THE GEOMETRY OF HEMI-SLANT SUBMANIFOLDS
OF A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

HAKAN METE TAŞTAN AND FATMA ÖZDEMİR

Abstract. In the present paper, we study hemi-slant submanifolds of a locally product Riemannian manifold. We prove that the anti-invariant distribution which is involved in the definition of hemi-slant submanifold is integrable and give some applications of this result. We get a necessary and sufficient condition for a proper hemi-slant submanifold to be a hemi-slant product. We also study this type submanifolds with parallel canonical structures. Moreover, we give two characterization theorems for the totally umbilical proper hemi-slant submanifolds. Finally, we obtain a basic inequality involving Ricci curvature and the squared mean curvature of a hemi-slant submanifold of a certain type locally product Riemannian manifold.

1. Introduction

Study of slant submanifolds was initiated by B.Y. Chen [8], as a generalization of both holomorphic and totally real submanifolds of a Kähler manifold. Slant submanifolds have been studied in different kind structures; almost contact [13], neutral Kähler [4], Lorentzian Sasakian [2] and Sasakian [6] by several geometers. N. Papaghiuc [14] introduced semi-slant submanifolds of a Kähler manifold as a natural generalization of slant submanifold. A. Carriazo [7], introduced bi-slant submanifolds of an almost Hermitian manifold as a generalization of semi-slant submanifolds. One of the classes of bi-slant submanifolds is that of anti-slant submanifolds which are studied by A. Carriazo [7]. However, B. Şahin [18] called these submanifolds as hemi-slant submanifolds because of that the name anti-slant seems to refer that it has no slant factor. We observe that a hemi-slant submanifold is a special case of generic submanifold which was introduced by G.S. Ronsse [16]. Since then many geometers have studied hemi-slant submanifolds in different kind structures; Kähler [3,18], nearly Kähler [21], generalized complex space form [20] and almost Hermitian [19]. We note that sometimes hemi-slant submanifolds are also studied under the name pseudo-slant submanifolds, see [11] and [21]. The submanifolds of a locally product Riemannian manifold have been studied by many geometers. For example, T. Adati [1] defined and studied invariant and anti-invariant submanifolds, while A. Bejancu [5] and G. Pitis [15] studied semi-invariant submanifolds. Slant and semi-slant submanifolds of a locally product Riemannian manifold are examined by B. Şahin [17] and H. Li and X. Liu [12]. In this paper, we study hemi-slant submanifolds of a locally product Riemannian manifold in detail.

2000 Mathematics Subject Classification. Primary 53B25; Secondary 53C55.
Key words and phrases. locally product manifold, hemi-slant submanifold, slant distribution.
2. Preliminaries

This section is devoted to preliminaries. Actually, in subsection 2.1 we present the basic background needed for a locally product Riemannian manifold. Theory of submanifolds and distributions related to the study are given in subsection 2.2.

2.1. Locally product Riemannian manifolds. Let $\bar{M}$ be an $m$-dimensional manifold with a tensor field of type $(1,1)$ such that

$$F^2 = I, (F \neq \pm I),$$

where $I$ is the identity morphism on the tangent bundle $T\bar{M}$ of $\bar{M}$. Then we say that $\bar{M}$ is an almost product manifold with almost product structure $F$. If an almost product manifold $(\bar{M}, F)$ admits a Riemannian metric $g$ such that

$$g(F\bar{U}, F\bar{V}) = g(\bar{U}, \bar{V})$$

for all $\bar{U}, \bar{V} \in T\bar{M}$, then $\bar{M}$ is called an almost product Riemannian manifold.

Next, we denote by $\nabla$ the Riemannian connection with respect to $g$ on $\bar{M}$. We say that $\bar{M}$ is a locally product Riemannian manifold, (briefly, l.p.R. manifold) if we have

$$(\nabla_{\bar{U}} F)\bar{V} = 0,$$

for all $\bar{U}, \bar{V} \in T\bar{M}$ [22].

2.2. Submanifolds. Let $M$ be a submanifold of a l.p.R. manifold $(\bar{M}, g, F)$. Let $\nabla, \nabla$, and $\nabla^\perp$ be the Riemannian, induced Riemannian, and induced normal connection in $\bar{M}, M$ and the normal bundle $T^\perp M$ of $M$, respectively. Then for all $U, V \in TM$ and $\xi \in T^\perp M$ the Gauss and Weingarten formulas are given by

$$(\nabla_U F)V = \nabla_U V + h(U, V)$$

and

$$(\nabla_U \xi) = -A_{\xi}U + \nabla^\perp_U \xi$$

where $h$ is the second fundamental form related to shape operator. A corresponding to the normal vector field $\xi$ is given by

$$g(h(U, V), \xi) = g(A_{\xi}U, V).$$

A submanifold $M$ is said to be totally geodesic if its second fundamental form vanishes identically, that is, $h = 0$, or equivalently $A_{\xi} = 0$. We say that $M$ is totally umbilical submanifold in $\bar{M}$ if for all $U, V \in TM$ we have

$$h(U, V) = g(U, V)H,$$

where $H$ is the mean curvature vector field of $M$ in $\bar{M}$. A normal vector field $\xi$ is said to be parallel, if $\nabla^\perp_U \xi = 0$ for each vector field $U \in TM$.

The Riemannian curvature tensor $\bar{R}$ of $\bar{M}$ is given by

$$\bar{R}(\bar{U}, \bar{V}) = [\nabla_{\bar{U}}, \nabla_{\bar{V}}] \bar{V} - \nabla_{[\bar{U}, \bar{V}]} \bar{V},$$

where $\bar{U}, \bar{V} \in T\bar{M}$.

Then the Codazzi equation is given by

$$(\bar{R}(U, V)W)^\perp = (\nabla_U h)(V, W) - (\nabla_V h)(U, W).$$
for all $U, V, W \in TM$. Here, $\perp$ denotes the normal component and the covariant derivative of $h$, denoted by $\nabla_U h$ is defined by

$$ (\nabla_U h)(V, W) = \nabla_U^\perp h(V, W) - h(\nabla_U V, W) - h(V, \nabla_U W). $$

Now, we write

$$ (2.11) \quad F U = TU + NU, $$

for any $U \in TM$. Here $TU$ is the tangential part of $FU$, and $NU$ is the normal part of $FU$. Similarly, for any $\xi \in T^\perp M$, we put

$$ (2.12) \quad F \xi = t\xi + \omega \xi, $$

where $t\xi$ is the tangential part of $F \xi$, and $\omega \xi$ is the normal part of $F \xi$.

A distribution $\mathcal{D}$ on a manifold $\bar{M}$ is called autoparallel if $\nabla_X Y \in \mathcal{D}$ for any $X, Y \in \mathcal{D}$ and called parallel if $\nabla_X X \in \mathcal{D}$ for any $X \in \mathcal{D}$ and $U \in TM$. If a distribution $\mathcal{D}$ on $\bar{M}$ is autoparallel, then it is clearly integrable, and by Gauss formula $\mathcal{D}$ is totally geodesic in $\bar{M}$. If $\mathcal{D}$ is parallel then the orthogonal complementary distribution $\mathcal{D}^\perp$ is also parallel, which implies that $\mathcal{D}$ is parallel if and only if $\mathcal{D}^\perp$ is parallel. In this case $\bar{M}$ is locally product of the leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$. Let $M$ be a submanifold of $\bar{M}$. For two distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on $M$, we say that $M$ is $(\mathcal{D}_1, \mathcal{D}_2)$ mixed totally geodesic if for all $X \in \mathcal{D}_1$ and $Y \in \mathcal{D}_2$ we have $h(X, Y) = 0$, where $h$ is the second fundamental form of $M$ [20, 22].

3. Hemi-slant submanifolds of a locally product Riemannian manifold

In this section, we define the notion of hemi-slant submanifold and observe its effect to the tangent bundle of the submanifold and canonical projection operators and start to study hemi-slant submanifolds of a locally product Riemannian manifold.

Let $(\bar{M}, g, F)$ be a locally product Riemannian manifold and let $M$ be a submanifold of $\bar{M}$. A distribution $\mathcal{D}$ on $M$ is said to be a slant distribution if for $X \in \mathcal{D}_p$, the angle $\theta$ between $FX$ and $\mathcal{D}_p$ is constant, i.e., independent of $p \in M$ and $X \in \mathcal{D}_p$. The constant angle $\theta$ is called the slant angle of the slant distribution $\mathcal{D}$. A submanifold $M$ of $\bar{M}$ is said to be a slant submanifold if the tangent bundle $TM$ of $M$ is slant [12, 17]. Thus, the $F-$invariant and $F-$anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold which is neither $F-$invariant nor $F-$anti-invariant is called a proper slant submanifold.

**Definition 3.1.** A hemi-slant submanifold $M$ of a locally product Riemannian manifold $\bar{M}$ is a submanifold which admits two orthogonal complementary distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ such that

(a) $TM$ admits the orthogonal direct decomposition $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta$
(b) The distribution $\mathcal{D}^\perp$ is $F-$anti-invariant, i.e., $F\mathcal{D}^\perp \subset T^\perp M$.
(c) The distribution $\mathcal{D}^\theta$ is slant with slant angle $\theta$. 
In this case, we call $\theta$ the slant angle of $M$. Suppose the dimension of distribution $D^\perp$ (resp. $D^\theta$) is $p$ (resp. $q$). Then we easily see that the following particular cases.

\begin{itemize}
  \item [(d)] If $q = 0$, then $M$ is an anti-invariant submanifold [1].
  \item [(e)] If $p = 0$ and $\theta = 0$, then $M$ is an invariant submanifold [1].
  \item [(f)] If $p = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then $M$ is a proper slant submanifold [17].
  \item [(g)] If $\theta = \frac{\pi}{2}$, then $M$ is an anti-invariant submanifold.
  \item [(h)] If $p \neq 0$ and $\theta = 0$, then $M$ is a semi-invariant submanifold [5].
\end{itemize}

We say that the hemi-slant submanifold $M$ is proper if $p \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$.

**Lemma 3.2.** Let $M$ be a proper hemi-slant submanifold of a l.p.R. manifold $\tilde{M}$. Then we have,

\begin{equation}
F(D^\perp) \perp N(D^\theta).
\end{equation}

**Proof.** For any $X \in D^\perp$ and $Z \in D^\theta$, using (2.2) and (2.11), we have $g(FX, NZ) = g(FX, FZ) = g(X, Z) = 0$. This completes the proof. \qed

In view of Lemma 3.2, for a hemi-slant submanifold $M$ of a l.p.R. manifold $\tilde{M}$, the normal bundle $T^\perp M$ of $M$ is decomposed as

\begin{equation}
T^\perp M = F(D^\perp) \oplus N(D^\theta) \oplus \mu,
\end{equation}

where $\mu$ is the orthogonal complementary distribution of $F(D^\perp) \oplus N(D^\theta)$ in $T^\perp M$ and it is invariant subbundle of $T^\perp M$ with respect to $F$.

The following facts follow easily from (2.1), (2.11) and (2.12) and will be used later.

\begin{equation}
(a) \quad T^2 + tN = I, \quad (b) \quad \omega^2 + Nt = I,
\end{equation}

\begin{equation}
(c) \quad NT + \omega N = 0, \quad (d) \quad Tt + t\omega = 0.
\end{equation}

As in a slant submanifold [17], for a hemi-slant submanifold $M$ of a l.p.R. manifold $\tilde{M}$, we have

\begin{equation}
T^2 Z = \cos^2 \theta Z,
\end{equation}

\begin{equation}
g(TZ, TW) = \cos^2 \theta g(Z, W)
\end{equation}

and

\begin{equation}
g(NZ, NW) = \sin^2 \theta g(Z, W),
\end{equation}

where $Z, W \in D^\theta$.

**Lemma 3.3.** Let $M$ be a proper hemi-slant submanifold of a l.p.R. manifold $\tilde{M}$. Then we have,

\begin{equation}
(T(D^\perp) = \{0\}, \quad (b) \quad T(D^\theta) = D^\theta).
\end{equation}

**Proof.** Since $D^\perp$ is anti-invariant with respect to $F$, (a) follows from (2.11). For any $Z \in D^\theta$ and $X \in D^\perp$, using (2.1), (2.2) and (2.11), we have $g(TZ, X) = g(FZ, X) = g(Z, FX) = 0$. Hence, we conclude that $T(D^\theta) \perp D^\perp$. Since $T(D^\theta) \subseteq TM$, it follows that $T(D^\theta) \subseteq D^\theta$. Let $W$ be in $D^\theta$. Then using (3.4), we have
$W = \frac{1}{\cos^2 \theta} (\cos^2 \theta W) = \frac{1}{\cos^2 \theta} T^2 W = \frac{1}{\cos^2 \theta} T(TW)$. So, we find $W \in T(D^\theta)$. It follows that $D^\theta \subseteq T(D^\theta)$. Thus, we get the assertion (b).

Thanks to Theorem 3.1 [17], we characterize hemi-slant submanifolds of a l.p.R. manifold.

**Theorem 3.4.** Let $M$ be a submanifold of a l.p.R. manifold $\bar{M}$. Then $M$ is a hemi-slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ and a distribution $D$ on $M$ such that

(a) $D = \{U \in TM \mid T^2 U = \lambda U\}$,
(b) for any $X \in TM$ orthogonal to $D$, $TX = 0$.

Moreover, in this case $\lambda = \cos^2 \theta$, where $\theta$ is the slant angle of $M$.

**Proof.** Let $M$ be a hemi-slant submanifold of $\bar{M}$. By the definition of hemi-slant submanifold, we have $D = D^\theta$ and $\lambda = \cos^2 \theta$. So, (a) follows. (b) follows from Lemma 3.3. Conversely, (a) and (b) imply $TM = D^\perp \oplus D$. Since $T(D) \subseteq D$, we conclude that $D^\perp$ is an anti-invariant distribution from (b).

**Example.** Consider the Euclidean 6-space $\mathbb{R}^6$ with usual metric $g$. Define the almost product structure $F$ on $(\mathbb{R}^6, g)$ by

$$F(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad F(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}, \quad i = 1, 2, 3.$$  

Where $(x_1, x_2, x_3, y_1, y_2, y_3)$ are natural coordinates of $\mathbb{R}^6$. Then $\bar{M} = (\mathbb{R}^6, g, F)$ be an almost product Riemannian manifold. Furthermore, it is easy to see that $\bar{M}$ is a l.p.R. manifold. Let $M$ be a submanifold of $\bar{M}$ defined by

$$f(u, v, w) = \left(\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}, \frac{u + v}{\sqrt{2}}, \frac{w}{\sqrt{2}}, \frac{w}{\sqrt{2}}, 0\right), \quad u \neq 0.$$  

Then, a local frame of $TM$ is given by

$$X = \frac{\partial}{\partial x_3},$$

$$Z = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3},$$

$$W = \frac{1}{\sqrt{2}} \frac{\partial}{\partial y_1} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial y_2}.$$  

By using the almost product structure $F$ above, we see that $FX$ is orthogonal to $TM$, thus $D^\perp = \text{span}\{X\}$. Moreover, it is not difficult to see that $D^\theta = \text{span}\{Z, W\}$ is a slant distribution with slant angle $\theta = \pi/3$. Thus, $M$ is a proper hemi-slant submanifold of $\bar{M}$.

4. Integrability

In this section, we give a necessary and sufficient condition for the integrability of the slant distribution of the hemi-slant submanifold. After that we prove that the anti invariant distribution of the hemi-slant submanifold is always integrable and give some applications of this result.
Let $M$ be a submanifold of a l.p.R. manifold $\tilde{M}$. For any $U, V \in TM$, we have $\nabla_U F V = F \nabla_U V$ from (2.3). Then, using (2.4-2.5), (2.11-2.12) and identifying the components from $TM$ and $T_M^\perp$, we have the following.

**Lemma 4.1.** Let $M$ be a submanifold of a l.p.R. manifold $\tilde{M}$. Then we have,

\begin{equation}
\nabla_UTV - A_NVU = T\nabla_UV + th(U, V),
\end{equation}

(4.1)

\begin{equation}
h(U, TV) + \nabla_U^\perp NV = N\nabla_UV + \omega h(U, V).
\end{equation}

(4.2)

for all $U, V \in TM$.

In a similar way, we have that:

**Lemma 4.2.** Let $M$ be a submanifold of a l.p.R. manifold $\tilde{M}$. Then we have,

\begin{equation}
\nabla_U t\xi - A_{\omega\xi}U = -TA_{\xi}U + t\nabla_U^\perp \xi,
\end{equation}

(4.3)

\begin{equation}
h(U, t\xi) + \nabla_U^\perp \omega\xi = -NA_{\xi}U + \omega\nabla_U^\perp \xi
\end{equation}

(4.4)

for any $U \in TM$ and $\xi \in T_M^\perp$.

**Theorem 4.3.** Let $M$ be a hemi-slant manifold of a l.p.R. manifold $\tilde{M}$. Then, the slant distribution $D^\theta$ is integrable if and only if

\begin{equation}
A_NZW - A_NWZ + \nabla_ZTW - \nabla_WTZ \in D^\theta
\end{equation}

(4.5)

for any $Z, W \in D^\theta$.

**Proof.** From (4.1), we have

\begin{equation}
\nabla_ZTW - A_NWZ = T\nabla_ZW + th(Z, V)
\end{equation}

(4.6)

and

\begin{equation}
\nabla_WTZ - A_NZW = T\nabla_WZ + th(W, Z)
\end{equation}

(4.7)

for any $Z, W \in D^\theta$. Since $h$ is a symmetric $(0, 2)$-type tensor field, from (4.6) and (4.7), we get

\begin{equation}
A_NZW - A_NWZ + \nabla_ZTW - \nabla_WTZ = T[Z, W].
\end{equation}

(4.8)

Thus, our assertion follows from (3.7-b) and (4.8).

The following we give an application of Theorem 4.3.

**Theorem 4.4.** Let $M$ be a hemi-slant manifold of a l.p.R. manifold $\tilde{M}$. If $M$ is $D^\theta$-totally geodesic, then the slant distribution $D^\theta$ is integrable.

**Proof.** Suppose that $M$ is $D^\theta$-totally geodesic, that is, for any $Z, W \in D^\theta$ we have

\begin{equation}
h(Z, W) = 0.
\end{equation}

(4.9)

Thus, from (4.1), using (4.9), we have

\begin{equation}
A_NZW - \nabla_WTZ = -T\nabla_WZ
\end{equation}

(4.10)

and similarly

\begin{equation}
A_NWZ - \nabla_ZTW = -T\nabla_ZW.
\end{equation}

(4.11)

From (4.10) and (4.11), using Lemma 3.3, we get

\begin{equation}
g(A_NZW - A_NWZ + \nabla_ZTW - \nabla_WTZ, X) = g(T[Z, W], X) = 0
\end{equation}

(4.12)
for any $X \in \mathcal{D}^\perp$. The last equation (4.12) says that

$$A_{NZ}W - A_{NW}Z + \nabla_ZTW - \nabla_WTZ \in \mathcal{D}^\theta$$

and by Theorem 4.3, we deduce that $\mathcal{D}^\theta$ is integrable. □

**Lemma 4.5.** Let $M$ be a hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then,

$$(4.13) \quad A_{NX}Y = -A_{NY}X$$

for any $X, Y \in \mathcal{D}^\perp$.

**Proof.** For any $X \in \mathcal{D}^\perp$ and $U \in TM$, using (3.7-a), we have

$$(4.14) \quad -T\nabla_U X = A_{NX}U + h(U, X)$$

from (4.1). Let $Y$ be in $\mathcal{D}^\perp$. Using (3.7-b), we obtain

$$(4.15) \quad 0 = -g(T\nabla_U X, Y) = g(A_{NX}U, Y) + g(h(U, X), Y)$$

from (4.14). On the other hand, using (2.2), (2.6), (2.11) and (2.12), we find

$$(4.16) \quad g(h(U, X), Y) = g(A_{NY}U, X).$$

Thus, from (4.15) and (4.16), we deduce that

$$(4.17) \quad g(A_{NX}Y + A_{NY}X, U) = 0.$$

This equation gives (4.13). □

**Theorem 4.6.** Let $M$ be a hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the anti-invariant distribution $\mathcal{D}^\perp$ is integrable if and only if

$$(4.18) \quad A_{NX}Y = A_{NY}X$$

for all $X, Y \in \mathcal{D}^\perp$.

**Proof.** From (4.1), using (3.7-a), we have

$$(4.19) \quad -A_{NY}X = T\nabla_X Y + h(X, Y)$$

for all $X \in \mathcal{D}^\perp$. By interchanging $X$ and $Y$ in (4.19), then subtracting it from (4.19) we obtain

$$(4.20) \quad A_{NX}Y - A_{NY}X = T[X, Y].$$

Because of (3.7-a), we know that $\mathcal{D}^\perp$ is integrable if and only if $T[X, Y] = 0$ for all $X, Y \in \mathcal{D}^\perp$. So, our assertion comes from (4.20). □

By Lemma 4.5 and Theorem 4.6, we have the following result.

**Corollary 4.7.** Let $M$ be a hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the anti-invariant distribution $\mathcal{D}^\perp$ is integrable if and only if

$$(4.21) \quad A_{NX}Y = 0$$

for all $X, Y \in \mathcal{D}^\perp$.

Now, we give main result of this section.

**Theorem 4.8.** Let $M$ be a hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the anti-invariant distribution $\mathcal{D}^\perp$ is always integrable.
Corollary 4.9. Let $\bar{M}$ be a l.p.R. manifold with Riemannian metric $g$ and almost product structure $F$. Define the symmetric $(0,2)$-type tensor field $\Omega$ by $\Omega(U, V) = g(FU, V)$ on the tangent bundle $TM$. It is not difficult to see that $(\nabla_U \Omega)(V, \bar{W}) = g((\nabla_U F)\bar{V}, \bar{W})$ on $TM$. Thus, because of (2.3), we deduce that

$$3d\Omega(V, \bar{W}; \bar{U}) = \mathcal{G}(\nabla_U \Omega)(V, \bar{W}) = 0$$

for all $U, \bar{V}, \bar{W} \in TM$, that is, $d\Omega \equiv 0$, where $\mathcal{G}$ denotes the cyclic sum over $U, \bar{V}, \bar{W} \in TM$. Next, for any $X, Y \in D^\perp$ and $U \in TM$ we have

$$0 = 3d\Omega(U, X, Y) = U \Omega(X, Y) + X \Omega(Y, U) + Y \Omega(U, X)$$

$$\quad - \Omega([U, X], Y) - \Omega([X, Y], U) - \Omega(Y, U) - \Omega(U, X)$$

$$\quad - g(T[Y, X], U).$$

It follows that $T[X, Y] = 0$ and because of (3.7-a), $[Y, X] \in D^\perp$. \qed

We remark that we used Tripathi’s technique [8] in the proof above.

Corollary 4.9. Let $M$ be a hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the following facts hold:

(4.22) $A_N D^\perp D^\perp = 0$

(4.23) $A_N X Z \in D^\theta$, i.e., $A_N D^\perp D^\theta \subseteq D^\theta$

and

(4.24) $g(h(TM, D^\perp), ND^\perp) = 0,$

where $X \in D^\perp$ and $Z \in D^\theta$. \newline

Proof. (4.22) follows from Corollary 4.7 and Theorem 4.8. (4.23) follows from (4.22). Finally, using (2.6), (4.22) gives (4.24). \qed

Next, we give another application of Theorem 4.8.

Theorem 4.10. Let $M$ be a proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. The anti-invariant distribution $D^\perp$ defines a totally geodesic foliation on $M$ if and only if $h(D^\perp, D^\perp) \perp ND^\theta$.

Proof. For $X, Y \in D^\perp$, we put $\nabla_X Y = \nabla^\perp_X Y + \theta \nabla_X Y$, where $\nabla^\perp_X Y$ (resp. $\theta \nabla_X Y$) denotes the anti-invariant (resp. slant) part of $\nabla_X Y$. Then using Lemma 3.3 and (3.5), for any $Z \in D^\theta$ we have

(4.25) $g(\nabla_X Y, Z) = g(\theta \nabla_X Y, Z) = \frac{1}{\cos \theta} g(T \theta \nabla_X Y, T Z) = \frac{1}{\cos \theta} g(T \nabla_X Y, T Z).$

On the other hand, from (4.1), we have

(4.26) $T \nabla_X Y + t h(X, Y) = -A_N Y X = 0$,

since the distribution $D^\perp$ is integrable. So, using (4.26), from (4.25), we get

(4.27) $g(\nabla_X Y, Z) = -\frac{1}{\cos \theta} g(T h(X, Y), T Z) = -\frac{1}{\cos \theta} g(F h(X, Y), T Z).$

Here, using (2.2), (2.11) and (3.4), we find

(4.28) $g(F h(X, Y), T Z) = g(h(X, Y), NT Z).$

From (4.27) and (4.28), we get

(4.29) $g(\nabla_X Y, Z) = -\frac{1}{\cos \theta} g(h(X, Y), NT Z).$
Since $TZ \in \mathcal{D}^\theta$, our assertion comes from (1.29). □

5. HEMI-SLANT PRODUCT

In this section, we give a necessary and sufficient condition for a proper hemi-slant submanifold to be a hemi-slant product.

**Definition 5.1.** A proper hemi-slant submanifold $M$ of a l.p.R. manifold $\bar{M}$ is called a hemi-slant product if it is locally product Riemannian of an anti-invariant submanifold $M_\perp$ and a proper slant submanifold $M_\theta$ of $\bar{M}$.

Now, we are going to examine the problem when a proper hemi-slant submanifold of a l.p.R. manifold is a hemi-slant product?

We first give a result which is equivalent to Theorem 4.10.

**Theorem 5.2.** Let $M$ be a proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the anti-invariant $\mathcal{D}^\perp$ defines a totally geodesic foliation on $M$ if and only if

$$g(A_{NY}Z, X) = -g(A_{NZ}Y, X),$$

where $X, Y \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$.

**Proof.** For any $X, Y \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$, using (2.4), (2.2), and (2.3), we have

$$g(\nabla_X Y, Z) = g(\nabla_X F Y, F Z).$$

Hence, using (2.11) and (3.1), we obtain

$$g(\nabla_X Y, Z) = -g(A_{NY}X, TZ) + g(\nabla_X Y, FNZ) + g(h(X, Y), FNZ).$$

Here, using (3.3)-c, (3.3)-a, (2.12) and (3.4), we have

$$FNZ = tNZ - NTZ$$
and
$$tNZ = Z - T^2 Z = \sin^2 \theta Z.$$

Thus, with the help of (2.6), we get

$$g(\nabla_X Y, Z) = -g(A_{NY}X, TZ) + \sin^2 \theta g(\nabla_X Y, Z) - g(A_{NZ}Y, X).$$

After some calculations, we find

$$\cos^2 \theta g(\nabla_X Y, Z) = -g(A_{NY}TZ, X) - g(A_{NZ}Y, X).$$

It follows that the distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation on $M$ if and only if

$$g(A_{NY}TZ, X) = -g(A_{NZ}Y, X).$$

Putting $Z = TZ$ in (5.2), we obtain (5.1) and vice versa. □

**Theorem 5.3.** Let $M$ be a proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the distribution $\mathcal{D}^\theta$ defines a totally geodesic foliation on $M$ if and only if

$$g(A_{NX}W, Z) = -g(A_{NW}X, Z),$$

where $X, Y \in \mathcal{D}^\perp$ and $Z, W \in \mathcal{D}^\theta$.

**Proof.** Using (2.4), (2.2), and (2.3), we have $g(\nabla_Z W, X) = g(\nabla_Z FW, FX)$ for any $Z, W \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^\perp$. Next, using (2.11) and (3.1), we obtain $g(\nabla_Z W, X) = -g(TW, \nabla_Z NX) - g(NW, \nabla_Z FX)$. Hence, using (2.5) and (2.4), we get $g(\nabla_Z W, X) = g(TW, A_{NX}Z) - g(FNW, \nabla_Z X)$. With the help of (2.12), (3.3)-(a), (3.3)-(c) and (2.4), we arrive at

$$g(\nabla_Z W, X) = -g(A_{NX}Z, TW) - \sin^2 \theta g(\nabla_Z X, W) + g(h(X, Z), NTW).$$
Upon direct calculation, we find
\[ \cos^2 \theta \ g(\nabla_Z W, X) = g(A_{NX} TW, Z) + g(A_{NTW} X, Z) \]
So, we deduce that the slant distribution \( \mathcal{D}^\theta \) defines a totally geodesic foliation if and only if
\[ g(A_{NX} TW, Z) = -g(A_{NTW} X, Z), \]
By putting \( W = TW \), we see that the last equation is equivalent to the equation (5.3).

Thus, from Theorems 5.2 and 5.3, we obtain the expected result.

**Corollary 5.4.** Let \( M \) be a proper hemi-slant submanifold of a l.p.R. manifold \( \bar{M} \). Then \( M \) is a hemi-slant product manifold \( M = M_\perp \times M_\theta \) if and only if
\[ A_{NX} Z = -A_{NZ} X, \]
where \( X \in \mathcal{D}^\perp \) and \( Z \in \mathcal{D}^\theta \).

6. Hemi-slant submanifolds with parallel canonical structures

In this section, we get several results for the hemi-slant submanifolds with parallel canonical structures using the previous results.

Let \( M \) be any submanifold of a l.p.R. manifold \( \bar{M} \) with the endomorphism \( T \) and the normal bundle valued 1-form \( N \) defined by (2.11). We put
\[ (\nabla_U T)V = \nabla_U TV - T \nabla_U V \]
and
\[ (\nabla_U N)V = \nabla_U^\perp NV - N \nabla_U V \]
for any \( U, V \in TM \). Then the endomorphism \( T \) (resp. 1-form \( N \)) is parallel if \( \nabla T \equiv 0 \) (resp. \( \nabla N \equiv 0 \)). From (6.1) and (6.2) we have
\[ (\nabla_U T)V = A_{NV} U + th(U, V) \]
and
\[ (\nabla_U N)V = \omega h(U, V) - h(U, TV), \]
respectively.

**Theorem 6.1.** Let \( M \) be any submanifold of a l.p.R. manifold \( \bar{M} \). Then \( T \) is parallel, i.e., \( \nabla T \equiv 0 \) if and only if
\[ A_{NV} U = -A_{NU} V, \]
for all \( U, V \in TM \).

**Proof.** For any \( U, V, W \in TM \) from (6.3), we have
\[ g((\nabla_W T)V, U) = g(A_{NVW} U, U) + g(th(W, V), U). \]
Hence, using (2.11), (2.2) and (2.11), we obtain
\[ g((\nabla_W T)V, U) = g(A_{NVW} U, U) + g(h(W, V), NU). \]
Since \( A \) is self-adjoint, with the help of (2.6), we get
\[ g((\nabla_W T)V, U) = g(A_{NVU} W, U) + g(A_{NU} V, W). \]
Thus, our assertion comes from (6.6). \( \square \)
Theorem 6.2. Let $M$ be a proper hemi-slant submanifold of a l.p.R. manifold $\mathcal{M}$. If $T$ is parallel, then $M$ is a hemi-slant product. The converse is true, if $h(D^\theta, D^\theta) \perp N D^\theta$.

Proof. Let $X$ be in $D^\perp$ and $Z$ in $D^\theta$. If $T$ is parallel, then from (6.9), we have

$$A_{NZ}Z = -A_{NZ}X.$$  

(6.7)

Thus, by Corollary 5.4, we conclude that $M$ is a hemi-slant product. Conversely, if $M$ is a hemi-slant product and $h(D^\theta, D^\theta) \perp N D^\theta$, then for any $Z, W$ and $V \in D^\theta$, we have $g(A_{NZ}W, V) = g(h(V, W), NZ) = 0$. It means that $A_{NZ}W \in D^\perp$. Now, let calculate $g(A_{NZ}W, X)$ for $X \in D^\perp$. Since $M$ is a hemi-slant product and $A$ is self-adjoint $g(A_{NZ}W, X) = g(A_{NZ}X, W) = -g(A_{NX}Z, W) = -g(A_{NX}W, Z) = -g(A_{NW}X, Z)$.

Hence, we deduce

$$A_{NZ}W = -A_{NW}Z,$$

for all $Z, W \in D^\theta$.

Thus, from (4.13), (6.7) and (6.8), we obtain (6.5) and by Theorem 6.1, $T$ is parallel. $\square$

Theorem 6.3. Let $M$ be a proper hemi-slant submanifold of $\mathcal{M}$. If $N$ is parallel, then

(a) $A_{\mu}D^\perp = 0$, (b) $A_{ND^\theta}D^\perp = 0$, (c) $A_{ND^\perp}D^\theta = 0$,

(d) $M$ is a hemi-slant product, (e) $M$ is $(D^\perp, D^\theta)$-mixed totally geodesic.

Proof. Let $N$ be parallel, it follows from (6.4) that

$$h(U, TV) = \omega h(U, V)$$

for any $U, V \in TM$. Then, for any $X \in D^\perp$, we have

$$\omega h(U, X) = 0$$

from (6.9). For any $\xi \in \mu$, using (2.11), (2.2) and (2.6), we have

$$g(\omega h(U, X), \xi) = g(h(U, X), F\xi) = g(A_{F\xi}X, U).$$

Thus, using (6.10) we get

$$g(A_{F\xi}X, U) = 0.$$  

(6.11)

Since $\mu$ is invariant with respect to $F$, the assertion (a) comes from (6.11). Now, take $Z \in D^\theta$, after some calculations, we find

$$g(A_{NZ}X, U) = g(\omega h(U, X), NZ).$$

So, using (6.10), we get $g(A_{NZ}X, U) = 0$, which is equivalent to the assertion (b). On the other hand, for any $X \in D^\perp$, using (2.2), (2.11), (2.12) and (6.9), we have

$$0 = g(h(U, Z), X) = g(Fh(U, Z), FX) = g(\omega h(U, Z), FX) = g(h(U, TZ), FX) = g(h(U, TZ), NX),$$

that is, $g(h(U, TZ), NX) = 0$. Putting $Z = TZ$ in last equation, we obtain

$$\cos^2 \theta g(h(U, Z), NX) = \cos^2 \theta g(A_{NX}Z, U) = 0.$$  

(6.12)

Since $\theta \neq \frac{\pi}{2}$, the assertion (c) follows. The assertion (d) follows from the assertions (b), (c) and (5.5). Lastly, using (5.4), from (6.9), we have
\[ \omega^2 h(X, Z) = \omega h(X, T Z) = h(X, T^2 Z) = \cos^2 \theta h(X, Z). \] On the other hand, using (3.7)-(a), we have \[ \omega^2 h(X, Z) = \omega^2 h(Z, X) = \omega h(Z, T X) = 0. \] Thus, we get \[ \cos^2 \theta h(X, Z) = 0. \] Since \( \theta \neq \frac{\pi}{2} \), we deduce that \( h(X, Z) = 0 \), which proves that the last assertion.

7. Totally Umbilical Hemi-Slant Submanifolds

In this section we shall give two characterization theorems for the totally umbilical proper hemi-slant submanifolds of a l.p.R. manifold. First we prove

**Theorem 7.1.** If \( M \) is a totally umbilical proper hemi-slant submanifold of a l.p.R. manifold \( \bar{M} \), then either the anti-invariant distribution \( \mathcal{D}^\perp \) is 1-dimensional or the mean curvature vector field \( H \) of \( M \) is perpendicular to \( F(\mathcal{D}^\perp) \). Moreover, if \( M \) is a hemi-slant product, then \( H \in \mu \).

**Proof.** Since \( M \) is a totally umbilical proper hemi-slant submanifold either \( \text{Dim}(\mathcal{D}^\perp) = 1 \) or \( \text{Dim}(\mathcal{D}^\perp) > 1 \). If \( \text{Dim}(\mathcal{D}^\perp) = 1 \), it is obvious. If \( \text{Dim}(\mathcal{D}^\perp) > 1 \), then we can choose \( X, Y \in \mathcal{D}^\perp \) such that \( \{X, Y\} \) is orthonormal. By using (2.11), (2.7), (2.8) and (4.22), we have

\[
(7.1) \quad g(H, FY) = g(h(X, X), NY) = g(A_{NY}X, X) = 0
\]

It means that

\[
(7.2) \quad H \perp F(\mathcal{D}^\perp).
\]

Moreover, if \( M \) is a hemi-slant product, for any \( Z \in \mathcal{D}^\theta \), using (5.5) and (2.7), we have

\[
(7.3) \quad H \perp N(\mathcal{D}^\theta).
\]

Thus, using (7.2) and (7.3) from (3.2), we get \( H \in \mu \).

Before giving the second result of this section, recall that the following fact about locally product Riemannian manifolds.

Let \( M_1(c_1) \) (resp. \( M_2(c_2) \)) be a real space form with sectional curvature \( c_1 \) (resp. \( c_2 \)). Then the Riemannian curvature tensor \( \bar{R} \) of the locally product Riemannian manifold \( \bar{M} = M_1(c_1) \times M_2(c_2) \) has the form

\[
(7.4) \quad \bar{R}(\bar{U}, \bar{V})\bar{W} = \frac{1}{4}(c_1+c_2) \left\{ g(\bar{V}, \bar{W})\bar{U} - g(\bar{U}, \bar{W})\bar{V} + g(F\bar{V}, \bar{W})F\bar{U} - g(F\bar{U}, \bar{W})F\bar{V} \right\}
\]

\[ + \frac{1}{4}(c_1-c_2) \left\{ g(F\bar{V}, \bar{W})\bar{U} - g(\bar{U}, \bar{W})\bar{V} + g(\bar{V}, \bar{W})F\bar{U} - g(F\bar{U}, \bar{W})F\bar{V} \right\},
\]

where \( \bar{U}, \bar{V}, \bar{W} \in TM \) \([22]\).

**Theorem 7.2.** Let \( M \) be a totally umbilical hemi-slant submanifold with parallel mean curvature vector field \( H \) of a l.p.R. manifold \( \bar{M} = M_1(c_1) \times M_2(c_2) \) with \( c_1 \neq c_2 \). Then, \( M \) can not be proper.
Proof. Let $X \in D^\perp$ and $Z \in D^\theta$ be two unit vector fields. Since $H$ is parallel, using (2.10) and (2.7) from the Codazzi equation (2.9), we have

\[(\mathcal{R}(X, Z)X)^\perp = -\nabla^\perp Z H = 0.\]

(7.5)

On the other hand, the equation (7.4) gives

\[\mathcal{R}(X, Z)X = -\frac{1}{4} \left( (c_1 + c_2)Z + (c_1 - c_2)FZ \right).\]

(7.6)

Taking the normal component of (7.6), we get

\[(\mathcal{R}(X, Z)X)^\perp = -\frac{1}{4}(c_1 - c_2)NZ,\]

which contradicts (7.5). \hfill \Box

We have immediately from Theorem 7.2. that:

**Corollary 7.3.** There exists no totally geodesic proper hemi-slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$ with $c_1 \neq c_2$.

8. Ricci curvature of hemi-slant submanifolds

In this section, we obtain a basic inequality involving Ricci curvature and the squared mean curvature of a hemi-slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. We first represent the following fundamental facts about this topic.

Let $\bar{M}$ be a $n$-dimensional Riemannian manifold equipped with a Riemannian metric $g$ and $\{e_1, ..., e_n\}$ be an orthonormal basis for $T_p\bar{M}$, $p \in \bar{M}$. Then the Ricci tensor $\overline{S}$ is defined by

\[\overline{S}(U, V) = \sum_{i=1}^{n} \mathcal{R}(e_i, U, V, e_i),\]

where $U, V \in T_p\bar{M}$. For a fixed $i \in \{1, ..., n\}$, the Ricci curvature of $e_i$, denoted by $\overline{\text{Ric}}(e_i)$, is given by

\[\overline{\text{Ric}}(e_i) = \sum_{i \neq j}^{n} \overline{K}_{ij},\]

where $\overline{K}_{ij} = g(\mathcal{R}(e_i, e_j)e_j, e_i)$ is the sectional curvature of the plane spanned by the plane spanned by $e_i$ and $e_j$ at $p \in \bar{M}$. Let $\Pi_k$ be a $k$-plane of $T_p\bar{M}$ and $\{e_1, ..., e_k\}$ any orthonormal basis of $\Pi_k$. For a fixed $i \in \{1, ..., k\}$, the $k$-Ricci curvature $\overline{\text{Ric}}_{\Pi_k}(e_i)$ of $\Pi_k$ at $e_i$, denoted by $\overline{\text{Ric}}_{\Pi_k}(e_i)$, is defined by

\[\overline{\text{Ric}}_{\Pi_k}(e_i) = \sum_{i \neq j}^{k} \overline{K}_{ij}.\]

(8.3)

It is easy to see that $\overline{\text{Ric}}_{(T_p\bar{M})}(e_i) = \overline{\text{Ric}}(e_i)$ for $1 \leq i \leq n$, since $\Pi_n = T_p\bar{M}$.

We now recall that the following basic inequality [10, Theorem 3.1] involving Ricci curvature and the squared mean curvature of a submanifold of a Riemannian manifold.
Theorem 8.1. ([10, Theorem 3.1]) Let $M$ be an $m$-dimensional submanifold of a Riemannian manifold $\bar{M}$. Then, for any unit vector $X \in T_pM$, we have

\begin{equation}
\text{Ric}(X) \leq \frac{1}{4}m^2\|H\|^2 + \overline{\text{Ric}}(\bar{T}_p\bar{M})(X)
\end{equation}

where $\text{Ric}(X)$ is the Ricci curvature of $X$.

Of course, the equality case of (8.4) was also discussed in [10], but we will not deal with the equality case in this paper.

Now, we are ready to state main result of this section.

Theorem 8.2. Let $M$ be an $m$-dimensional hemi-slant submanifold of a l.p.R. manifold $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$. Then, for unit vector $V \in T_p\bar{M}$, we have

\begin{equation}
4\text{Ric}(V) \leq m^2\|H\|^2 + (c_1 + c_2)\left\{(m - 1)\sum_{i=2}^{m} g(Te_i, e_i)g(TV, V)
- \|TV\|^2 + g(TV, V)\right\} + (c_1 - c_2)\left\{\sum_{i=2}^{m} g(Te_i, e_i) + (m - 1)g(TV, V)\right\}
\end{equation}

where $\{V, e_2, ..., e_m\}$ is an orthonormal basis for $T_p\bar{M}$.

Proof. Let $M$ be an $m$-dimensional hemi-slant submanifold of a l.p.R. manifold $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$. Then, for any unit vector $V \in T_pM$, using (7.4) and (2.11) from (8.3) we have

\begin{equation}
4\overline{\text{Ric}}_{\bar{T}_p\bar{M}}(V) = (c_1 + c_2)\left\{(m - 1)\sum_{i=2}^{m} g(Te_i, e_i)g(TV, V)
- \|TV\|^2 + g(TV, V)\right\} + (c_1 - c_2)\left\{\sum_{i=2}^{m} g(Te_i, e_i) + (m - 1)g(TV, V)\right\}
\end{equation}

Thus, using (8.6) in (8.4) we get (8.5). \hfill \Box

Remark 8.3. In general, $g(F\bar{V}, \bar{V}) \neq 0$ for any unit vector $\bar{V} \in T_p\bar{M}$ in a l.p.R. manifold $\bar{M}$, contrary to almost Hermitian $(g(J\bar{V}, \bar{V}) = 0)$ and almost contact ($(g(\varphi \bar{V}, \bar{V}) = 0)$) manifolds. However, we can establish that the almost product structure $F$ in a l.p.R. manifold $\bar{M}$ such that $g(F\bar{V}, \bar{V}) = 0$, for all $\bar{V} \in T_p\bar{M}$. In fact, if $\bar{M}$ is an even dimensional l.p.R. manifold with an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2n}\}$, then we can define $F$ by

\begin{equation}
F(e_i) = e_{n+j}, \quad F(e_{n+j}) = e_j, \quad j \in \{1, 2, ..., n\}.
\end{equation}

Hence, we observe easily that the almost product structure $F$ satisfies

\begin{equation}
g(Fe_j, e_j) = 0.
\end{equation}

For example, the almost product structure $F$ in example of section 3, satisfies the condition (8.7). On the other hand, because of Lemma 3.3 and the equation (3.5), we have $TV = 0$, if $V \in D^\perp$ and $\|TV\|^2 = \cos^2\theta$, if $V \in D^\theta$ and $\|V\| = 1$, respectively. Thus, by Theorem 8.2 we get the following two results.
Corollary 8.4. Let $M$ be an $m$-dimensional anti-invariant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. If the almost product structure $F$ of $\bar{M}$ satisfies the condition (8.7), then we have

$$4\text{Ric}(V) \leq m^2\|H\|^2 + (c_1 + c_2)(m - 1),$$

where $V \in T_pM$ is any unit vector.

Corollary 8.5. Let $M$ be an $m$-dimensional slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. If the almost product structure $F$ of $\bar{M}$ satisfies the condition (8.7), then we have

$$4\text{Ric}(Z) \leq m^2\|H\|^2 + (c_1 + c_2)((m - 1) - \cos^2\theta),$$

where $Z \in T_pM$ is any unit vector.

**References**

1. T. Adati, Submanifolds of an almost product manifold, *Kodai Math. J.* 4 (1981), no. 2, 327–343.
2. P. Alegre, Slant submanifolds of Lorentzian Sasakian and Para-Sasakian manifolds, *Taiwanese J. Math.* 17 (2013), no. 3, 897–910. DOI:10.11650/tjm.17.2013.2427.
3. F.R. Al-Solamy, M. A. Khan and S. Uddin, Totally umbilical hemi-slant submanifolds of Kähler manifolds, *Abstr. Appl. Anal.* 2011, Art. ID 987157, 9 pp.
4. K. Arslan, A. Carriazo, B. Y. Chen and C. Murathan, On slant submanifolds of neutral Kähler manifolds, *Taiwanese J. Math.* 17 (2010), no. 2, 561-584.
5. A. Bejancu, Semi-invariant submanifolds of locally product Riemannian manifolds, *An. Univ. Timișoara Ser. Științ. Mat.* 22 (1984), no. 1-2, 3–11.
6. J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, Slant submanifolds in Sasakian manifolds, *Glasgow Math. J.* 42 (2000), 125-138.
7. A. Carriazo, Bi-slant immersions, in: Proc. ICRAMS 2000, Kharagpur, India, 2000, 88–97.
8. B.Y. Chen, Geometry of slant submanifolds, *Katholieke Universiteit Leuven*, 1990.
9. B.Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions, *Glasgow Math. J.* 41 (1999), 33-41.
10. S. Hong, M.M. Tripathi, On Ricci curvature of submanifolds, *Internat. J. Pure Appl. Math. Sci.* 2 (2005), no. 2, 227–246.
11. V.A. Khan, M.A. Khan, Pseudo-slant submanifolds of a Sasakian manifold, *Indian J. Pure Appl. Math.,* 38 (2007), 31–42.
12. H. Li, X. Liu, Semi-slant submanifolds of a locally product manifold, *Georgian Math. J.* 12 (2005), no. 2, 273–282.
13. A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Română*, 39(1996), 183-198.
14. N. Papaghiuc, Semi-slant submanifolds of a Kählerian manifold, *Ann. Şt. Al. I. Cuza Univ. Iaşi*, 40 (1994), 55–61.
15. G. Pitis, On some submanifolds of a locally product manifold, *Kodai Math. J.* 9 (1986), 327–333.
16. G.S. Ronse, Generic and skew CR-submanifolds of a Kähler manifold, *Bull. Inst. Math. Acad. Sinica*, 18 (1990), 127–141.
17. B. Şahin, Slant submanifolds of an almost product Riemannian manifold, *Bull. Korean Math. Soc.* 43 (2006), no. 4, 717–732.
18. B. Şahin, Warped product submanifolds of a Kähler manifold with a slant factor, *Ann. Pol. Math.* 95 (2009), no. 3, 207–226.
19. H.M. Taşkân, The axiom of hemi-slant 3-spheres in almost Hermitian geometry, *Bull. Malays. Math. Sci. Soc.* (2) 37 (2) (2014), 555–564.
20. M.M. Tripathi, Generic submanifolds of generalized complex space forms, *Publ. Math. Debrecen*, 50 (1997), no. 3-4, 373–392.
21. S. Uddin, M. A. Khan and K. Singh, A note on totally umbilical pseudo-slant submanifolds of a nearly Kähler manifold, *Acta Univ. Apulensis Math. Inform.* No. 29 (2012), 279-285.
22. K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore, 1984.
İstanbul University, Department of Mathematics, Vezneciler, İstanbul, Turkey
E-mail address: hakmete@istanbul.edu.tr

Department of Mathematics, İstanbul Technical University, Maslak, İstanbul, Turkey
E-mail address: fozdemir@itu.edu.tr