1. INTRODUCTION

Convection plays an important and well-known role in the transport of energy in stellar interiors. It has also been argued that convection is important in a number of low-density astrophysical plasmas, such as the intracluster medium (ICM) in clusters of galaxies (Chandran & Rasera 2007) and accretion flows onto compact objects (Quataert & Gruzinov 2000). Although convection in stellar interiors has been thoroughly studied over the course of several decades, the theory of convection in low-density plasmas is still being developed, and investigations carried out during the last several years have led to some interesting surprises.

For many years, it was widely assumed that the convective stability criterion for a low-density, nonrotating, weakly magnetized plasma is the Schwarzschild criterion, \( ds/dz > 0 \), where \( s \) is the specific entropy of the plasma and the gravitational acceleration is in the \(-z\)-direction. However, Balbus (2000, 2001) showed that even weak magnetic fields strongly modify the convective stability criterion by causing heat to be conducted almost exclusively along magnetic field lines. This anisotropy in the thermal conductivity arises when the electron gyroradius is much less than the electron mean free path, a condition that is easily satisfied for realistic magnetic fields in most cases of interest. Balbus (2000, 2001) considered an equilibrium in which the magnetic field is in the \( xy \)-plane, and in which \( \beta = \frac{8\pi p}{B^2} \gg 1 \), where \( p \) is the pressure and \( B \) is the magnetic field strength. He showed that near marginal stability, the temperature of a rising fluid parcel is almost constant, essentially for two reasons. First, the parcel remains magnetically connected to material at its initial height. Second, near marginal stability a fluid parcel rises very slowly, so thermal conduction has enough time to approximately equalize the temperature along the perturbed magnetic field lines. As a result, the stability criterion becomes \( dT/dz > 0 \). When this criterion is satisfied, a rising fluid parcel is cooler than its surroundings, and hence denser at the same pressure, so that it falls back down to its initial height. Parrish & Stone (2005, 2007) carried out numerical simulations that validated Balbus’ analysis and extended it to the nonlinear regime.

An immediate question arises, namely, why doesn’t Balbus’s stability criterion apply to stars? Although stellar plasmas are magnetized, heat is conducted in stellar interiors primarily by photons. As discussed by Balbus (2000, 2001), the conductivity is thus almost isotropic, so it is the Schwarzschild criterion that applies. The reason that the Schwarzschild criterion applies, even if the isotropic conductivity is large, is somewhat subtle. If \( ds/dz < 0 \), then adiabatic expansion would cause a slowly rising fluid parcel to be hotter and lighter than its surroundings, and hence buoyant. The effect of isotropic conductivity is then to relax the temperature in the parcel toward that of the immediately surrounding fluid. However, because the conductivity is finite, the rising fluid parcel’s temperature is never decreased all the way to the temperature of its surroundings. The fluid parcel thus remains slightly hotter than its surroundings, and hence slightly lighter at the same pressure, and the fluid is convectively unstable. Although isotropic conductivity does not modify the Schwarzschild stability criterion, it does reduce the convective heat flux and the “efficiency of convection” in a convectively unstable fluid by decreasing the temperature difference \( \delta T \) between rising fluid parcels and their surroundings (Cox & Guli 1968).

Balbus’s analysis has been extended in two ways by recent studies. First, Chandran & Dennis (2006; hereafter referred to as CD06) investigated how the stability criterion is affected by the presence of cosmic rays that diffuse primarily along magnetic field lines. Like Balbus (2000, 2001), they assumed that \( \beta \gg 1 \) and took the equilibrium magnetic field to be in the
xy-plane. They showed that near marginal stability, the cosmic-ray pressure is nearly constant within a rising fluid element. This is because the fluid element remains connected to material at its initial height, and because fluid elements rise very slowly near marginal stability, so that there is plenty of time for cosmic-ray diffusion to approximately equalize the cosmic-ray pressure $p_{\text{cr}}$ along the perturbed magnetic field lines. CD06 showed analytically that the stability criterion in the presence of cosmic rays is $nk B d T / d z + d p_{\text{cr}} / d z > 0$, where $n$ is the total number density of thermal particles.

More recently, Quataert (2008) considered buoyancy instabilities in a low-density, high-$\beta$ plasma in the absence of cosmic rays, but allowing the equilibrium magnetic field to have a component in the $z$-direction, parallel or antiparallel to the direction of gravity. Since the temperature is a function of $z$, the $z$ component of the equilibrium magnetic field leads to an equilibrium heat flux. Quataert (2008) showed that this heat flux causes the plasma to become convectively unstable even if $d T / d z > 0$, so that the plasma is always convectively unstable if the magnetic field has a nonzero $z$ component and a nonzero component in the $xy$-plane. This heat–flux–buoyancy instability arises because of the geometry of the perturbed magnetic field lines in the plasma. For example, when the magnetic field is in the $z$-direction and a fluid element is displaced upwards at a 45º angle with respect to the $z$-axis, field lines converge as they enter the fluid element from “above” (i.e., from larger $z$). As a result, if $d T / d z > 0$ then the parallel heat flux converges within the fluid element, causing the fluid element to become hotter than its surroundings, and thus less dense at the same pressure. Buoyancy forces then cause the upwardly displaced fluid element to rise unstably (Quataert 2008). The nonlinear development of this instability was investigated numerically by Parrish & Quataert (2008).

One of the open questions in this area of research is whether the buoyancy instabilities identified in these previous studies for the $\beta \gg 1$ regime still operate when the magnetic field strength is increased to the point that $\beta \lesssim 1$. We address this question in this paper. We consider the equilibrium geometry investigated by Balbus (2000, 2001) and CD06, in which the magnetic field is in the $xy$-plane—in particular, we set $B_0 = B_0 \hat{y}$. We also allow for cosmic rays that diffuse along magnetic field lines, but now we allow $\beta$ to take any value. We focus on wave vectors in the “quasi-interchange” limit, in which $|k_x| \gg |k_y|, |k_\parallel| \gg |k_z|$, and $|k_x| H \gg 1$, where $H$ is the density scale height. This is the most unstable wave-vector regime for stratified adiabatic plasmas, because a small $k_x$ reduces the stabilizing effects of magnetic tension and a large $k_\parallel$ allows a rising fluid element to easily get out of the way of the next rising element beneath it by moving just a small distance in the $z$-direction (Parker 1967; Shu 1974; Ferriere et al. 1999). We show analytically that the stability criterion in this limit is

$$nk B d T / d z + d p_{\text{cr}} / d z + \frac{1}{8\pi} dB^2 / d z > 0,$$

and present a heuristic derivation of this stability criterion from physical arguments. We also derive approximate analytical solutions to the dispersion relation for small-amplitude perturbations to the equilibrium in different parameter regimes, and compare these solutions to numerical solutions of the full dispersion relation.

Our results are important for determining the convective stability of galaxy-cluster plasmas, in which cosmic rays are often produced by central radio sources. Convection in intracluster plasmas is of interest because it may provide a mechanism for regulating the temperature profiles of galaxy-cluster plasmas and offsetting radiative cooling, thereby solving the so-called “cooling-flow problem” (Chandran 2004, 2005; Parrish & Stone 2005, 2007; Chandran & Rasera 2007; Rasera & Chandran 2008; Sharma et al. 2008). Our results are also important for determining the conditions under which the Parker instability can operate in the interstellar medium (ISM). Previous treatments of the Parker instability assume an adiabatic thermal plasma (Parker 1966, 1967; Shu 1974; Ryu et al. 2003). Our results show that anisotropic thermal conductivity makes the ISM more unstable to the Parker instability, so that the instability can operate under a wider range of equilibrium profiles than was previously recognized.

The remainder of this paper is organized as follows. In Section 2 we outline the derivation of the general form of the dispersion relation. In Section 3 we specialize to the quasi-interchange limit, present our derivation of the necessary and sufficient condition for convective stability, and describe the properties of the unstable eigenmodes in plasmas that are very close to marginal stability. In Section 4 we present a heuristic, physical derivation of the stability criterion. We discuss the implications of our work for galaxy-cluster plasmas and the ISM in Sections 5 and 6, respectively. In Section 7 we summarize our results, and in Appendix A we present approximate analytical solutions and numerical solutions to the dispersion relation.

2. THE GENERAL DISPERSION RELATION

We begin with a standard set of two-fluid equations (Drury & Volk 1981; Jones & Kang 1990), which we modify to include thermal conduction along the magnetic field:

$$\frac{d \rho}{d t} = -\rho \nabla \cdot \mathbf{v},$$

$$\frac{d \mathbf{v}}{d t} = -\frac{1}{\rho} \nabla \left( p + p_{\text{cr}} \frac{B^2}{8\pi} \right) + \frac{1}{4\pi \rho} \mathbf{B} \cdot \nabla \mathbf{B} + \mathbf{g},$$

$$\frac{d \mathbf{B}}{d t} = -\mathbf{B} \nabla \cdot \mathbf{v} + \mathbf{B} \cdot \nabla \mathbf{v},$$

$$\frac{dp}{dt} = -\gamma \rho \nabla \cdot \mathbf{v} + (\gamma - 1) \nabla \cdot [\mathbf{b} \kappa_{\parallel} \mathbf{b} \cdot \nabla T],$$

$$\frac{dp_{\text{cr}}}{d t} = -\gamma_{\text{cr}} \rho_{\text{cr}} \nabla \cdot \mathbf{v} + \nabla \cdot [\mathbf{b} D_{\text{cr}} (\mathbf{b} \cdot \nabla p_{\text{cr}})],$$

where $d/dt = \partial / \partial t + \mathbf{v} \cdot \nabla$, and where $\rho$ is the plasma mass density, $\mathbf{v}$ is the velocity, $p$ is the plasma pressure, $p_{\text{cr}}$ is the cosmic-ray pressure, $\gamma$ is the ratio of specific heats of the plasma, $\gamma_{\text{cr}}$ is the effective ratio of specific heats for the cosmic rays, $\mathbf{B}$ is the magnetic field, $\mathbf{b}$ is a unit vector in the direction of the magnetic field, $\kappa_{\parallel}$ is the thermal conductivity along the direction of the magnetic field, $D_{\text{cr}}$ is the cosmic-ray diffusivity along the direction of the magnetic field, and $\mathbf{g}$ is the gravitational acceleration. We have ignored cross-field conduction and diffusion in Equations (5) and (6) since the gyroradii of the thermal particles are small compared to their Coulomb mean free path, and the gyroradii of the cosmic rays are small compared to the mean free path for cosmic-ray scattering. Equations (2)–(6) are closed via the equation of state for an ideal gas:

$$p = C_v (\gamma - 1) \rho T.$$
We take
\[ \mathbf{g} = -g \hat{z}, \]
and consider an equilibrium in which
\[ B_0 = B_0 \hat{y}. \]

All equilibrium quantities (denoted with a “0” subscript) are taken to be functions of \( z \) only, and we set \( v_0 = 0 \). These assumptions lead to a condition for hydrostatic equilibrium of the form
\[ \frac{d}{dz} \left( p_0 + p_{cr,0} + \frac{B_0^2}{8\pi} \right) = -\rho_0 g. \]

To introduce perturbations, we represent the variables in our two-fluid equations as sums of an equilibrium value and a small fluctuating quantity as follows:
\[ \rho = \rho_0 + \delta \rho, \]
\[ p = p_0 + \delta p, \]
\[ \ldots. \]

We employ local analysis, in which we take the fluctuating quantities to be proportional to \( e^{i(kr - \omega t)} \), with
\[ kH \gg 1, \]
where
\[ H = \left| \frac{d \ln \rho_0}{dz} \right|^{-1} \]
is the density scale height, which we take to be comparable to the length scales over which each of the equilibrium quantities varies. Substituting Equations (11) and (12), and analogous expressions for \( p_{cr}, B, \mathbf{v}, \hat{b}, \) and \( T \) into Equations (2)–(6), and into Equation (7), we obtain the following equations for the fluctuating quantities:
\[ -i \omega \frac{\delta \rho}{\rho_0} + i \mathbf{k} \cdot \delta \mathbf{v} + \delta \mathbf{v}_z \frac{d \ln \rho_0}{dz} = 0, \]
\[ -i \omega \delta \mathbf{v}_z = \frac{g}{\rho_0} \delta \rho - i \mathbf{k} \frac{\delta \mathbf{B}_0}{\rho_0} + \gamma^2 \left[ \frac{d \ln B_0}{dz} \frac{\delta B_z}{B_0} \hat{y} + ik \gamma \frac{\delta \mathbf{B}}{B_0} \right], \]
\[ -i \omega \frac{\delta B_z}{B_0} = ik \gamma \delta \mathbf{v}_z - \delta \mathbf{v}_z \frac{d \ln B_0}{dz} - \hat{y}(i \mathbf{k} \cdot \delta \mathbf{v}), \]
\[ -i \omega \left[ \frac{\delta p}{\rho_0} - \gamma \frac{\delta \rho}{\rho_0} + \delta \mathbf{v}_z \frac{d \ln (\rho p^{-\gamma})}{dz} \right] = D_{\text{cond}} \left[ i k \gamma \frac{d \ln T_0}{dz} \frac{\delta B_z}{B_0} - k^2 \frac{\delta T}{T_0} \right], \]
\[ -i \omega \frac{\delta p_{cr}}{p_{cr,0}} + \delta \mathbf{v}_z \frac{d \ln p_{cr,0}}{dz} + \nu_{cr} \mathbf{k} \cdot \delta \mathbf{v} = D_{\|} \left[ i k \gamma \frac{d \ln p_{cr,0}}{dz} \frac{\delta B_z}{B_0} - k^2 \frac{\delta p_{cr}}{p_{cr,0}} \right], \]
\[ \frac{\delta p}{\rho_0} = \frac{\delta \mathbf{v}_z}{T_0} \]
where
\[ D_{\text{cond}} = \frac{(\gamma - 1) \kappa T_0}{\rho_0} \]
and
\[ p_{\text{tot}} = p + p_{cr} + \frac{B_z^2}{8\pi}. \]

Equations (15)–(20) may be reduced to an expression of the form: \( M \cdot \delta \mathbf{v} = 0 \), where \( M \) is a 3 \times 3 matrix. For nontrivial solutions of this equation, we require \( |M| = 0 \), whence we obtain the dispersion relation
\[ A_{\omega \delta \mathbf{v}, \delta \mathbf{v}} + A_2 \omega^2 + A_4 \omega^2 + A_6 = 0, \]
where
\[ A_0 = 1, \]
\[ A_2 = -k^2 (u^2 + v^2_\Lambda) - k^2 v^2_\Lambda + g \frac{d \ln \rho_0}{dz}, \]
\[ A_4 = k^2 k^2 v^2_\Lambda (2u^2 + v^2_\Lambda) - (k^2 + k^2_y) \left[ g^2 + (u^2 + v^2_\Lambda) \frac{d \ln \rho_0}{dz} \right], \]
\[ A_6 = k^2 v^2_\Lambda \left[ -k^2 k^2 v^2_\Lambda u^2 + (k^2 + k^2_y) \left( g^2 + u^2 + v^2_\Lambda \frac{d \ln \rho_0}{dz} \right) \right], \]
and
\[ u^2 = \frac{1}{\rho_0} \left[ p_0 \left( \frac{\gamma \omega + i \eta}{\omega + i \nu} \right) + p_{cr} \frac{\gamma \omega}{\omega + i \nu} \right], \]
and where in Equation (28) we have introduced the quantities
\[ \eta = k^2 D_{\text{cond}}, \]
\[ v = k^2 D_{\|}, \]
\[ v^2_\Lambda = \frac{B_0^2}{8\pi \rho_0}, \]
where \( v_\Lambda \) is the Alfvén speed, and \( \eta \) and \( v \) are, respectively, the rates at which temperature fluctuations and cosmic-ray-pressure fluctuations are smoothed out along the magnetic field. Equations (23)–(28) represent the same result as that presented in Equations (26) and (27) of CD06 and we shall henceforth refer to this result as the “general dispersion relation.” As we shall see, this relation constitutes an eighth-order polynomial equation in \( \sigma = -i \omega \) (where the change of variables is made so as to make all of the polynomial coefficients real). It is worthwhile to note that in the absence of cosmic rays and thermal conductivity
\[ u^2 \rightarrow 0 \]
\[ \frac{\gamma \rho_0}{\rho_0} = c^2, \]
where \( c \) is the adiabatic sound speed, and that if we take this limit, together with the limit of no stratification and \( g \rightarrow 0 \), the general dispersion relation reduces to the well-known dispersion relation obtained in ideal MHD. Thus, the normal modes described by Equation (23) may be viewed as modifications of the Alfvén mode and the fast and slow magnetosonic modes of ideal MHD.

We now present the definitions of a number of frequencies that allow us to write the polynomial form of the dispersion relation more compactly. These are
\[ \omega^2 = \frac{k^2 p_0}{\rho_0}, \]
\[ \omega^2_\Lambda = k^2 v^2_\Lambda, \]
\[ \omega_0^2 = \frac{p_0}{\rho_0} g^2 \sin^2 \theta, \]
The quantities $\omega_1^2$ and $\omega_2^2$ are the squares of the Alfvén and isothermal sound-wave frequencies, respectively. The quantity $\omega_3^2$ is the square of the usual Brunt–Väisälä frequency for buoyancy oscillations in the limit of vanishing cosmic-ray pressure and magnetic field. As we shall see below, the quantities $\omega_1^2$ and $\omega_2^2$ serve to modify the frequency of these oscillations when the cosmic-ray pressure and magnetic field are non-vanishing. The quantities $\omega_2^2$ and $\omega_3^2$ are related to $\omega_1^2$ and $\omega_4^2$ through the following identities:

$$\omega_2^2 = \gamma \omega_1^2 + (\gamma - 1) g \sin^2 \theta \frac{d \ln \rho_0}{dz},$$

$$\omega_3^2 = \gamma \omega_1^2 + \gamma g \sin^2 \theta \frac{d \ln \rho_0}{dz}.$$  

We also define the quantities $W^2$ and $C$:

$$W^2 = \omega_2^2 + \chi \omega_1^2 + \frac{1}{\beta} \omega_3^2,$$

$$C = \omega_4^2 + W^2,$$

where in Equation (44), $\chi = p_{cr,0}/\rho_0$. Finally, noting the identity

$$-g \sin^2 \theta \frac{d \ln \rho_0}{dz} = \omega_2^2 - \omega_4^2 + C,$$

we find that the general dispersion relation may be written as

$$a_0 \sigma^8 + a_1 \sigma^7 + \cdots + a_7 \sigma + a_8 = 0,$$

where

$$a_0 = \omega_4^2,$$

$$a_1 = (\nu + \eta) \omega_4^2,$$

$$a_2 = \left[ (\nu \chi \gamma_\nu + \frac{2}{\beta} \gamma) + \nu \eta + \omega_1^2 - g \frac{d \ln \rho_0}{dz} \right] \omega_4^2,$$

$$a_3 = \left[ (\nu \chi \gamma_\nu + \frac{2}{\beta} \gamma) + \nu \eta + \omega_1^2 - g \frac{d \ln \rho_0}{dz} \right] \omega_4^2,$$

$$a_4 = \nu \eta \left[ (\nu \chi \gamma_\nu + \frac{2}{\beta} \gamma) + \omega_1^2 - g \frac{d \ln \rho_0}{dz} \right] \omega_4^2,$$

$$a_5 = \left[ (\nu \chi \gamma_\nu + \frac{2}{\beta} \gamma) + \nu \eta + \omega_1^2 - g \frac{d \ln \rho_0}{dz} \right] \omega_4^2,$$

$$a_6 = \nu \eta \left[ (\nu \chi \gamma_\nu + \frac{2}{\beta} \gamma) + \omega_1^2 - g \frac{d \ln \rho_0}{dz} \right] \omega_4^2,$$

$$a_7 = \nu \eta \left[ (\nu \chi \gamma_\nu + \frac{2}{\beta} \gamma) + \omega_1^2 - g \frac{d \ln \rho_0}{dz} \right] \omega_4^2,$$

$$a_8 = \nu \eta \omega_4^2.$$

We assume that $|d \ln \rho_0/dz|, |d \ln p_{cr,0}/dz|, |d \ln B_0^2/dz|$ are of order $H^{-1}$. We may thus conclude from Equation (10) that $g \sim p_{cr,0} H^{-1}/\rho_0$. We also assume that $p_{cr,0}$ is not much greater than $p$. Our assumption that $|k H \gg 1|$ then allows us to write that

$$\omega_4^2 \frac{d \ln \rho_0}{dz} \sim (k H)^{-1} \ll 1.$$  

This inequality enables us to drop the $\omega_4^2 \frac{d \ln \rho_0}{dz}$ terms in Equations (50)–(52).

As a check on the results of this section, we show in Appendix B that Equation (47) reduces properly to the results obtained by Parker (1966, 1967) and Shu (1974) when cosmic-ray diffusivity is taken to be infinite and thermal conduction is negligible, and when the results of Parker (1966, 1967) and Shu (1974) are considered in the short-wavelength limit.

### 3. THE QUASI-INTERCHANGE LIMIT

The most unstable modes in a gravitationally stratified adiabatic plasma threaded by a horizontal magnetic field are those for which $|k_z|$ is very large, so that

$$|k_z H| \gg 1,$$

$$|k_z| \gg |k_x|,$$

$$|k_z| \gg |k_x|,$$

$$|k_x| \gg |k_z|,$$

where

$$\sin^2 \theta \rightarrow 1$$

(Parker 1967; Shu 1974; Ferriere et al. 1999). We conjecture that the same is true when thermal conduction is taken into account, at least when the equilibrium magnetic field is horizontal, and thus we focus on this limit, which we call the "quasi-interchange limit."

For very large $|k_z|$, one set of modes consists of high-frequency magnetosonic-like waves. In the $\beta \gg 1$ limit, these waves are stable (CD06), and we assume they are stable here as well. We note, however, that in the presence of an equilibrium heat flux (i.e., $B_{0z} \neq 0$) anisotropic conduction can cause magnetosonic waves to become overstable (Socrates et al. 2008). To filter out these high-frequency waves, we assume that

$$\sigma \ll \omega_4.$$

We also assume that $|k_x|/|k_z|$ is sufficiently large that

$$\frac{\omega_4}{\omega_4} \ll 1 \quad \text{and} \quad \omega_4 \ll 1.$$
Using these inequalities and Equation (57), we can rewrite the general dispersion relation as a sixth-degree polynomial equation,

\[ b_0\sigma^6 + b_1\sigma^5 + b_2\sigma^4 + b_3\sigma^3 + b_4\sigma^2 + b_5\sigma + b_6 = 0, \]  

where

\[ b_0 = \gamma + \chi\gamma + \frac{2}{\beta}, \]

\[ b_1 = v \left( \gamma + \frac{2}{\beta} \right) + \eta \left( 1 + \chi\gamma + \frac{2}{\beta} \right), \]

\[ b_2 = (\gamma + \chi\gamma)(\omega_0^2 + \omega_0^4 + C) - \omega_0^2 \]

\[ + \frac{2}{\beta}(\omega_0^2 + C) + \nu \eta \left( 1 + \frac{2}{\beta} \right), \]

\[ b_3 = [\eta(1 + \chi\gamma) + v\gamma](\omega_0^2 + \omega_0^4 + C) \]

\[ + (v + \eta) \left[ \frac{2}{\beta}(\omega_0^2 + C) - \omega_0^2 \right], \]

\[ b_4 = \omega_0^2(\gamma + \chi\gamma)(\omega_0^2 + \omega_0^4) \]

\[ + \nu \eta \left[ \omega_0^2 + C + \frac{2}{\beta}(\omega_0^2 + C) \right], \]

\[ b_5 = \omega_0^2(\gamma + \chi\gamma)(\omega_0^2 + \omega_0^4) \]

\[ + (v + \eta)(1 + \chi\gamma)(\omega_0^2 + \omega_0^4) \]

\[ + \nu \eta b_1 \left[ \omega_0^2 + \frac{2}{\beta}\omega_0^2 \right] + \frac{2}{\beta}(v + \eta)(\omega_0^2 - \omega_0^4)^2, \]

\[ b_6 = v\eta \omega_0^2 C. \]  

### 3.1. Stability Criterion

To obtain the stability criterion for the modes described by Equation (66), we use the Routh–Hurwitz theorem (see, for example, Levinson & Redheffer 1970). To apply this theorem, we construct the matrix \( R \) from the (real) coefficients of the polynomial in Equation (66), where

\[
R = \begin{pmatrix}
  b_1 & b_3 & b_5 & 0 & 0 & 0 \\
  b_0 & b_2 & b_4 & b_6 & 0 & 0 \\
  0 & b_1 & b_3 & b_5 & 0 & 0 \\
  0 & b_0 & b_2 & b_4 & b_6 & 0 \\
  0 & 0 & b_1 & b_3 & b_5 & 0 \\
  0 & 0 & 0 & b_1 & b_3 & b_5
\end{pmatrix}.
\]  

The Routh–Hurwitz theorem then states that for the real parts of the roots of Equation (66) to all take on negative values, it is a necessary and sufficient condition that the determinants of the principle minor matrices \( M_i \) of \( R \) all be positive-definite. This necessary and sufficient condition is the stability criterion for our plasma. The determinants of the principle minors of \( R \) are

\[ \text{det}(1) = b_1, \]  

\[ \text{det}(2) = \begin{vmatrix} b_1 & b_3 \\ b_0 & b_2 \end{vmatrix}, \]  

\[ \text{det}(3) = \begin{vmatrix} b_1 & b_5 \\ b_0 & b_3 \end{vmatrix}, \]  

\[ \text{det}(4) = \begin{vmatrix} b_1 & b_3 & b_5 & 0 \\ b_0 & b_2 & b_4 & b_6 \\ 0 & b_1 & b_3 & b_5 \\ 0 & b_0 & b_2 & b_4 \end{vmatrix}, \]  

\[ \text{det}(5) = \begin{vmatrix} b_1 & b_3 & b_5 & 0 & 0 \\ b_0 & b_2 & b_4 & 0 \end{vmatrix}, \]  

\[ \text{det}(6) = |R|. \]  

After some algebra, we find that these determinants may be expressed as

\[ \text{det}(1) = v \left( \gamma + \frac{2}{\beta} \right) + \eta \left( 1 + \chi\gamma + \frac{2}{\beta} \right), \]  

\[ \text{det}(2) = b_1 \nu \eta \left( 1 + \frac{2}{\beta} \right) + J \left( \omega_0^2 + \frac{2}{\beta}\omega_0^2 \right), \]  

\[ \text{det}(3) = b_1 J \omega_0^2(\omega_0^2 + W^2) + JK \left( \omega_0^4 + \frac{2}{\beta}\omega_0^4 \right) \]

\[ + v\eta b_1 \left( \omega_0^2 + \frac{2}{\beta}\omega_0^2 \right) + \frac{2}{\beta}(v + \eta)(\omega_0^2 - \omega_0^4)^2, \]

\[ \text{det}(4) = \frac{2}{\beta} J^2 \omega_0^2\omega_0^2[W^2 + \omega_0^4]^2 \]

\[ + J \nu \eta \left\{ (v + \eta)(\omega_0^2 + \omega_0^4)^2 \right\} + K(C^2\omega_0^4 + C\omega_0^4) \]

\[ + \frac{2}{\beta}(v + \eta)(\omega_0^2 - \omega_0^4)^2 + \frac{2}{\beta}K(\omega_0^4 + \omega_0^4 - \omega_0^4)^2 + \omega_0^2 + \omega_0^2)^2 \]

\[ + (v\eta)^2 b_1 K \left( 1 + \frac{2}{\beta} \right) \omega_0^2 C \]

\[ + \frac{2}{\beta}(\omega_0^2 - \omega_0^4)^2, \]

\[ \text{det}(5) = \frac{2}{\beta} \omega_0^2\omega_0^2(W^2 + \omega_0^4)^2 \]

\[ \times \left[ b_1 C + \omega_0^2(v + \eta) + k(\omega_0^2 + \omega_0^4) + \frac{2}{\beta}\omega_0^2(v + \eta) \right] \]

\[ + J^2 \omega_0^2[K\omega_0^4 + (K + v + \eta)C], \]  

and

\[ \text{det}(6) = |R| = b_6 \text{det}(5). \]

The quantities \( J \) and \( K \) appearing in the above expressions are defined as

\[ J = (\gamma - 1)\eta + \chi\gamma v, \]

and

\[ K = (\gamma - 1)\nu + \chi\gamma v, \]

and are always positive.

We first consider the case \( k_\gamma \neq 0 \). In this case, \( J, K \), and the first two determinants are seen to be composed of sums of positive-definite quantities and so are themselves positive definite. By inspection, \( \text{det}(3) \) through \( \text{det}(6) \) are positive if \( C > 0 \), and thus \( C > 0 \) is a sufficient condition for stability. On the other hand, Equation (86) shows that if \( C < 0 \), then either \( \text{det}(5) \) or \( \text{det}(6) \) is negative. Therefore, \( C > 0 \) is also a necessary
condition for stability. If we fix the wave vector \( \mathbf{k} \), taking \( k_y \neq 0 \), the necessary and sufficient condition for modes at that \( \mathbf{k} \) to be stable is then

\[
C > 0. \tag{89}
\]

Since \( C = W^2 + k_y^2 v_A^2 \), the smallest value of \( C \) is obtained in the limit \( k_y \to 0 \). The necessary and sufficient condition for the plasma to be stable at all wave vectors in the quasi-interchange limit is thus

\[
W^2 > 0. \tag{90}
\]

Using the definition of \( W^2 \) given in Equation (44), and the definitions of the frequencies \( \omega_z^2 \), \( \omega_\gamma^2 \), and \( \omega_A^2 \) given in Equations (37), (39), and (40), respectively, we can rewrite Equation (90) as

\[
nk_B \frac{dT}{dz} + \frac{dp_c}{dz} + \frac{2}{8\pi} \frac{dB^2}{dz} > 0,
\]

where we have dropped the zero subscripts on the equilibrium quantities. Equation (91) shows that an “upwardly decreasing” temperature, cosmic-ray pressure, or magnetic pressure is destabilizing.

We next consider the special case, \( k_y = 0 \), which corresponds to pure interchanges. In this case, Equation (66) leads to the two non-trivial solutions,

\[
\sigma = \pm \sqrt{-\frac{b}{b_0}}, \tag{92}
\]

where \( b \) is what remains of the coefficient \( b_2 \) at \( k_y = 0 \). By inspection we see that the necessary and sufficient condition for these modes to be stable is \( b > 0 \). For a vanishing cosmic-ray pressure, the condition \( b > 0 \) reduces to

\[
- \frac{dp_c}{dz} > \frac{\rho^2 g}{\gamma p + B^2/4\pi}, \tag{93}
\]

where we have again dropped the zero subscripts on the equilibrium quantities. Equation (93) is the result of Tserkovnikov (1960) for the pure interchanges as quoted by Newcomb (1961). The criterion \( W^2 > 0 \) obtained above for the case \( k_y \neq 0 \) is more restrictive than the condition \( b > 0 \), since \( \gamma > 1 \). Thus, \( W^2 > 0 \) is the necessary and sufficient condition for the plasma to be stable to all modes in the quasi-interchange limit, including those with \( k_y = 0 \).

### 3.2. Eigenmodes Near Marginal Stability

In this section, we consider the properties of unstable modes very near the limit of marginal stability. We assume that \( k_y \neq 0 \), but take the limit \( k_y H \ll 1 \)—that is, the parallel wavelength is much longer than the scale height. Near marginal stability, the quantity \( b_6 \) in Equation (66) approaches zero. Thus, there exists a solution to the dispersion relation in which \( \sigma \) also approaches zero, for which the terms proportional to \( \sigma^2 \) through \( \sigma^5 \) in Equation (66) can be neglected. This solution satisfies the approximate equation

\[
\sigma \simeq -\frac{b_6}{b_5}. \tag{94}
\]

Making use of the fact that \( \sigma \to 0 \) for this mode, we can return to the results of Section 2 (in particular, the equation \( M \cdot \delta \mathbf{v} = 0 \)) and show that, to leading order in \( k_y H \),

\[
\frac{k_x \delta v_x}{k_y \delta v_y} \simeq \frac{i k_z p}{\rho g}, \tag{95}
\]

and

\[
\frac{k_x \delta v_x + k_y \delta v_y}{\rho g} \ll \frac{k_y \delta v_y}{\rho g}. \tag{96}
\]

so that

\[
|k_x \delta v_x + k_y \delta v_y| \ll |k_y \delta v_y|. \tag{97}
\]

Thus, for modes with \( |k_y H| \ll 1 \) near marginal stability, most of the compression or expansion of the plasma occurs in the direction of the magnetic field rather than perpendicular to the magnetic field, despite the fact that \( k_y \) is very small. We discuss the importance of this result further in the following section.

### 4. HEURISTIC DERIVATION OF STABILITY CRITERION

In this section, we present a way of understanding the stability criterion in Equation (1) in physical terms. We consider the same equilibrium discussed in Section 2, in which \( g = -g \hat{z} \) and \( B_0 = B_0 \hat{y} \), and we again take the plasma to be perfectly conducting, so that magnetic field lines are frozen-in to the fluid. However, now we assume that the equilibrium is very close to marginal stability. We then imagine some mode in the plasma that causes a long and narrow magnetic flux tube to rise upwards, as depicted in Figure 1. For simplicity, we assume that the ends of the flux tube are anchored at the flux tube’s initial height. We take the flux tube to be very long, so that magnetic tension forces are very weak. Because the medium is arbitrarily close to marginal stability, the growth time or oscillation time for the mode is arbitrarily long. Thus, even though the flux tube is long, there is plenty of time for conduction and diffusion to equalize \( T \) and \( p_{cr} \) along the perturbed magnetic field lines. We assume that the total pressure, \( p_{tot} = p + p_{cr} + B^2/8\pi \), at each point along the flux tube is equal to the total pressure just outside the flux tube at that point.\(^3\)

We define \( \Delta n \), \( \Delta T \), \( \Delta B^2 \), and \( \Delta p_{cr} \), respectively, as the difference between the density, temperature, field-strength-squared, and cosmic-ray pressure at the highest point in our flux tube and the immediately surrounding medium, at a point in time when the top of the flux tube is a small distance \( \Delta z \) above the flux tube’s initial height. The constancy of \( T \) and \( p_{cr} \) along the flux tube yields the relations (accurate to first-order in \( \Delta z/H \))

\[
\Delta T = -\frac{dT}{dz} \tag{98}
\]

\(^3\) Total-pressure variations are associated with high-frequency magnetosonic waves. These waves are stable at \( \beta \gg 1 \) (CD06), and we assume they are stable here as well. However, we note that Socrates et al. (2008) have shown that magnetosonic waves can become unstable in the presence of an equilibrium heat flux, when \( B_{0x} \neq 0 \).
and
\[ \Delta p_B = -\Delta \frac{d p_B}{d z} \, , \quad (99) \]
where \( dT/dz \) and \( dp_B/dz \) are the gradients of the equilibrium temperature and cosmic-ray pressure evaluated at the initial height of the flux tube.

Equations (98) and (99) tell us how to evaluate \( T \) and \( p_B \) in our flux tube. Evaluating \( B^2 \) in the flux tube is a little more involved. Assuming that the total pressure decreases with height, the fluid in the flux tube has to expand in order to achieve total-pressure balance. However, the manner in which the flux tube expands is not obvious. If the plasma expands primarily along the magnetic field, the cross-sectional area of the flux tube will be constant along the flux tube, and thus so will the magnetic field strength.

On the other hand, if the plasma expands perpendicular to the field, the magnetic field strength will decrease. Which type of expansion does the plasma favor? We answer this question analytically in Section 3.2, where we show that, near marginal stability, \( k_x \delta v_x \gg k_y \delta v_y \) for the low-frequency long-parallel-wavelength buoyancy instability in the large-\( k_z \) limit. Thus, for this mode, most of the expansion \( \nabla \cdot \mathbf{v} \) arises from the parallel motion. We note that this statement is stronger than the statement that \( |v_y| \gg |v_x| \), because we take \( |k_x| \gg |k_y| \).

How can we understand this result in physical terms? One way is by analogy to the \( \delta W \) analysis of the stability of ideal MHD plasmas, in the absence of thermal conduction (Bernstein et al. 1958; Friedberg 1987). In this analysis, it is shown that if a mode expands in the direction perpendicular to the magnetic field, additional work must be done on the surrounding magnetic field. This requirement makes the mode more stable. To find the stability criterion, we must seek out the most unstable mode, which in this case is a mode that keeps the cross-sectional area of the flux tube constant.

Taking the cross-sectional area of the flux tube to be constant, we can treat \( B^2 \) as constant along the flux tube. This allows us to write that
\[ \Delta B^2 = -\Delta \frac{d B^2}{d z} \, . \quad (100) \]
The condition that the total pressure inside the flux tube matches the total pressure outside the flux tube can be written as
\[ k_B T \Delta n + n k_B \Delta T + \Delta p_B + \frac{\Delta B^2}{8 \pi} = 0 \, . \quad (101) \]

Together, Equations (98)–(101) imply that
\[ k_B T \Delta n = \Delta \left( n k_B \frac{dT}{d z} + p_B \frac{d p_B}{d z} + \frac{1}{8 \pi} \frac{d B^2}{d z} \right) \, . \quad (102) \]
The stability criterion, Equation (1), is thus the condition that the material inside an upwardly displaced, long, and narrow flux tube is denser than the surrounding medium.

5. CONVECTION IN GALAXY-CLUSTER PLASMAS

In many galaxy-cluster cores, the radiative cooling time is much shorter than the cluster’s likely age (Fabian 1994). Nevertheless, high-spectral-resolution X-ray observations show that very little plasma actually cools to low temperatures (Böhringer et al. 2001; David et al. 2001; Molendi & Pizzolato 2001; Peterson et al. 2001, 2003; Tamura et al. 2001; Blanton et al. 2003). This finding, sometimes referred to as the “cooling-flow problem,” strongly suggests that plasma heating approximately balances radiative cooling in cluster cores.

A heating mechanism for cluster cores that has been studied extensively is heating by a central active galactic nucleus (AGN). The importance of such “AGN feedback” is suggested by the observation that almost all clusters with strongly cooling cores possess active, central radio sources (Burns 1990; Ball et al. 1993; Eilek 2004) and by the correlation between the X-ray luminosity from within a cluster’s cooling radius and the mechanical luminosity of a cluster’s central AGN (Birzan et al. 2004; Eilek 2004). One of the main unsolved problems regarding AGN feedback is to understand how AGN power is transferred to the diffuse ambient plasma. A number of mechanisms have been investigated, including Compton heating (Binney & Tabor 1995; Ciotti & Ostriker 1997, 2001; Ciotti et al. 2004; Sazonov et al. 2005), shocks (Tabor & Binney 1993; Binney & Tabor 1995), MHD wave-mediated plasma heating by cosmic rays (Böhringer & Moffill 1988; Rosner & Tucker 1989; Loewenstein et al. 1991), and cosmic-ray bubbles produced by the central AGN (Churazov et al. 2001, 2002; Reynolds 2003; Reynolds et al. 2005), which can heat intracluster plasma by generating turbulence (Loewenstein & Fabian 1990; Churazov et al. 2004; Cattaneo & Teyssier 2007) and sound waves (Fabian et al. 2003; Ruszkowski et al. 2004a, 2004b) and by doing \( pdV \) work (Begelman 2001, 2002; Ruszkowski & Begelman 2002; Hoef & Brüggen 2004).

Another way in which central AGNs may heat the ICM is by accelerating cosmic rays that mix with the intracluster plasma and cause the ICM to become convectively unstable. A steady-state, spherically symmetric, mixing-length model based on this idea was developed by Chandran (2004) and subsequently refined by Chandran (2005) and Chandran & Rasera (2007). In this model, a central supermassive black hole accretes hot intracluster plasma at the Bondi rate (Bondi 1952), and converts a small fraction of the accreted rest-mass energy into cosmic rays that are accelerated by shocks within some distance \( r_{\text{source}} \) of the center of the cluster. The resulting cosmic-ray pressure gradient leads to convection, which in turn heats the thermal plasma in the cluster core by advecting internal energy inwards and allowing the cosmic rays to do \( pdV \) work on the thermal plasma. The model also includes thermal conduction, cosmic-ray diffusion, and radiative cooling. By adjusting a single parameter in the model \( (r_{\text{source}}) \), Chandran & Rasera (2007) were able to achieve a good match to the observed density and temperature profiles in a sample of eight clusters.

The treatment of convective stability in the work of Chandran (2004, 2005) and Chandran & Rasera (2007) was based on the assumption that \( \beta = 8 \pi p / B^2 \gg 1 \). The present paper investigates convective stability for arbitrary \( \beta \). One of the motivations for this work is the possibility that some clusters with short central cooling times (cooling-core clusters) may be in the \( \beta \sim 1 \) regime. For a fully ionized plasma with a hydrogen mass fraction \( X = 0.7 \) and helium mass fraction \( Y = 0.29 \),
\[ \beta = 6.3 \times \left( \frac{n_e}{10^{-2} \text{ cm}^{-3}} \right) \left( \frac{k_B T}{3 \text{ keV}} \right) \left( \frac{B}{10 \mu \text{G}} \right)^{-2} \, . \quad (103) \]
Although many studies of the magnetic field strength in clusters of galaxies find \( B \) in the range of 1–5 \( \mu \text{G} \) (e.g., Kronberg 1994; Taylor et al. 2001; Eilek & Owen 2002), some studies of Faraday rotation in cooling-core clusters find much stronger magnetic fields (Taylor & Perley 1993; Kronberg 1994; Taylor et al. 2002). In the case of Hydra A, Taylor & Perley (1993) found a tangled magnetic field of \( \sim 30 \mu \text{G} \), and Taylor et al. (2002) found a tangled magnetic field of \( \sim 35 \mu \text{G} \). The analysis of X-ray observations of Hydra A
carried out by Kastra et al. (2004), when converted to a CDM cosmology (see Chandran & Rasera 2007), indicates that \( n_e \approx 0.01 \text{ cm}^{-3} \) and \( k_B T \approx 3.4 \text{ keV} \) in Hydra A at \( r = 50 \text{ kpc} \). Equation (103) thus shows that if \( B \) is indeed as large as 30 \( \mu \text{G} \) in the core of Hydra A, then \( \beta \) is of order unity. Values of \( \beta \sim 0.1-1 \) for cluster cores in several other galaxy clusters were reported by Eilek & Owen (2002). Although these studies suggest that \( \beta \sim 1 \) magnetic fields could be common in cooling-core clusters, some caution is warranted here. Vogt & Ensslin (2005) have re-analyzed the Faraday-rotation data for Hydra A using an updated plasma-density profile, and found an rms magnetic field of 7 \( \mu \text{G} \), which corresponds to \( \beta \approx 15 \) at \( r = 50 \text{ kpc} \) in Hydra A. In the remainder of this section, we explore the implications of the condition \( \beta \lesssim 1 \) on convective instability in clusters, but the above uncertainty in the value of \( \beta \) in cooling-core clusters should be born in mind.

In Section 3, we showed that the necessary and sufficient condition for stability for a mode with fixed nonzero \( k_z \) in the quasi-interchange limit (\(|k_x|, |k_y|, |k_z|\), and \( H^{-1} \)) is

\[
k_y^2 v_A^2 + g \left( \frac{d \ln T}{dz} + \frac{\rho c}{p} \frac{d \ln \rho c}{dz} + \frac{B^2}{8\pi p} \frac{d \ln B^2}{dz} \right) > 0. \tag{104}
\]

This equation shows that the magnetic field has two competing effects on convective stability. First, if the field strength decreases “upwards” (i.e., \( d B^2/dz < 0 \)), the \( g B^{-1} d \ln B^2/dz \) “magnetic-buoyancy term” in Equation (104) is destabilizing. On the other hand, the \( k_y^2 v_A^2 \) “magnetic-tension term” is stabilizing. We can estimate the relative importance of the different terms in Equation (104) by defining the length scales \( H_f \), \( H_B \), and \( H_p \) via the equations

\[
H_f^{-1} = \left| \frac{d \ln T}{dz} + \frac{\rho c}{p} \frac{d \ln \rho c}{dz} \right|, \tag{105}
\]

\[
H_B^{-1} = \left| \frac{d \ln B}{dz} \right|, \tag{106}
\]

and

\[
H_p^{-1} = \frac{\rho g}{p}. \tag{107}
\]

The ratio of the magnetic-tension term to the magnetic-buoyancy term is then

\[
\frac{k_y^2 v_A^2}{2gB^{-1}H_B^{-1}} = k_y^2 H_B H_p, \tag{108}
\]

while the ratio of the magnetic-tension term to the “fluid terms,”

\[ g[d \ln T/dz + (\rho c/p)d \ln \rho c/dz] \]

is

\[
k_y^2 v_A^2 \frac{H_f^{-1}}{g H_f^{-1}} = 2\beta^{-1} k_y^2 H_f H_f. \tag{109}
\]

The magnetic field turns off the buoyancy instability at wave vectors for which the magnetic-tension term dominates over both the magnetic-buoyancy term and the fluid terms. At \( \beta \sim 1 \), this happens for \( k_y^2 H_p H_f \gg 1 \) and \( k_z H \gg 1 \). If we take all the scale lengths to be comparable to the density scale height \( H \), then at \( \beta \sim 1 \) magnetic tension turns off the instability for \( k_z \sim 1 \), but it is negligible for \( k_z H \ll 1 \). At \( \beta \sim 1 \) and \( k_z H \sim 1 \), the tension, buoyancy, and fluid terms are all comparable, and magnetic buoyancy and magnetic tension to some extent cancel out. When \( \beta \ll 1 \), magnetic tension dominates for \( k_z \gg 1 \), magnetic buoyancy dominates for \( k_z \ll 1 \), and the two are comparable at \( k_z H \sim 1 \), again assuming that \( H_p \sim H_B \sim H_f \sim H \).

To apply our results to galaxy-cluster plasmas, we imagine some hypothetical spherical equilibrium, and consider local modes at a radius \( r \) at some location where the radial component of the magnetic field vanishes, and where all the scale lengths are of order \( r \). Our local analysis of a slab-symmetric equilibrium is strictly applicable only to modes with \( k_r \gg 1 \), that is, to modes with parallel wavelengths much less than the scale height. Our results show that such modes are stable when \( \beta \lesssim 1 \) because of the stabilizing effects of magnetic tension.

6. THE PARKER INSTABILITY IN THE ISM

The Parker instability is an unstable mode in a gravitationally stratified plasma that is driven by the buoyancy of the magnetic field and/or cosmic rays (Parker 1966, 1967). The Parker instability is thought to be important for the ISM for several reasons. It has been argued that this mode, acting alone or in concert with the thermal instability (Field 1965), contributes to the formation of molecular clouds (Blitz & Shu 1980; Parker & Jokipii 2000; Kosinski & Hanasz 2005, 2006, 2007). It has also been suggested that the Parker instability is a mechanism for regulating the transport of magnetic fields and cosmic rays in the direction perpendicular to the galactic plane, and for driving the Galactic dynamo (see e.g., Parker 1992; Hanasz & Lesch 2000; Hanasz et al. 2004).

The Parker instability is very similar to the instability that we have investigated in this paper. Standard analyses of the Parker instability consider an equilibrium in which \( q = -g \vec{z} \), \( \vec{B} \) is in the \( xy \)-plane, \( \rho \propto \exp(-z/H) \), and \( H, T, \beta \), and \( p_c/p \) are constant (Parker 1966, 1967; Shu 1974; Ryu et al. 2003). In early studies, the parallel cosmic-ray diffusion coefficient \( D_{\parallel} \) was taken to be infinite, since \( p_c \) was assumed to be constant along magnetic field lines (Parker 1966, 1967; Shu 1974). On the other hand, Ryu et al. (2003) considered the effects of finite \( D_{\parallel} \), as well as cosmic-ray diffusion perpendicular to magnetic field lines. All of these studies took the thermal plasma to be adiabatic.

The analysis of the present paper extends our understanding of the Parker instability in two ways. First, we allow the equilibrium values of \( T, \beta \), and \( p_c/p \) to vary with \( z \). Second, we consider the effects of anisotropic thermal conduction. By doing so, we show that the condition \( dT/dz < 0 \) makes a plasma more unstable to the Parker instability than when the equilibrium is isothermal. We also show that even if the equilibrium is isothermal, anisotropic thermal conduction makes a stratified plasma more unstable to the Parker instability than when the plasma is treated as adiabatic. The Parker stability criterion in the limit \( |k_x| \to \infty \) for the equilibrium described above can be obtained from the \( k_y \to 0 \) limit of Equation (73) of Shu (1974):

\[
\frac{B^2}{8\pi} + \frac{p_c}{\rho} < (\gamma - 1)p. \tag{110}
\]

Multiplying this equation by \(-1/H\) and making use of the assumptions that \( dB^2/dz = -B^2/H \), \( dp_c/dz = -p_c/H \), and \( dp/dz = -p/H \), we can rewrite Equation (110) as

\[
\frac{d}{dz} \left( \frac{B^2}{8\pi} + p_c \right) > -\left( \frac{\gamma - 1}{H} \right)p \tag{111}
\]

On the other hand, when anisotropic thermal conduction is taken into account, the stability criterion for this constant-temperature
equilibrium from Equation (91) is
\[ \frac{d}{dz} \left( \frac{B^2}{8\pi} + p_{\text{cr}} \right) > 0. \]  
(112)
Since \( \gamma > 1 \), Equation (112) is more restrictive than Equation (111), and anisotropic thermal conduction allows for instability under a larger range of equilibria than when the plasma is taken to be adiabatic. The reason for this is that as a fluid parcel rises and expands, anisotropic thermal conduction allows heat to flow up along the magnetic field lines into the rising fluid parcel. This heat flow increases the temperature of the rising fluid parcel relative to the adiabatic case and thereby lowers the density, making the fluid parcel more buoyant, as in the high-\( \beta \) zero-\( p_{\text{cr}} \) limit considered by Balbus (2000, 2001).

7. CONCLUSION

In this paper we derive the stability criterion for local buoyancy instabilities in a stratified plasma, with the equilibrium magnetic field in the \( \hat{y} \)-direction and gravity in the \( -\hat{z} \)-direction. We take into account cosmic-ray diffusion and thermal conduction along magnetic field lines and focus on the large-\(|k_y| \) limit, which is the most unstable limit for adiabatic plasmas. Our work extends the earlier work of Balbus (2000, 2001) and CD06 by allowing for arbitrarily strong magnetic fields. Applying our work to galaxy-cluster plasmas, we find that increasing the magnetic field to the point that \( k_y = k_{\text{crit}} \) we have \( \omega_{\text{ref}} = \eta \sim \nu \). These definitions and observations allow us to define two limits for which it is possible to derive approximate analytical solutions to Equation (66); the “long-parallel-wavelength” limit \(|k_y| \ll k_{\text{crit}} \) for which the diffusive frequencies \( \nu \) and \( \eta \) are small compared to the buoyancy frequencies, and the “short-parallel-wavelength” limit \(|k_y| \gg k_{\text{crit}} \) for which the diffusive frequencies are large compared to the buoyancy frequencies. In the following subsections, we consider each of these limits in turn, and as a further check we also compare their high-\( \beta \) limits with the results of CD06.

A.1. Long-Parallel-Wavelength Limit

As stated above, in the long-parallel-wavelength limit we have \( \eta, \nu \ll \omega_{\text{ref}}^2 \), while all of the other quantities listed in Equations (33)-(40) that remain in the dispersion relation (Equation 66) are \( \sim \omega_{\text{ref}}^2 \). We take advantage of this and apply the method of dominant balance (Bender & Orszag 1978). We set \( \sigma = \omega_{\text{ref}} \tilde{\sigma}, \omega_0 = \omega_{\text{ref}} \tilde{\omega}_0, \omega_1 = \omega_{\text{ref}} \tilde{\omega}_1, \omega_2 = \omega_{\text{ref}} \tilde{\omega}_2, \omega_3 = \omega_{\text{ref}} \tilde{\omega}_3, \omega_4 = \omega_{\text{ref}} \tilde{\omega}_4, \omega_5 = \omega_{\text{ref}} \tilde{\omega}_5, \omega_\Lambda = \omega_{\text{ref}} \tilde{\omega}_\Lambda, \eta = \epsilon \omega_{\text{ref}} \tilde{\eta}, \nu = \epsilon \omega_{\text{ref}} \tilde{\nu} \), where \( \epsilon \ll 1 \), and substitute these into Equation (66) leading to the result
\[ \tilde{b}_0 \sigma^6 + \tilde{b}_1 \sigma^5 + \tilde{b}_2 \sigma^4 + \tilde{b}_3 \sigma^3 + \tilde{b}_4 \sigma^2 + \tilde{b}_5 \sigma + \tilde{b}_6 = 0, \]  
(A3)
where the coefficients are now given by
\[ \tilde{b}_0 = \gamma + \chi \gamma_c + \frac{2}{\beta}, \]  
(A4)
\[ \tilde{b}_1 = \epsilon \left[ \tilde{v} \left( \gamma + \frac{2}{\beta} \right) + \tilde{\eta} \left( 1 + \chi \gamma_c + \frac{2}{\beta} \right) \right], \]  
(A5)
\[ \tilde{b}_2 = (\gamma + \chi \gamma_c) \left( \tilde{\omega}_0^2 + \tilde{\omega}_0^2 + \tilde{\eta}^2 \right) + \frac{2}{\beta} \left( \tilde{\omega}_0^2 + \tilde{\eta}^2 \right) \]  
(A6)
\[ \tilde{b}_3 = \epsilon \left[ \tilde{v} \tilde{\eta} \left( 1 + \chi \gamma_c \right) + \tilde{v} \gamma \left( \tilde{\omega}_0^2 + \tilde{\omega}_0^2 + \tilde{\eta}^2 \right) + \tilde{v}^2 \tilde{\eta} \right] \]  
(A7)
\[ \tilde{b}_4 = \tilde{\omega}_0^2 \left[ \left( \gamma + \chi \gamma_c \right) \left( \tilde{\omega}_0^2 + \tilde{\eta}^2 \right) + \tilde{\omega}_0^2 \right] + \epsilon^2 \tilde{v} \tilde{\eta} \]  
(A8)
\[ \tilde{b}_5 = \epsilon \tilde{\omega}_0^2 \left[ \left( \tilde{\omega}_0^2 + \tilde{\eta}^2 \right) \tilde{v} \gamma + \tilde{\eta} \left( 1 + \chi \gamma_c \right) + \tilde{v} \tilde{\eta} \tilde{\omega}_0^2 \right] \]  
(A9)
\[ \tilde{b}_6 = \epsilon^2 \tilde{v} \tilde{\eta} \tilde{\omega}_0^2 \tilde{\omega}_0^2 \tilde{\gamma}. \]  
(A10)
We may now solve for the two sets of approximate solutions to Equation (A3) by assuming that one set will satisfy \( \sigma \sim \omega_{\text{ref}} \), while the other satisfies \( \sigma \sim \epsilon \omega_{\text{ref}} \). We present each of these solutions in turn in the following subsections.

A.1.1. Adiabatic-Buoyancy Modes

We refer to the set of solutions satisfying \( \sigma \sim \omega_{\text{ref}} \) as the “adiabatic-buoyancy modes” since these are of the order of the buoyancy frequencies except when near the limit of marginal stability. To obtain expressions for these modes, we set
\[ \tilde{\sigma} = \tilde{\sigma}_0 + \epsilon \tilde{\sigma}_1 + \cdots, \]  
(A11)
We note for clarity that in the limiting case of high-\( \beta \) discussed in CD06, these modes were referred to as the “adiabatic convective/buoyancy modes.”

APPENDIX A

APPROXIMATE ANALYTICAL SOLUTIONS AND COMPARISONS TO NUMERICAL SOLUTIONS

In this Appendix, in order to provide further insight into buoyancy instabilities in the equilibrium described in Section 2, we derive a set of approximate analytical solutions to Equation (66). To check our analytical solutions, we compare them with numerical solutions of the general dispersion relation (Equation 47). We begin by defining the quantities \( \omega_{\text{ref}}^2 \) and \( k_{\text{crit}} \):
\[ \omega_{\text{ref}}^2 = \frac{\rho_0}{\rho_0 H^2}, \]  
(A1)
and
\[ k_{\text{crit}} = \sqrt{\frac{\omega_{\text{ref}}}{D_{\text{cond}}}}. \]  
(A2)
We assume that \( D_{\text{cond}} \sim D_1 \), and that the equilibrium cosmic-ray pressure is not very small compared to the equilibrium thermal pressure. The frequency \( \omega_{\text{ref}} \) is comparable to the frequency of buoyancy oscillations in the medium except when
where \( \bar{\sigma}_0 \) represents the lowest-order part of \( \bar{\sigma} \) and \( \bar{\sigma}_1 \) is the first-order correction. Substituting this expansion into Equation (A3), collecting terms that are of like-order in \( \epsilon \), and requiring sums of terms of like-order in \( \epsilon \) to vanish separately, we find for the lowest-order terms:

\[
\bar{a} \bar{\sigma}_0 + \bar{b} \bar{\sigma}_0^2 + \bar{c} = 0,
\]

where

\[
\bar{a} = \gamma + \chi \gamma_c + \frac{2}{\beta},
\]

\[
\bar{b} = \left( \gamma + \chi \gamma_c + \frac{2}{\beta} \right) \bar{\sigma}_1 + \left( (\gamma - 1) + \chi \gamma_c + \frac{2}{\beta} \right) \bar{\sigma}_0^2 + \left( (\gamma - 1) + \chi \gamma_c \right) \bar{\sigma}_0 \omega_0^2,
\]

\[
\bar{c} = \bar{\omega}_A^2 \left( (\gamma + \chi \gamma_c) \bar{\sigma}_1 + (\gamma - 1) + \chi \gamma_c \right) \bar{\sigma}_0^2.
\]

Restoring the dimensions, we write the solutions with the notation

\[
\bar{\sigma}_{0,\pm} \simeq \pm \sqrt{-\frac{b \pm \sqrt{b^2 - 4ac}}{2a}},
\]

where \( a, b, \) and \( c \) are the dimensional analogs of \( \bar{a}, \bar{b}, \) and \( \bar{c}, \) respectively, and where the left-most \( \pm \) subscript on \( \bar{\sigma} \) shall refer to the \( \pm \) symbol inside the radical, while the right-most refers to the \( \pm \) symbol outside the radical. We note that the solution \( \bar{\sigma}_{0,+} \) is unstable when \( c < 0 \). More explicitly,

\[
(\gamma + \chi \gamma_c) \bar{\sigma} + [(\gamma - 1) + \chi \gamma_c] \bar{\sigma}_0^2 < 0,
\]

which holds only when \( \bar{\sigma} \) is sufficiently negative. We define the buoyancy frequency \( N \) through the equation

\[
N^2 = \left[ \gamma \omega_1^2 + \chi \gamma_c \omega_1^2 + \frac{1}{\beta} \omega_2^2 \right] \left[ \gamma + \chi \gamma_c + \frac{2}{\beta} \right]^{-1},
\]

which is related to \( W^2 \) through the identity

\[
W^2 + \frac{[(\gamma - 1) + \chi \gamma_c] \omega_0^2}{\gamma + \chi \gamma_c} = \frac{(\gamma + \chi \gamma_c + 2/\beta) N^2}{\gamma + \chi \gamma_c}.
\]

In a high-\( \beta \) plasma in the absence of cosmic rays, \( N \) reduces to the Brunt–Väisälä frequency for buoyancy oscillations in a gravitationally stratified medium. With \( N \) defined as in Equation (A18), we may rewrite the condition in Equation (A17) as

\[
\frac{\omega_0^2}{1 + \frac{2}{\chi \gamma_c}} + N^2 < 0,
\]

which, in the limit of high-\( \beta \), reduces to the corresponding result obtained previously for these modes in CD06.

### A.1.2. Quasi-Isothermal Buoyancy Modes

We refer to the set of solutions which satisfy \( \bar{\sigma} \sim \epsilon \omega_{\text{ref}} \) as the “quasi-isothermal buoyancy modes,” since these modes are of the same order as the frequencies \( \eta \) and \( \nu \) in the long-parallel-wavelength limit. To obtain the quasi-isothermal buoyancy modes, we now set \( \bar{\sigma} = 0 \), so that \( \bar{\sigma} = \epsilon \bar{\sigma}_1 + O(\epsilon^2) \). Substituting into Equations (A3)–(A10) and retaining only lowest-order terms, we find

\[
a \bar{\sigma}_1^2 + b \bar{\sigma}_1 + c = 0,
\]

where now,

\[
a = (\gamma + \chi \gamma_c) C + [(\gamma - 1) + \chi \gamma_c] \omega_0^2,
\]

\[
b = f C + K \omega_0^2,
\]

\[
c = \nu \eta C,
\]

and where the definition of \( K \) is given in Equation (88) and we have introduced the positive-definite quantity:

\[
f = \gamma \nu + (1 + \chi \gamma_c).
\]

For notational convenience, we have again reverted to the dimensional form of our solution. The discriminant of Equation (A21) can be expressed in the form

\[
b^2 - 4ac = \left[ \gamma \nu - \eta (1 + \chi \gamma_c) C + [(\gamma - 1) \nu - \chi \gamma_c \eta] \omega_0^2 \right]^2
\]

\[
+ 4 \eta \nu (\gamma - 1) \chi \gamma_c (C + \omega_0^2)^2,
\]

which is non-negative so that the solutions,

\[
\sigma_1 = -b \pm \sqrt{b^2 - 4ac}
\]

\[
= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
\]

are both real.

To examine the stability of these solutions we first note that when all three coefficients of Equation (A21) are nonzero, both roots will be negative if and only if all the coefficients also have the same sign. The necessary and sufficient condition for this is \( ac > 0 \). Inspection of Equations (A22)–(A24) indicates that if \( c \) is positive, then both \( a \) and \( b \) are also positive. If instead, \( c < 0 \), then we must have \( a < 0 \) for both roots to have the same sign, in which case \( b < 0 \) as well, and the resulting roots are again both negative. Thus, an unstable mode results only when \( c < 0 < a \), or explicitly

\[
\nu \eta C < 0 \leq (\gamma + \chi \gamma_c) C + [(\gamma - 1) + \chi \gamma_c] \omega_0^2.
\]

Once again making use of Equation (A19) we may rewrite this result as

\[
\omega_0^2 + W^2 < 0 \leq \frac{\omega_0^2}{1 + \frac{2}{\chi \gamma_c}} + N^2,
\]

an expression which again reduces in the high-\( \beta \) limit to the result obtained for these modes in CD06 where they are referred to as the “quasi-isothermal convective” modes.

Note that when both inequalities in Equation (A28) are satisfied, the quasi-isothermal mode is unstable, while if only the first inequality is satisfied, the quasi-isothermal mode is stable and the adiabatic buoyancy mode is unstable so that we are guaranteed an unstable mode whenever \( C < 0 \). Since these results must hold for arbitrarily small finite values of \( k_y \), we conclude that these inequalities are consistent with our stability criterion \( W^2 > 0 \).

### A.1.3. Comparison With Numerical Solutions

As a final check of the solutions, we present numerical solutions to the general dispersion relation (Equation 47), using a suitably chosen set of parameters for comparison with the predictions of Equation (A16). For the particular set of parameters chosen, the unstable mode is the quasi-isothermal buoyancy mode. Results for this mode are presented for the unstable quasi-isothermal buoyancy mode in the top row of Figure 2 for the cases \( \beta = 1 \) and \( \beta = 100 \). For the entire
range of wave number shown, we have \( k_y \gg k_{\text{crit}} \), so we do not indicate the location of \( k_{\text{crit}} \) in the figure. The analytical solutions in these figures are seen to compare well with the numerical results, and the growth rates become negative where the quantity \( C \) passes from negative to positive values as indicated by the intersection of the vertical and horizontal dotted lines, demonstrating that the solutions honor the stability criterion \( C > 0 \). For the \( \beta = 1 \) case, the mode is unstable in the range \( 0 < k_y \lesssim 3.16 \text{ kpc}^{-1} \), and the maximum growth rate occurs where \( k_y = k_{y, \text{max}} \simeq 2.1 \text{ kpc}^{-1} \) and takes the value \( \sigma_{y, \text{max}} \simeq 0.114 \text{ Myr}^{-1} \), while for the \( \beta = 100 \) case, the range of unstable wave numbers is \( 0 < k_y \lesssim 0.316 \text{ kpc}^{-1} \), \( k_{y, \text{max}} \simeq 0.21 \text{ kpc}^{-1} \), and \( \sigma_{y, \text{max}} \simeq 5.44 \times 10^{-3} \text{ Myr}^{-1} \). These results illustrate that a dynamically significant magnetic field results in a significantly larger maximum rate of growth for the instability as well as a much larger range of unstable wave numbers for these modes.

**A.2. Short-Parallel-Wavelength Limit**

We now take up the case \( |k_y| \gg k_{\text{crit}} \), which we recall is the limit in which the rates of diffusion and conduction are large relative to the buoyancy frequencies. Once again we set \( \sigma = \omega_{\text{ref}} \hat{\sigma}, \omega_0 = \omega_{\text{ref}} \hat{\omega}_0, \omega_1 = \omega_{\text{ref}} \hat{\omega}_1, \omega_2 = \omega_{\text{ref}} \hat{\omega}_2, \omega_3 = \omega_{\text{ref}} \hat{\omega}_3, \omega_4 = \omega_{\text{ref}} \hat{\omega}_4, \omega_5 = \omega_{\text{ref}} \hat{\omega}_5, \omega_\Lambda = \omega_{\text{ref}} \hat{\omega}_\Lambda \), as before; but we scale the diffusive frequencies according to \( \omega = \epsilon \omega_0 \), and we define the scaled coefficients \( \tilde{b}_0 = b_0, \tilde{b}_1 = b_1 \omega_{\text{ref}}, \tilde{b}_2 = b_2 \omega_{\text{ref}}^2, \tilde{b}_3 = b_3 \omega_{\text{ref}}^3, \tilde{b}_4 = b_4 \omega_{\text{ref}}^4, \tilde{b}_5 = b_5 \omega_{\text{ref}}^5, \) \( \tilde{b}_6 = b_6 \omega_{\text{ref}}^6 \). We obtain

\[
\tilde{b}_0 \hat{\sigma}^6 + \tilde{b}_1 \hat{\sigma}^5 + \tilde{b}_2 \hat{\sigma}^4 + \tilde{b}_3 \hat{\sigma}^3 + \tilde{b}_4 \hat{\sigma}^2 + \tilde{b}_5 \hat{\sigma} + \tilde{b}_6 = 0, \tag{A30}
\]

with

\[
\tilde{b}_0 = \gamma + \chi \gamma \Lambda + \frac{2}{\beta}, \tag{A31}
\]

\[
\tilde{b}_1 = \epsilon^{-1} \left[ \tilde{\gamma} \left( \gamma + \frac{2}{\beta} \right) + \tilde{\eta} \left( 1 + \chi \gamma \Lambda + \frac{2}{\beta} \right) \right], \tag{A32}
\]

\[
\tilde{b}_2 = (\gamma + \chi \gamma \Lambda) \left( \tilde{\omega}_0^2 + \tilde{\omega}_0^2 + \tilde{\sigma} \right) - \tilde{\omega}_0^2 + \frac{2}{\beta} \left( \tilde{\omega}_0^2 + \tilde{\sigma} \right)
+ \epsilon^{-2} \tilde{\eta} \left( 1 + \frac{2}{\beta} \right), \tag{A33}
\]
where this time

\[ a = 1 + \frac{2}{\beta}, \quad (A45) \]

\[ b = \left( \omega_\lambda^2 + \frac{2}{\beta} \omega_0^2 \right) + \left( 1 + \frac{2}{\beta} \right) C, \quad (A46) \]

\[ c = \omega_0^2 C. \quad (A47) \]

In the high-\( \beta \) limit, Equation (A44) reduces to the results of CD06.

### A.2.3. Comparison With Numerical Solutions

Once again we compare our approximate analytical solutions to the solutions obtained numerically from Equation (47) for a set of parameters chosen to ensure that the solutions presented fall within the bounds of the short-parallel-wavelength limit. As before, we present two examples of the unstable (isothermal buoyancy) mode: one for the case \( \beta = 1 \), and one for the case \( \beta = 100 \). These are shown in the bottom row of Figure 2. For each mode, the location of the critical wave number, \( k_{\text{crit}} \), is marked with a vertical line. It is again seen that within the limits of validity of our analytical expressions (\( |k| \gg k_{\text{crit}} \)), the approximate results and numerical results are in good agreement. Also as before, we mark the location where \( C \) passes from negative to positive values and observe that these solutions again honor the stability criterion \( C > 0 \). Finally, we note that for both cases the range of unstable wavelengths is the same as for the long-parallel-wavelength limit, while in the \( \beta = 1 \) case, the maximum growth rate occurs for \( k_{y,\text{max}} \simeq 2.10 \text{ kpc}^{-1} \), where we find \( \sigma_{\pm,\text{max}} \simeq 0.625 \text{ Myr}^{-1} \), whereas for the \( \beta = 100 \) case we find \( k_{y,\text{max}} \simeq 0.112 \text{ kpc}^{-1} \) and \( \sigma_{\pm,\text{max}} \simeq 0.0325 \text{ Myr}^{-1} \). Here again we observe the importance of the magnetic field in determining both the rate of growth and the range of unstable wave numbers. Comparing these results to those in Section A.1.3, we also note that high rates of diffusivity lead to growth rates several times larger than when these rates are low.

### APPENDIX B

RELATION TO SHU’S ANALYSIS OF THE PARKER INSTABILITY

In this Appendix, we show that the general dispersion relation given by Equation (47) reduces properly to the results obtained by Parker (1966, 1967) and Shu (1974) when these results are restricted to the short-parallel-wavelength approximation (\( kH \gg 1 \)) assumed throughout this paper. In the limit considered by these authors, cosmic-ray diffusivity is taken to be infinite and thermal conduction vanishes, whence \( \eta \to 0 \) and \( v \to \infty \). Taking these limits in Equation (47) and once again neglecting the terms involving \( gd \ln \rho_0/dz \), our dispersion relation reduces to

\[ c_0 \sigma^6 + c_2 \sigma^4 + c_4 \sigma^2 + c_6 = 0, \quad (B1) \]

where the coefficients are now

\[ c_0 = 1, \quad (B2) \]

\[ c_2 = \omega_\lambda^2 + \left( \gamma + \frac{2}{\beta} \right) \omega_0^2, \quad (B3) \]

\[ c_4 = \omega_\lambda^2 \left( \gamma + \frac{2}{\beta} \right) \left( \omega_0^2 + C \right) - \omega_0^2 + \gamma \omega_\lambda^2, \quad (B4) \]

\[ c_6 = \omega_\lambda^2 \omega_0^2 \left( \gamma - 1 \right) \omega_0^2 + \gamma C, \quad (B5) \]
We compare this result to the expression obtained by Shu (1974) in the limit of no rotation and no shear which is given by Equation (53) of that paper. To do this we must take account of the differences in notation, the fact that Shu’s expression is given in dimensionless form, and most importantly that given the global equilibrium assumed in Shu (1974), the scale height, $H$, in that paper is consistent with our definition of $H$ given by Equation (14). It is also important to note that Shu’s perturbations include a multiplicative “envelope” function, which is an exponentially decreasing function of $z$ above the origin. To account for this, the quantity “$k$” as defined in Shu (1974) must be set to zero. With all of these requirements accounted for we find that Shu’s result may be expressed in our notation according to

$$s_0 \sigma^6 + s_2 \sigma^4 + s_4 \sigma^2 + s_6 = 0,$$  \hfill (B6)

where the coefficients are

$$s_0 = 1,$$  \hfill (B7)

$$s_2 = \omega^2_\lambda + \left( \gamma + \frac{2}{\beta} \right) \left( \omega^2_s - \frac{ik_z p_0}{\rho_0 H} \right),$$  \hfill (B8)

$$s_4 = \omega^4_\lambda \left[ \left( \gamma + \frac{2}{\beta} \right) (\omega^2_\lambda + C) - \omega^2_s + \gamma \omega^2_s - \frac{2 g}{\beta} \frac{k_z^2}{H^2} \right],$$  \hfill (B9)

$$s_6 = \omega^6_\lambda \omega^2_s \left[ (\gamma - 1) \omega^2_0 + \gamma C - \gamma \omega^2_s \left( \frac{ik_z}{k H} \right) \right].$$  \hfill (B10)

Because $k_z \ll k$, the terms in Equations (B9) and (B10) that involve the factor $1/kH$ must be neglected in the short-wavelength limit. Additionally, the term in Equation (B8) involving $k_z$ can be seen to be small compared to $\omega^2_s$ as follows:

$$\frac{ik_z p_0}{\rho_0 H \omega^2_s} \sim \frac{ik_z}{k^2 H} \sim \frac{ik_z}{k H} \ll 1,$$  \hfill (B11)

and similarly, the term in Equation (B9) involving $k_z^2$ can be seen to be small by comparison to $\gamma \omega^2_s$ according to

$$\frac{(2/\beta)(g/H)(k_z^2/k^2)}{\gamma \omega^2_s} = \frac{g}{H} \frac{\rho_0}{\rho_0} \frac{k_z}{k} \frac{1}{\gamma k^2 H^2} \ll 1,$$  \hfill (B12)

where the second-to-last relation uses the fact that $g \sim \rho_0 H^{-1}/\rho_0$. Thus, the terms by which the coefficients $s_i$ of Shu (1974) differ from the coefficients $c_i$ obtained from our general dispersion relation in the Parker/Shu limit and given by Equations (B2)–(B4) are just those that must be neglected in the short-wavelength limit to which we have restricted ourselves in this paper, so that in this limit the results we have obtained match those of Shu (1974).

We can informally recover the necessary and sufficient condition for stability in this limit by applying the method of dominant balance (Bender & Orszag 1978) which we have described in Appendix A, to the dispersion relation given by Equation (B1) with the quantity $\omega^2_s$ used to scale the solutions, and with the quantities $\omega^2_0$ and $C$ taken to be small compared to $\omega^2_s$. Thus, to lowest order we find that solutions $\sigma^2$ to Equation (B1) that are of order $\omega^2_s$ satisfy

$$\sigma^4_{0,s} + c_2 \sigma^2_{0,s} + \gamma \omega^2_s \sigma^2_s = 0.$$  \hfill (B13)

The discriminant of this expression can be expressed in a positive-definite form as

$$\left( \omega^4_{0,s} - \gamma \omega^2_{0,s} \right)^2 + 2\frac{2}{\beta} \omega^4_{s} \left( \omega^2_s + \gamma \omega^2_s \right) + \left( \omega^4_{s} \left( \frac{2}{\beta} \right)^2 \right)^2 > 0,$$  \hfill (B14)

and one can see by inspection of Equation (B3) that $c_2$ is also a positive-definite quantity. Thus, the solutions

$$\sigma^2_{0,s} = -c_2 \pm \frac{\sqrt{c^2_2 - 4\gamma \omega^2_s \omega^2_s}}{2}$$  \hfill (B15)

are always real and negative, and the modes of order $\sigma_{0,s} \sim \omega_s$ are always stable.

The potentially unstable modes $\sigma_{0,s}$ can be found by seeking solutions to Equation (B1) that satisfy $\sigma^2/\omega^2_s \sim \epsilon$ where $\epsilon \ll 1$, while continuing to assume that $\omega^2_0$ and $C$ are also small compared to $\omega^2_s$. One finds

$$\sigma_{0,s} = \pm \sqrt{-\frac{1}{\gamma} (\gamma - 1) \omega^2_0 + \gamma C},$$  \hfill (B16)

from which we conclude that these modes will be stable if and only if

$$\gamma - 1 \omega^2_0 + \gamma C > 0,$$  \hfill (B17)

which is identical to the necessary and sufficient condition given by Shu (1974) when the latter is evaluated in the limit $kH \gg 1$.

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