IN Variant $f$-STRUCTURES ON THE FLAG MANIFOLDS

$SO(N)/SO(2) \times SO(N - 3)$

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Abstract. We consider manifolds of oriented flags $SO(n)/SO(2) \times SO(n - 3)$ ($n \geq 4$) as 4- and 6-symmetric spaces and indicate characteristic conditions for invariant Riemannian metrics under which the canonical $f$-structures on these homogeneous $\Phi$-spaces belong to the classes Kill$_f$, NK$_f$, and $G_1f$ of generalized Hermitian geometry.

1. Introduction

An important place among homogeneous manifolds is occupied by homogeneous $\Phi$-spaces [8, 7] of order $k$ (which are also referred to as $k$-symmetric spaces [16]), i.e. the homogeneous spaces generated by Lie group automorphisms $\Phi$ such that $\Phi^k = id$. Each $k$-symmetric space has an associated object, the commutative algebra $A(\theta)$ of canonical affinor structures [6, 7]. In its turn, $A(\theta)$ contains well-known classical structures, in particular, $f$-structures in the sense of K.Yano [18]. It should be mentioned that an $f$-structure compatible with a (pseudo-)Riemannian metric is known to be one of the central objects in the concept of generalized Hermitian geometry [13].

From this point of view it is interesting to consider manifolds of oriented flags of the form

$SO(n)/SO(2) \times SO(n - 3)$ ($n \geq 4$)

as they can be generated by automorphisms of any even finite order $k \geq 4$. At the same time, it can be proved that an arbitrary invariant Riemannian metric on these manifolds is (up to a positive coefficient) completely determined by the pair of positive numbers $(s, t)$. Therefore, it is natural to try to find characteristic conditions imposed on $s$ and $t$ under which canonical $f$-structures on homogeneous manifolds (1) belong to the main classes of $f$-structures in the generalized Hermitian geometry. This question is partly considered in the paper.

The paper is organized as follows.

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Key words and phrases. Homogeneous $\Phi$-space, regular $\Phi$-space, $k$-symmetric space, invariant structure, canonical affinor structure, $f$-structure, nearly Kähler structure, flag manifold.
In Section 2, basic notions and results related to homogeneous regular $\Phi$-spaces and canonical affinor structures on them are collected. In particular, this section includes a precise description of all canonical $f$-structures on homogeneous $k$-symmetric spaces.

In Section 3, we dwell on the main concepts of generalized Hermitian geometry and consider the special classes of metric $f$-structures such as $\text{Kill}_f$, $\text{NK}_f$, and $G^1_f$.

In Section 4, we describe manifolds of oriented flags of the form

$$SO(n)/\underbrace{SO(2) \times \cdots \times SO(2)}_{m} \times SO(n-2m-1)$$

and construct inner automorphisms by which they can be generated.

In Section 5, we describe the action of the canonical $f$-structures on the flag manifolds of the form (1) considered as homogeneous $\Phi$-spaces of orders 4 and 6.

Finally, in Section 6, we indicate characteristic conditions for invariant Riemannian metrics on the flag manifolds (1) under which the canonical $f$-structures on these homogeneous $\Phi$-spaces belong to the classes $\text{Kill}_f$, $\text{NK}_f$, and $G^1_f$.

2. Canonical structures on regular $\Phi$-spaces

We start with some basic definitions and results related to homogeneous regular $\Phi$-spaces and canonical affinor structures. More detailed information can be found in [17], [8], [16], [7], [5] and some others.

Let $G$ be a connected Lie group, $\Phi$ its automorphism. Denote by $G^\Phi$ the subgroup consisting of all fixed points of $\Phi$ and by $G^\Phi_0$ the identity component of $G^\Phi$. Suppose a closed subgroup $H$ of $G$ satisfies the condition $G^\Phi_0 \subset H \subset G^\Phi$.

Then $G/H$ is called a homogeneous $\Phi$-space [3] [7].

Among homogeneous $\Phi$-spaces a fundamental role is played by homogeneous $\Phi$-spaces of order $k$ ($\Phi^k = \text{id}$) or, in the other terminology, homogeneous $k$-symmetric spaces (see [16]).

Note that there exist homogeneous $\Phi$-spaces that are not reductive. That is why so-called regular $\Phi$-spaces first introduced by N.A. Stepanov [17] are of fundamental importance.

Let $G/H$ be a homogeneous $\Phi$-space, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for $G$ and $H$, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - \text{id}$ and the Fitting decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to $A$, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote 0- and 1-component of the decomposition respectively. Further, let $\varphi = \varphi_s \varphi_u$ be the Jordan decomposition, where $\varphi_s$ and $\varphi_u$ is a semisimple and unipotent component of $\varphi$ respectively, $\varphi_s \varphi_u = \varphi_u \varphi_s$. Denote by $\mathfrak{g}^\gamma$ a subspace of all fixed points for a linear endomorphism $\gamma$ in $\mathfrak{g}$. It is clear that $\mathfrak{h} = \mathfrak{g}^\varphi = \text{Ker} A$, $\mathfrak{h} \subset \mathfrak{g}_0$, $\mathfrak{h} \subset \mathfrak{g}^{\varphi_s}$. 
Definition 1 [8, 17, 7, 5]. A homogeneous $\Phi$-space $G/H$ is called a regular $\Phi$-space if one of the following equivalent conditions is satisfied:

1. $h = g_0$.
2. $g = h \oplus A g$.
3. The restriction of the operator $A$ to $Ag$ is non-singular.
4. $A^2 X = 0 \implies AX = 0$ for all $X \in g$.
5. The matrix of the automorphism $\varphi$ can be represented in the form 
   \[
   \begin{pmatrix}
   E & 0 \\
   0 & B
   \end{pmatrix},
   \]
   where the matrix $B$ does not admit the eigenvalue 1.
6. $h = g^{\varphi^2}$.

A distinguishing feature of a regular $\Phi$-space $G/H$ is that each such space is reductive, its reductive decomposition being $g = h \oplus Ag$ (see [17]). $g = h \oplus Ag$ is commonly referred to as the canonical reductive decomposition corresponding to a regular $\Phi$-space $G/H$ and $m = Ag$ is the canonical reductive complement.

It should be mentioned that any homogeneous $\Phi$-space $G/H$ of order $k$ is regular (see [17]), and, in particular, any $k$-symmetric space is reductive.

Let us now turn to canonical $f$-structures on regular $\Phi$-spaces.

An affinor structure on a smooth manifold is a tensor field of type $(1, 1)$ realized as a field of endomorphisms acting on its tangent bundle. It is known that any invariant affinor structure $F$ on a homogeneous manifold $G/H$ is completely determined by its value $F_o$ at the point $o = H$, where $F_o$ is invariant with respect to $Ad(H)$. For simplicity, further we will not distinguish an invariant structure on $G/H$ and its value at $o = H$ throughout the rest of the paper.

Let us denote by $\theta$ the restriction of $\varphi$ to $m$.

Definition 2 [6, 7]. An invariant affinor structure $F$ on a regular $\Phi$-space $G/H$ is called canonical if its value at the point $o = H$ is a polynomial in $\theta$.

Remark that the set $\mathcal{A}(\theta)$ of all canonical structures on a regular $\Phi$-space $G/H$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on $G/H$. This subalgebra contains well-known classical structures such as almost product structures ($P^2 = \text{id}$), almost complex structures ($J^2 = -\text{id}$), $f$-structures ($f^3 + f = 0$).

The sets of all canonical structures of the above types were completely described in [6] and [7]. In particular, for homogeneous $k$-symmetric spaces the precise computational formulae were indicated. For future reference we cite here the result pertinent to $f$-structures and almost product structures only. Put

\[
u = \begin{cases}
  n & \text{if } k = 2n + 1, \\
  n - 1 & \text{if } k = 2n.
\end{cases}
\]

Theorem 1 [6, 7]. Let $G/H$ be a homogeneous $\Phi$-space of order $k$ ($k \geq 3$).
1) All non-trivial canonical $f$-structures on $G/H$ can be given by the operators

$$f(\theta) = \frac{2}{k} \sum_{m=1}^{u} \left( \sum_{j=1}^{u} \zeta_j \sin \frac{2\pi mj}{k} \right) (\theta^m - \theta^{k-m}),$$

where $\zeta_j \in \{1, 0, -1\}$, $j = 1, 2, \ldots, u$, and not all $\zeta_j$ are equal to zero.

2) All canonical almost product structures $P$ on $G/H$ can be given by polynomials $P(\theta) = \sum_{m=0}^{k-1} a_m \theta^m$, where:

- if $k = 2n + 1$, then
  $$a_m = a_{k-m} = \frac{2}{k} \sum_{j=1}^{u} \xi_j \cos \frac{2\pi mj}{k};$$

- if $k = 2n$, then
  $$a_m = a_{k-m} = \frac{1}{k} \left( \frac{2}{k} \sum_{j=1}^{u} \xi_j \cos \frac{2\pi mj}{k} + (-1)^m \xi_n \right).$$

Here the numbers $\xi_j$, $j = 1, 2, \ldots, u$, take their values from the set $\{-1, 1\}$.

The results mentioned above were particularized for homogeneous $\Phi$-spaces of smaller orders 3, 4, and 5 (see [6, 7]). Note that there are no fundamental obstructions to considering of higher orders $k$. Specifically, for future consideration we need the description of canonical $f$-structures and almost product structures on homogeneous $\Phi$-spaces of orders 4 and 6 only.

**Corollary 1** [6, 7]. Any homogeneous $\Phi$-space of order 4 admits (up to sign) the only canonical $f$-structure

$$f_0(\theta) = \frac{1}{2} (\theta - \theta^3)$$

and the only almost product structure

$$P_0(\theta) = \theta^2.$$

**Corollary 2.** On any homogeneous $\Phi$-space of order 6 there exist (up to sign) only the following canonical $f$-structures:

$$f_1(\theta) = \frac{1}{\sqrt{3}} (\theta - \theta^5), \quad f_2(\theta) = \frac{1}{2\sqrt{3}} (\theta - \theta^2 + \theta^4 - \theta^5),$$

$$f_3(\theta) = \frac{1}{2\sqrt{3}} (\theta + \theta^2 - \theta^4 - \theta^5), \quad f_4(\theta) = \frac{1}{\sqrt{3}} (\theta^2 - \theta^4)$$
and only the following almost product structures:

$$P_1(\theta) = -\text{id}, \quad P_2(\theta) = \frac{\theta}{3} + \frac{\theta^3}{3} + \frac{\theta^4}{3},$$

$$P_3(\theta) = \theta^3, \quad P_4(\theta) = -\frac{2\theta^2}{3} + \frac{\theta^3}{3} - \frac{2\theta^5}{3}.$$ 

3. Some important classes in generalized Hermitian geometry

The concept of generalized Hermitian geometry created in the 1980s (see [13]) is a natural consequence of the development of Hermitian geometry. One of its central objects is a metric $f$-structure, i.e. an $f$-structure compatible with a (pseudo-)Riemannian metric $g = \langle \cdot, \cdot \rangle$ in the following sense:

$$\langle fX, Y \rangle + \langle X, fY \rangle = 0 \text{ for any } X, Y \in \mathfrak{X}(M).$$

Evidently, this concept is a generalization of one of the fundamental notions in Hermitian geometry, namely, almost Hermitian structure $J$. It is also worth noticing that the main classes of generalized Hermitian geometry (see [13, 11, 12, 5, 4]) in the special case $f = J$ coincide with those of Hermitian geometry (see [10]).

In what follows, we will mainly concentrate on the classes $\text{Kill}_f$, $\text{NK}_f$, and $G_1f$ of metric $f$-structures defined below.

A fundamental role in generalized Hermitian geometry is played by a tensor $T$ of type $(2,1)$ which is called a composition tensor [13]. In [13] it was also shown that such a tensor exists on any metric $f$-manifold and it is possible to evaluate it explicitly:

$$T(X, Y) = \frac{1}{4}f(\nabla fX(f)Y - \nabla fY(f)X),$$

where $\nabla$ is the Levi-Civita connection of a (pseudo-)Riemannian manifold $(M, g)$, $X, Y \in \mathfrak{X}(M)$.

The structure of a so-called adjoint $Q$-algebra (see [13]) on $\mathfrak{X}(M)$ can be defined by the formula $X \ast Y = T(X, Y)$. It gives the opportunity to introduce some classes of metric $f$-structures in terms of natural properties of the adjoint $Q$-algebra. For example, if $T(X, X) = 0$ (i.e. $\mathfrak{X}(M)$ is an anticommutative $Q$-algebra) then $f$ is referred to as a $G_1f$-structure. $G_1f$ stands for the class of $G_1f$-structures.

A metric $f$-structure on $(M, g)$ is said to be a Killing $f$-structure if

$$\nabla X(f)X = 0 \text{ for any } X \in \mathfrak{X}(M)$$

(i.e. $f$ is a Killing tensor) (see [11, 12]). The class of Killing $f$-structures is denoted by $\text{Kill}_f$. The defining property of nearly Kähler $f$-structures (or $NKf$-structures) is

$$\nabla fX(f)X = 0.$$
This class of metric \( f \)-structures, which is denoted by \( \text{NKf} \), was determined in [4] (see also [1, 2]). It is easy to see that for \( f = J \) the classes \( \text{Kill} f \) and \( \text{NKf} \) coincide with the well-known class \( \text{NK} \) of nearly Kähler structures [9].

The following relations between the classes mentioned are evident:

\[ \text{Kill} f \subset \text{NKf} \subset G_1 f. \]

A special attention should be paid to the particular case of naturally reductive spaces. Recall that a homogeneous Riemannian manifold \((G/H, g)\) is known to be a naturally reductive space [14] with respect to the reductive decomposition \( g = \mathfrak{h} \oplus \mathfrak{m} \) if

\[ g([X, Y]_\mathfrak{m}, Z) = g(X, [Y, Z]_\mathfrak{m}) \]

for any \( X, Y, Z \in \mathfrak{m} \).

It should be mentioned that if \( G/H \) is a regular \( \Phi \)-space, \( G \) a semisimple Lie group then \( G/H \) is a naturally reductive space with respect to the (pseudo-)Riemannian metric \( g \) induced by the Killing form of the Lie algebra \( g \) (see [17]). In [1], [2], [3] and [4] a number of results helpful in checking whether the particular \( f \)-structure on a naturally reductive space belongs to the main classes of generalized Hermitian geometry was obtained.

4. MANIFOLDS OF ORIENTED FLAGS

In linear algebra a flag is defined as a finite sequence \( L_0, \ldots, L_n \) of subspaces of a vector space \( L \) such that

\[ L_0 \subset L_1 \subset \cdots \subset L_n, \]

\[ L_i \neq I_{i+1}, \ i = 0, \ldots, n-1 \ (\text{see [15]}). \]

A flag [2] is known to be full if for any \( i = 0, \ldots, n-1 \) \( \dim L_{i+1} = \dim L_i + 1 \). It is readily seen that having fixed any basis \( \{e_1, \ldots, e_n\} \) of \( L \) we can construct a full flag by setting \( L_0 = \{0\}, \ L_i = \mathcal{L}(e_1, \ldots, e_i), \ i = 1, \ldots, n. \)

We call a flag \( L_{i_1} \subset L_{i_2} \subset \cdots \subset L_{i_n} \) (here and below the subscript denotes the dimension of the subspace) oriented if for any \( L_{i_j} \) and its two bases \( \{e_1, \ldots, e_{i_j}\} \) and \( \{e_1', \ldots, e_{i_j}'\} \) \( \det A > 0 \), where \( e_t' = Ae_t \) for any \( t = 1, \ldots, i_j \). Moreover, for any two subspaces \( L_{i_k} \subset L_{i_j} \) their orientations should be set in accordance.

**Proposition 1.** The set of all oriented flags

\[ L_1 \subset L_3 \subset \cdots \subset L_{2m+1} \subset L_n = L \]

of a vector space \( L \) with respect to the action of \( \text{SO}(n) \) is isomorphic to

\[ \widetilde{\text{SO}(n)} / \text{SO}(2) \times \cdots \times \text{SO}(2) \times \text{SO}(n-2m-1) \]

**Proof.** Fix some basis \( \{e_1, \ldots, e_n\} \) in \( L_n \). Consider the isotropy subgroup \( I_o \) at the point

\[ o = (\mathcal{L}(e_1) \subset \mathcal{L}(e_1, e_2, e_3) \subset \cdots \subset \mathcal{L}(e_1, \ldots, e_{2m+1}) \subset \mathcal{L}(e_1, \ldots, e_n)). \]
By the definition for any $A \in I_o$

$$A : \mathcal{L}(e_1) \to \mathcal{L}(e_1),$$
$$A : \mathcal{L}(e_1, e_2, e_3) \to \mathcal{L}(e_1, e_2, e_3), \ldots,$$
$$A : \mathcal{L}(e_1, \ldots, e_{2m+1}) \to \mathcal{L}(e_1, \ldots, e_{2m+1}),$$
$$A : \mathcal{L}(e_1, \ldots, e_n) \to \mathcal{L}(e_1, \ldots, e_n).$$

As $\{e_1, \ldots, e_n\}$ is a basis, it immediately follows that

$$A : \mathcal{L}(e_1) \to \mathcal{L}(e_1),$$
$$A : \mathcal{L}(e_2, e_3) \to \mathcal{L}(e_2, e_3), \ldots,$$
$$A : \mathcal{L}(e_{2m}, e_{2m+1}) \to \mathcal{L}(e_{2m}, e_{2m+1}),$$
$$A : \mathcal{L}(e_{2m+2}, \ldots, e_n) \to \mathcal{L}(e_{2m+2}, \ldots, e_n).$$

Thus $L = L_n$ can be decomposed into the sum of $A$-invariant subspaces

$$L = \mathcal{L}(e_1) \oplus \mathcal{L}(e_2, e_3) \oplus \cdots \oplus \mathcal{L}(e_{2m}, e_{2m+1}) \oplus \mathcal{L}(e_{2m+2}, \ldots, e_n).$$

The matrix of the operator $A$ in the basis $\{e_1, \ldots, e_n\}$ is cellwise-diagonal:

$$A = \text{diag}\{A_1^{1 \times 1}, A_2^{3 \times 2}, \ldots, A_{2m+1}^{2 \times 2}, A_n^{(n-2m-1) \times (n-2m-1)}\}.$$

Since $A \in SO(n)$, its cells $A^1, A^3, \ldots, A^{2m+1}, A^n$ are orthogonal matrices. All the flags we consider are oriented, thus for any $i \in \{1, 3, \ldots, 2m+1, n\}$ $\det A^i > 0$. This proves that $A^1 = (1), A^3 \in SO(2), \ldots, A^{2m+1} \in SO(2), A^n \in SO(n-2m-1)$.

Therefore $I_o = \underbrace{SO(2) \times \cdots \times SO(2)}_{m} \times SO(n-2m-1).$ This completes the proof.

**Proposition 2.** The manifold of oriented flags

$$SO(n)/\underbrace{SO(2) \times \cdots \times SO(2)}_{m} \times SO(n-2m-1)$$

is a homogeneous $\Phi$-space. It can be generated by inner automorphisms $\Phi$ of any finite order $k$, where $k$ is even, $k > 2$ and $k \geq 2m-2$:

$$\Phi : SO(n) \to SO(n), A \to BAB^{-1},$$

$$B = \text{diag}\{1, \varepsilon_1, \ldots, \varepsilon_m, -1, \ldots, -1\},$$

$$\varepsilon_t = \begin{pmatrix} \cos \frac{2\pi t}{k} & \sin \frac{2\pi t}{k} \\ -\sin \frac{2\pi t}{k} & \cos \frac{2\pi t}{k} \end{pmatrix}.$$

**Proof.** Here $G = SO(n), H = \underbrace{SO(2) \times \cdots \times SO(2)}_{m} \times SO(n-2m-1)$.

We need to prove that the group of all fixed points $G^\Phi$ satisfies the condition $G^\Phi_0 \subset H \subset G^\Phi$. 
By definition $G^\Phi = \{A|BAB^{-1} = A\} = \{A|BA = AB\}$. Equating the corresponding elements of $AB$ and $BA$ and solving systems of linear equations it is possible to calculate that

\[ G^\Phi = \{\pm 1\} \times SO(2) \times \cdots \times SO(2) \times SO(n - 2m - 1). \]

\[ \square \]

5. Canonical $f$-structures on 4- and 6-symmetric space $SO(n)/SO(2) \times SO(n - 3)$

Let us consider $SO(n)/SO(2) \times SO(n - 3)$ ($n \geq 4$) as a homogeneous $\Phi$-space of order 4. According to Proposition 2 it can be generated by the inner automorphism $\Phi : A \rightarrow BAB^{-1}$, where

\[ B = \text{diag} \left\{ 1, \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), -1, \ldots, -1 \right\}. \]

Therefore $\{1\}$ is a reductive space. It is not difficult to check that the canonical reductive complement $m$ consists of matrices of the form

\[ S = \begin{pmatrix} 0 & s_{12} & s_{13} & s_{14} & \cdots & s_{1n} \\ -s_{12} & 0 & 0 & s_{24} & \cdots & s_{2n} \\ -s_{13} & 0 & 0 & s_{34} & \cdots & s_{3n} \\ -s_{14} & -s_{24} & -s_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ -s_{1n} & -s_{2n} & -s_{3n} & 0 & \cdots & 0 \end{pmatrix} \in m. \]

According to Corollary 1 the only canonical $f$-structure on this homogeneous $\Phi$-space is determined by the formula

\[ f_0(\theta) = \frac{1}{2} (\theta - \theta^3). \]

Its action can be written in the form:

\[ f_0 : S \mapsto \begin{pmatrix} 0 & s_{13} & -s_{12} & 0 & \cdots & 0 \\ -s_{13} & 0 & 0 & -s_{34} & \cdots & -s_{3n} \\ s_{12} & 0 & 0 & s_{24} & \cdots & s_{2n} \\ 0 & s_{34} & -s_{24} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & s_{3n} & -s_{2n} & 0 & \cdots & 0 \end{pmatrix}. \]

Now let us consider $\{1\}$ as a 6-symmetric space generated by the inner automorphism $\Phi : A \rightarrow BAB^{-1}$, where

\[ B = \text{diag} \left\{ 1, \left( \begin{array}{cc} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{array} \right), -1, \ldots, -1 \right\}. \]
Taking Corollary 2 into account we can represent the action of the canonical $f$-structures on this homogeneous Φ-space as follows:

$$f_1(\theta) = \frac{1}{\sqrt{3}} (\theta - \theta^5) : S \rightarrow \begin{pmatrix} 0 & s_{13} & -s_{12} & 0 & \ldots & 0 \\ -s_{13} & 0 & 0 & -s_{34} & \ldots & -s_{3n} \\ s_{12} & 0 & 0 & s_{24} & \ldots & s_{2n} \\ 0 & s_{34} & -s_{24} & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & s_{3n} & -s_{2n} & 0 & \ldots & 0 \end{pmatrix},$$

$$f_2(\theta) = \frac{1}{2\sqrt{3}} (\theta - \theta^2 + \theta^4 - \theta^5) : S \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -s_{34} & \ldots & -s_{3n} \\ 0 & 0 & 0 & s_{24} & \ldots & s_{2n} \\ 0 & s_{34} & -s_{24} & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & s_{3n} & -s_{2n} & 0 & \ldots & 0 \end{pmatrix},$$

$$f_3(\theta) = \frac{1}{2\sqrt{3}} (\theta + \theta^2 - \theta^4 - \theta^5) : S \rightarrow \begin{pmatrix} 0 & s_{13} & -s_{12} & 0 & \ldots & 0 \\ -s_{13} & 0 & 0 & 0 & \ldots & 0 \\ s_{12} & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$f_4(\theta) = \frac{1}{\sqrt{3}} (\theta^2 - \theta^4) : S \rightarrow \begin{pmatrix} 0 & s_{13} & -s_{12} & 0 & \ldots & 0 \\ -s_{13} & 0 & 0 & s_{34} & \ldots & s_{3n} \\ s_{12} & 0 & 0 & -s_{24} & \ldots & -s_{2n} \\ 0 & -s_{34} & s_{24} & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -s_{3n} & s_{2n} & 0 & \ldots & 0 \end{pmatrix}.$$

6. **Canonical $f$-structures and invariant Riemannian metrics on $SO(n)/SO(2) \times SO(n-3)$**

Let us consider manifolds of oriented flags of the form (1) as 4- and 6-symmetric spaces. Our task is to indicate characteristic conditions for invariant Riemannian metrics under which the canonical $f$-structures on these homogeneous Φ-spaces belong to the classes $\text{Kill} f$, $\text{NK} f$, and $\text{G}_1 f$.

We begin with some preliminary considerations.

**Proposition 3.** The reductive complement $\mathfrak{m}$ of the homogeneous space $SO(n)/SO(2) \times SO(n-3)$ admits the decomposition into the direct sum of $\text{Ad}(H)$-invariant irreducible subspaces $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$. 
**Proof.** The explicit form of the reductive complement of \(\Pi\) was indicated in Section 5. Put

\[
m_1 = \left\{ \begin{pmatrix}
0 & a_1 & a_2 & 0 & \ldots & 0 \\
-a_1 & 0 & 0 & 0 & \ldots & 0 \\
-a_2 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix} \mid a_1, a_2 \in \mathbb{R} \right\},
\]

\[
m_2 = \left\{ \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & c_1 & \ldots & c_{n-3} \\
0 & 0 & 0 & d_1 & \ldots & d_{n-3} \\
0 & -c_1 & -d_1 & 0 & \ldots & 0 \\
0 & -c_{n-3} & -d_{n-3} & 0 & \ldots & 0
\end{pmatrix} \mid c_1, \ldots, c_{n-3} \in \mathbb{R} \\
d_1, \ldots, d_{n-3} \in \mathbb{R} \right\},
\]

\[
m_3 = \left\{ \begin{pmatrix}
0 & 0 & 0 & b_1 & \ldots & b_{n-3} \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & -b_1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-b_{n-3} & 0 & 0 & 0 & \ldots & 0
\end{pmatrix} \mid b_1, \ldots, b_{n-3} \in \mathbb{R} \right\}.
\]

Since \(SO(2) \times SO(n-3)\) is a connected Lie group, \(m_i (i = 1, 2, 3)\) is \(Ad(H)\)-invariant iff \([\mathfrak{h}, m_i] \subseteq m_i\). It can easily be shown that this condition holds.

We claim that for any \(i \in \{1, 2, 3\}\) there exist no such non-trivial subspaces \(\mathfrak{m}_i\) and \(\hat{m}_i\) that \(m_i = \mathfrak{m}_i \oplus \hat{m}_i\) and \([\mathfrak{h}, \mathfrak{m}_i] \subseteq \mathfrak{m}_i\), \([\mathfrak{h}, \hat{m}_i] \subseteq \hat{m}_i\).

To prove this we identify \(m\) and

\[\{(a_1, a_2, b_1, \ldots, b_{n-3}, c_1, \ldots, c_{n-3}, d_1, \ldots, d_{n-3})\}\].

In what follows we are going to represent any \(H \in \mathfrak{h}\) in the form

\[H = \text{diag}\{0, H_1, H_2\}\]

where

\[H_1 = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix},\]

\[H_2 = \begin{pmatrix} 0 & h_{12} & \ldots & h_{1n-3} \\ -h_{12} & 0 & \ldots & h_{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ -h_{1n-3} & -h_{2n-3} & \ldots & 0 \end{pmatrix}.
\]

(3)

Put \(F(H)(M) = [H, M]\) for any \(H \in \mathfrak{h}, M \in \mathfrak{m}\). In the above notations we have

\[F(H)|_{\mathfrak{m}_1} : (a_1 a_2)^T \rightarrow H_1(a_1 a_2)^T,\]
First, let us prove that $m_3$ cannot be decomposed into the direct sum of $Ad(H)$-invariant subspaces.

The proof is by reductio ad absurdum. Suppose there exists an $Ad(H)$-invariant subspace $W \subset m_3$. This implies that for any $H_2$ of the form $c_1 \ldots c_n - 3, d_1 \ldots d_n - 3$, and $x = (x_1 \ldots x_{n-3})^T \in W$, $H_2 x$ belongs to $W$.

It is possible to choose a vector $v_1 = (\alpha_1 \ldots \alpha_{n-3})^T \in W$ such that $\alpha_1 \neq 0$. Indeed, the nonexistence of such a vector yields that for any $w = (w_1 \ldots w_{n-3})^T \in W$, $w_1 = 0$. Take such $w \in W$ that for some $1 < i \leq n-3$, $w_i \neq 0$ and the skew-symmetric matrix $K = \{k_{ij}\}$ with all elements except $k_{1i} = -k_{i1} = 1$ equal to zero. Then $K w = (w_i \ldots \ast) \notin W$.

Consider the following system of vectors $\{v_1, \ldots, v_{n-3}\}$, where

\[
v_2 = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
-1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix},
v_1 = (\alpha_2 - \alpha_1 0 \ldots 0)^T,
\]

\[
v_3 = \begin{pmatrix}
0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
-1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix},
v_1 = (\alpha_3 0 - \alpha_1 \ldots 0)^T, \ldots,
\]

\[
v_{n-3} = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
-1 & 0 & \ldots & 0 & 0
\end{pmatrix},
v_1 = (\alpha_n 0 \ldots 0 - \alpha_1)^T.
\]

Obviously, $\dim \mathcal{L}(v_1, \ldots, v_{n-3}) = \rank \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n-3} \\
\alpha_2 & -\alpha_1 & 0 & \ldots & 0 \\
\alpha_3 & 0 & -\alpha_1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{n-3} & 0 & 0 & \ldots & -\alpha_1
\end{pmatrix} = n - 3.$

This contradicts our assumption.

Continuing the same line of reasoning, we see that neither $m_1$ nor $m_2$ can be decomposed into the sum of $Ad(H)$-invariant summands.
It is not difficult to check that the space in question possesses the following property.

**Proposition 4.**

\[(m_i, m_{i+1}) \subset m_{i+2} \text{ (modulo 3)}.\]

Denote by \(g_0\) the naturally reductive metric generated by the Killing form \(B: g_0 = -B|_{m \times m}\). In our case \(B = -(n-1) \text{Tr} X^T Y, X, Y \in \mathfrak{so}(n)\).

**Proposition 5.** The decomposition \(\mathfrak{h} \oplus m_1 \oplus m_2 \oplus m_3\) is \(B\)-orthogonal.

**Proof.** For the explicit form of \(m\) and \(h\) see Section 5 and Section 6. It can easily be seen that for any \(X \in m, Y \in h\) \(\text{Tr} X^T Y = 0\). It should also be noted that it was proved in [17] that \(h\) is orthogonal to \(m\) with respect to \(B\).

For any almost product structure \(P\) put:

\[
m^- = \{X \in m | P(X) = -X\}, \quad m^+ = \{X \in m | P(X) = X\}.
\]

Clearly, \(m^-\) and \(m^+\) are orthogonal with respect to \(g_0\), since for any \(X \in m^+, Y \in m^-\):

\[
g_0(X, Y) = g_0(P(X), P(Y)) = g_0(X, -Y) = -g_0(X, Y).
\]

Let us consider the action of the canonical almost product structures on the 6-symmetric space (1). Here we use notations of Corollary 2.

For \(P_2(\theta) = \frac{1}{3} \theta + \theta^2 + \frac{1}{3} \theta^3 + \theta^4 + \frac{1}{3} \theta^5\) \(m^- = m_1 \cup m_2, m^+ = m_3\), therefore \(m_3 \perp m_1, m_3 \perp m_2\).

For \(P_3(\theta) = \theta^3\) \(m^- = m_1 \cup m_3, m^+ = m_2\), thus \(m_2 \perp m_1\). The statement is proved.

It can be deduced from Proposition 3 and Proposition 5 that any invariant Riemannian metric \(g\) on (1) is (up to a positive coefficient) uniquely defined by the two positive numbers \((s, t)\). It means that

\[(5) \quad g = g_0|_{m_1} + sg_0|_{m_2} + tg_0|_{m_3}.
\]

**Definition 3.** \((s, t)\) are called the characteristic numbers of the metric (5).

It should be pointed out that the canonical \(f\)-structures on the homogeneous \(\Phi\)-space (1) of the orders 4 and 6 are metric \(f\)-structures with respect to all invariant Riemannian metrics, which is proved by direct calculations.

Recall that in case of an arbitrary Riemannian metric \(g\) the Levi-Civita connection has its Nomizu function defined by the formula (see [14]):

\[(6) \quad \alpha(X, Y) = \frac{1}{2} [X, Y]_m + U(X, Y),
\]

where \(X, Y \in m\), the symmetric bilinear mapping \(U\) is determined by means of the formula

\[(7) \quad 2g(U(X, Y), Z) = g(X, [Z, Y]_m) + g([Z, X]_m, Y), \quad X, Y, Z \in m.
\]
Suppose \( g \) is an invariant Riemannian metric on the homogeneous \( \Phi \)-space \( U \) with the characteristic numbers \((s, t)\) \((s, t > 0)\). The following statement is true.

**Proposition 6.**

\[
U(X, Y) = \frac{t - s}{2} ([X_{m_2}, Y_{m_3}] + [Y_{m_2}, X_{m_3}]) + \frac{t - 1}{2s} ([X_{m_1}, Y_{m_3}] + [Y_{m_1}, X_{m_3}]) + \frac{s - 1}{2t} ([X_{m_1}, Y_{m_2}] + [Y_{m_1}, X_{m_2}]).
\]

**Outline of the proof.** First we apply (5) and the definition of \( g \) to (4). We take four matrices \( X = \{x_i\}, Y = \{y_i\}, Z = \{z_{ij}\} \) and \( U = \{u_{ij}\} \) and calculate the right-hand and left-hand side of the equality obtained. After that we can represent it in the form

\[
c_{12}z_{12} + c_{13}z_{13} + \sum_{i=1}^{n} c_{1i}z_{1i} + \sum_{i=1}^{n} c_{2i}z_{2i} + \sum_{i=1}^{n} c_{3i}z_{3i} = 0,
\]

where \( c_{12}, c_{13}, c_{1i}, c_{2i}, c_{3i} (i = 1, \ldots , n) \) depend on elements of the matrices \( X, Y \) and \( U \). As (9) holds for any \( Z \in \mathfrak{m} \), it follows in the standard way that

\[
c_{12} = c_{13} = c_{1i} = c_{2i} = c_{3i} = 0, \quad (i = 1, \ldots , n).
\]

Using (10), we calculate \( u_{ij} = u_{ij}(X, Y) \). To conclude the proof, it remains to transform the formula for \( U(X, Y) \) into (7), which is quite simple. \( \Box \)

In the notations of Section 2 we have the following statement.

**Theorem 2.** Consider \( SO(n)/SO(2) \times SO(n-3) \) as a 4-symmetric \( \Phi \)-space. Then the only canonical \( f \)-structure \( f_0 \) on this space is

1. a Killing \( f \)-structure iff the characteristic numbers of a Riemannian metric are \((1, \frac{t}{s})\);
2. a nearly Kähler \( f \)-structure iff the characteristic numbers of a Riemannian metric are \((1, t), t > 0\);
3. a \( G_1 \)-\( f \)-structure with respect to any invariant Riemannian metric.

**Proof.** Application of (5) to the definitions of the classes \( \text{Kill}_f, \text{NK}_f \) and \( \text{G}_1 f \) yields that

1. \( f \in \text{Kill}_f \) iff \( \frac{1}{2}[X, f f X]_m + U(X, f X) - f(U(X, X)) = 0; \)
2. \( f \in \text{NK}_f \) iff \( \frac{1}{2}[X, f^2 X]_m + U(f X, f^2 X) - f(U(f X, f X)) = 0; \)
3. \( f \in \text{G}_1 f \) iff \( f(2U(f X, f^2 X) - f(U(f X, f X)) + f(U(f^2 X, f^2 X))) = 0. \)

The proof is straightforward. For example, it is known that \( f_0 \) is a nearly Kähler \( f \)-structure in the naturally reductive case, which means that \( \frac{1}{2}[f_0 X, f_0^2 X]_m = 0 \) for any \( X \in \mathfrak{m} \) (see [4]). Making use of Proposition 4 and Proposition 6, we obtain \( U(f_0 X, f_0 X) \in \text{Ker} f_0 \) for any \( X \in \mathfrak{m} \), \( U(f_0 X, f_0^2 X) = 0 \) for any \( X \in \mathfrak{m} \) iff \( s = 1 \). Thus we have 2). Other statements are proved in the same manner. \( \Box \)

The similar technique is used to prove
Theorem 3. Consider $SO(n)/SO(2) \times SO(n-3)$ as a 6-symmetric space. Let $(s,t)$ be the characteristic numbers of an invariant Riemannian metric. Then

1) $f_1$ is a Killing $f$-structure iff $s = 1$, $t = \frac{4}{3}$; 
   $f_2$, $f_3$, $f_4$ do not belong to $\text{Kill} f$ for any $s$ and $t$.
2) $f_1$ is an NK$f$-structure iff $s=1$; 
   $f_2$ and $f_3$ are NK$f$-structures for any $s$ and $t$; 
   $f_4$ is not an NK$f$-structure for any $s$ and $t$.
3) $f_1$, $f_2$, $f_3$, $f_4$ are $G_1f$-structures for any $s$ and $t$.

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