Integrable spin-boson models descending from rational six-vertex models

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We construct commuting transfer matrices for models describing the interaction between a single quantum spin and a single bosonic mode using the quantum inverse scattering framework. The transfer matrices are obtained from certain inhomogeneous rational vertex models combining bosonic and spin representations of $SU(2)$, subject to non-diagonal toroidal and open boundary conditions. Only open boundary conditions are found to lead to integrable Hamiltonians combining both rotating and counter-rotating terms in the interaction. If the boundary matrices can be brought to triangular form simultaneously, the spectrum of the model can be obtained by means of the algebraic Bethe ansatz after a suitable gauge transformation; the corresponding Hamiltonians are found to be non-hermitian. Alternatively, a certain quasi-classical limit of the transfer matrix is considered where hermitian Hamiltonians are obtained as members of a family of commuting operators; their diagonalization, however, remains an unsolved problem.

I. INTRODUCTION

Models describing the interaction between bosonic modes and spin degrees of freedom play an important role in physics. The prototype application is provided by atom-radiation interactions [1], where the atomic dipole is described by an effective spin interacting with a single mode electric field [2, 3]. More recently, this class of models has been considered to investigate various aspects in mesoscopic physics where a given two-level system interacts with a bosonic environment [4]. Furthermore, we mention applications to analyze the decoherence in superconducting circuits [5, 6], and to describe ions in harmonic traps [7]. Both these quantum devices are of potential importance for quantum computation [8, 9]. Finally, applications have emerged in quasi-2D semiconductors in transverse magnetic field [10, 11, 12].

In this paper we focus on models describing a single spin $S$ interacting with a single mode boson field which were introduced originally by Jaynes, Tavis and Cummings [2, 3]. In two cases the corresponding model Hamiltonian is known to be exactly diagonalizable with elementary means: for $S \to \infty$ where the model is quadratic [13], and for weak or resonant interactions, where the Rotating Wave Approximation (RWA) can be applied [2, 3, 14]. In the traditional framework of laser physics, the RWA assumes that the relevant dynamics is given by coherent oscillations of the population of the atomic energy levels. Therefore only photon emissions accompanied by an excitation of the atom
(and vice versa), described by operators of the type \(a^\dagger S^-\), are taken into account in the atomic dipole coupling with electric field \(\vec{E} \cdot \vec{d} \sim (a + a^\dagger)S^x\). The so-called counter-rotating terms \(a^\dagger S^+ + a S^-\), although contained in the coupling, can be neglected at resonant cavity frequency \(\Omega_0\) since they induce a short time dynamics (compared to the rotating terms \(a S^+ + a^\dagger S^-\)); as a result of the approximation the operator \(S^z + a^\dagger a\) is conserved. Such a picture is robust for weak fields and as long as the cavity mode can be tuned conveniently. In the most recent applications, however, one or even both these conditions are not met. In superconducting circuits, for example, the effective spin-boson coupling to the electromagnetic field of the device can be naturally large, thereby involving multi-boson processes, which are characteristic for the presence of counter rotating terms [15]; furthermore numerical inspections indicate that such terms can bend the energy levels whose slope should be controlled reliably to study how the noise affects the coherence in the system [16]. Finally we mention that in semiconductors, rotating and counter rotating terms result from Rashba and Dresselhaus spin-orbit interactions; the physical origin of the spin transport in such systems is an open problem that is intensively studied [12].

For finite spin representations and generic spin-boson interaction the Hamiltonian is a non-trivial matrix of infinite rank. The main complication comes from the absence of a conserved “number operator” of the type \(S^z + a^\dagger a\) as in the RWA that would have made the problem finite dimensional – \(a^\dagger a\) is the bosonic number operator. In this sense, using the language of spin chains, the problem will have the complexity of an XYZ chain. We will speak of a spin-boson model to contain rotating terms only, if the Hamiltonian action only connects a subspace of finite dimension. It is worth noting that it is the simultaneous presence of so called rotating, \(S^\dagger a\), and counter-rotating terms \(S^\dagger a^\dagger\) that makes the Hamiltonian non-trivial in the sense that a reduction to a finite dimensional subspace is impossible.

In this work we study integrable spin-boson models that can be constructed by means of the quantum inverse scattering method (QISM) [17]. The central object is the transfer matrix, that it is a generating function of integrals of the motion, including the Hamiltonian by definition. The formalism includes a certain freedom to choose (twisted) periodic boundary conditions and open boundary conditions, where two boundary matrices can be included, representing boundary magnetic fields in the standard formalism for spin chains [18, 19]. Twisted boundary conditions have been investigated for XXX-symmetry, i.e. for models emerging from the XXX \(R\)-matrix in [22] and hermitian spin boson models have been created that still have a conserved number operator, indicating that the apparent counter-rotating terms are spurious. In the present work, open boundary conditions for XXX-type symmetry are analyzed, where two non-parallel boundary matrices are meant to break the XXX symmetry completely. Integrable open boundaries, corresponding to dynamic boundary matrices (i.e. depending on the spectral parameter) obey the reflection equation due to Sklyanin [19]. Two approaches are used. First, a bosonic Lax matrix, relative to the XXX \(R\)-matrix is employed and the QISM and algebraic Bethe ansatz is elaborated for this case. Second, the spin-boson Hamiltonian is obtained from a two-site spin chain with mixed representations by virtue of a 'large S limit' (algebraic contraction) of one of the spins to a bosonic degree of freedom.

The paper is laid out as follows. In the next section we summarize the ingredients of the reflection equations to deal with integrable models with generic boundaries. In the Section III we present the spin boson transfer matrix. The diagonalization will be pursued by a 'direct' algebraic Bethe ansatz (Section IV A) and by means of the algebraic contraction of a suitable auxiliary spin problem (Section IV B). The Hamiltonian models extracted from the transfer matrix will be presented in the Section V. The Section VI is devoted to the conclusions.
II. INTEGRABLE BOUNDARY CONDITIONS

In this section, we review the basic aspects of the quantum inverse scattering method providing integrable theories with non-trivial boundary conditions.

The central object of the method is the transfer matrix $t(\lambda)$, which can be conveniently written as the trace over an auxiliary space $\mathcal{A}$ of an ordered product of $L$ distinct Lax matrices $\mathcal{L}_j(\lambda)$ acting in the site $j$. It can be explicitly written as

$$ t(\lambda) = \text{tr}_\mathcal{A}[T(\lambda)], \quad T(\lambda) = \mathcal{L}_L(\lambda)\mathcal{L}_{L-1}(\lambda)\ldots\mathcal{L}_1(\lambda), $$

where $T(\lambda)$ is called monodromy matrix.

The $R$-matrix and Lax matrices $\mathcal{L}_j(\lambda)$ are solutions of the Yang-Baxter equations

$$ R_{12}(\lambda - \lambda')\mathcal{L}_j(\lambda)\mathcal{L}_j(\lambda') = \mathcal{L}_j(\lambda')\mathcal{L}_j(\lambda)R_{12}(\lambda - \lambda'), $$
$$ R_{12}(\lambda)R_{13}(\lambda + \lambda')R_{23}(\lambda') = R_{23}(\lambda')R_{13}(\lambda + \lambda')R_{12}(\lambda), $$

where the suffix $m$ in $\mathcal{L}_j^{(m)}$ indicates that in a multiple tensor product of auxiliary spaces, the Lax matrix acts non-trivially only on the $m$-th copy. A rational solution of (3) is the following $R$ matrix

$$ R(\lambda) = \begin{pmatrix} \lambda + \eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda + \eta \end{pmatrix}, $$

where $\eta$ is a $\mathbb{C}$-number named quasi-classical parameter. The Yang-Baxter equation for the Lax matrices (2) provides the condition to the commutation property of the transfer matrix (1) and gives rise to the fundamental commutation relations (FCR) for their operator entries.

In the realm of the QISM boundary terms can be considered. The simplest way to do it is to construct the monodromy matrix in terms of modified Lax matrices $\mathcal{L}(\lambda) \rightarrow \mathcal{G}\mathcal{L}(\lambda)$, $\mathcal{G}$ being a matrix of constants (with respect to the spectral parameter)\[18]. The corresponding theory is still integrable so far the quadratic relation (2) is satisfied. For that it suffices that

$$ [R_{12}(\lambda), \mathcal{G}^{(1)}\mathcal{G}^{(2)}] = 0 $$

For the case of $R$-matrix (1) this relation is satisfied by any $2 \times 2$ matrix with entries $g_{11}, g_{12}, g_{21}, g_{22}$ independent of the spectral parameter.

It turns out that such modified monodromy matrices correspond to theories with toroidal boundary conditions. In the case where the boundary matrix $\mathcal{G}$ is diagonal the corresponding transfer matrix (1) can be diagonalized with very little difference from the periodic case because it does not change in a drastic way the properties of the monodromy matrix elements. In general this does not occur when $\mathcal{G}$ is non-diagonal. Nevertheless, for the rational solutions of the Yang-Baxter equation (3), it is possible to apply a combination of suitable auxiliary and quantum space transformations which relate the non-diagonal boundary problem with a diagonal one\[27]. These transformations are the irreducible representations of the symmetry of the auxiliary and quantum space of the $\mathcal{L}$-operator $\mathcal{L}_j(\lambda)$. Such
invariance of the Lax matrix can be expressed by

$$\mathcal{L}_j^{(1)}(\lambda) = \left[G^{(1)}U_j\right]^{-1} \mathcal{L}_j^{(1)}(\lambda) \left[G^{(1)}U_j\right],$$

(6)

where $U_j$ stand for the irreducible representation of the symmetry of the quantum space $[27]$. For instance, if the quantum space is invariant by spin-$S$ representation of $SU(2)$, the relation $[28]$ is satisfied by

$$U_j = \begin{pmatrix}
    u_{S,S} & u_{S,S-1} & \cdots & u_{S,-S} \\
    u_{S-1,S} & u_{S-1,S-1} & \cdots & u_{S-1,-S} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{-S,S} & u_{-S,S-1} & \cdots & u_{-S,-S}
\end{pmatrix},$$

(7)

where some of the matrix elements $u_{i,j}$ are

$$u_{S,S-k} = g_{1,1}^{2S,-k} g_{1,2}^{k} \sqrt{\frac{(2S)!}{k!(2S - k)!}},$$

(8)

while the remaining satisfy the following recurrence relation

$$u_{S-k-1,S-n} = \frac{(g_{1,1}g_{2,2} - g_{1,2}g_{2,1})}{g_{1,1}^2} \sqrt{\frac{n(2S - n + 1)}{(k+1)(2S-k)}} u_{S-k,S-n+1} + \frac{g_{2,1}^2}{g_{1,1}} \sqrt{\frac{k(2S-k+1)}{(k+1)(2S-k)}} u_{S-k+1,S-n} + \frac{g_{2,1}}{g_{1,1}} \sqrt{\frac{2(S-k)}{(k+1)(2S-k)}} u_{S-k,S-n},$$

(9)

and $k, n$ are integers satisfying $k, n = 0, \ldots, 2S$.

As the transfer matrix $[28]$ gives origin to a commutative family of operators, it can be regarded as generating functional of the integrals of the motion of the quantum theory, among which one of them can be taken as the Hamiltonian. For example the logarithmic derivative of $t(\lambda)$ provided a protocol to demonstrate the integrability of a variety of local models (that is with nearest neighbor interactions) $[17]$. For later use we remark that an arbitrary function of $t(\lambda)$ or e.g. its coefficients in a Taylor series in $\lambda$ can be taken to be the Hamiltonian.

A different class of lattice models can be obtained focusing on the quasi-classical parameter $\eta$. Specifically, integrable long-range Hamiltonians, as the Gaudin and the BCS model, emerge from the so called quasi classical limit $\eta \to 0$ of vertex models $[23, 24, 25, 26]$.

More complicated boundary terms can be considered in the integrable theory via a procedure demonstrated by Sklyanin $[19, 30]$. To begin with, let’s impose the following constrains on the $R$-matrix

$$\text{Unitarity: } R_{12}(\lambda)R_{12}(-\lambda) = \zeta(\lambda) \mathbb{1};$$

$$\text{Parity invariance: } P_{12} R_{12}(\lambda) P_{12} = R_{12}(\lambda);$$

$$\text{Temporal invariance: } R_{12}(\lambda) \mathcal{L}_{12} = R_{12}(\lambda);$$

$$\text{Crossing symmetry: } R_{12}^\dagger(\lambda) R_{12}^\dagger(-\lambda - 2 \rho) = \zeta(\lambda + \rho) \mathbb{1};$$

(10)

(11)

(12)

(13)

where $\zeta(\lambda) = \eta^2 - \lambda^2$ and the crossing parameter $\rho = \eta$. Here $\mathbb{1}$ is the identity matrix, $P_{12}$ is the permutation operator, $t_\alpha$ denotes transposition on the $\alpha$-th space.

According to the Sklyanin procedure, open boundary conditions preserve the integrability of the model need to provide 2-particle scattering in the bulk ‘compatible’ with the scattering off the boundaries. Such boundary conditions
can be parametrized by boundary matrices $K(\lambda)$ which satisfy the so called reflection equations

$$R_{12}(\lambda - \lambda') \frac{1}{2} K_-(\lambda) R_{21}(\lambda + \lambda') \frac{2}{2} K_-(\lambda') = R_{21}(\lambda - \lambda') \frac{1}{2} K_-(\lambda') R_{12}(\lambda + \lambda') \frac{2}{2} K_-(\lambda),$$

$$R_{21}(-\lambda + \lambda') \frac{1}{2} K_+(\lambda) R_{12}(-\lambda - \lambda' - 2\eta) \frac{2}{2} K_+(\lambda') = K_+(\lambda') R_{21}(-\lambda - \lambda' - 2\eta) K_+(\lambda) R_{12}(-\lambda + \lambda'),$$

(14)

where $\frac{1}{2} K(\lambda) = K(\lambda) \otimes \mathbb{1}$, $\frac{2}{2} K(\lambda) = \mathbb{1} \otimes K(\lambda)$, $K^\dagger$ is the transpose of $K$, and $R_{12}$ is a solution of the Yang-Baxter equation (3).

General non-diagonal $\mathcal{C}$-number solutions in case of the rational $R$-matrix (11) are the $K$-matrices (31)

$$K_\pm(\lambda) = \left( \begin{array}{cc} \xi_\pm + \lambda & \lambda \mu_\pm \\ \lambda \nu_\pm & \xi_\pm - \lambda \end{array} \right).$$

(15)

Within this framework the analogue of the row-to-row transfer matrix (11) as generator of commuting integrals of motion in the case of open boundaries is the following operator (19)

$$t(\lambda) = \text{tr}_A \left[ K_+(\lambda + \eta) [T(-\lambda)]^{-1} K_-(\lambda) T(\lambda) \right].$$

(16)

often called 'double row transfer matrix' in the literature. For diagonal $K$-matrices the diagonalization of $t(\lambda)$ can be done by an extension of the algebraic Bethe Ansatz (19). For models based on the rational $R$-matrices (11) considered in this paper this approach can be extended to certain non-diagonal $K$-matrices by combination of suitable transformations in both quantum and auxiliary space (32, 33).

III. QUANTUM REVERSE SCATTERING FOR SINGLE MODE SPIN-BOSON MODEL

In this section we apply the general theory presented in the previous section to the spin-boson problem. Starting from the $R$-matrix (11) we consider two types of Lax matrices satisfying (2): the spin degree of freedom is described by

$$\mathcal{L}_j^{(s)}(\lambda) = \left( \begin{array}{cc} \lambda + \eta S_j^z & \eta S_j^- \\ \eta S_j^+ & \lambda - \eta S_j^z \end{array} \right),$$

(17)

where the operators $S_j^\pm$ and $S_j^z$ are chosen to be irreducible representations of $su(2)$. One way to construct a Lax matrix for the bosonic degree of freedom is using a representations of $su(2)$ in terms of Bose operators in (17), e.g. the Holstein-Primakoff or the Dyson-Maleev representation. Here, however, we choose the bosonic Lax matrix as (17, 38)

$$\mathcal{L}_j^{(b,1)}(\lambda) = \left( \begin{array}{cc} \lambda - \eta a_j^\dagger a_j & \beta a_j^\dagger \\ \gamma a_j & -\frac{\beta}{\eta} \end{array} \right); \quad \mathcal{L}_j^{(b,2)}(\lambda) = \left( \begin{array}{cc} \frac{\beta \gamma}{\eta} & \beta a_j^\dagger \\ \gamma a_j & \eta a_j^\dagger a_j + \lambda + \eta \end{array} \right).$$

(18)

with $[a_j, a_k^\dagger] = \delta_j k$. For the construction of the transfer matrix (16) we also need the inverse of the bosonic Lax operator which reads

$$\mathcal{L}_j^{(b,1)^{-1}}(\lambda) = \text{det}_q[\mathcal{L}_j^{(b,1)}]^{-1} \left( \begin{array}{cc} \frac{\beta \gamma}{\eta} & \beta a_j^\dagger \\ \gamma a_j & \eta a_j^\dagger a_j - \lambda + \eta \end{array} \right) = \text{det}_q[\mathcal{L}_j^{(b,1)}]^{-1} \mathcal{L}_j^{(b,2)}(-\lambda).$$

(19)

Here $\text{det}_q[\mathcal{L}_j^{(b,1)}] = -\beta \gamma (\lambda/\eta)$ is the quantum determinant of $\mathcal{L}_j^{(b,1)}$. 
The $R$-matrix (4) intertwines both the spin and bosonic Lax matrices; additionally it is $GL(2)$ invariant, namely it commutes with $G \otimes G$ for all $G \in GL(2)$ (see (5)). This $GL(2)$ invariance of the $R$-matrix implies that $GL(2)$-transformed Lax matrices $G_1 \mathcal{L} G_2$ are again solutions of the Yang-Baxter relation (2). At a formal level, $GL(2)$ transformations preserves the structure of the spin Lax matrix (17) and essentially correspond to a rotation of the local spin. We point out that this is not the case for the bosonic Lax matrix (15): though general $GL(2)$ transformations lead to canonical transformations of the boson degrees of freedom, only diagonal $GL(2)$ transformations preserve the structure of the Lax operator.

In what follows we restrict ourselves to setups with a single spin and a single bosonic mode, corresponding to a two-site monodromy matrix. Many relevant features are already contained in this simple case, and an extension to the multi-spin and multi-mode situation is technically straightforward. Integrable spin-boson models can be constructed e.g. from the transfer matrix for toroidal boundary conditions (periodic boundary with two boundary twists)

$$t_{\text{twist}}(\lambda) = \text{tr} \mathcal{A} \left[ G_s \mathcal{L}^{(s)}(\lambda - z_0) G_b \mathcal{L}^{(b)}(\lambda - z_1) \right]$$

where $G_s, G_b \in GL(2)$ do not depend on $\lambda$; the quantities $z_0, z_1$, shifting the spectral parameter locally, are known as 'impurities' in the jargon of QISM. Since $[t_{\text{twist}}(\lambda), t_{\text{twist}}(\mu)] = 0$, in particular all coefficients of a $\lambda$-expansion of $t_{\text{twist}}(\lambda)$ commute with each other. The first order coefficient of this $\lambda$-expansion of the transfer matrix is a linear combination of the spin- and boson operators occurring in the Lax matrix. This operator, where spin- and boson degrees of freedom are decoupled, consequently commutes with every Hamiltonian constructable from this transfer matrix. This implies that the problem can be effectively reduced to a dressed spin degree of freedom, in which the Hamiltonian is block-diagonal (11). This can be argued from the $GL(2)$ symmetry of the $R$-matrix discussed above, in that $\lambda$-independent twists are elements of $GL(2)$ and can hence be absorbed in a redefined spin and boson Lax operator. Note that this result is independent of the specific bosonic Lax matrix chosen; either choice of a bosonic Lax operator, as given in Eq. (15), or obtained from the spin Lax operator (17) inserting e.g. the Holstein-Primakoff or Dyson-Maleev bosonic representation of $su(2)$, will produce model Hamiltonians with a conserved number operator $S^z + a^\dagger a$. Consequently all Hamiltonians constructed in this way are of block diagonal form, with more or less complicated non linearity in the bosonic and the interaction part. Examples for such nonlinear generalizations of $H_{-ph}$ can be found in [37, 38, 39, 40, 41], where both twist matrices were chosen as the same diagonal matrix. Non-diagonal hermitian twist matrices have been used in [22].

For obtaining a model including counter rotating terms from the XXX symmetric $R$ matrix, we look at open boundary conditions. This amounts to using the transfer matrix (16) as the generator of integrable Hamiltonians. To this end we will consider the following monodromy matrix

$$\mathcal{T}(\lambda) = K_+ (\lambda + \eta) \mathcal{L}^{(s)} (\lambda - z_0) \mathcal{L}^{(b)} (\lambda - z_1) K_- (\lambda) \mathcal{L}^{(b)} (\lambda - z_1) \mathcal{L}^{(s)} (\lambda - z_0)$$

where the boundary matrices are taken from equation (16).

In the next section we discuss the exact solution of the eigenvalue problem for the transfer matrix $t(\lambda) = \text{tr} \mathcal{T}(\lambda)$ by means of the algebraic Bethe Ansatz following two approaches. First by 'direct' algebraic Bethe ansatz, i.e. including the bosonic Lax operator (15), and then by first solving an auxiliary spin-spin problem and then performing a suitable large-spin limit leading to the bosonic degree of freedom. The explicit construction of model Hamiltonians from (21) will be presented in Section V.
IV. ALGEBRAIC BETHE ANSATZ FOR THE SPIN-BOSON TRANSFER MATRIX

The starting point of the exact diagonalization of the transfer matrix by the algebraic Bethe ansatz is to identify a suitable eigenstate of the transfer matrix, the so-called pseudo-vacuum $|\Omega\rangle$. In the presence of non-diagonal boundary matrices, this state can be identified by a method proposed in Refs. 27, 32. Sketching their approach, the boundary matrices are brought to triangular form by a suitable similarity transformation of the monodromy matrix (21). This amounts to finding matrices $M_j = \begin{pmatrix} x_j & r_j \\ y_j & s_j \end{pmatrix}$ in

$$\hat{T}(\lambda) = M_0^{-1}T(\lambda)M_0$$

$$= M_0^{-1}K_+(\lambda+\eta)M_5M_5^{-1}L^{(s)^{-1}}(-\lambda-z_0)M_4M_4^{-1}L^{(b)^{-1}}(-\lambda-z_1)M_3M_3^{-1}K_-(\lambda)M_2 \times$$

$$\times M_2^{-1}L^{(b)}(\lambda-z_1)M_1M_1^{-1}L^{(s)}(\lambda-z_0)M_0$$

(24)

such that the lower left entry of both $M_0^{-1}K_+(\lambda+\eta)M_5$ and $M_3^{-1}K_-(\lambda)M_2$ vanishes. The remaining parameters in the $M_j$ have to be chosen such that the lower left entries of the transformed Lax operators $L^{(s,b)}$ annihilate the spin and bosonic vacuum states $|\omega_\alpha\rangle$ and $|\omega_b\rangle$ respectively. As a result we can choose the product state $|\Omega\rangle = |\omega_\alpha\rangle\otimes|\omega_b\rangle$ as pseudo vacuum for the algebraic Bethe Ansatz. Following Refs. 27, 32 for the spin Lax matrix we readily obtain $x_0/y_0 = x_1/y_1$ and $x_4/y_4 = x_5/y_5$. Note that having $M_j^{-1}L^{(a)}M_{j-1}|\omega_\alpha\rangle$ triangular implies the same for $\tilde{M}_j^{-1}L^{(a)^{-1}}(\lambda-z_0)\tilde{M}_j|\omega_\alpha\rangle$, where the matrices $\tilde{M}_j$ may differ from $M_j$ in their right column only. Since the transfer matrix is independent of this spurious freedom, we are free to choose $M_3 = M_2$, $M_4 = M_1$, and $M_5 = M_0$ without loss of generality. If all Lax matrices are spin Lax matrices, all $M_j$ can be chosen to be equal 27, 32, corresponding to a global rotation of the spin.

A. Algebraic Bethe ansatz with the bosonic Lax operator

In this section, we adapt this procedure to the spin-boson model for the two-site monodromy (21) and the choice $L^{(b)} = L^{(b,1)}$; the alternative choice $L^{(b)} = L^{(b,2)}$ is discussed below. The approach described next for open boundary conditions equally applies to toroidal boundary conditions.

The lower left entry of the transformed bosonic Lax matrix $\tilde{L}^{(b)}(\lambda) \equiv M_2^{-1}L^{(b)}(\lambda)M_1$ is (up to a rescaling of the spectral parameter)

$$-x_1y_2\lambda + \left(\frac{\beta}{\eta}x_2 + y_2a^1\right)(\eta x_1a - \beta y_1)$$

(25)

Clearly, this operator can annihilate a vacuum state for all $\lambda$ only if $x_1y_2 = 0$. The bosonic creation operator $a^1$ has no right eigenstate, therefore we have to choose $y_2 = 0$. This means that the gauge matrix left to the bosonic Lax matrix must already be in upper triangular form. Consequently, the pseudo-vacuum of the bosonic degree of freedom is the coherent state $(\eta x_1a - \beta y_1)|\omega_b\rangle = 0$ with

$$\frac{\beta}{\eta}x_0 = \frac{\beta}{\eta}x_1 \equiv \frac{\beta}{\eta}e^p .$$

(26)

We also comment that for the Eq. (26) the structure of two reference states found in [42] is lost in the spin-boson case.
Writing \( \hat{\mathcal{L}}^{(b)} = \begin{pmatrix} \hat{A}^{(b)} & \hat{B}^{(b)} \\ \hat{C}^{(b)} & \hat{D}^{(b)} \end{pmatrix} \) we have

\[
\begin{align*}
\hat{C}^{(b)} |\omega_b\rangle &= 0, \\
\hat{A}^{(b)} |\omega_b\rangle &= \frac{\kappa_+}{x_2} ((\lambda - z_1)e^p) |\omega_b\rangle, \\
\hat{D}^{(b)} |\omega_b\rangle &= \frac{\beta \gamma r_1 e^{-p} - s_1}{s_2} |\omega_b\rangle.
\end{align*}
\]

The characteristic factor \( e^p \) is obtained from the condition that \( K_+ = M^{-1}_0 K_+ M_0 \) be upper triangular

\[
e^{-p} = \frac{K_{+;21}}{\kappa_+ - K_{+;22}}
\]

where \( \kappa_+ \) is an eigenvalue of \( K_+ \).

Note that the parameters \( r_0, s_0, \tilde{r}_0, \tilde{s}_0 \) as well as \( r_2, s_2, \tilde{r}_2, \tilde{s}_2 \) can be chosen such that both boundary matrices \( K_+ \) and \( K_- \) are diagonal. These parameters enter both the Bethe equations and the expression for the eigenvalue \( \lambda(\lambda) \) of the transfer matrix, Eq. \((22)\). The transfer matrix and its spectrum, nevertheless, is independent of this choice. This freedom provides with a variety of isospectral Hamiltonians and corresponding Bethe equations. The transfer matrix, however, differs from that obtained from Eq. \((21)\) and corresponding diagonalized \( K_+ \) and \( K_- \). This is due to the fact that for \( r_j \), \( s_j \) not being all the same, the “gauge” transformations of the Lax operators do not induce canonical transformations on the spin- and bosonic degrees of freedom.

Here, rather than bringing the boundary matrices into diagonal form, we choose gauge transformations to simplify the Bethe ansatz equations. To be specific, let det \( M_j = x_j s_j - y_j r_j = 1 \) and \( x_j = r_j = 1 \) for \( j = 0, 1 \) giving \( s_j = y_j + 1 = e^{-p} + 1 \). Furthermore, the condition \( y_2 = 0 \) for the existence of the bosonic pseudo vacuum implies that the boundary matrix \( K_- \) in \((22)\) must be upper triangular (when bringing the monodromy matrix in lower triangular form instead, the particular role of \( x_j \) and \( y_j \) is taken by \( r_j \) and \( s_j \) respectively). Therefore, we are free to choose \( M_2 = I \) and the action \((27)\) of \( \hat{\mathcal{L}}^{(b)}(\lambda - z_1) \) on the bosonic coherent state simplifies

\[
\begin{align*}
\hat{C}^{(b)} |\omega_b\rangle &= 0, \\
\hat{A}^{(b)} |\omega_b\rangle &= (\lambda - z_1) |\omega_b\rangle, \\
\hat{D}^{(b)} |\omega_b\rangle &= - |\omega_b\rangle.
\end{align*}
\]

Similarly, after the similarity transformation, the action of the spin Lax operator \( \mathcal{L}^{(s)}(\lambda - z_0) \) for a \( 2S + 1 \) dimensional representation of \( su(2) \) on \( |\omega_s\rangle \) becomes

\[
\begin{align*}
\hat{C}^{(s)} |\omega_s\rangle &= 0, \\
\hat{A}^{(s)} |\omega_s\rangle &= (\lambda - z_0 + \eta(S + 1)) |\omega_s\rangle, \\
\hat{D}^{(s)} |\omega_s\rangle &= (\lambda - z_0 - \eta(S - 1)) |\omega_s\rangle,
\end{align*}
\]

while the boundary matrix \( K_+ \) is transformed into

\[
\hat{K}_+ = e^{-p} \left( \begin{array}{cc} -K_{+;21} e^{2p} + (2K_{+;11} - K_{+;12}) e^{p} + 2K_{+;12} & * \\ 0 & K_{+;21} e^{2p} - (K_{+;11} - 2K_{+;22}) e^{p} - 2K_{+;12} \end{array} \right)
\]

\[
= \begin{pmatrix} \kappa_+ & * \\ 0 & \kappa'_+ \end{pmatrix} = \begin{pmatrix} \xi_+ \pm \lambda \sqrt{1 + \nu_+ \mu_+} & * \\ 0 & \xi_+ \pm \lambda \sqrt{1 + \nu_+ \mu_+} \end{pmatrix}
\]

\[(31)\]
It is worth noticing that for a multi-site spin and multi-mode boson generalization of this model the actions of all consecutive Lax matrices on the pseudo-vacuum have to be in triangular form for the above procedure to work. Furthermore will we briefly discuss the diagonalization procedure, when the bosonic Lax matrix Eq. (19) is inserted into the transfer matrix, Eq. (16) – and not Eq. (18) as above. A straightforward calculation shows that then the gauge matrix \( M_1 \) to the right of the bosonic Lax matrix and all following Lax and boundary matrices must be in triangular form. As a consequence, \( K_+ \) instead of \( K_- \) must be triangular.

We now proceed to the result of the algebraic Bethe ansatz described above. The eigenvalues \( \Lambda(\lambda; \{\lambda_j\}) \) of the transfer matrix (16) are obtained within the standard formalism [19] as

\[
\Lambda(\lambda) = -\frac{[\lambda + \eta S + z_0](\lambda + \eta(S + 1) - z_0)}{[\lambda + z_0 + \eta S](\lambda + z_0 - \eta(S + 1))} \left[ \frac{\lambda - z_1}{\lambda + z_1} \right] \\
\times \frac{2(\lambda + \eta)(\lambda + \zeta_-)(\lambda + \zeta_+)}{(2\lambda + \eta)} \prod_{j=1}^{n} \left[ \frac{\lambda - \lambda_j - \eta}{\lambda - \eta} \right] \\
+ \frac{[\lambda - \eta S + z_0](\lambda - \eta(S - 1) - z_0)}{[\lambda + z_0 + \eta S](\lambda + z_0 - \eta(S - 1))} \left[ \frac{\lambda + z_1 + \eta}{\lambda + z_1} \right] \\
\times \frac{2\lambda(\lambda - \zeta_- + \eta)(\lambda - \zeta_+ + \eta)}{(2\lambda + \eta)} \prod_{j=1}^{n} \left[ \frac{\lambda - \lambda_j + \eta}{\lambda - \lambda_j - \eta} \right]
\]

(32)

The pseudo-vacuum \( |\Omega\rangle \) has eigenvalue \( \Lambda_0(\lambda) = \Lambda(\lambda; \emptyset) \). The consequence of choosing \( L^{(b,2)} \) as bosonic Lax operator consists in an overall minus sign and \( z_1 \to -z_1 \). Eigenstates are characterized by the roots \( \{\lambda_j\} \) of the Bethe equations

\[
-\frac{[\lambda_i + \eta(S - \frac{3}{2}) + z_0](\lambda_i + \eta(S - \frac{1}{2}) - z_0)}{[\lambda_i - \eta(S + \frac{1}{2}) + z_0](\lambda_i - \eta(S + \frac{3}{2}) - z_0)} \frac{\lambda_i - z_1 - \eta}{\lambda_i + z_1 + \eta} \\
\frac{[\lambda_i + \zeta_- - \frac{3}{2}](\lambda_i + \zeta_- - \frac{1}{2})}{[\lambda_i - \zeta_+ + \frac{1}{2}](\lambda_i - \zeta_+ + \frac{3}{2})}
\]

(33)

where the boundary matrix \( K_+ \) has been rescaled with \( \sqrt{T + \mu_+ \nu_+} \) and \( \zeta_+ := \frac{\xi_+}{\sqrt{T + \mu_+ \nu_+}} \) is the inverse of the effective boundary field. Eqs. (33) coincide with the Bethe equations for an open chain with diagonal boundary matrices after a rotation of the spin and bosonic degrees of freedom. Choosing \( L^{(b,2)} \) as the bosonic Lax operator induces the change \( z_1 \to -z_1 \).

We close this paragraph commenting on the general implications of the restriction to triangular \( K_- \). Though the boundary conditions do break the \( su(2) \) symmetry of the system, there is a remaining \( u(1) \) conserved charge in the system in this case. Hence, the spin boson models which can be diagonalized within this approach should not be expected to feature counter rotating terms. Nevertheless for non-diagonal triangular \( K_- \), the resulting model will show some features of counter rotating terms. We emphasize, however, that the triangularity of \( K_- \) is a constraint necessary only for this method for the diagonalization of the transfer matrix which is not related to the integrability of the corresponding spin-boson model.

### B. Solution by algebraic contraction of the spin-spin transfer matrix

It is well known that bosonic degrees of freedom can be realized as spin operators, in a certain limit. Such a limit constitutes an example of algebraic contraction [13]. In this section we demonstrate how this idea can be applied to the different stages of QISM method, thus allowing to obtain integrable spin-boson model from certain auxiliary integrable...
obtained by means of algebraic Bethe ansatz \[32\]. In this rephrasing the
matrix $k$ extending the theory for the toroidal boundaries\[22\]. At the level of the Lax matrices, the spin to boson mapping is achieved by the combination of two singular transformations, one in the auxiliary and the other in quantum space, to have a finite result in the limit. Explicitly, the bosonic Lax matrix is obtained as \[22\]

$$\mathcal{L}^{(b)}(\lambda) = \lim_{\epsilon \to \infty} k(\epsilon)\mathcal{L}^{(j)}(\lambda - \delta),$$

where the matrix $k(\epsilon)$ is a diagonal matrix

$$k(\epsilon) = \text{diag}\left(1, \frac{1}{\epsilon^2}\right), \delta = \eta + \eta \epsilon^2/2, J = \epsilon^2/2,$$

and the quantum space transformation is defined by \(\{J^-, \frac{1}{\sqrt{2}}J^+, J^z\} \to \{a^1, a, -a^\dagger a + \frac{\epsilon^2}{2}\}\). We have assumed $\beta = \gamma = \eta$ in the definition of the Bose Lax matrix, Eq. (18) for convenience.

We apply the Eq.\[34\] to an auxiliary double-row transfer matrix $t_a(\lambda)$ that in the limit of $\epsilon \to \infty$ is equal to $t(\lambda)$

$$t_a(\lambda) = \text{tr} \left\{ K_+(\lambda + \eta) \left[ \mathcal{L}^{(s)}(-\lambda - z_0) \right]^{-1} \left[ \mathcal{L}^{(j)}(-\lambda - \delta - z_1) \right]^{-1} [k(\epsilon)]^{-1} K_-(\lambda) \right. $$

$$\left. \times k(\epsilon)\mathcal{L}^{(j)}(\lambda - \delta - z_1)\mathcal{L}^{(s)}(\lambda - z_0) \right\}.$$

The above operator $t_a(\lambda)$ can be conveniently rotated into $\tilde{t}_a(\lambda)$ through a new transformation $U_j$\[7\] acting in the quantum spaces, along the line suggested by the property \[16\]. This simplifies the the effect of $k(\epsilon)$, acting on the auxiliary space. The transformed $\tilde{t}_a(\lambda)$ reads

$$\tilde{t}_a(\lambda) = U t_a(\lambda) U^{-1}$$

$$= \text{tr} \left\{ k(\epsilon)K_+(\lambda + \eta)(k(\epsilon))^{-1} \left[ \mathcal{L}^{(s)}(-\lambda - z_0) \right]^{-1} \left[ \mathcal{L}^{(j)}(-\lambda - \delta - z_1) \right]^{-1} K_-(\lambda) \right. $$

$$\left. \times \mathcal{L}^{(j)}(\lambda - \delta - z_1)\mathcal{L}^{(s)}(\lambda - z_0) \right\}.$$

Therefore, we can obtain the spectral properties of $t_a(\lambda)$ by the analysis of $\tilde{t}_a(\lambda)$, whose eigenvalues were previously obtained by means of algebraic Bethe ansatz \[32\]. In this rephrasing the $K$-matrix we need to take into account is

$$K_+(\lambda) = k(\epsilon)K_+(\lambda)(k(\epsilon))^{-1}$$

$$= \left( \begin{array}{c} \xi_+ + \lambda & e^2 \lambda \mu_+ \\ \frac{1}{\sqrt{2}} \lambda \nu_+ & \xi_+ - \lambda \end{array} \right).$$

Of course, the eigenvalues of the transfer matrix $t_a(\lambda)$ and $\tilde{t}_a(\lambda)$ are identical and given by the following expression

$$\Lambda_a(\lambda) = \left[ \frac{(\lambda + \eta S + z_0)(\lambda + \eta + \eta S - z_0)}{(\eta S + \lambda + z_0)(\eta S + \lambda - z_0)} \right] \left[ \frac{(\lambda - \delta + \eta J + \eta - z_1)(\lambda + \delta + \eta J + z_1)}{(\eta J + \lambda + \delta + z_1)(\eta J + \lambda - \delta - z_1)} \right]$$

$$\times \left[ \frac{2(\lambda + \eta)(\lambda + \xi_+)(\lambda + \xi_+)}{2(\lambda + \eta)} \right] \prod_{j=1}^n \left[ \frac{(\lambda - \lambda_j - \frac{e^2}{2}) (\lambda + \lambda_j - \frac{e^2}{2})}{(\lambda - \lambda_j + \frac{e^2}{2}) (\lambda + \lambda_j + \frac{e^2}{2})} \right]$$

$$- \left[ \frac{(\lambda - \eta S + z_0)(\lambda + \eta - \eta S - z_0)}{(\eta S + \lambda + z_0)(\eta S + \lambda - z_0)} \right] \left[ \frac{(\lambda - \delta - \eta J + \eta - z_1)(\lambda + \delta - \eta J + z_1)}{(\eta J + \lambda + \delta + z_1)(\eta J + \lambda - \delta - z_1)} \right]$$

$$\times \left[ \frac{2(\lambda - \zeta_+ + \eta)(\lambda - \zeta_+ + \eta)}{2(\lambda + \eta)} \right] \prod_{j=1}^n \left[ \frac{(\lambda - \lambda_j + \frac{e^2}{2}) (\lambda + \lambda_j + \frac{e^2}{2})}{(\lambda - \lambda_j - \frac{e^2}{2}) (\lambda + \lambda_j - \frac{e^2}{2})} \right].$$

spin model. The approach is pursued here in the case the integrable model is with open boundary conditions, thus extending the theory for the toroidal boundaries\[22\].
while the Bethe ansatz equations are given by

\[
\left[ \frac{\lambda_j - \frac{\eta}{2} - \eta S + z_0}{\lambda_j - \frac{\eta}{2} - \eta S + z_0} \right] \left[ \frac{\lambda_j + \delta - \frac{\eta}{2} + \eta J + z_1}{\lambda_j + \delta - \frac{\eta}{2} - \eta J + z_1} \right] = \left[ \frac{\lambda_j - \frac{\eta}{2} + \eta S - z_0}{\lambda_j - \frac{\eta}{2} + \eta S - z_0} \right] \left[ \frac{\lambda_j + \delta + \frac{\eta}{2} + \eta J - z_1}{\lambda_j + \delta + \frac{\eta}{2} - \eta J - z_1} \right]
\]

(40)

\[
\frac{\lambda_j - \zeta - \frac{\eta}{2}}{\lambda_j + \zeta - \frac{\eta}{2}} \prod_{i \neq j} \left( \frac{\lambda_j - \lambda_i - \eta}{\lambda_j - \lambda_i + \eta} \right)
\]

where \(\zeta = -\frac{\xi}{\sqrt{1 + \mu \pm \nu}}\) which do not depend on \(\epsilon\). For the algebraic Bethe ansatz, one of the two following assumptions on the parameters \(\xi, \mu, \text{ and } \nu\) has been made (see [32])

\[
\frac{1 + \sqrt{1 + \nu_+ \mu_+}}{\mu_-} = \frac{1 + \sqrt{1 + \nu_+ \mu_+}}{\mu_- \epsilon^2}, \quad \text{or}
\]

(41)

\[
\frac{1 + \sqrt{1 + \nu_- \mu_-}}{-\nu_-} = \frac{1 + \sqrt{1 + \nu_+ \mu_+}}{\mu_+ \epsilon^2}
\]

(42)

To obtain the eigenvalues and the Bethe ansatz equation of the spin-boson problem, associated to the transfer matrix \(t(\lambda)\) we take the limit \(\epsilon \to \infty\) of Eqs.(40). This leads to the same eigenvalue (32) and Bethe ansatz equations (33) as obtained from the 'direct' ansatz. In the limit of \(\epsilon \to \infty\) the constraints (41),(42) become \(\nu_- = 0\) and \(\nu_- \mu_- = 0\), respectively. This means that the boundary matrix \(K_-(\lambda)\) has to be in a triangular (or even diagonal) form already, and \(\zeta_- = -\zeta_-\).

Considering \(\mu_- = \nu_+ = 0\) in the eigenvalue expression and Bethe ansatz equation above we can recover the results obtained for the case of diagonal \(K_\pm(\lambda)\) \(K\)-matrices without use of the limit process.

As final remark we observe that a definite limit \(\epsilon \to 0\) exists, despite the matrices \(k(\epsilon)\) are singular in the limit; such singularity is compensated by the divergence in the “impurity” \(\delta\) (see Eq.(35)).

V. INTEGRABLE HAMILTONIANS

The transfer matrix \(t(\lambda)\) for rational solutions of the YBE is a polynomial expression in the spectral parameter. Because the transfer matrix is a commuting family of operators in \(\lambda\), the coefficients of such a polynomial are the integrals of the motion of the theory. In the following section we exploit this property of the transfer matrix to extract the Hamiltonian. In the section V B we obtain the Hamiltonian following an alternative procedure that is in the spirit of the quasi-classical expansion.

A. Integrable models derived from the transfer matrix

We analyze the integrals of the motion generated by the transfer matrix [10] and the bosonic Lax matrix \(L^{(b)} = L^{(b,2)}\) for spin-1/2. We therefore expand it in powers of the spectral parameter

\[
t(\lambda) =: \sum_{n=0}^{6} H_n \lambda^n
\]

(43)

where we set \(\eta = 1\).
There are only two independent constants of the motion in this problem and we find that $H_6$, $H_5$, and $H_0$ are constants and the relations

$$2H_4 - H_3 = 5\mu_-\nu_+$$  \hspace{1cm} (44)  
$$H_4 - H_2 + H_1 = 3\mu_-\nu_+$$  \hspace{1cm} (45)  

With this result for the general case we are now ready for a systematic case study.

At first, we focus on diagonal boundary matrices $K_{\pm}$. In this case we find up to a constant

$$H_4 = 4\beta\gamma(S^2 - \hat{n}) .$$  \hspace{1cm} (46)  

This constant of the motion readily unveils this Hamiltonian to be of the Jaynes-Cummings type in the RWA because the Hilbert space splits up in two-dimensional (spin-1/2) subspaces with fixed eigenvalue of $H_4$. Problems in absence of a conserved number operator in general will require more sophisticated techniques as employed for the diagonalization of the XYZ chain or the XXZ chain with non-diagonal boundary fields \cite{34,35}.

For diagonal $K_+$ and non-diagonal but triangular $K_-$, the expansion (43) leads to (up to a constant)

$$H_4 = \Omega_0(S^2 - \hat{n}) + C_1 \left( 1 - \frac{1}{2s} + 2S^2s \right) S^+ + C_2\hat{a} + C_3\hat{a}^\dagger(2S^2 - \hat{n})$$  \hspace{1cm} (47)  

where $s$ is the spin, $\Omega_0 = 4\beta\gamma$, $C_1 = \nu_- (2\xi_+ + 2z_0 - 1)$, $C_2 = 2\gamma\nu_- (z_1 + \xi_+)$, and $C_3 = 2\gamma\nu_-$. As we will show below, the Hilbert space of the non-hermitian Hamiltonian \cite{47} is not reduced to two-dimensional subspaces; therefore, it describes a resonant spin-boson model (i.e. with vanishing de-tuning of the spin-boson frequencies) containing counter-rotating terms. It can be treated by the method used above. A careful analysis reveals that the non-hermiticity of the Hamiltonian is inherent to this transfer matrix unless the spin degrees of freedom are decoupled from the boson. We attribute this feature to the structure of the bosonic Lax matrix \cite{18} used here.

In what follows we restrict ourselves to spin $s = 1/2$. A vacuum state for the Bethe ansatz on the Hamiltonian \cite{47}, then is $|0, \frac{1}{2}\rangle$, using the notation $|n, \sigma\rangle$ such that $S^z |n, \sigma\rangle = \sigma |n, \sigma\rangle$ and $\hat{n} |n, \sigma\rangle = n |n, \sigma\rangle$. Defining $C_2 =: C_1 + \delta$, $C_3 = C_1 + \delta + \Delta$, the creation operator $B(\lambda)$ has the form

$$B(\lambda) = [\Delta^2\lambda(\lambda + 1) - \delta^2 + 2\delta\Delta S^z] \{\hat{a}^\dagger(\hat{n} - \xi_-) + \hat{n}(\hat{n} - 1)\}$$

$$-\Delta [2\delta\hat{n} + 2\Delta(\Delta\lambda(\lambda + 1) - \delta)\hat{a} + 2\Delta\delta\hat{a}\hat{n})] S^-$$

$$-\Delta(\Delta(\lambda^2 + \lambda + \xi_-) - \delta\xi_-) S^-$$  \hspace{1cm} (48)  

A right-eigenstate $|\lambda_1, \ldots, \lambda_N\rangle := B(\lambda_1) \ldots B(\lambda_N) |0, \frac{1}{2}\rangle$ involves all states $|n, \sigma\rangle$ with $n \leq N$ except $|N, -\frac{1}{2}\rangle$. It is interesting to notice that for positive integer values of the inverse boundary field $\xi_-$ the Hilbert space accessible from the vacuum $|0, \frac{1}{2}\rangle$ is restricted to

$$\mathcal{H}_{\xi_- \in \mathbb{N}^+} := \{ |n, \sigma\rangle : n \leq \xi_- \}$$  \hspace{1cm} (49)  

Without loss of generality we set $g_1 = 1$ and $\mu_+ = \nu_+ = 0$. The latter corresponds to a diagonal boundary matrix $K_+ : = \xi_+ 1 + \lambda\sigma_z$; any non-diagonal $K_+$ can be diagonalized by a gauge transformation \cite{27,32} and proper $r_0, s_0, \tilde{r}_0, \tilde{s}_0$. For convenience, we choose $\gamma = \beta = 1$ and rename $g_2 =: g$. The most general Hamiltonian generated in this way is equivalent to

$$H = -\frac{g_1 H_4 + g_2 H_1}{4g_1} =: H_s + H_b + H_{sb} + H_{sbb}$$  \hspace{1cm} (50)
via suitable canonical transformations on the spin and bosonic degrees of freedom.

Our goal will be to have the spin and boson Hamiltonian together with major part of the spin-boson interaction hermitian. To this end, we set $\nu = \frac{\mu - \xi}{2\xi} (\frac{1}{g} + \xi z_1)$, which makes the occurring terms $\hat{a} S^z$ and $\hat{a}^\dagger S^z$ hermitian and the choice $\xi = 1 - z_1 - \frac{2z_0 - 2\xi + 1}{2z_0 + 2\xi + 1}$ realizes the same for $\hat{a} S^-$ and $\hat{a}^\dagger S^+$. $z_1 = 1/2$ eliminates a term $\hat{n} S^+$ and the pure spin Hamiltonian becomes hermitian if

$$g = 4 - \frac{4}{\xi} + 16 \frac{2\xi - 1}{2z_0 - 6\xi - 1} .$$  \hspace{1cm} (51)

The terms $\hat{a}$ and $\hat{a}^\dagger$ in the pure boson part of the Hamiltonian are obtained hermitian if

$$z_0 = \frac{1}{2} \pm \frac{\sqrt{1 - g(g + 2)\xi_+^2 + g^2(g + 1)\xi_+^4}}{\sqrt{g(\xi_+^2 - 1)}}$$  \hspace{1cm} (52)

and it turns out that both conditions (51) and (52) are compatible and fixes a one-dimensional manifold in $\xi_+ - z_0$ space (see figure 1).

Abbildung 1: Left: Solutions of the nonlinear equation in $\xi_+$ and $g$ emerging from inserting equation (52) into (51). Right: the corresponding values for $z_0$.

The resulting Hamiltonian is as follows

$$H_s = \left( \frac{4 - 4\xi_+^2}{\xi_+} - 4 \frac{1 + 2\xi_+}{2z_0 + 2\xi_+ - 1} - 8 \frac{\xi_+(2\xi_+ - 1)}{2z_0 - 6\xi_+ - 1} \right) S^z + 2\mu_-(\frac{\xi_+ - 1}{\xi_+}((2z_0 - 1)^2 - 4\xi_+^2) S^z$$  \hspace{1cm} (53)

$$H_b = g^2 \left( \frac{4\xi_+^2}{g\xi_+^2 - 1} \hat{n} - g \frac{\mu - \xi_+ (g\xi_+ + 2)}{g\xi_+^2 - 1} (\hat{a} + \hat{a}^\dagger) \right) + g^2 \frac{\mu - \xi_+^2}{g\xi_+^2 - 1} \hat{a} \hat{n}$$  \hspace{1cm} (54)

$$H_{sb} = -g\xi_+ \hat{n} S^z + 2\mu_-(g\xi_+ + 2) (\hat{a} + \hat{a}^\dagger) S^z + 2g(2z_0 - 2\xi_+ - 1) (\hat{a} S^- + \hat{a}^\dagger S^+)$$  \hspace{1cm} (55)

$$H_{sbb} = g\mu_+ \left( 2z_0 + 2\xi_+ - 1 \right) \frac{2g\xi_+ + 2}{2g\xi_+} \hat{a}^2 S^z - (2z_0 - 2\xi_+ - 1) \hat{a}^2 S^+ \right)$$

$$+ g(2z_0 + 2\xi_+ - 1)(\mu_+ \hat{n}^2 S^+ + 2 \hat{a}^\dagger \hat{n} S^+) - 4g\mu_+ \xi_+ \hat{a} \hat{n} S^z$$  \hspace{1cm} (56)

Besides the hermitian parts $H_s$ and $H_{sb}$, there are many non-hermitian terms, in particular nonlinear couplings as in $H_{sbb}$. It is worth focusing on the non-hermitian term in $H_b$ proportional to $\hat{a} \hat{n}$. This term has an immediate physical
interpretation: it describes a leakage of bosons/photons from the cavity, where the escape rate is proportional to the number of bosons/photons (i.e. the intensity of the cavity boson field).

### B. Integrable models derived from the quasi-classical limit of the transfer matrix

Another way to obtain an integrable Hamiltonian is to take the so called quasi-classical limit of the transfer matrix [20, 21, 29]. It consists in a series expansion in the ‘quantum parameter’ \( \eta \) of the transfer matrix around \( \eta = 0 \):

\[
\hat{t}(\lambda) = \hat{t}^{(0)}(\lambda) + \eta \hat{t}^{(1)}(\lambda) + \eta^2 \hat{t}^{(2)}(\lambda) + \cdots
\]

with the aim of creating a commuting family of quasi-classical transfer matrices \( \hat{\tau}^{(k)}(\lambda) \). This procedure may be particularly useful for extracting 'simple' though non-locally interacting Hamiltonians out of the transfer matrix. Examples are the Gaudin magnets and corresponding BCS-like models.

There is a wide freedom of introducing an \( \eta \)-dependence to the boundary matrix parameters in the theory, without destroying its integrability. For the case of integrable spin chains with diagonal boundaries [29], this procedure was a valid tool to define integrable one-parameter extension of Gaudin models in non uniform magnetic fields [25]; such models describe metallic grains with pairing and magnetic interaction [24].

In the present case the transfer matrix is the finite sum

\[
\hat{t}(\lambda) = \eta^{-k} \hat{t}^{(-k)} + \cdots + \hat{t}^{(0)} + \eta \hat{t}^{(1)}(\lambda) + \eta^2 \hat{t}^{(2)}(\lambda) + \cdots + \eta^m \hat{t}^{(m)}(\lambda)
\]

(57)

where \( k \) and \( m \) are integers. Expanding the commutator relation \([\hat{t}(\lambda), \hat{t}(\lambda')] = 0 \) in \( \eta \), we obtain

\[
[\hat{t}(\lambda), \hat{t}(\lambda')] = \sum_{l=-2k}^{2m} \eta^l C_l(\lambda, \lambda') = 0,
\]

which implies \( C_l(\lambda, \lambda') = 0 \) for all \( l \). The first relevant terms are

\[
\begin{align*}
C_{-2k}(\lambda, \lambda') &= [\hat{t}^{(-k)}(\lambda), \hat{t}^{(-k)}(\lambda')] , \\
C_{-2k+1}(\lambda, \lambda') &= [\hat{t}^{(-k)}(\lambda), \hat{t}^{(-k+1)}(\lambda')] + [\hat{t}^{(-k+1)}(\lambda), \hat{t}^{(-k)}(\lambda')] , \\
C_{-2k+2}(\lambda, \lambda') &= [\hat{t}^{(-k)}(\lambda), \hat{t}^{(-k+2)}(\lambda')] + [\hat{t}^{(-k+2)}(\lambda), \hat{t}^{(-k)}(\lambda')] + [\hat{t}^{(-k+1)}(\lambda), \hat{t}^{(-k+1)}(\lambda')] ,
\end{align*}
\]

(58)

From the expressions above, one finds that the first \( \hat{\tau}^{(n)}(\lambda) \) which is not a \( \mathcal{C} \)-number (times the identity) gives rise to a family of commuting operators. Generically, the lowest order \( \hat{\tau}^{(-k)} \) is a \( \mathcal{C} \)-number. Therefore the first class of integrable models were generated by \([\hat{t}^{(-k+1)}(\lambda), \hat{t}^{(-k+1)}(\lambda')] = 0 \). In the presence of boundary matrices, these are typically non-trivial operators but representing non-interacting Hamiltonians. The task is then to tune the free parameters such that \( \hat{t}^{(-k+1)}(\lambda) \) is also a \( \mathcal{C} \)-number, and that the lowest non-trivial order in \( \eta \), e.g. \( \hat{t}^{(-k+2)}(\lambda) \), be an interesting Hamiltonian.

Here we allow \( \xi_\pm, \mu_\pm, \nu_\pm \) to be generic functions of \( \eta \) whose form can be fixed to modify to our convenience the \( \eta \)-expansion of the transfer matrix, and ultimately the Hamiltonian. Specifically, we write the boundary parameters as

\[
x_\pm = x_\pm^{(-1)} \eta^{-1} + x_\pm^{(0)} + x_\pm^{(1)} \eta
\]

(59)

with \( x = \{ \xi, \mu, \kappa \} \), and set \( \mu_\pm^{(-1)} = \nu_\pm^{(-1)} = \xi_\pm^{(-1)} = 0 \). Then the lowest non-vanishing term in the expansion, proportional to \( \eta^{-2} \), already contains non trivial operators; they can be made vanishing by the choice \( \mu_\pm^{(0)} = \nu_\pm^{(0)} = 0 \).
Analogously, the next term in the expansion can be made proportional to the identity by \( \nu_+^{(0)} = \xi_+^{(0)} = 0 \) and \( \lambda_1 = 0 \). The next non-trivial term in the \( \eta \)-expansion \( \hat{\tau}^{(0)}(\lambda) \) can be used as Hamiltonian

\[
H \doteq \frac{1}{\lambda^2 \alpha_3} \hat{\tau}^{(0)}(\lambda) = 4 \Omega_0 a^\dagger a + 2 \alpha (a + a^\dagger) + \Delta \hat{S}^z + 4 \gamma \lambda_0 a \hat{S}_x \\
+ G \hat{S}^z a^\dagger a - 4 \gamma \lambda_0 (a \hat{S}^- + a^\dagger \hat{S}^+) + \gamma_s^2 [\hat{S}^+ \hat{S}^- + \hat{S}^- \hat{S}^+ - 2(\hat{S}^z)^2] + C \mathbb{I} 
\]

(60)

where \( \Omega_0 = - (\lambda^2 - z_0^2) \), \( \alpha = \beta \nu_-^{(1)} - \gamma \nu_+^{(1)} \) and \( \Delta = -4 \lambda^2 - 2 \gamma^2 (1 - 2 \xi_+^{(1)}) \), \( G = 4 \gamma^2 (1 - \gamma) \) and \( C = -\gamma \lambda_1 \{ \Omega_0 [2(2 + \xi_0^0 - \alpha \beta \nu_-^{(1)}) \lambda_0^2 + 2 \xi_0^0 \lambda_+^0] - 2(\lambda - \lambda_0)(\gamma - \lambda^2 \xi_0^0) \} \).

For \( S = 1/2 \) the Hamiltonian simplifies to

\[
H_{1/2} = \Omega_0 a^\dagger a + \Delta \hat{S}^z + G \hat{S}^z a^\dagger a + G \lambda \hat{S}^z (a + a^\dagger) - g (S^+ a^\dagger + S^- a) 
\]

(61)

where the bosonic operators have been displaced \( a \to a + a/2 \), \( \hat{\Delta} = \Delta + \lambda \lambda G \) and a term proportional to the identity was omitted. The Hamiltonian (61) display the rotating and counter-rotating simultaneously. This can be evidenced by a rotation \( R \) of the spin

\[
RH_{1/2} R^{-1} = \Omega_0 a^\dagger a + \Delta \hat{S}^z [\cos(2 \theta) \hat{S}^z - \sin(2 \theta) \hat{S}^x] + G [\cos(2 \theta) \hat{S}^z - \sin(2 \theta) \hat{S}^x] a^\dagger a \\
- g \frac{\cos(2 \theta) + 1}{2} (S^+ a^\dagger + S^- a) - g \frac{\cos(2 \theta) - 1}{2} (S^+ a + S^- a) .
\]

(62)

where \( R \doteq \exp[\theta(S^+ - S^-)] \) and \( \tan \theta = G \lambda / g \) has been chosen to eliminate the \( \hat{S}^z(a + a^\dagger) \) term.

### VI. CONCLUSIONS

By the Quantum Inverse scattering method we have constructed integrable models with twisted and open boundary conditions which contain rotating as well as counter rotating terms.

Twisted boundary conditions lead to Hamiltonians where solely rotating (or counter-rotating) terms appear.

In case of open boundaries, we diagonalize the models by algebraic Bethe ansatz when at least one of the boundary matrices has triangular form. To this end, we extended the procedure for spin chains to models which are derived from a particular bosonic Lax matrix besides the standard spin Lax matrix. We further analyzed generic boundary matrices in the sense that they cannot be brought both in triangular form at the same time. For this choice of boundary fields, the symmetry leading to conservation of the 'number operator' \( \hat{S}^z + \sum_j a_j^\dagger a_j \) is broken.

Taking the full transfer matrix the integrable Hamiltonians are non-hermitian. However, a parameter choice is found that shifts the non-hermiticity to non-linear terms in the spin-boson interaction and the bosonic part of the Hamiltonian. In the latter, the non-hermiticity can be interpreted as a density dependent leakage of photons out of the cavity. Different parameter choices could be analyzed in order that non-hermitian terms will appear elsewhere, e.g. in the spin part of the Hamiltonian. It is worth studying what type of non-hermiticity are most realistic in order to tailor the parameters closest to this situation. The diagonalization of the resulting Hamiltonians together with a discussion of the effects of the undesired non-hermitian parts is left to forthcoming work.

Interestingly enough, by a procedure that is close in spirit to the quasi-classical limit, also hermitian Hamiltonians can be extracted out of the transfer matrix. We have given an explicit example, which contains both rotating and counter-rotating terms. The evidence we have for this is that any eigenstate (i.e. coherent states) of the bosonic part is spread out all over the bosonic Hilbert space by means of the interaction term.
Though integrable counter-rotating spin-boson models have been formulated in this work, it must be emphasized that their diagonalization with e.g. the algebraic Bethe ansatz is still an open problem. It could be interesting to study general open boundary conditions still for the XXX model but based on the Holstein-Primakoff representation of $su(2)$. The advantage of such an approach is that the Hamiltonian is constructed in a straightforward manner from the existing solution of the open XXX spin chain Hamiltonian and hence also the resulting spin-boson Hamiltonian is hermitian by construction. The challenge again lies in the diagonalization of the transfer matrix. Finally, the analysis of spin-boson models fanning out from a bulk with XYZ symmetry (parametrized by elliptic functions) constitute an interesting challenge for future investigation.

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