In the quantum Hall effect, the density operators at different wave-vectors generally do not commute and give rise to the Girvin MacDonald Plazmann (GMP) algebra with important consequences such as ground-state center of mass degeneracy at fractional filling fraction, and \( W_{1+\infty} \) symmetry of the filled Landau levels. We show that the natural generalization of the GMP algebra to higher dimensional topological insulators involves the concept of a \( D \)-algebra formed by using the fully anti-symmetric tensor in \( D \)-dimensions. For insulators in even dimensional space, the \( D \)-algebra is isotropic and closes for the case of constant non-Abelian \( F(k) \wedge F(k) \ldots \wedge F(k) \) connection (\( D \)-Berry curvature), and its structure factors are proportional to the \( D/2 \)-Chern number. In odd dimensions, the algebra is not isotropic, contains the weak topological insulator index (layers of the topological insulator in one less dimension) and does not contain the Chern-Simons \( \theta \) form \((F \wedge A - 2/3 A \wedge A \wedge A)\) in 3 dimensions). The Chern-Simons form appears in a certain combination of the parallel transport and simple translation operator which is not an algebra. The possible relation to \( D \)-dimensional volume preserving diffeomorphisms and parallel transport of extended objects is also discussed.

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INTRODUCTION

Fractional topological insulators are the strongly correlated states that may appear when a narrow bandwidth bulk band of a topological insulator [1,2] is fractionally filled and subject to strong interactions. The plethora of new experimental facts and theoretical ideas discovered in the non-interacting topological insulators suggests that their interacting fractional counterparts will also exhibit new physical properties of topological phases, especially in space dimensions higher than 2. Evidence for their existence has been provided in a series of analytical and numerical works in two-dimensional Chern insulators [3–11] and time-reversal invariant topological insulators [12,13], although more investigation is needed to determine the nature of the ground-state in the time-reversal invariant systems [14].

The most studied fractional topological insulator, the Fractional Chern Insulator (FCI), exhibits many of the properties of its Fractional Quantum Hall counterpart if the Berry curvature is smooth enough in the Brillouin zone (BZ). As evidence for the FCI state, it was analytically shown in [15] that the counting of the quasiholes excitations of the FCI state (in both the energy and the entanglement spectrum) is identical to the counting of quasiholes in a FQH state plus a lattice folding from the many-body FQH Brillouin zone to that of the FCI. The existence of a many-body Brillouin zone and the center of mass degeneracy of \( q \) at filling \( p/q \) in the FQH, as well as the mapping of the counting of states from the FQH to the FCI, is a direct consequence of the Girvin-Plazmann-MacDonald (GMP) algebra [16] exhibited by the FQH translational operators. This algebra has a simple interpretation in that the commutator of two translation operators is non-zero and roughly proportional to the magnetic flux thru the area subtended by the translations. The Chern insulator in 2-dimensions satisfied a similar algebra for parallel transport translational operators in the Brillouin zone [17]. This algebra has far reaching consequences: it is identical to the algebra of area-preserving diffeomorphisms, thereby providing for an explanation of the edge modes of an integer quantum Hall liquid as shape deformations of the liquid droplet. It allows for the construction of nontrivial many-body symmetry operators of the Hilbert space, it provides for a center of mass degeneracy (exact in the FQH but approximate in the FCI), and is related to the Hall viscosity, \( q^2 \) form factor, as well as the edge dipole-moment [18].

In view of these consequences, the generalization of the GMP algebra to higher dimensions is desirable. In this paper, we present such a generalization. Realizing that the commutator of the projected densities \( \rho_q \) in the "valence" bands of a topological insulators for two wave-vectors is a simple contraction with the antisymmetric tensor in two dimensions, we generalize this structure in \( D \) spatial dimensions to a \( D \)-commutator by contracting with the antisymmetric tensor in \( D \)-dimensions. If this commutator is closed, the relation is called a \( D \)-algebra. We find that for topological insulators in even dimensions, the commutator is closed, and the algebra is isotropic, under a condition similar to that of the existence of the GMP in the 2-D Chern insulator [17]: the \( F(k) \wedge F(k) \ldots \wedge F(k) \) object (which we call the \( D \)-th Berry curvature since \( F(k) \) is the Berry curvature in the BZ) connection must be smooth and ideally proportional to the identity matrix. The structure factors of the algebra are then proportional to the \( D/2 \)-th Chern number. In odd space dimensions, where we would ideally expect the Chern-Simons form (which in 3 D reads
While contains layers of (D−1)-dimensional topological insulators. While part of the non-Abelian Chern-Simons form (and the entire Abelian Chern-Simons form) appears in the odd-D-commutator, once we try to close the commutator to obtain a closed algebra, the Chern-Simons term vanishes. We can, however, obtain the Chern-Simons term as a combination of parallel transport and simple translation operators, but they do not form an algebra. We close with comments and conjectures about the volume preserving diffeomorphisms and parallel transport of extended objects that the D-algebra structure suggests.

**FLAT-BAND HAMILTONIAN AND PROJECTED DENSITY OPERATORS**

We start by fixing notations and recalling some well known results about band structure and projected density operators for topological insulators. We consider a N band topological insulator described by a translationally invariant Hamiltonian, and we work on a D dimensional lattice with periodic boundary conditions. The Hamiltonian reads as

\[ H = \sum_{i,j,\alpha,\beta} c_{i\alpha}^\dagger h_{i-j,\alpha,\beta} c_{j\beta} \]  

(1)

where \( \alpha, \beta = 1, \ldots, N \) contain orbital and spin indices. We work in units where the lattice spacing is unity. The system being explicitly translation invariant, it decouples in momentum space

\[ H = \sum_{k,\alpha,\beta} c_{k\alpha}^\dagger h_{\alpha,\beta}(k) c_{k\beta} \]  

(2)

keeping in mind that momenta are restricted to the first Brillouin zone. The Bloch Hamiltonian matrix \( h_{\alpha,\beta}(k) = \sum_{\varphi} e^{-i k \cdot r_{\varphi}} h_{\alpha,\beta}(k) \) can be diagonalized \( \sum_{\beta} h_{\alpha,\beta}(k) u_{k\beta}^n = E_n(k) u_{k\alpha}^n \), where the eigenvectors \( u_{k\beta}^n \) are chosen to form an orthonormal basis. Consequently the Bloch Hamiltonian can be separated into normal mode operators \( \gamma_k^n \) at momentum \( k \) of the band \( n \)

\[ H = \sum_{k,n} E_n(k) \gamma_k^n \gamma_k^n \]  

(3)

where the normal modes can be written as a matrix rotation of the original electron operators \( \gamma_k^n = \sum_{\beta} u_{k\beta}^n c_{k\beta} \). We then define the projector into the band \( n \) at momentum \( k \) by

\[ P_{n,k} = \gamma_k^n \gamma_k^n \]  

(4)

From now on, we will consider the physics of the (possibly fractionally) occupied bands and look only at projectors into the occupied bands. The projection operator in the occupied bands is

\[ P = \sum_{k,n} P_{n,k} \]  

(5)

where the band index \( n = 1, \ldots, N_{\text{occ}} \) ranges over all occupied bands. The projected density operator in the occupied bands of a topological insulator is the following operator:

\[ \rho_q = P \sum_j \exp^{-i q j} c_{j\alpha}^\dagger c_{j\beta} P = \sum_{k,n,m} \langle u_k^n | u_{k+q}^m \rangle \gamma_k^n \gamma_k^m | 0 \rangle \langle 0 | \gamma_{k+q}^m \]  

(6)

where \( n,m \) ranges over the set of occupied bands. A simple calculation reveals that:

\[ [H, \rho_q] = \sum_{k,n,m} (E_n(k) - E_m(k+q)) \langle u_{k\alpha}^n | u_{k+q\beta}^m \rangle \gamma_k^n \gamma_k^m | 0 \rangle \langle 0 | \gamma_{k+q}^m \]  

(7)

It follows immediately that the projected density operators commute with the Hamiltonian if \( E_n(k) = E_m(k+q) \) for all occupied bands and all momenta. This is called the flat-band Hamiltonian:

\[ H_{FB} = - \sum_{n_1 < \mu} P_{n_1,k} + \sum_{n_2 > \mu} P_{n_2,k} \]  

(8)

Fractional topological insulators are usually constructed and observed in models with fractionally filled bands whose bandwidth is very small, such that interactions and not the kinetic energy dominate the physics. The ideal example of such an insulator is precisely the flat band one-body deformation (S).

We stress that projected density operators are an exact symmetry of the flat-band Hamiltonian (S), to which the true one body Hamiltonian (1) is adiabatically connected. As a matter of fact, this also holds for any projected operator that does not mix occupied and unoccupied bands (in particular any projected operator).

**THE TWO DIMENSIONAL CASE**

Before moving to higher dimensions, we quickly review what is known about the algebra of projected density operators in two dimensions, with an emphasis to its main characteristics. We focus on the appearance of the Chern number in the algebra and on the link between projected densities and parallel transport in the background of the Berry curvature.

**Density algebra and first Chern number**

At long wavelength \( q_1, q_2 \to 0 \), the reference [17] finds the following commutation relation:

\[ (\gamma_{k_1^n} | 0 \rangle \langle 0 | \gamma_{k_2^m}, (\gamma_{k_2^n} | 0 \rangle \langle 0 | \gamma_{k_1^m}) = 0 \]
\[ [\rho_{q_1}, \rho_{q_2}] = -i\theta_{\mu''}^\mu q_2 \sum_{k,n,m} F_{\mu''}^{n,m}(k) \gamma_k^{n+} \langle 0 | \gamma_{m+k+q_1}^{m} \langle 0 | \rho_{q_2} \rho_{q_1} \gamma_{m+k+q_2}^{m} \rangle \]

where the Einstein summation convention after repeating indices is assumed. \( F_{\mu''} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \) is the non-Abelian Berry field strength in the Brillouin zone, while the vector potential is \( A_{\mu}^{nm}(k) = i(u_k^{n+} \partial_{k^\mu} u_k^{m}) \). This result holds in any dimension.

In two dimensions \( F_{\mu''}(k) = B(k) \epsilon_{\mu''} \), and its integral over the whole Brillouin zone yields the first Chern number

\[ C_1 = \frac{1}{4\pi} \int_{BZ} d^2 k \epsilon^{\mu''} \epsilon^{\mu''} \text{Tr}(F_{\mu''}(k)). \]

The commutator of two densities has to be non-zero in a nontrivial Chern insulator. That is so because the Chern number \( C_1 \) of the bidimensional insulator can be expressed as a trace over the Brillouin zone of the density commutator

\[ \text{Tr}([\rho_{q_1}, \rho_{q_2}] \rho_{-q_1-q_2}) \sim_{q \to 0} \frac{1}{2\pi i} (q_1 \wedge q_2) C_1 \]

where \( q_1 \wedge q_2 = \epsilon_{\mu''}^\mu q_2^\mu q_2^\nu \). In the continuum limit of the Quantum Hall effect, the projected density algebra of the Lowest Landau Level is called the GMP algebra. Its generators are the area-preserving diffeomorphisms in two-dimensions. This result is recovered for bidimensional topological insulators with an Abelian U(1) uniform Berry curvature, in the long wavelength limit. As pointed out in [17] (see also [19] and [13]), if the local Berry curvature can be replaced by its average

\[ F_{xy}(k) = B \frac{C_1}{2\pi} \]

then

\[ [\rho_{q_1}, \rho_{q_2}] = -iB \gamma_{q_2} \gamma_{q_1} \gamma_{q_2} \]

Note that \( q_1 \wedge q_2 \) is the area enclosed in the parallelogram delimited by \( q_1 \) and \( q_2 \). This algebra is nothing but the bidimensional Aharonov-Bohm effect in momentum space, in the background of the “magnetic field” \( F_{xy} = B \).

This Abelian treatment applies to two-band models (insulators with one band below and above the gap) or to many-band insulators where the non-Abelian components of the field strength can be neglected (up to a prefactor \( N_{\text{occ}} \)).

Expanding the projected densities at long wave-vectors as \( \rho_q = 1 + i q \cdot R + O(q^2) \), the algebra of the guiding center is recovered:

\[ [R_{\mu}, R_{\nu}] = iB \epsilon_{\mu\nu} = i \frac{C_1}{2\pi} \]

The Chern number quantifies the non-commutativity of the guiding center operators.

We remark that in a two-band insulator, it is impossible to have a constant Berry curvature due to the no-hair theorem [20], although this seems possible in insulators with four or more bands [21].

**Regularized density operators**

Since projected density operators commute with the flat-band Hamiltonian [3], it would seem that they are the generators of a proper symmetry group of the system. However this is not quite true, as they suffer from a serious deficiency. Because of the projection, they are not unitary. The density operator translates states in momentum space but does not keep their norm:

\[ \rho_q |n, k\rangle = \sum_m (u_k^{m+} |u_k^{m}\rangle |m, k-q\rangle \]

where \( |n, k\rangle = \gamma_k^{n+} |0\rangle \). It is possible to replace the projected density operator \( \rho_q \) by a unitary operator \( \tilde{\rho}_q \), while not spoiling the long wavelength behavior from Eq. [14]. For a uniform Abelian Berry curvature, the answer is quite straightforward, and is simply the exponentiation of the guiding center operator. Doing so, one recovers the GMP algebra

\[ [\tilde{\rho}_{q_1}, \tilde{\rho}_{q_2}] = -2i \sin \left( \frac{B(q_1 \wedge q_2)}{2} \right) \tilde{\rho}_{q_1+q_2} \].

More generally, for a non-Abelian and non-uniform Berry field strength, the answer is parallel transport in the background of the Berry gauge potential \( A_\mu(k) \):

\[ \tilde{\rho}_q = \sum_{k,n,m} \left( p e^{-i \int_k^{k+q} A(k')dk'} \right)_{nm} \gamma_k^{n+} |0\rangle \langle m+k+q| \]

In the Abelian case this result was pointed out in [17]. Note that this operator also commutes with the Flat-Band Hamiltonian, and at small momenta we recover the projected density operator \( \rho_q = \rho_q + O(q^2) \).

**Density algebra in even-space dimensions**

The density commutator is natural in two dimensions. In higher space dimension \( D > 2 \), the commutator algebra Eq. [9] reveals whether a two dimensional quantum Hall effect exists on planes of the \( D \)-dimensional space.

For example, in \( D = 3 \), the existence of a commutator algebra Eq. [10] for \( q_1, q_2 \) in the \( xy \) planes for each \( z \) position (momentum) reveals the existence or absence of a 2-dimensional quantum Hall effect on those planes, trivially translated in the \( z \) direction. It is apparent then that
in $D > 2$ dimension the commutator algebra Eq.(19) cannot be isotropic. In order to find an isotropic algebraic structure in higher dimensions, we must look somewhere else. We first realize that the commutator $[\rho_{q_1}, \rho_{q_2}]$ is, in two dimensions, simply a re-writing of the operators $\varepsilon_{\alpha\beta} \rho_{q_\alpha} \rho_{q_\beta}$. This object can now be generalized to any dimension. In $D$ space dimensions, it is then suggestive to look at the operator:

$$[\rho_{q_{a_1}}, \rho_{q_{a_2}}, \cdots, \rho_{q_{a_D}}] = \varepsilon_{\alpha_1\alpha_2\cdots\alpha_D} \rho_{q_{a_1}} \rho_{q_{a_2}} \cdots \rho_{q_{a_D}}$$  \hspace{1cm} (18)

where $\varepsilon_{\alpha_1\alpha_2\cdots\alpha_D}$ is the totally antisymmetric tensor in $D$-dimensions. and $\alpha = 1 \ldots D$. These generalized commutators are called $D$-commutators. We will now compute this object in the long wavelength limit and find it is closed, thereby generating a $D$-algebra.

**$D$-commutator and Berry curvature density**

The density algebra in even space dimensions is simpler to obtain than in odd-space dimensions for reasons that will become apparent. In even space dimensions we have the Chern-insulator (QH)-classes, so we anticipate that the algebra closes. We first re-express the $D$-commutator as a product of 2-commutators:

$$[\rho_{q_1}, \rho_{q_2}, \cdots, \rho_{q_D}] = 2^{-D/2} \varepsilon_{\alpha_1\alpha_2\cdots\alpha_D-1\alpha_D} [\rho_{q_{a_1}}, \rho_{q_{a_2}}, \cdots, \rho_{q_{a_D-1}}, \rho_{q_{a_D}}]$$  \hspace{1cm} (19)

Using the long wavelength two dimensional algebra Eq.(1) and working at order $q^D$ we obtained :

$$[\rho_{q_1}, \rho_{q_2}, \cdots, \rho_{q_D}] = (-i/2)^{D/2} (q_1 \wedge q_2 \cdots \wedge q_D) \times \sum_{k,n,m} (F(k) \wedge \cdots \wedge F(k))_{n,m} \gamma^m_k \alpha^{n} 0 0 \gamma^m_{k+q_1+\cdots+q_D}$$  \hspace{1cm} (20)

where we used the identity:

$$\varepsilon_{\alpha_1\alpha_2\cdots\alpha_D} q^\alpha_{\alpha_1} q^\beta_{\alpha_2} \cdots q^\mu_{\alpha_D} = (q_1 \wedge q_2 \wedge \cdots \wedge q_D) \varepsilon^{\mu_1\mu_2\cdots\mu_D}$$  \hspace{1cm} (21)

This equation is the $D$-dimensional analogue of Eq.(1). In the $D$-commutator appear the matrix

$$F(k) \wedge \cdots \wedge F(k) = \varepsilon^{\mu_1\cdots\mu_D} F_{\mu_1\mu_2}(k) \cdots F_{\mu_{D-1}\mu_D}(k)$$  \hspace{1cm} (22)

which is the $D/2$'th Berry curvature density of the $D/2$'th Chern number:

$$C_{D/2} = \frac{1}{(D/2)!2^{D/2}(2\pi)^D} \int d^D k \text{Tr} (F(k) \wedge \cdots \wedge F(k))$$  \hspace{1cm} (23)

For even dimensional topological insulators, the $D/2$'th Chern number can be expressed as the the trace over the $D$-commutator of the projected density operator:

$$\text{Tr} \left( [\rho_{q_1}, \rho_{q_2}, \cdots, \rho_{q_D}] \right) \sim_{q \to 0} \frac{1}{(2\pi i)^{D/2}} \frac{C_{D/2}}{(2\pi)^{D/2}}$$

This is the exact analog of the bidimensional relation III.

**GMP in higher dimensions and $D$-algebra**

It is possible to obtain an analog of the GMP algebra in $D$-dimensions. As for topological insulators in two dimensions, this algebra holds when the Berry density $F(k) \wedge \cdots \wedge F(k)$ is uniform in the Brillouin zone, and proportional to the identity matrix. Then the projected density operators algebra becomes a $D$-algebra, in the long wavelength limit:

$$[\rho_{q_{a_1}}, \rho_{q_{a_2}}, \cdots, \rho_{q_{a_D}}] = (D/2)! \frac{1}{(2\pi i)^{D/2}} \frac{C_{D/2}}{(2\pi)^{D/2}}$$

and we recover a $D$-algebra. It is very tempting to expand the projected densities as $\rho_q = 1 + i q \mathbf{R} + O(q^2)$. As a consequence of Eq.(20) the “guiding center” operator $\mathbf{R}$ also form a $D$-algebra:

$$[R_{\mu_1}, R_{\mu_2}, \cdots, R_{\mu_D}] = (D/2)! \left( \frac{i}{2\pi} \right)^{D/2} \frac{C_{D/2}}{(2\pi)^{D/2}}$$

This algebra is most easily understood in the continuum limit. From Eq.(15) the guiding center operators are simply the covariant derivative with the Berry potential in momentum space:

$$R_{\mu}(k) = -i \partial_{k_\mu} - i A_\mu(k)$$  \hspace{1cm} (27)

Using the relation $[R_\mu, R_\nu] = i F_{\mu\nu}$, it is straightforward to obtain their $D$-commutator

$$[R_{\mu_1}(k), R_{\mu_2}(k), \cdots, R_{\mu_D}(k)] = (i/2)^{D/2} F(k) \wedge \cdots \wedge F(k)$$

This elementary derivation in the continuum is not plagued by the limitations of the projected density operators algebra, as there is no need to suppose the Berry density $F(k) \wedge \cdots \wedge F(k)$ to be uniform or proportional to the identity.

This $D$-algebra structure may be understood in two ways. On one hand, as was pointed out in [12], the projected position operators can be expressed in terms of the projected density operators. Therefore an immediate
interpretation of \([20]\) is the non commutativity of the coordinates of particles projected to the occupied bands of a topological insulator. This is the \(D\)-dimensional analog of Eq. (14) for the Quantum Hall Effect.

On the other hand, the GMP algebra also describes a bidimensional Aharonov-Bohm effect: the projected density operators implement parallel transport of point like objects in the background of the Berry curvature \(F\). In higher dimensions, an Aharonov-Bohm effect with respect to the \(D\)-form \(F \wedge \cdots \wedge F\) requires parallel transport of higher dimensional objects. Indeed, the natural objects that can couple to a \(D\)-form are \(D-2\) dimensional membranes \([21]\). Interestingly, the classical limit of the \(D\)-commutator is the Nambu-Poisson bracket \([22]\), which is a natural setup to describe the dynamics of classical membranes \([23]\). The appearance of extended objects in the field theory description of topological insulators in dimensions greater than three is also expected from the BF field theory description of topological insulators in dimensions \([23]\). The appearance of extended objects in the field theory description of topological insulators in dimensions greater than three is also expected from the BF proposal of \([24]\). We conjecture that the algebra \([25]\) is related to an Aharonov-Bohm effect involving extended excitations (membranes) coupled to the Berry curvature \(F \wedge F \wedge \cdots \wedge F\). However, unlike in two dimensions, it is not clear how to interpret the projected density operators as an implementation of membrane parallel transport.

**DENSITY ALGEBRA IN ODD-SPACE DIMENSIONS**

Pursuing the same strategy in odd dimensions leads to an impasse. The topological invariant in odd dimensions is defined as the integral over the Brillouin zone of a Chern-Simons form. For instance in three dimensions the \(Z_2\) topological invariant is given by

\[
P_3 = \frac{\theta}{2\pi} = \frac{1}{32\pi^2} \int d^3k e^{\mu\nu\rho} \text{Tr} \left[ F_{\mu\nu} A_\rho - \frac{2}{3} A_\mu A_\nu A_\rho \right]
\]

Defined for all odd dimensions, a characteristic feature of Chern-Simons form is that their integral is not necessarily invariant under large gauge transformations. However the variation has to be an integer \([25]\). Contrary to the even dimensional Chern numbers, the odd dimensional \(Z_2\) topological invariant is only gauge invariant modulo integers. Trying to obtain \(P_3\) through the gauge invariant trace

\[
\text{Tr}([\rho_{q_1}, \rho_{q_2}, \rho_{q_3}] [\rho_{-q_1-q_2-q_3}])
\]

is doomed to fail. A simple relation like Eq. (28) is ruled out in odd dimensions. Moreover \(D\)-commutators in odd dimensions are known \([26]\) to be more problematic than their even dimensional counterpart. For instance while even commutators involving the identity matrix do vanish, this is no longer the case for odd commutators. This is most easily seen in 3 dimensions:

\[
[A, B, 1] = [A, B] \neq 0
\]

Consequently, when expanding the 3-commutator of a projected density operator \(\rho_q = 1 + iq \cdot R + O(q^2)\), the lowest order contribution is of order \(q^2\) and not \(q^3\):

\[
[\rho_{q_1}, \rho_{q_2}, \rho_{q_3}] \sim -i (q_1^\mu q_2^\nu + q_3^\nu q_1^\mu + q_2^\mu q_3^\nu) F_{\mu\nu} \rho_{q_1+q_2+q_3}
\]

This term is reminiscent of the 2-commutator algebra Eq. (19), and accounts for a possible bidimensional topological structure in the 3D insulator. This would be the case for a weak 3D Chern insulator, obtained by stacking layers of the 2D Chern insulator. This structure remains true in all odd dimensions, where the \(D\)-commutator contains an anisotropic \(O(q^{D-1})\) term in contrast with the isotropic \(O(q^D)\) term appearing in Eq. (20) for even dimensions.

In order to illustrate explicitly the kind of problems that arise in odd dimensions, we focus in the following on topological insulators in three dimensions. If the Chern-Simons density \([21]\) is to appear at all in the triple commutator, this has to be a \(O(q^3)\) term. This term is sensitive to the \(O(q^2)\) correction to the projected density operator, and therefore it matters what kind of regularization we are considering. In order to be able to make a generic statement, we consider in the following the most general regularization up to level \(O(q^2)\):

\[
\rho_q \sim \sum_{k, n, m} \left( 1 - iq^\mu A_\mu - \frac{i}{2} q^\mu q^\nu B_{\mu\nu} \right)_{nm} \gamma^{n|}_{k} \langle 0 | \langle q^m_{k+q} \rangle
\]

For instance the parallel transport \(\tilde{\rho}_q\) from Eq. (17) would correspond to \(B_{\mu\nu} = \partial_\mu A_\nu - i A_\mu A_\nu\). Note that only the symmetric part of \(B_{\mu\nu}\) is relevant. We computed the subdominant term to the 3-commutator \([27]\). The result is the sum of two \(O(q^3)\) terms, corresponding to the two kinds of 3-tensors which are both \(O(q^3)\) and antisymmetric with respect to the three momenta \(q_i\):

\[
-\frac{1}{2} (q_1 \wedge q_2 \wedge q_3) \sum_{k, n, m} (F \wedge A)_{nm} \gamma^{n|}_k \langle 0 | \langle q^m_{k+q_1+q_2+q_3} \rangle
\]

and

\[
\epsilon_{\alpha_1 \alpha_2 \alpha_3} q^\mu_{\alpha_1} q^\nu_{\alpha_2} q^\rho_{\alpha_3} \sum_{k, n, m} (i D_\sigma B_{\mu\nu} - i \partial_\mu \partial_\nu A_\sigma)_{nm} \gamma^{n|}_k \langle 0 | \langle q^m_{k+q_1+q_2+q_3} \rangle
\]

While the first one \(\epsilon_{\alpha_1 \alpha_2 \alpha_3} q^\mu_{\alpha_1} q^\nu_{\alpha_2} q^\rho_{\alpha_3} = (q_1 \wedge q_2 \wedge q_3) \epsilon^{\mu\nu\rho}\) is familiar from the even dimensional case, the second one \(\epsilon_{\alpha_1 \alpha_2 \alpha_3} q^\mu_{\alpha_1} q^\nu_{\alpha_2} q^\rho_{\alpha_3}\) is only possible in odd dimensions. Note that this last tensor being symmetric under exchange of \(\mu\) and \(\nu\), it cannot yield the antisymmetric Chern-Simons term. At first glance the first term looks promising as it contain part of the Chern-Simons form.
(F ∧ A) although the term A ∧ A ∧ A is still missing. Before taking a trace though, we need to multiply the 3-commutator by ρ - q1 - q2 - q3, and one could hope that A ∧ A ∧ A will appear. This is not so, as the multiplication by ρ - q1 - q2 - q3 yield the additional term

$$\sum_{k,n,m} \left[ \epsilon_{\alpha_{1}\alpha_{2}\alpha_{3}}(q_{1}^{\mu} q_{2}^{\sigma} q_{3}^{\nu}) + \frac{1}{2} (q_1 \land q_2 \land q_3) \epsilon^{\mu\sigma\nu} \right]$$

$$(F_{\mu\nu} A_{\sigma})_{nm} \gamma_{k}^{m} |0\rangle_\gamma |0\rangle_\gamma |0\rangle_\gamma$$

(36)

which kills the term $$(q_1 \land q_2 \land q_3) \epsilon^{\mu\sigma\nu} F_{\mu\nu} A_{\sigma}$$. The final result is

$$-i \sum_{k,n,m} (q_{1}^{\mu} q_{2}^{\sigma} q_{3}^{\nu} + q_{2}^{\mu} q_{3}^{\sigma} q_{1}^{\nu} + q_{3}^{\mu} q_{1}^{\sigma} q_{2}^{\nu}) (F_{\mu\nu})_{nm} \gamma_{k}^{m} |0\rangle_\gamma |0\rangle_\gamma |0\rangle_\gamma$$

$$+ \epsilon_{\alpha_{1}\alpha_{2}\alpha_{3}} q_{1}^{\mu} q_{2}^{\sigma} q_{3}^{\nu} \frac{1}{2} \sum_{k,n,m} (C_{\mu\nu\sigma})_{nm} \gamma_{k}^{m} |0\rangle_\gamma |0\rangle_\gamma |0\rangle_\gamma$$

(37)

where the 3-tensor $C_{\mu\nu\sigma}$ has the fatal inconvenience to be symmetric under μ → ν:

$$C_{\mu\nu\sigma} = iD_{\mu} B_{\nu\sigma} - i\partial_{\mu} \partial_{\nu} A_{\sigma} - (A_{\mu} \partial_{\nu} + A_{\nu} \partial_{\mu}) A_{\sigma}$$

$$+ F_{\mu\nu} A_{\sigma} + F_{\mu\sigma} A_{\nu} + F_{\nu\sigma} A_{\mu}$$

(38)

The Chern-Simons term being fully antisymmetric, this calculation shows explicitly that the Berry curvature cannot appear in the algebra of projected density operators in three dimensions, no matter what is chosen for the regularization $R_{\mu\nu}$. For instance the parallel transport operator $\hat{\rho}_{q}$ yields

$$\text{Tr} ([\hat{\rho}_{q_1}, \hat{\rho}_{q_2}, \hat{\rho}_{q_3}] \hat{\rho}_{-q_1 - q_2 - q_3}) =$$

$$-i(q_{1}^{\mu} q_{2}^{\nu} + q_{2}^{\mu} q_{3}^{\nu} + q_{3}^{\mu} q_{1}^{\nu}) \int d^{2} k \text{Tr} (F_{\mu\nu}(k))$$

$$+ i\epsilon_{\alpha_{1}\alpha_{2}\alpha_{3}} q_{1}^{\mu} q_{2}^{\sigma} q_{3}^{\nu} \frac{1}{2} \int d^{2} k \text{Tr} (\partial_{\mu} F_{\sigma\nu}(k)) + O(q^{4})$$

(39)

A way to free ourself from this no-go theorem is to involve non gauge invariant operators, such as the pure translation:

$$T_{q} |n\rangle = |n, k - q\rangle$$

(40)

This can be used to generate the Chern-Simons form as

$$P_{3} = \frac{1}{32\pi^{2}} \epsilon_{ijk} \text{Tr} [\rho_{q_{1}} (\rho_{q_{2}} - T_{q_{1}}) (\rho_{q_{3}} - T_{q_{2}}) (\rho_{q_{1}} - T_{q_{3}}) (\rho_{q_{2}} - T_{q_{3}}) (\rho_{q_{3}} - T_{q_{1}})] - \frac{1}{3} (\rho_{q_{1}} - I_{q_{1}}) (\rho_{q_{2}} - T_{q_{1}}) (\rho_{q_{3}} - T_{q_{2}}) \rho_{q_{3}} - q_{1} - q_{2} - q_{3}$$

(41)

but the physical picture behind this relation is still unclear.

CONCLUSION

We have presented a generalization of the GMP algebra to D-dimensional topological insulators by generalizing the commutator, algebra and Berry phase to their higher-dimensional counterparts. This hints at a different group structure from the usual gauge theories. Particularly interesting candidates are higher gauge theories [21, 22]. In light of this, the recent proposal [24] to describe topological insulators by a BF theory [25] looks very promising. In this topological field theory coexist two gauge fields. In addition to the usual 1-form A, there is an extra D - 1 form B, to which D - 2 dimensional membranes can couple. At this level, the even and odd-dimensions are fundamentally different - in even dimensions, the structure factors of the algebra are proportional to the D/2th Chern number, while in odd dimensions they are not proportional to the expected Chern-Simons form. In two dimensions, the classical limit of the GMP algebra is isomorphic to the algebra of area preserving diffeomorphisms, and is related to incompressibility. A D-algebra on the other hand is related to volume preserving differentials [21, 22]. Indeed it is a quantization of the classical Nambu-Poisson bracket [24], which is known to be invariant under volume preserving diffeomorphisms. It would be interesting to make this connection more explicit and to understand its link to the incompressibility of TIs in higher dimensions. This however suggests that the correct "effective" description of the higher-dimensional topological insulators is in terms of parallel transport not of electrons but of extended objects, such as strings in 3 dimensions. We speculate the the Chern-simons term could appear when such algebras are constructed.

Note: One day prior to the posting of our paper, a related paper [30] by T. Neupert, L. Santos, S. Ryu, C. Chamon, and C. Mudry, containing similar results appeared on the arxiv. While most of our results are similar, our paper differs in a fundamental aspect. In the case of topological insulators in odd space dimensions, the authors of [30] claim that the $\theta$ term appears in the odd-dimensional algebras. The basis for this claim, their Eq. 2.71, is identical to our equation (44) and (45) when we particularize our non-Abelian result to their Abelian one. This equation contains the $\theta$ term, although only the $F \land A$ is present while the $A \land A \land A$ is missing in the non-Abelian Eq. (44). However, Eq. 2.71 of [30] is not an algebra, as the right hand side (RHS) does not yet contain the $\rho$ operator and is hence not closed, even when the conditions of constant Berry curvature imposed in [arXiv:1202.5188] are satisfied. In order to close the algebra one must recast the RHS of Eq 2.71 of [30] in terms of the density, as in our Eq. (47), the $\theta$ term cancels, and is not contain in any of the odd D-algebras.
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