Some Properties of Fuzzy Compact Algebra Fuzzy Normed Spaces and Finite Dimensional Algebra Fuzzy Normed Spaces

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Abstract. Our goal in this article is to recall the notion of algebra fuzzy normed space and its basic properties to introduce the notion of fuzzy compact algebra fuzzy normed space. Then some properties of fuzzy compact algebra fuzzy normed spaces are proved. After that, we study a finite dimensional algebra fuzzy normed space and we proved some properties that algebra fuzzy normed spaces do not admit it.

1. Introduction

In 2009 [1] the fuzzy topological structure of a fuzzy normed space was studied by Sadeqi and Kia. In 2011 [2] Kider introduced a fuzzy normed space. Also he proved this fuzzy normed space has a completion in [3]. Again in 2012 [4] Kider introduced a new type of fuzzy normed space. In 2013 [5] Bag and Samanta study some basic results on finite dimensional fuzzy normed linear spaces. In 2017 [6] Kider and Kadhum introduced the fuzzy norm for a fuzzy bounded operator on a fuzzy normed space and proved its basic properties then other properties were proved by Kadhum in 2017 [7]. In 2018 [8] Ali proved basic properties of complete fuzzy normed algebra. Again in 2018 [9] Kider and Ali introduced the notion of fuzzy absolute value and study some properties of finite dimensional fuzzy normed space.

The concept of general fuzzy normed space was presented by Kider and Gheeab in 2019 [10] [11] also they proved basic properties of this space and the general fuzzy normed space GFB (V, U). In 2019 [12] Kider and Kadhum introduced the notion fuzzy compact linear operator and proved its basic properties. In 2020 kider [13] introduced the notion fuzzy soft metric space after that he investigated and proved some basic properties of this space again kider in 2020 [14] introduced a new type of fuzzy metric space called algebra fuzzy metric space after that the basic properties of this space is proved.

In this paper, first we recall the notion algebra fuzzy absolute value space and its some basic properties that introduced by Khudhair and Kider in [15]. Then, we recall the notion algebra fuzzy normed space and its some basic properties that introduced by Khudhair and Kider in [15]. Our first aim is to define fuzzy compact algebra fuzzy normed space then we study this space and proved some of its basic properties.

The second aim is to assume that the algebra fuzzy normed space has a finite dimension and proved some basic results that this type of spaces must have.
2. Basic properties of algebra fuzzy normed space

Definition 2.1: [1]
Assume that $S \neq \emptyset$, a fuzzy set $\tilde{D}$ in $S$ is represented by $\tilde{D} = \{(s, \mu_{\tilde{D}}(s)) : s \in S, 0 \leq \mu_{\tilde{D}}(s) \leq 1\}$. Then $\mu_{\tilde{D}}(x) : S \rightarrow I$ is a membership function where $I = [0, 1]$.

Definition 2.2: [14]
The binary operation $\odot : I \times I \rightarrow I$ is said to be continuous t-conorm if it satisfies
(i) $r \odot s = s \odot r$, (ii) $r \odot [s \odot w] = [r \odot s] \odot w$, (iii) $\odot$ is continuous function
(iv) $s \odot 0 = 0$, (v) $(r \odot z) \leq (s \odot w)$ whenever $r \leq s$ and $z \leq w$. For all $r, s, z, w \in I = [0, 1]$.

Lemma 2.3: [14]
If $\odot$ is a continuous t-conorm on $[0, 1]$ then
(i) $1 \odot 1 = 1$, (ii) $0 \odot 1 = 1 \odot 0 = 1$, (iii) $0 \odot 0 = 0$, (iv) $p \odot p \geq p$ for all $p \in [0, 1]$.

Remark 2.4: [14]
If $\odot$ is a continuous t-conorm then
(i) For any $p, q \in (0, 1)$ with $p > q$ we have $w \in (0, 1)$ whenever $p > q \odot w$. In general for any $p, q \in (0, 1)$ with $p > q$ we can find $w_1, w_2, \ldots, w_k \in (0, 1)$ whenever $p > q \odot w_1 \odot w_2 \odot \ldots \odot w_k$ where $k \in \mathbb{N}$.
(ii) For any $r \in (0, 1) \exists s \in (0, 1)$ s. t. $s \odot s \leq r$. In general for any $r \in (0, 1)$ there exists $w_1, w_2, \ldots, w_k \in (0, 1)$ such that $w_1 \odot w_2 \odot \ldots \odot w_k \leq r$ where $k \in \mathbb{N}$.

Example 2.5: [14]
The algebra product $p \odot q = p + q - pq$ is a continuous t-conorm for all $p, q \in [0, 1]$.

Definition 2.6: [15]
Let $\odot$ be a continuous t-conorm and $a_{\mathbb{R}} : \mathbb{R} \rightarrow I$ be a fuzzy set then $a$ is called algebra fuzzy absolute value on $\mathbb{R}$ if
(1) $0 < a_{\mathbb{R}}(\alpha) \leq 1$, (2) $a_{\mathbb{R}}(\alpha) = 0$ if and only if $\alpha = 0$, (3) $a_{\mathbb{R}}(\alpha \beta) \leq a_{\mathbb{R}}(\alpha) \cdot a_{\mathbb{R}}(\beta)$
(4) $a_{\mathbb{R}}(\alpha + \beta) \leq a_{\mathbb{R}}(\alpha) \odot a_{\mathbb{R}}(\beta)$. For all $\alpha, \beta \in \mathbb{R}$. Then $(\mathbb{R}, a_{\mathbb{R}}, \odot)$ is called algebra fuzzy absolute value space.

Example 2.7: [15]
Let $|.|$ be absolute value on $\mathbb{R}$ and $\alpha \odot \beta = \alpha + \beta - \alpha \beta$ for all $\alpha, \beta \in \mathbb{I}$. Define $a_{\mathbb{I}}(\alpha) = \begin{cases} |\alpha| & \text{if } \alpha \neq 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$ for all $\alpha \in \mathbb{R}$. Then $(\mathbb{R}, a_{\mathbb{I}}, \odot)$ is algebra fuzzy absolute value space. Also $a_{\mathbb{I}}$ is called the standard algebra fuzzy absolute value on $\mathbb{R}$.

Example 2.8: [15]
Define $a_{\mathbb{I}} : \mathbb{R} \rightarrow I$ by $a_{\mathbb{I}}(r) = \begin{cases} 1 & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}$ for all $r \in \mathbb{R}$ and if $s \odot r = s + r - sr$ for all $s, r \in \mathbb{I}$. Then $(\mathbb{R}, a_{\mathbb{I}}, \odot)$ is algebra fuzzy absolute value space. This space is called the algebra fuzzy absolute value space induced by $|.|$.

Definition 2.9: [15]
Let $(\mathbb{R}, a_{\mathbb{R}}, \odot)$ be algebra fuzzy absolute value space and let $\{p_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, we say that $\{p_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit $p$ as $n$ approaches to $\infty$ if $\forall \ s \in (0, 1) \exists N \in \mathbb{N}$ s. t. $a_{\mathbb{R}}(p_n - p)$
<s, \forall n \geq N. If \( p_n \) is fuzzy approaches to the limit p we write \( \lim_{n \to \infty} p_n = p \) or \( p_n \to p \) as n approaches to \( \infty \) or \( \lim_{n \to \infty} a_R(p_n - p) = 0 \).

Definition 2.10:[15]
Let \( (\mathbb{R}, a_R, \circ) \) be algebra fuzzy absolute value space and let \( \{p_n\}_{n=1}^{\infty} \) be a sequence in \( \mathbb{R} \), we say that \( \{p_n\}_{n=1}^{\infty} \) is fuzzy Cauchy sequence in \( \mathbb{R} \) if \( \forall s \in (0, 1) \exists N \in \mathbb{N} \) s. t. \( a_R(p_k - p_m) < s, \forall k, m \geq N \).

Definition 2.11:[15]
Let \( (\mathbb{R}, a_R, \circ) \) be algebra fuzzy absolute value space. The sequence \( \{q_n\}_{n=1}^{\infty} \) in \( \mathbb{R} \) is called fuzzy bounded if \( \exists t \in (0, 1) \) s. t. \( a_R(q_n) < t, \forall n \in \mathbb{N} \).

Theorem 2.12:[15]
Let \( (\mathbb{R}, a_R, \circ) \) be algebra fuzzy absolute space. If \( \{q_n\}_{n=1}^{\infty} \) is a fuzzy Cauchy sequence in \( \mathbb{R} \) then \( \{q_n\}_{n=1}^{\infty} \) is fuzzy bounded.

Definition 2.13:[15]
The algebra fuzzy absolute value \( (\mathbb{R}, a_R, \circ) \) is called fuzzy complete if every fuzzy Cauchy sequence in \( \mathbb{R} \) fuzzy approaches to a real number in \( \mathbb{R} \).

Definition 2.14:[15]
Let \( U \) be a vector space over \( \mathbb{R} \) and let \( \circ \) be a continuous t-conorm. Let \( (\mathbb{R}, a, \circ) \) be algebra fuzzy absolute value space and n: \( U \to \mathbb{I} \) be a fuzzy set then n is called algebra fuzzy norm on \( U \) if
\[
1) 0 < n(u) \leq 1, 2) n(u) = 0 \text{ if and only if } u = 0, 3) n(\alpha u) \leq a(\alpha) n(u) \text{ for all } 0 \neq \alpha \in \mathbb{R},
\]
\[
4)n(u + v) \leq n(u \circ v) \text{ for all } u, v \in U. \text{ Then } (U, n, \circ) \text{ is called algebra fuzzy normed space.}
\]

Example 2.15:[15]
Let \( U = \mathbb{C}[p, b] \), \( \circ = s + t - ts \) for all t, s \( \in \mathbb{I} \) and \( (\mathbb{R}, a, \circ) \) is algebra fuzzy absolute space. Define n(r) = \( \max_{s \in [p, b]} a[r(s)] \) for all r \( \in U \). Then (U, n, \circ) is algebra fuzzy normed space.

Definition 2.16:[15]
Suppose that \( (U, n, \circ) \) is algebra fuzzy normed space and assume that \( \{u_k\} \in U \), we say that \( \{u_k\} \) is fuzzy converges to the limit u ask approaches to \( \infty \) if \( \forall s \in (0, 1) \exists N \in \mathbb{N} \) s. t. \( n(u_k - u) < s, \forall k \geq N \). If \( \{u_k\} \) is fuzzy approaches to the limit u we write \( \lim_{k \to \infty} u_k = u \) or \( u_k \to u \) as k approaches to \( \infty \) or \( \lim_{n \to \infty} n(u_k - u) = 0 \).

Definition 2.17:[15]
Let \( (U, n, \circ) \) be algebra fuzzy normed space . Then the open and closed fuzzy ball with the center u \( \in U \) and radius t, with t \( \in (0, 1) \) is defined by fb(u, r) = \{v \( \in U \): n(u−v) < t\} and fb[u, r] = \{v \( \in U \): n(u−v) \leq t\} respectively.

Definition 2.18:[15]
Let \( (U, n, \circ) \) be algebra fuzzy normed space and let \( \{u_k\} \) be a sequence in \( U \), we say that \( \{u_k\} \) is fuzzy Cauchy sequence in \( U \) if \( \forall s \in (0, 1) \exists N \in \mathbb{N} \) s. t. \( n(u_k - u_m) < s, \forall k, m \geq N \).

Lemma 2.19:[15]
If \( (U, n, \circ) \) is algebra fuzzy normed space then the function u \( \mapsto n(u) \) is a fuzzy continuous function from \( (U, n, \circ) \to (\mathbb{R}, n, \circ) \).
Lemma 2.20: [15]
If \((U, n, \odot)\) is algebra fuzzy normed space then \(n(u-v) = n(v-u)\) for all \(u, v \in U\).

Definitions 2.21: [15]
If \((U, n, \odot)\) is algebra fuzzy normed space and \(W \subseteq U\) is known as fuzzy open if \(fb(w, j) \subseteq W\) for any arbitrary \(w \in W\) and for some \(j \in (0,1)\). Also \(D \subseteq U\) is known as fuzzy closed if \(D^C\) is fuzzy open. Moreover the fuzzy closure of \(D, \tilde{D}\) is defined to be the smallest fuzzy closed set contains \(D\).

Definition 2.22: [15]
An algebra fuzzy normed space \((U, n, \odot)\) is known as fuzzy complete if \((s_k)\) is fuzzy Cauchy sequence in \(U\) then \(s_k \to s \in U\).

Theorem 2.23: [15]
If \(fb(s, j)\) is an open fuzzy ball in algebra fuzzy normed space \((U, n, \odot)\) then it is a fuzzy open set.

Definition 2.24: [15]
If \((U, n, \odot)\) is algebra fuzzy normed space then \(D \subseteq U\) is known as fuzzy dense in \(U\) if whenever \(\tilde{D} = U\).

Theorem 2.25: [15]
In algebra fuzzy normed space \((U, n, \odot)\) if \(u_k \to u \in U\) then \((u_k)\) is fuzzy Cauchy.

Theorem 2.26: [15]
In algebra fuzzy normed space \((U, n, \odot)\) when \(D \subset U\) then \(d \in \tilde{D}\) if and only if there is \((d_k) \in D\) with \(d_k \to d\).

3. Fuzzy compact algebra fuzzy normed space

Definition 3.1:
Let \((U, n, \odot)\) be algebra fuzzy normed space and \(P \subseteq U\). Let \(\Omega = \{O_j: j \in J\}\) where \(O_j\) is fuzzy open set in \(U\). Then \(\Omega\) is called a fuzzy open cover or a fuzzy open covering of \(P\) if \(P \subseteq \bigcup_{j \in J} O_j\) that is for each \(p \in P\) there is \(O_k \in \Omega\) such that \(p \in O_k\) for some \(k \in J\). A finite sub collection of \(\Omega\) which itself is a fuzzy open cover is called a finite fuzzy open sub cover or a finite fuzzy open sub covering of \(P\).

Definition 3.2:
An algebra fuzzy normed space \((U, n, \odot)\) is said to be fuzzy compact if every fuzzy open covering \(\Omega\) of \(U\) has a finite fuzzy open sub covering that is there is a finite sub collection \(\{O_1, O_2, O_3, \ldots, O_k\} \subseteq \Omega\) such that \(U = \bigcup_{j=1}^{k} O_j\).

Definition 3.3:
A nonempty subspace \(P\) of \(U\) is said to be fuzzy compact if every fuzzy open covering \(\Omega\) of \(P\) has a finite fuzzy open sub covering that is there is a finite sub collection \(\{O_1, O_2, O_3, \ldots, O_k\} \subseteq \Omega\) such that \(P = \bigcup_{j=1}^{k} O_j\).

Example 3.4:
The open interval \(\{r \in \mathbb{R}: 0<r<1\}\) in the algebra fuzzy normed space \((\mathbb{R}, a|_{[1]}, \odot)\) where \(a|_{[1]}(\alpha) = \frac{|\alpha|}{1+|\alpha|}\) is not fuzzy compact. Since the fuzzy open covering \(\{(\frac{1}{k},1): k = 2, 3, \ldots\}\) is fuzzy open covering of \((0, 1)\) from which no finite fuzzy open sub covering can be selected.

Remark 3.5:
If \(P\) is a finite subset of algebra fuzzy normed space \((U, n, \odot)\) then \(P\) is fuzzy compact.
Definition 3.6:
Assume that \((U, n, \ominus)\) is algebra fuzzy normed space and \(P\) be a subset of \(U\). Then \(P\) is called fuzzy totally bounded if for each \(0 < r < 1\), there is a finite set of points \(\{p_1, p_2, \ldots, p_k\} \subseteq P\) such that whenever \(u \in U\), \(n(u - p_j) < r\) for some \(p_j \in \{p_1, p_2, \ldots, p_k\}\). This set of points \(\{p_1, p_2, \ldots, p_k\}\) is called fuzzy r-net.

Proposition 3.7:
A fuzzy totally bounded algebra fuzzy normed space is fuzzy bounded.
Proof:
Let \((U, n, \ominus)\) be fuzzy totally bounded and suppose \(0 < r < 1\) is given. Then there exists a finite fuzzy r-net for \(U\), say \(A\). Since \(A\) is a finite set of points and \(0 < n(A) < 1\), where \(n(A) = \sup\{n(b-a): b, a \in A\}\). Now let \(u_1\) and \(u_2\) be any two points of \(U\). There exists points \(b\) and \(a\) in \(A\) such that \(n(u_1 - b) < r\) and \(n(u_2 - a) < r\). Now for \(n(A)\) and \(r\) there is \(t\), where \(0 < t < 1\) such that \(r \ominus n(A) \ominus r \leq t\). It follows that
\[
n(u_1 - u_2) \leq n(u_1 - b) \ominus n(b-a) \ominus n(a - u_2) \leq r \ominus n(A) \ominus r \leq t.
\]
So, \(n(U) = \sup\{n(u_1 - u_2): u_1, u_2 \in X\} \leq t\). Hence \(U\) is fuzzy bounded.

Theorem 3.8:
Let \((U, n, \ominus)\) be algebra fuzzy normed space and let \(P\) be a subset of \(U\). Then \(P\) is fuzzy totally bounded if and only if \(\forall (p_k)\) in \(P\) contains subsequence \((p_{k_1})\) which is fuzzy Cauchy.
Proof:
Let \(P\) be fuzzy totally bounded and \((p_k) \in P\). Consider a finite fuzzy \(\frac{1}{2}\)-net in \(P\). Then we can find fuzzy open ball of radius \(\frac{1}{2}\) with center in the fuzzy \(\frac{1}{2}\)-net contains infinite elements of \((p_k)\). Let \((p_{k_1})\) denote the subsequence contains these element. Again consider a finite fuzzy \(\frac{1}{4}\)-net in \(P\). Then we can find fuzzy open ball of radius \(\frac{1}{4}\) with center in the fuzzy \(\frac{1}{4}\)-net contains infinitely many elements of \((p_{k_1})\).

Let \((p_{k_2})\) be the subsequence. After many steps we get \((p_k) \supseteq (p_{k_1}) \supseteq (p_{k_2}) \supseteq \ldots\)
At the jth step members of \((p_{k_j})\) lie in the fuzzy open ball of radius \(\frac{1}{2^j}\) with center in the fuzzy \(\frac{1}{2^j}\)-net.
Now \((p_{jj})\) is a subsequence of \((p_{j})\). Assume that \(\exists r \in (0, 1)\). Consider \(N \in \mathbb{N}\) so large that
\[
\left(\frac{1}{2^j}\right) \ominus \left(\frac{1}{2^{j+1}}\right) \ominus \ldots \ominus \left(\frac{1}{2^{m-1}}\right) < r.
\]
Then for \(m > j > N\), we have
\[
n(p_{jj} - p_{mm}) \leq n(p_{jj} - p_{jj+j+1}) \ominus n(p_{jj+j+1} - p_{jj+2j+2}) \ominus \ldots \ominus n(p_{m-1m-1} - p_{mm})
\]
\[
n(p_{jj} - p_{mm}) \leq \left(\frac{1}{2^j}\right) \ominus \left(\frac{1}{2^{j+1}}\right) \ominus \ldots \ominus \left(\frac{1}{2^{m-1}}\right) < r.
\]
Thus the sequence \((p_{kk})\) is a fuzzy Cauchy sequence.
To prove the converse assume that any sequence \((p_k)\) in \(P\) contains subsequence \((p_{k})\) which is fuzzy Cauchy.
Let \(r \in (0, 1)\) and let \(p_1 \in P\). If \(P - fb(p_1, r) = \emptyset\), we have found a fuzzy r-net put it \(\{p_1\}\). Or consider \(p_2 \in P - fb(p_1, r)\). If \(P - [fb(p_1, r) \cup fb(p_2, r)] = \emptyset\). We have found a fuzzy r-net put it by \(\{p_1, p_2\}\). We must prove that this process after a finite number of steps stops. If it does not stops, we get \((p_1, p_2, \ldots, p_k)\) with \(n(p_j - p_m) \geq r, j \neq m\). Thus this implies \((p_k)\) does not contain subsequence \((p_{k})\) which is fuzzy Cauchy. This is a contradiction.

Proposition 3.9:
If \((U, n, \ominus)\) is a fuzzy compact algebra fuzzy normed space then \(U\) is fuzzy totally bounded.
Proof:
Let $0 < r < 1$ then \{ $fb(p, r) : p \in U$ \} is a fuzzy open cover of $U$. But $U$ is fuzzy compact this implies that \{ $fb(p, r) : p \in U$ \} contains \{ $fb(p_1, r), fb(p_2, r), \ldots, fb(p_k, r)$ \} such that $U = \bigcup_{j=1}^{k} fb(p_j, r)$. Hence for $0 < r < 1 \{ p_1, p_2, \ldots, p_k \} U$ is a finite fuzzy r-net for $U$. Hence $U$ is fuzzy totally bounded.

Proposition 3.10:
If $(U, n, C)$ is a fuzzy compact algebra fuzzy normed space then $(U, n, C)$ is fuzzy complete.
Proof:
Assume that $(U, n, C)$ is not fuzzy complete. Then we can find a fuzzy Cauchy $(p_k)$ in $U$ does not have a limit in $U$. Let $p \in U$, since $p_k \not\rightarrow p \exists 0 < r < 1$ such that $n(p_k - p) \geq r$ for $k=1, 2, \ldots$ but $(p_k)$ is fuzzy Cauchy $\exists N \in \mathbb{N}$ s. t. $n(p_j - p_m) < r$ $\forall j, m \geq N$. Choose $m \geq N$ for which $n(p_m - p) < r$. So, the fuzzy open ball $fb(p, t)$ contains \{ $p_1, p_2, \ldots, p_k$ \} where $k \in \mathbb{N}$. Now consider fuzzy \{ $fb(p_1, t(p_1)), fb(p_2, t(p_2)), \ldots, fb(p_k, t(p_k))$ \} where $0 < t(p_k) < 1$ and $U = \bigcup_{j=1}^{k} fb(p_j, t(p_j))$. But each $fb(p_j, t(p_j))$ contains $p_k$ for only a finite number of values so $U$, must contains $p_k$ for only a finite number of values of $k$. This is a contradiction. Hence $(U, n, C)$ must be fuzzy complete.

Theorem 3.11:
If $(U, n, C)$ is fuzzy totally bounded and fuzzy complete algebra fuzzy normed space then $(U, n, C)$ is fuzzy compact.
Proof:
Assume that $(U, n, C)$ is not fuzzy compact. Then we can find a fuzzy Cauchy $(p_k)$ in $U$ that does not have a fuzzy finite open sub covering. But $U$ is fuzzy totally bounded, so it is fuzzy bounded, hence consider $fb(p, r)$ for some $0 < r < 1$ and some $p \in U$, clearly $fb(p, r) \subseteq U$ if $U \subseteq fb(p, r)$ then we must have $U = fb(p, r)$.

Put $t_k = \frac{r}{2^k}$ but $U$ is fuzzy totally bounded this means that $U$ can be covered by finite fuzzy open balls of radius $t_k$. By our assumption at least one of these fuzzy open balls, say $fb(p_1, t_1)$ is itself fuzzy totally bounded. $\exists \ p_2 \in fb(p_1, t_1)$, s. t. $fb(p_2, t_2)$ $\not\subseteq \bigcup_{j=1}^{k} O_{\lambda_j}$. After many steps we get a sequence $(p_k)$ has the property that for each $k$, $fb(p_k, t_k)$ $\not\subseteq \bigcup_{j=1}^{k} O_{\lambda_j}$ and $p_{k+1} \not\subseteq fb(p_k, t_k)$. We next show that the sequence $(p_k)$ is convergent. Since $p_{k+1} \not\subseteq fb(p_k, t_k)$ it follows that $n(p_k - p_{k+1}) < t_k$. Let $0 < t < 1$ such that $t_k \bigcirc t_{k+1} \bigcirc \ldots \bigcirc t_m < t$.

Hence $n(p_k - p_m) \leq t_k \bigcirc t_{k+1} \bigcirc \ldots \bigcirc t_m < t$

So $(p_k)$ is a fuzzy Cauchy sequence in $U$ and since $U$ is fuzzy complete, it converges to $p \in U$. Because $p \in U$ $\exists \lambda_0 \in \mathbb{N}$ such that $O_{\lambda_0}$. Since $O_{\lambda_0}$ is fuzzy open it contains $fb(p, s)$ for some $0 < s < 1$. Let $N \in \mathbb{N}$ $n(p_k - p) < s$ then, for any $u \in U$ such that $n(u - p_k) < t_k$. It follows that $n(u - p_k) \leq n(u - p) \bigcirc n(p_k - p) \leq t_k \bigcirc s < r$.

for some $0 < r < 1$. So that $fb(p_k, t_n) \subseteq fb(p, r)$. Therefore $fb(p_k, t_k)$ has a finite fuzzy open sub covering, namely by the set $O_{\lambda_0}$. Since this contradicts $fb(p_k, t_k) \not\subseteq \bigcup_{j=1}^{k} O_{\lambda_j}$. The proof is complete.

Proposition 3.12:
Let $(U, n, C)$ be algebra fuzzy normed space. Then every set \{ $u_1, u_2, \ldots, u_k, \ldots$ \} in $(U, n, C)$ has at least one limit point in $U$ $\iff$ every sequence $(u_k)$ in $(U, n, C)$ contains $(u_k)$ such that $u_k \rightarrow u \in U$.
Proof:
Assume that every set \{ $u_1, u_2, \ldots, u_k, \ldots$ \} in $(U, n, C)$ has at least one limit point in $U$. Let $(u_k) \in U$. If \{ $u_1, u_2, \ldots, u_k$ \} then take $u_j \in \{ u_1, u_2, \ldots, u_k \}$ and $(u_j, u_j, \ldots) \subseteq (u_k)$, where $(u_j, u_j, \ldots) \rightarrow u_j$. Now consider \{ $u_1, u_2, \ldots, u_k, \ldots$ \}.
By our assumption the infinite set \( \{u_1, u_2, \ldots \} \) has at least one limit point \( u \in U \). Now we construct \( (u_{k_1}) \) as follows: Take \( k_1 \in \mathbb{N} \) s. t. \( n(u_{k_1} - u) > 0 \).

Having defined \( k_1 \), let \( k_{j+1} \) be the smallest integer such that \( k_{j+1} > k_j \) and \( n(u_{k_{j+1}} - u) < \frac{1}{j+1} \). Then the sequence \( (u_{k_j}) \) fuzzy converges to \( u \).

Conversely assume that every sequence \( (u_{k_j}) \) in \((U, n, \otimes)\) contains \( (u_{k_j}) \) such that \( u_{k_j} \rightarrow u \in U \). Let \( S=\{p_1, p_2, \ldots \} \subseteq U \) then \( \exists (p_k) \) in \( U \) where \( p_k \not= p_j \). By our assumption \( (p_k) \) contains a subsequence \( (p_{k_j}) \) where \( p_{k_j} \not= p_{k_i} \) and \( p_{k_j} \rightarrow p \in U \). Thus \( \forall \text{fb}(p, r) \) we have \( \{p_{k_1}, p_{k_2}, \ldots, p_{k_j}, \ldots\} \subseteq \text{fb}(p, r) \). But \( p_{k_j} \not= p_k \) hence \( p \in U \) is a limit point of \( S \).

**Theorem 3.13:**
Suppose that \((U, n, \otimes)\) is algebra fuzzy normed space then the following is equivalent

1. \((U, n, \otimes)\) is fuzzy compact;
2. \( \forall (u_{k_j}) \) in \((U, n, \otimes)\) contains \( (u_{k_j}) \) such that \( u_{k_j} \rightarrow u \in U \).

**Proof:**

Consider \((u_{k_j}) \) in \( U \) then by Theorem 3.12, it follows that \( (u_{k_j}) \subseteq (u_k) \) is a fuzzy Cauchy. But \( u_{k_j} \rightarrow u \in U \) since \( U \) is fuzzy complete by Theorem 3.11.

**(2) \Rightarrow (1)**
Since every fuzzy convergent sequence is fuzzy Cauchy and by using Theorem 3.11 we have \( U \) is fuzzy totally bounded. Let \( (p_k) \) be a fuzzy Cauchy sequence in \( U \). By (2) \( (p_{k_j}) \subseteq (p_k) \) and \( p_{k_j} \rightarrow p \in U \). To prove that \( p_k \rightarrow p \). Consider \( r \in (0, 1) \) by Remark 1.2, there is \( 0 < t < 1 \) such that \( r \otimes r < t \).

Now \( p_{k_j} \rightarrow p \), there exist \( N_1 \) such that \( n(p_{k_j} - p) < r \) for all \( k_j \geq N_1 \).

Since the sequence \( (p_k) \) is fuzzy Cauchy, there exists \( N_2 \) such that \( n(p_j - p_m) < r \) for all \( m, j \geq N_2 \).

Let \( N = \min\{N_1, N_2\} \) then \( n(p_k - p) \leq n(p_k - p_{k_j}) \otimes n(p_{k_j} - p) \leq r \otimes r < t \)

For all \( k \geq N \). Hence \( U \) is fuzzy complete and by Theorem 3.11 \( U \) is fuzzy compact.

**Corollary 3.14:**
If \( P \) is a fuzzy closed subset of the fuzzy compact algebra fuzzy normed space \((U, n, \otimes)\) then \( P \) is fuzzy compact.

**Proof:**

Let \( (p_k) \in P \). Then \( (p_k) \) in \( U \) and \( (p_{k_j}) \subseteq (p_k) \) and \( p_{k_j} \rightarrow p \in U \). Thus \( p \in P \) since \( P \) is fuzzy closed. Hence by Theorem 3.13 \( P \) is fuzzy compact.

**Theorem 3.15:**
Suppose that \((U, n, \otimes)\) is algebra fuzzy normed space and \( P \subseteq U \). If \( P \) is fuzzy compact then \( P \) is a fuzzy closed subset of \( U \).

**Proof:**

Consider \( p \in U \) is a limit point of \( P \). Then \( \exists (p_k) \) in \( P \) such that \( p_k \rightarrow p \). But then \( (p_k) \) is a fuzzy Cauchy sequence in \( P \). Because \( P \) is fuzzy complete \( p_k \rightarrow q \) in \( P \). Thus \( q = p \) and so \( q \in P \). Thus \( P \) fuzzy closed since it contains all its limit points.

4. When the dimension of algebra Fuzzy Normed Space is finite
In this section we deal with finite dimensional vector spaces with algebra fuzzy norm.
The following theorem plays the key role in the studying properties of finite dimensional algebra fuzzy normed linear spaces.
Theorem 4.1:
Let $(U, \alpha, \oplus)$ be algebra fuzzy normed space and $(\mathbb{R}, a_\mathbb{R}, \ominus)$ is algebra fuzzy absolute value space. Suppose that $\{u_1, u_2, \ldots, u_k\}$ is linearly independent set in $U$. Then there is $r \in (0, 1)$ such that $n_U[\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_k u_k] \geq r[a_\mathbb{R}(\alpha_1) \ominus a_\mathbb{R}(\alpha_2) \ominus \ldots \ominus a_\mathbb{R}(\alpha_k)]$.

Proof:
Assume that we cannot find $r \in (0, 1)$ such that $n_U[\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_k u_k] \geq r[a_\mathbb{R}(\alpha_1) \ominus a_\mathbb{R}(\alpha_2) \ominus \ldots \ominus a_\mathbb{R}(\alpha_k)]$.

Then we can find a sequence $(d_m)$ in $U$ where $d_m = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_k u_k$ such that $n_U(d_m) \to 0$ as $m \to \infty$.

Now for each fixed $j$ we have a sequence $\alpha_m = (\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jm}, \ldots)$ is fuzzy bounded since $0 \leq a_\mathbb{R}(\alpha_m) \leq 1$ so $(\alpha_j)$ has a convergent subsequence. Let $\alpha_j$ denote the limit of the subsequence $(\alpha_{jm})$ for each $1 \leq j \leq k$. Let $(u_{jm})$ denote the corresponding subsequence of $(d_m)$ where the corresponding subsequence of scalar $a_{jm}$ fuzzy converges to $\alpha_j$ for each $1 \leq j \leq k$.

Now put $d = \sum_{j=1}^{k} \alpha_j u_j$ then $(d_m)$ has a subsequence $(d_{jm})$ fuzzy converges to $d$ since $\{u_1, u_2, \ldots, u_k\}$ is linearly independent set so $d \neq 0$. Now $d_m \to d$ implies $n_U(d_{jm}) \to n_U(d)$ by fuzzy continuity of the algebra fuzzy norm.

But $n_U(d_m) \to 0$ as $m \to \infty$ by our assumption and $(d_{jm})$ is a subsequence of $(d_m)$. Thus $n_U(d_{jm}) \to 0$. Hence $n_U(d) = 0$ so $d = 0$. This contradicts $d \neq 0$.

Theorem 4.2:
Consider $(U, \alpha, \oplus)$ is algebra fuzzy normed space. If $D$ is a subspace of $U$ with finite dimension, then $D$ is fuzzy complete.

Proof:
Consider that $(d_m)$ is a algebra fuzzy Cauchy sequence in $D$. Let dim $D = k$ and $B = \{u_1, u_2, \ldots, u_k\}$ be any basis for $D$. Then each $d_m$ has a unique representation as $d_m = \alpha_{1m} u_1 + \alpha_{2m} u_2 + \ldots + \alpha_{km} u_k$ since $(d_m)$ is fuzzy Cauchy sequence so for every $t \in (0, 1)$ there is $N \in \mathbb{N}$ such that $n_U(d_m - d_j) \leq t$ for every $m, j \geq N$. Now by Theorem 4.1 we have some $r \in (0, 1)$ such that $r[a_\mathbb{R}(\alpha_{1m} - \alpha_{1j}) \ominus a_\mathbb{R}(\alpha_{2m} - \alpha_{2j}) \ominus \ldots \ominus a_\mathbb{R}(\alpha_{km} - \alpha_{kj})] \leq n_U(d_m - d_j) = n_U[\sum_{i=1}^{k}(\alpha_{im} - \alpha_{ij}) u_i]$.

This show that $(\alpha_{1j}, \alpha_{2j}, \ldots)$ is fuzzy Cauchy sequence in $\mathbb{R}$. Hence $\alpha_{jm} \to \alpha_j$ for each $1 \leq j \leq k$. Put $d = \sum_{j=1}^{k} \alpha_j u_j$. Clearly $d \in D$.

Also now for all $m > N$

$\lim_{m \to \infty} n_U(d_m - d) = n_U[\sum_{i=1}^{k}(\alpha_{im} - \alpha_{ij}) u_i] \leq a_\mathbb{R}(\alpha_{im} - \alpha_{ij}) u_i = a_\mathbb{R}(\alpha_{im} - \alpha_{ij}) u_i$. 

By taking limit to both sides as $m \to \infty$ we get $\lim_{m \to \infty} n_U(d_m - d) \leq 0 \ominus 0 \ominus \ldots \ominus 0 = 0$. Hence $d_m \to d$ and $D$ is fuzzy complete.

Definition 4.3:
Consider that $(U, n_1, \alpha)$ is algebra fuzzy normed space and $(u_k) \in U$. Then the algebra fuzzy norm $n_2$ is said to be equivalent to $n_1$ whenever $u_k \to u \in U$ in $(U, n_1, \alpha)$ if and only if $u_k \to u \in U$ in $(U, n_2, \alpha)$.

Theorem 4.4:
Suppose that $(U, n_1, \alpha)$ is algebra fuzzy normed space then the algebra fuzzy norm $n_2$ is equivalent to $n_1$ if there exists $p, q$ in $(0, 1)$ such that $pn_2(u) \leq n_1(u) \leq qn_2(u)$.

Proof:
Suppose that $u_k \to u \in U$ in $(U, n_1, \alpha)$ then for any $r \in (0, 1) \exists N \subseteq \mathbb{N}$ such that
Again by Theorem 4.1

Now assume that \( u_k \rightarrow u \in (U, n_2, \odot) \) for \( r \in (0, 1) \) \( \exists \ N \in \mathbb{N} \) such that \( n_2(u_k - u) \leq r \). But \( n_1(u_k - u) \leq q n_2(u_k - u) \) or \( n_1(u_k - u) \leq q \) put \( qr = t \in (0, 1) \) that is \( n_1(u_k - u) \leq t \). Hence we have \( u_k \rightarrow u \in (U, n_1, \odot) \).

Theorem 4.5:
Suppose that \( U \) is vector space with \( \dim U = k \) if \( n_1 \) and \( n_2 \) are two algebra fuzzy norms on \( U \) then \( n_1 \) is equivalent to \( n_2 \).

Proof:
Since \( \dim U = k \) and \( B = \{ u_1, u_2, \ldots, u_k \} \) be any basis for \( U \). Then for any \( u \in U \) we have \( u = \sum_{j=1}^{k} a_j u_j \). Now

\[
\begin{align*}
n_1(u) &= n_1 \left( \sum_{j=1}^{k} a_j u_j \right) \\
&= n_1(a_1 u_1) \odot n_1(a_2 u_2) \odot \ldots \odot n_1(a_k u_k) \\
&
\leq a_R(a_1) n_1(u_1) \odot a_R(a_2) n_1(u_2) \odot \ldots \odot a_R(a_k) n_1(u_k) \\
&
\leq t[a_R(a_1) \odot a_R(a_2) \odot \ldots \odot a_R(a_k)]
\end{align*}
\]

Where \( \max \{ n_1(u_1), n_1(u_2), \ldots, n_1(u_k) \} \) that is \( n_1(u) \leq [a_R(a_1) \odot a_R(a_2) \odot \ldots \odot a_R(a_k)] \) or dividing by \( t \)

\[
\frac{1}{t} n_1(u) \leq \frac{1}{t} n_2(u) \text{ this implies that } n_1(u) \leq \frac{t}{s} n_2(u) \text{ put } \frac{t}{s} = q \text{ we have } n_1(u) \leq q n_2(u)
\]

In similar way

\[
\begin{align*}
n_2(u) &= n_2 \left( \sum_{j=1}^{k} a_j u_j \right) \\
&= n_2(a_1 u_1) \odot n_2(a_2 u_2) \odot \ldots \odot n_2(a_k u_k) \\
&
\leq a_R(a_1) n_2(u_1) \odot a_R(a_2) n_2(u_2) \odot \ldots \odot a_R(a_k) n_2(u_k) \\
&
\leq [s a_R(a_1) \odot a_R(a_2) \odot \ldots \odot a_R(a_k)]
\end{align*}
\]

Where \( \max \{ n_2(u_1), n_2(u_2), \ldots, n_2(u_k) \} \) that is

\[
\frac{1}{s} n_2(u) \leq [s a_R(a_1) \odot a_R(a_2) \odot \ldots \odot a_R(a_k)] \text{ or dividing by } s
\]

Again by Theorem 4.1

\[
\begin{align*}
n_1(u) &= n_1 \left( \sum_{j=1}^{k} a_j u_j \right) \\
&= n_1(a_1 u_1) \odot n_1(a_2 u_2) \odot \ldots \odot n_1(a_k u_k) \\
&
\leq a_R(a_1) n_1(u_1) \odot a_R(a_2) n_1(u_2) \odot \ldots \odot a_R(a_k) n_1(u_k) \\
&
\leq [r a_R(a_1) \odot a_R(a_2) \odot \ldots \odot a_R(a_k)]
\end{align*}
\]

\[
\begin{align*}
n_2(u) &= n_2(a_1 u_1) \odot n_2(a_2 u_2) \odot \ldots \odot n_2(a_k u_k) \\
&
\leq [r a_R(a_1) \odot a_R(a_2) \odot \ldots \odot a_R(a_k)]
\end{align*}
\]

Again by Theorem 4.1

\[
\begin{align*}
n_1(u) &= n_1 \left( \sum_{j=1}^{k} a_j u_j \right) \\
&= n_1(a_1 u_1) \odot n_1(a_2 u_2) \odot \ldots \odot n_1(a_k u_k) \\
&
\leq a_R(a_1) n_1(u_1) \odot a_R(a_2) n_1(u_2) \odot \ldots \odot a_R(a_k) n_1(u_k) \\
&
\leq [s a_R(a_1) \odot a_R(a_2) \odot \ldots \odot a_R(a_k)]
\end{align*}
\]

Now from (4) and (5) we have

\[
\begin{align*}
\frac{1}{r} n_1(u) &\geq [a_R(a_1) \odot a_R(a_2) \odot \ldots \odot a_R(a_k)] \geq \frac{1}{s} n_2(u) \text{ or } \frac{1}{r} n_1(u) \geq \frac{1}{s} n_2(u) \text{ this implies that } \frac{1}{r} n_2(u) \leq \frac{s}{r} n_1(u) \text{ put } \frac{s}{r} = q \text{ we have } \frac{1}{s} n_2(u) \leq n_1(u) \text{ (6)}
\end{align*}
\]

Now from (3) and (6) we obtain

\[
\frac{1}{s} n_2(u) \leq n_1(u) \leq q n_2(u)
\]

Hence by Theorem 4.4, \( n_1 \) is fuzzy equivalent to \( n_2 \).

Theorem 4.6:
Suppose that \((U, n, \odot)\) is algebra fuzzy normed space with \(\dim U = k\) and \(D \subset U\). If \(D\) is fuzzy closed and fuzzy bounded, then \(D\) is fuzzy compact.

**Proof:**
Since \(\dim U = k\) and \(B = \{u_1, u_2, \ldots, u_k\}\) be any basis for \(U\). Consider the sequence \((d_m)\) in \(D\) then \(\forall r \in (0, 1) \exists N \in \mathbb{N}\) such that \(d_m = a_1 u_1 + a_2 u_2 + \ldots + a_k u_k\), since \(D\) is fuzzy bounded so, \((d_m)\) that is \(n_u(d_m) < t\) for all \(m\) and for some \(t \in (0, 1)\). Now by Theorem 4.1

\[
\|u\|_D = n_u(\sum_{j=1}^{k} a_{jm} u_j) \geq \|a_R(\alpha_1) \odot a_R(\alpha_2) \ldots \odot a_R(\alpha_k)\| \leq n_u(d_m) < t
\]

Thus \(r[a_R(\alpha_1) \odot a_R(\alpha_2) \odot \ldots \odot a_R(\alpha_k)] \leq n_u(d_m) < t\) or \(a_R(\alpha_{jm}) \leq [a_R(\alpha_1) \odot a_R(\alpha_2) \odot \ldots \odot a_R(\alpha_k)] < t\)

Hence the sequence \((a_{jm})\) for fixed \(j\) is fuzzy bounded so it has a limit point \(\alpha_j\) for each \(1 < j < k\). We see that \((d_m)\) has a subsequence \((z_m)\) which converges to \(z = \sum_{j=1}^{n} \alpha_j u_j\). Since \(D\) is closed so \(z \in D\). But \((d_m)\) is an arbitrary sequence in \(D\). Hence \(D\) is fuzzy compact.

**Lemma 4.7:**
Consider that \((U, n, \odot)\) is algebra fuzzy normed space and assume that \(D\) and \(Z\) two subspace of \(U\) with \(D \subset Z\) and \(D\) is fuzzy closed. Then for every \(t \in (0, 1)\) there is \(z \in Z\) such that \(n_u(z-d) \geq t\) for all \(d \in D\).

**Proof:**
Let \(u \in Z - D\) and put \(s = \inf_{d \in D} n_u(u-d)\). Clearly \(s > 0\) since \(D\) is fuzzy closed. Take \(t \in (0, 1)\) with \(s > t\) then by definition of infimum there is \(d_0 \in D\) such that \(s \leq n_u(u - d_0) \leq \frac{s}{t}\). Put \(z = u - d_0\). Now \(n_u(z-d) = n_u(u-d_0) = n_u(u-d_1)\) where \(d_1 = d_0 + d\). Hence \(n_u(z-d) = n_u(u-d_1) \geq s > t\).

**Theorem 4.8:**
Suppose that \((U, n, \odot)\) is algebra fuzzy normed space and if \(D = \{u \in U : 0 < n(u) \leq 1\}\) is a fuzzy closed in \(U\) and is compact then \(U\) must be finite dimension.

**Proof:**
Suppose that \(D\) is fuzzy compact and \(\dim U\) is not finite. Choose \(u_1 \in D\) and let \(U_1\) be the subspace of \(U\) with basis \(\{u_1\}\) so it is fuzzy closed. But \(U_1 \neq U\) since \(\dim U\) is not finite. Now by Lemma 4.7 there is \(u_2 \in D\) such that \(n(u_2 - u_1) \geq \frac{1}{2}\). Let \(U_2\) be the subspace of \(U\) with basis \(\{u_1, u_2\}\) since \(U_2 \neq U\) so there is \(u_3 \in D\) such that \(n(u_3 - u_2) \geq \frac{1}{2}\) and \(n(u_3 - u_1) \geq \frac{1}{2}\).

Now by induction we get \((u_k) \in D\) with \(n(u_m - u_{m-j}) \geq \frac{1}{2}\) where \(m \neq j\). This implies that \((u_k)\) does not contains a subsequence which is fuzzy converges. This contradicts the compactness of \(D\). Hence \(\dim U\) must be finite.

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