Nonasymptotic Gaussian Approximation for Inference with Stable Noise

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Abstract

The results of a series of theoretical studies are reported, examining the convergence rate for different approximate representations of $\alpha$-stable distributions. Although they play a key role in modelling random processes with jumps and discontinuities, the use of $\alpha$-stable distributions in inference often leads to analytically intractable problems. The LePage series, which is a probabilistic representation employed in this work, is used to transform an intractable, infinite-dimensional inference problem into a conditionally Gaussian parametric problem. A major component of our approach is the approximation of the tail of this series by a Gaussian random variable. Standard statistical techniques, such as Expectation-Maximization, Markov chain Monte Carlo, and Particle Filtering, can then be applied. In addition to the asymptotic normality of the tail of this series, we establish explicit, nonasymptotic bounds on the approximation error. Their proofs follow classical Fourier-analytic arguments, using Essén’s smoothing lemma. Specifically, we consider the distance between the distributions of: (i) the tail of the series and an appropriate Gaussian; (ii) the full series and the truncated series; and (iii) the full series and the truncated series with an added Gaussian term. In all three cases, sharp bounds are established, and the theoretical results are compared with the actual distances (computed numerically) in specific examples of symmetric $\alpha$-stable distributions. This analysis facilitates the selection of appropriate truncations in practice and offers theoretical guarantees for the accuracy of resulting estimates. One of the main conclusions obtained is that, for the purposes of inference, the use of a truncated series together with an approximately Gaussian error term has superior statistical properties and is likely a preferable choice in practice.

Index Terms

Linear model, central limit theorem, $\alpha$-stable distribution, LePage series representation, conditionally Gaussian distribution, Kolmogorov distance, inverse Fourier transform, smoothing lemma, nonasymptotic bound, Lévy process, Bayesian inference, Berry-Essén bound

I. INTRODUCTION

STATISTICAL modelling and inference for time series and random processes are of central importance in many areas of science and engineering. In applications, the time- or space-evolution of quantities of interest is often described through regression models that include random ‘noise’ components. These components may represent the inherent randomness in the underlying system, or the noise introduced by the observation process, or both.

Consider, for example, a simple discrete-time linear regression model for a time series $x := [x_1, \ldots, x_N]'$, expressed as,

$$x = G\lambda + u,$$

where the $P$-dimensional parameter vector $\lambda := [\lambda_1, \ldots, \lambda_P]'$ and the $N \times P$ matrix of known regressors $G$ describe the deterministic part of the system, and the random process $\{u_n\}$ describes the random noise component, $u := [u_1, \ldots, u_N]'$. This encompasses many models of current interest, including Fourier, wavelet and other expansions used in compressive sensing, communication systems, genomics, and signal processing.

Another common class of motivating examples is that of state-space models where the state evolves over time with random disturbances, as,

$$x_n = Ax_{n-1} + u_n, \quad 1 \leq n \leq N,$$

where $A$ is the autoregressive parameter, and observations may in addition contain noise:

$$y_n = bx_n + v_n, \quad 1 \leq n \leq N.$$

Here $b$ is an observation parameter and $\{v_n\}$ is the observation noise process.

Depending on the application at hand, there are many possible inference objectives; for example, state inference or prediction for $x_n$ in the state-space model, parameter estimation for $\lambda$, $A$ and $b$, and model choice to determine the structure and dimensionality of the model. A common modelling choice is to assume that the processes $\{u_n\}$ and $\{v_n\}$ are

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Gaussian: Since the driving noise process \( \{u_n\} \) can often be thought of as the sum of many small independent contributions, the Gaussian assumption is a natural consequence of the central limit theorem (CLT). Similarly, the measurement noise process \( \{v_n\} \) typically is the result of the sum of small independent perturbations, which again justifies the Gaussian assumption via the CLT. In these cases, standard methods are available for likelihood-based or Bayesian inference, using closed-form results combined with, for example, variational Bayes or Monte Carlo sampling.

However, many real-world cases exhibit extreme values much more frequently than the Gaussian model of (1) or (2) would allow. Examples of such abrupt changes include variations presented by stock prices or insurance gains/losses in financial applications, as studied extensively since the seminal works [54] and [24]; we refer to [62] for a more recent review. Further applications can be found in various fields of engineering, such as communications (see [6] for statistical modelling of channels, [18], [23] for capacity bounds, [50] for delay bounds in networks with \( \alpha \)-stable noise, and [71], [81] for signal detection), signal processing [57], image analysis [3], [4] and audio processing [52]. Sudden changes are studied also in the climatological sciences [35], [36], and in the medical sciences; see, e.g., [17] on brain connectivity representations. Moreover, in the field of sparse modelling and Compressive Sensing, a noise distribution is required that leads to sparse solutions (in transformed domains), a case much better dealt with using heavy-tailed models than the Gaussian case; see, e.g., [5], [15], [77]–[79] for a detailed review of modelling with sparse signals, and a connection between sparsity and heavy-tailed distributions, [53], [83] for the estimation of the degree of sparsity, and [2], [76] for compressed-sensing Bayesian methods based on heavy-tailed assumptions.

In many of these situations, the random phenomena considered can be still thought of as emerging from the combination of many independent perturbations. According to the generalized CLT [26, p. 162] [25, p. 576], whenever the sum of independent identically distributed (i.i.d.) random variables (RVs) converges in distribution, it converges to a member of many independent perturbations, which again justifies the Gaussian assumption via the CLT. In these cases, standard methods are available for likelihood-based or Bayesian inference, using closed-form results combined with, for example, variational Bayes or Monte Carlo sampling.

The main motivation for this work, as well as the main driving force for the large attention that the \( \alpha \)-stable laws have received in applications (see the extensive bibliography listed in [59]), both stem from the key role of the \( \alpha \)-stable distribution in the generalized CLT, and from the modelling flexibility offered by the class of \( \alpha \)-stable laws.

### A. \( \alpha \)-stable distributions

We adopt the standard notation of [69]. We write \( X \sim S_\alpha(\sigma, \beta, \mu) \) to denote that the RV \( X \) has an \( \alpha \)-stable distribution with parameters \( \sigma, \beta, \mu \), where \( \alpha \in (0, 2) \), is the scale parameter. Indeed, as consequence of the generalized CLT [25, Theorem XVII.5.1], when \( \mu = 0 \), the probability density function (PDF) \( p(x) \) of \( X \) has tails that decay like \( |x|^{1+\alpha} \),

\[
\lim_{|x| \to \infty} \frac{p(x)}{|x|^{1-\alpha}} = C(\alpha, \sigma, \beta),
\]

for some finite constant \( C(\alpha, \sigma, \beta) \). This asymptotic behaviour of the PDF corresponds to the presence of extreme values in the distribution, with more extreme values (and hence heavier tails) appearing more frequently for smaller values of \( \alpha \). The parameter \( \beta \in [-1, 1] \) is a measure of skewness: \( \beta = 0 \) corresponds to symmetric stable laws, while \( \beta = \pm 1 \) corresponds to the fully left or right skewed cases. Finally, \( \mu \in (-\infty, \infty) \) and \( \sigma > 0 \) are the location and scale parameters, respectively.

The characteristic function (CF) \( \phi_X(s) := \mathbb{E}[\exp(\imath sX)] \), for \( s \in \mathbb{R} \), of an \( \alpha \)-stable RV \( X \sim S_\alpha(\sigma, \beta, \mu) \) can be expressed [26] as,

\[
\log(\phi_X(s)) = \begin{cases} 
-\sigma^\alpha |s|^\alpha \left\{ 1 - i\beta \text{sign}(s) \tan \frac{\pi \alpha}{2} \right\} + i\mu s, & \alpha \neq 1, \\
-\sigma |s| \left\{ 1 + i\beta \text{sign}(s) \frac{2}{\pi} \log |s| \right\} + i\mu s, & \alpha = 1.
\end{cases}
\]  

(4)

Throughout, \( \log \) denotes the natural logarithm. Notice that this CF has a pole for \( \alpha = 1 \). For the sake of simplicity, throughout the paper we assume that \( \alpha \neq 1 \). Also, it is easy to see from (4) that the class of \( \alpha \)-stable distributions includes the Gaussian (\( \alpha = 2 \)), Cauchy (\( \alpha = 1, \beta = 0 \)), and Lévy (\( \alpha = 1/2, \beta = 1 \)) families. Unlike the CF, the PDF of \( \alpha \)-stable distributions cannot be expressed in closed form, except in the three special cases mentioned. This presents significant complications in the development of effective methodological tools for inference, when models involve \( \alpha \)-stable distributions. Nevertheless, as we describe next, a wide variety of relevant statistical tools have been proposed in the literature and have been applied in practice.

### B. Motivation: Inference with stable distributions

The simplest and most common class of inferential procedures is probably that of parameter estimation. For systems governed by \( \alpha \)-stable noise this may for example involve estimating the parameters \( \boldsymbol{A}, \boldsymbol{b} \) and \( \boldsymbol{A} \) in (1)-(3) above, as described in more detail in Section IX-A. Other inference examples can be found in the references to specific application domains provided above.
Numerous techniques have been developed for estimating the parameter vector $\theta = (\alpha, \sigma, \beta, \mu)$ from batch data. Common frequentist approaches include those based on the quantiles of the distribution [55], its logarithmic moments [39], the empirical CF [38], approximate maximum likelihood estimators [58], or block-maxima scaling [72]. However, since the $\alpha$-stable PDF is not available in closed form, all the above approaches are approximate. This issue similarly affects corresponding Bayesian methods aiming at computing the posterior distribution of the parameters, for which the likelihood function needs to be evaluated; see, e.g., [51].

On the other hand, $\alpha$-stable distributions admit representations involving latent variables, enabling (asymptotically) exact Bayesian inference. Schemes such as those based on marginal representations of the stable likelihood or on the conditional and pseudo-marginal samplers [13], [61], [68], belong to this class. Also, the product property [25] and the scale mixture of normals representation of symmetric stable distributions can be used, as in [27]–[29], [75]. The central object of interest in the present work is yet another latent variable model, the so-called Poisson series representation (PSR) of stable laws, previously employed in [41], [44], [46].

Although the PSR is an exact representation, it is an infinite series which itself needs to be approximated. For effective inference, it is then necessary to quantify the error incurred by such an approximation. The study of this approximation error is the main aim of this paper. Therefore, the present work provides a more firm theoretical foundation for the Bayesian parameter samplers mentioned above, and more generally for Bayesian inference in models involving $\alpha$-stable noise.

Studying the error incurred by an approximation to the PSR is also relevant to state inference in continuous-time stochastic dynamics through Bayesian methods such as sequential Monte-Carlo (SMC) [14], [20], [21]. When part of the state is (conditionally) linear and Gaussian, combining Kalman-filter steps [34] with SMC filters results in more efficient samplers in terms of Monte Carlo variance [21], [70]. As discussed in Section IX-B, the PSR extends to $\alpha$-stable Lévy processes and state space models driven by these processes, and it enables efficient SMC inference methods as implemented in [42], [41], [45], [67]. Note, however, that the results established here only pertain to $\alpha$-stable RVs and Lévy processes; continuous-time models driven by $\alpha$-stable Lévy processes will be examined in future work; also see Section IX-B for some additional remarks.

### C. Main contributions and paper organisation

The central object of interest in this work is the Poisson series representation (PSR) of an $\alpha$-stable RV, mentioned above. As described in Section II, the PSR is an infinite sum of RVs involving the arrival times of a Poisson process. Since it is impossible to compute the entire infinite series in practice, only approximate versions of the PSR can be employed for simulation and inference purposes. The starting point of our approach is the truncation of the PSR, followed by the approximation of the tail of the series (to which we refer as the residual series) by an appropriately chosen Gaussian RV. We recently noted that such an approximation is asymptotically exact, as the truncation point becomes larger [64]. This CLT is the first main contribution of this work, given in Section IV: Theorem 1 provides a precise version of the CLT together with a complete proof, under conditions weaker than those stated in [64].

We then investigate the nonasymptotic accuracy of the above approximation, as a function of the truncation point. The main tool in this investigation is Essén’s smoothing lemma; this is a classical Fourier-inversion inequality, used to translate information on the distance between two characteristic functions (CFs) to information about the distance between the two corresponding probability distributions; see, e.g., [63], and the discussion in Section V. The derivation of explicit expressions for the CFs of several quantities of interest is our second main contribution, given in Section VI.

Let $c$ denote the truncation parameter for the PSR. In Section VII we establish nonasymptotic, strong, explicit upper bounds on the distance between the distribution of the PSR residual and an appropriate Gaussian distribution. In the symmetric ($\beta = 0$) case, these results in particular imply that the convergence of the CLT in Section IV takes place at a rate $O(1/c)$, and that it is faster when $\alpha$ approaches 2: this is consistent with the numerical findings reported in [65]. We also establish a different bound that decays like $1/\sqrt{c}$ asymptotically, but which is tighter than our previous bound for relatively small values of $c$ and $\alpha$. This is the third main contribution of this paper.

The Gaussian approximation of the PSR residual suggests an elegant approximate representation of a stable RV. However, bounds on the distance between the residual and a Gaussian do not immediately translate to corresponding bounds between this approximate representation and the corresponding stable law. Obtaining such bounds is the fourth and perhaps most important contribution of this work. When using this approximate representation in the context of inference procedures with $\alpha$-stable models, having explicit bounds facilitates the selection of an appropriate value for the truncation parameter, in a way that also provides estimation error guarantees for the results. For the case of symmetric ($\beta = 0$) stable laws, our bounds are stated in Section VIII. These results, as well as the numerical study performed in [65], indicate that the approximated PSR is closer to the stable distribution for smaller values of $\alpha$, and that the rate of convergence now depends on $\alpha$.

There is earlier extensive work on the analysis of the convergence rate of the truncated PSR to the corresponding stable law – as opposed to the convergence of the distribution of the residual studied here; see, e.g., [8], [9], [31]–[33], [40]. In
Section VIII we review the most relevant of these results, and we derive bounds indicating that representation proposed in this work should typically yield a better approximation to the stable distribution than simply truncating the PSR.

Finally, in Section IX we illustrate the utility of our main results with an example of statistical inference. We recall an MCMC-based inference scheme for the parameters of the discrete-time linear models (1) and (2), and we discuss the use of our Gaussian approximation bounds in this setting. We also briefly describe potential extensions of our results to continuous-time systems and to multivariate stable distributions. In each of these cases, analogues of the PSR representation have already been established, and having a CLT for the residual and nonasymptotic bounds on the induced approximation, like the ones established in this work, would potentially be of significant interest in applications.

The proofs of most of the main results in the paper, together with the more technical lemmas, are given in the appendices.

D. Notation

Capital letters, e.g., \(X, Y\), are used for RVs, and ‘hats’ denote approximate versions, e.g., \(\hat{X}\) denotes a RV with a distribution which is close to that of \(X\). The following notation is used for some common distributions: \(\mathcal{N}(\mu_W, \sigma_W^2)\) is the normal distribution with mean \(\mu_W\) and variance \(\sigma_W^2\); \(\mathcal{U}(a,b)\) is the uniform distribution on the interval \((a,b)\); and Poisson\((t)\) is the Poisson distribution with mean \(t\).

Throughout, the sequence \(\{\Gamma_j\}\) will denote the successive arrival times of a unit rate Poisson process. If a RV \(X\) is defined as a series of random terms involving the sequence \(\{\Gamma_j\}\), then \(X_{(c,d)}\) will denote the sum of those terms corresponding to indices \(j\) such that \(\Gamma_j \in (c,d)\), for \(0 \leq c < d \leq \infty\). The number of terms in \(X_{(c,d)}\) is denoted by \(N_{(c,d)}\), with the convention that \(X_{(c,c)} = 0\) if \(N_{(c,c)} = 0\). A subscript notation is used for the moments of such RVs, e.g., \(\mu_W\) is the mean of \(W\), and \(m_{(c,d)}\) is the mean of \(X_{(c,d)}\).

The Kolmogorov distance between two RVs \(S\) and \(T\) with distribution functions \(F\) and \(G\), respectively, is denoted by \(\Delta(S,T) := \sup_x |F(x) - G(x)|\). Upper bounds on \(\Delta(S,T)\) derived from the smoothing lemma with a finite smoothing parameter will be denoted as \(I(S,T)\), and when the smoothing parameter goes to infinity the corresponding bounds will be denoted by \(\bar{I}(S,T)\). Numerically computed values for these bounds will be denoted by \(\bar{Q}(S,T)\) and \(\bar{Q}(S,T)\), respectively.

Some of the CFs considered in the paper are complex valued and, for a fixed argument, can be expressed in polar form as \(z = re^{i\theta}\), with \(r > 0\) and \(\theta \in \mathbb{R}\). To avoid any ambiguity, we adopt the convention that in all such expressions \(\theta\) is assumed to lie in the interval \((-\pi, \pi]\). Then it is possible to uniquely invert the exponential of \(z\), obtaining the principal value complex logarithm of a CF, \(\log(z) = \log(r) + i\theta\). Although not always necessary, for the sake of clarity we will always work with principal value complex logarithms.

Two complex-analytic functions that appear repeatedly in our analysis are the lower incomplete gamma function,

\[
\gamma(s,x) := \int_0^x t^{s-1}e^{-t}dt, \quad -s \notin \mathbb{N}, x > 0,
\]

and the upper incomplete gamma function,

\[
\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt, \quad -s \notin \mathbb{N}, x > 0.
\]

Then, for any \(x > 0\),

\[
\Gamma(s) = \gamma(s,x) + \Gamma(s,x), \quad -s \notin \mathbb{N},
\]

is the regular (complete) gamma function.

We use the symbol ‘\(\sim\)’ to denote the fact that a RV \(X \sim \mathcal{D}\) has distribution \(\mathcal{D}\), but also to denote the following asymptotic relationship: for two (real) functions \(f_1\) and \(f_2\), we say \(f_1(x) \sim f_2(x)\) when \(\lim_{x \to \infty} f_1(x)/f_2(x) = 1\). Finally, \(f_1(x) = O(f_2(x))\) as usual means that, \(\limsup_{x \to \infty} |f_1(x)/f_2(x)| < \infty\).
II. The Poisson Series Representation (PSR)

Let \( X \sim \mathcal{S}_\alpha(\sigma, \beta, 0) \) be an \( \alpha \)-stable RV for some \( \alpha \in (0, 2), \alpha \neq 1 \). The PSR, originally introduced by Lévy and formalised by LePage et al. [47]–[49], states that \( X \) admits the representation,

\[
X \overset{D}{=} \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} W_j - \mathbb{E}[W_1] b_j^{(\alpha)},
\]

where \( \overset{D}{=} \) denotes equality in distribution, and:

- \( \{\Gamma_j\}_j=1^\infty \) are the arrival times of a unit rate Poisson process, so that the differences \( \Gamma_j - \Gamma_{j-1}, j = 1, 2, \ldots \), are i.i.d. exponential RVs with mean 1;
- \( \{W_j\}_{j=1}^\infty \) are i.i.d. RVs independent of \( \{\Gamma_j\}_j=1^\infty \), with,
  \[
  \mathbb{E}[|W_1|^\alpha] < \infty;
  \]
- \( \{b_j^{(\alpha)}\}_j=1^\infty \) are constants that are non-zero only if \( \alpha \in (1, 2) \), given by:
  \[
  b_j^{(\alpha)} = \frac{\alpha}{\alpha - 1} \left( j \frac{\alpha-1}{\alpha} - (j - 1) \frac{\alpha-1}{\alpha} \right).
  \]

The exact representation in (8) can be found in [69, Theorem 1.4.5]. Observe that this is only valid for a strictly stable \( \mathcal{S}_\alpha(\sigma, \beta, \mu) \) RV, with \( \mu \neq 0 \) can simply be obtained from \( X \sim \mathcal{S}_\alpha(\sigma, \beta, 0) \) as \( Y \overset{D}{=} X + \mu \). We also observe that the constants \( b_j^{(\alpha)} \) have a telescoping nature, so that, for \( N \geq 1 \),

\[
\sum_{j=1}^{N} b_j^{(\alpha)} = \frac{\alpha}{\alpha - 1} N \frac{\alpha - 1}{\alpha} \mathbb{I}\left(\alpha \in (1, 2)\right),
\]

where \( \mathbb{I}(\cdot) \) denotes the indicator function, equal to 1 if \( \cdot \) is satisfied, and 0 otherwise.

Although the distribution of the RVs \( \{W_j\} \) above has not been explicitly described, the \( \alpha \)-th absolute moment of \( W_1 \) can be expressed in terms of \( \alpha, \beta \) and \( \sigma \) as follows:

\[
\begin{align*}
\sigma^\alpha &= \mathbb{E}[|W_1|^\alpha] / C_\alpha, \quad (11a) \\
\beta &= \mathbb{E}[|W_1|^\alpha \text{sign} W_1] / \mathbb{E}[|W_1|^\alpha], \quad (11b)
\end{align*}
\]

where,

\[
C_\alpha = \left( \int_0^{\infty} x^{-\alpha} \sin x \, dx \right)^{-1} = \frac{1 - \alpha}{\Gamma(2 - \alpha) \cos(\pi \alpha/2)}.
\]

See Appendix A for some more details about (11a) and (11b), particularly when \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \), a case of special interest since taking \( W_1 \sim \mathcal{N}(0, \sigma_W^2) \) corresponds to the symmetric stable distribution, namely, the case \( \beta = 0 \). Also note that, in view of the relations (11a), (11b) and (12), when \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \), there is a 1-1 relationship between the parameters \( (\alpha, \mu_W, \sigma_W) \) and \( (\alpha, \beta, \sigma) \); Table I contains a few numerical examples to which we will return later.

**TABLE I**

| \( \alpha \) | \( \mu_W \) | \( \sigma_W \) | \( \sigma \) | \( \beta \) |
|---|---|---|---|---|
| 0.8 | 0 | 1 | 1.16 | 0 |
| | 1 | 1 | 1.71 | 0.84 |
| | 1 | 0 | 1.42 | 1 |
| 1.2 | 0 | 1 | 1.37 | 0 |
| | 1 | 1 | 1.99 | 0.99 |
| | 1 | 0 | 1.80 | 1 |

Figure 1 shows a realization of the first 100 terms \( W_j \Gamma_j^{-1/\alpha} - \mu_W b_j^{(\alpha)} \) of the PSR with \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \), for different values of the parameters \( \alpha, \mu_W \) and \( \sigma_W \). The corresponding values of \( \sigma \) and \( \beta \) are shown in Table I. Since the sequence \( \{\Gamma_j\}_j=1^\infty \) is increasing with probability one, the terms \( \{\Gamma_j^{-1/\alpha}\}_j=1^\infty \) are decreasing, and the summands in the PSR are therefore stochastically decaying (in absolute value). This is indeed confirmed by the sample paths shown in Figure 1.
In most of the paper we will focus on the case \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \), since this allows a conditionally Gaussian representation of the \( \alpha \)-stable distribution which is useful for the inference tasks mentioned earlier. In fact, from the PSR (8) it follows that, if \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \), we can write an auxiliary variable model for \( X \) as,

\[
X | \{ \Gamma_j \}_{j=1}^{\infty} \sim \mathcal{N}(\mu_W m, \sigma_W^2 S^2),
\]

\[
m := \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} - b_j^{(\alpha)},
\]

\[
S^2 := \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}.
\]

In this model \( m \) and \( S^2 \) are treated as auxiliary RVs, and \( X \) has a conditionally Gaussian structure. This means that standard auxiliary variable methods for conditionally Gaussian models may be readily applied, for example blocked and collapsed Gibbs samplers [80], Monte Carlo EM [74] and Rao-Blackwellised particle filters [21], [70]. We note that the general framework here is a continuous scale and mean mixture of normals, since the density \( p \) of \( X \), can be expressed in obvious notation as:

\[
p = \int_{S \in \mathbb{R}^+} \int_{m \in \mathbb{R}} \mathcal{N}(\mu_W m, \sigma_W^2 S^2) p(m, S) \, dm \, dS.
\]
III. TRUNCATION OF THE PSR

While the exact representations of the stable law in (8) and (15) are very appealing, they are computationally intractable because of the infinite summations involved. Since the summands of the PSR (8) are stochastically decaying, a first intuitive approach would be to consider a large but finite number of summands so that the distribution of the truncated PSR,

\[ X_{(0,c)} := \sum_{j: \Gamma_j \in [0,c]} W_j \Gamma_j^{-1/\alpha} , \]  

is ‘close enough’ to that of \( X \), e.g., in terms of their Kolmogorov distance. A number of works in the literature have been devoted to the study of this approximation, and in Section VIII we will recall the main results that are used as a comparison with ours. Here we mention that the truncated PSR has been used in engineering applications by [6] for the task of generating stable variables, and the choice of the truncation parameter is also based on analysis in the frequency domain.

However, more accurate results can be obtained by taking into account the residual part of the series, at least approximately. Making this idea precise is the main aim of this paper: For a given truncation parameter \( c > 0 \), let \( X_{(0,c)} \) be defined as in (16) above, so that,

\[ X = X_{(0,c)} + R_{(c,\infty)} , \]  

where the PSR residual \( R_{(c,\infty)} \) is,

\[ R_{(c,\infty)} := \lim_{d \to \infty} R_{(c,d)} , \]  

with,

\[ R_{(c,d)} := \sum_{j: \Gamma_j \in (c,d]} W_j \Gamma_j^{-1/\alpha} - E[W_1] \sum_{j=1}^{[d]} \beta_j^{(\alpha)} , \]  

and \([\cdot]\) denoting the lower integer part. Our first main result will establish that the approximation of \( X \) by \( X_{(0,c)} + R_{(c,\infty)} \) is asymptotically (as \( c \to \infty \)) exact, as long as \( E[W_1^2] < \infty \).

To simplify the notation, from now on we will assume that \( d \) is integer so that \([d] = d\). Notice that the limit in (18) denotes convergence in distribution, and is guaranteed to exist by the fact that the full series is known to converge to an \( \alpha \)-stable RV [69, Theorem 1.4.5].

Next, we compute the first and second order moments of \( R_{(c,d)} \) and of \( R_{(c,\infty)} \). Since, conditioned on the Poisson number of events, the ordered arrival times \( \{ \Gamma_j \} \) of a unit rate Poisson process may equivalently be written as an unordered set of i.i.d. uniformly distributed random variables \( \{ U_j \} \) [37], a generative model for \( R_{(c,d)} \) is as follows,

\[ R_{(c,d)} = \sum_{j=1}^{N_{(c,d)}} Y_j - B , \]  

\[ N_{(c,d)} \sim \text{Poisson}(d-c) , \]

\[ Y_j := W_j U_j^{-1/\alpha} , \]

\[ W_j \sim F , \]

\[ U_j \sim \mathcal{U}(c,d) , \]

\[ B := E[W_1] \frac{\alpha}{\alpha - 1} d^{\frac{\alpha-1}{\alpha}} \mathbb{1} (\alpha \in (1,2)) , \]  

where Poisson\((d-c)\) is the Poisson distribution with mean \( d-c \), \( F \) is the distribution of \( W_1 \), which is assumed to satisfy (9), and the expression for \( B \) is obtained from (10). In other words, we can think of \( R_{(c,d)} \) as a compound Poisson process containing two sources of randomness: The random Poisson number of arrivals, \( N_{(c,d)} \), and the values of the RVs \( Y_j \) being summed. Based on this observation, the following lemma is proved in Appendix B

\[ \text{Lemma 1:} \text{ Suppose } E[W_1^2] < \infty, \text{ let } Y_1 \text{ be the RV defined in (21) with CF } \phi_{Y_1}(s), \text{ and let } B \text{ be the constant in (23). Then } \phi_{R_{(c,d)}}(s), \text{ the CF of } R_{(c,d)}, \text{ is:} \]

\[ \phi_{R_{(c,d)}}(s) = \exp \left( (d-c)(\phi_{Y_1}(s) - 1) - isB \right) . \]  

Moreover, the mean and variance of the residual \( R_{(c,d)} \) are:

\[ m_{(c,d)} = E[W_1] \frac{\alpha}{\alpha - 1} \left( d^{\frac{\alpha-1}{\alpha}} - c^{\frac{\alpha-1}{\alpha}} \right) - B , \]  

\[ S^2_{(c,d)} = E[W_1^2] \frac{\alpha}{\alpha - 2} \left( d^{\frac{\alpha-2}{\alpha}} - c^{\frac{\alpha-2}{\alpha}} \right) . \]
Note that, we can take the limit as \( d \to \infty \) in the last two expressions in the lemma to obtain,

\[
m_{c,\infty} := \lim_{d \to \infty} m_{c,d} = E[W_1] \frac{\alpha}{1 - \alpha} e^{\frac{\alpha}{\alpha - 1}},
\]

\[
S^2_{(c,\infty)} := \lim_{d \to \infty} S^2_{(c,d)} = E[W_1^2] \frac{\alpha}{2 - \alpha} e^{\frac{\alpha}{\alpha - 2}}.
\]

In the following section we will take a similar limit for the CF of \( R_{c,\infty} \),

\[
\phi_{R_{c,\infty}}(s) = \lim_{d \to \infty} \phi_{R_{c,d}}(s),
\]

where the existence of the limit is guaranteed by the existence of the PSR.
IV. ASYMPTOTIC NORMALITY OF THE PSR RESIDUAL

Although it is easy to see that \( R_{(c, \infty)} \) is not Gaussian, the following CLT-like result states its asymptotic normality as \( c \to \infty \). A first version of this result for the special case \( W_j \equiv 1 \) was presented in [44], and the general case with random \( W_j \) was stated in [64]. Here we provide a precise statement of the claim together with a complete proof, under milder moment conditions than those in [64].

**Theorem 1:** Let \( R_{(c, \infty)} \), \( m_{(c, \infty)} \) and \( S_{(c, \infty)}^2 \) be defined as in (18), (27) and (28), respectively. If \( \mathbb{E}[W_j^2] < \infty \), then,

\[
Z_{(c, \infty)} := \frac{R_{(c, \infty)} - m_{(c, \infty)}}{S_{(c, \infty)}} \xrightarrow{D} c \to \infty Z,
\]

where \( Z \sim \mathcal{N}(0, 1) \) and \( \xrightarrow{D} c \to \infty \) denotes convergence in distribution, as \( c \to \infty \).

**Proof:** The proof is based on the Lévy continuity theorem [25]: We will show that, for any fixed \( s \in \mathbb{R} \), the CF of \( Z_{(c, \infty)} \), \( \phi_{Z_{(c, \infty)}}(s) \), converges to the CF of \( Z \), \( \phi_Z(s) = \exp(-s^2/2) \), as \( c \to \infty \). First we express the CF of \( Z_{(c, \infty)} \) in terms of the CF of \( Y_1 \), defined in (21). Using (30), by a change of variables we have,

\[
\phi_{Z_{(c, \infty)}}(s) = \exp \left( -\frac{m_{(c, \infty)}}{S_{(c, \infty)}} s \right) \phi_{R_{(c, \infty)}} \left( \frac{s}{S_{(c, \infty)}} \right),
\]

and taking the limit as in (29) and using (24),

\[
\phi_{R_{(c, \infty)}}(s) = \lim_{d \to \infty} \phi_{R_{(c, d)}}(s) = \lim_{d \to \infty} \exp \left( (d - c) \left( \phi_{Y_1}(s) - 1 \right) - iBs \right),
\]

or, equivalently,

\[
\log(\phi_{R_{(c, \infty)}}(s)) = \lim_{d \to \infty} \left( (d - c) \left( \phi_{Y_1}(s) - 1 \right) - iBs \right). \tag{32}
\]

By Lemma 3.3.19 in [22],

\[
\left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^m}{m!} \right| \leq \min \left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}. \tag{33}
\]

Therefore, we have the following bound on the difference between \( \phi_{Y_1}(s) \) and its second-order Taylor expansion at zero based on (33) and Jensen’s inequality (applied to the absolute value function),

\[
\left| \phi_{Y_1}(s) - \sum_{k=0}^{2} \frac{i^k \mathbb{E}[Y_1^k]}{k!} s^k \right| \leq \mathbb{E} \left[ \min \left\{ \frac{|s|^3 |Y_1|^3}{3!}, s^2 Y_1^2 \right\} \right]
\]

\[
= \mathbb{E}_W \mathbb{E}_{U_1 \sim U(c,d)} \left[ \min \left\{ \frac{|s|^3 |W_1|^3}{3!} U_1^{-3/\alpha}, s^2 W_1^2 U_1^{-2/\alpha} \right\} \right]
\]

\[
\leq \mathbb{E}_W \left[ \min \left\{ \frac{|s|^3 |W_1|^3}{3!} \mathbb{E}_{U_1 \sim U(c,d)} \left[ U_1^{-3/\alpha} \right], s^2 W_1^2 \mathbb{E}_{U_1 \sim U(c,d)} \left[ U_1^{-2/\alpha} \right] \right\} \right], \tag{34}
\]

where in the equality we have made explicit the distributions with respect to which we are taking the expected value, based on the definition of \( Y_1 \) (21), and the second inequality trivially follows from \( \min\{a, b\} \leq a \) and \( \min\{a, b\} \leq b \), for any \( a, b \in \mathbb{R} \). In order to further bound the above right-hand side, we recall that from (69) and (70) in the proof of Lemma 1,

\[
(d - c) \mathbb{E}[Y_1^k] = \begin{cases} 
\frac{m_{(c, d)} + B}{S_{(c, d)}^2}, & k = 1, \\
\frac{S_{(c, d)}^2}{S_{(c, d)}^2}, & k = 2,
\end{cases} \tag{35}
\]

while, from (71),

\[
(d - c) \mathbb{E}[U_1^{-k/\alpha}] = \frac{\alpha}{\alpha - k} \left( d^{-\frac{k}{\alpha}} - c^{-\frac{k}{\alpha}} \right).
\]
Multiplying (34) by \((d-c)\) and substituting these yields,
\[
(d-c) \left( \phi_Y(s) - 1 - i(m_{(c,d)} + B)s + \frac{s^2 S_{(c,d)}^2}{2} \right) \leq \mathbb{E} \left[ \min \left\{ \frac{|s^3| W_1^3}{3!}, \frac{3\alpha}{\alpha-3} \left( d^{(\alpha-3)/\alpha} - c^{(\alpha-3)/\alpha} \right), s^2 W_1^2 \frac{\alpha}{\alpha-2} \left( d^{(\alpha-2)/\alpha} - c^{(\alpha-2)/\alpha} \right) \right\} \right],
\]
and taking the limit as \(d \to \infty\) and using (32), gives,
\[
\left| \log(\phi_{R(c,\infty)}(s)) - im_{(c,\infty)}s + \frac{s^2 S_{(c,\infty)}^2}{2} \right| \leq \mathbb{E} \left[ \min \left\{ \frac{|s^3| W_1^3}{3!}, \frac{3\alpha}{\alpha-3} c^{(\alpha-3)/\alpha}, s^2 W_1^2 \frac{\alpha}{\alpha-2} c^{(\alpha-2)/\alpha} \right\} \right]. \tag{36}
\]
Finally, replacing \(s\) by \(s/S_{(c,\infty)}\) in (36), and using (31) and (28), we obtain,
\[
\left| \log(\phi_{Z(c,\infty)}(s)) + \frac{s^2}{2} \right| \leq s^2 \mathbb{E} \left[ \min \left\{ \frac{|s| |W_1|^3}{3! \mathbb{E}[|W_1|^3]/2}, \frac{\alpha}{\alpha-2} \left( \frac{\alpha}{\alpha-2} \right)^{-1/2}, \frac{W_1^2}{\mathbb{E}[W_1^2]} \right\} \right].
\]
We can apply the dominated convergence theorem to the argument of the expectation, given that this is bounded by \(W_1^2\mathbb{E}[|W_1|^3]/2\) (which is integrable by assumption) and it vanishes as \(c \to \infty\). Thus the limit of the expectation exists and it is also zero. Hence
\[
\log(\phi_{Z(c,\infty)}(s)) \to \log \phi_Z(s) := -\frac{s^2}{2}, \tag{37}
\]
so that \(\phi_{Z(c,\infty)}(s) \to \phi_Z(s)\), as required. \(\blacksquare\)

**A. Gaussian approximation of the residual**

Theorem 1 offers an asymptotic justification for the Gaussian approximation of the PSR residual,
\[
\hat{R}_{(c,\infty)} \sim \mathcal{N}(m_{(c,\infty)}, S_{(c,\infty)}), \tag{38}
\]
discussed earlier in the context of practical inference procedures. Notice that this approximation matches the first two moments of \(\hat{R}_{(c,\infty)}\) to those of the exact residual \(R_{(c,\infty)}\) for any value of the truncation parameter \(c\), and that \(R_{(c,\infty)}\) converges in distribution to its Gaussian approximation as \(c \to \infty\). Then, by analogy with (17), we can introduce the following RV,
\[
\hat{X} := X(0,c) + \hat{R}_{(c,\infty)}, \tag{39}
\]
that converges in distribution to \(X \sim S_\alpha(\sigma, \beta, 0)\), as \(c \to \infty\).

Note that Theorem 1 does not assume that the \(W_j\) are Gaussian. However, when they are, we have the following overall approximate conditionally Gaussian structure for the model, which in part justifies our focus on the case \(W_j \sim \mathcal{N}(\mu_W, \sigma_W^2)\) in the rest of the paper.

**B. Approximate conditionally Gaussian representation**

Suppose \(W_j \sim \mathcal{N}(\mu_W, \sigma_W^2)\) and that only the finite collection of values \(\{\Gamma_j \leq c\}\) is known, a much more realistic assumption than knowing all the values in the infinite sequence \(\{\Gamma_j\}\). Then the \(\alpha\)-stable distributed RV \(X \sim S_\alpha(\sigma, \beta, 0)\) has the approximate conditionally Gaussian representation,
\[
X|\{\Gamma_j \leq c\} \overset{\text{approx}}{\sim} \mathcal{N} \left( m_{(0,c)} + m_{(c,\infty)}, S_{(0,c)}^2 + S_{(c,\infty)}^2 \right), \tag{40}
\]
\[
m_{(0,c)} := \mu_W \sum_{j: \Gamma_j \in [0,c]} \Gamma_j^{-1/\alpha},
\]
\[
S_{(0,c)}^2 := \sigma_W^2 \sum_{j: \Gamma_j \in [0,c]} \Gamma_j^{-2/\alpha},
\]
\[
S_{(c,\infty)}^2 := \sigma_W^2 \sum_{j: \Gamma_j \in [c,\infty)} \Gamma_j^{-2/\alpha}.
\]
where $m_{(c,\infty)}$ and $S^2_{(c,\infty)}$ are given in (27) and (28). Thus, by analogy with (15), the density $p$ of $X$ can be approximately represented as

$$p \approx \int_{S_{(0,c)} \in \mathbb{R}^+} \int_{m_{(0,c)} \in \mathbb{R}} \mathcal{N}\left(m_{(0,c)} + m_{(c,\infty)}, S^2_{(0,c)} + S^2_{(c,\infty)}\right) p(m_{(0,c)}, S_{(0,c)}) \, dm_{(0,c)} \, dS_{(0,c)},$$

(41)

implying that $X$ can also be approximated by a continuous mean and scale mixture of normals.

In Figure 2 we compare kernel density estimates for the density of $X \sim S_\alpha(\sigma, \beta, 0)$ obtained by three different sampling methods: (i) $X_{(0,c)}$ is the obvious approximation of $X$ by the truncated PSR (16); (ii) $X_{(0,c)} + \hat{R}_{(c,\infty)}$ is our proposed approximation for $X$ (39); and (iii) ‘CMS’ is the benchmark Chambers-Mallows-Stuck method for generating exact samples of stable RVs [16], [82]. The results shown, indicate that adding the Gaussian approximation of the residual to $X_{(0,c)}$ produces an approximate distribution that is much closer to the true stable law than that obtained by simple truncation of the PSR.

Therefore, in view of Theorem 1, existing inference methods for the exact PSR can be used for the approximation (39) and, indeed, the inference schemes in [41], [43], [45], [46], [64] mentioned in Section I-B are based on (40). Moreover, the quality of this approximation is controlled directly by the truncation parameter $c$, therefore, it is important to have some quantitative measure of the accuracy of the resulting approximation, and also of the nature of its dependence on $c$ and on the parameters $\{\alpha, \beta, \sigma\}$. These issues are addressed in the following sections.

C. Choice of the truncation parameter $c$

In order to quantify the approximation error in the representation $X_{(0,c)} + \hat{R}_{(c,\infty)}$, and also in order to be able to choose appropriate values for the truncation parameter $c$, the following considerations should be kept in mind:

- The distribution of $\hat{R}_{(c,\infty)}$ is closer to that of $R_{(c,\infty)}$ when $c$ is large, according to Theorem 1.
- On the other hand, the computational complexity of the approximate conditionally Gaussian model (40) increases with $c$; in fact, the expected cardinality of the set of latent RVs $\left\{T_j < c\right\}$ needed to compute $m_{(0,c)}$ and $S^2_{(0,c)}$ is $O(c)$.
- Even when the distribution of $R_{(c,\infty)}$ is far from Gaussian (in particular, when $c$ is small), its contribution to the PSR might be relatively small when compared to that of $X_{(0,c)}$.

We will consider a choice of $c$ to be ‘good’ if it makes the distribution of $\hat{X}$ close to that of $X$. Quantifying the distance between $X$ and $\hat{X}$ involves computing the distance between $R_{(c,\infty)}$ and $\hat{R}_{(c,\infty)}$, so we proceed by first estimating how far the distribution of the PSR residual is from the corresponding Gaussian, for finite $c$.

In view of the above discussion, our aim in the rest of the paper is to provide accurate bounds that can guide us in choosing appropriate values of $c$, given the distribution parameters. The main tools that we employ in the derivation of such bounds are based on classical Fourier-analytic techniques, summarized in the following section.
As before, let $c \geq 0$ be the value of the truncation parameter. Suppose $S$ and $T$ are RVs with CFs $\phi_S(s)$ and $\phi_T(s)$, $s \in \mathbb{R}$, respectively, let $F_S(x)$ and $F_T(x)$, $x \in \mathbb{R}$, be the corresponding CDFs, and assume that $\mathbb{E}[S] = \mathbb{E}[T] = 0$. Furthermore, assume that $F_T(x)$ has derivative $p_T(x)$ such that $|p_T(x)| \leq m < \infty$, $\forall x \in \mathbb{R}$. We write,

$$
\Delta(S, T) := \sup_{x \in \mathbb{R}} \left| F_S(x) - F_T(x) \right|,
$$

for the Kolmogorov distance between the distributions of $S$ and $T$. Esséen’s smoothing lemma [25, Lemma XVI.4.2] states that, for any $\Theta > 0$,

$$
\Delta(S, T) \leq \frac{1}{\pi} \int_{-\Theta}^{\Theta} \frac{\left| \phi_S(s) - \phi_T(s) \right|}{|s|} \, ds + \frac{24m}{\pi \Theta} := I(S, T),
$$

(42)

and letting $\Theta \to \infty$, we also have,

$$
\Delta(S, T) \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\left| \phi_S(s) - \phi_T(s) \right|}{|s|} \, ds := I(S, T),
$$

(43)

where (43) is meaningful only if the improper integral converges. Note that the integrals above have a removable singularity at $s = 0$, due to the zero-mean assumption on $S$ and $T$.

We will use the smoothing lemma to obtain bounds on the following: (i) The distance between the distribution of the standardized PSR residual and the standard normal,

$$
\Delta(Z_{(c, \infty)}, Z) := \sup_{x \in \mathbb{R}} \left| F_{Z_{(c, \infty)}}(x) - F_Z(x) \right|,
$$

(44)

where $F_{Z_{(c, \infty)}}(x)$ and $F_Z(x)$ denote the CDF of $Z_{(c, \infty)}$ and the standard normal CDF, respectively. (ii) The distance between the distribution of the approximately stable RV $\hat{X}$ and the exact stable law,

$$
\Delta(X, \hat{X}) := \sup_{x \in \mathbb{R}} \left| F_X(x) - F_{\hat{X}}(x) \right|,
$$

(45)

where $F_X(x)$ and $F_{\hat{X}}(x)$ are the CDFs of $X$ and $\hat{X}$, respectively. And (iii) The distance between the truncated PSR and the full PSR,

$$
\Delta(X, X_{(0,c)}) := \sup_{x \in \mathbb{R}} \left| F_X(x) - F_{X_{(0,c)}}(x) \right|,
$$

(46)

where $F_{X_{(0,c)}}(x)$ is the CDF of $X_{(0,c)}$ (16).

In order to apply the smoothing lemma to bound (44), (45) and (46), we need explicit expressions for the various CFs of interest. These are derived in the following section.
VI. CHARACTERISTIC FUNCTIONS

Recall that \( X = X_{(0,c)} + R_{(c,\infty)} \sim S_{(c,\infty)}(\sigma, \beta, 0) \) has CF \( \phi_X(s) \) given in (4), and that we approximate it by \( \hat{X} = X_{(0,c)} + \hat{R}_{(c,\infty)} \) as in (39). Since \( R_{(c,\infty)} \) and \( \hat{R}_{(c,\infty)} \) are independent of \( X_{(0,c)} \), we have,

\[ \phi_X(s) = \phi_{X_{(0,c)}}(s) \phi_{R_{(c,\infty)}}(s), \quad (47) \]

\[ \phi_{\hat{X}}(s) = \phi_{X_{(0,c)}}(s) \phi_{\hat{R}_{(c,\infty)}}(s). \quad (48) \]

Also, the CFs for the true and approximated residuals can be expressed in terms of the CFs of their normalised counterparts by a simple change of variable,

\[ \phi_{R_{(c,\infty)}}(s) = \phi_{Z_{(c,\infty)}}(S_{(c,\infty)} s) \exp \left( i s m_{(c,\infty)} \right), \quad (49) \]

\[ \phi_{\hat{R}_{(c,\infty)}}(s) = \phi_{Z_{(c,\infty)}}(S_{(c,\infty)} s) \exp \left( i s m_{(c,\infty)} \right), \quad (50) \]

where \( m_{(c,\infty)} \) and \( S_{(c,\infty)} \) are given in (27) and (28), and \( \phi_Z(\cdot) \) is the CF of the standard normal distribution (37).

Therefore, in order to use the smoothing lemma for (44) and (45), we need to obtain explicit expressions for \( \phi_{Z_{(c,\infty)}}(s), \phi_{R_{(c,\infty)}}(s), \) and \( \phi_{X_{(0,c)}}(s). \) These are derived in the following two subsections, in the case \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2). \) The proofs are given in the Appendix. For easy reference, the results are summarized in Table II at the end of this section.

A. CF expressions when \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \)

When \( W_1 \) is Gaussian, the following lemma shows that it is possible to write the CF of the PSR residual in terms of an infinite series.

Lemma 2: Let \( Z_{(c,\infty)} \) be defined as in (30), and let \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \). Then, for \( s \in \mathbb{R} \) and \( c > 0, \)

\[ \phi_{Z_{(c,\infty)}}(s) = \exp \left( -\frac{s^2}{2} + \sum_{k=3}^{\infty} \bar{z}_k s^k \right), \quad (51) \]

where,

\[ \bar{z}_k := \frac{i^k}{k!} \frac{\mathbb{E}[W_1^k] \alpha}{k-\alpha} \frac{\mathbb{E}[W_1^2]^{\alpha-2}}{\alpha-2} c^{1-k/2}, \quad k \geq 3. \quad (52) \]

An examination of the proof in Appendix C shows that, in the process of establishing the lemma, we also obtained expressions for the CF of the unnormalized residuals \( R_{(c,d)} \) and \( \hat{R}_{(c,\infty)} \); these are shown in Table II. We also note that these results hold not only in the case \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \), but also more generally for any distribution on \( W_1 \) that satisfies condition (72).

Alternatively, performing direct computations when \( W_1 \) is normally distributed, we obtain the following integral expressions for the CF of the residual.

Lemma 3: Let \( R_{(c,\infty)} \) be defined as in (18), and let \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \). Then, for \( s \in \mathbb{R}, \)

\[ \log(\phi_{R_{(c,\infty)}}(s)) = \alpha \int_0^{c^{-1/\alpha}} \left( e^{\iota s \mu_W - \sigma_W^2 s^2 t^2/2} - 1 - i s \mu_W t \right) t^{-\alpha-1} dt - i s \mu_W \frac{\alpha}{\alpha-1} e^{\frac{\alpha}{\alpha-1}}. \quad (53) \]

Lemma 3 is established in Appendix D, where we also derive analogous integral expressions for the CFs of \( X_{(0,c)} \) and of \( X \).

B. CF expressions when \( W_1 \sim \mathcal{N}(0, \sigma_W^2) \)

Next we obtain a more explicit expression for the CF of the normalized residual \( Z_{(c,\infty)} \) in the case when the mean \( \mu_W = 0 \). This expression was first derived, by summing the series (51), in [64]. A different proof, based on the integral representation in Lemma 3, is given in Appendix E.

Lemma 4: Suppose \( W_1 \sim \mathcal{N}(0, \sigma_W^2) \), and denote,

\[ a := \frac{\alpha}{2}, \quad \eta := \frac{1-a}{a}, \quad w := \frac{\eta s^2}{2c}, \quad u := ws_{(c,\infty)}, \quad (54) \]
for $\alpha \in (0, 2), \alpha \neq 1$, where $S^2_{(c, \infty)}$ is given as in (28) with $E[W_t^2] = \sigma^2_W$. Then,
\[
\phi_{Z_{(c, \infty)}}(s) = \psi_{Z_{(c, \infty)}}(w) = \exp \left( e(1 - e^{-w} - u^a \gamma(1 - a, w)) \right),
\]
where $\gamma(s, x)$ is the lower incomplete gamma function (5). Moreover,
\[
\phi_{\hat{X}_{(o, e)}}(s) = \omega_{\hat{X}_{(o, e)}}(u) = \exp(-c(1 - e^{-u} + u^a \Gamma(1 - a, u))),
\]
where $\Gamma(s, x)$ is the upper incomplete gamma function (6).

Observe that, using the change of variables (54), the CF of the standard normal RV $Z$ (37) can be written,
\[
\phi_Z(s) = \psi_Z(w) = \exp(-cw/\eta).
\]
Hence, using (50), when $\mu_W = 0$ we have
\[
\phi_{R_{(c, \infty)}}(s) = \omega_{R_{(c, \infty)}}(u) = \exp(-cu/\eta).
\]

Then, as a consequence of Lemma 4 and equation (48), it follows that, when $\mu_W = 0$, the CF $\phi_{\hat{X}}(s) = \omega_{\hat{X}}(u)$ of $\hat{X}$, the approximated stable distribution, satisfies,
\[
\log \omega_{\hat{X}}(u) = -c(1 - e^{-u} + u^a \Gamma(1 - a, u) + u/\eta).
\]

From now on and for the rest of the paper we restrict attention to the case $\mu_W = 0$ of the symmetric stable law, for which we have the above closed-form expressions for $\phi_{Z_{(c, \infty)}}(s)$.

### Table II

**Summary of the Logarithms of the CF Expressions Derived and Used.** Recall that $\alpha \in (0, 2), \alpha \neq 1$, and $c > 0$.

| Distribution $W_t$ | Skewness | RV | $\log(\phi(s))$ or $\log(\psi(w))$ or $\log(\omega(u))$, with $w$ and $u$ as in (54) | Equations |
|-------------------|-----------|-----------------|---------------------------------|------------|
| $N(\mu_W, \sigma^2_W)$ | $\beta \in [-1, 1]$ | $X$ | $-c - d + \alpha \int_0^{\alpha} e^{(ist\mu_W - \sigma^2_{1/2})t}q_{-\alpha-1} - ist\mu_W dt - i\lambda B$ | (78), (23) |
| $N(\mu_W, \sigma^2_W)$ | $\beta \in [-1, 1]$ | $R_{(c, d)}$ | $\alpha \int_0^{\alpha} e^{(ist\mu_W - \sigma^2_{1/2})t}q_{-\alpha-1} - ist\mu_W dt - i\lambda B$ | (79) |
| $N(\mu_W, \sigma^2_W)$ | $\beta = 0$ | $R_{(c, \infty)}$ | $\alpha \int_0^{\alpha} e^{(ist\mu_W - \sigma^2_{1/2})t}q_{-\alpha-1} - ist\mu_W dt - i\lambda B$ | (53) |
| $N(\mu_W, \sigma^2_W)$ | $\beta = 0$ | $Z_{(c, \infty)}$ | $c(1 - \exp(-u) - u^{a}\gamma(1 - a, u))$ | (84) |
| $N(\mu_W, \sigma^2_W)$ | $\beta = 0$ | $X_{(o, e)}$ | $-c(1 - \exp(-u) + u^{a}\Gamma(1 - a, u))$ | (56) |
| $N(\mu_W, \sigma^2_W)$ | $\beta = 0$ | $R_{(c, \infty)}$ | $-cu/\eta$ | (58) |
VII. Nonasymptotic Gaussian Bounds for the PSR Residual

A. Nonasymptotic bound of order \(O(1/c)\)

In this section we apply the smoothing lemma of Section V to derive explicit bounds on the distance \(\Delta(Z_{(c,\infty)}, Z)\), defined in (44). When \(\mu_W = 0\), the closed-form expression in (55) for \(\phi_{Z_{(c,\infty)}}(s)\) can be used to further bound above the term \(\bar{I}(Z_{(c,\infty)}, Z)\) in (43). The resulting bounds, first presented in [66], are stated in the following theorem and proved in Appendix H.

**Theorem 2:** Let \(W_j \sim \mathcal{N}(0, \sigma^2_W)\) and let \(\Delta(Z_{(c,\infty)}, Z)\) be the Kolmogorov distance between \(Z_{(c,\infty)}\) and \(Z\), as in (44). Let \(a = a(\alpha)\) and \(\eta = \eta(\alpha)\) as in (54), and define,

\[
g(w) := 1 - e^{-w} - w^a \gamma(1 - a, w), \quad w \geq 0.
\]

Let \(\gamma(s, x)\) and \(\Gamma(s, x)\) be the lower and upper incomplete gamma functions, (5) and (6) respectively, and write,

\[
\begin{align*}
\bar{\gamma}(a) &= \gamma(1 - a, 1), \\
\bar{\eta} &:= g(1), \\
K(a) &:= \frac{1}{\pi} \left[ \frac{a}{2(2 - a)} + \frac{1}{\eta^2} \right].
\end{align*}
\]

Then, for any \(c > 1\), \(\Delta(Z_{(c,\infty)}, Z)\) is bounded above by:

\[
B_1(c, \alpha) := cK(a) \left[ \frac{1}{(c-1)\bar{\eta}^2} + \left( \frac{1}{\bar{\eta}} - \frac{1}{(c-1)\bar{\eta}^2} \right) \exp \left( (c-1)\frac{1}{\bar{\eta}} \right) + \right.
\]

\[
\left. + \frac{(c-1)\exp\left( (c-1)(1 - e^{-1}) \right)}{a\left( (c-1)\bar{\gamma}(a) \right)^{2/a}} \Gamma \left( \frac{2}{a}, (c-1)\bar{\gamma}(a) \right) \right].
\]

**Remark 1:** Observe that the upper bound on the Kolmogorov distance of the PSR residual from its Gaussian approximation coincides with the upper bound on the the distance of the PSR standardized residual from the standard Gaussian,

\[
\bar{I}(R_{(c,\infty)}, R_{(c,\infty)}) = \bar{I}(Z_{(c,\infty)}, Z).
\]

In fact, the Kolmogorov distance itself is invariant under monotone transformations, thus in particular for translation and scaling.

**Remark 2:** Even though the PSR split was introduced in (17) for any \(c \geq 0\), Theorem 2 and the results below hold only for \(c > 1\). This is not a significant limitation because, in practice, we are indeed interested in scenarios where there are at least a few terms in \(X_{(c,\infty)}\).

The following corollary is a simple consequence of Theorem 2; its proof is given in Appendix I.

**Corollary 3:** Under the assumptions of Theorem 2, as \(c \to \infty\),

\[
B_1(c, \alpha) \sim \frac{K(a)}{\bar{\eta}^2} \left( \frac{1}{c-1} \right),
\]

so that,

\[
\Delta(Z_{(c,\infty)}, Z) = O\left( \frac{1}{c} \right).
\]

For values of \(\alpha\) greater than 0.4, \(B_1(c, \alpha)\) gives very good bounds, as shown on the left-hand side of Figure 3. But for \(\alpha\) below 0.4, the results deteriorate significantly; for example, for \(\alpha = 0.2\), \(B_1(c, \alpha)\) is below 1 (the maximum possible Kolmogorov distance) only for \(c > 115\).

B. Nonasymptotic bound of order \(O(1/\sqrt{c})\)

The following result, obtained by bounding \(I(Z_{(c,\infty)}, Z)\) in (42), gives an \(O(1/\sqrt{c})\) bound which is, of course, asymptotically inferior to that in Theorem 2, but which gives sharper results for small \(c\) and \(\alpha < 0.4\).

**Theorem 4:** Under the assumptions and in the notation of Theorem 2, for any \(\delta \in (0, 2)\),

\[
\Delta(Z_{(c,\infty)}, Z) \leq B_2(c, \alpha, \delta) := \frac{9.6\sqrt{\eta}}{\pi \sqrt{2(2-\delta)c}} + B_3(c, \alpha, \delta),
\]

so that,
Fig. 3. Bounds on $\Delta(Z(c,\infty), Z)$. Left: each curve represents the values of the bound $B_1(c,\alpha)$ for $\alpha = 0.4, 0.9, \ldots, 1.9, \alpha \neq 1$, plotted against $12 \leq c \leq 50$. Centre: each curve represents the values of the bound $\bar{B}_2(c,\alpha)$ for $\alpha = 0.1, 0.2, \ldots, 1.9, \alpha \neq 1$, plotted against $10 \leq c \leq 300$. Right: each curve represents $B_4(c,\alpha)$, for $\alpha = 0.1, 0.2, \ldots, 1.9, \alpha \neq 1$, plotted against $10 \leq c \leq 1000$. [The red horizontal line at 1 shows the maximum possible value of the Kolmogorov distance.]

where $B_3(c,\alpha,\delta)$ is the following $O(1/c)$ term:

$$B_3(c,\alpha,\delta) := \frac{K(a)}{c} \left( \frac{c(2-\delta)}{(c-1)g(2-\delta)} \right)^2 \times \left\{ 1 - \left[ 1 - g(2-\delta)(c-1) \right] \exp \left( g(2-\delta)(c-1) \right) \right\}.$$

The proof is given in Appendix J. Numerically minimizing the bound $B_2(c,\alpha,\delta)$ over $\delta$ yields $\bar{B}_2(c,\alpha)$, shown in the central part of Figure 3.

C. Combined bound and comparison with numerical results

Finally, we combine the results of Theorems 2 and 4, to obtain useful bounds essentially for all values of $\alpha \in (0, 2)$, $\alpha \neq 1$, and $c > 50$:

$$\Delta(Z(c,\infty), Z) \leq B_4(c,\alpha) := \min \left\{ B_1(c,\alpha), \bar{B}_2(c,\alpha) \right\}.$$

The resulting numerical bound is shown on the right-hand side of Figure 3 (on a log-log scale).

Figure 4 shows a comparison between the theoretical bound $B_4(c,\alpha)$ and the numerical estimate $\bar{Q}(Z(c,\infty), Z)$ of $\bar{I}(Z(c,\infty), Z)$, produced through the Matlab routine quadgk, which implements the Gauss-Kronrod method; see [65] and [66] for more details. This method also produces an approximate upper bound on the absolute error $|\bar{I}(Z(c,\infty), Z) - \bar{Q}(Z(c,\infty), Z)|$, which can be used to construct approximate error bands. But for $c \geq 3$ these are negligibly small, so we do not show them here. Observe that $B_4(c,\alpha)$ appears to have the exact same asymptotic rate as $\bar{Q}(Z(c,\infty), Z)$.

Fig. 4. Bounds on $\Delta(Z(c,\infty), Z)$: comparison between $\bar{Q}(Z(c,\infty), Z)$ and $B_4(c,\alpha)$, for $\alpha = 0.1, 0.2, \ldots, 1.9, \alpha \neq 1$, plotted for $3 \leq c \leq 300$. [The red horizontal line at 1 shows the maximum possible value of the Kolmogorov distance.]
VIII. BOUNDS ON $\alpha$-STABLE DISTRIBUTION APPROXIMATIONS

Here we develop bounds on the distances $\Delta(X, \hat{X})$ and $\Delta(X, X_{(0,c)})$ defined in (45) and (46), respectively. In terms of inference, ultimately, it is these Kolmogorov distances that we wish to make “small” by appropriately choosing the value of the parameter $c$.

A. Nonasymptotic bound on $\Delta(X, \hat{X})$

The bound on $\Delta(X, \hat{X})$ stated next, is established by using the smoothing lemma (43) and the bound in Theorem 2. Its proof is given in Appendix K.

**Theorem 5:** Let $\Delta(X, \hat{X})$ be the Kolmogorov distance between $X$ and $\hat{X}$, as in (45), under the same assumptions and in the same notation as Theorem 2. Let $N \geq 1$, and introduce $N$ arbitrary points $u_i$ on $[0,1]$,

$$0 := u_0 < u_1 < \cdots < u_N := 1,$$

together with the corresponding values of the logarithm of the CF $\omega_{X(0,c)}(u)$ defined in (56),

$$f_0 := 0, \quad f_i := \log(\omega_{X(0,c)}(u_i)), \quad i = 1, 2, \ldots, N.$$  

Also let,

$$m_i := \frac{f_{i+1} - f_i}{u_{i+1} - u_i}, \quad q_i := -m_i u_i + f_i, \quad (62)$$

Then, for any $c > 1$, $\Delta(X, \hat{X})$ is bounded above by,

$$B_5(c, \alpha, N) := cK(a) \times \left\{ \sum_{i=0}^{N-1} \frac{e^{\tilde{m}_i} u_{i+1} - 1}{m_i} - e^{\tilde{m}_i} u_i \left( u_i - \frac{1}{m_i} \right) \right\} + \frac{e^{\tilde{k}_{(1,\infty)}} \Gamma \left( \frac{2}{a}, \tilde{l}_{(1,\infty)} \right)}{a\tilde{l}_{(1,\infty)}}$$

where,

$$\tilde{m}_i := m_i + (c - 1)\bar{g},$$

$$k_{(1,\infty)} := -c(1 - \exp(-1)) + \Gamma(1 - a, 1) < 0,$$

$$\tilde{k}_{(1,\infty)} := k_{(1,\infty)} - (c - 1)(e^{1 - 1}),$$

$$\tilde{l}_{(1,\infty)} := (c - 1)\bar{g}.$$

(63)

and $\bar{g}$ as in (60) and $\bar{g}(a)$ as in (59).

The values $\{u_i\}$ and $\{f_i\}$ serve to define a piece-wise linear envelope on $\omega_{X(0,c)}(u)$ for $u \in [0,1]$, which is used in the proof; see Appendices G-B and K. Increasing $N$ improves (i.e., decreases) the value of $B_5(c, \alpha, N)$, but the improvement becomes negligible for $N \geq 10$ and logarithmically spaced points, as shown in Figure 5, where bounds with three different values of $N = 1, 2, 10$ are compared.

![Fig. 5. Bounds on $\Delta(X, \hat{X})$. Each curve shows the bound $B_5(c, \alpha, N)$ for $\alpha = 0.1, 0.9, \ldots, 1.9$, $\alpha \neq 1$, plotted against $1 \leq c \leq 10^5$. The $N$ points $u_0, \ldots, u_{N-1}$ are logarithmically spaced on $[0,1]$. [The red horizontal line at 1 shows the maximum possible value of the Kolmogorov distance.]](image_url)

In Figure 6 we compare the numerical estimates $\tilde{Q}(X, \hat{X})$ for $\tilde{I}(X, \hat{X})$ obtained in [65], with the bound $B_5(c, \alpha, N)$ of Theorem 5 with $N = 10$. Note that this bound correctly captures the dependence on $\alpha$, and that the approximation error is lower for smaller values of $\alpha$, a reversal of the trend shown in Figure 3. One reason for this is that, as $\alpha$ decreases, the relative significance of the residual term becomes smaller, when compared with the heavy-tailed initial terms in the PSR.
We also observe that the rate of convergence is dramatically better for smaller $\alpha$, again in contrast with the analysis of the residual approximation in Figure 3. Finally, it seems that $B_5(c, \alpha, N)$ has the same asymptotic behaviour as $B_1(c, \alpha)$ for $c \to \infty$, see also Remark 3 in Appendix K. However, these two bounds have reversed asymptotic ordering with respect to $\alpha$.

Fig. 6. Bounds on $\Delta(X, \hat{X})$. The blue solid lines represent $\bar{Q}(X, \hat{X})$; the black dashed lines represent $B_5(c, \alpha, N)$ with $N = 10$ and logarithmically spaced points. The values are plotted for $\alpha = 0.1, 0.2, \ldots, 1.9, \alpha \neq 1$, and for $1 \leq c \leq 5000$. [The red horizontal line at 1 shows the maximum possible value of the Kolmogorov distance.]

B. Nonasymptotic bound on $\Delta(X, X_{(0,c)}(c))$

The following bound on $\Delta(X, X_{(0,c)}(c))$ is similar to results obtained in [40]. Its proof, given in Appendix L, is based on direct computations and does not rely on the smoothing lemma.

**Proposition 6:** Let $\Delta(X, X_{(0,c)}(c))$ be defined as in (46). Under the same assumptions and notation as Theorem 2,

$$\Delta(X, X_{(0,c)}(c)) \leq B_6(c, \alpha) := \exp\left(-\frac{1}{2}\right) \sqrt{\frac{2\pi}{\alpha}} \times \sqrt{\Gamma\left(\frac{\alpha + 4}{\alpha}\right) \left(\frac{\alpha}{4 - \alpha} e^{\frac{\alpha - 4}{\alpha}} + \left(\frac{\alpha}{2 - \alpha} e^{\frac{\alpha - 2}{\alpha}}\right)^2\right)}.$$

It is easy to see that the bound $B_6(c, \alpha)$ is of $O(c^{(\alpha-2)/\alpha})$, and a corresponding lower bound of the same order is also established in [40]. Figure 7 illustrates this bound, and Figure 8 compares it to the numerical estimates $\bar{Q}(X, X_{(0,c)}(c))$ of $I(X, X_{(0,c)}(c))$ as in (43).

Fig. 7. Bounds on $\Delta(X, X_{(0,c)}(c))$. Each curve represents the values of the bound $B_6(c, \alpha)$ for $\alpha = 0.1, 0.9, \ldots, 1.9, \alpha \neq 1$, plotted against $1 \leq c \leq 10^2$. [The red horizontal line at 1 shows the maximum possible value of the Kolmogorov distance.]
Fig. 8. Bounds on $\Delta(X, X_{(0,c)})$. The blue solid lines represent $\bar{Q}(X, X_{(0,c)})$; the black dashed lines represent $B_6(c, \alpha)$. The values are plotted for $\alpha = 0.1, 0.5, 1.9$, and for $1 \leq c \leq 1000$. [The red horizontal line at 1 shows the maximum possible value of the Kolmogorov distance.]

C. Comparison of $\bar{I}(X, \hat{X})$ and $\bar{I}(X, X_{(0,c)})$

Finally, we establish a result that compares the approximation of an $\alpha$-stable RV $X$ by (i) the truncated PSR $X_{(0,c)}$, or (ii) by $\hat{X} = X_{(0,c)} + \hat{R}_{(0,c)}$, which is the truncated PSR plus a Gaussian approximation $\hat{R}_{(0,c)}$ to the residual $R_{(0,c)}$. Specifically, in Proposition 7, proved in Appendix M, we compare the bounds $\bar{I}(X, X_{(0,c)})$ and $\bar{I}(X, \hat{X})$. The result indicates that, in the symmetric case $W_1 \sim \mathcal{N}(0, \sigma^2_{W'})$, adding a Gaussian approximation will likely provide a better approximation to the $\alpha$-stable distribution than the truncated PSR alone, for most values of the truncation parameter.

**Proposition 7:** Let $\bar{I}(X, \hat{X})$ and $\bar{I}(X, X_{(0,c)})$ be defined as in (43), under the same assumptions and notation as in Theorem 2. Then, for any $\alpha \in (0, 2)$, $\alpha \neq 1$, we have that,

$$\bar{I}(X, \hat{X}) < \bar{I}(X, X_{(0,c)}),$$

for all,

$$c > c(\alpha) := \log(2) \frac{\log(2)}{\gamma(1 - \alpha/2, 1) + e^{-1} - 1}.$$

Proposition 7 suggests that the Gaussian residual approximation produces a smaller approximation error than simply truncating the series, a result also borne out by previous numerical results reported in [44, p. 56-57]; see also Figure 2. Although the result of the proposition is only valid for $c > c(\alpha)$, we note that this is not a severe restriction: $c(\alpha) < 16.5$ for all $\alpha > 0.1$ and $c(\alpha) < 1$ for $\alpha > 1$; see Figure 9. Moreover, the condition $c > c(\alpha)$ is only shown to be sufficient and, in fact, numerical estimates of the integrals $\bar{I}(X, \hat{X})$ and $\bar{I}(X, X_{(0,c)})$ show that,

$$\bar{Q}(X, \hat{X}) < \bar{Q}(X, X_{(0,c)}), \quad \text{for all } c > 1,$$

as shown in Figure 10.

We remark however that the results in this section only indicate, but do not prove, that $\hat{X}$ is closer in distribution to $X$ than $X_{(0,c)}$. This could be proved by providing lower bounds on the relative Kolmogorov distances, which is left to future studies.
Fig. 9. The function $c = c(\alpha)$ plotted against $\alpha = 0.1, \ldots, 1.9$. The red horizontal line corresponds to $c = 1$, which is the smallest minimum value of $c$ that we consider relevant for practical purposes.

Fig. 10. Numerical bounds on $\Delta(X, \hat{X})$ and on $\Delta(X, X_{(0,c)})$. The solid lines correspond to $\bar{Q}(X, \hat{X})$ and the dashed lines to $\bar{Q}(X, X_{(0,c)})$ for $\alpha = 0.1, 0.2, \ldots, 1.9$, $\alpha \neq 1$, plotted against $1 \leq c \leq 5000$. [The red horizontal line at 1 shows the maximum possible value of the Kolmogorov distance.]
IX. Inference for Regression Models

In order to illustrate the utility of the approximate conditionally Gaussian framework introduced in this paper, we give an example of a Bayesian inference scheme. Specifically, we consider the problem of estimating the parameters $\lambda$ of model (1), described in the Introduction, when $u := [u_1, \ldots, u_N]$ has symmetric $\alpha$-stable components.

A. Parameter inference in $\alpha$-stable regression models

Assume, for simplicity, that $x = G\lambda + u$ is fully observed, and that the matrix $G$ is known. We augment this model by introducing a set of latent vectors, $T := \{\Gamma_1, \ldots, \Gamma_N\}$, one for every element of $u = [u_1, \ldots, u_N]'$, as follows. Let $c$ be a truncation parameter, and for each $1 \leq n \leq N$, let $\Gamma_n := [\Gamma_{1,n}, \ldots, \Gamma_{N(0,c)(n),n}]$ be event times of a unit rate Poisson process, where $N_{(0,c)}(n)$ is the number of those $\Gamma_{j,n}$ that are smaller than $c$. We then use the approximate representation,

\[ u_n \approx \sum_{j=1}^{N_{(0,c)}(n)} W_{j,n} \Gamma^{-1/\alpha}_{j,n} + R_n, \]

where the $W_{j,n}$ are i.i.d. with distribution $\mathcal{N}(0, \sigma^2_W)$, and, according to (38), $R_n$ is an independent Gaussian with mean and variance $m_{\alpha}(c, \infty)$ and $S^2_{\alpha}(c, \infty)$ given by (27) and (28), respectively, so that $m_{\alpha}(c, \infty) = 0$ and $S^2_{\alpha}(c, \infty) = \sigma^2_W (\frac{\alpha}{2-\alpha}) c^{(\alpha-2)/\alpha}$. This, then, leads to the conditionally Gaussian representation of each $u_n$ as in (40),

\[ u_n | \{\Gamma_{j,n} \in [0,c]\} \approx \mathcal{N}(0, \sigma^2_n), \] (64)

where the variance $\sigma^2_n$ is,

\[ \sigma^2_n = \sigma^2_W \left[ \sum_{j: \Gamma_{j,n} \in [0,c]} \Gamma^{-2/\alpha}_{j,n} + \left( \frac{\alpha}{2-\alpha} \right) c^{(\alpha-2)/\alpha} \right]. \]

We assume that the truncation parameter $c$ has been chosen so that the Kolmogorov distance between the true distribution of $u_n$ and its approximation is below a certain threshold, based on Theorem 5.

Writing $\Sigma$ for the diagonal matrix with elements $\sigma^2_n, n = 1, \ldots, N$, (1) and (64) imply that the likelihood of $x$ can be approximated as,

\[ p(x|G\lambda, T) \approx \mathcal{N}(G\lambda, \Sigma). \]

Regular inference can then be carried out as for Gaussian models, by augmenting the set of parameters to be estimated to $\{\lambda, T\}$: A Metropolis-within-Gibbs sampler can be used, which in the $k$-th iteration draws,

\[ \lambda^{(k)} \sim p(\lambda | T^{(k-1)}, x), \]

\[ T^{(k)} \sim p(T | \lambda^{(k)}, x). \] (65) (66)

The Gibbs step is (65): Adopting a conjugate Gaussian prior leads to a Gaussian $p(\lambda | T^{(k-1)}, x)$. And for sampling $T$, the full conditional density in (66) can be targeted with a Metropolis step. The posterior distribution of the parameters $\lambda$ can then be estimated by looking at the first component of the chain $\{\lambda^{(k)}, T^{(k)}\}_{k=1}^M$, after it has been run for a sufficiently large number $M$ of iterations. We refer to [41], [43] for more details and simulation results from this scheme.

Under the assumption of Gaussian $W_j$, both the exact (15) and the approximate (41) representation of each term $u_n$ are continuous scale mixtures of normals. However, the difference between the two representations lies in the possibility of exactly sampling the mean and scale latent variables.

In the presented inference scheme, the distribution of the terms $u_n$ is symmetric. When $\alpha$ is known, an exact continuous scale mixture of normals representation can be deduced for the symmetric stable distribution from the product property [25, p. 176]. In the aforementioned case with $\alpha$ known, the scale latent variable has positive stable distribution, and can thus be sampled exactly via the method of [16]. This was used in [27]–[29], [75] for developing posterior samplers for the parameters of the stable distribution and the parameters of linear models with stable noise.

However, for asymmetric stable distributions, or for symmetric distributions with unknown $\alpha$, it appears not to be possible to sample the mean and scale latent variables in closed form. Consequently, in the cases above it is not possible to do inference based on the exact PSR. On the other hand, it is possible to do exact inference for the approximate representation. Exploiting this possibility is perhaps the most relevant continuation of this work.

Finally, we stress that our findings are indeed related to the stable likelihood, and further analytic studies are required in order to establish the quality of inference procedures based on truncating the PSR and approximating its residual. We refer

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where the variance $\sigma^2_n$ is,

\[ \sigma^2_n = \sigma^2_W \left[ \sum_{j: \Gamma_{j,n} \in [0,c]} \Gamma^{-2/\alpha}_{j,n} + \left( \frac{\alpha}{2-\alpha} \right) c^{(\alpha-2)/\alpha} \right]. \]

We assume that the truncation parameter $c$ has been chosen so that the Kolmogorov distance between the true distribution of $u_n$ and its approximation is below a certain threshold, based on Theorem 5.

Writing $\Sigma$ for the diagonal matrix with elements $\sigma^2_n, n = 1, \ldots, N$, (1) and (64) imply that the likelihood of $x$ can be approximated as,

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Regular inference can then be carried out as for Gaussian models, by augmenting the set of parameters to be estimated to $\{\lambda, T\}$: A Metropolis-within-Gibbs sampler can be used, which in the $k$-th iteration draws,

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The Gibbs step is (65): Adopting a conjugate Gaussian prior leads to a Gaussian $p(\lambda | T^{(k-1)}, x)$. And for sampling $T$, the full conditional density in (66) can be targeted with a Metropolis step. The posterior distribution of the parameters $\lambda$ can then be estimated by looking at the first component of the chain $\{\lambda^{(k)}, T^{(k)}\}_{k=1}^M$, after it has been run for a sufficiently large number $M$ of iterations. We refer to [41], [43] for more details and simulation results from this scheme.

Under the assumption of Gaussian $W_j$, both the exact (15) and the approximate (41) representation of each term $u_n$ are continuous scale mixtures of normals. However, the difference between the two representations lies in the possibility of exactly sampling the mean and scale latent variables.

In the presented inference scheme, the distribution of the terms $u_n$ is symmetric. When $\alpha$ is known, an exact continuous scale mixture of normals representation can be deduced for the symmetric stable distribution from the product property [25, p. 176]. In the aforementioned case with $\alpha$ known, the scale latent variable has positive stable distribution, and can thus be sampled exactly via the method of [16]. This was used in [27]–[29], [75] for developing posterior samplers for the parameters of the stable distribution and the parameters of linear models with stable noise.

However, for asymmetric stable distributions, or for symmetric distributions with unknown $\alpha$, it appears not to be possible to sample the mean and scale latent variables in closed form. Consequently, in the cases above it is not possible to do inference based on the exact PSR. On the other hand, it is possible to do exact inference for the approximate representation. Exploiting this possibility is perhaps the most relevant continuation of this work.

Finally, we stress that our findings are indeed related to the stable likelihood, and further analytic studies are required in order to establish the quality of inference procedures based on truncating the PSR and approximating its residual. We refer
to [44] for numerical insights on superior behaviour of Bayesian estimators based on the truncated PSR with accounted residual, with respect to those based on the simply truncated PSR.

As anticipated in the Introduction, the PSR and its approximation (39) are also relevant for inference of continuous-time Lévy processes and linear models driven by Lévy noise. These are more challenging in terms of inference than the discrete time model (1)–(3), but in some applications they are more realistic, e.g., when data are sampled at irregular time intervals. Potential extension of our work in this direction are described next.

### B. PSR for continuous-time stochastic processes and linear models

Consider the continuous-time version of the linear model corresponding to (1), where we assume that linear observations are made at discrete times \( \{ t_i \} \), as in (3),

\[
\begin{align*}
\mathrm{d}x(t) &= Ax(t) \, \mathrm{d}t + h \, \mathrm{d}\ell(t), \\
y(t_i) &= b'x(t_i) + v(t_i),
\end{align*}
\]

where \( x(t) = [x_1(t), \ldots, x_P(t)]' \) is the state, \( A \) is a \( P \times P \) matrix describing the interaction of the components of \( x(t) \), \( h \) is a \( P \)-dimensional vector describing the effects of the noise process \( \{ \mathrm{d}\ell(t) \} \), \( b \) is a \( P \)-dimensional vector, and \( \{ v(t_i) \} \) is the observation noise process. A wide range of results have been developed in the literature for the case when \( \{ \ell(t) \} \) is a Brownian motion [30], [60], but, as for the discrete-time case, such models are not appropriate for certain applications.

Large jumps and heavy tails in the state process, as often observed in applications, can be modelled by choosing \( \{ \ell(t) \} \) to be a (non-Gaussian) Lévy process; see [10]–[12] for a first formulation, [73] and [7] for a review, and [5], [77], [79] for more recent work from an engineering perspective. The sub-class of \( \alpha \)-stable Lévy processes [33], [69] is of special importance. In fact, the self-similarity of stable Lévy processes [69] implies that transition densities, although still intractable, all come from the same \( \alpha \)-stable family. Hence, as argued in the Introduction, \( \alpha \)-stable Lévy processes may be considered to be the natural first choice towards generalising the classical Gaussian process framework to the heavy-tailed case.

LePage series representations are available for Lévy processes. In the \( \alpha \)-stable symmetric case, \( \{ \ell(t) \} \) can be expressed,

\[
\ell(t) \overset{D}{=} \sum_{j=1}^{\infty} (\Gamma_j^{-1/\alpha} W_j) 1(V_j < t),
\]

where \( \{ \Gamma_j \}_{j=1}^{\infty} \) and \( \{ W_j \}_{j=1}^{\infty} \) are as before, and \( \{ V_j \}_{j=1}^{\infty} \) are i.i.d. uniform RVs on \( [0,T] \), where \( T \) is the time horizon considered. A similar representation is available for the asymmetric case, see [41, Lemma 4.1.1]. Like in discrete-time, \( \ell(t) \) is conditionally Gaussian,

\[
\ell(t)|\{ \Gamma_j, V_j \}_{j=1}^{\infty} \sim \mathcal{N}(m_k, S_k^2),
\]

with \( m_k \) and \( S_k^2 \) can be appropriately defined as series involving \( \{ \Gamma_j, V_j \}_{j=1}^{\infty} \). Once again, the series in (68) and in the definitions of \( m_k \) and \( S_k^2 \) cannot be computed exactly. However, defining \( W_j := W_j 1(V_j < t) \), we have that \( \mathbb{E}[W_j^2] < \infty \), hence Theorem 1 applies and the residual of (68) is asymptotically Gaussian. If the variables \( W_j \) are selected to be Gaussian, this implies that an overall approximate conditionally Gaussian representation of \( \ell(t) \) is again available. Furthermore, considering \( 1(V_j < t) \) as a thinning operation on the Poisson process associated with the RVs \( \{ \Gamma_j \} \), it is straightforward to extend our present results to the continuous-time setting of \( \alpha \)-stable Lévy processes.

The stable distribution of \( \ell(t) \) is inherited by \( x(t) \), so a PSR representation holds for the \( \alpha \)-stable vector \( x(t) \); a Gaussian approximation result for its residual is discussed in [67]. Note, however, that our present nonasymptotic results need to be further adapted to \( x(t) \), due to the fact that we would need to consider the structure of the solution to the stochastic differential equation (67), and the multivariate nature of \( x(t) \). Preliminary versions of these ideas have been implemented in [41], [42], [45], by accounting for the PSR residual in inference tasks, while theoretical studies on the choice of the threshold parameter \( c \) are left to future developments.
APPENDIX A
TRANSFORMATIONS FOR σ AND β

Suppose that $W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2)$ or, more generally, that $W_1$ belongs to a location-scale family, with location $\mu_W$, scale $\sigma_W^2$, and PDF $f_W(w)$, $w \in \mathbb{R}$. For a given value of the tail parameter $\alpha$, here we describe how any pair of values $\sigma > 0$ and $\beta \in [-1, 1]$ can be obtained via the mappings in (11a) and (11b), by appropriate choices of $\mu_W$ and $\sigma_W^2$.

We introduce the following auxiliary PDF:
\[
\pi(w) := \frac{|w|^\alpha f_W(w)}{\int_{\mathbb{R}} |w|^\alpha f_W(w') \, dw'} = \frac{\tilde{\pi}(w)}{I},
\]
where $\tilde{\pi}(w)$ and $I$ denote the unnormalized density and the normalizing constant, respectively. Then (11a), the transformation related to $\sigma$, can be rewritten as,
\[
\sigma = \frac{\int_{\mathbb{R}} \tilde{\pi}(w) \, dw}{C_\alpha} = \frac{I}{C_\alpha}.
\]
Since $C_\alpha > 0$, we have that $\sigma > 0$, and it is easy to see that it is possible to achieve any $\sigma > 0$, by appropriately choosing $\sigma_W^2$.

Similarly, we can express (11b), the transformation related to $\beta$, as,
\[
\beta = -\frac{\int_{-\infty}^{0} \tilde{\pi}(w) \, dw + \int_{0}^{\infty} \tilde{\pi}(w) \, dw}{I} = -\int_{-\infty}^{0} \pi(w) \, dw + \int_{0}^{\infty} \pi(w) \, dw = - (1 - I^+) + I^+ = 2I^+ - 1,
\]
where $I^+$ is the probability of $\mathbb{R}^+$ under $\pi$. Then any $\beta \in [-1, 1]$ can be obtained by choosing the parameters of the distribution of $W_1$ to give the required value of $I^+ \in [0, 1]$. Since that $|w|^\alpha$ is a symmetric function, it is clear from (11b) that $\beta = 0$ when $f_W(w)$ is an even function. In the Gaussian case, this corresponds to $\mu_W = 0$. Similarly, $I^+ > 0.5$ (i.e., $\mu_W > 0$) leads to positive skewness $\beta > 0$, while $I^+ < 0.5$ (i.e., $\mu_W < 0$) leads to $\beta < 0$. A combined choice of the scale and location parameters is required to achieve the limiting cases $\beta = -1$ ($\mu_W < 0, \sigma_W = 0$) and $\beta = 1$ ($\mu_W > 0, \sigma_W = 0$).

APPENDIX B
PROOF OF LEMMA 1

We make use of the observation that $R_{(c,d)}$ can be viewed as a compound Poisson process. Hence, to compute expectations with respect to its distribution, we first condition on $N_{(c,d)}$, the random number of terms in $R_{(c,d)}$, and take the expectation over $N_{(c,d)}$. Using the expression (20) for $R_{(c,d)}$,
\[
\phi_{R_{(c,d)}}(s) = \mathbb{E} \left[ \exp \left( isR_{(c,d)} \right) \right] = \mathbb{E} \left[ \exp \left( is \left( \sum_{j=1}^{N_{(c,d)}} W_j \Gamma_j^{-1/\alpha} - \mathbb{E}[W_1] \sum_{j=1}^{d} b_j^{(\alpha)} \right) \right) \right] = \mathbb{E} \left[ \exp \left( is \left( \sum_{j=1}^{N_{(c,d)}} W_j U_j^{-1/\alpha} - \mathbb{E}[W_1] \sum_{j=1}^{d} b_j^{(\alpha)} \right) \right) \right] = \mathbb{E} \left[ \sum_{n=0}^{\infty} \mathbb{E} \left[ e^{is \sum_{j=1}^{n} Y_j} \right] \mathbb{P} (N_{(c,d)} = n) \right] \times \exp \left( -is \mathbb{E}[W_1] \sum_{j=1}^{d} b_j^{(\alpha)} \right) = \sum_{n=0}^{\infty} \mathbb{E} \left[ \phi_{Y_1}(s)^n \right] \mathbb{P} (N_{(c,d)} = n) \times \exp \left( -isB \right) = \sum_{n=0}^{\infty} \phi_{Y_1}(s)^n \frac{(d-c)^n}{n!} e^{-(d-c)} \times \exp \left( -isB \right) = \exp \left( (d-c)(\phi_{Y_1}(s) - 1) - isB \right),
\]
where $\phi_{Y_1}(s)$ is the CF of $Y_1 = W_1 U_1^{-1/\alpha}$.
Since we assume \( \mathbb{E}[W_1^2] \) is finite, it follows that \( \mathbb{E}[Y_1^2] \) is finite, and hence the first and second moments of \( R_{(c,d)} \) are finite as well and can be computed [25, Lemma XV.4.2] by taking derivatives of its CF at zero:

\[
\begin{align*}
m_{(c,d)} &= \mathbb{E}[R_{(c,d)}] \\
&= (-i) \phi_{R_{(c,d)}}(0) \\
&= (d-c) \mathbb{E}[Y_1] - B, \\
S_{(c,d)}^2 &= \mathbb{E}[R_{(c,d)}^2] - (\mathbb{E}[R_{(c,d)}])^2 \\
&= (-i)^2 \phi''_{R_{(c,d)}}(0) - (\phi_{R_{(c,d)}}(0))^2 \\
&= (d-c) \mathbb{E}[Y_1^2].
\end{align*}
\]

(69)

Since \( U_1 \overset{i.i.d.}{\sim} \mathcal{U}(c, d) \), we have that, for \( k = 1, 2 \),

\[
\mathbb{E}[U_1^{-k/\alpha}] = \frac{1}{d-c} \int_c^d U^{-(k/\alpha)} \, dU
\]

\[
= \frac{1}{d-c} \left[ \frac{\alpha}{\alpha - k} U^{-(k/\alpha)+1} \right]_c^d
\]

\[
= \frac{1}{d-c} \frac{\alpha}{\alpha - k} \left( d^{\frac{k}{\alpha}} - c^{\frac{k}{\alpha}} \right),
\]

and since \( W_1 \) is independent of \( U_1 \),

\[
\mathbb{E}[Y_1^k] = \mathbb{E}[W_1^k] \frac{1}{d-c} \frac{\alpha}{\alpha - k} \left( d^{\frac{k}{\alpha}} - c^{\frac{k}{\alpha}} \right).
\]

(71)

Substituting (71) into the moment expressions (69) and (70) gives (25) and (26) as claimed.

**APPENDIX C**

**PROOF OF LEMMA 2**

As in the proof of Theorem 1 we note that the CF of \( Z_{(c, \infty)} \) can be expressed in terms of the CF of \( Y_1 = W_1 U_1^{-1/\alpha} \). Therefore, we begin by expanding \( \phi_{Y_1}(s) \) as a Taylor series.

Suppose \( W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2) \). Using (71) and the well-known formula for the moments of the normal distribution, it is easy to check that,

\[
\limsup_{k \to \infty} \frac{1}{k} \mathbb{E}[|Y_1|^k]^{1/k} < \infty.
\]

(72)

Therefore, \( \phi_{Y_1}(s) \) is analytic around \( s = 0 \) and admits the Taylor expansion [25, p. 514],

\[
\phi_{Y_1}(s) = 1 + i \mathbb{E}[Y_1] s - \frac{1}{2} \mathbb{E}[Y_1^2] s^2 + \sum_{k=3}^\infty \frac{i^k}{k!} \mathbb{E}[Y_1^k] s^k.
\]

(73)

Then, from (24), (35) and (71), for \( c > 0 \), we have,

\[
\phi_{R_{(c,d)}}(s) = \exp \left( ism_{(c,d)} - \frac{s^2 S_{(c,d)}^2}{2} + \sum_{k=3}^\infty r_k s^k \right),
\]

(74)

where,

\[
r_k := \frac{i^k}{k!} \mathbb{E}[W_1^k] \frac{\alpha}{\alpha - k} \left( d^{(\alpha-k)/\alpha} - c^{(\alpha-k)/\alpha} \right).
\]

(75)

Taking the limit as \( d \to \infty \),

\[
\phi_{R_{(c, \infty)}}(s) = \exp \left( ism_{(c, \infty)} - \frac{s^2 S_{(c, \infty)}^2}{2} + \sum_{k=3}^\infty \tilde{r}_k s^k \right),
\]

(76)

where,

\[
\tilde{r}_k := \frac{i^k}{k!} \mathbb{E}[W_1^k] \frac{\alpha}{k - \alpha} c^{(\alpha-k)/\alpha}, \quad k \geq 3.
\]

(77)
The justification for taking the term-by-term limit of the series in (73) follows a standard argument. Since it is a power series, it converges absolutely. Now, for any fixed $s$, the coefficients $r_k$ are continuous functions of $d$, and they are uniformly bounded in absolute value for $d > \max\{1, c\}$. Therefore, the series converges uniformly in $d$, which implies that we can take its term-by-term limit.

From (31) and (28),

$$\phi_{Z_{c,\infty}}(s) = \exp\left(-\frac{s^2}{2} + \lim_{d \to \infty} \sum_{k=3}^{\infty} z_k s^k\right)$$

$$= \exp\left(-\frac{s^2}{2} + \sum_{k=3}^{\infty} \tilde{z}_k s^k\right),$$

where the term-by-term limit can be justified as before,

$$z_k := \frac{i^k}{k!} \frac{(d-c)E[Y_1^k]}{(d-c)E[Y_1^2]^{k/2}}$$

$$= \frac{i^k}{k!} \frac{E[W_1^k]}{E[W_1^2]^{k/2}} \left(e^{(\alpha-k)/\alpha} - c_{(\alpha-k)/\alpha}\right)^{k/2},$$

and,

$$\tilde{z}_k := \frac{i^k}{k!} \frac{E[W_1^k]}{E[W_1^2]^{k/2}} \frac{\alpha}{k-\alpha} c^{1-k/2}, \quad k \geq 3.$$

**APPENDIX D**

**PROOF OF LEMMA 3**

**A. CF of $R_{c,\infty}$**

For convenience, we write $Y_1 = W_1 t(U_1)$, where $t(x) := x^{-1/\alpha}$. Then we have,

$$\phi_{Y_1}(s) = E[e^{isY_1}]$$

$$= E[e^{isW_1 t(U_1)}]$$

$$= \int_c^d \int_{\mathbb{R}} e^{iswt(u)} p(w, u) \, dw \, du$$

$$= \int_c^d \int_{\mathbb{R}} e^{iswt(u)} p(w) p(u) \, dw \, du$$

$$= \int_c^d \phi_w(st(u)) \, du,$$

where $p(w, u)$ is the joint density of the random vector $(W_1, U_1)$, and $p(w)$ and $p(u)$ are the respective marginals. Now substituting the expression for the CF of a Gaussian $W_1 \sim \mathcal{N}(\mu_W, \sigma_W^2)$,

$$\phi_{Y_1}(s) = \int_c^d \frac{\exp(i st(u)) \mu_W - \sigma_W^2 s^2 t(u)^2/2}{d-c} \, du$$

$$= \frac{1}{d-c} \int_{t(c)}^{t(d)} \frac{\exp(i st\mu_W - \sigma_W^2 s^2 t^2/2)}{dt/du} \, dt$$

$$= \frac{\alpha}{d-c} \int_{d^{-1/\alpha}}^{d^{-1/\alpha}} \exp(i st\mu_W - \sigma_W^2 s^2 t^2/2) t^{-\alpha-1} \, dt,$$

and recalling (24),

$$\log(\phi_{R_{c,d}}(s)) = c - d + \alpha \int_{-c^{-1/\alpha}}^{d^{-1/\alpha}} e^{i st\mu_W - \sigma_W^2 s^2 t^2/2} t^{-\alpha-1} \, dt - isB.$$  (78)

By simple algebra we can re-write,

$$\log(\phi_{R_{c,d}}(s)) = \alpha \int_{-c^{-1/\alpha}}^{d^{-1/\alpha}} \left(e^{i st\mu_W - \sigma_W^2 s^2 t^2/2} - 1 - is\mu_W t\right) t^{-\alpha-1} \, dt - is\mu_W \frac{\alpha}{\alpha-1} e^{\alpha-1},$$  (79)
and taking the limit $d \to \infty$,

$$
\log(\phi_{R_{(c,\infty)}}(s)) = \alpha \int_0^{c^{-1/\alpha}} \left( e^{(ist\mu_W - \sigma_W^2 s^2 t^2/2) - 1 - is\mu_W t} \right) t^{-\alpha - 1} dt - is\mu_W \frac{\alpha}{\alpha - 1} e^{\sigma_W^2 s^2 t^2/2}. 
$$

**B. CFs of $X_{(0,c)}$ and $X$**

Notice that no assumptions on the finiteness of the moments of $R_{(c,d)}$ or $Y_1$ were made above. Therefore, we can use the expression of the CF of $R_{(c,d)}$ with $c \downarrow 0$ to obtain an analogous expression for the CF of $X_{(0,c)}$. In this case, the term $iB$s appearing in (78) is included in the PSR residual $R_{(c,d)}$, see (19). Hence we have,

$$
\log(\phi_{X_{(0,c)}}(s)) = -c + \alpha \int_{c^{-1/\alpha}}^{\infty} \left( e^{(ist\mu_W - \sigma_W^2 s^2 t^2/2) - 1} \right) t^{-\alpha - 1} dt
= \alpha \int_{c^{-1/\alpha}}^{\infty} \left( e^{(ist\mu_W - \sigma_W^2 s^2 t^2/2) - 1} \right) t^{-\alpha - 1} dt. \quad (80)
$$

Finally, we obtain an integral expression for the CF of the full PSR $X$,

$$
\log(\phi_X(s)) = \log(\phi_{X_{(0,c)}}(s)) + \log(\phi_{R_{(c,\infty)}}(s))
= \alpha \int_0^{\infty} \left( e^{(ist\mu_W - \sigma_W^2 s^2 t^2/2) - 1} \right) t^{-\alpha - 1} dt. \quad (81)
$$

**APPENDIX E**

**PROOF OF LEMMA 4**

**A. CF of $Z_{(c,\infty)}$**

Starting from the expression for $\phi_{R_{(c,d)}}(s)$ in (78), we take $\mu_W = 0$ and perform the change of variables,

$$
v = \sigma_W^2 s^2 t^2/2, \quad (82)
$$

so that $t = \sqrt{2v/\sigma_W s}$, $dv/dt = \sigma_W^2 s^2 t = \sqrt{2v\sigma_W s}$, and the indefinite integral corresponding to the definite integral in (78) becomes,

$$
\int e^{(-\sigma_W^2 s^2 t^2/2) t^{-\alpha - 1}} dt = |s|^{\alpha} e^{\sigma_W^2 s^2 (\alpha + 2)/2} \int \exp(-v) v^{-\alpha - 1} dv
= |s|^{\alpha} e^{\sigma_W^2 s^2 (\alpha + 2)/2} \left[ -\frac{1}{a} v^{-a} e^{-v} - \int \frac{1}{a} v^{-a} e^{-v} dv \right]
= |s|^{\alpha} e^{\sigma_W^2 s^2 (\alpha + 2)/2} \frac{1}{a} \left( -\exp(-v) v^{-\alpha} - \gamma(1 - a, v) \right)
= \frac{t^{-\alpha}}{\alpha} \left( -\exp(-v) - v^a \gamma(1 - a, v) \right), \quad (83)
$$

where $\gamma(s, x)$ is the lower incomplete gamma function (5) and $a = \alpha/2$, as in (54).

We evaluate the indefinite integral (83) in the upper limit $t = c^{-1/\alpha}$ and $v = v_c := \sigma_W^2 s^2 c^{-2}/2$ and in the lower limit $t = d^{-1/\alpha}$ and $v = v_d := \sigma_W^2 s^2 d^{-2}/2$, respectively, as in (79); the definite integral then becomes,

$$
\frac{c}{\alpha} \left( -\exp(-v_c) - v_c^a \gamma(1 - a, v_c) \right) - \frac{d}{\alpha} \left( -\exp(-v_d) - v_d^a \gamma(1 - a, v_d) \right).
$$

Then, as $d \to \infty$, $v_d \to 0$ and $v_d^a \gamma(1 - a, v_d) \to 0$, hence the definite integral,

$$
\frac{c}{\alpha} \left( -\exp(-v_c) - v_c^a \gamma(1 - a, v_c) \right) + \frac{d}{\alpha},
$$

and substituting in (78), recalling that $B = 0$ when $\mu_W = 0$, we have,

$$
\phi_{R_{(c,\infty)}}(s) = \omega_{R_{(c,\infty)}}(v_c)
= \exp(c(1 - \exp(-v_c) - v_c^a \gamma(1 - a, v_c))). \quad (84)
$$

Notice that $v_c$ corresponds to $u$ in (54), in view of (28) and the fact that $\mathbb{E}[W_i^2] = \sigma_W^2$. Then the first part of the statement follows by recalling that, from (49), $\phi_{Z_{(c,\infty)}}(s) = \phi_{R_{(c,\infty)}}(s/s_{(c,\infty)})$. Finally, using the change of variables (54), $\phi_{R_{(c,\infty)}}(s/S_{(c,\infty)}) = \omega_{R_{(c,\infty)}}(u/S_{(c,\infty)^2}) = \psi_{R_{(c,\infty)}}(u)$. 


B. CF of $X_{(0, c)}$

To prove the second part of the lemma, we evaluate the indefinite integral (83) in the upper limit $t = \infty$ and $v = \infty$, and in the lower limit $t = c - 1/\alpha$ and $v = v_c = \sigma^2 W s^2 e^{-2/\alpha}/2$, as in (80). Recalling from (82) that $t^{-\alpha} v^\alpha = (|s| \sigma W / \sqrt{2})^\alpha$, where $\alpha = 2a$ as in (54), and also recalling the definition of the gamma function (7), the corresponding definite integral becomes,

$$-\frac{1}{\alpha} \left( \frac{|s| \sigma W}{\sqrt{2}} \right)^\alpha \Gamma(1 - a) - \frac{c}{\alpha} (-\exp(-v_c) - v_c^a \gamma(1 - a, v_c)),$$

and, substituting in (80) and recalling the definition of $v_c$, we obtain,

$$\phi_{X_{(0, c)}}(s) = \omega_{X_{(0, c)}}(v_c) = \exp(-c(1 - \exp(-v_c) - u_c^a \gamma(1 - a, v_c) + v_c^a \Gamma(1 - a)))$$

as claimed.

APPENDIX F

PROPERTIES OF GAMMA FUNCTIONS

A. Inequalities for $\gamma(s, x)$

Recall the definition of the lower incomplete gamma function in (5).

Lemma 5: [56, Theorem 4.1] For all $x > 0$ and $s \in (0, 1]$

$$\gamma(s, x) \leq \frac{x^s}{s(s + 1)}(1 + se^{-x}).$$

Lemma 6: [19, Ineq. (8.10.2)] For all $x > 0$ and $s > 0$

$$\gamma(s, x) \geq \frac{x^{s-1}}{s} (1 - e^{-x}).$$

Combining Lemmas 5 and 6 we obtain that

$$\lim_{x \to 0} \frac{\gamma(s, x)}{x^s} = \frac{1}{s},$$

and also the following:

Lemma 7: For all $x > 0$ and $s \in (0, 1]$

$$-\frac{x}{1 + s} \leq e^{-x} - s\gamma(s, x) \leq -\frac{x(1 - x)}{2}.$$

Proof: The bounds in Lemmas 5 and 6 immediately give,

$$\frac{1 - e^{-x}}{x} \leq \frac{s\gamma(s, x)}{x^s} \leq \frac{1 + se^{-x}}{1 + s},$$

and subtracting all three sides from $e^{-x}$ and simplifying,

$$\frac{e^{-x} - 1}{1 + s} \leq e^{-x} - s\gamma(s, x) \leq \frac{(x + 1)e^{-x} - 1}{x}.$$

Applying the elementary inequality $e^{-x} > 1 - x$, $x \geq 0$, to the left-hand side gives the lower bound in the statement, and similarly applying the inequality $e^{-x} \leq 1 - x + x^2/2$, $x \geq 0$, to the right-hand side gives the corresponding upper bound.
B. Inequalities for $\Gamma(s, x)$

Recall the definition of the upper incomplete gamma function in (6).

**Lemma 8:** [19, Eq. (8.6.7)] For all $s > 0, x > 0$, $\Gamma(s, x)$ admits the representation:

$$\Gamma(s, x) = x^s \int_0^\infty \exp(st - xe^t) \, dt.$$  

**Proof:** From Lemma 8,

$$\Gamma(s, x) = x^s \int_0^\infty \exp(st - xe^t + x) \, dt,$$

and since $e^t \geq t + 1$ for $t \geq 0$,

$$\Gamma(s, x) \leq \int_0^\infty \exp(-t(x - s)) \, dt = \frac{1}{x - s}.$$  

The bound in the last lemma applies when $x > s$. When $x$ may be smaller than $s$ we have a somewhat weaker bound, but this time uniformly in $x$:

**Lemma 10:** Let $\delta \in (0, 1)$ be arbitrary. For all $s > 0$ and all $x \geq \delta s$:

$$\frac{\Gamma(s, x)}{x^s e^{-x}} \leq \sqrt{\frac{2\pi}{\delta s}} \exp \left[ \frac{\delta s}{2} (\delta^{-1} - 1)^2 \right].$$

**Proof:** Starting again with (88) and noting that the integrand is decreasing in $x$, we have that, for $x \geq \delta s$,

$$\frac{\Gamma(s, x)}{x^s e^{-x}} \leq \int_0^\infty \exp \left[ s(t - \delta e^t + \delta) \right] \, dt,$$

and since $e^t \geq 1 + t + \frac{t^2}{2}$ for $t \geq 0$,

$$\frac{\Gamma(s, x)}{x^s e^{-x}} \leq \int_0^\infty \exp \left\{ -\frac{\delta s}{2} t^2 - 2t(\delta^{-1} - 1) \right\} \, dt.$$  

Completing the square in the exponent,

$$\frac{\Gamma(s, x)}{x^s e^{-x}} \leq \exp \left[ \frac{\delta s}{2} (\delta^{-1} - 1)^2 \right] \int_0^\infty \exp \left\{ -\frac{\delta s}{2} t^2 - 2t(\delta^{-1} - 1) \right\} \, dt$$

$$\leq \sqrt{\frac{2\pi}{\delta s}} \exp \left[ \frac{\delta s}{2} \left( \frac{1}{\delta} - 1 \right)^2 \right] F_Z \left( (\delta^{-1} - 1)(\delta s)^{1/2} \right),$$

where $F_Z(\cdot)$ denotes the standard normal CDF, which is, of course, no greater than 1.

C. Asymptotics and derivatives

When $x \in \mathbb{R}$, the upper incomplete gamma function (6) has the following asymptotic behaviour for $x \to \infty$,

$$\Gamma(s, x) \sim x^{s-1} e^{-x}. \quad (89)$$

This can be proved using series expansions of $\Gamma(s, x)$, see [1] or [19].

Finally, by the fundamental theorem of calculus and the definitions (5) and (6), we have,

$$\frac{d\gamma(s, x)}{dx} = -\frac{d\Gamma(s, x)}{dx} = x^{s-1} e^{-x}. \quad (90)$$

APPENDIX G

Preliminary results for the proofs of Section VII and VIII

Here we list a number of auxiliary lemmas that will be needed in the proofs of Theorems 2 and 4. The lemma below, stated without proof, is a simple calculus exercise.

**Lemma 11:** If $0 \leq x \leq C$ and $0 \leq y \leq C$, then for any (not necessarily integer) $n \geq 1$,

$$|x^n - y^n| \leq n|x - y|C^{n-1}.$$
A. The function $g(w)$

When $W_j \sim \mathcal{N}(0, \sigma^2_{W_j})$, the CF of the PSR standardized residual is given in (55). For convenience we write, $\psi_{Z(e, \infty)}(w) = f(w)^e$, with,

$$f(w) := \exp\left(1 - e^{-w} - w^a \gamma(1 - a, w)\right), \quad (91)$$

and we also define,

$$g(w) := \log f(w) = 1 - e^{-w} - w^a \gamma(1 - a, w). \quad (92)$$

First note that $g(0) = 0$ and that, from Lemma 6, we have $g(w) \leq 0$ for all $w \geq 0$ and in fact,

$$g(w) \leq -\frac{1 - e^{-w}}{\eta} \leq -\frac{w}{\eta} \left(1 - \frac{w}{2}\right), \quad (93)$$

where the second step follows from the fact that $e^{-x} \leq 1 - x + x^2/2$, for $x \geq 0$. Therefore,

$$g(w) \leq -\frac{w}{2\eta}, \quad 0 \leq w \leq 1. \quad (94)$$

For $w \geq 1$ we have the bound:

**Lemma 12:** For all $w \geq 1$,

$$g(w) \leq 1 - e^{-1} - \tilde{\gamma}(a)w^a, \quad (95)$$

where $\tilde{\gamma}(a) = \gamma(1 - a, 1)$.

**Proof:** The statement is equivalent to,

$$H(w) := \gamma(1 - a, w) - \gamma(1 - a, 1) - \frac{e^{-1} - e^{-w}}{w^a} \geq 0,$$

for all $w \geq 1$. Since $H(1) = 0$ and, using (90), the derivative,

$$H'(w) = aw^{-a-1}(e^{-1} - e^{-w}),$$

is always nonnegative, the result follows.

Differentiating,

$$g'(w) = -aw^{-a-1}\gamma(1 - a, w), \quad (96)$$

so that $g'(0+) = -1/\eta$ by (86), and,

$$0 \geq g'(w) \geq -\frac{1 + (1 - a)e^{-w}}{\eta(2 - a)} \geq -\frac{1}{\eta}, \quad (97)$$

for all $w \geq 0$, where the first inequality is obvious by (96), the second follows from Lemma 5, and the third from the fact that $e^{-w} \leq 1$ always. In particular, this implies that,

$$g(w) \geq -\frac{w}{\eta}, \quad w \geq 0. \quad (98)$$

Differentiating again,

$$g''(w) = -\frac{a}{w} \left[e^{-w} - (1 - a)w^a \gamma(1 - a, w)\right],$$

and applying Lemma 7, we have that,

$$-a \leq \frac{a(1 - w)}{2} \leq g''(w) \leq \frac{a}{2 - a}, \quad (99)$$

where the first inequality only holds for $0 \leq w \leq 3$; on the other hand, since (87) holds for all $w \geq 0$ and the function $x \mapsto (x + 1)e^{-x}$ is decreasing, for all $w \geq 1$ we have the simple lower bound,

$$g''(w) \geq -\frac{a}{w} \left(\frac{(w + 1)e^{-w} - 1}{w}\right) \geq \frac{a(1 - 2e^{-1})}{w^2}. \quad (100)$$

In particular, from (99) and (100) it follows that,

$$0 \leq g''(w) \leq \frac{a}{2 - a}, \quad \text{for all } w \geq 0. \quad (101)$$
This implies that the function \( g(w) \) is convex, and a tighter upper bound than (94) for \( w \in [0, 1] \) is the linear interpolation at the extremes,
\[
g(w) \leq \overline{g}w, \quad 0 \leq w \leq 1,
\]
where \( \overline{g} = g(1) \), as in (60). More generally, the linear interpolating upper bound for \( g(w) \) on \([0, \bar{w}]\) is
\[
g(w) \leq \frac{g(\bar{w})}{\bar{w}}w, \quad 0 \leq w \leq \bar{w}.
\]

**B. The function \( q(u) \)**

When \( W_j \sim \mathcal{N}(0, \sigma^2_{W}) \), the CF of the truncated PSR is given in (56). For convenience we write \( \omega_{X_{(0,c)}}(u) = r(u)^c \), with,
\[
r(u) := \exp(-(1 - e^{-u} + u^a \Gamma(1 - a, u))),
\]
and we also define:
\[
q(u) := \log r(u) = -(1 - e^{-u} + u^a \Gamma(1 - a, u)).
\]
It is clear that \( q(0) = 0 \) and \( q(u) < 0 \) for \( u > 0 \). In fact, using (90),
\[
q'(u) = -au^{a-1} \Gamma(1 - a, u) \leq 0,
\]
hence \( q(u) \) is monotonically decreasing. Moreover, by (89), \( q(u) \) is asymptotic to \(-1\) as \( u \to \infty \).

Since \( a - 1 < 0 \) and \( \Gamma(1 - a, u) \) is decreasing in \( u \) we have that \( q'(u) \) is monotonically increasing towards zero at \( u \to \infty \), since \( \Gamma(1 - a, u) \) tends to \( \Gamma(1 - a) \) as \( u \to 0 \). Note also that \( q'(0) \) diverges to \(-\infty\), which prevents a Taylor expansion around zero.

Differentiating once again,
\[
q''(u) = a((1 - a)u^{a-2} \Gamma(1 - a, u) + u^{-1}e^{-u}) \geq 0,
\]
so \( q'' \) is also monotonically decreasing, it decreases to zero at \( u \to \infty \), and it diverges to \(+\infty\) at \( u = 0 \).

Therefore, \( \log(\omega_{X_{(0,c)}}(u)) = cq(u) \) is convex, which means it can easily be bounded above by line segments. Indeed, the following construction will be useful in the proof of Theorem 5: We will employ a piecewise linear interpolating bound with \( N \) segments for \( u \in [0, 1] \), and a constant bound for \( u > 1 \). When \( N = 1 \), the bound is simply,
\[
\log(\omega_{X_{(0,c)}}(u)) = cq(u) \leq L^1(u) := \begin{cases} -cu((1 - \exp(-1)) + \Gamma(1 - a, 1)), & u \in [0, 1], \\ -c((1 - \exp(-1)) + \Gamma(1 - a, 1)), & u > 1, \\ m_0u, & u \in [0, 1], \\ k_{(1,\infty)}, & u > 1, \end{cases}
\]
where,
\[
m_0 = k_{(1,\infty)} := -c((1 - \exp(-1)) + \Gamma(1 - a, 1)) = \log(\omega_{X_{(0,c)}}(1)) = \log(cq(1)) < 0.
\]

Bounding with more than one line segment in the domain \([0, 1] \) turns out to be important for tightening the bound on \( \Delta(X, X) \) and for capturing its dependence on \( \alpha \), as observed via numerical integration results; cf. Section VIII and Appendix K. For \( N \geq 1 \), we select \( N \) points \( u_i \) in \([0, 1]\),
\[
0 =: u_0 < u_1 < \ldots < u_N := 1,
\]
and the respective values of \( \log \omega_{X_{(0,c)}} \),
\[
f_0 := 0 > f_1 := \log(\omega_{X_{(0,c)}}(u_1)) > \ldots > f_N := \log(\omega_{X_{(0,c)}}(u_N)).
\]
The equation of the \( i \)-th line segment, for \( i = 0, \ldots, N - 1 \) is \( y = (m_iu + q_i)\mathbb{I}_{A_i}(u) \), with \( A_i := [u_i, u_{i+1}] \) and,
\[
m_i := \frac{f_{i+1} - f_i}{u_{i+1} - u_i}, \quad q_i := -m_iu_i + f_i.
\]
The general upper bound on \( \log(\omega_{X_{(0,c)}}(u)) \) then becomes,
\[
\log(\omega_{X_{(0,c)}}(u)) \leq L^N(u) := \begin{cases} 
\sum_{i=0}^{N-1}(m_iu + q_i)\mathbb{I}_{A_i}(u), & u \in [0, 1], \\
\frac{\sum_{i=0}^{N-1}(m_iu + q_i)\mathbb{I}_{A_i}(u)}{k_{(1,\infty)}}, & u > 1,
\end{cases}
\]
with \( k_{(1,\infty)} \) as above.
APPENDIX H
PROOF OF THEOREM 2

Step I. Let \( W_j \sim N(0, \sigma^2_j) \) so that the CF of the PSR standardized residual \( Z_{(c, \infty)} \) is given in (55), and write \( \psi_{Z_{(c, \infty)}}(w) = f(w)^c \), with \( f(w) \) defined as in (91). We apply the smoothing lemma (42) to \( \Delta(Z_{(c, \infty)}, Z) \) as in (44). Given that the standard Gaussian PDF is uniformly bounded by \( m := 1/\sqrt{2\pi} < 2/5 \), for any \( \Theta > 0 \) equation (42) gives,

\[
\pi \Delta(Z_{(c, \infty)}, Z) \leq \int_{-\Theta}^{\Theta} |\phi_{Z_{(c, \infty)}}(s) - \phi_Z(s)| \frac{1}{|s|} \, ds + \frac{9.6}{\Theta},
\]

with \( \phi_Z(s) \) as in (37). Letting \( \Theta \to \infty \) and changing variables as in (54) we have,

\[
\pi \Delta(Z_{(c, \infty)}, Z) \leq \int_0^\infty |f(w)^c - \phi_Z(\sqrt{2w/\eta})^c| \frac{1}{w} \, dw
\]

\[
= \pi I(Z_{(c, \infty)}, Z).
\]

Step II. We apply Lemma 11 to the integrand in (106), with \( x = f(w) \), \( y = \phi_Z(\sqrt{2w/\eta}) = \exp(-w/\eta) \) by (57), and \( n = c \). Let \( g(w) = \log(f(w)) \) as in (92), and \( \tilde{\gamma}(a) \) and \( \tilde{\gamma} \) as in (59) and (60), respectively, and define,

\[
h(w) := \begin{cases} -\tilde{\gamma}w, & w \in [0, 1], \\ e^{-1} - 1 + \tilde{\gamma}(a)w^a, & w > 1. \end{cases}
\]

From (102) and (95) it follows that \( x \leq \exp(-h(w)) \) for all \( w \geq 0 \). Since, by (98), we have \( y \leq x \), for all \( w \geq 0 \), it follows that \( y \leq \exp(-h(w)) \), for all \( w \geq 0 \). Therefore, we can take, \( C = \exp(-h(w)) \) in Lemma 11, and substituting the resulting bound in (106) we get,

\[
\pi \Delta(Z_{(c, \infty)}, Z) \leq \int_0^\infty \frac{e}{w} \left| f(w) - \phi_Z \left( \sqrt{\frac{2w}{\eta}} \right) \right| e^{-(c-1)h(w)} \, dw.
\]

Step III. In order to bound the absolute difference in the integrand in (108), we write \( f \) as a quadratic Taylor expansion. Noting that \( f' = g' \exp(g) \) and \( f'' = (g'' + g^2) \exp(g) \), where \( g \) is defined in (92), and recalling that \( g(0) = 0 \) and \( g'(0) = -1/\eta \), we have,

\[
\left| f(w) - 1 + \frac{w}{\eta} \right| \leq \frac{w^2}{2} \sup_{v \geq 0} \left( |g''(v)| + g'(v)^2 \right) e^{g(v)}
\]

\[
\leq \frac{w^2}{2} \left[ \frac{a}{2-a} + \frac{1}{\eta^2} \right],
\]

where in the second inequality we used (97), (101), and the fact that \( g(w) \leq 0 \) for all \( w \geq 0 \). From the standard quadratic expansion for the exponential function we similarly have,

\[
\left| \exp \left( -\frac{w}{\eta} \right) - 1 + \frac{w}{\eta} \right| \leq \frac{w^2}{2\eta^2}.
\]

Combining the last two bounds,

\[
\left| f(w) - \phi_Z \left( \sqrt{\frac{2w}{\eta}} \right) \right| \leq \frac{w^2}{2} \left[ \frac{a}{2-a} + \frac{2}{\eta^2} \right].
\]

Step IV. Finally, substituting (109) in (108),

\[
\Delta(Z_{(c, \infty)}, Z) \leq \frac{c}{\pi} \left[ \frac{a}{2(2-a)} + \frac{1}{\eta^2} \right] \int_0^\infty w \exp \left( -\left( c-1 \right) h(w) \right) \, dw
\]

\[
= cK(a)[I^Z(c) + J^Z(c)],
\]

where \( K(c) \) is defined in (61), and where we first integrate over \([0, 1]\),

\[
I^Z(c) := \int_0^1 w \exp \left( (w-1)\tilde{\gamma} \right) \, dw
\]

\[
= \frac{1}{(c-1)^2\tilde{\gamma}^2} + \frac{1}{(c-1)\tilde{\gamma}} \left( 1 - \frac{1}{(c-1)\tilde{\gamma}} \right) \exp \left( (c-1)\tilde{\gamma} \right),
\]

(111)
and then over \([1, \infty)\),

\[
J^Z(c) := e^{(1-e^{-1})(c-1)} \int_1^\infty w \exp \left\{- (c-1)\bar{\gamma}(a) w^a\right\} \,dw \\
= \frac{e^{(1-e^{-1})(c-1)}}{a[(c-1)\bar{\gamma}(a)]^{2/a}} \int_{(c-1)\bar{\gamma}(a)}^\infty u^{2/a-1} e^{-u} \,du \\
= e^{(1-e^{-1})(c-1)} \frac{\Gamma(2/a, (c-1)\bar{\gamma}(a))}{a[(c-1)\bar{\gamma}(a)]^{2/a}},
\]

(112)

where \(\Gamma(s, x)\) is the upper incomplete gamma function defined in (6). Combining (110), (111) and (112) gives the claimed result, upon re-writing,

\[
B_1(c, \alpha) = \left( cK(a)(I^Z(c) + J^Z(c)) \right) \\
= \left( \frac{cK(a)}{c-1} ((c-1)I^Z(c) + (c-1)J^Z(c)) \right),
\]

(113)

**APPENDIX I**

**PROOF OF COROLLARY 3**

Clearly, it suffices to prove the first asymptotic assertion of the corollary. To that end, we examine each of the two terms in the expression (113) for \(B_1(c, \alpha)\). First we note that,

\[
(c-1)I^Z(c) = \frac{1}{(c-1)g^2} + \left( \frac{1}{g} - \frac{1}{(c-1)g} \right) \exp\left( (c-1)\bar{g} \right)
\]

since \(\bar{g} < 0\). And also,

\[
(c-1)J^Z(c) = \frac{(c-1) \exp\left\{ (c-1)(1-e^{-1}) \right\}}{a[(c-1)\bar{\gamma}(a)]^{2/a}} \times [ (c-1)\bar{\gamma}(a) ]^{2/a-1} \exp(- (c-1)\bar{\gamma}(a))
\]

\[
= \frac{1}{a\bar{\gamma}(a)} \exp\left\{ (c-1)(1-e^{-1}-\bar{\gamma}(a)) \right\},
\]

where we used the asymptotic property (89). Observe that the exponent in the last expression is negative: Using (85) and the fact that \(a \in (0, 1)\), we have,

\[
1-e^{-1}-\bar{\gamma}(a) < (1-e^{-1}) \left( 1 - \frac{1}{1-a} \right) < 0.
\]

Therefore, combining these two estimates with (113), we indeed have,

\[
B_1(c, \alpha) \sim K(a) \frac{1}{(c-1)g^2(1)}.
\]

**APPENDIX J**

**PROOF OF THEOREM 4**

As in the proof of Theorem 2, we start from (105) and perform the change of variables (54) to obtain that, for any \(\Theta > 0\),

\[
\pi \Delta(Z_{(c, \infty), Z}) \leq \int_0^{\eta^2 \Theta^2 / 2c} \left| f(w) e^{-\phi_Z \left( \sqrt{2w/\eta} \right)} \right| \frac{1}{w} \,dw + \frac{9.6}{\Theta}.
\]

(114)

But here, instead of letting \(\Theta \to \infty\), we choose \(\Theta = \sqrt{2(2-\delta)c/\eta}\), and we apply Lemma 11 to the integrand in (114), with \(x = f(w)\) and \(y = \phi_Z(\sqrt{2w/\eta}) = \exp(-w/\eta)\). From (98) \(y \leq x\), for all \(w \geq 0\), and using (103) with \(\bar{w} = 2-\delta\) we have that \(x \leq \exp\left( (2-\delta)\bar{w} \right)\). We can then take \(C = \exp\left( (2-\delta)\bar{w} \right)\) in Lemma 11 applied to (114), to obtain,

\[
\pi \Delta(Z_{(c, \infty), Z}) \leq \int_0^{2-\delta} \frac{e^c}{w} \left| f(w) - \phi_Z \left( \sqrt{2w/\eta} \right) \right| \exp\left( (c-1)g(2-\delta)w/2-\delta \right) \,dw.
\]

(115)
Recalling the earlier expansion (109), substituting in (115), and integrating, yields,
\[
\pi \Delta(Z_{c,\infty}, Z) = \frac{9.6\sqrt{7}}{\sqrt{2(2-\delta)}}c \leq c \left[ \frac{a}{2(2-a)} + \frac{1}{\eta^2} \right] \int_0^{2-\delta} w \exp \left( \frac{(c-1)g(2-\delta)w}{2-\delta} \right) \, dw
\]
\[
= \frac{1}{c} \left[ \frac{a}{2(2-a)} + \frac{1}{\eta^2} \right] \left( \frac{c(2-\delta)}{(c-1)g(2-\delta)} \right)^2 \times \\
\times \left\{ 1 - \left[ 1 - g(2-\delta)(c-1) \right] \exp \left( g(2-\delta)(c-1) \right) \right\},
\]
as claimed.

**APPENDIX K**

**PROOF OF THEOREM 5**

Starting from the smoothing lemma (43) stating that \(\Delta(X, \hat{X}) \leq \tilde{I}(X, \hat{X})\), we will proceed to bound \(\tilde{I}(X, \hat{X})\). Using the expressions (47) and (48) and performing the change of variables (54), we can express,
\[
\tilde{I}(X, \hat{X}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\phi_X(u)\phi_{\hat{X}}(s)|}{|s|} \, ds \\
= \frac{1}{\pi} \int_0^\infty \frac{\omega_X(u)}{u} \left| \omega_{\hat{X}}(u) - \frac{\omega_{R(c,\infty)}}{u} \right| \, du \\
= \frac{1}{\pi} \int_0^\infty \frac{\omega_X(u)}{u} \left| \psi_{Z_{c,\infty}}(u) - \psi_Z(u) \right| \, du,
\]
where \(\omega_X(u), \omega_{\hat{X}}(u)\) and \(\omega_{R(c,\infty)}(u)\) are defined in (56), (84) and (58) respectively; the last equality follows again by (54), with \(\psi_{Z_{c,\infty}}(u)\) as in (55) and \(\psi_Z(u)\) as in (57). The proof is in the following steps.

The term \(|\psi_{Z_{c,\infty}}(u) - \psi_Z(u)|\) in (116) has already been bounded over the intervals \([0, 1]\) and \((1, \infty)\) in the proof of Theorem 2; see Appendix H. Combining equations (108) and (109), and recalling the definition of \(K(a)\) in (61), we obtain,
\[
|\psi_{Z_{c,\infty}}(u) - \psi_Z(u)| \leq c u^2 \left( \frac{a}{2(2-a)} + \frac{1}{\eta^2} \right) \exp(-c-1)h(u) \\
= c u^2 \pi K(a) \exp(-c-1)h(u),
\]
(117)

with,
\[
h(u) = \begin{cases} 
-\bar{g}u, & u \in [0, 1] \\
\bar{u} - 1 + u^a\tilde{\gamma}(a), & u > 1.
\end{cases}
\]

For the term \(|\omega_X(u)|\) in (116), we recall the bounds obtained in Appendix G-B based on the fact that it is log-convex. From (104) and (117), the numerator of the integrand in (116) is bounded as,
\[
|\omega_X(u)| \left| \psi_{Z_{c,\infty}}(u) - \psi_Z(u) \right| \leq c u^2 \pi K(a) \exp(h(u)),
\]
with,
\[
h(u) = \begin{cases} 
\sum_{i=0}^{N-1} \left( (m_i + (c-1)\bar{g})u + q_i \right) \mathbb{I}_{A_i}(u), & u \in [0, 1] \\
\eta_{(1,\infty)} - (c-1)(e^{-1} - 1 + u^a\tilde{\gamma}(a)), & u > 1
\end{cases}
\]
\[
= \begin{cases} 
\sum_{i=0}^{N-1} \left( \bar{m}_i u + q_i \right) \mathbb{I}_{A_i}(u), & u \in [0, 1] \\
\tilde{\mathbb{H}}_{(1,\infty)} - \tilde{\mathbb{L}}_{(1,\infty)} u^a, & u > 1
\end{cases}
\]
where \(m_i\) and \(q_i\) are as in (62); \(\mathbb{H}_{(1,\infty)}\) is as in (63), and,
\[
\bar{m}_i := m_i + (c-1)\bar{g}, \\
\tilde{\mathbb{H}}_{(1,\infty)} = \mathbb{H}_{(1,\infty)} - (c-1)(e^{-1} - 1), \\
\tilde{\mathbb{L}}_{(1,\infty)} = (c-1)\tilde{\gamma}(a).
\]

Finally, substituting into the integral (116)
\[
\tilde{I}(X, \hat{X}) = cK(a) \int_0^\infty u \exp(-h(u)) \, du \\
= cK(a)(\hat{I}^X_X(c) + J_X(c)),
\]
(118)
where $I_N^X(c)$ and $J^X(c)$ denote the integrals over $(0, 1)$ and $(1, \infty)$, respectively. Computing the integral $I_N^X(c)$,

$$I_N^X(c) = \sum_{i=0}^{N-1} \int_{A_i} u \exp(\tilde{m}_i u + q_i) \, du$$

$$= \sum_{i=0}^{N-1} \exp(q_i) \int_{A_i} u \exp(\tilde{m}_i u) \, du$$

$$= \sum_{i=0}^{N-1} \exp(q_i) \left[ \exp(\tilde{m}_i u) \left( u - \frac{1}{\tilde{m}_i} \right) \right]_{u_i}^{u_i+1}$$

$$= \sum_{i=0}^{N-1} \exp(q_i) \left[ e^{\tilde{m}_i u_i+1} \left( u_i+1 - \frac{1}{\tilde{m}_i} \right) - e^{\tilde{m}_i u_i} \left( u_i - \frac{1}{\tilde{m}_i} \right) \right].$$

Observe that, when $N = 1$, $i = 0$, $q_0 = 0$, $u_0 = 0$, $u_1 = 1$, $\tilde{m}_0 = m_0 = (c - 1)\bar{g} = \log(\omega_{X(0,c)}(1)) + (c - 1)\bar{g}$, and the last equation becomes,

$$I_1^X(c) = \frac{1}{m_0} \left[ e^{\tilde{m}_0} \left( 1 - \frac{1}{\tilde{m}_0} \right) + \frac{1}{\tilde{m}_0} \right]$$

$$= \frac{1}{m_0} \left[ e^{\tilde{m}_0} + \frac{1}{\tilde{m}_0} \left( 1 - e^{\tilde{m}_0} \right) \right].$$

Similarly for the integral $J^X(c)$,

$$J^X(c) = \int_1^\infty u \exp \left( \tilde{k}(1, \infty) - \tilde{l}(1, \infty) u^a \right) \, du$$

$$= \exp \left( \tilde{k}(1, \infty) \right) \int_1^\infty u \exp \left( - \tilde{l}(1, \infty) u^a \right) \, du$$

$$= \frac{\exp \left( \tilde{k}(1, \infty) \right)}{a(\tilde{l}(1, \infty))^{2/a}} \int_1^\infty t^{2/a-1} \exp(-t) \, dt$$

$$= \frac{\exp \left( \tilde{k}(1, \infty) \right)}{a(\tilde{l}(1, \infty))^{2/a}} \Gamma \left( 2/a, \tilde{l}(1, \infty) \right).$$

Substituting these in (118) yields exactly the claimed bound.

**Remark 3:** Observe that the bounds $I_N^X(c)$ and $J^X(c)$ are smaller than $I^Z(c)$ and $J^Z(c)$, because the latter correspond to the former when $\omega_{X(0,c)}(u) \equiv 1$; Figure 11 illustrates their difference. The asymptotic rates of both $J^Z(c)$ and $J^X(c)$ depend on $\alpha$, and that larger values of $\alpha$ give smaller bounds. On the other hand, the asymptotic rates of both $I^Z(c)$ and $I_N^X(c)$ are independent on $\alpha$, but their nonasymptotic behaviour does depend on $\alpha$ for a wide range of values of $c$. Moreover, for $N > 1$, lower values of $\alpha$ now give lower values of the bound $I_N^X(c)$, whereas for $N = 1$, the dependence of $I_N^X(c)$ on $\alpha$ is the same as that of $I^Z(c)$. But even in this case, the dependence of the combined bound $B_1(c, \alpha, N)$ on $\alpha$ is the opposite to that of $B_1(c, \alpha)$, as illustrated in Figure 12. This can be justified by the fact that the growth of $I_N^X$ in $\alpha$ is much slower than that of $I^Z$, so while the coefficient $cK(a)$ rectifies the former, it fails to do so with the latter.

![Comparison of the terms $I^Z(c)$ and $I^X_N(c)$, and $J^Z(c)$ and $J^X(c)$ for $N = 1, 10$, and $\alpha = 0.1, 0.2, \ldots, 1.9$, plotted against $c \in [1, 10^9]$.](image-url)
Fig. 12. Left: The function $cK(a)$, with $a = \alpha/2$, plotted against $1 \leq c \leq 10^3$. Centre and right: comparison of $B_4(c, \alpha) = cK(a)(I^Z(c) + J^Z(c))$ and $B_5(c, \alpha, N) = cK(a)(I^X_N(c) + J^X_N(c))$, for $N = 1, 10$, and $\alpha = 0.1, 0.2, \ldots, 1.9$, plotted against $1 \leq c \leq 10^3$. [The red horizontal line at 1 shows the maximum possible value of the Kolmogorov distance.]

APPENDIX L

PROOF OF PROPOSITION 6

We adapt the proof strategy used for a similar result in [40] on the convergence of the truncated PSR with $W_1 \sim \mathcal{N}(0, 1)$.

The difference here is that the number of terms in the truncated PSR is random, and that we allow the variance $\sigma^2_W$ to not necessarily be equal to 1.

For $x > 0$,

$$\Delta(X, X_{(0,c)}) = \sup_{x \in \mathbb{R}} \left| F_X(x) - F_{X_{(0,c)}}(x) \right|$$

$$= \sup_{x \in \mathbb{R}} \left| E[1_{X \leq x}] - E[1_{X_{(0,c)} \leq x}] \right| .$$

Conditioning first on the number $N_{(0,c)}$ of terms in the truncated PSR and the Poisson event times $\{\Gamma_j\}$, we can expand,

$$E[1_{X \leq x}] = E \left[ 1_{\left( \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} W_j \leq x \right)} \right]$$

$$= E \left[ 1_{\left( \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} W_j \leq x \right)} \left| N_{(0,c)}, \{\Gamma_j\}_{j=1}^{\infty} \right. \right]$$

$$= E \left[ P \left( \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} W_j \leq x \left| N_{(0,c)}, \{\Gamma_j\}_{j=1}^{\infty} \right. \right. \right]$$

$$= E \left[ F_Z \left( \frac{x}{\sigma_W S_{N_{(0,c)}}} \right) \right] ,$$

where $F_Z(\cdot)$ is again the CDF of the standard normal distribution and $S^2$ is defined in (14). Similarly we can compute,

$$E[1_{X_{(0,c)} \leq x}] = E \left[ F_Z \left( \frac{x}{\sigma_W S_{N_{(0,c)}}} \right) \right] ,$$

where,

$$S^2_{N_{(0,c)}} := \sum_{j=1}^{N_{(0,c)}} \Gamma_j^{-2/\alpha} ,$$

with the convention that $S^2_{N_{(0,c)}} = 0$ if $N_{(0,c)} = 0$, and $F_Z(-\infty) = 0, F_Z(\infty) = 1$. Then, given that $\Gamma_j > 0$ for any $j \geq 1$ with probability 1, it follows that $S^2_{N_{(0,c)}} < S^2$, and, for $x > 0$,

$$\Delta(X, X_{(0,c)}) = \sup_{x \in \mathbb{R}} E \left[ F_Z \left( \frac{x}{\sigma_W S_{N_{(0,c)}}} \right) - F_Z \left( \frac{x}{\sigma_W S} \right) \right] .$$
The argument of the expectation above can be bounded as,

\[ F_Z \left( \frac{x}{\sigma_W S_{N(0,c)}} \right) - F_Z \left( \frac{x}{\sigma_W S} \right) = \frac{1}{\sqrt{2\pi}} \int_{x/(\sigma_W S)}^{\infty} \exp \left( -u^2/2 \right) du \]

\[ \leq \frac{x}{\sigma_W \sqrt{2\pi}} \left( \frac{1}{S_{N(0,c)}} - \frac{1}{S} \right) \exp \left( -\frac{x^2}{2\sigma_W^2 S^2} \right) \]

\[ \leq \frac{1}{\sigma_W \sqrt{2\pi}} \left( \frac{S - S_{N(0,c)}}{S_{N(0,c)} S} \right) S\sigma_W \exp(-1/2) \]

\[ = \exp(-1/2) \left( \frac{S^2 - S_{N(0,c)} S}{S_{N(0,c)} S} \right) \]

\[ \leq \exp(-1/2) \left( \frac{S^2 - S_{N(0,c)}^2}{S_{N(0,c)}^2} \right), \]

where in the first inequality we used the constant bound of the integrand on the integration domain; in the second inequality we used the fact that the mode of the Rayleigh distribution is achieved for \( x = \sigma_W S \); and in the third inequality we used (twice) the fact that \( S_{N(0,c)}^2 < S^2 \). Notice that this upper bound does not depend on \( \sigma_W \). Finally, taking the expectation,

\[ \Delta(X, X_{(0,c)}) \leq \frac{\exp(-1/2)}{\sqrt{2\pi}} \mathbb{E} \left[ S_{N(0,c)}^{-2} \left( S^2 - S_{N(0,c)}^2 \right) \right] \]

\[ \leq \frac{\exp(-1/2)}{\sqrt{2\pi}} \left( \mathbb{E} \left[ S_{N(0,c)}^{-4} \right] \right)^{1/2} \left( \mathbb{E} \left[ (S^2 - S_{N(0,c)}^2)^2 \right] \right)^{1/2} \]

\[ \leq \frac{\exp(-1/2)}{\sqrt{2\pi}} \left( \mathbb{E} \left[ \frac{1}{\Gamma^{4/\alpha}} \right] \right)^{1/2} \left( \mathbb{E} \left[ (S^2 - S_{N(0,c)}^2)^2 \right] \right)^{1/2} \]

\[ = \frac{\exp(-1/2)}{\sqrt{2\pi}} \left( \Gamma \left( \alpha + 4 \right) \right)^{1/2} \times \left( \frac{\alpha}{4 - \alpha} e^{\alpha/\alpha} + \left( \frac{\alpha}{2 - \alpha} e^{\alpha/\alpha} \right)^2 \right)^{1/2}, \]

where in the second inequality we used the Cauchy-Schwartz inequality; and in the third inequality we used the fact that \( S_{N(0,c)}^2 > \Gamma_1^{-2/\alpha} \), where \( \Gamma_1 \) is the smallest of the \( \Gamma_j \) variables, hence exponentially distributed. Thus,

\[ \mathbb{E} \left[ \Gamma^{4/\alpha} \right] = \int_0^\infty u^{(4/\alpha + 1)-1} e^{-u} du = \Gamma \left( \frac{4}{\alpha} + 1 \right), \]

with \( \Gamma(\cdot) \) the gamma function (7). On the other hand, \( S^2 - S_{N(0,c)}^2 = \sum_{j=N(0,c)+1}^{\infty} \Gamma_j^{-2/\alpha} = \lim_{d \to \infty} \sum_{j=1}^{\infty} \mathbb{E} \left[ \Gamma_j \right] \Gamma_j^{-2/\alpha} \), and we know that such limit exists through the PSR, being the second moment of the PSR residual with deterministic \( W_j = 1 \). Hence,

\[ \mathbb{E} \left[ (S^2 - S_{N(0,c)}^2)^2 \right] = \lim_{d \to \infty} \mathbb{E} \left[ \sum_{j=1}^{\infty} \mathbb{E} \left[ \Gamma_j^{-2/\alpha} \right] \sum_{\ell=1}^{N(c,d)} \Gamma_{i}^{-2/\alpha} \right] \]

\[ = \lim_{d \to \infty} \mathbb{E} \left[ \sum_{j=1}^{\infty} \mathbb{E} \left[ \Gamma_j^{-2/\alpha} \Gamma_{i}^{-2/\alpha} \right] N(c,d) \right] \]

\[ = \lim_{d \to \infty} \mathbb{E} \left[ \sum_{j=1}^{\infty} U_j^{-2/\alpha} \sum_{i=1}^{N(c,d)} U_i^{-2/\alpha} N(c,d) \right] \]

\[ = \lim_{d \to \infty} \mathbb{E} \left[ N(c,d) \mathbb{E}[U_1^{-4/\alpha}] + \mathbb{E}[N(c,d)^2] - N(c,d)(\mathbb{E}[U_1^{-2/\alpha}])^2 \right] \]

\[ = \frac{\alpha}{4 - \alpha} e^{\alpha/\alpha} + \left( \frac{\alpha}{2 - \alpha} e^{\alpha/\alpha} \right)^2, \]

where the \( \{U_j\} \) are i.i.d. uniformly distributed RVs on \( (c, d) \), as in (22).
APPENDIX M
PROOF OF PROPOSITION 7

Recall that $\bar{I}(X, \hat{X})$ is given by (116). For $\bar{I}(X, X_{(0,c)})$, using (47) and (48), and performing the change of variables (54), we similarly have,

$$\bar{I}(X, X_{(0,c)}) = \frac{1}{\pi} \int_{0}^{\infty} \left| \frac{\omega_{X_{(0,c)}}(u)}{u} \right| \psi_{Z(c,\infty)}(u) - 1 \right| du. \quad (119)$$

We proceed by comparing the integrands in (116) and (119). Write $\tilde{c}(a) = c(\alpha)$, with $a = \alpha/2$. It suffices to show that, for all $c > \tilde{c}(a)$ and all $u > 0$,

$$|\psi_{Z(c,\infty)}(u) - \psi_Z(u)| < |\psi_{Z(c,\infty)}(u) - 1|, \quad (120)$$

where $\log(\psi_Z(u)) = -cu/\eta$, and $\psi_{Z(c,\infty)}$, defined in (55), satisfies,

$$\log(\psi_{Z(c,\infty)}(u)) = c(1 - \exp(-u) - u^a \gamma(1 - a, u)) = cg(u),$$

with $g$ defined in (92). Using the fact that $g(u) < 0$, see (93), we have $|\omega_{R_{c,\infty}}(u) - 1| = 1 - \omega_{R_{c,\infty}}(u)$. Furthermore, using (98), we have $|\omega_{R_{c,\infty}}(u) - \omega_{\hat{R}}(u)| = \omega_{R_{c,\infty}}(u) - \omega_{\hat{R}}(u)$, and (120) becomes,

$$v(u) := 1 - 2 \exp(cg(u)) + \exp(-cu/\eta) > 0.$$ 

Using (94), we have that, for $u \in (0,1]$,

$$v(u) \geq (1 - 2 \exp(-cu/2\eta) + \exp(-cu/\eta)) = (1 - \exp(-cu/2\eta))^2 > 0,$$

hence (120) holds for any $c > 0$ when $u \leq 1$. We then consider the case $u > 1$. By the monotonicity of $g$, see (97), we have that $g(u) < g(1)$ for $u > 1$, leading to,

$$v(u) > 1 - 2 \exp(cg(1)) + \exp(-uc/\eta) := z(u), \quad u > 1,$$

where $z(u)$ is a lower bound on $v(u)$ for $u > 1$. We know that $z(1) = v(1) > 0$ as shown above, and also,

$$z'(u) = -c/\eta \exp(-uc/\eta) < 0,$$

implying that the lower bound $z(u)$ on $v(u)$ is decaying. Furthermore, $\lim_{u \to \infty} z(u) = 1 - 2 \exp(cg(1))$, so that, for all $u \geq 1$,

$$v(u) \geq 1 - 2 \exp(cg(1)),$$

and the right-hand side above is itself positive as long as $c > -\log(2)/g(1) = \tilde{c}(a)$, as required.
