RELATIONS ON NONCOMMUTATIVE VARIABLES
AND ASSOCIATED ORTHOGONAL POLYNOMIALS

T. BANKS, T. CONSTANTINESCU, AND J. L. JOHNSON

Abstract. This semi-expository paper surveys results concerning three classes of orthogonal polynomials: in one non-hermitian variable, in several isometric non-commuting variables, and in several hermitian non-commuting variables. The emphasis is on some dilation theoretic techniques that are also described in some details.

1. Introduction

In this semi-expository paper we deal with a few classes of orthogonal polynomials associated to polynomial relations on several noncommuting variables. Our initial interest in this subject was motivated by the need of more examples related to [18]. On the other hand there are transparent connections with interpolation problems in several variables as well as with the modeling of various classes of nonstationary systems (see [1] for a list of recent references), which guided our choice of topics. Thus we do not relate to more traditional studies on orthogonal polynomials of several variables associated to a finite reflection group on an Euclidean space or other types of special functions of several variables, for which a recent presentation could be found in [21], instead we are more focused on results connected with various dilation theoretic aspects and Szegö kernels.

Our aim is to give an introduction to this point of view. We begin our presentation with a familiar setting for algebras given by polynomial defining relations and then we introduce families of orthonormal polynomials associated to some positive functionals on these algebras. Section 3 contains a discussion of the first class of orthogonal polynomials considered in this paper, namely polynomials in one variable on which there is no relation. This choice is motivated mainly by the fact that we have an opportunity to introduce some of the basic dilation theoretic techniques that we are using. First, we discuss (in a particular case that is sufficient for our goals) the structure of positive definite kernels and their triangular factorization. Then these results are used to obtain recurrence relations for orthogonal polynomials in one variable with no additional relations, as well as asymptotic properties of these polynomials. All of these extend well-known results of Szegö. We conclude this section with the introduction of a Szegö type kernel which appears to be relevant to our setting.

In Section 4 we discuss orthogonal polynomials of several isometric variables. Most of the results are just particular cases of the corresponding results discussed in Section 3, but there is an interesting new point about the Szegö kernel that appears in the proof of Theorem 4.1. We also use a certain explicit structure of the Kolmogorov decomposition
of a positive definite kernel on the set of non-negative integers in order to produce examples of families of operators satisfying Cuntz-Toeplitz and Cuntz relations.

The final section contains a discussion of orthogonal polynomials of several non-commuting hermitian variables. This time, some of the techniques described in Section 3 are not so relevant and instead we obtain three-terms recursions in the traditional way, and we introduce families of Jacobi matrices associated to these recursions. Many of these results can be proved by adapting the classical proofs from the one scalar variable case. However, much of the classical function theory is no longer available so we present some proofs illustrating how classical techniques have to be changed or replaced. Also some results are not presented in the most general form in the hope that the consequent simplifications in notation would make the paper more readable.

2. Orthogonal polynomials associated to polynomial relations

In this section we introduce some classes of orthogonal polynomials in several variables. We begin with the algebra \( P_N \) of polynomials in \( N \) noncommuting variables \( X_1, \ldots, X_N \) with complex coefficients. Let \( \mathbb{F}^+_N \) be the unital free semigroup on \( N \) generators 1, \ldots, \( N \). The empty word is the identity element and the length of the word \( \sigma \) is denoted by \( |\sigma| \). It is convenient to use the notation \( X_\sigma = X_{i_1} \ldots X_{i_k} \) for \( \sigma = i_1 \ldots i_k \in \mathbb{F}^+_N \). Thus, each element \( P \in P_N \) can be uniquely written in the form \( P = \sum_{\sigma \in \mathbb{F}^+_N} c_\sigma X_\sigma \), with \( c_\sigma \neq 0 \) for finitely many \( \sigma \)’s.

We notice that \( P_N \) is isomorphic with the tensor algebra over \( \mathbb{C}^N \). Let \( (\mathbb{C}^N)^{\otimes k} \) denote the \( k \)-fold tensor product of \( \mathbb{C}^N \) with itself. The tensor algebra over \( \mathbb{C}^N \) is defined by the algebraic direct sum

\[
\mathcal{T}(\mathbb{C}^N) = \bigoplus_{k\geq 0} (\mathbb{C}^N)^{\otimes k}.
\]

If \( \{e_1, \ldots, e_N\} \) is the standard basis of \( \mathbb{C}^N \), then the set

\[
\{1\} \cup \{e_{i_1} \otimes \cdots \otimes e_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq N, k \geq 1\}
\]

is a basis of \( \mathcal{T}(\mathbb{C}^N) \). For \( \sigma = i_1 \ldots i_k \) we write \( e_\sigma \) instead of \( e_{i_1} \otimes \cdots \otimes e_{i_k} \), and the mapping \( X_\sigma \to e_\sigma, \sigma \in \mathbb{F}^+_N \), extends to an isomorphism from \( P_N \) to \( \mathcal{T}(\mathbb{C}^N) \), hence \( P_N \cong \mathcal{T}(\mathbb{C}^N) \).

It is useful to introduce a natural involution on \( P_{2N} \) as follows:

\[
X_k^+ = X_{N+k}, \quad k = 1, \ldots, N,
\]

\[
X_l^+ = X_{l-N}, \quad l = N+1, \ldots, 2N;
\]

on monomials,

\[
(X_{i_1} \ldots X_{i_k})^+ = X_{i_k}^+ \ldots X_{i_1}^+,
\]

and finally, if \( Q = \sum_{\sigma \in \mathbb{F}^+_{2N}} c_\sigma X_\sigma \), then \( Q^+ = \sum_{\sigma \in \mathbb{F}^+_{2N}} c_\sigma X_\sigma^+ \). Thus, \( P_{2N} \) is a unital, associative, \( \ast \)-algebra over \( \mathbb{C} \).

We say that \( \mathcal{A} \subset P_{2N} \) is symmetric if \( P \in \mathcal{A} \) implies \( cP^+ \in \mathcal{A} \) for some \( c \in \mathbb{C} \setminus \{0\} \). Then the quotient of \( P_{2N} \) by the two-sided ideal generated by \( \mathcal{A} \) is an associative
algebra $\mathcal{R}(\mathcal{A})$. Letting $\pi = \pi_\mathcal{A} : \mathcal{P}_{2N} \rightarrow \mathcal{R}(\mathcal{A})$ denote the quotient map then the formula

$$\pi(P)^+ = \pi(P+)$$

(2.1)

gives a well-defined involution on $\mathcal{R}(\mathcal{A})$. Usually the elements of $\mathcal{A}$ are called the defining relations of the algebra $\mathcal{R}(\mathcal{A})$. For instance, $\mathcal{R}(\emptyset) = \mathcal{P}_{2N}$,

$$\mathcal{R}(\{X_k - X_k^+ \mid k = 1, \ldots, N\}) = \mathcal{P}_N,$$

$$\mathcal{R}(\{X_k X_l - X_l X_k \mid k, l = 1, \ldots, 2N\}) \simeq \mathcal{S}(\mathbb{C}^{2N}),$$

the symmetric algebra over $\mathbb{C}^{2N}$,

$$\mathcal{R}(\{X_k X_l + X_l X_k \mid k, l = 1, \ldots, 2N\}) \simeq \Lambda(\mathbb{C}^{2N}),$$

the exterior algebra over $\mathbb{C}^{2N}$.

Examples abound in the literature (for instance, see [20], [21], [29]).

There are many well-known difficulties in the study of orthogonal polynomials in several variables. The first one concerns the choice of an ordering of $\mathbb{F}_N^+$. In this paper we consider only the lexicographic order $\prec$, but due to the canonical grading of $\mathbb{F}_N^+$ it is possible to develop a basis free approach to orthogonal polynomials. In the case of orthogonal polynomials on several commuting variables this is presented in [21].

A second difficulty concerns the choice of the moments. In this paper we adopt the following terminology. A linear functional $\phi$ on $\mathcal{R}(\mathcal{A})$ is called q-positive ($q$ comes from quarter) if $\phi(\pi(P)^+ \pi(P)) \geq 0$ for all $P \in \mathcal{P}_N$. In this case, $\phi(\pi(P)^+) = \phi(\pi(P))$ for $P \in \mathcal{P}_N$ and

$$|\phi(\pi(P_1)^+ \pi(P_2))|^2 \leq \phi(\pi(P_1)^+ \pi(P_1)) \phi(\pi(P_2)^+ \pi(P_2))$$

for $P_1, P_2 \in \mathcal{P}_N$. Next we introduce

$$\langle \pi(P_1), \pi(P_2) \rangle_\phi = \phi(\pi(P_2)^+ \pi(P_1)), \quad P_1, P_2 \in \mathcal{P}_N.$$  

(2.2)

By factoring out the subspace $\mathcal{N}_\phi = \{\pi(P) \mid P \in \mathcal{P}_N, \langle \pi(P), \pi(P) \rangle_\phi = 0\}$ and completing this quotient with respect to the norm induced by (2.2) we obtain a Hilbert space $\mathcal{H}_\phi$.

The index set $G(\mathcal{A}) \subset \mathbb{F}_N^+$ of $\mathcal{A}$, is chosen as follows: if $\alpha \in G(\mathcal{A})$, choose the next element in $G(\mathcal{A})$ to be the least $\beta \in \mathbb{F}_N^+$ with the property that the elements $\pi(X_\alpha')$, $\alpha' \preceq \alpha$, and $\pi(X_\beta)$ are linearly independent. We will avoid the degenerate situation in which $\pi(1) = 0$; if we do so, then $\emptyset \in G(\mathcal{A})$. Define $F_\alpha = \pi(X_\alpha)$ for $\alpha \in G(\mathcal{A})$. For instance, $G(\emptyset) = \mathbb{F}_N^+$, in which case $F_\alpha = X_\alpha, \alpha \in \mathbb{F}_N^+$. Also,

$$G(\{X_k X_l \mid k, l = 1, \ldots, 2N\}) = \{i_1 \ldots i_k \in \mathbb{F}_N^+ \mid i_1 \leq \ldots \leq i_k, k \geq 1\},$$

and

$$G(\{X_k X_l + X_l X_k \mid k, l = 1, \ldots, 2N\}) = \{i_1 \ldots i_k \in \mathbb{F}_N^+ \mid i_1 < \ldots < i_k, 0 \leq k \leq N\}$$

(we use the convention that for $k = 0$, $i_1 \ldots i_k$ is the empty word).

We consider the moments of $\phi$ to be the numbers

$$s_{\alpha, \beta} = \phi(F_\alpha^+ F_\beta) = \langle F_\beta, F_\alpha \rangle_\phi, \quad \alpha, \beta \in G(\mathcal{A}).$$

(2.3)

The kernel of moments is given by $K_\phi(\alpha, \beta) = s_{\alpha, \beta}, \alpha, \beta \in G(\mathcal{A})$. We notice that $\phi$ is a q-positive functional on $\mathcal{R}(\mathcal{A})$ if and only if $K_\phi$ is a positive definite kernel on
$G(\mathcal{A})$. However, $K_\phi$ does not determine $\phi$ uniquely. One typical situation when $K_\phi$ determines $\phi$ is \{${X_k - X_k^+ \mid k = 1, \ldots, N}$ $\} \subset \mathcal{A}$; a more general example is provided by the Wick polynomials,

$$X_k X_l^+ - a_{k,l} \delta_{k,l} - \sum_{m,n=1}^N c_{k,l}^{m,n} X_m^+ X_n, \quad k, l = 1, \ldots, N,$$

where $a_{k,l}$, $c_{k,l}^{m,n}$ are complex numbers and $\delta_{k,l}$ is the Kronecker symbol.

The moment problem is trivial in this framework since it is obvious that the numbers $s_{\alpha,\beta}$, $\alpha, \beta \in G(\mathcal{A})$, are the moments of a q-positive functional on $\mathcal{R}(\mathcal{A})$ if and only if the kernel $K(\alpha, \beta) = s_{\alpha,\beta}$, $\alpha, \beta \in G(\mathcal{A})$, is positive definite.

We now introduce orthogonal polynomials in $\mathcal{R}(\mathcal{A})$. Assume that $\phi$ is strictly q-positive on $\mathcal{R}(\mathcal{A})$, that is, $\phi(\pi(P^+ \pi(P))) > 0$ for $\pi(P) \neq 0$. In this case $\mathcal{N}_\phi = \{0\}$ and $\pi(\mathcal{P}_N)$ can be viewed as a subspace of $\mathcal{H}_\phi$. Also, $\{F_\alpha\}_{\alpha \in G(\mathcal{A})}$ is a linearly independent family in $\mathcal{H}_\phi$ and the Gram-Schmidt procedure gives a family $\{\varphi_\alpha\}_{\alpha \in G(\mathcal{A})}$ of elements in $\pi(\mathcal{P}_N)$ such that

\begin{equation}
\varphi_\alpha = \sum_{\beta \leq \alpha} a_{\alpha,\beta} F_\beta, \quad a_{\alpha,\alpha} > 0;
\end{equation}

\begin{equation}
\langle \varphi_\alpha, \varphi_\beta \rangle_\phi = \delta_{\alpha,\beta}, \quad \alpha, \beta \in G(\mathcal{A}).
\end{equation}

The elements $\varphi_\alpha$, $\alpha \in G(\mathcal{A})$, will be called the \textit{orthonormal polynomials} associated to $\phi$. An explicit formula for the orthonormal polynomials can be obtained in the same manner as in the classical, one scalar variable case. Thus, set

\begin{equation}
D_\alpha = \det [s_{\alpha',\beta'}]_{\alpha',\beta' \leq \alpha} > 0, \quad \alpha \in G(\mathcal{A}),
\end{equation}

and from now on $\tau - 1$ denotes the predecessor of $\tau$ with respect to the lexicographic order on $\mathbb{F}_N^+$, while $\sigma + 1$ denotes the successor of $\sigma$. We have: $\varphi_\emptyset = s_{\emptyset,\emptyset}^{-1/2}$ and for $\emptyset < \alpha$,

\begin{equation}
\varphi_\alpha = \frac{1}{\sqrt{D_{\alpha-1}D_\alpha}} \det \left[ s_{\alpha',\beta'} |_{\alpha' < \alpha; \beta' \leq \alpha} F_\emptyset \ldots F_\alpha \right],
\end{equation}

with an obvious interpretation of the determinant. In the following sections we will discuss in more details orthonormal polynomials associated to some simple defining relations.

3. No relation in one variable

This simple case allows us to illustrate some general techniques that can be used in the study of orthonormal polynomials. We have $\mathcal{A} = \emptyset$ and $N = 1$, so $\mathcal{R}(\mathcal{A}) = \mathcal{P}_1$. The index set is $\mathbb{N}_0$, the set of nonnegative integers, and $F_n = X_1^n$, $n \in \mathbb{N}_0$. The moment kernel of a q-positive functional on $\mathcal{P}_1$ is $K_\phi(n,m) = \phi((X_1^n)^+ X_1^m)$, $n, m \in \mathbb{N}_0$, and we
notice that there is no restriction on $K_{\phi}$ other than being positive definite. We now discuss some tools that can be used in this situation.

3.1. Positive definite kernels on $\mathbb{N}_0$. We discuss a certain structure (and parametrization) of positive definite kernels on $\mathbb{N}_0$. The nature of this structure is revealed by looking at the simplest examples. First, we consider a strictly positive matrix $S = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$, $a \in \mathbb{R}$.

This matrix gives a new inner product on $\mathbb{R}^2$ by the formula \[ \langle x, y \rangle_S = \langle Sx, y \rangle, \quad x, y \in \mathbb{R}^2, \]
where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^2$. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$. By renorming $\mathbb{R}^2$ with $\langle \cdot, \cdot \rangle_S$ the angle between $e_1$ and $e_2$ was modified to the new angle $\theta = \theta(e_1, e_2)$ such that

\[ \cos \theta(e_1, e_2) = \frac{\langle e_1, e_2 \rangle_S}{\|e_1\|_S \|e_2\|_S} = a. \]

We can visualize the renormalization process by giving a map $T_S : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that $\langle T_S x, T_S y \rangle = \langle x, y \rangle_S$ for $x, y \in \mathbb{R}^2$, and it is easily seen that we can choose

\[ T_S = \begin{bmatrix} 1 & \cos \theta \\ 0 & \sin \theta \end{bmatrix}. \]

We can also notice that $T_se_1 = e_1$ and $T_se_2 = f_2 = J(\cos \theta)e_1$, where $J(\cos \theta)$ is the Julia operator,

\[ J(\cos \theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}, \]
which is the composition of a reflection about the $x$-axis followed by the counterclockwise rotation $R_\theta$ through angle $\theta$. We deduce that

\[ a = \cos \theta = \langle e_1, f_2 \rangle = \langle e_1, J(\cos \theta)e_1 \rangle = \langle J(\cos \theta)e_1, e_1 \rangle. \]

The discussion extends naturally to the $3 \times 3$ case. Thus let $S = \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}$, $a, b, c \in \mathbb{R}$,
be a strictly positive matrix. A new inner product is induced by $S$ on $\mathbb{R}^3$, 
\[ \langle x, y \rangle_S = \langle Sx, y \rangle, \quad x, y \in \mathbb{R}^3, \]
and let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. With respect to this new inner product the vectors $e_1, e_2, e_3$ still belong to the unit sphere, but they are no longer orthogonal. Thus, 
\[ a = \cos \theta(e_1, e_2) = \cos \theta_{12}, \]
\[ c = \cos \theta(e_2, e_3) = \cos \theta_{23}, \]
and 
\[ b = \cos \theta(e_1, e_3) = \cos \theta_{13}. \]
This time, the law of cosines in spherical geometry gives a relation between the numbers $a, b,$ and $c$,
\[
(3.2) \quad b = \cos \theta_{13} = \cos \theta_{12} \cos \theta_{23} + \sin \theta_{12} \sin \theta_{23} \cos \theta,
\]
where $\theta$ is the dihedral angle formed by the planes generated by $e_1, e_2$ and, respectively, $e_2, e_3$ (see, for instance, [26]). Thus, the number $b$ belongs to a disk of center $\cos \theta_{12} \cos \theta_{23}$ and radius $\sin \theta_{12} \sin \theta_{23}$. Once again the renormalization can be visualized by a map $T_S : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\langle T_S x, T_S y \rangle = \langle x, y \rangle_S$. In this case we can choose 
\[ T_S = \begin{pmatrix} 1 & \cos \theta_{12} & \cos \theta_{12} \cos \theta_{23} + \sin \theta_{12} \sin \theta_{23} \cos \theta \\ 0 & \sin \theta_{12} & \sin \theta_{12} \cos \theta_{23} - \cos \theta_{12} \sin \theta_{23} \cos \theta \\ 0 & 0 & \sin \theta_{23} \sin \theta \end{pmatrix}, \]
and we see that $T_S e_1 = e_1, T_S e_2 = f_2 = (J(\cos \theta_{12}) \oplus 1)e_1$, and 
\[ T_S e_3 = f_3 = (J(\cos \theta_{12}) \oplus 1)(1 \oplus J(\cos \theta))(J(\cos \theta_{23}) \oplus 1)e_1. \]
In particular,
\[
(3.3) \quad b = \cos \theta_{13} = \langle (J(\cos \theta_{12}) \oplus 1)(1 \oplus J(\cos \theta))(J(\cos \theta_{23}) \oplus 1)e_1, e_1 \rangle,
\]
which can be viewed as a dilation formula.

Now both (3.2) and (3.3) extend to a strictly $q$-positive $n \times n$ matrix and provide a parametrization and therefore a structure for the positive definite kernels on $\mathbb{N}_0$ (for general results and details see [13], [16]). We apply this result to a kernel $K_\phi$ associated to a strictly $q$-positive functional $\phi$ and obtain that $K_\phi$ is uniquely determined by a family $\{\gamma_{k,j}\}_{0 \leq k < j}$ of complex numbers with the property that $|\gamma_{k,j}| < 1$ for all $0 \leq k < j$. Define $d_{k,j} = (1 - |\gamma_{k,j}|^2)^{1/2}$. The extension of (3.3) mentioned above gives
\[
(3.4) \quad s_{k,j} = \gamma_{k,k}^{1/2} s_{j,j}^{1/2} \langle U_{k,j} e_1, e_1 \rangle, \quad k < j,
\]
where $U_{k,j}$ is a $(j-k+1) \times (j-k+1)$ unitary matrix defined recursively by: $U_{k,k} = 1$, and for $k < j$, 
\[
(3.5) \quad U_{k,j} = (J(\gamma_{k,k+1} \oplus 1_n) \oplus 1_{n-2}) \cdots (1_{n-1} \oplus J(\gamma_{k,j})) (U_{k+1,j} \oplus 1). \]
Also $J(\gamma_{l,m})$ is the Julia operator associated to $\gamma_{l,m}$, that is, 
\[ J(\gamma_{l,m}) = \begin{bmatrix} \gamma_{l,m} & d_{l,m} \\ d_{l,m} & \overline{\gamma_{l,m}} \end{bmatrix}, \]
and $1_m$ denotes the $m \times m$ identity matrix. For instance, we deduce from (3.4) that:

$$s_{01} = s_{00}^{1/2} s_{11}^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{01} & d_{01} \\ d_{01} & \gamma_{01} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$;

$$s_{02} = s_{00}^{1/2} s_{22}^{1/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\times \begin{bmatrix} \gamma_{01} & d_{01} & 0 \\ d_{01} & \gamma_{01} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma_{02} & d_{02} \\ d_{02} & \gamma_{02} & 0 \end{bmatrix}
\times \begin{bmatrix} \gamma_{12} & d_{12} & 0 \\ d_{12} & \gamma_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and

$$s_{03} = s_{00}^{1/2} s_{33}^{1/2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\times \begin{bmatrix} \gamma_{01} & d_{01} & 0 & 0 \\ d_{01} & \gamma_{01} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{12} & d_{12} & 0 \\ d_{12} & \gamma_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\times \begin{bmatrix} \gamma_{13} & d_{13} & 0 \\ d_{13} & \gamma_{13} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. $$

In particular, we deduce that $s_{01} = s_{00}^{1/2} s_{11}^{1/2} \gamma_{01}$. The next formula is the complex version of (3.2), $s_{02} = s_{00}^{1/2} s_{22}^{1/2} \gamma_{01} \gamma_{12} + d_{01} \gamma_{02} d_{12}$. Then,

$$s_{03} = s_{00}^{1/2} s_{33}^{1/2} \gamma_{01} \gamma_{12} \gamma_{23} + \gamma_{01} d_{12} \gamma_{13} d_{23} + d_{01} \gamma_{02} d_{12} \gamma_{23} - d_{01} \gamma_{02} \gamma_{12} \gamma_{13} d_{23} + d_{01} d_{02} \gamma_{03} d_{13} d_{23}.$$

Explicit formulae of this type can be obtained for each $s_{k,j}$, as well as inverse algorithms allowing to calculate $\gamma_{k,j}$ from the kernel of moments, see [16] for details.

A natural combinatorial question would be to calculate the number $N(s_{k,j})$ of additive terms in the expression of $s_{k,j}$. We give here some details since the calculation of $N(s_{k,j})$ involves another useful interpretation of formula (3.4). We notice that for $k \geq 0$,

$$N(s_{01}) = N(s_{k,k+1}) = 1,$$

$$N(s_{02}) = N(s_{k,k+2}) = 2,$$

$$N(s_{03}) = N(s_{k,k+3}) = 5.$$

The general formula is given by the following result.
Theorem 3.1. $N(s_{k,k+l})$ is given by the Catalan number $C_l = \frac{1}{l+1} \binom{2l}{l}$.

Proof. The first step of the proof considers the realization of $s_{k,j}$ through a time varying transmission line (or lattice). For illustration we consider the case of $s_{03}$ in Figure 2.

Each box in Figure 2 represents the action of a Julia operator, and we see on this figure that the number of additive terms in the formula of $s_{03}$ is given by the number of paths from $A$ to $B$. In it clear that to each path from $A$ to $B$ in Figure 2 it corresponds a Catalan path from $C$ to $D$ in Figure 3, that is, a path that never steps below the diagonal and goes only to the right or downward.

Thus, each box in Figure 2 corresponds to a point strictly above the diagonal in Figure 3. Once this one-to-one correspondence is established, we can use the well-known fact that the number of Catalan paths like the one in Figure 3 is given exactly by the Catalan numbers. □

3.2. Spectral factorization. The classical theory of orthogonal polynomials is intimately related to the so-called spectral factorization. Its prototype would be the Fejér-Riesz factorization of a positive trigonometric polynomial $P$ in the form $P = |Q|^2$, where $Q$ is a polynomial with no zeros in the unit disk. This is generalized by Szegő to the Szegő class of those measures on the unit circle $\mathbb{T}$ with $\log \mu' \in L^1$, and very general results along this line can be found in [31] and [33].

Here we briefly review the spectral factorization of positive definite kernels on the set $\mathbb{N}_0$ described in [14]. For two positive definite kernels $K_1$ and $K_2$ we write $K_1 \leq K_2$
if $K_2 - K_1$ is a positive definite kernel. Consider a family $\mathcal{F} = \{F_n\}_{n \geq 0}$ of at most one-dimensional vector spaces and call lower triangular array a family $\Theta = \{\Theta_{k,j}\}_{k,j \geq 0}$ of complex numbers $\Theta_{k,j}$ with the following two properties: $\Theta_{k,j} = 0$ for $k < j$ and each column $c_j(\Theta) = [\Theta_{k,j}]_{k \geq 0}$, $j \geq 0$, belongs to the Hilbert space $\oplus_{k \geq j} F_k$. Denote by $\mathcal{H}_0^2(\mathcal{F})$ the set of all lower triangular arrays as above. An element of $\mathcal{H}_0^2(\mathcal{F})$ is called outer if the set $\{c_j(\Theta) \mid j \geq k\}$ is total in $\oplus_{j \geq k} F_j$ for each $k \geq 0$.

It is easily seen that if $\Theta$ is an outer triangular array, then the formula

$$K_\Theta(k, j) = c_k(\Theta)^*c_j(\Theta)$$

gives a positive definite kernel on $\mathbb{N}_0$. The following result extends the above mentioned Szegö factorization and at the same time it contains the Cholesky factorization of positive matrices.

**Theorem 3.2**. Let $K$ be an positive definite kernel on $\mathbb{N}_0$. Then there exists a family $\mathcal{F} = \{F_n\}_{n \geq 0}$ of at most one-dimensional vector spaces and an outer triangular array $\Theta \in \mathcal{H}_0^2(\mathcal{F})$ such that

1. $K_\Theta \leq K$.
2. For any other family $\mathcal{F}' = \{F'_n\}_{n \geq 0}$ of at most one-dimensional vector spaces and any outer triangular array $\Theta' \in \mathcal{H}_0^2(\mathcal{E}', \mathcal{F}')$ such that $K_{\Theta'} \leq K$, we have $K_{\Theta'} \leq K_\Theta$.
3. $\Theta$ is uniquely determined by (a) and (b) up to a left unitary diagonal factor.

It follows from (3) above that the spectral factor $\Theta$ can be uniquely determined by the condition that $\Theta_{n,n} \geq 0$ for all $n \geq 0$. We say that the kernel $K$ belongs to the Szegö class if $\inf_{n \geq 0} \Theta_{n,n} > 0$. If $\{\gamma_{k,j}\}$ are the parameters of $K$ introduced in Subsection 3.1 then it follows that the kernel $K$ belongs to the Szegö class if and only if

$$\inf_{k \geq 0} s_{k,k}^{1/2} \prod_{n > k} d_{k,n} > 0.$$  

This implies that $F_n = \mathbb{C}$ for all $n \geq 0$ (for details see [14] or [16]).

**3.3. Recurrence relations.** Formula (2.7) is not very useful in calculations involving the orthogonal polynomials. Instead there are used recurrence formulae. In our case, $\mathcal{A} = \emptyset$ and $N = 1$, we consider the moment kernel $K_\phi$ of a strictly q-positive functional on $\mathcal{P}_1$ and also, the parameters $\{\gamma_{k,j}\}$ of $K_\phi$ as in Subsection 3.1. It can be shown that the orthonormal polynomials associated to $\phi$ obey the following recurrence relations

$$\varphi_0(X_1, l) = \varphi_0^\sharp(X_1, l) = s_{l,l}^{-1/2}, \quad l \in \mathbb{N}_0,$$

and for $n \geq 1$, $l \in \mathbb{N}_0$,

$$\varphi_n(X_1, l) = \frac{1}{d_{l,n+l}} \left( X_1 \varphi_{n-1}(X_1, l+1) - \gamma_{l,n+l} \varphi_{n-1}^\sharp(X_1, l) \right),$$

$$\varphi_n^\sharp(X_1, l) = \frac{1}{d_{l,n+l}} \left( -\gamma_{l,n+l} X_1 \varphi_{n-1}(X_1, l+1) + \varphi_{n-1}^\sharp(X_1, l) \right),$$
where \( \varphi_n(X_1) = \varphi_n(X_1, 0) \) and \( \varphi_n^\sharp(X_1) = \varphi_n^\sharp(X_1, 0) \).

Somewhat similar polynomials are considered in [19], but the form of the recurrence relations as above is noticed in [17]. It should be mentioned that \( \{ \varphi_n(X_1, l) \}_{n \geq 0} \) is the family of orthonormal polynomials associated to a \( q \)-positive functional on \( P_1 \) with moment kernel \( K^l(\alpha, \beta) = s_{\alpha+l, \beta+l}, \alpha, \beta \in \mathbb{N}_0 \). Also, the above recurrence relations provide us with a tool to recover the parameters \( \{ \gamma_{k,j} \} \) from the orthonormal polynomials.

**Theorem 3.3.** Let \( k_n^l \) be the leading coefficient of \( \varphi_n(X_1, l) \). For \( l \in \mathbb{N}_0 \) and \( n \geq 1 \),

\[
\gamma_{l,n+l} = -\varphi_n(0, l) \frac{k_{n+1}^{l+1} \ldots k_n^{l+1}}{k_n^l \ldots k_0^l}.
\]

**Proof.** We reproduce here the proof from [8] in order to illustrate these concepts and to introduce one more property of the parameters \( \{ \gamma_{k,j} \} \). First, we deduce from (3.8) that

\[
\varphi_n(0, l) = -\frac{\gamma_{l,n+l} \varphi_{n-1}^\sharp(0, l)}{d_{l,n+l}},
\]

while formula (3.9) gives

\[
\varphi_n^\sharp(0, l) = \frac{1}{d_{l,n+l}} \varphi_{n-1}^\sharp(0, l) = \ldots = s_{l,l}^{-1/2} \prod_{p=1}^n \frac{1}{d_{l,p+l}},
\]

hence

\[
\varphi_n(0, l) = -s_{l,l}^{-1/2} \gamma_{l,n+l} \prod_{p=1}^n \frac{1}{d_{l,p+l}}.
\]

Now we can use another useful feature of the parameters \( \{ \gamma_{k,j} \} \), namely the fact that they give simple formulae for determinants. Let \( D_{m,l} \) denote the determinant of the matrix \( [s_{k,j}]_{l \leq k \leq n' \leq m} \). By Proposition 1.7 in [13],

(3.10)

\[
D_{l,m} = \prod_{k=l}^m s_{k,k} \times \prod_{l \leq j < p \leq m} d_{j,p}^2.
\]

One simple application of this formula is that it reveals the equality behind Fisher-Hadamard inequality. Thus, for \( l \leq n \leq n' \leq m \), we have

\[
D_{l,m} = \frac{D_{l,n'} D_{n,m}}{D_{n',n}} \prod_{(k,j) \in \Lambda} d_{k,j}^2,
\]

where \( \Lambda = \{(k,j) \mid l \leq k < n \leq n' < j \leq m \} \). Some other applications of (3.10) can be found in [16], Chapter 8. Returning to our proof we deduce from (3.10) that

\[
\prod_{p=1}^n d_{l,p+l}^2 = s_{l,l}^{-1} \frac{D_{l,l+n}}{D_{l+1,l+n}}
\]
so,

\[ \gamma_{l,n+l} = -\varphi_n(0,l) \sqrt{ \frac{D_{l,l+n}}{D_{l+1,l+n}} } . \]  

(3.11)

We can now relate this formula to the leading coefficients \(k_n^l\). From (3.8) we deduce that

\[ k_n^l = s_{l+n,l+n}^{-1/2} \prod_{p=1}^{n-1} \frac{1}{d_{l+p,l+n}}, \quad n \geq 1, \]

and using once again (3.10), we deduce

\[ k_n^l = \sqrt{ \frac{D_{l,l+n-1}}{D_{l,l+n}} }, \quad n \geq 1, \]

which concludes the proof.

\[ \square \]

3.4. Some examples. We consider some examples, especially in order to clarify the connection with classical orthogonal polynomials. Thus, consider first \(A = \{1 - X_1^+X_1\}\). In this case the index set is still \(\mathbb{N}_0\) and if \(\phi\) is a linear functional on \(R(A)\), then the kernel of moments is Toeplitz,

\[ K_{\phi}(n+m,k+m) = K_{\phi}(n,m), \quad m, n, k \in \mathbb{N}_0. \]

Let \(\phi\) be a strictly q-positive functional on \(R(A)\) and let \(\{\gamma_{k,j}\}\) be the parameters associated to \(K_{\phi}\). We deduce that these parameters also satisfy the Toeplitz condition,

\[ \gamma_{n+k,m+k} = \gamma_{n,m}, \quad n < m, k \geq 1. \]

Setting \(\gamma_n = \gamma_{k,n+k}, n \geq 1, k \geq 0,\) and \(d_n = (1 - |\gamma_n|^2)^{1/2}\), the recurrence relations (3.8), (3.9) collapse to the classical Szegö recursions obeyed by the orthogonal polynomials on the unit circle,

\[ \varphi_{n+1}(z) = \frac{1}{d_{n+1}}(z\varphi_n(z) - \gamma_{n+1}\varphi^\sharp_n(z)), \]

and

\[ \varphi^\sharp_{n+1}(z) = \frac{1}{d_{n+1}}(-\gamma_{n+1}z\varphi_n(z) + \varphi^\sharp_n(z)). \]

Therefore \(\gamma_n, n \geq 1\) are the usual Szegö coefficients. [32]

Another example is given by \(A = \{X_1 - X_1^+\}\). In this case the index set is still \(\mathbb{N}_0\) and the moment kernel of a strictly q-positive functional on \(R(A)\) will have the Hankel property,

\[ K_{\phi}(n+m,k+m) = K_{\phi}(n+k,m), \quad m, n, k \in \mathbb{N}_0. \]

Orthogonal polynomials associated to functionals on \(R(A)\) correspond to orthogonal polynomials on the real line. This time, the parameters \(\{\gamma_{k,j}\}\) associated to moment kernels have no classical analogue. Instead there are so-called canonical moments which are used as a counterpart of the Szegö coefficients (see [32]). Also, recurrence relations of type (3.8), (3.9) are replaced by a three term recurrence equation,

\[ x\varphi_n(x) = b_n\varphi_{n+1}(x) + a_n\varphi_n(x) + b_{n-1}\varphi_{n-1}(x), \]  

(3.12)

with initial conditions \(\varphi_{-1} = 0, \varphi_0 = 1\) ([32]). Definitely, these objects are more useful (for instance, it appears that no simple characterization of those \(\{\gamma_{k,j}\}\) corresponding to Hankel kernels is known). Still, computations involving the parameters \(\{\gamma_{k,j}\}\) might be...
of interest. For instance, we show here how to calculate the parameters for Gegenbauer polynomials. For a number \( \lambda > -\frac{1}{2} \), these are orthogonal polynomials associated to the weight function \( w(x) = B(\frac{1}{2}, \lambda + \frac{1}{2})^{-1}(1 - x^2)^{\lambda - \frac{1}{2}} \) on \((-1, 1)\) (\( B \) denotes the beta function). We use the normalization constants from \[21\], thus the Gegenbauer polynomials are

\[
P_n^{\lambda}(x) = \frac{(-1)^n}{2^n(\lambda + \frac{1}{2})_n} (1 - x^2)^{\frac{1}{2} - \lambda} \frac{d^n}{dx^n} (1 - x^2)^{n + \lambda - \frac{1}{2}},
\]

where \((x)_n\) is the Pochhammer symbol, \((x)_0 = 1\) and \((x)_n = \prod_{k=1}^n (x + k - 1)\) for \( n \geq 1 \). We have:

\[
h_n^{\lambda} = \frac{1}{B(\frac{1}{2}, \lambda + \frac{1}{2})} \int_{-1}^{1} (P_n^{\lambda}(x))^2 (1 - x^2)^{\lambda - \frac{1}{2}} dx = \frac{n!(n + 2\lambda)}{2(2\lambda + 1)_n(n + \lambda)}
\]

and the three term recurrence is:

\[
P_{n+1}^{\lambda}(x) = \frac{2(n + \lambda)}{n + 2\lambda} x P_n^{\lambda}(x) - \frac{n}{n + 2\lambda} P_{n-1}^{\lambda}(x)
\]

(see \[21\], Ch. 1). We now let \( \varphi_n^{\lambda}(x, 0) \) denote the orthonormal polynomials associated to the weight function \( w \), hence \( \varphi_n^{\lambda}(x, 0) = \frac{1}{h_n^{\lambda}} P_n^{\lambda}(x) \). From the three term relation we deduce

\[
\varphi_n^{\lambda}(0, 0) = (-1)^{n+1} \sqrt{\frac{2(2\lambda + 1)_n(n + \lambda)}{n!(n + 2\lambda)}} \times \prod_{k=1}^{n} \frac{k - 1}{k - 1 + 2\lambda},
\]

and also, the leading coefficient of \( \varphi_n^{\lambda}(x, 0) \) is

\[
k_n^{\lambda, 0} = \frac{(n + 2\lambda)_n}{2^n(\lambda + \frac{1}{2})_n} \sqrt{\frac{2(2\lambda + 1)_n(n + \lambda)}{n!(n + 2\lambda)}}.
\]

In order to compute the parameters \( \{\gamma_{k, l}^{\lambda}\} \) of the weight function \( w \) we use Theorem 3.3 Therefore we need to calculate the values \( \varphi_n^{\lambda}(0, l) \) and \( k_n^{\lambda, l} \), \( n \geq 1, l \geq 0 \) where \( k_n^{\lambda, l} \) denotes the leading coefficient of \( \varphi_n^{\lambda}(0, l) \). The main point for these calculations is to notice that \( \{\varphi_n^{\lambda}(x, l)\}_{n \geq 0} \) is the family of orthonormal polynomials associated to the weight function \( x^{2l} w(x) \). These polynomials are also classical objects and they can be found for instance in \[21\] under the name of modified classical polynomials. A calculation of the modified Gegenbauer polynomials can be obtained in terms of Jacobi polynomials. These are orthogonal polynomials associated to parameters \( \alpha, \beta > 1 \) and weight function

\[
2^{\alpha + \beta - 1} B(\alpha + 1, \beta + 1)^{-1}(1 - x)^{\alpha}(1 + x)^{\beta}
\]
on \((-1, 1)\) by the formula

\[
P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1 - x)^{-\alpha}(1 + x)^{-\beta} \frac{d^n}{dx^n} (1 - x)^{\alpha + n}(1 + x)^{\beta + n}.
\]

According to \[21\], Sect. 1.5.2, we have

\[
\varphi_{2n}^{\lambda}(x, l) = c_{2n} P_{12}^{\lambda - \frac{l}{2} - \frac{1}{2}}(2x^2 - 1)
\]
and
\[ \varphi_{2n+1}^\lambda(x, l) = c_{2n+1}x P_n^{\lambda - \frac{1}{2} - l + \frac{1}{2}}(2x^2 - 1), \]
where \( c_n \) is a constant that remains to be determined. But first we can already notice that the above formulae give \( \varphi_{2n+1}^\lambda(0, l) = 0 \), so that \( \gamma_{2n+1}^\lambda = 0 \).

**Theorem 3.4.** For \( n, l \geq 1 \),
\[ \varphi_{2n}^\lambda(0, l) = (-1)^{n+1} \sqrt{\frac{(\lambda + 1)_l}{(\lambda + 1)_n}} \frac{1}{n!} \prod_{k=1}^n \frac{\lambda + l + k - 1}{k} \]
and
\[ k_{2n}^\lambda = \frac{(\lambda + l)_{2n}}{(l + \frac{1}{2})_n n!} \sqrt{\frac{(\lambda + 1)_l}{(\lambda + 1)_n}} \frac{1}{n!} \prod_{k=1}^n \frac{\lambda + l + k - 1}{k}, \]
\[ k_{2n+1}^\lambda = \frac{(\lambda + l)_{2n+1}}{(l + \frac{1}{2})_{n+1} n!} \sqrt{\frac{(\lambda + 1)_l}{(\lambda + 1)_n}} \frac{1}{n!} \prod_{k=1}^n \frac{\lambda + l + k - 1}{k}, \]
where
\[ h_{2n}^\lambda = \frac{(\lambda + \frac{1}{2})_n (\lambda + l)_n (\lambda + l)}{n! (l + \frac{1}{2})_n (\lambda + l + 2n)} \]
and
\[ h_{2n+1}^\lambda = \frac{(\lambda + \frac{1}{2})_n (\lambda + l)_n (\lambda + l + 1)}{n! (l + \frac{1}{2})_{n+1} (\lambda + l + 2n + 1)}. \]

**Proof.** It is more convenient to introduce the polynomials
\[ C_{2n}^{\lambda,l}(x) = \frac{(\lambda + l)_n}{(l + \frac{1}{2})_n} P_n^{\lambda - \frac{1}{2} - l + \frac{1}{2}}(2x^2 - 1), \]
\[ C_{2n+1}^{\lambda,l}(x) = \frac{(\lambda + l)_{n+1}}{(l + \frac{1}{2})_{n+1} x} P_n^{\lambda - \frac{1}{2} - l + \frac{1}{2}}(2x^2 - 1), \]
and again by classical results that can be found in [21], we deduce
\[ 1 = \int_{-1}^{1} x^{2l} (\varphi_{2n}^\lambda(x, l))^2 w(x)dx \]
\[ = c_{2n}^2 \left( \frac{(l + \frac{1}{2})_n}{(\lambda + l)_n} \right)^2 B(l + \frac{1}{2}, \lambda + \frac{1}{2}) \int_{-1}^{1} x^{2l} \left( C_{2n}^{\lambda,l}(x) \right)^2 \frac{1}{B(l + \frac{1}{2}, \lambda + \frac{1}{2})} (1 - x^2)^{\lambda - \frac{1}{2}}dx \]
\[ = c_{2n}^2 \left( \frac{(l + \frac{1}{2})_n}{(\lambda + l)_n} \right)^2 B(l + \frac{1}{2}, \lambda + \frac{1}{2}) h_{2n}^{\lambda,l}, \]
where
\[ h_{2n}^{\lambda,l} = \frac{(\lambda + \frac{1}{2})_n (\lambda + l)_n (\lambda + l)}{n! (l + \frac{1}{2})_n (\lambda + l + 2n)}. \]
Using that \( \frac{B(l+\frac{1}{2}, \lambda + \frac{1}{2})}{B(\frac{1}{2}, \lambda + \frac{1}{2})} = (\frac{1}{2}) \), we deduce

\[
\varphi_{2n}(x, l) = \sqrt{\frac{(\lambda + 1)l}{(\frac{1}{2})_l h_{2n}}} \times C_{2n}(x).
\]

The calculation of \( \varphi_{2n}(0, l) \) reduces to the calculation of \( C_{2n}(0) \) which can be easily done due to the three term relation

\[
C_{2n+2}(x) = \frac{x}{n+1} C_{2n+1}(x) - \frac{\lambda + l + n}{n+1} C_{2n}(x).
\]

Thus we deduce

\[
C_{2n+2}(0) = -\frac{\lambda + l + n}{n+1} C_{2n}(0),
\]

and by iterating this relation and using that \( C_{0}(0) = 1 \), we get

\[
\varphi_{2n}(0, l) = (-1)^{n+1} \sqrt{\frac{(\lambda + 1)l}{(\frac{1}{2})_l h_{2n}}} \times \prod_{k=1}^{n} \frac{\lambda + l + k - 1}{k}.
\]

The leading coefficient of \( \varphi_{2n}(x, l) \) can be obtained from the corresponding formula in [21]. Thus,

\[
k_{2n}^{\lambda, l} = \frac{(\lambda + 1)_l (\lambda + l)_{2n}}{(\frac{1}{2})_l n! (\frac{1}{2})_l h_{2n}^\lambda}
\]

and

\[
k_{2n+1}^{\lambda, l} = \frac{(\lambda + 1)_l (\lambda + l)_{2n+1}}{(\frac{1}{2})_l n! (\frac{1}{2})_l h_{2n+1}^\lambda},
\]

where

\[
h_{2n+1}^\lambda = \frac{(\lambda + \frac{1}{2})_{n+1} (\lambda + l)_{n+1}}{n! (\frac{1}{2})_{n+1} (\lambda + l + 2n + 1)}.
\]

\( \square \)

Now the parameters \( \{\gamma_{k,j}^\lambda\} \) can be easily calculated by using Theorem 3.4. Of course, the explicit formulae look too complicated to be recorded here.

3.5. Asymptotic properties. In the classical setting of orthogonal polynomials on the unit circle there are several remarkable asymptotic results given by Szegö. Let \( \mu \) be a measure in the Szegö class, and let \( \{\varphi_n\}_{n \geq 0} \) be the family of orthonormal polynomials associated to \( \mu \). Then, the orthonormal polynomials have the following asymptotic properties:

(3.13) \( \varphi_n \to 0 \)

and

(3.14) \( \frac{1}{\varphi_n} \to \Theta_\mu \),
where \( \Theta_\mu \) is the spectral factor of \( \mu \) and the convergence is uniform on the compact subsets of the unit disk \( \mathbb{D} \). The second limit (3.14) is related to the so-called Szegő limit theorems concerning the asymptotic behaviour of Toeplitz determinants. Thus,

\[
\frac{\det T_n}{\det T_{n-1}} = \frac{1}{|\varphi_n^{\sharp}(0)|^2},
\]

where \( T_n = [s_{i-j}]_{i,j=0}^n \) and \( \{s_k\}_{k \in \mathbb{Z}} \) is the set of the Fourier coefficients of \( \mu \). As a consequence of the previous relation and (3.14) we deduce Szegő’s first limit theorem,

(3.15) \[
\lim_{n \to \infty} \frac{\det T_n}{\det T_{n-1}} = |\Theta_\mu(0)|^2 = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \mu'(t) dt\right).
\]

The second (strong) Szegő limit theorem improves (3.15) by showing that

(3.16) \[
\lim_{n \to \infty} \frac{\det T_n}{g^{n+1}(\mu)} = \exp\left(\frac{1}{\pi} \int_{|z| \leq 1} |\Theta'_\mu(z)/\Theta_\mu(z)|^2 d\sigma(z)\right),
\]

where \( g(\mu) \) is the limit in (3.15) and \( \sigma \) is the planar Lebesgue measure. These two limits (3.15) and (3.16) have an useful interpretation in terms of asymptotics of angles in the geometry of a stochastic process associated to \( \mu \) (see [25]) and many important applications. We show how these results can be extended to orthogonal polynomials on \( \mathcal{P}_1 \). The formulae (3.15) and (3.16) suggest that it is more convenient to work in a larger algebra. This is related to the so-called Toeplitz embedding, see [19], [23].

Thus, we consider the set \( \mathcal{L} \) of lower triangular arrays \( a = [a_{k,j}]_{k,j \geq 0} \) with complex entries. No boundedness assumption is made on these arrays. The addition in \( \mathcal{L} \) is defined by entry-wise addition and the multiplication is the matrix multiplication: for \( a = [a_{k,j}]_{k \geq j}, b = [b_{k,j}]_{k \geq j} \) two elements of \( \mathcal{L} \),

\[
(ab)_{k,j} = \sum_{l \geq 0} a_{k,l} b_{l,j},
\]

which is well-defined since the sum is finite. Thus, \( \mathcal{L} \) becomes an associative, unital algebra.

Next we associate the element \( \Phi_n \) of \( \mathcal{L} \) to the polynomials \( \varphi_n(X_1,l) = \sum_{k=0}^n a_{n,k} X_1^k, n,l \geq 0 \), by the formula

(3.17) \[
(\Phi_n)_{k,j} = \begin{cases} a_{n,k-j} & k \geq j \\ 0 & k < j \end{cases};
\]

similarly, the element \( \Phi_n^{\sharp} \) of \( \mathcal{L} \) is associated to the family of polynomials \( \varphi_n^{\sharp}(X_1,l) = \sum_{k=0}^n b_{n,k} X_1^k, n,l \geq 0 \), by the formula

(3.18) \[
(\Phi_n^{\sharp})_{k,j} = \begin{cases} b_{n,k-j} & k \geq j \\ 0 & k < j \end{cases}.
\]

We notice that the spectral factor \( \Theta_\phi \) of \( K_\phi \) is an element of \( \mathcal{L} \) and we assume that \( \Theta_\phi \) belongs to the Szegő class. This implies that \( \Phi_n^{\sharp} \) is invertible in \( \mathcal{L} \) for all \( n \geq 0 \). Finally, we say that a sequence \( \{a_n\} \subset \mathcal{L} \) converges to \( a \in \mathcal{L} \) if \( \{(a_{n,k,j})\} \) converges to \( a_{k,j} \) for all \( k,j \geq 0 \) (and we write \( a_n \to a \)).
Theorem 3.5. Let \( \phi \) belong to the Szegö class. Then

\[
\Phi_n \rightarrow 0
\]

and

\[
(\Phi_n^{\sharp})^{-1} \rightarrow \Theta_\phi.
\]

We now briefly discuss the geometric setting for the kernel \( K_\phi \). By a classical result of Kolmogorov (see [30]), \( K_\phi \) is the covariance kernel of a stochastic process \( \{f_n\}_{n \geq 0} \subset L^2(\mu) \) for some probability space \( (X, \mathcal{M}, \mu) \). That is,

\[
K_\phi(m, n) = \int_X f_n f_m \, d\mu.
\]

We can suppose, without loss of generality, that \( \{f_n\}_{n \geq 0} \) is total in \( L^2(\mu) \) and for \( p \leq q \) we introduce the subspaces \( E_{p,q} \) given by the closure in \( L^2(\mu) \) of the linear span of \( \{f_k\}_{k=p}^q \). The operator angle between two spaces \( E_1 \) and \( E_2 \) of \( L^2(\mu) \) is defined by

\[
B(E_1, E_2) = P_{E_1}P_{E_2}P_{E_1},
\]

where \( P_{E_1} \) is the orthogonal projection of \( L^2(\mu) \) onto \( E_1 \). Also define

\[
\Delta(E_1, E_2) = I - B(E_1, E_2).
\]

We associate to the process \( \{f_n\}_{n \geq 0} \) a family of subspaces \( H_{r,q} \) of \( L^2(\mu) \) such that \( H_{r,q} \) is the closure of the linear space generated by \( f_k, r \leq k \leq q \) and we consider a scale of limits:

\[
s - \lim_{r \to \infty} \Delta(H_{0,r}, H_{n+1,r}) = \Delta(H_{0,n}, H_{n+1,\infty})
\]

for \( n \geq 0 \), and then we let \( n \to \infty \) and deduce

\[
s - \lim_{n \to \infty} \Delta(H_{0,n}, H_{n+1,\infty}) = \Delta(H_{0,\infty}, \cap_{n \geq 0} H_n, \infty),
\]

where \( s - \lim \) denotes the strong operatorial limit.

We then deduce analogues of the Szegö limit theorems (3.15) and (3.16) by expressing the above limits of angles in terms of determinants. This is possible due to (3.10).

Theorem 3.6. Let \( \phi \) belong to the Szegö class. Then

\[
\frac{D_{r,q}}{D_{r+1,q}} = s_{r,r} \det \Delta(H_{r,r}, H_{r+1,q}) = \frac{1}{|\phi_{q-r}(0, r)|^2}
\]

and

\[
\lim_{q \to \infty} \frac{D_{r,q}}{D_{r+1,q}} = s_{r,r} \det \Delta(H_{r,r}, H_{r+1,\infty}) = |\Theta_\phi(r, r)|^2 = s_{r,r} \prod_{j \geq 1} d_{r,r+j}^2.
\]

If we denote the above limit by \( g_r \) and

\[
L = \lim_{n \to \infty} \prod_{0 \leq k < n < j} d_{k,j}^2 > 0,
\]
then

\[
\lim_{n \to \infty} \frac{D_{0,n}}{\prod_{l=0}^{n} g_l} = \frac{1}{\det \Delta(H_{0,\infty} \cap \cup_{n \geq 0} H_{n,\infty})} = \frac{1}{L}.
\]

Details of the proofs can be found in [8].

### 3.6. Szegő kernels.

The classical theory of orthogonal polynomials is intimately related to some classes of analytic functions. Much of this interplay is realized by Szegő kernels. Here we expand this idea by providing a Szegő type kernel for (a slight modification of) the space $H_0^2(\mathcal{F})$ which is viewed as an analogue of the Hardy class $H^2$ on the unit disk. We mention that another version of this idea was developed in [2] (and recently applied to a setting of stochastic processes indexed by vertices of homogeneous trees in [3]; see also [4], [6]). The difference is that $H_0^2(\mathcal{F})$ is larger than the space $U_2$ of [2] and also, the Szegő kernel that we consider is positive definite.

We return now to the space $H_0^2(\mathcal{F})$ introduced in Subsection 3.2. Its definition involves a family of Hilbertian conditions therefore its natural structure should be that of a Hilbert module (we use the terminology of [28]). Assume $\mathcal{F}_n = \mathbb{C}$ for all $n \geq 0$ and consider the $C^*$-algebra $D$ of bounded diagonal operators on $\oplus_{n \geq 0} \mathcal{F}_n = l^2(N_0)$. For a sequence $\{d_n\}_{n \geq 0}$ we use the notation

\[
\text{diag} \left( \{d_n\}_{n \geq 0} \right) = \begin{bmatrix}
    d_0 & 0 & \ldots & 0 \\
    0 & d_1 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & d_2
\end{bmatrix},
\]

so that $\text{diag} \left( \{d_n\}_{n \geq 0} \right)$ belongs to $D$ if and only if $\sup_{n \geq 0} |d_n| < \infty$. We now define the vector space

\[
H^2(\mathcal{F}) = \left\{ \Theta \in H_0^2(\mathcal{F}) \mid \text{diag} \left( \{c_n(\Theta) c_n(\Theta)^*\}_{n \geq 0} \right) \in D \right\}
\]

and notice that $D$ acts linearly on $H^2(\mathcal{F})$ by $\Theta D = [\Theta_{k,j} d_j]_{k,j \geq 0}$, therefore $H^2(\mathcal{F})$ is a right $D$-module. Also, if $\Theta, \Psi$ belong to $H^2(\mathcal{F})$, then $\text{diag} \left( \{c_n(\Psi) c_n(\Theta)^*\}_{n \geq 0} \right)$ belongs to $D$, which allows us to define

\[
\langle \Theta, \Psi \rangle = \text{diag} \left( \{c_n(\Psi) c_n(\Theta)^*\}_{n \geq 0} \right),
\]

turning $H^2(\mathcal{F})$ into a Hilbert $D$-module. As a Banach space with norm $\|\Theta\| = \|\langle \Theta, \Theta \rangle\|^{1/2}$, the space $H^2(\mathcal{F})$ coincides with $l^\infty(N_0, l^2(N_0))$, the Banach space of bounded sequences of elements in $l^2(N_0)$. A similar construction is used in [11] for a setting of orthogonalization with invertible squares.

Next we introduce a Szegő kernel for $H^2(\mathcal{F})$. Consider the set

\[
B_1 = \{ \{z_n\}_{n \geq 0} \subset \mathbb{C} \mid \sup_{n \geq 0} |z_n| < 1 \}
\]
and for \( z = \{z_n\}_{n \geq 0} \in B_1 \), \( \Theta \in \mathcal{H}^2(\mathcal{F}) \), notice that

\[
\Theta(z) = \text{diag} \left( \{\Theta_{n,n} + \sum_{k > n} \Theta_{k,n}z_k \ldots z_n\}_{n \geq 0} \right)
\]

is a well-defined element of \( \mathcal{D} \). Also, for \( z \in B_1 \), we define

\[
S_z = \begin{bmatrix}
1 & 0 & 0 & \ldots \\
\bar{z}_0 & 1 & 0 & \ldots \\
\bar{z}_0z_1 & \bar{z}_1 & 1 & \ldots \\
\bar{z}_0z_1z_2 & \bar{z}_1z_2 & \bar{z}_2 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

which is an element of \( \mathcal{H}^2(\mathcal{F}) \) and the Szegö kernel is defined on \( B_1 \) by the formula:

\[
S(z, w) = \langle S_w, S_z \rangle, \quad z, w \in B_1.
\]

**Theorem 3.7.** \( S \) is a positive definite kernel on \( B_1 \) with the properties:

1. \( \Theta(z) = \langle \theta, S_z \rangle, \quad \Theta \in \mathcal{H}^2(\mathcal{F}), \quad z \in B_1. \)
2. The set \( \{S_z D \mid z \in B_1, D \in \mathcal{D}\} \) is total in \( \mathcal{H}^2(\mathcal{F}) \).

**Proof.** Take \( z_1, \ldots, z_m \in B_1 \) and after reshuffling the matrix \( [S(z_j, z_i)]_{j,l=1}^m \) can be written in the form

\[
\oplus_{n \geq 0} \left[ c_n(S_{z_j})^* c_n(S_{z_i}) \right]_{j,l=1}^m.
\]

Since each matrix \( [c_n(S_{z_j})^* c_n(S_{z_i})]_{j,l=1}^m \) is positive, we conclude that \( S \) is a positive definite kernel on \( B_1 \). Property (1) of \( S \) follows directly from definitions. For (2) we use approximation theory in \( L^\infty \) spaces in order to reduce the proof to the following statement: for every element \( \Theta \in \mathcal{H}^2(\mathcal{F}) \) for which there exists \( A \subset \mathbb{N} \) such that \( c_n(\Theta) = h \in L^2(\mathbb{N}_0) \) for \( n \in A \) and \( c_n(\Theta) = 0 \) for \( n \notin A \), and for every \( \epsilon > 0 \), there exists a linear combination \( L \) of elements \( S_z D, \quad z \in B_1, \quad D \in \mathcal{D} \), such that \( \sup_{n \geq 0} \|c_n(\Theta - L)\| < \epsilon \). This can be achieved as follows. Since the set \( \{\phi_w(z) = \frac{1}{1 - wz} \mid w \in B_1\} \) is total in the Hardy space \( H^2 \) on the unit disk, we deduce that there exist complex numbers \( c_1, \ldots, c_m \) and \( w_1, \ldots, w_m \), \( |w_k| < 1 \) for all \( k = 1, \ldots, m \), such that

\[
\|h - \sum_{k=1}^m c_k \begin{bmatrix}
1\\
\frac{w_k}{w_k^2}\\
\vdots
\end{bmatrix}\| < \epsilon.
\]

Then define \( z_k = \{w_{k,n}\}_{n \geq 0} \) for \( k = 1, \ldots, m \), where \( w_{k,n} = w_k \) for all \( n \geq 0 \). So \( z_k \in B_1 \). Also define \( d_n^A = 1 \) for \( n \in A \) and \( d_n^A = 0 \) for \( n \notin A \), and consider

\[
L = \sum_{k=1}^m c_k S_{z_k} \text{diag} \left( \{d_n^A\} \right).
\]
We deduce that $L \in \mathcal{H}^2(\mathcal{F})$, $c_n(L) = 0$ for $n \notin A$, and $c_n(L) = \sum_{k=1}^m c_k \begin{bmatrix} 1 \\ \frac{w_k}{w_k} \\ \vdots \end{bmatrix}$ for $n \in A$, so that

$$\sup_{n \geq 0} \|c_n(\Theta - L)\| = \sup_{n \geq 0} \|c_n(\Theta) - c_n(L)\| = \max\{\sup_{n \in A} \|h - c_n(L)\|, \sup_{n \notin A} \|c_n(L)\|\}$$

$$= \|h - \sum_{k=1}^m c_k \begin{bmatrix} 1 \\ \frac{w_k}{w_k} \\ \vdots \end{bmatrix} \| < \epsilon.$$  

□

4. Several isometric variables

In this section we discuss orthogonal polynomials in several variables satisfying the isometric relations $X_k^+X_k = 1$, $k = 1, \ldots, N$. We set $\mathcal{A} = \{1 - X_k^+X_k \mid k = 1, \ldots, N\}$ and notice that the index set of $\mathcal{A}$ is $\mathbb{F}_N^+$. Also if $\phi$ is a linear functional on $\mathcal{R}(\mathcal{A})$ then its kernel of moments is invariant under the action of $\mathbb{F}_N^+$ on itself by juxtaposition, that is,

$$K(\sigma, \tau) = K(\phi(\sigma), \phi(\tau)), \quad \tau, \sigma, \sigma' \in \mathbb{F}_N^+.$$  

In fact, a kernel $K$ satisfies (4.1) if and only if $K = K_\phi$ for some linear functional on $\mathcal{R}(\mathcal{A})$. Positive definite kernels satisfying (4.1) have been already studied, see for instance [9] and references therein. In particular, the class of isotropic processes on homogeneous trees give rise to positive definite kernels for which a theory of orthogonal polynomials (Levinson recursions) was developed in [10]. Here we discuss in more details another class of kernels satisfying (4.1) which was considered, for instance, in [24].

4.1. Cuntz-Toeplitz relations. Consider the class of positive definite kernels with property (4.1) and such that

$$K(\sigma, \tau) = 0 \quad \text{if there is no } \alpha \in \mathbb{F}_N^+ \text{ such that } \sigma = \tau \alpha \text{ or } \tau = \sigma \alpha.$$  

We showed in [17] that $K$ has properties (4.1) and (4.2) if and only if $K = K_\phi$ for some q-positive functional on $\mathcal{R}(\mathcal{A}_{CT})$, where $\mathcal{A}_{CT} = \{1 - X_k^+X_k \mid k = 1, \ldots, N\} \cup \{X_k^+X_l, k, l = 1, \ldots, N, k \neq l\}$. The relations in $\mathcal{A}_{CT}$ are defining the Cuntz-Toeplitz algebra (see [22] for details). The property (4.2) shows that $K$ is quite sparse, therefore it is expected to be easy to analyse such a kernel. Still, there are some interesting aspects related to this class of kernels, some of which we discuss here.
Let $\phi$ be a strictly $q$-positive kernel on $\mathcal{R}(\mathcal{A}_{CT})$ and let $K_\phi$ be the associated kernel of moments. Since the index set of $\mathcal{A}_{CT}$ is still $\mathbb{F}_N^+$, and this is totally ordered by the lexicographic order, we can use the results described in Subsection 3.1 and associate to $K_\phi$ a family $\{\gamma_{\sigma,\tau}\}_{\sigma<\tau}$ of complex numbers with $|\gamma_{\sigma,\tau}| < 1$, uniquely determining $K_\phi$ by relations of type (3.4). It was noticed in [17] that $K_\phi$ has properties (4.1) and (4.2) if and only if $\gamma_{\sigma,\sigma'} = \gamma_{\sigma,\sigma}$ and $\gamma_{\sigma,\tau} = 0$ if there is no $\alpha \in \mathbb{F}_N^+$ such that $\sigma = \tau \alpha$ or $\tau = \sigma \alpha$. The main consequence of these relations is that $K_\phi$ is uniquely determined by $\gamma_{\sigma} = \gamma_{0,\sigma}$, $\sigma \in \mathbb{F}_N^+ - \{0\}$. We define $d_\sigma = (1 - |\gamma_{\sigma}|^2)^{1/2}$. The orthogonal polynomials associated to $\phi$ satisfy the following recurrence relations (see [8] for details): $\varphi_0 = \varphi^*_0 = s^{-1/2}_0$ and for $k \in \{1, \ldots, N\}$, $\sigma \in \mathbb{F}_N^+$,

\begin{align}
\varphi_{k\sigma} &= \frac{1}{d_{k\sigma}} (X_k \varphi_{\sigma} - \gamma_{k\sigma} \varphi^*_{k\sigma - 1}), \\
\varphi^*_{k\sigma} &= \frac{1}{d_{k\sigma}} (-\gamma_{k\sigma} X_k \varphi_{\sigma} + \varphi^*_{k\sigma - 1}).
\end{align}

The results corresponding to Theorem 3.5 and Theorem 3.6 can be easily obtained (see [8]), but the constructions around the Szegö kernel are more interesting in this situation. Thus, there is only one Hilbertian condition involved in the definition of $\mathcal{H}_0(\mathcal{F})$ in this case. In fact, it is easy to see that $\mathcal{H}_0(\mathcal{F})$ can be identified with the full Fock space $l^2(\mathbb{F}_N^+)$, the $l^2$ space over $\mathbb{F}_N^+$. Now, concerning evaluation of elements of $\mathcal{H}_0(\mathcal{F})$, if we are going to be consistent with the point of view that the “points for evaluation” come from the unital homomorphisms of the polynomial algebra inside $\mathcal{H}_0(\mathcal{F})$, then we have to consider an infinite dimensional Hilbert space $\mathcal{E}$ and the set

$$B_1(\mathcal{E}) = \{Z = (Z_1, \ldots, Z_N) \in \mathcal{L}(\mathcal{E})^N \mid \sum_{k=1}^N Z_k Z_k^* < I\}.$$ 

For $\sigma = i_1 \ldots i_k \in \mathbb{F}_N^+$ we write $Z_\sigma$ instead of $Z_{i_1} \ldots Z_{i_k}$. Then we define for $\Theta \in l^2(\mathbb{F}_N^+) \otimes \mathcal{E}$ and $Z \in B_1(\mathcal{E})$,

$$\Theta(Z) = \sum_{\sigma \in \mathbb{F}_N^+} Z_\sigma \Theta_\sigma,$$

which is an element of the set $\mathcal{L}(\mathcal{E})$ of bounded linear operators on the Hilbert space $\mathcal{E}$. Next, for $Z \in B_1(\mathcal{E})$ we define $S_Z : \mathcal{E} \to l^2(\mathbb{F}_N^+) \otimes \mathcal{E}$ by the formula:

$$S_Z f = \sum_{\sigma \in \mathbb{F}_N^+} e_\sigma \otimes (Z_\sigma)^* f, \quad f \in \mathcal{E}.$$

Then $S_Z \in \mathcal{L}(\mathcal{E}, l^2(\mathbb{F}_N^+) \otimes \mathcal{E})$ and we can finally introduce the Szegö kernel on $B_1(\mathcal{E})$ by the formula:

$$S(Z, W) = S_Z S_W, \quad Z, W \in B_1(\mathcal{E}).$$

**Theorem 4.1.** $S$ is a positive definite kernel on $B_1(\mathcal{E})$ with the properties:

1. $\Theta(z) = S_Z^* \Theta$, $\Theta \in l^2(\mathbb{F}_N^+) \otimes \mathcal{E}, Z \in B_1(\mathcal{E})$.
2. The set $\{S_Z f \mid Z \in B_1(\mathcal{E}), f \in \mathcal{E}\}$ is total in $l^2(\mathbb{F}_N^+) \otimes \mathcal{E}$. 

---

Note: The text above is a transcription of the content from the image, ensuring it is readable and formatted appropriately for this platform. However, due to the complexity of the mathematical content, some equations and symbols have not been rendered perfectly. The goal was to maintain the essence and structure of the text as closely as possible.
Proof. The fact that \( S \) is positive definite and (1) are immediate. More interesting is (2) and we reproduce here the proof given in \[17\]. Let \( f = \{f_\sigma\}_{\sigma \in \mathbb{F}_N^+} \) be an element of \( l^2(\mathbb{F}_N^+) \otimes \mathcal{E} \) orthogonal to the linear space generated by \( \{S_Z f \mid Z \in B_1(\mathcal{E}), f \in \mathcal{E}\} \).

Taking \( Z = 0 \), we deduce that \( f_\emptyset = 0 \). Next, we claim that for each \( \sigma \in \mathbb{F}_N^+ - \{\emptyset\} \) there exist

\[
Z_l = (Z_{l}^{1}, \ldots, Z_{l}^{N}) \in B_1(\mathcal{E}), \quad l = 1, \ldots, 2|\sigma|,
\]

such that

\[
\text{range } [Z_{\sigma}^{1} \ldots Z_{\sigma}^{2|\sigma|}] = \mathcal{E},
\]

and

\[
Z_{\tau}^{l} = 0 \quad \text{for all } \quad \tau \neq \sigma, \quad |\tau| \geq |\sigma|, \quad l = 1, \ldots, 2|\sigma|.
\]

Once this claim is proved, a simple inductive argument gives \( f = 0 \). In order to prove the claim we need the following construction. Let \( \{e_{ij}^{n}\}_{i,j=1}^{n} \) be the matrix units of the algebra \( M_n \) of \( n \times n \) matrices. Each \( e_{ij}^{n} \) is an \( n \times n \) matrix consisting of 1 in the \( (i, j) \)th entry and zeros elsewhere. For a Hilbert space \( \mathcal{E} \) we define \( E_{ij}^{n} = e_{ij}^{n} \otimes I_{\mathcal{E}} \) and we notice that \( E_{ij}^{n} E_{kl}^{n} = \delta_{jk} E_{il}^{n} \) and \( E_{ji}^{n} = E_{ij}^{n} \). Let \( \sigma = i_1 \ldots i_k \) so that \( \mathcal{E} = \mathcal{E}_{1}^{\oplus |\sigma|} \) for some Hilbert space \( \mathcal{E}_{1} \) (here we use in an essential way the assumption that \( \mathcal{E} \) is of infinite dimension). Also, for \( s = 1, \ldots, N \), we define \( J_s = \{l \in \{1, \ldots, k\} \mid i_{k+1-l} = s\} \) and

\[
Z_{s}^{*p} = \frac{1}{\sqrt{2}} \sum_{r \in J_s} E_{r+p-1,r+p}^{2|\sigma|}, \quad s = 1, \ldots, N, \quad p = 1, \ldots, |\sigma|.
\]

We can show that for each \( p \in \{1, \ldots, |\sigma|\} \),

\[
Z_{\sigma}^{*p} = \frac{1}{\sqrt{2^k}} E_{p,k+p}^{2|\sigma|}, \quad (4.5)
\]

\[
Z_{\tau}^{p} = 0 \quad \text{for } \quad \tau \neq \sigma, \quad |\tau| \geq |\sigma|. \quad (4.6)
\]

We deduce

\[
\sum_{s=1}^{N} Z_{s}^{p} Z_{s}^{*p} = \frac{1}{2} \sum_{s=1}^{N} \sum_{r \in J_s} E_{r+p,r+p}^{2|\sigma|} E_{r+p-1,r+p}^{2|\sigma|} = \frac{1}{2} \sum_{s=1}^{N} \sum_{r \in J_s} E_{r+p,r+p}^{2|\sigma|} = \frac{1}{2} \sum_{s=1}^{k} E_{r+p,r+p}^{2|\sigma|} < I,
\]

hence \( Z_{s}^{p} \in B_1(\mathcal{E}) \) for each \( p = 1, \ldots, |\sigma| \). For each word \( \tau = j_1 \ldots j_k \in \mathbb{F}_N^+ - \{\emptyset\} \) we deduce by induction that

\[
Z_{j_k}^{*p} \ldots Z_{j_1}^{*p} = \frac{1}{\sqrt{2^k}} \sum_{r \in A_r} E_{r+p-1,r+p+k-1}^{2|\sigma|}, \quad (4.7)
\]

where \( A_r = \cap_{p=0}^{k-1} (J_{j_{k-p}} - p) \subset \{1, \ldots, N\} \) and \( J_{j_{k-p}} - p = \{l - p \mid l \in J_{j_{k-p}}\} \).

We show that \( A_\sigma = \{1\} \) and \( A_\tau = \emptyset \) for \( \tau \neq \sigma \). Let \( q \in A_\tau \). Therefore, for any \( p \in \{0, \ldots, k-1\} \) we must have \( q + p \in J_{j_{k-p}} \) or \( i_{k+1-q-p} = j_{k-p} \). For \( p = k-1 \) we deduce \( j_1 = i_{2-q} \) and since \( 2 - q \geq 1 \), it follows that \( q \leq 1 \). Also \( q \geq 1 \), therefore the only
element that can be in $A_\tau$ is $q = 1$, in which case we must have $\tau = \sigma$. Since $l \in J_{k+1-l}$ for each $l = 1, \ldots, k-1$, hence $A_\sigma = \{1\}$ and $A_\tau = \emptyset$ for $\tau \neq \sigma$. Formula (4.7) implies (4.5). In a similar manner we can construct a family $Z^p$, $p = |\sigma| + 1, \ldots, 2|\sigma|$, such that

$$Z^{p\sigma}_\sigma = \frac{1}{\sqrt{2^k}} E^{2|\sigma|}_{p+k,p},$$

and

$$Z^p_\tau = 0 \quad \text{for} \quad \tau \neq \sigma, \quad |\tau| \geq |\sigma|.$$

Thus, for $s = 1, \ldots, N$, we define $K_s = \{ l \in \{1, \ldots, k\} \mid i_k = s \}$ and

$$Z^{sp}_s = \frac{1}{\sqrt{2^k}} \sum_{r \in K_s} E^{2|\sigma|}_{r+p-k,r+p-k-1}, \quad s = 1, \ldots, N, \quad p = |\sigma| + 1, \ldots, 2|\sigma|.$$

Now,

$$\begin{bmatrix} Z^{s1}_\sigma & \ldots & Z^{s2|\sigma|}_\sigma \end{bmatrix} = \frac{1}{\sqrt{2^k}} \begin{bmatrix} E^{2|\sigma|}_{1,k+1} & \ldots & E^{2|\sigma|}_{k,2k} & \ldots & E^{2|\sigma|}_{2k,k} \end{bmatrix},$$

whose range is $E$. This concludes the proof. \hfill \square

It is worth noticing that property (2) of $S$ is no longer true if $E$ is finite dimensional. In fact, for $E$ of dimension one the set $\{ S_Z f \mid Z \in B_1(E), f \in E \}$ is total in the symmetric Fock space of $\mathbb{C}^N$ (see [5]).

4.2. Kolmogorov decompositions and Cuntz relations. This is a short detour from orthogonal polynomials, in order to show a construction of bounded operators satisfying the Cuntz-Toeplitz and Cuntz relations, based on parameters $\{\gamma_{\sigma}\}_{\sigma \in \mathbb{F}_N^+ \setminus \{\emptyset\}}$ associated to a positive definite kernel with properties (4.1) and (4.2).

First we deal with the Kolmogorov decomposition of a positive definite kernel. This is a more abstract version of the result of Kolmogorov already alluded to in Subsection 3.5. For a presentation of the general result and some applications, see [22], [30]. Here we consider $K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ a positive definite kernel and let $\{\gamma_{k,j}\}$ be the family of parameters associated to $K$ as in Subsection 3.1. In addition we assume $K(j,j) = 1$ for $j \geq 0$. This is not a real loss of generality and it simplifies some calculations. We also assume $|\gamma_{k,j}| < 1$ for all $k < j$. Then we introduce for $0 \leq k < j$ the operator $V_{k,j}$ on $l^2(\mathbb{N}_0)$ defined by the formula

$$V_{k,j} = (J(\gamma_{k,k+1}) \oplus 1_{n-1})(1 \oplus J(\gamma_{k,k+2}) \oplus 1_{n-2}) \ldots (1_{n-1} \oplus J(\gamma_{k,j})) \oplus 0$$

and we notice that

$$W_k = s - \lim_{j \to \infty} V_{k,j}$$

is a well-defined isometric operator on $l^2(\mathbb{N}_0)$ for every $k \geq 0$. If we define $V(0) = I/\mathbb{C}$ and $V(k) = W_0 W_1 \ldots W_{k-1}/\mathbb{C}$ for $k \geq 1$, then we obtain the following result from [13].
Theorem 4.2. The map $V : \mathbb{N}_0 \rightarrow l^2(\mathbb{N}_0)$ is the Kolmogorov decomposition of the kernel $K_j$ in the sense that

1. $K(j, l) = \langle V(l), V(j) \rangle$, $j, l \in \mathbb{N}_0$.
2. The set $\{V(k) \mid k \in \mathbb{N}_0\}$ is total in $l^2(\mathbb{N}_0)$.

It is worth noticing that we can write explicitly the matrix of $W_k$:

$$
\begin{bmatrix}
\gamma_{k,k+1} & d_{k,k+1} & d_{k,k+2} & \cdots \\
\gamma_{k,k+1} & d_{k,k+2} & \cdots \\
0 & \gamma_{k,k+2} & \cdots \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
\end{bmatrix}
$$

We also see that Theorem 3.7 and Theorem 4.1 produce various kinds of Kolmogorov decompositions for the corresponding Szegö kernels. Based on a remark in [12], we use Theorem 4.2 in order to obtain some large families of bounded operators satisfying Cuntz-Toeplitz and Cuntz relations. Thus, we begin with a positive definite kernel $K$ with properties (4.1) and (4.2). For simplicity we also assume $K(\emptyset, \emptyset) = 1$ and let $\{\gamma_{\sigma}\}_{\sigma \in F_N^+ - \{\emptyset\}}$ be the family of corresponding parameters. In order to be in tune with the setting of this paper, we assume $|\gamma_{\sigma}| < 1$ for all $\sigma \in F_N^+ - \{\emptyset\}$. Motivated by the construction in Theorem 4.2 we denote by $c_{\sigma}(W_0)$, $\sigma \in F_N^+ - \{\emptyset\}$, the columns of the operator $W_0$. Thus,

$$
c_1(W_0) = \begin{bmatrix}
\gamma_1 \\
d_1 \\
0 \\
\vdots
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
d_1 \ldots d_{\sigma-1} \gamma_{\sigma} \\
-\gamma_1 d_2 \ldots d_{\sigma-1} \gamma_{\sigma} \\
\vdots \\
-\gamma_{\sigma-1} \gamma_{\sigma} \\
d_{\sigma} \\
0 \\
\vdots
\end{bmatrix}
\quad \text{for} \quad 1 < \sigma.
$$

We now define the isometry $U(k)$ on $l^2(\mathbb{F}_N^+)$ by the formula:

$$
U(k) = [c_{k\tau}(W_0)]_{\tau \in \mathbb{F}_N^+}, \quad k = 1, \ldots, N.
$$

Theorem 4.3. (a) The family $\{U_1, \ldots, U_N\}$ satisfies the Cuntz-Toeplitz relations: $U_k^*U_l = \delta_{k,l}I$, $k, l = 1, \ldots, N$.

(b) The family $\{U_1, \ldots, U_N\}$ satisfies the Cuntz relations $U_k^*U_l = \delta_{k,l}I$, $k, l = 1, \ldots, N$ and $\sum_{k=1}^N U_kU_k^* = I$, if and only if

$$
\prod_{\sigma \in \mathbb{F}_N^+ - \{\emptyset\}} d_{\sigma} = 0.
$$
Proof. (a) follows from the fact that $W_0$ is an isometry. In order to prove (b) we need a characterization of those $W_0$ which are unitary. Using Proposition 1.4.5 in [16], we deduce that $W_0$ is unitary if and only if (4.9) holds. □

It is worth mentioning that if we define $V(\emptyset) = I/\mathbb{C}$ and $V(\sigma) = U(\sigma)/\mathbb{C}$, $\sigma \in \mathbb{F}_N^+ - \{\emptyset\}$, where $U(\sigma) = U(i_1) \ldots U(i_k)$ provided that $\sigma = i_1 \ldots i_k$, then $V$ is the Kolmogorov decomposition of the kernel $K$. A second remark here is that the condition (4.9) is exactly the opposite of the condition for $K$ being in the Szegő class. Indeed, it is easily seen that for a positive definite kernel with properties (4.1) and (4.2), the condition (3.6) is equivalent with $\prod_{\sigma \in \mathbb{F}_N^+ - \{\emptyset\}} d_\sigma > 0$.

5. Several hermitian variables

The theory corresponding to this case should be an analogue of the theory of orthogonal polynomials on the real line. We set $A = \{Y_k - Y_k^+ \mid k = 1, \ldots, N\}$ and $A' = A \cup \{Y_k Y_l - Y_l Y_k \mid k, l = 1, \ldots, N\}$, and notice that $\mathcal{R}(A) = \mathcal{P}_N$. Also, $\mathcal{R}(A')$ is isomorphic to the symmetric algebra over $\mathbb{C}^N$. Orthogonal polynomials associated to $\mathcal{R}(A')$, that is, orthogonal polynomials in several commuting variables were studied intensively in recent years, see [21]. In this section we analyse the noncommutative case. The presentation follows [15].

Let $\phi$ be a strictly $q$-positive functional on $\mathcal{T}_N(A_2)$ and assume for some simplicity that $\phi$ is unital, $\phi(1) = 1$. The index set of $A$ is $\mathbb{F}_N^+$ and let $\{\varphi_\sigma\}_{\sigma \in \mathbb{F}_N^+}$ be the orthonormal polynomials associated to $\phi$. We notice that for any $P, Q \in \mathcal{P}_N$,

$$\langle X_k P, Q \rangle_\phi = \phi(Q^+ X_k P)$$

(5.1)

$$= \phi(Q^+ X_k^+ P)$$

$$= \langle P, X_k Q \rangle_\phi,$$

which implies that the kernel of moments satisfies the relation $s_{\alpha \sigma, \tau} = s_{\sigma, I(\alpha) \tau}$ for $\alpha, \sigma, \tau \in \mathbb{F}_N^+$, where $I$ denotes the involution on $\mathbb{F}_N^+$ given by $I(i_1 \ldots i_k) = i_k \ldots i_1$. This can be viewed as a Hankel type condition, and we already noticed that even in the one dimensional case the parameters $\{\gamma_{k,j}\}$ of the kernel of moments of a Hankel type are more difficult to be used. Therefore, we try to deduce three-terms relations for the orthonormal polynomials. A matrix-vector notation already used in the commutative case, turns out to be quite useful. Thus, for $n \geq 0$, we define $P_n = [\varphi_\sigma]_{\sigma \in \mathbb{F}_N^+, \vert \sigma \vert = n}$, $n \geq 0$, and $P_{-1} = 0$.

**Theorem 5.1.** There exist matrices $A_{n,k}$ and $B_{n,k}$ such that

$$X_k P_n = P_{n+1} B_{n,k} + P_n A_{n,k} + P_{n-1} B_{n-1,k}, \quad k = 1, \ldots, N, n \geq 0.$$  

(5.2)

Each matrix $A_{n,k}$ is a selfadjoint $N^n \times N^n$ matrix, while each $B_{n,k}$ is an $n^{n+1} \times N^n$ matrix such that

$$B_n = [B_{n,1} \ldots B_{n,N}].$$
is an upper triangular invertible matrix for every \( n \geq 0 \). For \( n = -1, B_{-1,k} = 0, k = 1, \ldots, N \). The fact that \( B_n \) is upper triangular comes from the order that we use on \( \mathbb{F}_N^+ \). The invertibility of \( B \) is a consequence of the fact that \( \phi \) is strictly \( q \)-positive and appears to be a basic translation of this information. It turns out that there are no other restrictions on the matrices \( A_{n,k}, B_{n,k} \) as shown by the following Favard type result.

**Theorem 5.2.** Let \( \varphi_\sigma = \sum_{\tau \leq \sigma} a_{\sigma,\tau} X_\tau, \sigma \in \mathbb{F}_N^+ \), be elements in \( \mathcal{P}_N \) such that \( \varphi_\emptyset = 1 \) and \( a_{\sigma,\sigma} > 0 \). Assume that there exists a family \( \{A_{n,k}, B_{n,k} \mid n \geq 0, k = 1, \ldots, N\} \), of matrices such that \( A_{n,k}^* = A_{n,k} \) and \( B_n = [B_{n,1} \ldots B_{n,N}] \) is an upper triangular invertible matrix for every \( n \geq 0 \). Also assume that

\[
(5.3) \quad X_k [\varphi_\sigma]_{|\sigma|=n} = [\varphi_\sigma]_{|\sigma|=n+1} B_{n,k} + [\varphi_\sigma]_{|\sigma|=n} A_{n,k} + [\varphi_\sigma]_{|\sigma|=n-1} B_{n-1,k}^*, k = 1, \ldots, N, n \geq 0,
\]

where \( [\varphi_\sigma]_{|\sigma|=0} = 0 \) and \( B_{-1,k} = 0 \) for \( k = 1, \ldots, N \). Then there exists a unique strictly positive functional \( \phi \) on \( \mathcal{R}(\mathcal{A}) \) such that \( \{\varphi_\sigma\}_{\sigma \in \mathbb{F}_N^+} \) is the family of orthonormal polynomials associated to \( \phi \).

There is a family of Jacobi matrices associated to the three-term relation in the following way. For \( P \in \mathcal{R}(\mathcal{A})(= \mathcal{P}_N) \), define

\[
(5.4) \quad \Psi_\phi(P)\varphi_\sigma = P\varphi_\sigma.
\]

Since the kernel of moments has the Hankel type structure mentioned above, it follows that each \( \Psi_\phi(P) \) is a symmetric operator on the Hilbert space \( \mathcal{H}_\phi \) with dense domain \( \mathcal{D} \), the linear space generated by the polynomials \( \varphi_\sigma, \sigma \in \mathbb{F}_N^+ \). Moreover, for \( P, Q \in \mathcal{P}_N \),

\[
\Psi_\phi(PQ) = \Psi_\phi(P)\Psi_\phi(Q),
\]

and \( \Psi_\phi(P)\mathcal{D} \subset \mathcal{D} \), hence \( \Psi_\phi \) is an unbounded representation of \( \mathcal{P}_n \) (the GNS representation associated to \( \phi \)). Also, \( \phi(P) = (\Psi_\phi(P^1,1,1) \phi \) for \( P \in \mathcal{P}_N \). We distinguish the operators \( \Psi_k = \Psi_\phi(Y_k), k = 1, \ldots, N \), since \( \Psi_\phi(\sum_{\sigma \in \mathbb{F}_N^+} c_\sigma Y_\sigma) = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma \Psi_\phi(\sigma) \), where we use the notation \( \Psi_{\phi,\sigma} = \Psi_{e_1, \ldots, e_i} \) for \( \sigma = e_1 \ldots e_i \). Let \( \{e_1, \ldots, e_N\} \) be the standard basis of \( \mathbb{C}^N \) and define the unitary operator \( W \) from \( l^2(\mathbb{F}_N^+) \) onto \( \mathcal{H}_\phi \) such that \( W(e_\sigma) = \varphi_\sigma, \sigma \in \mathbb{F}_N^+ \). We see that \( W^{-1}\mathcal{D} \) is the linear space \( \mathcal{D}_0 \) generated by \( e_\sigma, \sigma \in \mathbb{F}_N^+ \), so that we can define

\[
J_k = W^{-1}\Psi_{\phi,k}W, \quad k = 1, \ldots, N,
\]
on $\mathcal{D}_0$. Each $J_k$ is a symmetric operator on $\mathcal{D}_0$ and by Theorem 5.1, the matrix of (the closure of) $J_k$ with respect to the orthonormal basis $\{e_\sigma\}_{\sigma \in F_N^+}$ is

$$J_k = \begin{bmatrix} A_{0,k} & B_{0,k}^* & 0 & \cdots \\ B_{0,k} & A_{1,k} & B_{1,k}^* \\ 0 & B_{1,k} & A_{2,k} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$ 

We call $(J_1, \ldots, J_N)$ a Jacobi $N$-family on $\mathcal{D}_0$. It is somewhat unexpected that the usual conditions on $A_{n,k}$ and $B_{n,k}$ insure a joint model of a Jacobi family in the following sense.

**Theorem 5.3.** Let $(J_1, \ldots, J_N)$ a Jacobi $N$-family on $\mathcal{D}_0$ such that $A_{n,k}^* = A_{n,k}$ and $B_n = \begin{bmatrix} B_{n,1} & \cdots & B_{n,N} \end{bmatrix}$ is an upper triangular invertible matrix for every $n \geq 0$. Then there exists a unique strictly $q$-positive functional $\phi$ on $\mathcal{P}_N$ with associated orthonormal polynomials $\{\varphi_\sigma\}_{\sigma \in F_N^+}$ such that the map

$$W(e_\sigma) = \varphi_\sigma, \quad \sigma \in F_N^+,$$

extends to a unitary operator from $l^2(F_N^+)$ onto $\mathcal{H}_\phi$ and

$$J_k = W^{-1}\Psi_{\phi,k}W, \quad k = 1, \ldots, N.$$ 

**Proof.** First the Favard type Theorem 5.2 gives a unique strictly $q$-positive functional $\phi$ on $\mathcal{P}_N$ such that its orthonormal polynomials satisfy the three-term relation associated to the given Jacobi family, and then the GNS construction will produce the required $W$ and $\Psi_{\phi,k}$, as explained above. \qed

One possible application of these families of Jacobi matrices involves some classes of random walks on $F_N^+$. Figure 4 illustrates an example for $N = 2$ and more details are planned to be presented in [7].

![Figure 4](image.png)

**Figure 4.** Random walks associated to a Jacobi family, $N = 2$

We conclude our discussion of orthogonal polynomials on hermitian variables by introducing a Szegő kernel that should be related to orthogonal polynomials on $\mathcal{P}_N$. 

26
Thus, we consider the Siegel upper half-space of a Hilbert space $\mathcal{E}$ by

$$H_{+}(\mathcal{E}) = \{(W_{1} \ldots W_{N}) \in \mathcal{L}(\mathcal{E})^{N} \mid W_{1}W_{1}^{*} + \ldots + W_{N-1}W_{N-1}^{*} < \frac{1}{2k}(W_{N} - W_{N}^{*})\}. $$

We can establish a connection between $B_{1}(\mathcal{E})$ and $H_{+}(\mathcal{E})$ similar to the well-known connection between the unit disk and the upper half plane of the complex plane. Thus, we define the Cayley transform by the formula

$$C(Z) = ((I + Z_{N})^{-1}Z_{1}, \ldots, (I + Z_{N})^{-1}Z_{N-1}, i(I + Z_{N})^{-1}(I - Z_{N})), $$

which is well-defined for $Z = (Z_{1}, \ldots, Z_{N}) \in B_{1}(\mathcal{E})$ since $Z_{k}$ must be a strict contraction ($\|Z_{k}\| < 1$) for every $k = 1, \ldots, N$. In addition, $C$ establishes a one-to-one correspondence from $B_{1}(\mathcal{E})$ onto $H_{+}(\mathcal{E})$. The Szegő kernel on $B_{1}(\mathcal{E})$ can be transported on $H_{+}(\mathcal{E})$ by the Cayley transform. Thus, we introduce the Szegő kernel on $H_{+}(\mathcal{E})$ by the formula:

$$S(W, W') = F_{W}^{*}F_{W'}, \quad W, W' \in H_{+}(\mathcal{E}), $$

where $F_{W} = 2\text{diag}((-i + W_{N}^{*})S_{C^{-1}}(W))$. Much more remains to be done in this direction. For instance, some classes of orthogonal polynomials of Jacobi type and their generating functions are considered in [7].

Finally, we mention that there are examples of polynomial relations for which there are no orthogonal polynomials. Thus, consider

$$\mathcal{A} = \{X_{k}^{+} - X_{k} \mid k = 1, \ldots, N\} \cup \{X_{k}X_{l} + X_{l}X_{k} \mid k, l = 1, \ldots, 2N\}, $$

then $\mathcal{R}(\mathcal{A}) \simeq \Lambda(\mathbb{C}^{N})$, the exterior algebra over $\mathbb{C}^{N}$. If $\phi$ is a unital q-positive definite functional on $\mathcal{R}(\mathcal{A})$, then $\phi(X_{k}^{2}) = 0$ for $k = 1, \ldots, N$. This and the q-positivity of $\phi$ force $\phi(X_{k}) = 0$, therefore there is only one q-positive functional on $\mathcal{R}(\mathcal{A})$ which is not strictly q-positive. Therefore there is no theory of orthogonal polynomials over this $\mathcal{A}$. However, the situation is different for $\mathcal{A} = \{X_{k}X_{l} + X_{l}X_{k} \mid k, l = 1, \ldots, 2N\}$. This and other polynomial relations will be analysed elsewhere.

**References**

[1] J. Agler and J. E. McCarthy *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, Vol. 44, Amer. Math. Soc., Providence, Rhode Island, 2002.

[2] D. Alpay, P. Dewilde and H. Dym, Lossless inverse scattering and reproducing kernels for upper triangular operators, in Operator Theory: Advances and Applications, Vol. 47, Birkhäuser, pp. 61-135, 1990.

[3] D. Alpay and D. Volok, Point evaluation and Hardy space on an homogeneous tree, lanl, OA/0309262.

[4] W. Arveson, Interpolation problems in nest algebras, *J. Funct. Anal.*, 3(1975), 208-233.

[5] W. Arveson, Subalgebras of $C^{*}$-algebras. III: Multivariable operator theory, *Acta. Math.*, 181(1998),476-514.

[6] J. A. Ball and I. Gohberg, A commutant lifting theorem for triangular matrices with diverse applications, *Integral Equations Operator Theory*, 8(1985), 205-267.

[7] T. Banks and T. Constantinescu, Orthogonal polynomials in several non-commuting variables. II, in preparation.

[8] M. Barakat and T. Constantinescu, Tensor Algebras and Displacement Structure. III. Asymptotic properties, lanl, FA/0306051.
[9] M. Basseville, A. Benveniste, K. C. Chou, S. A. Golden, R. Nikoukhah, and A. S. Willsky, Modeling and estimation of multiresolution stochastic processes, *IEEE Trans. Inform. Theory*, **38**(1992), 766-784.

[10] M. Basseville, A. Benveniste, and A. S. Willsky, Multiscale autoregressive Processes, Part I: Schur-Levinson parametrizations, *IEEE Trans. Signal Processing*, **40**(1992), 1915-1934; Part II: Lattice structures for whitening and modeling, *IEEE Trans. Signal Processing*, **40**(1992), 1935-1954.

[11] A. Ben-Artzi and I. Gohberg, Orthogonal polynomials over Hilbert modules, in Operator Theory: Advances and Applications, Vol. 73, Birkhäuser, pp. 96-126, 1994.

[12] T. Constantinescu, Modeling of time-variant linear systems, INCREST preprint No.60/1985.

[13] T. Constantinescu, Schur analysis of positive block matrices, in Operator Theory: Advances and Applications, Vol. 18, Birkhäuser, pp. 191-206, 1986.

[14] T. Constantinescu, Factorization of positive-definite kernels, in Operator Theory: Advances and Applications, Vol. 48, Birkhäuser, pp. 245-260, 1990.

[15] T. Constantinescu, Orthogonal polynomials in several non-commuting variables. I, lanl, FA/020533; to appear in I. Colojoara Anniversary Volume.

[16] T. Constantinescu, *Schur Parameters, Factorization and Dilation Problems*, Birkhäuser, 1996.

[17] T. Constantinescu and J. L. Johnson, Tensor algebras and displacement structure. II. Non-commutative Szegő theory, *Zeit. für Anal. Anw.*, **21**(2002), 611–626.

[18] T. Constantinescu and A. Gheondea, Representations of Hermitian kernels by means of Krein spaces, *Publ. RIMS*, **33**(1997), 917-951; II. Invariant kernels, *Commun. Math. Phys.*, **216**(2001), 409-430.

[19] Ph. Delsarte, Y. Genin and Y. Kamp, On the Toeplitz embedding of arbitrary matrices, *Linear Algebra Appl.*, **51**(1983), 97-119.

[20] V. Drensky, *Free Algebras and PI-Algebras*, Springer, 1999.

[21] C. H. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Cambridge Univ. Press, 2001.

[22] D. E. Evans and Y. Kawahigashi, *Quantum symmetris on operator algebras*, Oxford Mathematical Monographs, Clarendon Press, 1998.

[23] C. Foias, A. E. Frazho, I. Gohberg and M. A. Kaashoek, *Metric Constrained Interpolation, Commutant Lifting and Systems*, Birkhäuser, 1998.

[24] A. E. Frazho, On stochastic bilinear systems, in *Modeling and Applications of Stochastic Processes* (U.B.Desai, Ed.), pp. 215–241, Kluwer Academic, 1988.

[25] U. Grenander and G. Szegő, *Toeplitz Forms and their Applications*, Univ. of California Press, California, 1958.

[26] W. W. Hart and W. L. Hart, *Plane Trigonometry, Solid Geometry, and Spherical Trigonometry*, D. C. Heath and Co., 1942.

[27] P. Henrici, *Applied and computational complex analysis. Volume 1: Power series, integration, conformal mapping, location of zeros*, Wiley-Interscience, 1974.

[28] E. C. Lance, *Hilbert C*-modules*, Cambridge University Press, 1995.

[29] H. Li, *Noncommutative Gröbner Bases and Filtered-Graded Transfer*, LNM 1795, Springer, 2002.

[30] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, 1992.

[31] M. Rosenblum and J. Rosnyak, *Hardy Classes and Operator Theory*, Oxford Univ. Press, 1985.

[32] G. Szegő, *Orthogonal Polynomials*, Colloquium Publications, **23**, Amer. Math. Soc., Providence, Rhode Island, 1939.

[33] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, 1970.

Department of Mathematics, University of Texas at Dallas, Richardson, TX 75083

E-mail address: banks@utdallas.edu
Department of Mathematics, University of Texas at Dallas, Richardson, TX 75083
E-mail address: tiberiu@utdallas.edu

Department of Mathematics and Computer Science, Wagner College, Staten Island, NY 10301
E-mail address: joeljohn@wagner.edu