ILL-POSEDNESS ISSUE FOR THE 2D VISCOUS SHALLOW WATER EQUATIONS IN SOME CRITICAL BESOV SPACES

QIONGLEI CHEN, YAO NIE

Abstract. We study the Cauchy problem of the 2D viscous shallow water equations in some critical Besov spaces $\dot{B}^{\frac{q}{p},r}_{p,1}(\mathbb{R}^2) \times \dot{B}^{\frac{q}{p},r-\frac{1}{p},q}_{p,1}(\mathbb{R}^2)$. As is known, this system is locally well-posed for large initial data as well as globally well-posed for small initial data in $\dot{B}^{\frac{q}{p},r}_{p,1}(\mathbb{R}^2) \times \dot{B}^{\frac{q}{p},r-\frac{1}{p},q}_{p,1}(\mathbb{R}^2)$ for $p < 4$ and ill-posed in $\dot{B}^{\frac{q}{p},r}_{p,1}(\mathbb{R}^2) \times \dot{B}^{\frac{q}{p},r-\frac{1}{p},q}_{p,1}(\mathbb{R}^2)$ for $p > 4$. In this paper, we prove that this system is ill-posed for the critical case $p = 4$ in the sense of “norm inflation”. Furthermore, we also show that the system is ill-posed in $\dot{B}^{\frac{q}{p},r}_{1,1}(\mathbb{R}^2) \times \dot{B}^{\frac{q}{p},r-\frac{1}{p},q}_{4,1}(\mathbb{R}^2)$ for any $q \neq 2$.

1. Introduction

The 2D viscous shallow water equations read as follows:

\[
\begin{cases}
\partial_t h + \text{div}(hu) = 0, & t > 0, x \in \mathbb{R}^2, \\
h(\partial_t u + u \cdot \nabla u) - \nu \nabla \cdot (h \nabla u) + h \nabla h = 0, & t > 0, x \in \mathbb{R}^2, \\
h(0, x) = h_0(x), u(0, x) = u_0(x), & x \in \mathbb{R}^2.
\end{cases}
\]

where $h(t, x)$ denotes the height of fluid surface, $u(t, x) = (u_1(t, x), u_2(t, x))$ represents the horizontal velocity field and $\nu > 0$ is the viscous coefficient.

The well-posedness of system (1.1) has been widely investigated during the past 30 years. Readers can refer to [2] for more details. Bui [3] proved the local existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem for system (1.1) with initial data in $C^{2+\alpha}$. Kloeden [12] and Sundbye [20] independently showed global existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem in Sobolev spaces for small initial data. Sundbye [21] established a global existence and uniqueness theorem of strong solutions for the Cauchy problem for equations (1.1) with small initial data. Wang and Xu [22] got the local existence of solution for all size initial data and global existence for small initial data $u_0$ if $h_0 - \bar{h}_0$ is small enough in $H^{2+s}$ for any $s > 0$. Chen, Miao and Zhang [4] introduced some kind of weighted Besov space to prove the existence and uniqueness of the solutions to a more general diffusion system with low regularity assumptions on the initial data as well as the initial height bounded away from zero.

For more results on well-posedness of system (1.1) in Besov spaces, readers can refer to [14, 15, 16].

For convenience, we take $\bar{h}_0 = 1$, $\nu = 1$, and substitute $h$ by $1 + h$ in equations (1.1), then it yields that

\[
\begin{cases}
\partial_t h + \text{div}(u + u \cdot \nabla h) = -h \text{div} u, & t > 0, x \in \mathbb{R}^2, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla h = \nabla (\ln(1 + h)) \cdot \nabla u, & t > 0, x \in \mathbb{R}^2, \\
h(0, x) = h_0(x), u(0, x) = u_0(x), & x \in \mathbb{R}^2.
\end{cases}
\]
With respect to compressible Navier-Stokes equations under barotropic condition, which possess similar structure of system (1.2), there has a vast mathematical literature on well-posedness and ill-posedness results. In terms of the critical Besov spaces, it has been proved that compressible Navier-Stokes system is well-posed in critical Besov spaces \( \dot{B}_{p,1}^{\frac{d-1}{2}}(\mathbb{R}^d) \times \dot{B}_{p,1}^{\frac{d-1}{2}}(\mathbb{R}^d) \) for \( 1 \leq p < 2d \) (see [5, 8, 9]). And this system is ill-posed for \( p \geq 2d \) (see [6, 11]). Following methods in [4, 5], one can obtain that system (1.2) is local well-posed for large initial data and global well-posed for small initial data in \( \dot{B}_{p,1}^{\frac{d}{2}}(\mathbb{R}^2) \times \dot{B}_{p,1}^{\frac{d}{2}-1}(\mathbb{R}^2) \) for \( 1 \leq p < 4 \). Recently, Li, Hong and Zhu [13] proved system (1.2) is ill-posed for \( p > 4 \). However, the question that whether system (1.2) is ill-posed or not for the endpoint case \( p = 4 \) has not been answered.

In this paper, we aim to prove the ill-posedness of shallow water equations (1.2) for the endpoint case \( p = 4 \). Motivated by [6, 11], we construct initial data in the Schwartz class which are arbitrarily small in \( \dot{B}_{4,1}^{\frac{1}{2}}(\mathbb{R}^2) \times \dot{B}_{4,1}^{\frac{1}{2}-1}(\mathbb{R}^2) \), meanwhile the corresponding solutions are arbitrarily large in \( \dot{B}_{4,1}^{\frac{1}{2}}(\mathbb{R}^2) \times \dot{B}_{4,1}^{\frac{1}{2}-1}(\mathbb{R}^2) \) after an arbitrarily short time. This phenomenon shows that the solution map \( (h_0,u_0) \mapsto (h[h_0],u[u_0]) \) is discontinuous in \( \dot{B}_{4,1}^{\frac{1}{2}}(\mathbb{R}^2) \times \dot{B}_{4,1}^{\frac{1}{2}-1}(\mathbb{R}^2) \). Moreover, we observe that the special nonlinear mechanism and \( L^2(\mathbb{R}^2) \hookrightarrow \dot{B}_{4,1}^{\frac{1}{2}-1}(\mathbb{R}^2) \) lead to the second iteration is continuous in \( \dot{B}_{4,1}^{\frac{1}{2}}(\mathbb{R}^2) \). Therefore, we generalize the ill-posedness results and show that system (1.2) is ill-posed in \( \dot{B}_{4,1}^{\frac{1}{2}}(\mathbb{R}^2) \times \dot{B}_{4,q}^{\frac{1}{2}}(\mathbb{R}^2) \) for any \( q \neq 2 \). Our main result is as follows:

**Theorem 1.1.** For \( 1 \leq q < 2 \), system (1.2) is ill-posed in critical spaces \( \dot{B}_{4,1}^{\frac{1}{2}}(\mathbb{R}^2) \times \dot{B}_{4,q}^{\frac{1}{2}}(\mathbb{R}^2) \). More precisely, for any \( \delta > 0 \), there exists an initial data \( u_0 \in \dot{B}_{4,q}^{\frac{1}{2}} \cap S \) satisfying

\[
\|u_0\|_{\dot{B}_{4,q}^{\frac{1}{2}}} \leq \delta,
\]

such that the corresponding solution \( (h, u) \) to system (1.2) satisfies

\[
\|u(\cdot,t)\|_{\dot{B}_{4,q}^{\frac{1}{2}}} > \frac{1}{\delta}, \quad \text{for some } 0 < t < \delta.
\]

**Remark 1.2.** Generally speaking, researchers are focused on the well-posedness and ill-posedness of system (1.2) in critical Besov spaces \( \dot{B}_{p,1}^{\frac{d}{2}}(\mathbb{R}^2) \times \dot{B}_{p,1}^{\frac{d}{2}-1}(\mathbb{R}^2) \). Our result not only shows the ill-posedness of system (1.2) in endpoint case \( \dot{B}_{4,1}^{\frac{1}{2}}(\mathbb{R}^2) \times \dot{B}_{4,1}^{\frac{1}{2}-1}(\mathbb{R}^2) \), but also generalizes the ill-posed results in more critical Besov spaces.

**Theorem 1.3.** For \( q > 2 \), system (1.2) is ill-posed in critical spaces \( \dot{B}_{4,1}^{\frac{1}{2}}(\mathbb{R}^2) \times \dot{B}_{4,q}^{\frac{1}{2}}(\mathbb{R}^2) \). More precisely, for any \( \delta > 0 \), there exists an initial data \( u_0 \in \dot{B}_{4,q}^{\frac{1}{2}} \cap S \) satisfying

\[
\|u_0\|_{\dot{B}_{4,q}^{\frac{1}{2}}} \leq \delta,
\]

such that the corresponding solution \( (h, u) \) to system (1.2) satisfies

\[
\|u(\cdot,t)\|_{\dot{B}_{4,q}^{\frac{1}{2}}} > \frac{1}{\delta}, \quad \text{for some } 0 < t < \delta.
\]
Remark 1.4. By our method, the condition \( q \neq 2 \) is sharp, because the second iteration, a bilinear operator,

\[(1.3) \quad B(f, g) := - \int_0^t e^{(t-s)\Delta} (e^{s\Delta} f \cdot \nabla e^{s\Delta} g + \nabla) \int_0^s \text{div} e^{r\Delta} f \, dt \cdot \nabla e^{r\Delta} g) \, ds \]

satisfies that there exists an absolute constant \( C \) such that for any \( f, g \in \dot{B}^{\frac{q}{2}}_{4,2}(\mathbb{R}^2) \), and \( t \geq 0 \),

\[ \|B(f, g)(t)\|_{\dot{B}^{\frac{q}{2}}_{4,2}} \leq C\|f\|_{\dot{B}^{\frac{q}{2}}_{4,2}} \|g\|_{\dot{B}^{\frac{q}{2}}_{4,2}}. \]

See Appendix for the proof. Therefore, extending our approach to the case \( q = 2 \) would require new ideas and we will consider it later.

Remark 1.5. In [11], the second iteration of compressible Navier-Stokes equations is

\[ \tilde{I}[f, g] = - \int_0^t e^{(t-s)\Delta} \{ e^{s\Delta} f \nabla e^{s\Delta} g + \frac{B}{A} (e^{s\Delta} f - f) \nabla e^{s\Delta} g \} \, ds, \]

which is different from \( B(f, g) \). In fact, their methods imply compressible Navier-Stokes equations is ill-posedness in \( \dot{B}^{\frac{q}{2}}_{p,1} \times \dot{B}^{\frac{q}{2}}_{p,1} \) for \( q < 2 \). We show system (1.2) is also ill-posed for \( q > 2 \), which implies that the mechanism between compressible of Navier-Stokes and system (1.2) is different.

2. Preliminaries

Lemma 2.1 ([1]). Let \( 1 \leq p \leq p_1 \leq \infty \) and \( s \in (-2 \min\{\frac{1}{p_1}, 1 - \frac{1}{p}\}, 1 + \frac{2}{p_1}] \). Let \( v \) be a vector field such that \( \nabla v \in L^1_T(\dot{B}^{\frac{2}{p_1}}_{p_1,1}(\mathbb{R}^2)) \). There exists a constant \( C \) depending on \( p, s, p_1 \) such that all solutions \( f \in L^\infty_T(\dot{B}^{s}_{p_1,1}(\mathbb{R}^2)) \) of the transport equation

\[ \partial_t f + v \cdot \nabla f = g, \quad f(0, x) = f_0(x), \]

with initial data \( f_0 \in \dot{B}^s_{p_1,1}(\mathbb{R}^2) \) and \( g \in L^1_T(\dot{B}^{s}_{p_1,1}(\mathbb{R}^2)) \), we have, for \( t \in [0, T] \),

\[ \|f\|_{L^\infty_T(\dot{B}^s_{p_1,1})} \leq e^{CV_1(t)} \left( \|f_0\|_{\dot{B}^s_{p_1,1}} + \int_0^t e^{-CV_1(\tau)} \|g(\tau)\|_{\dot{B}^s_{p_1,1}} \, d\tau \right), \]

where \( V_1(t) = \int_0^t \|\nabla v\|_{\dot{B}^{s_1}_{p_1,1}(\mathbb{R}^2)} \, ds \). Particularly, if \( \nabla v \in L^1_T(\dot{B}^0_{\infty,1}(\mathbb{R}^2)) \) for some \( \varepsilon > 0 \), we have

\[ \|f\|_{L^\infty_T(\dot{B}^s_{p_1,1})} \leq e^{CV(t)} \left( \|f_0\|_{\dot{B}^0_{p_1,1}} + \int_0^t \|g(\tau)\|_{\dot{B}^0_{p_1,1}} \, d\tau \right), \]

where \( V(t) = \int_0^t \|\nabla v\|_{\dot{B}^0_{\infty,1}} + \|\nabla v\|_{\dot{B}^0_{\infty,1}} \, ds \).

Lemma 2.2 ([9]). Let \( s \in \mathbb{R} \) and \( 1 \leq r_1, r_2, p, q \leq \infty \) with \( r_2 \leq r_1 \). Consider the heat equation

\[ \partial_t u - \Delta u = f, \quad u(0, x) = u_0(x). \]
Assume that \( u_0 \in \dot{B}^s_{p,q}(\mathbb{R}^2) \) and \( f \in L^r_T(\dot{B}^{s-2+\frac{2}{p}}_{p,q}(\mathbb{R}^2)) \). Then the above equation has a unique solution \( u \in L^r_T(\dot{B}^{s-2+\frac{2}{p}}_{p,q}(\mathbb{R}^2)) \) satisfying
\[
\|u\|_{L^r_T(\dot{B}^{s-2+\frac{2}{p}}_{p,q}(\mathbb{R}^2))} \leq C\left(\|u_0\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} + \|f\|_{L^r_T(\dot{B}^{s-2+\frac{2}{p}}_{p,q}(\mathbb{R}^2))}\right).
\]

**Lemma 2.3** ([1]). Let \( s > 0 \) and \( 1 \leq r, p \leq \infty \). Assume \( F \in W^{[\sigma]+3}_\text{loc}((\mathbb{R}) \) with \( F(0) = 0 \).
Then for any \( f \in L^\infty \cap \dot{B}^{s-1}_{p,1} \), we have
\[
\|F(f)\|_{L^r_T(\dot{B}^{s-1}_{p,1})} \leq C(1 + \|f\|_{L^\infty_T(\mathbb{R}^n)})^{[\sigma]+2}\|f\|_{L^r_T(\dot{B}^{s+1}_{p,1})}.
\]

**Lemma 2.4** ([11]). Define \( \dot{B}^\sigma_{2,1} := \dot{B}^\sigma_{2,1} \cap \dot{B}^\sigma_{\infty,1} \). For \( \sigma > 0 \) and \( 0 < \varepsilon < 1 \), we have
\[
\|uv\|_{\dot{B}^\sigma_{2,1}} \leq C\min\{\|u\|_{\dot{B}^\sigma_{2,1}}, \|v\|_{\dot{B}^\sigma_{2,1}}\}, \|u\|_{\dot{B}^0_{2,1}}, \|v\|_{\dot{B}^0_{2,1}} \}
\leq C\min\{\|u\|_{\dot{B}^\sigma_{2,1}}, \|v\|_{\dot{B}^\sigma_{2,1}}\}, \|u\|_{\dot{B}^0_{2,1}}, \|v\|_{\dot{B}^0_{2,1}} \}.
\]

**Lemma 2.5** ([7]). Let \( m \) be a smooth function satisfying that
\[
|\partial_{\xi,\eta}^\alpha m(\xi, \eta)| \leq C_\alpha (|\xi| + |\eta|)^{-|\alpha|}
\]
for all multi-index \( \alpha \). Assume \( p, p_1, p_2 \in (1, \infty)^2 \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), then
\[
\left\|F^{-1}\left[\int_{\mathbb{R}^d} m(\xi - \eta, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\eta\right]\right\|_{L^p} \leq C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.
\]

### 3. Ill-Posedness

In this section, we construct special initial data \((h_0, u_0)\) and obtain “norm inflation” of the corresponding solution. In order to show the local existence and uniqueness of system (1.2) for given initial data, firstly we provide the following proposition which involves in some properties we needed later.

**Proposition 3.1.** Fixed some \( 0 < \varepsilon < \frac{1}{4} \), let \( \delta > 0 \) such that \( 5\varepsilon + 3\delta < 1 \) and \( N \) be a large enough integer, if initial data \((h_0, u_0) \in \dot{B}^0_{2,\infty,1} \times \dot{B}^0_{2,\infty,1}\) is Schwarz function and satisfies
\[
\|h_0\|_{\dot{B}^0_{2,\infty,1}}, \|h_0\|_{\dot{B}^1_{2,\infty,1}} \leq C 2^{(\frac{3}{2} + \frac{1}{\delta})N}, \|u_0\|_{\dot{B}^0_{2,\infty,1}} \leq C 2^{(\frac{3}{2} + \frac{1}{\delta})N}
\]
there exist constants \( C_0 \) and \( N_0 \) such that for \( N > N_0 \) and \( T = (\ln N)^{-1}2^{2N} \), system (1.2) has a unique local solution \((h, u)\) associated with initial data \((h_0, u_0)\) satisfying
\[
\begin{align*}
&h \in C([0, T], \dot{B}^0_{2,\infty,1}) \cap L^\infty([0, T], \dot{B}^1_{2,\infty,1}), \\
u \in C([0, T], \dot{B}^0_{2,\infty,1}) \cap L^1([0, T], \dot{B}^2_{2,\infty,1}),
\end{align*}
\]
and the following estimates hold
\[
\begin{align*}
\|h\|_{L^\infty_T(\dot{B}^0_{2,\infty,1})} &\leq 2C_0^2e^{C_02^{(-\frac{1}{2} + \frac{1}{\delta})N}}, \\
\|h\|_{L^\infty_T(\dot{B}^1_{2,\infty,1})} &\leq 2C_0^2e^{(\frac{3}{2} + \frac{1}{\delta})N}, \\
\|u\|_{L^\infty_T(\dot{B}^0_{2,\infty,1})} + \|u\|_{L^1_T(\dot{B}^2_{2,\infty,1})} &\leq C_02^{(\frac{3}{2} + \frac{1}{\delta})N}.
\end{align*}
\]
Proof. First Step: Constructing Approximate Solutions
Starting from \((h^0, u^0) = (0, 0)\) and we define the approximate sequence \((h^n, u^n)_{n \in \mathbb{N}}\) of equations (1.2) by solving the following linear system:

\[
\begin{aligned}
\partial_t h^{n+1} + u^n \cdot \nabla h^{n+1} &= -(1 + h^n) \text{ div } u^n, & t > 0, x \in \mathbb{R}^2, \\
\partial_t u^{n+1} - \Delta u^{n+1} &= -u^n \cdot \nabla u^n - \nabla h^n + \nabla (\ln(1 + h^n)) \cdot \nabla u^n, & t > 0, x \in \mathbb{R}^2, \\
h^{n+1}(0, x) = h_0, u^{n+1}(0, x) = u_0(x), & x \in \mathbb{R}^2.
\end{aligned}
\]

(3.2)

Second Step: Uniform Bounds
It is easy to check that \((h^1, u^1) = (h_0, e^{t\Delta} u_0)\) and \(h_1\) satisfies estimates in (3.1). For \(u_1\), there exists a constant \(C_0 > 1\) such that

\[
\|u^1\|_{L^\infty_T(B^0_1(0, 1))} + \|u^1\|_{L^1_T(B^2_2(0, 1))} \leq C \|u_0\|_{\dot{B}^0_2(0, 1)} \leq C_0^2(\frac{1}{2} + \frac{4}{5} + \frac{1}{2})(1 + t)N.
\]

Assume estimates (3.1) hold for \((h^n, u^n)\), next we check it for \((h^{n+1}, u^{n+1})\). With aid of Lemma 2.1 and Lemma 2.4, we obtain that

\[
\|h^{n+1}\|_{L^\infty_T(\dot{B}^0_2(0, 1))} \leq e^{C \|u^n\|_{L^1_T(\dot{B}^1_1(0, 1) \cap \dot{B}^{1+\frac{\epsilon}{2}}_\infty(0, 1))}} \int_0^T \left( \|h^n\|_{\dot{B}^0_2(0, 1)} \|u^n\|_{\dot{B}^{1+\frac{\epsilon}{2}}_2(0, 1)} + \|u^n\|_{\dot{B}^{1+\frac{\epsilon}{2}}_2(0, 1)} \right) dt.
\]

Noting the fact that

\[
\|u^n\|_{L^1_T(\dot{B}^1_1(0, 1) \cap \dot{B}^{1+\frac{\epsilon}{2}}_\infty(0, 1))} \leq T^{\frac{1}{2}} \|u^n\|_{L^2_T(\dot{B}^1_1(0, 1))} + T^{\frac{1-\frac{\epsilon}{2}}{2}} \|u^n\|_{L_T^2(\dot{B}^{1+\frac{\epsilon}{2}}_\infty(0, 1))}
\]

\[
\leq C_0^2(\frac{1}{2} + \frac{4}{5} + \frac{1}{2})(1 + t)N + C_0^2(\frac{1}{2} + \frac{4}{5} + \frac{1}{2})(1 + t)N \cdot (\ln N)^{-\frac{1-\frac{\epsilon}{2}}{2}} 2^{(1+\frac{\epsilon}{2})N}
\]

\[
\leq 2C_0(\ln N)^{-\frac{1-\frac{\epsilon}{2}}{2}} 2^{(1+\frac{\epsilon}{2})N},
\]

and

\[
\int_0^T \|h^n\|_{\dot{B}^0_2(0, 1)} \|u^n\|_{\dot{B}^{1+\frac{\epsilon}{2}}_2(0, 1)} dt \leq e^{C \|u^n\|_{L^1_T(\dot{B}^1_1(0, 1) \cap \dot{B}^{1+\frac{\epsilon}{2}}_\infty(0, 1))}} \|h^n\|_{L^\infty_T(\dot{B}^0_2(0, 1))} \|u^n\|_{L_T^2(\dot{B}^{1+\frac{\epsilon}{2}}_2(0, 1))}
\]

\[
\leq 2C_0^3 e^{C_0(\ln N)^{-\frac{1-\frac{\epsilon}{2}}{2}} 2^{(\frac{1}{2} + \frac{\epsilon}{2})N}},
\]

it yields that for \(3\delta + 5\varepsilon < 1\), there exists a \(N_0\) such that for \(N > N_0\),

\[
\|h^{n+1}\|_{L^\infty_T(\dot{B}^0_2(0, 1))} \leq e^{C \|u^n\|_{L^1_T(\dot{B}^1_1(0, 1) \cap \dot{B}^{1+\frac{\epsilon}{2}}_\infty(0, 1))}} \|u^n\|_{L^\infty_T(\dot{B}^0_2(0, 1))} \|u^n\|_{L_T^2(\dot{B}^{1+\frac{\epsilon}{2}}_2(0, 1))}
\]

\[
\leq e^{C \|u^n\|_{L^1_T(\dot{B}^1_1(0, 1) \cap \dot{B}^{1+\frac{\epsilon}{2}}_\infty(0, 1))}} \left( 2C_0^2 e^{C_0(\ln N)^{-\frac{1-\frac{\epsilon}{2}}{2}} 2^{(\delta+\frac{\epsilon}{2})N}} + C_0(\ln N)^{-\frac{1}{2}} 2^{(\frac{1}{2} + \frac{\epsilon}{2})N} \right)
\]

\[
\leq 2C_0^2 e^{C_0(\ln N)^{-\frac{1-\frac{\epsilon}{2}}{2}} 2^{(\delta+\frac{\epsilon}{2})N}}.
\]

Similarly, for \(3\delta + 5\varepsilon < 1\), \(N > N_0\) and \(C_0 > C\), we have

(3.3)

\[
\|h^{n+1}\|_{L^\infty_T(\dot{B}^1_1(0, 1))}
\]

\[
\leq e^{C \|u^n\|_{L^1_T(\dot{B}^1_1(0, 1))}} \int_0^T \left( \|h^n\|_{\dot{B}^1_1(0, 1)} \|u^n\|_{\dot{B}^1_1(0, 1)} + \|u^n\|_{\dot{B}^2_2(0, 1)} + \|h^n\|_{\dot{B}^0_2(0, 1)} \|u^n\|_{\dot{B}^2_2(0, 1)} \right) dt
\]

\[
\leq e^{C \|u^n\|_{L^1_T(\dot{B}^1_1(0, 1) \cap \dot{B}^{1+\frac{\epsilon}{2}}_\infty(0, 1))}} \left( 2C_0^3 e^{C_0(\ln N)^{-\frac{1-\frac{\epsilon}{2}}{2}} 2^{(\frac{1}{2} + \frac{\epsilon}{2})N}} + 2C_0^2 e^{C_0(\ln N)^{-\frac{1-\frac{\epsilon}{2}}{2}} 2^{(\frac{1}{2} + \frac{\epsilon}{2})N}} \right) \leq C_0^2 e^{C_0(\ln N)^{-\frac{1-\frac{\epsilon}{2}}{2}} 2^{(\frac{1}{2} + \frac{\epsilon}{2})N}}.
\]

Using Lemma 2.2, we obtain that

\[
\|u^{n+1}\|_{L^\infty_T(\dot{B}^0_2(0, 1))} + \|u^{n+1}\|_{L^1_T(\dot{B}^2_2(0, 1))}
\]
there exists a subsequence of $\{h_n\}_{n \geq 0}$ such that, for all $N \geq N_0$, 
\[(h^n, u^n)_{n \geq N_0} \text{ satisfies uniformly estimates (3.1)}.\]

**Third Step: Time Derivatives**

Furthermore, the sequence $(h^n, u^n)_{n \geq 0}$ is uniformly bounded in 

\[ C^\frac{1}{2}(0, T; B^0_{[2,\infty],1}) \times C^2_T([0, T]; B^{-\varepsilon}_{[2,\infty],1}). \]

Indeed, $(h^n, u^n)_{n \geq 0}$ possesses uniformly bounds (3.1) and 

\[ \partial_t h^{n+1} = -u^n \cdot \nabla h^{n+1} - (1 + h^n) \text{ div } u^n, \]

the right-hand side is uniformly bounded in $L^2_T(B^0_{[2,\infty],1})$.

As regards to $(u^n)_{n \geq 0}$, this follows from the fact that 

\[ \partial_t u^{n+1} = \Delta u^{n+1} - u^n \cdot \nabla u^n - \nabla h^n + \nabla (\ln(1 + h^n)) \cdot \nabla u^n. \]

by using the fact that $(u^n)_{n \geq 0}$ and $(h^n)_{n \geq 0}$ are uniformly bounded in $L^\infty_T(\hat{B}^0_{[2,\infty],1}) \cap L^1_T(\hat{B}^2_{[2,\infty],1})$ and $L^\infty_T(\hat{B}^0_{[2,\infty],1} \cap \hat{B}^1_{[2,\infty],1})$, we easily deduce that the four terms on the right-hand side are in $L^{2-\varepsilon}_T(B^{-\varepsilon}_{[2,\infty],1})$.

**Fourth Step: Convergence and Uniqueness**

Let $\{\phi_m\}_{m \in \mathbb{N}}$ be a sequence of smooth functions with values in $[0, 1]$, supported in the ball $B(0, m + 1)$ and equal to 1 on $B(0, m)$. Taking advantage the uniformly estimates on $(h^n, u^n)_{n \geq 0}$, by Aubin-Lions lemma and the Cantor diagonal process, we obtain that there exists a subsequence of $(h^n, u^n)_{n \geq 0}$ (still denoted by $(h^n, u^n)_{n \geq 0}$) such that, for all $m \in \mathbb{N}$,

\[ (\phi_m h^n, \phi_m u^n) \to (\phi_m h, \phi_m u) \text{ in } C([0, T]; B^0_{[2,\infty],1} \times B^{-\varepsilon}_{[2,\infty],1}). \]

Therefore, $(h^n, u^n)$ tends to $(h, u)$ in $C([0, T] \times \mathbb{R}^2)$. Following the argument in [8], it is routine to verify that $(h, u)$ satisfies system (1.2) and the solution is continuous in terms of time in $\hat{B}^0_{[2,\infty],1}(\mathbb{R}^2) \times \hat{B}^0_{[2,\infty],1}(\mathbb{R}^2)$. Readers can refer to [4] to prove the uniqueness. □
Proposition 3.2. Let $\varepsilon$ and $\delta$ be defined in Proposition 3.1. The solution $(h, u)$ obtained in Proposition 3.1 satisfies the following estimates:

\begin{equation}
\|h + \int_0^t \text{div} \, U_0 \, ds\|_{L^2(T(B^1_{[2,\infty]}, 1)} \leq C2^{(\varepsilon + \delta - 1)N}.
\end{equation}

\begin{equation}
\|u - U_0\|_{L^2(T(B^0_{[2,\infty]}, 1)} + \|u - U_0\|_{L^2(T(B^2_{[2,\infty]}, 1)} \leq C2^{(\varepsilon + \delta)N}.
\end{equation}

\textbf{Proof.} Taking advantage of Lemma 2.2 and then we obtain

\begin{equation*}
\|u - U_0\|_{L^2(T(B^0_{[2,\infty]}, 1)} + \|u - U_0\|_{L^2(T(B^2_{[2,\infty]}, 1)} \\
\leq C(\|u \cdot \nabla u\|_{L^1_2(B^0_{[2,\infty]}, 1)} + \|\nabla h\|_{L^1_2(B^0_{[2,\infty]}, 1)} + \|\nabla (1 + h) \cdot \nabla u\|_{L^1_2(B^0_{[2,\infty]}, 1)}) \\
\leq CT \frac{1}{2}\|u\|_{L^2(T(B^0_{[2,\infty]}, 1))}^2 + CT\|h\|_{L^2(T(B^1_{[2,\infty]}, 1)} \\
+ CT \frac{1}{2}(1 + \|h\|_{L^\infty})^2 \|h\|_{L^2(T(B^1_{[2,\infty]}, 1))} \\
\leq C2^{(\varepsilon + \delta)N}.
\end{equation*}

Based on the above inequality, it holds that

\begin{equation*}
\|h + \int_0^t \text{div} \, U_0 \, ds\|_{L^2(T(B^0_{[2,\infty]}, 1)} \leq \left\| \int_0^t (\text{div}(u - U_0) + \text{div}(hu)) \, ds\right\|_{L^2(T(B^0_{[2,\infty]}, 1))} \\
\leq \|u - U_0\|_{L^2(T(B^1_{[2,\infty]}, 1)} + \|hu\|_{L^2(T(B^1_{[2,\infty]}, 1)} \leq C2^{(\varepsilon + \delta - 1)N}.
\end{equation*}

\textbf{3.1 Proof of Theorem 1.3 for $1 \leq q < 2$.}

Let $(\varphi_j)_{j \in \mathbb{Z}}$ be the Littlewood-Paley convolution functions. We introduce

$$
\Phi_{j, N} = \varphi(2^j(x - 2^j [2N - 1] e_1)),
$$

and initial data $(h_0, u_0)$ is defined by:

$$
h_0 = 0, \quad u_0 = \left( \frac{2^j}{N^\frac{3}{2} \ln N} \sum_{-\delta N \leq j \leq 0} 2^j \Phi_{j, N} \sin(2^N x_1), \frac{2^j}{N^\frac{3}{2} \ln N} \sum_{-\delta N \leq j \leq 0} 2^j \Phi_{j, N} \cos(2^N x_1) \right),
$$

where $\delta$ is consistent with that in Proposition 3.1. It is easy to check that $\text{supp} \, \hat{u}_0(\xi) \subset \{\xi \in \mathbb{R}^2 | 2^N - 1 \leq |\xi| \leq 2^{N+1}\}$. Hence

$$
\|u_0\|_{B^{\frac{1}{2}}_{4, 4}} \leq \frac{C}{N^\frac{3}{2} \ln N} \left\| \sum_{-\delta N \leq j \leq 0} 2^j \Phi_{j, N} \right\|_{L^4}.
$$

An easy computation yields that

$$
\left\| \sum_{-\delta N \leq j \leq 0} 2^j \Phi_{j, N} \right\|_{L^4}^4 \leq \sum_{-\delta N \leq j \leq 0} \int_{\mathbb{R}^2} 2^{2j} |\Phi_{j, N}|^4 \, dx + \sum_{(j_1, \ldots, j_4) \in \Lambda} \int_{\mathbb{R}^2} 2^{2j_1 + \ldots + j_4} \Phi_{j_1, N} \cdots \Phi_{j_4, N} \, dx,
$$

where the set $\Lambda$ is defined by

$$
\Lambda = \{(j_1, \ldots, j_4) \in [-N, 0]^4 \cap \mathbb{N}^4 | \exists 1 \leq k, \ell \leq 4 \text{ s.t. } -N \leq j_k \neq j_\ell \leq 0\}.
$$
From the definition of $\Phi_{j,N}$, we obtain that

\[ (3.7) \quad \sum_{-\delta N \leq j \leq 0} \int_{\mathbb{R}^2} 2^{2j} |\Phi_{j,N}|^4 \, dx = \sum_{-\delta N \leq j \leq 0} \int_{\mathbb{R}^2} |\varphi_0|^4 \, dx \leq CN. \]

Due to $\Phi_{j,N} \in \mathcal{S}$, for any $k > 0$, there exists a constant $C_k$ such that

\[ |\Phi_{j,N}| \leq C_k(1 + 2^j|x - 2^{j|+2N} e_1|)^{-k}. \]

Assume $j_1 \neq j_2$, from the above inequality, we have that

\[ \int_{\mathbb{R}^d} \left| \Phi_{j_1,N} \cdots \Phi_{j_4,N} \right| \, dx \leq C_k \int_{\mathbb{R}^d} \frac{1}{(1 + 2^{j_1}|x - 2^{j_1|+2N} e_1|)^k (1 + 2^{j_2}|x - 2^{j_2|+2N} e_1|)^k} \, dx \]

\[ \leq C_k \left( \int_{|x - 2^{j_1|+2N} e_1| \leq \frac{1}{2} 2^{N}} \frac{1}{(1 + 2^{j_1}|x - 2^{j_1|+2N} e_1|)^k (1 + 2^{j_2}|x - 2^{j_2|+2N} e_1|)^k} \, dx \right. 
\]

\[ + \int_{|x - 2^{j_1|+2N} e_1| > \frac{1}{2} 2^{N}} \frac{1}{(1 + 2^{j_1}|x - 2^{j_1|+2N} e_1|)^k (1 + 2^{j_2}|x - 2^{j_2|+2N} e_1|)^k} \, dx \). \]

Noting the fact that $||j_1| - |j_2|| \geq 1$, for $|x - 2^{j_1|+2N} e_1| \leq \frac{1}{2} 2^{2N}$, by triangle inequality we obtain that

\[ 2^{j_2}|x - 2^{j_2|+2N} e_1| \geq 2^{j_2}|2^{j_1|+2N} e_1 - 2^{j_2|+2N} e_1| = \frac{1}{2} 2^{j_2} 2^{2N} \]

\[ \geq 2^{j_2} 2^{2N} - \frac{1}{2} 2^{j_2} 2^{2N} \geq \frac{1}{2} 2^{j_2} 2^{2N}. \]

Therefore, taking $k > 2$, one yields that

\[ \int_{|x - 2^{j_1|+2N} e_1| \leq \frac{1}{2} 2^{2N}} \frac{1}{(1 + 2^{j_1}|x - 2^{j_1|+2N} e_1|)^k (1 + 2^{j_2}|x - 2^{j_2|+2N} e_1|)^k} \, dx \]

\[ \leq C 2^{-(j_2+2N-1)k} \int_{|x - 2^{j_1|+2N} e_1| \leq \frac{1}{2} 2^{2N}} \frac{1}{(1 + 2^{j_1}|x - 2^{j_1|+2N} e_1|)^k} \, dx \]

\[ \leq C 2^{-(j_2+2N-1)k} 2^{-2j_1} \leq C. \]

Similarly, we have

\[ \int_{|x - 2^{j_1|+2N} e_1| > \frac{1}{2} 2^{2N}} \frac{1}{(1 + 2^{j_1}|x - 2^{j_1|+2N} e_1|)^k (1 + 2^{j_2}|x - 2^{j_2|+2N} e_1|)^k} \, dx \]

\[ \leq C 2^{-(j_1+2N-1)k} 2^{-2j_2} \leq C. \]

Hence, by (3.7) and the above two inequalities, one gets

\[ \|u_0\|_{B_{4,q}} \leq \frac{C}{N+\ln N} \left( N + \sum_{(j_1, \cdots, j_4) \in \Lambda} 2^{\frac{1}{2}(j_1 + \cdots + j_4)} \right)^{\frac{1}{2}} \leq \frac{C}{\ln N}. \]

Now we decompose the solution $u$ into three parts:

\[ (3.8) \quad u = U_0 + U_1 + U_2, \]
where \( U_0 = e^{t\Delta}u_0 \), and

\[
U_1 := -\int_0^t e^{(t-s)\Delta} (U_0 \cdot \nabla U_0 + \nabla \int_0^s \text{div} U_0(\tau) \, d\tau \cdot \nabla U_0(s) - \nabla h_0 \cdot \nabla e^{s\Delta}u_0) \, ds.
\]

Next, we estimate \( \|U_i(t_0)\|_{B^{\frac{1}{2}}_{4,q}} \) \((i = 0, 1, 2)\) respectively, where \( t_0 = (\ln N)^{-1}2^{-2N} \).

**Estimates on \( \|U_0(t_0)\|_{B^{\frac{1}{2}}_{4,q}} \)**

From the above estimates on \( \|u_0\|_{B^{\frac{1}{2}}_{4,q}} \), it yields that

\[
(3.9) \quad \|U_0(t_0)\|_{B^{\frac{1}{2}}_{4,q}} \leq C \|u_0\|_{B^{\frac{1}{2}}_{4,q}} \leq \frac{C}{\ln N}.
\]

We denote the \( k-th \) component of \( U_1 \) by \( U_1^{(k)} \), then

\[
U_1^{(2)} = -\int_0^t e^{(t-s)\Delta} (U_0(s) \cdot \nabla U_0^{(2)}(s) + \nabla \int_0^s \text{div} U_0(\tau) \, d\tau \cdot \nabla U_0^{(2)}(s)) \, ds.
\]

**Estimates on \( \|U_1(t_0)\|_{B^{\frac{1}{2}}_{4,q}} \)**

By the definition of \( U_1 \) and \( h_0 = 0 \), we obtain that

\[
\|U_1(t_0)\|_{B^{\frac{1}{2}}_{4,q}} \geq \|U_1^{(2)}(t_0)\|_{B^{\frac{1}{2}}_{4,q}} \geq \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}q} \|\hat{\Delta}_j U_1^{(2)}(t_0)\|_{L^q}^q \right)^{\frac{1}{q}}
\]

\[
\geq \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}q} \|\hat{\Delta}_j \int_0^t e^{(t-s)\Delta} U_0 \cdot \nabla U_0^{(2)}(s) \, ds\|_{L^q}^q \right)^{\frac{1}{q}}
\]

\[
- \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}q} \|\hat{\Delta}_j \int_0^t e^{(t-s)\Delta} \nabla \int_0^s \text{div} U_0(\tau) \, d\tau \cdot \nabla U_0^{(2)}(s) \, ds\|_{L^q}^q \right)^{\frac{1}{q}}
\]

\[
(3.10) \quad \geq \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}q} \|\hat{\Delta}_j \int_0^t e^{(t-s)\Delta} (U_0^1 \partial_{x_1} U_0^2) \, ds\|_{L^q}^q \right)^{\frac{1}{q}}
\]

\[
- \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}q} \|\hat{\Delta}_j \int_0^t e^{(t-s)\Delta} (\nabla \int_0^s \text{div} U_0(\tau) \, d\tau \cdot \nabla U_0^{(2)}(s)) \, ds\|_{L^q}^q \right)^{\frac{1}{q}}.
\]

At the beginning, we give the upper bound of the second term on the right-hand side of the above inequality. Using Bernstein's inequality, we have

\[
\left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}q} \|\hat{\Delta}_j \int_0^t e^{(t-s)\Delta} (U_0^1 \partial_{x_1} U_0^2) \, ds\|_{L^q}^q \right)^{\frac{1}{q}} \leq \left( \sum_{-\delta N \leq j \leq 0} 2^{j} t_0 \|U_0\|^2_{L^\infty L^4} \right)^{\frac{1}{q}}.
\]

By Fourier transform, it yields that

\[
\left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}q} \|\hat{\Delta}_j \int_0^t e^{(t-s)\Delta} (U_0^1 \partial_{x_1} U_0^2) \, ds\|_{L^q}^q \right)^{\frac{1}{q}}
\]

\[
= \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}q} \|\mathcal{F}^{-1} (\phi_j(\xi) \int_{\mathbb{R}^2} e^{-t_0|\xi|^2} - e^{-t_0(|\xi-\eta|^2+|\eta|^2)} \frac{1}{|\xi-\eta|^2 + |\eta|^2 - |\xi|^2} u_0^1(\xi - \eta) i\eta_1 u_0^2(\eta) \, d\eta)\|_{L^q}^q \right)^{\frac{1}{q}}.
\]
Taking advantage of Taylor’s series, we have
\[
\frac{e^{-t_0|ξ|^2} - e^{-t_0(|ξ-η|^2 + |η|^2)}}{|ξ - η|^2 + |η|^2 - |ξ|^2} = t_0 e^{-t_0|ξ|^2} \sum_{k=1}^{∞} \frac{(-1)^{k+1} \left(t_0(|ξ - η|^2 + |η|^2 - |ξ|^2)\right)^{k-1}}{k!}
\]
\[= t_0 e^{-t_0|ξ|^2} + t_0 e^{-t_0|ξ|^2} \sum_{k=2}^{∞} \frac{(-1)^{k+1} \left(t_0(|ξ - η|^2 + |η|^2 - |ξ|^2)\right)^{k-1}}{k!} := t_0 e^{-t_0|ξ|^2} + G(ξ - η, η).
\]
Hence, it follows from the first term on the right-hand side of inequality (3.10) that
\[
\left( \sum_{-δN ≤ j ≤ 0} 2^{-\frac{j}{2} q} \int_{0}^{t_0} e^{(t_0-s)\Delta}(U_0^2 \partial_x U_0^2) \, ds \right)^{\frac{1}{q}} ≤ \left( \sum_{-δN ≤ j ≤ 0} 2^{-\frac{j}{2} q} \left| F^{-1}\left(φ_j(ξ)t_0 \int_{R^2} \hat{u}_0^1(ξ - η)iη \hat{u}_0^2(η) \, dη \right)\right|_{L^4}^{q}\right)^{\frac{1}{q}}
\]
\[− \left( \sum_{-δN ≤ j ≤ 0} 2^{-\frac{j}{2} q} \left| F^{-1}\left(φ_j(ξ)(t_0 e^{-t_0|ξ|^2} - t_0) \int_{R^2} \hat{u}_0^1(ξ - η)iη \hat{u}_0^2(η) \, dη \right)\right|_{L^4}^{q}\right)^{\frac{1}{q}}
\]
\[− \left( \sum_{-δN ≤ j ≤ 0} 2^{-\frac{j}{2} q} \left| F^{-1}\left(φ_j(ξ) \int_{R^2} G(ξ - η, η) \hat{u}_0^1(ξ - η)iη \hat{u}_0^2(η) \, dη \right)\right|_{L^4}^{q}\right)^{\frac{1}{q}} := \left( \sum_{-δN ≤ j ≤ 0} 2^{-\frac{j}{2} q} I_j^q\right)^{\frac{1}{q}} - \left( \sum_{-δN ≤ j ≤ 0} 2^{-\frac{j}{2} q} I_{III, j}^q\right)^{\frac{1}{q}}.
\]
For the term $I_j$, from the definition of initial data $u_0$, we get that
\[
I_j ≥ \frac{t_0 2^{2N}}{(ln N)^2 N^\frac{3}{2}} 2^j \left| \hat{Δ}_j \left(Φ_{j,N}^2 \sin^2(2^N x_1)\right)\right|_{L^4}
\]
\[− \frac{t_0 2^{2N}}{(ln N)^2 N^\frac{3}{2}} \sum_{-δN ≤ k \neq j ≤ 0} 2^k \left| \hat{Δ}_j \left(Φ_{k,N}^2 \sin^2(2^N x_1)\right)\right|_{L^4}
\]
\[− \frac{t_0 2^{2N}}{(ln N)^2 N^\frac{3}{2}} \sum_{-δN ≤ k \neq m ≤ 0} 2^{k + \frac{3}{2}} \left| \hat{Δ}_j \left(Φ_{k,N} Φ_{m,N} \sin^2(2^N x_1)\right)\right|_{L^4}
\]
\[− \frac{t_0 2^{2N}}{(ln N)^2 N^\frac{3}{2}} \sum_{-δN ≤ k ≤ m ≤ 0} 2^{k + \frac{3}{2}} \left| \hat{Δ}_j \left(Φ_{k,N} \partial_{x_1} (Φ_{m,N} \sin(2^N x_1) \cos(2^N x_1))\right)\right|_{L^4}
\]
\[:= I_{J,1} - I_{J,2} - I_{J,3} - I_{J,4}.
\]
Now we estimate $I_{j,i}(i = 1, 2, 3, 4)$ respectively. We define the set $E_j$ by
\[
E_j = \{ x \in R^2 | |x - 2^j + 2^N e_1| ≤ 2^{-j}\}.
\]
For $I_{J,1}$, by $\sin^2 x = \frac{1 - \cos 2x}{2}$ and triangle inequality, we have
\[
I_{J,1} ≥ \frac{t_0 2^{2N}}{2(ln N)^2 N^\frac{3}{2}} 2^j \left| \int_{R^2} 2^j φ_0(2^j (x - y)) φ_0^2(2^j (y - 2^j + 2^N e_1)) \, dy \right|_{L^4(E_j)}
\]
\[− \frac{t_0 2^{2N}}{2(ln N)^2 N^\frac{3}{2}} 2^j \left| \int_{R^2} 2^{2j} φ_0(2^j (x - y)) φ_0^2(2^j (y - 2^j + 2^N e_1)) \cos(2^{N+1} y_1) \, dy \right|_{L^4(E_j)}.
\]
Taking advantage of change of variables, the first term on the right-hand side of the above inequality can be bounded as follows.

\[
\frac{t_0 2^{N}}{2(\ln N)^2 N^\frac{1}{2}} 2^j \| 2^j \varphi_0(2^j (x - y)) \varphi_0^2(2^j (y - 2^{[j]+2N} e_1)) \|_{L^4(E_j)} \nabla \varphi_0(x - y) \varphi_0^2(y) \|_{L^4(|x| \leq 1)} = \frac{C 2^j}{(\ln N)^3 N^\frac{1}{2}}.
\]

By integration by parts, one yields that

\[
\frac{t_0 2^{N}}{2(\ln N)^2 N^\frac{1}{2}} 2^j \| \int_{\mathbb{R}^2} 2^j \varphi_0(2^j (x - y)) \varphi_0^2(2^j (y - 2^{[j]+2N} e_1)) \cos(2^{N+1} y_1) \|_{L^4(E_j)} \nabla \varphi_0(x - y) \varphi_0^2(y) \|_{L^4(E_j)} \leq \frac{C 2^{-N} 2^{\frac{3}{2}}}{(\ln N)^3 N^\frac{1}{2}}.
\]

Owing to \(-\delta N \leq j \leq 0\), from the above two estimates, we infer that

\[
I_{j1} \geq \frac{C 2^j}{(\ln N)^3 N^\frac{1}{2}}.
\]

Noting the fact that \(|\varphi_0(x)| \leq \frac{C_\beta}{(1 + |x|)^\beta}\) for \(\beta \in \mathbb{N}\), \(I_{j2}\) can be bounded by

\[
I_{j2} \leq \frac{t_0 2^{N}}{(\ln N)^2 N^\frac{1}{2}} \sum_{\delta N \leq k \neq j \leq 0} 2^k \| 2^j \int_{\mathbb{R}^2} |\varphi_0(2^j (x - y))| \varphi_0^2(2^j (y - 2^{[j]+2N} e_1)) \|_{L^4(E_j)} \nabla \varphi_0(x - y) \varphi_0^2(y) \|_{L^4(E_j)} \leq \frac{C t_0 2^{N}}{(\ln N)^2 N^\frac{1}{2}} \sum_{\delta N \leq k \neq j \leq 0} 2^{k+j} \| \int_{\mathbb{R}^2} \frac{1}{(1 + 2^j |x - y|)^\beta} \frac{1}{(1 + 2^k |y - 2^{[j]+2N} e_1|)^\beta} \|_{L^4(E_j)}.
\]

Dividing the integral region in terms of \(y\) into the following three parts to estimate:

\[
A_1 := \{ y | y - 2^{[j]+2N} e_1 | \leq 2^{N-1} \},
\]
\[
A_2 := \{ y | y - 2^{[j]+2N} e_1 | \geq 2^{N-1}, |y - 2^{[j]+2N} e_1 | \leq 2^{N-2} \},
\]
\[
A_3 := \{ y | y - 2^{[j]+2N} e_1 | \geq 2^{N-1}, |y - 2^{[j]+2N} e_1 | \leq 2^{N-2} \},
\]

we conclude that, for \(x \in E_j\) and \(y \in A_1\),

\[
|y - 2^{[j]+2N} e_1 | = |y - 2^{[j]+2N} e_1 + 2^{[j]+2N} e_1 - 2^{[j]+2N} e_1 | \geq 2^{[j]+2N} e_1 - 2^{[j]+2N} e_1 - |y - 2^{[j]+2N} e_1 | \geq 2^{N-1}.
\]

For \(x \in E_j\), \(y \in A_3\), it is easy to check that

\[
|x - y | = |x - 2^{[j]+2N} e_1 + 2^{[j]+2N} e_1 - 2^{[j]+2N} e_1 + 2^{[j]+2N} e_1 - y | \geq 2^{[j]+2N} e_1 - 2^{[j]+2N} e_1 - |y - 2^{[j]+2N} e_1 | - 2^{-j} \geq C 2^N.
\]

Therefore, for \(-\delta N \leq j \leq 0\), we obtain that

\[
I_{j2} \leq \frac{C t_0 2^{N}}{(\ln N)^2 N^\frac{1}{2}} \sum_{\delta N \leq k \neq j \leq 0} 2^{k+j} \left( \left\| \frac{2^{-2j}}{(2^k 2^{2N})^\beta} \right\|_{L^4(A_1)} + \left\| \frac{2^{-2k}}{(2^{2j} 2^{2N})^\beta} \right\|_{L^4(A_1)} \right) \leq \frac{C 2^{-N} 2^{-\frac{j}{2}}}{(\ln N)^3 N^\frac{1}{2}}.
\]
I. Similarly, we conclude that
\[
I_J \leq \frac{t_0 2^{2N}}{(\ln N)^2 N^{3/2}} \sum_{-\delta N \leq k \neq m \leq 0} 2^{\frac{1}{2}(k+m)} \left\| \varphi_0(2^k(x - 2^k|+2^N e_1)) \varphi_0(2^m(x - 2^m|+2^N e_1)) \right\|_{L^4(\mathbb{R}^2)}.
\]

For \( |x - 2^{|k|+2N} e_1| \geq 2^{2N-1} \), we have
\[
\left\| \frac{1}{(1 + 2^k|x - 2^{|k|+2N} e_1|) (1 + 2^m|x - 2^{|m|+2N} e_1|) \right\|_{L^4(\mathbb{R}^2)} \leq C 2^{-k} 2^{-2N} \frac{1}{2^k}.
\]

For \( |x - 2^{|k|+2N} e_1| \leq 2^{2N-1} \), then we derive
\[
|x - 2^{|m|+2N} e_1| \geq 2^{|k|+2N} e_1 - 2^{|m|+2N} e_1 - |x - 2^{|k|+2N} e_1| \geq 2^{2N-1},
\]

from which we obtain that
\[
\left\| \frac{1}{(1 + 2^k|x - 2^{|k|+2N} e_1|) (1 + 2^m|x - 2^{|m|+2N} e_1|) \right\|_{L^4(\mathbb{R}^2)} \leq C 2^{-m} 2^{-2N} \frac{1}{2^k}.
\]

Therefore, we conclude that
\[
I_J \leq C t_0 2^{2N} \sum_{-\delta N \leq k \neq m \leq 0} 2^{-2N} 2^{-\frac{1}{2}(k+m)} \leq C 2^{-2N} \frac{1}{(\ln N)^3 N^{3/2}}.
\]

Similarly, \( I_{J4} \) can be bounded by
\[
I_{J4} \leq C t_0 2^{2N} \sum_{-\delta N \leq k \neq m \leq 0} 2^{-2N} 2^{-\frac{1}{2}(k+m)} 2^{-N} 2^m \leq C 2^{-2N} \frac{1}{(\ln N)^3 N^{3/2}}.
\]

To sum up, we get that
\[
(3.11) \quad \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{1}{2} j} \gamma J_j^2 \right)^{\frac{1}{2}} \geq \frac{C}{(\ln N)^3 N^{3/2}} \left( \sum_{-\delta N \leq j \leq 0} 1 \right)^{\frac{1}{2}} = C N^{1/2} \frac{1}{(\ln N)^3}.
\]

Now we estimate the upper bound of \( II_J \). Due to \( t_0 e^{-t_0|\xi|^2} - t_0 = t_0 \sum_{k=1}^\infty \frac{(-t_0|\xi|^2)^{k}}{k!} \), then
\[
II_J \leq \sum_{k=1}^\infty \frac{k^{k+1}}{k!} \| \mathcal{F}^{-1} ((|\xi|^2)^k \phi_j(\xi) \int_{\mathbb{R}^2} \hat{u}_0^1(\xi - \eta) i \eta_1 \hat{u}_0^2(\eta) d\eta) \|_{L^4}
\]
\[
= \sum_{k=1}^\infty \frac{k^{k+1}}{k!} \| \left( \Delta^k \hat{u}_0^1(x_1, u_0^2) \right) \|_{L^4}
\]
\[
\leq \sum_{k=1}^\infty \frac{C \xi^{k+1}}{k!} 2^{2k} 2^N \| u_0^1 \|_{L^4} \| u_0^2 \|_{L^4} \leq C \xi^2 \frac{1}{(\ln N)^4}.
\]
Therefore, it is easy to obtain that
\[
\left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{\ell j}{4}} II_j^3 \right)^{\frac{1}{2}} \leq \left( \sum_{-\delta N \leq j \leq 0} \frac{C 2^{2j} q}{(\ln N)^4} \right)^{\frac{1}{2}} \leq \frac{C}{(\ln N)^4}.
\]

For \(III_j\), it is easy to check that
\[
(3.12) \quad G(\xi - \eta, \eta) = t_0 e^{-t_0 |\xi|^2} \sum_{k=1}^{\infty} \frac{(-1)^k t_0^k}{(k+1)!} \sum_{\ell=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \sum_{m=0}^{k} C^2_{\ell} C^m_{k-\ell} |\xi - \eta|^2 \eta^{2m} (-|\xi|^2)^{k-\ell-m}.
\]

Noting the fact that \(\text{supp } \hat{u}_0 \sim 2^N C\) and \(|\xi| \leq 2\), then \(G(\xi - \eta, \eta) = O((\ln N)^{-2} 2^{-2N})\).

Hence, by Lemma 2.5, for \(-\delta N \leq j \leq 0\), we obtain that
\[
(3.13) \quad III_j \leq C \sum_{k=1}^{\infty} \frac{t_0^{k+1}}{(k+1)!} \sum_{m=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \sum_{\ell=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \sum_{\ell=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{k!}{(k-\ell-m)!} \left\| \hat{\Delta}_j \hat{\eta}_0^0 \Delta^{k-\ell-m} \left( (-\Delta)^\ell u_0^{(1)} (-\Delta)^m \partial x_1 u_0^{(2)} \right) \right\|_{L^4}
\]
\[
\leq C \sum_{k=1}^{\infty} \frac{t_0^{k+1}}{(k+1)!} \sum_{m=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \sum_{\ell=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{k!}{(k-\ell-m)!} \left\| (-\Delta)^\ell u_0^{(1)} \right\|_{L^8} \left\| (-\Delta)^m \partial x_1 u_0^{(2)} \right\|_{L^8}
\]
\[
+ C \sum_{k=1}^{\infty} \frac{t_0^{k+1}}{(k+1)!} \left( \begin{array}{c} k \\ \ell \end{array} \right) \sum_{\ell=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{k!}{(k-\ell-m)!} \left\| \hat{\Delta}_j \left( \Delta^{\ell} u_0^{(1)} \Delta^{k-\ell} \partial x_1 u_0^{(2)} \right) \right\|_{L^4}
\]
\[
\leq C \sum_{k=1}^{\infty} \frac{t_0^{k+1}}{(k+1)!} \sum_{m=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \sum_{\ell=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{k!}{(k-\ell-m)!} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{2^{2N(\ell+m)} 2^{2N}}{N^2 (\ln N)^2} \left\| \hat{\Delta}_j \left( \Delta^{\ell} u_0^{(1)} \Delta^{k-\ell} \partial x_1 u_0^{(2)} \right) \right\|_{L^4}
\]
\[
+ C \sum_{k=1}^{\infty} \frac{t_0^{k+1}}{(k+1)!} \left( \begin{array}{c} k \\ \ell \end{array} \right) \sum_{\ell=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{k!}{(k-\ell-m)!} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{2^{2N(\ell+m)} 2^{2N}}{N^2 (\ln N)^2} \left\| \hat{\Delta}_j \left( \Delta^{\ell} u_0^{(1)} \Delta^{k-\ell} \partial x_1 u_0^{(2)} \right) \right\|_{L^4}
\]
\[
\leq C 2^{2j} \sum_{k=1}^{\infty} \frac{t_0^{k+1}}{(k+1)!} \left( \begin{array}{c} k \\ \ell \end{array} \right) \sum_{\ell=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{k!}{(k-\ell-m)!} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{2^{2N(k-1)} 2^{2N}}{N^2 (\ln N)^2} + C \sum_{k=1}^{\infty} \frac{t_0^{k+1}}{(k+1)!} \sum_{\ell=0}^{k-\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{k!}{(k-\ell-m)!} \left( \begin{array}{c} k \\ \ell \end{array} \right) \frac{2^{2N(k-1)} 2^{2N}}{N^2 (\ln N)^2} \left\| \hat{\Delta}_j \left( \Delta^{\ell} u_0^{(1)} \Delta^{k-\ell} \partial x_1 u_0^{(2)} \right) \right\|_{L^4}
\]
\[
\leq \frac{2^N}{N^2 (\ln N)^2} \left[ \sum_{\delta N \leq m \leq 0} 2^\frac{m}{2} \Phi_{m,N}(\Delta^\ell \sin(2^N x_1)) \sum_{-\delta N \leq n \leq 0} 2^\frac{m}{2} \Phi_{k,N}(\Delta^{k-\ell} \partial x_1 \cos(2^N x_1)) \right]_{L^4}
\]
\[
\leq \frac{2^N}{N^2 (\ln N)^2} \left[ \sum_{\delta N \leq m \leq 0} 2^\frac{m}{2} \Phi_{m,N}(\Delta^\ell \sin(2^N x_1)) \sum_{-\delta N \leq n \leq 0} 2^\frac{m}{2} \Phi_{m,N}(\Delta^{k-\ell} \sin^2(2^N x_1)) \right]_{L^4}
\]
\[
+ \sum_{-\delta N \leq m \neq n \leq 0} 2^\frac{m+n}{2} \left[ \sum_{\delta N \leq m \leq 0} 2^\frac{m}{2} \Phi_{m,N}(\Delta^\ell \sin^2(2^N x_1)) \right]_{L^4}.
\]
By the definition of $\Phi_{j,N}$, one yields that
\[ 2^j \| \hat{\Delta}_j (\Phi_{j,N}^2 \sin^2(2^N x_1)) \|_{L^4} \leq C 2^j \| \Phi_{j,N} \|_{L^2}^2 \leq C 2^{\frac{j}{2}}. \]

Following the methods on $I_{j2}$ and $I_{j3}$, it is easy to get that
\[ \sum_{-\delta N \leq m \neq n \leq 0} 2^m \| \hat{\Delta}_j (\Phi_{m,N}^2 \sin^2(2^N x_1)) \|_{L^4} + \sum_{-\delta N \leq m \neq n \leq 0} 2^{m+n} \| \hat{\Delta}_j (\Phi_{m,N} \Phi_{n,N} \sin^2(2^N x_1)) \|_{L^4} \leq C 2^{-N} 2^{-\frac{j}{2}} + C 2^{-2N}. \]

Therefore, we obtain that
\[ \frac{2^N}{N^\frac{3}{2} (\ln N)^2} \left\| \hat{\Delta}_j \left( \sum_{-\delta N \leq m \leq 0} 2^m \Phi_{m,N} (\Delta^\ell \sin(2^N x_1)) \sum_{-\delta N \leq n \leq 0} 2^n \Phi_{k,N} (\Delta^{k-\ell} \partial_{x_1} \cos(2^N x_1)) \right) \right\|_{L^4} \leq \frac{C 2^{2N(k+1)} 2^j}{N^\frac{3}{2} (\ln N)^2}. \]

This implies that
\[ III_j \leq \frac{C 2^{2j} 2^{-2N}}{N^\frac{3}{2} (\ln N)^4} + C 2^j \sum_{k=1}^{\infty} \frac{k(t_0 2^{2N})^{k+1}}{(k+1)!} \leq \frac{C (2^{2j} 2^{-2N} + 2^j)}{N^\frac{3}{2} (\ln N)^4}. \]

Therefore, we have that
\[ \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}} q \right)^\frac{1}{q} \leq \left( \frac{C N^{\frac{1}{q} - \frac{1}{2}}}{(\ln N)^4} \right)^\frac{1}{q}. \]

We utilize estimates on $I_j$ and $II_j$ to see that
\[ \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}} q \right)^\frac{1}{q} \geq C \left( \frac{N^{\frac{1}{q} - \frac{1}{2}}}{(\ln N)^3} - \frac{C}{(\ln N)^4} - \frac{C N^{\frac{1}{q} - \frac{1}{2}}}{(\ln N)^4} \right) \geq \frac{C N^{\frac{1}{q} - \frac{1}{2}}}{(\ln N)^3}. \]

Now we turn to estimate the third term on the right-hand side of inequality (3.10). An easy computation yields that
\[ \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}} q \right)^\frac{1}{q} \leq \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{j}{2}} q \right)^\frac{1}{q} \leq \frac{C N^{\frac{1}{q} - \frac{1}{2}}}{(\ln N)^3} - \frac{C}{(\ln N)^4} - \frac{C N^{\frac{1}{q} - \frac{1}{2}}}{(\ln N)^4} \geq \frac{C N^{\frac{1}{q} - \frac{1}{2}}}{(\ln N)^3}. \]
For the last three terms on the right-hand side of the above inequality, we obtain that

\[
\left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{1}{2}j} q \right) |T| \int_{0}^{t} e^{(t-s)\Delta} \left( ( \int_{0}^{s} \partial_{x_{1}} \partial_{x_{2}} U^{(2)}(0)(\tau) \, d\tau ) \partial_{x_{1}} U^{(2)}(s) \right) \, ds \left\| Q \right\|_{L_{4}^{1}}^{\frac{1}{2}} \]

\[
+ \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{1}{2}j} q \right) \left| T \right| \int_{0}^{t} e^{(t-s)\Delta} \left( ( \int_{0}^{s} \partial_{x_{2}} \partial_{x_{2}} U^{(2)}(0)(\tau) \, d\tau ) \partial_{x_{2}} U^{(2)}(s) \right) \, ds \left\| Q \right\|_{L_{4}^{1}}^{\frac{1}{2}} \]

\[
+ \left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{1}{2}j} q \right) |T| \int_{0}^{t} e^{(t-s)\Delta} \left( ( \int_{0}^{s} \partial_{x_{2}} \partial_{x_{2}} U^{(2)}(0)(\tau) \, d\tau ) \partial_{x_{2}} U^{(2)}(s) \right) \, ds \left\| Q \right\|_{L_{4}^{1}}^{\frac{1}{2}} \]

\[
\leq C N^{\frac{1}{4}} T^{2} ( \left\| \partial_{x_{1}} u_{0}^{(2)} \right\|_{L_{4}^{1}} \left\| \partial_{x_{1}} u_{0}^{(2)} \right\|_{L_{4}^{1}} + \left\| \partial_{x_{1}} u_{0}^{(2)} \right\|_{L_{4}^{1}} \left\| \partial_{x_{1}} u_{0}^{(2)} \right\|_{L_{4}^{1}} + \left\| \partial_{x_{2}} u_{0}^{(2)} \right\|_{L_{4}^{1}} \left\| \partial_{x_{2}} u_{0}^{(2)} \right\|_{L_{4}^{1}} )
\]

\[
\leq C N^{\frac{1}{2} - N} \left( \ln N \right)^{4}.
\]

By Fourier transform, we get that

\[
F \left( \int_{0}^{t} e^{(t-s)\Delta} \left( ( \int_{0}^{s} \partial_{x_{2}} \partial_{x_{2}} U^{(2)}(0)(\tau) \, d\tau ) \partial_{x_{1}} U^{(2)}(s) \right) \, ds \right)
\]

\[
= - \int_{0}^{t} e^{(t-s)\xi^{2}} \int_{\mathbb{R}^{2}} \int_{0}^{\infty} |\xi_{1} - \eta_{1}|^{2} e^{-|\xi - \eta|^{2}} u_{0}^{(2)}(\xi - \eta) \, d\tau \eta_{1} e^{-s|\eta|^{2}} u_{0}^{(2)}(\eta) \, d\eta \, ds
\]

\[
= \int_{\mathbb{R}^{2}} e^{(t-s)|\xi|^{2}} \left( \frac{e^{t(|\xi|^{2} - |\eta|^{2}) - 1}}{|\xi|^{2} - |\eta|^{2} - |\xi - \eta|^{2}} - \frac{e^{t(|\xi|^{2} - |\eta|^{2}) - 1}}{|\xi|^{2} - |\eta|^{2}} \right) |\xi_{1} - \eta_{1}|^{2} u_{0}^{(2)}(\xi - \eta) i\eta_{1} u_{0}^{(2)}(\eta) \, d\eta.
\]

Using Taylor’s series, it easily yields that

\[
\frac{e^{t(|\xi|^{2} - |\eta|^{2}) - 1}}{|\xi|^{2} - |\eta|^{2} - |\xi - \eta|^{2}} - \frac{e^{t(|\xi|^{2} - |\eta|^{2}) - 1}}{|\xi|^{2} - |\eta|^{2}} = \sum_{k=1}^{\infty} \frac{t^{k+1}(|\xi|^{2} - |\eta|^{2})^{2} - |\xi - \eta|^{2}}{(k+1)!} - \sum_{k=1}^{\infty} \frac{t^{k+1}(|\xi|^{2} - |\eta|^{2})^{2}}{(k+1)!}.
\]

Therefore, by (3.12) and estimates on $III_{j}$, we have

\[
\left\| \mathcal{D}_{j} \int_{0}^{t} e^{(t-s)\Delta} \left( ( \int_{0}^{s} \partial_{x_{2}} \partial_{x_{2}} U^{(2)}(0)(\tau) \, d\tau ) \partial_{x_{1}} U^{(2)}(s) \right) \, ds \right\|_{L_{4}^{1}}
\]

\[
\leq \left\| \mathcal{F}^{-1} \left( \hat{\varphi}_{j}(\xi) \int_{\mathbb{R}^{2}} e^{-(t-s)|\xi|^{2}} \left( \sum_{k=1}^{\infty} \frac{t^{k+1}(|\xi|^{2} - |\eta|^{2})^{2} - |\xi - \eta|^{2}}{(k+1)!} \right) \right) \right\|_{L_{4}^{1}}
\]

\[
+ \left\| \mathcal{F}^{-1} \left( \hat{\varphi}_{j}(\xi) \int_{\mathbb{R}^{2}} e^{-(t-s)|\xi|^{2}} \left( \sum_{k=1}^{\infty} \frac{t^{k+1}(|\xi|^{2} - |\eta|^{2})^{2}}{(k+1)!} \right) \right) \right\|_{L_{4}^{1}}
\]

\[
\leq \sum_{k=1}^{\infty} \frac{t^{k+1}}{(k+1)!} \sum_{m=0}^{k} \sum_{\ell=0}^{k} C_{k}^{\ell} C_{k-\ell}^{m} \left\| \mathcal{D}_{j} e^{\Delta} (-\Delta)^{k-\ell-m} \left( \Delta^{\ell-1} \partial_{x_{1}} U^{(2)}(0) \Delta^{m} \partial_{x_{1}} U^{(2)}(0) \right) \right\|_{L_{4}^{1}}
\]

\[
+ \sum_{k=1}^{\infty} \frac{t^{k+1}}{(k+1)!} \sum_{\ell=0}^{k} C_{k}^{\ell} \left\| \mathcal{D}_{j} e^{\Delta} (-\Delta)^{k-\ell} \left( \Delta^{\ell-1} \partial_{x_{1}} U^{(2)}(0) \Delta^{m} \partial_{x_{1}} U^{(2)}(0) \right) \right\|_{L_{4}^{1}} \leq \frac{C^{2} j^{2}}{N^{\frac{1}{2}} (\ln N)^{4}}.
\]
Hence, we obtain that
\[
\left( \sum_{-\delta N \leq j \leq 0} 2^{-\frac{1}{2}jq} \left\| \Delta_j \int_0^t e^{(t-s)\Delta} (\nabla \int_0^s \text{div} U_0(\tau) \, d\tau \cdot \nabla U_0^{(2)}(s)) \, ds \right\|_{L^4}^q \right)^{\frac{1}{q}} \leq \frac{CN^{\frac{1}{q}-\frac{1}{2}}}{(\ln N)^4} + \frac{CN^{\frac{1}{q}-\frac{1}{2}}2^{-N}}{(\ln N)^4}.
\]
To sum up,
\[
\|U_1(t_0)\|_{B^{\frac{1}{2}}_{4,1}} \geq \frac{CN^{\frac{1}{q}-\frac{1}{2}}}{(\ln N)^3} - C(\ln N)^{-2} - \frac{CN^{\frac{1}{q}-\frac{1}{2}}2^{-N}}{(\ln N)^4} \geq \frac{CN^{\frac{1}{q}-\frac{1}{2}}}{(\ln N)^3}.
\]
Now we need to estimate \(\|U_2(t_0)\|_{B^{0}_{2,1}}\). Based on Proposition 3.2, we obtain that
\[
\|u - U_0 - U_1\|_{L^\infty_T(B^{\frac{1}{2}}_{4,1})} \\
\leq C\|u - U_0\|_{L^\infty_T(B^{0}_{4,1})} + \|U_0 \cdot \nabla (u - U_0)\|_{L^1_T(B^{\frac{1}{2}}_{4,1})} + \|\nabla (\ln (1 + h) - h) \cdot \nabla u\|_{L^1_T(B^{\frac{1}{2}}_{4,1})} \\
+ T\|h\|_{L^\infty_T(B^{0}_{4,1})} + \int_0^t \int_0^s \text{div} U_0 \cdot \Delta u \, ds \, dt \\
+ \|\int_0^t \text{div} U_0 \cdot \Delta u\|_{L^\infty_T(B^{0}_{4,1})}.
\]
Actually, by inequality (3.6), we have
\[
\|u - U_0\|_{L^\infty_T(B^{0}_{4,1})} + \|U_0 \cdot \nabla (u - U_0)\|_{L^1_T(B^{0}_{4,1})} \\
\leq CT^{\frac{1}{2}} \|u - U_0\|_{L^\infty_T(B^{0}_{4,1})} + CT^{\frac{1}{2}} \|u\|_{L^\infty_T(B^{1+\varepsilon}_{4,1})} + \frac{CT^{\frac{1}{2}}}{\varepsilon} \|u\|_{L^\infty_T(B^{1+\varepsilon}_{4,1})} \\
\leq C2^{(-\frac{1}{2} + \frac{\delta}{2} + 2\varepsilon)N}.
\]
By bilinear estimates, it follows that
\[
\|\nabla (\ln (1 + h) - h) \cdot \nabla u\|_{L^1_T(B^{0}_{4,1})} = \left\| \frac{h}{1 + h} \nabla h \cdot \nabla u \right\|_{L^1_T(B^{0}_{4,1})} \\
\leq CT^{\frac{1}{2}} \left\| \frac{h}{1 + h} \right\|_{L^\infty_T(B^{2}_{4,1})} \|h\|_{L^\infty_T(B^{1}_{4,1})} \|u\|_{L^\infty_T(B^{1+\varepsilon}_{4,1})} \\
\leq CT^{\frac{1}{2}} \sum_{m=1}^\infty C^m \|h\|_{L^\infty_T(B^{2}_{4,1})} \|h\|_{L^\infty_T(B^{1}_{4,1})} \|h\|_{L^\infty_T(B^{1}_{4,1})} \|u\|_{L^\infty_T(B^{1+\varepsilon}_{4,1})} \\
\leq C2^{(-\frac{1}{2} + \frac{\delta}{2} + 2\varepsilon)N} \cdot 2^{(1+\delta)N} \cdot 2^{(1+\varepsilon)N} = C2^{(-\frac{1}{2} + 2\varepsilon + \frac{\delta}{2})N}.
\]
Moreover, it holds that
\[
\|\int_0^t e^{(t-s)\Delta} (\nabla (h + \int_0^s \text{div} U_0 \, d\tau) \cdot \nabla u) \, ds\|_{L^\infty_T(B^{\frac{1}{2}}_{4,1})} \\
\leq \|\int_0^t e^{(t-s)\Delta} \text{div} ((h + \int_0^s \text{div} U_0 \, d\tau) \Delta u) \, ds\|_{L^\infty_T(B^{\frac{1}{2}}_{4,1})} \\
+ \|\int_0^t e^{(t-s)\Delta} (h + \int_0^s \text{div} U_0 \, d\tau) \, ds\|_{L^\infty_T(B^{\frac{1}{2}}_{4,1})} \\
\leq C \|(h + \int_0^t \text{div} U_0 \, d\tau) \Delta u\|_{L^\infty_T(B^{\frac{1}{2}}_{4,1})} + \|(h + \int_0^t \text{div} U_0 \, d\tau) \, ds\|_{L^\infty_T(B^{\frac{1}{2}}_{4,1})}.
\]
\[ \leq C \|
abla \int_0^t \text{div } U_0 \, ds \cdot \nabla (u - U_0) \|_{L^1_t L^{\infty}_x(B_{[2,\infty],1})} \]

\[ \leq C \sum_{k=\lfloor \frac{3}{4} - 1 \rfloor}^{q} \int_0^t \text{div } U_0 \|_{L^1_t L^{\infty}_x(B_{[2,\infty],1})} \]

\[ \leq C2^{(\frac{3}{2} + \varepsilon(\frac{3}{2} - \frac{3}{2} - \frac{1}{2}))}N, \]

and

\[ \|
abla \int_0^t \text{div } U_0 \|_{L^1_t L^{\infty}_x(B_{[2,\infty],1})} \leq C \|
abla \int_0^t \text{div } U_0 \|_{L^1_t L^{\infty}_x(B_{[2,\infty],1})} \]

\[ \leq C T^{\frac{3}{2}} \| U_0 \|_{L^1_t L^{\infty}_x(B_{[2,\infty],1})} \| u - U_0 \|_{L^{1+\varepsilon}_t L^{1+\varepsilon}_x(B_{[2,\infty],1})} \]

\[ \leq C 2^{(-1 \varepsilon + \frac{3}{2} \delta)N}. \]

Noting the fact that \(3\delta + 5\varepsilon < 1\), we obtain that

\[ (3.15) \quad \| U_2(t_0) \|_{L^1_t L^{\infty}_x(B_{[2,\infty],1})} \leq C 2^{(\frac{3}{2} + \frac{5}{2} \varepsilon + \frac{3}{2} \delta)N} \leq C. \]

To sum up, combined with (3.9), (3.14) and (3.15), we get that for \(1 \leq q < 2\), \(t_0 = (\ln N)^{-1} 2^{-2N}\),

\[ \| u(t_0) \|_{B_{4,q}^\frac{1}{2}} \geq \| U_1(t_0) \|_{B_{4,q}^\frac{1}{2}} - \| U_0(t_0) \|_{B_{4,q}^\frac{1}{2}} - \| U_2(t_0) \|_{B_{4,q}^\frac{1}{2}} \]

\[ \geq C (\ln N)^{-3} N^{\frac{1}{2} - \frac{1}{2}} - C (\ln N)^{-1} - C \]

\[ \geq C (\ln N)^{-3} N^{\frac{1}{2} - \frac{1}{2}}, \]

with initial data \(\| u_0 \|_{B_{4,q}^\frac{1}{2}} \leq C (\ln N)^{-1}\).

### 3.2 Proof of Theorem 1.3 for \(q > 2\).

For \(q > 2\), we construct initial data \((h_0, u_0)\) as follows:

\[ h_0 = 0, \quad u_0 = \left( \frac{\ln N}{\sqrt{N}} \sum_{N \leq k \leq (1 + \delta)N} 2^k \varphi_0(x) \sin(2^k x_1), \frac{\ln N}{\sqrt{N}} \sum_{N \leq k \leq (1 + \delta)N} 2^k \varphi_0(x) \cos(2^k x_1) \right), \]

where \(\varphi_0\) is the function stemming from localization homogeneous operator \(\hat{\Delta}_0\) and \(\delta\) is defined in Proposition 3.1. For convenience, we set \(\varphi_0(x) = \mathcal{F}^{-1}(\hat{\phi}(\xi))\). It is easy to check that

\[ \| u_0 \|_{B_{4,q}^\frac{1}{2}} \leq C \left( \sum_{N \leq k \leq (1 + \delta)N} 2^k \| \hat{\Delta}_j(\varphi_0(x) \sin(2^k x_1)) \|_{L^1_t}^q \right)^{\frac{1}{q}} \leq C \| u_0 \|_{B_{4,q}^\frac{1}{2}} \]

Now we decompose the solution \(u\) into three parts:

\[ u = U_0 + U_1 + U_2, \]

here the decomposition is the same with (3.8), and next we respectively estimate \(\| U_i(t_0) \|_{B_{4,q}^\frac{1}{2}} (i = 0, 1, 2)\) for \(t_0 = (\ln N)^{-1} 2^{-2N}\).

**Estimates on \(\| U_0(t_0) \|_{B_{4,q}^\frac{1}{2}}**

From estimates on initial data \(u_0\), one yields that

\[ \| U_0(t_0) \|_{B_{4,q}^\frac{1}{2}} \leq C \| u_0 \|_{B_{4,q}^\frac{1}{2}} \leq C \ln N \frac{1}{N^{\frac{1}{2} - \frac{1}{q}}}. \]
We denote the \( k \)-th component of \( U_1 \) by \( U_1^{(k)} \). Then

\[
U_1^{(2)} = - \int_0^t e^{(t-s)\Delta} (U_0 \cdot \nabla U_0^{(2)}) \, ds = - \int_0^t e^{(t-s)\Delta} (U_0^{(1)} \partial_{x_1} U_0^{(2)} + \int_0^s \partial_{x_1}^2 U_0^{(1)} \, d\tau \partial_{x_1} U_0^{(2)}(s)) \, ds
\]

Estimates on \( ||U_1(t_0)||_{\dot{B}_{4,q}^{\frac{1}{2}}} \). Taking advantage of embedding \( \dot{B}_{4,q}^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow \dot{B}_{4,\infty}^{1} \), we can get that

\[
||U_1(t_0)||_{\dot{B}_{4,q}^{\frac{1}{2}}} \geq C||\Delta U_1(t_0)||_{L^\infty}
\]

By \( ||f||_{L^\infty} \geq |f(0)| = |\int_{\mathbb{R}^2} \tilde{f}(\xi) \, d\xi| \), we obtain that

(3.16)

\[
||U_1(t_0)||_{\dot{B}_{4,q}^{\frac{1}{2}}} \geq C|\int_{\mathbb{R}^2} \phi(\xi) \mathcal{F} \left( \int_0^t e^{(t-s)\Delta} (U_0^{(1)} \partial_{x_1} U_0^{(2)} + \int_0^s \partial_{x_1}^2 U_0^{(1)} \, d\tau \partial_{x_1} U_0^{(2)}(s)) \, ds \right) \, d\xi|
\]

According to the definition of \( U_0 \), one yields that

\[
\int_{\mathbb{R}^2} \phi(\xi) \mathcal{F} \left( \int_0^t e^{(t-s)\Delta} (U_0^{(1)} \partial_{x_1} U_0^{(2)} + \int_0^s \partial_{x_1}^2 U_0^{(1)} \, d\tau \partial_{x_1} U_0^{(2)}(s)) \, ds \right) \, d\xi
\]

\[
= \int_{\mathbb{R}^2} \phi(\xi) \mathcal{F} \left( \int_0^t e^{(t-s)\Delta} (U_0^{(1)} + \int_0^s \partial_{x_1}^2 U_0^{(1)} \, d\tau \partial_{x_1} U_0^{(2)}(s)) \, ds \right) \, d\xi
\]

\[
= \int_{\mathbb{R}^2} \phi(\xi) \left( \int_0^t e^{-s|\xi-\eta|^2} u_0(\xi - \eta) - (\xi_1 - \eta_1)^2 \int_0^s e^{-\tau|\xi-\eta|^2} u_0(\xi - \eta) \, d\tau \right.
\]

\[
\left. \times \, (\xi_1 - \eta_1)^2 u_0(\eta) \, d\eta \, d\xi \right) \, d\xi
\]

\[
= \int_{\mathbb{R}^2} \phi(\xi) \left( \int_0^t e^{-s|\xi-\eta|^2} \int_{\mathbb{R}^2} (e^{-s|\xi-\eta|^2} u_0(\xi - \eta) - \frac{|\xi_1 - \eta_1|^2}{|\xi - \eta|^2}(1 - e^{-s|\xi-\eta|^2}) u_0(\xi - \eta)) \, d\eta \right) \, d\xi
\]
By triangle inequality, we have

\[
\begin{align*}
|I_1| & \geq \int_{\mathbb{R}^2} \phi(\xi) \left( \int_0^{t_0} e^{-(t_0-s)|\xi|^2} \int_{\mathbb{R}^2} (2e^{-s|\xi-\eta|^2} - 1) \hat{u}_0^1(\xi-\eta)i\eta e^{-s|\eta|^2} \hat{u}_0^0(\eta) \, d\eta \, ds \right) d\xi \\
& \quad - \int_{\mathbb{R}^2} \phi(\xi) \left( \int_0^{t_0} e^{-(t_0-s)|\xi|^2} \int_{\mathbb{R}^2} \frac{|\xi_2 - \eta_2|^2}{|\xi - \eta|^2} (1 - e^{-s|\xi-\eta|^2}) \hat{u}_0^1(\xi-\eta)i\eta e^{-s|\eta|^2} \hat{u}_0^0(\eta) \, d\eta \, ds \right) d\xi \\
& := I - II.
\end{align*}
\]

By Fourier transform, we obtain that

\[
\hat{u}_0^1 = \frac{\ln N}{2\sqrt{N}} \sum_{N \leq k \leq (1+\delta)N} 2\pi (i\phi(\xi + 2^k e_1) - i\phi(\xi - 2^k e_1)),
\]

(3.17)

\[
\hat{u}_0^2 = \frac{\ln N}{2\sqrt{N}} \sum_{N \leq k \leq (1+\delta)N} 2\pi (\phi(\xi + 2^k e_1) + \phi(\xi - 2^k e_1)).
\]

Noting the fact that supp $\phi(\xi) = \{\xi \in \mathbb{R}^2 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, therefore

\[
I = \frac{(\ln N)^2}{4N} \sum_{N \leq k \leq (1+\delta)N} \int_{\mathbb{R}^2} \phi(\xi) \left( \int_0^{t_0} e^{-(t_0-s)|\xi|^2} \int_{\mathbb{R}^2} (2e^{-s|\xi-\eta|^2} - 1)e^{-s|\eta|^2} \right.
\]

\[
\times 2^k(\eta_1 \phi(\xi - \eta + 2^k e_1)\phi(\eta - 2^k e_1) - \eta_1 \phi(\xi - \eta - 2^k e_1)\phi(\eta + 2^k e_1)) \, d\eta \, ds \, d\xi.
\]

By triangle inequality, we have

\[
I \geq \frac{(\ln N)^2}{4N} \sum_{N + \log_4((\ln 4N)N) \leq k \leq (1+\delta)N} \int_{\mathbb{R}^2} \phi(\xi) \left( \int_0^{t_0} e^{-(t_0-s)|\xi|^2} \int_{\mathbb{R}^2} (2e^{-s|\xi-\eta|^2} - 1)e^{-s|\eta|^2} \right.
\]

\[
\times 2^k(\eta_1 \phi(\xi - \eta + 2^k e_1)\phi(\eta - 2^k e_1) - \eta_1 \phi(\xi - \eta - 2^k e_1)\phi(\eta + 2^k e_1)) \, d\eta \, ds \, d\xi \\
- \frac{(\ln N)^2}{4N} \sum_{N \leq k \leq N + \log_4((\ln 4N)N)} \int_{\mathbb{R}^2} \phi(\xi) \left( \int_0^{t_0} e^{-(t_0-s)|\xi|^2} \int_{\mathbb{R}^2} (2e^{-s|\xi-\eta|^2} - 1)e^{-s|\eta|^2} \right.
\]

\[
\times 2^k(\eta_1 \phi(\xi - \eta + 2^k e_1)\phi(\eta - 2^k e_1) - \eta_1 \phi(\xi - \eta - 2^k e_1)\phi(\eta + 2^k e_1)) \, d\eta \, ds \, d\xi \\]

\[
:= I_1 - I_2.
\]

Noting the support of $\phi(\eta \pm 2^k e_1)$ and $\phi(\xi - \eta \pm 2^k e_1)$, we have

$$\eta_1 \phi(\eta - 2^k e_1) \geq 0, \quad -\eta_1 \phi(\eta + 2^k e_1) \geq 0.$$  

Because $t_0 = (\ln N)^{-1}2^{-2N}$, it is easy to check that if $|\xi - \eta| \geq \frac{9}{10}2^k$, for any $k \geq N + \log_4((\ln 4N)N) (N \gg 1)$,

$$2e^{-|\xi-\eta|^2} \leq 2e^{-(\ln N)^{-1}2^{-2N}2^{2k}} = 2e^{-(\frac{9}{10})2^{k}((\ln N)^{-1}2^{-2N}2^{2N}\cdot \ln 4N)} = 2e^{-(\frac{9}{10})2^{k}\ln 4} \leq \frac{3}{4}.$$  

Therefore, due to $k \geq N \gg 1$, it is easy to check that $|\eta| \geq \frac{9}{10}2^k$ for $\eta \in \text{supp} \phi(\eta - 2^k e_1)$,

\[
I_1 \geq \frac{(\ln N)^2}{16N} \sum_{k = N + \log_4((\ln 4N)N)}^{(1+\delta)N} \int_{\mathbb{R}^2} \phi(\xi) \left( \int_0^{t_0} e^{-(t_0-s)|\xi|^2} \int_{\mathbb{R}^2} e^{-s|\eta|^2} \right.
\]

\[
\times 2^k\eta_1 \phi(\xi - \eta + 2^k e_1)\phi(\eta - 2^k e_1) \, d\eta \, ds \, d\xi.
\]
\[
\begin{aligned}
&= \frac{(\ln N)^2}{16N} \sum_{k=N}^{(1+\delta)N} \int \phi(\xi) \frac{e^{-t|\xi|^2} - e^{-t|\eta|^2}}{|\eta|^2 - |\xi|^2} 2^k \eta_1 \phi(\xi - \eta + 2^k e_1) \phi(\eta - 2^k e_1) \, d\eta \, d\xi \\
&\geq \frac{C(\ln N)^2}{N} \sum_{k=N}^{(1+\delta)N} \int \phi(\xi) \phi(\xi - \eta + 2^k e_1) \phi(\eta - 2^k e_1) \, d\eta \, d\xi \\
&\geq C\delta(\ln N)^2.
\end{aligned}
\]

For \(I_2\), from above analysis, it yields that
\[
I_2 \leq \frac{C(\ln N)^2}{N} \sum_{k=N}^{(1+\delta)N} \int \phi(\xi) \phi(\xi - \eta + 2^k e_1) \phi(\eta - 2^k e_1) \, d\eta \, d\xi \leq \frac{C(\ln N)^3}{N}.
\]

Taking advantage of (3.17), \(II\) can be written as
\[
II = \frac{(\ln N)^2}{N} \sum_{N\leq k \leq (1+\delta)N} \int_{\mathbb{R}^2} \phi(\xi) \left( \int_0^{t_0} e^{-t(\eta - \xi)^2} \int_{\mathbb{R}^2} \frac{\xi_2 - \eta_2}{|\xi - \eta|^2} (1 - e^{-t|\xi - \eta|^2}) e^{-t|\eta|^2} \right. \\
\times 2^k(\eta_1 \phi(\xi - \eta + 2^k e_1) \phi(\eta - 2^k e_1) - \eta_1 \phi(\xi - \eta - 2^k e_1) \phi(\eta + 2^k e_1)) \, d\eta \, dt \bigg) \, d\xi \\
\leq \frac{(\ln N)^2}{N} \sum_{N\leq k \leq (1+\delta)N} \int \phi(\xi) \frac{e^{-t_0|\xi|^2} - e^{-t_0|\eta|^2}}{|\eta|^2 - |\xi|^2} \frac{|\xi_2 - \eta_2|}{|\xi - \eta|^2} \\
\times 2^k(\eta_1 \phi(\xi - \eta + 2^k e_1) \phi(\eta - 2^k e_1) - \eta_1 \phi(\xi - \eta - 2^k e_1) \phi(\eta + 2^k e_1)) \, d\eta \, d\xi \\
\leq \frac{C(\ln N)^2}{N} \sum_{N\leq k \leq (1+\delta)N} 2^{-2k} \leq \frac{C(\ln N)^3}{N2^{\delta N}}.
\]

Now we need to estimate the last two term on the right-hand side of inequality (3.16).

At first, it holds that
\[
\|\partial_{x_1} U_0\|_{L_t^\infty L^\infty} \leq C2^{\frac{(1+\delta)}{2}N}, \quad \|\partial_{x_2} U_0\|_{L_t^\infty L^\infty} \leq C2^{\frac{(1+\delta)}{2}N}, \\
\|\partial_{x_1 x_2} U_0\|_{L_t^\infty L^\infty} \leq C2^{\frac{(1+\delta)}{2}N}, \quad \|\partial_{x_2}^2 U_0\|_{L_t^\infty L^\infty} \leq C2^{\frac{(1+\delta)}{2}N}.
\]

Based on these, we obtain that
\[
\left\| \Delta_0 \int_0^{t_0} e^{(t-s)\Delta} (U_0^{(2)} \partial_{x_2} U_0^{(2)} + \int_0^s \partial_{x_1 x_2} U_0^{(2)} \, d\tau \partial_{x_1} U_0^{(2)}(s)) \, ds \right\|_{L_t^\infty L^\infty} \\
\leq C t_0 \|U_0\|_{L_t^\infty L^\infty}^2 + t_0^2 \|\partial_{x_1 x_2} U_0\|_{L_t^\infty L^\infty} \|\partial_{x_1} U_0^2\|_{L_t^\infty L^\infty} \leq C2^{(3\delta-1)N},
\]

and
\[
\left\| \Delta_0 \int_0^{t_0} e^{(t-s)\Delta} \left( \int_0^s \partial_{x_2} (\partial_{x_1} U_0^{(1)} + \partial_{x_1} U_0^{(2)}) \, d\tau \partial_{x_2} U_0^{(2)}(s)) \, ds \right\|_{L_t^\infty L^\infty} \\
\leq C t_0^2 \|\partial_{x_1 x_2} U_0^{(1)}\|_{L_t^\infty L^\infty} + \|\partial_{x_2}^2 U_0^{(2)}\|_{L_t^\infty L^\infty} \|\partial_{x_2} U_0^{(2)}\|_{L_t^\infty L^\infty} \leq C2^{(3\delta-2)N}.
\]

Therefore, we have
\[
\|U_1\|_{B^\frac{\delta}{4}_4}^2 \geq C\delta(\ln N)^2 - \frac{C(\ln N)^3}{N} - \frac{C(\ln N)^2}{N2^{\delta N}} - C2^{(3\delta-1)N} \geq C\delta(\ln N)^2.
\]
Following the same method in (3.15), then we obtain
\[ \|U_2(t_0)\|_{B^{\frac{1}{2}}_{4,q}} \leq C\|U_2(t_0)\|_{B^{0}_{2,\infty,1}} \leq C^{2}(\frac{1}{2} + 2\varepsilon + \frac{3}{2}\delta)N. \]

Therefore, we conclude that, for \( 3\varepsilon + 5\delta < 1 \), and \( q > 2 \)
\[ \|u(t_0)\|_{B^{\frac{1}{2}}_{4,q}} \geq \|U_1(t_0)\|_{B^{\frac{1}{2}}_{4,q}} - \|U_0(t_0)\|_{B^{\frac{1}{2}}_{4,q}} - \|U_2(t_0)\|_{B^{\frac{1}{2}}_{4,q}} \]
\[ \geq C\delta (\ln N)^2 - C(\ln N)N^{\frac{\gamma}{q} - \frac{1}{2}} - C2^{(-\frac{1}{2} + 2\varepsilon + \frac{3}{2}\delta)N} \]
\[ \geq C\delta (\ln N)^2, \]
associated with initial data \( \|u_0\|_{B^{\frac{1}{2}}_{4,q}} \leq C(\ln N)N^{\frac{1}{\gamma} - \frac{1}{2}}. \)

4. Appendix

Proposition 4.1. Let \( B(f, g) \) be defined by (1.3). There exists an absolute constant \( C \) such that
\[ \sup_{t>0} \|B(f, g)(t)\|_{B^{\frac{1}{2}}_{4,2}} \leq C\|f\|_{B^{\frac{1}{2}}_{4,2}}\|g\|_{B^{\frac{1}{2}}_{4,2}}. \]

Proof. By Bony’s paraproduct decomposition, we have
\[ B(f, g) = \sum_{j \in \mathbb{Z}} B(\dot{S}_{j-1}f, \dot{\Delta}_j g) + \sum_{j \in \mathbb{Z}} B(\dot{\Delta}_j f, \dot{S}_{j-1} g) + \sum_{j \in \mathbb{Z}} B(\dot{\Delta}_j f, \dot{\Delta}_j g) \]
\[ := I + II + III. \]

For \( I \), using semi-group estimates and Bernstein’s inequality, it follows that
\[ \|I\|_{B^{\frac{1}{2}}_{4,2}} \leq \int_0^t (t-s)^{-\frac{1}{2}} \left\{ 2^{-\frac{3}{2}k} \sum_{|j-k| \leq 5} \dot{\Delta}_k((\dot{S}_{j-1}e^{s\Delta}f^i)(\dot{\Delta}_j\partial_ie^{s\Delta}g)) \right\}_{L^4 \ell^2(k \in \mathbb{Z})} ds \]
\[ + \int_0^t (t-s)^{-\frac{1}{2}} \left\{ 2^{-\frac{3}{2}k} \sum_{|j-k| \leq 5} \dot{\Delta}_k((\int_0^s \dot{S}_{j-1}e^{\tau\Delta}\partial_i \partial_m f^m d\tau)(\dot{\Delta}_j\partial_ie^{s\Delta}g)) \right\}_{L^4 \ell^2(k \in \mathbb{Z})} ds \]
\[ \leq \int_0^t (t-s)^{-\frac{1}{2}} \left\{ \sum_{|j-k| \leq 5} 2^{-\frac{3}{2}(k-j)}2^{-j} ||\dot{S}_{j-1}e^{s\Delta}f^i||_{L^\infty}2^{-\frac{3}{2}j} ||\dot{\Delta}_j\partial_ie^{s\Delta}g||_{L^4} \right\}_{\ell^2(k \in \mathbb{Z})} ds \]
\[ + \int_0^t (t-s)^{-\frac{1}{2}} \left\{ \sum_{|j-k| \leq 5} 2^{-\frac{3}{2}(k-j)}2^{-\frac{3}{2}j} \int_0^s ||\dot{S}_{j-1}e^{\tau\Delta}\partial_i\partial_m f^m||_{L^\infty} d\tau ||\dot{\Delta}_j\partial_ie^{s\Delta}g||_{L^4} \right\}_{\ell^2(k \in \mathbb{Z})} ds \]
\[ \leq C \int_0^t (t-s)^{-\frac{1}{2}} \left( ||f||_{B^{-1}_{\infty,2}}||\partial_ie^{s\Delta}g||_{B^{\frac{1}{2}}_{4,2}} + \int_0^s ||e^{\tau\Delta}\partial_i f||_{B^{-\frac{1}{2}}_{\infty,2}} ||\nabla e^{s\Delta}g||_{B^0_{4,2}} d\tau \right) ds \]
\[ \leq C \int_0^t (t-s)^{-\frac{1}{2}} \left( ||f||_{B^{\frac{1}{2}}_{4,2}} ||s^{-\frac{1}{2}}g||_{B^{\frac{1}{2}}_{4,2}} + \int_0^s ||e^{\tau\Delta}\partial_i f||_{B^{-\frac{1}{2}}_{4,2}} ||\nabla e^{s\Delta}g||_{B^0_{4,2}} d\tau \right) ds \]
\[ \leq C \int_0^t (t-s)^{-\frac{1}{2}} \left( ||f||_{B^0_{4,2}} ||s^{-\frac{1}{2}}g||_{B^0_{4,2}} + \int_0^s ||e^{\tau\Delta}\partial_i f||_{B^{-\frac{1}{2}}_{4,2}} ||\nabla e^{s\Delta}g||_{B^0_{4,2}} d\tau \right) ds \]
\[ \leq C \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds ||f||_{B^{\frac{1}{2}}_{4,2}} ||g||_{B^{\frac{1}{2}}_{4,2}} \leq C ||f||_{B^{\frac{1}{2}}_{4,2}} ||g||_{B^{\frac{1}{2}}_{4,2}}. \]
For $II$, owning to $\hat{B}^0_{1,2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, we have

$$\|II\|_{\hat{B}^1_{4,2}} \leq \int_0^t (t-s)^{-\frac{1}{2}} \left\{ \sum_{|j-k| \leq 5} \hat{\Delta}_k((\hat{S}_{j-1} \partial_t e^{s\Delta}g)(\hat{\Delta}_j e^{s\Delta}f^i)) \right\}_{L^4} \|e\|_{\ell^2(k)} ds$$

$$+ \int_0^t (t-s)^{-\frac{1}{2}} \left\{ \sum_{|j-k| \leq 5} \hat{\Delta}_k((\hat{S}_{j-1} \partial_t e^{s\Delta}g)(\int_0^s \Delta_j e^{r\Delta} \partial_r \partial_m f^i \, dr)) \right\}_{L^4} \|e\|_{\ell^2(k)} ds$$

$$\leq \int_0^t (t-s)^{-\frac{1}{2}} \left\{ \sum_{|j-k| \leq 5} 2^{-\frac{3}{2}(k-j)} 2^{-\frac{1}{2}j} \|\hat{S}_{j-1} \partial_t e^{s\Delta}g\|_{L^4} 2^{-j} \|\hat{\Delta}_j e^{s\Delta} f^i\|_{L^\infty} \right\}_{\ell^2(k)} ds$$

$$+ \int_0^t (t-s)^{-\frac{1}{2}} \left\{ \sum_{|j-k| \leq 5} 2^{-\frac{3}{2}(k-j)} 2^{-\frac{1}{2}j} \int_0^s \|\Delta_j e^{r\Delta} \partial_r f\|_{L^\infty} \, dr \|\hat{S}_{j-1} \partial_t e^{s\Delta}g\|_{L^4} \right\}_{\ell^2(k)} ds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|f\|_{\hat{B}^{-1}_{4,2}} \|\partial_t e^{s\Delta}g\|_{\hat{B}^{-\frac{1}{2}}_{4,2}} + \int_0^s \|e^{r\Delta} \partial_r f\|_{\hat{B}^{-\frac{1}{2}}_{4,2}} \, dr \|\nabla e^{s\Delta}g\|_{\hat{B}^0_{4,2}} ds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2}} ds \|f\|_{\hat{B}^{-\frac{1}{2}}_{4,2}} \|g\|_{\hat{B}^{-\frac{1}{2}}_{4,2}} \leq C \|f\|_{\hat{B}^{-\frac{1}{2}}_{4,2}} \|g\|_{\hat{B}^{-\frac{1}{2}}_{4,2}}.$$

For $III$, using Littlewood-Paley decomposition yields that

$$III = \sum_{m \in \mathbb{Z}} \sum_{j \geq m-3} \hat{\Delta}_m \int_0^t e^{(t-s)\Delta} (\hat{\Delta}_j e^{s\Delta} f \hat{\Delta}_j \nabla e^{s\Delta}g + \hat{\Delta}_j \int_0^s \nabla \Delta e^{r\Delta} f \hat{\Delta}_j \nabla e^{s\Delta}g) \, ds$$

$$= \sum_{m \in \mathbb{Z}} \sum_{m-3 \leq j < m+N_0} \hat{\Delta}_m \int_0^t e^{(t-s)\Delta} (\hat{\Delta}_j e^{s\Delta} f \hat{\Delta}_j \nabla e^{s\Delta}g + \hat{\Delta}_j \int_0^s \nabla \Delta e^{r\Delta} f \hat{\Delta}_j \nabla e^{s\Delta}g) \, ds$$

$$+ \sum_{m \in \mathbb{Z}} \sum_{j \geq m+N_0} \hat{\Delta}_m \int_0^t e^{(t-s)\Delta} (\hat{\Delta}_j e^{s\Delta} f \hat{\Delta}_j \nabla e^{s\Delta}g + \hat{\Delta}_j \int_0^s \nabla \Delta e^{r\Delta} f \hat{\Delta}_j \nabla e^{s\Delta}g) \, ds$$

$$:= III_1 + III_2.$$

For $III_1$, thanks to Young’s inequality, we obtain that

$$\|III_1\|_{\hat{B}^1_{4,2}} \leq \left\{ \sum_{|j-m| \leq 1} \sum_{|j-k| \leq N_0} \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{\Delta}_k \hat{\Delta}_m ((\hat{\Delta}_j e^{s\Delta} f^i)(\hat{\Delta}_j \partial_t e^{s\Delta}g)) \|_{L^4} \right\}_{\ell^2(k)}$$

$$+ \left\{ \sum_{|j-m| \leq 1} \sum_{|j-k| \leq N_0} \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{\Delta}_k \hat{\Delta}_m ((\int_0^s \hat{\Delta}_j e^{r\Delta} \partial_r \partial_m f^i \, dr)(\hat{\Delta}_j \partial_t e^{s\Delta}g)) \|_{L^4} \right\}_{\ell^2(k)}$$

$$\leq C \left\{ \sum_{|j-m| \leq N_0} \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{\Delta}_m ((\hat{\Delta}_j e^{s\Delta} f^i)(\hat{\Delta}_j \partial_t e^{s\Delta}g)) \|_{L^4} \right\}_{\ell^2(m)}$$

$$+ C \left\{ \sum_{|j-m| \leq N_0} \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{\Delta}_m ((\int_0^s \hat{\Delta}_j e^{r\Delta} \partial_r \partial_m f^i \, dr)(\hat{\Delta}_j \partial_t e^{s\Delta}g)) \|_{L^4} \right\}_{\ell^2(m)}$$

$$\leq C \left\{ \sum_{|j-m| \leq N_0} 2^{-\frac{3}{2}(m-j)} \int_0^t (t-s)^{-\frac{1}{2}} \|\hat{\Delta}_j e^{s\Delta} f \|_{L^\infty} 2^{-j} \|\hat{\Delta}_j \partial_t e^{s\Delta}g\|_{L^4} \right\}_{\ell^2(m)}.$$
\[ + C \left\{ \sum_{|j-m| \leq N_0} 2^{-\frac{3}{2}(m-j)} \int_0^t (t-s)^{-\frac{1}{2}} \int_0^s 2^{-\frac{1}{2}j} \| \hat{\Delta}_j e^{r_\Delta} \partial_t \partial_m f^m \|_{L^\infty} \, dt \right\} e^{(m)} \]
\[ \leq C \int_0^t (t-s)^{-\frac{1}{2}} \left( \| f \|_{B_{2,1}^{-1}} \| \partial_t e^{s \Delta} g \|_{B_{4,2}^{-1}} + \int_0^s \| e^{r_\Delta} \partial_t \nabla f \|_{B_{2,1}^{-\frac{3}{2}}} \, dr \right) \] \[ \leq C \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \| f \|_{B_{4,2}^{-\frac{1}{2}}} \| g \|_{B_{4,2}^1} \leq C \| f \|_{B_{4,2}^{-\frac{1}{2}}} \| g \|_{B_{4,2}^{\frac{1}{2}}}. \]

For \( III_2 \), due to \( L^2(\mathbb{R}^2) \hookrightarrow \dot{B}_{2,2}^0(\mathbb{R}^2) \hookrightarrow \dot{B}_{4,2}^{\frac{1}{2}}(\mathbb{R}^2) \) and Lemma 2.5, it yields that
\[ \| III_2 \|_{B_{4,2}^{\frac{1}{2}}} \leq \| III_2 \|_{L^2} \]
\[ \leq \sum_{j \in \mathbb{Z}} \sum_{m=j-N_0} \hat{\Delta}_m \int_0^t (e^{(t-s)})(\hat{\Delta}_j e^{s \Delta} f \hat{\Delta}_j \nabla e^{s \Delta} g + \hat{\Delta}_j \int_0^s \nabla \div e^{r_\Delta} f \hat{\Delta}_j \nabla e^{s \Delta} g) \, ds \|_{L^2} \]
\[ \leq \sum_{j \in \mathbb{Z}} \left\{ \int \int_{m \leq j-N_0} \hat{\varphi}_m(\xi) \frac{e^{-t|\xi|^2} - e^{-t(|\xi| - \eta)^2 + |\eta|^2}}{|\xi - \eta|^2 + |\eta|^2 - |\xi|^2} \hat{\varphi}_j(\xi - \eta) \hat{\varphi}_j(\eta) \, d\eta \right\} \|_{L^2} \]
\[ + \sum_{j \in \mathbb{Z}} \left\{ \int \int_{m \leq j-N_0} \hat{\varphi}_m(\xi) \frac{e^{-t|\xi|^2} - e^{-t|\eta|^2}}{|\eta|^2 - |\xi|^2} \frac{e^{-t(|\xi| - \eta)^2 + |\eta|^2}}{|\eta|^2 + |\xi - \eta|^2 - |\xi|^2} \right. \]
\[ \left. \times \frac{(\xi - \eta) \xi (\xi - \eta) \xi}{|\xi - \eta|^2} \hat{\varphi}_j(\xi - \eta) \hat{\varphi}_j(\eta) \, d\eta \right\} \|_{L^2} \leq C \sum_{j \in \mathbb{Z}} 2^{-2j} \| \hat{\Delta}_j f \|_{L^4} \| \hat{\Delta}_j \nabla g \|_{L^4} \leq C \sum_{j \in \mathbb{Z}} \sum_{|j' - j| \leq 1} 2^{-2j} 2^{j'} \| \hat{\Delta}_j f \|_{L^4} \| \hat{\Delta}_j' g \|_{L^4} \]
\[ = C \sum_{j \in \mathbb{Z}} \sum_{|j' - j| \leq 1} 2^{-\frac{3}{2}(j-j')} 2^{-\frac{1}{2}j} \| \hat{\Delta}_j f \|_{L^4} \| \hat{\Delta}_j' g \|_{L^4} \leq C \| f \|_{B_{4,2}^{-\frac{1}{2}}} \| g \|_{B_{4,2}^{\frac{1}{2}}} \]

\[ \square \]

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Institute of Applied Physics and Computational Mathematics, Beijing 100191, P.R. China

Email address: chen_qionglei@iapcm.ac.cn

Institute of Applied Physics and Computational Mathematics, Beijing 100191, P.R. China

Email address: nieyao930930@163.com