Quantization formula for singular reductions

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Abstract

In this note, we prove a quantization formula for singular reductions. The main result is obtained as a simple application of an extended quantization formula proved in [TZ2].

§0. Introduction

Let \((M, \omega)\) be a closed symplectic manifold. We make the assumption that there is a Hermitian line bundle \(L\) over \(M\) admitting a Hermitian connection \(\nabla^L\) with the property that \(\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2 = \omega\). When such a line bundle exists, it is usually called a prequantum line bundle over \(M\). Let \(J\) be an almost complex structure on \(TM\) so that \(g^{TM}(u, v) = \omega(u, Jv)\) defines a Riemannian metric on \(TM\). Then one can construct canonically a Spin\(^c\)-Dirac operator

\[
D^L : \Omega^{0,*}(M, L) \to \Omega^{0,*}(M, L),
\]

which gives the finite dimensional virtual vector space

\[
Q(M, L) = (\ker D^L) \cap \Omega^{0,\text{even}}(M, L) - (\ker D^L) \cap \Omega^{0,\text{odd}}(M, L).
\]

Next, suppose that \((M, \omega)\) admits a Hamiltonian action of a compact connected Lie group \(G\) with Lie algebra \(\mathfrak{g}\). Let \(\mu : M \to \mathfrak{g}^*\) be the corresponding

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moment map. Then a formula due to Kostant [Ko] (cf. [TZ2, (1.13)]) induces a natural \( g \) action on \( L \). We make the assumption that this \( g \) action can be lifted to a \( G \) action on \( L \). Then this \( G \) action preserves \( \nabla^L \). One can also assume, after an integration over \( G \) if necessary, that \( G \) preserves the Hermitian metric on \( L \), the almost complex structure \( J \) and thus also the Riemannian metric \( g^{TM} \). \( Q(M, L) \) then becomes a virtual representation of \( G \). We denote by \( Q(M, L)^G \) its \( G \)-invariant part.

Now let \( a \in g^* \) be a regular value of \( \mu \). Let \( O_a \subset g^* \) be the coadjoint orbit of \( a \). For simplicity, we assume that \( G \) acts on \( \mu^{-1}(O_a) \) freely. Then the quotient space \( M_{G,a} = \mu^{-1}(O_a)/G \) is smooth. Furthermore, \( \omega \) descends canonically to a symplectic form \( \omega_a \) on \( M_{G,a} \) so that one gets the Marsden-Weinstein reduction \( (M_{G,a}, \omega_a) \). The pair \( (L, \nabla^L) \) also descends canonically to a Hermitian line bundle \( L_{G,a} \) with a Hermitian connection denoted by \( \nabla^{L_{G,a}} \). The almost complex structure \( J \) also descends canonically to an almost complex structure on \( TM_{G,a} \). Thus once again one can construct canonically a Spin\( ^c \)-Dirac operator \( D^{L_{G,a}} \) as well as the corresponding virtual vector space \( Q(M_{G,a}, L_{G,a}) \).

The purpose of this short note is to give a direct proof of the following result as an immediate consequence of a result proved in [TZ2].

**Theorem 0.1.** If \( \mu^{-1}(0) \) is nonempty, then there exists an open neighborhood \( O \) of \( 0 \in g^* \) such that for any regular value \( a \in O \) of \( \mu \) with \( \mu^{-1}(a) \) nonempty, the following identity holds,

\[
\dim Q(M, L)^G = \dim Q(M_{G,a}, L_{G,a}).
\]

(0.3)

If \( 0 \in g^* \) is a regular value of \( \mu \), then one can take \( a = 0 \) in (0.3). This is the Guillemin-Sternberg geometric quantization conjecture [GS], which was proved in various generalities in [DGMW, GS, G, JK, M1, M2, V1, V2, TZ1, TZ2]. Thus the main feature of Theorem 0.1 is that it equally applies when \( 0 \in g^* \) is a singular value of \( \mu \).

A complete treatment of Theorem 0.1 when \( G \) is the circle group was given in [TZ3] using the spectral flow technique. A different treatment of

\footnote{However, if \( a \neq 0 \), then \( L_{G,a} \) is in general no longer a prequantum line bundle over \( (M_{G,a}, \omega_{G,a}) \).}
(0.3) which works for nonabelian actions has also been given by Meinrenken and Sjamaar [MS].

Theorem 0.1 still holds, for orbifolds, when $G$ does not act freely on $\mu^{-1}(O_a)$. Furthermore, if $(M, \omega)$ is Kähler and $G$ acts on $M$ holomorphically, then we can refine (0.3) to a system of holomorphic Morse type inequalities as in [TZ1, 2].

This note is organized as follows. In Section 1, we recall the result in [TZ2] to be used in this paper. In Section 2, we prove a key estimate which allows one to obtain Theorem 0.1 immediately.

§1. An extended quantization formula

We make the same assumption and use the same notation as in Introduction. We further assume in this section that $0 \in g^*$ is a regular value of $\mu$ and, for simplicity, that $G$ acts on $\mu^{-1}(0)$ freely. Then one can construct the canonical Marsden-Weinstein reduction $(M_G = \mu^{-1}(0)/G, \omega_G)$. We equip $g$ (and thus $g^*$ also) with an $\text{Ad} G$-invariant metric. Let $h_i$, $1 \leq i \leq \dim G$, be an orthonormal base of $g^*$. Let $V_i$, $1 \leq i \leq \dim G$, be the dual base of $h_i$, $1 \leq i \leq \dim G$. Then the moment map $\mu$ can be written as

$$\mu = \sum_{i=1}^{\dim G} \mu_i h_i, \quad (1.1)$$

with each $\mu_i$ a real function on $M$.

Let

$$H = |\mu|^2 \quad (1.2)$$

be the norm square of the moment map. It is clearly $G$-invariant.

For any $V \in g$, we use the same notation to denote the vector field it generates on $M$.

Let now $E$ be a Hermitian vector bundle over $M$ admitting a Hermitian connection $\nabla^E$. We make the assumption that the $G$ action on $M$ lifts to a $G$ action on $E$ which preserves the Hermitian metric and the Hermitian connection $\nabla^E$ on $E$. Then $E$ descends canonically to a Hermitian vector bundle $E_G$ over $M_G$ with Hermitian connection $\nabla^{E_G}$.

With these data one can construct canonically a Spin$^c$-Dirac operator $D^E$ acting on $\Omega^{0,*}(M, E) = \Gamma(\Lambda^{0,*}(T^*M) \otimes E)$ as well as the corresponding
virtual vector space $Q(M, E)$ as in (0.2), with $L$ there replaced now by $E$ (cf. [TZ2, Sect. 1]). Similarly, one can construct the Spin$^c$-Dirac operator $D^{E \mathcal{G}}$ on $M_\mathcal{G}$ as well as the corresponding virtual vector space $Q(M_\mathcal{G}, E_\mathcal{G})$. Furthermore, one verifies directly that the $G$ action commutes with $D^E$ so that $Q(M, E)$ is a virtual $G$-representation. Denote its $G$-invariant part by $Q(M, E)^G$.

Let $V \in \mathfrak{g}$. Set

$$r^E_V = L^E_V - \nabla^E_V,$$

where $L^E_V$ denotes the infinitesimal action of $V$ on $E$.

The following result was proved in Tian-Zhang [TZ2], where a direct analytic approach to the Guillemin-Sternberg geometric quantization conjecture [GS] was developed.

**Theorem 1.1.** ([TZ2, Theorem 4.2]). If $\mu^{-1}(0)$ is nonempty and if at each critical point $x \in M \setminus \mu^{-1}(0)$ of $\mathcal{H}$,

$$-1 \sum_{i=1}^{\dim G} \mu_i(x) r^E_{v_i}(x) \geq 0,$$

then the following identity holds,

$$\dim Q(M, E)^G = \dim Q(M_\mathcal{G}, E_\mathcal{G}).$$

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**§2. A proof of Theorem 0.1**

In this section, we prove Theorem 0.1 by using Theorem 1.1. In order to explain the basic idea clearer, we will first treat the case where $G$ is abelian in detail. Then as will be seen, the basic estimate in the $G$ nonabelian case, which is necessary for applying Theorem 1.1, can be reduced to the abelian case.

This section is organized as follows. In a), we deal with the $G$ abelian case. In b), we prove Theorem 0.1 for the $G$ nonabelian case. The final subsection c) contains some further remarks related to Theorem 0.1.

a) The abelian case
In this subsection, we assume that $G$ is a torus.

For any $a \in g^*$, set
\[ \mu_a = \mu - a. \]  \hspace{1cm} (2.1)

Then $\mu_a$ is again a moment map for the $G$-action on $(M, \omega)$. Let
\[ H_a = |\mu_a|^2 \]  \hspace{1cm} (2.2)

be the norm square of $\mu_a$.

Recall that $h_i, 1 \leq i \leq \dim G$, is an orthonormal base of $g^*$. Thus $a$ has the expression
\[ a = \sum_{i=1}^{\dim G} a_i h_i. \]  \hspace{1cm} (2.3)

The main result of this subsection can be stated as follows.

**Proposition 2.1.** There is an open neighborhood $O$ of $0 \in g^*$ such that for any $a \in O$, the following inequality holds at each critical point of $H_a$,
\[ \sum_{i=1}^{\dim G} (\mu_i - a_i)\mu_i \geq 0. \]  \hspace{1cm} (2.4)

**Proof.** We will use Kirwan’s geometric characterization [K1] of the critical point set of the norm square of moment maps to prove (2.4).

As in [K1, 3.4], denote by $A \subset g^*$ the finite set of weights associated to $\mu$, which consists of the values of $\mu$ taking on the fixed point set of $G$ on $M$. We also denote by $Y$ the set of convex hulls in $g^*$ generated by nonempty subsets of $A$. Then $Y$ consists naturally of two parts: the part $Y_I$ consists of all those convex hulls not containing $0$, and the rest part denoted by $Y_{II}$.

Now let $U_\delta$ be an open ball in $g^*$ with center $0$ and radius $\delta > 0$ such that the closure $\overline{U_\delta}$ does not intersect with any convex hull in $Y_I$. The existence of $U_\delta$ is clear. Set $O = U_\delta/2$.

Let $a \in O$. Let $A_a = \{A - a : A \in A\}$ be the finite set of weights associated to $\mu_a$, and $Y_a = \{Y - a : Y \in Y\}$ the associated set of convex hulls. Then $Y_a$ consists of two parts accordingly: $Y_{I,a} = \{Y - a : Y \in Y_I\}$ and $Y_{II,a} = \{Y - a : Y \in Y_{II}\}$. One verifies easily that the closure $\overline{O}$ does not intersect with any convex hull in $Y_{I,a}$.
Let $B_a$ be the open ball $B_a = \{y \in g^* : |y + \frac{a}{2}| < |\frac{a}{2}|\}$. Clearly, $B_a \subset O$. Thus $B_a$ does not intersect with any convex hull in $Y_{I,a}$.

Now take $Y \in Y_{II,a}$. Let $y \in Y$ be the (unique) point on $Y$ which is closest to 0. We claim that $y \in g^* \setminus B_a$.

To prove this claim, we suppose on the contrary that $y \in B_a$. Let $x \in \partial B_a$, with $x \neq -a$, lie in the straight line generated by $y$ and $-a \in \partial B_a \cap Y$, then it is easy to see that $x \in Y$. For if $x \notin Y$, then $y$ should lie in a face of $Y$ which does not contain $-a$. This would imply that $y$ lies in a convex hull in $Y_{I,a}$, a contradiction. But with such an $x \in \partial B_a \cap Y$, one encounters another contradiction $d(0, x) < d(0, y)$. Thus we should have $y \notin B_a$.

By this and by the result of Kirwan [K1, 3.12], one finds that if $y$ is a critical point of $H_a$, then

$$0 \leq |\mu_a(y) + \frac{a}{2}|^2 - |\frac{a}{2}|^2 = \sum_{i=1}^{\dim G} (\mu_i(y) - a_i)\mu_i(y),$$

(2.5)

which is exactly (2.4). □

We can now prove Theorem 0.1 in this abelian case. In fact, the Kostant formula [Ko] (cf. [TZ2, (1.13)]) implies, in using the notation of (1.3), that for each $1 \leq i \leq \dim G$

$$r_{Y_i}^T = -2\pi \sqrt{-1}\mu_i,$$

(2.6)

Thus one verifies via Proposition 2.1 that at any critical point of $H_a$, one has

$$\sqrt{-1} \sum_{i=1}^{\dim G} (\mu_i - a_i)r_{Y_i}^T = 2\pi \sum_{i=1}^{\dim G} (\mu_i - a_i)\mu_i \geq 0.$$

(2.7)

Theorem 0.1 then follows by applying Theorem 1.1 to $(M, \omega, \mu_a, L)$. □

b) The nonabelian case

In this subsection, we no longer assume that $G$ is abelian.

Let $T$ be a maximal torus of $G$, with Lie algebra $t$. Then $\mu_T = P_T \mu$, where $P_T$ is the orthogonal projection from $g^*$ to $t^*$, is the moment map of the induced $T$ action on $(M, \omega)$ (cf. [K1, 3.3]). Let $O_T \subset t^*$ be the open set
defined in a) for $\mu_T$. Then $O = \text{Ad}G(O_T) \subset g^*$ is an open neighborhood of $0 \in g^*$.

Let $a \in O$ be a regular value of $\mu$ and, for simplicity, assume that $G$ acts on $\mu^{-1}(O_a)$ freely. Recall that the coadjoint orbit $O_a$ admits a canonical symplectic (actually Kähler) form $\omega_a$ and the $\text{Ad}G$ action on $O_a$ is Hamiltonian with the moment map given by the canonical embedding $i_a : O_a \hookrightarrow g^*$ (cf. [McS, Chap. 5]).

We now form the symplectic product $(M \times O_a, \omega \times (-\omega_a))$. Then the induced action of $G$ on $M \times O_a$ is Hamiltonian with the moment map $\hat{\mu} : M \times O_a \to g^*$ given by

$$\hat{\mu}(x, b) = \mu(x) - b.$$ (2.8)

Then $0 \in g^*$ is a regular value of $\hat{\mu}$ and $G$ acts on $\hat{\mu}^{-1}(0)$ freely. Furthermore, one has the standard identification of the symplectic quotients

$$\hat{\mu}^{-1}(0)/G \equiv \mu^{-1}(O_a)/G = M_{G,a}$$ (2.9)

(cf. [McS, Chap. 5]).

Let

$$\hat{H} = |\hat{\mu}|^2$$ (2.10)

be the norm square of the moment map $\hat{\mu}$.

We now state the nonabelian extension of Proposition 2.1 as follows.

**Proposition 2.2.** If $(x, b) \in M \times O_a$ is a critical point of $\hat{H}$, then one has the following inequality for the inner product on $g^*$,

$$\langle \mu(x) - b, \mu(x) \rangle \geq 0.$$ (2.11)

**Proof.** Without loss of generality, we can assume that $a \in O_T$. By a result of Kirwan [K2, pp. 551], we know that for any critical point $(x, b) \in M \times O_a$ of $\hat{H}$, one can find $y \in t^*$ such that $(y, a)$ is a critical point of $\hat{H}$ in the $G$-orbit of $(x, b)$.

Now since $a \in O_T$, one finds from Proposition 2.1 and from Kirwan [K1, 3.3] that

$$\langle \mu(y) - a, \mu(y) \rangle \geq 0.$$ (2.12)

(2.11) then follows from the $\text{Ad}G$-invariance of the inner product on $g^*$.

\[ \square \]
Proof of Theorem 0.1. Denote by \(\pi\) the projection from \(M \times O_a\) to its first factor \(M\). Let \(L = \pi^* L\) be the pull-back Hermitian line bundle over \(M \times O_a\) with the pull-back Hermitian connection \(\nabla^L\) on \(L\) verifying that
\[
\frac{\sqrt{-1}}{2\pi} (\nabla^L)^2 = \pi^* \omega. \tag{2.13}
\]
Furthermore, the \(G\) action on \(L\) lifts canonically to an action on \(L\). In particular, for any \(V \in \mathfrak{g}\), its induced infinitesimal action on \(L\) is, via the Kostant formula \([\text{Ko}]\) (cf. \([\text{TZ2}, (1.13)]\)) for the \(\mathfrak{g}\)-action on \(L\), given by
\[
L^e_V = \nabla_V^e - 2\pi \sqrt{-1} \langle \mu, V \rangle, \tag{2.14}
\]
from which one has, in using the notation in (1.3), that
\[
r^e_V \mu_i = - 2\pi \sqrt{-1} \mu_i(x) \tag{2.15}
\]
at any point \((x, b) \in M \times O_a\). Thus by (2.15) and Proposition 2.2, one verifies that at any critical point \((x, b) \in M \times O_a\) of \(\hat{H}\),
\[
\sum_{i=1}^{\dim G} \sqrt{-1} \mu_i(x, b) r^e_{V_i}(x, b) = 2\pi \langle \mu(x) - b, \mu(x) \rangle \geq 0. \tag{2.16}
\]
One the other hand, one verifies directly that the induced line bundle \(L_G\) over \(\hat{\mu}^{-1}(0)/G\) is exactly the line bundle \(L_{G,a}\) over \(M_{G,a} = \mu^{-1}(O_a)/G\).

One can then apply Theorem 1.1 to \(M \times O_a\), \(\mu\) and \(L\) to get
\[
\dim Q(M \times O_a, L)^G = \dim Q(M_{G,a}, L_{G,a}). \tag{2.17}
\]
Furthermore, by the definition of \(L\), one verifies directly that
\[
\dim Q(M \times O_a, L)^G = \dim Q(M, L)^G \cdot \dim Q(O_a, C)^G, \tag{2.18}
\]
with
\[
\dim Q(O_a, C)^G = 1, \tag{2.19}
\]
which can also be verified directly. (0.3) follows from (2.17)-(2.19). \(\square\)

c) Further extensions and remarks
First of all, as have already been remarked in Introduction, Theorem 0.1 still holds for orbifolds when $G$ does not act on $\mu^{-1}(O_a)$ freely, and that in the holomorphic category one can refine (0.3) to a system of Morse type inequalities. We will not fill the easy details here.

The second point we want to remark is that by now it is clear that Theorem 1.1 has a very wide range of applicability, and Theorem 0.1 is just one manifestation of this. One can easily state and prove an extended version of Theorem 0.1 which works for general coefficients. In particular, one can take the trivial line bundle as coefficient to get as in [TZ2] that the Todd genus of $M$ equals to the Todd genus of $M_{G,a}$ for any regular value $a$ of $\mu$. This last result is also contained in [MS].

Last but not least, recall that we have extended the Guillemin-Sternberg geometric quantization conjecture [GS] to the case of manifolds with boundary in [TZ3]. Combining the methods there with our proof of Theorem 0.1, one can easily extend Theorem 0.1 to the case of manifolds with boundary as well. We again leave the details to the interested reader.

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