Obstructing Visibilities with One Obstacle

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Obstacle Number of a Graph

- **Outside obstacle:**
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\[
\text{obs}_{\text{out}}(G) = \text{Obstacle number using an outside obstacle}
\]

\[
\text{obs}_{\text{in}}(G) = \text{Obstacle number only using inside obstacles}
\]
Some Known Results

- \( \text{obs}(G) \leq \# \text{ of non-edges of } G = O(n^2) \)
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- $\text{obs}(G) \leq O(n \lg n)$ \hspace{1cm} [Balko, Chibulka and Valtr, GD’15]
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- \( \text{obs}(G) \leq O(n \log n) \) \hspace{1cm} [Balko, Chibulka and Valtr, GD’15]

- There are graphs that require \( \Omega(n/(\log \log n)^2) \) obstacles. \hspace{1cm} [Dujmović and Morin ’15]
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- There are graphs that require \( \Omega(n/(\log \log n)^2) \) obstacles. [Dujmović and Morin ’15]

- For each \( m \), there exists a graph \( G \) s.t. \( \text{obs}(G) = m \) [Mukkamala, Pach, Sariöz, WG’10]
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- **What is the smallest graph of obstacle number 2?**

- $\text{obs}(K^*_5,5) = 2$ [Pach, Sariöz, ’11]
Questions

- Graphs of obstacle number 0 are complete graphs.

- **What are the graphs of obstacle number 1?**

- The smallest graph of obstacle number 1 consists of 2 vertices and 1 non-edge.

- **What is the smallest graph of obstacle number 2?**

- \( \text{obs}(K_{5,5}^*) = 2 \) \[Pach, Sariöz, '11\]

- **Can an outside obstacle and an inside obstacle do different jobs?**
  i.e. \( \{ G : \text{obs}_{\text{out}}(G) = 1 \} \) vs. \( \{ G : \text{obs}_{\text{in}}(G) = 1 \} \)
Our Results

- Graphs whose longest cycle has length $\leq 6$ have obstacle number 1.
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• Graphs with at most 7 vertices have obstacle number 1.
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- Graphs whose longest cycle has length \( \leq 6 \) have obstacle number 1.
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- Smallest graph of obstacle number 2 has 8 vertices.
- \( \{ G : \text{obs}_{\text{out}}(G) = 1 \} \) and \( \{ G : \text{obs}_{\text{in}}(G) = 1 \} \) are incomparable.
- The single-obstacle graph sandwich problem is NP-hard. Given two graphs \( G \) and \( H \), it is NP-hard to decide the existence of a graph \( K \) s.t. \( G \subset K \subset H \) and \( \text{obs}(K) = 1 \).
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  Given two graphs \( G \) and \( H \), it is NP-hard to decide the existence of a graph \( K \) s.t. \( G \subset K \subset H \) and \( \text{obs}(K) = 1 \).
- The following problems are all NP-hard:
  - The outside-obstacle graph sandwich problem
  - The inside-obstacle graph sandwich problem
  - The simple-polygon visibility graph sandwich problem

Graphs with at most 7 vertices have obstacle number 1.
Graphs of Obstacle Number 1

**Thm.** Every outerplanar graph has an outside-obstacle representation. [Alpert, Koch, Laison, '09]

**Thm.** Graphs represented by 1 convex polygon are non-double covering circular arc graphs. [Alpert, Koch, Laison, '09]

- Circular arc graphs: intersection graphs for arcs in a circle
- Non-double covering: No two arcs cover the whole circle.
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- Circular arc graphs: intersection graphs for arcs in a circle
- Non-double covering: No two arcs cover the whole circle.

**Thm.** Any graph whose longest cycle has length \( \leq 6 \) has an outside-obstacle representation.

**Thm.** Any graph with at most 7 vertices has an outside-obstacle representation.
Co-bipartite Graphs

Let $G$ be a co-bipartite graph with a co-bipartition $Z, Z'$ with $\text{obs}_{\text{out}}(G) = 1$.

(A co-bipartite graph is the complement of a bipartite graph)

**Obs.** $\text{CH}(Z)$ and $\text{CH}(Z')$ cannot be pierced by the outside obstacle.
Co-bipartite Graphs

Let $G$ be a co-bipartite graph with a co-bipartition $Z, Z'$ with $\text{obs}_{\text{out}}(G) = 1$.

Def. $\text{CH}(Z)$ and $\text{CH}(Z')$ are $k$-crossing if $\text{CH}(Z) \setminus \text{CH}(Z')$ consists of $k + 1$ disjoint regions.
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Let $G$ be a co-bipartite graph with a co-bipartition $Z, Z'$ with $\text{obs}_{\text{out}}(G) = 1$.

**Def.** $\text{CH}(Z)$ and $\text{CH}(Z')$ are $k$-crossing if $\text{CH}(Z) \setminus \text{CH}(Z')$ consists of $k + 1$ disjoint regions.

**Lemma.** Suppose $\text{CH}(Z)$ and $\text{CH}(Z')$ are 1-crossing. If $G$ contains an induced 4-cycle $z_1 z'_1 z'_2 z_2$ where $\{z_1, z_2\} \subseteq Z$, $\{z'_1, z'_2\} \subseteq Z'$, then either $z_1$ and $z_2$ or $z'_1$ and $z'_2$ are in different petals.
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**Lemma.** Let $A, B$ be a co-bipartition of $K_6^*$. Then $\text{CH}(A)$ and $\text{CH}(B)$ are at least 1-crossing in any outside-obstacle representation. Moreover, if $G$ contains $K_6^*$ as an induced subgraph, then $\text{CH}(Z)$ and $\text{CH}(Z')$ are at least 1-crossing.
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Smallest Graph of Obstacle Number 2

**Thm.** The smallest graph of obstacle number 2 has 8 vertices.
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Proof. 1) $\text{obs}(G) \leq 2$.
2) Every graph with at most 7 vertices has obstacle number 1.
Thm. The smallest graph of obstacle number 2 has 8 vertices.

Proof. 3) $\text{obs}_{\text{out}}(G) > 1$

$\text{CH}(A)$ and $\text{CH}(B)$ are at least 1-crossing.
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Proof. 3) \( \text{obs}_{\text{out}}(G) > 1 \)

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Proof. 3) $\text{obs}_{\text{out}}(G) > 1$

$\text{CH}(A)$ and $\text{CH}(B)$ are 1-crossing.

Consider $G - \{v_4, v_8\}$. 
**Smallest Graph of Obstacle Number 2**

**Thm.** The smallest graph of obstacle number 2 has 8 vertices.

**Proof.** 3) $\text{obs}_{\text{out}}(G) > 1$

$\text{CH}(A)$ and $\text{CH}(B)$ are 1-crossing.
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Proof. 3) $\text{obs}_{\text{out}}(G) > 1$

$\text{CH}(A)$ and $\text{CH}(B)$ are 1-crossing.

Cannot add $v_4, v_8$. 
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Induced 4-cycle $v_1v_4v_8v_7$
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Cannot add $v_4, v_8$.

Induced 4-cycles $v_1v_4v_8v_7$, $v_1v_4v_8v_5$, $v_2v_4v_8v_6$, $v_2v_4v_8v_7$
**Smallest Graph of Obstacle Number 2**

**Thm.** The smallest graph of obstacle number 2 has 8 vertices.

![Graph Diagram]

**Proof.** 4) $\text{obs}_{\text{in}}(G) > 1$

The convex hull of $V(G)$ forms a cycle.

Case analysis on vertices on CH
\[ \{ G : \text{obs}_{\text{out}}(G) = 1 \} \not\subset \{ G : \text{obs}_{\text{in}}(G) = 1 \} \]
\{ G : \text{obs}_{\text{out}}(G) = 1 \} \not\supset \{ G : \text{obs}_{\text{in}}(G) = 1 \}

**Thm.** There is a graph $G$ such that $\text{obs}_{\text{in}}(G) = 1$ but $\text{obs}_{\text{out}}(G) > 1$. 
\( \{ G : \text{obs}_{\text{out}}(G) = 1 \} \not\supset \{ G : \text{obs}_{\text{in}}(G) = 1 \} \)

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\( \text{CH}(A) \) and \( \text{CH}(B) \) are exactly 1-crossing.
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**Thm.** There is a graph $G$ such that $\text{obs}_{\text{in}}(G) = 1$ but $\text{obs}_{\text{out}}(G) > 1$. 
NP-hardness

**Def.** In a graph sandwich problem for a property $\Pi$, given two graphs $G \subseteq H$ with the same vertex set, we ask for a graph $K$ s.t. $G \subseteq K \subseteq H$ and $K$ has the property $\Pi$.

**Thm.** The outside-obstacle graph sandwich problem is NP-hard. In other words, given two graphs $G \subseteq H$ with the same vertex set, it is NP-hard to decide if there is a graph $K$ s.t. $G \subseteq K \subseteq H$ and $\text{obs}_{\text{out}}(K) = 1$. 
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**Thm.** The inside-obstacle graph sandwich problem and the single-obstacle graph sandwich problem are NP-hard.

**Def.** The simple-polygon visibility graph problem asks to recognize the visibility graph of a simple polygon where the obstacle is the complement of the polygon.

**Thm.** The simple-polygon visibility graph sandwich problem is NP-hard.
NP-hardness

Reduction from MonotoneNotAllEqual3Sat where each clause contains 3 variables, not all of which are equal.
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\[
C_1 = \{v_1, v_2, v_4\} \\
C_2 = \{v_2, v_3, v_4\} \\
C_3 = \{v_1, v_4, v_5\}
\]
Summary and Open Problems

- Graphs of circumference at most 6 and graphs with at most 7 vertices have obstacle number 1.
- Smallest graph of obstacle number 2 has 8 vertices.
- \( \{ G : \text{obs}_{\text{out}}(G) = 1 \} \) and \( \{ G : \text{obs}_{\text{in}}(G) = 1 \} \) are incomparable.
- All of outside-, inside-, and single-obstacle graph sandwich problems are NP-hard. The simple-polygon visibility graph sandwich problem is also NP-hard.
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• An upper bound for \( \text{obs}_{\text{in}}(G) \) in terms of \( \text{obs}_{\text{out}}(G) \)?
  Shown to be tight: \( \text{obs}(G) \leq \text{obs}_{\text{out}}(G) \leq \text{obs}(G) + 1 \)
  \( \text{obs}_{\text{in}}(G) \geq \text{obs}_{\text{out}}(G) - 1 \)