Decay of scattering solutions to one-dimensional free Schrödinger equation

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March 18, 2010

Abstract

In this paper, we investigate the decay property of scattering solutions to the initial value problem for the free Schrödinger equation in $\mathbb{R}$. It becomes clear that the rate of time decay is essentially determined by the behavior of the Fourier transform of initial data near the origin. The proof is described by basic calculus.

1 Introduction

Let us consider the free Schrödinger equation

$$i \frac{\partial}{\partial t} u(t, x) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x)$$  \hspace{1cm} (1)

with initial data $u(0, x) = u_0(x)$, where $i = \sqrt{-1}$ and $u$ is a complex-valued function of $t \in \mathbb{R}$ and $x \in \mathbb{R}$. If $u_0$ is a squared-integrable function, then we can solve the equation (1), such as

$$u(t, x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{ix\xi} e^{-\frac{1}{2}it\xi^2} \hat{u}_0(\xi) \, d\xi,$$  \hspace{1cm} (2)

where $\hat{u}_0$ stands for the Fourier transform of $u_0$. For any squared-integrable function $v$ in $\mathbb{R}$, we define

$$\hat{v}(\xi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-ix\xi} v(x) \, dx.$$  \hspace{1cm} (3)

There are two types of solutions to the free Schrödinger equation classified by the behavior as $t \to \infty$. Let $\lambda$ be an eigenvalue of the operator $-\frac{1}{2} \frac{\partial^2}{\partial x^2}$ and $\psi$ be the corresponding eigenfunction, that is,

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} \psi(x) = \lambda \psi(x).$$  \hspace{1cm} (4)

Then, $u(t, x) = e^{-i\lambda t}\psi(x)$ is a solution to the equation although $u$ may not belong as a function of $x$, to the class of squared-integrable functions.
This \( u(t, x) \) does not decay in time as \( |t| \to \infty \), because the absolute value of \( e^{-i\lambda t} \) is one for any \( t \) in \( \mathbb{R} \). A solution of this type is called a bound-state solution.

On the other hand, a solution which decays in time as \( |t| \to \infty \) is called a scattering solution. As an example of scattering solutions, we give \( u_0(x) = e^{-\frac{1}{2}x^2} \), then we can explicitly solve the equation,

\[
    u(t, x) = \left\{ 2\pi(1 + it) \right\}^{-\frac{1}{2}} e^{-\frac{1}{2}itx^2},
\]

which decays at the rate of \( |t|^{-\frac{1}{2}} \) as \( |t| \to \infty \). In this paper, we investigate the decay property of scattering solution \( u(t, x) \) as \( |t| \to \infty \).

Before describing the main result, we define function spaces. Let \( \Omega \) be an open set in \( \mathbb{R} \). We say \( v \in L^2(\Omega) \) if the quantity

\[
    \int_{\Omega} |v(x)|^2 \, dx
\]

is finite. \( L^2(\Omega) \) is a Hilbert space with the inner product

\[
    (u,v) = \int_{\Omega} u(x)\overline{v(x)} \, dx
\]

for any \( u \in L^2(\Omega) \) and \( v \in L^2(\Omega) \). We use the norm \( \|u\|_{L^2(\Omega)} = \sqrt{(u,u)} \) throughout the paper. \( \|u\|_{L^2(\Omega)} \) is equal to the square root of (6).

Then, we can prove the following result.

**Theorem 1.** Let \( xu_0 \in L^2(\mathbb{R}) \). If

\[
    \xi^{-1}\hat{u}_0(\xi) \in L^2(|\xi| < 1)
\]

and

\[
    \xi^{-1}\left( \frac{\partial}{\partial \xi} \hat{u}_0(\xi) \right)(\xi) \in L^2(|\xi| < 1),
\]

then

\[
    |u(t, x)| \leq C(1 + |x|)|t|^{-\frac{1}{2}},
\]

for any \( t \in \mathbb{R} \) and \( x \in \mathbb{R} \), where \( C > 0 \) is a constant which is independent both of \( t \) and \( x \).

It should be remarked that the region \( |\xi| < 1 \) is not essential. This may be changed into any compact set which contains \( \xi = 0 \).

## 2 Lemmas

In this section, we discuss some lemmas in order to prove the main theorem later.

**Lemma 1.** Let \( u_0 \in L^2(\mathbb{R}) \), \( \xi^{-1}\hat{u}_0 \in L^2(|\xi| < 1) \) and

\[
    I_1(t, x) = \int_{-\infty}^{\infty} e^{ix\xi} e^{-\frac{1}{2}it\xi^2} \xi^{-1}\hat{u}_0(\xi) \, d\xi.
\]

Then

\[
    |I_1(t, x)| \leq \sqrt{2}\|\xi^{-1}\hat{u}_0(\xi)\|_{L^2(|\xi| < 1)} + \sqrt{2}\|u_0\|_{L^2(\mathbb{R})}.
\]
Proof. We have
\[ |I_1(t, x)| \leq \int_{-\infty}^{\infty} |\xi^{-1} \hat{u}_0(\xi)| \, d\xi \]
\[ = \int_{|\xi| < 1} |\xi^{-1} \hat{u}_0(\xi)| \, d\xi + \int_{|\xi| \geq 1} |\xi^{-1} \hat{u}_0(\xi)| \, d\xi. \]  \hspace{1cm} (13)
By Schwarz’ inequality, we obtain
\[ \int_{|\xi| < 1} 1 \cdot |\xi^{-1} \hat{u}_0(\xi)| \, d\xi \leq \left( \int_{|\xi| < 1} 1^2 \, d\xi \right) \left( \int_{|\xi| < 1} |\xi^{-1} \hat{u}_0(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \]
\[ = \sqrt{2} \| \xi^{-1} \hat{u}_0(\xi) \|_{L^2(|\xi| < 1)}. \]  \hspace{1cm} (14)
On the other hand, we have similarly
\[ \int_{|\xi| \geq 1} |\xi^{-1} \hat{u}_0(\xi)| \, d\xi \leq \left( \int_{|\xi| \geq 1} |\xi|^{-2} \, d\xi \right)^{\frac{1}{2}} \left( \int_{|\xi| \geq 1} |\hat{u}_0(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}. \]  \hspace{1cm} (15)
Planceler theorem implies
\[ \int_{|\xi| \geq 1} |\hat{u}_0(\xi)|^2 \, d\xi \leq \| \hat{u}_0 \|_{L^2(\mathbb{R})}^2 = \| u_0 \|_{L^2(\mathbb{R})}^2, \] \hspace{1cm} (16)
which concludes our desired estimate.  \hfill \box

**Lemma 2.** Let \( u_0 \in L^2(\mathbb{R}), \) \( \xi^{-2} \hat{u}_0 \in L^2(|\xi| < 1) \) and
\[ I_2(t, x) = \int_{-\infty}^{\infty} e^{ix\xi} e^{-\frac{1}{2}it\xi^2} \xi^{-2} \hat{u}_0(\xi) \, d\xi. \] \hspace{1cm} (17)
Then
\[ |I_2(t, x)| \leq \sqrt{2} \| \xi^{-2} \hat{u}_0(\xi) \|_{L^2(|\xi| < 1)} + C_2 \] \hspace{1cm} (18)
for some \( C_2 > 0, \) independent both of \( t \) and \( x. \)

**Proof.** By similar discussion to the previous lemma, we obtain
\[ |I_2(t, x)| \leq \int_{-\infty}^{\infty} |\xi^{-2} \hat{u}_0(\xi)| \, d\xi \leq \sqrt{2} \| \xi^{-2} \hat{u}_0(\xi) \|_{L^2(|\xi| < 1)} + C_2 \] \hspace{1cm} (19)
where
\[ C_2 = \int_{|\xi| \geq 1} |\xi^{-2} \hat{u}_0(\xi)| \, d\xi, \] \hspace{1cm} (20)
which completes the proof of the lemma.  \hfill \box

In the following lemma, \( \partial_\xi \) denotes \( \frac{\partial}{\partial \xi}. \)

**Lemma 3.** Let \( xu_0 \in L^2(\mathbb{R}), \) \( \xi^{-1} \partial_\xi \hat{u}_0 \in L^2(|\xi| < 1) \) and
\[ I_3(t, x) = \int_{-\infty}^{\infty} e^{ix\xi} e^{-\frac{1}{2}it\xi^2} \xi^{-1} (\partial_\xi \hat{u}_0)(\xi) \, d\xi. \] \hspace{1cm} (21)
Then
\[ |I_3| \leq \sqrt{2} \| \xi^{-1} (\partial_\xi \hat{u}_0)(\xi) \|_{L^2(|\xi| < 1)} + C_3 \] \hspace{1cm} (22)
for some \( C_3 > 0, \) independent both of \( t \) and \( x. \)
is a positive constant which is independent both of $t$ and $x$. By the theory of Fourier transformation, $xu_0 \in L^2(\mathbb{R})$ implies that there exists $\partial_\xi \hat{u}_0$ in $L^2(\mathbb{R})$, so that

$$|I_3(t, x)| \leq \sqrt{2}||\xi^{-1}(\partial_\xi \hat{u}_0)(\xi)||_{L^2(|\xi| < 1)} + C_3,$$  \hspace{1cm} (24)

where

$$C_3 = \int_{|\xi| \geq 1} |\xi^{-1}(\partial_\xi \hat{u}_0)(\xi)| \, d\xi$$  \hspace{1cm} (25)

is a positive constant which is independent both of $t$ and $x$. \qed

3 Proof of the main theorem

We can prove the main theorem by basic technic in calculus. The key idea of the proof is

$$\frac{\partial}{\partial \xi} e^{-\frac{1}{2} it \xi^2} = (-it \xi) e^{-\frac{1}{2} it \xi^2},$$  \hspace{1cm} (26)

which implies

$$u(t, x) = (2\pi)^{-\frac{1}{2}} (-it)^{-1} \int_{-\infty}^\infty e^{it \xi \frac{1}{2}} \frac{\partial}{\partial \xi} [e^{-\frac{1}{2} it \xi^2}] \hat{u}_0(\xi) \, d\xi,$$  \hspace{1cm} (27)

integrating the above integral by part,

$$\int_{-\infty}^\infty e^{it \xi \frac{1}{2}} \frac{\partial}{\partial \xi} [e^{-\frac{1}{2} it \xi^2}] \hat{u}_0(\xi) \, d\xi = - \int_{-\infty}^\infty \frac{\partial}{\partial \xi} [e^{it \xi \frac{1}{2}} \hat{u}_0(\xi)] e^{-\frac{1}{2} it \xi^2} \, d\xi,$$  \hspace{1cm} (28)

if $u_0 \in L^2(\mathbb{R})$. It follows from Leibniz’ rule that

$$u(t, x) = (2\pi)^{-\frac{1}{2}} t^{-1} (-xI_1(t, x) - iI_2(t, x) + iI_3(t, x))$$  \hspace{1cm} (29)

where

$$I_1(t, x) = \int_{-\infty}^\infty e^{it \xi \frac{1}{2}} \hat{u}_0(\xi) \, d\xi$$
$$I_2(t, x) = \int_{-\infty}^\infty e^{it \xi \frac{1}{2}} \xi^{-1} \hat{u}_0(\xi) \, d\xi$$
$$I_3(t, x) = \int_{-\infty}^\infty e^{it \xi \frac{1}{2}} \xi^{-2} (\partial_\xi \hat{u}_0)(\xi) \, d\xi.$$  \hspace{1cm} (30)

From the results discussed in the previous section, we obtain

$$|u(t, x)| \leq \left(\frac{2\pi}{|t|}\right)^{-\frac{3}{2}} \left(\sqrt{2}||\xi^{-1}\hat{u}_0(\xi)||_{L^2(|\xi| < 1)} + \sqrt{2}||u_0||_{L^2(\mathbb{R})} |x| \right)$$
$$+ (\sqrt{2}||\xi^{-2}\hat{u}_0(\xi)||_{L^2(|\xi| < 1)} + C_2)$$
$$+ (\sqrt{2}||\xi^{-1}\partial_\xi \hat{u}_0(\xi)||_{L^2(|\xi| < 1)} + C_3) \right)$$  \hspace{1cm} (31)

when $\xi^{-2} \hat{u}_0(\xi) \in L^2(|\xi| < 1)$ and $\xi^{-1}(\partial_\xi \hat{u}_0)(\xi) \in L^2(|\xi| < 1)$. Hence,

$$|u(t, x)| \leq C(1 + |x|)|t|^{-1},$$  \hspace{1cm} (32)

where $C > 0$ is a constant which depends only on the behavior of $\hat{u}$ near $\xi = 0$. This statement completes the proof of the theorem.
References

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