Realization and characterization of a 2-photon 4-qubit linear cluster state

Giuseppe Vallone$^{1,*}$, Enrico Pomarico$^{1,*}$, Paolo Mataloni$^{1,*}$, Francesco De Martini$^{1,*}$, Vincenzo Berardi$^{2}$

$^{1}$Dipartimento di Fisica dell'Università “La Sapienza” and Consorzio Nazionale Interuniversitario per le Scienze Fisiche della Materia, Roma, 00185 Italy

$^{2}$Dipartimento Interateneo di Fisica, Università e Politecnico di Bari and Consorzio Nazionale Interuniversitario per le Scienze Fisiche della Materia, Bari, 70126 Italy

We report on the experimental realization of a 4-qubit linear cluster state via two photons entangled both in polarization and linear momentum. This state was investigated by performing tomographic measurements and by evaluating an entanglement witness. By use of this state we carried out a novel nonlocality proof, the so-called “stronger two observer all versus nothing” test of quantum nonlocality.

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Multipartite graph states and, in particular, cluster states, have been recently introduced by Briegel and Raussendorf as a fundamental resource aimed at the linear optics one way quantum computation [1,2], and at the realization of important quantum information tasks, such as quantum error correction and quantum communication protocols [3,4]. Recently, the experimental feasibility of one way quantum computation by four photon cluster states was demonstrated [5,6]. Besides the applications to quantum computation, cluster states are powerful tools for performing nonlocality tests [7,8]. It is well known that the adoption of an increasing number of internal degrees of freedom, i.e., in a higher dimensional Hilbert space, leads to a stronger violation of local realism [9].

In this letter we report the experimental realization of a high fidelity 2-photon 4-qubit linear cluster state by a linear optical technique consisting of the entanglement of the polarization ($\pi$) and momentum ($k$) degrees of freedom of one of the two photons belonging to an hyperentangled state [10]. It is worth noting that, at variance with the cluster states, hyperentangled, or double entangled states, are bi-separable and do not represent genuine four-qubit entangled states.

In the present experiment the 2-photon 4-qubit linear (SPDC) method already described in details in other papers [13,14]. A thin type I $\beta$-barium-borate BBO crystal slab operating under the double (back and forth) excitation of a cw $Ar^+$ laser ($\lambda_p = 364$ nm) generated the $\pi$-entangled state $|\Phi^-(\pi)\rangle$, obtained by the superposition of two perpendicularly polarized SPDC cones emerging from the crystal at the degenerate wavelength $\lambda = 728$ nm. The $k$-entangled state $|\psi^+(k)\rangle$ was realized by selecting two pairs of correlated $k$-modes, $r_A,\ell_B$ and $\ell_A,\ell_B$, belonging to the conical emission of the crystal. Because of the “phase-preserving” character of the SPDC process, the relative phase between the two pair emissions was set to the value $\phi = 0$. By adoption of hyperentangled states several AVN tests of quantum nonlocality were recently proposed [15] and carried out [16].

In the present experiment the 2-photon 4-qubit linear

![Diagram](image)

FIG. 1: Generation of the linear cluster state by a source of polarization-momentum hyperentangled 2-photon state. The state $|\Xi\rangle = |\Phi^-(\pi)\rangle \otimes |\psi^+(k)\rangle$ corresponds to two separate 2-qubit clusters. The $HW$ acts as a Controlled-Phase (CP) thus generating the 4-qubit linear cluster $|C_4\rangle$. 
FIG. 2: Interferometer and measurement apparatus. a) The mode pairs $r_A-\ell_B$ and $\ell_A-\ell_B$ are matched on the BS. The phase shifters $\phi_A$ and $\phi_B$ (thin glass plates) are used for the measurement of momentum observables. The polarization analyzers on each of BS output modes are shown (QWP/HWP=Quarter/Half-Wave Plate, PBS=Polarized Beam Splitter). b) Same configuration as in a) with BS and glasses removed.

FIG. 3: State characterization. a) Coincidence rates versus path delay $\Delta x$ showing the interference pattern between the two pairs $r_A-\ell_B$ and $\ell_A-\ell_B$. The dip(peak) FWHM and the coherence time ($\sim 150 fsec$) of the photons are determined by the bandwidth (6nm) of the interference filter used. b) The two thin glass plates inserted on the right modes ($\phi_A$ and $\phi_B$ in Fig. 2a) and the BS transform the input states in the following way: $\frac{1}{\sqrt{2}}(|\ell_A| + e^{-i\phi_B}|r_B|) \rightarrow |\ell_A|$, $\frac{1}{\sqrt{2}}(|\ell_B| - e^{-i\phi_A}|r_A|) \rightarrow |r_B|$ with $i = A, B$. Note that these are single photon transformations: in fact the single BS shown in Fig. 2a is equivalent to two BS’s, one for each (A or B) particle. The reason why we used the single BS apparatus resides on its higher phase stability.

The photons associated with the BS output modes $\ell_A$, $r_B$, $\ell_B$, $r_A$ are analyzed each by a quarter-wave plate (QWP), half-wave plate (HWP), a polarizing beam splitter (PBS). The optical path delay $\Delta x$ can be simultaneously changed for both $\ell_A$ and $\ell_B$ modes by using a trombone mirror assembly. The null value delay ($\Delta x = 0$) corresponds to the exact superposition on the BS between $r_A-\ell_B$ and $\ell_A-\ell_B$, i.e., when the right ($r$) and left ($\ell$) optical paths are equal [13]. The two thin glass plates inserted on the right modes ($\phi_A$ and $\phi_B$ in Fig. 2a) and the BS transform the input states in the following way: $\frac{1}{\sqrt{2}}(|\ell_A| + e^{-i\phi_B}|r_B|) \rightarrow |\ell_A|$, $\frac{1}{\sqrt{2}}(|\ell_B| - e^{-i\phi_A}|r_A|) \rightarrow |r_B|$ with $i = A, B$. Note that these are single photon transformations: in fact the single BS shown in Fig. 2a is equivalent to two BS’s, one for each (A or B) particle. The reason why we used the single BS apparatus resides on its higher phase stability.

The photons associated with the BS output modes $\ell_A$, $r_B$, $\ell_B$, $r_A$ are analyzed each by a quarter-wave plate (QWP), half-wave plate (HWP), a polarizing beam splitter (PBS) and detected by single photon avalanche detectors. Two thin glass plates on modes $r_A$ and $r_B$ are properly set for measuring momentum observables. The analysis setup shown in Fig. 2a) is obtained from Fig. 2a) by removing the interferometric apparatus and allows the measurement of several relevant observables that will be introduced later in the paper.

We characterized the state |11⟩ by measuring the interference between the mode pairs $r_A-\ell_B$ and $\ell_A-\ell_B$ as a function of the delay $\Delta x$. The dip-peak graph (88% average visibility) for $H$ polarized photons, corresponding to the k-entangled state $|\psi^+⟩$, is shown in Fig. 3a).

The cluster state

$$|C_4⟩ = \frac{1}{2}(|Hr⟩_A|H\ell⟩_B + |Vr⟩_A|V\ell⟩_B + |H\ell⟩_A|Hr⟩_B - |V\ell⟩_A|Vr⟩_B)$$

$$= \frac{1}{\sqrt{2}}(|\Phi^+⟩|r⟩_A|\ell⟩_B + |\Phi^-⟩|\ell⟩_A|r⟩_B) ,$$

where $|\Phi^+⟩ = \frac{1}{\sqrt{2}}(|Hr⟩_A|H\ell⟩_B + |Vr⟩_A|V\ell⟩_B)$, was created by inserting in the $r_A$ (right-Alice) mode a zero order half wave plate (HWP) with the optical axis oriented along the vertical direction (see Fig. 1). The $HW$ left the state $|\Phi^+⟩|r⟩_A|\ell⟩_B$ unchanged, while the transformation $|\Phi^-⟩|r⟩_A|\ell⟩_B$ transformed $|1⟩$ into $|0⟩$ and $|0⟩$ into $|1⟩$, i.e. when the photon is polarized in the $H$ direction (right-Alice) mode $m$ a zero order half wave plate (HWP) is enough. Indeed, the $HW$ only, while it is “nonlocal” for the two qubits associated to photon $A$ itself. Indeed, the $HW$ operates as a Controlled Phase (CP) between the target qubit 2 and the control qubit 3 (i.e. the polarization and the momentum degree of freedom of photon $A$), thus entangling the four qubits together. Moreover this operation does not require any kind of post-selection.

Let’s consider the measurement setup shown in Fig. 2b). The mode pairs $r_A-\ell_B$ and $\ell_A-\ell_B$ are there spatially and temporally superimposed by means of a $50\%$ beam splitter (BS). The optical path delay $\Delta x$ can be simultaneously changed for both $\ell_A$ and $\ell_B$ modes by using a trombone mirror assembly. The null value delay ($\Delta x = 0$) corresponds to the exact superposition on the BS between $r_A-\ell_B$ and $\ell_A-\ell_B$, i.e., when the right ($r$) and left ($\ell$) optical paths are equal [13]. The two thin glass plates inserted on the right modes ($\phi_A$ and $\phi_B$ in Fig. 2a) and the BS transform the input states in the following way: $\frac{1}{\sqrt{2}}(|\ell⟩_A + e^{-i\phi_B}|r⟩_B) \rightarrow |\ell⟩_A$, $\frac{1}{\sqrt{2}}(|\ell⟩_B - e^{-i\phi_A}|r⟩_A) \rightarrow |r⟩_B$ with $i = A, B$. Note that these are single photon transformations: in fact the single BS shown in Fig. 2a) is equivalent to two BS’s, one for each (A or B) particle. The reason why we used the single BS apparatus resides on its higher phase stability.

The photons associated with the BS output modes $\ell_A$, $r_B$, $\ell_B$, $r_A$ are analyzed each by a quarter-wave plate (QWP), half-wave plate (HWP), a polarizing beam splitter (PBS) and detected by single photon avalanche detectors. Two thin glass plates on modes $r_A$ and $r_B$ are properly set for measuring momentum observables. The analysis setup shown in Fig. 2a) is obtained from Fig. 2a) by removing the interferometric apparatus and allows the measurement of several relevant observables that will be introduced later in the paper.

We characterized the state |11⟩ by measuring the interference between the mode pairs $r_A-\ell_B$ and $\ell_A-\ell_B$ as a function of the delay $\Delta x$. The dip-peak graph (88% average visibility) for $H$ polarized photons, corresponding to the k-entangled state $|\psi^+⟩$, is shown in Fig. 3a).
and lower cases refer to momentum operators \( \pi \).

The experimental setups for measuring the \( \pi \)-entangled states are shown in Fig. 3. By removing the BS (Fig. 3a)), we performed a quantum tomographic analysis on the mode sets \( r_A - r_B \) and \( r_A + r_B \), corresponding to the \( \pi \)-entangled states \( |\Phi^+\rangle \) (Fig. 3a)) and \( |\Phi^-\rangle \) (Fig. 3b)) respectively. The tomographic reconstructions were obtained by the “Maximum Likelihood Estimation” method described in 17. The corresponding fidelities are \( F_{|\Phi^+\rangle} = 0.9068\pm 0.0035 \) and \( F_{|\Phi^-\rangle} = 0.9314\pm 0.0032 \). Note that the path interference measurement shown in Fig. 3b demonstrates the quantum superposition between the two states \( |\Phi^+\rangle |r\rangle_A |\ell\rangle_B \) and \( |\Phi^-\rangle |\ell\rangle_A |r\rangle_B \) of Fig. 3a), leading to the linear cluster state 11.

The genuine multipartite 4-qubit entanglement was verified by measuring the entanglement witness 11

\[
W = \frac{1}{2} \left[ 41 - Z_A Z_B - Z_A x_A x_B + X_A z_A X_B \right. \\
\left. + z_A z_B - x_A Z_B x_B - X_A X_B z_B \right]
\]

where upper cases refer to polarization operators

\[
\begin{align*}
Z_i &= |H\rangle_i |H\rangle_i - |V\rangle_i |V\rangle_i \\
Y_i &= i |V\rangle_i |H\rangle_i - i |H\rangle_i |V\rangle_i \\
X_i &= |H\rangle_i |V\rangle_i + |V\rangle_i |H\rangle_i
\end{align*}
\]

and lower cases refer to momentum operators

\[
\begin{align*}
z_i &= |\ell\rangle_i |\ell\rangle_i - |r\rangle_i |r\rangle_i \\
y_i &= i |r\rangle_i |\ell\rangle_i - i |\ell\rangle_i |r\rangle_i \\
x_i &= |\ell\rangle_i |r\rangle_i + |r\rangle_i |\ell\rangle_i
\end{align*}
\]

The experimental setups for measuring the polarization observables for either Alice or Bob photon are shown in Fig. 4. Note that the eigenvectors of \( x_i \) and \( y_i \) can be written in the form \( 1/\sqrt{2} |\ell\rangle_i \pm e^{-i\phi} |r\rangle_i \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Observable} & \text{Value} & \text{W} & \text{S} & \text{C} \\
\hline
Z_A Z_B & +0.9283 \pm 0.0032 & ✓ & ✓ & ✓ \\
Z_A x_A x_B & +0.8194 \pm 0.0049 & ✓ & ✓ & ✓ \\
X_A z_A X_B & -0.9074 \pm 0.0037 & ✓ & ✓ & ✓ \\
z_A z_B & -0.9951 \pm 0.0009 & ✓ & ✓ & ✓ \\
x_A Z_B x_B & +0.8110 \pm 0.0050 & ✓ & ✓ & ✓ \\
Z_A y_A y_B & +0.8071 \pm 0.0050 & ✓ & ✓ & ✓ \\
Y_A z_A Y_B & +0.8948 \pm 0.0040 & ✓ & ✓ & ✓ \\
X_A X_B z_B & +0.9074 \pm 0.0037 & ✓ & ✓ & ✓ \\
Y_A y_B z_B & -0.8936 \pm 0.0041 & ✓ & ✓ & ✓ \\
X_A x_A Y_B y_B & +0.8177 \pm 0.0055 & ✓ & ✓ & ✓ \\
Y_A x_A X_B y_B & +0.7959 \pm 0.0055 & ✓ & ✓ & ✓ \\
\hline
\end{array}
\]

TABLE I: Experimental values of the observables used for measuring the entanglement witness \( W \) and the expectation value of \( S \) on the cluster state \( |C_4\rangle \). The last column \( C \) refers to the control measurements needed to verify that \( X_A, X_B, x_A, X_B, y_B, y_B \) and \( z_B \) can be considered as elements of reality. Each experimental value corresponds to a measure lasting an average time of 10 sec. In the experimental errors we considered the poissonian statistic and the uncertainties due to the manual setting of the polarization analysis wave plates.

Those states can be discriminated, as previously explained, by the glass plates and the BS.

The expectation value of \( W \) is positive for any separable state (for instance it is equal to 1 for the hyper-entangled state \( |\Xi\rangle \)), whereas its negative value detects 4-party entanglement close to the cluster state 11. A perfect cluster state gives \( -1 \) as expectation value.

The experimental values of the observables of eq. 3 are shown in Table I. The perfect correlations were due to the impurity of the states \( |\Phi^+\rangle \) and \( |\Phi^-\rangle \), as well as to imperfections in the polarization and momentum analysis devices. The resulting experimental value of \( W \) is

\[
\text{Tr}[W \rho_{\text{exp}}] = -0.6843 \pm 0.0094,
\]

demonstrating the genuine multipartite entanglement of our cluster state, whose \( \rho_{\text{exp}} \) represents the experimental density matrix.

From the projector-based entanglement witness 11

\[
\tilde{W} = \frac{1}{2} - |C_4\rangle \langle C_4|,
\]

could obtain information about the fidelity \( F_{|C_4\rangle} \) of the state through the equation \( F_{|C_4\rangle} = \frac{1}{2} - \text{Tr}[(W \rho_{\text{exp}})] \). As shown in 11, the following relation holds between \( W \) and \( \tilde{W} \): \( W - 2\tilde{W} \geq 0 \). Hence the lower bound of the experimental fidelity \( F_{|C_4\rangle} \) is:

\[
F_{|C_4\rangle} \geq \frac{1}{2} - \frac{1}{2} \text{Tr}[W \rho_{\text{exp}}] \geq 0.84,
\]

giving a further evidence of the cluster generation.

Finally, we tested the nonlocal character of our cluster state by using the “stronger two observer AVN” proof of...
local realism, recently introduced in [12]. It is based on the following eigenvalue equations:

\[ X_{A}z_{A}X_{B}|C_{4}\rangle = -|C_{4}\rangle \]  
\[ z_{A}z_{B}|C_{4}\rangle = -|C_{4}\rangle \]  
\[ x_{A}z_{B}x_{B}|C_{4}\rangle = +|C_{4}\rangle \]  
\[ Z_{A}Y_{A}Y_{B}|C_{4}\rangle = +|C_{4}\rangle \]  
\[ Y_{A}z_{A}Y_{B}|C_{4}\rangle = +|C_{4}\rangle \]  
\[ X_{A}X_{B}z_{B}|C_{4}\rangle = +|C_{4}\rangle \]  
\[ Y_{A}Y_{B}z_{B}|C_{4}\rangle = -|C_{4}\rangle \]  
\[ X_{A}x_{A}Y_{B}Y_{B}|C_{4}\rangle = +|C_{4}\rangle \]  
\[ Y_{A}x_{A}X_{B}Y_{B}|C_{4}\rangle = +|C_{4}\rangle \]  

(8a) 
(8b) 
(8c) 
(8d) 
(8e) 
(8f) 
(8g) 
(8h) 
(8i)

The first seven equalities demonstrate that the local observables \( X_{A}, Y_{A}, x_{A}, X_{B}, Y_{B}, y_{B} \) and \( z_{B} \) are elements of reality in the EPR sense [13]. The last four equalities are used in the AVN proof through the following quantum mechanical expectation value of the cluster state \( \Pi \):

\[
\langle S \rangle = \langle C_{4} | X_{A}X_{B}z_{B} - Y_{A}Y_{B}z_{B} + X_{A}x_{A}Y_{B}Y_{B} + Y_{A}x_{A}X_{B}Y_{B} | C_{4} \rangle = 4
\]

(9)

In any local realistic theory based on the previously defined elements of reality, the upper bound of the expected value for eq. (9) is 2.

From the experimental values given in Table II we obtain

\[
\text{Tr}[S_{\rho_{\text{exp}}}] = 3.4145 \pm 0.0095,
\]

(10)

which violates the classical bound by 148 standard deviations. Note that this result provides another enhanced discrepancies between the quantum versus classical predictions (4 versus 2) with respect to the standard CHSH inequality (\(2\sqrt{2} \) versus 2) [10].

In summary, in this letter we have presented the experimental realization of a high fidelity linear cluster state consisting of four entangled qubits by adoption of 2-photon polarization-momentum hyperentanglement within a linear optical method. The cluster state was generated by applying a CP gate between the polarization and momentum qubits of one photon of the hyperentangled state. The genuine entangled character of the cluster state was experimentally demonstrated. Its non-local behaviour was also tested by a novel AVN quantum mechanical test proposed for 2-photon linear cluster state.

Other kinds of cluster states can be easily produced by the same technique presented here. Apart for the relevance of these states for fundamental physics, two photon cluster states may be good candidates to realize important quantum information tasks because of their high purity and the relatively high generation rate. Whether or not these states may also represent a useful resource for linear optics quantum computation is as yet unclear. In fact our method could be used in probabilistic quantum computation with the advantage of high counting rates. Moreover, two \( |C_{4}\rangle \) states generated by the same laser source could be linked together by a suitable CP gate to form an 8-qubit linear cluster state.

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*Electronic address: [http://quantumoptics.phys.uniroma1.it/](http://quantumoptics.phys.uniroma1.it/)

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[18] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47, 777 (1935).
[19] The state \( |\Phi^{+}\rangle \) is equivalent to the linear cluster state \( |\Phi^{+}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle) \), generated in \( \mathcal{R} \), up to a \( \sigma_{z} \) operation on the third qubit.
[20] The cluster state \( |\Xi\rangle \) can be also written as

\[
|\Xi\rangle = \frac{1}{\sqrt{2}}(|\Phi^{+}\rangle_{A}|\Phi^{+}\rangle_{B} - |\Phi^{-}\rangle_{A}|\Phi^{-}\rangle_{B})
\]

where \( |\Phi^{-}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{A}|1\rangle_{B} - |1\rangle_{A}|0\rangle_{B}) \), then dips and peaks are flipped for \( V \) photons.