Exact resonance A-D-E S-matrices and their renormalization group trajectories

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Abstract

We introduce the A-D-E resonance factorized models as an appropriate analytical continuation of the Toda S-matrices to the complex values of their coupling constant. An investigation of the associated Casimir energy, via the thermodynamic Bethe ansatz, reveals a rich pattern of renormalization group trajectories interpolating between the central charges of the $G_1 \otimes G_k/G_{k+1}$ GKO coset models. We have also constructed the simplest resonance factorized model satisfying the "$\phi^3$"-property. From this resonance scattering, we predict new flows in non-unitary minimal models.

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1 Introduction

Conformal invariance has played an important role in our understanding of the properties of two-dimensional systems at the critical point [1, 2]. Away from the critical point, the corresponding field theory can be considered as a conformal model perturbed by an appropriated relevant operator. The important insight added by A. Zamolodchikov [3] was that certain perturbations may possess infinitely many non-trivial conserved charges, hence integrable even in the off-critical regime. This observation allowed him to construct the on-mass-shell solution, i.e. the mass spectrum and the exact S-matrices, of the Ising model in a magnetic field [4]. In this context, Al. Zamolodchikov [5] has pointed out that by applying the thermodynamic Bethe ansatz (TBA) [6] one is able to recover the ultraviolet properties of the ground state from the respective S-matrices. In this sense, the TBA makes an interesting bridge between the properties of the infrared and the ultraviolet regimes. In principle, this technique can be applied to any factorized scattering theory, even for those that a priori lack a background conformal field theory. An a posteriori analysis of the behaviour of the finite-size corrections to the ground state may reveal amazing features not previously known. Recently, one example has been put forward by Al. Zamolodchikov [7] in the case of a resonance S-matrix consisting of a single particle scattering through the following amplitude,

\[ S(\theta, \theta_0) = \frac{\sinh(\theta) - i \cosh(\theta_0)}{\sinh(\theta) + i \cosh(\theta_0)} \]  

where \( \theta_0 \) is the resonance parameter.

Let us briefly summarize Al. Zamolodchikov’s results [7]. Although the ultraviolet limit associated with this S-matrix is always governed by a theory with central charge \( c = 1 \), at intermediate distances (for \( \theta_0 >> 1 \)) the pattern of the renormalization group (RG) trajectories is surprisingly rich. The RG trajectories evolve to the infrared regime by first interpolating between the central charges \( c_p = 1 - 6/p(p + 1) \) of the minimal models. Each fixed point \( c_p \) takes approximately \( \frac{\theta_0}{2} \) of the RG “time” before crossing over to the next lower critical point \( c_{p-1} \). At this point we recall that such crossover phenomena can be
induced by the perturbation of the least relevant field operator $\phi_{1,3}$ of the minimal models \cite{8,9,10}. Recently, it has also been argued in ref. \cite{11} that the field $\phi_{1,3}$ can produce crossover behaviour even in non-unitary minimal models.

The purpose of this paper \cite{1} is to present and analyze a large class of resonance factorized scattering theories based on the simply laced Lie algebra G. These S-matrices will define a family of one parameter field theories $G(\theta_0)$, depending on the resonance parameter $\theta_0$.

We point out a connection between these resonance S-matrices and those from the A-D-E Toda field theories \cite{16,17,18,19,20} through an appropriate analytical continuation of the Toda coupling constant to complex values. This allows us to express the resonance S-matrices amplitudes in a rather simple and useful form. We study the Casimir energy $E_0(R, \theta_0)$ at moderate distances R via the TBA approach. The behaviour of the associated RG trajectories are then related to flows \cite{13} of certain deformed GKO coset models \cite{12}. Finally, we propose a simplest resonance factorized model satisfying the “$\phi^3$”-property. We argue that the respective RG trajectories can be related to new flows in the non-unitary minimal models.

The paper is organized as follows. In sect.2 we introduce the A-D-E resonance models and write down the respective TBA equations. Sect.3 is devoted to our numerical checks of the behaviour of the ground state energy $E_0(R, \theta_0)$ for finite values of the volume R. Sect.4 describes the “$\phi^3$”-resonance model and our arguments to predict new flows in non-unitary minimal models. Sect.5 contains a discussion of our results. In appendix A we summarize some useful relations in order to compute the logarithmic corrections to the Casimir energy when $R \to 0$.

\footnote{A brief account of this paper has recently been presented in ref.\cite{14}}
2 The A-D-E resonance S-matrices and the TBA equations

We start this section by describing the basic properties of the resonance scattering that we have constructed. We assume that the S-matrices are diagonal and factorizable, i.e., there is no particle production. In this case, the only constraints are due to the unitarity and crossing conditions of the two-body amplitude $S_{a,b}$,

$$S_{a,b}(\theta)S_{a,b}(-\theta) = 1, \quad S_{a,b}(\theta) = S_{b,a}(i\pi - \theta)$$  \hspace{1cm} (2)$$

where $\bar{a}$ stands for an antiparticle.

We also assume that the resonance amplitude $S_{a,b}(\theta, \theta_0)$ can be written as a product of a background amplitude $S_{a,b}^{\text{min}}(\theta)$ and the resonance part $S_{a,b}^{\text{res}}(\theta, \theta_0)$, namely $S_{a,b}(\theta, \theta_0) = S_{a,b}^{\text{min}}(\theta)S_{a,b}^{\text{res}}(\theta, \theta_0)$. The physical meaning of this hypothesis is that the background and the resonance interactions appear in different spatial regions, and therefore the phase-shifts are simply added. More complicated cases will involve correlations that will not be considered in this paper. However, as we have already stated, even this simple ansatz will lead us to a rather rich pattern of RG trajectories appearing in deformed conformal field theory.

In order to introduce the A-D-E structure we assume that the particle spectrum and the background S-matrices, $S_{a,b}^{\text{min}}(\theta)$, are given by the so-called minimal solution of Eq.(2). For details and respective properties see ref. [16, 17, 18, 19, 20]. The simplest solution of Eq.(2) to the resonance amplitude $S_{a,b}^{\text{res}}(\theta, \theta_0)$, invariant by the symmetry $\theta_0 \rightarrow -\theta_0$, appears in the case of the $A_{N-1}$ Lie algebra. The S-matrix for the fundamental particle is given by

$$S_{1,1}^{\text{res}}(\theta, \theta_0) = \frac{\sinh \frac{1}{2}(\theta - \theta_0 - i\frac{\pi}{N}) \sinh \frac{1}{2}(\theta + \theta_0 - i\frac{\pi}{N})}{\sinh \frac{1}{2}(\theta - \theta_0 + i\frac{\pi}{N}) \sinh \frac{1}{2}(\theta + \theta_0 + i\frac{\pi}{N})},$$  \hspace{1cm} (3)$$

where the resonance poles are located at $\theta = \pm \theta_0 - i\frac{\pi}{N}$. The other amplitudes can be computed by a straightforward bootstrap. For $N=2$, we recover Al. Zamolodchikov’s S-matrix, Eq.(1).

\footnote{In our case the antiparticle appears as a pole in the particle-particle scattering amplitude, and therefore the reflection amplitude is null in order to satisfy factorizability [15].}
Similar expressions can be found for the D and E groups, by consistently solving Eq.(2) in analogous manner as pursued in refs. \[16, 17, 18, 19, 20\]. However, we prefer to take a short route. First, we notice that the amplitude $S_{1,1}^{res}(\theta, \theta_0)$ above is related in a simple way to the so-called Z-factors of the $A_{N-1}$ Toda field theory \[16\]. These factors encode the dependence of the coupling constant $\alpha$ present in the Lagrangean, through a function $b(\alpha)$ \[16\]. Then, by simply taking $b(\alpha) = \pm i\theta_0 + \frac{\pi}{N}$ in the Toda $Z_{1,1}(\theta, b(\alpha))$-factor we recover the amplitude of Eq.(3) \[4\]. In the general case, we have verified that the resonance S-matrix satisfying the properties mentioned above is given by \[4\].

$$S_{a,b}^{res}(\theta, \theta_0) = Z_{a,b}(\theta, b(\alpha) = \pm i\theta_0 + \frac{\pi}{h})$$  \[4\]
where $Z_{a,b}(\theta, b(\alpha))$ are the Z-factors appearing in the A-D-E Toda field theory \[16, 17, 18, 19, 20\] and $h$ is the Coxeter number.

Using Eq.(4) we can write down a closed expression for all the $S_{a,b}(\theta, \theta_0)$ amplitudes. First it is convenient to define the functions,

$$\psi_{a,b}(\theta, \theta_0) = -i \frac{d}{d\theta} \log S_{a,b}(\theta, \theta_0)$$  \[5\]

which will also be important in the TBA equations. In terms of its Fourier component $\tilde{\psi}_{a,b}(k, \theta_0)$ defined by,

$$\tilde{\psi}_{a,b}(k, \theta_0) = \int_{-\infty}^{\infty} e^{ik\theta} \psi_{a,b}(\theta, \theta_0)$$  \[6\]
we find the remarkable matrix identity,

$$\left[ \delta_{a,b} - \frac{\tilde{\psi}_{a,b}(k, \theta_0)}{2\pi} \right]^{-1} = \frac{\delta_{a,b} \cosh\left[\frac{\pi k}{h}\right] - l_{a,b}/2}{\cosh\left[\frac{\pi k}{h}\right] - \cos(k\theta_0)}$$  \[7\]
where $l_{a,b} = 2 - C_{a,b}$ is the incident matrix of the G=A,D,E Lie algebra. Finally, the amplitude $S_{a,b}(\theta, \theta_0)$ can be written in terms of a standard integral representation,

$$S_{a,b}(\theta, \theta_0) = \exp\left( \frac{i}{\pi} \int_{0}^{\infty} \tilde{\psi}_{a,b}(k, \theta_0) \frac{\sin(\theta k)}{k} dk \right)$$  \[8\]

\[3\]The case $N=2$ was first noticed by Al. Zamolodchikov \[7\] in the context with the sinh-Gordon model.

\[4\]Here we have followed the notation of ref. \[16\]. There are other similar notations in the literature \[16, 18, 19, 20\]. For example, in the case of ref. \[18\], by taking $B(\alpha) = 1 \pm i\theta_0 \frac{\pi}{N}$ we achieve the same results.
We now start to discuss the TBA equations and their properties. The TBA approach allows one \[5\] to compute the Casimir energy \( E_0(R, \theta_0) \) on a radius of length \( R \). The ground state energy is written in terms of pseudoenergies \( \epsilon_a(\theta) \), and for diagonal S-matrices there is one such function \( \epsilon_a(\theta) \) for each stable mass \( m_a \) of the spectrum. The \( E_0(R, \theta) \) is given by,

\[
E_0(R, \theta) = -\frac{1}{2\pi} \sum_{a=1}^{N} m_a \int_{-\infty}^{\infty} d\theta \cosh(\theta) L_a(\theta) \tag{9}
\]

where \( L_a(\theta) = \ln(1 + e^{-\epsilon_a(\theta)}) \), and \( N \) is the number of particles.

The volume dependence is encoded by a set of integral equations for the pseudoenergies \( \epsilon_a(\theta) \), depending on the kernel \( \psi_{a,b}(\theta, \theta_0) \) as follows,

\[
\epsilon_a(\theta) + \frac{1}{2\pi} \sum_{b=1}^{N} \int_{-\infty}^{\infty} d\theta' \psi_{a,b}(\theta - \theta', \theta_0) L_b(\theta') = \nu_a(\theta) \quad a = 1, 2, ..., N \tag{10}
\]

where \( \nu_a(\theta) = m_a R \cosh(\theta) \) and in our case \( m_a \) are the mass gaps of the A-D-E spectrum \[16, 17, 18, 19, 20\]. It is possible to rewrite Eq.(10) in a more suggestive way, adopting an approach similar to that of ref. \[21\]. Using Eq.(7) and the following identity \[21\],

\[
\nu_a(\theta + \frac{i\pi}{h}) + \nu_a(\theta - \frac{i\pi}{h}) = \sum_{b=1}^{N} l_{a,b} \nu_b(\theta) \tag{11}
\]

we then obtain a set of functional equations for the functions \( Y_a(\theta) = e^{\epsilon_a(\theta)} \),

\[
Y_a(\theta + \frac{i\pi}{h})Y_a(\theta - \frac{i\pi}{h}) = \prod_{b \in G} [1 + Y_b(\theta)]^{l_{a,b}} \left[1 + Y_a^{-1}(\theta + \theta_0)\right]^{l_{a,b}^{-1}} \left[1 + Y_a^{-1}(\theta - \theta_0)\right]^{-1} \tag{12}
\]

Motived by Al. Zamolodchikov’s discussion of the \( A_1 \) \[7\] it is possible to rewrite Eq.(12) as an interesting discrete equation if we restrict the rapidities \( \theta \) on the lattice variables \( \theta_{m,n} = \Theta + m\theta_0 + \frac{i\pi}{h}n \). Using the identification \( Y_a^{m,n} = Y_a(\theta_{m,n}) \) we have,

\[
Y_a^{m,n+1}Y_a^{m,n-1} = \prod_{b \in G} [1 + Y_b^{m,n}]^{l_{a,b}} \prod_{c \in A} \left[1 + \frac{1}{Y_c^{m,n}}\right]^{-l_{m,c}} \tag{13}
\]

where \( b \) is an index characterizing the nodes of the Dynkin diagram of G=A,D,E Lie algebra and \( c \) is the same index for the A Lie algebra.

Eq.(13) is defined on the product space of the Lie algebras G (A,D,E) and A, namely \( G \otimes A \). Eq.(13) also assumes a form similar to those obtained from certain perturbed GKO
coset models [24, 25]. It is worth mentioning that a similar equation can also be defined in the dual space $A \otimes G$, provide that we take $Y_a \rightarrow Y_a^{-1}$ in Eq.(12). Therefore they are self-dual when $G \equiv A$. In this sense these two pairs of equations are richer than the TBA equation that we start with, Eq.(10). However, the TBA equations can be easily recovered from Eq.(12) by specifying the initial conditions of the functions $\nu_a(\theta)$.

In the next section, we show that even the simple guess for the energies nodes $\nu_a$, i.e. $\nu_a(\theta) = m_a R \cosh(\theta)$, will generate a rather interesting pattern of RG trajectories. It turns out that these trajectories resemble those of unstable particles decaying to intermediate states until reaching the vacuum. At each intermediate step the particle has a life-time of $\frac{\theta}{2}$ (the RG “time”) before decaying again to the next state.

3 The numerical work

We begin by analyzing the ultraviolet behaviour of Eqs.(9,10). Standard manipulations of the TBA equations [3] give us the following behaviour of the ground state energy,

$$E_0(R, \theta_0) \simeq -\frac{\pi r}{6R}, R \rightarrow 0$$

(14)

where $r$ is the rank of the respective Lie algebra.

From Eq.(14) we conclude that the background conformal field theory has central charge $c = r$ [22]. In appendix A we discuss the next to the leading order in $R$ corrections to $E_0(R, \theta_0)$. The first correction is logarithmic in $R$, in contrast with the fractional power behaviour of the typical non-resonance factorized theories [21]. In terms of the function $c(R, \theta_0) = -\frac{6R}{\pi} E_0(R, \theta_0)$ our result reads

$$c(R, \theta_0) = r - \frac{3(\theta^2 + \frac{\pi^2}{k^2})}{X^2} \sum_{a,b} C_{a,b}^{-1}$$

(15)

where $X = \ln(\frac{m_1 R}{2})$.

The behaviour of the function $c(R, \theta_0)$ becomes particularly interesting at intermediate lengths of $R$. In this case one needs to solve numerically the integral equations (10) using,
e.g., standard iterative procedures. We start by analyzing the simplest case of our proposed resonance S-matrices, namely the $A_2$ theory. In fig. 1(a,b,c,d) we show the behaviour of the function $c(R, \theta_0)$ for $\theta_0 = 0, 10, 20, 40$. $c(R, 0)$ behaves as a smooth function interpolating from the ultraviolet regime, $c(R \to 0, 0) = 2$, to the infrared region. On the contrary, by increasing $\theta_0$, it is clear that $c(R, \theta_0)$ starts to form plateaux around the following fixed points, $c_p = 2(1 - 12/p(p + 1))$, $p = 4, 5, 6, \ldots$. These critical points correspond to the conformal field theories possessing a $Z(3)$-symmetry [23]. Since the numbers of plateaux increase with $\theta_0$ one expects that when $\theta_0 \to \infty$, all the fixed points will be visited by the RG trajectories until finally reaching the infrared region.

It is also fruitful to examine the same pattern in terms of the $\beta$-function along the RG trajectory. In this case, the plateaux will correspond to zeros of the beta function. Borrowing Al.Zamolodchikov’s definition [3] of the $\beta$-function, we have

$$\beta(g) = -\frac{\partial}{\partial X} c(R, \theta_0), \quad g = r - c(R, \theta_0)$$

where the “coupling constant” $g$ has been normalized to zero in the ultraviolet regime.

In Fig. 2(a,b,c) we show $\beta(g)$ for $\theta_0 = 10, 20, 40$. One can notice that the zeros of $\beta(g)$ are precisely at the points $g = 24/p(p+1)$, $p = 4, 5, 6, \ldots$. By increasing $\theta_0$, more zeros are going to appear in $\beta(g)$, however keeping the general shape of $\beta(g)$ for those that have already been formed. We stress that our findings are very much in agreement with Al. Zamolodchikov’s results for the $A_1$ case. In general, for the proposed A-D-E scattering theories, the function $c(R, \theta_0)$ will form plateaux around the values of the central charge $c^r_p$ of the $G_1 \otimes G_{p-h}/G_{p-h+1}$ GKO [12] coset construction, namely $c^r_p = r(1 - h(h + 1)/p(p + 1))$, $p = h + 1, h + 2, \ldots$ . In Fig. 3(a) we illustrate this pattern for the first plateaux in the models $A_2, A_3, D_4$, which share some common central charges. In Fig. 3(b) we plot the respective $\beta(g)$ function.

What we can learn from this numerical computation is as follows. Each time that $X \simeq -(p - h)\theta_0/2$ the function $c(R, \theta_0)$ crosses over from its value $c^r_p$ to the next (up) value $c^r_{p+1}$ [4]. This indicates that at these values of $X$ the TBA equations may present a rather

5 A similar study has been performed by Al. Zamolodchikov in the case of $A_1$ [3].

6 From our figures 1(c),1(d),3(a) we notice that the “RG time” $\theta_0/2$ accounts for the plateau and for
special behaviour. Indeed, the linearization of these equations around $X \simeq -(p-h)\theta_0/2$ (or better the functional Eq.(12)) gives us a set of $\theta_0$-independent equations\footnote{One should notice that, e.g., the kernel $\psi_{1,1}(\theta,\theta_0)$ has deep peaks at $\theta = \pm \theta_0, 0$, appearing as the interacting (A Lie algebra structure) and the diagonal (G Lie algebra structure) of the linearized TBA equations.} describing the RG flows in the $G_1 \otimes G_{p-h}/G_{p-h+1}$ coset models perturbed (positive perturbation) by an operator $\Phi_p$ with conformal weight $\Delta_{\Phi_p} = 1 - h/(p+1)$\footnote{For $p = h+1$ this field is not present in the Kac table and it is replaced by the spinless combination $TT\bar{T}$ of the stress energy tensor $T$.}. On the other hand, the infrared finite size corrections will be dominated by the dual field $\tilde{\Phi}_p$ with conformal dimension $\Delta_{\tilde{\Phi}_p} = 1 + h/(p-1)$\footnote{For $p = h+1$ this field is not present in the Kac table and it is replaced by the spinless combination $TT\bar{T}$ of the stress energy tensor $T$.}. Therefore, for each plateau “p” the ultraviolet behaviour is governed by $\Phi_p$ and finally attracted to the infrared region by $\tilde{\Phi}_p$. It is tempting to propose that such behaviour may be reproduced if one considers the critical theory perturbed by the combination $\lambda \Phi_p + \tilde{\lambda} \tilde{\Phi}_p$ ($\lambda > 0$)\footnote{For $p = h+1$ this field is not present in the Kac table and it is replaced by the spinless combination $TT\bar{T}$ of the stress energy tensor $T$.}. In fact, in the case of the minimal models, Lässig\footnote{For $p = h+1$ this field is not present in the Kac table and it is replaced by the spinless combination $TT\bar{T}$ of the stress energy tensor $T$.} has identified such a phase for $\tilde{\lambda} < 0$, by using perturbative RG calculations\footnote{For $p = h+1$ this field is not present in the Kac table and it is replaced by the spinless combination $TT\bar{T}$ of the stress energy tensor $T$.}. The generalization of this approach to all A-D-E resonance models are technically complicated, because the necessary set of structure constants of the cosets models are not (up to now) explicitly known. However, we point out that the field $\tilde{\Phi}_p$ has the correct conformal dimensional to satisfy some of the arguments used in ref.\footnote{For $p = h+1$ this field is not present in the Kac table and it is replaced by the spinless combination $TT\bar{T}$ of the stress energy tensor $T$.}. Moreover, at least for the ground state, our numerical results are consistent with this picture.

Finally, we remark on some numerical observations concerning the behaviour of the functions $L_\alpha(\theta)$. In fig. 4 we show this behaviour for the $A_3$ model. The $Z(4)$-symmetry assures that $L_1(\theta) = L_3(\theta)$ and therefore we only plot the functions $L_1(\theta)$ and $L_2(\theta)$ for $\theta > 0$ ($L_\alpha(\theta)$ are even functions). Roughly it seems that these functions differ only by an overall constant (for each “p”) . This fact may be expected considering that these functions are smooth and should assume constant values at the plateaux. In fig. 5 we have plotted the first derivative (in the variable $\theta$) of $L_1(\theta)$ and $L_2(\theta)$ for $X = -70$. We believe that this
result may motivate us to derive more exact results from the TBA equation (10). Anyhow, for the moment, this information has been used (Appendix A) to estimate the logarithmic correction to the Casimir energy in the ultraviolet limit.

4 The “$\phi^3$” resonance S-matrix

Let us consider a solution of Eq.(2) possessing a single particle $a$. We also assume that the pole structure in the scattering amplitude produces the particle $a$ itself. Besides Eq.(2) the S-matrix $S_{a,a}$ should satisfy the “$\phi^3$” bootstrap condition,

$$S_{a,a}(\theta,\theta_0) = S_{a,a}(\theta + i\frac{\pi}{3},\theta_0)S_{a,a}(\theta - i\frac{\pi}{3},\theta_0)$$  \hspace{1cm} (17)

Following the considerations of section 2, the simplest solution of Eqs. (2,17) are given by,

$$S_{a,a}(\theta,\theta_0) = \frac{\tanh \frac{1}{2}(\theta + i\frac{\pi}{3})\tanh \frac{1}{2}(\theta - \theta_0 - i\frac{\pi}{3})\tanh \frac{1}{2}(\theta + \theta_0 - i\frac{\pi}{3})}{\tanh \frac{1}{2}(\theta - i\frac{\pi}{3})\tanh \frac{1}{2}(\theta - \theta_0 + i\frac{\pi}{3})\tanh \frac{1}{2}(\theta + \theta_0 + i\frac{\pi}{3})}$$  \hspace{1cm} (18)

The background part of $S_{a,a}(\theta,\theta_0)$ is the S-matrix [27] of the perturbed Yang-Lee edge singularity [28]. The Toda related field theory is that one analyzed by Arinshtein et al [16] and known as the Shabat-Mikhailov model. This theory has only one field and the interaction is through the potential $V(\phi) = e^{\alpha\phi} + e^{-2\alpha\phi}$, $\alpha$ being the coupling constant.

As has been shown by Mikhailov [29], this model is integrable and can be considered as a “reduction” of the $A_2$ Toda field theory provided that the two fields $\phi_1, \phi_2$ in the $A_2$ model satisfy the relation $\phi_1 = -\phi_2 = \phi$ (see ref. [29]). From the S-matrix point of view, this can be seen by defining the amplitude $S_{a,a}(\theta,\theta_0)$ as \footnote{A similar reduction as in Eq.(19) is possible for any $A_{N-1}$ theory, $N$ odd. The next task is, however, the identification of the reduced model for any kind of crossover behaviour in non-unitary models.}

$$S_{a,a}(\theta,\theta_0) = S_{1,1}(\theta,\theta_0)S_{1,2}(\theta,\theta_0)$$  \hspace{1cm} (19)

where $S_{1,1}(\theta,\theta_0)$ and $S_{1,2}(\theta,\theta_0)$ are the $A_2$ amplitudes.

It follows from Eq.(19) and Eq.(10) that the pseudoenergie ($\epsilon(\theta)$) of the model defined by Eq.(18) and those from the $A_2$ Toda theory ($\epsilon_1(\theta), \epsilon_2(\theta)$) are equal, namely $\epsilon(\theta) = \epsilon_1(\theta) = \epsilon_2(\theta)$.
\( \epsilon_2(\theta) \). Hence the ground state energy associated to the S-matrix of Eq.(18) is precisely half of that of the \( A_2 \) model analyzed in sect.3. Therefore, the plateaux will now form around the values \( c_p = 1 - 12/p(p + 1), p = 4, 5, \ldots \). At this point it is important to recall that the Casimir energy in non-unitary minimal models \( M_{2q} \) depends on both the central charge and the lowest conformal dimension \( \Delta_{min} \) (effective central charge) \([30]\),

\[
E_0 \simeq -\frac{\pi}{6R}(-24\Delta_{min} + c) \simeq -\frac{\pi}{6R}c_{ef}
\]

where \( c_{ef} = 1 - 6/(pq) \).

This fact strongly suggests that we are dealing with non-unitary minimal models. Indeed, comparing \( c_{ef} \) with \( c_p = 1 - 12/p(p + 1), p = 4, 5, \ldots \) we identify the following series of non-unitary minimal models: \( M_{2q/2q+1} \) (q=p/2, p even) and \( M_{2q+1/2q+1} \) (q=(p-1)/2, p odd). We also recall that for these models the field \( \phi_{1,2} \) has the minimal conformal dimension. Motivated by these analogies with the \( A_2 \) resonance factorized theory it is also interesting to find the fields that may drive the system to the staircase pattern of section 3. From the expected finite size corrections to the ground state we identify the fields \( \phi_{2,1} \) and \( \phi_{1,5} \) as those that can induce these models to the crossover phenomenon. More precisely, we expect the following behaviour for \( M_{q+1/2q+1} \),

\[
M_{q+1/2q+1} + \phi_{2,1} \rightarrow M_{q/2q+1} \quad q = 2, 3, \ldots
\]

and for \( M_{q/2q+1} \),

\[
M_{q/2q+1} + \phi_{1,5} \rightarrow M_{(q-1)+1/2(q-1)+1} \quad q = 3, 4, \ldots
\]

Moreover, the infrared regime is also governed by the operators \( \phi_{2,1} \) and \( \phi_{1,5} \). An exception is the flow \( M_{3/7} + \phi_{1,5} \rightarrow M_{3/5} \), where the field \( \phi_{1,5} \) is not present in the \( M_{3/5} \) Kac-table.

In this case we think that \( \phi_{1,5} \) should be replaced by the level 2 descendent of \( \phi_{1,3} \). To the best of our knowledge this pattern in which the fields \( \phi_{2,1} \) and \( \phi_{1,5} \) interchange their roles as relevant and irrelevant operators along the RG trajectory is new in the literature. In fig.(6) we show this predicted behaviour \([3]\).\footnote{It is notable that the central charge increases only in the flow defined by Eq.(22). Since these models}
In order to give more support to this prediction we have numerically studied the spectrum of the Hamiltonian of the simplest flow, namely $M_{3/5} + \phi_{2,1} \to M_{2/5}$. We have constructed the Hamiltonian of this perturbed conformal field theory on a torus of radius $R$ using the truncated conformal approach \cite{33,34,35}. In fig. (7) we show the evolution of the first 40 eigenvalues as a function of the volume $R$. The $\phi_{2,1}$ perturbation allows us to divide the Hilbert space in two sectors of the fields $[I, \phi_{2,1}]$ (dashed lines) $[\phi_{1,2}, \phi_{1,3}]$ (solid lines) and their descendents. The striking feature is the absence of any crossing levels, that makes it difficult to interpret fig.(7) as a consistent integrable massive theory. We recall that in an integrable massive theory, there will be an abundance of level crossings between states that are distinguished by their particle content \cite{33,36}. For example, this will be the case for momentum lines accumulating in the lowest particle threshold and other n-particle states present in the theory. Moreover an a posteriori analysis of the (possible) mass gaps are not consistent (within our numerical precision) with, e.g., the exponential split of the ground state. Although, we are not able to estimate the infrared region using this approach, fig.(7) has a rather remarkable resemblance to typical massless crossover spectrum. We believe that these observations bring some extra support to the prediction of Eqs. (21,22). Finally, some additional comments are in order.

A first remark concerns a possible Ginsburg-Landau \cite{32} interpretation of the simplest flow $M_{3/5} + \phi_{2,1} \to M_{2/5}$. The fusion algebra for the operator $\phi_{1,2}$ with the lowest dimension in the $M_{3/5}$ model is,

$$
\phi_{1,2} \otimes \phi_{1,2} \sim I + \phi_{1,3}; \quad \phi_{1,2} \otimes \phi_{1,3} \sim \phi_{2,1} + \phi_{1,2}; \quad \phi_{1,2} \otimes \phi_{2,1} \sim \phi_{1,3} \quad (23)
$$

By identifying the operator $\phi_{1,2}$ with the elementary field $\Phi$ of the Ginsburg-Landau theory, it follows from Eq.(23) that $\phi_{1,3} \equiv \Phi^2$ and $\phi_{2,1} \equiv \Phi^3$. The first descendent $L_{-1}\bar{L}_{-1}\phi_{1,2}$ being an odd field should be identified with $\Phi^5$. The Ginsburg-Landau potential consistent with this last identification is $V(\Phi) = \Phi^6$, which is in accordance with the $Z(2)$ symmetry of are non-unitary they do not present any contradiction with Zamolodchikov’s c-theorem \cite{31}. For other interesting behaviour in polymer systems see the discussion in ref. \cite{37}.
$M_{3/5}$ model. Therefore, perturbing with the field $\phi_{2,1}$ one can drive the $M_{3/5}$ model to a $\Phi^3$ class of universality, which is precisely that of the Yang-Lee edge singularity [28]. It seems important to find the Ginsburg-Landau picture of other non-unitary models. However, we have noticed that in models with several negative and positive dimensions it seems quite difficult to find a consistent description in terms of only one basic field $\Phi$.

A second observation concerns the ambiguity of sign of the perturbing coupling constant in the case of the $\phi_{1,5}$ operator. The convention that is assumed in unitary models is that the positive sign is related to the massless crossover and the negative sign to a massive behaviour. Since these non-unitary models are somewhat related (ground state) to an unitary theory ($A_2$ model) one may expect that the same picture holds. However, in the case of non-unitary theories we may also have phases in which the system starts to present complex eigenvalues after reaching a certain value of $R$ [33]. We believe that a further investigation is needed to resolve this issue, which may end up in discovering new S-matrices for non-unitary models.

5 Conclusions

We have studied resonance factorized models based on the A-D-E Lie algebra. By using the thermodynamic Bethe ansatz we have analyzed the respective Casimir energy on a radius of length $R$. At moderate distances of $R$ we find a rich pattern of RG trajectories associated with typical flows in the deformed GKO coset models. We have pointed out a connection between the A-D-E resonance scattering and one from the Toda field theory when its coupling constant assumes special complex values. However, the meaning of this analytical continuation in terms of a well defined background quantum field theory is still to be found. Moreover, it would be interesting to investigate whether this is the only possible way leading to factorized resonance S-matrices which can be interpreted as a deformed conformal field theory. We have obtained a general functional equation for the functions $Y_a(\theta) = e^{i\theta}$ and

---

11 We also recall that the identification of the operator $\phi_{1,3}$ is in reasonable agreement with a $Z(2)$ spontaneously broken symmetry (Ising-like) found in its spectrum [36].
one may start with this equation instead of considering the A-D-E resonance S-matrices. We recall that similar functional equations appear in integrable lattice models in the context of the inversion relations to the transfer-matrix \[38\]. It seems also that these equations hide a large amount of information yet to be explored \[39\].

The construction of a resonance factorized S-matrix satisfying the “φ³”-property has led us to predict new flows in the non-unitary minimal models. In the case of the simplest crossover, the spectrum presented in section 4 may give us a guess on how the fields in the ultraviolet regime will evolve to the infrared region. However, in the more general cases considered in this paper, this property has still to be disentangled. It would be also important to identify the lattice models possessing such properties, the \(A_2^2\)-RSOS models being a probable guess \[40\].

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Note added: After we had submitted this paper to publication, we received a copy of the preprint \[41\] by P. Dorey and F. Ravanini. There is some overlap with the present paper concerning the A-D-E generalization of ref.[7].
6 Appendix A

This appendix is concerned with the logarithmic corrections appearing in Eq.(15). Our analysis is fairly parallel with that of ref.[7], in order to generalize it to the case of many pseudoenergies. First we notice that Eq.(7) permits us to write the expansion

\[ \tilde{\psi}_{a,b} = \sum_{n=0}^{\infty} \frac{(-1)^n \tilde{\psi}_{a,b}^{(2n)}}{(2n)!} k^{2n} \tag{24} \]

where up to order \( k^2 \) we have,

\[ \tilde{\psi}_{a,b}^{(0)} = \delta_{a,b}, \quad \tilde{\psi}_{a,b}^{(2)} = 2C_{a,b}^{-1} \left( \theta_0^2 + \frac{\pi^2}{h^2} \right) \tag{25} \]

Using this expansion in Eq.(7), in the regime \( X \to -\infty \), we find

\[ m_a e^{\theta} + \ln(1 - e^{-L_a(y)}) = \sum_{b} \sum_{n=1}^{\infty} \frac{\tilde{\psi}_{a,b}^{(2n)}}{(2n)!} L_b^{(2n)}(\theta) \tag{26} \]

where \( L_b^{(2n)}(\theta) = \frac{d^{2n}}{d\theta^{2n}} L_b(\theta) \).

Following ref. [7], the central charge in the region \( X \sim y, \theta - 2X >> 1, y << 0 \), satisfies the differential equation,

\[ \frac{\pi^2}{6} \frac{\partial}{\partial y} c(X,y) = - \sum_{a} m_a e^{y} L_a(y) \tag{27} \]

and one possible ansatz for its solution is given by,

\[ \frac{\pi^2}{6} = - \frac{1}{2} \sum_{a,b} \sum_{n=1}^{\infty} \frac{\tilde{\psi}_{a,b}^{(2n)}}{2(2n)!} \sum_{k=1}^{2n-1} (-1)^k L_a^{(k)}(y) L_b^{(2n-k)}(y) - \sum_{a} \int_{0}^{L_a(y)} \ln(1 - e^{-t}) dt - \sum_{a} m_a e^{y} L_a(y) \tag{28} \]

The leading logarithmic correction is obtained by expanding \( \ln(1 - e^{-t}) = -e^{-t} + ... \), and by ignoring the term \( e^\theta \) in Eq.(26). The final result is,

\[ c(X,y) = r + \frac{3}{2\pi^2} \sum_{a,b} \tilde{\psi}_{a,b}^{(2)} L_a^{(1)} L_b^{(1)} - \frac{6}{\pi^2} \sum_{a} e^{-L_a(y)} \tag{29} \]

provided that the \( L_a(\theta) \) satisfy,

\[ \sum_{b} \frac{\tilde{\psi}_{a,b}^{(2)}}{2} \frac{d^2}{d\theta^2} L_b(\theta) + e^{-L_a(\theta)} = 0 \tag{30} \]
Based on our numerical results, it is a reasonable approximation to consider that the first derivatives of the $L_a(\theta)$ are index $a$ independent. Therefore, we can have the following ansatz as a solution of Eq.(30),

$$L_a(\theta) = \ln \left[ \frac{\sin^2 \lambda (\theta - \gamma)}{\lambda^2 \sum_b \tilde{\psi}^{(2)}_{a,b}} \right]$$  \hspace{1cm} (31)

where $\lambda$ is fixed by periodicity to be $\lambda = \frac{\pi}{2(X - \gamma_0)}$, $\gamma_0$ is a constant (not determined by this approach) such that when $X \to \infty$, $\gamma \to \gamma_0$. Substituting Eq.(31) in Eq(29) we have,

$$c = r - \frac{3}{2} \sum_{a,b} \tilde{\psi}^{(2)}_{a,b} X^2, \hspace{0.5cm} X \to \infty$$  \hspace{1cm} (32)

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Figure Captions

Fig. 1(a,b,c,d) The scaling function \( c(R, \theta_0) \) for four values of the resonance parameter \( \theta_0 \). (a) \( \theta_0 = 0 \), (b) \( \theta_0 = 10 \), (c) \( \theta_0 = 20 \) and (d) \( \theta_0 = 40 \).

Fig. 2(a,b,c) The beta function \( \beta(g) \) for three values of the resonance parameter \( \alpha_0 \). (a) \( \theta_0 = 10 \), (b) \( \theta_0 = 20 \) and (c) \( \theta_0 = 40 \).

Fig.3(a) The first RG plateaux for the models \( A_2 \) (short dashed line), \( A_3 \) (solid line) and \( D_4 \) (long dashed line).

Fig.3(b) The beta-function \( \beta(g) \) for the models \( A_2 \) (short dashed line), \( A_3 \) (solid line) and \( D_4 \) (long dashed line).

Fig.4 The functions \( L_1(\theta) \) (solid line) and \( L_2(\theta) \) (dashed line) of the \( A_3 \) model for \( \theta_0 = 40 \) and \( X = -70 \).

Fig.5 The first derivative of the functions \( L_1(\theta) \) (solid line) and \( L_2(\theta) \) (dashed line) of the \( A_3 \) model for \( \theta_0 = 40 \) and \( X = -70 \).

Fig.6 The flow pattern in the non-unitary minimal models \( M_{\frac{q}{2q+1}} ; M_{\frac{q+1}{2q+1}} \), \( q = 2, 3, \ldots \). The horizontal(vertical) arrows represent the relevant(irrelevant) operators defining the ultraviolet(infrared) corrections to the fixed point.

Fig.7 The first 40 levels of the spectrum of the model \( M_{3/5} + \phi_{2,1} \). The dashed (solid) lines correspond to the sector \([I, \phi_{2,1}] ([\phi_{1,2}, \phi_{1,3}]) \) and its descendents.