Examples of non connective $C^*$-algebras

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Abstract

We give an example of two infinite families of not connective groups. Both of them are generalization of the 3-dimensional Hantzsche-Wendt group.

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1 Introduction

For a Hilbert space $\mathcal{H}$, we denote by $L(\mathcal{H})$ the $C^*$-algebra of bounded and linear operators on $\mathcal{H}$. The ideal of compact operators is denoted by $\mathcal{K} \subset L(\mathcal{H})$. For the $C^*$-algebra $A$, the cone over $A$ is defined as $CA = C_0[0,1) \otimes A$, the suspension of $A$ as $SA = C_0(0,1) \otimes A$.

Definition 1. Let $A$ be a $C^*$-algebra and $n \in \mathbb{N}, n \geq 1$. $A$ is connective if there is a $*$-monomorphism

$$\Phi : A \rightarrow \prod_n CL(\mathcal{H}) / \bigoplus_n CL(\mathcal{H})$$

which is liftable to a completely positive and contractive map $\phi : A \rightarrow \prod_n CL(\mathcal{H})$.

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For a discrete group $G$, we define $I(G)$ to be the augmentation ideal, i.e. the kernel of the trivial representation $C^*(G) \to \mathbb{C}$. $G$ is called connective if $I(G)$ is a connective $C^*$-algebra. From definition (see [5, p. 492]) connectivity of $G$ may be viewed as a stringent topological property that accounts simultaneously for the quasidiagonality of $C^*(G)$ and the verification of the Kadison-Kaplansky conjecture for certain classes of groups. Here we can referring to conjecture from 2014 [2, p. 166]. If $G$ is a discrete, countable, torsion-free, amenable group, then the natural map

$$[[I(G), \mathcal{K}] \to KK(I(G), \mathcal{K}) \cong K^0(I(G))$$

is an isomorphism of groups. Where $KK(I(G), \mathcal{K})$ is the Kasparov group and $[[I(G), \mathcal{K}]]$ is a group of the homotopy classes of asymptotic morphisms. In 2017 M. Dadarlat found an amenable and not connective group $G_2$ for which the above conjecture fails [4, Cor. 3.2].

Connective groups must be torsion-free, [3, Remark 2.8 and 4.4]. Here is a short list of such groups:

1. a countable torsion free nilpotent groups, [3, Th.4.3];
2. let $0 \to N \to G \to H \to 0$ be a central extension of discrete countable amenable groups where $N$ is torsion-free. If $H$ is connected then so does $G$; [3, Th. 4.1];
3. wreath product of connected groups is a connected group [7, Th.3.2];
4. a torsion-free crystallographic group is connective if and only if is locally indicable if and only if is diffuse (see below) and [4].

A discrete group $G$ is called locally indicable if every finitely generated non-trivial subgroup $L$ of $G$ has an infinite abelianization. The group $G$ is called diffuse if every non-empty finite subset $A$ of $G$ has an element $a \in A$ such that for any $g \in G$, either $ga$ or $g^{-1}a$ is not in $A$. [4], [8].

More examples of nonabelian connective groups were exhibited in [4], [5], [7].

The above group $G_2$ is a 3-dimensional, torsion-free crystallographic group, where a crystallographic group $\Gamma$, of dimension $n$ is a cocompact and discrete subgroup of the isometry group $E(n) = O(n) \ltimes \mathbb{R}^n$ of the Euclidean space $\mathbb{R}^n$. $\Gamma$ is cocompact if and only if the orbit space $E(n)/\Gamma$ is compact. From
Bieberbach theorems (see [10, Chapter 1]) any crystallographic group $\Gamma$ defines a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to H \to 0,$$

where a free abelian group $\mathbb{Z}^n$ is a maximal abelian subgroup and $H$ is a finite group. $H$ is sometimes called a holonomy group of $\Gamma$. The above group $G_2$ is isomorphic to the subgroup $E(3)$ and is generated by

$$G_2 \cong \text{gen}\{A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, (1/2, 1/2, 0)), B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, (0, 1/2, 1/2)))\}.$$

A torsion-free crystallographic group is called a Bieberbach group. The orbit space $\mathbb{R}^n/\Gamma$ of a Bieberbach group is a $n$-dimensional closed flat Riemannian manifold $M$ with holonomy group isomorphic to $H$.

A general characterization of connective Bieberbach groups is given in [4]. The following two theorems give us a landscape of them.

**Theorem 1.** ([4, Theorem 1.2]) Let $\Gamma$ be a Bieberbach group. The following assertions are equivalent.

1. $\Gamma$ is connective
2. Every nontrivial subgroup of $\Gamma$ has a nontrivial center.
3. $\Gamma$ is a poly-$\mathbb{Z}$ group
4. $\hat{G} \setminus \{\iota\}$ has no nonempty compact open subsets.

The unitary dual $\hat{G}$ of $G$ consists of equivalence classes of irreducible unitary representations of $G$. $\iota$ denotes the trivial representation.

**Theorem 2.** ([4, Theorem 1.1]) A Bieberbach group with a finite abelianization is not connective.

In our note we give an example of two infinite families of not connective groups. Both of them are generalization of the 3-dimensional Hantzsche-Wendt group $G_2$. 


2 Examples

Example 1. ([10] Definition 9.1]) Let $n \geq 3$. By generalized Hantzsche-Wendt (GHW for short) group we shall understand any torsion-free crystallographic groups of rank $n$ with a holonomy group $(\mathbb{Z}_2)^{n-1}$.

Example 2. ([1] Definition], [11] Definition 1]) Let $n \geq 0$. A group

$$G_n = \{x_1, x_2, \ldots, x_n \mid x_i^{-1}x_j^2x_i, \forall i \neq j \}$$

we shall call a combinatorial Hantzsche-Wendt group.

For the properties of GHW groups we refer to [10] Chapter 9]. We have $G_0 = 1$ and $G_1 \cong \mathbb{Z}$. Combinatorial Hantzsche-Wendt groups are torsion-free, see [1] Theorem 3.3] and for $n \geq 2$ are nonunique product groups. A group $G$ is called a unique product group if given two nonempty finite subset $X, Y$ of $G$, there exists at least one element $g \in G$ which has a unique representation $g = xy$ with $x \in X$ and $y \in Y$. We are ready to present our main result.

Proposition 1. Generalized Hantzsche-Wendt groups with trivial center and nonabelian, combinatorial Hantzsche-Wendt groups are not connective.

Proof: From [3] Remark 2.8 (i] the connectivity property is inherited by subgroups. Let $G$ be any group from family of GHW groups or family of combinatorial Hantzsche-Wendt groups. In both cases a group $G_2$ is a subgroup of $G$. In the first case it follows from [10] Proposition 9.7]. In the second case it follows from definition, see [1] Prop. 3.4]. Note that in the case of GHW groups we can also use Theorem 2, since all these groups have a finite abelianizations.

Remark 1. From [11], for $n \geq 3$, $G_n$ has a non-abelian free subgroup. Hence is not amenable.

Remark 2. The counterexample to the Kaplansky unit conjecture was given in 2021 by G. Gardam [9]. It was found in the group ring $\mathbb{F}_2[G_2]$. The Kaplansky unit conjecture states that every unit in $K[G]$ is of the form $kg$ for $k \in K \setminus \{0\}$ and $g \in G$.

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