The Langevin Monte Carlo algorithm in the non-smooth log-concave case

Joseph Lehec

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Abstract

We prove non-asymptotic polynomial bounds on the convergence of the Langevin Monte Carlo algorithm in the case where the potential is a convex function which is globally Lipschitz on its domain, typically the maximum of a finite number of affine functions on an arbitrary convex set. In particular the potential is not assumed to be gradient Lipschitz, in contrast with most existing works on the topic.

Keywords: Statistical Sampling, Markov Chain Monte Carlo, Convexity.

1 Introduction

Setting. Sampling from a high-dimensional log-concave probability measure is a problem dating back to the early nineties and the seminal work of Dyer, Frieze and Kannan [13] and which has many applications to various fields such as Bayesian statistics, convex optimization and statistical inference. This problem is always addressed via Markov Chain Monte Carlo methods, but there is a large variety of those: Metropolis-Hastings type random walks (ball walk), Glauber like dynamics (hit and run) or Hamiltonian Monte Carlo. In this article, we will consider the so-called Langevin algorithm, which is defined as follows. Given a probability measure $\mu$ on $\mathbb{R}^n$ we let $\varphi$ be its potential, namely $\mu$ has density $e^{-\varphi}$ with respect to the Lebesgue measure. The Langevin diffusion associated to $\mu$ is the solution $(X_t)$ of the following stochastic differential equation

$$dX_t = dB_t - \frac{1}{2} \nabla \varphi(X_t) \, dt,$$

where $(B_t)$ is a standard $n$-dimensional Brownian motion. The Langevin algorithm is the Euler scheme associated to this diffusion: Given a time step parameter $\eta$ we let $(\xi_k)_{k \geq 1}$ be a sequence of i.i.d. centered Gaussian vectors with covariance $\eta \text{Id}$ and set

$$x_{k+1} = x_k + \xi_{k+1} - \frac{\eta}{2} \nabla \varphi(x_k).$$

We shall focus on the log-concave case, namely the case where the potential $\varphi$ is convex. One originality of this work is that we will consider the constrained case, allowing the measure $\mu$ to be supported on a set $K$ different from $\mathbb{R}^n$. In other words the potential $\varphi$ is allowed to take the value
outside some set $K$. Notice that the log-concavity assumption implies that $K$ is convex. In the constrained case the Langevin diffusion (1) becomes

$$dX_t = dB_t - \frac{1}{2} \nabla \varphi(X_t) \, dt - d\Phi_t,$$

where $(\Phi_t)$ is a process that repels $(X_t)$ inward when it reaches the boundary of $K$, see the next section for a precise definition. The discretization then becomes

$$x_{k+1} = \mathcal{P} \left(x_k + \xi_{k+1} - \frac{n}{2} \nabla \varphi(x_k)\right),$$

where $\mathcal{P}$ is the projection on $K$: For $x \in \mathbb{R}^n$ the point $\mathcal{P}_K(x)$ is the closest point to $x$ in $K$. This is the algorithm we will study throughout the article. It was first introduced in our joint paper with Bubeck and Eldan [5] and to the best of our knowledge it has not been investigated since.

The second originality of this work is that we do not assume the potential $\varphi$ to be smooth. More precisely we will assume that the gradient of $\varphi$ (or rather its subdifferential) is uniformly bounded on $K$, but we do not assume it to be Lipschitz or even continuous. Let us point out that this is by no means an exotic situation, the reader could think for instance of $\varphi$ being the maximum of a finite number of affine functions on $K$. We do not make any assumption whatsoever on the convex set $K$.

The drawback of this very generic situation and of our approach is that we are only able to get convergence estimates in Wasserstein distance. Recall that the Wasserstein distance $W_2$ between two probability measures $\mu$ and $\nu$ is defined as

$$W_2^2(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \{ \mathbb{E}[|X - Y|^2]\}.$$

By a slight abuse of notation, if $X, Y$ are random vectors we also write $W_2(X, Y)$ for the Wasserstein distance between the law of $X$ and that of $Y$.

**Main results.** Our main result is the following bound between the Langevin algorithm (4) after $k$ steps and its corresponding point $X_{k\eta}$ in the true Langevin diffusion (3).

**Theorem 1.** Assume that $\mu$ is log-concave, with globally Lipschitz potential $\varphi$ on its support $K$ and let $L$ be the Lipschitz constant. Assume that the time step $\eta$ satisfies $\eta < nL^{-2}$ and suppose that the Langevin algorithm and diffusion are initiated at the same point $x_0$. Then for every integer $k$ we have

$$\frac{1}{n} W_2^2(X_{k\eta}, x_k) \leq A k \eta^{3/2}$$

where

$$A = \left(2e^{1/2} + 1\right) \frac{(1 + \sigma_0) (n + 2 \log k)^{1/2}}{r_0} + \frac{7}{6} \frac{L}{n^{1/2}},$$

and

$$r_0 = d(x_0, \partial K) \quad \text{and} \quad \sigma_0 = \frac{1}{n} \left(\varphi(x_0) - \inf_K \{\varphi(y)\}\right).$$

**Remark.** The transport cost $W_2^2$ behaves additively when taking tensor products, so the Wasserstein distance between any two probability measures on $\mathbb{R}^n$ is typically of order $\sqrt{n}$. Therefore $\frac{1}{n} W_2^2$ is of order 1, which explains why we wrote the theorem this way. The reader should thus have in mind that the theorem provides some nontrivial information as soon as the right-hand side of (5) is smaller than some small constant $\epsilon$. 

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The result depends on the starting point via the parameters $r_0$ and $\sigma_0$. In order to get a meaningful bound the algorithm should not be initiated too close to the boundary of $K$ or at a point where the potential $\varphi$ is too large. Let us also point out that the theorem is also valid when there is no support constraint, namely when $K = \mathbb{R}^n$. One just replaces $r_0$ by $+\infty$, so that $A = O(Ln^{-1/2})$ in this case. Let us comment also on the parameter $\sigma_0$. Obviously $\sigma_0 = 0$ if the potential is minimal at $x_0$. Fradelizi’s theorem [16, Theorem 4] asserts that if $\mu$ is log-concave on $\mathbb{R}^n$ with density $f$ and has its barycenter at $x_0$ then

$$\sup_{x \in \mathbb{R}^n} \{f(x)\} \leq e^n f(x_0).$$

In terms of the parameter $\sigma_0$ this means that if $\mu$ has its barycenter at $x_0$ then $\sigma_0 \leq 1$. Since $\varphi$ is assumed to be Lipschitz with constant $L$, if $x_0$ is at $O(nL^{-1})$ distance from the barycenter then again $\sigma_0$ is order 1. In general we shall think of $\sigma_0$ as a parameter of order 1. Also we are never going to take more than $\operatorname{poly}(n)$ steps so $\log k$ will always be negligible compared to $n$. Under the previous assumptions the parameter $A$ thus satisfies

$$A = O\left(\max\left(\frac{n^{1/2}}{r_0}; \frac{L}{n^{1/2}}\right)\right).$$

In order to estimate the complexity of the Langevin algorithm, we need to combine the previous theorem with some estimate on the speed of convergence of the Langevin diffusion $(X_t)$ towards its equilibrium measure $\mu$. For this we shall use two functional inequalities, the Poincaré inequality and the logarithmic Sobolev inequality. Recall that the measure $\mu$ is said to satisfy the logarithmic Sobolev inequality if for every probability measure $\nu$ on $\mathbb{R}^n$ we have

$$D(\nu \mid \mu) \leq C_{LS} I(\nu \mid \mu) \quad (8)$$

where $D(\nu \mid \mu)$ and $I(\nu \mid \mu)$ denote respectively the relative entropy and the relative Fisher information of $\nu$ with respect to $\mu$:

$$D(\nu \mid \mu) = \int_{\mathbb{R}^n} \log \left(\frac{d\nu}{d\mu}\right) d\nu \quad \text{and} \quad I(\nu \mid \mu) = \int_{\mathbb{R}^n} \left|\nabla \log \left(\frac{d\nu}{d\mu}\right)\right|^2 d\nu.$$

The smallest constant $C_{LS}$ for which (8) holds true is called the log-Sobolev constant of $\mu$. The factor $\frac{1}{2}$ is just a matter of convention, with this normalization the log-Sobolev constant of the standard Gaussian measure is 1, in any dimension. It is well-known that the log-Sobolev inequality is stronger than the Poincaré inequality. More precisely, letting $C_P$ be the best constant in the Poincaré inequality:

$$\text{var}_\mu(f) \leq C_P \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,$$

we have $C_P \leq C_{LS}$.

**Theorem 2.** Again assume that $\mu$ is log-concave with globally Lipschitz potential on its support, with Lipschitz constant $L$. Let $x_0$ be a point in the support of $\mu$ and recall the definition (7) of $\sigma_0$ and $r_0$. Assume in addition that the measure $\mu$ satisfies the log-Sobolev inequality with constant $C_{LS}$. Then after $k$ steps of the Langevin algorithm started at $x_0$ with time step parameter $\eta < nL^{-2}$ we have

$$\frac{1}{n} W_2^2(x_k; \mu) \leq 2B e^{-k\eta/2C_{LS}} + 2A k\eta^{3/2}.$$
where again $A$ is given by (6) and

$$B = 4C_{LS} \left( 1 + \log \left( \frac{\max(C_{LS}, 1) n}{\min(r_0, 1)} \right) + \sigma_0 + \frac{L}{n} \right).$$

Remark. Note that we initiate the Langevin algorithm at a Dirac point mass, we do not need any warm start hypothesis. The starting point only plays a role through the parameters $r_0$ and $\sigma_0$.

Let us describe what the theorem gives in terms of the complexity of the Langevin algorithm. Say we want \( \frac{1}{n} W_2^2(x_k, \mu) \leq \varepsilon \) for some small $\varepsilon$. The first term of the right-hand side is a little bit intricate, so let us assume that all parameters of the problem are at most polynomial in the dimension. Then that term is just $\text{poly}(n) \exp(-k\eta/2C_{LS})$, which is negligible as soon as $k\eta = \Omega(C_{LS} \log n)$. Let us also assume that $\sigma_x = O(1)$ (see the discussion above). Then the theorem shows that choosing $\eta = \Theta^*(\varepsilon^2 C_{LS}^2 \min(r_0^2/n, L^2))$ and running the algorithm for

$$k = \Theta^* \left( C_{LS}^3 \frac{\sigma^2}{\varepsilon^2} \max \left( \frac{n}{r_0^2}; \frac{n}{L^2} \right) \right)$$

steps produces a point $x_k$ satisfying $\frac{1}{n} W_2^2(x_k, \mu) \leq \varepsilon$. The notation $\Theta^*$ hides universal constants as well as possible $\text{polylog}(n)$ dependencies, this is a common practice in this field.

Note in particular that if we treat all parameters other than the dimension as constants, then we already get a non trivial information after a number of steps of the algorithm which is nearly linear in the dimension.

Of course not every log-concave measure satisfies log-Sobolev, simply because log-Sobolev implies sub-Gaussian tails. However there are a number of interesting cases in which the log-Sobolev inequality is known to hold true, which we list below.

1. If the potential $\varphi$ is $\alpha$-uniformly convex for some $\alpha > 0$, in the sense that $x \mapsto \varphi(x) - \frac{\alpha}{2} |x|^2$ is convex, then $\mu$ satisfies log-Sobolev with constant $1/\alpha$. This is the celebrated Bakry-Émery criterion, see [1]. See also [3] for an alternate proof based on the Prékopa-Leindler inequality.

2. If $\mu$ is log-concave and is supported on a ball of radius $R$, then $\mu$ satisfies log-Sobolev with constant $R^2$, up to a universal constant. This follows trivially from E. Milman’s result that, within the class of log-concave measures, Gaussian concentration and the log-Sobolev inequality are equivalent, see [21, Theorem 1.2.], or [18, Theorem 2].

3. If $\mu$ is log-concave, supported on a ball of radius $R$ and isotropic, in the sense that its covariance matrix is the identity matrix, then Lee and Vempala [19] have shown that $\mu$ satisfies log-Sobolev with constant $R$, up to a universal factor. Note that the isotropy condition implies that $R \geq \sqrt{n}$, so this improves greatly upon the previous result in the isotropic case.

In the first case, notice that since the potential is at the same time globally Lipschitz and uniformly convex, the support of $\mu$ must be bounded. Actually if the potential is globally Lipschitz it cannot grow fast enough at infinity to insure log-Sobolev. So if we insist on assuming that the potential is Lipschitz and on using log-Sobolev then we have to assume that the support is bounded.

One way around this issue is to use the Poincaré inequality rather than log-Sobolev. Indeed every log-concave measure satisfies the Poincaré inequality. Kannan, Lovasz and Simonovits [17]...
proved that the Poincaré constant of an isotropic log-concave measure on $\mathbb{R}^{n}$ is $O(n)$ and conjectured that it should actually be bounded. This conjecture, which was the major open problem in the field of asymptotic convex geometry, was recently nearly solved by Yuansi Chen [8], who proved an $n^{o(1)}$ bound for the Poincaré constant of an isotropic log-concave vector in dimension $n$. The result of Chen relies on a technique invented by Eldan [14] which was also used by Lee and Vempala [19] to prove a $O(\sqrt{n})$ bound for the KLS constant, as well as the aforementioned log-Sobolev result. Recall that if $\nu$ is a probability measure, absolutely continuous with respect to $\mu$, the chi-square divergence of $\nu$ with respect to $\mu$ is defined as

$$\chi^2(\nu \mid \mu) = \int_{\mathbb{R}^n} \left( \frac{d\nu}{d\mu} - 1 \right)^2 d\mu.$$ 

Our next theorem then states as follows.

**Theorem 3.** Assume that $\mu$ is a log-concave probability measure with globally Lipschitz potential $\varphi$ on its domain, with constant $L$. Then after $k$ steps of the Langevin algorithm initiated at a random point $x_0$ taking values in the domain, and with time step parameter $\eta$ satisfying $\eta \leq nL^{-2}$, we have

$$\frac{1}{n}W_2^2(x_k, \mu) \leq \frac{4}{n} C_P \chi^2(x_0 \mid \mu) e^{-k\eta/C_P} + 2Ak\eta^{3/2},$$

where $C_P$ is the Poincaré constant of $\mu$ and where

$$A = (2e^{1/2} + 1)(n + 2 \log k)^{1/2} \mathbb{E} \left[ \frac{1 + \sigma_0}{r_0} \right] + \frac{7}{6} \frac{L}{n^{1/2}}.$$ 

Note that $r_0$ and $\sigma_0$ are random here.

So the price we have to pay for using Poincaré rather than log-Sobolev is a warm start hypothesis: the algorithm must be initiated at a random point $x_0$ whose chi-square divergence to $\mu$ is finite. In the unconstrained case, namely when $\mu$ is supported on the whole $\mathbb{R}^n$, a natural choice for a warm start is an appropriate Gaussian measure. One can indeed get the following estimate.

**Lemma 4.** Suppose $\mu$ is log-concave, supported on the whole $\mathbb{R}^n$, with globally Lipschitz potential, with Lipschitz constant $L$. Let $\gamma$ be the Gaussian measure centered at a point $x_0$ and with covariance $\frac{L^2}{n^2} \text{Id}$. Then

$$\log \chi^2(\gamma \mid \mu) \leq n(1 + \sigma_0) + \frac{n}{2} \log \left( \frac{L^2C_P}{n} \right),$$

where $C_P$ is the Poincaré constant of $\mu$.

In particular when $\sigma_0 = O(1)$ and all other parameters of the problem are at most polynomial in $n$, we get $\log \chi^2(\gamma \mid \mu) \leq O(n \log n)$. With this choice of a warm start, and observing that in the unconstrained case the parameter $A$ is just $O(L/\sqrt{n})$, the previous theorem gives $\frac{1}{n}W_1^2(x_k, \mu) \leq \varepsilon$ after

$$k = \Theta^*(\frac{C_P^2L^2n^2}{\varepsilon^2})$$
steps, with $\eta$ chosen appropriately. Also in the constrained case, one can get a non trivial complexity estimate from Theorem 3 by choosing the uniform measure on a ball contained in the support as a warm start. We leave this annoying computation to the reader.
Finally let us point out that it is also possible to obtain interesting bounds from our result when the potential is not globally Lipschitz, simply by restricting the measure to a large ball. For simplicity let us only spell out the argument when the measure $\mu$ is supported on the whole $\mathbb{R}^n$ and when we have a linear control on the gradient of the potential, but the method could give non trivial bounds in more general situations. So let $\mu$ be a log-concave measure supported on the whole $\mathbb{R}^n$, let $\varphi$ be its potential, and consider the Langevin algorithm associated to the measure $\mu$ conditioned on the ball of radius $R$:

$$x_{k+1} = \mathcal{P}\left(x_k + \sqrt{\eta}\xi_{k+1} - \frac{\eta}{2} \nabla \varphi(x_k)\right),$$

where $\mathcal{P}$ is the orthogonal projection on the ball of radius $R$:

$$\mathcal{P}(x) = \begin{cases} x & \text{if } |x| \leq R \\ \frac{Rx}{|x|} & \text{otherwise} \end{cases}$$

In this special case, Theorem 2 yields the following complexity for the Langevin algorithm.

**Theorem 5.** Assume that $\mu$ is log-concave, supported on the whole $\mathbb{R}^n$ and that the gradient of the potential $\varphi$ grows at most linearly:

$$|\nabla \varphi(x)| \leq \beta(|x| + 1),$$

for all $x \in \mathbb{R}^n$ and for some $\beta > 0$. Assume that the Langevin algorithm is initiated at 0, that $\sigma_0 = O(1)$, that $\int |x|^2 d\mu = O(n)$, and let $C_{LS}$ be the log-Sobolev constant of $\mu$, with the convention that it equals $+\infty$ if $\mu$ does not satisfy log-Sobolev. Then choosing $R = \Theta^\ast(\sqrt{n})$, $\eta = \Theta^\ast(\varepsilon^2 \max(\beta, 1)^{-2} \min(C_{LS}, n)^{-2})$ and running the algorithm (9) initiated at 0 for

$$k = \Theta^\ast\left(\frac{\min(C_{LS}, n)^3 \max(\beta, 1)^2}{\varepsilon^2}\right)$$

steps produces a point $x_k$ satisfying $\frac{1}{n} W_2^2(x_k, \mu) \leq \varepsilon$.

Note in particular that in the case where $C_{LS}$ and $\beta$ are of constant order the complexity does not depend on the dimension.

**Related works.** We end this introduction with a discussion on a short selection of related works. Let us first mention that as far as we know, the Langevin algorithm with support constraint was only investigated in our previous paper with Bubeck and Eldan [5]. In this paper the potential was assumed to be gradient Lipschitz. In all the works that we could find on the Langevin Monte Carlo algorithm the potential is always assumed to be smooth, most of the time gradient Lipschitz. This hypothesis is somewhat relaxed in the recent article [7], but the authors analyze the Langevin algorithm for a smoothed out approximation of $\mu$, and in any case they still require the gradient of the potential to be Hölder continuous. The present work appears to be the first were $\nabla \varphi$ is allowed to be discontinuous.

Let us give the state of the art convergence bounds for the Langevin algorithm in the smooth, unconstrained case. The first quantitative result appears to be Dalalyan’s article [9]. The result is in total variation distance rather than Wasserstein but as in the present work the strategy consists in writing

$$TV(x_k, \mu) \leq TV(x_k, X_{k\eta}) + TV(X_{k\eta}, \mu)$$

(10)
and estimating both terms separately. His assumption is that the potential $\varphi$ satisfies

$$\alpha \text{Id} \leq \nabla^2 \varphi \leq \beta \text{Id}$$

pointwise on the whole $\mathbb{R}^n$, where $\alpha$ and $\beta$ are positive constants. Actually a closer look at his argument shows that he does not really use log-concavity. Indeed, his main contribution is a bound for the relative entropy of the Langevin algorithm at time $k$ with respect to the corresponding point in the Langevin diffusion. That part of the argument is a nice application of Girsanov’s formula and does not use log-concavity at all, only the fact that $\nabla \varphi$ is Lipschitz is needed. Dalalyan only uses strict log-concavity to estimate how fast the diffusion $(X_t)$ converges to $\mu$. But that only requires Poincaré for an exponentially fast decay in chi-square divergence or log-Sobolev for a decay in relative entropy. Dalalyan’s theorem can thus be rewritten as follows: if $d\mu = e^{-\varphi} \, dx$ is supported on the whole $\mathbb{R}^n$, if $\nabla \varphi$ is Lipschitz with constant $\beta$ and if $\mu$ satisfies the log-Sobolev inequality with constant $C_{LS}$ then after $k$ steps of the Langevin algorithm with times step parameter $\eta$ we have

$$TV(x_k, \mu) \leq D(x_0 \mid \mu)^{1/2} e^{-kn/2C_{LS}} + \beta n^{1/2} (1 + E[\sigma_0])^{1/2} k^{1/2} \eta,$$

where again $\sigma_0 = \frac{1}{n} (\varphi(x_0) - \min_{\mathbb{R}^n} \{\varphi\})$. The result depends on a warm start hypothesis, the algorithm must be initiated from a random point $x_0$ whose relative entropy to the target measure is finite. On the other hand, it is not hard to see that one can find a Gaussian measure whose relative entropy to $\mu$ is $O^*(n)$. As a result, it follows from the previous bound that if $\eta$ is chosen appropriately then one has $TV(x_k, \mu) \leq \varepsilon$ after

$$k = \Theta^* \left( \frac{C_{LS}^2 \beta^2 n}{\varepsilon^2} \right)$$

steps of the algorithm.

Durmus and Moulines [12] have the same set of hypothesis as Dalalyan but they prove a result in Wasserstein distance rather than total variation. As opposed to Dalayan they really use the hypothesis $\nabla^2 \varphi \geq \alpha \text{Id}$ for some positive $\alpha$. Also their approach is a bit different from that of Dalalyan: instead of bounding $W_2(x_k, X_{kn})$ and $W_2(X_{kn}, \mu)$ separately they directly obtain a recursive inequality for $W_2(x_k, \mu)$. Their approach essentially yields the following result: Suppose that $\alpha \text{Id} \leq \nabla^2 \varphi \leq \beta \text{Id}$ pointwise on the whole $\mathbb{R}^n$ for some positive constants $\alpha, \beta$. Assume also that the time step parameter $\eta$ satisfies $\eta \leq \frac{1}{2\beta}$. Then

$$W_2(x_k, \mu) \leq \left( 1 - \frac{\alpha \eta}{2} \right)^k W_2(x_0, \mu) + \frac{2\beta}{\alpha} n^{1/2} \eta^{1/2}. \quad (11)$$

Actually, the result of Durmus and Moulines is a bit more involved, for a (short) proof of that very statement see [10, Theorem 1.1].

This result implies that $\frac{1}{n} W_2^2(x_k, \mu) \leq \varepsilon$ after a number of steps $k = \Theta^* \left( \frac{\beta^2}{\alpha^2 \varepsilon} \right)$, with time step parameter of order $\varepsilon \alpha^2 / \beta^2$. This should be compared to the complexity given by Theorem 5 in this case. Indeed, observe that the hypothesis $\alpha \text{Id} \leq \nabla^2 \varphi \leq \beta \text{Id}$ implies that the log-Sobolev constant is $1/\alpha$ at most and that $\nabla \varphi$ grows linearly. Therefore Theorem 5 applies, and it gives the following complexity: $k = \Theta^* \left( \frac{\beta^2}{\alpha^2 \varepsilon^2} \right)$. The dependence in $\varepsilon$ is thus worse ($\varepsilon^2$ rather $\varepsilon$) but the dependence in the other parameters is the same, which is quite remarkable given the fact that Theorem 5 holds under considerably weaker assumptions.

Lastly, let us also mention Vempala and Wibisono’s work [25] whose approach is similar in spirit to that of Durmus and Moulines but gives a result closer to Dalalyan’s. They prove that if
∇φ is Lipschitz with constant β, if μ satisfies log-Sobolev with constant \( C_{LS} \), and if the time step parameter satisfies \( \eta \leq 1/(4C_{LS}\beta^2) \) then after \( k \) steps of the algorithm one has

\[
D(x_k \mid \mu) \leq e^{-k\eta/C_{LS}} D(x_0 \mid \mu) + 8n\beta^2 C_{LS}\eta. \tag{12}
\]

As in the result of Dalalyan the measure μ is not assumed to be log-concave, only log-Sobolev is required. Let us note that this result recovers Dalalyan’s by Pinsker’s inequality. Let us also remark that combining it with the transport inequality \( W_2^2(x_k, \mu) \leq 2C_{LS} D(x_k \mid \mu) \), which is a consequence of log-Sobolev, one gets

\[
W_2^2(x_k, \mu) \leq 2C_{LS} D(x_0 \mid \mu) e^{-k\eta/C_{LS}} + 16 n \beta^2 C_{LS}^2 \eta.
\]

This pretty much recovers (11) under a weaker hypothesis: Log-Sobolev rather than uniform convexity of the potential, with two caveats: The hypothesis on the time step parameter is a bit more restrictive, and this is from a warm start in the relative entropy sense.

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2 The Langevin diffusion with reflected boundary condition

In section we define properly the Langevin diffusion with reflection at the boundary of \( K \):

\[
dX_t = dB_t - \frac{1}{2} \nabla \varphi(X_t) - d\Phi_t.
\]

Recall that \( d\mu = e^{-\varphi}dx \) is assumed to be log-concave, which means that \( \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is convex. Actually, it will be slightly more convenient for our purposes to assume that the domain of \( \varphi \) is the whole \( \mathbb{R}^n \) and that the measure \( \mu \) is given by \( \mu(dx) = 1_K(x)e^{-\varphi(x)}dx \) where \( K \) is a convex subset of \( \mathbb{R}^n \) with non empty interior. Since we do not assume the potential \( \varphi \) to be everywhere differentiable, the expression \( \nabla \varphi(X_t) \) needs to be clarified. Let us agree on the convention that in the sequel \( \nabla \varphi(x) \) stands for the element of the subdifferential of \( \varphi \) at point \( x \) whose Euclidean norm is minimal. Since \( \varphi \) is assumed to be a convex function whose domain is the whole \( \mathbb{R}^n \), for every \( x \in \mathbb{R}^n \) the subdifferential of \( \varphi \) at \( x \) is a non empty closed convex set, so that the Euclidean norm does uniquely attain its minimum on this set.

According to Tanaka [24, Theorem 3.1], given a continuous semi-martingale \( (W_t) \) taking values in \( \mathbb{R}^n \) and satisfying \( W_0 \in K \), there exists a unique couple \( (X_t, \Phi_t) \) of continuous semi-martingales such that, almost surely

1. \( X_t \in K \) for all \( t \in \mathbb{R}_+ \),
2. \( X_t = W_t - \Phi_t \) for all \( t \in \mathbb{R}_+ \),
3. \( (\Phi_t) \) is of the form \( \Phi_t = \int_0^t \nu_s \, d\ell_s \) where \( \ell \) is a measure on \( \mathbb{R}_+ \) which is finite on bounded intervals and supported on the set \( \{ t \in \mathbb{R}_+ : X_t \in \partial K \} \), and for any such \( t \) the vector \( \nu_t \) is an outer unit normal to the boundary of \( K \) at \( X_t \).
In words the process \((\Phi_t)\) is a finite variation and continuous process which repels \((X_t)\) inwards when it reaches the boundary of \(K\). In the sequel we shall say that the process \((X_t)\) is the reflection of \((W_t)\) at the boundary of \(K\) and that the process \((\Phi_t)\) is associated to \((X_t)\). The process \((\ell_t)\) is called the local time of \((X_t)\) at the boundary of \(K\).

Now given a standard Brownian motion \((B_t)\) and a starting point \(x \in K\), we want to argue that there exists a unique process \((X_t)\) such that \((X_t)\) is the reflection at the boundary of \(K\) of the semi-martingale

\[
x + B_t - \frac{1}{2} \int_0^t \nabla \varphi(X_s) \, ds.
\]

In other words we want

\[
dx_t = dB_t - \frac{1}{2} \nabla \varphi(X_t) \, dt - d\Phi_t,
\]

where \((\Phi_t)\) is associated to \((X_t)\). This is a stochastic differential equation with reflected boundary condition. If \(\nabla \varphi\) is Lipschitz continuous then again Tanaka [24, Theorem 4.1] shows that this equation admits a unique strong solution. Let us now explain why Tanaka’s result remains valid in the present context, even though \(\nabla \varphi\) is not assumed to be continuous.

First of all, notice that since \(\nabla \varphi\) is the gradient of a convex function, it is a monotone map, in the sense that

\[
\langle x - y, \nabla \varphi(x) - \nabla \varphi(y) \rangle \geq 0, \quad \forall x, y \in \mathbb{R}^n.
\]

This property immediately implies pathwise uniqueness for the equation. Indeed suppose that \(X\) and \(\tilde{X}\) are two solutions of the equation and that \(\Phi\) and \(\tilde{\Phi}\) are the associated processes. Then

\[
d|X_t - \tilde{X}_t|^2 = -\frac{1}{2} \langle X_t - \tilde{X}_t, \nabla \varphi(X_t) - \nabla \varphi(\tilde{X}_t) \rangle \, dt - \langle X_t - \tilde{X}_t, d\Phi_t \rangle - \langle \tilde{X}_t - X_t, d\tilde{\Phi}_t \rangle.
\]

The first term of the right-hand side is non positive by monotony of \(\nabla \varphi\). The second term is also non positive. Indeed the fact that \(\Phi\) is associated to \(X\) and \(\tilde{X}\) takes values in \(K\) imply that \(\langle X_t - \tilde{X}_t, d\Phi_t \rangle \geq 0\) for all \(t\). Similarly \(\langle \tilde{X}_t - X_t, d\tilde{\Phi}_t \rangle \geq 0\). Therefore the quantity \(|X_t - \tilde{X}_t|\) is almost surely non increasing, which obviously implies that the equation has the pathwise uniqueness property. To get existence, one option is to approximate \(\varphi\) by a smooth convex function and pass to the limit, as in [6]. That paper is a little involved and is written French so let us give an alternative argument for completeness. This argument only works when \(\nabla \varphi\) is bounded, but this is the only case we shall consider here. When \(\nabla \varphi\) is bounded, it is well known that an application of Girsanov yields the existence of a solution to the equation. Indeed, let \(X\) be the reflection at the boundary of \(K\) of the process \(x + B\). We have

\[
dx_t = dB_t - d\Phi_t,
\]

where \(\Phi\) is associated to \(X\). Since \(\nabla \varphi\) is bounded the process \((D_t)\) given by

\[
D_t = \exp \left( -\frac{1}{2} \int_0^t \langle \nabla \varphi(X_s), dB_s \rangle - \frac{1}{8} \int_0^t |\nabla \varphi(X_s)|^2 \, ds \right)
\]

is a positive martingale with expectation 1. If we fix a time horizon \(T > 0\) and define a new probability measure by \(d\mathbb{Q} = D_T \, d\mathbb{P}\), then by Girsanov, the process \((\tilde{B}_t)_{t \in [0,T]}\) given by

\[
\tilde{B}_t = B_t + \frac{1}{2} \int_0^t \nabla \varphi(X_s) \, ds,
\]
is a standard Brownian motion under the new measure $Q$. Since
\[ dX_t = d\tilde{B}_t - \frac{1}{2} \nabla \varphi(X_t) dt + d\Phi_t, \]
where $\Phi$ is associated to $X$ this shows that under $Q$ the process $X$ solves the equation driven by $\tilde{B}$. This proves weak existence of a solution, in the sense that we had to change the probability space and the Brownian motion. However it is well known that weak existence and pathwise uniqueness altogether imply strong existence, see for instance [15, Chapter IV, Theorem 1.1]. Strictly speaking this only shows strong existence on a finite time interval $[0, T]$. But we can eventually let $T$ tend $+\infty$ and use pathwise uniqueness again to get strong existence of a solution defined for all time. Details are left to the reader.

The solution $(X_t)$ of the equation is a Markov process, whose semigroup is denoted $(P_t)$ in the sequel: For every test function $f: \mathbb{R}^n \to \mathbb{R}$ and every $x \in K$
\[ P_t f(x) = \mathbb{E}_x[f(X_t)] \]
where the subscript $x$ next to the expectation denotes the starting point of $X_t$. By Itô’s formula, if $f$ is $C^2$-smooth in a neighborhood of $K$ then
\[ df(X_t) = \langle \nabla f(X_t), dB_t \rangle - \frac{1}{2} \langle \nabla f(X_t), \nabla \varphi(X_t) \rangle dt + \frac{1}{2} \Delta f(X_t) dt - \langle \nabla f(X_t), d\Phi_t \rangle. \]

Here we are using Itô’s formula for a continuous semi-martingale having a finite variation part, see for instance [23, Chapter IV, Corollary 32.10]. If $f$ satisfies the Neumann boundary condition: $\langle \nabla f(x), \nu \rangle$ for every $x \in \partial K$ and every $\nu$ that is normal to the boundary of $K$ at $x$ then the last term of the right-hand side vanishes. Taking expectation we then see that the generator of the semigroup $(P_t)$ is
\[ \frac{1}{2} Lf := \frac{1}{2} (\Delta f - \langle \nabla f, \nabla \varphi \rangle) \]
with Neumann boundary condition. Also, an integration by part then shows that
\[ \int_K (Lf) g \, d\mu = - \int_K \langle \nabla f, \nabla g \rangle \, d\mu, \]
for every $f, g$ in the domain of $L$. In particular the operator $L$ is symmetric in $L^2(\mu)$, which implies that $\mu$ is a reversible measure for the semigroup $(P_t)$.

3 Discretization of the Langevin diffusion

In this section we prove Theorem 1. This is the main contribution of the article. We begin with a bound on the local time $(\ell_t)$ of the diffusion $(X_t)$ at the boundary of $K$. We need to show that $\ell_t = O(t)$. That lemma is essentially taken from our previous work with Bubeck and Eldan [5], except that we have simplified the proof and improved the result a bit.

**Lemma 6.** Assume that the Langevin diffusion $(X_t)$ is initiated at point $x_0$ in the interior of $K$ and recall the definition (7) of $r_0$ and $\sigma_0$. Then for every $t > 0$ we have
\[ \mathbb{E}[\ell_t^{1/2}] \leq \frac{n(1 + \sigma_0)t}{r_0}. \]
Proof. By Itô’s formula we have
\[
d|X_t - x_0|^2 = 2\langle X_t - x_0, dB_t \rangle - \langle X_t - x_0, \nabla \varphi(X_t) \rangle dt - 2\langle X_t - x_0, d\Phi_t \rangle + n dt.
\]
Recall that \( d\Phi_t = \nu_t d\ell_t \) where \( \nu_t \) is an outer unit normal at \( X_t \). By definition of \( (\Phi_t) \), \( r_0 \) and \( \ell_t \) we have
\[
\langle X_t - x_0, d\Phi_t \rangle \geq \sup_{x \in K} \langle x - x_0, d\Phi_t \rangle \geq r_0 d\ell_t.
\]
Also by convexity of \( \varphi \)
\[
-\langle X_t - x_0, \nabla \varphi(X_t) \rangle \leq \varphi(x_0) - \varphi(X_t) \leq n\sigma_0.
\]
We thus obtain
\[
|X_t - x_0|^2 + 2r_0\ell_t \leq n(1 + \sigma_0) t + \int_0^t \langle X_s - x_0, dB_s \rangle.
\]
Taking expectation already gives a bound on the first moment of \( \ell_t \). To get a bound on the second moment observe that (13) implies that
\[
4r_0^2 \mathbb{E}[\ell_t^2] \leq n^2(1 + \sigma_0)^2 t^2 + \mathbb{E} \left[ \left( \int_0^t \langle X_s - x_0, dB_s \rangle \right)^2 \right].
\]
By Itô’s isometry and using (13) again, this time to bound \( \mathbb{E}[|X_t - x_0|^2] \), we get
\[
\mathbb{E} \left[ \left( \int_0^t \langle X_s - x_0, dB_s \rangle \right)^2 \right] = \mathbb{E} \left[ \int_0^t |X_s - x_0|^2 ds \right] \leq n(1 + \sigma_0) t^2 / 2.
\]
Plugging this into the previous display yields the desired inequality. \qed

We also need the following elementary bound on the maximum of Gaussian vectors. We provide a proof for completeness.

**Lemma 7.** Let \( G_1, \ldots, G_k \) be standard Gaussian vectors on \( \mathbb{R}^n \). Then
\[
\mathbb{E} \left[ \max_{i \leq k} \{|G_i|^2\} \right] \leq e(n + 2 \log k).
\]

**Proof.** Set \( \chi_i = |G_i|^2 \) for every \( i \). The \( p \)-th moment of \( \chi_i \) satisfies
\[
\mathbb{E}[\chi_i^p] = \frac{2^p \Gamma \left( \frac{n}{2} + p \right)}{\Gamma \left( \frac{n}{2} \right)} \leq (n + 2(p - 1))^p,
\]
at least when \( p \) is an integer. Therefore
\[
\mathbb{E} \left[ \max_{i \leq k} \chi_i \right] \leq \left( \sum_{i \leq k} \mathbb{E}[\chi_i^p] \right)^{1/p} \leq k^{1/p} (n + 2(p - 1))
\]
Choosing \( p \) to be the smallest integer larger than \( \log k \) yields the result. \qed

We are now in a position to prove the main result.
Proof of Theorem 1. We first couple the diffusion \((X_t)\) and its discretization \((x_k)\) in the most natural way one could think of, by choosing the sequence \((\xi_k)\) as follows:

\[
\xi_k = B_{k\eta} - B_{(k-1)\eta}, \quad k \geq 1.
\]  

(14)

Observe that for any \(x \in K\) and \(y \in \mathbb{R}^n\) we have \(|x - \mathcal{P}_K(y)| \leq |x - y|\). Therefore

\[
|X_{(i+1)\eta} - x_{i+1}|^2 = \left| X_{(i+1)\eta} - \mathcal{P} \left( x_i + \xi_{i+1} - \frac{\eta}{2} \nabla \varphi (x_i) \right) \right|^2 \\
\leq \left| X_{(i+1)\eta} - x_i - \xi_{i+1} + \frac{\eta}{2} \nabla \varphi (x_i) \right|^2.
\]

Let \((\tilde{X}_t)\) be the process defined by

\[
\tilde{X}_t = x_i + B_t - B_{i\eta} - \frac{t - i\eta}{2} \nabla \varphi (x_i)
\]

for all \(t\) between \(i\eta\) and \((i+1)\eta\). Then \(\tilde{X}_{i\eta} = x_i\) and the previous display can be rewritten as

\[
|X_{(i+1)\eta} - x_{i+1}|^2 \leq |X_{(i+1)\eta} - \tilde{X}_{(i+1)\eta}|^2.
\]

The process \((X_t - \tilde{X}_t)\) is continuous with finite variation on \([i\eta, (i+1)\eta]\) (the Brownian part cancels out). Therefore, on that interval we have

\[
d|X_t - \tilde{X}_t|^2 = 2 \langle X_t - \tilde{X}_t, dX_t - d\tilde{X}_t \rangle \\
= -\langle X_t - \tilde{X}_t, \nabla \varphi (X_t) - \nabla \varphi (x_i) \rangle dt - 2 \langle X_t - \tilde{X}_t, d\Phi_t \rangle.
\]

Again by monotony of \(\nabla \varphi\) we have \(\langle X_t - x_i, \nabla \varphi (X_t) - \nabla \varphi (x_i) \rangle \geq 0\). Also since \(x_i \in K\) and \((\Phi_t)\) is associated to \((X_t)\) we have \(\langle X_t - x_i, d\Phi_t \rangle \geq 0\). Plugging this back in the previous display yields

\[
d|X_t - \tilde{X}_t|^2 \leq \langle \tilde{X}_t - x_i, (\nabla \varphi (X_t) - \nabla \varphi (x_i)) dt + 2d\Phi_t \rangle.
\]

Now we replace \(\tilde{X}_t\) by its value, and we integrate between \(i\eta\) and \((i+1)\eta\). We get

\[
|X_{(i+1)\eta} - x_{i+1}|^2 \leq |X_{i\eta} - x_i|^2 \\
+ \int_{i\eta}^{(i+1)\eta} \langle B_t - B_{i\eta} - \frac{t - i\eta}{2} \nabla \varphi (x_i), (\nabla \varphi (X_t) - \nabla \varphi (x_i)) dt + 2d\Phi_t \rangle.
\]

We now take expectation. Note that the martingale property of the Brownian motion implies that \(\mathbb{E}[\langle B_t - B_{i\eta}, \nabla \varphi (x_i) \rangle] = 0\) and that

\[
\mathbb{E}[\langle B_t - B_{i\eta}, d\Phi_t \rangle] = \mathbb{E}[\langle B_{(i+1)\eta} - B_{i\eta}, d\Phi_t \rangle].
\]

We also use the hypothesis \(|\nabla \varphi| \leq L\) and the inequality \(\mathbb{E}[|B_t - B_{i\eta}|] \leq n^{1/2} (t - i\eta)^{1/2}\). We obtain

\[
\mathbb{E} \left[ |X_{(i+1)\eta} - x_{i+1}|^2 \right] \leq \mathbb{E} \left[ |X_{i\eta} - x_i|^2 \right] + \frac{2}{3} L\eta^{3/2} n^{1/2} \\
+ 2 \mathbb{E} \left[ \langle \xi_{i+1}, \Phi_{(i+1)\eta} - \Phi_{i\eta} \rangle \right] + \frac{1}{2} L^2 \eta^2 + L\eta \mathbb{E}[\ell_{(i+1)\eta} - \ell_{i\eta}],
\]

(15)
where $\xi_{i+1} = B_{(i+1)\eta} - B_{i\eta}$. By Cauchy-Schwarz and Lemma 7
\[
\mathbb{E} \left[ \sum_{i=0}^{k-1} \langle \xi_{i+1}, \Phi_{(i+1)\eta} - \Phi_{i\eta} \rangle \right] \leq \mathbb{E} \left[ \max_{i \leq k} \{|\xi_i|\} \ell_{k\eta} \right] \\
\leq e^{1/2}(n + 2 \log k)^{1/2} \eta^{1/2} \mathbb{E}[(\ell_{k\eta})^{1/2}].
\]
Summing (15) over $i$ thus yields
\[
\mathbb{E} \left[ |X_{k\eta} - x_k|^2 \right] \leq 2e^{1/2}(n + 2 \log k)^{1/2} \eta^{1/2} \mathbb{E}[(\ell_{k\eta})^{1/2}]
\]
\[
+ L\eta \mathbb{E}[(\ell_{k\eta})] + \frac{2}{3}Ln^{1/2}k\eta^{3/2} + \frac{1}{2}L^2k\eta^2.
\]
Now recall from Lemma 6 that
\[
\mathbb{E}[(\ell_{k\eta})^{1/2}] \leq \frac{n(1 + \sigma_0)k\eta}{r_0}.
\]
Lastly, we use the assumption $\eta < n/L^2$ to simplify the inequality a bit. We finally obtain
\[
\mathbb{E} \left[ |X_{k\eta} - x_k|^2 \right] \leq (2e^{1/2} + 1)(n + 2 \log k)^{1/2} \frac{n(1 + \sigma_0)}{r_0}k\eta^{3/2} + \frac{7}{6}Ln^{1/2}k\eta^{3/2},
\]
which is the result.

4 Convergence of the algorithm under log-Sobolev

In this section we prove Theorem 2. Observe first that the Wasserstein distance is indeed a distance, so it satisfies the triangle inequality and we have
\[
\frac{1}{n}W_2^2(x_k, \mu) \leq \frac{2}{n}W_2^2(x_k, X_{k\eta}) + \frac{2}{n}W_2^2(X_{k\eta}, \mu).
\]
The first term of the right-side is handled by Theorem 1, so we only need to bound the second term. This is the purpose of the following lemma. In this lemma $(P_t)$ stands for the semi-group of the Langevin diffusion (3). In other words $\nu P_t$ denotes the law of $X_t$ when $X_0$ has law $\nu$.

Lemma 8. Assume that $\mu$ is log-concave with globally Lipschitz potential on its support, with Lipschitz constant $L$. Assume also that $\mu$ satisfies log-Sobolev with constant $C_{LS}$. Then for every $x_0$ in the interior of the support of $\mu$ and every $t > 0$ we have
\[
\frac{1}{n}W_2^2(\delta_{x_0} P_t, \mu) \leq 4C_{LS} \left( 1 + \log \left( \frac{\max(C_{LS}, 1) n}{\min(r_0, 1)} \right) \right) + \sigma_0 + \frac{L}{n} e^{-t/2C_{LS}},
\]
where again the parameters $\sigma_0$ and $r_0$ are defined by (7).

Proof. If $\mu$ satisfies the logarithmic Sobolev inequality then it satisfies the transport inequality:
\[
W_2^2(\nu; \mu) \leq 2C_{CLS} D(\nu \mid \mu)
\]
for every probability measure $\nu$. This is due to Otto and Villani [22], see also [2]. The log-Sobolev inequality also implies that the relative entropy decays exponentially fast along the semigroup $(P_t)$:
\[
D(\nu P_t \mid \mu) \leq e^{-t/C_{LS}} D(\nu \mid \mu).
\]
This is really folklore, one just need to observe that the derivative of the entropy is the Fisher information, and combine log-Sobolev with a Gronwall type argument. Combining the two inequalities yields

\[ W^2_2(\nu P_t, \mu) \leq 2C_{LS} e^{-t/C_{LS}} D(\nu \mid \mu). \]

This cannot be applied directly to a Dirac point mass. However, observe that the convexity of \( \varphi \) implies that the Wasserstein distance is non increasing along the semigroup: For any two probability measures \( \nu_0, \nu_1 \) and every time \( t \) we have

\[ W^2_2(\nu_0 P_t, \nu_1 P_t) \leq W^2_2(\nu_0, \nu_1). \]

This is a well-known fact, which is easily seen using parallel coupling. See the proof of the pathwise uniqueness property in section 2. Combining this with the triangle inequality for \( W^2_2 \) we thus get

\[ W^2_2(\delta_{x_0} P_t, \mu) \leq W^2_2(\delta_{x_0} P_t, \nu P_t) + 2W^2_2(\nu P_t, \mu) \]

\[ \leq 2W^2_2(\delta_{x_0}, \nu) + 4C_{LS} e^{-t/C_{LS}} D(\nu \mid \mu). \] (16)

This is valid for every \( \nu \) and it is natural to take \( \nu \) to be the measure \( \mu \) conditioned to the ball \( B(x_0, \delta) \) for some small \( \delta > 0 \). Then \( W^2_2(\delta_{x_0}, \nu) \leq \delta^2 \) and

\[ D(\nu \mid \mu) = \log \left( \frac{1}{\mu(B(x_0, \delta))} \right). \]

If \( \delta \leq r_0 \) then \( B(x_0, \delta) \) is included in the support of \( \mu \) and we have

\[ \log \left( \frac{1}{\mu(B(x_0, \delta))} \right) \leq \max_{B(x_0, \delta)} \{ \varphi \} + n \log \left( \frac{1}{\delta} \right) + \log \left( \frac{1}{v_n} \right), \]

where \( v_n \) is the Lebesgue measure of the unit ball in dimension \( n \). Recall that \( \log \left( \frac{1}{v_n} \right) \leq n \log n. \)

Also

\[ \max_{B(x_0, \delta)} \{ \varphi \} \leq \varphi(x_0) + L\delta \leq \min_K \{ \varphi \} + n\sigma_0 + L\delta. \]

Moreover

\[ \min_K \{ \varphi \} \leq \int_{\mathbb{R}^n} \varphi d\mu = S(\mu), \]

where \( S \) denotes the Shannon entropy. It is well-known that among measures of fixed covariance the Gaussian measure maximizes the Shannon entropy (this is just Jensen actually). Therefore

\[ S(\mu) = \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det(\text{cov}(\mu)) \]

\[ \leq \frac{n}{2} \log(2\pi e C_{LS}). \]

The last inequality is just a consequence of the fact that the log-Sobolev inequality implies Poincaré, which in turn implies a bound on the covariance matrix. Plugging everything back in (16) we get

\[ \frac{1}{n} W^2_2(\delta_{x_0} P_t, \mu) \leq \frac{2\delta^2}{n} + 4C_{LS} \left( \frac{3}{2} + \log \left( \frac{n}{\delta} \right) + \sigma_0 + \frac{L}{n} \right) e^{-t/C_{LS}} \]

for every \( \delta \leq \min(r_0, 1) \). Choosing \( \delta = \min \left( (2nC_{LS})^{1/2} e^{-t/2C_{LS}}, r_0, 1 \right) \) and using the inequality \( xe^{-x} \leq e^{-x/2} \) yields the result. \( \square \)
5 A convergence result using Poincaré only

In this section we prove Theorem 3. Again the idea is to write

$$\frac{1}{n} W_2^2(x_k, \mu) \leq \frac{2}{n} W_2^2(x_k, X_{k\eta}) + \frac{2}{n} W_2^2(X_{k\eta}, \mu),$$

and to bound the first term using Theorem 1. Actually, note that here we allow $x_0 = X_0$ to be random, so we rather condition on $x_0$, apply Theorem 1 and then take expectation again.

Therefore it is enough to bound the second term. This is where the Poincaré inequality enters the picture. Note that this part of the argument does not rely on the log-concavity of $\mu$. We shall use the following transport/chi-square divergence inequality: If $\mu$ satisfies Poincaré with constant $C_P$ then for every probability measure $\nu$ on $\mathbb{R}^n$ we have

$$W_2^2(\nu, \mu) \leq 2 C_P \chi^2(\nu \mid \mu).$$

It seems that this was first proved by Ding [11], with a worst constant. The result with constant 2 is due to Liu [20]. His argument is combined together the Langevin diffusion and the Hamilton-Jacobi semigroup.

On the other hand it is well-known that under Poincaré the chi-square divergence decays exponentially fast along the Langevin diffusion. Letting $(P_t)$ be the semigroup of the Langevin diffusion associated to $\mu$ we have

$$\chi^2(\nu P_t \mid \mu) \leq e^{-t/C_P} \chi^2(\nu \mid \mu).$$

We thus get the following:

**Lemma 9.** Suppose that $\mu$ satisfies Poincaré with constant $C_P$. Then for every probability measure $\nu$ on $\mathbb{R}^n$ and every $t > 0$ we have

$$W_2^2(\nu P_t, \mu) \leq 2 C_P \chi^2(\nu \mid \mu) e^{-t/C_P}.$$

This finishes the proof of Theorem 3.

We end this section with a simple estimate of the chi-square divergence of an appropriate Gaussian measure to $\mu$ in the unconstrained case.

**Proof of Lemma 4.** Recall that $\mu$ is assumed to be supported on the whole $\mathbb{R}^n$ with convex and globally Lipschitz potential $\varphi$. Let $\gamma$ be the Gaussian measure centered at a point $x_0$ and with covariance $\alpha \text{Id}$ for some $\alpha > 0$. Then

$$\chi^2(\gamma \mid \mu) \leq (2\pi \alpha)^{-n} \int_{\mathbb{R}^n} e^{-\frac{1}{\alpha} |x-x_0|^2 + \varphi(x)} dx$$

$$\leq (2\pi \alpha)^{-n} \int_{\mathbb{R}^n} e^{-\frac{1}{\alpha} |x-x_0|^2 + \varphi(x_0) + L|x-x_0|} dx$$

$$\leq (2\pi \alpha)^{-n} \int_{\mathbb{R}^n} e^{-\frac{1}{2\alpha} |x-x_0|^2 + \varphi(x_0) + \frac{1}{2}L^2x_0} dx = (2\pi \alpha)^{-n/2} e^{\varphi(x_0) + \frac{1}{2}L^2x_0}.$$

Also, reasoning along the same lines as in the previous section we get

$$\varphi(x_0) \leq \min_{\mathbb{R}^n} \{\varphi\} + n \sigma_0 \leq \frac{n}{2} \log(2\pi e) + \frac{n}{2} \log C_P + n \sigma_0.$$
Putting everything together and choosing $\alpha = n/L^2$ yields
\[
\log \chi^2(\gamma | \mu) \leq \frac{n}{2} \left(-\log \alpha + 1 + \log C_p + 2\sigma_0\right) + \frac{1}{2} L^2 \alpha \\
= n \left(1 + \sigma_0 + \frac{1}{2} \log \left(\frac{L^2 C_p}{n}\right)\right),
\]
which is the result.

6 An extension to the non-globally Lipschitz case

We begin this section with a simple lemma about the Wasserstein distance of $\mu$ to $\mu$ restricted to a large ball.

**Lemma 10.** Let $\mu$ be a log-concave measure on $\mathbb{R}^n$, and let $\mu_R$ be the measure $\mu$ conditioned on the ball centered at $0$ of radius $R$. There exists a universal constant $C$ such that
\[
W_2^2(\mu, \mu_R) \leq C M \exp \left(-\frac{R}{C \sqrt{M}}\right), \quad \forall R \geq C \sqrt{M}
\]
where $M = \int_{\mathbb{R}^n} |x|^2 \, d\mu$.

**Proof.** Let $X$ have law $\mu$. Note that by Borell’s lemma [4, Lemma 3.1] we have $P(|X| \geq t) \leq e^{-t/C_0 \sqrt{M}}$ for all $t \geq C_0 \sqrt{M}$ for some universal constant $C_0$. This also implies that $E[|X|^4] \leq C_1 M^2$ for some $C_1$. Now assume that $R \geq C_0 \sqrt{M}$, let $\tilde{X}$ have law $\mu_R$ and be independent of $X$ and let
\[
Y = \begin{cases} 
X & \text{if } |X| \leq R \\
\tilde{X} & \text{otherwise.}
\end{cases}
\]
Then $Y$ also has law $\mu_R$, so that
\[
W_2^2(\mu, \mu_R) \leq E[|X - Y|^2] = E[|X - \tilde{X}|^2; |X| > R] \\
\leq 4E[|X|^4]^{1/2} P(|X| > R)^{1/2} \leq C_1 M e^{-R/(C_0 M^{1/2})},
\]
which is the result. \qed

**Proof of Theorem 5.** Assuming that $\int_{\mathbb{R}^n} |x|^2 \, d\mu = O(n)$, the previous lemma shows that $\frac{1}{n} W_2^2(\mu, \mu_R)$ will be negligible as soon as $R$ is a sufficiently large multiple of $\sqrt{n} \log n$. Now we apply Theorem 2 to $\mu_R$ and initiating the Langevin algorithm at $0$. Then the parameter $r_0$ is of order $\sqrt{n} \log n$. Moreover the hypothesis
\[
|\nabla \varphi(x)| \leq \beta(|x| + 1)
\]
show that the potential of $\mu_R$ is Lipschitz with constant $O^*(\beta \sqrt{n})$. Therefore the constant $A$ defined by (6) satisfies $A = O^*(\max(\beta, 1))$ in this case. On the other hand since $\mu_R$ is log-concave and supported on a ball of radius $O^*(\sqrt{n})$ its log-Sobolev constant is $O^*(n)$ at most. Also, if $\mu$ satisfies log-Sobolev, then the log-Sobolev constant of $\mu_R$ cannot be larger than a constant factor times that of $\mu$. This follows easily from the fact that within log-concave measures log-Sobolev and Gaussian concentration are equivalent, see [21, Theorem 1.2]. To sum up, the log-Sobolev
constant of $\mu_R$ is $O^*(\min(n, C_{LS}))$ where $C_{LS}$ is the log-Sobolev constant of $\mu$ (which is possibly infinite). Applying Theorem 2 we see that after

$$k = \Theta^* \left( \frac{\min(C_{LS}, n)^3 \max(\beta, 1)^2}{\varepsilon^2} \right)$$

of the Langevin algorithm for $\mu_R$ initiated at 0, and with appropriate time step parameter we have $\frac{1}{n} W_2^2(x_k, \mu_R) \leq \varepsilon$. Since $\frac{1}{n} W_2^2(\mu_R, \mu)$ is negligible this implies $\frac{1}{n} W_2^2(x_k, \mu) \leq 2\varepsilon$. \square

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