Dynamics of interlacing peakons (and shockpeakons) in the Geng–Xue equation

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We consider multipeakon solutions, and to some extent also multishockpeakon solutions, of a coupled two-component integrable PDE found by Geng and Xue as a generalization of Novikov’s cubically nonlinear Camassa–Holm type equation. In order to make sense of such solutions, we find it necessary to assume that there are no overlaps, meaning that a peakon or shockpeakon in one component is not allowed to occupy the same position as a peakon or shockpeakon in the other component. Therefore one can distinguish many inequivalent configurations, depending on the order in which the peakons or shockpeakons in the two components appear relative to each other. Here we are particularly interested in the case of interlacing peakon solutions, where the peakons alternatingly occur in one component and in the other. Based on explicit expressions for these solutions in terms of elementary functions, we describe the general features of the dynamics, and in particular the asymptotic large-time behaviour (assuming that there are no antipeakons, so that the solutions are globally defined). As far as the positions are concerned, interlacing Geng–Xue peakons display the usual scattering phenomenon where the peakons asymptotically travel with constant velocities, which are all distinct, except that the two fastest peakons (the fastest one in each component) will have the same velocity. However, in contrast to many other peakon equations, the amplitudes of the peakons will not in general tend to constant values; instead they grow or decay exponentially. Thus the logarithms of the amplitudes (as functions of time) will asymptotically behave like straight lines, and comparing these lines for large positive and negative times, one observes phase shifts similar to those seen for the positions of the peakons (and also for the positions of solitons in many other contexts). In addition to these $K + K$ interlacing pure peakon solutions, we also investigate $1 + 1$ shockpeakon solutions, and collisions leading to shock formation in a $2 + 2$ peakon–antipeakon solution.

Keywords: Geng–Xue equation; peakons; shockpeakons.

1. Introduction

The Geng–Xue equation [1], also known as the two-component Novikov equation, is the following integrable two-component PDE in $1 + 1$ dimensions:

\[
\begin{align*}
\frac{m_t}{m} + (m_u + 3mu)v &= 0, \\
\frac{n_t}{n} + (n_v + 3nv)u &= 0,
\end{align*}
\]

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where \( m = u - u_{xx} \) and \( n = v - v_{xx} \) are auxiliary quantities associated to the two unknown functions \( u(x, t) \) and \( v(x, t) \), and where subscripts denote partial derivatives, as usual. It is a close mathematical relative of the Degasperis–Procesi and Novikov equations, which are in turn offspring of the Camassa–Holm shallow water wave equation; see Section 1.4.

1.1 Peakon solutions of the Geng–Xue equation

Our primary subject in this article is a particular class of solutions of the Geng–Xue equation (1) known as peakons (peaked solitons)—weak solutions formed by nonlinear superposition of \( e^{-|x|} \)-shaped waves:

\[
\begin{align*}
  u(x, t) &= N \sum_{k=1}^{N} m_k(t) e^{-|x - x_k(t)|}, \\
  v(x, t) &= N \sum_{k=1}^{N} n_k(t) e^{-|x - x_k(t)|}.
\end{align*}
\]

We will always label the variables in increasing order,

\[ x_1 < x_2 < \cdots < x_N. \]

As will be explained in Section 2 below, the ansatz (2) satisfies the PDE (1) in a certain distributional sense if and only if the functions \( x_k(t), m_k(t) \) and \( n_k(t) \) satisfy the ODEs

\[
\begin{align*}
  \dot{x}_k &= u(x_k) v(x_k), \\
  \dot{m}_k &= m_k (u(x_k) v(x_k) - 2u(x_k) v(x_k)), \\
  \dot{n}_k &= n_k (u(x_k) v(x_k) - 2u(x_k) v(x_k)),
\end{align*}
\]

for \( k = 1, 2, \ldots, N \), with the additional constraint that the two types of peakons (those belonging to \( u \) and to \( v \)) occupy different sites \( x_k \); in other words, for each \( k \), either peakon number \( k \) belongs to \( u \),

\[ m_k \neq 0 \quad \text{and} \quad n_k = 0, \]

or else it belongs to \( v \),

\[ m_k = 0 \quad \text{and} \quad n_k \neq 0. \]

(Note that these conditions are preserved by the ODEs. Also note that we may assume that \( m_k = n_k = 0 \) doesn’t happen, since in that case we can just discard the \( k \)th terms in the ansatz.) See Fig. 1.
Fig. 1. A non-overlapping peakon configuration (2), meaning that the peakons in $u(x, t)$ and $v(x, t)$ occupy different sites: since $n_1$ is nonzero, $m_1$ must be zero, and since $m_2$ is nonzero, $n_2$ must be zero, and so on.

The ODEs (3) have been written using a convenient shorthand notation, where $u(x_k)$, $u_x(x_k)$, $v(x_k)$ and $v_x(x_k)$ are nothing but abbreviations defined as follows:

$$u(x_k) := \sum_{i=1}^{N} m_i e^{-|x_k - x_i|},$$

$$u_x(x_k) := -\sum_{i=1}^{N} m_i \text{sgn}(x_k - x_i) e^{-|x_k - x_i|},$$

$$v(x_k) := \sum_{i=1}^{N} n_i e^{-|x_k - x_i|},$$

$$v_x(x_k) := -\sum_{i=1}^{N} n_i \text{sgn}(x_k - x_i) e^{-|x_k - x_i|},$$

where the convention $\text{sgn} 0 = 0$ is understood. The formulas for $u(x_k)$ and $u_x(x_k)$ in (4) result from formally substituting $x = x_k$ into the ansatz $u(x) = \sum_{i=1}^{N} m_i e^{-|x - x_i|}$ and its derivative $u_x(x) = -\sum_{i=1}^{N} m_i \text{sgn}(x - x_i) e^{-|x - x_i|}$, respectively. However, note that $u_x(x_k)$ does not exist in the ordinary sense if $m_k \neq 0$, so what we have denoted by $u_x(x_k)$ is actually the average of the the left and right limits of $u_x$ at $x = x_k$:

$$u_x(x_k) = \langle u_x \rangle(x_k) = \frac{1}{2} \left( \lim_{x \to x_k^-} u_x(x) + \lim_{x \to x_k^+} u_x(x) \right).$$

The Geng–Xue peakon ODEs (3) constitute a Lax integrable system, as we will explain in detail later in this article, and the general solution can be written down explicitly in terms of elementary functions.

**Remark 1.1** Our detailed study of the Geng–Xue equation is partly motivated by our interest in understanding the structure of peakon equations from some unifying principle. One of the most delicate issues with Lax integrable peakon equations is that the Lax equations in the peakon sector are distributional equations and as such require special care when dealing with nonlinear operations involving distributions with singular support. This makes peakons an intriguing ‘borderline’ case of Lax integrability. In this article, we take a conservative approach: the Lax pair with $m = u - u_{xx}$ and $n = v - v_{xx}$ being distributions with non-overlapping singular supports is a well-defined distributional pair requiring only the standard operation of multiplying measures by continuous functions. This, along with the distributional compatibility condition, dictates a unique way of defining a distributional Geng–Xue equation.
Fig. 2. A non-overlapping shockpeakon configuration (5), meaning that the shockpeakons (or ordinary peakons as a special case) in \( u(x, t) \) and \( v(x, t) \) occupy different sites. Compared to Fig. 1, shocks \( s_2 > 0 \) and \( s_5 > 0 \) have been added to \( u(x, t) \) (but \( s_3 = 0 \)), and a shock \( r_4 > 0 \) has been added to \( v(x, t) \) (but \( r_1 = 0 \)).

1.2 Shockpeakon solutions of the Geng–Xue equation

Remarkably, the Geng–Xue equation also admits a weaker type of solution, shockpeakons, given by the more general ansatz

\[
\begin{align*}
    u(x, t) &= \sum_{k=1}^{N} \left( m_k(t) - s_k(t) \text{ sgn}(x - x_k(t)) \right) e^{-|x-x_k(t)|}, \\
v(x, t) &= \sum_{k=1}^{N} \left( n_k(t) - r_k(t) \text{ sgn}(x - x_k(t)) \right) e^{-|x-x_k(t)|}.
\end{align*}
\]  

(5)

See Fig. 2. If \( s_k(t) \neq 0 \), then the function \( x \mapsto u(x, t) \) has a jump of size \( -2s_k(t) \) the point \( x = x_k(t) \), and similarly for \( x \mapsto v(x, t) \) if \( r_k(t) \neq 0 \), so shockpeakon solutions are only piecewise continuous. The ansatz (5) satisfies the Geng–Xue equation (again in a distributional sense explained in Section 2) if and only if the functions \( x_k(t), m_k(t), n_k(t), s_k(t) \) and \( r_k(t) \) satisfy the ODEs

\[
\begin{align*}
    \dot{x}_k &= u(x_k) v(x_k), \\
    \dot{m}_k &= m_k \left( u(x_k) v(x_k) - 2u(x_k) v(x_k) \right) + s_k \left( u(x_k) v(x_k) + u(x_k) v(x_k) \right), \\
    \dot{s}_k &= s_k \left( 2u(x_k) v(x_k) - u(x_k) v(x_k) \right), \\
    \dot{n}_k &= n_k \left( u(x_k) v(x_k) - 2u(x_k) v(x_k) \right) + r_k \left( u(x_k) v(x_k) + u(x_k) v(x_k) \right), \\
    \dot{r}_k &= r_k \left( 2u(x_k) v(x_k) - u(x_k) v(x_k) \right),
\end{align*}
\]  

(6)

for \( k = 1, 2, \ldots, N \), with the non-overlapping constraint that, for each \( k \), either

\( (m_k \neq 0 \text{ or } s_k \neq 0) \quad \text{and} \quad n_k = r_k = 0, \)

or

\( m_k = s_k = 0 \quad \text{and} \quad (n_k \neq 0 \text{ or } r_k \neq 0). \)
(See Theorem 2.1.) In (6), the previous shorthand notation (4) has been extended as follows:

\[ u(x_k) := \sum_{i=1}^{N} (m_i - s_i \text{ sgn}(x_k - x_i)) e^{-|x_k - x_i|}, \]

\[ u_s(x_k) := \sum_{i=1}^{N} (s_i - m_i \text{ sgn}(x_k - x_i)) e^{-|x_k - x_i|}, \]

\[ v(x_k) := \sum_{i=1}^{N} (n_i - r_i \text{ sgn}(x_k - x_i)) e^{-|x_k - x_i|}, \]

\[ v_s(x_k) := \sum_{i=1}^{N} (r_i - n_i \text{ sgn}(x_k - x_i)) e^{-|x_k - x_i|}. \]

We emphasize that these formulas are nothing but abbreviations which are convenient when writing down the shockpeakon ODEs (6); when interpreting the solutions distributionally, the precise value assigned to \( u \) at a jump discontinuity is irrelevant, and the derivative \( u_s \) doesn’t exist at such a point. However, we do have

\[ u(x_k) = \frac{1}{2} \left( \lim_{x \to x_k^-} u(x) + \lim_{x \to x_k^+} u(x) \right), \quad u_s(x_k) = \frac{1}{2} \left( \lim_{x \to x_k^-} u_s(x) + \lim_{x \to x_k^+} u_s(x) \right), \]

and similarly for \( v(x_k) \) and \( v_s(x_k) \).

**Remark 1.2** Of course, if \( s_k = r_k = 0 \) for all \( k \), then the shockpeakon ansatz (5), the ODEs (6) and the shorthand notation (7) reduce to their ordinary peakon counterparts (2), (3), (4), respectively.

**Remark 1.3** Shockpeakons were first introduced for the Degasperis–Procesi equation by Lundmark in [2], where it was found that it is necessary to have \( s_i \geq 0 \) in order to obtain an entropy solution as defined by Coclite and Karlsen [3, 4]; i.e. the jump at each shock must go downwards, from high on the left to low on the right. This condition will automatically be satisfied whenever a shockpeakon forms at a peakon–antipeakon collision in the Degasperis–Procesi equation.

Most likely, it will turn out natural to impose the corresponding restriction \( s_i \geq 0 \) and \( r_i \geq 0 \) on Geng–Xue shockpeakons as well, although we will not attempt here to define entropy solutions.

**Example 1.4** In the 1 + 1 case, with one shockpeakon in \( u(x, t) \) at \( x = x_1(t) \), and another in \( v(x, t) \) at \( x = x_2(t) \), where \( x_1 < x_2 \), the governing ODEs (6) take the following form:

\[ \dot{x}_1 = m_1 (n_2 + r_2) E_{12}, \]

\[ \dot{x}_2 = (m_1 - s_1) n_2 E_{12}, \]

\[ \dot{m}_1 = (m_1^2 - m_1 s_1 + s_1^2) (n_2 + r_2) E_{12}, \]

\[ \dot{m}_2 = -(m_1 - s_1) (n_2^2 + n_2 r_2 + r_2^2) E_{12}, \]

\[ \dot{s}_1 = s_1 (2m_1 - s_1) (n_2 + r_2) E_{12}, \]

\[ \dot{r}_2 = -r_2 (m_1 - s_1) (2n_2 + r_2) E_{12}, \]
Fig. 3. An interlacing peakon configuration (11) with $K = 3$, meaning that there are three peakons in each component, with the first peakon belonging to $u(x,t)$, the second to $v(x,t)$, and so on, alternatingly.

where $E_{12} = e^{-|x_1 - x_2|} = e^{x_1 - x_2}$. We have managed to integrate these equations explicitly (see Section 6), but we don’t know how to solve the shockpeakon ODEs (6) for $K \geq 2$. In fact, even for $K = 1$ it is not currently clear to us whether one can interpret the solution of (9) in terms of Lax integrability.

1.3 Interlacing peakon configurations

In this article, our main focus will be peakons rather than shockpeakons, and more specifically the particular case of peakon solutions which are interlacing in the sense that there are $N = 2K$ sites

$$x_1 < x_2 < \cdots < x_{2K},$$

with $u$ and $v$ containing $K$ peakons each, located at the odd-numbered sites $x_{2a-1}$ in the case of $u$, and at the even-numbered sites $x_{2a}$ in the case of $v$; see Fig. 3. Moreover, unless stated otherwise, we will assume that the nonzero amplitudes are positive (i.e. that there are no antipeakons with negative amplitude). In other words, we assume

$$m_{2a-1} > 0, \quad m_{2a} = 0,$$
$$n_{2a-1} = 0, \quad n_{2a} > 0,$$  \hspace{1cm} (10)

for $1 \leq a \leq K$, so that

$$u(x,t) = \sum_{a=1}^{K} m_{2a-1}(t) e^{-|x-x_{2a-1}(t)|},$$
$$v(x,t) = \sum_{a=1}^{K} n_{2a}(t) e^{-|x-x_{2a}(t)|}.$$  \hspace{1cm} (11)

Drawing heavily on the groundwork from our previous article [5], we will use inverse spectral methods to derive explicit formulas for these interlacing $(K + K)$-peakon solutions, and then explore their dynamical properties.

Remark 1.5 Explicit solution formulas for the general non-interlacing case, where the number of peakons in $u$ and $v$ need not be equal, and where they may appear in arbitrary order, can be obtained by starting from a larger interlacing case and performing certain limiting procedures in order to drive selected amplitudes to zero. However, the details are rather technical, and will be saved for a separate article.
Example 1.6 The case $K = 1$ is exceptional, and also very simple. There is one peakon in $u$ and one in $v$:

$$
u (x, t) = m_1(t) e^{-|x-x_1(t)|}, \quad v(x, t) = n_2(t) e^{-|x-x_2(t)|}.$$ 

Details will be given in Section 5. With initial data $m_1(0) > 0$, $n_2(0) > 0$ and $x_1(0) < x_2(0)$, the solution is

$$x_1(t) = x_1(0) + ct,$$
$$x_2(t) = x_2(0) + ct,$$
$$m_1(t) = m_1(0) e^{ct},$$
$$n_2(t) = n_2(0) e^{-ct},$$

where $c = m_1(0) n_2(0) e^{x_1(0) - x_2(0)} > 0$.

Example 1.7 When $K = 2$, the interlacing peakon solutions take the form

$$u(x, t) = m_1(t) e^{-|x-x_1(t)|} + m_3(t) e^{-|x-x_3(t)|},$$
$$v(x, t) = n_2(t) e^{-|x-x_2(t)|} + n_4(t) e^{-|x-x_4(t)|},$$

where it is understood that $x_1(t) < x_2(t) < x_3(t) < x_4(t)$ so that the solution really is interlacing. Such solutions are studied in Section 7, mainly for pedagogical reasons. (All the results for $2 + 2$ interlacing peakon solutions in Section 7 are special cases of the statements for $K + K$ interlacing peakon solutions in Section 9, but the proofs for arbitrary $K$ require a fair amount of additional notation.)

The ODEs governing the dynamics of the eight variables

$$x_1(t), x_2(t), x_3(t), x_4(t), m_1(t), n_2(t), m_3(t), n_4(t)$$

are given in equation (83), and the general solution of these ODEs is written out in complete detail in equations (85a) and (85b). These solution formulas contain five constant parameters

$$\lambda_2 > \lambda_1 > 0, \quad \mu_1 > 0, \quad b_\infty > 0, \quad b^*_\infty > 0,$$

and three time-dependent quantities

$$a_1(t) = a_1(0) e^{t/\lambda_1}, \quad a_2(t) = a_2(0) e^{t/\lambda_2}, \quad b_1(t) = b_1(0) e^{t/\mu_1}$$

determined by their initial values

$$a_1(0) > 0, \quad a_2(0) > 0, \quad b_1(0) > 0.$$
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Fig. 4. Spacetime plot of the peakon trajectories $x = x_1(t)$, $x = x_2(t)$, $x = x_3(t)$ and $x = x_4(t)$ for a $2 + 2$ interlacing pure peakon solution of the Geng–Xue equation: $u(x, t) = m_1 e^{-|x-x_1|} + m_3 e^{-|x-x_3|}$ and $v(x, t) = n_2 e^{-|x-x_2|} + n_4 e^{-|x-x_4|}$, with $x_1 < x_2 < x_3 < x_4$ and with $m_1, m_2, m_3, n_4$ positive. The parameters used in the solution formulas (85) are given in Example 1.7, together with a description of the noteworthy features in this picture. The blue curves $x = x_1(t)$ and $x = x_3(t)$ refer to the peakons in $u(x, t)$, while the red curves $x = x_2(t)$ and $x = x_4(t)$ refer to the peakons in $v(x, t)$.

Figure 4 shows a plot of the peakon trajectories

$$x = x_1(t), \quad x = x_2(t), \quad x = x_3(t), \quad x = x_4(t)$$

given by the formulas (85a), for the parameter values

$$\lambda_1 = \frac{1}{3}, \quad \lambda_2 = 3, \quad \mu_1 = 1, \quad a_1(0) = a_2(0) = 1, \quad b_1(0) = 100, \quad b_\infty = 1000, \quad b^*_\infty = 100.$$  \tag{13}$$

It is apparent in the picture, and will be proved in Section 7, that the trajectories approach certain straight lines asymptotically, as $t \to \pm \infty$. More precisely, there are three distinct asymptotic velocities

$$c_1 = \frac{1}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\mu_1} \right) = \frac{1}{2} \left( \frac{1}{1/3} + \frac{1}{1} \right) = 2,$$

$$c_2 = \frac{1}{2} \left( \frac{1}{\lambda_2} + \frac{1}{\mu_1} \right) = \frac{1}{2} \left( \frac{1}{3} + \frac{1}{1} \right) = \frac{2}{3},$$

$$c_3 = \frac{1}{2} \left( \frac{1}{\lambda_4} \right) = \frac{1}{2} \left( \frac{1}{1/6} \right) = \frac{1}{6}.$$  

As $t \to -\infty$, the two leftmost curves $x = x_1(t)$ and $x_2(t)$ both approach the line

$$x = c_1 t + \frac{1}{2} \ln \frac{2 a_1(0) b_1(0)}{\lambda_1 + \mu_1} + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{\lambda_2 (\lambda_2 + \mu_1)} = 2t + \frac{1}{2} \ln 150 + \frac{1}{2} \ln \frac{16}{27}. $$
the curve \( x = x_3(t) \) approaches the line
\[
x = c_2 t + \frac{1}{2} \ln \frac{2a_2(0)b_1(0)}{\lambda_2 + \mu_1} = \frac{2t}{3} + \frac{1}{2} \ln 50,
\]
and the curve \( x = x_4(t) \) approaches the line
\[
x = c_3 t + \frac{1}{2} \ln \frac{2a_2(0)b_1(0)}{\lambda_2 + \mu_1} = \frac{t}{6} + \frac{1}{2} \ln 2000.
\]
(These formulas are taken from (94), which is the special case \( K = 2 \) of the general formulas for the \( K + K \) case given in Theorem 9.3.) As \( t \to +\infty \), it is instead the two rightmost curves \( x = x_3(t) \) and \( x_4(t) \) that pair up; they both approach the line
\[
x = c_1 t + \frac{1}{2} \ln \frac{2a_1(0)b_1(0)}{\lambda_1 + \mu_1} = \frac{2t}{3} + \frac{1}{2} \ln 50 + \frac{1}{2} \ln 16,
\]
while the curve \( x = x_2(t) \) approaches
\[
x = c_4 t + \frac{1}{2} \ln \frac{2a_1(0)b_1(0)}{\lambda_1 + \mu_1} = 2t + \frac{1}{2} \ln 150,
\]
and the curve \( x = x_1(t) \) approaches
\[
x = c_5 t + \frac{1}{2} \ln \frac{2a_1(0)b_1(0)}{\lambda_1 + \mu_1} + \frac{1}{2} \ln \frac{\lambda_2}{\lambda_1(\lambda_1 + \mu_1)} = \frac{t}{6} + \frac{1}{2} \ln \frac{1}{100} + \frac{1}{2} \ln \frac{16}{3}.
\]
(This is proved in (88) and more generally in Theorem 9.3.) A comparison of the two lines of the form \( x = c t + \text{const.} \) shows that the second (outgoing) line is shifted relative to the first (incoming) one by the amount
\[
-\frac{1}{2} \ln \frac{\lambda_1 - \lambda_2}{\lambda_2(\lambda_1 + \mu_1)} = -\frac{1}{2} \ln \frac{16}{27}
\]
in the \( x \) direction, and the corresponding shifts for the other pairs of incoming and outgoing asymptotic lines are also easily computed (see Section 7.4 and Corollary 9.5).

As for the amplitudes given by the formulas (85b), Fig. 5 shows logarithmic plots
\[
y = \ln m_1(t), \quad y = -\ln n_2(t), \quad y = \ln m_3(t), \quad y = -\ln n_4(t),
\]
again with the same parameters (13). The reason for plotting the logarithms is that the amplitudes themselves grow or decay exponentially as \( t \to \pm \infty \), so that the logarithmic plots will asymptotically approach
Fig. 5. Plot of the curves $y = \ln m_1(t)$, $y = -\ln n_2(t)$, $y = \ln m_3(t)$ and $y = -\ln n_4(t)$ for the same solution as in Fig. 4. See Example 1.7 for further explanation. The blue curves $x = \ln m_1(t)$ and $x = \ln m_3(t)$ refer to the peakons in $u(x,t)$, and the red curves $x = -\ln n_2(t)$ and $x = -\ln n_4(t)$ refer to the peakons in $v(x,t)$.

The blue curves $x = \ln m_1(t)$ and $x = \ln m_3(t)$ refer to the peakons in $u(x,t)$, and the red curves $x = -\ln n_2(t)$ and $x = -\ln n_4(t)$ refer to the peakons in $v(x,t)$.

straight lines, and the purpose of the extra minus signs on the even-numbered curves is to highlight certain relations between the slopes of these lines. More precisely, there are three distinct asymptotic slopes

\[
d_1 = \frac{1}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right) = \frac{1}{2} \left( \frac{1}{1/3} - 1 \right) = 1,
\]

\[
d_2 = \frac{1}{2} \left( \frac{1}{\lambda_2} - \frac{1}{\mu_1} \right) = \frac{1}{2} \left( \frac{1}{3} - 1 \right) = -\frac{1}{3},
\]

\[
d_3 = \frac{1}{2} \left( \frac{1}{\lambda_2} \right) = \frac{1}{2} \left( \frac{1}{3} \right) = \frac{1}{6}.
\]

As $t \to -\infty$, the four curves approach, respectively, the four lines

\[
y = d_1 t + \ln \frac{\mu_1}{\lambda_1} + \frac{1}{2} \ln \frac{a_1(0)}{2b_1(0)} \left( \frac{\lambda_1 + \mu_1}{\lambda_1} \right) + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2 (\lambda_2 + \mu_1)}{\lambda_2^2} = t + \ln 3 + \frac{1}{2} \ln \frac{3}{800} + \frac{1}{2} \ln \frac{256}{243},
\]

\[
y = d_1 t + \ln \mu_1 - \frac{1}{2} \ln \frac{b_1(0)}{a_2(0)} \left( \frac{\lambda_1 + \mu_1}{\lambda_2} \right) + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2 (\lambda_2 + \mu_1)}{\lambda_2^3} = t - \frac{1}{2} \ln \frac{200}{3} + \frac{1}{2} \ln \frac{256}{243},
\]

\[
y = d_2 t - \ln \lambda_2 + \frac{1}{2} \ln \frac{a_2(0)}{2b_1(0)} \left( \frac{\lambda_2 + \mu_1}{\lambda_2} \right) = -\frac{t}{3} - \ln 3 + \frac{1}{2} \ln \frac{1}{50},
\]

\[
y = d_3 t - \frac{1}{2} \ln \frac{b_\infty}{2a_2(0)} = \frac{t}{6} - \frac{1}{2} \ln 500,
\]
according to (96), or more generally Theorem 9.8 in the $K + K$ case. As $t \to +\infty$, they approach instead the lines

$$
y = d_1 t + \frac{1}{2} \ln \frac{b_2(0)}{\lambda_1 \lambda_2} + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} = \frac{1}{6} + \frac{1}{2} \ln 100 + \frac{1}{2} \ln \frac{16}{3},$$

$$
y = d_2 t + \ln \mu_1 - \frac{1}{2} \ln \frac{b_1(0) \lambda_1 + \mu_1}{2a_2(0)} + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 + \mu_1)}{\lambda_1}$$

$$= -\frac{t}{3} - \frac{1}{2} \ln 200 + \frac{1}{2} \ln 256,$$

$$
y = d_3 t - \ln \lambda_1 + \frac{1}{2} \ln \frac{a_1(0) \lambda_1 + \mu_1}{2b_1(0)} = t - \ln \frac{1}{3} + \frac{1}{2} \ln \frac{1}{150},$$

$$
y = d_4 t - \frac{1}{2} \ln \frac{b_1(0)}{2a_1(0) \lambda_1 + \mu_1} = t - \frac{1}{2} \ln \frac{75}{2},$$

according to (91) or Theorem 9.8. Here, too, phase shifts between incoming and outgoing asymptotic lines with the same slope are easily computed; see Section 7.5 and Corollary 9.9.

Note that even though $u(x, t)$ and $v(x, t)$ exhibit exponential growth, their product $u(x, t) v(x, t)$ stays bounded as $t \to \pm \infty$; it is this quantity which determines the velocity of the peakons, according to the ODEs (3):

$$\dot{x}_3 = u(x_3) v(x_3).$$

Consequently,

$$uv|_{x=x_1(t)} \sim c_1, \quad uv|_{x=x_2(t)} \sim c_1, \quad uv|_{x=x_3(t)} \sim c_2, \quad uv|_{x=x_4(t)} \sim c_3,$$

as $t \to -\infty$, and

$$uv|_{x=x_1(t)} \sim c_3, \quad uv|_{x=x_2(t)} \sim c_2, \quad uv|_{x=x_3(t)} \sim c_1, \quad uv|_{x=x_4(t)} \sim c_1,$$

as $t \to +\infty$. This is clearly seen in Fig. 6, which shows the graph of the function $u(x, t) v(x, t)$.

For the general $K + K$ interlacing pure peakon solution, the solution formulas are given in terms of abbreviated notation defined in Section 3; the statement is given in Theorem 3.12 and Corollary 4.2. Already in the $3 + 3$ case,

$$u(x, t) = m_1(t) e^{-|x-x_1(t)|} + m_3(t) e^{-|x-x_3(t)|} + m_5(t) e^{-|x-x_5(t)|},$$

$$v(x, t) = n_2(t) e^{-|x-x_2(t)|} + n_4(t) e^{-|x-x_4(t)|} + n_6(t) e^{-|x-x_6(t)|},$$

the solution formulas contain so many terms that we have chosen not write them out here in the expanded form that we give for $K = 2$ in (85); however, this is partly done in [5, Ex. 4.11]. The general solution for the $2K$ positions and the $2K$ amplitudes depends on $4K$ parameters whose values determine the behaviour
Fig. 6. Because of the exponential growth and decay of the amplitudes $m_1, n_2, m_3, n_4$, it is difficult to make meaningful plots of the individual components $u(x,t)$ and $v(x,t)$, but their product $u(x,t)v(x,t)$ is well-behaved, and this product is graphed here for the same solution as in Figs 4 and 5. Note that it is the product $uv$ which determines the velocity of the peakons, according to the governing ODE $\dot{x}_k = u(x_k)v(x_k)$. The domain shown is $-20 \leq x \leq 20$, $-10 \leq t \leq 10$, and the function is sampled at time values 1/4 units apart. The projection is orthogonal, and the vertical scale is exaggerated by a factor of 2.

of the solution; $2K - 1$ of them are eigenvalues of certain boundary value problems coming from the two Lax pairs of the Geng–Xue equation,

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K, \quad 0 < \mu_1 < \mu_2 < \cdots < \mu_{K-1},$$

another $2K - 1$ of them are residues of the associated Weyl functions,

$$a_1(0), a_2(0), \ldots, a_K(0) \in \mathbb{R}_+, \quad b_1(0), b_2(0), \ldots, b_{K-1}(0) \in \mathbb{R}_+,$$

and there are also two additional parameters,

$$b_\infty, b^*_\infty \in \mathbb{R}_+,$$

where $-b_\infty$ is the limit at infinity of the second Weyl function, and $-b^*_\infty$ is the corresponding quantity for a Weyl function associated to an adjoint spectral problem. Just like in Example 1.7 above, the solutions exhibit scattering as $t \to \pm \infty$: the peakon trajectories $x = x_k(t)$ asymptotically approach certain straight lines, whose slopes turn out to be determined by the eigenvalues $\{\lambda_i, \mu_j\}$ only. Each amplitude grows or decays exponentially as $t \to \pm \infty$ (or tends to a constant, in borderline cases), so the curves $y = \ln m_{2k-1}(t)$ and $y = -\ln n_{2k}(t)$ will approach straight lines, whose slopes also are determined by the eigenvalues only. Proof of this asymptotic behaviour for general $K$ is given in Section 9.

1.4 A brief history of peakons

A few words about the history of this problem are perhaps in order, but since the story has been told many times, we will keep it short and refer to our earlier articles for more details. Peaked solitons of the form

$$u(x,t) = \sum_{k=1}^{N} m_k(t) e^{-|x-x_k(0)|}$$

were introduced by Camassa and Holm in 1993 [6] as solutions to their shallow water equation

$$m_t + m_x u + 2mu_x = 0, \quad m = u - u_{xx}. \quad (15)$$
The ansatz (14) is a weak solution of the Camassa–Holm equation (15) if and only if the positions $x_k(t)$ and amplitudes $m_k(t)$ satisfy the canonical Hamiltonian system generated by

$$H(x_1, \ldots, x_n, m_1, \ldots, m_n) = \frac{1}{2} \sum_{i,j=1}^{N} m_i m_j e^{-|x_i - x_j|},$$

namely, using the shorthand notation (4),

$$\dot{x}_k = \frac{\partial H}{\partial m_k} = u(x_k), \quad \dot{m}_k = -\frac{\partial H}{\partial x_k} = -m_k u_x(x_k). \quad (16)$$

The general solution of (16) (for arbitrary $N$) was computed by Beals, Sattinger and Szmigielski using inverse spectral methods; it is given completely explicitly in terms of elementary functions [7, 8]. Later, other similar integrable PDEs with explicitly computable multipeakon solutions were discovered, in particular the Degasperis–Procesi equation from 1998 [9–12],

$$m_t + m u_x + 3m u u_x = 0, \quad m = u - u_{xx}, \quad (17)$$

and V. Novikov’s cubically nonlinear equation from 2008 [13–15],

$$m_t + u(m u + 3m u_x) = 0, \quad m = u - u_{xx}, \quad (18)$$

both of which were found through mathematical (rather than physical) considerations, namely the use of integrability tests to isolate interesting equations similar in form to the Camassa–Holm equation. The Degasperis–Procesi equation has later appeared in the context of hydrodynamics [16, 17], but we are not aware of any physical applications for the Novikov equation so far. Geng and Xue [1] found their integrable two-component peakon equation (1) in 2009 by modifying the Lax pair for Novikov’s equation that was found by Hone and Wang [14].

The literature on the Camassa–Holm equation is enormous, and we will not attempt to survey it here. There are also plenty of articles devoted to the Degasperis–Procesi equation, so we only mention a few additional references particularly close to the topic of this article: [18–22]. Novikov’s equation is more recent, but it is beginning to attract attention; see for example [1, 14, 23–34]. Concerning the Geng–Xue equation, we are only aware of a few studies: [1, 5, 35–39]. A bihamiltonian $2n$-component system which reduces to the Geng–Xue equation when $n = 1$ is constructed in [40].

1.5 Outline of the article

In Section 2, we begin our study by deriving the ODEs for peakons and shockpeakons, and explaining the distributional sense in which we consider them to be solution of the Geng–Xue equation. Most of the computations are postponed to Appendix A, where it is also verified that the Lax formulation of the
The Geng–Xue equation is compatible with the peakon ODEs. (We do not know at present whether this can be extended to cover the shockpeakon case as well.)

Section 3 is a review of notation and results from our previous article [5] about an inverse spectral problem associated with the Geng–Xue Lax pairs. This technical foundation allows us to fairly easily derive the explicit solution formulas for the interlacing peakon solutions in Section 4.

Some readers may want to skip most of Section 3, since the abbreviated notation defined there will not really be needed until we come to $K + K$ interlacing peakon solutions for arbitrary $K$ in Section 9 at the end of the article; before that, we only deal with smaller cases where all formulas can be written out in full detail. Specifically, Section 5 deals with the simple but somewhat exceptional case of $1 + 1$ peakon solutions, in Section 6 we show how to integrate the $1 + 1$ shockpeakon ODEs and take a brief look at some properties of the solution, and Section 7 studies the dynamics of $2 + 2$ interlacing pure peakon solutions (as already illustrated in Example 1.7 above).

Mixed peakon–antipeakon solutions are only considered in Remark 5.1 for the case $K = 1$, where they cause no problems, and in Section 8 for the case $K = 2$, where it is found in one particular example that there is a collision after finite time where one of the components of the solution forms a jump discontinuity, while the other component loses a peakon at the corresponding location (the amplitude of the peakon tends to zero). A natural continuation past this singularity is given by a solution with one peakon and one shockpeakon; such a solution is a special case of the $1 + 1$ shockpeakon solutions studied in Section 6.

Finally, in Section 9 we derive the large time asymptotics for $K + K$ interlacing pure peakon solutions. This is somewhat more technical notation-wise, but the outcome is that the features seen already in the case $K = 2$ persist also for $K > 2$.

Section 10 rounds off the article with a summary and a few remarks about open questions for future research.

2. Peakons and shockpeakons as weak solutions of the Geng–Xue equation

Our first item of business is to explain in which sense the shockpeakon ansatz (5), and hence also the peakon ansatz (2), is a solution of the Geng–Xue equation.

For smooth functions $u$ and $v$, the Geng–Xue equation (1) is equivalent to

$$
\begin{align*}
    m_t + v \cdot (4 - \partial_x^2) \partial_x (\frac{1}{2} u^2) &= 0, \\
    n_t + u \cdot (4 - \partial_x^2) \partial_x (\frac{1}{2} v^2) &= 0,
\end{align*}
$$

(19)

where $m = u - u_{xx}$ and $n = v - v_{xx}$, as before. This rewriting is inspired by the fact that the Degasperis–Procesi equation (17) can be written as

$$
    m_t + (4 - \partial_x^2) \partial_x (\frac{1}{2} u^2) = 0.
$$

(20)

In (19) and (20), the equalities can be interpreted in a distributional sense. Assuming that the functions $x \mapsto u(x, t)^2$ and $x \mapsto v(x, t)^2$ are locally integrable for each fixed $t$, we can consider them as distributions in the space $\mathcal{D}'(\mathbb{R})$. Then the derivatives with respect to $x$ in (19) can be taken in the sense of distributions, while the time derivatives are defined as limits (in $\mathcal{D}'(\mathbb{R})$) of difference quotients in the $t$ direction; see (A.3) in Section A.1.
In the notation of Appendix A, where we use $D_x$ for the distributional derivative and $D_t$ for the time derivative as just explained, the interpretation that we propose is thus

$$D_t(u - D_x^2 u) + v \cdot (4 - D_x^2) D_x (\frac{1}{2} u^2) = 0,$$

$$D_t(v - D_x^2 v) + u \cdot (4 - D_x^2) D_x (\frac{1}{2} v^2) = 0. \tag{21}$$

However, for (21) to make sense, it is necessary that the distribution $(4 - D_x^2) D_x (\frac{1}{2} u^2)$ can be multiplied by the function $v$, and similarly with $u$ and $v$ interchanged. To ensure that this is possible in the context of peakons and shockpeakons, without having to make any ad hoc assignments of values at jump discontinuities we need to impose the non-overlapping condition mentioned in Section 1.2: the component $v$ must not have a peakon or a shockpeakon at any point where the other component $u$ has a peakon or a shockpeakon, and vice versa. Then, since $u^2$ is piecewise continuous, $(4 - D_x^2) D_x (\frac{1}{2} u^2)$ will involve nothing worse than Dirac deltas and their first and second derivatives, and this will be multiplied by a function $v$ which is smooth in a neighbourhood of the support of these singular distributions, so the products can be evaluated using the rules

$$f(x) \delta_a = f(a) \delta_a,$$

$$f(x) \delta'_a = f(a) \delta'_a - f'(a) \delta_a,$$

$$f(x) \delta''_a = f(a) \delta''_a - 2f'(a) \delta'_a + f''(a) \delta_a. \tag{22}$$

**Theorem 2.1** The shockpeakon ansatz (5) is a solution of the Geng–Xue equation (21), in the distributional sense just described, if and only if it is non-overlapping and satisfies the ODEs (6). As a special case, the peakon ansatz (2) is a solution of the Geng–Xue equation if and only if it is non-overlapping and satisfies the ODEs (3).

**Proof.** See Section A.2 in the appendix. □

**Remark 2.2** It is understood here that the ordering assumption $x_1 < \cdots < x_n$ must be fulfilled. If this condition holds at time $t = 0$, then it will hold at least in some neighbourhood of $t = 0$, so the ODEs always provide a local solution of the PDE. We will see that for pure peakon solutions, the ordering is automatically preserved for all $t$, so that the solution is global, whereas for peakon–antipeakon or shockpeakon solutions this may not be the case.

**Remark 2.3** Because of our assumption of non-overlapping we cannot perform the reduction $u = v$ which for smooth solutions turns the Geng–Xue equation into two copies of the Novikov equation. But since the Novikov equation does admit peakon solutions, we do not rule out the possibility that there is some way of defining solutions which would allow overlapping peakons or even shockpeakons. This is an interesting question and we leave it for future research. Let us just remark that in a multipeakon ansatz with overlapping, the distribution $D_x(4 - D_x^2)(\frac{1}{2} u^2)$ is a linear combination of $\delta$ and $\delta'$, while $v_j$ jumps at the location of those singular terms, so with our distributional approach we would need to assign some value to $v_j(x_k)$ in the multiplication

$$v(x) \delta'_k = v(x_k) \delta'_k - v_j(x_k) \delta_{x_k},$$

and of course also to $u_j(x_k)$ in the other equation.
Remark 2.4 Tang and Liu [39] study solutions of the Geng–Xue equation with $u(\cdot,t)$ and $v(\cdot,t)$ in the Besov space $B^{3/2}_{2,1}(\mathbb{R})$, and write it as

$$
\begin{align*}
    u_t + uu_x v + (1 - \partial_x^2)^{-1} (3uu_x v + 2u^2 v_x + uu_{xx} v_x + uu_{x} v_{xx}) &= 0, \\
    v_t + vv_x u + (1 - \partial_x^2)^{-1} (3vv_x u + 2v^2 u_x + vv_{xx} u_x + vv_{x} u_{xx}) &= 0,
\end{align*}
$$

(23)

where $(1 - \partial_x^2)^{-1}$ means convolution with $\frac{1}{2} e^{-|\cdot|}$. (See their equations (1.6)–(1.8).) A similar formulation was used by Mi, Mu and Tao [38, eq. (56)]. Since these formulations require the derivatives $u_{xx}$ and $v_{xx}$ to be in $L^1$, they are not general enough to incorporate peakon solutions.

Remark 2.5 For the Degasperis–Procesi equation $D_t(u - D_x^2 u) + D_x(4 - D_x^2)\frac{1}{2} u^2 = 0$, the computation in the proof of Theorem 2.1 provides a derivation of the shockpeakon ODEs

$$
\begin{align*}
    \dot{x}_k &= u(x_k), \\
    \dot{m}_k &= -2m_k u_x(x_k) + 2s_k u(x_k), \\
    \dot{s}_k &= -s_k u_x(x_k)
\end{align*}
$$

(24)

which is simpler than the one originally given in [2]: just identify coefficients in (A.12) and (A.14).

3. Preliminaries: The map to spectral variables, and its inverse

The main technical work needed for analyzing the $K + K$ interlacing peakon solutions was done in our previous article [5]. In this section, we summarize the relevant material from that article; it will be crucial in the following sections.

Throughout this section, we will assume implicitly that $K \geq 2$. The case $K = 1$ is exceptional and will be treated separately in Section 3.3.

3.1 The forward spectral map for $K \geq 2$

First we describe the forward spectral map, a change of variables which takes the $4K$ ‘physical’ variables describing the positions and amplitudes of an interlacing peakon solution,

$$
\begin{align*}
    x_1 < x_2 < \cdots < x_{2K-1} < x_{2K}, \quad m_1, m_3, \ldots, m_{2K-1} \in \mathbb{R}_+, \quad n_2, n_4, \ldots, n_{2K} \in \mathbb{R}_+,
\end{align*}
$$

(25)

to a set of $4K$ spectral variables

$$
\begin{align*}
    0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K, \quad 0 < \mu_1 < \mu_2 < \cdots < \mu_{K-1}; \\
    a_1, a_2, \ldots, a_K \in \mathbb{R}_+, \quad b_1, b_2, \ldots, b_{K-1} \in \mathbb{R}_+, \quad b_{\infty}, b_{\ast\infty} \in \mathbb{R}_+.
\end{align*}
$$

(26)

It was shown in [5] that this map is a bijection, and the inverse map (which is much more explicit) will be described in Section 3.2. Combining this with the time dependence for the spectral variables, derived in Section 4, we get explicit formulas for the general interlacing solution to the peakon ODEs (3). Using these formulas, the dynamics of interlacing peakons will be analyzed in Section 5 (for the case $K = 1$), Section 7 (for $K = 2$), and Section 9 (for arbitrary $K$).
As shown in [1], the Geng–Xue equation (1) arises as the compatibility condition of the Lax pair

$$\frac{\partial}{\partial x} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 & zm & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix},$$

(27a)

$$\frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} -v_xu & vz^{-1} - vumn & \frac{v_xu_x}{v} \\ \frac{uxz^{-1}}{v} & v_xu - vux - z^{-2} & -uxz^{-1} - vumn \\ -uv & \frac{ux}{v} & \frac{uxz^{-1}}{v} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix},$$

(27b)

where \( z \) is the spectral parameter. However, because of the obvious symmetry in the Geng–Xue equation, it also arises as the compatibility condition of a different Lax pair, obtained by interchanging \( u \) and \( v \):

$$\frac{\partial}{\partial x} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix} = \begin{pmatrix} 0 & zm & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix},$$

(28a)

$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix} = \begin{pmatrix} -u_xv & uz^{-1} - uvnz & \frac{u_xv_x}{u} \\ \frac{vxz^{-1}}{u} & u_xv - uvx - z^{-2} & -vxz^{-1} - uvnz \\ -uv & \frac{vx}{u} & \frac{vxz^{-1}}{u} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix}.$$

(28b)

(In the case \( u = v \), when also \( m = u - u_x \) and \( n = v - v_x \) coincide, these Lax pairs reduce to the one found by Hone and Wang [14] for Novikov’s equation (18), and the Geng–Xue equation reduces to two copies of Novikov’s equation.)

Since we are dealing with the interlacing case, with the first (leftmost) peakon appearing in \( u \), the second in \( v \), etc., the setup is not symmetric, and spectral data from both Lax pairs must be used in order to solve the inverse spectral problem which will let us compute the peakon positions and amplitudes.

When \( u \) and \( v \) are given by the interlacing peakon ansatz (11), \( m \) and \( n \) are discrete measures, as explained in Appendix A. Interpreting the derivatives in the Lax equations in a suitable distributional sense, and imposing boundary conditions on (27a) and (28a) which are compatible with the time evolution given by (27b) and (28b), we obtain finite-dimensional eigenvalue problems which define our spectral data. Here we will keep the exposition brief, and merely state the resulting formulas which are necessary for defining the spectral data. For details, see [5], in particular Appendix B.

Consider equation (27a) for a fixed \( t \) (which we will suppress in the notation). Since \( m \) and \( n \) are zero away from the points \( x = x_i \), it follows that \( \psi_2(x; z) \) is piecewise constant, and \( \psi_1(x; z) \) and \( \psi_3(x; z) \) are piecewise linear combinations of \( e_x \) and \( e^{-x} \). We impose the following boundary condition on the left:

$$\begin{pmatrix} \psi_1(x; z) \\ \psi_2(x; z) \\ \psi_3(x; z) \end{pmatrix} = \begin{pmatrix} e^x \\ 0 \\ 0 \end{pmatrix}, \quad x < x_1.$$

(29)

Then we get on the right

$$\begin{pmatrix} \psi_1(x; z) \\ \psi_2(x; z) \\ \psi_3(x; z) \end{pmatrix} = \begin{pmatrix} A(-z^2)e^x + z^2C(-z^2)e^{-x} \\ 2zB(-z) \\ A(-z^2)e^x - z^2C(-z^2)e^{-x} \end{pmatrix}, \quad x > x_N.$$

(30)
with polynomials $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$, of degrees $K$, $K - 1$ and $K - 1$, respectively, defined by
\[
\begin{pmatrix}
A(\lambda) \\
B(\lambda) \\
C(\lambda)
\end{pmatrix} = S_{2K}(\lambda)S_{2K-1}(\lambda) \cdots S_2(\lambda)S_1(\lambda) \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\] (31)
where, for $a = 1, \ldots, K$,
\[
S_k(\lambda) = \begin{cases}
\begin{pmatrix}
1 & 0 & 0 \\
m_k e^{i\lambda} & 1 & \lambda m_k e^{-i\lambda} \\
0 & 0 & 1
\end{pmatrix}, & k = 2a - 1, \\
\begin{pmatrix}
1 & -2\lambda n_k e^{-i\lambda} & 0 \\
0 & 1 & 0 \\
0 & 2m_k e^{i\lambda} & 1
\end{pmatrix}, & k = 2a.
\end{cases}
\] (32)

The second Lax equation (28a) is similar, with $m$ and $n$ swapped. So with
\[
\begin{pmatrix}
\tilde{\psi}_1(x; z) \\
\tilde{\psi}_2(x; z) \\
\tilde{\psi}_3(x; z)
\end{pmatrix} = \begin{pmatrix}
e^x \\
0 \\
e^x
\end{pmatrix}, \quad x < x_1,
\] (33)
we get
\[
\begin{pmatrix}
\tilde{\psi}_1(x; z) \\
\tilde{\psi}_2(x; z) \\
\tilde{\psi}_3(x; z)
\end{pmatrix} = \begin{pmatrix}
\tilde{A}(-z^2)e^x \pm z^2 \tilde{C}(-z^3)e^{-x} \\
2z\tilde{B}(-z^2) \\
\tilde{A}(-z^2)e^x - z^2 \tilde{C}(-z^3)e^{-x}
\end{pmatrix}, \quad x > x_N,
\] (34)
with polynomials $\tilde{A}(\lambda)$, $\tilde{B}(\lambda)$ and $\tilde{C}(\lambda)$ of degrees $K - 1$, $K - 1$ and $K - 2$, respectively, defined by
\[
\begin{pmatrix}
\tilde{A}(\lambda) \\
\tilde{B}(\lambda) \\
\tilde{C}(\lambda)
\end{pmatrix} = \tilde{S}_{2K}(\lambda)\tilde{S}_{2K-1}(\lambda) \cdots \tilde{S}_2(\lambda)\tilde{S}_1(\lambda) \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\] (35)
where
\[
\tilde{S}_k(\lambda) = \begin{cases}
\begin{pmatrix}
1 & -2\lambda n_k e^{-i\lambda} & 0 \\
0 & 1 & 0 \\
0 & 2m_k e^{i\lambda} & 1
\end{pmatrix}, & k = 2a - 1, \\
\begin{pmatrix}
1 & 0 & 0 \\
0 & n_k e^{i\lambda} & 1 \\
0 & 0 & 1
\end{pmatrix}, & k = 2a.
\end{cases}
\] (36)
If all masses \( m_{2a-1} \) and \( n_{2a} \) are positive, then it turns out that the polynomial \( A \) has positive simple zeros

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K,
\]

and likewise \( \tilde{A} \) has positive simple zeros

\[
0 < \mu_1 < \mu_2 < \cdots < \mu_{K-1},
\]

and we will refer to these zeros \( \{\lambda_i, \mu_j\} \) as the eigenvalues of the spectral problems above. Moreover, we define residues

\[
a_1, a_2, \ldots, a_K; \quad b_1, b_2, \ldots, b_{K-1}; \quad b_\infty
\]

from the partial fractions decomposition of Weyl functions \( W \) and \( \tilde{W} \):

\[
W(\lambda) = \frac{-B(\lambda)}{A(\lambda)} = \sum_{i=1}^{K} \frac{a_i}{\lambda - \lambda_i}, \quad \tilde{W}(\lambda) = \frac{-\tilde{B}(\lambda)}{\tilde{A}(\lambda)} = -b_\infty + \sum_{j=1}^{K-1} \frac{b_j}{\lambda - \mu_j}.
\]  

The coefficients of the polynomials \( \tilde{A}(\lambda) \) and \( B(\lambda) \) can be worked out from the defining matrix products (35); in particular, the highest coefficients are given in equation (B.23) in [5], from which it follows that

\[
b_\infty = n_{2k} e^{12k} (1 - E_{2k-1,2k}).
\]

So there is a sort of duality between the roles played by \( b_\infty \) and \( b_\infty^* \).

**Remark 3.2** It can be verified directly by differentiation of the expressions in (38) and (39) that both \( b_\infty \) and \( b_\infty^* \) are constants of motion for the Geng–Xue peakon ODEs (3). (Cf. Theorem 4.1.)
3.2 The inverse spectral map for $K \geq 2$

The main result that we need from our previous work is the explicit formulas for the inverse spectral map, taken from Corollary 4.5 in [5]. These formulas are quoted in Theorem 3.12 below, but first we need to define a fair amount of notation.

**Definition 3.3** Given spectral data as in (26), let

$$\alpha = \sum_{i=1}^{K} a_i \delta_{\lambda_i}, \quad \beta = \sum_{j=1}^{K-1} b_j \delta_{\mu_j},$$ (40)

where $\delta$ is the Dirac delta. These two discrete measures on the positive real axis $\mathbb{R}_+$ will be called the spectral measures.

**Remark 3.4** For the application to Geng–Xue peakons, the measures (40) are the only ones that we will have in mind, but Definition 3.7 below makes sense whenever $\alpha$ and $\beta$ are measures on $\mathbb{R}_+$ with finite moments

$$\alpha_k = \int x^k \, d\alpha(x) < \infty, \quad \beta_k = \int y^k \, d\beta(y) < \infty,$$ (41)

and finite bimoments with respect to the Cauchy kernel $1/(x+y)$,

$$I_{ab} = \int \int \frac{x^a y^b}{x+y} \, d\alpha(x) \, d\beta(y) < \infty.$$ (42)

**Definition 3.5** For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\Delta(x)$ denote the Vandermonde determinant

$$\Delta(x) = \Delta(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j)$$ (43)

and $\Gamma(x)$ its counterpart with only plus signs,

$$\Gamma(x) = \Gamma(x_1, \ldots, x_n) = \prod_{i<j} (x_i + x_j);$$ (44)

in both cases the right-hand side is interpreted as 1 (the empty product) if $n = 0$ or $n = 1$. Moreover, for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, let

$$\Gamma(x; y) = \Gamma(x_1, \ldots, x_n; y_1, \ldots, y_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x_i + y_j).$$ (45)

**Remark 3.6** We will not really need $\Gamma(x)$ here, only $\Gamma(x; y)$. But we have included the definition of $\Gamma(x)$ anyway, as it occurs often in the study of peakons solutions of the Degasperis–Procesi and Novikov equations, and also in basic identities such as

$$\Gamma(x_1, \ldots, x_n; y_1, \ldots, y_n) = \frac{\Gamma(x_1, \ldots, x_n, y_1, \ldots, y_n)}{\Gamma(x_1, \ldots, x_n) \, \Gamma(y_1, \ldots, y_n)}.$$
With $\sigma_n$ denoting the sector in $\mathbb{R}^n_n$ defined by the inequalities $0 < x_1 < \cdots < x_n$, let

$$ J_{rs}^{nm} = \int_{\sigma_n \times \sigma_m} \frac{\Delta(x)^2 \Delta(y)^2 \left(\prod_{i=1}^n x_i\right)^r \left(\prod_{j=1}^n y_j\right)^s}{\Gamma(x,y)} d\alpha^n(x) d\beta^m(y), \quad (46) $$

for $n$ and $m$ positive. We also consider the degenerate cases

$$ J_{rs}^{nm} = \int_{\sigma_n} \frac{\Delta(x)^2 \left(\prod_{i=1}^n x_i\right)^r}{\Gamma(x)} d\alpha^n(x), \quad (n > 0), $$

$$ J_{rs}^{nm} = \int_{\sigma_m} \frac{\Delta(y)^2 \left(\prod_{j=1}^m y_j\right)^s}{\Gamma(x,y)} d\beta^m(y), \quad (m > 0), $$

$$ J_{00}^{nm} = 1. \quad (47) $$

**Remark 3.8** In particular, $J_{00}^{nm} = \alpha_r$ and $J_{01}^{nm} = \beta_s$ are the moments (41) of the measures $\alpha$ and $\beta$, and $J_{11}^{nm} = I_{rs}$ is the Cauchy bimoment (42).

**Remark 3.9** The integrals $J_{nm}^{rs}$ arise as evaluations of certain bimoment determinants occurring naturally in the theory of Cauchy biorthogonal polynomials; see comments in Appendix A.3 of [5]. This is similar to Heine’s evaluation of the Hankel determinant of moments from the classical theory of orthogonal polynomials,

$$ \det(\alpha_{i+j})_{i,j=0}^{n-1} = \frac{1}{n!} \int_{\mathbb{R}^n} \Delta(x)^2 d\alpha^n(x). $$

If we now specialize to the case when $\alpha$ and $\beta$ are the discrete measures (40), the moments can be written as sums instead of integrals,

$$ \alpha_r = \int x^r d\alpha(x) = \sum_{i=1}^K \lambda_i^r a_i, \quad \beta_s = \int y^s d\beta(y) = \sum_{j=1}^{K-1} \mu_j^s b_j, \quad (48) $$

and likewise for the bimoments,

$$ I_{rs} = \iint \frac{x^r y^s}{x+y} d\alpha(x) d\beta(y) = \sum_{i=1}^K \sum_{j=1}^{K-1} \frac{\lambda_i^r \mu_j^s}{\lambda_i + \mu_j} a_i b_j. \quad (49) $$

The integrals $J_{nm}^{rs}$ also turn into sums,

$$ J_{nm}^{rs} = \sum_{\ell \in \binom{[K]}{2}} \sum_{\ell' \in \binom{[K-1]}{2}} \Psi_{\ell \ell'} \lambda_i^r a_i \mu_j^s b_j, \quad (50) $$

where we have used yet some more notation, defined as follows:
DEFINITION 3.10 The binomial coefficient \( \binom{n}{k} \) denotes the collection of \( n \)-element subsets of the set \( S \), and \([k]\) denotes the integer interval \( \{1, 2, 3, \ldots, k\} \). We always label the elements of a set \( I \in \binom{[k]}{n} \) in increasing order: \( I = \{i_1 < i_2 < \cdots < i_n\} \). Moreover,

\[
\lambda'_i a_i \mu'_j b_j = \left( \prod_{i \in I} \lambda'_i a_i \right) \left( \prod_{j \in J} \mu'_j b_j \right)
\]

(51)

and

\[
\Psi_{IJ} = \frac{\Delta^2_I \tilde{\Delta}^2_J}{\Gamma_{IJ}},
\]

(52)

where

\[
\Delta^2_I = \Delta(\lambda_{i_1}, \ldots, \lambda_{i_n})^2 = \prod_{a < b} (\lambda_a - \lambda_b)^2,
\]

\[
\tilde{\Delta}^2_J = \Delta(\mu_{j_1}, \ldots, \mu_{j_m})^2 = \prod_{a < b} (\mu_a - \mu_b)^2,
\]

\[
\Gamma_{IJ} = \Gamma(\lambda_{i_1}, \ldots, \lambda_{i_n}; \mu_{j_1}, \ldots, \mu_{j_m}) = \prod_{i \in I, j \in J} (\lambda_i + \mu_j).
\]

(53)

Degenerate cases: we let \( [0] = \emptyset \), and consider empty products to be equal to 1, so that

\[
\Delta^2_0 = \Delta^2_{[\emptyset]} = \tilde{\Delta}^2_0 = \tilde{\Delta}^2_{[\emptyset]} = \Gamma_{\emptyset\emptyset} = \Gamma_{[\emptyset][\emptyset]} = 1.
\]

REMARK 3.11 In the discrete case (50), we have \( J_{nm} > 0 \) if \( 0 \leq n \leq K \) and \( 0 \leq m \leq K - 1 \), otherwise \( J_{nm} = 0 \).

THEOREM 3.12 (Explicit formulas for the inverse spectral map) For \( K \geq 2 \), the inverse spectral map from the spectral variables (26) to the ‘physical’ variables (25) is given by the following formulas, in terms of the sums (48), (49) and (50) above.

The even-numbered quantities are

\[
x_{2(K+1-j)} = \frac{1}{2} \ln \left( \frac{2 J_{j-2}^{10}}{J_{j-1}^{11}} \right),
\]

(54)

\[
h_{2(K+1-j)} = \frac{J_{j-1,j-2}^{10}}{J_{j-1,j-2}^{01} J_{j-2}^{11}} \sqrt{\frac{J_{j-1,j-2}^{00} J_{j-1,j-2}^{11}}{2}},
\]

(55)
for \( j = 2, \ldots, K \), together with
\[
\begin{align*}
x_{2K} &= \frac{1}{2} \ln 2(I_{00} + b_{\infty}a_0), \\
n_{2K} &= \frac{1}{a_0} \sqrt{\frac{I_{00} + b_{\infty}a_0}{2}}.
\end{align*}
\]

The odd-numbered quantities are
\[
\begin{align*}
x_{2(K+1-j)-1} &= \frac{1}{2} \ln \left( \frac{2 \mathcal{J}_{j-1}^{00}}{\mathcal{J}_{j-1,j-1}^{11}} \right), \\
m_{2(K+1-j)-1} &= \frac{\mathcal{J}_{j-1}^{01}}{\mathcal{J}_{j-1,j-1}^{10}} \sqrt{\frac{\mathcal{J}_{j-1,j-1}^{11} \mathcal{J}_{j-1,j-1}^{00}}{2}},
\end{align*}
\]
for \( j = 1, \ldots, K - 1 \), together with
\[
\begin{align*}
x_{1} &= \frac{1}{2} \ln \left( \frac{2 \mathcal{J}_{K,K-1}^{00}}{\mathcal{J}_{K,K-2}^{11} + b_{\infty}L} \right), \\
m_{1} &= \frac{M/L}{\mathcal{J}_{K,1,K-1}^{10}} \sqrt{\frac{\mathcal{J}_{K,K-1}^{00}}{2} \left( \frac{\mathcal{J}_{K,K-2}^{11} + \frac{2b_{\infty}L}{M} \mathcal{J}_{K-1,K-1}^{10}}{2} \right)}.
\end{align*}
\]

where
\[
L = \prod_{i=1}^{K} \lambda_i, \quad M = \prod_{j=1}^{K-1} \mu_j.
\]

**Remark 3.13** Let us write \( j' = K + 1 - j \). For example, \( x_{2j'} \) then corresponds to \( x_{2K}, x_{2K-2}, x_{2K-4}, \ldots \), as \( j = 1, 2, 3, \ldots \), i.e. it is the position of the \( j \)th of the even-numbered peakon if we count them from the right. Then we can express the formulas in Theorem 3.12 in the following more compact way:

\[
\frac{1}{2} \exp 2x_{2j'} = \begin{cases} 
\mathcal{J}_{11}^{00} + b_{\infty} \mathcal{J}_{10}^{00}, & j = 1, \\
\mathcal{J}_{j-1,j-1}^{00}, & j = 2, \ldots, K, \\
\mathcal{J}_{11}^{00}, & j = 1, \ldots, K - 1, \\
\mathcal{J}_{11}^{00}, & j = K,
\end{cases}
\]

\[
\frac{1}{2} \exp 2x_{2j'-1} = \begin{cases} 
\mathcal{J}_{j-1,j-1}^{00}, & j = 2, \ldots, K - 2, \\
\mathcal{J}_{K,1,K-1}^{00}, & j = K - 1, \\
\mathcal{J}_{K-1,K-1}^{00}, & j = K.
\end{cases}
\]
and

\[ 2n_{2j} \exp(-x_{2j}) = \begin{cases} 
\frac{1}{J_{10}^{10}}, & j = 1, \\
\frac{J_{1j}^{11}}{J_{1j-2}^{10}} J_{1j-1}^{10}, & j = 2, \ldots, K, \\
\frac{1}{J_{1j-2}^{10}} J_{1j-1}^{10}, & j = 2, \ldots, K, \\
1, & j = 0.
\end{cases} \]  

(64)

\[ 2m_{2j-1} \exp(-x_{2j-1}) = \begin{cases} 
\frac{J_{1j}^{11}}{J_{1j-2}^{10}} J_{1j-1}^{10}, & j = 1, \ldots, K - 1, \\
\frac{M J_{K-1,K-2}^{11}}{L J_{K-1,K-1}^{10}} + 2b_\infty^*, & j = K.
\end{cases} \]

3.3 The forward and inverse spectral map for \( K = 1 \)

In the case \( K = 1 \), the correspondence between peakon variables \((x_1, x_2, m_1, n_2)\) and spectral variables \((\lambda_1, a_1, b_\infty, b_\infty^*)\) reduces to equation (4.51) in [5]:

\[ \begin{align*}
\frac{1}{2} e^{2x_2} &= a_1 b_\infty, \\
\frac{1}{2} e^{-2x_1} &= (2a_1)^{-1} \lambda_1 b_\infty^*, \\
2n_2 e^{-x_2} &= \frac{1}{a_1}, \\
2m_1 e^{x_1} &= \frac{2a_1}{\lambda_1}. 
\end{align*} \]  

(65)

These formulas define a bijection from the pure peakon sector (where \( m_1 > 0, n_2 > 0 \) and \( x_1 < x_2 \)) to the region where \( \lambda_1, a_1, b_\infty \) and \( b_\infty^* \) are all positive (as usual) and in addition satisfy a nonlinear constraint particular to the case \( K = 1 \):

\[ 1 < 2 \lambda_1 b_\infty b_\infty^*. \]  

(66)

The inverse spectral map is immediately found by solving (65) for the peakon variables:

\[ \begin{align*}
x_1 &= \frac{1}{2} \ln \frac{a_1}{\lambda_1 b_\infty^*}, \\
x_2 &= \frac{1}{2} \ln 2a_1 b_\infty, \\
m_1 &= \sqrt{\frac{a_1 b_\infty^*}{\lambda_1}}, \\
n_2 &= \sqrt{\frac{b_\infty}{2a_1}}. 
\end{align*} \]  

(67)

See also Section 5, in particular Remark 5.3.

4. Time dependence of the spectral variables

If the positions \( x_i \) and the amplitudes \( m_{2i-1} \) and \( n_{2i} \) depend on time, then the spectral map described in Section 3, being a bijection, will induce a time dependence for the spectral variables (26) as well. The point of the spectral map is that it transforms the complicated time dependence given by the Geng–Xue peakon ODEs into a very simple dependence for the spectral variables.
Theorem 4.1 The ODEs (3) for interlacing peakons are equivalent to the following linear ODEs for the spectral variables:

\[
\frac{d\lambda_i}{dt} = 0, \quad \frac{da_i}{dt} = a_i, \quad \frac{d\mu_j}{dt} = 0, \quad \frac{db_j}{dt} = b_j, \quad \frac{db_\infty}{dt} = 0, \quad \frac{db^*_\infty}{dt} = 0.
\] (68)

Hence, the variables \(\{\lambda_i, \mu_j, b_\infty, b^*_\infty\}\) are constant, while \(\{a_i, b_j\}\) have the time dependence

\[
a_i(t) = a_i(0) e^{y/\lambda_i}, \quad b_j(t) = b_j(0) e^{y/\mu_j}.
\] (69)

Proof. As we carefully verify in Appendix A, the peakon ODEs (3) are equivalent to the Lax equations (27a) and (27b) with \(u\) and \(v\) given by the interlacing \(K + K\) peakon ansatz (11). For \(x < x_1\) we have

\[
\begin{align*}
  u &= u_x = M_- e^x, \quad \text{where} \quad M_- = \sum_{a=1}^{K} m_{a-1} e^{-x_{2a-1}}, \\
  v &= v_x = N_- e^x, \quad \text{where} \quad N_- = \sum_{a=1}^{K} n_{2a} e^{-x_{2a}}.
\end{align*}
\]

This makes both sides of (27b) vanish when \((\psi_1, \psi_2, \psi_3) = (e^x, 0, e^x)\), showing that this boundary condition (which was imposed when defining the spectral data; see (29)) is indeed consistent with the time evolution.

For \(x > x_{2K}\) we get instead

\[
\begin{align*}
  u &= -u_x = M_+ e^{-x}, \quad \text{where} \quad M_+ = \sum_{a=1}^{K} m_{2a-1} e^{x_{2a-1}}, \\
  v &= -v_x = N_+ e^{-x}, \quad \text{where} \quad N_+ = \sum_{a=1}^{K} n_{2a} e^{x_{2a}}.
\end{align*}
\]

Inserting this into (27b) together with the expression (29) for \((\psi_1, \psi_2, \psi_3)\), identifying coefficients of the linearly independent functions \((e^x, 1, e^{-x})\), and setting \(\lambda = -z^2\), we find

\[
\frac{\partial A}{\partial t}(\lambda) = 0, \quad \frac{\partial B}{\partial t}(\lambda) = \frac{B(\lambda) - A(\lambda) M_+}{\lambda}, \quad \frac{\partial C}{\partial t}(\lambda) = \frac{2(B(\lambda) - A(\lambda) M_+) N_+}{\lambda}.
\]

This shows that the polynomial \(A(\lambda)\) is constant in time, hence so are its zeros \(\lambda_1, \ldots, \lambda_K\). The time evolution of the Weyl function defined in (37),

\[
W(\lambda) = -\frac{B(\lambda)}{A(\lambda)} = \sum_{i=1}^{K} \frac{a_i}{\lambda - \lambda_i},
\]
is
\[
\frac{\partial W}{\partial t}(\lambda) = \frac{\partial}{\partial t} \left( -\frac{B(\lambda)}{A(\lambda)} \right) = -\frac{\partial B(\lambda)}{\partial t} \frac{A(\lambda)}{\lambda A(\lambda)} = \frac{M_+ + W(\lambda)}{\lambda},
\]
and taking residues of both sides of this equality at \( \lambda = \lambda_i \) we obtain
\[
\frac{da_i}{dt} = \frac{a_i}{\lambda_i}.
\]
(As an aside, the residue at \( \lambda = 0 \) is zero, since \( W(0) = -M_+ \); see equation (B.16) in [5].)

An entirely similar computation, using the other pair of Lax equations (28a) and (28b), shows that
\[
\frac{\partial \tilde{A}}{\partial t}(\lambda) = 0, \quad \frac{\partial \tilde{B}}{\partial t}(\lambda) = \frac{\tilde{B}(\lambda) - \tilde{A}(\lambda) N_+}{\lambda}, \quad \frac{\partial \tilde{C}}{\partial t}(\lambda) = \frac{2(\tilde{B}(\lambda) - \tilde{A}(\lambda) N_+) M_+}{\lambda}.
\]
It follows that \( \mu_1, \ldots, \mu_{K-1} \) are constant in time, and that
\[
\tilde{W}(\lambda) = -\frac{\tilde{B}(\lambda)}{\tilde{A}(\lambda)} = -b_\infty + \sum_{j=1}^{K-1} \frac{b_j}{\lambda_j - \mu_j}
\]
satisfies
\[
\frac{\partial \tilde{W}}{\partial t}(\lambda) = \frac{N_+ + \tilde{W}(\lambda)}{\lambda},
\]
so that \( \frac{db_\infty}{dt} = \frac{b_\infty}{\mu} \). Taking the limit \( \lambda \to \infty \), we may also deduce from this that \( \frac{db_\infty}{dt} = 0 \), i.e. \( b_\infty \) is a constant of motion. However, this was already noticed in Remark 3.2, where we also saw that \( b_\infty^* \) is a constant of motion.

**Corollary 4.2** (Solution formulas for interlacing Geng–Xue peakons) The formulas in Theorem 3.12 give the general solution to the peakon ODEs (3) in the interlacing \( K + K \) case (with \( K \geq 2 \), and all amplitudes \( m_{2n-1} \) and \( n_{2n} \) positive), if we let the parameters
\[
\{\lambda_i, \mu_j, b_\infty, b_\infty^*\}
\]
be constants with
\[
0 < \lambda_1 < \cdots < \lambda_K, \quad 0 < \mu_1 < \cdots < \mu_{K-1}, \quad b_\infty > 0, \quad b_\infty^* > 0,
\]
and let \( \{a_i, b_j\} \) have the time dependence
\[
a_i(t) = a_i(0) e^{i\lambda_i}, \quad b_j(t) = b_j(0) e^{i\mu_j},
\]
where \( \{a_i(0), b_j(0)\} \) are positive constants. These solutions are globally defined, and satisfy \( x_1(t) < \cdots < x_{2K}(t) \) for all \( t \in \mathbb{R} \).

**Remark 4.3** The solution for the case \( K = 1 \) is derived in Section 5 below.
5. Dynamics of 1 + 1 peakon solutions

With all the preliminaries out of the way, we can finally begin analyzing the properties of the $K + K$ interlacing peakon solutions of the Geng–Xue equation. The governing ODEs are (3), and we have explicit formulas for the general solution for any $K$, as described in Corollary 4.2. However, these formulas are fairly involved, so we will warm up by first looking at the case $K = 1$ (which is somewhat exceptional) in this section, and the case $K = 2$ in Section 7. The general case $K \geq 2$ will be treated in Section 9.

The ODEs governing 1 + 1 peakon solutions

\[
u(x, t) = m_1(t) e^{-|x-x_1(t)|}, \quad v(x, t) = n_2(t) e^{-|x-x_2(t)|}
\]

with $x_1(t) < x_2(t)$ are

\[
\dot{x}_1 = \dot{x}_2 = \frac{\dot{m}_1}{m_1} = -\frac{\dot{n}_2}{n_2} = m_1 n_2 E_{12},
\]

where $E_{12} = e^{x_1 - x_2}$. It is immediately verified by differentiation that $m_1 n_2 E_{12}$ is a constant of motion; denote its value by $c$. (If we impose the pure peakon assumption that $m_1 > 0$ and $n_2 > 0$, then obviously $c > 0$.) Direct integration then gives the solution

\[
x_1(t) = x_1(0) + ct, \\
x_2(t) = x_2(0) + ct, \\
m_1(t) = m_1(0) e^{ct}, \\
n_2(t) = n_2(0) e^{-ct},
\]

with constants

\[
x_1(0) < x_2(0), \quad m_1(0) > 0, \quad n_2(0) > 0.
\]

So the two peakons travel with the same (constant) velocity

\[
c = m_1(0) m_2(0) e^{x_1(0) - x_2(0)}.
\]

As $t \to \infty$, the amplitude $m_1$ of the left peakon tends to infinity and the amplitude $n_2$ of the right peakon tends to zero, in such a way that the product $m_1 n_2$ stays constant, and the other way around as $t \to -\infty$.

The actual peakon wave profiles are

\[
u(x, t) = m_1(t) e^{-|x-x_1(t)|} = \begin{cases} 
  m_1(0) e^{x-x_1(0)}, & x \leq x_1(0) + ct, \\
  m_1(0) e^{2ct+x_1(0)-x}, & x \geq x_1(0) + ct,
\end{cases}
\]

and

\[
v(x, t) = n_2(t) e^{-|x-x_2(t)|} = \begin{cases} 
  n_2(0) e^{-2ct+x_2(0)-x}, & x \leq x_2(0) + ct, \\
  n_2(0) e^{x_2(0)-x}, & x \geq x_2(0) + ct.
\end{cases}
\]
This means that the function \( u(x, t) v(x, t) \) will be a stump-shaped travelling wave with velocity \( c \):

\[
u(x, t) = \begin{cases} 
  ce^{2(x-ct-x_1(0))}, & x \leq x_1(0) + ct, \\
  c, & x_1(0) + ct \leq x \leq x_2(0) + ct, \\
  ce^{-2(x-ct-x_2(0))}, & x \geq x_2(0) + ct.
\end{cases}
\]

**Remark 5.1** Here in the case \( K = 1 \), it is not really necessary to assume that \( m_1 \) and \( n_2 \) are positive. If we allow negative amplitudes (antipeakons), the solution will still be given by the same formulas and it will be globally defined; the only difference is that the constant of motion \( c = m_1 n_2 e^{x_1-x_2} \) will be negative if \( m_1 \) and \( n_2 \) have opposite signs.

However, for \( K \geq 2 \), mixed peakon–antipeakon solutions may exhibit collisions and finite-time blowup; see Section 8.

**Remark 5.2** For Camassa–Holm and Degasperis–Procesi peakons, the simplest integral of motion is \( \sum_{i=1}^{N} m_i \), corresponding to the conserved quantity \( \int_{\mathbb{R}} m \, dx \). For Novikov peakons, the simplest integral of motion is

\[
\sum_{i,j=1}^{N} m_i m_j E_{ij} = \sum_{i,j=1}^{N} m_i m_j e^{-|x_i-x_j|},
\]

and Geng–Xue peakons admit two analogous integrals of motion:

\[
\mathcal{M} = \sum_{1 \leq i < j \leq N} m_i m_j E_{ij} = m_1 n_2 E_{12} + m_1 n_4 E_{14} + \cdots + m_1 n_{2K} E_{1,2K} + \cdots + m_{2K-1} n_{2K} E_{2K-1,2K}
\]

and

\[
\tilde{\mathcal{M}} = \sum_{1 \leq j < i \leq N} n_i n_j E_{ij} = n_2 m_3 E_{23} + n_2 m_5 E_{25} + \cdots + n_2 m_{2K-1} E_{2,2K-1} + \cdots + n_{2K-2} m_{2K-1} E_{2K-2,2K-1},
\]

In the case \( K = 1 \), clearly \( \mathcal{M} \) reduces to the constant of motion \( m_1 n_2 E_{12} \) that we saw above, while \( \tilde{\mathcal{M}} \) is identically zero.

**Remark 5.3** As we just saw, the \( 1 + 1 \) peakon solution is easy to find directly, but it is reassuring to check that the inverse spectral map from Section 3.3 works correctly here as well. Inserting the time
dependence \(a_1(t) = a_1(0) e^{(2/\lambda_1 t)}\) into (67) yields

\[
\begin{align*}
    x_1(t) &= \frac{t}{2\lambda_1} + \frac{1}{2} \ln \frac{a_1(0)}{\lambda_1 b_\infty^*}, \\
    x_2(t) &= \frac{t}{2\lambda_1} + \frac{1}{2} \ln 2 a_1(0) b_\infty, \\
    m_1(t) &= \sqrt{\frac{a_1(0) b_\infty^*}{\lambda_1}} \exp \left( \frac{t}{2\lambda_1} \right), \\
    n_2(t) &= \sqrt{\frac{b_\infty}{2 a_1(0)}} \exp \left( -\frac{t}{2\lambda_1} \right),
\end{align*}
\]  

(73)

which is just another way of writing the solution (72); in particular, the velocity \(c\) corresponds to \((2\lambda_1)^{-1}\).

The distance

\[
x_2(t) - x_1(t) = \frac{1}{2} \ln 2 \lambda_1 b_\infty b_\infty^*
\]

is positive due to the constraint (66).

6. Dynamics of 1 + 1 shockpeakon solutions

We will continue the study of interlacing peakon solutions in Section 7 below, but first we will show how to integrate the 1 + 1 shockpeakon ODEs which were given in Example 1.4, and are repeated here for convenience:

\[
\begin{align*}
    \dot{x}_1 &= m_1(n_2 + r_2)E_{12}, \\
    \dot{x}_2 &= (m_1 - s_1)n_2E_{12}, \\
    \dot{m}_1 &= (m_1^2 - m_1 s_1 + s_1^2)(n_2 + r_2)E_{12}, \\
    \dot{n}_2 &= -(m_1 - s_1)(n_2^2 + n_2 r_2 + r_2^2)E_{12}, \\
    \dot{s}_1 &= s_1(2m_1 - s_1)(n_2 + r_2)E_{12}, \\
    \dot{r}_2 &= -r_2(m_1 - s_1)(2n_2 + r_2)E_{12},
\end{align*}
\]  

(74)

where \(E_{12} = e^{-|x_1 - x_2|} = e^{x_1 - x_2}\). We will restrict ourselves to looking for solutions with \(s_1 \geq 0\) and \(r_2 \geq 0\) (cf. Remark 1.3). Also recall that in writing down the ODEs, we assumed that \(x_1(t) < x_2(t)\). If we take initial data \(x_1(0) < x_2(0)\), this assumption will at least continue to hold in some neighbourhood of \(t = 0\), but in general not globally (see Remark 6.3); the solution formulas derived below will only be valid until the time of the first collision \(x_1(t) = x_2(t)\).

To begin with, it is straightforward to verify, simply by differentiating the expressions with respect to \(t\) and using the ODEs (74), that the quantities

\[
K = (m_1 - s_1)(n_2 + r_2)E_{12}, \quad M = (m_1 + s_1)e^{-x_1}, \quad N = (n_2 - r_2)e^x
\]  

(75)
are constants of motion, and that the quantities
\[ S = s_1 e^{-x_1}, \quad R = r_2 e^{x_2} \] (76)
have the time dependence
\[ \dot{S} = KS, \quad \dot{R} = -KR. \]
Since \( K \) is constant, this means that
\[ S(t) = S(0) e^{Kt}, \quad R(t) = R(0) e^{-Kt}. \] (77)
In the same way one may check that the quantities
\[ X = (m_1 - s_1)e^{x_1}, \quad Y = (n_2 + r_2)e^{-x_2} \] (78)
satisfy
\[ \dot{X} = 2KX, \quad \dot{Y} = -2KY, \]
so that
\[ X(t) = X(0) e^{2Kt}, \quad Y(t) = Y(0) e^{-2Kt}. \] (79)
The fact that \( X(t) \) keeps its sign means that if \( m_1(0) - s_1(0) \neq 0 \), then \( m_1(t) - s_1(t) \neq 0 \) for all \( t \) such that the solution remains valid, i.e. up until the first collision. For these \( t \) we then get
\[
e^{2x_1(t)} = \frac{(m_1(t) - s_1(t)) e^{x_1(t)}}{(m_1(t) - s_1(t)) e^{-x_1(t)}} = \frac{X(t)}{M - 2S(t)} = \frac{e^{2x_1(0)} (m_1(0) - s_1(0)) e^{Kt}}{(m_1(0) + s_1(0)) e^{-x_1(0)} - 2s_1(0) e^{-x_1(0)} e^{Kt}} = \frac{e^{2x_1(0)} (m_1(0) - s_1(0)) e^{Kt}}{(m_1(0) + s_1(0)) e^{-Kt} - 2s_1(0)},
\]
and hence
\[ x_1(t) = x_1(0) + \frac{Kt}{2} + \frac{1}{2} \ln \frac{m_1(0) - s_1(0)}{(m_1(0) + s_1(0)) e^{-Kt} - 2s_1(0)}. \] (80a)
Similarly, if \( n_2(0) + r_2(0) \neq 0 \), we find
\[
e^{-2x_2(t)} = \frac{(n_2(t) + r_2(t)) e^{-x_2(t)}}{(n_2(t) + r_2(t)) e^{x_2(t)}} = \frac{Y(t)}{N + 2R(t)} = \frac{e^{-2x_2(0)} (n_2(0) + r_2(0)) e^{-Kt}}{(n_2(0) - r_2(0)) e^{Kt} + 2r_2(0)}.\]
so that
\[ x_2(t) = x_2(0) + \frac{Kt}{2} - \frac{1}{2} \ln \left( \frac{n_2(0) + r_2(0)}{n_2(0) - r_2(0)} \right) e^{Kt} + 2r_2(0). \]  

(80b)

From this we obtain
\[ s_1(t) = S(t) e^{s_1(t)} = s_1(0) e^{-s_1(0)} e^{Kt} e^{s_1(t)} \]
\[ = s_1(0) e^{3Kt/2} \sqrt{\frac{(m_1(0) - s_1(0))}{(m_1(0) + s_1(0)) e^{-Kt} - 2s_1(0)}} \]  

(80c)

and
\[ r_2(t) = R(t) e^{-s_2(t)} = r_2(0) e^{s_2(0)} e^{-Kt} e^{-s_2(t)} \]
\[ = r_2(0) e^{-3Kt/2} \sqrt{\frac{(n_2(0) + r_2(0))}{(n_2(0) - r_2(0)) e^{Kt} + 2r_2(0)}}. \]  

(80d)

Finally,
\[ m_1(t) = M e^{s_1(t)} - s_1(t) = (m_1(0) + s_1(0)) e^{-s_1(0)} e^{s_1(t)} - s_1(t) \]
\[ = (m_1(0) + s_1(0)) e^{Kt/2} - s_1(0) e^{3Kt/2} \sqrt{\frac{(m_1(0) - s_1(0))}{(m_1(0) + s_1(0)) e^{-Kt} - 2s_1(0)}} \]  

(80e)

and
\[ n_2(t) = N e^{-s_2(t)} + r_2(t) = (n_2(0) - r_2(0)) e^{s_2(0)} e^{-s_2(t)} + r_2(t) \]
\[ = (n_2(0) - r_2(0)) e^{-Kt/2} + r_2(0) e^{-3Kt/2} \sqrt{\frac{(n_2(0) + r_2(0))}{(n_2(0) - r_2(0)) e^{Kt} + 2r_2(0)}}. \]  

(80f)

The formulas (80) give the solution of (6) in the generic case \( m_1(0) - s_1(0) \neq 0 \) and \( n_2(0) + r_2(0) \neq 0 \).

If \( m_1(0) - s_1(0) = 0 \), we get instead \( X(0) = 0 \), hence \( X(t) = 0 \), meaning that \( m_1(t) = s_1(t) \) for all \( t \).

According to the ODEs (6), this immediately implies that \( \dot{x}_2 = \dot{n}_2 = \dot{r}_2 = 0 \), so the second shockpeakon doesn't move or change at all. For the first shockpeakon we then have
\[ \dot{x}_1(t) = m_1(t) (n_2(0) + r_2(0)) e^{s_1(t)} e^{-s_2(t)} = Y(0) m_1(t) e^{s_1(t)}, \]
\[ \dot{m}_1(t) = m_1(t)^2 (n_2(0) + r_2(0)) e^{s_1(t)} e^{-s_2(t)} = Y(0) m_1(t)^2 e^{s_1(t)}. \]

With \( m_1 = s_1 \), the constant of motion \( M \) reduces to \( M = 2m_1 e^{-s_1}, \) which is positive because we are assuming that \( s_1 \geq 0 \) and \( (m_1, s_1) \neq (0, 0) \). Thus,
\[ \dot{m}_1(t) = \frac{2Y(0)}{M} m_1(t)^3 = \frac{A}{2m_1(0)^2} m_1(t)^3, \]
where

\[ A = \frac{4Y(0) m_1(0)^2}{M} = 2m_1(0) (n_2(0) + r_2(0)) e^{x_1(0) - x_2(0)}, \]  

(81a)

which gives

\[ m_1(t) = s_1(t) = \frac{m_1(0)}{\sqrt{1 - At}}, \]  

(81b)

and consequently

\[ e^{x_1(t)} = \frac{2m_1(t)}{M} = \frac{2m_1(t)}{2m_1(0) e^{-x_1(0)}} = \frac{e^{x_1(0)}}{\sqrt{1 - At}}, \]  

so that

\[ x_1(t) = x_1(0) - \frac{1}{2} \ln(1 - At). \]  

(81c)

In the case \( n_2(0) + r_2(0) = 0 \), a similar computation shows that \( \dot{x}_1 = \dot{m}_1 = \dot{s}_1 = 0 \),

\[ n_2(t) = -r_2(t) = \frac{n_2(0)}{\sqrt{1 + Bt}}, \]  

(82a)

and

\[ x_2(t) = x_2(0) + \frac{1}{2} \ln(1 + Bt) \]  

(82b)

where

\[ B = 2(m_1(0) - s_1(0)) n_2(0) e^{x_1(0) - x_2(0)}. \]  

(82c)

**Remark 6.1** Note that if both conditions \( m_1(0) - s_1(0) = 0 \) and \( n_2(0) + r_2(0) = 0 \) hold at the same time, then the solution is completely time-independent:

\[ \dot{x}_1 = \dot{m}_1 = \dot{s}_1 = \dot{x}_2 = \dot{n}_2 = \dot{r}_2 = 0. \]

**Remark 6.2** We may also point out that \( \dot{x}_2 = K \) if \( s_1 = 0 \), i.e. if the first shockpeakon is in fact an ordinary peakon, then the second shockpeakon travels with constant speed. Similarly, \( \dot{x}_1 = K \) if \( r_2 = 0 \). As a special case, when \( x_1 = r_2 = 0 \), we recover the result from the Section 5 that in the 1+1 peakon solution, both peakons travel with equal and constant speed.

**Remark 6.3** The formulas above show that in many cases there will be a collision \( x_1(t) = x_2(t) \) after finite time, with \( (m_1, s_1) \neq (0, 0) \) and \( (n_2, r_2) \neq (0, 0) \) at the instant of collision. For example, if \( B < 0 \) in (82b), then \( x_2(t) \to -\infty \) as \( t \to (1/B)^- \), so by continuity there must be a collision time \( t_0 \in (0, 1/B) \) such that \( x_2(t_0) \) equals the constant \( x_1(t) = x_1(0) = x_1(t_0) \). Thus, the non-overlapping condition is not
preserved at shockpeakon–shockpeakon collisions. (This is in contrast to the peakon–antipeakon collision scenario described in Section 8.) It is not clear to us at present if, and in that case how, such a solution can be continued past the collision.

7. Dynamics of 2 + 2 interlacing pure peakon solutions

In this section, we leave shockpeakons and return to the interlacing pure peakon solutions. We take a detailed look at the case $K = 2$, as a preparation for the general case $K \geq 2$ treated in Section 9. Except for Section 8, we will only consider pure peakon solutions, i.e. we will assume that the amplitudes $m_1, n_2, m_3, n_4$ are all positive.

7.1 Governing ODEs and explicit solution formulas

The 2 + 2 interlacing peakon solutions of the Geng–Xue equation are governed by the ODEs

$$
\begin{align*}
\dot{x}_1 &= (m_1 + m_3 E_{13})(n_2 E_{12} + n_4 E_{14}), \\
\dot{x}_2 &= (m_1 E_{12} + m_3 E_{23})(n_2 + n_4 E_{24}), \\
\dot{x}_3 &= (m_1 E_{13} + m_3)(n_2 E_{23} + n_4 E_{34}), \\
\dot{x}_4 &= (m_1 E_{14} + m_3 E_{34})(n_2 E_{24} + n_4), \\
\dot{m}_1 &= (m_1 + m_3 E_{13})(n_2 E_{12} + n_4 E_{14}) - 2m_3 E_{13}(n_2 E_{12} + n_4 E_{14}), \\
\dot{m}_2 &= (-m_1 E_{12} + m_3 E_{23})(n_2 + n_4 E_{24}) - 2(m_1 E_{12} + m_3 E_{23})n_4 E_{24}, \\
\dot{m}_3 &= (m_1 E_{13} + m_3)(-n_2 E_{23} + n_4 E_{34}) + 2m_1 E_{13}(n_2 E_{23} + n_4 E_{34}), \\
\dot{m}_4 &= (-m_1 E_{14} - m_3 E_{34})(n_2 E_{24} + n_4) + 2(m_1 E_{14} + m_3 E_{34})n_4 E_{24},
\end{align*}
$$

(83)

where $E_{ij} = e^{-|i-j|} = e^{i \pi /2}$ for $i < j$. It is apparent that for $K = 2$ the equations are already much more complicated than for $K = 1$; cf. equation (71). There seems to be little hope of integrating this system by any direct methods, so we proceed without further ado to the general solution formulas derived by inverse spectral methods (Corollary 4.2). These formulas are stated in terms of the sums $J_{mn}^t$ defined in Section 3.2, but here in the case $K = 2$ the expressions are still small enough to allow us to write out in detail what these sums actually are. Then the sought quantities $x_1(t), x_2(t), x_3(t), x_4(t), m_1(t), n_2(t), m_3(t), n_4(t)$ will be directly expressed in terms of the spectral variables

$$
\lambda_1, \lambda_2, \mu_1, a_1, a_2, b_1, b_\infty, b_\infty^*,
$$

where $\lambda_1, \lambda_2, \mu_1, b_\infty$ and $b_\infty^*$ are arbitrary positive constants with $\lambda_1 < \lambda_2$, and where $a_1, a_2$ and $b_1$ have the time dependence

$$
\begin{align*}
a_1(t) &= a_1(0) e^{\nu_1}, \\
a_2(t) &= a_2(0) e^{\nu_2}, \\
b_1(t) &= b_1(0) e^{\nu_1},
\end{align*}
$$

(84)
with arbitrary positive constants \(a_1(0), a_2(0)\) and \(b_1(0)\). (This time dependence will mostly be suppressed in the notation; whenever we write just \(a_i\), we mean \(a_i(t)\).)

In terms of the quantities from Remark 3.13, the general solution of (83) is then given (completely explicitly in terms of elementary functions) by

\[
\frac{1}{2} e^{2s_3} = I_{00} + b_\infty a_0 = \frac{a_1 b_1}{\lambda_1 + \mu_1} + \frac{a_2 b_1}{\lambda_2 + \mu_1} + b_\infty(a_1 + a_2),
\]

\[
\frac{1}{2} e^{2s_3} = \frac{\mathcal{J}_{11}^{00}}{\mathcal{J}_{11}^{10}} = \frac{I_{00}}{I} = \frac{a_1 b_1}{\lambda_1 + \mu_1} + \frac{a_2 b_1}{\lambda_2 + \mu_1},
\]

\[
\frac{1}{2} e^{2s_2} = \frac{\mathcal{J}_{11}^{00}}{\mathcal{J}_{11}^{11}} = \frac{\mathcal{J}_{21}^{00}}{\mathcal{J}_{11}^{11}} = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_1)} a_1 a_2 b_1
\]

\[
\frac{\mathcal{J}_{11}^{00}}{\mathcal{J}_{11}^{11}} + \frac{2 b_\infty^* L}{M \mathcal{J}_{11}^{10}} = \frac{2 b_\infty^* \lambda_1 \lambda_2}{\lambda_1 a_1 + \lambda_2 a_2 + \frac{2 b_\infty^* \lambda_1 \lambda_2}{\lambda_1 + \mu_1} \left(\frac{\lambda_1 a_1 b_1}{\lambda_1 + \mu_1} + \frac{\lambda_2 a_2 b_1}{\lambda_2 + \mu_1}\right)}
\]

and

\[
2n_4 e^{-s_4} = \frac{1}{a_0} = \frac{1}{a_1 + a_2},
\]

\[
2m_3 e^{-s_3} = \frac{\mathcal{J}_{11}^{11}}{\mathcal{J}_{11}^{10} \mathcal{J}_{10}^{10}} = \frac{1}{\mathcal{J}_{11}^{10} \mathcal{J}_{10}^{10}} = \frac{a_1 + a_2}{\lambda_1 a_1 b_1 + \frac{\lambda_2 a_2 b_1}{\lambda_2 + \mu_1}}
\]

\[
2n_5 e^{-s_2} = \frac{\mathcal{J}_{11}^{11} \mathcal{J}_{11}^{10}}{\mathcal{J}_{11}^{10} \mathcal{J}_{21}^{10}} = \frac{(\lambda_1 + \lambda_2) a_1 + \lambda_2 a_2}{\lambda_1 a_1 b_1 + \frac{\lambda_2 a_2 b_1}{\lambda_2 + \mu_1}}
\]

\[
2m_5 e^{-s_2} = \frac{M \mathcal{J}_{11}^{11}}{L \mathcal{J}_{11}^{10}} + 2 b_\infty^* = \frac{\mu_1 (\lambda_1 a_1 + \lambda_2 a_2)}{\lambda_1 a_1 b_1 + \frac{\lambda_2 a_2 b_1}{\lambda_2 + \mu_1}} + 2 b_\infty^*.
\]

### 7.2 Asymptotics as \(t \to +\infty\)

We will now derive formulas for the large time asymptotics of the \(2 + 2\) interlacing peakon solution (85). We remind the reader that these features were illustrated with graphics in Example 1.7, and it may be helpful to revisit that example at this point.
We begin with the case $t \to +\infty$. In this case, the factors $a_1(t), a_2(t)$ and $b_1(t)$ will all diverge to $+\infty$, and $a_1(t)$ will grow much faster than $a_2(t)$ since we are assuming that $0 < \lambda_1 < \lambda_2$; indeed, setting

$$
\delta = \frac{1}{\lambda_1} - \frac{1}{\lambda_2} > 0
$$

we have

$$
\frac{a_2(t)}{a_1(t)} = \frac{a_2(0) e^{t/\lambda_2}}{a_1(0) e^{t/\lambda_1}} = \frac{a_2(0)}{a_1(0)} e^{-\delta t} \to 0, \quad t \to +\infty.
$$

Factoring out the dominant terms, we get

$$
\frac{1}{2} e^{2x_4} = a_1 b_1 \left( \frac{1}{\lambda_1 + \mu_1} + \frac{\mu_1}{\lambda_2} + b_\infty \left( \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_2} \right) \right),
$$

$$
\frac{1}{2} e^{2x_3} = a_1 b_1 \left( \frac{1}{\lambda_1 + \mu_1} + \frac{\mu_1}{\lambda_2 + \mu_1} \right),
$$

$$
(\lambda_1 - \lambda_2)^2
$$

$$
\frac{1}{2} e^{2x_2} = a_2 b_1 \cdot \frac{(\lambda_1 + \mu_1)(\lambda_2 + \mu_1)}{\lambda_1 + \lambda_2 \frac{\mu_1}{\lambda_1}}
$$

$$
\frac{1}{2} e^{2x_1} = a_2 \cdot \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_1) + 2b_\infty \lambda_1 \lambda_2 \mu_1 \left( \frac{1}{\lambda_1 + \mu_1} + \frac{\lambda_2 \frac{\mu_1}{\lambda_1}}{\lambda_2 + \mu_1} \right)}
$$

and

$$
2n_4 e^{-x_4} = \frac{1}{a_1} \cdot \frac{1}{1 + \frac{\mu_1}{\lambda_1}}
$$

$$
2m_3 e^{-x_3} = \frac{1}{b_1} \cdot \frac{1 + \frac{\mu_1}{\lambda_1}}{\lambda_1 + \frac{\lambda_2 \frac{\mu_1}{\lambda_1}}{\lambda_2 + \mu_1}},
$$

$$
2n_2 e^{-x_2} = \frac{1}{a_2} \cdot \frac{(\lambda_1 + \lambda_2 \frac{\mu_1}{\lambda_1})}{(\lambda_1 + \mu_1)} \left( \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2 \frac{\mu_1}{\lambda_1}}{\lambda_2 + \mu_1} \right)
$$

$$
(1 + \frac{\mu_1}{\lambda_1}) \frac{\mu_1 (\lambda_1 - \lambda_2)^2}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_1)}
$$

$$
2n_1 e^{-x_1} = \frac{1}{a_2} \cdot \frac{\mu_1 (\lambda_1 + \lambda_2 \frac{\mu_1}{\lambda_1})}{\lambda_1 \lambda_2} \left( \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2 \frac{\mu_1}{\lambda_1}}{\lambda_2 + \mu_1} \right) + 2b_\infty
$$

(86)
Writing $o(1)$ for terms which tend to zero (exponentially fast, actually), we therefore have, as $t \to +\infty$,

$$x_4(t) = \frac{t}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\mu_1} \right) + \frac{1}{2} \ln \frac{2a_1(0) b_1(0)}{\lambda_1 + \mu_1} + o(1),$$

$$x_3(t) = \frac{t}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\mu_1} \right) + \frac{1}{2} \ln \frac{2a_1(0) b_1(0)}{\lambda_1 + \mu_1} + o(1),$$

$$x_2(t) = \frac{t}{2} \left( \frac{1}{\lambda_2} + \frac{1}{\mu_1} \right) + \frac{1}{2} \ln \frac{2a_2(0) b_1(0)}{\lambda_2 + \mu_1} + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1(\lambda_1 + \mu_1)} + o(1),$$

$$x_1(t) = \frac{t}{2} \left( \frac{1}{\lambda_2} + \frac{1}{\mu_1} \right) + \frac{1}{2} \ln \frac{\mu_1 a_2(0)}{\lambda_1 \lambda_2 b_2^*} + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1(\lambda_1 + \mu_1)} + o(1) \quad (88)$$

and

$$n_4(t) = (1 + o(1)) \sqrt{\frac{b_1(0)}{2a_1(0)(\lambda_1 + \mu_1)}} \exp \left( \frac{t}{2} \left( \frac{1}{\mu_1} - \frac{1}{\lambda_1} \right) \right),$$

$$m_3(t) = (1 + o(1)) \frac{1}{\lambda_1} \sqrt{\frac{a_1(0) (\lambda_1 + \mu_1)}{2b_1(0)}} \exp \left( \frac{t}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right) \right),$$

$$n_2(t) = (1 + o(1)) \frac{1}{\mu_1} \sqrt{\frac{b_1(0) (\lambda_2 + \mu_1) \lambda_1^3}{2a_2(0) (\lambda_1 - \lambda_2)^2 (\lambda_1 + \mu_1)}} \exp \left( \frac{t}{2} \left( \frac{1}{\mu_1} - \frac{1}{\lambda_2} \right) \right),$$

$$m_1(t) = (1 + o(1)) \sqrt{\frac{b_2^* a_2(0) \mu_1 (\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2 (\lambda_1 + \mu_1)}} \exp \left( \frac{t}{2} \left( \frac{1}{\lambda_2} \right) \right). \quad (89)$$

In other words, what (88) says is that the peakons asymptotically travel along straight lines in $(x, t)$ space as $t \to +\infty$, with the asymptotic velocities

$$\dot{x}_1 \sim c_3 = \frac{1}{2} \left( \frac{1}{\lambda_2} \right),$$

$$\dot{x}_2 \sim c_3 = \frac{1}{2} \left( \frac{1}{\lambda_2} + \frac{1}{\mu_1} \right),$$

$$\dot{x}_3, \dot{x}_4 \sim c_1 = \frac{1}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\mu_1} \right) \quad (90)$$

satisfying $0 < c_3 < c_2 < c_1$. So the two rightmost peakons asymptotically have the same velocity $c_1$, and in fact we see from (88) that $x_4(t) - x_3(t) = o(1)$, i.e. the distance between them approaches zero as $t \to +\infty$.

Looking at (89), we note that the amplitude of the leftmost peakon always diverges: $m_1(t) \to \infty$ as $t \to +\infty$. The other amplitudes decay exponentially to zero, or grow exponentially to infinity, or tend to some positive constant value, depending on the relative sizes of the eigenvalues $\lambda_i$ and $\mu_j$. It is easier to
visualize the situation if we take logarithms in (89):

\[-\ln n_4(t) = \frac{t}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right) - \frac{1}{2} \ln \frac{b_1(0)}{2a_1(0)(\lambda_1 + \mu_1)} + o(1),\]

\[\ln m_3(t) = \frac{t}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right) - \ln \lambda_1 + \frac{1}{2} \ln \frac{a_1(0)(\lambda_1 + \mu_1)}{2b_1(0)} + o(1),\]

\[-\ln n_2(t) = \frac{t}{2} \left( \frac{1}{\lambda_2} - \frac{1}{\mu_1} \right) + \ln \mu_1 - \frac{1}{2} \ln \frac{b_1(0)(\lambda_2 + \mu_1)}{2a_2(0)} + o(1),\]

\[\ln m_1(t) = \frac{t}{2} \left( \frac{1}{\lambda_2} \right) + \frac{1}{2} \ln \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2} \right)^2 (\lambda_1 + \mu_1) + \frac{1}{2} \ln \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 (\lambda_2 + \mu_1)} \right)^2 + o(1).\]  

(91)

Setting

\[d_1 = \frac{1}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right), \quad d_2 = \frac{1}{2} \left( \frac{1}{\lambda_2} - \frac{1}{\mu_1} \right), \quad d_3 = \frac{1}{2} \left( \frac{1}{\lambda_2} \right).\]  

(92)

we thus have

\[-\ln n_4(t) = d_1 t + \text{constant} + o(1),\]

\[\ln m_3(t) = d_1 t + \text{constant} + o(1),\]

\[-\ln n_2(t) = d_2 t + \text{constant} + o(1),\]

\[\ln m_1(t) = d_3 t + \text{constant} + o(1),\]  

(93)

so the graphs of the quantities on the left-hand side approach straight lines as \( t \to +\infty \). Note that \( d_3 > 0 \) always, but \( d_1 \) and \( d_2 \) may be positive, negative, or zero.

7.3 Asymptotics as \( t \to -\infty \)

As \( t \to -\infty \), we can carry out similar calculations, but now the roles are reversed; the factors \( a_1(t), a_2(t) \) and \( b_1(t) \) will all tend to zero, and \( a_1(t) \) does so much faster than \( a_2(t) \):

\[\frac{a_1(t)}{a_2(t)} = \frac{a_1(0)}{a_2(0)} e^{2t} \to 0, \quad t \to -\infty.\]

So now the dominant terms to be factored out are not the same as in (86) and (87); we get

\[\frac{1}{2} e^{2x_4} = a_2 \left( \frac{a_1}{a_2} b_1 \frac{1}{\lambda_1 + \mu_1} + \frac{b_1}{\lambda_2 + \mu_1} + b_\infty \left( \frac{a_1}{a_2} + 1 \right) \right),\]

\[\frac{1}{2} e^{2x_3} = a_2 b_1 \left( \frac{a_1}{a_2} + \frac{1}{\lambda_2 + \mu_1} \right),\]
and so on. Taking this into account, we obtain the following asymptotics as $t \to -\infty$:

$$x_4(t) = \frac{t}{2} \left( \frac{1}{\lambda_2} \right) + \frac{1}{2} \ln \left( 2a_2(0) b_\infty \right) + o(1),$$

$$x_3(t) = \frac{t}{2} \left( \frac{1}{\lambda_2} + \frac{1}{\mu_1} \right) + \frac{1}{2} \ln \frac{2a_2(0) b_1(0)}{\lambda_2 + \mu_1} + o(1),$$

$$x_2(t) = \frac{t}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\mu_1} \right) + \frac{1}{2} \ln \frac{2a_1(0) b_1(0)}{\lambda_1 + \mu_1} + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{\lambda_2(\lambda_2 + \mu_1)} + o(1),$$

$$x_1(t) = \frac{t}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\mu_1} \right) + \frac{1}{2} \ln \frac{2a_1(0) b_1(0)}{\lambda_1 + \mu_1} + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{\lambda_2(\lambda_2 + \mu_1)} + o(1) \quad (94)$$

and

$$n_4(t) = (1 + o(1)) \sqrt{\frac{b_\infty}{2a_2(0)}} \exp \left( -\frac{t}{2} \left( \frac{1}{\lambda_2} \right) \right),$$

$$m_3(t) = (1 + o(1)) \frac{1}{\lambda_2} \sqrt{\frac{a_2(0) (\lambda_2 + \mu_1)}{2b_1(0)}} \exp \left( -\frac{t}{2} \left( \frac{1}{\mu_1} - \frac{1}{\lambda_2} \right) \right),$$

$$n_2(t) = (1 + o(1)) \frac{1}{\mu_1} \sqrt{\frac{b_1(0) (\lambda_1 + \mu_1) \lambda_1^3}{2a_1(0) (\lambda_1 - \lambda_2)^2 (\lambda_2 + \mu_1)}} \exp \left( -\frac{t}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right) \right),$$

$$m_1(t) = (1 + o(1)) \frac{1}{\lambda_1} \sqrt{\frac{a_1(0) (\lambda_1 - \lambda_2)^2 (\lambda_2 + \mu_1)}{2b_1(0) (\lambda_1 + \mu_1) \lambda_2^3}} \exp \left( -\frac{t}{2} \left( \frac{1}{\mu_1} - \frac{1}{\lambda_1} \right) \right) \quad (95)$$

Taking logarithms, we can write (95) as

$$-\ln n_4(t) = \frac{t}{2} \left( \frac{1}{\lambda_2} \right) - \frac{1}{2} \ln \frac{b_\infty}{2a_2(0)} + o(1),$$

$$\ln m_3(t) = \frac{t}{2} \left( \frac{1}{\lambda_2} - \frac{1}{\mu_1} \right) - \ln \lambda_2 + \frac{1}{2} \ln \frac{a_2(0) (\lambda_2 + \mu_1)}{2b_1(0)} + o(1),$$

$$-\ln n_2(t) = \frac{t}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right) + \ln \mu_1 - \frac{1}{2} \ln \frac{b_1(0) (\lambda_1 + \mu_1)}{2a_1(0)}$$

$$+ \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2 (\lambda_2 + \mu_1)}{\lambda_2^3} + o(1),$$

$$\ln m_1(t) = \frac{t}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right) + \ln \frac{\mu_1}{\lambda_1} + \frac{1}{2} \ln \frac{a_1(0)}{2b_1(0) (\lambda_1 + \mu_1)}$$

$$+ \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2 (\lambda_2 + \mu_1)}{\lambda_2^3} + o(1). \quad (96)$$
From (94), we conclude that the peakons asymptotically travel along straight lines in \((x, t)\) space also in the case \(t \to -\infty\), with the same asymptotic velocities \(c_k\) (defined in equation (90)) as in the case \(t \to +\infty\), but in the opposite order:

\[
\begin{align*}
\dot{x}_1, \dot{x}_2 &\sim c_1 = \frac{1}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\mu_1} \right), \\
\dot{x}_3 &\sim c_2 = \frac{1}{2} \left( \frac{1}{\lambda_2} + \frac{1}{\mu_1} \right), \\
\dot{x}_4 &\sim c_3 = \frac{1}{2} \left( \frac{1}{\lambda_1} \right).
\end{align*}
\] (97)

Now it is the two leftmost peakons that asymptotically have the same velocity, and from (94) we get \(x_2(t) - x_1(t) = o(1)\), i.e. the distance between them approaches zero as \(t \to -\infty\).

Moreover, the formulas (96) have the structure

\[
\begin{align*}
-\ln n_4(t) &= d_3 t + \text{constant} + o(1), \\
\ln m_3(t) &= d_2 t + \text{constant} + o(1), \\
-\ln n_2(t) &= d_1 t + \text{constant} + o(1), \\
\ln m_1(t) &= d_1 t + \text{constant} + o(1),
\end{align*}
\] (98)

which is similar to what we had in the case \(t \to +\infty\); see (93). The constants \(d_k\) defined in (92) appear here too, but in the opposite order. The amplitude of the rightmost peakon always diverges: \(n_4(t) \to \infty\) as \(t \to -\infty\), since \(d_3 > 0\). The behaviour of the other amplitudes depends on the relative sizes of the eigenvalues \(\lambda_i\) and \(\mu_j\).

### 7.4 Phase shifts in positions

As usual in soliton theory, we can identify ‘phase shifts’ in the positions of solitons, by comparing the asymptotics as \(t \to +\infty\) and \(t \to -\infty\).

Thus, we may compare the straight line approached by the curves \(x = x_3(t)\) and \(x = x_4(t)\) as \(t \to +\infty\),

\[
x = c_1 t + \frac{1}{2} \ln \frac{2 a_1(0) b_1(0)}{\lambda_1 + \mu_1},
\]

to the parallel line approached by the curves \(x = x_1(t)\) and \(x = x_2(t)\) as \(t \to -\infty\),

\[
x = c_1 t + \frac{1}{2} \ln \frac{2 a_1(0) b_1(0)}{\lambda_1 + \mu_1} + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{\lambda_2 (\lambda_2 + \mu_1)},
\]

and say that during the complete course of the evolution, as \(t\) runs from \(-\infty\) to \(+\infty\), ‘the pair of fast peakons with asymptotic velocity \(c_1\)’ experiences a shift in the \(x\) direction of size

\[
\lim_{t \to +\infty} (x_{3,4}(t) - c_1 t) - \lim_{t \to -\infty} (x_{1,2}(t) - c_1 t) = -\frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{\lambda_2 (\lambda_2 + \mu_1)}. \tag{99}
\]
Similarly, comparing the line approached by the curve \( x = x_2(t) \) as \( t \to +\infty \),
\[
x = c_2 t + \frac{1}{2} \ln \frac{2 a_2(0) b_1(0)}{\lambda_2 + \mu_1} + \frac{1}{2} \ln \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 (\lambda_1 + \mu_1)} \right),
\]
to the line approached by the curve \( x = x_3(t) \) as \( t \to -\infty \),
\[
x = c_2 t + \frac{1}{2} \ln \frac{2 a_2(0) b_1(0)}{\lambda_2 + \mu_1},
\]
we see that ‘the peakon with asymptotic velocity \( c_2 \)’ experiences the phase shift
\[
\lim_{t \to +\infty} (x_2(t) - c_2 t) - \lim_{t \to -\infty} (x_3(t) - c_2 t) = \frac{1}{2} \ln \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 (\lambda_1 + \mu_1)} \right).
\] (100)

And finally, comparing the line approached by the curve \( x = x_1(t) \) as \( t \to +\infty \),
\[
x = c_3 t + \frac{1}{2} \ln \frac{\mu_1 a_2(0)}{\lambda_1 \lambda_2 b_\infty} + \frac{1}{2} \ln \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 (\lambda_2 + \mu_1)} \right),
\]
to the line approached by the curve \( x = x_4(t) \) as \( t \to -\infty \),
\[
x = c_3 t + \frac{1}{2} \ln (2 a_2(0) b_\infty),
\]
it’s clear that the phase shift of ‘the slow peakon with asymptotic velocity \( c_3 \)’ is
\[
\lim_{t \to +\infty} (x_1(t) - c_3 t) - \lim_{t \to -\infty} (x_4(t) - c_3 t) = \frac{1}{2} \ln \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 (\lambda_2 + \mu_1)} \right) - \frac{1}{2} \ln \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 (\lambda_2 + \mu_1)} \right).
\] (101)

### 7.5 Phase shifts in logarithms of amplitudes

Geng–Xue peakons also exhibit a ‘phase shift’ of a more unusual type, not seen in systems where the amplitudes simply tend to constant values as \( t \to \pm \infty \). Here, the amplitudes instead have exponential growth or decay as \( t \to \pm \infty \), so their logarithms asymptotically behave like \( d_i t + \) constant, with the same coefficients \( d_i \) appearing at \( +\infty \) and \( -\infty \) (see (91)/(93) and (96)/(98)). Therefore, we can make a similar comparison as we did for positions above. This gives the following formulas:

\[
\lim_{t \to +\infty} \left( -\ln n_4(t) - d_1 t \right) - \lim_{t \to -\infty} \left( \ln m_1(t) - d_1 t \right) = \ln \frac{2\lambda_1 (\lambda_1 + \mu_1)}{\mu_1} - \frac{1}{2} \ln \left( \frac{\lambda_1 - \lambda_2}{\lambda_2^3 (\lambda_2 + \mu_1)} \right),
\] (102)

\[
\lim_{t \to +\infty} \left( \ln m_3(t) - d_1 t \right) - \lim_{t \to -\infty} \left( -\ln n_2(t) - d_1 t \right) = -\ln \frac{2\lambda_1 \mu_1}{\lambda_1 + \mu_1},
\] (103)
lim_{t \to +\infty} \left( -\ln n_2(t) - d_2t \right) - \lim_{t \to -\infty} \left( \ln m_3(t) - d_2t \right) = \ln \frac{2\lambda_2\mu_1}{\lambda_2 + \mu_1} + \frac{1}{2} \ln \left( \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^3} \right), \quad (104)

and

lim_{t \to +\infty} \left( \ln m_1(t) - d_3t \right) - \lim_{t \to -\infty} \left( -\ln n_4(t) - d_3t \right) = \frac{1}{2} \ln \frac{\mu_1}{2\lambda_1\lambda_2} + \frac{1}{2} \ln (b_\infty b_\infty^*) + \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1(\lambda_2 + \mu_1)}. \quad (105)

8. Shock formation in a $2+2$ mixed peakon–antipeakon case

Throughout the article, except in this section, we are assuming that all the peakon amplitudes $m_k$ and $n_k$ are positive, i.e. we are only considering what is known as ‘pure peakon solutions’. The experience from other peakon equations (Camassa–Holm, Degasperis–Procesi, Novikov) is that pure peakon solutions are globally defined, whereas mixed peakon–antipeakon solutions, with some amplitudes positive and some negative, lead to collisions, finite-time blowup, and subtle questions of how to continue the solution past singularities [2, 41–45].

For the Geng–Xue equation, negative amplitudes cause no problems as long as all the peakons in $u$ have the same sign and all the peakons in $v$ have the same sign. In fact, from the governing ODEs (3) it is immediate that if

$$x_k = \xi_k(t), \quad m_k = \mu_k(t), \quad n_k = \nu_k(t)$$

is a pure peakon solution, then

$$x_k = \xi_k(t), \quad m_k = -\mu_k(t), \quad n_k = -\nu_k(t)$$

is a pure antipeakon solution, while

$$x_k = \xi_k(-t), \quad m_k = -\mu_k(-t), \quad n_k = \nu_k(-t)$$

is a solution with antipeakons in $u$ and peakons in $v$, and the other way around for

$$x_k = \xi_k(-t), \quad m_k = \mu_k(-t), \quad n_k = -\nu_k(-t).$$

This is the reason why there was nothing remarkable about the $1+1$ interlacing peakon–antipeakon case considered in Remark 5.1.

But when we mix peakons and antipeakons within one component of a solution, there will indeed be complications. Just to get an idea of what may happen, we will spend the rest of this section looking at the $2+2$ interlacing case with $m_1$, $n_2$, $m_3$ positive and $n_4$ negative. It will emerge that it is possible for the solution $(u(x, t), v(x, t))$ to form a jump discontinuity after finite time, meaning that one is forced to consider shockpeakons in order to provide a meaningful continuation past the singularity; this is similar to what happens for the Degasperis–Procesi equation [2].
When negative amplitudes are involved, the spectral variables will not lie in the usual positive sector (26), so we must begin by investigating their signs. From (31) we compute

\[ A(\lambda) = 1 - 2\lambda(m_1 n_2 E_{12} + m_1 n_4 E_{14} + m_3 n_4 E_{34}) + 4\lambda^2 m_1 n_2 m_3 n_4 (1 - E_{12}^2)E_{34}, \]

\[ B(\lambda) = e^{\alpha} \left( m_1 E_{13} + m_3 - 2\lambda m_1 n_2 m_3 E_{12} (1 - E_{12}^2) \right), \]

while (35) gives

\[ \tilde{A}(\lambda) = 1 - 2\lambda n_2 m_3 E_{23}, \]
\[ \tilde{B}(\lambda) = e^{\alpha} \left( n_2 E_{24} + n_4 - 2\lambda n_2 m_3 (1 - E_{12}^2) \right). \]

Since the polynomial \( A(\lambda) = (1 - \lambda/\lambda_1)(1 - \lambda/\lambda_2) \) has negative \( \lambda^2 \)-coefficient, its zeros will be of opposite sign, say \( \lambda_1 < 0 < \lambda_2 \), and the single zero of \( \tilde{A}(\lambda) \) is \( \mu_1 = (2n_2 m_3 E_{23})^{-1} > 0 \). From (38) and (39), we see that \( b_\infty < 0 \) and \( b_1^* > 0 \).

For the sake of simplicity, we will now consider some concrete numerical values.

**Example 8.1** The following initial data are designed to give simple spectral data:

\[ x_1(0) = 0, \quad x_2(0) = \frac{1}{2} \ln \frac{9}{4}, \quad x_3(0) = \frac{1}{2} \ln \frac{7}{2}, \quad x_4(0) = \frac{1}{2} \ln \frac{11}{2} \]

and

\[ m_1(0) = \frac{9}{5}, \quad n_2(0) = \frac{5}{6}, \quad m_3(0) = \frac{1}{5} \sqrt{2}, \quad n_4(0) = -\frac{1}{2} \sqrt{11} \frac{2}{7}. \]

The Weyl functions at time \( t = 0 \) are

\[ \omega(\lambda; 0) = -\frac{B(\lambda; 0)}{A(\lambda; 0)} = -\frac{-\frac{3}{2} - \frac{1}{2} \lambda}{\frac{1}{2} \lambda - \frac{1}{2} \lambda^2} = \frac{-2}{\lambda - (-1)} + \frac{1}{\lambda - 2} = \frac{a_1(0)}{\lambda - \lambda_1} + \frac{a_2(0)}{\lambda - \lambda_2}, \]

and

\[ \tilde{\omega}(\lambda; 0) = -\frac{\tilde{B}(\lambda; 0)}{\tilde{A}(\lambda; 0)} = -\frac{-\frac{3}{2} + \frac{1}{2} \lambda}{\frac{1}{2} \lambda - \frac{1}{2} \lambda^2} = 1 + \frac{-1}{\lambda - 2} = -b_\infty + \frac{b_1(0)}{\lambda - \mu_1}, \]

so that we have

\[ \lambda_1 = -1, \quad \lambda_2 = 2, \quad \mu_1 = 2, \quad a_1(0) = -2, \quad a_2(0) = 1, \quad b_1(0) = -1, \quad b_\infty = -1, \]
and also $b_\infty^* = 1$ from (38). Then, since $a_1 = a_1(t) = a_1(0) e^{t/\lambda_1} = -2 e^{-t}$, and so on, the quantities appearing in the peakon solution formulas (85) are

$$a_1 + a_2 = -2 e^{-t} + e^{t/2} < 0 \iff t < \frac{1}{2} \ln 2,$$

$$\lambda_1 a_1 + \lambda_2 a_2 = 2 e^{-t} + 2 e^{t/2} > 0,$$

$$\frac{a_1 b_1}{\lambda_1 + \mu_1} + \frac{a_2 b_1}{\lambda_2 + \mu_1} = 2 e^{-t/2} - \frac{1}{2} e^t > 0 \iff t < 2 \ln 2,$$

$$\frac{\lambda_1 a_1 b_1}{\lambda_1 + \mu_1} + \frac{\lambda_2 a_2 b_1}{\lambda_2 + \mu_1} = -2 e^{-t/2} - \frac{1}{2} e^t < 0,$$

$$\frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_1)} a_1 a_2 b_1 = \frac{9}{2} > 0.$$

Two of these quantities change their sign, and the first one to become zero is $a_1 + a_2$, which changes sign from negative to positive when

$$t = t_0 := \frac{1}{2} \ln 2.$$

The fact that the solution formulas really satisfy the peakon ODEs is purely algebraic, and does not depend on the signs of the spectral variables, as long as everything is defined and the ordering $x_1 \leq x_2 \leq x_3 \leq x_4$ is preserved. From (85a) we obtain

$$\frac{e^{x_4} - e^{x_3}}{e^{3/2} - e^{x_2}} = b_\infty(a_1 + a_2)
- 2 e^{-t} + e^{t/2},$$

$$\frac{e^{x_3} - e^{x_2}}{e^{3/2} - e^{x_2}} = \frac{a_1 + a_2}{\lambda_1 a_1 + \lambda_2 a_2}
\left( \frac{\lambda_1 a_1 b_1}{\lambda_1 + \mu_1} + \frac{\lambda_2 a_2 b_1}{\lambda_2 + \mu_1} \right)
= \frac{-2 e^{-t} + e^{t/2}}{2 e^{-t} + 2 e^{t/2}},$$

$$2 e^{-x_1} - 2 e^{-x_2} = \frac{2 b_\infty^* \lambda_1 \lambda_2}{\mu_1}
\left( \frac{\lambda_1 a_1 b_1}{\lambda_1 + \mu_1} + \frac{\lambda_2 a_2 b_1}{\lambda_2 + \mu_1} \right)
= -2 \left( -2 e^{-t/2} - \frac{1}{2} e^t \right),$$

so the ordering is indeed preserved up until $t = t_0$, when a (triple) collision

$$\frac{1}{2} \ln \frac{3}{e^{x_2} + e^{x_2}} = x_1(t_0) < x_2(t_0) = x_3(t_0) = x_4(t_0) = \frac{1}{2} \ln \frac{3}{x_4^*},$$

occurs, and looking at where the factor $a_1 + a_2$ occurs in (85b), we also see that

$$n_2(t) \to \infty, \quad m_3(t) \to 0, \quad n_4(t) \to -\infty, \quad \text{as } t \to t_0^-.$$

The component $u(x, t)$ simply converges to $m_1(t_0) e^{-|x-x_1(t_0)|}$, where

$$m_1(t_0) = \left( \frac{3}{2} (1 + 2^{1/3}) \right)^{1/2},$$
Because of this cancellation, these two expressions have finite limits as claimed:

$$
\lim_{t \to t_0^-} u(x, t) = \left( \frac{1}{2} \left( 1 + 2^{3/2} \right) \right)^{1/2} e^{-\frac{1}{2} \ln \left| \frac{3}{2} \right|}.
$$

(106)

The behaviour of \( v(x, t) \) is more subtle. The solution formulas (85) imply that

\[
v(x_2(t), t) = n_2 + n_4 E_{24}
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{1}{2} e^{2x_2} \right)^{1/2} (2n_2 e^{-x_2} + 2n_4 e^{-x_4})
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \mu_1)(\lambda_2 + \mu_1)} \right)^{1/2}
\]

\[
\times \left( \frac{\lambda_1^2 a_1 + \lambda_2^2 a_2}{(\lambda_1 - \lambda_2)^2 \mu_1 a_1 a_2} \left( \frac{\lambda_1}{\lambda_1 + \mu_1} + \frac{\lambda_2^2}{\lambda_2 + \mu_1} \right) \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( 2 e^{-t/2} + 2 e^{t/2} \right)^{1/2} \frac{-e^{t/2}}{9} (-2 e^{-t/2} - 1).
\]

Here it is important that the expression \( 2n_2 e^{-x_2} + 2n_4 e^{-x_4} \) has been simplified by cancelling a common factor \( a_1 + a_2 \) from the numerator and the denominator. The factor \( a_1 + a_2 \) likewise cancels when computing

\[
v(x_4(t), t) = n_2 E_{24} + n_4
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{1}{2} e^{2x_4} \right)^{-1/2} (2n_2 e^{-x_2} - \frac{1}{2} e^{2x_2} + 2n_4 e^{-x_4} - \frac{1}{2} e^{2x_4})
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{a_1 b_1}{\lambda_1 + \mu_1} + \frac{a_2 b_1}{\lambda_2 + \mu_1} + b_\infty (a_1 + a_2) \right)^{-1/2} \left( \frac{b_1}{\mu_1} + b_\infty \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( 2 e^{-t/2} - \frac{1}{2} e^{t/2} - e^{t/2} - \frac{1}{2} e^{-t/2} - 1 \right).
\]

Because of this cancellation, these two expressions have finite limits as \( t \to t_0^- \),

\[
\lim_{t \to t_0^-} v(x_2(t), t) = 0, \quad \lim_{t \to t_0^-} v(x_4(t), t) = -\frac{3(1 + 2^{3/2})}{4\sqrt{2}} < 0,
\]

which means that \( v(x, t) \) converges to a shockpeakon-type wave profile at the instant of collision, as claimed:

\[
\lim_{t \to t_0^-} v(x, t) = \begin{cases} 
\lim_{t \to t_0^-} v(x_2(t), t) e^{-x_0}, & x < x_0 \\
\lim_{t \to t_0^-} v(x_4(t), t) e^{x_0}, & x > x_0 \\
0, & x = x_0 
\end{cases}
\]

\[
= \begin{cases} 
0, & x < x_0 \\
-\frac{3(1 + 2^{3/2})}{4\sqrt{2}} e^{x_0}, & x > x_0 
\end{cases}
\]

(107)
where \( x_0 = x_3(t_0) = x_4(t_0) = \tfrac{1}{2} \ln \frac{3}{2 \sqrt{3}} \) is the site of the triple collision. (The fact that \( v \) becomes identically zero for \( x < x_0 \) is clearly a bit of a coincidence; it’s just that for our particular spectral data, the factor \( \frac{\lambda_1^2 + \mu_1^2}{\lambda_1 + \mu_1} + \frac{\lambda_2^2 + \mu_2^2}{\lambda_2 + \mu_2} \) happens to equal \( a_1 + a_2 \), and hence it vanishes at the collision.)

Since the second peakon in \( u \) vanishes at the instant of the collision, the non-overlapping condition is actually preserved automatically, and we can continue the solution for \( t \geq t_0 \) by taking the limiting wave profiles (106) and (107) as initial data \( u(x, t_0) \) and \( v(x, t_0) \) for a 1 + 1-shockpeakon solution starting at \( t = t_0 \), i.e. a solution of the type in Section 6, with

\[
x_1(t_0) = \frac{1}{2} \ln \frac{3}{2 \sqrt{3}}, \quad x_2(t_0) = \frac{1}{2} \ln \frac{3}{2 \sqrt{3}},
\]
\[
m_1(t_0) = \left( \frac{3}{2} (1 + 2^{1/3}) \right)^{1/2}, \quad s_1(t_0) = 0, \quad n_2(t_0) = -r_2(t_0) = -\frac{3(1 + 2^{2/3})}{8 \sqrt{2}}.
\]

**Remark 8.2** Apart from collisions, another technical complication with mixed peakon–antipeakon solutions is that there may be resonant cases where some \( \lambda_i + \mu_i \) vanishes. In such cases there will be division by zero in the usual solution formulas, so they will not be valid at all, not even before the blowup. However, this can be handled by limiting arguments. Just to give one example, consider the initial data

\[
x_1(0) = 0, \quad x_2(0) = \ln 2, \quad x_3(0) = \ln 4, \quad x_4(0) = \ln 8
\]

and

\[
m_1(0) = n_2(0) = m_3(0) = 1, \quad n_4(0) = -1.
\]

Then

\[
\omega(\lambda; 0) = -\frac{B(\lambda)}{A(\lambda)} = -\frac{5 - 3 \lambda}{1 + \frac{1}{4} \lambda - \frac{3}{4} \lambda^2} = \frac{-32/7}{\lambda - (-1)} + \frac{4/7}{\lambda - \frac{1}{2}} = \frac{a_1(0)}{\lambda - \lambda_1} + \frac{a_2(0)}{\lambda - \lambda_2}
\]

and

\[
\tilde{\omega}(\lambda; 0) = \frac{\tilde{B}(\lambda)}{A(\lambda)} = -\frac{6 + 6 \lambda}{1 - \lambda} = 6 + \frac{0}{\lambda - 1} = -b_\infty + \frac{b_1(0)}{\lambda - \mu_1},
\]

so that \( \lambda_1 = -1, \lambda_2 = \frac{1}{2}, \mu_1 = 1 \), and in particular \( \lambda_1 + \mu_1 = 0 \). However, this is balanced in the peakon solution formulas (85) by the fact that \( b_1(0) = 0 \) also. In fact, if we take \( n_4(0) = -(1 + \varepsilon) \) instead, we find

\[
\lambda_1 = \frac{1 + 5 \varepsilon - \sqrt{49 + 58 \varepsilon + 25 \varepsilon^2}}{6(1 + \varepsilon)}, \quad \mu_1 = 1, \quad b_1(0) = -2 \varepsilon,
\]

so that

\[
\frac{b_1(t)}{\lambda_1 + \mu_1} = \frac{b_1(0) e^{t/\mu_1}}{\lambda_1 + \mu_1} \rightarrow \frac{-7}{4} e^t, \quad \text{as} \ \varepsilon \rightarrow 0.
\]
Thus the correct solution formulas in this limiting case are given by replacing the expression $b_1(t)/(\lambda_1 + \mu_1)$ with $-\frac{1}{2}e^t$ wherever it occurs in (85).

9. Dynamics of $K + K$ interlacing pure peakon solutions

With the detailed examples of the previous sections (in particular Section 7) under our belt, we are now ready to tackle the asymptotics of the general $K + K$ interlacing peakon solution (described in Corollary 4.2). Since the exceptional case $K = 1$ has been treated in Section 5, we will assume in this section that $K \geq 2$. Moreover, we will only consider pure peakon solutions, i.e. we will assume that all the amplitudes $m_k$ and $n_k$ are positive.

9.1 Preparations

First we establish the large time asymptotic behaviour of the sum $J_{nm}^r$ defined by equation (50). We recall the definition here, for convenience:

$$J_{nm}^r = \sum_{i \in [K]_n} \sum_{j \in [K-1]_m} \Psi_{ij} \lambda_i^r a_i \mu_j^r b_j.$$ 

Determining the asymptotics of this sum is quite easy, since the dominant contribution comes from a single term, with all other terms being exponentially small in comparison. (If we write the sum with the index sets $I$ and $J$ in lexicographic order, then the first term dominates as $t \to +\infty$, and the last term dominates as $t \to -\infty$.)

We remind the reader that our notation was defined in Section 3.2; in particular, $[k]$ denotes the integer interval $\{1, 2, \ldots, k\}$ if $k \geq 1$, and the empty set if $k = 0$.

Lemma 9.1 Suppose $0 \leq n \leq K$ and $0 \leq m \leq K - 1$. Given the time evolution of the spectral data described by Theorem 4.1, the leading long-time behaviour of $J_{nm}^r$ is given by

$$J_{nm}^r = (1 + o(1)) \Psi_{AB} \lambda_A^r a_A b_B, \quad t \to +\infty,$$  

(108)

where $A = [n]$ and $B = [m]$, and by

$$J_{nm}^r = (1 + o(1)) \Psi_{CD} \lambda_C^r a_C b_D, \quad t \to -\infty,$$  

(109)

where $C = [K] \setminus [K - n]$ and $D = [K - 1] \setminus [K - 1 - m]$. (Thus, $C = [K - n + 1, \ldots, K]$ if $n \geq 1$ and $C = \emptyset$ if $n = 0$, and similarly for $D$.)

Proof. Remember (see (26)) that we always label the eigenvalues so that they are ordered:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K, \quad 0 < \mu_1 < \mu_2 < \cdots < \mu_{K-1}.$$ 

This implies that $a_i(t) \gg a_2(t) \gg \cdots \gg a_K(t)$ and $b_1(t) \gg b_2(t) \gg \cdots \gg b_{K-1}(t)$ as $t \to +\infty$. What we mean by this is that if $i > j$, then

$$\frac{a_i(t)}{a_j(t)} = \frac{a_i(0)}{a_j(0)} \exp \left( t \left( \frac{1}{\lambda_i} - \frac{1}{\lambda_j} \right) \right) \to 0, \quad \frac{b_i(t)}{b_j(t)} = \frac{b_i(0)}{b_j(0)} \exp \left( t \left( \frac{1}{\mu_i} - \frac{1}{\mu_j} \right) \right) \to 0,$$ 

as $t \to +\infty$. \hfill \Box
as \( t \to +\infty \). Then it is clear that the term \( a_A b_B \), which contains the smallest indices, will be dominant as \( t \to +\infty \); if we factor it out from the sum (50), what remains is the constant \( \Psi_{1B} \lambda_A \mu_B \) plus terms which tend to zero (exponentially fast) as \( t \to +\infty \).

The case \( t \to -\infty \) is similar, taking into account that \( a_1(t) \ll a_2(t) \ll \cdots \ll a_K(t) \) and \( b_1(t) \ll b_2(t) \ll \cdots \ll b_{K-1}(t) \) as \( t \to -\infty \). □

9.2 Asymptotics for positions

We can now easily derive the general formulas for the asymptotics of the positions \( x_k \) and their derivatives \( \dot{x}_k \). In order to make these formulas more readable, we use the notation \( c_1, \ldots, c_{2K-1} \) for the asymptotic velocities, and we also introduce abbreviations for certain other combinations of the eigenvalues \( \lambda_i \) and \( \mu_j \).

**Definition 9.2** Define \( c_1 > c_2 > \cdots > c_{2K-1} > 0 \) by

\[
c_{2j} = \frac{1}{2} \left( \frac{1}{\lambda_j} + \frac{1}{\mu_j} \right), \quad j = 1, \ldots, K - 1, \\
c_{2j-1} = \begin{cases} 
\frac{1}{2} \left( \frac{1}{\lambda_j} + \frac{1}{\mu_j} \right), & j = 1, \ldots, K - 1, \\
\frac{1}{2} \left( \frac{1}{\lambda_j} \right), & j = K.
\end{cases}
\]

Moreover, let

\[
R'_y = \frac{1}{2} \ln \frac{(\lambda_r - \lambda_j)^2}{\lambda_r (\lambda_r + \mu_j)}, \\
R''_y = \frac{1}{2} \ln \frac{(\lambda_r - \lambda_j)^2}{\lambda_r (\lambda_r + \mu_{j-1})}, \\
S'_y = \frac{1}{2} \ln \frac{(\mu_s - \mu_j)^2}{(\lambda_j + \mu_s) \mu_s}, \\
S''_y = \frac{1}{2} \ln \frac{(\mu_s - \mu_{j-1})^2}{(\lambda_j + \mu_s) \mu_s}.
\]

**Theorem 9.3 (Asymptotics for positions and velocities)** In terms of the abbreviations of Definition 9.2, the positions and velocities in the \( K + K \) interlacing Geng–Xue peakon solution with \( K \geq 2 \) satisfy the following asymptotic formulas (where empty sums, such as \( \sum_{j=1}^{K-1} \) when \( j = K \), should be interpreted as zero).

Asymptotic velocities as \( t \to -\infty \):

\[
\dot{x}_i \sim \begin{cases} 
c_1, & i = 1, \\
c_{i-1}, & i = 2, 3, \ldots, 2K.
\end{cases}
\]
Asymptotic velocities as $t \to +\infty$:

\[ \dot{x}_{2K+1-i} \sim \begin{cases} 
  c_1, & i = 1, \\
  c_{i-1}, & i = 2, 3, \ldots, 2K.
\end{cases} \tag{112b} \]

Asymptotics for positions as $t \to -\infty$:

\[
x_{2j-1}(t) = \begin{cases} 
  c_1 t + \frac{1}{2} \ln \frac{2 a_1(0) b_1(0)}{\lambda_1 + \mu_1} \\
  \quad + \sum_{r=2}^{K} R_{rj}^- + \sum_{s=2}^{K-1} S_{sj}^- + o(1), & j = 1,
\end{cases} \tag{112c} \]

and

\[
x_{2j}(t) = \begin{cases} 
  c_{2j-1} t + \frac{1}{2} \ln \frac{2 a_j(0) b_j(0)}{\lambda_j + \mu_{j-1}} \\
  \quad + \sum_{r=1}^{j-1} R_{rj}^+ + \sum_{s=1}^{j-2} S_{sj}^+ + o(1), & j = 2, \ldots, K-1,
\end{cases} \tag{112d} \]

and

\[
x_{2K-1}(t) + \frac{1}{2} \ln (2 a_K(0) b_{\infty}) + o(1), & j = K. \tag{112e} \]

Asymptotics for positions as $t \to +\infty$:

\[
x_{2(2K+1-j)}(t) = \begin{cases} 
  c_1 t + \frac{1}{2} \ln \frac{2 a_1(0) b_1(0)}{\lambda_1 + \mu_1} + o(1), & j = 1,
\end{cases} \tag{112e} \]

\[
\quad + \sum_{r=1}^{j-1} R_{rj}^- + \sum_{s=1}^{j-2} S_{sj}^- + o(1), & j = 2, \ldots, K, \tag{112e} \]
positions here. If we look at the curve and only depend on the eigenvalues (see Definition 9.2). The coefficients velocities of the peakon in question, and they belong to the set \( x^{j} \) and so on.

The third peakon from the right (number 2 = \( x^{2} \)) affects \( x^{3} \) and \( x^{1} \). As \( \lim_{t \to -\infty} x^{j} = x^{1} \) (see Theorem 3.12). This is the reason for the division into cases in the asymptotic formulas before we delve into the details of the proof. (Cf. the discussion of the case \( K = 2 \) in Section 7.)

Remark 9.4 The formulas in Theorem 9.3 are somewhat involved, so let us say a few words about their structure before we delve into the details of the proof. The coefficients \( A \) and \( C \) are the asymptotic velocities of the peakon in question, and they belong to the set \{c_1, c_2, \ldots, c_{2K-1}\}. So even though there are \( 2K \) peakons, there are only \( 2K - 1 \) asymptotic velocities, which are numbered in decreasing order,

\[
c_1 > c_2 > \cdots > c_{2K-1},
\]

and only depend on the eigenvalues (see Definition 9.2). The coefficients \( B \) and \( D \) are given by more complicated expressions involving all the spectral variables.

As \( t \to -\infty \), the two leftmost peakons both asymptotically have the fastest velocity \( c_1 \), and in fact we see from the cases \( j = 1 \) in (112c) and (112d) that the curves \( x = x_1(t) \) and \( x = x_2(t) \) approach the same asymptotic line

\[
x = c_1 t + B,
\]

where

\[
B = \frac{1}{2} \ln \frac{2 a_1(0) b_1(0)}{\lambda_1 + \mu_1} + \sum_{r=2}^{K} R_{r}^{*} + \sum_{s=2}^{K-1} S_{s}^{*},
\]

i.e. the distance \( x_2(t) - x_1(t) \) tends to zero. The third peakon has asymptotic velocity \( c_2 \), the fourth one \( c_3 \), and so on.

Similarly, as \( t \to +\infty \), we see from the cases \( j = 1 \) in (112f) and (112g) that the curves \( x = x_{2K-1}(t) \) and \( x = x_{2K}(t) \) have the same asymptote

\[
x = c_1 t + D,
\]

where

\[
D = \frac{1}{2} \ln \frac{2 a_1(0) b_1(0)}{\lambda_1 + \mu_1},
\]

so the two rightmost peakons approach each other and both asymptotically have the fastest velocity \( c_1 \). The third peakon from the right (number \( 2K - 2 \)) has asymptotic velocity \( c_2 \), and so on.

Recall that \( x_1(t) \) and \( x_{2K}(t) \) are given by formulas which look different from the ones for the other positions \( x_j(t) \); in particular, the spectral variable \( b_{\infty} \) only enters in the formula for \( x_{2K} \), and \( b_{\infty}^{*} \) only affects \( x_1 \) (see Theorem 3.12). This is the reason for the division into cases in the asymptotic formulas here. If we look at the curve \( x_1(t) \) and its asymptote \( x = At + B \) as \( t \to -\infty \), we see that both its velocity
\( A = c_1 \) and the coefficient \( B \) deviate completely from the pattern followed by the other odd-numbered peakons \( x_{2j-1} \). However, in the asymptote \( x = Ct + D \) for \( x_1(t) \) as \( t \to +\infty \) (the case \( j = K \) in (112f)), it is only \( D \) which is exceptional; the velocity \( C = c_{2K-1} \) follows the general pattern \( x_{2(K+1)-1-j} \to c_{2j-1} \). Similar remarks apply to \( x_{2j}(t) \): it has the expected velocity \( c_{2K-1} \) as \( t \to -\infty \), but apart from that, its asymptotics deviate from the pattern followed by the other even-numbered peakons.

**Proof of Theorem 9.3.** We will work out the details for \( x_{2(K+1)-j} \) with \( 2 \leq j \leq K \) as \( t \to +\infty \). The proofs for the other cases are entirely similar and will be omitted. The formulas for velocities follow from the ones from positions, because the \( o(1) \) terms and their derivatives are actually bounded by factors of the form \( e^{\pm \delta t} \) with \( \delta > 0 \), as \( t \to \pm \infty \); see the proof of Lemma 9.1.

From Theorem 3.12 we have the exact formula for the solution:

\[
x_{2(K+1)-j}(t) = \frac{1}{2} \ln \left( \frac{2 J_{j-1}^0(t)}{J_{j-1,j-2}^1(t)} \right).
\]

The asymptotic behaviour of the factors as \( t \to +\infty \) is given by Lemma 9.1:

\[
J_{j-1}^0(t) = \left(1 + o(1)\right) \Psi_{[j][j-1]} a_{j-1}(t) b_{j-1}(t)
\]

and

\[
J_{j-1,j-2}^1 = \left(1 + o(1)\right) \Psi_{[j-1][j-2]} \lambda_{j-1} \mu_{j-2} a_{j-1}(t) b_{j-2}(t).
\]

It follows that

\[
x_{2(K+1)-j}(t) = \frac{1}{2} \ln \left( 1 + o(1) \right) \frac{2 \Psi_{[j][j-1]} a_{j-1}(t) b_{j-1}(t)}{\Psi_{[j-1][j-2]} \lambda_{j-1} \mu_{j-2} a_{j-1}(t) b_{j-2}(t)}
\]

\[
= \frac{1}{2} \ln \left( \frac{2 \Psi_{[j][j-1]} a_{j}(t) b_{j-1}(t)}{\Psi_{[j-1][j-2]} \lambda_{j-1} \mu_{j-2}} \right) + \frac{1}{2} \ln \left(1 + o(1)\right)
\]

\[
= \frac{1}{2} \ln \left( \frac{2 \Psi_{[j][j-1]} a_{j}(0) b_{j-1}(0) e^{\delta_j t}}{\Psi_{[j-1][j-2]} \lambda_{j-1} \mu_{j-2}} \right) + o(1)
\]

\[
= \frac{t}{2} \left( \frac{1}{\lambda_{j}} + \frac{1}{\mu_{j-1}} \right) + \frac{1}{2} \ln \frac{2 \Psi_{[j][j-1]} a_{j}(0) b_{j-1}(0)}{\Psi_{[j-1][j-2]} \lambda_{j-1} \mu_{j-2}} + o(1).
\]

We now obtain the claimed formula by expanding the definition of \( \Psi_{IJ} \) and cancelling all common factors from the ratios involving \( \Delta_{j}^2, \tilde{\Delta}_{j}^2 \) and \( \Gamma_{IJ}^2 \):

\[
\frac{\Psi_{[j][j-1]} \lambda_{j-1} \mu_{j-2}}{\Psi_{[j-1][j-2]} \lambda_{j-1} \mu_{j-2}} = \frac{\Delta_{j}^2 \tilde{\Delta}_{j}^2}{\Delta_{j-1}^2 \tilde{\Delta}_{j-2}^2} \times \frac{\Gamma_{[j-1][j-2]} \lambda_{j-1} \mu_{j-2}}{\Gamma_{[j][j-1]} \lambda_{j-1} \mu_{j-2}} \times \frac{1}{\lambda_{j-1} \mu_{j-2}}
\]
\[
\begin{align*}
\prod_{r=1}^{j-1} (\lambda_r - \lambda_j)^2 \left( \prod_{r=1}^{j-2} (\mu_j - \mu_{j-1})^2 \right) \\
= \frac{\left( \prod_{r=1}^{j-1} (\lambda_r + \mu_{j-1}) \right) \left( \prod_{s=1}^{j-2} (\lambda_s + \mu_{s+1}) \right) \left( \prod_{r=1}^{j-1} \lambda_r \right) \left( \prod_{r=1}^{j-2} \mu_r \right)}{\prod_{r=1}^{j-1} (\lambda_r + \mu_{j-1}) \left( \prod_{s=1}^{j-2} (\lambda_s + \mu_{s+1}) \right) \left( \prod_{r=1}^{j-1} \lambda_r \right) \left( \prod_{r=1}^{j-2} \mu_r \right)} \\
= \left( \prod_{r=1}^{j-1} (\lambda_r - \lambda_j)^2 \right) \left( \prod_{s=1}^{j-2} (\mu_j - \mu_{j-1})^2 \right) \frac{1}{\lambda_j + \mu_{j-1}}.
\end{align*}
\]

(The empty product appearing when \(j = 2\) should be read as having the value 1.)

\[\square\]

**Corollary 9.5** (Phase shifts for positions) The following formulas hold for the \(K + K\) interlacing Geng–Xue peakon solution with \(K \geq 2\):

\[
\begin{align*}
\lim_{t \to +\infty} (x_a(t) - c_1t) - \lim_{t \to -\infty} (x_b(t) - c_1t) &= -\sum_{r=2}^{K-1} R_{ji} - \sum_{s=2}^{j-1} S_{ji}, \quad \text{for } a \in \{2K - 1, 2K\} \text{ and } b \in \{1, 2\}, \tag{113a} \\
\lim_{t \to +\infty} (x_{2(K+1-j)}(t) - c_{2j-1}t) - \lim_{t \to -\infty} (x_{2j}(t) - c_{2j-1}t) &= \sum_{r=1}^{j-1} R_{ji} - \sum_{r=1}^{K} R_{ji} + \sum_{s=1}^{j-1} S_{ji} - \sum_{s=1}^{K-1} S_{ji}, \quad \text{for } j = 1, \ldots, K - 1, \tag{113b} \\
\lim_{t \to +\infty} (x_{2(K+1-j)}(t) - c_{2j-2}t) - \lim_{t \to -\infty} (x_{2j-1}(t) - c_{2j-2}t) &= \sum_{r=1}^{j-1} R_{ji} - \sum_{r=1}^{K} R_{ji} + \sum_{s=1}^{j-2} S_{ji} - \sum_{s=1}^{K-1} S_{ji}, \quad \text{for } j = 2, \ldots, K - 1, \tag{113c}
\end{align*}
\]

and

\[
\begin{align*}
\lim_{t \to +\infty} (x_1(t) - c_{2K-1}t) - \lim_{t \to -\infty} (x_{2K}(t) - c_{2K-1}t) &= \frac{1}{2} \ln \frac{M}{2L} - \frac{1}{2} \ln(b_{\infty} b_\infty^*) + \frac{1}{2} \sum_{r=1}^{K-1} \ln \frac{\lambda_r - \lambda_h^2}{\lambda_r(\lambda_h + \mu_r)}. \tag{113d}
\end{align*}
\]

**Remark 9.6** This way of writing the formulas is slightly redundant, since the case \(j = 1\) of (113b) is already included in (113a) as the case \(a = 2K - 1, b = 2\). The purpose of including \(j = 1\) in (113b) is to show that the pattern persists, and the purpose of (113a) is to emphasize that the curves \(x = x_1(t)\) and \(x = x_{2j}(t)\) have a common asymptote as \(t \to -\infty\), which is parallel to the common asymptote of \(x = x_{2K-1}(t)\) and \(x = x_{2K}(t)\) as \(t \to +\infty\).
9.3 Asymptotics for amplitudes

Next, we turn to the asymptotics of the amplitudes \( m_k \) and \( n_k \), which exhibit exponential growth or decay (or tend to constant values in borderline cases), so that their logarithms asymptotically behave like straight lines. To obtain concise formulas, we make definitions similar to Definition 9.2 above.

**Definition 9.7** Let

\[
\begin{align*}
    d_{2j} &= \frac{1}{2} \left( \frac{1}{\lambda_{j+1}} - \frac{1}{\mu_j} \right), \quad j = 1, \ldots, K - 1, \\
    d_{2j-1} &= \begin{cases} 
        \frac{1}{2} \left( \frac{1}{\lambda_j} - \frac{1}{\mu_j} \right), & j = 1, \ldots, K - 1, \\
        \frac{1}{2} \left( \frac{1}{\lambda_K} \right), & j = K,
    \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
    P'_{ij} &= \frac{1}{2} \ln \left( \frac{(\lambda_i - \lambda_j)^2 (\lambda_i + \mu_j)}{\lambda_i^3} \right), \\
    P''_{ij} &= \frac{1}{2} \ln \left( \frac{(\mu_i - \mu_j)^2 (\lambda_i + \mu_{j-1})}{\mu_i^3} \right), \\
    Q'_{ij} &= \frac{1}{2} \ln \left( \frac{(\mu_i - \mu_j)^2 (\lambda_j + \mu_i)}{\mu_i^3} \right), \\
    Q''_{ij} &= \frac{1}{2} \ln \left( \frac{(\mu_i - \mu_{j-1})^2 (\lambda_j + \mu_i)}{\mu_i^3} \right).
\end{align*}
\]

**Theorem 9.8** (Asymptotics for amplitudes) In terms of the abbreviations of Definition 9.7, the amplitudes in the \( K + K \) interlacing Geng–Xue peakon solution with \( K \geq 2 \) satisfy the following asymptotic formulas.

Asymptotics for amplitudes as \( t \to -\infty \):

\[
\ln m_{2j-1}(t) = \begin{cases} 
    d_1 t + \ln \frac{\mu_i}{K} + \frac{1}{2} \ln \frac{a_i(0)}{b_i(0)} \left( \lambda_i + \mu_i \right), & j = 1, \\
    d_{2j-2} t - \ln \lambda_j + \frac{1}{2} \ln \frac{a_i(0)}{b_{j-1}(0)} \left( \lambda_j + \mu_{j-1} \right) + \sum_{r=j+1}^{K-1} P'_{ij} - \sum_{s=j}^{K-1} Q'_{ij} + o(1), & j = 2, \ldots, K,
\end{cases}
\]

(116a)
and

\[
-\ln n_2(t) = \begin{cases} 
  d_{2j-1} t + \ln \mu_j + \frac{1}{2} \ln \frac{a_j(0)}{b_j(0)} \left( \lambda_j + \mu_j \right) 
  + \sum_{r=j+1}^{K} P'_{ij} - \sum_{s=j+1}^{K-1} Q'_{sj} + o(1), & j = 1, \ldots, K - 1, \\
  d_{2K-1} t + \frac{1}{2} \ln \frac{2 a_K(0)}{b_{\infty}} + o(1), & j = K,
\end{cases}
\]

(116b)

Asymptotics for amplitudes as \( t \to +\infty \):

\[
-\ln n_{2(K+1-j)}(t) = \begin{cases} 
  d_1 t + \frac{1}{2} \ln \frac{2 a_{j}(0)}{b_{1}(0)} \left( \lambda_{1} + \mu_{1} \right) + o(1), & j = 1, \\
  d_{2j-2} t + \ln \mu_{j-1} + \frac{1}{2} \ln \frac{2 a_{j}(0)}{b_{j-1}(0)} \left( \lambda_{j} + \mu_{j-1} \right) 
  + \sum_{r=1}^{j-1} P'_{ij} - \sum_{s=1}^{j-2} Q'_{sj} + o(1), & j = 2, \ldots, K,
\end{cases}
\]

(116c)

and

\[
\ln m_{2(K+1-j)-1}(t) = \begin{cases} 
  d_{2j-1} t - \ln \lambda_j + \frac{1}{2} \ln \frac{a_j(0)}{b_j(0)} \left( \lambda_j + \mu_j \right) 
  + \sum_{r=1}^{j-1} P'_{ij} - \sum_{s=1}^{j-1} Q'_{sj} + o(1), & j = 1, \ldots, K - 1, \\
  d_{2K-1} t + \frac{1}{2} \ln \left( b_{\infty} a_k(0) \frac{M}{L} \right) 
  + \sum_{r=1}^{K-1} \ln \frac{(\lambda_r - \lambda_K)^2}{\lambda_r(\lambda_k + \mu_r)}, & j = K.
\end{cases}
\]

(116d)

Proof. Calculations very similar to those in the proof of Theorem 9.3. We omit the details. \( \square \)

Corollary 9.9 (Phase shifts for amplitudes) The following formulas hold for the \( K + K \) interlacing Geng–Xue peakon solution with \( K \geq 2 \):

\[
\lim_{t \to +\infty} \left( \ln m_K(t) - d_1 t \right) - \lim_{t \to -\infty} \left( \ln m_1(t) - d_1 t \right) 
= \ln \frac{2 \lambda_1(\lambda_1 + \mu_1)}{\mu_1} - \sum_{r=2}^{K} P'_{r1} + \sum_{s=2}^{K-1} Q'_{s1},
\]

(117a)
\[
\lim_{t \to +\infty} \left( -\ln n_{2(K+1-j)}(t) - d_{2j-2} t \right) - \lim_{t \to -\infty} \left( \ln m_{2j-1}(t) - d_{2j-2} t \right) \\
= \ln \left( \frac{2\lambda_j}{\lambda_j + \mu_j} \right) + \sum_{r=1}^{j-1} P''_{rj} - \sum_{r=1}^{j-1} P''_{rj} \\
- \sum_{s=1}^{j-2} Q''_{sj} + \sum_{s=1}^{j-1} Q''_{sj}, \quad \text{for } j = 2, \ldots, K,
\] (117b)

\[
\lim_{t \to +\infty} \left( \ln m_{2(K+1-j)-1}(t) - d_{2j-2} t \right) - \lim_{t \to -\infty} \left( -\ln n_{2}(t) - d_{2j-2} t \right) \\
= -\ln \left( \frac{2\lambda_j}{\lambda_j + \mu_j} \right) + \sum_{r=1}^{j-1} P'_{rj} - \sum_{r=1}^{j-1} P'_{rj} \\
- \sum_{s=1}^{j-1} Q'_{sj} + \sum_{s=1}^{j-1} Q'_{sj}, \quad \text{for } j = 1, \ldots, K-1,
\] (117c)

and

\[
\lim_{t \to +\infty} \left( \ln m_{1}(t) - d_{2K-1} t \right) - \lim_{t \to -\infty} \left( -\ln n_{2K}(t) - d_{2K-1} t \right) \\
= \frac{1}{2} \ln \frac{M}{2L} + \frac{1}{2} \ln (b_\infty b_\infty^*) + \frac{1}{2} \sum_{r=1}^{K-1} \ln \left( \frac{\lambda_r - \lambda_K}{\lambda_r (\lambda_r + \mu_r)} \right).
\] (117d)

10. Concluding remarks

In this article, we have given a fairly complete treatment of interlacing pure \((K+K)\)-peakon solutions of the two-component Geng–Xue equation. To our knowledge, this is the first multi-component peakon equation for which the peakon ODEs have been solved explicitly. The third-order inverse spectral problem used for deriving the explicit solution formulas involves two Lax pairs and correspondingly two spectra [5]. (There are some similarities to the \(3 \times 3\) inverse problems studied by Kaup and collaborators, for example in [46], and it might be interesting to investigate whether there are any deeper connections.) This is the first application to peakon equations of the theory of Cauchy biorthogonal polynomials [18, 19] in its full generality, where the polynomials are biorthogonal with respect to two \textit{independent} spectral measures.

The interlacing peakon solutions display the Toda-like asymptotic properties typical of other peakon equations, with the peakons having the same asymptotic velocities when \(t \to +\infty\) as when \(t \to -\infty\), but in the opposite order, and the velocities depending only on the eigenvalues in the two spectra. A noteworthy feature, however, is that the peakons don’t scatter completely; instead, the two fastest peakons have the same asymptotic velocity, and in fact the distance between them tends to zero. Another interesting phenomenon is that the amplitudes of the peakons grow or decay exponentially, instead of just approaching constant values as is usually the case for peakons, and that the logarithms of the amplitudes exhibit similar scattering and phase shifts as the positions.

In the case of non-interlacing peakon configurations, to be studied in a separate article, we anticipate that there will be even less scattering, and more peakons clustering together with the same asymptotic velocity.
We have also reported on the discovery that the Geng–Xue equation admits discontinuous shock-peakon solutions, like the Degasperis–Procesi equation. Although we were able to integrate the $1+1$ shockpeakon ODEs, there are still several questions open for further investigation, concerning for example the status of the Lax pairs in the context of shockpeakons, formation of shockpeakons at peakon–antipeakon collisions, continuation of solutions past singularities, and the possibility of allowing overlapping peakon or shockpeakon solutions.

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Appendix A. Distributional interpretation of the Lax pair

The aim of this appendix is to derive the Geng–Xue shockpeakon ODEs (6), which of course include the peakon ODEs (3) as a special case, and to verify that the Lax pairs (27) and (28) are valid not only for smooth solutions, but also for peakon solutions. In other words, we want to show that these Lax pairs, if given an appropriate distributional interpretation, really are equivalent to the peakon ODEs (3) when $u(x, t)$ and $v(x, t)$ are given by the peakon ansatz (2) with non-overlapping peakons. The analysis is similar to the one done for Novikov peakons in Appendix B of our previous article [15].

A.1 Notation

The functions $u$ and $v$ considered in this article, as well as all their partial derivatives in the classical sense, belong to a certain family of piecewise smooth functions which we will denote by $PC^\infty$. We say that a function $f(x, t)$ belongs to $PC^\infty$ if there are finitely many smooth curves $\{x = x_k(t), t \in \mathbb{R}\}_{k=1}^N$ such that $x_1(t) < \cdots < x_N(t)$, so that they divide the $(x, t)$ plane into $N+1$ open regions $\{\Omega_k\}_{k=0}^N$, and if the restriction of $f$ to each $\Omega_k$ is a smooth function of $x$ and $t$ which can be extended to a smooth function on a neighbourhood of the closure $\overline{\Omega}_k$. We do not require that $f(x, t)$ is defined on the curves $x = x_k(t)$. However, the assumptions imply that the left and right limits of $f$ exist at every $x$, and they will be denoted by $f(x^-)$ and $f(x^+)$, respectively. Hence we can define the jump function and the average function,

$$[f](x) := f'(x^+) - f'(x^-), \quad \langle f \rangle(x) := \frac{f'(x^+) + f'(x^-)}{2},$$  \hspace{1cm} (A.1)

which belong to $PC^\infty$, are defined everywhere, and satisfy $\langle f \rangle = f$ and $[f] = 0$ away from the curves $x = x_k(t)$. In addition, we have the product rules

$$[fg] = [f][g] + [f][g], \quad \langle fg \rangle = \langle f \rangle \langle g \rangle + \frac{1}{2} [f][g].$$  \hspace{1cm} (A.2)

The class $PC^\infty$ is closed under partial differentiation in the classical sense, and we will use subscripts to denote such partial derivatives, for example $f_t$ or $f_{xtt}$.

On the other hand, we can also interpret functions in $PC^\infty$ as distributions: for each fixed $t$, the function $x \mapsto f(x, t)$ defines a regular distribution in the class $D'(\mathbb{R})$, by acting on test functions $\varphi(x)$ in
the usual way,

\[ \langle f, \varphi \rangle = \int_{\mathbb{R}} f(x, t) \varphi(x) \, dx. \]

The notation \( D_x f \) will denote the distributional derivative:

\[ \langle D_x f, \varphi \rangle = - \langle f, \varphi_x \rangle. \]

Note that we view \( f \) and \( D_x f \) as distributions with respect to the variable \( x \), depending only parametrically on \( t \). Therefore the time derivative \( D_t f \) is defined differently, as a limit in the space \( \mathcal{D}'(\mathbb{R}) \):

\[ D_t f(\cdot, t) = \lim_{\tau \to 0} \frac{f(\cdot, t + \tau) - f(\cdot, t)}{\tau}. \]  

This limit, provided it exists, commutes with \( D_x \) by the continuity of \( D_x \) on \( \mathcal{D}'(\mathbb{R}) \).

If \( f \in PC^\infty \) with (possible) jump discontinuities at \( \{x = x_k(t)\}_{k=1}^N \), then

\[ D_x f(\cdot, t) = f_x(\cdot, t) + \sum_{k=1}^N [f](.., x_k(t)) \delta_{x_k(t)}, \]

or

\[ D_x f = f_x + \sum_{k=1}^N [f](x_k) \delta_{x_k} \]

for short. (Here, of course, \( \delta_a \) denotes the Dirac delta at the point \( x = a \).) Moreover,

\[ D_t f = f_t - \sum_{k=1}^n \dot{x}_k [f](x_k) \delta_{x_k}, \]

where \( \dot{x}_k = dx_k/dt \).

We also note that

\[ \frac{d}{dt} f(x_k(t)^\pm, t) = f_t(x_k(t)^\pm, t) \dot{x}_k(t) + f(x_k(t)^\pm, t), \]

which gives

\[ \frac{d}{dt} (f(x_k)) = [f_t](x_k) \dot{x}_k + [f](x_k), \]

\[ \frac{d}{dt} ([f](x_k)) = [f_t](x_k) \dot{x}_k + [f_t](x_k). \]  

Finally, we remark that the discussion above generalizes easily to the case of matrix-valued functions with entries in \( PC^\infty \). For example, if \( A \) and \( B \) are two matrices with entries from \( PC^\infty \), and the matrix product \( AB \) is defined, then

\[ [AB] = [A][B] + [A][B], \]

\[ \langle AB \rangle = \langle A \rangle [B] + \frac{1}{4} [A][B]. \]

Likewise, equation (A.6) generalizes to matrices.
A.2 Derivation of the shockpeakon ODEs

In this section we prove Theorem 2.1, which says that the shockpeakon ansatz (5) is a solution of the distributional Geng–Xue equation (21) if and only if it is non-overlapping and satisfies the shockpeakon ODEs (6).

Proof of Theorem 2.1. A function \( u \) given by the shockpeakon ansatz is piecewise of the form

\[
A e^x + B e^{-x},
\]

which implies that \( \frac{1}{2} u^2 \) is piecewise of the form

\[
\frac{1}{2} A^2 e^{2x} + AB + \frac{1}{2} B^2 e^{-2x},
\]

which lies in the kernel of the differential operator \((4 - D_x^2)D_x\). Thus, the expression \((4 - D_x^2)D_x(\frac{1}{2} u^2)\) vanishes identically away from the points \( x = x_k \), and will therefore be a purely singular distribution: a linear combination of \( \{\delta_{x_k}, \delta'_{x_k}, \delta''_{x_k}\}_{k=1}^N \) resulting from differentiating the jump discontinuities of \( \frac{1}{2} u^2 \) at those points.

Using the notations for jump \( [u] \) and average \( \langle u \rangle \) as in (A.1), we immediately have

\[
[u](x_k) = -2s_k, \quad [u_x](x_k) = -2m_k,
\]  

(A.7)

and we also recall from (7) and (8) that we defined the notation \( u(x_k) \) and \( u_x(x_k) \) simply as abbreviations for the averages,

\[
u(x_k) := \langle u \rangle(x_k), \quad u_x(x_k) := \langle u_x \rangle(x_k).
\]  

(A.8)

From the rules (A.2), we then find the jump in \( \frac{1}{2} u^2 \) at \( x = x_k \):

\[
\left[\frac{1}{2} u^2\right](x_k) = \left[u\right](x_k) \cdot \langle u \rangle(x_k)
\]

\[
= (-2s_k) \cdot u(x_k).
\]

Each such jump contributes a Dirac delta to the distributional derivative, so the singular part of the distribution \( D_x(\frac{1}{2} u^2) \) is

\[
-2 \sum_{k=1}^N s_k u(x_k) \delta_{x_k},
\]

and the regular part is just the function \( uu_x \) (the classical partial derivative of \( \frac{1}{2} u^2 \), defined away from the points \( x = x_k \)). Thus, as a distribution,

\[
D_x(\frac{1}{2} u^2) = uu_x - 2 \sum_{k=1}^N s_k u(x_k) \delta_{x_k}.
\]  

(A.9)
In the next step, we need
\[ [uu_x](x_k) = [u](x_k) \cdot (u_x)(x_k) + [u](x_k) \cdot [u_x](x_k) \]
\[ = (-2s_k) \cdot u_x(x_k) + u(x_k) \cdot (-2m_k), \]
together with the fact that \( u_{xx} = u \) for \( x \neq x_k \) (since \( u \) is piecewise of the form \( Ae^x + Be^{-x} \)). Using this, we find when differentiating (A.9) that, as a distribution,
\[ D_x^2 \left( \frac{1}{2} u^2 \right) = u_x^2 + uu_{xx} + \sum_{k=1}^N [uu_x](x_k) \delta_{x_k} - 2 \sum_{k=1}^N s_k u(x_k) \delta''_{x_k} \]
\[ = u_x^2 + u_x^2 - 2 \sum_{k=1}^N (m_k u(x_k) + s_k u_x(x_k)) \delta_{x_k} - 2 \sum_{k=1}^N s_k u(x_k) \delta''_{x_k}. \]
(A.10)

For the final differentiation, we can reuse the result (A.9) for \( D_x (u^2) = 2D_x \left( \frac{1}{2} u^2 \right) \), and we also need
\[ [u^2_x](x_k) = 2[u_x](x_k) \cdot (u_x)(x_k) \]
\[ = 2 \cdot (-2m_k) \cdot u_x(x_k). \]

Upon differentiating (A.10), this gives
\[ D_x^3 \left( \frac{1}{2} u^2 \right) = 2D_x (\frac{1}{2} u^2) + 2uu_{xx} + \sum_{k=1}^N [u^2_x](x_k) \delta_{x_k} \]
\[ - 2 \sum_{k=1}^N (m_k u(x_k) + s_k u_x(x_k)) \delta''_{x_k} - 2 \sum_{k=1}^N s_k u(x_k) \delta'''_{x_k} \]
\[ = 4uu_x - 4 \sum_{k=1}^N (s_k u(x_k) + m_k u_x(x_k)) \delta_{x_k} \]
\[ - 2 \sum_{k=1}^N (m_k u(x_k) + s_k u_x(x_k)) \delta''_{x_k} - 2 \sum_{k=1}^N s_k u(x_k) \delta'''_{x_k}. \]
(A.11)

Combining (A.9) and (A.11), we get
\[ (4 - D_x^2)D_x (\frac{1}{2} u^2) = 4D_x (\frac{1}{2} u^2) - D_x^3 (\frac{1}{2} u^2) \]
\[ = 4 \left( uu_x - 2 \sum_{k=1}^N s_k u(x_k) \delta_{x_k} \right) \]
\[ - \left( 4u_{uu_x} - 4 \sum_{k=1}^N (s_k u(x_k) + m_k u_x(x_k)) \delta_{x_k} \right) \]
\[ - 2 \sum_{k=1}^N (m_k u(x_k) + s_k u_x(x_k)) \delta''_{x_k} - 2 \sum_{k=1}^N s_k u(x_k) \delta'''_{x_k} \]
\[
\begin{align*}
&= 4 \sum_{k=1}^{N} (-s_k u(x_k) + m_k u_t(x_k)) \delta_{x_k} \\
&+ 2 \sum_{k=1}^{N} (m_k u(x_k) + s_k u_t(x_k)) \delta_{x_k}' + 2 \sum_{k=1}^{N} s_k u(x_k) \delta_{x_k}''.
\end{align*}
\]

(A.12)

(Notice that the regular parts cancel out, as predicted.)

Indices \(k\) for which \(m_k = s_k = 0\) give no contribution to the sums here. Let \(\mathcal{K} \subset \{1, 2, \ldots, N\}\) be the set of the remaining indices (those that do contribute); then we can replace \(\sum_{k=1}^{N}\) with \(\sum_{k \in \mathcal{K}}\) above.

Because of the non-overlapping assumption, \(v\) is smooth near \(x_k\) for \(k \in \mathcal{K}\), so the values \(v(x_k), v_t(x_k)\) and \(v_{xx}(x_k) = v(x_k)\) exist for \(k \in \mathcal{K}\) (and they coincide with the averages \(\langle v(x_k)\rangle\) and \(\langle v_t(x_k)\rangle\), so there is no conflicting notation). Thus, multiplying (A.12) by \(v(x, t)\) according to the rules

\[
\begin{align*}
&v(x) \delta_u = v(a) \delta_u, \\
&v(x) \delta'_u = v(a) \delta'_u - v'(a) \delta_u, \\
&v(x) \delta''_u = v(a) \delta''_u - 2v'(a) \delta'_u + v''(a) \delta_u,
\end{align*}
\]

we obtain

\[
\begin{align*}
&v \cdot (4 - D^2_\tau)D_\tau \left( \frac{u}{2} \right) = 4 \sum_{k \in \mathcal{K}} \left(-s_k u(x_k) + m_k u_t(x_k)\right) v(x_k) \delta_{x_k} \\
&\quad + 2 \sum_{k \in \mathcal{K}} \left(m_k u(x_k) + s_k u_t(x_k)\right) \left(v(x_k) \delta''_{x_k} - v_t(x_k) \delta_{x_k}'\right) \\
&\quad + 2 \sum_{k \in \mathcal{K}} s_k u(x_k) \left(v(x_k) \delta''_{x_k} - 2v_t(x_k) \delta'_x + v_{xx}(x_k) \delta_{x_k}\right) \\
&= 2 \sum_{k \in \mathcal{K}} \left(-s_k u(x_k) + 2m_k u_t(x_k)\right) v(x_k) \\
&\quad - \left(m_k u(x_k) + s_k u_t(x_k)\right) v_t(x_k) \delta_{x_k} \\
&\quad + 2 \sum_{k \in \mathcal{K}} \left(m_k u(x_k) + s_k u_t(x_k)\right) v(x_k) \\
&\quad - 2s_k u(x_k) v_t(x_k) \delta'_{x_k} \\
&\quad + 2 \sum_{k \in \mathcal{K}} s_k u(x_k) v(x_k) \delta''_{x_k}. \quad \text{(A.13)}
\end{align*}
\]

The requirement of the distributional Geng–Xue equation (21) is that this should equal \(-D_t m\), where \(m = u - D^2_\tau u\). From

\[
D_t u = u_t + \sum_{k \in \mathcal{K}} [u](x_k) \delta_{x_k} = u_t - 2 \sum_{k \in \mathcal{K}} s_k \delta_{x_k}
\]
and

\[ D_x^2 u = u_x + \sum_{k \in K} [u_x](x_k) \delta_{x_k} - 2 \sum_{k \in K} s_k \delta'_{x_k} \]

\[ = u - 2 \sum_{k \in K} m_k \delta_{x_k} - 2 \sum_{k \in K} s_k \delta'_{x_k} \]

we get

\[ m = 2 \sum_{k \in K} m_k \delta_{x_k} + 2 \sum_{k \in K} s_k \delta'_{x_k}, \]

and thus

\[ -D_m = -2 \sum_{k \in K} \dot{m}_k \delta_{x_k} - 2 \sum_{k \in K} \dot{s}_k \delta'_{x_k} + 2 \sum_{k \in K} m_k \dot{x}_k \delta'_{x_k} + 2 \sum_{k \in K} s_k \dot{x}_k \delta''_{x_k}. \]  

(A.14)

Identifying coefficients in (A.13) and (A.14), we find, for \( k \in K \),

\[ -\dot{m}_k = (-s_k u(x_k) + 2m_k u_x(x_k)) v(x_k) - (m_k u(x_k) + s_k u_x(x_k)) v_x(x_k), \]

\[ -\dot{s}_k + m_k \dot{x}_k = (m_k u(x_k) + s_k u_x(x_k)) v(x_k) - 2s_k u_x(x_k) v_x(x_k), \]

\[ s_k \dot{x}_k = s_k u(x_k) v(x_k). \]

If \( s_k \neq 0 \), the third equation implies that \( \dot{x}_k = u(x_k) v(x_k) \), and then the second equation reduces to \(-\dot{s}_k = s_k (u_x(x_k) v(x_k) - 2u(x_k) v_x(x_k)) \). If \( s_k = 0 \), then \( m_k \neq 0 \) (since \( k \in K \)), and the second equation shows that \( \dot{s}_k = u(x_k) v(x_k) \). So in either case, we can simplify the equations to

\[ \dot{x}_k = u(x_k) v(x_k), \]

\[ \dot{m}_k = (s_k u(x_k) - 2m_k u_x(x_k)) v(x_k) + (m_k u(x_k) + s_k u_x(x_k)) v_x(x_k), \]

\[ = m_k (u(x_k) v_x(x_k) - 2u_x(x_k) v_x(x_k)) + s_k (u(x_k) v(x_k) + u_x(x_k) v_x(x_k)), \]  

(A.15)

\[ \dot{s}_k = s_k (2u(x_k) v_x(x_k) - u_x(x_k) v_x(x_k)), \]

for \( k \in K \), in agreement with the claimed shockpeakon ODEs (6). (And for \( k \notin K \), we can include the same equations for \( m_k \) and \( s_k \) if we like, since they are consistent with \( m_k = s_k = 0 \).)

By symmetry (simply interchanging the roles of \( u \) and \( v \)), we immediately obtain the corresponding equations for \( x_k, n_k, r_k \) with \( k \notin K \). \qed

**Remark A.1** Using (A.7), one can write the shockpeakon ODEs (6) for \( k \in K \) as

\[ \frac{d}{dt} x_k = [u] v, \]

\[ \frac{d}{dt} [u] = [u] \left( 2[u] v_x - [u] v \right), \]

\[ \frac{d}{dt} [u_1] = [u] \left( [u] v + [u] v_x \right) + [u_x] \left( [u] v_x - 2 [u_x] v \right). \]  

(A.16)
where all evaluations of jumps and averages are carried out at $x_k$, and similarly for $k \notin K$ but with $u$ and $v$ interchanged. For comparison, the Degasperis–Procesi shock peakon ODEs are

\[
\begin{align*}
\frac{dx_k}{dt} &= \langle u \rangle, \\
\frac{du}{dt} &= -\{u\} \{u\}, \\
\frac{du}{dt} &= 2\{u\} - 2\{u\}. 
\end{align*}
\]  

(A.17)

A.3 Verification of the Lax pair for peakon solutions

The peakon solutions considered in this article are of the form

\[
\begin{align*}
u(x,t) &= \sum_{k=1}^{K} m_k(t) e^{-|x-x_k(t)|}, \\
v(x,t) &= \sum_{k=1}^{K} n_k(t) e^{-|x-y_k(t)|}. 
\end{align*}
\]  

(A.18)

(Here we have found it convenient to change the notation from the main text, and use $x_k$ and $y_k$ instead of $x_2k-1$ and $x_2k$, and $m_k$ and $n_k$ instead of $m_2k-1$ and $n_2k$.)

The functions $u$ and $v$ both belong to the piecewise smooth class $PC^\infty$. They are continuous and satisfy

\[
\begin{align*}
D_x u &= u_x = \sum_{k=1}^{K} m_k \text{sgn}(x_k - x) e^{-|x-x_k|}, \\
D_x^2 u &= D_x(u_x) = u_{xx} + \sum_{k=1}^{K} \{u_x\}(x_k) \delta_{x_k} = u + \sum_{k=1}^{K} (-2m_k) \delta_{x_k}, 
\end{align*}
\]

with analogous formulas for $v$. These formulas imply

\[
\begin{align*}
m &= u - D_x^2 u = 2 \sum_{k=1}^{K} m_k \delta_{x_k}, \\
n &= v - D_x^2 v = 2 \sum_{k=1}^{K} n_k \delta_{y_k}. 
\end{align*}
\]  

(A.19)

The Lax pair (27a)–(27b) will involve the functions $u$, $v$, $u_x$, $v_x$, as well as the purely singular distributions $m$ and $n$. We will take $\psi_1$, $\psi_2$, $\psi_3$ to be functions in $PC^\infty$, and separate the regular (function) part from the singular (Dirac delta) part. Writing $\Psi = (\psi_1, \psi_2, \psi_3)'$, the formulation obtained in this way reads

\[
D_t \Psi = \hat{L} \Psi, \quad D_t \Psi = \hat{A} \Psi,
\]  

(A.20)

where

\[
\begin{align*}
\hat{L} &= L + 2z \left( \sum_{k=1}^{K} n_k \delta_{y_k} \right) E_{12} + 2z \left( \sum_{k=1}^{K} m_k \delta_{x_k} \right) E_{23}, \\
L &= \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad E_{12} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad E_{23} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\end{align*}
\]  

(A.21)
and

\[
\hat{A} = A - 2z \left( \sum_{k=1}^{K} n_k u(x_k) v(x_k) \right) E_{12} - 2z \left( \sum_{k=1}^{K} m_k u(x_k) v(x_k) \right) E_{23},
\]

(A.22)

First we make a few general comments. Note that (A.20) involves multiplying \((\psi_2, 0, 0)^t\) by \(\delta_{y_k}\) and multiplying \((0, \psi_3, 0)^t\) by \(\delta_{y_k}\). In the second case there is no problem, since the function \(x \mapsto \psi_3(x, t)\) will automatically be continuous according to the third component of the vector equation \(D_t \Psi = \hat{L} \Psi\).

But in the first case, the function \(x \mapsto \psi_2(x, t)\) may have a jump at \(x = y_k\), so some value \(\psi_2(y_k)\) must be assigned in order for this operation to be well-defined, and this assignment must be consistent with \(D_x D_t \Psi = D_x^2 \Psi\). However, this is only a problem if one tries to consider the general case of overlapping supports. In this article, the supports of \(m\) and \(n\) are assumed to be disjoint, and as a result \(\psi_2\) is continuous at the points of support of \(n\), and no other assignment is needed.

**Theorem A.2** Let \(u, v, m, n\) be given by (A.18)–(A.19) and assume that the supports of \(m\) and \(n\) are disjoint. Then (A.21)–(A.22) form a weak Lax pair whose compatibility condition is given by the peakon ODEs

\[
\begin{align*}
\dot{x}_k &= u(x_k) v(x_k), & \dot{m}_k &= m_k \left( u(x_k) v(x_k) - 2(u(x_k) v(x_k)) \right), \\
\dot{y}_k &= u(y_k) v(y_k), & \dot{n}_k &= n_k \left( v(y_k) u(y_k) - 2(v(y_k) u(y_k)) \right).
\end{align*}
\]

**Proof.** An essential simplification is to observe that we can localize our computations to a vicinity of one of the points of support of the measures, by considering test functions which are zero except in a small neighbourhood of such a point. Since the supports of \(m\) and \(n\) are disjoint, we will only deal with one type of computation; either involving \(E_{23}\) (for \(n\)) or \(E_{23}\) (for \(m\)). Let us localize our computation around \(y_k\).

Then we can omit the summation, as well as completely ignore the contribution coming from \(m\) but not from \(u\). (Some of the equalities below are a slight abuse of notation, which should be understood in the light of this remark.)

We observe that \(\psi_3\) is continuous at all the points of supports of both measures; in particular \([\psi_3](y_k) = 0\). Likewise, \([\psi_2](y_k) = 0\), even though \([\psi_2](x_k) = 2z m_k \psi_3(x_k)\). Clearly, \(\psi_1\) is not defined at \(y_k\). Even though this will not impact the computation, we will set \(\Psi(y_k) = (\Psi)(y_k)\) for the duration of the computation.

Next, we compute the derivatives of (A.20):

\[
\begin{align*}
D_x(D_t \Psi) &= D_x(D_x \Psi) = D_x(L \Psi + 2z n_k \delta_{y_k} E_{12} \Psi) \\
&= L(\hat{A} \Psi) + 2z \left( \sum_{k=1}^{K} n_k \delta_{y_k} E_{12} \Psi \right) - 2z E_{12} E_{12} \Psi \delta_{y_k} \\
D_x(D_t \Psi) &= D_x(D_x \Psi) = D_x(A \Psi - 2z n_k u(y_k) v(y_k)) E_{12} \Psi \delta_{y_k} \\
&= (A \Psi)_x + [A \Psi](y_k) \delta_{y_k} - 2z n_k E_{12} \Psi(v_k) u(y_k) v(y_k) \delta_{y_k}.
\end{align*}
\]
The regular part of (A.20) gives $\Psi_1 = L\Psi$, so that $(A\Psi)_1 = A_1 \Psi + A L\Psi$, and it is easily verified that $LA = A_1 + AL$ holds identically (since $u_{xx} = u$). This implies that the regular parts of the two expressions above are equal, and the terms involving $\delta'_k$ are also equal provided $\dot{y}_k = u(y_k) v(y_k)$. Therefore the compatibility condition $D_t(D_x\Psi) = D_x(D_t\Psi)$ reduces to an equality between the coefficients of $\delta_k$,

$$-2zn_k u(y_k) v(y_k) L E_{12} \Psi(y_k) + 2z E_{12}\frac{d}{dt}(n_k \Psi(y_k)) = [A\Psi](y_k). \tag{A.23}$$

Using the product rule (A.2), $[\Psi](y_k) = 2zn_k E_{12}\Psi(y_k)$ and $[v](y_k) = -2n_k$, we find that the right-hand side of (A.23) equals

$$\langle A \rangle (y_k) 2zn_k E_{12}\Psi(y_k) + [A](y_k)[\Psi](y_k)$$

$$= 2zn_k \begin{pmatrix} 0 & -u(v_k) & 0 \\ 0 & u/z & 0 \\ 0 & -uv & 0 \end{pmatrix} \Psi(y_k) + 2n_k \begin{pmatrix} u & -1/z & -u \ \\ 0 & -u & 0 \\ 0 & 0 & 0 \end{pmatrix} \Psi(y_k). \tag{A.24}$$

The $(3,2)$ entry $-uv$ in the matrix in the first term will cancel against the whole first term on the left-hand side of (A.23), since the only nonzero entry of $LE_{12}$ is $(LE_{21})_{32} = 1$. Likewise, the $(2,2)$ entries sum up to 0. Thus (A.23) is equivalent to

$$\dot{n}_k E_{12}\Psi(y_k) + n_k E_{12}\frac{d}{dt}\Psi(y_k) = n_k \begin{pmatrix} u/z & -u(v_k) - 1/z^2 & -u v_k/z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{yk} \Psi(y_k). \tag{A.25}$$

Computing $\frac{d}{d t}\Psi(y_k)$ using (A.6), $\Psi_1 = L\Psi$ and $\Psi_1 = A\Psi$, we obtain

$$E_{12}\frac{d}{d t}\Psi(y_k) = E_{12}[L\Psi](y_k) \dot{y}_k + E_{12}[A\Psi](y_k)$$

$$= E_{12} \left( Lu(y_k) v(y_k) + \langle A \rangle (y_k) \right) \Psi(y_k) + E_{12} \frac{1}{2} [A](y_k) [\Psi](y_k)$$

$$= \begin{pmatrix} u/z & \langle v \rangle u - u_v v - 1/z^2 & -u z_k \ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{yk} \Psi(y_k)$$

$$+ \frac{1}{2} E_{12}[A(y_k)]E_{12} 2zn_k \Psi(y_k).$$

After cancelling common terms in (A.25) we arrive at

$$\left( \dot{n}_k + n_k (2u(y_k) \langle v \rangle(y_k) - u_{xx}(y_k) v(y_k)) \right) \Psi_2(y_k) = 0,$$

which gives the claimed equation for $\dot{n}_k$.

To prove the statement for $x_k$ and $m_k$ one can either repeat an analogous computation for the coefficient of $\delta_k$, involving $E_{23}$ rather than $E_{12}$, or simply use the twin Lax pair which immediately produces the result via the symmetry $u \leftrightarrow v, m \leftrightarrow n, y_k \leftrightarrow x_k$. \hfill $\square$
References

1. Geng, X. & Xue, B. (2009) An extension of integrable peakon equations with cubic nonlinearity. *Nonlinearity*, 22(8), 1847–1856.
2. Lundmark, H. (2007) Formation and dynamics of shock waves in the Degasperis–Procesi equation. *J. Nonlinear Sci.*, 17(3), 169–198.
3. Coclite, G. M. & Karlsen, K. H. (2006) On the well-posedness of the Degasperis–Procesi equation. *J. Funct. Anal.*, 233(1), 60–91.
4. Coclite, G. M. & Karlsen, K. H. (2007) On the uniqueness of discontinuous solutions to the Degasperis–Procesi equation. *J. Differ. Equ.*, 234(1), 142–160.
5. Lundmark, H. & Szmigielski, J. (2016) An inverse spectral problem related to the Geng–Xue two-component peakon equation. *Mem. Am. Math. Soc.*, 244(1155).
6. Camassa, R. & Holm, D. D. (1993) An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11), 1661–1664.
7. Beals, R., Sattlinger, D. H. & Szmigielski, J. (1999) Multi-peakons and a theorem of Stieltjes. *Inverse Probl.*, 15(1), L1–L4.
8. Beals, R., Sattlinger, D. H. & Szmigielski, J. (2000) Multipeakons and the classical moment problem. *Adv. Math.*, 154(2), 229–257.
9. Degasperis, A. & Procesi, M. (1999) Asymptotic integrability. *Symmetry and Perturbation Theory (Rome, 1998)* (A. Degasperis, & G. Gaeta eds). River Edge, NJ: World Scientific Publishing, pp. 23–37.
10. Degasperis, A., Holm, D. D. & Hone, A. N. W. (2002) A new integrable equation with peakon solutions. *Theor. Math. Phys.*, 133(2), 1463–1474.
11. Lundmark, H. & Szmigielski, J. (2003) Multi-peakon solutions of the Degasperis–Procesi equation. *Inverse Probl.*, 19(6), 1241–1245.
12. Lundmark, H. & Szmigielski, J. (2005) Degasperis–Procesi peakons and the discrete cubic string. *IMRP Int. Math. Res. Pap.*, 2005(2), 53–116.
13. Novikov, V. (2009) Generalizations of the Camassa–Holm equation. *J. Phys. A: Math. Theor.*, 42(34), 342002 (14 pages).
14. Hone, A. N. W. & Wang, J. P. (2008) Integrable peakon equations with cubic nonlinearity. *J. Phys. A: Math. Theor.*, 41(37), 372002 (10 pages).
15. Hone, A. N. W., Lundmark, H. & Szmigielski, J. (2009) Explicit multipeakon solutions of Novikov’s cubically nonlinear integrable Camassa–Holm type equation. *Dynam. Part. Differ. Equ.*, 6(3), 253–289.
16. Constantin, A. & Lannes, D. (2009) The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. *Arch. Ration. Mech. Anal.*, 192(1), 165–186.
17. Johnson, R. S. (2003) The classical problem of water waves: a reservoir of integrable and nearly-integrable equations. *J. Nonlinear Math. Phys.*, 10(suppl. 1), 72–92.
18. Bertola, M., Gekhtman, M. & Szmigielski, J. (2009) Cubic string boundary value problems and Cauchy biorthogonal polynomials. *J. Phys. A: Math. Theor.*, 42(45), 454006 (13 pages).
19. Bertola, M., Gekhtman, M. & Szmigielski, J. (2010) Cauchy biorthogonal polynomials. *J. Approx. Theory*, 162(4), 832–867.
20. Kohlenberg, J., Lundmark, H. & Szmigielski, J. (2007) The inverse spectral problem for the discrete cubic string. *Inverse Probl.*, 23(1), 99–121.
21. Szmigielski, J. & Zhou, L. (2013) Colliding peakons and the formation of shocks in the Degasperis–Procesi equation. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 469(2158), 20130379 (19 pages).
22. Szmigielski, J. & Zhou, L. (2013) Peakon-antipeakon interactions in the Degasperis–Procesi equation. *Algebraic and Geometric Aspects of Integrable Systems and Random Matrices*. Contemp. Math., vol. 593. Providence, RI: Amer. Math. Soc., pp. 83–107.
23. Chen, G., Chen, R. M. & Liu, Y. (2015) Existence and uniqueness of the global conservative weak solutions for the integrable Novikov equation. arXiv:1509.08569 [math.AP].
24. GRAYSHAN, K. (2013) Peakon solutions of the Novikov equation and properties of the data-to-solution map. *J. Math. Anal. Appl.*, **397**(2), 515–521.
25. HIMONAS, A. A. & HOLLIMAN, C. (2012) The Cauchy problem for the Novikov equation. *Nonlinearity*, **25**(2), 449–479.
26. JIANG, Z. & NI, L. (2012) Blow-up phenomenon for the integrable Novikov equation. *J. Math. Anal. Appl.*, **385**(1), 551–558.
27. LAI, S., LI, N. & WU, Y. (2013) The existence of global strong and weak solutions for the Novikov equation. *J. Math. Anal. Appl.*, **399**(2), 682–691.
28. MI, Y. & MU, C. (2013) On the Cauchy problem for the modified Novikov equation with peakon solutions. *J. Differ. Equ.*, **254**(3), 961–982.
29. MOHAIER, K. & SZMIGIELSKI, J. (2012) On an inverse problem associated with an integrable equation of Camassa–Holm type: explicit formulas on the real axis. *Inverse Probl.*, **28**(1), 015002 (13 pages).
30. NI, L. & ZHOU, Y. (2011) Well-posedness and persistence properties for the Novikov equation. *J. Differ. Equ.*, **250**(7), 3002–3021.
31. TGLAY, F. (2011) The periodic Cauchy problem for Novikov’s equation. *Int. Math. Res. Not.*, **2011**(20), 4633–4648.
32. WU, X. & GUO, B. (2016) Global well-posedness for the periodic Novikov equation with cubic nonlinearity. *Appl. Anal.*, **95**(2), 405–425.
33. WU, X. & YIN, Z. (2012) Well-posedness and global existence for the Novikov equation. *Ann. Sc. Norm. Super. Pisa.*, **11**(3), 707–727.
34. YAN, W., LI, Y. & ZHANG, Y. (2012) The Cauchy problem for the integrable Novikov equation. *J. Differ. Equ.*, **253**(1), 298–318.
35. BAROSTICHI, R. F., HIMONAS, A. A. & PETRONILHO, G. (2016) Autonomous Ovsyannikov theorem and applications to nonlocal evolution equations and systems. *J. Funct. Anal.*, **270**(1), 4633–4648.
36. LI, N. & LIU, Q. P. (2013) On bi-Hamiltonian structure of two-component Novikov equation. *Phys. Lett. A.*, **377**(3–4), 257–261.
37. LI, N. & NIU, X. (2014) A reciprocal transformation for the Geng–Xue equation. *J. Math. Phys.*, **55**(5), 053505.
38. MI, Y., MU, C. & TAO, W. (2013) On the Cauchy problem for the two-component Novikov equation. *Adv. Math. Phys.*, **2013**, 810725 (11 pages).
39. TANG, H. & LIU, Z. (2015) The Cauchy problem for a two-component Novikov equation in the critical Besov space. *J. Math. Anal. Appl.*, **423**(1), 120–135.
40. LI, H., LI, Y. & CHEN, Y. (2014) Bi-Hamiltonian structure of multi-component Novikov equation. *J. Nonlinear Math. Phys.*, **21**(4), 509–520.
41. BRESSAN, A. & CONSTANTIN, A. (2007) Global conservative solutions of the Camassa–Holm equation. *Arch. Ration. Mech. Anal.*, **183**(2), 215–239.
42. BRESSAN, A. & CONSTANTIN, A. (2007) Global dissipative solutions of the Camassa–Holm equation. *Anal. Appl. (Singap.)*, **5**(1), 1–27.
43. HOLDEN, H. & RAYNAUD, X. (2007) Global conservative multipeakon solutions of the Camassa–Holm equation. *J. Hyperbol. Differ. Equ.*, **4**(1), 39–64.
44. HOLDEN, H. & RAYNAUD, X. (2007) Global conservative solutions of the Camassa–Holm equation – a Lagrangian point of view. *Commun. Part. Differ. Equ.*, **32**(10), 1511–1549.
45. HOLDEN, H. & RAYNAUD, X. (2008) Global dissipative multipeakon solutions of the Camassa–Holm equation. *Commun. Part. Differ. Equ.*, **33**(11), 2040–2063.
46. KAUP, D. J. & VAN GORDER, R. A. (2010) The inverse scattering transform and squared eigenfunctions for the nondegenerate $3 \times 3$ operator and its soliton structure. *Inverse Probl.*, **26**(5), 055005 (34 pages).