Notions of Lawvere theory *

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Abstract

Categorical universal algebra can be developed either using Lawvere theories (single-sorted finite product theories) or using monads, and the category of Lawvere theories is equivalent to the category of finitary monads on \( \text{Set} \). We show how this equivalence, and the basic results of universal algebra, can be generalized in three ways: replacing \( \text{Set} \) by another category, working in an enriched setting, and by working with another class of limits than finite products.

Classical universal algebra begins with structures of the following type: a set \( X \) equipped with operations \( X^n \to X \) for various natural numbers \( n \), subject to equations between induced operations. For example the structure of group can be encoded using a binary operation \( m : X^2 \to X \) (multiplication), a unary operation \( (\cdot)^{-1} : X \to X \) (inverse) and a nullary one \( i : 1 \to X \) (unit). Then there is an equation encoding associativity, two equations for the two unit laws, and two equations for the inverses.

There are two main ways for treating such structures categorically. In each case, one ends up giving all the operations generated, in a suitable sense, by a presentation as in the previous paragraph. Thus the structure of group (as opposed to some particular group) becomes a mathematical object in its own right.

On the one hand, for each such type of structure there is a small category \( \mathcal{T} \) with finite products, such that a particular instance of the structure is a finite-product-preserving functor from \( \mathcal{T} \) to \( \text{Set} \). The objects of \( \mathcal{T} \) will be exactly the powers \( T^n \) of a fixed object \( T \). Such a category is called

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a Lawvere theory. In this case, the operations in the structure are seen as the morphism in the theory $T$.

On the other hand, for each such type of structure there is a monad $T$ on $\text{Set}$, such that a particular instance of the structure is an algebra for the monad. The basic idea this time is to describe the structure in question by saying what the free algebras are. Since we have been considering only finitary operations (operations $X^n \to X$ where $n$ is finite) the resulting monad $T$ will also be finitary. This means that it is determined by its behaviour on finite sets; formally, this means that the endofunctor $T$ preserves filtered colimits.

There is a very well developed theory of Lawvere theories and of finitary monads, and of the equivalence between the two notions. In this paper we are interested in what happens when we move beyond the context of structures borne by a set, to consider structures borne by other objects. Lawvere theories are single-sorted theories; much of the time we shall spend on the many-sorted case, or rather the “unsorted” case, where a theory consists of an arbitrary small category with finite products, and a model is just a finite-product-preserving functor.

We shall consider three different ways in which the classical setting could be adapted; the three ways are to some extent independent of each other.

Replace the base category $\text{Set}$ by some other suitable category $\mathcal{K}$. We could consider structures borne not by sets but by objects of other categories. For example we could take $\mathcal{K}$ to be the category $\text{Gph}$ of (directed) graphs, and consider structures borne by graphs. An obvious example is a category: this is a graph equipped with a composition law and identities.

From the point of view of monads, this change is straightforward enough: one can simply consider monads on $\text{Gph}$ rather than monads on $\text{Set}$. As for theories, there is no problem considering models of a Lawvere theory $T$ in $\text{Gph}$, one simply takes finite-product-preserving functors from $T$ to $\text{Gph}$ rather than from $T$ to $\text{Set}$. But often this is too restrictive; for example it would not include the case of categories. One needs to allow more general arities than natural numbers; in fact it is (certain) objects of $\text{Gph}$ which will serve as arities.

Deal not with ordinary categories but with categories enriched over some suitable monoidal category $\mathcal{V}$. A monoidal category can clearly be thought of as a category with extra structure, of an algebraic sort. When we come to think of the morphisms between such structures, this naive algebraic point of view would lead us to take as morphisms the strict monoidal functors (those which preserve the unit and tensor product in the strict sense of equality: $F(X \otimes Y) = FX \otimes FY$). Although such strict monoidal functors have an important role to play, much more important are various “weak” notions, where the monoidal structure is preserved up to coherent isomorphism, or perhaps only up to a coherent non-invertible comparison map. These weaker notions of homomorphism can be recovered by working not over the category of small categories and functors, but over the 2-category of small categories, functors, and natural transformations. In other words, we work with categories enriched in $\text{Cat}$.

More generally one could consider many other monoidal categories $\mathcal{V}$, such as abelian groups, chain complexes, simplicial sets, and many more: see [11].

When we are working over $\mathcal{V}$, the “default” choice of $\mathcal{K}$ becomes $\mathcal{V}$ rather than $\text{Set}$, but many other $\mathcal{V}$-categories may also be suitable.
Replace finite products by some suitable class $\Phi$ of limits. This is probably the most important aspect for this paper.

Some structures cannot be defined with finite products alone. Once again the structure of categories is a good example. An internal category consists of an object-of-objects $C_0$, an object-of-morphisms $C_1$, with domain and codomain maps $d, c : C_1 \rightarrow C_0$, together with identities $i : C_0 \rightarrow C_1$ and composition $m : C_2 \rightarrow C_1$ satisfying various equations, where $C_2$ is object-of-composable-pairs, given by the pullback

$$
\begin{array}{ccc}
C_2 & \rightarrow & C_1 \\
\downarrow & & \downarrow \\
C_1 & \rightarrow & C_0.
\end{array}
$$

So to define internal categories one needs a class $\Phi$ of limits which contains pullbacks.

Once again, the limits in $\Phi$ should be weighted limits in the $\mathcal{V}$-enriched sense.

Our main new example of such a class is where $\Phi$ consists of what we call strongly finite limits. This is a class intermediate between finite products and finite limits. An object $X \in \mathcal{V}$ is strongly finite if the internal hom functor $[X, -] : \mathcal{V} \rightarrow \mathcal{V}$ preserves sifted colimits (see [1] or Section 4.2 below). The strongly finite limits are those which can be constructed using finite products and powers (cotensors) by strongly finite objects. These correspond to monads which preserve sifted colimits. As observed for example in [21], every monad on a cocomplete symmetric monoidal closed category arising from an operad preserves filtered colimits and reflexive coequalizers; thus, using Proposition 3.2, it preserves sifted colimits.

It is worth observing that as well as being the free completion of the category $\text{Set}_{\mathcal{F}}$ of finite sets under filtered colimits, $\text{Set}$ is also the free completion of $\text{Set}_{\mathcal{F}}$ under sifted colimits. Thus an endofunctor of $\text{Set}$ preserves filtered colimits if and only if it preserves sifted colimits. Thus the well-known equivalence between Lawvere theories and finitary monads on $\text{Set}$ could also be seen as an equivalence between Lawvere theories and sifted-colimit-preserving monads on $\text{Set}$. This is a very special property of $\text{Set}$; for other base categories, one therefore potentially has (at least) two ways to generalize the equivalence.

So altogether we have three changes

(A) replace the base category $\text{Set}$ by some other suitable category $\mathcal{K}$

(B) deal not with ordinary categories but with categories enriched over some suitable monoidal category $\mathcal{V}$

(C) replace finite products by some suitable class $\Phi$ of limits

and these appear to be independent to some extent, apart from the obvious restrictions that $\mathcal{K}$ should be a $\mathcal{V}$-category and the limits in $\Phi$ should be weighted $\mathcal{V}$-limits. But the real relationship between these conditions appears when we try to say what “suitable” should mean in each case; doing this will be one of the main aims of the paper.

We’ll see that a key aspect is that it gives a sort of decomposition or factorization of all limits into two parts. In fact it’s easier to think in terms of colimits. For example if $\Phi$ consists of the finite limits, then we have the decomposition of arbitrary colimits as filtered colimits of finite ones.
Then again, if \( \Phi \) consists of the finite products we have a corresponding decomposition into sifted colimits of finite coproducts (see Section 3 for more about sifted colimits).

There is a trade-off between different levels of generality of theory. As usual, the more general the notion of theory, the more general the categories of models that can be described, but the less that can be proved about them. There are also more subtle effects. For example, consider the case \( \mathcal{V} = \mathbf{Set} \) of unenriched categories and compare finite limit theories with finite product theories. Finite limit theories are of course more general. On the other hand, in the first instance one needs a category with finite limits as the category \( \mathcal{K} \) in which models are taken; whereas for a finite product theory, \( \mathcal{K} \) need only have finite products. As one goes further, however, it turns out that for the strongest results, \( \mathcal{K} \) should itself be the category of models for a theory of the given type, and this is a stronger condition on \( \mathcal{K} \) when we use finite products than it is when we use finite limits.

The main results of universal algebra that we should like to obtain in our general setting are:

**Φ-algebraic functors have left adjoints.** If \( G : \mathcal{I} \to \mathcal{I} \) is morphism of theories, then there is an induced map \( G^* : \text{Mod}(\mathcal{I}) \to \text{Mod}(\mathcal{I}) \), and such a \( G^* \) we call \( \Phi \)-algebraic (or just algebraic if \( \Phi \) is understood). Our first basic fact is that such a \( G^* \) has a left adjoint; or rather, we give an explicit construction of a left adjoint in terms of colimits in the base category \( \mathcal{K} \)— the existence of the left adjoint is known for general reasons.

This includes the existence (and construction) of free algebras for single-sorted theories.

**The reflectiveness of models.** For any theory \( \mathcal{I} \), the category \( \text{Mod}(\mathcal{I}, \mathcal{K}) \) of algebras in \( \mathcal{K} \) is reflective in \([\mathcal{I}, \mathcal{K}]\). This means that \( \text{Mod}(\mathcal{I}, \mathcal{K}) \) will be complete and cocomplete provided that \( \mathcal{K} \) is so, as we shall usually suppose to be the case. But we shall do more than just prove the reflectivity, we shall construct a reflection in terms of colimits in \( \mathcal{K} \), and so obtain a description of colimits in \( \text{Mod}(\mathcal{I}, \mathcal{K}) \) in terms of colimits in \( \mathcal{K} \).

In the final chapter of [11], the reflectiveness was proved under extremely general conditions, but without an explicit construction. Our setting will be much more restrictive, but will allow an explicit construction (as explicit as is our knowledge of colimits in \( \mathcal{K} \)).

Although we can obtain an explicit construction of all colimits in \( \text{Mod}(\mathcal{I}, \mathcal{K}) \), it is particularly simple for the class of colimits which commute in \( \mathcal{K} \) with \( \Phi \)-limits, since these colimits are formed in \( \text{Mod}(\mathcal{I}, \mathcal{K}) \) as in \([\mathcal{I}, \mathcal{K}]\), and so no reflection is required.

Once again we see the trade-off: the smaller the class of limits in \( \Phi \), the closer \( \text{Mod}(\mathcal{I}, \mathcal{K}) \) is to \([\mathcal{I}, \mathcal{K}]\), and so the greater our knowledge of colimits in \( \text{Mod}(\mathcal{I}, \mathcal{K}) \).

For example, if we take \( \mathcal{K} = \mathcal{V} = \mathbf{Set} \) and compare the case where \( \Phi \) consists of all finite limits with that where it consists only of finite products, in the first case the inclusion \( \text{Mod}(\mathcal{I}, \mathbf{Set}) \to \text{Mod}(\mathcal{I}, \mathcal{K}) \) preserves filtered colimits, while in the second case it also preserves reflexive coequalizers.

It is perhaps worth noting that the reflectivity of models is a special case of the existence of left adjoints to algebraic functors. Let \( \mathcal{I} \mathcal{I} \) be obtained by freely adding \( \Phi \)-limits to \( \mathcal{I} \). Since \( \mathcal{I} \) already has \( \Phi \)-limits, the canonical inclusion \( \mathcal{I} \to \mathcal{I} \mathcal{I} \) has a right adjoint \( R \), and now the induced algebraic functor \( R^* : \text{Mod}(\mathcal{I}, \mathcal{K}) \to \text{Mod}(\mathcal{I} \mathcal{I}, \mathcal{K}) \) may be identified with the inclusion \( \text{Mod}(\mathcal{I}, \mathcal{K}) \to \text{Mod}(\mathcal{K}, \mathcal{K}) \), and so the left adjoint to \( R^* \) gives the desired reflection.

On the other hand, we also have a sort of converse: the left adjoint to \( G^* : \text{Mod}(\mathcal{I}) \to \text{Mod}(\mathcal{I}) \) can be obtained by first taking the left Kan extension along \( G \) and then reflecting into the subcategory of models.
The correspondence between (single-sorted) theories and monads. For suitable \( \mathcal{V} \) and \( \Phi \), we consider the class of all colimits which commute in \( \mathcal{V} \) with \( \Phi \)-limits; these will be called \( \Phi \)-flat. We shall say that a \( \mathcal{V} \)-functor is \( \Phi \)-accessible if it preserves \( \Phi \)-flat colimits, and that a \( \mathcal{V} \)-monad is \( \Phi \)-accessible if its underlying \( \mathcal{V} \)-functor is so.

For suitable \( \mathcal{V} \)-categories \( \mathcal{K} \), we shall describe a notion of \( \mathcal{K} \)-based \( \Phi \)-theory, or \( \Phi \)-theory in \( \mathcal{K} \), and prove that the category of these is equivalent to the category of \( \Phi \)-accessible \( \mathcal{V} \)-monads on \( \mathcal{K} \), and furthermore that the algebras for a \( \Phi \)-accessible monad are the same as the models for the corresponding \( \Phi \)-theory. This generalizes the classical equivalence between Lawvere theories and finitary monads on \( \text{Set} \), and is one of the main results of the paper.

Outline of paper

We begin in Section 1 with a review of the enriched category theory that will be needed in the paper. In Section 2 we describe the basic assumptions we shall make connecting our monoidal category \( \mathcal{V} \), our base \( \mathcal{V} \)-category \( \mathcal{K} \), and the class of limits \( \Phi \) to be considered. Section 3 is largely a review of various ideas relating to sifted colimits: these are the colimits which commute in \( \text{Set} \) with finite products. In Section 4 we describe the various possible classes of limits, and the corresponding requirements on \( \mathcal{V} \) (and on \( \mathcal{K} \)). The last two sections contain our main results; we have divided these into those which are independent of the sorts, in Section 5, and those which relate specifically to single-sorted theories, in Section 6.

1 Review of enriched category theory

We work over a symmetric monoidal closed category \( \mathcal{V} = (\mathcal{V}_0, \otimes, I) \) whose underlying ordinary category \( \mathcal{V}_0 \) is complete and cocomplete. The general results in [11] on reflectivity of models, referred to above, used the further assumption that \( \mathcal{V} \) is locally bounded, in the sense of [11, Chapter 6]. This includes all the key examples of [11], including the categories of sets, pointed sets, abelian groups, modules over a commutative ring, chain complexes, categories, groupoids, simplicial sets, compactly generated spaces (Hausdorff or otherwise, pointed or otherwise), Banach spaces, sheaves on a site, truth values, and Lawvere’s poset of extended non-negative real numbers. We shall often make stronger assumptions on \( \mathcal{V} \).

If the base \( \mathcal{V} \) is clear, we generally omit the prefix “\( \mathcal{V} \)-” and speak simply of a category, functor, or natural transformation.

A weight will be a presheaf \( F : \mathcal{A}^{\text{op}} \to \mathcal{V} \) which is a small colimit of representables. If \( \mathcal{A} \) is small, any presheaf on \( \mathcal{A} \) is a small colimit of representables, so there is no restriction. We sometimes say that \( F \) is small to mean that it is a small colimit of representables. See [6] for more on small functors.

For a \( \mathcal{V} \)-functor \( S : \mathcal{A}^{\text{op}} \to \mathcal{K} \), the limit \( \{F, S\} \) of \( S \) weighted by \( F \) is defined by a natural isomorphism

\[
\mathcal{K}(X, \{F, S\}) \cong [\mathcal{A}^{\text{op}}, \mathcal{V}](F, \mathcal{K}(X, S))
\]

while for a \( \mathcal{V} \)-functor \( R : \mathcal{A} \to \mathcal{K} \), the colimit \( F \ast R \) of \( R \) weighted by \( F \) is defined by a natural isomorphism

\[
\mathcal{K}(F \ast R, X) \cong [\mathcal{A}^{\text{op}}, \mathcal{V}](F, \mathcal{K}(R, X)).
\]
Note that the previous two displayed equations appear to involve hom-objects in \([\mathcal{A}^{\text{op}}, \mathcal{V}]\). If \(\mathcal{A}\) is large then \([\mathcal{A}^{\text{op}}, \mathcal{V}]\) does not exist as a \(\mathcal{V}\)-category; nonetheless, the desired hom-object will exist as an object of \(\mathcal{V}\), since \(F\) is small (see [6] for example).

In particular, if \(F\) and \(G\) are both small, then \([\mathcal{A}^{\text{op}}, \mathcal{V}]\)(\(F, G\)) exists as an object of \(\mathcal{V}\), and so we do have a \(\mathcal{V}\)-category \(\mathcal{P}A\) of all small presheaves on \(\mathcal{A}\). This is the free completion of \(\mathcal{A}\) under colimits [17].

A class \(\Phi\) of limits means a class of weights; then \(\Phi\)-completeness or \(\Phi\)-continuity means the existence or the existence and preservation of all limits with weights in \(\Phi\).

An important special case is the weight \(C : \mathcal{I}^{\text{op}} \to \mathcal{V}\), where \(\mathcal{I}\) is the unit \(\mathcal{V}\)-category consisting of a single object \(*\) with \(\text{hom } \mathcal{I}(*,*) = I\). Then to give the weight is just to give an object \(C \in \mathcal{V}\).

The \(C\)-weighted limit of a \(\mathcal{V}\)-functor \(S : \mathcal{I}^{\text{op}} \to \mathcal{K}\) (that is, of an object \(S \in \mathcal{K}\) is defined by a natural isomorphism

\[
\mathcal{K}(X, C \ast S) \cong \mathcal{V}(C, [X, S])
\]

has traditionally been called a cotensor, but we shall simply call a power, or \(C\)-power where necessary. The corresponding colimit, written \(C \cdot S\), used to be called a tensor, but we shall call a copower.

The ordinary, unweighted notion of limit can be seen as a special case. Let \(D\) be an ordinary category, and let \(\mathcal{F}D\) be the free \(\mathcal{V}\)-category on \(D\). Then \(\mathcal{V}\)-functors \(\mathcal{F}D \to \mathcal{K}\) correspond to functors \(D \to \mathcal{K}_0\), and we define the limit in \(\mathcal{K}\) of a functor \(S : D \to \mathcal{K}_0\) to be the limit of the corresponding \(R : \mathcal{F}D \to \mathcal{K}\) weighted by the terminal weight \(\Delta 1 : \mathcal{F}D \to \mathcal{K}\). Limits of this form are called conical.

The universal property of \({\Delta 1, R}\) involves an isomorphism in \(\mathcal{V}\), and is strictly stronger than the universal property of \(\text{lim } S\), which involves only a bijection of sets. Nonetheless, the universal property of \(\text{lim } S\) does serve to identify \({\Delta 1, R}\) if the latter is known to exist. Thus if \(\mathcal{K}\) and \(\mathcal{L}\) have the relevant limits, then to say that \(F : \mathcal{K} \to \mathcal{L}\) preserves a particular conical limit is equivalent to saying that \(F_0 : \mathcal{K}_0 \to \mathcal{L}_0\) does so.

We shall need the following basic result:

**Proposition 1.1** Let \(F : \mathcal{A}^{\text{op}} \to \mathcal{V}\) be a weight, and \(J : \mathcal{A} \to \mathcal{B}\) and \(S : \mathcal{B} \to \mathcal{C}\) functors. Then

\[
F \ast SJ \cong \text{Lan}_J F \ast S
\]

either side existing if the other does.

**Proof:** Here \(\text{Lan}_J F\) denotes the left Kan extension of \(F : \mathcal{A}^{\text{op}} \to \mathcal{V}\) along \(J : \mathcal{A} \to \mathcal{B}^{\text{op}}\). The result follows from the calculation

\[
\mathcal{C}(F \ast SJ, C) \cong [\mathcal{A}^{\text{op}}, \mathcal{V}](F, \mathcal{C}(SJ, C)) \cong [\mathcal{B}^{\text{op}}, \mathcal{V}](\text{Lan}_J F, \mathcal{C}(S, C)) \cong \mathcal{C}(\text{Lan}_J F \ast S, C).
\]

\[\Box\]

**Definition 1.2** A class \(\Phi\) of weights is said to be saturated if for any diagram \(S : \mathcal{D} \to [\mathcal{A}^{\text{op}}, \mathcal{V}]\) in a presheaf category, with each \(SD : \mathcal{A}^{\text{op}} \to \mathcal{V}\) in \(\Phi\), and for any \(F : \mathcal{D}^{\text{op}} \to \mathcal{V}\) in \(\Phi\), the colimit \(F \ast S \in [\mathcal{A}^{\text{op}}, \mathcal{V}]\) is also in \(\Phi\).
The original reference [4] used the word “closed” in place of “saturated”, but the latter is now standard. The basic result about a saturated class Φ is that the full subcategory of \([A^{\text{op}}, V]\) consisting of the presheaves in Φ is the free completion of \(A\) under \(\Phi\)-colimits (provided that \(A\) is small).

A \(V\)-functor \(F : A \to B\) with small domain is said to be dense [11, Chapter 5] if the induced \(V\)-functor \(B(F, 1) : B \to [A^{\text{op}}, V]\) sending \(B \in B\) to \(B(F(-), B) : A^{\text{op}} \to V\) is fully faithful.

Let \(F : A^{\text{op}} \to V\) and \(G : B^{\text{op}} \to V\) be weights, and \(K\) a category with \(F\)-limits and \(G\)-colimits. For any \(S : A^{\text{op}} \otimes B \to K\), and any \(B \in B\), we can form the limit \(\{F, S(-, B)\}\) in \(K\), and this defines the object part of a functor \(\{F, S\} : B \to K : B \mapsto \{F, S(-, B)\}\), to which we can now apply \(G*\) to obtain \(G*\{F, S\} \in K\). Similar we can form a functor \(G* S : A^{\text{op}} \to K\) and then \(\{F, G* S\} \in K\), and there is a canonical comparison map

\[
G* \{F, S\} \to \{F, G* S\}
\]
in \(K\). If this is invertible for all \(S\), we say that \(F\)-limits commute in \(K\) with \(G\)-colimits. The following observation was made in [14] in the case \(K = V\):

**Proposition 1.3** Let the \(V\)-category \(K\) have all \(F\)-limits and all \(G\)-colimits. Then the following are equivalent:

1. \(F\)-limits commute in \(K\) with \(G\)-colimits;
2. \(\{F, -\} : [A, K] \to K\) is \(G\)-cocontinuous;
3. \(G* - : [B^{\text{op}}, K] \to K\) is \(F\)-continuous.

More generally, if \(\Phi\) and \(\Psi\) are classes of weights, we say that \(\Phi\)-limits commute in \(K\) with \(\Psi\)-colimits if this is so for all \(F \in \Phi\) and all \(G \in \Psi\).

If \(G\) commutes in \(V\) with \(\Phi\)-limits we follow [14] in calling \(G\) is \(\Phi\)-flat, or just \(F\)-flat if \(\Phi = \{F\}\).

This is by analogy with the case where \(V = \text{Ab}\) and \(\Phi\) consists of the finite conical limits. Then a one-object \(\text{Ab}\)-category \(B\) is a ring, and a weight \(G : B^{\text{op}} \to \text{Ab}\) is a \(B\)-module, while \(G* -\) corresponds to tensoring over \(B\). A module is flat exactly when tensoring with that module preserves finite limits.

In [14, Proposition 5.4], the class of \(\Phi\)-flat weights is shown to be saturated. An important part of this is the following:

**Proposition 1.4** If \(F : A^{\text{op}} \to V\) if \(\Phi\)-flat and \(G : A^{\text{op}} \to B^{\text{op}}\) is arbitrary, then \(\text{Lan}_G F : B^{\text{op}} \to V\) is \(\Phi\)-flat.

**Proof:** To say that \(\text{Lan}_G F\) is \(\Phi\)-flat is to say that \(\text{Lan}_G F* -\) is \(\Phi\)-continuous. But by Proposition 1.1, this \(\text{Lan}_G F* -\) is given by the composite

\[
[\mathcal{B}, V] \xrightarrow{[J, \mathcal{V}]} [A, V] \xrightarrow{F* -} V
\]

and \([J, \mathcal{V}]\) preserves all limits, since limits in presheaf categories are calculated pointwise, while \(F* -\) is \(\Phi\)-continuous since \(F\) is \(\Phi\)-flat. \(\square\)
2 Key requirements

The first requirement involves $\Phi$ and $\mathcal{V}$. It is convenient to suppose that $\Phi$ is \textit{locally small} [14] in the sense that for any small $\mathcal{A}$, the closure of the representables in $[\mathcal{A}^{\text{op}}, \mathcal{V}]$ under $\Phi$-colimits is again small; typically this will happen because $\mathcal{V}_0$ is locally presentable and all weights in $\Phi$ are $\alpha$-presentable for some regular cardinal $\alpha$.

\textbf{Axiom A.} If $\mathcal{A}$ is a small $\mathcal{V}$-category with $\Phi$-limits, and $F : \mathcal{A} \to \mathcal{V}$ is $\Phi$-continuous, then so is $F^* - : [\mathcal{A}^{\text{op}}, \mathcal{V}] \to \mathcal{V}$.

Note that $F^* -$ is the left Kan extension $\text{Lan}_Y F$ of $F$ along the Yoneda embedding. This condition has been considered by many different authors in various special cases, and some of these are listed below when we turn to examples. In particular, it holds in the case $\mathcal{V} = \text{Set}$ if $\Phi$ consists of either finite products or finite limits. It was considered, still in the case $\mathcal{V} = \text{Set}$, for a general class of conical limits in [2], and in full generality in [14]. It could equivalently be stated as

$\Phi$-limits commute in $\mathcal{V}$ with colimits that have $\Phi$-continuous weights

or

All $\Phi$-continuous weights are $\Phi$-flat.

It is this condition which allows us to “decompose” colimits, in analogy with the finite/filtered decomposition, where now $\Phi$-colimits play the role of “finite”, and $\Phi$-flat colimits play the role of “filtered”. This can be done thanks to the following, which is a restatement of parts of [14, Theorems 8.9, 8.11]:

\textbf{Theorem 2.1} The following condition on the class $\Phi$ of weights are equivalent:

1. If $F : \mathcal{A} \to \mathcal{V}$ is $\Phi$-continuous then so is $F^* - : [\mathcal{A}^{\text{op}}, \mathcal{V}] \to \mathcal{V}$ (Axiom A)

2. The category $\Phi\text{-Cts}(\mathcal{A}, \mathcal{V})$ of $\Phi$-continuous presheaves is the free completion of $\mathcal{A}^{\text{op}}$ under $\Phi$-flat colimits

3. Any presheaf $F : \mathcal{A} \to \mathcal{V}$ is a $\Phi$-flat colimit of presheaves in $\Phi$

where in each case $\mathcal{A}$ is allowed to be any small $\mathcal{V}$-category, $\Phi$-complete in the first two cases.

\textbf{Proposition 2.2} Let $\mathcal{A}$ and $\mathcal{B}$ be small $\mathcal{V}$-categories with $\Phi$-limits, and $G : \mathcal{A} \to \mathcal{B}$ an arbitrary $\mathcal{V}$-functor. If $M : \mathcal{A} \to \mathcal{V}$ preserves $\Phi$-limits then so does $\text{Lan}_G M : \mathcal{B} \to \mathcal{V}$.

\textbf{Proof:} There is a functor $\mathcal{B}(G, 1) : \mathcal{B} \to [\mathcal{A}^{\text{op}}, \mathcal{V}]$ sending $B \in \mathcal{B}$ to $\mathcal{B}(G-, B) : \mathcal{A}^{\text{op}} \to \mathcal{V}$ which in turn sends an object $A \in \mathcal{A}$ to the hom-object $\mathcal{B}(GA, B)$. This functor preserves all existing limits, and its composite with $\text{Lan}_Y M : [\mathcal{A}^{\text{op}}, \mathcal{V}] \to \mathcal{V}$ is $\text{Lan}_G M$.

We now turn to the requirements on $\mathcal{K}$. It is possible to define models in any $\mathcal{V}$-category with $\Phi$-limits, but in order to develop the theory, somewhat more is required. We shall consider two levels of generality (more precise conditions will be given later).
Axiom B1. \( \mathcal{K} \) is locally \( \Phi \)-presentable: this is equivalent to saying that \( \mathcal{K} \) itself has the form \( \Phi \text{-Cts}(\mathcal{I}, \mathcal{V}) \) for some small \( \mathcal{V} \)-category \( \mathcal{I} \) with \( \Phi \)-limits. It follows that \( \mathcal{K} \) is reflective in \( [\mathcal{I}, \mathcal{V}] \), and so is complete and cocomplete.

Axiom B2. \( \mathcal{K} \) has \( \Phi \)-limits, and the inclusion \( y : \mathcal{K} \to \mathcal{P} \mathcal{K} \) has a \( \Phi \)-continuous left adjoint.

By Theorem 2.1, the Axiom B1 is equivalent to saying that \( \mathcal{K} \) is the free completion under \( \Phi \)-flat colimits of a small \( \Phi \)-cocomplete \( \mathcal{V} \)-category. Axiom B2 implies in particular that \( \mathcal{K} \) is cocomplete; note that \( \mathcal{P} \mathcal{K} \) has \( \Phi \)-limits by [6, Proposition 4.3] and Axiom A. Axiom B2 is a strong exactness condition, related to lex-totality [23]: when \( \mathcal{V} \) is \( \text{Set} \), and \( \Phi \) consists of the finite limits, it holds in any Grothendieck topos.

Remark 2.3 Both axioms imply that \( \Phi \)-limits commute in \( \mathcal{K} \) with \( \Phi \)-flat colimits, since this is true in \( \mathcal{V} \), and so in both \( [\mathcal{I}, \mathcal{V}] \) and \( \mathcal{P} \mathcal{K} \), since the the limits and colimits are computed pointwise there. Now \( \Phi \text{-Cts}(\mathcal{I}, \mathcal{V}) \) is closed in \( [\mathcal{I}, \mathcal{V}] \) under limits and \( \Phi \)-flat colimits, so the desired commutativity remains true there. In the case of Axiom B2, both \( \Phi \)-limits and arbitrary colimits may be computed in \( \mathcal{K} \) by passing to \( \mathcal{P} \mathcal{K} \) (where they commute) and then reflecting back into \( \mathcal{K} \).

Proposition 2.4 A presheaf category \( \mathcal{K} = [\mathcal{C}, \mathcal{V}] \) satisfies Axioms B1 and B2.

Proof: Axiom B1 is easy: if \( \mathcal{I} \) is the free \( \mathcal{V} \)-category with \( \Phi \)-limits on \( \mathcal{C} \), then \( \Phi \text{-Cts}(\mathcal{I}, \mathcal{V}) \cong [\mathcal{C}, \mathcal{V}] \).

As for Axiom B2, since \( [\mathcal{C}, \mathcal{V}] \) is cocomplete, the Yoneda functor \( \text{ev}_C : [\mathcal{C}, \mathcal{V}] \to \mathcal{P}[\mathcal{C}, \mathcal{V}] \) certainly has a left adjoint \( L \dashv Y \). Explicitly, for a small presheaf \( G : [\mathcal{C}, \mathcal{V}]^{\text{op}} \to \mathcal{V} \), the reflection \( LG \in [\mathcal{C}, \mathcal{V}] \) is the functor sending \( C \in \mathcal{C} \) to \( G(C, -) \).

Writing \( \text{ev}_C \) for the functor \( [\mathcal{C}, \mathcal{V}] \) given by evaluation at \( C \), and \( yC \) for the representable \( \mathcal{C}(C, -) \), we may therefore characterize \( L \) by the isomorphisms \( \text{ev}_C L \cong \text{ev}_yC \), natural in \( C \).

Let \( F : \mathcal{D} \to \mathcal{V} \) be in \( \Phi \), and \( S : \mathcal{D} \to \mathcal{P}[\mathcal{C}^{\text{op}}, \mathcal{V}] \). We must show that \( L \) preserves the limit \( \{F, S\} \); but this will be true if and only if \( \text{ev}_C L \) preserves the limit for each \( C \in \mathcal{C} \). This we show as follows:

\[
\text{ev}_C L \{F, S\} \cong \text{ev}_yC \{F, S\} \\
\cong \{F, \text{ev}_yS\} \\
\cong \{F, \text{ev}_C LS\}
\]

using the fact that limits are preserved by evaluation functors. \( \square \)

3 Sifted colimits and locally strongly finitely presentable categories

A key notion will be that of sifted colimit [1, 16]. A small category \( \mathcal{D} \) is said to be sifted if \( \mathcal{D} \)-colimits commute in \( \text{Set} \) with finite products; equivalently, if \( \mathcal{D} \) is non-empty and for all objects \( A, B \in \mathcal{D} \), the category of cospans from \( A \) to \( B \) is connected. Reflexive coequalizers and filtered colimits are both sifted, but Adámek [3] has given an example of a category with reflexive coequalizers and filtered colimits but not all sifted colimits.
Much of the theory of filtered colimits, involving finitely presentable objects and locally finitely presentable categories, has analogues involving sifted colimits. This was developed in [1], and put into a more general setting in [2].

For example, a functor $F : \mathcal{A} \to \text{Set}$ is said to be sifted-flat [1] if the left Kan extension $\text{Lan}_Y F : [\mathcal{A}^{\text{op}}, \text{Set}] \to \text{Set}$ preserves finite products. Since $\text{Lan}_Y F$ is also the functor $F * - : [\mathcal{A}^{\text{op}}, \text{Set}] \to \text{Set}$ calculating the $F$-weighted colimit of a functor $\mathcal{A}^{\text{op}} \to \text{Set}$, to say that $F$ is sifted-flat is equivalent to say that $F$-weighted colimits commute in $\text{Set}$ with finite products. In particular, if $\mathcal{A}$ is sifted then $\Delta_1 : \mathcal{A}^{\text{op}} \to \text{Set}$ is sifted-flat. The sifted-flat functors were characterized in [1, Theorem 2.6] as the presheaves which are sifted colimits of representables.

An object $X$ of a category $\mathcal{K}$ with sifted colimits is called strongly finitely presentable [1] if the representable functor $\mathcal{K}(X, -) : \mathcal{K} \to \text{Set}$ preserves sifted colimits.

In the following theorem, the equivalence between (i) and (ii) is a special case of the general characterization of free completions under colimits [11, Proposition 5.62], while the equivalence between (ii) and (iii) is [1, Theorem 3.10], but is essentially already in [7, Proposition 5.52].

**Theorem 3.1** For a category $\mathcal{K}$ the following conditions are equivalent:

(i) $\mathcal{K}$ is cocomplete and is the free completion under sifted colimits of a small category $\mathcal{G}$

(ii) $\mathcal{K}$ is cocomplete and has a small full subcategory $\mathcal{G}$ consisting of strongly finitely presentable objects, such that every object of $\mathcal{K}$ is a sifted colimit of objects in $\mathcal{G}$

(iii) $\mathcal{K}$ is equivalent to the category $\text{FPP}(\mathcal{G}^{\text{op}}, \text{Set})$ of finite-product-preserving functors from a small category $\mathcal{G}$ with finite coproducts to $\text{Set}$.

The strongly finitely presentable objects will be the closure under retracts of the category $\mathcal{G}$ in each case.

Such a category $\mathcal{K}$ is called locally strongly finitely presentable. The locally strongly finitely presentable categories are the (possibly multisorted) varieties.

We saw above that reflexive coequalizers and filtered colimits are not enough to guarantee all sifted colimits. On the other hand, the following proposition shows that preservation of reflexive coequalizers and filtered colimits is enough to guarantee preservation of sifted colimits, provided that all colimits actually exist. The history of this result is slightly complicated. The first-named author knew the result and its proof from soon after the time of the first papers on sifted colimits, but did not know that it was regarded as an important open problem, and did not publish it. An analogue in the context of quasicategories was recently proved by Joyal [9], and inspired by this, the result itself was proved by Adámek [3].

**Proposition 3.2** Let $\mathcal{K}$ be a cocomplete category, and $F : \mathcal{K} \to \mathcal{L}$ a functor. Then $F$ preserves sifted colimits if and only if it preserves reflexive coequalizers and filtered colimits.

**Proof:** Since reflexive coequalizers and filtered colimits are both sifted colimits, one direction is immediate. For the converse, suppose that $F$ preserves reflexive coequalizers and filtered colimits. Let $\mathcal{D}$ be a sifted category, and $S : \mathcal{D} \to \mathcal{K}$ a diagram. We must show that $F$ preserves the colimit of $S$.

Let $\text{Fam}\mathcal{D}$ be the free completion of $\mathcal{D}$ under finite coproducts, and $J : \mathcal{D}^{\text{op}} \to (\text{Fam}\mathcal{D})^{\text{op}}$ the canonical inclusion. Let $G = \text{Lan}_J \Delta_1$ be the left Kan extension of the terminal functor $\Delta_1 : \mathcal{D}^{\text{op}} \to \text{Set}$.
Set along $J$. Since $\mathcal{D}$ is sifted, $\Delta 1 : \mathcal{D}^{op} \to \textbf{Set}$ is sifted-flat, hence so by Proposition 1.4 is its left Kan extension $G$, and so $G$ is a sifted colimit of representables. But Fam$\mathcal{D}$ has finite coproducts, and so sifted colimits can be constructed using reflexive coequalizers and filtered colimits by [1, Example 2.3(2)]; thus $G$ can be constructed from the representables using reflexive coequalizers and filtered colimits. We conclude that any functor preserving reflexive coequalizers and filtered colimits also preserves $G$-weighted colimits.

For a functor $R$ with domain Fam$\mathcal{D}$, we have (see Proposition 1.1) canonical isomorphisms

$$G * R = \text{Lan}_J \Delta 1 * R \cong \Delta 1 * RJ \cong \text{colim}(RJ)$$

with all terms existing if any one of them does. If $R : \text{Fam}\mathcal{D} \to \mathcal{K}$ is the finite-coproduct-preserving functor extending $S$, then we get $G * R \cong \text{colim}(RJ) \cong \text{colim}(S)$. Since $F$ preserves reflexive coequalizers and filtered colimits, it preserves $G$-weighted colimits, and now

$$F \text{colim}(S) \cong F(G * R) \cong G * FR \cong \Delta 1 * FRJ \cong \text{colim}(FRJ) \cong \text{colim}(FS).$$

□

4 Possible choices for the class of limits

4.1 Finite limits

Fix a symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ with underlying category $\mathcal{V}_0$ complete and complete. As usual, an object $x \in \mathcal{V}_0$ is called finitely presentable if the representable functor $\mathcal{V}_0(x, -) : \mathcal{V}_0 \to \textbf{Set}$ preserves filtered colimits.

Kelly [10] defines $\mathcal{V}$ to be locally finitely presentable as a closed category if $\mathcal{V}_0$ is locally finitely presentable in the usual sense, and the finitely presentable objects are closed under the monoidal structure: the unit $I$ is finitely presentable, and the tensor product of any two finitely presentable objects is finitely presentable.

Remark 4.1 In fact all the key results of [10] remain true if we drop the assumption that $I$ is finitely presentable: see Remark 4.5 below. What is lost is the fact that finite presentability in $\mathcal{V}$ is the same as finite presentability in $\mathcal{V}_0$: one only knows that every finitely presentable object in $\mathcal{V}_0$ is finitely presentable in $\mathcal{V}$, not the converse. This possible generalization seems to be of limited interest — we know of no important new examples — but we mention it here, because a similar generalization will be important when we move from the locally finitely presentable case to the locally strongly finitely presentable case.

This now gives a good notion of finite limit in the $\mathcal{V}$-enriched sense. First of all an object $X$ of a cocomplete $\mathcal{V}$-category $\mathcal{K}$ is said to be finitely presentable if the hom-functor $\mathcal{K}(X, -) : \mathcal{K} \to \mathcal{V}$ preserves filtered colimits.

For an object $X$ of a cocomplete $\mathcal{V}$-category $\mathcal{K}$, there is in general no relation between the property of being finitely presentable in $\mathcal{K}$ and the property of being finitely presentable in the underlying ordinary category $\mathcal{K}_0$ of $\mathcal{K}$. But if $\mathcal{V}$ is locally finitely presentable as a closed category, then these two notions agree for $\mathcal{K} = \mathcal{V}$ (and more generally for any locally finitely presentable $\mathcal{V}$-category $\mathcal{K}$).
The finite limits are now those in the saturation of the class of finite conical limits and \( V_f \)-powers, where \( V_f \)-powers are powers (cotensors) by finitely presentable objects of \( V \). We now take these finite limits to be our class \( \Phi \). The fact that Axiom A holds is Theorem 6.12 of [10]. Furthermore, Axiom B will hold if \( \mathcal{K} \) is any locally finitely presentable \( V \)-category, in the sense of [10]; in other words, if \( \mathcal{K} \) is a full reflective subcategory of a presheaf category \([\mathcal{C}, V]\) which is closed in \([\mathcal{C}, V]\) under filtered colimits; or, equivalently, if \( \mathcal{K} \) is the category \( \text{Lex}(\mathcal{T}, V) \) of models of a small \( V \)-category \( \mathcal{T} \) with finite limits.

Many examples of locally finitely presentable \( V \) were given in [10]; they include the closed categories of sets, pointed sets, abelian groups, modules over a commutative ring, chain complexes, categories groupoids, and simplicial sets.

Most of the general results we shall prove about monads and theories for \( \mathcal{K} = V \) were obtained in this case by Kelly in [10], but for the part involving monads on, and single-sorted theories in, \( V \) see [19]. The treatment of monads and single-sorted theories for more general \( \mathcal{K} \) (still with \( \Phi \) the class of finite limits) appeared in [18].

4.2 Finite products

In [5] the notion of \( \pi \)-category was introduced as a suitable setting for universal algebra. Explicitly, this is a complete and cocomplete symmetric monoidal category \( V \), such that Axiom A holds for \( \Phi \) the class of finite products, and furthermore, the functors \(- \times X : V \to V\) and \( X \times - : V \to V\) preserve reflexive coequalizers and filtered colimits for all objects \( X \). By Proposition 3.2, this condition on \(- \times X\) and \( X \times -\) is equivalent to saying that they preserve sifted colimits.

**Proposition 4.2** Let \( \mathcal{K} \) be a cocomplete category with finite products. The following conditions are equivalent:

(i) \( X \times - : \mathcal{K} \to \mathcal{K} \) preserves sifted colimits for all \( X \)

(ii) \( - \times X : \mathcal{K} \to \mathcal{K} \) preserves sifted colimits for all \( X \)

(iii) \( \times : \mathcal{K} \times \mathcal{K} \to \mathcal{K} \) preserves sifted colimits

(iv) sifted colimits commute in \( \mathcal{K} \) with finite products

In the setting of [5], the \( V \)-category we are calling \( \mathcal{K} \) is always \( V \) itself. Under these assumptions, the various results we have considered were proved in [5] (for the case \( \mathcal{K} = V \)): left adjoints to algebraic functors, reflectiveness of models, correspondence between monads and theories, and so on.

Now finite products commute with sifted colimits in \( \text{Set} \), and more generally in any locally strongly finitely presentable category.

If \( V_0 \) is locally strongly finitely presentable then \( - \times X \) and \( X \times - \) will preserve sifted colimits; and now if Axiom A holds, \( V \) will be a \( \pi \)-category.

In the case \( V = \text{Cat} \), the \( \Phi \)-accessible monads on \( \text{Cat} \) will be the strongly finitary 2-monads of [12]. These correspond to (a finitary version of) the discrete Lawvere theories of [20].

In the following section these sifted colimits will play a still more central role, and we shall see how they allow a more expressive notion of theory than that of [5], although with somewhat greater restrictions on \( V \).
4.3 Strongly finite limits

In this section, which is one of the main original contributions of the paper, we adapt the setting of [10] using sifted colimits in place of filtered ones. The notion of strongly finite limit, introduced below, reduces to that of finite product in the case \( \mathcal{V} = \text{Set} \), but not in general.

Suppose as usual that \( \mathcal{V} = (\mathcal{V}_0, \otimes, I) \) is a complete and cocomplete symmetric monoidal closed category. This time we suppose that \( \mathcal{V}_0 \) is locally strongly finitely presentable, and so has the form \( \text{FPP}(\mathcal{G}^{\text{op}}, \text{Set}) \) for a category \( \mathcal{G} \) with finite coproducts, which we may take to be the category of strongly finitely presentable objects of \( \mathcal{V}_0 \).

The directly analogous approach to [10] would be to suppose that \( \mathcal{G} \) was closed under the monoidal structure; unfortunately this is not true for key examples such as \( \mathcal{V} = \text{Set} \), but not that it contains the unit \( I \). This time we suppose that \( \mathcal{V}_0 \) is strongly finitely presentable objects of \( \mathcal{F}^{\text{op}} \).

We weaken the assumption slightly by supposing that \( \mathcal{G} \) is closed under the tensor product, but not that it contains the unit \( I \). We then say that \( \mathcal{V} \) is locally strongly finitely presentable as a \( \otimes \)-category.

Given such a \( \mathcal{V} \), we can now develop the theory of locally strongly finitely presentable categories in the \( \mathcal{V} \)-enriched context. We say that an object \( X \) of a cocomplete \( \mathcal{V} \)-category \( \mathcal{K} \) is strongly finitely presentable if the hom-functor \( \mathcal{K}(X, -) : \mathcal{K} \to \mathcal{V} \) preserves sifted colimits, and write \( \mathcal{K}_{sf} \) for the full subcategory of \( \mathcal{K} \) consisting of such objects.

Just as in [10], it is important to distinguish between \( X \) being strongly finitely presentable in \( \mathcal{K} \), in the sense that \( \mathcal{K}(X, -) : \mathcal{K} \to \mathcal{V} \) preserves sifted colimits, and \( X \) being strongly finitely presentable in \( \mathcal{K}_0 \), in the sense that \( \mathcal{K}_0(X, -) : \mathcal{K}_0 \to \text{Set} \) preserves such sifted colimits. (On the other hand, since \( \mathcal{K} \) and \( \mathcal{V} \) do have sifted colimits, to say that the \( \mathcal{V} \)-functor \( \mathcal{K}(X, -) : \mathcal{K} \to \mathcal{V} \) preserves sifted colimits is no different to saying that the underlying ordinary functor \( \mathcal{K}(X, -)_0 : \mathcal{K}_0 \to \mathcal{V}_0 \) preserves sifted colimits.) If \( I \in \mathcal{V}_0 \) were strongly finitely presentable, then every strongly finitely presentable object in \( \mathcal{K} \) would be strongly finitely presentable in \( \mathcal{K}_0 \), but we are not assuming this. What we do have is:

Lemma 4.3 If \( X \in \mathcal{V} \) is strongly finitely presentable as an object of \( \mathcal{V}_0 \), in the sense that \( \mathcal{V}_0(X, -) : \mathcal{V}_0 \to \text{Set} \) preserves sifted colimits, then it is strongly finitely presentable in \( \mathcal{V} \).

Proof: Suppose that \( X \) is strongly finitely presentable in \( \mathcal{V}_0 \). Then it is a retract of a finite coproduct of objects in \( \mathcal{G} \). If we now tensor \( X \) by an arbitrary \( G \in \mathcal{G} \), the resulting \( G \otimes X \) is again a retract of a finite coproduct of objects in \( \mathcal{G} \), since \( \mathcal{G} \) is closed under tensoring. Thus \( G \otimes X \) is again strongly finitely presentable in \( \mathcal{V}_0 \), and so \( \mathcal{V}_0(G \otimes X, -) \) preserves sifted colimits. But this means that \( \mathcal{V}_0(G, \mathcal{V}(X, -)) \) preserves sifted colimits. Since the \( \mathcal{V}_0(G, -) \) preserve and jointly reflect sifted colimits, it follows that \( \mathcal{V}(X, -) \) preserves sifted colimits, and so that \( X \) is strongly finitely presentable in \( \mathcal{V} \).

The converse is false: we shall see in Example 4.14 below that if \( \mathcal{V} \) is the cartesian closed category \( \mathcal{G}^{\text{op}} \) of graphs, then the terminal object is strongly finitely presentable in \( \mathcal{V} \) but not in \( \mathcal{V}_0 \).

As a further indication of the distinction between the properties of being strongly finitely presentable in \( \mathcal{V} \) or \( \mathcal{V}_0 \), notice that although we had to assume that the strongly finitely presentable objects of \( \mathcal{V}_0 \) were closed under tensoring, this is automatic for the strongly finitely presentable objects of \( \mathcal{V} \), since \( \mathcal{V}(X, \mathcal{V}(Y, -)) \cong \mathcal{V}(X \otimes Y, -) \) and sifted-colimit-preserving functors are closed under composition.
An important technical result is:

**Proposition 4.4** \( \mathcal{V}_{sf} \) is (equivalent to) a small \( \mathcal{V} \)-category.

**Proof:** Since \( \mathcal{V}_0 \) is locally strongly finitely presentable, it is also locally finitely presentable. Thus there is a regular cardinal \( \alpha \) for which \( I \) is \( \alpha \)-presentable; and \( \mathcal{V}_0 \) is still locally \( \alpha \)-presentable. The \( \alpha \)-presentable objects of \( \mathcal{V}_0 \) are the \( \alpha \)-colimits of the finitely presentable ones, and these are closed under tensoring, and by assumption they contain the unit object \( I \). It now follows, just as in [10, 5.2, 5.3] that an object is \( \alpha \)-presentable in \( \mathcal{V} \) if and only if it is \( \alpha \)-presentable in \( \mathcal{V}_0 \). An object of \( \mathcal{V}_{sf} \) is certainly \( \alpha \)-presentable in \( \mathcal{V} \); thus \( (\mathcal{V}_{sf})_0 \) is a full subcategory of \( (\mathcal{V}_0)_\alpha \), which is small, and so \( \mathcal{V}_{sf} \) too is small. \( \square \)

**Remark 4.5** All the key results of [10] remain true if \( \mathcal{V}_0 \) is locally finitely presentable and the tensor product of two finitely presentable objects is finitely presentable. The reason for assuming that the unit \( I \) is finitely presentable is that then the notions of finite presentability in \( \mathcal{V} \) and in \( \mathcal{V}_0 \) agree. But this is only used at one point: in the proof of 7.1, in order to prove that the full subcategory \( \mathcal{V}_f \) of finitely presentable objects in \( \mathcal{V} \) is small. However this can be obtained alternatively as follows. As observed in [13], if \( \mathcal{V}_0 \) is locally finitely presentable, then \( \mathcal{V} \) is locally \( \alpha \)-presentable as a closed category for some regular cardinal \( \alpha \) — by the same argument that was used in the previous proposition. Then the full subcategory \( \mathcal{V}_\alpha \) of \( \mathcal{V} \) consisting of the \( \alpha \)-presentable objects is small, by the same argument as in [10]; but \( \mathcal{V}_f \) is clearly contained in \( \mathcal{V}_\alpha \) and so is also small.

Having fixed our monoidal category \( \mathcal{V} \), we now turn to the class \( \Phi \) of weights. We take for \( \Phi \) the saturation of the class of finite products and \( \mathcal{V}_{sf} \)-powers (powers by objects of \( \mathcal{V}_{sf} \)).

**Proposition 4.6** If \( \mathcal{K} \) is cocomplete, the strongly finitely presentable objects of \( \mathcal{K} \) are closed under \( \Phi \)-colimits.

**Proof:** It suffices to show that they are closed under finite coproducts and under \( \mathcal{V}_{sf} \)-copowers. If \( X_1, \ldots, X_n \) are strongly finitely presentable, then

\[
\mathcal{K}(X_1 + \ldots + X_n, -) \cong \mathcal{K}(X_1, -) \times \ldots \times \mathcal{K}(X_n, -)
\]

and each \( \mathcal{K}(X_i, -) \) preserves sifted colimits since \( X_i \) is strongly finitely presentable, while finite products of sifted-colimit-preserving functors into \( \mathcal{V} \) still preserve sifted colimits, since finite products commute with sifted colimits in \( \mathcal{V} \). This proves that \( X_1 + \ldots + X_n \) is strongly finitely presentable.

Similarly if \( X \in \mathcal{K} \) is strongly finitely presentable, and \( G \in \mathcal{V} \) is strongly finitely presentable, then

\[
\mathcal{K}(G \cdot X, -) \cong \mathcal{V}(G, \mathcal{K}(X, -))
\]

which preserves sifted colimits since \( \mathcal{K}(X, -) \) and \( \mathcal{V}(G, -) \) do, thus \( G \cdot X \) is strongly finitely presentable. \( \square \)

**Corollary 4.7** If \( F : \mathcal{A}^{\text{op}} \to \mathcal{V} \) is in \( \Phi \) then it is strongly finitely presentable as an object of \( [\mathcal{A}^{\text{op}}, \mathcal{V}] \).
Proof: This is immediate from the case \( \mathcal{K} = [\mathcal{A}^{\text{op}}, \mathcal{V}] \) of the proposition, given that representables are strongly finitely presentable, and \( F \) is a \( \Phi \)-colimit of representables, via the Yoneda isomorphism \( F \cong F * Y \). \qed

The following theorem, adapted from [10, Theorem 6.11], implies in particular that Axiom A holds:

**Theorem 4.8** Let \( \mathcal{T} \) be a small \( \mathcal{V} \)-category with \( \Phi \)-limits. For a \( \mathcal{V} \)-functor \( F : \mathcal{T} \to \mathcal{V} \) the following are equivalent:

1. \( F \) is a sifted colimit of representables;
2. \( F \) is \( \Phi \)-flat;
3. \( F \) is \( \Phi \)-continuous.

**Proof:** Representables are \( \Phi \)-flat, and sifted colimits commute in \( \mathcal{V} \) with \( \Phi \)-limits, thus (1) implies (2). To see that (2) implies (3), observe that if \( F \) is \( \Phi \)-flat then \( \text{Lan}_Y F \) is \( \Phi \)-continuous, but \( Y \) is \( \Phi \)-continuous, hence so is \( F = (\text{Lan}_Y F)Y \).

So it remains to prove that (3) implies (1). Suppose then that \( F \) is \( \Phi \)-continuous. Consider the underlying ordinary functor \( F_0 : \mathcal{T}_0 \to \mathcal{V}_0 \), and the induced \( \mathcal{V}_0(\mathcal{T}, F_0) : \mathcal{T}_0 \to \text{Set} \). Like any \text{Set}-valued functor, this is canonically a colimit of representables, We form the category of elements \( E \) and the induced \( \mathcal{V}_0(\mathcal{T}, F_0) : \mathcal{T}_0 \to \text{Set} \) which sends an object \((T, x : \mathcal{T}(T, -) \to F)\) of \( E \) to \( T \). Then \( \mathcal{V}_0(\mathcal{T}, F_0) \) is the colimit of the composite

\[
E \xrightarrow{P} \mathcal{T}_0^{\text{op}} \xrightarrow{Y} [\mathcal{T}_0, \text{Set}]
\]

Since \( F \) preserves finite products, so does \( \mathcal{V}_0(\mathcal{T}, F_0) \); it follows that \( E \) has finite coproducts, and so is sifted. Thus we have expressed \( \mathcal{V}_0(\mathcal{T}, F_0) \) as a sifted colimit of representables.

The idea is to adapt this to obtain \( F \) and not just \( \mathcal{V}_0(\mathcal{T}, F_0) \). Consider now the composite

\[
E \xrightarrow{P} \mathcal{T}_0^{\text{op}} \xrightarrow{Y_0} [\mathcal{T}, \mathcal{V}]_0
\]

which sends an object \((T, x : \mathcal{T}(T, -) \to F)\) of \( E \) to \( \mathcal{T}(T, -) \). We shall show that it has colimit \( F \), and so that \( F \) is a sifted colimit of representables. There is an evident cocone \( \gamma : Y_0P \to \Delta F \), whose coprojection at \((T, x : \mathcal{T}(T, -) \to F)\) is just \( x \). We must show that this is a colimit. It will be a colimit if and only if it gives a colimit after evaluating at all \( S \in \mathcal{T} \); in other words, if the \( E \)-indexed diagram

\[
\mathcal{T}(T, S) \longrightarrow FS
\]

is a colimit (in \( \mathcal{V} \), or equivalently in \( \mathcal{V}_0 \)) for all \( S \). Now the hom-functors \( \mathcal{V}_0(G, -) : \mathcal{V}_0 \to \text{Set} \) for \( G \in \mathcal{G} \) jointly reflect colimits, since \( \mathcal{G} \) is a strong generator, and they preserve sifted colimits, by assumption on \( \mathcal{G} \). Thus we are reduced to showing that the \( E \)-indexed diagram

\[
\mathcal{V}_0(G, \mathcal{T}(T, S)) \longrightarrow \mathcal{V}_0(G, FS)
\]

is a colimit in \( \text{Set} \), for all \( S \in \mathcal{T} \) and all \( G \in \mathcal{G} \). But by the universal property of powers, and the fact that \( F \) preserves \( \mathcal{G} \)-powers, this diagram is equivalently

\[
\mathcal{V}_0(I, \mathcal{T}(G \downharpoonright S)) \longrightarrow \mathcal{V}_0(I, F(G \downharpoonright S))
\]
and this is a colimit since

\[ \mathcal{V}_0(I, \mathcal{V}(T, -)) \longrightarrow \mathcal{V}_0(I, F) \]

is one. \( \square \)

We now turn to examples of \( \mathcal{V} \) which are locally strongly finitely presentable as \( \otimes \)-categories. An important special case is where \( \mathcal{V} \) is a presheaf category, equipped with the cartesian closed structure. First we prove:

**Lemma 4.9** In a presheaf category \([C^{\text{op}}, \text{Set}]\), an object is a retract of a finite coproduct of representables if and only if it is a finite coproduct of retracts of representables.

**Proof:** If \( R_i \) is a retract of the representable \( yC_i \) for \( i = 1, \ldots, n \) then \( \sum_i R_i \) is a retract of \( \sum_i yC_i \). Thus a finite coproduct of retracts of representables is a retract of a finite coproduct of representables.

Conversely, suppose that \( R \) is a retract of a finite coproduct \( \sum_i yC_i \) of representables. By extensivity of \([C^{\text{op}}, \text{Set}]\), we can write the inclusion \( R \to \sum_i yC_i \) as a coproduct \( \sum_i R_i \to \sum_i yC_i \) where each \( R_i \to yC_i \) is the inclusion of a retract. \( \square \)

**Corollary 4.10** If idempotents split in \( C \), then the retracts of finite coproducts of representables are just the finite coproducts of representables.

**Corollary 4.11** A retract of a finite coproduct of retracts of finite coproducts of representables is a retract of a finite coproduct of representables.

**Proposition 4.12** If \( \mathcal{V} = [C^{\text{op}}, \text{Set}] \), equipped with the cartesian closed structure, then \( \mathcal{V} \) is locally strongly finitely presentable as a \( \otimes \)-category if and only if the product of any two representables is a retract of a finite coproduct of representables.

**Proof:** In this case \((\mathcal{V}_0)_{sf}\) is obtained from \( C \) by freely adjoining finite coproducts and then splitting idempotents; in other words it may be identified with the full subcategory of \([C^{\text{op}}, \text{Set}]\) consisting of the retracts of finite coproducts of representables. Then \( \mathcal{V} \) will be locally strongly finitely presentable as a \( \otimes \)-category if and only if this subcategory is closed under binary products.

This certainly implies that the product of any two representables is in the subcategory, but in fact it is equivalent: if \( R \) and \( S \) are retracts of finite coproducts \( \sum_i C_i \) and \( \sum_j D_j \) of representables, then \( R \times S \) is a retract of the finite sum \( \sum_i \sum_j C_i \times D_j \) of products of representables. If each \( C_i \times D_j \) is a retract of a finite coproduct of representables, then so is \( R \times S \) by Corollary 4.11. \( \square \)

**Corollary 4.13** \([C^{\text{op}}, \text{Set}]\) is locally strongly finitely presentable as a \( \otimes \)-category if \( C \) has binary products.

**Example 4.14** The cartesian closed category \( \text{Gph} \) of (directed) graphs is locally strongly finitely presentable as a \( \otimes \)-category. There are two representables: the free-living edge \( E \), and the free-living vertex \( V \). It is easy to check that \( V \times V \cong V \), \( V \times E \cong E \times V \cong V + V \), and \( E \times E \cong V + E + V \), so that the product of any two representables is a finite coproduct of representables. On the other hand, the terminal graph \( 1 \) (which is of course the unit for the cartesian monoidal structure) is not a finite coproduct of representables, and so \( \text{Gph} \) is not locally strongly finitely presentable as a **closed** category.
We can also see this more directly by exhibiting a sifted colimit in \( \mathbf{Gph} \) which \( \mathbf{Gph}(1, -) : \mathbf{Gph} \to \mathbf{Set} \) fails to preserve. Now \( \mathbf{Gph}(1, -) \) sends a graph \( G \) to a vertex \( v \in G \) with a chosen loop. Consider the parallel pair

\[
E + V \xrightarrow{f} E \xleftarrow{g} E
\]

where \( f \) and \( g \) act as the identity on the \( E \) component, while on the \( V \) component \( f \) picks out the source of the edge in \( E \), and \( g \) picks out the target. The coequalizer is formed by identifying the source and target, which now gives a loop: in other words, the coequalizer is \( 1 \). This pair is clearly reflexive, and so its coequalizer is a sifted colimit. But it is not preserved by \( \mathbf{Gph}(1, -) \), since \( \mathbf{Gph}(1, 1) \) has one element, while \( \mathbf{Gph}(1, E) \) is empty.

**Example 4.15** The cartesian closed category \( \mathbf{RGph} \) of reflexive graphs is not locally strongly finitely presentable as a \( \otimes \)-category. In this case there are two representables: the terminal graph \( 1 \) with a single loop, and the reflexive graph \( E \) generated by a single edge. It follows that any strongly finitely presentable reflexive graph can only have these graphs as its connected components. But if we take the product of two copies of \( E \) we obtain the reflexive graph

\[
\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \longrightarrow & \bullet
\end{array}
\]

which, by Corollary 4.10, is not strongly finitely presentable. (This time the terminal object is not just strongly finitely presentable, but representable.)

**Example 4.16** Let \( \mathbb{I} \) be a skeletal category of finite sets and injections, and \( \mathcal{V} \) the cartesian closed category \( [\mathbb{I}, \mathbf{Set}] \). This is locally strongly finitely presentable as a \( \otimes \)-category: given representables \( \mathbb{I}(m, -) \) and \( \mathbb{I}(n, -) \) we have the formula

\[
\mathbb{I}(m, -) \times \mathbb{I}(n, -) \cong \sum_{k = \max(m, n)}^{m+n} \left( \begin{array}{c} m \\ m+n-k \end{array} \right) \times \left( \begin{array}{c} n \\ m+n-k \end{array} \right) \cdot \mathbb{I}(k, -)
\]

exhibiting \( \mathbb{I}(m, -) \times \mathbb{I}(n, -) \) as a finite coproduct of representables.

The category \( [\mathbb{I}, \mathbf{Set}] \) is one of models giving a denotational semantics for the \( \pi \)-calculus [22]. Since all operations used there have strongly finitely presentable arities, our formalism might be useful there.

It is now possible to develop a theory of locally strongly finitely presentable \( \mathcal{V} \)-categories as one might expect. We say that a \( \mathcal{V} \)-category \( \mathcal{K} \) is locally strongly finitely presentable if

(i) \( \mathcal{K} \) is cocomplete

(ii) \( \mathcal{K} \) has a small full subcategory \( \mathcal{G} \) consisting of strongly finitely presentable objects

(iii) every object of \( \mathcal{K} \) is a sifted colimit of objects in \( \mathcal{G} \)

We shall provide various characterizations below; in the meantime we prove:
**Theorem 4.17** Any locally strongly finitely presentable \( \mathcal{V} \)-category \( \mathcal{K} \) is equivalent to the category of \( \Phi \)-continuous \( \mathcal{V} \)-functors from \( \mathcal{T} \) to \( \mathcal{V} \) for some small \( \Phi \)-complete \( \mathcal{V} \)-category \( \mathcal{T} \).

**Proof:** Let \( \overline{\mathcal{G}} \) be the closure of \( \mathcal{G} \) in \( \mathcal{K} \) under \( \Phi \)-colimits; that is, under finite coproducts and \( \mathcal{V}_{sf} \)-copowers. Since \( \mathcal{V}_{sf} \) is small, it follows that \( \overline{\mathcal{G}} \) is still small. By assumption it has \( \Phi \)-colimits, so \( \mathcal{T} = \overline{\mathcal{G}}^{op} \) is a small \( \mathcal{V} \)-category with \( \Phi \)-limits. The inclusion of \( \overline{\mathcal{G}} \to \mathcal{K} \) induces a \( \mathcal{V} \)-functor \( W : \mathcal{K} \to \left[ \overline{\mathcal{G}}^{op}, \mathcal{V} \right] = \left[ \mathcal{T}, \mathcal{V} \right] \). Since \( \Phi \)-colimits of strongly finitely presentable objects are strongly finitely presentable, the objects of \( \overline{\mathcal{G}} \) are strongly finitely presentable, and so the functor \( W : \mathcal{K} \to \left[ \overline{\mathcal{G}}^{op}, \mathcal{V} \right] \) preserves sifted colimits.

Since \( \mathcal{G} \) is contained in \( \mathcal{K}_{sf} \) it follows by Proposition 4.6 that \( \mathcal{G} \) is so too, and so that \( W \) actually lands in the category \( \Phi \text{-Cts}(\mathcal{T}, \mathcal{V}) \) of \( \Phi \)-continuous functors. Since \( \mathcal{G} \) is dense [11, Theorem 5.35] it follows that \( \mathcal{G} \) is dense and so that \( W \) is fully faithful [11, Theorem 5.13]. It remains to show that \( \Phi \text{-Cts}(\mathcal{T}, \mathcal{V}) \) is the image of \( W \); in other words that every \( \Phi \)-continuous \( F : \mathcal{T} \to \mathcal{V} \) has the form \( W A \) for some \( A \in \mathcal{K} \).

Suppose then that \( F : \overline{\mathcal{G}}^{op} \to \mathcal{V} \) is \( \Phi \)-continuous; by Theorem 4.8 it is a sifted colimit of representables, say \( F = \text{colim}_i yC_i \). Write \( J : \overline{\mathcal{G}} \to \mathcal{K} \) for the inclusion. Since \( \mathcal{K} \) is cocomplete, we may form the colimit \( F \ast J = (\text{colim}_i yC_i) \ast J \cong \text{colim}_i (yC_i \ast J) \cong \text{colim}_i (JC_i) \) which will be preserved by \( W \), since \( W \) preserves sifted colimits. Then \( W(F \ast J) \cong F \ast WJ \cong F \ast Y \cong F \), and so \( F \) is indeed in the image of \( W \). \( \Box \)

We shall see below that the converse is also true, but we shall prove this using some of the more general theory developed in Section 5 below.

### 4.4 Sound doctrines

We now show the case of finite limits and of strongly finite limits fit into a common framework. The treatment follows that of Section 4.3 very closely, so we leave out some of the details.

Recall from [2] the notion of sound doctrine. In that paper a doctrine consisted of a small collection \( \mathcal{D} \) of small categories \( \mathcal{D} \). Then a category \( \mathcal{C} \) was said to be \( \mathcal{D} \)-filtered if \( \mathcal{D} \)-colimits commute in \( \text{Set} \) with \( \mathcal{D} \)-limits. It follows that for any \( \mathcal{D} \in \mathcal{D} \) and any diagram \( S : \mathcal{D}^{op} \to \mathcal{C} \), the category of cocones is connected. If conversely, the connectedness of the category of cocones of a diagram \( \mathcal{D}^{op} \to \mathcal{C} \) implies that \( \mathcal{C} \) is \( \mathcal{D} \)-filtered, then the doctrine \( \mathcal{D} \) is said to be *sound*.

The first main theorem about such doctrines [2, Theorem 2.4] includes in particular:

**Theorem 4.18** For a sound doctrine \( \mathcal{D} \) and a functor \( F : \mathcal{A} \to \text{Set} \) with \( \mathcal{A} \) small, the following are equivalent:

1. \( \text{Lan}_Y F \) is \( \mathcal{D} \)-continuous
2. \( F \) is a \( \mathcal{D} \)-filtered colimit of representables

and if \( \mathcal{A} \) has \( \mathcal{D} \)-limits then these are further equivalent to

3. \( F \) is \( \mathcal{D} \)-continuous.

We recover the setting of Sections 4.1 and 4.3 by taking for \( \mathcal{D} \), respectively, all finite categories and all finite discrete categories.
A category is said to be **locally** \( \mathcal{D} \)-**presentable** [2] if it is equivalent to the category of \( \mathcal{D} \)-continuous functors from \( \mathcal{A} \) to \( \text{Set} \), for a small category \( \mathcal{A} \) with \( \mathcal{D} \)-limits. There are analogues for all the main results about locally finitely presentable categories, and these results are then recovered on taking \( \mathcal{D} \) to be the finite categories.

Suppose that \( \mathcal{D} \) is sound, and let \( \mathcal{V}_0 \) be locally \( \mathcal{D} \)-presentable, with the subcategory \( (\mathcal{V}_0)_{\mathcal{D}} \) of \( \mathcal{D} \)-presentable objects closed under the tensor product in \( \mathcal{V} \). We then say that \( \mathcal{V} \) is **locally** \( \mathcal{D} \)-**presentable** as a \( \otimes \)-category.

Let \( \mathcal{V}_{\mathcal{D}} \) be the full sub-\( \mathcal{V} \)-category of \( \mathcal{V} \) consisting of the \( \mathcal{D} \)-presentable objects of \( \mathcal{V} \): those objects \( G \in \mathcal{V} \) for which \([G, -]: \mathcal{V} \to \mathcal{V}\) preserves \( \mathcal{D} \)-filtered colimits.

**Lemma 4.19** If \( G \) is \( \mathcal{D} \)-presentable in \( \mathcal{V}_0 \) then it is \( \mathcal{D} \)-presentable in \( \mathcal{V} \).

**Proof:** We know that \( \mathcal{V}_0 \) is (equivalent to) the category of \( \mathcal{D} \)-continuous functors from \( \mathcal{I} \) to \( \text{Set} \) for some small category \( \mathcal{I} \) with \( \mathcal{D} \)-limits. If \( \mathcal{D} \)-limits do not already include the splittings of idempotents then we may split the idempotents of \( \mathcal{I} \) without changing the \( \mathcal{D} \)-continuous functors. A \( \mathcal{D} \)-presentable object of \( \mathcal{V}_0 \) is then a representable functor \( y_T = \mathcal{I}(T, -) \). We must show that the internal hom \([y_T, -]: \mathcal{V} \to \mathcal{V}\) preserves \( \mathcal{D} \)-filtered colimits, or equivalently that the \( \mathcal{V}_0(\mathcal{V}_0, -): \mathcal{V}_0 \to \mathcal{V}_0 \) does. But the \( \mathcal{V}_0(\mathcal{V}_0, [y_T, -]): \mathcal{V}_0 \to \text{Set} \) preserve and detect \( \mathcal{D} \)-filtered colimits, so it suffices to show that the \( \mathcal{V}_0(\mathcal{V}_0, [y_T, -]) \) preserve \( \mathcal{D} \)-filtered colimits. Finally \( \mathcal{V}_0(\mathcal{V}_0, [y_T, -]) \cong \mathcal{V}_0(yS \otimes yT, -) \), but \( \mathcal{V}_0(yS \otimes yT, -) \) preserves \( \mathcal{D} \)-filtered colimits since \( yS \otimes yT \) is \( \mathcal{D} \)-presentable by assumption. \( \square \)

**Proposition 4.20** \( \mathcal{V}_{\mathcal{D}} \) is (equivalent to) a small \( \mathcal{V} \)-category.

**Proof:** Any locally \( \mathcal{D} \)-presentable category is locally \( \alpha \)-presentable for some regular cardinal \( \alpha \) (see [2, Theorem 5.5]): we may take \( \alpha \) to be larger than the cardinality of any \( \mathcal{D} \in \mathcal{D} \) and such that the unit \( I \) is \( \alpha \)-presentable. It then follows that \( \mathcal{V} \) is locally \( \alpha \)-presentable as a closed category, and so that \( \mathcal{V}_\alpha \) is small; but \( \mathcal{V}_{\mathcal{D}} \) is contained in \( \mathcal{V}_\alpha \). \( \square \)

Let \( \Phi \) be the saturation of the class of all (conical) \( \mathcal{D} \)-limits and \( \mathcal{V}_{\mathcal{D}} \)-powers. These limits certainly commute in \( \mathcal{V} \) with \( \mathcal{D} \)-filtered colimits.

**Proposition 4.21** If \( \mathcal{K} \) is a cocomplete \( \mathcal{V} \)-category, the \( \Phi \)-presentable objects of \( \mathcal{K} \) are closed under \( \Phi \)-colimits. In particular, if \( F: \mathcal{A}^{\text{op}} \to \mathcal{V} \) is in \( \Phi \), then it is \( \Phi \)-presentable in \( \mathcal{K} \).

We now prove following analogue of Theorem 4.8, which shows in particular that Axiom A is satisfied.

**Theorem 4.22** Let \( \mathcal{I} \) be a small \( \mathcal{V} \)-category with \( \Phi \)-limits. For a \( \mathcal{V} \)-functor \( F: \mathcal{I} \to \mathcal{V} \) the following are equivalent:

1. \( F \) is a \( \mathcal{D} \)-filtered colimit of representables;
2. \( F \) is \( \Phi \)-flat;
3. \( F \) is \( \Phi \)-continuous.
Proof: Since \( D \)-filtered colimits commute in \( \mathcal{V} \) with \( \Phi \)-limits, the \( \Phi \)-flat weights are closed under \( D \)-filtered colimits. Since representables are certainly \( \Phi \)-flat, we deduce that (1) implies (2). Of course (2) implies (3) since the Yoneda embedding preserves all existing limits. So it remains to show that (3) implies (1).

Suppose then that \( \mathcal{T} \) is a small \( \mathcal{V} \)-category with \( \Phi \)-limits and that \( F : \mathcal{T} \to \mathcal{V} \) is \( \Phi \)-continuous. Just as in the proof of Theorem 4.8, we consider the ordinary functor

\[
\mathcal{T}_0 \xrightarrow{F_0} \mathcal{V}_0 \xrightarrow{\Phi_0, (-)} \text{Set}
\]

its category of elements \( \mathcal{E} \) and the the induced \( P : \mathcal{E} \to \mathcal{T}_0^{\text{op}} \), and observe that \( \mathcal{V}_0(I, F_0) \) is canonically the colimit of

\[
\mathcal{E} \xleftarrow{P} \mathcal{T}_0^{\text{op}} \xrightarrow{\Phi_0} [\mathcal{T}, \text{Set}]_0.
\]

Since \( F \) preserves \( D \)-limits, so does \( \mathcal{V}_0(I, F_0) : \mathcal{T}_0 \to \text{Set} \); it follows that \( \mathcal{E} \) is \( D \)-filtered. If the colimit of

\[
\mathcal{E} \xleftarrow{P} \mathcal{T}_0^{\text{op}} \xrightarrow{\Phi_0} [\mathcal{T}, \mathcal{V}]_0
\]

is \( F \), then \( F \) will be a \( D \)-filtered colimit of representables, as required.

The verification goes exactly as in the proof of Theorem 4.8. □

4.5 Finite connected limits

This is the case where \( \mathbb{D} \) consists of the finite connected categories. A category \( \mathcal{K} \) is locally \( \mathbb{D} \)-presentable if and only if it is locally finitely presentable and furthermore its category of finitely presentable objects is itself the free completion under finite colimits of a full subcategory. The \( \mathbb{D} \)-presentable objects are those which are both finitely presentable and connected.

We suppose that \( \mathcal{V}_0 \) is such a category and that the finitely presentable connected objects are closed under tensoring. This time we take \( \Phi \) to be the saturation of the class of all finite connected conical limits and all \( \mathcal{G} \)-powers.

Example 4.23 The cartesian closed categories \( \text{Gph}, \text{RGph}, \text{Cat}, \text{Gpd}, \text{SSet}, \) and \( \text{CGTop} \) of graphs, reflexive graphs, categories, groupoids, simplicial sets, and compactly generated spaces are all examples; so is any presheaf topos (such as \( \text{Gph}, \text{RGph}, \) and \( \text{SSet} \)), and so, of course, is \( \text{Set} \).

A monad will be \( \Phi \)-accessible if and only if it is finitary and preserves coproducts.

5 Many-sorted theories

Let \( \mathcal{V} \) and \( \Phi \) be given, satisfying Axiom A. To start with, we allow \( \mathcal{K} \) to be an arbitrary \( \mathcal{V} \)-category with \( \Phi \)-limits, but before long we shall suppose that it satisfies Axiom B1 or B2. The most important case is \( \mathcal{K} = \mathcal{V} \), which satisfies both Axiom B1 and B2.

5.1 Theories and models

A small \( \mathcal{V} \)-category \( \mathcal{T} \) with \( \Phi \)-limits is called a \( \Phi \)-theory in \( \mathcal{V} \), or just a theory when \( \Phi \) and \( \mathcal{V} \) are understood. A model of \( \mathcal{T} \) in \( \mathcal{K} \) is a \( \Phi \)-continuous \( \mathcal{V} \)-functor from \( \mathcal{T} \) to \( \mathcal{K} \).
The \(\mathcal{V}\)-category of models of \(\mathcal{T}\) in \(\mathcal{K}\) is the full subcategory \(\Phi\text{-Cts}(\mathcal{T}, \mathcal{K})\) of the functor category \([\mathcal{T}, \mathcal{K}]\) consisting of the models. When \(\mathcal{K} = \mathcal{V}\), we write simply \(\Phi\text{-Mod}(\mathcal{T})\) for \(\Phi\text{-Cts}(\mathcal{T}, \mathcal{V})\).

### 5.2 Left adjoints to algebraic functors

A *morphism of theories* is a \(\Phi\)-continuous \(\mathcal{V}\)-functor \(G : \mathcal{S} \to \mathcal{T}\). Composition with \(G\) induces a \(\mathcal{V}\)-functor \(G^* : \Phi\text{-Mod}(\mathcal{T}) \to \Phi\text{-Mod}(\mathcal{S})\); such a \(\mathcal{V}\)-functor is called \(\Phi\)-algebraic, or just algebraic.

Such algebraic functors have left adjoints: given a model \(M : \mathcal{S} \to \mathcal{V}\) we may form the left Kan extension \(\text{Lan}_G M : \mathcal{T} \to \mathcal{V}\) of \(M\) along \(G\) and by Proposition 2.2 this is \(\Phi\)-continuous, and so is a model of \(\mathcal{T}\); it is easy to see that it has the required universal property.

(In fact the existence of a left adjoint holds much more generally; the point here is that it can be constructed via left Kan extension.)

We now turn to the case of a general \(\mathcal{K}\). Once again \(G\) induces a \(\mathcal{V}\)-functor \(G^* : \Phi\text{-Cts}(\mathcal{T}, \mathcal{K}) \to \Phi\text{-Cts}(\mathcal{S}, \mathcal{K})\); such a \(G\) might be called “\(\Phi\)-algebraic relative to \(\mathcal{K}\)”.

**Proposition 5.1** Let \(\mathcal{A}\) and \(\mathcal{B}\) be small \(\mathcal{V}\)-categories with \(\Phi\)-limits, and \(G : \mathcal{A} \to \mathcal{B}\) an arbitrary \(\mathcal{V}\)-functor. If \(\mathcal{K}\) satisfies Axiom B2 and \(M : \mathcal{T} \to \mathcal{K}\) is \(\Phi\)-continuous, then \(\text{Lan}_G M : \mathcal{T} \to \mathcal{K}\) is also \(\Phi\)-continuous.

**Proof:** Let \(Y : \mathcal{K} \to \mathcal{P}\mathcal{K}\) be the Yoneda embedding and \(L \dashv Y\) its \(\Phi\)-continuous left adjoint. Of course \(Y\) preserves \(\Phi\)-limits (and any other existing limits).

Since \(L\) is cocontinuous, it preserves left Kan extensions, and so \(\text{Lan}_G M \cong \text{Lan}_G LY M \cong LL\text{Lan}_G Y M\). Now \(L\) is \(\Phi\)-continuous by assumption, thus it will suffice to show that \(\text{Lan}_G Y M\) is.

In other words, we can work with \(\mathcal{P}\mathcal{K}\) rather than \(\mathcal{K}\). But in \(\mathcal{P}\mathcal{K}\) both the left Kan extensions and the \(\Phi\)-limits are computed pointwise, and so we actually need only consider the case \(\mathcal{K} = \mathcal{V}\); this is Proposition 2.2.

Thus when Axiom B2 holds we can once again construct left adjoints to algebraic functors by Kan extension. Once again, the existence of the left adjoint holds much more generally, certainly whenever \(\mathcal{K}\) is locally presentable.

We shall see below that left adjoints to algebraic functors include in particular free models for single-sorted theories.

### 5.3 Reflectiveness of models

Let \(\mathcal{T}\) be the free completion of \(\mathcal{T}\) under \(\Phi\)-limits. Since \(\mathcal{T}\) has \(\Phi\)-limits, the canonical inclusion \(J : \mathcal{T} \to \mathcal{T}\) has a right adjoint \(R\), and the algebraic functor \(R^* : \Phi\text{-Mod}(\mathcal{T}) \to \Phi\text{-Mod}(\mathcal{T}) \simeq [\mathcal{T}, \mathcal{V}]\) has a left adjoint by the previous result. But this is just the full inclusion \(\Phi\text{-Mod}(\mathcal{T}) \to [\mathcal{T}, \mathcal{V}]\). Thus the models form a full reflective subcategory of the functor category, and in particular \(\Phi\text{-Mod}(\mathcal{T}, \mathcal{V})\) is complete and cocomplete.

**Theorem 5.2** \(\Phi\text{-Mod}(\mathcal{T})\) is reflective in \([\mathcal{T}, \mathcal{V}]\) and so is complete and cocomplete. It is closed in \([\mathcal{T}, \mathcal{V}]\) under all limits and under \(\Phi\)-flat colimits.
In fact the models will be reflective much more generally (see [11, Chapter 6]), but our framework
gives a simple construction, which can be computed in practice (provided that colimits in the base
category \( \mathcal{V} \) can be computed).

Note also that whenever the models are reflective we also have adjoints to algebraic functors:
given a morphism of theories \( G : \mathcal{T} \rightarrow \mathcal{T} \) and a model \( M : \mathcal{T} \rightarrow \mathcal{K} \), first take the left Kan
extension \( \text{Lan}_G M : \mathcal{T} \rightarrow \mathcal{V} \) and then reflect into models.

More generally, \( \Phi \text{-Mod}(\mathcal{T}, \mathcal{K}) \) will be reflective in \([\mathcal{T}, \mathcal{V}]\) if \( \mathcal{K} \) satisfies Axiom B2.

5.4 Characterization

In this section we characterize \( \mathcal{V} \)-categories of models in \( \mathcal{V} \).

Let \( \mathcal{M} \) be a \( \mathcal{V} \)-category with \( \Phi \)-flat colimits. We define an object \( M \in \mathcal{M} \) to be \( \Phi \)-presentable
if the hom-functor \( \mathcal{M}(M, -) : \mathcal{M} \rightarrow \mathcal{V} \) preserves \( \Phi \)-flat colimits.

We define a \( \mathcal{V} \)-category \( \mathcal{M} \) to be locally \( \Phi \)-presentable if it is cocomplete and has a small full
subcategory \( \mathcal{G} \) consisting of \( \Phi \)-presentable objects such that every object of \( \mathcal{M} \) is a \( \Phi \)-flat colimit
of objects in \( \mathcal{G} \).

It follows that \( \mathcal{M} \) is the free completion of \( \mathcal{G} \) under \( \Phi \)-flat colimits.

Let \( \mathcal{F} \) be the closure of \( \mathcal{G} \) in \( \mathcal{M} \) under \( \Phi \)-colimits. This is a small dense full subcategory
consisting of \( \Phi \)-presentable objects, and it is \( \Phi \)-cocomplete by construction. The inclusion \( J : \mathcal{F} \rightarrow \mathcal{M} \) induces a \( \mathcal{V} \)-functor \( W : \mathcal{M} \rightarrow [\mathcal{G}^{\text{op}}, \mathcal{V}] \) which is fully faithful since \( \mathcal{F} \) is dense. It has a
left adjoint sending \( F : [\mathcal{F}^{\text{op}}, \mathcal{V}] \rightarrow \mathcal{M} \) to the colimit \( F * J \in \mathcal{M} \). The composite \( WJ \) is the Yoneda
embedding.

Explicitly, \( W \) sends an object \( M \in \mathcal{M} \) to \( \mathcal{M}(J - , M) \) which is \( \Phi \)-continuous since the inclusion
\( J : \mathcal{F} \rightarrow \mathcal{M} \) is \( \Phi \)-cocontinuous. Thus \( W \) lands in \( \Phi \text{-Mod}(\mathcal{F}^{\text{op}}) \). Furthermore, \( W \) preserves \( \Phi \)-flat colimits, since \( \mathcal{F} \) consists of \( \Phi \)-presentable objects.

Proposition 5.3 The \( \mathcal{V} \)-functor \( W : \mathcal{M} \rightarrow \Phi \text{-Mod}(\mathcal{F}^{\text{op}}) \) is an equivalence.

Proof: We already know that \( W \) is fully faithful, so it will suffice to show that it is essentially
surjective on objects. Suppose then that \( F : [\mathcal{F}^{\text{op}}, \mathcal{V}] \rightarrow \mathcal{M} \) is \( \Phi \)-continuous. By Axiom A it is also
\( \Phi \)-flat, and so \( W \) preserves \( F \)-weighted colimits. Now every presheaf is a colimit of representables,
weighted by itself, and so we have

\[ F \cong F * Y \cong F * WJ \cong W(F * J) \]

which completes the proof. \( \square \)

Thus any locally \( \Phi \)-presentable \( \mathcal{V} \)-category is the category of models in \( \mathcal{V} \) for a small theory.
Conversely, let \( \mathcal{T} \) be a theory and consider \( \Phi \text{-Mod}(\mathcal{T}) \). This is reflective in \([\mathcal{T}, \mathcal{V}]\) and so
cocomplete. The representables provide a small dense subcategory

Since \( \Phi \)-limits commute in \( \mathcal{V} \) with \( \Phi \)-flat colimits, the inclusion \( \Phi \text{-Mod}(\mathcal{T}) \rightarrow [\mathcal{T}, \mathcal{V}] \) preserves \( \Phi \)-flat colimits, which is equivalent to saying that the representables are \( \Phi \)-presentable in
\( \Phi \text{-Mod}(\mathcal{T}) \). Finally if \( F : \mathcal{T} \rightarrow \mathcal{V} \) is \( \Phi \)-continuous then it is \( \Phi \)-flat by Axiom A, and so once again
\( F \cong F * Y \cong F * WJ \cong W(F * J) \) shows that \( F \) is a \( \Phi \)-flat colimit in \( \Phi \text{-Mod}(\mathcal{T}) \) of representables.
This proves that \( \Phi \text{-Mod}(\mathcal{T}) \) is locally \( \Phi \)-presentable, and so gives:

Theorem 5.4 A category \( \mathcal{M} \) is locally \( \Phi \)-presentable if and only if it is equivalent to a category of
\( \Phi \)-continuous \( \mathcal{V} \)-functors from \( \mathcal{T} \) to \( \mathcal{V} \) for a small \( \Phi \)-complete \( \mathcal{V} \)-category \( \mathcal{T} \).
As observed above, if $\Phi$ consists of the finite limits, then this was proved in [10]; and, in the case $\mathcal{V} = \mathbf{Set}$, is of course due to Gabriel-Ulmer [8]. We now return to the special case of Section 4.3.

**Theorem 5.5** Let $\mathcal{V}$ be locally strongly finitely presentable as a $\otimes$-category and let $\Phi$ be the saturation of the finite products and $\mathcal{V}_{sf}$-powers. For a $\mathcal{V}$-category $\mathcal{M}$ the following are equivalent:

(i) $\mathcal{M}$ is locally strongly finitely presentable

(ii) $\mathcal{M}$ is locally $\Phi$-presentable

(iii) There is a small $\mathcal{V}$-category $\mathcal{T}$ with finite products and $\mathcal{V}_{sf}$-powers for which $\mathcal{M}$ is equivalent to the full subcategory of $[\mathcal{T}, \mathcal{V}]$ consisting of the $\mathcal{V}$-functors which preserve these limits.

**Proof:** The equivalence of (ii) and (iii) is a special case of the previous theorem. The fact that (i) implies (iii) is Theorem 4.17. To see that (iii) implies (i), observe that a $\mathcal{M}$ satisfying (iii) is cocomplete, since (iii) implies (ii), and now consider the small full subcategory consisting of the representables $\mathcal{T} \to \mathcal{V}$. Certainly these are strongly finitely presentable. It remains to show that every $\Phi$-continuous $F : \mathcal{T} \to \mathcal{V}$ is a sifted colimit of representables. But this is Theorem 4.8. □

This reduces to Theorem 3.1 in the case $\mathcal{V} = \mathbf{Set}$; see the discussion before that theorem for the history of the result in that case.

### 6 Lawvere theories

In this section we turn to theories which can be thought of as single-sorted, and see how to extend the classical correspondence between such theories and monads.

A major difference is that the resulting theories need not have all the limits under consideration. This was observed in [18] in the case where $\Phi$ consists of the finite limits, and our approach is modelled on that of [18]. As a consequence, in this more general setting, a Lawvere theory need not be a theory; we shall see, however, how a Lawvere theory does generate a theory with the same models.

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories with $\Phi$-flat colimits. We shall say that a $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ is $\Phi$-accessible if it preserves $\Phi$-flat colimits. A monad on $\mathcal{A}$ will be called $\Phi$-accessible if its underlying endofunctor is so. We write $\text{Mnd}_\Phi(\mathcal{A})$ for the category of $\Phi$-accessible monads on $\mathcal{A}$.

#### 6.1 Lawvere $\Phi$-theories

Let $\mathcal{K}$ be a $\mathcal{V}$-category satisfying Axiom B1. Let $J : \mathcal{K}_\Phi \to \mathcal{K}$ be the full subcategory of $\mathcal{K}$ consisting of the $\Phi$-presentable objects; then $\mathcal{K}_\Phi$ has $\Phi$-colimits and $J$ preserves them. Furthermore, $\mathcal{K}$ is equivalent to the category of $\Phi$-continuous $\mathcal{V}$-functors from $\mathcal{K}_\Phi^{\text{op}}$ to $\mathcal{V}$.

Let $T = (T, m, i)$ be a $\Phi$-accessible $\mathcal{V}$-monad on $\mathcal{K}$, write $\mathcal{K}^T$ for the Eilenberg-Moore $\mathcal{V}$-category, with forgetful functor $U^T : \mathcal{K}^T \to \mathcal{K}$ and left adjoint $F^T \dashv U^T$.

We may factorize the composite $F^T J : \mathcal{K}_\Phi \to \mathcal{K}^T$ as an identity-on-objects $\mathcal{V}$-functor $E : \mathcal{K}_\Phi \to \mathcal{G}$ followed by a fully faithful $H : \mathcal{G} \to \mathcal{K}^T$. The opposite of the resulting $\mathcal{V}$-category $\mathcal{G}$ will become the Lawvere theory $\mathcal{L}$ corresponding to $T$.

Now $\mathcal{K}_\Phi$ has $\Phi$-colimits, preserved by $J$, while $F^T$ preserves all colimits, so the composite $HE = F^T J : \mathcal{V}_\Phi \to \mathcal{V}^T$ preserves $\Phi$-colimits. It follows that $E$ preserves $\Phi$-colimits, but it does
not follow that \mathcal{G} has all \Phi-colimits. It does have \Phi-colimits of diagrams in the image of \mathcal{E}, but need not have \Phi-colimits in general.

Note, however, that \mathcal{G} will have \( F \)-weighted colimits for any \( F \in \Phi \) which is a weight for coproducts or copowers, since these involve only the objects of \mathcal{G}, and so the resulting diagrams will be in the image of the identity-on-object \( \mathcal{V} \)-functor \( \mathcal{E} \).

**Remark 6.1** Thus in the special case of Section 4.3 where all weights in \( \Phi \) are of this type, \( \mathcal{G} \) will have \( \Phi \)-colimits. The same is true if \( \Phi \) consists of just the finite products.

But in general, we make the following definition, given in [18] for the case where \( \Phi \) is the finite limits.

**Definition 6.2** A Lawvere \( \Phi \)-theory in \( \mathcal{V} \) is an identity-on-object \( \mathcal{V} \)-functor \( \mathcal{E} : \mathcal{V}^{\text{op}} \to \mathcal{L} \) which preserves \( \Phi \)-limits. A morphism of Lawvere \( \Phi \)-theories is a commutative triangle of (identity-on-object) \( \mathcal{V} \)-functors; we write \( \text{Law}_\Phi(\mathcal{H}) \) for the resulting category of Lawvere \( \Phi \)-theories on \( \mathcal{H} \).

We cannot in general simply define a model to be a \( \Phi \)-continuous \( \mathcal{V} \)-functor with domain \( \mathcal{L} \), since \( \mathcal{L} \) may not have all \( \Phi \)-limits. Instead we define the \( \mathcal{V} \)-category of models of \( \mathcal{L} \) by the following pullback in \( \mathcal{V} \)-Cat

\[
\begin{array}{ccc}
\Phi\text{-Cts}(\mathcal{L}, \mathcal{V}) & \to & [\mathcal{L}, \mathcal{V}] \\
\downarrow U & & \downarrow [E, \mathcal{V}] \\
\mathcal{K}(J, 1) & \to & [\mathcal{K}_{\Phi}^{\text{op}}, \mathcal{V}]
\end{array}
\]

**Remark 6.3** As observed in [15] in the case of finite limits, since \( \mathcal{K} \) is equivalent to the \( \mathcal{V} \)-category of \( \Phi \)-continuous \( \mathcal{V} \)-functors from \( \mathcal{K}_{\Phi}^{\text{op}} \) to \( \mathcal{V} \), up to an equivalence, a model of \( \mathcal{L} \) is just a \( \mathcal{V} \)-functor \( M : \mathcal{L} \to \mathcal{V} \) whose restriction along \( \mathcal{E} \) is \( \Phi \)-continuous.

**Remark 6.4** If \( \Phi \) consists only of finite products and/or powers, then \( \Phi \)-limits in \( \mathcal{L} \) are determined by those in \( \mathcal{K}_{\Phi}^{\text{op}} \), and so the restriction of \( M \) along \( \mathcal{E} \) is \( \Phi \)-continuous if and only if \( M \) is \( \Phi \)-continuous.

**Proposition 6.5** \( U \) is monadic via a \( \Phi \)-accessible monad.

**Proof:** \( [\mathcal{L}, \mathcal{V}] \) has all coequalizers, \( [E, \mathcal{V}] \) has both adjoints (given by left and right Kan extension), and \( [E, \mathcal{V}] \) is also conservative because \( E \) is the bijective on objects. An easy application of Beck’s theorem shows that \( [E, \mathcal{V}] \) is monadic. The pullback \( U \) of \( [E, \mathcal{V}] \) will still satisfy the conditions of Beck’s theorem, and so be monadic, provided that it has a left adjoint.

Now \( \mathcal{K}(J, 1) : \mathcal{K} \to [\mathcal{K}_{\Phi}^{\text{op}}, \mathcal{V}] \) is fully faithful and has a left adjoint, \( [E, \mathcal{V}] \) has a left adjoint, and the inclusion \( \Phi\text{-Cts}(\mathcal{L}, \mathcal{V}) \to [\mathcal{L}, \mathcal{V}] \) has a left adjoint, thus \( U \) does indeed have a left adjoint and so is monadic.

It remains to show that the monad is \( \Phi \)-accessible, or equivalently that \( U \) preserves \( \Phi \)-flat colimits. But this follows because the other three functors in the definition of \( \Phi\text{-Cts}(\mathcal{L}, \mathcal{V}) \) preserve \( \Phi \)-flat colimits, and \( \mathcal{K}(J, 1) \) is fully faithful so also reflects them. \( \square \)

Thus to every Lawvere theory we have associated a \( \Phi \)-accessible monad. This gives the object-part of a functor \( \text{mnd} : \text{Law}_\Phi(\mathcal{H}) \to \text{Mnd}_\Phi(\mathcal{H}) \).

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Conversely, for a \( \Phi \)-accessible monad \( T \) on \( \mathcal{K} \), the inclusion \( H : \mathcal{L}^{\text{op}} = \mathcal{G} \to \mathcal{K}^{\text{op}} \) induces a \( \mathcal{V} \)-functor \( \mathcal{K}^{\text{op}}(H, 1) : \mathcal{K}^{\text{op}} \to [\mathcal{L}, \mathcal{V}] \).

**Theorem 6.6** The \( \mathcal{V} \)-functor \( \mathcal{K}^{\text{op}}(H, 1) : \mathcal{K}^{\text{op}} \to [\mathcal{L}, \mathcal{V}] \) restricts to an isomorphism of \( \mathcal{V} \)-categories

\[ \mathcal{K}^{\text{T}} \simeq \Phi \text{-Cts}(\mathcal{L}, \mathcal{V}). \]

**Proof:** Composition with \( E : \mathcal{K}^{\text{op}}(H, 1) \to [\mathcal{L}, \mathcal{V}] \) induces a \( \mathcal{V} \)-functor

\[ [\mathcal{L}, \mathcal{V}] \xrightarrow{E} [\mathcal{L}, \mathcal{V}] \]

which has both adjoints, given by left and right Kan extension. Since \( E \) is bijective on objects, \( [\mathcal{L}, \mathcal{V}] \) is conservative, and now by the Beck theorem, it is monadic. Write \( S \) for the induced monad.

Consider what happens when we restrict the induced monad \( S \) along the fully faithful \( \mathcal{K}(J, 1) : \mathcal{K} \to [\mathcal{K}^{\text{op}}, \mathcal{V}] \).

We have

\[
(Lan_E \mathcal{K}(J, X))E \cong \int^c \mathcal{G}(E-, Ec) \cdot \mathcal{K}(Jc, X) \\
\cong \int^c \mathcal{K}^{\text{T}}(HE-, HEc) \cdot \mathcal{K}(Jc, X) \quad \text{(because \( H \) is fully faithful)} \\
eq \int^c \mathcal{K}^{\text{T}}(F^TJ-, F^TJc) \cdot \mathcal{K}(Jc, X) \\
\cong \int^c \mathcal{K}(J-, TJc) \cdot \mathcal{K}(Jc, X) \quad \text{(by adjointness)} \\
\cong \mathcal{K}(J, X) * (\mathcal{K}(J, 1)T) \quad \text{(because \( \mathcal{K}(J, 1) \) preserves \( \Phi \)-flat colimits and \( \mathcal{K}(J, X) \) is \( \Phi \)-flat)} \\
\cong \mathcal{K}(J, 1)(TX) \quad \text{(because \( T \) is \( \Phi \)-accessible)} \\
= \mathcal{K}(J, TX)
\]

Thus the functor part of \( S \) restricts to \( T \). In fact the monad itself restricts, and so we conclude that a \( T \)-algebra is an \( S \)-algebra (an object of \( [\mathcal{L}, \mathcal{V}] \)) whose underlying object (restriction along \( E \)) is in the image of \( \mathcal{K}(J, 1) \). But this is exactly the definition of models of \( \mathcal{L} \).

Similarly, a morphism of \( T \)-algebras is the same as a morphism of the corresponding \( S \)-algebras, whence the result. \( \square \)

We have associated a Lawvere \( \Phi \)-theory \( \mathcal{L} \) to every \( \Phi \)-accessible \( \mathcal{V} \)-monad on \( \mathcal{K} \). This process is clearly functorial, giving a functor \( \text{th} : \text{Mnd}_\Phi(\mathcal{K}) \to \text{Law}_\Phi(\mathcal{K}) \).

**Theorem 6.7** The functors \( \text{mnd} \) and \( \text{th} \) form an equivalence of categories \( \text{Mnd}_\Phi(\mathcal{K}) \simeq \text{Law}_\Phi(\mathcal{K}) \).

**Proof:** The previous theorem gives an isomorphism \( \text{mnd} \circ \text{th} \cong 1 \). For the other isomorphism \( \text{th} \circ \text{mnd} \cong 1 \), let \( E : \mathcal{K}^{\text{op}} \to \mathcal{L} \) be a Lawvere theory, and \( T \) the induced monad \( \text{mnd}(\mathcal{L}) \).

Then \( \text{th}(\text{mnd}(\mathcal{L})) \) can be obtained by factorizing \( F^TJ : \mathcal{K} \to \Phi \text{-Cts}(\mathcal{L}, \mathcal{V}) \) as an identity-on-object functor followed by a fully faithful one. Now to form \( F^TJc \), for \( c \in \mathcal{K} \), we send \( Jc \) to
\( \mathcal{K}(J,JC) : \mathcal{K}^\text{op}_\Phi \to \mathcal{V} \), then to its left Kan extension \( \text{Lan}_E \mathcal{K}(J,JC) : \mathcal{L} \to \mathcal{V} \), and then reflect this into \( \Phi\text{-Cts}(\mathcal{L},\mathcal{V}) \). But \( \mathcal{K}(J,JC) \cong \mathcal{K}_\Phi(-,c) \), whose left Kan extension along \( E \) is \( \mathcal{L}(-,Ec) \), and this is already in \( \Phi\text{-Cts}(\mathcal{L},\mathcal{V}) \). But since \( F^\top J \) sends \( c \) to \( \mathcal{L}(-,Ec) \), its identity-on-object/fully-faithful factorization gives just \( \mathcal{L} \).

We now turn to the \( \Phi \)-theory generated by a Lawvere \( \Phi \)-theory \( \mathcal{L} \). Every representable functor \( \mathcal{L}(L,-) : \mathcal{L} \to \mathcal{V} \) is a model of \( \mathcal{L} \), and so we get a fully faithful embedding \( Y : \mathcal{L}^\text{op} \to \Phi\text{-Cts}(\mathcal{L},\mathcal{V}) \). Form the closure of the representables in \( \Phi\text{-Cts}(\mathcal{L},\mathcal{V}) \) under \( \Phi \)-colimits. This gives fully faithful \( K : \mathcal{L} \to \mathcal{T} \) and \( P : \mathcal{T} \to \Phi\text{-Cts}(\mathcal{L},\mathcal{V})^\text{op} \). Clearly \( H \) preserves \( \Phi \)-limits, while \( P \) preserves those \( \Phi \)-limits in the image of \( E \).

Now \( \mathcal{T} \) is a small \( \mathcal{V} \)-category with \( \Phi \)-limits; that is, a \( \Phi \)-theory. Furthermore, it has the same models as \( \mathcal{L} \). This is really a special case of [11, Proposition 6.23], but we outline here the argument.

First of all the composite
\[
\mathcal{L}^\text{op} \xrightarrow{K} \mathcal{T}^\text{op} \xrightarrow{P} \Phi\text{-Cts}(\mathcal{L},\mathcal{V})
\]
is dense, and both \( K \) and \( P \) are fully faithful. It follows by [11, Theorem 5.13] that both \( K \) and \( P \) are dense, and that \( P \cong \text{Lan}_K(PK) \). Since \( P \) is dense, the induced functor \( \Phi\text{-Cts}(\mathcal{L},\mathcal{V})(P,1) : \Phi\text{-Cts}(\mathcal{L},\mathcal{V}) \to [\mathcal{T},\mathcal{V}] \) is fully faithful. We must show that its image is exactly the \( \Phi \)-continuous functors.

Now \( P : \mathcal{T}^\text{op} \to \Phi\text{-Cts}(\mathcal{L},\mathcal{V}) \) is \( \Phi \)-cocontinuous, by construction of \( \mathcal{T}^\text{op} \), and so the induced \( \Phi\text{-Cts}(\mathcal{L},\mathcal{V})(P-,M) : \mathcal{T} \to \mathcal{V} \) will be \( \Phi \)-continuous for all models \( M \). This proves that \( \Phi\text{-Cts}(\mathcal{L},\mathcal{V})(P,1) \) takes values among the \( \Phi \)-continuous functors. Conversely, let \( G : \mathcal{T} \to \mathcal{V} \) be \( \Phi \)-continuous. Then \( GKE \) is \( \Phi \)-continuous, and so \( GK \) is (isomorphic to) a model; and indeed the model can be calculated as \( GK * P \). Thus
\[
GK \cong \Phi\text{-Cts}(\mathcal{L},\mathcal{V})(PK-,GK * P).
\]

But \( G \) and \( \Phi\text{-Cts}(\mathcal{L},\mathcal{V})(P-,GK * P) \) are both \( \Phi \)-continuous \( \mathcal{V} \)-functors \( \mathcal{T} \to \mathcal{V} \), and \( \mathcal{T} \) is the closure of \( \mathcal{L} \) under \( \Phi \)-limits, thus \( G \) will be isomorphic to \( \Phi\text{-Cts}(\mathcal{L},\mathcal{V})(P-,GK * P) \) provided their restrictions along \( K \) are isomorphic; but this is the previous displayed equation.

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