Optimal Error Estimates for Semidiscrete Galerkin approximations to the Equations of Motion Described by Kelvin-Voigt Viscoelastic Fluid Flow Model

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Abstract

In this paper, the finite element Galerkin method is applied to the equations of motion arising in the Kelvin-Voigt viscoelastic fluid flow model, when the forcing function is in $L^\infty (L^2)$. Some a priori estimates for the exact solution, which are valid uniformly in time as $t \to \infty$ and even uniformly in the retardation time $\kappa$ as $\kappa \to 0$, are derived. It is shown that the semidiscrete method admits a global attractor. Further, with the help of a priori bounds and Sobolev-Stokes projection, optimal error estimates for the velocity in $L^\infty (L^2)$ and $L^\infty (H^1)$-norms and for the pressure in $L^\infty (L^2)$-norm are established. Since the constants involved in error estimates have an exponential growth in time, therefore, in the last part of the article, under certain uniqueness condition, the error bounds are established which are valid uniformly in time. Finally, some numerical experiments are conducted which confirm our theoretical findings.

Keywords: Kelvin-Voigt viscoelastic model, a priori bounds, global attractor, semidiscrete Galerkin approximations, optimal error estimates, uniqueness condition.

AMS 1991 Classification:

1 Introduction

Consider the following system of partial differential equations arising in the Kelvin-Voigt’s model

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \kappa \Delta \mathbf{u} + \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(x,t), \ x \in \Omega, \ t > 0, \tag{1.1}$$

and incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \ x \in \Omega, \ t > 0, \tag{1.2}$$

with initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \text{ in } \Omega, \quad \mathbf{u} = 0, \text{ on } \partial \Omega, \ t \geq 0, \tag{1.3}$$
where, $\Omega$ is a bounded convex polygonal or polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$ with boundary $\partial \Omega$. Here, $\nu$ is the coefficient of kinematic viscosity and $\kappa$ is the retardation time or the time of relaxation of deformations. In the context of viscoelastic fluid, this model was first introduced by Pavlovskii [16], who called it as a model describing the motion of weakly concentrated water-polymer solutions. It was called Kelvin-Voigt model by Oskolkov [20] and his collaborators. Subsequently, Cao et. al. [6] proposed it as a smooth, inviscid regularization of the 2D and 3D-Navier-Stokes equations. For applications of such models in organic polymer and food industry, and in the mechanisms of diffuse axonal injury, etc., we refer to [4], [5] and [7].

Earlier, based on the analysis of Ladyzenskaya [15] in the context of Navier Stokes equations, Oskolkov [21]-[22] have proved existence of a unique ‘almost’ classical solution in finite time interval for the problem (1.1)-(1.3). Subsequently, further investigations on solvability were continued by group members of Oskolkov, see [24] and [25].

On numerical analysis of such problems, Oskolkov et a. [23] have discussed the convergence analysis of the spectral Galerkin approximation for all $t \geq 0$ assuming that the exact solution is asymptotically stable as $t \to \infty$. Subsequently, Pani et a. [17] have applied a variant of nonlinear semidiscrete spectral Galerkin method and optimal error estimates are proved. It is, further, shown that a priori error estimates are valid uniformly in time under uniqueness assumption. Recently, Bajpai et al. [1] have applied finite element Galerkin methods for the problem (1.1)-(1.3) with the forcing function $f = 0$. They have proved a priori bounds for the exact solution in 3D and established exponential decay property. With an introduction of the Sobolev-Stokes projection, they have derived optimal error estimates, which again preserve the exponential decay property. In [2], completely discrete schemes which are based on both backward Euler and second order backward difference methods are analyzed and optimal error bounds which again preserve exponential decay property are established. For related articles in the context of Oldroyd viscoelastic model, we refer to [10]-[12], [18, 19], [26]-[29].

In this paper, we, further, continue the investigation on finite element approximation to the problem (1.1)-(1.3) when the non-zero forcing function $f \in L^\infty(L^2)$. This is crucial particularly in the study of the dynamical system (1.1)-(1.3), when the forcing function is assumed to be time independent. The major results obtained in this paper are summarized as follows:

(i) New regularity results for the solution of (1.1)-(1.3) even in 3D, which are valid uniformly in time are derived and as a consequence, existence of a global attractor is proved. It is further shown that these estimates hold uniformly in $\kappa$ as $\kappa \to 0$.

(ii) When $f$ is independent of time, it is, further, established that the semi-discrete finite element method admits a discrete global attractor.

(iii) Based on the Sobolev-Stokes projection introduced earlier in [1], optimal error estimates for the semidiscrete Galerkin approximations to the velocity in $L^\infty(L^2)$-norm as well as in $L^\infty(H^1_0)$-norm and to the pressure in $L^\infty(L^2)$-norm are derived with error bounds depending on exponential in time.

(iv) Moreover, it is proved under uniqueness assumption that error estimates are valid uniformly in time.

Note that for (i), exponential weight functions in time are used which help us to derive regularity result for all $t > 0$. A special care is taken to show that these estimates are valid uniformly in $\kappa$ as $\kappa \to 0$. When $f$ is independent of time, based on uniform estimates in time existence of a global attractor is shown for the semidiscrete scheme. For (iii), a use of Sobolev-Stokes projection as an intermediate projection helps us to retrieve optimal error estimates for the velocity vector in
$L^\infty(\mathbf{L})$-norm. When either $f = 0$ or $f = O(e^{-\alpha t})$, we derive, as in [1], exponential decay property not only for the solution, but also for error estimates.

This paper is organized as follows. In Section 2, we discuss the weak formulation and state some basic assumptions. Section 3 is devoted to development of a priori bounds for the exact solutions. In Section 4, we describe the semidiscrete Galerkin approximations and derive a priori estimates with discrete global attractor for the semidiscrete solutions. In Section 5, we establish optimal error estimates for the velocity. Section 6 deals with the optimal error estimates for the pressure. In Section 7, results of numerical experiments, which confirm our theoretical estimates, are established.

2 Preliminaries and Weak formulation

In this section, we define $\mathbb{R}^d$, $(d = 2, 3)$-valued function spaces using boldface letters as

$$
\mathbf{H}_0^1 = (H_0^1(\Omega))^d, \quad \mathbf{L}^2 = (L^2(\Omega))^d \quad \text{and} \quad \mathbf{H}^m = (H^m(\Omega))^d,
$$

where $L^2(\Omega)$ is the space of square integrable functions defined in $\Omega$ with inner product $(\phi, \psi) = \int_\Omega \phi(x)\psi(x)\, dx$ and norm $||\phi|| = \left(\int_\Omega |\phi(x)|^2\, dx\right)^{1/2}$. Further, $H^m(\Omega)$ denotes the standard Hilbert Sobolev space of order $m \in \mathbb{N}^+$ with norm $||\phi||_m = \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha \phi|^2\, dx\right)^{1/2}$. Note that $\mathbf{H}_0^1$ is equipped with a norm

$$
||\nabla v|| = \left(\sum_{i,j=1}^d (\partial_j v_i, \partial_j v_i)\right)^{1/2} = \left(\sum_{i=1}^d (\nabla v_i, \nabla v_i)\right)^{1/2}.
$$

Further, introduce divergence free spaces :

$$
\mathbf{J}_1 = \{\phi \in \mathbf{H}_0^1 : \nabla \cdot \phi = 0\}
$$

and

$$
\mathbf{J} = \{\phi \in \mathbf{L}^2 : \nabla \cdot \phi = 0 \text{ in } \Omega, \quad \phi \cdot n|_{\partial \Omega} = 0 \text{ holds weakly}\},
$$

where $n$ is the outward normal to the boundary $\partial \Omega$ and $\phi \cdot n|_{\partial \Omega} = 0$ should be understood in the sense of trace in $H^{-1/2}(\partial \Omega)$, see [7]. Let $H^m/\mathbb{R}$ be the quotient space with norm $||p||_{H^m/\mathbb{R}} = \inf_{c \in \mathbb{R}} ||p + c||_m$. For a Banach Space $X$ with norm $\cdot \|X$, let $L^p(0,T;X)$ denote the space of measurable $X$- valued functions $\phi$ on $(0,T)$ such that $\int_0^T ||\phi(t)||_{X}^p\, dt < \infty$ if $1 \leq p < \infty$ and for $p = \infty$, \text{ess sup}_{0 < t < T} ||\phi(t)||_{X} < \infty$. Now, set $P : \mathbf{L}^2 \rightarrow \mathbf{J}$ as the $\mathbf{L}^2$- orthogonal projection.

Throughout this paper, the following assumptions are made.

\textbf{(A1).} Setting $-\Delta = -P \Delta : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \rightarrow \mathbf{J}$ as the Stokes operator, assume that the following regularity result holds:

$$
||v||_2 \leq C||\tilde{\Delta}v|| \quad \forall v \in \mathbf{J}_1 \cap \mathbf{H}^2. \tag{2.1}
$$

The above assumption is valid as the domain $\Omega$ is a convex polygon or convex polyhedron. Note that the following Poincaré inequality [13] holds true:

$$
||v||_2 \leq \lambda_1^{-1}||\nabla v||_2 \quad \forall v \in \mathbf{H}_0^1(\Omega), \tag{2.2}
$$

3
where $\lambda_1^{-1}$, is the best possible positive constant depending on the domain $\Omega$. Further, observe that
\begin{equation}
\|\nabla v\|^2 \leq \lambda_1^{-1} \|\tilde{\Delta} v\|^2 \quad \forall v \in J_1 \cap H^2. \tag{2.3}
\end{equation}

(A2). There exists a positive constant $M$ such that the initial velocity $u_0$ and the external force $f, f_t$ satisfy for $t \in (0, \infty)$
\[ u_0 \in H^2 \cap J_1, \quad f, \ f_t \in L^\infty(0, \infty; L^2) \] with $\|u_0\|_2 \leq M$, $\text{ess sup} \|f(\cdot, t)\| \leq M$.

Now, the weak formulation of (1.1)-(1.3) is to seek a pair of functions $(u(t), p(t)) \in H^1_0 \times L^2/\mathbb{R}$ with $u(0) = u_0$, such that for all $t > 0$
\begin{equation}
\begin{aligned}
(u_t, \phi) + \kappa(\nabla u, \nabla \phi) + \nu(\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) &= (p, \nabla \cdot \phi) + (f, \phi) \quad \forall \phi \in H^1_0, \\
(\nabla \cdot u, \chi) &= 0, \quad \forall \chi \in L^2.
\end{aligned}
\tag{2.4}
\end{equation}

Equivalently, find $u(t) \in J_1$ with $u(0) = u_0$ such that for $t > 0$
\begin{equation}
(u_t, \phi) + \kappa(\nabla u_t, \nabla \phi) + \nu(\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) = (f, \phi) \quad \forall \phi \in J_1. \tag{2.5}
\end{equation}

Define the trilinear form $b(\cdot, \cdot, \cdot)$ as
\[ b(v, w, \phi) := \frac{1}{2} (v \cdot \nabla w, \phi) - \frac{1}{2} (v \cdot \nabla \phi, w), \quad v, w, \phi \in H^1_0. \]

Note for $v \in J_1, w, \phi \in H^1_0$ that $b(v, w, \phi) = (v \cdot \nabla w, \phi)$. Because of antisymmetric property of the trilinear form, it is easy to check that for,
\begin{equation}
\begin{aligned}
b(v, w, w) &= 0, \quad \forall v, w \in J_1.
\end{aligned}
\tag{2.6}
\end{equation}

3  A priori estimates for the exact solution

In this section, some a priori bounds for the solution $(u, p)$ of (2.4) are derived. Since these results differ from [1] in the sense that $0 \neq f \in L^\infty(L^2)$ in the present article, therefore, the major differences in the analysis are indicated.

**Lemma 3.1.** Let the assumptions (A1)-(A2) hold true, and let $0 < \alpha < \frac{\nu \lambda_1}{4(1 + \kappa \lambda_1)}$. Then, the solution $u$ of (2.5) satisfies for all $t > 0$
\begin{equation}
\begin{aligned}
\left( \|u(t)\|^2 + \kappa \|\nabla u(t)\|^2 \right) + \beta e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla u(s)\|^2 \, ds &
\leq e^{-2\alpha t} (\|u_0\|^2 + \kappa \|\nabla u_0\|^2) + \left( \frac{1 - e^{-2\alpha t}}{2\nu \lambda_1 \alpha} \right) \|f\|^2_{L^\infty(L^2)} :=: K_0(t) \\
&\leq (\|u_0\|^2 + \kappa \|\nabla u_0\|^2) + \left( \frac{1}{2\nu \lambda_1 \alpha} \right) \|f\|^2_{L^\infty(L^2)} :=: K_{0, \infty}, \quad t > 0. \tag{3.1}
\end{aligned}
\end{equation}

where $\beta = \nu - 2\alpha(\kappa + \lambda_1^{-1}) \geq \nu/2 > 0$, and $K_{0, \infty} = \sup_{t \in [0, \infty)} K_0(t)$. Moreover,
\begin{equation}
\limsup_{t \to \infty} \|\nabla u(t)\| \leq \left( \frac{1}{\lambda_1 \nu^2} \right) \|f\|_{L^\infty(0, \infty; L^2)}. \tag{3.2}
\end{equation}
Proof. Set \( \hat{u}(t) = e^{\alpha t}u(t) \) for some \( \alpha > 0 \) in (2.5). Then, choose \( \phi = \hat{u} \) in (2.6) and use (2.6) in the resulting equation to arrive at
\[
\frac{1}{2} \frac{d}{dt} (||\hat{u}||^2 + \kappa ||\nabla \hat{u}||^2) + (\nu - \alpha (\kappa + \lambda_1^{-1})) ||\nabla \hat{u}||^2 \leq (\hat{f}, \hat{u}). \tag{3.3}
\]
Now, estimate the right-hand side of (3.3) as
\[
||\hat{f}, \hat{u}|| \leq ||\hat{f}||||\hat{u}|| \leq \frac{1}{\sqrt{\lambda_1}} ||\hat{f}||||\nabla \hat{u}|| \leq \nu ||\nabla \hat{u}||^2 + \frac{1}{2\nu \lambda_1} ||\hat{f}||^2. \tag{3.4}
\]
Substitute (3.4) in (3.3), use kickback argument and \( \beta = \nu - 2\alpha (\kappa + \lambda_1^{-1}) = \nu/2 - (\nu/2 - 2\alpha (\kappa + \lambda_1^{-1})) \geq \nu/2 > 0 \) to obtain
\[
\frac{d}{dt} (||\hat{u}||^2 + \kappa ||\nabla \hat{u}||^2) + \beta ||\nabla \hat{u}||^2 \leq \frac{1}{\nu \lambda_1} ||\hat{f}||^2 \tag{3.5}
\]
Integrate with respect to time from 0 to \( t \), then multiply by \( e^{-2\alpha t} \) and use the assumption (A2) as well as the fact that
\[
e^{-2\alpha t} \int_0^t e^{2\alpha s} ds = \frac{1}{2\alpha} (1 - e^{-2\alpha t}) \tag{3.6}
\]
to complete the proof of (3.1).

Note that the second term on the left had side of (3.1) is nonnegative and hence, it can be dropped. Then taking limit superior as \( t \to \infty \) for the remaining terms on both sides, we arrive at
\[
\lim_{t \to \infty} \sup (||u(t)||^2 + \kappa ||\nabla u(t)||^2) \leq \left( \frac{1}{2\nu \lambda_1} \right) ||f||^2_{L^\infty(L^2)}. \tag{3.7}
\]
For (3.2), we rewrite (3.3) as :
\[
\frac{1}{2} \frac{d}{dt} (||u||^2 + \kappa ||\nabla u||^2) + \nu ||\nabla u||^2 \leq (\hat{f}, \hat{u}) + \alpha (||\hat{u}||^2 + \kappa ||\nabla \hat{u}||^2). \tag{3.3a}
\]
Integrate with respect to time and then, divide the resulting equation by \( e^{-2\alpha t} \) to arrive at
\[
(||u(t)||^2 + \kappa ||\nabla u(t)||^2) + \nu e^{-2\alpha t} \int_0^t e^{2\alpha s} ||\nabla u(s)||^2 ds \leq e^{-2\alpha t}(||u_0||^2 + \kappa ||\nabla u_0||^2) \tag{3.8}
\]
\[
+ \frac{||f||^2_{L^\infty(L^2)}}{2\alpha \lambda_1 \nu} (1 - e^{-2\alpha t}) + 2\alpha e^{-2\alpha t} \int_0^t e^{2\alpha s} (||u(s)||^2 + \kappa ||\nabla u(s)||^2) ds.
\]
Now, the first term on the left hand side of (3.8) is nonnegative which can then be dropped. Taking limit superior on the both sides of (3.8) for the remaining terms and using L’Hospital rule, we note that
\[
\lim_{t \to \infty} 2\alpha e^{-2\alpha t} \int_0^t e^{2\alpha s} (||u(s)||^2 + \kappa ||\nabla u(s)||^2) ds = \lim_{t \to \infty} (||u(t)||^2 + \kappa ||\nabla u(t)||^2), \tag{3.9}
\]
\[
\lim_{t \to \infty} \nu e^{-2\alpha t} \int_0^t e^{2\alpha s} ||\nabla u(s)||^2 ds = \frac{\nu}{2\alpha} \lim_{t \to \infty} ||\nabla u(t)||^2, \tag{3.10}
\]
and hence, using (3.7) we arrive at
\[
\lim_{t \to \infty} ||\nabla u(t)|| \leq \left( \frac{1}{\lambda_1 \nu^2} \right) ||f||_{L^\infty(0, \infty; L^2)}.
\]
This completes the rest of the proof. \( \square \)
Remark 3.1. As a consequence of Lemma 3.1, we obtain from (3.5) with \( \alpha = 0 \) the following estimate

\[
\frac{d}{dt}(\|u\|^2 + \kappa \|\nabla u\|^2) + \nu \|\nabla u\|^2 \leq \frac{1}{\nu \lambda_1} \|f\|^2. \tag{3.11}
\]

On integration with respect to time from \( t \) to \( t + T_0 \), and using (3.1) of Lemma 3.1, we obtain for fixed \( T_0 > 0 \) and \( t \geq 0 \)

\[
\nu \int_t^{t+T_0} \|\nabla u\|^2 \, ds \leq K_0(t) + \frac{T_0}{\nu \lambda_1} \|f\|^2,
\]

\[
\leq K_{0,\infty} + \frac{T_0}{\nu \lambda_1} \|f\|^2. \tag{3.12}
\]

Taking limit superior on both sides of (3.12), we now arrive at

\[
\nu \limsup_{t \to \infty} \int_t^{t+T_0} \|\nabla u\|^2 \, ds \leq K_{0,\infty} + \frac{T_0}{\nu \lambda_1} \|f\|^2. \tag{3.13}
\]

Remark 3.2. Note that if \( f \in L^\infty(H^{-1}) \), where \( H^{-1} \) is the topological dual of \( H_0^1 \), then following the proof of the Lemma 3.1, obtain

\[
\|u(t)\|^2 + \kappa \|\nabla u(t)\|^2 + \beta e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla u(s)\|^2 \, ds
\]

\[
\leq e^{-2\alpha t}(\|u_0\|^2 + \kappa \|\nabla u_0\|^2) + \left(\frac{1 - e^{-2\alpha t}}{2\nu \alpha}\right) \|f\|^2_{L^\infty(H_0^1)^*} = K_0^*(t) \leq K_{0,\infty}, \quad t > 0. \tag{3.14}
\]

Remark 3.3. Earlier, Oskolkov [22] has proved the existence of a unique weak solution to the problem (1.1)-(1.3) for finite time, but the proof cannot be extended to all \( t > 0 \) as the constants involved in a priori estimates depend on exponentially in time. Now, using Bubnov Galerkin method with a priori bounds in Lemma 3.1 and standard weak compactness arguments, it can be shown that there exists a unique global weak solution \( u \) to the problem (2.5) for all \( t > 0 \). Further, it is easy to check that the problem (2.5) generates a continuous semigroup \( S(t) : J_1 \to J_1 \), \( t \in [0, \infty) \). Therefore, the result of [14] shows that if \( f \in L^\infty(H^{-1}) \), then the semigroup \( S(t) \) has an absorbing ball

\[
B_\rho(0) : \{v \in J_1 : \left(\|v\|^2 + \kappa \|\nabla v\|^2\right)^{1/2} \leq \rho\}
\]

with \( \rho \) given by

\[
\rho^2 = \left(\frac{1}{\alpha \nu}\right) \|f\|^2_{L^\infty(H_0^1)^*}.
\]

Hence, it may be easily shown that the problem has a global attractor \( A_1 \subset J_1 \).

Lemma 3.2. Let assumptions (A1)-(A2) hold true. Then, for \( 0 < \alpha < \frac{\nu \lambda_1}{4(1 + \lambda_1 \kappa)} \) and for all \( t > 0 \)

\[
\|\nabla u(t)\|^2 + \kappa \|\nabla u(t)\|^2 + \beta e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\Delta u(s)\|^2 \, ds \leq e^{-2\alpha t}(\|\nabla u(0)\|^2 + \kappa \|\Delta u(0)\|^2)
\]

\[
+ C(\nu, \alpha) \left(\frac{K_{0,\infty}^{t+2}}{\kappa^\ell}(1 - e^{-2\alpha t}) + (1 - e^{-2\alpha t}) \|f\|^2_{L^\infty(L^2)}\right) = K_1(t) \leq K_{1,\infty}
\]

holds, where \( \beta = \nu - 2\alpha(\kappa + \lambda_1^{-1}) \geq \nu/2 > 0 \), for \( d = 2, \, \ell = 1 \), and when \( d = 3, \, \ell = 3 \).
Proof. Set \( \hat{u} = e^{\alpha t} u \) and use the definition of the Stokes operator \( \tilde{\Delta} \) to rewrite (3.15) as

\[
\langle \hat{u} \rangle_t - \alpha \hat{u} - \kappa \tilde{\Delta} \hat{u} + \kappa \alpha \tilde{\Delta} \hat{u} - \nu \tilde{\Delta} \hat{u} = -e^{-\alpha t}(\hat{u} \cdot \nabla \hat{u}) + \hat{f} \quad \forall \phi \in J_1. \tag{3.15}
\]

Multiply (3.15) by \(-\tilde{\Delta} \hat{u}\) and integrate over \(\Omega\). A use of integration by parts with (2.2) and \(-(\hat{u}_t, \tilde{\Delta} \hat{u}) = \frac{1}{2} \frac{d}{dt} \|\nabla \hat{u}\|^2\) leads to

\[
\frac{1}{2} \frac{d}{dt}(\|\nabla \hat{u}\|^2 + \kappa \|\tilde{\Delta} \hat{u}\|^2) + (\nu - \alpha(\kappa + \lambda^{-1})) \|\tilde{\Delta} \hat{u}\|^2 = e^{-\alpha t}(\hat{u} \cdot \nabla \hat{u}, \tilde{\Delta} \hat{u}) + (\hat{f}, -\tilde{\Delta} \hat{u}) = I_1 + I_2. \tag{3.16}
\]

For \(I_1\), we note by generalized \(H^\lambda\)-Holder’s inequality that

\[
|I_1| \leq e^{-\alpha t} \|\hat{u}\|_{L^4} \|\nabla \hat{u}\|_{L^4} \|\tilde{\Delta} \hat{u}\|. \tag{3.17}
\]

When \(d = 2\), a use of Ladyzhenskaya’s inequality:

\[
\|\hat{u}\|_{L^4} \leq C \|\hat{u}\|^{\frac{1}{2}} \|\nabla \hat{u}\|^{\frac{1}{2}} \quad \text{and} \quad \|\nabla \hat{u}\|_{L^4} \leq \|\nabla \hat{u}\|^{\frac{1}{2}} \|\Delta \hat{u}\|^{\frac{1}{2}}.
\]

in (3.17) with the Young’s inequality with \(p = 4\), \(q = \frac{4}{3}\), \(\epsilon = \frac{2\nu}{3}\) yields

\[
|I_1| \leq Ce^{-\alpha t} \|\hat{u}\|^{\frac{1}{2}} \|\nabla \hat{u}\| \|\tilde{\Delta} \hat{u}\|^{\frac{2}{3}} \leq C \left(\frac{1}{\nu}\right)^3 e^{2\alpha t} \|u\|^2 \|\nabla u\|^4 + \frac{\nu}{6} \|\tilde{\Delta} \hat{u}\|^2. \tag{3.18}
\]

When \(d = 3\), a use of Ladyzhenskaya’s inequality:

\[
\|\hat{u}\|_{L^4} \leq C \|\hat{u}\|^{\frac{1}{2}} \|\nabla \hat{u}\|^{\frac{1}{2}} \quad \text{and} \quad \|\nabla \hat{u}\|_{L^4} \leq \|\nabla \hat{u}\|^{\frac{1}{2}} \|\Delta \hat{u}\|^{\frac{1}{2}}.
\]

in (3.17) with the Young’s inequality with \(p = 8/7\), \(q = 8\), \(\epsilon = \frac{4\nu}{21}\) shows

\[
|I_1| \leq Ce^{-\alpha t} \|\hat{u}\|^{\frac{1}{2}} \|\nabla \hat{u}\| \|\tilde{\Delta} \hat{u}\|^{\frac{7}{8}} \leq C \left(\frac{1}{\nu}\right)^7 e^{2\alpha t} \|u\|^2 \|\nabla u\|^8 + \frac{\nu}{6} \|\tilde{\Delta} \hat{u}\|^2. \tag{3.20}
\]

For \(I_2\), an application of the Cauchy-Schwarz inequality with the Young’s inequality leads to

\[
|I_2| = \|\hat{f}, -\tilde{\Delta} \hat{u}\| \leq \|\hat{f}\| \|\tilde{\Delta} \hat{u}\| \leq \frac{\nu}{3} \|\tilde{\Delta} \hat{u}\|^2 + \frac{3}{2\nu} \|\hat{f}\|^2. \tag{3.21}
\]

Substitute (3.18) and (3.21) in (3.16) to find at

\[
\frac{d}{dt}(\|\nabla \hat{u}\|^2 + \kappa \|\tilde{\Delta} \hat{u}\|^2) + (\nu - 2\alpha(\kappa + \lambda^{-1})) \|\tilde{\Delta} \hat{u}\|^2 \leq C(\nu) \left(e^{2\alpha t} \|u\|^2 \|\nabla u\|^{2(\ell+1)} + \|\hat{f}\|^2\right), \tag{3.22}
\]

where \(\ell = 1\), when \(d = 2\) and for \(d = 3\), \(\ell = 3\). Integrate (3.22) with respect to time from 0 to \(t\). Then, use Lemma 3.1 and \(\beta = \nu - 2\alpha(\kappa + \lambda^{-1}) \geq \nu/2 > 0\) to arrive at

\[
\|\nabla u(t)\|^2 + \kappa \|\tilde{\Delta} u(t)\|^2 + \beta e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\tilde{\Delta} u(s)\|^2 ds \leq e^{-2\alpha t}(\|\nabla u_0\|^2 + \kappa \|\tilde{\Delta} u(0)\|^2)
+ C(\nu) e^{-2\alpha t} \int_0^t e^{2\alpha s} \|u(s)\|^2 \|\nabla u(s)\|^2 \|\nabla u(s)\|^{2\ell} ds
+ C(\nu, \alpha)(1 - e^{-2\alpha t}) \|\hat{f}\|^2_{L^\infty(L^2)}. \tag{3.23}
\]
For the second term one the right hand side of (3.23), apply Lemma 3.1 to obtain
\[ \|\nabla u(t)\|^2 + \kappa \|\tilde{\Delta} u(t)\|^2 + \beta e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\tilde{\Delta} u(s)\|^2 ds \leq e^{-2\alpha t} (\|\nabla u_0\|^2 + \kappa \|\tilde{\Delta} u_0\|^2) \]
\[ + C(\nu, \alpha) \left( \frac{K^{t+2}_{0,\infty}}{\kappa^t} (1 - e^{-2\alpha t}) + \|f\|^2_{L^\infty(L^2)} (1 - e^{-2\alpha t}) \right). \]

This completes the rest of the proof. \( \square \)

Note that results in Lemma 3.2 are valid uniformly in time for both 2D and 3D problems. However, constants in those bounds depend on \( 1/\kappa \), which blow up as \( \kappa \) tends to zero. Therefore, in the following Lemma, we propose to discuss results which are valid for all time, but their bounds are independent of \( 1/\kappa \).

**Lemma 3.3.** Let assumptions (A1)-(A2) hold true. Then, there exists a positive constant \( K_{12} = K_{12}(\nu, \alpha, \lambda_1, M) \) such that for \( 0 < \alpha < \frac{\nu \lambda_1}{4(1 + \lambda_1 \kappa)} \) and for all \( t > 0 \),
\[ \|\nabla u(t)\|^2 + \kappa \|\tilde{\Delta} u(t)\|^2 + \beta e^{-2\alpha t} \leq K_{12}, \quad (3.24) \]
where \( \beta = \nu - 2\alpha (\kappa + \lambda_1^{-1}) \geq \nu/2 > 0 \). For \( d = 3 \), the estimate (3.24) holds true under smallness assumption on \( M \), that is, on the data.

**Proof.** When \( d = 2 \), we note from (3.23) that
\[ \|\nabla \hat{u}(t)\|^2 + \kappa \|\tilde{\Delta} \hat{u}(t)\|^2 + \int_0^t e^{2\alpha s} \|\tilde{\Delta} \hat{u}(s)\|^2 ds \leq (\|\nabla u_0\|^2 + \kappa \|\tilde{\Delta} u_0\|^2) \]
\[ + C(\nu) \int_0^t \|\tilde{f}(s)\|^2 ds + C(\nu) \int_0^t \|u(s)\|^2 \|\nabla u(s)\|^2 \|\nabla \hat{u}(s)\|^2 ds. \quad (3.25) \]

An application of Gronwall’s lemma leads to
\[ \|\nabla \hat{u}(t)\|^2 + \kappa \|\tilde{\Delta} \hat{u}(t)\|^2 + \beta \int_0^t e^{2\alpha s} \|\tilde{\Delta} \hat{u}(s)\|^2 ds \leq \{ (\|\nabla u(0)\|^2 + \kappa \|\tilde{\Delta} u(0)\|^2) \]
\[ + C(\nu) \int_0^t \|\tilde{f}(s)\|^2 ds \} \times \exp \left( C(\nu) \int_0^t \|u(s)\|^2 \|\nabla u(s)\|^2 ds \right). \quad (3.26) \]

Apply assumption (A2) in (3.26) to obtain
\[ \|\nabla u(t)\|^2 + \kappa \|\tilde{\Delta} u(t)\|^2 + \beta \int_0^t e^{2\alpha s} \|\tilde{\Delta} u(s)\|^2 ds \leq C(\nu, \alpha, K_{0,\infty}) \exp \left( C(\nu) \int_0^t \|u(s)\|^2 \|\nabla u(s)\|^2 ds \right). \quad (3.27) \]

A use of estimate (3.1) of Lemma 3.1 with estimate (3.13) in (3.27) shows that for all finite but fixed \( 0 < T_0 \) with \( 0 < t \leq T_0 \) and for \( d = 2 \)
\[ \|\nabla u(t)\|^2 + \kappa \|\tilde{\Delta} u(t)\|^2 + \beta \int_0^t e^{2\alpha s} \|\tilde{\Delta} u(s)\|^2 ds \leq C(\nu, \alpha, K_{0,\infty}, T_0). \quad (3.28) \]
Since the inequality (3.28) is valid for all finite, but fixed \( T_0 \), now a use of the following result (3.2) from Lemma 3.1
\[ \limsup_{t \to \infty} \|\nabla u\| \leq C \]
leads to the boundedness of $\|\nabla u(t)\|$ for all $t > 0$. This completes the proof for $d = 2$.

When $d = 3$, that is, the problem in 3D, we observe from (3.28) with $\ell = 3$ after multiplying with $e^{-2\alpha t}$ both sides and using (3.1) that

$$\|\nabla u(t)\|^2 + \kappa \|\Delta u(t)\|^2 + \beta e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\Delta u(s)\|^2 ds \leq e^{-2\alpha t} (\|\nabla u_0\|^2 + \kappa \|\Delta u_0\|^2)$$

$$+ \frac{3}{\nu} e^{-2\alpha t} \int_0^t \|f(s)\|^2 ds + C(\nu) e^{-2\alpha t} \int_0^t \|u(s)\|^2 \|\nabla u(s)\|^8 ds$$

$$\leq C_1(K_{0,\infty}) + C_2(K_{0,\infty}) \int_0^t \|\nabla u(s)\|^8 ds$$

(3.29)

Setting $\Psi = \|\nabla u(t)\|^2$ and dropping the last two terms on the left hand side of (3.29) as these are nonnegative, then we arrive at

$$\Psi(t) \leq C_1(K_{0,\infty}) + C_2(K_{0,\infty}) \int_0^t \Psi(s) ds$$

(3.30)

This integral inequality holds true for all finite time $t > 0$ provided both $C_1(K_{0,\infty})$ and $C_2(K_{0,\infty})$ are sufficiently small, that is, under the assumption that the condition $\text{(A}_2\text{)}$ is valid for sufficiently small $M$. Therefore, the boundedness of $\|\nabla u(t)\|$ is proved for all finite, but fixed $t > 0$ and for sufficiently smallness assumption on both initial data and forcing function. The rest of the analysis follows as in 2D case, that is, when $d = 2$, using the estimate (3.2). This completes the rest of the proof. □

**Lemma 3.4.** Under assumptions (A1)-(A2), there exists a constant $C = C(\nu, \alpha, \lambda_1, M)$ such that the following holds true for $0 < \alpha < \frac{\nu \lambda_1}{4 (1 + \lambda_1 \kappa)}$ and for all $t > 0$

$$e^{-2\alpha t} \int_0^t e^{2\alpha s} (\|u(s)\|^2 + \kappa \|\nabla u(s)\|^2) ds + \nu \|\nabla u(t)\|^2 \leq C.$$

**Proof.** Choose $\phi = e^{2\alpha t} u_t$ in (2.5) to arrive at

$$e^{2\alpha t} (\|u_t\|^2 + \kappa \|\nabla u_t\|^2) + \frac{\nu}{2} e^{2\alpha t} \frac{d}{dt} \|\nabla u\|^2 = e^{2\alpha t} (f, u_t) - e^{2\alpha t} (u, \nabla u, u_t).$$

(3.31)

For the nonlinear term on the right hand side of (3.31), use Sobolev imbedding theorem to obtain

$$|\langle u, \nabla u, u_t \rangle| \leq C \|u\|_{L^4} \|\nabla u\|_{L^4} \|u_t\| \leq C \|\nabla u\| \|\Delta u\| \|u_t\|.$$  

(3.32)

Use (3.32) in (3.31), then integrate the resulting inequality with respect to time from 0 to $t$ and apply the Young’s inequality. Then, multiply the resulting equation by $e^{-2\alpha t}$ to arrive at

$$e^{-2\alpha t} \int_0^t e^{2\alpha s} (\|u(s)\|^2 + 2\kappa \|\nabla u(s)\|^2) ds + \nu \|\nabla u(t)\|^2 \leq C e^{-2\alpha t} \|\nabla u_0\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla u(s)\|^2 ds$$

$$+ e^{-2\alpha t} \int_0^t e^{2\alpha s} |f(s)|^2 ds + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla u(s)\|^2 \|\Delta u(s)\|^2 ds.$$  

(3.33)

A use of Lemmas 3.1 with 3.3 leads to the desired result and this concludes the proof. □

**Lemma 3.5.** Let the assumptions (A1)-(A2) hold true. Then, there exists a positive constant $C = C(\nu, \alpha, \lambda_1, M)$ such that for all $t > 0$

$$\|u_t(t)\|^2 + \kappa \|\nabla u_t(t)\|^2 + \nu e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla u_t(s)\|^2 ds \leq C.$$
\textbf{Proof.} Differentiate (2.5) with respect to time to obtain
\[ (u_t, \phi) + \kappa(\nabla u_t, \nabla \phi) + \nu(\nabla u_t, \nabla \phi) = -(u_t \cdot \nabla u, \phi) - (u \cdot \nabla u_t, \phi) + (f, \phi) \quad \forall \phi \in J_1. \] (3.34)

Choose \( \phi = u_t \) in (3.34) with \((u \cdot \nabla u_t, u_t) = 0\) to find that
\[ \frac{1}{2} \frac{d}{dt} \left( \|u_t\|^2 + \kappa \|\nabla u_t\|^2 \right) + \nu \|\nabla u_t\|^2 = -(u_t \cdot \nabla u, u_t) + (f, u_t). \] (3.35)

Apply the Ladyzenskaya’s inequality (3.19) for \( d = 3 \) and the Young’s inequality (with \( p = 8 \) and \( q = 8/7 \)) to arrive at
\[ (u_t \cdot \nabla u, u_t) \leq C \|u_t\|^{1/4} \|\nabla u\| \|\nabla u_t\|^{7/4} \leq C(\nu) \|\nabla u\|^8 \|u_t\| + \frac{\nu}{4} \|\nabla u_t\|^2. \] (3.36)

A use of the Cauchy-Schwarz inequality with the Young’s inequality leads to
\[ (f, u_t) \leq \|f\| \|u_t\| \leq \frac{1}{\sqrt{\lambda_1}} \|f\| \|\nabla u_t\| \leq \frac{1}{\lambda_1 \nu} \|f\|^2 + \frac{\nu}{4} \|\nabla u_t\|^2. \] (3.37)

Substitute (3.36)-(3.37) in (3.35) and then multiply by \( e^{2at} \). An application of \textit{a priori} estimates from Lemma 3.3 yields
\[ \frac{d}{dt} e^{2at} \left( \|u_t\|^2 + \kappa \|\nabla u_t\|^2 \right) + \nu e^{2at} \|\nabla u_t\|^2 \leq C(\nu, \lambda_1) e^{2at} (\|u_t\|^2 + \|f\|^2) + 2 \alpha e^{2at} (\|u_t\|^2 + \kappa \|\nabla u_t\|^2). \] (3.38)

Integrate (3.38) from 0 to \( t \) with respect to time to obtain
\[ \|u_t\|^2 + \kappa \|\nabla u_t\|^2 + \nu e^{-2at} \int_0^t e^{2as} \|\nabla u_t(s)\|^2 ds \leq e^{-2at} (\|u_t(0)\|^2 + \kappa \|\nabla u_t(0)\|^2) + C e^{-2at} \int_0^t e^{2as} (\|u_t(s)\|^2 + \|f(s)\|^2) ds + 2 \alpha e^{-2at} \int_0^t e^{2as} (\|u_t(s)\|^2 + \kappa \|\nabla u_t(s)\|^2) ds. \] (3.39)

From (2.5), it may be observed that
\[ \|u_t\|^2 + \kappa \|\nabla u_t\|^2 \leq C(||\tilde{\Delta} u||^2 + ||f||^2 + ||u||^2 ||\nabla u||^4) \leq C(\lambda_1)(||\tilde{\Delta} u||^2 + ||f||^2). \] (3.40)

Using (3.40) (see, the proof in [13] pp 285, eq (2.19)), we can define (3.40) at \( t = 0 \). A use of Lemma 3.4 with (A2) and (3.40) in (3.39) establishes the desired estimates. This completes the rest of the proof. \( \square \)

\textbf{Lemma 3.6.} Let assumptions (A1)-(A2) hold. Then, there exists a positive constant \( C = C(\nu, \alpha, \lambda_1, M) \) such that for \( 0 < \alpha < \frac{\nu \lambda_1}{4(1 + \lambda_1 \kappa)} \) and for all \( t > 0 \),
\[ \nu \|\tilde{\Delta} u(t)\|^2 + e^{-2at} \int_0^t e^{2as} (\|\nabla u_t(s)\|^2 + \kappa \|\tilde{\Delta} u_t(s)\|^2) ds \leq C. \] (3.41)

Moreover, the following estimate hold:
\[ \kappa \|\tilde{\Delta} u(t)\| \leq C. \] (3.42)
Proof. Rewrite (2.5) as
\[ u_t - \kappa \Delta u_t - \nu \Delta u + u \cdot \nabla u = f \quad \forall \phi \in J_1. \] (3.43)

Form \( L^2 \) inner-product between (3.43) and \(-e^{2 \alpha t} \Delta u_t \) to obtain
\[ \frac{\nu}{2} \frac{d}{dt}\|\Delta u\|^2 + e^{2 \alpha t} \left( \|\nabla u_t\|^2 + \kappa \|\Delta u_t\|^2 \right) = e^{2 \alpha t} (f, -\Delta u_t) + e^{2 \alpha t} (u \cdot \nabla u, \Delta u_t) \]
\[ + \nu \alpha \|\Delta u\|^2 = I_1 + I_2 + \nu \alpha \|\Delta u\|^2. \] (3.44)

Now, integrate (3.44) with respect to time from 0 to \( t \) and then, multiply by \( 2e^{-2 \alpha t} \) to arrive at
\[ \nu \|\Delta u\|^2 + 2e^{-2 \alpha t} \int_0^t e^{2 \alpha s} (\|\nabla u_t\|^2 + \kappa \|\Delta u_t\|^2) \, ds \leq \nu e^{-2 \alpha t} \|\Delta u_0\|^2 \]
\[ + 2e^{-2 \alpha t} \int_0^t \left( I_1(s) + I_2(s) \right) \, ds + 2\nu \alpha e^{-2 \alpha t} \int_0^t e^{2 \alpha s} \|\Delta u(s)\|^2 \, ds. \] (3.45)

For \( I_2 \) on the right hand side of (3.44), rewrite it as
\[ I_2 = e^{2 \alpha t} (u \cdot \nabla u, \Delta u_t) = \frac{d}{dt} \left( e^{2 \alpha t} (u \cdot \nabla u, \Delta u) \right) - 2\alpha e^{2 \alpha t} (u \cdot \nabla u, \Delta u) \]
\[ - e^{2 \alpha t} (\nabla u, \Delta u) - e^{2 \alpha t} (u \cdot \nabla u, \Delta u). \] (3.46)

Note that an application of the Ladyzhenskaya’s inequality (3.19) with the Young’s inequality shows that
\[ e^{2 \alpha t} (u \cdot \nabla u, \Delta u) \leq Ce^{2 \alpha t} \|u\|^{1/4} \|\nabla u\| \|\Delta u\|^{7/4} \leq C(\nu) e^{2 \alpha t} \|u\|^2 \|\nabla u\|^8 + \frac{\nu}{2} e^{2 \alpha t} \|\Delta u\|^2. \] (3.47)

From (3.43), we observe using bounds from Lemmas 3.3 and 3.5 that
\[ \|\Delta u\| \leq \frac{1}{\nu} \left( \|u_t\| + \|u\| \|\nabla u\| + \|f\| + \kappa \|\Delta u_t\| \right) \leq C(\nu, \alpha, \lambda_1, M) + \frac{1}{\nu} \kappa \|\Delta u_t\|. \] (3.48)

For the third term on the right hand side of (3.47), we again employ Ladyzhenskaya’s inequality (3.19) with estimates from Lemmas 3.3, 3.5, (3.48) and the Young’s inequality to obtain
\[ e^{2 \alpha t} (\nabla u, \Delta u) \leq C e^{2 \alpha t} \|u_t\|^{1/4} \|\nabla u_t\|^{3/4} \|\Delta u\|^{7/4} \]
\[ \leq C e^{2 \alpha t} \|u_t\|^{1/4} \|\nabla u_t\|^{3/4} \left( C(\nu, \alpha, \lambda_1, M) + \kappa \|\Delta u_t\| \right)^{7/4} \]
\[ \leq C(\nu, \alpha, \lambda_1, M) e^{2 \alpha t} \|u_t\|^{1/4} \|\nabla u_t\|^{3/4} \]
\[ + C(\nu, \alpha, \lambda_1, M) e^{7/8} \|\nabla u_t\|^{3/4} \left( \sqrt{\kappa} \|\Delta u_t\| \right)^{7/4} \]
\[ \leq C(\nu, \alpha, \lambda_1, M) e^{2 \alpha t} \left( 1 + \|\nabla u_t\|^2 \right) \]
\[ + C(\nu, \alpha, \lambda_1, M) e^{2 \alpha t} \|u_t\|^2 \kappa^4 \left( \kappa \|\nabla u_t\| \right)^3 \]
\[ + \frac{1}{4} e^{2 \alpha t} \kappa \|\Delta u_t\|^2 \]
\[ \leq C(\nu, \alpha, \lambda_1, M) e^{2 \alpha t} \left( 1 + \|\nabla u_t\|^2 + \kappa^4 \|u_t\|^2 \right) + \frac{1}{4} e^{2 \alpha t} \kappa \|\Delta u_t\|^2. \] (3.49)

Moreover for the last term on the right hand side of (3.47), a use of following Agmon inequality (see, [8] which is valid for 3D)
\[ \|u\|_{L^\infty} \leq C \|\nabla u\|^{1/2} \|\Delta u\|^{1/2}, \] (3.50)
with estimates from Lemmas 3.3, 3.5, 3.8 and the Young’s inequality yields

\[ e^{2\alpha t} (u \cdot \nabla u_t, \tilde{\Delta} u) \leq C e^{2\alpha t} \|u\|_{L^\infty} \|\nabla u_t\| \|\tilde{\Delta} u\| \leq C e^{2\alpha t} \|\nabla u\|^{1/2} \|\tilde{\Delta} u\|^{1/2} \|\nabla u_t\| \|\tilde{\Delta} u\| \]

\[ \leq C e^{2\alpha t} \|\nabla u\|^{1/2} \|\nabla u_t\| \left( C(\nu, \alpha, \lambda_1, M) + \kappa \|\tilde{\Delta} u_t\| \right)^{3/2} \]

\[ \leq C e^{2\alpha t} \left( 1 + \|\nabla u_t\|^2 \right) + C(\nu, \alpha, \lambda_1, M) e^{2\alpha t} \|\nabla u_t\|^3 \|\tilde{\Delta} u\|^3 \]

\[ \leq C e^{2\alpha t} \left( 1 + \|\nabla u_t\|^2 \right) + C \kappa e^{2\alpha t} \left( \kappa \|\nabla u_t\|^2 \right)^2 + \frac{1}{4} \kappa \|\tilde{\Delta} u_t\|^2. \tag{3.51} \]

Substituting (3.49) and (3.51) in \( I_2 \) and integrating with respect to time, use a priori bounds in Lemmas 3.3, 3.5 to arrive for the second term on the right hand side of (3.45) at

\[ 2e^{-2\alpha t} \int_0^t I_2(s) \, ds \leq C(\nu, \alpha, \lambda_1, M) + C e^{-2\alpha t} \int_0^t e^{2\alpha s} \left( 1 + (1 + \kappa) \|\nabla u_t\|^2 + \|u_t\|^2 + \|\tilde{\Delta} u\|^2 \right) \, ds \]

\[ + \frac{\nu}{4} \|\tilde{\Delta} u(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \left( \|\nabla u_t\|^2 + \kappa \|\tilde{\Delta} u_t\|^2 \right) \, ds \]

\[ \leq C(\nu, \alpha, \lambda_1, M) + \frac{\nu}{4} \|\tilde{\Delta} u(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \left( \|\nabla u_t\|^2 + \kappa \|\tilde{\Delta} u_t\|^2 \right) \, ds \tag{3.52} \]

For \( I_1 \) term, again rewrite it

\[ I_1 = e^{2\alpha t} (f, \tilde{\Delta} u_t) = \frac{d}{dt} \left( e^{2\alpha t} (f, \tilde{\Delta} u) \right) - 2\alpha e^{2\alpha t} (f, \tilde{\Delta} u) - e^{2\alpha t} (f, \tilde{\Delta} u). \tag{3.53} \]

Now integrate \( I_1 \) with respect to time and then multiply by \( 2e^{-2\alpha t} \). Then, a use of assumption (A2) shows

\[ 2e^{-2\alpha t} \int_0^t I_1(s) \, ds = (f, \tilde{\Delta} u) - e^{-2\alpha t} (f_0, \tilde{\Delta} u_0) \]

\[ - 2e^{-2\alpha t} \int_0^t \alpha e^{2\alpha s} \left( 2\alpha (f, \tilde{\Delta} u) + (f, \tilde{\Delta} u) \right) \, ds \]

\[ \leq C(M) + \frac{\nu}{4} \|\tilde{\Delta} u(t)\|^2 + C(\alpha) e^{-2\alpha t} \int_0^t \alpha e^{2\alpha s} \left( \|f\|^2 + \|f_t\|^2 \right) \, ds \]

\[ + C e^{-2\alpha t} \int_0^t \alpha e^{2\alpha s} \|\tilde{\Delta} u\|^2 \, ds. \tag{3.54} \]

Substitute (3.52) and (3.54) in (3.45) and use Lemmas 3.1, 3.3, 3.5 with assumption (A2) and standard kickback argument to arrive at the desired estimate (3.41). To prove (3.42), we note from (3.43) using Lemmas 3.3, 3.5 with estimate (3.19) and (3.41) that

\[ \kappa \|\Delta u_t(t)\| \leq \|u_t\| + \nu \|\tilde{\Delta} u\| + \|u \cdot \nabla u\| + \|f\| \]

\[ \leq \left( \|u_t\| + \nu \|\tilde{\Delta} u\| + C \|u\|^{1/4} \|\nabla u\| \|\tilde{\Delta} u\|^{3/4} + \|f\| \right) \leq C. \]

This completes the rest of the proof. \( \square \)

The following Lemma 3.7 deals with a priori bounds of the pressure term.

**Lemma 3.7.** Under assumptions (A1)-(A2), there exists a positive constant \( C = C(\nu, \lambda_1, \alpha, M) \) such that for \( 0 < \alpha < \frac{\nu \lambda_1}{4 (1 + \lambda_1 \kappa)} \) and for all \( t > 0 \), the following estimate holds true:

\[ \|p(t)\|_{L^2/\mathbb{R}}^2 + \|p(t)\|_{H^1/\mathbb{R}}^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|p(s)\|_{H^1/\mathbb{R}}^2 \, ds \leq C. \]
An application of Lemmas 3.3, 3.4 and 3.6 leads to proof.

from Lemmas 3.1, 3.3-3.7.

A use of Lemmas 3.3, 3.5 and 3.6 in (3.60) yields (A2)

∥∇φ∥ ≤ C(ν)∥u∥ + κ∥∆u∥ + ∥u∥∥Δu∥^{3/4} + ∥f∥)(∥φ∥)

and hence,

∥∇p∥ ≤ C(ν)(∥u∥ + κ∥∆u∥ + ∥u∥∥Δu∥^{3/4} + ∥f∥).

A use of Lemmas 3.3, 3.5 and 3.6 in (3.60) yields

∥p(t)∥_{H^1/R} ≤ C.

Take square of both sides of (3.60). Then, multiply the resulting equation by e^{2αt} and integrate from 0 to t with respect to time to obtain

∫^t_0 e^{2αs}∥∇p(s)∥^2 ds ≤ C(ν)\left(∫^t_0 e^{2αs}(∥u(s)∥^2 + κ∥∆u∥^2) ds + ∫^t_0 e^{2αs}∥f(s)∥^2 ds \right).

An application of Lemmas 3.3, 3.4 and 3.6 leads to

e^{-2αt}∫^t_0 e^{2αs}∥∇p(s)∥^2 ds ≤ C.

A use of (3.57), (3.61) and (3.63) would lead to the desired result. This concludes the rest of the proof.

The main Theorem of this section is stated below without proof as its proof follows easily from Lemmas 3.1, 3.3, 3.7.

**Theorem 3.1.** Let the assumptions (A1) and (A2) hold. Then, there exists a positive constant C = C(ν, α, λ1, M) such that for 0 ≤ α < \frac{νλ1}{2(1 + λ1 κ)} the following estimates hold true:

∥u(t)∥^2 + ∥p(s)∥^2_{L^2/R} + e^{-2αt}∫^t_0 e^{2αs}(∥u(s)∥^2 + ∥p(s)∥^2_{H^1/R}) ds ≤ C,

∥u^r(t)∥^2 + κ∥u^r(t)∥^2 + ∥p(s)∥^2_{H^1/R} + e^{-2αt}∫^t_0 e^{2αs}(∥u^r(s)∥^2 + κ∥u^r(s)∥^2_{H^1/R}) ds ≤ C.
the following estimate: for $0 < t > 0$ and even for small $\kappa$ in 2D and for 3D with data small. As a result, we can take limit of the equations \((2.4)\) as $\kappa$ tends to zero which may result in the convergence of the Kelvin-Voigt system to the Navier-Stokes system.

Note that an application of Lemmas \(3.1, 3.2, 3.3, 3.7\) instead of Lemma \(3.7\) would easily provide results of Theorem \(3.1\) which are valid for both 2D and 3D without data small, but with constant $C$ in the Theorem \(3.1\) now depending on $1/\kappa$.

### Remark 3.5

If $f \in L^2(0, \infty; L^2)$, Theorem \(3.1\) holds uniformly in time with $\alpha = 0$. When $f(t) = O(e^{-\alpha t})$, then simple modifications in all Lemmas show exponential decay property which is of order $O(e^{-\alpha t})$, where $\alpha = \min(\alpha, \alpha_0)$ in Theorem \(3.1\).

### 4 The semidiscrete scheme

With $h > 0$ as a discretization parameter, let $H_h$ and $L_h$, $0 < h < 1$ be finite dimensional subspaces of $H^1_0$ and $L^2$, respectively, and be such that, there exist operators $i_h$ and $j_h$ satisfying the following approximation properties:

\((B1)\). For each $v \in J_1 \cap H^2$ and $q \in H^1/\mathbb{R}$, there are approximations $i_h v \in J_h$ and $j_h q \in L_h$ such that

$$
\|v - i_h v\| + h \|
abla(v - i_h v)\| \leq K_0 h^2 \|v\|_2, \quad \|q - j_h q\|_{L^2/\mathbb{R}} \leq K_0 h \|q\|_{H^1/\mathbb{R}}.
$$

For defining the Galerkin approximations, for $v, w, \phi \in H^1_0$, set $a(v, \phi) = (\nabla v, \nabla \phi)$ and $b(v, w, \phi)$ as in Section 2. Note that, the operator $b(\cdot, \cdot, \cdot)$ preserves the antisymmetric properties of the original nonlinear term, i.e.,

$$
b(v_h, w_h, w_h) = 0 \quad \forall v_h, w_h \in H^1_0.
$$

The discrete analogue of the weak formulation \((2.4)\) is to find $u_h(t) \in H_h$ and $p_h(t) \in L_h$ such that $u_h(0) = u_{0h}$ and for $t > 0$,

$$
(u_{ht}, \phi_h) + \kappa a(u_{ht}, \phi_h) + \nu a(u_h, \phi_h) + b(u_h, u_h, \phi_h) - (p_h, \nabla \cdot \phi_h) = (f, \phi_h) \quad \forall \phi_h \in H^1_0,
$$

\((\nabla \cdot u_h, \chi_h) = 0 \quad \forall \chi_h \in L_h,
$$

\(4.1\)

where $u_{0h} \in H^1_0$ is a suitable approximation of $u_0 \in J_1$ to be defined later.

We now introduce $J_h$ as

$$
J_h = \{v_h \in H^1_0 : (\chi_h, \nabla \cdot v_h) = 0 \quad \forall \chi_h \in L_h\}.
$$

Note that, $J_h$ is not a subspace of $J_1$. Now, the semidiscrete approximation in $J_h$ is to seek $u_h(t) \in J_h$ such that $u_h(0) = u_{0h} \in J_h$ and for $t > 0$

$$
(u_{ht}, \phi_h) + \kappa a(u_{ht}, \phi_h) + \nu a(u_h, \phi_h) = -b(u_h, u_h, \phi_h) + (f, \phi_h) \quad \forall \phi_h \in J_h.
$$

\(4.2\)

Since $J_h$ is finite dimensional, the equation \((4.2)\) leads to a system of nonlinear ordinary differential equations. Therefore, an application of Picard’s theorem ensures existence of a unique solution $u_h$ for $(0, t^*_h)$ for some $t^*_h > 0$. For global existence, we need to use continuation argument provided the discrete solution is bounded for all $t > 0$. Following the argument in the proof of Lemma \(3.1\), it is easy to prove the following estimate: for $0 < \alpha < \frac{\nu \lambda_1}{4 (1 + \kappa \lambda_1)}$ and for all $t > 0$

$$
\|u_h(t)\|^2 + \kappa \|\nabla u_h(t)\|^2 + \beta e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla u_h(s)\|^2 ds \leq e^{-2\alpha t} (\|u_{0h}\|^2 + \kappa \|\nabla u_{0h}\|^2) + \left(\frac{1 - e^{-2\alpha t}}{2 \nu \lambda_1 \alpha}\right) \|f\|_{L^\infty(L^2)}^2.
$$

\(4.3\)
where \( \beta = \nu - 2\alpha(\kappa + \lambda_1^{-1}) > \nu/2 > 0 \). This completes the proof of existence and uniqueness of a global discrete solution for all \( t > 0 \).

As a consequence of (4.3), the following result on existence of a discrete global attractor is derived.

**Lemma 4.1.** There exists a bounded absorbing set

\[ B_{\rho_0}(0) = \{ u_h \in J_h : \left( \| u_h \|^2 + \kappa \| \nabla u_h \|^2 \right)^{1/2} \leq \rho_0 \} \]

with \( \rho_0 \) given by

\[ \rho_0^2 = \left( \frac{1}{\sqrt{\alpha \nu \lambda_1}} \right) \| f \|_{L^\infty(L^2)}^2. \]

Further, the problem (4.2) has a global attractor \( A_h \subset J_h \), which attracts bounded sets in \( J_h \).

**Proof.** To prove the first part, we need to show an existence of \( \rho_1 > 0 \) such that for any \( u_{0h} \in J_h \), there exists a time \( t^* := t^* (\| u_{0h} \|^2 + \kappa \| \nabla u_{0h} \|^2)^{1/2} \) such that for \( t \geq t^* \) the discrete solution \( u_h(t) \) of (4.2) satisfies \( u_h(t) \in B_{\rho_1}(0) \). For any ball \( B_{\rho_1}(0) \), \( \rho_1 > \rho_0/2 \) with the initial condition \( u_{0h} \in B_{\rho_1}(0) \), it follows from (4.3) that

\[
\left( \| u_h(t) \|^2 + \kappa \| \nabla u_h(t) \|^2 \right)^{1/2} \leq e^{-2\alpha t} \rho_1^2 + \frac{1}{2} \rho_0^2 \left( 1 - e^{-2\alpha t} \right) \]

(4.4)

\[
= e^{-2\alpha t} \left( \rho_1^2 - \frac{1}{2} \rho_0^2 \right) + \frac{1}{2} \rho_0^2.
\]

To complete the proof, we claim that

\[ e^{-2\alpha t} \left( \rho_1^2 - \frac{1}{2} \rho_0^2 \right) \leq \frac{1}{2} \rho_0^2. \]

This can be achieved if

\[ t \geq \frac{1}{\alpha \log \left( \frac{2 \rho_1^2 - \rho_0^2}{\rho_0^2} \right)} =: t^* > 0, \]

that is, for \( t \geq t^* \), \( B_{\rho_1}(0) \subset B_{\rho_0}(0) \). Note that for \( \rho_1 \leq \rho_0/2 \), it is trivially satisfied for all \( t > 0 \). Hence, \( B_{\rho_1}(0) \) is an absorbing ball and it further follows that the problem (4.2) has a discrete global attractor \( A_h \subset J_h \), which attracts bounded sets in \( J_h \). This completes the rest of the proof. \( \square \)

Define the quotient space \( L_h/N_h \), where

\[ N_h = \{ q_h \in L_h : (q_h, \nabla \cdot \phi_h) = 0, \forall \phi_h \in H_h \} \]

with its norm given by

\[ \| q_h \|_{L^2/N_h} = \inf_{\chi_h \in N_h} \| q_h + \chi_h \|. \]

Furthermore, assume that the pair \((H_h, L^2/N_h)\) satisfies the following uniform inf-sup condition: (B2). For every \( q_h \in L_h \), there exist a non-trivial function \( \phi_h \in H_h \) and a positive constant \( K_1 \), independent of \( h \), such that

\[ |(q_h, \nabla \cdot \phi_h)| \geq K_1 \| \nabla \phi_h \| \| q_h \|_{L^2/N_h}. \]
As a consequence of conditions (B1)-(B2), we have the following properties of the $L^2$ projection $P_h : L^2 \to J_h$. For $\phi \in J_1$, we note that, (see [9, 13]),  
\[ \|\phi - P_h\phi\| + h\|\nabla P_h\phi\| \leq C h\|\nabla\phi\|, \tag{4.5} \]
and for $\phi \in J_1 \cap H^2$  
\[ \|\phi - P_h\phi\| + h\|\nabla (\phi - P_h\phi)\| \leq C h^2\|D\phi\|. \tag{4.6} \]
We may define the discrete operator $\Delta_h : H^1 \to H^1$ through the bilinear form $a(\cdot, \cdot)$ as  
\[ a(v_h, \phi_h) = (-\Delta_h v_h, \phi) \quad \forall v_h, \phi_h \in H_h. \tag{4.7} \]

Set the discrete analogue of the Stokes operator $\tilde{\Delta} = P\Delta$ as $\tilde{\Delta}_h = P_h\Delta_h$. Examples of subspaces $H_h$ and $L_h$ satisfying assumptions (B1) and (B2) can be found in [9] and [13].

Next in the following Lemma, a priori bounds for the discrete solution $u_h$ of (4.2), which will be helpful in establishing the error estimates, are stated. The proof can be obtained following the similar steps as in the proofs of Lemma 3.1 and 3.3.

**Lemma 4.2.** For all $t > 0$, the semi-discrete Galerkin approximation $u_h$ for the velocity satisfies  
\[ \|u_h(t)\|^2 + \kappa\|\tilde{\Delta}_h u_h(t)\|^2 + \|\tilde{\Delta}_h u_h(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s}(\|\nabla u_h\|^2 + \|\tilde{\Delta}_h u_h\|^2 + \|\nabla u_{ht}\|^2) \, ds \leq C. \]

5 \quad Error estimates for the velocity

In this section, we analyze the error occurred due to the Galerkin approximation for the velocity term.

Since $J_h$ is not a subspace of $J_1$, the weak solution $u$ satisfies  
\[ (u_t, \phi_h) + \kappa a(u_t, \phi_h) + \nu a(u, \phi_h) = -b(u, u, \phi_h) + (p, \nabla \cdot \phi_h) + (f, \phi_h) \quad \forall \phi_h \in J_h. \tag{5.1} \]
Set $e = u - u_h$. Then, from (5.1) and (4.2), we obtain  
\[ (e_t, \phi_h) + \kappa a(e_t, \phi_h) + \nu a(e, \phi_h) = \Lambda(\phi_h) + (p, \nabla \cdot \phi_h), \tag{5.2} \]
where $\Lambda(\phi_h) = -b(u, u, \phi_h) + b(u_h, u_h, \phi_h)$. Below, we derive an optimal error estimate of $\|\nabla e(t)\|$, for $t > 0$.

**Lemma 5.1.** Let assumptions (A1)-(A2) and (B1)-(B2) be satisfied. With $u_{0h} = P_h u_0$, then, there exists a positive constant $C$ depending on $\lambda_1, \nu, \alpha$ and $M$, such that, for fixed $T > 0$ with $t \in (0, T)$ and for $0 \leq \alpha < \frac{\nu \lambda_1}{4(1 + \lambda_1 \kappa)}$, the following estimate holds true :  
\[ \|(u - u_h)(t)\|^2 + \kappa\|\nabla(u - u_h)(t)\|^2 \leq C h^2 e^{CT}. \]

**Proof.** On multiplying (5.2) by $e^{\alpha t}$ with $\phi_h = P_h \hat{e} = \hat{e} + (P_h \hat{u} - \hat{u})$, it follows that  
\[ e^{\alpha t} e_t + \kappa a(e^{\alpha t} e_t, \hat{e}) + \nu a(e, \hat{e}) = e^{\alpha t} \Lambda(P_h \hat{e}) + (\hat{p}, \nabla \cdot P_h \hat{e}) + (e^{\alpha t} e_t, \hat{u} - P_h \hat{u}) + \kappa a(e^{\alpha t} e_t, \hat{u} - P_h \hat{u}) + \nu a(e, \hat{u} - P_h \hat{u}). \tag{5.3} \]
Note that

\[
(e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{e}}) + \kappa a(e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{e}}) = \frac{1}{2} \frac{d}{dt}(\|\hat{\mathbf{e}}\|^2 + \kappa \|\nabla \hat{\mathbf{e}}\|^2) - \alpha(\|\hat{\mathbf{e}}\|^2 + \kappa \|\nabla \hat{\mathbf{e}}\|^2),
\]

and using \(L^2\)-projection \(P_h\), we find that

\[
(e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{u}} - P_h \hat{\mathbf{u}}) = (e^{\alpha t}(\mathbf{e}_t - P_h \mathbf{e}_t), \hat{\mathbf{u}} - P_h \hat{\mathbf{u}}) - \alpha(e^{\alpha t}(\mathbf{e}_t - P_h \mathbf{e}_t), \hat{\mathbf{u}} - P_h \hat{\mathbf{u}})
\]

\[
= \frac{1}{2} \frac{d}{dt}\|\hat{\mathbf{u}} - P_h \hat{\mathbf{u}}\|^2 - \alpha\|\hat{\mathbf{u}} - P_h \hat{\mathbf{u}}\|^2.
\]

A use of (2.2) with (5.4) and (5.5) in (5.3) yields

\[
\frac{d}{dt}(\|\hat{\mathbf{e}}\|^2 + \kappa \|\nabla \hat{\mathbf{e}}\|^2) + (2\nu - 2\alpha(\kappa + \lambda_1^{-1}))\|\nabla \hat{\mathbf{e}}\|^2 \leq 2e^{\alpha t}A(P_h \hat{\mathbf{e}}) + 2(\hat{\mathbf{p}}, \nabla \cdot \hat{\mathbf{e}})
\]

\[
+ \frac{d}{dt}\left(\|\hat{\mathbf{u}} - P_h \hat{\mathbf{u}}\|^2 + 2\kappa a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h \hat{\mathbf{u}})\right) - 2\kappa a(\hat{\mathbf{e}}, e^{\alpha t}(\mathbf{u}_t - P_h \mathbf{u}_t))
\]

\[
- 2\alpha\|\hat{\mathbf{u}} - P_h \hat{\mathbf{u}}\|^2 + 2\nu a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h \hat{\mathbf{u}}).
\]

For the last three terms on the right hand side of (5.6), apply the Cauchy-Schwarz inequality with Poincaré inequality and Young inequality to bound it as

\[
|2\alpha(\|\hat{\mathbf{u}} - P_h \hat{\mathbf{u}}\|^2 + \kappa a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h \hat{\mathbf{u}})) + 2\nu a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h \hat{\mathbf{u}}) + 2\kappa a(\hat{\mathbf{e}}, e^{\alpha t}(\mathbf{u}_t - P_h \mathbf{u}_t))|
\]

\[
\leq C(\alpha, \lambda_1, \nu, \epsilon)\left(\|\nabla(\hat{\mathbf{u}} - P_h \hat{\mathbf{u}})\|^2 + \kappa^2\|e^{\alpha t}(\mathbf{u}_t - P_h \mathbf{u}_t)\|^2 + \frac{\epsilon}{2}\|\nabla \hat{\mathbf{e}}\|^2\right).
\]

For the second term on the right-hand side of (5.6), a use of approximation property (B1) with discrete in compressibility condition and \(H^1\) stability of the \(L^2\)- projection \(P_h\) shows

\[
2|\langle \hat{\mathbf{p}}, \nabla \cdot P_h \hat{\mathbf{e}} \rangle| = |\langle \hat{\mathbf{p}} - j\mathbf{h}, \nabla \cdot P_h \hat{\mathbf{e}} \rangle| \leq C\|\hat{\mathbf{p}} - j\mathbf{h}\| \|\nabla P_h \hat{\mathbf{e}}\|
\]

\[
\leq C(\epsilon)h^2 \|\nabla \hat{\mathbf{p}}\|^2 + \frac{\epsilon}{2}\|\nabla \hat{\mathbf{e}}\|^2.
\]

To estimate the first term on the right-hand side of (5.6), use anti-symmetric property (2.6) of the trilinear form \(b(\cdot, \cdot, \cdot)\) and the property of \(P_h\) to obtain

\[
2e^{\alpha t}A(P_h \hat{\mathbf{e}}) = -2e^{-\alpha t}\left(b(\hat{\mathbf{e}}, \hat{\mathbf{e}} - P_h \hat{\mathbf{u}}) + b(\hat{\mathbf{e}}, \hat{\mathbf{u}}, P_h \hat{\mathbf{e}}) + b(\hat{\mathbf{u}}, \hat{\mathbf{e}}, P_h \hat{\mathbf{e}})\right).
\]

Then, using the generalized Hölder inequality, the Agmon inequality (5.5), the Young inequality, the Sobolev embedding theorem, (2.1) and (1.5), we arrive at

\[
2e^{\alpha t}|A(P_h \hat{\mathbf{e}})| \leq 2e^{-\alpha t}(\|\hat{\mathbf{u}}\|_{L^\infty} \|\nabla \hat{\mathbf{e}}\| \|P_h \hat{\mathbf{e}}\| + \|\nabla \hat{\mathbf{e}}\| \|\nabla \hat{\mathbf{u}}\| \|P_h \hat{\mathbf{u}}\| + \|\nabla \hat{\mathbf{e}}\| \|\nabla \hat{\mathbf{e}}\| \|\nabla(\hat{\mathbf{u}} - P_h \hat{\mathbf{u}})\|)
\]

\[
\leq 2e^{-\alpha t}\left(\|\nabla \hat{\mathbf{u}}\|^{\frac{3}{2}} \|\Delta \hat{\mathbf{u}}\|^\frac{1}{2} + \|\nabla \hat{\mathbf{u}}\|\|\nabla \hat{\mathbf{e}}\| + \|\nabla \hat{\mathbf{u}}\| \|\nabla \hat{\mathbf{e}}\| \|\nabla(\hat{\mathbf{u}} - P_h \hat{\mathbf{u}})\|\right)
\]

\[
\leq C(\epsilon)e^{-2\alpha t}\left(\|\nabla \hat{\mathbf{u}}\| \|\Delta \hat{\mathbf{u}}\| + \|\Delta \hat{\mathbf{u}}\|^2\right)\|\hat{\mathbf{e}}\|^2 + \|\nabla(\hat{\mathbf{u}} - P_h \hat{\mathbf{u}})\|^2 + \frac{\epsilon}{2}\|\nabla \hat{\mathbf{e}}\|^2.
\]

Integrating (5.6) with respect to time from 0 to \(t\), use bounds (5.7), (5.8) and (5.10) with \(\epsilon = \frac{\delta}{3}\), to arrive at

\[
\|\hat{\mathbf{e}}(t)\|^2 + \kappa \|\nabla \hat{\mathbf{e}}(t)\|^2 + \beta \int_0^t \|\nabla \hat{\mathbf{e}}\|^2 ds \leq C(\|e(0)\|^2 + \|\nabla e(0)\|^2)
\]

\[
+ C(\alpha, \nu, \lambda_1, M)\left(\|\nabla(\hat{\mathbf{u}} - P_h \hat{\mathbf{u}})\|^2 + \int_0^t (\|\nabla(\hat{\mathbf{u}} - P_h \hat{\mathbf{u}})\|^2 + \kappa^2 \|\nabla(\hat{\mathbf{u}} - P_h \hat{\mathbf{u}})\|^2 + \|\nabla \hat{\mathbf{p}}\|^2)ds\right) + C \int_0^t (\|\nabla \hat{\mathbf{u}}\| \|\Delta \hat{\mathbf{u}}\| + \|\Delta \mathbf{u}\|^2)\|\hat{\mathbf{e}}\|^2 ds.
\]
A use of (4.6) and (B1) in (5.11) yields
\[ ||\hat{e}(t)||^2 + \kappa||\nabla \hat{e}(t)||^2 + \beta \int_0^t ||\nabla \hat{e}||^2 ds \leq C h^2 \left( ||u_0||^2 + ||\hat{u}||^2 + \int_0^t (||\hat{u}||^2 + ||\hat{v}||^2 + ||\hat{p}(t)||^2_{H^1(\mathbb{R})}) ds \right) + C \int_0^t (||u|| ||\Delta u|| + ||\Delta u||^2)(||\hat{e}||^2 + \kappa||\nabla \hat{e}||^2) ds. \]

From the \textit{a priori} bounds of \( u, u_t \) and \( p \) in Theorem 5.1 we arrive using the Gronwall lemma at
\[ ||\hat{e}(t)||^2 + \kappa||\nabla \hat{e}(t)||^2 + \beta \int_0^t ||\nabla \hat{e}||^2 ds \leq C(\nu, \alpha, \lambda_1, M) h^2 \exp \left( \int_0^t (||\Delta u||^2 + ||\nabla u|| ||\Delta u||) ds \right). \]

A use of \textit{a priori} bounds given in Lemma 3.3 yields
\[ \int_0^t (||\nabla u|| ||\Delta u|| + ||\Delta u||^2) ds \leq C t, \quad (5.12) \]
and hence, we find that
\[ ||(u - u_h)(t)||^2 + \kappa||\nabla (u - u_h)(t)||^2 \leq C h^2 e^{C t}. \]

This concludes the proof. \( \square \)

Observe that the Lemma 5.1 provides a suboptimal error estimates for the velocity in \( L^\infty(\mathbb{R}^2) \)-norm. Therefore, in the remaining part of this section, we derive an optimal error estimate for the velocity in \( L^\infty(\mathbb{L}^2) \)-norm.

Introduce an intermediate solution \( v_h \) which is a finite element Galerkin approximation to a linearized Kelvin-Voigt equation, that is, \( v_h \) satisfies
\[ (v_{ht}, \phi_h) + \kappa a(v_{ht}, \phi_h) + \nu a(v_h, \phi_h) = (f, \phi_h) - b(u, u, \phi_h) \quad \forall \phi_h \in J_h, \quad (5.13) \]
with \( v_h(0) = P_h u_0 \).

Now, we split \( e \) as
\[ e := u - u_h = (u - v_h) + (v_h - u_h) = \xi + \eta. \]

Note that \( \xi \) is the error committed by approximating a linearized Kelvin-Voigt equation \( (5.13) \) and \( \eta \) represents the error due to the non-linearity in the equation. Now, subtract \( (5.13) \) from \( (5.1) \) to write an equation in \( \xi \) as
\[ (\xi_t, \phi_h) + \kappa a(\xi_t, \phi_h) + \nu a(\xi, \phi_h) = (p, \nabla \phi_h) \quad \forall \phi_h \in J_h. \quad (5.14) \]

For deriving optimal error estimates of \( \xi \) in \( L^\infty(\mathbb{L}^2) \) and \( L^\infty(\mathbb{H}^1) \)-norms, we introduce, as in \( \Pi \), the following Sobolev-Stokes’s projection \( V_h u : [0, \infty) \rightarrow J_h \) satisfying
\[ \kappa a(u_t - V_h u_t, \phi_h) + \nu a(u - V_h u, \phi_h) = (p, \nabla \phi_h) \quad \forall \phi_h \in J_h, \quad (5.15) \]
where \( V_h u(0) = P_h u_0 \). In other words, given \( (u, p) \), find \( V_h u : [0, \infty) \rightarrow J_h \) satisfying \( (5.15) \). Since \( J_h \) is finite dimensional, for a given \( u \) the problem \( (5.15) \) leads to a linear system of ODEs. Then, an application of Picard’s theorem with continuation argument ensures existence of a unique solution in \([0, \infty)\). With \( V_h u \) defined as above, we now split \( \xi \) as
\[ \xi := (u - V_h u) + (V_h u - v_h) = : \zeta + \rho. \]
To obtain estimates for $\xi$, first of all, we state estimates of $\zeta$ in Lemmas 5.2 and 5.3. Then, we proceed to estimate $\|\rho\|$ and $\|\nabla \rho\|$ in Lemma 5.4. Combining these results, we obtain estimates for $\xi$ in $L^\infty(L^2)$ and $L^\infty(H^1_0)$-norms in Lemma 5.5. Finally, we derive an estimate for $\eta$ to complete the proof of our main Theorem 5.1.

Below, we briefly state the proofs of the above lemmas. The proofs are along similar lines as in the proofs of Lemmas 5.2-5.7 in [1]. The difference occur only in applying \textit{a priori} estimates as they do not decay exponentially in time. Therefore, in the following proofs, we briefly indicate the differences.

\textbf{Lemma 5.2.} Assume that (A1)-(A2) and (B1)-(B2) are satisfied. Then, there exists a positive constant $C = C(\nu, \lambda_1, \alpha, M)$ such that for $0 < \alpha < \frac{\nu \lambda_1}{4(1 + \kappa \lambda_1)}$, the following estimate holds true:

$$\kappa \|\nabla \zeta(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla \zeta(s)\|^2 ds \leq C h^2.$$ 

\textit{Proof.} We first multiply (5.15) by $e^{\alpha t}$ with $\zeta = u - V_h u$ and then choose $\phi_h = P_h \zeta = \hat{\zeta} - (\hat{u} - P_h \hat{u})$ to arrive at

$$\kappa \frac{d}{dt} \|\nabla \zeta\|^2 + 2(\nu - \kappa \alpha) \|\nabla \zeta\|^2 = 2\kappa \frac{d}{dt} (\hat{\zeta}, \hat{\zeta} - (\hat{u} - P_h \hat{u})) - 2\kappa a(\zeta, \frac{d}{dt} (\hat{u} - P_h \hat{u})) + 2(\nu - \kappa \alpha) a(\zeta, \hat{u} - P_h \hat{u}) + 2(\hat{p}, \nabla \cdot P_h \zeta). \quad (5.16)$$

Integrating (5.16) with respect to time from 0 to $t$, a use of (4.5) along with the Youngs inequality yields

$$\kappa \|\nabla \zeta\|^2 + (\nu - \kappa \alpha) \int_0^t \|\nabla \zeta\|^2 ds \leq C(\nu, \alpha) \left( \|\nabla (u_0 - P_h u_0)\|^2 + e^{2\alpha t} \|\nabla (u - P_h u)\|^2 \right)$$

$$+ \int_0^t e^{2\alpha s} (\|\nabla u_t - P_h u_t\|^2 + \|\nabla (u - P_h u)\|^2 + \|\nabla p\|^2) ds. \quad (5.17)$$

Now, use (4.6) and (B1) in (5.17) to obtain

$$\kappa \|\nabla \zeta\|^2 + (\nu - \kappa \alpha) \int_0^t \|\nabla \zeta\|^2 ds \leq C(\nu, \alpha) h^2 \left( \|\tilde{\Delta} u_0\|^2 + e^{2\alpha t} \|\tilde{\Delta} u\|^2 + \int_0^t e^{2\alpha s} \|\nabla p\|^2 ds \right. $$

$$\left. + \int_0^t e^{2\alpha s} (\|\tilde{\Delta} u_t\|^2 + \|\tilde{\Delta} u\|^2) ds \right). \quad (5.18)$$

From \textit{a priori} bounds for $u$ and $p$ derived in Lemmas 3.2, 3.6 and 3.7 we arrive at the desired result. This completes the rest of the proof. \hfill \Box

Below, we state a lemma without proof. The proof can be obtained in a similar fashion as in [1] and applying now \textit{a priori} estimates derived in Theorem 3.1.

\textbf{Lemma 5.3.} Under the assumptions (A1)-(A2) and (B1)-(B2), there exists a positive constant $C = C(\nu, \lambda_1, \alpha, M)$ such that for $0 < \alpha < \frac{\nu \lambda_1}{4(1 + \kappa \lambda_1)}$, the following estimate holds true for $t > 0$:

$$\kappa \|\zeta(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \left( \|\zeta(s)\|^2 + \kappa \|\zeta_t(s)\|^2 + \kappa h^2 \|\nabla \zeta_t(s)\|^2 \right) ds \leq C h^4.$$ 

In the following Lemma, estimates of $\rho$ are derived.
Lemma 5.4. **Under the assumptions (A1)-(A2) and (B1)-(B2), there exists a positive constant C = C(ν, λ_1, α, M) such that for 0 < α < \frac{ν λ_1}{4(1 + κ λ_1)}, the following estimate holds true:**

\[
κ(∥ρ∥^2 + κ∥∇ρ∥^2) + 2κβe^{-2αt} \int_0^t e^{2αs}∥ρ(s)∥^2 ds ≤ C(ν, λ_1, α, M)h^4.
\]

**Proof.** Subtract (5.15) from (5.14) and substitute \(φ_h\) by \(e^{αt}ρ\) to obtain

\[
(e^{αt}ρt, \tilde{ρ}) + κ a(e^{αt}ρt, \tilde{ρ}) + ν∥∇\tilde{ρ}∥^2 = -(e^{αt}ζt, \tilde{ρ}) \quad ∀φ_h ∈ J_h.
\]  (5.19)

Applying the Cauchy-Schwarz inequality, (2.2) with the Young inequality in (5.19) and integrating with respect to time from 0 to \(t\) to arrive at

\[
∥\tilde{ρ}∥^2 + κ∥∇\tilde{ρ}∥^2 + 2β \int_0^t ∥∇\tilde{ρ}∥^2 ds ≤ C(α, λ_1) \int_0^t e^{αs}ζt(s)^2 ds.
\]  (5.20)

The desired result follows after a use of Lemma 5.3 in (5.20). □

We now derive an estimate of \(ξ\) in \(L^∞(L^2)\) and \(L^∞(H^1_h)\)-norms.

Lemma 5.5. **Let the assumptions (A1)-(A2) and (B1)-(B2) be satisfied. Then, there exists a positive constant C = C(ν, λ_1, α, M) such that for 0 < α < \frac{ν λ_1}{4(1 + κ λ_1)}, the following estimate holds:**

\[
κ∥ξ(t)∥^2 + κ∥∇ξ(t)∥^2 + e^{-2αt} \int_0^t e^{2αs}∥ξ(s)∥^2 ds ≤ C(ν, λ_1, α, M)h^4.
\]

**Proof.** A use of the triangle inequality along with Lemmas 5.2 and 5.4 leads to the desired result. □

Lemma 5.6. **Let the assumptions (A1)-(A2) and (B1)-(B2) hold true. Let \(u_h(t) ∈ J_h\) be a solution of (4.19) with initial condition \(u_h(0) = P_h u_0\), where \(u_0 ∈ J_1\). Then there exist a constant C such that for 0 < \(T < ∞\) with \(t \in (0, T]\)

\[
e^{-2αt} \int_0^t e^{2αs}∥e∥^2 ≤ Ce^{CT}h^4.
\]

**Proof.** In view of Lemma 5.3, we only need to prove the estimate for \(η\). From (5.13) and (4.22), the equation in \(η\) becomes

\[
(η_t, φ_h) + κ a(η_t, φ_h) + ν a(η, φ_h) = Λ_h(φ_h), \quad ∀φ_h ∈ J_h.
\]  (5.21)

where

\[
Λ_h(φ_h) = b(u_h, u_h, φ_h) - b(u, u, φ_h) - b(e, u_h, φ_h) - b(u, e, φ_h).
\]  (5.22)

Substitute \(φ_h = e^{2αt}(\hat{Δ}_h^{-1} η)\) in (5.21) to obtain

\[
\frac{1}{2} \frac{d}{dt}(∥\tilde{η}∥^2 + κ∥η∥^2) - α∥η∥^2_2 + (ν - κα)∥η∥^2 = e^{αt}Λ_h(\tilde{η}).
\]  (5.23)

We recall that \(∥w_h∥_1 := ∥(−\hat{Δ}_h)^{-1/2}w_h∥\) for \(w_h ∈ J_h\). Again for \(v ∈ J_1\) and \(φ, ξ ∈ J_h\)

\[
|b(v, φ, ξ)| ≤ C∥v∥^{1/2}∥∇v∥^{1/2}∥φ∥∥∇ξ∥^{1/2}∥\hat{Δ}_hξ∥^{1/2}.
\]  (5.24)

20
For \( v, \phi, \xi \in \mathcal{J}_h \)
\[
|b(v, \phi, \xi)| \leq ||v|| ||\nabla \phi||^{1/2} ||\Delta_h \phi||^{1/2} (||\xi||^{1/2} ||\nabla \xi||^{1/2} + ||\nabla \xi||).
\] (5.25)

Now, a use of \( e = \xi + \eta \), along with (5.24) and (5.25) leads to
\[
|e^{\alpha t} \Lambda_h(\hat{\Delta}_h^{-1} \hat{\eta})| \leq C \left( ||\nabla u_h|| + ||u_h||^{1/2} ||\nabla u_h||^{1/2} + ||u||^{1/2} ||\nabla u||^{1/2} \right) \left( ||\eta||_{-1}^{1/2} ||\hat{\eta}||^{3/2} + ||\eta|| ||\hat{\xi}|| \right)
\leq \epsilon ||\hat{\eta}||^2 + C(\epsilon) \left( ||\nabla u_h||^2 + ||u_h|| ||\nabla u_h|| + ||u|| ||\nabla u|| \right) ||\xi||^2 + C(\epsilon) ||\hat{\eta}||_{-1}^{2}
\left( ||\nabla u_h||^4 + ||u_h||^2 ||\nabla u_h||^2 + ||u||^2 ||\nabla u||^2 \right).
\] (5.26)

Put \( \epsilon = \frac{\nu}{2} \) in (5.26) and use Lemmas 3.1 and 4.2 to obtain
\[
\frac{d}{dt}(||\hat{\eta}||^2_{-1} + \kappa ||\hat{\eta}||^2) + (\nu - \kappa \alpha) ||\hat{\eta}||^2 \leq C||\hat{\xi}||^2 + (C(\nu) + 2\alpha)||\hat{\eta}||_{-1}^2.
\] (5.27)

Integrate (5.27) with respect to time and observe that \( \eta(0) = 0 \)
\[
||\hat{\eta}||^2_{-1} + \kappa ||\hat{\eta}||^2 + (\nu - \kappa \alpha) \int_0^t ||\hat{\eta}||^2 ds \leq C \int_0^t ||\hat{\xi}||^2 ds + (C(\nu) + 2\alpha) \int_0^t ||\hat{\eta}||_{-1}^2 ds.
\] (5.28)

Apply Gronwall’s Lemma in (5.28) and use Lemma 5.5. Now, a use of triangular inequality completes the rest of proof.

Now, we derive the main Theorem 5.1 of this section.

**Theorem 5.1.** Let the assumptions (A1)-(A2) and (B1)-(B2) be satisfied. Further, let the discrete initial velocity \( u_{0h} = P_h u_0 \). Then, there exists a positive constant \( C = C(\nu, \lambda_1, \alpha, M) \) such that, for all \( t \in (0, T] \) and for \( 0 \leq \alpha < \frac{\nu \lambda_1}{4(1 + \lambda_1 \kappa)} \), the following estimate holds:
\[
||u - u_h(t)|| + h||\nabla (u - u_h)(t)|| \leq C e^{CT} \kappa^{-1/2} h^2.
\] (5.29)

**Proof.** Since \( e = u - u_h = (u - v_h) + (v_h - u_h) = \xi + \eta \) and the estimate of \( \xi \) is derived in Lemma 5.5, therefore to complete the proof, it is enough to estimate \( \eta \).

With a choice of \( \phi_h = e^{2\alpha t} \eta \) in (5.21), we apply (2.2) to arrive at
\[
\frac{1}{2} \frac{d}{dt}(||\hat{\eta}||^2 + \kappa ||\nabla \hat{\eta}||^2) + (\nu - \alpha(\kappa + \frac{1}{\lambda_1})) ||\nabla \hat{\eta}||^2 = e^{\alpha t} \Lambda_h(\hat{\eta}),
\] (5.30)
where \( \Lambda_h(\phi_h) \) is given as in (5.22). For the term on the right hand side of (5.30), we first rewrite it as
\[
e^{\alpha t} \Lambda_h(\hat{\eta}) = e^{-\alpha t} \left( -b(\hat{e}, \hat{u}_h, \hat{\eta}) + b(\hat{u}, \hat{\eta}, \hat{e}) \right).
\]
An application of the Hölder inequality with the Poincaré inequality, the Agmon inequality (3.50) and the discrete Sobolev inequality (see, Lemma 4.4 in [13]) shows
\[
e^{\alpha t} ||\Lambda_h(\hat{\eta})|| \leq C e^{-\alpha t} (||\hat{e}|| ||\nabla \hat{u}_h||_{L^6} ||\hat{\eta}||_{L^3} + ||\hat{u}|| \sup_{|\hat{e}|} ||\nabla \hat{\eta}|| ||\hat{e}||)
\leq C (e^{\alpha t} ||\Delta_h u_h|| ||\nabla \eta|| ||\hat{e}|| + ||\nabla u|| ||\Delta \hat{u}|| \frac{1}{2} ||\nabla \hat{\eta}|| ||\hat{e}||)
\leq C(\nu) e^{-2\alpha t} (||\Delta_h \hat{u}_h||^2 + ||\nabla \hat{u}|| ||\Delta \hat{u}|| ||\hat{e}||^2 + \nu \frac{1}{2} ||\nabla \hat{\eta}||^2).
\] (5.31)
Remark 5.1. We observe that in the above proof the presence of the exponential term on the right-hand side of (5.32) now yields
\[ \frac{d}{dt}(\|\hat{\eta}\|^2 + \kappa\|\nabla \hat{\eta}\|^2) + (\beta + \nu)\|\nabla \hat{\eta}\|^2 \leq C(\nu)e^{-2\alpha t}((\|\hat{\xi}\|^2 + \|\hat{\eta}\|^2)\|\Delta_h \hat{u}_h\|^2 + (\|\hat{\xi}\|^2 + \|\hat{\eta}\|^2)\|\nabla \hat{u}\|^2) + \nu\|\nabla \hat{\eta}\|^2. \] (5.33)

Integrate (5.33) with respect to time from 0 to t and apply Lemmas 3.2 and 4.2 to arrive at
\[ \|\hat{\eta}\|^2 + \kappa\|\nabla \hat{\eta}\|^2 + \beta\int_0^t \|\nabla \hat{\eta}(s)\|^2 ds \leq C(\nu, \alpha, \lambda_1, M)h^4e^{2\alpha t} + \int_0^t \|\hat{\eta}(s)\|^2(\|\nabla u\|\|\Delta u\| + \|\Delta_h u_h\|^2) ds. \] (5.34)

Then, use Gronwall’s Lemma and then multiply by $e^{-2\alpha t}$ to obtain
\[ \|\eta\|^2 + \kappa\|\nabla \eta\|^2 + \beta e^{-2\alpha t}\int_0^t \|\nabla \eta(s)\|^2 ds \leq Ct^2h^4\exp\left(\int_0^T (\|\nabla u\|\|\Delta u\| + \|\Delta_h u_h\|^2) ds\right). \] (5.35)

For the integral on the right hand side of (5.35), apply Lemmas 3.2 and 4.2 to arrive at
\[ \int_0^T (\|\nabla u\|\|\Delta u\| + \|\Delta_h u_h\|^2) ds \leq CT. \] (5.36)

Apply (5.36) in (5.35) to derive estimates for $\eta$ as
\[ \|\eta\|^2 + \kappa\|\nabla \eta\|^2 + 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla \eta(s)\|^2 ds \leq C \nu e^{CT}. \] (5.37)

A use of triangle inequality along with (5.37) and Lemma 5.5 completes the rest of the proof.  

**Remark 5.1.** We observe that in the above proof the presence of the exponential term on the right-hand side of the estimate (5.32) is due to the estimate of $\eta$, as the estimate $\xi$ is uniform in time. In fact, the contribution of the exponential term comes from the Lemma 5.6. If $u_0$ and $f$ are sufficiently small with respect to the norms in the assumptions (A2) so that
\[ \nu - (\kappa\alpha + C(K, \nu) + 2\alpha) \geq 0. \] (5.38)

then, from (5.27), we have
\[ \frac{d}{dt}(\|\eta\|^2 + \kappa\|\nabla \eta\|^2) + (\nu - (\kappa\alpha + (C(K, \nu) + 2\alpha))\|\hat{\eta}\|^2 \leq C(K, \nu)\|\hat{\xi}\|^2. \]

Integrate (5.39) with respect to time 0 to t and use $\eta(0) = 0$ to arrive at
\[ (\|\eta\|^2 + \kappa\|\nabla \eta\|^2) + (\nu - (\kappa\alpha + (C(K, \nu) + 2\alpha))\int_0^t \|\hat{\eta}\|^2 ds \leq C(K, \nu)\int_0^t \|\hat{\xi}\|^2 ds. \]

We can now avoid Gronwall’s Lemma and use Lemma 5.5 with triangle inequality to obtain
\[ e^{-2\alpha t}\int_0^t e^{2\alpha s}\|e\|^2 ds \leq C\kappa^{-1}h^4. \]

Following similar lines of proof, one can show the estimate of $\|e(t)\|$ for all $t > 0$ from Theorem 5.1 provided the assumption (5.35) is satisfied.
Remark 5.2. When \( f \in L^2(0, \infty; L^2(\Omega)) \), all the error estimates are valid uniformly in time as all the a priori bounds hold true for \( \alpha = 0 \) and therefore, the estimate \((5.12)\) bounded uniformly in time. Moreover, if \( f = 0 \) or \( f = O(e^{-\alpha t}) \), we have as in \((5.11)\) exponential decay property for the solution as well as for the error estimates.

**Uniform in time estimates for the velocity:** We now derive uniform (in time) error estimate for the velocity term under the following uniqueness condition

\[
\frac{N}{\nu^2} \| f \|_{L^\infty(0,\infty; L^2(\Omega))} < 1 \quad \text{and} \quad N = \sup_{u,v,w \in V} \frac{|b(u,v,w)|}{\| \nabla u \| \| \nabla v \| \| \nabla w \|}. \tag{5.39}
\]

When \( f = 0 \) or \( \| f(t) \| = O(e^{-\alpha t}) \) for some \( \alpha_0 > 0 \), \((5.39)\) satisfies trivially.

**Theorem 5.2.** Under the assumption of Theorem 5.1 and the uniqueness condition \((5.39)\), there exist a positive constant \( C \), independent of time and \( \kappa \), such that for all \( t > 0 \)

\[
\| (u - \mathbf{u}_h)(t) \| + h \| (\mathbf{u} - \mathbf{u}_h)(t) \| \leq C \kappa^{-1/2} h^2. \tag{5.40}
\]

**Proof.** In order to derive estimates, which are valid uniformly for all \( \alpha \), we need derive a different estimate for the nonlinear term \( \Lambda_h(\mathbf{\eta}) \) with the help of the uniqueness condition \((5.39)\). Therefore, we rewrite

\[
\Lambda_h(\mathbf{\eta}) = -[b(\mathbf{\xi}, \mathbf{u}_h, \mathbf{\eta}) + b(\mathbf{\eta}, \mathbf{u}_h, \mathbf{\eta}) + b(\mathbf{u}, \mathbf{\xi}, \mathbf{\eta})]. \tag{5.41}
\]

Using uniqueness condition, it follows that

\[
|b(\mathbf{\eta}, \mathbf{u}_h, \mathbf{\eta})| \leq N \| \nabla \mathbf{\eta} \|^2 \| \nabla \mathbf{u}_h \|. \tag{5.42}
\]

Apply \((5.24)\) and \((5.25)\) to find that

\[
|b(\mathbf{\xi}, \mathbf{u}_h, \mathbf{\eta}) + b(\mathbf{u}, \mathbf{\xi}, \mathbf{\eta})| \leq C \left( \| \Delta \mathbf{u} \|^2 + \| \nabla \mathbf{u}_h \|^2 \right) \| \nabla \mathbf{\eta} \| \| \mathbf{\xi} \|. \tag{5.43}
\]

Substitute \((5.42)\), \((5.43)\) in \((5.42)\) and use Lemma 5.5 to obtain

\[
|\Lambda_h(\mathbf{\eta})| \leq N \| \nabla \mathbf{\eta} \|^2 \| \nabla \mathbf{u}_h \| + Ch^2 \| \nabla \mathbf{\eta} \|. \tag{5.44}
\]

Now, we modify the proof of Theorem 5.1 as follows

\[
\frac{1}{2} \frac{d}{dt} (\| \mathbf{\eta} \|^2 + \kappa \| \nabla \mathbf{\eta} \|^2) + (\nu - N \| \nabla \mathbf{u}_h \|) \| \nabla \mathbf{\eta} \|^2 \leq \alpha (\| \mathbf{\eta} \|^2 + \kappa \| \nabla \mathbf{\eta} \|^2) + Ch^2 \| \nabla \mathbf{\eta} \|. \tag{5.45}
\]

An integration with respect to time with multiplication by \( e^{2\alpha t} \) leads to

\[
\| \mathbf{\eta}(t) \|^2 + \kappa \| \nabla \mathbf{\eta}(t) \|^2 + 2e^{-2\alpha t} \int_0^t e^{2\alpha s} (\nu - N \| \nabla \mathbf{u}_h \|) \| \nabla \mathbf{\eta}(s) \|^2 \leq 2\alpha e^{-2\alpha t} \int_0^t e^{2\alpha s} (\| \mathbf{\eta}(s) \|^2 + \kappa \| \nabla \mathbf{\eta}(s) \|^2) ds + Ch^2 e^{-2\alpha t} \int_0^t e^{2\alpha s} \| \nabla \mathbf{\eta}(s) \| ds. \tag{5.46}
\]

Letting \( t \to \infty \), we obtain

\[
\frac{1}{\nu} \left( 1 - N \nu^{-2} \| f \|_{L^\infty(0,\infty; L^2(\Omega))} \right) \limsup_{t \to \infty} \| \nabla \mathbf{\eta}(t) \| \leq C h^2. \tag{5.47}
\]
Then, we conclude from the uniqueness condition (5.39) that
\[
\limsup_{t \to \infty} \|\nabla \eta(t)\| \leq Ch^2,
\]
and hence,
\[
\limsup_{t \to \infty} \|\eta(t)\| \leq Ch^2.
\]
Now the uniform estimate of \(\xi\) combined with (5.49) leads to
\[
\limsup_{t \to \infty} \|e(t)\| \leq C \kappa^{-1/2} h^2.
\]
Note that \(C\) is valid uniformly for all \(t > 0\), and this complete the rest of the proof. \(\square\)

6 Error estimate for the pressure

In this section, the optimal error estimate for the Galerkin approximation \(p_h\) of the pressure \(p\) is derived. Further, under the uniqueness condition (5.39), the estimate is shown to be valid uniformly in time. The main theorem of this section is stated as follows:

**Theorem 6.1.** Under the hypotheses of Theorem 5.1 there exists a positive constant \(C\) depending on \(\nu\), \(\lambda_1\), \(\alpha\) and \(M\), such that for \(T > 0\) with \(0 < t \leq T\)
\[
\|(p - p_h)(t)\|_{L^2/N_h} \leq C e^{CT} \kappa^{-1/2} h.
\]
We prove the theorem 6.1 with help of Lemmas 6.1 and 6.2. From (B2), it follows that
\[
\|(j_h p - p_h)(t)\|_{L^2/N_h} \leq C \left( \|j_h p - p\| + \sup_{\phi_h \in \mathbf{H}_h/\{0\}} \left\{ \frac{(p - p_h, \nabla \cdot \phi_h)}{\|\nabla \phi_h\|} \right\} \right).
\]
We observe that the estimate of the first term on the right hand side of (6.1) follows from the approximation property stated in (B1). To complete the proof, it is sufficient to estimate the second term in (6.1). Use (4.1) and (5.1) to find that for \(\phi_h \in \mathbf{H}_h\)
\[
(p - p_h, \nabla \cdot \phi_h) = (e_t, \phi_h) + \kappa a(e_t, \phi_h) + \nu a(e, \phi_h) - \Lambda_h(\phi_h) \forall \phi_h \in \mathbf{H}_h,
\]
where \(\Lambda_h(\phi_h)\) is given as in (5.22). A use of generalized Hölders inequality with Sobolev imbedding, Lemmas 3.1 and 4.2 leads to
\[
|\Lambda_h(\phi_h)| \leq C(\|\nabla u_h\| + \|\nabla u\|)\|\nabla e\|\|\nabla \phi_h\| \leq C\|\nabla e\|\|\nabla \phi_h\|.\]
Thus,
\[
(p - p_h, \nabla \cdot \phi_h) \leq C(\nu) \left( \|e_t\|_{-1,h} + \kappa \|\nabla e_t\| + \kappa \|\nabla e\| \right)\|\nabla \phi_h\|
\]
where
\[
\|e_t\|_{-1,h} = \sup_{\phi_h \in \mathbf{H}_h/\{0\}} \left\{ \frac{(e_t, \phi_h)}{\|\nabla \phi_h\|} \right\}.
\]
Altogether, we derive the following result.

**Lemma 6.1.** The semidiscrete Galerkin approximation \(p_h\) of the pressure \(p\) satisfies for all \(t \in (0, T]\)
\[
\|(p - p_h)(t)\|_{L^2/N_h} \leq C(\|e_t\|_{-1,h} + \kappa \|\nabla e_t\| + \|\nabla e\|).\]
An application of the Cauchy-Schwarz inequality to (6.13) with estimates (6.8) and (6.9) shows
\[ B_1 \]
and hence, a use of (6.12)
Proof of Theorem 6.1
□
To complete the rest of the proof, observe from (5.2) that
\[ (e_t, \phi_h) = -\kappa a(e_t, \phi_h) - \nu a(e, \phi_h) + \Lambda(\phi_h) + (p, \nabla \cdot \phi_h) \]  
(6.13)
An application of the Cauchy-Schwarz inequality to (6.13) with estimates (6.8) and (6.9) shows
\[ (e_t, \phi_h) \leq \left( \kappa \|\nabla e_t\| + \nu \|\nabla e\| + C \|\nabla e\| + \|\nabla e_t\| \right) \|\nabla \phi_h\|, \]  
(6.14)
and hence, a use of (B1) with theorem 5.1 and estimate (6.12) yields the estimate of \|e_t\|_{1,h}. This concludes the proof. □

Proof of Theorem 6.7 The proof follows from Lemmas 6.1 and 6.2 with the approximation property (B1) of \( j_h \). □
Remark 6.1. Under uniqueness condition (5.39), an appeal to (6.3) and (6.17) leads to the error estimate for the pressure, which is valid for all time \( t > 0 \):

\[
\|(p - p_h)(t)\|_{L^2/N_h} \leq K \kappa^{-1/2} h,
\]

and this provides optimal error estimate for pressure term, which is valid uniformly in time.

Remark 6.2. In Theorems 5.1, 5.2 and 5.3, if we choose \( \kappa^{1/2} = O(h^{\delta}) \), where \( \delta > 0 \) can be taken sufficiently small, then we obtain the following quasi-optimal order of convergence:

\[
\|(u - u_h)(t)\| + h \left( \|\nabla (u - u_h)(t)\| + \|(p - p_h)(t)\|_{L^2/N_h} \right) = O(h^{2-\delta}).
\]

7 Numerical Experiments

In this section, three numerical examples using mixed finite element space \( P_2-P_0 \) for spatial discretization and backward Euler method for temporal discretization are discussed with computed orders of convergence, which confirm our theoretical findings. Moreover, it is shown through numerical experiments that orders of convergence do not deteriorate with \( \kappa \) small which again matches with theory. For all three examples, consider the domain \( \Omega = (0,1) \times (0,1) \), \( T = 1 \), \( \kappa = \nu = 1 \). Choose approximating spaces \( H_h \) and \( L_h \) for velocity and pressure, respectively, as

\[
H_h = \{ v \in \left( C(\Omega) \right)^2 : v|_K \in (P_2(K))^2, K \in \tau_h \} \quad \text{and} \quad L_h = \{ q \in L^2(\Omega) : q|_K \in P_0(K), K \in \tau_h \},
\]

where \( \tau_h \) denotes an admissible triangulation of \( \Omega \) into closed triangles with mesh size \( h \). Let \( 0 = t_0 < t_1 < \cdots < t_N = T \), be a uniform subdivision of the time interval \( (0,T) \) with \( t_n = nk \) and \( k = t_n - t_{n-1} \). The fully discrete backward Euler method can be formulated as: given \( U_{n-1} \), find the pair \( (U^n, P^n) \) approximating the pair \( (u,p) \) at \( t = t_n = nk \) satisfying

\[
(\tilde{\partial}_t U^n, v_h) + \kappa a(\tilde{\partial}_t U^n, v_h) + \nu a(U^n, v_h) + b(U^n, U^n, v_h) + (v_h, \nabla P^n) = (f^n, v_h), \quad \forall v_h \in H_h,
\]

\[
(\nabla \cdot U^n, w_h) = 0, \quad \forall w_h \in L_h,
\]

where \( \tilde{\partial}_t U^n = \frac{U^n - U^{n-1}}{h} \).

Example 7.1. The convergence rates of the approximate solution is verified by choosing the right hand side function \( f \) in such a way that the exact solution \( (u,p) = ((u_1,u_2),p) \) of (1.1)-(1.3) is given as

\[
u_1 = 10 \cos t x^2(x-1)^2y(y-1)(2y-1), \quad u_2 = -10 \cos t y^2(y-1)^2x(x-1)(2x-1), \quad p = 40 \cos t xy.
\]

The theoretical analysis proves the convergence rates \( O(h^2) \) for velocity in \( L^2 \) norm, \( O(h) \) for velocity in \( H^1 \) norm and \( O(h) \) for pressure in \( L^2 \) norm. Figure 1 provides convergence rates obtained on successively refined meshes with time step size \( k = h^2 \). These results agree with the optimal theoretical convergence rates obtained in Theorems 5.1, 6.1. Figure 2 depicts that the approximate solution for the data in Example 7.1 is bounded. Note that, here the right hand side function is bounded for all time. Further, Tables 1, 2, 3 represent that the order of convergence for the velocity and pressure errors in Theorems 5.1 and 6.1 hold true in the limit \( \kappa \to 0 \).
\begin{align*}
\text{Table 1: Numerical convergence rates for velocity error with variation in } \kappa \text{ for Example 7.1} \\
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{S No} & h & \|u(t_n) - U^n\|_{L^2} & \|u(t_n) - U^n\|_{L^2} & \|u(t_n) - U^n\|_{L^2} & \|u(t_n) - U^n\|_{L^2} \\
\kappa = 1 & \kappa = 10^{-3} & \kappa = 10^{-6} & \kappa = 10^{-9} \\\n\hline
1 & 1/4 & 1.28476 & 1.46678 & 1.46699 & 1.46699 \\
2 & 1/8 & 1.66634 & 1.71546 & 1.71552 & 1.71552 \\
3 & 1/16 & 1.84754 & 1.86060 & 1.86062 & 1.86062 \\
4 & 1/32 & 1.93052 & 1.93390 & 1.93391 & 1.93391 \\
\hline
\end{array}
\end{align*}

\begin{align*}
\text{Table 2: Numerical convergence rates for velocity error with variation in } \kappa \text{ for Example 7.1} \\
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{S No} & h & \|u(t_n) - U^n\|_{H^1} & \|u(t_n) - U^n\|_{H^1} & \|u(t_n) - U^n\|_{H^1} & \|u(t_n) - U^n\|_{H^1} \\
\kappa = 1 & \kappa = 10^{-3} & \kappa = 10^{-6} & \kappa = 10^{-9} \\\n\hline
1 & 1/4 & 0.52668 & 0.70916 & 0.70938 & 0.70938 \\
2 & 1/8 & 0.80620 & 0.85510 & 0.85516 & 0.85516 \\
3 & 1/16 & 0.91745 & 0.93032 & 0.93033 & 0.93033 \\
4 & 1/32 & 0.96385 & 0.96716 & 0.96716 & 0.96716 \\
\hline
\end{array}
\end{align*}
| S No | $h$  | $\|p(t_n) - P^n\|_{\kappa = 1}$ | $\|p(t_n) - P^n\|_{\kappa = 10^{-3}}$ | $\|p(t_n) - P^n\|_{\kappa = 10^{-6}}$ | $\|p(t_n) - P^n\|_{\kappa = 10^{-9}}$ |
|------|------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1    | 1/4  | 1.25307                       | 1.25165                       | 1.25164                       | 1.25164                       |
| 2    | 1/8  | 1.12462                       | 1.11394                       | 1.11393                       | 1.11393                       |
| 3    | 1/16 | 1.06496                       | 1.05938                       | 1.05937                       | 1.05937                       |
| 4    | 1/32 | 1.02882                       | 1.02663                       | 1.02663                       | 1.02663                       |

Table 3: Numerical convergence rates for pressure error with variation in $\kappa$ for Example 7.1

Example 7.2. In this example, the initial velocity is chosen as

$$u_1 = 10 \, x^2(x - 1)^2y(y - 1)(2y - 1), \quad u_2 = -10 \, y^2(y - 1)^2x(x - 1)(2x - 1), \quad p = 40 \, xy$$

with $\nu = 1$, $\kappa = 1$ and $f = 0$. In this case, to obtain the error estimates the exact solution $\mathbf{u}$ is replaced by finite element solution obtained in a refined mesh.

The convergence rates presented in Figure 3 are in agreement with the results obtained for $f = 0$, that is, the convergence rate for velocity in $L^2$ norm is $O(h^2)$, for velocity in $H^1$-norm is $O(h)$ and for pressure in $L^2$ norm is $O(h)$. In Figure 4, the exponential decay property for the approximate solution $\|\mathbf{U}^n\|$ is shown which verifies theoretical estimates for $f = 0$.

Example 7.3. This example demonstrates the exponential decay property of the discrete solution. Here, $\nu = 1$, $\kappa = 1$ and $f = 0$ with $\mathbf{u}_0 = (\sin^2(3\pi x)\sin(6\pi y), -\sin^2(3\pi y)\sin(6\pi x), \sin(2\pi x)\sin(2\pi y))$ in (1.1)-(1.3). Once again, the error estimates are achieved by considering refined finite element solution as an exact solution.

The order of convergence is shown in Table 4. Figure 5 represents the exponential decay property of $\|\mathbf{U}^n\|$ as time varies which is expected from theoretical analysis for right hand side function $f = 0$.

8 Conclusion

This article in its first part deals with a priori estimates for the weak solution of (1.1)-(1.3) which are valid uniformly in time as $t \rightarrow \infty$ and also uniformly for all $\kappa$ as $\kappa \rightarrow 0$. While estimates hold for 2D,
that is, $d = 2$, and for 3D, that is, $d = 3$, estimates are valid with smallness assumption on the data. In the second part, semidiscrete optimal error estimates of order $O(\kappa^{-1/2}h^m)$ are derived for the velocity in $L^\infty(L^2)$-norm when $m = 2$ and for the velocity in $L^\infty(H^1_0)$-norm, when $m = 1$. Moreover for the pressure term, optimal order estimate $L^\infty(L^2)$-norm, which is of order $O(\kappa^{-1/2}h)$ is established. In all these error analyses, constants appeared in the error estimates depend exponentially on $T$. But, under the uniqueness assumption, it is shown that optimal error estimates are valid uniformly for all time $t > 0$. Further, with $\kappa = O(h^{2\delta})$, $\delta > 0$ very small, quasi-optimal error estimates are derived which are valid uniformly in $\kappa$ as $\kappa \to 0$. All the above results hold true for 2D, but for 3D with smallness assumption on the data. However, in stead of applying Lemma 3.3, if we apply Lemma 3.2, then regularity results like in Theorem 3.1 can be obtained now with constants depending on $1/\kappa$, but all results are valid for 3D without assumption of smallness on the data. Similar conclusion for optimal error estimates can be derived, but with constants depending on $1/\kappa$. Finally, numerical

| S No | $h$  | $\|u - U^n\|_{L^2}$ | Convergence Rate | $\|u - U^n\|_{H^1}$ | Convergence Rate |
|------|------|---------------------|-----------------|---------------------|-----------------|
| 1    | 1/4  | 0.430939            |                 | 6.83152943841204   |                 |
| 2    | 1/8  | 0.203398            | 1.083175531775576 | 5.967502741440636 | 0.195424        |
| 3    | 1/16 | 0.065544            | 1.633758732735566 | 3.67410879088224  | 0.699614        |
| 4    | 1/32 | 0.017502            | 1.904904362752530 | 1.917811790943292 | 0.938051        |

Table 4: Numerical errors and Convergence rates with $k = h^2$ for Example 7.3
experiments are conducted to confirm our theoretical findings.

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References

[1] Bajpai S., Nataraj N., Pani A. K., Damazio P. and Yuan J. Y., Semidiscrete Galerkin method for equations of motion arising in Kelvin-Voigt model of viscoelastic fluid flow, Numer. Meth. PDE. 29 (2013), 857-883.

[2] Bajpai S., Nataraj N. and Pani A. K., On fully discrete finite element schemes for equations of motion of Kelvin-Voigt fluids, Int. J. Numer. Anal. Mod. 10 (2) (2013), 481-507.

[3] Brezzi, F. and Fortin, M., Mixed and hybrid finite element methods, Springer-Verlag, New York, 1991.

[4] Burtscher, M. and Szczyrba, I., Numerical Modeling of Brain Dynamics in Traumatic Situations - Impulsive Translations, The 2005 International Conference on Mathematics and Engineering Techniques in Medicine and Biological Sciences (2005), pp. 205-211.

[5] Burtscher, M. and Szczyrba, I., Computational Simulation and Visualization of Traumatic Brain Injuries, 2006 International Conference on Modeling, Simulation and Visualization Methods (2006), pp. 101-107.

[6] Cao, Y., Lunasin, E. and Titi, E.S., Global wellposedness of the three dimensional viscous and inviscid simplified Bardina turbulence models, Commun. Math. Sci., 4 (2006), pp. 823-848.

[7] Cotter, C.S., Smolarkiewicz, P.K. and Szezyrba, I. N., A viscoelastic model from brain injuries, Intl. J. Numer. Meth. Fluids 40 (2002), pp. 303-311.

[8] Foias, C., Manley, O., Rosa, R. and Temam, R., Navier-Stokes equations and turbulence, Cambridge Univ. Press, Cambridge, 2001.

[9] Girault, V. and Raviart, P. A., Finite element approximation of the Navier-Stokes equations, Lecture notes in Mathematics, No. 749, Springer, New York, 1980.

[10] Goswami, D. and Pani, Amiya K., A priori error estimates for semidiscrete finite element approximations to equations of motion arising in Oldroyd fluids of order one, International J. Numer. Anal. and Modeling (IJNAM) 8 (2011), pp. 324-352.

[11] He, Y., Lin, Y., Shen, S. S. P. and Tait, R., On the convergence of viscoelastic fluid flows to a steady state, Advances in Differential Equations 7 (2002), pp. 717-742.

[12] He Y., Lin Y., Shen S. S. P., Sun W. and Tait R., Finite element approximation for the viscoelastic fluid motion problem, J. Comp. Appl. Mathematics 155 (2003), pp. 201-222.

[13] Heywood, J. G. and Rannacher, R., Finite element approximation of the nonstationary Navier-Stokes problem: I. Regularity of solutions and second order error estimates for spatial discretization, SIAM J. Numer. Anal. 19 (1982), pp. 275-311.
[14] Kalantarov, V.K. and Titi, E., *Global attractors and determining modes for the 3D Navier-Stokes-Voight equations*, Chinese Annals of Math. Ser. B. **30** (2009), pp. 697-714.

[15] Ladyzenskaya, O. A., *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York, 1969.

[16] Pavlovskii, V.A., *To the question of theoretical description of weak aqueous polymer solutions*, Sov. Phy. Dokl. **200** (1971), pp.809-812.

[17] Pani, A. K., Pany, A.K., Damazio, P. and Yuan, J. Y. , *A modified nonlinear spectral Galerkin method for the equations of motion arising in the Kelvin-Voigt fluids*, Applicable Anal. **93** (2014), pp. 1587-1610.

[18] Pani, A. K. and Yuan, J. Y., *Semidiscrete finite element Galerkin approximations to the equations of motion arising in the Oldroyd model*, IMA J. Numerical Analysis **25** (2005), pp. 750-782.

[19] Pani, A. K., Yuan, J. Y. and Damazio, P., *On a linearized backward Euler method for the equations of motion arising in the Oldroyd fluids of order one*, SIAM J. Numer. Anal. **44** (2006), pp. 804-825.

[20] Oskolkov, A. P., *The uniqueness and global solvability for boundary value problems for the equations of motion of water solutions of polymers*, Zapiski Nauch. Sem. POMI, **38** (1973), pp.98-136.

[21] Oskolkov, A. P., *Theory of nonstationary flows of Kelvin-Voigt fluids*, J. Math. Sciences **28** (1985), pp. 751-758.

[22] Oskolkov, A. P., *Initial-boundary value problems for equations of motion of Kelvin-Voigt fluids and Oldroyd fluids*, Proc. Steklov Inst. Math. **2** (1989), pp. 137-182.

[23] Oskolkov, A. P., *On an estimate, uniform on the semiaxis t ≥ 0, for the rate of convergence of Galerkin approximations for the equations of motion of Kelvin-Voigt fluids*, J. Math. Sciences **62**(3) (1992), pp. 2802-2806.

[24] Oskolkov, A. P. and Shadiev, R. D., *Non local problems in the theory of the motion equations of Kelvin-Voigt fluids*, J. Math. Sciences **59**(6)(1992), pp. 1206-1214.

[25] Oskolkov, A. P. and Shadiev, R. D., *Towards a theory of global solvability on [0,∞] of initial-boundary value problems for the equations of motion of Oldroyd and Kelvin-Voigt fluids*, J. Math. Sciences **68**(2) (1994), pp. 240-253.

[26] Wang, Kun, He, Y. and Shang, Y., *Fully discrete finite element method for the viscoelastic fluid motion equations*, Discrete. Contin. Dyn. Sys. Ser. B **13** (2010), pp. 665-684.

[27] Wang, K., He, Y. and Feng, X., *On error estimates of the penalty method for the viscoelastic flow problem I: Time discretization*, Applied Mathematical Modelling **34** (2010), pp. 4089-4105.

[28] Wang, K., He, Y. and Feng, X., *On error estimates of the fully discrete penalty method for the viscoelastic flow problem*, Int. J. Comput. Math. **88** (2011), pp. 2199-2220.

[29] Wang, K., Lin, Y. and He, Y., *Asymptotic analysis of the equations of motion for viscoelastic oldroyd fluid*, Discrete Contin. Dyn. Syst. **32** (2012), pp. 657-677.
exponential decay

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