WEIGHTED GJMS OPERATORS ON SMOOTH METRIC MEASURE SPACES

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Abstract. We construct weighted GJMS operators on smooth metric measure spaces, and prove that they are formally self-adjoint. We also provide factorization formulas for them in the case of quasi-Einstein spaces and under Gover–Leitner conditions.

1. Introduction

Graham–Jenne–Mason–Sparling operators, commonly abbreviated as GJMS operators, are an important class of formally self-adjoint [17] conformally covariant operators [16]. In this paper, we construct a weighted analogue of these operators for smooth metric measure spaces.

A smooth metric measure space is a five-tuple \((M^d, g, f, m, \mu)\), where \(M^d\) is a Riemannian manifold of dimension \(d\), \(f\) a smooth function defined on \(M\), \(m \in \mathbb{R}\) a dimensional parameter, and \(\mu \in \mathbb{R}_+\) an auxiliary curvature parameter [3, 4]. If \(m \in \mathbb{N}\), a smooth metric measure space may be thought of as the warped product \((M^d \times F^m(\mu), g \oplus f^2 h)\), where \((F^m(\mu), h)\) is the \(m\)-dimensional simply connected spaceform of constant curvature \(\mu\) [4].

The space of conformal densities of weight \(w \in \mathbb{R}\) is denoted by \(\mathcal{E}[w]\). Also, a pointwise conformal transformation of \((M^d, g, f, m, \mu)\) with respect to a smooth function \(\sigma \in C^\infty(M)\) is the map

\[ (M^d, g, f, m, \mu) \mapsto (M^d, e^{2\sigma} g, e^{\sigma} f, m, \mu). \]

Weighted GJMS operators are known in orders two and four [2], and were formally defined by Case and Chang [6] to study fractional GJMS operators [17] via a curved analogue of the Caffarelli–Silvestre extension [1, 5, 8, 9]. In this paper, we give a rigorous definition of the weighted GJMS operators, and develop some of their properties.

The ambient metric is a key tool in defining weighted GJMS operators, and a weighted analogue of the ambient metric has recently been defined by Case and the author [7]. By adapting the arguments in [16], we construct the weighted GJMS operators.

Theorem 1.1. If \(d + m \notin 2\mathbb{N}\), then for each positive integer \(k\) there is a conformally invariant operator

\[ L_{2k,\phi}^m : \mathcal{E}[-\frac{1}{2}(d + m) + k] \to \mathcal{E}[-\frac{1}{2}(d + m) - k], \]

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with leading term \((\Delta \phi)^k\). If \(d + m \in 2\mathbb{N}\), the same result holds with the restriction \(1 \leq k \leq \frac{1}{2}(d + m)\).

Graham–Zworski proved the formal self-adjointness of GJMS operators [17]. Other proofs for formal self-adjointness are also known [10, 12, 18]. In this article, we prove that weighted GJMS operators are formally self-adjoint.

**Theorem 1.2.** \(L_{2k,\phi}^m\) is a formally self-adjoint operator.

GJMS operators factor nicely for Einstein metrics [11, 13]. Moreover, Case and Chang [6] have also formally proved a factorization of these operators for products of negatively-curved Einstein manifolds with positively-curved Einstein manifolds, using the explicit ambient metric for such spaces found by Gover and Leitner [15]. In this paper, we provide factorization formulas of GJMS operators for quasi-Einstein spaces [4] and under Gover–Leitner conditions [15]. Before we state our result, we provide the relevant definitions and formulas.

Let \((M^d, g, f, m, \mu)\) be a smooth metric measure space. When \(m > 0\), we set \(\phi := -m \log f\), so that
\[
dv_{\phi} = e^{-\phi} \, dvol_g.
\]
In terms of \(\phi\), it holds that
\[
\mathbf{Ric}_\phi^m = \mathbf{Ric} + \nabla^2 \phi - \frac{1}{m} d\phi \otimes d\phi,
\]
\[
R^m_\phi = R + 2\Delta \phi - \frac{m+1}{m} \left| \nabla \phi \right|^2 + m(m-1)\mu e^{2\phi/m}.
\]
Recall that weighted Schouten tensor \(P^m_\phi\) and the weighted Schouten scalar \(J^m_\phi\) of \((M^d, g, f, m, \mu)\) are
\[
P^m_\phi := \frac{1}{d + m - 2} (\mathbf{Ric}_\phi^m - J^m_\phi g),
\]
\[
J^m_\phi := \frac{1}{2(d + m - 1)} P^m_\phi.
\]
A quasi-Einstein space [4] is a smooth metric measure space such that for some \(\lambda \in \mathbb{R}\),
\[
P^m_\phi = \lambda g, \quad J^m_\phi = (d + m)\lambda.
\]
In this paper, we prove a factorization formula of the weighted GJMS operator for quasi-Einstein spaces.

**Theorem 1.3.** The weighted GJMS operator
\[
L_{2k,\phi}^m : \mathcal{E}\left[-\frac{d + m}{2} + k\right] \to \mathcal{E}\left[-\frac{d + m}{2} - k\right]
\]
can be factorized as
\[
\prod_{l=0}^{k-1} \left( \Delta \phi + 2\lambda \left( -\frac{d + m}{2} + k - 2l \right) \left( \frac{d + m}{2} + k - 1 - 2l \right) \right)
\]
for quasi-Einstein spaces.

**Weighted Gover–Leitner conditions** are a generalization to \(m \notin \mathbb{N}_0\) of the Gover–Leitner conditions defined in [14], and are defined as
\[
f(x) = 1 \quad \mu = 1, \quad (\mathbf{Ric}_\phi^m)_{ij} = (-d + 1)g.
\]
In this paper, we also prove a factorization formula of the weighted GJMS operator for smooth metric measure spaces under weighted Gover–Leitner conditions. We thus provide a rigorous proof of the factorization formula constructed for Poincaré-Einstein spaces by Case and Chang [6].

Theorem 1.4. The weighted GJMS operator

\[ L_{2k,\phi}^m : \mathcal{E} \left[ -\frac{d+m}{2} + k \right] \to \mathcal{E} \left[ -\frac{d+m}{2} - k \right] \]

can be factorized as

\[ L_{2k,\phi}^m = \prod_{j=0}^{k-1} \left[ \Delta + \frac{(2k-4j-d-m)(2-d+m-2k+4l)}{4} \right] \]

under Gover–Leitner conditions.

This article is organized as follows: in Section 2, we discuss some properties of the weighted ambient space and the commutation relations satisfied by the differential operators relevant to this article. In Section 3, we derive weighted GJMS operators with the help of the weighted ambient space. In Section 4, we show that weighted GJMS operators have a power of the weighted Laplacian as the leading part. In Section 5, we derive the factorization formulas for weighted GJMS operators in the case of quasi-Einstein spaces, and also under Gover–Leitner conditions. In Section 6, we prove that weighted GJMS operators are formally self-adjoint.

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2. The weighted ambient space

For a smooth metric measure space \((M^d, g, f, m, \mu)\), consider the \((d+2)\)-dimensional space \(\mathbb{R}^+ \times M \times \mathbb{R}\) with coordinates \((t, x, \rho)\). Then the corresponding straight and normal weighted ambient space is

\[ \tilde{g} := t^2g_{\rho} + 2pdf^2 + 2tdtd\rho, \]
\[ \tilde{f} := tf_{\rho}, \]

such that \(\tilde{\text{Ric}}_{\phi}^m, \tilde{F}_{\phi}^m = O(\rho^j)\), where \(j = (d + m - 2)/2\) or \(\infty\) depending on whether \(d + m \in 2\mathbb{N}\) or \(d + m \notin 2\mathbb{N}\). The existence, and the uniqueness of \((g, f)\) up to order \(O(\rho^j)\) and of \([2^{-1}g^{kl}(g_{\rho})_{kl} + mf^{-1}(f_{\rho})]\) up to order \(O(\rho^{j+1})\), has been proven in [7]. We denote \(\mathbb{R}^+ \times M \times \mathbb{R}\) as \(\tilde{G}\), \(X^I\) as the position vector on \(\tilde{G}\), and \(\tilde{G}\) as \(\tilde{G}\) for \(t = 0\).

2.1. Commutation relations. Equation (1) implies that

\[ \tilde{\nabla}_J X_I = \tilde{g}_{IJ}. \]

Note that Equation (2) can also be deduced from the fact that Equation (1) is a straight weighted ambient metric (cf. [7]). From Equation (2), we get that

\[ \tilde{\nabla}_K \tilde{\nabla}_J X_I - \tilde{\nabla}_J \tilde{\nabla}_K X_I = 0. \]

Hence,

\[ \tilde{R}_{LJKI} X^L = 0. \]
Now set \( Q = X^I X_I \). From Equation (1), we conclude that it is a defining function for \( \mathcal{G} \). From Equation (2), we compute that

\[
\tilde{\nabla}_I Q = 2X_I.
\]

Let us now define the following operators acting on functions on \( \tilde{\mathcal{G}} \).

\[
x = -\frac{1}{4}Q, \quad y = \tilde{\Delta}_\phi, \quad h = X + \frac{1}{2}(d + m + 2).
\]

Here \( \tilde{\Delta}_\phi \) is the weighted Laplacian, \( \tilde{\Delta}_\phi := \tilde{\Delta} - \tilde{\nabla}_I \nabla^I \). Note that the degree of homogeneity of \( f \) with respect to \( t \) is 1, and \( X = t \partial_t \).

**Theorem 2.1.** The operators \( x, y, z \) satisfy the commutation relations

\[
[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.
\]

**Proof.** First, we prove that \([x, y] = h\). Let \( F \) be a function defined on \( \tilde{\mathcal{G}} \). Using Equation (2), Equation (4) and the fact that \( \tilde{g}^I J \tilde{g}_{IJ} = d + 2 \), we get

\[
[x, y] F = [X + \frac{1}{2}(d + 2)](F) - \frac{1}{4}(\tilde{\nabla}_I \phi \tilde{\nabla}^I Q)F.
\]

As \( \tilde{\nabla}_I \phi \nabla^I Q = -\frac{2m}{\tilde{T}} \nabla_X f \), we have \(-\frac{1}{4}(\tilde{\nabla}_I \phi \tilde{\nabla}^I Q)F = \frac{1}{2}mF \). Consequently, \([x, y] = h\).

Second, we prove that \([h, x] = 2x\). This is easily seen using Equation (4).

Third, we prove that \([h, y] = -2y\). Using Equation (3), we conclude that

\[
[X, \tilde{\Delta}](F) = -2\tilde{\Delta}(F),
\]

\[
[\tilde{\nabla}_\phi, X](F) = \tilde{\nabla}_\phi(F) - X^I(\tilde{\nabla}_I \phi)(\tilde{\nabla}_J F).
\]

On commuting \( X_I \) with \( \tilde{\nabla}_J \), using Equation (2), and noting that \( \nabla_X \phi = -m \), we get \([\tilde{\nabla}_\phi, X](F) = 2\tilde{\nabla}_\phi(F) \). Hence, \([h, y] = -2y\). \( \square \)

Using an induction argument, we get the following commutation relations (cf. [16]).

\[
[y^k, x] = -ky^{k-1}(h - k + 1),
\]

\[
[x^k, y] = ky^{k-1}(h + k - 1),
\]

\[
h^{k-1}x^{k-1} = (-1)^{k-1}(k - 1)!h(h + 1)\ldots(h + k - 2) + xZ_k,
\]

for some polynomial \( Z_k \) in \( x, y, h \).

### 3. Weighted conformally invariant operators

In this section, we construct two weighted GJMS operators, and then prove that they are the same up to a constant.

In the rest of the paper, \( w = -(d + m)/2 + k \).

**Theorem 3.1.** Let \( k \in \mathbb{N} \) and \( F \in \mathcal{E}[w] \). Then \( \tilde{\Delta}_\phi^F |_{\mathcal{G}} \) is independent of the choice of smooth extension \( F \) to \( \mathcal{G} \). Thus, \( L_{2k,\phi}^m : \mathcal{E}[w] \to \mathcal{E}[w - 2k] \), defined as \( L_{2k,\phi}^m F := \tilde{\Delta}_\phi^F |_{\mathcal{G}} \), is a conformally invariant operator.
Theorem 3.3. Any two extensions of \( F \) differ by a function of the form \( QH \), where \( H \in C^\infty(\mathcal{G}) \) is homogeneous of weight \( w - 2 \). Now by Equation (6), we have

\[
\tilde{\Delta}_\phi(QH) = Q\tilde{\Delta}_\phi^k(H) + 4k\tilde{\Delta}_\phi^{k-1}(w + \frac{1}{2}(d + m) - k)H = Q\tilde{\Delta}_\phi^k(H).
\]

As \( \tilde{\Delta}_\phi \) reduces the degree of homogeneity by 2, it holds that \( \tilde{\Delta}_\phi^k(F) \big|_\mathcal{G} \) belongs to \( \mathcal{E}[w - 2k] \).

We now study the obstruction to constructing harmonic extensions of smooth conformal densities on \( \mathcal{G} \).

Theorem 3.2. For \( F \in \mathcal{E}[w] \),

1. if \( k \notin \mathbb{N} \), then \( F \) has a unique formal harmonic extension to \( \tilde{\mathcal{G}} \), homogeneous of degree \( w \);
2. if \( k \in \mathbb{N} \), then \( F \) has a homogeneous extension \( \tilde{F} \) uniquely determined modulo \( O(Q^k) \) by the requirement that \( \tilde{\Delta}_\phi\tilde{F} = 0 \) modulo \( O(Q^{k-1}) \). The obstruction to a harmonic extension is \( Q^{1-k}\tilde{\Delta}_\phi^k\tilde{F} \big|_\mathcal{G} \), which is independent of the extension \( \tilde{F} \) mod \( Q^k \), and hence is conformally invariant.

Proof. Assume that we have an extension \( \tilde{F}_{l-1} \) of \( F \) such that \( \tilde{\Delta}_\phi\tilde{F}_{l-1} = 0 \mod Q^{l-1} \). Now let \( \tilde{F}_l = \tilde{F}_{l-1} + Q^lH \), where \( H \) is of weight \( w - 2l \). We have

\[
\tilde{\Delta}_\phi\tilde{F}_l = \tilde{\Delta}_\phi\tilde{F}_{l-1} + \tilde{\Delta}_\phi(Q^lH) \equiv \tilde{\Delta}_\phi\tilde{F}_{l-1} + 4lQ^{l-1}(k - l)H \mod Q^l.
\]

If \( k \) is not a positive integer, we can choose a unique function \( H \) for each \( l \) such that the above expression is \( 0 \mod Q^l \). On the other hand, if \( k \) is a positive integer, then \( \tilde{\Delta}_\phi(\tilde{F}_k) \equiv \tilde{\Delta}_\phi(\tilde{F}_{k-1}) \mod Q^k \). Note that \( Q^{1-k}\tilde{\Delta}_\phi(\tilde{F}) \big|_\mathcal{G} \) depends only on \( F \), and is homogeneous of degree \( w - 2k \).

We now show that the two conformally invariant operators constructed above are scalar multiples of one another.

Theorem 3.3. If \( k \in \mathbb{N} \), then

\[
(6) \quad \tilde{\Delta}_\phi^k\tilde{F} \big|_\mathcal{G} = (-4)^{k-1}(k - 1)!^2Q^{1-k}\tilde{\Delta}_\phi\tilde{F} \big|_\mathcal{G},
\]

where \( \tilde{F}_{k-1} \) is the extension of \( F \) such that \( \tilde{\Delta}_\phi\tilde{F}_{k-1} = 0 \mod Q^{k-1} \).

Proof. Let \( L = Q^{1-k}\tilde{\Delta}_\phi \big|_\mathcal{G} \) be as in Theorem 3.2. Then \( \tilde{\Delta}_\phi\tilde{F}_{k-1} = Q^{k-1}L\tilde{F} \mod Q^k \). Now from Equation (6), we know that

\[
\tilde{\Delta}_\phi^k\tilde{F}_{k-1} = \tilde{\Delta}_\phi^{k-1}(Q^{k-1}LF) = 4^{k-1}(k - 1)!h(h + 1)\ldots(h + k - 2)L\tilde{F} \mod Q.
\]

But \( hL\tilde{F} = -(k - 1)L\tilde{F} \). Using this, we verify Equation (6). \( \square \)

In the rest of the paper, we shall denote \( \tilde{\Delta}_\phi^k\tilde{F} \big|_\mathcal{G} \) as \( L^m_{2k,h} \).
4. The leading order term

Let $k \in \mathbb{N}$. For $\psi \in C^\infty(M)$, let $t^w \psi(x, \rho)$ be a homogeneous extension of weight $w$. The weighted Laplacian with respect to the ambient metric measure structure of the form Equation (1) is

\begin{align}
\Delta_{\phi}(t^w \psi) &= t^{w-2} \left[ -2 \rho \phi'' + (2w + d + m - 2 - \rho g^{ij} g_{ij}) \phi' \right. \\
& \quad + \Delta_{\phi} \phi + \frac{1}{2} w \psi g^{ij} g'_{ij} + \frac{m}{f} f' (w \psi - 2 \rho \phi') ,
\end{align}

where $\psi = \psi(x, \rho)$, the prime denotes $\partial_\rho$, and the $\Delta_{\phi}$ on the right-hand side refers to the weighted Laplacian with respect to $(g_0, f_0)$.

On differentiating Equation (7) a total of $l$ times, where $l < k - 1$, and setting $\rho = 0$, we get

\begin{align}
(8) \quad 2(l + 1 - k)(\partial_\rho)^{{l + 1}}|_{\rho = 0} \psi &= (\partial_\rho)^l |_{\rho = 0} \left[ \Delta_{\phi} \psi - 2 \rho \psi' \left( \frac{1}{2} g^{ij} g'_{ij} + \frac{m}{f} f' \right) \\
& \quad + w \psi \left( \frac{1}{2} g^{ij} g'_{ij} + \frac{m}{f} f' \right) \right] .
\end{align}

For $l = k - 1$, we get

\begin{align}
(9) \quad c_k L^k_{2k, \phi} \psi &= (\partial_\rho)^{k-1} |_{\rho = 0} \left[ \Delta_{\phi} \psi - 2 \rho \psi' \left( \frac{1}{2} g^{ij} g'_{ij} + \frac{m}{f} f' \right) + w \psi \left( \frac{1}{2} g^{ij} g'_{ij} + \frac{m}{f} f' \right) \right],
\end{align}

where $c_k = (-1)^{k-1} (2k - 1)!^{-1}$.

Let $d + m \in 2\mathbb{N}$. In Equation (8), as $l < k - 2$, we have at most $k - 1$ derivatives of $(\frac{1}{2} g^{ij} (g_0)_{ij} + \frac{m}{f} f_0)$ on the right. Hence, if $k - 1 < (d + m)/2$, Equation (8) is well-defined. However, in Equation (9), there are at most $k - 1$ derivatives of $\psi$ and $k$ derivatives of $(\frac{1}{2} g^{ij} (g_0)_{ij} + \frac{m}{f} f_0)$ at $\rho = 0$. Hence, both Equations (8) and (9) are well-defined only for $k < (d + m)/2$. Also, note that $w = 0$ for $k = (d + m)/2$.

Equation (9) also tells us that $L^k_{2k, \phi}$ has leading part $(\Delta_{\phi})^k$. This completes the proof of Theorem 1.1.

5. Factorization formulas

We now prove factorization formulas of the weighted GJMS operator $\tilde{\Delta}_{\phi}^k|_{\G}$ under quasi-Einstein conditions and Gover–Leitner conditions.

5.1. Quasi-Einstein conditions.

Proof of Theorem 5.1.3. Let

\begin{align}
    g(x, \rho) &= (1 + \lambda \rho)^2 g(x), \quad f(x, \rho) = (1 + \lambda \rho) f(x).
\end{align}

We know from [7, Section 7] that $(\tilde{g}, \tilde{f})$ of the form Equation (1), with $(g_0, f_0)$ as given in Equation (10), is a weighted ambient space. Also, for $\psi \in C^\infty(M)$, let $\tilde{\psi}(t, x, \rho) = t^w (1 + \lambda \rho)^w \psi(x)$. Equation (7) becomes

\begin{align}
\tilde{\Delta}_{\phi} \tilde{\psi} &= t^{w-2} (1 + \lambda \rho)^{w-2} \left[ \Delta_{\phi} + 2 \lambda (w - 2l) (w + d + m - 1 - 2l) \right] \psi.
\end{align}

By induction, we obtain

\begin{align}
\tilde{\Delta}_{\phi}^k \tilde{\psi} &= t^{w-2k} (1 + \lambda \rho)^{w-2k} \prod_{l=0}^{k-1} \left[ \Delta_{\phi} + 2 \lambda (w - 2l) (w + d + m - 1 - 2l) \right] \psi(x).
\end{align}
With \( w = -(d + m)/2 + k \), restricting to \( \rho = 0 \), and using the fact that \( \Delta_k \tilde{\psi} \)|\( \mathcal{G} \) is independent of the choice of extension \( \tilde{\psi} \) to \( \tilde{\mathcal{G}} \), we get
\[
L_{2k, \phi}^m(\psi) = \prod_{l=0}^{k-1} \left[ \Delta_\phi + 2l \left( -\frac{d + m}{2} + k - 2l \right) \left( \frac{d + m}{2} + k - 1 - 2l \right) \right] \psi. \tag{\text{□}}
\]

The idea for such an argument originated in [19].

5.2. Gover–Leitner conditions.

Proof of Theorem 1.4. Let \((g_\rho, f_\rho)\) be defined as
\[
g(x, \rho) = \left( 1 - \frac{1}{2} \rho \right)^2 g(x), \quad f(x, \rho) = \left( 1 + \frac{1}{2} \rho \right).
\]
We know from [7, Section 7] that \((\tilde{g}, \tilde{f})\) of the form Equation (11), with \((g_\rho, f_\rho)\) as given in Equation (10), is a weighted ambient space. Now if \( \tilde{\psi}(x, \rho, t) = t^v (1 - \rho/2)^w \psi(x) \) for \( w = -(d + m)/2 + k \), on using Equation (7) and induction we get
\[
\Delta_k \tilde{\psi} \mid_{\mathcal{G}} = t^{- \frac{d + m}{2} - k} \prod_{j=0}^{k-1} \left[ \Delta + \frac{(2k - 4j - d - m)(2 - d + m - 2k + 4l)}{4} \right] \psi.
\]
Since \( \Delta_k \tilde{\psi} \mid_{\mathcal{G}} \) is independent of the choice of extension \( \tilde{\psi} \) to \( \tilde{\mathcal{G}} \), we get
\[
L_{2k, \phi}^m = \prod_{j=0}^{k-1} \left[ \Delta + \frac{(2k - 4j - d - m)(2 - d + m - 2k + 4l)}{4} \right]. \tag{\text{□}}
\]

6. Formal self-adjointness

We now prove Theorem 12. To begin, we identify the weighted GJMS operators in terms of the weighted Poincaré space [7].

**Theorem 6.1.** Let \((X^{d+1}, g_+, f_+, m, \mu)\) be a weighted Poincaré space [7] for the smooth metric measure space \((M^d, g, f, m, \mu)\), and let \( v \in C^\infty(M) \). Also, let \( k \in \mathbb{N} \) and \( s = (d + m)/2 + k \), with \( k \leq (d + m)/2 \) if \( d + m \in 2\mathbb{N} \). Then there is a formal solution to the equation
\[
((\Delta_{\phi^+})_{g_+} - s(d + m - s)) u = O (r^{2k \log r})
\]
of the form
\[
u = r^{\frac{d + m}{2} - k} \left( V + d_k L_{2k, \phi}^m v r^{2k \log r} \right),
\]
where the function \( V \in C^\infty(X) \) is uniquely determined by \( g, f \) and \( v \) modulo \( O (r^{2k}) \), \( V|_M = v \), and \( d_k = \lfloor 2^{2k-1} k!(k-1)! \rfloor^{-1} \).

**Proof.** For a chosen \((g, f)\), we may assume without loss of generality [7, Proposition 5.3] that \((g_+, f_+) = (r^{-2} (g_r + dr^2), r^{-1} f_r)\). A straightforward calculation shows that \((\Delta_{\phi^+})_{g_+} - s(d + m - s) \circ r^{d+m-s} = r^{d+m-s+1} D_s\), where
\[
D_s = -r \partial_r^2 + \left[ 2s - d - m - 1 - r \left( \frac{1}{2} g^{ij} g_{ij} + \frac{m}{f} f' \right) \right] \partial_r
\]
\[
- (d + m - s) \left( \frac{1}{2} g^{ij} g_{ij} + \frac{m}{f} f' \right) + r(\Delta_\phi)_{g_r}
\]
Here \((g_{ij}, f)\) denotes \((g_r, f_r)\) with \(r\) fixed, and \((g'_{ij}, f') = (\partial_{r}g_{ij}, \partial_{r}f)\). For \(v_j \in C^\infty(M)\) and \(s = (d + m)/2 + k\), one has

\[
D_s(v_j r^j) = j(2k - j)v_j r^{j-1} + O(r^j).
\]

Beginning with \(V_0 = v\), define \(v_j, V_j\) for \(j \geq 1\) by

\[
j(2k - j)v_j = -(r^{j-1}D_s(V_j - 1))|_{x=0},
\]

\[
V_j = V_{j-1} + v_j r^j.
\]

Observe that since \((g_r, f_r)\) is even in \(r\) modulo \(O(r^j)\), where \(j = \infty\) or \(d + m\) for \(d + m \notin 2N\) or \(d + m \in 2N\) respectively [7], \(D_s\) maps even functions to odd and vice versa modulo \(O(r^j)\). Therefore, \(v_j = 0\) for \(j\) odd and \(j < d + m\).

For \(j = 2k\), there is an obstruction to solving for a smooth function \(V\). However, observe that

\[
D_s(p_j r^j \log r) = j(2k - j)p_j r^{j-1} \log r + 2(k - j)p_j r^{j-1} + O(r^j \log r).
\]

Therefore, if we take

\[
p_{2k} = (2k)^{-1}(r^{1-2k}D_s(V_{2k-1}))|_{r=0},
\]

then we have \(D_sV_{2k} = O(r^{2k} \log r)\). We can deduce from Equation (11) that \(p_{2k} = d_k P_{m, k, \phi}^v\), where \(P_{m, k, \phi}^v\) is a differential operator with principal part \(\Delta_{\phi}^k\), and \(d_k = (2k - 1)!!|^{-1}\).

Now we show that \(P_{m, k, \phi}\), when defined in terms of the ambient metric, is the same as the differential operator \(L_{m, k, \phi}\).

Let \(x = \sqrt{-2\rho}\) and \(v = x t\). Then from [7, Proposition 5.6|] we know that

\[
\vec{g} = -dv^2 + v^2 g_+,
\]

\[
\vec{f} = v f_+,
\]

where \((\vec{g}, \vec{f})\) is a straight and normal weighted ambient space and \((g_+, f_+)\) is a normal weighted Poincaré space. For function \(\vec{F}\) of weight \(w\), we have

\[
\vec{\Delta}_{\phi}\vec{F} = v^{-2}[(\Delta_{\phi_+})_{g_+} + w(w + d + m)]\vec{F}.
\]

Let \(s = w + d + m\). Then this equation can be written as

\[
\vec{\Delta}_{\phi}\vec{F} = v^{-2}[(\Delta_{\phi_+})_{g_+} - s(d + m - s)]\vec{F}.
\]

Let \(u\) be the restriction of \(\vec{F}\) to the Poincaré-Einstein space \(v = 1\). Then \(\vec{F}\) can be recovered from \(u\) by \(\vec{F} = s^u u = t^x x^u u\). Also, in order for \(\vec{F}\) to be smooth up until \(\rho = 0\), we require that \(u\) be smooth until the boundary. Thus, the two extension problems are equivalent, and the normalized obstruction operators must agree. \(\square\)

**Theorem 6.2.** Let \((X^{d+1}, g_+, f_+, m, \mu)\) be a weighted Poincaré space for the smooth metric measure space \((M^d, g, f, m, \mu)\). Let \(k \in \mathbb{N}\), \(k \leq (d + m)/2\) for \(d + m \in 2\mathbb{N}\), and set \(s = (d + m)/2 + k\). Let \(u_1, u_2 \in C^\infty(M)\) and let \(u_1, u_2\) denote the corresponding solutions of

\[
((\Delta_{\phi_+})_{g_+} - s(d + m - s)) u = O(x^{2k} \log r)
\]
given by Theorem 6.1. Then for fixed small \( x_0 > 0 \)

\[
(12) \quad \int_{\epsilon < r < x_0} [(du_1, du_2) - s(d + m - s)u_1u_2] (dv^m_{g_f})_{g_f} \\
= -d_k \int_M \left[ \left( \frac{d + m}{2} + k \right) v_1 L_{2k, \phi}^m v_2 + \left( \frac{d + m}{2} - k \right) v_2 L_{2k, \phi}^m v_1 \right] (dv^m_{\phi})_h,
\]

where \( \int \) denotes the coefficient of log \( \epsilon \) in the asymptotic expansion of the integral as \( \epsilon \to 0 \), \( d_k = [2^{2k-1}k!(k-1)]^{-1} \), and \( dv^m_{\phi} = e^\phi d\text{vol} \) denotes the weighted volume element. In particular, \( L_{2k, \phi}^m \) is formally self-adjoint.

**Proof.** For \( (g, f) \), we may assumed without loss of generality \([7, \text{Proposition 5.3}]\) that \( (g_+, f_+) = (r^{-2}(dr^2 + g_r), r^{-1}f_r) \). Green’s identity gives

\[
\int_{\epsilon < r < r_0} [(du_1, du_2) - s(d + m - s)u_1u_2] (dv^m_{g_f})_{g_f} \\
= -\epsilon^{1-d-m} \int_{r=\epsilon} u_1 \partial_r u_2 (dv^m_{\phi})_{g_r} + O(1).
\]

Substituting

\[
u_1 = r \frac{d\text{vol}}{2^m} \quad \left( V_i + d_k L_{2k, \phi}^m v_i i^{2k \log r} \right),
\]

\[ (dv^m_{\phi})_{g_r} = (1 + (v_{\phi})^2 r^2 + \text{(even powers) + ...}) (dv^m_{\phi}), \]

and expanding shows that the coefficient of log \( \epsilon \) in the expansion of this expression is

\[-d_k \int_M \left[ \left( \frac{d + m}{2} + k \right) v_1 L_{2k, \phi}^m v_2 + \left( \frac{d + m}{2} - k \right) v_2 L_{2k, \phi}^m v_1 \right] (dv^m_{\phi})_h.\]

The symmetry of the left-hand side of Equation 12 yields the final conclusion. \( \square \)

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