COMPLETE $\kappa$-REDUCIBILITY OF PSEUDOVARIETIES OF THE FORM DRH

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ABSTRACT. We denote by $\kappa$ the implicit signature that contains the multiplication and the $(\omega - 1)$-power. It is proved that for any completely $\kappa$-reducible pseudovariety of groups $H$, the pseudovariety $\text{DRH}$ of all finite semigroups whose regular $\mathcal{R}$-classes are groups in $H$ is completely $\kappa$-reducible as well. The converse also holds. The tools used by Almeida, Costa, and Zeitoun for proving that the pseudovariety of all finite $\mathcal{R}$-trivial monoids is completely $\kappa$-reducible are adapted for the general setting of a pseudovariety of the form $\text{DRH}$.

1. Introduction

The study of finite semigroups goes back to the beginning of the 1950’s, having its roots in Theoretical Computer Science. It was strongly motivated and developed by Eilenberg in collaboration with Schützenberger and Tilson in the mid 1970’s [19, 20]. In particular, Eilenberg [20, Chapter VII] established a correspondence between varieties of rational languages and pseudovarieties of semigroups, which has made possible to study combinatorial properties of the former through the study of algebraic properties of the latter. As a result, it became of interest to study the decidability of the membership problem for pseudovarieties. That means to prove either that there exists an algorithm deciding whether a given finite semigroup belongs to a certain pseudovariety, in which case the pseudovariety is said to be decidable; or to prove that such an algorithm does not exist, being thus in the presence of an undecidable pseudovariety. Considering some natural operators on pseudovarieties $V$ and $W$, such as the join $V \lor W$, the semidirect product $V \rtimes W$, the two-sided semidirect product $V \rtimes^* W$, or the Mal’cev product $V \oplus W$, it is also relevant to decide the membership problem for the resulting pseudovariety. It turns out that none of these operators preserves decidability [11, 22]. Aiming to guarantee the decidability of pseudovarieties obtained through the application of $\ast$, from a stronger property for the involved pseudovarieties, Almeida [3] introduced the notion of hyperdecidability. This property consists of a generalization of inevitability for finite groups introduced by Ash in [13]. Since then, other notions like tameness and reducibility and some other variants were also considered [4].

On the other hand, Brzozowski and Fich [16] conjectured that $\mathcal{SI} \ast \mathcal{L} = \mathcal{GLT}$ and established the inclusion $\mathcal{SI} \ast \mathcal{L} \subseteq \mathcal{GLT}$. Motivated by this problem, Almeida and Weil [11] considered the dual of the pseudovariety $\mathcal{L}$, the pseudovariety $\mathcal{R}$ of $\mathcal{R}$-trivial finite semigroups, and described the structure of the free pro-$\mathcal{R}$ semigroup. Later on, it was proved

2010 Mathematics Subject Classification. Primary 20M07, Secondary 20M05.
Keywords and phrases: pseudovariety, free profinite semigroup, $\mathcal{R}$-class, pseudoequation, implicit signature, complete reducibility.
by Almeida and Silva [8] that the pseudovariety \( R \) is \( SC \)-hyperdecidable for the canonical implicit signature \( \kappa \), and by Almeida, Costa and Zeitoun [6] that \( R \) is completely \( \kappa \)-reducible. In this paper, we generalize the results obtained in [6] for pseudovarieties of the form \( DRH \), where \( H \) is a pseudovariety of groups and \( DRH \) is the pseudovariety of semigroups whose regular \( R \)-classes are groups lying in \( H \). More precisely, we prove that \( DRH \) is a completely \( \kappa \)-reducible pseudovariety if and only if the pseudovariety of groups \( H \) is completely \( \kappa \)-reducible as well. Of course, the latter condition holds for every locally finite pseudovariety \( H \). However, so far, the unique known instance of a completely \( \kappa \)-reducible non-locally finite pseudovariety is \( Ab \), the pseudovariety of abelian groups [7]. Hence, the pseudovariety \( DRAb \) is completely \( \kappa \)-reducible. On the contrary, since neither the pseudovarieties \( G \) and \( G_p \) (respectively, of all finite groups, and of all finite \( p \)-groups, for a prime \( p \)) nor proper non-locally finite subpseudovarieties of \( Ab \) are completely \( \kappa \)-reducible [17] [14] [18], we obtain a family of pseudovarieties of the form \( DRH \) that are not completely \( \kappa \)-reducible.

In Section 2 we introduce the basic concepts and set up the notation used later. Section 3 is devoted to general facts on the structure of the free pro-\( DRH \) semigroup \( \Omega_A DRH \) already known from [11]. In particular, we describe members of \( \Omega_A DRH \) by means of certain decorated reduced \( A \)-labeled ordinals. Section 4 contains a generalization of a periodicity phenomenon over pseudovarieties of the form \( DRH \) that was proved for \( R \) in [6]. Some simplifications concerning the class of systems of equations that we must consider in order to achieve complete \( \kappa \)-reducibility of \( DRH \) are introduced in Section 5 while in Sections 6 and 7 we redefine the tools used in [6], adapting them for the context of the pseudovarieties \( DRH \). Finally, in Section 8 we prove the main theorem, that is, we prove that \( DRH \) is completely \( \kappa \)-reducible provided so is \( H \), whose converse amounts to a simple observation.

2. General definitions and notation

For the basic concepts and results on (pro)finite semigroups the reader is referred to [2] [5]. The required topological tools may be found in [23].

The symbols \( \mathcal{R}, \leq_{\mathcal{R}}, \mathcal{D}, \) and \( \mathcal{H} \) denote some of Green’s relations. Given a semigroup \( S \), we denote by \( S^I \) the monoid whose underlying set is \( S \cup \{I\} \), where \( S \) is a subsemigroup and \( I \) plays the role of a neutral element. Given \( n \) elements \( s_1, \ldots, s_n \) of a semigroup \( S \), we use the notation \( \prod_{i=1}^{n} s_i \) for the product \( s_1 s_2 \cdots s_n \). Given a sequence \((s_n)_{n \geq 1}\) of a semigroup \( S \) we call infinite product the sequence \( (\prod_{i=1}^{n} s_i)_{n \geq 1} \).

If nothing else is said, then we use \( V \) and \( W \) for denoting arbitrary pseudovarieties of semigroups. Some pseudovarieties referred in this paper are \( S \), the pseudovariety of all finite semigroups; \( Sl \), the pseudovariety of all finite semilattices; \( G \), the pseudovariety of all finite groups; \( G_p \), the pseudovariety of all \( p \)-groups (for a prime number \( p \)); and \( Ab \), the pseudovariety of all finite Abelian groups. We denote arbitrary subpseudovarieties of \( G \) by \( H \). Our main focus are the pseudovarieties of the form \( DRH \), that is, the class of all finite semigroups whose regular \( R \)-classes are groups lying in \( H \), and hence, are also \( \mathcal{H} \)-classes. If \( H \) is the trivial pseudovariety of groups \( I = [x = y] \), then \( DRH = DRI \) is the pseudovariety \( R \) of all finite \( R \)-trivial semigroups.

We reserve the letter \( A \) to denote a finite alphabet. Then, \( \Omega_A V \) is the free \( A \)-generated pro-\( V \) semigroup. If the pseudovariety \( V \) contains at least one non-trivial semigroup, then the generating mapping \( i : A \rightarrow \Omega_A V \) is injective. So, we often identify the elements of \( A \) with their images under \( i \). In the monoid \( (\Omega_A V)^I \), we sometimes call \( I \) the empty
(pseudo)word. Also, if $B \subseteq A$, then the inclusion mapping induces an injective continuous homomorphism $\overline{B}V \to \overline{A}V$. Hence, we look at $\overline{B}V$ as a subsemigroup of $\overline{A}V$. On the other hand, if $W$ is another pseudovariety contained in $V$, then $\rho_{V,W}$ represents the natural projection of $\overline{A}V$ onto $\overline{A}W$. We shall write $\rho_W$ when $V$ is clear from the context. In the case where $W = S$ we denote $\rho_S$ by $c$ and call it the content function.

Given a pro-$V$ semigroup $S$ and $u \in \overline{A}V$, we denote by $u_S : S^A \to S$ the interpretation in $S$ of the implicit operation induced by $u$. An implicit signature, usually denoted $\sigma$, is a set of implicit operations on $S$ containing the multiplication. Of course, every implicit signature $\sigma$ endows $\overline{A}V$ with a structure of $\sigma$-algebra under the interpretation of each one of its symbols. We denote by $\Omega^\sigma_A V$ the $\sigma$-subalgebra of $\overline{A}V$ generated by $A$. The implicit signature $\kappa = \{ \cdot : \cdot \cdot \cdot \cdot \}$ is the canonical implicit signature, where $x^{\cdot^{-1}} = \lim_{n \to 1} x^{n^{-1}}$.

Elements of $\overline{A}S$ are called pseudowords, while elements of $\Omega^\sigma_A S$ are $\sigma$-words.

A formal equality $u = v$, with $u, v \in \overline{A}S$ is called a pseudoidentity. Expressions like $V$ satisfies $u = v$, $u = v$ holds modulo $V$, and $u = v$ holds in $V$ mean that the interpretations of $u$ and $v$ coincide on every semigroup $S \in V$. If that is the case, then we may write $u =_V v$. We have $u =_V v$ if and only if $\rho_V(u) = \rho_V(v)$.

Let $X$ be a finite set of variables and $P$ a finite set of parameters, disjoint from $X$. A pseudoequation is a formal expression $u = v$ with $u, v \in \overline{XUP}S$. If $u, v \in \Omega^\sigma_{XUP}S$, then $u = v$ is said to be a $\sigma$-equation, and if $u, v \in (X \cup P)^+$, then it is called a word equation. A finite system of pseudoequations (respectively, $\sigma$-equations, word equations) is a finite set
\begin{equation}
\{ u_i = v_i : i = 1, \ldots, n \},
\end{equation}
where each $u_i = v_i$ is a pseudoequation (respectively, $\sigma$-equation, word equation). For each variable $x \in X$, we consider a constraint given by a pair $(\varphi, \nu)$, where $\varphi : \overline{A}S \to S$ is a continuous homomorphism into a finite semigroup $S$, and $\nu : X \to S$ is a function. The evaluation of the parameters in $P$ is given by a map $\ev : P \to \overline{A}S$. A solution modulo $V$ of the system (I) satisfying the given constraints and subject to the evaluation of the parameters is a continuous homomorphism $\delta : \overline{XUP}S \to \overline{A}S$ such that the following conditions are satisfied:
\begin{enumerate}
\item[(S.1)] $\delta(u_i) =_V \delta(v_i)$, for $i = 1, \ldots, n$;
\item[(S.2)] $\varphi(\delta(x)) = \nu(x)$, for $x \in X$;
\item[(S.3)] $\delta(p) = \ev(p)$, for $p \in P$.
\end{enumerate}

Without loss of generality, we assume from now on that the semigroup $S$ has a content function (see [10] Proposition 2.1]).

If $\delta(X \cup P) \subseteq \Omega^\sigma_A S$, then we say that $\delta$ is a solution modulo $V$ of (I) in $\sigma$-words. In particular, the existence of a solution in $\sigma$-words implies, by (S.3), that $\ev$ evaluates the parameters in $\sigma$-words as well. Let $\mathcal{C}$ be a class of finite systems of $\sigma$-equations. We say that $V$ is $\sigma$-reducible with respect to $\mathcal{C}$ if any system in $\mathcal{C}$ which has a solution modulo $V$ also has a solution modulo $V$ in $\sigma$-words. The pseudovariety $V$ is said to be completely $\sigma$-reducible if it is $\sigma$-reducible with respect to the class of all finite systems of $\sigma$-equations.

3. Structural aspects of the free pro-DRH semigroup

3.1. Preliminaries. Before describing how to represent pseudowords over DRH conveniently, we need to introduce a few concepts.
Suppose that $S \subseteq V$ and let $u \in \Omega_A V$. A left basic factorization of $u$ is a factorization of the form $u = u_1 a u_r$, where $u_1, u_r \in (\Omega_A V)\dagger$ and $c(u) = c(u_1) \uplus \{a\}$. For certain pseudovarieties such a factorization always exists and is unique.

**Proposition 3.1** ([11][12]). Let $V \in \{\text{DRH}, S\}$. Then, every element $u \in \Omega_A V$ admits a unique factorization of the form $u = u_1 a u_r$ such that $a \notin c(u_1)$ and $c(u) = c(u_1)$.

Applying inductively Proposition 3.1 to the leftmost factor of the left basic factorization of a pseudoword over $V \in \{\text{DRH}, S\}$, we obtain the following result.

**Corollary 3.2.** Let $V \in \{\text{DRH}, S\}$ and $u$ be a pseudoword over $V$. Then, there exists a unique factorization $u = a_1 u_1 a_2 u_2 \cdots a_n u_n$ such that $a_i \notin c(a_1 u_1 \cdots a_{i-1} u_{i-1})$, for every $i = 2, \ldots, n$, and $c(u) = \{a_1, \ldots, a_n\}$.

We refer to the factorization described in Corollary 3.2 as the first-occurrences factorization of $u$.

For a pseudoword $u$ over $V \in \{\text{DRH}, S\}$, we may also iterate the left basic factorization of $u$ to the right as follows. We set $u'_0 = u$ and, for each $k \geq 1$, whenever $u'_k \neq I$, we let $u'_{k-1} = u_k a_k u'_k$ be the left basic factorization of $u'_k$. Then, for every such $k$, the equality $u = u_1 a_1 u_2 a_2 \cdots u_k a_k u'_k$ holds. Moreover, the content of each factor $u_k a_k$ decreases as $k$ increases. Since the alphabet $A$ is finite, the sequence of contents $(c(u_k a_k))_{k \geq 1}$ is either finite or it stabilizes. The cumulative content of $u$ is the empty set if the sequence is finite, and is the set $c(u_m a_m)$ if $c(u_m a_m) = c(u_k a_k)$ for every $k \geq m$. We denote the cumulative content of a pseudoword $u$ by $\overline{c}(u)$. If $\overline{c}(u) \neq \emptyset$ and $m$ is the least integer such that $\overline{c}(u) = c(u_m a_m)$, then we say that $u'_m$ is the regular part of $u$. It may be proved that an element $u$ of $\Omega_A \text{DRH}$ is regular if and only if its content coincides with its cumulative content ([11] Corollary 6.1.5), that is, if $u$ is its own regular part. If $\overline{c}(u) = \emptyset$, then we set $\overline{c}(u) = k$ if $u_k = I$. Otherwise, we set $\overline{c}(u) = \infty$. We also write $\lbf(\infty)(u)$ for the sequence $(u_1 a_1, \ldots, u_1 a_1, u_1 a_1, I, I, \ldots)$ if $\overline{c}(u) = \emptyset$, and for the sequence $(u_k a_k)_{k \geq 1}$ otherwise. We denote the $k$-th element of $\lbf(\infty)(u)$ by $\lbf_k(u)$.

**Remark 3.3.** It is not hard to check that if $V \in \{\text{DRH}, S\}$ satisfies the pseudoidentity $uu_0 = u$, then $\lbf(\infty)(uu_0) = \lbf(\infty)(u)$ and $c(u_0) \subseteq \overline{c}(u)$. Conversely, if $c(u_0) \subseteq \overline{c}(u)$, then the equality $\lbf(\infty)(u) = \lbf(\infty)(uu_0)$ holds modulo $V$.

Suppose that the iteration of the left basic factorization of $u \in \Omega_A \text{DRH}$ to the right runs forever. Since $\Omega_A \text{DRH}$ is a compact monoid, the infinite product $(\lbf_1(u) \cdots \lbf_k(u))_{k \geq 1}$ has, at least, one accumulation point. Plus, any two accumulation points are $R$-equivalent (cf. [11] Lemma 2.1.1). If, in addition, $u$ is regular, then the $R$-class containing the accumulation points of the mentioned sequence is regular [11] Proposition 2.1.4] and hence, it is a group. In that case, we may define the idempotent designated by the infinite product $(\lbf_1(u) \cdots \lbf_k(u))_{k \geq 1}$ to be the identity of the group to which its accumulation points belong. It further happens that each regular $R$-class of $\Omega_A \text{DRH}$ is homeomorphic to a free pro-$H$ semigroup. This claim consists of a particular case of the next proposition, which is behind the results on the representation of elements of $\Omega_A \text{DRH}$ presented in [11], some of which we state later. We use DO and $\overline{H}$ to denote the pseudovarieties consisting, respectively, of all finite semigroups whose regular $D$-classes are orthodox semigroups, and of all finite semigroups whose subgroups belong to $H$.

**Proposition 3.4** ([11] Proposition 5.1.2]). Let $V$ be a pseudovariety such that the inclusions $H \subseteq V \subseteq \text{DO} \cap \overline{H}$ hold. If $e$ is an idempotent of $\Omega_A V$ and $H_e$ is its $K$-class, then
Letting \( \psi_e(a) = eae \) for each \( a \in c(e) \) defines a unique homeomorphism \( \psi_e : \overline{\Omega}_c(e)H \to H_e \) whose inverse is the restriction of \( \rho_H \) to \( H_e \).

The following is an important consequence of Proposition 3.1 which we use later on.

**Corollary 3.5.** Let \( u \) be a pseudoword and \( v, w \in (\Omega_A S)^I \) be such that \( c(v) \cup c(w) \subseteq \overline{c}(u) \) and \( v =_H w \). Then, the pseudovariety \( DRH \) satisfies \( uv = vw \).

We now have all the necessary ingredients to describe the elements of \( \overline{\Omega}_A DRH \) by means of the so-called “decorated reduced \( A \)-labeled ordinals”, which we do along the next subsection. The construction is based on [11].

### 3.2. Decorated reduced \( A \)-labeled ordinals

A decorated reduced \( A \)-labeled ordinal is a triple \((\alpha, \ell, g)\) where

- \( \alpha \) is an ordinal.
- \( \ell : \alpha \to A \) is a function. For a limit ordinal \( \beta \leq \alpha \), we let the cumulative content of \( \beta \) with respect to \( \ell \) be given by
  \[
  \overline{c}(\beta, \ell) = \{ a \in A : \exists (\beta_n)_{n \geq 1} \mid \cup_{n \geq 1} \beta_n = \beta, \ \beta_n < \beta \ \text{and} \ \ell(\beta_n) = a \}.
  \]
- \( g : \{ \beta : \beta \text{ is a limit ordinal} \} \to \overline{\Omega}_A H \) is a function such that \( g(\beta) \in \overline{\Omega}_{\overline{c}(\beta, \ell)} H \).

We denote the set of all decorated reduced \( A \)-labeled ordinals by \( rLO_H(A) \).

To each pseudoword \( u \) over \( DRH \), we assign an element of \( rLO_H(A) \) as follows. Let us say that the product \( ua \) is end-marked if \( a \notin \overline{c}(u) \). It is known that the set of all end-marked pseudowords over a finite alphabet constitutes a well-founded forest under the partial order \( \leq_R \) [6, Proposition 4.8]. Then, \( \alpha_u \) is the unique ordinal such that there exists an isomorphism (also unique)

\[
\theta_u : \alpha_u \to \{ \text{end-marked prefixes of } u \}
\]

such that \( \theta_u(\beta) \geq_R \theta_u(\gamma) \) whenever \( \beta < \gamma \). We let \( \ell_u : \alpha_u \to A \) be the function sending each ordinal \( \beta \leq \alpha \) to the letter \( a \) if \( \theta_u(\beta) = va \).

**Remark 3.6.** We point out that, for every limit ordinal \( \beta \leq \alpha \) such that \( \theta_u(\beta) = va \), we have \( \overline{c}(v) = \overline{c}(\beta, \ell_u) \).

It remains to define \( g_u \). Let \( \beta \leq \alpha_u \) be a limit ordinal. By definition of \( \theta_u \), if \( \theta_u(\beta) = va \), then the regular part of \( v \) is nonempty. Then, we set \( g_u(\beta) \) to be the projection onto \( \overline{\Omega}_A H \) of the regular part of \( v \). Observe that, by Remark 3.6, \( g_u(\beta) \) defined in that way belongs to \( \overline{\Omega}_{\overline{c}(\beta, \ell_u)} H \). Hence, \((\alpha_u, \ell_u, g_u)\) is indeed a decorated reduced \( A \)-labeled ordinal. We call \( F \) the mapping thus defined:

\[
F : \overline{\Omega}_A DRH \to rLO_H(A)
\]

\[
u \mapsto (\alpha_u, \ell_u, g_u).
\]

It turns out that \( F \) is a bijection [11, Theorem 6.1.1]. In fact, it is possible to define an algebraic structure on \( rLO_H(A) \) that turns \( F \) into an isomorphism. We do not include such construction since we make no explicit use of it.
Let \( u \in \Omega_A S \). Sometimes we abuse notation and write \( \alpha_u \) to refer to \( \alpha_{\rho_{DRH}(u)} \).

**Notation 3.7.** Let \( u \in \Omega_A S \) and take ordinals \( \beta \leq \gamma \leq \alpha_u \). Let \( \theta_u(\beta) = va \) and \( \theta_u(\gamma) = wb \). If \( \beta < \gamma \), then we denote by \( u[\beta,\gamma[ \) the product \( az \), where \( z \) is the unique pseudoword such that \( w = vaz \). We set \( u[\beta,\beta[ = I \).

If \( u \) is a \( \kappa \)-word, then the factors of \( u \) of the form \( u[\beta,\gamma[ \) are \( \kappa \)-words as well. This fact arises as a consequence of the following lemma when we iterate it inductively.

**Lemma 3.8** ([12 Lemma 2.2]). Let \( u \in \Omega^\kappa_A S \) and let \( (u_\ell,a,u_r) \) be its left basic factorization. Then, \( u_\ell \) and \( u_r \) are \( \kappa \)-words.

### 3.3. Further properties of pseudowords over DRH

We proceed with the statement of some structural results to handle pseudowords modulo DRH. Although we could not find the exact statement that fits our purpose, they seem to be already used in the literature. For that reason, we do not include any proof. They may be found in [15].

We first characterize \( R \)-classes of \( \Omega_A DRH \) by means of iteration of left basic factorizations to the right.

**Lemma 3.9.** Let \( u,v \) be pseudowords over DRH. Then, \( u \) and \( v \) lie in the same \( R \)-class if and only if \( lbf_{\infty}(u) = lbf_{\infty}(v) \).

As a consequence, we have the following:

**Corollary 3.10.** Let \( u,v \in \Omega_A DRH \). Then, the relation \( R \) holds if and only if \( \alpha_u = \alpha_v \), \( \ell_u = \ell_v \) and \( g_u|\{\beta<\alpha_u: \beta \text{ is a limit ordinal}\} = g_v|\{\beta<\alpha_v: \beta \text{ is a limit ordinal}\} \).

We also have a kind of left cancellative law over DRH.

**Corollary 3.11.** Let \( u \) and \( v \) be pseudowords over DRH that are \( R \)-equivalent. Suppose that they admit factorizations \( u = u_1au_2 \) and \( v = v_1bv_2 \) such that \( u_1a \) and \( v_1b \) are end-marked. If \( \alpha_u = \alpha_v \), then \( a = b \), \( u_1 = v_1 \), and \( u_2 \equiv v_2 \). If, in addition, the equality \( u = v \) holds, then also \( u_2 = v_2 \).

The following result is just a rewriting of the previous corollary that we state for later reference.

**Corollary 3.12.** Let \( u,v \) be pseudowords that are \( R \)-equivalent modulo DRH. Take ordinals \( \beta < \gamma < \alpha_u = \alpha_v \). Then, the pseudovariety DRH also satisfies \( u[\beta,\gamma[ = v[\beta,\gamma[ \) and \( u[\gamma,\alpha_u[ \equiv v[\gamma,\alpha_v[ \). Moreover, if \( u \equiv_{DRH} v \), then \( u[\gamma,\alpha_u[ \equiv_{DRH} v[\gamma,\alpha_v[ \).

The next lemma can be thought as the key ingredient when proving our main result. It becomes trivial when \( DRH = R \).

**Lemma 3.13.** Let \( u,v \in \Omega_A DRH \) and \( u_0,v_0 \in (\Omega_A DRH)^I \) be such that \( c(u_0) \subseteq c(u) \) and \( c(v_0) \subseteq c(v) \). Then, the equality \( uu_0 = vv_0 \) holds if and only if \( u \equiv R v \) and if, in addition, the pseudovariety \( H \) satisfies \( uu_0 = vv_0 \). In particular, by taking \( u_0 = I = v_0 \), we get that \( u = v \) if and only if \( u \equiv R v \) and \( u \equiv_H v \).

### 4. Periodicity modulo DRH

Now, we state and prove two results concerning a certain periodicity of members of \( \Omega_A DRH \). We first need a few auxiliary lemmas.
Lemma 4.1 (cf. [6] Lemma 5.1). Let \( u, v \) be pseudowords over \( \text{DRH} \) such that \( uv^\omega \R v^\omega \). If \( c(u) \subseteq c(v) \), then equality \( uv = v \) holds.

Proof. Let \( a \) be a letter in \( c(v) \setminus c(u) \). By Corollary 3.12 we may factorize \( v = v_1 a v_2 \) with \( a \notin c(v_1) \). Then, the equality \( uv^\omega = v^\omega \) may be rewritten as \( uv_1 a v_2 v^\omega - 1 = v_1 a v_2 v^\omega - 1 \). Since \( a \notin c(v_1) \), again Corollary 3.12 implies \( uv_1 = v_1 \), resulting in turn that \( uv = v \). \( \square \)

We also recall a lemma related with the pseudovariety \( R \) that may be used to prove a weaker similar result for \( \text{DRH} \).

Lemma 4.2 ([6] Lemma 5.2). If \( u, v \in \Omega_A R \) are such that \( vu^2 = u^2 \), then \( vu = u \).

We say that the product \( uv \) of two pseudowords is reduced if \( v \) is not the empty word and its first letter does not belong to the cumulative content of \( u \).

Corollary 4.3. If \( u, v \in \Omega_A \text{DRH} \) are such that \( vu^2 = u^2 \) and the product \( u \cdot u \) is reduced, then the equality \( vu = u \) holds.

Proof. Since \( R \subseteq \text{DRH} \), the pseudovariety \( R \) satisfies \( vu^2 = u^2 \) and Lemma 4.2 yields that it also satisfies \( vu = u \). Therefore, from Corollary 3.10 we conclude that \( \alpha_{vu} = \alpha_u \). As the product \( u \cdot u \) is reduced, it follows that \( u^2[0, \alpha_u] = u \). On the other hand, Corollary 3.12 yields the identity \( vu^2[0, \alpha_{vu}] = vu^2[0, \alpha_u] = u^2[0, \alpha_u] \). Moreover, either \((vu) \cdot u\) is a reduced product and \( vu^2[0, \alpha_{vu}] = vu \), or we may write \( u = u_1 \cdot u_2 \), with \( u_2 = I \) or \( u_1 \cdot u_2 \) a reduced product and \( c(u_1) \subseteq c(uv) \), and then \( vu^2[0, \alpha_{vu}] = vu u_1 \). In any case, \( vu \) and \( u \) are \( R \)-equivalent. Also, the inclusion \( H \subseteq \text{DRH} \) implies that \( vu^2 = u^2 \) modulo \( H \) and so, \( vu = u \) modulo \( H \). Finally, it follows from Lemma 3.13 that \( vu = u \). \( \square \)

Now, we are ready to prove the announced results on the periodicity in \( \Omega_A \text{DRH} \).

Lemma 4.4 (cf. [6] Lemma 5.4). Let \( x \) and \( y \) be pseudowords such that \( x^\omega = y^\omega \) modulo \( \text{DRH} \). If the products \( x \cdot x \) and \( y \cdot y \) are reduced, then there are pseudowords \( u \in \Omega_A \text{S} \) and \( v, w \in (\Omega_A \text{S})^I \), and positive integers \( k, \ell \) such that the following pseudoidentities hold in \( \text{DRH} \)

\[
\begin{align*}
x &= u^k v, \\
y &= u^\ell w, \\
u &= vu = wu,
\end{align*}
\]

and all the products \( u \cdot u, u \cdot v, u \cdot w, v \cdot u, \) and \( v \cdot w \) are reduced.

Proof. We argue by transfinite induction on \( \alpha = \max\{\alpha_x, \alpha_y\} \).

If \( \alpha_x = \alpha_y \), since the products \( x \cdot x \) and \( y \cdot y \) are reduced, we then have \( x = y \) in \( \text{DRH} \), by Corollary 3.12. So, we may choose \( u = x, v = w = I \), and \( k = \ell = 1 \).

From now on, we assume that the pseudovariety \( \text{DRH} \) does not satisfy \( x = y \). Suppose, without loss of generality, that \( \alpha_x < \alpha_y = \alpha \). Again, by Corollary 3.12 \( \text{DRH} \) satisfies

\[
y = y^\omega[0, \alpha_y] = x^\omega[0, \alpha_y] = x^\omega[0, \alpha_x][x^\omega[\alpha_x, \alpha_y] = x^\omega[\alpha_x, \alpha_y]
\]

and so, \( x \) is a prefix of \( y \) modulo \( \text{DRH} \). Thus, the set

\[
P = \{ m \geq 1 : \exists (y_1, \ldots, y_m) \in (\Omega_A \text{S})^m \text{ such that } y_i \leq_R y_1 \cdots y_m \text{ and } y_i = \text{DRH} x, \text{ for } i = 1, \ldots, m \}
\]

is nonempty. If it were unbounded then, since \( x \cdot x \) is a reduced product and by definition of cumulative content, every letter of \( c(x) = c(y_i) \) would be in the cumulative content
of $y$, so that $\overline{c}(y) = c(x) = c(y)$, a contradiction with the hypothesis that $y \cdot y$ is a reduced product. Take $m = \max(P)$ and let $y = y_1 \cdots y_m y'$, with $y_i =_{\text{DRH}} x$, for $i = 1, \ldots, m$. Since $x^\omega =_{\text{DRH}} y^\omega$, we deduce that DRH satisfies

$$x^\omega = y^\omega = y_1 \cdots y_m y'^{\omega - 1} = x^m y'^{\omega - 1}$$

which in turn, since the involved products are reduced, implies that DRH also satisfies

$$x^\omega - m = y'^{\omega - 1}.$$ 

In particular, as $y^\omega = x^\omega$ in DRH (and so, $c(x) = c(y)$), we may conclude that DRH satisfies

$$x^\omega = y'^{\omega - 1} x^m = y' x^{\omega - 1} x^m \in \mathcal{R} y' x^\omega. \tag{2}$$

We now distinguish two cases.

- If $c(y') \subseteq c(x)$ then, by Lemma 4.1, the pseudovariety DRH satisfies $x = y' x$, so that we may choose $u = x$, $v = I$, $k = 1$, $w = y'$, and $\ell = m$.
- If $c(y') = c(x)$ then, successively multiplying by $y'$ on the left the leftmost and rightmost sides of (2), we get that the relation $x^\omega \in \mathcal{R} y'^{\omega - 1} x^m = y^{\omega - 1} x^m$ holds in DRH. As $x^\omega$ and $y^\omega$ are both the identity in the same regular $\mathcal{R}$-class, hence in the same group, the mentioned relation is actually an equality: $x^\omega =_{\text{DRH}} y^\omega$. Furthermore, the product $y' \cdot y'$ is reduced because so is $y \cdot y$. Indeed, $\overline{c}(y') = \overline{c}(y)$, the first letters of $y'$ and $x$ coincide and, in turn, the first letter of $x$ is the first letter of $y$. Consequently, $y'$ and $x$ verify the conditions of applicability of the lemma and have associated a smaller induction parameter. In fact, maximality of $m$ guarantees that $\alpha_{y'} \leq \alpha_x < \alpha_y = \alpha$. By induction hypothesis, there exist $u \in \overline{\Omega}_A S$, $v, w \in (\overline{\Omega}_A S)^I$, and $k, \ell > 0$ such that the identities

$$x = u^k v, \tag{3}$$

$$y' = u^\ell w,$$

$$u = vu = wu$$

are valid in DRH, and where all products, including $u \cdot u$ are reduced. The computation

$$y = x^m y' = (u^k v)^m u^\ell w = u^{km + \ell} w$$

modulo DRH justifies that, except for the value of $\ell$, which now is $km + \ell$, the choice in (3) also fits the original pair $x, y$.

The proof of the next result consists of an induction argument that is similar to the one used in the proof of [6 Proposition 5.5]. Here, the induction basis is given by Lemma 4.4 and Corollary 4.3 plays the role of [6 Lemma 5.2].

**Proposition 4.5.** Let $x_0, x_1, \ldots, x_n \in \overline{\Omega}_A S$ be such that $x_0^\omega = x_1^\omega = \cdots = x_n^\omega$ modulo DRH and suppose that, for $i = 0, 1, \ldots, n$, the product $x_i \cdot x_i$ is reduced. Then, there exist pseudowords $u \in \overline{\Omega}_A S$, $v_0, v_1, \ldots, v_n \in (\overline{\Omega}_A S)^I$, and positive integers $p_0, p_1, \ldots, p_n$ such that the pseudovariety DRH satisfies

$$x_i = u^{p_i} v_i, \quad \text{for } i = 0, 1, \ldots, n,$$

$$u = v_i u, \quad \text{for } i = 0, 1, \ldots, n,$$

and all the products $u \cdot u$, $u \cdot v_i$, and $v_i \cdot u$ are reduced.
5. Some simplifications concerning reducibility

Almeida, Costa and Zeitoun [6] proved that, in order to achieve complete \(\kappa\)-reducibility, it is enough to consider systems of \(\kappa\)-equations with empty set of parameters (in fact, they proved the result more generally, for any implicit signature \(\sigma\)).

**Proposition 5.1** ([6 Proposition 3.1]). Let \(V\) be an arbitrary pseudovariety. If \(V\) is \(\kappa\)-reducible for systems of \(\kappa\)-equations without parameters, then \(V\) is completely \(\kappa\)-reducible.

A pseudovariety \(V\) is said to be **weakly cancellable** if whenever \(V\) satisfies \(u_1a_1u_2 = v_1a_1v_2\) with \(a\) not belonging to any of the sets \(c(u_1), c(u_2), c(v_1),\) and \(c(v_2)\), it also satisfies \(u_1 = u_2\) and \(v_1 = v_2\). When \(V\) is a weakly cancellable pseudovariety, we may restrict our study to systems consisting of one single \(\kappa\)-equation without parameters.

**Proposition 5.2** ([6 Proposition 3.2]). Let \(V\) be a weakly cancellable pseudovariety. If \(V\) is \(\kappa\)-reducible for systems consisting of just one \(\kappa\)-equation without parameters, then \(V\) is completely \(\kappa\)-reducible.

Of course, the pseudovariety \(\text{DRH}\) is weakly cancellable. Indeed, weak cancellability is a particular instance of uniqueness of the first-occurrences factorization (recall Corollary 3.2). Actually, we may go even further and, similarly to the case of \(R\) (see [6] Lemmas 6.1 and 6.2), we prove that, in order to obtain complete \(\kappa\)-reducibility of a pseudovariety \(\text{DRH}\), it suffices to consider systems of word equations (without parameters).

**Lemma 5.3.** Let \(u, v \in \Omega_A\). Then, \(\text{DRH}\) satisfies the pseudoidentity \(u = v^{\omega-1}\) if and only if \(c(u) = c(v)\), and the pseudoidentities \(uw = u\) and \(uv = vu\) hold in \(\text{DRH}\).

*Proof.* Suppose that \(\text{DRH}\) satisfies \(u = v^{\omega-1}\). Since the semigroup \(\Omega_A\text{DRH}\) has a content function, we have \(c(u) = c(v^{\omega-1}) = c(v)\). In order to verify that the pseudoidentities \(uw = u\) and \(uv = vu\) are valid in \(\text{DRH}\), we may perform the following computations:

\[
\begin{align*}
  u &=_{\text{DRH}} v^{\omega-1} = v^{\omega-1} (vv^{\omega-1}) =_{\text{DRH}} uvu, \\
  uv &=_{\text{DRH}} v^{\omega-1}v = vv^{\omega-1} =_{\text{DRH}} vu.
\end{align*}
\]

Conversely, suppose that \(\text{DRH}\) satisfies the pseudoidentities \(uw = u\) and \(uv = vu\), and \(c(u) = c(v)\). Then, the following pseudoidentities are valid in \(\text{DRH}\):

\[
\begin{align*}
  v^{\omega-1} &= v^{\omega-1}u^{\omega} \quad \text{by Corollary 3.5} \\
  &= v^{\omega-1}u^{\omega-1}u = (uv)^{\omega-1}u \quad \text{because } uv =_{\text{DRH}} vu \\
  &= (uv)u \quad \text{because } uv =_{\text{DRH}} u \text{ implies } (uv)^{\omega-1} =_{\text{DRH}} uv \\
  &= u.
\end{align*}
\]

This concludes the proof. \(\square\)

Lemma 5.3 allows us to transform each \(\kappa\)-equation into a finite system of word equations. Therefore, by Proposition 5.2 in order to prove the complete \(\kappa\)-reducibility of \(\text{DRH}\), it is enough to consider systems consisting of a single word equation. We do not include the details of that step, as it is entirely analogous to [6 Proposition 6.2].

**Proposition 5.4.** The pseudovariety \(\text{DRH}\) is completely \(\kappa\)-reducible if and only if it is \(\kappa\)-reducible for a single word equation without parameters.
Let \( u, v \in X^+ \) and \( \delta : \overline{\Omega}_X S \to \overline{\Omega}_A S \) be a solution modulo \( \text{DRH} \) of \( u = v \), subject to the constraints given by the pair \( (\varphi : \overline{\Omega}_A S \to S, \nu : X \to S) \). The last simplification consists in transforming the word equation \( u = v \) into a more convenient system of equations, namely, into a system that we denote by \( S_{u=v} \) and that is the union of systems \( \{u' = v'\}, S_1 \) and \( S_2 \) with variables in \( X' \). We construct \( S_{u=v} \) inductively as follows.

We use an auxiliary system \( S_0 \) and start with \( S_0 = S_1 = S_2 = \emptyset \), \( X' = X \), \( u' = u \), and \( v' = v \). Since \( \text{DRH} \) is a weakly cancellable pseudovariety, the word equation \( u = v \) is equivalent to the equation \( u\# = v\# \), where \( \# \notin A \) is a parameter evaluated to itself. Suppose that, whenever \( xy \) is a factor of \( u\#v\# \) \( (x, y \in X) \), the product \( \delta(x) \cdot \delta(y) \) is reduced. Then, we say that the solution \( \delta \) is reduced with respect to the equation \( u = v \). If \( \delta \) is not reduced with respect to \( u = v \), then we pick a factor \( xy \) such that \( \delta(x) \cdot \delta(y) \) is not a reduced product and we distinguish between two situations:

- If \( c(\delta(y)) \subseteq \overline{c}(\delta(x)) \), then we add a new variable \( z \) to \( X' \) and we put the equation \( xy = z \) in \( S_1 \). We also redefine \( u' \) and \( v' \) by substituting each occurrence of the product \( xy \) in the equation \( u'\#v'\# \) by the variable \( z \).
- If \( c(\delta(y)) \not\subseteq \overline{c}(\delta(x)) \), then we add three new variables \( y_1, y_2, \) and \( z \) to \( X' \) and we put the equations \( y = y_1y_2 \) and \( z = xy_1 \) in \( S_0 \) and \( S_1 \), respectively. We also redefine \( u' \) and \( v' \) by substituting the product \( xy \) in the equation \( u'\#v'\# \) by the product of variables \( zy_2 \).

In both situations, we can factorize \( \delta(y) = \delta(y_1)\delta(y_2) \), with \( \delta(y_2) \) possibly an empty word, such that \( c(\delta(y_1)) \subseteq \overline{c}(\delta(x)) \) and the product \( \delta(x)(\delta(y_1)) \cdot \delta(y_2) \) is reduced if \( \delta(y_2) \neq I \). We extend \( \delta \) to \( \overline{\Omega}_X S \) by letting \( \delta(z) = \delta(x)\delta(y_1) \) and, whenever we are in the second situation, by letting \( \delta(y_i) = \delta(y_i) \) \( (i = 1, 2) \). Of course, \( \delta \) is a solution modulo \( \text{DRH} \) of the new system of equations \( \{u' = v'\} \cup S_0 \cup S_1 \).

We repeat the described process until the extended solution \( \delta \) is reduced with respect to the equation \( u' = v' \). Since \( u \) and \( v \) are both words, we have for granted that this iteration eventually ends. Yet, the extension of \( \delta \) to \( \overline{\Omega}_X S \) (which is a solution modulo \( \text{DRH} \) of \( \{u' = v'\} \cup S_0 \cup S_1 \)) has the property of being reduced with respect to the equation \( u' = v' \).

We further observe that the resulting system \( S_1 \) may be written as \( S_1 = \{x(i)y(i) = z(i)\}_1^N \) and its extended solution \( \delta \) satisfies \( c(\delta(y(i))) \subseteq \overline{c}(\delta(x(i))) \). For each variable \( x \in X' \), we set \( A_x = \overline{c}(\delta(x)) \) and define \( S_2 = \{xa^e = x : a \in A_x\}_{x \in X'} \). The homomorphism \( \delta \) is a solution modulo \( \text{DRH} \) of \( S_2 \). Finally, since \( \text{DRH} \) is weakly cancellable and all the products \( \delta(y_1) \cdot \delta(y_2) \) are reduced, we may assume that the satisfaction of the equations in \( S_0 \) by \( \delta \) is a consequence of the satisfaction of the equation \( u' = v' \) by \( \delta \), without losing the reducibility of \( \delta \) with respect to \( u' = v' \). More specifically, if \( y = y_1y_2 \) is an equation of \( S_0 \), then we take for \( u' \) the word \( u'\#y \) and for \( v' \) the word \( v'\#y_1y_2 \), where \( \# \) is a new symbol, working as a parameter evaluated to itself. In the same fashion, we may also assume that all the variables of \( X' \) occur in \( u' = v' \). Although at the moment it may not be clear to the reader why we wish that all the variables in \( X' \) occur in the equation \( u' = v' \), that becomes useful later, when dealing with certain systems of equations modulo \( H \) that intervene in the so-called “systems of boundary relations”. The resulting system \( \{u' = v'\} \cup S_1 \cup S_2 \) is the one that we denote by \( S_{u'=v'} \) and it also has a solution modulo \( \text{DRH} \). The constraints for the variables in \( X' \) are those defined by the described extension of \( \delta \) to \( \overline{\Omega}_X S \), namely, we put \( \nu(x) = \varphi(\delta(x)) \) for each \( x \in X' \).

Conversely, suppose that \( S_{u=v} \) has a solution modulo \( \text{DRH} \) in \( \kappa \)-words, say \( \varepsilon \). Then, it is easily checked that, by construction, the restriction of \( \varepsilon \) to \( \overline{\Omega}_X S \) is a solution modulo
DRH of the original equation \( u = v \). Moreover, by definition of \( S_2 \), this solution is such that \( \overline{c}(\varepsilon(x)) = \overline{c}(\delta(x)) \), for all \( x \in X' \). As, in addition, \( S \) has a content function, the satisfaction of the constraints yields that \( c(\varepsilon(y_{(i)})) = c(\delta(y_{(i)})) \) and, in particular, the inclusion \( c(\varepsilon(y_{(i)})) \subseteq \overline{c}(\varepsilon(x_{(i)})) \) holds for all the equations \( x_{(i)}y_{(i)} = z_{(i)} \) in \( S_1 \).

Taking into account Proposition \( 5.4 \), we have just proved the following result in which we use the above notation.

**Proposition 5.5.** Suppose that the pseudovariety \( DRH \) is \( \kappa \)-reducible for systems of equations of the form

\[
S_{a=v} = \{u' = v'\} \cup S_1 \cup S_2,
\]

where \( u' = v' \) is a word equation, \( S_1 = \{x_{(i)}y_{(i)} = z_{(i)}\}_{i=1}^{N} \) and \( S_2 = \{x\omega = x : a \in A_x \}_{x \in X} \), which have a solution \( \delta \) modulo \( DRH \) that is reduced with respect to the equation \( u' = v' \) and satisfies \( \overline{c}(\varepsilon(y_{(i)})) \subseteq \overline{c}(\varepsilon(x_{(i)})) \), for \( i = 1, \ldots, N \). Then, the pseudovariety \( DRH \) is completely \( \kappa \)-reducible.

**Remark 5.6.** It is sometimes more convenient to allow \( \delta \) to take its values in \((\overline{\Pi}_A S)^I\). For this purpose, we naturally extend the function \( \varphi \) to a continuous homomorphism \( \varphi^I : (\overline{\Pi}_A S)^I \rightarrow S^I \) by letting \( \varphi^I(I) = I \). It is worth noticing that this assumption does not lead us to trivial solutions since the constraints must be satisfied. We allow ourselves some flexibility in this point, adopting each scenario according to each particular situation, without further mention. In the case where we consider the homomorphism \( \varphi^I \), we abuse notation and denote it by \( \varphi \).

We end this section with a result regarding reducibility of pseudovarieties of groups that is later used to derive reducibility properties of \( DRH \).

**Lemma 5.7.** Let \( H \) be a completely \( \kappa \)-reducible pseudovariety of groups and \( S \) a finite system of \( \kappa \)-equations with constraints given by the pair \((\varphi : (\overline{\Pi}_A S)^I \rightarrow S^I, \nu : X \rightarrow S^I)\), and with \( \delta : \overline{\Pi}_X S \rightarrow (\overline{\Pi}_A S)^I \) as a solution modulo \( H \). Then \( S \) has a solution modulo \( H \) in \( \kappa \)-words, say \( \varepsilon \), such that \( \overline{c}(\varepsilon(x)) = \overline{c}(\delta(x)) \) for all \( x \in X \).

**Proof.** We prove the result by induction on \( m = \max\{|c(\delta(x))| : x \in X\} \). If \( m = 0 \), then there is nothing to prove. Otherwise, let \( x \) be a variable of \( X \). Given \( i \leq \floor{\delta(x)} \), we denote \( \text{lbf}_i(\delta(x)) \) by \( \delta(x),\alpha_{x,i} \) and write \( \delta(x) = \text{lbf}_1(\delta(x)) \cdots \text{lbf}_i(\delta(x))\delta(x)_i' \). If \( \overline{c}(\delta(x)) \) is the empty set, then we have

\[
\varphi(\delta(x)) = \varphi(\text{lbf}_1(\delta(x)) \cdots \text{lbf}_i(\delta(x))\delta(x)_i').
\]

For the remaining variables, since \( X, A, \) and \( S \) are finite, there are integers \( 1 < k < \ell \) such that

\[
\overline{c}(\delta(x)) = c(\text{lbf}_{k+1}(\delta(x))); \quad \varphi(\text{lbf}_1(\delta(x)) \cdots \text{lbf}_k(\delta(x))) = \varphi(\text{lbf}_1(\delta(x)) \cdots \text{lbf}_\ell(\delta(x))),
\]

for all \( x \in X \) with \( \overline{c}(\delta(x)) \neq \emptyset \). In particular, from the second equality we deduce

\[
\varphi(\delta(x)) = \varphi(\text{lbf}_1(\delta(x)) \cdots \text{lbf}_k(\delta(x)))\varphi(\text{lbf}_{k+1}(\delta(x)) \cdots \text{lbf}_\ell(\delta(x)))^\omega \varphi(\delta(x)_i').
\]

We consider a new set of variables \( X' \) given by

\[
X' = \{y_{x,1}, b_{x,1}, \ldots, y_{x,[\delta(x)]}, b_{x,[\delta(x)]} : x \in X \text{ and } \overline{c}(\delta(x)) = \emptyset\}
\]

\[
\cup \{y_{x,1}, b_{x,1}, \ldots, y_{x,\ell}, b_{x,\ell}, y'_{x} : x \in X \text{ and } \overline{c}(\delta(x)) \neq \emptyset\}
\]
and a new system of equations $S'$ with variables in $X'$ obtained from $S$ by substituting each variable $x$ by the product
\begin{equation}
    P_x = y_1b_1 \cdots y_{x,[\delta(x)]}b_{x,[\delta(x)]},
\end{equation}
whenever $c(\delta(x)) = \emptyset$, and by the product
\begin{equation}
    P_x = y_1b_1 \cdots y_{x,k}b_{x,k}(y_{x,k+1}b_{x,k+1} \cdots y_{x,l}b_{x,l})^\omega y_x',
\end{equation}
otherwise. Let us define the constraints for the variables in $X'$. Let $a \in A$ be a letter. Since $\{a\}$ is a clopen subset of $\Omega_A S$, by Hunter's Lemma there exists a continuous homomorphism $\varphi_a : \Omega_A S \rightarrow S_a$ such that $\{a\} = \varphi^{-1}(\varphi(\{a\}))$. Representing by $\prod_{a \in A} S_a$ the direct product of the semigroups $S_a$, we let the constraints be given by the pair $(\varphi', \nu')$, where $\varphi'$ is the following continuous homomorphism
\begin{equation}
\varphi' : \Omega_A S \rightarrow S' \times \prod_{a \in A} S_a
\end{equation}
\[u \mapsto (\varphi(u), (\varphi_a(u))_{a \in A}),\]
and $\nu'$ is the mapping
\begin{equation}
\nu' : X' \rightarrow S' \times \prod_{a \in A} S_a
\end{equation}
\[y_{x,i} \mapsto \varphi'(\delta(x))_i,\]
\[y_x' \mapsto \varphi'(\delta(x'))_i,\]
\[b_{x,i} \mapsto \varphi'(a_{x,i}).\]
Since $H$ satisfies $\delta'(P_x) = \delta(x)$, for every variable $x \in X$ (check (7) and (8)), the homomorphism $\delta'$ is a solution modulo $H$ of $S'$. Therefore, as we are assuming that the pseudovariety $H$ is completely $\kappa$-reducible, there is a solution $\varepsilon' : \Omega_A S \rightarrow \Omega_A S$ modulo $H$ of $S'$ such that $\varepsilon'(X') \subseteq \Omega_A S$. On the other hand, this homomorphism $\varepsilon'$ defines a solution in $\kappa$-words modulo $H$ of the original system $S$, namely, by letting $\varepsilon(x) = \varepsilon'(P_x)$ for each $x \in X$. Moreover, by definition of $(\varphi', \nu')$, we necessarily have $\varepsilon'(b_{x,i}) = a_{x,i}$ and the fact that $S$ has a content function entails that $c(\varepsilon'(y_{x,i})) = c(\delta'(y_{x,i})) = c(\delta(x))$, and similarly, that $c(\varepsilon'(y_x')) = c(\delta'(y_x')) = c(\delta(x')))$. In particular, $a_{x,i}$ does not belong to $c(\delta(x))$. So, the iteration of left factorization to the right of $\varepsilon(x)$ is the one induced by the product $P_x$, implying that $\varepsilon(x) = \varphi(\delta(x))$ as intended. Finally, we verify that the constraints on $X$ are satisfied by $\varepsilon$. Taking into account that the definition of $(\varphi', \nu')$ yields the equalities
\begin{equation}
\varepsilon(x) = \varepsilon'(y_{x,i} b_{x,i}) = \varphi(\text{bf}_1(\delta(x))) \quad \text{and} \quad \varepsilon(y_x') = \varphi(\text{bf}_k(\delta(x'))) \quad \text{(for } x \in X \text{ and } i = 1, \ldots, k),\end{equation}
we may compute
\begin{align*}
\varepsilon(x) &= \begin{cases}
\varphi(\varepsilon'(y_{x,i} b_{x,i} \cdots y_{x,[\delta(x)]} b_{x,[\delta(x)]})) & \text{if } \varepsilon(\delta(x)) = \emptyset \\
\varphi(\varepsilon'(y_{x,i} b_{x,i} \cdots y_{x,k} b_{x,k} y_{x,k+1} b_{x,k+1} \cdots y_{x,l} b_{x,l} \cdots y_x')^\omega y_x')) & \text{otherwise}
\end{cases} \\
&= \begin{cases}
\varphi(\text{bf}_1(\delta(x)) \cdots \text{bf}_1(\delta(x))) & \text{if } \varepsilon(\delta(x)) = \emptyset \\
\varphi(\text{bf}_1(\delta(x)) \cdots \text{bf}_k(\delta(x))) & \text{if } \varepsilon(\delta(x)) = \emptyset
\end{cases} \\
&= \varepsilon(\delta(x)).
\end{align*}
Hence, the homomorphism $\varepsilon$ plays the desired role.
6. Systems of boundary relations and their models

In this section, we define some tools that turn out to be useful when proving that DRH is completely $\kappa$-reducible. The original notion of a boundary equation was given by Makanin [21] and it was later adapted by Almeida, Costa and Zeitoun [6] to deal with the problem of complete $\kappa$-reducibility of the pseudovariety $R$. Here, we extend the definitions used in [6] to the context of the pseudovariety DRH, for any pseudovariety of groups $H$, and use them to prove that, under certain conditions, the pseudovariety DRH is completely $\kappa$-reducible.

From hereon, we fix a word equation $u = v$ and a solution $\delta : \Omega_X S \to \Omega_A S$ modulo DRH of $S_{u=v}$ (recall (4)), subject to the constraints given by the pair $(\phi : \Omega_A S \to S, \nu : X \to S)$. By a system of boundary relations we mean a tuple $S = (X, J, \zeta, \chi, \text{right}, B, B_H)$ where

- $X$ is a finite set equipped with an involution without fixed points $x \mapsto \bar{x}$, whose elements are called variables;
- $J$ is a finite set equipped with a total order $\leq$, whose elements are called indices. If $i$ and $j$ are two consecutive indices, then we write $i < j$ and we denote $i$ by $j^{-}$;
- $\zeta : \{(i, j) \in J \times J : i < j\} \to 2^S \times S$ is a function that is useful to deal with the constraints;
- $M : \{(i, j, s) \in J \times J \times (S \times S^f) : i < j, s \in \zeta(i, j)\} \to \omega \setminus \{0\}$ is a function that determines the number of different factorizations in $\Omega_A S$ modulo DRH that we assign to each variable of $X$;
- $\chi : \{(i, j) \in J \times J : i < j\} \to 2^A$ is a function whose aim is to fix the cumulative content of each variable;
- right : $X \to J$ is a function that helps in defining the relations we need to attain our goal;
- $B$ is a subset of $J \times X \times J \times X$, whose elements are of the form $(i, x, j, \bar{x})$. Moreover, if $(i, x, j, \bar{x})$ is an element of $B$, then so is $(j, \bar{x}, i, x)$. The elements of $B$ are called boundary relations and the boundary relation $(j, \bar{x}, i, x)$ is said to be the dual boundary relation of $(i, x, j, \bar{x})$. The pairs $(i, x)$ and $(j, \bar{x})$ are boxes of $B$. Together with the right function, the set $B$ encodes the relations we want to be satisfied in DRH;
- finally, for each pair of indices $i, j$ such that $i < j$, we consider a symbol $(i \mid j)$ and, for each pair $(\bar{s}, \mu) \in \zeta(i, j)$, we consider another symbol $(i \mid j)_{\bar{s}, \mu}$. These symbols are understood as variables and we denote by $X_{(J, \zeta, M)}$ the set of those variables:

$$X_{(J, \zeta, M)} = \{(i \mid j) : i, j \in J, i < j\} \cup \{(i \mid j)_{\bar{s}, \mu} : (i, j, \bar{s}) \in \text{Dom}(M) \text{ and } \mu \in M(i, j, \bar{s})\}$$

(9)

Then, $B_H$ is a finite set of $\kappa$-equations with variables in $X_{(J, \zeta, M)}$ whose solutions are meant to be taken over $H$. If $i_0 < \cdots < i_n$ is a chain of indices in $J$, then we denote by $(i_0 \mid i_n)$ the product of variables $\prod_{k=1}^n(i_{k-1} \mid i_k)$.

Given a variable $x \in X$, the left of $x$ is the index

$$\text{left}(x) = \{i \in I : \text{there exists a box } (i, x) \text{ in } B\}.$$ 

We let prod : $\prod A S \times (\Omega_A S)^f \to \Omega_A S$ be the function sending each pair of pseudowords $(u, v)$ to its product $uv$.

A model of the system of boundary relations $S$ is a triple $M = (w, \iota, \Theta)$, where
• $w$ is a possibly empty pseudoword;
• $i : J \to \alpha_w + 1$ is an injective function that preserves the order and such that, if $J$ is not the empty set then $i$ sends $\min(J)$ to 0 and $\max(J)$ to $\alpha_w$;
• for each triple $(i, j, \vec{s})$ in $\text{Dom}(M)$ and each $\mu$ in $M(i, j, \vec{s})$, $\Theta(i, j, \vec{s}, \mu)$ is a pair $(\Phi(i, j, \vec{s}, \mu), \Psi(i, j, \vec{s}, \mu))$ of $\prod_\mathcal{A}S \times (\prod_\mathcal{A}S)^I$ such that $c(\Psi(i, j, \vec{s}, \mu)) \subseteq \bar{c}(\Phi(i, j, \vec{s}, \mu))$.

**Notation 6.1.** When there exists a map $i : J \to \alpha_w + 1$ as above, we may write $w(i, j)$ instead of $w[i(i), i(j)]$ (recall Notation 3.7).

Moreover, the following properties are required for $\mathcal{M}$:
(M.1) if $(i, j, \vec{s}) \in \text{Dom}(M)$ and $\mu \in M(i, j, \vec{s})$, then $\text{prod} \circ \Theta(i, j, \vec{s}, \mu) = \text{DRH} w(i, j)$;
(M.2) if $(i, j, \vec{s}) \in \text{Dom}(M)$, $\vec{s} = (s_1, s_2)$, and $\mu \in M(i, j, \vec{s})$, then
$$\varphi(\Phi(i, j, \vec{s}, \mu)) = s_1 \text{ and } \varphi(\Psi(i, j, \vec{s}, \mu)) = s_2;$$
(M.3) if $i \prec j$, then $\bar{c}(w(i, j)) = \chi(i, j)$;
(M.4) if $(i, x, j, \vec{\pi}) \in \mathcal{B}$, then $\text{DRH}$ satisfies $w(i, \text{right}(x)) \mathcal{R} w(j, \text{right}(\vec{\pi}))$;
(M.5) let $\mathcal{C} := (J, \iota, M, \Theta)$ and $\delta_{w, \mathcal{C}} : \prod_{X, \iota, (\mathcal{C}, M)} S \to \prod_\mathcal{A}S$ be the unique continuous homomorphism defined by

$$(10) \quad \delta_{w, \mathcal{C}}(i | j \vec{s}, \mu) = \Psi(i, j, \vec{s}, \mu).$$

Then, $\delta_{w, \mathcal{C}}$ is a solution modulo $H$ of $\mathcal{B}_H$.

We say that $\mathcal{M}$ is a *model of $S$ in $\kappa$-words* if $w \in (\Omega^*_\mathcal{A}S)^I$ and the coordinates of $w$ are given by $\kappa$-words. By Proposition 5.5 to prove that $\text{DRH}$ is completely $\kappa$-reducible, it is enough to prove that $\text{DRH}$ is $\kappa$-reducible for certain systems of equations of the form $S_{u,v}$. With that in mind, we associate to such a system $S_{u,v}$ a system of boundary relations, denoted $\overline{S}_{u,v}$. Then, we construct a model of $\overline{S}_{u,v}$ and prove that the existence of a model in $\kappa$-words entails the existence of a solution of the original system $S_{u,v}$ also in $\kappa$-words (Theorem 5.3).

Let $\delta : \prod_X S \to \prod_\mathcal{A}S$ be a solution modulo $\text{DRH}$ of $S_{u,v} = \{u' = v'\} \cup S_1 \cup S_2$ such that $\delta$ is reduced with respect to $u' = v'$ and for every equation $xy = z$ of $S_1$ we have $c(\delta(x)) \subseteq \overline{c}(\delta(x))$ (recall Proposition 5.5). Suppose that $u' = x_1 \cdots x_r$ and $v' = x_{r+1} \cdots x_t$, and write $S_1 = \{x(i)y(i) = z(i)\}_{i=1}^N$ and $S_2 = \{xa^\omega = x : a \in A_x\}_{x \in X}$. Let $G$ be an undirected graph whose vertices are given by the set $\{1, \ldots, t\}$ and that has an edge connecting the vertices $p$ and $q$ if and only if $p \neq q$ and either $x_p = x_q$ or $\{x_p, x_q\} = \{x(i), z(i)\}$ for a certain $i$. Let $\widehat{G}$ be a spanning forest for $G$. We define

$$(11) \quad \overline{S}_{u,v} = (X, J, \iota, M, \chi, \text{right}, \mathcal{B}, \mathcal{B}_H)$$
as follows:

• the set of variables is $X = \{(p, q) : \text{there is an edge in } \widehat{G} \text{ connecting } p \text{ and } q\} \cup \{1\} \cup \{r\}$, and the involution in $X$ is given by $|(p, q)| = (q, p)$ and by $1 = r$;
• the set of indices is $J = \{i_0, \ldots, i_t\}$ with $i_0 < \cdots < i_t$;
• the function $\iota$ is defined by $\iota(p-1, i_p) = \{(\nu(x_p), I)\}$ for every $p = 1, \ldots, t$;
• we set $M(i_{p-1}, i_p, (\nu(x_p), I)) = 1$ for every $p = 1, \ldots, t$;
• the function $\chi$ sends each pair $(i_{p-1}, i_p)$ to the set $A_{x_p}$.
the right function is given by \( \text{right}(p,q) = i_p, \text{right}(l) = i_r, \) and \( \text{right}(r) = i_t; \)

- the set of boundary relations \( B \) contains the boundary relations \((i_0, 1, i_r, r), \) and \((i_r, r, i_0, 1)\) plus all the boundary relations of the form \((i_{p-1}, (p,q), i_{q-1}, (q,p)),\) where \((p,q) \in \mathcal{X};\)

- we put in \( B_H \) the equations \((i_0 | i_r) = (i_r | i_t) \) and \((i_{p-1} | i_p) = (i_{q-1} | i_q),\) whenever \( x_p = x_q, \) and the equation \((i_{p-1} | i_p)(i_{m-1} | i_m) = (i_{q-1} | i_q)\) for each \( x_p x_m = x_q \in \mathcal{S}_1.\)

Example 6.2. Let \( \mathcal{X} = \{x, y, z, u = x y z, v = x^2 z, \) and let \( \delta : \overline{\Pi}_X S \to \overline{\Pi}_A S \) be defined by \( \delta(x) = a, \delta(y) = (ab)^\omega, \) and \( \delta(z) = (ba)^\omega. \) Clearly, the homomorphism \( \delta \) is a solution modulo DRH of \( u = v \) and the system \( S_{u=v} = \{ u' = v' \} \cup S_1 \cup S_2 \) is given by \( u' = x t y z \# y \# , \) \( v' = x^2 z \# y \# , S_1 = \{ t y z = y x \}, \) and \( S_2 = \{ y a^\omega = y, y b^\omega = y, t y x a^\omega = t y x, t y x b^\omega = t y x \}. \) The extended solution \( \delta \) is obtained by letting \( \delta(t y z) = (ab)^\omega a. \) Then, the set of indices is \( J = \{ i_0, i_1, \ldots, i_{11} \}. \) Although the graph \( G \) is unique, there are several possibilities for \( \tilde{G} \), so that the set of variables \( \mathcal{X} \) is not uniquely determined. One of the possible choices of \( \tilde{G} \) produces the following \( \mathcal{X}: \)

\[
\mathcal{X} = \{(1, 6), (6, 1), (6, 7), (7, 6), (2, 4), (4, 2), (3, 9), (9, 3), (5, 11), (11, 5), l, r\}.
\]

We schematize the set of boundary relations \( B \) in Fig. 1.

Finally, the set \( B_H \) contains the equations \((i_0 | i_1) = (i_5 | i_6) = (i_6 | i_7), (i_2 | i_3) = (i_8 | i_9), (i_4 | i_5) = (i_{10} | i_{11}), (i_0 | i_5) = (i_5 | i_{11}), \) and \( (i_1 | i_2) = (i_3 | i_4) (i_0 | i_1). \)

A candidate to be a model of \( \overline{S}_{u=v} \) is \( M_{u=v} = (w, t, \Theta), \) where

- \( w = \delta(uv); \)
- \( t : J \to \alpha_w + 1 \) is given by \( t(i_0) = 0, \) and \( t(i_p) = \alpha_{\delta(x_1 \ldots x_p)}, \) for each \( p = 1, \ldots, t; \)
- \( \Theta(i_{p-1}, i_p, (\nu(x_p), I), 0) = (\delta(x_p), I), \) for \( p = 1, \ldots, t. \)

Proposition 6.3. The tuple \( \overline{S}_{u=v} \) in (11) is a system of boundary relations which has \( M_{u=v} \) as a model. Moreover, if \( \overline{S}_{u=v} \) admits a model in \( \kappa \)-words, then the system of equations \( S_{u=v} \) has a solution modulo DRH in \( \kappa \)-words.

**Proof.** For the first part, we notice that the Properties \([\text{M.1}][\text{M.3}]\) of the requirements for being a model are given for free from the construction. Let \((i, x, j, x)\) be a boundary relation. Since each equation \( x(k) y(l) = z(k) \) of \( S_1 \) is such that the inclusion \( c(\delta(y(k))) \subseteq \delta(c(x(k))) \) holds, whenever an edge in the graph \( \tilde{G} \) links two indices \( p \) and \( q, \) the elements \( \delta(x_p) = \Phi(i_{p-1}, i_p, (\nu(x_p), I), 0) \) and \( \delta(x_q) = \Phi(i_{q-1}, i_q, (\nu(x_q), I), 0) \) are \( R \)-equivalent modulo DRH. Therefore, unless \((i, x, j, x)\) is one of the relations \((i_0, 1, i_r, r)\) or \((i_r, r, i_0, 1),\) the
Property \(\text{[M.4]}\) is trivially satisfied. For those relations, we just need to observe that \(w(i_0, \text{right}(l)) = \delta(u)\) and \(w(i_r, \text{right}(r)) = \delta(v)\). The last Property \(\text{[M.5]}\) translates into the verification of pseudoidentities modulo \(H\) that are satisfied by the pseudovariety \(\text{DRH}\) by construction. This proves that \(\mathcal{M}_{u=v}\) is a model of \(\text{DRH}\).

For the second assertion, we consider a model of \(\mathcal{M}' = (u', t', \Theta')\), and we let \(\varepsilon : \overline{\Omega}_X \mathcal{S} \to \overline{\Omega}_A \mathcal{S}\) be the continuous homomorphism that sends the variable \(x\) to \(\prod \circ \Theta(i_{p-1}, i_p, (\nu(x_p), I), 0)\), where \(p\) is such that \(x_p = x\). Such an \(x_p\) exists for every variable since we are assuming that all the variables occur in \(u' = v'\). It is worth to mention that the value modulo \(H\) we assign to \(\varepsilon(x)\) when \(x = x_p\) for some \(p\) does not depend on the chosen \(p\). By Property \(\text{[M.2]}\) all the constraints imposed by \(\mathcal{S}_{u=v}\) are satisfied by \(\varepsilon\). The following computation shows that \(\text{DRH}\) satisfies \(\varepsilon(u') = \varepsilon(v')\):

\[
\varepsilon(u') = \varepsilon(x_1 \cdots x_r) = \varepsilon(x_1) \cdots \varepsilon(x_r)
= \prod \circ \Theta(i_0, i_1, (\nu(x_1), I), 0) \cdots \prod \circ \Theta(i_{r-1}, i_r, (\nu(x_r), I), 0)
= \prod \circ \Theta(i_0, i_1, ) \cdots w'(i_{r-1}, i_r) = w'(i_0, i_r)
= \prod \circ \Theta(i_{r-1}, i_r, (\nu(x_r), I), 0) \cdots \prod \circ \Theta(i_{t-1}, i_t, (\nu(x_t), I), 0)
= \varepsilon(x_{r+1}) \cdots \varepsilon(x_t) = \varepsilon(x_{r+1} \cdots x_t) = \varepsilon(v').
\]

The reason for \((*)\) is the fact that the relation \((i_0, 1, i_r, r)\) belongs to \(\mathcal{B}\) and the equation \((i_0 | i_r) = (i_r | i_0)\) to \(\mathcal{B}_H\), together with Properties \(\text{[M.4]}\) and \(\text{[M.5]}\) and with Lemma 3.13. For the system \(\mathcal{S}_2\), we point out that its only aim is to fix the cumulative content of the variables and Property \(\text{[M.3]}\) ensures that. Finally, let \(x_p x_m = x_q\) be an equation of \(\mathcal{S}_1\). Since for such an equation, we have a relation \((i_{p-1}, (p, q), i_{q-1}, (q, p))\) in \(\mathcal{B}\) and an equation \((i_{p-1} | i_p) (i_{m-1} | i_m) = (i_{q-1} | i_q)\) in \(\mathcal{B}_H\), from \(\text{[M.4]}\) we deduce that \(\varepsilon(x_p)\) and \(\varepsilon(x_q)\) are \(\mathcal{R}\)-equivalent in \(\text{DRH}\) and from \(\text{[M.5]}\) that \(\varepsilon(x_p) \varepsilon(x_m) = \varepsilon(x_q)\) is a valid pseudoidentity in \(H\). In addition, the assumption that \(S\) has a content function together with Property \(\text{[M.2]}\) yield that \(c(\delta(x)) = c(\varepsilon(x))\). In turn, we already observed that \(\overline{c}(\delta(x)) = \overline{c}(\varepsilon(x))\). Therefore, as by construction of \(\mathcal{S}_{u=v}\) we know that \(c(\delta(x_m)) \subseteq \overline{c}(\delta(p)),\) we have \(\varepsilon(x_p) \varepsilon(x_m) \in \mathcal{R} \varepsilon(x_q)\) modulo \(\text{DRH}\), and from Lemma 3.13 we obtain that \(\text{DRH}\) satisfies \(\varepsilon(x_p) \varepsilon(x_m) = \varepsilon(x_q)\).

The following criterion for having complete \(\kappa\)-reducibility of a pseudovariety \(\text{DRH}\) follows from Proposition 3.5 together with Proposition 6.3.

**Corollary 6.4.** If every system of boundary relations which has a model also has a model in \(\kappa\)-words, then \(\text{DRH}\) is a completely \(\kappa\)-reducible pseudovariety.

7. Factorization schemes

A **factorization scheme** for a pseudoword \(w\) is a tuple \(\mathcal{C} = (J, \iota, M, \Theta)\), where:

- \(J\) is a totally ordered finite set;
- \(\iota : J \to \alpha_w + 1\) is an injective function that preserves the order;
- \(M : \{(i, j, s) \in J \times J \times (\mathcal{S} \times \mathcal{S})\} \to \omega \setminus \{0\}\) is a partial function;
- \(\Theta : \{(i, j, s, \mu) : (i, j, s) \in \text{Dom}(M), \mu \in M(i, j, s)\} \to \overline{\Omega}_A \mathcal{S} \times (\overline{\Omega}_A \mathcal{S})^I\) is a function that sends each tuple \((i, j, s, \mu)\) to a pair \((\Phi(i, j, s, \mu), \Psi(i, j, s, \mu))\) and satisfies \(c(\Psi(i, j, s, \mu)) \subseteq \overline{c}(\Phi(i, j, s, \mu))\).
Moreover, if \((i, j, \vec{s}) \in \text{Dom}(M)\) and \(\mu \in M(i, j, \vec{s})\), then the following properties should be satisfied:

\begin{align*}
\text{(FS.1)} \quad \text{prod} \circ \Theta(i, j, \vec{s}, \mu) &= \text{DRH}\ w[i(i), \nu(j)]; \\
\text{(FS.2)} \quad \text{if } \vec{s} = (s_1, s_2), \text{ then } \varphi(\Phi(i, j, \vec{s}, \mu)) &= s_1 \text{ and } \varphi(\Psi(i, j, \vec{s}, \mu)) = s_2.
\end{align*}

We say that \(\mathcal{C}\) is a factorization scheme in \(\kappa\)-words if the coordinates of \(\Theta\) take \(\kappa\)-words as values. It is easy to check that, given a system of boundary relations \(\mathcal{S}\) and a model \(\mathcal{M}\) for \(\mathcal{S}\), the pair \((\mathcal{S}, \mathcal{M})\) determines a factorization scheme for \(w\), namely \((J, \iota, M, \Theta)\), which we denote by \(\mathcal{C}(\mathcal{S}, \mathcal{M})\). Furthermore, a factorization scheme \(\mathcal{C}\) for a pseudoword \(w\) induces functions \(\zeta_{w, \mathcal{C}}\) and \(\chi_{w, \mathcal{C}}\) as follows

\begin{align*}
\zeta_{w, \mathcal{C}} : \{(i, j) \in J \times J : i < j\} &\to 2^{S \times S^f} \\
(i, j) &\mapsto \{\vec{s} : (i, j, \vec{s}) \in \text{Dom}(M)\},
\end{align*}

and

\begin{align*}
\chi_{w, \mathcal{C}} : \{(i, j) \in J \times J : i < j\} &\to 2^A \\
(i, j) &\mapsto \mathcal{C}(w[i(i), \nu(j)]).
\end{align*}

The reason for using this notation becomes clear with the following lemma, whose proof we leave to the reader.

**Lemma 7.1.** Let \(\mathcal{S} = (X, J, \zeta, M, \chi, \text{right}, B, B_{\mu})\) be a system of boundary relations, \(w\) a pseudoword, and \(\mathcal{C} = (J, \iota, M, \Theta)\), a factorization scheme for \(w\). We define \(\mathcal{M} = (w, \nu, \Theta)\) as a candidate for a model of \(\mathcal{S}\). If \(\zeta = \zeta_{w, \mathcal{C}}\) and \(\chi = \chi_{w, \mathcal{C}}\), then the Properties \([M.1]\)–\([M.3]\) are satisfied.

For \(k = 1, 2\), let \(\mathcal{C}_k = (J_k, \iota_k, M_k, \Theta_k)\) be a factorization scheme for \(w\). We say that \(\mathcal{C}_1\) is a refinement of \(\mathcal{C}_2\) if the following properties are satisfied:

(R.1) \(\text{Im}(\iota_2) \subseteq \text{Im}(\iota_1)\);

(R.2) there exists a function

\[
\Lambda : \{(i, j, \vec{s}, \mu) : (i, j, \vec{s}) \in \text{Dom}(M_2) \land \mu \in M_2(i, j, \vec{s})\} \to \bigcup_{k \geq 1} (S \times S^f)^k \times \omega
\]

such that, if \(\Lambda(i, j, \vec{s}, \mu) = ((\vec{t}_1, \ldots, \vec{t}_n), \mu')\), then the following holds:

(R.2.1) there are \(n + 1\) elements \(i_0, \ldots, i_n\) in \(J_1\) such that \(i_0 < \cdots < i_n\), \(\iota_2(i) = \iota_1(i_0)\), and \(\iota_2(j) = \iota_1(i_n)\);

(R.2.2) \((m, m, \vec{t}_m) \in \text{Dom}(M_1)\) for \(m = 1, \ldots, n\);

(R.2.3) if \(\vec{s} = (s_1, s_2)\) and \(\vec{t}_m = (t_{m, 1}, t_{m, 2})\) for \(m = 1, \ldots, n\), then the equalities \(s_1 = t_{1, 1} t_{2, 1} \cdots t_{n-1, 1} t_{n-1, 2} \cdots t_{n, 2}\) hold;

(R.2.4) \(\mu' \in M_1(i_{n-1}, i_n, \vec{t}_n)\) and \(\Psi_2(i, j, \vec{s}, \mu) = \Psi_1(i_{n-1}, i_n, \vec{t}_n, \mu')\) modulo \(H\).

We call the function \(\Lambda\) in \([R.2]\) a refining function from \(\mathcal{C}_2\) to \(\mathcal{C}_1\).

**Proposition 7.2** (cf. [6 Proposition 8.1]). Let \(\mathcal{C}_k = (J_k, \iota_k, M_k, \Theta_k) \quad (k = 1, 2)\) be factorization schemes for a given pseudoword \(w\). Then, there is a factorization scheme \(\mathcal{C}_3 = (J_3, \iota_3, M_3, \Theta_3)\) for \(w\) which is a common refinement of \(\mathcal{C}_1\) and \(\mathcal{C}_2\). Moreover, if \(\mathcal{C}_1\) and \(\mathcal{C}_2\) are both refinement schemes in \(\kappa\)-words, then we may choose \(\mathcal{C}_3\) with the same property.
Proof. Let $J_3 = i_1(J_1) \cup i_2(J_2)$ and $\iota_3 : J_3 \to \alpha_w + 1$ be the inclusion of ordinals. Starting with $\Theta_3$ defined nowhere, we extend it inductively as follows. Fix $k, \ell \in \{1, 2\}$ with $k \neq \ell$, and let $i < j$ in $J_k$. Let $p_1, \ldots, p_m \in J_k$ be the indices that are sent by $i_\ell$ to an ordinal between $\iota_k(i)$ and $\iota_k(j)$ and suppose that $\{\beta_0, \beta_1, \ldots, \beta_n\} = \iota_k(\{i, j\}) \cup \iota_k(\{p_1, \ldots, p_m\})$ with $\beta_0 < \cdots < \beta_n$. Then, for $r = 1, \ldots, n$, the relation $\beta_{r-1} < \beta_r$ holds in $J_3$. We fix $\bar{s} \in \zeta_w, \zeta_k(i, j)$, with $\bar{s} = (s_1, s_2)$. For each $r < n$, let
\[
\bar{t}_r = (\varphi(\Phi_k(i, j, \bar{s}, 0)|\beta_{r-1}, \beta_r], I),
\]
and let $\mu_r = \{\bar{p} : \Theta_3(\beta_{r-1}, \beta_r, \bar{t}_r, \bar{p}) \text{ is defined} \} + 1$.

We set
\[
\Theta_3(\beta_{r-1}, \beta_r, \bar{t}_r, \mu_r) = \left(\text{prod} \circ \Theta_k(i, j, \bar{s}, 0)|\beta_{r-1}, \beta_r[0, I] \right).
\]

For $r = n$, we take
\[
\bar{t}_n = (\varphi(\Phi_k(i, j, \bar{s}, \mu)|\beta_n, \beta_n], s_2).
\]

Then, for each $\mu \in M(i, j, \bar{s})$, we set
\[
\Theta_3(\beta_{n-1}, \beta_n, \bar{t}_n, \mu) = (\Phi_k(i, j, \bar{s}, \mu)|\beta_{n-1}, \beta_n], \Psi_k(i, j, \bar{s}, \mu)),
\]
\[
\Lambda_k(i, j, \bar{s}, \mu) = ((\bar{t}_1, \ldots, \bar{t}_n), \mu'),
\]
where
\[
\mu' = \{\bar{p} : \Theta_3(\beta_{n-1}, \beta_n, \bar{t}_n, \bar{p}) \text{ is defined} \} + 1.
\]

We repeat this process for all possible choices of $k, \ell, i$, and $j$. Finally, we set $M_3(\beta, \gamma, \bar{t}) = \{\mu : \Theta_3(\beta, \gamma, \bar{t}, \mu) \text{ is defined} \}$ whenever $\Theta_3(\beta, \gamma, \bar{t}, 0)$ is defined.

Then, the way the construction was performed guarantees not only that $\zeta_3$ is a factorization scheme for $w$, but also that it is a common refinement of $\zeta_1$ and $\zeta_2$. Moreover, it follows from Lemma 3.8 that if $\zeta_1$ and $\zeta_2$ are both factorization schemes in $\kappa$-words, then so is $\zeta_3$.

If $\zeta_1 = (J_1, \iota_1, M_1, \Theta_1)$ is a factorization scheme for $w$, then it induces a set of factorizations for $w$. However, it might be useful to consider the set of factorizations that we obtain by multiplying some of the adjacent factors. To this end, we define what is a candidate for a refining function to $\zeta_1$ with respect to $J_2$: given a totally ordered finite set $J_2$ and an order preserving injective function $\iota_2 : J_2 \to \alpha_w + 1$ such that $\text{Im}(\iota_2) \subseteq \text{Im}(\iota_1)$, it consists of a partial function $\Lambda : \{(i, j, \bar{s}, \mu) \in J_2 \times J_2 \times (S \times S^I) \times \omega : i < j \} \to \bigcup_{k \geq 1} (S \times S^I)^k \times \omega$

such that
\begin{enumerate}
  \item[(C.1)] $\text{Dom}(\Lambda)$ is finite;
  \item[(C.2)] if $(i, j, \bar{s}, \mu) \in \text{Dom}(\Lambda)$ and $\mu' \in \mu$, then $(i, j, \bar{s}, \mu') \in \text{Dom}(\Lambda)$;
  \item[(C.3)] If $(i, j, \bar{s}, \mu) \in \text{Dom}(\Lambda)$ and $\Lambda(i, j, \bar{s}, \mu) = ((\bar{t}_1, \ldots, \bar{t}_n), \mu')$, then
    \begin{enumerate}
      \item[(C.3.1)] there exist $n + 1$ elements $i_0, \ldots, i_n \in J_1$, such that $i_0 < \cdots < i_n$, $\iota_2(i) = \iota_1(i_0)$, and $\iota_2(j) = \iota_1(i_n)$;
      \item[(C.3.2)] if $\bar{s} = (s_1, s_2)$ and $\bar{t}_m = (t_m, t_m)$ for $m = 1, \ldots, n$, then the equalities $s_1 = t_{1,1} t_{1,2} \cdots t_{n-1,1} t_{n-1,2} t_{n,1}$ and $s_2 = t_{n,2}$ hold;
      \item[(C.3.3)] for $m = 1, \ldots, n$, $(i_{m-1}, i_m, \bar{t}_m) \in \text{Dom}(M_1)$ and $\mu' \in M_1(i_{m-1}, i_m, \bar{t}_m)$.
    \end{enumerate}
\end{enumerate}

Given a candidate $\Lambda$ for a refining function to $\zeta_1$ with respect to $J_2$, we define the tuple $\zeta_2 = (J_2, \iota_2, M_2, \Theta_2)$ as follows:
Then, the homomorphism \( \delta \)
- we let \( \text{Dom}(M_2) = \{(i,j,\bar{s}) : \exists \mu \in \omega \mid (i,j,\bar{s},\mu) \in \text{Dom}(\Lambda)\}; \)
- if \( (i,j,\bar{s}) \in \text{Dom}(M_2) \), then we let \( M_2(i,j,\bar{s}) = \{\mu : (i,j,\bar{s},\mu) \in \text{Dom}(\Lambda)\}; \)
- let \( (i,j,\bar{s}) \in \text{Dom}(M_2) \) and \( \mu \in M(i,j,\bar{s}) \). If \( \Lambda(i,j,\bar{s},\mu) = ((\bar{t}_1,\ldots,\bar{t}_n),\mu') \) and \( i_0 < \cdots < i_n \) in \( J_1 \) are such that \( t_2(i) = t_1(i_0) \) and \( t_2(j) = t_1(i_n) \), then we define

\[
\Phi_2(i,j,\bar{s},\mu) = \left( \prod_{m=1}^{n-1} \text{prod} \circ \Theta_1(i_{m-1},i_m,\bar{r}_m,0) \right) \cdot \Phi_1(i_{n-1},i_n,\bar{r}_n,\mu');
\]

\[
\Psi_2(i,j,\bar{s},\mu) = \Psi_1(i_{n-1},i_n,\bar{r}_n,\mu').
\]

We put \( \Theta_2(i,j,\bar{s},\mu) = (\Phi_2(i,j,\bar{s},\mu),\Psi_2(i,j,\bar{s},\mu)) \).

We say that \( \mathcal{C}_2 \) is the restriction of \( \mathcal{C}_1 \) to \( J_2 \) with respect to \( \Lambda \). The following result justifies this terminology. It is a routine matter to prove it.

**Proposition 7.3.** Let \( \mathcal{C}_1, \mathcal{C}_2 \) and \( \Lambda \) be as above. Then,
- (a) \( \mathcal{C}_2 \) is a factorization scheme for \( w \);
- (b) \( \mathcal{C}_1 \) is a refinement of \( \mathcal{C}_2 \);
- (c) \( \Lambda \) is a refining function from \( \mathcal{C}_2 \) to \( \mathcal{C}_1 \).

Moreover, if \( \mathcal{C}_1 \) is a factorization scheme in \( \kappa \)-words, then so is \( \mathcal{C}_2 \).

We proceed with a few notes describing general situations that appear repeatedly later.

**Remark 7.4.** Let \( w \) be a pseudoword and \( \mathcal{C} = (J,\iota,M,\Theta) \) a factorization scheme for \( w \). Suppose that \( \mathcal{C}_1 = (J_1,\iota_1,M_1,\Theta_1) \) is a refinement of the factorization scheme \( \mathcal{C} \) and let \( \Lambda \) be a refining function from \( \mathcal{C} \) to \( \mathcal{C}_1 \). Finally, suppose that \( \mathcal{C}_1 = (J_1,\iota_1',M_1,\Theta_1') \) is a factorization scheme for another pseudoword \( w' \). The function \( \Lambda \) is clearly a candidate for a refining function to \( \mathcal{C}_1' \) with respect to \( J \). Moreover, if \( \mathcal{C}' = (J,\iota',M',\Theta') \) is the restriction of \( \mathcal{C}_1' \) with respect to \( \Lambda \), then \( M' = M \).

**Notation 7.5.** Suppose that \( S = (X,J,\zeta,\chi,\text{right},\mathcal{B},\mathcal{B}_H) \) is a system of boundary relations that has \( M = (w,\iota,\Theta) \) as a model. Let \( \mathcal{C}_1 = (J_1,\iota_1,M_1,\Theta_1) \) be a refinement of \( \mathcal{C}(S,M) \) and let \( \Lambda \) be a refining function from \( \mathcal{C}_1 \) to \( \mathcal{C}(S,M) \). Define \( \xi = \iota_1^{-1} \circ \iota \). We denote by \( \xi_\Lambda(\mathcal{B}_H) \) the system of \( \kappa \)-equations with variables in \( X(J_1,\zeta_1,M_1) \) (recall [9] and [12]) obtained from \( \mathcal{B}_H \) by substituting each variable \( (i \mid j) \) by \( (\xi(i) \mid \xi(j)) \) and each variable \( \{i \mid j\}_{\bar{z},\mu} \) by \( \{\xi(j) \mid \xi(j)\}_{\bar{r},\mu'} \), where \( \Lambda(i,j,\bar{s},\mu) = ((\bar{t}_1,\ldots,\bar{t}_n),\mu') \).

**Remark 7.6.** Using the notation above, the homomorphism \( \delta_{\xi_\Lambda,\mathcal{C}_1} \) (recall [10]) is a solution modulo \( H \) of the system \( \xi_\Lambda(\mathcal{B}_H) \).

**Remark 7.7.** Keeping again the notation, suppose that we are given a pseudoword \( w_1' \) and a factorization scheme \( \mathcal{C}_1' = (J_1,\iota_1',M_1,\Theta_1') \) for \( w_1' \), such that \( \delta_{w_1',\mathcal{C}_1} \) is a solution modulo \( H \) of \( \xi_\Lambda(\mathcal{B}_H) \). Further assume that there exists a factorization scheme of the form \( \mathcal{C}' = (J,\iota',M,\Theta') \) for another pseudoword \( w' \) such that \( \zeta_{w',\mathcal{C}'} = \zeta \) and the following pseudoidentities are valid in \( H \), for every \( (i \mid j), \{i \mid j\}_{\bar{z},\mu} \in X(J,\zeta,M) \):

\[
w'(i,j) = w_1'(\xi(i),\xi(j));
\]

\[
\Psi'(i,j,\bar{s},\mu) = \Psi_1'(\xi(j),\bar{r}_n,\mu').
\]

Then, the homomorphism \( \delta_{w',\mathcal{C}'} \) is a solution modulo \( H \) of \( \mathcal{B}_H \).
8. Complete $\kappa$-reducibility of the pseudovarieties DRH

Suppose that DRH is a completely $\kappa$-reducible pseudovariety and consider a finite system of $\kappa$-equations $S = \{u_i = v_i\}_{i=1}^n$ with variables in $X$ and constraints given by the pair $(\varphi : \overline{\Omega} AS \to S, \nu : X \to S)$. Let $\delta : \overline{\Omega} AX \to \overline{\Omega} AS$ be a solution modulo $H$ of $S$. For a new variable $x_0 \notin X$, we consider a new finite system of $\kappa$-equations given by $S' = \{x_0u_i = x_0v_i\}_{i=1}^n$ and, writing $A = \{a_1, \ldots, a_k\}$, we set the constraints on $X \cup \{x_0\}$ to be given by the pair $(\varphi, \nu')$, where $\nu'|X = \nu$ and $\nu'(x_0) = \varphi((a_1 \cdots a_k)\omega)$. By Corollary 3.5, the continuous homomorphism $\delta'$ defined by

$$\delta' : \overline{\Omega} AX(x_0) \to \overline{\Omega} AS$$

$$x \mapsto \delta(x), \text{ if } x \in X$$

$$x_0 \mapsto (a_1 \cdots a_k)^\omega$$

is a solution modulo DRH of $S'$. Since we are assuming that DRH is completely $\kappa$-reducible, there exists a solution in $\kappa$-words modulo DRH of $S'$. Of course, any solution modulo DRH of $S'$ provides a solution modulo $H$ of $S$, by restriction to $\overline{\Omega} AX$. Hence, we proved the following.

**Proposition 8.1.** If DRH is a completely $\kappa$-reducible pseudovariety, then $H$ is completely $\kappa$-reducible as well.

It is known that neither any proper non-locally finite subpseudovariety of $\text{Ab}$ [18] nor the pseudovarieties $G$ [17] and $G_p$ [14] are completely $\kappa$-reducible. Hence, we have the following.

**Corollary 8.2.** Let $H$ be either a proper non-locally finite subpseudovariety of $\text{Ab}$, or one of the pseudovarieties $G$ and $G_p$. Then, DRH is not completely $\kappa$-reducible.

In fact, it may be proved that both DRH, for $H \subseteq \text{Ab}$ non-locally finite, and $DGR_p$ are not even $\kappa$-reducible [15], meaning that they are not $\kappa$-reducible with respect to the class of systems of equations that may be obtained from finite graphs (see [9] for details).

Our next goal is to prove that $H$ being completely $\kappa$-reducible also suffices for so being DRH. With that in mind, throughout this section we fix a pseudovariety of groups $H$ that is completely $\kappa$-reducible. In view of Corollary 6.4, we should prove the following.

**Theorem 8.3.** Let $S$ be a system of boundary relations that has a model. Then, $S$ has a model in $\kappa$-words.

We fix the pair $(S, M)$, where

$$S = (X, J, \zeta, M, \chi, \text{right}, B, \mathcal{B}_H)$$

is a system of boundary relations,

$$M = (w, t, \Theta)$$

is a model of $S$,

and we define the parameter

$$[S, M] = (\alpha, n),$$

where $\alpha$ is the largest ordinal of the form $\kappa(c)$ such that there exists a box $(i, x)$ with $\text{right}(x) = c$ if $B \neq \emptyset$, and is 0 otherwise, and $n$ is the number of boxes $(i, x)$ such that $\kappa(\text{right}(x)) = \alpha$. We denote by $r$ the index $\kappa^{-1}(\alpha)$. In order to prove Theorem 8.3 we argue by transfinite induction on the parameter $[S, M]$, where the pairs $(\alpha, n)$ are ordered lexicographically. The induction step amounts to associating to each pair $(S, M)$ a new pair $(S_1, M_1)$ such that the following properties are satisfied:
(P.1) \([S_1, M_1] < [S, M]\);
(P.2) if \(S_1\) has a model in \(\kappa\)-words, then \(S\) also has a model in \(\kappa\)-words.

Depending on the set of boundary relations \(B\), we consider the following cases:

Case 1: There is a box \((i, x)\) in \(B\) such that \(i = r = \text{right}(x)\).
Case 2: There is a boundary relation \((i, x, i, \overline{x})\) such that \(\text{right}(x) = r = \text{right}(\overline{x})\).
Case 3: There is a boundary relation \((i, x, j, \overline{x})\) such that \(i < j\), \(\text{right}(x) = r = \text{right}(\overline{x})\), and the inclusion \(c(w(i, j)) \subset c(w(i, \text{right}(x)))\) holds.
Case 4: There is a boundary relation \((i, x, j, \overline{x})\) such that \(\text{right}(x) < \text{right}(\overline{x}) = r\).
Case 5: There is a boundary relation \((i, x, j, \overline{x})\) such that \(i < j\), \(\text{right}(x) = r = \text{right}(\overline{x})\), and \(c(w(i, j)) = c(w(i, \text{right}(x)))\).

In each case, we assume that all the preceding cases do not apply. In [6] Section 9, where the analogous result for the pseudovariety \(R\) is proved, the cases that are considered are similar. However, the difference in definition of the induction parameter \([15]\) justifies the fact of needing to deal with one less case in the present work.

8.1. Induction basis. If the induction parameter \([S, M]\) is \((0, 0)\), then either \(B = \emptyset\) or all the boundary relations of \(B\) are of the form \((\text{min}(J), x, \text{min}(J), \overline{x})\) with \(\text{right}(x) = \text{min}(J) = \text{right}(\overline{x})\). In any case, Property \([M.4]\) for a model of \(S\) becomes trivial. Hence, having a model in \(\kappa\)-words amounts to having, for each \((i, j, \overline{s}) \in \text{Dom}(M)\) and each \(\mu \in M(i, j, \overline{s})\), a pair of \(\kappa\)-words \((\Phi(i, j, \overline{s}, \mu), \Psi(i, j, \overline{s}, \mu))\) such that the Properties \([M.1]\)–\([M.5]\) are satisfied. Note that the Property \([M.1]\) means that we should have

\[\Phi(i, j, \overline{s}_1, \mu_1)\Psi(i, j, \overline{s}_1, \mu_1) =_{\text{DRH}} \Phi(i, j, \overline{s}_2, \mu_2)\Psi(i, j, \overline{s}_2, \mu_2)\]

for all \((i, j, \overline{s}_k) \in \text{Dom}(M)\) and \(\mu_k \in M(i, j, \overline{s}_k)\), \(k = 1, 2\). We formalize that in the following proposition.

Proposition 8.4. Suppose that \(H\) is a completely \(\kappa\)-reducible pseudovariety of groups. Let \(S_1 = \{x_i, y_{i, 1} = \cdots = x_i, n_i, y_{i, n_i}\}_{i=1}^N\) and let \(S_2\) be a finite system of \(\kappa\)-equations (possibly with parameters in \(P\)). Let \(X\) be the set of variables occurring in \(S_1\) and \(S_2\) and suppose that the constraints for the variables are given by the pair \((\varphi, \nu)\). Let \(\delta : \Omega_{X, \mu}S \rightarrow (\Omega_A S)^I\) be a solution modulo \(\text{DRH}\) of \(S_1\) which is also a solution modulo \(H\) of \(S_2\) and such that, for \(i = 1, \ldots, N\) and \(p = 1, \ldots, n_i\), \(c(\delta(y_{i, p})) \subseteq c(\delta(x_{i, p}))\). Then, there exists a continuous homomorphism \(\varepsilon : \Omega_{X, \mu}S \rightarrow (\Omega_A S)^I\) such that

(a) \(\varepsilon(X) \subseteq (\Omega_A S)^I\);
(b) \(\varepsilon\) is a solution modulo \(\text{DRH}\) of \(S_1\);
(c) \(\varepsilon\) is a solution modulo \(H\) of \(S_2\);
(d) \(\varepsilon(\delta(x)) = c(\delta(x))\), for all the variables \(x \in X\).

Proof. We argue by induction on \(m = \max\{|c(\delta(x_{i, p}))| : i = 1, \ldots, N; p = 1, \ldots, n_i\}\).

Note that, if \(\delta(x_{i, 1}) = I\), then we may discard the equations \(x_{i, 1, y_{i, 1}} = \cdots = x_{i, n_i, y_{i, n_i}}\). Hence, when \(m = 0\), the result amounts to proving the existence of \(\varepsilon\) satisfying \([a]\)–\([c]\) and \([d]\).

But that comes for free from the fact that \(H\) is completely \(\kappa\)-reducible, together with Lemma \([5, 7]\).

Now, assume that \(m \geq 1\) and suppose that \(\delta(x_{i, p}) \neq I\), for all \(i, p\). For each variable \(x\) and each \(k \geq 1\) such that \(\text{lb}_{f_k}(\delta(x))\) is defined we write

\[\text{lb}_{f_k}(\delta(x)) = \delta(x)_ka_{x, k},\]

\[\delta(x) = \text{lb}_1(\delta(x)) \cdots \text{lb}_{f_k}(\delta(x))\delta(x)_k.\]
Since $X$, $A$ and $S$ are finite, there exist $1 \leq k < \ell$ such that, for all $x \in X$ with $c(\delta(x)) \neq \emptyset$, the following equalities hold:

$$c(\delta(x)) = c(lbf_{k+1}(\delta(x)));$$

$$\varphi(lbf_1(\delta(x)) \cdots lbf_k(\delta(x))) = \varphi(lbf_1(\delta(x)) \cdots lbf_{\ell}(\delta(x))).$$

In particular, the latter equality yields

$$\varphi(\delta(x)) = \varphi(lbf_1(\delta(x)) \cdots lbf_k(\delta(x)))\varphi(lbf_{k+1}(\delta(x)) \cdots lbf_{\ell}(\delta(x))) \varphi(\delta(x)_k).$$

For $i = 1, \ldots, N$, set

$$\ell_i = \begin{cases} \ell, & \text{if } c(\delta(x_{i,1})) \neq \emptyset, \\ \delta(x_{i,1}), & \text{otherwise.} \end{cases}$$

We consider a new set of variables $X'$ given by

$$X' = X \uplus \{x_{i,p,j}: i = 1, \ldots, N; p = 1, \ldots, n_i; j = 1, \ldots, \ell_i\} \uplus \{x'_{i,p}: i = 1, \ldots, N; p = 1, \ldots, n_i; c(\delta(x_{i,p})) \neq \emptyset\},$$

where the variables $x_{i,p,j}$ and $x_{i,q,j}$, and the variables $x'_{i,p}$ and $x'_{i,q}$ (if defined) are the same, whenever the variables $x_{i,p}$ and $x_{i,q}$ are also the same. We also consider the following systems of equations with variables in $X'$:

- $S'_1 = \{x_{i,1,j} = \cdots = x_{i,n,j}: i = 1, \ldots, N; j = 1, \ldots, \ell_i\};$
- $S'_2$ is the system of equations obtained from $S_2$ by substituting each one of the variables $x_{i,p}$ by the product $P_{i,p}$ given by

  $$P_{i,p} = \begin{cases} x_{i,p,1}a_{x_{i,p,1}} \cdots x_{i,p,\ell_i}a_{x_{i,p,\ell_i}}, & \text{if } c(\delta(x_{i,p})) \neq \emptyset, \\ x_{i,p,1}a_{x_{i,p,1}} \cdots x_{i,p,\ell_i}, & \text{otherwise;} \end{cases}$$

- $S''_2 = \{x'_{i,1}z_{i,1} = \cdots = x'_{i,n_i}z_{i,n_i}: i = 1, \ldots, N; c(\delta(x_{i,1})) \neq \emptyset\}$, where we take

  $$z_{i,p} = \begin{cases} P_{j,q}, & \text{if } y_{i,p} = x_{j,q} \text{ for some } j = 1, \ldots, N; q = 1, \ldots, n_j, \\ y_{i,p}, & \text{otherwise.} \end{cases}$$

In the systems $S'_2$ and $S''_2$ the letters in $A$ work as parameters evaluated to themselves, so that the system of equations $S'_2 \cup S''_2$ has parameters in $P' = P \cup A$. We let the constrains for the variables be given by the pair $(\varphi, \nu')$, where the map $\nu'$ is given by

$$\nu' : X' \to S$$

$$x \mapsto \nu(x), \quad \text{if } x \in X;$$

$$x_{i,p,j} \mapsto \varphi(\delta(x_{i,p,j})), \quad \text{if } x_{i,p,j} \in X' \setminus X;$$

$$x'_{i,p} \mapsto \varphi(\delta(x'_{i,p})), \quad \text{if } x'_{i,p} \in X' \setminus X;$$

Let $\delta' : \overline{\Omega}(X \cup P)S \to \overline{\Omega}_A S$ be the continuous homomorphism defined by

$$\delta'(y) = \delta(y), \quad \text{if } y \in X \cup P;$$

$$\delta'(x_{i,p,j}) = \delta(x_{i,p,j}), \quad \text{if } i = 1, \ldots, N; p = 1, \ldots, n_i; j = 1, \ldots, \ell_i;$$

$$\delta'(x'_{i,p}) = \delta(x'_{i,p}), \quad \text{if } i = 1, \ldots, N; p = 1, \ldots, n_i; c(\delta(x_{i,p})) \neq \emptyset;$$

$$\delta'(a) = a, \quad \text{if } a \in A.$$
Then, $\delta'$ is a solution modulo DRH of $S'_1$ which is also a solution modulo $H$ of $S_2' \cup S''_2$. Since we decreased the induction parameter and the pair $(S'_1, S'_2 \cup S''_2)$ satisfies the hypothesis of the proposition, we may invoke the induction hypothesis to derive the existence of a solution in $\kappa$-words modulo DRH of $S'_1$, and modulo $H$ of $S'_2 \cup S''_2$, satisfying condition $[d]$. Now, we define the continuous homomorphism $\varepsilon : \Omega_{X \cup P} S \to \Omega_{A} S$ by:

$$
\varepsilon(x_i) = \begin{cases}
\varepsilon'(x_i;1) & \text{if } \overline{c}(\delta(x_i)) \neq \emptyset; \\
\varepsilon'(P_i), & \text{if } \overline{c}(\delta(x_i)) = \emptyset;
\end{cases}
$$

$$
\varepsilon(x) = \varepsilon'(x), \quad \text{otherwise.}
$$

Clearly, $\varepsilon(X) \subseteq \Omega_{A} S$. Moreover, since we are assuming that $S$ has a content function, it follows from $\varphi \circ \varepsilon' = \varphi \circ \delta'$ that $\overline{c}(\varepsilon(x_i)) = \overline{c}(\delta(x_i))$, for all $i$, $p$. For the other variables $x \in X$, the condition $[d]$ for $\varepsilon$ follows from the same condition for $\varepsilon'$.

Let us verify that $\varepsilon$ is a solution modulo DRH of $S_1$ and a solution modulo $H$ of $S_2$. Since $\varepsilon'$ is a solution modulo DRH of $S'_1$, for every pair of variables $x_i,p$, $x_{i,q}$, DRH satisfies $\varepsilon'(x_{i,p,q}) = \varepsilon(x_{i,q,i,j})$, for $j = 1, \ldots, \ell_i$. Further, since $\delta$ is a solution modulo DRH of $S_1$ we also have $a_{x_i,p,j} = a_{x_i,q,j}$. Thus, we get

$$
\varepsilon(x_{i,p}) = \begin{cases}
\varepsilon'(x_{i,p};1)_{a_{x_i,p,k}a_{x_i,p,k}} & \text{if } \overline{c}(\delta(x_{i,p})) \neq \emptyset; \\
\varepsilon'(x_{i,p};1_{a_{x_i,p,k}a_{x_i,p,k}}) & \text{if } \overline{c}(\delta(x_{i,p})) = \emptyset;
\end{cases}
$$

In the second situation, when $\overline{c}(\delta(x_{i,p})) = \emptyset$, since $c(\delta(y_{i,p})) \subseteq \overline{c}(\delta(x_{i,p}))$, it follows that DRH satisfies $\varepsilon(x_{i,p,y_{i,p}}) = \varepsilon(x_{i,p}) = \varepsilon(x_{i,q}) = \varepsilon(x_{i,q,y_{i,q}})$. Otherwise, if $\overline{c}(\delta(x_{i,p})) \neq \emptyset$, the above equalities imply the relation $\mu(x_{i,p,y_{i,p}}) \otimes \varepsilon(x_{i,q,y_{i,q}})$ modulo DRH. Also, since $\varepsilon'$ is a solution modulo $H$ of $S'_2$, we may use Lemma 3.13 to conclude that DRH satisfies $\varepsilon(x_{i,p,y_{i,p}}) = \varepsilon(x_{i,q,y_{i,q}})$. Thus, the homomorphism $\varepsilon$ is a solution modulo DRH of $S_1$. On the other hand, the pseudovariety $H$ satisfies $\varepsilon(P_{i,p}) = \varepsilon(x_{i,p})$. By definition of $S'_2$ it follows that $\varepsilon$ is a solution modulo $H$ of $S_2$. Finally, due to [16] and [17], the constraints for the variables of $X$ are satisfied by $\varepsilon$. 

8.2. Factorization of a pair $(S, M)$. Instead of repeating the same argument several times, we use this subsection to describe a general construction that is performed later in some of the considered cases.

Let $E$ be a subset of $B$ such that, if $(i, x, j, \overline{e}) \in E$, then $(j, \overline{e}, i, x) \notin E$. Suppose that we are given a set of pairs of ordinals $\Delta = \{(\beta_e, \gamma_e)\}_{e \in E}$ such that, for each boundary relation $e = (i_e, e, j_e, \overline{e}_e) \in E$, the following properties are satisfied:

- $\ell(i_e) < \beta_e < \ell(right(x_e))$ and $\ell(j_e) < \gamma_e < \ell(right(x_e)$;
- $w[\ell(i_e), \beta_e] = \text{DRH} w[\ell(j_e), \gamma_e]$. 

We say that the factorization of $(S, M)$ with respect to $(E, \Delta)$ is the pair $(S_0, M_0)$, where

$S_0 = (X_0, J_0, \zeta_0, M_0, \chi_0, \text{right}_0, B_0, \text{DRH})_0$ and $M_0 = (w_0, \iota_0, \Theta_0)$,

are defined as follows:
• the set of variables $\mathcal{X}_0$ contains all the variables from $\mathcal{X}$ and a pair of new variables $\gamma e, \bar{\gamma}_e$ for each relation $e \in \mathcal{E}$;
• we take $w_0 = w$;
• we let $J_0$, $\iota_0$, $M_0$ and $\Theta_0$ be given by the factorization scheme $\mathcal{C}_0 = (J_0, \iota_0, M_0, \Theta_0)$, which is chosen to be a common refinement of the factorization schemes $\mathcal{C}(\mathcal{S}, \mathcal{M})$ and $\{(\gamma e, \bar{\gamma}_e) : (\beta e, \bar{\gamma}_e) \rightarrow \alpha_0 + 1, \emptyset, \emptyset\}$ for $w$. We denote by $\ell_0$ and $k_0$ the indices $\iota_0^{-1}(\beta e)$ and $\iota_0^{-1}(\gamma e)$ in $J_0$, respectively, by $\xi$ the composite function $\iota_0^{-1} \circ \iota$, and we let $A : \{ (i, j, \bar{s}, \mu) : (i, j, \bar{s}) \in \text{Dom}(M), \mu \in M(i, j, \bar{s}) \} \rightarrow \bigcup_{k \geq 0} (S \times S^I)^k \times \omega \setminus \{0\}$ be a refining function from $\mathcal{C}(\mathcal{S}, \mathcal{M})$ to $\mathcal{C}_0$;
• the maps $\xi_0$ and $\chi_0$ are, respectively, $\iota_{e_0, e_0}$ and $\chi_{e_0, e_0}$ (recall (12) and (13));
• the right$_0$ function assigns $\xi(\text{right}(x))$ to each variable $x \in \mathcal{X}$ and, for each $e \in \mathcal{E}$, we let $\text{right}_0(y_e) = \ell_e$ and $\text{right}_0(\bar{\gamma}_e) = \bar{\gamma}_e$;
• the set of boundary relations $\mathcal{B}_0$ is obtained by putting the boundary relation $(\xi(i), x, \xi(j), \overline{\gamma})$ whenever $(i, x, j, \overline{\gamma})$ neither belongs to $\mathcal{E}$ nor is the dual of a boundary relation of $\mathcal{E}$, and the boundary relations $(\xi(i_e), y_e, \xi(j_e), \overline{\gamma}_e), (\ell_e, x, \bar{\gamma}_e)$ and their duals for each $e \in \mathcal{E}$;
• the set $\langle \mathcal{B}_H \rangle_0$ contains $\xi(\mathcal{B}_H)$ as well as the equation $(\xi(i_e) \mid \ell_e) = (\xi(j_e) \mid \bar{\gamma}_e)$, for each $e \in \mathcal{E}$.

The way we construct $\mathcal{B}_0$ is illustrated in Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{factorization.png}
\caption{Factorization of $(\mathcal{S}, \mathcal{M})$, when $\mathcal{E} = \{e, f\}$.}
\end{figure}

**Proposition 8.5.** The triple $M_0$ is a model of $S_0$ such that $[S_0, M_0] = [S, M]$ and the Property \([P.2]\) is satisfied.

**Proof.** The facts that $M_0$ is a model of $S_0$ and $[S_0, M_0] = [S, M]$ are easy to derive from the construction.

For Property \([P.2]\) we suppose that $M'_0 = (w'_0, \iota'_0, \Theta'_0)$ is a model of $S_0$ in $\kappa$-words and we take $M' = (w', \iota', \Theta')$, where $w' = w'_0$, $\iota' = \iota'_0 \circ \xi$, and $\Theta'$ is given by the factorization scheme $\mathcal{C}' = (J, \iota', M, \Theta')$ corresponding to the restriction of $\mathcal{C}(S_0, M_0)$ with respect to $A$ (cf. Remark 7.4). We claim that $M'$ is a model of $S$ (in $\kappa$-words by Proposition 7.3). Properties \([M.1]\) and \([M.2]\) are a consequence of $\mathcal{C}'$ being a factorization scheme for $w'$. A simple computation shows that $\overline{\chi}(w'(i, j)) = \chi(i, j)$, so that we have \([M.3]\). Property \([M.4]\) is straightforward for all boundary relations except for the relations $(\iota_e, x, j, \overline{\gamma}_e)$ and their duals. In this case, since $(\xi(i_e), y_e, \xi(j_e), \overline{\gamma}_e)$ belongs to $\mathcal{B}_0$, $(\xi(\iota_e) \mid \ell_e) = (\xi(j_e) \mid k_e)$ belongs to $(\mathcal{B}_H)_0$, and $M'_0$ is a model of $S_0$, we invoke Lemma 8.13 to conclude that DRH satisfies $w'_0(\xi(i_e), \ell_e) = w'_0(\xi(j_e), \bar{\gamma}_e)$. On the other hand, the relation $(\ell_e, x, \bar{\gamma}_e)$ also belongs to $\mathcal{B}_0$, so that the relation $w'_0(\ell_e, \text{right}_0(x)) \mathcal{R} w'_0(\bar{\gamma}_e, \text{right}_0(\overline{\gamma}_e))$ holds modulo
Thus, we obtain $w'(i, \text{right}(x_i)) \mathcal{R} w'(j, \text{right}(\overline{\pi}_e))$ modulo DRH. Finally, since $\xi_\Lambda(\mathcal{B}_\Lambda) \subseteq (\mathcal{B}_H)_0$, we may use Remark 7.7 to conclude that in order to prove Property (M.5) it is enough to show that the following identities hold in $\xi$

It is enough to show that the following identities hold in $\xi$

The first one follows from the definition of $\Psi'$ and the fact that $\mathcal{C}'$ is the restriction of $\mathcal{C}'_0$ with respect to $\Lambda$.

8.3. Case 1. When we are in Case 1, we have at least one empty box $(r, x)$. Since for every pseudoword $w$ we have $w(r, \text{right}(x)) = w(r, r) = I$, we may delete the boundary relations involving empty boxes. In this way we obtain a new system of boundary relations $\mathcal{S}_1$ which has exactly the same models as $\mathcal{S}$ and so, Property (P.2) is satisfied. Moreover, the parameter associated to $(\mathcal{S}_1, \mathcal{M})$ is smaller than the parameter associated to $(\mathcal{S}, \mathcal{M})$ since we removed some boxes ending at $r$. Therefore, Property (P.1) also holds.

8.4. Case 2. In this case, there exists a boundary relation $(i, x, i, \overline{\pi})$ with $\text{right}(x) = r = \text{right}(\overline{\pi})$. Since such a boundary relation yields a trivial relation in (M.4) we may argue as in the previous case and simply delete $(i, x, i, \overline{\pi})$ and its dual from $\mathcal{S}$ obtaining thus a new pair $(\mathcal{S}_1, \mathcal{M})$ satisfying (P.1) and (P.2).

8.5. Case 3. This is the case where we assume the existence of a boundary relation $(i_0, x_0, j_0, \overline{\pi}_0)$ such that $i_0 < j_0$, $\text{right}(x_0) = r = \text{right}(\overline{\pi}_0)$ and $c(w(i_0, j_0)) \subseteq c(w(i_0, \text{right}(x_0)))$.

Let $a \in c(w(i_0, r)) \setminus c(w(i_0, j_0))$. Since $i_0 < j_0$, the letter $a$ also belongs to $w(j_0, r)$. Therefore, by Corollary 3.2 there are unique factorizations given by $w(i_0, r) = u_i a v_i$ and $w(j_0, r) = u_j a v_j$ such that $a \notin c(u_i) \cup c(u_j)$ and DRH satisfies the equality $u_i = u_j$ and the relation $v_i \mathcal{R} v_j$. Thus, the decreasing of the induction parameter in this case is achieved by discarding the segment $[\ell(i_0) + \alpha_{u_i}, \ell(r)]$ in the boundary relation $(i_0, x_0, j_0, \overline{\pi}_0)$ as it is outlined in Fig. 3 below.

![Figure 3](image-url)

**Figure 3.** Discarding the segment $[\ell(i_0) + \alpha_{u_i}, \ell(r)]$ in the boundary relation $(i_0, x_0, j_0, \overline{\pi}_0)$.

Let $\mathcal{E} = \{(i_0, x_0, j_0, \overline{\pi}_0)\}$ and $\Delta = \{[\ell(i_0) + \alpha_{u_i}, \ell(i_0) + \alpha_{u_i}]\}$. By the above, the pair $(\mathcal{E}, \Delta)$ satisfies (F.1) and (F.2). Let $(\mathcal{S}_0, \mathcal{M}_0)$ be the factorization of $(\mathcal{S}, \mathcal{M})$ with respect to $(\mathcal{E}, \Delta)$. Then, the pair $(\mathcal{S}_0, \mathcal{M}_0)$ is covered by Case 2 and we may use it in order to decrease the induction parameter.

Before proceeding with Cases 4 and 5 we perform an auxiliary step that is useful in both of the remaining cases.
8.6. Auxiliary step. We are interested in modifying some of the boundary relations of the form \((i, x, j, \overline{x})\) such that \(i < j\) and \(\text{right}(x) = r = \text{right}(\overline{x})\), so we assume that there exists at least one. For each \(i_0 \in \{i \in J : i < r\}\), let \(E(S, i_0) = \{(i, x, j, \overline{x}) : \text{right}(x) = r = \text{right}(\overline{x}), i < j, i \leq i_0\}\). Our goal is to prove the existence of a new pair \((S_1, M_1)\) that keeps the induction parameter unchanged, satisfies Property \((P.2)\) and such that \(E(S_1, i_0) = \emptyset\). We first construct a pair \((S_0, M_0)\) satisfying the first two properties and such that \(|E(S_0, i_0)| < |E(S, i_0)|\). Then we argue by induction to conclude the existence of such a pair \((S_1, M_1)\).

If \(E(S, i_0) \neq \emptyset\), then we fix a boundary relation \((k_0, x_0, k_1, \overline{x}_0) \in E(S, i_0)\). Property \((M.4)\) yields \(w(k_0, k_1)w(k_1, r) = w(k_0, r) \mod \text{DRH}\), which in turn implies that \(\text{DRH}\) satisfies \(w(k_0, r) \mod \text{DRH}\), which in turn implies that \(\text{DRH}\) satisfies \(w(k_0, k_1) \mod \text{DRH}\). As we are assuming that the Case 3 does not hold, the contents of \(w(k_0, k_1)\) and \(w(k_0, r)\) are the same, and so, \(\text{DRH}\) satisfies \(w(k_0, k_1) \mod \text{DRH}\). Moreover, since the product \(w(k_0, k_1) \cdot w(k_0, k_1)\) is reduced, we may use Corollary \(3.10\) to obtain \(\alpha_{w(k_0, r)} = \alpha_{w(k_0, k_1)^{\omega}} = \alpha_{w(k_0, k_1)} \cdot \omega\). In particular, setting \(\beta_p = \alpha_{w(k_0, k_1)} \cdot p\) for every \(p \geq 0\), the inequality \(\beta_p < \alpha = \iota(r)\) holds. On the other hand, as \(k_0 \leq i_0 < r\), we also have \(\alpha_{w(k_0, i_0)} < \alpha_{w(k_0, r)} = \alpha_{w(k_0, k_1)} \cdot \omega\) and therefore there exists an integer \(n \geq 1\) such that \(\alpha_{w(k_0, i_0)} < \alpha_{w(k_0, k_1)} \cdot n\). We fix such an \(n\) and we take \(E = \{(k_0, x_0, k_1, \overline{x}_0)\}\) and \(\Delta = \{\beta_{n+1}\}\). It is easy to check that the pair \((E, \Delta)\) satisfies both \((F.1)\) and \((F.2)\). So, we let \((S_0, M_0)\) be the factorization of \((S, M)\) with respect to \((E, \Delta)\). Intuitively, the transformation performed in the step \((S, M) \rightarrow (S_0, M_0)\) is represented in pictures 4 (before) and 5 (after).

![Figure 4. Original relation \((k_0, x_0, k_1, \overline{x}_0)\) in the system of boundary relations \(S\).](image)

![Figure 5. Factorization of the relation \((k_0, x_0, k_1, \overline{x}_0)\) in the new system of boundary relations \(S_0\).](image)

The definition of \((S_0, M_0)\) yields the following:

**Lemma 8.6.** Let \((S_0, M_0)\) be the pair defined above. Then the following holds:

(a) Cases 2 and 3 do not apply to the system of boundary relations \(S_0\);

(b) the inequality \(|E(S_0, i_0)| < |E(S, i_0)|\) holds.

Recall that, by Proposition 8.5, we also have \([S_0, M_0] = [S, M]\) and Property \((P.2)\) is satisfied by \((S_0, M_0)\). Thus, arguing by induction, we may assume, without loss of generality, that given a system \(S\) in Cases 4 or 5 we have \(E(S, i_0) = \emptyset\), for all \(i_0 < r \in J\).

8.7. Case 4. In this case we suppose that the Cases 1, 2 and 3 do not hold and that there is a boundary relation \((i, x, j, \overline{x})\) such that \(\text{right}(\overline{x}) < \text{right}(x) = r\). Consider the index \(\ell = \min\{\text{left}(x) : \text{right}(\overline{x}) < \text{right}(x) = r\}\). By the auxiliary step in Subsection 8.6 we may assume without loss of generality that all boundary relations \((i, x, j, \overline{x})\) with \(\text{right}(x) =
\[ r = \text{right}(\pi) \] are such that \( i, j > \ell \). Let \( x_0 \in \mathcal{X} \) be such that \( \text{left}(x_0) = \ell \) and \( \text{right}(\pi_0) < \text{right}(x_0) = r \), and let \( \ell^* \in \mathcal{J} \) be such that \( (\ell, x_0, \ell^*, \pi_0) \in \mathcal{B} \). We set \( r^* = \text{right}(\pi_0) \). Since Case 3 does not hold, we know that \( \ell < r \). The intuitive idea consists in transferring all the information comprised in the factor \( w(\ell, r) \) to the factor \( w(\ell^*, r^*) \) in order to decrease the induction parameter by discarding the factors \( w(\ell^-, r) \) and \( w(\ell^*, r^*) \) intervening in the boundary relation \( (\ell, x_0, \ell^*, \pi_0) \). See Fig. 6.

![Figure 6](image)

**Figure 6.** Transferring the segment \((\ell, r)\) to the segment \((\ell^*, r^*)\) and discarding the final segments of the boxes \((\ell, x_0)\) and \((\ell^*, \pi_0)\).

More formally, we define the set of transport positions by

\[ T = \{ i \in J : \exists \text{ box } (i, x) \text{ such that } \text{right}(x) = r \} \cup \{ r^-, r \}. \]

Observe that \( \min(T) = \ell \) and \( \max(T) = r \). Hence, for \( i \in T \) we may define the index \( i^o = \iota(\ell^*) + (\iota(i) - \iota(\ell)) \). Some useful properties of \( \iota^o \) are stated in the next lemma.

**Lemma 8.7.** The function \( \iota^o : T \to \alpha_w + 1 \) satisfies the following:

(a) it preserves the order and is injective;
(b) for every \( i < j \) in \( T \), the pseudovariety \( \mathcal{D} \mathcal{R} \mathcal{H} \) satisfies the equality \( w[i^o, j^o] = w(i, j) \) if \( j < r \) and the relation \( w[i^o, r^o] \mathcal{R} w(i, r) \);
(c) for every \( i \in T \), the inequality \( i^o < \iota(i) \) holds.

**Proof.** We omit the proofs of assertions (a) and (c) since they express properties of ordinal numbers and thus, are entirely analogous to the proofs of the corresponding properties in [Lemma 9.3].

Let us prove (b). Since \((\ell, x_0, \ell^*, \pi_0)\) is a boundary relation in \( \mathcal{B} \) and \( \mathcal{M} \) is a model of \( \mathcal{S} \), we have \( w(\ell, r) = w(\ell, \text{right}(x_0)) \mathcal{R} w(\ell^*, \text{right}(\pi_0)) = w(\ell^*, r^*) \) modulo \( \mathcal{D} \mathcal{R} \mathcal{H} \). Further, the fact that \( \ell^o = \iota(\ell^*) \) and \( r^o = \iota(r^*) \), implies that \( \mathcal{D} \mathcal{R} \mathcal{H} \) satisfies \( w(\ell, r) \mathcal{R} w[\ell^o, r^o] \).

On the other hand, since \( j^o - i^o = \iota(j) - \iota(i) \), we may use Corollary [3.12] twice to first conclude that, for \( j < r \), \( \mathcal{D} \mathcal{R} \mathcal{H} \) satisfies \( w(\ell, j) = w[\ell^o, j^o] \) and then, that it satisfies the desired identity \( w(i, j) = w[i^o, j^o] \). Similarly, when \( j = r \), we get that \( \mathcal{D} \mathcal{R} \mathcal{H} \) satisfies \( w[i^o, j^o] \mathcal{R} w(i, r) \).

Before defining a new pair \((\mathcal{S}_1, \mathcal{M}_1)\), we still need to consider a factorization scheme for the pseudoword \( w \), in order to memorize the information on constraints that we lose when transforming \( \mathcal{S} \) according to Fig. 6. We let \( \mathcal{C}_0 = (J_0, \iota_0, M_0, \Theta_0) \) be defined as follows:

- \( J_0 = \{ i^o : i \in T \} \);
- \( \iota_0 : J_0 \to \alpha_w + 1 \) is the inclusion of ordinals;
- By Lemma [3.12] the pseudowords \( w(\ell^-, r) \) and \( w[(\ell^-)^o, r^o] \) are \( \mathcal{R} \)-equivalent modulo \( \mathcal{D} \mathcal{R} \mathcal{H} \). Therefore, since Property [M.1] holds for \((\mathcal{S}, \mathcal{M})\), given \( \vec{s} \in \zeta(\ell^-, r) \) and \( \mu \in M(\ell^-, r, \vec{s}) \) the pseudowords \( \Phi(\ell^-, r, \vec{s}, \mu) \) and \( w[(\ell^-)^o, r^o] \) are \( \mathcal{R} \)-equivalent.
modulo DRH as well. For each such pair \((\vec{s}, \mu)\), we fix a pseudoword \(v_{\vec{s}, \mu} \in (\overline{\prod} A S)^I\) such that

\[
w[(r^-)^\circ, r^\circ[ =_{\text{DRH}} \Phi(r^-, r, \vec{s}, \mu) v_{\vec{s}, \mu}.
\]

In particular, it follows that \(\Phi(r^-, r, \vec{s}, \mu) v_{\vec{s}, \mu}\) and \(\Phi(r^-, r, \vec{s}, \mu)\) are \(R\)-equivalent modulo DRH. Combining Remark 3.3 with Lemma 3.9 we may deduce the inclusion \(c(v_{\vec{s}, \mu}) \subseteq \overline{c}(\Phi(r^-, r, \vec{s}, \mu)) = \overline{c}(w[(r^-)^\circ, r^\circ[).\)

Since \(\zeta(r^-, r)\) is a finite set, we may write \(\zeta(r^-, r) = \{\vec{s}_1, \ldots, \vec{s}_m\}\). Let \(\vec{s}_p = (s_{p,1}, s_{p,2})\) and denote by \(\vec{t}_{p,\mu}\) the pair \((s_{p,1}, \varphi(v_{\vec{s}_p, \mu}))\) for each \(\vec{s}_p \in \zeta(r^-, r)\) and \(\mu \in M(r^-, r, \vec{s}_p)\). We define \(\Theta_0\) inductively as follows:

- start with \(\Theta_0 = \emptyset\);
- for each \(p \in \{1, \ldots, m\}\) and \(\mu \in M(r^-, r, \vec{s}_p)\), we set

\[
\mu_{p, \mu} = (\overline{p}: \Theta_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu) \text{ is defined});
\]

\[
\Theta_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p, \mu}) = (\Phi(r^-, r, \vec{s}_p, \mu), v_{\vec{s}_p, \mu}.
\]

- the map \(M_0\) is given by \(M_0((r^-)^\circ, r^\circ, \vec{t}) = \{\mu': \Theta_0(r^-, r, \vec{t}, \mu') \text{ is defined}\}\), whenever \(\vec{t} = \vec{t}_{p,\mu}\) for certain \(p = 1, \ldots, m\) and \(\mu \in M(r^-, r, \vec{s}_p)\). Observe that we may have \(\vec{t}_{p,\mu} = \vec{t}_{p',\mu'}\) with \((p, \mu) \neq (p', \mu')\).

Lemma 8.8. The tuple \(C_0\) just constructed is a factorization scheme for \(w\).

Proof. Since \(r^- \prec r\) in \(J\), Lemma 8.7(a) yields \((r^-)^\circ \prec r^\circ\) in \(J_0\). Therefore, the domain of \(\Theta_0\) is compatible with the definition of factorization scheme. Moreover, the definition of \(M_0\) guarantees that the relationship between the domains of \(\Theta_0\) and of \(M_0\) is the correct one. To prove \([\text{FS.1}]\) let \(\vec{s}_p \in \zeta(r^-, r)\) and \(\mu \in M(r^-, r, \vec{s}_p)\). In DRH, we have

\[
\prod \circ \Theta_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p, \mu}) \overset{\text{def.}}{=} \Phi(r^-, r, \vec{s}_p, \mu) v_{\vec{s}_p, \mu}
\]

\[= w[(r^-)^\circ, r^\circ[.
\]

as intended. On the other hand, recalling that \(\vec{t}_{p,\mu} = (s_{p,1}, \varphi(v_{\vec{s}_p, \mu}))\) and that \(M\) is a model of \(S\), Property \([\text{M.2}]\) yields

\[
\varphi(\Phi_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p, \mu})) = \varphi(\Phi(r^-, r, \vec{s}_p, \mu)) = s_{p,1}
\]

and by construction,

\[
\varphi(\Psi_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p, \mu})) = \varphi(v_{\vec{s}_p, \mu}.
\]

This proves \([\text{FS.2}]\) \(\square\)

We are now ready to proceed with the construction of the new pair \((S_1, M_1)\), where

\[
S_1 = (X_1, J_1, \zeta_1, M_1, \chi_1, \text{right}_1, B_1, (B_H)_1) \text{ and } M = (w_1, \iota_1, \Theta_1).
\]

We take as set of variables \(X_1\) the old set \(X\) together with a pair of new variables \(y_i\) and \(\overline{y}_i\), for each \(i \in T \setminus \{r\}\). The pseudoword \(w_1\) is \(w\). Let \(C_1 = (J_1, \iota_1, M_1, \Theta_1)\) be a common refinement of \(C(S, M)\) and \(C_0\). The elements \(J_1, M_1, \iota_1\) and \(\Theta_1\) are those given by \(C_1\). To simplify the notation, we set \(\xi = \iota_1^{-1} \circ \iota\) and \(\iota^* = \iota_1^{-1} (\iota^\circ)\). The refining functions from \(C(S, M)\) to \(C_1\) and from \(C_0\) to \(C_1\) are given, respectively, by \(\Lambda\) and \(\Lambda_0\). The functions \(\zeta_1\) and...
\(\chi_1\) are the ones induced by \(G_1\), namely \(\zeta_1 = \zeta_{w_1,c_1}\) and \(\chi_1 = \chi_{w_1,c_1}\) (recall [12] and [13]). The right\(_1\) function is given by

\[
\text{right}_1 : X_1 \to J_1
\]
\[
x \mapsto \xi(\text{right}(x)), \quad \text{if } x \in X \text{ and } \text{right}(x) < r;
\]
\[
x \mapsto r^*, \quad \text{if } x \in X \text{ and } \text{right}(x) = r;
\]
\[
y_i \mapsto \xi(i), \quad \text{if } i \in T \setminus \{r\};
\]
\[
\overline{y}_i \mapsto r^*, \quad \text{if } i \in T \setminus \{r\}.
\]

We define \(B_1\) iteratively by:

1. Set \(B' = B \setminus \{((\ell, x_0, \ell^*, \overline{x}_0), (\ell^*, \overline{x}_0, \ell, x_0))\};
2. Start with \(B_1 = \{(\xi(\ell), y_i, \ell^*, \overline{y}_i), (\ell^*, \overline{y}_i, \xi(\ell), y_i) : i \in T \setminus \{r\}\};
3. For each variable \(x \in X\) such that \(\text{right}(x) = r\) and for each boundary relation \((i, x, j, \overline{x}) \in B'\), we add to \(B_1\) two new boundary relations as follows:
   a. If \(\text{right}(\overline{x}) < r\), then add the relations \((i^*, x, \xi(j), \overline{x})\) and \((\xi(j), \overline{x}, i^*, x)\);
   b. If \(\text{right}(\overline{x}) = r\), then add the relations \((i^*, x, j^*, \overline{x})\) and \((j^*, \overline{x}, i^*, x)\);
4. For each variable \(x \in X\) such that \(\text{right}(x) < r\) and \(\text{right}(\overline{x}) < r\) and for each boundary relation \((i, x, j, \overline{x}) \in B'\), we add to \(B_1\) the boundary relations \((\xi(i), x, \xi(j), \overline{x})\) and \((\xi(j), \overline{x}, \xi(i), x)\).

Finally, in \((B_1, \backslash)\) we include all the equations of the set \(\xi_{\Lambda}(B_H)\) as well as the following:

1. \(\xi(\ell) | \xi(r^*) = (\ell^* | (r^*)^\omega)\);
2. \(\xi(r^-) | \xi(r) = ((r^-)^\omega | r^*) \cdot \{((r^-)^\omega | r^*)^\omega \cdot (\xi(r^-) | \xi(r))\}_{\overline{s}_p, \mu'}\), for each \(\overline{s}_p \in \zeta(r^-, r)\) and \(\mu \in \mu(M(r^-, r, \overline{s}_p))\). Here, we are writing
   \[
   \Lambda(r^-, r, \overline{s}_p, \mu) = ((r^-)^\omega, \mu');
   \]
   \[
   \Lambda_0((r^-)^\omega, \overline{s}_p, \mu') = ((r^-)^\omega, \overline{\mu}').
   \]

Proposition 8.9. The tuple \(M_1\) is a model of the system of boundary relations \(S\).

Proof. Properties \([M.1] [M.3]\) are satisfied as a consequence of Lemma 7.1. It is a routine computation to check \([M.4]\). By Remark 7.6, the homomorphism \(\delta_{w_1,c_1} = \delta_{w_1,c_1}\) is a solution modulo \(H\) of \(\xi_{\Lambda}(B_H)\). Also, the homomorphism \(\delta_{w_1,c_1}\) is a solution modulo \(H\) of the equation \((\xi(\ell) | \xi(r^-)) = (\ell^* | (r^-)^\omega)\) as a consequence of the fact that DRH satisfies \(w_1(\xi(\ell), \xi(r^-)) = w_1(\ell^*, (r^-)^\omega)\), which follows from \([M.4]\). Finally, the equations of the form \((\xi(r^-) | \xi(r)) = ((r^-)^\omega | r^*) \cdot \{((r^-)^\omega | r^*)^\omega \cdot (\xi(r^-) | \xi(r))\}_{\overline{s}_p, \mu'}\) are satisfied by \(\delta_{w_1,c_1}\) modulo \(H\) since the following pseudo-identities are valid in \(H\):

\[
\delta_{w_1,c_1}((r^-)^\omega | r^*) \cdot \{((r^-)^\omega | r^*)^\omega | r^*\}_{\overline{s}_p, \mu'} = \delta_{w_1,c_1}((r^-)^\omega | r^*) \cdot \{((r^-)^\omega | r^*)^\omega | r^*\}_{\overline{s}_p, \mu'}.
\]
With this, we may conclude that $M_1$ is a model of $\mathcal{S}_1$. \hfill \qed

**Proposition 8.10.** Properties $[P.1]$ and $[P.2]$ are satisfied by the pairs $(S, M)$ and $(S_1, M_1)$.

**Proof.** Property $[P.1]$ is trivial. For Property $[P.2]$ we may let $M'_1 = (w'_1, \iota'_1, \Theta'_1)$ be a model of $\mathcal{S}_1$ in $\kappa$-words and we construct a new triple $M' = (w', \iota', \Theta')$ as follows. We fix a pair $(\bar{s}_q, \mu_0) \in \zeta(r, r) \times M(r, r, \bar{s}_q)$, for a certain $q \in \{1, \ldots, m\}$. We write $\Lambda(r, r, \bar{s}_q, \mu_0) = ((\bar{s}'_q), \mu'_0)$ and $\Lambda_0((r')^0, r', \bar{\ell}_{q, \mu_0}, \mu_{q, \mu_0}) = ((\bar{s}'_q), \mu'_0)$. The $\kappa$-word $w'$ is given by

$$w' = w'_1[0, \iota'_1(\xi(r))] \cdot w'_1((r^*)^0, r^*)\Phi'_1((r^*)^0, r^*, \bar{\ell}_{q, \mu_0}, \mu_{q, \mu_0})^{\omega-1} \Psi'_1(\xi(r), \xi(r), \bar{s}'_q, \mu'_0) \cdot w'_1[1(\xi(r)), \alpha_{w'_1}].$$

Note that Lemma 3.8 yields that $w'$ is indeed a $\kappa$-word. For $i \in J$, we let $\iota'(i)$ be given by

$$\iota'(i) = \begin{cases} 
\iota'_1(\xi(i)), & \text{if } i \leq r; \\
\iota'(r) + (\iota'_1(r^*) - \iota'_1((r^*)^0)), & \text{if } i = r; \\
\iota'(r) + (\iota'_1(\xi(i)) - \iota'_1(\xi(r))), & \text{if } i > r.
\end{cases}$$

Finally, we define $\Theta'$. For $i < j < r$ or $r < i < j$ in $J$, $\bar{s} \in \zeta(i, j)$ and $\mu \in M(i, j, \bar{s})$, let $\Lambda(i, j, \bar{s}, \mu) = ((\bar{t}_1, \ldots, \bar{t}_n), \mu')$ and $\xi(i) = i_0 < i_1 < \cdots < i_n = \xi(j)$. Then, we take

$$\Phi'(i, j, \bar{s}, \mu) = \left( \prod_{k=1}^{n-1} \text{prod } \circ \Theta'_1(i_{k-1}, i_k, \bar{t}_k, 0) \right) \Phi'_1(\xi(j)^0, \xi(j), \bar{t}_n, \mu'),$$

$$\Psi'(i, j, \bar{s}, \mu) = \Psi'_1(\xi(j)^0, \xi(j), \bar{t}_n, \mu').$$

On the other hand, when $(i, j) = (r, r)$, $\bar{s}_p \in \zeta(r, r)$, and $\mu \in M(r, r, \bar{s}_p)$, we write $\Lambda(r, r, \bar{s}_p, \mu) = ((\bar{t}_1, \ldots, \bar{t}_n), \mu')$, $\Lambda_0((r')^0, r', \bar{\ell}_{p, \mu}, \mu_{p, \mu}) = ((\bar{t}_1, \ldots, \bar{t}_n), \mu'_{p, \mu})$, and we let $(r^*)^0 = i_0 < i_1 < \cdots < i_n = r^*$. We define

$$\Phi'(r, r, \bar{s}_p, \mu) = \left( \prod_{k=1}^{n-1} \text{prod } \circ \Theta'_1(i_{k-1}, i_k, \bar{t}_k, 0) \right) \Phi'_1((r^*)^0, r^*, \bar{t}_n, \mu'_{p, \mu}),$$

$$\Psi'(r, r, \bar{s}_p, \mu) = \Psi'_1(\xi(r)^0, \xi(r), \bar{t}_n, \mu').$$

It is worth observing that, since each component of $\Theta'_1$ is a $\kappa$-word, the components of $\Theta'$ are $\kappa$-words as well.

Let us verify that $M'$ is a model of $\mathcal{S}$. For Properties $[M.1]$ and $[M.2]$ take an element $(i, j, \bar{s}) \in \text{Dom}(M)$ and let $\mu \in M(i, j, \bar{s})$. Property $[M.1]$ follows from the same property for the pair $(\mathcal{S}_1, M'_1)$ and when $(i, j) \neq (r, r)$. Property $[M.2]$ follows from the same property for $(\mathcal{S}_1, M'_1)$ and from Property $[R.2.3]$ for $\Lambda$. To prove $[M.1]$ when $(i, j) = (r, r)$ is more delicate. We suppose that $\bar{s} = \bar{s}_p$. Using the construction of $\Theta'$ and the Property $[M.1]$ for $(\mathcal{S}_1, M'_1)$, it may be derived that DRH satisfies

$$\text{prod } \circ \Theta'(r, r, \bar{s}_p, \mu) = w'_1((r^*)^0, r^*)\Psi'_1((r^*)^0, r^*, \bar{t}_n, \mu'_{p, \mu})^{\omega-1} \Psi'_1(\xi(r)^0, \xi(r), \bar{t}_n, \mu').$$
In turn, since $c(\Psi_1^e((r)^-,r^*,\bar{r}_n',\mu_{p,\mu})^{w-1}\Psi_1^e((\xi(r)^-,\xi(r),\bar{r}_n,\mu')) \subseteq c(w'_1((r)^-,r^*))$, DRH also satisfies
\[ w'_1((r)^-,r^*)\Psi_1^e((r)^-,r^*,\bar{r}_n',\mu_{p,\mu})^{w-1}\Psi_1^e((\xi(r)^-,\xi(r),\bar{r}_n,\mu') \in \mathcal{R} w'_1((r)^-,r^*)) \tag{20} \]

On the other hand, since the equations
\[ (\xi(r^-) | \xi(r)) = ((r^-) | r^*) \cdot \{(r^*)^- | r^*\}^{w-1}_{\bar{r}_n,\mu_{p,\mu}} \cdot \{(\xi(r^-) | \xi(r))_{\bar{r}_n,\mu'} \tag{21} \]
\[ (\xi(r^-) | \xi(r)) = ((r^-) | r^*) \cdot \{(r^*)^- | r^*\}^{w-1}_{\bar{r}_{q,\mu_0},\mu_{q,\mu_0}} \cdot \{(\xi(r^-) | \xi(r))_{\bar{s}_q,\mu_0} \tag{22} \]

belong to $(\mathcal{B}_1 \mathcal{H})$, H satisfies
\[ \text{prod} \circ \Theta'(r^-,r,\bar{s}_p,\mu) \overset{19}{=} w'_1((r^-)^*,r^*) \]
\[ \cdot \Psi_1^e((r^-)^*,r^*,\bar{r}_n',\mu_{p,\mu})^{w-1}\Psi_1^e((\xi(r^-),\xi(r),\bar{r}_n,\mu') \overset{21}{=} w'_1((r^-)^*,\xi(r)) \]
\[ \overset{22}{=} w'_1((r^-)^*,r^*) \]
\[ \cdot \Psi_1^e((r^-)^*,r^*,\bar{r}_{q,\mu_0},\mu_{q,\mu_0})^{w-1}\Psi_1^e((\xi(r^-),\xi(r),\bar{s}_q,\mu_0) \overset{24}{=} w'(r^-,r). \]

Using $(19)$, $(20)$, $(24)$ and Lemma 3.13 we finally get that DRH satisfies the pseudoidentity $\text{prod} \circ \Theta'(r^-,r,\bar{s}_p,\mu) = w'(r^-,r)$, obtaining $[M.1]$. For Property $[M.3]$ let $i < j$ in $J$. Then, we have
\[ c(w'(i,j)) = \begin{cases} c(w'_1((\xi(i),\xi(j))), & \text{if } (i,j) \neq (r^-,r); \\ c(w'_1((r^-)^*,r^*)), & \text{if } (i,j) = (r^-,r); \end{cases} \]
\[ = \chi_1(\xi(i),\xi(j)), \quad \text{if } (i,j) \neq (r^-,r); \]
\[ = \chi_1((r^-)^*,r^*), \quad \text{if } (i,j) = (r^-,r); \quad \text{by } [M.3] \text{ for } (S_1,M_1') \]
\[ = c(w[i_1(\xi(i)),i_1(\xi(j))], \quad \text{if } (i,j) \neq (r^-,r); \]
\[ = c(w[i_1((r^-)^*),i_1(r^*)], \quad \text{if } (i,j) = (r^-,r); \quad \text{by definition } 13 \]
\[ = \chi(i,j), \quad \text{if } (i,j) \neq (r^-,r); \]
\[ = c(w[r^-,r^*]), \quad \text{if } (i,j) = (r^-,r); \quad \text{by definition } \omega^* \text{ and } \xi \]
\[ = \chi(i,j), \quad \text{if } (i,j) \neq (r^-,r); \]
\[ = c(w[r^-,r]), \quad \text{if } (i,j) = (r^-,r); \quad \text{by Lemma 8.4(b)} \]
\[ = \chi(i,j) \quad \text{by } [M.3] \text{ for } (S,M). \]

To prove that Property $[M.4]$ holds, we first notice that, for every $i < j < r$ in $T$,
\[ w'_1((\xi(i),\xi(j)) =_{DRH} w'_1((i^*,j^*). \tag{25} \]
Now, let \((i, x)\) be a box in \(B'\). Using the definitions of \(w'\) and of \(i'\) we may compute

\[
  w'(i, \text{right}(x)) = \begin{cases} 
    w'_1(\xi(i), \xi(\text{right}(x))), & \text{if right}(x) \leq r^-; \\
    w'_1(\xi(i), \xi(r^-)) w'_1((r^-)^*, r^*) \\
    \cdot \Psi'_1((r^*)^-, r^*, \ell'_{q,\mu_0}, \mu'_{q,\mu_0})) \omega^{-1} \Psi'_1(\xi(r^-), \xi(r), s'^q, \mu'_0), & \text{otherwise}; \\
  \end{cases}
\]

\[= w'_1(\ell'^*, (r^-)^*), \]

then

\[
  \Psi'_1((r^*)^-, r^*, \ell'_{q,\mu_0}, \mu'_{q,\mu_0})) \omega^{-1} \Psi'_1(\xi(r^-), \xi(r), s'^q, \mu'_0)
\]

\[= w'_1(\ell'^*, r^*), \]

\[
  \mathcal{R} w'_1(\ell'^*, r^*), \quad \text{because}
\]

\[c(\Psi'_1((r^*)^-, r^*, \ell'_{q,\mu_0}, \mu'_{q,\mu_0})) \omega^{-1} \Psi'_1(\xi(r^-), \xi(r), s'^q, \mu'_0) \subseteq \mathcal{C}(w'_1(\ell'^*, r^*))
\]

\[
= w'_1(\ell'^*, r^*)
\]

\[
= w'_1(\ell'^*, r^*).
\]

Taking into account the steps \([2]\) and \([3]\) in the construction of \(B_1\), it is now easy to deduce that \([M.4]\) holds for all the relations added in those steps. It remains to verify that \(w'(\ell, r)\) and \(w'(\ell^*, r^*)\) are \(\mathcal{R}\)-equivalent modulo DRH. For that purpose, we show that the following relations hold in DRH:

\[
w'(\ell, r) = w'(\ell, r^-)w'(r^-, r)
\]

\[
def. \quad w'_1(\xi(\ell), \xi(r^-))
\]

\[
\cdot w'_1((r^-)^*, r^*) \Psi'_1((r^*)^-, r^*, \ell'_{q,\mu_0}, \mu'_{q,\mu_0})) \omega^{-1} \Psi'_1(\xi(r^-), \xi(r), s'^q, \mu'_0)
\]

\[= w'_1(\ell'^*, (r^-)^*), \]

\[
\mathcal{R} w'_1(\ell'^*, r^*),
\]

\[
\mathcal{C}(\Psi'_1((r^*)^-, r^*, \ell'_{q,\mu_0}, \mu'_{q,\mu_0})) \omega^{-1} \Psi'_1(\xi(r^-), \xi(r), s'^q, \mu'_0) \subseteq \mathcal{C}(w'_1(\ell'^*, r^*))
\]

\[
= w'_1(\ell'^*, r^*)
\]

\[
= w'_1(\ell'^*, r^*).
\]

Finally, since \(\xi_\Lambda(B_H) \subseteq (B_H)_1\), in Remark \([7]\) we observed that, in order to prove that Property \([M.5]\) is satisfied, it is enough to prove that \(H\) satisfies

\[w'(i, j) = w'_1(\xi(i), \xi(j))\]

\[\Psi'(i, j, s', \mu) = \Psi'_1(\xi(j)^-, \xi(j), s'^q, \mu'),\]

for every \((i, j, s', \mu) \in \text{Dom}(M) \times M(i, j, s', \mu)\), where \(\Lambda(i, j, s', \mu) = ((..., s'), \mu')\). The pseudoidentity \([26]\) follows straightforwardly from the definition of \(w'\), except when \((i, j) = (r^-, r)\). In that case, by computing \([26]\) modulo \(H\), we get

\[
w'(r^-, r) = w'_1((r^-)^*, r^*) \Psi'_1((r^*)^-, r^*, \ell'_{q,\mu_0}, \mu'_{q,\mu_0})) \omega^{-1} \Psi'_1(\xi(r^-), \xi(r), s'^q, \mu'_0)
\]

\[
= w'_1(\xi(r^-), \xi(r)),
\]

where the last equality holds because the equation

\[(\xi(r^-) | \xi(r)) = (r^-^* | r^*) \cdot ((r^*)^-^* | r^*) \omega^{-1} \ell'_{q,\mu_0, \mu'_0} \cdot (\xi(r^-) | \xi(r)) s'^q, \mu'_0\]
belongs to \((B_1)_1\) and \(M'_1\) is a model of \(S_1\). Lastly, the pseudoidentity \((27)\) corresponds precisely to the definition of \(\Theta'\). Thus, \(M'\) is a model of \(S\) in \(\kappa\)-words and so, Property \((P.2)\) holds for the pair \((S_1,M'_1)\).

### 8.8. Case 5

Finally, it remains to consider the case where \(B\) has a boundary relation of the form \((i,j,\overline{x},\overline{y})\) with \(\text{right}(x) = r = \text{right}(\overline{y})\) and none of the Cases [1]–[4] hold. In particular, the non occurrence of Cases [2]–[4] implies that all the boundary relations \((i,j,\overline{x},\overline{y})\) verifying \(i \leq j\) and \(\text{right}(\overline{x}) = r\) are such that \(i < j\), \(\text{right}(x) = r\) and the equality \(c(w(i,j)) = c(w(i,r))\) holds.

We consider the index

\[
c = \max\{\min(J), \max\{\text{right}(x) : \text{right}(x) < r\}, \max\{i \in J : i < r \text{ and } \overline{x}\text{ a box } (i,x)\}\}
\]

and we let \(E = \{(i,j,\overline{x},\overline{y}) \in B : i < j, \text{right}(x) = r = \text{right}(\overline{y})\}\). By the auxiliary step, we may assume that all the boundary relations of \(E\) are such that \(c < i,j < r\). Since the auxiliary step consists in successively factorizing a boundary relation from \(E\) with respect to a pair of ordinals both greater than \(\iota(c)\) (recall Fig. [5] and Lemma 8.6), it follows that for every index \(c < i < r\) there exists a box \((i,x)\) such that \(\text{right}(x) = r\). Observe that the choice of \(c\) guarantees that all the indices in the original set of boundary relations already satisfy this condition. Moreover, since \(E\) contains all the boxes ending in \(r\), if \((i,x)\) is a box such that \(\text{right}(x) = r\), then \(c < i < r\).

Now, we let \(\ell = \max\{i \in J : \text{there exists } (i,j,\overline{x},\overline{y}) \in E\}\). Using the construction presented in Subsection 8.6 to align the left of each variable intervening in \(E\) (as schematized in Fig. [7]), we may assume, without loss of generality, that the set \(E\) defined above is given by \(E = \{(\ell,x_1,j_1,\overline{x_1}), \ldots, (\ell,x_n,j_n,\overline{x_n})\}\), with \(j_1 \leq j_2 \leq \cdots \leq j_n\). We notice that, by definition of the index \(c\), we have \(j_n < r\) in \(J\). Since \(M\) is a model of \(S\), DRH satisfies \(w(\ell,j_m)w(\ell,r) \bowtie w(\ell,j_m)w(j_m,r) = w(\ell,r)\), for \(m = 1, \ldots, n\). Multiplying successively by \(w(\ell,j_m)\) on the left, we get that DRH satisfies \(w(\ell,j_m)\omega w(\ell,r) \bowtie w(\ell,r)\). Since \(\overline{c}(w(\ell,j_m)\omega) = c(w(\ell,j_m)) = c(w(\ell,r))\), it follows that DRH satisfies

\[
(28) \quad w(\ell,r) \bowtie w(\ell,j_1)\omega \cdot \cdots \cdot \bowtie w(\ell,j_n)\omega.
\]

But all the pseudowords \(w(\ell,j_m)\omega\) represent the identity in the same maximal subgroup of \(\overline{\Omega}_A\)DRH where they belong (recall Proposition 3.4). Therefore, all the elements \(w(\ell,j_m)\omega\) are the same over DRH. Then, Proposition 3.5 applied to the elements \(w(\ell,j_1), \ldots, w(\ell,j_n)\) guarantees the existence of pseudowords \(u \in \overline{\Omega}_A S, v_1, \ldots, v_n \in (\overline{\Omega}_A S)^f\) and of positive integers \(h_1, \ldots, h_n\) such that, for \(m = 1, \ldots, n\) we have

\[
(29) \quad w(\ell,j_m) = \text{DRH} \overset{\text{DRH}}{w^{h_m}v_m}, \quad v_mu = \text{DRH} u,
\]
where all the products $u \cdot u$, $u \cdot v_m$ and $v_m \cdot u$ are reduced. Note that $h_n$ is the maximum of $\{h_1, \ldots, h_0\}$.

We observe that the pseudoidentities in (29) imply that every finite power of $u$ is a prefix of $w(\ell, j_n)\omega$, which in turn, by (28), is $\mathcal{R}$-equivalent to $w(\ell, r)$ modulo $\text{DRH}$. Since the semigroup $S$ where the constraints are defined is finite, this allows us to find some periodicity on them. With this in mind, to deal with the constraints, we consider a big enough direct power of the semigroup $S$, more specifically, the semigroup $T = S^K$, with $K = \sum \bar{s} \in \zeta(j_n, r) M(j_n, r, \bar{s})$, and we take $N = |T| + 2$. Let us construct a new pair $(S_1, M_1)$ as follows:

$$S_1 = (X_1, J_1, \zeta_1, M_1, \chi_1, \text{right}_1, B_1, (B_H)_1) \text{ and } M_1 = (w_1, \epsilon_1, \Theta_1),$$

where

- the set of variables is $X_1 = X \uplus \{y_q, \bar{y}_q\}_{q=1}^h \uplus \{z_m, \bar{z}_m\}_{m=1}^n \uplus \{f_1, \bar{f}_1\}_{\ell=1}^N$, where variables with different names are assumed to be distinct;
- the pseudoword in the model is $w_1 = w$;
- let $O$ be the set containing the following ordinals:
  - $\beta_0 = \iota(\ell)$;
  - $\beta_q = \beta_0 + \alpha_u \cdot q$, for $q = 1, \ldots, h_n + 1$;
  - $\gamma_m = \beta_0 + (\iota(j_m) - \beta_{h_n})$, for $m = 1, \ldots, n$;
  - $\delta_p = \beta_0 + \alpha_u \cdot h_n p$, for $p = 0, \ldots, N$.

We let $C_1 = (J_1, \iota_1, M_1, \Theta_1)$ be a common refinement of the factorization schemes $C(S, M)$ and $(O, O \mapsto \alpha_w + 1, 0, 0)$ for $w$ and

$$\Lambda : \{(i, j, \bar{s}, \mu) : (i, j, \bar{s}) \in \text{Dom}(M), \mu \in M(i, j, \bar{s})\} \rightarrow \bigcup_{k \in \mathbb{N}} (S \times S^I)^k \times \omega$$

be a refining function from $C(S, M)$ to $C_1$. The factorization scheme $C_1$ supplies the items $J_1$, $\iota_1$, $M_1$ and $\Theta_1$ and the items $\zeta_1$ and $\chi_1$ by taking $\zeta_1 = \zeta_{w_1, \epsilon_1}$ and $\chi_1 = \chi_{w_1, \epsilon_1}$ (recall (12) and (13)). We denote $b_q = \iota_1^{-1}(\beta_q)$, $c_m = \iota_1^{-1}(\gamma_m)$, $d_p = \iota_1^{-1}(\delta_p)$, and $\xi = \iota_1 \circ \iota$;
- the function $\text{right}_1$ is given by

$$\text{right}_1(x) = \begin{cases} 
\xi(\text{right}(x)), & \text{if } x \in X; \\
b_q, & \text{if } x = y_q; \\
b_{q+1}, & \text{if } x = \bar{y}_q; \\
b_{h_n+1}, & \text{if } x \in \{z_m, \bar{z}_m\}; \\
d_p, & \text{if } x = f_p; \\
d_{p+1}, & \text{if } x = \bar{f}_p;
\end{cases}$$

- in the set $B_1$ we include the following boundary relations:
  - $(\xi(i), x, \xi(j), x)$, if $(i, x, j, x) \in B \setminus (E \cup \{\text{dual of } e : e \in E\})$;
  - $(b_{h_n-1}, y_q, b_q, \bar{y}_q)$ and $(b_q, \bar{y}_q, b_{q-1}, y_q)$, for $q = 1, \ldots, h$;
  - $(b_{h_n}, z_m, \xi(j_m), \bar{z}_m)$ and $(\xi(j_m), \bar{z}_m, b_{h_n}, z_m)$, for $m = 1, \ldots, n$;
  - $(d_{p-1}, f_p, d_p, \bar{f}_p)$ and $(d_p, \bar{f}_p, d_{p+1}, f_p)$, for $p = 1, \ldots, N - 1$;
- the set $(B_H)_1$ consists of the following equations:
  - all the equations of $\xi_\Lambda(B_H)$;
  - $(b_0 | b_1) = (b_1 | b_2) = \cdots = (b_{h_n} | b_{h_n+1})$;
  - $(b_{h_n} | b_{h_n+1}) = (\xi(j_m) | b_{h_n+1})$, for $m = 1, \ldots, n$;
In order to ease the notation, for each \( i \) component of \( \Lambda(\cdot) \), we define integers \( 1 \leq H < M < N \) that later play an essential role.

Recall that, for \( i,1 \) have \( \xi(j_n) \prec b_{j_n+1} \leq d_2 \prec d_3 \prec \cdots \prec d_N \prec \xi(r) \), where \( b_{j_n+1} = d_2 \) if and only if \( h_n = 1 \). Therefore, for each \( (\vec{s},\mu) \in \zeta(j_{n},r) \times \mathcal{M}(j_{n},r,\vec{s}) \), the first component of \( \Lambda(j_{n},r,\vec{s}) \) belongs to \((S \times S')^N\) if \( h_n = 1 \) or to \((S \times S')^{N+1}\), otherwise. We assume that \( h_n > 1 \). The same argument can be used when \( h_n = 1 \), simply by working with \( N \) instead of \( N + 1 \). We may write

\[
\Lambda(j_{n},r,\vec{s},\mu) = \left( \tilde{t}_{i,1}^{(\vec{s},\mu)}, \ldots, \tilde{t}_{N+1}^{(\vec{s},\mu)} \right),
\]

with \( \tilde{t}_{i}^{(\vec{s},\mu)} = \left( t_{i,1}^{(\vec{s},\mu)}, t_{i,2}^{(\vec{s},\mu)} \right) \). Let \( \tilde{t}_{1}, \ldots, \tilde{t}_{N+1} \in T \) satisfy the following properties:

- each element \( \tilde{t}_{i} \) is a tuple whose coordinates are of the form \( t_{i,1}^{(\vec{s},\mu)}, t_{i,2}^{(\vec{s},\mu)} \), for certain \( \vec{s} \in \zeta(j_{n},r) \) and \( \mu \in \mathcal{M}(j_{n},r,\vec{s}) \);
- for \( i \in \{1, \ldots, K\} \) and \( k_1 \neq k_2 \), if the \( k_1 \)-th coordinate of \( \tilde{t}_{i} \) is \( t_{i,1}^{(\vec{s},\mu)} \) and the \( k_2 \)-th coordinate of \( \tilde{t}_{i} \) is \( t_{i,2}^{(\vec{s},\mu)} \), then \( (\vec{s},\mu) \neq (\vec{s},\mu) \);
- for \( i \in \{2, \ldots, K\} \), if the \( k \)-th coordinate of \( \tilde{t}_{i} \) is \( t_{i,1}^{(\vec{s},\mu)} \) and the \( k \)-th coordinate of \( \tilde{t}_{i} \) is \( t_{i,2}^{(\vec{s},\mu)} \), then the \( k \)-th coordinate of \( \tilde{t}_{i} \) is \( t_{i,1}^{(\vec{s},\mu)} \).

Since \( N - 1 > |T| \), there exist \( 1 \leq H < M < N \) such that \( \tilde{t}_{1} \cdots \tilde{t}_{H} = \tilde{t}_{1} \cdots \tilde{t}_{M} \), which implies that

\[
\tilde{t}_{1} \cdots \tilde{t}_{H} \tilde{t}_{H+1} \cdots \tilde{t}_{M} = \tilde{t}_{1} \cdots \tilde{t}_{M} (\tilde{t}_{H+1} \cdots \tilde{t}_{M})^\omega.
\]

In order to ease the notation, for each \( \vec{s} = (s_1, s_2) \in \zeta(j_{n},r) \) and \( \mu \in \mathcal{M}(j_{n},r,\vec{s}) \), we define

\[
\begin{align*}
\tilde{s}_{1}^{(\vec{s},\mu)} & = t_{1,1}^{(\vec{s},\mu)} \cdot t_{1,2}^{(\vec{s},\mu)} \cdot \cdots \cdot t_{H,1}^{(\vec{s},\mu)} \cdot t_{H,2}^{(\vec{s},\mu)}, \\
\tilde{s}_{2}^{(\vec{s},\mu)} & = t_{H+1,1}^{(\vec{s},\mu)} \cdot t_{H+1,2}^{(\vec{s},\mu)} \cdot \cdots \cdot t_{M,1}^{(\vec{s},\mu)} \cdot t_{M,2}^{(\vec{s},\mu)}, \\
\tilde{s}_{3,1}^{(\vec{s},\mu)} & = t_{M+1,1}^{(\vec{s},\mu)} \cdot t_{M+1,2}^{(\vec{s},\mu)} \cdot \cdots \cdot t_{N,1}^{(\vec{s},\mu)} \cdot t_{N,2}^{(\vec{s},\mu)}, \\
\tilde{s}_{3,2}^{(\vec{s},\mu)} & = t_{N+1,1}^{(\vec{s},\mu)}.
\end{align*}
\]

Then, since \( \Lambda \) satisfies (R.2.3) we have \( s_1 = \tilde{s}_{1}^{(\vec{s},\mu)} \cdot \tilde{s}_{2}^{(\vec{s},\mu)} \cdot \omega^{\alpha_{s_1}} \cdot \tilde{s}_{3,1}^{(\vec{s},\mu)} \) and \( s_{3,2}^{(\vec{s},\mu)} = s_2 \).

Next, we verify that Property (P.2) is satisfied, as claimed before.

**Proposition 8.12.** Suppose that there exists a model \( \mathcal{M}' = (w_1', t_1', \Theta_1') \) of \( S_1 \) in \( \kappa \)-words. Then, there is a model of \( S \) in \( \kappa \)-words as well.

**Proof.** Let \( \mathcal{M}' = (w_1', t_1', \Theta_1') \) be constructed as follows. The \( \kappa \)-word \( w_1' \) is set to be

\[
w_1' = w_1'[0, t_1'(d_M) \cdot (w_1'(d_H,d_M))^{\alpha_{w_1'}}].
\]

The map \( t_1' : J \to \alpha_{w_1'} + 1 \) is given by \( t_1'(i) = i_1'(\xi(i)) \) if \( i < r \), \( t_1'(r) = \alpha_{w_1'} \cdot (w_1'(d_H,d_M))^{\alpha_{w_1'}} \), and \( t_1'(i) = t_1'(r) + (i_1'(\xi(i)) - i_1'(\xi(r))) \) if \( i > r \). In order to define \( \Theta_1' \), we first consider the following auxiliary pseudowords:
Now, we verify that 
\[
\phi_{0}(i, j, \vec{s}, \mu) = \prod_{m=1}^{k-1} \prod \Theta_{0}(i_{m-1}, i_{m}, \vec{t}_{m}, 0) \cdot \Phi_{0}(\xi(j), t_{k}, \mu');
\]
\[
\psi_{0}(i, j, \vec{s}, \mu) = \Psi_{0}(\xi(j), \vec{t}_{k}, \mu');
\]
- for each \(i < j < j_{n}\) and each \(r \leq i < j\) in \(J\), each \(\vec{s} \in \zeta(i, j)\) and each \(\mu \in M(i, j, \vec{s})\), if \(\Lambda(i, j, \vec{s}, \mu) = (\vec{t}_{1}, \ldots, \vec{t}_{k}, \mu')\) and \(\xi(i) = i_{0} < i_{1} < \cdots < i_{k} = \xi(j)\), then we take

\[
\begin{align*}
\Phi_{0}(i, j, \vec{s}, \mu) &= \prod_{m=1}^{k-1} \prod \Theta_{0}(i_{m-1}, i_{m}, \vec{t}_{m}, 0) \cdot \Phi_{0}(\xi(j), t_{k}, \mu');
\Psi_{0}(i, j, \vec{s}, \mu) &= \Psi_{0}(\xi(j), \vec{t}_{k}, \mu');
\end{align*}
\]
- for each \(s \in \zeta(j_{n}, r)\) and \(\mu \in M(j_{n}, r, \vec{s})\), we set (recall the notation in (30))

\[
\begin{align*}
\Phi_{0}(j_{n}, H, \vec{s}, \mu) &= \prod \Theta_{0}(\xi(j_{n}), b_{h+1}, \vec{t}_{1}^{(\vec{s}, \mu)}), 0)
\cdot \prod \Theta_{0}(d_{m-1}, d_{m}, \vec{t}_{m}^{(\vec{s}, \mu)}), 0);
\Phi_{0}(H, M, \vec{s}, \mu) &= \prod \Theta_{0}(d_{m-1}, d_{m}, \vec{t}_{m}^{(\vec{s}, \mu)}), 0);
\Phi_{0}(M, r, \vec{s}, \mu) &= \prod \Theta_{0}(d_{m} \cdot \vec{t}_{m}^{(\vec{s}, \mu)}), 0);
\Psi_{0}(M, r, \vec{s}, \mu) &= \Psi_{0}(d_{N}, \vec{t}_{N+1}^{(\vec{s}, \mu)}), 0).
\end{align*}
\]
Now, for \(i < j\) in \(J\), \((i, j, \vec{s}) \in \text{Dom}(M)\) and \(\mu \in M(i, j, \vec{s})\) we define

\[
\begin{align*}
\Theta'(i, j, \vec{s}, \mu) &= \Phi_{0}(i, j, \vec{s}, \mu), \Psi_{0}(i, j, \vec{s}, \mu), \text{ whenever } j \neq r;
\Theta'(j_{n}, r, \vec{s}, \mu) &= \Phi_{0}(j_{n}, H, \vec{s}, \mu), \Phi_{0}(M, r, \vec{s}, \mu).
\end{align*}
\]
Now, we verify that \(\mathcal{M}'\) just defined is a model of \(\mathcal{S}\). Let \((i, j, \vec{s}) \in \text{Dom}(M)\) be such that \(\vec{s} = (s_{1}, s_{2})\), and \(\mu \in M(i, j, \vec{s})\). Suppose that \(j \neq r\), write \(\Lambda(i, j, \vec{s}, \mu) = (\vec{t}_{1}, \ldots, \vec{t}_{k}, \mu')\), and let \(\xi(i) = i_{0} < i_{1} < \cdots < i_{k} = \xi(j)\). Then, using the definition of \((\Phi_{0}, \Psi_{0})\), it is easy to derive \((\mathcal{M}'\), using the same property for the pair \((S_{1}, M_{1}')\). Similarly, invoking Property \((\mathcal{M}'_{2})\) for the pair \((S_{1}, M_{1}')\) and writing \(t_{m} = (t_{m,1}, t_{m,2})\), we may deduce the equalities \(\varphi(\Phi'(i, j, \vec{s}, \mu)) = (\prod_{m=1}^{k-1} t_{m,1}^{t_{m,2}}) \cdot t_{k,1}\) and \(\varphi(\Psi'(i, j, \vec{s}, \mu)) = t_{k,2}\). In turn, Property \((\mathcal{R}_{2.3})\) for \(\Lambda\) yields Property \((\mathcal{M}'_{2})\) for the pair \((S, M')\). We justify \((\mathcal{M}'_{3})\) with the following computation:

\[
\begin{align*}
\tilde{c}(w'(i, j)) &= \tilde{c}(w_{1}'(\xi(i), \xi(j))) \text{ by definition of } w' \text{ and } w';
\end{align*}
\]
\[
\begin{align*}
&= \chi_{1}(\xi(i), \xi(j)) \text{ by \((\mathcal{M}'_{1})\) for } (S_{1}, M_{1}')
&= \tilde{c}(w_{1}(\xi(i), t_{1}(\xi(j)))) \text{ by definition \((\mathcal{R}_{1})\) of } \chi_{1} = \chi_{w_{1}} e_{1}
&= \tilde{c}(w(i, j)) \text{ by \((\mathcal{M}'_{1})\) for } (S, M).
\end{align*}
\]
Now, consider the case where \(i = j_{n}\) and \(j = r\). Again, we may use Property \((\mathcal{M}'_{1})\) for \((S_{1}, M_{1}')\) to obtain the identity \(\prod \Theta'(j_{n}, r, \vec{s}, \mu) = w'(j_{n}, r)\) in DRH, thereby proving
(M.1) In order to prove (M.2), we use the same property for the pair \((S_1, M'_1)\) to derive the following equalities (recall (31)):

\[
\varphi(\Phi'(j_n, r, s, \mu')) = s'_1, \quad \varphi(\Phi'(j_n, r, s, \mu')) = s'_2.
\]

To establish (M.3), we observe that, since \(S\) has a content function and thanks to Property (M.2) for both pairs \((S, M)\) and \((S_1, M'_1)\), the content of the corresponding segments in \(w\) and in \(w'_1\) does not change. Therefore, the equalities

\[
\begin{align*}
\varphi(w'(j_n, r)) &= \varphi(w'(j_n, r)) = \varphi(w(j_n, r)) = \chi_j,
\end{align*}
\]

hold. Thus, we also have

\[
\begin{align*}
\varphi(w'(j_n, r)) &= \varphi(w'(j_n, r)) = \varphi(w(j_n, r)) = \chi_j.
\end{align*}
\]

It remains to verify that (M.4) and (M.5) are satisfied. For Property (M.4) all boundary relations but the ones of the form \(\ell, x, j_m, \bar{y}, \bar{x}_m\) are immediate. For those relations, we already observed in (32) that \(c(w'(j_m, d_M)) = c(w'(d_M, \xi(r)))\), so that, \(w'(j_m, r)\) and \(w'(\xi(j_m), d_M) \cdot w'(d_M, \xi(r))\) lie in the same \(R\)-class modulo DRH. Hence, the pseudovariety DRH satisfies

\[
\begin{align*}
\varphi(w'(\ell, r)) &= \varphi(w'(d_0, d_M)) \cdot w'(d_H, d_M) \cdot w'(d_M, \xi(r)) \\
&= \varphi(w'(j_m, d_M)) \cdot w'(d_H, d_M) \\
&= \varphi(w'(j_m, d_M)) \cdot w'(d_H, d_M) \\
&= \varphi(w'(j_m, d_M)) \cdot w'(d_H, d_M).
\end{align*}
\]

The validity of step (*) is justified in view of \(S_1\) having \(M'_1\) as a model. More precisely, it follows from Property (M.4) for the relation \((b_{h_m}, \bar{b}_m, \bar{x}_m)\) and from Property (M.5) for the equation \((b_{h_m} \mid b_{h_m + 1}) = (\xi(j_m) \mid b_{h_m + 1})\), together with Lemma 3.13. Finally, as the inclusion \(\xi_\Lambda(\mathcal{B}_H) \subseteq (\mathcal{B}_H)_1\) holds, by Remark 7.6 it is enough to show that for all \((i, j, s) \in \text{Dom}(M)\) and \(\mu \in M(i, j, s)\), if \(\Lambda(i, j, s, \mu) = ((\ldots, \ell), \mu')\), then the pseudoidentities

\[
\begin{align*}
\varphi(w'(i, j)) &= \varphi(w'(\xi(i), \xi(j))) \\
\Psi'(i, j, s, \mu) &= \Psi'(\xi(j), \ell, \mu')
\end{align*}
\]

are valid in \(H\). Analyzing the construction of \(\Psi'\), the second pseudoidentity becomes clear, since it is actually an equality of pseudowords. The first pseudoidentity \(w'(i, j) = w'(\xi(i), \xi(j))\) is also immediate, whenever \(j \neq r\), after noticing that \(w'(i, j) = \varphi(w'(\xi(i), \xi(j)))\).
It remains to prove that \( w'(j_n, r) = w'_1(\xi(j_n), r) \) modulo \( H \). That is made clear in the next computation modulo \( H \):
\[
\begin{align*}
w'(j_n, r) &= w'_1(\xi(j_n), d_M) \cdot w'_1(d_H, d_M) \omega \cdot w'_1(d_M, \xi(r)) \\
&= w'_1(\xi(j_n), d_M) \cdot w'_1(d_M, \xi(r)) = w'_1(\xi(j_n), \xi(r)).
\end{align*}
\]
This completes the proof. \( \square \)

We have just completed the analysis of all the Cases [1]. Thus, we proved Theorem [8.3]. The announced result follows from Corollary [6.4].

**Theorem 8.13.** Let \( H \) be a pseudovariety of groups. Then, the pseudovariety \( DRH \) is completely \( \kappa \)-reducible if and only if the pseudovariety \( H \) is completely \( \kappa \)-reducible.

Now, we are able to supply a family of examples of completely \( \kappa \)-reducible pseudovarieties.

**Corollary 8.14.** The pseudovariety \( DRH \) is completely \( \kappa \)-reducible for every locally finite pseudovariety of groups \( H \).

**Corollary 8.15.** The pseudovariety \( DRA_b \) is completely \( \kappa \)-reducible.

**Acknowledgments**

The work of both authors was supported, in part, by CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020. The second author was also partially supported by the FCT doctoral scholarship (SFRH/BD/75977/2011), with national (MEC) and European structural funds through the program POCH.

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