CR-ANALOGUE OF SIU-$\overline{\partial}$-FORMULA AND APPLICATIONS TO RIGIDITY PROBLEM FOR PSEUDO-HERMITIAN HARMONIC MAPS

SONG-YING LI AND DUONG NGOC SON

Abstract. We give several versions of Siu’s $\overline{\partial}$-formula for maps from a strictly pseudoconvex pseudo-Hermitian manifold $(M^{2m+1}, \theta)$ into a Kähler manifold $(N^n, g)$. We also define and study the notion of pseudo-Hermitian harmonicity for maps from $M$ into $N$. In particular, we prove a CR version of Siu Rigidity Theorem for pseudo-Hermitian harmonic maps from a pseudo-Hermitian manifold with vanishing Webster torsion into a Kähler manifold having strongly negative curvature.

1. Introduction

Let $(M, h)$ and $(N, g)$ be Riemannian manifolds and $f: M \to N$. Then $f$ is said to be harmonic if

$$\Delta_M f^\alpha + \Gamma^\alpha_{\beta\gamma}(f) \frac{\partial f^\beta}{\partial y^i} \frac{\partial f^\gamma}{\partial y^j} h^{ij} = 0,$$

where $y^i$ are the coordinates on $M$, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of $N$, and $\Delta_M$ is the Laplace-Beltrami operator of $M$. In [16], Siu proved the following theorem for harmonic maps between Kähler manifolds which implies his celebrated strong rigidity theorem.

**Theorem 1.1** (Siu Rigidity Theorem). Suppose that $f: M \to N$ is a harmonic map between Kähler manifolds. If $M$ is a closed manifold, $N$ has strongly negative curvature in the sense of Siu, and $df$ has real rank at least 4 at some point $p \in M$, then $f$ is either holomorphic or anti-holomorphic.

This theorem, together with existence theorem for harmonic maps by Eells and Sampson [5], gives the following strong rigidity result for compact Kähler manifolds of strongly negative curvature: suppose that $M$ and $N$ are two Kähler manifolds of complex dimension at least 2 with $M$ being closed and suppose that $N$ has strongly negative curvature. Then $M$ and $N$ are topologically equivalent if and only if they are biholomorphically equivalent.

For strictly pseudoconvex pseudo-Hermitian CR manifolds, beside Laplace-Beltrami operator associated with the Webster metric, there are other notions of Laplacian (e.g., the sub-Laplacian and Kohn-Laplacian), which lead to several notions of harmonicity for maps from CR manifolds. For instance, the pseudoharmonic maps defined by using sub-Laplacian have been studied by many authors (see, e.g, [1, 4, 13]). (However, our notion of pseudo-Hermitian harmonic maps defined below does not coincide with the notion of pseudoharmonic maps into Riemannian manifolds as defined in [4]). Motivated by these research and our own work with X. Wang [11] on Kohn-Laplacian, we define the notion of pseudo-Hermitian harmonic maps, using Kohn-Laplacian on CR manifolds, and study the rigidity analogous to Siu’s strong rigidity.

Date: July 2, 2019.

2000 Mathematics Subject Classification. 32Q05, 30Q15, 32V20.
Let \((M^{2m+1}, \theta)\) be a strictly pseudoconvex pseudo-Hermitian manifold with a pseudo-Hermitian structure \(\theta\) and the Levi metric \(h\), and let \((\mathcal{N}^n, g)\) be a Kähler manifold with Kähler metric \(g\). For any differentiable map \(f : M \rightarrow N\), we define the total \(\bar{\partial}_b\)-energy functional of \(f\) by
\[
E[f] = \int_M g_{\alpha\beta} f^\alpha_i \bar{f}^\beta_j h^{ij} \theta \wedge (d\theta)^m. \tag{1.2}
\]
We say that \(f\) is pseudo-Hermitian harmonic if \(f\) is a critical point of the functional \(E[\cdot]\). Write \(e_\alpha = \partial/\partial z^\alpha\), where \(z^\alpha\) is a local holomorphic coordinate on \(N\), and let
\[
\Gamma^\alpha_{\beta\gamma} = \frac{\partial g_{\alpha\delta}}{\partial z^\beta} g^{\delta\gamma}
\tag{1.3}
\]
be the corresponding Christoffel symbols. Then, the Euler-Lagrange equation for \(E[\cdot]\) is
\[
\tau[f] = h^{ij} \left( f^\alpha_{ij} + \Gamma^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j \right) e_\alpha = 0. \tag{1.4}
\]
The key ingredient in Siu’s proof is his celebrated \(\bar{\partial}\bar{\partial}_{\bar{\partial}}\)-formula, which does not involve Ricci curvature of the source manifold \(M\). In order to prove a Siu-type theorem in CR geometry, the important step is to find an analogue of the \(\bar{\partial}\bar{\partial}_{\bar{\partial}}\)-formula. However, since the Tanaka-Webster connection always has torsion, our formula should involve torsion of \(M\). To be more precise, let \(f^\alpha_{ij[kl]}\) be the components of \(DD\bar{\partial}_b f\), where \(D\) is the connection induced by Tanaka-Webster connection on \(M\) and the pull-back of the complexified Levi-Civita connection on \(N\) (see Section 3 for detail). Following Graham and Lee \([8]\), we define the second and third order operators
\[
f^\alpha_{ij} = f^\alpha_{ij} + N^\alpha_{\sigma j\bar{k}} f^\sigma_{\bar{k}l}; \tag{1.5}
\]
\[
B^\alpha_{ij} f^\beta = f^\alpha_{ij} - \frac{1}{m} (f^\alpha_{kh} h^{ki} h_{ij}); \tag{1.6}
\]
\[
P^i f^\alpha = f^\alpha_{[ij]} h^{ij} + m \sqrt{-1} A^i f^\alpha; \tag{1.7}
\]
\[
P f = (P^i f^\alpha) \theta^i \otimes e_\alpha. \tag{1.8}
\]
Thus, \(Pf\) is a \(f^*T^{1,0}N\)-valued \((1,0)\)-form on \(M\). We can contract \(Pf\) with \(\bar{\partial}_b \bar{f}\) to obtain a scalar, namely
\[
\langle Pf, \bar{\partial}_b \bar{f} \rangle = g_{\alpha\beta} (P^i f^\alpha) \bar{f}^\beta_i h^{ij}. \tag{1.9}
\]
We also define the norm of the tensor \(B^\alpha_{ij} f\) by
\[
|B^\alpha_{ij} f^\beta|^2 = g_{\alpha\beta} (B^\alpha_{ij} f^\alpha) (\bar{B}^{\beta\sigma}_{kl} f^\sigma_{\bar{k}l}) h^{ik} h^{lj}. \tag{1.10}
\]
We say that \(f\) is a CR-pluriharmonic map if \(B^\alpha_{ij} (f^\alpha) = 0\) for all \(1 \leq i, j \leq m\) and \(1 \leq \alpha \leq n\). Now we can state our main theorem.

**Theorem 1.2.** Let \((M^{2m+1}, \theta)\) be a closed pseudo-Hermitian CR manifold and let \((\mathcal{N}^n, g)\) be a Kähler manifold. Let \(f : M \rightarrow N\) be a smooth map. Then
\[
- \frac{m - 1}{m} \int_M \langle Pf, \bar{\partial}_b \bar{f} \rangle = \int_M |B^\alpha_{ij} f^\alpha|^2 + \int_M R^i_{\rho\sigma\bar{\alpha}} f^\alpha_i \bar{f}^\beta_j (f^\gamma_i \bar{f}^{\bar{\gamma}}_j - f^\gamma_j \bar{f}^{\bar{\gamma}}_i) h^{ij} h^{kl}, \tag{1.11}
\]
and
\[
- \int_M \langle Pf, \bar{\partial}_b \bar{f} \rangle = \int_M \langle \tau[f], \bar{\tau}[f] \rangle - \sqrt{-1} m \int_M g_{\alpha\beta} A^i \bar{f}^\alpha_i \bar{f}^\beta_j. \tag{1.12}
\]
Here, the curvature tensor $R_{\alpha\beta\gamma\delta}$ is given by

$$R_{\alpha\beta\gamma\delta} = \frac{\partial^2 g_{\gamma\delta}}{\partial z^\alpha \partial \bar{z}^\beta} - g^{\mu\nu} \frac{\partial g_{\gamma\delta}}{\partial z^\mu} \frac{\partial g_{\mu\nu}}{\partial \bar{z}^\beta}. \tag{1.13}$$

In [16], the author introduced the following definition: the curvature tensor $R_{\alpha\beta\gamma\delta}$ is said to be strongly negative (resp. strongly seminegative) if $R_{\alpha\beta\gamma\delta}(A^\alpha B^\beta - C^\alpha D^\beta)(A^\delta B^\gamma - C^\delta D^\gamma)$ is positive (resp. nonnegative) for arbitrary complex numbers $A^\alpha, B^\alpha, C^\alpha, D^\alpha$, when $A^\alpha B^\beta - C^\alpha D^\beta \neq 0$ for at least one pair of indices $\alpha, \beta$. Using argument in [16], we obtain the following corollary.

**Corollary 1.3.** Let $(M^{2m+1}, \theta)$ be a closed pseudo-Hermitian CR manifold of dimension at least 5, and $(N^n, g)$ a Kähler manifold. Then

(a) If $N$ has strongly semi-negative curvature, then

$$\int_M \langle Pf, \bar{\partial}_b \bar{f} \rangle =: \int_M g_{\alpha\beta}(P_f f^\alpha) \bar{f}_j \bar{f}_\beta \theta \wedge (d\theta)^m \leq 0 \tag{1.14}$$

for any smooth map $f: M \to N$. The equality holds if and only if $f$ is CR-pluriharmonic.

(b) If $N$ has vanishing pseudohermitian torsion, $N$ has strongly negative curvature, and if $f$ is pseudo-Hermitian harmonic with $df$ having rank at least 4 at a dense set of $M$, then $f$ must be CR-holomorphic or anti CR-holomorphic.

We remark that part (a) in the corollary generalizes a result in [6] about positivity of the operator $P$ for maps; and part (b) gives a CR version of Siu’s rigidity theorem.

### 2. Harmonic map equations

For basic notions in pseudohermitian geometry, we refer the reader to [7, 8] or [17, 9, 12] and [11]. Let $(M^{2m+1}, \theta)$ be a $(2m+1)$-dimensional, strictly pseudoconvex pseudo-Hermitian manifold and let $(N^n, g)$ be a Kähler manifold. Suppose that $f: M \to N$ is a smooth map. We can define the **pointwise $\bar{\partial}_b$-energy** of $f$ as follows. Suppose $p \in M$ and $q = f(p)$. We choose a local coordinate chart $V$ of $N$ near $q$. Near $p$, we choose a local holomorphic frame $\{Z_i\}$ and let $h_{ij}$ be the Levi-form with respect to $\{Z_i\}$. That is,

$$d\theta = \bar{h}_{ij} \theta^i \wedge \bar{\theta}^j, \tag{2.1}$$

where, $\{\theta^i\}$ is a holomorphic coframe dual to $\{Z_i\}$ and $\bar{\theta}^i = \bar{\partial}^i$. Then the $\bar{\partial}_b$-energy density $e(f)(p)$ is

$$e(f)(p) = \bar{g}_{\alpha\beta} f^\alpha_i f^\beta_j h_{ij}, \tag{2.2}$$

where the summation convention is used. The $\bar{\partial}_b$-energy functional of $f$ is

$$E[f] = \int_M e(f)(p) = \int_M e(f) \theta \wedge (d\theta)^m. \tag{2.3}$$

A critical point of the functional $E$ satisfies $\tau(f) = 0$, where

$$\tau^\alpha[f] =: \bar{h}^{ij} \left( f^\alpha_{ij} + \Gamma^{\alpha}_{\beta\gamma} f^\beta_j f^\gamma_i \right), \quad 1 \leq \alpha \leq n. \tag{2.4}$$

Here, $\Gamma^{\alpha}_{\beta\gamma}$, or more precisely, $\Gamma^{\alpha}_{\beta\gamma} \circ f$, is the Christoffel symbols of $N$ evaluated at the point $f(p)$, with respect to coordinates $\{z^\alpha\}$ and $f^\alpha_{ij}$ is the second order covariant derivative of $f^\alpha$ with respect to the Tanaka-Webster connection on $M$. In fact, given $p \in M$, we choose a local chart $V$ of $f(p)$ and consider the family $f_t$ defined as $f_t^\alpha = f^\alpha + t\psi^\alpha$, where $\psi^\alpha$ are smooth
functions with compact supports in a neighborhood of \( p \). Then, by a standard calculation for \( \frac{d}{dt} E(f_t) |_{t=0} = 0 \), one has Euler-Lagrange equation \((2.4)\).

**Definition 2.1.** Let \( (M^{2m+1}, \theta) \) be a \((2m+1)\)-dimensional pseudohermitian manifold and let \((N, g)\) is a Kähler manifold with Kähler metric \( g \). Suppose that \( f: M \to N \) is a smooth map.

(i) We say that \( f \) is pseudohermitian harmonic map if it satisfies \( \tau[f] = 0 \). That is,

\[
 h^{ij} \left( f^\alpha_{ij} + \Gamma^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j \right) e_\alpha = 0
\]

(ii) We say that \( f \) is \( \bar{\partial}_b \)-pluriharmonic map if

\[
 \left( f^\alpha_{ij} + \Gamma^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j \right) \theta^i \wedge \theta^j \otimes e_\alpha = 0
\]

(iii) We say that \( f \) is CR-pluriharmonic if \( m \geq 2 \) and

\[
 f^\alpha_{ij} + \Gamma^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j = \frac{\tau^\alpha[f]}{m} h_{ij}.
\]

**Remark 1.** (a) It is easy to see that if \( f \) is \( \bar{\partial}_b \)-pluriharmonic map and only if \( f \) is both CR-pluriharmonic and pseudohermitian harmonic.

(b) If \( f \) is a CR map, then regardless of the Kähler metric on \( N \) and pseudohermitian structure on \( M \), \( f \) is \( \bar{\partial}_b \)-pluriharmonic. On the other hand, a conjugate (anti) CR map is \( \bar{\partial}_b \)-pluriharmonic if and only if \( df(T) = 0 \), where \( T \) is the Reeb vector field associated to the contact form \( \theta \).

(c) When \( N = \mathbb{C}^n \) and \( m \geq 2 \), from a well-known result by Bedford, Ferderbush \([2, 3]\) and Lee \([8]\), \( f \) is CR-pluriharmonic if and only if for any holomorphic local coordinates \( \{z^\alpha\} \) on \( N \), the real and imaginary parts of \( f^\alpha := z^\alpha \circ f \) are locally real parts of CR functions.

**Example 1.** Let \( N = \mathbb{B}_n \) be the unit ball in \( \mathbb{C}^n \) with Bergman metric given in standard coordinates by

\[
 g_{\alpha\overline{\beta}} = (1 - |z|^2)^{-2} \left( \overline{z^\beta \partial_z} + (1 - |z|^2) \delta_{\alpha\beta} \right).
\]

The corresponding Christoffel symbols are

\[
 \Gamma^\alpha_{\beta\gamma} = (1 - |z|^2)^{-1} (\overline{z^\beta \delta_{\alpha\gamma}} + \overline{z^\gamma \delta_{\alpha\beta}}).
\]

Therefore, a map \( f: M \to \mathbb{B}_n \) is pseudohermitian harmonic if and only if \( (f^\alpha) \) satisfies the following system.

\[
 - \square_b f^\alpha + (1 - |z|^2)^{-1} h^{ij} \sum_\beta \left( f^\alpha_i f^\beta_j + f^\alpha_j f^\beta_i \right) = 0,
\]

where \( \square_b \) is the Kohn-Laplacian. It is well-known that the Bergman metric has strongly negative curvature \([10]\). Corollary \([1, 3]\) implies that any smooth embedding of the sphere \( S^{2m+1} \) \((m \geq 2)\) into \( \mathbb{B}_n \) satisfying \((2.10)\) must be a CR embedding.

The following proposition shows that the CR-pluriharmonicity is CR invariant (i.e. does not depend on the pseudo-Hermitian structures on \( M \)). When \( N = \mathbb{C}^n \), the proposition follows directly from aforementioned result in \([8]\).

**Proposition 2.2.** Let \( (M, \theta) \) and \( (N, g) \) be a pseudohermitian manifold and a Kähler manifold, respectively. If \( f: M \to N \) is CR-pluriharmonic with respect to \( \theta \), then it is CR-pluriharmonic with respect to any \( \bar{\theta} = e^{2\sigma} \theta \).
Proof. Locally, we can choose a local holomorphic frame \( \{Z_i\} \) and its dual admissible coframe \( \{\theta^i\} \) for \( \theta \). As in [7], we choose
\[
\hat{\theta}^k = e^\sigma (\theta^k + 2\sqrt{-1}\sigma^k \theta).
\]
(2.11)
Then \( \{\hat{\theta}^k\} \) is an admissible coframe for \( \hat{\theta} \), with the same matrix \( \hat{h}_{ij} = h_{ij} \), and dual to the holomorphic frame \( \{\hat{Z}_k = e^{-\sigma} Z_k\} \). The Webster connection forms \( \hat{\omega}^i \) is given by [7]
\[
\hat{\omega}^i = \omega^i + 2(\sigma_i \theta^l - \sigma^l \theta_i) + \delta^i_l (\sigma_k \theta^k - \sigma^k \theta_k) + \sqrt{-1}(\sigma_i^l + \sigma^l_i + 4\sigma_i \sigma^l + 4\delta^i_l \sigma_k \sigma^k) \theta.
\]
(2.12)
Therefore,
\[
\hat{\omega}^i(\hat{Z}_j) = e^{-\sigma} (\omega^i(Z_j) - 2\sigma^l h_{ij} - \delta^i_l \sigma_j).
\]
(2.13)
In local frame \( \hat{Z}_k \), we write \( \hat{\nabla}_i = \hat{\nabla}_{Z_i} \), etc.,
\[
\hat{\nabla}_j \hat{\nabla}_i f^\alpha = \hat{Z}_j \hat{Z}_i f^\alpha - \hat{\omega}^i(\hat{Z}_j) \hat{Z}_i f^\alpha
= e^{-2\sigma} (Z_j Z_i f^\alpha) - \sigma_j Z_i f^\alpha
- \omega^i_l (Z_j Z_i f^\alpha + 2\sigma^l h_{ij} Z_i f^\alpha + \delta^i_l \sigma_j Z_i f^\alpha)
= e^{-2\sigma} \nabla_j \nabla_i f^\alpha + 2e^{-2\sigma} \sigma^i_l f_i^\alpha h_{ij}.
\]
(2.14)
Suppose that \( f \) is CR-pluriharmonic, i.e.\( (2.7) \) holds, then we obtain
\[
\hat{\nabla}_j \hat{\nabla}_i f^\alpha + \Gamma^\beta_{\gamma i} (\hat{Z}_j f^\beta)(\hat{Z}_i f^\gamma) = e^{-2\sigma} \left( \frac{\tau^\alpha[f]}{m} + 2\sigma^i_l f_i^\alpha \right) h_{ji},
\]
(2.15)
which implies that \( f \) is CR-pluriharmonic with respect to \( \hat{\theta} \), as desired. \( \square \)

3. PROOF OF THEOREM 1.2

In this section, we give a proof of Theorem 1.2. For convenience, we introduce several notations similar to those in [14]. Let \( \partial_b f \) be the \( f^* T^{0,1} N \)-valued one form represented by \( (f^\alpha_i) \). Then the covariant derivative \( \hat{D} \partial_b f \) in [16] has components
\[
f^\alpha_{ij} =: f^\alpha_{ij} + \Gamma^\alpha_{\beta i} f^\beta_j f^\gamma_i,
\]
(3.1)
where, \( f^\alpha_{ij} \) denotes covariant derivative with respect to Tanaka-Webster connection on \( M \). Thus, \( f \) is \( \partial_b \)-pluriharmonic if and only if \( \hat{D} \partial_b f = 0 \), and \( f \) is pseudohermitian harmonic if and only if
\[
\tau^\alpha[f] =: h^{ji} f^\alpha_{ij} = 0.
\]
(3.2)
The covariant derivative \( \hat{D} \hat{D} \partial_b f \) is formed similarly, namely
\[
f^\alpha_{ijk} = Z_k (f^\alpha_{ij}) - \Gamma^\alpha_{ki} f^\gamma_j f^\gamma_i + \Gamma^\alpha_{ki} f^\gamma_j f^\gamma_i = (f^\alpha_{ij})_k + \Gamma^\alpha_{ki} f^\gamma_j f^\gamma_i.
\]
(3.3)
Here, the covariant derivatives with respect to Tanaka-Webster connection on \( M \) are denoted by indexes preceded by commas. Also,
\[
f^\alpha_{ij[k]} = (f^\alpha_{ij})_k + \Gamma^\alpha_{ki} f^\gamma_j f^\gamma_i \quad \text{and} \quad f^\alpha_{ij[k]} = (f^\alpha_{ij})_k + \Gamma^\alpha_{ki} f^\gamma_j f^\gamma_i.
\]
(3.4)
In what follows, we will denote the curvatures on \( M \) and \( N \) by \( R \) with Latin indices and Greek indices, respectively. Thus, \( R^i_{jkl} \) and \( A_{ij} \) are components of Webster curvature and torsion on \( M \), respectively, while \( R^a_{\beta\gamma\delta} \) are components of curvature on \( N \).
Lemma 3.1. We have the following commutation relations

\begin{align*}
f_{ij} - f_{ji} &= \sqrt{-1}f_0h_{ij}, \quad f_{ij} = f_{ji}; \quad (3.5) \\
f^{0}_{ij} - f^{0}_{ji} &= A_i^k f^0_{ki}; \quad (3.6) \\
f^{0}_{ij|k} - f^{0}_{ji|k} &= \sqrt{-1} \left( h_{ij} A_k^l - h_{ik} A_j^l \right) f^0_l + R^{0}_{\beta\gamma} f^\beta_i (f_j^\gamma f_k^\gamma - f_k^\gamma f_j^\gamma); \quad (3.7) \\
f^{\alpha}_{ij|k} - f^{\alpha}_{ji|k} &= \sqrt{-1} \left( h_{ij} A_k^\alpha - h_{ik} A_j^\alpha \right) f^{\alpha}_l + R^{\alpha}_{\beta\gamma} f^{\beta}_i (f_j^{\gamma} f_k^{\gamma} - f_k^{\gamma} f_j^{\gamma}); \quad (3.8) \\
f^{\alpha}_{ij|k} - f^{\alpha}_{ji|k} &= \sqrt{-1} h_{jk} f^\alpha_{i\bar{j}0} + R^l_{ij} f^{\alpha}_l + R^{\alpha}_{\beta\gamma} f^{\beta}_i (f_j^{\gamma} f_k^{\gamma} - f_k^{\gamma} f_j^{\gamma}); \quad (3.9)
\end{align*}

Proof. The proof use the usual commutation relations for functions on CR manifolds as derived \cite{8} and \cite{12}. For the first relation in (3.5), since \( \Gamma^\alpha_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} \), we deduce from (3.1) that

\[ f^{0}_{ij} - f^{0}_{ji} = f^3 - f^3 = \sqrt{-1} f_0 h_{ij}, \quad (3.10) \]

as desired. The proof of (4.5) is similar. To prove (3.7), we compute, at a point \( p \) under the assumption that the Christoffel symbols of \( M \) at \( p \) and those of \( N \) at \( f(p) \) vanish,

\begin{align*}
f^{\alpha}_{ij|k} &= (f^{\alpha}_{ij})_k \quad \text{(covariant derivative)} \\
&= f^{\alpha}_{ijk} + (\Gamma^{\alpha}_{\gamma\beta} f^{\beta}_i f^{\gamma}_j)_k \\
&= f^{\alpha}_{ijk} + \partial_k \Gamma^{\alpha}_{\beta\gamma} f^{\beta}_i f^{\gamma}_j + R^{\alpha}_{\beta\gamma} f^{\beta}_j f^{\gamma}_k f^{\gamma}_j. \quad (3.11)
\end{align*}

Since the Christoffel symbols vanish at \( f(p) \), one has \( \partial_k \Gamma^{\alpha}_{\beta\gamma} = \partial_k \Gamma^{\alpha}_{\gamma\beta} \) at \( f(p) \), and thus

\[ f^{\alpha}_{ij|k} - f^{\alpha}_{ji|k} = f^{\alpha}_{ijk} - f^{\alpha}_{ikj} + R^{\alpha}_{\beta\gamma} f^{\beta}_i (f_j^{\gamma} f_k^{\gamma} - f_k^{\gamma} f_j^{\gamma}), \quad (3.12) \]

which, by the commutation relations for Tanaka-Webster–covariant derivatives (see \cite{6}), implies (3.7).

Similarly, with \( \Gamma^{\alpha}_{\beta\gamma}(f(p)) = 0 \), one has

\begin{align*}
f^{\alpha}_{ji|k} - f^{\alpha}_{ij|k} &= f^{\alpha}_{ij|k} - f^{\alpha}_{ij|k} \\
&= \sqrt{-1} (h_{ij} A_k^\alpha - h_{ik} A_j^\alpha) f^{\alpha}_l + R^{\alpha}_{\beta\gamma} f^{\beta}_j f^{\gamma}_k f^{\gamma}_j \\
&= \sqrt{-1} (h_{ij} A_k^\alpha - h_{ik} A_j^\alpha) f^{\alpha}_l + R^{\alpha}_{\beta\gamma} f^{\beta}_j f^{\gamma}_k f^{\gamma}_j.
\end{align*}

This proves (3.8). The proof of (3.9) is similar and is omitted. \( \square \)

Proof of Theorem 1.2. We compute covariant derivative, using (3.7), (3.8) and (3.9),

\begin{align*}
D_k (B_{ij} f^{\alpha}) &= f^{\alpha}_{ij|k} - \frac{1}{m} f^{\alpha}_{ijk} h^{lp} \\
&= f^{\alpha}_{ij|k} + \sqrt{-1} f^{\alpha}_{0|k} h_{ij} - \frac{1}{m} f^{\alpha}_{ijk} h^{lp} h_{ij} \\
&= f^{\alpha}_{ij|k} + \sqrt{-1} \left( h_{ij} A_k^l - h_{ik} A_j^l \right) f^{\alpha}_l \\
&\quad + R^{\alpha}_{\beta\gamma} f^{\beta}_j f^{\gamma}_k f^{\gamma}_j + \sqrt{-1} f^{\alpha}_{0|k} h_{ij} - \frac{1}{m} f^{\alpha}_{ijk} h^{lp} h_{ij} \\
&= f^{\alpha}_{ij|k} + \sqrt{-1} \left( h_{ij} A_k^l - h_{ik} A_j^l \right) f^{\alpha}_l \\
&\quad + R^{\alpha}_{\beta\gamma} f^{\beta}_j f^{\gamma}_k f^{\gamma}_j - \frac{1}{m} f^{\alpha}_{ijk} h^{lp} h_{ij}. \quad (3.13)
\end{align*}
Taking the trace over $k$ and $j$ (using the Levi matrix $h^{kj}$),

$$h^{kj} D_k(B_{ij} f^\alpha) = \frac{m-1}{m} P_i f^\alpha + R^\alpha_{\rho\gamma\beta} f^\rho_j (f^\gamma_k f^\beta_i - f^\beta_k f^\gamma_i) h^{kj}$$  \hfill (3.14)

From (3.11), we see that if the target manifold $N$ is flat and $m \geq 2$, then any CR-pluriharmonic map satisfies $P_i f^\alpha = 0$. This is Graham and Lee’s result in [6].

As in [6], we consider the tensor $E$ on $M$ defined by

$$E_j = g_{\alpha\beta} f_{ij}^\alpha (B_{ij} f^\alpha) h^{ji}. \hfill (3.15)$$

Then, the divergent $\delta E$ is

$$E_j^\beta = g_{\alpha\beta} f_{ij}^\alpha (B_{ij} f^\alpha) h^{ji} h^{k\beta} + g_{\alpha\beta} f_{ij}^\alpha (B_{ij} f^\alpha) h^{ji} h^{k\beta}$$

$$= |B_{ij} f^\alpha|^2 + \frac{m-1}{m} \langle P f, \bar{\partial}_b \bar{f} \rangle + R^\alpha_{\rho\gamma\beta} f_{ij}^\rho (f^\gamma_k f^\beta_i - f^\beta_k f^\gamma_i) h^{ji} h^{k\beta}. \hfill (3.16)$$

Here, we used the fact that $B_{ij} f^\alpha$ is trace-free. Taking integral both sides over closed manifold $M$, we obtain

$$- \frac{m-1}{m} \int_M \langle P f, \bar{\partial}_b \bar{f} \rangle = \int_M |B_{ij} f^\alpha|^2 + \int_M R^\alpha_{\rho\gamma\beta} f_{ij}^\rho (f^\gamma_k f^\beta_i - f^\beta_k f^\gamma_i) h^{ji} h^{k\beta}. \hfill (3.17)$$

This proves (1.11). To prove (1.12), we consider

$$F_i = g_{\alpha\beta} f_{ij}^\alpha [f^\beta_i h^{kj}]. \hfill (3.18)$$

The divergent $\delta F$ is

$$F_i^\gamma = g_{\alpha\beta} f_{ij}^\alpha [f^\beta_i h^{kj} h^{ji} + g_{\alpha\beta} f_{ij}^\alpha [f^\beta_i h^{kj} h^{ji}$$

$$= g_{\alpha\beta} \left( P_i f^\alpha - \sqrt{-1} m A^k f^\alpha_k \right) f_i^\nu h^{k\beta} h^{ji} + g_{\alpha\beta} [f^\beta_i h^{kj} h^{ji}. \hfill (3.19)$$

Taking integration over $M$, we obtain the desired equality.

\hfill $\square$

4. Rigidity theorem/Proof of Corollary 1.3

In this section, we prove Corollary 1.3. For the part (a), suppose $N$ has strongly semi-negative curvature, then

$$R^\rho_{\gamma\beta\delta} f_{ij}^\rho (f^\gamma_k f^\beta_i - f^\beta_k f^\gamma_i) h^{ji} h^{k\beta}$$

$$= \frac{1}{2} R^\rho_{\gamma\beta\delta} (f^\gamma_k f^\beta_i - f^\beta_k f^\gamma_i) (f^\gamma_k f^\delta_i - f^\delta_k f^\gamma_i) h^{k\beta} h^{ji} \geq 0. \hfill (4.1)$$

This and (1.11) imply that

$$\int_M \langle P f, \bar{\partial}_b \bar{f} \rangle \leq 0. \hfill (4.2)$$

The equality holds if and only if both terms in the right of (1.11) vanish; in particular,

$$B_{ij} f^\alpha = 0, \hfill (4.3)$$

and therefore, $f$ is CR-pluriharmonic.

Part (b) follows from the following more general theorem.
Theorem 4.1. Let \((M^{2m+1}, \theta)\) be a closed pseudo-Hermitian CR manifold with CR dimension \(m\), \((N, g)\) a Kähler manifold, and \(f: M \rightarrow N\) a pseudo-Hermitian harmonic map. Suppose also that \(M\) has vanishing Webster torsion, \(N\) has strongly negative curvature, and \(df|_H\) has (real) rank at least 4 on a dense set of \(M\), then \(f\) is a CR or anti CR map. Furthermore, if \(f\) is an immersion, then \(f\) must be CR. More generally, if the curvature of \(N\) is negative of order \(k\) and \(df|_H\) has rank at least \(2k\) on a dense set, then the same conclusion holds.

Here, as defined in [16], the curvature tensor \(R_{\alpha\beta\gamma\delta}\) of \(N\) is said to be negative of order \(k\) if it is strongly semi-negative and satisfies the following: If \(A = (A^\alpha_i)\) and \(B = (B^\alpha_j)\) are any two \(m \times k\) matrices with
\[
\text{rank} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = 2k,
\]
and if
\[
\sum_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \xi^{\alpha\beta}_{ij} \xi^{\gamma\delta}_{ij} = 0, \quad \text{where} \quad \xi^{\alpha\beta}_{ij} = A^\alpha_i B^\beta_j - A^\beta_i B^\alpha_j
\]
for all \(1 \leq i, j \leq k\), then either \(A = 0\) or \(B = 0\). As proved in [16] Lemma 2, if the curvature of \(N\) is strongly negative, then it is negative of order 2.

Proof of Theorem 4.1. The proof follows the lines in [16] and is included here for completeness. Let \(U\) be an open connected open subset of \(M\) such that the rank of \(df|_H\) over \(\mathbb{R}\) is at least \(2k\) on \(U\). Fix \(p \in M\), we shall prove that either \(\partial_b f(p) = 0\), or \(\partial_b f(p) = 0\); since \(U\) is dense in \(M\), it suffices to prove in the case \(p \in U\). Let \(K\) be the kernel of \(df|_H: H_p \rightarrow T_p M\). Then \(\dim_k K \leq 2n - 2k\). By Lemma 1 in [16], there exists a basis \(g_1, g_2, \ldots, g_n\) of \(H_p\) over \(\mathbb{C}\) such that for a tube of \(k\) indices \(1 \leq i_1 < \cdots < i_k \leq n\) the intersection of \(K\) with the complex vector subspace of \(H_p\) spanned by \(g_{i_1}, \ldots, g_{i_k}\) is zero. Choose a holomorphic frame \(\{Z_j: 1 \leq j \leq n\}\) in a neighborhood of \(p\) such that \(g_j = Z_j + \bar{Z}_j\) and let \(W\) be the complex vector subspace of \(T^1_p \oplus T^{0,1}_p\), spanned by \(\{Z_j|_p, \bar{Z}_j|_p \mid j = 1, 2, \ldots, k\}\). Then \(df|_W\) has rank at least \(2k\). Let \(A = (f^\alpha_i)\) and \(B = (f^\alpha_j)\) at \(p\). Then
\[
\text{rank} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = 2k \quad \text{(4.4)}
\]
Since \(M\) has vanishing torsion, i.e., \(A^{ij} = 0\), and \(\tau[f] = 0\), from (1.12), we deduce that \(f\) is \(\partial_b\)-pluriharmonic, and
\[
\sum_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \xi^{\alpha\beta}_{ij} \xi^{\gamma\delta}_{ij} = 0 \quad \text{at} \quad q', \quad \text{at} \quad q' \quad \text{(4.5)}
\]
where
\[
\xi^{\alpha\beta}_{ij} = f^\alpha_i(p) f_j^\beta(p) - f^\alpha_j(p) f_i^\beta(p) = A^\alpha_i B^\beta_j - A^\beta_i B^\alpha_j. 
\]
Since \(R_{\alpha\beta\gamma\delta}\) is negative of order \(k\), we deduce that for \(1 \leq i_1 < \cdots < i_k \leq n\), either \(f^\alpha_{ij} = 0\) for all \(\alpha\) and \(j = i_1, i_2, \ldots, i_k\) or \(f^\alpha_{ij} = 0\) for all \(\alpha\) and \(j = i_1, i_2, \ldots, i_k\). Because \(k \geq 2\), it must follows that either \(f^\alpha_{ij} = 0\) for all \(\alpha\) and all \(j\), or \(f^\alpha_{ij} = 0\) for all \(\alpha\) and all \(j\).

To finish the proof, we need the following lemma which generalizes a previous result for sphere in \(\mathbb{C}^{m+1}\) in [10] Theorem 3.1.

Lemma 4.2. Let \(M\) be a strictly pseudoconvex CR manifold of dimension at least 5. Suppose that \(M\) admits a (local) transversal infinitesimal CR automorphism \(X\) at every point. Assume that \(g\) is a twice differentiable function on \(M\) such that for any \(p \in M\), either \(\partial_b g(p) = 0\), or \(\partial_b g(p) = 0\). Then \(g\) is either CR, or anti CR.
Proof. We shall follow the idea in [10, Theorem 3.1]. Let $A_0$ and $B_0$ be the closures of the interiors of the sets $\{\partial_b g = 0\}$ and $\{\partial_b \hat{g} = 0\}$, respectively. It follows that $A_0 \cup B_0 = M$. If either $A_0$ or $B_0$ is empty, then the conclusion of the lemma is clear. Thus suppose that both sets are nonempty. We shall show that $g$ is a constant. By connectedness, it suffices to show that $g$ is locally constant. Therefore, we can assume that there is a frame $\{Z_i, Z_i, T\}$ on $M$ such that $T = X$ is the given infinitesimal CR automorphism, $\{Z_i\}$ is a holomorphic frame, and $Z_i = \overline{Z_i}$.

We have the following identities

$$[Z_j, Z_i] = \sqrt{-1} h_{ij} T + \Gamma^l_{ji} Z_l - \Gamma^l_{ij} Z_l,$$  \hspace{1cm} (4.7)

$$[Z_j, Z_i] = \Gamma^k_{ji} Z_k - \Gamma^k_{ij} Z_k,$$  \hspace{1cm} (4.8)

$$[Z_j, T] = -\Gamma^k_{ij} Z_k,$$  \hspace{1cm} (4.9)

where the $h_{ij}$ and the Christoffel symbols are the Levi-form and the symbols corresponding to the pseudohermitian structure $\theta$ for which $T$ is the associated Reeb vector field [7, p. 418]. Notice that the vanishing of the components of the Webster torsion follows from the assumption that $T$ is an infinitesimal CR automorphism.

Let $K = A_0 \cap B_0$ and $p \in K$. Then for each $j$, $Z_j g$ and $Z_j \hat{g}$ both vanish at $p$. Moreover, by continuity, $Z_k Z_j g = 0$ on $B_0$ and $Z_j Z_k \hat{g} = 0$ on $A_0$. Therefore, from (4.7) we find that $T g = 0$ on $K$.

Let

$$G(p) = \begin{cases} 0, & \text{if } p \in B_0, \\ T g(p), & \text{if } p \in A_0. \end{cases}$$  \hspace{1cm} (4.10)

Then $G$ is continuous on $M$. We claim that $G$ is anti CR on $M$. In fact, on the interior of $A_0$, one computes, using (4.9),

$$Z_j (T g) = T (Z_j g) - \Gamma^k_{ij} Z_k g = 0.$$

Then by continuity, $Z_j (G)$ vanishes on $A_0$ and so on the whole $M$; the claim immediately follows.

By well-known unique continuation for anti CR functions (see, e.g., [11] for a detail proof, notice that $M$ is locally embeddable into $\mathbb{C}^{m+1}$), it follows that $G \equiv 0$ on $M$. Whence, $T g$ vanishes on $A_0$. Using similar argument for $\hat{g}$, we find that $T g = \overline{T \hat{g}}$ also vanishes on $B_0$. Therefore,

$$T g = 0 \quad \text{on } M. \hspace{1cm} (4.11)$$

From (4.7) and (4.11), we find that, for all $i, j$,

$$g_{ij} = g_{ji}. \hspace{1cm} (4.12)$$

Since the left hand side vanishes on $A_0$, while the right hand side vanishes on $B_0$, both must vanish on $M$. Hence $g$ must be a constant.

Applying the lemma for each component $f^\alpha$ (noticing that the vanishing of Webster torsion implies that the Reeb vector field $T$ is a transversal CR automorphism), we conclude that $f$ is either CR or anti-CR, as desired.

Finally, if $f$ is anti CR, then $f^\alpha_j = 0$. Therefore,

$$f^\alpha_0 = \sqrt{-1} \frac{1}{m} (f^\alpha_{ij} - f^\alpha_{ji}) h^i_{\bar{j}} = 0. \hspace{1cm} (4.13)$$

This equation cannot happen when $f$ is an immersion. \qed
References

[1] Barletta, E., Dragomir, S., and Urakawa, H.: *Pseudoharmonic maps from nondegenerate CR manifolds to Riemannian manifolds*, Indiana University Mathematics Journal, 50(2)(2001): 719-746.

[2] Bedford, E.: *(∂∂̅) and the real parts of CR functions*, Indiana Univ. Math. J. 29 (1980), no. 3, 333–340.

[3] Bedford, E., and Federbush, P.: *Pluriharmonic boundary values*, Tôhoku Mathematical Journal, Second Series 26.4 (1974): 505-511.

[4] Dragomir, S., and Kamishima, Y.: *Pseudoharmonic maps and vector fields on CR manifolds*, Journal of the Mathematical Society of Japan 62.1 (2010): 269-303.

[5] Eells, J., and Sampson, J.-H.: *Harmonic mappings of Riemannian manifolds*, American Journal of Mathematics (1964): 109-160.

[6] Graham, C. R., and Lee, J. M.: *Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains*, Duke Math. J 57.3 (1988): 697-720.

[7] Lee, J. M.: *The Fefferman metric and pseudo-Hermitian invariants*, Transactions of the American Mathematical Society 296 (1986): 411-429.

[8] Lee, J. M.: *Pseudo-Einstein Structures on CR Manifolds*, American Journal of Mathematics (1988): 157-178.

[9] Li, S.-Y., and Luk, H.-S.: *An explicit formula for the Webster pseudo Ricci curvature on real hypersurfaces and its application for characterizing balls in C^n*, Communications in Analysis and Geometry 14 (2006), 673-701.

[10] Li, S.-Y., Ni, L.: *On the holomorphicity of proper harmonic maps between unit balls with the Bergman metrics*, Mathematische Annalen 316 (2000): 333-354

[11] Li, S.-Y., Son, D. N, and Wang, X.: *A New Characterization of the CR Sphere and the sharp eigenvalue estimate for the Kohn Laplacian*, Advances in Mathematics, Volume 281 (2015), pp. 1285-1305.

[12] Li, S.-Y., and Wang, X.: *An Obata-type Theorem in CR Geometry*, Journal of Differential Geometry, 95(2013), 483–502.

[13] Petit, R.: *Harmonic maps and strictly pseudoconvex CR manifolds*, Communication in Analysis and Geometry, 10 (2002): 575-610.

[14] Sampson, J. H.: *Harmonic maps in Kähler geometry*, Harmonic Mappings and Minimal Immersions. Springer Berlin Heidelberg, 1985. 193-205.

[15] Sampson, J. H.: *Applications of harmonic maps to Kähler geometry*, Contemporary Mathematics, 49 (1986): 125-134.

[16] Siu, Y.-T.: *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Annals of Mathematics (1980): 73-111.

[17] Webster, S. M.: *Pseudo-Hermitian structures on a real hypersurface*, Journal of Differential Geometry, 13.1 (1978): 25-41.

Department of Mathematics, University of California, Irvine, CA 92697

E-mail address: sli@math.uci.edu

Department of Mathematics, University of California, Irvine, CA 92697

E-mail address: snduong@math.uci.edu