Consistent Hashing with Bounded Loads

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Abstract

In dynamic load balancing, we wish to allocate a set of clients (balls) to a set of servers (bins) with the goal of minimizing the maximum load of any server and also minimizing the number of moves after adding or removing a server or a client. We want a hashing-style solution where we given the ID of a client can efficiently find its server in a distributed dynamic environment. In such a dynamic environment, both servers and clients may be added and/or removed from the system in any order. The most popular solutions for such dynamic settings are Consistent Hashing [KLL+97, SML+03] or Rendezvous Hashing [TR98]. However, the load balancing of these schemes is no better than a random assignment of clients to servers, so with $n$ of each, we expect many servers to be overloaded with $\Theta(\log n/\log \log n)$ clients. In this paper, we aim to design hashing schemes that achieve any desirable level of load balancing, while minimizing the number of movements under any addition or removal of servers or clients.

In particular, we consider a problem with $m$ balls and $n$ bins, and given a user-specified balancing parameter $c = 1 + \varepsilon > 1$, we aim to find a hashing scheme with no load above $\lceil cm/n \rceil$, referred to as the capacity of the bins. Our algorithmic starting point is the consistent hashing scheme where current balls and bins are hashed to a unit cycle, and a ball is placed in the first bin succeeding it in clock-wise order. In order to cope with given capacity constraints, we apply the idea of linear probing by forwarding the ball on the circle to the first non-full bin. We show that in our hashing scheme when a ball or bin is inserted or deleted, the expected number of balls that have to be moved is within a multiplicative factor of $O\left(\frac{1}{\varepsilon^2}\right)$ of the optimum for $\varepsilon \leq 1$ (Theorem 3) and within a factor $1 + O\left(\frac{\log c}{c}\right)$ of the optimum for $\varepsilon \geq 1$ (Theorem 2). Technically, the latter bound is the most challenging to prove. It implies that we for superconstant $c$, we only pay a negligible cost in extra moves. We also get the same bounds for the simpler problem where we instead of a user-specified balancing parameter have a fixed bin capacity $C$ for all bins, and define $c = 1 + \varepsilon = Cn/m$. 

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$^1$Throughout this paper, we use balls as clients, and also bins as servers, interchangeably.
1 Introduction

Load balancing in dynamic environments is a central problem in designing several networking systems and web services \cite{SML03,KLL97}. We wish to allocate clients (also referred to as balls) to servers (also referred to as bins) in such a way that none of the servers gets overloaded. Here, the load of a server is the number of clients allocated to it. We want a hashing-style solution where we given the ID of a client can efficiently find its server. Both clients and servers may be added or removed in any order, and with such changes, we do not want to move too many clients. Thus, while the dynamic allocation algorithm has to always ensure a proper load balancing, it should aim to minimize the number of clients moved after each change to the system. For every update in the system, we need to change the allocation of clients to servers. For simplicity, we assume that the updates (ball and bin additions and removals) do not happen simultaneously and will be operated one at a time, so that we have time to finish changing the allocation before we get another update.

Note that such allocation problems become even more challenging when we face hard constraints in the capacity of each server, that is, each server has a capacity and the load may not exceed this capacity. Typically, we want capacities close to the average loads.

There is a vast literature on solutions in the much simpler case where the set of servers is fixed and only the client set is updated, and we shall return to some of these in Section 1.4. For now, we focus on solutions that are known to work in our fully-dynamic case where both clients and servers can be added and removed in an arbitrary order. This also rules out solutions where only the last added server may be removed\footnote{This rules out the external memory techniques \cite{Lar88} where blocks (playing the role of fixed capacity servers) can only be added to and removed from the top of the current memory.}

The above problem formulation is very general, and does not assume anything about the ratio between the number \(m\) of clients and the number \(n\) of servers, e.g., processors are cheap, so one can imagine systems with a large number of servers, but there could also be many clients. One could also imagine a balanced system in which \(m = n\).

The classic solution to the scenario where both clients and servers can be added and removed is Consistent Hashing \cite{SML03,KLL97} where the current clients are assigned in a random way to the current servers. While consistent hashing schemes minimize the expected number of movements, they may result in hugely overloaded servers, and they do not allow for explicit capacity constraints on the servers. The basic point is that the load balancing of consistent hashing \cite{KLL97,SML03} is no better than a random assignment of clients to servers. The same issue holds for Rendezvous or Highest Random Weight Hashing \cite{TR98}. Hence, with \(m\) clients and \(n\) servers, we expect a good load balancing if \(m/n = \Omega(\log n)\), but the balance is lost with smaller loads, e.g., with \(m = n\), we expect many servers to be overloaded with \(\Theta(\log n/\log \log n)\) clients. In this paper, with \(n\) clients and \(n\) servers, we get a guaranteed max-load of 2, while only moving an expected constant number of clients each time a client or server is added or removed. Our general result is described below.

In this paper, we present an algorithm that works with arbitrary capacity constraints on the servers. For the purpose of load balancing, the system designer can specify a balancing parameter \(c = 1 + \varepsilon\), guaranteeing that the maximum load is at most \(\lceil cm/n \rceil\). While maintaining this hard balancing constraint, we limit the expected number of clients to be moved when clients or servers are inserted or removed.

Even without capacity constraints, the obvious general lower bounds for moves are as follows. When a client is added or removed, at least we have to move that client. When a server is added or removed, at least

\[ \frac{m}{n} = \Omega(\log n) \]

...
we have to move the clients belonging to it. On the average, we therefore have to move least \( \frac{m}{n} \) clients when a server is added or removed.

With our solution, while guaranteeing a balancing parameter \( c = 1 + \varepsilon \leq 2 \), when a client is added or removed, the expected number of clients moved is \( O(\frac{1}{\varepsilon^2}) \). When a server is added or removed, the expected number of clients moved is \( O(\frac{m}{\varepsilon n}) \) (Theorem 3). These numbers are only a factor \( O(\frac{1}{\varepsilon^2}) \) worse than the general lower bounds without capacity constraints. For balancing parameter \( c \geq 2 \), our expected number of moves is increased by a factor \( 1 + O(\frac{\log c}{c}) \) over the lower bounds (Theorem 2). The bound for \( c \geq 2 \) is the most challenging to prove. It implies that for superconstant \( c \), we only expect to pay a negligible cost in extra moves.

From a more practical perspective, it is useful that our algorithm provides a simple knob, the load balancing parameter \( c = 1 + \varepsilon \), which captures the tradeoff between load balancing and stability upon changes in the system. This gives a more direct control to the system designer in meeting explicit balancing constraints.

We get the same bounds for the simpler problem where we instead of a user specified balancing parameter have a fixed bin capacity \( C \) for all bins and define \( c = 1 + \varepsilon = Cn/m \). An interesting case of light loads is the unit capacity case where each server can only handle a single client at the time.

**Applications.** Consistent hashing has found numerous applications [OV11, GF04] and early work in this area [KLL+97, SMK+01, SML+03] has been cited more than ten thousand times. To highlight the wide variety of areas in which similar allocation problems might arise, we just mention a few more important references: content-addressable networks [RFH+01], peer-to-peer systems and their associated multicast applications [RD01, CDKR02]. Our algorithm is very similar to consistent hashing, and should work for most of the same applications, bounding the loads whenever this is desired. In fact, our work has already found two quite different industrial applications; namely Google’s cloud system [MTZ16] and Vimeo’s video streaming [Rod16]. Both systems had to handle the lightly loaded case. Also, in both cases, load balancing was not an objective to maximize, but rather a hard constraint, e.g., in the Vimeo blog post [Rod16]. Rodland describes how no server is allowed to be overloaded, and how he found a load balancing parameter \( c = 1.25 \) to be satisfactory for Vimeo’s video streaming. We shall return to this later.

### 1.1 Background: Consistent hashing

The standard solution to our fully-dynamic allocation problem is consistent hashing [SML+03, KLL+97].

**Simple consistent hashing.** In the simplest version of consistent hashing, we hash the active balls and bins onto a unit circle, that is, we hash to the unit interval, using the hash values to create a circular order of balls and bins. Assuming no collisions, a ball is placed in the bin succeeding it in the clockwise order around the circle. One of the nice features of consistent hashing is that it is history-independent, that is, we only need to know the IDs of the balls and the bins and the hash functions, to compute the distribution of balls in bins. If a bin is closed, we just move its balls to the succeeding bin. Similarly, when we open a new bin, we only have to consider the balls from the succeeding bin to see which ones belong in the new bin.

With \( m \) balls, \( n \) bins, and a fully random hash function \( h \), each bin is expected to have \( m/n \) balls. This is also the number of balls we expect to move when a bin is opened or closed.

One problem with simple consistent hashing as described above is that the maximum load is likely to be \( \Theta(\log n) \) times bigger than the average. This has to do with a big variation in the coverage of the bins. We say that bin \( b \) covers the interval of the cycle from the preceding bin \( b' \) to \( b \) because all balls hashing to this
interval land in \( b \). When \( n \) bins are placed randomly on the unit cycle, on the average, each bin covers an interval of size \( 1/n \), but we expect some bins to cover intervals of size \( \Theta(\log n/n) \), and such bins are expected to get \( \Theta(m\log n) \) balls. The maximum load is thus expected to be a factor \( \Theta(\log n) \) above the average.

A related issue is that the expected number of balls landing in the same bin as any given ball is almost twice the average. More precisely, consider a particular ball \( x \). Its expected distance to the neighboring bin on either side is exactly \( 1/(n+1) \), so the expected size of the interval between these two neighbors is \( 2/(n+1) \). All balls landing in this interval will end in the same bin as \( x \); namely the bin \( b \) succeeding \( x \). Therefore we expect \( 2(m-1)/(n+1) \approx 2m/n \) other balls to land with \( x \) in \( b \). Thus each ball is expected to land in a bin with load almost twice the average. If the load determines how efficiently a server can serve a client, the expected performance is then only half what it should be.

In [KLL+97] they addressed the above issue using virtual bins as described below.

**Uniform bin covers.** To get a more uniform bin cover, [KLL+97] suggests the use of virtual bins. The virtual bin trick is that the ball contents of \( d = O(\log n) \) bins is united in a single super-bin. The \( d \) bins making up a super bin are called virtual bins. We have only \( n' = n/d \) super bins and these super bins represent the servers. A super bin covers the intervals covered by its \( d \) virtual bins. The point is that for any constant \( \varepsilon > 0 \), if we pick a large enough \( d = O(\log n) \), then with high probability, each super bin covers a fraction \((1 \pm \varepsilon)/n'\) of the unit cycle.

We note that many other methods have been proposed to maintain such a uniform bin cover as bins are added and removed (see, e.g., [BSS00, GH05, Man04, KM05, KR06]). Different implementations have different advantages depending on the computational model, but for the purpose of our discussion below, it does not matter much which of these methods is used.

We also note that what corresponds to a perfectly uniform bin cover can be obtained using Rendezvous or Highest Random Weight Hashing [TR98]. A hash function assigns a random weight \( h(q, b) \) to each ball \( q \) and bin \( b \). A ball \( q \) is placed in the current bin \( b \) that maximizes \( h(q, b) \). If \( h \) is min-wise independent [BCFM00], then \( b \) is uniformly random among the current bins, corresponding to a perfectly uniform bin cover. An issue with this scheme is that to place the ball \( q \), we need to know all the current bins.

With a uniform bin cover, balls distribute uniformly between bins. However, with \( n \) balls and \( n \) bins, we still expect many bins with \( \Theta((\log n)/(\log \log n)) \) balls even though the average is 1. On the positive side, in the heavily loaded case when \( m/n \) is large, e.g., \( m/n = \omega(\log n) \), all loads are \((1 \pm \varepsilon)m/n\) w.h.p. However, in this paper, we want a good load balancing for all possible load levels.

### 1.2 Our algorithmic solution: Respecting bin capacities via forwarding.

We describe our full algorithm in this subsection and provide some intuition behind its logic. An important part of our solution to the load balancing problem is to introduce a capacity for each bin that may not be exceeded by the load of the bin. In general, such a capacity could be a direct result of a resource constraint on the bin, but here we will use it to enforce a strict load balancing, guaranteeing for some balancing parameter \( c > 1 \), that no bin has more than \( \lceil cm/n \rceil \) balls.

In the description below, we will first describe the system for the case where each bin comes with its own fixed capacity. At the end of the section, we will elaborate on the exact vector of capacities to achieve desirable load balancing.

Our starting point is simple consistent hashing (without virtual bins). Bins and balls are hashed to a unit circle with a fixed hash function that is not changed once the system is is started. For efficiency, the hash function should be fixed at random, but the system described below is well-defined for any given hash
function. For now, we assume that balls and bins all hash to distinct locations so that we get a complete cyclic ordering of the balls and bins along the cycle. A ball hashes to the bin following it in clockwise order, but if the number of balls hashing to a bin exceed its capacity, then some of the balls have to be placed in other bins.

**Simple insertions via forwarding.** Suppose for now that we are only adding balls to a fixed set of bins. To deal with bin capacities, if a ball hashes to a full bin, then it is forwarded around the circle until it finds a bin that is not full, and then it is placed in that bin.

This forwarding is the basic idea behind linear probing. It appears not to have been suggested before in the context of Consistent Hashing.

**General system invariant between updates.** The location or placement of a ball may depend on the order in which the balls are added, but regardless of the insertion order, the forwarding from full bins maintains a simple invariant that we discuss more generally before discussing other updates to the system.

If a ball $q$ hash to a bin $b$ but is located in some other bin $b'$, then we say that $q$ passes all the bins from $b$ to the bin preceeding $b'$ in clockwise order. The above forwarding from full bins maintains.

**Invariant 1.** No ball passes a non-full bin.

In Section 2.1 we prove that Invariant 2.1 uniquely determines the load of each bin. This is for a given set of balls and bins with given capacities, but independent of history.

As balls and bins are added and removed we will maintain Invariant 2.1. More precisely, after each of these updates, we assume we have time to settle the system, moving balls so as to restore Invariant 2.1 and satisfy all capacity constraints before we get another update to the system.

**Searching a ball in the bins** Assuming Invariant 2.1, it is easy to find a ball $q$ among the bins. First we search the bin $b$ that $q$ hashes to. If $q$ is not found and if $b$ is full, we move to the next bin $b'$ and recurse until either we find $q$ or we reach a bin $b'$ without $q$ that is not full. In the latter case, by Invariant 2.1 we conclude that $q$ is not in the system. If $q$ is to be inserted, it should be inserted in $b'$.

**General insertions and forwarding from overfull bins** Above, when we inserted a ball, we just placed it in the first non-empty bin. We refer to this as simple insertions. We shall here allow for more flexible non-deterministic insertions.

Temporarily, we allow a bin to be overfull in the sense that the load exceeds the capacity. When a ball arrives, it is first placed in the bin it hashes to, which may become overfull. If the system has an overfull bin, we forward any of its balls to the next bin.

In relation to Invariant 2.1, we view an overfull bin as full, not non-full. This means that Invariant 2.1 is never violated in the above process, so we are done when there are no overfull bins.

The previously described simple insertions correspond to always forwarding the most recent arrival, but now we are free to choose which ball to forward. For example this allows a history independent system using the idea from [BG07]. It assumes a linear order on all balls which is independent of the order in which they are inserted. When placing balls in bins, we place them as if they were inserted lowest to highest using simple insertions. To maintain this history independence with general insertions, we always forward to the highest balls from an overfull bin.

When we later analyze the efficiency of our system, we will allow for general insertions with no restrictions on which balls get forwarded from overfull bins.
We will generally use the number forwardings from overfull bins as an upper bound on the number of balls moved, but note that the actual number of moves may be much smaller, e.g., with the simple insertion, we are just moving the inserted ball directly to the first non-full bin.

**Removing a bin** Removing a bin has the same effect as setting its capacity to 0. It is now overfull, and then we forward balls from overfull bins arbitrarily until no bin is overfull. Invariant was never violated.

**Deleting balls and filling holes** When we delete a ball from a bin, we may have to replace it. It is convenient to talk about it as filling holes, where the number of holes in a bin is the number of balls it is missing to get full. When we delete a ball, we create a new hole.

Our concern about a new hole in a bin \(b\) is if \(b\) was full and passed by a ball \(q\) in some later bin \(b'\). The hole renders \(b\) non-full and then Invariant 1 is violated. We can then fill the hole by moving \(q\) to \(b\), but this creates a new hole in \(b'\) that we may have to fill recursively. Invariant 1 is restored when no hole is passed. The process never violates any capacity constraint, so we are done.

Inspired by deletions in linear probing, an efficient way to fill holes, starting from the first hole in some bin \(b\), is to scan the bins one by one. When we get to a bin \(b'\) with a ball \(q\) that hash to or before \(b\), we fill \(b\) with \(q\), and proceed from \(b'\). This way we never consider the same bin twice, and we can stop when we meet a non-full bin. We shall discuss an even more efficient implementation in Section 5 where we maintain the number of balls passing each bin. This allows us to stop as soon as we create a hole in a bin that is not passed. For now, however, we are focussed on bounding the number of moves.

**Adding a bin** When a bin is added, it starts with as many holes as its capacity. We then keep filling holes as long as possible so Invariant 1 is restored. No capacity constraint is violated in this process.

**Changing Capacities for Load balancing in Dynamic Environments.** We will now show how to set and change capacities to achieve good load balancing. For a given load balancing parameter \(c = 1 + \varepsilon > 1\), we want to guarantee that no bin has more than \(\lceil \frac{cm}{n} \rceil\) balls. One possibility would be to just say that all bins had capacity \(\lceil \frac{cm}{n} \rceil\), but then adding a single ball could force us to increase the capacity of all bins, completely changing the configuration. As a result, we need to be careful about enforcing the above capacity constraints across all bins. In particular, to minimize the number of capacity changes when balls are inserted or deleted, we aim for a total bin capacity of \(\lceil cm \rceil\). Assuming a linear ordering of the bins, we let the lowest \(\lceil cm \rceil - n \lfloor cm/n \rfloor\) bins have capacity \(\lceil cm/n \rceil\) while the rest have capacity \(\lfloor cm/n \rfloor\). We refer to the former bins as big bins and the latter bins as small bins, though the difference is only 1. Moreover, as an exception to the above rule, we will never let the capacity drop below 1, that is, if \(cm < n\), then all bins have capacity 1.

For a given set of balls and bins, with a given linear order on the bins, the above protocol tells us the exact capacity of each bin. When we add or removing a ball, it affects the capacity of at most \(\lceil c \rceil\) bins: if the ball is added, we increment capacity, and if it is deleted, we decrement capacity. If a bin is inserted or deleted, it changes at most at \(\lceil cm/n \rceil\) capacities. If capacities are increased, we may have to fill holes, and if capacities are decreased, we may have to forward balls.

**Separating capacity changes from ball and bin updates.** For the sake of later analysis and efficiency, we make the rule that if a ball is inserted or a bin is deleted, we first do all the required capacity increases, one by one, settling the system after each capacity increase by hole filling. When the capacities have all settled, we add the ball or remove the bin, upon which we do the final settlement by forwarding from overfull bins.
Conversely, when a ball is deleted or a bin is inserted, we settle by hole filling before we decrease any capacity. Next we do the required capacity decreases one by one, forwarding from overfull bins after each, until we are back to the desired settled system satisfying Invariant 1 and all capacity constraints.

**Concrete random hash functions** The system as described above always works correctly with a fixed hash function, but an adversary knowing our hash function could force us to make way too many moves. In order to claim efficiency, we use a hash function that is fixed randomly and independent of the operations performed on the system. Thus we can think of the updates and searches for balls in bins as chosen in advance by an adversary that is oblivious to what goes on inside the system.

In practice, we work with hash functions with a limited range \([r] = \{0, ..., r - 1\}\). Mapping this range to a circle, position 0 succeeds \(r - 1\). We assume that the hash function for balls is independent of the hash function for bins. Unless otherwise stated, the bounds in this paper only assume that \(r \geq n\) and that both the ball hash function and the bin hash function are 5-independent (simple tabulation \[PT12\] will also work even though it is only 3-independent). With limited-range hash functions, we may have collisions. To get a complete cyclic order, we do the following tie breaking: if two balls or two bins hash to the same location, then the one with the lower ID precedes the one with the higher ID. Moreover, if a ball and a bin hash to the same location, the ball precedes the bin. This implies that the bins hashing to a given position \(x\) will always be filled bottom-up.

### 1.3 Our Results: Theoretical and Empirical

In this subsection, we state our main theorems and other results on the above system. Subject to the capacity constraints, our main focus in this paper is the expected number of balls that have to be moved when a ball or bin is inserted or deleted. Our bounds will hold no matter which balls we decide to move when a bin is overfull or a hole has to be filled, as long as we follow the above description. We shall discuss how to efficiently identify the balls to be moved in Section 5.

Mathematically, the most interesting case is when \(c = \omega(1)\). We note that inserting a ball results in up to \(\lceil c \rceil\) bins increasing their capacity. Nevertheless, besides placing the new ball, we will prove that the expected number of ball moves is \(O((\log c)/c) = o(1)\).

The general result is

**Theorem 2.** For a given load balancing parameter \(c \geq 2\), the expected number of bins visited in a search is \(1 + O((\log c)/c)\). When a ball is inserted or deleted, the expected number of other balls that have to be moved between bins is \(O((\log c)/c)\). When a bin is inserted or deleted, besides moving \(O(m/n)\) expected balls hashing directly to the bin, we expect to move \(O((m/n)(\log c)/c)\) other balls.

For the insertion and deletion of bins, Theorem 2 implies that the expected number of moves is \(O(m/n)\). We distinguish the balls hashing directly to the bin so as to allow a direct comparison with simple consistent hashing without capacity constraints \[KLL+97\]. For simple consistent hashing, the balls affected by the insertion or deletion of a bin are exactly the balls hashing to it. We expect \(O(m/n)\) such balls (previously, we have said that exactly \(m/n\) balls were expected to hash to any given bin, but that was assuming ideal fully random hash functions). With our capacity constraints, we only expect to move \(O((m/n)(\log c)/c)\) other balls. The price we pay for guaranteeing a maximum load of \(\lceil cm/n \rceil\) is thus only a multiplicative factor \(1 + O((\log c)/c) = 1 + o(1)\) in the expected number of ball moves.

For \(c \in (1, 2]\), we parameterize by \(\varepsilon = c - 1 > 0\).
Theorem 3. For a given load balancing parameter \( c = 1 + \varepsilon \in (1, 2] \), the expected number of bins visited in a search is \( O(1/\varepsilon^2) \). When a ball is inserted or deleted, the expected number of other balls that have to be moved between bins is \( O(1/\varepsilon^2) \). When a bin is inserted or deleted the expected number of balls that have to be moved is \( O(m/(m\varepsilon^2)) \).

The bounds of Theorem 3 are similar to those obtained in [PT12] for linear probing. The challenge here is to deal with the fact that bins are randomly placed, as opposed to linear probing where every hash location has a bin of size 1. Nevertheless we will be able to reuse some of the key lemmas from [PT12]. The proof of Theorem 2 is far more challenging, and the main focus of this paper.

Remark 4. The bounds from Theorems 2 and 3 also hold in the simpler case where all bins have a fixed capacity \( C \) and we define \( c = 1 + \varepsilon = Cn/m \). We note that our updates change the value of \( m \) and \( n \), hence of \( c = Cn/m \). For the bounds to apply, we always use the smaller value of \( c \) in connection with each update. Thus, for the bounds on the moves in connection with a ball insertion or bin removal, we use the value of \( c \) before the update. For the bounds on the moves in connection with a ball deletion or bin addition, we use the value of \( c \) after the update.

Additional results. In this paper, we are also going to discuss how to efficiently find the balls to be moved in connection with system updates. We are also going to discuss high probability bounds that are, essentially, a factor \( O(\log n) \) worse than the above expected bounds. Finally, we note that in practice, it may be relevant to study our problem with weighted balls. In this case, the sum of the weights of the balls assigned to a bin should not exceed its capacity. With weights, the results get less clean, e.g., a single weight might be bigger than the allowed capacity. Also, there may be situations where we have to move a large number of light balls to maintain balance. However, if we use integer capacities as described above, and if no balls has weight above 1, then the above results hold if we replace “number of balls” with “weight of balls”. We note that in the non-weighted case, the probability bounds in this paper are about sums of binary 0–1 variables. However, the extend easily to sums of real variables in range \([0, 1]\). To avoid corner cases, we should assume no ball has weight zero, i.e. all weights are bounded below from zero by some constant.

Empirical Results and Industrial use. We confirm effectiveness of our algorithm in various practical settings via an empirical study in section 7. Furthermore, we note that versions of our algorithm have been deployed in a number of industrial applications. The first industrial application of our algorithm was in Google’s Cloud Pub/Sub as described in [MZ17]. This application needed good load balancing in a variety of instances with different characteristics including the lightly loaded case. History independence was also necessary in that application.

Surprisingly, within a few months of the first release of this paper on arXiv [MTZ16], our algorithm got picked up by the video streaming company Vimeo. In a Vimeo blog post [Rod16], citing our paper, Rodland describes how Vimeo used our algorithmic idea to solve the scaling issues they had in handling almost a billion requests per day (they have 170 million users). They implemented a version of algorithm with balancing parameter 1.25, and then the problems vanished as seen in Figure 1. In [Rod16], they also explain that they had given up on traditional consistent hashing and power of two choices because of load balancing problems. Finally [Rod16] explains that an open source version of the code has been released HAPProxy 1.7.0, and we expect that other companies will start using it.

The Vimeo story is a good example of how theory can have impact. We offer a simple algorithmic solution to the load balancing issue. We are not claiming to have a theoretical model that captures all the important aspects of performance since it depends on the concrete implementation context. What we do
Figure 1: This figure is copied from [Rod16]. It shows Vimeo’s use of shared cache bandwidth over time. The bandwidth problems disappeared when they switched to a version of our algorithm on October 26.

offer is a theoretical analysis, showing that for every possible input, the algorithm has very good expected performance on important combinatorial parameters, including guaranteed load balancing for all possible inputs. This is something that can never be verified by tests, but it is very valuable for the trust in an online dynamic system that is meant to run “forever” not knowing future inputs.

**Note about simplicity and general applicability.** Consistent hashing is a simple versatile scheme that has been implemented in many different systems with different constraints and performance measures [OV11, GF04]. Our consistent hashing scheme, respecting capacities via forwarding, is almost as simple, and should work with most of these implementations. The classic implementation of consistent hashing is the distributed system Chord [SMK01, SML03] which has more than ten thousand citations. The Chord papers [SMK01, SML03] give a thorough description of the many issues affecting the design. Below we only consider the aspects affecting the message complexity of finding a ball among the bins.

In Chord, they have a system of pointers so that given an arbitrary point on the cycle, they can find the next bin in the clockwise order using $O(\log n)$ messages. This is how they find the bin a ball hashes to. In simple consistent hashing, this is where the ball is to be found. With our forwarding, starting from the bin a ball hashes to, we need to visit succeeding bins until we either find the ball or a non-full bin. Chord does maintain explicit successor pointers between neighboring bins, so we only have to pay $O(1)$ extra messages per bin visited. By Theorem 3, we expect to visit $O(1/\varepsilon^2)$ bins, so our total expected message cost is $O(\log n + 1/\varepsilon^2)$. The extra $O(1/\varepsilon^2)$ is negligible if $\varepsilon = \omega(1/\sqrt{\log n})$.

Of course, there may be other systems with different costs associated, but generally we assume it to be significantly more expensive to find the bin a ball hashes to than it is to go from one bin to its succeeding neighbor. Since the costs depend on the implementation context, our focus here is on fundamental combinatorial properties like loads, the number of bins searched, and the number of balls that are moved when the system is updated.

1.4 Power of multiple choices as an alternative?

We have proposed applying the idea of forwarding in the style of linear probing to bound the loads in consistent hashing. The reader may be wondering if alternative ideas in dynamic load balancing can be applied to deal with our natural problem. Here, we discuss them, and highlight their drawbacks or challenges
to convert them to working solution. In particular, we discuss the possibility of instead using the power of multiple choices [ABKU99, PR01, Mit01, BCSV00, TW14, Vöc03]. The discussion is speculative in that none of these techniques have been analyzed in our dynamic context where both balls and bins can be added and removed. We can try to guess what may happen, but have limited knowledge without a proper analysis.

The most basic form is that we have a fixed set of \( n \) bins. Balls are added, one by one. Each ball is given \( d \) uniformly random bins to choose from, and picks the least loaded, breaking ties arbitrarily. With \( m \) balls, w.h.p., we end up with a maximum load of \( m/n + \ln n \over m + d + \Theta(1) \) [BCSV00]. An interesting twist suggested by Vöcking is that if several bins have the same smallest load, we always pick the left-most [Vöc03]. Surprisingly, this reduces the max load to \( m/n + \Theta(1 + \ln n \over d) \) [Vöc03]. We note that to get a constant ratio between maximum and average load when the average load is constant, we do need a super constant number of choices, e.g., \( d = \Omega(\log \log n) \) with left-most choice.

The above mentioned bounds are proved in the ideal case where we pick uniformly between the bins. Consider now first the case of simple consistent hashing where both balls and bins are placed at random on a unit circle, and a ball goes to the succeeding bin in clockwise order. This case was studied in [BCM03, BCM04], where it was proved that if \( m = n \), then the maximum load is \( O(\log \log n) \). However, with a concrete example, [Wie07] showed that we cannot in general hope to get max-load \( m/n + O(\log \log n) \) when \( m \gg n \log n \). This is again for the case of simple consistent hashing where bins are just placed randomly on the circle. However, using, e.g., virtual bins, we know that that we can obtain a more uniform bin cover such that each bin represents a fraction \( (1 \pm \epsilon) / n \) of the unit cycle and where a ball lands in a bin with this probability. With \( \epsilon = 1/2 \), the main result from [Wie07] implies that using the power of \( d \) choices, w.h.p., we get a maximum load of \( m/n + O(\log \log n \over \log d) \).

We still have to consider what happens in our dynamic case where balls may be deleted and bins can be added and/or removed. The results from [CFM+98] indicate that to delete a ball, it may suffice to just remove it without moving any other balls. However, if a bin is removed, we have to move all its balls. In order to claim any bounds, we would need a careful analysis, but the best we can possibly hope for is to match the above bounds for the basic case where \( m \) balls are just added picking uniformly between \( n \) fixed bins.

**Cuckoo hashing** We will now discuss how the power of multiple choices could possibly be used in the style of Cuckoo hashing [PR04] to provide balancing guarantees like those in our solution. With user specified load balancing parameter \( c = (1 + \epsilon) \), we assigned capacities \( \lfloor cm/n \rfloor \) or \( \lceil cm/n \rceil \) to each bin in such a way that the total capacity was \( cm \) and such that we only changed at most \( \lfloor c \rceil \) bin capacities in connection with each ball update. We then used forwarding in the style of linear probing to respect these capacities. Obviously, one could try to adapt many other hash table schemes to respect these capacities. Most notably, we could hope to adapt Cuckoo hashing [PR04].

We now review the basic results for Cuckoo hashing in the ideal case where we have a fixed set of \( n \) bins, all of the same capacity \( C \), and where each of \( m \) balls gets \( d \) uniformly independent choices between these bins. The basic feasibility question is how large \( C \) has to be before we expect to be able to place all balls. This question is studied in [FKP16], but here we also want to update the system efficiently as balls and bins are inserted and removed. In the original Cuckoo hashing [PR04], the capacities are all 1. With 2 choices, the balls are placed into \( n = (2 + \epsilon)m \) bins for any constant \( \epsilon \). However, [FPSS05] proves that using \( d = O(\log(1/\epsilon)) \) choices, we can place the balls in only \( n = (1 + \epsilon)m \) bins, so the maximum load of 1 is at most \( (1 + \epsilon) \) times the average load, as desired for load balance \((1 + \epsilon)\). For efficient insertions, [FPSS05] prove that if \( d \geq 5 + 3 \ln(1/\epsilon) \), then a ball can be placed in \( d^{O(\log(1/\epsilon))} = (1/\epsilon)^{O(\log \log(1/\epsilon))} \) expected time and in \( o(n) \) time with high probability. The high probability bound on the insertion time is improved to a
polylog in \[\text{FPS13}\]. Moreover, using a larger value of \(d = O\left(\frac{\log(1/\varepsilon)}{\varepsilon}\right)\), it was recently shown \[\text{FJ17}\] that the expected search time can be reduced to \(O(1)\) independent of \(\varepsilon\). In \[\text{DW07}\] they consider having only \(d = 2\) choices, but instead they increase the capacity. They show that a bin capacity of size \(O(\log(1/\varepsilon))\) suffices to distribute the balls with load balance \((1 + \varepsilon)\). For efficient insertions, they prove that if the capacity is above \(16 \ln(1/\varepsilon)\), then a ball can be placed in \(\log(1/\varepsilon)\) expected time. The bounds require fairly complex hash functions, but \[\text{DW07}\] states that the hash functions can be implemented in \(O(n^{5/6})\) space. A further discussion of hash functions for Cuckoo hashing is found in \[\text{ADW14}\].

We note that it is would require a full analysis to figure out to which degree the above bounds transfer to our fully dynamic case where bins can be added and removed, and where capacities adapt to the load based on the balancing parameter. Also, we would most likely need to use virtual bins to get a reasonably uniform bin cover. The best we can hope for is to match the above bounds for the simpler case with uniformly distributed choices between a fixed set of bins. This may indeed be possible, and would be interesting.

Stepping back, the first contribution of this paper is to raise the problem of getting consistent hashing to obey a user specified load balance parameter \(c = (1 + \varepsilon)\) for all possible inputs. We present the first solution with proven bounds on the efficiency of searches, ball and bin updates. When \(\varepsilon\) is a positive constant, our expected bounds are only a constant factor from the general lower bounds.

It may very well be possible to get a different solution based on Cuckoo hashing instead of linear probing, getting bounds similar to those reviewed above for a fixed set of bins, all with the same capacity. However, even if this can be done, it would not always be better than our solution. One of the attractions of \(d \geq 2\) choices is that a ball only has to be searched in \(d\) bins, but these are \(d\) random bins, as opposed to the single segment of consecutive bins searched with our solutions. If we, as Vimeo, use balancing parameter \(1 + \varepsilon = 1.25\), then, by Theorem\[3\] we only expect to consider a constant number of consecutive bins. Thus, if, as in the Chord implementation, the cost of accessing a random bin is much bigger than that of getting from a bin to its successor, then our search is nearly twice as efficient as checking 2 random choices (for successful searches, in expectation, we can bring 2 down to 1.5). With Cuckoo hash tables, it is often seen as an advantage that the \(d\) choices can be checked in parallel, but here we imagine a large system processing many requests in parallel, and then the total number of messages is important. With updates we have the same advantage of only working within a single segment of bins whereas Cuckoo hashing has to consider arbitrary bins. However, the Cuckoo updates are still to be defined and analyzed for us to understand their efficiency when bins are inserted and deleted and capacities change. Moreover, for the implementation of Cuckoo hashing, we probably need the added complexity of, e.g., virtual bins to get a more uniform bin cover. Thus, we expect our linear forwarding solution to remain relevant regardless of future developments with multiple choices. The situation is like linear probing versus Cuckoo hashing for regular hash tables. Both are very important solutions to an extremely important problem, and each has its advantages in different contexts, e.g., with Cuckoo hashing, the queries are easy while the updates are more challenging. For our problem of consistent hashing with bounded loads, we prove here that the forwarding of linear probing works in a simple practical solution. A corresponding analysis for Cuckoo hashing is yet to be done.

2 High Level Analysis

To analyze the expected number of moves, we shall use a general probabilistic understanding of configurations encountered. By a configuration, we refer to the situation between updates where the allocation has settled to satisfy Invariant\[1\] and with no overfull bins.
2.1 Uniqueness of loads

As a first step in understanding configurations, we argue that Invariant 1 uniquely determines the load of all bins. This is for a given set of balls and bins with given capacities, but independent of history. This is a very simple generalization of the argument that the set of cells that are filled by linear probing is unique and history independent.

First we note that Invariant 1 implies that which bins are full is independent of the order in which the balls are inserted. It only depends on which balls and bins are present and where they hash to. More precisely, we have

**Lemma 5.** A bin \( b \) is full if and only if there is an interval of consecutive bins \( B = b_1, \ldots, b_k \) ending in \( b = b_k \) such that the total number of balls hashing to these bins is at least as big as their total capacity.

**Proof.** If \( b \) is full, we take \( B = b_1, \ldots, b_k \) to be the maximum interval of full bins, that is, the bin \( b' \) preceding \( b_1 \) is not full. By Invariant 1, this means that no balls hashing to or before \( b' \) can end in \( B \), so \( B \) must be filled with balls hashing to \( B \). In the other direction, the result is trivial if all balls hashing to \( B \) end in \( B \), since there are enough balls to fill all bins. However, if a ball hashing to \( B \) ends up after \( b_k \), then \( b_k \) is full by Invariant 1.

In fact, the hashing of balls and bins determines completely the number of balls landing in any given bin \( b \). If the bin is full, this follows from Lemma 5. Otherwise, the number of balls landing in \( b \) is determined by Lemma 6 below.

**Lemma 6.** If a bin \( b \) is not full, consider the longest interval \( b_1, \ldots, b_k \) of full bins leading to \( b \), that is, \( b \) succeeds \( b_k \) and the predecessor \( b' \) of \( b_1 \) is not full. If the predecessor of \( b \) is empty, with have \( k = 0 \), hence no bins in \( b_1, \ldots, b_k \).

Then the balls landing in \( b_1, \ldots, b_k, b \) are exactly the balls hashing to these bins. If \( s \) balls hash to \( b_1, \ldots, b_k, b \) and \( b_1, \ldots, b_k \) have capacities \( C_1, \ldots, C_k \), then we have exactly \( s - \sum_{i=1}^{k} C_k \) balls landing in \( b \).

**Proof.** The result follows by Invariant 1 together with the fact that \( b \) and \( b' \) are not full.

Summing up the two previous lemmas, we have

**Lemma 7.** The set of balls and the set of bins with their capacities uniquely determine how many balls land in each bin.

2.2 Expected distance to non-full bin

Suppose we have \( m \) balls and \( n \) bins that are currently in the system. We refer to these balls and bins as active. We will also study passive balls and bins that are not currently in the system, yet which have hash values that will be used if they get inserted. For some \( \bar{c} > 1 \), the total capacity will be exactly \( \bar{c} m \). Since no bin has capacity below 1, we always have \( \bar{c} m / n \geq 1 \). In our analysis, we will only assume that each bin has a capacity between \( \bar{c} m / (2n) \) and \( 2 \bar{c} m / n \), and that the concrete capacities are independent of the hashing of balls and bins. We note that this is always satisfied when bin capacities are at least 1 and differ by at most 1.

Theorem 8 below gives our main technical understanding of configurations for larger \( c \). It does not make any assumptions about how we reached the configuration as long as Invariant 1 is satisfied with the desired bin capacities. Theorem 8 may seem too complicated, but this is exactly the technical challenge of this paper: *the devil in the details*. We need to exploit every detail of the theorem to get the bounds we have
claimed in the introduction. The proof of Theorem 8 is based on a very delicate analysis of the interaction between the balls and bins.

**Theorem 8.** Consider a configuration with \( m \) active balls and \( n \) active bins and total capacity \( \bar{c}m \) for some \( \bar{c} \geq 2 \). Suppose, moreover, that each bin has capacity between \( \bar{c}m/(2n) \) and \( 2\bar{c}m/n \). Then

(a) Starting from the hash location of a given passive ball or active or passive bin, the expected number of consecutive full bins is \( O(1/\bar{c}) \).

(b) If we start from a given active bin of capacity at least 2, the expected number of consecutive full bins is \( O((\log \bar{c})/\bar{c}^2) \).

(c) The expected number of balls hashing directly to any given active bin is \( O(m/n) \). The expected number of balls forwarded into the bin is \( O((m/n)(\log \bar{c})/\bar{c}^2) \). Finally, if a bin \( i \) is not active, and its active successor \( i' \) is given an extra capacity of one, then the expected number of full bins starting from \( i' \) is \( O((\log \bar{c})/\bar{c}^2) \).

The above statements are satisfied if the balls and bins are hashed independently, each using 5-independent hash functions or simple tabulation hashing. The statement of (c) may seem a bit cryptic, and will make more sense in the context of the analysis it is used in below.

The worst-case for our bounds is when the capacities are 1 and 2. This case explains why (a) would not work for an active ball since an active ball by itself could fill a bin of capacity 1. However, when a ball \( q \) is inserted, we do forwarding to the nearest non-full bin in the configuration before \( q \) is insertion where \( q \) is still passive, and therefore Theorem 8 (a) gives the expected number of full bins passed.

**Corollary 9.** With balancing parameter \( c = 1 + \varepsilon \in (1, 2] \), the expected number of bins visited in a search is \( 1 + O((\log c)/c) \) if \( c \geq 2 \), and \( O(1/\varepsilon^2) \) if \( c \leq 2 \).

**Proof.** If the ball \( q \) searched is not in the system, then the search is to only up to and including the first non-full bin. However, if \( q \) is in the system, the latest it can be placed is if was added last, which corresponds is the first non-full bin if in the system without \( q \). So in both cases, the expected search is bounded as one plus the number of consecutive full bins starting from an passive ball as in Theorem 8 (a).

### 2.3 Bounding the expected number of moves

We are now going to prove Theorem 2 applying Theorem 8 to settled configurations. If the settled configurations has \( m \) active balls, \( n \) active bins, and total capacity \( \bar{c}m \), then we need to make sure that \( \bar{c} \geq c \) where \( c \) is the fixed parameter chosen to control the bin capacity. The bounds only get better with larger \( \bar{c} \). We know that that before and after every update we have total capacity \( \lceil cm/n \rceil = \bar{c}m/n \) so \( \bar{c} \geq c \). However, we will also argue about intermediate configurations, e.g., when we insert a ball, we first implement capacity increases one by one, settling into a valid configuration after each capacity increase, but these capacity increases will only increase \( \bar{c} \) since \( m \) has not increased yet.

The moves have already been described in Subsection 1.2, but we review them below for the sake of analysis.

First we discuss plain insertions and deletions without any changes to bin capacities.
**Plain insertion** Consider an insertion of a ball $q$ hashing to some bin $i$. For now, we ignore that some capacities have to be increased. First we place $q$ in $i$, which may become overfull. If so, we forward some ball to the succeeding bin $i'$, repeating until we reach a bin that has room because it was not full. The important thing to note is that we move at most one ball per full bin encountered, and that we stop when we meet a bin that was not full before the insertion. By Theorem 8 (a) applied to the configuration before the insertion, the expected number of moves, excluding the initial placement, is $O(1/c) = O(1/c)$.

**Plain deletion** When we delete a ball $q$ from a bin $i$, we create a hole in $i$ that we try to fill with a ball $q'$ that has passed bin $i$ in the sense that $q'$ resides in a later bin $i'$ but hashes to bin $i$ or some earlier bin. If we succeed, we recursively try to fill the new hole left by $q'$ in $i'$. Suppose the last hole created is in bin $i''$. We have now completed the deletion with a new valid configuration. We know that all bins from $i$ to the bin preceding $i''$ are full while $i''$ is not full. The number of moves, including the initial deletion of $b$, is bounded by the number of bins from $i$ to $i''$. By Theorem 8 (a) applied to the configuration after the deletion, it follows that the expected number of moves, including the final removal, is $1 + O(1/c)$ but now $c$ is measured right after the deletion where we have only $m-1$ balls so $c = \lceil cm/n\rceil m/(m-1) > c$.

**Full deletion** To complete a deletion, we have to do $c \pm 1$ capacity decreases. Since the ball has been removed, we have $m' = m-1$ balls during the capacity decreases that all together bring the the capacity down to $\lceil cm'\rceil$. Doing one capacity decrease at the time, we know that every configuration encountered has total capacity $D \geq \lceil cm'\rceil$, and hence $\bar{c} = D/m' \geq c$. We recall that the bins picked for capacity decreases are chosen independently of the random choices made by the hash functions.

When we decrease the capacity of a bin $i$, if the bin now has one ball more than its capacity, we perform exactly the same process as when we hashed a ball to a full bin, forwarding a ball until we reach a bin that was not already full. The number of moves is bounded by the number of consecutive full bins starting from $i$ in the configuration before the capacity of bin $i$ was decreased.

A crucial observation is that since the lowest possible capacity is 1, bin $i$ had capacity at least two before the decrease, so by Theorem 8 (b) applied to the configuration before the capacity decrease, the expected number of moves is $O((\log c)/c^2) = O((\log c)/c^2)$. We are doing at most $\lceil c \rceil$ such capacity decreases, so the total expected number of moves, including the plain deletion of $q$ itself, is bounded by

$$1 + O(1/c) + (1 + c)O((\log c)/c^2) = 1 + O((\log c)/c).$$

**Full insertion** A full insertion is symmetric to a full deletion, doing $c \pm 1$ capacity increases before we do the plain insertion. The expected number of moves, including the addition of $q$ itself is $1 + O((\log c)/c)$.

**Bin updates and forwarding** When we do bin updates, there may be a lot possibilities for which balls to move. We will argue that the concrete choices do not matter for our overall bounds.

Note that for the placement of balls in bins according to Invariant 11 we can think of all passive bins as active bins with capacity zero. Then adding a bin with capacity $C$ is like increasing the capacity to $C$, while removing it is like decreasing its capacity to 0. The lemma below considers the general impact of changing the capacity of a bin.

**Lemma 10.** Suppose we have a valid configuration, and decrease the capacity of some bin $i$ from $C^+$ to $C^-$ while leaving all other capacities unchanged. We assume that the total capacity remains strictly bigger than the total load. Then the total number of forwardings done from overfull bins to reach a new valid
configuration depends only on which balls are in the system. The number of forwardings also bounds the maximal number of moves done if we conversely increase the capacity from $C^-$ to $C^+$, filling holes to restore Invariant\textsuperscript{[7]}

\textbf{Proof.} The proof of Lemma\textsuperscript{[10]} is fairly standard, thinking of forwarding as a flow of balls.

We know from Lemma\textsuperscript{[7]} that the balls and bins with capacities uniquely determine the load of any bin. In particular, this determines all bin loads before we do the capacity decrease.

When the capacity of bin $i$ is decreased to from $C^+$ to $C^-$, bin $i$ may become overfull, and then we have to forward some balls. Then, repeatedly, we take an arbitrary ball from any overfull bin and forward it to the next bin, until no overfull bin remains. When we forward from a ball from an overfull bin it remains at least full, so if $i'$ is the last bin we forward from, then all bins from $i$ to $i'$ are full. Since the total capacity remains strictly bigger than the total load, we cannot keep forwarding the whole way around the cycle.

Let $i_0, \ldots, i_k$ be the bins from $i$ to $i'$. For $j = 0, \ldots, k - 1$, let $Q_j$ be the set of balls that got forwarded from $i_j$. Then $Q_0$ are all balls forwarded from $i_0$ so $|Q_0|$ is exactly the number of balls which $b_0$ exceeded the new capacity $C^-$. For $j = 1, \ldots, k - 1$, $Q_j$ consists of balls that either started in bin $i_j$ or came from $Q_{j-1}$. Since bin $i_j$ ends up full, $|Q_j| - |Q_{j-1}|$ is exactly the number of balls bin $i_j$ needed to fill its capacity. We have $|Q_j| = |Q_{j-1}|$ if bin $i_j$ was full before the capacity decrease; otherwise $|Q_j| < |Q_{j-1}|$.

The process stops at bin $i_k$ because adding $|Q_{k-1}|$ balls to bin $i_k$ does not exceed its capacity.

From the above description, it follows that the numbers $|Q_0|, \ldots, |Q_{k-1}|$ are uniquely determined by the bin loads before the capacity decrease, and these were uniquely determined by the balls and bins with capacities. In particular, this determines the total number of forwardings $\sum_{j=1}^{k-1} |Q_j|$.

We now consider the reverse operation, increasing the capacity of bin $i$ from $C^-$ to $C^+$. Even before the capacity increase, the total capacity exceeds the total number of balls, so there is some first non-full bin $i''$ following $i$ in the clockwise order. By Invariant\textsuperscript{[11]} there is no ball located after bin $i''$ that hash to bin $i''$ or before.

After increasing the capacity of bin $i$, we get $C^+ - C^-$ new holes in bin $i$, and then start trying to fill new holes recursively: if we have a new hole in some bin $i'$, we look for a ball hashing to a later bin that hash to bin $i'$ or an earlier bin. We know that we never have to look past bin $i''$. Let $i'$ be the bin furthest from $i$ that loses a ball, creating a new hole that cannot be filled.

Let $i_0, \ldots, i_k$ be the bins from $i$ to $i'$. For $j = 0, \ldots, k - 1$, let $Q_j$ be the set of balls that were used to fill a hole in bins $i_0, \ldots, i_j$ with a ball starting from bins $i_{j+1}, \ldots, i_k$. In particular $Q_0$ is the set of balls moved to fill the holes in $i$, and then $\sum_{j=0}^{k-1} |Q_j|$ is an upper bound on the number of moves to fill holes.

We now observe that $Q_0, \ldots, Q_{k-1}$ is a forwarding sequence from a valid configuration after the capacity increase for bin $i$ and to a valid configuration before the capacity increase, that is, we get back to the start if we for $j = 1, \ldots, k - 1$, forward the balls in $Q_j$ from bin $i_j$ to $i_{j+1}$. As we saw above, this implies that the balls and bins with capacities uniquely determines $|Q_0|, \ldots, |Q_{k-1}|$, hence also our upper bound $\sum_{j=1}^{k-1} |Q_j|$ on the number of hole filling moves.

\textbf{Closing a bin} When closing a bin $i$, we are going to lose its capacity $C \leq \lceil cm/n \rceil$. Our first action is to increase by one the capacities of $C$ other bins. Doing the increases first, we make sure that the total capacity before every increase is always at least $cm$. An increase is the inverse of the decrease that we studied.
under full deletions, so the expected number of moves for each increase is bounded by $O((\log c)/c^2)$. The expected total number of moves resulting from all $C$ bin increases is thus

$$C \cdot O((\log c)/c^2) = O((m/n)(\log c)/c).$$

We are now ready to start closing $i$, having made sure that the total capacity remains above $cn$. The closing have the same effect as setting the capacity to zero, which may become overfull. We now have to forward balls from overfull bins. By Lemma 10 when bounding the number of forwarding, we can perform them in any order that is convinient for the analysis.

The first forwarding we do in our analysis is to transfer all the balls in $i$ to its successor $i'$. The balls from $i$ either hashed directly to $i$, or they were forwarded from the predecessor of $i$. By Theorem 8 (c), we expect to have $O(m/n)$ balls that hashed directly to $i$ and $O((m/n)(\log c)/c)$ forwarded to $i$ from its predecessor. The balls from $i$ are now all moved to $i'$. Bin $i$ is no longer active.

Now $i'$ may be overfull, and then we repeatedly forward balls from overfull bins until no overfull bins remain and Invariant 1 is restored. Let $i''$ be the last bin receiving a ball from this forwarding. Then the total number of forwardings is at most $C$ times the number of bins from $i'$ to the predecessor of $i''$. All bins from $i'$ to $i''$ are now full, but if we gave $i$ an extra capacity of one, then $i''$ would have received one less ball, hence not be full. Applying by Theorem 8 (c), to this situation, we know that the expected number full bins starting from $i'$ to the predecessor of $i''$ is $O((\log c)/c^2)$, so the expected number of forwardings is $C O((\log c)/c^2) \leq [(m/n)O((\log c)/c^2) = O((m/n)(\log c)/c)$. It might seem that we could here replace $C$ with its expectation $O(m/n)$, but then we would be multiplying two expectations that are not independent.

Summing up, we have proved that when closing $i$, besides the transfer to $i$ of $O(m/n)$ expected balls hashing directly to $i$, we have $O((m/n)(\log c)/c)$ ball moves. Here $n$ is the number of bins before the closing.

**Opening a bin** When we open a bin it is like increasing its capacity from zero to the desired capacity. Thanks to Lemma 10, we can bound the number of moves simply refering to the bound from above on the number of forwards used when we above closed a bin, decreasing its capacity to zero. The subsequent unit capacity decreases are also symmetric to the unit capacity increases we did when we closed a bin, so get the same overall bounds for opening a bin as we did for closing a bin, that is, besides the transfer to $i$ of $O(m/n')$ expected balls hashing directly to $i$, we have $O((m/n')(\log c)/c)$ ball moves. Here $n'$ is the number of bins after the opening of bin $i$, but this only makes our bound better than if we used the number $n = n' - 1$ of balls before the opening.

This completes the proof of Theorem 2 assuming the correctness of Theorem 8.

### 2.4 Small capacities

In the same way that we proved Theorem 2 assuming Theorem 8, we can prove Theorem 9 using the following theorem.

**Theorem 11.** Consider a configuration with $m$ active balls and $n$ active bins and total capacity $(1 + \bar{\varepsilon})m$ for some $\bar{\varepsilon} \in (0, 1]$. Suppose, moreover, that each bin has capacity between $(1 + \bar{\varepsilon})m/(2n)$ and $2(1 + \bar{\varepsilon})m/n$. Then starting from the hash location of a given passive ball or active or passive bin, the expected number of consecutive full bins is $O(1/\bar{\varepsilon}^2)$.

Here we assume that balls and bins are hashed independently, each using 5-independent hash functions or simple tabulation hashing.
As in the analysis for Theorem 2, we will be operating with an \( \bar{\epsilon} \geq \epsilon \) such that the total capacity is exactly \((1 + \bar{\epsilon})m\). Applying Theorem 11 will then yield a bound of \( O(1/\bar{\epsilon}^2) = O(1/\epsilon^2) \). However, it could be that while \( \epsilon \leq 1 \), we end up with \( \bar{\epsilon} > 1 \). In this case, we will instead apply Theorem 8, and get a bound of \( O((\log(1 + \bar{\epsilon}))/(1 + \bar{\epsilon})) = O(1) = O(1/\epsilon^2) \).

We note that Theorem 11 is much simpler than Theorem 8. The point is that Theorem 8 is used to prove a loss by a factor \( 1 + o(1) \) relative to simple consistent hashing without capacities. For Theorem 3, the loss factor is \( O(1/\epsilon^2) \), which is much less delicate to deal with, e.g., for the number of balls in a bin, instead of analyzing the expected number, which is \( O(m/n) \), we can just use the hard capacity, which is at most \( 2(1 + \bar{\epsilon})m/n = O(m/n) \).

3 Analysing expectations with large capacities

In this section, we are going to prove Theorem 8.

3.1 Basic probability bounds

We first briefly review the probability bounds we will use, and what demands they put on the hash functions used. The general scenario is that we have some bounded random variables \( X_1, \ldots, X_n = O(1) \) and \( X = \sum_{i=1}^{n} X_i \). Let \( \mu = \mathbb{E}[X] \). Assuming that the \( X_i \) are 4-independent, we have the fourth moment bound

\[
\Pr[|X - \mu| \geq x] = O \left( (\mu + \mu^2)/x^4 \right).
\]

(1)

Deriving (1) is standard (see, e.g., [PPR09]). We will typically have the variable \( X_i \) denoting that a ball (or a bin) hash to a certain interval, which may be defined based on the hash of a certain query ball \( q \). If the hash function is 5-independent, the \( X_i \) are 4-independent.

The fourth moment bound is very useful when \( \mu \geq 1 \). For smaller \( \mu \), we are typically looking for upper bounds \( x \) on \( X \), and there we have much better bounds in the combinatorial case where \( X_i \) indicates if ball (or bin) \( i \) lands in some interval. Suppose we know that the \( X_i \) are \( a \)-independent for some \( a \leq x \). The probability that \( a \) given balls land in the interval is \( p^a \), so the expected number of such \( a \)-sets is \( \binom{m}{a} p^a \). If we get \( x \) or more balls in the interval, then this includes at least \( \binom{x}{a} \) such \( a \)-sets in the interval. Thus, by Markov’s inequality, with independence \( a \leq x \),

\[
\Pr[X \geq x] = (\mu^a/a!)/(x^a/a!) = O((\mu/x)^a) \quad \text{for } a = O(1)
\]

(2)

where \( x^a \) is defined to be \( x(x - 1) \cdots (x - a + 1) \). We shall only use (2) with \( a \leq 3 \). One advantage to this is that our results will hold, not only with 5-independent hashing, but also with simple tabulation hashing. The point is that in [PT12] it is shown that while simple tabulation is only 3-independent, it does satisfy the fourth moment bound (1) even with a fixed hashing of a given query ball. According to the experiments [PT12], simple tabulation hashing is an order of magnitude faster than 5-independent hashing implemented via a degree-6 polynomial.

3.2 Proof of Theorem 8

We now focus on the proof of Theorem 8. Cheating a bit, we will assume \( \bar{\epsilon} \geq 64 \). We will instead handle \( \bar{\epsilon} < 64 \) as a small capacity in Section 4 where we consider any \( \bar{\epsilon} = 1 + \bar{\epsilon} = O(1) \).

Also, to increase readability, since the parameter \( c \leq \bar{\epsilon} \) does not appear in the analysis, we will just write \( c \) instead of \( \bar{\epsilon} \) below. First we focus on proving Theorem 8(a) and (b) as restated in the following lemma.
Lemma 12. Starting from the hash location of a given \( q \), which is either a passive ball or an active or passive bin, the expected number of consecutive full bins is \( O(1/c) \). If \( q \) is a given bin with capacity at least 2, the expected number of consecutive full bins is \( O((\log c)/c^2) \).

Proof. We bound the expected number \( d \) of consecutive full bins around \( h(q) \). These should include \( q \) if \( q \) is a bin, and the bin \( q \) would hash to if \( q \) is a passive ball. If this bin is not full, then \( d = 0 \) and we are done. Otherwise, let \( I = (a, b) \ni h(q) \) be the interval covered by these full bins, that is, \( a \) is the location of non-full bin preceding the first full bin, and \( b \) is the location of the last full bin. In our analysis, we assume that \( h(q) \) is fixed before the hashing of any other balls and bins.

We will study the event that \( t^- \leq d < t^+ \) and \( I \leq \ell^+ \). First we note that \( d \geq t^- \) and \( |I| \leq \ell^+ \) imply the event:

\[ A(t^-, \ell^+) : \text{Enough balls to fill } t^- \text{ bins hash to } (h(q) - \ell^+, h(q) + \ell^+). \]

Also \( d < t^+ \) implies that at most \( t^+ - 2 \) bins land in \( I \setminus \{b\} = (a, b) \). We are discounting position \( b \), because we could have additional bins hashing to that are not full. Note that we could have \( h(q) = b \). No matter how \( I = (a, b) \ni h(q) \) is placed, we have either \((a, b) \supseteq [h(q) - \lfloor \ell^-/2 \rfloor, h(q)) \), or \((a, b) \supseteq [h(q), h(q) + \lfloor \ell^-/2 \rfloor) \). Thus \( d < t^+ \) and \( |I| > \ell^- \) imply the event:

\[ B(t^+, \ell^-) : \text{Either at most } t^+ - 2 \text{ bins hash to } [h(q) - \lfloor \ell^-/2 \rfloor, h(q)) \text{ or at most } t^+ - 2 \text{ bins hash to } [h(q), h(q) + \lfloor \ell^-/2 \rfloor). \]

Since balls and bins hash independently, after \( h(q) \) has been fixed, the events \( A(t^-, \ell^+) \) and \( B(t^+, \ell^-) \) are independent, so

\[ \Pr[t^- \leq d < t^+ \wedge \ell^- < |I| \leq \ell^+] \leq \Pr[A(t^-, \ell^+)] \Pr[B(t^+, \ell^-)]. \]

For \( i = 0, 1, \ldots \), let \( t = 2^i \), \( t^- = t \), and \( t^+ = 2t \). Moreover, define \( \ell(t) = \lceil 8tr/n \rceil \). Recall here that \( r \geq n \) is the range we hash balls and bins to. We will bound \( \Pr[t \leq d < 2t] \) as follows.

\[
\begin{align*}
\Pr[t \leq d < 2t] &\leq \Pr[t \leq d < 2t \wedge |I| \leq \ell(t)] \\
&+ \sum_{j=0}^{\lceil \log_2 c \rceil} \Pr[t \leq d < 2t \wedge 2^j \ell(t) < |I| \leq 2^{j+1} \ell(t)] \\
&+ \Pr[t \leq d < 2t \wedge c \ell(t) < |I|] \\
&\leq \Pr[A(t, \ell(t))] \\
&+ \sum_{j=0}^{\lceil \log_2 c \rceil} \Pr[A(t, 2^{j+1} \ell(t))] \Pr[B(2t, 2^j \ell(t))] \\
&+ \Pr[B(2t, c \ell(t))].
\end{align*}
\]

The main motivation for the definition of \( \ell(t) \) is that with \( \ell^- = s \ell(t), s \geq 1 \), we can get a good fourth moment bound on \( \Pr[B(2t, \ell^-)] \).

We consider the case where among all bins different from \( q \) (which may be a ball or a bin), at most \( x = t^+ - 2 = 2(t - 1) \) hash to \([h(q) - \lfloor \ell^-/2 \rfloor, h(q)) \). We note that there are at least \( n - 1 \) bins that are different from \( q \). We will pay a factor 2 in probability to cover the equivalent case where they hash to \([h(q), h(q) + \lfloor \ell^-/2 \rfloor) \). The expected number of bins different from \( q \) hashing to \([h(q) - \lfloor \ell^-/2 \rfloor, h(q)) \) is
\[ \mu \geq (n-1)[\ell^--/2]/r, \text{ and we want this to be at least } 2x = 4(t-1). \] This is indeed the case with \( \ell^-- \geq \ell(t) \) since \( \ell(t) = [8r/n] \geq 8(t-1)r/(n-1). \) Since \( n \geq 2, \) we have
\[ \mu \geq (n-1)[\ell^--/2]/r \geq n s\ell(t)/(4r) \geq st \geq 1. \]

Applying the fourth moment bound (1), we now get
\[ \Pr[B(2t, s\ell(t))] = O((\mu + \mu^2)/(\mu - 2(t-1))^4) = O(1/\mu^2) = (1/(st)^2). \] (4)

Next, we will develop different bounds for \( \Pr[A(t, s\ell(t))] \) that are all useful in different contexts. The interval \( (h(b) - \ell(t), h(b) + \ell(t)) \) has length \( 2s\ell(t) - 1, \) so the expected number of active balls hashing to it is \( \mu < 2s\ell(t)n/r = O(stm/n). \) However, to satisfy \( \Pr[A(t, s\ell(t))] \), we need enough balls to fill \( t \) bins in that interval. Their total capacity is \( x \geq tcm/(2n). \) Suppose we also know that \( x \geq a \) and that the balls hash \( a \)-independently. Since we only assume \( 3\)-independence, \( a \leq 3. \) However, if \( q \) is a ball, with \( h(q) \) fixed, we are left with \( 2\)-independence. Now, from (2), we get the bound
\[ \Pr[A(t, s\ell(t))] = O((\mu/x)^a) = O((s/c)^a). \] (5)

Thus, from (3), we get
\[
\begin{align*}
\Pr[t \leq d < 2t] & \leq \Pr[A(t, \ell(t))] + \left( \sum_{j=0}^{[\log_2 c]} \Pr[A(t, 2^{j+1} \ell(t))] \Pr[B(2t, 2^j \ell(t))] \right. \\
& \quad + \left. \Pr[B(2t, c\ell(t))] \right) \\
& = O(1/c^a + \left( \sum_{j=0}^{[\log_2 c]} (2^{2j+1}/c)^a (1/(2^j t)^2) \right) + 1/(ct)^2 \\
& = \begin{cases} 
O(1/c) & \text{if } a = 1 \\
O(1/c^2 + (\log c)/(ct)^2) & \text{if } a = 2 \\
O(1/c^3 + 1/(ct)^2) & \text{if } a = 3 
\end{cases} \quad (6)
\end{align*}
\]

When we start from the hash of an passive ball or bin, or an active bin whose capacity might be only one, we can use \( a = 1 \) for \( t = 1 \) bin and \( a = 2 \) for \( t \in [2, c]. \) We now get
\[
\begin{align*}
\mathbb{E}[d \cdot [d \leq c]] & \leq \sum_{t=2^j, j=0,\ldots,[\log_2 c]} 2t \Pr[t \leq d < 2t] \\
& = O(1/c) + \sum_{t=2^j, j=1,\ldots,[\log_2 c]} t O(1/c^2 + (\log c)/(ct)^2) \\
& = O(1/c).
\end{align*}
\]

Above, for a Boolean expression \( B, \) we have \([B] = 1\) if \( B \) is true, and \([B] = 0\) if \( B \) is false.

In the case where we start from a bin \( q \) with capacity at least \( 2, \) we can use \( a = 2 \) for \( t = 1 \) bin. After the fixing of \( h(q), \) the balls are still hashed \( 3\)-independently and the capacity is at least \( 3 \) with \( t \geq 2 \) bins, so for \( t \in [2, c], \) we can use \( a = 3. \) Thus, when \( q \) is a bin with capacity at least \( 2, \) we get
\[
\begin{align*}
\mathbb{E}[d \cdot [d \leq c]] & = \sum_{t=2^j, j=0,\ldots,[\log_2 c]} 2t \Pr[t \leq d < 2t] \\
& = O((\log c)/c^2) + \sum_{t=2^j, j=1,\ldots,[\log_2 c]} t O(1/c^3 + 1/(ct)^2) \\
& = O((\log c)/c^2).
\end{align*}
\]
For \( t \geq c \), we are instead going to use a fourth moment bound on \( \Pr[A(t, \ell(t))] \). To satisfy \( A(t, \ell(t)) \), we need \( x \geq tc/m/(2n) \) active balls to hash to \((h(q) - \ell(t), h(q) + \ell(t))\) but the expectation is only \( \mu = (2\ell(t) - 1)n/r \leq 2[8tr/n]/n/r \leq 16tm/n \). Since \( c \geq 64 \), we have \( x \geq 2\mu \). Moreover, \( \mu \geq 16 \) since \( cm/n \geq 1 \) and \( t \geq c \). Hence, by (1), we get

\[
\Pr[A(t, \ell(t))] = O((\mu + \mu^2)/(t - \mu)^4) = O(\mu^2/t^4) = O((tm/n)^2/((tc/m/n)^4)) = O(1/(tc)^2) \tag{7}
\]

For \( t \geq c \), this replaces the 1/c\(^a\) term in (6), so we get

\[
\Pr[t \leq d < 2t] \leq \Pr[A(t, \ell(t)] + \sum_{j=0}^{\lfloor \log_2 c \rfloor} \Pr[A(t, 2^j+1 \ell(t))] \Pr[B(2^j, 2^j \ell(t))] + \Pr[B(2^j, c \ell(t))]
\]

\[
= O(1/(ct)^2) + \sum_{j=0}^{\lfloor \log_2 c \rfloor} (2^{j+1}/c^a(1/(2^j t)^2 + 1/(ct)^2)
\]

\[
= \begin{cases} 
O((\log c)/(ct)^2) & \text{if } a = 2 \\
O(1/(ct)^2) & \text{if } a = 3
\end{cases}
\]

In the general case where we start from any passive ball, or active or passive bin, we have \( a = 2 \) for \( t \geq 2 \), so we get

\[
E[d \cdot [d \geq c]] = \sum_{t=2^j c, j=0, \ldots, \infty} 2t \Pr[t \leq d < 2t]
\]

\[
= \sum_{t=2^j c, j=0, \ldots, \infty} 2t O((\log c)/(ct)^2) = O(((\log c)/c^2),
\]

and then

\[
E[d] = E[d \cdot [d < c]] + E[d \cdot [d \geq c]] = O(1/c) + O(((\log c)/c^2) = O(1/c).
\]

This completes the proof of the first statement of the theorem.

For the case where \( q \) is a bin with capacity at least 2, we have \( a = 3 \) for \( t \geq 2 \), and therefore

\[
E[d \cdot [d \geq c]] = \sum_{t=2^j c, j=0, \ldots, \infty} 2t \Pr[t \leq d < 2t]
\]

\[
= O(\sum_{t=2^j c, j=0, \ldots, \infty} t O(1/(ct)^2)
\]

\[
= O(1/c^2).
\]

Therefore, when \( q \) is a bin with capacity at least 2,

\[
E[d] = E[d \cdot [d < c]] + E[d \cdot [d \geq c]] = O(((\log c)/c^2) + O(1/c^2) = O(((\log c)/c^2).
\]

This completes the proof of the theorem.

\[\square\]

We will now prove Theorem 8(c) restated below.

**Lemma 13.** The expected number of balls hashing directly to any given active bin \( q \) is \( O(m/n) \). The expected number of balls forwarded into \( q \) from its predecessor \( q^- \) is \( O(m/n(\log c)/c^2) \). Finally, if a bin is not active, and its active successor \( q^+ \) is given an extra capacity of one, then the expected number of full bins starting from \( q^- \) is \( O((\log c)/c^2) \).
Proof. For the first statement, we note that the expected number of balls hashing to each location \( h(q) - i \) is \( n/r \) for any \( 0 \leq i \leq r \). These are not added to \( q \) if some bin hashes to \([h(q) - i, h(q))\), which is an independent event because balls and bins hash independently. The expected number of bins hashing to \([h(q) - i, h(q))\) is \( \mu = i(n-1)/r \). For \( i \geq r/(n-1) \), we have \( \mu \geq 1 \), and then, by (1), the probability of getting no bins in \([h(q) - i, h(q))\) is \( O((\mu + \mu^2)/(\mu - 0)^4) = O(1/\mu^2) = O((r/(ni))^2) \). The expected number of balls hashing directly to \( q \) is thus bounded by

\[
\frac{n}{r} \cdot \left( \left\lceil \frac{r}{n-1} \right\rceil + \sum_{i=\left\lceil \frac{r}{n-1} \right\rceil+1}^{\infty} \left( \frac{r}{n i} \right)^2 \right) = O(m/n).
\]

We also have to consider the probability that the preceding bin \( q^- \) forwarded balls to \( q \). For this to happen, we would need \( q^- \) to be filled even if we increased its capacity by 1, and then \( q^- \) would have capacity at least 2. This is bounded by the probability of having an interval \( I \ni h(q) - 1 \) with \( d \geq 1 \) consecutive full bins including one with capacity at least 2. This is what we analyzed in the proof of Lemma 12, so we get

\[
\Pr[d \geq 1] \leq \mathbf{E}[d] = O((\log c/e^2)).
\]

By the capacity constraint, the maximal number of balls that can be forwarded to and end in \( q \) is \( 2cm/n \), so the expected number is

\[
O((\log c/e^2)2cm/n = O((m/n)(\log c)/e).
\]

Next we ask for the expected number \( d \) of full bins starting from the active successor \( q^+ \) of a given passive bin \( q \), when \( q^+ \) is given an extra capacity of one. Again this implies that \( q^+ \) has capacity at least 2, and then the analysis from the proof of Lemma 12 implies that \( \mathbf{E}[d] = O((\log c/e^2)) \).

\[\square\]

4 Small capacities

In this section we will prove Theorem 11 restated below with \( \varepsilon \) instead of \( \bar{\varepsilon} \) and allowing any positive \( \varepsilon \) instead of just \( \varepsilon \in (0, 1] \).

Lemma 14. Consider a configuration with \( n \) active balls and \( n \) bins and total capacity \((1 + \varepsilon)m\) for some \( \varepsilon = O(1) \). Suppose, moreover, that each bin has capacity between \((1 + \varepsilon)n/(2n)\) and \(2(1 + \varepsilon)m/n\). Then starting from the hash location of a given passive ball or active or passive bin, the expected number of consecutive full bins is \( O(1/\varepsilon^2) \).

Here we assume that balls and bins are hashed independently, each using 5-independent hash functions or simple tabulation hashing.

We shall sometimes use a weighted count for number \( Y_I \) of bins hashing to an interval \( I \), where the weight of a bin is its capacity divided by the average capacity \((1 + \varepsilon)m/n\). These weights are between 1/2 and 2. The total capacity of the bins hashing into \( I \) is precisely \( Y_I (1 + \varepsilon)m/n \).

Below we choose \( \delta \) such that \((1 + \delta)/(1 - \delta) = (1 + \varepsilon)\). For \( \varepsilon \leq 1 \), we have \( \delta \geq \varepsilon/3 \), but we also have for any \( \varepsilon = O(1) \) that \( \delta = \Omega(\varepsilon) \). For \( d \geq 6 \), our goal is to show that the probability of getting \( d \) consecutive full bins is

\[
O(1/d^2\delta^4) = O(1/d^2\varepsilon^4).
\]

It will then follow that the expected number of full bins is \( O(1/\varepsilon^2) \), as required for Lemma 14.

Before doing the probabilistic analysis, we consider the combinatorial consequences of having \( d \) consecutive full bins.

20
Lemma 15. Let $p$ be the hash location of an open or closed bin or an inactive ball from which the number of successive full bins is $d \geq 2$. If we do not have $d$ balls hashing directly to $p$, there is an interval $I$ containing $p$ of length at least $\ell = \lfloor dr/(2n) \rfloor$, where either

(i) the number balls $X_I$ landing in $I$ is $X_I \geq (1+\delta)m|I|/r$, or

(ii) the weighted number of bins $Y_I$ in $I$ is $Y_I \leq (1-\delta)n|I|/r$.

Proof. Let $I = (a, b)$ be the longest interval covered by consecutive full bins around $p$. More precisely, $a$ is the hash location of the non-full bin preceding the first full bin in the interval. We note here that the first full bins hash strictly after $a$ because bins hashing to the same location always get filled bottom-up. Since the preceding bin at $a$ is not full, bins in $I$ can only be filled with balls hashing to $I$.

Since we do not have $d$ bins hashing to $p$, and we had $d$ full bins starting from $p$, we must have $p < b$, so $p \in (a, b)$. Also, we have at least $d$ full bins in $I$, so $I$ must contain at least $d(1+\varepsilon)n/(2n)$ balls. Suppose $|I| \leq \ell$. We can then expand $I$ in either end to an interval $I^+ \supseteq I$ of length $\ell = \lfloor dr/(2n) \rfloor$, and then $X_{I^+} \geq X_I \geq (1+\varepsilon)dn/(2n) \geq (1+\varepsilon)n|I^+|/r > (1+\delta)n|I^+|/r$, so (i) is satisfied for $I^+$. Thus we may assume that $|I| \geq \ell + 1$.

We now look at the interval $I^- = (a, b)$ which contains $p$ and is of length at least $\ell$. All bins in $I^-$ are full. Even though our last full bin hashed to $b$, we have to exclude $b$ because we might also have non-full bins hashing to $b$.

Since all bins hashing to $I^-$ are filled with balls hashing to $I^-$, we have $X_{I^-} \geq (1+\varepsilon)Y_{I^-}m/n$. We now reach a contradiction if (ii) and (i) are false for $I^-$, for then

$$X_{I^-} < (1+\delta)m|I^-|/r = (1+\varepsilon)(1-\delta)m|I^-|/r < (1+\varepsilon)Y_{I^-}m/n.$$  

□

We note that applying (2) with $a = 2$, we get that the probability of $d$ bins hashing to $p$ is $O(1/d^2) = O(1/d^2\delta^4)$. Hence it suffices to bound the probability of (i) and (ii). We will do this using a technical result from [PT12].

To state the result from [PT12], we first reconsider the standard fourth moment bound (1). We are hashing $m$ balls into $[r]$. Let $\alpha = n/r$ be the density of the balls in $r$. Let $I$ be an interval that may depend on the hash location of the inactive query ball, and let $X_I$ be the number of active balls hashing to $I$. Then $\mathbb{E}[X_I] = \alpha|I|$, and hence we can state (1) as

$$\Pr[|X_I - \alpha|I|| \geq \delta\alpha|I|] = O\left(\frac{\alpha|I| + (\alpha|I|)^2}{(\delta\alpha|I|)^4}\right)$$

(9)

Now [PT12] considered the general event $\mathcal{D}_{\ell,\delta,\rho}$ that, for a given $p \in [r]$ that may depend on the hash value of an inactive query ball, there exists an interval $I$ containing $p$ of length at least $\ell$ such that the number of keys $X_I$ in $I$ deviates from the mean $\alpha|I|$ by at least $\delta\alpha|I|$. As a perfect generalization of (1), [PT12] proved

$$\Pr[\mathcal{D}_{\ell,\delta,\rho}] = O\left(\frac{\alpha\ell + (\alpha\ell)^2}{(\delta\alpha\ell)^4}\right)$$

(10)

The proof of (10) in [PT12] only assumes (9), even though (9) only considers a single interval whereas (10) covers all intervals around a given point of some minimal length.

The bound (10) immediately covers the probability that there exists an interval $I \ni p$ of length at least $\ell$ which satisfies (i). As stated earlier, we may assume that $d \geq 6$, and since $r \geq n$, we get that
\( \ell = \left\lfloor \frac{dr}{2n} \right\rfloor \geq \frac{dr}{3n} \). Moreover, the total capacity is \((1+\varepsilon)m \leq O(m)\), and the minimum bin capacity is 1, so \(n \leq O(m)\), and therefore \(\alpha \ell \geq \frac{dn}{(3n) = \Omega(1)}\). Thus the probability of (i) is bounded by

\[ O \left( \left( \frac{\alpha \ell}{(1 + \varepsilon) m} \right)^2 / \left( \delta \alpha \ell \right)^4 \right) = O(1/(\delta^4 d^2)) = O(1/(\delta^4 d^2)). \]

The analysis above could be made tighter if \(m \gg n\), but then the error probability would just be dominated by that for (ii) on bins studied in the next part.

We now want to limit the probability of getting too few bins, that is, for some \(I \supseteq p\) containing \(p\) of length at least \(\ell \geq \frac{dr}{3n}\),

(iii) the weighted number of bins in \(I\) is \(Y_I \leq (1 - \delta)n|I|/r\)

It makes no difference to [10] from [PT12] that we apply it to bins instead of balls. We note the bin counts are weighed with weights below 2. This is not an issue because weights where considered in [PT12], and the equations (1) and (10) both hold for weights bounded by a constant. Another technical issue is if we in Lemma 14 start from an active query bin at \(p\). Then this bin is always included in the interval \(I \supseteq p\). It should only have been included with probability \(|I|/r\). A simple solution is to only count the start bib with this probability, yielding a slightly smaller value \(Y_I' \leq Y_I\). We then instead bound the probability that \(Y_I' \leq (1 - \delta)n|I|/r\), which is implied by (ii). Now the bin density is exactly \(\alpha = m/r\) and \(\alpha \ell \geq (n/r)\frac{dr}{3n} \geq \frac{d}{3} \geq 2\), so the probability of (ii) is bounded by

\[ O \left( \left( \frac{\alpha \ell}{(1 + \varepsilon) m} \right)^2 / \left( \delta \alpha \ell \right)^4 \right) = O(d^2/(\delta d)^4) = O(1/(\delta^4 d^2)). \]

We have thus shown that \(O(1/(\delta^4 d^2))\) bounds the probability of both (i) and (ii) in Lemma [15] hence the probability of getting \(d \geq 1/\delta^2\) successive full bins. It follows that the expected number of successive bins is bounded by

\[ 1/\delta^2 + \sum_{d = \lceil 1/\delta^2 \rceil}^{\infty} O(1/(\delta^4 d^2)) = O(1/\delta^2) = O(1/\varepsilon^2) \]

This completes the proof of Lemma [14] and hence of Theorem [11].

5 Computing moves when the system is updated

In this paper, we have not been concerned about computing the moves that have to be done. In many applications this is a non-issue since we can afford to recompute the configuration from scratch before and after each update. These are also the applications where history independence matters. Our more efficient computation of moves is based on a global simulation of the system in RAM. Our implementation is tuned to yield the best theoretical bounds.

We will now first describe an efficient implementation when balls are just inserted in historical order, much like in the standard implementation of linear probing. We will describe how to compute the moves associated with an update in expected time proportional to the bounds we gave above for the expected number of moves.

Given a ball \(q\), we want to find the bin it hashes to in expected constant time. This is the bin succeeding \(q\) clockwise when bins and balls are hashed to the cycle. For that we use an array \(B\) of size \(t = \Theta(n)\). Entry \(i\) is associated with the interval \(I_i = [i/t, (i + 1)/t)\). In \(B[i]\), we store the head of a list with the bins hashing to \(I_i\) in sorted order. The list has expected constant size. When a ball \(q\) comes, we compute the interval \(I_i \supseteq h(q)\), and check \(B[i]\) for its succeeding bin. If \(L[i]\) has no bin hashing after \(q\), we go to \(L_{i+1}\).
The expected time is $O(1)$. As bins are inserted and deleted, we use standard doubling/halving techniques in the back ground to make sure that $L$ always have $\Theta(n)$ entries.

For each bin, we store the balls landing in it in a hash table. Recall here that when a ball is inserted, we first find the bin it hashes to, but if it is full, we have to place it in the first non-full succeeding bin. Such insertions are trivially implemented in time proportional to the number of bins considered, which is exactly what we bounded when we considered the number of moves.

We now turn to deletions. A deletion is essentially like a deletion in linear probing. When we take out a ball $q$ from a bin $b$, we try to refill the hole by looking at succeeding bins $b'$ for a ball $q'$ that hash to or before $b$. We then move $q'$ to $b$, and recurse to fill the hole left by $q'$ in $b'$. The last hole created is the bin where $q$ would land if $q$ was inserted last in the current configuration. We are willing to pay time proportional to number of bins from the one $q$ hashed to, and to the one it would land in if inserted last.

To support efficient deletions, we let each bin $b$ have a forwarding count for the number of balls that hashed preceding $b$ but landed in a bin succeeding $b$. Also, we divide the balls landing in $b$ according to which bins they originally hashed to. More precisely, we maintain a doubly-linked list with the bins that balls landing in $b$ hashed to, sorted in the same order as the bins appear on the cycle. With each of these preceding bins $b^-$, we maintain a list with the balls that hashed to $b^-$ and landed in $b$. The total space for these lists is proportional to the number of balls landing in $b$, so the total space remains linear. We assume that we can access the sorted bin list from each end.

Now, if a deletion of a ball $q$ creates a hole in $b$, then this hole can be filled if and only if the forwarding counter of $b$ is non-zero. If so, we consider the succeeding bins $b'$ one by one. In $b'$, we go to the beginning of the sorted bin list to find the first bin $b^-$ that a ball landing in $b'$ hashed to. If $b^-$ equals or precedes $b$, we can use any ball $q'$ from the list of $b^-$ in $b'$ and fill the hole in $b$. Checking the bin $b'$ and possibly moving a ball takes constant time. If a hole is created in $b'$ and $b'$ has non-zero forwarding counter, we recurse.

Another issue is that we have is to locate a ball to be deleted in one of the above lists. For that we employ an independent hash table for the current balls point to their location. The hash table could be implemented with linear probing which works in expected constant time using the same 5-independent or simple tabulation hash function that we used to map balls to the cycle [PPR09, PT12]. Below it is assumed that we always update the hash table with the location of the balls as we move them around. An alternative solution would be that we searched the ball starting from the bin it hashed to, and then only used a local hash table for each bin.

When a ball $q$ is inserted, we have to update the above information. If it hashes to a bin $b$ and lands in a bin $b'$, then we have to increment the forwarding counter for all bins starting from $b$ and going to the bin preceding $b'$—if $b = b'$, no forwarding counter is incremented. Now inside $b'$, we have to go backwards in the list of bins that balls landing in $b'$ hashed to, searching for $b$, inserting $b$ if necessary. Next we add $q$ to the list of $b$ in $b'$. The bins considered inside $b'$ is a subset of the bins we passed from $b$ to $b'$, so the time bound is not increased.

When bins are inserted and deleted, we implement the effect using insertion and deletion of the affected bins as described above.

The most interesting challenge is that when a ball is inserted or deleted, we have to change the capacity of $c$ bins. This becomes hard when $c = \omega(1)$ where we only have $O(1)$ expected time available.

As the system is described above, we let the lowest $\lceil cm \rceil - n \lfloor cm/n \rfloor$ bins have large capacity $\lfloor cm/n \rfloor$ while the rest have small capacity $\lceil cm/n \rceil$. However, we only have to guarantee that no bin has capacity above $\lfloor cm/n \rfloor$.

We now relax things a bit. With no history independence, we just maintain the list of current bins in the order they were inserted (if one is deleted and reinserted, we count it as new). The large bins form a prefix
of this list. We also relax the requirement on the number of large bins to be \( \lfloor cm \rfloor - n \lfloor cm/n \rfloor \pm c \). This means that when a ball is inserted or deleted, it doesn’t have to be exactly \( c \) bins whose capacity we change. Next we partition the list of current bins into groups of length at most \( c \) with the property that the combined length of any two consecutive groups is at least \( c \). This partition is easily maintained in constant time per bin update, including pointers so that we can from each bin can find the head of the group it belongs to.

The changes to bin capacities are now done for one or two groups at the time, telling only the group heads what the capacity is. Thus, to check if a bin is full, we have to ask the group head about the capacity.

Now, when we change the capacity of a group, we find out if the change in capacity means that some bins need changes to their information. A bin requiring action is called critical. More precisely, a large bin is critical if it is full. A small bin is critical if it is full and its forwarding count is non-zero. This implies that it would be full if it became large. By Lemma 12, a bin is critical with probability \( O((\log c)/c^2) \), so we only expect \( O((\log c)/c) = o(1) \) critical bins in a group. To identify critical bins efficiently, we divide groups into subgroups of size at most \( \sqrt{c} \), using the same algorithm as we used for dividing into groups. For each group and for each subgroup, we count the number of critical bins. Now a group only has a critical bin with probability \( O((\log c)/c) \), which is our expected cost to check if there are critical bins. If so, we check which subgroups have a positive count of critical bins. If a subgroup has critical bins, we find them by checking all the bins in the subgroup. Altogether, we pay \( O(\sqrt{c}(\log c)/c^2) \) time per critical bin, so the expected time to identify the critical bins is \( O(\sqrt{c}(\log c)/c^2) \) = \( o(1) \).

We next consider an implementation with history independence. For history independence, we first do a random permutation of the bin and ball IDs. We can use the classic \( \pi(x) = ax + b \mod p \) where \( p \) is a random prime and \( a \) and \( b \) are uniformly random in \( \mathbb{Z}_p \). These permuted IDs are still history independent. The point now is that if we have a set \( X \) of \( \Theta(k) \) permuted IDs, then we can maintain order in this set by bucketing based on \( \lfloor \pi(x)/k \rfloor \). The buckets are then in relative sorted order, and we can easily maintain order within each bucket since each ID is expected to end in bucket with \( O(1) \) other IDs. As usual, we can use doubling/halving if the set \( X \) grows or decreases by a constant factor. Now it is easy to maintain an ordered list of permuted bin IDs where a prefix of these are large bins, just like in our original description. Also, for each bin \( b \), we maintain the list of balls that hash to it, ordered based on the permuted IDs. We note that the balls in this list are extracted based on the hashing to the unit cycle which is independent of the permutation of the ball IDs. The balls hashing to \( b \) are placed in \( b \) and succeeding bins based on the ordering of this list, that is, first we fill \( b \) with the balls with the lowest permuted IDs. In particular, the list of balls hashing to \( b \) but landing in bin \( b' \) is just a segment of the sorted list of balls hashing to \( b \).

6 High Probability Bounds

We are now going to present high probability bounds, that is, bounds that holds with probability \( 1 - 1/n^\gamma \) for any desired constant \( \gamma \). The analysis assumes fully random hashing. The 5-independent hashing that sufficed for our expected bounds does not suffice for high probability. However, some of our high probability bounds can also be obtained with simple tabulation [PTT2].

**Theorem 16.** With balancing parameter \( c = (1+\varepsilon) = O(1) \), w.h.p., the maximal number of bins considered in connection with a search is \( O((\log n)/\varepsilon^2) \). This bound also holds for the number of balls that have to be moved when a ball is added or removed. When a bin is added or removed, w.h.p., the number of balls to move is \( O((m/n)(\log n)/\varepsilon^2) \).

With balancing parameter \( c = \omega(1) \), w.h.p., the maximal number of bins considered in connection with a search is \( 1 + O((\log n)/c) \) if \( m > n/2 \), and \( 1 + O\left(\frac{\log n}{(cm/n)(\log(n/m))}\right) \) if \( m \leq n/2 \). This bound also
holds for the number of balls that have to be moved when a ball is added or removed. When a bin is added or removed, w.h.p., the number of balls to move is and \( O((m/n) \log n) \) if \( m > n/2 \), and \( O\left(\frac{\log n}{\log(n/m)}\right) \) if \( m \leq n/2 \).

If \( m = \Omega(n) \), the above bounds are at most a log-factor worse than the expected bounds. In the very lightly loaded case where \( m = o(n) \), things have to get worse, e.g., even for simple consistent hashing without forwarding, we expect some bins to have \( \Theta\left(\frac{\log n}{\log(n/m)}\right) \) balls, matching our high probability bound for the number of balls to be moved when a bin is closed.

The rest of this section is devoted to proving Theorem \ref{thm:large-caps}. We will often study some run \( R \) of consecutive bins full bins, preceeded by an non-full bin at some position \( x \). Let \( y \) be the position of the last bin in \( R \). Because the preceeding bin is not full, all balls landing in the bins of \( R \) must hash to the interval \((x, y]\) covered by \( R \). We note that the bin succeeding \( R \) could be in position \( y \), colliding with the last bin of \( R \).

**Large capacities** We first study the case with super constant balancing parameter \( c = \omega(1) \). What makes this case tricky to analyze is that we in connection with ball updates have to make up to \( \lceil c \rceil \) capacity changes. Recall that bins have capacity \( cm/n \) rounded up or down. Also, recall that \( cm/n \geq 1 \).

For a given number \( d \), we will bound the probability that a given bin \( q \) is in a maximal run \( R \) of \( d \) full bins. Based on \( d \) we are going to pick a threshold \( \ell \). If the run covers an interval \( I \) of length at most \( \ell \), then \( I \subseteq (h(q) - \ell, h(q) + \ell] \), and then we know that at least \( dcm/(2n) \) hash to \((h(q) - \ell, h(q) + \ell] \). We call this the ball event. On the other hand, if the \( R \) covers interval of length bigger \( \ell \), then, either we have at most \( d \) bins landing in \([h(q) - \lceil \ell/2 \rceil, h(q)] \) or at most \( d \) bins landing in \([h(q), h(q) + \lceil \ell/2 \rceil] \). We call this the left and right bin events.

It is convenient to reparameterize with \( s = \ell n/(dr) \), and we will always have \( s \geq 6 \).

We will now bound the probability of the ball event. The expected number of balls that hash to \((h(q) - \ell, h(q) + \ell] \) is \( \mu_A = 2\ell m/r = 2dsm/n \). Using the Poisson bound, the probability that at least \( x_A = dcm/(2n) \) hash to this interval is bounded by

\[
P_A = (e^{\mu_A}/x_A)^{x_A} = ((e^{2dsm/n})/(dcm/(2n)))^{dcm/(2n)} = ((4es/c)^{dcm/(2n)}).
\]

(11)

Next we consider any one of the two bin events. In both cases, we want at most \( d \) balls to hash into an interval \( I \) of length at \( \ell/2 \). Ignoring the bin \( q \), we expect at least \( \mu_B = (\ell/2)(n - 1)/r = (ds/3) \) bins to land in \( I \), and \( (ds/3) \geq 2 \). Therefore, by the Chernoff bounds, the probability of each bin event is bounded by

\[
P_B = \exp(-\mu_B/8) = \exp(-ds/24).
\]

(12)

Now, if \( n = \Omega(m) \), we pick \( s = c/(8e) \), which is bigger than 6 since \( c = \omega(1) \). With this parameter choice, the probability that any of our events happen is bounded by \( \exp(-\Omega(dc)) \), which hence also bounds the probability that bin \( q \) is in a maximal run of length \( d \). It follows that, w.h.p., the maximal run length is \( O((\log n)/c) \). In particular, it follows that for some sufficiently large \( c = O(\log n) \), w.h.p., there is no full bin.

If \( n = o(m) \), we pick \( s = 24(cm/n) \ln(n/m) \). Then

\[
P_A = (4es/c)^{cm/(2n)} = (96cm/n \ln(n/m))^{cm/(2n)} = (m/n)^{\Omega(dcm/n)}
\]

while

\[
P_B = \exp(-ds/24) = (m/n)^{dcm/n}
\]

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Thus we conclude that the probability that a bin $q$ is in a maximal run of length $d$ is bounded by $(m/n)^{Ω(cm/n)}$. It follows that w.h.p., the maximal run length is $O((log n)/(cm/n log(n/m))).$

Above we studied the maximal run length, which also bounds the number of bins we have to search if the first bin is full. It also bounds the number of balls that have to be moved when a ball is added or removed, except for the effect of changing $\lceil c \rceil$ bin capacities, each by $1$. Recall here that adding and removing a ball has the same cost by symmetry, so it suffices to consider a ball removal where capacities are decreased by $1$.

$\lceil c \rceil$ capacity decreases We will now study the number of moves in connection with at most $\lceil c \rceil$ capacity decreases. Let $C$ be the set of affected bins. The cost is bounded by the sum over the bins $p \in C$ of the length of the maximal run of full bins containing $p$. We already have a bound on the maximal run length that we can multiply by $c$, but we will show that, w.h.p., the sum of these run lengths is only a constant factor bigger than the maximal run length. Since one bin makes no difference here, we can assume that $c = \lceil c \rceil$.

The argument below assumes that $c \leq \sqrt{m}$. The case $c > \sqrt{n}$ is quite extreme and can be handled with a different argument.

Based on the previous analysis for a single run, we argue that any run of full bins is contained in an interval of length at most $O((log n)r/n))$. To see this, recall that we gave high probability bounds on (1) the maximum length $d$ of a run of full bins, and (2) the length $\ell$ such that intervals of length $\ell$ have more than $d$ bins. Here (1) and (2) implies that no run of full bin can span an interval longer than $\ell$. For the case where $n = \Omega(m)$, we had $s = c/(4e)$ and $d = O((log n)/c)$, hence

$$\ell = sdr/n = (c/(4e)O((log n)/c)r/n = O((log n)r/n).$$

For the case, $n = \omega(m)$, we had $s = 24(cm/n)\ln(n/m)$ and $O((log n)/(cm/n log(n/m)))$, hence

$$\ell = sdr/n = 24((cm/n)\ln(n/m))O((log n)/(cm/n log(n/m)))r/n = O((log n)r/n).$$

We fix the hashing of the bins in $C$ before we start hashing any of the other balls and bins. We say that two bins from $C$ are well-separated if they are at least $4\ell$ apart on the cycle. We will partition $C$ into a constant number of sets, $C_1, ..., C_{O(1)}$ so that bins in the same set are all well-separated. To do that we partition the cycle into intervals of length $4\ell$ and $8\ell$. Consider one of these intervals $I$. A bin from $C$ lands in it with probability $O((log n)/n)$ and $c = |C| \leq \sqrt{n}$, so with high probability, we get only a constant number of bins from $C$ in any $I$. To get our first set $C_1$, we pick an arbitrary bin from every other interval, and likewise we get $C_2$ picking a bin from each of the other intervals. We have now picked a bin from every interval, so if we repeat this a constant number of times, we get the desired partitioning $C_1, ..., C_{O(1)}$.

We now focus on any one of the well-separated $C_k$. Let $c_k = |C_k| \leq c$. For each bin $p \in C_k$, we have a run of full bins, and we want to bound the probability that the total length of these runs is $d$.

The run of full bins that includes $p \in C_k$ covers some interval $I_p$ of length $\ell_p$, and let $\hat{\ell}^* = \sum_{p \in C_k} \ell_p$. As when we studied the run of a single bin, we are going to define a threshold length $\ell = O((log n)r/n)$ and $s = \ell n/(dr)$. Our combined ball event is that there exists intervals $I_p$ of total length $\hat{\ell}^* = \ell$ with at least $d cm/(2n)$ hashing to them. Our new combined bin event is that there exists intervals $I_p$ of total length $\hat{\ell}^* = \ell$ with at most $d$ bins hashing inside of them (not including the last position). If both events fail, then the total run length cannot be $d$.

For each $p$, we will use a coarse overestimate $\ell^*_p$ of $\ell_p$. We say that $\ell_p$ is on level $0$ if $\ell_p \leq \ell/c_k$.

Otherwise $\ell_p$ is on level $j$ where $2^{j-1}\ell/c_k < \ell_p \leq 2^j \ell/c_k$. We set $\ell^*_p = 2^j \ell/c_k$. With these overestimates, we have $I_p \subseteq I^*_p = (h(p) - \ell^*_p, h(p) + \ell^*_p)$ for all $p$. The bins in $C_k$ are well-separated so that there is no overlap between any two $I^*_p$. Moreover, $\sum_{p \in C_k} \ell^*_p \leq \ell + \sum_{p \in C_k} 2\ell_p \leq 3\hat{\ell}^*$, so $|\sum_{p \in C_k} I^*_p| \leq 6\hat{\ell}^*$. Our combined ball event implies that $\hat{\ell}^* \leq \ell$ and that $d cm/(2n)$ balls hash to $\bigcup_{p \in C_k} I^*_p = \bigcup_{p \in C_k} I_p$. 

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Assume the $I^+_p$ were fixed. They have total length at most $6\ell$, so the expected number of balls hashing to $\bigcup_{p \in C_k} I^+_p$ is at most $m(6\ell/r) = 6ds/m/n$. The probability that at least $dcm/(2n)$ balls hash to $\bigcup_{p \in C_k} I^+_p$ is therefore at most
\[
\left(\frac{6ds/m}{n}\right)/(dcm/(2n))^{dcm/(2n)} = (12es/c)^{dcm/(2n)}.
\] (13)
The important point here is that there are only $4^k$ possible choices for the $I^+_p$. More precisely, for each level $j \geq 0$, we have a bit map telling which of bins from level $j$ that are on level $> j$. There are at most $ck/2^{j-1}$ bins on level $\geq j$, which is the size of the bit map for that level. The total number of bits needed to describe all $I^+_p$ is thus at most $4ck$, so there at most $2^{4ck}$ combinations. This means that even when we quantify over all possible combinations of the intervals $I^+_p$, as long as their total length is $\ell$, the total probability of the combined ball event is bounded by
\[
2^{4ck}((12es/c)^{dcm/(2n)}).
\]
For the bins, we use the same idea, but starting from level $-1$ when $\ell/(2ck) \leq \ell_p < \ell/c_k$. We now use an underestimates $\ell^-_p$ of $\ell_p$. We set $\ell^-_p = 0$ if $\ell_p < \ell/(2ck)$. Otherwise, if $\ell_p$ is on level $j \geq -1$, we set $\ell^-_p = 2^{j-2}\ell/c_k$. We also have, for each bin a bit that tells if the interval $I_p$ is mostly before or mostly after $h(q)$. If before, then $I_p$ contains $I^-_p = [h(p) - \ell^-_p, h(p)]$. Otherwise $I_p$ contains $I^-_p = [h(p), h(p) + \ell^-_p]$. Including the before/after bitmaps, we can completely describe all the $I^-_p$ using $6ck$ bits, so there are at most $2^{6ck}$ combinations. Moreover, the total size of the small $\ell_p < \ell/(2ck)$ is at most $\ell/2$. For the $\ell_p \geq \ell/(2ck)$, we get $\ell^-_p \geq \ell_p/4$, so we conclude that $\sum_{p \in C_k} \ell^-_p \geq \ell/8$. Assume that the $I^-_p$ are fixed. The combined bin event implies that at most $d$ bins hash to them. Considering only the $n - c$ bins outside $C$, we expect at least $(n - c)(\ell/8)/r = (n - c)sd/(8n) \leq sd/10$ bins. With $s \geq 20$, the probability of the bin event is $\exp(-sd/10)/8 = \exp(-sd/80)$. Considering all possible configurations of the bins $I^-_p$, we get that the probability of the combined bin event is bounded by $2^{6ck}\exp(-sd/80)$.

If $n = \Omega(m)$, we pick $s = c_k/(24e)$ and conclude that the probability of combined ball and bin events is bounded by $2^{O(c_k)}\exp(-\Omega(dck)) = \exp(-\Omega(dck))$ for $d = \omega(1)$. This bounds the probability that the sum of run lengths from the bins in $C_k$ is $d$, so we conclude that, w.h.p., the total run length is $O((\log n)/c)$. There are only $O(1)$ sets $C_k$ in $C$, so we get the same bound on the total run length over all bins in $C$.

We would like to follow the same idea in the lightly loaded case where $n = o(m)$. If we set $s = 80(cm/n)\ln(n/m)$, we get that the probability of the combined ball and bin events is bounded by $2^{O(c_k)(m/n)^{\Omega(dcm/n)}}$. However, this time, the $2^{O(c_k)}$ may be so large. One issue is that $c_k$ may be much larger than $d$.

We now note that if $d$ is total number of bins in the maximal runs of full bins involving bins in $C_k$, then at most $d$ bins from $C_k$ can be full. Let us assume we guess the number $d' \leq d$ of full bins from $C_k$, and let $C' \subseteq C_k$ be the subset of these bins. We can then apply the preceding argument to $C'$ instead of $C_k$ with $d'$ instead of $c_k$. Note that we are not conditioning on the bins from $C'$ all being full. We are merely studying the probability that $d$ is the total number of bins in the maximal runs of full bins involving bins in $C'$. To get bounds for $C_k$, we have to sum over all $d' \leq d$ over all $\binom{c_k}{d'}$ choices of $C' \subseteq C_k$. If $d' = O(1)$, we are done, for then we involve only $O(1)$ maximal runs, so we may assume that $d' = \omega(1)$.

When $C'$ is fixed, we only have $2^{nd'}$ choices for the coarse intervals, so for given $d' = \omega(1)$, the total number of configurations we have to consider is
\[
\binom{c_k}{d'}2^{nd'} < (c_k^{d'}/d'!)2^{6d'} < c_k^{d'}.
\]
We will now sharpen our probability bounds for the combined bin and ball events. Setting $s = 160(cm/n)\ln(n/m)$, we get that the probability of the combined bin event is $\exp(-sd/80) =$
\((m/n)^{2\Delta m/n}\). For \(n = \omega(m)\), this upper bound is maximized when \(m/n\) is minimized, and we know that \(m/n \geq 1/c\), so \((m/n)^{2\Delta m/n} \leq (1/c)^{2d}\) which is at least the square of \(c_k^{d'}\). Thus, covering all combinations, we conclude that the combined bin event has probability at most \((m/n)^{\Delta m/n}\).

When it comes to the ball bound, we need to exploit a subtle point that we also used in our expected bound; namely that when we decrease the capacity, it only has effect if the bin was full before the decrease, and then the capacity was at least 2. Our ball event assuming that all bins in \(C'_k\) are full thus implies that at least \(2d'\) balls hash to the intervals \(I_p^+\) for \(p \in C'_k\). We also know that we need \(\Delta m/(2n)\) to fill \(d\) bins. Poisson bound from (13) can be refined to

\[
P = (12e\epsilon/c)^{\max\{\Delta m/(2n),2d'\}} \leq (12\epsilon 160(c/n) \ln(n/m)/c)^{\max\{\Delta m/(2n),2d'\}}
\]

\[
= O((m/n) \log(n/m))^{\max\{\Delta m/(2n),2d'\}}.
\]

For \(n/m = \omega(1)\) and \(n/m \leq c/4\), this bound grows with \(n/m\), so in this case,

\[
P \leq O(\log(c/e)^{2d} \leq 1/e^{1.5d}.
\]

On the other hand, for \(c/4 < n/m \leq c\), we use the \(2d'\) bound, and conclude that

\[
P \leq O(\log(c/e)^{2d'} \leq 1/e^{1.5d'}.
\]

Thus, including our factor \(c^{d'}\) from the union over all configurations, we get the probability of the ball event is bounded by

\[
c^{d'} P \leq P^{1/3} \leq (O(m/n) \log(n/m))^{\max\{\Delta m/(2n),2d'\}/3} \leq (m/n)^{\Omega(\Delta m/(n))}.
\]

This completes the proof that when \(n = \omega(n)\), w.h.p., the total run length over all bins in \(C\) is \(O((\log n)/((cm/n) \log(n/m)))\).

**Bin updates** The last thing we need to analyze is the number of balls that may be moved when a bin is added or removed. The two are symmetric, but the easiest to think about is a bin removal. The direct effect is that the balls in the bin are forwarded to the next bin, but if it is full, they are further forwarded. Then the total number of affected balls is the bin capacity \([cn/m]\) times 1 plus the maximal number of full bins in a run. We already have high probability bounds on the maximal run length. In addition, we have to do \([cn/m]\) bin capacity changes. We already gave high probability bounds on the number of balls that had to be moved with up to \(c\) capacity changes, so now we just have to multiply that number with \([[cn/m]/c]\) \(\leq 1 + n/m\).

Thus, when a bin is added or removed, if \(m = \Omega(n)\), then, w.h.p., the number of balls moved is bounded by

\[
(1 + cm/n)(1 + O((\log n)/c)) + (1 + n/m)O((\log n)/c) = O(cm/n + (n/m)(\log n)).
\]

If \(m = o(n)\), then, w.h.p., the number of balls moved is bounded by

\[
(1 + cn/m) \left(1 + O\left(\frac{\log n}{(cm/n) \log(n/m)}\right)\right) + (1 + n/m)O\left(\frac{\log n}{(cm/n) \log(n/m)}\right).
\]

However, if we get no full bins, then there is no forwarding, so if a bin is removed, then this only affects the ball that were in the bin that are now transferred to the successor. Our high probability bound on the
maximal run length was, in fact, based on a bound on how many balls we could have in an interval, large enough to have several bins, with high probability, so multiplying that number with the capacity, we get a high probability upper bound on how many balls hash directly to any given bin. For $m = \Omega(n)$, the bound is
\[
\left\lfloor \frac{cm}{n} \right\rfloor O \left( \frac{\log n}{c(\log n/m)} \right) = O \left( \frac{\log n}{\log(n/m)} \right).
\]
If this bound is below $\left\lfloor \frac{cm}{n} \right\rfloor$, then there are no bins, so this bound replaces the previous $cm/n$ term. Thus for $m = \Omega(n)$, we conclude, w.h.p., that if a bin added or removed, then the total number of balls moved is $O \left( \frac{\log n}{c(\log n/m)} \right)$, as claimed in Theorem 16.

Likewise, for $m = o(n)$, w.h.p., the maximal number of balls hashing to a bin is
\[
\left\lfloor \frac{cm}{n} \right\rfloor O \left( \frac{\log n}{\log(n/m)} \right) = O \left( \frac{\log n}{\log(n/m)} \right).
\]
Again this replaces the previous $cm/n$. Thus for $m = o(n)$, we conclude, w.h.p., that if a bin added or removed, then the total number of balls moved is $O \left( \frac{\log n}{\log(n/m)} \right)$, as claimed in Theorem 16. This completes the proof of Theorem 16 when $c = \omega(1)$.

**Small capacities** For the cases where $c = O(1)$, we get all he bounds from Theorem 16 follows if we prove, w.h.p., that the longest run of full bins has length $O((\log n)/\varepsilon^2)$.

As in Section 4, we say that the weight of a bin is its capacity divided by the $(1+\varepsilon)m/n$. These weights are between $1/2$ and 2. Suppose there a run of full bins of total weight at least $d$. We consider the shortest prefix $R$ of this run of length at weight at least $d$. The bins have weight less than $d + 2$. Let $(x, y)$ be the interval covered by these bins. Here $x$ is the location of the first bin and $y$ is the location of the first bin after $R$ (for technical reasons, we made the tie breaking such that if a ball and bin landed in the same position, then the ball proceeds the bin). We now have at least $d(1 + \varepsilon)m/n$ balls landing in $[x, y]$. We also know that the total weight of the bins landing in $(x, y)$ is at most $d + 2$.

Let $\ell$ be any value. If $x - y \leq \ell$, then we conclude that we have $d(1 + \varepsilon)m/n$ balls landing in some interval of length $\ell + 1$. On the other hand, if $x - y > \ell$, then bins of weight at most $d + 2$ landing in an interval of length at most $\ell - 1$.

We divide the cycle into at most $2n$ parts of length at most $r/n \geq 1$. Inside an interval of length $\ell - 1$, we can have an interval of length at least $\ell^- = \ell - 1 - 2r/n \geq \ell - 3r/n$. This interval can be chosen in at most $2^{2n}$ ways, and it that consists of a whole number of parts.

We pick $\ell_d$ as a function of $d$.

Define $\delta$ such that $(1 + \varepsilon) = (1 + \delta)^2/(1 - \delta)^2$. Then $\delta = \Omega(\varepsilon)$. We set $\ell = d(r/n)/(1 - \delta)^2$.

We will have $d \geq (\log n)/\delta^2$. Then $\ell^- \geq (d + 2)(r/n)/(1 - \delta)$. It follows that the expected weight of bins in such an interval $I^-$ is $(d + 2)/(1 - \delta)$. Using standard Chernoff bounds, since each bin has weight below 2, the probability of this event is
\[
\exp(-((d + 2)/2\delta^2)/2)) \leq \exp(-\Omega(d\varepsilon^2)).
\]
We can pick $d = O((\log n)/\varepsilon^2)$ large enough that the probability of an underfull interval of length $\ell^-$ is $1/n^\gamma$.

We now consider the other error event: that there is an interval of length $\ell^+$ with too many balls. The bad interval should consist of a minimal number of pieces so as to have length at least $\ell + 1$. It will have length at most $\ell^+ = \ell + 1 + 2r/n \leq \ell + 3r/n \leq \ell(1 + \delta)$. The number of balls expected in such an interval is $m\ell(1 + \delta)/r = d(1 + \delta)(m/bins)/(1 - \delta)^2$. The error event is that we getting $d(1 + \varepsilon)m/n$, which is
\((1 + \delta)\) times bigger than the expectation. By definition, \((1 + \varepsilon)m/n \geq 1\), so by Chernoff bounds, the error probability is
\[
\exp\left(-\left(d/(1 - \delta)\right)\delta^2/3\right) \leq \exp(-\Omega(d\varepsilon^2)).
\]
Again, we can drive the error probability down for the at most \(2n\) possible combinations. This completes our proof of Theorem 16.

**Tabulation hashing** The bounds from Theorem 16 with load balancing \(c = (1 + \varepsilon) = O(1)\) also hold with simple tabulation, which has somewhat weaker Chernoff bounds [PT12]. These Chernoff bounds are weaker in the exponent by a constant factor, and this only costs us a constant factor in the maximal number of consecutive full bins, and that suffices for \(c = (1 + \varepsilon) = O(1)\). For \(c = \omega(1)\), we have to deal with bin capacity changes, which may not be sufficiently independent with simple tabulation, but for \(c = (\log n)^{1+o(1)}\), we can still get the bounds of Theorem 16 if we use twisted tabulation [PT13].

### 7 Simulation Results

To validate the consistency property of our hashing scheme which is theoretically analysed in Theorems 2 and 3, we present the following empirical results. We generated thousands of instances, and tracked the number of ball movements in each operation, and the distribution of bin sizes. We picked the number of bins, \(n\), the average balls per bin ratio \(r = m/n\), and \(\varepsilon\) as follows:

- \(n \in \{10, 20, 40, 70, 100, 150, 200, 300, 450, 600, 800, 1000, 2000\}\).
- \(r = \frac{m}{n} \in \{0.5, 0.8, 1, 1.2, 1.5, 2, 3, 5, 10\}\).
- \(\varepsilon \in \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1.2, 1.5, 1.8, 2, 2.3, 2.5, 2.8, 3\}\).

Figures 2 and 3 show the distribution of bin loads for our algorithm and Consistent Hashing algorithm respectively. Figures 4 and 5 depict the average number of movements in our algorithm for each value of \(\varepsilon\). We start with Figure 2 that shows the distribution of bin loads. The three plots represent three values of \(\varepsilon \in \{0.1, 0.3, 0.9\}\). We expect the load of each bin to be at most \([(1 + \varepsilon)m/n]\). To unify the results of various simulations with different average loads \((m/n)\), we divide the loads of bins by \(m/n\), and sort the normalized bin loads in a decreasing order (breaking ties arbitrarily). The \(y\) coordinate is the normalized load, and the \(x\) coordinate shows the fractional rank of the bin in this order. For instance if a bin’s load is greater than 35% of other bins, its \(x\) coordinate will be 35%. As we expect, no bin has normalized load more than 1 + \(\varepsilon\). A significant fraction of bins have normalized load 1 + \(\varepsilon\), and the rest have normalized loads distributed smoothly in range \([0, 1 + \varepsilon]\). The smaller the \(\varepsilon\) is, the more we expect bins to have normalized loads equal to 1 + \(\varepsilon\), and consequently having a more uniform load distribution.
As shown above, the maximum load never exceeds $1 + \varepsilon$ times the average load in our algorithm. However, you can see below that there is no constant upper bound on maximum load for Consistent Hashing algorithm. We simulate consistent hashing with $n$ balls and $n$ bins for three different values of $n \in \{200, 1000, 8000\}$. As expected, the maximum load grows with $n$, it is expected to be around $\log(n)/\log\log(n)$. The percentage axis is rescaled to highlight the more interesting bin sizes.

Figure 3: Bin loads divided by average load for Consistent Hashing algorithm.

Figure 4 depicts the number of movements per ball operation. Theorems 2, and 3 suggests that the expected number of movements per ball operation is $O(\log(\varepsilon)/\varepsilon)$, and $O(1/\varepsilon^2)$ respectively where $c$ is $1 + \varepsilon$. The solid red curve in Figure 4 depict the average normalized ball movements in all simulations for each value of $\varepsilon$. The bars show the standard deviation of these normalized movements. The dashed black line is the upper bound on these numbers of movements predicted by Theorems 2 and 3 with the following formula:

$$f(\varepsilon) = \begin{cases} 
\varepsilon^2 & \text{if } \varepsilon < 1 \\
1 + \log(1 + \varepsilon)/(1 + \varepsilon) & \text{if } \varepsilon \geq 1 
\end{cases}$$
Our results predict that unlike the ball operations, the average movements per bin operation (insertion or deletion of a bin) is proportional to average density of bins \( r = \frac{m}{n} \). Therefore we designate Figures 5 and 4 to bin and ball operations respectively. We start by elaborating on Figure 5. Theorems 2 and 3 suggest that the average number of movements per bin operation is \( O(r \log(c)/c) \), and \( O(r/\varepsilon^2) \) respectively where \( c = 1 + \varepsilon \). To unify the results of all our simulations, we normalize the number of movements in bin operations with dividing them by \( r \). The solid red curve in Figure 5 depict the average normalized bin movements in all simulations for each value of \( \varepsilon \). The bars show the standard deviation of these normalized movements. The dashed black line is the upper bound function, \( f(\varepsilon) \) (defined above), on these numbers of movements predicted by Theorems 2 and 3. Similarly Figure 4 shows the relation between the number of movements and \( \varepsilon \) for ball insertions and deletions. These are the actual number of movements and are not normalized by the average density \( r \).

Figure 4: Number of movements per ball operation for different values of \( \varepsilon \).

Figure 5: Normalized number of movements per bin operation for different values of \( \varepsilon \).
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