Quantum Criticality in Resonant Andreev Conduction

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Motivated by recent experiments with proximitized nanowires, we study a mesoscopic s-wave superconductor connected via point contacts to normal-state leads. We demonstrate that at energies below the charging energy the system is described by the two-channel Kondo model, which can be brought to the quantum critical regime by varying the gate potential and conductances of the contacts.

The prediction of and search for Majorana physics in hybrid semiconductor-superconductor structures[1] touched off a rapid progress in the technology of such devices[2][19]. In particular, the pairing gap induced in semiconductor wires by the proximity effect is already in hybrid semiconductor-superconductor structures[1]. Indeed, it turns out that tunable proximitized devices that described above, but is interesting in its own right. The second term in Eq. (1), \( H_S \), describes an isolated superconductor. In the conventional BCS framework, it is given by[29]

\[
H_S = \sum_{n \sigma} \sqrt{\Delta^2 + \varepsilon_n^2} c_{n \sigma}^\dagger \gamma_n \gamma_{n \sigma},
\]

where \( \Delta \) is the superconducting gap, \( \gamma_{n \sigma} \) is the fermionic quasiparticle operator and \( \varepsilon_n \) are single-particle energies characterized by the mean level spacing \( \delta \ll \Delta \). The third term in Eq. (1) originates in electrostatics and is given by

\[
H_C = E_C(N - N_g)^2,
\]

where \( E_C \ll \Delta \) is the charging energy, \( N_g \) is the dimensionless gate potential, and \( N \) is an operator with integer eigenvalues representing the number of electrons in the superconductor. Finally, \( H_T \) describes the tunneling,

\[
H_T = \sum_{\alpha n} t_{\alpha n} c_{\alpha \sigma}^\dagger(0) d_{n \sigma}^\dagger N - 1)(N) + \text{H.c.}
\]

Here \( t_{\alpha n} \) is the tunneling amplitude, the operator \( c_{\alpha \sigma}^\dagger(0) = L^{-1/2} \sum_k c_{\alpha k \sigma} \) creates an electron with spin \( \sigma \) at point contact \( \alpha \) (\( L \) is the size of the system that will be taken to infinity in the thermodynamic limit), \( d_{n \sigma} = u_n \gamma_{n \sigma} - \sigma v_n \gamma_{n, -\sigma} \), where the BCS coherence factors \( u_n \) and \( v_n \) satisfy[29]

\[
u_n^2 = 1 - \varepsilon_n^2 = \frac{1}{2} \left( 1 - \varepsilon_n / \sqrt{\Delta^2 + \varepsilon_n^2} \right), \text{and } |N| \text{ is eigenvector of } \tilde{N} \text{ with eigenvalue } N.
\]

At low temperatures \( T \ll \Delta \), the superconductor favors states with an even number of electrons \( N \). Taking into account virtual transitions to states with odd \( N \)[20][30] in the second order of perturbation theory, we obtain

\[
H = H_0 + H_C + H_A, \text{ where}
\]

\[
H_A = \sum_{\text{odd } N} \sum_{\alpha \sigma} J_{\alpha \sigma} c_{\alpha \sigma}^\dagger(0)c_{\alpha \sigma}(0)|N+1\rangle(N-1)+\text{H.c.}
\]
describes Andreev processes \[31\] in which electrons tunnel into and out of the superconductor in pairs.

In the leading order in $E_C/\Delta \ll 1$ the two-particle tunneling amplitudes in Eq. (5) are given by \[30, 32\]

$$J_{\alpha \alpha'} = \sum_{n} \frac{\Delta}{\Delta^2 + \varepsilon_{n}^2} t_{\alpha n} t_{\alpha' n}^*, \quad \text{(6)}$$

and are subject to mesoscopic fluctuations. Provided that the motion of electrons inside the superconductor is chaotic, such fluctuations can be analyzed using the standard random matrix theory-based prescriptions (see, e.g., Refs. \[33\] and references therein). In this approach, the single-particle tunneling amplitudes $t_{\alpha n}$ are statistically independent of each other and of the single-particle energies $\varepsilon_{n}$. Accordingly, the sum in Eq. (6) consists of a large (of order $\Delta/\delta \gg 1$) number of statistically independent random contributions.

The central limit theorem then suggests that the distribution of $J_{\alpha \alpha'}$ is Gaussian. Using $\langle t_{\alpha n} \rangle = 0$ and $\langle t_{\alpha n} t_{\alpha' n} \rangle = \langle t_{\alpha n} t_{\alpha' n}^* \rangle = (2\pi)^{-1} \delta(y_{\alpha \beta} \delta_{\alpha \beta} \delta_{mn})$ \[33\] \[34\], where the double angular brackets denote averaging over the mesoscopic fluctuations and $g_{\alpha \beta}$ is the dimensionless (in units of $2e^2/h$) conductance of contact $\alpha$, and replacing the summation over $n$ by the integration, we find

$$\langle J_{\alpha \alpha'} \rangle = J_{\alpha} \delta_{\alpha \alpha'}, \quad J_{\alpha} = g_{\alpha \alpha} v \quad \text{(7)}$$

and

$$\langle J_{\alpha \alpha'} J_{\alpha' \alpha} \rangle - J_{\alpha} J_{\alpha'} = \frac{1}{2\pi} \frac{\delta}{\Delta} (1 + \delta_{\alpha \alpha'}) J_{\alpha} J_{\alpha'}, \quad \text{(8a)}$$

$$\langle J_{\alpha \alpha'}^2 \rangle_{\alpha \alpha'} = \frac{1}{2\pi} \frac{\delta}{\Delta} J_{\alpha} J_{\alpha'}. \quad \text{(8b)}$$

These equations show that both the off-diagonal elements of the $2 \times 2$ matrix $J_{\alpha \alpha'}$ and the fluctuations of the diagonal elements are parametrically suppressed at $\delta/\Delta \ll 1$, and can be neglected. Note that $\delta/\Delta \ll 1$ is the limit when the BCS description of the superconductor employed in the above derivation is accurate [35].

The charge states $|N \pm 1\rangle$ in Eq. (3) are discriminated by electrostatics, see Eq. (3). For almost all values of the gate potential $N_g$, the ground state of $H_C$ is nondegenerate. Exceptions are narrow intervals of $N_g$ around odd integers $N_g^a$, where states with $N = N_g \pm 1$ electrons have almost identical electrostatic energies. Accordingly, at $T \ll E_C$ and $|N_g - N_g^a| \ll 1$ the Hamiltonian can be simplified further by discarding all but the two almost degenerate charge states $|N_g^a \pm 1\rangle$, which can be viewed as two eigenstates of spin-1/2 operator $S_z |N_g^a \pm 1\rangle \rightarrow |\rangle$ and $|N_g^a - 1\rangle \rightarrow |\rangle$. Upon performing the particle-hole transformation \[32\] \[33\] $c_{\alpha k \uparrow} \rightarrow c_{\alpha, -k \downarrow}$, and taking into account Eq. (7), we arrive at the Hamiltonian of the anisotropic two-channel Kondo model \[36\] \[40\]

$$H = H_0 + B S^z + \sum_{\alpha} J_{\alpha} [s_{\alpha}^+(0) S^+ + s_{\alpha}^-(0) S^-],$$

where $s_{\alpha}^+(0) = c_{\alpha \uparrow}^+(0) c_{\alpha \downarrow}(0)$, $s_{\alpha}^-(0) = [s_{\alpha}^+(0)]^\dagger$, and $B = 4E_C (N_g^a - N_g)$. (In writing Eq. (9), we changed the sign of the exchange term with the help of the unitary transformation $e^{i\pi S^z} H e^{-i\pi S^z}$.)

Importantly, the exchange constants $J_{\alpha}$ in Eq. (9) are controlled independently by the conductances of the point contacts [see Eq. (7)], and, therefore, can be easily tuned to be equal. Similarly, the “magnetic field” $B$ describes departures from the charge degeneracy and can be tuned to zero by changing the gate potential $N_g$. Such remarkable tunability allows one to fully explore various parameter regimes of the two-channel Kondo model \[9\].

At $B = 0$ and $J_L = J_R$ [these equations define a line in the three-dimensional parameter space $(B, J_L, J_R)$] observable quantities exhibit a non-Fermi liquid behavior \[36\] \[40\], whereas anywhere away from this critical line they behave at lowest temperatures as prescribed by the Fermi-liquid theory. On crossing the critical line at $T = 0$, the system undergoes a quantum phase transition between two Fermi-liquid states that are adiabatically connected to each other by going around the critical line. At the transition, observable quantities exhibit singularities. For example, at $J_L = J_R$ the susceptibility $dS_z/dB$, associated with the correlation function $\langle S_z(t) S_z(0) \rangle$, diverges logarithmically \[38\] at $B \rightarrow 0$.

Our observable of choice, the linear conductance, is given by the Kubo formula \[41\]

$$\frac{G}{G_0} = \lim_{\omega \rightarrow 0} \frac{\pi}{\omega} \int_0^\infty dt \, e^{i\omega t} \langle [I(t), I(0)] \rangle \quad \text{(10)}$$

with $G_0 = 2e^2/h$ and with the particle current operator given by

$$I = \frac{d}{dt} \langle N_R - N_L \rangle,$$

where $N_{\alpha}$ is the total number of electrons in the lead $\alpha$. In terms of the Kondo model \[9\], it reads

$$N_{\alpha} = N_{\alpha 1} - N_{\alpha \downarrow}, \quad N_{\alpha \sigma} = \sum_{k \sigma} c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma}. \quad \text{(12)}$$

With time dependence governed by the Hamiltonian \[9\], we find $I = i [J_L s_{\alpha}^+(0) - J_R s_{\alpha}^-(0)] S^- + H.c.$ Accordingly, the conductance \[10\] provides direct access to the correlation functions of the type $\langle S^+(t) S^-(0) s_{\alpha}^R(0) \rangle$.

We discuss first the temperature dependence of the conductance when the parameters of the Kondo model \[9\] are tuned precisely to the critical line. In other words, we consider exact charge degeneracy ($B = 0$) and equal conductances of the contacts ($J_L = J_R = J = g_0 v/2$). Writing the rate equations result \[24\] \[39\] in terms of exchange constants in Eq. (9), we find $G/G_0 = 2\pi^2 (\nu J)^2$ for the conductance in the lowest order in $\nu J < 1$ [here $\nu = (2\pi v)^{-1}$ is the density of states per length]. The Kondo effect can be accounted for in the rate equations formalism \[42\] by replacing $J$ with its renormalized
value reached when the bandwidth $D$ of conduction electrons in Eq. (9) is reduced from its initial value $D \simeq E_C$ to $D \sim T$. In the scaling limit $T_K \ll D \ll E_C$ we have $\nu J(D) = [2 \ln(D/T_K)]^{-1}$, and the conductance assumes the form

$$
\frac{G}{G_0} = \frac{\pi^2}{2 \ln^2(T/T_K)},
$$

(13)

where $T_K \sim E_C e^{-\pi^2/g}$ is the Kondo temperature. The temperature dependence of the conductance in the strong-coupling regime ($T \ll T_K$) can be found using the technique of Ref. [53], which yields

$$
\frac{G}{G_0} = 1 - \frac{a T}{T_K}, \quad a \sim 1.
$$

(14)

(The value of $a$ depends on the precise definition of $T_K$). This result can be derived by considering the least-relevant perturbation of the Emery-Kivelson Hamiltonian, which is the same perturbation as the one producing the correct low-temperature asymptote of the specific heat in the two-channel Kondo model. A standard perturbative calculation of the conductance then yields the linear-in-$T$ dependence of $G$. Alternatively, Eq. (14) can be obtained by mapping our problem onto that of a resonant tunneling of a Luttinger liquid with the Luttinger-liquid parameter $K = 1/2$ through a double-barrier structure. Accounting for the least-relevant (at $K > 1/3$) perturbation identified in Ref. [49] the correction to the conductance scales as $G(0) - G(T) \propto T^{2K}$, in agreement with Eq. (14). Note that $G(T)$ we found differs from that in the two-channel Kondo device proposed in Ref. [52] and realized experimentally in Ref. [53]. The difference arises because $G$ in the device of Refs. [52] is proportional to the single-particle $t$-matrix $g$, whereas $G(0) - G(T) \propto \sqrt{T}$, whereas in our case $G$ is given by the two-particle correlation function.

According to Eq. (14), the conductance at zero temperature is exactly half of the conductance of an ideal single-channel interface between a normal conductor and a superconductor $4e^2/h$. Such halving of the ideal conductance is one of the manifestations of quantum criticality. This property is reminiscent of the predicted and observed behavior of inelastic cotunneling of spin-polarized electrons through a Coulomb-blockaded normal-state island with vanishing single-particle level spacing. Indeed, in this case the zero-temperature conductance at the charge-degeneracy point is $e^2/2h$, which again is exactly half of the ideal conductance of a single-channel point contact $e^2/h$.

Finite zero-temperature conductance in our model is the hallmark of the non-Fermi-liquid behavior. Any departure from the critical line restores the Fermi liquid: at finite $B$, $J_L - J_R$, or both, the conductance scales as $G \propto T^2$ at lowest temperatures instead of Eq. (14). The origin of this behavior is easy to understand in the limit of large $B \gg T_K$. In this limit, the entire dependence $G(T)$ can be found by perturbation theory. At $T \ll B$ transitions $|\downarrow \rightarrow | \uparrow \rangle$ are virtual, and their role reduces to merely generating a residual local exchange interaction between conduction electrons [36, 59]. The contribution giving rise to nonzero current reads

$$
H_{\text{int}} = V \left[ s_R^+(0)s_L(0) + s_R^+(0)s_L^-(0) \right].
$$

(15)

In the second order of perturbation theory, the interaction constant in Eq. (15) is given by $V = -J_B^2/B$. With $J$ here replaced with its renormalized value at $D \sim B$, Eq. (15) is applicable at all $B$ in the range $T_K \ll B \ll E_C$. The particle current [see Eqs. (11) and (12)] evaluated with the Hamiltonian $H = H_0 + H_{\text{int}}$ reads $I = 2iV \left[ (s_L^+(0)s_R(0) - s_R^+(0)s_L^-(0)) \right]$. The Kubo formula [10] then yields $G/G_0 = [(2\pi)^4/3]V^2T^2$, leading to the asymptote

$$
\frac{G}{G_0} = \frac{\pi^4T^2}{3B^2 \ln^2(B/T_K)}
$$

(16)

at $T \ll B$. On the other hand, in the opposite limit $T \gg B$ the conductance is still described by Eq. (13). Hence, the dependence $G(T)$ is nonmonotonic, with a maximum at $T \sim B$.

The channel asymmetry also leads to a nonmonotonic temperature dependence of the conductance. If the contact conductances are small but very different, the conductance reaches its maximum in the regime accessible by perturbative renormalization. To be definite, we consider the case when $J_L \ll J_R$. Evaluating the conductance with the help of the rate equations [24, 30], we find $G/G_0 = 8\pi^2(\nu J_L)^2$. When considering perturbative renormalization of $J_L$ [30, 37, 43, 44], it is important to take into account, in addition to the usual second-order term $J_L^2$, the dominant next-order contribution. This contribution is proportional to $J_L J_R^2$ and is negative, leading to a nonmonotonic dependence $J_R(D)$ [43]. In the scaling limit, it is convenient to express the results in terms of $T_K^o \sim E_C \exp(-\pi^2/g_0)$ ($T_K$ is the Kondo temperature in the limit when the conductance of contact $\alpha$ is finite, whereas the second contact is completely shut off). The conductance reaches its maximum

$$
\left( \frac{G}{G_0} \right)_{\text{max}} = \frac{2\pi^2}{\ln^2(T_K^o/T_K)}
$$

(17)

at $T \sim T_K^o \exp\sqrt{\ln(T_K^o/T_K)}$ [43], which belongs to the perturbative domain ($T \gg T_K^o \sim T_K$) provided that $|g_R - g_L| \gg g_0$. The dependence on temperature near the maximum is weak, see the left panel in Fig. 1 and $G(T)$ crosses over from Eq. (17) to the Fermi-liquid low-temperature asymptote $G \propto T^2$ at $T \sim T_K^o$.

In the opposite limit of small deviations from the critical line in the parameter space $(B, J_L, J_R)$, the system
Width is lowered, the corresponding dimensionless coupling dimension \( i \) (i.e., distance to the charge degeneracy point) and the width of order unity at \( T \approx T_F \) of the Coulomb blockade peak in the dependence \( T \sim T_F \ll T_K \), see Eq. 18.

Upon lowering the temperature first enters the strong-coupling non-Fermi-liquid regime [see Eq. (14)], and then crosses over at \( T \sim T_F \ll T_K \) to the limiting Fermi-liquid behavior. The crossover is described by \[ G \approx f \left( \frac{\pi T}{T_F} \right), \quad f(x) = 1 - \frac{1}{2x} \Psi \left( \frac{1 + x}{2x} \right), \quad (18) \]

where \( \Psi(z) = d^2 \ln \Gamma(z)/dz^2 \) is the trigamma function. The universal function \( f(x) \) interpolates between \( f(x) = 1 - \pi^2/4x \) at \( x \gg 1 \) and \( f(x) = x^2 \) at \( x \ll 1 \). The latter limit corresponds to the Fermi-liquid regime. The characteristic crossover scale \( T_F \) in Eq. (18) is set by the distance of the system parameters to the critical line. This scale can be estimated by scaling analysis. Near the critical line, both the magnetic field (i.e., distance to the charge degeneracy point) and the channel asymmetry are relevant perturbations with scaling dimension \( 1/2 \) [60][63]. Therefore, as the bandwidth is lowered, the corresponding dimensionless coupling constants grow at \( D \leq T_K \) as \( (T_K/D)^{1/2} \), becoming of order unity at \( D \sim T_F \). Taking into account that \( B \) at \( D \sim T_K \) is of order of its bare value [13], we obtain \[ 37[32] T_F \sim B^2/T_K \sim (E_K/T_K)(N_{\theta} - N_{\theta}^*)^2 \] for channel-symmetric setup [64]. Accordingly, the width of the Coulomb blockade peak in the dependence \( G(N_{\theta}) \) scales as \( \sqrt{T} \) with temperature. The above estimate of \( T_F \) and Eq. (15) are applicable as long as \( T_F \ll T_K \), i.e., close to the charge degeneracy point. Further away from this point, the conductance is described by Eqs. (13) and (16) at \( T \gg B \) and \( T \ll B \), respectively.

Interestingly, Eq. (18) also describes conductance in a device with almost open contacts, i.e., in the limit when \( 1 - g_\alpha \ll 1 \) and the tunneling Hamiltonian description of the contacts [see Eq. (4)] is inapplicable. In fact, it was originally derived [42] in this limit in the context of the closely related problem of inelastic cotunneling. For almost open contacts the crossover scale \( T_F \) also scales as \( (N_{\theta} - N_{\theta}^*)^2 \) in the vicinity of the charge degeneracy point [42], hence, the width of the Coulomb blockade peak is again proportional to \( \sqrt{T} \). However, in this limit the number of electrons in the Coulomb-blockaded region is not quantized. Strong charge fluctuations render the reduction to the Kondo model [cf. Eq. (6)] impossible. As a result, the temperature dependence of the conductance is characterized by only two energy scales, \( E_C \) and \( T_F \) [42]. In the symmetric case, despite the absence of the intermediate scale \( T_K \), the linear-in-\( T \) correction to the conductance at the degeneracy point remains valid, except that \( T_K \) is replaced by \( E_C \) in Eq. (14).

In conclusion, conduction through a Coulomb-blockaded mesoscopic s-wave superconductor is facilitated by Andreev processes. In the vicinity of the charge degeneracy points these processes can be mapped onto exchange terms in the effective two-channel Kondo model. Unlike in the case of inelastic cotunneling through a normal-state island [22][61][67][58], the mapping does not rely on the smallness of the single-particle level spacing in the Coulomb-blockaded region in comparison with temperature. The critical two-channel-Kondo regime corresponds to the limit when conductances of the point contacts connecting the superconductor to the normal-state leads are equal. In such symmetric setup conductance at the Coulomb blockade peak increases with the decrease of temperature, reaching \( 2e^2/h \) at zero temperature, whereas the width of the peak decreases as \( \sqrt{T} \).

Our theory is valid provided that the induced superconducting gap \( \Delta \) is large compared with both the charging energy \( E_C \) and the single-particle level spacing \( \delta \). These parameters are set by the device geometry and properties of the materials used. Experiments [14] on the already existing 1.5 \( \mu m \)-long aluminum-coated InAs wires yielded \( \Delta \approx 0.2 \) meV and \( E_C \approx 20 \) \( \mu \)eV. Estimating \( \delta \) with the help of the results of Ref. [24], we find \( \delta \sim 3 \) \( \mu \)eV. Accordingly, parameters of these wires fall well within the desired range. Estimated values of these parameters for the prospective devices [65] of the type studied in Ref. [68] with normal metal NiGeAu replaced by superconducting In had \( \Delta \sim 1 \) meV, \( E_C \sim 20 \) \( \mu \)eV, and \( \delta \sim 10^{-5} \) \( \mu \)eV, thus promising a much larger value of the ratio \( \Delta/\delta \). Unlike \( \Delta \), \( E_C \), and \( \delta \), the Kondo temperature \( T_K \) and the crossover scale \( T_F \) are tunable by varying conductances of the contacts and the gate potential. The tunability makes it possible to explore experimentally all the regimes discussed above and crossovers between them on a single device.

We thank Harold Baranger, Christophe Moro, and Eran Sela for pointing out the correct temperature dependence of Eq. (14), which corrects a previous version of the manuscript, and for helping us understand its origin. We are grateful to Fabrizio Nichele and Frederic Pierre for discussions and correspondence. This work is supported by ONR Grant Q00704 (BvH) and by DOE contract DEFG02-08ER46482 (LG).
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Quantum Criticality in Resonant Andreev Conduction

Supplemental Material

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1. Two-channel Kondo model in the weak coupling regime

We consider the two-channel Kondo model

$$H = \sum_{\alpha} \int dx : c_{\alpha}(x) (-iv\partial_x) c_{\alpha}(x) : + BS^z$$ (1.1)

$$+ \sum_{\alpha} \left\{ J_{z\alpha} s^z_{\alpha}(0) S^z + J_{\perp\alpha} \left[ s^+_{\alpha}(0) S^- + s^-_{\alpha}(0) S^+ \right] \right\},$$

where $c_{\alpha}(x) = L^{-1/2} \sum_k e^{ikx} c_{\alpha k}$, the colons denote the normal ordering, and

$$s^z_{\alpha}(0) = \frac{1}{2} \left[ \{ c_{\alpha}^\dagger(0) c_{\alpha}(0) : - : c_{\alpha}^\dagger(0) c_{\alpha}(0) \} \right],$$ (1.2a)

$$s^z_{\alpha}(0) = \left[ s_{\alpha}(0) \right] = e^{i\alpha} c_{\alpha}(0).$$ (1.2b)

The exchange amplitudes corresponding to the initial bandwidth $D_0 \sim E_C$ are given by

$$J_{z\alpha} = 0, \quad J_{\perp\alpha} = \frac{g_{\alpha} v}{2},$$ (1.3)

see Eq. (7) in the paper. Upon reduction of the bandwidth $D$, the dimensionless exchange amplitudes

$$I_{z\alpha} = \frac{J_{z\alpha}}{2\pi v}, \quad I_{\perp\alpha} = \frac{J_{\perp\alpha}}{\pi v},$$ (1.4)

evolve according to the weak-coupling renormalization group equations [3]

$$\frac{d}{d\zeta} I_{z\alpha} = I_{z\alpha}^2 - \frac{1}{2} I_{z\alpha} \sum_{\beta} I_{z\beta}^2 + \ldots,$$ (1.5a)

$$\frac{d}{d\zeta} I_{\perp\alpha} = I_{\perp\alpha} I_{\perp\alpha} - \frac{1}{4} I_{\perp\alpha} \sum_{\beta} (I_{z\beta}^2 + I_{z\beta}^2) + \ldots,$$ (1.5b)

where $\zeta = \ln(D_0/D)$. Neglecting the cubic terms in the right-hand sides of Eqs. (1.5), we obtain [3]

$$I_{z\alpha}(\zeta) = \frac{I_{z\alpha}(0)}{\sin[I_{z\alpha}(0)(\zeta_\alpha - \zeta)]},$$ (1.6a)

$$I_{\perp\alpha}(\zeta) = I_{\perp\alpha}(0) \cot[I_{\perp\alpha}(0)(\zeta_\alpha - \zeta)],$$ (1.6b)

with

$$\zeta_\alpha = \frac{\pi}{2I_{\perp\alpha}(0)}.$$ (1.7)

The equation $\ln(D_0/T_K^\alpha) = \zeta_\alpha$ gives the estimate of the Kondo temperature in the channel $\alpha$ when the other channel is completely shut off,

$$T_K^\alpha \sim D_0 e^{-\zeta_\alpha} \sim E_C e^{-\pi^2/4g_{\alpha}}.$$ (1.8)

[Corrections to $T_K^\alpha$ come from the cubic and higher-order terms in the right-hand sides of Eqs. (1.5) neglected in the derivation of Eqs. (1.6).]

In the scaling limit defined by

$$1 \ll \zeta_\alpha - \zeta \ll \zeta_\alpha$$ (1.9)

Eqs. (1.6) simplify to

$$I_{z\alpha}(\zeta) = I_{z\alpha}(\zeta) = I_{\alpha}(\zeta) = \frac{1}{\zeta_\alpha - \zeta} = \frac{1}{\ln(D/(T_K^\alpha))},$$ (1.10)

indicating a restoration of $SU(2)$ symmetry. In the channel-symmetric case we have $\zeta_\alpha = \zeta_K$ and $T_K^\alpha = T_K$. In terms of $D$ and $T_K$, the condition (1.9) then reads

$$T_K \ll D \ll E_C,$$ (1.11)

and Eq. (1.10) assumes the form

$$I_{\alpha}(D) = \frac{1}{\ln(D/T_K)}.$$ (1.12)

With the identification $\nu J_\alpha = I_{\alpha}/2$ [see Eq. (1.4)], this expression is used in Eqs. (13) and (16) in the paper.

Renormalization of the magnetic field $B$ in Eq. (1.1) is governed by the equation

$$\frac{d}{d\zeta} B = -\frac{B}{2} \sum_{\alpha} I_{z\alpha}(\zeta) + \ldots.$$ (1.13)

Taking into account Eqs. (1.6), we find

$$\ln \frac{B(0)}{B(\zeta)} = \frac{1}{2} \sum_{\alpha} I_{z\alpha}(\zeta).$$ (1.14)

For symmetric channels, Eq. (1.14) reduces in the scaling limit to

$$\ln \frac{B(0)}{B(\zeta)} = \frac{1}{2\zeta_K - \zeta}.$$ (1.15)
The right-hand side of Eq. (1.15) is small at all $\zeta \ll \zeta_K$, becoming of order unity only at $\zeta_K - \zeta \sim 1$, when Eqs. (1.5) and (1.13) cease to be applicable. Therefore, the renormalization of $B$ throughout the weak coupling regime $\zeta_K - \zeta \sim 1$ can be ignored in the first approximation. This lack of renormalization is taken into account in writing Eq. (16) in the paper.

If the asymmetry between the channels is strong, e.g., $J_L \ll J_R$ or, equivalently, $\zeta_L \gg \zeta_R$, the exchange amplitudes grow with $\zeta$ according to Eqs. (1.6) until $\zeta$ reaches the value $\zeta^*$ at which the cubic terms in Eqs. (1.5) for the weaker coupled channel become compatible with the quadratic ones. If $\zeta^*$ belongs to the scaling limit [see Eq. (1.9)], this leads to the equation $I_L(\zeta^*) = I_R^2(\zeta^*)$, where $I_\alpha(\zeta)$ are given by Eq. (1.10) with $D = T^*$. Taking into account that $\zeta_L - \zeta^* \gg \zeta_R - \zeta^* \gg 1$, we obtain

\begin{align}
I_L(\zeta) &= \frac{1}{\ln(T_K^R/T_K^L)}, \\
\ln(T_*/T_K^R) &= \left[\ln(T_K^R/T_K^L)\right]^{1/2}.
\end{align}

With further reduction of the bandwidth, at $\zeta^* \ll \zeta \ll \zeta_R - \zeta_K$, the smaller exchange amplitude $I_L$ evolves according to the equation

\begin{equation}
\frac{d}{d\zeta} I_L = -\frac{1}{2} I_L I_R^2.
\end{equation}

which describes a downward renormalization of $I_L(\zeta)$ similar to that of $B(\zeta)$. Accordingly, $I_L(\zeta)$ is nonmonotonic, with $\max\{I_L(\zeta)\} = I_L(\zeta^*)$. The estimates (1.16) are used in Eq. (17) in the paper.

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