String theory dualities from M theory

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We analyze how string theory dualities may be described in M theory. T dualities arise from scalar-vector dualities in the worldvolume of the membrane of M theory. “Electric-magnetic” dualities arise from a duality transformation in M theory compactified on a 3-torus, which takes the membrane into a fivebrane wrapped around the 3-torus.

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1. Introduction

The conjecture of the existence of an 11 dimensional M theory has led to a better understanding of many non-perturbative effects in string theory \[1\]. The fundamental formulation of this theory is not yet known, but many of its properties may be derived just from the fact that its low-energy limit is 11 dimensional supergravity. The type IIA string theory and the heterotic $E_8 \times E_8$ string theory are described by compactifying M theory on $S^1$ \[1,2\] and $S^1/Z_2$ \[4\], respectively. The other consistent string theories, the type IIB and the $SO(32)$ theories, can be reached from these by T duality transformations, so they can only be straightforwardly described by M theory when they are compactified on a circle \[3\]. All $p$-brane states (for $p \leq 6$) of these string theories may be identified in M theory by starting with a membrane and a fivebrane in 11 dimensions \[1,2,3\], and their actions (or at least the field content of their worldvolume theories) may also be derived from the action (field content) of the membrane and the fivebrane \[5,6,7\]. Classically, the interactions of these $p$-branes which can be seen in weakly coupled string theory may also be derived from simple interactions of the membranes and fivebranes in 11 dimensions \[3\].

Since M theory is supposed to unify all string theories, it should be possible to understand the origin of all string theory dualities within M theory. The simplest string theory dualities, the T dualities on a circle, are in some sense trivially incorporated into M theory since they serve as the M theory “definitions” of the type IIB and $SO(32)$ string theories. We will discuss in section 2 exactly how the type IIB string action arises from M theory by an appropriate worldvolume duality transformation of the membrane. Other dualities, such as the $SL(2,\mathbb{Z})$ duality of the type IIB string theory, have a simple geometrical origin in M theory \[3,4\]. Most of this paper is devoted to a discussion of the third type of dualities, which appear to be related to electric-magnetic duality in M theory. These include, for instance, the various string-string dualities in 6 dimensions. So far the formulation of this electric-magnetic duality in M theory has not been clear, since it does not seem to exist in the low-energy 11 dimensional supergravity theory, due to the existence of the $C \wedge G \wedge G$ term in the action (where $C$ is the 3-form field of 11 dimensional supergravity and $G = dC$ is its field strength).

We would like to propose that the proper setting for electric-magnetic duality in M theory is 8 dimensions, where one can naturally define a membrane-membrane duality as originally proposed by Townsend \[10\]. One argument supporting this is that all known
string theory dualities above 8 dimensions transform one string theory into another, and, therefore, they do not necessarily correspond to any symmetry of M theory. Only in 8 dimensions do we have symmetries (in particular, T dualities) which transform a string theory into itself, and these must indeed be symmetries also of M theory. Another justification for this point of view comes from looking at the U duality groups of the supergravity theories we get by toroidally reducing 11 dimensional supergravity [11]. Above 8 dimensions, all these groups have a natural interpretation in M theory, either as “complex structure” deformations (for the \( SL(2, \mathbb{Z}) \) group in 9 dimensions) or as a parity transformation (which is the same parity transformation used by Hořava and Witten in M theory [4]). In 8 dimensions, the U duality group is expected to be \( SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) [11]. The first factor has an obvious geometrical interpretation in M theory, but the second does not. As we will show, the \( \tau \to -1/\tau \) transformation in this \( SL(2, \mathbb{Z}) \) group takes the membrane of M theory into a fivebrane wrapped around \( T^3 \). Upon further compactification and orbifolding, this transformation gives rise to all of the known “electric-magnetic” dualities. We conjecture that this \( SL(2, \mathbb{Z}) \) transformation group, together with the standard \( SL(n, \mathbb{Z}) \) rotations (which do not generally commute with it below 8 dimensions) and \( \mathbb{Z}_2 \) parity transformations, generates all symmetries of M theory.

Our analysis is purely classical, assuming only that in the classical low-energy limit M theory is correctly described by the supermembrane action. The actual quantum theory may be a theory of strings for which the membrane is the target space [12], a theory of strings for which the fivebrane is the target space [13], or something completely different that we have not yet been able to imagine. As shown in [14], quantum corrections are definitely needed in order to properly define M theory. At our present level of understanding, we can only hope that in the quantum theory the membrane and wrapped fivebrane will also be equivalent, and the duality we describe will survive. The existence of a duality relating membranes and fivebranes may suggest that they should be related already in the formulation of the theory, as suggested in [13]. Quantum effects are believed to break the classical U duality groups of supergravity to discrete subgroups [11], in a way which has not yet been completely understood. We will not discuss this issue here.

In section 2 we discuss the scalar-vector duality in the membrane of M theory and its relation to T duality in string theory. In section 3 we analyze a particular duality

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3 Except for the \( SL(2, \mathbb{Z}) \) duality of the type IIB string which has a geometrical origin in M theory as mentioned above.
transformation in type II string theory compactified on a torus, and see how it acts on the 11 dimensional fields. In section 4 we show that this transformation exchanges the action of the membrane of M theory (compactified on $T^3$) with the action of the fivebrane wrapped around $T^3$. In section 5 we show that from this 8 dimensional duality we can derive all of the known string dualities which have a straightforward M theory interpretation.

2. T duality in M theory

The simplest “derivation” of T duality in string theory comes from looking at it as a scalar-scalar duality in the 1+1 dimensional worldsheet (string theory T duality is described in [15] and references within). If none of the background fields depend on one of the spacetime dimensions, say $X_9$, then it enters into the worldsheet lagrangian only through its derivative $\partial_\alpha X_9$. We can replace this derivative by a worldsheet gauge field $V_\alpha$, if we also add another term to the string action ensuring that the associated gauge field strength is zero. This term is just $\Lambda \epsilon^{\alpha\beta} \partial_\alpha V_\beta$, where $\Lambda$ is a Lagrange multiplier. By integrating out the Lagrange multiplier $\Lambda$ we find that $V_\alpha$ is a total derivative, and we return to the original action. If, however, we integrate over $V_\alpha$ instead, we find a dual formulation of the action, in which $\Lambda$ becomes a dynamical scalar field. We can only perform this integration simply if the derivatives appear in the action quadratically, so that we should use the Polyakov form of the action and not the Nambu-Goto form. By performing this sort of 1+1 dimensional scalar-scalar dualities we can derive any T duality transformation.

In M theory, strings generally arise by wrapping a membrane around a compact dimension [5]. This suggests that T duality in M theory should be a duality transformation in the worldvolume of the membrane. Particularly, in 2+1 dimensions there is a duality transformation transforming a scalar field (on which the action depends only through its derivative) into a vector field, and this goes over to the scalar-scalar duality described above when we dimensionally reduce the membrane to the string. For instance, let us describe the T duality of type IIA theory on a circle in M theory. We begin with M theory on a torus, and perform a duality transformation on the circle of the type IIA theory, exchanging $\partial_\alpha X_9$ by $V_\alpha$, adding a Lagrange multiplier term $\epsilon^{\alpha\beta\gamma} \Lambda_\alpha \partial_\beta V_\gamma$, and integrating out $V_\alpha$. Again, we can do this in a simple way only in a formulation of the supermembrane action in which the derivatives enter quadratically, and luckily such a formulation indeed exists [16]. This transformation is, in fact, known to give the action of the D-2-brane.
Next, to go over to the string theory, we dimensionally reduce this theory along the eleventh dimension $X_{10}$, by setting one of the membrane coordinates $\xi_2$ to be exactly proportional to $X_{10}$. The membrane gauge field $\Lambda_\alpha$ now becomes a gauge field on the string worldsheet and a scalar $\Lambda_2$. The gauge field has no dynamical degrees of freedom (as in the D-string), and the scalar becomes an additional, tenth, dimension. It is easy to check that the metric of the ten new scalars is just the T-dual of the original metric. Starting with the supermembrane action in 11 dimensions (whose dimensional reduction gives the type IIA string action [5]), we end up after the duality with a type IIB string action [17]. Thus, T duality in M theory is simply scalar-vector duality in the membrane worldvolume.

The procedure described above can easily be performed once, but becomes more complicated when we try to perform it for more scalar fields, which have couplings between them (through the metric or 3-form fields). Unlike the string action, the membrane action is generally not quadratic in the fields $\partial_\alpha X^\mu$. Therefore, for general background fields (which we always assume not to depend on the compact coordinates we want to dualize), we can only perform explicitly two duality transformations. After that, if the 3-form field corresponding to the 3 directions we want to dualize does not vanish, the action for the fields we want to integrate out includes higher than quadratic terms, and we do not know how to integrate them out in a simple way. Of course, we can always leave the auxiliary fields in and be left with a more complicated description of the dual theory. Upon dimensional reduction to string theory these problems disappear, since the string action is always quadratic in the fields $\partial_\alpha X^\mu$.

3. T duality in 8 dimensions

The “single” T duality transformation described above is not really a symmetry of M theory, since it exchanges one type of membrane theory (which has only scalar fields) with a different type of theory (which has also a vector field). This is not surprising, since this is not really a symmetry in string theory as well, where it exchanges different types of string theories. However, once we compactify two dimensions in string theory, we have duality transformations which leave us in the same string theory, and these should be genuine symmetries of M theory (they should certainly be symmetries at least of eleven dimensional supergravity). In this section we will analyze the simplest transformation of this type, which is the duality inverting the area of a torus in type IIA string theory. Since
we know how this duality acts on the fields of the low-energy type IIA supergravity, we can find how it acts on the fields of the 11 dimensional supergravity, because we know how the two are related. As we will show, this duality is actually an 8 dimensional electric-magnetic duality in M theory, exchanging the 3-form field with its dual. Then, in the next section, we will show that in M theory the duality exchanges the membrane action expressed in terms of the original background fields, with the action of a fivebrane wrapped around the 3-torus in the dual background fields.

We begin by finding the transformation of this duality on the fields of the low-energy 11 dimensional supergravity. Since we know how T duality acts on the low-energy fields and on the D-branes in string theory, all we need is to translate the transformation of these fields and p-branes to M theory. In fact, the exact action of T duality on all RR fields has not been computed as far as we know (for 9 dimensional T duality it is given in [18]), but we will use approximations in which the transformations of these fields are simple. For simplicity, we will begin by taking the type IIA string theory to be compactified on a torus with a diagonal metric, with radii \( r_8 \) and \( r_9 \) in the string metric (we will denote string theory fields and radii with small letters, and M theory fields and radii with capital letters). We will also work only to leading order in the off diagonal fields \( G_{\mu i}, C_{\mu ij} \) and \( C_{\mu \nu i} \), where \( \mu, \nu = 0, \cdots, 7 \) and \( i, j = 8, 9, 10 \) (this is the notation we will generally use in this paper). The exact expressions are known, at least for the NS-NS fields, but they are much more cumbersome and do not seem to involve any new issues. We will discuss here only the transformations of the bosonic fields. The transformations of the fermionic fields are related to these by supersymmetry.

T duality transformations on a torus act naturally on the Kähler structure parameter \( \tau = b_{89} + i r_8 r_9 \) (which is often denoted by \( \rho \)). The T duality group includes \( SL(2, \mathbb{Z}) \) transformations of this parameter, and we will be interested in the transformation taking \( \tau \to -1/\tau \), which (for \( b_{89} = 0 \)) inverts both radii of the torus. In a diagonal metric, this transformation takes \( r_8 \) to \( r_8/|\tau| \) and \( r_9 \) to \( r_9/|\tau| \) (when \( b_{89} = 0 \) we can regard this transformation as a T duality on \( r_8 \), followed by a T duality on \( r_9 \), followed by a rotation exchanging the two coordinates). The string coupling \( \lambda \) transforms as \( \lambda \to \lambda/|\tau| \). Next, we should translate these results to M theory. The string coupling is related to the radius \( R_{10} \) of the eleventh dimension by \( \lambda = R_{10}^{3/2} \), while the relation between the string theory and M theory metrics sets \( r_8 = R_8 \sqrt{R_{10}} \) and \( r_9 = R_9 \sqrt{R_{10}} \). The tensor fields are related by \( b_{\mu \nu} = C_{\mu \nu (10)} \). Thus, in M theory \( \tau \) is simply given by \( \tau = C_{89(10)} + i R_8 R_9 R_{10} \), with the
same transformation law $\tau \rightarrow -1/\tau$ (as also noted by Sen \[19\]), while the 11 dimensional radii transform as

$$
R_8 \rightarrow \frac{R_8}{|\tau|^{2/3}} \quad R_9 \rightarrow \frac{R_9}{|\tau|^{2/3}} \quad R_{10} \rightarrow \frac{R_{10}}{|\tau|^{2/3}}.
$$

(3.1)

The fact that these transformations are symmetrical in the three compact dimensions suggests that this particular T duality may indeed be given by a simple transformation in M theory. The other T duality transformations are the shift in the $b_{89} = C_{89(10)}$ field, and the $SL(2, \mathbb{Z})$ transformations of the complex structure, which are both obviously expected to be symmetries in M theory as well.

Next, let us examine how the duality acts on the 8 dimensional metric. The string metric $g_{\mu\nu}$ does not change under the duality (in the leading order approximation we are working in), but the radius of the eleventh dimension does change (by equation (3.1)). Therefore, the 8 dimensional metric in 11 dimensional units changes by $G_{\mu\nu} \rightarrow G_{\mu\nu} |\tau|^{2/3}$.

We will discuss the transformation of the off-diagonal metric elements below, but first let us analyze the transformation of the 3-form field, $C_{\mu\nu\lambda}$. In M theory this couples to the membrane, so in the type IIA theory on a torus it couples to a 2-brane which is not wrapped around any cycle of the torus. After the T duality, this becomes a 4-brane which is wrapped around both cycles of the torus, which in M theory is described by a fivebrane wrapped around $T^3$. In 11 dimensions the fivebrane couples to the dual $\tilde{C}$ of $C_{\mu\nu\lambda}$. Thus, in 8 dimensions the fivebrane wrapped around $T^3$ couples to the 8 dimensional dual $\tilde{C}'_{\mu\nu\lambda}$ of $C_{\mu\nu\lambda}$ (we will discuss the exact definition of $\tilde{C}'$ in the next section). Therefore, the T duality transformation exchanges the 8 dimensional 3-form field with its electric-magnetic dual, and in this sense this duality is a membrane-membrane duality as originally proposed by Townsend \[10\]. By doing the transformation more carefully we find that in fact it is given, to leading order, by

$$
C_{\mu\nu\lambda} \rightarrow C_{89(10)} C_{\mu\nu\lambda} + R_8 R_9 R_{10} \tilde{C}_{\mu\nu\lambda}.
$$

(3.2)

More general $SL(2, \mathbb{Z})$ transformations will mix all of the dyonic membranes found in \[20\]. As we will see below, other components of the 11 dimensional 3-form field do not transform into their duals, so that this duality does not seem to be related (at least directly) to an 11 dimensional electric-magnetic (membrane-fivebrane) duality. In fact, we already saw above that $C_{89(10)}$ has a simple transformation, which is not related to electric-magnetic duality.
Let us now check the transformation of the off-diagonal components of the 3-form field, starting with the fields $C_{\mu\nu i}$ (for $i = 8, 9, 10$). If $i = 8$, for instance, this field couples to a 2-brane wrapped around $r_8$. When $b_{89} = 0$, we can do a T duality transformation on $r_8$, which turns this into an unwrapped 1-brane, then a T duality transformation on $r_9$, which turns this into a 2-brane wrapped around $r_9$, and finally a rotation exchanging the two circles of the torus, which returns this to a 2-brane wrapped around $r_8$. Thus, $C_{\mu\nu 8}$ is actually invariant under the duality. $C_{\mu\nu(10)}$ is just the 2-form field $b_{\mu\nu}$ of the string theory, which is also invariant under the duality (to leading order in the off-diagonal background fields). Therefore, the duality leaves the fields $C_{\mu\nu i}$ invariant (to leading order in the off-diagonal fields).

The fields $C_{\mu ij}$, on the other hand, are not invariant. Let us begin, for instance, with $C_{\mu 8(10)}$. In the string theory this is just $b_{\mu 8}$, which transforms under the T duality to $(b_{89} b_{\mu 8} - g_{88} g_{\mu 9})/|\tau|^2$. The transformation of $C_{\mu 9(10)}$ is analogous. The other 8 dimensional vector field, $C_{\mu 89}$, is part of the RR 3-form in the type IIA string theory, which couples to a 2-brane wrapped around both $r_8$ and $r_9$. Under T duality this becomes a 0-brane, coupling to the RR gauge field $A_\mu$, which in 11 dimensional terms is proportional to $G_\mu(10)$ (with the constant of proportionality equal to $R^{2}_{10}$). In fact, $C_{\mu 89}$ mixes with $A_\mu b_{89}$, and the actual transformation is slightly more complicated. Translating all this to 11 dimensions, we find that the transformation of these fields is given by

$$C_{\mu ij} \to \frac{1}{|\tau|^2} \left( -\epsilon^{ijk} G_{\mu k} R^2_i R^2_j - C_{89(10)} C_{\mu ij} \right). \quad (3.3)$$

Analogously, one can compute the transformation of the fields $G_{\mu i}$, and find that their transformation, in terms of 11 dimensional fields, is given (to leading order in the off-diagonal fields) by

$$G_{\mu i} \to \frac{1}{|\tau|^{4/3}} \left( \frac{1}{2} R^2_i \epsilon^{ijk} C_{\mu jk} - C_{89(10)} G_{\mu i} \right). \quad (3.4)$$

Thus, the duality exchanges a membrane wrapped around two cycles of the torus $T^3$ with a momentum mode around the third cycle, as in [19]. Note that by performing the T duality transformation twice, the 8 dimensional vector fields change sign, so that we get (in 11 dimensional terms) a parity transformation on the 3 compact dimensions, together with a change in the sign of the 3-form field $C$. This can also be seen by the fact that $C_{\mu\nu\lambda}$ transforms essentially by electric-magnetic duality, which squares to $(-1)$ in 8 dimensions. All other fields are invariant under the double transformation. This $\mathbb{Z}_2$ is, of course,
expected to be a symmetry of M theory, like any change of sign in an odd number of dimensions and in the $C$ field.

This completes the transformations of all fields in the eleven dimensional low-energy supergravity theory, so let us summarize our results. We have performed a T duality transformation in the type IIA theory on a torus, involving the inversion of the Kähler structure of the torus. Then, we translated this transformation to M theory. Obviously this duality should be a symmetry of M theory as well. Generalizing our previous results to a general metric on the torus, we find that the transformation of all bosonic fields of the 11 dimensional supergravity is given, to leading order in the off-diagonal fields, by:

$$
G_{ij} \rightarrow \frac{G_{ij}}{|\tau|^{4/3}} \\
C_{89(10)} \rightarrow -\frac{C_{89(10)}}{|\tau|^2} \\
C_{\mu\nu\lambda} \rightarrow C_{89(10)}C_{\mu\nu\lambda} + \sqrt{\det(G_{ij})} \tilde{C}_{\mu\nu\lambda} \\
C_{\mu\nu i} \rightarrow C_{\mu\nu i} \\
C_{\mu ij} \rightarrow 1 \left( -\epsilon^{i'j'k'} G_{ii'}G_{jj'}G_{k'} - C_{89(10)}G_{ij} \right) \\
G_{\mu\nu} \rightarrow G_{\mu\nu}|\tau|^{2/3} \\
G_{\mu i} \rightarrow \frac{1}{|\tau|^{4/3}} \left( \frac{1}{2} \epsilon^{i'j'k'} G_{ii'}C_{\mu j'k'} - C_{89(10)}G_{ii} \right).
$$

(3.5)

4. Membrane-membrane duality in 8 dimensions

We would now like to identify the symmetry described in the previous section in M theory. For a membrane wrapped around any cycle of the torus, it is obviously just a T duality. This is clear for a membrane wrapped around $x_{10}$ from our definition of the transformation, but since our results are symmetric it is true for any wrapped membrane. Thus, we should only identify how an unwrapped membrane transforms. The transformation of the 3-form field suggests that it should transform into a fivebrane wrapped around $T^3$. In this section we will examine how the transformations (3.5) indeed relate the action of the membrane with the action of the completely wrapped fivebrane. Fivebranes which are wrapped around less than three cycles of the 3-torus transform into themselves according to (3.5), and we will not discuss them further.

First, let us write down the action of the membrane. We will use the Howe-Tucker form of the supermembrane action [16], which has an auxiliary metric $\gamma_{\alpha\beta}$ on the worldvolume.
of the membrane, and write only the purely bosonic terms throughout this section (the addition of the fermionic terms is not expected to change our results, since they should be determined by supersymmetry). Separating the compact and non-compact directions, this action is

\[
S_M = \int d^3 \xi \left\{ -\frac{1}{2} \sqrt{-\gamma} \left[ \gamma^{\alpha\beta} \left( \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} + 2 \partial_\alpha X^\mu \partial_\beta X^i G_{\mu i} + \partial_\alpha X^i \partial_\beta X^j G_{ij} \right) - 1 \right] \right. \\
- \frac{1}{6} \epsilon^{\alpha\beta\gamma} \left( C_{\mu\nu\rho} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho + 3 C_{\mu\nu i} \partial_\alpha X^\mu \partial_\beta X^i \partial_\gamma X^j + 3 C_{\mu ij} \partial_\alpha X^\mu \partial_\beta X^i \partial_\gamma X^j + C_{ijk} \partial_\alpha X^i \partial_\beta X^j \partial_\gamma X^k \right) \right\}. 
\]

(4.1)

Note that throughout this paper we use conventions in which epsilon symbols with upper indices are equal to ±1.

We would like to compare this result with the action for the M theory fivebrane wrapped around the 3-torus. However, we do not know how to write an action for this fivebrane, due to the existence of a self-dual 2-form field \( B_{ab} \) in its worldvolume\(^4\). Townsend has suggested\(^6\) an action for the fivebrane (at least in the low-energy limit) which is gauge invariant, and may describe correctly at least some properties of the fivebrane if the self-duality condition is added by hand to the equations of motion. It is given, in 11 dimensions, by

\[
S_F^{(0)} = -\frac{1}{2} \int d^6 \xi \sqrt{-\gamma} \left[ \gamma^{ab} G_{MN} \partial_a X^M \partial_b X^N - 4 \right. \\
+ \frac{1}{2} \gamma^{ad} \gamma^{be} \gamma^{cf} \left( F_{abc} - C_{MNP} \partial_a X^M \partial_b X^N \partial_c X^P \right) \\
\left( F_{def} - C_{M'N'P'} \partial_d X^{M'} \partial_e X^{N'} \partial_f X^{P'} \right) \right], 
\]

(4.2)

where \( F_{abc} \) is the field strength associated with the 2-form field \( B_{ab} \). This is gauge invariant if the gauge transformation \( C_{MNP} \rightarrow C_{MNP} + \partial_{[M} \Lambda_{NP]} \) is accompanied by a shift in the 2-form field of the fivebrane, \( B_{ab} \rightarrow B_{ab} + \Lambda_{NP} \partial_a X^N \partial_b X^P \). The self-duality condition must also be changed from stating that \( F_{abc} \) is self-dual to stating that \( F_{abc} - \hat{C}_{abc} \) is self-dual (where \( \hat{C} \) is the pullback of \( C \) to the fivebrane worldvolume, \( \hat{C}_{abc} = C_{MNP} \partial_a X^M \partial_b X^N \partial_c X^P \)), since only this combination is gauge invariant. The equations of motion arising from this action are consistent with the self-duality condition after we add an additional term to the action as described below. Upon dimensionally reducing three dimensions from the fivebrane, the self-duality conditions may be trivially

\(^4\) We thank P. K. Townsend for discussions on this issue.
resolved (as described below). Thus, we can hope that this action, together with the correction described below, may indeed be a correct description for the wrapped fivebrane, even though in 11 dimensions the self-duality condition has to be added to it by hand.

We expect the fivebrane action to also include another term, describing the coupling to the dual $\tilde{C}$ of the 3-form field. Before discussing this we should define exactly what we mean by $\tilde{C}$, since the equation of motion of the $C$ field in 11 dimensional supergravity is $d(*dC) = dC \wedge dC$ (we ignore numerical constants in this paragraph), and we cannot in general define a field $\tilde{C}$ by $*dC = d\tilde{C}$. However, since $d(*dC - C \wedge dC) = 0$, we can define $d\tilde{C} = *dC - C \wedge dC$, and this is the definition we will be using for $\tilde{C}$. This definition means that $\tilde{C}$ is not invariant under gauge transformations of $C$. If $C \rightarrow C + d\Lambda$, $\tilde{C}$ transforms by $\tilde{C} \rightarrow \tilde{C} + d\Lambda \wedge C$. Thus, we cannot write a Lagrangian for the fivebrane with a term $\int d^6\xi \hat{\tilde{C}}$ (where $\hat{\tilde{C}}$ is the pullback of the 6-form field $\tilde{C}$ to the worldvolume of the fivebrane), as we would like to, since this is not gauge invariant. We cannot fix this by a term proportional to $\hat{\tilde{C}} \wedge \hat{\tilde{C}}$, since this vanishes. Instead, the only gauge invariant lagrangian which can write seems to be $S^{(1)}_F = \int d^6\xi (\hat{\tilde{C}} - F \wedge \hat{\tilde{C}})$, and, therefore, the complete lagrangian we will use for the fivebrane is $S_F = S^{(0)}_F - S^{(1)}_F$. Fortunately, the equations of motion derived from this action are consistent with the self-duality condition we described above, and $S_F$ seems to be a consistent action. The 11 dimensional supergravity theory also seems to be consistent with this form of the fivebrane action, as discussed in [21].

Let us now discuss the field content we obtain when reducing this action to the action of a membrane in 8 dimensions. To perform the dimensional reduction we choose three of the fivebrane coordinates to equal the coordinates of the torus, $\xi_3 = X_8, \xi_4 = X_9, \xi_5 = X_{10}$, and then we can perform the integration over these coordinates (since the background fields do not depend on them). We can do this in a simple way only if $G_{\mu i} = 0$, and we will assume this from here on in the fivebrane theory (according to (3.7), when $C_{89(10)} = 0$ this is dual to assuming that in the membrane action $C_{\mu ij} = 0$). We will be left with a membrane action with 8 scalar fields $X_\mu (\mu = 0, \cdots, 7)$, and with what remains of the self-dual 2-form. As discussed above, the self-duality condition on the 3-form field strength is

$$F_{abc} - \hat{\tilde{C}}_{abc} = \frac{1}{6\sqrt{-\gamma}}\gamma_{aa'}\gamma_{bb'}\gamma_{cc'}\epsilon^{a'b'c'def}(F_{def} - \hat{\tilde{C}}_{def}). \quad (4.3)$$

When none of the fields depend on the last 3 coordinates of the fivebrane, this expression simplifies considerably. For $a, b, c = 0, 1, 2$ we find that $(F_{012} - \hat{\tilde{C}}_{012}) \propto (F_{345} - \hat{\tilde{C}}_{345}) = -\hat{\tilde{C}}_{345}$, since $F_{345} = 0$. Thus, $F_{012}$ is no longer an independent dynamical field. For
a, b, c = 0, 1, 3 (for instance), we find that \((\partial_0 B_{13} - \partial_1 B_{03} - \check{C}_{013}) \propto (\partial_2 B_{54} - \check{C}_{254})\) (assuming for the moment that the metric \(\gamma\) is diagonal). Hence, the scalar field \(B_{45}\) is determined in terms of the vector field \(B_{\alpha 3}\) (and vice versa). The analysis of the other components of the self-duality equations is analogous, and we find that we can remain either with three independent vector fields \(B_{\alpha a}\) (where \(\alpha = 0, 1, 2\) is a membrane worldvolume index and \(a = 3, 4, 5\)) in the membrane worldvolume, or with three independent scalar fields \(B_{ab}\) \((a, b = 3, 4, 5)\). We will choose to remain with the scalar fields, since we will show that these may be identified in a simple way with the scalar fields in the membrane theory. Leaving the vector fields would lead to an action which is related to the scalar action by a triple scalar-vector duality of the type described in section 2.

After the reduction, we can just throw away the terms in the action involving \(F_{\alpha \beta \gamma}\) (from here on \(\alpha, \beta, \gamma = 0, 1, 2\) and \(a, b, c = 3, 4, 5\)), since the equations of motion of these vector fields just give, when using the self-duality equation, the Bianchi identity for the scalar fields (which is \(\epsilon^{\alpha \beta \gamma} \epsilon^{abc} \partial_\alpha \partial_\beta B_{ab} = 0\)). Thus, retaining these terms does not add any new information. In the same way we can throw away the terms involving \(F_{012}\), since they also give trivial equations of motion (when using the self-duality condition). Hence, the only terms involving \(F\) which remain in the action are those involving \(F_{\alpha ab}\). The field content we find is, therefore, the same as the field content in the membrane action (4.1)\[10\].

We would now like to identify the terms in the action (4.1) with the terms in the reduction of (4.2), according to the transformation (3.5). Let us begin with the metric field \(G_{\mu \nu}\). In both (4.1) and (4.2) this field appears canonically, but we should recall that we have (implicitly) set the tensions of the membrane and the fivebrane to one in the above formulas, and these determine the length scale by which the metric is measured. In 11 dimensional units, the tensions are related by \[3\] \(T_5 = (T_2)^2\) (up to numerical factors which we suppress). When \(C_{89(10)} = 0\), the tension of the wrapped fivebrane is \(T_5 \sqrt{\det(G_{ij})}\), and we see that in units in which the membrane tension is one, it is just the volume of the 3-torus, \(\sqrt{\det(G_{ij})}\). Thus, the scales of the two theories are related by a factor of \((\det(G_{ij}))^{1/6}\). Since the metric has dimensions of length squared, this gives exactly the relation in (3.3) between the 8 dimensional metrics, as expected (recall that generally \(\tau = C_{89(10)} + i\sqrt{\det(G_{ij})}\)). For a non-vanishing \(C_{89(10)}\) field, the relation is more complicated due to the presence of a \(C_{89(10)}^2\) term in the wrapped fivebrane action. To simplify the equations we will assume from here on that \(C_{89(10)} = 0\).

Next, we can easily compare the terms which are linear in the 8 dimensional 3-form fields. In the wrapped fivebrane action the \(\check{C}\) term in the action becomes
\[-\frac{1}{6}e^{\alpha\beta\gamma} \tilde{C}_{\mu\nu\rho} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho.\]

Identifying this with the 3-form term in (4.2), using the relation described above between the membrane and fivebrane metrics, we find exactly the relation (3.3) between the background fields in the 2 actions, as desired.

The relation between the other terms in the two actions is slightly more complicated. We claim that the actions are related by the transformation (3.3) if we identify the scalars \(\frac{1}{2}e^{abc}B_{ab}(a, b, c = 3, 4, 5)\) in the fivebrane action with \(X^{c+5}\) in the membrane action (4.1).

Note that this identification exchanges the gauge transformation of the 3-form which shifts \(B_{ab}\) with the isometry corresponding to a shift in \(X^i\), as is also evident from (3.3). In order to perform the comparison we should replace the metric \(\gamma_{ab}\) in the internal directions by its classical value, according to the equations of motion. In the absence of the second term in the action (4.2), this is just \(\gamma_{ab} = G_{(a+5)(b+5)}\), but generally there are corrections to this, arising from the \((F - \hat{C})^2\) term. In the leading order approximation in which we analyzed the transformation in section 3, it is justified to ignore these corrections, since they are of the same order as the terms which we ignored. Plugging in this solution, and the relation between \(B_{ab}\) and \(X^i\), we find that the quadratic term in the \(B_{ab}\) fields becomes exactly the term \(-\frac{1}{2}\sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j G_{ij}\) in the membrane action. The term in (4.2) linear in \(B_{ab}\) becomes, using (3.3), the term \(-\sqrt{-\gamma} \gamma^{\nu\mu} \partial_\nu X^\mu \partial_\gamma X^i G_{\mu i}\) in the membrane action, while the term linear in \(B_{ab}\) from the \(F \wedge \hat{C}\) term becomes just \(-\frac{1}{2}e^{\alpha\beta\gamma} C_{\mu\nu i} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^i\), which we equate with the same term in (4.1) (since \(C_{\mu\nu i}\) is invariant under the duality). Since we have taken \(G_{\mu i} = 0\) in the fivebrane action and ignored all higher order terms, these are the only terms we can compare. Presumably, when doing the exact transformation instead of (3.3) and plugging in the exact solution for \(\gamma_{ab}\), the other terms will match as well.

We conclude that, to the extent that we have checked it, the membrane action in the original background seems to be the same as the wrapped fivebrane action in the dual background given by (3.3). At least in the approximation we used, a simple form of the fivebrane action, supplemented by the self-duality condition, seems to describe the fivebrane worldvolume theory in a way consistent with the duality.

Let us end this section with a comment about the relevance of this duality to the issue of the length scales in M theory. When \(C_{89(10)} = 0\), the duality transformation inverts the 3-volume of the torus \(V = \sqrt{\det(G_{ij})}\). In M theory there does not seem to exist a minimal length scale, at least classically. Taking M theory on a radius much smaller than the 11 dimensional Planck scale just leads to a weakly coupled string theory. However, the existence of the duality transformation we described suggests that the 11 dimensional...
Planck scale may serve as a minimal 3-volume scale in M theory, since M theory on a 3-torus of volume $V$ is equivalent to M theory on a 3-torus of volume $1/V$. This is perhaps natural in some sense if M theory is indeed a theory of membranes. Unfortunately, this interpretation is not clear-cut, since the duality also changes the metric in the remaining 8 directions, unlike T duality in string theory. In any case, we hope that understanding this duality symmetry may shed some light on the problem of understanding the length scales in string theory and in M theory \cite{22}.

5. String dualities from membrane-membrane duality

In this section we will derive many of the known string dualities from the M theory duality described in the previous two sections. Our general strategy will be to take the original M theory, perhaps compactified along more directions, and orbifold it by some discrete symmetries. The relation (3.5) between the variables of the original and of the dual theories will allow us to identify these discrete symmetries in the dual theory, so that we will know how to perform the orbifold also in the dual theory. In this way we will obtain a duality transformation relating two, generally different, orbifolds of M theory.

First, we can easily get the T duality of the heterotic string on a torus from the duality transformation of the previous section, by simply orbifolding by $x_{10} \leftrightarrow -x_{10}$, together with $C \leftrightarrow -C$ (in the membrane worldvolume this involves a parity transformation as well). This transformation is invariant under the duality. Thus, on both sides we get the heterotic $E_8 \times E_8$ string on $T^2$, and the relation between them is just the usual T duality of this theory on the torus.

Less trivial dualities arise if we add more compact dimensions and orbifold by different symmetries. Without orbifolding, or by performing only the orbifold of the previous paragraph, we can of course get more complicated T duality transformations for the type II and heterotic strings. The orbifolds we discuss below are all of type 2(b) in the classification proposed by Sen \cite{23} of orbifolds and dualities. Therefore, we expect the duality to commute with the orbifolding in these cases, and indeed we will always find a pair of dual theories. Let us begin by adding another compact dimension $x_7$, and orbifolding by the symmetry which changes the sign of all four compact dimensions. In the original theory we would thus get M theory on $T^4/Z_2$, which is an orbifold limit of K3. Using (3.3) (or working directly in the worldvolume of the membrane), we find that in the dual theory this symmetry changes the sign of $x_7$ and of the 3-form field $C$, and we get the heterotic string
theory on $T^3$. Thus, the 8 dimensional membrane-membrane duality naturally leads to a duality between M theory on K3 and the heterotic string theory on $T^3$.

By adding another compact dimension we can easily derive from this the duality between type IIA theory on K3 and the heterotic string theory on $T^4$. If we then perform another orbifold, by the symmetry which changes the sign of $x_6$ and $C$ in the original theory, and changes the sign of $x_6, x_8, x_9, x_{10}$ in the dual theory, this leads to the heterotic-heterotic duality of the heterotic string theory on K3. One can easily check that the transformation rules (3.5) indeed exchange the fivebrane wrapped around K3 in the original theory with the membrane wrapped around $x_7$ in the dual theory, as expected. Thus, these string-string dualities may also be derived directly from the membrane-membrane duality of M theory (at least in the orbifold limit of K3). We do not yet know how to define (from first principles) the twisted sectors in orbifolds of M theory. However, once we know how to do this, the description given above of the duality transformation may allow us to find the exact relation between the twisted fields before and after the duality transformation. We do not know how to generalize these dualities to generic K3 manifolds in our framework, but presumably this should also be possible.

Other dualities in string theory involve the type IIB string theory on various backgrounds (including “F theory” backgrounds) which do not include a circle. The description of these in M theory seems to be singular, because we must take the limit in which the area of a torus goes to zero in order to get the type IIB theory in ten dimensions. Since there is no clear evidence for a minimal length scale in M theory, it is not clear that this limit is indeed singular. In any case, we can try to derive these dualities upon compactifying on an additional $S^1$, when the type IIB theory is identical to the type IIA theory. For instance, let us derive the duality between M theory on $(S^1)^5/Z_2$ and type IIB theory on K3, when compactified on an additional circle $S^1$. We add three more compact dimensions, and orbifold the original theory by the symmetry which changes the sign of $x_5, x_6, x_7, x_8, x_9$ and $C$, leading to M theory on $(S^1)^5/Z_2 \times S^1$. In the dual theory, we find that this symmetry corresponds to changing the sign of $x_5, x_6, x_7$ and $x_{10}$, leading to the M theory on $T^4/Z_2 \times T^2$, which is an orbifold limit of the type II theory on $K3 \times S^1$. Thus, this duality also arises from the membrane-membrane duality in 8 dimensions, but in order to get it without the additional $S^1$ we must take the limit $R_{10} \to \infty$ (in the original theory). Note that if we do not add an additional $S^1$, and divide the original theory by the symmetry changing the sign of $x_6, x_7, x_8, x_9, x_{10}$ and $C$, we find the same symmetry in the dual theory, so this is just a T duality of M theory on $(S^1)^5/Z_2$. Translating this
T duality to the type IIB theory on $T^4/Z_2$, we find that it is just the exchange of two of the circles in $T^4/Z_2$.

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References

[1] P. K. Townsend, “The eleven dimensional supermembrane revisited”, Phys. Lett. 350B (1995) 184, hep-th/9501068
[2] E. Witten, “String theory dynamics in various dimensions”, Nucl. Phys. B443 (1995) 85, hep-th/9503124
[3] J. H. Schwarz, “An SL(2, Z) multiplet of type II superstrings”, Phys. Lett. 360B (1995) 13, hep-th/9508143; “Superstring dualities”, hep-th/9509148; “The power of M theory”, Phys. Lett. 367B (1996) 97, hep-th/9510086; “M theory extensions of T duality”, hep-th/9601077
[4] P. Hořava and E. Witten, “Heterotic and type I string dynamics from eleven dimensions”, Nucl. Phys. B460 (1996) 506, hep-th/9510203
[5] M. J. Duff, P. S. Howe, T. Inami and K. S. Stelle, “Superstrings in D=10 from supermembranes in D=11”, Phys. Lett. 191B (1987) 70
[6] P. K. Townsend, “D-branes from M-branes”, hep-th/9512062
[7] C. Schmidhuber, “D-brane actions”, hep-th/9601003
[8] O. Aharony, J. Sonnenschein and S. Yankielowicz, “Interactions of strings and D-branes from M theory”, hep-th/9603009
[9] P. S. Aspinwall, “Some relationships between dualities in string theory”, hep-th/9508154
[10] P. K. Townsend, “String-membrane duality in seven dimensions”, Phys. Lett. 354B (1995) 247, hep-th/9504005
[11] C. M. Hull and P. K. Townsend, “Unity of superstring dualities”, Nucl. Phys. B438 (1995) 109, hep-th/9410167
[12] D. Kutasov and E. Martinec, “New principles for string/membrane unification”, hep-th/9602049; D. Kutasov, E. Martinec and M. O'Loughlin, “Vacua of M theory and N = 2 strings”, hep-th/9603116
[13] R. Dijkgraaf, E. Verlinde and H. Verlinde, “BPS spectrum of the fivebrane and black hole entropy”, hep-th/9603126; “BPS quantization of the fivebrane”, hep-th/9604055
[14] P. Hořava and E. Witten, “Eleven dimensional supergravity on a manifold with a boundary”, hep-th/9603142
[15] A. Giveon, M. Porrati and E. Rabinovici, “Target space duality in string theory”, Phys. Rep. 244 (1994) 77, hep-th/9401139
[16] E. Bergshoeff, E. Sezgin and P. K. Townsend, “Supermembranes and eleven dimensional supergravity”, Phys. Lett. 189B (1987) 75; “Properties of the eleven dimensional supermembrane theory”, Ann. Phys. 185 (1988) 330
[17] M. Dine, P. Huet and N. Seiberg, “Large and small radius in string theory”, Nucl. Phys. B322 (1989) 301; J. Dai, R. G. Leigh and J. Polchinski, “New connections between string theories”, Mod. Phys. Lett. A4 (1989) 2073
[18] E. Bergshoeff, C. Hull and T. Ortin, “Duality in the type II superstring effective action”, Nucl. Phys. B451 (1995) 547, hep-th/9504081
[19] A. Sen, “T duality of p-branes”, hep-th/9512203
[20] J. M. Izquierdo, N. D. Lambert, G. Papadopoulos and P. K. Townsend, “Dyonic membranes”, Nucl. Phys. B460 (1996) 560, hep-th/9508177
[21] E. Witten, “Fivebranes and M theory on an orbifold”, hep-th/9512219
[22] S. H. Shenker, “Another length scale in string theory?”, hep-th/9509132
[23] A. Sen, “Duality and orbifolds”, hep-th/9604070
[24] M. J. Duff, R. Minasian and E. Witten, “Evidence for heterotic/heterotic duality”, hep-th/9601036
[25] C. Vafa, “Evidence for F-theory”, hep-th/9602022, D. R. Morrison and C. Vafa, “Compactifications of F-theory on Calabi-Yau threefolds - I,II”, hep-th/9602114, hep-th/9603161; E. Witten, “Phase transitions in M theory and F theory”, hep-th/9603150
[26] K. Dasgupta and S. Mukhi, “Orbifolds of M theory”, hep-th/9512196