Origin of Small Cosmological Constant in Brane-World

Kazuo Ghoroku

Fukuoka Institute of Technology, Wajiro, Higashi-ku
Fukuoka 811-0295, Japan

Masanobu Yahiro

Department of Physics and Earth Sciences, University of the Ryukyus,
Nishihara-chou, Okinawa 903-0213, Japan

Abstract

We address the relation among the parameters of accelerating brane-universe embedded in five dimensional bulk space. It is pointed out that the tiny cosmological constant of our world can be obtained as quantum corrections around a given brane-solution in the bulk theory or in the field theory on the boundary from the holographic viewpoint. Some implications to the cosmology and constraints on the parameters are also given.
1 Introduction

It is quite expectable that our four dimensional world can be regarded as a brane like the one proposed by Randall and Sundrum (RS brane) in [1, 2]. It might be formed in the process of compactification from the ten-dimensional superstring theory. The D-brane approach is getting very useful for the study of such theory. In particular, there is some interest in the geometry obtained from the D3-brane of type IIB theory. Near the horizon of the stacked D3-branes, the configuration $AdS_5 \times S^5$ is realized and the string theory on this background describes the four-dimensional SUSY Yang-Mills theory which lives on the boundary of $AdS_5$ [3, 4, 5, 6]. This holographic correspondence of the five dimensional gravity and the field theory on the boundary has attracted many interest.

It would be very suggestive that a thin three-brane like RS brane could be embedded in $AdS_5$ as our world. This idea gives an alternative to the standard Kaluza-Klein (KK) compactification via the localization of the zero mode of the graviton [2]. Brane approach opened also a new way to the construction of the hierarchy between four-dimensional Planck mass and the electro-weak scale, and also for realization of the small observable cosmological constant with lesser fine-tuning [7, 8].

The localization of the graviton on such a brane with a cosmological constant has also been confirmed when it is embedded in $AdS_5$ [9, 10] or in $dS_5$ [10]. The theory of the gravity under consideration is five-dimensional. However, it could be considered as a part of 10 dimensional theory with compact $S^5$. This picture has been supported in the case of flat brane through the KK mode contribution to the Newton’s law on the brane via AdS/CFT correspondence [11].

Being stimulated by the recent cosmological observation, many approaches to the cosmology with a small cosmological constant have been given from the viewpoint of brane universe (for example [12, 4]). Up to now many solutions for such brane-world have been given, but the cosmological constant of our world is given by hand. However its value should be determined by some dynamical reason in the bulk theory. It is a challenging problem to resolve this point.

Here we propose a clue to the resolution of this issue by pointing out that the 4d cosmological constant should be determined by the quantum corrections in the bulk theory. Those corrections can be regarded as the breaking of the conformal symmetry of the bulk theory for such a brane-solution with a tiny cosmological constant.

In Section 2, we give a set-up of accelerating brane solutions obtained previously on the basis of a simple ansatz imposed on the bulk metric. In section 3, the 4d cosmological constant is estimated by considering the effective 4d action, which is obtained from the bulk theory. And the necessity of quantum corrections is discussed. In section 4, phenomenological constraints on the parameter are given, and the models of the brane-world are restricted. In the final section, the summary and speculations are given.
2 Brane solutions for accelerating universe

We start from a simple five-dimensional gravitational action. It is given in the Einstein frame as

\[
S_5 = \frac{1}{2\kappa^2} \left\{ \int d^5 X \sqrt{-G} (R - 2\Lambda + \cdots) + 2 \int d^4 x \sqrt{-g} K \right\},
\]

(1)

where the dots denote the matter part, \( K \) being the extrinsic curvature on the boundary. This term is necessary to construct the effective 4-dimensional brane action as shown in the next section, but it plays no role in solving 5d Einstein equation here. The fields represented by the dots are not needed to construct the background. The other ingredient is the brane action,

\[
S_b = -\tau \int d^4 x \sqrt{-g},
\]

(2)

which is added to \( S_5 \). And the Einstein equation is written as

\[
R_{MN} - \frac{1}{2} g_{MN} R = \kappa^2 T_{MN}
\]

(3)

where \( \kappa^2 T_{MN} = -\left( \Lambda + \frac{1}{6} \delta \right) g_{MN} \) and \( b = \sqrt{-g} / \sqrt{-G} \).

We solve the equation (3) in the following Friedmann-Robertson-Walker type (FRW) metric,

\[
ds^2 = A^2(y) \left\{ -dt^2 + a^2(t) \gamma_{ij}(x^i) dx^i dx^j \right\} + dy^2,
\]

(4)

where the coordinates parallel to the brane are denoted by \( x^\mu = (t, x^i) \), \( y \) being the coordinate transverse to the brane. The position of the brane is taken at \( y = 0 \). In this case, the three-dimensional metric \( \gamma_{ij} \) is described in Cartesian coordinates as,

\[
\gamma_{ij} = \left( 1 + k \delta_{mn} x^m x^n / 4 \right)^{-2} \delta_{ij},
\]

where the parameter values \( k = 0, 1, -1 \) correspond to a flat, closed, or open universe respectively.

For the metric (4), we obtain from Eq.(3) the following reduced equations [10]:

\[
\left( \frac{a_0'}{a_0} \right)^2 = \lambda - k = \frac{\kappa^4 \tau^2}{36} + \lambda / 6,
\]

(5)

where we set \( a(0) = 1 \), but this normalization does not affect the generality of our analysis. Then \( \lambda \) is given by

\[
\lambda = \kappa^4 \tau^2 / 36 + \Lambda / 6.
\]

(6)

The first equation of (3) is solved for each \( k \), but they are abbreviated since we do not use them here. The solutions for \( \lambda > 0 \) of the second equation are obtained under the following boundary condition,

\[
A'(0+) - A'(0-) = -\frac{\kappa^2 \tau}{3}.
\]

(7)

\footnote{Here we take the following definition, \( R^\mu_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\nu\sigma} - \cdots \), \( R_{\nu\sigma} = R^\mu_{\nu\mu\sigma} \) and \( \eta_{AB} = \text{diag}(-1,1,1,1,1) \). Five dimensional suffices are denoted by capital Latin and the four dimensional one by the Greek ones.}
In the following, we give the solutions for $\lambda > 0$ since they are used hereafter. Such solutions are obtained for both cases of $\Lambda > 0$ and $\Lambda < 0$.

For $\Lambda < 0$, $A(y)$ is solved as

$$A(y) = \frac{\sqrt{-\Lambda}}{\mu} \sinh[\mu(y_H - |y|)]$$

where $\mu = \sqrt{-\Lambda/6}$, $\sinh(\mu y_H) = \mu/\sqrt{-\Lambda}$ and $y_H$ represents the position of the horizon in $AdS_5$. This solution represents a brane at $y = 0$. The configuration is taken to be $Z_2$ symmetric with respect to the reflection, $y \to -y$.

When $\Lambda$ is positive, the solution for $a_0(t)$ is the same as above, but $A(y)$ is different. One has

$$A(y) = \frac{\sqrt{\Lambda}}{\mu_d} \sin[\mu_d(y_H - |y|)].$$

Here $\mu_d = \sqrt{\Lambda/6}$, $\sin(\mu_d y_H) = \mu_d/\sqrt{\Lambda}$ and $y_H$ represents the position of the horizon in the bulk $dS_5$, where there is no spatial boundary as in $AdS_5$. This configuration represents a brane with $dS_4$ embedded in the bulk $dS_5$ at $y = 0$. The $Z_2$ symmetry is also imposed.

### 3 4d Cosmological Constant

The solutions given above are obtained by solving 5-dimensional Einstein equation, and the parameters of the theory have no restriction. In other words, a favorite solution is realized by imposing an appropriate relation on the parameters by hand. For example, we can set the relation, $\kappa^2 \tau/6 = \sqrt{-\Lambda/6}$, to obtain the original Randall-Sundrum brane solution, and this relation is known as the fine tuning. Up to now, no one has given any satisfactory explanation to this relation in terms of some dynamical reason or symmetries.

On the other hand, it is well known that the solutions are severely restricted when the theory has some symmetry and this symmetry should be preserved for the solutions obtained. In this sense, the above relation for the RS brane solution might be considered as a result of such a symmetry. In fact, we can see below that this relation can be regarded as a reflection of conformal symmetry in the bulk theory or in the field theory on the boundary in the sense of AdS/CFT correspondence. Then, we could get a solution with a finite $\lambda$ when this symmetry is broken.

The understanding is confirmed as follows. The four dimensional part of the bulk five dimensional equations (3) should also be obtained in the same form from the effective 4d brane action ($S_{b}^{\text{eff}}$). And this action $S_{b}^{\text{eff}}$ can be obtained from the bulk theory via path integral. Although the five dimensional theory is our starting point, it would be natural to consider this is also derived from ten dimensional superstring
theory. However, we start from the five dimensional theory for simplicity. Then we can write $S_{\text{eff}}^b$ as,

$$S_{\text{eff}}^b = \frac{1}{2} S_b + \ln Z_5(g)$$

(10)

$$Z_5(g) = \int_{G|y=0=g} DGD\psi e^{iS_5},$$

(11)

where $S_b$ and $S_5$ are given in the previous section. The other fields contained in $S_5$ are denoted by $\psi$. We notice here (i) the path integration in (11) is performed for the field defined in the region $y \geq 0$, half of the whole region. This is justified by the $Z_2$ symmetry of the solution. (ii) The bulk fields take their boundary values on the brane at $y = 0$. (iii) In the sense of (i), a half of the brane action is considered.

In the case of $\Lambda < 0$, the bulk space has a boundary where a dual field theory can be considered in the context of AdS/CFT holographic correspondence. But this holography would not be appropriate here since the AdS$_5$ is deformed by a small cosmological constant $\lambda$ and there is no evidence of the conformal invariance in this case. So, we should consider some quantum field theory (QFT), in which the conformal invariance would be slightly broken, instead of a CFT on the boundary. Further we must introduce counter terms [13, 14], $S_{\text{CT}}$, which are needed to regularize the action for a class of classical solutions like the given here in the bulk. Then it is possible to write as [13, 14]

$$\ln Z_5(g) = S_{\text{CT}} + S_{\text{QFT}},$$

(12)

where $S_{\text{QFT}}$ represents the action for the boundary field theory, and the explicit forms of these actions are given below. One more important point is that the boundary is pulled to the position of the brane ($y = 0$). Then QFT in $\ln Z_5(g)$ is replaced by a cutoff theory since $y$ has the meaning of the energy scale of the QFT. In this case, we expect extra counter terms in $S_{\text{eff}}^b$ due to the loop corrections coming from the cut-off QFT which couples with the boundary metric. It is shown below that this corrections play an important role. The action, $S_{\text{CT}}$, is divergent at the boundary ($y = -\infty$) since they have been introduced to cancel out the divergences which appear there. However they are finite at the brane position, and they are rewritten in terms of the induced metric on the brane into the form of the part of 4d gravity action.

The alternative way to get $S_{\text{eff}}^b$ is performing the path integral directly in the expression (11). We firstly consider this method since we like to estimate $\lambda$ also for the case of $\Lambda > 0$ where there is no boundary in the bulk space. The simplest estimation is given by the semi-classical approximation, and it is obtained by substituting the classical solutions given in the previous section into $S_5$. Then the effective 4-dimensional action is obtained by integrating over the fifth coordinate $y$.

Although our solutions are given in the form of (4), we can write the following more general form for the metric,

$$ds^2 = A^2(y)g_{\mu\nu}(x)dx^\mu dx^\nu + dy^2.$$

(13)
And the resultant effective 4-dimensional action is given as follows,

\[ S_{\text{eff}}^b = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} (R^{(4)} - 2\Lambda_4 + \cdots), \quad (14) \]

\[ \frac{1}{2\kappa_4^2} = \frac{1}{2\kappa^2} \int_0^{y_H} A^2 dy \quad (15) \]

\[ \Lambda_4 = 4 \int_0^{y_H} dy A^2 (A A'' + \frac{3}{2} A^2 + \frac{\Lambda_4}{4} A^2) + \partial_y A|_{y=0} + \frac{\kappa^2}{8}. \quad (16) \]

Here, the dots in (14) denote other 4-dimensional modes remained.

The results given by (14) \sim (16) are correct if we need no other terms coming from the loop corrections. Such a case would be realized when the five dimensional theory is constructed in a conformal invariant form, and the classical solution is consistent with this invariance. When the above action (14) is correct, we should obtain the same equation with the 4d part of the 5d equations, which is given in the first equation in (5). In \( S_{\text{eff}}^b \), two parameters, \( \kappa_4 \) and \( \Lambda_4 \), are derived. The corresponding parts are read as, \( \left( \frac{\dot{a}_0}{a_0} \right)^2 = \lambda + \frac{2\kappa^4 \tau \rho_b(t)}{a_0^2} (1 + \frac{2\kappa^4}{2\tau} - \frac{\kappa^4}{a_0^2} \frac{\rho_b(t)}{a_0^2} ) \) from the first equation of (3) by the replacement \( y \rightarrow \tau + \rho_b(t) \), where \( \rho_b(t) \) denotes the matter density on the brane. We should have taken into account of this term, \( \rho_b(t) \), at the starting point, but there is no problem here since it is taken to be zero in our solution. The resultant correspondences are obtained as,

\[ 3\lambda = \Lambda_4, \quad (17) \]

\[ \frac{\kappa^4 \tau}{3} = \kappa_4^2. \quad (18) \]

The second relation (18) however should be remained as a useful one.

Our next task is to see the consistency of the above two relations by using (15) and (16) for our solution of \( \lambda > 0 \). After a little calculation, we can see that the first relation (17) is satisfied for both our solutions of \( \Lambda > 0 \) and \( \Lambda < 0 \). In other words, no extra constraint is needed from (17). However the second condition is non-trivial, and we obtain a new constraint on the parameter.

For the solution of \( \Lambda < 0 \), (18) can be written as

\[ \frac{1}{\sqrt{1 + x}} = \sqrt{1 + x} - x \ln \left( \frac{1 + \sqrt{1 + x}}{\sqrt{x}} \right), \quad (19) \]

where \( x = \lambda/\mu^2 \). This equation has only one solution, \( x = 0 \) or \( \lambda = 0 \). At a glance, this result seems to be inconsistent since we have used the solution for \( \lambda > 0 \) in this analysis. However we should notice the following points. (i) The solution used covers the one of \( \lambda = 0 \) as a limit, and (ii) the above \( S_{\text{eff}}^b \) in (14) is obtained without including any loop-correction. We could interpret this result, \( \lambda = 0 \), as the reflection of the conformal invariance of the bulk theory since the second point mentioned above is reasonable when the loop-corrections are cancelled out. In other words, the Poincare
invariant solution (RS solution) can be obtained by imposing the conformal invariance of the theory not by the fine-tuning of the parameters.

Therefore, in order to obtain a non-zero $\lambda$ from (18), we must include quantum corrections in deriving $S_{b}^{\text{eff}}$ through the path-integral over $G_{\mu\nu}$ and other fields. They modify the form of (19). To consider in this way is quite natural since there is no reason to consider the conformal invariance of the bulk theory for the background with small $\lambda$.

When the conformal invariance is slightly broken, the corrections would appear generally in the forms of cosmological, Einstein terms and other general coordinate invariant forms,

$$\Delta S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g}(\epsilon_{2}(\lambda)/\mu R^{(4)} + 8\epsilon_{0}(\lambda)\mu + \cdots),$$

where $\epsilon_{2}(\lambda)$ and $\epsilon_{0}(\lambda)$ are the dimensionless correction terms scaled by the physical dimension-full parameter $\mu$. The other corrections are denoted by $\cdots$.

We didn’t perform an explicit calculation of the loop corrections, but the corrections could be obtained as functions of $A(y)$. And they can be integrated over $y$ as in the case of tree approximation. In general, the corrections are also dependent of the parameters contained in $A(y)$, so they are expressed as $\epsilon_{2}(\lambda)$ and $\epsilon_{0}(\lambda)$. In this sense, it is impossible to see the precise constraint on $\lambda$ from (18) without knowing the explicit form of $\epsilon_{2}(\lambda)$ and $\epsilon_{0}(\lambda)$. However, we can see the possibility that the quantum corrections in the bulk theory could give the small $\lambda$. For a while we consider $\epsilon_{2}$ as a small constant, then it appears on the right hand side of (19) as $\sqrt{1 + x - x \ln(x + \sqrt{1 + x} + \epsilon_{2})}$. And we find a non-trivial solution in this case by solving the approximated equation,

$$x \ln x + 4\epsilon_{2} = 0,$$

for small $x$. We also obtain a non-trivial solution from (17) when the quantum corrections are included, and a relation of the corrections would be obtained.

In the following, we can see these points by using an alternative formulation where the correction terms are independent of the parameter $\lambda$. The formulation is based on the AdS/CFT correspondence, and $\lambda$ appears as a result of conformal symmetry breaking in the quantum field theory on the boundary. Before showing it, we examine the case of $\Lambda > 0$.

In the case of $\Lambda > 0$, the bulk space has no spacial boundary as in the case of negative $\Lambda$, so we can not expect the correspondence of the gravity and the field theory on the boundary. But we can calculate $S_{b}^{\text{eff}}$ as above and study the solution of (18) as above. Using the explicit form of the solution, (18) leads to the following equation,

$$\frac{1}{\sqrt{x_{d} - 1}} = x_{d} \sin^{-1}(\frac{1}{\sqrt{x_{d} - 1}}) - \sqrt{x_{d} - 1},$$

where $x_{d} = \lambda / \mu_{A}^{2}$. This equation has no solution for $x_{d} \geq 1$, which is the allowed value of $x_{d}$. Then quantum corrections are essential to get a non-trivial solution in this case.
In fact, there is no supersymmetry in $dS_5$ \cite{17}, so there is no conformal symmetry. When we consider the corrections given above, the equation (22) is modified by adding $\epsilon_2$ on the right hand side. Assuming that $\epsilon_2$ is independent of $x_d$ and small, we obtain the solution $\sqrt{x_d} = 1/(3\epsilon_2)$. Then it seems to be possible to have a reasonable solution for $\Lambda > 0$ when quantum corrections are taken into account. However we can see that this result is inconsistent with the analysis given in the next section.

We now return to the case of $\Lambda < 0$, where the effective 4-dimensional action, $S_b^{\text{eff}}$, can also be obtained from the holographic viewpoint mentioned above. According to the formulation given above, we can write the effective action as

$$S_b^{\text{eff}} = S_{b2} + S_{\text{QFT}}.$$  \hspace{2cm} (23)

Here, the QFT part is given as

$$S_{\text{QFT}} = \int d^4x (L_{\text{QFT}} + \Sigma_i \lambda_i O^i),$$  \hspace{2cm} (24)

where $O^i$ are the composite operators of the fields contained in $L_{\text{QFT}}$ and $\lambda_i$ are their corresponding sources which are given as the boundary values of the bulk fields at the brane position. And the first term is defined as

$$S_{b2} = \frac{1}{2} S_b + S_{\text{CT}},$$  \hspace{2cm} (25)

where $S_{\text{CT}}$ denotes the counter term stated above. This counter term has been obtained for the metric given in the form \cite{13, 14},

$$ds^2 = \frac{L^2}{\rho^2} \left\{ g_{\mu\nu}(x, \rho) dx^\mu dx^\nu + d\rho^2 \right\},$$  \hspace{2cm} (26)

where $g_{\mu\nu}(x, \rho)$ is expanded near the boundary ($\rho = 0$) in the series of $\rho$ as

$$g_{\mu\nu}(x, \rho) = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(2)} \rho^2 + \cdots,$$  \hspace{2cm} (27)

where the higher order terms are denoted by $\cdots$. Our solution of $\Lambda < 0$ is written in this form by taking as

$$L = \frac{1}{\mu}, \quad g_{\mu\nu}(x, \rho) = (1 - \frac{\lambda}{4\mu^2})^2 g_{\mu\nu}(x)$$  \hspace{2cm} (28)

$$\rho = \frac{2}{\sqrt{\lambda}} \tanh\left(\frac{\sqrt{\lambda} z}{2}\right),$$  \hspace{2cm} (29)

where $z = \text{sgn}(y)(\lambda)^{-1/2} \ln(\coth[\mu(\mu - |y|)/2])$ and the position of the brane $z_0$ ($y = 0$) is given by

$$z_0 = \frac{1}{\sqrt{\lambda}} \text{arsinh}\left(\frac{\sqrt{\lambda}}{\mu}\right).$$  \hspace{2cm} (30)
Hence we find \( g^{(0)}_{\mu\nu} = g_{\mu\nu}(x) \), \( g^{(2)}_{\mu\nu} = -\frac{\lambda}{2} g^{(0)}_{\mu\nu} \), and \( g^{(4)}_{\mu\nu} = \frac{\lambda^2}{16} g^{(0)}_{\mu\nu} \). This is consistent with the results given in \([13, 14]\) when we take the four dimensional Riemann tensor in the de Sitter form, \( R_{\mu\nu\lambda\sigma} = -\lambda (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) \). So we obtain the same form of counter terms with the one given in \([13, 14]\).

The counter terms are written by the induced metric on the brane. It is easily seen that the induced metric on the brane (at \( z = z_0 \)) is equal to \( g_{\mu\nu}(x) \), i.e.

\[
L^2 \rho^2 g_{\mu\nu}(x, \rho) \big|_{z = z_0} = g_{\mu\nu}(x).
\]

This is independent on \( \lambda \), and we obtain

\[
S_{b_2} = \int dx^4 \sqrt{-g} \left\{ L_{\text{brane}} - \left( \frac{\tau}{2} + b_0 \right) - b_2 R - b_4 R_2 \right\},
\]

where \( b_0 = -(6/L)/(2\kappa^2) \), \( b_2 = -(L/2)/(2\kappa^2) \), \( b_4 = 2L^3/(2\kappa^2) \) and

\[
R_2 = -\frac{1}{8} R_{\mu\nu} R^{\mu\nu} + \frac{1}{24} R^2.
\]

From the above result, we obtain

\[
\frac{1}{2\kappa^4} = -b_2, \quad -2\Lambda_4 = \frac{b_0 + \tau/2}{b_2}.
\]

Then we obtain the same result from both \([17]\) and \([18]\),

\[
\frac{\kappa^2 \tau}{6} = \mu,
\]

where \( \mu = 1/L \). Hence, we find \( \lambda = 0 \) again. This result is consistent with the analysis given above in the case of the conformal invariant bulk theory. In this case, QFT is equivalent to CFT in the sense that any quantum corrections to \( b_0 \) and \( b_2 \) parts are not added. Even if we consider CFT, \( b_4 \) should be modified by the anomaly term coming from the loop corrections in CFT. There would be a possibility to find 4d de Sitter solution within CFT by considering the higher derivative gravity with the anomaly \([19]\).

As in the previous analysis, we should consider QFT, in which conformal symmetry is slightly broken, to find a solution with small \( \lambda \). Then we include the quantum corrections in the effective four-dimensional action, especially for \( b_0 \) and \( b_2 \), as discussed above. Then we should modify \( b_0 \) and \( b_2 \) as follows,

\[
b_0 = -(6/L)/(2\kappa^2) + \bar{\epsilon}_0 \mu^4, \quad b_2 = -(L/2)/(2\kappa^2) + \bar{\epsilon}_2 \mu^2.
\]

where \( \bar{\epsilon}_0 \) and \( \bar{\epsilon}_2 \) represent the dimensionless loop-correction terms derived from \( S_{\text{QFT}} \). In this case, \( \bar{\epsilon}_0 \) and \( \bar{\epsilon}_2 \) are independent on \( A(y) \), then \( \lambda \) can be estimated by using \([17]\) and \([18]\) in terms of \( \bar{\epsilon}_0 \) and \( \bar{\epsilon}_2 \). And we obtain

\[
\lambda = \bar{\epsilon}_2 \frac{\mu^3}{M^2} \mu^2,
\]
where $1/2\kappa^2 = M^3$. The coefficient $\bar{\epsilon}$ is given as $\bar{\epsilon} = 4\bar{\epsilon}_2$ and $\bar{\epsilon} = \bar{\epsilon}_0/6$ from (17) and (18) respectively. Then the following relation of the corrections,

$$\bar{\epsilon}_0 = 24\bar{\epsilon}_2,$$

(38)

is needed from the consistency. Hence a small $\lambda$ is obtained as a quantum correction from the boundary QFT. Then, the conformal symmetry should be broken slightly in the boundary QFT.

This implies that the gravity on the bulk manifold, which is described by our solutions with a small $\lambda$, would describe a non-conformal quantum field theory on the boundary. It will be an interesting problem to see what kind of field theory can be seen on the boundary. The relation (38) given above could be a clue to this problem.

4 Observational constraint on bulk space

In this section, the 5d bulk space is constrained from observational information. The cosmological constant $\lambda$ and the Planck mass $M_{pl}$ are set to the measured values, $\lambda_{\text{obs}} \sim 10^{-122}M_{pl}^2$ and $M_{pl} \sim 10^{19}\text{GeV}$. As a merit of this setting, quantum corrections to these are taken into account implicitly. Especially, quantum corrections to $\lambda$ are essential, as mentioned in the previous section. Inserting Eq. (18) into Eq.(6) leads to $\Lambda = \frac{6\lambda_{\text{obs}} - 3M^6/(2M_{pl}^4)}{< -10^{-99}M_{pl}^2}$, because of $1/2\kappa^2 = M^3$, $1/2\kappa_4^2 = M_{pl}^2$. The upper limit of $\Lambda$ is determined by the observational constraint $M > 10^4\text{GeV}$, which comes from the condition that in the effective 4d Friedmann equation derived from the 5d theory the $\rho_b$ term should be larger than the $\rho_b^2$ term at the epoch of Big Bang Nucleosynthesis [18, 20]. The upper limit is negative, and its absolute value is quite small. This indicates that the $dS_5$ space is prohibited, while the $AdS_5$ space is allowed for almost all negative $\Lambda$. In the previous section, it is found that there exists a solution $\lambda = 0$ for $\Lambda < 0$ but none for $\Lambda > 0$, when the classical limit is taken. Quantum corrections to $\lambda$ and $M_{pl}$ relax the situation so that a small positive $\lambda$ can exist for each of $\Lambda > 0$ and $\Lambda < 0$. The possibility of the small $\lambda$ for the case of $\Lambda > 0$, however, is excluded by the observational constraint on $\Lambda$.

It is possible to make a similar analysis with Eq. (15) instead of Eq.(18), since both are considered to be identical. The two analyses should give a consistent constraint on $\Lambda$. This consistency is confirmed, as follows. When $\Lambda > 0$, the solution (9) is inserted into Eq. (15). This leads to a condition for $\alpha = 1/\sqrt{x_d} = \mu_d/\sqrt{\lambda_{\text{obs}}}$, $2\sqrt{\lambda_{\text{obs}}M_{pl}}/M^3 = \{\arcsin(\alpha) - \alpha\sqrt{1 - \alpha^2}\}/\alpha^3$. The left hand side contains quantum corrections implicitly. The left hand side is smaller than $10^{-16}$ because of $M > 10^4\text{GeV}$, but the right hand side is larger than $2/3$ in the range $0 \leq \alpha \leq 1$ determined from Eq.(6). Hence, no $\alpha$ satisfies the equation, as expected above.

Now we consider quantum corrections to the right hand side by adding $\Delta S$ defined in Eq. (20) to $S_{\text{eff}}$. The resultant equation is the same as the equation shown above, except that a correction term, $\epsilon_2(\lambda)/\alpha$, is added to the right hand side. As an important
property, the correction term is positive, as shown in the previous section. The right hand side is still larger than 2/3. Therefore, no $\alpha$ satisfies the equation. The $dS_5$ bulk space is thus prohibited for the small observable cosmological constant.

Also in the case of $\Lambda < 0$, the corresponding condition is obtained for $\alpha = 1/\sqrt{x} = \mu/\sqrt{x_{obs}}$ by inserting the solution (8) into Eq. (13) and considering the quantum correction. The condition is $2\sqrt{x_{obs}}M_{pl}^2/M^3 = \{^-\text{arcsinh}(\alpha) + \alpha\sqrt{1 + \alpha^2}\}/\alpha^3 + \epsilon_2(\lambda)/\alpha$. In the classical limit, where $\epsilon_2(\lambda) = 0$, this equation is satisfied when $\alpha > 10^{16}$, because the left hand side is less than $10^{-16}$. The allowed range of $\alpha$ corresponds to $x < 10^{-32}$ or $|\Lambda| > 10^{32}x_{obs} \sim 10^{-90}M_{pl}^2$. The quantum correction makes the lower limit of $|\Lambda|$ go up, since it is positive. But the shift is quite small, because so is $\epsilon_2(\lambda)$. The allowed range of $\Lambda$ is consistent with the one mentioned above.

5 Summary and Cosmological Implications

From the viewpoint of 5-dimensional brane-world, we have examined the 4-dimensional universe with a small cosmological constant. The three brane considered here has been embedded in $AdS_5$, and it can be considered as an extended part near the horizon of the configuration realized by the stack of D3 branes in the type IIB superstring theory. In this sense, supersymmetry could be preserved in the bulk and other properties of the superstring theory would be expected. Then we can expect that the quantum corrections for such a configuration would be cancelled out when the conformal symmetry remains. In this case, we could arrive at our conclusion of $\lambda = 0$, which has been actually derived without any quantum corrections comming from the bulk theory. This result would be important in the sense that any finite $\lambda$ should be forbidden by the conformal invariance in the bulk and we would need not any fine-tuning for the Poincare invariance in the 4d space-time on the brane.

Hence, some symmetry breaking is expected for producing the quantum corrections given above in order to get a small $\lambda$. In fact, our solutions used here are deformed from the $AdS_5$ by the metric of three space, $g_{ij} = a_0(t)^2\gamma_{ij}$, due to non-zero $\lambda$. The supersymmetric domain wall solutions are known in 5d supergravity by considering flux condensation in the form of $AdS_5$. Then, it would be an intersting work to see whether some supersymmetries are preserved or not for our solutions in the bulk 5d supergravity. It will be remained as a future work to see a possible solution with some small conformal symmetry breaking and to estimate $\lambda$ quantitatively in terms of such a solutions derived from more concrete theory.

In any case, the universe with small $\lambda$ would be explained by assuming a five dimensional theory with a weak breaking of the conformal symmetry. This symmetry breaking can be considered as a reflection of the deformation of AdS space as mentioned above. In the context of AdS/CFT correspondence, this deformation would break also the conformal invariance of the field theory on the boundary (QFT). In fact, we could show that a small cosmological constant on the brane is obtained by considering quantum corrections coming from the QFT which couples with the gravity on the
brane in the framework of holography. The cosmological constant on the brane and the conformal symmetry breaking in the bulk theory or in QFT are related intimately to each other.

As a natural statement, we can say that this small cosmological constant causes the observed acceleration in the present universe. Its amount is controlled by the symmetry not on the brain but in the bulk. For the brane-world with observable cosmological constant, the $dS_5$ bulk space seems to be prohibited because of the observational constraint on $M_5$. In contrast, the $AdS_5$ bulk space is allowed for almost all negative $\Lambda$. It is thus highly expected that our universe is embedded not in $dS_5$ but in $AdS_5$.

While, some amount of dark matter is also expected from the recent observations. This would be also explainable from the brane-world viewpoint. From our analysis, which would be reported soon in a separate paper, the localization of massive scalar with mass smaller than $3\sqrt{\lambda}/2$ in the bulk would be trapped on the brane and this scalar would interact with matters on the brane only through gravitation. So this scalar could be considered as a candidate for the cold dark matter. As an important fact, it should be stressed that the phenomena occur only when the positive cosmological constant exists.

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