A juxtaposition of data driven and stochastic finite element analyses for problems with noisy material data

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In data driven modeling the constitutive model is replaced by a data set and therefore yields a model-free computation. In this context the data set also can be the input for noisy material data and describe material uncertainty. This contribution presents a juxtaposition of different finite element techniques to describe material uncertainties, namely the stochastic finite element method and a data driven finite element method.

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1 Motivation

Often deterministic simulations with an idealized parameter lead to simulations which are not realistic. In the actual material there are always defects or impurities so that the measured material responses from different samples, which are treated exactly the same way, are not identical. To describe this behavior in numerical simulations the usual way is a stochastic description, often done by adding a normal distributed variable, see right hand side of Fig. 1. However, the description of the stochastic variable is often arbitrary and not verifiable. The only information which is at hand are experimental results. The data-driven approach uses this data as measured as an input for the simulation directly. Thus, the empirical material modeling step is skipped.

Fig. 1: From the measured data (left) to a deterministic material law without (middle) or with an additional stochastic description (right).

2 Governing equations

For our comparison we stay in the well known finite element method framework of linear elasticity. The equation which needs to be solved for the unknown displacements \( u \) is therefore \( Ku = f \) with stiffness matrix \( K \) and force vector \( f \),

\[
K = \int_\Omega B^T C B \, d\Omega, \quad f = \int_\Omega N^T b \, d\Omega + \int_{\Gamma_1} N^T t \, d\Gamma, \tag{1}
\]

where \( \Omega \) is the domain of the body, \( B \) is the matrix which relates the displacements to the strains, \( C \) comprises the material tensor, \( N \) the vector of shape functions, \( b \) the body forces and \( t \) the traction.

In the stochastic finite element method (SFEM) we add a stochastic random field \( \chi(x) \) to the material stiffness tensor

\[
C_{\text{SFEM}}(x) = C^\mu(x)(1 + \chi(x))
\]

Therefore the entities of the stiffness matrix \( K \) change from expression (1) to

\[
K = \int_\Omega B^T C^\mu B \, d\Omega + \int_\Omega B^T \chi C^\mu B \, d\Omega = K^\mu + K^\chi
\]

The force vector \( f \) is assumed to be deterministic here, cf. [1].

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In the data driven framework we have to solve the following two systems of equations for the nodal displacements $u$ and the Lagrange multipliers $\lambda$ which both have the same stiffness matrix

$$
\begin{align*}
K u &= \int_{\Omega} B^T C^0 \epsilon^* \, d\Omega, \\
K \lambda &= f - \int_{\Omega} B^T \sigma^* \, d\Omega
\end{align*}
$$

where $(\epsilon^*, \sigma^*)$ are the assigned data points and $K$ is now defined as

$$
K = \int_{\Omega} B^T C^0 B \, d\Omega
$$

with $C^0$ as a purely numerical stiffness tensor. For more and detailed information towards the different methods, see [2, 3].

### 3 Simulation examples

As simulation examples we use a cantilever beam which is loaded on top and a plate with a hole which is pulled apart. For both problems we used a Young’s modulus of $85 \cdot 10^9$ Pa, a Poisson’s ratio of 0.3 and a Gaussian white noise as random field. The white noise is not the most realistic assumption but suffices for comparison reasons. The $2m \times 0.5m \times 0.1m$ cantilever beam is loaded by $p_0 = 4 \cdot 10^6$ N/m$^2$.

![Fig. 2: Bending stress $\sigma_x$ of two data-driven solutions (left) and stress distribution for the data-driven (middle) and SFEM (right) approach.](image)

In Fig. 2 we see on the left hand side two data-driven solutions of the cantilever beam problem, 151^3 and 31^3 data points were used. Note that for one data set we get one solution of the data-driven solver as it is not a stochastic method. Only if we use many data sets we gain a distribution of solutions. For example, in the middle of Fig. 2 the normalized stress $\sigma_x$ at a point on the upper edge is plotted. On the right hand side of Fig 2 we see the corresponding SFEM solution. From both distributions we can compute probabilities of failure.

The computed plate of Fig. 3 is $2m \times 2m$ and pulled apart by $p_0 = 10^7$ N/m$^2$. For symmetry reasons we compute only one fourth of the plate. On the upper side the standard FEM and the analytical solution are displayed, while on the lower side the data-driven and the SFEM solution are displayed. The data-driven and SFEM solution show the local varying material behavior. Summarizing we can say that the data-driven approach enables us to get the same output variables without modeling the material and its stochastic deviations which are hard to quantify. Therefore we think that a data-driven approach can be an alternative to the common SFEM method.

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