However the situation in 2D is different. In this case there is a microcanonical description for all values of energies, through the canonical approach exists only above some critical temperature.

We shall study some properties of these gravitating systems by introducing a toy-model (originally used in Padmanabhan, 1990), based on a simple Hamiltonian, describing two particles of finite size, confined inside a box. This system shows several important properties of more complicated systems studied for example in D. Lynden-Bell and R. M. Lyndel-Bell [1977]. We will study this toy model in both the microcanonical and canonical approach.

In section 2 we will overview the properties of the toy model in 3D, [which was earlier done in Padmanabhan, (1990)] for providing the background needed for comparison with the 2D case, which will be studied in section 3. In section 4, we compare the nature of this system in 3D and 2D.

An interesting feature of this 2D system is that the thermodynamic functions are all calculable analytically. This contrasts with the thermodynamical behaviour of 3D confined binary system. By studying the “isothermal cylinders” we find that these systems are remarkably similar to the simple toy model and that the system cannot exist at \(T < T_\text{c}\), where \(T_\text{c}\) is given by \(T_\text{c} = (1/2)Gm^2\). We also find that by introducing a short distance cutoff, in contrast to 3D case, the specific heat of the system would become negative in some intermediate temperatures.

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2 OVERVIEW OF 3D CASE

In this section we introduce and review a toy model which was first given by Padmanabhan (1990). In this model, by constructing the “statistical mechanics” of two particles of finite size confined inside a box and interacting via 3D gravity are described. This system exhibits several important properties of more complicated gravitating systems in spite of the fact that it has only two particles. In particular, this system exhibits the following two features, which seem to be generic to all gravitating systems: (i) When studied using the microcanonical ensemble, the system shows evidence for two different phases: a high temperature phase, dominated by kinetic energy, and a low temperature phase dominated by the potential energy and stabilised by some short distance cutoff which is of non-gravitational origin. Both these phases have positive specific heat. These two phases are connected at intermediate temperatures by a region of negative specific heat; this is precisely the range in which the kinetic and potential energies of the system are comparable and the system is in virial equilibrium. (ii) If the same system is studied using the canonical ensemble, the intermediate region of negative specific heat is replaced by a sharp phase transition.

Then, in order to show that this toy model exhibits several important properties of more complicated gravitating systems, we shall compare the equilibrium of a gravitating system in the mean field limit, by evaluating the partition function for this system. In the absence of any short distance cutoff to gravitational interaction, the mean field solution is given by an isothermal sphere. It turns out that this system is remarkably similar to our simple toy model studied earlier. In fact, this mean field analysis confirms all the conjectures based on the toy model.

2.1 Microcanonical description

We will begin by studying the statistical mechanics of our toy model described by the Hamiltonian

\[ H(P,Q,p,r) = \frac{p^2}{2M} + \frac{P^2}{2\mu} - \frac{Gm^2}{r}, \]

where \((Q,P)\) are coordinates and momenta of the centre of mass, \((r,p)\) are the relative coordinates and momenta, \(M=2m\) is the total mass, \(\mu=(m/2)\) is the reduced mass and \(m\) is the mass of individual particles. The range of \(r\) varies in the interval \((a,R)\). This is equivalent to assuming that the particles are hard spheres of radius \((a/2)\) and that the system is confined to a spherical box of radius \(R\). In microcanonical distribution corresponding to this toy model (which is the relevant ensemble to the gravitating systems) we define the entropy \(S(E)\) and temperature \(T(E)\) through the relations

\[ S(E) = \ln g(E), \quad T(E) = (\partial S(E)/\partial E)^{-1} = (\partial \ln g(E)/\partial E)^{-1}. \]

For our system the phase volume \(g(E)\) of the constant energy surface \(H=E\) is

\[ g(E) = \begin{cases} \frac{4}{3} R^3 (-E)^{-1} (1 + aE/Gm^2)^3, & -Gm^2/a < E < -Gm^2/R, \\ \frac{4}{3} R^3 (-E)^{-1} [(1 + RE/Gm^2)^3 - (1 + aE/Gm^2)^3], & -Gm^2/R < E < \infty. \end{cases} \]

This function \(g(E)\) is continuous and smooth at \(E = (-Gm^2/R)\). Now the thermodynamic properties of the system can be analyzed from the \(T(E)\) curve. In the range when \(-Gm^2/a < E < -Gm^2/R\) we can write \(T(E)\) in the dimensionless form

\[ t(\varepsilon) = \left( \frac{3}{1 + \varepsilon - \frac{1}{\varepsilon}} \right)^{-1}, \]

where \(t = (aT/Gm^2)\) and \(\varepsilon = (aE/Gm^2)\).

At \(\varepsilon = -1\) which corresponds to the lowest energy admissible for the system, the dimensionless temperature \(t\) vanishes. It is obvious that in this case of \(\varepsilon \approx -1\), \(t(\varepsilon)\) dominated by the first term of \(\frac{3}{1 + \varepsilon - \frac{1}{\varepsilon}}\). As we increase the energy of the system, the temperature increases, which is the normal behaviour for the system. This trend continues up to

\[ \varepsilon = \varepsilon_1 = - \frac{1}{2} (\sqrt{3} - 1) \approx -0.36, \]

at which point the \(t(\varepsilon)\) curve reaches a maximum and turns around. As we increase the energy further the temperature decreases. The system exhibits negative specific heat in this range.

As one can see from (3), for realistic systems, \(R \gg a\) only a small region in the range of \(-Gm^2/a\) to \(-0.36Gm^2/a\) we will have positive specific heat; for the rest of the region the specific heat is negative. In fact the existence of the positive specific heat region is due to nonzero short distance cutoff. In the absence of this nonzero short distance cutoff, the first term in (3) will vanish and we will get \(t \propto -\varepsilon^{-1}\) and negative specific heat in this entire region.

For high energy limits, \(E \geq -Gm^2/R\), the second expression in (3) for \(g(E)\) will give

\[ t(\varepsilon) = \left( \frac{3[(1 + \varepsilon)^2 - (R/a)[1 + (R/a)e]^2]}{(1 + \varepsilon)^3 - [1 + (R/a)e]^3} - \frac{1}{\varepsilon} \right)^{-1}. \]
This function will match with (4) at $\varepsilon = -(a/R)$. It will decreases as we increase the energy, for a while, and then very soon it starts to increases as energy increases at some $\varepsilon = \varepsilon_2$. Thus system will enter another phase with positive specific heat. The form of $t(\varepsilon)$ is shown in fig.1. The specific heat is positive along the portions $AB$ and $CD$ and is negative along $BC$.

For $E \gg E_2 = -(Gm^2/R)$, gravity is not strong enough to keep $r < R$ and the system behaves like a gas confined by the container; we have high temperature phase with positive specific heat. As the energy decreases to $E \leq E_2$, the effects of gravity begin to be felt. For $E_1 = -(Gm^2/R) < E < E_2$, the system is unaffected by either the box or the short distance cutoff; this is the domain dominated entirely by gravity and we have negative specific heat. As system goes to $E \simeq E_1$, the hard core nature of the particles begins to be felt and gravity is again resisted. This give rise to a low temperature phase with positive specific heat.

It is also interesting to study (i) the effect of increasing $R$, keeping $a$ and $E$ fixed, and (ii) the effect of varying $a$, keeping $R$ fixed. In former case, it is amusing to note that, if $2 < R/a < (\sqrt{3} + 1)$, there is no region of negative specific heat. As we increase $R$, this negative specific heat region appears. In the latter case, at first one should rescale variables using $(Gm^2/R)$. This can be easily done and fig.2 shows the behaviour of the $T(E)$ curve as the lower cutoff $a$ is changed. As $a$ is lowered, the negative specific heat region becomes more and more pronounced. If $a$ is zero, we have negative specific heat for all $E < -(Gm^2/R)$ (see fig.3).

2.2 Canonical description

It is of interest to look at our system from the point of view of the canonical distribution. To do this we have to compute the partition function

$$Z(\beta) = \int d^3P \, d^3p \, d^3Q \, d^3r \, \exp(-\beta H),$$

which after integrating over $P, p$ and $Q$ and omitting an overall constant, which is unimportant, in dimensionless form becomes

$$Z(t) = t^3(R/a)^3 \int_1^{R/a} dx \, x^2 \, \exp(1/xt),$$

where $t$ is the dimensionless temperature defined before. One can show that $Z(t)$ can be well approximated by the expression

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Figure 2. The $T(E)$ curve on the short distance cutoff $a$. For $a > (3^{1/2} + 1)^{-1}R$ there is no region of negative specific heat. As $a$ is lowered, the negative specific heat region becomes more and more pronounced.

Figure 3. The $T(E)$ curve for $a = 0$. The low temperature region with positive specific heat does not exist in the absence of the short distance cutoff. The system also exhibits a lower bound on the temperature.
Figure 4. Comparison of the $T(E)$ relations for the canonical and microcanonical distributions. The negative temperature region of the microcanonical distribution is replaced by a phase transition in the canonical distribution. The microcanonical temperature is replaced by a factor $2/3$ for convenience of comparison.

\[
Z \approx \begin{cases} 
(R/a)^3 t^4 (1 - 2t)^{-1} \exp(1/t), & \text{for } t < t_c, \\
\frac{1}{3} t^3 (R/a)^6 (1 + 3a/2Rt), & \text{for } t > t_c.
\end{cases}
\] 

(9)

where $t_c = [3 \ln(R/a)]^{-1}$, is the critical temperature at which the transition occurs. Given $Z(\beta)$ one can compute $E(\beta)$ by the relation $E(\beta) = -(\partial \ln Z/\partial \beta)$. This relation can converted to give $T(E)$, which can be compared with the $T(E)$ obtained earlier using the microcanonical distribution. From (9) we get

\[
\varepsilon \approx \begin{cases} 
aE/Gm^2 = 4t - 1, & \text{for } t < t_c, \\
3t - 3a/2R, & \text{for } t > t_c.
\end{cases}
\] 

(10)

Near $t \approx t_c$ is a rapid variation of the energy and we cannot use either asymptotic form. The system undergoes a phase transition at $t = t_c$ absorbing a large amount of energy,

\[
\Delta \varepsilon \approx 1 - \frac{1}{3 \ln(R/a)}.
\] 

(11)

The specific heat is, of course, positive throughout the range. This is to be expected because the canonical description cannot lead to negative specific heats.

The exact $T(E)$ curves obtained from the canonical and microcanonical distributions are shown in fig.4. The descriptions match very well in the regions of positive specific heat. The negative specific heat region of the microcanonical distribution is replaced by a phase transition in the canonical description.

The physics of canonical distribution is best understood by studying $E$ as a function of $T$. As we increase the temperature from zero to, the energy increases from the ground state value $-(Gm^2/a)$ in accordance with first equation of (10). As the temperature approaches $T_c$ and cross it, a phase transition occurs in the system and the energy increases rapidly. The latent heat in the system is large enough to push the system into the high temperature phase. At still higher temperatures, the energy increases steadily with the temperature in accordance with second equation of (10).

We can now compare the canonical and microcanonical descriptions for our system. At both very low and very high temperatures, the descriptions match. The crucial difference occurs at the intermediate energies and temperatures. The microcanonical description predicts a negative specific heat and a reasonably slow variation of energy with temperature. The
Figure 5. The $u - v$ curve for the isothermal sphere without any short distance cutoff.

canonical description, on the other hand, predicts a phase transition with a rapid variation of energy with temperature. Such phase transitions are accompanied by large fluctuations in the energy, which is the main reason for disagreement between the two descriptions.

In order to confirm all the conjectures made earlier based on the toy model we shall briefly explain the equilibrium of a gravitating system in the mean field limit. In the absence of any short distance cutoff to gravitational interaction, the mean field solution is given by an isothermal sphere. In this case we may write,

$$\nabla^2 \phi = 4\pi G \rho_c e^{-\beta[\phi(x) - \phi_c]}, \quad (12)$$

where $\rho_c = \rho(0)$.

One may introduce the following dimensionless variables in order to write the isothermal equation,

$$x \equiv r/L_0, \quad n \equiv \rho/\rho_c, \quad m \equiv M(r)/M_0, \quad y \equiv \beta[\phi - \phi_c]. \quad (13)$$

Then in terms of $y(x)$ the isothermal equation becomes

$$\frac{1}{x^2} \frac{d}{dx} \left( x^3 \frac{dy}{dx} \right) = e^{-y}, \quad (14)$$

with the boundary condition $y(0) = y'(0) = 0$. By defining the new variables as

$$v \equiv m/x, \quad u \equiv nx^3/m = nx^2/v, \quad (15)$$

one can express the isothermal equation as the following:

$$\frac{u}{v} \frac{dv}{du} = -\frac{u - 1}{u + v - 3}, \quad (16)$$

with the boundary conditions $v = 0$ at $u = 3$ and $dv/du = -(5/3)$ at $(u, v) = (3, 0)$.

The solution curve starts at $(3, 0)$ and spirals indefinitely around the point $(1, 2)$ as $x$ tends to infinity (see fig. 5). All isothermal spheres must necessarily lie on the this curve.

The $u - v$ curve also implies a bound on the temperature of the system [Lynden-Bell and Wood (1968)]. It is clear that for any isothermal sphere $v$ is bounded from above, i.e., $v < v_{\text{max}}$, where $v_{\text{max}} \approx 2.5$. Since

$$v = m/x = (M/M_0)(R/L_0) = (GM/R)\beta, \quad (17)$$
Confined binary system

Figure 6. The $T(E)$ curve for the isothermal sphere without any short distance cutoff. The specific heat is positive along $AB$ and negative along $BC$.

We immediately get

$$T > T_{\text{min}}, \quad T_{\text{min}} \simeq 0.4GM^2 / R. \quad (18)$$

The $T(E)$ curve for the isothermal sphere can be determined by combining the relations

$$T = (GM^2/R)u^{-1}, \quad E = (GM^2/R)u^{-1}(u - \frac{3}{2}). \quad (19)$$

The form of the curve is shown in fig. 6. The specific heat is positive along the $AB$ (which represents a high temperature phase) and is negative along $BC$. The branch $CD$ is unstable and is not physically realisable. Since we have not introduced any short distance cutoff in the system we do not have a low temperature phase with positive specific heat. If we do that, a short distance cutoff makes the curve turn to the left on the $CD$ branch and produces a second region of positive specific heat. Figure 6 is analogous to fig. 3, which describes the $T(E)$ curve for a binary model in the absence of a short distance cutoff. The modification of fig. 6 in the presence of a short distance cutoff will be very similar to the modification of fig. 3 into fig. 3 (For example see fig. 4.11 in Padmanabhan, 1990).

3 THE CASE OF 2D BINARY

We shall now derive corresponding results for the 2D confined binary system. We will see that the thermodynamic functions are all analytically calculable for this case. It is also interesting to contrast the thermodynamical behaviour of the 3D confined binary system with the different, but eventful, 2D one.

3.1 Microcanonical approach

Let us consider two particles interacting via 2D gravitational force and moving in a 2D region of radius $R$. The potential in 2D gravity satisfies the Poisson equation:

$$\nabla^2 \phi = 2\pi G \rho. \quad (20)$$

For point particle $\phi$ will be logarithmic; the potential energy of interaction between two such particles will be,

$$U(x_1, x_2) = Gm^2 \ln \frac{|x_2 - x_1|}{R}. \quad (21)$$
Figure 7. The $T(E)$ curve for the low energies region for 2D. In spite of existence of short distance cutoff, there is a region which specific heat is negative, although very small.

We shall first study the microcanonical distribution corresponding to our system. The phase volume $g(E)$, which is the volume of the constant energy surface, in the phase space:

$$g(E) = \int \prod_{i=1}^{2} d^2x_i d^2p_i \delta(E - H), \quad (22)$$

where, $H$, is the Hamiltonian for a 2D gravitating system of 2 particles with logarithmic potential, defined as,

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + Gm^2 \ln \left( \frac{r}{R} \right), \quad (23)$$

where, as in the 3D case, $(Q, P)$ are the coordinates and momenta of the centre of mass, $(r, p)$ are the relative coordinates and momenta, $M = 2m$, is the total mass, $\mu = (m/2)$ is the reduced mass and $m$ is the mass of the individual particles, we shall further restrict the range of the coordinate $r$ to the interval $(a, R)$. Where, $a$, is the short distance cutoff equivalent to assuming that particles are hard spheres of radius $(a/2)$, while, $R$, is the large distance cutoff equal to the confining radius of the system. Therefore, the phase volume $g(E)$ becomes,

$$g(E) = AR^2 \int_{a}^{r_{\text{max}}} r \left[ E - Gm^2 \ln \left( \frac{r}{R} \right) \right] dr. \quad (24)$$

With the range of integration in $(24)$, limited to the region in which the expression in parentheses is positive, i.e., $[E - Gm^2 \ln (r/R)] > 0$. Therefore,

$$r_{\text{max}} = \begin{cases} R \exp (E/Gm^2), & -Gm^2 \ln (R/a) < E < 0, \\ R, & 0 < E < \infty. \end{cases} \quad (25)$$

$A$ is some constant which is irrelevant to our discussion.

The integration, then, will yield the following result:

$$\frac{g(E)}{Gm^2} = \begin{cases} \frac{1}{4} AR^2 \{ R^2 e^{2E/Gm^2} - a^2 (2E/Gm^2 + 2 \ln (R/a) + 1) \}, & -Gm^2 \ln (R/a) < E < 0, \\ \frac{1}{3} AR^2 \{ R^2 (2E/Gm^2 + 1) - a^2 (2E/Gm^2 + 2 \ln (R/a) + 1) \}, & 0 < E < \infty. \end{cases} \quad (26)$$
Figure 8. The $T(E)$ curve. Note that the negative specific heat region is not visible due to its smallness compared to the high energy region.

Figure 9. The $T(E)$ curve for different short distance cutoff $a$. As one decreases $R$, the negative specific heat will become more and more pronounced.
Figure 10. The $T(E)$ curve for the special case, $a = 0$. The system does not exhibit any negative specific heat. Compare this figure with the fig. 3.

It is obvious that $g(E)$ is continuous and smooth at $E = 0$. We shall study the thermodynamics of the system, which can be now studied using $g(E)$, in the two regimes given above.

The entropy of the system, $S(E) = \ln g(E)$ in the case of very low energies, i.e., $Gm^2 \ln(a/R) < E < 0$ will be,

$$S(E) = \frac{1}{4} AGm^2 R^2 + \ln \left[ R^2 e^{2E/Gm^2} - a^2 \left( \frac{2E}{Gm^2} + 2 \ln \left( \frac{R}{a} \right) + 1 \right) \right],$$

and the temperature of the system, $T(E) = \frac{1}{S'}$ with the help of (27) is,

$$T(E) = Gm^2 \left( \frac{1}{2} - \frac{[E/Gm^2 + \ln(R/a)]}{(R/a)^2 e^{2E/Gm^2} - 1} \right),$$

or in dimensionless form as,

$$t(\varepsilon) = \frac{1}{2 \ln(R/a)} + \frac{(\varepsilon + 1)}{1 - (R/a)^{2(1-\varepsilon)}}$$

where we have defined $t(\varepsilon) = T(E)/Gm^2 \ln(R/a)$, and $\varepsilon = E/Gm^2 \ln(R/a)$. At the lowest energy admissible for our system, which corresponds to $\varepsilon = -1$, the temperature is $t(\varepsilon) = \frac{1}{2 \ln(R/a)}$. It is clear that in (29) for $\varepsilon \approx -1$ the first term dominates. So as we increase the energy of the system, the temperature decreases. This behaviour continues up to $\varepsilon = 0$ at which the point $t(\varepsilon)$ curve reaches to it’s minimum, $t(\varepsilon) = 1/(2 \ln(R/a)) - a^2/(R^2 - a^2)$. Therefore, we obtain a negative specific heat region for $-1 < \varepsilon < 0$ (see fig. 8).

For $E \geq 0$ we should use the second expression in (26) for $g(E)$. In this case the proper expression for $S(E)$ would be,

$$S(E) = \frac{1}{4} AGm^2 R^2 + \ln \left[ R^2 \left( \frac{2E}{Gm^2} + 1 \right) - a^2 \left( \frac{2E}{Gm^2} + 2 \ln \left( \frac{R}{a} \right) + 1 \right) \right],$$

and thus we get

$$t(\varepsilon) = \varepsilon + \frac{1}{2 \ln(R/a)} + \frac{1}{1 - (R/a)^2}$$

This function, clearly, matches with (29) at $\varepsilon = 0$. As we increase the energy, the temperature continues to increase. Thus in high temperature phase we enter to the positive specific heat region. The form of $t(\varepsilon)$ is shown in fig. 8. (Due to smallness of negative specific heat region compared to high energy limit, the negative specific heat region is not visible in the figure). However, as we decrease the long distance cutoff $R$, the negative specific heat region becomes more and more pronounce, see
If we set $a = 0$, we easily get,

$$
\frac{T(E)}{Gm^2} = \begin{cases} 
1/2, & -\infty < E < 0, \\
1/2 + E/Gm^2, & 0 < E < \infty. 
\end{cases}
$$

(32)

It is obvious that in the absence of the short distance cutoff, there will be no negative specific heat region(fig.10). Comparing this result with its counterpart in 3D case for $a = 0$ shows that negative specific heat replaced by constant temperatures. In fact, the effect of short distance cutoff in 2D case is just destabilising the effect of gravitational potential energy, unlike its effect in 3D potential. By putting a short distance cutoff, we actually distort the phase space. In other words, having a short distance cutoff in the system is equivalent to removing some part of phase space. This distortion in phase space may cause many unexpected consequences. In 2D, for instance, removing some part of phase space causes an unexpected region of negative specific heat, whereas for 3D, we get region of positive specific heat. Comparison of 2D and 3D with short distance cut off, though, suggests the $E - T$ graph in 2D in many aspects is similar to its 3D counterpart, except that in 2D there is no initial positive specific heat region. In other words, the $E - T$ graph in 2D is almost same as the $E - T$ graph in 3D, but as if the graph has been shifted to the right: the starting point of the 2D graph almost coincides with the maximum of 3D one.

### 3.2 Canonical approach

Let us now consider the partition function $Z(\beta)$ of the system, which is given by the integral,

$$
Z(\beta) = \int_{-\infty}^{+\infty} dE \, g(E)e^{-\beta E} = \int_{-\infty}^{0} dE \, g(E)e^{-\beta E} + \int_{0}^{+\infty} dE \, g(E)e^{-\beta E} \equiv Z_1 + Z_2
$$

(33)

The range of integration is from $(-\infty)$ to $(+\infty)$, since negative values of $E$ are allowed. Thus in our case, that interval will break to two separate areas. We have for $Z_1$:

$$
Z_1(\beta) = \frac{1}{4}AR^4Gm^2\int_{-\infty}^{0} dE \, exp\left[2/Gm^2 - \beta E\right]
$$

The above integrand will diverge in the lower limit, unless,

$$
\beta < \frac{2}{Gm^2}.
$$

(35)

Then we obtain,

$$
Z_1(\beta) = \frac{1}{4}AR^4\frac{(Gm^2)^2}{2 - Gm^2\beta}.
$$

(36)

Since $Z_1(\beta)$ diverges at $Gm^2\beta_c = 2$, the system can exist only at $\beta < \beta_c$. As for $Z_2$,

$$
Z_2(\beta) = \frac{1}{4}AR^4\int_{0}^{+\infty} dE(Gm^2 + 2E)e^{-\beta E}
$$

$$
= \frac{1}{2}A\left(\frac{R^2}{\beta}\right)^2\left(1 + \frac{Gm^2}{2}\right),
$$

(37)

which is the $Z(\beta)$ as one can obtain from saddle-point approximation [Padmanabhan 1991]. Thus, we see that the saddle-point approximation is accurate at high temperatures, i.e. for $Gm^2\beta \ll 1$, and the mean field approximation must break down at low temperatures. If now we introduce the short distance cutoff, a, the corresponding $Z_1$ and $Z_2$ will be modified as

$$
Z_1(\beta) = \frac{1}{4}AR^4\frac{(Gm^2)^2}{2 - Gm^2\beta} - \frac{1}{2}A\left(\frac{Ra}{\beta}\right)^2\left[1 + \frac{1}{2}Gm^2[2\ln(R/a) + 1]\beta\right],
$$

(38)

and

$$
Z_2(\beta) = \frac{1}{4}AR^4\int_{0}^{+\infty} dE(Gm^2 + 2E)e^{-\beta E}
$$

$$
= \frac{1}{2}A\left(\frac{R^2}{\beta}\right)^2\left(1 + \frac{Gm^2}{2}\right) - \frac{1}{2}A\left(\frac{Ra}{\beta}\right)^2\left[1 + \frac{1}{2}Gm^2[2\ln(R/a) + 1]\beta\right].
$$

(39)

Given $Z(T)$, one can compute the mean energy of the system, which is given by

$$
E(T) = T^2(\partial \ln Z/\partial T),
$$

(40)
Figure 11. The $T/T_c$ as a function of the total energy of two dimensional gravitating system. There is no region of negative specific heat but the system exhibits a lower bound on the temperature.

or in dimensionless form as

$$\varepsilon = t^2 (\partial \ln Z / \partial t),$$

where for high energies, regardless of whether there is a short distance cutoff, the dimensionless energy scales as

$$\varepsilon_2 \simeq \frac{2t^2}{2t + 1} \sim t.$$

However at $t \approx t_c = 1/[2 \ln(R/a)]$, energy diverges. Thus, again at $Gm^2\beta_c = 2$, the partition function will blows up even in the presence of short distance cutoff $a$ and in this case the system cannot exist at $\beta > \beta_c$, as well. In other words, the canonical description of the system exists only at sufficiently high temperatures (Katz and Lynden-Bell 1978). Note that there is no phase transition for this two dimensional system either with or without short distance cutoff. As the energy of the system is lowered, the temperature continuously decreases and asymptotically approaches $T_c = (1/\beta_c)$.

Comparison of the canonical and microcanonical descriptions of our model shows that at very high temperatures, the descriptions match. The main difference occurs at low temperatures and energies. The microcanonical description predicts either infinite or negative specific heat, depend on whether there is a short distance cutoff, at low energies, whereas in canonical approach, there is no physical state below a critical temperature. Therefore, for low energies the two approaches disagree.

As we compared the result of 3D binary with isothermal sphere, we shall outline the result of studying isothermal cylinders in order to confirm our earlier claim based on 2D binary.

An isothermal self-gravitating cylinder is described by Poisson’s equation, (20), in 2D,

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\varphi}{dr} \right) = 2\pi G\rho(r),$$

with $\rho(r)$ given by

$$\rho(r) = A\exp[-\beta\varphi(r)],$$

where the constants $A$, and $\beta$ are to be determined in terms of total mass $M$ and energy $E$ of the system. Regular solution of (3) have been given in Ostriker (1964), and Stodolkiewicz (1963):

$$\varphi(r) = GM \ln R + 2\beta^{-1} \ln \left[ 1 - \frac{1}{4} GM\beta(1 - r^2/R^2) \right] + \text{constant},$$

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where $R$ is the radius of the confining box. The potential on the axes, $R = 0$, is real if $(1/4)GM\beta$ is smaller than unity and becomes infinite in the limit $(1/4)GM\beta = 1$. Thus, there exist a lower bound, $T_c$. Now all other physical variables like density $\rho(r)$, pressure on the confining wall $P(R) = \beta^{-1}\rho(r)$, and the energy

$$E = M\beta^{-1} + \frac{1}{2} \int \rho \phi dx,$$

(46)

can be computed from (45). We get

$$E = \frac{1}{4} GM^2 \left[ 2 \ln R + 2(T/T_c) + (T/T_c)^2 \ln(1 - T_c/T) + \text{constant} \right],$$

(47)

$$PV = N(T - T_c),$$

(48)

where $T_c = (1/4)NGM^2$ is the same critical temperature as that found earlier (for $N = 2$) in the canonical approach. It is clear the system can not exist for $T < T_c$, the pressure becomes negative, and the potential at the origin diverges.

The $T(E)$ curve for this system is shown in fig.1. It is clear that there is no region of negative specific heat for this two dimensional system. As $E$ is lowered, the $T(E)$ continuously decreases, however, $T$ will never reach $T_c$ since for that to happen, an infinite amount of energy has to be given away. As $T$ tends to $T_c$ the pressure on the wall tends to zero, see equation (48), the density in the centre grows continuously as

$$\rho(0) = \frac{M}{V}(1 - T_c/T)^{-1},$$

(49)

and the density contrast grows as,

$$\rho(0)/\rho(R) = (1 - T_c/T)^{-2}.$$

(50)

The system, after shrinking, collapses to thin dense string. Therefore, isothermal cylinders in contact with heat bath whose temperature $T$ is slightly smaller than $T_c$ are unstable and giving up an unlimited amount of energy (Katz and Lynden-Bell 1978). However, a comparison with the case of $3D$ binaries shows that, two-dimensional systems are more stable than three-dimensional ones. As we obtained in eq. (43) there also exists a $T_c$ for isothermal spheres, in which there is no equilibrium. If an isothermal sphere has a density contrast less than 32 (Lynden-Bell and Wood 1968) and temperature slightly hotter than $T_c$, while the density contrast grows, it will lose energy to heat bath and cool down. As the density contrast keeps growing above 32, the specific heat becomes negative. When density contrast reaches the value 709 (Katz and Lynden-Bell 1978), and becomes very hot, the system is unstable and not physically realisable. Then after, isothermal starts collapsing in which the centre of the system becomes smaller and hotter whereas giving up energy to the outside parts of the isothermal sphere.

Instabilities in isothermal spheres are mainly due to “wiggling” of potential around singular solution (see, e.g. fig.6). There is no singular solution and thus wiggling of potential in $2D$ (see fig.12).

4 CONCLUSION

It seems a binary system exhibits several important properties of more complicated gravitating systems in spite of the fact it has only two degrees of freedom. In particular, this system exhibits the following two features, which seems to be generic to all gravitating systems: (i) When studied using the microcanonical ensemble, the system shows evidence for two different phases: a high temperature phase, dominated by kinetic energy, and a low temperature phase dominated by the potential energy and (de)stabilised by some short distance cutoff in $(2)3D$ which is of non-gravitational origin, mainly due to distortion of phase space. Both these phases have positive specific heat in $3D$ whereas in $2D$ the latter phase has negligible negative specific heat for the case of $(a/R) \ll 1$. In $3D$ case, these two phases are connected at intermediate temperatures by a region of negative specific heat; this is precisely the range in which the kinetic and potential energies are comparable and the system is in virial equilibrium. (ii) If the system is studied using canonical ensemble, the intermediate region of negative specific heat in $3D$ is replaced by a sharp phase transition releasing a large amount of latent heat. This suggests the following analogy: Gravitating systems in virial equilibrium are similar to normal systems (with short range forces) at the verge of phase transition. For the case of $2D$, however, the canonical description leads to the completely different picture in low energies. The system does not exist below some critical picture $T_c = (1/2)Gm^2$.

On the other hand, the isothermal considerations reveal the similarity of results in this context with these simple toy model binaries. In fact, the mean field analysis confirms all the conjectures made earlier based on the binaries.

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Figure 12. The isothermal cylinder density curve. There is no singularity in contrast to isothermal sphere density.

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