Complex variable solution for mode-III quadratically varying PS model in piezoelectric media

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Abstract. Complex variable technique and extended Stroh formalism approach are applied to develop the closed-form solutions for mode-III quadratically varying polarization saturation (PS) model in an infinite piezoelectric media. A semipermeable centre crack problem is considered for study under the influence of out of plane far-field mechanical loading and in-plane electrical displacement loading. Similar to PS model, the electrical yielding zones or saturated zones are considered just infront of the crack-tips in the form of a strip. But in place of constant PS condition imposed on yielding zones, the quadratically varying PS condition is considered for study. Mathematically, this problem reduces into simultaneous non-homogeneous Riemann-Hilbert problems which are solved by finding the solutions of involved singular integrals. Hence, some standard fracture parameters are derived in explicit forms such as saturated zone lengths, crack sliding displacement, crack potential drop and local stress intensity factor. Also, the comparative studies for quadratically varying PS model and PS model are presented graphically.

Keywords : Complex variable, Center Crack, Piezoelectric, Polarization Saturation, Local Stress Intensity Factor.

1. Introduction

Complex variable methods are the most elegant mathematical techniques to solve the engineering and allied sciences problems in explicit forms. Muskhelishvili \cite{1} presented the complex representation of general solutions of the equations of the plane theory of elasticity and developed complex variable solutions for many engineering problems such as infinite plane with circular problems, problems of half-plane and for the plane with straight cuts, torsion of bars, and etc. He \cite{2} had also provided the solutions to many cracks problems using Riemann-Hilbert approach and singular integrals. England \cite{3} presented the detailed solutions for cracks problems, half plane problems, regions with circular and curvilinear boundaries using complex variable methods. Theocaris \cite{4} studied the two unequal collinear cracks problem in elastic-perfectly plastic infinite plane based on Dugdales model and employing complex variable technique. Collins and Cartwright \cite{5} derived the analytical solutions for strip yield two equal collinear cracks problems based on complex variable and singular integrals. Bhargava and Agrawal \cite{6} implemented

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the complex variable technique for studying the quadratically varying strip yield model in elastic perfectly plastic infinite plane having two equal collinear cracks with coalesced zones case. Hasan [7, 8] developed the analytical solutions for two pairs of collinear cracks with coalesced yield zones and four collinear cracks with coalesced yield zones based on modified Dugdales model and using complex variable technique. Explicit expressions for some standard fracture parameters were derived. This technique has also been extended to study the cracks problems in piezoelectric materials. In analogues to Dugdales model, Gao et al. [9] proposed the PS model in piezoelectric domain and obtained the closed form solutions for saturated zone length and local energy release rate using complex variable method. Ru [10] presented the complex variable methodology for finding the analytical solution of generalized case of PS model in piezoelectric media. A generalized saturated condition imposed on the electrical yielding zone was considered for the study. Bhargava and Jangid [11, 12] extended the complex variable approach to study strip saturation models for two collinear cracks problems in semipermeable piezoelectric media subjected to mode-I case. Singh et al. [13] derived the new analytical solutions for modified PS models in piezoelectric media using complex variable and Stroh formalism approach. A mode-I centre crack problem was considered for the studies with polynomial types of variable saturation conditions. Singh et al. [14] also extended their modified PS models to two equal collinear cracks problems in semipermeable piezoelectric media. Explicit forms of expressions were derived for fracture parameters using complex variable approach.

In literature, fracture mechanics problems were discussed not only for mode-I case (means under tensile loading) but also for mode-III case (out of plane loading). Zhang and Hack [15] applied complex variable approach to study the mode-III cracks in piezoelectric materials. Li and Kardomateas [16] studied the mode-III interface crack in piezo-electro-magneto-elastic dissimilar materials. Closed form solutions were derived for impermeable and permeable conditions implementing complex variable technique and extended Stroh formalism. Wang and Gao [17] presented the analytical solution for a mode III fracture problem of edge cracks originating from a circular hole in an infinite piezoelectric solid based on complex variable method combined with the method of conformal mapping. Kim et al. [18] obtained a semi-analytical solution via an integro-differential equation of first kind and studied the contribution of surface elasticity to the anti-plane deformations of a linearly elastic bi-material with Mode-III interface crack. Zhang and Kim [19] addressed the asymptotic full crack-tip fields for a Mode-III stationary crack in an anisotropic functionally graded material using complex variable technique. A moving mode-III crack at the interface between two dissimilar piezoelectric materials was discussed by Li et al. [20]. Xia and Zhong [21] analyzed a mode III Yoffe-type interfacial crack propagating sub-sonically under the moving strip magneto-electric saturation model. Bhargava and Verma [22] applied the Fourier transform technique to study the modified strip-yield-saturation-induction model solution for mode-III cracked piezo-electromagnetic plate. Bhargava et al. [23] extended their methodology of complex variable to study a mode-III strip saturation model for two collinear semipermeable cracks in piezoelectric media. Recently, Verma and Verma [24] applied the Fourier series approach to study the poling effects on mode-III strip-saturated two equal collinear cracks problem.

In order to get insight into the modified PS models in piezoelectric materials, the mode-III quadratically varying PS model in center crack problem is investigated here by using complex variable technique. This paper is organized as follows.

Section 2 provides some basic fundamental equations for piezoelectric media. The mathematical formulation of the problem and its closed-form solutions are provided in Section 3. The derivation of some standard fracture parameters are derived in Section 4. In Section 5, numerical studies are presented graphically for the obtained fracture parameters with respect to variations in electrical loading, polarization angle and crack-face conditions. The studies based upon de-
rived analytical solutions are concluded in Section 6.

2. Basic Equations and Crack-face Conditions

The well known plane problem, displacement components \( u_i(i = x, y, z) \) are given as
\[
\begin{align*}
u_x(x, y, z) &= 0, \\
u_y(x, y, z) &= 0, \\
u_z(x, y, z) &= w(x, y)
\end{align*}
\]
and for in-plane electric field problem, the electric field component \( E_i(x, y, z) \) may be defined as
\[
\begin{align*}
E_x(x, y, z) &= -\phi_x, \\
e_y(x, y, z) &= -\phi_y
\end{align*}
\]
where \( \phi \) denotes the electric potential.

Gradient equations
\[
\varepsilon_{zi} = w, \quad \text{and} \quad E_i = -\phi, \quad \text{where} \quad i = x, y
\] (1)

Constitutive equations
\[
\sigma_{ij} = c_{44}w_i + e_{15}\phi_i, \quad D_i = k_{11}\phi_i
\] (2)

where \( c_{44}, e_{15}, \) and \( k_{11} \), denoted for the elastic constants, the piezoelectric constants, and dielectric constants, respectively.

Equilibrium equations
The equilibrium equations under the absence of body force and body charge are as
\[
\sigma_{ij,j} = 0, \quad D_{i,i} = 0
\] (3)

Crack face boundary conditions
The following three crack-face boundary conditions are mainly available in literature[13-15] which can be described based on the permittivity constant \( \kappa_c \) of the medium between crack faces. Mathematically, all are expressed as
\[
\sigma_{ij}n_j = 0, \quad D^+ = D^- = -k_c\frac{\phi(x)}{\Delta w(x)}
\] (4)

where the indices ”+” and ”-” represent the upper and lower crack surfaces, \( \Delta w(x) \) and \( \Delta \phi(x) \) are the crack sliding displacement and electrical potential jump respectively.

The impermeable, semipermeable and permeable conditions can be defined by considering \( \kappa_c = 0, \kappa_c = 8.85 \times 10^{-12} Fm^{-1} \) i.e. the permittivity of air and \( \kappa_c = 10000 \times 8.85 \times 10^{-12} Fm^{-1} \).

3. Mathematical Formulation of Mode-III Quadratically Varying PS Model

An infinite isotropic piezoelectric domain with arbitrary poling direction and occupying the \( xy \)-plane is considered. The domain is cut along straight crack with crack length equals to \( 2a \) occupying the intervals \([-a, a]\) on x-axis. Now in analogues to PS model, the proposed model has also considered under the assumption that the electrical yielding zones (saturated zones) develop just infront of the crack-tips in the form of a strip subject to the influence of far-field out of plane mechanical loading \( \sigma_{23} = \tau_0 \) and in-plane electric displacement \( D^\infty \). The saturated zones are represented on the intervals \( a < |x| < c \) along x-axis. The domain is studied subjected to arbitrary poling direction and impermeable & semipermeable crack-face boundary conditions. The geometrical configuration of the problem is shown in Fig. 1.

The physical boundary conditions of the problem can be mathematically written as
\[
(i) \quad \sigma_{2j}^+ = \sigma_{2j}^-, \quad D_2^+ = D_2^-, \quad \Phi_{1}^+ = \Phi_{1}^-, \quad -B = -[0, 0, \tau_0, D^\infty - D_2^0], \quad |x| < a
\]
Figure 1: Schematic representation of the quadratically varying PS model

(ii) \( \sigma_{ij} = 0, \quad D_2 = 0 \quad \text{for } |y| \to \infty \)

(iii) \( \Phi_1^+ = \Phi_1^-, \quad u_j^+ = u_j^-, \quad D_2^+ = D_2^- = (\frac{c^2}{c^2} D_s - D_2^0) - (D_2^\infty - D_2^0), \quad a < |x| < c \)

The general solution of Eq.(1) to Eq.(3) may be expressed in terms of generalized stress function \( \Phi \) and generalized displacement vector \( u = [w, \phi]^T \) such that

\[
\begin{align*}
u_{,1} &= SF(z) + \bar{SF}(z) \quad (5) \\
\Phi_{,1} &= MF(z) + \bar{MF}(z) \quad (6)
\end{align*}
\]

where \( z = (x + iy), \quad F(z) = \frac{df(z)}{dz}, \quad f(z) \) being an analytic function and \( S, M \) stand for material constant matrices and defined as \( S = I \) is the identity matrix of \( 2 \times 2 \) and

\[
M = i \begin{bmatrix} c_{44}^* & c_{15}^* \\ c_{15}^* & -k_{11}^* \end{bmatrix}
\]

. The transformed material constants \( (C^*, e^*, k^*) \) are obtained with the help of transformed matrices and the generalized material constants matrices \( (C, e, k) \). The details of these matrices can be found in [13]. Considering the continuity of \( \Phi_{,1}(x) \) on the whole real axis and Eq. (6), one can write as

\[
\begin{align*}
MF^+(x) + \bar{MF}^+(x) &= MF^-(x) + \bar{MF}^-(x), \quad -\infty < x < \infty, \\
[MF^+(x) - MF^-(x)] - [\bar{MF}^-(x) - \bar{MF}^+(x)] &= 0, \quad -\infty < x < \infty. \quad (7a)
\end{align*}
\]

According to Musklishvili[2], the solution of Eq.(7) may be written as:

\[
MF(z) - \bar{MF}(z) = 0 \quad (8)
\]

Let \( l(z) = MF(z) = \bar{MF}(z) \)
where \( F(z) \) is the complex function of the in-homogeneous field. Appying the boundary condition (1) in Eq. (6), we have

\[
l^+(x) + l^-(x) = -B, \quad |x| < a
\]  

(10)

Introducing new complex function as \( W(z) = [W_3(z), W_4(z)] \)

\[
W(z) = K^R MF(z)
\]  

(11)

where \( K^R = 2Re[Y] \), \( Y = iSM^{-1} \).

Eq. (10) can be express in its component as:

\[
\Omega_{11}[W_3^+(x) + W_3^-(x)] + \Omega_{12}[W_4^+(x) + W_4^-(x)] = -\tau_0, \quad |x| < a
\]  

(13)

\[
\Omega_{21}[W_3^+(x) + W_3^-(x)] + \Omega_{22}[W_4^+(x) + W_4^-(x)] = D_\sigma^2 - D_\Omega^2, \quad |x| < a
\]  

(14)

Solving Eqs. (13) and (14), we have

\[
\Omega_\sigma[W_3^+(x) + W_3^-(x)] = -T_3^\infty, \quad |x| < a
\]  

(15)

where \( \Omega_\sigma = \Omega_{12}\Omega_{21} - \Omega_{22}\Omega_{11}, \quad T_3^\infty = (\Omega_{12}(D_\sigma^\infty - D_\Omega^2) - \Omega_{22}\tau_0) \).

Using singlevaluedness condition of the mechanical displacement, the solution of above Eq. (15) is

\[
\Omega_\sigma W_3(z) = \frac{T_3^\infty}{2} \left[ \frac{z}{\sqrt{z^2 - a^2}} - 1 \right]
\]  

(16)

Now from Eq. (14)

\[
[W_4^+(x) + W_4^-(x)] = -\left( \frac{\Omega_{21}}{\Omega_{22}}[W_3^+(x) + W_3^-(x)] + \frac{D_\sigma^\infty - D_\Omega^2}{\Omega_{22}} \right), \quad |x| < a
\]  

(17)

Applying the boundary condition (3) in Eq. (14), one can write

\[
[W_3^+(x) + W_3^-(x)] = -\left( \frac{\Omega_{21}}{\Omega_{22}}[W_3^+(x) + W_3^-(x)] + \frac{D_\sigma^\infty - D_\Omega^2}{\Omega_{22}} \right) + \frac{\frac{x^2}{\sigma^2}D_s - D_\sigma^2}{\Omega_{22}}, \quad a < |x| < c
\]  

(18)

Solution of Eqs. (17) and (18) after considering the condition of single valuedness of electrical potential is given as:

\[
W_4(z) = -\frac{D_s z g_3(z, c)}{2\pi \Omega_{22}\sqrt{z^2 - c^2}} - \frac{1}{\pi \Omega_{22}} D_\sigma^2 g_2(z, c) + \frac{(D_\sigma^\infty - D_\Omega^2)}{2\Omega_{22}} \left( \frac{z}{\sqrt{z^2 - c^2}} - 1 \right) - \frac{\Omega_{21}}{\Omega_{22}} W_3(z),
\]  

(19)

where 

\[
g_2(z, c) = \cos^{-1} \frac{a}{c} - \frac{1}{2} \sqrt{z^2 - c^2} \tan^{-1} \left( \frac{\sqrt{z^2 - c^2}}{a} \right), \quad g_3(z, c) = \frac{a}{c} \sqrt{1 - \frac{a^2}{c^2} - \left( 1 - \frac{2z^2}{c^2} \right) \cos^{-1} \frac{a}{c} - \frac{2z^2}{c^2} \tan^{-1} \left( \frac{\sqrt{z^2 - a^2}}{z} \right)}.
\]
4. Analytical Solutions of the Fracture Parameters

We express various applications of the problem by finding saturation zone lengths, CSD, CPD and local stress intensity factor.

The jump displacement vector $\Delta u$ is defined as

$$i \Delta u_1 = DR[MF^+(x) - MF^-(x)] = W^+(x) - W^-(x) \tag{20}$$

4.1. Saturation Zone

The electric displacement is obtained from Eq. (6) as

$$\Phi_{i,1} = MF^+(x) - MF^-(x) = \Omega[W^+(x) - W^-(x)], \quad |x| > c \tag{21}$$

Taking the second component of above equation, we have

$$D_2(x) = \frac{Dc_1(x,c)}{2\pi \Omega_2 \sqrt{x^2 - c^2}} - 2Dc_2(x,c) + \frac{(Dc^\infty_c - Dc^\infty_2)}{2\Omega_2} \left(\frac{x}{\sqrt{x^2 - c^2}} - 1\right) \tag{22}$$

Now, considering the finiteness condition of electric displacement at the extended crack-tip ($c$), the saturation zones length $c - a$ can be evaluated from the following equation:

$$\frac{c}{a} = Sec(\pi \frac{Dc^\infty_c - Dc^\infty_2}{Dc_c - 2Dc^\infty_c}) - \frac{a}{c} \sqrt{1 - \frac{a^2}{c^2} \frac{Dc_c}{(Dc_c - 2Dc^\infty_c)}} \tag{23}$$

From the above equation, one can get the critical value of applied electrical loading. It means the maximum electrical loading that one could apply in the proposed PS model. This could be evaluated by considering $\frac{c}{a} \rightarrow \infty$ in the above Eq. (24).

$$\Rightarrow \frac{Dc^\infty_c}{Dc_c} = 0.5. \tag{25}$$

4.2. Crack Sliding Displacement (CSD)

Crack sliding displacement is the relative crack face opening between the two surfaces of the crack. It can be used as a measure of the toughness of the materials under mode-III deformation. CSD, $\Delta w$ can be determined after substituting $W_3(z)$ from Eq. (16) into Eq. (20) and then integrating the obtained expression as

$$\Delta w_{m,1} = i[W_3^+ - W_3^-] \tag{26}$$

$$\Delta w_m(x) = \frac{1}{\pi} (\tau_0 - (Dc^\infty_c - Dc^\infty_2) \frac{\Omega_1 \Omega_2}{\Omega_2}) \sqrt{a^2 - x^2} \tag{27}$$

4.3. Crack Potential Drop (CPD)

The relative crack potential drop of the crack faces $\Delta \varphi(x)$ can be evaluated after substituting $W_4(z)$ from Eq. (19) into Eq. (20) and then integrating, we have
Fig. 2 shows the variation in normalized saturation zone length (displacement is taken as $D$ material constants used for the study are given in Table 1. The saturated value for electric equals to $\tau$ for the study. The domain is under the influence of far-field out-of-plane mechanical loading solutions for mode-III quadratically varying PS model in piezoelectric media. Explicit In this paper, complex variable technique is successfully implemented to obtain the analytical results shown in Fig. 3 clearly depict the effects of variation in saturation condition on CPD. However, due to the basic definitions of CSD and LSIF, they are independent of the polarization PS model.

5. Numerical Studies:
This section presents the numerical studies for mode-III quadratically varying PS model subjected to variations in electrical loading, polarization angle and crack-face conditions. A center crack problem of half-crack length $a$ is considered for the study. The domain is under the influence of far-field out-of-plane mechanical loading equals to $\tau_0 = 10MPa$ and in-plane electrical loading $D_2^\infty = 0.06c/m^2$ (if not defined). The material constants used for the study are given in Table 1. The saturated value for electric displacement is taken as $D_s = 0.3c/m^2$ for the studies.

Fig. 2 shows the variation in normalized saturation zone length ($\frac{c-a}{a}$) with increasing $D_2^\infty$ evaluated for proposed quadratically varying PS model and PS model. The results are plotted for two cases of polarization angles i.e. $\alpha = 0^\circ$ and $45^\circ$ subjected to impermeable and semipermeable crack-face conditions. Effects of crack-face conditions and polarization angle are found similar to the mode-I case as established in [13]. It is also observed that the saturation zone length obtained under proposed quadratically varying PS model has higher values than PS model irrespective of the crack-face condition and polarization angle. This is also in agreement with the outcomes of the study of [13].

Results shown in Fig. 3 clearly depict the effects of variation in saturation condition on CPD. Here also, higher numerical values of CPD are obtained for proposed PS model than in case of PS model. However, due to the basic definitions of CSD and LSIF, they are independent of the polarization saturation condition. But both the fracture parameters are significantly dependent on the crack-face conditions and polarization angle. These can be seen from the results plotted in Figs. 4 and 5.

6. Conclusions:
In this paper, complex variable technique is successfully implemented to obtain the analytical solutions for mode-III quadratically varying PS model in piezoelectric media. Explicit

$$\Delta \Phi(x) = \begin{cases} \frac{D_s}{3c^2\pi^{12}} (a\sqrt{(c^2 - a^2)(c^2 - x^2)} + (c^2 - x^2)^{3/2} \cos^{-1}\left(\frac{x}{c}\right)) + a^3 \csc^{-1}\left(\frac{\sqrt{c^2 - x^2}}{\sqrt{c^2 - a^2}}\right), \\ -x^3 \csc^{-1}\left(\frac{\sqrt{c^2 - x^2}}{x\sqrt{c^2 - a^2}}\right)), \quad \text{if } |x| \leq a \\ -\frac{2D_s}{\pi^{12}} (a \coth^{-1}\left(\frac{\sqrt{c^2 - x^2}}{x\sqrt{c^2 - a^2}}\right) - x \csc^{-1}\left(\frac{a\sqrt{c^2 - x^2}}{x\sqrt{c^2 - a^2}}\right)) - \Delta w_m(x), \quad \text{if } a \leq |x| \leq c \\ \end{cases}$$

$$4.4. \text{Local Stress Intensity Factor (LSIF):}$$
The local stress intensity factor (LSIF), $K^\tau(a)$ is determined at the actual crack-tip $x = a$ as $K^\tau(a) = \lim_{x \to a^+} \sqrt{2\pi(x - a)\sigma_{23}(x)}$

$$= (\tau_0 - (D_2^\infty - D_2^\infty)\frac{\Omega_{12}}{\Omega_{22}})\sqrt{a\pi}$$

$$29$$
Figure 2: Variations in saturation zone lengths w.r.t $D_2^\infty$ subjected to polarization angle $\phi = 0$ & $\phi = 45^0$ and impermeable and semipermeable crack face boundary conditions.

Figure 3: Variations in CPD w.r.t. $(c - x)/(c - a)$ under different poling directions and impermeable & semipermeable crack face boundary conditions.

Figure 4: Variations in CSD w.r.t. $(a - x)/a$ under different poling directions and impermeable & semipermeable crack face boundary conditions.
Table 1: The material constants of PZT-4H used for the analysis

| Properties          | PZT-4H                        |
|---------------------|-------------------------------|
| Elastic constants   | $c_{11} = 126\text{GPa}$, $c_{12} = 55\text{GPa}$, $c_{13} = 53\text{GPa}$, $c_{33} = 117\text{GPa}$, $c_{44} = 35.3\text{GPa}$ |
| Piezoelectric       | $e_{15} = 17\text{C/m}^2$, $e_{31} = -6.5\text{C/m}^2$, $e_{33} = 23.3\text{C/m}^2$ |
| Permittivity        | $k_{11} = 15.1\text{nC/Vm}$, $k_{33} = 13.1\text{nC/Vm}$ |

expressions for saturated zone length, CSD, CPD and LSIF are mathematically derived for the proposed model. The numerical studies presented in Section 5 concludes that the saturated zone length and CPD significantly dependent on the saturated condition whereas LSIF and CSD are independent of this condition. Moreover, it is observed that for a particular electrical loading the saturated zone length and CPD have higher numerical values obtained for quadratically varying PS model than in comparison to PS model. Also, the critical value of applied electrical loading obtained here is $0.5D_s$ which is same to the mode-I crack solution. Since, the applied electrical loading in the proposed model cannot be greater than $0.5D_s$ therefore this study can also be considered as a study on new crack arrest model in piezoelectric media. The studies also show that although the explicit expressions for the fracture parameters obtained for mode-III and mode-I cracks problems are different but the effects of electrical loading, poling direction and crack-face conditions on these parameters are the same.

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