OPTICAL SOLITONS IN HIGHER ORDER NONLINEAR
SCHRÖDINGER EQUATION

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Abstract
We show the complete integrability and the existence of optical solitons of higher order nonlinear Schrödinger equation by inverse scattering method for a wide range of values of coefficients. This is achieved first by invoking a novel connection between the integrability of a nonlinear evolution equation and the dimensions of a family of matrix Lax pairs. It is shown that Lax pairs of different dimensions lead to the same evolution equation only with the coefficients of the terms in different integer ratios. Optical solitons, thus obtained by inverse scattering method, have been found by solving an n dimensional eigenvalue problem.
The propagation of optical solitons in an optical fibre draws a lot of attention in recent times because of its plausible applications in telecommunication. It is known that the propagation of optical pulses of very short duration in a fibre is described by the dynamics of a nonlinear evolution equation, namely higher order Schrödinger equation (HNLS) [1-5]:

\[ \partial_z E = i[\alpha_1 \partial_\tau \tau E + \alpha_2 |E|^2 E] + \epsilon[\alpha_3 \partial_\tau \tau \tau E + \alpha_4 \partial_\tau (|E|^2 E) + \alpha_5 \partial_\tau (|E|^2) E] \] (1)

where, \( E \) represents an envelop of electric fields, \( z \) is the direction of propagation of optical pulse and \( \tau \) is time. The terms on the r.h.s. of (1) describe respectively the effects of group velocity dispersion (GVD), self-phase modulation (SPM), third order dispersion (TOD), self-steepening (SS) and self-frequency shifting via stimulated Raman scattering (SRS).

If \( \epsilon = 0 \), (1) reduces to nonlinear Schrödinger equation (NLS) and in this case optical soliton can be produced due to a balance between GVD and SPM. But the optical soliton thus produced, cannot propagate over a long distance with their shapes undistorted, due to the propagation loss, as they travel along the optical fibre. This propagation loss in the fibre, however, can be compensated by utilising Raman effect, which is achieved by transmitting a pump light wave simultaneously through the fibre. It is interesting to note that the Raman process which takes care of propagation loss in the fibre, already exists within the spectrum of a soliton in HNLS equation (1) [1,2,8]. It is the last term, which is responsible for stimulated Raman scattering. No additional pump light wave is needed to compensate the propagation loss for the ultrashort optical pulses.
through the fibre. Moreover, for ultrashort pulses effects of higher order terms (last three terms) in (1) cannot be neglected. It is these reasons the study of the soliton solutions of HNLS equation becomes an important issue for the propagation of ultrashort pulses in the fibre. But, unfortunately the complete integrability of HNLS equation for arbitrary coefficients $\alpha$ is not well understood. It is our aim, in this paper, to study the Lax integrability and consequently to obtain soliton solutions of HNLS equation (1) by inverse scattering method (IST) for a wide range of values of the coefficients $\alpha$. Integrability of HNLS equation (1) by IST has so far been shown only for two fixed ratios of the coefficients [9,10].

Now, in order to obtain soliton solutions of HNLS equation by IST, it is convenient to study a gauge equivalent nonlinear evolution equation,

$$\frac{\partial}{\partial t} q(x,t) = \epsilon [\partial_{xxx} q(x,t) + \gamma_1 |q(x,t)|^2 \partial_x q(x,t) + \gamma_2 \partial_x (|q(x,t)|^2) q(x,t)]$$ (2)

which is related to the HNLS equation (1) explicitly through the following gauge transformation

$$E(z, \tau) = \exp[-i px + i \epsilon p^3 t] q(x, t)$$ (3a)

$$x = \tau + \alpha_1 p z$$ (3b)

$$t = \alpha_3 z$$ (3c)

with $p = \alpha_1/(3\epsilon \alpha_3) = \alpha_2/(\epsilon \alpha_4)$, $\gamma_1 = \alpha_4/\alpha_3$, and $\gamma_2 = \alpha_5/\alpha_3$. The equation (2) is known as complex modified KdV (cmKdV) equation. Thus there is an one to one relation between the solitons of cmKdV equation and those of HNLS
equation through the relation (3a). Integrability of (2) is shown by many authors by various methods [9-13] only for two cases, when the coefficients of the terms on the r.h.s. are in the ratios (i) 1:6:0 and (ii) 1:6:3. Recently N solitary wave solutions of HNLS equation has been obtained for one parameter family of \( \gamma \) parameters [5]. But the Painleve analysis restricts the integrability of HNLS equation further to a subset of \( \gamma \) parameters, when the terms in the r.h.s. of (2) are in the ratio 1:6:3. In this paper we show the complete integrability of HNLS equation by IST for all values of the \( \gamma \) parameters. This is achieved by exploiting a novel connection between the dimensionality of the Lax pair and the integrability of HNLS equation. It is interesting to observe that the Lax pairs of different dimensions lead to the same equation of motion (3), but \( \gamma_1 \) and \( \gamma_2 \) in different ratios. Subsequently, we generalise the IST method for the \( n \)-dimensional Lax pair and obtain soliton solutions for HNLS equation for all integer valued ratios of the coefficients.

In order to show the Lax integrability of HNLS equation vis a vis cmKdV equation, let us start with an \( n \) dimensional Lax equations,

\[
\begin{align*}
\partial_x \Psi &= U(x,t,\lambda) \Psi \\
\partial_t \Psi &= V(x,t,\lambda) \Psi
\end{align*}
\]  

(4a)

(4b)

where, \( \Psi(x,t) \) is an \( n \) dimensional auxilliary field and the Lax operators \( U(x,t) \) and \( V(x,t) \) are \( n \times n \) matrices of the form

\[
U = -i\lambda \Sigma + A
\]  

(5a)
\[ V = \epsilon A_{xxx} + \epsilon (A_x A - A A_x) + 2\epsilon A^3 \]  

(5b) 

\[-2i\epsilon \lambda \Sigma (A^2 - A_A) + 4\epsilon \lambda^2 A - 4i\epsilon \lambda^3 \Sigma\]

In (5), \( \Sigma \) is a c-no. diagonal matrix and the matrix \( A \) consists of dynamical fields, \( q(x,t) \) and \( q^*(x,t) \) only. The explicit form of \( \Sigma \) and \( A \) may be given as 

\[ \Sigma = \sum_{i=1}^{n-1} e_{ii} - e_{nn} \]  

(6a) 

\[ A(x,t) = \sum_{i=1}^{n-1} \alpha_i(x,t)e_{in} - \sum_{i=1}^{n-1} \alpha_i^*(x,t)e_{ni} \]  

(6b) 

where, \( e_{ij} \) is an \( n \times n \) matrix whose only \((ij)\)th. element is unity, the rest elements being zero and \( \alpha_i(x,t) \) represent the dynamical fields. They may be chosen either as \( q(x,t) \) or as \( q^*(x,t) \). It is interesting to note that different choices of \( \alpha_i \) lead to the same cmKdV equation (2) only with the coefficients of the last two terms in different ratios. To obtain the equation of motion for the dynamical fields \( q \) and \( q^* \), we first notice that 

\[ \Sigma^2 = 1, \quad \Sigma A + A \Sigma = 0 \]  

(7) 

The equation of motion will then follow from the compatibility condition of (4), namely 

\[ \partial_t U(x,t) - \partial_x V(x,t) + [U(x,t), V(x,t)] = 0 \]

and by using the relation (7) as 

\[ \partial_t A = \epsilon \partial_{xxx} A - 3\epsilon (A^2(\partial_x A) + (\partial_x A)A^2) \]  

(8) 

The above relation gives a nonlinear evolution equation for the matrix \( A \). It is now straightforward to obtain evolution equations for \( q(x,t) \) or \( q^*(x,t) \), by
choosing an explicit form of \( A \) \( (6b) \). We will see the coefficients \( \gamma_1 \) and \( \gamma_2 \) depend not only on the dimensions of the matrix \( A \), but also on specific choices of the dynamical fields \( \alpha_i \) as \( q \) or \( q^* \). Let us consider some specific examples to clarify this point.

If we consider a two dimensional Lax operator, \( \Sigma \) and \( A \) in \( (6) \) would be of the form

\[
\Sigma = e_{11} - e_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

and

\[
A = \alpha_1 e_{12} - \alpha_1^* e_{21} = \begin{pmatrix} 0 & \alpha_1 \\ -\alpha_1^* & 0 \end{pmatrix}
\]

and in this case only one dynamical field, \( \alpha_1 \) exists. Therefore, in two dimensional case the choice of Lax operator is unique, \( \alpha_1 = q \) and substituting \( A \) in \( (8) \), it is found that the equation of motion reduces to the well known Hirota equation \( (9) \),

\[
\partial_t q = \epsilon \partial_{xxx} q + 6|q|^2 \partial_x q
\]

where, the coefficients are in the ratio 1:6:0.

However, for three dimensional Lax operator \( \Sigma \) and \( A \) are of the form,

\[
\Sigma = e_{11} + e_{22} - e_{33} \quad A = \alpha_1 e_{13} + \alpha_2 e_{23} - \alpha_1^* e_{31} - \alpha_2^* e_{32}
\]
We have, therefore, two possible choices of the Lax operators since two independent dynamical fields, $\alpha_1$ and $\alpha_2$ are present in $A$. In one case, we may choose both $\alpha_1$ and $\alpha_2$ as $q$ and consequently the equation of motion (8) becomes

$$\partial_t q = \epsilon \partial_{xxx} q + 12\epsilon |q|^2 \partial_x q$$

which reduces to Hirota equation (9) by appropriate scaling of the fields, $q$ and $q^*$. In another possible case, we may choose $\alpha_1$ as $q$, but $\alpha_2$ as $q^*$ and then both the Lax operator and the equation of motion become identical to Sasa Satsuma case [10]:

$$\partial_t q = \epsilon \partial_{xxx} q + 6\epsilon |q|^2 \partial_x q + 3\epsilon \partial_x (|q|^2) q$$

where the coefficients are in the ratio 1:6:3. These two specific cases are already studied from various points of view. Moreover, no other ratios of the coefficients $\gamma$ emerge up to three dimensions from the family of Lax operators, considered here. We will see, however, some interesting consequences as we consider four dimensional Lax operators. It follows from (6) that for four dimensional Lax operators, $\Sigma$ and $A$ takes the form

$$\Sigma = e_{11} + e_{22} + e_{33} - e_{44}$$

$$A = \alpha_1 e_{14} + \alpha_2 e_{24} + \alpha_3 e_{34} - \alpha_1^* e_{41} - \alpha_2^* e_{42} - \alpha_3^* e_{43}$$

and eventually, three different choices of the Lax operators are manifest in this case. One possibility, of course, may be considered when all the $\alpha_1$, $\alpha_2$ and $\alpha_3$ are chosen as $q$. The evolution equation (8) then, once again, reduces to
the known one, namely Hirota equation (9) by appropriate scaling of the fields.

Interestingly, however, if we choose both $\alpha_1$ and $\alpha_2$ as $q$, but $\alpha_3$ as $q^*$, the evolution equation (8) becomes

$$\partial_t q = \epsilon \partial_{xxx} q + 12\epsilon |q|^2 \partial_x q + 3\epsilon \partial_x(|q|^2)q$$

(10)

where the coefficients are in the ratio 1:12:3. On the other hand, if we consider another possibility, where $\alpha_1$ is chosen as $q$, but both $\alpha_2$ and $\alpha_3$ as $q^*$, the evolution equation turns out to be

$$\partial_t q = \epsilon \partial_{xxx} q + 6\epsilon |q|^2 \partial_x q + 6\epsilon \partial_x(|q|^2)q,$$

(11)

the coefficients being in the ratio 1:6:6. The last two cases are definitely new ones, which will be shown to be integrable. It is now clear that as we go to higher and higher dimensions, more and more new ratios of the coefficients appear for which cmKdV equation will be integrable. If fact, in general for an $n$ dimensional Lax operator, as is evident from (6), $(n-1)$ number of dynamical fields, $\alpha_i$ exist. We may, therefore, identify $l$ number of $\alpha$’s as $q$ and the rest $(n-l-1)$ ones as $q^*$. Consequently, the evolution equation (8) yields as

$$\partial_t q = \epsilon \partial_{xxx} q + 6\epsilon |q|^2 \partial_x q + 3(n-l-1)\epsilon \partial_x(|q|^2)q,$$

(12)

which is nothing but cmKdV equation, where the coefficients are in the ratio 1:6$l$:3$(n-l-1)$, i.e

$$\gamma_1 = 6l, \quad \gamma_2 = 3(n-l-1).$$

(13)
cmKdV equation is, thus, Lax integrable for all possible integer values of coefficients, \( \gamma_1 \) and \( \gamma_2 \), which can be demonstrated by appropriately scaling the dynamical fields, \( q \) and \( q^* \) in each case. It is evident now, for a given \( n \) dimensional matrix, \( (n-1) \) Lax pairs can be constructed and each Lax pair of a given dimension, leads to \( (n-1) \) different ratios of the coefficients. This is a significant achievement over the previous works. The existence of Lax pair, although, is a strong evidence for the integrability of cmKdV equation, it remains to show that the family of Lax pairs (5) admit soliton solutions through IST.

In order to obtain soliton solutions through IST in our case, we have generalised the \( 3 \times 3 \) AKNS type eigenvalue problem [10,14] to \( n \times n \) eigenvalue problem. Notice that asymptotically the auxiliary field, \( \Psi(x,t) \) obeys a simple relation, viz.

\[
\Psi(x,t) = e^{\{ -i\lambda x - 4i\epsilon\lambda^3 t \} \Sigma},
\]

for the whole family of the Lax pairs (5) and it depends only on the dimensions of the matrix Lax pair. Consequently the scattering data matrix, which, by definition, connects the Jost functions, \( \phi^{(i)}(x,\lambda)\big|_{x=-\infty} \) to the Jost functions, \( \psi^{(i)}(x,\lambda)\big|_{x=\infty} \), for \( i = 1, 2, \cdots, n \), [10,15] evolves with time in some universal form. Finally, solving \( n \) coupled Gelfand Levitan Marchenko equations one soliton solution may be expressed in a simple form

\[
q(x,t) = (\eta/\sqrt{(n-1)})sechA(x,t)exp(iB(x,t))
\]
with

\[ A(x,t) = \eta x - \epsilon(\eta^3 - 3\xi^2\eta)t - \gamma - \frac{1}{2} \ln(n-1) \]

\[ B(x,t) = \xi x + \epsilon(\xi^3 - 3\xi^2\eta)t + \delta \]

where \(\gamma\) and \(\delta\) are determined by the initial conditions and we assume that the simple pole is situated in the upper half plane at \(\lambda_1 = \frac{1}{2}(\xi + i\eta)\) for one soliton solution. A detailed calculation of soliton solutions by IST method for an \(n\) dimensional eigenvalue problem is considered in [15] for a more general case than HNLS equation and will be published elsewhere. It is straightforward to obtain optical soliton in terms of the original parameters by substituting the relations (3) in (15) and it turns out to be

\[ E(z,\tau) = \left(\eta/\sqrt{(n-1)}\right) sech\tilde{A}(z,\tau) exp(i\tilde{B}(z,\tau)) \]  \hspace{1cm} (16)

with

\[ \tilde{A}(z,\tau) = \eta\tau - [\epsilon\alpha_3(\eta^3 - 3\xi^2\eta) - \frac{\alpha^2_1}{3\epsilon\alpha_3}\eta]z - \gamma - \frac{1}{2} \ln(n-1) \]

\[ \tilde{B}(z,\tau) = (\xi - \frac{\alpha^2_1}{3\epsilon\alpha_3})\tau + [\epsilon\alpha_3(\xi^2 - 3\eta^2) + \frac{\alpha^2_2}{3\epsilon^2\alpha^2_3}\xi]z + \delta \]

It is interesting to observe from (16) that the envelop of one soliton solution admits a simple \textit{sech} type shape, which can be easily produced by a mode-locked laser. Moreover, the intensity, \(I_s = \eta^2/(n-1)\) and the width, \(\Gamma_s = \eta^{-1}\) for one soliton in (16) are related by the expression \(I_s\Gamma^2_s = 1/(n-1)\). In terms of \(\gamma\) parameters, given in (13), the expression \(I_s\Gamma^2_s\) becomes \(I_s\Gamma^2_s = 6/\{3\gamma_1 + 2(\gamma_2 - \gamma_1)\}\), which is in full agreement with the result of [3].
To conclude, the exact integrability of the cmKdV equation is shown for the integer valued ratios of the coefficients by establishing an intriguing relationship of the coefficients $\gamma$ with the dimensionality of the Lax pairs. The family of Lax pairs, so obtained, is shown to admit soliton solutions. One soliton solutions are found by solving an $n \times n$ eigenvalue problem through IST. For one soliton, the relation between the intensity and the width turns out to be simple and depends only on the diemnions of the Lax pairs.

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