Isotopy of Morin singularities

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Abstract

We define an equivalence relation called $\mathcal{A}$-isotopy between finitely determined map-germs, which is a strengthened version of $\mathcal{A}$-equivalence. We consider the number of $\mathcal{A}$-isotopy classes of equidimensional Morin singularities, and some other well-known low-dimensional singularities. We also give an application to stable perturbations of simple equi-dimensional map-germs.

1 Introduction

There are various groups which act on the set $C^\infty(m,n)$ of map-germs $(\mathbb{R}^m,0) \to (\mathbb{R}^n,0)$. The group $\mathcal{A}$ will denote the group of changes of coordinates in the source and target, which act on $C^\infty(m,n)$ by $\tau \circ f \circ \sigma$, where $f \in C^\infty(m,n)$ and $\sigma$ and $\tau$ are, respectively, diffeomorphism-germs in the source and target. Two map-germs $f,g \in C^\infty(m,n)$ are $\mathcal{A}$-equivalent if they belong to the same orbit. In this paper, we define an equivalence relation called $\mathcal{A}$-isotopy, which is a strengthened version of $\mathcal{A}$-equivalence. Let $r$ be a natural number. A map-germ $f \in C^\infty(m,n)$ is said to be $r$-determined if any $g \in C^\infty(m,n)$ satisfying $j^r f(0) = j^r g(0)$ is $\mathcal{A}$-equivalent to $f$, where $j^r f(0)$ is the $r$-jet of $f$ at $0$.

Definition 1.1. Let $f,g \in C^\infty(m,n)$ be $\mathcal{A}$-equivalent map-germs that are $r$-determined. Then $f$ and $g$ are $\mathcal{A}$-isotopic if there exist continuous curves $\sigma : I \to \text{Diff}^r(m) \subset J^r(m,m)$ and $\tau : I \to \text{Diff}^r(n) \subset J^r(n,n)$ such that $\sigma(0)$, $\tau(0)$ are both the identity, and

$$j^r(g)(0) = j^r(\tau(1) \circ f \circ \sigma(1))(0)$$

holds, where $I = [0,1]$ and $\text{Diff}^r(m)$ denotes the set of the $r$-jets of diffeomorphism-germs $(\mathbb{R}^m,0) \to (\mathbb{R}^m,0)$.

Namely, $f$ and $g$ are $\mathcal{A}$-isotopic if and only if $j^r f(0)$ and $j^r g(0)$ are located on the same arc-wise connected component of the $r$-jet of the $\mathcal{A}$-orbit of $j^r f(0)$. Since the set $\text{Diff}^{r+}(m)$ of $r$-jets of orientation-preserving diffeomorphism-germs is arc-wise connected, $f$ and $g$ are $\mathcal{A}$-isotopic if and only if there exist orientation-preserving diffeomorphism-germs $\sigma^+ : (\mathbb{R}^m,0) \to (\mathbb{R}^m,0)$ and $\tau^+ : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ such that $j^r g(0) = j^r(\tau^+ \circ f \circ \sigma^+)(0)$ holds.

In this paper, we study the number of $\mathcal{A}$-isotopy classes of equidimensional Morin singularities. Morin singularities are stable, and conversely, corank one and stable germs are Morin singularities. This means that Morin singularities are fundamental and frequently appear as singularities of maps from a manifold to another. We show that for an $n$-Morin singularity $f : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$, there are four (respectively, two) $\mathcal{A}$-isotopy classes in $\mathcal{A}(f)$ if $n = 4i$.

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(respectively, \( n \neq 4i \)) with \( i \in \mathbb{N} \) (see section 2), where \( \mathcal{A}(f) \) stands for the \( \mathcal{A} \)-orbit of \( f \). For a \( k \)-Morin singularity \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) \((k < n)\), there are two \( \mathcal{A} \)-isotopy classes (respectively, is one \( \mathcal{A} \)-isotopy class) in \( \mathcal{A}(f) \) if \( k = 2i \) (respectively, \( k \neq 2i \)) (see section 2). The tables in section 3 summarize the invariants and normal forms for the \( \mathcal{A} \)-isotopy classes of these Morin singularities. In section 4 we consider the same problem for some other well-known low-dimensional singularities. As an application, we consider in section 5 \( \mathcal{A} \)-isotopy classes of \( n \)-Morin singularities appearing on stable perturbations of simple map-germs \((\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\). We remark that homotopy types of the \( \mathcal{A}^2 \)-orbit of the fold are considered by Ando [2, 3], thus we are mainly interested in the case \( k \geq 2 \). We also remark that this type of problem is asked by Nishimura [24].

2 \( \mathcal{A} \)-isotopy of Morin singularities

Let \( f \in C^\infty(n, n) \) be an equidimensional map-germ. Then \( f \) is a Morin singularity of \( \Sigma^{(k, 0)} \)-type (or shortly, a \( k \)-Morin singularity) if \( f \) is \( \mathcal{A} \)-equivalent to the germ

\[
(x_1, \ldots, x_n) \mapsto (x_1 x_2 + x_1^2 x_3 + \cdots + x_1^{k-1} x_k + x_1^{k+1}, x_2, \ldots, x_n)
\]

at the origin, where \( k \leq n \). For the meaning of the notation, and further details, see [22]. It is well known that a \( k \)-Morin singularity is \((k+1)\)-determined. There are recognition criteria for \( k \)-Morin singularities [31]. Let \( f \in C^\infty(n, n) \) and \( \lambda \) be the determinant of the Jacobi matrix of \( f \). Let 0 be a singular point of \( f \), namely \( \lambda(0) = 0 \), then the singular point 0 is non-degenerate if \( d\lambda(0) \neq 0 \). Let 0 be a non-degenerate singular point of \( f \), then there exists a never-vanishing vector field \( \eta \) around 0 on \( \mathbb{R}^n \) such that \( \eta(p) \in \ker df(p) \) for \( p \in S(f) \), where \( S(f) \) is the set of singular points of \( f \). We call \( \eta \) the null-vector field. Then the following theorem holds.

Theorem 2.1. [31] Theorem A1, page 746] Let \( f \) and \( \lambda \) be as above. Then \( f \) at 0 is a \( k \)-Morin singularity if and only if

\[
\lambda(0) = \eta\lambda(0) = \cdots = \eta^{k-1}\lambda(0) = 0, \quad \eta^k\lambda(0) \neq 0
\]

and

\[
\text{rank } d(\lambda, \eta\lambda, \ldots, \eta^{k-1}\lambda)(0) = k
\]

hold, where \( d(\lambda, \eta\lambda, \ldots, \eta^{k-1}\lambda) \) denotes the differential of the map

\[
(\lambda, \eta\lambda, \ldots, \eta^{k-1}\lambda) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0),
\]

and \( \eta\lambda \) denotes the directional derivative of \( \lambda \) with respect to \( \eta \), and \( \eta^k\lambda = \eta \cdots \eta \lambda \), \( k \)-times.

We have the following lemma.

Lemma 2.2. Let \( f \in C^\infty(n, n) \) be a \( k \)-Morin singularity. Assume \( k \neq 1 \). Then \( f \) is \( \mathcal{A} \)-isotopic to

\[
f^{k}_{(\varepsilon_1, \varepsilon_2)}(x) = \left( \varepsilon_1 (\varepsilon_2 x_2 x_3 x_1 + x_3 x_1 x_1^2 + \cdots + x_k x_1^{k-1} + x_1^{k+1}), \varepsilon_2 x_2, x_3, \ldots, x_n \right),
\]

where \( x = (x_1, \ldots, x_n) \), \( \varepsilon_1 = \pm 1 \) and \( \varepsilon_2 = \pm 1 \). If \( k = 1 \), then \( f \) is \( \mathcal{A} \)-isotopic to \( f^1_{\varepsilon_1} = (\varepsilon_1 x_1^2, x_2, \ldots, x_n) \) where \( \varepsilon_1 = \pm 1 \).

In what follows, we use the following notation: For a given map-germ \((\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\), the small letters \( x = (x_1, \ldots, x_n) \) denote the coordinate system on the source space, and the capital letters \( X = (X_1, \ldots, X_n) \) denote that of the target space. Following a characterization of Morin singularities given in [22], and taking care to use only orientation-preserving diffeomorphism-germs, one can easily prove Lemma 2.2.
Proof of Lemma \[2.2\] Assume that \( f(x) = (f_1(x), \ldots, f_n(x)) \) is a \( k \)-Morin singularity. Since \( d\lambda(0) \neq 0 \), we have \( \text{rank}(df)(0) = n - 1 \). Then by a rotation, we may assume that

\[
\{\dot{t}(1, 0, \ldots, 0), \grad(f_2)(0), \ldots, \grad(f_n)(0)\}
\]

forms a positive basis of \( \mathbb{R}^n \), where \( \grad(h) = \dot{t}(h_{x_1}, \ldots, h_{x_n}) \) and \( h_{x_1} = \partial h / \partial x_1 \), for example, and \( \dot{t}(\cdot) \) means the transpose matrix. Moreover, since \( \{ A \in O(n) \mid \det A > 0 \} \) is arc-wise connected, we may assume that

\[
\left(\dot{t}(1, 0, \ldots, 0), \grad(f_2)(0), \ldots, \grad(f_n)(0)\right) = E,
\]

where \( E \) is the identity matrix. Then the map-germ \( x \mapsto (x_1, f_2(x), \ldots, f_n(x)) \) is an orientation-preserving diffeomorphism-germ. Hence we may assume that

\[
f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), x_2, \ldots, x_n).
\]

Then we can take the null vector field \( \eta = \partial x_1 \). Since \( \lambda = (f_1)_{x_1} \) and \( \eta^k \lambda(0) \neq 0 \), it holds that

\[
f_1(x_1, 0, 0, \ldots, 0) = a x_1^{k+1} + \cdots + (a \neq 0).
\]

Then by the Malgrange preparation theorem, there exist functions \( a_0, \ldots, a_k \) of \((X_1, \ldots, X_n)\) such that

\[
x_1^{k+1} = a_0(f(x)) - a_1(f(x)) x_1 + \cdots + a_k(f(x)) x_1^k \quad (x = (x_1, \ldots, x_n)) \quad (2.3)
\]

holds. Considering an orientation-preserving diffeomorphism-germ

\[
\varphi(x) = (x_1 + \frac{1}{k} a_k(f(x)), x_2, \ldots, x_n)
\]

and set \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) = \varphi(x) \). Then by a direct calculation, there exist functions \( b_0, \ldots, b_{k-1} \) such that

\[
\tilde{x}_1^{k+1} = b_0 \left( f \circ \varphi^{-1}(\tilde{x}) \right) - \sum_{i=1}^{k-1} \tilde{x}_i b_i \left( f \circ \varphi^{-1}(\tilde{x}) \right).
\]

Differentiating the equation (2.4) by \( \tilde{x}_1 \), we see that \( b_0(0) = \cdots = b_{k-1}(0) = 0 \). Furthermore, setting \( \tilde{x}_2 = \cdots = \tilde{x}_k = 0 \) in (2.4) and expanding both sides in powers of \( \tilde{x}_1 \), we see that \( (b_1)_{X_1}(0) \neq 0 \). Thus

\[
\Phi(X) = (\varepsilon_0 b_0(X), X_2, \ldots, X_n) \quad (X = (X_1, \ldots, X_n))
\]

is an orientation-preserving diffeomorphism-germ, where, \( \varepsilon_0 = \text{sgn}( (b_1)_{X_1}(0) ) = \pm 1 \). Then we see that \( f \) is \( \mathcal{A} \)-isotopic to

\[
\Phi \circ f \circ \varphi^{-1}(\tilde{x}) = \left( \varepsilon_0 \left( \tilde{x}_1^{k+1} - \sum_{i=1}^{k-1} \tilde{x}_i b_i \left( f \circ \varphi^{-1}(\tilde{x}) \right) \right), \tilde{x}_2, \ldots, \tilde{x}_n \right).
\]

If \( k = 1 \), we have the assertion. We assume \( k > 1 \) in what follows. Since the condition \( \text{rank} d(\lambda, \eta, \lambda, \ldots, \lambda)(0) = k \) does not depend on the coordinate system, we may assume that

\[
\lambda = (\partial / \partial \tilde{x}_1) \left\{ \tilde{x}_1^{k+1} - \sum_{i=1}^{k-1} \tilde{x}_i b_i \left( f \circ \varphi^{-1}(\tilde{x}) \right) \right\}
\]

and \( \eta = \partial \tilde{x}_1 \). Thus we see that

\[
\grad \left( b_1 \left( f \circ \varphi^{-1}(\tilde{x}) \right) \right)(0), \ldots, \grad \left( b_{k-1} \left( f \circ \varphi^{-1}(\tilde{x}) \right) \right)(0), \grad \tilde{x}_1(0)
\]

are linearly independent. Thus

\[
\psi(\tilde{x}) = \left( \tilde{x}_1, \varepsilon_2 b_1 \left( f \circ \varphi^{-1}(\tilde{x}) \right), \ldots, b_{k-1} \left( f \circ \varphi^{-1}(\tilde{x}) \right), \tilde{x}_{k+1}, \ldots, \tilde{x}_n \right)
\]

\[
\Psi(X) = (\varepsilon_3 b_0(X), b_1(X), \ldots, b_{k-1}(X), X_{k+1}, \ldots, X_n)
\]

are orientation-preserving diffeomorphism-germs for some \( \varepsilon_2 = \pm 1 \) and \( \varepsilon_3 = \pm 1 \), and we see that \( f \) is \( \mathcal{A} \)-isotopic to \( \Psi \circ f \circ \varphi^{-1} \circ \psi^{-1} \). Setting \( \varepsilon_1 = \varepsilon_0 \varepsilon_3 \), we complete the proof. \( \blacksquare \)
Let $f$ be a map-germ of the form (2.2). Since it holds that
\[ \lambda = \varepsilon_1 \varepsilon_2 \frac{\partial}{\partial x_1}(x_2 x_2 x_1 + x_3 x_1^2 + \cdots + x_k x_1^{k-1} + x_1^{k+1}), \ \eta = \partial x_1, \]
we have
\[ \text{sgn } (\eta^k \lambda(0)) = \varepsilon_1 \varepsilon_2, \ \text{sgn } \det(\lambda, \eta, \ldots, \eta^{n-1}) = (-1)^{n-1} \varepsilon_1^n \varepsilon_2^{n+1}, \quad (2.6) \]
where $\det(\lambda, \eta, \ldots, \eta^{n-1}) = (\det \lambda, \det \eta, \ldots, \det \eta^{n-1})$. By orientation-preserving diffeomorphism-germs on source and target, $\lambda$ is multiplied by a positive function. On the other hand, reversing the direction of $\eta$, the sign of $\eta \lambda$ changes. Summarizing the above arguments, we have the following lemma.

**Lemma 2.3.** Let $f \in C^\infty(n, n)$ be a $k$-Morin singularity. If $k$ is even, then $\text{sgn}(\eta^k \lambda) = \varepsilon_1 \varepsilon_2$ is an invariant of $A$-isotopy. If $k = n$ and $1 + \cdots + n - 1 = (n - 1)n/2$ is even, then $\text{sgn } \det(\lambda, \ldots, \eta^{n-1}) = (-1)^{n-1} \varepsilon_1^n \varepsilon_2^{n+1}$ is an invariant of $A$-isotopy. Furthermore, if $k = n$ and $n$ and $(n - 1)n/2$ are both odd, then
\[ \text{sgn } (\eta^k \lambda \cdot \det(\lambda, \eta, \ldots, \eta^{k-1})) = (-1)^{n-1} \varepsilon_1^n \varepsilon_2^{n+1} \]
is an invariant of $A$-isotopy.

Now we consider $A$-isotopy of $f^k_{(\varepsilon_1, \varepsilon_2)}(x_1, \ldots, x_n)$. By the above lemma, in the case of $k = n$, we consider four cases $k = n = 4l, 4l + 1, 4l + 2, 4l + 3$.

### 2.1 The case $k = n = 4l$

By Lemma 2.3 we see that $\varepsilon_1 \varepsilon_2$ and $-\varepsilon_1 \varepsilon_2^{4l+1} = -\varepsilon_2$ are invariants of $A$-isotopy. Thus if $(\varepsilon_1, \varepsilon_2) \neq (\varepsilon_1', \varepsilon_2')$, then $f^k_{(\varepsilon_1, \varepsilon_2)}$ and $f^k_{(\varepsilon_1', \varepsilon_2')}$ are not $A$-isotopic. We remark that $\varepsilon_1 \varepsilon_2$ is known as the local degree of $f$. The algebraic sum of it is related to the topology of the source and the target manifolds (See [11, 27, 28] for the $Z_2$-case, and [32] for the $Z$-case. See also [8]).

### 2.2 The case $k = n = 4l + 1$ and $l \neq 0$

Here, we use the following terminology: Let $I$ be a set of indices such that $\#I$ is even. Then the $\pi$-rotations of $I$ stands for the diffeomorphism-germ $(x_1, \ldots, x_k) \mapsto (\tilde{x}_1, \ldots, \tilde{x}_k)$, where $\tilde{x}_j = \varepsilon x_j$ if $j \in I$, and $\tilde{x}_j = x_j$ if $j \notin I$, with $\varepsilon = -1$. We see that applying $\pi$-rotations on the source space and on the target space does not change the $A$-isotopy class.

We assume that $\varepsilon_2 = -1$. Since $l \neq 0$, the number $\#\{1, 4, 6, \ldots, 4l\}$ is even. By $\pi$-rotations of $\{1, 4, 6, \ldots, 4l\}$ on the source space, we see that $f^k_{(\varepsilon_1, x_2)}$ is $A$-isotopic to
\[ (\varepsilon_1(x_2 x_1 + x_3 x_1^2 + \cdots + x_k x_1^{k-1} + x_1^{k+1}), \varepsilon_2 x_2, x_3, \varepsilon x_4, x_5, \ldots, \varepsilon x_{4l}, x_{4l+1}). \quad (2.7) \]
Then by $\pi$-rotations of $\{2, 4, \ldots, 4l\}$ (even number) on the target space, we see that $f^k_{(\varepsilon_1, x_2)}$ is $A$-isotopic to $f^k_{(\varepsilon_1, x_1) \cdot \varepsilon \cdot \varepsilon_1}$. On the other hand, by Lemma 2.3 we see that $(-1)^{4l} \varepsilon_1^{4l+1} \varepsilon_2^{4l} = \varepsilon_1$ is an invariant of $A$-isotopy. Thus if $\varepsilon_1 \neq \varepsilon_1'$, then $f^k_{(\varepsilon_1, 1)}$ and $f^k_{(\varepsilon_1', 1)}$ are not $A$-isotopic.

### 2.3 The case $k = n = 1$

We see that the sign of $f^1_{(\varepsilon_1, x_1)}(0)$ is an invariant of $A$-isotopy. Thus for a given $1$-Morin singularity $f : (R^1, 0) \to (R^1, 0)$, if $f^1_{(\varepsilon_1, x_1)}(0) > 0$ (respectively, $f^1_{(\varepsilon_1, x_1)}(0) < 0$), $f$ is $A$-isotopic to $x_1^2$ (respectively, $-x_1^2$).
2.4 The case $k = n = 4l + 2$

We assume that $\varepsilon_2 = -1$. By $\pi$-rotations of $\{1, 2, 3, 5, \ldots, 4l + 1\}$ on the source space, we see that $f^k_{(\varepsilon_1, \varepsilon_2)}$ is $A$-isotopic to

$$\left(\varepsilon_1 \varepsilon_2 (x_2 x_1 + x_3 x_1^2 + \cdots + x_k x_1^{k-1} + x_1^{k+1}), \ x_2, \varepsilon_2 x_3, \ x_4, \varepsilon_2 x_5, \ldots, \varepsilon_2 x_{4l+1}, \ x_{4l+2}\right).$$

Then by $\pi$-rotations of $\{3, 5, \ldots, 4l + 1\}$ on the target space, it is $A$-isotopic to $f^k_{(\varepsilon_1, \varepsilon_2)}$. On the other hand, by Lemma 2.3 we see that if $\varepsilon_1 \varepsilon_2$ is an invariant of the $A$-isotopy. Hence $f^k_{(\varepsilon_1)}$ and $f^k_{(\varepsilon', 1)}$ are not $A$-isotopic if $\varepsilon \neq \varepsilon'$, where $\varepsilon, \varepsilon' \in \{\pm 1\}$. Like as in the case of $k = n = 4l$, the invariant $\varepsilon_1 \varepsilon_2$ is known as the local degree of $f$, and algebraic sum of it is related to the topology of the source and target manifolds ([8, 11, 27, 28, 32]).

2.5 The case $k = n = 4l + 3$

We assume that $\varepsilon_1 = -1$. By $\pi$-rotations of $\{1, 2, 4, \ldots, 4l + 2\}$ (even number) on the source space, $f^k_{(\varepsilon_1, \varepsilon_2)}$ is $A$-isotopic to

$$\left(\varepsilon_1 (\varepsilon_2 x_2 x_1 + x_3 x_1^2 + \cdots + x_k x_1^{k-1} + x_1^{k+1}), \varepsilon_1 \varepsilon_2 x_2, \ x_3, \varepsilon_1 x_4, \ x_5, \ldots, \varepsilon_1 x_{4l+2}, \ x_{4l+3}\right).$$

Again by $\pi$-rotations of $\{1, 2, 4, \ldots, 4l + 2\}$ on the target space, we see that $f^k_{(\varepsilon_1, \varepsilon_2)}$ is $A$-isotopic to $f^k_{(1, \varepsilon_2)}$. On the other hand, by Lemma 2.3 the sign $\varepsilon_1 \varepsilon_2 \cdot (-1)^{4l+2} \varepsilon_1^{l+3} \varepsilon_2^{l+4} = \varepsilon_2$ is an invariant of $A$-isotopy. Thus if $\varepsilon_2 \neq \varepsilon'_2$, then $f^k_{(1, \varepsilon_2)}$ and $f^k_{(1, \varepsilon'_2)}$ are not $A$-isotopic. This invariant is related to the Vassiliev type invariants of singularities, since we consider isotopy (see [14]). In [14], its global properties are also investigated. See [4] for another interpretation.

2.6 The case $n > k$

If $n > k$, then by a $\pi$-rotation $\{2, n\}$ on the source space, and by a $\pi$-rotation $\{1, n\}$ on the target space, we see that $f^k_{(\varepsilon_1, \varepsilon_2)}$ is $A$-isotopic to $f^k_{(\varepsilon_1, \varepsilon_2, 1)}$. If $k$ is even, by Lemma 2.3 $f^k_{(\varepsilon, 1)}$ is $A$-isotopic to $f^k_{(\varepsilon', 1)}$ if and only if $\varepsilon = \varepsilon'$. If $k$ is odd and $k = 4l + 1$ ($l \neq 0$) (respectively, $k = 4l + 3$), assume that $\varepsilon = -1$. Then by $\pi$-rotations $\{1, 2, 4, \ldots, 4l, n\}$ (respectively, $\{1, 2, 4, \ldots, 4l + 2\}$) on the source space, we see that $f^k_{(\varepsilon, 1)}$ is $A$-isotopic to

$$\left(\varepsilon (x_2 x_1 + x_3 x_1^2 + \cdots + x_{4l+1} x_1^{4l} + x_1^{4l+2}), \varepsilon x_2, \ x_3, \varepsilon x_4, \ldots, \varepsilon x_{4l}, \ x_{4l+1}, \ldots, \varepsilon x_{n-1}, \varepsilon x_n\right)$$

if $k = 4l + 1$, and

$$\left(\varepsilon (x_2 x_1 + x_3 x_1^2 + \cdots + x_{4l+1} x_1^{4l} + x_1^{4l+2}), \varepsilon x_2, \ x_3, \varepsilon x_4, \ldots, \varepsilon x_{4l+2}, \ x_{4l+3}, \ldots, \varepsilon x_n\right)$$

if $k = 4l + 3$. Then we easily see that these germs are $A$-isotopic to $f^k_{(1, 1)}$. Furthermore, in the case of $n > k = 1$, one can easily see that $f^1_{(1)}$ is $A$-isotopic to $f^1_1$.

We remark that in the case of $k = 1$ and $n > 1$, 1-Morin singularities are also called folds. Thus all folds are $A$-isotopic to $(x_1^2, x_2, \ldots, x_n)$. This is a special case of Ando’s result which claims that the homotopy types of the set of $r$-jets of folds are $O(n)$ ([2] p.169).
3 Normal forms and invariants

We summarize the normal forms and invariants for each case. The case of \( k = n \) is shown in Table 3.1 and the case of \( k < n \) is shown in Table 3.2, where \( \# \) indicates the number of \( \mathcal{A} \)-isotopy classes.

### Table 3.1: \( \mathcal{A} \)-isotopy classes of \( n \)-Morin singularities in \( \mathcal{C}^{\infty}(n, n) \).

| name         | \( k \) | normal form | invariants                      | \# |
|--------------|--------|-------------|---------------------------------|----|
| fold         | 1      | \( f^1_1 \) | \( \eta^2 f \)                  | 2  |
| cusp         | 2      | \( f^k_{(\varepsilon_1, 1)} \) | \( \eta^2 \lambda = \varepsilon_1 \) | 2  |
| swallowtail  | 3      | \( f^k_{(\varepsilon_1, \varepsilon_2)} \) | \( \eta^3 \lambda \det \text{grad}(\lambda, \eta \lambda, \eta^2 \lambda) = \varepsilon_2 \) | 2  |
| butterfly    | 4      | \( f^k_{(\varepsilon_1, \varepsilon_2)} \) | \( (\eta^4 \lambda, \det \text{grad}(\lambda, \ldots, \eta^4 \lambda)) \) | 4  |
|              |        |             |                                 |    |
|              | 5      | \( f^k_{(\varepsilon_1, 1)} \) | \( \det \text{grad}(\lambda, \ldots, \eta^3 \lambda) = \varepsilon_1 \) | 2  |
|              |        |             |                                 |    |
|              | \( 4l \) | \( f^k_{(\varepsilon_1, \varepsilon_2)} \) | \( (\eta^4 \lambda, \det \text{grad}(\lambda, \ldots, \eta^4 \lambda)) \) | 4  |
|              |        |             |                                 |    |
|              | \( 4l + 1 \) | \( f^k_{(\varepsilon_1, 1)} \) | \( \det \text{grad}(\lambda, \ldots, \eta^4 \lambda) = \varepsilon_1 \) | 2  |
|              |        |             |                                 |    |
|              | \( 4l + 2 \) | \( f^k_{(\varepsilon_1, 1)} \) | \( \eta^4 \lambda = \varepsilon_1 \) | 2  |
|              |        |             |                                 |    |
|              | \( 4l + 3 \) | \( f^k_{(\varepsilon_1, \varepsilon_2)} \) | \( \eta^4 \lambda \det \text{grad}(\lambda, \ldots, \eta^4 \lambda) = \varepsilon_2 \) | 2  |

### Table 3.2: \( \mathcal{A} \)-isotopy classes of \( k \)-Morin singularities in \( \mathcal{C}^{\infty}(n, n) \).

| name                     | \( k \) | normal form | invariants | \# |
|--------------------------|--------|-------------|------------|----|
| fold (\( \times \) intervals) | 1      | \( f^1_1 \) | -          | 1  |
| cusp (\( \times \) intervals) | 2      | \( f^k_{(\varepsilon_1, 1)} \) | \( \eta^2 \lambda = \varepsilon_1 \) | 2  |
| swallowtail (\( \times \) intervals) | 3      | \( f^1_{(1, 1)} \) | -          | 1  |
|                          |        |             |            |    |
|                          | \( 2m \) | \( f^k_{(\varepsilon_1, 1)} \) | \( \eta^{2m} \lambda = \varepsilon_1 \) | 2  |
|                          |        |             |            |    |
|                          | \( 2m + 1 \) | \( f^k_{(1, 1)} \) | -          | 1  |

4 Other singularities

In this section, we consider \( \mathcal{A} \)-isotopy for other well-known low-dimensional singularities.

4.1 Codimension one map-germs from the plane into the plane

Classification up to \( \mathcal{A} \)-equivalence for map-germs from the plane into the plane is given by Rieger [26]. He classified map-germs \((\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)\) with corank one and \( \mathcal{A}_e \)-codimension \( \leq 6 \). Table 4.1 shows the list of the \( \mathcal{A}_e \)-codimension \( \leq 1 \) local singularities obtained in [26]. Folds and cusps are Morin singularities. Recognition criteria for other singularities are given in [29]:

**Lemma 4.1.** Let \( f \in \mathcal{C}^{\infty}(2, 2) \) be a map-germ.

1. \( f \) is \( \mathcal{A} \)-equivalent to a lips if and only if \( d\lambda = 0, \det \text{Hess} \lambda > 0 \) at 0.
(2) $f$ is $A$-equivalent to a beaks if and only if $d\lambda = 0$, $\det \Hess \lambda < 0$, $\eta \eta \lambda \neq 0$ at 0.

(3) $f$ is $A$-equivalent to a (planar) swallowtail if and only if $d\lambda \neq 0$, $\lambda = \eta \eta \lambda = 0$, $\eta \eta \lambda \neq 0$ at 0.

We have the following theorem.

**Theorem 4.2.** Let $f \in C^\infty(2, 2)$ be a map-germ.

1. If $f$ is $A$-equivalent to a lips, and $\text{sgn} \, \eta \eta \lambda = \varepsilon$, then $f$ is $A$-isotopic to $(\varepsilon x_1(x_1^2 + x_2^2), x_2)$. Moreover these two map-germs are not $A$-isotopic.

2. If $f$ is $A$-equivalent to a beaks, and $\text{sgn} \, \eta \eta \lambda = \varepsilon$, then $f$ is $A$-isotopic to $(\varepsilon x_1(x_1^2 - x_2^2), x_2)$. Moreover these two map-germs are not $A$-isotopic.

3. If $f$ is $A$-equivalent to a (planar) swallowtail, and $\text{sgn}(\xi \lambda \eta \eta \lambda) = \varepsilon$, then $f$ is $A$-isotopic to $(\varepsilon x_1 x_2 + x_1^3, x_2)$, where $\xi$ is a vector field such that $(\xi, \eta)$ is a positive frame at 0. Moreover these two map-germs are not $A$-isotopic.

**Proof.** By the same method as in the proof of Lemma 2.2 we may assume $f$ has the form $f(x_1, x_2) = (f_1(x_1, x_2), x_2)$. There exist functions $g_1(x_1, x_2)$ and $g_2(x_2)$ such that $f_1(x_1, x_2) = x_1 g_1(x_1, x_2) + g_2(x_2)$. Thus we may assume that $f(x_1, x_2) = (x_1 g_1(x_1, x_2), x_2)$ in all cases. (Proofs of (1) and (2).) Since the function $\lambda$ satisfies that $\lambda(0) = 0$ and $d\lambda(0) = 0$, $f$ can be written as

$$a x_1^2 + bx_1 x_2 + c x_2^2 + h(x_1, x_2),$$

where $h(x_1, x_2)$ is a function which order is greater than 3. Since $\eta \eta \lambda \neq 0$, it holds that $c \neq 0$. Thus by an orientation-preserving diffeomorphism-germ $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2 + 2b x_1/3c$ ($t \in [0, 1]$), and by a suitable scaling change, $f$ is $A$-isotopic to the map-germ $(x_1, x_2^2 \pm 2x_1^2 \pm x_2^2) + h(x_1, x_2))$, where $h(x_1, x_2)$ is a function whose order is greater than 3. It is well known that lips and beaks are three-determined ([26] Lemma 3.1.3), and the proof of it contains that $(x_1, x_2^2 \pm 2x_1^2)$ and $(x_1, x_2^2 \pm 2x_1^2 \pm x_2^2 + h(x_1, x_2))$ are $A$-isotopic (see [17] Section 3, see also [19] Section 3). Since $\eta \eta \lambda$ does not change by positive coordinate changes on the source and target, the second assertion of the theorem is obvious.

**(Proof of (3).)** We can write

$$x_1 g_1(x_1, x_2) = a_{20} x_1^2 + a_{11} x_1 x_2 + a_{30} x_2^3 + a_{21} x_1^2 x_2 + a_{12} x_1 x_2^2 + a_{40} x_1^4 + a_{31} x_1^3 x_2 + a_{22} x_1^2 x_2^2 + a_{13} x_1 x_2^3 + h(x_1, x_2),$$

where $h(x_1, x_2)$ is a function whose order is greater than 4. Since $\eta \lambda(0) = \eta \eta \lambda(0) = 0$ and $d\lambda(0) \neq 0$, it holds that $a_{20} = a_{30} = 0$ and $a_{11} \neq 0$, thus by an orientation-preserving diffeomorphism-germ $\tilde{x}_1 = |a_{11}| x_1 + a_{21} x_1^2 + a_{12} x_1 x_2 + a_{31} x_1^3 + a_{22} x_1^2 x_2 + a_{13} x_1 x_2^2$ and by a suitable scaling change, $f$ is $A$-isotopic to the map-germ $(x_1, x_2^2 \pm x_1^2 \pm h(x_1, x_2))$. By the same argument as just above, we see that $f$ is $A$-isotopic to $(x_1, x_2^2 \pm x_1^2)$. Since the sign of the product $\xi \lambda \eta \eta \lambda$ does not depend on the choice of $(\xi, \eta)$, the second assertion is obvious. \[\square\]
4.2 Whitney umbrellas and $S_1$-singularities

Classification for map-germs from the plane into the 3-space up to $A$-equivalence is given by Mond [23]. He classified simple map-germs $(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$. Table 4.2 shows the list of the $A_c$-codimension $\leq 1$ local singularities obtained in [23]. In the list, $S^\pm_1$ singularities are also

| name          | normal form          | $A_c$-codimension |
|---------------|----------------------|-------------------|
| Whitney umbrella | $(x_1^2, x_1x_2, x_2)$ | 0                 |
| $S^+_1$       | $(x_1^2, x_1(x_1^2 + x_2^2), x_2)$ | 1                 |
| $S^-_1$       | $(x_1^2, x_1(x_1^2 - x_2^2), x_2)$ | 1                 |

Table 4.2: Classification of $C^\infty(2, 3)$

called Chen-Matsumoto-Mond $\pm$-singularities [6]. Recognition criteria for them are given in [30]. Let $f \in C^\infty(2, 3)$ be a corank one map-germ at 0 and $\eta$ a non-zero vector field such that $\eta(0) \in \ker(df)(0)$. Let $\xi$ be a vector field such that $\xi, \eta$ are linearly independent. We set

$$w = \det(\xi f, \eta f, \eta\eta f).$$

(4.1)

Then $f$ is a Whitney umbrella if and only if $dw \neq 0$ at 0. Furthermore, $f$ is an $S^+_1$ singularity (respectively, $S^-_1$ singularity) if and only if $dw = 0$ and $\det \text{Hess } w(0) > 0$ (respectively, $dw = 0$, $\det \text{Hess } w(0) < 0$ and $\eta\eta w(0) \neq 0$) [30 Theorem 2.2]. For $A$-isotopy, we have the following theorem:

**Theorem 4.3.** Let $f \in C^\infty(2, 3)$ be a corank one map-germ at 0.

1. If $f$ is $A$-equivalent to a Whitney umbrella then $f$ is $A$-isotopic to $(x_1^2, x_1x_2, x_2)$.
2. If $f$ is $A$-equivalent to a $S^+_1$ singularity, and $\sgn \eta\eta w = \varepsilon$, then $f$ is $A$-isotopic to $(x_1^2, \varepsilon x_1(x_1^2 + x_2^2), x_2)$. Moreover these two map-germs are not $A$-isotopic.
3. If $f$ is $A$-equivalent to a $S^-_1$ singularity, and $\sgn \eta\eta w = \varepsilon$, then $f$ is $A$-isotopic to $(x_1^2, \varepsilon x_1(x_1^2 - x_2^2), x_2)$. Moreover these two map-germs are not $A$-isotopic.

Here, $\varepsilon = \pm 1$.

**Proof.** Let $f \in C^\infty(2, 3)$ be a corank one map-germ at 0. Then one can easily see that $f$ is $A$-isotopic to the map-germ of the form $(x_1^2, x_1 h(x_1^2, x_2), x_2)$ for some function $h$ satisfying $h(0) = 0$ (see [30 p72], for example). We may choose $\xi = \partial x_2$ and $\eta = \partial x_1$. Then the function $w$ defined in (4.1) is

$$w(x_1, x_2) = 8x_1^2hx_1(x_1^2, x_2) + 8x_1hx_1x_1(x_1^2, x_2) - 2h(x_1^2, x_2).$$

Thus we have

$$\xi w(0) = -2h_{x_2}(0), \ \det \text{Hess } w(0) = -24h_{x_2x_2}(0)h_{x_1}(0), \ \eta\eta w = 12h_{x_1}(0).$$

Hence (1) is obvious. We prove (2) and (3). We assume that $dw(0) = 0$, $\det \text{Hess } w(0) \neq 0$ and $\eta\eta w(0) \neq 0$. Since $h(0) = h_{x_2}(0) = 0$, $h_{x_2x_2}(0) \neq 0$ and $h_{x_1}(0) \neq 0$, there exist functions $\tilde{h}$ and $\bar{h}$ satisfying $h(0) = \tilde{h}(0) = 0$ such that

$$h(x_1, x_2) = ax_1^2(1 + \tilde{h}(x_1^2, x_2)) + \beta x_2^2(1 + \bar{h}(x_2)).$$

We remark that $\det \text{Hess } w(0) = -48a\beta$. Thus by a coordinate change

$$(x_1, x_2) \rightarrow \left(x_1\sqrt{1 + \tilde{h}(x_2)}, \ x_2\sqrt{1 + \bar{h}(x_1^2, x_2)}\right)$$

on the source, and a suitable scale change, we see the first assertions of (2) and (3). The second assertions are obvious. □
5 Perturbation of simple singularities

Consider a simple corank 1 singularity \( f \in C^\infty(n, n) \) and a small stable perturbation \( \tilde{f} \) of \( f \). Since all stable corank 1 singularities are Morin singularities, \( \tilde{f} \) has some \( n \)-Morin singularities. In the complex case, the number of \( n \)-Morin singularities appearing in \( \tilde{f} \) is constant, but in the real case, it is not constant and the maximal number has been studied \([10, 12]\). It is denoted by \( c(f) \), and it represents a geometric property of \( f \). In the present paper, we divide \( \mathcal{A} \)-classes into \( \mathcal{A} \)-isotopy classes, and we can study \( c(f) \) more precisely using \( \mathcal{A} \)-isotopy. In this section, we observe the \( \mathcal{A} \)-isotopy classes of \( n \)-Morin singularities appearing on some perturbations of simple singularities. Since the numbers of \( \mathcal{A} \)-isotopy classes of \( n \)-Morin singularities has a periodicity 4 with respect to \( n \), we consider the cases of \( n = 2, 3, 4 \) and 5 here. For the sake of simplicity, in the case of \( n = 4 \), we call the invariant \( \eta^4 \lambda \) the first invariant, and the invariant \( \det \text{grad}(\lambda, \ldots, \eta^3 \lambda) \) the second invariant.

5.1 Classification of simple corank 1 map-germs of \( C^\infty(n, n) \)

Let \( f \) be a corank 1 map-germ of \( C^\infty(n, n) \). Then \( f \) is \( \mathcal{A} \)-equivalent to the map-germ

\[ (t, x_2, \ldots, x_n) \mapsto (f_1(t, x_2, \ldots, x_n), x_2, \ldots, x_n). \]

The function \( f_1(t, 0, \ldots, 0) \) is called the genotype of \( f \). If \( f \) is simple and \( n \geq 3 \), then the genotype of \( f \) is \( t^{i+1} \) \((i \leq n+1)\). If \( f \) is simple and \( n = 2 \), then the genotype of \( f \) is \( t^{i+1} \) \((i \leq 4)\). Thus, if \( f \) is simple, one can show that \( f \) is \( \mathcal{A} \)-equivalent to

\[ (t, x_2, \ldots, x_n) \mapsto \left( t^{i+1} + \sum_{j=1}^{i-1} p_j(x_2, \ldots, x_n)t^j, x_2, \ldots, x_n \right). \]

(5.1)

We denote the map-germ of the form (5.1) by \([p_1, \ldots, p_{i-1}]\).

Classification of simple map-germs of \( C^\infty(2, 2) \) is given by Rieger \([26]\), and classification of simple map-germs of \( C^\infty(3, 3) \) is given by Marar and Tari \([21]\) using the method of complete transversal \([9]\). Using the same method as in \([21]\), we can find three families of simple map-germs as in Table 5.1. It should be remarked that these families can be also obtained by the augmentation of map-germs \([7, 15, 16]\), because \([x_2, \ldots, x_n] \) is a stable map-germ.

| Family | genotype map-germ | \( t^2 \) | \( t^4 \) | \( t^6 \) | \( t^8 \) |
|--------|------------------|--|--|--|--|
| A      | \([x_2^2]\)      | \([x_2, x_3^2]\) | \([x_2, x_3, x_4^2]\) | \([x_2, x_3, x_4, x_5]\) |
| B      | \([x_2, 0]\)     | \([x_2, x_3, 0] \) | \([x_2, x_3, x_4, 0]\) | \([x_2, x_3, x_4, x_5, 0]\) |
| C      | \([x_2^2, x_2]\) | \([x_2, x_3^2, x_3]\) | \([x_2, x_3, x_4, x_5]\) | \([x_2, x_3, x_4, x_5^2, x_5]\) |

Table 5.1: Families of map-germs \((l \geq 2)\).

Observing the invariants of \( n \)-Morin singularities \( \tilde{f} \) for the families A, B and C, we may clarify the difference between these families with respect to \( n \).
5.2 Family A

A versal unfolding of a map-germ in family A is \( F_u(t, x) = (q(t, x, u), x_2, \ldots, x_n) \),

\[
q(t, x, u) = \begin{cases} 
  t^3 + \bar{q}(x_2, u)t, & (n = 2), \\
  t^4 + x_2t + \bar{q}(x_3, u)t^2, & (n = 3), \\
  t^5 + x_2t + x_3t^2 + \bar{q}(x_4, u)t^3, & (n = 4), \\
  t^6 + x_2t + x_3t^2 + x_4t^3 + \bar{q}(x_5, u)t^4, & (n = 5), 
\end{cases}
\]

\[
\bar{q}(x_n, u) = x_n^2 + u_0 + u_1x_n + \cdots + u_{-2}x_n^{l-2},
\]

where \( x = (x_2, \ldots, x_n) \) and \( u = (u_0, \ldots, u_{-2}) \) ∈ \( \mathbb{R}^{l-1} \). For versal unfolding, see [18, Chapter XIV], for example. Since \( \lambda = q_t(t, x, u) \) and \( \eta = \partial t \), we have Table 5.2 where “n-M” means the condition for n-Morin singularity, and “inv” means the value of the invariants mentioned in Lemma 2.3. Looking at Table 5.2, we observe in the case \( n = 2 \), where all n-Morin singularities are \( \mathcal{A} \)-isotopic, and in the cases \( n = 3 \) and \( 5 \), there are two kinds of n-Morin singularities, and in the case \( n = 4 \), there are two kinds of n-Morin singularities, but in all cases, the first and second invariants coincide.

| \( n \) | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| \( c(f) \) | \( l \) | \( l \) | \( l \) | \( l \) |
| n-M | \( t = 0, \bar{q} = 0, \bar{q}_{x_2} \neq 0 \) | \( t = x_2 = \cdots = x_{n-1} = 0, \bar{q} = 0, \bar{q}_{x_n} \neq 0 \) | \( 1 \) | \( (1, \bar{q}_{x_4}) \) | \( \bar{q}_{x_5} \) |

Table 5.2: Maximal number of Morin singularities and their invariants (family A)

5.3 Family B

A versal unfolding of a map-germ in family B is \( F_u(t, x) = (q(t, x, u_0), x_2, \ldots, x_n) \),

\[
q(t, x, u_0) = \begin{cases} 
  t^4 + x_2t + u_0t^2 & (n = 2), \\
  t^5 + x_2t + x_3t^2 + u_0t^3 & (n = 3), \\
  t^6 + x_2t + x_3t^2 + x_4t^3 + u_0t^4 & (n = 4), \\
  t^7 + x_2t + x_3t^2 + x_4t^3 + x_5t^4 + u_0t^5 & (n = 5), 
\end{cases}
\]

where \( u_0 \in \mathbb{R} \). Since \( \lambda = q_t(t, x, u_0) \) and \( \eta = \partial t \), we have Table 5.3. Looking at Table 5.3, we observe that in the case \( n = 2 \), all n-Morin singularities are \( \mathcal{A} \)-isotopic, and in the cases \( n = 3 \) and \( 5 \), there are two kinds of n-Morin singularities, and in the case \( n = 4 \), there are two kinds of n-Morin singularities, but in all cases, the first and second invariants coincide.

| \( n \) | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| \( c(f) \) | 2 | 2 | 2 | 2 |
| n-M | \( t = \pm \sqrt{-\frac{u_0}{6}}, x_2 = 8t^3, t \neq 0 \) | \( t = \pm \sqrt{-\frac{u_0}{10}}, x_2 = 105t^4, x_3 = -40t^3, t \neq 0 \) | \( t = \pm \sqrt{-\frac{u_0}{15}}, x_2 = 24t^5, x_3 = -45t^4, x_4 = 40t^3, t \neq 0 \) | \( t = \pm \sqrt{-u_0/21}, x_2 = -35t^6, x_3 = 84t^5, x_4 = -105t^4, x_5 = 70t^3, t \neq 0 \) |
| inv | \( t \) | \( t^2 \) | \( (t, t) \) | \( t \) |

Table 5.3: Maximal number of Morin singularities and their invariants (family B)
5.4 Family C

A versal unfolding of a map-germ in family C is \( F_u(t, x) = (q(t, x, u), x_2, \ldots, x_n) \),

\[
q(t, x, u) = \begin{cases} 
  t^4 + (x_2^2 + u_0 + u_1x_2)t + x_2t^2 & (n=2), \\
  t^3 + x_2t^2 + (x_2^2 + u_0 + u_1x_3)t^2 + x_3t^3 & (n=3), \\
  t^2 + x_2t + x_3t^2 + (x_2^2 + u_0 + u_1x_4)t^3 + x_4t^4 & (n=4), \\
  t + x_2t + x_3t^2 + x_4t^3 + (x_2^2 + u_0 + u_1x_5)t^4 + x_5t^5 & (n=5),
\end{cases}
\]

where \( u = (u_0, u_1) \in \mathbb{R}^2 \). Since \( \lambda = q_t(t, x, u_0, u_1) \) and \( \eta = \partial t \), we have Table 5.4. Here, equations \( C_2 \) stands for the equations \( 36t^4 - 8t^3 - 6u_1t^3 + u_0 = 0, x_2 = -6t^2, t \neq 0 \), equations \( C_3 \) stands for the equations \( 100t^4 - 20t^3 - 10u_1t^2 + u_0 = 0, x_2 = 25t^4 - 200t^3 + 20u_1t^3 - 2uqt, x_3 = -10t^2, t \neq 0 \), equations \( C_4 \) stands for the equations \( 255t^4 - 40t^3 - 15t^2u_1 + u_0 = 0, x_2 = 2(225t^6 - 32t^5 - 15t^4u_1 + u_0t^2), x_3 = -3(225t^5 - 25t^4 - 15t^3u_1 + tu_0), x_4 = -15t^2, t \neq 0 \) and equations \( C_5 \) stands for the equations \( 441t^4 - 70t^3 - 2tt^2u_1 + u_0 = 0, x_2 = -4u_0t^3 + 84u_1t^5 + 2456 - 1764t^7, x_3 = 2640t^6 - 336t^5 - 126u_1t^4 + 6u_0t^2, x_4 = -1764t^5 + 175t^4 + 84u_1t^4 - 4u_0t, x_5 = -21t^2, t \neq 0 \).

| \( n \) | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| \( c(f) \) | 4 | 4 | 4 | 4 |
| \( n-M \) | equations | equations | equations | equations |
| \( C_2 \) | \( C_3 \) | \( C_4 \) | \( C_5 \) |
| inv | \( t \) | \(-20t^2 + 3t + u_1 \) | \((t, t(30t^2 - 4t - u_1)) \) | \(-42t^2 + 5t + u_1 \) |

Table 5.4: Maximal number of Morin singularities and their invariants (family C)

6 Criteria for \( \Sigma^{2,0} \) singularities and isotopy

In this section, we consider a corank two singularity for \( C^\infty(4, 4) \). 1-Morin, 2-Morin, 3-Morin singularities and \( \Sigma^{2,0} \)-singularities are stable (equivalently, generic) singularities for maps from 4-manifolds to 4-manifolds. Let \( f \in C^\infty(4, 4) \) be a stable map-germ such that the origin is a singular point of \( f \). Then \( f \) is \( A \)-equivalent to 1-Morin, 2-Morin, 3-Morin singularity or the following map-germ:

\[
\Sigma^{2,0} \text{hyp} : (x_1, x_2, x_3, x_4) \mapsto (x_1^2 + x_2x_3, x_2^2 + x_1x_4, x_3, x_4)
\]

\[
\Sigma^{2,0} \text{ell} : (x_1, x_2, x_3, x_4) \mapsto (x_1^2 - x_2^2 + x_1x_3 + x_2x_4, x_1x_2 + x_1x_4 - x_2x_3, x_3, x_4).
\]

The germ \( \Sigma^{2,0} \text{hyp} \) (respectively \( \Sigma^{2,0} \text{ell} \)) is also called the hyperbolic umbilic (respectively, the elliptic umbilic), and \( I_{2,2}^- \) (respectively, \( I_{2,2}^- \)) \cite{13, 20}. Moreover, we define “signed” umbilics as follows:

\[
\Sigma^{2,0}_{\text{hyp}, \varepsilon_1} : (x_1, x_2, x_3, x_4) \mapsto (x_1^2 + x_2x_3, x_2^2 + \varepsilon_1x_1x_4, x_3, x_4)
\]

\[
\Sigma^{2,0}_{\text{ell}, \varepsilon_1, \varepsilon_2} : (x_1, x_2, x_3, x_4) \mapsto (x_1^2 - x_2^2 + \varepsilon_1x_1x_3 + x_2x_4, \\
\varepsilon_1x_1x_2 + \varepsilon_1x_1x_4 - x_2x_3, x_3, x_2x_4),
\]

where \( \varepsilon_1 = \pm 1 \) and \( \varepsilon_2 = \pm 1 \). Then we have the following theorem.

**Theorem 6.1.** Let \( f \in C^\infty(4, 4) \) be a map-germ such that \( \text{rank}(df)(0) = 2 \) holds. Then \( f \) is \( A \)-isotopic to \( \Sigma^{2,0}_{\text{hyp}, \varepsilon_1} \) (respectively, \( \Sigma^{2,0}_{\text{ell}, \varepsilon_1, \varepsilon_2} \)) if and only if for a coordinate system \( (X_1, X_2, X_3, X_4) \) on the target satisfying that \( d(X_1 \circ f)(0) = d(X_2 \circ f)(0) = 0 \) and a pair of vector fields \( \langle \xi, \eta \rangle \) on the source satisfying that \( \langle \xi(0), \eta(0) \rangle = \ker(df)(0) \), it holds that

1. \( \det \text{Hess}_{\langle \xi, \eta \rangle} \lambda(0) < 0 \) (respectively, \( \det \text{Hess}_{\langle \xi, \eta \rangle} \lambda(0) > 0 \)),

2. \( \text{Hess}_{\langle \xi, \eta \rangle} \lambda(0) \).
symmetric matrix. Since rank(ξ) holds at 0. Thus the independency of the conditions for Hessian matrix is proven. Moreover, 

\[
\text{Hess}(ξ,η) \lambda = \begin{pmatrix} ξξλ & ξηλ \\ ηξλ & ηηλ \end{pmatrix}.
\]

We remark that since ξ(0) and η(0) belong to the kernel of (df)(0), it holds that Hess(ξ,η) λ is a symmetric matrix. Since rank(df)(0) = 2, it holds that λ has a critical point at 0. The proof of this theorem is given as follows.

**Lemma 6.2.** Conditions (1) and (2) of Theorem 6.1 do not depend on the choice of vector fields spanning ker(df)(0) at 0.

**Proof.** Let ξ, η, ζ, ω be a quadruple of vector fields of (R^4, 0) such that ξ, η, ζ, ω are linearly independent. Let a function λ : (R^4, 0) → (R, 0) has a critical point at 0. We remark that ξηλ = ηξλ holds at 0. Let ξ, η, ζ, ω be another quadruple of vector fields such that

\[
\lambda(ξ, η, ζ, ω) = A \lambda(ξ, η, ζ, ω),
\]

where A = (a_{ij})_{i,j=1,...,4} and a_{13} = a_{14} = a_{23} = a_{24} = 0 hold at 0. Here, \(\lambda^t\) means the transpose matrix. Then it holds that

\[
\begin{pmatrix} ξξλ & ξηλ \\ ηξλ & ηηλ \end{pmatrix} = A \begin{pmatrix} ξξλ & ξηλ \\ ηξλ & ηηλ \end{pmatrix} A^t
\]

at 0. Thus the independency of the conditions for Hessian matrix is proven. Moreover,

\[
\begin{align*}
(a_{11} d(ξf_1) + a_{12} d(ηf_1), & \quad a_{11} d(ξf_2) + a_{12} d(ηf_2), \\
& \quad a_{21} d(ξf_1) + \frac{a_{22}}{d(ηf_1)}, & \quad a_{21} d(ξf_2) + \frac{a_{22}}{d(ηf_2)})
\end{align*}
\]

holds at 0, where dh means grad h, for the sake of simplicity. We have

\[
\begin{align*}
\det(a_{11} d(ξf_1) + a_{12} d(ηf_1), & \quad a_{11} d(ξf_2) + a_{12} d(ηf_2), \\
& \quad a_{21} d(ξf_1) + a_{22} d(ηf_1), & \quad a_{21} d(ξf_2) + a_{22} d(ηf_2))
\end{align*}
\]

\[
= (a_{11} a_{22} - a_{12} a_{21})^2 \det(d(ξf_1), d(ξf_2), d(ηf_1), d(ηf_2)).
\]

Thus the conditions (1) and (2) do not depend on the choice of vector fields. □

**Lemma 6.3.** Conditions (1) and (2) of Theorem 6.1 do not depend on the choice of the coordinate system of the target.

**Proof.** Let f(x) = (f_1, f_2, f_3, f_4)(x) ∈ C^∞(4, 4) (x = (x_1, x_2, x_3, x_4)) be a map-germ such that rank(df)(0) = 2, f_{x_1} = f_{x_2} = 0 and df_1 = df_2 = 0 holds at 0. Let Φ = (Φ_1, Φ_2, Φ_3, Φ_4) be a diffeomorphism-germ of (R^4, 0) such that df(Φ_1 o f) = df(Φ_2 o f) = 0. We show that the condition is the same for both f and Φ o f. Since (Φ o f)_{x_1} = (Φ o f)_{x_2} = 0 holds at 0, ker df is the same for f and Φ o f. Since the difference of the determinant of the Jacobi matrix is a positive function between f and Φ o f, thus we see the independence of conditions for Hess(ξ,η) λ.
Hence it is enough to show that if \( d((f_1)_x), d((f_1)_x), d((f_2)_x), d((f_2)_x) \) is a positive frame, then \( d((\Phi_1 \circ f)_x), d((\Phi_1 \circ f)_x), d((\Phi_2 \circ f)_x), d((\Phi_2 \circ f)_x) \) is a positive frame.

Firstly we detect the condition for \( d(\Phi_1 \circ f) = d(\Phi_2 \circ f) = 0 \). By \( d(f_1)(0) = d(f_2)(0) = 0 \), it holds that \((f_1)_x, (f_2)_x, (f_3)_x, (f_4)_x\)(0) = 0 \((i = 1, 2)\). Since

\[
d(\Phi_i \circ f)(0) = \left( (\Phi_i \circ f)_x, (\Phi_i \circ f)_x, (\Phi_i \circ f)_x, (\Phi_i \circ f)_x \right)(0) = 0 \quad (i = 1, 2),
\]

we have

\[
(\Phi_1)_x(f_1)_x + (\Phi_2)_x(f_2)_x + (\Phi_3)_x(f_3)_x + (\Phi_4)_x(f_4)_x = 0 \quad (i = 1, 2, j = 3, 4) \tag{6.1}
\]
at 0. Substituting \((f_1)_x, (f_2)_x, (f_3)_x, (f_4)_x\)(0) = 0 \((i = 1, 2)\) into \(6.1\), we have

\[
(\Phi_1)_x(f_3)_x + (\Phi_2)_x(f_4)_x = 0 \quad \text{and} \quad (\Phi_1)_x(f_3)_x + (\Phi_2)_x(f_4)_x = 0 \quad (i = 1, 2)
\]
at 0. Since \(\text{rank}(df)(0) = 2\), it holds that \((f_3)_x, (f_4)_x\)(0) = 0 \((i = 1, 2)\) and \(f_{x_1} = f_{x_2} = 0\) at 0, we see that \((f_3)_x, (f_4)_x\)(0) and \((f_3)_x, (f_4)_x\)(0) are linearly independent. Thus \((\Phi_1)_x = (\Phi_1)_x = (\Phi_2)_x = (\Phi_2)_x = 0\) at 0. On the other hand,

\[
d(\Phi_i \circ f)_x = d \left( \sum_{l=1}^{4}(\Phi_i)_x(f_l)_x \right) = \sum_{l=1}^{4}(\Phi_i)_x d((f_l)_x) = \sum_{l=1}^{2}(\Phi_i)_x d((f_l)_x)
\]

\((i, j = 1, 2)\) hold at 0. Thus

\[
\left( d((\Phi_1 \circ f)_x), d((\Phi_2 \circ f)_x), d((\Phi_3 \circ f)_x), d((\Phi_4 \circ f)_x) \right)
= \left( d((f_1)_x), d((f_2)_x), d((f_3)_x), d((f_4)_x) \right) \begin{pmatrix} J\Phi & O \\ O & J\Phi \end{pmatrix},
\]

where

\[
J\Phi = \begin{pmatrix} \Phi_1_x & \Phi_2_x \\ \Phi_3_x & \Phi_4_x \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

holds at 0. This shows the desired result. \(\square\)

Next we study the relation between the condition (1) and the quotient ring \(Q(f)\). Let \(E_n\) be the local ring of function-germs \((R^n, 0) \to (R, )\). Let \(f = (f_1, \ldots, f_4) \in C^\infty(4, 4)\) and \(Q(f)\) denote the quotient ring \(E_4/\langle f_1, f_2, f_3, f_4 \rangle_{E_4}\).

**Lemma 6.4.** Let \(f \in C^\infty(4, 4)\) satisfies that \(\text{rank} df_0 = 2\). Then \(Q(f) = E_2/\langle x_1^2, x_2^2 \rangle_{E_2}\) (respectively, \(Q(f) = E_2/\langle x_1^2, x_2^2, x_1 x_2 \rangle_{E_2}\)) is equivalent to that \(\det \text{Hess}_{(\xi, \eta)}(0) < 0\) (respectively, \(\det \text{Hess}_{(\xi, \eta)}(0) > 0\)) holds, where \((\xi, \eta)\) is a pair of vector fields on the source such that \(\xi, \eta\) are the basis of \(\ker df\) at 0.

**Proof.** Since the condition and the conclusion do not depend on the choice of coordinate systems and the choice of vector fields, we may assume that

\[
f(x_1, x_2, x_3, x_4) = (f_1(x_1, x_2, x_3, x_4), f_2(x_1, x_2, x_3, x_4), x_3, x_4), \tag{6.2}
\]

\((df_1)_0 = (df_2)_0 = 0, \xi = \partial x_1\) and \(\eta = \partial x_2\). Then

\[
Q(f) = E_2/\langle (f_1(x_1, x_2, 0, 0), f_2(x_1, x_2, 0, 0) \rangle_{E_2}.
\]

Let us assume \(Q(f) = E_2/\langle x_1^2, x_2^2 \rangle_{E_2}\), and let \(i\) be an isomorphism \(i : Q(f) \to E_2/\langle x_1^2, x_2^2 \rangle_{E_2}\). Then the conclusion is obvious by considering a coordinate change \(i(x_1), i(x_2)\) to \(X_1, X_2\) on the
target. We show the converse. We set the second order terms of \( f_1(x_1, x_2, 0, 0), f_2(x_1, x_2, 0, 0) \) as
\[
a_1x_1^2 + 2b_1x_1x_2 + c_1x_2^2, \quad a_2x_1^2 + 2b_2x_1x_2 + c_2x_2^2.
\] (6.3)
Then we have
\[
det\text{Hess} \lambda/16 = 4a_2b_1b_2c_1 - 4a_1b_2^2c_1 - a_2^2c_1^2 - 4a_2b_1^2c_2 + 4a_1b_1b_2c_2 + 2a_1a_2c_1c_2 - a_1^2c_2^2.
\]
By a suitable coordinate change, we may assume \( b_1 = 0 \) and \( a_2 = 0 \) in (6.3). If \( b_2 \neq 0 \), then by the coordinate change \( \tilde{x}_1 = 2b_2x_1 + c_2x_2, \tilde{x}_2 = x_2 \), it holds that
\[
\langle f_1(x_1, x_2, 0, 0), f_2(x_1, x_2, 0, 0) \rangle_{\mathcal{E}_2} = \langle \tilde{a}_1\tilde{x}_1^2 + 2b_1\tilde{x}_1\tilde{x}_2 + \tilde{c}_1\tilde{x}_2^2 + O(3), \tilde{x}_1\tilde{x}_2 + O(3) \rangle_{\mathcal{E}_2} = \langle \tilde{a}_1\tilde{x}_1^2 + \tilde{c}_1\tilde{x}_2^2 + O(3), \tilde{x}_1\tilde{x}_2 + O(3) \rangle_{\mathcal{E}_2}
\]
for some \( \tilde{a}_1(\neq 0), \tilde{b}_1, \tilde{c}_1 \in \mathbb{R} \), where \( O(3) \) means the terms which consist of terms whose degrees are higher than 3. In the case \( \det\text{Hess} \lambda < 0 \), since \( \det\text{Hess} \lambda = -64a_1\tilde{c}_1 \), we have \( \tilde{a}_1\tilde{c}_1 > 0 \). Thus it holds that
\[
\langle \tilde{a}_1\tilde{x}_1^2 + \tilde{c}_1\tilde{x}_2^2, \tilde{x}_1\tilde{x}_2 \rangle_{\mathcal{E}_2} = \langle \tilde{x}_1^2 + \tilde{x}_2^2, \tilde{x}_1\tilde{x}_2 \rangle_{\mathcal{E}_2} = \langle \tilde{x}_1^2, \tilde{x}_2^2 \rangle_{\mathcal{E}_2},
\]
where we omit \( O(3) \). If \( b_2 = 0 \), since \( a_1c_2 \neq 0 \), it holds that
\[
\langle f_1(x_1, x_2, 0, 0), f_2(x_1, x_2, 0, 0) \rangle_{\mathcal{E}_2} = \langle x_1^2 + O(3), x_2^2 + O(3) \rangle_{\mathcal{E}_2}.
\]
Thus in both cases \( Q(f) = \mathcal{E}_2/\langle x_1^2 + O(3), x_2^2 + O(3) \rangle_{\mathcal{E}_2} \) holds. One can show that this is isomorphic to \( \mathcal{E}_2/\langle x_1^2, x_2^2 \rangle_{\mathcal{E}_2} \).

In the case, \( \det\text{Hess} \lambda > 0 \), we have \( \tilde{a}_1\tilde{c}_1 < 0 \). Thus
\[
\langle f_1(x_1, x_2, 0, 0), f_2(x_1, x_2, 0, 0) \rangle_{\mathcal{E}_2} = \langle x_1^2 - x_2^2 + O(3), x_1x_2 + O(3) \rangle_{\mathcal{E}_2}
\]
holds. If \( b_2 = 0 \), then it holds that \( \det\text{Hess} \lambda = -16a_1^2c_2^2 \). This means that this case does not occur. Hence \( Q(f) = \mathcal{E}_2/\langle x_1^2 - x_2^2 + O(3), x_1x_2 + O(3) \rangle_{\mathcal{E}_2} \) holds. Since \( x_1^2, x_2^2 \in \langle x_1^2 - x_2^2, x_1x_2 \rangle_{\mathcal{E}_2} \) holds, this is isomorphic to \( \mathcal{E}_2/\langle x_1^2 - x_2^2, x_1x_2 \rangle_{\mathcal{E}_2} \).

The 1-jet extension \( j^1f \) of the map-germ \( f \) of the form \([6.2] \) is transverse to the set \( \Sigma^2 = \{ j^1f \mid \text{rank}(df)(0) = 2 \} \) at 0 if and only if
\[
det\left( d((f_1)_{x_1}), d((f_2)_{x_1}), d((f_1)_{x_2}), d((f_2)_{x_2}) \right)(0) \neq 0,
\]
where \( \Sigma^2 = \{ j^1f \mid d((f_1)_{x_1}) = d((f_2)_{x_1}) = d((f_1)_{x_2}) = d((f_2)_{x_2}) = 0 \} \). Summarizing the above arguments, and by following the same arguments as in [13] p183–186, and taking care to use orientation-preserving diffeomorphism-germs, one can complete the proof of Theorem 6.1.

Theorem 6.1 shows that there are two \( \mathcal{A} \)-isotopy classes in the \( \mathcal{A} \)-class of \( \Sigma^2_{\text{hyp}} \), and there are four \( \mathcal{A} \)-isotopy classes in that of \( \Sigma^2_{\text{ell}} \). The invariant \( \varepsilon_2 \) is equal to the mapping degree. See [8] Corollary 5.13 for its global meaning. See also [9] Theorem 2.5, Remark 2.6, [23] page 398, [33] Theorem 1. As a corollary, we get an \( \mathcal{A} \)-criterion for \( \Sigma^2_{\text{hyp}} \)-singularities:

**Corollary 6.5.** Let \( f \in C^{\infty}(4,4) \) be a map-germ such that \( \text{rank}(df)(0) = 2 \) holds. Then \( f \) is \( \mathcal{A} \)-equivalent to \( \Sigma^2_{\text{hyp}} \) (respectively, \( \Sigma^2_{\text{ell}} \)) if and only if for a coordinate system \( (X_1, X_2, X_3, X_4) \) on the target satisfying that \( d(X_1 \circ f)(0) = d(X_2 \circ f)(0) = 0 \) and a pair of vector fields \( (\xi, \eta) \) on the source satisfying that \( (\xi(0), \eta(0)) = \ker(df)(0) \), it holds that
(1) $\det \text{Hess}_{(\xi,\eta)} \lambda(0) < 0$ (respectively, $\det \text{Hess}_{(\xi,\eta)} \lambda(0) > 0$), and

(2) $\det(d(\xi g_1), d(\xi g_2), d(\eta g_1), d(\eta g_2)) \neq 0$.

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