ON CANONICAL BASES OF LETZTER ALGEBRA $U^i(\mathfrak{sl}_2)$

YIQIANG LI†

INTRODUCTION

Let $U^i \equiv U^i(\mathfrak{sl}_2)$ be Letzter’s coideal subalgebra of quantum $\mathfrak{sl}_2$ corresponding to the symmetric pair $(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C})$ ([Le02]). As a subalgebra of quantum $\mathfrak{sl}_2$, $U^i$ is generated by the sum $E + vKF + K$ of standard generators, and hence can be identified with the polynomial ring $\mathbb{Q}(v)[t]$. In [BW18] and [LW18], two distinguished bases, called canonical bases, are constructed inside the modified form of $U^i$ via algebraic and geometric approaches respectively. The modified form of $U^i$ can be identified with a direct sum of two copies of $U^i \cong \mathbb{Q}(v)[t]$ itself. An explicit and elegant formula, as a polynomial in $t$, of algebraic basis elements is conjectured in [BW18] and proved in [BeW18]. The purpose of this short paper is to show itself. An explicit and elegant formula, as a polynomial in $t$, of algebraic basis elements is conjectured in [BW18] and proved in [BeW18]. The purpose of this short paper is to show that the geometric basis in [LW18] admits the same description and, consequently, that the two bases coincide. Notice that the proofs are within the scope of loc. cit. and [BKLW], whose notations shall be adopted here.

THE DESCRIPTION

Set $[n] = (v^n - v^{-n})/(v - v^{-1})$ and $[n]! = \prod_{i=1}^{n}[i]$. Let $U^j(\mathfrak{sl}_3)$ be an associative algebra over $\mathbb{Q}(v)$ generated by $e, f, k, k^{-1}$ and subject to the following defining relations.

$$kk^{-1} = 1, \hspace{1cm} ke = v^3ek, \hspace{1cm} kf = v^{-3}fk,$$

$$e^2f - [2]efe + fe^2 = -[2]e(vk + v^{-1}k^{-1})$$

$$f^2e - [2]efe + ef^2 = -[2](vk + v^{-1}k^{-1})f.$$  

Let $e^{(n)} = e^n/[n]!$ and $f^{(n)} = f^n/[n]!$. By an induction argument and making use the above inhomogeneous Serre relations, we have the following formula in $U^j(\mathfrak{sl}_3)$.

**Lemma 1.** We have $fe^{(n+1)} = e^{(n)}(fe - vef - [n](v^nk + v^{-n}k^{-1}) + v^{n+1}e^{(n+1)}f$.

Let $S^j_{3,d}$ be $S^j$ in [BKLW] 3.1 for $n = 1$ and $e, f, k = d_1d_2^{-1}$ its generators. The assignments $e \mapsto e, f \mapsto f$ and $k^{\pm 1} \mapsto k^{\pm 1}$ define an algebra homomorphism $U^j(\mathfrak{sl}_3) \rightarrow S^j_{3,d}$. We set

$$A_{a,b} = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & a \end{bmatrix}, \hspace{1cm} a, b \in \mathbb{N}.$$  

Let $j_d = [A_{d,0}]$, an idempotent in $S^j_{3,d}$ (loc. cit. 3.17). We consider the subalgebra $S^j_{3,d} = j_dS^j_{3,d}j_d$ and its integral form $\mathcal{A}S^j_{2,d}$ with $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ ([LW18] 4.1)]. Let

$$t_d = \left( \frac{fe + k - k^{-1}}{v - v^{-1}} \right) j_d \in S^j_{2,d}.$$  

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Lemma 2. In $S^1_{2, d}$, one has $f^{(n)}e^{(n)}j_d = (t_d + [d - 1])(t_d + [d - 3]) \cdots (t_d + [d - 2n + 1])/[n]!$.

Proof. When $n = 1$, this is the defining relation for $t_d$ in [BKLW, Thm. 3.7(a)]. Assume the statement holds for $n$. Due to Lemma 1, $f_j = 0$ and $k_d = v^{1-d}$, we have

$$f^{(n+1)}e^{(n+1)}j_d = \frac{1}{[n+1]}f^{(n)}f_j e^{(n)}j_d = \frac{1}{[n+1]}f^{(n)}e^{(n)}(fe - [n](v^n v^{1-d} + v^{-n} v^{d-1}))j_d$$

$$= f^{(n)}e^{(n)}(t_d + [d - 1] - [n](v^n v^{1-d} + v^{-n} v^{d-1}))j_d/[n + 1]$$

$$= (t_d + [d - 1])(t_d + [d - 3]) \cdots (t_d + [d - 2n + 1])/[n + 1]!.$$

Lemma follows by induction. □

Lemma 2 provides a characterization of $S^1_{2, d}$ as follows.

Proposition 3. The algebra $S^1_{2, d}$ is isomorphic to the quotient algebra of $Q(v)[t]$ by the ideal generated by the polynomial $(t + [d - 1])(t + [d - 3]) \cdots (t + [-d - 1])$.

Proof. The map $t \mapsto t_d$ defines a surjective algebra homomorphism $\phi_d : Q(v)[t] \to S^1_{2, d}$. Due to $f^{(d+1)}e^{(d+1)}j_d = 0$ and Lemma 1, the polynomial above is zero in $S^1_{2, d}$ for $t = t_d$. So $\phi_d$ factors through the desired quotient. Clearly the dimensions of $S^1_{2, d}$ and the quotient algebra are the same, so they must be isomorphic. The proposition is thus proved. □

Define an equivalence relation on the set $\{A_{a,b} | a, b \in \mathbb{N}\}$ by $A_{a,b} \sim A_{a',b'}$ if $a \equiv a' \pmod{2}$ and $b = b'$. Let $\tilde{A}_{a,b}$ be the equivalence class of $A_{a,b}$. As a $Q(v)$-vector space, the modified form $\tilde{U}^i$ of $U^i$ is spanned by the canonical basis elements $b_{\tilde{A}_{0,d}}$ and $b_{\tilde{A}_{1,d}}$ for $d \in \mathbb{N}$. Let $\tilde{U}_0^i = \text{Span}\{b_{\tilde{A}_{0,d}}, b_{\tilde{A}_{1,d+1}} | d \text{ even}\}$ and $\tilde{U}_1^i = \text{Span}\{b_{\tilde{A}_{0,d}}, b_{\tilde{A}_{1,d-1}} | d \text{ odd}\}$. Then $\tilde{U}^i = \tilde{U}_0^i \oplus \tilde{U}_1^i$ as algebras. We have an isomorphism $\tilde{U}_0^i \to Q(v)[t]$ (resp. $\tilde{U}_1^i \to Q(v)[t]$) via $b_{\tilde{A}_{1,i}} \mapsto t$ (resp. $b_{\tilde{A}_{0,i}} \mapsto t$). The isomorphisms are compatible with $\phi_d$. We have the following explicit description of geometrically-defined canonical basis elements of $\tilde{U}^i$.

Theorem 4. The canonical basis elements of $\tilde{U}^i \equiv U^i(sI_2)$ in [LW18] are of the form

$$(1) \quad b_{\tilde{A}_{0,d}} = \frac{(t + [d - 1])(t + [d - 3]) \cdots (t + [d - 3])(t + [-d + 1])}{[d]!}, \quad \forall d \in \mathbb{N};$$

$$(2) \quad b_{\tilde{A}_{1,d+1}} = \frac{t \cdot (t + [d - 1])(t + [d - 3]) \cdots (t + [d - 3])(t + [-d + 1])}{[d + 1]!}, \quad \forall d \in \mathbb{N}.$$

Proof. Let us denote the polynomial in (1) by $P_{0,d}(t)$ and that in (2) by $P_{1,d+1}(t)$. By Lemma 2 we have $f^{(d)}e^{(d)}j_d = P_{0,d}(t_d)$. Observe that the element $f^{(d)}e^{(d)}j_d$ is a canonical basis element in $S^1_{2, d}$, corresponding to the constant sheaf on the product of maximal isotropic Grassmannians. Indeed, by [BKLW Thm. 3.7(a)], we have $f^{(d)}e^{(d)}j_d = \sum_{i=0}^{d} v^{-(\frac{i}{2})}[A_{i,d-i}]$. By [LW18 Prop. 6.3], the canonical basis elements of $\tilde{U}^i$ are sent to canonical basis elements in $S^1_{2, d}$ or zero via $\phi_d$. So $b_{\tilde{A}_{0,d}} = P_{0,d}(t)$, and (1) holds.

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Now consider the element $P_{1,d+1}(t_{d+2})$ in $S^i_{2,d+2}$. By rewriting the factor $t$ in $P_{1,d+1}(t)$ as $(t + [d + 1]) - [d + 1]$ and combining with the remaining terms, there is

$$P_{1,d+1}(t_{d+2}) = f^{(d+1)}e^{(d+1)}j_{d+2} - P_{0,d}(t_{d+2}).$$

Hence $P_{1,d+1}(t_{d+2})$ is in the integral form $A S^i_{2,d}$. Further, the polynomial $P_{1,d+1}(t_{d+2})$ can be rewritten as

$$P_{1,d+1}(t_{d+2}) = P_{0,d}(t_{d+2}) + \left( (t_{d+2} + [d - 1])(t_{d+2} + [d - 3]) \cdots (t_{d+2} + [d - 1]) \right).$$

With the above expression, Proposition 3 and the property of the transfer map $\phi^i_{d+2,d}$ in [LW18 (6.6)] and [FL19], we must have

$$\phi^i_{d+2,d}(P_{1,d+1}(t_{d+2})) = P_{0,d}(t_d).$$

By the definition of $P_{1,d+1}(t)$, there is $P_{1,d+1}(t_{d+2}) = \frac{1}{[d+1]}t_{d+2} \ast \{A_{2,d}\}$ and so, in light of the positivity property in [LW18 Theorem 6.12], it leads to

$$P_{1,d+1}(t_{d+2}) = \{A_{1,d+1}\} + \sum_{i > 1} c_i \{A_{i,d+2-i}\}, \quad c_i \in \mathbb{N}[v, v^{-1}].$$

The positivity of $c_i$ together with (3) and loc. cit. (6.6) implies that $c_i = 0$ and so

$$P_{1,d+1}(t_{d+2}) = \{A_{1,d+1}\} \in S^i_{2,d+2}.$$

Therefore, we have $b_{A_{1,d+1}} = P_{1,d+1}(t)$ and (2) holds. The proof is finished.

We conclude the paper with a remark.

**Remark 5.** (1) The canonical basis of $S^i_{2,d}$ is $\{P_{0,d-2i}(t_d), P_{1,d-2i-1}(t_d)\}$.

(2) Clearly $b_{A_{0,1}} \ast b_{A_{0,d}} = [d+1]\{A_{1,d+1}\} = \{[d+2]\{A_{0,d+2}\} + [d+1]\{A_{0,d}\}\}.

(3) The $U^i$ in [BW18] and [LW18] differs by an involution $(E, F, K) \mapsto (F, E, K^{-1})$ on quantum $sl_2$. Hence the canonical bases constructed therein coincide.

**References**

[BW18] H. Bao and W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs*, Astérisque. 402 (2018).

[BKLW] H. Bao, J. Kujawa, Y. Li and W. Wang, *Geometric Schur duality of classical type*, Transform. Groups 23 (2018), no. 2, 329-389. Available at [arXiv:1404.4000](https://arxiv.org/abs/1404.4000).

[BeW18] C. Berman and W. Wang, *Formulae of $i$-divided powers in $U_q(sl_2)$*, J. Pure Appl. Algebra 222 (2018), no. 9, 2667-2702.

[FL19] Z. Fan and Y. Li, *Positivity of canonical bases under comultiplication*, IMRN, available at [here](https://arxiv.org/abs/1511.02434)

[Le02] G. Letzter, *Coideal subalgebras and quantum symmetric pairs*, New directions in Hopf algebras (Cambridge), MSRI publications, vol. 43, Cambridge Univ. Press, 2002, pp. 117-166.

[LW18] Y. Li and W. Wang, *Positivity vs negativity of canonical bases*, Bull. Inst. Math. Acad. Sin. (N.S.) 13 (2018), no. 2, 143-198. Available at [arXiv:1501.00688](https://arxiv.org/abs/1501.00688)

**University at Buffalo, The State University of New York**

**E-mail address:** yiqiang@buffalo.edu