ON THE COBAR CONSTRUCTION OF A BIALGEBRA

TORNIKE KADEISHVILI

Abstract. We show that the cobar construction of a DG-bialgebra is a homotopy G-algebra. This implies that the bar construction of this cobar is a DG-bialgebra as well.

1. Introduction

The cobar construction $\Omega C$ of a DG-coalgebra $(C, d : C \to C, \Delta : C \to C \otimes C)$ is, by definition, a DG-algebra. Suppose now that $C$ is additionally equipped with a multiplication $\mu : C \otimes C \to C$ turning $(C, d, \Delta, \mu)$ into a DG-bialgebra. How this multiplication reflects on the cobar construction $\Omega C$?

It was shown by Adams [1] that in mod 2 situation in this case the multiplication of $\Omega C$ is homotopy commutative: there exists a $\sim_1$ product

$$\sim_1 : \Omega C \otimes \Omega C \to \Omega C$$

which satisfies the standard condition

$$d(a \sim_1 b) = da \sim_1 b + a \sim_1 db + a \cdot b + b \cdot a,$$

(1)

(since we work mod 2 the signes are ignored in whole paper).

In this note we show that this $\sim_1$ gives rise to a sequence of operations

$$E_{1,k} : \Omega C \otimes (\Omega C)^\otimes k \to \Omega C, \ k = 1, 2, 3, ...$$

which form on the cobar construction $\Omega C$ of a DG-bialgebra a structure of homotopy G-algebra (hGa) in the sense of Gerstenhaber and Voronov [8].

There are two remarkable examples of homotopy G-algebras.

The first one is the cochain complex of 1-reduced simplicial set $C^*(X)$. The operations $E_{1,k}$ here are dual to cooperations defined by Baues in [2], and the starting operation $E_{1,1}$ is the classical Steenrod’s $\sim_1$ product.

The second example is the Hochschild cochain complex $C^*(U, U)$ of an associative algebra $U$. The operations $E_{1,k}$ here were defined in [11] with the purpose to describe $A(\infty)$-algebras in terms of Hochschild cochains although the properties of those operations which were used as defining ones for the notion of homotopy G-algebra in [8] did not appear there. These operations where defined also in [9]. Again the starting operation $E_{1,1}$ is the classical Gerstenhaber’s circle product which is sort of $\sim_1$-product in the Hochschild complex.

In this paper we present the third example of homotopy G-algebra: we construct the operations $E_{1,k}$ on the cobar construction $\Omega C$ of a DG-bialgebra $C$, and the starting operation $E_{1,1}$ is again classical, it is Adams’s $\sim_1$-product.

The notion of hGa was introduced in [8] as an additional structure on a DG-algebra $(A, d, \cdot)$ that induces a Gerstenhaber algebra structure on homology. The

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source of the defining identities and the main example was Hochschild cochain complex $C^i(U, U)$. Another point of view is that $hGa$ is a particular case of $B(\infty)$-algebra. This is an additional structure on a DG-algebra $(A, d, \cdot)$ that induces a DG-bialgebra structure on the bar construction $BA$.

We emphasize the third aspect of $hGa$: this is a structure which measures the noncommutativity of $A$. There exists the classical tool which measures the noncommutativity of a DG-algebra $(A, d, \cdot)$, namely the Steenrod’s $\smile_1$ product, satisfying the condition (1). The existence of such $\smile_1$ guarantees the commutativity of $H(A)$, but $\smile_1$ product satisfying just the condition (1) is too poor for most of applications. In many constructions some deeper properties of $\smile_1$ are needed, for example the compatibility with the dot product of $A$ (the Hirsch formula)

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0.$$  

(2)

For a $hGa (A, d, \cdot, \{E_{i,k}\})$ the starting operation $E_{1,1}$ is a kind of $\smile_1$ product: it satisfies the conditions (1) and (2). As for the symmetric expression

$$a \smile_1 (b \cdot c) + b \cdot (a \smile_1 c) + (a \smile_1 b) \cdot c,$$

it is just homotopical to zero and the appropriate homotopy is the operation $E_{1,2}$. The defining conditions of a $hGa$ which satisfy higher operations $E_{1,k}$ can be regarded as generalized Hirsch formulas. So we can say that a $hGa$ is a DG-algebra with a ”good” $\smile_1$ product.

2. Notation and preliminaries

We work over $\mathbb{Z}_2$. For a graded $\mathbb{Z}_2$-module $M$ we denote by $sM$ the suspension of $M$, i.e. $(sM)^i = M^{i-1}$. Respectively $s^{-1}M$ denotes the desuspension of $M$, i.e. $(s^{-1}M)^i = M^{i+1}$.

A differential graded algebra (DG-algebra) is a graded $\mathbb{R}$-module $C = \{C^i\}$, $i \in \mathbb{Z}$, with an associative multiplication $\mu : C^i \otimes C^j \to C^{i+j}$ and a homomorphism (a differential) $d : C^i \to C^{i+1}$ with $d^2 = 0$ and satisfying the Leibniz rule $d(x \cdot y) = dx \cdot y + x \cdot dy$, where $x \cdot y = \mu(x \otimes y)$. We assume that a DG-algebra contains a unit $1 \in C^0$. A non-negatively graded DG-algebra $C$ is connected if $C^0 = \mathbb{Z}_2$. A connected DG-algebra $C$ is $n$-reduced if $C^i = 0, 0 \leq i \leq n$. A DG-algebra is commutative if $\mu = \mu T$, where $T(x \otimes y) = y \otimes x$.

A differential graded coalgebra (DG-coalgebra) is a graded $\mathbb{Z}_2$-module $C = \{C_i\}, i \in \mathbb{Z}$, with a coassociative counimultiplication $\Delta : C \to C \otimes C$ and a homomorphism (a differential) $d : C_i \to C_{i+1}$ with $d^2 = 0$ and satisfying $\Delta d = (d \otimes id + id \otimes d) \Delta$. A DG-coalgebra $C$ is assumed to have a counit $\epsilon : C \to \mathbb{Z}_2$, $(\epsilon \otimes id) \Delta = (id \otimes \epsilon) \Delta = id$. A non-negatively graded dgc $C$ is connected if $C_0 = \mathbb{Z}_2$. A connected DG-coalgebra $C$ is $n$-reduced if $C_i = 0, 0 \leq i \leq n$.

A differential graded bialgebra (DG-bialgebra) $(C, d, \mu, \Delta)$ is a DG-coalgebra $(C, d, \Delta)$ with a morphism of DG-coalgebras $\mu : C \otimes C \to C$ turning $(C, d, \mu)$ into a DG-algebra.

2.1. Cobar and Bar constructions. Let $M$ be a graded $\mathbb{Z}_2$-vector space with $M^{\leq 0} = 0$ and let $T(M)$ be the tensor algebra of $M$, i.e. $T(M) = \oplus_{i=0}^{\infty} M^\otimes i$. $T(M)$ is a free graded algebra: for a graded algebra $A$ and a homomorphism $\alpha : M \to A$ of degree zero there exists its multiplicative extension, a unique morphism of graded algebras $f_\alpha : T(M) \to A$ such that $f_\alpha(a) = \alpha(a)$. The map $f_\alpha$ is given by $f_\alpha(a_1 \otimes ... \otimes a_n) = \alpha(a_1) \cdot ... \cdot \alpha(a_n)$. 
Dually, let $T^c(M)$ be the tensor coalgebra of $M$, i.e. $T^c(M) = \oplus_{i=0}^{\infty} M^\otimes i$, and the comultiplication $\nabla : T^c(M) \to T^c(M) \otimes T^c(M)$ is given by

$$\nabla(a_1 \otimes \ldots \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes \ldots \otimes a_k) \otimes (a_{k+1} \otimes \ldots \otimes a_n).$$

$(T^c(M), \nabla)$ is a cofree graded coalgebra: for a graded coalgebra $C$ and a homomorphism $\beta : C \to M$ of degree zero there exists its comultiplicative extension, a unique morphism of graded coalgebras $g_\beta : C \to T^c(M)$ such that $p_1 g_\beta = \beta$, here $p_1 : T^c(M) \to M$ is the clear projection. The map $g_\beta$ is given by

$$g_\beta(c) = \sum_n \beta(c^{(1)}) \otimes \ldots \otimes \beta(c^{(n)}),$$

where $\Delta^n(c) = c^{(1)} \otimes \ldots \otimes c^{(n)}$ and $\Delta^n : C \to C^\otimes n$ is $n$-th iteration of the diagonal $\Delta : C \to C \otimes C$, i.e. $\Delta^1 = id$, $\Delta^2 = \Delta$, $\Delta^n = (\Delta^{n-1} \otimes id) \Delta$.

Let $(C, d_C, \Delta)$ be a connected DG-coalgebra and $\Delta = id \otimes 1 + 1 \otimes id + \Delta'$. The (reduced) cobar construction $\Omega C$ on $C$ is a DG-algebra whose underlying graded algebra is $T(sC^{>0})$. An element $(sC^1 \otimes \ldots \otimes sC^n) \in (sC)^\otimes n \subset T(sC^{>0})$ is denoted by $[c_1, \ldots, c_n] \in \Omega C$. The differential on $\Omega C$ is the sum $d = d_1 + d_2$ which for a generator $[c] \in \Omega C$ is defined by $d_1[c] = [dC(c)]$ and $d_2[c] = \sum [c', c'']$ for $\Delta'(c) = c' \otimes c''$, and extended as a derivation.

Let $(A, d_A, \mu)$ be a 1-reduced DG-algebra. The (reduced) bar construction $BA$ on $A$ is a DG-coalgebra whose underlying graded coalgebra is $T(s^{-1} A^{>0})$. Again an element $(s^{-1} a_1 \otimes \ldots \otimes s^{-1} a_n) \in (s^{-1} A)^\otimes n \subset T(s^{-1} A^{>0})$ we denote as $[a_1, \ldots, a_n] \in BA$. The differential of $BA$ is the sum $d = d_1 + d_2$ which for an element $[a_1, \ldots, a_n] \in BA$ is defined by

$$d_1[a_1, \ldots, a_n] = \sum_{i=1}^{n} [a_1, \ldots, d_A a_i, \ldots, a_n], d_2[a_1, \ldots, a_n] = \sum_{i=1}^{n-1} [a_1, \ldots, a_i \cdot a_{i+1}, \ldots, a_n].$$

### 2.2. Twisting cochains.

Let $(C, d, \Delta)$ be a dge, $(A, d, \mu)$ a dga. A twisting cochain [5] is a homomorphism $\tau : C \to A$ of degree $+1$ satisfying the Browns’ condition

$$d\tau + \tau d = \tau - \tau,$$  

(3)

where $\tau \sim \tau' = \mu_A(\tau \otimes \tau') \Delta$. We denote by $T(C, A)$ the set of all twisting cochains $\tau : C \to A$.

There are universal twisting cochains $C \to \Omega C$ and $BA \to A$ being clear inclusion and projection respectively. Here are essential consequences of the condition (3):

(i) The multiplicative extension $f_\tau : \Omega C \to A$ is a map of DG-algebras, so there is a bijection $T(C, A) \leftrightarrow Hom_{DG-Alg}(\Omega C, A)$;

(ii) The comultiplicative extension $g_\tau : C \to BA$ is a map of DG-coalgebras, so there is a bijection $T(C, A) \leftrightarrow Hom_{DG-Coalg}(C, BA)$.

### 3. Homotopy G-algebras

#### 3.1. Products in the bar construction.

Let $(A, d, \cdot)$ be a 1-reduced DG-algebra and $BA$ it’s bar construction. We are interested in the structure of a multiplication

$$\mu : BA \otimes BA \to BA,$$

turning $BA$ into a DG-bialgebra, i.e. we require that
(i) $\mu$ is a DG-coalgebra map;
(ii) is associative;
(iii) has the unit element $1_A \in \Lambda \subset BA$.

Because of the cofreeness of the tensor coalgebra $BA = T^c(s^{-1}A)$, a map of graded coalgebras
\[ \mu : BA \otimes BA \to BA \]
is uniquely determined by the projection of degree $+1$
\[ E = \text{pr} \cdot \mu : BA \otimes BA \to BA \to A. \]
Conversely, a homomorphism $E : BA \otimes BA \to A$ of degree $+1$ determines its coextension, a graded coalgebra map $\mu_E : BA \otimes BA \to BA$ given by
\[ \mu_E = \sum_{k=0}^{\infty} (E \otimes \ldots \otimes E) \nabla_{BA \otimes BA}^k, \]
where $\nabla_{BA \otimes BA} : BA \otimes BA \to (BA \otimes BA)^{\otimes k}$ is the k-fold iteration of the standard coproduct of tensor product of coalgebras
\[ \nabla_{BA \otimes BA} = (id \otimes T \otimes id)(\nabla \otimes \nabla) : BA \otimes BA \to (BA \otimes BA)^{\otimes 2}. \]

The map $\mu_E$ is a chain map (i.e. it is a map of DG-coalgebras) if and only if $E$ is a twisting cochain in the sense of E. Brown, i.e. satisfies the condition
\[ dE + Ed_{BA \otimes BA} = E \circ E. \tag{4} \]

Indeed, again because of the cofreeness of the tensor coalgebra $BA = T^c(s^{-1}A)$ the condition $d_{BA} \mu_E = \mu_E d_{BA \otimes BA}$ is satisfied if and only if it is satisfied after the projection on $A$, i.e. if $\text{pr} \cdot d_{BA} \mu_E = \text{pr} \cdot \mu_E d_{BA \otimes BA}$ but this condition is nothing else than the Brown’s condition (4).

The same argument shows that the product $\mu_E$ is associative if and only if $\text{pr} \cdot \mu_E(\mu_E \otimes id) = \text{pr} \cdot \mu_E(id \otimes \mu_E)$, or, having in mind $E = \text{pr} \cdot \mu_E$
\[ E(\mu_E \otimes id) = E(id \otimes \mu_E). \tag{5} \]

A homomorphism $E : BA \otimes BA \to A$ consists of components
\[ \{ \tilde{E}_{p,q} : (s^{-1}A)^{\otimes p} \otimes (s^{-1}A)^{\otimes q} \to A, p, q = 0, 1, 2, \ldots \}, \]
where $\tilde{E}_{pq}$ is the restriction of $E$ on $(s^{-1}A)^{\otimes p} \otimes (s^{-1}A)^{\otimes q}$. Each component $\tilde{E}_{p,q}$ can be regarded as an operation
\[ E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A, p, q = 0, 1, 2, \ldots. \]
The value of $E_{p,q}$ on the element $(a_1 \otimes \ldots \otimes a_p) \otimes (b_1 \otimes \ldots \otimes b_q)$ we denote by $E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q)$.

It is not hard to check that the multiplication $\mu_E$ induced by $E$ (or equivalently by a collection of multioperations $\{E_{p,q}\}$) has the unit $1_A \in \Lambda \subset BA$ if and only if
\[ E_{0,1} = E_{1,0} = id; \ E_{0,k} = E_{k,0} = 0, k > 1. \tag{6} \]

So we can summarize:

**Proposition 1.** The multiplication $\mu_E$ induced by a collection of multioperations $\{E_{p,q}\}$ turns $BA$ into a DG-bialgebra, i.e. satisfies (i-iii), if and only if the conditions (4), (5), and (6) are satisfied.
Let us interpret the condition (4) in terms of the components \( E_{pq} \).

The restriction of (4) on \( A \otimes A \) gives
\[
dE_{1,1}(a; b) + E_{1,1}(da; b) + E_{1,1}(a; db) = a \cdot b + b \cdot a.
\] (7)

This condition coincides with the condition (1), i.e. the operation \( E_{1,1} \) is sort of \( \sim_1 \) product, which measures the noncommutativity of \( A \). Below we denote \( E_{1,1}(a; b) = a \sim_1 b \).

The restriction on \( A^2 \otimes A \) gives
\[
dE_{2,1}(a; b; c) + E_{2,1}(da; b; c) + E_{2,1}(a, db; c) + E_{2,1}(a, b; dc) = (a \cdot b) \sim_1 c + a \cdot (b \sim_1 c) + (a \sim_1 c) \cdot b,
\] this means, that this \( \sim_1 \) satisfies the left Hirsch formula (2) up to homotopy and the appropriate homotopy is the operation \( E_{2,1} \).

The restriction on \( A \otimes A^2 \) gives:
\[
dE_{1,2}(a; b; c) + E_{1,2}(da; b; c) + E_{1,2}(a; db; c) + E_{1,2}(a; b; dc) = a \sim_1 (b \cdot c) + (a \sim_1 b) \cdot c + b \cdot (a \sim_1 c),
\] this means, that this \( \sim_1 \) satisfies the right Hirsch formula (2) up to homotopy and the appropriate homotopy is the operation \( E_{1,2} \).

Generally the restriction of (4) on \( A^m \otimes A^n \) gives:
\[
dE_{m,n}(a_1, ..., a_m; b_1, ..., b_n) + \sum_{i=1}^m E_{m,n}(a_1, ..., da_i, ..., a_m; b_1, ..., b_n) + \sum_{i=1}^n E_{m,n}(a_1, ..., a_m; b_1, ..., db_i, ..., b_n) = a_1 \cdot E_{m-1,n}(a_2, ..., a_m; b_1, ..., b_n) + E_{m-1,n}(a_1, ..., a_m; b_1-1; b_1, ..., b_n) \cdot a_m
\]
\[+b_1 \cdot E_{m,n-1}(a_1, ..., a_m; b_1, ..., b_n) + E_{m,n-1}(a_1, ..., a_m; b_1, ..., b_{n-1}) \cdot b_m + \sum_{i=1}^m E_{m-1,n}(a_1, ..., a_i \cdot a_{i+1}, ..., a_m; b_1, ..., b_n) + \sum_{i=1}^n E_{m,n-1}(a_1, ..., a_m; b_1, ..., b_i \cdot b_{i+1}, ..., b_n)
\]
\[
= \sum_{p=1}^{m-1} \sum_{q=1}^{n-1} E_{p,q}(a_1, ..., a_p; b_1, ..., b_q) \cdot E_{m-p,n-q}(a_{p+1}, ..., a_m; b_{q+1}, ..., b_n).
\] (10)

Now let us interpret the associativity condition (5) in terms of the components \( E_{p,q} \). The restriction of (5) on \( A \otimes A \otimes A \) gives
\[
(a \sim_1 b) \sim_1 c + (b \sim_1 c) \sim_1 a = E_{1,2}(a; b; c) + E_{1,2}(a; c; b) + E_{2,1}(a; b; c) + E_{2,1}(b; a; c).
\] (11)

Generally the restriction of (5) on \( A^k \otimes A^l \otimes A^m \) gives
\[
= \sum_{r=1}^{l+m} \sum_{l_1 + \ldots + l_r = l, m_1 + \ldots + m_r = m} E_{k,r}(a_1, ..., a_k; E_{l_1,m}(b_1, ..., b_{l_1}; c_1, ..., c_{m_1}), ...,
\]
\[E_{l,r,m}(b_1 + \ldots + b_{r-1} + 1, ..., b_1, c_1 + \ldots + c_{m_r-1} + 1, ..., c_m)
\]
\[E_{k,m}(E_{k_1,l_1}(a_1, ..., a_{k_1}; b_1, ..., b_{l_1}), ...,
\]
\[E_{k_2,l_2}(a_{k_1} + \ldots + a_{k_r-1} + 1, ..., a_k; b_{l_1} + \ldots + b_{l_r-1} + 1, ..., b_l); c_1, ..., c_m)
\] (12)

We define a Hirsch algebra as a DG-algebra \( (A,d,\cdot) \) endowed with a sequence of multioperations \( \{E_{p,q}\} \) satisfying (6), (10). This name is inspired by the fact that the defining condition (10) can be regarded as generalizations of classical Hirsch formula (2). This notion was used in [12], [13].

A Hirsch algebra we call associative if in addition the condition (12) is satisfied.

This structure is a particular case of \( B_{\infty} \)-algebra, see below. Moreover turn the notion of homotopy \( G \)-algebra, described below, is a particular case of an associative Hirsch algebra.
3.2. Some particular cases. For a Hirsch algebra \((A, d, \cdot, \{E_{p,q}\})\) the operation \(E_{1,1} \sim 1\) satisfies (1), so this structure can be considered as a tool which measures the noncommutativity of the product \(a \cdot b\) of \(A\). We distinguish various levels of "noncommutativity" of \(A\) according to the form of \(\{E_{p,q}\}\).

**Level 1.** Suppose for the collection \(\{E_{p,q}\}\) all the operations except \(E_{0,1} = id\) and \(E_{1,0} = id\) are trivial. Then it follows from (7) that in this case \(A\) is a *strictly* commutative DG-algebra.

**Level 2.** Suppose all operations except \(E_{0,1} = id\), \(E_{1,0} = id\) and \(E_{1,1}\) are trivial. In this case \(A\) is endowed with a "strict" \(\sim 1\) product \(a \sim 1 b = E_{1,1}(a; b)\): the condition (10) here degenerates to the following 4 conditions

\[
\begin{align*}
    d(a \sim 1 b) &= da \sim 1 b + a \sim 1 db + a \cdot b + b \cdot a, \\
    (a \cdot b) \sim 1 c + a \cdot (b \sim 1 c) + (a \sim 1 c) \cdot b &= 0, \\
    a \sim 1 (b \cdot c) + b \cdot (a \sim 1 c) + (a \sim 1 b) \cdot c &= 0, \\
    (a \sim 1 c) \cdot (b \sim 1 d) &= 0.
\end{align*}
\]

The condition (12) degenerates to the associativity \(\sim 1\)

\[
a \sim 1 (b \sim 1 c) = (a \sim 1 b) \sim 1 c.
\]

As we see in this case we have very strong restrictions on \(\sim 1\)-product. An example of DG-algebra with such strict \(\sim 1\) product is \((H^*(SX, Z_2), d = 0)\) with \(a \sim 1 b = 0\) if \(a \neq b\) and \(a \sim 1 a = Sq^{[a]} - 1 a\), another example is \(C^*(SX, CX)\), where \(SX\) is the suspension and \(CX\) is the cone of a space \(X\) (see [18]).

**Level 3.** Suppose all operations except \(E_{0,1} = id\), \(E_{1,0} = id\) and \(E_{1,k}, k = 1, 2, 3,...\) are trivial. In this case the condition (10) degenerates into two conditions: at \(A \otimes A^\otimes k\)

\[
dE_{1,k}(a; b_1, ..., b_k) + E_{1,k}(da; b_1, ..., b_k) + \sum_i E_{1,k}(a; b_1, ..., db_i, ..., b_k) = \sum_i E_{1,k-1}(a; b_1, ..., b_i, b_{i+1}, ..., b_k),
\]

and at \(A^\otimes 2 \otimes A^\otimes k\)

\[
E_{1,k}(a_1 \cdot a_2; b_1, ..., b_k) = a_1 \cdot E_{1,k}(a_2; b_1, ..., b_k) + E_{1,k}(a_1; b_1, ..., b_k) \cdot a_2 + \sum_{p=1}^{k-1} E_{1,p}(a_1; b_1, ..., b_p) \cdot E_{1,m-p}(a_2; b_{p+1}, ..., b_k); (14)
\]

moreover at \(A^\otimes n > 2 \otimes A^\otimes k\) the condition is trivial. In particular the condition (8) here degenerates to Hirsch formula (2).

The associativity condition (12) in this case looks as

\[
E_{1,n}(E_{1,m}(a_1; b_1, ..., b_n); c_1, ..., c_n) = \sum_{0 \leq m_1 \leq ... \leq m_n \leq n} \sum_{0 \leq n_1 + ... + n_n \leq n} c_{i_1}, E_{1,n_1}(b_1; c_{i_1+1}, ..., c_{i_1+n_1}, c_{i_1+n_1+1}, ..., c_{i_2}, E_{1,n_2}(b_2; c_{i_2+1}, ..., c_{i_2+n_2}, c_{i_2+n_2+1}, ..., c_{i_m}, E_{1,n_m}(b_m; c_{i_m+1}, ..., c_{i_m+n_m}, c_{i_m+n_m+1}, ..., c_n), (15)
\]

In particular the condition (11) here degenerates to

\[
(a \sim 1 b) \sim 1 c + a \sim 1 (b \sim 1 c) = E_{1,2}(a; b, c) + E_{1,2}(a, c; b). (16)
\]

The structure of this level coincides with the notion of *Homotopy G-algebra*, see below.

**Level 4.** As the last level we consider a Hirsch algebra structure with no restrictions. An example of such structure is the cochain complex of a 1-reduced cubical set. Note that it is a *nonassociative* Hirsch algebra.
3.3. $B_{\infty}$-algebra. The notion of $B_{\infty}$-algebra was introduced in [2], [10] as an additional structure on a DG-algebra $(A, \cdot, d)$ which turns the tensor coalgebra $T^c(s^{-1}A) = BA$ into a DG-bialgebra. So it requires a new differential

$$\tilde{d} : BA \to BA$$

(which should be a coderivation with respect to standard coproduct of $BA$) and a new associative multiplication

$$\tilde{\mu} : (BA, \tilde{d}) \otimes (BA, \tilde{d}) \to (BA, \tilde{d})$$

which should be a map of DG-coalgebras, with $1_{\Lambda} \in \Lambda \subset BA$ as the unit element.

It is known that such $\tilde{d}$ specifies on $A$ a structure of $A_{\infty}$-algebra in the sense of Stasheff [19], namely a sequence of operations \{$m_i : \otimes^i A \to A, i = 1, 2, 3, ...$\} subject of appropriate conditions.

As for the new multiplication $\tilde{\mu}$, it follows from the above considerations, that it is induced by a sequence of operations \{$E_{pq}$\} satisfying (6), (12) and the modified condition (10) with involved $A_{\infty}$-algebra structure \{$m_i$\}.

Thus the structure of associative Hirsch algebra is a particular $B_{\infty}$-algebra structure on $A$ when the standard differential of the bar construction $d_B : BA \to BA$ does not change, i.e. $\tilde{d} = d_B$ (in this case the corresponding $A_{\infty}$-algebra structure is degenerate: $m_1 = d_A, m_2 = \cdot, m_3 = 0, m_4 = 0, ...$).

Let us mention, that a twisting cochain $E$ satisfying (6) and (4), (but not (5) i.e. the induced product in the bar construction is not strictly associative), was constructed in [14] for the singular cochain complex of a topological space $C^*(X)$ using acyclic models. The condition (6) determines this twisting cochain $E$ uniquely up to standard equivalence (homotopy) of twisting cochains in the sense of N. Berikashvili [4].

3.4. Strong homotopy commutative algebras. The notion of strong homotopy commutative algebra (shc-algebra), as a tool for measuring of noncommutativity of DG-algebras, was used in many papers: [17], [20], etc.

A shc-algebra is a DG-algebra $(A, d, \cdot)$ with a given twisting cochain $\Phi : B(A \otimes A) \to A$ which satisfies appropriate up to homotopy conditions of associativity and commutativity. Compare with the Hirsch algebra structure which is represented by a twisting cochain $E : BA \otimes BA \to A$. Standard contraction of $B(A \otimes A)$ to $BA \otimes BA$ allows to establish connection between these two notions.

3.5. DG-Lie algebra structure in a Hirsch algebra. A structure of an associative Hirsch algebra on $A$ induces on the homology $H(A)$ a structure of Gerstenhaber algebra (G-algebra) (see [6], [8], [21]) which is defined as a commutative graded algebra $(H, \cdot)$ together with a Lie bracket of degree -1

$$[\cdot, \cdot] : H^p \otimes H^q \to H^{p+q-1}$$

(i.e. a graded Lie algebra structure on the desuspension $s^{-1}H$) that is a biderivation: $[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c]$.

The existence of this structure in the homology $H(A)$ is seen by the following argument.

Let $(A, d, \cdot, \{E_{pq}\})$ be an associative Hirsch algebra, then in the desuspension $s^{-1}A$ there appears a structure of DG-Lie algebra: although the $\sim_1 = E_{1,1}$ is not associative, the condition (11) implies the pre-Jacobi identity

$$a \sim_1 (b \sim_1 c) + (a \sim_1 b) \sim_1 c = a \sim_1 (c \sim_1 b) + (a \sim_1 c) \sim_1 b$$
This condition guarantees that the commutator \([a, b] = a \circ b + b \circ a\) satisfies the Jacobi identity. Besides the condition (7) implies that \([\cdot, \cdot] : A^p \otimes A^q \to A^{p+q-1}\) is a chain map. Thus on \(s^{-1}H(A)\) there appears the structure of graded Lie algebra.

The up to homotopy Hirsh formulae (8) and (9) imply that the induced Lie bracket is a biderivation.

3.6. **Homotopy G-algebra.** An associative Hirsch algebra of level 3 in the literature is known as **Homotopy G-algebra**.

A **Homotopy G-algebra** in [8] and [21] is defined as a DG-algebra \((A, d, \cdot)\) with a given sequence of multibraces \(a\{a_1, \ldots, a_k\}\) which, in our notation, we regard as a sequence of operations

\[ E_{1,k} : A \otimes (\otimes^k A) \to A, \quad k = 0, 1, 2, 3, \ldots \]

which, together with \(E_{01} = \text{id}\) satisfies the conditions (6), (13), (14) and (15).

The name **Homotopy G-algebra** is motivated by the fact that this structure induces on the homology \(H(A)\) the structure of G-algebra (as we have seen in the previous section such a structure appears even on the homology of an associative Hirsch algebra).

The conditions (13), (14), and (15) in [8] are called higher homotopies, distributivity and higher pre-Jacobi identities respectively. As we have seen the first two conditions mean that \(E : BA \otimes BA \to A\) is a twisting cochain, or equivalently \(\mu_E : BA \otimes BA \to BA\) is a chain map, and the third one means that this multiplication is associative.

3.7. **Operadic description.** Appropriate language to describe such huge sets of operations is the operadic language. Here we use surjection operad \(\chi\) and Barratt-Eccles operad \(E\) which are most convenient \(E_\infty\) operads. For definitions we refer to [3].

The operations \(E_{1,k}\) forming hGa have nice description in the surjection operad, see [15], [16], [3]. Namely, to the dot product corresponds the element \((1, 2) \in \chi_0(2)\), to \(E_{1,1} = -1\) product corresponds \((1, 2, 1) \in \chi_1(2)\), to the operation \(E_{1,2}\) the element \((1, 2, 1, 3) \in \chi_2(3)\), etc. Generally to the operation \(E_{1,k}\) corresponds the element

\[ E_{1,k} = (1, 2, 1, 3, \ldots, 1, k, 1, k+1, 1) \in \chi_k(k+1). \]  

We remark here that the defining conditions of a hGa (13), (14), (15) can be expressed in terms of operadic structure (differential, symmetric group action and composition product) and the elements (17) satisfy these conditions already in the operad \(\chi\). This in particular implies that any \(\chi\)-algebra is automatically a hGa. Note that the elements (17) together with \((1, 2)\) generate the suboperad \(F_2\chi\) which is equivalent to the little square operad. This fact and a hGa structure on the Hochschild cochain complex \(C^*(U, U)\) of an algebra \(U\) are used by many authors to prove so called Deligne conjecture about the action of the little square operad on \(C^*(U, U)\).

Now look at the operations \(E_{p,q}\) which define a structure of Hirsch algebra. They can not live in \(\chi\): it is enough to mention that the Hirsch formula (2), as a part of defining conditions of hGa, is satisfied in \(\chi\), but for a Hirsch algebra this condition is satisfied up to homotopy \(E_{2,1}\), see (8). We belive that \(E_{p,q}\)-s live in
the Barratt-Eccles operad $\mathcal{E}$. In particular direct calculation shows that

$$E_{1,1} = ((1, 2), (2, 1)) \in \mathcal{E}_1(2);$$
$$E_{1,2} = ((1, 2, 3), (2, 1, 3), (2, 3, 1)) \in \mathcal{E}_2(3);$$
$$E_{2,1} = ((1, 2, 3), (1, 3, 2), (3, 1, 2)) \in \mathcal{E}_2(3);$$
$$E_{1,3} = ((1, 2, 3, 4), (2, 1, 3, 4), (2, 3, 1, 4), (2, 3, 4, 1)) \in \mathcal{E}_3(4);$$
$$E_{3,1} = ((1, 2, 3, 4), (1, 2, 4, 3), (1, 4, 2, 3), (4, 1, 2, 3)) \in \mathcal{E}_3(4);$$

and in general

$$E_{1,k} = ((1, 2, \ldots, k + 1), \ldots, (2, 3, \ldots, i, 1, i + 1, \ldots, k + 1), \ldots, (2, 3, \ldots, k + 1, 1));$$
$$E_{k,1} = ((1, 2, \ldots, k + 1), \ldots, (1, 2, \ldots, i, k + 1, i + 1, \ldots, k), \ldots, (k + 1, 1, 2, \ldots, k)).$$

As for other $E_{p,q}$ s we can indicate just

$$E_{2,2} = \left( (1, 2, 3, 4), (1, 3, 4, 2), (3, 1, 4, 2), (3, 4, 1, 2) \right) + \left( (1, 2, 3, 4), (1, 2, 3, 4), (1, 3, 4, 2), (3, 4, 1, 2) \right) + \left( (1, 2, 3, 4), (1, 3, 2, 4), (1, 3, 4, 2), (1, 4, 1, 2) \right) + \left( (1, 2, 3, 4), (1, 3, 2, 4), (3, 1, 2, 4), (3, 1, 4, 2) \right).$$

We remark that the operadic table reduction map $TR : \mathcal{E} \to \chi$, see [3], maps $E_{k\geq 1,1}$ and $E_{2,2}$ to zero, and $E_{1,k} \in \mathcal{E}_k(k + 1)$ to $E_{1,k} \in \chi_k(k + 1)$.

4. Adams $\sim_1$-product in the cobar construction of a bialgebra

Here we present the Adams $\sim_1$-product $\sim_1 : \Omega A \otimes \Omega A \to \Omega A$ on the cobar construction $\Omega A$ of a DG-bialgebra $(A, d, \Delta : A \to A \otimes A, \mu : A \otimes A \to A)$ (see [1]). This will be the first step in the construction of hoGa structure on $\Omega A$.

This $\sim_1$ product satisfies the Steenrod condition (1) and the Hirsch formula (2).

First we define the $\sim_1$-product of two elements $x = [a], y = [b] \in \Omega A$ of length 1 as $[a] \sim_1 [b] = [a \cdot b]$. Extending this definition by (2) we obtain

$$[a_1, a_2] \sim_1 [b] = ([a_1] \cdot [a_2]) \sim_1 [b] = [a_1] \cdot ([a_2] \sim_1 [b]) + ([a_1] \sim_1 [b]) \cdot [a_2] = [a_1] \cdot [a_2] \cdot [b] + [a_1] \cdot [a_2] \cdot [b] + [a_1 \cdot b, a_2].$$

Further iteration of this process gives

$$[a_1, \ldots, a_n] \sim_1 [b] = \sum_i [a_1, \ldots, a_{i-1}, a_i \cdot b, a_{i+1}, \ldots, a_n].$$

Now let’s define $[a] \sim_1 [b_1, b_2] = [a^{(1)} \cdot b, a^{(2)} \cdot b]$ where $\Delta a = a^{(1)} \otimes a^{(2)}$ is the value of the diagonal $\Delta : A \to A \otimes A$ on $[a]$. Inspection shows that the condition (1) for short elements

$$d([a] \sim_1 [b]) = d[a] \sim_1 [b] + [a] \sim_1 d[b] + [a] \cdot [b] + [b] \cdot [a].$$

is satisfied.

Generally we define the $\sim_1$ product of an element $x = [a] \in \Omega A$ of length 1 and an element $y = b_1, b_n] \in \Omega A$ of arbitrary length by

$$[a] \sim_1 [b_1, \ldots, b_n] = [a^{(1)} \cdot b_1, \ldots, a^{(n)} \cdot b_n],$$

here $\Delta^n a = a^{(1)} \otimes \ldots \otimes a^{(n)}$ is the n-fold iteration of the diagonal $\Delta : A \to A \otimes A$ and $a \cdot b = \mu(a \otimes b)$ is the product in $A$. 
Extending this definition for the elements of arbitrary lengths \([a_1, ..., a_m] \sim_1 [b_1, ..., b_n]\) by the Hirsch formula (2) we obtain the general formula

\[
[a_1, ..., a_m] \sim_1 [b_1, ..., b_n] = \sum_k [a_1, ..., a_{k-1}, a_k^{(1)} \cdot b_1, ..., a_k^{(n)} \cdot b_n, a_{k+1}, ..., a_m].
\]  

(18)

Of course so defined \(\sim_1\) satisfies the Hirsch formula (2) automatically. It remains to prove the

**Proposition 2.** This \(\sim_1\) satisfies Steenrod condition (1)

\[
d_0([a_1, ..., a_m] \sim_1 [b_1, ..., b_n]) =
\]

\[
d_0[a_1, ..., a_m] \sim_1 [b_1, ..., b_n] + [a_1, ..., a_m] \sim_1 d_0[[b_1, ..., b_n] +
\]

\[
[a_1, ..., a_m, b_1, ..., b_n] + [b_1, ..., b_n, a_1, ..., a_m].
\]

**Proof.** Let us denote this condition by \(\text{Steen}_{m,n}\). The first step consists in direct checking of the conditions \(\text{Steen}_{1,n}\) by induction on \(m\). Furthermore, assume that \(\text{Steen}_{m,n}\) is satisfied. Let us check the condition \(\text{Steen}_{m+1,n}\) for \([a, a_1, ..., a_m] \sim_1 [b_1, ..., b_n]\). We denote \([a_1, ..., a_m] = x, [b_1, ..., b_n] = y\). Using the Hirsch formula (2), \(\text{Steen}_{m,n}\), and \(\text{Steen}_{n+1,n}\) we obtain:

\[
d([a, a_1, ..., a_m] \sim_1 [b_1, ..., b_n]) =
\]

\[
d([a, x \sim_1 y] = d([a] \cdot (x \sim_1 y) + ([a] \cdot y \sim_1 x) =
\]

\[
d([a] \cdot (x \sim_1 y) + [a] \cdot (y \sim_1 x) + (a \cdot y \sim_1 x) + (a \cdot x \sim_1 y) dx =
\]

\[
d([a] \cdot (x \sim_1 y) + [a] \cdot (y \sim_1 x) + (a \cdot y \sim_1 x) + (a \cdot x \sim_1 y) dy + (a \cdot y \sim_1 x) dx + (a \cdot x \sim_1 y) dy =
\]

\[
d([a] \cdot (x \sim_1 y) + [a] \cdot (x \sim_1 y) + [a] \cdot (y \sim_1 x) + (a \cdot y \sim_1 x) + (a \cdot x \sim_1 y) dy + (a \cdot y \sim_1 x) dx + (a \cdot x \sim_1 y) dy + (a \cdot y \sim_1 x) dx +
\]

Besides, using Hirsch (2) formula we obtain

\[
d([a, a_1, ..., a_m] \sim_1 [b_1, ..., b_n]) =
\]

\[
d([a] \cdot (x \sim_1 y) + (a \cdot y \sim_1 x) + (a \cdot y \sim_1 x) + (a \cdot x \sim_1 y) dy + (a \cdot x \sim_1 y) dy =
\]

\[
d([a] \cdot (x \sim_1 y) + (a \cdot y \sim_1 x) + (a \cdot y \sim_1 x) + (a \cdot x \sim_1 y) dx + (a \cdot y \sim_1 x) dx + (a \cdot x \sim_1 y) dx +
\]

and

\[
[a, a_1, ..., a_m] \sim_1 d[b_1, ..., b_n] =
\]

\[
([a] \cdot (x \sim_1 y) + (a \cdot (x \sim_1 y) + (a \cdot x \sim_1 y) \cdot x,
\]

now it is evident that \(\text{Steen}_{m+1,n}\) is satisfied. This completes the proof.

5. **Homotopy G-algebra structure on the cobar construction of a bialgebra**

Below we present a sequence of operations

\[
E_{1,k} : \Omega A \otimes (\Omega A)^{\otimes k} \to \Omega A,
\]

which extends the above described \(E_{1,1} = \sim_1\) to a structure of homotopy G-algebra on the cobar construction of a DG-bialgebra. This means that \(E_{1,k}\)-s satisfy the conditions (13), (14) and (15).

For \(x = [a] \in \Omega A\) of length 1, \(y_i \in \Omega A\) and \(k > 1\) we define \(E_{1,k}([a]; y_1, ..., y_k) = 0\) and extend for an arbitrary \(x = [a_1, ..., a_n]\) by (14). This gives

\[
E_{1,k}([a_1, ..., a_n]; y_1, ..., y_k) = 0
\]

for \(n < k\) and

\[
E_{1,k}([a_1, ..., a_k]; y_1, ..., y_k) = [a_1 \circ y_1, ..., a_k \circ y_k],
\]
here we use the notation \( a \circ (b_1, ..., b_s) = (a^{(1)} \cdot b_1, ..., a^{(s)} \cdot b_s) \), so using this notation 
\[ [a] \sim_1 [b_1, ..., b_s] = [a \circ (b_1, ..., b_s)] \]. Further iteration by (14) gives the general formula

\[
E_{1,k}([a_1, ..., a_n]; y_1, ..., y_k) = \sum [a_1, ..., a_{i-1}, a_i \circ y_1, a_{i+1}, ..., a_{k-1}, a_k \circ y_k, a_{k+1}, ..., a_n],
\]

(19)

where the summation is taken over all \( 1 \leq i_1 < ... < i_k \leq n \).

Of course so defined operations \( E_{1,k} \) automatically satisfy the condition (14). It remains to prove the

**Proposition 3.** The operations \( E_{1,k} \) satisfy the conditions (13) and (15).

**Proof.** The condition (13) is trivial for \( x = [a] \) of length 1 and \( k > 2 \). For \( x = [a] \) and \( k = 2 \) this condition degenerates to

\[ E_{1,1}([a]; y_1 \cdot y_2) + y_1 \cdot E_{1,1}([a]; y_2) - E_{1,1}([a]; y_1) \cdot y_2 = 0 \]

and this equality easily follows from the definition of \( E_{1,1} = \sim_1 \). For a long \( x = [a_1, ..., a_m] \) the condition (13) can be checked by induction on the length \( m \) of \( x \) using the condition (14).

Similarly, the condition (15) is trivial for \( x = [a] \) of length 1 unless the case \( m = n = 1 \) and in this case this condition degenerates to

\[ E_{1,1}(E_{1,1}(x; y); z) = E_{1,1}(x; E_{1,1}(y; z)) + E_{1,2}(x; y, z) + E_{1,2}(x; z, y). \]

This equality easily follows from the definition of \( E_{1,1} = \sim_1 \). For a long \( x = [a_1, ..., a_m] \) the condition (15) can be checked by induction on the length \( m \) of \( x \) using the condition (14).

**Remark 1.** For a DG-coalgebra \( (A, d, \Delta : A \to A \otimes A) \) there is a standard DG-coalgebra map \( g_A : A \to B\Omega A \) from \( A \) to the bar of cobar of \( A \). This map is the coextension of the universal twisting cochain \( \phi_A : A \to \Omega A \) defined by \( \phi(a) = [a] \) and is a weak equivalence, i.e. it induces an isomorphism of homology. Suppose \( A \) is a DG-bialgebra. Then the constructed sequence of operations \( E_{1,k} \) define a multiplication \( \mu_E : B\Omega A \otimes B\Omega A \to B\Omega A \) on the bar construction \( B\Omega A \) so that it becomes a DG-bialgebra. Direct inspection shows that \( g_A : A \to B\Omega A \) is multiplicative, so it is a weak equivalence of DG-bialgebras. Dualizing this statement we obtain a weak equivalence of DG-bialgebras \( \Omega B\Omega A \to A \) which can be considered as a free (as an algebra) resolution of a DG-bialgebra \( A \).

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kade@rmi.acnet.ge

A. Razmadze Mathematical Institute of the Georgian Academy of Sciences
M. Alexidze str. 1, Tbilisi, 380093, Georgia