Tripartite quantum correlations is a special subclass of Svetlichny-box polytope

C. Jebarathinam

1Department of Physical Sciences
Indian Institute of Science Education and Research Mohali,
Sector-81, S.A.S. Nagar, Manauli 140306, India.
(Dated: July 22, 2014)

Here we define Svetlichny discord (SD) and Mermin discord (MD) which are the multipartite generalization of the two measures introduced in arXiv:1407.3170. We find that Svetlichny-box polytope which is a subpolytope of the full nonsignaling polytope suffices to characterize tripartite quantum correlations. We show that any correlations in Svetlichny-box polytope can be decomposed into the convex mixture of Svetlichny box, a maximal genuine two-way nonlocal box and a Mermin local box which has zero SD and zero MD. We observe that maximal MD with maximal local randomness is a unique nonclassical feature of quantum theory.

I. INTRODUCTION

Quantum correlations between outcomes of local measurements on entangled states can lead to nonlocality which can not be explained by local hidden variable (LHV) theories [1]. In the multipartite scenario, distinct types of local hidden variable theories exist. In the tripartite case, Svetlichny showed that quantum correlations can lead to genuine tripartite nonlocality which can not be explained by hybrid local-nonlocal hidden variable (HLHV) theory [2]. Genuine multipartite nonlocal correlations lead to the violation of a Svetlichny inequality (SI) [3]. Quantum correlations cannot lead to maximal violation of the SI. Quantum theory is a subclass of a multipartite generalized nonsignaling theory that predicts maximal violation of the SI [4].

Multiparticle genuine nonlocality has applications to multipartite quantum information tasks [5]. It is natural to consider that the Svetlichny local correlations can not have genuine multipartite nonclassicality. The correlations associated with GHZ paradox [6] is Svetlichny local, however, it leads to weaker form of genuine nonclassicality [7]. In [6], it has been shown that the Bell local correlations exhibit Bell discord and Mermin discord which quantify irreducible PR-box component and irreducible Mermin box component in any bipartite NS correlations. In this paper, we generalize these measures to the multipartite scenario.

In this work, we are interested in the tripartite nonsignaling polytope with two inputs and two outputs [9], in particular, we consider Svetlichny-box polytope to investigate quantum correlations in tripartite quantum systems. Svetlichny-box polytope can be divided into three regions which are Svetlichny nonlocal region, two-way nonlocal region and Mermin local polytope. The quantum correlations forms a subset of Svetlichny-box polytope which allows to decompose any quantum correlations as the convex sum of the extremal boxes of the Svetlichny-box polytope. We define Svetlichny discord and Mermin discord using Svetlichny operators and Mermin operators which put upper bound on the correlations under the constraints of the HLHV model [2] and fully LHV model [7]. Svetlichny boxes which violate a SI inequality maximally have maximal Svetlichny discord, whereas the extremal Mermin discordant correlations lie in the two-way nonlocal region and violate a Mermin inequality (MI) [7]. Svetlichny box polytope can be divided into three nonclassical regions and a purely classical region which has zero Svetlichny discord and zero Mermin discord. The violation of a SI and the violation of a MI are only a sufficient condition for Svetlichny discord and Mermin discord respectively. There are pure genuinely entangled states which do not violate a SI or a MI [10][11]. All the pure genuinely entangled states and all the mixed genuinely quantum-quantum states can have Svetlichny discord or Mermin discord or both of them simultaneously.

The paper is organized as follows. In Sec. II, we review the tripartite nonsignaling polytope with two inputs and two outputs and discuss motivation for studying a restricted NS polytope. In Sec. II, we define Svetlichny-box polytope. In Sec. IV, we define Svetlichny discord and Mermin discord and we study the canonical decomposition of any correlation in the Svetlichny box polytope. In Sec. V, we investigate the quantum correlations using these measures. We present conclusions in Sec. VI.

II. PRELIMINARIES

Consider the tripartite correlation scenario in which three spatially separated parties share a nonsignaling (NS) box which has two inputs and two outputs per party. The correlations between the outputs is characterized by joint probability distributions (JPD), \( P(a_m, b_n, c_o | A_i, B_j, C_k) \). \( m, n, o, i, j, k \in \{0, 1\} \). In addition to positivity and normalizations, the JPD satisfy nonsignaling constraints:

\[
\sum_m P(a_m, b_n, c_o | A_i, B_j, C_k) = P(b_n, c_o | B_j, C_k) \quad \forall n, o, i, j, k \tag{1}
\]

and the permutations. The set of such NS boxes forms a convex polytope, \( \mathcal{N} \), in a 26 dimensional space [4].

Pironio et al. [9] found that \( \mathcal{N} \) has 53856 extremal boxes, which belong to 46 classes. The vertices in each class are equivalent in that they can be converted into each other through local reversible operations (LRO) and permutations of the parties. These 46 classes of vertices can be classified.
into local, two-way local and 44 classes of three-way nonlocal vertices.

**Local vertices.** There are 64 fully deterministic boxes which are given as follows,

\[ P^{\text{byc}}_{D} = \begin{cases} 1, m = \alpha \otimes \beta, n = \gamma \otimes \epsilon, o = \zeta \otimes \eta & \text{otherwise} \\ 0, & \text{otherwise} \end{cases} \]  

(2)

Here \( \alpha, \beta, \gamma, \epsilon, \zeta, \eta \in \{0, 1\} \) and \( \otimes \) denotes addition modulo 2.

These local vertices are the extremal boxes of the set of local boxes which can be explained by the LHV theory, i.e., the correlations can be written as follows,

\[ P(a_m, b_n, c_o | A_i, B_j, C_k) = \sum_A n P_A(a_m | A_i) P_A(b_n | B_j) P_A(c_o | C_k). \]  

(3)

**Two-way local vertices.** There are 48 two-way local vertices which are the extremal boxes of the set of local boxes which can be decomposed into the hybrid local-nonlocal form,

\[ P(a_m, b_n, c_o | A_i, B_j, C_k) = p_1 \sum_A n P^{ABC}_A + p_2 \sum_A n P^{ACB}_A + p_3 \sum_A n P^{A\betaC}_A, \]  

(4)

where \( P^{ABC} = P_B(a_m, b_n | A_i, B_j) P_A(c_o | C_k) \), \( P^{ACB} \) and \( P^{A\betaC} \) are similarly defined. The two-way local vertices are bipartite PR-boxes shared between two parties with the third party deterministic: they are 16 PR-boxes shared between \( A \) and \( B \) which are given as follows,

\[ P^{\text{byc}}_{12} = \begin{cases} \frac{1}{2}, n = \gamma \otimes \epsilon \otimes \beta \otimes \alpha & \mathrm{and} \quad o = ek \text{ otherwise}, \\ 0, & \text{otherwise}, \end{cases} \]  

(5)

and the other 48 PR-boxes, \( P^{\text{byc}}_{13} \) and \( P^{\text{byc}}_{23} \), in which PR-box is shared \( A \) and \( C \), and \( B \) and \( C \) can be similarly defined.

**Three-way nonlocal vertices.** The correlations that do not admit the decomposition in Eq. (4) exhibits three-way nonlocality. In this paper, we are interested in one particular class of 3-way nonlocal vertices which are 16 Svetlichny boxes,

\[ P^{\text{byc}}_{Sv} = \begin{cases} \frac{1}{2}, m \otimes n \otimes o = i j \otimes ik \otimes jk \otimes \alpha \otimes \beta \otimes \gamma \otimes \epsilon \text{ otherwise}, \\ 0, & \text{otherwise}, \end{cases} \]  

(6)

These boxes violate one of the Svetlichny inequalities,

\[ S^{\text{byc}} = \sum_{ijk} (-1)^{i+j+k+l} P(a_m, b_n, c_o | A_i, B_j, C_k) \leq 4, \]  

(7)

to its algebraic maximum of 8, here \( \langle A_i B_j C_k \rangle = \sum_{mno} (-1)^{m+n+o} P(a_m, b_n, c_o | A_i, B_j, C_k) \).

In the quantum scenario, the parties generate the JPD by making spin projective measurements, \( A_i = \hat{a}_i \cdot \hat{\sigma}, B_j = \hat{b}_j \cdot \hat{\sigma}, \) and \( C_k = \hat{c}_k \cdot \hat{\sigma} \), on an ensemble of three-qubit systems described by the density matrix \( \rho \) in the Hilbert space \( \mathcal{H}_2^A \otimes \mathcal{H}_2^B \otimes \mathcal{H}_2^C \). The correlation predicted by quantum theory is defined as follows,

\[ P(a_m, b_n, c_o | A_i, B_j, C_k) = \text{Tr} \left( \rho \Pi_{a_i}^m \otimes \Pi_{b_j}^n \otimes \Pi_{c_k}^o \right), \]  

(8)

where \( \Pi_{a_i}^m = 1/2 \left( \mathbb{1} + a_i \hat{a}_i \cdot \hat{\sigma} \right) \), \( \Pi_{b_j}^n = 1/2 \left( \mathbb{1} + b_j \hat{b}_j \cdot \hat{\sigma} \right) \) and \( \Pi_{c_k}^o = 1/2 \left( \mathbb{1} + c_k \hat{c}_k \cdot \hat{\sigma} \right) \) are the projectors generating binary outcomes \( a_m, b_n, c_o \in \{-1, 1\} \). Svetlichny showed that genuine nonlocality in quantum theory is limited by \( 4 \sqrt{2} \). Consider the following Svetlichny inequality,

\[ S := \langle A_0 B_0 C_0 \rangle + \langle A_0 B_0 C_1 \rangle + \langle A_0 B_1 C_0 \rangle - \langle A_0 B_1 C_1 \rangle - \langle A_1 B_0 C_0 \rangle - \langle A_1 B_1 C_0 \rangle - \langle A_1 B_1 C_1 \rangle \leq 4. \]  

(9)

The GHZ state, \( |\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \), gives rise to \( S = 4 \sqrt{2} \), for the measurements, \( A_0 = \sigma_x, A_1 = \sigma_y, B_0 = \sigma_x, B_1 = \sigma_y, C_0 = \sigma_x, \) and \( C_1 = \sigma_y \), the correlations arising from GHZ state maximally violates the MI.

\[ M := \langle A_0 B_0 C_0 \rangle - \langle A_0 B_1 C_1 \rangle - \langle A_1 B_0 C_1 \rangle - \langle A_1 B_1 C_0 \rangle \leq 2, \]  

(11)

i.e., it gives \( M = 4 \). The correlations can be decomposed as follows,

\[ P_M = \frac{1}{4} \sum_{i=1}^4 P_i(a_m | A_i) P_i(b_n, c_o | B_j, C_k), \]  

(12)

where \( P_1(a_m | A_i) = \hat{a}_i \cdot \hat{\sigma}, P_2(a_m | A_i) = \hat{a}_i \cdot \hat{\sigma}, P_3(a_m | A_i) = \delta_{i=1}, P_4(a_m | A_i) = \delta_{i=2}, P_1(b_n, c_o | B_j, C_k) = P_{\text{PR}}^0, P_2(b_n, c_o | B_j, C_k) = P_{\text{PR}}^0, P_3(b_n, c_o | B_j, C_k) = P_{\text{PR}}^1, \) and \( P_4(b_n, c_o | B_j, C_k) = P_{\text{PR}}^1 \).

Thus Svetlichny locality is nonconvex in that convex combination of certain two-way local vertices can give rise to genuine two-way nonlocality. The correlations can also be decomposed as the convex combination of the two Svetlichny boxes,

\[ P_M = \frac{1}{2} \left( P_M^{000} + P_M^{111} \right). \]  

(13)

Thus this box is the tripartite generalization of the bipartite quantum Mermin box [8]. There are 16 quantum Mermin boxes which maximally violate one of the Mermin inequalities,

\[ M_{\text{byc}} \leq 2, \]  

(14)

where,
\[ M_{\alpha\beta\gamma} := \left(\alpha \oplus \beta \oplus \gamma \oplus 1\right) \left\{ (-1)^{\alpha\beta\gamma} \langle A_0 B_0 C_1 \rangle + (-1)^{\alpha\beta\gamma} \langle A_0 B_1 C_0 \rangle + (-1)^{\alpha\beta\gamma} \langle A_1 B_0 C_0 \rangle + (-1)^{\alpha\beta\gamma} \langle A_1 B_1 C_1 \rangle \right\} \\
+ \left(\alpha \oplus \beta \oplus \gamma \right) \left\{ (-1)^{\alpha\beta\gamma} \langle A_1 B_0 C_0 \rangle + (-1)^{\alpha\beta\gamma} \langle A_1 B_1 C_1 \rangle \right\}. \] (15)

Svetlichny local polytope can be divided into two-way non-local region and Mermin local polytope, \( L \). Mermin local polytope is a convex hull of the 64 deterministic boxes, i.e., if \( P \in L \),

\[ P = \sum_{j=0}^{63} t_j P_{jD}; \quad \sum_{j} t_j = 1. \] (19)

If a correlation does not admit the local deterministic strategy in Eq. (19), it violates a Mermin inequality. The set of Mermin inequalities in Eq. (15) is invariant under LRO. Therefore this complete set of Mermin inequalities is satisfied if the correlations has the local deterministic strategy.

Recently, it has been shown that any bipartite NS correlation can be decomposed as the convex mixture of a PR-box, a maximal Mermin discordant box, and a Bell local box which has zero Bell discord and zero Mermin discord [8].

\[ P = G' P_{PR}^p + Q P_{Q2} + (1 - G' - Q') P_{L}^{G'=Q=0}. \] (20)

The tripartite generalization of this decomposition is achieved in the Svetlichny-box polytope.

### III. Svetlichny-Box Polytope

Svetlichny-box polytope, \( \mathcal{R} \), is a restricted NS polytope in which we discard in total 53856 – 128 = 53728 extremal boxes. The 128 extremal boxes of \( \mathcal{R} \) are the Svetlichny boxes, the bipartite PR-boxes and the deterministic boxes. Svetlichny-box polytope is convex, i.e., if \( P \in \mathcal{R} \),

\[ P = \sum_{i=0}^{15} p_i P_{i1D} + \sum_{i=0}^{15} q_i P_{i12} + \sum_{i=0}^{15} r_i P_{i13} + \sum_{i=0}^{15} s_i P_{i23} + \sum_{j=0}^{63} t_j P_{jD}, \] (16)

with \( \sum_i p_i + \sum_i q_i + \sum_i r_i + \sum_i s_i + \sum_j t_j = 1 \), \( i = \alpha\beta\gamma \) and \( j = \alpha\beta\gamma\epsilon\eta \). Svetlichny-box polytope can be divided into Svetlichny nonlocal region and Svetlichny local polytope.

Svetlichny local polytope, \( L_2 \), is a convex hull of the 48 two-way local vertices and the 64 deterministic boxes, i.e., if \( P \in L_2 \),

\[ P = \sum_{i=0}^{15} q_i P_{i12} + \sum_{i=0}^{15} r_i P_{i13} + \sum_{i=0}^{15} s_i P_{i23} + \sum_{j=0}^{63} t_j P_{jD}, \]

\[ \sum_i q_i + \sum_i r_i + \sum_i s_i + \sum_j t_j = 1. \] (17)

The Svetlichny local correlations that admit this decomposition satisfy the Svetlichny inequalities in Eq. (15). Svetlichny inequality can be interpreted as bipartite Bell-CHSH inequality between any two combined system and the third system which can be readily seen by rewriting Svetlichny operator as bipartite Bell-CHSH operator, for instance,

\[ S_{0000} = \langle (A_0 B_0 + A_1 B_0)(C_0 + C_1) - (A_0 B_0 - A_1 B_1)(C_0 - C_1) \rangle. \]

Here we have considered the combined system AB as a single subsystem. In the bipartite scenario, the complete set of Bell-CHSH inequalities,

\[ B_{\alpha\beta\gamma} := (-1)^{\alpha} \langle A_0 B_0 \rangle + (-1)^{\alpha\beta\gamma} \langle A_0 B_1 \rangle + (-1)^{\alpha\beta\gamma} \langle A_1 B_0 \rangle + (-1)^{\alpha\beta\gamma} \langle A_1 B_1 \rangle \leq 2, \] (18)

serve as the necessary and sufficient condition for the correlations to belong to the Bell polytope [12] and is invariant under LRO. The set of Svetlichny inequalities in Eq. (15) is complete since it is invariant under LRO. Therefore the complete set of Svetlichny inequalities is a neccessary and sufficient condition for the correlations to belong to the Svetlichny local polytope.

### IV. Svetlichny Discord and Mermin Discord

The violation of a Svetlichny inequality is monogamous i.e., if a Svetlichny inequality in Eq. (7) is violated, then the rest of the Svetlichny inequalities cannot be violated. This is a consequence of Svetlichny function monogamy,

\[ S_i + S_j \leq 8, \quad \forall i, j. \] (21)

where \( S_i \) and \( S_j \) are any two Svetlichny functions defined as follows:

\[ S_{\alpha\beta\gamma} = \left| \langle \{j\} \rangle \right| \left| \langle A_i B_j C_k \rangle \right|. \]

The isotropic Svetlichny box,

\[ P = p P_{Sv}^\alpha + (1 - p) P_N, \] (22)

have a special property that only one of \( S_{\alpha\beta\gamma} \) is nonzero and the rest of them are zero, which is due to the Svetlichny function monogamy of the irreducible Svetlichny box in the decomposition.

We define Svetlichny discord which quantifies irreducible Svetlichny box in any correlations in \( \mathcal{R} \) as follows,

\[ \mathcal{G} = \min\{G_1, ..., G_9\}, \] (23)

where

\[ G_1 = \left| S_{000} - S_{001} \right| - \left| S_{010} - S_{011} \right| - \left| S_{100} - S_{101} \right| - \left| S_{110} - S_{111} \right|. \]
and the rest of the $G_i$ are obtained by permuting $S_{\text{qff}}$ in $G_1$. $G$ is invariant under LRO, since the set $\{G_i\}$ is invariant under LRO. Here $0 \leq G \leq 8$; $G = 8$ for the Svetlichny boxes, whereas the bipartite PR-boxes and the deterministic boxes have $G = 0$ since they take the value of 4 for even number of Svetlichny functions.

The set of $G = 0$ correlations forms a subpolytope of the Svetlichny local polytope. The $G = 0$ polytope is nonconvex in that certain convex combination of the $G = 0$ correlations can go to the $G > 0$ region.

We now show that any correlation in $R$ can be written as the convex mixture a reducible Svetlichny box and a $G = 0$ box. If we maximize the Svetlichny box components in Eq. (16) over all possible decompositions, the resulting correlation can be written as the convex combination of the 16 Svetlichny boxes and a Svetlichny local box that does not have the Svetlichny box components,

$$ P = \sum_i g_i P_{S_v}^i + \left(1 - \sum_i g_i\right) P_{S_vL}.$$ (24)

The unequal mixture of the two Svetlichny boxes can be written as the convex mixture of a single Svetlichny box and a Svetlichny local box which has the uniform mixture of the two Svetlichny boxes. Therefore, the convex combination of the 16 Svetlichny boxes in Eq. (24) can be written as the convex combination of a single Svetlichny box and the 15 Svetlichny local boxes. The largest component of the Svetlichny box which is unequal to any other Svetlichny box components in Eq. (24) gives rise to the single Svetlichny box component, $G'$:

$$ \sum_i g_i P_{S_v}^i = G' P_{S_v}^r + \sum_{i=1}^{15} p_i P_{S_vL}.$$ (25)

Here $G'$ is obtained by minimizing the component of the single Svetlichny box overall possible decompositions to ensure that this component is irreducible. This minimization corresponds to the minimization in Eq. (23). Substituting Eq. (25) in Eq. (24), we get the canonical decomposition for any correlations in $R$,

$$ P = G' P_{S_v}^r + (1 - G') P_{S_vL},$$ (26)

where $P_{S_vL} = \frac{1}{1 - G'} \left( \sum_i p_i P_{S_vL} + (1 - \sum_i g_i) P_{S_vL} \right)$. Here $P_{S_vL}$ is a $G = 0$ box which follows from the geometry of the convex polytope that any point in the polytope lies along the line joining two points: The measure $G$ divides the Svetlichny local polytope into $G > 0$ region and $G = 0$ polytope. Since $P_{S_vL}$ is from the $G > 0$ region, the other point must be from $G = 0$ polytope.

A. Monogamy of genuine two-way nonlocality and Mermin discord

Svetlichny boxes and the bipartite PR-boxes do not show monogamy of Mermin inequality violation, since two Mermin inequalities are maximally violated by them. However, genuine two-way nonlocal correlations show monogamy of Mermin inequality. The extremal genuine two-way nonlocal boxes are the tripartite Mermin boxes. As in the bipartite case, there are two types of Mermin boxes which are distinguished by their marginals. There are 16 quantum Mermin boxes which have maximally mixed marginals and 64 non-quantum Mermin boxes. A maximally mixed Mermin box is a convex mixture of the two non-maximally mixed Mermin boxes which violate the same Mermin inequality, further, they can also be decomposed as the convex combination of the two Svetlichny boxes. Not all the uniform mixture of two Svetlichny boxes can give rise to Mermin box, for instance, white noise can be decomposed as the convex combination of two Svetlichny boxes. In a quantum Mermin box, the uniform mixture of two Svetlichny boxes destroys Svetlichny nonlocality, however, the perfect correlations left in them have contextuality which leads to the GHZ argument of nonlocality [6].

For any Mermin box, only one of the Mermin functions,

$$ M_{\text{qff}} := (\alpha \otimes \beta \otimes \gamma \otimes 1) \left\{ (-1)^y \langle A_0 B_0 C_1 \rangle + (-1)^{\alpha \otimes \beta} \langle A_0 B_1 C_0 \rangle + (-1)^{\alpha \otimes \gamma} \langle A_1 B_0 C_0 \rangle + (-1)^{2 \alpha \otimes \beta \otimes \gamma} \langle A_1 B_1 C_1 \rangle \right\} + (\alpha \otimes \beta \otimes \gamma) \left\{ (-1)^{\alpha \otimes \beta \otimes \gamma} \langle A_1 B_1 C_0 \rangle + (-1)^{\alpha \otimes \gamma} \langle A_1 B_0 C_0 \rangle + (-1)^{\beta \otimes \gamma} \langle A_1 B_0 C_1 \rangle + (-1)^{\alpha \otimes \beta \otimes \gamma} \langle A_0 B_0 C_0 \rangle \right\},$$ (27)

takes the maximum and the rest of the $M_{\text{qff}}$ are zero. We define Mermin discord which quantifies irreducible Mermin box component in any correlation in $R$ as follows,

$$ Q = \min \{Q_1, ..., Q_9\},$$ (28)

where

$$ Q_1 = \left| |M_{000} - M_{001}| - |M_{010} - M_{011}| \right|,$$

and the rest of the $Q_i$ are obtained by permuting $M_{\text{qff}}$ in $Q_1$. $Q$ is invariant under LRO, since the set $\{Q_i\}$ is invariant under LRO. Here $0 \leq Q \leq 4$. The $Q = 0$ correlations forms a non-convex polytope whose vertices are the deterministic boxes, the bipartite PR-boxes and the Svetlichny boxes. $Q = 4$ for the Mermin boxes and the convex combination of the three Mermin boxes which violate the same Mermin inequality:

$$ P_{Q=4} = u P_M^{mm} + v P_M^{mm} + w P_M^{mm},$$

where $P_M^{mm} = \frac{1}{2} \left( P_M^{mm} + P_M^{mm} \right)$ is a maximally mixed Mermin box and $P_M^{mm}$ and $P_M^{mm}$ are the
two non-maximally mixed Mermin boxes.

All the Mermin boxes have \( G = 0 \), i.e., they lie in the \( G = 0 \) polytope. \( G = 0 \) polytope can be divided into two regions \( Q > 0 \) region and \( Q = 0 \) polytope whose vertices are the deterministic boxes and the bipartite PR-boxes. This division allows to decompose the Svetlichny local box in Eq. (35) as the convex mixture of a \( Q = 4 \) box and a restricted Mermin local box,

\[
P_{\text{SV}}^{i} = q_{M} P_{i}^{Q = 4} + (1 - q_{M}) P_{i}^{Q = 0}.
\]

Here the Mermin local box, \( P_{\text{SV}}^{i} \), has \( G = Q = 0 \) follows from the geometry of the \( G = 0 \) polytope.

**B. Monogamy between the measures**

Here we show that the correlations which have simultaneously irreducible Svetlichny box and irreducible Mermin box components cannot have arbitrarily Svetlichny discord and Mermin discord. Substituting Eq. (29) in Eq. (26) implies the following decomposition for any correlations in \( R \),

\[
P = G + P_{Q = 4}^{\text{SV}} + Q + P_{Q = 0}^{\text{SV}} - (G - Q)
\]

where \( Q' = (1 - G) q_{M} \). The evaluation of Svetlichny discord and Mermin discord for this decomposition gives \( G = 8Q' \) and \( Q = 4Q' \) respectively. Since \( G' + Q' = 1 \), the following monogamy relationship between the measures follows,

\[
G + 2Q \leq 8.
\]

The above monogamy implies that if a correlation has maximal Mermin discord, it cannot have Svetlichny discord.

### V. Quantum Correlations

Here we study the correlations arising from the three-qubit states. The states which can not have Svetlichny discord and Mermin discord can be decomposed as,

\[
\rho_{i}^{123} = \sum p_{i} \rho_{i}^{AB} \otimes \rho_{i}^{BC} \tag{32}
\]

or

\[
\rho_{i}^{123} = \sum p_{i} \rho_{i}^{AB} \otimes \rho_{i}^{BC} \tag{33}
\]

or

\[
\rho_{i}^{132} = \sum p_{i} \rho_{i}^{AC} \otimes \rho_{i}^{BC} \tag{34}
\]

where \( \rho_{i}^{AB}, \rho_{i}^{AC}, \) and \( \rho_{i}^{BC} \) are in general quantum-quantum states which are neither classical-quantum nor quantum-classical states and there is no restriction on \( \rho_{i}^{A}, \rho_{i}^{B}, \text{ and } \rho_{i}^{C} \).

In the tripartite scenario, these states define classical-quantum (CQ) states and quantum-classical (QC) states with respect to the measures \( G \) and \( Q \).

We now show that \( G = Q = 0 \) for all classical-quantum and quantum classical states for all measurements.

**Proof.** Consider the quantum-classical state as given in Eq. (33). For this state, the expectation value factorizes as follows,

\[
\langle A, B, C \rangle = \sum_{i} p_{i} \langle A, B \rangle_{i} \langle C \rangle_{i},
\]

which implies that the Svetlichny operators in \( G_{i} \) factorize as follows,

\[
G_{i} = \left| \sum_{i} p_{i} \left[ \mathcal{B}_{000}^{i} (C_{0})_{i} + \mathcal{B}_{111}^{i} (C_{1})_{i} \right] - \sum_{i} p_{i} \left[ \mathcal{B}_{000}^{i} (C_{0})_{i} - \mathcal{B}_{111}^{i} (C_{1})_{i} \right] \right|
\]

here \( \mathcal{B}_{\text{off}}^{i} \) and \( \langle C_{k} \rangle_{i} \) are evaluated for \( \rho_{i}^{AB} \) and \( \rho_{i}^{C} \) in Eq. (33).

Let us now try to maximize \( G_{i} \) for the quantum-classical state in which \( \rho_{i}^{AB} \) are QQ states. Consider an optimal settings which gives nonzero for only one of \( \mathcal{B}_{\text{off}}^{i} \) in Eq. (36) and the rest of them are zero. For this settings, Eq. (36) implies that \( G_{i} = 0 \). Similarly, we can prove \( Q = 0 \) by using the factorization property in Eq. (35).

Since \( G \) and \( Q \) are symmetric under the permutations of the parties, they are also zero for the states in Eq. (32) and Eq. (33) for all measurements.

A mixed state is genuinely quantum-quantum if it admits the following decomposition,

\[
\rho = p_{1} \sum_{i} q_{1} \rho_{i}^{A} \otimes \rho_{i}^{BC} + p_{2} \sum_{i} q_{2} \rho_{i}^{AB} \otimes \rho_{i}^{B} + p_{3} \sum_{i} q_{3} \rho_{i}^{AC} \otimes \rho_{i}^{C},
\]

(37)
with at least two of the three coefficients $p_1$, $p_2$, and $p_3$ are nonzero. Here $\rho_{ij}^{AB}$, $\rho_{ij}^{AC}$, and $\rho_{ij}^{BC}$ must be the bipartite quantum-quantum states.

In the subsequent sections, we will choose the following four measurement settings:

\[ \hat{a}_0 = \hat{x}, \quad \hat{a}_1 = \hat{y}, \quad \hat{b}_j = \frac{1}{\sqrt{2}} \left( \hat{x} + (-1)^j \hat{y} \right), \quad \hat{c}_0 = \hat{x}, \quad \hat{c}_1 = \hat{y} \]

(38)

\[ \hat{a}_0 = \hat{z}, \quad \hat{a}_1 = \hat{x}, \quad \hat{b}_j = \frac{1}{\sqrt{2}} \left( \hat{z} + (-1)^j \hat{x} \right), \quad \hat{c}_0 = \hat{z}, \quad \hat{c}_1 = \hat{x} \]

(39)

\[ \hat{a}_0 = \hat{x}, \quad \hat{a}_1 = \hat{y}, \quad \hat{b}_0 = \hat{x}, \quad \hat{b}_1 = \hat{y}, \quad \hat{c}_0 = \hat{x}, \quad \hat{c}_1 = \hat{y} \]

(40)

\[ \hat{a}_0 = \hat{z}, \quad \hat{a}_1 = \hat{x}, \quad \hat{b}_0 = \hat{z}, \quad \hat{b}_1 = \hat{x}, \quad \hat{c}_0 = \hat{z}, \quad \hat{c}_1 = \hat{x} \]

(41)

for studying correlations arising from the genuinely quantum-quantum states. The first two settings correspond to Svetlichny discordant correlations which have $G > 0$ and $Q = 0$, whereas the last two settings correspond to Mermin discordant correlations which have $Q > 0$ and $G = 0$. We denote the bipartite discords by $g_{ij}$ and $q_{ij}$, here $i = 2, 3$.

## A. GHZ-class states

The GHZ-class states, which have bipartite entanglement between $A$ and $B$, can be written as,

\[ |\psi_G\rangle = \cos \theta |000\rangle + \sin \theta |111\rangle \left( \cos \theta |1\rangle \langle 0| + \sin \theta |1\rangle \langle 1| \right), \quad (42) \]

The genuine tripartite entanglement is quantified by the tangle \[ \tau = \frac{1}{2} \left( |\langle 123| \rangle^2 - 1 - |\langle 12\rangle|^2 - |\langle 13\rangle|^2 - |\langle 23\rangle|^2 \right), \]

and the bipartite entanglement is quantified by the tangle, \[ \tau_{12} = (\sin 2\theta \cos \theta)^2. \]

### 1. Svetlichny discordant box

The settings given in Eq. (38) maximizes Svetlichny discord for the GHZ-class states. For this settings, the correlations can be decomposed as follows,

\[ P = \sqrt{\tau_3} \left[ \frac{1}{\sqrt{2}} P_{Sv}^{000} + \left( 1 - \frac{1}{\sqrt{2}} \right) P_N \right] + \left( 1 - \sqrt{\tau_3} \right) P_{SvL}^{G=0}, \quad (43) \]

where,

\[ P_{SvL}^{G=0} = \frac{\sqrt{\tau_{12}}}{\sqrt{2}} P_{PR}^{000} P_N + \left( 1 - \frac{\sqrt{\tau_{12}}}{\sqrt{2}} \right) P_N. \]

The correlations gives $G_{000} = 4 \sqrt{\tau_3}$ and the rest of the Svetlichny functions are zero, which implies $G = 4 \sqrt{\tau_3}$ and $Q = 0$. The correlations violates the Svetlichny inequality if $\tau > \frac{1}{2}$. When $0 < \tau < \frac{1}{2}$, the correlations are Svetlichny local and Svetlichny discordant due to the incompatible measurements performed on the genuinely entangled states. In addition to the genuine tripartite discord, the correlations has bipartite discord between $A$ and $B$, $G_{12} = 2 \sqrt{\tau_{12}}$.

### 2. Mermin discordant box

The settings in Eq. (40), maximizes tripartite Mermin discord for the GHZ-class states since only one of the Mermin functions is nonzero. The correlations can be decomposed as the convex mixture of the Mermin box and the Mermin local box:

\[ P = \frac{\sqrt{\tau_3}}{2} \left( P_{Sv}^{000} + P_{Sv}^{111} \right) + \left( 1 - \sqrt{\tau_3} \right) P_{ML}^{G=0}. \]

(44)

Here the Mermin local box is given as follows,

\[ P_{ML}^{G=0} = \frac{\sqrt{\tau_{12}}}{2} \left( P_{PR}^{000} + P_{PR}^{111} \right) P_N + \left( 1 - \sqrt{\tau_{12}} \right) P_N. \]

The correlations gives $Q = 4 \sqrt{\tau_3}$ and $Q_{12} = 2 \sqrt{\tau_{12}}$. The correlations violates the Mermin inequality if $\tau > \frac{1}{4}$, however, $Q > 0$ if $\tau > 0$.

### 3. Svetlichny-Mermin-GHZ box

The GHZ state, $|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$, is simultaneously Svetlichny and Mermin discordant for the settings:

\[ \hat{a}_0 = \hat{x}, \hat{a}_1 = \hat{y}, b_0 = \sqrt{\tau} \hat{z} - \sqrt{1 - \tau} \hat{p}, b_1 = \sqrt{\tau} \hat{p} + \sqrt{1 - \tau} \hat{p}, \]

$\hat{c}_0 = \hat{x}$ and $\hat{c}_1 = \hat{y}$, which gives

\[ G = 4 \left| \sqrt{\tau} + \sqrt{1 - \tau} - \sqrt{\tau} - \sqrt{1 - \tau} \right| \]

and

\[ Q = 4 \left| \sqrt{p} - \sqrt{1 - p} \right|, \]

where $0 \leq p \leq \frac{1}{2}$. When $p$ is not equal to 0 and $\frac{1}{2}$, the correlations has components of irreducible Svetlichny box and irreducible Mermin box since it has $G > 0$ and $Q > 0$.

## B. W-class states

We now study the correlations arising from the W-class states,

\[ |\psi_w\rangle = \alpha |100\rangle + \beta |010\rangle + \gamma |001\rangle, \quad (45) \]

The genuine tripartite entanglement of the class is quantified by the three nonvanishing bipartite concurrences $C_{12} = 2 \alpha \beta$, $C_{13} = 2 \alpha \gamma$ and $C_{23} = 2 \beta \gamma$. Since the W-class states have genuine tripartite entanglement as well as three bipartite entanglement, if the correlations has genuine discord, it will also have three bipartite discords arising from the three bipartite entanglement.

### 1. Svetlichny discordant box

When the optimal GHZ-class settings given in Eq. (38), which is in the $xy$-plane, is chosen, Svetlichny discord is minimized, $G_{123} = 0$, whereas the three bipartite discords are...
maximized, $G_{12} = 2\sqrt{2C_{12}^2}$, $Q_{13} = 2C_{13}$ and $G_{23} = 2\sqrt{2C_{23}^2}$. Svetlichny discord is maximized for the settings given in Eq. (39), which gives,

$$G = \sqrt{2}\left|\left|1 + C_{12} + C_{13} + C_{23} \right|- \left|1 + C_{12} - C_{13} - C_{23}\right|\right|
- \left|\left|1 - C_{12} - C_{13} + C_{23}\right|- \left|1 - C_{12} + C_{13} - C_{23}\right|\right|
> 0 \text{ iff } C_{ij}C_{jk} > 0,$$  \hspace{1cm} (46)

$G_{12} = \sqrt{2}\left|\left|-(\alpha^2 - \beta^2 + \gamma^2) + C_{12}\right|- \left|-(\alpha^2 - \beta^2 + \gamma^2) - C_{12}\right|\right|
> 0 \text{ iff } C_{12} > 0,$

$Q_{13} = 2\left|\left|-(\alpha^2 + \beta^2 - \gamma^2) + C_{13}\right|- \left|-(\alpha^2 + \beta^2 - \gamma^2) - C_{13}\right|\right|
> 0 \text{ iff } C_{13} > 0,$

and,

$G_{23} = \sqrt{2}\left|\left|\alpha^2 - \beta^2 - \gamma^2 + C_{23}\right|- \left|\alpha^2 - \beta^2 - \gamma^2 - C_{23}\right|\right|
> 0 \text{ iff } C_{23} > 0.$

When $C_{12} + C_{13} + C_{23} \leq 2\sqrt{2} - 1$, the correlations do not violate a Svetlichny inequality or a Mermin inequality i.e., the correlations admit local deterministic model, however, the correlation is Svetlichny discordant whenever the state is genuinely entangled. The correlation can be decomposed as the convex sum of a tripartite Mermin discord and a Svetlichny discord when the settings chosen lies in the $\{000\}$-plane. The Werner states are separable if $p \leq 0.2$, biseparable if $0.2 < p \leq 0.429$ and genuinely entangled if $p > 0.429$.

### C. Mixture of GHZ state with white noise

Here we study the correlations arising from the following Werner states,

$$\rho_W = p|\psi_{GHZ}\rangle\langle\psi_{GHZ}| + (1-p)\frac{1}{4},$$  \hspace{1cm} (49)

where $|\psi_{GHZ}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle)$. The Werner states are separable if $p \leq \frac{1}{\sqrt{3}}$. The correlations violate the Svetlichny inequality if $p > \frac{1}{\sqrt{3}}$. When $p \leq \frac{1}{\sqrt{3}}$, the correlations admit local deterministic model.

#### 1. Svetlichny discordant box

For the settings given in Eq. (39), the Werner states have Svetlichny discord if $p > 0$ since the correlations admit the following decomposition,

$$P = pP_{S_v}^0 + (1-p)P_N,$$  \hspace{1cm} (50)

which gives $G = 4p\sqrt{2}$. The correlations violate the Svetlichny discord when $p > \frac{1}{\sqrt{2}}$. When $p \leq \frac{1}{\sqrt{2}}$, the correlations admit local deterministic model.

#### 2. Mermin discordant box

The correlations can be decomposed as the convex combination of the Mermin box and white noise,

$$P = p\left(\frac{P_{S_v}^0 + P_{N}^{111}}{2}\right) + (1-p)P_N,$$  \hspace{1cm} (51)

for the optimal settings that gives maximal Mermin discord for the GHZ-class states in Eq. (40) is chosen. The correlations has $G = 4p$ and violates the Mermin inequality if $p > \frac{1}{2}$.

### D. Biseparable W states

Let us now study discord of the following biseparable states,

$$\rho = \frac{1}{3}|\psi_{AB}\rangle\langle\psi_{AB}| + \frac{1}{3}|\psi_{AC}\rangle\langle\psi_{AC}| + \frac{1}{3}|\psi_{BC}\rangle\langle\psi_{BC}|,$$  \hspace{1cm} (52)

where $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\psi_{AC}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $|\psi_{BC}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Discord for the above biseparable state can be achieved only when the settings in the $xz$-plane is chosen, for instance, the settings given in Eq. (39) gives the following decomposition for the correlation:

$$P = \frac{1}{3}\left[\frac{1}{\sqrt{2}}P_{AB} + \left(\frac{1}{\sqrt{2}}P_{N}^0\right)\right]P_A + \frac{1}{3}\left(\frac{P_{AB}^0 + P_{N}^{111}}{2}\right)P_B
+ \frac{1}{3}P_A\left[\frac{1}{\sqrt{2}}P_{AB} + \left(\frac{1}{\sqrt{2}}P_{N}^0\right)\right],$$  \hspace{1cm} (53)

which gives $G = \frac{4\sqrt{2}}{3}$. Here $P_A$, $P_B$ and $P_C$ are the correlations arising from the state, $|0\rangle$. The state can not have genuine discord when the settings chosen lies in the $xy$-plane, i.e., the state is a convex combination of a W state and a $G = Q = 0$ state.
E. Mixture of GHZ state and W state

Consider the following genuinely entangled state, which is a mixture of the maximally entangled GHZ state and the symmetric W state,

$$\rho = p |\psi_{GHZ}\rangle\langle\psi_{GHZ}| + q |\psi_{W}\rangle\langle\psi_{W}|.$$  (54)

The correlations arising from these states can be written as the convex sum of correlation arising from the GHZ state and correlation arising from the W state:

$$P = pP(|\psi_{GHZ}\rangle) + qP(|\psi_{W}\rangle).$$  (55)

When the optimal settings given in Eq. (38) is chosen, $\mathcal{G}$ detects the fraction of the GHZ state. For this settings, $P(|\psi_{GHZ}\rangle) = \frac{1}{8} P_{s} + \left(1 - \frac{1}{8}\right) P_{N}$ and $P(|\psi_{W}\rangle) = P_{s}^{0}$, which is a Svetlichny local box that has zero Svetlichny discord and $\mathcal{G}_{12} = \mathcal{G}_{13} = \mathcal{G}_{23} = \frac{3}{2}$, i.e., $P(|\psi_{W}\rangle)$ behaves like white noise.

For the optimal settings given in Eq. (39) that gives Svetlichny discord of the W state is chosen, $P(|\psi_{GHZ}\rangle) = P_{N}$ and $P(|\psi_{W}\rangle)$ is given in Eq. (47). For this settings, the GHZ state behaves as white noise.

F. Mixture of two complete sets of GHZ states

Consider the mixture of two complete sets of GHZ states,

$$\rho = \sum_{i=1}^{8} p_{i} |\psi_{i}^{z}_{GHZ}\rangle\langle\psi_{i}^{z}_{GHZ}| + \sum_{j=1}^{8} q_{j} |\phi_{j}^{x}_{GHZ}\rangle\langle\phi_{j}^{x}_{GHZ}|,$$  (56)

These states can be decomposed as the convex mixture of the 16 Mermin boxes,

$$P = \sum_{i} p_{i} P_{i}^{M} + \sum_{j} q_{j} P_{j}^{M}.$$  (57)

Thus the states in Eq. (56) simulate the entire polytope whose vertices are the 16 Mermin boxes which have maximally mixed marginals. The convex hull of the 16 quantum Mermin boxes, which is represented by Eq. (57), is a subpolytope of the maximally mixed marginals Svetlichny local polytope. In addition to maximal Mermin discord, these Mermin boxes have maximal local randomness. Since non-quantum Mermin boxes do not have maximally mixed marginals, maximal Mermin discord with maximal local randomness is a unique nonclassical feature of quantum theory. Therefore, finding the constraint of the quantum Mermin box polytope given in Eq. (57) would single out quantum theory.

VI. CONCLUSION

Svetlichny-box polytope generalizes the bipartite PR-box polytope to the multipartite scenario. We observed that Svetlichny-box polytope suffices to characterize the tripartite quantum correlations. We introduced the measures Svetlichny discord and Mermin discord to quantify nonclassicality of Svetlichny local correlations. We showed that any correlation in Svetlichny-box polytope can be decomposed as the convex mixture of a Svetlichny box, a maximal Mermin discordant box, and a Mermin local box which has zero Svetlichny discord and zero Mermin discord. Svetlichny discord in quantum theory is limited, however, quantum correlations can have maximal Mermin discord. We found that maximal Mermin discord with maximal local randomness is a unique nonclassical feature of quantum theory.

ACKNOWLEDGEMENTS

I thank Pranaw Rungta for his helpful guidance and discussions while carrying out this work. I thank S. Aravinda, Sibash Ghosh, and R. Srikanth for discussions.
[15] G. M. Ziegler, Lectures on Polytopes, Graduate Texts in Mathematics, Vol. 152 (Springer, Berlin, 1995).
[16] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. 86, 419 (2014).
[17] Ll. Masanes, A. Acín, and N. Gisin, Phys. Rev. A 73, 012112 (2006).
[18] A. Ferraro, L. Aolita, D. Cavalcanti, F. M. Cucchietti and A. Acín, Phys. Rev. A 81, 052318 (2010).
[19] W. Dür, G. Vidal, and J. Cirac, Phys. Rev. A 62, 062314(2000).
[20] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
[21] O. Gühne and M. Seevinck, New J. Phys. 12, 053002 (2010).
[22] Y. Xiang and W. Ren, arXiv:1101.2971 (2011).
[23] Y. Xiang and W. Ren, J. Phys. A: Math. Theor. 44, 325305 (2011).
[24] N. D. Mermin, Phys. Rev. Lett. 65, 3373 (1990).
[25] S. Popescu and D. Rohrlich, Found. Phys. 24, 379 (1994).
[26] A. Acín, S. Massar, and S. Pironio, Phys. Rev. Lett. 108, 100402 (2012).
[27] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. 82, 665 (2010).
[28] P. Skrzypczyk, N. Brunner, and S. Popescu, Phys. Rev. Lett. 102, 110402 (2009).
[29] M. T. Quintino, T. Vértesi, and N. Brunner, arXiv:1406.6976 (2014).
[30] F. Hirsch, M. T. Quintino, J. Bowles, and N. Brunner, Phys. Rev. Lett. 111, 160402 (2013).
[31] T. Fritz, A.B. Sainz, R. Augusiak, J Bohr Brask, R. Chaves, A. Leverrier, and A. Acín, Nat Comms 4, 2263 (2013).
[32] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués Phys. Rev. Lett. 109, 070401 (2012).
[33] J-D. Bancal, J. Barrett, N. Gisin, and S. Pironio, Phys. Rev. A 88, 014102 (2013).
[34] E. G. Cavalcanti, Q. Y. He, M. D. Reid, and H. M. Wiseman, Phys. Rev. A 84, 032115 (2011).