COMPLEXITY OF ACTIONS OVER PERFECT FIELDS

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Abstract. Let $G$ be a connected reductive group over a perfect field $k$ acting on an algebraic variety $X$ and let $P$ be a minimal parabolic subgroup of $G$. For $k$-spherical $G$-varieties we prove finiteness result for $P$-orbits that contain $k$-points. This is a consequence of an equality on $P$-complexities of $X$ and of any $P$-invariant $k$-dense subvariety in $X$, which generalizes a corresponding result of E.B.Vinberg in the case of algebraically closed field $k$. Also we introduce an action of the restricted Weyl group $W$ on the set of $k$-dense $P$-invariant closed subvarieties of $X$ of maximal $P$-complexity and $k$-rank in the case of char $k = 0$ and on the set of all $k$-dense $P$-orbits in the case of real spherical variety which generalizes the action on $B$-orbits introduced by F.Knop in the algebraically closed field case. We also introduce a little Weyl group related with this action and describe its generators in terms of the generators of $W$ which generalize the description of M.Brion in algebraically closed field case.

To the memory of Èrnest Vinberg

1. Introduction

Let $k$ be a ground field, let $G$ be a connected reductive algebraic group defined over $k$, and let $X$ be a $G$-variety. For $k$ algebraically closed and of characteristic zero, the following theorem was proved independently by M.Brion [Bri86], and E.B.Vinberg [Vin86] (the latter based on the paper of V.L.Popov [Pop86]). Later T.Matsuki, [Mat91], gave a completely different and much simpler argument which also works in positive characteristic.

1.1. Theorem. Assume that a Borel subgroup $B$ of $G$ has an open orbit in $X$ (in which case $X$ is called spherical). Then the number of $B$-orbits in $X$ is finite.

The theorem is sharp in two ways. First, it does not hold if $B$ is replaced by $G$ or any other non-solvable parabolic subgroup of $G$. Secondly, the theorem is not true for $B$-actions which are not the restriction of a $G$-action.

The aim of this paper is to generalize the Theorem 1.1 above to more general ground fields. The obvious way is to replace the Borel subgroup which may not be defined over $k$ by a minimal parabolic $k$-subgroup $P$. However this is not enough. Masuki [Mat91, Rmk. 7] gave for $k = \mathbb{R}$ an example of a $k$-variety $X$ having infinitely many $P$-orbits despite having an open $P$-orbit. Most of these orbits do not contain $k$-points, though. So, Matsuki conjectured [Mat91, Conj. 2] that in the presence of an open $P$-orbit, $X(\mathbb{R})$ has only finitely many $P(\mathbb{R})$-orbits. In the same paper, Matsuki showed how to reduce his conjecture to the case of groups of real rank one. For that case, Kimelfeld [Kim87] had already classified all relevant spaces three years earlier but his paper was overlooked. Matsuki’s conjecture was subsequently proved independently by Bien [Bie93] and Krötz-Schlichtkrull [KS16].
It is now tempting to replace in Matsuki’s conjecture $\mathbb{R}$ by an arbitrary field $k$ but in that form it is not true. The reason is simply that there may be $P$-orbits $Y$ where $Y(k)$ decomposes into infinitely many $P(k)$-orbits. The easiest example is $k = \mathbb{Q}, G = X = G_m$ with “square” action $a \ast x := a^2 x$. Here $X(k)/P(k) = \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is infinite. This difficulty is overcome by only considering $P$-orbits which contain a $k$-point. Now our generalization of Theorem 1.1 is:

1.2. **Theorem.** Let $k$ be an infinite perfect field and assume that a minimal parabolic $k$-subgroup $P$ of $G$ has an open orbit in $X$ (in which case $X$ is called $k$-spherical). Then the number of $P$-orbits $Y$ of $X$ with $Y(k) \neq \emptyset$ is finite.

The theorem also obviously also true for finite fields but in that case Theorem 1.1 is stronger (since then $P = B$). So we excluded that case. The perfection assumption comes from Kempf’s instability theorem [Kem78]. We don’t know whether the theorem holds for non-perfect fields.

Borel-Serre [BS64] have proved that over local fields of characteristic zero for every $P$-orbit $Y$ the set $Y(k)/P(k)$ is finite. This immediately implies Matsuki’s conjecture for local fields:

1.3. **Corollary.** Let $k$ be a local field of characteristic zero and let $X$ be a $k$-spherical $G$-variety. Then $X(k)/P(k)$ is finite.

Vinberg [Vin86] has proved an even more general version of Theorem 1.1 which works for any $G$-variety. To this end, he introduced the complexity $c(Y)$ of a $B$-invariant subvariety $Y$ as the dimension of a birational quotient $Y/B$. So $c(Y) = 0$ means that $B$ has an orbit in $Y$. For this notion, Vinberg proved $c(Y) \leq c(X)$ for any $B$-stable subvariety. Theorem 1.1 is just the special case of $c(X) = 0$. In this paper we prove an analogous extension of Theorem 1.2. The idea is to consider $P$-stable subvarieties $Y$ for which $Y(k)$ is dense. For the precise statement see Theorem 4.2 below.

The original proofs of Vinberg and Brion used a reduction argument where $X$ is deformed to a horospherical variety. We do not know how to adapt this to our situation since it seems to be impossible to control $k$-points under a deformation of $X$. Instead we use the aforementioned reduction argument of Matsuki to semisimple groups. Over algebraically closed fields there is basically only one, namely $SL(2, k)$. In general, the structure of semisimple rank-1-groups is much more involved so that we cannot simply resort to a classification. Instead, we use some general structure theorems for the action of anisotropic groups which may interesting in their own right.

After proving the finiteness theorem it suggests itself to generalize also the results of [Kno95]. Assume for simplicity that $X$ is $k$-spherical and let $\mathfrak{B}(X)$ be the finite set of $P$-orbits $Y \subseteq X$ such that $Y(k) \neq \emptyset$. In the setting of $k$ being algebraically closed fields of characteristic not 2, a canonical action of the Weyl group $W$ of $G$ on $\mathfrak{B}(X)$ has been constructed. For general fields, we replace $W$ by the restricted Weyl group $W_k$. For every simple reflection of $W_k$ be define an involution of $\mathfrak{B}(X)$ and we conjecture that these involutions extend to an action of $W_k$ on $\mathfrak{B}(X)$. The problem is of course the verification of the braid relations.

Here we have only partial results. Apart from algebraically closed fields we show that the conjecture holds for the field $k = \mathbb{R}$ of real numbers. The proof uses the same Hecke algebra approach as in [Kno95].
Otherwise, we show that for char \( k = 0 \), the \( W_k \)-action exists on a certain subset \( \mathcal{B}_0(X) \) of \( \mathcal{B}(X) \) where \( \mathcal{B}_0(X) \) consists of the \( P \)-orbits of maximal rank or, equivalently, of those \( P \)-orbits \( Y \) for which \( \dim Y / \text{Rad}_P P \) is maximal. This subset contains the open \( P \)-orbit \( (X) \) and we show that \( W_k \circ (X) = \mathcal{B}_0(X) \). Moreover, we prove that the stabilizer \( W(X) \) of the open \( P \)-orbit is related to the little Weyl group \( W_k(X) \) of \( X \) as defined in [KK16].

To this end we use, as in [Kno95], the momentum map on the cotangent bundle of \( X \).

Finally, we also generalize Brion’s approach [Bri01] for describing generators of \( W(X) \). These are either reflections or products of commuting reflections in \( W_k \).

We indicate the structure of the paper. In Section 3 we study the case where the semisimple \( k \)-rank of \( G \) is one. Section 4 is dedicated to deriving an inequality for complexities in the general case. In Section 5 we consider Hecke algebra approach to the action of the Weyl group on the set of \( P \)-orbits of a real spherical variety. In Section 6 we study equivariant geometry of cotangent vector bundle and it relation with the action of a Weyl group on the \( k \)-dense sheets of \( P \)-orbits with maximal complexity, rank and homogeneity. Section 7 is dedicated to comparison of the actions of the Weyl groups under field extensions. Section 8 is dedicated to the proof of the fact that little Weyl group is generated by the set of reflections in the roots of \( G \) and by the products of commuting reflections. In the Section 9 for a real homogeneous spherical variety that have wonderful compactification we provide the action of fundamental group of an associated flag variety (a unique closed orbit in the wonderful compactification) on the set of connected components of the open \( P \)-orbit (which are \( P(R)_c \)-orbits). This action is important since it preserves the set of \( G(R) \)-orbits.

One of the natural questions that arise in relation with our work is studying the possibility of extension of our action of the restricted Weyl group in the real case on the set of principal orbits to the action on to the set of \( B(R) \)-orbits in \( X(R) \). S.Cupit-Foutou and D.A.Timashev first raised this question in [CFT] in the case of spherical varieties of the real split groups and provided the extension of a subgroup of the Weyl group under some technical assumptions on the class of varieties. These assumptions were necessary, since they also discovered an example where the generators of their action generate a group which is different from the Weyl group.

Finally, let us mention that our work has started during Vinberg’s 80\(^{th} \) anniversary conference in Moscow 2017 and was announced officially on Brion’s 60\(^{th} \) anniversary conference in Lyon 2018. We would like express our warmest wishes and greatest respect to Michel Brion. Unfortunately, Êrnest Vinberg had passed away on 12th May 2020, so we wish to dedicate this paper to his blessed memory.

### 2. Complexities

Let \( k \) be an infinite perfect field with algebraic closure \( K \). A (not necessarily irreducible) \( k \)-variety \( X \) is called \( k \)-dense if \( X(k) \) is Zariski dense in \( X \). Note that for \( k \)-dense varieties, irreducibility is equivalent to absolute irreducibility.

Let \( H \) be a linear algebraic \( k \)-group. An \( H \)-variety \( X \) is called \( H \)-irreducible if \( H \)-span of the set of \( k \)-points is Zariski dense in \( X \), i.e. \( \overline{H \cdot X(k)} = X \). When \( H \) is connected then \( H \)-irreducibility is equivalent to irreducibility.
Let $k(X)$ be the total ring of fractions of $X$. If $X$ is $H$-irreducible then $k(X)^H$ is a field and we define the $H$-complexity of $X$ as

\begin{equation}
(2.1) \quad c(X/H) := \text{trdeg}_k k(X)^H.
\end{equation}

The notation “$X/H$” is purely formal and does not mean that an orbit space $X/H$ exists. If it does, though, then $c(X/H) = \dim X/H$. If $X^o \subseteq X$ is any irreducible component and $H^o \subseteq H$ the component of unity then clearly $c(X/H) = c(X^o/H^o)$.

The most important instance of $H$-complexity is as follows: Assume $X$ is a $k$-dense, irreducible $G$-variety where $G$ is a connected reductive $k$-group. Let $P \subseteq G$ be a be a minimal parabolic $k$-subgroup. Then the $k$-complexity of $X$ is defined as its $P$-complexity:

\begin{equation}
(2.2) \quad c_k(X) := c(X/P).
\end{equation}

By $\mathfrak{B}(X)$ let us denote the set of $k$-dense closed subvarieties $Y \subset X$ such that $c_k(Y) = c_k(X)$. If $X$ is $k$-spherical we shall prove that $\mathfrak{B}(X)$ coincides with the set of $P$-orbit closures $P\pi x$ where $x \in X(k)$.

More generally, given an $H$-variety $X$, we are interested in the set $\mathfrak{Irr}(X/H)$ of all closed $H$-irreducible subvarieties $Z \subseteq X$.

2.1. Lemma. Let $G$ be a connected reductive group, let $P \subseteq G$ be a $k$-parabolic subgroup, and $H \subseteq G$ a closed $k$-subgroup. Put $X := G/H$ and $Y := G/P$ inducing the diagram

\begin{equation}
(2.3) \quad \begin{array}{ccc}
G/H & \xleftarrow{\pi_H} & G/P \\
\downarrow{\pi_H} & & \downarrow{\pi_P} \\
G/H & \xrightarrow{\pi_P} & G/P
\end{array}
\end{equation}

with $\pi_H(g) = gH$ and $\pi_P(g) = g^{-1}P$. Then there is an injective inclusion preserving map

\begin{equation}
(2.4) \quad \mathfrak{Irr}(Y/H) \to \mathfrak{Irr}(X/P) : Z \mapsto Z' := \pi_H(\pi_P^{-1}(Z))
\end{equation}

which also preserves complexities:

\begin{equation}
(2.5) \quad c(Z/H) = c(Z'/P).
\end{equation}

Proof. The problem is that the preimage of a $k$-dense subvariety is, in general, not $k$-dense. But this is true for $\pi_P$ since $G(k) \to Y(k)$ is surjective and $P$ is $k$-dense. This implies that $\pi_P^{-1}$ induces a bijection $\mathfrak{Irr}(Y/H) \to \mathfrak{Irr}(G/(H \times P))$. The latter maps via $\pi_H$ into $\mathfrak{Irr}(X/P)$. Since this map preserves codimension and relative codimension of the subsets, it is clear that complexities are preserved.

2.2. Remarks. 1. Observe that the case $c = 0$ means that the correspondence maps orbit closures to orbit closures.

2. Observe that $\mathfrak{Irr}(Y/H)$ and its inclusion into $\mathfrak{Irr}(X/P)$ depends only on $G(k)$-conjugacy class of $H$. It does not just depend on $X$, though. A typical example would be $k = \mathbb{R}$ and $X = SL(3)/SO(3)$, the space of unimodular quadratic forms. Then $X(\mathbb{R})$ has two $G(\mathbb{R})$-orbits with isotropy groups $H_1 = SO(3, \mathbb{R})$ and $H_2 = SO(1, 2, \mathbb{R})$, respectively. Since $H_1$ acts transitively on $Y(\mathbb{R})$ we have $\mathfrak{Irr}(Y/H_1) = \{Y\}$. On the other hand, $H_2(\mathbb{C})$ has four orbits in $Y(\mathbb{C})$ and all of them are $\mathbb{R}$-dense. Thus, $|\mathfrak{Irr}(Y/H_2)| = 4$. 

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3. The rank-1-case

In the following, $k$ is an arbitrary infinite perfect field. We start with a codimension-1-variant of a well-known theorem of Rosenlicht.

3.1. Lemma. In the following, everything should be defined over $k$. Let $P$ be a connected group and let $R \subseteq P$ a connected normal subgroup. Let $X$ be a smooth $P$-variety such that all stabilizers $R_x$ are finite. Let, moreover, $Y \subseteq X$ be a $P$-stable prime divisor. Then there is a $P$-stable open subset $X_0 \subseteq X$ with $X_0 \cap Y \neq \emptyset$, a smooth $P$-variety $Z$, and a surjective $R$-invariant $P$-morphism $\varphi : X_0 \to Z$ such that every fiber is a finite union of $R$-orbits.

Proof. By Rosenlicht’s theorem [Ros56, Thm. 2], there is a $P$-stable open subset $X_1 \subseteq X$ such that the orbit space $Z_1 = X_1/R$ exists. We may assume that $Z_1$ is smooth. If $X_1 \cap Y \neq \emptyset$ we are done. So assume that $X_1 \cap Y = \emptyset$. The divisor $Y$ induces a $P$-invariant valuation $v_Y$ of $K(X)$. Its restriction $v' = v_Y|_{K(Z_1)}$ is $P/R$-invariant and induces an equivariant open embedding $Z_1 \subseteq Z = Z_1 \cup Y'$ where $Z$ is a smooth $P$-variety and $Y' \subset Z$ is a $P$-invariant prime divisor. The morphism $X_1 \to Z_1$ extends, by construction, to a $P$-equivariant $k$-morphism $\varphi : X_0 \to Z$ where $X_0 \subseteq X$ is open $P$-stable with $X_0 \supseteq X_1$ and $Y_0 := X_0 \cap Y \neq \emptyset$. Since the map $Y_0 \to Y'$ between divisors is dominant (again by construction), we may shrink $Z$ such that $\varphi$ is surjective and equidimensional. By assumption, all $R$-orbits in $X$ have dimension $\dim R$ which is also the dimension of the fibers of $\varphi$. The $R$-invariance of $\varphi$ implies that every fiber is a finite union of $R$-orbits. □

Next we extend a well-known property of torus actions.

3.2. Lemma. Let $G$ be an elementary $k$-group acting on a locally linear $k$-variety $X$. Then every $x \in X(k)$ is contained in a $G$-stable, affine, open subset $X_0 \subseteq X$.

Proof. We have $G = MA$ with $M$ anisotropic and $A$ a split central torus. By local linearity, we may assume that $X$ is a subvariety of $P(V)$ where $V$ is a $G$-module. Let $Z := X \setminus X$ be the boundary and let $\tilde{Z} \subseteq V$ be the affine cone over $Z$. Let, moreover, $0 \neq \tilde{x} \in V(k)$ be a lift of $x$. Since $M$ is anisotropic, the orbit $M\tilde{x}$ is closed in $V$ by Kempf’s theorem (see [Kem78, Remark after Cor. 4.4]) and does not meet $\tilde{Z}$. It follows that there is a homogeneous $M$-invariant $f$ on $V$ which vanishes on $\tilde{Z}$ and with $f(M\tilde{x}) = 1$. Let $f = \sum f_x$ be the decomposition of $f$ into $A$-eigenfunctions. Since $Z$ is $G$-stable, all $f_x$ are $G$-eigenfunction which vanish on $\tilde{Z}$. One of them, say $f_{x_0}$, is non-zero in $\tilde{x}$. Then $X_0 = X \setminus \{f_{x_0} = 0\}$ is clearly a $G$-stable, affine, open subset of $X$ containing $x$. □

Let $P$ be a quasi-elementary group with Iwasawa decomposition $P = L_{an}AP_u$, where $L_{an}$ is maximal anisotropic subgroup, $P_u$ is the unipotent radical and $A$ is the maximal split torus. For a $P$-invariant subset $Y$. By shrinking the rational quotient $Y/AP_u$ we can assume that it is smooth and by previous theorem we may assume that is affine. In the case of char $k = 0$ by Richardson theorem (see [PV94]) there exists the stabilizer of general position for $L_{an}$ on rational quotient $Y/AP_u$ that we will denote by $M_Y$. By $s_k(Y)$ let us denote the dimension of $L_{an}$-orbit on $Y/AP_u$ of general position.

3.3. Lemma. Let $P$ be a quasi-elementary group (e.g. a minimal parabolic) acting on a smooth $k$-dense variety $X$. Let $Y \subseteq X$ be a proper $P$-stable subvariety such that the set of $k$-points $x \in Y(k)$ with anisotropic stabilizer $P_x$ is dense in $Y$. Then $c(Y/P) < c(X/P)$, $\text{rk}_k(X) \geq \text{rk}_k(Px)$ and $s_k(Y) \leq s_k(X)$. 

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Proof. By removing $Y_{\operatorname{sing}}$ from $X$ and $Y$, we may assume that $Y$ is smooth, as well.

Let $\pi : \tilde{X} \to X$ be the blow-up of $X$ in $Y$ and let $\tilde{Y} = \pi^{-1}(Y)$ be the exceptional divisor. Then $\tilde{X}$ and $\tilde{Y}$ are smooth and every fiber of $\pi|_D : \tilde{Y} \to Y$ over a $k$-point is a projective space. If $\tilde{y} \in \tilde{Y}$ we have $P_{\tilde{y}} \subseteq P_y$. This implies that also $\tilde{Y}$ has a dense set of $k$-points with an anisotropic stabilizer. Since $c(\tilde{Y}/P) \geq c(Y/P)$ it suffices to prove first assertion for $(\tilde{X}, \tilde{Y})$. Since we have a rational map between $\tilde{Y}/AP_u \to Y/AP_u$ we also have $s_k(\tilde{Y}) \geq s_k(Y)$. This reduces the problem to the case when $Y$ is of codimension 1 in $X$.

Let $R \subseteq P$ be the split radical of $P$, i.e., $R$ is normal, split solvable and $M = P/R$ is anisotropic. Since $R$ does not contain any nontrivial anisotropic subgroup, we have that $R_x$ is finite for every $x \in Y(k)$ with $P_x$ anisotropic. These points form a dense subset of $Y$. It follows that $R$ acts locally freely (i.e. with finite stabilizers) on an open subset $X'$ of $X$ with $X' \cap Y \neq \emptyset$. Replacing $X$ by $X'$ we may assume that the action of $R$ on $X$ is locally free.

Now let $X_0 \to Z$ be as Lemma 3.1 and let $Y' \subseteq Z$ be the image of $X_0 \cap Y$ which is an $M$-stable prime divisor. Since clearly $c(X/P) = c(Z/M)$ and $c(Y/P) = c(Y'/M)$. So after replacing $(X,Y)$ with $(Z,Y')$ we may assume that $P = M$ is anisotropic.

Now Lemma 3.2 allows us to replace $X$ by an affine open subset. Since $Mx$ is closed in $X$ for $x \in X(k)$ we conclude that the generic orbits in $X$ and $Y$ are closed. Hence, $Y/M$ is a proper subvariety of $X/M$ which implies $c(Y/P) = \dim Y/M < \dim X/M = c(X/P)$.

The inequality $s_k(X) \geq s_k(\tilde{Y})$ now follows from the fact that $Y/P_u A$ is a subset of $X/P_u A$ and from the semicontinuity of the dimension of $L_{\text{an}}$-orbits. The same argument applied to $A$-orbits on the quotient by $P_u^{\text{Lan}}$ gives $\operatorname{rk}_k(X) \geq \operatorname{rk}_k(Px)$.

3.4. Corollary. Let $P$ be a quasi-elementary $k$-group acting on a smooth $k$-variety $X$. Assume that $X$ contains an open $P$-orbit $X_0$. Then the stabilizer $P_x$ of every point $x \in X(k) \setminus X_0(k)$ is isotropic.

In the rest of this section, let $H^\circ$ be the reduced, connected component of an algebraic group $H$.

3.5. Lemma. Let $G$ be a reductive group and $P = L_{an} AP_u \subseteq G$ a parabolic subgroup. Let $X$ be a homogeneous $G$-variety. Let $Z \subseteq X$ be the union of all closed $P$-orbits. Then:

i) $x \in Z$ if and only if $P_x^0$ is parabolic in $G_x^\circ$.

ii) $Z$ is closed in $X$.

iii) Let $x \in Z$ and let $T_0 \subseteq P_x$ be a maximal torus. Assume $P_u^{T_0} = \{1\}$. Then $Px$ is a connected component of $Z$.

Proof. Let $H := G_x$. Then $X = G/H$. Now consider the diagram

\[
\begin{array}{ccc}
G/H & \xrightarrow{\pi_H} & G/P \\
\pi_P & \searrow & \\
& & \\
\end{array}
\]

where $\pi_H(g) = gH$ and $\pi_P(g) = g^{-1}P$. Then $Z \mapsto \pi_H \pi_P^{-1}(Z)$ furnished an inclusion preserving bijection between closed $H$-stable subsets of $G/P$ and closed $P$-stable subsets
of $G/H$. In particular, closed $H$-orbits correspond to closed $P$-orbits. Moreover, $Px \subseteq X$ corresponds to $Hy \subseteq G/P$ where $y = eP$.

i) Since $G/P$ is complete, $Hy$ is closed if and only if it is complete if and only if $H^o = (H \cap P)^o = P_x^e$ is parabolic in $H^o = G^o_x$.

ii) Let $B_H \subseteq H^o$ be a Borel subgroup. Then an $H$-orbit is closed if and only if it contains a $B_H$-fixed point. Thus, the union of closed $H$-orbits is $H(G/P)^B_H$ which is closed since it is the image of the proper map $H/B_H \times (G/P)^B_H \to G/P$.

iii) The torus $T_0$ is contained in a Levi complement $L$ of $P$. Let $P^-$ be opposite to $P$ with respect to $L$ and let $P^-_x$ be its unipotent radical. Then, as an $L$-variety, an open neighborhood of $y$ in $G/P$ is isomorphic to $P^-_x$. Since $P^u_{\psi} = 1$ also $(P^u)^{\psi} = 1$. This means that $y$ is an isolated $T_0$-fixed point. Since $T_0 \subseteq P_x = H \cap P$ we may choose $B_H$ such that $T_0 \subseteq B_H$. Then, a fortiori, $y$ is an isolated point of $(G/P)^B_H$, as well. Since each closed $H$-orbit being a flag variety contains a unique $B_H$-fixed point the $H^0$-spans of different irreducible components of $(G/P)^B_H$ are closed and do not intersect. Therefore $Hy$ and $Px$ are open and closed in their respective union of all closed orbits.

A $P$-orbit as in iii) will be called isolated. Observe, that there exist only finitely many of them.

3.6. Lemma. Let $G$ be a semisimple group with $\text{rk}_k G = 1$ and with minimal parabolic $P \subseteq G$. Let $X$ be an affine homogeneous $G$-variety and let $x \in X(k)$ a be $k$-point whose stabilizer $P_x$ is isotropic. Then $Px$ is isolated and $\text{rk}_k(X) > \text{rk}_k(Px)$.

Proof. Since $X$ is affine, the stabilizer $H := G_x$ is reductive, as well. It contains the isotropic subgroup $P_x$ so is also isotropic. Therefore $\text{rk}_k H = 1$ and $H$ contains a split torus $A_0$ of dimension 1. Let $\lambda : G_m \to A_0$ be an isomorphism. Then $\lambda$ induces parabolic subgroups $P^\pm_H := P^\pm_H(\lambda) \subseteq H^o$ and $P^\pm_G := P^\pm_G(\lambda) \subseteq G$ with $P^\pm_H = P^\pm_G \cap H^o$. Observe that $P^\pm_H$ and $P^\pm_G$ are minimal parabolics of $H$ and $G$, respectively.

We claim that the stabilizer $P^o_x = (H \cap P)^o$ is parabolic in $H$. Since it is isotropic by assumption, it contains a subgroup $S$ which is either isomorphic to $G_m$ or $G_m$. By replacing $A_0$ by $hA_0h^{-1}$ with a suitable $h \in H(k)$ we may assume that $S$ either lies in $R_u(P_H)$ or is equal to $A_0$.

Let $Y := G/P_G$ and $y_0 := eP_G \in Y(k)$. From the Bruhat decomposition

\[
Y(k) = \{y_0\} \cup P_G(y_0) y_0
\]

we see that the set of $S$-fixed points in $Y(k)$ is either $\{y_0\}$ or $\{y_0, sy_0\}$. On the other hand, there is $g \in G(k)$ with $P = gP_Gg^{-1}$. Then $P_x = H \cap P$ is equal to $H_y$ with $y := gP_G \in Y(k)$. From $S \subseteq H_y$ we get $y \in \{y_0, sy_0\}$. But then $H^o y \cong H^o / P^\pm_x$ is closed in $Y$ which shows that $P^o_x = H^o = P^\pm_H$ is parabolic in $H$.

Finally, since $P^o_x$ is parabolic in $H^o$ it contains a split torus $A \cong G_m$ which acts with non-zero weights on $P_u$. The same is true for a maximal torus $T_0$ of $P_x$. This shows $P^u_{\psi} T_0 = 1$ and we conclude with Lemma 3.5. The proof of the inequality for the rank is postponed to the section where we calculate the rank lattices.

Following Serre, a subgroup $H$ of $G$ is called irreducible if is not contained in a proper parabolic subgroup of $G$. In our settings we shall use more general definition and reserve a different name for it:
3.7. Definition. A $G$-variety over $k$ is homogeneously irreducible over $k$ if there is no $G$-morphism $\varphi : X_0 \to Z$ with $X_0 \subseteq X$ open $G$-stable and $Z$ complete, homogeneous, and positive dimensional where $\varphi$, $X_0$, and $Z$ are all defined over $k$. Otherwise, $X$ will be called homogeneously reducible.

3.8. Lemma. Let $G$ be a connected reductive $k$-group acting on a $k$-dense variety $X$. Assume that $X$ is homogeneously irreducible over $k$. Then the generic orbits of $X$ are affine.

Proof. By Sumihiro’s theorem, we may assume that $X$ is a subvariety of $P(V)$ where $V$ is a finite dimensional $G$-module. Clearly it suffices to prove the assertion for the closure of $X$. Thus, we may assume that $X$ is closed. Let $\hat{X} \subseteq V$ be the affine cone of $X$.

Suppose, for a generic point $x \in X$ the orbit $Gx$ is not affine or, equivalently, that the stabilizer $G_x$ is not reductive. Let $G'$ be the derived subgroup of $G$. Then also $G_x'$ is not reductive. Let $\hat{x} \in \hat{X}$ be a lift of $x$. Then the stabilizer $G_x'$ can’t be reductive, either. This in turn implies that the orbit $G'\hat{x}$ is not closed in $\hat{X}$. Thus, a generic $G'$-orbit in $\hat{X}$ is not closed.

Let $m$ be the dimension of a generic $G'$-orbit in $\hat{X}$. Then the union $Y$ of all $G'$-orbits of dimension $< m$ is closed. It is a proper subset of $\hat{X}$ and hence, by the above, the property that $Y$ meets the closure of a generic $G'$-orbit.

Let $W' \subseteq k[V]$ be a finite dimensional rational submodule generating the ideal $I(Y)$. Observe that, since $Y$ is a cone, $W'$ can be chosen to be homogeneous. Then the inclusion $W' \hookrightarrow k[V]$ induces a homogeneous $G$-morphism $\pi : V \to W$ where $W := (W')^*$ with the property that $\pi^{-1}(0) = Y$.

Let $X' \subseteq W$ be the closure of $\pi(\hat{X})$. Then, by construction, the generic $G'$-orbit of $X'$ contains 0 in its closure. Thus, Kempf’s instability theory applies: there is a 1-parameter subgroup $\lambda : G_m \to G'$ which is optimal for the generic $G'$-orbit in $X'$. More precisely, let $X^+$ be the set of points $x \in X'$ such that $\lim_{t \to 0} \lambda(t)x = 0$. Then $G'X^+ = X$. Moreover, $X^+$ is stable under $P := P(\lambda)$, the parabolic of $G$ attached to $\lambda$ inducing a surjective morphism $\psi : G *_{P(X^+)} X' \to X'$.

Observe that $P \neq G$ since $\lambda(G_m)$ must be a non-trivial subgroup of $G'$.

The main property, though, is that $G_x' \subseteq P \cap G'$ for all generic $x \in X^+$. This immediately implies that also $G_x \subseteq P$ which means that $\psi$ is generically bijective, hence purely inseparable. Let $G_n \subseteq G$ be the $n$-th Frobenius kernel and $Z := G/G_nP$. Then $Z$ is a complete $G$-variety such that for $n \gg 0$ there is a rational morphism $\pi'$ completing the following square:

$$
\begin{array}{ccc}
G *_{P(\lambda)} X^+ & \xrightarrow{\psi} & X' \\
\downarrow & & \downarrow \pi' \\
G/P & \longrightarrow & Z
\end{array}
$$

(3.3)

Observe that $\pi'$ is defined over $k$ since $X'$ is $k$-dense and therefore $\lambda$ is Galois invariant.

\footnote{where for the groups $G \supset H$ and a quasiprojective variety $Z$ by $G *_{H} Z$ we denote the quotient of $G \times Z$ by the $H$-action defined by $(g, z) \mapsto (gh^{-1}, hz)$. There is a projection $G *_{H} Z \to G/H$ given by $(g, z) \mapsto gH$. If $Z$ is $H$-invariant subset of $G$-variety $X$ there is an action map $\psi : G *_{H} Z \to X$ given by $\psi(g, z) = gz$ whose image is $GZ$.}

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Finally, composition with $\hat{X} \rightarrow X'$ yields a rational map $\hat{X} \rightarrow Z$. This morphism is invariant for the scalar $G_m$-action on $\hat{X}$. Therefore it descends to a rational $G$-morphism $X \rightarrow Z$. \hfill \Box

3.9. Proposition. Let $G$ be a connected reductive group with $\mathrm{rk}^{ss}_k G = 1$ and with minimal parabolic subgroup $P$. Let $X$ be a $G$-variety and let $Y \subset X$ be a $k$-dense $P$-stable proper subvariety such that $GY$ is dense in $X$. Then

i) either $c(Y/P) < c_k(X)$ or $c(Y/P) = c_k(X) = c(X/G)$.

ii) $\mathrm{rk}_k(Y) \leq \mathrm{rk}_k(X)$, if moreover $\mathrm{rk}_k(Y) = \mathrm{rk}_k(X)$, then $X$ is $k$-horospherical.

iii) The inequality $s_k(Y) > s_k(X)$ can hold only in the case when $X$ is homogeneously irreducible and in this case $\mathrm{rk}_k(Y) < \mathrm{rk}_k(X)$.

3.10. Corollary. In the lexicographic order we have

$$(c_k(Y), \mathrm{rk}_k(Y), s_k(Y)) \leq (c_k(X), \mathrm{rk}_k(X), s_k(X)).$$

3.11. Remark. The inequality $s_k(Y) > s_k(X)$ can occur. This happens for suitable $n, m$ when $Y$ is the closed $P$-orbit in a homogeneous space $X = SO(1, n)/SO(1, m) \times SO(n-m)$ over $k = \mathbb{R}$. However in this case $\mathrm{rk}_k(Y) < \mathrm{rk}_k(X)$.

3.12. Definition. The closed $P$-variety $Y$ (the family of generic $P$-orbits in $Y$) is called principal if $(c_k(Y), \mathrm{rk}_k(Y), s_k(Y))$ is maximal in the lexicographic order. We denote the set of principal varieties by $\mathcal{B}_0(X)$.

Proof. Let $Z(G) \subseteq G$ be the center of $G$. By Rosenlicht, $X$ contains an open $G$-stable subset $X^0$ such that the orbit spaces $X^0/G$ and $X^0/Z(G)$ exist. It follows from $X = GY$ that $Y^0 = X^0 \cap Y \neq \emptyset$ and that $Y^0/G$ exists. Moreover, dividing by $Z(G)$ does not change complexities. So we may replace $G$, $P$, $X$, $Y$ by $G/Z(G)$, $P/Z(G)$, $X^0$, $Y^0$ (or any smaller open subset) and assume that $G$ is semisimple of $k$-rank 1 and that $X/G$ exists.

Since $X = GY$, the map $Y \rightarrow X/G$ is surjective. Assume $c(Y/P) \geq c_k(X) \geq c(X/G)$.

Assume first that the generic orbit of $X$ is affine. After shrinking $X$, if necessary, we may assume that all orbits are affine and that the union $Z \subseteq X$ of all closed $P$-orbits is closed. Let $Z_0 \subseteq Z$ be the union of all components such that $Z_0/P \rightarrow X/G$. After shrinking $X$, we may assume that the intersection $Z_0 \cap Gx$ with every orbit is the union of all isolated $P$-orbits of $Gx$. Hence all $x \in X(k)$ with isotropic stabilizer $P_x$ lie in $Z_0$ by Lemma 3.6. If the set of points $x \in Y(k)$ with an anisotropic stabilizer $P_x$ is dense in $Y$ by Lemma 3.3 we get $c(Y/P) < c_k(X)$. Otherwise $Y$ is the closure of the set of points $x \in Y(k)$ with isotropic stabilizer and $Y \subset Z_0$. Then $c(Y/P) \leq c(Z_0/P) = c(X/G) \leq c(X/P)$ and the equality can hold iff $P$ has an open $k$-dense orbit in $Gx$. By Corollary 3.4 there is only one $k$-dense orbit with an anisotropic stabilizer and the remaining $k$-dense orbits are isolated (cf. case (RI), (N), (RT)).

Now assume that the generic orbit of $X$ is not affine. After shrinking $X$, Lemma 3.8 yields an equivariant $k$-morphism $\pi : X \rightarrow Z$ where $Z$ is a non-trivial complete homogeneous $G$-variety. For some $x \in X(k)$ let $y := \pi(x) \in Z(k)$. Then $G^\text{red}_y$ is a proper parabolic subgroup of $G$. Since it has to be conjugate to $P$ (recall $\mathrm{rk}^{ss}_k G = 1$) we obtain a bijective equivariant morphism $G/P \rightarrow Z$. Put $\tilde{X} := X \times_Z G/P$ and $\tilde{Y} := Y \times_Z G/P$. Then $\tilde{X} \rightarrow X$ is bijective, as well, which, $k$ being perfect, induces a bijection $\tilde{X}(k) \rightarrow X(k)$.
Since also the complexities \( c(*)/P \) are unchanged we may replace \( X, Y \) by \( \tilde{X}, \tilde{Y} \). Thus we may assume that there is an equivariant \( k \)-morphism \( \pi : X \to G/P \).

Put \( y_0 := eP \in G/P \). Then we have \( X = G \ast P X_0 \) where \( X_0 := \pi^{-1}(y_0) \). Observe that \( X_0 \) must be irreducible and \( k \)-dense. Moreover, the map \( X_0/P \to X/G \) is bijective. Since then \( c(X_0/P) = c(X/G) \leq c_k(X) \) for \( Y \subseteq X_0 \).

Let \( s \in G(k) \) with \( sa^{-1} = a^{-1} \) for all \( a \in A \). Then Bruhat decomposition of \( (G/P)(k) \) (see (3.2)) implies that all \( k \)-points of \( X \) are either contained in the closed set \( X_0 \) or in the open set \( X_1 := \pi^{-1}(Psy_0) \). Since \( Y \) is \( k \)-dense this implies that \( Y_1 := Y \cap X_1 \neq \emptyset \). As a \( P \)-variety, we have \( X_1 = P \ast L \ sX_0 \). Thus \( Y_1 = P_u \times sY_0 \) where \( Y_0 = s^{-1}Y_1 \cap X_0 \) is \( L \)-stable and \( k \)-dense.

Suppose first that \( Y_0 \subseteq X_0^A \). Let \( x \in Y_0(k) \). Since then \( A \) is a maximal split torus of \( P_x \) all of which are conjugate and since \( N_P(A) = L \) it follows that \( (Px)^A = Lx \). By continuity, this holds for \( x \) in a dense open subset of \( Y_0 \). From this we get

\[
(3.4) \quad c(Y/P) = c(Y_0/L) \leq c(X_0/P) = c(X/G)
\]

contrary to our assumption. Thus \( Y_0 \cap X^k_0 := (X_0 \setminus X^A_0) \neq \emptyset \). Now observe that the \( L \)-stabilizer of any \( x \in X^k_0(k) \) is anisotropic. This implies that the set of \( x \in Y(k) \) with anisotropic \( P \)-stabilizer is dense. We conclude with Lemma 3.3.

Let \( H \) be the stabilizer of \( x_0 \in \pi^{-1}(y_0) \), then \( Gx_0 \cap \pi^{-1}(y_0) = Px_0 \cong P/H \). Suppose the equality \( c(Y/P) = c_k(X) \) holds. Then also \( c_k(X) = c(X/G) \) and for generic point \( x_0 \in \pi^{-1}(y_0) \) \( P \)-has a dense orbit in \( Gx_0 \), which lies in \( Gx_0 \cap \pi^{-1}(Psy_0) = PsPx_0 \cong PsP/H \). By assumptions \( H \) has an open orbit on \( PsP/P \). And in particular the adjoint action of \( H/H_a \) has an open orbit on \( P_u/H_u \) (where \( H_u \) is the unipotent radical of \( H \)).

If \( H \) does not contain a split torus then \( H/H_u \) is anisotropic and its orbits of the points from \( P_u/H_u(k) \) are closed. Combined together this implies the transitivity of the action. Since the point \( eH_u \) is fixed, this can happen only for \( H_u = P_u \) and we are in the case of \((U)\)-type.

When \( H \) contain a group \( A \) and \( H_u \neq P_u \) (otherwise we are in \((U)\)-type) the set of \( P \)-orbits contain two orbits with non-anisotropic stabilizers: \( Px_0 \), and the orbit \( Psx_0 \) that correspond to the set \( (Px_0)^A = Lx \). Since the remaining orbits have anisotropic stabilizer and there is an open orbit with this property, by Corollary 3.4 there is only one such orbit. Moreover when \( H_u \) is nontrivial (otherwise we are in type \((RT)\)) the orbit \( Psx_0 \) contains \( Px_0 \)in its closure. Indeed these orbits correspond to the \( H \)-orbits \( H_uPsP/P \) and \( eP/P \) in \( G/P \) respectively and \( H_uPsP/P \) is closed in \( PsP/P \). The orbit \( G_aPsP/P \) of \( G_a \subset H_u \) is closed in the affine orbit \( H_uPsP/P \) but not closed in \( G/P \). The closure of \( G_aPsP/P \) in \( G/P \) gives a \( k \)-point in the closure of \( H_uPsP/P \) that is equal to \( eP/P \), which follows from Bruhat decomposition (this describes type \((TU)\)).

The calculation of the rank we postpone until next section, where we calculate the lattice. In order to prove (3) by Lemma 3.3 we have to prove inequality \( s_k(Y) > s_k(X) \) in the homogeneously reducible case. In this case for \( x \in Y \subseteq G/H \) the \( k \)-dense orbit \( Px \) maps to \( G/P(k) = Psy_0 \cap P^*_y \). If it maps to \( Psy_0 \), \( P_u \) is acting locally freely and we apply Lemma 3.3. Otherwise \( Px \) is isomorphic to \( P/H \), and the stabilizer for the action of \( L_{an} \) on \( P_u \setminus P/H \) is equal to \( H/\text{Rad} \) \( H \) that has maximal possible dimension. That implies \( s_k(Y) \geq s_k(X) \). □

In the course of the proof of Proposition 3.9 we proved the following propositions.
3.13. **Proposition.** Let $G$ be semisimple group of semisimple $k$-rank 1 and let $X$ be a $k$-dense homogeneous $G$-variety. Then one of the following cases holds:

1. $X$ is $k$-spherical and homogeneously irreducible over $k$. Then $\mathfrak{B}(X) = \{X, Y_1, \ldots, Y_r\}$ with $r \geq 0$.

2. $X$ is $k$-spherical homogeneously reducible over $k$.

   2.1 $X$ is horospherical. Then $\mathfrak{B}(X) = \{X, Y\}$

   2.2 $X$ is not horospherical. Then $\mathfrak{B}(X) = \{X, Y_1, Y_2\}$

3. $X$ is not $k$-spherical. Then $\mathfrak{B}(X) = \{X\}$.

In the following table we summarize the types of raise for the $P$-orbit and we define the action of the group $W_\alpha := \langle s_\alpha \rangle$. We also note that the introduced action is consistent with the action on the character lattices.

3.14. **Proposition.** Let $G$ be semisimple group of $k$-rank 1 and $H$ is the $k$-subgroup. Then for $H$-orbits on $Y = G/P$ one of the following cases holds:

1. $H$ spherical, not reductive.

   (U) $H$ is horospherical; $\mathfrak{B}(Y/H) = \{Y > Z\}; \text{rk}_k Z = \text{rk}_k Y; s_k(Z) = s_k(Y); s \circ [Y] = [Z]$.

   (TU) $H$ is not horospherical; $\mathfrak{B}(Y/H) = \{Y > Z_1 > Z_2\}; \text{rk}_k Z_i = \text{rk}_k Y - 1; s_k(Z_1) = s_k(Z_2); s \circ [Y] = [Y]; s \circ [Z_1] = [Z_2]$.

2. $H$ is spherical, reductive

   (A) $H^0$ homogeneously irreducible, anisotropic; $\mathfrak{B}(Y/H) = \{Y\}; s \circ [Y] = [Y]$.

   (RT) $H$ is homogeneously reducible. $H = L_0.A$ with $L_0 \subseteq L_{an}$; $\mathfrak{B}(Y/H) = \{Y > Z_1, Z_2\}; \text{rk}_k Z_i = \text{rk}_k Y - 1; s \circ [Y] = [Y], s \circ [Z_1] = [Z_2]$.

   (RI) $H$ is homogeneously irreducible, isotropic; $\mathfrak{B}(Y/H) = \{Y > Z\}; \text{rk}_k Z = \text{rk}_k Y - 1; s \circ [Y] = [Y]; s \circ [Z] = [Z]$.

   (N) $H$ is homogeneously irreducible, $H^0$ is homogeneously reducible. $H = L_0.A.(s)$ with $s \in N_G(a) \setminus C_G(A)$; $\mathfrak{B}(Y/H) = \{Y > Z\}; \text{rk}_k Z = \text{rk}_k Y - 1; s \circ [Y] = [Y], s \circ [Z] = [Z]$.

3. $H$ is not spherical.

Our next aim is to describe how the character lattices of $P$-orbits are related with the restricted Weyl group action in the semi-simple split rank one case. Let $P_0$ be the maximal normal subgroup of $P$ containing $[P, P]$ such that $\Xi_k(P) = \Xi_k(P/P_0)$. If $P_x$ is the stabilizer of $x$ then the calculation of $\Xi_k(P/P_x)$ reduces to study of the $A$-semiinvariant functions on the $P_0$-orbit space of $P/P_x$ that reduces to the calculation of the quotient space of $P/P_0 \cong A$ by the image of $P_x$ in $A$. 

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3.1. Affine irreducible case: types \((RI), (N)\). Let \(X_0\) be the open \(P\)-orbit in \(X\) and \(y_i \in Y_i\), is the point in the closed orbit with the stabilizer equal to \(P_H = P \cap H\). By Lemma 3.6 for \(x \in X_0\) any subgroup of \(P_x\) which is \(k\)-isomorphic to \(\mathbb{G}_a\) belong to \(Z(G)\). Then the reduced connected component of the image of \(P_x\) in \(A\) is \(s\)-fixed, thus

\[ s\Xi_k(X)_Q = \Xi_k(X)_Q. \]

Let us recall that for \(s\) we can take an element of \(H\), such that \(s\lambda(t)s^{-1} = \lambda(-t)\). Since \(P_x\) contains \(\lambda\) and \(Z(G)\) is fixed by \(s\) then the image of \(P_x\) in \(P/P_0\) is fixed by \(s\). Then

\[ s\Xi_k(Y)_Q = \Xi_k(Y)_Q. \]

3.2. Reducible case: types \((U), (TU), (RT)\). After shrinking \(X\) we have a \(G\)-morphism \(\pi : X \to G/P\).

Type \((U)\). When \(X\) is horospherical i.e. \(H \subset P\) and \(H_u = P_u\), \(X\) consists of the closed \(Py := \pi^{-1}(eP)\) and the open orbit \(\pi^{-1}(P_sP)\). Since \(sPsP \supset P\), the orbit \((sPs)y\) contains \(Py\) and by the next lemma each orbit \(P_uy\) is contained in \((sP_uy)\). We have a map

\[ k[X]^{(P)}_\chi \to k[Y]^{(P)}_s\chi \]

obtained by applying \(s\) to \(f\) to get a \(sPs\)-semimvariant function of weight \(s\chi\), and restricting it to \(Y\). The resulting function \(sf|_Y\) is \(P_0\)-invariant since \(sf\) is constant on \(sP_0s\)-orbits and for each \(y \in Y\) the orbit \(P_0y\) is contained in \((sP_0y)\) that is verified by the following lemma.

3.15. Lemma. Let \(X\) be a variety with an action of a minimal parabolic group \(P\). Let be a \(k\)-group \(H\) group with anisotropic quotient \(H/H_u\) with the \(k\)-action on \(X\) such that for some \(x \in X_k\) we have \(Hx \subset PX\). Then we have \(Hux \subset Pu\) and \(Hx \subset Pu\).

Proof. Since \(Pu\) is a normal subgroup in \(P\) the elementary group \(LA\) acts transitively on the set of \(Pu\)-orbits in \(PX\), moreover there exists the geometric quotient \(\Pi_{Pu} : PX \to Z\) by \(Pu\) where \(Z\) is the homogeneous space for \(LA\). To prove the lemma we need to show that \(\Pi_{Pu}(Hux)\) is a point. Since \(H_u\) is generated by one dimensional additive \(k\)-subgroups \(G_a\), it is sufficient to prove the lemma for \(H_u = G_a\), which is the assertion next lemma.

3.16. Lemma. There are no nontrivial \(k\)-morphisms from affine line \(A^1_k\) to homogeneous space of an elementary \(k\)-group \(G\).

If \(Hx \subset Pu\) does not hold, we have nontrivial map from \(Hx\) to homogeneous space of the split torus \(A\) (since \(A\) acts transitively of the set of \(Pu\)-orbits in \(PX\)). Since the \(\Pi_{Pu}(Hux)\) is already a point. This map gives a nontrivial map from \(Hux\) which is a homogeneous space of \(H/H_u\) to a homogeneous space of the split torus \(A\). Which is impossible since by Rosenlicht theorem there are no invertible non-constant \(k\)-regular functions (which are \(k\)-characters) on the anisotropic group. \(\square\)

Types \((TU), (RT)\). \(X = G/H\) is not horospherical, \(H \subset P\) and \(H_u \subset Pu\). In this case we can assume that there is a split non-central torus \(A_0\) contained in \(H\). After conjugation of \(A_0\) by \(p \in P_H\) we can take one-parameter subgroup \(\lambda_0 \subset A_0\) with \(P(\lambda_0) = P\) and such that \(M = Z_G(\lambda_0)\) (We note that it may be different from initial \(\lambda\) that defines \(s\), but these groups have the same centralizer). This allows to define the Levi decomposition of \(H = LH/H_u\) (where \(L_H = Z_H(\lambda_0)\)). We have \(p((G/H)^A_0) = eP \cup sP\) and \(Y_0 = p^{-1}(eP) = P/H \cong L/L_H \times Pu/H_u\) set of \(A_0\)-fixed points can be identified with \(L/L_H\).
since \((P_u/H_u)^A_0 = e\). Also the \(p^{-1}(sP)^A_0 \cong L/L_H\) is obtained by applying \(s\) to \(Y_0^{A_0}\) is contained in \(Y_1\) and not in \(X_0\). Comparing stabilizers of this points we get
\[s\Xi_k(Y_0)_Q = \Xi_k(Y_1)_Q.\]
If \(S \subseteq P\) is one-dimensional non-central torus of \(G\) then \((X_0 \cap p^{-1}(sP))^S = \emptyset\) (see Lemma 3.3). Since \(k\)-split tori that are contained in \(P_x\) for \(x \in X_0\) are central in \(G\), then image of the projection of \(P_x\) to \(A_{X,k} \cong P/P_0\) is fixed under \(s\) and
\[s\Xi_k(X)_Q = \Xi_k(X)_Q.\]

4. The general case

We continue to assume that \(k\) is an infinite perfect field.

4.1. Lemma. Let \(P\) be a connected linear algebraic \(k\)-group and let \(X\) be a \(k\)-dense \(P\)-variety. Assume that \(c(Y/P) \leq c(X/P)\) for all \(k\)-dense \(P\)-stable closed subvarieties \(Y \subseteq X\). Then there are at most finitely many \(k\)-dense \(P\)-stably closed subvarieties \(Y\) with \(c(Y/P) = c(X/P)\).

Proof. Suppose there are infinitely many \(Y\) with \(c(Y/P) = c(X/P)\). Call these \(Y\) exceptional. Then there is some \(d \in \mathbb{N}\) such that there are infinitely many exceptional \(Y\) with \(\dim Y = d\). Let \(Z^0 \subseteq X\) be the closure of their union. Then one of its irreducible components, say \(Z\), will be the closure of infinitely many exceptional proper subvarieties. Since \(Z\) is a \(P\)-stable \(k\)-dense subvariety we have by assumption \(c(Z/P) \leq c(X/P)\). Let \(Z^0 \subseteq Z\) be the open sheet. Then every proper \(P\)-stably closed subvariety \(Y\) of \(Z^0\) has \(c(Y/P) < c(Z/P) \leq c(X/P)\). In particular, its closure \(\overline{Y}\) is not exceptional. Thus \(Z\) can’t be the closure of a set of exceptional subvarieties. \(\square\)

4.2. Theorem. Let \(G\) be a connected reductive \(k\)-group with minimal parabolic \(P \subseteq G\). Let \(X\) be a locally linear \(k\)-dense \(G\)-variety. Then \(c(Y/P) \leq c_k(X)\) for all \(k\)-dense \(P\)-stable subvarieties \(Y \subseteq X\). Moreover, the set of all closed \(Y\) with \(c(Y/P) = c_k(X)\) is finite.

Proof. Suppose there is \(Y\) with \(c(Y/P) > c_k(X)\). Then choose one of maximal dimension. Clearly \(Y \neq X\). Suppose \(GY\) is a proper subvariety of \(X\). Then \(c_k(GY) \leq c_k(X)\) by [KK] and \(c(Y/P) \leq c_k(GY)\) by induction on \(\dim X\). This contradiction shows that \(GY = X\). Hence, there is a subminimal parabolic \(P_\alpha\) (for some \(\alpha \in S_k\)) with \(Y \neq Z := P_\alpha Y\). Then \(c(Z/P) \leq c_k(X)\) by maximality.

Let \(U_\alpha\) be the unipotent radical of \(P_\alpha\). It is also contained and normal in \(P\). The group \(\overline{G} := P_\alpha/U_\alpha\) is of semisimple \(k\)-rank 1 and contains \(\overline{P} := P/U_\alpha\) as a minimal parabolic. Rosenlicht’s theorem asserts that \(Z\) contains a \(P_\alpha\)-stable open subset \(Z^0 \subseteq Z\) such that the orbit space \(\overline{Z} := Z^0/U_\alpha\) exists. Every \(P_\alpha\)-orbit meets \(Y\). So the intersection \(Y^0 := Y \cap Z^0\) is non-empty and the quotient \(\overline{Y} := Y^0/U_\alpha\) exists. Clearly, both \(\overline{Z}\) and \(\overline{Y}\) are \(k\)-dense. Moreover, \(\overline{G}\) acts on \(\overline{Z}\) and \(\overline{Y}\) is a \(\overline{P}\)-stable subvariety with \(c(\overline{Z}/\overline{P}) = c(Z/P) \leq c_k(X)\) and \(c(\overline{Y}/\overline{P}) = c(Y/P)\). Thus, the first assertion follows from Proposition 3.9. The second assertion is now implied by Lemma 4.1. \(\square\)

4.3. Corollary. Let \(G\) be a connected reductive \(k\)-group with minimal parabolic \(P \subseteq G\). Let \(H \subseteq G\) be a \(k\)-subgroup and let \(Z \subseteq Y := G/P\) be a closed \(H\)-stably \(k\)-dense subvariety. Then \(c(Y/H) \leq c(Y/H) = c_k(G/H)\). Equality holds for only finitely many \(Y\).
4.4. Corollary. A $k$-spherical $G$-variety contains only finitely many $P$-orbits $Y$ with $P(k) \neq \emptyset$.

4.5. Corollary. Let $k$ be a local field of characteristic 0 and let $X$ a $k$-spherical variety. Then $P(k) \backslash X(k)$ is finite. Moreover, if $X = G/H$ is homogeneous then $P(k) \backslash G(k)/H(k)$ is finite.

Proof. By Corollary 4.4, the set $X(k)$ is covered by finitely many $P$-orbits $Y_1, \ldots, Y_n$. Now each intersection $Y_i \cap X(k) = Y_i(k)$ decomposes into finitely many $P(k)$-orbit by [BS64, Cor. 6.4]. The second assertion follows from the first and $G(k)/H(k) \subseteq X(k)$. □

Finally consider the group $F(W_k)$ defined by the set of generators $\tilde{s}_\alpha$ for all $\alpha \in \Pi_k$ (where $\Pi_k$ is the set of simple roots of the restricted root system $\Delta_k$) and by the set of relations $\tilde{s}_\alpha^2 = e$ for all $\alpha \in \Pi_k$. Let us introduce the action of $F(W_k)$ on the set $\mathfrak{B}(X)$ as follows. For any $\alpha \in \Pi_k$ and $Y \in \mathfrak{B}(X)$ the set $P_\alpha Y$ is also a $k$-dense closed subset. Consider a space $P_\alpha Y / \text{Rad}_u(P_\alpha)$ that is a rational $P_\alpha$-equivariant quotient by $\text{Rad}_u(P_\alpha)$. Since the set of $k$-points of the homogeneous space of unipotent group over a perfect field $k$ is always non-empty the preimage induces the inclusion $\mathfrak{B}(P_\alpha Y / \text{Rad}_u(P_\alpha)) \subset \mathfrak{B}(P_\alpha Y)$. Also by Theorem 4.2 $c_k(P_\alpha Y) = c_k(Y) = c_k(X)$, thus $\mathfrak{B}(X) \supset \mathfrak{B}(P_\alpha Y)$. We have already defined the action of $\tilde{s}_\alpha$ on $\mathfrak{B}(P_\alpha Y / \text{Rad}_u(P_\alpha))$ according to Proposition 3.14, so let us define $\tilde{s}_\alpha \circ [Y]$ in a consistent way.

1. Conjecture. For a field $k$ such that $\text{char } k \neq 2$ the action of $F(W_k)$ on the set $\mathfrak{B}(X)$ factors through the restricted Weyl group $W_k$.

As it was already mentioned this conjecture is known for algebraically closed fields (cf. [Kno95]). We prove this conjecture in the real case in full generality in Section 5 and for the action on the subset $\mathfrak{B}_0(X) \subset \mathfrak{B}(X)$ of principal families of $P$-orbits for char $k = 0$ in Section 6.

4.6. Remark. Let $X = G/H$ for some $k$-group $H$. Then we can also define the action of $F(W_k)$ on the set $\mathfrak{B}_H(G/P)$ (i.e. on the set of $k$-dense orbits with maximal $H$-complexity) in the following way. For each $\alpha \in \Pi_k$ let us fix a morphism $\pi_\alpha : G/P \to G/P_\alpha$. For $Hy \subset G/P$ denote by $\tilde{P}, \tilde{P}_\alpha$ the stabilizers in $G$ of $y$ and $\pi_\alpha(y)$ respectively. By the surjectivity of the map $G(k) \to (G/P_\alpha)(k)$, the set $\mathfrak{B}_H(\pi_\alpha^{-1}(\pi_\alpha(Hy)))$ is bijective to the set $\mathfrak{B}_H \cap \tilde{P}_\alpha(\tilde{P}_\alpha/\tilde{P})$. Consider the image $H \cap \tilde{P}_\alpha$ in $\tilde{P}_\alpha / \text{Rad}_u(P_\alpha)$. This reduces the situation to a semisimple split rank one case (see Proposition 3.14) and we define the action of $\tilde{s}_\alpha$ on $\mathfrak{B}_H(\pi_\alpha^{-1}(\pi_\alpha(Hy)))$ to be consistent with the action in the rank one case. Despite the fact that the identification of the considered sets $\mathfrak{B}_H$ and $\mathfrak{B}_H \cap \tilde{P}_\alpha$ is non-canonical and depends on the choice of $y$ a brief look at the cases in Proposition 3.14 shows that the introduced action does not depend on it. As a result we have $F(W_k)$-equivariant embedding of $\mathfrak{B}_H(G/P)$ to $\mathfrak{B}(G/H)$. Moreover $\mathfrak{B}(G/H) = \bigcup_{x \in (G/H)(k)} \mathfrak{B}_{Hx}(G/P)$.

5. $W_k$-action on $\mathfrak{B}(X)$ in the real case

In [Kno95] the spherical case the $W$-action on the set of $B$-orbits was introduced by two different methods via the Hecke algebra and via equivariant geometry of cotangent bundle that worked only for the orbits of maximal rank. Here we give the way to generalize the first method to the $W_k$-action on the set of $\mathfrak{B}(X)$ in the real case i.e. $k = \mathbb{R}$. 

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For an $H$-variety $X$ consider the category of $(H, \text{Gal}(\mathbb{C}/\mathbb{R}))$-equivariant sheaves of $\mathbb{F}_2$-vector spaces on $X$ (i.e., on $X(\mathbb{C})$) that are constructible with respect to the étale topology and denote its Grothendieck group by $\mathcal{G}(X, H)$. Nevertheless, we consider the sheaves as sheaves with respect to the Hausdorff topology. Accordingly, we use singular cohomology.

We recall that for a $H$-equivariant morphism $f : X \to Y$ the push forward $f_! : \mathcal{G}(X, H) \to \mathcal{G}(Y, H)$ is defined as $f_![\mathcal{F}] := \sum_{i=0}^n (-1)^i [R^i f_! \mathcal{F}]$. For $\mathcal{F}_1 \in \mathcal{G}(X_1 \times X_2, G)$ and $\mathcal{F}_2 \in \mathcal{G}(X_2 \times X_3, G)$ we define a product by

$$F_1 \circ F_2 = p_{13}^* [p_{12}^* F_1 \otimes p_{23}^* F_2] \in \mathcal{G}(X_1 \times X_3, G).$$

For $H \subseteq G$ and an $H$-variety $X$ we have an induction functor

$$\text{ind}^H_G : \mathcal{G}(X, H) \to \mathcal{G}(G \ast_H X, G) : \mathcal{F} \mapsto G \ast_H \mathcal{F} = (p*q^* \mathcal{F})^H,$$

for the quotient $p : G \times X \to G \ast_H X$ and the action morphism $q : G \times X \to X$.

When $X$ is already a $G$-variety then there is an isomorphism $G \ast_H X \cong G/H \times X$ given by $(g, x) \to (gH, gx)$. Thus, we have an isomorphism

$$\mathcal{G}(X, H) \cong \mathcal{G}(G \ast_H X, G) \cong \mathcal{G}(G/H \times X, G)$$

which provides $\mathcal{G}(X, H)$ with the structure of a $\mathcal{G}(G/H, H) \cong \mathcal{G}(G/H \times G/H, G)$-module. For an inclusion $i : Z \subset X$ by $\mathbb{F}_Z$ we denote the pushforward $i_* \mathbb{F}_Z$ of a constant sheaf on $Z$.

The next step is to take into account only those sheaves that have nonzero stalks only on $\mathbb{R}$-dense $H$-orbits. Let $\mathcal{G}(X, H)_0 \subseteq \mathcal{G}(X, H)$ be the subgroup generated by all sheaves $\mathcal{F}$ such that the stalk $\mathcal{F}_x$ is even dimensional for every $x \in X(\mathbb{R})$. And let $S(X, H) := \mathcal{G}(X, H)/\mathcal{G}(X, H)_0$ which is $\mathbb{F}_2$-module since $2[\mathcal{F}] = [\mathcal{F}^{\oplus 2}]$.

5.1. Lemma. Let $\mathcal{F} \in \mathcal{G}(X, H)_0$. Then:

- If $f : X \to Y$ is an $H$-equivariant $\mathbb{R}$-morphism. Then $f_! [\mathcal{F}] \in \mathcal{G}(Y, H)_0$.
- If $H \subseteq G$ such that If $G(\mathbb{R}) \to (G/H)(\mathbb{R})$ is surjective then $\text{ind}^H_G [\mathcal{F}] \in \mathcal{G}(G \ast_H X, G)_0$.

Proof. We mention only the nontrivial steps. First notice that since $G(\mathbb{R}) \to (G/H)(\mathbb{R})$ is surjective, the condition $Y(\mathbb{R}) = \emptyset$ for an $H$-invariant $Y \subset X$ is equivalent to $(G \ast_H Y)(\mathbb{R}) = \emptyset$ (Indeed if $y \in (G \ast_H Y)(\mathbb{R})$ one can find $g \in G(\mathbb{R})$ in such that $gy$ maps to $eH$ under projection to $G/H$ i.e. $gy \in Y(\mathbb{R})$). This implies that if $\mathcal{F} \in \mathcal{G}(X, H)_0$ then $\text{ind}^H_G [\mathcal{F}] \in \mathcal{G}(G \ast_H X, H)_0$. Also an exterior tensor product of a sheaf without $k$-points in its support gives a sheaf without $k$-points in its support.

For any closed subset $Z \subset X$ and the complement $U = X \setminus Z$ there is a long exact sequence in cohomology with compact supports and a corresponding long exact sequence for higher direct images:

$$\ldots \to R^i_Z f_* \mathcal{F} \to R^i f_* \mathcal{F} \to R^i_U f_* \mathcal{F} \to R^{i+1}_Z f_* \mathcal{F} \to \ldots$$

This allows us to express a push forward in $K$-theory as

$$f_! [\mathcal{F}] = \sum_{i=0}^n (-1)^i [R^i_Z f_* \mathcal{F}] + \sum_{i=0}^n (-1)^i [R^i_U f_* \mathcal{F}].$$
Extending this to any stratification \( X \supset \ldots \supset Z_i \supset Z_{i+1} \supset \ldots \) by the closed subsets such that \( \mathcal{F}|_{Z_i \setminus Z_{i+1}} \) is a constant sheaf of rank \( r_i \), we get

\[
f_i[\mathcal{F}] = \sum_{j=0}^{n} (-1)^j [R^j_{Z_i/Z_{i+1}} f_i \mathcal{F}].
\]

We may choose the stratification in such a way that \( \mathcal{F} \) is trivial on \( Z_i \setminus Z_{i+1} \) and since it has even rank, then \( R^i_{Z_i/Z_{i+1}} f_i \mathcal{F} \) is also a sheaf of even rank. Thus \( f_i[\mathcal{F}] \in \mathfrak{S}(Y, H)_{ev} \).

5.2. Proposition. Assume that \( \Gamma \) is a finite group acting freely on the manifold \( X \). And \( \mathcal{F} \) is the \( \Gamma \)-equivariant locally constant sheaf on \( X \). Then \( \chi(X, \mathcal{F}) = |\Gamma| \chi(X/\Gamma, \mathcal{F}/\Gamma) \).

Let us notice that the \( G \)-orbits for diagonal action on \( G/P \times X \) are in bijective correspondence with \( P \)-orbits in \( X \). Since \( G(k)/P(k) = (G/P)(k) \) the \( G(k) \) orbits on the set of \( k \)-points of \( G/P \times X \) are in bijection correspondence with \( P(k) \)-orbits in \( X(k) \). This identifies \( S(G/P \times X, G) \) with \( S(X, P) \). It also provides the group \( S(G/P, P) \) with the structure of algebra and the group \( S(X, P) \) with the structure of \( S(G/P, P) \)-module. For a group algebra \( \mathbb{F}_2[W_k] \) we have the following group\(^2\) isomorphism \( S(G/P, P) \cong \mathbb{F}_2[W_k] \), so we have a map \( \mathbb{F}_2[F(W_k)] \to S(G/P, P) \).

We are going to define the action group \( F(W_k) \) on the set \( \mathfrak{S}(X) \) by associating to each \( Z \subset \mathfrak{S}(X) \) the corresponding element \( \mathbb{F}_2 Z^o \in S(X, P) \) (where \( Z^o \) is the open \( P \)-orbit in \( Z \)) and by assigning to \( s_\alpha \) the action of \( s_\alpha \in S(G/P, P) \) on \( S(X, P) \) that is defined as multiplication by the sheaf \( \mathbb{F}_{P s_\alpha P/P} \in S(G/P, P) \) supported on \( P s_\alpha P/P \subset G/P \). This action can be described by the following formula:

\[
s_\alpha \circ [\mathcal{F}] = \psi[\text{ind}^P_{P^o} s_\alpha \mathcal{F}] - [\mathcal{F}],
\]

which is a corollary of a Bruhat decomposition of \((P_\alpha/P)(\mathbb{R})\) and the following diagram.

\[
\begin{array}{ccc}
G/P \times X & \xrightarrow{\psi} & P_\alpha/P \times X & \xleftarrow{\psi} & P_\alpha \ast_P Y \\
P_\alpha Y & \xleftarrow{\psi} & & & \\
\end{array}
\]

where the horizontal right map is given by \( [g, y] \mapsto (gP, gy) \).

The \( P \)-equivariant locally constant sheaf with the support on \( Pz \) is defined by the irreducible representation of \( \rho : P_z/P^o_z \to GL(V) \), let us denote such sheaf by \([z, \rho]\). Then 5.1 can be rewritten in the following way:

\(^2\)Later we shall prove that it is the isomorphism of algebras.
We have the following table for the $s_\alpha$-action according to the type of raise. Let us notice that due to Bruhat decomposition $(P_\alpha/P)(\mathbb{R})$ is smooth one point compactification of a disk and thus is diffeomorphic to the sphere $S^n$. In the calculations below we used the equalities $\chi(S^n) = 0 \mod 2$ and $\chi(X(\mathbb{C}), F) = \chi(X, F) = 0 \mod 2$.

(U) In this case $Z$ is a point. Thus
\[
s_\alpha \circ [y] = (\chi(S^n) - \chi(pt))F_{S^n} - [y] = [z],
\]
\[
s_\alpha \circ [z] = \chi(pt)F_{S^n} - [z] = [y].
\]

(TU) The closure of $Z_1$ is homeomorphic to sphere $S^k$ and $Z_2 = pt$.
\[
s_\alpha \circ [y] = (\chi(S^n) - \chi(S^k))F_{S^n} - [y] = [y],
\]
\[
s_\alpha \circ [z_1] = (\chi(S^n) - \chi(pt))F_{S^n} - [z_1] = [y] + [z_2],
\]
\[
s_\alpha \circ [z_2] = \chi(pt)F_{S^n} - [z_2] = [y] + [z_1].
\]

(A) Here we have $s_\alpha \circ [y] = [y]$.

(RT) Here $Z_i = pt$, thus
\[
s_\alpha \circ [y] = (\chi(S^n) - 2\chi(pt))F_{S^n} - [y] = [y],
\]
\[
s_\alpha \circ [z_1] = \chi(pt)F_{S^n} - [z_1] = [y] + [z_2],
\]
and the same for the other closed orbit.

(RI) In this case $Z = H/P_H$, and again by Bruhat decomposition $H/P_H \cong S^k$.
\[
s_\alpha \circ [y] = (\chi(S^n) - \chi(S^k))F_{S^n} - [y] = [y],
\]
\[
s_\alpha \circ [z] = \chi(S^k)F_{S^n} - [z] = [z],
\]

(N) Here $Z$ is the union of two points which is actually $S^0$.
\[
s_\alpha \circ [y] = (\chi(S^n) - \chi(S^0))F_{S^n} - [y] = [y],
\]
\[
s_\alpha \circ [z] = \chi(S^0)F_{S^n} - [z] = [z].
\]

In order to make out of this action the action of the restricted Weyl group we consider the decreasing filtration by the dimension of the support of the sheaf
\[
S(X, P)_i := \{\text{generated by } i_s F_Z \in S(X, P)\mid \text{for closed } i : Z \subset X \text{ and dim } Z \geq i\}.
\]

An inspection of the above table shows the product by generators $s_\alpha$ of $S(G/P, P)$ is consistent with this filtration (i.e. $s_\alpha \circ S(X, P)_i \subset S(X, P)_{i-1}$), thus Gr $S(X, P) = \sum_i S(X, P)_i / S(X, P)_{i-1}$ is the associated graded $S(G/P, P)$-module supplied with the action of $F(W_k)$ which is consistent with our ad hoc definition from previous section.

Finally the action of $F(W_k)$ on Gr $S(X, P)$ factors through the action of $W_k$ that follows from the following proposition.

5.4. Proposition. The Hecke algebra $S(G/P, P)$ is isomorphic to a group algebra $\mathbb{F}_2[W_k]$. 17
**Proof.** Let us recall the $P$-orbits with $k$-points in $G/P$ are parametrized by the elements $w \in W_k$. Also we know that $P_{\alpha}PwP = \overline{Ps_{\alpha}wP}$ for $\ell(s_{\alpha}w) > \ell(w)$ and the raise from $P_{\alpha}PwP$ to $\overline{Ps_{\alpha}wP}$ is of type $(U)$ since all $k$-dense $P$-orbits are of the same $k$-rank. By construction, the action of $s_{\alpha} \in F(W_k)$ on the element $[PwP]$ is given by $s_{\alpha} \circ [PwP] = [Ps_{\alpha}wP]$, which is multiplication by $s_{\alpha}$. This shows that $S(G/P, P)$ is identified with $\mathbb{F}_2[W_k]$. \[\square\]

Finally we proved.

5.5. **Theorem.** There is an action of $W_k$ on $\mathcal{B}(X)$ which factors the action of $F(W_k)$ on $\mathcal{B}(X)$ and that is obtained by restriction of the action $S(G/P, P)$ on $Gr S(X, P)$, where $W_k$ is realized as a subgroup in $S(G/P, P)$ by the map on the generators $s_{\alpha} \rightarrow [F_{Ps_{\alpha}P/P}]$ and $\mathcal{B}(X)$ is mapped to the set of lines in $Gr S(X, P)$ by assigning to $Z \in \mathcal{B}(X)$ the sheaf $[\mathbb{F}_{Z_\ast}] \in S(X, P)$.

5.6. **Remark.** It is important to notice that $P_{\alpha}Y(\mathbb{R})$ can decompose as the union of open $P(\mathbb{R})$-orbits with different type of the stabilizer. However the $P$-equivariance of the corresponding constructible sheaves shows that the action of $s_{\alpha}$ does not depend on the choice of $P(\mathbb{R})$-orbit.

6. **Action of $W_k$ on the principal families of $P$-orbits.**

From now on let us assume that char $k = 0$. Without loss of generality we can pass to the $G$-invariant subset of smooth points of $X$ and then by Sumihiro theorem we can find the covering $X$ by $G$-invariant quasi-projective varieties. This allows to assume that $X$ is quasi-projective so by an argument from [Kno94, Section 5] (cf. [KK16]) we can pass from $X$ to the total space of some very ample $G$-linearized line bundle (or just take an affine cone), so we can assume that $X$ is non-degenerate in the sense of [Kno94] or even quasi-affine.

The aim of this section is to define the action of the restricted Weyl group on the principal families of $P$-orbits by means of equivariant geometry of cotangent vector bundle. Let us recall that for $Y \in \mathcal{B}_0(X)$ the action of $\tilde{s}_{\alpha} \in F(W_k)$ is defined in the following way: For a simple reflection $s_{\alpha} \in W_k$ consider the corresponding subminimal parabolic subgroup $Q \supset P$ containing preimage of this reflection in $N_G(A)$. Assume first that we have $\dim QY > \dim Y$. If the raise from $Y$ to $QY$ is of type $(U)$, we put $\tilde{s}_{\alpha} \circ Y = QY$ otherwise we put $\tilde{s}_{\alpha} \circ Y = Y$. When $Y = QY'$, in the case of existence of $Y' \in \mathcal{B}_0(X)$ such that $Y = QY'$ we put $\tilde{s}_{\alpha} \circ Y = Y'$ (by Theorem 4.2 this raise is of type $(U)$), otherwise we put $\tilde{s}_{\alpha} \circ Y = Y$. We are going to prove that this action of $F(W_k)$ factors through the action of the restricted Weyl group $W_k$.

Let us denote by $\mu_X/G$ the composition map $T_X^* \rightarrow g^* \rightarrow g^*/G \cong t^*/N_G(T)$, where the last equality is the Chevalley isomorphism, and consider the morphism:

$$\tilde{T}_X := T_X^* \times_{t^*/W} t^* \rightarrow g^* \times_{t^*/W} t^*,$$

that coincides with the moment map on the first factors and with identity morphism on the second factors. For algebraically closed field we have the following theorem of F.Knop.

6.1. **Lemma.** [Kno95] All irreducible components of $\tilde{T}_X := T_X^* \times_{t^*/W} t^*$ have the same dimension. They map onto $T_X^*$ and $W$ acts transitively on them via its natural action on the second factor.
For algebraically non-closed fields these results can be refined in the following way. Consider the following sequence of finite maps:

\[
t^* \xrightarrow{W_L} \Gamma / L \xrightarrow{W_k} \Gamma / N_G(L) \rightarrow g^* / G \cong t^* / W,
\]

and let us take into account that \( N_G(L) = N_G(A), \) \( \Gamma = (g^*)^A \) and that the map \( \Gamma / N_G(L) \rightarrow g^* / G \) is finite by the result of Luna (see [PV94, Thm.6.16]).

There are at least two candidates for the substitute of \( \tilde{T}_X \) in an algebraically non-closed case:

\[
\tilde{T}_{X,P} := T_X^* \times_{V/W} \Gamma / L, \quad \tilde{T}_{X,P'} := T_X^* \times_{V/W} a^*.
\]

The first one is better related with geometry of cotangent bundle. And the second is important being related with the sections of conormal bundle to \( p_u \)-orbits which are defined by the semi-invariant functions. These objects are related by the following sequence of maps, where the right down vertical arrow is induced by \( L \)-equivariant splitting \( a^* \subset \Gamma. \)

\[
\begin{array}{ccc}
T_X^* \times_{V/W} t^* & \xrightarrow{W_k} & T_X^* \times_{V/W} \Gamma / L \xrightarrow{W_k} T_X^* \times_{V/W} a^* \xrightarrow{\pi} T_X^* \\
\end{array}
\]

From [Kno95] we know that families of \( B \)-orbits of maximal rank and complexity defined over \( K \) (denoted by \( B_{00}(X) \)) correspond to connected components of \( \tilde{T}_X. \) Our aim is to associate to each element of \( B_0(X) \) the corresponding irreducible components of \( \tilde{T}_{X,P} \) and \( \tilde{T}_{X,P'} \) and to prove that the action of \( W_k \) on the irreducible components of \( \tilde{T}_{X,P} \) and \( \tilde{T}_{X,P'} \) via its action on the right factor correspond to the action of \( F(W_k) \) on \( B_0(X) \) introduced above. We have to consider both varieties \( \tilde{T}_{X,P} \) and \( \tilde{T}_{X,P'} \) simultaneously since it is not a priory clear that the natural maps between them are inclusions on the irreducible components which is due to the fact that \( \tilde{T}_{X,P'} \to T_X^* \) is not surjective. The first variety is important because it is more closely related with the geometry of the moment map. The second one is interesting since it is related with the geometry of the sections for \( p_u \)-orbits.

To study the components of \( \tilde{T}_{X,P} \) let us consider the composition of maps:

\[
T_X^* \times_{V/W} \Gamma / L \rightarrow g^* \times_{V/W} \Gamma / L \rightarrow g^*
\]

Let us notice that the last map admits a section over \( p_u^+ \) defined by

\[
\tau : p_u^+ \to g^* \times_{V/W} \Gamma : \lambda \mapsto (\lambda, |\lambda|_V).
\]

This formula is well defined by the following lemma.

6.2. Lemma. Let \( p \to 1 \) be an \( L \)-equivariant projection to the Levi subgroup. Then for \( \xi \in 1 \) the fiber \( \xi + p_u \) is contained in the fiber of quotient map \( g^* \to g^* / G. \) Moreover if \( \xi_0 \in \xi + p_u \) is semisimple then \( \xi_0 \in P_u \xi. \)

Proof. Let \( \lambda(t) \in \Lambda(A) \) be a one-parameter subgroup of \( A \) adapted to \( P. \) Then we have \( \lim_{t \to 0} \lambda(t) \xi_0 = \xi \) for all \( \xi_0 \in \xi + p_u, \) that implies the first claim. If \( \xi_0 \in \xi + p_u \) is semisimple, then there exists an element \( p_u \in P_u \) such that \( \text{Ad}(p_u) \xi_0 \in 1. \) Since \( 1 \) is fixed by \( \lambda \) and \( \lim_{t \to 0} \lambda(t)p_u \lambda(t)^{-1} = e \) we have \( \text{Ad}(p_u) \xi_0 = \lim_{t \to 0} \text{Ad}(\lambda(t)p_u) \xi_0 = \text{Ad}(\lim_{t \to 0} \lambda(t)p_u \lambda(t)^{-1}) \lim_{t \to 0} \text{Ad}(\lambda(t)) \xi_0 = \xi. \)
This gives us the embedding of $\mu^{-1}(p_u^\perp)$ in $\widetilde{T}_{X,P}$ via $\widetilde{\tau}: \eta \mapsto (\eta, \mu(\eta)|_{\tau})$. The set $X$ also defines the distinguished component of $T^r_{X} \times_{\mathcal{G}/W} \mathcal{G}/N_G(L)$ which is denoted by $T^r_{X}$. The main result of this section is the following.

6.3. **Theorem.** There exists the action of $W_k$ on the set $\mathfrak{B}_0(X)$ of the principal families of $P$-orbits, and $\mathfrak{B}_0(X)$ is isomorphic to the set of those $k$-irreducible components of $\widetilde{T}_{X,P}$ (resp. $\widetilde{T}_{X,P'}$) that map to the distinguished component of $T^r_{X} \times_{\mathcal{G}/W} \mathcal{G}/N_G(L)$. The $W_k$-action on this set of components is induced by the action on the second factor of $T^r_{X} \times_{\mathcal{G}/W} \mathcal{G}/L$.

The variety $\mu^{-1}(p_u^\perp)$ is the union of the conormal bundles to all $P_u$-orbits in $X$. For an $H$-invariant subvariety $Y$ by $\mathcal{N}(H,Y)$ (or briefly by $\mathcal{N}(Y)$ when $H = P_u$) we denote the conormal bundle to the family of generic $H$-orbits in $Y$. By $\widetilde{\mathcal{N}}(Y, P)$ (resp. $\widetilde{\mathcal{N}}(Y, P')$) we denote the image of $\mathcal{N}(Y, P_u)$ (resp. $\mathcal{N}(Y, P')$) in $\widetilde{T}_{X,P}$ (resp. $\widetilde{T}_{X,P'}$). Since each element of $\mathfrak{g}^*$ is $G$-conjugate to the element in $\mathfrak{p}$, we get that for each component of $\widetilde{T}_{X,P}$ there is an irreducible component $\widetilde{\mathcal{N}}$ of $\mu^{-1}(p_u^\perp)$ whose $G$-span is dense. Denoting by $Y$ the image of projection of $\widetilde{\mathcal{N}}$ to $X$ we get $\mathcal{N} = \mathcal{N}(Y)$.

Note that the dimension of the family of $P_u$-orbits is equal to $c(Y/P) + r_k(Y) + s_k(Y)$ for every $P$-invariant $k$-dense subvariety $Y$. The dimension for the corresponding conormal bundle to the foliation is equal to $\dim X + c(Y/P) + r_k(Y) + s_k(Y)$, which is the same for all principal families.

6.4. **Definition.** The elementary radical $\operatorname{Rad}_{el} H$ (anisotropic radical $\operatorname{Rad}_{an} H$) is the smallest normal subgroup such that $H/\operatorname{Rad}_{el} H$ ($H/\operatorname{Rad}_{an} H$) is elementary (anisotropic) subgroup.

6.5. **Definition.** Let $H$ be a connected $k$-group acting on a $k$-dense variety $X$. Then the action is called **elementary (anisotropic)** if the elementary radical $\operatorname{Rad}_{el} H$ (the anisotropic radical $\operatorname{Rad}_{an} H$) acts trivially on $X$.

6.6. **Remark.** It is easy to prove that $\operatorname{Rad}_{el} H$ is generated by $k$-defined unipotent subgroups of $H$, and $\operatorname{Rad}_{an} H$ is generated by $\operatorname{Rad}_{el} H$ and $k$-defined subgroups $\mathbb{G}_m$ of $H$.

Let us recall the definition of the normalizer of generic $P$-orbit in $X$, which is a parabolic subgroup of $G$.

$$P_{X,k} := \{ g \in G \mid gPx = Px \text{ for } x \text{ in a dense open subset of } X \}$$

When we work over the fixed field $k$ we use the simplified notation $P_X$. We shall need the following version of the Local Structure Theorem proved by F.Knop and B.Krotz [KK16, Prop. 4.6].

6.7. **Theorem (Generic Structure Theorem).** Let $X$ be a $k$-dense $G$-variety and let $P_X = L_X P_{X,u}$ be the Levi decomposition in which $L_X$ is normalized by $A$. Then there exists a smooth affine $L_X$-subvariety $X_{el} \subseteq X$ such that

i) the action of $L_X$ on $X_{el}$ is elementary, all orbits are closed, and the categorical quotient $X_{el} \to X_{el}/L_X$ is a locally trivial fiber bundle in the etale topology. In particular there exists the stabilizer $M_X$ of general position for this action.

---

3 For the split case the dimension of the component of conormal bundle is equal to $\dim X + c(Y/P) + r_k(Y)$. 20
ii) the natural morphism $P_{X,u} \times X_{el} = P_X \ast_{L_X} X_{el} \to X$ is an open embedding.

The slice $X_{el}$ is unique up to a unique $L_X$-equivariant birational isomorphism. More precisely, its field of rational functions can be computed as

$$k(X_{el}) = k(X)^{P_{X,u}} = k(X)^{P_u}.$$  

By $L_0, T_0, A_0$ let us denote subgroups of $L_X, T, A$ which are the kernels of the characters of all $P$-semi-invariant functions on $X$. By $a_X$ let us denote the orthocomplement to $a$ in $a$.

6.8. Remark. The above theorem implies the existence for the stabilizer of general position for the action $P$ on $X$ as well as existence of stabilizer of general position $L(X) := M_X \cap L$ for the action of $L$ on $X_{an}$. We also have $s_k(X) = \dim L/L(X)A$.

6.9. Lemma. Let $Y \in \mathcal{B}_0(X)$. Then there exists a sequence $P_1, \ldots, P_i$ of subminimal parabolic subgroups containing $P$ such that in the sequence $Y_i := P_i \ldots P_1 Y$ the raise from $Y_{i−1}$ to $Y_i$ is of type $(U)$ and $P_1 \ldots P_i Y = X$. Moreover there exists a stabilizer of general position for the action $P$ on $Y_i$ and there exists a stabilizer of general position $L(Y)$ for the action of $L$ on $Y_i$; $P_u$ which is conjugate to $L(X)$ by the action of $w := s_1 \ldots s_{i−1} \in N_G(A)_k$.

Proof. Let us argue by decreasing induction on the dimension of $Y_i$. The base of induction is given by the local structure theorem 6.7. Given $Y \in \mathcal{B}_0(X)$, consider a parabolic minimal subgroup $P_1$ containing $P$ for which $P_1 Y \neq Y$ (Such $P_1$ exists since $G$ is generated by subminimal parabolic subgroups and according to [KK16] if $G Y \neq X$ we have $c_k(Y) < c_k(X)$). Since the $(c_k(Y), rk_k(Y)) = (c_k(X), rk_k(X))$ the raise from $Y$ to $P_1 Y$ is of type $U$. Moreover there is a $P_1$-equivariant map $\varphi : P_1 y \to P_1 / P$. By induction on the dimension we can assume that the assertion of the lemma is true for $P_1 Y$ and $P_1 \ldots P_i Y = X$. Let us recall that for a general $y \in Y$ the set of $k$-dense orbits of $P_1 y$ consists of the open orbit $P_1 s_1 P_y = P s_1 y = \varphi^{-1}(P s_1 P y_0)$ and the orbit $P_y = \varphi^{-1}(y_0)$ in its closure. Meantime applying $s_1$ to $P_1 y$ we get that the same decomposition in the union of $s_1 P s_1$-orbits. The orbit $s_1 P_1 y$ is closed and $s_1 P_1 s_1 P_y$ is open so it is also equal to $s_1 P s_1 y$, which follows from $y_0 \in s_1 P s_1 P y_0$ and $\varphi^{-1}(s_1 P s_1 y_0) = \varphi^{-1}(s_1 P s_1 y_0)$. Let us also notice that $s_1 P s_1 P y$ contains $P y$.

Since $(s_1 P s_1 y)$ contains $P y$ by lemma 3.15 the orbit $P s_1 y$ is contained in $(s_1 P u s_1 y)$. In particular if $Z$ is the $L$-invariant section for $P_u$-orbits in $Y$ (that always exists), then it is also the section for the family of $s_1 P s_1$ in $P_1 Y$ and then $s_1 Z$ is the section for the family of $P_u$-orbits in $P_1 Y$. Taking into account that $s_1 \in N_G(S)_k$ normalizes $L$, this proves that stabilizer of general position for the action of $L$ on $Y_i$; $P_u$ is equal to $L(Y) = s_1 L(P_1 Y)s_1 = w L(X)^{w^{-1}}$.

6.10. Corollary. Let $P_1$ be a subminimal $k$-parabolic subgroup properly containing $P$, $s \in N_{P_1}(A)$ be a representative of a simple reflection. Let $Y \in \mathcal{B}_0(X)$ be such that $P_1 Y$ be the span of type $(U)$. Then for all $y \in Y$ we have

$$s \mathcal{N}(P_1 y, P_u) = \mathcal{N}(s P s y, s P u s) \subset (s P s \cap P_1) \mathcal{N}(P y, P_u),$$

Proof. Since $P y \subset s P s y$ for a general point of $y \in Y$, the Lemma 3.15 implies that $P s y \subset s P u s y$. Thus the restriction of $\mathcal{N}(s P s y, s P u s)$ to $P y$ is contained in $\mathcal{N}(P y, P_u)$ and we have $s \mathcal{N}(P_1 y, P_u) = \mathcal{N}(s P s y, s P u s) \subset (s P s \cap P_1) \mathcal{N}(P y, P_u)$. 

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6.11. Proposition. There is a bijection \( Y \mapsto \mathcal{N}(Y, P_u) \) between \( Y \in \mathfrak{B}_0(X) \) and the \( k \)-dense irreducible components of \( \mu^{-1}(p^+_u) \) that dominate \( \mu(T^*_X) \) and have dimension \( \dim X + c(X/P) + \text{rk}_k(X) + s_k(X) \) which is maximal for the components with this property.

The following proposition whose proof is postponed until the end of the section finishes the proof of Theorem 6.3.

6.12. Proposition. The action of the group \( F(W_k) \) on the set of \( \mathfrak{B}_0(X) \) factors through \( W_k \) and is isomorphic to the action on the set of the \( k \)-irreducible components of \( \bar{T}_{X,P} \) (resp. \( \bar{T}_{X,P'} \)) that map to the distinguished component of \( T^*_X \times_{\mathfrak{t}/w} \mathfrak{t}^* \). Where the bijection on the sets is given by \( Y \mapsto G\bar{N}(Y, P_u) \).

By [KK16] for an quasi-affine \( X \) there exists a \( \chi \in \Lambda_k(X) \) that satisfies \( \langle \chi, \alpha \rangle \neq 0 \) for any \( \alpha \in \Delta_{p_{X,u}} \). Such \( \chi \) can also be characterized by the property \( s_\alpha \chi \neq \chi \) for \( s_\alpha \in N_G(A) \setminus L_X \).

We consider \( \Lambda_k(X) \) as a lattice in \( \mathfrak{a}_X^* \). The collection of such \( \chi \in \mathfrak{a}_X^* \) which satisfies the above condition is Zariski dense in \( \mathfrak{a}_X^* \) and is denoted by \( \mathfrak{a}_{p_{X,u}}^* \). By \( A_0 \) let us denote the common kernel in \( A \) of characters of all \( P \)-semi-invariant functions \( (a = a_0 + a_\chi) \).

Consider a \( P_X \)-semi-invariant function \( f_\chi \in k(X)^{(p)} \), following Knop ([Kno94]) we can define a \( P_X \)-equivariant map \( \psi_\chi : X \setminus \text{div}(f_\chi) \to \mathfrak{g}^* \), where \( l_x(\xi) = \frac{\xi f_\chi}{f_\chi}(x) \).

The map \( \Psi_\chi : X^0 \to T^*_X \), where \( X^0 = X \setminus \text{div}(f_\chi) \), that maps \( y \in X^0 \) to the value of the section \( f_\chi^{-1}(df_\chi) \) in the point \( y \), produces the following commutative diagram:

\[
\begin{array}{ccc}
X^0 & \xrightarrow{\psi_\chi} & \mathfrak{g}^* \\
\downarrow{\Psi_\chi} & & \downarrow{\mu} \\
T^*_X & \xrightarrow{\mu} & \mathfrak{g}^*
\end{array}
\]

Let us notice that \( \Psi_\chi(X) \) is the subset of \( \mathcal{N}(X) \), since the differential form \( f_\chi^{-1}(df_\chi) \) annihilated on the action vector field of \( p_u \). We also recall from [KK16, (4.5)] that the image \( \psi_\chi(X^0) \) is equal to \( \xi + p_{X,u} \mathfrak{g} \). The variety \( \mathcal{K}_\chi = \psi_\chi^{-1}(X^0) \subset X^0 \) (or simply \( \mathcal{K} \)) provides the section for \( P_u \)-orbits (which are also \( P_{X,u} \)-orbits) in \( X^0 \). The \( P_X \)-equivariant map \( \Psi_\chi \) embeds \( \mathcal{K} \) in \( \mathcal{N}(X) \), we put \( \tilde{\mathcal{K}} := \Psi_\chi(\mathcal{K}) \).

Before stating next proposition recall that by reductivity of \( L \) we have an isomorphism \( \mathfrak{l} \cong \mathfrak{t}^* \) and a splitting \( \mathfrak{g}^* \cong \mathfrak{t}^* \oplus \mathfrak{l}^* \) which are both \( L \)-equivariant.

6.13. Proposition. Let \( G \) be \( k \)-group acting on \( k \)-dense quasiaffine variety \( X \). For \( X_{el} \) which is birational to \( X \), consider \( \mu_t : T^*_{X,el} \to \mathfrak{t}^* \) the corresponding moment map. For \( Y \in \mathfrak{B}_0(X) \) and for the presentation of the element \( w := s_1 \ldots s_l \in W \) consider \( Y_i := P_1 \ldots P_i Y \) and assume that each the raise from \( Y_i \) to \( Y_{i+1} \) is of type \((U)\). Then

(i) \( G\mathcal{N}(Y) \) is dense in \( T^*_X \).

(ii) \( \mu(\mathcal{N}(Y, P_u)) = P w \mu_t(T^*_{X,el}) \) and \( \mu(\mathcal{N}(Y, P^u)) = P w \mathfrak{a}_X \).

(iii) \( w \mathcal{K} \) is the \( L \)-invariant rational section for generic \( P_u \)-orbits in \( Y \).
Proof. For the action of $L_X$ on $X_{el}$ by Luna slice theorem there exists a slice $Z$ (which we can assume is defined over $k$), so we have an excellent map: $L_X *_{M_X} Z \to X_{el}$, and the corresponding map $P_X *_{M_X} Z \to X^o$. The $L_X$-orbits on $X_{el}$ coincide with the $L$-orbits. So we have

$$L *_{L(X)} Z \to X_{el}, \quad P_{X,a} \times (L *_{L(X)} Z) \to X^o,$$

which gives the following $L$-equivariant etale map to $T_X^*$:

$$P_{X,a} \times p_{X,a} \times (L *_{L(X)} l(X)^+) \times T_Z^*.$$

To embed $T_Z^*$ into $T_Z^*$ we used an isomorphism $\mathcal{N}(X,P_X)|_Z \cong T_Z^*$, obtained by the restriction of the elements of $\mathcal{N}(X,P_X) \subset T_Z^*$ to $T_Z$. Also we have

$$T^*_{L *_{L(X)} Z} \cong \mathcal{N}(X,P_{X,a})|_{L *_{L(X)}} Z \subset T_X^*.$$

The action of $Rad_{el} L_X$ is trivial on $X_{el}$. After taking quotient by $P_{X,u}$ we obtain that $P/P_{X,u}$ is minimal parabolic for $L_X$. Since anisotropic group $L_X/Rad_{an} L_X$ does not contain unipotent elements, the image of $P$ in $L_X/Rad_{an} L_X$ coincide with the image of $L_{an}$ and this restriction map is surjective. We denote by $t_{X,0}$ the lie algebra which is the kernel of this map. The lie algebra which is the direct summand of $l$ and which coincide with the image of $l$ in the lie algebra of $L_X/Rad_{an} L_X$ is denoted by $t_{X,eff}$.

The $P_{u}$-orbits in $X$ are also normalized by $P_{u}(L \cap Rad_{el} L_{X}) A_{0}$. Then the conormal bundle $\mathcal{N}(X)$ to $P_{u}$-orbits in $X$ is also conormal bundle to $P_{u}(L \cap Rad_{el} L_{X}) A_{0}$-orbits and its image under the moment map belong to $(p_{u}+t_{X,0}+a_{0})^+ = a_{X}+t_{X,eff}+p_{X,u}$.

Let us notice that the restriction of the moment map to the subset $\mathcal{N}(X)$ and subalgebra $I$ can be obtained by composition of the moment map $\mathcal{N}(X) \to t_{X,eff}+p_{X,u}$ and the projection to $t_{X,eff}$. Recall that we have the etale map

$$P_{X,u} \times (L *_{L(X)} l(X)^+) \times T_Z^* \to \mathcal{N}(X,P_{u}).$$

In particular $\dim \mathcal{N}(X,P_{u}) = \dim T_{X}^* - \dim P_{X,u}$. Let $t_{0}^+$ be the orthocomplement in $\mathfrak{t}$ of the Lie algebra $t_{0}$ that annihilates $B$-semi-invariant rational $K$-functions on $X$. Then by [Kno94] the closure of the image of the moment map $\mu_{t}: T_{X,el}^* \to I$ is equal to $L_{t_{0}^+}^*$. Taking into account that $\langle \chi, \alpha \rangle \neq 0$ for $\chi \in t_{0}^+$ and $\alpha \in \Delta_{P_{X,u}}$ we get $P_{X,u} \chi = \chi + p_{X,u}$. So the closure of the image of the map $\mathcal{N}(X,P_{u}) \to \mathfrak{g}^*$ contains $P_{X} t_{0}^+ = L_{t_{0}^+}^* + p_{X,u}$. It cannot be larger since $\mu_{t}(\mathcal{N}(X,P_{u})) = L_{t_{0}^+}^*$ which is the projection of $\mu(\mathcal{N}(X,P_{u}))$ to $I$.

Since $P_{X,u} \chi = \chi + p_{X,u}$ is transversal to $L_{t_{0}^+}^* + p_{X,u}$ in the point $\chi$, then it is also true for generic point $L_{t_{0}^+}^* + p_{X,u}$. By looking at differential of $d\mu$ we get that $P_{X,u}^\chi y$ is transversal to $\mathcal{N}(X,P_{u})$ in the generic point $y \in \mathcal{N}(X,P_{u})$. This implies that $dim P_{X,u}^\chi \mathcal{N}(X,P_{u}) = dim P_{X,u}^\chi + dim \mathcal{N}(X,P_{u}) = dim T_{X}^*$ and that $P_{X,u}^\chi \mathcal{N}(X,P_{u})$ is dense in $T_{X}^*$.

To prove (i) and (iii) let us argue by descending induction on the dimension of $Y$. Assume that $GN(P_{1}Y,P_{u})$ is dense in $T_{X}^*$. By corollary 6.10 we have $s\mathcal{N}(P_{1}Y,P_{u}) \subset (sPs \cap P_{1}) \mathcal{N}(Y,P_{u})$, that implies the density of $GN(Y,P_{u})$ in $T_{X}^*$.

The construction of $K$ gives the base of the induction for the proof of (iii). By taking a rational quotient and passing to the open subset of $Y$ we can assume that existence of a quotient morphism from $Y_{1} := P_{1}Y$ to the smooth variety $Y_{1}/(P_{1})_{u}$. For the conormal bundle $\mathcal{N}(Y,P_{1})_{u}$ to the foliation of the $(P_{1})_{u}$-orbits in $Y_{1}$ consider a natural map to $T_{Y_{1}/(P_{1})_{u}}^*$, which is a composition of natural projection to $T_{Y_{1}}^*$ and the quotient by $(P_{1})_{u}$. 23
We note that the conormal bundles $\mathcal{N}(Y_1, P_u)$ and $\mathcal{N}(Y, P_u)$ to the foliations of $P_u$-orbits are preimages under this map of the conormal bundles to the foliations of the $\overline{F}$-orbits in $Y_1/(P_1)_u$ and $Y/(P_1)_u$. We have a following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{N}(Y_1, (P_1)_u) & \overset{(p_1)_u^+ \subset g^*}{\longrightarrow} & \mathcal{N}(Y, (P_1)_u) \\
\downarrow & & \downarrow \\
T^*_u/(P_1)_u & \overset{m^*}{\longrightarrow} & m^*
\end{array}
\]

By induction assumption we have the section $\mathcal{K}_1 = s_1 w \mathcal{K}$ for the set of general $P_u$-orbits in $Y_1$ and its lifting $\tilde{\mathcal{K}}_1 \subset \mathcal{N}(Y_1, P_u)$ such that $\mu(\tilde{\mathcal{K}}_1) = \chi \in \Lambda(Y_1)^{pr}$. We have to prove that $s_1 \mathcal{K}_1$ is the section for the $P_u$-orbits in $Y$. From the proof of the Corollary 6.10 we know that $s_1 \mathcal{K}_1$ is the section for $s_1 P_u s_1$-orbits in $Y_1$ and each generic orbit contain a unique corresponding $P_u$-orbit of $Y$. Suppose that $s_1 \mathcal{K}_1$ is not a section for the set of $P_u$-orbits in $Y$, so we have $z \in \mathcal{K}_1$ and its lifting $\tilde{z} \in \mathcal{K}_1$ such that $s_1 z \notin Y$. Let us take $u^- \in (P_u)^- \cap P_1$ such that $u^- s_1 z \in Y$. The set of $P_u$-orbits in $Y_1$ is not normalized by $(P_u)^- \cap P_1$.

6.14. Lemma. The general weight $\chi_1 = s_1 w \chi$ of $\Lambda(Y_1)$ is non-degenerate with respect to $P_1$.

Proof. The set of $P$-orbits in $Y_1$ is not stabilized by $P_1$ and in particular not stabilized by $s_1$. Then the points of the section $\mathcal{K}_1 = s_1 w \mathcal{K}$ are not stabilized by $s_1$. Recalling that the stabilizer of $\mathcal{K}_1$ is equal to $s_1 w L_X w^{-1} s_1$ which is the centralizer of $s_1 w \chi$ thus $s_1$ do not fix it. □

Since $s_1 \chi_1 \neq \chi_1$, we have $\mu(u^- s_1 \tilde{z}) = s_1 \chi_1 + \eta_-$, where $\eta \in (p_u)^- \cap p_1$ and $\eta \neq 0$. We have the inclusion $s_1 \mathcal{K}_1 \subset \mathcal{N}(Y_1, s P_u s)$ Moreover since $u^- s_1 z \in Y$ the element $u^- s_1 \tilde{z}$ belongs to the restriction of $\mathcal{N}(Y_1, s P_u s)$ to $Y$. By the Corollary 6.10 we have $u^- s_1 \tilde{z} \in \mathcal{N}(Y, U)$ and in particular $\mu(u^- s_1 \tilde{z}) \in p$. We get a contradiction since $s_1 \alpha + \eta_- \notin p$, thus we get $u^- = e$ and $s_1 \mathcal{K}_1 \subset Y$. As a corollary we also get that $\mu(\mathcal{N}(Y)) \supset P \mathcal{K}$. Moreover the points of $w \mathcal{K}$ are stabilized by $w(L(X) A_0 \text{Rad}_{an} P_X) w^{-1}$.

6.15. Lemma. Let $Y \in \mathcal{B}(X)$ be such that the raise from $Y$ to $Y_1 := P_1 Y$ is not of type $(U)$. Then the action of $s_1 \in W_{P_1}$ on the component of $\overline{T}_{X,P}$ (resp. $\overline{T}_{X,P'}$) corresponding to $\mathcal{N}(Y_1, P_u)$ (resp. $\mathcal{N}(Y_1, P')$) is trivial.

Proof. By the overline let us denote the corresponding images of the groups in $L_1 \cong P_1/(P_1)_u$. In the following diagram we have to prove that for L1-variety $\overline{Y}_1 := Y_1/(P_1)_u$ the variety $T^*_{\overline{Y}_1} \times_{\overline{W}_1} t^*$ (resp. $T^*_{\overline{Y}_1} \times_{\overline{W}_1} s^*$) is irreducible.

\[
\begin{array}{ccc}
T^*_{\overline{Y}_1} \times_{\overline{W}_1} s^* & \overset{P_1 \mathcal{N}(Y_1, P')}{\longrightarrow} & T^*_{X} \times_{W_1} s^* \\
\downarrow & & \downarrow \\
T^*_{\overline{Y}_1} \times_{\overline{W}_1} \Gamma / L & \overset{\mathcal{N}(Y_1, (P_1)_u)}{\longrightarrow} & T^*_{X} \times_{W_1} \Gamma / L
\end{array}
\]

By the assumption on the raise the triple $(c_\Gamma, \text{rk}_\Gamma, s_\Gamma)$ takes strictly lower value on $\overline{F}$-invariant subvariety of $\overline{Y}_1$ which we denote by $\overline{Y}$. In case $\text{rk}_\Gamma(Y) < \text{rk}_\Gamma(\overline{Y}_1)$ we get that
\[ \mu(\mathcal{N}(Y, \overline{P}_u)) \subset a_Y + p' \] so the closure of \( G\mathcal{N}(Y, \overline{P}_u) \) is not equal to \( T_{Y_1}^\perp \) (by the similar argument \( \mathcal{G} \mathcal{T}_{\mathcal{N}(Y, \overline{P})} \subset T_{X}^\perp \times_{\mathcal{V}/W} a_Y \) does not provide a component of \( T_{X}^\perp \times_{\mathcal{V}/W} s^* \)).

Otherwise the sum \( c_{\mathcal{F}} + r_{\mathcal{F}} + s_{\mathcal{F}} \) (resp. \( c_{\mathcal{F}} + r_{\mathcal{F}} \)) takes strictly lower value on \( \overline{Y} \) that gives \( \dim \mathcal{N}(Y, \overline{P}_u) < \dim \mathcal{N}(Y_1, \overline{P}_u) \) (resp. \( \dim \mathcal{N}(Y, \overline{P}) < \dim \mathcal{N}(Y_1, \overline{P}) \)). Taking into account that \( \dim \mathcal{N}(Y_1, \overline{P}_u) = 2 \dim Y_1 - \dim \overline{P}_u \), we get

\[ \dim(L_{11}(\mathcal{N}(\overline{Y}, \overline{P}_u))) = \dim(\mathcal{P}_u \mathcal{N}(\overline{Y}, \overline{P}_u) < \dim T_{Y_1}^\perp \]

(resp. \( \dim(L_{11}(\mathcal{N}(\overline{Y}, \overline{P})) = \dim(\mathcal{P}_u \mathcal{N}(\overline{Y}, \overline{P})) \)).

Thus \( L(\mathcal{N}(\overline{Y}, \overline{P}_u)) \) is not dense in \( T_{Y_1}^\perp \) (resp. \( L(\mathcal{N}(\overline{Y}, \overline{P})) \)) is not dense in \( \tilde{T}_{X,P^*} \). Recalling that irreducible components of \( \tilde{T}_{X}^\perp \times_{\mathcal{V}/W} t^* \) have the same dimension and each one contain \( L(\mathcal{N}(\overline{Y}, \overline{P}_u)) \) as a dense subset for some \( Y \), we see that there is only one such component.

6.16. Lemma. Let \( Y \in \mathcal{B}_0(X) \). Then the equality of the closures of \( G\tilde{\mathcal{N}}(Y, P') \) and \( G\tilde{\mathcal{N}}(X, P') \) which are components of \( \tilde{T}_{X,P'} \) implies \( Y = X \).

Proof. If the equality holds then for general \( \xi \in \mathcal{N}(X, P') \) and \( \eta \in \mathcal{N}(Y, P') \) we have \( g(\xi, \mu(\xi)) = (g(\eta, \mu(\eta)) \).

6.17. Lemma. Let \( Y \in \mathcal{B}_0(X) \). Then the equality of the closures of \( G\tilde{\mathcal{N}}(Y, P_u) \) and \( G\tilde{\mathcal{N}}(X, P_u) \) which are components of \( \tilde{T}_{X,P_u} \) implies \( Y = X \).

Proof. If the equality holds then for general \( \xi \in \mathcal{N}(X) \) and \( \eta \in \mathcal{N}(Y) \) we have \( g(\xi, \mu(\xi)|_\mathcal{N}) = (g(\eta, \mu(\eta)|_\mathcal{N}) \).

6.18. Lemma. Let \( Y \in \mathcal{B}_0(X) \) such that \( Y_1 := P_1Y \neq Y \), then the raise from \( Y \) to \( P_1Y \) is of type (U). The action of \( s \in W_{P_1} \) on the set of components of \( \tilde{T}_{X,P} \) (resp. \( \tilde{T}_{X,P^*} \)) permutes the corresponding to \( Y \) and \( Y_1 \).

Proof. The varieties \( \mathcal{N}(Y, P_u) \) and \( \mathcal{N}(Y_1, P_u) \) (resp. \( \mathcal{N}(Y, P') \) and \( \mathcal{N}(Y_1, P') \)) provide the components of \( \tilde{T}_{X,P} \) (resp. components of \( \tilde{T}_{X,P^*} \)). By the reduction to \( Y_1/(P_1)_{\eta} \), described in Lemma 6.15 and by Lemma 6.16 and 6.17 these components are distinct. Moreover since \( Y_1/(P_1)_{\eta} \) is the variety for the group \( L_1 \) of split rank 1, there are no other subsets of \( Y_1/(P_1)_{\eta} \) of maximal \( k \)-rank and \( k \)-complexity. Since \( s \) acts nontrivially on \( \mathfrak{g} \), it must permute these components.

To prove \((\text{ii})\) let us notice that since \( P' \)-orbits in \( Y \) are normalized by \( T_Y \) we have \( \mu(\mathcal{N}(Y, P')) \subset a_Y + p_u \). Since \( G\tilde{\mathcal{N}}(Y, P') \) is dense in \( G\tilde{\mathcal{N}}(X, P') \) the general element
in \(\mu(N(Y,P'))\) is conjugate to \(Ga^{pr}\) and in particular semisimple. All semisimple elements of \(a_Y + p_u\) are \(P\)-conjugate to the elements of \(a_Y\). This implies \(\mu(N(Y,P')) \subset P\). The map \(\mu|_a : N(Y,P') \to s^*\) factors through \(N(Y,P') \to T_Y \to s^*\), where the first map is a quotient by sub-bundle \(N_X/Y \subset N(Y,P')\). Moreover by Lemma 6.10 for a \(\chi \in \Lambda(Y)\) we have \(f_\chi \in \mathbb{K}(Y)^{(P)}\), thus \(df_\chi \in T_Y^{*,(P)}\) provides a rational section of the quotient of \(N(Y,P')\) by \(N_X/Y\). The points of this section map into \(\chi + p_u\) under \(\mu\). From the above we get \(\mu(N(Y,P')) = P\).

Let us notice that \(P_u\)-orbits in \(Y\) are normalized by \(T_Y\) and by \(P_uL_{X,0}\). In particular \(\mu(N(Y,P_u)) \subset a_Y + \text{Ad}(w)L_{X,eff} + p_u\), where \(a_Y = w\) and \(wl_{X,eff} \subset I\) (since \(w \in N_G(L)\)). This family has the section \(wK\) which is \(wLw^{-1}\)-invariant, has stabilizer of general position \(L(Y) = wL(X)w^{-1}\) and the slice \(wZ\). Since \(G\tilde{N}(Y,P_u)\) is dense in \(T_X\) the general element \(\xi \in \mu(N(Y,P_u))\) is semisimple. By Lemma 6.2 the element \(\xi\) is \(P_u\)-conjugate to its projection to \(a_Y + w_{eff}\). Moreover the image of the element from \(I\) and its \(P_u\)-conjugate have the same projection to \(I\). This implies that image of moment map of \(N(Y,P_u)\) as well as its projection to \(I\) can be recovered from the moment map of the restriction of \(N(Y,P_u)\) to \(wK\). The map \(\mu|_a\) factors as \(N(Y,P_u)|_{wK} \to T^*(wK) \to W\) whose image is \(w_{eff}(T_{X,eff})\) gives the equality \(\mu(N(Y,P_u)) = Pw_{eff}(T_{X,eff})\) and proves (ii).

6.19. Corollary. Let the raise from \(Y_1 \in \mathfrak{B}_0(X)\) to \(Y = P_1Y_1\) be not of type \((U)\). Then the transform \(s_1wK_\xi\) of the section \(wK_\xi\) for sufficiently general \(\xi\) is also a section for \(Y\).

Proof. By Lemma 6.15 \(G\tilde{N}(Y,P')\) is dense in some component of \(\tilde{T}_{X,P'}\) and \(G\tilde{N}(Y_1,P')\) is not dense in any component for all \(Y' \in \mathfrak{B}_0(Y)\). Since \(\mu(N(Y,P')) = P\), the \(P\) holds for the \(G\)-span of the lift \(\tilde{C}_Y \subset \tilde{T}_{X,P'}\) of \(C_Y := N(Y,P') \cap \mu^{-1}(a_Y)\). Since \(s_1C_Y\) is contained in the union of \(N(Y,P')\) for \(Y' \in \mathfrak{B}_0(Y)\), and in the same time \(G\tilde{C}_Y\) is dense in \(\tilde{T}_{X,P'}\), the latter implies that \(s_1C_Y = C_Y\) as well as \(s_1a_Y = a_Y\). Since \(C_Y = \bigcup_{\xi \in \mathfrak{a}_X} wK_\xi\), we have \(s_1wK_\xi = w K_{w^{-1}s_1w\xi}\).

Proof of Proposition 6.12. Given \(Y \in \mathfrak{B}_0(X)\) we associate the corresponding \(k\)-irreducible component by taking the \(G\)-span of the lifting of \(N(Y,P_u)\) (resp. \(N(Y,P')\)) to \(\tilde{T}_{X,P}\) (resp. \(\tilde{T}_{X,P'}\)). By Lemma 6.18 and 6.15 the action of generators of \(F(W_k)\) on \(Y \in \mathfrak{B}_0(X)\) is consistent with the action of generators of \(W_k\) on the set of corresponding components, that gives homomorphism. By Lemma 6.16 (6.17) the map from the set of principal families \(\mathfrak{B}_0(X)\) to the components of \(\tilde{T}_{X,P}\) (resp. \(\tilde{T}_{X,P'}\)) is injective and by Proposition 6.11 it is surjective.

7. Functoriality properties of the Weyl group actions for field extensions.

In this part we are interested in studying functoriality properties of the Weyl group actions under the field extensions \([E : F]\). All the objects defined before will have additional index, for example, \(P_E\) denotes a minimal parabolic defined over \(E\) and \(P_E(E)\) denotes the set of its \(E\)-points. The set of principal closed \(P_E\)-subvarieties in \(X\) is denoted by \(\mathfrak{B}_0(X,E)\), and the action of \(w \in W_E\) on \(Y \in \mathfrak{B}_0(X,E)\) is denoted by \(w \circ_E Y\). For the subgroup \(H \subset G\) defined over \(E\) and normalized by \(A_E\) by \(\Delta_{H,E}^\pm, \Pi_{H,E}\) we denote the set of positive, negative and simple \(A_E\) roots where the positivity is related to the
There is a sequence of supminimal parabolic $P_E$. The functorial properties for the field extensions can be summarized in the following diagram.

\[ \begin{array}{c}
T_X^* \times_{t'/W} s_E^* \\
\downarrow \\
T_X^* \times_{t'/W} t^* \\
\downarrow \\
T_X^* \times_{t'/W} t^* \\
\downarrow \\
T_X^* \times_{t'/W} t^* \\
\downarrow \\
T_X^* \times_{t'/W} t^* \\
\end{array} \]

We shall define the action of some extension of $\widetilde{W}_F \subset W_E$ of $W_F$ on the set $\mathfrak{B}_0(X, E)$ by choosing a lifting of the generators of $W_F$ to $N_G(A_E)$ defined over $E$ and generating a subgroup $\widetilde{W}_F \subset W_E$ by these generators. This can be done as follows.

Let $s \in W_F$ be the generator. Over $E$ we may choose the minimal parabolic subgroup $P_E \subset P_F$ as the preimage of a minimal parabolic subgroup in $L_F := P_F/(P_F)_a$. Since $s$ normalize $L_F$, the group $sP_Es^{-1} \cap L_F$ is also a minimal parabolic $E$-subgroup of $L_F$ over $E$ and it is conjugate to $P_E \cap L_F$ by some $l \in L_F(E)$. The maximal $E$-split tori in $P_E$ are conjugate by element of $P_E(E)$, so we may specify the choice of $l$ in such a way that $\tilde{s} = ls$ normalizes $A_E$. This defines $l$ up to the element of $L_E := P_E \cap N_G(A_E)$ so $l$ defines a unique element of $W_E$. Since the lifting $\tilde{s}$ normalize $P_E \cap L_F$ it also permutes the supminimal parabolic subgroups of $L_F$ containing $P_E \cap L_F$. These supminimal parabolics correspond to the set of simple $A_E$-roots and, in particular, if $\alpha$ is the simple $A_E$-root of $L_F$, then $\beta := s\alpha s^{-1}$ is also a simple $A_E$-root.

This can be restated as follows. Consider a simple root $\gamma \in \Delta_F$ and let $(A_F)_\gamma$ be the torus that stabilizes the corresponding root subspace $g_\gamma$. Considering the group $Z_G((A_F)_\gamma)$ instead of $G$ we can assume that $G$ is of semisimple split rank 1. Since $P_F$ and $P_F$ are opposite minimal parabolic subgroups defined over $F$ then they are conjugate by the element $s \in G$ that preserves $L_F$. We can assume that $s$ maps $(P_F)_a$ into $(P_F^-)_a$ and $P_E \cap L_F$ maps into itself. As above, this defines the element of $\tilde{s} \in W_E$ whose the length in $W_E$ that is the number of positive $A_E$-roots made negative by applying $s$ and this number is equal to the number of $A_E$-root subspaces in $(p_F)_a$. By this definition we have $\tilde{s} = w_{L_F,E}w_E$, where $w_E$ is the longest element of $W_E$ i.e. a unique element such that $w_E\Delta^+_F = \Delta^-_F$ and $w_{L_F,E}$ is the longest element for $L_F$ (related with the root system of $A_E$) i.e. a unique element such that $w_{L_F,E}\Delta^+_F = \Delta^-_{L_F,E}$.

7.1. Theorem. The set $Y \in \mathfrak{B}_0(X, F)$ is $P_E$-invariant and provides the principal family of $P_E$-orbits, i.e. the element of $\mathfrak{B}_0(X, E)$.

Proof. Let recall that by Lemma 6.9 there is a sequence of supminimal parabolic subgroups $P_i$ defined over $F$ and a sequence of subvarieties $Y_i := P_{F,1}Y$ such that the raise from $Y_{i-1}$ to $Y_i$ is of type $(U)$ and $Y_i = X$. Let us notice that if we take the decomposition of minimal length $s_i = s_{a_1} \ldots s_{a_i}$ for the lifting of $s_i$ to $W_E$, where $a_\alpha \in \Pi_E$ are simple $A_E$-roots, the raise from $Y_{i-1,j} = P_{E,a_1} \ldots P_{E,a_1} Y_{i-1}$ to $Y_{i-1,j+1} = P_{E,a_1} \ldots P_{E,a_1} Y_{i-1}$ is also of type $(U)$. The raise of type $(U)$ for $P_E$-invariant subvarieties is characterized by the property there exists a generic point $y \in Y_i$ such that the image of $(P_F)_y$ in contains
the unipotent subgroup \((U_F)_{\gamma_i}\) corresponding to the root \(\gamma_i \in \Pi_F\). In the same time the words \(w_j := s_{\alpha_1} \ldots s_{\alpha_i}\) have minimal length considered as the elements of \(W_E\). The roots \(\alpha_j\) are the roots of the semisimple part of \(P_{E,i}\). Since \(\ell(s_{\alpha_1+w_j}) = \ell(w_j) + 1\) the \(w_j(D_E^+ \cap w^{-1}D_E)\) consists of a root \(\alpha_{j+1}\) which is a unique \(A_E\)-root that changes the sign after application of \(s_{\alpha_{i+1}}\). Since the set of \(A_E\)-roots from the set \(D_E^+ \cap w^{-1}D_E\) are precisely the \(A_E\)-roots of the group \((U_F)_{\gamma_i}\), the stabilizer of \(w_j\) also contains the root subspace \((U_F)_{\alpha_{j+1}}\) and the corresponding raise from \(Y_{i-1,j}\) to \(Y_{j+1,i-1}\) is of type \((U)\). Thus we get that \(Y\) is connected to \(X\) by the sequence of raises defined over \(E\) of type \((U)\). Since the \((P_E)_u\)-complexity does not increase after the raise of type \((U)\) this proves that \(Y\) also has maximal \((P_E)_u\)-complexity i.e. \(Y \in \mathfrak{B}_0(X, E)\).

Let us fix the following notation till the end of the section. Let \(\tilde{s} \in W_E\) be the lift of simple reflection \(s \in W_F\) corresponding to the simple root \(\gamma \in \Pi_F\). Let \(P_{F,\gamma}\) be a corresponding supminimal parabolic. Let us take \(\alpha \in \Pi_{L,F,E}\) and \(\beta := \tilde{s}\alpha\tilde{s}^{-1} \in \Pi_{L,F,E}\).

7.2. Proposition. Let \(Y_F \in \mathfrak{B}_0(X, F)\), \(Y \in \mathfrak{B}_0(Y_F, E)\) and \(P_{F,\gamma}\) be a supminimal parabolic such that the raise from \(Y_F\) to \(P_{F,\gamma}Y_F\) is of type \((U)\). Then the word \(\tilde{s}_\alpha = s_{\alpha_1} \ldots s_{\alpha_i}\) defines the decomposition \(P_{F,\gamma} = P_{F,\alpha_1} \ldots P_{F,\alpha_i}\) that allows to describe the raise from \(Y\) to \(P_{F,\gamma}Y\) as the composition of the raises of type \((U)\).

Proof. Let us notice that by \([KK16, \text{Prop. 4.13}]\) we have \(c(Y/(P_E)_u) \leq c(P_FY/(P_E)_u)\) and the equality can take place only if \(Y_F = P_FY\). Let us notice that the raise from \(Y_F\) to \(P_{F,\gamma}Y_F\) is of type \((U)\), iff the stabilizer of some \(y \in Y_F\) contains the unipotent subgroup \((U_F)_{\gamma}\). This property is preserved after conjugation by the element of \(l \in L_F\), and since \(Y_F = L_FY\) this property holds for some point of \(Y\) as well. Now we can finish the proof similar to the proof of Theorem 7.1.

7.3. Proposition. Let \(Y_F \in \mathfrak{B}_0(X, F)\) and \(Y \in \mathfrak{B}_0(Y_F, E)\). Let \(\tilde{s} \in W_E\) be the lift of simple reflection \(s \in W_F\) corresponding to the simple root \(\gamma \in \Pi_F\), \(\alpha \in \Pi_{L,F,E}\) and \(\beta = \tilde{s}\alpha\tilde{s}^{-1}\). Then the type of the raise from \(Y\) to \(P_{E,\alpha}Y\) is the same as the type of the raise \(\tilde{s} \circ Y\) to \(P_{E,\beta}(\tilde{s} \circ Y)\).

Proof. Given the sets \(Y_F\) and \(P_{F,\gamma}Y_F \in \mathfrak{B}_0(X, F)\) we can pass to the some open subset of \(Y_F\), in order to assume existence of the \(L_F\)-varieties \(Y_F/(P_F)_u\) and \(P_{F,\gamma}Y_F/(P_F)_u\). Let us note that for \(y \in Y_F\) and its image \(\overline{y} \in Y_F/(P_F)_u\), the stabilizer \((L_F)_{\overline{y}}\) is equal to the image of \((P_F)_y\) in \(L_F\). If we fix the section \(K_E\) defined over \(E\) for the subset \(Y \in \mathfrak{B}_0(Y_F, E)\), we get the following diagram:

\[
\begin{array}{cccccc}
P_{F,\gamma}Y & \longrightarrow & P_{F,\gamma}P_{E,\alpha}Y = P_{E,\beta}P_{F,\gamma}Y & \longrightarrow & P_{F,\gamma}Y/(P_F)_u \\
Y & \longmapsto & P_{E,\alpha}Y & \longmapsto & Y_F/(P_F)_u \\
\end{array}
\]

Which can be rewritten in terms of sections:

\[
\begin{array}{cccccc}
P_{F}\tilde{s}K_E & \longrightarrow & P_{F}\tilde{s}s_{\alpha}K_E = P_{F}s_{\beta}\tilde{s}K_E & \longrightarrow & P_{F,\gamma}Y/(P_F)_u \\
P_{F}K_E & \longrightarrow & P_{F}s_{\alpha}K_E & \longrightarrow & Y_F/(P_F)_u \\
\end{array}
\]

The type of the raise from \(Y\) to \(P_{E,\alpha}Y\) is the same as the type of the raise for their images in \(Y_F/(P_F)_u\) and is defined by the type of the stabilizer \((L_F)_{\overline{y}} \cap P_{E,\alpha}\) (where
\( y \in \mathcal{K}_E \) and \( \overline{y} \) is its image in \( Y_F/(P_F)_a \). In the same time the type the raise from \( P_{F,\gamma} Y \) to \( P_{E,\beta} P_{F,\gamma} Y \) is defined by the type of the stabilizer \( (L_F)_{\overline{y}} \cap P_{E,\beta} \) which is obtained from \( (L_F)_{\overline{y}} \cap P_{E,\alpha} \) via conjugation by \( \tilde{s} \).

7.4. **Remark.** Assume that for the element \( \tilde{s} \in W_E \) (which is a lift of \( s \in W_F \)) and \( Y_F \in \mathfrak{B}_0(X, F) \) we have \( \tilde{s} \circ Y_F \subset Y_F \). Then \( \tilde{s} \) acts on the graph of \( P_E \cap L_F \)-orbit inclusions for the \( L_F \)-variety \( Y_F/(P_F)_a \) by the automorphism that \( \tilde{s} \) induces on \( L_F \) (on the set of simple roots of \( L_F \)) and \( \tilde{s} \circ Y_F = Y_F \).

The set \( Y_F \) is irreducible over \( E \) since it contains a Zariski dense subset of \( F \)-points. In particular it contains a unique subset of \( (P_E)_a \)-orbits of maximal dimension that gives the corresponding subset of \( (P_E)_a \cap L_F \)-orbits in \( Y_F/(P_F)_a \) which maps into itself by automorphism \( \tilde{s} \). So we have the following corollary:

7.5. **Corollary.** Suppose that the action of \( s \in W_F \) on the irreducible over \( E \) subset \( Y_F \in \mathfrak{B}_0(X, F) \) is not of type \( (U) \). Then the \( W_E \)-action of the lift \( \tilde{s} \in W_E \) on the subset \( Y_F \) considered as the element of \( \mathfrak{B}_0(X, E) \) fixes \( Y_F \).

Combining the above results we get the following theorem:

7.6. **Theorem.** The action of \( W_E \) on \( \mathfrak{B}_0(X, E) \) induces the action of \( W_F \) on \( \mathfrak{B}_0(X, F) \) via the lift defined above.

8. Generators of the little Weyl group.

For \( Y \in \mathfrak{B}_0(X) \) let \( W(Y) \subset W \) be the stabilizer of \( Y \) with respect to the Weyl group action on the principal \( P \)-invariant closed irreducible subsets of \( X \). The proof of the following theorem is the smart combination of the proofs of M. Brion [Bri01, Prop.4] and F. Knop [Kno95, §7] with some new ingredients.

8.1. **Theorem.** The subgroup \( W(X) \subset W_k \) is generated by the reflections \( s_\alpha \in W_k \) or the products of reflections \( s_\alpha s_\beta \) such that \( s_\alpha, s_\beta \in W_k, \alpha \perp \beta \) and \( \alpha + \beta \in \alpha \) is not a root of \( \Delta_k \).

**Proof.** We shall first make several observations that are essential for the proof of the theorem. Let \( Y \in \mathfrak{B}_0(X) \) and \( \alpha, \beta \in \Pi \) be the simple roots such that \( \dim P_\alpha Y > \dim Y \) and \( \dim P_\beta Y > \dim Y \). Then since the \( k \)-rank, \( k \)-complexity and \( k \)-homogeneity are maximal, then the raise from \( Y \) to \( P_\alpha Y \) (resp. to \( P_\beta Y \)) is of type \( (U) \), that also implies that for \( P_\alpha \) (resp. for \( P_\beta \)) we have

\[
\dim P_\alpha Y = \dim Y + \dim P_\alpha/P.
\]

Let \( P_{\alpha \beta} \) be the subgroup of \( G \) generated by \( P_\alpha \) and \( P_\beta \). Consider the \( P \)-invariant subvariety \( Y_m^{(\alpha)} := P_\alpha P_\beta \ldots Y \) of \( X \) obtained by \( m \)-subsequent applications of \( P_\alpha \) and \( P_\beta \). Let us notice that if \( Y_\alpha \) is also \( P_\beta \) stable then \( Y_m = P_{\alpha \beta} Y \). Similarly define \( Y_m^{(\beta)} := P_\beta P_\alpha \ldots Y \)

For a simple \( k \)-root \( \alpha \) let us denote \( n_\alpha = \dim P_\alpha/P \). This implies that unless \( Y_m^{(\alpha)} = P_{\alpha \beta} Y \) we have \( \dim Y_m^{(\alpha)} = \dim Y_{m-1}^{(\alpha)} + n_\alpha \) and \( \dim Y_m^{(\beta)} = \dim Y_{m-1}^{(\beta)} + n_\beta \). Let us notice that there exist \( m_\alpha, m_\beta \) for which:

\[
P_{\alpha \beta} Y = P_{\alpha \beta} \ldots Y = P_{\beta \alpha} \ldots Y
\]
Comparing the dimensions of the terms of this equality, we get that \( m_\alpha = m_\beta \), and this number should be even in the case \( n_\alpha \neq n_\beta \).

This equality implies that \((s_\alpha s_\beta)^m \cdot \Lambda(Y) = \Lambda(Y)\).

Assume that \((s_\alpha s_\beta)^m \neq e\). Since \( P_\alpha \) do not normalize \( Y \) and the raise is of type (U) by Claim 6.14 we have \( \alpha \notin \Delta_Y \). Thus the roots \( \alpha, \beta \) are not orthogonal to \( a_Y \).

Let \( \text{Rad}(P_\alpha) \) be the radical of \( P_\alpha \), \( W_\alpha \in W \) be the subgroup generated by \( s_\alpha \) and \( s_\beta \), and \( G_{\alpha,\beta} := P_\alpha / \text{Rad}(P_\alpha) \).

8.2. Lemma. There is an isomorphism of the set \( \mathfrak{B}_0(P_\alpha Y, P) \) and the set \( \mathfrak{B}_0(P_\alpha Y / \text{Rad}(P_\alpha), P / \text{Rad}(P_\alpha)) \) equipped with the action of \( W_{\alpha,\beta} \).

This lemma reduces our situation to the study of homogeneous space \( X := G_{\alpha,\beta}/H \). For brevity put \( G := G_{\alpha,\beta} \).

8.3. Lemma. The plane \( H_{\alpha,\beta} \) is not contained in \( a_Y \) unless \( G/H \) is horospherical and \((s_\alpha s_\beta)^m = e\).

Proof. After taking quotient by the radical of \( P_{\alpha,\beta} \) we reduce the situation to the homogeneous space of the group of rank 2 that we denote by \( G \). Since \( Y \) is not stable under \( P_\alpha, P_\beta \) there are no orbits that are raised to \( Y \). In particular, all \( P \)-orbits of the points from \( G/H(k) \) have maximal rank equal to 2, and in particular the stabilizer \( g H g^{-1} \cap P \) does not contain nontrivial \( k \)-split torus for all \( g \in G_k \). Since all \( k \)-split torus is conjugate to the subtorus of \( P \) by some element \( g \in G_k \). This implies that \( H \) do not contain \( k \)-split tori and in particular \( H/\text{Rad}_u H \) is anisotropic. Moreover since the raise from \( Y \) to \( P_\alpha Y \) is of type (U), \( H \) has a subgroup isomorphic to \( \mathbb{G}_a \), that implies \( \text{Rad}_u H \) is non-trivial. We can assume \( H \) belong to one of the parabolic groups \( P : = P_\alpha, P_\beta \) and \( H_u \) belong to the corresponding unipotent radical \( (P_i)_u \). Also we can assume after conjugation by \( p \in P_\alpha(k) \) that a Levi subgroup \( L_H \) of \( H \) (which is elementary) is contained in the Levi subgroup \( L_i \) of \( P_i \). Consider the equivariant map \( G/H \rightarrow G/P_i \). The \( P_\alpha \)-orbits on \( G/H \) that map to \( P_u w P_i / H \) (where \( w \in W_k \)) correspond to orbits \( w^{-1} P_u w \cap P_u \) on \( P_u / H \times L_i / L_H \) (which correspond to orbits of stabilizer \( w P_u w^{-1} \cap P_u \) on fiber \( w P_i / H \)). For the open cell \( P_u w_0 P_i / H \) this stabilizer is trivial and the dimension of such family is equal to \( \dim(P_i/H) \).

Since the raise from \( Y_{m-1}^\alpha \) and from \( Y_{m-1}^\beta \) to \( G/H \) is of type (U) then the dimension of general \( P_u \)-orbits in \( Y_{m-1}^\alpha \) and \( Y_{m-1}^\beta \) is strictly smaller than \( \dim P_\alpha \) so these divisors do not intersect \( P_u w_0 P_i / H \), and moreover \( \text{codim}\_X Y_{m-1}^\alpha = \text{codim}\_G/P B_{\beta,\beta} w_0 P_i / P = n_\beta \). If \( m = 1 \), then \( n_\alpha = n_\beta \) and \( Y \) is equal to preimage of one of the Schubert variety of codimension \( n_\alpha \) which is impossible since \( Y \) is neither \( P_\alpha \) nor \( P_\beta \)-stable and the preimage of the Schubert divisor is stable with respect to one of these groups.

In the case \( Y_{m-1}^\alpha \) and \( Y_{m-1}^\beta \) are \( k \)-dense and have codimension \( n_\beta \) and \( n_\alpha \) in \( Y \) respectively and do not contain in the closure of each other. On the other hand their images to \( G/P_i \) coincide with \( k \)-dense Schubert varieties \( P_u w P_i / H \) for \( w \in W_k, \ w \neq e \). But the minimal codimension for such Schubert varieties that do not lie in the closure of larger variety with this property is equal to \( n_\alpha \) and \( n_\beta \) which is also equal to codimension in \( X \) of their preimages under the equidimensional map to \( G/P_i \).
Thus $P_1 = P$ and $Y^\alpha_{m-1}$ and $Y^\beta_{m-1}$ are equal to $P_\alpha s_\beta w_0 P/H$ and $P_\alpha s_\beta w_0 P/H$ which are the preimages of different $k$-dense Schubert varieties. Since the raise of type $(U)$ preserve the $P_\alpha$-complexity i.e. $c(X/P_\alpha) = c(Y^\alpha_{m-1}/P_\alpha) = c(Y^\beta_{m-1}/P_\alpha)$ then the complexity of the actions of $w_0^{-1}s_\alpha P_\alpha s_\alpha w_0 \cap P_\alpha$ and $w_0^{-1}s_\beta P_\alpha s_\beta w_0 \cap P_\alpha$ (which are the root subgroups $(P_\alpha)_\beta$ and $(P_\alpha)_\alpha$) on $P/H$ is equal to $\dim(P/H)$, that implies the triviality of these actions.

In particular the triviality of the action of root subgroups $P_\alpha \beta$ and $P_\alpha \alpha$ implies the triviality of the action of $P_\alpha$ on $P/H$, thus $P_\alpha = H_\alpha$. In this case $W_{(X)}$ is trivial since the families of orbits of maximal $P_\alpha$-complexity equal to $\dim(L/L_H)$ form the subsets to $P_\alpha w P/H$ for $w \in W_k$, and $W_k$ acts transitively on them (by 6.1). Thus we have $(s_\alpha s_\beta)^m = e$ that contradicts with our assumptions. 

The element $(s_\alpha s_\beta)^m$ is the rotation in the two dimensional plane $H_{\alpha\beta}$ spanned by $\alpha$ and $\beta$ which preserves $a_Y$ (and fixes $H_{\alpha\beta}^\perp$). By the previous Lemma and since $\alpha, \beta$ are not orthogonal to $a_Y$ the subspace $a_Y \cap H_{\alpha\beta}$ is one dimensional and fixed by rotation $(s_\alpha s_\beta)^m$ that implies:

8.4. Proposition. $(s_\alpha s_\beta)^m = -\operatorname{id}|_{H_{\alpha\beta}}$ and the restriction of $(s_\alpha s_\beta)^m$ to $a_Y$ is the reflection in the hyperplane orthogonal to $H_{\alpha\beta} \cap a_Y$.

This can happen only in the following cases: (i) $m = 1$, $\alpha \perp \beta$, (ii) $m = 2$, $\alpha, \beta$ generate the root system of type $B_2$, $BC_2$, (iii) $m = 3$, $\alpha, \beta$ generate the root system of type $G_2$.

8.5. Proposition. Let $G$ be a group of rank two, then we have the following possibilities: either $G/H \cong PGL_2 \times PGL_2 / PGL_2$ and $a_X = \langle \alpha + \beta \rangle$, or $m > 1$ and $(s_\alpha s_\beta)^m$ is equal to a reflection $s_\gamma \in W$, where $\gamma \in \Xi(Y_{\alpha})$.

Proof. Let us prove first that either $G/H$ satisfies the conditions of proposition or it is spherical. The subset $G(k)x_0 \cong G(k)/H(k)$ for $x_0 \in X(k)$ is Zariski dense in $G/H$ and correspond to the set $G(k)/P(k)$ see Lemma 2.1. Since a general $P$-orbit has $k$-rank $r$ there exists a split torus $A_0$ of dimension $\operatorname{rk}_k(G/H) - r$ such that for each $P$-orbit of $x = gx_0$, after conjugation by $p \in P(k)$ we have $A_0 \subset gHg^{-1} \cap P$. We can also assume that $A_0$ is contained in some fixed maximal split torus $A_H$ of $H$. Let us notice that the set of $g \in G$ such that $g^{-1}A_0 g \subset A_H$ is closed and in particular has finite number of connected components. Since the a subtorus of $A_H$ cannot vary continuously, we can assume that there exists $A' \subset A_H$ which is $H(k)$-conjugate to $g^{-1}A_0 g$ where $g \in G(k)$. In this way for a generic $H$-orbit we have associated a subset $Y' \in (G/P(k))^{A'}$, and for those $H$-orbits that contain a point from $G/P(k)$ we have associated a point in $Y'(k) \in (G/P(k))^{A'}$. In particular if $G/H$ is not spherical $Y'(k)$ is infinite.

Assume that $\Lambda(X)$ is not generated by a root. Then $(G/P(k))^{A'} = \{sP, eP\}$ since this set of points coincide with the fixed points of $A$. In this case $G/H$ is spherical.

In the case, assume that $[L_{X,k}, L_{X,k}]$ be nonanisotropic. Then the root system $\Delta_{L_{X,k}}$ is generated by $\alpha$ or $\beta$, $a_{X_{\alpha\beta}}$ is generated by the corresponding orthogonal root and $(s_\alpha s_\beta)^m$ is the reflection with respect to this root.

When $[L_{X,k}, L_{X,k}]$ is anisotropic, by the local structure theorem $P_\alpha$ acts freely on the points of open $P$-orbit. The following lemma finishes the proof of the proposition.

8.6. Lemma. Let $G_{\alpha\beta}/H$ be a spherical homogeneous space with a locally free action of $P_\alpha$ and such that $H$ contains regular one-dimensional split torus $A_0$ (i.e. $Z_{G_{\alpha\beta}}(A_0) = L$). Then either $G_{\alpha\beta}$ is of type $A_1 + A_1$ and $a_{X_{\alpha\beta}} = \langle \alpha + \beta \rangle$ or $G_{\alpha\beta}/H$ is horospherical.
Proof. By the local structure theorem we have a choice of $P$ and $H$ such that $p + h = g$ and $p_\alpha \cap h = \{0\}$. Moreover we may assume that $\Xi(X_{\alpha\beta})$ is contained in the positive Weyl chamber. This can be done by passing to quasiaffine $G/H$ via taking an affine cone over $G/H$ and extending $G_{\alpha\beta}$ and $H$ by dilatations. The torus $a_0$ is regular and orthogonal subspace $\Xi(X)$ belong to the interior of the Weyl chamber. Denote by $\lambda_0$ a one-parameter subgroup corresponding to $t_0$ in the case when $A_0$ has different weights on all the roots subspaces of $g_{\alpha\beta}$, $h$ is normalized by $A_0$ and it is the direct sum of $A_0$ subspace of $l$ and subspaces of distinct root subspaces. The above equalities imply $h = \ell + a_0 + p_u$ for $\ell \in [0, l]$. The direct check shows that $\lambda_0$ has the same pairing with different roots only in the following cases: 

(i) $\Delta$ of type $A_1 + A_1$, $a_{\lambda_0} = \alpha + \beta$, and $\lambda_0 = \alpha - \beta$ has the same pairing with the roots $\alpha$ and $-\beta$. This is the precisely the statement of the Lemma. (ii) $\Delta$ of type $B_2$, $a_{\lambda_0} = 2\beta + 3\alpha$, and $\lambda_0 = \alpha - \beta$ has the same pairing with the roots $-\alpha - \beta$ and $2\alpha + \beta$ and the same holds for the opposite pair, (iii) $\Delta$ of type $G_2$, $a_{\lambda_0} = 5\alpha + 3\beta$, and $\lambda_0 = \alpha - \beta$ has the same pairing on the roots $-2\alpha - \beta$ and $3\alpha + 2\beta$. But in the cases (ii) and (iii) the pairing of $\lambda_0$ with the roots $-\alpha$, $-\beta$ are distinct and differ from the pairings with other roots. Since $p + h = g_{\alpha\beta}$, $h$ and $p$ is normalized by $T_0$ and the root subspaces of $-\alpha$, $-\beta$ do not belong to $p$ they are contained in $h$. Since subspaces $g_{-\alpha}$ and $g_{-\beta}$ generate $p_u$ this proves the lemma.

Since the space $G/H$ is horospherical, it has a trivial little Weyl group and also trivial $W(X)$ due to triviality of $[L, L]$. Thus $(s_\alpha s_\beta)^m = 1$, which finishes the proof of proposition.

Let $w \in W(X)$ and let $s_{\alpha_1} \cdots s_{\alpha_i}$ be a reduced decomposition of $w$. Let us put $w_i = s_{\alpha_i} \cdots s_{\alpha_1}$ and $Y_i = w_i \cdot X$.

We finish the proof of the theorem by the induction on the following three parameters:

- $m_{cd} = \{\text{the maximal codimension of } Y_i \text{ in } X \text{ for all } i\}$,
- $n_{cd} = \{\text{the number of } Y_i | \text{codim}_Y X = m_{cd}\}$,
- $l_w = \{\text{length of the given decomposition of } w\}$.

On each step of the induction we shall either decrease $m_{cd}$ (possibly increasing $l_w$ and $n_{cd}$), or decrease $n_{cd}$ not changing $m_{cd}$ (possibly increasing $l_w$), or decrease $l_w$, not changing $m_{cd}$ and $n_{cd}$.

Consider the minimal $i$ for which $Y_i$ has maximal codimension in $X$.

**Case 1** Let $P_{\alpha_i+1} Y_i = Y_i$ (in particular $s_{\alpha_i+1} \in W(Y_i)$) then we have two cases: there either exists or not exists $Y' \in \mathfrak{B}_0(X)$ with dim $Y' = \text{dim } Y_i - 1$, such that $P_{\alpha_i+1} Y' = Y_i$. In the second case $P_{\alpha_i+1}$ is the normalizer of the general $B$-orbits in $Y_i$ and $\alpha_i+1 \in \Delta_Y$. In the first case the type of the pair $(Y_i, Y')$ is different from the type (U), which implies that the vector space spanned by the character lattice $\Lambda(Y_i)$ is $s_{\alpha_i+1}$-stable. Let us notice that $\alpha_i+1 \notin \Delta_Y$, so $\alpha_i+1$ is not orthogonal to $\Lambda(Y_i)$ (due to non-degeneracy assumption). In particular this implies that $\alpha_i+1 \in \Lambda(Y_i)$. Since $w_i^{-1} s_{\alpha_i+1} w_i \in W(X)$, the element $w(w_i^{-1} s_{\alpha_i+1} w_i) = s_{\alpha_i} \cdots s_{\alpha_1+1} \cdots s_{\alpha_1}$ also belongs to $W(X)$ and has the smaller length than $w$.

**Case 2** Let dim $P_{\alpha_i+1} Y_i > \text{dim } Y_i$. From the first part of our proof we see that there exists $m$ such that $(s_{\alpha_i+1} s_{\alpha_i})^m \in W(Y_i)$ and either $s_\beta = (s_{\alpha_i+1} s_{\alpha_i})^m$ is the reflection of $W$...
and \( \beta \in \Lambda(Y_i) \) or \( m = 1 \) and \( \alpha_{i+1} \) and \( \alpha_i \) are orthogonal roots whose sum is not a root of \( \Delta \). In the last case \( \alpha_{i+1} + \alpha_i \in \Lambda(Y_i) \). As before \( w_i^{-1}(s_{\alpha_{i+1}}s_{\alpha_i})^mw_i \in W(X) \), consider the element of \( W(X) \)

\[
ww_i^{-1}(s_{\alpha_{i+1}}s_{\alpha_i})^mw_i = s_{\alpha_i} \ldots s_{\alpha_{i+2}}(s_{\alpha_i}s_{\alpha_{i+1}})^{m-1} s_{\alpha_{i-1}} \ldots s_{\alpha_1}.
\]

It will have the larger minimal \( i \) for which \( Y_i \) has maximal codimension in \( X \) and the number of such \( Y_i \) of maximal codimension will be decreased. This proves the theorem. \( \square \)

9. Monodromy action on the set of real orbits

In this section we focus again on the case when \( k = \mathbb{R} \), and the variety \( X \) is homogeneous and admits the so-called \( k \)-wonderful compactification. For the definition of the concept of \( k \)-wonderful variety and the required details see [KK16]. Also we shall need the following theorem:

9.1. Theorem. [KK16, Thm.13.7] Let \( X = G/H \) be a homogeneous \( \mathbb{R} \)-spherical variety. Assume that \( X \) is \( \mathbb{R} \)-wonderful and let \( X \rightarrow X^{st} \) be its wonderful embedding. Let \( Y \subseteq X \) be the closed \( G \)-orbit isomorphic to \( G/P_X \). Then

i) \( X^{st}(\mathbb{R}) \) is a compact connected manifold.

ii) \( Y(\mathbb{R}) \) is the only closed \( G(\mathbb{R})^0 \)-orbit of \( X^{st}(\mathbb{R}) \). In particular, it is connected and \( G(\mathbb{R}) \)-stable.

iii) \( P(\mathbb{R})^0 \) has at most \( 2^{2\text{rk}_k X} \) open orbits in \( X(\mathbb{R}) \) (or, equivalently, in \( X^{st}(\mathbb{R}) \)). They all contain \( Y(\mathbb{R}) \) in their closure.

Let us also recall that we can calculate the topological fundamental group of \( Y(\mathbb{R}) \) using the fact that for the set \( Y(\mathbb{R}) \) we have a Bruhat decomposition which is also a cell decomposition of the corresponding CW-complex (cf. [Wig98] where \( \pi_1(Y(\mathbb{R})) \) is described by generators and relations). The generators of \( \pi_1(Y(\mathbb{R})) \) correspond to one dimensional cells in \( Y(\mathbb{R}) \) which in turn correspond to the simple roots \( \alpha \in \Delta_k \setminus \Delta_{L_X} \) such that \( \dim P_{\alpha}/P = 1 \) (or equivalently \( g_{2\alpha} = 0 \) and \( \dim g_\alpha = 1 \)). The loop is represented by applying \( \exp(\pi t Z_\alpha) \) to a \( P \)-fixed point \( y_0 \in Y \) for \( t \in [0, 1] \), where \( Z_\alpha = e_\alpha + e_{-\alpha} \).

Let us notice that the set \( X(\mathbb{R}) \) is obtained by throwing away \( \mathbb{R} \)-points of complete intersection divisor (its smooth components we denote by \( E_i \)) from the compact manifold \( X^{st}(\mathbb{R}) \). And in turn the set of real points in the open \( P \)-orbit is obtained by throwing away the set of real points from \( P \)-invariant but not \( G \)-invariant divisors (we denote them by \( D_i \) and by \( D^s_i \) the corresponding dense \( P \)-orbit). Any connected component of \( G \)-orbit is defined by the the finite union of open \( P(\mathbb{R}) \)-orbits in the real topology. Without the loss of generality we may assume that the real path \( \gamma(t) \) joining any two points in the open \( G(\mathbb{R})^0 \)-orbit do not pass through the real points of the orbits of codimension \( \leq 2 \) in particular we can assume that the path lie in the open \( P \)-orbit for almost all points and intersect \( D_i \) in the finite set of points. Assume that \( P_\alpha \) raises \( D_i \), then the neighborhood of the intersection point lies entirely in \( D_i \). In order to understand which \( P(\mathbb{R})^s \)-orbits form the connected component of the \( G(\mathbb{R}) \)-orbit it is sufficient to study only the case when the split semi-simple rank is 1 and the closed or intermediate orbit have codimension 1.
Let us fix a unique $P^\ast$-invariant point $y_0 \in Y$, and let $N_{X/Y,y_0}$ be the fiber over $y_0$ of a normal bundle to $Y$. The open $\varepsilon$-neighborhood which we denote by $N_{X/Y,\varepsilon}$ of $Y$ in the real topology is diffeomorphic to the normal bundle $G \ast_{P^\ast} N_{X/Y,y_0}$ and the intersection $N_{X/Y,\varepsilon} \cap E_i$ defines the corresponding coordinate plane $H_i$ in $N_{X/Y,y_0}$. The connected components of $N_{X/Y,\varepsilon}^0 = N_{X/Y,\varepsilon} \cup \cup_i \Gamma^X E_i$ are marked by the components of $G(\mathbb{R})$-orbits to which they belong. Since the divisors $E_i$ form a complete intersection along $Y$ the element of $\pi_1(Y(\mathbb{R}))$ defines the monodromy of the fiber $N_{X/Y,\varepsilon} \cup \cup_i \Gamma^X E_i$ over $y_0$ which is isomorphic to $N_{X/Y,y_0}^0 := N_{X/Y,y_0} \cup \cup_i \Gamma^X H_i$. Observe that the generators of $\pi_1(Y(\mathbb{R}))$ are represented by the loops $\exp(\pi t Z_\alpha)y_0$ and $\exp(\pi t Z_\alpha) \in G(\mathbb{R})$. For a point $x_0 \in N_{X/Y,\varepsilon}^0$ that defines a connected component we can represent the monodromy action by considering the path $\exp(\pi t Z_\alpha)x_0$ which lie entirely in the $G(\mathbb{R})^\circ$-orbit and by taking the corresponding connected component of $N_{X/Y,y_0}^0$. This actually proves that the monodromy action reduces to the action of $\exp(\pi Z_\alpha)$ on $N_{X/Y,y_0}^0$.

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