Quantum parameter estimation via dispersive measurement in circuit QED

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Abstract
We investigate the quantum parameter estimation in circuit quantum electrodynamics via dispersive measurement. Based on the Metropolis–Hastings algorithm and the Markov chain Monte Carlo (MCMC) integration, a new algorithm is proposed to calculate the Fisher information by the stochastic master equation. The Fisher information is expressed in the form of log-likelihood functions and further approximated by the MCMC integration. Numerical results show that the evolution of the Fisher information can approach the quantum Fisher information in a short time interval. These results demonstrate the effectiveness of the proposed algorithm. Finally, based on the proposed algorithm, we consider the effects of the measurement operator and the measurement efficiency on the Fisher information.

Keywords Quantum Fisher information · Quantum parameter estimation · Stochastic master equation

1 Introduction
The problem of accurately estimating unknown parameters in quantum system is of fundamental and practical importance. According to the parameter estimation theory [1–4], in classical system the estimation precision is limited by the standard quantum limit (SQL) [5–9], $1/\sqrt{N}$, where $N$ refers to the number of experiments. In quantum system, Refs. [10,11] showed that with the help of squeezed state technique the parameter estimation accuracy can exceed the SQL and even approach the Heisenberg limit (HL) [12,13], $1/N$. The classical Fisher information (FI) is a tool widely used to calculate the parameter estimation accuracy, and the Cramér–Rao bound states that the inverse of the Fisher information is a tight lower bound on the variance of any unbiased...
estimation parameter [14–17]. By explicitly maximizing the Fisher information over all possible measurement strategies, one can obtain the quantum Fisher information (QFI) [18,19].

Over the past decades, parameter estimation via continuous weak measurement in quantum system caused a wide range of interests [20–23]. Reference [22] showed that weak measurements have a rich structure, based on which more novel strategies for quantum-enhanced parameter estimation can be constructed. Reference [23] experimentally demonstrated a new robust method for precision phase estimation based on quantum weak measurement [24]. Reference [25] evaluated the ultimate quantum limits based on time-continuous monitoring of the lightly coupled atomic ensemble. The stochastic master equation with quantum weak measurement was also derived for quantum parameter estimation [26]. Moreover, the likelihood function and the statistical properties of the measurement output were demonstrated to be effective resources for quantum parameter estimation [27,28]. Although much progress has been made in quantum parameter estimation, how to effectively calculate the Fisher information (or the estimation precision) based on continuous weak measurement is still with remarkable difficulty. To figure out this problem, one needs to represent the Fisher information in computable forms and take effective measures to prior estimate the parameter of interest. A preliminary work [29] to calculate the Fisher information based on various weak measurements in linear Gaussian quantum system has been reported recently. Here, we propose an efficient algorithm to calculate the Fisher information based on the quantum stochastic master equation in circuit quantum electrodynamics (circuit QED) [30,31]. Circuit QED is widely regarded as an excellent platform for quantum estimation and quantum control [3,32–35]. Due to the randomness of the measurement record, the numerical differentiation approach is used to calculate the derivative of the log-likelihood function, and a series of parameters of interest is randomly generated by the Metropolis–Hastings (MH) algorithm [36]. Finally, the calculable Fisher information is approximated by the Markov chain Monte Carlo (MCMC) integration [37,38].

This paper is organized as follows. In Sect. 2, a brief introduction of quantum parameter estimation is presented. In Sect. 3, we discuss the weak measurement in circuit QED. The reduced stochastic master equation and the measurement record are exhibited in this section. An efficient algorithm to calculate the Fisher information is introduced in Sect. 4. Numerical simulations in circuit QED demonstrate the feasibility and effectiveness of the proposed algorithm. In addition, the evolutions of the Fisher information with the proposed algorithm for various measurement operators and measurement efficiencies are also discussed. We summarize our conclusion in Sect. 5.

2 Quantum parameter estimation

Suppose $\theta$ is an unknown parameter that needs to be estimated in a quantum system. As we mentioned above, the precision of the unbiased parameter estimation is always indicated by the Cramér–Rao inequality [16,39,40], i.e.,
\[
\left\langle (\delta \theta)^2 \right\rangle \geq \frac{1}{NI(\theta)},
\]
(1)

where \( I(\theta) \) is the Fisher information of \( \theta \), \( \delta \theta \) is the estimation error, and \( N \) is the number of measurements.

Let \( D \) be the measurement output, which is conditioned on the value of the unknown parameter \( \theta \). The estimation performance depends on the probability of observing the output given the parameters \( P(D|\theta) \), which can be characterized by the Fisher information, i.e.,

\[
I(\theta) = E \left[ \left( \frac{\partial \ln P(D|\theta)}{\partial \theta} \right)^2 \right],
\]
(2)

where \( E[\cdot] \) refers to the expectation value with respect to independent realizations of the measurement results \( D \). Sometimes the probability density \( P(D|\theta) \) is also defined as a likelihood function. In addition, the theory that tackles the probability distribution of the measurement resource is the same as the one for the classical problems with stochastic measurement outcomes, while the underlying dynamics of the system and \( P(D|\theta) \) may be dominated by the laws of quantum physics [41].

By maximizing \( I(\theta) \) over all possible quantum measurements on the system, one can obtain the quantum Fisher information (QFI). Simply, if a quantum pure state \( \rho_\theta = |\psi_\theta\rangle \langle \psi_\theta| \) evolves in a closed quantum system, the quantum Fisher information of the parameter is given by

\[
\mathcal{T} = 4 \left| \langle \psi_\theta | \psi_\theta' \rangle - \langle \psi_\theta | \psi_\theta' \rangle^2 \right|.
\]
(3)

where \( |\psi_\theta'\rangle \) stands for the derivative of \( |\psi_\theta\rangle \) with respect to the parameter \( \theta \).

### 3 Dispersive measurement in circuit QED

We consider a circuit QED system with a superconducting qubit coupled to a microwave readout cavity and driven by two external drives: (i) a readout drive with amplitude \( \epsilon_d(t) \) and frequency \( \omega_d \) close to the cavity resonance frequency \( \omega_c \), and (ii) a Rabi drive with amplitude \( \epsilon_r(t) \) and frequency \( \omega_r \) close to the frequency of the qubit \( \omega_q \), [32]. The Hamiltonian of the entire system can be written as

\[
H_{\text{total}} = \hbar \omega_c a^\dagger a + \hbar \frac{\omega_q}{2} \sigma_z + \hbar g \left( a^\dagger \sigma_- + a \sigma_+ \right) \\
+ \hbar \left[ \epsilon_d(t) e^{-i \omega_d t} a^\dagger + \epsilon_r(t) e^{-i \omega_r t} a^\dagger + \text{h.c.} \right],
\]
(4)

where \( a^\dagger \) and \( a \) are the creation and annihilation operators for the microwave readout cavity, \( \sigma_+ \) and \( \sigma_- \) are the raising and lowering operators of the superconducting qubit, and \( g \) is the coupling strength between the cavity and the qubit. In the dispersive regime [32], \( |\Sigma| = |\omega_q - \omega_c| \gg g \), by applying the dispersive shift
\[ U = \exp\left[ g(a\sigma_+ - a^\dagger \sigma_-) / \Sigma \right], \] and moving to the rotating frames for both the qubit and cavity, \( U_c = \exp(i a^\dagger \omega d t), \) \( U_q = \exp(i \sigma_z \omega_r t / 2), \) with the rotating-wave approximation, the Hamiltonian in Eq. (4) becomes

\[
H_{\text{eff}} = \hbar \Delta_c a^\dagger a + \hbar \chi a^\dagger a \sigma_z + \hbar \frac{\Omega_R}{2} \sigma_x + \hbar (\epsilon_d + \epsilon_r) \left( a^\dagger + a \right),
\] (5)

where \( \Delta_c = \omega_c - \omega_d, \chi = g^2 / \Sigma, \) \( \Omega_R = 2\epsilon_r(t)g / \Sigma, \) and the Lamb-shifted qubit transition frequency \( \tilde{\omega}_q = \omega_q - \omega_r + \chi. \)

Without loss of generality, we describe the Hamiltonian of the superconducting qubit as

\[
H = \frac{\Omega}{2} \sigma_x + \frac{\Delta}{2} \sigma_z,
\] (6)

where \( \Omega \) is the Rabi frequency and \( \Delta \) is the detuning \([30,42]\). By applying a displacement transformation and tracing over the resonator state, we can eliminate the cavity degrees of freedom and get a reduced stochastic master equation \([43–46]\) with weak measurement (\( \hbar = 1 \))

\[
d\tilde{\rho}_t = -i \left[ H, \tilde{\rho}_t \right] dt + \eta D [ F ] \tilde{\rho}_t dt + \sqrt{\eta} M (\tilde{\rho}_t) dY_t,
\] (7)

where \( D [ A ] \rho = A \rho A^\dagger - \frac{1}{2} \left( A^\dagger A \rho + \rho A^\dagger A \right) \) and \( M (\rho) = A \rho + \rho A^\dagger \). Here, \( \tilde{\rho}_t \) is the un-normalized state, \( F \) is the measurement operator, and \( \eta \) is the quantum measurement efficiency with the continuous weak measurement constraint, i.e., \( \eta \ll 1 \). Also, \( dY_t \) is the independent and infinitesimal increment which represents the measurement output. A detailed derivation of the stochastic master equation was presented in our previous works \([35,44]\).

In generally, the parameters to be estimated might be the Rabi frequency, the detuning, the dissipation rates, the measurement strength, and so on. In this paper, we focus on estimating a single unknown parameter in the Hamiltonian, which is located in the interval \([a, b]\). The measurement process is assumed to be Markovian. Due to the relationship between a normalized state \( \rho_t \) and an un-normalized quantum state \( \tilde{\rho}_t \), namely \( \rho_t = \tilde{\rho}_t / \text{Tr}(\tilde{\rho}_t) \) \([27]\), together with Eq. (7), the increment \( dY_t \) has the form

\[
dY_t = \sqrt{\eta} \text{Tr}(M(\tilde{\rho}_t)) dt + dW_t,
\] (8)

where \( dW_t \) is the Wiener increment with zero mean and variance \( dt \). Equation (8) describes the quantum fluctuations of the continuous output signal. Define \( L_t = \text{Tr}(\tilde{\rho}_t) \) as a likelihood function. Owing to Eq. (7), the derivative of the likelihood function \( L_t \) with respect to time \( t \) can be written as \([22,27]\)

\[
dL_t = \text{Tr}(d\tilde{\rho}_t) = \sqrt{\eta} \text{Tr}(M(\tilde{\rho}_t)) dY_t = \sqrt{\eta} \text{Tr}(M(\rho_t)) L_t dY_t.
\] (9)

Combining Eq. (7) with Eq. (9), we can get the normalized quantum stochastic master equation by means of the multi-dimensional Itô formula (one can refer to “Appendix A” for details):
\[
d\rho_t = -i [H, \rho_t] dt + \eta D[F] \rho_t dt + \sqrt{\eta} \mathcal{H}[F] \rho_t dW_t, \quad (10)
\]

where \( \mathcal{H}[F] \rho = \mathcal{M}(\rho) - \rho \text{Tr}(\mathcal{M}(\rho)) \).

4 Quantum parameter estimation in circuit QED

In this section, we propose an efficient algorithm to calculate the Fisher information by the measurement output and the likelihood function in circuit QED.

4.1 Algorithm

Below, we use \( l_t \) to denote the log-likelihood function, i.e., \( l_t = \ln L_t \) [22]. From Eq. (9), the derivative of \( l_t \) with respect to time \( t \) is described by

\[
dl_t = d \ln \mathcal{L}_t = \frac{d\mathcal{L}_t}{\mathcal{L}_t} = \sqrt{\eta} \text{Tr}(\mathcal{M}(\rho_t)) dY_t. \quad (11)
\]

Therefore, according to Eq. (2), the Fisher information for single parameter estimation can be written as

\[
I(\theta) = E \left[ \left( \frac{\partial \ln \mathcal{L}_t}{\partial \theta} \right)^2 \right] = E \left[ \left( \frac{\partial l_t}{\partial \theta} \right)^2 \right]. \quad (12)
\]

Substituting Eq. (11) into Eq. (12), we can obtain an analytic form of the Fisher information.

From the Fisher information Eq. (12), it is easy to find that \( \theta \) is not an independent variable of the likelihood function. In other words, there does not exist an explicit expression of \( l_t \) with respect to \( \theta \), which results in many difficulties. In order to efficiently calculate the Fisher information, we propose a numerical algorithm with the help of the MH algorithm [36] and the MCMC integration [38].

In the beginning, we set a series of the parameter \( \{\theta_i\} \) satisfying

\[
\theta_{i+1} = \theta_i + d\theta, \quad i = 0, 1, 2, \ldots, N_p, \quad (13)
\]

where the interval \( d\theta \) is a small constant. For each \( \theta_i \), there exists a log-likelihood function, say \( l_{t}^i \), corresponding to \( \theta_i \). Here, the collection of log-likelihood functions \( \{l_{t}^0, l_{t}^1, \ldots, l_{t}^{N_p}\} \) is a set of functions of time with \( t \in [0, T] \). From Eq. (12), it is clear that the calculation of the Fisher information requires the derivative of \( l_t \) with respect to \( \theta \). However, the noise induced by the measurement process makes it improper to use the ordinary numerical derivation to compute \( \partial l_{t} / \partial \theta \). To deal with it, it is natural to use the evolution of the ensemble to eliminate the impact of measurement noise. Since \( d\theta \) can be infinitesimal, the derivative of \( l_{t} \) with respect to \( \theta \) can be given by the Newton’s backward difference quotient with infinitesimal errors, i.e.,
\[
\frac{\partial l_i^t}{\partial \theta} \approx \frac{l_i^t - l_{i-1}^t}{\theta_i - \theta_{i-1}} = \frac{l_i^t - l_{i-1}^t}{d\theta}, \quad i = 1, 2, \ldots, N_p. \tag{14}
\]

Next, we generate a cluster of \(\theta\) by the MH algorithm (one can refer to “Appendix B” for details), whose prior probability distribution is assumed to satisfy a certain distribution. Denote such generated cluster of \(\theta\) by
\[
\hat{\theta} = \left\{ \hat{\theta}_j \mid j = 1, 2, \ldots, N_M \right\}, \tag{15}
\]
where \(N_M\) is the Monte Carlo number. Note that the number of candidate points \(N_A\) that used to generate random samples is chosen to be larger than the Monte Carlo number, i.e., \(N_M \leq N_A\). In the set of \(\hat{\theta}\), the fluctuation of the pre-estimated parameter values is rather small. This process makes the following numerical calculation as close to the analytic result as possible. For simplicity, one may anticipate the initial value of the sequence generating \(\hat{\theta}\) to be a constant value. It is easy to look for a point \(\theta_j^*\) such that its distance from \(\hat{\theta}_j\) is minimized, where the distance is defined as \(|\theta_j^* - \hat{\theta}_j|\) for any \(\theta_j^* \in \{\theta_i\}_{i=1}^{N_p}\). As a result, \(\{(\partial l_j^* / \partial \theta)^2 \mid j = 1, 2, \ldots, N_M\}\) could be picked out from the collection \(\{(\partial l_1^i / \partial \theta)^2, (\partial l_2^i / \partial \theta)^2, \ldots, (\partial l_N^p / \partial \theta)^2\}\).

Finally, calculating the Fisher information means to acquire the expectation value \(E[(\partial l_t / \partial \theta)^2]\) from the sample \(\{(\partial l_j^* / \partial \theta)^2 \mid j = 1, 2, \ldots, N_M\}\) owing to Eq. (12). By the Makov chain Monte Carlo integration (see “Appendix C” for details), the Fisher information can be approximated as
\[
E \left[ (\partial l_t / \partial \theta)^2 \right] \approx \frac{1}{N_M} \sum_{j=1}^{N_M} \left( \frac{\partial l_j^*}{\partial \theta} \right)^2. \tag{16}
\]

It should be noted that the expectation value is calculated over different realization of the measurement results \(D\). In our algorithm, even if each realization is evaluated for a different value of \(\hat{\theta}\) that generated by the MH algorithm, the approximation principle \(\theta_j^* = \arg \min |\theta_j - \hat{\theta}_j|\) and the large number of \(N_M\), \(N_M \to +\infty\), would make sure that the approximative Fisher information is evaluated at the mean value \(\overline{\theta}\) (see “Appendix B” and Fig. 6 for detailed explanation). As a conclusion, the procedure of calculating the Fisher information is shown in Fig. 1.

4.2 Numerical simulations

Let \(\Delta\) in Hamiltonian (6) be an unknown parameter that requires estimating. We denote the normalized quantum state \(\rho_t\) by
\[
\rho_t = \frac{1}{2} \begin{pmatrix} 1 + z & x - iy \\ x + iy & 1 - z \end{pmatrix}, \tag{17}
\]
and the initial state is \(x(0) = y(0) = 0, \ z(0) = 1\). The other parameters in stochastic master Eq. (10) are \(\Omega = 1.73\) and \(\eta = 0.01\), and the measurement operator is

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Fig. 1 The flowchart of calculating the Fisher information via quantum weak measurement
Fig. 2 The top panel shows the evolution of the three components of the Bloch vector with $\Omega = 1, \Delta = 1.73, \kappa = 1,$ and $\eta = 0.01$. The curves are $x = \text{Tr} (\sigma_x \rho_t)$ (the red dotted curve), $y = \text{Tr} (\sigma_y \rho_t)$ (the blue dashed curve), and $z = \text{Tr} (\sigma_z \rho_t)$ (the green solid curve). The middle one represents the measurement output $Y_t$. The bottom panel shows the log-likelihood function $l_t$ (Color figure online).

Fig. 3 Time evolutions of the Fisher information and the quantum Fisher information

![Graph showing the evolution of the Fisher information and the quantum Fisher information](image)

given by $F = \sqrt{\kappa} \sigma_y$ with the measurement strength $\kappa = 1$. For convenience, we define $\tau = \Delta t$ throughout this section. Suppose that the mean of the unknown parameter $\Delta$ and the initial value of the sequence generating $\hat{\Delta}$ are set to be 1, then the sequence $\hat{\Delta}$ can be obtained by proceeding the MH algorithm when the stationary distribution and proposal distribution are assumed to satisfy the normal distributions $N(0, 1)$ and $N(0, d \tau)$, respectively. Figure 2 shows the evolution of the normalized quantum state with dispersive measurement according to Eq. (10). The output $Y_t$ and the log-likelihood function $l_t$ are also plotted.

Based on the proposed algorithm and stochastic master Eq. (10), we show the evolution of the Fisher information for quantum parameter estimation in Fig. 3. The blue dash dotted curve in Fig. 3 represents the evolution with 500 dispersive measurements in circuit QED, and the red solid curve is the evolution of the quantum Fisher information calculated by Eq. (3). The quantum Fisher information, obtained with the assumption that there is only unitary evolution, always represents the upper bound of the Fisher information. From Fig. 3, we find that the evolution of the Fisher information can approach the quantum Fisher information in a short time interval. These results demonstrate the effectiveness of the proposed algorithm. This proposed algorithm is inspired by Mølmer’s works [27,28,41]. Reference [27] demonstrated that the
Fisher information can be identified by stochastic master equation simulations. The Fisher information matrix was calculated with the assumption that the derivative of the Hamiltonian and damping terms with respect to θ were known. Based on the MH algorithm and the MCMC integration, we propose a new algorithm to calculate the Fisher information and apply this method to the superconducting circuit QED system with dispersive measurement. We do not need priori information about the derivative of the system Hamiltonian and the damping terms with respect to θ. Moreover, Ref. [27] considered jump-type measurements and diffusion-type measurements. Here, we consider the quantum parameter estimation via the dispersive measurement with a more general stochastic master equation.

Furthermore, we plot the evolutions of the Fisher information with the proposed algorithm for various measurement operators in Fig. 4. The green dashed, blue dot dashed and purple dotted curves are with the $\sqrt{\kappa}\sigma_x$, $\sqrt{\kappa}\sigma_y$ and $\sqrt{\kappa}\sigma_z$ measurement operators with $\kappa = 1$ in circuit QED, respectively. Finally, we consider the effect of the measurement efficiency on the Fisher information with the help of the proposed algorithm. In Fig. 5, we plot the time evolutions of the Fisher information for various efficiencies. Here, the measurement operator is $\sqrt{\kappa}\sigma_y$ with measurement efficient
\( \kappa = 1 \). From Fig. 5, it is easy to find that the Fisher information increases with the detection efficiency.

## 5 Conclusion

We considered the quantum parameter estimation in circuit QED via dispersive measurement and the stochastic master equation. Based on the Metropolis–Hastings algorithm and the Markov chain Monte Carlo integration, a new algorithm was proposed to calculate the Fisher information. We discussed the evolutions of the Fisher information with the proposed algorithm for various measurement operators and measurement efficiencies. Simulation results showed that the Fisher information increases with the weak measurement efficiency.

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**Appendix A: The lemma of the multi-dimensional Itô formula**

In the multi-dimensional Itô formula, it is worth noting that if \( x(t) \) were continuously differentiable with respect to time \( t \), then the term \( \frac{1}{2} \mathrm{d}x^T(t) V_x(x(t), t) \mathrm{d}x(t) \) would not appear owing to the classical calculus formula for total derivatives. For example, if \( V(x_1, x_2) \) is continuously differentiable with respect to \( t \), e.g., \( V(x_1, x_2) = x_1(t) x_2(t) \), then its derivation should be \( \mathrm{d}V(x_1, x_2) = x_1 \mathrm{d}x_2 + x_2 \mathrm{d}x_1 + \mathrm{d}x_1 \mathrm{d}x_2 \).

**Appendix B: Metropolis–Hastings algorithm**

In Markov chains, suppose we generate a sequence of random variables \( X_1, X_2, \ldots, X_n \) with Markov property, namely the probability of moving to the next state depends only on the present state instead of the previous state:

\[
\Pr \{ X_{n+1} = x \mid X_1 = x_1, \ldots, X_n = x_n \} = \Pr \{ X_{n+1} = x \mid X_n = x_n \}. \tag{18}
\]

Then, for a given state \( X_t \), the next state \( X_{t+1} \) does not depend further on the hist of the chain \( X_1, X_2, \ldots, X_{t-1} \), but comes from a distribution which only depends on the current state of the chain \( X_t \). For any time instant \( t \), if the next state is the first sample reference point \( Y \) obeying distribution \( q(\cdot \mid X_t) \) which is called the transition kernel of the chain, then obviously it depends on the current state \( X_t \). In generally, \( q(\cdot \mid X_t) \) may be a multi-dimensional normal distribution with mean \( X \), so the candidate point \( Y \) is accepted with probability \( \alpha(X_t, Y) \) where

\[
\alpha(X, Y) = \min \left( 1, \frac{\pi(Y) q(X \mid Y)}{\pi(X) q(Y \mid X)} \right). \tag{19}
\]
Fig. 6 Illustration of the Metropolis–Hasting algorithm with the initial value $X(1) = -10$. a Stationary distribution $N(0, 0.1)$, and in b, 500 iterations from Metropolis–Hastings algorithm with the stationary distribution $N(0, 1)$ and proposal distribution $N(0, 0.1)$ are plotted.

Here, $\pi (A)$ stands for a function only depends on $A$. If the candidate point is accepted, the next state becomes $X_{t+1} = Y$. If the candidate point is rejected, it means that the chain does not move, and the next state will be $X_{t+1} = X_t$. We illustrate this sampling process with a simple example (see Fig. 6). Here, the initial value is $X(1) = -10$. Figure 6a represents the stationary distribution $N(0, 0.1)$. In Fig. 6b, we plot 500 iterations from Metropolis–Hastings algorithm [36] with the stationary distribution $N(0, 1)$ and proposal distribution $N(0, 0.1)$. Obviously, sampling data selecting from the latter part would be better.

**Appendix C: Markov Chain Monte Carlo integration**

In Markov chain, the Monte Carlo integration [38] can be used to evaluate $E[f(X)]$ by drawing samples $\{X_1, \ldots X_n\}$ from the Metropolis–Hastings algorithm. Here,

$$E[f(X)] \approx \frac{1}{n} \sum_{i=1}^{n} f(X_i)$$  \hspace{1cm} (20)

means that the population mean of $f(X)$ is approximated by the sample mean. When the sample $X_t$ is independent, the law of large numbers ensures that the approximation can be made as accurate as desired by increasing the sample. Note that here $n$ is not the total amount of samples by Metropolis–Hastings algorithm but the length of drawing samples.
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