DISCRETE COCOMPACT SUBGROUPS OF $G_{5,3}$
AND RELATED $C^*$-ALGEBRAS

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Abstract. The discrete cocompact subgroups of the 5-dimensional Lie group $G_{5,3}$ are determined up to isomorphism. Each of their group $C^*$-algebras is studied by determining all of its simple infinite dimensional quotient $C^*$-algebras. The $K$-groups and trace invariants of the latter are also obtained.

§1. Introduction.
Consider the Lie group $G_{5,3}$ equal to $\mathbb{R}^5$ as a set with multiplication given by

$$(h, j, k, m, n)(h', j', k', m', n') = (h + h' + nj' + m'n(n - 1)/2 + mk', j + j' + nm', k + k', m + m', n + n').$$

and inverse

$$(h, j, k, m, n)^{-1} = (-h + nj + mk - mn(n - 1)/2, -j + nm, -k, -m, -n).$$

The group $G_{5,3}$ is one of only six nilpotent, connected, simply connected, 5-dimensional Lie groups; it seemed the most tractable of them for our present purposes. (Our notation is as in Nielsen [7], where a detailed catalogue of Lie groups like this one is given.) In [5, Section 3] the authors have studied a natural discrete cocompact subgroup $H_{5,3}$, the lattice subgroup $H_{5,3} = \mathbb{Z}^5 \subset G_{5,3}$. In section 2 of this paper we study the group $G_{5,3}$ more closely, determining the isomorphism classes of all its discrete cocompact subgroups (Theorem 1). These are given by five integer parameters $\alpha, \beta, \gamma, \delta, \epsilon$ that satisfy certain conditions (see (*) and (**) of Theorem 1), and are denoted by $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$. It is shown that each such subgroup is isomorphic to a cofinite subgroup of $H_{5,3} = H_{5,3}(1, 0, 1, 1, 0)$. Conversely, each cofinite subgroup of $H_{5,3} \subset G_{5,3}$ is a discrete cocompact subgroup of $G_{5,3}$. In sections 3 and 4 the group $C^*$-algebras of the $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$’s are examined by obtaining their simple infinite dimensional quotients. These are shown to be crossed products of certain types of Heisenberg $C^*$-algebras (in Packer’s terminology [10]) and the rest are matrix algebras over irrational rotation algebras (Theorem 5). In section 5 the $K$-groups of the simple quotients are calculated (Theorem 6) as are their trace invariants (Theorem 8). The paper ends with a discussion of the classification of the simple quotients.

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We use the conventional notation for crossed products as in, for example, [11] or [16]. Hence, if a discrete group $G$ acts on a $C^*$-algebra $A$, we write $C^*(A,G)$ to denote the associated $C^*$-crossed product algebra. We use a similar notation for twisted crossed products, i.e. when there is a cocycle instead of an action (as in Theorem 2). (See the Preliminaries of [5] for more details.)

§2. Determination of the Discrete Cocompact Subgroups.

1. **Theorem.** Every discrete cocompact subgroup $H$ of $G_{5,3}$ has the following form: there are integers $\alpha, \beta, \gamma, \delta$ and $\epsilon$ satisfying $\alpha, \gamma, \delta > 0$, and

\[
0 \leq \epsilon \leq \gcd\{\gamma, \delta\}/2 \quad \text{and}
\]

\[
0 \leq \beta \leq \gcd\{\alpha, \gamma, \delta, \epsilon\}/2,
\]

yielding $H \cong H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ ( = $\mathbb{Z}^5$ as a set) with multiplication

\[
(h, j, k, m, n)(h', j', k', m', n') =
\]

\[
\begin{cases}
(h + h' + \gamma n j' + \alpha \gamma m' n (n-1)/2 + \beta n m' + \delta m k' + \epsilon n k', \\
j + j' + \alpha m m', k + k', m + m', n + n').
\end{cases}
\]

Different choices for $\alpha, \beta, \gamma, \delta$ and $\epsilon$ give non-isomorphic groups. Each such group is, in fact, isomorphic to a cofinite subgroup of $H_{5,3}$ (the lattice subgroup of $G_{5,3}$), and each cofinite subgroup of $H_{5,3}$ is isomorphic to some $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$.

**PROOF.** Using the discreteness and cocompactness as in [6], the second commutator subgroup of $H$ tells us that there is a member (with entries that don’t need to be identified indicated by *)

$e_5 = (\ast, \ast, \ast, a, z)$

of $H$, where $z > 0$ is the smallest positive number that can appear as the last coordinate of a member of $H$. Continuing in this vein, we get

$e_4 = (\ast, \ast, \ast, y, 0)$,

$e_3 = (\ast, b, x, 0, 0)$,

$e_2 = (\ast, w, 0, 0, 0)$ and

$e_1 = (v, 0, 0, 0, 0)$,

where $x > 0$ is the smallest positive number that can appear as the 3rd coordinate of a member of $H$ whose last 2 coordinates are 0, and similarly for $v, w$ and $y$. Also, all other coordinates are $\geq 0$, and the bottom non-zero coordinate in each column is greater than the coordinates above it, e.g., $w > b > 0$ and $w$ is also greater than the 2nd coordinate of $e_5$ or of $e_4$. These considerations show that the map

$\pi : (h, j, k, m, n) \mapsto e_1^h e_2^j e_3^k e_4^m e_5^n, \ \mathbb{Z}^5 \rightarrow H,$

is 1 − 1 and onto. We want the multiplication (m) for $\mathbb{Z}^5$ that makes $\pi$ a homomorphism (hence an isomorphism); (m) is determined using the commutators,

\[
\begin{aligned}
[e_5, e_4] &= (\ast, zy, 0, 0, 0) = e_1^\beta e_2^\alpha, \quad [e_5, e_3] = (zb + xa, 0, 0, 0, 0) = e_1^\epsilon, \\
[e_5, e_2] &= (zw, 0, 0, 0) = e_1^\gamma, \quad \text{and} \quad [e_4, e_3] = (xy, 0, 0, 0) = e_1^\delta,
\end{aligned}
\]
for some integers \( \alpha, \beta, \gamma, \delta, \epsilon \) (other pairs of \( e \)'s commuting). Using the commutators to collect terms in
\[
(e_1^b e_2^j e_3^k e_4^m e_5^n)(e_1^{b'} e_2^{j'} e_3^{k'} e_4^{m'} e_5^{n'})
\]
gives the multiplication formula \((m)\) for \( \mathbb{Z}^5 \), and also the equation
\[
e_5^n e_4^{m'} = e_1^{\alpha \gamma m'n(n-1)/2+\beta nm'} e_2^{\alpha m'n} e_4^{m'} e_5^n,
\]
which the reader may find helpful in checking computations later.

For a start in putting the restrictions on \( \alpha, \beta, \gamma, \delta, \epsilon \), \((C)\) tells us that \( \alpha, \gamma, \delta > 0 \) (since \( v, w, x, y \) and \( z > 0 \)). Let \( Z \) denote the center of \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \), \( Z = (\mathbb{Z}, 0, 0, 0, 0) \).
Then, as for \( G_4 \), with quotients and subgroups it is shown that different (positive) \( \alpha, \gamma, \delta \) give non-isomorphic groups, e.g., \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)/Z \) gives \( \alpha \), then \( Z \) modulo the second commutator subgroup gives \( \gamma \), and also, with \( K_3, K_4 \subset H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \) as defined below,
\[
Z \supset (\delta \mathbb{Z}, 0, 0, 0, 0) = \{ xy^{-1} y^{-1} \mid x \in K_3, \ y \in K_4 \}
\]
and \( \mathbb{Z}/(\delta \mathbb{Z}, 0, 0, 0, 0) = \mathbb{Z}_\delta \), the cyclic group of order \( \delta \).

Then we have an isomorphism of \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \) onto \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon + d\gamma + e\delta) \), which is simpler to give in terms of generators,
\[
(\otimes) \quad e_3 \mapsto e_3', \ e_5 \mapsto e_5' = e_4^e e_5, \text{ and } e_i \mapsto e_i' = e_i \text{ otherwise}.
\]
Here we are merely changing the basis for \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \), and the only commutator (using \((m)\) and \((C)\)) that changes is \([e_5', e_3'] = e_1^{\epsilon + \delta + d\gamma} \), so the resulting isomorphism is of \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \) onto \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon + d\gamma + e\delta) \), which shows we can require
\[
0 \leq \epsilon < \gcd \{ \gamma, \delta \}.
\]
This, accompanied by another isomorphism,
\[
(\otimes') \quad (h, j, k, m, n) \mapsto (-h, -j, k, -m, n), \quad H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \rightarrow H_{5,3}(\alpha, \beta, \gamma, \delta, -\epsilon),
\]
assures that we can have
\[
(\ast) \quad 0 \leq \epsilon \leq \gcd \{ \gamma, \delta \}/2,
\]
the required range for \( \epsilon \).

Now, to control \( \beta \),
\[
(\dagger) \quad \begin{cases} e_1 \mapsto e_1' = e_1', & e_2 \mapsto e_1^{-q} e_2 = e_2', \ e_3 \mapsto e_3 = e_3', \\ e_4 \mapsto e_4^r e_3^q e_4 \text{ and } e_5 \mapsto e_3^{-f} e_5 = e_5' & \end{cases}
\]
is an isomorphism of \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \) onto \( H_{5,3}(\alpha, \beta + q\alpha + r\gamma + f\delta + g\epsilon, \gamma, \delta, \epsilon) \), which yields
\[
0 \leq \beta < \gcd\{\alpha, \gamma, \delta, \epsilon\}.
\]
Then the isomorphism

\[(\phi') \quad (h, j, k, m, n) \mapsto (-h, j, k, -m, -n)\]

of $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ onto $H_{5,3}(\alpha, -\beta + \alpha \gamma, \gamma, \delta, \epsilon)$ leads to the conclusion

\[(**): \quad 0 \leq \beta \leq \gcd \{\alpha, \gamma, \delta, \epsilon\}/2.\]

It must still be shown that changing $\epsilon$ or $\beta$ within the allowed limits (namely, $\epsilon$ and $\beta$ must satisfy $(*)$ and $(**)$, respectively) gives a non-isomorphic group.

So, suppose that $\varphi : H = H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \to H_{5,3}(\alpha, \beta', \gamma, \delta, \epsilon') = H'$ is an isomorphism. Then

\[\varphi : Z = K_1 = (\mathbb{Z}, 0, 0, 0, 0) \to (\mathbb{Z}, 0, 0, 0, 0) = K'_1 = Z',\]

\[K_2 = (\mathbb{Z}, \mathbb{Z}, 0, 0, 0) \to (\mathbb{Z}, \mathbb{Z}, 0, 0, 0) = K'_2,\]

\[K_3 = (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, 0, 0) \to (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, 0, 0) = K'_3,\]

and

\[K_4 = (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}, 0) \to (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}, 0) = K'_4,\]

since the $Z'$s are the centers, the $K_2$'s consist of those $s \in H$ for which $s^r$ is in the commutator subgroup of $H$ for some $r \in \mathbb{Z}$, the $K_3$'s are the largest subsets for which all commutators are central (e.g., $xyx^{-1}y^{-1} \in Z$ for all $x \in K_3$ and $y \in H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$), and the $K_4$'s are the centralizers of the commutator subgroups. So we must have

\[\varphi(0, 0, 0, 0, 1) = (*, *, -f, e, a) = S_5 \text{ with } a = \pm 1,\]

\[\varphi(0, 0, 0, 1, 0) = (*, r, g, b, 0) = S_4 \text{ with } b = \pm 1, \text{ and}\]

\[\varphi(0, 0, 1, 0, 0) = (*, d, c, 0, 0) = S_3 \text{ with } c = \pm 1;\]

furthermore, commutators give

\[\varphi(\beta, \alpha, 0, 0, 0) = [S_5, S_4] = S_5S_4S_5^{-1}S_4^{-1} = (*, \alpha ab, 0, 0, 0),\]

hence $\varphi(0, 1, 0, 0, 0) = (q, ab, 0, 0, 0) = S_2$, and

\[\varphi(\gamma, 0, 0, 0, 0) = [S_5, S_2] = (\gamma a^2 b, 0, 0, 0, 0),\]

so $\varphi(1, 0, 0, 0, 0) = (b, 0, 0, 0, 0) = S_1$, but also

\[\varphi(\delta, 0, 0, 0, 0) = [S_4, S_3] = (\delta bc, 0, 0, 0, 0),\]

so $c = 1$. Furthermore, $\varphi(\epsilon, 0, 0, 0, 0) = [S_5, S_3] = (ae + e\delta + ad \gamma, 0, 0, 0, 0)$, which shows that the manipulations at $(\otimes)$ and $(\otimes')$ above give the only way of changing $\epsilon$ in $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$; that is, if

\[(*) \quad 0 \leq \epsilon, \epsilon' \leq \gcd \{\gamma, \delta\}/2\]
and $\epsilon = \pm \epsilon' + a_1 \delta + a_2 \gamma$ with $a_1, a_2 \in \mathbb{Z}$, then $\epsilon = \epsilon'$. Now consider
\[
\varphi(h, j, k, m, n) = \varphi((h, 0, 0, 0, 0)(0, j, 0, 0, 0)(0, k, 0, 0, 0)(0, 0, 0, m, 0)(0, 0, 0, 0, n))
= (hS_1) \cdot (jS_2) \cdot (kS_3) \cdot S_4^m \cdot S_5^n = hS_1 + jS_2 + kS_3 + S_4^m S_5^n \in H'.
\]
Note that $S_5^n \neq nS_5$, but $S_5^n = (\ast, \ast, -nf, ne, na)$, and also $S_4^m = (\ast, mr, mg, mb, 0)$; further, the $(jS_2)$ term puts a $jq$ in the first entry of $\varphi(h, j, k, m, n)$, so also $(j + j' + \alpha nm')q$ in the first entry of $\varphi(h, j, k, m, n) \cdot \varphi(h', j', k', m', n')$ (product in $H_{5,3}(\alpha, \beta', \gamma, \delta, \epsilon)$). Then, equating the coefficients of the $nm'$ terms in the first entry of
\[
\varphi(e_5^n e_4^{m'}) \quad \text{and} \quad \varphi(e_4^n e_3^{m'}) = S_5^n S_4^{m'}
\]
gives
\[
b(-\alpha \gamma / 2 + \beta) + qa = ab\beta' - ab\alpha \gamma / 2 + a\gamma \epsilon + ar\gamma + (eg + bf)\delta, \quad \text{or}
\]
\[
\beta = \pm \beta' + a_1 \alpha + a_2 \gamma + a_3 \delta + a_4 \epsilon \quad \text{for some} \quad a_i \in \mathbb{Z}, \quad 1 \leq i \leq 4,
\]
which shows that the manipulations at \((\dagger)\) and \((\dagger')\) above give the only way of changing just $\beta$ in $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$.

Here is an isomorphism $\varphi$ of $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ into the lattice subgroup $H_{5,3} = \mathbb{Z}^5 \subset G_{5,3}$ in terms of generators; $H_{5,3}$ has multiplication
\[
(h, j, k, m, n)(h', j', k', m', n') = \begin{cases}
(h + h' + nj + m'n(n - 1)/2 + mk', \\
\quad j + j' + nm'n, k + k', m + m', n + n')
\end{cases}
\]
(i.e., $\alpha = \gamma = \delta = 1$ and $\beta = \epsilon = 0$). First suppose $\epsilon > 0$. Then, with $\vartheta = \alpha \gamma \epsilon$ and generators
\[
e_1 = (1, 0, 0, 0, 0), \quad e_2 = (0, 1, 0, 0, 0), \quad \ldots, \quad e_5 = (0, 0, 0, 0, 1)
\]
for $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ satisfying
\[
(C) \quad [e_5, e_4] = e_4^\delta e_2^\alpha, \quad [e_5, e_3] = e_1^\epsilon, \quad [e_5, e_2] = e_1^\gamma, \quad \text{and} \quad [e_4, e_3] = e_1^\delta,
\]
$\varphi$ is given by
\[
\varphi : e_1 \mapsto e_1' = (\delta \vartheta^2, 0, 0, 0, 0), \quad e_2 \mapsto e_2' = (\gamma \delta \vartheta (\vartheta - 1)/2, \gamma \delta \vartheta, 0, 0, 0),
\]
\[
e_3 \mapsto e_3' = (0, \delta \vartheta, \delta \epsilon \vartheta, 0, 0), \quad e_4 \mapsto e_4' = (0, \beta \delta \vartheta, 0, \alpha \gamma \delta, 0),
\]
and $e_5 \mapsto e_5' = (0, 0, 0, 0, 0)$. That $\varphi$ is an isomorphism is verified by showing that $\{e_1', e_2', e_3', e_4', e_5'\} \subset H_{5,3}$ satisfy $(C)$. (Here $\varphi$ is given by
\[
(h, j, k, m, n) \mapsto (\delta \vartheta^2 h + (\gamma \delta \vartheta (\vartheta - 1)/2) j, \gamma \delta \vartheta j + \delta \epsilon \vartheta k + \beta \delta \vartheta m, \delta \epsilon \vartheta k, \alpha \gamma \delta m, \vartheta n).
\]
When $\epsilon = 0$, use $\vartheta = \alpha \gamma$ and $e_3' = (0, 0, \delta \vartheta, 0, 0)$. 
It is easy to see that the image \( H_1 = \varphi(H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)) \) is cofinite in \( H_{5,3} \). Consider the coset \( sH_1 \) for \( s = (h, j, k, m, n) \in H_{5,3} \); since \( e'_5 = (0, 0, 0, 0, 0) \), we can choose \( r_5 \in \mathbb{Z} \) so that \( se'_5r_5 \) has its last coordinate in \([0, \delta]\). Then choose \( r_4 \in \mathbb{Z} \) so that \( se'_5r_5e'_4r_4 \) has its second last coordinate in \([0, \alpha \gamma \delta]\). Continuing like this, we arrive at

\[
s e'_5r_5 e'_4r_4 e'_3r_3 e'_2r_2 e'_1 r_1 \in K = \left([0, \delta^2] \times [0, \alpha \gamma \delta]\right) \cap \mathbb{Z}^5 \subset H_{5,3}
\]

so every coset \( sH_1, s \in H_{5,3} \), has a representative in \( K \), which is a finite set. It follows that the quotient map \( H_{5,3} \to H_{5,3}/H_1 \) maps \( K \) onto \( H_{5,3}/H_1 \), which is therefore finite. (A similar argument shows that \( G_{5,3}/H_1 \) is cocompact.)

Finally, note that since any cofinite subgroup of \( H_{5,3} \) is also a discrete cocompact subgroup of \( G_{5,3} \), it must therefore be isomorphic to some \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \). This completes the proof. \( \square \)

**REMARKS.**

1. The image \( H_1 = \varphi(H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)) \) above is not a normal subgroup of \( H_{5,3} \), e.g.,

\[
(0, 0, 1, 0, 0)e'_5(0, 0, -1, 0, 0) = (\delta, 0, 0, 0, 0) \notin H_1.
\]

This makes it seem unlikely that \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \) can be embedded in \( H_{5,3} \) as a normal subgroup; however, the existence of such an embedding is still a possiblity.

2. The theorem gives an isomorphism \( \varphi \) of \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \) into \( H_{5,3} \); conversely, there is always an isomorphism \( \varphi' \) of \( H_{5,3} \) into \( H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \), and as for \( \varphi \), it is easier to give \( \varphi' \) in terms of the generators \( e_i, 1 \leq i \leq 5 \), of \( H_{5,3} \), which satisfy

\[
(C') \quad [e_5, e_4] = e_2, \quad [e_5, e_2] = e_1 = [e_4, e_3].
\]

Then

\[
\varphi' : e_1 \mapsto e'_1 = (\alpha \gamma^2 \delta^2, 0, 0, 0, 0), \quad e_2 \mapsto e'_2 = (\alpha \gamma^2 \delta(\delta - 1)/2, \alpha \gamma \delta, 0, 0, 0),
\]

\[
e_3 \mapsto e'_3 = (0, -\alpha \delta \epsilon, \alpha \delta \gamma, 0, 0), \quad e_4 \mapsto e'_4 = (0, -\beta, 0, \gamma, 0), \quad \text{and} \quad e_5 \mapsto e'_5 = (0, 0, 0, 0, \delta).
\]

That \( \varphi' \) is an isomorphism is verified by showing that \( \{e'_1, e'_2, e'_3, e'_4, e'_5\} \subset H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \) satisfy \( (C') \). (Here \( \varphi' \) is given by

\[
(h, j, k, m, n) \mapsto (\alpha \gamma^2 \delta^2 h + j \alpha \gamma^2 \delta(\delta - 1)/2, \alpha \gamma \delta j - \alpha \delta \epsilon k - \beta m, \alpha \gamma \delta k, \gamma m, \delta n).
\]

So, as for the 3-dimensional groups \( H_3(p) \) and the 4-dimensional groups \( H_4(p_1, p_2, p_3) \), here we have an infinite family of non-isomorphic groups, each of which is isomorphic to a subgroup of any other one.
§3. Infinite Dimensional Simple Quotients of $C^*(H_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon))$.

We begin by obtaining concrete representations on $L^2(\mathbb{T}^2)$ of the faithful simple quotients (i.e., those arising from a faithful representation of $H_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$), and consider first the case $\epsilon = 0$. In this case $H_{5,3}(\alpha,\beta,\gamma,\delta,0)$ has an abelian normal subgroup $N = (\mathbb{Z},\mathbb{Z},0,\mathbb{Z},0)$, with quotient

$$H_{5,3}(\alpha,\beta,\gamma,\delta,0)/N \cong (0,0,\mathbb{Z},0,\mathbb{Z}) = \mathbb{Z}^2,$$

also abelian and embedded in $H_{5,3}(\alpha,\beta,\gamma,\delta,0)$ as a subgroup, so that $H_{5,3}(\alpha,\beta,\gamma,\delta,0)$ is isomorphic to a semidirect product $N \times \mathbb{Z}^2$; in this situation, the simple quotients of $C^*(H_{5,3}(\alpha,\beta,\gamma,\delta,0))$ can be presented as $C^*$-crossed products using flows from commuting homeomorphisms, as follows.

Let $\lambda = e^{2\pi i \theta}$ for an irrational $\theta$, and consider the flow $\mathcal{F}' = (\mathbb{Z}^2, \mathbb{T}^2)$ generated by the commuting homeomorphisms

$$\psi_1' : (w,v) \mapsto (\lambda^\gamma w, \lambda^\beta w^\alpha v) \quad \text{and} \quad \psi_2' : (w,v) \mapsto (w, \lambda^{-\delta} v).$$

$\mathcal{F}'$ is minimal, so the $C^*$-crossed product $C' = C^*(C(\mathbb{T}^2), \mathbb{Z}^2)$ is simple [1, Corollary 5.16].

Let $v$ and $w$ denote (as well as members of $\mathbb{T}$) the functions in $C(\mathbb{T}^2)$ defined by

$$(w,v) \mapsto v \text{ and } w,$$

respectively. Define unitaries $U$, $V$, $W$ and $X$ on $L^2(\mathbb{T}^2)$ by

$$(U') \quad U : f \mapsto f \circ \psi_1', \quad V : f \mapsto vf, \quad W : f \mapsto f \circ \psi_2' \quad \text{and} \quad X : f \mapsto wf.$$  

These unitaries satisfy

$$(CR') \quad UV = \lambda^\beta X^\alpha VU, \quad UX = \lambda^\gamma XU, \quad \text{and} \quad VW = \lambda^\delta WV$$

(other pairs of unitaries commuting), equations which ensure that

$$\pi : (h,j,k,m,n) \mapsto \lambda^h X^j W^k V^m U^n$$

is a representation of $H_{5,3}(\alpha,\beta,\gamma,\delta,0)$. Denote by $A_{\theta}^{5,3}(\alpha,\beta,\gamma,\delta,0)$ the $C^*$-subalgebra of $B(L^2(\mathbb{T}^2))$ generated by $\pi$, i.e., by $U, V, W$ and $X$. Since $A_{\theta}^{5,3}(\alpha,\beta,\gamma,\delta,0)$ is generated by a representation of $H_{5,3}(\alpha,\beta,\gamma,\delta,0)$, it is a quotient of the group $C^*$-algebra $C^*(H_{5,3}(\alpha,\beta,\gamma,\delta,0))$. It follows readily that $A_{\theta}^{5,3}(\alpha,\beta,\gamma,\delta,0)$ is isomorphic to the simple $C^*$-crossed product $C'$ above, and hence is simple.

However, when $0 < \epsilon \leq \gcd \{\gamma,\delta\}/2$ (which implies $\gamma > 1$, by $\ast$), $H_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ is only an extension $(\mathbb{Z},\mathbb{Z},0,\mathbb{Z},0) \times (0,0,\mathbb{Z},0,\mathbb{Z}) = N \times \mathbb{Z}^2$, and not a semidirect product. Nonetheless, we can modify the flow $\mathcal{F}'$ representing $A_{\theta}^{5,3}(\alpha,\beta,\gamma,\delta,0)$ above to get a concrete representation of $A_{\theta}^{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$. Consider the flow $\mathcal{F} = (\mathbb{Z}^2, \mathbb{T}^2)$ generated by the commuting homeomorphisms

$$\psi_1 : (w,v) \mapsto (\lambda w, \lambda^\beta w^\alpha v) \quad \text{and} \quad \psi_2 : (w,v) \mapsto (w, \lambda^{-\delta} v).$$
$\mathcal{F}$ is minimal, so the $C^*$-crossed product $\mathcal{C} = C^*(\mathbb{C}(\mathbb{T}^2), \mathbb{Z}^2)$ is simple. Define unitaries on $L^2(\mathbb{T}^2)$ by

$$(U) \quad U : f \mapsto f \circ \psi_1, \quad V : f \mapsto \psi f, \quad W : f \mapsto w^f \circ \psi_2 \text{ and } X : f \mapsto w^\gamma f.$$  

These unitaries satisfy

$$(CR) \quad UV = \lambda^\beta X^\alpha U, \quad UX = \lambda^\gamma XU, \quad WV = \lambda^\delta WV \text{ and } UW = \lambda^\epsilon WU,$$

equations which ensure that

$$\pi : (h, j, k, m, n) \mapsto \lambda^h X^j W^k V^m U^n$$

is a representation of $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$. Denote by $A^5,3_{\theta}(\alpha, \beta, \gamma, \delta, \epsilon)$ the $C^*$-subalgebra of $B(L^2(\mathbb{T}^2))$ generated by $\pi$. Now $A^5,3_{\theta}(\alpha, \beta, \gamma, \delta, \epsilon)$ is isomorphic only to a subalgebra of $\mathcal{C}$ (as may be shown using conditional expectations); a unitary that is missing is $X' : f \mapsto w f$ (since $\gamma > 1$).

**NOTE.** The reason we did not use $\mathcal{F}$ when $\epsilon = 0$ (and $\gamma > 1$) is that $A^5,3_{\theta}(\alpha, \beta, \gamma, \delta, 0)$ seems to be isomorphic only to a subalgebra of $\mathcal{C}$ in that case too, whereas with $\mathcal{F}'$, $A^5,3_{\theta}(\alpha, \beta, \gamma, \delta, 0) \cong \mathcal{C}'$.

Since the flow method can no longer be used to prove the simplicity of the algebra $A^5,3_{\theta}(\alpha, \beta, \gamma, \delta, \epsilon)$ (when $0 < \epsilon \leq \gcd \{\gamma, \delta\}/2$), we use the strong result of Packer [9].

2. **Theorem.** Let $\lambda = e^{2\pi i \theta}$ for an irrational $\theta$.

(a) There is a unique (up to isomorphism) simple $C^*$-algebra $A^5,3_{\theta}(\alpha, \beta, \gamma, \delta, \epsilon)$ generated by unitaries $U$, $V$, $W$ and $X$ satisfying

$$(CR) \quad UV = \lambda^\beta X^\alpha U, \quad UX = \lambda^\gamma XU, \quad WV = \lambda^\delta WV \text{ and } UW = \lambda^\epsilon WU,$$

Furthermore, for a suitable $\mathbb{C}$-valued cocycle on $H_3(\alpha) \times \mathbb{Z}$,

$$A^5,3_{\theta}(\alpha, \beta, \gamma, \delta, \epsilon) \cong C^*(\mathbb{C}, H_3(\alpha) \times \mathbb{Z}).$$

(b) Let $\pi'$ be a representation of $H'_{5,3} = H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ such that $\pi = \pi'$ (as scalars) on the center $(\mathbb{Z}, 0, 0, 0, 0)$ of $H'_{5,3}$, and let $A$ be the $C^*$-algebra generated by $\pi'$. Then $A \cong A^5,3_{\theta}(\alpha, \beta, \gamma, \delta, \epsilon) = A^{5,3}_{\theta}$ (say) via a unique isomorphism $\omega$ such that the following diagram commutes.

$$\begin{array}{ccc}
H'_{5,3} & \xrightarrow{\pi} & A_{6,3}^{5,3} \\
\pi' \downarrow & \searrow & \downarrow \omega \\
A & & A
\end{array}$$

**PROOF.** To use Packer's result, we regard $H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ as an extension

$$H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \cong \mathbb{Z} \times (0, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \times (H_3(\alpha) \times \mathbb{Z})$$

(with $H_3(\alpha) \cong (0, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}) \subset H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$); this extension has cocycle

$$[s, s'] = [(j, k, m, n), (j', k', m', n')] = \lambda^{mj' + \alpha mn(n-1)/2 + \beta nm' + \delta mk' + \epsilon nk'},$$
(H_3(\alpha) \times \mathbb{Z}, H_3(\alpha) \times \mathbb{Z}) \to \mathbb{T}.

The application of Packer’s result requires the consideration of the related function
\[ \chi^{s'}(s) = [s', s][s, s^{-1}s'] \quad s, s' \in (0, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}) \cong H_3(\alpha) \times \mathbb{Z}. \]

It must be shown that \( \chi^{s'} \) is non-trivial on the centralizer of \( s' \) in \( H_3(\alpha) \times \mathbb{Z} \) if \( s' \) has finite conjugacy class in \( H_3(\alpha) \times \mathbb{Z} \); this is easy because the only elements of \( H_3(\alpha) \times \mathbb{Z} \) that have finite conjugacy class are in the center \( Z_3 = (\mathbb{Z}, \mathbb{Z}, 0, 0) \) of \( H_3(\alpha) \times \mathbb{Z} \), so their centralizer is all of \( H_3(\alpha) \times \mathbb{Z} \). Thus the \( C^* \)-crossed product \( C^*(\mathbb{C}, H_3(\alpha) \times \mathbb{Z}) \) is simple; it is isomorphic to \( A_{5,3}^3(\alpha, \beta, \gamma, \delta, \epsilon) \) because, with basis members
\[ e_1 = (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0) \quad \text{and} \quad e_4 = (0, 0, 0, 1) \]

for \( H_3(\alpha) \times \mathbb{Z} \), the unitaries
\[ U' = \delta_{e_4}, \quad V' = \delta_{e_4}, \quad W' = \delta_{e_2} \quad \text{and} \quad X' = \delta_{e_1} \]
in \( \ell_1(H_3(\alpha) \times \mathbb{Z}) \subset C^*(\mathbb{C}, H_3(\alpha) \times \mathbb{Z}) \) satisfy (CR). \( \Box \)

The theorem showed \( A_{5,3}^3(\alpha, \beta, \gamma, \delta, \epsilon) \) was simple by showing it was isomorphic to the simple \( C^* \)-crossed product \( C^*(\mathbb{C}, H_3(\alpha) \times \mathbb{Z}) \). It follows that \( A_{5,3}^3(\alpha, \beta, \gamma, \delta, \epsilon) \) is isomorphic to a number of other \( C^* \)-crossed products (much as in [3; Theorem 3]), one of which has been derived from a flow at the beginning of the section for the case \( \epsilon = 0 \). Here are 2 other \( C^* \)-crossed products that can be used to establish the simplicity of \( A_{5,3}^3(\alpha, \beta, \gamma, \delta, \epsilon) \). The variable change for the second one will be used again, in the proof of Theorem 3 below.

1. Take the \( C^* \)-algebra \( B \) generated by \( U, V \) and \( X \) from (CR), satisfying
\[ UV = \lambda^2 X^\alpha VU, \quad UX = \lambda^\gamma UX, \quad \text{and} \quad VX = XV. \]
The algebra \( B \) is a faithful simple quotient of a discrete cocompact subgroup \( H_4(\beta, \alpha, \gamma) \) of \( H_4 \), the connected 4-dimensional nilpotent group [6; Theorem 2]. Then the rest of (CR) gives an action of \( \mathbb{Z} \) on \( B \) generated by \( \text{Ad}_W \); it follows that \( A_{5,3}^3(\alpha, \beta, \gamma, \delta, \epsilon) \cong C^*(B, \mathbb{Z}) \). The simplicity of \( C^*(B, \mathbb{Z}) \) can be proved directly by showing that \( (\text{Ad}_W)^r = \text{Ad}_{W^r} \) is outer on \( B \) if \( r \neq 0 \) [8].

2. First, we change the variables in (CR). Pick relatively prime integers \( c, d \) such that \( d\delta + ac = 0 \) and let \( a, b \) be integers such that \( ad - bc = 1 \). Put
\[ U' = U^aV^b \quad \text{and} \quad V' = U^cV^d. \]

Then keeping \( X \) and \( W \) the same, (CR) becomes
\[(\text{CR}') \begin{cases} U'V' = \lambda^{\beta'} X^\alpha V'U', \quad U'X = \lambda^{\gamma} Xu', \\ U'W = \lambda^{\delta'} Wu', \quad \text{and} \quad V'X = \lambda^{\epsilon'} XV' \end{cases} \]

(other pairs of unitaries commuting) for some integer \( \beta' \) and \( \delta' = b\delta + ac \). Note that \( A_{5,3}^3(\alpha, \beta, \gamma, \delta, \epsilon) \) is generated by \( U', V', W, X \) (since \( ad - bc = 1 \)). Consider the \( C^* \)-algebra \( B' = A_{5,3}^3 \otimes \mathcal{C}(T) \) generated by unitaries \( V', W \) and \( X_1 \) satisfying
\[ V'X = \lambda^{\gamma} XV', \quad V'W = WV' \quad \text{and} \quad WX_1 = X_1W; \]
here \( e^{2\pi i \theta'} = \lambda^{\gamma} \). Then the rest of (CR') gives an action of \( \mathbb{Z} \) on \( B' \) generated by \( \text{Ad}_{W'} \), and it follows that \( C^*(B', \mathbb{Z}) \cong A_{5,3}^3(\alpha, \beta, \gamma, \delta, \epsilon) \). One can prove the simplicity of \( C^*(B', \mathbb{Z}) \) directly; the method of proof is to show that the Connes spectrum of \( \text{Ad}_{U'} \) is \( \mathbb{T} \), which follows from Theorem 2 and [11; 8.11.12].
§4. Other Simple Quotients of \( C^*(H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)) \).

Now assume that \( \lambda \) is a primitive \( q \)th root of unity and that \( U, V, W \) and \( X \) are unitaries generating a simple quotient \( A \) of \( C^*(H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)) \), i.e., they satisfy

\[
(UV = \lambda^\beta X^\alpha VU, \quad UX = \lambda^\gamma XU, \quad VW = \lambda^\delta WV \quad \text{and} \quad UW = \lambda^\epsilon UW).
\]

We may assume that \( A \) is irreducibly represented. Then, if

\[
\begin{align*}
q_1 & \text{ is the order of } \lambda^\gamma \text{ and} \\
q_2 & \text{ is the lcm of the orders of } \lambda^\delta \text{ and } \lambda^\epsilon,
\end{align*}
\]

\( W^{q_2} \) and \( X^{q_1} \) are scalar multiples of the identity (by irreducibility). Since \( W \) can be multiplied by a scalar without changing (CR), we may assume \( W^{q_2} = 1 \). However, \( X^{q_1} = \mu' \), a multiple of the identity. Put \( X = \mu X_1 \) for \( \mu^{q_1} = \mu' \), so that \( X_1^{q_1} = 1 \), and substitute \( X = \mu X_1 \) in (CR) to get

\[
(UV = \lambda^\beta \mu^\alpha X_1^{q_1} VU, \quad UX_1 = \lambda^\gamma X_1 U, \quad V'X_1^{q_1} = 1, \quad UV' = \lambda^\gamma X_1 U' \quad \text{and} \quad W^{q_2} = 1 = X_1^{q_1}.
\]

1. If \( \mu \) is also a root of unity, then (CR) (along with irreducibility) shows that \( U \) and \( V \), as well as \( W \) and \( X \), are (multiples of) unipotent unitaries, so \( A \) is finite dimensional.

2. If \( \mu \) is not a root of unity, the flow \( F = (\mathbb{Z}^2, \mathbb{T}^2) \) used above to get a concrete representation of \( A_5^{\alpha, \beta, \gamma, \delta, \epsilon} \) can be modified to get a concrete representation of \( A \) on \( L^2(\mathbb{Z}_{q_1} \times \mathbb{T}) \) (where \( \mathbb{Z}_{q_1} \) is the subgroup of \( \mathbb{T} \) with \( q_1 \) elements). The proof of the simplicity of \( A \) comes next.

First consider the universal \( C^* \)-algebra \( \mathfrak{A} \) generated by unitaries satisfying

\[
(UV = \lambda^\beta \mu^\alpha X_1^{q_1} VU, \quad UX_1 = \lambda^\gamma X_1 U, \quad V'X_1^{q_1} = 1, \quad UV' = \lambda^\gamma X_1 U' \quad \text{and} \quad W^{q_2} = 1 = X_1^{q_1}.
\]

A change of variables is useful. Pick relatively prime integers \( c, d \) such that \( d\delta + ce = 0 \) and let \( a, b \) be integers such that \( ad - bc = 1 \). Put

\[
U' = U^a V^b \quad \text{and} \quad V' = U^c V^d.
\]

Then keeping \( X \) and \( W \) the same, (CR) becomes

\[
(UV' = \xi X_1^{q_1} V'U', \quad UX_1 = \lambda^\gamma X_1 U', \quad U'X = \lambda^{c\gamma} X' V' \quad \text{and} \quad W^{q_2} = 1 = X_1^{q_1}\]

(other pairs of unitaries commuting), where \( \xi = \lambda^\beta \mu^\alpha \lambda^s \) for some integer \( s \), and \( \delta' = b\delta + a\epsilon \).

It is clear that \( \lambda^{\delta'} \) is a primitive \( q_2 \)-th root of unity and that the algebra \( \mathfrak{A} \) is generated by \( U', V', W \) and \( X_1 \), since \( ad - bc = 1 \).
Let $B = C^*(X_1, V')$ and let $C(Z_{q_2}) = C^*(W)$ be the $C^*$-algebra generated by $W$. Since $W$ commutes with $X_1$ and $V'$, we can form the tensor product algebra $B \otimes C(Z_{q_2}) = C^*(X_1, V', W)$. The automorphism $\text{Ad}_{U'}$ acts on this tensor product as $\sigma \otimes \tau$, where $\sigma$ and $\tau$ are automorphisms of $B$ and $C(Z_{q_2})$, respectively, given by 

$$\sigma(X_1) = \lambda^{a\gamma}X_1, \quad \sigma(V') = \xi_1X_1^\alpha V' \quad \text{and} \quad \tau(W) = \zeta W.$$ 

Therefore, by the universality of $\mathfrak{A}$ and of the $C^*$-crossed product $C^*(B \otimes C(Z_{q_2}), Z)$, these algebras are isomorphic. By Rieffel’s Proposition 1.2 [14], the latter of these is isomorphic to $M_{q_2}(D)$, where $D = C^*(B, Z) = C^*(X_1, V', U'^{q_2})$, and the action of $Z$ on $B$ is generated by $\sigma^{q_2}$.

Now, the unitaries $X_1, V'$ and $U'^{q_2}$ generating $D$ satisfy 

$$(*) \quad \left\{ \begin{array}{lcl} U'^{q_2}V' &=& \xi_2^{q_2}\lambda^{s'}X_1^{a^{q_2}v'}U'^{q_2}, \\ U'^{q_2}X_1 &=& \lambda^{\gamma q_2}X_1U'^{q_2} \quad \text{and} \quad X_1^{q_1} = 1, \end{array} \right.$$ 

for some $s' \in \mathbb{Z}$.

Now we apply another change of variables. Choose relatively prime integers $c', d'$ such that $cd' + aq_2c' = 0$, then pick integers $a', b'$ with $a'd' - b'c' = 1$, and put 

$$U'' = U'^{q_2a'}V'^{b'} \quad \text{and} \quad V'' = U'^{q_2c'}V'^{d'}.$$ 

Then $(*)$ becomes (keeping $X_1$ the same) 

$$(***) \quad \left\{ \begin{array}{lcl} U''V'' &=& \xi_1X_1^{a^{q_2}v''}U'', \\ U''X_1 &=& \lambda'X_1U'' \quad \text{and} \quad X_1^{q_1} = 1, \end{array} \right.$$ 

where $\xi_1 = \xi^{q_2}\lambda^{s'}$ for some integer $s'$, $\lambda' = \lambda^{\gamma (aq_2a'+cb')}$ has order $q_3$ dividing $q_1$ (the order of $\lambda^\gamma$), and perhaps $q_3 \neq q_1$.

Now, with $\mathbb{Z}_{q_1} \subset \mathbb{T}$ representing the subgroup with $q_1$ members, one observes that $D$ is isomorphic to the crossed product of $C^*(C(\mathbb{Z}_{q_1} \times \mathbb{T}), \mathbb{Z})$ from the flow generated by $\phi(w, v) = (\lambda'w, \xi_1\lambda^{-aq_2v})$. (Note that the flow is not minimal unless the order of $\lambda'$ is exactly $q_1$.) This proves the following.

3. Theorem. The universal $C^*$-algebra $\mathfrak{A}$ generated by unitaries $U, V, W$ and $X_1$ satisfying $(\text{CR}_1)$ as for 2 (see also $(c')$) is isomorphic to $M_{q_2}(D)$, where $D = C^*(C(\mathbb{Z}_{q_1} \times \mathbb{T}), \mathbb{Z})$, as above.

Therefore, we now obtain all simple algebras satisfying $(\text{CR}_1)$.

4. Corollary. Every simple $C^*$-algebra generated by unitaries satisfying $(\text{CR}_1)$ is isomorphic to a matrix algebra over an irrational rotation algebra.

PROOF. By Theorem 3, any such simple algebra $Q$ is a quotient of $M_{q_2}(D)$. Hence $Q = M_{q_2}(Q')$ where $Q'$ is a simple quotient of $D$. But such a $Q'$ is generated by unitaries satisfying $(***)$, but with $X_1$ (of order $q_1$) replaced by another unitary $X_2$, which after suitable rescaling, has order equal to the order of $\lambda'$. But this algebra is known to be a matrix algebra over an irrational rotation algebra (see for example Theorem 3 of [4]). \[\square\]
We state

5. Theorem. A $C^*$-algebra $A$ is isomorphic to a simple infinite dimensional quotient of $C^*(H_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon))$ if and only if $A$ is isomorphic to $A_5^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ for an irrational $\theta$, or to an algebra as in Corollary 4.

The matrix algebra presentation for the simple $C^*$-algebra $A$ generated by unitaries satisfying (CR$_1$) is not given in a definite form in the proof of Corollary 4. Here is an explicit matrix presentation. First, as near the beginning of the section, change the variables so that (CR$_1$) becomes

$$
\begin{align*}
(CR_2) & \quad U'V' = \xi X_1^{\alpha} V' U', \quad U' X_1 = \lambda^{a\gamma} X_1 U', \\
& \quad U' W = \lambda^{\delta} W U', \quad V' X = \lambda^{c\gamma} X V' \quad \text{and} \quad W^{q_2} = 1 = X_1^{q_1}
\end{align*}
$$

Now we shall present the algebra $A$ by unitaries in a matrix algebra as follows. Consider the $C^*$-algebra $B_1$ generated by unitaries $u, v$ and $x$ enjoying the relations

$$
uv = \xi' x^{q_2} v u, \quad ux = \lambda^{q_2 a\gamma} x u, \quad vx = \lambda^{c\gamma} x v \quad \text{and} \quad x^{q_3} = 1,
$$

where $q_3$ is the least common multiple of the orders of $\lambda^{q_2 a\gamma}$ and $\lambda^{c\gamma}$, and $\xi'$ is to be determined below. Clearly, $q_3$ divides $q_1$ so that also $x^{q_1} = 1$. It was shown in the proof of Theorem 6.4 of [6] that $B_1$ is isomorphic to a $q_3 \times q_3$ matrix algebra over an irrational rotation algebra, when $\xi'$ is not a root of unity. Hence it will suffice to show that $A$ is isomorphic to $M_{q_2}(B)$ (so that $Q = q_2 q_3$). Indeed, let

$$
U' = \begin{pmatrix}
0 & 0 & \cdots & 0 & u \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
$$

(so $U' \text{diag}(K_1, K_2, \ldots, K_{q_2}) U'^* = \text{diag}(u K_{q_2} u^*, K_1, K_2, \ldots, K_{q_2-1})$),

$$
V' = \text{diag}(\tau_1 v, \tau_2 x^{-\alpha} v, \tau_3 x^{-2\alpha} v, \ldots, \tau_{q_2} x^{-(q_2-1)\alpha} v),
$$

$$
W = \text{diag}(1, \xi^{-1}, \xi^{-2}, \ldots, \xi^{-(q_2-1)}),
$$

$$
X = \text{diag}(x, \lambda^{-a\gamma} x, \lambda^{-2a\gamma} x, \ldots, \lambda^{-(q_2-1)a\gamma} x),
$$

where

$$
\tau_j = \lambda^a \xi^j \tau_1^{-j+1}, \quad 1 \leq j \leq q_2, \quad \text{and} \quad \tau_{q_2} \xi' \lambda^{q_2(q_2-1)a^2\gamma} = \xi_1.
$$

One now checks that these unitaries satisfy (CR$_2$). It is also evident that they generate $M_{q_2}(B)$. □
§5. K-Theory and the Trace Invariant. In this section we shall calculate the K-groups of the C*-algebra $A := A^5_0(\alpha, \beta, \gamma, \delta, \epsilon)$ by means of the Pimsner-Voiculescu six term exact sequence [13]. Since one of the groups in the sequence turns out to have torsion elements, the application of this result requires careful examination.

6. Theorem. For the C*-algebra $A^5_0(\alpha, \beta, \gamma, \delta, \epsilon)$, one has $K_0 = K_1 = \mathbb{Z}^6 \oplus \mathbb{Z}_\alpha$.

PROOF. To prove this theorem, we combine two applications of the PV sequence corresponding to two presentations P1 and P2 of $A$ as follows.

P1. In view of (CR), let $B_1 = \mathcal{C}(X, V, U)$ and let $\text{Ad}_W$, with

$$\text{Ad}_W(X) = X, \quad \text{Ad}_W(V) = \lambda^{-\delta}V, \quad \text{Ad}_W(U) = \lambda^{-\epsilon}U,$$

generate an action of $\mathbb{Z}$ on $B_1$, so that $A = \mathcal{C}(B_1, \mathbb{Z})$. Applying the PV sequence to $B_1$, viewed as the crossed product of $\mathcal{C}(\mathbb{T}^2) = \mathcal{C}(X, V)$ by the automorphism $\text{Ad}_U$, it is not hard to see that $K_0(B_1) = \mathbb{Z}^3$ and $K_1(B_1) = \mathbb{Z}^3 \oplus \mathbb{Z}_\alpha$. Since $\text{Ad}_W$ is homotopic to the identity, the PV sequence immediately gives

$$K_1(A) = \mathbb{Z}^6 \oplus \mathbb{Z}_\alpha.$$

However, since in the short exact sequence

$$0 \longrightarrow K_0(B_1) \xrightarrow{i} K_0(A) \xrightarrow{\delta} K_1(B_1) \longrightarrow 0$$

$K_1(B_1)$ has torsion, we cannot readily obtain $K_0(A)$. For this, the next presentation will help.

P2. In view of (CR), we can also let $B_2 = \mathcal{C}(X, V, W) = \mathcal{C}(\mathbb{T}) \otimes A_{\delta \theta}$, where $\mathcal{C}(\mathbb{T}) = \mathcal{C}(X)$ and $A_{\delta \theta} = \mathcal{C}(V, W)$. Let $\sigma = \text{Ad}_U$, with

$$\sigma(X) = \lambda^\gamma X, \quad \sigma(V) = \lambda^3 X^\alpha V, \quad \sigma(W) = \lambda^\epsilon W,$$

generate an action of $\mathbb{Z}$ on $B_2$, so that $A = \mathcal{C}(B_2, \mathbb{Z})$. In this case the PV sequence becomes

$$K_0(B_2) \xrightarrow{id_* - \sigma_*} K_0(A) \xrightarrow{i_*} K_0(A) \xrightarrow{\delta_0} K_1(B_2) \xrightarrow{id_* - \sigma_*} K_1(B_2)$$

(*)

It is not hard to see that a basis for $K_1(B_2) = \mathbb{Z}^4$ is given by $\{[[X], [V], [W], [\xi]]\}$ where $\xi = X \otimes e + 1 \otimes (1 - e)$ and $e = e(V, W)$ is a Rieffel projection in $A_{\delta \theta}$ of trace $\delta \theta$ mod 1. Also, a basis of $K_0(B_2) = \mathbb{Z}^4$ is given by $\{[1], [e], B_{XV}, B_{WX}\}$ where $B_{XV} = [P_{XV}] - [1]$ is the Bott element in $X, V$ and $P_{XV}$ the usual Bott projection in the commuting variables $X, V$. The action of $id_* - \sigma_*$ on $K_1(B_2)$ is given by

$$id_* - \sigma_* : \begin{align*}
[X] &\mapsto 0, & [V] &\mapsto -\alpha[X], & [W] &\mapsto 0, & [\xi] &\mapsto m\alpha[X]
\end{align*}$$

for some integer $m$, as shown by the following lemma. The action of $id_* - \sigma_*$ on $K_0(B_2)$ is given by

$$id_* - \sigma_* : \begin{align*}
[1] &\mapsto 0, & [e] &\mapsto \alpha B_{XW}, & B_{XW} &\mapsto 0, & B_{XV} &\mapsto 0.\n\end{align*}$$

The action on $[e]$ is also shown in the following

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7. **Lemma.** We have $\sigma[e] = [e] - \alpha B_{XW}$ in $K_0(B_2)$, and $\sigma[\xi] = [\xi] + m\alpha[X]$ for some integer $m$.

**PROOF.** The proof of the first equality can be established using an argument quite similar to that of the proof of Lemma 4.2 of [15]. Hence the kernel of $id_* - \sigma_*$ on $K_0(B_2)$ is $Z^3$. For the second equality, let $\eta = (id_* - \sigma_*)[\xi]$. From P1 and (*) we have

$$Z^6 \oplus Z_\alpha = K_1(A) = Z^3 \oplus \text{Im}(i_*) = Z^3 \oplus \frac{K_1(B_2)}{\text{Im}(id_* - \sigma_*)} = Z^3 \oplus \frac{K_1(B_2)}{Z\alpha[X] + Z\eta}.$$ 

Thus

$$\frac{K_1(B_2)}{Z\alpha[X] + Z\eta} = Z^3 \oplus Z_\alpha.$$ 

But since $K_1(B_2) = Z^4$, it follows that the subgroup $Z\alpha[X] + Z\eta$ must have rank one.\(^1\) Therefore, $Z\alpha[X] + Z\eta = Zd[X]$ for some integer $d$. Substituting this into (***) one gets $d = \alpha$ and so $\eta \in Z\alpha[X]$. □

It now follows that in $K_1(B_2)$ one has $\text{Im}(id_* - \sigma_*) = Z\alpha[X]$ and that $\text{Ker}(id_* - \sigma_*) = Z^3$ whether $m$ is zero or not. Therefore, from the exactness of (*) we obtain $\text{Im}(\delta_0) = Z^3$ and hence by Lemma 7

$$K_0(A) = Z^3 \oplus \text{Im}(i_*) = Z^3 \oplus \frac{K_0(B_2)}{\text{Im}(id_* - \sigma_*)} = Z^3 \oplus \frac{K_0(B_2)}{Z\alpha B_{XW}} = Z^6 \oplus Z_\alpha$$

which completes the proof of Theorem 6. □

**The Trace Invariant.**

8. **Theorem.** The range of the unique trace on $K_0(A_g^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon))$ is $Z + Z\rho \theta + Z\gamma \delta \theta^2$ where $\rho = \gcd\{\gamma, \delta, \epsilon\}$.

Note that this agrees with the the trace invariant $Z + Z\theta + Z\theta^2$ of the algebra $A_g^{5,3}$ as done in [15], section 2, in the case $(\alpha, \beta, \gamma, \delta, \epsilon) = (1, 0, 1, 1, 0)$.

**PROOF.** First we make an appropriate change of variables for the unitary generators of the algebra $A_g^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$. Referring back to the defining relations (CR), pick integers $a, b, c, d$ such that $b\delta + a\epsilon = 0$, $ad - bc = 1$, and let

$$U' = U^aV^b, \quad V' = U^cV^d.$$ 

Then the commutation relations (CR), with $W$ remaining the same and $X$ suitably scaled, become

$$U'V' = X^\alpha V'U', \quad U'X = \lambda^\alpha\gamma XU', \quad V'W = \lambda^{d\delta + e\epsilon}WV',$$

$$U'W = WU', \quad V'X = \lambda^{\gamma} XV', \quad WX = XW$$

\(^1\)If $0 \to F_1 \to G \to F_2 \oplus H \to 0$ is a short exact sequence of finitely generated Abelian groups, where $F_1, F_2$ are free groups and $H$ is torsion, then $\text{rank}(G) = \text{rank}(F_1) + \text{rank}(F_2)$. This can be seen from the naturally obtained short exact sequence $0 \to F_1 \oplus F_2 \to G \to H \to 0$, from which the result follows. (If $G$ has rank greater than that of a subgroup $K$, then $G/K$ contains a non-torsion element.)
Let $B = C^*(X, U', V')$. It is isomorphic to the crossed product of $C^*(X, U') = A_{a\gamma\theta}$ by $\mathbb{Z}$ and automorphism $\text{Ad}_{V'}$. An easy application of Pimsner’s trace formula shows that

$$\tau_*K_0(B) = \mathbb{Z} + \mathbb{Z}a\gamma\theta + \mathbb{Z}c\gamma\theta = \mathbb{Z} + \mathbb{Z}\gamma\theta,$$

since $(a,c) = 1$. Next, it is not hard to see that an application of the Pimsner-Voiculescu sequence to the above crossed product presentation of $B$ gives the basis $\{[X],[V'],[U'],[\xi]\}$ for $K_1(B)$, where $[X]$ has order $\alpha$, $\xi = 1 - e + ew^*V'^*e$ is a unitary in $B$, $e$ is a Rieffel projection in $A_{a\gamma\theta}$ of trace $(a\gamma\theta)$mod1, and $w$ is a unitary in $A_{a\gamma\theta}$ such that $V'^*eV' = wew^*$ (which exists by Rieffel’s Cancellation Theorem [14]). The underlying connecting homomorphism $\partial : K_1(B) \to K_0(A_{a\gamma\theta})$ gives $\partial[\xi] = [c]$ and $\partial[V'] = [1]$, the usual basis of $K_0(A_{a\gamma\theta})$.

To apply Pimsner’s trace formula, one calculates the usual “determinant” on the aforementioned basis, since the kernel of $id_* - (\text{Ad}_W)_*$ is all of $K_1(B)$ (since $\text{Ad}_W$ is homotopic to the identity). It is easy to see that this determinant (whose values are in $\mathbb{R}/\tau_*K_0(B)$) on the elements $[X],[V'],[U']$ gives the respective values 1, $(d\delta + ce)\theta, 1$. For the $\xi$, since now $\text{Ad}_W$ fixes $A_{a\gamma\theta}$ (and in particular $e$ and $w$), one obtains

$$\text{Ad}_W(\xi)\xi^* = (1 - e + \lambda^{d\delta + ce}ew^*V'^*e)(1 - e + eV'we) = 1 - e + \lambda^{d\delta + ce}e.$$ 

Now a simple homotopy path connecting this element to 1 is just $t \mapsto 1 - e + e^{2\pi i\theta(d\delta + ce)}t e$, and the corresponding determinant gives the value $(d\delta + ce)\theta\tau(e)$. Since $\tau(e) = a\gamma\theta \mod 1$, the range of the trace is

$$\tau_*K_0(A) = \mathbb{Z} + \mathbb{Z}\gamma\theta + \mathbb{Z}(d\delta + ce)\theta + \mathbb{Z}(d\delta + ce)\theta^2.$$ 

Now $a(d\delta + ce) = ad\delta + ace - c(b\delta + ae) = \delta$, and similarly $-b(d\delta + ce) = \epsilon$, thus showing that $d\delta + ce = \gcd\{\delta, \epsilon\}$. Therefore, one gets $\tau_*K_0(A) = \mathbb{Z} + \mathbb{Z}\gcd\{\gamma, \delta, \epsilon\}\theta + \mathbb{Z}\gamma\delta\theta^2$. □

Discussion of Classification.

Next, let us consider briefly the classification of the algebras $A^{5,3}_{\theta}(\alpha, \beta, \gamma, \delta, \epsilon)$. First, it is easy to show that $A^{5,3}_{\theta}(\alpha, \beta, \gamma, \delta, \epsilon) \cong A^{5,3}_{\theta'}(\alpha, \beta, \gamma, \delta, \epsilon)$. Second, we note that the simple quotients $A^{5,3}_{\theta} = A^{5,3}_{\theta}(1, 0, 1, 1, 0)$ have been almost completely classified in [15]; specifically, they have been classified for all non-quartic irrationals (which are those that are not zeros of any polynomial of degree at most 4 with integer coefficients). But generally, with $\lambda = e^{2\pi i\theta}$ for an irrational $\theta$, the operator equations

$$(\text{CR}) \quad UV = \lambda^\beta X^\alpha U, \quad UX = \lambda^\gamma XU, \quad VW = \lambda^\delta WV \quad \text{and} \quad UW = \lambda^\epsilon UW,$$

for $A^{5,3}_{\theta}(\alpha, \beta, \gamma, \delta, \epsilon)$ can be modified by changing some of the variables, i.e., by substituting $X_0 = e^{2\pi i\theta\beta/\alpha}X$ and putting $\lambda_0 = \lambda^p$, where $p = \gcd\{\gamma, \delta, \epsilon\}$, and then $\gamma_0 = \gamma/p$, $\delta_0 = \delta/p$ and $\epsilon_0 = \epsilon/p$ with $\gcd\{\gamma_0, \delta_0, \epsilon_0\} = 1$. The equations (CR) become

$$(\text{CR}_0) \quad \begin{cases} UV = X_0^\alpha U, & UX_0 = \lambda_0^\alpha X_0 U, & VW = \lambda_0^\delta^* WV \quad \text{and} \quad UW = \lambda_0^\epsilon^* WU \quad \text{with} \quad \gcd\{\gamma_0, \delta_0, \epsilon_0\} = 1, \end{cases}$$
which are the equations for $A^{5,3}_{\rho\theta}(\alpha, 0, \gamma_0, \delta_0, \epsilon_0)$, so

$$A^{5,3}_{\rho\theta}(\alpha, \beta, \gamma, \delta, \epsilon) \cong A^{5,3}_{\rho\theta}(\alpha, 0, \gamma_0, \delta_0, \epsilon_0)$$

where $\gcd\{\gamma_0, \delta_0, \epsilon_0\} = 1$. This reduces the classification to the class of algebras $A^{5,3}_{\rho\theta}(\alpha, 0, \gamma, \delta, \epsilon)$ where $\gcd\{\gamma, \delta, \epsilon\} = 1$.

If two such $C^*$-algebras $A_j = A^{5,3}_{\rho_j\theta_j}(\alpha_j, 0, \gamma_j, \delta_j, \epsilon_j), \ j = 1, 2$, are isomorphic, where now $\rho_j = \gcd\{\gamma_j, \delta_j, \epsilon_j\} = 1$, what constraints must hold between their respective parameters? As we observed in Theorem 6, one must have $\alpha_1 = \alpha_2$. By Theorem 8, one has

$$Z + Z\theta_1 + Z\gamma_1 \delta_1 \theta_1^2 = Z + Z\theta_2 + Z\gamma_2 \delta_2 \theta_2^2.$$ 

One can show that if one assumes that $\theta_j$ are non-quadratic irrationals, then these trace invariants are equal if, and only if, there is a matrix $S \in \text{GL}(2, \mathbb{Z})$ such that

$$\begin{pmatrix} \theta_2 \\ \gamma_2 \delta_2 \theta_2^2 \end{pmatrix} = S \begin{pmatrix} \theta_1 \\ \gamma_1 \delta_1 \theta_1^2 \end{pmatrix} \mod \left(\frac{\mathbb{Z}}{\mathbb{Z}}\right).$$

Further, one can more easily show that if $\theta_j$ are non-quartic irrationals (i.e., not roots of polynomials over $\mathbb{Z}$ of degree at most four), then the trace invariants are equal if, and only if,

$$\theta_2 = (\pm \theta_1) \mod 1, \quad \text{and} \quad \gamma_2 \delta_2 \theta_2^2 = (\pm \gamma_1 \delta_1 \theta_1^2 + m \theta_1) \mod 1,$$

for some integer $m$. An interesting special situation can be considered. For example, if one fixes $\theta$ (assumed non-quartic for simplicity) and varies the other parameters, then the above shows that $\gamma_1 \delta_1 = \gamma_2 \delta_2$ will follow from $A_1 \cong A_2$. At this point it is not clear if the parameters can be determined more precisely than this. For example, is it possible, if $\theta$ is held fixed, that the equalities $\gamma_1 = \gamma_2$ and $\delta_1 = \delta_2$ could fail to hold? This is unclear. However, the following heuristic argument (based only on canonical considerations) suggests that perhaps $\gamma_j, \delta_j$ are uniquely determined.

Let us attempt to apply a canonical transformation of the unitary generators of the form

$$U_1 = U^{a_1} V^{b_1} W^{c_1}, \quad V_1 = U^{a_2} V^{b_2} W^{c_2}, \quad W_1 = U^{a_3} V^{b_3} W^{c_3},$$

in the hope of changing $\gamma, \delta$, by working out the commutation relations and ensuring that they are preserved. (We have kept $X$ the same since it is the only auxiliary unitary that can occur if one looks at the most general transformation of the form $U_1 = U^{a_1} V^{b_1} W^{c_1} X^{d_1}, V_1 = U^{a_2} V^{b_2} W^{c_2} X^{d_2}$ — in fact, the commutator $[U_1, V_1]$ is a scalar multiple of $X^{a(a_1 b_2 - a_2 b_1)}$.)

The 3 by 3 matrix $T$ with rows $a_j, b_j, c_j$ should have determinant $\pm 1$ for the new unitaries to generate the same $C^*$-algebra. The first relation in (CR) demands that $a_1 b_2 - a_2 b_1 = 1$ so as to keep $X^\alpha$ the same. Also, since $VX = VX$ and $WX = WX$ must be preserved, we should have $V_1 X = X V_1$ and $W_1 X = X W_1$. However, it is easy to see that these imply that $a_2 = 0$ and $a_3 = 0$, respectively (since $U$ does not commute with $X$). Since the relation between $V_1$ and $W_1$ does not contain $X$, one must have $a_2 b_3 - a_3 b_2 = 0$, which is already satisfied. Similarly, for the relation between $U_1$ and $W_1$ one must have $a_1 b_3 - a_3 b_1 = 0$. But
since $a_3 = 0$ this gives $b_3 = 0$. This means that the matrix $T$ is upper triangular with $1$ or 
$-1$ on its diagonal, hence the transformation is reduced to

$$U_1 = U^\pm V^{b_1}W^{c_1}, \quad V_1 = V^\pm W^{c_2}, \quad W_1 = W^\pm.$$

(where the third ± here is independent of the first two, which should both be $1$ or both $-1$).

In view of this transformation, however, the new commutation relations are now forced to take the following form

$$U_1V_1 = X^\alpha V_1U_1, \quad U_1X = \lambda^{\pm \gamma} XU_1, \quad V_1W_1 = \lambda^{\pm \delta} W_1V_1, \quad U_1W_1 = \lambda^{\delta b_1 \pm \epsilon} W_1U_1,$$

(and of course $V_1X = XV_1$, $W_1X = XW_1$, and after one rescales $X$). These are exactly in the same form as the relations (CR). In particular, the integer parameters $\gamma$ and $\delta$, since they are assumed to be positive, have remained unchanged. (Also unchanged is $\epsilon$, since it is, by $(\ast)$ of Theorem 1, smaller than $\delta$.) This seems to suggest that $\gamma$ and $\delta$ (and hence also $\epsilon$) are uniquely determined in an isomorphism classification theorem. The broad scope of this classification problem, however, must be left to another time; the fact that $\gamma$ and $\delta$ are not clearly singled out in the invariants considered here, but appear mixed, seems to present an obstacle to the classification of these $C^*$-algebras. (The authors doubt that the Ext invariant of Brown-Douglas-Fillmore contains any more information, though they have not checked this in detail.)
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