Asymptotics for two-dimensional vectorial Allen-Cahn systems

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Abstract

The formation of codimension-one interfaces for multi-well gradient-driven problems is well-known and established in the scalar case, where the equation is often referred to as the Allen-Cahn equation. The proofs rely for a large on a monotonicity formula for the energy density, which is itself related to the vanishing of the so-called discrepancy function. The vectorial case in contrast is quite open. This lack of results and insight is to a large extend related to the absence of known appropriate monotonicity formula. In this paper, we focus on the elliptic case in two dimensions, and introduce methods, relying on the analysis of the partial differential equation, which allow to circumvent the lack of monotonicity formula for the energy density. In the last part of the paper, we recover a new monotonicity formula which relies on a new discrepancy relation. These tools allow to extend to the vectorial case in two dimensions most of the results obtained for the scalar case. We emphasize also some specific features of the vectorial case.

1 Introduction

1.1 Statement of the main result

Let Ω be a smooth bounded domain in \( \mathbb{R}^2 \). In the present paper we investigate asymptotic properties of families of solutions \((u_\varepsilon)_{\varepsilon > 0}\) of the systems of equations having the general form

\[
- \Delta u_\varepsilon = - \varepsilon^{-2} \nabla u V(u_\varepsilon) \quad \text{in} \quad \Omega \subset \mathbb{R}^2,
\]

as the parameter \( \varepsilon > 0 \) tends to zero. The function \( V \), usually termed the potential, denotes a smooth scalar function on \( \mathbb{R}^k \), where \( k \in \mathbb{N} \) is a given integer. Given \( \varepsilon > 0 \), the function \( v_\varepsilon \) represents a function defined on the domain \( \Omega \) with values into the euclidian space \( \mathbb{R}^k \), so that equation (1) is a system of \( k \) scalar partial differential equations for each of the components of the map \( u_\varepsilon \). The equation (1) and its parabolic version have been have been introduced as models in the physics and material literature (see e.g. [16] and the references therein, in particular [8]).

Equation (1) corresponds to the Euler-Lagrange equation of the energy functional \( E_\varepsilon \) which is defined for a function \( u : \Omega \mapsto \mathbb{R}^k \) by the formula

\[
E_\varepsilon(u) = \int_{\Omega} e_\varepsilon(u) = \int_{\Omega} \varepsilon \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon} V(u). \tag{2}
\]

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We assume that the potential $V$ is bounded below, so that we may impose, without loss of generality and changing possibly $V$ by a suitable additive constant, that

$$\inf V = 0. \quad (3)$$

We introduce the set $\Sigma$ of minimizers of $V$, sometimes called the vacuum manifold, that is the subset of $\mathbb{R}^k$ defined

$$\Sigma \equiv \{ y \in \mathbb{R}^k, V(y) = 0 \}.$$

Properties of solutions to (1) crucially depend on the nature of $\Sigma$. In this paper, we will assume that the vacuum manifold is finite, with at least two distinct elements, so that

$$(H_1) \quad \Sigma = \{ \sigma_1, \ldots, \sigma_q \}, \quad q \geq 2, \quad \sigma_i \in \mathbb{R}^k, \quad \forall i = 1, \ldots, q.$$

We impose furthermore a condition on the behavior of $V$ near its zeroes, namely:

$$(H_2) \quad \text{The matrix } \nabla^2 V(\sigma_i) \text{ is positive definite at each point } \sigma_i \text{ of } \Sigma, \text{ in other words, if } \lambda^-_i \text{ denotes its smallest eigenvalue, then } \lambda^-_i > 0. \quad \text{We denote by } \lambda^+_i \text{ its largest eigenvalue.}$$

Finally, we also impose a growth conditions at infinity:

$$(H_3) \quad \text{There exists constants } \alpha_\infty > 0 \text{ and } R_\infty > 0 \text{ such that}$$

$$\begin{cases} 
 y \cdot \nabla V(y) \geq \alpha_\infty |y|^2, \text{ if } |y| > R_\infty \text{ and} \\
 V(x) \to +\infty \text{ as } |x| \to +\infty.
\end{cases} \quad (4)$$

A potential $V$ which fulfill the conditions $(H_1)$, $(H_2)$ and $(H_3)$ is termed throughout the paper a potential with multiple equal depth wells (see Figure 1).

A typical example is provided in the scalar case $k = 1$ by the potential, often termed Allen-Cahn or Ginzburg-Landau potential,

$$V(u) = \frac{(1-u^2)^2}{4}, \quad (5)$$

whose infimum equals 0 and whose minimizers are $+1$ and $-1$, so that $\Sigma = \{ +1, -1 \}$. It is used as an elementary model for phase transitions for materials with two equally preferred states, the minimizers $+1$ and $-1$ of the potential $V$.

Important efforts have been devoted so far to the study of solutions of the stationary Allen-Cahn equations, i.e. solutions to (1) for the potentials similar to (5), or to the corresponding parabolic evolution equations, in the asymptotic limit $\varepsilon \to 0$, in arbitrary dimension $N$ of the domain $\Omega$. The mathematical theory for this question is now well advanced and may be considered as satisfactory. The results found there provide a sound mathematical foundation to the intuitive idea that the domain $\Omega$ decomposes into regions where the solution takes values either close to $+1$ or close to $-1$, the regions being separated by interfaces of width of order $\varepsilon$. These interfaces, are expected to converge to hypersurfaces of codimension 1, which are shown to be generalized minimal surfaces in the stationary case, or moved by mean curvature for the parabolic evolution equations.

Several of the arguments rely on integral methods and energy estimates. For instance in [25], T. Ilmanen proved convergence to motion by mean curvature in the weak sense of
Figure 1: Graph of a potential with several minimizers.

Brakke, a notion relying on the language, concepts and methods of geometric measure theory. In the elliptic case considered in this paper, convergence to minimal surfaces was established by Modica and Mortola in their celebrated paper [28], F. Hutchinson and Y. Tonegawa in [24] established related results for non-minimizing solutions in [24]. More references will be provided in Subsection 1.3.

Remark 1. The case of minimizing solutions was treated in the vectorial case in by Baldo, on one hand (see [7]), and Fonseca and Tartar on the other (see [20]), where they obtained quite similar results to [28]. The approaches rely on ideas from Gamma convergence, and do not rely on monotonicity formulas, as for general stationary solutions or solutions of the corresponding evolution equations in the scalar case.

The purpose of the present paper is to show that, to some extent, the results obtained in the scalar case, can be transposed to the vectorial case for potentials $V$ which fulfill conditions (H$_1$), (H$_2$) and (H$_3$), that is potentials with multiple equal depth wells, if we restrict ourselves to two dimensional domains. Let us emphasize that, prior to the present paper, no monotonicity formula similar to (33) was known in the vectorial case, so that different arguments have to be worked out. Several of them rely strongly on some specificities of dimension two.

We assume that we are given a constant $M_0 > 0$ and a family $(u_\varepsilon)_{0<\varepsilon\leq 1}$ of solutions to the equation (1) for the corresponding value of the parameter $\varepsilon$, satisfying the natural energy bound

$$E_\varepsilon(u_\varepsilon) \leq M_0, \ \forall \varepsilon > 0.$$  \hfill (6)

Assumption (6) is rather standard in the field, since it corresponds to the energy magnitude required for the creation of $(N-1)$-dimensional interfaces. Our main result is the following:

**Theorem 1.** Let $(u_\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of solutions to (1) satisfying (6). There exist a subset $\mathcal{S}_*$ of $\Omega$ and a subsequence of $(\varepsilon_n)_{n \in \mathbb{N}}$, still denoted $(\varepsilon_n)_{n \in \mathbb{N}}$ for sake of simplicity, such that the following properties hold:
\text{i) } \mathcal{S}_* \text{ is a closed 1 dimensional rectifiable subset of } \Omega \text{ such that}

\[ \mathcal{H}^1(\mathcal{S}_*) \leq C_H M_0, \]  

where \( C_H \) is a constant depending only on the potential \( V \).

\text{ii) Set } U_* = \Omega \setminus \mathcal{S}_*, \text{ and let } (U_{i*})_{i \in I} \text{ be the connected components of } U_*. \text{ For each } i \in I \text{ there exists an element } \sigma_i \in \Sigma \text{ such that}

\[ u_{\varepsilon n} \rightarrow \sigma_i \text{ uniformly on every compact subset of } U_* \text{ as } n \rightarrow +\infty. \]  

Similar to the results obtained for the scalar case, Theorem 1 expresses, for the vectorial case in dimension two, the fact that the domain can be decomposed into subdomains, where, for \( n \) large, the maps \( u_{\varepsilon n} \) takes values close to an element of the vacuum set \( \Sigma \) (see Figure 2). This subdomains which are separated by a closed one-dimensional subdomain, the set \( \mathcal{S}_* \), on which the map \( u_{\varepsilon n} \) might possibly undergo a transition from one element of \( \Sigma \) to another. Notice that Theorem 1 result extends also to non-minimizing solutions the results \(^1\) of \([7, 20]\) (see Remark 1).

An important property of the set \( \mathcal{S}_* \) stated in Theorem 1 is its rectifiability. Recall that a Borel set \( \mathcal{S} \subset \mathbb{R}^2 \), and is rectifiable of dimension 1 if its one-dimensional Hausdorff dimension is locally finite, and if there there is a countable family of \( C^1 \) one-dimensional submanifolds of \( \mathbb{R}^2 \) which cover \( \mathcal{H}^1 \) almost all of \( \mathcal{S} \). Rectifiability of \( \mathcal{S} \) implies in particular, that the set \( \mathcal{S} \) has an approximate tangent line at \( \mathcal{H}^1 \)-almost every point \( x_0 \in \mathcal{S} \). More precisely, there exists a set \( \mathfrak{A}_* \) with \( \mathcal{H}^1(\mathfrak{A}_*) = 0 \) such that, if \( x_0 \in \mathcal{S}_* \setminus \mathfrak{A}_* \), then we have

\[ \lim_{r \to 0} \frac{\mathcal{H}^1(\mathcal{S}_*(\mathbb{D}^2(x_0, r)))}{2r} = 1, \]  

and there exists a unit vector \( \vec{e}_{x_0} \) (depending on the point \( x_0 \)) with the following property: For any number \( \theta > 0 \) we have

\[ \lim_{r \to 0} \frac{\mathcal{H}^1(\mathcal{S}_* \cap (\mathbb{D}^2(x_0, r) \setminus \mathcal{C}_{\text{cone}}(x_0, \vec{e}_{x_0}, \theta))))}{r} = 0, \]  

where, for a unit vector \( \vec{e} \) and \( \theta > 0 \), the set \( \mathcal{C}_{\text{cone}}(x_0, \vec{e}, \theta) \) is the cone given by

\[ \mathcal{C}_{\text{cone}}(x_0, \vec{e}, \theta) = \left\{ y \in \mathbb{R}^2 \mid |\vec{e}^\perp \cdot (y - x_0)| \leq \tan \theta |\vec{e} \cdot (y - x_0)| \right\}, \]  

\( \vec{e}^\perp \) being a unit vector orthonormal to \( \vec{e} \) (see e.g. \([32]\)). A point \( x_0 \in \mathcal{S}_* \setminus \mathfrak{A}_* \) is termed a regular point of \( \mathcal{S}_* \).

In the minimizing case, it is established in \([7, 20]\) that the interface \( \mathcal{S}_* \) is a co-dimension one minimal surface, which hence reduces, in dimension two, to the union of segments. Our next result shows that, in dimension two, the same kind of result holds for non-minimizing solutions.

To order to state the result, and since the notion of minimality is also related in our context to the presence of densities of measures, we specify first which measures we have in mind.

\(^1\)This result hold however in arbitrary dimension and yield stronger, in particular minimizing, properties for \( \mathcal{S}_* \).
To that aim, we introduce a limiting measure for the potential term: Consider the positive measure $\zeta_\varepsilon$ defined on $\Omega$ by

$$\zeta_\varepsilon \equiv \frac{V(u_\varepsilon)}{\varepsilon} \, dx,$$

so that $\zeta_\varepsilon(\Omega) \leq M_0$. \hfill (12)

Since the family $(\zeta_\varepsilon)_{\varepsilon>0}$ is uniformly bounded, passing possibly to a further subsequence, we have the convergence

$$\zeta_{\varepsilon_n} \equiv \frac{V(u_{\varepsilon_n})}{\varepsilon_n} \, dx \rightharpoonup \zeta_\star, \text{ in the sense of measures on } \Omega, \text{ as } n \to +\infty,$$

\hfill (13)

**Theorem 2.** There exists a set $\mathcal{E}_\star \subset \mathcal{S}_\star$ such that $\mathcal{H}^1(\mathcal{E}_\star) = 0$, such that $\mathcal{A}_\star \subset \mathcal{E}_\star$ and such that, for $x_0 \in \mathcal{S}_\star \setminus \mathcal{E}_\star$, the set $\mathcal{S}_\star$ is locally near $x_0$ a segment. More precisely, there exists a unit vector $\vec{e}_{x_0}$ and a radius $r_0 > 0$, depending on $x_0$, such that

$$\mathcal{S}_\star \cap \mathbb{D}^2(x_0, r_0) = (x_0 - r_0\vec{e}_{x_0}, x_0 + r_0\vec{e}_{x_0}).$$

\hfill (14)

Moreover the restriction of the measure $\zeta_\star$ to $\mathbb{D}^2(x_0, r_0)$ is proportional to the $\mathcal{H}^1$ measure of $(x_0 - r_0\vec{e}_{x_0}, x_0 + r_0\vec{e}_{x_0})$, that is there exists a number $c_{x_0} > 0$, depending on $x_0$, such that

$$\zeta_\star \sqcap \mathbb{D}^2(x_0, r_0) = c_{x_0} \left( \mathcal{H}^1 \sqcap (x_0 - r_0\vec{e}_{x_0}, x_0 + r_0\vec{e}_{x_0}) \right).$$

\hfill (15)

The number $c_{x_0}$ are bounded below, that is, there exists a constant $\eta_0(d(x)) > 0$, depending only on $V$, $M_0$ and $d(x_0) \equiv \text{dist}(x, \partial \Omega)$ such that such that

$$c_{x_0} \geq \eta_0(d(x)) \text{ for any } x_0 \in \mathcal{S}_\star \setminus \mathcal{E}_\star.$$\hfill (16)

Notice that, as a consequence of (16), for any $x_0 \in \mathcal{S}_\star \setminus \mathcal{E}_\star$, the one dimensional density $\Theta_\star$ defined by

$$\Theta_\star(x) = \liminf_{r \to 0} \frac{\zeta_\star(\mathbb{D}^2(x, r))}{2r}$$

\hfill (17)

is bounded below by $\eta_0(d(x))$, hence away from zero, and is locally constant, equal to $c_{x_0} = \Theta_\star(x_0)$.

Theorem 2 expresses local stationarity properties of the set $\mathcal{S}_\star$ and the measure $\zeta_\star$. As we will discuss later, the set $\mathcal{S}_\star$ may have singularities, and hence $\mathcal{E}_\star$ is not empty. However, more global stationary properties are also available. In order to state these properties, the abstract language of varifolds is the most appropriate. An important preliminary step is to establish that the measure $\zeta_\star$ concentrate on the set $\mathcal{S}_\star$, i.e. its restriction the $\Omega \setminus \mathcal{S}_\star$ vanishes (see Theorem 1), and that it is absolutely continuous with respect to the $\mathcal{H}^1$-measure on $\mathcal{S}_\star$ (see Theorem 4). In particular, this property implies that the measure $\zeta_\star$ is completely determined by the set $\mathcal{S}_\star$ and the density $\Theta_\star$, and we have

$$\zeta_\star = \Theta_\star(\mathcal{H}^1 \sqcap \mathcal{S}_\star) = \Theta_\star d\lambda, \text{ where } d\lambda = \mathcal{H}^1 \sqcap \mathcal{S}_\star.$$\hfill (18)

We have:

**Theorem 3.** The rectifiable one-varifold $V(\mathcal{S}_\star, \Theta_\star)$ corresponding to the measure $\zeta_\star$ is stationary.
The domain $\Omega$ is divided in subdomains where $u_\varepsilon$ is nearly constant. The interfaces are union of segments.

The theory of varifolds has been developed in the context of minimal surfaces, but it turns out to be also an important tool in the study of singular limits (see e.g. [32] for a general presentation of the theory of varifolds). The fact that $V(\mathcal{G}_*, \Theta_*)$ is a stationary varifold is equivalent to the following statement: Given any smooth vector field $\vec{X} \in C_c(\Omega, \mathbb{R}^2)$ on $\Omega$ with compact support, the following identity holds

$$\int_\Omega \text{div}_{T_x e_x} \vec{X} \, d\zeta_* = 0. \quad (19)$$

Here, for $x \in \mathcal{G}_* \setminus \mathcal{E}_*$, the number $\text{div}_{T_x e_x} \vec{X}(x)$ is defined by

$$\text{div}_{T_x e_x} \vec{X}(x) = \left( \vec{e}_x \cdot \vec{\nabla} \vec{X}(x) \right) \cdot \vec{e}_x, \text{ for } x \in \mathcal{G}_*. \quad (20)$$

The structure on one-dimensional varifolds with densities bounded away from zero was thoroughly investigated by Allard and Almgren in [5]. They showed that such varifolds have a graph structure and are the sum of segments with densities. Theorem 2 may therefore be deduced from Theorem 3 invoking the results of Section 3 in [5]. In the present paper, we provide however a simple self-contained proof, based on several results which are worked out independently.

One-dimensional varifolds may have singularities, which are characterized by the fact that the density is not constant in their neighborhood. The simplest example of such a singular varifold in the whole plane with a singularity at 0 is provided by the union of a finite numbers of distinct half-lines, intersecting at the origin, with appropriate constant densities. More precisely, consider an integer $d > 2$, and let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d$ be $d$ distinct unit vectors in $\mathbb{R}^2$. Set

$$\mathcal{S}_* = \bigcup_{i=1}^d \mathbb{H}_i, \text{ where for } i = 1, \ldots, d, \text{ we set } \mathbb{H}_i = \{ t\vec{e}_i, t \geq 0 \}, \quad (21)$$

Figure 2: The domain $\Omega$ is divided in subdomains where $u_\varepsilon$ is nearly constant. The interfaces are union of segments.
and let $\theta_1, \ldots, \theta_d$ be $d$ positive numbers. If $\theta_i$ represents the density $\Theta$ of $\mathcal{S}_\ast$ on $\mathbb{H}_i$ (which is hence constant there), then $V(\mathcal{S}_\ast, \Theta)$ is a stationary one-dimensional rectifiable varifold if and only if
\begin{equation}
\sum_{i=1}^{d} \theta_i \vec{e}_i = 0. \tag{22}
\end{equation}

Singularities $x_0$ which behave locally as (21)-(22) are termed of finite type. It turns out that singularities of finite type appear in the asymptotics of the vectorial Allen Cahn equation, even in the minimizing case, and are actually an intrinsic part in the problem. A first trivial example is provided by an uncoupled system of two scalar Allen-Cahn equation, taking for instance as a potential $V : \mathbb{R}^2 \to \mathbb{R}$ the potential $V(u_1, u_2) = \frac{1}{4} \left[ (1 - u_1)^2 + (1 - u_2)^2 \right]$. For this potential, the map $u_\varepsilon$ defined on $\mathbb{R}^2$ by
\begin{equation}
u_\varepsilon(x_1, x_2) = \left( \tanh \left( \frac{x_1}{\sqrt{2\varepsilon}} \right), \tanh \left( \frac{x_2}{\sqrt{2\varepsilon}} \right) \right), \quad \text{for } (x_1, x_2) \in \mathbb{R}^2,
\end{equation}
is a solution to (1) on the whole plane. The limiting interface $\mathcal{S}_\ast$ for $\varepsilon \to 0$ is then given as the union of the lines $x_1 = 0$ and $x_2 = 0$, so that 0 is a singularity where these lines cross with right angles. One may actually construct similar examples where the angle between the two lines is arbitrary.

A more involved example is constructed in [15], where a sequence of minimizing solutions is constructed on the entire plane, for a potential with three minimizers and equilateral symmetry. The set $\mathcal{S}_\ast$ then consists of three half lines with equal angles and equal densities, yield a singularity at zero with triple junction (see Figure 3). The appearance of triple junctions in general minimizing problems is discussed in [34] and analyzed through Gamma-convergence results.

**Remark 2.** Singularities of finite type have also been constructed as limits of scalar Allen-Cahn problem (see [17, 22]). In these constructions, the number $d$ of half-lines in (21) is even.

Besides singularities with a locally finite sum of segments as in (21), an example of a singularity of a stationary varifold with an infinite complexity is produced in [5]. It is however shown in [5] that the occurrence of such singularities is ruled out if the set of densities is discrete. As we will see later, there are examples of potential such that the possible set of densities is infinite, so that of singularities of infinite type cannot be excluded a priori in the limits of solutions to (1).

### 1.2 Comparing results in the scalar and vectorial cases

Although the results stated in Theorems 1, 2 and 3 for the vectorial Allen-Cahn equation are somewhat parallel with the results obtained so far in the literature for the scalar case, it is worthwhile to stress some major differences between the scalar and the vectorial case.

**The one-dimensional case**

Distinct behaviors are already observed for the one-dimensional case. Indeed, for $\Omega = \mathbb{R}$, equation (1) reduces to the ordinary differential equation
\begin{equation}
- \frac{d^2w_\varepsilon}{ds^2} = \varepsilon^{-2} \nabla w V(w_\varepsilon) \quad \text{on } \mathbb{R}. \tag{23}
\end{equation}
Finite energy solutions necessarily connect at ±∞ two minimizers σ− and σ+: They are called profiles or heteroclinic connections, if σ− ̸= σ+. Multiplying (23) by $w_\varepsilon$, we are led to the conservation law
\[
d \frac{d}{dx} \left( \frac{1}{\varepsilon} V(w_\varepsilon) - \varepsilon \frac{|\dot{w}_\varepsilon|^2}{2} \right) = 0,
\]
so that for profiles one derives the identity
\[
\varepsilon |\dot{w}_\varepsilon| = \sqrt{V(w_\varepsilon)} \text{ on } \mathbb{R}.
\] (25)

In the scalar case, the first order equation (25) is easily integrated by separation of variables, so that profiles connect only nearby minimizers σ− and σ+ of the potential, and are essentially unique, up to translations and symmetries. For instance, in the case of the Allen-Cahn potential (5), the solution is given up to translation and symmetry, by $w_\varepsilon(s) = \tanh \left( \frac{s}{\sqrt{2} \varepsilon} \right)$, for $s \in \mathbb{R}$.

The situation is very different in the vectorial case, since relation (25) is less constraining: Under additional assumptions on the potential $V$, one may find several profiles connecting two minimizers of the potential (see e.g [3] and references therein). The search for such solution is still an active field of research (see for instance [4, 35, 29]). As we will see next, the genuine non-uniqueness of one-dimensional profiles is a first source of important difference also in the higher dimensional case, in particular concerning the conservation law (25).

The higher dimensional case
The higher-dimensional theory in the scalar case is rather advanced and a very satisfactory theory has been set up in any dimension $N \geq 2$. As mentioned, the existence of a $(N - 1)$-dimensional set $\mathcal{S}_*$ is established in [25, 24]. Moreover, it is shown there that the $(N - 1)$-rectifiable set $\mathcal{S}_*$, equipped with the energy density corresponding to the measure $\nu_*$ defined in [37] is a stationary rectifiable varifold. The results in [24] embody the intuitive idea that locally, the equation reduces to a one-dimensional problem. More precisely, typically, in
dimension two, the expected situation reduces, \textit{locally near some point} $x_0$, to the case
\[ u_\varepsilon(x) \simeq w_\varepsilon(x_0), \text{ with } x = (x_1, x_2) \in \mathbb{R}^2, \]  
(26)
where the coordinates are chosen so that the tangent to $\mathfrak{S}_\varepsilon$ at $x_0$ has equation $x_2 = 0$, and where $w_\varepsilon$ stands for a solution to the one-dimension problem (23) (see Figure ??). Notice that the possibility of gluing of several such solutions is not excluded, but we will not discuss this here. Ultimately, the results in [25, 24] provide a rather simple picture of the solutions. They involve a minimal surface, the solution may be represented as a one-dimensional profiles glued to the surface in the transversal direction, so that one is tempted to write the correspondence
\[ \text{solutions to } [1] \sim \text{ minimal surface + glued profiles.} \]  
(27)
The general structure of solution is hence fairly well understood (see Figure 4). As a matter of fact, the correspondance goes to some extent in either way, since, conversely, given a minimal surface, one may construct solutions to the scalar Allen-Cahn equation having the previous behavior (see [30]). This should be also connected with the famous De Giorgi conjecture (18) (see [31], and references therein).

Figure 4: Interface near a regular point $x_0$ in the scalar case, with an Allen-Cahn type potential.

The picture in the vectorial case is more complex. Firstly, as we have already seen, the set of one dimensional profiles is much larger, it could perhaps be even infinite. Besides this, there are solutions which \textit{cannot be reduced to one dimensional profiles}, in view of results in [1] and [14], and are hence genuinely multi-dimensional, so that a property similar the (26) or (27) \textit{cannot not be expected in full generality.}

In [14], it is shown that, under specific conditions on the potential $V$, one may \textit{mountain-pass solutions} to $-\Delta u = \nabla u V(u)$ on the cylinder $\Lambda_L = [-L, L] \times \mathbb{R}$ provided $L > 0$ is sufficiently large, with periodic boundary conditions in the $x_1$ direction, namely such that
\[ u(-L, x_2) = u(L, x_2) \text{ and } \frac{\partial u}{\partial x_1}(-L, x_2) = \frac{\partial u}{\partial x_1}(L, x_2), \text{ for any } x_2 \in \mathbb{R}. \]  
(28)
The solution obtained in [14] is \textit{not a one-dimensional profile}, since one may show that there are also \textit{tangential contributions}: Indeed, we have
\[
\frac{\partial u}{\partial x_1} \neq 0 \text{ on } \Lambda_L. \tag{29}
\]
One then considers the scaled map on \( \mathbb{R}^2 \) defined for \( x = (x_1, x_2) \) by
\[
u_\varepsilon(x) = u\left(x - \frac{N\varepsilon\mathbf{e}_1}{\varepsilon}\right), \text{ if } x_1 \in [N\varepsilon, (N + 1)\varepsilon] \tag{30}
\]
which solves (1) on \( \mathbb{R}^2 \) (see Figure 5). Moreover, it follows from (29), that for the transversal derivative, we have
\[
\varepsilon \left| \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 \to \mu_{*,1,1} \neq 0, \text{ where } \mu_{*,1,1} = c\mathcal{H}^1(D) \text{ with } D = \{(x_1,0), x_1 \in \mathbb{R}\}, \tag{31}
\]
for some constant \( c > 0 \). Finally it can be shown that the set of densities obtained for such solution is infinite, choosing various values for the constant \( L > 0 \).

1.3 Comparing the methods in the scalar and vectorial cases

Monotonicity for the energy in the scalar case

A large part of the arguments developed for the scalar theory, as well as actually in the present paper, rely on integral estimates, starting with the energy, but also the integral of the potential. In the present context, we set for an arbitrary subdomain \( G \in \Omega \),
\[
E_\varepsilon(u_\varepsilon, G) = \int_{\mathcal{U}} e_\varepsilon(u)dx \text{ and } V_\varepsilon(u, G) = \frac{1}{\varepsilon} \int_{\mathcal{U}} V(u)dx. \tag{32}
\]
Monotonicity formula play a distinguished role in the field. We recall that the monotonicity formula
\[
d \left( \frac{1}{r^{N-2}} E_\varepsilon(u_\varepsilon, \mathbb{B}^N(x_0, r)) \right) \geq 0, \text{ for any } x_0 \in \Omega,
\]
holds for arbitrary potentials, and is relevant if one wants to establish concentration on $N - 2$ dimensional sets, as it occurs in Ginzburg-Landau theory. If one wants instead to establish concentration on $N - 1$ dimensional sets, then the stronger monotonicity formula

$$\frac{d}{dr} \left( \frac{1}{r^{N-1}} E_\varepsilon (u_\varepsilon, B^N(x_0, r)) \right) \geq 0,$$

for any $x_0 \in \Omega$, (33)

seems more appropriate. As a matter of fact, we have, in dimension $N = 2$, the identity (see Remark 3.6)

$$\frac{d}{dr} \left( \frac{E_\varepsilon (u_\varepsilon, D^2(r))}{r} \right) = \frac{1}{r^2} \int_{S^2(\varepsilon)} \xi_\varepsilon (u_\varepsilon) dx + \frac{1}{r} \int_{S^1(\varepsilon)} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 d\ell,$$

where $\xi_\varepsilon (u_\varepsilon)$ denotes the discrepancy function given by

$$\xi_\varepsilon (u_\varepsilon) = \frac{1}{\varepsilon} V(u_\varepsilon) - \varepsilon \frac{|\nabla u|^2}{2}.$$  (35)

Notice that, in view of (25), the discrepancy function vanishes for one-dimensional profiles, a property which allows to compute solution in the scalar case as seen before.

Formula (33) has been established in [25] in the scalar case. The proof provided in [25] relies strongly on the positivity of the discrepancy function $\xi_\varepsilon$, a property obtained thanks to the maximum principle. The fact that $\xi_\varepsilon$ is positive for scalar solutions of (1) was observed first by L. Modica in [27] for entire solutions. It is actually proved in [24] that the discrepancy $\xi_\varepsilon$ vanishes asymptotically as $\varepsilon \to 0$.

Inequality (33) is the cornerstone of the scalar theory, as developed in [25, 24]. It yields both upper and lower bounds for the concentration of the energy. A large part of the arguments deal with properties of limiting measures, obtained as $\varepsilon \to 0$. Instead of the measure $\zeta_\varepsilon$ which appears both in Theorem 1 and Theorem 2, obtained as a limit of the potential (see (12) and (13)), the central tool in the scalar case is the corresponding measure for the full energy. More precisely, consider the family $(\nu_\varepsilon)_{0 < \varepsilon \leq 1}$ of measures defined on $\Omega$ by

$$\nu_\varepsilon \equiv e_\varepsilon (u_\varepsilon) \, dx \text{ on } \Omega,$$

(36)

so that, in view of (6), the total mass of the measures is bounded by $M_0$, that is $\nu_\varepsilon (\Omega) \leq M_0$. By compactness, there exists a decreasing subsequence $(\varepsilon_n)_{n \in \mathbb{N}}$ tending to 0 and a limiting measure $\nu_*$ on $\Omega$ with $\nu_*(\Omega) \leq M_0$, such that

$$\nu_{\varepsilon_n} \rightharpoonup \nu_* \text{ in the sense of measures on } \Omega \text{ as } n \to +\infty.$$

(37)

A first straightforward consequence of the monotonicity formula (33) for the energy is that the one-dimensional density of the measure $\nu_*$ is bounded from above. This property then implies that the the concentration set $\mathcal{S}_*$ of $\nu_*$ has at least dimension one. Combining the monotonicity (33) with a weak form of the clearing-out property, similar to Proposition 6.1 in the present paper, the monotonicity formula yields also a lower bound on the density of $\nu_*$ which is hence bounded away from zero. This property implies that the concentration set $\mathcal{S}_*$ of $\nu_*$ has at most dimension one, hence its dimension is exactly one. The previous discussion therefore the concentration property of $\nu_*$ is a direct consequence of (33).
Notice also that the previous arguments show that the measure $\nu_*$ is absolutely continuous with respect to $d\lambda$, the $H^{N-1}$ measure on $\mathfrak{S}$, so that one may write $\nu_* = e \, d\lambda$, where $e$ is a integrable function on $\mathfrak{S}$. Going to the limit $\varepsilon \rightarrow 0$ in (35), we obtain, since $\xi_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$,

$$2\zeta_* = \nu_*,$$  \hspace{1cm} (38)

a relation which in some sense extends (25) to the high-dimensional setting. We will see, in contrast, that relation (38) does not extend to the vectorial case.

**Remark 3.** As already mentioned, it has been proven in [25, 24] that, in the scalar case, the rectifiable varifolds $V(\mathfrak{S}, e)$ corresponding to the measure $\nu_*$ is stationary. In view of relation (38) this implies that the rectifiable varifold $V(\mathfrak{S}, \zeta_*)$ corresponding to the measure $\zeta_*$ is also stationary, which is hence consistent with Theorem 3 of the present paper.

**Circumventing lack of monotonicity for the energy in the two-dimensional vectorial case.**

Concerning the vectorial case, non-negativity of the discrepancy as well as the monotonicity formula are known to fail for some solutions of the Ginzburg-Landau system, so that the question whether they might still hold under some possible additional conditions on the potential or the solution itself is widely open to our knowledge (see [2] for a discussion of these issues and for additional references).

In order to circumvent the lack of monotonicity formula for the energy, we have to work out new results on the level of solutions to PDE (termed in the paper the $\varepsilon$-level), which will be present in Subsection 1.4. The clearing-out result given in Theorem 6 is central in our analysis: It implies, as in the scalar case, that the set $\mathfrak{S}$ has dimension at most one. Combining with several other results for the PDE, we are able to deduce most of the properties developed in Theorem 1.

For the proofs of Theorems 2 and 3, the fact that the measures $\zeta_*$ and $\nu_*$ are absolutely continuous with respect to the $H^1$ measure of $\mathfrak{S}$ is crucial. We will show, in the last part of this paper:

**Theorem 4.** The measures $\nu_*$ and $\zeta_*$ have support on the set $\mathfrak{S}$ defined in Theorem 7 and are absolutely continuous with respect to $d\lambda = H^1 \mathfrak{S}$, the one-dimensional Hausdorff measure on $\mathfrak{S}$. Let $e_*$ and $\Theta_*$ denote the densities of $\nu_*$ and $\zeta_*$ with respect to $d\lambda$ respectively, so that $\nu_* = e_* d\lambda$ and $\zeta_* = \Theta_* d\lambda$. We have the inequalities, for $x \in \mathfrak{S}$,

$$\begin{align*}
\eta_1 \leq e_*(x) \leq K_{\text{dens}} \left(d(\mathfrak{S})\right) \Theta(x), \\
\Theta_*(x) \leq \frac{M_0}{d(x)},
\end{align*}$$  \hspace{1cm} (39)

where $\eta_1 > 0$ is some constant depending only on $V$, $d(x) = \text{dist}(x, \partial \Omega)$ and $K_{\text{dens}}(d(x))$ denotes a constant depending only on $V$, $M_0$ and $d(x)$.

Notice that we have also the straightforward inequality $\zeta_* \leq \nu_*$, so that $\Theta_* \leq e$. It follows from the inequalities (39) that the densities $e_*$ and $\Theta_*$ are locally bounded from above and away from zero.

**A new discrepancy relation**
Our arguments require to split the energy, in particular the gradient term, into its components, leading to several other measures. For a given orthonormal basis \((\vec{e}_1, \vec{e}_2)\), we consider, for \(i, j = 1, 2\), the quadratic gradient terms \(\varepsilon u_{\varepsilon x_i} \cdot u_{\varepsilon x_j}\), and pass to the limit \(\varepsilon \to 0\), extracting possibly a further subsequence

\[ \varepsilon u_{\varepsilon n x_i} \cdot u_{\varepsilon n x_j} \rightrightarrows \mu_{*,i,j} \text{ in the sense of measures on } \Omega, \] as \(n \to +\infty\), for \(i, j = 1, 2\). \hspace{1cm} (40)

where \(\mu_{*,i,j}\) denotes a bounded (signed) Radon measure on \(\Omega\). Notice that

\[ -2\nu_* \leq \mu_{i,j} \leq 2\nu_* \] and \(\mu_{*,j,i} = \mu_{*,i,j}\). \hspace{1cm} (41)

In the scalar case, the fact that solutions essentially reduce to the one-dimensional profile with respect to the transversal direction, also implies the vanishing of the tangential contributions to the gradient terms. More precisely, we may write, in view of Theorem 4 since \(\nu_*\) is absolutely continuous with respect to \(d\lambda\)

\[ \mu_{*,i,j} = m_{*,i,j} d\lambda, \] \hspace{1cm} (42)

where \(m_{*,i,j}\) is an integrable function on \(S_*\). The definition and values of \(m_{*,i,j}\) strongly depend on the choice of orthonormal frame. In order to derive some more intrinsic objects, we may work in a moving frame associated to \(S_*\). More precisely if \(x_0 \in S_* \setminus E_*\), and if the orthonormal frame \((\vec{e}_1, \vec{e}_2)\) is chosen so that \(\vec{e}_1 = \vec{e}_{x_0}\), then we set

\[ m_{*,\perp,\perp}(x_0) = m_{*,2,2}(x_0), \quad m_{*,\|,\|}(x_0) = m_{*,1,1}(x_0) \] and \(m_{*,\perp,\|}(x_0) = m_{*,1,2}(x_0)\), \hspace{1cm} (43)

and define the measures

\[ \mu_{*,\perp,\perp} = m_{*,\perp,\perp} d\lambda, \quad \mu_{*,\|,\|} = m_{*,\|,\|} d\lambda, \quad \text{and } \mu_{*,\perp,\|} = m_{*,\perp,\|} d\lambda. \] \hspace{1cm} (44)

In the scalar case, the fact that the tangential contributions vanish (see [24]) can be expressed as

\[ \left\{ \begin{array}{ll} \mu_{*,\|,\|}(x_0) &= 0 \text{ when } k = 1 \text{ (i.e. in the scalar case)} \\ \mu_{*,\perp,\|} &= 0 \text{ when } k = 1 \text{ (i.e. in the scalar case)} \end{array} \right. \] \hspace{1cm} (45)

On the other hand, vanishing of the discrepancy leads to (see [24] once more)

\[ 2\zeta_* = \mu_{*,\perp,\perp}, \text{ when } k = 1 \text{ (i.e. in the scalar case)}. \] \hspace{1cm} (46)

It turns out that the relation (46) does not hold in general for the vectorial case. Indeed, for the map constructed in [14] and given in (30), we have \(\nu_{*,\|,\|} \neq 0\), so that the first relation in (45) is not satisfied. We will see later that the second one is always satisfied, whereas the discrepancy relation (46) is not, in general. Our next result provides a generalization of (46) for the vectorial case.

**Theorem 5.** We have the identities

\[ 2\zeta_* = \mu_{*,\perp,\perp} - \mu_{*,\|,\|} \quad \text{and } \quad \mu_{*,\|,\|} = 0. \] \hspace{1cm} (47)
Notice that, in view of identities (45), the discrepancy identity (46) appears as a special case of (47).

**Recovering monotonicity**

So far, we have introduced in Theorems 1, 2, 3, 4 and 5 the main results of this paper. As mentioned, many arguments have to be carried out without monotonicity formula, in particular Theorem 1. However, in order to obtain the proofs of Theorems 2 to 5, we rely ultimately on a new monotonicity formula, which we describe at the end of this subsection.

Before doing so, let us emphasize that, in order to prove Theorem 4 and several intermediate results, we rely in an essential way on Lebesgue’s decomposition theorem for measures which assert that measures at hand can be decomposed into an absolutely continuous part and a singular part with respect to the one-dimensional measure Hausdorff measure on \( S \). More precisely, we decompose the measure \( \nu \) and \( \zeta \) as

\[
\nu = \nu^{ac} + \nu^s, \quad \text{and} \quad \zeta = \zeta^{ac} + \zeta^s,
\]

where the measures \( \nu^{ac} \) and \( \zeta^{ac} \) are absolutely continuous with respect to the measure \( H^1 \mid \mathcal{S}_* \), that is

\[
\nu^{ac} \ll H^1 \mid \mathcal{S}_* \quad \text{and} \quad \zeta^{ac} \ll H^1 \mid \mathcal{S}_*.
\]

and

\[
\nu^s \perp \nu^{ac} \quad \text{and} \quad \zeta^s \perp \zeta^{ac}. \quad (49)
\]

We are then in position to write, prior to the proof of Theorem 4, \( \nu^{ac} = e_\ast d\lambda \) and \( \zeta^{ac} = \Theta d\lambda \). An important intermediate step in the paper, is a preliminary version of Theorem 5 (see Proposition 5) established only for the absolutely continuous parts of the measures.

In order to show that \( \nu^s = \zeta^s = 0 \), the cornerstone of the argument is an alternate differential inequality for solutions of the (1). We have indeed, for any \( x_0 \in \Omega \) such that \( D(x_0, r) \subset \Omega \) (see Subsection 3.6 for the proof), the differential relation

\[
\frac{1}{\varepsilon^2} \frac{d}{dr} \left( \frac{V \left( u_\varepsilon, D^2(x_0, r) \right)}{r} \right) = \frac{1}{4r} \int_{D^2(x_0, r)} \left( \frac{2}{\varepsilon^2} V(u_\varepsilon) - r^2 \left| \frac{\partial u_\varepsilon}{\partial \theta} \right|^2 + \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 \right) d\tau. \quad (50)
\]

Although this does not transpire from the formula above, we will see that the right hand side has, in an asymptotic limit \( \varepsilon \to 0 \), an appropriate sign, yielding monotonicity for the measure \( \zeta \ast \): As a matter of fact, it turns out that the function \( \zeta \ast (D^2(x_0, r))/r \) is non-decreasing (see Proposition 6). This yields and upper bound for the density of \( \zeta \ast \), so that the singular part vanishes.

In the next subsections, we provide more details on the structure of the proof.

**1.4 Elements in the proof of Theorem 1: PDE analysis**

As mentioned, many of our main results, dealing with the limiting measures, are derived from corresponding results at the \( \varepsilon \)-level for the map \( u_\varepsilon \), for given \( \varepsilon > 0 \), which rely on PDE methods. We describe nexts these PDE results.
1.4.1 Scaling invariance of the equation

As a first preliminary remark, we notice the invariance of the equation by translations as well as scale changes, which plays an important role in our later arguments. Given any fixed \( r > 0 \) and \( \varepsilon > 0 \), we introduce the corresponding scaled parameter \( \tilde{\varepsilon} = \frac{\varepsilon}{r} \). For a given map \( u_\varepsilon : \mathbb{D}^2(x_0, r) \rightarrow \mathbb{R}^k \), we consider the scaled (and translated) map \( \tilde{u}_\varepsilon \) defined on the unit disk \( \mathbb{D}^2 \) by

\[
\tilde{u}_\varepsilon(x) = u_\varepsilon(rx + x_0), \forall x \in \mathbb{D}^2.
\]

If the map \( u_\varepsilon \) is a solution to (1), then the map \( \tilde{u}_\varepsilon \) is a solution to (1) with the parameter \( \varepsilon \) changed into \( \tilde{\varepsilon} \). The scale invariance of the energy is given by the relation

\[
e_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon)(x) = re_{\varepsilon}(u)(rx + x_0), \forall x \in \mathbb{D}^2.
\]  

(51)

Integrating this identity, we obtain the integral relations

\[
E_{\varepsilon}(u_\varepsilon, \mathbb{D}^2(r)) = rE_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon, \mathbb{D}^2(1)) \quad \text{and} \quad V_{\varepsilon}(u_\varepsilon, \mathbb{D}^2(r)) = rV_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon, \mathbb{D}^2(1)),
\]

(52)

where we have made use of the notation (32). It follows from the previous discussion that the parameter \( \varepsilon \) as well as the energy \( E_{\varepsilon} \) behave, according to the previous scaling laws, essentially as lengths. In this loose sense, inequality (52) shows that the quantity \( \varepsilon^{-1}E_{\varepsilon} \) is scale invariant.

1.4.2 The \( \varepsilon \)-clearing-out Theorem

We next provide clearing-out results for solutions of the PDE (1). In view of the assumptions \((H_1), (H_2) \) and \((H_3) \) on the potential \( V \), we may choose some constant \( \mu_0 > 0 \) sufficiently small so that

\[
\begin{cases}
B^k(\sigma_i, 2\mu_0) \cap B^k(\sigma_j, 2\mu_0) = \emptyset \quad \text{for all } i \neq j \text{ in } \{1, \ldots, q\} \\
\frac{1}{2} \lambda_i^- \text{Id} \leq \nabla^2 V(y) \leq 2\lambda_i^+ \text{Id} \quad \text{for all } i \in \{1, \ldots, q\} \text{ and } y \in B(\sigma_i, 2\mu_0).
\end{cases}
\]  

(53)

We then have:

**Theorem 6.** Let \( 0 < \varepsilon \leq 1 \) and \( u_\varepsilon \) be a solution of (1) on \( \mathbb{D}^2 \). There exists some constant \( \eta_1 > 0 \) such that if

\[
E_{\varepsilon}(u_\varepsilon, \mathbb{D}^2) \leq 2\eta_1,
\]

(54)

then there exists some \( \sigma \in \Sigma \) such that

\[
|u_\varepsilon(x) - \sigma_0| \leq \frac{\mu_0}{2}, \text{ for every } x \in \mathbb{D}^2(\frac{3}{4}),
\]

(55)

where \( \sigma_0 \) is defined in (53). Moreover, we have the energy estimate, for some constant \( C_{\text{arg}} > 0 \) depending only on the potential \( V \)

\[
E_{\varepsilon}\left(u_\varepsilon, \mathbb{D}^2\left(\frac{5}{8}\right)\right) \leq C_{\text{arg}} \varepsilon E_{\varepsilon}(u_\varepsilon, \mathbb{D}^2).
\]

(56)

Theorem 6 is the main PDE result of the present paper: It paves the way to the concentration of measures on set \( \mathcal{G}_* \), and will be used to show that its dimensional is at most one. The main ingredient in the proof of Theorem 6 is provided by the following estimate:
Proposition 1. Let $0 < \varepsilon \leq 1$ and $u_\varepsilon$ be a solution of (1) on $\mathbb{D}^2$. There exists a constant $C_{\text{dec}} > 0$ depending only on $V$ such that

$$\int_{\mathbb{D}^2(\frac{\varepsilon}{16})} e_\varepsilon(u_\varepsilon) \, dx \leq C_{\text{dec}} \left[ \left( \int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon) \, dx \right)^{\frac{3}{2}} + \varepsilon \int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon) \, dx \right]. \quad (57)$$

From a technical point of view, Proposition 1 is perhaps the main new ingredient provided by the present paper: When both $E_\varepsilon(u_\varepsilon)$ and $\varepsilon$ are small, it provides a fast decay of the energy on smaller balls.

Combining the result (57) of proposition 1 with the scale invariance properties of the equation given in subsection 1.4.1, we obtain corresponding results for arbitrary discs $\mathbb{D}^2(x_0,r)$. Indeed, applying Proposition 1 to the map $\tilde{u}_\varepsilon$ with parameter $\tilde{\varepsilon} = \varepsilon/r$ and expressing the corresponding inequality (57) we obtain, provided $\tilde{\varepsilon} \leq 1$, i.e. $\varepsilon \leq r$,

$$E_{\tilde{\varepsilon}} \left( \tilde{u}_\varepsilon, \mathbb{D}^2 \left( x_0, \frac{9}{16} \right) \right) \leq C_{\text{dec}} \left[ E_{\tilde{\varepsilon}} (\tilde{u}_\varepsilon) \tilde{\varepsilon}^2 + \tilde{\varepsilon} E_{\tilde{\varepsilon}} (\tilde{u}_\varepsilon) \right].$$

Since $E_{\tilde{\varepsilon}} (\tilde{u}_\varepsilon) = r^{-1} E_\varepsilon \left( u_\varepsilon, \mathbb{D}^2(r) \right)$ and $E_{\tilde{\varepsilon}} \left( \tilde{u}_\varepsilon, \mathbb{D}^2(x_0,9/16) \right) = r^{-1} E_\varepsilon \left( u_\varepsilon, \mathbb{D}^2(9r/16) \right)$ we are led, provided $\varepsilon \leq r$, to the inequality

$$E_\varepsilon \left( u_\varepsilon, \mathbb{D}^2 \left( x_0, \frac{9r}{16} \right) \right) \leq C_{\text{dec}} \left[ \frac{1}{\sqrt{r}} \left( E_\varepsilon \left( u_\varepsilon, \mathbb{D}^2(x_0,r) \right) \right)^{\frac{3}{2}} + \frac{\varepsilon}{r} E_\varepsilon \left( u_\varepsilon, \mathbb{D}^2(x_0,r) \right) \right]. \quad (58)$$

Iterating this decay estimate on concentric discs centered at $x_0$, and combining with elementary properties of the solution $u_\varepsilon$, we eventually obtain the proof of Theorem 6.

Invoking once more the scale invariance properties of the equation given in subsection 1.4.1, we obtain, for arbitrary discs $\mathbb{D}^2(x_0,r)$, the scaled version of Theorem 6 writes then as follows:

Proposition 2. Let $x_0 \in \Omega$ and $0 < \varepsilon \leq r$ be given, assume that $\mathbb{D}^2(x_0,r) \subset \Omega$ and let $u_\varepsilon$ be a solution of (1) on $\Omega$. If

$$\frac{E_{\varepsilon} \left( u_\varepsilon, \mathbb{D}^2 \left( x_0, \frac{1}{r} \right) \right)}{r} \leq \eta_1, \quad (59)$$

then there exist some $\sigma \in \Sigma$ such that

$$|u_\varepsilon(x) - \sigma| \leq \frac{\mu_0}{2}, \text{ for } x \in \mathbb{D}^2 \left( x_0, \frac{3r}{4} \right) \text{ and }$$

$$E_{\varepsilon} \left( u_\varepsilon, \mathbb{D}^2 \left( x_0, \frac{5r}{8} \right) \right) \leq C_{\text{arg}} \frac{\varepsilon}{r} E_{\varepsilon} \left( u_\varepsilon, \mathbb{D}^2 \left( x_0, r \right) \right). \quad (60)$$

The proof of Proposition 2 is a straightforward consequence of Theorem 6 and the scaling properties given in subsection 1.4.1 in particular identities (52).

1.4.3 Other results at the $\varepsilon$-level

The analysis of the limiting measures require some other ingredients, in particular concerning the interplay between the measure $\zeta_\varepsilon$ and $\nu_\varepsilon$, leading to the relations (39) on the limiting densities.

The connectedness of $\mathcal{S}_*$ also requires results at the $\varepsilon$-level, in particular we will rely on Proposition 4.6.

We next present the main tools for handling the measures and the concentration set $\mathcal{S}_*$. 

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1.5 Elements in the proof of Theorem 1: construction of $\mathcal{S}_*$ and clearing out for the measure $\nu_*$

The set $\mathcal{S}_*$ in Theorem 1 and Proposition 4 is obtained as a concentration set of the energy. The properties stated in Theorem 1 are, for a large part, consequences of the two results we present next. These results are deduced from corresponding properties of solutions to (1), and presented in the previous subsection.

The first result we present next represents a classical form of a clearing-out result for the measure $\nu_*$ and leads directly to the fact that energy concentrates on sets which are at most one-dimensional.

**Theorem 7.** Let $x_0 \in \Omega$ and $r > 0$ be given such that $D^2(x_0, r) \subset \Omega$. There exists a constant $\eta_1 > 0$ such that, if we have

$$\nu_*(D^2(x_0, r)) < \eta_1,$$

then it holds

$$\nu_*(D^2(x_0, \frac{r}{2})) = 0. \quad (61)$$

The previous statement leads to consider the one-dimensional lower density of the measure $\nu_*$ defined, for $x_0 \in \Omega$, by

$$e_*(x_0) = \liminf_{r \to 0} \frac{\nu_*(D^2(x_0, r))}{r},$$

so that $e_*(x_0) \in [0, +\infty]$. We define the set $\mathcal{S}_*$ as the concentration set for the measure $\nu_*$. More precisely, we set

$$\mathcal{S}_* = \{x \in \Omega, e_*(x_0) \geq \eta_1\}, \quad (62)$$

where $\eta_1 > 0$ is the constant provided by Theorem 7. The fact that $\mathcal{S}_*$ is closed of finite one-dimensional Hausdorff measure is then a rather direct consequence of the clearing-out property for the measure $\nu_*$ stated in Theorem 7.

**Remark 4.** Let us emphasize once more that the previous definition of $\mathcal{S}_*$ directly leads, by construction and in view of Theorem 7 to concentration of the measure $\nu_*$ and $\zeta_*$ on the set $\mathcal{S}_*$ and also a lower bound on the density of $\nu_*$. The upper bound requires different arguments, in particular a monotonicity formula.

The connectedness properties of $\mathcal{S}_*$ stated in Theorem 1 part ii) require a different type of clearing-out result. Its statement involves general regular subdomains $U \subset \Omega$, and, for $\delta > 0$, the related sets (see Figure 6)

$$\mathcal{U}_\delta = \{x \in \Omega, \text{dist}(x, U) \leq \delta\} \supset U \quad \text{and} \quad \mathcal{V}_\delta = U \setminus \mathcal{U} = \{x \in \Omega, 0 \leq \text{dist}(x, U) \leq \delta\}. \quad (63)$$

**Theorem 8.** Let $U \subset \Omega$ be a open subset of $\Omega$ and $\delta > 0$ be given. If we have

$$\nu_*(\mathcal{V}_\delta) = 0, \quad \text{then it holds} \quad \nu_*(\overline{U}) = 0. \quad (64)$$

In other terms, if the measure $\nu_*$ vanishes in some neighborhood of the boundary $\partial U$, then it vanishes on $\overline{U}$. This result will allow us to establish connectedness properties of $\mathcal{S}_*$. For instance, we will prove the following local connectedness property:
Figure 6: The sets $U_\delta$ and $V_\delta$.

Figure 7: The tangent cone property, as given in Proposition 4.

**Proposition 3.** Let $x_0 \in \Omega$, $r > 0$ such that $D^2(x_0, 2r) \subset \Omega$. There exists a radius $\rho_0 \in (r, 2r)$ such that $S^* \cap D^2(x_0, \rho_0)$ contains a finite union of path-connected components.

This connectedness property in Proposition 3 implies the rectifiability of $S^*$, invoking classical results on continua of bounded one-dimensional Hausdorff measure (see e.g [19]). The proof of Theorem 1 is then a combination of the results in Theorem 7 and Proposition 7.

For the set $S^*$ given by Theorem 1, the approximate tangent line property (10) can actually be strengthened as follows (see Figure 7):

**Proposition 4.** Let $x_0$ be a regular point of $S^*$. Given any $\Theta > 0$ there exists a radius $R_{\text{cone}}(\theta, x_0)$ such that

$$S^* \cap D^2(x_0, r) \subset C_{\text{cone}}(x_0, \bar{e}_{x_0}, \theta), \text{ for any } 0 < r \leq R_{\text{cone}}(\theta, x_0).$$

(65)
1.6 A useful tool: The limiting Hopf differential $\omega_s$

We introduce the complex-valued measure referred to as the \textit{limiting Hopf differential}

$$\omega_s = (\mu_{s,1,1} - \mu_{s,2,2}) - 2i\mu_{s,1,2},$$

where the measures $\mu_{i,j}$ have been defined in \cite{40}. Since the measures $\nu_{s,i,j}$ depend on the choice of orthonormal basis, the expression of the Hopf differential also strongly depends on this choice. The measures $\zeta_s$ and $\omega_s$ are strongly related in view of our next result.

\textbf{Lemma 1.} \textit{We have, in the sense of distributions}

$$\frac{\partial \omega_s}{\partial \bar{z}} = 2 \frac{\partial \zeta_s}{\partial z}$$

in $D'(\Omega)$.

Relation (67) is the two-dimensional analog of the conservation law (25). It expresses the fact that the energy of the solution $u_s$ is stationary with respect to variations of the domain. Since the measures $\nu_s$ and $\zeta_s$ are supported by $S_s$, identity (67) also expresses a stationary condition, when integrated against a test function, for the set $S_s$ and the measure $\nu_s$. As a matter of fact, identity (67), is the starting point of the proofs of Theorems 2, 3, 4 and 5.

Taking the real and imaginary parts of this relation, we obtain, in the sense of distributions, the \textit{modified Cauchy-Riemann relations}

\begin{equation}
\begin{cases}
\frac{\partial}{\partial x_2}(2\mu_{s,1,2}) = \frac{\partial}{\partial x_1}(2\zeta_s - \mu_{s,1,1} + \mu_{s,2,2}) \\
\frac{\partial}{\partial x_1}(2\mu_{s,1,2}) = \frac{\partial}{\partial x_2}(2\zeta_s + \mu_{s,1,1} - \mu_{s,2,2}),
\end{cases}
\end{equation}

the second relation being in some sense the closest to (25).

Our next result involve the decomposition of the measures into absolutely continuous parts with respect to $d\lambda = H^1 \subseteq S_s$ and singular part and describe properties of the absolutely continuous part. Besides \cite{48}, we may also decompose the measures $\mu_{s,i,j}$, writing

$$\mu_{s,i,j} = \mu_{s,i,j}^\text{ac} + \mu_{s,i,j}^\perp$$

where the measures $\mu_{s,i,j}^\text{ac}$ is absolutely continuous with respect to the measure $d\lambda = H^1 \subseteq S_s$. Relations \cite{48} and \cite{69} imply that there exists a set $B_s \subset S_s$ such that $H^1 \subseteq S_s(B_s) = 0$ and

$$\nu_s(B_s \setminus S_s) = 0, \zeta_s(S_s \setminus B_s) = 0, \text{ and } \mu_{s,i,j}(S_s \setminus B_s) = 0, \text{ for } i, j = 1, 2.$$  

(70)

Since the measures $\zeta_s$, $\nu_s$ and $\mu_{s,i,j}$ are absolutely continuous with respect to $d\lambda$ there exists functions $\Theta_s, e_s$ and $m_{s,i,j}$ defined on $S_s$, such that we have

$$\zeta_{s}^{ac} = \Theta_s d\lambda, \nu_{s}^{ac} = e_s d\lambda, \text{ and } \mu_{s,i,j}^{ac} = m_{s,i,j} d\lambda.$$  

(71)

Besides $A_s$ and $B_s$, we introduce a third class of exceptional points, the set $\mathcal{C}_s$, defined the complementary of the set of Lebesgue points for the densities of the measures $\mu_{s,i,j}^{ac}$, $\zeta_{s}^{ac}$, $\mu_{s,i,j}^{ac}$ with respect to $d\lambda = H^1 \subseteq S_s$. The set $S_s \setminus \mathcal{C}_s$, then corresponds to the intersection of the set of Lebesgue’s points of the functions $\Theta_s, e$ and $m_{s,i,j}$. We consider the union of all exceptional points

$$\mathcal{E}_s = A_s \cup B_s \cup \mathcal{C}_s,$$  

(72)

which is precisely the set appearing in Theorems 1 and 2.
Proposition 5. Let \( x_0 \in \mathcal{S}_* \setminus \mathcal{E}_* \). Assume that the orthonormal frame \((\vec{e}_1, \vec{e}_2)\) is chosen so that \( \vec{e}_1 = \vec{e}_{x_0} \). We have the identity, for the functions \( \Theta, \, m_{i,j} \) defined in [(71)],

\[
2 \Theta_*(x_0) = m_{*,2,2}(x_0) - m_{*,1,1}(x_0) \quad \text{and} \quad m_{*,2,1}(x_0) = 0. \tag{73}
\]

Let \( \omega_*^{ac} = (\mu_{*,1,1}^{ac} - \mu_{*,2,2}^{ac}) - 2i\mu_{*,1,2}^{ac} \) denote the absolutely continuous part of \( \omega_* \) with respect to \( d\lambda \). The previous result yields, after change of orthonormal basis:

Lemma 2. For a given orthonormal basis \((\vec{e}_1, \vec{e}_2)\), we have the identity

\[
\omega_*^{ac} = -2 \exp(-2i\gamma_*)\xi_*^{ac} = -2(\cos 2\gamma_* - i \sin 2\gamma_*)\xi_*^{ac}, \tag{74}
\]

where, \( \gamma_*(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) denotes for \( x \in \mathcal{S}_* \setminus \mathcal{E}_* \), the angle between \( \vec{e}_1 \) and \( \vec{e}_{x_0} \).

Remark 5. Changing possibly \( \vec{e}_{x_0} \) into \( -\vec{e}_{x_0} \) we may indeed always choose \( \gamma_*(x_0) \) in an interval of length \( \pi \), here \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

We present next some arguments involved in the proof of Proposition 5. We work near a regular point \( x_0 = (x_{0,1}, x_{0,2}) \in \mathcal{S}_* \setminus \mathcal{E}_* \), where \( \mathcal{E}_* \) is defined in [(72)], and choose the orthonormal basis so that \( \vec{e}_1 = \vec{e}_{x_0} \). In the neighborhood of the point \( x_0 \), the measure \( \nu_* \) hence concentrates near the line \( x_2 = x_{0,2} \), and we may follow the approach of [6], eliminating the derivatives according to the transversal direction, that is eliminating the \( x_2 \)-variable, in order to obtain a one-dimensional problem: For that purpose, we integrate along "vertical" lines. The general idea would be to consider integrals of the form

\[
I_{i,j}(s) = \int_{(x_{0,2}, -3/4r)}^{(x_{0,2} + 3/4r)} \mu_{*,i,j}(s, x_{0,2}) \, dx_2 \quad \text{or} \quad W(s) = \int_{(x_{0,2}, -3/4r)}^{(x_{0,2} + 3/4r)} \xi_{*,i,j}(s, x_{0,2}) \, dx_2.
\]

However, since at this stage of our argument we don’t know that the measures are absolutely continuous with respect to the Lebesgue measure, one has to be a little more careful in order to define the previous integrals. To that aim, we introduce for \( s > 0 \), the segments \( J_* = [s - r, s + r] = B^1(s, r) \) and the square \( Q_r(x_0) = J_r(x_0, 1) \times J_r(x_0, 2) \) and consider the localized measures

\[
\tilde{\mu}_{*,i,j} = 1_{Q_r(x_0)}(\mu_{*,i,j}) \quad \text{and} \quad \tilde{\xi}_{*,i,j} = 1_{Q_r(x_0)}(\xi_{*,i,j}).
\]

We introduce also the orthogonal projection \( \mathbb{P} \) onto the tangent line \( D_{x_0}^1 = \{x_0 + s\vec{e}_1, \, s \in \mathbb{R}\} \), and the pushforward measures on \( D_{x_0}^1 \) of the localized measures we have introduced so far, namely the measures on \( D_{x_0}^1 \)

\[
\tilde{\mu}_{*,i,j} = \mathbb{P}_x(\tilde{\mu}_{*,i,j}) \quad \text{and} \quad \tilde{\xi}_{*,i,j} = \mathbb{P}_x(\tilde{\xi}_{*,i,j}), \tag{75}
\]

defined for every Borel set \( A \) of \( D_{x_0}^1 \) by

\[
\begin{cases}
\tilde{\mu}_{*,i,j}(A) = \mu_{*,i,j}(\mathbb{P}^{-1}(A) \cap Q_r(x_0)) = \mu_{*,i,j}(\mathbb{R} \times Q_r(x_0)) \quad \text{and} \\
\tilde{\xi}_{*,i,j}(A) = \xi_{*,i,j}((A \times \mathbb{R}) \cap Q_r(x_0)).
\end{cases}
\]

We then consider the measure \( \mathbb{L}_{x_0, r} \) defined on \( J_r(x_0, 1) \) by

\[
\begin{cases}
\mathbb{L}_{x_0, r} = \mathbb{P}_x(2\tilde{\xi}_{*,i,j} + \tilde{\mu}_{*,1,1} - \tilde{\mu}_{*,2,2}) = 2\tilde{\xi}_{*,i,j} - \tilde{\mu}_{*,1,1} + \tilde{\mu}_{*,2,2} \quad \text{and} \\
N_{x_0, r} = \mathbb{P}_x(2\tilde{\xi}_{*,i,j} + \tilde{\mu}_{*,1,1} - \tilde{\mu}_{*,2,2}) = 2\tilde{\xi}_{*,i,j} + \tilde{\mu}_{*,1,1} - \tilde{\mu}_{*,2,2}. \tag{76}
\end{cases}
\]

Multiplying [1] by appropriate test functions and integrating, we are led to the somewhat remarkable properties of these measures, expressed in Propositions 8.1, 8.4, 8.5 and 8.6 leading to the completion of the proof of Proposition 5.
1.7 Monotonicity for $\zeta_*$ and its consequences

The next important step in the proofs of Theorem 2, 3, 4, and 5 is to show that the singular part of all measures introduced so far vanish. We first establish this statement for the measure $\zeta_*$. Our argument involves a new ingredient, the monotonicity formula for $\zeta_*$, which actually directly yields the absolute continuity of $\zeta_*$ with respect to $\mathcal{H}^1 \subset \mathcal{G}_*$.

**Proposition 6.** Let $x_0 \in \Omega$, let $\rho > 0$ be such that $\mathbb{D}^2(x_0, \rho) \subset \Omega$. If $0 < r_0 < r_1 \leq \rho$, then we have

$$\frac{\zeta_*(\mathbb{D}^2(x_0, r_1))}{r_1} \geq \frac{\zeta_*(\mathbb{D}^2(x_0, r_0))}{r_0}. \quad (77)$$

For every $x_0 \in \Omega$ the quantity $\frac{\zeta_*(\mathbb{D}^2(x_0, r))}{r}$ has a limit when $r \to 0$ and we have the estimate

$$\Theta_*(x_0) = \lim_{r \to 0} \frac{\zeta_*(\mathbb{D}^2(x_0, r))}{r} \leq \frac{\zeta_*(\mathbb{D}^2(x_0, \rho))}{\rho} \leq \frac{M_0}{d(x, \partial \Omega)}. \quad (78)$$

The measure $\zeta_*$ is hence absolutely continuous with respect to the $\mathcal{H}^1$-measure on $\mathcal{G}_*$.

The starting point of the proof of Proposition 6 is the monotonicity formula (50) for the potential term $V$, which may be written, after integration, for a solution $u_\varepsilon$ of (1) on $\Omega$ and $0 < r_0 < r_1 \leq \rho$ such that $D(x_0, \rho) \subset \Omega$

$$\frac{\zeta_*(\mathbb{D}^2(x_0, r_1))}{r_1} - \frac{\zeta_*(\mathbb{D}^2(x_0, r_0))}{r_0} = \int_{\mathbb{D}^2(x_0, r) \setminus \mathbb{D}^2(x_0, \rho)} \frac{1}{4r} \, d\mathcal{N}_\varepsilon, \quad (79)$$

where we have set

$$\mathcal{N}_\varepsilon = \left\{ \frac{2}{\varepsilon} V(u_\varepsilon) - \varepsilon r^2 \left| \frac{\partial u_\varepsilon}{\partial \theta} \right|^2 + \varepsilon \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 \right\} \, dx,$$

and where $(r, \theta)$ denote radial coordinates so that $x_1 - x_{0,1} = r \cos \theta$ and $x_2 - x_{0,2} = r \sin \theta$. Passing to the limit $\varepsilon \to 0$ in identity (79), we are led to:

**Lemma 3.** Let $x_0 \in \Omega$, let $\rho > 0$ and assume that $\mathbb{D}^2(x_0, \rho) \subset \Omega$. For almost every radii $0 < r_0 < r_1 \leq \rho$, we have the identity

$$\frac{\zeta_*(\mathbb{D}^2(x_0, r_1))}{r_1} - \frac{\zeta_*(\mathbb{D}^2(x_0, r_0))}{r_0} = \int_{\mathbb{D}^2(r_1) \setminus \mathbb{D}^2(r_0)} \frac{1}{4r} \, d\mathcal{N}_*$ \quad (80)$$

where $\mathcal{N}_* = 2\zeta_* - r^2 \mu_{*,\theta,\theta} + \mu_{*,r,r}$, with

$$\begin{cases} \mu_{*,r,r} = 2 \mu_{*,1,1} + \mu_{*,1,2} + 2 \mu_{*,2,2} + 2 \sin \theta \cos \theta \mu_{1,1,2} \text{ and} \vspace{0.2cm} \\
\mu_{*,\theta,\theta} = 2 \mu_{*,1,1} + \mu_{*,1,2} + 2 \mu_{*,2,2} - 2 \sin \theta \cos \theta \mu_{1,2}. \end{cases} \quad (81)$$

Notice that we may verify that

$$\left| \frac{\partial u_{\varepsilon n}}{\partial \theta} \right|^2 \to_{n \to +\infty} \mu_{*,r,r} \text{ and } \left| \frac{\partial u_{\varepsilon n}}{\partial r} \right|^2 \to_{n \to +\infty} \mu_{*,\theta,\theta} \text{ as measures.}$$

The next step in the proof of Proposition 6 is the fact that, as a consequence of Proposition 5, the absolutely continuous part of $\mathcal{N}_*$ is non-negative. We them perform a few manipulation...
which allow to get rid of the singular part in (80), and lead to the completion of the proof of Proposition 6.

In order to prove that $\nu_\star$ is also absolutely continuous with respect to $d\lambda$, we will invoke the fact that $\nu_\star$ is "dominated" by the measure $\zeta_\star$. Whereas the reverse statement is straightforward, since we have the inequality $\zeta_\star \leq \nu_\star$, the fact that $\nu_\star$ is "dominated" by the measure $\zeta_\star$ is a consequence of several estimates at the $\varepsilon$-level, requiring some PDE analysis (in particular Proposition 4.5). Theorem 5 is then an direct consequence of Theorem 4 and Proposition 5.

1.8 On the proof on Theorem 2 and 3

The proof of Theorem 3 is a direct consequence of Lemma 1 combined with Theorem 5. Theorem 2 could be deduced from Theorem 3 combined with the result of [5], but we provide here a self contained proof.

1.9 Open questions and conclusion

As already mentioned, one of the main unsolved open problems in the present paper, i.e. in the two dimensional elliptic context, is the existence or not of singularities of "infinite type" in the limiting varifold. If such singularities do exist, their actual construction may turn out to be extremely difficult.

Although the paper focuses exclusively on the two-dimension case, it is quite tempting to believe that a large part of the results might extend to higher dimensions. However, it is not clear how the arguments presented in this paper, in paper the PDE part, can be adapted in higher dimensions. Indeed, as the previous presentation hopefully shows, many of our arguments rely on the fact that we work in two dimensions., and do not seem to have natural extensions in higher dimensions.

Another challenging problem is the related parabolic two-dimensional equation, which has already attracted attention (see e.g. [16] or more recently [26]). One might express the hope that some of the methods introduced in this paper extend also to this case.

1.10 Plan of the paper

The outline of the paper merely follows the description given in Subsections 1.4 to 1.8. As a matter of fact, the presentation of arguments is divided into three parts. Part I is a preliminary part which presents various properties of the energy functional and consists of a single section, Section 2. It presents some consequences of the energy bound, starting with estimates on one-dimensional sets, as well as consequences of the co-area formula. Part II, which runs from Section 3 to Section 6, gathers all properties of solutions to the PDE (1), including standard one. For a large part, in both parts, special emphasis is put on energy estimates on level sets of some appropriate simple scalar functions (see (2.7)). Section 5 presents the proof of Proposition 1. The last part, Part III, describes the properties of the limiting set $\mathcal{S}_\star$ and the limiting measures measures, and contains therefore the proofs to the main results of the paper.
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Part I : Properties of the functional

2 First consequences of the energy bounds

The results in this section are based on variants of an idea of Modica and Mortola (see [28]), adapted to the vectorial case as in [7, 20]. We also present some applications of the co-area formula in connection with the functional. The results in this section apply to maps having a suitable bound on their energy $E_ε$, of the type of the bound (6). We stress in particular $BV$ type bounds obtained under these energy bound. None of the results in this section involves the PDE. We start with simple consequences of assumptions $(H_1)$, $(H_2)$ and $(H_3)$ for the potential with multiple equal depth wells (see Figure 1).

2.1 Properties of the potential

It follows from the definition of $µ_0$ and property (53) that we have the following behavior near the points of the vacuum manifold $Σ$:

**Proposition 2.1.** For any $i = 1, \ldots, q$ and any $y \in \mathbb{R}^k(σ_i, 2µ_0)$, we have the local bound

$$
\begin{align*}
\frac{1}{4}λ_i^-|y - σ_i|^2 &\leq V(y) \leq λ_i^+|y - σ_i|^2 \\
\frac{1}{2}λ_i^-|y - σ_i|^2 &\leq \nabla V(y) \cdot (y - σ_i) \leq 2λ_i^+|y - σ_i|^2,
\end{align*}
$$

(2.1)

Choosing possibly an even smaller constant $µ_0$, we have

$$
V(y) \geq α_0 \equiv \frac{1}{2}λ_0µ_0^2 \text{ on } \mathbb{R}^k \setminus \bigcup_{i=1}^{q} \mathbb{B}^k(σ_i, µ_0),
$$

(2.2)

where we have set $λ_0 = \inf\{λ_i^-, i = 1, \ldots, q\}$

The proof relies on a straightforward integration of (53) and we therefore omit it. Proposition 2.1 shows that the potential $V$ essentially behaves as a positive definite quadratic function near points of the vacuum manifolds $Σ$. This will be used throughout as a guiding thread. Proposition 2.1 leads to a first elementary observation:

**Lemma 2.1.** Let $y \in \mathbb{R}^k$ be such that $V(y) < α_0$. Then there exists some point $σ \in Σ$ such that

$$
|y - σ| \leq µ_0.
$$

Moreover, we have the upper bound

$$
|y - σ| \leq \sqrt{4λ_0^{-1}V(y)}.
$$
We next turn to the behavior at infinity. For that purpose, we introduce the radius
\[ R_0 = \sup\{|\sigma|, \sigma \in \Sigma\} \tag{2.3} \]
and study the properties of \( V \) on the set \( \mathbb{R}^k \setminus \mathbb{B}^k(2R_0) \).

**Proposition 2.2.** There exists a constant \( \beta_\infty > 0 \) such that
\[ V(y) \geq \beta_\infty |y|^2 \text{ for any } y \text{ such that } |y| \geq 2R_0. \tag{2.4} \]

**Proof.** Integrating assumption \( H_3 \) we obtain that, for some constant \( C_\infty > 0 \), we have
\[ V(y) \geq \alpha_\infty \frac{|y|^2}{2} - C_\infty, \text{ for any } y \in \mathbb{R}^k. \tag{2.5} \]

It follows that
\[ V(y) \geq \frac{\alpha_\infty |y|^2}{4}, \text{ provided } |y| \geq R'_0 \equiv \sup \left\{ 2\sqrt{\frac{C_\infty}{\alpha_\infty}}, 4R_0 \right\}. \tag{2.6} \]

On the other hand, by assumption, we have
\[ \frac{V(y)}{|y|^2} > 0 \text{ for } y \in \mathbb{B}^k(R'_0) \setminus \mathbb{B}^k(2R_0), \]
so that, by compactness, we deduce that there exist some constant \( \alpha'_\infty > 0 \), such that
\[ V(y) \geq \alpha'_\infty |y|^2 \text{ for } y \in \mathbb{B}^k(2R'_0) \setminus \mathbb{B}^k(2R_0). \]

Combining the last inequality with (2.6), the conclusion follows, choosing \( \beta = \inf\left\{ \frac{\alpha_\infty}{4}, \alpha'_\infty \right\} \). \( \square \)

### 2.2 Modica-Mortola type inequalities

Let \( \sigma_i \) be an arbitrary element in \( \Sigma \). We consider the function \( \chi_i : \mathbb{R}^k \rightarrow \mathbb{R}^+ \) defined by
\[ \chi_i(y) = \varphi(|y - \sigma_i|) \text{ for } y \in \mathbb{R}^k, \]
where \( \varphi \) denotes a function \( \varphi : [0, +\infty[ \rightarrow \mathbb{R}^+ \) such that \( 0 \leq \varphi' \leq 1 \) and
\[ \varphi(t) = t \text{ if } 0 \leq t \leq \frac{\mu_0}{2} \text{ and } \varphi(t) = \frac{3\mu_0}{4} \text{ if } t \geq \mu_0. \]

Given a function \( u : \Omega \rightarrow \mathbb{R}^k \) we finally define the scalar function \( w_i \) on \( \Omega \) as
\[ w_i(x) = \chi_i(u(x)), \forall x \in \Omega. \tag{2.7} \]

First properties of the map \( w_i \) are summarized in the next Lemma.
Lemma 2.2. Let \( w_i \) be as above. We have
\[
\begin{align*}
&\begin{cases}
  w_i(x) = |u(x) - \sigma_i|, & \text{if } |u(x) - \sigma_i| \leq \frac{\mu_0}{2}, \\
  w_i(x) = \frac{3\mu_0}{4}, & \text{hence } \nabla w_i = 0 \text{ if } |u(x) - \sigma_i| \geq \mu_0,
\end{cases} \\
&|\nabla w_i| \leq |\nabla u| \text{ on } \Omega,
\end{align*}
\]
and
\[
|\nabla (w_i)^2| \leq 4\sqrt{\lambda_0^{-1}J(u)(x)},
\]
where we have set
\[
J(u) = |\nabla u|\sqrt{V(u)}.
\]

Proof. Properties (2.8) is a straightforward consequence of the definition (2.7). For (2.9), we notice that, in view of (2.8), we may restrict ourselves to the case \( u(x) \in B^k(\sigma_i, \mu_0) \), since otherwise \( \nabla w_i = 0 \), and inequality (2.9) is hence straightforwardly satisfied. In that case, it follows from (2.1), we have
\[
|w_i(x)| \leq |u(x) - \sigma_i| \leq \sqrt{4\lambda_0^{-1}V(u(x))}, \text{ for all } x \text{ such that } u(x) \in B^k(\sigma_i, \mu_0),
\]
so that
\[
|\nabla (w_i)^2(x)| = 2|w_i(x)| \cdot |\nabla |w_i(x)|| \leq 2|\nabla u|\sqrt{4\lambda_0^{-1}V(u(x))} \leq 4\sqrt{\lambda_0^{-1}J(u)(x)},
\]
and the proof is complete.

Lemma 2.3. We have, for any \( x \in \Omega \), the inequality
\[
J(u(x)) \leq e_\varepsilon(u(x)).
\]

Proof. We have, by definition of the energy \( e_\varepsilon(u) \),
\[
J(u(x)) = (\varepsilon|\nabla u(x)|).\sqrt{\varepsilon^{-1}V(u(x))}
\]
We invoke next the inequality \( ab \leq \frac{1}{2}(a^2 + b^2) \) to obtain
\[
J(u(x)) \leq \frac{1}{2}(\varepsilon|\nabla u(x)|^2 + \varepsilon^{-1}V(u(x)))
\]
which yields the desired result.

2.3 The one-dimensional case

In dimension 1 estimate (2.9) directly leads to uniform bound on \( w_i \), as expressed in our next result. For that purpose, we consider, for \( r > 0 \), the circle \( S^1(r) = \{ x \in \mathbb{R}^2, |x| = r \} \) and maps \( u : S^1(r) \to \mathbb{R}^k \).
Lemma 2.4. Let $0 < \varepsilon \leq 1$ and $\varepsilon < r \leq 1$ be given. There exists a constant $C_{unf} > 0$, depending only on $V$, such that, for any given map $u : S^1(r) \to \mathbb{R}^k$, there exists an element $\sigma_{main} \in \Sigma$ such that

$$|u(\ell) - \sigma_{main}| \leq C_{unf} \sqrt{\int_{S^1(r)} \frac{1}{2} (J(u(\ell)) + r^{-1}V(u(\ell))) d\ell}, \quad \text{for all } \ell \in S^1(r),$$

(2.14)

and hence

$$|u(\ell) - \sigma_{main}| \leq C_{unf} \int_{S^1(r)} \varepsilon(u) d\ell.$$  

(2.15)

Proof. By the mean-value formula, there exists some point $\ell_0 \in S^1(r)$ such that

$$V(u(\ell_0)) = \frac{1}{2\pi r} \int_{S^1(r)} V(u(\ell)) d\ell.$$  

(2.16)

We distinguish two cases.

Case 1. The function $u$ satisfies additionnally the estimate

$$\frac{1}{2\pi r} \int_{S^1(r)} V(u(\ell)) \, d\ell < \alpha_0,$$  

(2.17)

where $\alpha_0$ is the constant introduced in Lemma 2.1. Then, we deduce from inequality (2.17) that

$$V(u(\ell_0)) \leq \frac{1}{2\pi r} \int_{S^1(r)} V(u(\ell)) \, d\ell < \alpha_0.$$  

It follows from Lemma 2.1 that there exists some $\sigma_{main} \in \Sigma$ such that

$$|u(\ell_0) - \sigma_{main}|^2 \leq 4\lambda_0^{-1}V(u(\ell_0)) \leq \frac{2\lambda_0^{-1}}{\pi r} \int_{S^1(r)} V(u) d\ell.$$  

On the other hand, we deduce, integrating the bound (2.9), that, for any $\ell \in S^1(r)$, we have

$$| |u - \sigma_{main}|^2 (\ell) - |u - \sigma_{main}|^2 (\ell_0)| \, d\ell \leq 4\sqrt{\lambda_0^{-1}} \int_{S^1(r)} J(u).$$

Combining the two previous estimates, we obtain the desired result in case 1, using the fact that $\varepsilon \leq 1$ and provided the constant $C_{unf}$ satisfies the bound

$$C_{unf}^2 \geq 4\sqrt{\lambda_0^{-1}} + 2\lambda_0^{-1}.$$  

Case 2. Inequality (2.17) does not hold. In that case, we have hence

$$\frac{1}{2\pi r} \int_{S^1(r)} V(u(\ell)) \, d\ell \geq \alpha_0.$$  

(2.18)

We consider the number $R_0 = \sup\{|\sigma|, \sigma \in \Sigma\}$, introduced in definition (2.5) of the proof of Proposition 2.2 and discuss next three subcases.
Subcase 2a: For any $\ell \in S^1(r)$, we have
\[ u(\ell) \in B^k(2R_0). \]

Then, in this case, for any $\sigma \in \Sigma$, we have
\[ |u(\ell) - \sigma|^2 \leq 9R_0^2 = \left( \frac{9R_0^2}{\alpha_0} \right) \alpha_0 \leq \left( \frac{9R_0^2}{\alpha_0} \right) \frac{1}{2\pi r} \int_{S^1(r)} V(u(\ell))d\ell. \tag{2.19} \]

Hence, inequality (2.14) is immediately satisfied, whatever the choice of $\sigma_{\text{main}}$, provided we impose the additional condition
\[ C_{\text{unf}}^2 \geq \frac{9R_0^2}{2\alpha_0}. \tag{2.20} \]

Subcase 2b: There exists some $\ell_1 \in S^1(r)$, and some $\ell_2 \in S^1(r)$ such that, we have
\[ u(\ell_1) \in B^k(2R_0) \text{ and } u(\ell_2) \not\in B^k(2R_0). \]

Let $\ell \in S^1(r)$. If $u(\ell) \in B^k(2R_0)$, then we argue as in subcase 2a, so that we obtain inequality (2.19) as before, hence (2.14), and we are done. Otherwise, by continuity, there exists some $\ell' \in S^1(r)$ such that $u(\ell') \in \partial B^k(2R_0)$ and such for any point $a \in C(\ell, \ell')$ we have $u(a) \not\in B^k(2R_0)$, where $C(\ell, \ell')$ denotes the arc on $S^1(r)$ joining counterclockwise the points $\ell$ and $\ell'$. We have, by integration, using the fact that $|u(\ell)| \geq 2R_0$ for $a \in C(\ell, \ell')$ together with inequality (2.4),
\[ |u(\ell)|^2 - |u(\ell')|^2 \leq 2 \int_{\ell}^{\ell'} |u(a)| \cdot |\nabla u(a)| \, da \]
\[ \leq R_0^{-1} \int_{\ell}^{\ell'} |u(a)|^2 \cdot |\nabla u(a)| \, da \]
\[ \leq \frac{1}{\beta_0 R_0} \int_{\ell}^{\ell'} V(u(a)) |\nabla u(a)| \, da \leq \frac{1}{\beta_0 R_0} \int_{S^1(r)} J(u(a)) \, da. \]

Since $|u(\ell')| = 2R_0$, we obtain, for any $\sigma \in \Sigma$,
\[ |u(\ell) - \sigma|^2 \leq 2 \left( |u(\ell)|^2 + |\sigma|^2 \right) \leq 2 \left( |u(\ell)|^2 + R_0^2 \right) \]
\[ \leq 2 \left( \frac{1}{\beta_0 R_0} \int_{S^1(r)} J(u(a)) \, da + R_0^2 + |u(\ell')|^2 \right) \]
\[ \leq \left( \frac{2}{\beta_0 R_0} \int_{S^1(r)} J(u(a)) \, da + 10R_0^2 \right) \]
\[ \leq \left( \frac{2}{\beta_0 R_0} \int_{S^1(r)} J(u(a)) \, da + 10 - \frac{R_0^2}{\alpha_\infty} \right) \]
\[ \leq \left( \frac{2}{\beta_0 R_0} \int_{S^1(r)} J(u(a)) \, da + 10 \frac{R_0^2}{2\pi \alpha_0 r} \int_{S^1(r)} V(u(\ell))d\ell, \right) \]
so that the conclusion (2.14) follows for any choice of $\sigma_{\text{main}} \in \Sigma$, imposing again an appropriate lower bound on the constant $C_{\text{unf}}$. 

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Subcase 2c: For any $\ell \in S^1(r)$, we have

$$|u(\ell)| \geq 2R_0.$$ 

Let $\ell_0$ satisfy (2.3), so that, in view of Proposition 2.2

$$|u(\ell_0)|^2 \leq \frac{1}{\beta_\infty} V(u(\ell_0)) = \frac{1}{\beta_\infty} \left( \frac{1}{2\pi r} \int_{S^1(r)} V(u(\ell)) \right).$$

We obtain hence, for any arbitrary $\sigma \in \Sigma$

$$|u(\ell_0) - \sigma|^2 \leq 2 \left( |u(\ell_0)|^2 + |\sigma|^2 \right) \leq \frac{2}{\beta_\infty} \left( \frac{1}{2\pi r} \int_{S^1(r)} V(u(\ell)) d\ell + R_0^2 \beta_\infty \right)$$

$$\leq \frac{2}{\beta_\infty} \left( \frac{1}{2\pi r} \int_{S^1(r)} V(u(\ell)) d\ell + \alpha_0 \left( \frac{R_0^2 \beta_\infty}{\alpha_0} \right) \right)$$

$$\leq \frac{1}{\pi \beta_\infty} \left( 1 + \left( \frac{2R_0^2 \beta_\infty}{\alpha_0} \right) \right) \left( r^{-1} \int_{S^1(r)} V(u(\ell)) d\ell \right).$$

This yields again (2.14) for an arbitrary choice of $\sigma_{main} \in \Sigma$ and imposing an additional suitable lower bound on $C_{unf}$.

We have hence established for upper bound (2.14) in all three possible cases 2a, 2b and 2c, for a suitable an arbitrary choice of $\sigma_{main} \in \Sigma$ and imposing an additional suitable lower bound on $C_{unf}$. It is hence established in case 2. Since we alreaday establishes it in Case 1, the proof of (2.14) is complete.

Turning to inequality (2.15), we first observe that, since by assumption $r \geq \varepsilon$, we have

$$r^{-1} \int_{S^1(r)} V(u(\ell)) d\ell \leq \int_{S^1(\varepsilon)} \varepsilon^{-1} V(u(\ell)) d\ell \leq \int_{S^1(\varepsilon)} e_\varepsilon(u(\ell)) d\ell. \quad (2.22)$$

Combining (2.14) with (2.13) and (2.22), we obtain the desired result (2.15). \qed

2.4 Controlling the energy on circles

When working on two-dimensional disks, the tools developed in the previous section allow to choose radii with appropriate control on the energy, invoking a standard mean-value argument. More precisely, we have:

**Lemma 2.5.** Let $\varepsilon \leq r_0 < r_1 \leq 1$ and $u : \mathbb{D}^2 \to \mathbb{R}^k$ be given. There exists a radius $r_\varepsilon \in [r_0, r_1]$ such that

$$\int_{S^1(t_\varepsilon)} e_\varepsilon(u) d\ell \leq \frac{1}{r_1 - r_0} E_\varepsilon(u, \mathbb{D}^2(r_1)).$$

The proof is based on a classical mean-value argument, therefore we omit it.

In the sequel, we will often make use of Lemma 2.5 combined with the uniform bounds obtained in dimension one. For instance, it follows from Lemma 2.4 that there exists some point $\sigma_{t_\varepsilon} \in \Sigma$, depending on $t_\varepsilon$, such that

$$|u(\ell) - \sigma_{t_\varepsilon}| \leq \frac{C_{unf}}{\sqrt{r_1 - r_0}} \sqrt{E_\varepsilon(u, \mathbb{D}^2(r_1))}, \quad \text{for all } \ell \in S^1(t_\varepsilon). \quad (2.23)$$
Moreover, it follows from (2.11) that
\[ \int_{S^1(r_\varepsilon)} |J(u)| \leq \frac{1}{r_1 - r_0} \int_{D^2(r_1)} e_\varepsilon(u_\varepsilon) \, dx. \] (2.24)

2.5 BV estimates and the coarea formula

The right-hand side of estimate (2.15), in particular the term involving \( J(u) \), may be inter-
preted as a BV estimate (as in [28]). In dimension 1, as expected, it yields used a uniform
estimates. In higher dimensions of course, this is no longer true. Nevertheless our BV-
estimates lead to useful estimates for the measure of specific level sets. In order to state the
kind of results we have in mind, we consider more generally an arbitrary smooth function
\( \varphi : \Omega \rightarrow \mathbb{R} \), where \( \Omega \subset \mathbb{R}^N \) is a arbitrary
N-dimensional domain, and introduce, for a given
number \( s \in \mathbb{R} \), the level set
\[ \varphi^{-1}(s) = \{ s \in \Omega, \text{ such that } \varphi(x) = s \}. \]

If \( w \) is assumed to be sufficiently smooth, then Sard’s theorem asserts that
\( w^{-1}(s) \) is a regular
submanifold of dimension \( (N - 1) \), for almost every \( s \in \mathbb{R} \), and the coarea formula relates the
integral of the total measures of these level sets to the BV-norm through the formula
\[ \int_{\mathbb{R}} \mathcal{H}^{N-1}(\varphi^{-1}(s)) \, ds = \int_{\Omega} |\nabla \varphi(x)| \, dx. \] (2.25)

We specify this formula to our needs in the specific case
\( N = 2, \Omega = D^2(r) \), for some \( r > \varepsilon \), and
\( \varphi = (w_i)^2 : \Omega \rightarrow \mathbb{R} \), where \( i \in \{1, \ldots, q\} \) and where \( w_i : \Omega \rightarrow \mathbb{R} \) is the map constructed in (2.7) for a given \( u : \Omega \rightarrow \mathbb{R}^k \). Combining (2.25) with (2.9) and (2.13), we are led to the
inequality, for the level sets \((w_i^2)^{-1}(s) \subset \Omega = D^2(r)\),
\[ \int_{\mathbb{R}^+} \mathcal{L}( (w_i^2)^{-1}(s) ) \, ds \leq 4 \sqrt{\lambda_0}^{-1} \int_{D^2(r)} J(u(x)) \, dx \]
\[ \leq 4 \sqrt{\lambda_0}^{-1} \int_{D^2(r)} e_\varepsilon(u) \, dx = 4 \sqrt{\lambda_0}^{-1} E_\varepsilon (u, D^2(r)), \] (2.26)

where \( \mathcal{L} = \mathcal{H}^1 \) denotes length. In most places, we will invoke this inequality jointly with a
mean value argument. This yields:

**Lemma 2.6.** Let \( u, w_i \) and \( r > \varepsilon \) be as above. Given any number \( A > 0 \), there exists some
\( A_0 \in [\frac{A}{2}, A] \) such that \( w_i^{-1}(s_0) \) is a regular curve in \( D^2(r) \) and such that
\[ \mathcal{L}(w_i^{-1}(A_0)) \leq \frac{8}{\sqrt{\lambda_0} A^2} \int_{D^2(r)} e_\varepsilon(u) \, dx \leq \frac{8 E_\varepsilon (u, D^2(r))}{\sqrt{\lambda_0} A^2}. \] (2.27)

**Proof.** In view of Definition 2.7, the map \( w_i \) takes values in the interval \([0, \frac{3\mu_0}{4}]\), so that
\( w_i^{-1}(s) = \emptyset \), if \( s > \frac{3\mu_0}{4} \). Hence, it remains only to consider the case \( A \leq \frac{3\mu_0}{4} \). We introduce
to that aim the domain \( \Omega_{i,A} = \{ x \in \mathbb{D}^2(r), \frac{A}{2} \leq |u(x) - \sigma_i| \leq A \} \). Using formula (2.26) on this domain, we are led to the inequality

\[
\int_{\frac{1}{2}^2} \mathcal{L}(w_i^2(s)^{-1}) ds \leq 4 \sqrt{\lambda_0}^{-1} \int_{\Omega_{i,A}} e_\varepsilon(u) dx \leq 4 \sqrt{\lambda_0}^{-1} E_\varepsilon(u, \mathbb{D}^2(r)).
\]

The conclusion that follows by a mean-value argument.

2.6 Controlling uniform bounds on good circles

Whereas in subsection 2.4 we have selected radii with controlled energy for the map \( u \), in this subsection, we select radii with appropriate uniform bounds on \( u \). We assume throughout this subsection that we are given a radius \( \varrho \in \left[ \frac{1}{2}, 1 \right] \), a number \( 0 < \kappa < \frac{\lambda_0}{2} \), a smooth map \( u : \overline{\mathbb{D}^2(\varrho)} \rightarrow \mathbb{R}^k \) and an element \( \sigma \in \Sigma \) such that

\[
|u - \sigma| < \kappa \text{ on } \partial \mathbb{D}^2(\varrho). \tag{2.28}
\]

We introduce the subset \( I(u, \kappa) \) of radii \( r \in \left[ \frac{1}{2}, \varrho \right] \) such that

\[
I(u, \kappa) = \left\{ r \in \left[ \frac{1}{2}, \varrho \right] \text{ such that } |u(\ell) - \sigma| \leq \kappa, \forall \ell \in S^1(r) \right\}. \tag{2.29}
\]

We have:

**Proposition 2.3.** Assume that (2.28) holds. Then, we have the lower bound

\[
|I(u, \kappa)| \geq \varrho - \frac{9}{16}, \tag{2.30}
\]

provided we have the lower bound on \( \kappa \)

\[
\kappa^2 \geq \frac{1}{32 \sqrt{\lambda_0}} E_\varepsilon(u, \mathbb{D}^2(\varrho)). \tag{2.31}
\]

**Proof.** We consider the number \( A_0 \in [\kappa, 2\kappa] \) provided by Lemma 2.6 with the choice \( r = \varrho \) and \( A = \kappa \), so that \( w^{-1}(A_0) \subset \mathbb{D}^2(\varrho) \) is smooth and

\[
\mathcal{L}(w^{-1}(A_0)) \leq \frac{8 E_\varepsilon(u, \mathbb{D}^2(\varrho))}{4 \sqrt{\lambda_0} \kappa^2} = \frac{2 E_\varepsilon(u, \mathbb{D}^2(\varrho))}{\sqrt{\lambda_0} \kappa^2}.
\]

If moreover (2.31) is satisfied, then we have

\[
\mathcal{L}(w^{-1}(A_0)) < \frac{1}{16}. \tag{2.32}
\]

We introduce the auxiliary set

\[
\begin{align*}
\mathcal{J}(u, \kappa) &= \{ r \in \left[ \frac{1}{2}, \varrho \right], \text{ such that } |u_\varepsilon(\ell) - \sigma| < A_0, \forall \ell \in S^1(r) \}, \text{ and } \\
\mathcal{Z}(u, \kappa) &= \{ r \in \left[ \frac{1}{2}, \varrho \right], \text{ such that } |u_\varepsilon(\ell) - \sigma| > A_0, \forall \ell \in S^1(r) \}.
\end{align*}
\]
We first show that
\[ Z(u, \kappa) = \emptyset. \] (2.33)
Indeed, assume by contradiction that (2.33) does not hold, so that there exists some radius 
\[ \frac{1}{2} \leq r_0 \leq \rho \] in \( Z(u, \kappa) \). In view of the definition of \( Z(u, \kappa) \), we have therefore
\[ |u - \sigma| > A_0 \text{ on } \partial \mathbb{D}^2(r_0). \] (2.34)
On the other hand, in view of assumption (2.28), we have
\[ |u - \sigma| < \kappa < A_0 \text{ on } \partial \mathbb{D}^2(\rho). \]
Combining (2.34) and (2.28), it follows from the intermediate value theorem that there exists
some smooth domain \( V \) such that \( u(x) = A_0 \) for \( x \in \partial V \), so that \( \partial V \subset w^{-1}(A_0) \), and hence
is smooth, and such that
\[ \mathbb{D}^2(r_0) \subset V \subset \mathbb{D}^2(\rho). \] (2.35)
We deduce from (2.35) that, since by assumption \( 1/2 \leq r \leq \rho \),
\[ \partial V \subset w^{-1}(A_0) \text{ and } \mathcal{L}(\partial V) \geq 2\pi r \geq \pi, \]
Hence, we obtain, in view of (2.35),
\[ \mathcal{L}(w^{-1}(A_0)) \geq \pi. \]
This however contradicts inequality (2.32) and hence establishes (2.33).

We next show that
\[ |J(u, \kappa)| \geq \rho - \frac{9}{16}, \] (2.36)
For that purpose, consider an arbitrary radius \( \frac{1}{2} \leq r \leq \rho \) such that \( r \notin J(u, \kappa) \) (see Figure 8). It follows from the definition of \( J(u, \kappa) \) that there exists some \( \ell_r \in S^1(r) \) such that
\[ |u_\ell(\ell_r) - \sigma| \geq A_0. \]
We deduce from identity (2.33) and the intermediate value theorem that
\[ w^{-1}(A_0) \cap S^1(r) \neq \emptyset, \forall r \notin J(u, \kappa). \]
This relation implies, by Fubini’s theorem, that
\[ \mathcal{L}(w^{-1}(A_0)) \geq \left( \rho - \frac{1}{2} \right) - |J(u, \kappa)|, \]
so that
\[ |J(u, \kappa)| \geq \left( \rho - \frac{1}{2} \right) - \mathcal{L}(w^{-1}(A_0)) \geq \rho - \frac{9}{16}, \] (2.37)
where we made use of estimate (2.32). This establishes (2.36). Since \( 0 < \kappa \leq A_0 \) by
construction, we have
\[ J(u, \kappa) \subset I(u, \kappa), \text{so that } |J(u, \kappa)| \leq |I(u, \kappa)|. \]
Combining with inequality (2.36), we obtain the desired inequality (2.30).
2.7 Revisiting the control of the energy on concentric circles

Using the results of the previous section, we work out variants of the Lemma 2.5. For that purpose, given a radius \( \varrho \in [\frac{3}{4}, 1] \), a number \( 0 < \kappa \leq \frac{\mu_0}{2} \), a smooth map \( u : \mathbb{D}^2(\varrho) \to \mathbb{R}^k \) and an element \( \sigma \in \Sigma \) such that (2.28) holds, we introduce the set

\[
\Upsilon_{\sigma}(u, \varrho, \kappa) = \{ x \in \mathbb{D}^2(\varrho), \text{ such that } |u(x) - \sigma| \leq \kappa \}.
\]

(2.38)

The following result is a major tool in the proof of our main results:

**Lemma 2.7.** Let \( u, \varrho \) and \( \kappa \) be as above and assume that the bound (2.31) holds. Assume that \( \varrho \geq \frac{3}{4} \). There exists a radius \( \tau_\varepsilon \in [\frac{5}{8}, \varrho] \) such that \( S_{1}(\tau_\varepsilon) \subset \Upsilon_{\sigma}(u, \varrho, \kappa) \), i.e.

\[
|u(\ell) - \sigma| \leq \kappa, \text{ for any } \ell \in S_{1}(\tau_\varepsilon),
\]

and such that

\[
\int_{S_{1}(\tau_\varepsilon)} e_\varepsilon(u)d\ell \leq \frac{1}{\varrho - \frac{9}{16}} E_\varepsilon(u, \Upsilon_{\sigma}(u, \varrho, \kappa)).
\]

**Proof.** In view of definition (2.38) of \( \Upsilon_{\sigma}(u, \varrho, \kappa) \) and the definition (2.29) of \( \mathcal{I}(u, \kappa) \), we have \( S_{1}(r) \subset \Upsilon_{\sigma}(u, \varrho, \kappa) \) for any \( r \in \mathcal{I}(u, \kappa) \), so that, by Fubini’s theorem, we have

\[
\int_{\mathcal{I}(u, \kappa)} \left( \int_{S_{1}(\ell)} e_\varepsilon(u)d\ell \right) d\varrho \leq \int_{\Upsilon_{\sigma}(u, \varrho, \kappa)} e_\varepsilon(u)d\varrho.
\]

Since we assume that the bound (2.31) holds, it follows from Proposition 2.3 that

\[
|\mathcal{I}(u, \kappa)| \geq \varrho - \frac{9}{16} \text{ and hence } |\mathcal{I}(u, \kappa) \cap [\frac{5}{8}, \varrho]| \geq \varrho - \frac{11}{16}.
\]
Hence by a mean value argument that there exists some radius $\tau \in \left[\frac{5}{8}, \varrho\right] \cap I_{\epsilon}$ such that
\[
\int_{S^1(\tau_{\epsilon})} e_\epsilon(u_{\epsilon}) d\ell \leq \frac{1}{\varrho - \frac{11}{16}} \int_{I_{\sigma}(u, \varrho, \kappa)} e_\epsilon(u_{\epsilon}) dx,
\]
which is precisely the conclusion. \hfill \square

**Comment.** The result above will be used in connection with the estimates for $u$ when $u$ is the solution to (1). Thanks to the equation, we will be able to estimate the growth of $E_\epsilon(u, I_{\sigma}(u, \varrho, \kappa))$ with $\kappa$. We will choose $\kappa$ as small as possible to satisfy (2.31), which amounts to choose of the magnitude of $\sqrt{E_\epsilon(u)}$, as we will see in (5.1).

### 2.8 Gradient estimates on level sets

Given a arbitrary smooth function $\varphi : \Omega \to \mathbb{R}$, where $\Omega$ denotes a denote of $\mathbb{R}^N$, and an arbitrary integrable function $f : \Omega \to \mathbb{R}$, the coarea formula (2.25) generalized as
\[
\int_{\mathbb{R}} \left( \int_{\varphi^{-1}(s)} f(\ell) d\ell \right) ds = \int_{\Omega} |\nabla \varphi(x)| f(x) dx.
\] (2.39)

Given a smooth function $u : \Omega \to \mathbb{R}^k$, we specify identity (2.39) with choices $\varphi = |u|$ and $f = |\nabla u|$. We are led to the identity
\[
\int_{\mathbb{R}} \left( \int_{|u|^{-1}(s)} |\nabla u(\ell)| d\ell \right) ds = \int_{\Omega} |\nabla u(x)| |\nabla |u|| dx,
\] (2.40)

\[
\leq \int_{\Omega} |\nabla u(x)|^2 dx.
\]

We specify furthermore this formula, as in Subsection 2.5, for a given map $u$ defined on a disk $D^2(r)$ and $w_i$ being the corresponding maps $w_i$ defined on $D^2(r)$ by formula (2.7). We introduce the subdomain
\[
\Theta(u, r) = \left\{ x \in D^2(r) \text{ such that } u(x) \in D^2(r) \setminus \bigcup_{i=1}^{q} B^k(\sigma_i, \frac{\mu_0}{2}) \right\}
\]
\[
= u^{-1} \left( D^2(r) \setminus \bigcup_{i=1}^{q} B^k(\sigma_i, \frac{\mu_0}{2}) \right) = \bigcup_{i=1}^{q} I_{\sigma_i}(u, r, \frac{\mu_0}{2}).
\] (2.41)

We have:

**Lemma 2.8.** Let $u$ be as above. There exists some number $\bar{\mu} \in \left[\frac{\mu_0}{2}, \mu_0\right]$, where $\mu_0$ denotes the constant introduced Paragraph 2.7, such that
\[
\sum_{i=1}^{q} \int_{w_i^{-1}(\bar{\mu})} |\nabla u(\ell)| d\ell \leq \frac{2}{\mu_0} \int_{\Theta(u, r)} |\nabla u|^2 \leq \frac{4}{\mu_0^2} E_\epsilon(u, \Theta(u, r)).
\] (2.42)

**Proof.** It follows from identity (2.40), applied to $u - \sigma_i$, that
\[
\sum_{i=1}^{q} \int_{w_i^{-1}(s)} |\nabla u| d\ell \leq \frac{2}{\mu_0} \int_{\Theta(u, r)} |\nabla u|^2 \leq \frac{4}{\mu_0^2} E_\epsilon(u, \Theta(u, r)).
\] (2.43)

We conclude once more by a mean-value argument. \hfill \square
3 Some properties of the PDE

In this section, we recall first several classical properties of the solutions to the equation (1). We then provide some energy and potential estimates (see e.g. [11]).

3.1 Uniform bound through the maximum principle

The following uniform upper bound is standard:

**Proposition 3.1.** Let \( u_\varepsilon \in H^1(\Omega) \) be a solution of (1). Then we have the uniform bound, for \( x \in \Omega \)

\[
|u(x)|^2 \leq \frac{4C_{\text{unf}}}{\text{dist}(x, \partial \Omega)} E_\varepsilon(u_\varepsilon) + 2 \sup \{|\sigma|^2; \sigma \in \Sigma\}.
\]

**Proof.** We argue as in [10]. We compute, using equation (1)

\[
\Delta |u_\varepsilon|^2 = u_\varepsilon \cdot \Delta u_\varepsilon + |\nabla u_\varepsilon|^2 = \varepsilon^{-2} u_\varepsilon \cdot \nabla V_u(u_\varepsilon) + |\nabla u_\varepsilon|^2 \\
\geq \varepsilon^{-2} u_\varepsilon \cdot \nabla V(u_\varepsilon), \text{ on } \Omega.
\]

On the other hand, it follows from assumption (4), see (2.5), that there exists some constant \( C_\infty \geq 0 \) such that

\[
y \cdot \nabla V(y) \geq \alpha_\infty |y|^2 - C_\infty \text{ for any } y \in \mathbb{R}^N.
\]

Hence, combining (3.2) and (3.3) we obtain the inequality

\[
-\Delta |u_\varepsilon|^2 + \alpha_\infty \varepsilon^{-2} \left(|u_\varepsilon|^2 - \frac{C_\infty}{\alpha_\infty}\right) \leq 0 \text{ on } \Omega.
\]

We introduce the function \( W_\varepsilon = |u_\varepsilon|^2 - \frac{C_\infty}{\alpha_\infty} \). We are led to the differential inequality for \( W_\varepsilon \)

\[
-\Delta W_\varepsilon + \alpha_\infty \varepsilon^{-2} W_\varepsilon \leq 0 \text{ on } \Omega.
\]

Let \( x \in \Omega \) and set \( R_x = \text{dist}(x, \partial \Omega) \), so that \( D^2(x, R_x) \subset \Omega \). It follows from Lemma 2.5 and inequality (2.23) that there exists some radius \( \tau \in [\frac{R_x}{2}, R_x] \) and some element \( \sigma \in \Sigma \) such that

\[
|u_\varepsilon(\ell) - \sigma| \leq \frac{2C_{\text{unf}}}{\sqrt{R_x}} \sqrt{E_\varepsilon(u_\varepsilon, D^2(R_x))} \leq \frac{2C_{\text{unf}}}{\sqrt{R_x}} \sqrt{E_\varepsilon(u_\varepsilon)}, \text{ for all } \ell \in S^1(\tau).
\]

Hence, we deduce from the previous inequality that

\[
W_\varepsilon(\ell) = |u_\varepsilon(\ell)|^2 - \frac{C_\infty}{\alpha_\infty} \leq \frac{4C_{\text{unf}}}{R_x} E_\varepsilon(u_\varepsilon) + 2 \sup \{|\sigma|^2; \sigma \in \Sigma\} - \frac{C_\infty}{\alpha_\infty}, \text{ for all } \ell \in S^1(x, \tau),
\]

where \( S^1(x, \tau) = \{\ell \in \mathbb{R}^2; |\ell - x| = \tau\} \). Since \( W_\varepsilon \) satisfies inequality (3.4), we may apply the maximum principle to assert that

\[
W_\varepsilon(y) \leq \sup \{W_\varepsilon(\ell), \ell \in S^1(\tau)\} \text{ for } y \in D^2(x, \tau),
\]
so that, combining (3.5) and (3.6) and the definition of $R_x$, we obtain,

$$W_\varepsilon(y) \leq \frac{4C_{unf}}{\text{dist}(x, \partial \Omega)} E_\varepsilon(u_\varepsilon) + 2 \sup \{|\mathbf{\sigma}|^2\} - \frac{c_\infty}{\alpha_\infty} \quad \text{for all } y \in \mathbb{D}^2(x, \frac{R}{2}).$$

Choosing $y = x$, the conclusion follows. \qed

### 3.2 Regularity and gradient bounds

The next result is a standard consequence of the smoothness of the potential, the regularity theory for the Laplacian and the maximum principle.

**Proposition 3.2.** Let $u_\varepsilon \in H^1(\Omega)$ be a solution of (1) and $\delta > 0$. Set $O_\delta = \{x \in \Omega, \text{dist}(x, \partial \Omega) \geq \delta\}$. Then $u_\varepsilon$ is smooth on $\Omega$ and there exists a constant $C_{gd}\left(\|u\|_{L^\infty(O_{\delta/2})}, \delta\right)$, depending only on $V$, $\|u\|_{L^\infty(O_{\delta/2})}$ and $\delta$ such that

$$|\nabla u_\varepsilon|(x) \leq \frac{C_{gd}\left(\|u\|_{L^\infty(O_{\delta})}, \delta\right)}{\varepsilon}, \quad \text{if } \text{dist}(x, \partial \Omega) \geq \delta. \quad (3.7)$$

**Proof.** Estimate (3.7) is a consequence of Lemma A.1 of [10]. It asserts that, if $v$ is a solution on some domain $U$ of $\mathbb{R}^n$ of $-\Delta v = f$, then we have the inequality

$$|\nabla v|^2(x) \leq C\left(\|f\|_{L^\infty(U)}\|v\|_{L^\infty(U)} + \frac{1}{\text{dist}(x, \partial U)^2}\|v\|_{L^\infty(U)}^2\right), \quad \text{for all } x \in U. \quad (3.8)$$

We apply inequality (3.8) to the solution $u_\varepsilon$, with source term $f = \varepsilon^{-2}\nabla u V(u_\varepsilon)$ on the domain $U = O_{\frac{\delta}{2}}$. This yields (3.7). We invoke the uniform estimates provided by Proposition 3.1. \qed

Whereas the result of Proposition 3.2 involves the uniform norm of $u_\varepsilon$, the next results provide a related results, involving the energy $E_\varepsilon(u_\varepsilon)$.

**Proposition 3.3.** Let $u_\varepsilon \in H^1(\Omega)$ be a solution of (1), $\delta > 0$, $M > 0$, and assume that $E_\varepsilon(u_\varepsilon) \leq M$. There exists some constant $K_{dr}(M, \delta) > 0$, depending only on the potential $V$, $M$ and $\delta$, such that,

$$|\nabla u_\varepsilon|(x) \leq \frac{K_{dr}(M, \delta)}{\varepsilon}, \quad \text{if } \text{dist}(x, \partial \Omega) \geq \delta. \quad (3.9)$$

**Proof.** We invoke the uniform estimates provided by Proposition 3.1. We have, indeed, in view of (3.1), the uniform upper bounds, for $u_\varepsilon$ and $f = \varepsilon^{-2}\nabla u V(u_\varepsilon)$,

$$\begin{cases}
|u(x)|^2 \leq C\left(\frac{M}{\delta} + 1\right), \quad \text{for } x \in O_{\frac{\delta}{2}} \\
|f(x)| \leq \varepsilon^{-2}C(M, \delta), \quad \text{for } x \in O_{\frac{\delta}{2}}.
\end{cases}$$

Combining again these bounds with (3.8) and arguing as in Proposition 3.2, we derive the conclusion. \qed
3.3 Gradient term versus potential term: First estimates

Major ingredients in the proof of our main PDE result, namely Proposition 1, are provided in Proposition 4.2 and Proposition 4.4, which we will state below and prove a little later. They roughly states that the total energy, which involves both a gradient term and a potential terms, can "essentially" be bounded by the integral of the sole potential term. In order to derive these results, we are led to divide domains into two regions: the region where the solution is near the set of potential wells Σ, and the region where it is far. Whereas the region where the solution is near the potential wells requires some further analysis, the region where the solution is far from the wells can be handled thanks to the results of the previous subsection, in particular the gradient bound described in Proposition 3.2.

Restricting ourselves to the case \( u_\varepsilon \) is defined on \( \Omega = \mathbb{D}^2 \), we introduce for \( r > 0 \) the set

\[
\Xi_\varepsilon(r) \equiv \Xi(u_\varepsilon, r) = \left\{ x \in \mathbb{D}^2(r) \text{ such that } u_\varepsilon(x) \in \mathbb{R}^k \setminus \bigcup_{i=1}^{q} \mathbb{B}^k(\sigma_i, \frac{\mu_0}{4}) \right\}.
\]

(3.10)

The sets \( \Xi_\varepsilon \) are aimed to describe region where the solution is far from \( \Sigma \). Indeed, we have, by definition

\[
\text{dist}(u(x), \Sigma) \geq \frac{\mu_0}{4} \text{ for } x \in \Xi_\varepsilon(r).
\]

(3.11)

The integral of the energy on the set \( \Xi_\varepsilon \) can be estimated by the integral of the potential as follows:

**Lemma 3.1.** \( u_\varepsilon \in H^1(\mathbb{D}^2) \) be a solution of \( (1) \). There exist a constant \( C_{\text{pt}}(\|u\|_{L^\infty(\mathbb{D}^2(4/5))}) \) depending only on \( V \) and \( \|u\|_{L^\infty(\mathbb{D}^2(4/5))} \) such that

\[
e_\varepsilon(u_\varepsilon) \leq C_{\text{pt}} \left( \|u\|_{L^\infty(\mathbb{D}^2(4/5))} \right) \frac{V(u_\varepsilon)}{\varepsilon} \quad \text{on } \Xi_\varepsilon\left(\frac{3}{4}\right).
\]

(3.12)

Let \( M > 0 \) and assume that \( E(u_\varepsilon) \leq M \). There exists a constant \( C_T \) depending only on the potential \( V \) and on \( M \) such that

\[
e_\varepsilon(u_\varepsilon) \leq C_T(M) \frac{V(u_\varepsilon)}{\varepsilon} \quad \text{on } \Xi_\varepsilon\left(\frac{3}{4}\right).
\]

(3.13)

**Proof.** It follows from the definition of \( \Xi_\varepsilon \) and in view of inequality (2.2) that

\[
V(u_\varepsilon(x)) \geq \frac{\alpha_0}{16} \quad \text{for } x \in \Xi_\varepsilon.
\]

Since, by definition \( \Xi_\varepsilon \subset \mathbb{D}^2(4/5) \), we have \( \text{dist}(x, \partial \mathbb{D}^2) = 1/5 \), for \( x \in \Xi_\varepsilon \). We may therefore invoke inequality (3.9) of Proposition 3.2 with \( \delta = 1/20 \), we obtain, for \( x \in \Xi_\varepsilon \)

\[
\varepsilon |\nabla u_\varepsilon|^2(x) \leq C_{gd}^2(\|u\|_{L^\infty(\mathbb{D}^2(4/5), 1/20)}) \varepsilon^{-1} = \frac{\alpha_0}{4\varepsilon} \left( \frac{4C_{gd}^2}{\alpha_0} \right) \leq \frac{4C_{gd}^2}{\alpha_0} \frac{V(u_\varepsilon(x))}{\varepsilon}.
\]

(3.14)
Set $L = \|u\|_{L^\infty((\Omega^2(4/5)))}$. Inequality (3.14) yields

$$e(u_\varepsilon) \leq \left( \frac{2C_{gd}(L, 1/20)^2}{\alpha_0} + 1 \right) \frac{V(u_\varepsilon)}{\varepsilon}.$$  

The conclusion (4.21) follows choosing the constant $C_{pt}$ as $C_{pt} = \left( \frac{4C_{gd}(L, 1/20)^2}{\alpha_0} \right)$. For (3.13), we combine (4.21) with the uniform bound (3.1).

### 3.4 The stress-energy tensor

The stress-energy tensor is an important tool in the analysis of singularly perturbed gradient-type problems. In dimension two, its expression is simplified thanks to complex analysis.

**Lemma 3.2.** Let $u_\varepsilon$ be a solution of (1) on $\Omega$. Given any vector field $\vec{X} \in D(\Omega, \mathbb{R}^2)$ we have

$$\int_{\Omega} A_\varepsilon(u_\varepsilon)_{i,j} \frac{\partial X_i}{\partial x_j} \, dx = 0$$

where $A_\varepsilon(u_\varepsilon) = e_\varepsilon(u_\varepsilon)\delta_{ij} - \varepsilon \frac{\partial u_\varepsilon}{\partial x_i} \cdot \frac{\partial u_\varepsilon}{\partial x_j}$. (3.15)

The proof is standard (see [13] and references therein): It is derived multiplying the equation (1) by the function $v = \sum X_i \partial_i u_\varepsilon$ and integrating by parts on $\Omega$. The $2 \times 2$ stress-energy matrix $A_\varepsilon$ may be decomposed as

$$A_\varepsilon \equiv A_\varepsilon(u_\varepsilon) = T_\varepsilon(u_\varepsilon) + \frac{V(u_\varepsilon)}{\varepsilon} I_2,$$

where the matrix $T_\varepsilon(u)$ is defined, for a map $u : \Omega \to \mathbb{R}^2$, by

$$T_\varepsilon(u) = \frac{\varepsilon}{2} \begin{pmatrix} |u_{x_1}|^2 - |u_{x_2}|^2 & -2u_{x_1} \cdot u_{x_2} \\ -2u_{x_1} \cdot u_{x_2} & |u_{x_1}|^2 - |u_{x_2}|^2 \end{pmatrix}.$$ (3.17)

**Remark 3.1.** Formula (3.2) corresponds to the first variation of the energy when one performs deformations of the domain induced by the diffeomorphism related to the vector field $\vec{X}$. More precisely, it can be derived from the fact that

$$\frac{d}{dt} E_\varepsilon(u_\varepsilon \circ \Phi_t) = 0,$$

where, for $t \in \mathbb{R}$ $\Phi_t : \Omega \to \omega$ is a diffeomorphism such that

$$\frac{d}{dt} \Phi_t(x) = \vec{X}(\Phi_t(x)), \forall x \in \Omega.$$

In dimension two, one may use complex notation to obtain a simpler expression of $T_{ij} \frac{\partial X_i}{\partial x_j}$. Setting $X = X_1 + iX_2$ we consider the complex function $\omega_\varepsilon : \Omega \to \mathbb{C}$ defined by

$$\omega_\varepsilon = \varepsilon \left( |u_{\varepsilon x_1}|^2 - |u_{\varepsilon x_2}|^2 - 2i \varepsilon x_1 \cdot u_{\varepsilon x_2} \right),$$

the quantity $\omega_\varepsilon$ being usually termed the *Hopf differential* of $u_\varepsilon$. We obtain the identities

$$T_{ij}(u_\varepsilon) \frac{\partial X_i}{\partial x_j} = \text{Re} \left( -\omega_\varepsilon \frac{\partial X_i}{\partial z} \right) \text{ and } \delta_{ij} \frac{\partial X_i}{\partial x_j} = 2\text{Re} \left( \frac{\partial X_i}{\partial z} \right).$$
Identity (3.15) is turned into
\[ \int_{\Omega} \text{Re} \left( \omega_{\varepsilon} \frac{\partial X}{\partial \bar{z}} \right) = \frac{2}{\varepsilon} \int_{\Omega} V(u_{\varepsilon}) \text{Re} \left( \frac{\partial X}{\partial \bar{z}} \right) = \frac{1}{\varepsilon} \int_{\Omega} V(u_{\varepsilon}) \text{div} \bar{X}. \quad (3.19) \]

Remark 3.2. Recall that the Dirichlet energy is invariant by conformal transformation. Such transformation are locally obtained through vector-fields $\bar{X}$ which are holomorphic.

### 3.5 Pohozaev’s identity on disks

Identity (3.19) allows to derive integral estimates of the potential $V(u_{\varepsilon})$ using a suitable choice of test vector fields. We restrict ourselves to the special case the domain is $\Omega = \mathbb{D}^2(r)$, for some $r > 0$. We notice that for the vector field $X = z$, we have

\[ \frac{\partial X}{\partial \bar{z}} = 0 \text{ and } \frac{\partial X}{\partial z} = 1. \]

However $X = z$ is not a test vector field, since it does not have compact support, so that we consider instead vector fields $X_{\delta}$ of the form

\[ X_{\delta} = z \varphi_{\delta}(|z|), \]

where $0 < \delta < \frac{1}{2}$ is a small parameter and $\varphi_{\delta}$ is a scalar function defined on $[0, r]$ such that $\varphi_{\delta}(s) = 1$ for $s \in [0, r - \delta)$, $|\varphi'(s)| \leq 2\delta$ for $s \in [r - \delta, r]$ and $\varphi(s) = 0$ on $[r - \delta/4, r]$, (3.20) so that $\varphi_{\delta}(r) = 0$. A short computation shows that

\[ \frac{\partial \varphi_{\delta}(|z|)}{\partial \bar{z}} = \frac{z}{2|z|} \varphi'_{\delta}(|z|) \text{ and } \frac{\partial \varphi_{\delta}(|z|)}{\partial z} = \frac{\bar{z}}{2|z|} \varphi'_{\delta}(|z|), \]

so that

\[ \frac{\partial X_{\delta}}{\partial \bar{z}} = \frac{z^2}{2|z|} \varphi'_{\delta}(|z|) \text{ and } \frac{\partial X_{\delta}}{\partial z} = \frac{|z|}{2} \varphi'_{\delta}(|z|) + \varphi_{\delta}(|z|) \in \mathbb{R}. \]

We drop the subscript $\varepsilon$ and simply write $u = u_{\varepsilon}$. Using polar coordinates $(r, \theta)$ such that $(x_1, x_2) = (r \cos \theta, r \sin \theta)$, we have $u_{x_1} = \cos \theta u_r - r^{-1} \sin \theta u_{\theta}$ and $u_{x_2} = \sin \theta u_r + r^{-1} \cos \theta u_{\theta}$. After some computations, this leads to the formula

\[ \omega_{\varepsilon} = \varepsilon (\cos 2\theta - i \sin 2\theta) \left[ (|u_r|^2 - r^{-1}|u_{\theta}|^2) - 2i u_r u_{\theta} \right] \]

\[ = \frac{\varepsilon^2}{|z|^2} \left[ (|u_r|^2 - r^{-2}|u_{\theta}|^2) - 2i u_r u_{\theta} \right]. \]

Combining the previous computations, we obtain

\[ \left\{ \begin{array}{l}
\text{Re} \left( \omega_{\varepsilon} \frac{\partial X_{\delta}}{\partial \bar{z}} \right) = \frac{\varepsilon}{2} (|u_r|^2 - r^2|u_{\theta}|^2) |z| \varphi'_{\delta}(|z|) \text{ and } \\
\text{Re} \left( \frac{\partial X_{\delta}}{\partial z} \right) = \frac{1}{2} |z| \varphi'_{\delta}(|z|) + \varphi_{\delta}(|z|) \text{ on } \mathbb{D}^2(r). \end{array} \right. \quad (3.21) \]

We check that, as expected, we have

\[ \frac{\partial X_{\delta}}{\partial \bar{z}} = 0 \text{ and } \frac{\partial X_{\delta}}{\partial z} = 1 \text{ on } \mathbb{D}^2(r - \delta). \]

Inserting these relations into (3.19) and passing to the limit $\delta \to 0$ yields the following identity, usually termed Pohozaev’s identity:
Lemma 3.3. Let \( u_\varepsilon \) be a solution of \((1)\) on \( \mathbb{D}^2 \). We have, for any radius \( 0 < r \leq 1 \)
\[
\frac{1}{\varepsilon^2} \int_{\mathbb{D}^2(r)} V(u_\varepsilon) = \frac{r}{4} \int_{\partial \mathbb{D}^2(r)} \left( \frac{\partial u_\varepsilon}{\partial \tau} \right)^2 - \left| \frac{\partial u_\varepsilon}{\partial r} \right|^2 + \frac{2}{\varepsilon^2} V(u_\varepsilon) \, d\tau. \tag{3.22}
\]

Proof. using the vector field \( X_\delta \) in \((3.19)\), we obtain, in view of identities \((3.21)\)
\[
\frac{2}{\varepsilon^2} \int_{\mathbb{D}^2(r)} V(u_\varepsilon) \left[ \frac{1}{2} |x| \varphi_\delta(|x|) + \varphi_\delta(|x|) \right] \, dx = \int_{\mathbb{D}^2(r)} \frac{1}{2} \left( |u_r|^2 - r^{-2} |u_\theta|^2 \right) |x| \varphi_\delta'(|x|) \, dx.
\]
so that
\[
\frac{2}{\varepsilon} \int_{\mathbb{D}^2(r)} V(u_\varepsilon) \varphi_\delta(|z|) \, dx = \frac{1}{2} \int_{\mathbb{D}^2(r)} \left( |u_r|^2 - r^{-2} |u_\theta|^2 - \frac{2}{\varepsilon} V(u_\varepsilon) \right) |x| \varphi_\delta'(|x|) \, dx. \tag{3.23}
\]
Next we observe that
\[
\begin{align*}
\varphi_\delta(|\cdot|) & \to 1_{\mathbb{D}^2(r)} & \text{as } \delta \to 0 \text{ in the sense of measures,} \\
|\cdot| \varphi_\delta'(|\cdot|) & \to -r \, d\tau & \text{as } \delta \to 0 \text{ in } \mathcal{D}'(\mathbb{R}^2),
\end{align*}
\]
where \( d\tau \) denotes the \( \mathcal{H}^1 \) measure on \( S^1(r) \). the conclusion follows.

Identity \((3.22)\) is central in the paper, in particular it leads to the monotonicity for \( \zeta_\varepsilon \). This identity has the remarkable property that it yields an identity of the integral of the potential inside the disk involving only energy terms on the boundary. A straightforward consequence of Lemma 3.3 is the estimate:

Proposition 3.4. Let \( u_\varepsilon \) be a solution of \((1)\) on \( \mathbb{D}^2 \). We have, for any \( 0 < r \leq 1 \)
\[
\frac{1}{\varepsilon} \int_{\mathbb{D}^2(r)} V(u_\varepsilon) \leq \frac{r}{2} \int_{S^1(r)} \varepsilon_\varepsilon(u_\varepsilon) \, d\ell. \tag{3.24}
\]

Proposition 3.4 follows immediately from Lemma 3.3 noticing that the absolute value of the integrand on the left hand side is bounded by \( 2\varepsilon^{-1} \varepsilon_\varepsilon(u_\varepsilon) \).

Besides Proposition 3.4, we notice that Pohozaev's identity leads directly to remarkable consequences: For instance, all solutions which are constant with values in \( \Sigma \) on \( \mathbb{D}^2(r) \) are necessarily constant.

Remark 3.3. The previous results are specific to dimension 2, however the use of the stress-energy tensor yields other results in higher dimensions (for instance monotonicity formulas).

3.6 Proofs of the ”monotonicity” formula for \( \zeta_\varepsilon \)

We provide here a proof of formula \((50)\), which is actually not a real monotonicity, since there is no evidence that the right hand side is non negative (only the asymptotic version is a monotonicity formula). The proof relies on Lemma 3.3 identity \((3.22)\). We have indeed, by Leibnitz rules
\[
\frac{d}{dr} \left( \frac{\nabla \varepsilon(u_\varepsilon, \mathbb{D}^2(r))}{r} \right) = -\frac{1}{r^2} \nabla \varepsilon(u_\varepsilon, \mathbb{D}^2(r)) + \frac{1}{r} \frac{d}{dr} \left( \frac{\nabla \varepsilon(u_\varepsilon, \mathbb{D}^2(r))}{r} \right). \]
By Fubini’s theorem, we have
\[ \frac{d}{dr} \left( V_{\varepsilon} \left( u_{\varepsilon}, D^2(r) \right) \right) = \frac{1}{\varepsilon} \int_{S^1(r)} V(u_{\varepsilon})d\tau, \]
so that, combining the previous identities, we obtain
\[ \frac{d}{dr} \left( \frac{V_{\varepsilon} \left( u_{\varepsilon}, D^2(r) \right)}{r} \right) = \frac{1}{r^2} \int_{D^2(r)} \varepsilon V(u_{\varepsilon})d\tau + \frac{1}{r} \int_{S^1(r)} \varepsilon V(u_{\varepsilon})d\tau \]
\[ = \frac{1}{4r} \int_{S^1(r)} \left( \varepsilon |u_r|^2 - \varepsilon |u_\tau|^2 - 2V_\varepsilon(u) \right)d\tau + \frac{1}{r} \int_{S^1(r)} \varepsilon V(u_{\varepsilon})d\tau \]
\[ = \frac{1}{4r} \int_{S^1(r)} \left( \varepsilon |u_r|^2 - \varepsilon |u_\tau|^2 + 2V_\varepsilon(u) \right)d\tau \]
where we have used (3.22) for the second line. Hence, identity (50) is established.

### 3.7 Proof of formula (34)

For the identity (34), we have similarly
\[ \frac{d}{dr} \left( E_{\varepsilon} \left( u_{\varepsilon}, D^2(r) \right) \right) = -\frac{1}{r^2} \int_{D^2(r)} \varepsilon |\nabla u|^2 dx + \frac{1}{r} \int_{S^1(r)} \varepsilon V(u_{\varepsilon})d\tau \]
\[ = -\frac{1}{2r^2} \int_{D^2(r)} \varepsilon |\nabla u|^2 dx - \frac{1}{r^2} \int_{D^2(r)} \varepsilon V(u_{\varepsilon})d\tau \]
\[ + \frac{1}{2r} \int_{S^1(r)} \left( \varepsilon |u_r|^2 + |u_\tau|^2 \right) + 2\varepsilon V(u_{\varepsilon})d\tau \]
We may decompose \( \varepsilon |\nabla u|^2 \) as \( \varepsilon |\nabla u|^2 = 2\varepsilon^{-1}V(u) - 2\xi_\varepsilon(u) \), where the discrepancy \( \xi_\varepsilon(u_{\varepsilon}) \) is defined in (35), so that the second line in (3.25) may be written as
\[ -\frac{1}{2r^2} \int_{D^2(r)} \varepsilon |\nabla u|^2 dx - \frac{1}{r^2} \int_{D^2(r)} \varepsilon^{-1}V(u_{\varepsilon})d\tau = \int_{D^2(r)} \xi_\varepsilon(u) - \frac{2}{r^2} \int_{D^2(r)} \varepsilon^{-1}V(u_{\varepsilon})d\tau \]
Combining (3.25), (3.26) with (3.22), we obtain a nice cancelation which yields (34).

### 3.8 Pohozaev’s type inequalities on general subdomain

We present in this subsection a related tool which will be of interest in the proof of Theorem 8. We consider a solution \( u_{\varepsilon} \) of (1) on a general domain \( \Omega \), a subdomain \( \mathcal{U} \) of \( \Omega \) and for \( \delta > 0 \) the domain \( \mathcal{U}_\delta \) introduced in (63). As a variant of Proposition 3.4, we have:

**Proposition 3.5.** Let \( u_{\varepsilon} \) be a solution of (1) on \( \Omega \). We have, for any \( 0 < \delta \)
\[ \frac{1}{\varepsilon} \int_{\mathcal{U}_\delta} V(u_{\varepsilon})d\tau \leq C(\mathcal{U}, \delta) \int_{\mathcal{V}_\delta} e_\varepsilon(u_{\varepsilon})d\tau, \]
where the constant \( C(\mathcal{U}, \delta) > 0 \) depends on \( \mathcal{U}, \delta \) and \( V \).
The main difference with Proposition 3.4 is that, in the case of a disk, the form of the $C(U, \delta) > 0$ is determined more accurately.

**Proof of Proposition 3.5.** Turning back to identity (3.19), we choose once more a test vector field $\vec{X}_\delta$ of the form $X_\delta(z) = z\chi_\delta(z)$, where the function $\chi_\delta$ is a smooth scalar positive function such that $\chi_\delta(z) = 1$ for $z \in U_\delta$ and $\chi_\delta(z) = 0$ for $z \in \mathbb{R}^2 \setminus U_\delta$.

so that $\nabla \chi_\delta = 0$ on the set $U_\delta$ and hence

$$\frac{\partial X_\delta}{\partial \bar{z}} = 0 \quad \text{and} \quad \frac{\partial X_\delta}{\partial z} = 1 \quad \text{on} \quad U_\delta.$$

Inserting these relations into (3.19), we are led to inequality (3.27).

4 Energy estimates

4.1 First energy estimates on levels sets close to $\Sigma$

In this subsection, we estimate the energy on domains where the solution is close to one of minimizers of the potential $\sigma \in \Sigma$. Near such a point, the potential is locally convex, close to a quadratic potential. In such a situation, solutions to the equation behave, at first order, as solution to the linear equation of the type

$$-\Delta v + \varepsilon^{-2} \nabla^2 V(\sigma) \cdot v \simeq 0,$$

for which energy estimates can be obtained directly by multiplying the equation by the solution itself and integration by parts, provided estimates on the boundary are available. More precisely, we consider again for given $0 < \varepsilon \leq 1$ a solution $u_\varepsilon: \mathbb{D}^2 \to \mathbb{R}^k$ to (1) and assume that we are given a radius $\rho_\varepsilon \in [\frac{1}{2}, \frac{3}{4}]$, a number $0 < \kappa < \frac{\mu_0}{2}$, where $\mu_0 > 0$ is the constant provided in (53). We introduce the subdomain $\Upsilon_\varepsilon(\rho_\varepsilon, \kappa)$ defined by

$$\Upsilon_\varepsilon(\rho_\varepsilon, \kappa) = \{ x \in \mathbb{D}^2(\rho_\varepsilon) \mid \| u_\varepsilon(x) - \sigma_i \| < \kappa, \quad \text{for some} \ i = 1 \ldots q \} \quad \text{(4.1)}$$

where we have set

$$\Upsilon_{\varepsilon,i}(\rho_\varepsilon, \kappa) = w_i^{-1}([0, \kappa) \cap \mathbb{D}^2(\rho_\varepsilon) = \Upsilon_{\sigma_i}(u_\varepsilon, \rho_\varepsilon, \kappa) = \{ x \in \mathbb{D}^2(\rho_\varepsilon), \| u_\varepsilon - \sigma_i \| \leq \kappa \}.$$

Notice that sets of the above form have already been introduced in (2.38) for general maps $u$ and are denotes there $\Upsilon_{\sigma}(u, \rho, \kappa)$. The set $\Upsilon_\varepsilon(\rho_\varepsilon, \kappa)$ corresponds hence to a truncation of the domain $\mathbb{D}^2(\rho_\varepsilon)$, where points with values far from the set $\Sigma$ have been removed, whereas the set $\Upsilon_{\varepsilon,i}(\rho_\varepsilon, \kappa)$ corresponds to a truncation of the domain $\mathbb{D}^2(\rho_\varepsilon)$ where points with values far from the point $\sigma_i \in \Sigma$ have been removed.

The main result of the present section is to establish an estimate on the integral of the energy on the domain $\Upsilon_\varepsilon(\rho_\varepsilon, \kappa)$ in terms of the integral of the potential as well as boundary integrals. As a matter of fact, we choose here a fixed value of $\kappa$, namely

$$\kappa = \mu_1 = \frac{\mu_0}{4}. \quad \text{(4.2)}$$
However, many elements in the proof carry out for a full range of values of $\kappa$, and will be used later in Subsection 4.2.

**Proposition 4.1.** Let $u_\varepsilon$ be a solution of $[1]$ on $\mathbb{D}^2$, let $L > 0$ be given and assume that

$$\|u_\varepsilon\|_{L^\infty} \leq L. \quad (4.3)$$

Let $\varrho_\varepsilon \in [\frac{1}{2}, \frac{3}{4}]$. We have, for some constant $K_T(L) > 0$, depending only on the potential $V$ and $L$, the inequality

$$\int_{\Upsilon_\varepsilon(\varrho_\varepsilon, u_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)dx \leq K_T(L) \left[ \int_{D^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon} dx + \varepsilon \int_{\partial D^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon) dl \right]. \quad (4.4)$$

The proof will be divided in several results of independznt interest. Firstly, since $u_\varepsilon$ is smooth and in view of Sard’s Lemma, the boundary $\partial \Upsilon_\varepsilon(\varrho_\varepsilon, \kappa)$ is smooth and a finite union of smooth curves for almost every $\kappa$, which we will assume throughout. Hence, for $i = 1, \ldots, q$ the set $\partial \Upsilon_{\varepsilon,i}$ is an union of smooth curves intersecting the boundary $\partial D^2(\varrho_\varepsilon)$ transversally. For $i = 1, \ldots, q$, we define the curves $\Gamma^i_\varepsilon$ and $\Pi^i_\varepsilon$ as

$$\left\{ \begin{array}{l}
\Gamma^i_\varepsilon(\varrho_\varepsilon, \kappa) \equiv \partial \Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa) \cap D^2(\varrho_\varepsilon) = w^{-1}_i(\kappa) \cap D^2(\varrho_\varepsilon) \quad \text{for } i = 1 \ldots q, \\
\Pi^i_\varepsilon(\varrho_\varepsilon, \kappa) \equiv \Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa) \cap \partial D^2(\varrho_\varepsilon) = [w^{-1}_i([0, \kappa]) \cap \partial D^2(\varrho_\varepsilon)] ,
\end{array} \right. \quad (4.5)$$

so that

$$\partial \Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa) = \Gamma^i_\varepsilon(\varrho_\varepsilon, \kappa) \cup \Pi^i_\varepsilon(\varrho_\varepsilon, \kappa). \quad (4.6)$$

In view of $[4.1]$, we introduce, for $i = 1, \ldots, q$, the integral quantities

$$\Omega^i_\varepsilon(\varrho_\varepsilon, \kappa) = \int_{\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa)} \varepsilon |\nabla u_\varepsilon|^2 + \varepsilon^{-1} \nabla_u V(u_\varepsilon) \cdot (u_\varepsilon - \sigma_i). \quad (4.7)$$

We first notice that:

**Lemma 4.1.** We have, for every $\kappa \in [0, \mu_0]$, the inequality

$$\int_{\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa)} e_\varepsilon(u_\varepsilon)dx \leq \frac{2\lambda_{\max}}{\lambda_0} \Omega^i_\varepsilon(\varrho_\varepsilon, \kappa). \quad (4.8)$$

**Proof.** Since, by the definition of $\Upsilon_{\varepsilon,i}$, we have $|u - \sigma_i| \leq \kappa \leq \frac{\mu_0}{2}$, we are in position to invoke estimates $[2,1]$, which yields, for $i \in \{1, \ldots, q\}$,

$$\frac{\lambda_0}{2\lambda_{\max}} V(u_\varepsilon) \leq \frac{1}{2} \lambda_0 |u_\varepsilon - \sigma_i|^2 \leq \nabla V(u_\varepsilon) \cdot (u_\varepsilon - \sigma_i) \quad \text{on } \Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa), \quad (4.9)$$

where $\lambda_{\max} = \sup \{\lambda^+_i, i = 1, \ldots, q\}$. Multiplying the previous inequality by $2\lambda_{\max}/\lambda_0$ and integrating on $\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa)$, we are led to

$$\int_{\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa)} \varepsilon^{-1} V(u_\varepsilon)dx \leq \frac{2\lambda_{\max}}{\lambda_0} \int_{\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa)} \varepsilon^{-1} \nabla_u V(u_\varepsilon) \cdot (u_\varepsilon - \sigma_i). \quad (4.10)$$

The conclusion then follows from the definitions of $e_\varepsilon$ and $\Omega^i_\varepsilon(\varrho_\varepsilon, \kappa)$. \qed
Indeed, by definition

\[
\text{Remark 4.1.} \quad \text{Notice that we have the inequality}
\]

\[\partial \frac{|u_\varepsilon - \sigma_i|}{\partial \bar{n}} \geq 0 \quad \text{on } \Gamma_i^\varepsilon(q_\varepsilon, \kappa). \tag{4.14}\]

Indeed, by definition \(|u_\varepsilon - \sigma_i| = \kappa\) on \(\Upsilon_{\varepsilon,i}(q_\varepsilon, \kappa)\), so that we are on a level set and the normal derivative \(\bar{n}(\ell)\) is pointing towards the outside.

The next result will also be used extensively in Subsection 4.2

**Lemma 4.3.** Assume that \(0 < \varepsilon \leq 1\) and that \(u_\varepsilon\) is a solution to (1) on \(D^2(1)\). Let \(q_\varepsilon\) be in \([1/2, 3/4]\). We have, for every \(\kappa \in [0, \mu_0]\), the inequality

\[
\int_{\Upsilon_{\varepsilon,i}(q_\varepsilon, \kappa)} e_\varepsilon(u_\varepsilon)dx \leq C \varepsilon \left[ \kappa \sum_{i=1}^q \int_{\Gamma_i^\varepsilon(q_\varepsilon, \kappa)} \frac{\partial |u_\varepsilon(\ell) - \sigma_i|}{\partial \bar{n}(\ell)} d\ell + \int_{\partial D^2(q_\varepsilon)} e_\varepsilon(u_\varepsilon(\ell))d\ell \right]. \tag{4.15}\]

where \(C > 0\) is some constant depending only on the potential \(V\) and where \(\bar{n}(\ell)\) denotes the unit vector normal to \(\Gamma_{\varepsilon,i} \cup \Pi_{\varepsilon,i}\) pointing in the direction increasing \(|u_\varepsilon - \sigma_i|\).

**Remark 4.2.** Let us emphasize that in this statement, \(\kappa\) is not constrained by 4.2 and may actually take arbitrary small values.
Proof. The proof relies on a combination of the results of Lemmas 4.2 and 4.3. We first estimate the second term on the r.h.s of (4.11). Since by definition, we have the inclusion \( \Pi_\varepsilon(q, \kappa) \subset S^1(q, \varepsilon) \), it follows that \( \bar{u} (\ell) = e_r \) on \( \Pi_\varepsilon(q, \kappa) \), so that

\[
\left| \frac{\partial u_\varepsilon - \sigma_i}{\partial n} \right| = \left| \frac{\partial u_\varepsilon - \sigma_i}{\partial r} \right| \leq \left| \nabla u_\varepsilon \right|, \text{ on } \Pi_\varepsilon(q, \kappa). \tag{4.16}
\]

On the other hand, in view of Proposition 2.1 as well as the fact that \( |u_\varepsilon(\ell) - \sigma_i| \leq \kappa \leq \mu_0 \) on \( \Pi_\varepsilon(q, \kappa) \), we have

\[
|u_\varepsilon - \sigma_i| \leq \frac{2}{\lambda_0} \sqrt{V(u_\varepsilon)} \text{ on } \Pi_\varepsilon(q, \kappa). \tag{4.17}
\]

Combining (4.16) with (4.17) and integrating on \( \Pi_\varepsilon(q, \kappa) \), we obtain the estimate

\[
\int_{\Pi_\varepsilon(q, \kappa)} |u_\varepsilon - \sigma_i| \frac{\partial u_\varepsilon - \sigma_i}{\partial n} \, d\ell \leq \frac{2}{\lambda_0} \int_{\Pi_\varepsilon(q, \kappa)} \sqrt{V(u_\varepsilon)} |\nabla u_\varepsilon| \, d\ell \leq \frac{2}{\lambda_0} \int_{\Pi_\varepsilon(q, \kappa)} e_\varepsilon(u) \, d\ell, \tag{4.18}
\]

where, for the second inequality, we used Lemma 2.3 and the fact that \( \Pi_\varepsilon(q, \kappa) \subset S^1(q, \varepsilon) \).

Combining (4.18) with (4.11) and (4.8), we obtain the desired conclusion (4.15) for the choice of constant \( C = \frac{2\lambda_{\max}}{\lambda_0} (1 + \frac{2}{\lambda_0}) \).

Our next results allows to obtain, for a suitable choice of \( \kappa \), a bound on the first term on the right hand side of (4.15):

**Lemma 4.4.** Assume that \( 0 < \varepsilon \leq 1 \) and that \( u_\varepsilon \) is a solution to (1) on \( \mathbb{D}^2(1) \). Let \( q, \varepsilon, \mu_0, \kappa \in \left[ \frac{1}{2}, \frac{3}{4} \right] \).

There exists some number \( \tilde{\mu}_\varepsilon \in \left[ \frac{\mu_0}{4}, \frac{\mu_0}{2} \right] \) such that

\[
\varepsilon \int_{\Gamma_{\varepsilon}(q, \tilde{\mu}_\varepsilon)} \frac{\partial |u_\varepsilon - \sigma_i|}{\partial n} \, d\ell \leq \varepsilon \int_{\Gamma_{\varepsilon}(q, \tilde{\mu}_\varepsilon)} \left| \nabla u_\varepsilon \right| \, d\ell \leq \frac{1}{\mu_0} E_\varepsilon(u, \Xi(u_\varepsilon, q_\varepsilon)), \tag{4.19}
\]

where \( \Xi(u_\varepsilon, q_\varepsilon) \) is defined in (2.41).

**Proof.** We invoke first Lemma 2.8 with the choices \( r = q, \varepsilon, u = u_\varepsilon \). This yields directly a number \( \tilde{\mu}_\varepsilon \in \left[ \frac{\mu_0}{4}, \frac{\mu_0}{2} \right] \) such that (4.19) is satisfied, so that the proof is complete.

**Proof of Proposition 4.4 completed.** We combine (4.15) for \( \kappa = \tilde{\mu}_\varepsilon \) with (4.19). This yields

\[
\int_{\Gamma_{\varepsilon}(q, \tilde{\mu}_\varepsilon)} e_\varepsilon(u) \, dx \leq C_{\varepsilon} \left[ \frac{\tilde{\mu}_\varepsilon}{\mu_0} E_\varepsilon(u_\varepsilon, \Xi(u_\varepsilon, q_\varepsilon)) + \int_{\partial \mathbb{D}^2(q_\varepsilon)} e_\varepsilon(u_\varepsilon(\ell)) \, d\ell \right]. \tag{4.20}
\]

On the other hand, it follows from assumption (4.3) and Lemma 3.1 that

\[
e_\varepsilon(u_\varepsilon) \leq C_{pt} (L) \frac{V(u_\varepsilon)}{\varepsilon} \text{ on } \Xi(u_\varepsilon, \frac{3}{4}) \supset \Xi(u_\varepsilon, q_\varepsilon), \tag{4.21}
\]

so that

\[
E_\varepsilon(u_\varepsilon, \Xi(u_\varepsilon, q_\varepsilon)) = \int_{\Xi(u_\varepsilon, q_\varepsilon)} e_\varepsilon(u) \, dx \leq C_{pt} (L) \int_{\Xi(u_\varepsilon, q_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon} \, dx \leq C_{pt} (L) \int_{\mathbb{D}^2(q_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon} \, dx. \tag{4.22}
\]

Combining (4.22) with (4.20), we obtain (4.4) for \( K_T(L) = C C_{pt} (L) \).
4.2 Refined estimates on level sets close to $\Sigma$

Whereas we obtained in Proposition 4.2 an energy estimate on a fixed level set $\Upsilon_\varepsilon(\varrho_\varepsilon, \mu_1)$, we derive here an energy estimate on the set $\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa)$ allowing the value of $\kappa$ to vary and in particular to be small. This will be completed at the cost of an additional assumption. Indeed, we will assume that there exists an element $\sigma_{\text{main}} \in \Sigma$ such that
\[
|u_\varepsilon - \sigma_{\text{main}}| < \kappa \text{ on } \partial\mathbb{D}^2(\varrho_\varepsilon). \tag{4.23}
\]

The main result of this subsection is:

**Proposition 4.2.** Let $u_\varepsilon$ be a solution of (1) on $\mathbb{D}^2$, $M > 0$, $0 < \kappa < \frac{\mu_0}{4}$ and $\varrho_\varepsilon \in \left[\frac{1}{2}, \frac{3}{4}\right]$. Assume that (4.23) is satisfied and that
\[
E(u_\varepsilon) \leq M. \tag{4.24}
\]

We have, for some constant $C_{\Upsilon}(M) > 0$, depending only on the potential $V$ and on $M$,
\[
\int_{\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa)} \theta(u_\varepsilon)(x)dx \leq C_{\Upsilon}(M) \left[ \kappa \int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}dx + \varepsilon \int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} \theta(u_\varepsilon)d\ell \right]. \tag{4.25}
\]

Of major importance in estimate (4.25) is the presence of the term $\kappa$ in front of the integral of the potential, so that the energy on $\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa)$ grows essentially at most linearly with respect to $\kappa$. Proposition 4.2 will be used in the proof of the clearing-out result, so that we will use it for small $M$.

We may assume without loss of generality that $\sigma_{\text{main}} = \sigma_1$, so that it follows from assumption (4.23) that
\[
|u_\varepsilon(\ell) - \sigma_1| < \kappa \text{ for } \ell \in \partial\mathbb{D}^2(\varrho_\varepsilon). \tag{4.26}
\]

We deduce from inequality (4.26) that $\partial\mathbb{D}^2(\varrho_\varepsilon) \subset \overline{\Upsilon_{\varepsilon,1}(\varrho_\varepsilon, \kappa)}$, and that, for $i = 2, \ldots, q$, we have
\[
\partial\mathbb{D}^2(\varrho_\varepsilon) \cap \partial\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa) = \emptyset.
\]

In particular, we have the identities
\[
\begin{cases}
\Gamma_\varepsilon^i(\varrho_\varepsilon, \kappa) \equiv \partial\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa) = w_1^{-1}(\kappa) \cap \mathbb{D}^2(\varrho_\varepsilon) & \text{for } i = 2 \ldots q, \\
\Pi_\varepsilon^i(\varrho_\varepsilon, \kappa) = \emptyset, \text{ for } i = 2 \ldots q, \\
\Gamma_\varepsilon^i(\varrho_\varepsilon, \kappa) \equiv \partial\Upsilon_{\varepsilon,1}(\varrho_\varepsilon, \kappa) \setminus \mathbb{D}^2(\varrho_\varepsilon) = \left(\left[w_1^{-1}(\kappa) \cap \mathbb{D}^2(\varrho_\varepsilon)\right]\right) \setminus \mathbb{D}^2(\varrho_\varepsilon) & \text{and} \\
\Pi_\varepsilon^1(\varrho_\varepsilon, \kappa) = \partial\mathbb{D}^2(\varrho_\varepsilon).
\end{cases} \tag{4.27}
\]

As for Proposition 4.1, we will deduce Proposition 4.2 from Lemma 4.3. For that purpose, we will make use of a new ingredient, given by the following monotonicity formula:

**Lemma 4.5.** Let $\kappa_1 \geq \kappa_0 \geq \kappa$ be given. If $u_\varepsilon$ satisfies condition (4.23), then we have, for $i = 1, \ldots, q$, the inequality
\[
0 \leq \int_{\Gamma_\varepsilon^i(\varrho_\varepsilon, \kappa_0)} \frac{\partial|u_\varepsilon(\ell) - \sigma_i|}{\partial n(\ell)}d\ell \leq \int_{\Gamma_\varepsilon^i(\varrho_\varepsilon, \kappa_1)} \frac{\partial|u_\varepsilon(\ell) - \sigma_i|}{\partial n(\ell)}d\ell. \tag{4.28}
\]
Proof. The proof involves again Stokes formula, now on the domain

$$C(\kappa_0, \kappa_1) = \mathcal{Y}_{\epsilon,i}(\varrho_\epsilon, \kappa_1) \setminus \mathcal{Y}_{\epsilon,i}(\varrho_\epsilon, \kappa_0).$$

It follows from assumption (4.23) that

$$C(\kappa_0, \kappa_1) \cap \partial \mathbb{D}^2(\varrho_\epsilon) = \emptyset,$$

so that

$$\partial C(\kappa_0, \kappa_1) = \partial \mathcal{Y}_{\epsilon,i}(\varrho_\epsilon, \kappa_1) \cup \partial \mathcal{Y}_{\epsilon,i}(\varrho_\epsilon, \kappa_0).$$

We multiply the equation (4.1) by \(\frac{u_\epsilon - \sigma_i}{|u_\epsilon - \sigma_i|}\) which is well defined on \(C(\kappa_0, \kappa_1)\) and integrate by parts. Since, on \(\Gamma_{\epsilon,i}(\varrho_\epsilon, \kappa_1)\), we have

$$\frac{\partial u_\epsilon}{\partial \tilde{n}} \cdot \frac{u_\epsilon - \sigma_i}{|u_\epsilon - \sigma_i|} = \frac{\partial (u_\epsilon - \sigma_i)}{\partial \tilde{n}} \cdot \frac{u_\epsilon - \sigma_i}{|u_\epsilon - \sigma_i|} = \frac{\partial |u_\epsilon - \sigma_i|}{\partial \tilde{n}},$$

whereas on \(C(\kappa_0, \kappa_1)\), we have

$$\nabla u_\epsilon \cdot \nabla \left( \frac{u_\epsilon - \sigma_i}{|u_\epsilon - \sigma_i|} \right) = \nabla (u_\epsilon - \sigma_i) \cdot \nabla \left( \frac{u_\epsilon - \sigma_i}{|u_\epsilon - \sigma_i|} \right)$$

$$= \frac{1}{|u_\epsilon - \sigma_i|} [\nabla (u_\epsilon - \sigma_i)]^2 + [\nabla (u_\epsilon - \sigma_i) \cdot (u_\epsilon - \sigma_i)] \cdot \nabla \left( \frac{1}{|u_\epsilon - \sigma_i|} \right)$$

$$= \frac{1}{|u_\epsilon - \sigma_i|} \left[ |\nabla (u_\epsilon - \sigma_i)|^2 - |\nabla u_\epsilon - \sigma_i|^2 \right],$$

integration by parts thus yields

$$\int_{\Gamma_{\epsilon,i}(\varrho_\epsilon, \kappa_1)} \frac{\partial |u_\epsilon - \sigma_i|}{\partial \tilde{n}} - \int_{\Gamma_{\epsilon,i}(\varrho_\epsilon, \kappa_0)} \frac{\partial |u_\epsilon - \sigma_i|}{\partial \tilde{n}} = \int_{C(\kappa_0, \kappa_1)} \frac{1}{|u - \sigma_i|} \left[ |\nabla u_\epsilon|^2 - |\nabla |u_\epsilon - \sigma_i||^2 \right]$$

$$+ \int_{C(\kappa_0, \kappa_1)} \varepsilon^{-2} V(u_\epsilon) \cdot \frac{u_\epsilon - \sigma_i}{|u_\epsilon - \sigma_i|}. \quad (4.29)$$

Since

$$|\nabla u_\epsilon|^2 - |\nabla u_\epsilon - \sigma_i|^2 = |\nabla (u_\epsilon - \sigma_i)|^2 - |\nabla u_\epsilon - \sigma_i|^2 \geq 0,$$

it follows that the r.h.s of inequality (4.29) is positive. Hence, we deduce (4.28). \(\square\)

Lemma 4.6. Assume that \(0 < \epsilon \leq 1\) and that \(u_\epsilon\) is a solution to (4.1) which satisfies (4.23) and (4.24). Then, there exits a constant \(C(M) > 0\) depending only on \(V\) and \(M\) such that have

$$0 \leq \varepsilon \int_{\Gamma_{\epsilon,i}(\varrho_\epsilon, \kappa)} \frac{\partial |u_\epsilon - \sigma_i|}{\partial \tilde{n}(\ell)} d\ell \leq C(M) \int_{\mathbb{D}^2(\varrho_\epsilon)} \frac{V(u)}{\epsilon} dx \leq C(M) V(u_\epsilon, \mathbb{D}^2(\frac{3}{4})), \quad (4.30)$$

where, for a point \(\ell \in \Gamma_{\epsilon,i}\), \(\tilde{n}(\ell)\) denotes the unit vector perpendicular to \(\Gamma_{\epsilon,i}\) and oriented in the direction which increases \(|u - \sigma_i|\).
Proof. By Lemma 4.4, there exists a number \( \bar{\mu}_\varepsilon \in \left[ \frac{\mu_0}{4}, \frac{\mu_0}{2} \right] \) such that

\[
\varepsilon \int_{\Gamma_{\varepsilon,i}(\varrho,\bar{\mu}_\varepsilon)} \frac{\partial|u_\varepsilon - \sigma_i|}{\partial n(\ell)} \, d\ell \leq \varepsilon \int_{\Gamma_{\varepsilon,i}(\varrho,\bar{\mu}_\varepsilon)} |\nabla u_\varepsilon| \, d\ell \leq \frac{1}{\mu_0} E_\varepsilon(u, \Xi(u_\varepsilon, \varrho_\varepsilon)) \quad (4.31)
\]

On the level set \( \Xi(u_\varepsilon, \varrho_\varepsilon) \), we may however bound point-wise the energy in terms of the potential, as stated in Lemma 3.1, inequality (3.13). This yields by integration

\[
E_\varepsilon(u, \Xi(u_\varepsilon, \varrho_\varepsilon)) \leq C_T(M) V(u_\varepsilon, \Xi(u_\varepsilon, \varrho_\varepsilon)).
\]

Combining the two previous inequalities, we obtain

\[
\varepsilon \int_{\Gamma_{\varepsilon,i}(\varrho,\bar{\mu}_\varepsilon)} \frac{\partial|u_\varepsilon - \sigma_i|}{\partial n(\ell)} \, d\ell \leq \frac{C_T(M)}{\mu_0} V(u_\varepsilon, \Xi(u_\varepsilon, \varrho_\varepsilon)). \quad (4.32)
\]

On the other hand, we invoke to Lemma 4.5 with the \( \kappa_1 = \bar{\mu}_\varepsilon \) and \( \kappa_0 = \kappa \) to deduce that

\[
\int_{\Gamma_{\varepsilon,i}(\varrho,\bar{\mu}_\varepsilon)} \frac{\partial|u_\varepsilon - \sigma_i|}{\partial n(\ell)} \, d\ell \leq \int_{\Gamma_{\varepsilon,i}(\varrho,\bar{\mu}_\varepsilon)} \frac{\partial|u_\varepsilon - \sigma_i|}{\partial n(\ell)} \, d\ell,
\]

which together with (4.32) leads to the desired result (4.30). \( \square \)

Proof of Proposition 4.2 completed. We go back to Lemma 4.3 and combine (4.15) with (4.30): This yields the desired inequality (4.25).

4.3 Bounding the total energy by the integral of the potential

The main result of the present paragraph is the following result:

**Proposition 4.3.** Let \( u_\varepsilon \) be a solution of (1) on \( \mathbb{D}^2 \) and let \( L > 0 \) be given and assume that

\[
\|u_\varepsilon\|_{L^\infty(\mathbb{D}^2(1/2))} \leq L. \quad (4.33)
\]

There exists some constant \( K_{\text{pot}}(L) \) depending only on \( V \) and \( L \) such that

\[
\int_{\mathbb{D}^2(1/2)} e_\varepsilon(u_\varepsilon)(x) \, dx \leq K_{\text{pot}}(L) \left[ \int_{\mathbb{D}^2(1/2)} \frac{V(u_\varepsilon)}{\varepsilon} \, dx + \varepsilon \int_{\mathbb{D}^2 \setminus \mathbb{D}^2(1/2)} e_\varepsilon(u_\varepsilon) \, dx \right]. \quad (4.34)
\]

In the context of the present paper, the main contribution of the r.h.s of inequality (4.34) is given by the potential terms, so that Proposition 4.3 yields an estimate of the energy by the integral of potential, provided the later is sufficiently small, according to assumption (4.33).

Before turning to the proof of Proposition 4.3, we observe, as a preliminary remark, that the result of proposition 4.3 is, at first sight, rather close to the result of Proposition 4.1. However, let us emphasize that estimate (4.25) yields only an energy bound only for the domain where the value of \( u_\varepsilon \) is close to one of the wells whereas (4.34) yield an estimate for the full domain \( \mathbb{D}^2(1/2) \).

The first step in the proof of Proposition 4.3 is:
Lemma 4.7. Let $\varrho_\varepsilon \in \left[\frac{1}{2}, \frac{3}{4}\right]$, let $u_\varepsilon$ be a solution of (1) on $\mathbb{D}^2$ and assume that (4.33) is satisfied. We have, for some constant $C_{\text{pot}}(L) > 0$ depending only on the potential $V$ and the value of $L$ such that

$$\int_{\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)(x) dx \leq C_{\text{pot}}(L) \left[ \int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon} dx + \frac{\varepsilon}{4} \int_{\partial \mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon) d\ell \right]. \quad (4.35)$$

Proof. We observe first that

$$\mathbb{D}^2(\varrho_\varepsilon) = \Xi(\varrho_\varepsilon) \cup \Upsilon_\varepsilon(\varrho_\varepsilon, \frac{\mu_0}{4}). \quad (4.36)$$

In view of Lemma 3.1, we have

$$\int_{\Xi(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon) dx \leq C_T(L) \int_{\Xi(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon} dx,$$

whereas Proposition 4.1 yields

$$\int_{\Upsilon_\varepsilon(\varrho_\varepsilon, \frac{\mu_0}{4})} e_\varepsilon(u_\varepsilon) dx \leq K_T(L) \left[ \int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon} dx + \varepsilon \int_{\partial \mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon) d\ell \right].$$

The proof of (4.35) then follows straightforwardly from our first observation (4.36).

Proof of Proposition 4.3 completed. As usual, a mean-value argument allows us to choose some radius $\varrho_\varepsilon \in \left[\frac{1}{2}, \frac{3}{4}\right]$ such that

$$\int_{\partial \mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon) d\ell \leq 8 \int_{\mathbb{D}^2(\frac{3}{4}) \backslash \mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon) dx. \quad (4.37)$$

Combining with Lemma 4.7, we are led to

$$\int_{\mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon)(x) dx \leq \int_{\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)(x) dx$$

$$\leq C_{\text{pot}} \left[ \int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon} dx + \frac{\varepsilon}{4} \int_{\partial \mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon) d\ell \right] \leq C_{\text{pot}} \left[ \int_{\mathbb{D}^2(\frac{3}{4})} \frac{V(u_\varepsilon)}{\varepsilon} dx + \varepsilon \int_{\partial \mathbb{D}^2(\frac{3}{4})} e_\varepsilon(u_\varepsilon) d\ell \right]. \quad (4.38)$$

The proof of Proposition 4.3 is hence complete.

We will also invoke the following variant of Proposition 4.3.

Proposition 4.4. Let $u_\varepsilon$ be a solution of (1) on $\mathbb{D}^2$, let $M > 0$ be given and assume that (4.24) holds. There exists some constant $C_{\text{pot}}(M)$ depending only on $V$ and $M$ such that

$$\int_{\mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon)(x) dx \leq C_{\text{pot}}(M) \left[ \int_{\mathbb{D}^2(\frac{1}{2})} \frac{V(u_\varepsilon)}{\varepsilon} dx + \varepsilon \int_{\mathbb{D}^2 \backslash \mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon) dx \right]. \quad (4.39)$$
Proof. If \( u_\varepsilon \) satisfies (4.24), then it follows from Lemma 3.1
\[
\| u_\varepsilon \|_{L^\infty(D^2(x_0, \frac{4\varrho}{5}))} \leq L_M \equiv 5C_{unf}M + 2\sup\{\| \sigma \|^2, \sigma \in \Sigma\}. \tag{4.40}
\]
Invoking Proposition 4.3, inequality (4.39) follows with
\[
C_{pot}(M) = K_{pot}(L_M) = K_{pot}(5C_{unf}M + 2\sup\{\| \sigma \|^2, \sigma \in \Sigma\}).
\]

In the course of the paper, we will invoke the scaled versions of Proposition 4.3 and 4.4. Given \( \varrho > \varepsilon > 0 \) and \( x_0 \in \Omega \), we consider a solution \( u_\varepsilon \) on \( \Omega \) and assume it satisfies the bound (4.33) or the bound
\[
E(u_\varepsilon, D^2(x_0, \varrho)) \leq M\varrho, \tag{4.41}
\]
then, thanks to the relations (52), we have the scaled version of (4.34) or (4.39) respectively, namely
\[
\int_{D^2(x_0, \frac{\varrho}{\varepsilon})} e_\varepsilon(u_\varepsilon)dx \leq K_{pot}(L) \left[ \int_{D^2(x_0, \frac{3\varrho}{5})} \frac{V(u_\varepsilon)}{\varepsilon} dx + \frac{\varepsilon}{\varrho} \int_{D^2(x_0, \varrho) \setminus D^2(x_0, \frac{\varrho}{\varepsilon})} e_\varepsilon(u_\varepsilon)dx \right], \tag{4.42}
\]
and
\[
\int_{D^2(x_0, \frac{\varrho}{\varepsilon})} e_\varepsilon(u_\varepsilon)dx \leq C_{pot}(M) \left[ \int_{D^2(x_0, \frac{3\varrho}{5})} \frac{V(u_\varepsilon)}{\varepsilon} dx + \frac{\varepsilon}{\varrho} \int_{D^2(x_0, \varrho) \setminus D^2(x_0, \frac{\varrho}{\varepsilon})} e_\varepsilon(u_\varepsilon)dx \right]. \tag{4.43}
\]
These relations lead to:

**Proposition 4.5.** Let \( M_0 > 0 \) and \( \varepsilon > 0 \) be given. Let \( u_\varepsilon \) be a solution of (1) on \( \Omega \) such that \( E_\varepsilon(u_\varepsilon) \leq M_0 \), and \( x_0 \in \Omega \) and \( \varrho > \varepsilon > 0 \) such that \( D^2(x_0, \varrho) \subset \Omega \). Then, we have
\[
\int_{D^2(x_0, \frac{\varrho}{\varepsilon})} e_\varepsilon(u_\varepsilon)dx \leq K_V(\text{dist}(x_0, \partial\Omega)) \left[ \int_{D^2(x_0, \frac{3\varrho}{5})} \frac{V(u_\varepsilon)}{\varepsilon} dx + \frac{\varepsilon}{\varrho} \int_{D^2(x_0, \varrho) \setminus D^2(x_0, \frac{\varrho}{\varepsilon})} e_\varepsilon(u_\varepsilon)dx \right],
\]
where the constant \( K_V(\text{dist}(x_0, \partial\Omega)) \) depends only on \( V, M_0 \) and \( \text{dist}(x_0, \partial\Omega) \).

Proof. Since \( D^2(x_0, \varrho) \subset \Omega \), we have \( \text{dist} \left( D^2(x_0, \frac{4\varrho}{5}), \partial\Omega \right) \geq \frac{\varrho}{5} \). It therefore follows from Lemma 3.1 that
\[
\| u \|_{L^\infty(D^2(x_0, \frac{4\varrho}{5})))} \leq L_0 \equiv \frac{20C_{unf}M_0}{\text{dist}(x_0, \partial\Omega)} + 2\sup\{\| \sigma \|^2, \sigma \in \Sigma\}.
\]
The conclusion then follows directly from (4.43) with the choice \( K_V(\text{dist}(x_0, \partial\Omega)) = K_{pot}(L_0) \).
4.4 Bounds energy by integrals on external domains

Our next result paves the way for the proof of Theorem 8. As there, we consider a open subset $U$ of $\Omega$ and define $U_\delta$ and $V_\delta$ according to (63).

**Proposition 4.6.** Let $u_\varepsilon$ be a solution of (1) on $\Omega$, $U$ be an open bounded subset of $\Omega$ and $1 > \delta > \varepsilon > 1 > 0$ be given such that $U_\delta \subset \Omega$. Assume that

$$\int_{V_\delta} e_\varepsilon(u_\varepsilon) \, dx \leq K_{\text{ext}}(U_\delta, \varepsilon) \tag{4.44}$$

where $K_{\text{ext}}(U, \delta) > 0$ denotes some constant depending possibly on $U$ and $\delta$. Then, we have the bound, for some constant $C_{\text{ext}}(U, \delta)$ depending possibly on $U$ and $\delta$

$$\int_{U_\delta} e_\varepsilon(u_\varepsilon) \, dx \leq C_{\text{ext}}(U, \delta) \left( \int_{V_\delta} e_\varepsilon(u_\varepsilon) \, dx \right) \tag{4.45}$$

**Proof.** The proof combines Proposition 4.4, Proposition 3.5 with a standard covering by disks. We first bound the potential on the set $U_\delta$ thanks to Proposition 3.5, which yields

$$\frac{1}{\varepsilon} \int_{U_\frac{\delta}{2}} V(u_\varepsilon) \, dx \leq C(U, \varepsilon) \int_{V_\delta} e_\varepsilon(u_\varepsilon) \, dx \leq C(U, \delta) K_{\text{ext}}(U, \delta). \tag{4.46}$$

In inequality (4.46), we have assumed that the bound (4.44) is fulfilled for some constant $K_{\text{ext}}(U, \delta)$, which we choose now as

$$K_{\text{ext}}(U, \delta) = \frac{K_{\text{pot}}(M_0) \delta}{8C(U, \delta)}. \tag{4.47}$$

Inequality (4.46) then yields

$$\frac{1}{\varepsilon} \int_{U_\frac{\delta}{2}} V(u_\varepsilon) \, dx \leq \frac{\delta}{8} K_{\text{pot}}(M_0). \tag{4.48}$$

This bound will allow us to apply inequality (4.41) on disks of radius $\frac{\delta}{8}$ covering $U_\frac{\delta}{4}$. In this direction, we claim that there exists a finite collections of disks $\left\{ D^2 \left( x_i, \frac{\delta}{8} \right) \right\}_{i \in I}$ such that

$$U_\frac{\delta}{4} \subset \bigcup_{i \in I} D^2 \left( x_i, \frac{\delta}{8} \right) \text{ and } x_i \in \overline{U_\frac{\delta}{4}}, \text{ for any } i \in I. \tag{4.49}$$

Indeed, such a collections may be obtained invoking the collection of disks $\left\{ D^2 \left( x, \frac{\delta}{8} \right) \right\}$ with $x \in \overline{U_\frac{\delta}{4}}$ and then extracting a finite subcover thanks to Lebesgue’s Theorem. Notice that we also have

$$\bigcup_{i \in I} D^2 \left( x_i, \frac{\delta}{4} \right) \subset U_\frac{\delta}{4}. \tag{4.50}$$
On each of the disks $\mathbb{D}^2(x_i, \frac{\delta}{4})$, we have, thanks to (4.48)
\[ \frac{1}{\varepsilon} \int_{\mathbb{D}^2(x_i, \frac{\delta}{4})} V(u_\varepsilon) dx \leq \frac{\delta}{8} K_{\text{pot}}(M_0), \]
so that we may apply the scaled version (4.43) of Proposition 4.4 on the disk $\mathbb{D}^2(x_i, \frac{\delta}{4})$: This yields the estimate
\[ \int_{\mathbb{D}^2(x_i, \frac{\delta}{4})} e_\varepsilon(u_\varepsilon)(x) dx \leq C_{\text{pot}} \left[ \int_{\mathbb{D}^2(x_i, \frac{\delta}{4})} \frac{V(u_\varepsilon)}{\varepsilon} dx + \frac{\varepsilon}{\delta} \int_{\mathbb{D}^2(x_i, \frac{\delta}{4})} e_\varepsilon(u_\varepsilon) dx \right]. \]

Adding these relations for $i \in I$ and invoking relations (4.49) and (4.50) we are led to
\[ \int_{U_{\frac{\delta}{4}}} e_\varepsilon(u_\varepsilon)(x) dx \leq \sharp(I) C_{\text{pot}} \left[ \int_{U_{\frac{\delta}{2}}} \frac{V(u_\varepsilon)}{\varepsilon} dx + \frac{\varepsilon}{\delta} \int_{U_{\frac{\delta}{2}}} e_\varepsilon(u_\varepsilon) dx \right]. \]

Invoking again the first inequality in (4.46) we may bound the potential term on the right hand side, so that we obtain
\[ \int_{U_{\frac{\delta}{4}}} e_\varepsilon(u_\varepsilon)(x) dx \leq \sharp(I) C_{\text{pot}} \left[ C(U, \delta) \int_{V_\delta} e_\varepsilon(u_\varepsilon) dx + \frac{\varepsilon}{\delta} \int_{U_{\frac{\delta}{2}}} e_\varepsilon(u_\varepsilon) dx \right]. \]

This inequality finally leads to the conclusion (4.45).

5 Proof of the energy decreasing property

The purpose of this section is to provide a proof to Proposition 1, which is a major step in the proofs of the main theorems of the paper.

5.1 An improved estimate of the energy on level sets

In this paragraph, we consider again for given $0 < \varepsilon \leq 1$ a solution $u_\varepsilon : \mathbb{D}^2 \to \mathbb{R}^k$ to (1) and specify the result of Proposition 4.2 for special choices of $\kappa$ and $\varrho_\varepsilon$. More precisely, we choose
\[ \varrho_\varepsilon = r_\varepsilon \quad \text{and} \quad \kappa_\varepsilon = C_{bd} \sqrt{E_\varepsilon(u_\varepsilon)}, \]
where $\frac{3}{4} \leq r_\varepsilon \leq 1$ is the radius introduced in subsection 2.4 Lemma 2.5 for the choice $r_1 = 1, r_0 = \frac{3}{4}$ and where the constant $K_{bd}$ is choosen as
\[ C_{bd} = \sup \left\{ 2 C_{\text{unf}}, \sqrt{\frac{1}{16 \sqrt{\lambda_0}}} \right\}, \]
$C_{\text{unf}}$ being the constant provided in Lemma 2.4. With this choice, we have
\[ \kappa_\varepsilon^2 \geq \frac{1}{16 \sqrt{\lambda_0}} E_\varepsilon(u_\varepsilon), \]
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so that the bound \((2.31)\) is satisfied for \(\kappa = \kappa_\varepsilon\). We notice, in view of \((2.23)\), the definition \((5.1)\) of \(\kappa_\varepsilon\) and the definition \((5.2)\) of \(C_{\text{bd}}\), that there exists some element \(\sigma_{\text{main}} \in \Sigma\) such that

\[
|u(\ell) - \sigma_{\text{main}}| \leq 2C_{\text{unf}} \sqrt{E_\varepsilon(u_\varepsilon, D^2)} \leq \kappa_\varepsilon, \quad \text{for all } \ell \in \hat{S}^1(\tilde{r}_\varepsilon),
\]

so that condition \((4.23)\) is automatically fulfilled in view of our choice our choices of parameters. The main result of this subsection is the following:

**Proposition 5.1.** Assume that \(0 < \varepsilon \leq 1\) and that \(u_\varepsilon\) is a solution of \((1)\) on \(D^2\). There exists a constant \(K_\Upsilon > 0\) such

\[
\int_{\Upsilon_\varepsilon(\tau_\varepsilon, \kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)dx \leq K_\Upsilon \left[ \left( \int_{D^2} e_\varepsilon(u_\varepsilon)(x)dx \right)^{\frac{3}{2}} + \varepsilon \int_{D^2} e_\varepsilon(u_\varepsilon)(x)dx \right].
\]

**Proof.** Notice first that the result \((5.5)\) is non trivial only when the energy is small, otherwise it is obvious, for a suitable choice of constant. We introduce therefore the smallness condition on the energy

\[
\int_{D^2} e_\varepsilon(u_\varepsilon)dx \leq \nu_1 \equiv \frac{K_\Upsilon^2}{4C_{\text{bd}}^2},
\]

and distinguish two cases.

**Case 1:** Inequality \((5.6)\) does not hold, that is \(E(u_\varepsilon) \geq \nu_1\). In this case \((5.5)\) is straightforwardly satisfied, provided we choose the constant \(K_\Upsilon\) sufficiently large so that

\[
K_\Upsilon \geq \frac{1}{\sqrt{\nu_1}}.
\]

Indeed, we obtain, since \((5.6)\) is not satisfied,

\[
K_\Upsilon \left( \int_{D^2} e_\varepsilon(u_\varepsilon)(x)dx \right)^{\frac{3}{2}} \geq K_\Upsilon (\nu_1)^{\frac{3}{2}} \int_{D^2} e_\varepsilon(u_\varepsilon)(x)dx \geq \int_{D^2} e_\varepsilon(u_\varepsilon)(x)dx \geq \int_{\Upsilon_\varepsilon(\tau_\varepsilon, \kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)dx.
\]

**Case 2:** Inequality \((5.6)\) does hold. Since assumption \((4.23)\) is satisfied for \(\varrho_\varepsilon = \tau_\varepsilon\) thanks to \((5.4)\), we are in position to apply Proposition \(4.2\). It yields

\[
\int_{\Upsilon_\varepsilon(\tau_\varepsilon, \kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)dx \leq C_\Upsilon(\nu_1) \left[ \kappa_\varepsilon \int_{D^2(\tau_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon} dx + \varepsilon \int_{\partial D^2(\tau_\varepsilon)} e_\varepsilon(u_\varepsilon)dl \right].
\]

We choose the constant \(K_\Upsilon\) so that

\[
K_\Upsilon \geq \sup\{C_\Upsilon(\nu_1)C_{\text{bd}}, \frac{1}{\sqrt{\nu_1}}, 1\}.
\]

Inequality \((5.5)\) then follows directly from \((5.8)\) in view of the definition \(\kappa_\varepsilon = C_{\text{bd}} \sqrt{E_\varepsilon(u_\varepsilon)}\) of \(\kappa_\varepsilon\) and the fact that, by definition of the energy, we have the point-wise inequality

\[
\frac{V(u_\varepsilon)}{\varepsilon} \leq e_\varepsilon(u_\varepsilon).
\]
At this stage, we have already derived an inequality very close to (57), namely inequality (5.5) of Proposition 5.1. However it holds only on a domain where points on which the value of \(|u_\varepsilon - \sigma_i|\) is large in some suitable sense have been removed. To go further and obtain an estimate on a full disk, we invoke improved estimates on the potential \(V\) which are derived in the next subsection.

5.2 Improved potential estimates

**Proposition 5.2.** Assume that \(0 < \varepsilon \leq 1\) and that \(u_\varepsilon\) is a solution of (1) on \(D^2\). There exists a constant \(C_V > 0\) such that

\[
\frac{1}{\varepsilon} \int_{D^2(\frac{5}{8})} V(u_\varepsilon) dx \leq C_V \left[ \left( \int_{D^2} e_\varepsilon(u_\varepsilon)(x) dx \right)^{\frac{3}{2}} + \varepsilon \int_{D^2} e_\varepsilon(u_\varepsilon)(x) dx \right].
\]

(5.9)

**Proof.** The proof combines the energy estimates of Proposition 5.1, the avering argument of Lemma 2.7 together with the Pohozaev type potential estimate provided in Proposition 3.4.

We first apply Proposition 2.7 with the choice \(\varrho = \tau_\varepsilon\) and \(\kappa = \kappa_\varepsilon\), where \(\tau_\varepsilon\) and \(\kappa_\varepsilon\) have been defined in (5.1). Since in view of definitions (5.1), (5.2) and (5.3) the lower-bound (2.31) is verified for \(\kappa_\varepsilon\), we may invoke Proposition 2.7 to assert that there exists some radius \(\tau_\varepsilon \in [r_\varepsilon, \varrho]\) such that

\[
\int_{S^1(\tau_\varepsilon)} e_\varepsilon(u_\varepsilon) d\ell \leq \frac{1}{\varrho_\varepsilon - \frac{25}{16}} E_\varepsilon(u_\varepsilon, \Upsilon(\tau_\varepsilon, \kappa_\varepsilon)) \leq 16 E_\varepsilon(u, \Upsilon_\varepsilon(\tilde{\tau}_\varepsilon, \kappa_\varepsilon)).
\]

(5.10)

Invoking inequality (5.5) of Proposition 5.1 are led to

\[
\int_{S^1(\tau_\varepsilon)} e_\varepsilon(u_\varepsilon) d\ell \leq 16K \left[ \left( \int_{D^2} e_\varepsilon(u_\varepsilon)(x) dx \right)^{\frac{3}{2}} + \varepsilon \int_{D^2} e_\varepsilon(u_\varepsilon)(x) dx \right].
\]

(5.10)

On the other hand, thanks to Proposition 3.4 we have

\[
\frac{1}{\varepsilon} \int_{D^2(\tau_\varepsilon)} V(u_\varepsilon) dx \leq 2\tau_\varepsilon \int_{S^1(\tau_\varepsilon)} e_\varepsilon(u_\varepsilon) d\ell \leq 2 \int_{S^1(\tau_\varepsilon)} e_\varepsilon(u_\varepsilon) d\ell.
\]

(5.11)

Combining (5.10) and (5.11) with the fact that \(\tau_\varepsilon \geq \frac{5}{8}\), we derive (5.9) with

\[C_V = 32K_T.\]

The proof is hence complete.

5.3 Proof of Proposition 1 completed

We introduce first a new radius \(\tilde{\tau}_\varepsilon \in \left[\frac{9}{16}, \frac{5}{8}\right]\) corresponding to the intermediate radius defined in Lemma 2.5 for the choice \(r_1 = \frac{9}{16}, r_0 = \frac{7}{8}\) so that it satisfies

\[
\int_{S^1(\tilde{\tau}_\varepsilon)} e_\varepsilon(u) d\ell \leq 16 E_\varepsilon(u, D^2(\frac{5}{8})).
\]

(5.12)
It follows as above from Lemma 2.4 that there exists some element \( \sigma_{bis} \in \Sigma \), possibly different from \( \sigma_{main} \) defined in (5.4), such that

\[
|u(\ell) - \sigma_{bis}| \leq 4C_{unf}\sqrt{E_{\varepsilon}(u_\varepsilon, D^2(\tilde{\varepsilon}))}, \text{ for all } \ell \in S^1(\tilde{\varepsilon}). \tag{5.13}
\]

In order to apply Proposition 4.7, we introduce once more a smallness condition on the energy, namely

\[
E_{\varepsilon}(u_\varepsilon) \leq \frac{\eta_2}{256C_{unf}^2}. \tag{5.14}
\]

We then distinguish two cases:

**Case 1:** The smallness condition (5.14) holds. In this case, we have, in view of (5.13)

\[
|u(\ell) - \sigma_{bis}| \leq 4C_{unf}\sqrt{\eta_2} = \frac{\mu_0}{4}, \text{ for all } \ell \in S^1(\tilde{\varepsilon}),
\]

so that condition (4.23) holds for \( \varrho_\varepsilon = \tilde{\varepsilon} \) (with \( \sigma_{main} \) replaced by \( \sigma_{bis} \)). We are therefore in position to apply Lemma 4.7 on the disk \( D^2(\tilde{\varepsilon}) \), which yields

\[
\int_{D^2(\tilde{\varepsilon})} e_\varepsilon(u_\varepsilon)(x)dx \leq C_{pot}(L_M) \left[ \int_{D^2(\tilde{\varepsilon})} \frac{V(u_\varepsilon)}{\varepsilon}dx + \varepsilon \int_{\partial D^2(\tilde{\varepsilon})} e_\varepsilon(u_\varepsilon)dl \right], \tag{5.15}
\]

where \( L_M \) is defined in (4.40), so that \( \|u_\varepsilon\|_{L^\infty(D^2(\frac{1}{2}))} \leq L_M \). Invoking Proposition 5.2 and inequality (5.12) we are hence led to

\[
\int_{D^2(\tilde{\varepsilon})} e_\varepsilon(u_\varepsilon)(x)dx \leq C_{pot}(L_M) C_V \left( \int_{D^2} e_\varepsilon(u_\varepsilon)(x)dx \right)^{\frac{3}{4}} + C_{pot}(L_M) (C_V + 16) \varepsilon \int_{D^2} e_\varepsilon(u_\varepsilon)(x)dx,
\]

which yields (57), for a suitable choice of the constant \( C_{dec} \).

**Case 2:** The smallness condition (5.14) does not hold. In this case, inequality (57) is straightforwardly fulfilled, provided we choose

\[
C_{dec} \geq \eta_2^{-\frac{3}{2}}.
\]

The proof is hence complete in both cases.

6 Proof of the clearing-out theorem

The purpose of this section is to provide the proof of the clearing-out property stated in Theorem 6, a major step being the uniform bound (55). We first introduce a very weak form of the clearing-out theorem.
6.1 A very weak form of the clearing-out

The following result is classical in the field (see e.g. [25][11]).

**Proposition 6.1.** Let $u_\varepsilon$ be a solution of (1) on $\mathbb{D}^2$ with $0 < \varepsilon \leq 4$. There exists a constant $\eta_3 > 0$ such that if $E_\varepsilon(u) \leq \eta_3 \varepsilon$, then we have, for some $\sigma \in \Sigma$, the bound

$$|u_\varepsilon(0) - \sigma| \leq \frac{\mu_0}{2}.$$ 

**Remark 6.1.** In the scalar case, Lemma [10.3] combined with the monotonicity formula for the energy directly yields the proof of Theorem 6.

**Proof.** Assume that the bound $E_\varepsilon(u) \leq \eta_3 \varepsilon$ holds, for some constant $\eta_2$ to be determined later. Imposing first $\eta_2 \leq 1$, it follows from Proposition 3.1 and Proposition 3.2 that there exists a constant $C_0 > 0$ depending only on $V$ such that

$$|\nabla u_\varepsilon(x)| \leq \frac{C_0}{\varepsilon} \text{ and } |u_\varepsilon(x)| \leq C_0, \text{ for } x \in \mathbb{D}^2\left(\frac{7}{8}\right).$$

Since the potential $V$ is smooth, and hence its gradient is bounded on the disc $\mathbb{B}^k(C_0)$, we deduce that there exists a constant $C_1$ such that

$$|\nabla V(u_\varepsilon)(x)| \leq \frac{C_1}{\varepsilon} \text{ for } x \in \mathbb{D}^2\left(\frac{7}{8}\right). \quad (6.1)$$

Since $E_\varepsilon(u_\varepsilon) \leq \eta_2 \varepsilon$, we deduce from the definition of the energy that

$$\int_{\mathbb{D}^2\left(\frac{3}{4}\right)} V(u_\varepsilon(x))dx \leq \int_{\mathbb{D}^2} V(u_\varepsilon(x))dx \leq \eta_3 \varepsilon^2. \quad (6.2)$$

We claim that

$$V(u_\varepsilon(x)) \leq \alpha_0 \text{ for any } x \in \mathbb{D}^2\left(\frac{3}{4}\right). \quad (6.3)$$

Indeed, assume by contradiction that there exists some $x_0 \in \mathbb{D}^2\left(\frac{3}{4}\right)$ such that $V(u(x_0)) > \alpha$. Invoking the gradient bound (6.1), we deduce that

$$V(u_\varepsilon(x)) \geq \frac{\alpha_0}{2} \text{ for } x \in \mathbb{D}^2\left(x_0, \frac{\alpha_0 \varepsilon}{2C_1}\right).$$

Without loss of generality, we may assume that $C_1$ is chosen sufficiently large so that $\frac{4\alpha_0}{2C_1} \leq \frac{1}{8}$ and hence $\mathbb{D}^2\left(x_0, \frac{\alpha_0 \varepsilon}{2C_1}\right) \subset \mathbb{D}^2\left(\frac{7}{8}\right)$. Integrating (6.3), on the disk $\mathbb{D}^2\left(x_0, \frac{\alpha_0 \varepsilon}{2C_1}\right)$, we are led to

$$\int_{\mathbb{D}^2\left(\frac{7}{8}\right)} V(u_\varepsilon(x))dx \geq \int_{\mathbb{D}^2\left(x_0, \frac{\alpha_0 \varepsilon}{2C_1}\right)} V(u_\varepsilon(x))dx \geq \pi \frac{\alpha_0^3}{8C_1^2} \varepsilon^2.$$ 

This yields a contradiction with (6.2), provided we impose the upper bound on $\eta_3$ given by

$$\eta_3 \leq \pi \frac{\alpha_0^3}{8C_1^2} \varepsilon^2, \quad (6.4)$$

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and established the claim \([6.3]\). To complete the proof, we may invoke Lemma \([2.1]\) and the continuity of the map \(u_\varepsilon\) to asserts that there exists some \(\sigma \in \Sigma\) such that
\[
|u_\varepsilon(x) - \sigma| \leq \mu_0 \quad \text{for any } x \in \mathbb{D}^2(\frac{3}{4}). \tag{6.5}
\]
This yields almost estimate \((55)\), except that we still have to replace \(\mu_0\) by \(\mu_0/2\) on the right-hand side of \((6.5)\). In order to improve the constant, we merely rely on the same type of argument. Arguing as above by contradiction, let us assume that there exists a point \(x_1 \in \mathbb{D}^3(3/4)\) such that
\[
|u_\varepsilon(x_1) - \sigma| > \frac{\mu_0}{2} \quad \text{and hence } V(u_\varepsilon(x_1)) > \frac{\lambda_0 \mu_0^2}{16}, \tag{6.6}
\]
the second inequality in \((6.6)\) being a consequence of the second statement in Lemma \([2.1]\). Invoking again the gradient bound \((6.1)\), we deduce that
\[
V(u_\varepsilon(x)) \geq \frac{\lambda_0 \mu_0^2}{32}, \quad \text{for } x \in \mathbb{D}^2\left(x_1, \frac{\lambda_0 \mu_0^2}{32C}\right). \tag{6.7}
\]
Integrating the previous inequality, we obtain
\[
\int_{\mathbb{D}^2(\frac{7}{8})} V(u_\varepsilon(x))dx \geq \int_{\mathbb{D}^2(x_0, \frac{3\sigma}{2C})} V(u_\varepsilon(x))dx \geq \pi \frac{\lambda_0^2 \mu_0}{32768 C_1} \varepsilon^2,
\]
a contradiction with \((6.2)\), provided we impose that \(\eta_2\) is sufficiently small. \(\Box\)

### 6.2 Confinement near a well of the potential

Our next result is the main step in the proof of Theorem \([4]\). It shows that, if the energy is sufficiently small near, that \(0\) takes values inside a well of the potential.

**Proposition 6.2.** Let \(0 < \varepsilon \leq 1\) and \(u_\varepsilon\) be a solution of \((1)\) on \(\mathbb{D}^2\). There exists a constant \(\eta_2 > 0\) such that if
\[
E_\varepsilon(u_\varepsilon, \mathbb{D}^2) \leq \eta_2 \tag{6.7}
\]
then, we have, for some \(\sigma \in \Sigma\), the bound \(|u_\varepsilon(0) - \sigma| \leq \frac{\mu_0}{2}\).

The proof of the Proposition \([6.1]\) relies on inequality \((58)\) of Proposition \([1]\) a standard scaling argument combined with an iteration procedure.

**Step 1: A scaled version of inequality \((58)\).** Set, for \(0 < r \leq 1\), \(E_\varepsilon(r) = E_\varepsilon(u_\varepsilon, \mathbb{D}^2(r))\), and assume that
\[
E_\varepsilon(r) \geq \frac{\varepsilon^2}{r}. \tag{6.8}
\]
Then, we have
\[
E_\varepsilon\left(\frac{r}{2}\right) \leq 2C_{\text{dec}} \frac{E_\varepsilon(r)^{3/2}}{\sqrt{r}}, \quad \text{provided } r \geq \varepsilon. \tag{6.9}
\]
Indeed, scaling inequality \((58)\), we obtain
\[
E_\varepsilon\left(\frac{r}{2}\right) \leq C_{\text{dec}} \left[\frac{1}{\sqrt{r}} E_\varepsilon(r)^{3/2} + \frac{\varepsilon}{r} E_\varepsilon(r)\right], \quad \text{provided } r \geq \varepsilon, \tag{6.10}
\]
which yields (6.9).

**Step 2: The iteration procedure.** We consider the sequence \((r_n)_{n \in \mathbb{N}}\) of decreasing radii \(r_n\) defined as \(r_n = \frac{1}{2^n}\), for \(n \in \mathbb{N}\), and set \(E_n^\varepsilon = E_\varepsilon(r_n) = E_\varepsilon(\frac{1}{2^n})\), dropping the superscript in case this induces no ambiguity. We introduce the number

\[
n_\varepsilon = \sup \left\{ n \in \mathbb{N}, \text{ such that } E_n^\varepsilon \geq 2^n \varepsilon^2 \text{ and } r_n = \frac{1}{2^n} \geq \varepsilon \right\}. \tag{6.11}
\]

If we impose that \(\eta_2 \leq 1\), then condition (6.7) implies that \(E_\varepsilon(u_\varepsilon) \leq 1\), so that 0 belongs to the set of the r.h.s of (6.11), which is hence not empty. On the other hand, since \(2^n\) tends to infinity as \(n\) tends to infinity, and since the sequence \((E_n)_{n \in \mathbb{N}}\) is non-increasing, hence bounded by \(E_0\), the set of the r.h.s of (6.11) is a finite set of sequential number and the number \(n_\varepsilon\) is hence a well-defined integer. In view of the definition of \(n_\varepsilon\), inequality (6.8) is straightforwardly satisfied for every \(r_n < r_{n_\varepsilon}\). We deduce therefore from Step 1 and the definition of \(r_n\) that we have the inequality

\[
E_{n+1} \leq 2\sqrt{2^n} C_\text{dec} (E_n)^{\frac{3}{2}}, \text{ for } n = 0, \ldots n_\varepsilon. \tag{6.12}
\]

We introduce, for \(n \in \mathbb{N}\), the number \(A_n = -\log E_n\). Inequality (6.12) for \(E_n\) is turned into the inequality for \(A_n\) given by

\[
A_{n+1} \geq \frac{3}{2} A_n - \frac{(\log 2)}{2} n - \log(2C_\text{dec}), \text{ for } n = 0, \ldots n_\varepsilon. \tag{6.13}
\]

In order to study the sequence \((A_n)_{n \in \mathbb{N}}\), we will invoke the next elementary result.

**Lemma 6.1.** Let \(n_\ast \in \mathbb{N}^\ast\), \((a_n)_{n \in \mathbb{N}}\) and \((f_n)_{n \in \mathbb{N}}\) be two sequences of numbers such that

\[
a_{n+1} \geq c_0 a_n - f_n, \text{ for all } n \in \mathbb{N}, n \leq n_\ast, \tag{6.14}
\]

where \(c_0 > 1\) represents a given constant. Then we have the inequality,

\[
a_n \geq c_0^n \left( a_0 - \sum_{k=0}^{n} \frac{1}{c_{k+1}^{n+k+1}} f_k \right) \text{ for } n \in \mathbb{N}^\ast n \leq n_\ast. \tag{6.15}
\]

We postpone the proof of Lemma 6.1 and complete first the proof of Proposition 6.1.

**Step 3: Choice of \(\eta_2\) and energy decay estimates.** We apply Lemma 6.1 to the sequences \((A_n)_{n \in \mathbb{N}}\) and \((f_n)_{n \in \mathbb{N}}\) with \(f_n = \frac{(\log 2)}{2} n + \log(2C_\text{dec})\), for any \(n \in \mathbb{N}\), so that inequality (6.14) is satisfied with \(c_0 = \frac{3}{2}\) and \(n_\ast = n_\varepsilon\). Inequality (6.15) then yields, for \(n = 0, \ldots n_\varepsilon\),

\[
A_n = -\log E_n \geq \left( \frac{3}{2} \right)^n \left[ \log \left( \frac{1}{E_\varepsilon(u_\varepsilon)} \right) - \gamma_0 \right] \geq \left( \frac{3}{2} \right)^n \left[ -\log \eta_2 - \gamma_0 \right]. \tag{6.16}
\]

Here we have used, for the second inequality, assumption (6.7) and we have set

\[
\gamma_0 = \sum_{k=0}^{\infty} \left( \frac{2}{3} \right)^{k+1} \left( \frac{(\log 2)}{2} k + \log(2C_\text{dec}) \right) < +\infty.
\]
We impose a first constraint on the constant $\eta_2$, namely we impose

$$\eta_2 \leq \exp\left[-(1 + \gamma_0)\right] \text{ so that } - \log \eta_2 \geq 1 + \gamma_0, \quad (6.17)$$

It follows that inequality (6.16) yields, provided inequality (6.7) holds,

$$E_n \leq \exp\left[-\left(\frac{3}{2}\right)^n\right] \text{ for } n = 0, \ldots n_\varepsilon - 1. \quad (6.18)$$

**Step 4: Estimating $n_\varepsilon$ and $r_{n_\varepsilon}$.** It follows from (6.18) and the definition of $n_\varepsilon$ that

$$\varepsilon^2 = \exp(2 \log \varepsilon) \leq r_n E_n = 2^{-n} E_n \leq \exp\left[-\left(\frac{3}{2}\right)^n - n \log 2\right] \text{ for } n = 0, \ldots n_\varepsilon,$$

so that we are led to the inequality

$$\left(\frac{3}{2}\right)^{n_\varepsilon} + n_\varepsilon \log 2 \leq 2|\log \varepsilon|,$$

and hence

$$\left(\frac{3}{2}\right)^{n_\varepsilon} \leq 2|\log \varepsilon|.$$ 

Taking the logarithm of both sides, we obtain the bound for $n_\varepsilon$

$$n_\varepsilon \leq \frac{\log(2|\log \varepsilon|)}{\log 3 - \log 2}. \quad (6.19)$$

It yields a lower bound for $r_{n_\varepsilon}$, namely

$$r_{n_\varepsilon} = 2^{-n_\varepsilon} = \exp(-\log 2) n_\varepsilon \geq \exp\left(-\log(2|\log \varepsilon|) \frac{\log 2}{\log 3 - \log 2}\right) \geq \exp(-\gamma_1 \log(2|\log \varepsilon|)) \geq (2|\log \varepsilon|)^{-\gamma_1}. \quad (6.20)$$

Here we have set

$$\gamma_1 = \frac{\log 2}{\log 3 - \log 2}, \text{ so that } 1 \leq \gamma_1 \leq 2.$$ 

We notice that $(2|\log \varepsilon|)^{-\gamma_1} \ll \varepsilon$, so that there exists some $0 < \varepsilon_1 \leq 1$ such that

$$r_{n_\varepsilon} \geq 2 \varepsilon, \text{ provided } 0 < \varepsilon \leq \varepsilon_1. \quad (6.20)$$

Going back to the definition of $n_\varepsilon$, we deduce from (6.20) and (6.19) that

$$E_{n_\varepsilon + 1}^\varepsilon \leq \varepsilon^2 r_{n_\varepsilon + 1}^{-1} = 2^{n_\varepsilon + 1} \varepsilon^2 \leq 8|\log \varepsilon|^{\gamma_1} \varepsilon^2, \text{ if } 0 < \varepsilon \leq \varepsilon_1. \quad (6.21)$$

**Step 5: change of scale.** We consider the scaled map $\tilde{u}_\varepsilon$ and the scaled parameter $\tilde{\varepsilon} \geq \varepsilon$ defined by

$$\tilde{u}_\varepsilon(x) = u_\varepsilon(r_{n_\varepsilon + 1}x), \text{ for } x \in \mathbb{D}^2, \text{ and the scaled parameter } \tilde{\varepsilon} = r_{n_\varepsilon + 1}^{-1} \varepsilon = 2r_{n_\varepsilon}^{-1} \varepsilon.$$
Turning back to (52), we are led to the identity, for the energy
\[ E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon) = r_{n+1}^{-1} E_{\varepsilon}(u_\varepsilon, D^2(r_{n+1})) = r_{n+1}^{-1} E_{\varepsilon_{n+1}}, \]
so that, in view of (6.19) and (6.21), we have
\[
\begin{cases}
E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon) \leq 16 |\log \varepsilon|^2 \gamma_1 \varepsilon^2, & \text{if } \varepsilon \leq \varepsilon_1, \\
E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon) = E_{\varepsilon}(u_\varepsilon, D^2(r_{n+1})) \leq 8 |\log \varepsilon| \gamma_1 \varepsilon, & \text{if } \varepsilon \leq \varepsilon_1.
\end{cases}
\] (6.22)

Since the map \( s \to |\log s| \gamma_1 s \) is decreasing on the interval \((0, e^{-\gamma_1})\), assuming that the constant \( \eta_2 \) is chosen to be sufficiently small, there exists a unique number \( \varepsilon_2 \in (0, e^{-\gamma_1}) \), such that
\[ 8 |\log \varepsilon| \gamma_1 \varepsilon \leq \eta_2 \text{ and } \varepsilon_2 \leq \varepsilon_1. \] (6.23)

**Step 6: Proof of Proposition 6.1 completed.** We conclude invoking the weak clearing-out property stated in Proposition 6.1. For that purpose, we distinguish two cases:

**Case 1:** \( 0 < \varepsilon \leq \varepsilon_2 \). It follows in this case from the definition (6.23) of \( \varepsilon_2 \) and the first inequality in (6.22) that
\[
\begin{cases}
E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon) \leq \eta_2 \tilde{\varepsilon} \text{ and } \\
\tilde{\varepsilon} \leq 1.
\end{cases}
\] (6.24)

In view of (6.24), we are hence in position to apply Proposition 6.1 to the map \( \tilde{u}_\varepsilon \) with parameter \( \tilde{\varepsilon} \): Hence there exists some point \( \sigma \in \Sigma \) such that
\[ |\tilde{u}_\varepsilon(0) - \sigma| \leq \frac{\mu_0}{2}. \]

since \( u_\varepsilon(0) = \tilde{u}_\varepsilon(0) \) the conclusion of Proposition 6.1 follows.

**Case 2:** \( 1 \geq \varepsilon > \varepsilon_2 \). Here we apply directly Proposition 6.1 to \( u_\varepsilon \). Besides (6.17) we impose the additional condition \( \eta_2 \leq \eta_3 \varepsilon_2 \) on \( \eta_3 \), so that we finally may choose the constant \( \eta_2 \) as
\[ \eta_2 = \inf\{\eta_3 \varepsilon_2, \exp\{-(1 + \gamma_1), 1\}\}. \] (6.25)

With this choice, we have, in view of assumption (6.17), for \( \varepsilon \geq \varepsilon_2 \),
\[ E_{\varepsilon}(u_\varepsilon) \leq \eta_1 \leq \eta_2 \varepsilon_2 \leq \eta_2 \varepsilon. \]

Hence \( u_\varepsilon \) fulfills the assumptions of Proposition 6.1 so that its conclusion yields again the existence of an element \( \sigma \in \Sigma \) such that \( |u_\varepsilon(0) - \sigma| \leq \frac{\mu_0}{2} \).

In both cases, we have hence established the conclusion of Proposition 6.1 so that the proof is complete.

In the course of the proof, we have used Lemma 6.1 which has not been proved yet.

**Proof of Lemma 6.1.** We introduce, inspired by the method of variation of constant, the sequence \((b_n)_{n \in \mathbb{N}}\) defined by \( a_n = c_0^0 b_n \), for any \( n \in \mathbb{N} \). Substituting into (6.14), we obtain
\[ c_0^{k+1} b_{k+1} \geq c_0^{k+1} b_k - f_k, \text{ for all } k \in \{0, \ldots, n_\star\}, \]
so that
\[ b_{k+1} - b_k \geq -\frac{1}{c_0} f_k, \text{ for all } k \in \{0, \ldots, n\}. \]
Let \( n \in \mathbb{N} \), \( n \leq n \). Summing these relations for \( k = 0 \) to \( k = n - 1 \), we are led to
\[ b_n \geq b_0 - \sum_{k=0}^{n} \frac{1}{c_0^{k+1}} f_k = a_0 - \sum_{k=0}^{n} \frac{1}{c_0^{k+1}} f_k, \]
which, in view of the definition of \( b_n \), yields the desired conclusion (6.15). \qed

A direct consequence of Proposition 6.1 is the following:

**Corollary 6.1.** Let \( 0 < \varepsilon \leq 1 \) and \( u_\varepsilon \) be a solution of (1). Set \( \eta_1 = \inf \left\{ \frac{1}{8} \eta_2, \frac{1}{8} \eta_3 \right\} \) and assume that
\[ E_\varepsilon(u_\varepsilon, D^2) \leq 2\eta_1. \tag{6.26} \]
then, there exists some \( \sigma \in \Sigma \) such that
\[ |u_\varepsilon(x) - \sigma| \leq \frac{\mu_0}{2}, \text{ for any } x \in D^2 \left( \frac{3}{4} \right). \tag{6.27} \]

**Proof.** Let \( x_0 \in D^2(\frac{3}{4}) \) be an arbitrary point. We consider the scaled parameter \( \tilde{\varepsilon} = 4\varepsilon \) and the scaled and translated map \( \tilde{u}_\varepsilon \) defined on \( D^2 \) by
\[ \tilde{u}_\varepsilon(x) = u_\varepsilon \left( x_0 + \frac{1}{4} x \right) \text{ for every } x \in D^2, \]
so that
\[ E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon) = 4E_{\varepsilon} \left( u_\varepsilon, D^2(x_0, \frac{1}{4}) \right) \leq 4E_{\varepsilon}(u_\varepsilon) \leq 8\eta_1 \leq \eta_2, \tag{6.28} \]
where we have used assumption (6.26) and the definition of \( \nu_1 \) for the last inequality. As above, we distinguish two cases.

**Case 1:** \( \varepsilon \leq \frac{1}{4} \). In this case \( \tilde{\varepsilon} \leq 1 \), so that, in view of (6.28), we are in position to apply Proposition 6.1. It yields an element \( \sigma_{x_0} \in \Sigma \), depending possibly on the point \( x_0 \), such that
\[ |\tilde{u}_\varepsilon(0) - \sigma_{x_0}| \leq \frac{\mu_0}{2}. \]
Since \( \tilde{u}_\varepsilon(0) = u_\varepsilon(x_0) \), we conclude that
\[ |u_\varepsilon(x_0) - \sigma_{x_0}| \leq \frac{\mu_0}{2}. \tag{6.29} \]
Since inequality (6.29) holds for any point \( x_0 \in D^2(3/4) \), a continuity argument shows that the point \( \sigma_{x_0} \) does not depend on \( x_0 \), so that the proof of Proposition 6.1 is complete in Case 1.

**Case 2:** \( 1 \geq \varepsilon \geq \frac{1}{4} \). In this case \( 1 \leq \tilde{\varepsilon} \leq 4 \). In view of the definition of \( \eta_1 \), we have \( 16\eta_1 \leq \eta_3 \). It then follows from assumption (6.26) that
\[ E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon) = 4E_{\varepsilon} \left( u_\varepsilon, D^2 \left( x_0, \frac{1}{4} \right) \right) \leq 4E_{\varepsilon}(u_\varepsilon) \leq 8\eta_1 \leq \eta_3 \leq \eta_3 \tilde{\varepsilon}. \tag{6.30} \]
Hence, we are once more in position to apply Proposition 6.1 so that there exists an element \( \sigma_{x_0} \in \Sigma \), depending possibly on the point \( x_0 \) such that \( |\tilde{u}_\varepsilon(0) - \sigma_{x_0}| \leq \frac{\mu_0}{2} \). Since \( \tilde{u}_\varepsilon(0) = u_\varepsilon(x_0) \), we conclude that
\[
|u_\varepsilon(x_0) - \sigma_{x_0}| \leq \frac{\mu_0}{2}.
\]
The proof of Corollary 6.1 is hence complete.

\[\Box\]

6.3 Proof of Theorem 6 completed

We have determined so far the value of \( \eta_1 \) in Corollary 6.1, which as matter of fact provides the proof of (55). The only remaining unproved assertion is the energy estimate (56), which we establish next. For that purpose, we notice that the restriction of the map \( u_\varepsilon \) to the smaller disk \( \mathbb{D}^2(3/4) \) takes values into one of the wells, so that the functionals behaves there as a convex functional.

The proof is parallel and actually much easier then our earlier energy estimate. We first invoke Lemma 2.5 with \( r_1 = \frac{3}{4} \) and \( r_0 = \frac{5}{8} \): This yields a radius \( r_\varepsilon \in \left[ \frac{5}{8}, \frac{3}{4} \right] \) and an element \( \sigma \in \Sigma \) such that
\[
\int_{\partial \mathbb{S}(r_\varepsilon)} e_\varepsilon(u_\varepsilon) \leq 8 E_\varepsilon(u, \mathbb{D}^2) \quad \text{and} \quad \int_{\mathbb{S}(r_\varepsilon)} |u_\varepsilon - \sigma||\nabla u_\varepsilon| \leq 16 \sqrt{\lambda_0^{-1} E_\varepsilon(u_\varepsilon, \mathbb{D}^2)}.
\] (6.31)

We multiply the equation by \( (u_\varepsilon - \sigma) \) and integrate on the disk \( \mathbb{D}^2(r_\varepsilon) \) which yields, as in (4.29)
\[
\int_{\mathbb{D}^2(r_\varepsilon)} \varepsilon |\nabla u_\varepsilon|^2 + \varepsilon^{-1} \nabla u V(u_\varepsilon) \cdot (u_\varepsilon - \sigma) = \varepsilon \int_{\mathbb{S}(r_\varepsilon)} \frac{\partial u_\varepsilon}{\partial r} \cdot (u_\varepsilon - \sigma).
\] (6.32)

We deduce from (6.31) that
\[
\int_{\mathbb{S}(r_\varepsilon)} \frac{\partial u_\varepsilon}{\partial r} \cdot (u_\varepsilon - \sigma) \leq \int_{\mathbb{S}(r_\varepsilon)} |u_\varepsilon - \sigma||\nabla u_\varepsilon| \leq 16 \sqrt{\lambda_0^{-1} E_\varepsilon(u_\varepsilon, \mathbb{D}^2)}.
\] (6.33)

We use next the fact that, in view of assertion (55), we have \( |u_\varepsilon - \sigma| \leq \frac{\mu_0}{2} \) on the disk \( \mathbb{D}^2(r_\varepsilon) \). Arguing as in (4.9), we have the point-wise inequality
\[
\varepsilon |\nabla u_\varepsilon|^2 + \varepsilon^{-1} \nabla u V(u_\varepsilon) \cdot (u_\varepsilon - \sigma) \geq \frac{\lambda_0}{2 \lambda_{\max}} e_\varepsilon(u).
\] (6.34)

Combining (6.32) with (6.34) and (6.33), we obtain
\[
\int_{\mathbb{D}^2(r_\varepsilon)} e_\varepsilon(u_\varepsilon) dx \leq 16 \lambda_0^{-\frac{3}{2}} \lambda_{\max} e_\varepsilon(U, \mathbb{D}^2)),
\]
Which yields the energy estimate (56) choosing \( C_{\text{nerg}} = 16 \lambda_0^{-\frac{3}{2}} \lambda_{\max} \). The proof of Theorem 6 is hence complete.

Part III: Analysis of the limiting sets and measures
7 Properties of the concentration set $\mathcal{S}_*$

The purpose of this section is to provide the proof of assertion i) of Theorem 4. We start
with the proof of Theorem 7, the clearing-out property for the measure $\nu_*$. 

7.1 Proof of Theorem 7

Recall that $\nu_*$ is the weak limit of the measure $\nu_{\varepsilon_n}$ defined in (36) by $\nu_{\varepsilon} = \varepsilon(u_{\varepsilon})dx$, so that $E_{\varepsilon}(u, \mathbb{D}^2(x_0, r)) = \nu_{\varepsilon}(\mathbb{D}^2(x_0, r)) = \nu_{\varepsilon}(\mathbb{D}^2(x_0, r))$. Let $x_0 \in \Omega$ and $r > \rho > 0$ be such that $\mathbb{D}^2(x_0, r) \subset \Omega$. Since $\mathbb{D}^2(x_0, \rho)$ is an closed set, we have, by standard properties of weak convergence

$$\limsup_{n \to +\infty} \nu_{\varepsilon_n}(\mathbb{D}^2(x_0, \rho)) \leq \nu_*(\mathbb{D}^2(x_0, \rho)) \leq \nu_*(\mathbb{D}^2(x_0, r))$$

(7.1)

Next, let $x_0$ and $r > 0$ are such that $\nu_*(\mathbb{D}^2(x_0, r)) < \eta_1 r$. It follows from (7.1) that, for given $\rho < r$, there exists some $n(\rho) \in \mathbb{N}$ such that, if $n \geq n(\rho)$,

$$\nu_{\varepsilon_n}(\mathbb{D}^2(x_0, \rho)) \leq \frac{5}{4} \eta_1 r.$$  

(7.2)

We choose $\rho = \frac{8r}{9}$. We obtain, inserting in (7.2),

$$\nu_{\varepsilon_n}(\mathbb{D}^2(x_0, \rho)) = \nu_{\varepsilon_n}(\mathbb{D}^2(x_0, \frac{8r}{9})) \leq \frac{5}{4} \frac{8r}{9} \eta_1 = \frac{10}{9} \eta_1 < 2 \eta_1.$$  

(7.3)

Hence, for sufficiently large $n$, we are in position to apply Theorem 6 so that

$$\nu_{\varepsilon_n} \left( \mathbb{D}^2 \left( x_0, \frac{5\rho}{9} \right) \right) = \nu_{\varepsilon_n} \left( \mathbb{D}^2 \left( x_0, \frac{5\rho}{8} \right) \right) \leq C_{\text{erg}} \frac{\varepsilon_n}{r} E_{\varepsilon_n} \left( u_{\varepsilon_n}, \mathbb{D}^2(x_0, \rho) \right)$$

$$\leq \frac{5}{4} \varepsilon_n \eta_1 \to 0$$

as $n \to +\infty$.

(7.4)

Letting $n \to +\infty$, it follows that $\nu_* \left( \mathbb{D}^2(x_0, \frac{r}{2}) \right) = 0$ and the proof is complete.

7.2 Elementary consequences of the clearing-out property

We present here some simple consequences of the definition of $\mathcal{S}_*$, as well as of the clearing out property stated in Proposition 7.2.

Proposition 7.1. The set $\mathcal{S}_*$ is a closed subset of $\Omega$.

Proof. It suffices to prove that its complement, the set $\mathcal{U}_* = \Omega \setminus \mathcal{S}_*$ is an open subset of $\Omega$. This property is actually a direct consequence of the clearing out property stated in Theorem 7. Indeed let $x_0$ be an arbitrary point in $\mathcal{U}_*$. It follows from the definition (62) of $\mathcal{S}_*$ that $\theta_*(x_0) < \eta_1$, so that there exists some radius $r_0 > 0$ such that $\mathbb{D}^2(x_0, r_0) \subset \Omega$ and such that

$$\nu_*(\mathbb{D}^2(x_0, r_0)) < r_0 \eta_1.$$
In view of Theorem 7, we deduce that $\nu_\star(D^2(x_0, r_0/2)) = 0$. Hence, for any point $x \in D^2(x_0, r_0/4)$, we have $\theta_\star(x) = 0$ and therefore

$$D^2(x_0, r_0/4) \subset \mathcal{U}_\star.$$ 

Hence, $\mathcal{U}_\star$ is an open set.

**Proposition 7.2.** The set $\mathcal{S}_\star$ has finite one-dimensional Hausdorff dimension. There exist a constant $C_H > 0$ depending only on the potential $V$ such that

$$\mathcal{H}^1(\mathcal{S}_\star) \leq C_H M_0.$$ 

**Proof.** The proof relies on a standard covering argument. Let $0 < \rho < \frac{1}{4}$ be given, and consider the set

$$\Omega_\rho = \{x \in \Omega, \text{dist}(x, \partial \Omega) \geq \rho\}.$$ 

Next let $0 < \delta < \rho/4$ be given. Consider the points $x_i$ on a uniform square lattice of $\mathbb{R}^2$, with nearest neighbour at distance $\delta/2$. We obtain for a subfamily $I$ a standard finite covering of $\Omega_\rho$ of size $\delta$, that is such that

$$\Omega_\rho \subseteq \bigcup_{j \in I} D^2(x_j, \delta) \quad \text{and} \quad D^2(x_i, \delta/2) \cap D^2(x_j, \delta/2) = \emptyset \quad \text{for} \ i \neq j \in I.$$ 

We introduce then the set of indices

$$I_\delta = \{i \in I, \text{ such that } D^2(x_i, \delta) \cap \mathcal{S}_\star \neq \emptyset\},$$

so that given any arbitrary index $i \in I_\delta$, there exists a point $y_i \in \mathcal{S}_\star \cap D^2(x_i, \delta)$. It follows from the definition of $\mathcal{S}_\star$ that

$$\theta_\star(y_i) \geq \eta_1.$$ 

(7.5)

We claim that, for any $0 < r \leq \delta$, we have

$$\nu_\star(D^2(y_i, r)) \geq \eta_1 r.$$ 

(7.6)

Indeed, if (7.6) were not true, then we would be in position to apply Theorem 7 which would imply that $\nu_\star(B(y_i, \delta/2)) = 0$, and hence that $\theta_\star(y_i) = 0$, a contradiction which (7.5). Inequality (7.6) is therefore established. Since $D^2(y_i, \delta) \subset D^2(x_i, 2\delta)$, we deduce from (7.6) that

$$\nu_\star(D^2(x_i, 2\delta)) \geq \eta_1 \delta.$$ 

(7.7)

Since the points $x_i$ are on a uniform grid, we notice that a given point $x \in \mathbb{R}^2$ belongs to at most 25 distinct balls of the collection $D^2(x_i, 2\delta)$. We have therefore

$$\sharp(I_\delta) \eta_1 \delta \leq \sum_{i \in I_\delta} \nu_\star(D^2(x_i, 2\delta)) \leq 25 \nu_\star(\Omega) \leq 25 M_0.$$ 

(7.8)

It follows therefore that

$$\sharp(I_\delta) \delta \leq \frac{25 M_0}{\eta_1}. $$
Therefore, letting $\delta \to 0$, we deduce, as a consequence of the definition of the one-dimensional Hausdorff measure that
\[
\mathcal{H}^1(\mathcal{S} \cap \Omega) \leq \liminf_{\delta \to 0} 2^\varepsilon (I_\delta) \delta \leq \frac{50 M_0}{\eta_1}.
\]
We conclude letting $\rho \to 0$, choosing $C_H = \frac{50}{\eta_1}$.

### 7.3 Proof of Theorem 8

Theorem 8 is a direct consequence of Proposition 4.6 which has actually been tailored for this purpose. Indeed, since $\nu_\star(\mathcal{V}_\delta) = 0$, we have the convergence
\[
\int_{\mathcal{V}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n}) \, dx \to 0 \text{ as } n \to +\infty,
\]
so that condition (4.44) is fulfilled for $\varepsilon = \varepsilon_n$ and the map $u_{\varepsilon_n}$, provided $n$ is sufficiently large, say larger than some given value $n_0$. We are therefore in position to conclude, thanks to Proposition 4.6 provided $n \geq n_0$ is sufficiently large, that
\[
\int_{\mathcal{U}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n}) \, dx \leq C_{\text{ext}}(\mathcal{U}, \delta) \left( \int_{\mathcal{V}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n}) \, dx + \varepsilon_n \int_{\mathcal{U}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n}) \, dx \right) \leq C_{\text{ext}}(\mathcal{U}, \delta) \left( \int_{\mathcal{V}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n}) \, dx + \varepsilon_n M_0 \right).
\]
It follows that
\[
\int_{\mathcal{U}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n}) \, dx \to 0 \text{ as } n \to +\infty,
\]
so that the proof is complete.

### 7.4 Connectedness properties of $\mathcal{G}_\star$

The purpose of the present section is, among other things, to provide the proof of Proposition 3. Given $r > 0$ and $x_0 \in \Omega$ such that $\mathbb{D}^2(x_0, 2r) \subset \Omega$, we consider the closed set
\[
\mathcal{G}_{\star, r} = \mathcal{G}_{\star, r}(x_0) = \mathcal{G}_{\star} \cap \mathbb{D}^2(x_0, r) \text{ for } r \in [0, 2r).
\]

The proof of Proposition 8 relies on several intermediate properties we present next.

**Proposition 7.3.** Let $r > 0$ and $x_0 \in \Omega$ be as above. The closed set
\[
\Omega_{\star, r}(x_0) = \mathcal{G}_{\star, r}(x_0) \cup \mathbb{S}^2(x_0, r)
\]
is a continuum, that is, it is compact and connected.

**Proof.** The proof of compactness of $\Omega_{\star, r}(x_0)$ is a straightforward consequence of Proposition 7.1 since both sets composing the union (7.9) are compact. The proof of connectedness of $\Omega_{\star, r}(x_0)$ is more involved, and strongly relies on Theorem 8 as we will see next. In order to invoke Theorem 8, a first step is to approximate $\mathcal{G}_{\star, r}$ by sets $\mathcal{G}_{\delta, r}$ with a simpler structure.
Definition of the approximating sets $\mathcal{G}_{\delta,r}$. These sets are defined using a Besicovitch covering of $\mathcal{G}_{\ast,r}$. Let

$$\delta_{x_0,r} = \text{dist}(\mathbb{D}^2(x_0,r), \partial\Omega) > 0.$$  

For given $0 < \delta < \delta_{x_0,r}$, we consider the covering of $\mathcal{G}_{\ast,r}$ by the collection of open disks $\{\mathbb{D}^2(x_0,\delta)\}_{x \in \mathcal{G}_{\ast,r}}$, which is obviously a covering of $\mathcal{G}_{\ast,r}$, and actually a Besicovitch covering.

We may therefore invoke Besicovitch covering theorem to assert that there exists a universal constant $p$, depending only on the dimension $N = 2$, and $p$ families of points $\{x_i\}_{i \in A_1}, \{x_i\}_{i \in A_2}, \ldots, \{x_i\}_{i \in A_p}$, such that $x_i \in \mathcal{G}_{\ast,r}(x_0)$, for any $i \in A \equiv A_1 \cup A_2 \ldots \cup A_p$.

$$\mathcal{G}_{\ast,r} \subseteq \mathfrak{G}_{\delta,r} \equiv \bigcup_{\ell=1}^{p} \bigcup_{i \in A_\ell} \mathbb{D}^2(x_i,\delta) = \bigcup_{i \in A} \mathbb{D}^2(x_i,\delta),  \quad (7.10)$$

and such that the balls in each collection $\{\mathbb{D}^2(x_i,\delta)\}_{i \in A_\ell}$ are disjoint, that is, for any $\ell = 1, \ldots, p$, we have

$$\mathbb{D}^2(x_i,\delta) \cap \mathbb{D}^2(x_j,\delta) = \emptyset \text{ for } i \neq j \text{ with } i, j \in A_\ell.  \quad (7.11)$$

As a consequence of the above constructions, a point $x \in \mathfrak{G}_{\delta,r}$, where $\mathfrak{G}_{\delta,r}$ is defined in (7.10), belongs to at most $p$ distinct disks of the collection $\{\mathbb{D}^2(x_i,\delta)\}_{i \in A}$. We define the set $\mathcal{G}_{\delta,r}$ as the closure of the set $\mathfrak{G}_{\delta,r}$ that is

$$\mathcal{G}_{\delta,r} \equiv \overline{\mathfrak{G}_{\delta,r}} = \bigcup_{\ell=1}^{p} \bigcup_{i \in A_\ell} \overline{\mathbb{D}^2(x_i,\delta)},$$

Notice that, by construction, the total number $\sharp(A)$ of distinct disks is finite. Actually, we have the bound

$$\sharp(A) \leq \frac{4p r^2}{\delta^2}.  \quad (7.12)$$

Indeed, since the families of balls $\{\mathbb{D}^2(x_i,\delta)\}_{i \in A_\ell}$ are disjoint disks of radius $\delta$ which are included in a ball of radius $2r$, we have

$$\sharp(A_\ell) \leq \frac{4r^2}{\delta^2} \text{ for } \ell = 1, \ldots, p,$$

so that (7.12) follows by summation.

We next consider the set

$$\Omega_{\delta,r} = \mathcal{G}_{\delta,r} \cup \mathbb{S}^2(x_0,r)$$

and its distinct connected components $\{\mathfrak{T}_{\delta,r}^k\}_{k \in \mathcal{J}_\delta}$. In view of the structure of $\mathfrak{T}_{\delta,r}$, which is an union of $\sharp(A)$ disks with a circle, the total number of connected components $\sharp(\mathfrak{J}_\delta)$ is finite and actually bounded by $\sharp(A) + 1$, hence the number on the right hand side of inequality (7.12) plus one. As a matter of fact, we claim

$$\text{The set } \Omega_{\delta,r} \text{ is simply connected, so that } \sharp(\mathfrak{J}_\delta) = 1.  \quad (7.13)$$

Proof of the claim (7.13). We assume by contradiction that $\Omega_{\delta,r}$ has at least two distinct connected components and denote by $\Omega_{1}^{\delta,r}$ the connected component which contains the circle $\mathbb{S}^1(x_0,r)$. Let $\Omega_{2}^{\delta,r}$ be a connected component distinct from $\Omega_{1}^{\delta,r}$, and set

$$\beta \equiv \inf \left\{ \text{dist}(\Omega_{2}^{\delta,r}, \Omega_{j}^{\delta,r}), j \in \mathcal{J}_\delta, j \neq 2 \right\} > 0.$$


We consider the open set
\[ U = \{ x \in \mathbb{R}^2, \operatorname{dist} (x, \Omega_{\delta,r}^2) < \beta/4 \} \subset \mathbb{D}^2(x_0, r) \cup \cup_{j \in J_\delta \{2\}} \Omega_{\delta,r}^j, \]
so that using the notation (63), we have
\[ U_{\beta/4} = \{ x \in \mathbb{R}^2, \operatorname{dist} (x, U) < \beta/4 \} \subset \mathbb{D}^2(x_0, r) \cup \cup_{j \in J_\delta \{2\}} \Omega_{\delta,r}^j \]
and
\[ V_{\beta/4} \equiv U_{\beta/4} \setminus \{ x \in \mathbb{R}^2, \beta/4 \leq \operatorname{dist} (x, \Omega_{\delta,r}^2) \leq \beta/2 \} \]
combining (7.14) with the definition of \( \beta \), we obtain
\[ V_{\beta/4} \cap \mathcal{S} = \emptyset \] and \( \nu (V_{\beta/4}) = 0. \) (7.15)
We are therefore in position to apply Theorem 3 to assert that \( \nu (U) = 0. \) However, since by definition \( \Omega_{\delta,r}^2 \subset U \), it follows that \( U \cap \mathcal{S} \neq \emptyset \), so that \( \nu (U) > 0. \) We have hence reached a contradiction, which establishes the proposition.

**Proof of Proposition 7.3 completed.** It follows from the definition of \( \mathcal{S}_{\delta,r} \) that
\[ \operatorname{dist}(\Omega_{\delta,r}, \mathcal{S}_{\delta,r}) \leq \delta, \] where \( \mathcal{S}_{\delta,r} = \mathcal{S}_{\delta,r} \cup \mathbb{S}^2(x_0, r), \)
so that \( \Omega_{\delta,r} \) converges as \( \delta \to 0 \) to \( \mathcal{S}_{\delta,r} \) in the Hausdorff metric. Since for every \( \delta \), the set \( \mathcal{S}_{\delta,r} \) is a continuum, it then follows (see e.g. [19], Theorem 3.18) that the Hausdorff limit \( \mathcal{S}_{\delta,r} \) is also a continuum and the proof is complete.

We deduce as a consequence of Proposition 7.3:

**Corollary 7.1.** The set \( \Omega_{\delta,r} \) is arcwise connected.

**Proof.** Indeed, any continuum with finite one-dimensional Hausdorff dimension is arcwise connected, see e.g. [19], Lemma 3.12, p 34. \( \square \)

**7.4.1 Proof of Proposition 3**

Invoking Fubini’s theorem together with a mean value argument, we may choose some radius \( r_0 \in [r, 2r] \) such that the number of points in \( \mathcal{S} \cap \partial \mathbb{D}^2(x_0, r_0) \) is finite, more precisely
\[ m_0 \equiv \# (\mathcal{S} \cap \partial \mathbb{D}^2(x_0, r_0)) \leq \frac{C_H}{r} M_0, \]
where we have used estimate (7) of the \( \mathcal{H}^1 \) measure of \( \mathcal{S}. \) We may hence write
\[ \mathcal{S} \cap \partial \mathbb{D}^2(x_0, r_0) = \{ a_1, \ldots, a_{m_0} \}. \] (7.16)
Next, we claim that for any point \( y \in \mathcal{S}_{\delta,r_0} \), there exists a continuous path \( p : [0, 1] \to \mathcal{S}_{\delta,r_0} \) connecting the point \( y \) to one of the points \( a_1, \ldots, a_{m_0} \), that is such that
\[ p(0) = y \] and \( p(1) \in \{ a_1, \ldots, a_{m_0} \}. \) (7.17)
Proof of the claim (7.17). If \(|y - x_0| = r_0\), then \(y \in \mathfrak{S}_* \cap \partial D^2(x_0, r_0)\), and it therefore suffices to choose \(p(s) = y\), for all \(s \in [0, 1]\). Otherwise, since, in view of Corollary 7.1 applied at \(x_0\) with radius \(r_0\), the set \(\mathfrak{S}_{*, r_0} \cup \partial D^2(x_0, r_0)\) is path-connected, there exists a continuous path \(\tilde{p} : [0, 1] \to \mathfrak{S}_{*, r_0} \cup \partial D^2(x_0, r_0)\) such that
\[
\tilde{p}(0) = y \quad \text{and} \quad \tilde{p}(1) \in \partial D^2(x_0, r_0).
\]
By continuity, there exists some number \(s_0 \in [0, 1]\) such that
\[
|\tilde{p}(s)| < r_0, \quad \text{for } 0 \leq s < s_0 \quad \text{and} \quad |\tilde{p}(s_0)| = r_0.
\]
It follows that
\[
\tilde{p}(s_0) \in \mathfrak{S}_* \cap \partial D^2(x_0, r_0) = \{a_1, \ldots, a_{m_0}\}.
\]
We then set
\[
p(s) = \tilde{p}(s), \quad \text{for } 0 \leq s < s_0, \quad \text{and} \quad p(s) = \tilde{p}(s_0), \quad \text{for } s_0 \leq s \leq 1,
\]
and verify that \(p\) has the desired property, so that the proof of the claim is complete.

Proof of Proposition 3 completed. It follows from the claim (7.17) that any point \(y \in \mathfrak{S}_{*, r_0}\) is connected to one of the points \(a_1, \ldots, a_{m_0}\) given in (7.16). Hence \(\mathfrak{S}_{*, r_0}\) has at most \(m_0\) connected components and the proof is complete.

7.5 Rectifiability of \(\mathfrak{S}_*\)

In this section, we prove:

Theorem 7.1. The set \(\mathfrak{S}_*\) is rectifiable.

Proof. The result is actually an immediate consequence of Proposition 7.3 and the fact that any 1-dimensional continuum is rectifiable, a result due to Wazewski and independently Besicovitch (see e.g [19], Theorem 3.12). Indeed, given any \(x_0 \in \Omega, r > 0\) such that \(D^2(x_0, r) \subset \Omega\), the set \(\mathfrak{S}_{*, r} \cup S^2(x_0, r)\) is a continuum, hence rectifiable in view of the result quoted above, and hence so is the set \(\mathfrak{S}_{*, \tau}\). Since rectifiability is a local property, the conclusion follows.

7.6 Proof of Theorem 1 completed

All statements in Theorem 1 have been obtained so far. Indeed, assertions i) follows combining several result in Section 7, namely Proposition 7.1, Proposition 7.2, Proposition 7.3, Proposition 3 and Theorem 7.1.

7.7 On the tangent line at regular points of \(\mathfrak{S}_*\)

In this subsection, we provide the proof to Proposition 4. It relies on the following Lemma, which is actually a weaker statement:

Lemma 7.1. Let \(x_0\) be a regular point of \(\mathfrak{S}_*\) and \(\vec{e}_{x_0}\) be a unit tangent vector to \(\mathfrak{S}_*\) at \(x_0\). Given any \(\theta > 0\) there exists a radius \(R_{cone}(\theta, x_0)\) such that
\[
\mathfrak{S}_* \cap \left( D^2(x_0, \tau) \setminus D^2 \left( x_0, \frac{\tau}{2} \right) \right) \subset C_{cone}(x_0, \vec{e}_{x_0}, \theta), \quad \text{for any } 0 < \tau \leq R_{cone}(\theta, x_0).
\]
\textit{Proof.} Since we have the inclusion
\[ C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \theta \right) \subset C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \theta' \right) \]
for \( 0 \leq \theta \leq \theta' \), it suffices to establish the statement for \( \theta \) arbitrary small. For a given regular point \( x_0 \) of \( \mathcal{S}_* \), we may invoke the convergence \[10\] to assert that there exists some \( r_1 > 0 \) such that for \( 0 < \tau \leq r_1 \) we have
\[ \mathcal{H}^1 \left( \mathcal{S}_* \cap \mathbb{D}^2 (x_0, 2\tau) \setminus C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \frac{\theta}{2} \right) \right) \leq \frac{\theta \tau}{8}. \] (7.19)

We set
\[ A(x_0, \tau, \theta) = (\mathcal{S}_* \cap \mathbb{D}^2 (x_0, \tau)) \setminus \left( C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \theta \right) \cup \mathbb{D}^2 \left( x_0, \frac{\tau}{2} \right) \right), \]
and have to prove that \( A(x_0, \tau, \theta) \) is empty, if \( \tau \) is sufficiently small. We assume by contradiction that \( A(x_0, \tau, \theta) \neq \emptyset \) for small \( \tau \), and will show that we obtain a contradiction. We have, in view of the definition of \( A(x_0, \tau, \theta) \)
\[ A(x_0, \tau, \theta) \cap C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \frac{\theta}{2} \right) = \emptyset \text{ and } \mathcal{H}^1 \left( A(x_0, \tau, \theta) \right) \leq \frac{\theta \tau}{8}. \] (7.20)

we notice that, if \( A(x_0, \tau, \theta) \) is not empty, then we have
\[
\begin{cases}
\text{dist} \left( A(x_0, \tau, \theta), C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \frac{\theta}{2} \right) \right) \geq \frac{\tau}{2} \sin \left( \frac{\theta}{2} \right) \\
\text{dist} \left( A(x_0, \tau, \theta), \partial \mathbb{D}^2 (x_0, 2\tau) \right) \geq \tau,
\end{cases}
\]
so that, if \( \theta > 0 \) is sufficiently small
\[ \text{dist} \left( A(x_0, \tau, \theta), C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \frac{\theta}{2} \right) \cup \partial \mathbb{D}^2 (x_0, 2\tau) \right) \geq \frac{\tau}{2} \sin \left( \frac{\theta}{2} \right). \] (7.21)

Since, by assumption, the set \( A(x_0, \tau, \theta) \) is not empty, there exists some point \( x_1 \in A(x_0, \tau, \theta) \). We consider the set \( \mathcal{Q}_{*,2\tau} (x_0) = \mathcal{S}_* \cup \partial \mathbb{D}^2 (x_0, 2\tau) \) introduced in \[7.9\]. In view of Proposition \[7.3\] and Corollary \[7.1\] the set \( \mathcal{Q}_{*,2\tau} (x_0) \) is path-connected: Hence, there exists a continuous path \( p \) joining \( x_1 \) to some point \( x_2 \in \partial \mathbb{D}^2 (x_0, 2\tau) \) which stays inside \( \mathcal{S}_* \cup \partial \mathbb{D}^2 (x_0, 2\tau) \). On the other hand, since \( x_1 \in \mathbb{D}^2 (x_0, \tau) \) the length \( \mathcal{H}^1 (p) \) of this path is larger than \( \tau \). We claim that
\[ p \cap C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \frac{\theta}{2} \right) \neq \emptyset. \] (7.22)

Otherwise, indeed, \( p \) would be a path inside \( \mathcal{S}_* \cap \mathbb{D}^2 (x_0, 2\tau) \setminus C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \frac{\theta}{2} \right) \). Since its length is larger then \( \tau \), this would contradict \( \text{(7.19).} \) Next, combining \( \text{(7.22)} \) and \( \text{(7.21)} \), we obtain
\[ \mathcal{H}^1 \left( p \cap C_{\text{one}} \left( x_0, \bar{e}_{x_0}, \frac{\theta}{2} \right) \right) \geq \frac{\tau}{2} \sin \left( \frac{\theta}{2} \right) \sim \frac{\tau \theta}{4}. \]

Since \( p \) is a path inside \( \mathcal{S}_* \cup \partial \mathbb{D}^2 (x_0, 2\tau) \) this contradicts \( \text{(7.19),} \) provided \( \theta \) is chosen sufficiently small. This completes the proof of the Lemma, choosing \( R_{\text{cone}}(\theta, x_0) = r_1. \) \hfill \qed
Proof of Proposition 4 completed. Given $\tau < R_1$, we apply Lemma 7.1, the sequence of radii $(\tau_k)_{k \in \mathbb{N}}$ given by

$$\tau_k = \frac{\tau}{2^k} \text{ for } k \in \mathbb{N},$$

so that

$$\mathcal{G}_* \cap \left( \mathbb{D}^2(x_0, \tau_k) \setminus \mathbb{D}^2(x_0, \tau_{k+1}) \right) \subset \mathcal{C}_\text{one}(x_0, \varepsilon_{x_0}, \theta), \text{ for any } k \in \mathbb{N}.$$  

We take the union of these sets on the left hand side, we obtain

$$\mathcal{G}_* \setminus \{x_0\} = \bigcup_{k \in \mathbb{N}} \mathcal{G}_* \cap \left( \mathbb{D}^2(x_0, \tau_k) \setminus \mathbb{D}^2(x_0, \tau_{k+1}) \right) \subset \mathcal{C}_\text{one}(x_0, \varepsilon_{x_0}, \theta).$$

This yields the result.

8 Behavior near points in $\mathcal{G}_* \setminus \mathcal{E}_*$

In this section, we analyze more precisely the behavior of the measures $\zeta_*$ and $\mu_{*,i,j}$ in the vicinity of good points, that is points $x_0$ in $\mathcal{G}_* \setminus \mathcal{E}_*$, in particular points having the Lebesgue property for the absolutely continuous part of the measure. One of our main goals is to provide the proof to Proposition 5 and Lemma 2. The results in this section also pave the way to the proof of Theorem 2 provided in Section 10.

8.1 The limiting Hopf differential

The Hopf differential

$$\omega_\varepsilon \equiv \varepsilon \left( |(u_\varepsilon)_{x_1}|^2 - |(u_\varepsilon)_{x_2}|^2 - 2i(u_\varepsilon)_{x_1} \cdot (u_\varepsilon)_{x_2} \right)$$

defined in (3.18) has turned out to be a central tool in our analysis so far. We combine it in the present subsection with the rectifiability properties and Proposition 4 to derive new properties near good points. Recall that we have defined $\omega_*$ in (66) as

$$\omega_* = (\mu_{*,1,1} - \mu_{*,2,2}) - 2i\mu_{*,1,2}.$$

So that, in view of the definition (40) of the measures $\mu_{*,i,j}$, we have

$$\omega_{\varepsilon_n} \rightharpoonup \omega_*, \text{ in the sense of measures on } \Omega, \text{ as } n \to +\infty. \quad (8.1)$$

8.2 The limiting differential relation for $\omega_*$ and $\zeta_*$

In this paragraph, we provide a prove to Lemma 1. First, passing to the limit in (3.19), we are led to:

**Lemma 8.1.** Let $(u_{\varepsilon_n})_{n \in \mathbb{N}}$ be a sequence of solutions to (1) on $\Omega$ with $\varepsilon_n \to 0$ as $n \to +\infty$ and assume that (6) holds. Let $\omega_*$ and $\zeta_*$ be the bounded measures on $\Omega$ given by (8.1) and (13) respectively. Then, we have, in the sense of distributions

$$\text{Re} \left( \left\langle \omega_*, \frac{\partial X}{\partial \bar{z}} \right\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} \right) = \left\langle \zeta_*, \text{Re} \left( \frac{\partial X}{\partial \bar{z}} \right) \right\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}, \text{ for any } X \in C_0^\infty(\Omega, \mathbb{C}). \quad (8.2)$$
Lemma 8.1 is actually our main tool in the rest of the discussion, and will be used with vector fields $X$ of various types.

Proof of Lemma 1. Using $iX$ as test function in (3.2) and the fact that $\text{Re}(iz) = -\text{Im}(z)$ for any complex number $z \in \mathbb{C}$, we obtain likewise

$$\text{Im} \left( \left\langle \left( \omega, \frac{\partial X}{\partial \bar{z}} \right) \right\rangle_{D'((\Omega),\mathcal{D}(\Omega))} \right) = 2 \left\langle \zeta, \text{Im} \left( \frac{\partial X}{\partial \bar{z}} \right) \right\rangle_{D'((\Omega),\mathcal{D}(\Omega))},$$

for any $X \in C_0^\infty(\Omega, \mathbb{C})$. (8.3)

Combining (8.2) and (8.3), we are hence led to the simple identity

$$\left\langle \omega, \frac{\partial X}{\partial \bar{z}} \right\rangle_{D'((\Omega),\mathcal{D}(\Omega))} = 2 \left\langle \zeta, \frac{\partial X}{\partial \bar{z}} \right\rangle_{D'((\Omega),\mathcal{D}(\Omega))},$$

for any $X \in C_0^\infty(\Omega, \mathbb{C})$, (8.4)

which yields (67) in the sense of distributions. \hfill \Box

We describe next some additional properties of the measures $\omega \ast \nu$ et $\zeta \ast \nu$, mostly bases on Lemma 8.1, choosing various kinds of test vector fields $\vec{X}$. Whereas we have used so far mainly vector fields yielding dilatations of the domain (see e.g. Lemma 3.3), we consider also vector fields of different nature. Given a point $x_0 = (x_{0,1}, x_{0,2}) \in \Omega$, $\rho > 0$ such that $B^2(x_0, 2\rho) \subset \Omega$, the fields we will consider in the next paragraphs are of the form

$$\vec{X}_f(x_1, x_2) = f_1(x_1)f_2(x_2)\vec{e}_j = if_1(x_1)f_2(x_2),$$

with $j = 1, 2$. (8.5)

where, $f_i$ represents, for $i = 1, 2$ an arbitrary function in $C^\infty_c((x_{0,i} - \rho, x_{0,i} + \rho))$. These vector fields have hence support on the square $Q_\rho(x_0)$, defined by

$$Q_\rho(x_0) = I_r(x_{0,1}) \times I_\rho(x_{0,2}),$$

where $I_r(s) = [s - \rho, s + \rho] = B(s, r)$, for $s > 0$. (8.6)

We consider also the subset $R_\rho(x_0)$ of $Q_\rho(x_0)$ given by

$$R_\rho(x_0) \equiv I_\rho(x_{0,1}) \times I_{3\rho/4}(x_{0,2}) \subset Q_\rho(x_0),$$

so that $Q_\rho(x_0) \setminus R_\rho(x_0)$ is the union of two disjoint rectangles

$$Q_\rho(x_0) \setminus R_\rho(x_0) = \left( I_\rho(x_{0,1}) \times (x_{0,2} + 3\rho/4, x_{0,2} + \rho) \right) \cup \left( I_\rho(x_{0,1}) \times (x_{0,2} - \rho, x_{0,2} + 3\rho/4) \right).$$

In several places, we will assume that the following conditions holds

$$\nu_\ast(Q_\rho(x_0) \setminus R_\rho(x_0)) = 0,$$

which means that the measure $\nu_\ast$ concentrates, locally, in a neighborhood of the segment $(x_0 - \rho\vec{e}_1, x_0 + \rho\vec{e}_1)$. 

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8.3 Projecting the measures on the tangent line

In the above framework, the $\vec{e}_1$ direction plays a distinguished role: Integrating various quantities with respect to the $x_2$-variable, we obtain one-dimensional quantities, treated as measures on the interval $I_{\rho}(x_0,1) = (x_{0,1} - \rho, x_{0,1} + \rho)$. Using appropriate test functions, relation (8.2) is then turned into a differential equation.

Given a Radon measure $\nu$ on $Q_{\rho}(x_0)$, and a test function $\varphi \in C_c(Q_{\rho}(x_0), \mathbb{C})$, we define the Radon measure $(\nu \varphi)^{x_1} = \mathbb{P}_x(\varphi \nu)$ defined on $I_{\rho}(x_0,1)$ as follows: For any Borel set $A$ of $I_{\rho}(x_0,1)$, we have

$$ (\nu \varphi)^{x_1}(A) = (\nu \varphi)\left(\mathbb{P}^{-1}(A) \cap Q_{\rho}(x_0)\right) = \varphi \nu \left((A \times \mathbb{R}) \cap Q_{\rho}(x_0)\right). $$

so that

$$ (\nu, \varphi) = (\nu \varphi)(Q_{\rho}(x_0)) = \int_{Q_{\rho}(x_0)} \varphi \nu = \int_{I_{\rho}(x_0)} d(\nu \varphi)^{x_1}. \quad \text{(8.9)} $$

We mainly will make use of test functions $\varphi$ of the form

$$ \varphi(x_1, x_2) = g_1(x_1)g_2(x_2), \quad \text{(8.10)} $$

where $g_1$ and $g_2$ are defined on the intervals $I_{\rho}(x_0,1)$ and $I_{\rho}(x_0,1)$ respectively. If $\varphi$ is of the form (8.10), then (8.9) becomes

$$ \langle \nu, \varphi \rangle_{D'(Q_{\rho}(x_0)),D(Q_{\rho}(x_0))} = \int_{I_{\rho}(x_0)} g_1(x_1)d(g_2(x_2)\nu)^{x_1} $$$$ = \langle (dg_2(x_2)\nu)^{x_1}, g_1 \rangle_{D'(I_{\rho}(x_0,1)),D(I_{\rho}(x_0,1))}. \quad \text{(8.11)} $$

In the case where $\nu(Q_{\rho}(x_0)\setminus R_{\rho}(x_0)) = 0$ and $g_2(s) = 1$ for $s \in I_{\rho}(x_0,2)$, then we have $g(x_2)\nu = \nu$, so that identity (8.11) becomes

$$ \langle \nu, \varphi \rangle = \int_{I_{\rho}(x_0,1)} g_1(x_1)d\nu^{x_1}. \quad \text{(8.12)} $$

We will make use of this formulas in several places for a corresponding formulas for the Radon measures $\tilde{\mu}_{*,i,j}$, for $i = 1, 2$, $\nu_*$, and $\zeta_*$ and also related measures, obtained by multiplication and sums of the previous ones.

8.4 Some quantities of interest

The measures $\mathbb{L}_{x_0,\rho}$, $\mathbb{N}_{x_0,\rho}$, defined on $I_{\rho}(x_0)$ as well as the measures $\tilde{\mu}_{*,i,j}^{x_1}$ already introduced in the introduction in (76) and correspond to the description in the previous paragraph. Our computations will also involve some auxiliary ”moment ” measures, defined for $k \in \mathbb{N}$, by

$$ \mathbb{J}_{k,x_0,\rho} \equiv \mathbb{J}_{k,\rho} = \mathbb{P}_x\left((x_2-x_{0,2})^k\tilde{\mu}_{*,1,2}\right) $$$$ \mathbb{L}_{k,x_0,\rho} \equiv \mathbb{L}_{k,\rho} = \mathbb{P}_x\left((x_2-x_{0,2})^k\left[2\zeta_{*,i,j} - \tilde{\mu}_{*,1,1} + \tilde{\mu}_{*,2,2}\right]\right) $$$$ \mathbb{N}_{k,x_0,\rho}(s) \equiv \mathbb{N}_{k,\rho} = \mathbb{P}_x\left((x_2-x_{0,2})^k\left[2\zeta_{*,i,j} + \tilde{\mu}_{*,1,1} - \tilde{\mu}_{*,2,2}\right]\right), \quad \text{(8.13)} $$
With this notation (dropping the subscript $x_0$), we have $\tilde{\mu}_{*,1,2} = J_0, L_0, \rho$ and $N_\rho = N_{0,\rho}$.

We also consider the measures, for $k \in \mathbb{N}$,

\[
\mathbb{H}_{k,x_0,\rho}(s) = \frac{1}{4}(N_{k,\rho} + L_{k,\rho}) = \mathbb{P}_s\left((x_2 - x_0)^k \tilde{z}_\epsilon\right) \quad (8.14)
\]

The main result of this section is:

**Proposition 8.1.** Assume that (8.8) holds. Then, the measures $L_{x_0,\rho}$ and $J_{x_0,\rho}$ are proportional to the Lebesgue measure on $\mathcal{I}_\rho(x_0,1)$. Moreover, we have the differential relations

\[
\begin{cases}
\frac{d}{ds} J_{k,\rho} = kN_{k-1,\rho} \text{ in } \mathcal{D}'((x_{0,1} - \rho, x_{0,1} + \rho)) \\
-\frac{d}{ds} L_{k,\rho} = kJ_{k-1,\rho} \text{ in } \mathcal{D}'((x_{0,1} - \rho, x_{0,1} + \rho)).
\end{cases} \quad (8.15)
\]

In the case $k = 1$, we obtain hence the relations

\[
\begin{cases}
\frac{d}{ds} J_{1,\rho} = N_{\rho} \text{ in } \mathcal{D}'((x_{0,1} - \rho, x_{0,1} + \rho)) \text{ and } \\
-\frac{d}{ds} L_{1,\rho} = J_{\rho} \text{ in } \mathcal{D}'((x_{0,1} - \rho, x_{0,1} + \rho)).
\end{cases} \quad (8.16)
\]

Notice the following consequence of Proposition 8.1:

**Corollary 8.1.** For any $k \in \mathbb{N}$, the measures $J_{k,\rho}$ and $L_{k,\rho}$ are absolutely continuous with respect to the Lebesgue measure $dx_1$. Hence there exist measurable functions $J_{k,\rho}$ and $L_{k,\rho}$ on $\mathcal{I}_\rho(x_0,1)$ such that

\[
J_{k,\rho} = J_{k,\rho} dx_1 \text{ and } L_{k,\rho} = L_{k,\rho} dx_1. \quad (8.17)
\]

Moreover, the functions $J_{k,\rho}$ and $L_{k,\rho}$ are bounded on $\mathcal{I}_\rho(x_0,1)$.

**Proof.** The result is an immediate consequence of the fact that the measures $N_{k-1,\rho}$ and $J_{k-1,\rho}$ are bounded, so that, $J_{k,\rho}$ and $L_{k,\rho}$ represent BV functions on $\mathcal{I}_\rho(x_0)$, and hence are bounded.

The proof of Proposition 8.1 corresponds to the use of different kinds of vector fields of the form (8.5) in (8.2) that we will describe next in details. The proof of Proposition 8.1 is completed in Subsection 8.7.

### 8.5 Shear vector fields

We use in this section vector fields of the form (8.5), specifying $j = 2$. More precisely, we consider here vector fields of the form

\[
\vec{X}_f(x_1,x_2) = f_1(x_1)f_2(x_2)\hat{e}_2 = if_1(x_1)f_2(x_2). \quad (8.18)
\]

A short computation shows that

\[
\begin{cases}
\frac{\partial X_f}{\partial z} = \frac{1}{2}f_1(x_1)f_2'(x_2) + \frac{i}{2}f_1'(x_1)f_2(x_2), \\
-\frac{\partial X_f}{\partial \bar{z}} = -\frac{1}{2}f_1(x_1)f_2'(x_2) + \frac{i}{2}f_1'(x_1)f_2(x_2),
\end{cases} \quad (8.19)
\]
and hence
\[
\begin{aligned}
\zeta_* \text{Re} \left( \frac{\partial X_I}{\partial \bar{z}} \right) &= \frac{1}{2} f_1(x_1) f'_2(x_2) \zeta_* \quad \text{and} \\
\text{Re} \left( \omega_* \frac{\partial X_I}{\partial \bar{z}} \right) &= -\frac{\text{Re}(\omega_*)}{2} f_1(x_1) f'_2(x_2) - \frac{\text{Im}(\omega_*)}{2} f'_1(x_1) f(x_2).
\end{aligned}
\tag{8.20}
\]

Identity [8.2] then becomes
\[
\langle (\text{Re}(\omega_*) + 2\zeta_*), f'_2(x_2) f_1(x_1) \rangle - \langle \omega_*, f(x_2) f'_1(x_1) \rangle_{\nu'(\Omega), \nu'(\Omega)} = 0.
\tag{8.21}
\]

In view of [8.11], we have
\[
\begin{aligned}
\begin{cases}
\langle (\text{Re}(\omega_*) + 2\zeta_*), f'_2(x_2) f_1(x_1) \rangle = \int_{I_r(x_0, 1)} f_1(s) d\left[ f'_2(x_2) \left( \text{Re}(\tilde{\omega}_*) + 2\tilde{\zeta}_* \right) \right]^x_1 \\
\text{and} \\
\langle \omega_*, f(x_2) f'_1(x_1) \rangle_{\nu'(\Omega), \nu'(\Omega)} = -\int_{I_r(x_0, 1)} f'_1(s) d\left[ f(x_2) \tilde{\mu}_*, i,j \right]^x_1.
\end{cases}
\end{aligned}
\tag{8.22}
\]

We choose, in this subsection as functions $f_1, f_2$ in [8.18] $f_1 = f$, where $f$ is an arbitrary function in $C^\infty(I_{I_{\rho}}(x_0))$ and, for $f_2$, a function of the form
\[
f_2(x_2) = \chi\left( \frac{x_2 - x_{0,2}}{\rho} \right),
\]
where $\chi$ is a non-negative given smooth plateau function such that
\[
\chi(s) = 1, \quad \text{for } s \in \left[ -\frac{3}{4}, \frac{3}{4} \right], \quad \text{and } \varphi(s) = 0, \quad \text{for } |s| \geq 1.
\tag{8.23}
\]

In particular, we have
\[
f'_2(x_2) = 0, \quad \text{if } |x_2 - x_{0,2}| \leq \frac{3\rho}{4},
\tag{8.24}
\]

Such a vector field corresponds somewhat to shear vector field. Using these shear vector fields, as test vector fields in [8.2], we obtain:

**Proposition 8.2.** Assume that [8.8] holds. Then the measure $\mathcal{I}_r$, defined on $I_{I_{\rho}}(x_0)$ by [8.13], is proportional to the Lebesgue measure, that is $\mathcal{I}_r = J_0, \rho dx$, for some number $J_0, \rho \in \mathbb{R}$.

**Proof.** We first show that, for any function $f \in C^\infty_c(I_r(x_0, 1))$, we have
\[
\langle \tilde{\mu}_{*,1,2}, f'_1(x_1) \rangle_{\nu'(\Omega), \nu'(\Omega)} = 0.
\tag{8.25}
\]

Indeed, identity [8.25] follows combining [8.21] and [8.24], together with the fact that $\nu_*(Q_{r}(x_0) \setminus R_{r}(x_0)) = 0$, so that $f'_2$ vanishes on the support of vanish on the support $\text{Re}(\omega_*) + 2\zeta_*$. It follows that have
\[
f'_2(x_2) \left( \text{Re}(\tilde{\omega}_*) + 2\tilde{\zeta}_* \right) = 0 \quad \text{and therefore} \quad \left( f'_2(x_2) \left( \text{Re}(\tilde{\omega}_*) + 2\tilde{\zeta}_* \right) \right)^x_1 = 0.
\]

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In view of the first identity in (8.22), the first term on the left hand side of (8.21) vanishes, so that
\[ \text{Im} \langle \omega^*, f(x_2) f'_1(x_1) \rangle_{D'(\Omega), D(\Omega)} = 0. \] (8.26)
We notice that \( \omega^* f(x_2) = -2\mu^* f_1,2 \), so that we are led to the identity
\[ \langle \tilde{\mu}, f_1,2 \rangle = 0. \]
We invoke now identity in (8.22), together with the fact that \( [f(x_2)\tilde{\mu},i,j] = \tilde{\mu} \), to deduce from (8.26) that
\[ \langle J, f' \rangle_{D'(I_{\rho}(x_0), D(I_{\rho}(x_0)))} = \int_{I_{\rho}(x_0)} f'(s) d\rho = 0, \] (8.27)
We have hence, in the sense of distributions
\[ \frac{d}{ds} J = 0. \]
A classical results in distribution theory then shows that \( J \) is proportional to the uniform Lebesgue measure.

8.5.1 Stretching vector fields

In this subsection, we assume that \( f_1 = f \), where \( f \) is an arbitrary function in \( C_\infty(I_{\rho}(x_0)) \) as above, and, that \( f_2 \) is given by, for \( k \in N^* \), by
\[ f_2(x_2) = (x - x_{0,2})^k \chi \left( \frac{x_2 - x_{0,2}}{\rho} \right), \]
where \( \chi \) is a non-negative given smooth plateau function such that (8.23) holds. With this choice, we have now
\[ f'(x_2) = k((x - x_{0,2})^{k-1} \text{ if } |x_2 - x_{0,2}| \leq \frac{3\rho}{4}. \] (8.28)
Combining as above (8.21) and (8.28), we obtain:

**Lemma 8.2.** Assume that (8.8) holds. We have, for any function \( f \in C_\infty(I_{\rho}(x_0)) \)
\[ \left\langle 1_{Q_1}, k(x_2 - x_{0,2})^{k-1} ((\text{Re}(\omega^*) + 2\zeta^*), f(x_1)) \right\rangle + \left\langle 1_{Q_1}, (x_2 - x_{0,2})^k \text{Im}(\omega^*), f'(x_1) \right\rangle = 0. \]

The identity of Lemma 8.2 becomes, using (8.13), (8.11) and (8.11), and arguing as in the proof of Proposition 8.2
\[ \langle kN_{k-1}, f \rangle + \langle J_k, f' \rangle = 0, \text{ for } f \in C_\infty((x_{0,1} - r, x_{0,1} + r), \]
so that, in the sense of distributions
\[ \frac{d}{ds} J_k = kN_{k-1} \text{ in } D'((x_{0,1} - r, x_{0,1} + r)), \text{ for } k \in N^*. \] (8.29)
8.6 Dilation vector fields

We use here as test vector fields in (8.2), vector fields of the form

\[ \vec{X}_d(x_1, x_2) = f_1(x_1) f_2(x_2) \vec{e}_1. \]

Computations similar to (8.19) yield

\[
\begin{aligned}
\frac{\partial X_d}{\partial z} &= \frac{1}{2} f'_1(x_1) f_2(x_2) - \frac{i}{2} f_1(x_1) f'_2(x_2), \\
\frac{\partial X_d}{\partial \bar{z}} &= \frac{1}{2} f'_1(x_1) f_2(x_2) + \frac{i}{2} f_1(x_1) f'_2(x_2).
\end{aligned}
\]

Relation (8.2) then becomes

\[ \langle (\text{Re}(\omega_*) - 2\zeta_*), f_2(x_2) f'_1(x_1) \rangle_{\mathcal{D}'(\mathcal{D}(\Omega))} + \langle \text{Im}(\omega_*), f'_2(x_2) f_1(x_1) \rangle_{\mathcal{D}'(\mathcal{D}(\Omega))} = 0, \quad (8.30) \]

Arguing as for (8.22), we obtain the relations

\[ \begin{aligned}
\langle (\text{Re}(\omega_*) - 2\zeta_*), f_2(x_2) f'_1(x_1) \rangle_{\mathcal{D}'(\mathcal{D}(\Omega))} &= \int_{I_{\rho}(x_{0,1})} f'_1(s) d f_2(x_2) \left( \text{Re}(\tilde{\omega}_*) - 2\tilde{\zeta}_* \right)^{x_1} \\
\text{and} \quad \text{Im}(\omega_*, f'(x_2) f_1(x_1))_{\mathcal{D}'(\mathcal{D}(\Omega))} &= -\int_{I_{\rho}(x_{0,1})} f_1(s) d [f'_2(x_2) \tilde{\mu}_{*,i,j}]^{x_1}. \end{aligned} \quad (8.31) \]

We next choose test vectors fields \( \vec{X}_d \), with \( f_2 \) of the form \( f_2(x_2) = \chi \left( \frac{x_2 - x_{0,2}}{\rho} \right) \), so that (8.24) holds. With this choice, we obtain

\[ [f'_2(x_2) \tilde{\mu}_{*,i,j}]^{x_1} = 0 \text{ and } \left( f_2(x_2) \left( \text{Re}(\tilde{\omega}_*) - 2\tilde{\zeta}_* \right) \right)^{x_1} = -\mathbb{L}_\rho. \]

Inserting into (8.30), we derive the relation

\[ \langle \mathbb{L}_\rho, f' \rangle = 0, \text{ for any } f_1 \in C_c(I_{\rho}(x_{0,1})). \quad (8.32) \]

Arguing as in the proof of Proposition 8.2, we derive from (8.30) and (8.24) that:

**Proposition 8.3.** Assume that (8.8) holds. Then, the measure \( \mathbb{L}_{x_0,\rho} \) defined on \( I_{\rho}(x_0) \) by (8.13) is proportional to the uniform Lebesgue measure, that is \( \mathbb{L}_\rho = L_{0,\rho} \, dx \), for some number \( L_{0,\rho} \in \mathbb{R} \).

We finally use test vectors fields \( \vec{X}_d \), with \( f_2 \) of the form

\[ f_2(x_2) = (x_2 - x_{0,2})^k \varphi \left( \frac{x_2 - x_{0,2}}{\rho} \right), k \in \mathbb{N}^* \]

so that (8.28) holds. Inserting into (8.30), and setting \( f = f_1 \), we are led to

\[ \langle k \mathbb{J}_{k-1}, f \rangle_{\mathcal{D}'(I_{\rho}), \mathcal{D}'(I_{\rho})} + \langle \mathbb{L}_k(s), f' \rangle_{\mathcal{D}'(I_{\rho}), \mathcal{D}(I_{\rho})} = 0, \text{ for } f \in C_c^\infty((x_{0,1} - \rho, x_{0,1} + \rho)). \]

Hence, we have, in the sense of distributions, for \( k \in \mathbb{N}^* \)

\[ - \frac{d}{ds} \mathbb{L}_k = k \mathbb{J}_{k-1} \text{ in } \mathcal{D}'((x_{0,1} - \rho, x_{0,1} + \rho)). \quad (8.33) \]
8.7 Proof of Proposition 8.1 completed

The proof of Proposition 8.1 follows combining Proposition 8.2, Proposition 8.3 together with identities (8.29) and (8.33).

8.8 Behavior near regular points

We specify Proposition 8.1 to regular points.

8.8.1 Property (8.8) is satisfied near regular points

We have:

**Proposition 8.4.** Assume that $x_0 \in \mathcal{S}_* \setminus \mathcal{A}_*$. Then, there exists $\rho_0 > 0$ such that property (8.8) is satisfied. Consequently, the measures $L_{x_0,\rho_0}$ and $J_{x_0,\rho_0}$ are proportional to the Lebesgue measure on $I_{\rho_0}(x_{0,1})$ and the differential relations (8.15) hold for $\rho = \rho_0$.

Let $x_0 \in \Omega$ and $r > 0$ be such that $\mathbb{D}^2(x_0, r) \subset \Omega$. We assume that $x_0$ is a regular point of $\mathcal{S}_*$ and choose the orthonormal basis so that $\vec{e}_1 = \vec{e}_{x_0}$ is a unit tangent vector to $\mathcal{S}_*$ at $x_0$. In view of Proposition 4, we have, for any $\theta \in [0, \frac{\pi}{2}]$ and $0 < \vartheta \leq R_{\text{cone}}(\theta, x_0)$

$$\mathcal{S}_* \cap \mathbb{D}^2(x_0, \vartheta) \subset C_{\text{cone}}(x_0, \vec{e}_{x_0}, \theta) = C_{\text{cone}}(x_0, \vec{e}_1, \theta),$$

(8.34)

Since we have $Q_{\sqrt{2}}(x_0) \subset \mathbb{D}^2(x_0, \vartheta)$, we obtain, for $0 \leq r \leq \rho_0 \equiv \sqrt{2}^{-1} R_{\text{cone}}(\theta, x_0)$

$$\mathcal{S}_* \cap Q_r(x_0) \subset C_{\text{cone}}(x_0, \vec{e}_1, \theta).$$

(8.35)

Specifying (8.35) with $\theta = \frac{\pi}{8}$, we obtain, for $0 \leq r \leq \rho_0 \equiv \sqrt{2}^{-1} R_{\text{cone}}(\frac{\pi}{8}, x_0)$

$$\mathcal{S}_* \cap Q_r(x_0) \subset C_{\text{cone}}(x_0, \vec{e}_1, \frac{\pi}{8}) \cap Q_r(x_0) \subset \mathcal{R}_r(x_0).$$

(8.36)

It follows that, if $r \leq \rho_0$, then (8.8) holds. In particular, we are in position to apply Proposition 8.1 at the point $x_0$. This yields immediately, for the number $\rho_0 > 0$ provided by the discussion above, the fact that the functions $L_{x_0,\rho_0}$ and $J_{x_0,\rho_0}$ are constant on the interval $(x_{0,1} - \rho_0, x_{0,1} + \rho_0)$, and relations (8.15) hold. The proof of the proposition is hence complete.

8.8.2 Some additional properties near regular points

We derive next some additional properties for regular points, in connection with the singular part of the measures. We introduce therefore the set

$$B_* = \{ s \in \mathbb{R} \text{ such that } (\{ s \} \times \mathbb{R}) \cap \mathcal{B}_* \neq \emptyset \} = \mathcal{P}(\mathcal{B}_*).$$

where $\mathcal{B}_*$ is defined in (70) and represents the set where the singular part of the measures concentrates. Notice that, since $\mathcal{H}^1(\mathcal{B}_*) = 0$, the Lebesgue measure of the set $B_*$ vanishes likewise. Recall, in view of Corollary 8.1, that we have $J_{1,r} = J_{1,r}dx_1$ and $L_{1,r} = L_{1,r}dx_1$, where the function $L_{1,r}$ and $J_{1,r}$ are bounded. We have first:
Lemma 8.3. Let \( x_0 = (x_{0,1}, x_{0,2}) \) be a regular point in \( \mathfrak{H} \) \( \setminus \mathfrak{A} \) and let \( \rho_0 \) be given by Proposition 8.4. Let \( \theta \in \left[ 0, \frac{\pi}{8} \right) \). We have, for any \( r \leq \frac{1}{2} \inf \{ \text{Rcone}(\theta, x_0, \rho_0) \}, \)

\[
\int_{x_{0,1} - 2r}^{x_{0,1} + 2r} |J_{1, \rho_0}(s)| ds \leq 4r \sin \theta \nu_{\star}^{ac}(\mathbb{D}^2(x_0, 2r)) \quad \text{and} \quad \int_{x_{0,1} - 2r}^{x_{0,1} + 2r} |L_{1, \rho_0}(s)| ds \leq 8r \sin \theta \nu_{\star}^{ac}(\mathbb{D}^2(x_0, 2r)).
\]

(8.37)

Proof. If \( 2r \leq \text{Rcone}(\theta, x_0) \), it follows from (65) that we have \( \nu_\star(\mathbb{R}_2(x_0) \setminus \mathcal{C}_\text{cone}(x_0, \varepsilon_{x_0}, \theta)) = 0 \). On the other hand, we have

\[
|x_2 - x_{0,2}| \leq 2r \sin \theta \quad \text{for } x = (x_1, x_2) \in \mathbb{R}_2(x_0) \cap \mathcal{C}_\text{cone}(x_0, \varepsilon_{x_0}, \theta).
\]

Multiplying by \( \text{Im}(\omega_\star) \) and integrating on the set \( \mathbb{R}_2(x_0) \setminus \mathbb{B}_\star \times \mathbb{R} \), we are led to

\[
\int_{\mathbb{R}_2(x_0) \setminus \mathbb{B}_\star \times \mathbb{R}} d|\mu_\star,1,2(\omega_\star)(x_2 - x_{0,2})| \leq 4r \sin \theta \nu_\star(\mathbb{D}^2(x_0, 2r) \setminus \mathbb{B}_\star \times \mathbb{R}) \leq 4r \sin \theta \nu_{\star}^{ac}(\mathbb{D}^2(x_0, 2r))
\]

(8.38)

For the last inequality, we invoke the fact that we have the inclusion \( \mathbb{D}^2(x_0, 2r) \setminus \mathbb{B}_\star \times \mathbb{R} \subset \mathbb{D}^2(x_0, 2r) \setminus \mathbb{B}_\star \), so that

\[
\nu_\star(\mathbb{D}^2(x_0, 2r) \setminus \mathbb{B}_\star \times \mathbb{R}) \leq \nu_\star(\mathbb{D}^2(x_0, 2r) \setminus \mathbb{B}_\star) = \nu_{\star}^{ac}(\mathbb{D}^2(x_0, 2r)).
\]

Since, by definition \( J_{1, r} = \mathbb{D}_\star(\mu_\star,1,2(x_2 - x_{0,2})) \), we have hence, in view of (41)

\[
\int_{(x_{0,1} - 2r, x_{0,1} + 2r) \setminus \mathbb{B}_\star} |J_{1, \rho_0}(s)| ds \leq \int_{\mathbb{R}_2(x_0) \setminus \mathbb{B}_\star \times \mathbb{R}} d|\mu_\star,1,2(\omega_\star)(x_2 - x_{0,2})|.
\]

(8.39)

Combining (8.38), (8.39), together with the fact that \( \mathbb{B}_\star \) has zero Lebesgue measure and the function \( J_{1, \rho_0} \) is bounded, thus integrable, we deduce the first inequality in (8.37). The second is established invoking similar arguments. 

\[\square\]

Lemma 8.4. Let \( x_0 = (x_{0,1}, x_{0,2}) \) be a regular point \( \mathfrak{H} \) \( \setminus \mathfrak{A} \), and and let \( \rho_0 \) be given by Proposition 8.4. Let \( \theta \in \left[ 0, \frac{\pi}{8} \right) \). For any \( 0 < r < \frac{1}{2} \inf \{ \text{Rcone}(\theta, x_0, \rho_0) \} \), there exists some \( \rho_r \in [r, 2r] \) such that

\[
\left\{ \begin{array}{c}
\int_{x_{0,1} - \rho_r}^{x_{0,1} + \rho_r} |J_{\rho_0}(s)| ds \\
\int_{x_{0,1} - \rho_r}^{x_{0,1} + \rho_r} |L_{\rho_0}(s)| ds
\end{array} \right\} \leq 16 \sin \theta \nu_{\star}^{ac}(\mathbb{D}^2(x_0, 2r)) \quad \text{and} \quad \int_{x_{0,1} - \rho_r}^{x_{0,1} + \rho_r} \text{d}\mathcal{N}_{\rho_0} \leq 8 \sin \theta \nu_{\star}^{ac}(\mathbb{D}^2(x_0, 2r)).
\]

(8.40)

Proof. The proof of (8.40) follows from (8.37) integrating the differential equations (8.15) for \( k = 1 \). Indeed, for almost every \( \rho \in [r, 2r] \), \( x_{0,1} - \rho \) and \( x_{0,1} + \rho \) are Lebesgue points of \( J_{1, \rho_0}, L_{1, \rho_0}, J_{\rho_0} \) and the absolutely continuous part of \( \mathcal{N}_{\rho_0} \). We choose next a sequence of test functions \( \{ \psi_m \}_{m \in \mathbb{N}} \) such that such that \( 0 \leq \psi_m \leq 1 \), for any \( m \in \mathbb{N} \), and

\[
\psi_m \rightarrow 1_{(x_{0,1} - \rho, x_{0,1} + \rho)} \quad \text{in } L^1(L_r(x_{0,1})).
\]

(8.41)
In view of the differential equation (8.37) we have, for any \( m \in \mathbb{N} \)

\[
\begin{align*}
\left\{ \begin{array}{l}
\int_{x_0,1-\varrho}^{x_0,1+\varrho} \psi_m(s) J_{\rho_0}(s) ds = -\int_{x_0,1-\varrho}^{x_0,1+\varrho} \psi_m(s) L_{1,\rho_0}(s) ds \\
\int_{x_0,1-\varrho}^{x_0,1+\varrho} \psi_m \lambda \rho_0 = 0
\end{array} \right.
\end{align*}
\]  
(8.42)

Passing to the limit, we obtain, using the Lebesgue properties of the points \( x_0,1-\varrho \) and \( x_0,1+\varrho \)

\[
\begin{align*}
\int_{x_0,1-\varrho}^{x_0,1+\varrho} \lambda \rho_0 = J_{1,\rho_0}(x_0,1+\varrho) - J_{1,\rho_0}(x_0,1-\varrho)
\end{align*}
\]  
(8.43)

Next, we use a mean value argument to deduce that there exists some number \( \varrho_r \in [\varrho,2\varrho] \), such that \( x_0,1-\varrho_r \) and \( x_0,1+\varrho_r \) are Lebesgue points of \( J_{1,\rho_0}, L_{1,\rho_0}, J_{\rho_0} \) and the absolutely continuous part of \( \lambda \rho_0 \) and such that

\[
\begin{align*}
\left\{ \begin{array}{l}
|J_{1,\rho_0}(x_0,1+\varrho_r)| + |J_{1,\rho_0}(x_0,1-\varrho_r)| \leq \frac{2}{\varrho} \int_{x_0,1-2\varrho}^{x_0,1+2\varrho} |J_{1,\rho_0}(s)| ds \\
|L_{1,\rho_0}(x_0,1+\varrho_r)| + |L_{1,\rho_0}(x_0,1-\varrho_r)| \leq \frac{2}{\varrho} \int_{x_0,1-2\varrho}^{x_0,1+2\varrho} |L_{1,\rho_0}(s)| ds
\end{array} \right.
\end{align*}
\]  
(8.44)

Combining (8.43), (8.44) with (8.37), we obtain the desired result. \( \square \)

### 8.9 Behavior near Lebesgue points: Proofs to Proposition 5 and Lemma 2

Recall that, at this stage we already know that, if \( x_0 \in \mathcal{S}_* \setminus \mathcal{A}_* \), in view of Propositions 8.2 and 8.3

\[
\mathbb{L}_{x_0,\rho_0} = L_{0,\rho_0} dx_1 \text{ and } \mathbb{P}_{\rho_1}(\hat{\mu}_*,\rho_1,2) = J_{0,\rho_0} dx_1,
\]

where \( L_{0,\rho_0} \in \mathbb{R} \) and \( J_{0,\rho_0} \in \mathbb{R} \). We derive here additional properties in the case \( x_0 \notin \mathcal{C}_* \), leading eventually to the proof of Proposition 5.

#### 8.9.1 Additional properties of \( J_{x_0,\rho_0} \) and \( \lambda_{x_0,\rho_0} \) at Lebesgue points

Let \( x_0 \in \mathcal{S}_* \) and \( \rho_0 > 0 \). We impose in this paragraph the additional condition that \( x_0 \notin \mathcal{C}_* \), i.e. \( x_0 \) is a regular point, which is not on the support of the singular part, and is moreover a Lebesgue point for the densities of the absolutely continuous part for all measures of interest. More precisely, this means that

\[
\begin{align*}
\lim_{r \to 0} \frac{1}{r} \int_{x_0,1-2\varrho}^{x_0,1+2\varrho} \left| \Theta_*(\tau) - \Theta_*(x_0) \right| d\tau = 0 \\
\lim_{r \to 0} \frac{1}{r} \int_{x_0,1-2\varrho}^{x_0,1+2\varrho} \left| \phi_*(\tau) - \phi_*(x_0) \right| d\tau = 0, \text{ and} \\
\lim_{r \to 0} \frac{1}{r} \int_{x_0,1-2\varrho}^{x_0,1+2\varrho} \left| m_{\tau,i,j}(\tau) - m_{\tau,i,j}(x_0) \right| d\tau = 0
\end{align*}
\]  
(8.45)
As a first direct consequence, we deduce that, for some constant $K = K(x_0) > 0$ depending on $x_0$, we have
\begin{equation}
\nu^a(x; \mathbb{D}^2(x_0, r)) \leq Kr \text{ for any } 0 < r < R, \tag{8.46}
\end{equation}
and also that
\begin{align*}
&\lim_{r \to 0} \frac{1}{r} \int_{\mathbb{S}_r \cap \mathbb{D}^2(x_0, r)} \Theta_*(\tau) d\tau = \Theta_*(x_0), \\
&\lim_{r \to 0} \frac{1}{r} \int_{\mathbb{S}_r \cap \mathbb{D}^2(x_0, r)} \nu_*(\tau) d\tau = \nu_*(x_0), \text{ and} \tag{8.47}
\end{align*}
\begin{align*}
&\lim_{r \to 0} \frac{1}{r} \int_{\mathbb{S}_r \cap \mathbb{D}^2(x_0, r)} m_{*,i,j}(\tau) d\tau = m_{*,i,j}(x_0).
\end{align*}

At this stage, we already know that $J_{\rho_0}$ is a constant map. Concerning $N_{\rho_0}$ we may decompose this measure on $I_{\rho_0}(x_0, 1)$ as a sum of an absolutely continuous part and a singular part
$$N_{\rho_0} = N_{\rho_0}^{ac} + N_{\rho_0}^s,$$
so that there exists a set $F_{\rho_0} \subset I_{\rho_0}(x_0, 1)$ such that $\mathcal{H}^1(F_{\rho_0}) = 0$ and $N_{\rho_0}^s(I_{\rho_0}(x_0, 1 \setminus F_{\rho_0})) = 0$.

Let $x_0 \in \mathcal{B}_* \setminus \mathcal{E}_*$ and $\rho_0 > 0$ be given by Proposition 8.4 so that (8.8) holds for $\rho = \rho_0$. Choose the orthonormal basis so that $\vec{e}_1 = \vec{e}_{x_0}$ is a unit tangent vector to $\mathcal{S}_*$ at $x_0$. Then, $x_{0, 1} \not\in F_{\rho_0}$ and is a Lebesgue point for $N_{\rho_0}$ and $J_{\rho_0}(x_0)$. We have the identities, at the point $x_0$,
\begin{align*}
\begin{cases}
N_{\rho_0}(x_{0, 1}) = 2\Theta_*(x_0) - m_{*,2,2}(x_0) + m_{*,1,1}(x_0), \\
J_{\rho_0}(x_{0, 1}) = m_{*,1,2}(x_0) \text{ and} \\
L_{\rho_0}(x_{0, 1}) = 2\Theta_*(x_0) - m_{*,1,1}(x_0) + m_{*,2,2}^a(x_0).
\end{cases} \tag{8.48}
\end{align*}

**Proof.** We go back to the definition (72) of $\mathcal{E}_*$. Since $x_0 \not\in \mathcal{E}_*$, and hence $x_0 \not\in \mathcal{B}_*$ (see (70)), we have by definition of the set $\mathcal{B}_*$
\begin{align*}
D_\lambda(\nu_*)(x_0) = \lim_{r \to 0} \frac{\nu_*(\mathbb{D}^2(x_0, r))}{\lambda(\mathbb{D}^2(x_0, r))} < +\infty \text{ and } D_\lambda(\nu_*)^a(x_0) = \lim_{r \to 0} \frac{\nu_*^a(\mathbb{D}^2(x_0, r))}{\lambda(\mathbb{D}^2(x_0, r))} = 0, \tag{8.49}
\end{align*}
where $\lambda$ represents the one-dimensional Hausdorff measure on $\mathcal{S}_*$. On the other hand, since $x_0$ is a regular point, we have, in view of (8)
\begin{align*}
\lim_{r \to 0} \frac{\lambda(\mathbb{D}^2(x_0, r))}{2r} = 1,
\end{align*}
so that
\begin{align*}
D_\lambda(\nu_*)(x_0) = D_\lambda(\nu_*^a)(x_0) = \lim_{r \to 0} \frac{\nu_*(\mathbb{D}^2(x_0, r))}{2r} < +\infty. \tag{8.50}
\end{align*}

Turning to the measure $\tilde{\nu}_*^{\mathcal{E}_1}$, we have $\nu_*^{\mathcal{E}_1}(I_{\mathcal{F}_1}(x_{0, 1})) = \tilde{\nu}_*(I_{\mathcal{F}_1}(x_{0, 1}) \times I_{\mathcal{T}(x_{0, 2}))}$. In view of Proposition 4, given $\theta > 0$, we have, for $r \leq R_{\text{cone}}(\theta, x_0)$, the inclusion $\mathcal{S}_* \cap \mathbb{D}^2(x_0, r) \subset \mathcal{C}_{\text{cone}}(x_0, \vec{e}_{x_0}, \theta)$. On the other hand, we have also the chain of inclusions
\begin{align*}
\mathbb{D}^2(x_0, r) \subset (I_{\mathcal{T}(x_{0, 1})} \times I_{\mathcal{T}(x_{0, 2}))} \cap \mathcal{C}_{\text{cone}}(x_0, \vec{e}_{x_0}, \theta) \subset \mathbb{D}^2(x_0, \frac{r}{\cos \theta}), \tag{8.51}
\end{align*}
so that combining the previous relations, we are led to the bounds
\[
\nu_*(\mathbb{D}^2(x_0, r)) \leq \nu_*^x(\mathcal{I}_r(x_{0,1})) \leq \nu_*\left(\mathbb{D}^2(x_0, \frac{r}{\cos \theta})\right). \tag{8.52}
\]
Letting \( \theta \) and \( r \) go to zero, we deduce from (8.49) and (8.52) the identity
\[
\lim_{r \to 0} \frac{\nu_*^x(\mathcal{I}_r(x_{0,1}))}{2r} = D_\lambda(\nu_*)(x_0) = D_\lambda(\nu_*^{ac})(x_0) < +\infty,
\]
and similarly, for \( i, j = 1, 2 \)
\[
\begin{align*}
\lim_{r \to 0} \frac{\xi_1^x(\mathcal{I}_r(x_{0,1}))}{2r} &= D_\lambda(\zeta_*^{ac})(x_0) = \Theta_*(x_0) \quad \text{and} \\
\lim_{r \to 0} \frac{\mu_1^x(x_{0,1})}{2r} &= D_\lambda(\mu_{*,i,j})^x(x_0) = m_{*,i,j}(x_0) < +\infty.
\end{align*}
\]
It follows that, in view of the definition (76) of \( \mathbb{N}_{\rho_0} \), we have
\[
\lim_{r \to 0} \frac{\mathbb{N}_{\rho_0}(\mathcal{I}_r(x_{0,1}))}{2r} = 2D_\lambda(\zeta_{*,i,j}^{ac})(x_0) - D_\lambda(\mu_{2,2}^{ac})(x_0) + D_\lambda(\mu_{1,1}^{ac})(x_0) \in \mathbb{R}
\]
\[
= 2\Theta_{*,i,j}^{ac}(x_0) - m_{2,2}^{ac}(x_0) + m_{1,1}^{ac}(x_0).
\]
We deduce that \( x_{0,1} \notin F_{\rho_0} \) and that we have
\[
\lim_{r \to 0} \frac{\mathbb{N}_{\rho_0}^{ac}(\mathcal{I}_r(x_{0,1}))}{2r} = \lim_{r \to 0} \frac{\mathbb{N}_{\rho_0}^{ac}(\mathcal{I}_r(x_{0,1}))}{2r} = 2\Theta_{*,i,j}^{ac}(x_0) - m_{2,2}^{ac}(x_0) + m_{1,1}^{ac}(x_0).
\]
We prove using similar arguments that \( x_{0,1} \) is a Lebesgue point for the map \( N_{\rho_0} \), so that the first identity in (8.48) is established. Turning to the maps \( J_{\rho_0} \) and \( L_{\rho_0} \), we observe that, since these maps are constant, \( x_{0,1} \) is obviously a Lebesgue point for them. The two last identities in (8.48) are established using the same arguments.

We compute next \( J_{\rho_0}(x_0) \) and \( N_{\rho_0}(x_0) \) in a different way.

**Proposition 8.6.** Let \( x_0 \) and \( \rho_0 > 0 \) be as in Proposition 8.5. We have
\[
\begin{align*}
J_{x_0, \rho_0}(s) &= 0 \quad \text{for } s \in (x_{0,1} - \rho_0, x_{0,1} + \rho_0) \quad \text{and} \\
N_{x_0, \rho_0}(x_{0,1}) &= 0. \tag{8.53}
\end{align*}
\]

In order to proof Proposition 8.6, we rely on an intermediate result:

**Lemma 8.5.** Let \( x_0 \in \mathcal{E}_* \setminus \mathcal{E}_r \) and \( \rho_0 > 0 \) be given by Proposition 8.4. Choose the orthonormal basis so that \( \bar{e}_1 = \bar{e}_{x_0} \) is a unit tangent vector to \( \Theta_* \) at \( x_0 \). For \( \varrho < \rho_0 \), let \( \varrho_0 > 0 \) be given by Lemma 8.4. Then, we have
\[
\lim_{r \to 0} \frac{1}{2\varrho_0} \int_{x_{0,1} - \varrho_0}^{x_{0,1} + \varrho_0} \frac{d\mathbb{N}_r(s)}{ds} = 0 \quad \text{and} \quad \lim_{r \to 0} \frac{1}{2\varrho_0} \int_{x_{0,1} - \varrho_0}^{x_{0,1} + \varrho_0} J_r(s) ds = 0. \tag{8.54}
\]
We first let \( \rho = \rho_0 \) with (8.54), we obtain (73) and the proof is complete.

Since \( x \in \mathbb{R} \), we obtain (8.54). Combining with the first identity in (8.54), we are led to (8.55).

We first let \( r \to 0 \), so that \( \varrho \to 0 \) as \( r \to 0 \), and then let \( \theta \to 0 \) in (8.56), which yields (8.55).

Proof of Proposition [8.6] completed. We first consider \( J_{\rho_0} \). We already know that the function \( J_{\rho_0} \) is constant on \( I_{\rho_0}(x_{0,1}) \), so that

\[
\frac{1}{2\varrho} \int_{x_{0,1} - \varrho r}^{x_{0,1} + \varrho r} J_r(s) ds = J_r(x_{0,1}),
\]

we deduce therefore from the second relation in (8.54) that \( J_{\rho_0}(x_{0,1}) = 0 \).

We turn to \( N_{\rho_0} \). Since \( x_0 \notin F_{\rho_0} \), we have \( D\lambda(N_{\rho_0}^s)(x_{0,1}) = 0 \), that is

\[
\lim_{r \to 0} \frac{N_{\rho_0}^s(I_{\varrho r}(x_{0,1}))}{2\varrho} = 0.
\]

Combining with the first identity in (8.54), we are led to

\[
\lim_{r \to 0} \frac{1}{2\varrho} \int_{x_{0,1} - \varrho r}^{x_{0,1} + \varrho r} N_{\rho_0}(s) ds = \lim_{r \to 0} \frac{1}{2\varrho} \int_{x_{0,1} - \varrho r}^{x_{0,1} + \varrho r} dN_{\rho_0}^s(s) = 0. \quad (8.57)
\]

Since \( x_{0,1} \) is a Lebesgue point for \( N_{\rho_0} \), we derive that \( N_{\rho_0}(x_{0,1}) = 0 \), so that the proof is complete.

8.9.2 Proof of Proposition [5] completed

Since \( x_0 \in \mathcal{G}_* \setminus \mathcal{E}_* \), we are in position to apply Propositions [8.5] and [8.6] Combining (8.48) with (8.54), we obtain (73) and the proof is complete.

8.9.3 Change of orthonormal basis for the Hopf differential

Recall that we have assumed in Proposition [5] that the orthonormal basis is chosen so that \( \mathbf{e}_1 \) is tangent to \( \mathcal{G}_* \) at \( x_0 \). However, the definition of the Hopf differential clearly depends on the choice of coordinates, and we will need to compute it in various basis, for instance a moving frame on \( \mathcal{G}_* \) or a frame related to polar coordinates. For that purpose, and for a given angle \( \theta \in \mathbb{R} \), let \( (\mathbf{e}_1^\theta, \mathbf{e}_2^\theta) \) be a new orthonormal basis such that

\[
\begin{align*}
\mathbf{e}_1^\theta &= \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2, \\
\mathbf{e}_2^\theta &= -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2.
\end{align*}
\]
Let \((x_{\theta,1}, x_{\theta,2}) = (\cos \theta x_1 + \sin \theta x_2, -\sin \theta x_1 + \cos \theta x_2)\) denote the coordinates related to the new basis and \(\omega_{\theta}^\gamma\) the corresponding Hopf differential. Then, for any map \(u : \Omega \to \mathbb{R}^2\), we have the identities \(u_{x_{\theta,1}} = u x_1 \cos \theta + u x_2 \sin \theta\) and \(u_{x_{\theta,2}} = -u x_1 \sin \theta + u x_2 \cos \theta\), so that
\[
\begin{align*}
\left\{ \begin{array}{l}
|u_{x_{\theta,1}}|^2 - |u_{x_{\theta,2}}|^2 = \cos 2\theta \left( |u x_1|^2 - |u x_2|^2 \right) + 2 \sin 2\theta u x_1 \cdot u x_2 \\
2u_{x_{\theta,1}} \cdot u_{x_{\theta,2}} = -\sin 2\theta \left( |u x_1|^2 - |u x_2|^2 \right) + 2 \cos 2\theta u x_1 \cdot u x_2.
\end{array} \right.
\tag{8.59}
\]
We are hence led to the transformation law
\[
\begin{align*}
\omega_{\epsilon, \theta} &= (\cos 2\theta + i \sin 2\theta) \omega_{\epsilon} = \exp(2i\theta) \omega_{\epsilon} \quad \text{and} \\
\omega_{\epsilon, \theta}^\gamma &= (\cos 2\theta + i \sin 2\theta) \omega_{\epsilon}^\gamma = \exp(2i\theta) \omega_{\epsilon}^\gamma.
\tag{8.60}
\end{align*}
\]
It follows in particular from the above relations that, if the limits \([8.1]\) and \([13]\) exist for a given orthonormal basis, then they exist also for any other one.

### 8.9.4 Proof of Lemma 2 completed

In view of \([42]\), we may write, in the basis \((\vec{e}_1, \vec{e}_2)\)
\[
\omega_{\epsilon}^{ac} = ((m_{*,1,1} - m_{*,2,2}) - 2im_{1,2}) d\lambda.
\tag{8.61}
\]
Next let \(x_0 \in \mathcal{G}_* \setminus \mathcal{E}_*, \vec{e}_{x_0}\) be a tangent vector at \(x_0\) to \(\mathcal{G}_*\), so that the angle of \(\vec{e}_1\) with \(\vec{e}_{x_0}\) is given by \(\gamma_*(x_0) \in [-\pi/2, \pi/2]\). In view of the notation \([8.58]\), we have \(\vec{e}_{x_0} = \vec{e}_1^{\gamma_*(x)}\). It follows from \([8.60]\) that
\[
\omega_{\epsilon, \gamma_*(x)}^{ac} = \exp(2i\gamma_*) \omega_{\epsilon}^{ac} = \exp(2i\gamma_*) ((m_{*,1,1} - m_{*,2,2}) - 2im_{1,2}) d\lambda.
\tag{8.62}
\]
Applying Proposition \([5]\) at \(x_0\) in the basis \((\vec{e}_1^{\gamma_*(x)}, \vec{e}_2^{\gamma_*(x)})\), we are led to the identity
\[
\exp(-2i\gamma_*) ((m_{*,1,1}(x_0) - m_{*,2,2}(x_0)) - 2im_{1,2}(x_0)) = -2\Theta_*(x_0),
\]
so that, for any \(x \in \mathcal{G}_* \setminus \mathcal{E}_*,\) we have
\[
(m_{*,1,1}(x) - m_{*,2,2}(x)) - 2im_{1,2}(x) = -2 \exp(2i\gamma_*(x)) \Theta(x).
\]
Going back to \([8.61]\), we obtain hence that
\[
\omega_{\epsilon}^{ac} = -2 \exp(-2i\gamma_*(x)) \Theta d\lambda = -2 \exp(-2i\gamma_*) \zeta_{\epsilon}^{ac}.
\]
The proof is hence complete.

### 9 Monotonicity for \(\zeta_*\) and its consequence

The purpose of present section is to establish Proposition \([3]\)
9.1 Proof of Lemma 9.1

Since \(\xi_n \to \xi_*\), as \(n \to +\infty\), weakly in the sense of measures, we have for any Borel set \(A\) such that \(\xi_*(\partial A) = 0\), the convergence \(\xi_n(A) \to \xi_*(A),\) as \(n \to +\infty\). Since \(\nu_*(\partial D^2(x_0, r)) = 0\) for almost every \(r \in (0, \rho)\), we have

\[
\begin{array}{l}
\xi_n(D^2(x_0, r_i)) \to \xi_*(D^2(x_0, r_i)) \text{ for almost every } r_i \in (0, \rho), \ i = 0, 1 \text{ and }
\int_{D^2(x_0, r_1) \setminus D^2(x_0, r_0)} \frac{1}{r} d\nu_* \to \int_{D^2(x_0, r_1) \setminus D^2(x_0, r_0)} \frac{1}{r} d\nu_*.
\end{array}
\]

Passing to the limit in (79), we obtain the identity (80).

9.2 First properties of \(\mathcal{M}_*\)

Let \(\mu_{*, \theta, \theta}\) and \(\mu_{*, r, r}\) be defined by (81) on \(D^2(x_0, \rho)\). We denote by \(\mu_{*, \theta, \theta, r, r}\) the absolutely continuous parts of these measures with respect to the \(\mathcal{H}^1\)-Hausdorff measure \(\lambda\) on \(\mathcal{S}_* \cap D^2(x_0, \rho)\). We prove in this subsection:

Lemma 9.1. We have the relations

\[
\mathcal{M}_*^{ac} = (2\xi_*^{ac} - r^{-2}\mu_{*, \theta, \theta}^{ac} + \mu_{*, r, r}^{ac}) \perp D^2(x_0, \rho) = 4\sin^2(\gamma_* - \theta)(\xi_*^{ac} \perp D^2(x_0, \rho)) \geq 0,
\]

where \(\theta\) denotes the polar angle with respect to the \(x_0\).

Remark 9.1. Let \(\nabla r\) denote the gradient of the function \(r : (x_1, x_2) \mapsto \sqrt{x_1^2 + x_2^2}\), so that \(\nabla r(x) = (x_1/r, x_2/r)\). For given \(x \in (\mathcal{S}_* \setminus \mathcal{E}_*) \cap D^2(x_0, \rho)\), we denote by \(\nabla^\perp r(x)\), the projection of \(\nabla r(x)\) onto the orthogonal line to the tangent to \(\mathcal{S}_*\) at the point \(x\). We compute

\[
|\nabla^\perp r(x)| = |\sin(\gamma_*(x) - \theta)|.
\]

Formula (9.2) can therefore be rewritten as

\[
\mathcal{M}_*^{ac} = |\nabla^\perp r|^2 \xi_*^{ac} \geq 0.
\]

Proof of Lemma 9.1. We may write

\[
\mu_{*, r, r}^{ac} = m_{*, r, r} d\lambda \text{ and } r^2 \mu_{*, \theta, \theta}^{ac} = r^2 m_{*, \theta, \theta} d\lambda,
\]

where, similar to (81), we have set, for \(x \in (\mathcal{S}_* \setminus \mathcal{E}_*) \cap D^2(x_0, \rho)\),

\[
\begin{aligned}
m_{*, r, r}(x) &= \cos^2 \theta(x)m_{*, 1,1}(x) + \sin^2 \theta(x)m_{*, 2,2}(x) + 2\sin \theta \cos \theta(x)m_{*, 1,2}(x) \\
r^{-2}m_{*, \theta, \theta}(x) &= \sin^2 \theta(x)m_{*, 1,1}(x) + \cos^2 \theta(x)m_{*, 2,2}(x) - 2\sin \theta \cos \theta(x)m_{*, 1,2}(x).
\end{aligned}
\]

where \((r, \theta)\) denote the polar coordinates of \(x = (x_1, x_2)\), so that \(x_1 - x_0,1 = r \cos \theta\) and \(x_2 - x_0,2 = r \sin \theta\). We have, in view of Proposition 2 and relations (8.60)

\[
\omega_*^{ac} = -2 \exp(2i(\gamma_*(x) - \theta))\xi_*^{ac}.
\]

Since \(\omega_*^{ac,0}\) is absolutely continuous with respect to \(d\lambda\), we may write \(\omega_*^{ac} = w_{*, \theta, 0} d\lambda\), where \(w_{*, \theta}\) is a function on \(\mathcal{S}_* \cap D^2(x_0, \rho)\). Concerning the measure \(\mathcal{M}_*^{ac}\), we have

\[
\mathcal{M}_*^{ac} = (2\Theta - r^{-2}m_{*, \theta, \theta} + m_{*, r, r}) d\lambda.
\]
It follows from the definitions (9.5) and (9.4), that we have the identity
\[ w_{*,\theta}(x) = (m_{*,r,r}(x) - r^{-2}m_{*,\theta,\theta}(x)) - 2ir^{-1}m_{*,r,\theta}(x). \] (9.8)

Combining (9.6) and (9.8), we are hence led to
\[ m_{*,r,r}(x) - r^{-2}m_{*,\theta,\theta}(x) = -2 \cos (\gamma_*(x) - \theta) \Theta_*(x), \]
so that
\[ (2\Theta(x) - r^2m_{*,\theta,\theta}(x) + m_{*,r,r}(x)) = 2(1 - \cos (\gamma_*(x) - \theta) \Theta_*(x)) \]
\[ = 4\sin^2 (\gamma_*(x) - \theta) \Theta_*(x). \]

Going back to (9.7), we deduce that
\[ N_{ac}^\ast = 4\sin^2 (\gamma_*(x) - \theta) \zeta_{ac}^\ast, \]
so that (9.2) is established. \qed

9.3 Integrating on growing disks

Let \( \rho > \delta > 0 \) be given. We introduce and study in this section the functions \( V, F \) and \( G_\delta \) defined on the interval \([\delta, \rho]\), by
\[
\begin{align*}
V(r) & = \zeta_*(B^2(x_0, r)), \\
F(r) & = \frac{V(r)}{r} \quad \text{and} \\
G_\delta(r) & = \int_{D^2(x_0, r) \setminus D^2(x_0, \delta)} \frac{1}{r} dN_*, \quad \text{for } x \in [\delta, \rho].
\end{align*}
\] (9.9)

The three functions defined in (9.9) are clearly bounded on the interval \([\delta, \rho]\), since \( 0 \leq V(r) \leq V(\rho) \), \( 0 \leq F(r) \leq F(\rho) / \delta \) and \( |G_\delta(r)| \leq 2v_* (D^2(x_0, \rho)) / \delta \). Moreover, the function \( V \) is clearly non-decreasing. We will show below that these functions have bounded variation.

In order to relate these functions and their derivatives to the measures on \( D^2(x_0, \rho) \) introduced so far, we have to eliminate the polar angle \( \theta \). For that purpose, we consider the map \( \Pi : D^2(x_0, \rho) \setminus \{0\} \to (0, \rho) \) defined by
\[ \Pi(x) = r = \sqrt{(x_1 - x_{0,1})^2 + (x_2 - x_{0,2})^2}, \quad \text{for } x = (x_1, x_2) \in D^2(x_0, \rho), \]
so that \( \Pi^{-1}(\rho) = S^1(x_0, \rho) \), for any \( \rho \in (0, \rho) \). We define the measures \( \tilde{\zeta}_* \) and \( \tilde{N}_* \) on \([\delta, \rho]\) by
\[ \tilde{\zeta}_* = \Pi_* (\zeta_*) \quad \text{and} \quad \tilde{N}_* = \Pi_* (N_*). \]

More precisely, for any Borel subset of \((\delta, \rho)\), we have
\[ \tilde{\zeta}_*(A) = \zeta_*(\Pi^{-1}(A)) \quad \text{and} \quad \tilde{N}_*(A) = (\Pi^{-1}(A)). \] (9.10)

We first have:

Lemma 9.2. The function \( V \) and \( G_\delta \) have bounded variation. We have
\[ \frac{d}{dr} V = \tilde{\zeta}_* \geq 0, \quad \text{and} \quad \frac{d}{dr} G_\delta = r^{-1} \tilde{N}_* \quad \text{in the sense of distributions } D'(\delta, \rho). \] (9.11)
Proof. We first observe that, as a consequence of the definition (9.9) of $f(r)$, we have the identities

$$V(r) = \zeta_\star(0,r) = \int_0^r d\zeta_\star = \int_0^\rho 1_{(0,r)}d\zeta_\star. \tag{9.10}$$

The desired result (9.11) is then a direct consequence of Fubini’s Theorem. Indeed, let $\varphi \in C_c(\delta,\rho)$. We have

$$\int_\delta^\rho \varphi'(r)V(r)dr = \int_0^\rho \varphi'(r) \left[ \int_0^\rho d\zeta_\star \right] dr = \int_0^\rho \varphi'(r)1_{(0,r)}d\zeta_\star dr = \int_0^\rho \varphi(r)d\zeta_\star, \tag{9.12}$$

which establishes the first identity in (9.12). The second is proved using the same argument. Finally, since $\zeta_\star$ and $r^{-1}N_\star$ are bounded measures, it follows that the functions $V$ and $G_\delta$ have bounded variation.

For the proof of Proposition 6, we will make use of the fact that the derivative of $F$ may be written in two different ways, as stated in the next Lemma.

**Lemma 9.3.** The function $F$ has bounded variation. We have the identities

$$\frac{d}{dr} F = \frac{1}{r} \zeta_\star - \frac{1}{r^2} V = \frac{1}{r} \tilde{N}_\star, \text{ in the sense of distributions } \mathcal{D}'(\delta, \rho). \tag{9.13}$$

**Proof.** The first identity in (9.13) corresponds to the Leibnitz rule applied to the product $F = V$ of the measure $V$, handled as a distribution on $(\delta, \rho)$, by the smooth function $r \mapsto \frac{1}{r}$. It yields

$$\frac{d}{dr} F = -\frac{V}{r^2} + \frac{1}{r} \frac{d}{dr} V, \text{ in the sense of distributions,}$$

so that the first identity in (9.13) follows, in view of the first identity in (9.12).

For the second identity, we invoke Lemma 3 which asserts that, for almost every $r \in (\delta, \rho)$, we have

$$F(r) - F(\delta) = \int_{\mathbb{B}^2(x_0,r) \setminus \mathbb{B}^2(x_0,\delta)} \frac{1}{4\pi r} d\tilde{N}_\star = G_\delta(r). \tag{9.14}$$

Taking the derivative, in the sense of distributions, of this identity, the second identity in (9.13) then follows from the second identity in (9.12). \qed

**9.4 Refined analysis of the derivative of $F$: Proof of Proposition 6**

In this subsection, we make use of the two different forms of the derivative $F' = r^{-1}_{*}N_\star$ provided by Lemma 9.3 in order to show that this distribution is actually a non-negative measure. We first have:

**Lemma 9.4.** Set $\mathcal{B}_\rho = \Pi^{-1}(\mathfrak{C}_* \cap \mathbb{B}^2(\rho))$. We have $\mathcal{H}^1(\mathcal{B}_\rho) = 0$ and

$$\mathcal{N}_* \ll ((0, \rho) \setminus \mathcal{B}_\rho) \geq 0. \tag{9.15}$$
Proof. Since $\mathcal{H}^1(\mathcal{E}_*;\mathbb{E}) = 0$, we deduce that $\mathcal{H}^1(\mathcal{B}_\rho;\mathbb{E}) = 0$. Recall that

$$\mathcal{N}_* = \mathcal{N}_*^{ac} \text{ on } \mathbb{D}^2(\rho) \setminus \mathcal{E}_*, \quad (9.16)$$

whereas in view of Lemma 9.1, we have $\mathcal{N}_*^{ac} \geq 0$. Combining this inequality with (9.16) we obtain

$$\mathcal{N}_* \subseteq (\mathbb{D}^2(x_0, \rho) \setminus \mathcal{E}_*) \geq 0.$$  

In view of the definition of $\mathcal{N}_*$, we obtain hence (9.15).

It remains to study $\mathcal{N}_*^{\prime} \subseteq \mathbb{B}_\rho$. We have:

**Lemma 9.5.** The restriction of $\mathcal{N}_*$ to $\mathbb{B}_\rho$ is non-negative, i.e.

$$\mathcal{N}_* \subseteq \mathbb{B}_\rho \geq 0. \quad (9.17)$$

Proof. Recall, that, in view of Lemma 9.3, we have in the sense of distributions

$$\mathcal{N}_* = \zeta_* - \frac{V}{r \ln \rho} \text{ in } \mathcal{D}^\prime(\delta, \rho). \quad (9.18)$$

Since both sides of (9.18) are bounded measures, the identity in (9.18) is also an identity of measures. Since $V$ is a bounded function, it follows from the fact that $\mathbb{B}_\rho$ has vanishing one-dimensional Lebesgue measure that

$$\frac{V}{r \ln \rho} \subseteq \mathbb{B}_\rho = 0 \text{ and hence } \mathcal{N}_* \subseteq \mathbb{B}_\rho = \zeta_* \subseteq \mathbb{B}_\rho \geq 0.$$

Proof of Proposition 6 completed. Combining (9.15) and (9.17), we obtain that

$$\mathcal{N}_* \geq 0 \text{ on } (0, \rho).$$

Since $F(r) = r^{-1} \mathcal{N}_*$, we deduce that $F(r) \geq 0$ on $(0, \rho)$, so that $F$ is non-decreasing. Inequality (80) follows. The other statements of Proposition 6 are then straightforward, so that the proof is complete.

### 9.5 Proofs of Theorems 4 and 5

Recall that at this stage we already know, thanks to Proposition 6, that the measure $\zeta_*$ is absolutely continuous with respect to the measure $d\lambda$. We next derive the same statement for the measure $\nu_*$, thanks to a comparison with the measure $\zeta_*$ relying on our PDE analysis developed in Part II.

#### 9.5.1 An upper bound for the measure $\nu_*$

In follows from the very definition of the measures $\zeta_*$ and $\nu_*$ that we have the inequality $\zeta_* \leq \nu_*$. Indeed, we have for every $\epsilon > 0$, the straightforward inequality $\zeta_\epsilon \leq \nu_\epsilon$. We next present a reverse inequality:
Lemma 9.6. Let \( x_0 \in \Omega \) and \( r > 0 \) be such that \( D^2(x_0, r) \subset \Omega \). Then we have
\[
\nu_\star \left( D^2(x_0, \frac{r}{2}) \right) \leq K_V (d(x_0)) \zeta_\star \left( D^2(x_0, r) \right),
\]
where \( d(x_0) = \text{dist}(x_0, \partial \Omega) \) and where the constant \( K_V > 0 \) depends only on \( V, M_0 \) and \( d(x_0) \).

Proof. The result is an immediate consequence of Proposition 4.5. Indeed, for \( n \in \mathbb{N} \), we have the inequality
\[
\nu_{\varepsilon_n} \left( D^2(x_0, \frac{r}{2}) \right) \leq K_V (\text{dist}(x_0, \partial \Omega)) \left[ \zeta_{\varepsilon_n} \left( D^2(x_0, \frac{3r}{4}) \right) + \frac{\varepsilon_n}{r} \nu_{\varepsilon_n} \left( D^2(x_0, \frac{r}{2}) \right) \right].
\]
Letting \( n \to +\infty \), we are led to the inequality
\[
\nu_\star \left( D^2(x_0, \frac{r}{2}) \right) \leq K_V (\text{dist}(x_0, \partial \Omega)) \zeta_\star \left( D^2(x_0, \frac{3r}{4}) \right),
\]
which yields (9.19).

9.5.2 Proof of Theorem 4

In view of Proposition 6, we know that \( \zeta_\star \) is absolutely continuous with respect to the measure \( d\lambda = H^1 \ll \mathcal{G}_\star \). Moreover, we have, writing \( \nu_\star = e_\star d\lambda \), for \( \lambda \)-almost every \( x \in \mathcal{G}_\star \),
\[
e_\star(x) \leq K_V (\text{dist}(x, \partial \Omega)) \Theta(x).
\]

Proof. We have, for every \( x_0 \in \mathcal{G}_\star \), the identity
\[
\mathcal{D}_\lambda(\nu_\star)(x_0) \equiv \limsup_{r \to 0} \frac{\nu_\star \left( D^2(x_0, r) \right)}{2r} = \limsup_{r \to 0} \frac{\nu_\star \left( D^2(x_0, \frac{r}{2}) \right)}{r} \leq K_V (d(x_0)) \limsup_{r \to 0} \frac{\zeta_\star \left( D^2(x_0, r) \right)}{r} = 2K_V (d(x_0)) \Theta_\star(x_0),
\]
where we used Lemma 9.6 for the second line. It follows that \( \mathcal{D}_\lambda(\nu_\star)(x_0) \) is locally bounded for every \( x_0 \in \Omega \), so that \( \nu_\star \) is absolutely continuous with respect to \( \lambda \). Since
\[
e_\star(x) = \mathcal{D}_\lambda(\nu_\star)(x_0),
\]
for \( \lambda \)-almost every \( x_0 \in \mathcal{G}_\star \), (9.20) follows from (9.21).

9.5.3 Proof of Theorem 5

Theorem 5 is an immediate consequence of Proposition 5 combined with the fact that all measures are absolutely continuous with respect to the measure \( H^1 \ll \mathcal{G}_\star \) (so that the singular parts actually vanish).
10 Proof of Theorem 2

The argument consists, for a large part, in revisiting the analysis provided in Section 8 taking however now into account the fact that all measures at stake are absolutely continuous with respect to $H^1 \subseteq \mathcal{G}_*$. We first present several observations which are relevant for the proof. In particular, combining Lemma 2 with Theorem 3 we obtain, for $\omega_*(\mu_{*,1,1} - \mu_{*,2,2}) = 2\mu_{*,1,2}$

$$\omega_* = -2 \exp(-2i\gamma_*)\zeta_* = -2(\cos 2\gamma_* - i\sin 2\gamma_*)\zeta_*,$$

so that

$$\mu_{*,1,1} - \mu_{*,2,2} = -2(\cos 2\gamma_*)\zeta_* \text{ and } \mu_{*,1,2} = (\sin 2\gamma_*)\zeta_*.$$  \hspace{1cm} (10.1)

We will make use of these identities in several relations the obtained in Section 8.

10.1 Preliminary observations

**Lemma 10.1.** Given any orthonormal basis $(\vec{e}_1, \vec{e}_2)$, we have the relations

$$2\zeta_* - \mu_{*,2,2} + \mu_{*,1,1} = 4\sin^2 \gamma_* \zeta_* \geq 0.$$  \hspace{1cm} (10.2)

**Proof.** The proof is an immediate consequence of (10.1) since $2\zeta_* - \mu_{*,2,2} + \mu_{*,1,1} = 2(1 - \cos^2 \gamma_*)\zeta_*$. \hfill \Box

Next, consider a point $x_0 \in \mathcal{G}_* \setminus \mathcal{G}_*$, so that a tangent exists, and we assume moreover that the orthonormal basis $(\vec{e}_1, \vec{e}_2)$ is chosen so that $\vec{e}_1 = \vec{e}_{x_0}$.

**Lemma 10.2.** Let $x_0 \in \mathcal{G}_* \setminus \mathcal{G}_*$ and $\rho_0 > 0$ be the number provided by Proposition 8.4. Then the function $J_{1,\rho_0}$ defined on $I_{\rho_0}(x_0,1)$ by identity (8.17) in Corollary 8.1 is non-decreasing.

**Proof.** Let $\mathcal{P}$ be the orthogonal projection onto the line the tangent line $D^1 \mathcal{E}_0 = \{x_0 + s\vec{e}_1, s \in \mathbb{R}\}$. Recall that, in view of the definition (76), we have $N_{x_0,\rho_0} = \mathcal{P}_\gamma(2\zeta_* + \tilde{\mu}_{*,1,1} - \tilde{\mu}_{*,2,2})$ so that it follows from (10.2) that

$$N_{x_0,\rho_0} \geq 0.$$  \hspace{1cm} (10.3)

The conclusion is that an immediate consequence of the first differential relation in (8.15) for $k = 1$. \hfill \Box

For $s \in I_{\rho_0}(x_0,1)$, we introduce the set $\Lambda(s) = \mathcal{P}^{-1}(s) \cap Q_{\rho_0}(x_0)$, the set of points in the square $Q_{\rho_0}$ whose orthogonal projection onto the line $x_0 + \mathbb{R}\vec{e}_1$ is the point $(s, x_0,1)$. Let $\mathcal{Z}(s) = \sharp(\Lambda(s))$ be the numbers of elements in $\Lambda(s)$. An important step in the proof is to prove that $\mathcal{Z}(s) = 1$. Since $\mathcal{G}_*$ is connected, we have

$$\Lambda(s) \neq \emptyset \text{ and hence } \mathcal{Z}(s) \geq 1 \text{ for } s \in I_{\rho_0}(x_0,1).$$  \hspace{1cm} (10.4)

We provide next a few simple observations.

**Lemma 10.3.** For almost every $s \in I_{\rho_0}(x_0,1)$, the number $\mathcal{Z}(s)$ is finite. If $\Lambda(s)$ is finite, then we have, for $k \in \mathbb{N}$

$$J_{k,\rho_0}(s) = 2 \sum_{a(s) = (x_0,1),a_2(s) \in \Lambda(s)} (x_{0,2} - a_2(s))^k \sin(\gamma_*(a(s)))\Theta(a(s)).$$  \hspace{1cm} (10.5)
Combining (10.10) and (10.11), we deduce that
\[ J_{k, \rho_0} = \mathbb{E}_x \left( (x_{0,2} - x_2)^{k} \tilde{u}_{1,2} \right) = \mathbb{E}_x \left( (x_{0,2} - x_2)^{k} 1_{\mathcal{Q}_{\rho_0}(x_0)} \sin(2\gamma_*) \Theta_* \right). \] (10.6)

If \( \Lambda(s) \) is finite, and given any point \( a(s) \in \Lambda(s) \), we may find some arbitrary small number \( \delta > 0 \) such that \((\mathcal{G}_* \cap \mathbb{B}^2(x_0, \delta)) \cap \Lambda(s) = \{ a(s) \} \). If \( a(s) \notin \mathcal{E}_* \), then the angle of the tangent to \( \mathcal{G}_* \) at the point \( a(s) \) with the vector \( \mathbf{e}_1 \) is \( \gamma(a(s)) \) so that, if \( \gamma(a(s)) \neq \pm 1/2 \), then we have
\[ \frac{d\mathbb{P}_x (1_{\mathbb{B}^2(a(s))} d\lambda)}{ds} = \frac{1}{\cos(\gamma(a(s)))}. \] (10.7)

Since \( \sin(2\gamma(a(s))) = \sin(\gamma(a(s))) \cos(\gamma(a(s))) \), the conclusion follows combining (10.6) and (10.7).

**Lemma 10.4.** Let \( s = \mathcal{I}_{\rho_0}(x_0, 1) \) be such that \( \mathcal{Z}(s) = 1 \). Then, we have \( J_{k, \rho_0}(s) = 0 \), for any \( k \in \mathbb{N} \).

**Proof.** In view of the assumption of Lemma 10.4, \( \Lambda(s) \) contains a unique element \( a(s) = (x_{0,1} + s, a_2(s)) \), so that
\[ J_{k, \rho_0}(s) = (x_{0,2} - a_2(s))^k \sin(\gamma_*(a(s))) \Theta(a(s)) = (x_{0,2} - a_2(s))^k J_{0, \rho_0}(s). \]

In view of Proposition 8.6 we have \( J_{0, \rho_0}(s) = 0 \), for any \( s \in [x_{0,1} - \rho_0, x_{0,1} + \rho_0] \) (see the first identity in (8.53)), so that the conclusion follows.

Next consider for \( 0 < \rho \leq \rho_0 \), the set \( \mathcal{G}(\rho) = \{ s \in [x_{0,1} - \rho, x_{0,1} + \rho], \text{ such that } \mathcal{Z}(s) = 1 \} \).

We have:

**Lemma 10.5.** There exists \( 0 < \rho_1 \leq \rho_0 \), such that we have the upper bound
\[ |\mathcal{G}(\rho)| \geq \frac{5\rho}{3}, \text{ for any } 0 < \rho \leq \rho_1. \]

**Proof.** We first notice that, since \( \mathbb{P}_* \) is a contraction, that for any \( \rho \leq \rho_0 \), we have
\[ \int_{x_{0,1} - \rho}^{x_{0,1} + \rho} \mathcal{Z}(s) ds \leq \mathcal{H}^1(\mathcal{G}_* \cap \mathcal{Q}_\rho(x_0)) \leq \mathcal{H}^1(\mathcal{G}_* \cap \mathbb{B}^2(x_0, \frac{r}{\cos \gamma/2}) \leq \mathcal{H}^1(\mathcal{G}_* \cap \mathbb{B}^2(x_0, \frac{10r}{9})), \] (10.8)

where we used (8.51). On the other hand, in view of (9), there exists some \( 0 < \varrho_1 \leq \rho_0 \), such that, for \( \rho \leq \varrho_1 \), we have
\[ \mathcal{H}^1(\mathcal{G}_* \mathbb{B}^2(x_0, \rho)) \leq \frac{21\rho}{10}, \] (10.9)

Combining (10.8) and (10.9), we obtain hence, for \( \rho \leq \rho_1 \equiv \frac{9}{10} \varrho_1 \),
\[ \int_{x_{0,1} - \rho}^{x_{0,1} + \rho} \mathcal{Z}(s) ds \leq \frac{21\rho}{9} = \frac{7\rho}{3}. \] (10.10)

We introduce the set \( \mathcal{K}(\rho) = \{ s \in [x_{0,1} - \rho, x_{0,1} + \rho], \text{ such that } \mathcal{Z}(s) \geq 2 \} \). We have
\[ \int_{x_{0,1} - \rho}^{x_{0,1} + \rho} \mathcal{Z}(s) ds = \int_{\mathcal{G}(\rho)} \mathcal{Z}(s) ds + \int_{\mathcal{K}(\rho)} \mathcal{Z}(s) ds \leq |\mathcal{G}(\rho)| + 2|\mathcal{K}(\rho)| = 2\rho + |\mathcal{K}(\rho)|. \] (10.11)

Combining (10.10) and (10.11), we deduce that \( |\mathcal{K}(\rho)| \leq \frac{\rho}{3} \) and the conclusion follows. 

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10.2 Proof of Theorem 2 completed

For the sake of simplicity, we assume that the origin has been chosen so that $x_0 = 0$. We first apply Lemma 10.5 so that there exists $0 < \rho_1 \leq \rho_0$ such that $|G(\rho_1)| \geq 5/3 \rho_1$. Hence there exists two numbers $\rho^+_2 > 0$ and $\rho^-_2 > 0$ such that

$$
-\rho_1 \leq -\rho^-_2 \leq -\frac{2\rho_1}{3} < 0 < \frac{2\rho_1}{3} \leq \rho^+_2 \leq \rho_1 \text{ and such that } \{\rho^-_2, -\rho^-_2\} \subset G(\rho).
$$

Since $Z(-\rho^-_2) = Z(\rho^+_2) = 1$, we may apply Lemma 10.4 to $-\rho^-_2$ et $\rho^+_2$ to assert that

$$
J_{1,\rho_0}(-\rho^-_2) = J_{1,\rho_0}(\rho^+_2) = 0.
$$

Since, in view of Lemma 10.2, the function $J_{1,\rho_0}$ is monotone on $I_{\rho_0}$, we deduce that

$$
J_{1,\rho_0}(s) = 0 \text{ on } [-\rho^-_2, \rho^+_2] \text{ and hence } N_{\rho_0} = \frac{d}{ds}J_{1,\rho_0} = 0 \text{ in } \mathcal{D}'([-\rho^-_2, \rho^+_2]). \tag{10.12}
$$

It follows from the second identity in (10.12), the definition (76) of $N_{\rho_0}$ and (10.2), that the restriction of the measure $(2\hat{\zeta}_* + \tilde{\mu}_{*,1,1} - \tilde{\mu}_{*,2,2})$ to $I_{\rho_0}(0) \times [-\rho^-_2, \rho^+_2]$ vanishes. This implies, that, for any $k \in \mathbb{N}$, we have

$$
N_{k,\rho_0} \, [[-\rho^-_2, \rho^+_2]] = 0, \tag{10.13}
$$

where $N_{k,\rho_0}$ is defined in (8.13). In view of the first differential equation in (8.15), we have hence

$$
\frac{d}{ds}J_{k-1,\rho_0} = 0 \text{ on } [-\rho^-_2, \rho^+_2].
$$

Since $J_{k,\rho_0}(0) = 0$, for $k \geq 1$, it follows that

$$
J_{k,\rho_0}(s) = 0 \text{ for every } s \in [-\rho^-_2, \rho^+_2]. \tag{10.14}
$$

Similarly, invoking the second relation in (8.15), that is $-\frac{d}{ds}l_{k,r} = kJ_{k-1,r}$, (10.14) and the fact that $L_{k,\rho_0}(0) = 0$, we deduce that

$$
L_{k,\rho_0} = 0 \text{ for every } s \in [-\rho^-_2, \rho^+_2], \text{ for } k \in \mathbb{N}^*. \tag{10.15}
$$

Combining (10.13) and (10.15) with (8.14) we deduce that (since $x_0 = 0$)

$$
\begin{cases}
\frac{1}{4} L_0 ds \, \mathbb{L} \, [-\rho^-_2, \rho^+_2] = P_z \left( \hat{\zeta}_* \right) \, \mathbb{L} \, [-\rho^-_2, \rho^+_2] \text{ for } k = 0 \text{ and } \\
\frac{1}{4} (N_{k,r} + L_{k,r}) \, \mathbb{L} \, [-\rho^-_2, \rho^+_2] = P_z \left( x^k \, \zeta_* \right) \, \mathbb{L} \, [-\rho^-_2, \rho^+_2] = 0, \text{ for } k \in \mathbb{N}^*. \tag{10.16}
\end{cases}
$$

The first identity in (10.16) shows hence that $P_z \left( \hat{\zeta}_* \right) \text{ is constant on } [-\rho^-_2, \rho^+_2]$. Next, we choose $k = 2$. The second identity in (10.16) implies that $x^2 \, \zeta_* = 0$ on $[-\rho^-_2, \rho^+_2] \times I_{\rho_0}(0)$. these relations yield

$$
\zeta_* \, \mathbb{L} \, ([0, \rho^-_2, \rho^+_2] \setminus \{0\}) \times I_{\rho_0}(0) = 0. \tag{10.17}
$$

We choose next $r_0 = \inf \{\rho^-_2, \rho^+_2\} > 0$. Combining (10.17) with the first relation in (10.16) we are led to

$$
\zeta_* \, \mathbb{L} \, \mathbb{D}^2(r_0) = L_0 d\ell \text{ where } d\ell \text{ is the Lebesgue measure on } (-r_0, r_0). \tag{10.18}
$$

Invoking Theorem 4, we complete the proof of Theorem 2.
11 Proof of Theorem 3

Inserting identities (10.1) into the system (68), we are led to the system of first-order equations

\[
\begin{aligned}
-\frac{\partial}{\partial x_2} [\sin 2\gamma_* \zeta_*] &= \frac{\partial}{\partial x_1} [(1 + \cos 2\gamma_*) \zeta_*] \quad \text{and} \\
-\frac{\partial}{\partial x_1} [\sin 2\gamma_* \zeta_*] &= \frac{\partial}{\partial x_2} [(1 - \cos 2\gamma_*) \zeta_*].
\end{aligned}
\]

(11.1)

We are going to show next that these relations are equivalent, in the sense of distributions, to (19). For that purpose, let \( \vec{X} = (X_1, X_2) \) be a vector-field in \( C^\infty_c(\Omega, \mathbb{R}^2) \). We have, for any \( x \in \mathcal{S} \setminus \mathcal{E}_* \), since by definition \( \vec{e}_{x_0} = \cos \gamma(x_0) \vec{e}_1 + \sin \gamma(x_0) \vec{e}_2 \)

\[
\text{div}_{\vec{e}_x} \vec{X}(x) = \left( \vec{e}_x \cdot \nabla \vec{X}(x) \right) \cdot \vec{e}_x
\]

\[
= \left( \cos \gamma_*(x) \frac{\partial X_1}{\partial x_1}(x) + \sin \gamma_*(x) \frac{\partial X_2}{\partial x_2}(x) \right) \cdot \left( \cos \gamma_*(x) \vec{e}_1 + \sin \gamma_*(x) \vec{e}_2 \right)
\]

\[
= \cos^2 \gamma_*(x) \frac{\partial X_1}{\partial x_1}(x) + \sin^2 \gamma_*(x) \frac{\partial X_2}{\partial x_2}(x)
\]

\[
= \sin \gamma_*(x) \cos \gamma_*(x) \left[ \frac{\partial X_2}{\partial x_1}(x) + \frac{\partial X_1}{\partial x_2}(x) \right].
\]

Using this computation, we may expand relation (19) as

\[
\left\langle \zeta_*, \cos^2 \gamma_* \frac{\partial X_1}{\partial x_1} + \sin^2 \gamma_* \frac{\partial X_2}{\partial x_2} + \sin \gamma_* \cos \gamma_* \left[ \frac{\partial X_2}{\partial x_1} + \frac{\partial X_1}{\partial x_2} \right] \right\rangle = 0
\]

(11.2)

Integrating by parts in the sense of distributions, we obtain hence, for every \( X_1 \in C^\infty_c(\Omega, \mathbb{R}) \) and any \( X_2 \in C_c(\Omega, \mathbb{R}) \), the relation

\[
\left\langle \frac{\partial}{\partial x_1} (\cos^2 \gamma_* \zeta_*) + \frac{\partial}{\partial x_2} (\sin \gamma_* \cos \gamma_* \zeta_*), X_1 \right\rangle + \left\langle \frac{\partial}{\partial x_2} (\sin^2 \gamma_* \zeta_*) + \frac{\partial}{\partial x_1} (\sin \gamma_* \cos \gamma_* \zeta_*), X_2 \right\rangle = 0.
\]

Since \( X_1 \) and \( X_2 \) can be chosen independently, we are led to the system, in the sense of distributions,

\[
\begin{aligned}
-\frac{\partial}{\partial x_2} [\sin \gamma_* \cos \gamma_* \zeta_*] &= \frac{\partial}{\partial x_1} [(\cos^2 \gamma_*) \zeta_*] \quad \text{and} \\
-\frac{\partial}{\partial x_1} [\sin \gamma_* \cos \gamma_* \zeta_*] &= \frac{\partial}{\partial x_2} [(\sin^2 \gamma_*) \zeta_*].
\end{aligned}
\]

(11.3)

Since \( 2 \sin \gamma_* \cos \gamma_* = \sin 2\gamma_* \), \( 1 + \cos 2\gamma_* = 2 \cos^2 \gamma_* \), and \( 1 - \cos^2 \gamma_* = 2 \sin^2 \gamma_* \), we verify that (11.3) is equivalent to (11.1), so that the system (68) is equivalent to (19). The varifold \( \mathcal{V}(\mathcal{S}_*, \mathcal{E}_*) \) is hence stationary. The proof of Theorem 3 is complete.

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