Conformally flat metrics on 4-manifolds

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Abstract

We prove that for each closed smooth spin 4-manifold $M$ there exists a closed smooth 4-manifold $N$ such that $M\#N$ admits a conformally flat Riemannian metric.

Contents

1 Introduction 1

2 Definitions and notation 2

3 Reflection groups in $S^4$ with prescribed combinatorics of the fundamental domains 4

4 Proof of Theorem 1.1 6

Bibliography 9

1 Introduction

The goal of this note is to prove

**Theorem 1.1.** Let $M^4$ be an closed connected smooth spin 4-manifold. Then there exists a closed orientable 4-manifold $N$ such that $M\#N$ admits a conformally flat Riemannian metric.

Our motivation comes from the following beautiful theorem of C. Taubes $^8$:

**Theorem 1.2.** Let $M$ be a smooth closed oriented 4-manifold. Then there exists a number $k$ so that the connected sum of $M$ with $k$ copies of $\overline{\mathbb{CP}^2}$ admits a half-conformally flat structure.

Here $\overline{\mathbb{CP}^2}$ is the complex-projective plane with the reversed orientation. Recall that a Riemannian metric $g$ on $M$ is *anti self-dual* (or half-conformally flat) if the self-dual part $W_+$ of the Weyl tensor vanishes. Vanishing of both self dual and
anti self-dual parts of the Weyl tensor (i.e., vanishing of the entire Weyl tensor) is equivalent to local conformal flatness of the metric $g$.

Note that the assumption that $M$ is spin is equivalent to vanishing of all Stiefel-Whitney classes, which in turn is equivalent to triviality of the tangent bundle of $M' = M \setminus \{p\}$. According to the Hirsch-Smale theory (see for instance [7]), $M' := M \setminus \{p\}$ is parallelizable iff $M'$ admits an immersion into $\mathbb{R}^4$. Thus, by taking $M$ to be simply-connected with nontrivial 2-nd Stiefel-Whitney class, one sees that $M \# N$ does not admit a flat conformal structure for any $N$: otherwise the developing map would immerse $M'$ into $\mathbb{R}^4$. Therefore the vanishing condition is, to some extent, necessary. Note also that (unlike in Taubes’ theorem) one cannot expect $N$ to be simply-connected since the only closed conformally flat simply-connected Riemannian manifold is the sphere with the standard conformal structure.

Sonjong Hwang in his thesis [4], has proven a 3-dimensional version of Theorem 1.1; moreover, he proves that one can use a connected sum of Haken manifolds as the manifold $N$. Similar arguments can be used to prove an analogous theorem in the context of locally spherical CR structures on 3-manifolds.

The arguments in both 3-dimensional and 4-dimensional cases, in spirit (although, not in the technique), are parallel to Taubes’: one starts with a singular conformally-flat metric on $M$, where the singularity is localized in a ball $B \subset M$. The singular metric is obtained by pull-back of the standard metric on the 4-sphere under a branched covering $M \to S^4$. Then, by attaching another manifold to $M$ along the singular locus (which accounts for taking the connected sum $M \# N$), one “resolves the singularity” and constructs the desired flat conformal structure.

**Conjecture 1.3.** Let $M^n$ be a closed connected smooth $n$-manifold, whose Stiefel-Whitney classes all vanish. Then there exists a smooth closed orientable $n$-manifold $N$ such that $M \# N$ admits a conformally flat Riemannian metric.

It seems very likely that this conjecture can be proven by methods analogous to the ones used in the present paper. The main technical problem is that the reflection groups used in the present paper do not exists in the higher dimensions. However, it is plausible that one can use instead arithmetic groups containing large number of reflections, as it is done in [2] in a different context.

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## 2 Definitions and notation

We let $\text{Mob}(S^4)$ denote the full group of Moebius transformations of $S^4$, i.e. the group generated by inversions in round spheres. Equivalently, $\text{Mob}(S^4)$ is the restriction of the full group of isometries $\text{Isom}(\mathbb{H}^5)$ to the 4-sphere $S^4$ which is the ideal boundary of $\mathbb{H}^5$. We will regard $S^4$ as 1-point compactification $\mathbb{R}^4 \cup \{\infty\}$ of then Euclidean 4-space.

**Definition 2.1.** Let $Q$ be a unit cube in $\mathbb{R}^4$. We define the PL inversion $J$ in the boundary of $Q$ as follows. Let $h : S^4 \to S^4$ be a PL homeomorphism which sends
Definition 2.2. A Moebius or a flat conformal structure on a smooth 4-manifold $M$ is an atlas $\{(V_\alpha, \varphi_\alpha), \alpha \in A\}$ which consist of diffeomorphisms $\varphi_\alpha: V_\alpha \to U_\alpha \subset S^4$ so that the transition mappings $\varphi_\alpha \circ \varphi_\beta^{-1}$ are restrictions of Moebius transformations.

Equivalently, one can describe Moebius structures on $M$ are conformal classes of conformally-Euclidean Riemannian metrics on $M$. Each conformal structure on $M$ gives rise to a local conformal diffeomorphism, called a developing map, $d: M \to S^4$, where $\tilde{M}$ is the universal cover of $M$. If $M$ is connected, the mapping $d$ is equivariant with respect to a holonomy representation $\rho: \pi_1(M) \to \text{Mob}(S^4)$, where $\pi_1(M)$ acts on $\tilde{M}$ as the group of deck-transformations. Given a pair $(d, \rho)$, where $\rho$ is a representation of $\pi_1(M)$ into Mob($S^4$) and $d$ is a $\rho$-equivariant local diffeomorphism from $\tilde{M}$ to $S^4$, one constructs the corresponding Moebius structure on $M$ by taking a pull-back of the standard flat conformal structure on $S^4$ to $\tilde{M}$ via $d$ and then projecting the structure to $M$.

Analogously, one defines a complex-projective structures on complex 3-manifold $Z$: it is a $\mathbb{CP}^3$-valued holomorphic atlas on $Z$ so that the transition mappings belong to $PGL(3, \mathbb{C})$.

The concept of Moebius structure generalizes naturally to the category of orbifolds:
A 4-dimensional Moebius orbifold $O$ is a pair $(X, \mathcal{A})$, where $X$ is a Hausdorff topological space, the underlying space of the orbifold, $\mathcal{A}$ is a family of local parameterizations $\psi_\alpha: U_\alpha \to U_\alpha/\Gamma_\alpha = V_\alpha$, where $\{V_\alpha, \alpha \in A\}$ is an open covering of $X$, $U_\alpha$ are open subsets in $S^4$, $\Gamma_\alpha$ are finite groups of Moebius automorphisms of $U_\alpha$ and the mappings $\psi_\alpha$ satisfy the usual compatibility conditions:

If $V_\alpha \to V_\beta$ is the inclusion map then we have a Moebius embedding $U_\alpha \to U_\beta$ which is equivariant with respect to a monomorphism $\Gamma_\alpha \to \Gamma_\beta$, so that the diagram

$$
\begin{array}{ccc}
U_\alpha & \to & U_\beta \\
\downarrow & & \downarrow \\
V_\alpha & \to & V_\beta
\end{array}
$$

is commutative. The groups $\Gamma_\alpha$ are the local fundamental groups of the orbifold $O$.

For a Moebius orbifold one defines a developing mapping $d: \tilde{O} \to S^4$ (which is a local homeomorphism from the universal cover $\tilde{O}$ of $O$) and, if $O$ is connected, a holonomy homeomorphism $\rho: \pi_1(O) \to \text{Mob}(S^4)$, which satisfy the same equivariance condition as in the manifold case. Again, given a pair $(d, \rho)$, where $d$ is a $\rho$-equivariant homeomorphism, one defines the corresponding Moebius structure via pull-back.

Example 2.3. Let $G \subset \text{Mob}(S^4)$ be a subgroup acting properly discontinuously on an open subset $\Omega \subset S^4$. Then quotient space $\Omega/G$ has a natural Moebius orbifold structure. The local charts $\phi_\alpha$ appear in this case as restrictions of the projection $p: \Omega \to \Omega/G$ to open subsets with finite stabilizers.

In particular, suppose that $G$ is a finite subgroup of Mob($S^4$) generated by reflections, the quotient $Q := \Omega/G$ can be identified with the intersection of a fundamental
domain of $G$ with $\Omega$. The Moebius structures on 4-dimensional manifolds and orbifolds constructed in this paper definitely do not arise this way. A more interesting example is obtained by taking a manifold $M$ and a local homeomorphism $h : M \to S^4$, so that $Q \subset h(M)$. Then we can pull-back the Moebius orbifold structure on $Q$ to an appropriate subset $X$ of $M$, to get a 4-dimensional Moebius orbifold. As another example of a pull-back construction, let $O$ be a Moebius orbifold and $M \to O$ be an orbifold cover such that $M$ is a manifold. Then one can pull-back the Moebius orbifold structure from $O$ to a usual Moebius structure on $M$.

3 Reflection groups in $S^4$ with prescribed combinatorics of the fundamental domains

![Figure 1: Barycentric subdivision of a square.](image)

Consider the standard cubulation $Q$ of $\mathbb{R}^4$ by the unit Euclidean cubes and let $X$ denote the 2-skeleton of this cubulation. Given a collection of round balls $\{B_i, i \in I\}$ in $\mathbb{R}^4$, with the nerve $\mathcal{N}$, we define the canonical simplicial mapping $f : \mathcal{N} \to \mathbb{R}^4$ by sending each vertex of $\mathcal{N}$ to the center of the corresponding ball and extending $f$ linearly to the simplices of $\mathcal{N}$. For a subcomplex $K \subset Q$ define its barycentric subdivision $\beta(K)$ to be the following simplicial complex. Subdivide each edge of $K$ by its midpoint. Then inductively subdivide each $k$-cube $Q$ in $K$ by coning off the barycentric subdivision of $\partial Q$ from the center of $Q$.

**Proposition 3.1.** Suppose that $K \subset X$ is a 2-dimensional compact subcomplex such that each vertex belongs to a 2-cell. Then there exists a collection of open round 4-balls $B_i$, $i = 1, ..., k$, centered at the vertices of $\beta(K)$, so that:

1. The Moebius inversions $R_i$ in the round spheres $S_i = \partial B_i$ generate a discrete reflection group $G \subset \text{Mob}(S^4)$. 


Figure 2:

(2) The complement $S^4 \setminus \bigcup_{i=1}^k B_i$ is a fundamental domain $\Phi$ of $G$.

(3) The canonical mapping from the nerve of \{\(B_i, i = 1, ..., k\)\} to $\mathbb{R}^4$ is a simplicial isomorphism onto $\beta(K)$.

Proof: We begin by constructing the family of spheres $S_i, i \in \mathbb{N}$ centered at certain points of $X$. For each square $Q$ in $X$ we pick 9 points $x_1, ..., x_9$: $x_5, ..., x_8$ are the vertices of $Q$, $x_1, ..., x_4$ are midpoints of the edges of $Q$ and $x_9$ is the center of $Q$. We then take spheres $S(x_i, r)$ of radius $r = \frac{1}{\sqrt{6}}$ centered at the points $x_4, ..., x_9$ and the spheres $S(x_i, \rho)$ of radius $\rho = \frac{1}{\sqrt{12}}$ centered at the points $x_1, ..., x_4$. See Figure 2. The reader will verify that:

1. The spheres $S(x_i, r)$ and $S(x_j, r)$ are disjoint provided that $i \neq j \in \{1, ..., 4\}$ and $i \neq j \in \{5, ..., 8\}$.

2. The spheres $S(x_i, r)$ and $S(x_9, r)$ intersect at the right angle, $i = 1, ..., 4$; the spheres $S(x_i, r)$ and $S(x_9, r)$ intersect at the (exterior) angle $\frac{\pi}{3}$, $i = 5, ..., 8$.

3. The spheres $S(x_i, r)$ and $S(x_j, r)$ intersect at the (exterior) angle $\frac{\pi}{3}$, provided that $1 \leq i \leq 4$, $1 \leq j \leq 8$ and $j = 5 + i, 6 + i$, and are disjoint otherwise.

Suppose now that $Q^4$ is a unit 4-cube, apply the above construction to each 2-face of $Q^4$. The reader will verify that the properties 1–3 of the spheres $S_i$, ensure that the covering $\{B_i\}$ of $(Q^4)^{(2)}$ has the nerve $\mathcal{N}_{Q^4}$ such that the canonical mapping $\mathcal{N}_{Q^4} \to \beta((Q^4)^{(2)})$ is a simplicial isomorphism.

Now we are ready to construct the covering $\{B_i : i = 1, ..., k\}$ of the 2-complex $K$. For each 2-face $Q$ of $K$ introduce the family of nine round spheres $S_i$ constructed above, consider the inversions $R_i$ is these spheres; the spheres $S_i$ bound balls $\{B_i : i = 1, ..., k\}$. The fact that for each 4-cube $Q^4$ the mapping $\mathcal{N}_{Q^4} \to \beta((Q^4)^{(2)})$
is a simplicial isomorphism, implies that the mapping from the nerve of the covering \( \{ B_i : i = 1, \ldots, k \} \) to \( K \) is a simplicial isomorphism as well. Thus the exterior angles of intersections between the spheres equal \( \frac{\pi}{2} \) and \( \frac{3\pi}{2} \), thus we can apply Poincare’s fundamental polyhedron theorem \([6]\) to ensure that the intersection of the complements to the balls \( B_i \) is a fundamental domain for the Moebius group \( G \) generated by the above reflections.

\[ \square \]

**Remark 3.2.** Instead of collections of round balls based on a cubulation of \( \mathbb{R}^4 \) one can use a periodic triangulation of \( \mathbb{R}^4 \), however in this case the construction of a collection of balls covering the 2-skeleton of a 4-simplex is slightly more complicated.

### 4 Proof of Theorem 1.1

Recall that vanishing of all Stiefel-Whitney classes of the manifold \( M \) implies that the manifold \( M' = M \setminus \{ p \} \) is parallelizable; hence, by \([7]\), there exists an immersion \( f : M' \to \mathbb{R}^4 \). Let \( B \) denote a small open round ball centered at \( p \) and let \( M'' \) denote the complement \( M \setminus B \). We retain the notation \( f \) for the restriction \( f|_{M''} \).

We next convert to the piecewise-linear setting, since in dimension 4 the categories of PL and smooth structures are the same it does not limit our discussion: cubulate the manifold \( M \) so that \( \partial B \) is a subcomplex of the cubulation and that the restriction of \( f \) to each 4-cube \( Q' \subset M'' \) is a diffeomorphism onto a cube in the standard (unit cube) cubulation of \( \mathbb{R}^4 \). Without loss of generality we may assume that the mapping \( f \) preserves the orientation.

We now borrow the standard arguments from the proof of Alexander’s theorem which states that each closed \( n \)-dimensional PL manifold is a branched cover over the \( n \)-sphere, see e.g. \([3]\). Extend the map \( f \) to a map \( F \) on the ball \( B \) so that the restriction of \( F \) to each 4-cube is a diffeomorphism onto a cube in the standard cubulation of \( \mathbb{R}^4 \). Now, for each cube \( Q' \subset M \) such that \( F|Q' \) is orientation-reversing we replace \( F|Q' \) with the composition \( J \circ F|Q' \), where \( J \) is the PL inversion in the boundary of the unit cube \( F(\partial Q') \) (see Definition 2.1). The resulting mapping \( h : M \to S^4 \) has the property that it is a local PL homeomorphism away from a 2-dimensional subcomplex \( L \subset B \). (Note that \( L \) has dimension 2 near every point: each vertex in \( L \) belongs to a 2-cube.) Thus the mapping \( h \) is a branched covering over \( S^4 \) with the singular locus \( L \) contained in the ball \( B \), the branch-locus of \( h \) is the compact subcomplex \( K = h(L) \subset \mathbb{R}^4 \). The branched covering \( h \) has the property that for each point \( x \in K \) there exists a neighborhood \( U(x) \subset \mathbb{R}^4 \) such that \( h^{-1}(U(x)) \) is a disjoint union of balls \( V(y) \), \( y \in h^{-1}(x) \subset B \), (whose interiors contain \( y \)), so that for each \( y \in h^{-1}(x) \), the restriction \( h|V(y) \) is a branched covering onto \( U(x) \). Moreover, each branched covering \( h|V(y) \) is obtained by coning off a branched covering from the 3-sphere \( \partial V(y) \) to the 3-sphere \( U(x) \).

Let \( T \) denote a regular neighborhood of \( K \) in \( \mathbb{R}^4 \), so that \( U(x) \subset T \) for each \( x \in K \). Next, subdivide the cubulation of \( \mathbb{R}^4 \) and scale the subdivision up to the standard unit cubulation, so that the discrete group \( G \) and the collection of balls \( \{ B_j, j = 1, \ldots, k \} \) associated with the subcomplex \( K \) in section 3 have the properties:

1. \( T \subset \bigcup_{i=1}^{k} B_k \).
2. Each ball $B_j$, $j = 1, ..., k$, (centered at $x_j \in K$) is contained in the neighborhood $U(x_j)$.

We now use the branched covering $h$ to introduce a Moebius orbifold structure $O$ on the complement $X_O$ to an open tubular neighborhood $N^0(L)$ of $L$ in $M$ as follows:

For each ball $B_j \subset U(x_j)$ centered at $x_j \in K$ and for each $y_j \in h^{-1}(x_j) \cap L$, such that the restriction $h|V(y_j)$ is not a homeomorphism onto its image, we let $\tilde{B}(y_j)$ denote the inverse image $h^{-1}(B_j) \cap V(y_j)$. It follows that each $\tilde{B}(y_j)$ is a polyhedral 4-ball in $M$ and the union of these balls is a tubular neighborhood $N(L)$ of $L$. The boundary of $N(L)$ has a natural partition into subcomplexes: “vertices”, “edges”, “2-faces” and “3-faces”:

- The “vertices” are the points of triple intersections of the 3-spheres $\partial \tilde{B}(y_j)$, $\partial \tilde{B}(y_i)$, $\partial \tilde{B}(y_h)$.
- The “2-faces” are the connected components of the double intersections of the 3-spheres $\partial \tilde{B}(y_j)$, $\partial \tilde{B}(y_i)$.
- The “3-faces” are the connected components of the complements $\partial \tilde{B}(y_j) \setminus \bigcup_{i \neq j} \tilde{B}(y_i)$.

We declare each “3-face” a boundary reflector of the orbifold $O$. The dihedral angles between the balls $B_j$ define the dihedral angles between the boundary reflectors in $O$. Since the restriction $h|M \setminus L$ is a local homeomorphism, this construction defines a Moebius orbifold $O$. The mapping $h|X_O$ is the projection of the developing mapping $\tilde{h} : \tilde{O} \rightarrow S^4$ of this Moebius orbifold. Let $O'$ denote the orbifold with boundary $B \cap O$; let $Q$ be the closed orbifold obtained by attaching 4-disk $D^4$ along the boundary sphere $S^3 = \partial B$.

We now convert back to the smooth category. It is clear from the construction that the orbifold $O$ is obtained by (smooth) gluing of the manifold with boundary $M \setminus B$ and the orbifold with boundary $O'$. Hence $O$ is diffeomorphic to the connected sum of the manifold $M$ with the orbifold $Q$. We also note that all local fundamental groups of $O$ embed naturally into $\pi_1(O)$.

It remains to construct a finite manifold covering $\tilde{M}$ over the orbifold $O$, so that $M \setminus B$ lifts homeomorphically to $\tilde{M}$; the construction is analogous to the one used by M. Davis in [1]. The fundamental group $\pi_1(O)$ is the free product $\pi_1(M) \ast \pi_1(Q)$. We have holonomy homomorphism

$$\phi : \pi_1(O) \rightarrow G,$$

the subgroup $\pi_1(M)$ is contained in the kernel of this homomorphism; by construction, the kernel of $\phi$ contains no elements of finite order. The Coxeter group $G$ is virtually torsion-free, let $\theta : G \rightarrow A$ be a homomorphism onto a finite group $A$, so that $\text{Ker}(\theta)$ is torsion-free and orientation-preserving. Then the kernel of the homomorphism $\psi = \theta \circ \phi : \pi_1(O) \rightarrow A$ is a torsion-free finite index subgroup of $\pi_1(O)$, which contains $\pi_1(M)$. Let $\tilde{M} \rightarrow O$ denote the finite orbifold cover corresponding to the subgroup $\text{Ker}(\psi)$. Then $\tilde{M}$ is a smooth oriented conformally flat manifold, the submanifold $M \setminus B$ lifts
diffeomorphically into $M \setminus B \subset \hat{M}$. Thus the connected sum decomposition $O = M \# Q$ also lifts to $\hat{M}$, so that the latter manifold is diffeomorphic to the connected sum of $M$ and a 4-manifold $N$.

We observe that the proof of Theorem 1.1 can be modified to prove the following:

**Theorem 4.1.** Suppose that $M$ is a closed smooth 4-manifold with vanishing second Stiefel-Whitney class. Then there exists a closed smooth 4-manifold $N$ so that $\hat{M} = M \# N$ admits a conformally-Euclidean Riemannian metric.

**Proof:** The difference with Theorem 1.1 is that $M$ can be nonorientable. Let $\hat{M} \rightarrow M$ be the orientable double cover with the deck-transformation group $D \cong \mathbb{Z}/2$. Then all Stiefel-Whitney classes of $\hat{M}$ are trivial. As before, let $p \in M$, $\{p_1, p_2\}$ be the preimage of $\{p\}$ in $M$. Consider a Euclidean reflection $\tau$ in $\mathbb{R}^4$ and an epimorphism $\theta : D \rightarrow \langle \tau \rangle$. Then, arguing as in the proof of Phillips’ theorem, one gets a $\theta$-equivariant immersion $f : M \setminus \{p_1, p_2\} \rightarrow \mathbb{R}^4$. This yields a $D$-invariant flat conformal structure on $M \setminus \{p_1, p_2\}$ via pull-back of the flat conformal structure from $\mathbb{R}^4$. Let $B_1 \cup B_2$ be a $D$-invariant disjoint union of open balls around the points $p_1, p_2$. Then the rest of the proof of Theorem 1.1 goes through: replace the ball $B_1$ with a manifold with boundary $N_1$ so that the flat conformal structure on $M \setminus (B_1 \cup B_2)$ extends over $N_1$. Then glue a copy of $N_1$ along the boundary of $B_2$ in $D$-invariant fashion. Note that the quotient of the manifold $P := (\hat{M} \setminus (B_1 \cup B_2)) \cup (N_1 \cup N_2)$ by the group $D$ is diffeomorphic to a closed manifold $M \# N$, where $N$ is obtained from $N_1$ by attaching the 4-ball along the boundary. Finally, project the $D$-invariant Moebius structure on $P$ to a Moebius structure on the manifold $M \# N$.

As a corollary of Theorem 1.1 we get:

**Corollary 4.2.** Let $\Gamma$ be a finitely-presented group. Then there exists a 3-dimensional complex manifold $Z$ which admits a complex-projective structure, so that the fundamental group of $Z$ splits as $\Gamma \ast \Gamma'$.

**Proof:** Our argument is similar to the one used to construct (via Taubes’ theorem) 3-dimensional complex manifolds with the prescribed finitely-presented fundamental group, see [3]. We first construct a smooth closed oriented 4-dimensional spin manifold $\hat{M}$ with the fundamental group $\Gamma$. This can be done for instance as follows. Let $\langle x_1, ..., x_n | R_1, ..., R_t \rangle$ be a presentation of $\Gamma$. Consider a 4-manifold $X$ which is the connected sum of $n$ copies of $S^3 \times S^1$. This manifold is clearly spin. Pick a collection of disjoint embedded smooth loops $\gamma_1, ..., \gamma_t$ in $X$, which represent the conjugacy classes of the words $R_1, ..., R_t$ in the free group $\pi_1(X)$. Consider the pair $(S^4, \gamma)$, where $\gamma$ is an embedded smooth loop in $S^4$. For each $i$ pick a diffeomorphism $f_i$ between a tubular neighborhood $T(\gamma_i)$ of $\gamma_i$ in $S^4$ and a tubular neighborhood $T(\gamma_i)$ of $\gamma_i$ in $X$. We can choose $f_i$ so that it matches the spin structures of $T(\gamma_i)$ and $T(\gamma_i)$. Now, attach $n$ copies of $S^4 \setminus T(\gamma)$ to $X \setminus \cup_i T(\gamma_i)$ via the diffeomorphisms $f_i$. The result is a smooth spin 4-manifold $M$ with the fundamental group $\Gamma$.

Next, by Theorem 1.1 there exists a smooth 4-manifold $N$ (with the fundamental group $\Gamma'$) such that $M = M \# N$ admits a conformally-Euclidean Riemannian metric. Applying the twistor construction to the manifold $\hat{M}$ we get a complex 3-manifold $Z$ which is an $S^2$-bundle over $\hat{M}$ and the flat conformal structure on $\hat{M}$ lifts to a complex-projective structure on $Z$, see for instance [3]. Clearly, $\pi_1(Z) \cong \pi_1(\hat{M}) = \Gamma \ast \Gamma'$. 

\[
\pi_1(Z) \cong \pi_1(\hat{M}) = \Gamma \ast \Gamma'.
\]
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