Averages of ratios of characteristic polynomials in circular $\beta$-ensembles and super-Jack polynomials

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Abstract

We study the averages of ratios of characteristic polynomials over circular $\beta$-ensembles, where $\beta$ is a positive real number. Using Jack polynomial theory, we obtain three expressions for ratio averages. Two of them are given as sums of super-Jack polynomials and another one is given by a hyperdeterminant. As applications, we give dualities for ratio averages between $\beta$ and $4/\beta$.

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1 Introduction

Products and ratios of characteristic polynomials averaged over classical groups are closely related to number theory. Keating and Snaith [KS1, KS2] calculate moments of characteristic polynomials over classical groups by using Selberg’s integrals. They conjecture that those moments derive important coefficients appeared in mean values of the Riemann zeta function and $L$-functions. This conjecture, called the Keating-Snaith conjecture, and its extensions to product and ratio averages in [CFKRS2, CFZ2] are motivations for the study of characteristic polynomial averages of random matrices.

Conrey, Farmer, Keating, Rubinstein, and Snaith [CFKRS1] provide the product of characteristic polynomials averaged over classical groups. Furthermore, the averages of the ratio are calculated in [BG, CFZ1, CFS, HPZ]. We note that Bump and Gamburd [BG] evaluate those averages by applying symmetric function theory. Borodin and Strahov [BS] evaluate product and ratio averages in orthogonal, unitary, and symplectic matrix ensembles in terms of determinants and Pfaffians. Borodin and Olshanski [BO] consider products of a natural analogue of the characteristic polynomial in random partitions.

In the present paper, we deal with the circular $\beta$-ensemble ($C\beta E$) for a positive real parameter $\beta > 0$. This is a much-studied random ensemble, which generalize well-known orthogonal ($\beta = 1$), unitary ($\beta = 2$), and symplectic ($\beta = 4$) ensembles of random unitary matrices. In the $C\beta E$, the moments of characteristic polynomials are first calculated in [KSI] by using Selberg’s integral evaluation. Their results are very recently generalized in [M1] (see also Proposition 2.1 in the present paper) by applying Jack polynomial theory. Specifically, the average of products
of characteristic polynomials over the $C\beta E$ is given by a Jack polynomial with a rectangular-shaped Young diagram. Our aim in this paper is to extend this result to the ratio case. To do it, we employ super-Jack polynomials studied in [KOO, O, SV], which are Jack polynomials in two sets of variables. Remark that Ryckman [R] studies the distribution of some characteristic polynomials in the $C\beta E$.

We will give three kinds of expressions for the ratio average. Those have different advantages each other.

The first expression (Theorem 4.1) derives a dual relation identity for ratio averages between the $\beta$-ensemble and $\beta'$-ensemble, with $\beta' = 4/\beta$ for general $\beta > 0$. This duality is implied by a simple duality for super-Jack polynomials. A similar relation for Gaussian $\beta$-ensembles is recently obtained by Desrosiers [D], and therefore our duality is seen to be its analogue for circular ensembles.

The second expression (Theorem 4.3) derives an asymptotic duality of the ratio averages between $\beta$ and $\beta'$ as the number $n$ of variables goes to the infinity. As particular cases of this expression, the product average of characteristic polynomials is written as a multivariate hypergeometric function based on Jack polynomials, introduced by Yan [Y].

Both two expressions explained above are given by a sum of super-Jack polynomials. The super-Jack polynomials, first appeared in [KOO], are defined from the usual Jack symmetric functions via a homomorphism on the algebra of symmetric functions. As proved in [M2], Jack polynomials associated with rectangular diagrams and with special parameters are expressed as a hyperdeterminant. The hyperdeterminant ([BBL, LT, M2]) is a simple generalization of the determinant and is defined for a multi-dimensional array of the form $(A(i_1, i_2, \ldots, i_k))_{1 \leq i_1, i_2, \ldots, i_k \leq n}$. Combining this fact with the first result of us (Theorem 4.1), we obtain a hyperdeterminantal expression for the ratio average, that is our third expression (Theorem 4.6).

The present paper is organized as follows. In Section 2, we review circular $\beta$-ensembles and the previous result for the product average of characteristic polynomials. In Section 3, Jack and super-Jack polynomials are reviewed. Finally, in Section 4, we give our main results, three expressions for the average of ratios of characteristic polynomials over circular $\beta$-ensembles. In addition, in Section 4.4 we remark a Pfaffian expression for ratio averages over the circular orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) ensemble.

Throughout this paper, we let

$$T = \{ z \in \mathbb{C} \mid |z| = 1 \}, \quad D = \{ z \in \mathbb{C} \mid |z| < 1 \},$$

the unit circle and unit open disc. Unless otherwise stated, we let $\beta$ to be a positive real number.

## 2 Circular $\beta$-ensembles

Dyson’s circular $\beta$-ensemble $(C\beta E_n)$ is the probability space $T^n$ with the probability density proportional to

$$\Delta(z; 2/\beta) dz = \prod_{1 \leq i < j \leq n} |z_i - z_j|^\beta \, dz$$
for \( z = (z_1, \ldots, z_n) \in \mathbb{T}^n \). Here \( dz \) is the Haar measure on \( \mathbb{T}^n \) such that \( \int_{\mathbb{T}^n} dz = 1 \), or

\[
\int_{\mathbb{T}^n} F(z) dz = \int_0^{2\pi} \cdots \int_0^{2\pi} F(e^{i\theta_1}, \ldots, e^{i\theta_n}) \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n}
\]

for any continuous function \( F \) on \( \mathbb{T}^n \). The average of \( F \) over the \( C/\beta E_n \) is defined by

\[
\langle F \rangle_{C/\beta E_n} = \langle F(z) \rangle_{z \in C/\beta E_n} = \frac{\int_{\mathbb{T}^n} F(z) \Delta(z; 2/\beta) dz}{\int_{\mathbb{T}^n} \Delta(z; 2/\beta) dz}.
\]

The denominator is explicitly given by a quotient of gamma functions ([AAR §8.7]):

\[
(2.1) \quad \int_{\mathbb{T}^n} \Delta(z; 2/\beta) dz = \frac{\Gamma(\frac{\beta}{2}n + 1)}{\Gamma(\frac{\beta}{2} + 1)^n}.
\]

Consider typical cases \( \beta \in \{1, 2, 4\} \). In those cases, the \( C/\beta E_n \) give eigenvalue distributions for random unitary matrices as follows. See also the standard reference [Me] and [F, Chapter 2].

Let \( S^2(n) = U(n) \), the group of all complex unitary matrices of size \( n \). The probability space \( S^2(n) \) with the Haar measure \( \mu_2(dM) \) is called the circular unitary ensemble (CUE). By Weyl’s integration formula, we see that the probability density distribution for \( n \) eigenvalues \( z_1, \ldots, z_n \) of a matrix in \( S^2(n) \) is given by \( \frac{1}{n!} \Delta(z; 2/\beta) dz \) with \( \beta = 2 \).

Let \( S^1(n) \) be the space of all symmetric matrices in \( U(n) \). The space \( S^1(n) \) is isomorphic to the compact symmetric space \( U(n)/O(n) \), where \( O(n) \) is the orthogonal group of degree \( n \). We equip to \( S^1(n) \equiv U(n)/O(n) \) the probability measure \( \mu_1(dM) \) induced from the Haar measure on \( U(n) \). The probability space \( S^1(n) \) with the measure \( \mu_1(dM) \) is called the circular orthogonal ensemble (COE). The eigenvalue density distribution for matrices in \( S^1(n) \) is proportional to \( \Delta(z; 2/\beta) dz \) with \( \beta = 1 \).

Let \( S^4(n) \) be the space of all Hermitian matrices in \( U(n, \mathbb{H}) \), where \( \mathbb{H} \) is the (noncommutative) field of quaternions and \( U(n, \mathbb{H}) \) is the group of unitary quaternion matrices. The space \( S^4(n) \) is isomorphic to \( U(2n)/Sp(2n) \), where \( Sp(2n) \) is the symplectic group in \( U(2n) \). As the COE, the probability measure \( \mu_4(dM) \) on \( S^4(n) \) is defined. The probability space \( S^4(n) \) with \( \mu_4(dM) \) is called the circular symplectic ensemble (CSE). Each matrix in \( S^4(n) \) is diagonalizable with respect to unitary quaternion matrices and all \( n \) eigenvalues of it belong to \( \mathbb{T} \). The density distribution is proportional to \( \Delta(z; 2/\beta) dz \) with \( \beta = 4 \).

Let \( S^0(n) \) be the set of all diagonal unitary matrices. Then \( S^0(n) \) is obviously identified with \( \mathbb{T}^n \), and the eigenvalue density of the matrix in \( S^0(n) \) is uniform. This is regarded as the \( \beta = 0 \) case in \( C/\beta E_n \).

For general \( \beta > 0 \), Killip and Nenciu [KN] define matrix models whose eigenvalues distributed according to \( \Delta(z; 2/\beta) \). Suppose \( \beta = 1, 2, \) or \( 4 \) again. For a symmetric function \( F \) on \( \mathbb{T}^n \), we let \( \tilde{F} \) the function on \( S^\beta(n) \) given by \( \tilde{F}(M) = F(z_1, \ldots, z_n) \), where the \( z_i \) are eigenvalue of \( M \in S^\beta(n) \). Then the
average of $\tilde{F}$ over $S^3(n)$ equals that of $F$ over the $\mathrm{C}\beta\mathrm{E}_n$:

$$
\langle F(z) \rangle_{z \in \mathrm{C}\beta\mathrm{E}_n} = \int_{S^3(n)} F(M) \mu_\beta(dM) = \begin{cases} 
\langle \tilde{F}(M) \rangle_{M \in \mathrm{COE}_n} & \text{if } \beta = 1, \\
\langle \tilde{F}(M) \rangle_{M \in \mathrm{CUE}_n} & \text{if } \beta = 2, \\
\langle \tilde{F}(M) \rangle_{M \in \mathrm{CSE}_n} & \text{if } \beta = 4.
\end{cases}
$$

Consider the function

$$
\Psi(z; x) = \prod_{j=1}^{n} (1 + xz_j)
$$

for $z = (z_1, \ldots, z_n) \in \mathbb{T}^n$ and $x \in \mathbb{C}$. If $\beta \in \{1, 2, 4\}$, the function $\Psi$ is the characteristic polynomial

$$
\Psi(z; x) = \det(I + xM)
$$

for $M \in S^3(n)$, where $z_1, \ldots, z_n$ are eigenvalues of $M$. Here we should notice that $\det(I + xM)$ for $M \in S^4(n)$ is a quaternion determinant, see [Mac] for detail.

For all $\beta > 0$, the average of products of “characteristic polynomials” $\Psi$ is calculated in [M1] as follows. Let $P^{(\alpha)}_{\lambda}$ be the Jack $P$-polynomial (see the next section).

**Proposition 2.1** ([M1]). Let $\beta$ be any positive real number. Let $x_1, x_2, \ldots, x_L, x_{L+1}, \ldots, x_{L+K}$ be complex numbers. Assume that $x_i$, $1 \leq i \leq L$ are nonzero. Then it holds that

$$
\langle \prod_{l=1}^{L} \Psi(z_l; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(z; x_{L+k}) \rangle_{z \in \mathrm{C}\beta\mathrm{E}_n} = (x_1 x_2 \ldots x_L)^{-n} \cdot P^{(1/\alpha)}_{(n^L)}(x)
$$

with $\alpha = 2/\beta$ and $x = (x_1, x_2, \ldots, x_L, x_{L+1}, \ldots, x_{L+K})$. Here $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$.

**Example 2.1.** For $x \in \mathbb{C} \setminus \{0\}$ and $y \in \mathbb{C}$, we have

$$
\langle \Psi(z; x^{-1}) \Psi(z; y) \rangle_{z \in \mathrm{C}\beta\mathrm{E}_n} = x^{-n} P^{(1/\alpha)}_{(n)}(x, y) = \frac{n!}{(\alpha)_n} \sum_{j=0}^{n} \frac{(\alpha)_j (\alpha)_{n-j}}{j!(n-j)!} x^{-j} y^j,
$$

with $\alpha = 2/\beta$. Here $(\alpha)_k = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + k - 1)$ is the Pochhammer symbol.

Consider the limit $\beta \to 0$ in (2.2). Then the left hand side on (2.2) is the average with respect to the uniform measure on $\mathbb{T}^n$. The limit of Jack $P$-polynomial $P^{(\alpha)}_{\lambda}$ as $\alpha \to 0$ is well-defined and is given by ([Mac VI.10])

$$
P^{(0)}_{\lambda} = e_{\lambda'} = e_{\lambda'_1} e_{\lambda'_2} \cdots,
$$

where $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the conjugate partition of $\lambda$ (see the next section) and $e_k$ is the elementary symmetric polynomial defined in (3.3) below. Hence the right hand side of (2.2) is equal to $(x_1 \cdots x_L)^{-n} e_L(x_1^n, x_2^n, \ldots, x_{L+K}^n)$.

Consider the limit $\beta \to \infty$ in (2.2). This is the so-called low temperature limit, see [F §3.6.1]. Then, since $P^{(\infty)}_{\lambda} = m_\lambda$ ([Mac VI.10]), the right hand side in (2.2) equals

$$
(x_1 \cdots x_L)^{-n} m_{(n^L)}(x) = (x_1 \cdots x_L)^{-n} e_L(x_1^n, x_2^n, \ldots, x_{L+K}^n).$$
Here \( m_\lambda \) is the monomial symmetric polynomial defined in (3.1).

Our main purpose in this paper is to extend the identity (2.2) to the ratio of the form
\[
\left( \prod_{l=1}^{L} \Psi(z; x_{l-1}) \cdot \prod_{k=1}^{K} \Psi(z; x_{L+k}) \right) \left( \prod_{s=1}^{S} \Psi(z; x_{s})^{\beta/2} \cdot \prod_{t=1}^{T} \Psi(z; x_{t})^{\beta/2} \right)_{z \in \mathbb{C}, \beta \in \mathbb{E}_n}.
\]

But, in some cases, we regard it with letting \( S = 0 \). We remark that, in the denominator on the average (2.3), we deal with the \( \beta/2 \)-power of characteristic polynomials. This is a consequence of an application of Jack polynomial theory. In the case of Gaussian ensembles, the same power of characteristic polynomials is appeared, see [D, Proposition 2]. See Remark 3.2 below.

3 Jack and super-Jack functions

3.1 Jack functions

Following to Macdonald’s book [Mac], we recall Young diagrams and Jack functions. For a Young diagram \( \lambda = (\lambda_1, \lambda_2, \ldots) \), identified with the partition, we write as \( \lambda' = (\lambda'_1, \lambda'_2, \ldots) \) the transposed diagram. Equivalently, the number \( \lambda_i \) and \( \lambda'_j \) is the length of the \( i \)-th row and of the \( j \)-th column in the diagram \( \lambda \), respectively. Define
\[
z_\lambda = \prod_{k \geq 1} k^{m_k} m_k!,
\]
where \( m_k = m_k(\lambda) \) is the number of rows whose lengths equal to \( k \): \( m_k = \# \{ i \geq 1 \mid \lambda_i = k \} \).

We sometimes express \( \lambda \) as \( \lambda = (1^{m_1} 2^{m_2} \ldots) \). We put
\[
|\lambda| = \sum_{i \geq 1} \lambda_i = \sum_{k \geq 1} km_k \quad \text{and} \quad \ell(\lambda) = \lambda'_1 = \sum_{k \geq 1} m_k.
\]

These values are the weight and length of \( \lambda \). For two diagrams \( \lambda, \mu \), we write \( \lambda \subset \mu \) if \( \lambda_i \leq \mu_i \) for all \( i \).

Let \( \alpha \) be a positive real number and let \( \Lambda = \Lambda(\alpha) \) be the \( \mathbb{Q}(\alpha) \)-algebra of symmetric functions (Mac 1.2]). Define the scalar product \( \langle \cdot, \cdot \rangle_\alpha \) on \( \Lambda \) by
\[
\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda \mu} \alpha^{\ell(\lambda)} z_\lambda.
\]

Here the \( p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \) are Newton power-sum symmetric functions \( p_k(x_1, x_2, \ldots) = x_1^k + x_2^k + \cdots \), and form a basis of \( \Lambda \). Let \( m_\lambda \) be the monomial symmetric function: if \( \ell(\lambda) \leq n \),
\[
m_\lambda(x_1, \ldots, x_n) = \sum_{(\gamma_1, \ldots, \gamma_n)} x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n},
\]
summed over all distinct permutations \( (\gamma_1, \ldots, \gamma_n) \) of \( \lambda = (\lambda_1, \ldots, \lambda_n) \). For each diagram \( \lambda \), the Jack \( P \)-function \( P^{(\alpha)}_\lambda \) is defined as a unique symmetric function such that
\[
P^{(\alpha)}_\lambda = m_\lambda + \sum_{\mu < \lambda} u^{(\alpha)}_{\lambda \mu} m_\mu, \quad u^{(\alpha)}_{\lambda \mu} \in \mathbb{Q}(\alpha)
\].
and

\[(P^{(\alpha)}_\lambda, P^{(\alpha)}_\mu)_\alpha = 0 \quad \text{if } \lambda \neq \mu.\]

Here \(\mu < \lambda\) stands for the dominance order for diagrams:

\[
\mu \leq \lambda \quad \text{def} \quad |\lambda| = |\mu| \quad \text{and} \quad \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \quad \text{for any } i \geq 1.
\]

We do not distinguish the terminology “Jack functions” from “Jack polynomials”.

Let \(b^{(\alpha)}_\lambda = \langle P^{(\alpha)}_\lambda, P^{(\alpha)}_\lambda \rangle^{-1}\) and \(Q^{(\alpha)}_\lambda = b^{(\alpha)}_\lambda P^{(\alpha)}_\lambda\). The value \(b^{(\alpha)}_\lambda\) is given by

\[
b^{(\alpha)}_\lambda = c_\lambda(\alpha)/c_\lambda'(\alpha), \quad c_\lambda(\alpha) = \prod_{s \in \lambda} (\alpha a_\lambda(s) + l_\lambda(s) + 1), \quad c_\lambda'(\alpha) = \prod_{s \in \lambda} (\alpha a_\lambda(s) + l_\lambda(s) + \alpha),
\]

where \(s = (i, j) \in \lambda\) runs over all squares in \(\lambda\), i.e., \(1 \leq i \leq \ell(\lambda)\) and \(1 \leq j \leq \lambda_i\). Here \(a_\lambda(s) = \lambda_i - j\) and \(l_\lambda(s) = \lambda_j' - i\) may be called the arm-length and leg-length of \(s\), respectively. Observe that \(c_\lambda'(\alpha) = \alpha \mid c_\lambda(\alpha^{-1})\).

Let \(\lambda = (k) = (k, 0, 0, \ldots)\). Then \(Q^{(\alpha)}_{(k)} = g^{(\alpha)}_k\), where \(g^{(\alpha)}_k\) is given by

\[
g^{(\alpha)}_k(x_1, x_2, \ldots) = \sum_{k_1, k_2, \ldots \geq 0 \atop k_1 + k_2 + \cdots = k} \frac{(\alpha^{-1})_{k_1}(\alpha^{-1})_{k_2} \cdots}{k_1!k_2!\cdots} x_1^{k_1}x_2^{k_2}\cdots,
\]

with \((u)_k = u(u + 1) \cdots (u + k - 1)\). For convenience, we let \(g^{(\alpha)}_k = 0\) for negative integers \(k\). The generating function for \(b^{(\alpha)}_k\) is

\[
\sum_k g^{(\alpha)}_k(x_1, x_2, \ldots)z^k = \prod_{i \geq 1} (1 - x_i z)^{-1/\alpha}.
\]

In symmetric function theory, the generating function is regarded as a formal power series but in this paper we will suppose that \(x_i \in \mathbb{D}\) for all \(i \geq 1\) and that \(x_i = 0\) for \(i\) large enough.

Let \(\lambda = (1^k) = (1, 1, \ldots, 1)\). Then \(P^{(\alpha)}_{(1^k)} = m_{(1^k)} = e_k\), where

\[
e_k(x_1, x_2, \ldots) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k} x_{j_1}x_{j_2}\cdots x_{j_k}
\]

is the elementary symmetric function. In particular, the Jack \(P\)-function of a 1-column diagram is independent with \(\alpha\). Letting

\[
[u]^{(\alpha)}_\lambda = \prod_{i \geq 1} \frac{\Gamma(u - (i - 1)/\alpha + \lambda_i)}{\Gamma(u - (i - 1)/\alpha)} = \prod_{s \in \lambda} (u - l'_{\lambda}(s)/\alpha + a'_{\lambda}(s)),
\]

with \(a'_{\lambda}(s) = j - 1\) and \(l'_{\lambda}(s) = i - 1\), we have

\[
[u]^{(1/\alpha)}_{\lambda'\lambda} = (-\alpha)^{\mid \lambda \mid} (-u/\alpha)_{\lambda}^{(\alpha)}.
\]
For partitions \( \lambda, \mu \) such that \( \ell(\lambda), \ell(\mu) \leq n \), it holds \((\text{Mac} \ VI, (10.20) \text{ and } (10.37))\)

\[
P^{(\alpha)}_{\lambda}(1^n) = \frac{a^{[\lambda]}_{\lfloor n/\alpha \rfloor}^{(\alpha)}_{\lambda}}{c^{(\alpha)}_{\lambda}},
\]

(3.5)

\[
\left\langle P^{(\alpha)}_{\lambda}(z) \bar{Q}^{(\alpha)}_{\mu}(z) \right\rangle_{z \in C_{\alpha}E_n} = \delta_{\lambda \mu} \frac{[n/\alpha]^{(\alpha)}_{\lambda}}{[1 + (n-1)/\alpha]^{(\alpha)}_{\lambda}},
\]

(3.6)

with \( \alpha = 2/\beta \).

Let \( \omega_{\alpha} \) be the automorphism on \( \Lambda \) defined by

\[
\omega_{\alpha}(p_r) = (-1)^{r-1} \alpha p_r, \quad r \geq 1.
\]

(3.7)

Then \( \omega_{\alpha}(P^{(\alpha)}_{\lambda}) = Q^{(1/\alpha)}_{\lambda'} \) and \( \omega_{\alpha}(Q^{(\alpha)}_{\lambda}) = P^{(1/\alpha)}_{\lambda'} \) \((\text{Mac} \ VI (10.17))\).

The following identities are known as the Cauchy identity and the dual one for Jack functions \((\text{Mac} \ VI, (4.13) \text{ and } (5.4))\):

\[
\sum \lambda P^{(\alpha)}_{\lambda}(x_1, x_2, \ldots) Q^{(\alpha)}_{\lambda}(y_1, y_2, \ldots) = \prod_{i, j \geq 1} (1 - x_i y_j)^{-1/\alpha},
\]

(3.8)

\[
\sum \lambda \bar{P}^{(\alpha)}_{\lambda}(x_1, x_2, \ldots) P^{(1/\alpha)}_{\lambda'}(y_1, y_2, \ldots) = \prod_{i, j \geq 1} (1 + x_i y_j).
\]

(3.9)

We will use the following reduction formula \([\text{Mac} \ VI (4.17)]\): for a partition \( \lambda \) of length \( n \),

\[
P^{(\alpha)}_{\lambda}(x_1, \ldots, x_n) = x_1 \cdots x_n P^{(\alpha)}_{\mu}(x_1, \ldots, x_n)
\]

where \( \mu = (\lambda_1 - 1, \ldots, \lambda_n - 1) \).

### 3.2 Super-Jack functions

To calculate the ratio average of characteristic polynomials in the next section, we need Jack polynomials in two sets of variables.

Define the coefficients \( f^\lambda_{\mu \nu}(\alpha) \in \mathbb{Q}(\alpha) \) via

\[
P^{(\alpha)}_{\mu} P^{(\alpha)}_{\nu} = \sum \lambda f^\lambda_{\mu \nu}(\alpha) P^{(\alpha)}_{\lambda},
\]

or equivalently,

\[
f^\lambda_{\mu \nu}(\alpha) = f^\lambda_{\nu \mu}(\alpha) = \langle Q^{(\alpha)}_{\lambda}, P^{(\alpha)}_{\mu} P^{(\alpha)}_{\nu} \rangle_{\alpha}.
\]

The following properties are seen in \([\text{Mac} \ VI.7 \text{ and 10}]\).

**Lemma 3.1.** The numbers \( f^\lambda_{\mu \nu}(\alpha) \) satisfy the following properties.

1. \( f^\lambda_{\mu \nu}(\alpha) = 0 \) unless \( |\lambda| = |\mu| + |\nu| \) and \( \lambda \supset \mu, \lambda \supset \nu \).

2. \( f^\lambda_{\mu \nu}(1) = c^\lambda_{\mu \nu} \), where \( c^\lambda_{\mu \nu} \) are Littlewood-Richardson coefficients \((\text{e.g. } [\text{Mac} \ I.5])\).
Lemma 3.2. The functions $f_{\mu\nu}^\lambda(\alpha) = f_{\mu'\nu'}(1/\alpha) \frac{b_{\alpha}^{(\alpha)}}{b_{\alpha'}^{(\alpha')}}$.

Define skew Jack functions by

$$P_{\lambda/\mu}^{(\alpha)} = \sum_{\nu} f_{\mu\nu}^\lambda(\alpha) P_{\nu}^{(\alpha)}, \quad Q_{\lambda/\mu}^{(\alpha)} = \sum_{\nu} f_{\mu\nu}^\lambda(\alpha) Q_{\nu}^{(\alpha)}.$$  

Then we have $Q_{\lambda/\mu}^{(\alpha)} = \frac{b_{\alpha}^{(\alpha)}}{b_{\alpha'}^{(\alpha')}} P_{\lambda/\mu}^{(\alpha)}$ and the duality

$$\omega_{\alpha}(P_{\lambda/\mu}^{(\alpha)}) = Q_{\lambda'/\mu'}^{(1/\alpha)}, \quad \omega_{\alpha}(Q_{\lambda/\mu}^{(\alpha)}) = P_{\lambda'/\mu'}^{(1/\alpha)}.$$  

Let $x, y$ be the set of (possibly infinite many or empty) variables. Define

$$\tilde{Q}_{\lambda}^{(\alpha)}(x; y) := \sum_{\mu, \nu} f_{\mu\nu}^\lambda(\alpha) Q_{\mu}^{(\alpha)}(x) P_{\nu}^{(1/\alpha)}(y) = \sum_{\nu} Q_{\lambda/\nu}^{(\alpha)}(x) P_{\nu}^{(1/\alpha)}(y),$$

$$\tilde{P}_{\lambda}^{(\alpha)}(x; y) := \sum_{\mu, \nu} f_{\mu\nu}^\lambda(\alpha) P_{\mu}^{(\alpha)}(x) Q_{\nu}^{(1/\alpha)}(y) = \sum_{\nu} P_{\lambda/\nu}^{(\alpha)}(x) Q_{\nu}^{(1/\alpha)}(y).$$

We call these functions super-Jack functions (polynomials). This terminology is first appeared in [O] as far as the author knows. Sergeev and Veselov [SV] proved that these functions are the eigenfunctions for a deformed Calogero-Moser-Sutherland operator.

Remark 3.1. Our definition for the super-Jack function slightly differs from that in [O]. Specifically, the function $P_{\lambda}(x|y; \theta)$ in [O Eq.(2.7)] is given by $P_{\lambda}(x|y; \theta) = P_{\lambda}^{(1/\theta)}(x; \theta y)$ with $\theta > 0$, where $\theta y = (\theta y_1, \theta y_2, \ldots)$.

Note that $\tilde{Q}_{\lambda}^{(1)}(x; y) = \tilde{P}_{\lambda}^{(1)}(x; y)$ is the Littlewood-Shur symmetric function (or super-Shur function) $\text{LS}_\lambda(x; y) = \sum_{\nu} s_{\lambda/\nu}(x) s_{\nu}(y)$ (see [BG] Section 3 and [Mac] I-5, Ex.23). It is obvious that

$$\tilde{Q}_{\lambda}^{(\alpha)}(x; \emptyset) = Q_{\lambda}^{(\alpha)}(x), \quad \tilde{P}_{\lambda}^{(\alpha)}(x; \emptyset) = P_{\lambda}^{(\alpha)}(x),$$

$$\tilde{Q}_{\lambda}^{(\alpha)}(\emptyset; y) = P_{\lambda}^{(1/\alpha)}(y), \quad \tilde{P}_{\lambda}^{(\alpha)}(\emptyset; y) = Q_{\lambda}^{(1/\alpha)}(y).$$

If $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_q)$, since $Q_{\lambda/\mu}^{(\alpha)}(x_1, \ldots, x_p) = 0$ unless $0 \leq \lambda_i' - \mu_i' \leq p$ for each $i \geq 1$ (see [Mac] VI (10.18)), it follows that $Q_{\lambda}^{(\alpha)}(x_1, \ldots, x_p; y_1, \ldots, y_q) = 0$ unless $\lambda_p + 1 \leq q$. A partition $\lambda$ satisfying $\lambda_{p+1} \leq q$ is said to be a fat $(p, q)$-hook.

Lemma 3.2. The functions $\tilde{P}_{\lambda}^{(\alpha)}$ and $\tilde{Q}_{\lambda}^{(\alpha)}$ have the following properties.

1. $\tilde{Q}_{\lambda}^{(\alpha)}(x; y) = b_{\lambda}^{(\alpha)} \tilde{P}_{\lambda}^{(\alpha)}(x; y)$.

2. $\tilde{Q}_{\lambda}^{(\alpha)}(x; y) = \tilde{P}_{\lambda/\nu}^{(1/\alpha)}(y; x)$.  

Lemma 3.3. Let \( \phi(3.13) \lambda \) for any partitions \( \omega \). Denote by \( Q \) independent over \( \{\text{Mac, VI.} (7.9)\} \)

\[ \text{implies a dual relation for ratio averages of characteristic polynomials in circular } \beta \text{ -ensembles, see Corollary 4.2 below.} \]

The second identity in Lemma 3.2 is the duality for super-Jack functions. This property implies a dual relation for ratio averages of characteristic polynomials in circular \( \beta \)-ensembles, see Corollary 4.2 below.

The Jack Q-function \( Q^{(a)}_\lambda (x \sqcup y) \) in variables \( x \sqcup y = (x_1, y_1, x_2, y_2, \ldots) \) is expressed as (\[\text{Mac, VI.} (7.9)\])

\[ Q^{(a)}_\lambda (x \sqcup y) = \sum_{\nu} Q^{(a)}_{\lambda \nu} (x) Q^{(a)}_{\nu} (y). \]

Denote by \( \omega^y_\mu \) the homomorphism \( \omega_\alpha \) acting on symmetric functions of the \( y \) variables. Observe that

\[ (3.12) \; \; \; \hat{Q}^{(a)}_\lambda (x; y) = \sum_{\nu} Q^{(a)}_{\lambda \nu} (x) \omega^y_\mu (Q^{(a)}_{\nu} (y)) = \omega^y_\mu Q^{(a)}_\lambda (x \sqcup y). \]

This fact implies that a super-Jack function is a specialization of the Jack function. In fact, the algebra \( \Lambda \) of symmetric functions is generated by \( \{g^{(a)}_k \mid k \geq 1\} \) and the \( g^{(a)}_k \) are algebraically independent over \( Q(\alpha). \) Define the homomorphism \( \phi_\alpha \) on \( \Lambda \) by

\[ \phi_\alpha (g^{(a)}_k) = \hat{g}^{(a)}_k (x; y) := \omega^y_\mu g^{(a)}_k (x \sqcup y) = \sum_{l=0}^{k} g^{(a)}_l (x) e_{k-l} (y). \]

Equivalently, we define \( \phi_\alpha \) by \( \phi_\alpha (p_k) = p_k (x) + (-1)^{k-1} \alpha p_k (y) \) for power-sums \( p_k. \) Then

\[ (3.13) \; \; \; \phi_\alpha (Q^{(a)}_\lambda) = \hat{Q}^{(a)}_\lambda (x; y) \]

for any partitions \( \lambda. \)

The following is the Cauchy identity for super-Jack functions.

Lemma 3.3. Let \( x = (x_i), y = (y_j), u = (u_i), v = (v_i) \) be sequences of variables. Then

\[
\sum_{\lambda} \hat{P}^{(a)}_\lambda (x; u) \hat{Q}^{(a)}_\lambda (y; v) \\
= \prod_{i,j \geq 1} (1 - x_i y_j)^{-1/\alpha} \cdot \prod_{i,j \geq 1} (1 + x_i v_j) \cdot \prod_{i,j \geq 1} (1 + u_i y_j) \cdot \prod_{i,j \geq 1} (1 - u_i v_j)^{-\alpha}.
\]
In particular,

\[ (3.14) \quad \sum_{\lambda} P^{(\alpha)}_{\lambda}(x) Q^{(\alpha)}_{\lambda}(y; v) = \prod_{i,j \geq 1} (1 - x_i y_j)^{-1/\alpha} \cdot \prod_{i,j \geq 1} (1 + x_i v_j). \]

Proof. We apply \( \omega^u_\alpha \omega^v_\alpha \) to the Cauchy identity (recall (3.8))

\[ \sum_{\lambda} P^{(\alpha)}_{\lambda}(x \uplus u) Q^{(\alpha)}_{\lambda}(y \uplus v) = \prod_{i,j \geq 1} (1 - x_i y_j)^{-1/\alpha} \cdot \prod_{i,j \geq 1} (1 - u_i v_j)^{-1/\alpha} \cdot \prod_{i,j \geq 1} (1 - u_i v_j)^{-1/\alpha}. \]

Then the desired identity follows from (3.12). Here we have used

\[ \omega^u_\alpha \omega^v_\alpha \prod_{i,j \geq 1} (1 - u_i v_j)^{-1/\alpha} = \omega^u_\alpha \omega^v_\alpha \sum_{\lambda} Q^{(\alpha)}_{\lambda}(u) P^{(\alpha)}_{\lambda}(v) \]

\[ = \sum_{\lambda} P^{(1/\alpha)}_{\lambda'}(u) Q^{(1/\alpha)}_{\lambda'}(v) = \prod_{i,j \geq 1} (1 - u_i v_j)^{-\alpha}. \]

\[ \square \]

4 Averages of ratios of characteristic polynomials

Our purpose is to obtain three kinds of expressions for the ratio average (2.3) and for its degenerations.

4.1 First expression and a duality between \( \beta \) and \( 4/\beta \)

For two diagrams \( \lambda \) and \( \mu \), we define \( \lambda + \mu \) to be the sum of the sequences: \( (\lambda + \mu)_i = \lambda_i + \mu_i \). Also we define \( \lambda \cup \mu \) to be the diagram whose rows are those of \( \lambda \) and \( \mu \), arranged in descending order. Observe that \( (\lambda + \mu)' = \lambda' \cup \mu' \).

**Theorem 4.1.** Let

\[ x = (x_1, \ldots, x_L, x_{L+1}, \ldots, x_{L+K}), \quad u = (u_1, \ldots, u_S), \quad v = (v_1, \ldots, v_T) \]

be sequences of complex numbers. Assume that \( x_i \neq 0 \) for \( 1 \leq i \leq L \) and \( u_s, v_t \in \mathbb{D} \) for \( 1 \leq s \leq S \) and \( 1 \leq t \leq T \). Then we have

\[ \left\langle \prod_{s=1}^L \Psi(z; x_i^{-1}) \cdot \prod_{k=1}^K \Psi(z; x_{L+k}) \prod_{i=1}^L \Psi(z; -u_s)^{2/\beta} \prod_{t=1}^T \Psi(z; -v_t)^{2/\beta} \right\rangle \]

\[ = (x_1 \cdots x_L)^{-n} \sum_{\mu, \mu' \leq n} \frac{[-n(1/\alpha)]}{[-n + 1 - \alpha(1/\alpha)]} \frac{[-n - \alpha L_{\mu}(1/\alpha)]}{[-n + 1 - (L + 1)\alpha(1/\alpha)]} \hat{Q}^{(1/\alpha)}_{(nL)_{\mu}}(x; v) Q^{(\alpha)}_{\mu'}(u) \]

with \( \alpha = 2/\beta \).
Proof. First we see that
\[
\prod_{l=1}^{L} \Psi(\mathbf{z}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; x_{L+k}) = (x_1 \cdots x_L)^{-n} \cdot (z_1 \cdots z_n)^{-L} \cdot \prod_{k=1}^{L+K} \Psi(\mathbf{z}; x_k).
\]
By Cauchy identities (3.8) and (3.11), we have
\[
\frac{\prod_{s=1}^{S} \Psi(\mathbf{z}; -u_s)^{\beta/2} \cdot \prod_{t=1}^{T} \Psi(\mathbf{z}; -v_t)^{\beta/2}}{\prod_{s=1}^{S} \Psi(\mathbf{z}; -u_s)^{\beta/2} \cdot \prod_{t=1}^{T} \Psi(\mathbf{z}; -v_t)^{\beta/2}} = \sum_{\lambda, \mu} \hat{Q}_{\lambda}^{(\alpha)}(v; x)Q_{\mu}^{(\alpha)}(u)P_{\lambda}^{(\alpha)}(z)P_{\mu}^{(\alpha)}(\mathbf{z}).
\]
Since \( P_{(L^n)}^{(\alpha)}(\mathbf{z}) = (z_1 \cdots z_n)^{L} \) by (3.9), we see that
\[
\frac{\prod_{l=1}^{L} \Psi(\mathbf{z}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; x_{L+k})}{\prod_{s=1}^{S} \Psi(\mathbf{z}; -u_s)^{\beta/2} \cdot \prod_{t=1}^{T} \Psi(\mathbf{z}; -v_t)^{\beta/2}} = \sum_{\lambda, \mu} \hat{Q}_{\lambda}^{(\alpha)}(v; x)Q_{\mu}^{(\alpha)}(u)P_{\lambda}^{(\alpha)}(z)P_{\mu}^{(\alpha)}(\mathbf{z})\zeta_{\mathbf{C}_B E_n}.
\]
By using (3.9) and the orthogonality (3.6), we have
\[
\left\langle P_{\lambda}^{(\alpha)}(z)P_{\mu}^{(\alpha)}(\mathbf{z})P_{(L^n)}^{(\alpha)}(\mathbf{z}) \right\rangle_{\zeta_{\mathbf{C}_B E_n}} = \left\langle P_{\lambda}^{(\alpha)}(z)P_{\mu}^{(\alpha)}(\mathbf{z})P_{(L^n)}^{(\alpha)}(\mathbf{z}) \right\rangle_{\zeta_{\mathbf{C}_B E_n}},
\]
for partitions \( \lambda \) and \( \mu \) such that \( \ell(\lambda) \leq n \) and \( \ell(\mu) \leq n \). Therefore the desired average is equal to
\[
\prod_{l=1}^{L} x_l^{-n} \cdot \sum_{\mu: \mu(\mu) \leq n} \frac{[n/\alpha]_{\mu+(L^n)}}{[1 + (n-1)/\alpha]_{\mu+(L^n)}} \hat{P}_{\mu+(L^n)}^{(\alpha)}(v; x)Q_{\mu}^{(\alpha)}(u).
\]
Using equation (3.4) and the second claim in Lemma 3.2 and replacing \( \mu \) by \( \mu' \), it equals
\[
\prod_{l=1}^{L} x_l^{-n} \cdot \sum_{\mu: \mu(\mu) \leq n} \frac{[-n]_{(nL)\cup \mu}}{-n + \alpha_{(nL)\cup \mu}} \hat{Q}_{(nL)\cup \mu}^{(1/\alpha)}(x; v)Q_{\mu}^{(\alpha)}(u).
\]
Observing
\[
[u]_{(nL)\cup \mu}^{(1/\alpha)} = [u]_{(nL)}^{(1/\alpha)} \cdot [u - \alpha L]_{\mu}^{(1/\alpha)},
\]
we obtain the claim. \( \square \)

As a special case of Theorem 4.1, we have the following: Letting \( u = \emptyset \), since
\[
\frac{[-n]_{(nL)}^{(1/\alpha)}}{-n + \alpha_{(nL)}^{(1/\alpha)}} = b_{(L^n)}^{(\alpha)},
\]
we have

\[
\left( \prod_{i=1}^{L} \frac{\Psi(z; x_i^{-1}) \prod_{k=1}^{K} \Psi(z; x_{L+k})}{\prod_{l=1}^{T} \Psi(z; -v_l)^{\beta/2}} \right)_{z \in \mathbb{C}\beta E_n} = (x_1 \cdots x_L)^{-n} \cdot \hat{P}_{(nL)}^{(1/\alpha)}(\mathbf{x}; \mathbf{v})
\]

Moreover, if we let \( \mathbf{v} = \emptyset \), we recover Proposition \[2.1\].

The equation \( \hat{Q}_{\lambda}^{(2)}(\mathbf{v}; \mathbf{x}) = \hat{P}_{\lambda'}^{(1/\alpha)}(\mathbf{x}; \mathbf{v}) \) derives the following duality relation for ratio averages between the \( \beta \)-ensemble and another \( \beta' \)-ensemble with \( \beta' = 4/\beta \). A similar result for Gaussian \( \beta \)-ensembles is seen in \[D\].

**Corollary 4.2.** Let \( \beta \) be a positive real number and set \( \beta' = 4/\beta \). Put

\[ \mathbf{x} = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+K}), \quad \mathbf{v} = (v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+T}) \]

Assume \( x_i, v_j \in \mathbb{D} \) for all \( i, j \). Then we have

\[
\left( \prod_{i=1}^{m} \frac{\Psi(z; x_i^{-1}) \prod_{k=1}^{K} \Psi(z; x_{m+k})}{\prod_{l=1}^{n} \Psi(z; -v_l)^{\beta/2}} \right)_{z \in \mathbb{C}\beta E_n} = (v_1 \cdots v_n)^m \cdot \left( \prod_{j=1}^{n} \frac{\Psi(w; v_j^{-1}) \prod_{t=1}^{T} \Psi(w; v_{n+t})}{\prod_{k=1}^{m+K} \Psi(w; -x_k)^{\beta'/2}} \right)_{w \in \mathbb{C}\beta E_m}.
\]

Here \( \Psi(w; x) = \prod_{j=1}^{m} (1 + xw_j) \) for \( \mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{T}^m \).

**Proof.** By \( (4.1) \), the left hand side on the equation in the statement is equal to \( b_{(n^m)}^{(\beta/2)} \cdot \hat{P}_{(n^m)}^{(\beta/2)}(\mathbf{x}; \mathbf{v}) = \hat{Q}_{(n^m)}^{(\beta/2)}(\mathbf{x}; \mathbf{v}) \). On the other hand, the right hand side is equal to \( \hat{Q}_{(n^m)}^{(\beta/2)}(\mathbf{x}; \mathbf{v}) \) by \( (4.2) \). \( \square \)

Roughly speaking, there exists a duality:

\[
\begin{pmatrix} \beta \\ n \\ m \\ x \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} 4/\beta \\ m \\ n \\ u \\ v \end{pmatrix}.
\]

### 4.2 Second expression and its asymptotic behavior as \( n \to \infty \).

We obtain the second expression for the ratio average.

**Theorem 4.3.** Let

\[ \mathbf{x} = (x_1, \ldots, x_L), \quad \mathbf{y} = (y_1, \ldots, y_K), \quad \mathbf{u} = (u_1, \ldots, u_S), \quad \mathbf{v} = (v_1, \ldots, v_T) \]
be sequences of complex numbers. Assume $u_s, v_t \in \mathbb{D}$ for all $s, t$. Then

$$
\left< \frac{\prod_{i=1}^{L} \Psi_{\mathbf{z}}(x_i) \cdot \prod_{k=1}^{K} \Psi_{\mathbf{z}}(y_k)}{\prod_{s=1}^{S} \Psi_{\mathbf{z}}(-u_s)^{\beta/2} \cdot \prod_{t=1}^{T} \Psi_{\mathbf{z}}(-v_t)^{\beta/2}} \right>_{\mathbf{z} \in C_\beta E_n}
$$

(4.3)

$$
= \sum_{\lambda, \lambda_1 \leq n} \frac{[-n]^{(1/\alpha)}_{\lambda}}{[-n + 1 - \alpha]^{(1/\alpha)}_{\lambda}} \hat{\lambda}^{(1/\alpha)}_{\lambda}(x; u) \hat{\lambda}^{(1/\alpha)}_{\lambda}(y; v)
$$

(4.4)

with $\alpha = 2/\beta$.

**Proof.** By the generalized Cauchy identity (3.14),

$$
\left< \frac{\prod_{i=1}^{L} \Psi_{\mathbf{z}}(x_i) \cdot \prod_{k=1}^{K} \Psi_{\mathbf{z}}(y_k)}{\prod_{s=1}^{S} \Psi_{\mathbf{z}}(-u_s)^{\beta/2} \cdot \prod_{t=1}^{T} \Psi_{\mathbf{z}}(-v_t)^{\beta/2}} \right>_{\mathbf{z} \in C_\beta E_n}
$$

$$
= \sum_{\lambda, \mu} \hat{\lambda}^{(\alpha)}_{\lambda}(u; x) \hat{\lambda}^{(\alpha)}_{\mu}(v; y) \left< \prod_{\lambda}^{(\alpha)}_{\lambda}(z) \prod_{\mu}^{(\alpha)}_{\mu}(z) \right>_{\mathbf{z} \in C_\beta E_n}.
$$

From orthogonality (3.6) it equals

$$
\sum_{\lambda, \mu \in (\lambda) \leq n} \frac{[n/\alpha]^\alpha_{\lambda}}{1 + (n - 1)/\alpha} \hat{\lambda}^{(\alpha)}_{\lambda}(u; x) \hat{\lambda}^{(\alpha)}_{\mu}(v; y).
$$

(4.5)

Using equation (3.4) and the second claim in Lemma 3.2, we obtain our statement. \qed

The multivariate hypergeometric functions in two sets of variables are defined by (Y)

$$
\begin{align*}
F_q^{(\alpha)}(a_1, \ldots, a_p; b_1, \ldots, b_q; x_1, \ldots, x_n; y_1, \ldots, y_n) &= \sum_{\lambda} \prod_{i=1}^{p} [a_i]^{(\alpha)}_{\lambda} \prod_{i=1}^{q} [b_i]^{(\alpha)}_{\lambda} c_{\lambda}^{(\alpha)}(\alpha) P_{\lambda}^{(\alpha)}(1^n) \\
&= \sum_{\lambda} \prod_{i=1}^{p} [a_i]^{(\alpha)}_{\lambda} \prod_{i=1}^{q} [b_i]^{(\alpha)}_{\lambda} \frac{[n/\alpha]^{\alpha}_{\lambda}}{\prod_{i=1}^{n} [1]}.
\end{align*}
$$

The function $F_0^{(\alpha)}(-; -; x_1, \ldots, x_n; y_1, \ldots, y_n)$ is called multivariate Bessel function in [O] and plays important roles for Gaussian $\beta$-ensembles, see [BF, D]. By Theorem 4.3, we have the following hypergeometric expression for the product average:

$$
\left< \prod_{j=1}^{n} \Psi_{\mathbf{z}}(x_j) \cdot \prod_{j=1}^{n} \Psi_{\mathbf{z}}(y_j) \right>_{\mathbf{z} \in C_\beta E_n} = 2F_1^{(\beta/2)}(-n, 2n/\beta; -n + 1 - 2/\beta; x_1, \ldots, x_n; y_1, \ldots, y_n)
$$

for all complex numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$. Similarly, it follows from (4.5) that

$$
\left< \prod_{j=1}^{n} \Psi_{\mathbf{z}}(-u_j)^{-\beta/2} \cdot \prod_{j=1}^{n} \Psi_{\mathbf{z}}(-v_j)^{-\beta/2} \right>_{\mathbf{z} \in C_\beta E_n}
$$

$$
= 2F_1^{(2/\beta)}(n\beta/2, n\beta/2; \beta/2(n - 1) + 1; u_1, \ldots, u_n; v_1, \ldots, v_n)
$$
for any \( u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{D} \). We note that another hypergeometric expression for moments of characteristic polynomials is given in [PK] and [M1 (4.8)].

As a corollary of Theorem 4.3, we have the following asymptotic duality for ratio averages as in the limit \( n \to \infty \).

**Corollary 4.4.** Let

\[
\mathbf{x} = (x_1, \ldots, x_L), \quad \mathbf{y} = (y_1, \ldots, y_K), \quad \mathbf{u} = (u_1, \ldots, u_S), \quad \mathbf{v} = (v_1, \ldots, v_T)
\]

be sequences of complex numbers in \( \mathbb{D} \). Let \( \beta \) be a positive real number and set \( \beta' = 4/\beta \). Then

\[
\lim_{n \to \infty} \left\langle \frac{\prod_{l=1}^L \Psi(\mathbf{x}; x_l) \cdot \prod_{s=1}^S \Psi(\mathbf{z}; -u_s)^{\beta/2} \cdot \prod_{k=1}^K \Psi(\mathbf{z}; y_k)^{\beta/2}}{\prod_{l,t} (1 - x_l y_k)^{-2/\beta} \cdot \prod_{s,k}(1 + u_s y_k) \cdot \prod_{s,t} (1 - u_s v_t)^{-\beta/2}} \right\rangle_{\mathbf{z} \in \mathbb{C}^\beta \mathbb{E}_n} = \prod_{l,k} (1 - x_l y_k)^{-2/\beta} \cdot \prod_{s,k}(1 + u_s y_k) \cdot \prod_{s,t} (1 - u_s v_t)^{-\beta/2}.
\]

**Proof.** Let \( \alpha = 2/\beta \). Observe that, as \( n \to \infty \),

\[
\frac{[-n(1/\alpha)]_{\lambda}}{[-n + 1 - \alpha(1/\alpha)]_{\lambda}} \to 1
\]

for \( \lambda \) fixed, and therefore, by Theorem 4.3, two ratio averages in the corollary converge to sums

\[
\sum_{\lambda} \tilde{P}_{\lambda}(1/\alpha)_{\mathbf{x}; \mathbf{u}} \tilde{Q}_{\lambda}^{(1/\alpha)}_{\mathbf{y}; \mathbf{v}} \quad \text{and} \quad \sum_{\lambda} \tilde{P}_{\lambda}(1/\alpha')_{\mathbf{v}; \mathbf{y}} \tilde{Q}_{\lambda}^{(1/\alpha')}_{\mathbf{x}; \mathbf{u}}
\]

with \( \alpha' = 2/\beta' \) respectively but these are equal each other. The product expression in our statement is obtained from Lemma 3.3. \( \square \)

Roughly speaking, there exists a duality:

\[
\begin{pmatrix}
    \beta \\
    \mathbf{x} \\
    \mathbf{y} \\
    \mathbf{u} \\
    \mathbf{v}
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
    4/\beta \\
    \mathbf{u} \\
    \mathbf{v} \\
    \mathbf{x} \\
    \mathbf{y}
\end{pmatrix}
\]

in \( n \to \infty \).

**Remark 4.1.** The product expression in Corollary 4.4 follows directly from the following strong Szegö-type formula, which is algebraically proved in [M1, Theorem 3.4 and (4.10)]: Let \( \phi(z) = \exp(\sum_{k \in \mathbb{Z}} c(k) z^k) \) be a function on \( \mathbb{T} \) and assume \( \sum_{k \in \mathbb{Z}} |c(k)| < \infty \) and \( \sum_{k \in \mathbb{Z}} k |c(k)|^2 < \infty \). Then

\[
\lim_{n \to \infty} e^{-nc(0)} \left\langle \prod_{j=1}^n \phi(z_j) \right\rangle_{\mathbf{z} \in \mathbb{C}^\beta \mathbb{E}_n} = \exp \left( \alpha \sum_{k=1}^\infty k c(k) c(-k) \right).
\]
Indeed, if we take
\[ \phi(z) = \prod_l (1 + x_l/z) \prod_k (1 + y_k z^{-1}) \prod_s (1 - u_s/z)^{-1/\alpha} \prod_t (1 - v_t z)^{-1/\alpha}, \]
then a simple calculation gives \( c(0) = 0 \) and
\[ c(k) = \left\{ \frac{(-1)^k p_k(y) - \alpha^{-1} p_k(v)}{k} \right\}, \quad c(-k) = \left\{ \frac{(-1)^k p_k(x) - \alpha^{-1} p_k(u)}{k} \right\} \quad \text{for } k > 0, \]
and therefore we obtain Corollary 4.4 again.

4.3 Third expression involving hyperdeterminants

For a multi-dimensional array \( A = (A(i_1, i_2, \ldots, i_{2p}))_{1 \leq i_1, i_2, \ldots, i_{2p} \leq N} \), the hyperdeterminant (see e.g. [LT, M2]) of \( A \) is defined by the multi-alternating sum
\[ \det^{[2p]}(A) = \frac{1}{N!} \sum_{\sigma_1, \sigma_2, \ldots, \sigma_{2p} \in S_N} \sgn(\sigma_1) \sgn(\sigma_2) \cdots \sgn(\sigma_{2p}) \]
\[ \times \prod_{i=1}^{N} A(\sigma_1(i), \sigma_2(i), \ldots, \sigma_{2p}(i)). \]

If \( p = 1 \), this definition is reduced to the determinant of an \( N \times N \) matrix \( A = (A(i, j))_{1 \leq i, j \leq N} \).

In [M2] (see also [BBL]), the following hyperdeterminantal expression for rectangular Jack functions is obtained.

**Proposition 4.5.** For positive integers \( p, a, b \),
\[ Q_{(1/p)}^{(1/a)} = \frac{b!(p!)^b}{(pb)!} \det^{[2p]}(g_{a+i_1+i_2+\cdots+i_p-i_{p+1}-\cdots-i_{2p}})^{1 \leq i_1, \ldots, i_{2p} \leq b}. \]

Therefore \( \hat{Q}_{(1/p)}^{(1/a)}(x; y) = \frac{b!(p!)^b}{(pb)!} \det^{[2p]}(g_{a+i_1+i_2+\cdots+i_p-i_{p+1}-\cdots-i_{2p}}(x; y))^{1 \leq i_1, \ldots, i_{2p} \leq b} \) for any sequences \( x, y \) of variables.

From corollaries (4.1) and (4.2) of Theorem 4.1 we now derive the third expression for the ratio average. We must assume here that either parameter \( \beta \) or its dual one \( \beta' = 4/\beta \) is an even integer.

**Theorem 4.6.** Let
\[ x = (x_1, \ldots, x_L, x_{L+1}, \ldots, x_{L+K}), \quad v = (v_1, \ldots, v_T) \]
be sequences of complex numbers. Assume that \( x_i \neq 0 \) for \( 1 \leq i \leq L \) and \( v_t \in \mathbb{D} \) for \( 1 \leq t \leq T \). Suppose that \( \beta \) is an even positive integer:
\[ \beta = 2p. \]
Then we have
\[
\left< \frac{\prod_{l=1}^{L} \Psi(x; x^{-1}_l) \cdot \prod_{k=1}^{K} \Psi(z; x_{L+k})}{\prod_{t=1}^{T} \Psi(z; -v_t)^{\beta/2}} \right>_{z \in C/\beta E_n}
\]
\[
= \frac{n! (p!)^n}{(pm)!} (x_1 \cdots x_L)^{-n} \det [2p] (g^{(1/p)}_{n+i_1, \ldots, l_{p+1}, \ldots, l_{2p}} (v; x))_{1 \leq i_1, \ldots, i_{2p} \leq n}.
\]
(4.6)

We notice that the entries \( g^{(1/p)}_{r} (v; x) \) of the hyperdeterminant on (4.6) are ratio averages over \( T = U(1) = C/\beta E_1 \) with respect to the uniform probability measure:
\[
g^{(1/p)}_{r} (v_1, \ldots, v_T; x_1, \ldots, x_R) = x_1 \cdots x_r \left< \frac{\prod_{l=1}^{T} \Psi(x; x^{-1}_l) \cdot \prod_{k=1}^{R} \Psi(z; x_k)}{\prod_{t=1}^{T} \Psi(z; -v_t)^{p}} \right>_{z \in C/\beta E_1}
\]
\[
= x_1 \cdots x_r \int_T \frac{\prod_{l=1}^{r} (1 + x^{-1}_l z)^{-1} \cdot \prod_{k=r}^{R} (1 + x_k z)}{\prod_{t=1}^{T} (1 - v_t z)^p} \, dz
\]
for \( R \geq r \).

**Theorem 4.7.** Let \( x \) and \( v \) be as in Theorem 4.6. Suppose that
\[
\beta = 2/p
\]
with positive integer \( p \) (i.e. \( 4/\beta \) is the even integer \( 2p \)). Then we have
\[
\left< \frac{\prod_{l=1}^{L} \Psi(x; x^{-1}_l) \cdot \prod_{k=1}^{K} \Psi(z; x_{L+k})}{\prod_{t=1}^{T} \Psi(z; -v_t)^{\beta/2}} \right>_{z \in C/\beta E_n}
\]
\[
= b^{(p)}_{(L)} (x_1 \cdots x_L)^{-n} \det [2p] (g^{(1/p)}_{n+i_1, \ldots, l_{p+1}, \ldots, l_{2p}} (v; x))_{1 \leq i_1, \ldots, i_{2p} \leq L}.
\]

**Remark 4.2.** Define a deformed super-Jack function by
\[
\tilde{P}^{(\alpha)}_{\lambda} (x; y) = \sum_{\nu} \tilde{P}^{(\alpha)}_{\lambda/\nu} (x) \omega_1^{\nu} (P^{(\alpha)}_{\nu} (y)),
\]
where \( \omega_1^{\nu} \) is the homomorphism \( \omega_\alpha \) with \( \alpha = 1 \), acting the \( y \) variables. Compare this with (3.12). Then, for example, one may obtain the following ratio average evaluation similar to (4.1):
\[
\left< \frac{\prod_{l=1}^{L} \Psi(x; x^{-1}_l) \cdot \prod_{k=1}^{K} \Psi(z; x_{L+k})}{\prod_{t=1}^{T} \Psi(z; -v_t)^{\beta/2}} \right>_{z \in C/\beta E_n}
\]
\[
= (x_1 \cdots x_L)^{-n} \cdot \tilde{P}^{(\beta/2)}_{(nL)} (x; v).
\]
In particular, if \( \beta = 2/p \) with a positive integer \( p \), we have
\[
\tilde{P}^{(\beta/2)}_{(nL)} (x; v) = \tilde{P}^{(\beta/2)}_{(nL)} (x; v^{\times p})
\]
with
\[
v^{\times p} := (v_1, \ldots, v_1, v_2, \ldots, v_2, \ldots, v_T, \ldots, v_T).
\]
On the denominator of the ratio of the left hand side in (4.7), the power of characteristic polynomials is 1 instead of \( \beta/2 \). However, if one considers such a ratio average, our dualities given above are lost.
4.4 Pfaffian expressions for ratio averages over the COE and CSE

Consider the COE ($\beta = 1$) and CSE ($\beta = 4$). Let $\Psi(M; x) = \det(I + xM)$ be the characteristic polynomial of a unitary matrix $M$ in $S^1(n)$ or $S^4(n)$. Theorem 4.6 and Theorem 4.7 give hyperdeterminantal expressions for the ratio average in the COE and CSE. On the other hand, in those cases, the ratio averages can be expressed by a Pfaffian as follows. We need a Pfaffian expression for Jack functions at $\alpha = 1/2$ with rectangular diagrams. Such an expression is seen in [AKV, M2] and we rewrite its proof in the appendix below.

**Proposition 4.8.** Let $x$ and $v$ be as in Theorem 4.6. Then

$$\left\langle \prod_{l=1}^{L} \Psi(M; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(M; x_{L+k}) \right\rangle_{M \in \text{COE}_n} = \prod_{j=0}^{L-1} \frac{(n + 2j + 1)^{-1} \cdot (x_1 \cdots x_L)^{-n} \cdot \text{Pf}((j - i)g^{(1/2)}_{n+2L+1-i-j}(x; v)))_{1 \leq i, j \leq 2L}}{\prod_{t=1}^{T} \Psi(M; -v_t)^{1/2}}$$

and

$$\left\langle \prod_{l=1}^{L} \Psi(M; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(M; x_{L+k}) \right\rangle_{M \in \text{CSE}_n} = \frac{1}{(2n - 1)!!} \cdot (x_1 \cdots x_L)^{-n} \cdot \text{Pf}((j - i)g^{(1/2)}_{L+2n+1-i-j}(v; x)))_{1 \leq i, j \leq 2n}.$$ 

**Proof.** Consider the COE case. By (4.1) our ratio average equals $(x_1 \cdots x_L)^{-n}P_{(nL)}^{(1/2)}(x; v)$. Since

$$g^{(1/2)}_{(nL)}(2L - 1)!! = \prod_{j=0}^{L-1} (n + 2j + 1),$$

the desired Pfaffian expression follows from (4.13) and Proposition A.1 below with $\lambda = (nL)$. The proof of the CSE case is similar. \qed

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**Appendix**

A Pfaffians and rectangular Jack functions at $\alpha = 1/2$

For any $2n \times 2n$ antisymmetric matrix $B = (b_{ij})_{1 \leq i,j \leq 2n}$, the Pfaffian of $B$ is defined by

$$\text{Pf}(B) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^{n} b_{\sigma(2i-1), \sigma(2i)}.$$
More generally, given an upper triangular array \((b_{ij})_{1 \leq i < j \leq 2n}\), we denote by \(\text{Pf}(b_{ij})_{1 \leq i < j \leq 2n}\) the Pfaffian of the antisymmetric matrix \(B = (b_{ij})_{1 \leq i,j \leq 2n}\) whose diagonal and lower triangular entries are defined by \(b_{ii} = 0\) and \(b_{ji} = -b_{ij}\) for \(j > i\).

Proposition A.1 below gives the Pfaffian expression for the Jack function of a rectangular diagram with parameter \(\alpha = 1/2\). This Pfaffian expression is reminiscent of the Jacobi-Trudi identity for Schur functions: \(s_{\lambda} = \text{det}(h_{\lambda_i - i + j})_{1 \leq i,j \leq n}\), where \(h_k = g_k^{(1)}\) are complete symmetric functions and \(n\) is any integer such that \(n \geq \ell(\lambda)\).

The following proposition for \(n = \ell(\lambda)\) is proved in [AKV, M2]. Note that in [M2] we give a hyperpfaffian expression for rectangular Jack functions with parameter \(\alpha = 1/(2k)\) with any positive integer \(k\). We abbreviate one-row \(Q\)-functions \(g_k^{(\alpha)} = Q^{(\alpha)}_{(k)}\) with \(\alpha = 1/2\) as \(g_k\).

**Proposition A.1.** Let \(\lambda\) be a rectangular Young diagram and let \(n \geq \ell(\lambda)\). Then

\[
Q^{(1/2)}_{\lambda} = \frac{1}{(2n - 1)!!} \text{Pf}((j - i)g_{\gamma_j + 2n + 1 - i - j})_{1 \leq i < j \leq 2n}.
\]

Here the \(\gamma_j\) are given by

\[
(\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1}, \ldots, \gamma_{2n}) = (\lambda_0, \lambda_n - 1, \ldots, \lambda_1, \lambda_1, \ldots, \lambda_n).
\]

Also we have

\[
P^{(2)}_{\lambda} = \frac{1}{(2n - 1)!!} \text{Pf}((j - i)e_{\gamma_j + 2n + 1 - i - j})_{1 \leq i < j \leq 2n}
\]

by the automorphism \(\omega_{1/2}\) defined in (3.7).

**Example A.1.** Let \(\lambda = (\lambda_1, \lambda_2, \lambda_3)\) be a rectangular diagram of length \(\leq 3\), i.e., \(\lambda = (L, 0, 0)\), \((L, L, 0)\) or \((L, L, L)\) for a positive integer \(L\). Then

\[
Q^{(1/2)}_{\lambda} = \frac{1}{15} \cdot \text{Pf} \begin{pmatrix} 0 & g_{\lambda_2 + 4} & 2g_{\lambda_1 + 3} & 3g_{\lambda_1 + 2} & 4g_{\lambda_2 + 1} & 5g_{\lambda_3} \\ -g_{\lambda_2 + 4} & 0 & g_{\lambda_1 + 2} & 2g_{\lambda_1 + 1} & 3g_{\lambda_2} & 4g_{\lambda_3} \\ -3g_{\lambda_1 + 3} & -2g_{\lambda_1 + 2} & 0 & g_{\lambda_1} & 2g_{\lambda_2 - 1} & 3g_{\lambda_3} \\ -4g_{\lambda_2 + 1} & -4g_{\lambda_2} & -2g_{\lambda_2 - 1} & -g_{\lambda_1} & 0 & g_{\lambda_2 - 2} \\ -5g_{\lambda_3} & -4g_{\lambda_3 - 1} & -3g_{\lambda_3 - 2} & -2g_{\lambda_3 - 3} & -g_{\lambda_3 - 4} & 0 \end{pmatrix}.
\]

**Proof of Proposition A.1.** First we prove the case where \(n = \ell(\lambda)\):

\[
Q^{(1/2)}_{(\alpha^{(n)})}(x) = \frac{1}{(2n - 1)!!} \text{Pf}((j - i)g_{a + 2n + 1 - i - j}(x))_{1 \leq i,j \leq 2n}.
\]

Here we may suppose that complex variables \(x_i\) are in \(\mathbb{D}\) and that \(x_i\) is zero if \(i\) is large enough.

Recall the de Bruijn formula [1]: For a family \(\{\phi_j, \psi_j \mid 1 \leq j \leq 2n\}\) of functions on a measure space \((X, \mu(du))\),

\[
\int_X \frac{1}{n!} \det(\phi_j(u_k) | \psi_j(u_k))_{1 \leq j \leq 2n, 1 \leq k \leq 2n} \mu(du_1) \cdots \mu(du_n)
\]

\[
= \text{Pf} \left( \int_X (\phi_j(u)\psi_k(u) - \phi_k(u)\psi_j(u)) \mu(du) \right)_{1 \leq j, k \leq 2n}.
\]
Here \( \det(a_{jk} \mid b_{jk})_{1 \leq j \leq 2n, 1 \leq k \leq n} \) stands for the determinant of the matrix whose \( j \)-th row is \((a_{j1}, b_{j1}, a_{j2}, b_{j2}, \ldots, a_{jn}, b_{jn})\). Apply this formula to functions

\[
\phi_j(z) = \prod_{i \geq 1} (1 - x_i z)^{-2} \cdot z^{-a+j-n-1/2}, \quad \psi_j(z) = (j - n - 1/2) z^{j-n-1/2}
\]

on \( \mathbb{T} \) with the normalized Haar measure \( dz \). Then, since (e.g. [Me, pp.216])

\[
\det(z_k^{j-n-1/2} \mid (j - n - 1/2) z_k^{j-n-1/2})_{1 \leq j \leq 2n, 1 \leq k \leq n} = \prod_{1 \leq i < j \leq n} |z_i - z_j|^4,
\]

the integral in (A.2) is

\[
\int_{\mathbb{T}^n} (z_1 \cdots z_n)^{-a} \prod_{i \geq 1} \prod_{j=1}^n (1 - x_i z_j)^{-2} \cdot \prod_{1 \leq i < j \leq n} |z_i - z_j|^4 \, dz
\]

\[= c_n \left( \prod_{i \geq 1} \Psi(z_i; -x_i)^{-2} \right) , \quad z \in C4E_n \]

Here the constant \( c_n \) defined by \( c_n = \int_{\mathbb{T}^n} \prod_{i < j} |z_i - z_j|^4 \, dz \), is explicitly given as \( c_n = (2n)! / 2^n \) by (2.1). It follows by the reduction formula (3.9) and the Cauchy identity (3.3) that

\[
\left( \prod_{i \geq 1} \Psi(z_i; -x_i)^{-2} \right) \bigg|_{z \in C4E_n} = \sum_{\lambda} P^{(1/2)}(\lambda) \left( P^{(1/2)}(\lambda) Q^{(1/2)}(\lambda) \right) \bigg|_{z \in C4E_n},
\]

and, by the orthogonality (3.6), it equals

\[
\frac{[2n]^{(1/2)}_{(2n^2)}}{[1 + 2(n - 1)]^{(1/2)}_{(2n^2)}} P^{(1/2)}(\lambda) = i^{(1/2)}_{(2n^2)} P^{(1/2)}(\lambda) = Q^{(1/2)}(\lambda).
\]

Finally, the equation (A.2) is \((2n - 1)! Q^{(1/2)}(\lambda(x))\). On the other hand, (A.3) equals

\[
Pf \left( \int_{\mathbb{T}} (j - i) z^{-a-2n-1+i+j} \prod_{k \geq 1} (1 - x_k z) \, dz \right)_{1 \leq i, j \leq 2n}
\]

by (3.2) and we have proven (A.1).

Next we prove the case \( n > \ell(\lambda) \). To do it, it is enough to prove that, if \( \lambda_n = 0 \),

(A.4) \( Pf((j - i) g_{\gamma_i + 2n+1-i-j})_{1 \leq i < j \leq 2n} = (2n - 1) Pf((j - i) g_{\tilde{\gamma}_i + 2(n-1) + 1-i-j})_{1 \leq i < j \leq 2(n-1)} \),

where \( \tilde{\gamma}_j \) are given by \((\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{2n-2}) = (\lambda_{n-1}, \ldots, \lambda_2, \lambda_1, \lambda_1, \ldots, \lambda_{n-1})\). But, expanding the pfaffian on the left hand side of (A.4) with respect to the last column, we can obtain (A.4) immediately. \( \square \)
It is a natural question whether the Jack function $Q^{(1/2)}_\lambda$ for any $\lambda$ has a Pfaffian expression. Consider partitions $\lambda = (\lambda_1, \lambda_2)$ of length $\leq 2$. Then, since (see e.g. [11])

$$Q^{(1/2)}_{(\lambda_1, \lambda_2)} = g_{\lambda_1}g_{\lambda_2} - 2\frac{\lambda_1 - \lambda_2 + 2}{\lambda_1 - \lambda_2 + 3}g_{\lambda_1+1}g_{\lambda_2-1} + \frac{\lambda_1 - \lambda_2 + 1}{\lambda_1 - \lambda_2 + 3}g_{\lambda_1+2}g_{\lambda_2-2},$$

one can write as

$$Q^{(1/2)}_{(\lambda_1, \lambda_2)} = \frac{1}{\lambda_1 - \lambda_2 + 3}\text{Pf}
\begin{pmatrix}
0 & g_{\lambda_1+2} & (\lambda_1 - \lambda_2 + 3)g_{\lambda_2} \\
0 & g_{\lambda_1} & (\lambda_1 - \lambda_2 + 2)g_{\lambda_2-1} \\
0 & 0 & (\lambda_1 - \lambda_2 + 1)g_{\lambda_2-2}
\end{pmatrix}.$$

Here we abbreviate lower entries of the Pfaffian. We could not find such an expression for general partitions $\lambda$ of length $> 2$.

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