Correlation function of spectral staircase and particle number fluctuations in integrable systems

R. A. Serota

Department of Physics, University of Cincinnati,
Cincinnati, OH 45244-0011, serota@physics.uc.edu

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Abstract

We evaluate the correlation function of the spectral staircase and use it to evaluate the mesoscopic particle number fluctuations in integrable systems.
I. INTRODUCTION

In the preceding paper [1], we addressed the question of mesoscopic fluctuations vis-a-vis thermal fluctuations in classically integrable and chaotic systems. Mesoscopic fluctuations can be expressed in terms of the semiclassical expression for the level density correlation function and thermal fluctuations in terms of the fluctuations of thermal occupancy. We illustrated our formalism for both classically chaotic and integrable systems by calculating the particle number fluctuations and fluctuations of the specific heat in a two-dimensional, non-interacting electron gas. The results are particularly interesting in the classically integrable case where the ensemble of hard-wall rectangles with equal areas but varying aspect ratios was considered as a model system. [2] The key feature of such systems is the near absence of correlations between energy levels, with mean level spacing $\Delta$, on scales less then $E_m \sim \sqrt{\varepsilon \Delta}$ in proximity of level energy $\varepsilon$ (Fermi energy $\varepsilon_F$ in this case), and strong correlations on larger scales. Level correlations were described in terms of the correlation function of level density using a simplified ansatz, which neglects their long-range oscillatory behavior [2],[3] but is valid at finite temperature. [1] As a result, we found that the specific heat fluctuations grow linearly with temperature $\propto T/\Delta$ for $T \ll E_m$ and fall off exponentially for $T \gg E_m$. For the particle number fluctuations at a fixed chemical potential (which is exponentially close to Fermi energy), the zero-temperature contribution $\propto \sqrt{\varepsilon_F/\Delta}$ comes from the variations of the number of levels over the Fermi sea. For $T \ll E_m$, the leading temperature-dependent term in mesoscopic fluctuations is negative, $\propto -T/\Delta$, and cancels out the fluctuations due to fluctuations of thermal occupancy, $\propto T/\Delta$. As a result, the overall temperature-dependent part of the fluctuations is quadratic, $\propto T^2/(E_m \Delta)$. For $T \gg E_m$, on the other hand, the temperature-dependent part of the fluctuations is dominated by the fluctuations due to thermal occupancy fluctuations, $\propto T/\Delta$.

In this work, we reexamine the mesoscopic particle number fluctuations. Towards this end, we derive a general semiclassical expression for the correlation function of the spectral staircase [4] and show that it can be expressed concisely in terms of the level number variance. It is equivalent to the semiclassical expression for the level density correlation function [1] but lends itself more readily to thermal averaging in calculations of the fluctuations of thermodynamic quantities. We use this expression to evaluate the particle number fluctuations in a two-dimensional, non-interacting electron gas in a rectangular box and find the
results consistent with those in [1], which utilized a simplified ansatz for the level density

correlation function.

II. CORRELATION FUNCTION OF SPECTRAL STAIRCASE

In the quasi-continuous limit, $\varepsilon \gg \Delta$, the spectral staircase is defined as

$$N(\varepsilon) = \int_0^\varepsilon d\varepsilon \rho(\varepsilon)$$

(1)

where $\rho(\varepsilon)$ is the level density. It is well known that rescaling the energy variable $\varepsilon \to N(\varepsilon)$ allows for universal representation of the staircase $N(\varepsilon) = \varepsilon \overline{\rho}$ that eliminates the particular shape of $\overline{N}$ and reduces the spectrum into that of an infinite two-dimensional system with constant $\overline{\rho}$. Consequently, in what follows we limit our consideration to the thus reduced spectrum. Further below, we make an approximation that its properties are applicable to the size-quantized 2D systems, such as a hard-wall wells, where $\overline{\rho}(\varepsilon) = \Delta^{-1} \left[ 1 + O\left( \sqrt{\Delta/\varepsilon} \right) \right]$; neglecting corrections, we have

$$\overline{\rho} = \frac{mA}{2\pi\hbar^2}$$

(2)

and $\overline{N}(\varepsilon) = \varepsilon/\Delta$, where $A$ is the system area.

The correlation function of spectral staircase can be written using semiclassical expression for the staircase [4] as

$$\frac{\delta N(\varepsilon_1) \delta N(\varepsilon_2)}{2} = \sum_j \frac{A_j^2}{T_j} \cos \left( \frac{\omega T_j}{\hbar} \right) = \frac{\delta N(\varepsilon)^2}{2} - \frac{4}{\hbar^{N-1}} \sum_j \frac{A_j^2}{T_j^2} \sin^2 \left( \frac{\omega T_j}{2\hbar} \right)$$

(3)

where $\delta N = N - \overline{N}$, $\hbar \omega = \varepsilon_2 - \varepsilon_1 \ll \varepsilon = (\varepsilon_2 + \varepsilon_1)/2$, $2N$ is the dimension of the phase space, $A_j(\varepsilon)$ and $T_j(\varepsilon)$ are the amplitudes and periods of the periodic orbits, and

$$\frac{\delta N(\varepsilon)^2}{\hbar^{N-1}} = \sum_j \frac{A_j^2}{T_j^2}$$

(4)

Over-bars are used above to denote ensemble averaging. Using notations of [3], eq. (3) can be rewritten as

$$\frac{\delta N(\varepsilon_1) \delta N(\varepsilon_2)}{2} = \frac{1}{2} \left( \Sigma^\infty(\varepsilon) - \Sigma(\varepsilon, |\hbar\omega|) \right)$$

(5)

where. $\Sigma(\varepsilon, |\hbar\omega|)$ is the level number variance on the interval $|\hbar\omega|$ and $\Sigma^\infty(\varepsilon)$ is the saturation level number variance, averaged over the oscillations.
A quick check confirms that

\[ \frac{\partial}{\partial \varepsilon_1} \frac{\partial}{\partial \varepsilon_2} \delta N(\varepsilon_1) \delta N(\varepsilon_2) = \frac{2}{\hbar^{N+1}} \sum_j A_j^2 \cos \left( \frac{\omega T_j}{\hbar} \right) = \delta \rho(\varepsilon_1) \delta \rho(\varepsilon_2) \]  

where \( \delta \rho = \rho - \bar{\rho} \). In what follows, we set \( \hbar = 1 \). Generally, the level density correlation function can be written as

\[ \overline{\delta \rho(\varepsilon_1) \delta \rho(\varepsilon_2)} = \frac{1}{\Delta} \delta(\omega) - K(\varepsilon, \omega) \]  

where \( \delta \)-function is universally present and indicates absence of level correlations and \( K(\varepsilon, \omega) \) describes level repulsion, is system-specific and becomes important for \( \omega \) greater than the scale for onset of level rigidity. Per (5), the spectral staircase correlation function reduces to the properties of the level number variance. For small \(|\omega|\), the universal result for \( \Sigma \) is

\[ \Sigma(\varepsilon, |\omega|) = \frac{|\omega|}{\Delta} \]  

and is equivalent to the \( \delta \)-function term in the correlation function of level density (7), as is also seen from (5) and (6). For integrable systems, for large \(|\omega| > E_m\), \( \Sigma(\varepsilon, |\omega|) \) oscillates persistently around \( \Sigma^\infty(\varepsilon) \), and for a rectangular box is shown in Fig. 4 of [2]. Per (5), this proves corresponding oscillations around zero of the correlation function of spectral staircase, which, in turn, points to the persistence of correlations for \(|\omega| \gg E_m\).

### III. THERMODYNAMIC MESOSCOPIC FLUCTUATIONS.

As was shown in [1], the fluctuations of a thermodynamic quantity \( G \) that can be expressed as an integral over the spectrum

\[ G = \int d\varepsilon \rho(\varepsilon) f(\varepsilon) g(\varepsilon) \]  

\[ \overline{G} = \int d\varepsilon \rho(\varepsilon) \overline{f(\varepsilon) g(\varepsilon)} \]  

\[ \overline{\delta G}^2 = \overline{\delta G}^2_\rho + \overline{\delta G}^2_f \]  

\[ \overline{\delta G}^2_\rho = \int \int d\varepsilon_1 d\varepsilon_2 \delta \rho(\varepsilon_1) \delta \rho(\varepsilon_2) \overline{f(\varepsilon_1) f(\varepsilon_2) g(\varepsilon_1) g(\varepsilon_2)} \]  

\[ \overline{\delta G}^2_f = T \left( \frac{\partial G}{\partial \mu} \right)^2 \left( \frac{\partial N}{\partial \mu} \right)^{-1} + \left( \frac{\partial G}{\partial T} \right)^2 T^2 \frac{C}{\Delta} \]  

\[ T = \frac{4}{3} \]
where \( f(\varepsilon) \) is the thermal occupancy, \( \mu \) is the chemical potential, \( C \) is the specific heat, \( N \) is the particle number in the system, and \( \delta G = G - \overline{G} \). In what follows, we consider only Fermi thermal occupancy since we are interested in properties of a non-interacting electron gas in a 2D hard-wall potential. In 2D, \( \mu \) and \( \varepsilon_F \) are exponentially close, 
\[
\mu - \varepsilon_F \sim -T \exp \left( -\frac{\varepsilon_F}{T} \right),
\]
and the fluctuations of thermal factors can be easily evaluated using \[9\]
\[
\frac{\partial N}{\partial \mu} \approx \Delta, \quad \overline{C} \approx \frac{\pi^2}{3} \frac{T}{\Delta} \tag{13}
\]
and, in particular,
\[
\delta N^2 \rho \approx \frac{T}{2}, \quad \delta C^2 \rho \approx \overline{C} \tag{14}
\]
for the fluctuations of the number of particles and specific heat.\[1\]

Further, integration of (11) by parts, with the use of (6), yields an alternative form of mesoscopic fluctuations \( \delta G^2 \rho \)
\[
\overline{\delta G^2 \rho} = \int \int d\varepsilon_1 d\varepsilon_2 \delta N(\varepsilon_1) \delta N(\varepsilon_2) \frac{\partial f(\varepsilon_1)}{\partial \varepsilon_1} \frac{\partial f(\varepsilon_2)}{\partial \varepsilon_2} g(\varepsilon_1) g(\varepsilon_2) \tag{15}
\]
Notice, that (11) and (15) represent a combined ensemble and thermal averaging. Further manipulations with Fermi factors yield
\[
\frac{\partial f(\varepsilon_1)}{\partial \varepsilon_1} \frac{\partial f(\varepsilon_2)}{\partial \varepsilon_2} = T^{-2} \mathcal{H} \left( \frac{\varepsilon - \mu}{T}, \frac{\omega}{2T} \right)
\]
\[
\mathcal{H}(u, v) \equiv [2 (u \text{csch} u + v \text{csch} v)]^2 \tag{17}
\]
where \( \mathcal{H} \) has the following properties:
\[
h(u) \equiv \int_{-\infty}^\infty \mathcal{H}(u, v) dv = \frac{1}{2} (-1 + u \coth u) \cosh^2 u, \quad \int_{-\infty}^\infty h(u) du = \frac{1}{2} \tag{18}
\]
Substituting (3), (16) and (17) into (11) yields
\[
\overline{\delta G^2 \rho} = \int \int d\varepsilon d\omega \frac{1}{2} \left( \Sigma^\infty(\varepsilon) - \Sigma(\varepsilon, |\omega|) \right) T^{-2} \mathcal{H} \left( \frac{\varepsilon - \mu}{T}, \frac{\omega}{2T} \right) g \left( \frac{\varepsilon + \omega}{2} \right) g \left( \frac{\varepsilon - \omega}{2} \right) \tag{19}
\]
\[
\approx \int \int d\varepsilon d\omega \frac{1}{2} \left( \Sigma^\infty(\varepsilon) - \Sigma(\varepsilon, |\omega|) \right) T^{-2} \mathcal{H} \left( \frac{\varepsilon - \varepsilon_F}{T}, \frac{\omega}{2T} \right) g \left( \frac{\varepsilon + \omega}{2} \right) g \left( \frac{\varepsilon - \omega}{2} \right) \tag{20}
\]
In particular, substituting \( g = 1 \), we find
\[
\delta N^2 \rho \approx T^{-1} \int d\varepsilon \Sigma^\infty(\varepsilon) h \left( \frac{\varepsilon - \varepsilon_F}{T} \right) - \frac{1}{2} \int d\varepsilon d\omega (\Sigma(\varepsilon, |\omega|)) T^{-2} \mathcal{H} \left( \frac{\varepsilon - \varepsilon_F}{T}, \frac{\omega}{2T} \right) \tag{21}
\]
for mesoscopic particle number fluctuations.
In terms of the zero-temperature fluctuations,

$$ (\delta N_{\rho}^2)_0 = \frac{\Sigma^\infty (\varepsilon_F)}{2} $$

the latter can be rewritten as

$$ \delta N_{\rho}^2 \simeq (\delta N_{\rho}^2)_0 - \frac{1}{2} \int \int d\varepsilon d\omega (\Sigma (\varepsilon, |\omega|)) T^{-2} \mathcal{H} \left( \frac{\varepsilon - \varepsilon_F}{T}, \frac{\omega}{2T} \right) + O \left( \frac{T^2}{\varepsilon_F^2} \right) \tag{23} $$

Given that structure of the function $\mathcal{H}$, $\Sigma (\varepsilon, |\omega|)$ can be approximated by $\Sigma (\varepsilon_F, |\omega|)$ and we find, neglecting the terms of order $T^2/\varepsilon_F^2$,

$$ \delta N_{\rho}^2 \simeq (\delta N_{\rho}^2)_0 - \int_{-\infty}^\infty du (\Sigma (\varepsilon_F, 2T |u|)) h (u) \tag{24} $$

We proceed to apply this expression for a particular form of $\Sigma$ in a rectangular box.\[2\]

### IV. FLUCTUATIONS IN A 2D ELECTRON GAS IN A RECTANGULAR BOX

Rectangular box is a particular case of a hard-wall potential, whose main distinction is that it represents a "generic" classically integrable system, that is, with no degeneracies that may be caused by extra symmetries. The ensemble is understood as a collection of rectangular boxes of equal area $A = L_1 L_2$ and varying aspect ratio $\alpha_{asp} = L_2/L_1$.\[2\] In this case, we have \[2\]-\[4\]

$$ \Sigma (\varepsilon, E) = \sqrt{\frac{\varepsilon}{\pi^5 \Delta}} \sum_{M_1=0}^\infty \sum_{M_2=0}^\infty 4\delta_M \left[ E \sqrt{(\pi/\varepsilon \Delta)} \left( M_1^{1/2} \alpha_{asp}^{1/2} + M_2^{1/2} \alpha_{asp}^{-1/2} \right) \right] \left[ (M_1^{1/2} \alpha_{asp} + M_2^{1/2} \alpha_{asp}^{-1/2})^{3/2} \right] $$

$$ \sim 1 \rightarrow \sqrt{\frac{\varepsilon}{\pi^5 \Delta}} \sum_{M_1=0}^\infty \sum_{M_2=0}^\infty 4\delta_M \left[ E \sqrt{(\pi/\varepsilon \Delta)} \left( M_1^{1/2} + M_2^{1/2} \right) \right] \left[ (M_1^{1/2} + M_2^{1/2})^{3/2} \right] \tag{25} $$

$$ \Sigma (\varepsilon) = \sqrt{\frac{\varepsilon}{\pi^5 \Delta}} \sum_{M_1=0}^\infty \sum_{M_2=0}^\infty \frac{2\delta_M \alpha_{asp}^{1/2}}{(M_1^{1/2} \alpha_{asp} + M_2^{1/2} \alpha_{asp})^{3/2}} \rightarrow \sqrt{\frac{\varepsilon}{\pi^5 \Delta}} \sum_{M_1=0}^\infty \sum_{M_2=0}^\infty \frac{2\delta_M \alpha_{asp}^{1/2}}{(M_1^{1/2} + M_2^{1/2})^{3/2}} $$

$$ \delta_M = 1/4 \quad M_{1,2} = 0, M_{2,1} \neq 0 $$

$$ \delta_M = 1 \quad M_{1,2} \neq 0, M_{2,1} = 0 $$

where for simplicity we assumed $\alpha_{asp} \approx 1$, as is customary in analytical evaluations of the sums and numerical calculations.\[4,2\]
Numerical evaluation of the sum in (27) is straightforward and we find

\[(\delta N^2_{\rho})_0 = \frac{\Sigma^\infty (\varepsilon F)}{2} \approx \frac{3.31}{2\pi^{3/2}} \sqrt{\frac{\varepsilon F}{\Delta}}\]  

(29)

Also, in the limit \(T \ll E_m\), where (8) can be used, we find

\[\delta N^2_{\rho} \approx (\delta N^2_{\rho})_0 - \int_{-\infty}^{\infty} du \frac{2T|u|}{\Delta} h(u) = (\delta N^2_{\rho})_0 - \frac{T}{\Delta}\]  

(30)

in complete agreement with [1].

Insofar as the integral in (24) is concerned, we could not find its closed form. However, a very good approximation for (18) can be obtained by replacing

\[h(u) \rightarrow \frac{a^2}{\pi^2} \frac{u}{\sinh (au)}, \ a \sim 1\]  

(31)

For instance, \(a \approx 2\) describes its asymptotic behavior at \(u \rightarrow \infty\), while \(a \approx 3/2\) is suitable for the origin. Using (31), the integral in (24) can be easily evaluated and we find

\[\delta N^2_{\rho} \approx (\delta N^2_{\rho})_0 - \frac{a^2}{2\pi^2} \frac{u}{\sinh (au)} \]  

(32)

To proceed further with the evaluation of the sum, we must now consider two limits: \(T \ll E_m\) and \(T \gg E_m\). In the first limit, we can introduce continuous variables \(x, y \propto (T/E_m) M_{1,2}\) and approximate summation with integration, which, in polar coordinates, yields a convergent integral \(\int_0^\infty d\rho (\tanh \rho/\rho)^2\). Consequently, we find \(\delta N^2_{\rho} \approx (\delta N^2_{\rho})_0 - \text{const} \ (T/\Delta)\), as was already obtained above (30) [10]. In the opposite limit, \(\tanh \rightarrow 1\) and we find \(\delta N^2_{\rho} \approx (\delta N^2_{\rho})_0 - \text{const} \ (E_m/\Delta)\), again, in complete agreement with [1].

It must be pointed out that the results in [1] were obtained using a simplified ansatz for the level density correlation function [2]

\[\frac{\delta \rho (\varepsilon) \delta \rho (\varepsilon + \omega)}{\Delta} \approx \frac{\delta (\omega)}{\Delta} - \frac{\sin (2\pi \omega/E_m)}{\pi \omega \Delta}\]  

(33)

Correspondingly, it is clear that in the finite-temperature limit the oscillations of \(\Sigma (\varepsilon, E)\) as a function of \(E\), per (25), can be neglected and a simplified ansatz

\[\Sigma (\varepsilon, |\omega|) = \frac{|\omega|}{\Delta} D \left(\frac{|\omega|}{E_m}\right)\]  

(34)

\[D(x) \equiv 1 - 2 \frac{\sin^2 (\pi x) - \pi x \sin (2\pi x)}{\pi^2 x}\]  

(35)

where Si is the sine integral, can be used, which yields results identical to those in [1].
V. SUMMARY

We derived the correlation function of the spectral staircase in terms of the level number variance, eq. (5). We also showed that the oscillatory behavior of the level number variance can be neglected at finite temperatures and the simplified ansatz (34) can be used for evaluation of mesoscopic fluctuations of thermodynamic quantities, such as particle number fluctuation.

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