A NOTE ON LOG-TYPE GCD SUMS AND DERIVATIVES OF THE
RIEMANN ZETA FUNCTION

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Abstract. In [Yan22a], we defined so-called “log-type” GCD sums and proved the lower bounds
\( \Gamma_1^{(\ell)}(N) \gg_{\ell} (\log \log N)^{2+2\ell} \). We will establish the upper bounds \( \Gamma_1^{(\ell)}(N) \ll_{\ell} (\log \log N)^{2+2\ell} \) in this
note, which generalizes Gál’s theorem on GCD sums (corresponding to the case \( \ell = 0 \)). This result
will be proved by two different methods. The first method is unconditional. We establish sharp
upper bounds for spectral norms along \( \alpha \)-lines when \( \alpha \) tends to 1 with certain fast rates. As a
corollary, we obtain upper bounds for log-type GCD sums. The second method is conditional. We
prove that lower bounds for log-type GCD sums \( \Gamma_1^{(\ell)}(N) \) can produce lower bounds for large values
of derivatives of the Riemann zeta function on the 1-line. So from conditional upper bound for
\( [\zeta^{(\ell)}(1+it)] \), we obtain upper bounds for log-type GCD sums.

1. Introduction

The main goal of this note is to establish the following result on log-type GCD sums.

Theorem 1.1. Fix \( \ell \in [0, \infty) \). For \( N \geq 100 \), we have\(^1\)
\[
(1) \quad N (\log \log N)^{2+2\ell} \ll_{\ell} \sup_{|\alpha| = N} \sum_{m,n \in \mathscr{M}} \frac{(m,n)}{[m,n]} \log^\ell \left( \frac{m}{(m,n)} \right) \log^\ell \left( \frac{n}{(m,n)} \right) \ll_{\ell} N (\log \log N)^{2+2\ell},
\]
where the supremum is taken over all subsets \( \mathscr{M} \subset \mathbb{N} \) with size \( N \).

Theorem 1.1 generalizes Gál’s theorem [Gál49] on GCD sums (see (6)), which corresponding to
the case \( \ell = 0 \) in Theorem 1.1. Thus we only need to prove the theorem when \( \ell \in (0, \infty) \).

Let \( \ell \in [0, \infty) \) and \( \sigma \in \mathbb{R} \) be given, define the normalized log-type GCD sums \( \Gamma_{\sigma}^{(\ell)}(N) \) as we did in [Yan22a]:
\[
\Gamma_{\sigma}^{(\ell)}(N) := \sup_{|\alpha| = N} \frac{1}{N} \sum_{m,n \in \mathscr{M}} \frac{\sigma}{[m,n]} \log^\ell \left( \frac{m}{(m,n)} \right) \log^\ell \left( \frac{n}{(m,n)} \right).
\]

In [Yan22a], we proved that \( \Gamma_1^{(\ell)}(N) \gg_{\ell} (\log \log N)^{2+2\ell} \), for \( \forall \ell \in (0, \infty) \). In this note, we will
establish the corresponding upper bounds \( \Gamma_1^{(\ell)}(N) \ll_{\ell} (\log \log N)^{2+2\ell} \) by two different methods.
One method is unconditional and for \( \forall \ell \in (0, \infty) \), while another method is conditional and only
for \( \forall \ell \in \mathbb{N} \).

We have the following new result for spectral norms, which is a key ingredient for the unconditional
method.

Theorem 1.2. Let \( A \in (0, \infty) \) be fixed. Let \( c = (c(1), c(2), \ldots, c(n), \ldots,) \in \mathbb{C}^\mathbb{N} \) and let \( \mathscr{M} \subset \mathbb{N} \). Then,
for \( \alpha = \alpha(N) = 1 - \frac{A}{\log(\log N) \log(N)} \), we have
\[
\sup_{|\alpha| = N} \sum_{m,n \in \mathscr{M}} \frac{(m,n)^{\alpha c} c(m)}{[m,n]^{\alpha}} \leq \left( \frac{\exp(2\gamma + 2e^A - 2)}{\zeta(2)} + o(1) \right) \cdot (\log \log N)^2, \quad \text{as } N \to \infty.
\]

\(^1\)Here \( (m,n) \) denotes the greatest common divisor of \( m \) and \( n \), and \( [m,n] \) denotes the least common multiple of
\( m \) and \( n \).
**Remark 1.1.** We write \( \log_j \) for the \( j \)-th iterated logarithm, so for example, \( \log_2 N := \log \log N \), \( \log_3 N := \log \log \log N \).

The constant \( \exp\left(\frac{2\gamma+2e^4-2}{\zeta(2)}\right) \) in Theorem 1.2 is sharp since we have the following corresponding lower bounds on GCD sums.

**Theorem 1.3.** Let \( A \in (0, \infty) \) be fixed. Let \( \mathcal{M} \subset \mathbb{N} \). Then, for \( \alpha = \alpha(N) = 1 - \frac{A}{\log \log N} \), we have

\[
\sup_{|\mathcal{M}|=N} \sum_{n,m \in \mathcal{M}} \frac{(n,m)^{\alpha}}{|n,m|^\alpha} \geq \left( \frac{\exp\left(2\gamma+2e^4-2\right)}{\zeta(2)} + o(1) \right) \cdot N (\log \log N)^2, \quad \text{as } N \to \infty.
\]

An application of Theorem 1.2 is to prove the following results concerning upper bounds for a modified version of log-type GCD sums.

**Corollary 1.** Let \( \ell \in (0, \infty) \) be fixed. Let \( \mathcal{M} \subset \mathbb{N} \). Then

\[
\sup_{|\mathcal{M}|=N} \frac{1}{N} \sum_{n,m \in \mathcal{M}} \frac{(n,m)^{\alpha}}{|n,m|^\alpha} \log^{2\ell} \left( \frac{|n,m|}{\alpha N} \right) \leq (a^{\ell} + o(1)) \cdot (\log \log N)^{2+2\ell}, \quad \text{as } N \to \infty.
\]

where \( a^{\ell} \) is the positive constant defined by

\[
a^{\ell} := \min_{\lambda > 0} \frac{\exp\left(2\gamma+2e^{2\lambda A}-2\right)}{\zeta(2)A^{2\ell}}.
\]

By the above corollary, we have the following result.

**Corollary 2.** Let \( \ell \in (0, \infty) \) be fixed. Then

\[
\Gamma_1^{(\ell)}(N) \leq \left(4^{1-\ell} \cdot a^{\ell} + o(1)\right) \cdot (\log \log N)^{2+2\ell}, \quad \text{as } N \to \infty,
\]

where \( a^{\ell} \) is the positive constant defined as in Corollary 1.

**Remark 1.2.** Asymptotically, \( \log \left(4^{-\ell} \cdot a^{\ell}\right) \sim 2\ell \log \ell \), as \( \ell \to \infty \). On the other hand, numerical computations give \( 4^{-1}a_1 \approx 27.6 \), \( 4^{-2}a_2 \approx 861.5 \), and \( 4^{-3}a_3 \approx 43087 \).

In [Yan22a], the motivation of the study of such log-type GCD sums was to produce large values of \( |\zeta^{(\ell)}(1+it)| \). Our second method comes from the explicit connections between log-type GCD sums and \( |\zeta^{(\ell)}(1+it)| \), which is presented in the following Proposition 1. We will prove Proposition 1 using the resonance methods (see [Vor88, Sou08, Hil09, Ais16, BS17, BS18a, BS18b, dIBT19]).

**Proposition 1.** Fix \( \epsilon > 0 \), \( \beta \in [0, 1) \), \( \kappa \in (0, 1 - \beta) \) and \( \ell \in \mathbb{N} \).

Assume that \( c^{\ell} \) is a positive constant for which there exists an infinite sequence of positive integers \( N_1 < N_2 < \cdots < N_n < \cdots \) such that

\[
\Gamma_1^{(\ell)}(N_n) \geq c^{\ell} (\log \log N_n)^{2+2\ell}, \quad \forall n \in \mathbb{N}.
\]

Then for all sufficiently large \( n \in \mathbb{N} \), we can find a real number \( t \) with \( N_n^2 \leq t^{\kappa} \leq N_n \), such that

\[
|\zeta^{(\ell)}(1+it)| \geq \left( \sqrt{c^{\ell}} \cdot \diamondsuit - \epsilon \right) (\log \log t)^{\ell+1},
\]

where \( \diamondsuit \) is the positive constant defined as in Theorem 1.4.

**Remark 1.3.** In [Yan22a], we mentioned that we could use log-type GCD sums to establish lower bounds for the maximum of \( |\zeta^{(\ell)}(1+it)| \), but without giving such a proof. Instead, in [Yan22a] we used a different proposition to establish lower bounds for the maximum of \( |\zeta^{(\ell)}(1+it)| \) on the shorter interval \( \left[\frac{T}{2}, T\right] \).

The following Theorem is a corollary of Proposition 1.
**Theorem 1.4.** Fix $\epsilon > 0$ and $\ell \in \mathbb{N}$. For all sufficiently large $N \in \mathbb{N}$, we have
\begin{equation}
\Gamma_1^{(\ell)}(N) \leq \left( \frac{D_\ell}{\star} + \epsilon \right) (\log \log N)^{2+2\ell},
\end{equation}
where $D_\ell$ and $\star$ are defined by
\begin{align*}
D_\ell := \limsup_{t \to \infty} & \left| \frac{\zeta^{(\ell)}(1+it)}{(\log \log t)^{\ell+1}} \right|^2, \\
\star := \max_{0 < x < 2} & \frac{e^{-x}}{1 + 2 \sum_{n=0}^{\infty} e^{-xn^2}}.
\end{align*}
Furthermore, we have the following conditional result on the Riemann Hypothesis (RH).

**Proposition 2.** Assume RH. Fix $\ell \in \mathbb{N}$. For large $t \in \mathbb{R}$, we have $|\zeta^{(\ell)}(1+it)| \ll_{\ell} (\log \log t)^{\ell+1}$.

So when assuming RH, we will have $D_\ell < \infty$, and thus obtain the conditional upper bound $\Gamma_1^{(\ell)}(N) \ll_{\ell} (\log \log N)^{2+2\ell}$. On the other hand, if we assume the following unproven conjecture (which is similar to a conjecture of Granville and Soundararajan on character sums [GS01]), then we will have $\max_{T \leq t \leq 2T} |\zeta^{(\ell)}(1+it)| \sim Y_\ell (\log_2 T)^{\ell+1}$, as $T \to \infty$, by [Yan22b]. Here, $Y_\ell = \int_0^\infty u^\ell \rho(u) du$ and $\rho(u)$ denotes the Dickman function.

**Conjecture 1.1 ([Yan22b]).** There exists a constant $A > 0$ such that for any $1 \leq x \leq T$, $2T \leq t \leq 5T$, we have, uniformly,
\begin{equation}
\sum_{n \leq x} \frac{1}{n^t} = \sum_{n \leq x \atop P(n) \leq y} \frac{1}{n^t} + o(\Psi(x,y)), \quad \text{as } T \to \infty,
\end{equation}
where $y = (\log T + \log^2 x)(\log \log T)^A$. Here, $P(n)$ denotes the largest prime factor of $n$ and $\Psi(x,y)$ denotes the number of integers smaller than $x$ with $P(n) \leq y$.

**Remark 1.4.** By the conditional asymptotic formula for $\max_{T \leq t \leq 2T} |\zeta^{(\ell)}(1+it)|$, we will have $D_\ell = Y_\ell^2$ when assuming Conjecture 1.1. And $\log Y_\ell \sim \ell \log \ell$ [Yan22b], so conditionally we obtain $\log D_\ell \sim 2\ell \log \ell$, as $\ell \to \infty$.

**Remark 1.5.** We compute that $0.14149 < \star < 0.14151$, by using the following inequality
\begin{equation}
\frac{e^{-x}}{1 + 2 \sum_{n=0}^{\infty} e^{-xn^2}} < \frac{e^{-x}}{1 + 2 \sum_{n=0}^{\infty} e^{-xn^2}} < \frac{e^{-x}}{1 + 2 \sum_{n=0}^{\infty} e^{-xn^2}}, \quad x > 0.
\end{equation}
And one can compute that $Y_1 = e^\gamma$, $Y_2 = 3e^\gamma/2$ and $Y_3 = 17e^\gamma/6$ [Yan22b]. So if we assume Conjecture 1.1, then $D_1/\star \approx 22.4$, $D_2/\star \approx 50.4$ and $D_3/\star \approx 180$. In this case, the unconditional bound (2) is weaker than the conditional bound (4).

We will give two different proofs for Proposition 2. And the implied constants are effectively computable. In Sections 8, 9, after the proof, we give examples of computations for the implied constants. In particular, one of methods gives $|\zeta''(1+it)| \leq 20e^\gamma (\log \log t)^3 + O((\log \log t)^2)$ on RH. These constants are further improved in [Yan22b]. The main goal is not to sharp the implied constants but to present different approaches. In particular, one of the proof of Proposition 2 relies on the following Proposition 3, which could be viewed as analogs of Theorem 1.2.

**Proposition 3.** Assume RH and let $A > 0$ be fixed. When $t \to \infty$, we have
\begin{equation}
\log |\zeta(\sigma + it)| \leq \begin{cases} 
\log |\zeta(1+it)| + e^{2A} - 1 + o(1), & \text{if } 1 - \frac{A}{\log \log t} \leq \sigma \leq 1, \\
\log |\zeta(1+it)| + 2A + o(1), & \text{if } 1 \leq \sigma \leq 1 + \frac{A}{\log \log t}.
\end{cases}
\end{equation}
And we have the following result, which is similar to Theorem 1.3.
Theorem 1.5. Let $A \in (0, \infty)$ be fixed. Then, for $\sigma = \sigma(T) = 1 - \frac{A}{\log \log T}$, we have
\[
\max_{T \leq t \leq 2T} |\zeta(\sigma + it)| \geq (\exp(\gamma + e^A - 1) + o(1)) \log \log T, \text{ as } T \to \infty.
\]

We mention that Zaitsev [Zai00], Kalmygin [Kal18] and Bondarenko-Seip [BS18a], also investigated large values of $|\zeta(\sigma + it)|$, when $\sigma \to 1^-$. By Proposition 3, the problem reduces to give upper bounds for $|\zeta(1 + it)|$. Littlewood’s classical result on RH states that $|\zeta(1 + it)| \leq (2e^\gamma + o(1)) (\log \log t)$, as $t \to \infty$. In [LLS15], Lamzouri-X. Li-Soundararajan obtained the following result (on RH)
\[
|\zeta(1 + it)| \leq 2e^\gamma \left( \log \log t - \log 2 + \frac{1}{2} + \frac{1}{\log \log t} \right), \quad \forall t \geq 10^{10}.
\]

In [Yan22a], the author studied extreme values for $|\zeta^{(\ell)}(\sigma + it)|$ when $\sigma \in \left[\frac{1}{2}, 1\right]$. In this context, we also consider conditional upper bounds for $|\zeta^{(\ell)}(\sigma + it)|$ when $\sigma \in \left[\frac{1}{2}, 1\right]$. The following Proposition 4 is an easy consequence of the work of Chandee-Soundararajan [CS11] and Carneiro-Chandee [CC11].

**Proposition 4.** Assume RH. Fix $\epsilon > 0$, $\ell \in \mathbb{N}$ and $\sigma_0 \in \left(\frac{1}{2}, 1\right)$. Let $t$ be sufficiently large, then

(A) \[
\left|\zeta^{(\ell)} \left(\frac{1}{2} + it\right)\right| \leq \exp \left\{ \left( \frac{\log 2}{2} + \epsilon \right) \frac{t}{\log t} \right\},
\]

(B) \[
\left|\zeta^{(\ell)}(\sigma_0 + it)\right| \leq \exp \left\{ \left( \frac{1}{2} + \frac{2\sigma_0 - 1}{\sigma_0(1 - \sigma_0)} + \epsilon \right) \frac{(\log t)^{2 - 2\sigma_0}}{\log t} \right\}.
\]

In [Yan22a], when $\ell \in \mathbb{N}$ and $\sigma \in \left[\frac{1}{2}, 1\right]$ are given, we use GCD sums (rather than log-type GCD sums) to produce large values of $|\zeta^{(\ell)}(\sigma + it)|$. The reason is that when $\ell$ is fixed, there is no significant difference between GCD sums and log-type GCD sums. One can easily prove the following Proposition 5 based on the work of de la Bretèche-Tenenbaum [dBT19] and Aistleitner-Berkes-Seip [ABS15]. Therefore, the most interesting case for log-type GCD sum is the case when $\sigma = 1$.

**Proposition 5.** Fix $\ell \in [0, \infty)$ and $\sigma \in \left(\frac{1}{2}, 1\right)$.

(A) As $N \to \infty$, we have
\[
\Gamma^{(\ell)}_{\frac{1}{2}}(N) = \exp \left\{ (2\sqrt{2} + o(1)) \sqrt{\log N \log_3 N} \right\}.
\]

(B) There exists positive constants $c_\sigma$ and $C_\sigma$ depending on $\sigma$ such that for sufficiently large $N$, we have
\[
\exp \left\{ c_\sigma \cdot \frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma} \right\} \leq \Gamma^{(\ell)}_{\sigma}(N) \leq \exp \left\{ C_\sigma \cdot \frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma} \right\}.
\]

Let $\sigma \in (0, 1]$ be given and let $\mathcal{M} \subset \mathbb{N}$ be a finite set. The greatest common divisors (GCD) sums $S_\sigma(\mathcal{M})$ of $\mathcal{M}$ are defined as follows:
\[
S_\sigma(\mathcal{M}) := \sum_{m,n \in \mathcal{M}} \frac{(m,n)^\sigma}{|m,n|^\sigma}.
\]

The case $\sigma = 1$ was studied by Gál [Gál49], who proved that
\[
(\log \log N)^2 \ll \sup_{|\mathcal{M}|=N} \left| S_1(\mathcal{M}) \right| \ll (\log \log N)^2.
\]
Gál’s proof is a difficult combinatorial argument and his argument highly depends on the fact that \( \frac{\langle m, n \rangle}{|m, n|} \) is multiplicative. It’s not clear that Gál’s combinatorial method can be used to establish the upper bounds \( \Gamma^{(\ell)}_1(N) \ll_\ell (\log \log N)^{2+2\ell} \) since the log-type GCD sums are highly non-multiplicative.

The asymptotically sharp constant in (6) was found by Lewko and Radziwiłł in [LR17], where they proved that

\[
\left( \frac{6e^{2\gamma}}{\pi^2} + o(1) \right) (\log \log N)^2 \leq \sup_{\|\mathcal{M}\| = N} S_1(\mathcal{M}) \leq \left( \frac{6e^{2\gamma}}{\pi^2} + o(1) \right) (\log \log N)^2.
\]

Given \( \sigma \in (\frac{1}{2}, 1) \), Aistleitner, Berkes, and Seip [ABS15] proved the following result for GCD sums \( S_\sigma(\mathcal{M}) \), where \( c_\sigma \) and \( C_\sigma \) are positive constants only depending on \( \sigma \):

\[
\exp \left( c_\sigma \cdot \frac{(\log N)^{1-\sigma}}{(\log_2 N)^{\sigma}} \right) \leq \sup_{\|\mathcal{M}\| = N} S_\sigma(\mathcal{M}) \leq \exp \left( C_\sigma \cdot \frac{(\log N)^{1-\sigma}}{(\log_2 N)^{\sigma}} \right).
\]

For \( \sigma = \frac{1}{2} \), based on constructions of [BS15, BS17], de la Bretèche and Tenenbaum [dlBT19] proved the following result, improving early results of Bondarenko-Seip [BS15, BS17].

\[
\sup_{\|\mathcal{M}\| = N} \frac{S_\frac{1}{2}(\mathcal{M})}{|\mathcal{M}|} = \exp \left( (2\sqrt{2} + o(1)) \frac{\log N \log_3 N}{\log_2 N} \right), \quad \text{as } N \to \infty.
\]

By theorems of Aistleitner-Berkes-Seip and Gál on GCD sums, one can find that \( \sigma = 1 \) is a transition point for the GCD sums. Our Theorem 1.2 show that as long as \( \sigma \to 1^- \) with sufficiently fast rates, the optimal GCD sums on the \( \sigma \)-line will have the same size as the optimal GCD sums on the 1-line. So Theorem 1.2 could be interesting in this point of view. And by Proposition 3 and Theorem 1.5, we see that this phenomenon also happens for the Riemann zeta function (when assuming RH), i.e., the maximal size of \( |\zeta(\sigma + it)| \) has similar behavior as GCD sums, when \( \sigma \to 1^- \) with sufficiently fast rates.

2. The Random Zeta-Function \( \zeta(s, X) \) and Expectation Estimates

Let \( \{X(p)\}_p \) be a sequence of independent random variables (one for each prime \( p \)), uniformly distributed on the unit circle \( \{z \in \mathbb{C} : |z| = 1\} \). For an integer \( n \), we let

\[
X(n) := \prod_{p \nmid n} X(p)^\alpha.
\]

The random zeta-function \( \zeta(s, X) \) is defined as the following:

\[
\zeta(s, X) := \prod_p \left( 1 - \frac{X(p)}{p^s} \right)^{-1} = \sum_{n=1}^{\infty} \frac{X(n)}{n^s}.
\]

The product and series both converge almost surely when \( \Re(s) > \frac{1}{2} \) (for instance, see [Sou, page 4 and 6]). Note that

\[
\mathbb{E}[X(n)X(m)] = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}
\]

See [Lam11, LR17, Sou] for more information and applications of \( \zeta(s, X) \). Here, we only need estimates on the expectation of large powers of \( |\zeta(s, X)| \), which is related to upper bounds for GCD sums. The lemma below is proved by Lewko-Radziwiłł [LR17], using ideas from Lamzouri [Lam11, Lemma 2.1], who proved upper bounds for \( \log \mathbb{E}[|\zeta(\alpha, X)|^{2\gamma}] \) for fixed \( \alpha \in (\frac{1}{2}, 1) \).
Lemma 1 (Lemma 6 of [LR17]). We have the following bound,
\[ \log E[\zeta(1, X)^{2Y}] \leq 2Y(\log \log Y + \gamma) + O\left(\frac{Y}{\log Y}\right). \]

3. Preliminary Results on the Riemann zeta function \( \zeta \)

3.1. Lemmas on RH. In this subsection, we collect several conditional upper bounds for the Riemann zeta function, which will be used in later sections.

Lemma 2 (Littlewood, Thm 13.13 [MV06]). Assume RH. Then
\[ \left| \frac{\zeta'}{\zeta}(\sigma + it) \right| \leq \sum_{n \leq (\log t)^2} \frac{\Lambda(n)}{n^\sigma} + O\left( (\log t)^{2-2\sigma} \right), \]
uniformly for \( \frac{1}{2} + \frac{1}{\log \log t} \leq \sigma \leq \frac{3}{2}, |t| \geq 10. \)

Lemma 3 (Chandee, Soundararajan [CS11]). Assume RH. For large real numbers \( t \), we have
\[ \left| \zeta \left( \frac{1}{2} + it \right) \right| \leq \exp \left\{ \frac{\log 2}{2} \frac{\log t}{\log \log t} + O\left( \frac{\log t \log \log \log t}{(\log \log t)^2} \right) \right\}. \]

Lemma 4 (Carneiro, Chandee [CC11]). Let \( \alpha = \alpha(t) \) be a real-valued function with \( \frac{1}{2} < \alpha \leq 1. \) Assume RH. For large real numbers \( t \), we have
\[ \log |\zeta(\alpha + it)| \leq \begin{cases} \log \left( 1 + (\log t)^{1-2\alpha} \right) \frac{\log \log t}{2 \log \log \log t} + O\left( \left( \frac{(\log t)^{2-2\alpha}}{(\log \log t)^2} \right) \right), & \text{if } (\alpha - \frac{1}{2}) \log \log t = O(1); \\ \log (\log \log t) + O(1), & \text{if } (1 - \alpha) \log \log t = O(1); \\ \left( \frac{1}{2} + \frac{2\alpha - 1}{\alpha(1-\alpha)} \right) \frac{(\log t)^{2-2\alpha}}{\log \log t} + \log(2 \log \log t) + O\left( \frac{(\log t)^{2-2\alpha}}{(1-\alpha)^2(\log \log t)^2} \right), & \text{otherwise.} \end{cases} \]

3.2. Bell polynomials and Faà di Bruno’s formula. In this subsection, we present a formula for \( \frac{e^{\zeta(s)}}{\zeta(s)} \) by using Bell polynomials and Faà di Bruno’s formula. We will apply this formula for \( s = 1 + it \) in Section 9.

Definition 1. [Com74, page 134] The partial Bell polynomials \( B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \) are defined by
\[ B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) := \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \frac{x_1^{j_1}}{1!} \frac{x_2^{j_2}}{2!} \cdots \frac{x_{n-k+1}^{j_{n-k+1}}}{(n-k+1)!}, \]
where the summation takes place over all sequences \( j_1, j_2, j_3, \ldots, j_{n-k+1} \) of non-negative integers such that the following two conditions are satisfied:
\[ j_1 + j_2 + j_3 + \cdots + j_{n-k+1} = k; \]
\[ j_1 + 2j_2 + 3j_3 + \cdots + (n-k+1)j_{n-k+1} = n. \]
The \( n \)-th complete exponential Bell polynomial \( B_n \) is defined by the following sum:
\[ B_n(x_1, x_2, \ldots, x_n) := \sum_{k=1}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}). \]

In particular, we can compute \( B_1(x_1) = x_1, B_2(x_1, x_2) = x_1^2 + x_2 \) and \( B_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3. \)

Faà di Bruno’s formula [Com74, page 137] is a chain rule on higher derivatives, which can be expressed in terms of Bell polynomials as follows:
\[ \frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)). \]
Applying Faà di Bruno’s formula to \( f(s) = e^s \) and \( g(s) = \log \zeta(s) \) ([PRZZ20, page 19]), one can get
\[
\zeta^{(n)}(s) = \sum_{k=1}^{\infty} \zeta(s) \mathcal{B}_{n,k} \left( \frac{\zeta'}{\zeta}(s), \frac{d}{ds} \frac{\zeta'}{\zeta}(s), \ldots, \frac{d^{n-k}}{ds^{n-k}} \frac{\zeta'}{\zeta}(s) \right),
\]
(11)\[
\frac{\zeta^{(n)}}{\zeta}(s) = \sum_{k=1}^{\infty} \mathcal{B}_{n,k} \left( \frac{\zeta'}{\zeta}(s), \frac{d}{ds} \frac{\zeta'}{\zeta}(s), \ldots, \frac{d^{n-k}}{ds^{n-k}} \frac{\zeta'}{\zeta}(s) \right) = \mathcal{B}_{n} \left( \frac{\zeta'}{\zeta}(s), \frac{d}{ds} \frac{\zeta'}{\zeta}(s), \ldots, \frac{d^{n-1}}{ds^{n-1}} \frac{\zeta'}{\zeta}(s) \right).
\]

4. Proof of Theorem 1.2 and Corollary 1 and 2

4.1. Proof of Theorem 1.2. We will use the method of Lewko-Radziwiłł [LR17] to prove the Theorem. By [LR17, page 287-288], if \(|c| = N \) and \( \|c\|_2^2 = \sum_{n=1}^{\infty} |c(n)|^2 = 1 \), then we have
\[
(\zeta(2\alpha) \cdot \sum_{n,m \in \mathcal{M}} (n,m)^\alpha \cdot c(n)c(m) \leq e^{2Y} + N \cdot \mathbb{E}[|\zeta(\alpha,X)|^{2Y+2}] \cdot e^{-2Y}, \quad \forall Y, V > 0.
\]

So the problem now reduces to give suitable upper bounds for the expectation \( \mathbb{E}[|\zeta(\alpha,X)|^{2Y+2}] \), when \( \alpha \to 1^{-1} \) with certain converging rates. Before stating such upper bounds, we give the following lemma, which will be helpful for us to bound error terms when using the prime number theorem.

**Lemma 5.** Let \( A \in (0, \infty) \) be fixed. We have the following bound,
\[
\text{Int}(\alpha; A) := \int_2^{\exp(\frac{A}{\log A})} \frac{1}{t^\alpha} \exp\left(-\sqrt{\log t}\right) dt \ll_A 1, \quad \forall \alpha \in \left[\frac{1}{2}, 1\right] \cap [1 - \frac{A}{\log 2}, \infty).
\]

**Proof.** Note that \( \exp\left(-\sqrt{\log t}\right) \ll 1/(\log t)^2 \) and \( \frac{d}{dt} (-1/\log t) = 1/(t \log^2 t) \). So
\[
\text{Int}(\alpha; A) \ll \int_2^{\exp(\frac{A}{\log A})} \frac{t}{t^\alpha} \frac{d}{dt} \left(\frac{-1}{\log t}\right) dt.
\]

Now integration by parts gives
\[
\text{Int}(\alpha; A) \ll -\frac{e^A}{A}(1 - \alpha) + \frac{e^A}{\log 2} \ll_A 1.
\]

The following lemma is a key ingredient for the proof of Theorem 1.2.

**Lemma 6.** Let \( A \in (0, \infty) \) be fixed. We have the following bound,
\[
\log \mathbb{E}[|\zeta(\alpha,X)|^{2Y}] \leq 2Y(\log \log Y + \gamma + e^A - 1) + O_A \left( \frac{Y}{\log Y} \right), \quad \text{for} \quad \alpha = 1 - \frac{A}{\log Y}.
\]

**Proof.** Note that
\[
\mathbb{E}[|\zeta(\alpha,X)|^{2Y}] = \prod_p E_Y(p, \alpha) \text{ with } E_Y(p, \alpha) = \mathbb{E}\left[\left|1 - \frac{X(p)}{p^\alpha}\right|^{-2Y}\right].
\]

The crucial point is that the following inequality is valid for all real \( \theta \) and all positive \( \alpha \leq 1 \)
\[
\left(1 - \frac{e^{i\theta}}{p}\right)\left(1 - \frac{e^{-i\theta}}{p}\right) \leq \left(1 - \frac{1}{p}\right)^2.
\]
By the above observation and the inequality \(\int_{-\pi}^{\pi} f(\theta) d\theta / \int_{-\pi}^{\pi} g(\theta) d\theta \leq \sup_{\theta \in [-\pi, \pi]} f(\theta) / g(\theta)\) (which holds for any two positive functions \(f\) and \(g\)), we have
\[
\frac{E_Y(p, \alpha)}{E_Y(p, 1)} \leq \left( \frac{1 - \frac{1}{p}}{1 - \frac{1}{p^\alpha}} \right)^{2Y}.
\]

As a result, we obtain
\[
\log \frac{E_Y(p, \alpha)}{E_Y(p, 1)} \leq 2Y \log \left( \frac{1 - \frac{1}{p}}{1 - \frac{1}{p^\alpha}} \right) \leq 2Y (1 - \alpha) \left( \frac{\log p}{p^\alpha} + \frac{\log p}{(p^\alpha - 1)p^\alpha} \right).
\]

When \(Y/p^{2\alpha} \ll 1\), we have (also see [GS06, Lemma 4])
\[
E_Y(p, \alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( 1 - \frac{e^{i\theta}}{p^\alpha} \right)^{-Y} \left( 1 - \frac{e^{-i\theta}}{p^\alpha} \right)^{-Y} d\theta = I_0 \left( \frac{2Y}{p^\alpha} \right) \left( 1 + O \left( \frac{Y}{p^{2\alpha}} \right) \right),
\]
where \(I_0(t)\) is the 0-th modified Bessel function defined as
\[
I_0(t) := \frac{1}{\pi} \int_0^\pi e^{t \cos \theta} d\theta = \sum_{n=0}^{\infty} \frac{(t/2)^{2n}/(n!)^2}{2^n \pi}, \quad \forall t \in \mathbb{R}.
\]

From the expansion of \(I_0(t)\), we have \(0 < I_0(t) \ll t\) for \(0 < t < 2\). Thus when \(p > Y^{1/\alpha}\), we get
\[
\log I_0 \left( \frac{2Y}{p^\alpha} \right) - \log I_0 \left( \frac{2Y}{p} \right) = \int \frac{2Y}{p} \frac{dI_0(t)}{I_0(t)} dt \ll \int \frac{2Y}{p} dt \ll Y \left( \frac{Y}{p^{1/\alpha} - 1} \right) \ll Y^2 (1 - \alpha) \frac{\log p}{p^{2\alpha}}.
\]

Combining these bounds gives
\[
\log \mathbb{E}[\zeta(\alpha, X)|Z(2Y)] \leq 2Y (1 - \alpha) \sum_{p < Y^{1/\alpha}} \frac{\log p}{p^{1/\alpha}} + 2Y (1 - \alpha) \sum_{p < Y^{1/\alpha}} \frac{\log p}{(p^{1/\alpha} - 1)p^{1/\alpha}}
\]
\[
+ O(1) \cdot \sum_{p > Y^{1/\alpha}} Y^2 (1 - \alpha) \frac{\log p}{p^{2\alpha}} + O(1) \cdot \sum_{p > Y^{1/\alpha}} \frac{Y}{p^{2\alpha}}.
\]

By the prime number theorem and Lemma 5, the first term is bounded by
\[
\lesssim 2Y (e^A - 1) + O_A \left( \frac{Y}{\log Y} \right),
\]
and the other three terms are bounded by
\[
\ll_A \frac{Y}{\log Y}.
\]

The proof now follows from Lemma 1. \(\square\)

Now in the inequality (12), we let
\[
V = \log_3 N + \gamma + e^A - 1 + \frac{2}{\log_3 N} \quad \text{and} \quad Y = \log_3 N \cdot \log N,
\]
then
\[
\alpha = 1 - \frac{A}{\log Y}.
\]

With the choice of \(Y\) and \(V\), by Lemma 6 we have \(\mathbb{E}[\zeta(\alpha, X)|Z(2Y + 2)] \cdot e^{-2YV} \ll N^{-1}\). Clearly, \(\zeta(2\alpha) \rightarrow \zeta(2)\) when \(N \rightarrow \infty\). The claim of the theorem follows immediately.
4.2. Proof of Corollary 1. By the inequality $\log X \leq \frac{1}{\epsilon} X^\epsilon$, $(\forall \epsilon > 0, \forall X \geq 1)$, we obtain

$$\sum_{n,m \in \mathcal{M}} \frac{(n,m)}{[n,m]} \log^{2\ell} \left( \frac{[n,m]}{(n,m)} \right) \leq \left( \frac{1}{\epsilon} \right)^{2\ell} \sum_{n,m \in \mathcal{M}} \frac{(n,m)^{1-2\ell}}{[n,m]^{1-2\ell}}.$$

Let $|\mathcal{M}| = N$. Let $c(n) = 1/\sqrt{N}$ if $n \in \mathcal{M}$, and $c(n) = 0$ if $n \notin \mathcal{M}$. Take $\epsilon = A/\log (\log N \cdot \log_3 N)$, where $A$ is a positive number to be chosen later. Then by Theorem 1.2, we have

$$\frac{1}{N} \sum_{n,m \in \mathcal{M}} \frac{(n,m)^{1-2\ell}}{[n,m]^{1-2\ell}} \leq \left( \frac{1}{\epsilon} \right)^{2\ell} \exp \left( 2\gamma + 2e^{2\ell A} - 2 \right) + o(1) \cdot (\log \log N)^2, \text{ as } N \to \infty.$$

By our choice of $\epsilon$, we have

$$\left( \frac{1}{\epsilon} \right)^{2\ell} = \left( \frac{1}{A^{2\ell}} + o(1) \right) (\log \log N)^2.$$

Combining the above two inequalities and choosing $A$ to minimize the expression

$$\frac{\exp \left( 2\gamma + 2e^{2\ell A} - 2 \right)}{\zeta(2) A^{2\ell}},$$

we are done.

4.3. Proof of Corollary 2. Note that

$$\log \frac{m}{(m,n)} + \log \frac{n}{(m,n)} = \log \frac{m}{(m,n)} \cdot \frac{n}{(m,n)} = \log \frac{[m,n]}{(m,n)}.$$

By the inequality $ab \leq \left( \frac{a+b}{2} \right)^2$, we obtain

$$\sum_{m,n \in \mathcal{M}} \frac{(m,n)}{[m,n]} \log^\ell \left( \frac{m}{(m,n)} \right) \log^\ell \left( \frac{n}{(m,n)} \right) \leq \frac{1}{4\ell} \sum_{n,m \in \mathcal{M}} \frac{(n,m)}{[n,m]} \log^{2\ell} \left( \frac{[n,m]}{(n,m)} \right).$$

Now Corollary 2 immediately follows from Corollary 1.

5. Proof of Theorem 1.3

Let $\mathcal{P}(r, b) = p_1^{h_1} \cdots p_r^{h_r}$, where $p_n$ denotes the $n$-th prime. Define $\mathcal{M}$ to be the set of divisors of $\mathcal{P}(r, b)$, then $|\mathcal{M}| = b^r$. Then we have the following Gáll’s identity$^2$ [Gáll49],

$$\sum_{m,n \in \mathcal{M}} \frac{(m,n)^\alpha}{[m,n]^\beta} = \prod_{p \leq p_r} \left( b + 2 \sum_{k=1}^{b-1} \frac{b-k}{p^{k\alpha}} \right). \tag{13}$$

To prove the above identity, we write $m = p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}$, $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$ and compute the left hand side of (13) as follows

$$\sum_{m,n \in \mathcal{M}} \frac{(m,n)^\alpha}{[m,n]^\beta} = \sum_{\beta_0=0}^{b-1} \cdots \sum_{\beta_r=0}^{b-1} \sum_{\nu_0=0}^{b-1} \sum_{\nu_r=0}^{b-1} \left( p_1^{-|\nu_1-\beta_1|} p_2^{-|\nu_2-\beta_2|} \cdots p_r^{-|\nu_r-\beta_r|} \right)^\alpha$$

$$= \prod_{p \leq p_r} \left( \sum_{\beta_0=0}^{b-1} \sum_{\nu_0=0}^{b-1} \left( p^{-|\nu-\beta|} \right)^\alpha \right)$$

$$= \prod_{p \leq p_r} \left( b + 2 \sum_{k=1}^{b-1} \frac{b-k}{p^{k\alpha}} \right),$$

which is the right hand side of (13).

$^2$It was stated for $\alpha = 1$ in [Gáll49].
Let \( r = \lceil \log N / \log \log N \rceil \), then \( p_r \sim \log N \) by the prime number theorem. Let \( b \) be the integer satisfying that
\[
b^r \leq N < (b + 1)^r,\
\]
then \( b^r \sim N \), as \( N \to \infty \). Choose any set \( \mathcal{M} \subset \mathbb{N} \) such that \( \mathcal{M} \subset \mathcal{M}' \) and \( |\mathcal{M}'| = N \). Then the GCD sum over \( \mathcal{M}' \) is at least as large as the GCD sum over \( \mathcal{M} \).

Following Lewko-Radziwiłł in \([LR17]\), we use Gál’s identity for the GCD sum and split the product into several parts:
\[
\sum_{m,n \in \mathcal{M}} \frac{(m,n)\alpha}{(m,n)} = b^r \prod_{p \leq p_r} \left( 1 + 2 \sum_{k=1}^{b-1} \frac{1}{p^{k\alpha}} \right) \left( 1 - \frac{k}{b} \right)\
\]
(14)
\[
\geq (1 + o(1)) N \prod_{p \leq p_r} \left( \frac{1 - \frac{1}{p^\alpha}}{1 - \frac{1}{p^r}} \right)^2 \times \prod_{p \leq p_r} \left( 1 - \frac{1}{p} \right)^{-2} \times \prod_{p \leq p_r} \left( 1 + 2 \sum_{k=1}^{b-1} \frac{1}{p^{k\alpha}} \cdot \left( 1 - \frac{k}{b} \right) \left( 1 - \frac{1}{p^\alpha} \right)^2 \right).
\]

By Mertens’ third theorem, the second product is asymptotically equal to \((e^\gamma \log p_r)^2 \sim (e^\gamma \log \log N)^2\) as \( N \to \infty \). And when \( N \to \infty \), the last product converges to
\[
\prod_p \left( 1 + 2 \sum_{k=1}^{\infty} \frac{1}{p^k} \right) \left( 1 - \frac{1}{p} \right)^2 = \frac{1}{\zeta(2)}.
\]

So it remains to prove that the first product converges to \(\exp(2e^A - 2)\), which follows from the following Lemma 7, since \( p_r \sim \log N \), when \( N \to \infty \).

**Lemma 7.** Fix \( A > 0 \) and let \( \alpha = \alpha(N) = 1 - \frac{A}{\log \log N} \). Assume that \( \log X \sim \log \log N \), as \( N \to \infty \). Then we have
\[
\sum_{p \leq X} \log \left( \frac{1 - \frac{1}{p}}{1 - \frac{1}{p^\alpha}} \right) = e^A - 1 + o(1), \quad \text{as} \quad N \to \infty.
\]
(15)

**Proof.** Let
\[
J_\alpha(p) := \log \left( \frac{1 - \frac{1}{p}}{1 - \frac{1}{p^\alpha}} \right) = \int_\alpha^1 \frac{\log p}{p^x - 1} \, dx,
\]
then by the integral representation of \( J_\alpha(p) \) we have
\[
\frac{1}{p^\alpha} - \frac{1}{p} \leq J_\alpha(p) \leq \frac{(1 - \alpha) \log p}{p^\alpha} + \frac{(1 - \alpha) \log p}{p^\alpha (p^\alpha - 1)}.
\]

By the prime number theorem and Lemma 5, we obtain
\[
\sum_{p \leq X} \frac{(1 - \alpha) \log p}{p^\alpha} = e^A - 1 + o(1), \quad \text{as} \quad N \to \infty.
\]

For sufficiently large \( N \), we have
\[
\sum_{p \leq X} \frac{(1 - \alpha) \log p}{p^\alpha (p^\alpha - 1)} \ll (1 - \alpha) \sum_{p \leq X} \frac{\log p}{p^{2\alpha}} \ll A \frac{1}{\log \log N}.
\]

Let
\[
\Delta_\alpha(p) := \left( \frac{1}{p^\alpha} - \frac{1}{p} \right) - \frac{(1 - \alpha) \log p}{p^\alpha} = -\frac{1}{p^\alpha} \left( \frac{1}{p^{1-\alpha}} - 1 - \log \left( \frac{1}{p^{1-\alpha}} \right) \right),
\]
then by Taylor expansion of \( \log(1 - (1 - \frac{1}{p^{1-\alpha}})) \), we obtain
\[
|\Delta_\alpha(p)| = -\Delta_\alpha(p) \leq \frac{1}{2} \left( 1 - \frac{1}{p^{1-\alpha}} \right)^2 p^{1-2\alpha}.
\]

So when \( p \leq X \), we have the following estimates
\[
p^{1-2\alpha} = p^{1-2\left(1 - \frac{\Delta}{\log \log N}\right)} \leq \frac{1}{p} \exp \left( \log X - \frac{2A}{\log \log N} \right) \leq \frac{1}{p} e^{2A+o(1)}.
\]

Uniformly for \( t \geq 2 \) and \( 0 \leq \delta \leq \frac{1}{2} \), we have \( 1 - \frac{1}{t} \leq \delta \), which follows from Taylor expansion of \( \exp(-\delta \log 2) \). So for sufficiently large \( N \), we find that
\[
\sum_{p \leq X} |\Delta_\alpha(p)| \leq \frac{1}{2} e^{2A+o(1)} (1 - \alpha)^2 \sum_{p \leq X} \frac{1}{p} \ll A \log_3 N \left( \frac{\log \log N}{p} \right)^2 ,
\]
by Mertens’ second theorem. Therefore,
\[
J_\alpha(p) = \sum_{p \leq X} \frac{(1 - \alpha) \log p}{p^\alpha} + o(1) = e^A - 1 + o(1), \quad \text{as} \quad N \to \infty,
\]
which gives (15).

\[\square\]

6. Proof of Proposition 1

Without loss of generality, assume that \( \kappa + 2\epsilon < 1 \). Let \( N \in \{ N_1, N_2, \cdots, N_n, \cdots \} \) and \( T = N^{\frac{1}{2}} \). Let \( \mathcal{M} \subset \mathbb{N} \) with \( |\mathcal{M}| = N \). We will construct a resonator \( R(t) \), following ideas from [Ais16], [BS17] and [dlBT19]. Define
\[
\mathcal{M}_u := \left[ (1 + \frac{\log T}{T})^u, (1 + \frac{\log T}{T})^{u+1} \right) \cap \mathcal{M} \quad (u \geq 0).
\]
Let \( \mathcal{U} \) be the set of integers \( u \) such that \( \mathcal{M}_u \neq \emptyset \) and let \( m_u \) be the minimum of \( \mathcal{M}_u \) for \( u \in \mathcal{U} \). We then set
\[
\mathcal{M}^\prime := \{ m_u : u \in \mathcal{U} \} \quad \text{and} \quad r(m_u) := \sqrt{\sum_{m \in \mathcal{M}_u} 1} = \sqrt{|\mathcal{M}_u|}
\]
for \( m_u \in \mathcal{M}^\prime \). Then the resonator \( R(t) \) is defined as follows:
\[
R(t) := \sum_{m \in \mathcal{M}^\prime} \frac{r(m)}{m^t}.
\]

By Cauchy’s inequality, one has the following estimates [dlBT19]:
\[
R(0)^2 \leq N \sum_{m \in \mathcal{M}^\prime} r(m)^2 \leq N|\mathcal{M}|.
\]

Let \( A \) be a positive number. Let \( \Phi(t) := \frac{1}{\sqrt{4\pi}} e^{-\frac{t^2}{4}} \) with the Fourier transform \( \hat{\Phi} \) defined by
\[
\hat{\Phi}(\xi) := \int_{-\infty}^{\infty} \Phi(t) e^{-ix\xi} \, dt = e^{-A\xi^2}.
\]
Define the moments as follows:

\[ M_1(R, T) = \int_{T^3}^T |R(t)|^2 \Phi\left(\frac{t \log T}{T}\right) dt, \]
\[ M_2(R, T) = \int_{T^3}^T |\zeta(t)(1 + it)|^2 |R(t)|^2 \Phi\left(\frac{t \log T}{T}\right) dt, \]
\[ \hat{M}_2(R, T) = \int_{T^3}^T \left| \sum_{k \leq T} \frac{\log k}{k^{1+it}} \right|^2 |R(t)|^2 \Phi\left(\frac{t \log T}{T}\right) dt. \]

By the above definitions, we have

\[ \max_{T^3 \leq t \leq T} |\zeta(t)(1 + it)|^2 \geq \frac{M_2(R, T)}{M_1(R, T)}. \]  

From the proof Lemma 5 of [BS18b] (replacing \( T \) by \( T/\log T \)), one can obtain that

\[ M_1(R, T) \leq \int_{-\infty}^\infty |R(t)|^2 \Phi\left(\frac{t \log T}{T}\right) dt \leq \left(1 + 2 \sum_{n=0}^\infty \hat{\phi}(n) + \epsilon\right) \frac{T}{\log T} |\mathcal{M}|. \]

Define

\[ I(R, T) = \int_{-\infty}^\infty \left| \sum_{k \leq T} \frac{\log k}{k^{1+it}} \right|^2 |R(t)|^2 \Phi\left(\frac{t \log T}{T}\right) dt. \]

Then

\[ I(R, T) = \frac{T}{\log T} \sum_{j, k \leq T} \frac{(\log j \cdot \log k)^\ell}{jk} \sum_{u, \nu \in \mathcal{U}} r(m_u)r(m_\nu) \int_{-\infty}^\infty \frac{jm_\nu}{km_u}^it \Phi\left(\frac{t \log T}{T}\right) dt \]
\[ = \frac{T}{\log T} \sum_{j, k \leq T} \frac{(\log j \cdot \log k)^\ell}{jk} \sum_{u, \nu \in \mathcal{U}} r(m_u)r(m_\nu) \hat{\phi}\left(\frac{T}{\log T} \log \frac{km_u}{jm_\nu}\right). \]  

When \( j, k \) are fixed, for any \( u, \nu \in \mathcal{U} \), one has

\[ \sum_{m \in \mathcal{M}_u, n \in \mathcal{M}_\nu \atop mk=nj} 1 \leq \min\{|\mathcal{M}_u|, |\mathcal{M}_\nu|\} \leq r(m_u)r(m_\nu). \]

Note that when \( mk = nj \), we have \( \frac{n m_u}{m_\nu} = \frac{km_u}{jm_\nu} \). Multiplying by \( \hat{\phi}\left(\frac{T}{\log T} \log \frac{km_u}{jm_\nu}\right) \) in (21) and summing index over \( u, \nu \in \mathcal{U} \) give

\[ \sum_{u, \nu \in \mathcal{U}} r(m_u)r(m_\nu) \hat{\phi}\left(\frac{T}{\log T} \log \frac{km_u}{jm_\nu}\right) \geq \sum_{u, \nu \in \mathcal{U}} \sum_{m \in \mathcal{M}_u, n \in \mathcal{M}_\nu \atop mk=nj} \hat{\phi}\left(\frac{T}{\log T} \log \frac{nm_u}{m_\nu}\right) \geq \sum_{m, n \in \mathcal{M} \atop mk=nj} \hat{\phi}(1), \]

where the last inequality follows from \( 0 \leq \frac{m}{m_u} - 1 \leq \frac{\log T}{\log j} \) and \( 0 \leq \frac{n}{m_\nu} - 1 \leq \frac{\log T}{\log j} \). Returning to (20), we obtain

\[ I(R, T) \geq \frac{T}{\log T} \sum_{j, k \leq T} \frac{(\log j \cdot \log k)^\ell}{jk} \sum_{m, n \in \mathcal{M} \atop mk=nj} \hat{\phi}(1). \]
When \(j = \frac{m}{(m,n)}\) and \(k = \frac{n}{(m,n)}\), we have \(mk = \frac{mn}{(m,n)} = [m,n] = nj\). Thus we further get the following lower bound

\[
I(R,T) \geq \tilde{\Phi}(1) \frac{T}{\log T} \sum_{\substack{m,n \in \mathcal{M} \\
(m,n) \leq T}} \left( \log \frac{m}{(m,n)} \cdot \log \frac{n}{(m,n)} \right)^{\ell}
\]

\[
\geq \tilde{\Phi}(1) \frac{T}{\log T} S(\mathcal{M}; \ell) - E(\mathcal{M}; T),
\]

where \(S(\mathcal{M}; \ell)\) and \(E(\mathcal{M}; T)\) are defined as follows:

\[
S(\mathcal{M}; \ell) := \sum_{m,n \in \mathcal{M}} \left( \frac{m}{(m,n)} \right) \log^{\ell} \left( \frac{m}{(m,n)} \right) \log^{\ell} \left( \frac{n}{(m,n)} \right),
\]

\[
E(\mathcal{M}; T) := 2\tilde{\Phi}(1) \frac{T}{\log T} \sum_{\substack{m,n \in \mathcal{M} \\
(m,n) > T}} \left( \log \frac{m}{(m,n)} \right) \log^{\ell} \left( \frac{m}{(m,n)} \right) \log^{\ell} \left( \frac{n}{(m,n)} \right).
\]

Using Rankin’s trick, the inequality \(\log X \ll_{\epsilon} X^{\epsilon}\) (which holds for all \(X \geq 1\)) and the upper bound in \((7)\), we bound \(E(\mathcal{M}; T)\) by

\[
E(\mathcal{M}; T) \ll \frac{T}{\log T} \frac{1}{\sqrt{T}} \sum_{m,n \in \mathcal{M}} \sqrt{\frac{n}{(m,n)}} \left( \frac{m}{(m,n)} \right) \log^{\ell} \left( \frac{m}{(m,n)} \right) \log^{\ell} \left( \frac{n}{(m,n)} \right).
\]

\[
\ll \frac{T}{\log T} \frac{1}{\sqrt{T}} S_T^\frac{\gamma}{2} (\mathcal{M})
\]

\[
\ll \frac{T}{\log T} \frac{1}{\sqrt{T}} |\mathcal{M}| \exp \left\{ C_T^\frac{\gamma}{2} \left( \log \log T \right)^{\frac{\gamma}{2}} \right\} \ll \frac{T}{\log T} \frac{1}{\sqrt{T}} |\mathcal{M}|.
\]

So we have proved that

\[
I(R,T) \geq \tilde{\Phi}(1) \frac{T}{\log T} S(\mathcal{M}; \ell) + O \left( \frac{T}{\log T} \frac{1}{\sqrt{T}} |\mathcal{M}| \right).
\]

In the following steps, we will bound \(\left| I(R,T) - \tilde{M}_2(R,T) \right|\) and \(\left| M_2(R,T) - \tilde{M}_2(R,T) \right|\).

First, note that

\[
\int_{|t| \leq T^\beta} \left| \sum_{n \leq T} \frac{(\log n)^{\ell}}{n^{1+\epsilon}} \right|^2 |R(t)|^2 \Phi \left( \frac{t \log T}{T} \right) dt \ll R(0)^2 T^\beta \cdot (\log T)^{2\ell+2} \ll T^{\kappa+\beta} (\log T)^{2\ell+2} |\mathcal{M}|.
\]

On the other hand, by the fast decay of \(\Phi\), we find that

\[
\int_{|t| \geq T} \left| \sum_{n \leq T} \frac{(\log n)^{\ell}}{n^{1+\epsilon}} \right|^2 |R(t)|^2 \Phi \left( \frac{t \log T}{T} \right) dt \ll T^\kappa (\log T)^{2\ell+2} |\mathcal{M}| \int_{|t| \geq T} \Phi \left( \frac{t \log T}{T} \right) dt \ll o(1) |\mathcal{M}|.
\]

As a result, we have

\[
I(R,T) = \tilde{M}_2(R,T) + O \left( T^{\kappa+\beta} (\log T)^{2\ell+2} |\mathcal{M}| \right).
\]

Next, let

\[
E_1 = \frac{\ell! T^\epsilon}{\epsilon^t}.
\]
By Hardy-Littlewood’s approximation formula (see [Tit86, Thm 4.11]) for \( \zeta(s) \) and Cauchy’s integral formula for derivatives, we have

\[
(24) \quad \zeta^{(\ell)}(1 + it) = (-1)^\ell \sum_{k \leq T} \frac{(\log k)^\ell}{k^{1+it}} + O\left(E_1\right), \quad T^\beta \leq t \leq T.
\]

From (24) one can get

\[
\left| \zeta^{(\ell)}(1 + it) \right|^2 - \left| \sum_{k \leq T} \frac{(\log k)^\ell}{k^{1+it}} \right|^2 \ll |E_1|^2 + \sum_{k \leq T} \frac{(\log k)^\ell}{k^{1+it}} \cdot |E_1|, \quad T^\beta \leq t \leq T.
\]

We will estimate the contributions of \( E_1 \) in the integrals.

\[
\int_{T^\beta}^T |E_1|^2 |R(t)|^2 \Phi\left(\frac{t \log T}{T}\right) dt \ll T^{2\epsilon + \kappa} |\mathcal{M}|,
\]

where the implied constants depend on \( \ell \) and \( \epsilon \) only.

By the Cauchy-Schwarz inequality and (19),

\[
\int_{T^\beta}^T \sum_{k \leq T} \frac{(\log k)^\ell}{k^{1+it}} \cdot |E_1| \cdot |R(t)|^2 \Phi\left(\frac{t \log T}{T}\right) dt
\]

\[
\leq \sqrt{M_2(R, T)} \sqrt{\int_{T^\beta}^T |E_1|^2 |R(t)|^2 \Phi\left(\frac{t \log T}{T}\right) dt}
\]

\[
\ll (\log T)^{\ell+1} \sqrt{M_1(R, T)} \sqrt{T^{2\epsilon + \kappa} |\mathcal{M}|}
\]

\[
\ll (\log T)^{\ell+\frac{1}{2}} T^{\frac{1}{2} + \epsilon + \frac{\kappa}{2}} |\mathcal{M}|.
\]

As a result, we obtain

\[
(25) \quad \left| M_2(R, T) - \tilde{M}_2(R, T) \right| \ll T^{2\epsilon + \kappa} |\mathcal{M}| + (\log T)^{\ell+\frac{1}{2}} T^{\frac{1}{2} + \epsilon + \frac{\kappa}{2}} |\mathcal{M}|.
\]

(25) together with (22) and (23) give

\[
(26) \quad M_2(R, T) \geq \tilde{\Phi}(1) \frac{T}{\log T} S(\mathcal{M}; \ell) + o\left(\frac{T}{\log T} |\mathcal{M}|\right).
\]

By the assumption (3) and (18), (19), (26), we obtain

\[
\max_{T^\beta \leq t \leq T} \left| \zeta^{(\ell)}(1 + it) \right|^2 \geq \frac{M_2(R, T)}{M_1(R, T)} \geq c_\ell \frac{\tilde{\Phi}(1)}{1 + 2 \sum_{n=0}^{\infty} \Phi(n) + 2\epsilon} (\log \log T)^{2\ell + 2}.
\]

7. Proof of Theorem 1.5

Following [Sou08], let \( \Phi : \mathbb{R} \to \mathbb{R} \) be a smooth function, compactly supported in \([1, 2]\), with \( 0 \leq \Phi(y) \leq 1 \) for all \( y \), and \( \Phi(y) = 1 \) for \( 5/4 \leq y \leq 7/4 \). Set \( N = \lfloor T^{1/2} \rfloor \) and let \( R(t) := \sum_{n \leq N} r(n)n^{-it} \). Define

\[
M_1(R, T) = \int_{-\infty}^{+\infty} |R(t)|^2 \Phi\left(\frac{t}{T}\right) dt,
\]

\[
M_2(R, T) = \int_{-\infty}^{+\infty} \left( \sum_{k \leq T} \frac{1}{k^{\sigma+it}} \right) |R(t)|^2 \Phi\left(\frac{t}{T}\right) dt.
\]

Then with a little computations (for instance, see [Sou08, BS18a]), we obtain...
\[ M_1(R, T) = T \Phi(0) \left( 1 + O \left( T^{-1} \right) \right) \sum_{n \leq N} |r(n)|^2, \]
\[ M_2(R, T) = T \Phi(0) \sum_{mk=n \leq N} \frac{r(m)r(mk)}{k^\sigma} + O \left( T^{-1} \right) \sum_{n \leq N} |r(n)|^2. \]

By Hardy-Littlewood’s approximation formula, we have

\[ \max_{T \leq t \leq 2T} |\zeta(\sigma + it)| \geq \frac{|M_2(R, T)|}{M_1(R, T)} + O(1). \]

Set \( x = (\log T)/(3 \log \log T) \) and \( b = [\log \log T] \). As in [BS18a, page 128-129], define the function \( r : \mathbb{N} \to \{0, 1\} \) to be the characteristic function of a set \( \mathcal{M} \), where \( \mathcal{M} \) is the set of divisors of the integer \( K := \prod_{p \leq x} p^{b-1} \). Then we have

\[
\left| \sum_{mk=n \leq N} \frac{r(m)r(mk)}{k^\sigma} \right| \leq \left( \sum_{n \leq N} |r(n)|^2 \right)^{1/2} \prod_{p \leq x} \left( 1 + \sum_{k=1}^{b-1} \left( 1 - \frac{k}{b} \right) p^{-k\sigma} \right)
\]
\[
= \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \leq x} \left( 1 - \frac{1}{p^\sigma} \right) \left( 1 + \sum_{k=1}^{b-1} \left( 1 - \frac{k}{b} \right) p^{-k\sigma} \right).
\]

By Mertens’ third theorem, the second product is asymptotically equal to \( e^\gamma (\log \log T) \), as \( T \to \infty \).

And the last product converges to 1, when \( T \to \infty \). By Lemma 7, the first product converges to \( \exp(e^A - 1) \), as \( T \to \infty \). Now the theorem follows from (27).

### 8. Proof of Proposition 3 and the First Proof of Proposition 2

#### 8.1. The proof

We will use the identity

\[ \log |\zeta(\sigma + it)| - \log |\zeta(1 + it)| = \Re (\log \zeta(\sigma + it) - \log \zeta(1 + it)) = \Re \left( - \int_{\sigma}^{1} \frac{\zeta'(\alpha + it)}{\zeta(\alpha + it)} d\alpha \right). \]

Let \( A \) be a positive number. Consider the case \( 1 - \frac{A}{\log \log t} \leq \sigma \leq 1 \) first.

By the estimate \( \int_{\sigma}^{1} \frac{1}{n^\sigma} d\alpha \leq (1 - \sigma) \frac{1}{n^\sigma} \) and Lemma 2, we obtain

\[ \left| \log |\zeta(\sigma + it)| - \log |\zeta(1 + it)| \right| \leq \left| \int_{\sigma}^{1} \frac{\zeta'(\alpha + it)}{\zeta(\alpha + it)} d\alpha \right| \leq \left| \int_{\sigma}^{1} \left( \sum_{n \leq (\log t)^2} \frac{\Lambda(n)}{n^\sigma} + O \left( (\log t)^{2-2\alpha} \right) \right) d\alpha \right| \leq (1 - \sigma) \sum_{n \leq (\log t)^2} \frac{\Lambda(n)}{n^\sigma} + O(1) \int_{\sigma}^{1} (\log t)^{2-2\alpha} d\alpha \leq e^{2A} - 1 + O \left( \frac{1}{\log \log t} \right), \]

where the last inequality follows from partial summation and the prime number theorem on the Riemann Hypothesis \( \sum_{n \leq x} \Lambda(n) = x + O \left( \sqrt{x \log^2 x} \right) \).
So we have

\[(28) \quad \log |\zeta(\sigma + it)| \leq \log |\zeta(1 + it)| + e^{2A} - 1 + o(1), \quad \text{if} \quad 1 - \frac{A}{\log \log t} \leq \sigma \leq 1.\]

For the other case \(1 \leq \sigma \leq 1 + \frac{A}{\log \log t}\), we use the estimate \(\int_1^\sigma \frac{1}{n^s} \, d\alpha \leq (\sigma - 1) \frac{1}{n}\) instead and similarly prove that

\[(29) \quad \log |\zeta(\sigma + it)| \leq \log |\zeta(1 + it)| + 2A + o(1), \quad \text{if} \quad 1 \leq \sigma \leq 1 + \frac{A}{\log \log \log t}.\]

From (28) and (29), we have

\[(30) \quad |\zeta(\sigma + it)| \leq \exp \left( e^{2A} - 1 + o(1) \right) \cdot |\zeta(1 + it)|, \quad \text{if} \quad |\sigma - 1| \leq \frac{A}{\log \log t}.\]

Now set \(\delta = \frac{A}{\log \log t}\). Note that \(|(\sigma + it) - (1 + it)| \leq \delta\) implies \(|\sigma - 1| \leq \delta\) and \(|t - t| \leq \delta\).

By Cauchy’s integral formula for derivatives and Littlewood’s classical result on RH, we obtain

\[
\left| \zeta'(1 + it) \right| \leq \frac{\ell!}{\delta^\ell} \max_{|\sigma - 1| \leq \delta, |t - t| \leq \delta} \left| \zeta(\sigma + it) \right|
\leq \frac{\ell!}{A^\ell} \left( \log \log t \right)^\ell \exp \left( e^{2A} - 1 + o(1) \right) \cdot (2e^{\gamma} + o(1)) (\log \log t)
\leq \exp \left( e^{2A} + \gamma - 1 + o(1) \right) \frac{2\ell!}{A^\ell} (\log \log t)^{\ell + 1}.
\]

8.2. Some Examples. In order to optimize the constant, we let \(A\) be the solution of the equation \(2e^{2x} = \ell x^{-1}\). Then numerical computations give \(|\zeta'(1 + it)| \leq 15.2e^{\gamma} (\log \log t)^2\), \(|\zeta''(1 + it)| \leq 84.6e^{\gamma} (\log \log t)^3\), and \(|\zeta^{(3)}(1 + it)| \leq 531.5e^{\gamma} (\log \log t)^4\), for all sufficiently large \(t\).

9. The Second Proof of Proposition 2

9.1. The proof. Let \(x = (\log t)^2, \ t \geq 10, \ \sigma \geq \sigma_0 > \frac{1}{2}\).

Similar to the first formula in the proof of Lemma 2.6 of [LLS15], we have

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma + it}} \log \left( \frac{x}{n} \right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left( -\frac{\zeta'}{\zeta} \right) (\sigma + it + s) \frac{x^s}{s^2} \, ds.
\]

Moving the line of integration to the left gives that

\[(31) \quad \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma + it}} \log \left( \frac{x}{n} \right) = - (\log x) \frac{\zeta'}{\zeta} (\sigma + it) - \left( \frac{\zeta'}{\zeta} \right)' (\sigma + it)
- \sum_{\gamma} \frac{x^{\frac{1}{2} - \sigma + i(\gamma - t)}}{(\frac{1}{2} - \sigma + i(\gamma - t))^2} - \sum_{n=1}^{\infty} \frac{x^{-2n-it - \sigma}}{(2n + it + \sigma)^2} + \frac{x^{1-\sigma-it}}{(1-\sigma-it)^2},
\]

where \(\gamma\) denotes the imaginary part of nontrivial zeros of \(\zeta\) (only in this subsection 9.1, where it should not be confused with the Euler constant).

Applying \(\frac{d}{ds}\) on both sides of (31) gives

\[(32) \quad \sum_{n \leq x} \frac{(-\log n)^\ell}{n^{\sigma + it}} \Lambda(n) \log \left( \frac{x}{n} \right) = - (\log x) \left( \frac{\zeta'}{\zeta} \right) (\sigma + it) - \left( \frac{\zeta'}{\zeta} \right)' (\sigma + it + \ell+1),
\]

\(\tilde{E}\),
where by Cauchy’s integral formula for derivatives and the estimate \( \sum_{\gamma \frac{1}{x}} \frac{1}{\gamma + (\gamma - 1)x} \ll \max\{\frac{1}{b}, 1\} \log t \) \( (\forall b > 0) \), we can bound \( \tilde{E} \) by

\[
\begin{aligned}
\tilde{E} & \ll \frac{\ell!}{\epsilon^{\ell}} x^{\frac{1}{2} - \sigma - \epsilon} \frac{1}{(\frac{1}{2} - \sigma - \epsilon)^{\frac{\ell}{2}}} \log t + \frac{\ell!}{\epsilon^{\ell}} x^{-\sigma - \epsilon} + \frac{\ell!}{\epsilon^{\ell}} x^{1 - \sigma - \epsilon} \frac{1}{t^2}.
\end{aligned}
\]

Now we take \( \sigma = 1 \) and \( \epsilon = \frac{1}{\log \log t} \), and obtain

\[
\sum_{n \leq x} \frac{(-\log n)^{\ell} \Lambda(n)}{n^{1 + it}} \log \left( \frac{x}{n} \right) = - (\log x) \left( \frac{\zeta'}{\zeta} \right)^{(\ell)} (1 + it) - \left( \frac{\zeta'}{\zeta} \right)^{(\ell + 1)} (1 + it) + O_t \left( (\log \log t)^{\ell} \right).
\]

By partial summation and the prime number theorem on RH, we have

\[
\left| \sum_{n \leq x} \frac{(-\log n)^{\ell} \Lambda(n)}{n^{1 + it}} \log \left( \frac{x}{n} \right) \right| \leq \sum_{n \leq x} \frac{(\log n + 1) \Lambda(n)}{n} \log \left( \frac{x}{n} \right) = (\log x)^{\ell + 2} (\ell + 1) (\ell + 2) + O_t (\log x).
\]

Furthermore, by the triangle inequality, we get

\[
\left| \left( \frac{\zeta'}{\zeta} \right)^{(\ell + 1)} (1 + it) \right| \leq 2 (\log \log t) \left( \left( \frac{\zeta'}{\zeta} \right)^{(\ell)} (1 + it) \right) + \frac{(2 \log \log t)^{\ell + 2}}{(\ell + 1) (\ell + 2)} + O_t \left( (\log \log t)^{\ell} \right).
\]

We use the convention that the 0-th derivative of a function is the function itself. Using again the estimate \( \sum_{\gamma \frac{1}{x}} \frac{1}{\gamma + (\gamma - 1)x} \ll \max\{\frac{1}{b}, 1\} \log t \) \( (\forall b > 0) \), one can check that (33) also holds for \( \ell = 0 \).

Moreover, by Lemma 2,

\[
\left| \left( \frac{\zeta'}{\zeta} \right)^{(0)} (1 + it) \right| = \left| \frac{\zeta'}{\zeta} (1 + it) \right| \leq 2 \log t + O(1).
\]

By induction and (33), (34), we obtain

\[
\left| \left( \frac{\zeta'}{\zeta} \right)^{(\ell)} (1 + it) \right| \leq \left( 2^{\ell + 2} - \frac{2^{\ell + 1}}{\ell + 1} \right) (\log \log t)^{\ell + 1} + O_t \left( (\log \log t)^{\ell} \right).
\]

Let \( B_\ell \) be the \( \ell \)-th complete exponential Bell polynomial, then (11) and the triangle inequality imply

\[
\left| \frac{\zeta'}{\zeta} (1 + it) \right| \leq B_\ell \left( \left| \frac{\zeta'}{\zeta} (1 + it) \right|, \left| \left( \frac{\zeta'}{\zeta} \right)^' (1 + it) \right|, \left| \left( \frac{\zeta'}{\zeta} \right)^{(2)} (1 + it) \right|, \ldots, \left| \left( \frac{\zeta'}{\zeta} \right)^{(\ell - 1)} (1 + it) \right| \right).
\]

By property (10) of Bell polynomials and (35), we get

\[
\left| \frac{\zeta'}{\zeta} (1 + it) \right| \leq c(\ell) (\log \log t)^{\ell} + O_t \left( (\log \log t)^{\ell - 1} \right),
\]

where the positive constant \( c(\ell) \) is defined as

\[
c(\ell) := B_\ell \left( 2^2 - \frac{2^1}{1}, 2^3 - \frac{2^2}{2}, \ldots, 2^{\ell + 1} - \frac{2^\ell}{\ell} \right).
\]

Thus (5) and (37) give

\[
\left| \zeta^{(\ell)} (1 + it) \right| = |\zeta (1 + it)| \cdot \left| \frac{\zeta'}{\zeta} (1 + it) \right| \leq 2e^\gamma c(\ell) (\log \log t)^{\ell + 1} + O_t \left( (\log \log t)^{\ell} \right).
\]
9.2. Some Examples. From $B_1(x_1) = x_1, B_2(x_1, x_2) = x_1^2 + x_2, B_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3$ and

$$B_1(2) = 2, \quad B_2(2, 6) = 2^2 + 6 = 10, \quad B_3(2, 6, \frac{40}{3}) = 2^3 + 3 \times 2 \times 6 + \frac{40}{3} = \frac{172}{3},$$

we obtain $|\zeta'(1 + it)| \leq 4e^\gamma (\log \log t)^2 + O(\log \log t), |\zeta''(1 + it)| \leq 20e^\gamma (\log \log t)^3 + O((\log \log t)^2), \quad \text{and} \quad |\zeta'''(1 + it)| \leq \frac{344}{7}e^\gamma (\log \log t)^4 + O((\log \log t)^3).

10. Proof of Proposition 4

10.1. Proof of Proposition 4 (A). Let $A$ be a positive number, to be chosen later. By Lemma 3 and Lemma 4, we have

$$|\zeta(\sigma + it)| \leq \exp \left\{ \frac{\log 2}{2} \frac{\log t}{\log \log t} + O \left( \frac{\log t \log \log \log t}{(\log \log t)^2} \right) \right\}, \quad \text{if} \quad 0 \leq \sigma - \frac{1}{2} \leq \frac{A}{\log \log t}.$$

When $\frac{1}{2} - \frac{A}{\log \log t} \leq \sigma < \frac{1}{2}$, by the functional equation $\zeta(s) = \chi(s)\zeta(1 - s)$ and the asymptotic relation (see [Tit86, Page 95]) that $|\chi(s)| \sim \left( \frac{t}{2\pi} \right)^{\frac{1}{2} - \sigma}$, as $t \to \infty$, we have

$$|\zeta(\sigma + it)| \leq (1 + o(1)) \left( \frac{t}{2\pi} \right)^{\frac{1}{2} - \sigma} |\zeta(1 - \sigma + it)|$$

$$\leq \exp \left\{ \left( \frac{1}{2} - \sigma \right) \log \left( \frac{t}{2\pi} \right) + \frac{\log 2}{2} \frac{\log t}{\log \log t} + O \left( \frac{\log t \log \log \log t}{(\log \log t)^2} \right) \right\}$$

$$\leq \exp \left\{ \left( A + \frac{\log 2}{2} \right) \frac{\log t}{\log \log t} + O \left( \frac{\log t \log \log \log t}{(\log \log t)^2} \right) \right\}, \quad \text{if} \quad 1 - \frac{A}{\log \log t} \leq \sigma < \frac{1}{2}.$$

Set $\delta = \frac{A}{\log \log t}$ and let $A = \frac{1}{2} \epsilon$. By Cauchy’s formula we obtain that

$$\left| \zeta(t) \left( \frac{1}{2} + it \right) \right| \leq \frac{\ell!}{\delta^t} \max_{|\sigma - \frac{1}{2}| \leq \delta} \left| \zeta(\sigma + i\bar{t}) \right|$$

$$\leq \frac{\ell!}{A^t} (\log \log t)^t \exp \left\{ \left( A + \frac{\log 2}{2} \right) \frac{\log t}{\log \log t} + O \left( \frac{\log t \log \log \log t}{(\log \log t)^2} \right) \right\}$$

$$\leq \exp \left\{ \left( \frac{\log 2}{2} + \epsilon \right) \frac{\log t}{\log \log t} \right\}.$$

10.2. Proof of Proposition 4 (B). Let $A$ be a positive number, to be chosen later. By Lemma 4, we have

$$|\zeta(\sigma + it)| \leq \exp \left\{ \left( \frac{1}{2} + \frac{2\sigma - 1}{\sigma(1 - \sigma)} \right) \frac{\log t}{\log \log t} + O \left( \frac{\log t^{2 - 2\sigma}}{1 - \sigma} \frac{\log \log t^2}{(\log \log t)^2} \right) \right\}, \quad \text{if} \quad |\sigma - \sigma_0| \leq \frac{A}{\log \log t}.$$
Set \( \delta = \frac{A}{\log \log t} \). Let \( A = \frac{1}{2} \log (1 + \varepsilon) \) with \( \varepsilon \) satisfying \( \left( \frac{1}{2} + \frac{2\sigma_0 - 1}{\sigma_0(1 - \sigma_0)} \right) (1 + \varepsilon) = \frac{1}{2} + \frac{2\sigma_0 - 1}{\sigma_0(1 - \sigma_0)} + \frac{1}{2} \varepsilon \). By Cauchy’s formula we obtain that

\[
|\zeta(\sigma_0 + it)| \leq \frac{\ell!}{\delta^\ell} \max_{|\sigma - \sigma_0| \leq \delta, |t| \leq \delta} |\zeta(\sigma + it)| \\
\leq \frac{\ell!}{\delta^\ell} \left( \log \log t \right)^\ell \exp \left\{ \left( \frac{1}{2} + \frac{2\sigma_0 - 1}{\sigma_0(1 - \sigma_0)} \right) \frac{(\log t)^{2 - 2\sigma_0}(\log t)^{2\delta}}{\log \log t} + O_{\sigma_0, \ell, \varepsilon} \left( \frac{(\log t)^{2 - 2\sigma_0}(\log t)^{2\delta}}{\log \log t} \right) \right\} \\
\leq \exp \left\{ \left( \frac{1}{2} + \frac{2\sigma_0 - 1}{\sigma_0(1 - \sigma_0)} + \varepsilon \right) \frac{(\log t)^{2 - 2\sigma_0}}{\log \log t} \right\}.
\]

11. Proof of Proposition 5

11.1. Proof of Proposition 5 (A). We first prove the upper bound. Let \( \mathcal{M} \subset \mathbb{N} \) with \( |\mathcal{M}| = N \). Clearly, \( m \leq [m, n] \) and \( n \leq [m, n] \). By the inequality \( \log 2^\ell X \leq (200 \ell)^{2\ell} X^{\frac{1}{100}}, (\forall X \geq 1) \), Rankin’s trick and (8), we have

\[
\sum_{m, n \in \mathcal{M}} \sqrt{n} \log \left( \frac{m}{[m, n]} \right) \log \left( \frac{n}{[m, n]} \right) \\
\leq \sum_{m, n \in \mathcal{M}} \sqrt{\frac{m}{[m, n]}} \log 2^\ell \left( \frac{[m, n]}{m} \right) \\
\leq (200 \ell)^{2\ell} \sum_{m, n \in \mathcal{M}} \sqrt{\frac{m}{[m, n]}} \left( \frac{[m, n]}{m} \right)^{\frac{1}{100}} + \sum_{\frac{m, n \in \mathcal{M}}{[m, n] \leq N^{10}}} \sqrt{\frac{m}{[m, n]}} \log 2^\ell \left( \frac{[m, n]}{m} \right) \\
\leq (200 \ell)^{2\ell} N^{2 + 0.1 - 5} + \left( \log 2^\ell (N^{10}) \right) N \exp \left\{ (2\sqrt{2} + o(1)) \sqrt{\frac{(\log N \log_3 N)}{\log_2 N}} \right\} \\
\leq N \exp \left\{ (2\sqrt{2} + o(1)) \sqrt{\frac{(\log N \log_3 N)}{\log_2 N}} \right\}.
\]

Now we prove the lower bound. Let \( \mathcal{M} \) be defined as in [dIBT19, page 109], then \( \mathcal{M} \) is a divisor-closed set with \( |\mathcal{M}| \leq N \) and

\[
\frac{S_{\frac{1}{2}}(\mathcal{M})}{|\mathcal{M}|} \geq \exp \left\{ (2\sqrt{2} + o(1)) \sqrt{\frac{(\log N \log_3 N)}{\log_2 N}} \right\}.
\]
Since $\mathcal{M}$ is divisor-closed and $|\mathcal{M}| \leq N$, by [dlBT19, Lemma 5.1], we have

$$
\tilde{E}_1(\mathcal{M}) := \sum_{m, n \in \mathcal{M}, \text{ gcd}(m, n) \neq 1} \sqrt{\frac{(m, n)}{[m, n]}} 
\leq 2 \sum_{\substack{m, n \in \mathcal{M} \setminus \mathcal{M}^* \colon \text{ gcd}(m, n) \neq 1}} \sqrt{\frac{(m, n)}{[m, n]}} = 2 \sum_{\substack{m, n \in \mathcal{M} \setminus \mathcal{M}^* \colon \text{ gcd}(m, n) \neq 1}} \sqrt{\frac{n}{m}} \leq |\mathcal{M}| \exp \left\{ \frac{2 + o(1)}{\sqrt{2}} \log \frac{N}{\log_2 N} \right\}.
$$

We thus have

$$
\sum_{\substack{m, n \in \mathcal{M} \setminus \mathcal{M}^* \colon \text{ gcd}(m, n) \neq 1}} \sqrt{\frac{(m, n)}{[m, n]}} \geq |\mathcal{M}| \exp \left\{ \frac{2 \sqrt{2} + o(1)}{\sqrt{2}} \log \frac{N}{\log_2 N} \right\}.
$$

Now we return to the log-type GCD sum and get

$$
\sum_{m, n \in \mathcal{M}} \sqrt{\frac{(m, n)}{[m, n]}} \log^\ell \left( \frac{m}{(m, n)} \right) \log^\ell \left( \frac{n}{(m, n)} \right) 
\geq \sum_{\substack{m, n \in \mathcal{M} \setminus \mathcal{M}^* \colon \text{ gcd}(m, n) \neq 1}} \sqrt{\frac{(m, n)}{[m, n]}} \log^\ell \left( \frac{m}{(m, n)} \right) \log^\ell \left( \frac{n}{(m, n)} \right)
\geq (\log 2)^{2k} \cdot |\mathcal{M}| \cdot \exp \left\{ \frac{2 \sqrt{2} + o(1)}{\sqrt{2}} \log \frac{N}{\log_2 N} \right\}
\geq |\mathcal{M}| \exp \left\{ \frac{2 \sqrt{2} + o(1)}{\sqrt{2}} \log \frac{\log_2 |\mathcal{M}|}{\log_2 |\mathcal{M}|} \right\}.
$$

11.2. **Proof of Proposition 5 (B).** The proof for the upper bound is almost the same as in Subsection 11.1. The only difference is to use the upper bound in (7) instead of using (8).

Now we consider the proof for the lower bound. Without loss of generality, assume that $N$ is a power of 2, i.e., $N = 2^k$ for $k \in \mathbb{N}$ (since any positive integer is between two powers of 2, it suffices to prove the statement for this case).

As in [ABS15, page 1526], let $\mathcal{M}$ be the set of all square-free integers composed of the first $k$ primes (following ideas of [Gál49]), then $|\mathcal{M}| = 2^k = N$ and

$$
S_\sigma(\mathcal{M}) = 2^k \prod_{i=1}^k \left( 1 + \frac{1}{p_i^\sigma} \right) \geq N \cdot \exp \left\{ \frac{-\tilde{c}}{1-\sigma} \cdot \frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma} \right\}
$$

for some positive constant $\tilde{c}$, by the prime number theorem.

The multiplicative structure of $\mathcal{M}$ implies that

$$
\tilde{E}_\sigma(\mathcal{M}) := \sum_{\substack{m, n \in \mathcal{M} \setminus \mathcal{M}^* \colon \text{ gcd}(m, n) \neq 1}} \frac{(m, n)^\sigma}{[m, n]^\sigma} \leq 2 \sum_{\substack{m, n \in \mathcal{M} \setminus \mathcal{M}^* \colon \text{ gcd}(m, n) \neq 1}} \frac{(m, n)^\sigma}{[m, n]^\sigma} = 2 \sum_{\substack{m, n \in \mathcal{M} \setminus \mathcal{M}^* \colon \text{ gcd}(m, n) \neq 1}} \frac{n^\sigma}{m^\sigma} = 2^{k+1} \prod_{i=1}^k \left( 1 + \frac{1}{2p_i^\sigma} \right).
$$

By the prime number theorem, we have

$$
\frac{\tilde{E}_\sigma(\mathcal{M})}{S_\sigma(\mathcal{M})} \leq 2 \prod_{i=1}^k \left( 1 + \frac{1}{2p_i^\sigma} \right) = \exp \left( - \frac{1}{2} \sum_{i=1}^k \frac{1}{p_i^\sigma} + O \left( \sum_{i=1}^k \frac{1}{p_i^{2\sigma}} \right) \right) \rightarrow 0, \text{ when } k \rightarrow \infty.
$$
From this, we obtain
\[ \sum_{\substack{m, n \in \mathcal{M} \setminus \{(m, n)\} \geq 2}} |(m, n)|^\sigma \geq (1 + o(1)) N \exp\left( -\frac{c}{1 - \sigma} \cdot \frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma} \right). \]

The remaining steps can be done as in Subsection 11.1.

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