On formation of singularity of the full compressible magnetohydrodynamic equations with zero heat conduction

Xin Zhong

Abstract

We are concerned with the formation of singularity and breakdown of strong solutions to the Cauchy problem of the three-dimensional full compressible magnetohydrodynamic equations with zero heat conduction. It is proved that for the initial density allowing vacuum, the strong solution exists globally if the deformation tensor $D(u)$ and the pressure $P$ satisfy $\|D(u)\|_{L^1(0,T;L^\infty)} + \|P\|_{L^\infty(0,T,L^\infty)} < \infty$. In particular, the criterion is independent of the magnetic field. The logarithm-type estimate for the Lamé system and some delicate energy estimates play a crucial role in the proof.

Keywords: full compressible magnetohydrodynamic equations; strong solutions; blow-up criterion; zero heat conduction.

Math Subject Classification: 76W05; 35B65

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a domain, the motion of a viscous, compressible, and heat conducting magnetohydrodynamic (MHD) flow in $\Omega$ can be described by the full compressible MHD equations

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \text{div} u + \nabla P &= b \cdot \nabla b - \frac{1}{2} \nabla |b|^2, \\
c_v [(\rho \theta)_t + \text{div}(\rho u \theta)] + P \text{div} u - \kappa \Delta \theta &= 2\mu |D(u)|^2 + \lambda (\text{div} u)^2 + \nu |\text{curl} b|^2, \\
b_t - b \cdot \nabla u + u \cdot \nabla b + b \text{div} u &= \nu \Delta b, \\
\text{div} b &= 0.
\end{align*}
\]  

(1.1)

Here, $t \geq 0$ is the time, $x \in \Omega$ is the spatial coordinate, and $\rho, u, P = R\rho \theta$ ($R > 0$), $\theta, b$ are the fluid density, velocity, pressure, absolute temperature, and the magnetic field respectively; $D(u)$ denotes the deformation tensor given by

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

The constant viscosity coefficients $\mu$ and $\lambda$ satisfy the physical restrictions

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$  

(1.2)

Positive constants $c_v, \kappa,$ and $\nu$ are respectively the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, and the magnetic diffusive coefficient.

There is huge literature on the studies about the theory of well-posedness of solutions to the Cauchy problem and the initial boundary value problem (IBVP) for the compressible MHD system due to the physical importance, complexity, rich phenomena and mathematical challenges, refer
proved that Recently, for the Cauchy problem and the IBVP of 3D full compressible MHD system, Huang-Li \[9\] obtained the following Serrin type criterion here \(T^*\) is the finite blow up time. For the compressible isentropic MHD system, Xu-Zhang \[29\] obtained the following Serrin type criterion

\[
\lim_{T \to T^*} \|u\|_{L^s(0,T;L^r)} = \infty, \quad \text{for } \frac{2}{s} + \frac{3}{r} = 1, \ 3 < r \leq \infty,
\]

(1.3)

here \(T^*\) is the finite blow up time. For the compressible isentropic MHD system, Xu-Zhang \[29\] obtained the following Serrin type criterion

\[
\lim_{T \to T^*} \left( \|\text{div } u\|_{L^1(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)} \right) = \infty,
\]

(1.4)

where \(r\) and \(s\) as in \[163\]. This criterion is similar to \[12\] for 3D compressible isentropic Navier-Stokes equations, which shows that the mechanism of blow-up is independent of the magnetic field. For the 3D full compressible MHD system, Lu-Du-Yao \[17\] proved that

\[
\lim_{T \to T^*} \left( \|\nabla u\|_{L^1(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)} \right) = \infty
\]

(1.5)

under the assumption

\[
\mu > 4\lambda.
\]

(1.6)

Recently, for the Cauchy problem and the IBVP of 3D full compressible MHD system, Huang-Li \[9\] proved that

\[
\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)} \right) = \infty, \quad \text{for } \frac{2}{s} + \frac{3}{r} \leq 1, \ 3 < r \leq \infty.
\]

(1.7)

For more information on the blow-up criteria of compressible flows, we refer to \[1, 3, 9, 10, 12, 16, 22, 24, 25, 27, 31\] and the references therein.

It should be noted that all the results mentioned above on the blow-up of strong (or classical) solutions of viscous, compressible, and heat conducting MHD flows are for \(\kappa > 0\). Very recently, Huang-Xin \[14\] obtained blow-up criteria for the non-isentropic compressible Navier-Stokes equations without heat-conductivity. Therefore, it seems to be an interesting question to ask what the blow-up criterion is for the the system \([1\,1\,1]\) with zero heat conduction. In fact, this is the main aim of this paper.

When \(\kappa = 0\), and without loss of generality, take \(c_v = R = 1\), the system \([1\,1\,1]\) can be written as

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \text{div } u + \nabla P &= b \cdot \nabla b - \frac{1}{2} \nabla |b|^2, \\
P_t + \text{div}(P u) + P \text{div } u &= 2\mu |\nabla (u)|^2 + \lambda (\text{div } u)^2 + \nu |\text{curl } b|^2, \\
b_t - b \cdot \nabla u + u \cdot \nabla b + b \text{div } u &= \nu \Delta b, \\
\text{div } b &= 0.
\end{align*}
\]

(1.8)
The present paper is aimed at giving a blow-up criterion of strong solutions to the Cauchy problem of the system \([1.8]\) with the initial condition
\[
(\rho, u, P, b)(x, 0) = (\rho_0, u_0, P_0, b_0)(x), \quad x \in \mathbb{R}^3,
\]
and the far field behavior
\[
(\rho, u, P, b)(x, t) \to (0, 0, 0, 0), \quad \text{as} \ |x| \to +\infty, \ t > 0.
\]
Before stating our main result, we first explain the notations and conventions used throughout this paper. We denote by
\[
\int_0^T \cdot \, dx = \int_\mathbb{R}^3 \cdot \, dx.
\]
For \(1 \leq p \leq \infty\) and integer \(k \geq 0\), the standard Sobolev spaces are denoted by:
\[
\begin{align*}
L^p &= L^p(\mathbb{R}^3), \\
W^{k,p} &= W^{k,p}(\mathbb{R}^3), \\
H^k &= H^{k,2}(\mathbb{R}^3), \\
D_0^1 &= \{u \in L^0|\nabla u \in L^2\}, \\
D_0^{k,p} &= \{u \in L_1^1|\nabla^k u \in L^p\}.
\end{align*}
\]
Now we define precisely what we mean by strong solutions to the problem \([1.8]–[1.10]\).

**Definition 1.1 (Strong solutions)** \((\rho, u, P, b)\) is called a strong solution to \([1.8]–[1.10]\) in \(\mathbb{R}^3 \times (0,T)\), if for some \(q_0 > 3\),
\[
\begin{align*}
\rho &\geq 0, \; \rho \in C([0,T];L^1 \cap H^1 \cap W^{1,q_0}), \; \rho_t \in C([0,T];L^{q_0}), \\
(u, b) &\in C([0,T];D_0^1 \cap D^{2,2}) \cap L^2(0,T;D^{2,q_0}), \; b \in C([0,T];H^2), \\
(u_t, b_t) &\in L^2(0,T;D^{1,2}), \; (\sqrt{\rho} u_t, b_t) \in L^\infty(0,T;L^2), \\
P &\geq 0, \; P \in C([0,T];L^1 \cap H^1 \cap W^{1,q_0}), \; P_t \in C([0,T];L^{q_0}),
\end{align*}
\]
and \((\rho, u, P, b)\) satisfies both \([1.8]\) almost everywhere in \(\mathbb{R}^3 \times (0,T)\) and \([1.9]\) almost everywhere in \(\mathbb{R}^3\).

Our main result reads as follows:

**Theorem 1.1** For constant \(\tilde{q} \in (3, 6]\), assume that the initial data \((\rho_0 \geq 0, u_0, P_0 \geq 0, b_0)\) satisfies
\[
\begin{align*}
(\rho_0, P_0) &\in L^1 \cap H^1 \cap W^{1,\tilde{q}}, \; u_0 \in D_0^1 \cap D^{2,2}, \\
(\sqrt{\rho_0} u_0) &\in L^2, \; b_0 \in H^2, \; \text{div} b_0 = 0,
\end{align*}
\]
and the compatibility conditions
\[
- \mu \Delta u_0 - (\lambda + \mu) \text{div} u_0 + \nabla P_0 - (\text{curl} b_0) \times b_0 = \sqrt{\rho_0} g
\]
for some \(g \in L^2(\Omega)\). Let \((\rho, u, P, b)\) be a strong solution to the problem \([1.8]–[1.10]\). If \(T^* < \infty\) is the maximal time of existence for that solution, then we have
\[
\lim_{T \to T^*} \left( \| \mathcal{D}(u) \|_{L^1(0,T;L^\infty)} + \| P \|_{L^\infty(0,T;L^\infty)} \right) = \infty
\]
provided that
\[
3\mu > \lambda.
\]
Several remarks are in order.

**Remark 1.1** The local existence of a strong solution with initial data as in Theorem 1.1 can be established in the same manner as [4][20]. Hence, the maximal time \(T^*\) is well-defined.
Remark 1.2 It is worth noting that the blow-up criteria [1.13] is independent of the magnetic field. Moreover, compared with [7] for the non-isentropic Navier-Stokes equations with \( \kappa = 0 \), according to [1.13], the \( L^\infty \) bound of the temperature \( \theta \) is not the key point to make sure that the solution \((\rho, u, P, b)\) is a global one, and it may go to infinity in the vacuum region within the life span of our strong solution.

Remark 1.3 In [7], to obtain higher order derivatives of the solutions, the restriction \( \mu > 4\lambda \) plays a crucial role in the analysis. In fact, the condition \( \mu > 4\lambda \) is only used to get the upper bound of \( \int \rho |u|^r dx \) for some \( r \geq 4 \) (see [7, Lemma 3.2]). Here, we derive the upper bound of \( \int \rho |u|^4 dx \) under the assumption \( 3\mu > \lambda \) (see Lemma 3.4), which is weaker than \( \mu > 4\lambda \).

We now make some comments on the analysis of this paper. We mainly make use of continuation argument to prove Theorem 1.1. That is, suppose that (1.13) were false, i.e.,

\[
\lim_{T \to T^*} \left( \|D(u)\|_{L^1(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) \leq M_0 < \infty.
\]

We want to show that

\[
\sup_{0 \leq t \leq T^*} \left( \|\rho, P\|_{H^1 \cap W^{1,\delta}} + \|\nabla u\|_{H^1} + \|b\|_{H^2} \right) \leq C < +\infty.
\]

Since the magnetic field is strongly coupled with the velocity field of the fluid in the compressible MHD system, some new difficulties arise in comparison with the problem for the compressible Navier-Stokes equations studied in [14]. The following key observations help us to deal with the interaction of the magnetic field and the velocity field very well. First, we prove (see Lemma 3.3) that a control of \( L^1_t L^\infty_x \)-norm of the deformation tensor implies a control on the \( L^\infty_t L^\infty_x \)-norm of the magnetic field \( b \). Then, motivated by [14, 25], we derive a priori estimates of \( \|u\|\nabla u \) in both space and time, which is the second key observation in this paper (see Lemma 3.4). Finally, the a priori estimates on the \( L^\infty_t L^q_x \)-norm of \( (\nabla \rho, \nabla P) \) and the \( L^1_t L^\infty_x \)-norm of the velocity gradient can be obtained (see Lemma 5.4) simultaneously by solving a logarithm Gronwall inequality based on a logarithm estimate for the Lamé system (see Lemma 2.3) and the a priori estimates we have derived.

The rest of this paper is organized as follows. In Section 2 we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.1.

## 2 Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We begin with the following Gronwall’s inequality, which plays a central role in proving a priori estimates on strong solutions \((\rho, u, P, b)\).

**Lemma 2.1** Suppose that \( h \) and \( r \) are integrable on \((a, b)\) and nonnegative a.e. in \((a, b)\). Further assume that \( y \in C[a, b], y' \in L^1(a, b) \), and

\[
y'(t) \leq h(t) + r(t)y(t) \quad \text{for a.e.} \ t \in (a, b).
\]

Then

\[
y(t) \leq \left[ y(a) + \int_a^t h(s) \exp \left( - \int_a^s r(\tau)d\tau \right) d\tau \right] \exp \left( \int_a^t r(s)ds \right), \quad t \in [a, b].
\]

**Proof.** See [23, pp. 12–13].

Next, the following Gagliardo-Nirenberg inequality will be used later.

**Lemma 2.2** Let \( 1 \leq p, q, r \leq \infty \), and \( j, m \) are arbitrary integers satisfying \( 0 \leq j < m \). Assume that \( v \in C_c^\infty(\mathbb{R}^n) \). Then

\[
\|D^j v\|_{L^p} \leq C\|v\|_{L^q}^{1-a}\|D^m v\|_{L^r}^a,
\]
where

\[-j + \frac{n}{p} = (1 - a)\frac{n}{q} + a\left(-m + \frac{n}{r}\right),\]

and

\[a \in \begin{cases} \left[\frac{j}{m}, 1\right), & \text{if } m - j - \frac{n}{r} \text{ is an nonnegative integer,} \\
\left[\frac{1}{m}, 1\right], & \text{otherwise.} \end{cases}\]

The constant $C$ depends only on $n, m, j, q, r, a$.

Proof. See [20, Theorem].

Next, the following logarithm estimate will be used to estimate $\|\nabla u\|_{L^\infty}$.

**Lemma 2.3** For $q \in (3, \infty)$, there is a constant $C(q) > 0$ such that for all $\nabla v \in L^2 \cap D^{1,q}$, it holds that

\[
\|\nabla v\|_{L^\infty} \leq C \left( \|\text{div } v\|_{L^\infty} + \|\text{curl } v\|_{L^\infty} \right) \log(e + \|\nabla^2 v\|_{L^q}) + C \|\nabla v\|_{L^2} + C. \tag{2.1}
\]

Proof. See [12, Lemma 2.3].

Finally, we consider the following Lamé system

\[
\begin{cases}
-\mu \Delta U - (\lambda + \mu) \text{div } U = F, & x \in \mathbb{R}^3, \\
U \to 0, & \text{as } |x| \to \infty,
\end{cases}
\]

where $U = (U^1, U^2, U^3)$, $F = (F^1, F^2, F^3)$, and $\mu, \lambda$ satisfy (1.2).

The following logarithm estimate for the Lamé system (2.2) will be used to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla \rho\|_{L^2 \cap L^q}$.

**Lemma 2.4** Let $\mu, \lambda$ satisfy (1.2). Assume that $F = \text{div } g$ where $g = (g_{kj})_{3 \times 3}$ with $g_{kj} \in L^2 \cap L^r \cap D^{1,q}$ for $k, j = 1, \ldots, 3$, $r \in (1, \infty)$, and $q \in (3, \infty)$. Then the Lamé system (2.2) has a unique solution $U \in D^1_0 \cap D^{1,r} \cap D^{2,q}$, and there exists a generic positive constant $C$ depending only on $\mu, \lambda, q$, and $r$ such that

\[
\|\nabla U\|_{L^r} \leq C\|g\|_{L^r},
\]

and

\[
\|\nabla U\|_{L^\infty} \leq C \left( 1 + \log(e + \|\nabla g\|_{L^r})\|g\|_{L^\infty} + \|g\|_{L^r} \right). \tag{2.3}
\]

Proof. See [9, Lemma 2.3].

### 3 Proof of Theorem 1.1

Let $(\rho, u, P, b)$ be a strong solution described in Theorem 1.1. Suppose that (1.13) were false, that is, there exists a constant $M_0 > 0$ such that

\[
\lim_{T \to T^*} \left( \|\mathcal{D}(u)\|_{L^1(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) \leq M_0 < \infty. \tag{3.1}
\]

First, the upper bound of the density could be deduced directly from (1.8) and (3.1) (see [12, Lemma 3.4]).

**Lemma 3.1** Under the condition (3.1), it holds that for any $T \in [0, T^*)$,

\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^1(0,T;L^\infty)} \leq C, \tag{3.2}
\]

where and in what follows, $C, C_1, C_2$ stand for generic positive constants depending only on $M_0, \lambda, \mu, \nu, T^*$, and the initial data.

Next, we have the following standard estimate.
Lemma 3.2 Under the condition \((3.1)\), it holds that for any \(T \in [0, T^*)\),
\[
\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho}u\|_{L^2}^2 + \|P\|_{L^1 \cap L^\infty} + \|b\|_{L^2}^2 \right) + \int_0^T \left( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) dt \leq C. \tag{3.3}
\]
Proof. It follows from \((1.8)\) that
\[
P_t + u \cdot \nabla P + 2P \text{ div } u = F = 2\mu|\nabla(u)|^2 + \lambda|\text{div } u|^2 + \nu|\text{curl } b|^2 \geq 0. \tag{3.4}
\]
Due to \((3.1)\), we can always define particle path before blowup time
\[
\begin{cases}
  \frac{d}{dt} X(x, t) = u(X(x, t), t), \\
  X(x, 0) = x.
\end{cases}
\]
Thus, along particle path, we obtain from \((3.4)\) that
\[
\frac{d}{dt} P(X(x, t), t) = -2P \text{ div } u + F,
\]
which implies
\[
P(X(x, t), t) = \exp \left( -2 \int_0^t \text{div } u ds \right) \left[ P_0 + \int_0^t \exp \left( 2 \int_0^s \text{div } u d\tau \right) F ds \right] \geq 0.
\]
As a result, we deduce from \((3.1)\) that
\[
\sup_{0 \leq t \leq T} \|P\|_{L^1 \cap L^\infty} \leq C. \tag{3.5}
\]
Multiplying \((1.8)\) and \((1.8)\) by \(u\) and \(b\) respectively, then adding the two resulting equations together, and integrating over \(\mathbb{R}^3\), we obtain after integrating by parts that
\[
\frac{1}{2} \frac{d}{dt} \int (\rho |u|^2 + |b|^2) dx + \int [\mu |\nabla u|^2 + (\lambda + \mu)(\text{div } u)^2 + \nu|\nabla b|^2] dx
\]
\[
= \int P \text{ div } u dx
\]
\[
\leq (\lambda + \mu) \int (\text{div } u)^2 dx + C(\lambda, \mu) \int P^2 dx, \tag{3.6}
\]
which combined with \((3.5)\) gives
\[
\frac{1}{2} \frac{d}{dt} \left( \|\sqrt{\rho}u\|_{L^2}^2 + \|b\|_{L^2}^2 \right) + \mu\|\nabla u\|_{L^2}^2 + \nu\|\nabla b\|_{L^2}^2 \leq C \tag{3.7}
\]
Integrating \((3.7)\) with respect to \(t\) and applying \((3.5)\) lead to the desired \((3.3)\). This completes the proof of Lemma 3.2. \(\square\)

Inspired by \([6]\), we have the following the upper bound of the magnetic field \(b\).

Lemma 3.3 Under the condition \((3.1)\), it holds that for any \(T \in [0, T^*)\),
\[
\sup_{0 \leq t \leq T} \|b\|_{L^2 \cap L^\infty} \leq C. \tag{3.8}
\]
Proof. Multiplying \((1.8)\) by \(q|b|^{q-2}b\) \((q \geq 2)\) and integrating the resulting equation over \(\mathbb{R}^3\), we derive
\[
\frac{d}{dt} \int |b|^q dx + \nu q(q - 1) \int |b|^{q-2} \left| \nabla b \right|^2 dx \leq q \int (b \cdot \nabla u - u \cdot \nabla b - b \text{ div } u) \cdot |b|^{q-2} b dx
\]
\[
= q \int (b \cdot \nabla(u) - u \cdot \nabla b - b \text{ div } u) \cdot |b|^{q-2} b dx. \tag{3.9}
\]
By the divergence theorem and \((3.8)\), we get
\[-q \int (\mathbf{u} \cdot \nabla)\mathbf{b} \cdot |\mathbf{b}|^{q-2} \mathbf{b} dx = \int \text{div} \mathbf{u} |\mathbf{b}|^q dx,
\]
which together with \((3.9)\) yields
\[
\frac{d}{dt} \|\mathbf{b}\|_L^q + \nu q (q - 1) \int |\mathbf{b}|^{q-2} |\nabla \mathbf{b}|^2 dx \leq (2q + 1) \|\mathcal{D} \mathbf{u}\|_{L^\infty} \int |\mathbf{b}|^q dx. \tag{3.10}
\]
Consequently, from \(q \geq 2\) and \((3.10)\), we immediately have
\[
\frac{d}{dt} \|\mathbf{b}\|_L^q \leq \frac{2q + 1}{q} \|\mathcal{D} \mathbf{u}\|_{L^\infty} \|\mathbf{b}\|_L^q \leq 3 \|\mathcal{D} \mathbf{u}\|_{L^\infty} \|\mathbf{b}\|_L^q.
\]
Then Gronwall’s inequality and \((3.1)\) imply that for any \(q \geq 2\),
\[
\sup_{0 \leq t \leq T} \|\mathbf{b}\|_L^q \leq C, \tag{3.11}
\]
where \(C\) is independent of \(q\). Thus, letting \(q \to \infty\) in \((3.11)\) leads to the desired \((3.8)\) and finishes the proof of Lemma \(3.3\).

Motivated by \([14, 25]\), we can improve the basic estimate obtained in Lemma \(3.1\).

**Lemma 3.4** Under the condition \((3.1)\), it holds that for any \(T \in (0, T^*)\),
\[
\sup_{0 \leq t \leq T} \|\rho^{1/4} \mathbf{u}\|_L^4 + \int_0^T \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2 dt \leq C. \tag{3.12}
\]
provided that
\[3\mu > \lambda.\]

**Proof.** Multiplying \((1.8)\)_2 by \(4|\mathbf{u}|^2 \mathbf{u}\) and integrating the resulting equation over \(\mathbb{R}^3\) yield that
\[
\frac{d}{dt} \int \rho |\mathbf{u}|^4 dx + 4 \int \left[ \mu |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 + (\lambda + \mu) |\mathbf{u}|^2 (\text{div} \mathbf{u})^2 + \frac{\mu |\nabla |\mathbf{u}|^2|^2}{2} \right] dx
\]
\[
= 4 \int \text{div}(|\mathbf{u}|^2 \mathbf{u}) P dx - 8(\lambda + \mu) \int \text{div} \mathbf{u} |\mathbf{u}| \mathbf{u} \cdot \nabla |\mathbf{u}| dx
\]
\[
+ 4 \int |\mathbf{u}|^2 \mathbf{u} \cdot \left( \text{div}(\mathbf{b} \otimes \mathbf{b}) - \nabla \left( \frac{|\mathbf{b}|^2}{2} \right) \right) dx
\]
\[
\leq 4 \int \text{div}(|\mathbf{u}|^2 \mathbf{u}) P dx - 8(\lambda + \mu) \int \text{div} \mathbf{u} |\mathbf{u}| \mathbf{u} \cdot \nabla |\mathbf{u}| dx
\]
\[
+ C \int |\mathbf{u}|^2 |\nabla \mathbf{u}| |\mathbf{b}|^2 dx. \tag{3.13}
\]
For the last term of the right-hand side of \((3.13)\), one obtains from Hölder’s inequality, Sobolev’s inequality, and \((3.3)\) that, for any \(\varepsilon_1 \in (0, 1)\),
\[
C \int |\mathbf{u}|^2 |\nabla \mathbf{u}| |\mathbf{b}|^2 dx \leq 4\mu \varepsilon_1 \int |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 dx + C(\varepsilon_1) \int |\mathbf{u}|^2 |\mathbf{b}|^4 dx
\]
\[
\leq 4\mu \varepsilon_1 \int |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 dx + C(\varepsilon_1) \|\mathbf{u}\|_{L^6}^2 \|\mathbf{b}\|_{L^6}^4
\]
\[
\leq 4\mu \varepsilon_1 \int |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 dx + C(\varepsilon_1) \|\nabla \mathbf{u}\|_{L^2}^2,
\]
which together with \((3.13)\) leads to
\[
\frac{d}{dt} \int \rho |\mathbf{u}|^4 dx + 4 \int \left[ \mu (1 - \varepsilon_1) |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 + (\lambda + \mu) |\mathbf{u}|^2 (\text{div} \mathbf{u})^2 + \frac{\mu |\nabla |\mathbf{u}|^2|^2}{2} \right] dx
\]
Direct calculations give that for \( x \in \mathbb{R}^3 \cap \{|u| > 0\}, \)

\[
|u|^2 |\nabla u|^2 = |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |u|^2 |\nabla u|^2, \tag{3.15}
\]

\[
|u| \text{div} u = |u|^2 \text{div} \left( \frac{u}{|u|} \right) + u \cdot \nabla |u|. \tag{3.16}
\]

For \( \varepsilon_1, \varepsilon_2 \in (0, 1) \), we now define a nonnegative function as follows:

\[
k(\varepsilon_1, \varepsilon_2) = \begin{cases} 
\frac{\mu \varepsilon_2 (3 - \varepsilon_1)}{\lambda + \varepsilon_1 \mu}, & \text{if } \lambda + \varepsilon_1 \mu > 0, \\
0, & \text{if } \lambda + \varepsilon_1 \mu \leq 0.
\end{cases} \tag{3.17}
\]

We prove (3.12) in two cases.

**Case 1:** we assume that

\[
\int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 dx \leq k(\varepsilon_1, \varepsilon_2) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^2 |\nabla u|^2 dx. \tag{3.18}
\]

It follows from (3.14) that

\[
\frac{d}{dt} \int \rho |u|^4 dx + 4 \int_{\mathbb{R}^3 \cap \{|u| > 0\}} \Psi dx \leq 4 \int_{\mathbb{R}^3 \cap \{|u| > 0\}} \text{div}(|u|^2 u) P dx + C(\varepsilon_1) |\nabla u|^2_{L^2}, \tag{3.19}
\]

where

\[
\Psi = \mu (1 - \varepsilon_1) |u|^2 |\nabla u|^2 + (\lambda + \mu) |u|^2 (\text{div} u)^2 + 2 \mu |u|^2 |\nabla u|^2 + 2 (\lambda + \mu) \text{div} u |u| \cdot \nabla |u|.
\]

Employing (3.15) and (3.16), we find that

\[
\Psi = \mu (1 - \varepsilon_1) |u|^4 |\nabla u|^2 + (\lambda + \mu) |u|^2 (\text{div} u)^2 + 2 \mu |u|^2 |\nabla u|^2 \\
+ 2 (\lambda + \mu) |u|^2 \text{div} \left( \frac{u}{|u|} \right) u \cdot \nabla |u| + 2 (\lambda + \mu) |u| \cdot \nabla |u| \\
= \mu (1 - \varepsilon_1) \left| u \right|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |u|^2 |\nabla u|^2 \right) + (\lambda + \mu) \left| u \right|^2 \text{div} \left( \frac{u}{|u|} \right) + u \cdot \nabla |u| \right|^2 \\
+ 2 \mu |u|^2 |\nabla u|^2 + 2 (\lambda + \mu) |u|^2 \text{div} \left( \frac{u}{|u|} \right) u \cdot \nabla |u| + 2 (\lambda + \mu) |u| \cdot \nabla |u| \\
= \mu (1 - \varepsilon_1) |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + \mu (3 - \varepsilon_1) |u|^2 |\nabla u|^2 - \frac{\lambda + \mu}{3} |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 \\
+ 3 (\lambda + \mu) \left( \frac{2}{3} |u|^2 \text{div} \left( \frac{u}{|u|} \right) + u \cdot \nabla |u| \right)^2 \\
\geq - (\lambda + \varepsilon_1 \mu) |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + \mu (3 - \varepsilon_1) |u|^2 |\nabla u|^2.
\]
Here we have used the facts that $\lambda + \mu > 0$ and
\[
\left| \text{div} \left( \frac{\mathbf{u}}{|\mathbf{u}|} \right) \right|^2 \leq 3 \left| \nabla \left( \frac{\mathbf{u}}{|\mathbf{u}|} \right) \right|^2.
\]

Then we derive from (3.13) and (3.17) that
\[
4 \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} \Psi dx \geq -4(\lambda + \varepsilon_1 \mu)k(\varepsilon_1, \varepsilon_2) + 4\mu(3 - \varepsilon_1) \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx
\]
\[
\geq 4\mu(3 - \varepsilon_1)(1 - \varepsilon_2) \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx.
\]

(3.20)

Thus, substituting (3.20) into (3.19) and using (3.13), (3.15), and (3.18) yield
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} \rho |\mathbf{u}|^4 dx + 4\mu(3 - \varepsilon_1)(1 - \varepsilon_2) \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx \right]
\]
\[
\leq 4 \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} \text{div}(|\mathbf{u}|^2 \mathbf{u}) \mathbf{P} dx + C(\varepsilon_1) \|
\nabla |\mathbf{u}|^2 \|_{L^2}^2
\]
\[
\leq C \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx + C(\varepsilon_1) \|
\nabla |\mathbf{u}|^2 \|_{L^2}^2
\]
\[
\leq \varepsilon(1 + k(\varepsilon_1, \varepsilon_2)) \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx + C(\varepsilon_1, \varepsilon_2) \|
\nabla |\mathbf{u}|^2 \|_{L^2}^2.
\]

Taking $\varepsilon = \frac{2\mu(3 - \varepsilon_1)(1 - \varepsilon_2)}{1 + k(\varepsilon_1, \varepsilon_2)}$, we have
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} \rho |\mathbf{u}|^4 dx + 2\mu(3 - \varepsilon_1)(1 - \varepsilon_2) \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx \right] \leq C(\varepsilon_1, \varepsilon_2) \|
\nabla |\mathbf{u}|^2 \|_{L^2}^2,
\]
which combined with (3.13) and (3.18) implies
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} \rho |\mathbf{u}|^4 dx + \varepsilon \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx \right] \leq C(\varepsilon_1, \varepsilon_2) \|
\nabla |\mathbf{u}|^2 \|_{L^2}^2.
\]

(3.21)

Case 2: we assume that
\[
\int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^4 \left| \nabla \left( \frac{\mathbf{u}}{|\mathbf{u}|} \right) \right|^2 dx > k(\varepsilon_1, \varepsilon_2) \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx.
\]

(3.22)

It follows from (3.11) that
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} \rho |\mathbf{u}|^4 dx + 4 \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} \left[ |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 + (\lambda + \mu) |\mathbf{u}|^2 (\text{div} \mathbf{u})^2 + \frac{\mu |\nabla |\mathbf{u}|^2|^2}{2} \right] dx \right]
\]
\[
\leq 4 \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} \text{div}(|\mathbf{u}|^2 \mathbf{u}) \mathbf{P} dx - 8(\lambda + \mu) \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} \text{div} \mathbf{u} \mathbf{u} \cdot \nabla |\mathbf{u}| dx + C(\varepsilon_1) \|
\nabla |\mathbf{u}|^2 \|_{L^2}^2
\]
\[
\leq C \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx + 4(\lambda + \mu) \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 |\nabla |\mathbf{u}|^2 dx
\]
\[
+ 4(\lambda + \mu) \int_{\mathbb{R}^3 \cap \{ |\mathbf{u}| > 0 \}} |\mathbf{u}|^2 (\text{div} \mathbf{u})^2 dx + C(\varepsilon_1) \|
\nabla |\mathbf{u}|^2 \|_{L^2}^2.
\]

\footnote{From (1.2) and $3\mu - \lambda > 0$, we have $5\mu + 2\lambda > 0$. Then by (1.2) again one gets $7\mu + 5\lambda > 0$, which combined with (1.2) again implies $9\mu + 8\lambda > 0$. This together with (1.2) once more gives $11\mu + 11\lambda > 0$. Thus the result follows.}
which implies that

\[
\frac{d}{dt} \int \rho |\mathbf{u}|^4 dx + 4\mu(1 - \varepsilon_1) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 dx + 4(\mu - \lambda) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 dx \\
\leq C \int_{\mathbb{R}^3 \cap \{|u| > 0\}} P|\mathbf{u}|^2 |\nabla \mathbf{u}| dx + C(\varepsilon_1) \|\nabla \mathbf{u}\|_{L^2}^2.
\]

(3.23)

Inserting (3.15) into (3.23) yields

\[
\frac{d}{dt} \int \rho |\mathbf{u}|^4 dx + [8\mu - 4(\varepsilon_1 \mu + \lambda)] \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 dx \\
+ 4\mu(1 - \varepsilon_1) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |\mathbf{u}|^4 \left( |\nabla \left( \frac{\mathbf{u}}{|\mathbf{u}|} \right) |^2 \right) dx \\
\leq C \int_{\mathbb{R}^3 \cap \{|u| > 0\}} P|\mathbf{u}|^2 |\nabla \mathbf{u}| dx + C(\varepsilon_1) \|\nabla \mathbf{u}\|_{L^2}^2 \\
+ 4(\mu - \lambda) \varepsilon_3 \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |\mathbf{u}|^4 \left( |\nabla \left( \frac{\mathbf{u}}{|\mathbf{u}|} \right) |^2 \right) dx
\]

with \(\varepsilon_3 \in (0, 1)\). Hence we have

\[
\frac{d}{dt} \int \rho |\mathbf{u}|^4 dx + [8\mu - 4(\lambda + \varepsilon_1 \mu)] \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 dx \\
+ 4\mu(1 - \varepsilon_1)(1 - \varepsilon_3) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |\mathbf{u}|^4 \left( |\nabla \left( \frac{\mathbf{u}}{|\mathbf{u}|} \right) |^2 \right) dx \\
\leq C \int_{\mathbb{R}^3 \cap \{|u| > 0\}} P|\mathbf{u}|^2 |\nabla \mathbf{u}| dx + C(\varepsilon_1, \varepsilon_3) \|\nabla \mathbf{u}\|_{L^2}^2.
\]

This together with (3.22) and (3.11) leads to

\[
\frac{d}{dt} \int \rho |\mathbf{u}|^4 dx + k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 dx \\
+ k_2(\varepsilon_1, \varepsilon_3, \varepsilon_4) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |\mathbf{u}|^4 \left( |\nabla \left( \frac{\mathbf{u}}{|\mathbf{u}|} \right) |^2 \right) dx \\
\leq C \int_{\mathbb{R}^3 \cap \{|u| > 0\}} P|\mathbf{u}|^2 |\nabla \mathbf{u}| dx + C(\varepsilon_1, \varepsilon_3) \|\nabla \mathbf{u}\|_{L^2}^2,
\]

where

\[
k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = 4\mu(1 - \varepsilon_1)(1 - \varepsilon_3)(1 - \varepsilon_4)k(\varepsilon_1, \varepsilon_2) + 8\mu - 4(\lambda + \varepsilon_1 \mu), \\
k_2(\varepsilon_1, \varepsilon_3, \varepsilon_4) = 4\mu(1 - \varepsilon_1)(1 - \varepsilon_3)\varepsilon_4,
\]

with \(\varepsilon_i \in (0, 1), \quad i = 1, 2, 3, 4\).

Since \(k_2(\varepsilon_1, \varepsilon_3, \varepsilon_4) > 0\) for all \((\varepsilon_1, \varepsilon_3, \varepsilon_4) \in (0, 1) \times (0, 1) \times (0, 1)\), we only need to show that there exists \((\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in (0, 1) \times (0, 1) \times (0, 1) \times (0, 1)\) such that

\[
k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) > 0.
\]
In fact, if \( \lambda < 0 \), take \( \varepsilon_1 = -\frac{1}{m\mu} \in (0, 1) \), with the positive integer \( m \) large enough, then we have
\[
\varepsilon_1 \mu + \lambda = \frac{m - 1}{m} \lambda < 0,
\]
which implies that \( k(\varepsilon_1, \varepsilon_2) = 0 \), and hence
\[
k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = 8\mu - 4(\lambda + \varepsilon_1 \mu) > 8\mu > 0.
\]
If \( \lambda = 0 \), then \( \lambda + \varepsilon_1 \mu > 0 \), which implies that
\[
k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \frac{4\mu(1 - \varepsilon_1)(1 - \varepsilon_3)(1 - \varepsilon_4)(3 - \varepsilon_1)\varepsilon_2}{\varepsilon_1} + 8\mu - 4\varepsilon_1 \mu > 4\mu > 0.
\]
If \( 0 < \lambda < 3\mu \), then we have \( \lambda + \varepsilon_1 \mu > 0 \) and then
\[
k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \frac{4\mu^2(1 - \varepsilon_1)(1 - \varepsilon_3)(1 - \varepsilon_4)(3 - \varepsilon_1)\varepsilon_2}{\lambda + \varepsilon_1 \mu} + 8\mu - 4(\lambda + \varepsilon_1 \mu).
\]
Notice that \( k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \) is continuous over \([0, 1] \times [0, 1] \times [0, 1] \times [0, 1]\), and
\[
k_1(0, 1, 0, 0) = \frac{12\mu^2}{\lambda} + 8\mu - 4\lambda = 4\lambda^{-1}(\lambda + \mu)(3\mu - \lambda) > 0,
\]
so there exists \((\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in (0, 1) \times (0, 1) \times (0, 1) \times (0, 1)\) such that
\[
k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) > 0.
\]
So we have
\[
\frac{d}{dt} \int \rho |u|^4 \, dx + k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \int |u|^2 \left| \nabla |u| \right|^2 \, dx
\]
\[
+ k_2(\varepsilon_1, \varepsilon_3, \varepsilon_4) \int |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 \, dx
\]
\[
\leq \frac{k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)}{2} \int |u|^2 \left| \nabla |u| \right|^2 \, dx
\]
\[
+ C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \|u\|_{L^6}^2 \|P\|_{L^3}^2 + C(\varepsilon_1, \varepsilon_3) \|\nabla u\|_{L^2}^2
\]
\[
\leq \frac{k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)}{2} \int |u|^2 \left| \nabla |u| \right|^2 \, dx + C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \|\nabla u\|_{L^2}^2.
\]
Therefore,
\[
\frac{d}{dt} \int \rho |u|^4 \, dx + \frac{k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)}{2} \int |u|^2 \left| \nabla |u| \right|^2 \, dx + k_2(\varepsilon_1, \varepsilon_3, \varepsilon_4) \int |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 \, dx
\]
\[
\leq C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \|\nabla u\|_{L^2}^2.
\]
From (3.21), (3.24), and (3.15), we conclude that if \( 3\mu > \lambda \), there exists a constant \( \bar{C} > 0 \) such that
\[
\frac{d}{dt} \int \rho |u|^4 \, dx + \bar{C} \int |u|^2 \left| \nabla |u| \right|^2 \, dx \leq C \|\nabla u\|_{L^2}^2,
\]
which together with (3.3) and Gronwall’s inequality gives the desired (3.12). 

Let \( E \) be the specific energy defined by
\[
E = \theta + \frac{|u|^2}{2},
\]
Let $G$ be the effective viscous flux, $\omega$ be vorticity given by
\[
G = (\lambda + 2\mu) \text{div } u - \left( P + \frac{|b|^2}{2} \right), \quad \omega = \text{curl } u. \tag{3.26}
\]
Then the momentum equations (1.8) can be rewritten as
\[
\rho \dot{u} - b \cdot \nabla b = \nabla G - \text{curl } \omega, \tag{3.27}
\]
where $\dot{u} \triangleq u_t + u \cdot \nabla u$.

The following lemma gives the estimates on the spatial gradients of both the velocity and the magnetic field, which are crucial for deriving the higher order estimates of the solution.

**Lemma 3.5** Under the condition (3.1), it holds that for any $T \in [0,T^*)$,
\[
\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \int_0^T \left( \int \sqrt{\rho} \left| \nabla u \right|^2 + \|b\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) dt \leq C. \tag{3.28}
\]

**Proof.** Multiplying (1.8) by $u_t$ and integrating the resulting equation over $\mathbb{R}^3$ give rise to
\[
\frac{1}{2} \frac{d}{dt} \int (\rho |\nabla u|^2 + (\lambda + \mu)(\text{div } u)^2) \, dx + \int \rho |\dot{u}|^2 \, dx
\]
\[
= \int \rho \dot{u} \cdot (u \cdot \nabla) u \, dx + \int \left( P + \frac{|b|^2}{2} \right) \text{div } u \, dx - \int (b \otimes b) : \nabla u \, dx
\]
\[
\leq \eta \int \rho |u|^2 \, dx + C(\eta) \int |u|^2 |\nabla u|^2 \, dx + \frac{d}{dt} \int \left( P + \frac{|b|^2}{2} \right) \text{div } u - (b \otimes b) : \nabla u \, dx
\]
\[
- \int \left( P + \frac{|b|^2}{2} \right) \text{div } u - (b \otimes b) : \nabla u - \frac{1}{\lambda + 2\mu} \int \left( P + \frac{|b|^2}{2} \right) G \, dx + \int (b \otimes b) : \nabla u \, dx. \tag{3.29}
\]
It follows from (1.8) that $E$ satisfies
\[
\left( \rho E + \frac{|b|^2}{2} \right)_t + \text{div}(\rho E) = \text{div } H \tag{3.30}
\]
with
\[
H = (u \times b) \times b + \nu(\text{curl } b) \times b + (2\mu \mathcal{D}(u) + \lambda \text{div } u^3) u - Pu.
\]
Then we infer from (3.29), (3.30), and (1.8) that
\[
- \int \left( P + \frac{|b|^2}{2} \right)_t G \, dx = - \int \left( \rho E + \frac{|b|^2}{2} \right)_t G \, dx + \frac{1}{2} \int (\rho |u|^2)_t G \, dx
\]
\[
= \int \text{div}(\rho u E - H) G \, dx + \frac{1}{2} \int \rho_t |u|^2 G \, dx + \int \rho u \cdot u G \, dx
\]
\[
= - \int (\rho u E - H) \cdot \nabla G \, dx - \frac{1}{2} \int \text{div}(\rho u) |u|^2 G \, dx + \int \rho u \cdot u G \, dx
\]
\[
= - \int (Pu - H) \cdot \nabla G + \frac{1}{2} \int \rho u \cdot \nabla (|u|^2) G \, dx + \int \rho u \cdot u G \, dx
\]
\[
\triangleq \sum_{i=1}^3 I_i. \tag{3.31}
\]
From Hölder’s inequality, Sobolev’s inequality, (3.3), and (3.8), we have

\[ I_1 \leq \int (P|u| + |H|)|\nabla G| dx \]
\[ \leq C \int (P|u| + |u||b|^2 + |b||\nabla b| + |u||\nabla u|)|\nabla G| dx \]
\[ \leq \eta_1 ||\nabla G||^2_{L^2} + C(\eta_1) \left( ||P||^2_{L^2} ||u||^2_{L^6} + ||u||^2_{L^6} ||b||^4_{L^6} + ||b||^2_{L^6} ||\nabla b||^2_{L^6} + ||u||||\nabla u||^2_{L^6} \right) \]
\[ \leq \eta_1 ||\nabla G||^2_{L^2} + C(\eta_1) \left( ||\nabla u||^2_{L^4} + ||\nabla b||^2_{H^1} + ||u||||\nabla u||^2_{L^6} \right). \] (3.32)

By (3.20), (3.22), (3.23), (3.25), Hölder’s inequality, and Sobolev’s inequality, one gets

\[ I_2 \leq \int \rho|u|^2||\nabla u||G| dx \]
\[ \leq C \int \rho|u|^2||\nabla u||(|\nabla u| + |b|^2 + P) dx \]
\[ \leq C \int (|u|^2||\nabla u|^2 + |u|^2||\nabla u||b|^2 + \rho|u|^2||\nabla u||) dx \]
\[ \leq C(||u||||\nabla u||^2_{L^2} + ||u||||\nabla u||||u||||\nabla u||^2_{L^2} + ||u||||\nabla u||||u||||u||||\nabla u||_{L^6} \]
\[ \leq C(||u||||\nabla u||^2_{L^2} + C||\nabla u||^2_{L^6} + ||u||||\nabla u||^2_{L^2}). \] (3.33)

Similarly to \( I_2 \), we find that

\[ I_3 = \int \left[ \rho u \cdot \dot{u} - \rho u \cdot (u \cdot \nabla u) \right] G dx \]
\[ \leq C \int \rho(|u||u| + |u|^2||\nabla u||)(|\nabla u| + |b|^2 + P) dx \]
\[ \leq C(||\sqrt{\rho}||u||^2_{L^2} + ||u||||\nabla u||^2_{L^2} + ||\sqrt{\rho}||u||^2_{L^6} + ||u||||\nabla u||^2_{L^2} + ||u||||\nabla u||||u||||\nabla u||^2_{L^6} + ||u||||\nabla u||||u||||u||^2_{L^6} \]
\[ \leq \eta_1 ||\sqrt{\rho}||^2_{L^2} + C(\eta_1)(||u||||\nabla u||^2_{L^2} + ||u||||\nabla u||^2_{L^6}). \] (3.34)

Inserting (3.32), (3.33) into (3.31), we arrive at

\[ - \int \left( P + \frac{|b|^2}{2} \right)_t G dx \leq \eta_1 ||\nabla G||^2_{L^2} + \eta_1 ||\sqrt{\rho}||^2_{L^2} \]
\[ + C(\eta_1)(||\nabla u||^2_{L^2} + ||\nabla u||^2_{H^1} + ||u||||\nabla u||^2_{L^6}). \] (3.35)

In view of (3.20), we obtain that

\[ \Delta G = \text{div}(\rho \dot{u} - b \cdot \nabla b). \]

Then from the standard elliptic estimates, (3.22), and (3.8), we deduce that

\[ ||\nabla G||^2_{L^2} \]
\[ \leq C(||\rho u||^2_{L^2} + ||b \cdot \nabla b||^2_{L^2}) \]
\[ \leq C(||\sqrt{\rho}||u||^2_{L^2} + ||b||^2_{L^6}||\nabla b||^2_{L^6}) \]
\[ \leq C(||\sqrt{\rho}||^2_{L^2} + ||\nabla b||^2_{H^1}), \] (3.36)

which combined with (3.35) implies that

\[ - \int \left( P + \frac{|b|^2}{2} \right)_t G dx \leq C \eta_1 ||\sqrt{\rho}||^2_{L^2} + C(\eta_1)(||\nabla u||^2_{L^2} + ||\nabla b||^2_{H^1} + ||u||||\nabla u||^2_{L^6}). \] (3.37)

For the last term on the right-hand side of (3.20), we obtain from Hölder’s inequality and (3.8) that

\[ \int (b \otimes b) : \nabla u dx \leq C \int |b||b||\nabla u| dx \]
\[ \leq C\|b_t\|_{L^2}\|b\|_{L^\infty}\|\nabla u\|_{L^2} \]
\[ \leq \tilde{\eta}\|b_t\|_{L^2}^2 + C(\tilde{\eta})\|\nabla u\|_{L^2}^2. \]  

(3.38)

Inserting (3.37) and (3.38) into (3.29) and choosing \( \eta_1 \) suitably small, we have

\[ \frac{d}{dt} \int \Phi dx + \|\sqrt{\rho}u\|_{L^2}^2 \leq \tilde{\eta}\|b_t\|_{L^2}^2 + C_2\|\nabla^2 b\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|u\|\nabla u\|_{L^2}^2), \]  

(3.39)

where

\[ \Phi = \mu\|\nabla u\|^2 + \mu(\lambda + \lambda)(\nabla u)^2 - (2P + |b|^2) \nabla u + 2(b \otimes b) : \nabla u + \frac{(P + \frac{|b|^2}{2})^2}{\lambda + 2\mu} \]

satisfies

\[ \frac{\mu}{2}\|\nabla u\|_{L^2}^2 - C \leq \int \Phi dx \leq \mu\|\nabla u\|_{L^2}^2 + C \]  

deue to (3.33) and (3.8).

It follows from (1.8), Hölder’s inequality, and (3.8) that

\[ \nu \frac{d}{dt} \int |\nabla b|^2 \, dx + \int |b_t|^2 \, dx + \nu^2 \int |\Delta b|^2 \, dx \]
\[ = \int |b_t - \nu \Delta b|^2 \, dx \]
\[ = \int |b \cdot \nabla u - u \cdot \nabla b - b \div u|^2 \, dx \]
\[ \leq C \int |b|^2 |\nabla u|^2 \, dx + C \int |u|^2 |\nabla b|^2 \, dx \]
\[ \leq C \|b\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 + C \|u\|_{L^6}^3 \|\nabla b\|_{L^3}^2 \]  
\[ \leq C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla b\|_{L^2} \|\nabla b\|_{L^2} \]  
\[ \leq \eta_2 \|\nabla b\|_{L^2}^2 + C(\eta_2)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + 1). \]  

(3.41)

Noting that the standard \( L^2 \) estimate of elliptic system gives

\[ \|\nabla b\|_{L^2} \leq C_3 \|\Delta b\|_{L^2}, \]

hence we deduce after choosing \( \eta_2 \) suitably small that

\[ 2\nu \frac{d}{dt} \int |\nabla b|^2 \, dx + 2\|b_t\|_{L^2}^2 + C_3^{-1}\nu^2 \|\nabla^2 b\|_{L^2}^2 \]
\[ \leq C(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + 1). \]  

(3.42)

Then adding (3.42) to (3.39) and choosing \( \tilde{\eta} \) small enough, we have

\[ \frac{d}{dt} \int (\Phi + 2C_4 \nu|\nabla b|^2) \, dx + \frac{1}{2}(\|\sqrt{\rho}u\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2) \]
\[ \leq C(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + 1) \]
\[ + C(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|u\|\nabla u\|_{L^2}^2). \]

Then we obtain the desired (3.28) after using Gronwall’s inequality, (3.3), (3.12), and (3.40). This completes the proof of Lemma 3.5. \( \square \)

Next, we have the following estimates on the material derivatives of the velocity which are important for the higher order estimates of strong solutions.
Lemma 3.6 Under the condition (3.1), it holds that for any $T \in [0, T^*)$,
\[
\sup_{0 \leq t \leq T} (\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2) + \int_0^T (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla b_t\|_{L^2}^2) \, dt \leq C. \tag{3.43}
\]

Proof. By the definition of $\dot{u}$, we can rewrite (3.8) as follows:
\[
\rho \dot{u} + \nabla P = \mu \Delta u + (\lambda + \mu) \nabla \text{div} u + \text{curl} b \times b. \tag{3.44}
\]
Differentiating (3.44) with respect to $t$ and using (3.48), we have
\[
\rho \dot{u}_t + \rho u \cdot \nabla \dot{u} + \nabla P_t = \mu \Delta \dot{u} + (\lambda + \mu) \nabla \text{div} \dot{u} - \mu \Delta (u \cdot \nabla u) - (\lambda + \mu) \text{div}(u \cdot \nabla u) + \text{div}(\rho \dot{u} \otimes u).
\]
Multiplying (3.45) by $\dot{u}$ and integrating by parts over $\mathbb{R}^3$, we get
\[
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 \, dx + \mu \int |\nabla \dot{u}|^2 \, dx + (\lambda + \mu) \int |\nabla u|^2 \, dx
= \int P_t \div \dot{u} + (\nabla P \otimes u) : \nabla \dot{u} \, dx - \int [\text{div}(\text{curl} b \times b) \otimes u - (\text{curl} b \times b)_t] \cdot \dot{u} \, dx
+ \mu \int [\text{div}(\Delta u \otimes u) - \Delta (u \cdot \nabla u)] \cdot \dot{u} \, dx + (\lambda + \mu) \int [(\nabla \text{div} u) \otimes u - \nabla \text{div}(u \cdot \nabla u)] \cdot \dot{u} \, dx
\leq \sum_{i=1}^4 J_i, \tag{3.46}
\]
where $J_i$ can be bounded as follows.

It follows from (3.8) that
\[
J_1 = \int (- \text{div}(Pu) \div \dot{u} - P \text{div} u \div \dot{u} + T(u) : \nabla u \div \dot{u} + \nu |\text{curl} b|^2 \div \dot{u} + (\nabla P \otimes u) : \nabla \dot{u}) \, dx
= \int (Pu \nabla \div \dot{u} - P \text{div} u \div \dot{u} + T(u) : \nabla u \div \dot{u} + \nu |\text{curl} b|^2 \div \dot{u}) \, dx
- \int (P \nabla u^\top : \nabla u + Pu \nabla \div \dot{u}) \, dx
= \int (-P \text{div} u \div \dot{u} + T(u) : \nabla u \div \dot{u} + \nu |\text{curl} b|^2 \div \dot{u} - P \nabla u^\top : \nabla \dot{u}) \, dx
\leq C \int (|\nabla u| |\nabla \dot{u}| + |\nabla u|^2 |\nabla \dot{u}| + |\nabla b|^2 |\nabla \dot{u}|) \, dx
\leq C(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^4}^2 + \|\nabla b\|_{L^4}^2) \|\nabla u\|_{L^2}, \tag{3.47}
\]
where $T(u) = 2\mu \mathcal{Q}(u) + \lambda \text{div} u 3$. Integrating by parts leads to
\[
J_2 = \int \left[ \text{div}(b \otimes b)_t - \nabla \left( \frac{|b|^2}{2} \right) - \text{div}(\text{curl} b \times b) \otimes u \right] \cdot \dot{u} \, dx
\leq C \int (|b| |b_t| + |b| |\nabla b| |u|) |\nabla \dot{u}| \, dx
\leq C(\|b\|_{L^6} \|b_t\|_{L^3} + \|b\|_{L^6} \|\nabla b\|_{L^6} \|u\|_{L^6}) \|\nabla \dot{u}\|_{L^2}
\leq C(\|b_t\|^2_{L^2} \|\nabla b\|_{L^2}^2 + \|\nabla b\|_{H^1}^2) \|\nabla \dot{u}\|_{L^2}. \tag{3.48}
\]
For $J_3$ and $J_4$, notice that for all $1 \leq i, j, k \leq 3$, one has
\[
\partial_j(\partial_{kk} u_i u_j) - \partial_{kk}(u_j \partial_{ij} u_i) = \partial_k(\partial_{ij} u_j \partial_{ki} u_i) - \partial_k(\partial_{k} u_j \partial_{ij} u_i) - \partial_{ij}(\partial_{ki} u_j \partial_{kj} u_i),
\]
\[ \partial_j (\partial_k u_k u_j) - \partial_j (u_k \partial_k u_j) = \partial_t (\partial_j u_j \partial_k u_k) - \partial_t (\partial_j u_k \partial_k u_j) - \partial_t (\partial_j u_k \partial_j u_j). \]

So integrating by parts gives
\[
J_3 = \mu \int [\partial_t (\partial_j u_j \partial_k u_k) - \partial_t (\partial_j u_k \partial_k u_j) - \partial_t (\partial_j u_k \partial_j u_j)] u_t \, dx \\
\leq C \| \nabla u \|_{L^2}^2 \| \nabla \dot{u} \|_{L^2}, \tag{3.49}
\]
\[
J_4 = (\lambda + \mu) \int [\partial_t (\partial_j u_j \partial_k u_k) - \partial_t (\partial_j u_k \partial_k u_j) - \partial_t (\partial_j u_k \partial_j u_j)] u_t \, dx \\
\leq C \| \nabla u \|_{L^2}^2 \| \nabla \dot{u} \|_{L^2}. \tag{3.50}
\]

Inserting (3.47)–(3.50) into (3.46) and applying (3.28) lead to
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \mu \| \nabla \dot{u} \|_{L^2}^2 + (\lambda + \mu) \| \text{div} \dot{u} \|_{L^2}^2 \\
\leq C (\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 + \| \partial_t \|_{L^2}^2 + \| \nabla b \|_{H^1}^2 ) \| \nabla u \|_{L^2} \\
\leq \delta_1 \| \nabla u \|_{L^2}^2 + \delta_2 \| \nabla b \|_{L^2}^2 + C (\delta_1, \delta_2) (\| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 + \| \nabla^2 b \|_{L^2}^2 + 1). \tag{3.51}
\]

From (1.8), the standard regularity estimate of elliptic equations to (1.8), (3.5), and (3.28), we get
\[
\| \nabla^2 b \|_{L^2}^2 \leq C (\| b_t \|_{L^2}^2 + \| u \| \| \nabla b \|_{L^2}^2 + \| b \| \| \nabla u \|_{L^2}^2 ) \\
\leq C (\| b_t \|_{L^2}^2 + \| u \|_{L^2}^2 \| \nabla b \|_{L^2}^2 + \| b \|_{L^2}^2 \| \nabla u \|_{L^2}^2 ) \\
\leq C (\| b_t \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \| \nabla b \|_{L^2}^2 \| \nabla b \|_{H^1} + 1 ) \\
\leq \frac{1}{2} \| \nabla^2 b \|_{L^2}^2 + C \| b_t \|_{L^2}^2 + C,
\]
which implies that
\[
\| \nabla^2 b \|_{L^2}^2 \leq C \| b_t \|_{L^2}^2 + C. \tag{3.52}
\]

Differentiating (1.8) with respect to \( t \), we have
\[
b_{tt} - \nu \Delta b_t = b_t \cdot \nabla u - u \cdot \nabla b_t - b_t \text{div} u + b \cdot \nabla u_t - u_t \cdot \nabla b - b \text{div} u_t. \tag{3.53}
\]

Multiplying (3.53) by \( b_t \) and integrating by parts lead to
\[
\frac{1}{2} \frac{d}{dt} \int |b_t|^2 \, dx + \nu \int |\nabla b_t|^2 \, dx = \int (b_t \cdot \nabla u - u \cdot \nabla b_t - b_t \text{div} u) \cdot b_t \, dx \\
+ \int (b \cdot \nabla u_t - u_t \cdot \nabla b - b \text{div} u_t) \cdot b_t \, dx \\
\triangleq L_1 + L_2. \tag{3.54}
\]

Integrating by parts implies that
\[
L_1 = \int \left( b_t \cdot \nabla u \cdot b_t - \frac{1}{2} |b_t|^2 \text{div} u \right) \, dx \\
\leq C \| b_t \|_{L^2}^2 \| \nabla u \|_{L^2}^2 \\
\leq C \| b_t \|_{L^2}^2 \| \nabla b_t \|_{L^2}^2 \\
\leq \delta_1 \| \nabla b_t \|_{L^2}^2 + C (\delta_1) \| b_t \|_{L^2}^2, \tag{3.55}
\]
and
\[
L_2 = \int (b \cdot \nabla u - u \cdot \nabla b - b \text{div} u) \cdot b_t \, dx \tag{3.56}
\]
\[-\int (b \cdot \nabla (u \cdot \nabla u) - (u \cdot \nabla u) \cdot \nabla b - b \text{ div}(u \cdot \nabla u)) \cdot b_t\]
\[= \int (b \cdot \nabla \dot{u} - \dot{u} \cdot \nabla b - b \text{ div} \dot{u}) \cdot b_t \, dx\]
\[+ \int (u \cdot \nabla u) \cdot (b \cdot \nabla b_t) + (u \cdot \nabla u) \cdot \nabla b_t \cdot b_t \, dx\]
\[\leq C \int |b||b_t||\nabla \dot{u}| + |\dot{u}||\nabla b||b_t| + |u||\nabla u||b||\nabla b_t| \, dx\]
\[\leq C(\||b||L^6||b_t||L^L||\nabla \dot{u}||L^2 + ||\dot{u}||L^6||\nabla b||L^2||b_t||L^3 + ||u||L^6||\nabla u||L^6||b||L^6||\nabla b_t||L^2)\]
\[\leq C(\||b||L^6||\nabla \dot{u}||L^2 + ||\nabla u||L^6||\nabla b_t||L^2)\]
\[\leq \delta_1||\nabla b||^2_{L^2} + \delta_2||\nabla \dot{u}||^2_{L^2} + C(\delta_1, \delta_2)||b_t||_{L^2} + C(\delta_1)||\nabla u||^2_{L^6}. \quad (3.56)\]

Inserting (3.55) and (3.56) into (3.54), we have
\[\frac{1}{2} \frac{d}{dt}||b_t||^2_{L^2} + \nu||\nabla b_t||^2_{L^2} \leq 2\delta_1||\nabla b_t||^2_{L^2} + \delta_2||\nabla \dot{u}||^2_{L^2} + C(\delta_1, \delta_2)||b_t||_{L^2} + C(\delta_1)||\nabla u||^2_{L^6}. \quad (3.57)\]

Adding (3.57) to (3.51) and applying (3.52), we obtain after choosing \(\delta_1, \delta_2\) suitably small that
\[\frac{d}{dt} \left( ||\sqrt{\rho} \dot{u}||^2_{L^2} + ||b_t||^2_{L^2} \right) + \dot{C} \left( ||\nabla \dot{u}||^2_{L^2} + ||\nabla b_t||^2_{L^2} \right)\]
\[\leq C(\||b||^2_{L^2} + ||\nabla u||^2_{L^6} + ||\nabla b||^2_{L^4} + ||\nabla u||^2_{L^6} + 1). \quad (3.58)\]

To estimate \(||\nabla u||_{L^6}\), let \(u = v + w\) such that
\[\left\{ \begin{array}{l}
\mu \Delta v + (\lambda + \mu) \nabla \text{ div } v = \nabla \left( P + \frac{|b|^2}{2} \right), \\
v(x, t) \to 0, \quad \text{as } |x| \to +\infty;
\end{array} \right.\]
and
\[\left\{ \begin{array}{l}
\mu \Delta w + (\lambda + \mu) \nabla \text{ div } w = \rho \dot{u} - b \cdot \nabla b, \\
w(x, t) \to 0, \quad \text{as } |x| \to +\infty,
\end{array} \right.\]
which implies that
\[||\nabla v||_{L^6} \leq C \left| P + \frac{|b|^2}{2} \right|_{L^6} \leq C,\]
and
\[||\nabla w||_{L^6} + ||\nabla^2 w||_{L^2} \leq C(||\rho \dot{u}||_{L^2} + ||b \cdot \nabla b||_{L^2}) \leq C(||\sqrt{\rho} \dot{u}||_{L^2} + ||\nabla b||_{H^1}).\]

Then we have
\[||\nabla u||_{L^6}^2 \leq ||\nabla v||_{L^6}^2 + ||\nabla w||_{L^6}^2 \leq C(||\sqrt{\rho} \dot{u}||_{L^2} + ||\nabla b||_{H^1}) + C \quad (3.59)\]

By (3.59), (3.28), and (3.52), one has
\[||\nabla u||_{L^6}^2 \leq C(||\sqrt{\rho} \dot{u}||_{L^2}^3 + ||\nabla w||_{H^1}^2 + 1) \leq C(||\sqrt{\rho} \dot{u}||_{L^2}^3 + ||b_t||_{L^2}^3 + 1). \quad (3.60)\]

It follows from Hölder’s inequality, (3.28), and (3.60) that
\[||\nabla u||_{L^4}^4 \leq C||\nabla u||_{L^2}||\nabla u||_{L^6}^3 \leq C(||\sqrt{\rho} \dot{u}||_{L^2}^3 + ||b_t||_{L^2}^3 + 1)\]
\[\leq C \left( 1 + ||\sqrt{\rho} \dot{u}||_{L^2}^3 + ||b_t||_{L^2}^3 \left( ||\sqrt{\rho} \dot{u}||_{L^2}^3 + ||b_t||_{L^2}^3 \right) \right. + C. \quad (3.61)\]

Similarly, we get
\[||\nabla b||_{L^2}^4 \leq C||\nabla b||_{L^2}||\nabla b||_{H^1}^3\]
for some $\alpha$

For some $\beta$

By virtue of Gagliardo-Nirenberg inequality, Sobolev’s inequality, (3.28), and (3.65), we arrive at which combined with Gagliardo-Nirenberg inequality implies

It follows from (3.59), (3.43), and (3.64) that

Applying the standard $L^p$-estimate of elliptic system to (3.27), (3.2), and (3.64) yield

Similarly,

Applying the standard $L^p$-estimate of elliptic system to (3.27), (3.2), and (3.64) yield

which combined with Gagliardo-Nirenberg inequality implies

for some $\beta \in (0, 1)$.

Employing the standard $L^p$-estimate of elliptic system to (1.8) leads to

for some $\alpha \in (0, 1)$. This together with Lemma 3.6 gives

\[
\|\nabla u\|_{L^\infty} \leq C \left(1 + \|\nabla \hat{u}\|_{L^2}^{1-\beta}\right) \log (e + \|\nabla \hat{u}\|_{L^2} + \|\nabla P\|_{L^p}) + C \|\nabla \hat{u}\|_{L^2}.
\]
Applying the standard $L^p$-estimate to (3.74) yields
\[
\|\nabla^2 b\|_{L^p} \leq C \left(\|b_t\|_{L^p} + \|u\|\|\nabla b\|_{L^p} + \|b\|\|\nabla u\|_{L^p}\right)
\]
\[
\leq C \left(\|b_t\|_{L^p} \frac{6-p}{2p} \|\nabla b\|_{L^p \frac{3p-6}{p}} + \|u\|_{L^\infty} \|\nabla b\|_{L^p} + \|b\|_{L^\infty} \|\nabla u\|_{L^p}\right)
\]
\[
\leq C \left(1 + \|\nabla b\|_{L^p \frac{3p-6}{p}}\right).
\]
(3.71)

It follows from Lemma 2.2 that
\[
\|\nabla b\|_{L^\infty} \leq C(\|\nabla^2 b\|_{L^p} + 1).
\]
(3.72)

Substituting (3.71) and (3.72) into (3.67)–(3.68) yields that
\[
f'(t) = Cg(t)f(t)\log f(t) + Cg(t)f(t) + Cg(t),
\]
where
\[
f(t) \triangleq e + \|\nabla \rho\|_{L^p} + \|\nabla P\|_{L^p},
g(t) \triangleq (1 + \|\nabla u\|_{L^2}) \log(e + \|\nabla u\|_{L^2}) + \|\nabla b\|_{L^2}^2.
\]

This yields
\[
(\log f(t))' \leq Cg(t) + Cg(t) \log f(t)
\]
due to $f(t) > 1$. Thus it follows from (3.74), (3.43), and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \|\nabla \rho, \nabla P\|_{L^p} \leq C,
\]
(3.75)

which, combined with (3.70) and (3.43) gives that
\[
\int_0^T \|\nabla u\|_{L^\infty}^2 dt \leq C.
\]
(3.76)

Taking $p = 2$ in (3.74), one can get by using (3.70), (3.69) and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \|\nabla \rho, \nabla P\|_{L^2} \leq C,
\]
(3.77)

which together with (3.69) yields that
\[
\sup_{0 \leq t \leq T} \|\nabla^2 u\|_{L^2} \leq C.
\]

This combined with (3.75), (3.77), (3.28), and (3.64) finishes the proof of Lemma 3.7.

With Lemmas 3.1–3.7 at hand, we are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We argue by contradiction. Suppose that (1.13) were false, that is, (3.11) holds. Note that the general constant $C$ in Lemmas 3.1–3.7 is independent of $t < T^*$, that is, all the a priori estimates obtained in Lemmas 3.1–3.7 are uniformly bounded for any $t < T^*$. Hence, the function
\[
(\rho, u, P, b)(x, t^*) \triangleq \lim_{t \to T^*} (\rho, u, P, b)(x, t)
\]
satisfy the initial condition (1.11) at $t = T^*$.

Furthermore, standard arguments yield that $\rho \dot{u} \in C([0, T]; L^2)$, which implies
\[
\rho \dot{u}(x, T^*) = \lim_{t \to T^*} \rho \dot{u}(x, t) \in L^2.
\]
Hence,
\[-\mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla P - \text{curl} \times b \big|_{t=T^*} = \sqrt{\rho(x, T^*)}g(x)\]
with
\[
g(x) \triangleq \begin{cases} 
\rho^{-1/2}(x, T^*)(\rho \dot{u})(x, T^*), & \text{for } x \in \{x|\rho(x, T^*) > 0\}, \\
0, & \text{for } x \in \{x|\rho(x, T^*) = 0\},
\end{cases}
\]
satisfying \(g \in L^2\) due to (3.63). Therefore, one can take \((\rho, u, P, b)(x, T^*)\) as the initial data and extend the local strong solution beyond \(T^*\). This contradicts the assumption on \(T^*\).

Thus we finish the proof of Theorem 1.1. \(\square\)

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