Fractional double phase Robin problem involving variable order-exponents without Ambrosetti–Rabinowitz condition

Reshmi Biswas, Sabri Bahrouni and Marcos L. Carvalho

Abstract. We consider a fractional double phase Robin problem involving variable order and variable exponents. The non-linearity \( f \) is a Carathéodory function satisfying some hypotheses which do not include the Ambrosetti–Rabinowitz-type condition. By using a variational methods, we investigate the multiplicity of solutions.

Mathematics Subject Classification. 35R11, 35S15, 47G20, 47J30.

Keywords. Variable-order fractional \( p(\cdot)-\text{Laplacian} \), Double phase problem, Robin boundary condition, Variational methods.

1. Introduction

In the last few decades, problems involving \( p(x)\)-Laplacian, defined as \((-\Delta)_{p(x)}u := (|\nabla u|^{p(x)-2}\nabla u), \ x \in \mathbb{R}^N \), where \( p : \mathbb{R}^N \to [1, \infty) \) is continuous function, have been studied intensively due to its major real-world appearances in several mathematical models, for e.g., electrorheological fluid flow, image restorations, etc. (see [1,15,44,52]). Various parametric boundary value problems with variable exponents can be found in the book of Rădulescu-Repovš [41] and also one can refer to the book by Diening et al. [18] for the properties of such operator and associated variable exponent Lebesgue spaces and variable exponent Sobolev spaces. In addition, regarding the elliptic problems with nonstandard growth conditions and corresponding different type of nonuniformly elliptic operators, one can refer to [38].

On the other hand, recently, study of fractional and nonlocal operators of elliptic type takes a great attention, both for pure mathematical research and in view of concrete real-world applications (see [45]). In most of these applications, a fundamental tool to treat these types of problems is the so-called fractional-order Sobolev spaces. The literature on nonlocal operators and on their applications is very interesting and, up to now, quite large. We also refer to the recent monographs [19,37] for several motivations concerning nonlocal problems.

A bridge between fractional-order theories and Sobolev spaces with variable settings is first provided in [28]. In that paper, the authors defined the Fractional Sobolev spaces with variable exponents and introduce the corresponding fractional \( p(\cdot)\)-Laplacian as

\[
(-\Delta)^s_{p(\cdot)}u(x) := P.V. \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))}{|x-y|^{N+sp(x,y)}} \, dy, \ x \in \Omega,
\]

where P.V. denotes Cauchy’s principal value, \( p : \Omega \times \Omega \to \mathbb{R} \) is a continuous function with \( 1 < p(x,y) < \infty \) and \( 0 < s < 1 \), where \( \Omega \) is a smooth domain. The idea of studying such spaces and the associated operator defined in (1.1) arises from a natural inquisitiveness to see what results can be recovered when the standard local \( p(x)\)-Laplace operator is replaced by the fractional \( p(\cdot)\)-Laplacian. Continuing with this thought and inspired by the vast applications of variable-order derivative (see for, e.g., [30,31,33,34,43,47]
and references there in), Biswas and Tiwari [12] introduced the variable-order fractional Sobolev spaces with variable exponent and corresponding variable-order fractional \( p(\cdot) \)-Laplacian by imposing variable growth on the fractional order \( s \), given in (1.1), to study some elliptic problems. In fact, results regarding fractional \( p(\cdot) \)-Laplace equations and variable-order fractional \( p(\cdot) \)-Laplace equations are in progress, for example, we refer to [5, 10, 25, 26] and [7, 13, 50], respectively.

In this paper, we are interested in the following problem:

\[
\begin{align*}
\begin{cases}
\mathcal{L}^s_{p_1,p_2}(u) + |u|^{p_1(x)-2}u + |u|^{p_2(x)-2}u = f(x,u) & \text{in } \Omega, \\
N^s_{p_1,p_2}(u) + \beta(x)(|u|^{p_1(x)-2}u + |u|^{p_2(x)-2}u) = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{L}^s_{p_1,p_2}(u) & := (-\Delta)^{s_1(\cdot)}(u) + (-\Delta)^{s_2(\cdot)}(u), \\
N^s_{p_1,p_2}(u) & := \mathcal{N}^{s_1(\cdot)}_{p_1(\cdot)}(u) + \mathcal{N}^{s_2(\cdot)}_{p_2(\cdot)}(u)
\end{align*}
\]

and

\[
\begin{align*}
(-\Delta)^{s_1(\cdot)} u(x) & = \text{P.V.} \int_{\Omega} \frac{|u(x) - u(y)|^{p_i(x,y)}}{|x-y|^{N+s(x,y)p_i(x,y)}} \, dy, \quad i = 1, 2, \quad \text{for } x \in \Omega, \\
\mathcal{N}^{s_1(\cdot)}_{p_1(\cdot)} u(x) & = \int_{\Omega} \frac{|u(x) - u(y)|^{p_i(x,y)-2}(u(x) - u(y))}{|x-y|^{N+s(x,y)p_i(x,y)}} \, dy \quad \text{for } x \in \mathbb{R}^N \setminus \Omega.
\end{align*}
\]

Here \( \text{P.V.} \) denotes the Cauchy’s principal value, \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( s, p_1, p_2 \) are continuous functions such that \( p_i(x) = p_i(x, x) \), \( i = 1, 2 \), \( s(x, x) := s(x, x) \) with appropriate assumptions described later. The variable exponent \( \beta \) verifies the assumption

\[
\beta \in L^\infty(\mathbb{R}^N \setminus \Omega) \quad \text{and} \quad \beta \geq 0 \text{ in } \mathbb{R}^N \setminus \Omega. \tag{\( \beta \)}
\]

The operator, defined in (1.2), is called double phase-type operator which has some important applications in biophysics, plasma physics, reaction-diffusion, etc. (see [14, 23, 48], for e.g.). For more details on applications of such operators in constant exponent setup, that is, \((p, q)\)-Laplace equations, we refer to the survey article [35], see also [9, 42] for the nonconstant case. In [4], the author studied some nonlinear applications of such operators in constant exponent setup, that is, \((p, q)\)-Laplace equations, we refer to [4, 26, 36] and the references therein. But if either \( p_1 \) or \( p_2 \) is a non-constant function, then (1.2) has a more complicated structure, due to its non-homogeneity and to the presence of several nonlinear terms, only few recent works deal with these problems. For instance, in [51], the authors generalize the double phase problem involving a local version of the fractional operator with variable exponents, discussed in [17], and studied the problem involving variable-order fractional \( p(\cdot) \& q(\cdot) \)-Laplacian but with homogeneous Dirichlet boundary datum, that is, \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \).

Now we consider some notations as follows. For any set \( \mathcal{D} \) and any function \( \Phi : \mathcal{D} \to \mathbb{R} \), we fix

\[
\Phi^- := \inf_{\mathcal{D}} \Phi(x) \quad \text{and} \quad \Phi^+ := \sup_{\mathcal{D}} \Phi(x).
\]

We define the function space

\[
C_+ (\mathcal{D}) := \{ \Phi : \mathcal{D} \to \mathbb{R} \text{ is uniformly continuous } : 1 < \Phi^- \leq \Phi^+ < \infty \}.
\]

We consider the following hypotheses on the variable order \( s \) and on the variable exponents \( p_1, p_2 : \)

\[
(H_1) \quad s : \mathbb{R}^N \times \mathbb{R}^N \to (0, 1) \text{ is a uniformly continuous and symmetric function, i.e., } s(x, y) = s(y, x) \text{ for all } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ with } 0 < s^- \leq s^+ < 1.
\]
\((H_2)\) \(p_i \in C_+(\mathbb{R}^N \times \mathbb{R}^N)\) are uniformly continuous and symmetric functions, i.e., \(p_i(x, y) = p_i(y, x), i = 1, 2\) for all \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N\) with \(1 < p_1^- < p_2^- \leq p_2^+ < +\infty\) such that \(s^+p_1^+ < N\).

First, we study our problem without assuming the well-known Ambrosetti–Rabinowitz (AR, in short)-type condition on the nonlinearity \(f\), which is given as

\[
\exists \theta > p_2^+ \text{ s.t. } f(x, t) > \theta F(x, t), \forall |t| > 0. \quad (AR)
\]

As known, under \((AR)\), any Palais–Smale sequence of the corresponding energy functional is bounded, which plays an important role of the application of variational methods. In our problem, the nonlinearity \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function such that \(f(x, 0) = 0\) for a.e. \(x \in \Omega\). The further assumptions on \(f\) are given below.

\((f_1)\) There exists \(a \in L^\infty(\Omega)\) such that \(|f(x, t)| \leq a(x)(1 + |t|^{r(x)-1})\), for a.e. \(x \in \Omega\) and for all \(t \in \mathbb{R}\), where \(r \in C_+(\mathbb{R}^N)\) with \(p_2^- < r^- \leq r(x) < \frac{Np_2^+}{N-p_2^+} := p_2^+(x)\).

\((f_2)\) If \(F(x, t) := \int_0^t f(x, s)ds\), then \(\lim_{|t| \to +\infty} \frac{F(x, t)}{|t|^{p_2^+}} = 0\) uniformly for a.e \(x \in \Omega\).

\((f_3)\) \(\lim_{|t| \to 0} \frac{f(x, t)}{|t|^{p_2^+ - 2}} = 0\) uniformly for a.e \(x \in \Omega\).

\((f_4)\) Let \(\mathcal{F}(x, t) = tf(x, t) - p_2^+F(x, t)\). Then there exists \(b \in L^1(\Omega)\) such that \(\mathcal{F}(x, t) \leq \mathcal{F}(x, \tau) + b(x)\) for a.e. \(x \in \Omega\), all \(0 \leq t \leq \tau\) or all \(\tau \leq t \leq 0\).

Consider the following function

\[
g(x, t) = t|t|^\frac{p_2^+}{2} - 2 \log(1 + |t|).
\]

One can check that \(g\) does not satisfy \((AR)\) but it satisfies \((f_1)-(f_4)\). Therefore, by dropping \((AR)\) condition, not only we invite complications in the compactness of Palais–Smale sequence but also we include larger class of nonlinearities. To overcome such aforementioned difficulty, we analyze the Cerami condition (see Definition 4.1), which is more appropriate for the setup of our problem. Finally, we are in a position to state the main results of this article.

**Theorem 1.1.** Let hypotheses \((H_1)-(H_2), \ (\beta)\) and \((f_1)-(f_4)\) hold. Then there exists a non-trivial weak solution of \((1.2)\).

Next, for the odd nonlinearity \(f(x, t)\), we state the existence results of infinitely many solutions using the Fountain theorem and the Dual fountain theorem, respectively.

**Theorem 1.2.** Let hypotheses \((H_1)-(H_2), \ (\beta)\) and \((f_1)-(f_4)\) hold. Also let \(f(x, -t) = -f(x, t)\). Then the problem \((1.2)\) has a sequence of nontrivial weak solutions with unbounded energy.

**Theorem 1.3.** Let hypotheses \((H_1)-(H_2), \ (\beta)\) and \((f_1)-(f_4)\) hold. Also let \(f(x, -t) = -f(x, t)\). Then the problem \((1.2)\) has a sequence of nontrivial weak solutions with negative critical values converging to zero.

We prove the next theorem using the symmetric mountain pass theorem.

**Theorem 1.4.** Let hypotheses \((H_1)-(H_2), \ (\beta)\) and \((f_1)-(f_4)\) hold. Also let \(f(x, -t) = -f(x, t)\). Then the problem \((1.2)\) has a sequence of nontrivial weak solutions with unbounded energy characterized by a minmax argument.

In the next theorem, we consider the following concave- and convex-type nonlinearity \(f:\)

\((f_5)\) For \(\lambda > 0\) and \(q, r \in C_+(\Omega)\) with \(1 < q^- \leq q^+ < p_2^- \) and \(p_2^+ < r^-\)

\[
f(x, t) = \lambda |t|^{q(x)-2}t + |t|^{r(x)-2}t
\]

**Theorem 1.5.** Let hypotheses \((H_1)-(H_2), \ (\beta)\) and \((f_5)\) hold. Then for all \(\lambda > 0\), the problem \((1.2)\) has a sequence of nontrivial weak solutions converging to 0 with negative energy.
It is noteworthy to mention that we are the first (as per the best of our knowledge) to study the above existence results for the problem (1.2) driven by double phase variable-order fractional $p_1(\cdot)$\&$p_2(\cdot)$-Laplacian involving Robin boundary condition and non-AR-type nonlinearities.

**Remark 1.6.** Throughout this paper, $C$ represents generic positive constant which may vary from line to line.

2. Preliminaries results

2.1. Variable exponent Lebesgue spaces

In this section, first we recall some basic properties of the variable exponent Lebesgue spaces, which we will use to prove our main results.

For $q \in C_+(\Omega)$, define the variable exponent Lebesgue space $L^{q(\cdot)}(\Omega)$ as

$$L^{q(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ is measurable : } \int_{\Omega} |u(x)|^{q(x)} \, dx < +\infty \right\}$$

which is a separable, reflexive, uniformly convex Banach space (see [18, 20]) with respect to the Luxemburg norm

$$\|u\|_{L^{q(\cdot)}(\Omega)} := \inf \left\{ \eta > 0 : \int_{\Omega} \left| \frac{u(x)}{\eta} \right|^{q(x)} \, dx \leq 1 \right\}.$$ 

Define the modular $\rho_{\Omega}^q : L^{q(\cdot)}(\Omega) \to [0,\infty]$ as

$$\rho_{\Omega}^q(u) := \int_{\Omega} |u|^{q(x)} \, dx, \text{ for all } u \in L^{q(\cdot)}(\Omega).$$

**Proposition 2.1.** ([20]) Let $u_n, u \in L^{q(\cdot)}(\Omega) \setminus \{0\}$, then the following properties hold:

(i) $\eta = \|u\|_{L^{q(\cdot)}(\Omega)}$ if and only if $\rho_{\Omega}^q(u/\eta) = 1$.

(ii) $\rho_{\Omega}^q(u) > 1 \ (1; < 1)$ if and only if $\|u\|_{L^{q(\cdot)}(\Omega)} > 1 \ (1; < 1)$, respectively.

(iii) If $\|u\|_{L^{q(\cdot)}(\Omega)} > 1$, then $\|u\|_{L^{q(\cdot)}(\Omega)} \leq \rho_{\Omega}^q(u) \leq \rho_{p_1}^q(u)$.

(iv) If $\|u\|_{L^{q(\cdot)}(\Omega)} < 1$, then $\rho_{\Omega}^q(u) \leq \|u\|_{L^{q(\cdot)}(\Omega)}$.

(v) $\lim_{n \to +\infty} \|u_n - u\|_{L^{q(\cdot)}(\Omega)} = 0 \iff \lim_{n \to +\infty} \rho_{\Omega}^q(u_n - u) = 0$.

Let $q'$ be the conjugate function of $q$, that is, $1/q(x) + 1/q'(x) = 1$.

**Proposition 2.2.** (Hölder inequality) ([20]) For any $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have

$$\left| \int_{\Omega} uv \, dx \right| \leq \|u\|_{L^{q(\cdot)}(\Omega)} \|v\|_{L^{q'(\cdot)}(\Omega)}.$$ 

**Lemma 2.3.** ([24, Lemma A.1]) Let $\vartheta_1(x) \in L^{\infty}(\Omega)$ such that $\vartheta_1 \geq 0$, $\vartheta_1 \not\equiv 0$. Let $\vartheta_2 : \Omega \to \mathbb{R}$ be a measurable function such that $\vartheta_1(x)\vartheta_2(x) \geq 1$ a.e. in $\Omega$. Then for every $u \in L^{\vartheta_1(\cdot)\vartheta_2(\cdot)}(\Omega)$,

$$\|u\|_{L^{\vartheta_1(\cdot)}(\Omega)} \leq \|u\|_{L^{\vartheta_1(\cdot)\vartheta_2(\cdot)}(\Omega)} \leq \|u\|_{L^{\vartheta_1(\cdot)\vartheta_2(\cdot)}(\Omega)}.$$
2.2. Variable-order fractional Sobolev spaces with variable exponents

Next, we define the fractional Sobolev spaces with variable order and variable exponents (see [12]). Define

$$W = W^{s(\cdot), \overline{p}(\cdot), p(\cdot)}(\Omega) := \left\{ u \in L^{\overline{p}(\cdot)}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy < \infty, \text{ for some } \eta > 0 \right\}$$

endowed with the norm

$$\|u\|_W := \inf \left\{ \eta > 0 : \rho_W \left( \frac{u}{\eta} \right) < 1 \right\},$$

where

$$\rho_W(u) := \int_\Omega |u|^{\overline{p}(x)} \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy$$

is a modular on $W$. Then, $(W, \| \cdot \|_W)$ is a separable reflexive Banach space (see [12,25]). On $W$, we also make use of the following norm:

$$|u|_W := \|u\|_{L^{\overline{p}(\cdot)}(\mathbb{R}^N)} + [u]_W,$$

where the seminorm $[\cdot]_W$ is defined as follows:

$$[u]_W := \inf \left\{ \eta > 0 : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy < 1 \right\}.$$

Note that $\| \cdot \|_W$ and $| \cdot |_W$ are equivalent norms on $W$ with the relation

$$\frac{1}{2} \|u\|_W \leq |u|_W \leq 2\|u\|_W \quad \text{for all } u \in W. \quad (2.1)$$

The following embedding result is studied in [12]. We also refer to [25] where the authors proved the same result when $s(x,y) = s$, constant.

**Theorem 2.4.** (Sub-critical embedding) Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Let $s$ and $p$ satisfy $(H_1)$ and $(H_2)$, respectively, and $\gamma \in C_+ (\Omega)$ satisfy $1 < \gamma(x) < p_*^s(x)$ for all $x \in \Omega$. In addition, when $\Omega = \mathbb{R}^N$, $\gamma$ is uniformly continuous and $p(x) < \gamma(x)$ for all $x \in \mathbb{R}^N$ and $\inf_{x \in \mathbb{R}^N} (p_*^s(x) - \gamma(x)) > 0$. Then, it holds that

$$W \hookrightarrow L^{\gamma(\cdot)}(\Omega). \quad (2.2)$$

Moreover, the embedding is compact.

**Notations:**

- $\delta^p_{\Omega}(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy$.
- For any measurable set $S$, $|S|$ denotes the Lebesgue measure of the set.
3. Functional setting

Now, we give the variational framework of problem (1.2). Let $s, p$ satisfy $(H_1), (H_2)$, respectively. We set

$$\|u\|_{X_p} := \|u\|_{s(p), \mathbb{R}^{2N} \setminus (C \Omega)^2} + \|u\|_{L^{p(\cdot)}(\Omega)} + \left\| \beta \frac{|u|}{p(x)} \right\|_{L^{p(\cdot)}(C \Omega)},$$

where $C \Omega = \mathbb{R}^N \setminus \Omega$ and

$$X^{s(\cdot)}_{p(\cdot)} := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable : } \|u\|_{X_p} < \infty \right\}.$$

By following standard arguments, it can be seen that $X^{s(\cdot)}_{p(\cdot)}$ is reflexive Banach space with respect to the norm $| \cdot |_{X_p}$ (see [8, Proposition 3.1]).

Note that the norm $| \cdot |_{X_p}$ is equivalent on $X^{s(\cdot)}_{p(\cdot)}$ to the following norm:

$$\|u\|_{X_p} = \inf \left\{ \eta \geq 0 : \rho_p \left( \frac{u}{\eta} \right) \leq 1 \right\} = \inf \left\{ \eta \geq 0 : \int_{\mathbb{R}^{2N} \setminus (C \Omega)^2} |u(x) - u(y)|^{p(x,y)} \frac{1}{p(x,y)} \frac{1}{|x-y|^{N+s(x,y)p(x,y)}} \, dx \, dy + \frac{\beta(x)}{\eta p(x)} \int_{C \Omega} \frac{|u|^p(x)}{p(x)} \, dx \leq 1 \right\},$$

where the modular $\rho_p : X^{s(\cdot)}_{p(\cdot)} \to \mathbb{R}$ is defined by

$$\rho_p (u) = \int_{\mathbb{R}^{2N} \setminus (C \Omega)^2} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)} \frac{1}{|x-y|^{N+s(x,y)p(x,y)}} \, dx \, dy + \frac{\beta(x)}{p(x)} \int_{C \Omega} |u|^p(x) \, dx.$$

The following lemma will be helpful in later considerations. The proof of this lemma follows using the similar arguments as in [20].

**Lemma 3.1.** Let $s, p$ and $\beta$ satisfy $(H_1), (H_2)$ and $(\beta)$, respectively, and let $u \in X^{s(\cdot)}_{p(\cdot)}$. Then the following hold:

(i) For $u \neq 0$ we have: $\|u\|_{X_p} = \eta$ if and only if $\rho_p (\frac{u}{\eta}) = 1$;

(ii) If $\|u\|_{X_p} < 1$ then $\|u\|_{X_p}^{p^+} \leq \rho_p (u) \leq \|u\|_{X_p}^{p^-}$;

(iii) If $\|u\|_{X_p} > 1$ then $\|u\|_{X_p}^{p^-} \leq \rho_p (u) \leq \|u\|_{X_p}^{p^+}$.

**Lemma 3.2.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$. Let $s$ and $p$ satisfy $(H_1)$ and $(H_2)$, respectively, and $(\beta)$ hold. Then for any $\gamma \in C^+ (\overline{\Omega})$ satisfying $1 < \gamma(x) < p^*_s(x)$ for all $x \in \overline{\Omega}$, there exists a constant $C(s, p, N, \gamma, \Omega) > 0$ such that

$$\|u\|_{L^\gamma(\cdot)(\Omega)} \leq C(s, p, N, \gamma, \Omega) \|u\|_{X_p} \text{ for all } u \in X,$$

moreover this embedding is compact.

**Proof.** It can easily be seen that $\|u\|_W \leq \|u\|_{X_p}$. Now by applying Theorem 2.4, we get our desired result. □
In order to deal with fractional $p_1(\cdot)$-$p_2(\cdot)$-Laplacian problems, we consider the space

$$X := X^{s(\cdot)}_{p_1(\cdot)} \cap X^{s(\cdot)}_{p_2(\cdot)}$$

endowed with the norm

$$|u|_X = \|u\|_{X^{p_1(\cdot)}} + \|u\|_{X^{p_2(\cdot)}}.$$ 

Clearly $X$ is reflexive and separable Banach space with respect to the above norm. It is not difficult to see we can make use of another norm on $X$ equivalent to $| \cdot |_X$ given as

$$\|u\| := \|u\|_X = \inf \left\{ \eta \geq 0 \mid \rho \left( \frac{u}{\eta} \right) \leq 1 \right\},$$

where the modular $\rho : X \to \mathbb{R}$ is defined as

$$\rho(u) = \rho_{p_1}(u) + \rho_{p_2}(u)$$

such that $\rho_{p_1}$, $\rho_{p_2}$ are described as in (3.2).

**Lemma 3.3.** Let hypotheses $(H_1)$–$(H_2)$ and $(\beta)$ be satisfied and let $u \in X$. Then the following hold:

(i) For $u \neq 0$, we have: $\|u\| = \eta$ if and only if $\rho(u) = 1$;

(ii) If $\|u\| < 1$ then $\|u\|_{p_2^*} \leq \rho(u) \leq \|u\|_{p_1^*}$;

(iii) If $\|u\| > 1$ then $\|u\|_{p_1^*} \leq \rho(u) \leq \|u\|_{p_2^*}$.

**Lemma 3.4.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$. Let $s$ and $p_i$ satisfy $(H1)$ and $(H2)$, respectively, for $i = 1, 2$ and $(\beta)$ hold. Then for any $\gamma \in C_+(\bar{\Omega})$ satisfying $1 < \gamma(x) < p_{s_+}(x)$ for all $x \in \Omega$, there exists a constant $C(s, p_i, N, \gamma, \Omega) > 0$ such that

$$\|u\|_{L^{\gamma(\cdot)}(\Omega)} \leq C(s, p_i, N, \gamma, \Omega)\|u\| \text{ for all } u \in X,$$

moreover this embedding is compact.

**Proof.** The proof directly follows from the definition of $\|u\|$ and Lemma 3.2. \qed

Throughout this article, $X^*$ represents the topological dual of $X$.

**Lemma 3.5.** Let hypotheses $(H_1)$–$(H_2)$ and $(\beta)$ be satisfied. Then $\rho : X \to \mathbb{R}$ and $\rho' : X \to X^*$ have the following properties:

(i) The function $\rho$ is of class $C^1(X, \mathbb{R})$ and $\rho' : X \to X^*$ is coercive, that is,

$$\frac{\langle \rho'(u), u \rangle}{\|u\|} \to +\infty \text{ as } \|u\| \to +\infty.$$

(ii) $\rho'$ is strictly monotone operator.

(iii) $\rho'$ is a mapping of type $(S_+)$, that is, if $u_n \to u$ in $X$ and $\limsup_{n \to +\infty} \langle \rho'(u_n), u_n - u \rangle \leq 0$, then $u_n \to u$ strongly in $X$.

**Proof.** The proof of this result is similar to the proof of [10, Lemma 4.2], just noticing that, the quantities $\mathbb{R}^{2N} \setminus (\mathcal{C}\Omega)^2$ and $\Omega \times \Omega$ play a symmetrical role. \qed

As proved in [8, Proposition 3.6], the following integration by parts formula arises naturally for $u \in C^2$ functions:

$$\frac{1}{2} \int_{\mathbb{R}^{2N} \setminus (\mathcal{C}\Omega)^2} \frac{\|u(x) - u(y)\|^{p(x, y) - 2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + s(x, y)p(x, y)}} \, dx \, dy \leq \int_{\Omega} v(-\Delta)^{s(\cdot)}_{p(\cdot)} u \, dx + \int_{\partial \Omega} v_{N \cdot s(\cdot)}^{p(\cdot)} \, dx. \tag{3.3}$$

The previous integration by parts formula leads to the following definition:
**Definition 3.6.** We say that $u \in X$ is a weak solution to (1.2) if for any $v \in X$ we have

$$
\mathcal{H}(u, v) = \frac{1}{2} \int_{\mathbb{R}^{2N} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^{p_1(x,y)} - 2(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+s(x,y)p_1(x,y)}} \, dx \, dy + \int_{\Omega} |u|^{\bar{p}_1(x)} \, dx + \int_{\Omega} |u|^{\bar{p}_2(x)} \, dx + \int_{\Omega} \frac{1}{\bar{p}_1(x)} |u|^{\bar{p}_1(x)} \, dx
$$

Thus, the weak solutions of (1.2) are precisely the critical points of

$$
\mathcal{I}(u) = \int_{\mathbb{R}^{2N} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^{p_1(x,y)}}{p_1(x,y)|x - y|^{N+s(x,y)p_1(x,y)}} \, dx \, dy + \int_{\mathcal{C}\Omega} \frac{1}{p_1(x)} |u|^{p_1(x)} \, dx
$$

A direct computation from [8, Proposition 3.8] shows that the functional $\mathcal{I}$ is well defined on $X$ and $\mathcal{I} \in C^1(X, \mathbb{R})$ with $
\langle \mathcal{I}'(u), v \rangle = \mathcal{H}(u, v) \quad \text{for any} \quad v \in X.$

Thus, the weak solutions of (1.2) are precisely the critical points of $\mathcal{I}.$

**4. Proof of Theorem 1.1**

**4.1. Abstract results**

**Definition 4.1.** Let $E$ be a Banach space and $E^*$ be its topological dual. Suppose that $\Phi \in C^1(E).$ We say that $\Phi$ satisfies the Cerami condition at the level $c \in \mathbb{R}$ (the $(C)_c$-condition for short) if the following is true:

“every sequence $(u_n)_{n \in \mathbb{N}} \subseteq E$ such that $\Phi(u_n) \to c$ and $(1 + \|u_n\|_E) \Phi'(u_n) \to 0$ in $E^*$ as $n \to +\infty$ admits a strongly convergent subsequence”.

If this condition holds at every level $c \in \mathbb{R}$, then we say that $\Phi$ satisfies the Cerami condition (the $C$-condition for short).
The \((C)_{c}\)-condition is weaker than the \((PS)_{c}\)-condition. However, it was shown in [29] that from \((C)_{c}\)-condition it can obtain a deformation lemma, which is fundamental in order to get some minimax theorems. Thus, we have

**Theorem 4.2.** If there exist \(e \in E\) and \(r > 0\) such that
\[
\|e\| > r, \quad \max(\Phi(0), \Phi(e)) \leq \inf_{\|x\|=r} \Phi(x),
\]
and \(\Phi : E \to \mathbb{R}\) satisfies the \((C)_{c}\)-condition with
\[
c = \inf_{\gamma \in \Gamma, t \in (0,1)} \Phi(\gamma(t)),
\]
where
\[
\Gamma = \{ \gamma \in C((0,1), E) : \gamma(0) = 0, \gamma(1) = e \}.
\]
Then \(c \geq \inf_{\|x\|=r} \Phi(x)\) and \(c\) is a critical value of \(\Phi\).

### 4.2. Geometric condition

**Lemma 4.3.** Let \((H_1)-(H_2)\), \((\beta)\) and \((f_1)-(f_4)\) hold. Then

(i) There exist \(\alpha > 0\) and \(R > 0\) such that
\[
I(u) \geq \beta > 0 \quad \text{for any } u \in X \quad \|u\| = \alpha.
\]

(ii) There exists \(\varphi \in X\) such that \(I(\varphi) < 0\).

**Proof.** (i) For any \(e > 0\), by the assumptions \((f_1)-(f_3)\), we have
\[
F(x,t) \leq e|t|^{p_1^+} + C(e)|t|^{r(x)}, \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.
\]
Hence, using Theorem 2.4, Lemma 3.3, Lemma 2.3 and Lemma 3.4 for any \(u \in X\) with \(\|u\| < 1\) (i.e., \(\|u\|_{X_{\pi_i}} < 1, i = 1, 2\)), we obtain
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^2 \setminus (\Omega)^2} \frac{|u(x) - u(y)|^{p_1(x,y)}}{p_1(x,y)} dx dt + \int_{\Omega} \frac{1}{P_1(x)} |u|^{p_1(x)} dx
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^2 \setminus (\Omega)^2} \frac{|u(x) - u(y)|^{p_2(x,y)}}{p_2(x,y)} dx dt + \int_{\Omega} \frac{1}{P_2(x)} |u|^{p_2(x)} dx
\]
\[
+ \int_{\partial \Omega} \frac{\beta(x) |u|^{p_1(x)}}{p_1(x)} dx + \int_{\partial \Omega} \frac{\beta(x) |u|^{p_2(x)}}{p_2(x)} dx
\]
\[
- \int_{\Omega} F(x,u) dx
\]
\[
\geq \frac{1}{2} \rho(u) - \epsilon \int_{\Omega} |u|^{p_2^+} dx - C(\epsilon) \int_{\Omega} |u|^{r(x)} dx
\]
\[
\geq \frac{1}{2} \|u\|^{p_2^+} - \epsilon \|u\|_{L_{p_2^+}^+} - C(\epsilon) \left\{ \|u\|^{r_+}_{L_{r_+}^+} + \|u\|^{r^-}_{L_{r^-}} \right\}
\]
\[
\geq \frac{1}{2} \|u\|^{p_2^+} - \epsilon C \|u\|^{p_2^+} - C'(\epsilon) \|u\|^{r^-}
\]
\[
= \left(1 - \epsilon C\right) \|u\|^{p_2^+} - C'(\epsilon) \|u\|^{r^-},
\]
where $C'(e) > 0$ is a constant. Consider

$$0 < \varepsilon < \frac{1}{4C}.$$ 

Since $p_2^+ < r^-$, we can choose $\alpha \in (0, 1)$ sufficiently small such that for all $u \in X$ with $\|u\| = \alpha$

$$\mathcal{I}(u) \geq \alpha^{p_2^+} \left( \frac{1}{2} - \varepsilon C \right) - C'(e)\alpha^{r^-} = R > 0.$$ 

The proof of $(i)$ is complete.

$(ii)$ It follows from $(f_1)$ and $(f_2)$ that for any positive constant $M$, there exists a corresponding positive constant $C_M$ such that

$$F(x, t) \geq M|t|^{p_2^+} - C_M.$$ 

Let $e \in X$, $e > 0$ with $\|e\| = 1$ and $\int |e|^{p_2^+} \, dx > 0$ and $t > 1$. Then, using Lemma 3.3 and (4.2), we get

$$I(te) = \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} tp_1(x, y) \frac{|e(x) - e(y)|^{p_1(x, y)}}{2p_1(x, y)|x - y|^{N+p_1(x, y)}} \, dx \, dy + \int_{\partial \Omega} tp_1(x) \beta(x) \frac{|e|^{p_1(x)}}{p_1(x)} \, dx$$

$$+ \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} tp_2(x, y) \frac{|e(x) - e(y)|^{p_2(x, y)}}{2p_2(x, y)|x - y|^{N+p_2(x, y)}} \, dx \, dy + \int_{\partial \Omega} tp_2(x) \beta(x) \frac{|e|^{p_2(x)}}{p_2(x)} \, dx$$

$$+ \int_{\Omega} tp_2(x) \frac{|e|^{p_2(x)}}{p_2(x)} \, dx - \int_{\Omega} F(x, te) \, dx$$

$$\leq t^{p_2^+} \rho(e) - t^{p_2^+} M \int_{\Omega} |e|^{p_2^+} \, dx + |\Omega|C_M$$

$$= t^{p_2^+} \left[ 1 - M \int_{\Omega} |e|^{p_2^+} \, dx \right] + |\Omega|C_M$$

We choose $M$ sufficiently large so that

$$\lim_{t \to +\infty} I(te) = -\infty.$$ 

Hence, there exists some $t_0 > 0$ such that $\mathcal{I}(\varphi) < 0$, where $\varphi = t_0 e$. Thus, the proof of $(ii)$ is complete.

4.3. Cerami condition

**Proposition 4.4.** If hypotheses $(H_1)$–$(H_2)$, $(\beta)$ and $(f_1)$–$(f_4)$ hold, then the functional $\mathcal{I}$ satisfies the $(C)_c$-condition for any $c \in \mathbb{R}$.

**Proof.** In this proof, the value of the constant $C$ changes from line to line. We consider a sequence $(u_n)_{n \geq 1} \subset X$ such that

$$|\mathcal{I}(u_n)| \leq C \quad \text{for some} \quad C > 0 \quad \text{and for all} \quad n \geq 1,$$ 

$$(1 + \|u_n\|)\mathcal{I}'(u_n) \to 0 \quad \text{in} \quad X^* \quad \text{as} \quad n \to +\infty.$$ 


From (4.4), we have

$$|\mathcal{H}(u_n, v)| \leq \frac{\epsilon_n \|v\|}{1 + \|u_n\|},$$

(4.5)

for all \(v \in X\) with \(\epsilon_n \to 0\).

In (4.5), we choose \(v = u_n \in X\) and obtain for all \(n \in \mathbb{N}\)

$$- \frac{1}{2} \delta_{\mathbb{R}^N \setminus (\Omega)}^p(u_n) - \frac{1}{2} \delta_{\mathbb{R}^N \setminus (\Omega)}^p(u_n) - \rho_{\Omega}^{p_1}(u_n) - \rho_{\Omega}^{p_2}(u_n)
- \int_{\Omega} \beta(x)|u_n|^p_1(x)dx - \int_{\Omega} \beta(x)|u_n|^p_2(x)dx + \int_{\Omega} f(x, u_n(x))u_n(x)dx
\leq \epsilon_n.$$

(4.6)

Also, by (4.3) we have for all \(n \in \mathbb{N}\),

$$\frac{1}{2p_1} \delta_{\mathbb{R}^N \setminus (\Omega)}^p(u_n) + \frac{1}{2p_2} \delta_{\mathbb{R}^N \setminus (\Omega)}^p(u_n) + \frac{1}{p_1} \rho_{\Omega}^{p_1}(u_n) + \frac{1}{p_2} \rho_{\Omega}^{p_2}(u_n)
+ \frac{1}{p_1} \int_{\Omega} \beta(x)|u_n|^p_1(x)dx + \frac{1}{p_2} \int_{\Omega} \beta(x)|u_n|^p_2(x)dx
- \int_{\Omega} F(x, u_n(x))dx \leq C.$$

(4.7)

Adding relations (4.6) and (4.7), we obtain

$$\int_{\Omega} \mathcal{F}(x, u_n(x))dx \leq C \quad \text{for some } C > 0 \quad \text{and all } n \in \mathbb{N}.$$

(4.8)

**Claim:** The sequence \((u_n)_{n \geq 1} \subset X\) is bounded.

We argue by contradiction. Suppose that the claim is not true. We may assume that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty.$$

(4.9)

We set \(w_n := \frac{u_n}{\|u_n\|}\) for all \(n \in \mathbb{N}\). Then \(\|w_n\| = 1\), for all \(n \in \mathbb{N}\). Using reflexivity of \(X\) and Lemma 3.4, up to a subsequence, still denoted by \((w_n)_{n \geq 1}\), as \(n \to +\infty\), we get

$$w_n \rightharpoonup w \quad \text{weakly in } X \quad \text{and} \quad w_n \to w \quad \text{strongly in } L^{\gamma}(\Omega), \quad 1 < \gamma(x) < p_{2^*}^*(x).$$

(4.10)

We claim that \(w = 0\). Indeed, if not then the set \(\hat{\Omega} := \{x \in \Omega : w(x) \neq 0\}\) has positive Lebesgue measure, i.e., \(|\hat{\Omega}| > 0\). Hence, \(|u_n(x)| \to +\infty\) for a.e. \(x \in \hat{\Omega}\) as \(n \to +\infty\). On account of hypothesis \((f_2)\), for a.e. \(x \in \hat{\Omega}\) we have

$$\frac{F(x, u_n(x))}{\|u_n\|^{p_2^*}} = \frac{F(x, u_n(x))}{|u_n(x)|^{p_2^*}} |w_n(x)|^{p_2^*} \to +\infty \quad \text{as } n \to +\infty.$$

(4.11)

Then by Fatou’s lemma, we obtain

$$\int_{\hat{\Omega}} \frac{F(x, u_n(x))}{\|u_n\|^{p_2^*}}dx \to +\infty \quad \text{as } n \to +\infty.$$

(4.12)

Hypotheses \((f_1)-(f_2)\) imply there exists \(K > 0\) such that

$$\frac{F(x, t)}{|t|^{p_2^*}} \geq 1 \quad \text{for a.e. } x \in \Omega, \text{ all } |t| > K.$$

(4.13)
By (f1), there exists a positive constant $\tilde{C} > 0$ such that
\[ |F(x, t)| \leq \tilde{C}, \quad \text{for all } (x, t) \in \overline{\Omega} \times [-K, K]. \]

(4.14)

Now from (4.13) and (4.14), we get
\[ F(x, t) > C_0 \quad \text{for all } (x, t) \in \overline{\Omega} \times \mathbb{R}, \]

(4.15)

where $C_0 \in \mathbb{R}$ is a constant. The above relation implies
\[ \frac{F(x, u_n(x)) - C_0}{\|u_n\|^{p_2^+}_2} \geq 0 \quad \text{for all } x \in \overline{\Omega}, \quad \text{for all } n \in \mathbb{N}. \]

(4.16)

By (4.3), (4.9), (4.12), (4.16) and using the fact $\|w_n\| = 1$, Lemma 3.1 and Fatou’s lemma, we have

\[ +\infty = \int_{\overline{\Omega}} \liminf_{n \to +\infty} \frac{F(x, u_n(x))|w_n(x)|^{p_2^+}_2}{\|u_n\|^{p_2^+}_2} \, dx - \int_{\overline{\Omega}} \limsup_{n \to +\infty} \frac{C_0}{\|u_n\|^{p_2^+}_2} \, dx \]

\[ = \int_{\overline{\Omega}} \liminf_{n \to +\infty} \left[ \frac{F(x, u_n(x))|w_n(x)|^{p_2^+}_2}{\|u_n\|^{p_2^+}_2} - \frac{C_0}{\|u_n\|^{p_2^+}_2} \right] \, dx \]

\[ \leq \int_{\overline{\Omega}} \liminf_{n \to +\infty} \left[ \frac{F(x, u_n(x))|w_n(x)|^{p_2^+}_2}{\|u_n\|^{p_2^+}_2} - \frac{C_0}{\|u_n\|^{p_2^+}_2} \right] \, dx \]

\[ \leq \liminf_{n \to +\infty} \int_{\overline{\Omega}} \frac{F(x, u_n(x))|w_n(x)|^{p_2^+}_2}{\|u_n\|^{p_2^+}_2} \, dx - \limsup_{n \to +\infty} \int_{\overline{\Omega}} \frac{C_0}{\|u_n\|^{p_2^+}_2} \, dx \]

\[ = \liminf_{n \to +\infty} \int_{\overline{\Omega}} \frac{F(x, u_n(x))}{\|u_n\|^{p_2^+}_2} \, dx \]

\[ = \liminf_{n \to +\infty} \left[ \int_{\mathbb{R}^N \setminus (\Omega)^2} \frac{1}{\|u_n\|^{p_2^+ - p_1(x, y)}_2} \frac{|w_n(x) - w_n(y)|^{p_1(x, y)}_2}{|x - y|^{N + s(x, y)p_1(x, y)}} \, dx \right] \]

\[ + \frac{1}{2} \int_{\mathbb{R}^N \setminus (\Omega)^2} \frac{1}{\|u_n\|^{p_2^+ - p_2(x, y)}_2} \frac{|w_n(x) - w_n(y)|^{p_2(x, y)}_2}{|x - y|^{N + s(x, y)p_2(x, y)}} \, dx \]

\[ + \int_{\mathcal{C} \Omega} \frac{1}{\|u_n\|^{p_2^+ - p_1(x)}_2} \frac{|w_n|^{p_1(x)}_2}{p_1(x)} \, dx + \int_{\mathcal{C} \Omega} \frac{1}{\|u_n\|^{p_2^+ - p_2(x)}_2} \frac{|w_n|^{p_2(x)}_2}{p_2(x)} \, dx \]

\[ + \int_{\mathcal{C} \Omega} \frac{\beta(x)|w_n|^{p_1(x)}_2}{\|u_n\|^{p_2^+ - p_1(x)}_2 p_1(x)} \, dx + \int_{\mathcal{C} \Omega} \frac{\beta(x)|w_n|^{p_2(x)}_2}{\|u_n\|^{p_2^+ - p_2(x)}_2 p_2(x)} \, dx - \int_{\mathcal{C} \Omega} \mathcal{I}(u_n) \]

\[ \leq \liminf_{n \to +\infty} \rho(w_n) = 1. \]

(4.17)
Thus, we arrive at a contradiction. Hence, \( w = 0 \). Let \( \mu \geq 1 \) and set \( \kappa := (2\mu)^{-\frac{1}{p^*_2}} \geq 1 \) for all \( n \in \mathbb{N} \). Evidently, from (4.10) we have

\[
\lim_{n \to 0} w_n = 0 \quad \text{strongly in} \quad L^{\gamma}(\Omega), \quad 1 < \gamma(x) < p^*_2(x)
\]

which combining with \((f_1)\)–\((f_3)\) and Lebesgue dominated convergence theorem yields that

\[
\int_{\Omega} F(x, \kappa w_n) \, dx \to 0 \quad \text{as} \quad n \to +\infty.
\]  

(4.18)

We can find \( t_n \in [0, 1] \) such that

\[
I(t_n u_n) = \max_{0 \leq t \leq 1} I(t u_n).
\]  

(4.19)

Because of (4.9), for sufficiently large \( n \in \mathbb{N} \), we have

\[
0 < (2\mu)^{-\frac{1}{p^*_2}} \leq 1.
\]  

(4.20)

Using (4.18), (4.19) and (4.20) and recalling that \( \|w_n\| = 1 \), for sufficiently large \( n \in \mathbb{N} \), it follows that

\[
\|w_n\| = \|u_n\| = \|w_n\| = 1,
\]

for sufficiently large \( n \in \mathbb{N} \).

(4.21)

From hypothesis \((f_4)\), we obtain for all \( n \in \mathbb{N} \),

\[
\mathcal{F}(x, t_n u_n) \leq \mathcal{F}(x, u_n) + b(x) \quad \text{for a.e} \quad x \in \Omega,
\]

that is,

\[
f(x, t_n u_n)(t_n u_n) \leq \mathcal{F}(x, u_n) + b(x) + p^*_2 F(x, t_n u_n) \quad \text{for a.e} \quad x \in \Omega.
\]  

(4.26)
Combining (4.25) and (4.26), we deduce
\[
\frac{1}{2} \delta_{\mathbb{R}^N \setminus (\mathbb{O})^2}(t_n u_n) + \rho_{\mathbb{O}}(t_n u_n) + \int_{\mathbb{O}} \beta(x)|t_n u_n|^p_2(x) \, dx \\
+ \frac{1}{2} \delta_{\mathbb{R}^N \setminus (\mathbb{O})^2}(t_n u_n) + \rho_{\mathbb{O}}(t_n u_n) + \int_{\mathbb{O}} \beta(x)|t_n u_n|^p_2(x) \, dx - p_{\mathbb{O}}^+ \int_{\mathbb{O}} F(x, t_n u_n) dx \\
\leq \int_{\mathbb{O}} F(x, u_n) dx + \|b\|_{L^1(\mathbb{O})} \text{ for all } n \in \mathbb{N},
\]
and hence by (4.8), we get
\[
p_{\mathbb{O}}^+ \mathcal{I}(t_n u_n) \leq C \text{ for all } n \in \mathbb{N}.
\] (4.27)

We compare (4.21) and (4.27) and arrive at a contradiction. Thus, the claim follows.

On account of this claim, we may assume that
\[
u_n \rightarrow u \text{ weakly in } X \text{ and } u_n \rightarrow u \text{ strongly in } L^\gamma(\mathbb{O}), \ 1 < \gamma(x) < p_{\mathbb{O}}^*(x).
\] (4.28)

We show in what follows that
\[
u_n \rightarrow u \text{ in } X.
\]
Using (4.28), we have
\[
o_n(1) = \langle \mathcal{I}'(u_n), u_n - u \rangle \geq \frac{1}{2} \langle \rho'(u_n), u_n - u \rangle - \int_{\mathbb{O}} f(x, u_n)(u_n - u) dx.
\] (4.29)

Now by (f_1), Hölder inequality, (4.28), boundedness of (u_n)_n and Lemma 2.3, we obtain
\[
\left| \int_{\mathbb{O}} f(x, u_n)(u_n - u) dx \right| \\
\leq \|a\|_{L^\infty(\mathbb{O})} \left[ \int_{\mathbb{O}} |u_n - u| |\xi| \, dx + \int_{\mathbb{O}} |v_n|^\gamma(x) \xi v_n - v_0 \, dx \right] \\
\leq \|a\|_{L^\infty(\mathbb{O})} \left[ \|u_n - u\|_{L^\gamma(\mathbb{O})} (1 + |\mathbb{O}|)^{\gamma^+} + \|v_n - v_0\|_{L^{\gamma}(\mathbb{O})} \|u_n\|_{L^{\gamma}(\mathbb{O})}^{\gamma-1} \right] \\
\leq C \left[ \|u_n - u\|_{L^\gamma(\mathbb{O})} + \|u_n - u\|_{L^\gamma(\mathbb{O})} \right] \rightarrow 0 \text{ as } n \rightarrow +\infty.
\] (4.30)

Hence, combining (4.29) and (4.30) and using the \(S_+\) property of \(\rho'\) (see Lemma 3.5), we have \(u_n \rightarrow u\) strongly in \(X\) as \(n \rightarrow +\infty\), which shows that the \((C)_e\)-condition is satisfied. The proof is now complete.}

5. Proof of Theorem 1.2

To prove the Theorem 1.2, we use the Fountain theorem of Bartsch [6, Theorem 2.5]; (see also [49, Theorem 3.6]). To this end, we will start by recalling the following lemma (see [21]).

Lemma 5.1. Let \(E\) be a reflexive and separable Banach space. Then there are \(\{e_n\} \subset E\) and \(\{g^n_n\} \subset E^*\) such that
\[
E = \text{span}\{e_n : n = 1, 2, 3, \ldots\}, \ E^* = \text{span}\{g^n_n : n = 1, 2, 3, \ldots\}.
\]
and
\[ \langle g_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \]

Let us denote
\[ E_n = \text{span}\{e_n\}, \quad X_k = \bigoplus_{n=1}^k E_n \quad \text{and} \quad Y_k = \bigoplus_{n=k}^\infty E_n. \tag{5.1} \]

Now we recall the following variant of the well-known Fountain Theorem from [2]:

**Theorem 5.2.** (Fountain theorem) Assume that \( \Phi \in C^1(E, \mathbb{R}) \) satisfies the Cerami condition \((C)_c\) for all \( c \in \mathbb{R} \) and \( \Phi(-u) = \Phi(u) \). If for each sufficiently large \( k \in \mathbb{N} \), there exists \( \varrho_k > \delta_k > 0 \) such that

\[ (A_1) \quad b_k := \inf \{ \Phi(u) : u \in Y_k, \|u\|_E = \delta_k \} \to +\infty, \text{ as } k \to +\infty, \]

\[ (A_2) \quad a_k := \max \{ \Phi(u) : u \in X_k, \|u\|_E = \varrho_k \} \leq 0. \]

Then \( \Phi \) has a sequence of critical points \((u_k)_k\) such that \( \Phi(u_k) \to +\infty \).

**Proof of Theorem 1.2.** Define \( X_k \) and \( Y_k \) as in (5.1) for the reflexive, separable Banach space \( X \). Due to the hypothesis of Theorem 1.2, we obtain that \( I \) is even and satisfies Cerami condition \((C)_c\) for all \( c \in \mathbb{R} \) (see Lemma 4.4). Let us first prove that the conditions \((A_1)-(A_2)\) hold for our problem.

\((A_1)\) : For large \( k \in \mathbb{N} \), set
\[ \alpha_k = \sup_{u \in Y_k, \|u\|_E = 1} \|u\|_{L^\gamma(\Omega)}, \tag{5.2} \]
where \( \gamma \in C_+(\overline{\Omega}) \) such that for all \( x \in \overline{\Omega}, 1 < \gamma(x) < p^*_a(x). \) So,
\[ \lim_{k \to +\infty} \alpha_k = 0. \tag{5.3} \]

Suppose, contrary to our claim, that there exist \( \epsilon' > 0, k_0 \geq 0 \) and a sequence \((u_k)_k\) in \( Y_k \) such that
\[ \|u_k\|_E = 1 \quad \text{and} \quad \|u_k\|_{L^\gamma(\Omega)} \geq \epsilon' \]
for all \( k \geq k_0. \) Since \((u_k)_k\) is bounded in \( X \), there exists \( u_0 \in E \) such that up to a subsequence, still denoted by \((u_k)_k\), we have \( u_k \rightharpoonup u_0 \) weakly in \( E \) as \( k \to +\infty \) and
\[ \langle g_j^*, u_0 \rangle = \lim_{k \to +\infty} \langle g_j^*, u_k \rangle = 0 \]
for \( j = 1, 2, 3, \ldots \). Thus, we conclude that \( u = 0. \) Furthermore, by Lemma 3.4 we obtain
\[ \epsilon' \leq \lim_{k \to +\infty} \|u_k\|_{L^\gamma(\Omega)} = \|u_0\|_{L^\gamma(\Omega)} = 0, \]
which is impossible. Consequently, (5.3) holds true. Let \( u \in Y_k \) with \( \|u\| > 1 \). Note that (5.3) implies \( \alpha_k < 1 \) for large \( k \in \mathbb{N}. \) Thus, using Lemma 2.3, Lemma 3.3 and (5.2) and (4.1) with \( \epsilon = 1 \) for \( k \in \mathbb{N} \) large enough, we get
\[ I(u) \geq \frac{1}{2} \rho(u) - \int_\Omega |u|^{p^*_a} \, dx - C(1) \int_\Omega |u|^{r(x)} \, dx \]
\[ \geq \frac{1}{2} \|u\|^{p^*_a} - \|u\|_{L^{p^*_a}(\Omega)}^{p^*_a} - C(1) \left\{ \|u\|_{L^{r}(\Omega)}^{r} + \|u\|_{L^{r}(\Omega)}^{r^*} \right\} \]
\[ \geq \frac{1}{2} \|u\|^{p^*_a} - \alpha_k^{p^*_a} C_1 \|u\|^{p^*_a} - C_2 \left\{ \alpha_k^{r^*} \|u\|^{r^*} + \alpha_k^{r^*} \|u\|^{r^*} \right\} \]
\[ \geq \frac{1}{2} \|u\|^{p^*_a} - C \alpha_k \|u\|^{r^*}, \tag{5.4} \]
where \( C, C_1, C_2 \) are some positive constants.
Define the function \( G : \mathbb{R} \to \mathbb{R} \),
\[
G(t) = \frac{1}{2} t^{p_1^-} - C \alpha_k t^{r^+}.
\]

The function \( G \) attains its global maximum at
\[
\delta_k = \left( \frac{p_1^-}{2r^+ \alpha_k} \right)^{1/(r^+ - p_1^-)}
\]
and the maximum value of \( G \) is
\[
G(\delta_k) = \frac{1}{2} \left( \frac{p_1^-}{2r^+ \alpha_k} \right)^{p_1^-/(r^+ - p_1^-)} - C \alpha_k \left( \frac{p_1^-}{2r^+ \alpha_k} \right)^{r^+/(r^+ - p_1^-)}
\]
\[
= \left( \frac{1}{2} \right)^{r^+/(r^+ - p_1^-)} \left( \frac{1}{C \alpha_k} \right)^{p_1^-/(r^+ - p_1^-)} \left( \frac{p_1^-}{r^+} \right)^{\delta_k} \left( 1 - \frac{p_1^-}{r^+} \right).
\]

Since \( p_1^- < r^+ \) and \( \alpha_k \to 0 \) as \( k \to +\infty \), we obtain
\[
G(\delta_k) \to +\infty \text{ as } k \to +\infty. \tag{5.5}
\]

Again, (5.3) infers \( \delta_k \to +\infty \) as \( k \to +\infty \). Thus for \( u \in Y_k \) with \( \|u\| = \delta_k \), taking into account (5.4) and (5.5), it follows that as \( k \to +\infty \)
\[
b_k = \inf_{u \in Y_k, \|u\| = \delta_k} I(u) \to +\infty.
\]

(\( A_2 \)): Suppose the assertion \( (A_2) \) is false for some \( k \). Thus, there exists a sequence \( (u_n)_n \subset X_k \) such that
\[
\|u_n\| \to +\infty \text{ and } I(u_n) \geq 0. \tag{5.6}
\]

Let us take \( w_n := \frac{u_n}{\|u_n\|} \), then \( w_n \in X \) and \( \|w_n\| = 1 \). Since \( X_k \) is of finite dimension, there exists \( w \in X_k \setminus \{0\} \) such that up to a subsequence, still denoted by \( (w_n)_n \), \( w_n \to w \) strongly and \( w_n(x) \to w(x) \) a.e. \( x \in \Omega \) as \( n \to +\infty \). If \( w(x) \neq 0 \) then \( |u_n(x)| \to +\infty \) as \( n \to +\infty \). It follows by the same method as in (4.11) that
\[
\frac{F(x, u_n(x))}{|u_n(x)|^{p_2^+}} |w_n(x)|^{p_2^+} \to +\infty, \text{ a.e. } x \in \Omega, \tag{5.7}
\]
holds. Hence, using (5.6) and applying Fatou’s lemma, we have
\[
\frac{1}{\|u_n\|^{p_2^+}} \int_{\Omega} F(x, u_n) \, dx = \int_{\Omega} \frac{F(x, u_n)}{|u_n(x)|^{p_2^+}} |w_n(x)|^{p_2^+} \, dx \to +\infty \text{ as } \tag{5.8}
\]

Since \( \|u_n\| > 1 \) for large \( n \in \mathbb{N} \), from Proposition 3.3 and (5.8), we obtain as \( n \to +\infty \)
\[
I(u_n) \leq \rho(u_n) - \int_{\Omega} F(x, u_n) \, dx
\]
\[
\leq \|u_n\|^{p_2^+} - \int_{\Omega} F(x, u_n) \, dx
\]
\[
= \left( 1 - \frac{1}{\|u_n\|^{p_2^+}} \right) \int_{\Omega} F(x, u_n) \, dx \|u_n\|^{p_2^+} \to -\infty.
\]

This contradicts (5.6). Thus, for sufficiently large \( k \in \mathbb{N} \), we can get \( \varrho_k > \delta_k > 0 \) such that \( (A_2) \) holds for \( u \in X_k \) with \( \|u\| = \varrho_k \). \( \square \)
6. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. To this end, we first recall the Dual Fountain Theorem due to Bartsch and Willem (see [49, Theorem 3.18]). Considering Lemma 5.1 and using the reflexivity and separability of the Banach space $X$, we can define $X_k$ and $Y_k$ appropriately.

**Definition 6.1.** For $c \in \mathbb{R}$, we say that $I$ satisfies the $(C)_c^*$ condition (with respect to $Y_k$) if any sequence $(u_k)_k$ in $X$ with $u_k \in Y_k$ such that

$$I(u_k) \to c \text{ and } \|I|_{Y_k}(u_k)\|_{E^*}(1 + \|u_k\|) \to 0, \text{ as } k \to +\infty$$

contains a subsequence converging to a critical point of $I$, where $X^*$ is the dual of $X$.

**Theorem 6.2.** (Dual fountain Theorem) Let $\Phi \in C^1(E, \mathbb{R})$ satisfy $\Phi(-u) = \Phi(u)$. If for each $k \geq k_0$ there exist $\varrho_k > \delta_k > 0$ such that

(B1) $a_k = \inf\{\Phi(u) : u \in Z_k, \|u\|_E = \varrho_k\} \geq 0$;

(B2) $b_k = \sup\{\Phi(u) : u \in Y_k, \|u\|_E = \delta_k\} < 0$;

(B3) $d_k = \inf\{\Phi(u) : u \in Z_k, \|u\|_E \leq \varrho_k\} \to 0$ as $k \to +\infty$;

(B4) $\Phi$ satisfies the $(C)_c^*$ condition for every $c \in [d_k, 0]$.

Then $\Phi$ has a sequence of negative critical values converging to 0.

**Remark 6.3.** In [49], whenever the energy functional associated to the problem satisfies $(PS)_c^*$ condition, the Dual fountain theorem is obtained using Deformation theorem which is still valid under Cerami condition. Therefore, like many critical point theorems the Dual fountain theorem holds under $(C)_c^*$ condition.

Next lemma is due to [27, Lemma 3.2]

**Lemma 6.4.** Suppose that the hypotheses in Theorem 1.3 hold, then $I$ satisfies the $(C)_c^*$ condition.

**Proof of Theorem 1.3.** Since $X$ is a reflexive, separable Banach space, we can define $X_k$ and $Y_k$ as in (5.1). From the assumptions, we have that $I$ is even and by Lemma 6.4 we get that $I$ satisfies Cerami condition $(C)_c^*$ for all $c \in \mathbb{R}$. Thus, it remains to prove the conditions (B1)–(B3).

(B1): Let $u \in Y_k$ be a function such that $\|u\| < 1$, arguing similarly as we did for obtaining (5.4), we can derive

$$I(u) \geq \frac{1}{2} [\rho_{p_1}(u) + \rho_{p_2}(u)] - \int_\Omega F(x, u) \, dx$$

$$\geq \frac{1}{2} \rho(u) - \left[ \int_\Omega |u|^{p_2^+} \, dx - C(1) \int_\Omega |u|^{r(x)} \, dx \right]$$

$$\geq \frac{1}{2} \|u\|^{p_2^+} - \|u\|^{p_2^+}_{L^{p_2^+}}(\Omega) - C(1) \left\{ \|u\|^{r^-}_{L^r(\Omega)} + \|u\|^{r^+}_{L^r(\Omega)} \right\}$$

$$\geq \frac{1}{2} \|u\|^{p_2^+} - \alpha_k^{p_2^+} C_1 \|u\|^{p_2^+} - C_2 \{\alpha_k^{r^-} \|u\|^{r^-} + \alpha_k^{r^+} \|u\|^{r^+} \}$$

$$\geq \frac{1}{2} \|u\|^{p_2^+} - C_4 \alpha_k \|u\|, \quad (6.1)$$

Let us choose $\varrho_k = (C_4 \alpha_k/2)^{1/(p_2^+ - 1)}$. Since $p_2^+ > 1$, (5.2) infers that

$$\varrho_k \to 0 \text{ as } k \to +\infty. \quad (6.2)$$

Thus, for $u \in Y_k$ with $\|u\| = \varrho_k$ and for sufficiently large $k \in \mathbb{N}$, from (6.1) we have $I(u) \geq 0.$
(B₂): Suppose assertion (B₂) is false for some given $k \in \mathbb{N}$. Then there exists a sequence $(v_n)_n$ in $X_k$ such that

$$
\|v_n\| \to +\infty, \quad I(v_n) \geq 0.
$$

(6.3)

Now arguing in a similar way as in the proof of assertion (A₂) of Theorem 5.2, we obtain (5.7) and (5.8) which combining with Lemma 3.3 imply that

$$
I(v_n) \leq \rho_k \|v_n\| - \int_\Omega F(x, v_n) \, dx
$$

$$
= \left(1 - \frac{1}{\|v_n\|^{p^*_2}} \int_\Omega F(x, v_n) \, dx\right) \|v_n\|^{p^*_2} \to -\infty,
$$

which contradicts (6.3). So, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $1 > \varrho_k > \delta_k > 0$ such that for $u \in X_k$ with $\|u\| = \delta_k$ the condition (B₂) holds true.

(B₃): Since $X_k \cap Y_k \neq \emptyset$, we get that $d_k \leq b_k < 0$. Now for $u \in Y_k$ with $\|u\| \leq \varrho_k$ by (6.1), we get

$$
I(u) \geq -C_4 \alpha_k \|u\| \geq -C_4 \alpha_k \varrho_k.
$$

Therefore, combining (5.2) and (6.2), we obtain

$$
d_k \geq -C_4 \alpha_k \varrho_k \to 0 \text{ as } k \to +\infty.
$$

Since $d_k < 0$, it follows that $\lim_{k \to +\infty} d_k = 0$. This finishes the proof. □

7. Proof of Theorem 1.4

First we state the following $\mathbb{Z}_2$-symmetric version of mountain pass theorem due to [40, Theorem 9.12]. Here again we want to mention that in [40] this theorem is proved using (PS)-condition, which can also be proved using C-condition.

**Theorem 7.1.** (Symmetric Mountain pass Theorem): Let $E$ be a real infinite dimensional Banach space and $\Phi \in C^1(E, \mathbb{R})$ be an even functional satisfying the $(C)_c$ condition. Also let $\Phi$ satisfy the following:

(\textbf{D}₁) $\Phi(0) = 0$ and there exist two constants $\nu, \mu > 0$ such that $\Phi(u) \geq \mu$ for all $u \in E$ with $\|u\|_E = \nu$.

(\textbf{D}₂) For all finite dimensional subspaces $\hat{E} \subset E$ there exists $\overline{R} = \overline{R}(\hat{E}) > 0$ such that $\Phi(u) \leq 0$ for all $u \in \hat{E} \setminus B_{\overline{R}}(\hat{E})$, where $B_{\overline{R}}(\hat{E}) = \{u \in \hat{E} : \|u\|_E \leq \overline{R}\}$.

Then $\Phi$ poses an unbounded sequence of critical values characterized by a minimax argument.

**Proof of Theorem 1.4.** From the hypotheses of the theorem, it follows that $I$ is even and we have $I(0) = 0$. Now we will prove that $I$ satisfies the assertions in Theorem 7.1.

(\textbf{D}₁): It follows from Lemma 4.3(i).

(\textbf{D}₂): To show this, first claim that for any finite dimensional subspace $Y$ of $X$ there exists $\overline{R}_0 = \overline{R}_0(Y)$ such that $I(u) < 0$ for all $u \in E \setminus B_{\overline{R}_0}(Y)$, where $B_{\overline{R}_0}(Y) = \{u \in X : \|u\| \leq \overline{R}_0\}$. Fix $u \in$
X, \|u\| = 1. For t > 1 using (4.2) and Lemma 3.3, we get
\[
\mathcal{I}(tu) \leq \rho(tu) - \int_{\Omega} F(x,u)dx \\
\leq t^{p_2^*} \rho(u) - t^{p_2^*} M \int_{\Omega} |u|^{p_2^*} dx + |\Omega| C_M \\
= t^{p_2^*} \left[ 1 - M \|u\|_{L^{p_2^*}(\Omega)}^{p_2^*} \right] + |\Omega| C_M. \tag{7.1}
\]
Since Y is finite dimensional, all norms are equivalent on Y, which infers that there exists some constant \( C(Y) > 0 \) such that \( C(Y) \|u\| \leq \|u\|_{L^{p_2^*}(\Omega)} \). Therefore, from (7.1), we obtain
\[
\mathcal{I}(tu) \leq t^{p_2^*} \left[ 1 - M (C(Y))^{p_2^*} \|u\|^{p_2^*} \right] + |\Omega| C_M \\
= t^{p_2^*} \left[ 1 - M (C(Y))^{p_2^*} \right] + |\Omega| C_M.
\]
Now by choosing \( M \) sufficiently large such that \( M > \frac{1}{(C(Y))^{p_2^*}} \), from the last relation we yields that
\[
\mathcal{I}(u) \to -\infty \text{ as } t \to +\infty.
\]
Hence, there exists \( R_0 > 0 \) large enough such that \( \mathcal{I}(u) < 0 \) for all \( u \in X \) with \( \|u\| = R \) and \( R \geq R_0 \). Therefore, \( \mathcal{I} \) verifies \((D_2)\). \hfill \Box

8. Proof of Theorem 1.5

First, we recall a new variant of Clark’s theorem (see [32, Theorem 1.1]).

**Theorem 8.1.** Let \( E \) be a Banach space, \( \Phi \in C^1(E,\mathbb{R}) \). Let \( \Phi \) be even and \( \Phi(0) = 0 \). Also assume \( \Phi \) satisfies the (PS)-condition and bounded from below. If for any \( k \in \mathbb{N} \), there exists a \( k \)-dimensional subspace \( E^k \) of \( E \) and \( \beta_k > 0 \) such that \( \sup_{E^k \cap B_{\beta_k}} \Phi(u) < 0 \), where \( B_{\beta_k} = \{ u \in E : \|u\|_E = \beta_k \} \), then at least one of the following conclusions holds:

\((M_1)\) There exists a sequence of critical points \((u_k)_k\) satisfying \( \Phi(u_k) < 0 \) for all \( k \) and \( \|u_k\|_E \to 0 \) as \( k \to +\infty \).

\((M_2)\) There exists \( l > 0 \) such that for any \( 0 < b < l \) there exists a critical point \( u \) such that \( \|u\|_E = b \) and \( \Phi(u) = 0 \).

The corresponding energy functional is given as
\[
\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}\setminus \mathcal{C}^2} \frac{|u(x) - u(y)|^{p_1(x,y)}}{p_1(x,y)|x-y|^{N+s(x,y)p_1(x,y)}} dx dy + \int_{\Omega} \frac{1}{p_1(x)} |u|^{p_1(x)} dx \\
+ \frac{1}{2} \int_{\mathbb{R}^{2N}\setminus \mathcal{C}^2} \frac{|u(x) - u(y)|^{p_2(x,y)}}{p_2(x,y)|x-y|^{N+s(x,y)p_2(x,y)}} dx dy + \int_{\Omega} \frac{1}{p_2(x)} |u|^{p_2(x)} dx \\
+ \int_{\mathcal{C}^{\Omega}} \frac{\beta(x)|u|^{p_1(x)}}{p_1(x)} dx + \int_{\mathcal{C}^{\Omega}} \frac{\beta(x)|u|^{p_2(x)}}{p_2(x)} dx \\
- \lambda \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} dx - \int_{\Omega} \frac{|u|^{r(x)}}{r(x)} dx.
\]
Next, we will prove the following lemma:

**Lemma 8.2.** Suppose the hypotheses in Theorem 1.5 hold. Then $I$ satisfies $(PS)_{c}$ for any $c \in \mathbb{R}$.

**Proof.** Let $(v_n)_n$ be a sequence in $X$ such that

$$I(v_n) \rightarrow c \text{ and } I'(v_n) \rightarrow 0 \text{ in } X^* \text{ as } n \rightarrow +\infty.$$  

(8.1)

Therefore,

$$\langle I'(v_n), v_n - v_0 \rangle \rightarrow 0 \text{ as } n \rightarrow +\infty.$$  

(8.2)

Hence, we have that $(v_n)_n$ is bounded in $X$. If not, then $v_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Using (8.1) and (8.2) and $(f_5)$, we deduce

$$1 + C + \|v_n\| \geq I(v_n) - \frac{1}{q^*} \langle I'(v_n), v_n \rangle$$

$$\geq \frac{1}{2} \left[ \rho(v_n) - \int_{\Omega} F(x, v_n) dx - \frac{1}{q^*} \rho(v_n) + \frac{1}{q^*} \int_{\Omega} f(x, v_n) v_n dx \right]$$

$$\geq \frac{1}{2} \left[ (1 - \frac{1}{q^*}) \|v_n\|^{p^*} + \left( \frac{1}{q^*} - \frac{1}{r} \right) \int_{\Omega} |v_n|^{r(x)} dx \right]$$

$$\geq \frac{1}{2} \left( 1 - \frac{1}{q^*} \right) \|v_n\|^{p^*},$$  

(8.3)

which is a contradiction to the fact that $v_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Now, since $X$ is reflexive, up to a subsequence, still denoted by $(v_n)_n$, we have $v_n \rightharpoonup v_0$ weakly as $n \rightarrow +\infty$. Therefore, as $n \rightarrow +\infty$ by Theorem 3.4, arguing similar as in (4.30), we obtain

$$v_n \rightarrow v_0 \text{ strongly in } L^\gamma(\Omega), 1 < \gamma(x) < p^*_s(x) \text{ and } v_n(x) \rightarrow v_0(x) \text{ a.e. in } \Omega.$$  

(8.4)

By $(f_5)$, Hölder inequality, (8.4), boundedness of $(v_n)_n$ and Lemma 2.3, arguing in a similar fashion as (4.30), we obtain

$$\left| \int_{\Omega} f(x, v_n)(v_n - v_0) dx \right| \rightarrow 0 \text{ as } n \rightarrow +\infty.$$  

(8.5)

Hence, combining (8.2) and (8.5) and using the $(S_+)$ property of $\rho'$, we have $v_n \rightarrow v_0$ strongly in $X$ as $n \rightarrow +\infty$. \hfill $\square$

**Proof of Theorem 1.5.** From the hypotheses, we have that $I$ is even and $I(0) = 0$. Also Lemma 8.2 ensures that $I$ satisfies $(PS)$-condition. But note that, $I$ is not bounded from below on $X$. Hence, we will use a truncation technique. For that choose $h \in C^\infty([0, \infty), [0, 1])$ such that

$$h(t) = \begin{cases} 
1 & \text{if } t \in [0, l_0] \\
0 & \text{if } t \in [l_1, \infty),
\end{cases}$$
where \( l_0 < l_1 \) and set \( \Psi(u) := h(\|u\|) \). Now we define the truncated functional \( J \) as:

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^{2N} \setminus (C_1)^2} \frac{|u(x) - u(y)|^q_1(x,y)}{p_1(x,y)} \, dx \, dy + \int_{\Omega} \frac{1}{p_1(x)} |u|^q_1(x) \, dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^{2N} \setminus (C_1)^2} \frac{|u(x) - u(y)|^q_2(x,y)}{p_2(x,y)} \, dx \, dy + \int_{\Omega} \frac{1}{p_2(x)} |u|^q_2(x) \, dx
\]

\[
+ \int_{C_1} \beta(x) |u|^{q_1(x)}_{\frac{1}{p_1(x)}} \, dx + \int_{C_1} \beta(x) |u|^{q_2(x)}_{\frac{1}{p_2(x)}} \, dx
\]

\[
- \Psi(u) \left( \lambda \int_{\Omega} \frac{|u|^q(x)}{q(x)} \, dx - \int_{\Omega} \frac{|u|^r(x)}{r(x)} \, dx \right).
\]

Then \( J \in C^1(X, \mathbb{R}) \) and \( J(0) = 0 \). Also \( J \) is even. Moreover, from Lemma 8.2, it can be shown that \( J \) satisfies (PS)-condition. Now we will show \( J \) is bounded from below. For \( \|u\| > 1 \), using Lemma 3.3, we get

\[
J(u) \geq \frac{1}{2} \rho(u) \geq \frac{1}{2} \|u\|^{p^*_1} \to +\infty
\]

as \( \|u\| \to +\infty \), that is \( J(u) \) is coercive and hence bounded below on \( X \). Next, we claim that \( J \) verifies the assertion \((M_1)\) of Theorem 8.1. For any \( k \in \mathbb{N} \) and \( 0 < R_k < l_0 < 1 \) let us set

\[
B_{R_k} = \{ u \in X : \|u\| = R_k \}.
\]

Also consider the \( k \)-dimensional subspace \( X^k \) of \( X \). Then for \( u \in X^k \cap B_{R_k} \), there are some constants \( K_1, K_2 > 0 \) such that

\[
J(u) \leq \rho(u) - \int_{\Omega} F(x, u) \, dx
\]

\[
\leq \|u\|^{p^*_1} - \frac{\lambda}{q^*} \int_{\Omega} |u|^q(x) \, dx - \frac{1}{r^*} \int_{\Omega} |u|^r(x) \, dx
\]

\[
\leq \|u\|^{p^*_1} - K_1^q \frac{\lambda}{q^*} \|u\|^{q^*}, \tag{8.6}
\]

since \( X^k \cap B_{R_k} \) being of finite dimension all norms on it are equivalent. Now by letting \( R_k \to 0 \) as \( k \to +\infty \) from (8.6), we get \( \sup_{X^k \cap B_{R_k}} J(u) < 0 \) since \( p^*_1 < q^* \). Furthermore, for a given \( u \in X \) from (8.6) it follows that \( J(tu) < 0 \) for \( t \to 0^+ \), that is \( J(u) \neq 0 \). Thus, \( J \) does not satisfy \((M_2)\). Therefore, by appealing Theorem 8.1, there exists a sequence of critical points \((v_k)_k \) of \( J \) in \( X \) such that \( \|v_k\| \to 0 \) as \( k \to +\infty \). So, for \( l_0 > 0 \) there exists \( \hat{k}_0 \in \mathbb{N} \) such that for all \( k \geq \hat{k}_0 \) we have \( \|u\| < l_0 \) which infers that \( J(u_k) = I(u_k) \) for all \( k > \hat{k}_0 \). Since the critical values of \( I \) are the solutions to (1.2), the theorem follows.

\[\square\]

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
[30] Kikuchi, K., Negoro, A.: On Markov processes generated by pseudodifferential operator of variable order. Osaka J. Math. 34, 319–335 (1997)
[31] Leopold, H.G.: Embedding of function spaces of variable order of differentiation. Czechoslovak Math. J. 49, 633–644 (1999)
[32] Liu, Z., Wang, Z.-Q.: On Clark’s theorem and its applications to partially sublinear problems. Ann. Inst. H. Poincaré Anal. Non Linéaire 32, 1015–1037 (2015)
[33] Lorenzo, C.F., Hartley, T.T.: Initialised fractional calculus. Int. J. Appl. Math. 3, 249–265 (2000)
[34] Lorenzo, C.F., Hartley, T.T.: Variable order and distributed order fractional operators. Nonlinear Dyn. 29, 57–98 (2002)
[35] Marano, S., Mosconi, S.: Some recent results on the Dirichlet problem for $(p,q)$-Laplacian equation. Discrete Contin. Dyn. Syst. Ser. S 11, 279–291 (2018)
[36] Molica Bisci, G., Rădulescu, V.: Ground state solutions of scalar field fractional Schrödinger equations. Calc. Var. Partial. Differ. Equ. 54(3), 2985–3008 (2015)
[37] Molica Bisci, G., Rădulescu, V., Servadei, R.: Variational Methods for Nonlocal Fractional Problems, Encyclopedia of Mathematics and its Applications, vol. 162, Cambridge University Press, Cambridge (2016)
[38] Mingione, G., Rădulescu, V.: Recent developments in problems with nonstandard growth and nonuniform ellipticity. J. Math. Anal. Appl. (2021). https://doi.org/10.1016/j.jmaa.2021.125197
[39] Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Existence and multiplicity of solutions for double-phase Robin problems. Bull. Lond. Math. Soc. (2020). https://doi.org/10.1112/blms.12347
[40] Rabinowitz, P.H.: Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Regional Conference Series in Mathematics, vol. 65. American Mathematical Society, Providence (1986)
[41] Rădulescu, V., Repovš, D.: Partial Differential Equations with Variable Exponents. CRC Press, Boca Raton (2015)
[42] Rădulescu, V.: Isotropic and anistropic double-phase problems: old and new. Opuscula Math. 39(2), 259–279 (2019)
[43] Ruiz-Medina, M.D., Anh, V.V., Angulo, J.M.: Fractional generalized random fields of variable order. Stoch. Anal. Appl. 22, 775–799 (2004)
[44] Ružička, M.: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000)
[45] Segatti, A., Vázquez, J.L.: On a fractional thin film equation. Adv. Nonlinear Anal. 9(1), 1516–1558 (2020)
[46] Shi, X., Rădulescu, V.D., Repovš, D.D., Zhang, Q.: Multiple solutions of double phase variational problems with variable exponent. Adv. Calc. Var. (2018). https://doi.org/10.1515/acv-2018-0003
[47] Sun, H., Chen, W., Wei, H., Chen, Y.Q.: A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems. Eur. Phys. J. Spec. Top. 193, 185–192 (2011)
[48] Wilhelmsson, H.: Explosive instabilities of reaction–diffusion equations. Phys. Rev. A 36 (1987)
[49] Willem, M.: Minimax Theorems, vol. 24. Birkhäuser, Boston (1996)
[50] Zuo, J., Fiscella, A.: A critical Kirchhoff type problem driven by a $p(\cdot)$-fractional Laplace operator with variable $s(\cdot)$-order. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6813
[51] Zuo, J., Fiscella, A., Bahrouni, A.: Existence and multiplicity results for $p(\cdot)$&$q(\cdot)$ fractional Choquard problems with variable order. Complex Var. Elliptic Equ. (2020). https://doi.org/10.1080/17476933.2020.1835878
[52] Zhikov, V.V.: Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR, Izv. 29(1), 33–36 (1987)
Marcos L. Carvalho
Mathematics Institute
Universidade Federal de Goiás
Goiânia
Brazil
e-mail: marcos_leandro_carvalho@ufg.br

(Received: February 16, 2021; revised: February 5, 2022; accepted: March 3, 2022)