Helicity-supersymmetry of dyons

F. BLOORE
DAMTP, The University of Liverpool
P. O. Box 147, LIVERPOOL L69 3BX (U.K.)

P. A. Horváthy
Laboratoire de Mathématiques et de Physique Théorique
Université de Tours.
Parc de Grandmont. F-37 200 TOURS (France).

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Abstract

The 'dyon' system of D'Hoker and Vinet consisting of a spin $\frac{1}{2}$ particle with anomalous gyromagnetic ratio 4 in the combined field of a Dirac monopole plus a Coulomb plus a suitable $1/r^2$ potential (which arises in the long-range limit of a self-dual monopole) is studied following Biedenharn’s approach to the Dirac-Coulomb problem: the explicit solution is obtained using the ‘Biedenharn-Temple operator’, $\Gamma$, and the extra two-fold degeneracy is explained by the subtle supersymmetry generated by the 'Dyon Helicity' or generalized 'Biedenharn-Johnson-Lippmann' operator $R$. The new SUSY anticommutes with the chiral SUSY discussed previously.

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1 Introduction

In a recent series of papers [1] D’Hoker and Vinet studied the strange 'dyon' system consisting of a charged, spin $\frac{1}{2}$ particle with anomalous gyromagnetic ratio 4 in a combined Dirac monopole + Coulomb potential + inverse-square potential field, described by the Hamiltonian

$$H_1 = \pi^2 + q^2(1 - \frac{1}{r})^2 - 2q\frac{\sigma \cdot r}{r^3}. \quad (1.1)$$

Here $\pi = -i\partial - eA$, $A$ being the vector-potential of a Dirac monopole of strength $g$, rot $A = -gr/r^3$, $q = eg > 0$ without loss of generality. The surprising dynamical and supersymmetries allow to calculate the spectrum [1] (and the $S$-matrix [2]),

$$E_p = q^2\left(1 - \frac{q^2}{p^2}\right), \quad p = q, q + 1, \ldots \quad (1.2)$$

whose multiplicity is $2(p^2 - q^2)$ for $p \geq q + 1$, and $2q$ for $p = q$.

The Hamiltonian $H_1$ is similar to that of the 'MIC-Zwanziger' [3] system,

$$H_0 = \pi^2 + q^2(1 - \frac{1}{r})^2. \quad (1.3)$$

\[e-mail : horvathy@univ-tours.fr\]
representing a spin 0 particle in the same field. Doubling $H_0$ yields a spin $\frac{1}{2}$ particle but with gyromagnetic ratio $g = 0$, which has the same symmetries, spectrum and multiplicities as (1.1) (except for the 0-energy ground states: for MIC-Zwanziger, $p \geq q + 1$ only) [4].

This is explained by that $H_1$ and $H_0$ are superpartners [1,4]: they both arise as the long-distance limits of a Dirac particle in the field of a self-dual monopole which has a chiral supersymmetry,

$$
\begin{pmatrix}
H_1 \\
H_0
\end{pmatrix} = D^2 \quad \text{for} \quad D = \begin{pmatrix}
Q & Q^\dagger \\
\sigma \cdot \pi + i \Phi & \sigma \cdot \pi - i \Phi
\end{pmatrix},
$$

(1.4)

where $\Phi = \Phi(1 - 1/r)$ is the long-range tail of the Higgs field. The 'upper' and 'lower' sectors (i.e. the $\pm 1$ eigensectors of the chirality operator $\rho_3$) are related by the unitary transformations

$$
U = Q^\dagger \frac{1}{\sqrt{H_0}} \quad U^{-1} = U^\dagger = \frac{1}{\sqrt{H_0}} Q,
$$

(1.5)

which intertwine $H_1 = Q^\dagger Q$ and $H_0 = QQ^\dagger$, $H_1 = UH_0U^\dagger$. If $\Psi_0$ is an $H_0$-eigenfunction with eigenvalue $E > 0$, then

$$
\begin{pmatrix}
U\Psi_0 \\
\pm \Psi_0
\end{pmatrix}
$$

(1.6)

is a $D$-eigenfunction with eigenvalues $\pm \sqrt{E}$. The arousal of zero-energy ground states for $H_1$ (but not for $H_0$) as solutions of $Q\Psi = 0$ [4] is a nice manifestation of supersymmetry. The multiplicity $2q$ is consistent with the index-theoretical calculations in a self-dual monopole background [5].

Although this approach provides an insight into the mysteries found by D’Hoker and Vinet, explicit calculations are rather complicated, because $U$ mixes the radial and angular parts. In another paper [6] we introduced two operators,

$$
x = \sigma \cdot L + 1, \quad y = \sigma \cdot L + 1 + 2q\sigma \cdot \hat{r},
$$

(1.7)

where $L = \ell - q\hat{r}$, $\ell = r \times \pi$ is the orbital angular momentum. $y$ (resp. $x$) are constants of the motion for the $H_1$ (resp $H_0$) dynamics, and satisfy

$$
y^2 = x^2 = J^2 + 1/4.
$$

(1.8)

Both $y$ and $x$ have therefore eigenvalues $\pm (j + 1/2)$ and bring $H_1$ and $H_0$ into a non-relativistic Coulomb form (cf. Section 3),

$$
-(\partial_r + \frac{1}{r})^2 - \frac{2q^2}{r^2} + q^2 + \frac{1}{r^2} \left( (j - \frac{1}{2})(j + \frac{1}{2}) + (j + \frac{1}{2})(j + \frac{3}{2}) \right)
$$

(1.9)

whose spectrum is shown on FIGs. 1 and 2.

It is tempting to think that 'supersymmetry' means just $x \mapsto -y$. However, $U^\dagger xU \neq -y$ which seemingly contradicts our expectations. But we show below that our system admits another supersymmetry encountered before in Biedenharn's approach to the Dirac-Coulomb problem [7-9].

This second SUSY is actually more convenient then the chiral one, because it does respect the angular decomposition.

The method is based on two operators, namely the 'Biedenharn-Temple' operator $\Gamma$ [7, 10] and the 'Biedenharn-Johnson-Lippmann' [11] or 'Coulomb Helicity' [7-9] operator $R$. The first of these allows us to write the iterated Dirac equation in a non-relativistic Coulomb form, but
Figure 1: The dyon spectrum in the $g = 0$ sector. The sign refers to that of $(-x)$. Each $j \geq q + 1/2$ sector is doubly degenerate. For $j = q - 1/2$ there are no $(-x) = -q$ states. The energy only depends on the principal quantum number $= L(\gamma) + 1 + n$.

Figure 2: The dyon spectrum in the $g = 4$ sector. The sign refers to that of $(-y)$. Each $j \geq q + 1/2$ sector is doubly degenerate. For $j = q - 1/2$ there are no $(-y) = +q$ states but $E = 0$ ground states arise for $(-y) = -q$. 
with an irrational angular momentum $\ell(\gamma)$ (where $\gamma$ is an eigenvalue of the Biedenharn-Temple operator), yielding the well-known spectrum. The irrationality of $\ell(\gamma)$ results in shifting the different angular momentum sectors with respect to each other i.e. in the fine structure. The two-fold degeneracy of each $j = \text{const.}$ sector, (except for the ground state) can be viewed as indicating SUSY, the intertwining transformation being Biedenharn’s ’Coulomb Helicity operator’ $R$ [8, 12].

In this paper we extend the Biedenharn method to the dyon problem. Remarkably, things work even better than in the Dirac-Coulomb case: one gets again a Coulomb-type equation, but with (half)integer angular momentum $L(\gamma)$: The fine structure is automatically suppressed, without having to be removed by hand, as Biedenharn did for his ‘symmetric Hamiltonian’ [7].

We show first that it reproduces the $‘x – y’$ picture of Ref. [4]. Then we focus our attention to the $j = \text{const.}$ sectors of FIG. 1-2. The pattern is reminiscent to that of a supersymmetric system - except that the ground states have non-zero energy $E_0^{(j)}$ cf. (4.1). But this can easily be cured: it is enough to subtract $E_0^{(j)}$ from the Hamiltonian and consider rather the new Hamiltonians $K_1$ and $K_0$ defined by

$$\begin{pmatrix} K_1 \\ K_0 \end{pmatrix} = \mathcal{D}^2 - E_0^{(j)} = \begin{pmatrix} H_1 - q^2 + \frac{q^4}{(j+1/2)^2} & 0 \\ 0 & H_0 - q^2 + \frac{q^4}{(j+1/2)^2} \end{pmatrix}. \quad (1.10)$$

In Section 4 we show that this indeed yields a SUSY system : following Biedenharn, we exhibit a new conserved operator,

$$S = \begin{pmatrix} S_H \\ S \end{pmatrix} = \begin{pmatrix} i\sigma.\pi - \frac{q}{2} + (\sigma.\hat{r})\frac{q^2}{2} \\ i\sigma.\pi + \frac{q}{2} + (\sigma.\hat{r})\frac{q^2}{2} \end{pmatrix} \quad (1.11)$$

which we call 'Dyon Helicity', and show that it is a supersymmetry operator for the new Hamiltonian,

$$S^2 = \begin{pmatrix} K_1 \\ K_0 \end{pmatrix}. \quad (1.11)$$

The need for an extra supersymmetry is understood by noting that giving the energy, the total angular momentum and the third component, $E, j, \text{ and } \mu,$ respectively, do not specify completely a state: one has to give also the sign of $\gamma – and this is exactly this sign which labels the new SUSY sectors.

Our Dyon Helicity operator $S$ also allows us to derive, along the lines indicated in Refs. [8, 9], the $S$-matrix (Section 5). Thing again work better as for the Dirac-Coulomb problem, where an arbitrary phase has to be chosen in each $j = \text{const.}$ sector [8,9]. Here, since the dyon quantum numbers are half-integers rather then irrational, it is enough to chose a phase in one single sector.

Another simplification with respect to the Dirac-Coulomb problem is that we work with the mass 0 Dirac equation – albeit in higher dimension – and mass enters only after dimensional reduction.

Let us mention that the same technique applies to a particle in the field of a charged monopole [13-17]. The difference with dyons comes from that the Coulomb potential belongs to the time coordinate with Minkowski signature, in contrast as for dyons, where the Higgs field appears in an extra euclidean dimension. This changes a sign and quantities simply add, yielding perfect squares rather than irrational (or even complex) values.
2 The Biedenharn-Temple operator for dyons.

Let us consider a massless Dirac particle in the long-distance field $B = -qr/r^3$, $\Phi = q(1 - 1/r)$ of a Bogomolny-Prasad-Sommerfield monopole. Identifying $\Phi$ with the fourth component of a gauge field we get a self-dual Yang-Mills field in four dimensions. This leads to the Dirac Hamiltonian

$$\mathcal{H} = \rho_1(\sigma \cdot \pi) - \rho_2 \Phi = \begin{pmatrix} Q \cr Q^\dagger \end{pmatrix} = \begin{pmatrix} \sigma, \pi + i \Phi \\ \sigma, \pi - i \Phi \end{pmatrix}. \tag{2.1}$$

Unlike in the Coulomb case, the scalar term $\rho_2 \Phi$ is now off-diagonal, because it comes from the fourth, euclidean, direction rather than from the time coordinate.

The total angular momentum,

$$J = L + \frac{\sigma, \pi}{2}, \quad L = \ell - q \mathbf{r}, \quad \ell = r \times \pi \tag{2.2}$$

is conserved. Set

$$w = \sigma, \mathbf{r}, \quad z = \sigma, \ell + 1 \quad \mathcal{K} = -\rho_2 z = -\rho_2 (\sigma, \ell + 1). \tag{2.3}$$

Note that $w^2 = 1$ and that $z$ anticommutes with $w$ and $\sigma, \pi$,

$$\{z, w\} = 0 \quad \{z, \sigma, \pi\} = 0. \tag{2.4}$$

Since $z$ anticommutes with the first term in eqn. (2.1) and commutes with the second, $\mathcal{K}$ commutes with the Dirac Hamiltonian $\mathcal{H}$. Using

$$(\sigma \cdot L)^2 = L^2 + i\sigma (L \times L) = L^2 - \sigma \cdot L$$

one proves that

$$\mathcal{K}^2 = z^2 = J^2 + \frac{1}{4} - q^2, \tag{2.5}$$

so that $z$ (and $\mathcal{K}$) have irrational$^3$ eigenvalues,

$$\kappa = \sqrt{(j + 1/2)^2 - q^2}, \tag{2.6}$$

($J^2 = j(j + 1)$). Since $j \geq q - 1/2$, $\mathcal{K}$ is hermitian, but for $j = q - 1/2$ its eigenvalue $\kappa$ vanishes and thus $\mathcal{K}$ is not invertible.

The Dirac operator (2.1) is, as in any even dimensional space, chiral-supersymmetric: for $Q$ and $Q^\dagger$ in (1.4), $\{Q, Q^\dagger\}$ is a SUSY Hamiltonian and the SUSY sectors are the $\pm 1$ eigenspaces of the chirality operator $\rho_3$. The supercharges $Q$ and $Q^\dagger$ can be written as

$$Q = -iw \left( \partial_r + \frac{1}{r} - \frac{z + qw}{r} + qw \right) = -i \left( \partial_r + \frac{1}{r} + \frac{z - qw}{r} + qw \right)w, \tag{2.7}$$

and

$$Q^\dagger = iw \left( - (\partial_r + \frac{1}{r} + \frac{z - qw}{r} + qw) \right) = i \left( - (\partial_r + \frac{1}{r}) - \frac{z + qw}{r} + qw \right)w. \tag{2.8}$$

$^1$Here $\rho_1 = \begin{pmatrix} 1_2 \cr 1_2 \end{pmatrix}$, $\rho_2 = \begin{pmatrix} i1_2 \cr -i1_2 \end{pmatrix}$, $\rho_3 = \begin{pmatrix} 1_2 \cr -1_2 \end{pmatrix}$.

$^2$Notice that $z$ is $\sigma, \mathbf{L} + 1 + qw$ [13], rather than $\sigma \cdot \mathbf{L} + 1$, as in the Dirac case [17].

$^3$In the Dirac-Coulomb case the eigenvalues of $\mathcal{K}$ are half-integers.
The square $D^2$ of the Dirac Hamiltonian (2.1) is thus

$$
\begin{pmatrix}
Q\dagger Q & QQ\dagger
\end{pmatrix} = -(\partial_r + \frac{1}{r})^2 - \frac{2q^2}{r^2} + q^2 + \frac{z^2 + q^2}{r^2} - \frac{1}{r^2} \left( z + qw \quad z - qw \right).
$$

Let us now introduce the Biedenharn-Temple operator

$$
\Gamma = -(z + \rho_3qw) = -(\sigma.\ell + 1 + \rho_3qw).
$$

Although $\Gamma$ does not commute with the Dirac Hamiltonian $D$, it commutes with its square $D^2$: it is thus conserved for the quadratic dynamics $H_0$ and $H_1$ but not for the Dirac Hamiltonian. In terms of $\Gamma$, $D^2$ becomes

$$
D^2 = -(\partial_r + \frac{1}{r})^2 + \frac{\Gamma(\Gamma + 1)}{r^2} - \frac{2q^2}{r^2} + q^2.
$$

Eqns. (2.5) and (2.10) imply

$$
\Gamma^2 = z^2 + q^2 = J^2 + \frac{1}{4},
$$

so that the eigenvalues of $\Gamma$ are (half)integers,

$$
\gamma = \pm (j + 1/2) \quad \text{sign } \gamma = \text{sign } \kappa.
$$

Hence for a $\Gamma$-eigenfunction (constructed in the next Section),

$$
\Gamma(\Gamma + 1) = L(\gamma)(L(\gamma) + 1) \quad \text{where} \quad L(\gamma) = j \pm \frac{1}{2},
$$

(The sign is plus or minus depending on the sign of $\gamma$). Observe that $L(\gamma)$ is now a (half)integer.

Using the notations $x = z - qw$ and $y = z + qw$, cf. (1.7), the supercharges are written as

$$
Q = -iw\left(\partial_r + \frac{1}{r} - \frac{y}{r} + qw\right) = -i\left(\partial_r + \frac{1}{r} + \frac{x}{r} + qw\right)w
$$

and

$$
Q^\dagger = iw\left( - (\partial_r + \frac{1}{r}) + \frac{x}{r} + qw \right) = i\left( - (\partial_r + \frac{1}{r}) - \frac{y}{r} + qw \right).
$$

One can also write

$$
\Gamma = -\begin{pmatrix}
\sigma \cdot L + 1 + 2qw & 0 \\
0 & \sigma \cdot L + 1
\end{pmatrix} = \begin{pmatrix}
-y & 0 \\
0 & -x
\end{pmatrix}.
$$

$x$ and $y$ are self-adjoint, $x = x^\dagger$, $y = y^\dagger$, $w = w^\dagger$. In terms of $x$ and $y$ the lower (resp. upper) two components of (2.9) are exactly the Hamiltonians $H_0$ and $H_1$ in [4].

## 3 An explicit solution.

In order to find an explicit solution, we first construct angular 2-spinors $\varphi_{\pm}^\mu$ and $\Phi_{\pm}^\mu$, which are both eigenfunctions of $J^2$ and $J_3$ with eigenvalues $j(j + 1)$ and $\mu$, and which diagonalize the operators $x$ and $y$:

$$
x \varphi_{\pm}^\mu = \mp | \gamma | \varphi_{\pm}^\mu \quad y \Phi_{\pm}^\mu = \mp | \gamma | \Phi_{\pm}^\mu.
$$

In the lower ($\rho_3 = -1$) sector the gyromagnetic ratio is $g = 0$, so the $x$-eigenspinors $\varphi$ are obtained by the prescription valid in the Coulomb case except for that the ordinary spherical

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For Dirac-Coulomb the eigenvalues of $\Gamma$ are irrational, yielding the fine structure.
harmonics should be replaced by the Wu and Yang [19] ‘monopole’ harmonics. The coefficient of the \( r^{-2} \) term here is the square of the orbital angular momentum,

\[
x(x - 1) = L^2 = L(\gamma)(L(\gamma) + 1),
\]

so that \( L(\gamma) \) is just the orbital angular quantum number. Due to the addition theorem of the angular momentum, if \( j \geq q + 1/2 \), \( L(\gamma) = j \pm 1/2 \) but for \( j = q − 1/2 \) the only allowed value of \( L(\gamma) \) is \( L(\gamma) = j + 1/2 \).

In detail, for \( j \geq q + 1/2 \) consider therefore the spinorial functions

\[
\varphi_{\pm}^\mu = \sqrt{\frac{L(\gamma) + 1/2 ± \mu}{2L(\gamma) + 1}} Y_{L(\gamma)}^{\mu - 1/2}(1) \pm \sqrt{\frac{L(\gamma) + 1/2 ± \mu}{2L(\gamma) + 1}} Y_{L(\gamma)}^{\mu + 1/2}(0),
\]

where the \( Y \)'s are monopole spherical harmonics and the sign \( ± \) refers to the sign of \( \gamma \). The \( \varphi \)'s satisfy \(^5\)

\[
J^2 \varphi_{\pm} = j(j + 1) \varphi_{\pm}, \quad J_3 \varphi_{\pm} = \mu \varphi_{\pm}, \quad \mu = -j, \cdots, j, \quad L^2 \varphi_{\pm} = L(\gamma)(L(\gamma) + 1) \varphi_{\pm}.
\]

Since \( L_3 = J^2 - L^2 - 3/4 \),

\[
x_\varphi = (L_3 + 1) \varphi_{\pm} = ± |\gamma| \varphi_{\pm},
\]

as wanted.

For \( j = q - 1/2 \) no \( \varphi_{\mp} \) (i.e. \( L(\gamma) = q - 1 \) state is available but eqn. (3.3) still yields \( (2q) \) \( \varphi_{\mp} \)-states with \( L(\gamma) = q \):

\[
\varphi_{\mp}^\mu = \sqrt{\frac{q + 1/2 ± \mu}{2q + 1}} Y_q^{\mu - 1/2}(1) \pm \sqrt{\frac{q + 1/2 ± \mu}{2q + 1}} Y_q^{\mu + 1/2}(0),
\]

(where \( \mu = -(q - 1/2), \cdots, (q - 1/2) \)) are eigenstates of \( x \) with eigenvalue \( -q \).

The \( y \)-eigenspinors \( \Phi \) of the upper (i.e. \( \rho_3 = 1 \)) sector are constructed indirectly. Assume first that one can find angular spinors \( \chi_{\pm} \) which diagonalize \( z = \sigma \cdot \ell + 1 \),

\[
z \chi_{\pm} = ± |\kappa| \chi_{\pm},
\]

and also satisfy \( J^2 \chi_{\pm}^\mu = j(j + 1) \chi_{\pm}^\mu, j = q - 1/2, q + 1/2, \cdots, J_3 \chi_{\pm}^\mu = \mu \chi_{\pm}^\mu, \mu = -j, \cdots, j \) and\n
\[
w \chi_{\pm}^\mu = \chi_{\mp}^\mu.
\]

In the subspace spanned by the \( \chi_{\pm} \)'s, \( x = z - qw \) and \( y = z + qw \) have the remarkably symmetric matrix representations

\[
[x] = \begin{pmatrix} |\kappa| & −q \\ −q & −|\kappa| \end{pmatrix}, \quad [y] = \begin{pmatrix} |\kappa| & q \\ q & −|\kappa| \end{pmatrix}.
\]

The eigenvectors \( \varphi_{\pm} \) and \( \Phi_{\pm} \) of \( x \) and \( y \) with eigenvalues \( ± |\gamma| \) are thus

\[
\varphi_{\pm} = (|\kappa| + |\gamma||)\chi_{\pm} - q\chi_{\pm} \quad \varphi_{\mp} = q\chi_{\pm} + (|\kappa| + |\gamma||)\chi_{\mp}
\]

\[
\Phi_{\pm} = (|\kappa| + |\gamma||)\chi_{\pm} + q\chi_{\pm} \quad \Phi_{\mp} = −q\chi_{\pm} + (|\kappa| + |\gamma||)\chi_{\mp}.
\]

\(^5\)the superscript \( \mu \) is dropped for the sake of simplicity.
Expressing the \( \chi \)'s from the upper two equations in terms of the \( x \)-eigenspinors \( \varphi \) yield the \( z \)-eigenspinors

\[
\chi_+ = \frac{1}{2 |\gamma|} \left( \varphi_+ + \frac{q}{|\gamma| + |\kappa|} \varphi_- \right) \quad \chi_- = \frac{1}{2 |\gamma|} \left( - \frac{q}{|\gamma| + |\kappa|} \varphi_+ + \varphi_- \right),
\]

which do indeed satisfy (3.5). For \( j = q - 1/2 \), \( \chi_- \) is missing and \( \chi_+ \) is proportional to the lowest \( \varphi_+ \) as expressed in (3.4), since no \( \varphi_- \) is available.

Eliminating the \( \chi \)'s allows to deduce the \( y \)-eigenspinors \( \Phi \) from the \( x \)-eigenspinors \( \varphi \) according to

\[
\Phi_+ = \frac{1}{|\gamma|} \left( |\kappa| \varphi_+ + q \varphi_- \right) \quad \Phi_- = \frac{1}{|\gamma|} \left( - q \varphi_+ + |\kappa| \varphi_- \right)
\]

which (by construction) satisfy \( J^2 \Phi_\pm = j(j+1) \Phi_\pm \), \( J_0 \Phi_\pm = \mu \Phi_\pm \), \( \mu = -j, \ldots, j \) and

\[
y \Phi_\pm = \mp |\gamma| \Phi_\pm.
\]

Finally, \( w = \sigma \cdot \hat{r} \) interchanges the \( x \) and \( y \) eigenspinors,

\[
w \varphi^\mu_\pm = \Phi^\mu_\mp.
\]

In contrast to what happens in the ‘lower’ (i.e. \( \rho_3 = -1 \)) sector, in the ‘upper’ (i.e. \( \rho_3 = 1 \)) sector \( y(y+1) = L^2 - 2 \sigma \cdot \hat{r} \) is not the square of an angular momentum and hence we do have \( L(\gamma) = q - 1 \) states: \( |\gamma| = q, \kappa = 0 \) for the lowest value of total angular momentum, \( j = q - 1/2 \), and for \( \gamma = -q \) eqn. (3.8) yields (3.4),

\[
\Phi_0 (= \Phi_-) = \varphi_+,
\]

while the entire \( \Phi_+ \) -tower is missing. By (3.6), this is a \((-1)\) eigenstate of \( w \),

\[
w \Phi_0 = -\Phi_0.
\]

Since \( \varphi_+ \) is a \((-q)\) eigenstate of \( x \), \( \Phi_0 \) is indeed an eigenstate of \( y = x + 2qw \) with eigenvalue \((+q)\). Since

\[
\Gamma(\Gamma + 1) \Phi_0^\mu = L(\gamma)(L(\gamma) + 1) \Phi_0^\mu, \quad \Gamma(\Gamma + 1) \varphi^\mu_\gamma = L(\gamma)(L(\gamma) + 1) \varphi^\mu_\gamma,
\]

by construction, for \( j \geq q + 1/2 \) the eigenfunctions of \( D^2 \) are found as

\[
\Psi_{\pm |\gamma|} = u_{\pm} \begin{pmatrix} \Phi_{\pm} \\ 0 \end{pmatrix} \quad \text{for } \rho_3 = 1,
\]

\[
\psi_{\pm |\gamma|} = u_{\pm} \begin{pmatrix} 0 \\ \varphi_{\pm} \end{pmatrix} \quad \text{for } \rho_3 = -1
\]

if \( j \geq q + 1/2 \),

where the radial functions \( u_{\pm}(r) \) solve the non-relativistic Coulomb-type equations

\[
\left[ - (\partial_r + \frac{1}{r})^2 + \frac{L(\gamma)(L(\gamma) + 1)}{r^2} - \frac{2q^2}{r} + q^2 \right] u_{\pm} = E^2 u_{\pm}.
\]

By (2.14), these are just the upper (resp. lower) equations in (1.9), and hence

\[
u_{\pm}(r) \propto r^{L(\gamma)} e^{ikr} F(L(\gamma) + 1 - i\frac{q^2}{k} ; 2L(\gamma) + 2, -2ikr),
\]

(3.17)
where \( k = \sqrt{E^2 - q^2} \). For \( j = q - 1/2 \) we get the \((2q)\) spinors

\[
\psi_+ = u_+ \begin{pmatrix} 0 \\ \varphi_+ \end{pmatrix}, \quad \text{sign } \gamma = +1
\]

(3.18)

in the \( \rho_3 = -1 \) sector with \( L(\gamma) = q^6 \), with \( u_+ \) still as in (3.17).

The energy levels are obtained from the poles of \( F \), \( L(\gamma) + 1 - iq^2/k = -n \), \( n = 0, 1, \ldots \). Introducing the principal quantum number \( p = L(\gamma) + 1 + n \geq q + 1 \) we conclude that, in both \( \rho_3 \) sectors,

\[
E_p = q^2 \left( 1 - \left( \frac{q}{p} \right)^2 \right), \quad p = q + 1, \ldots
\]

(3.19)

The same energy is obtained if \( L + n = L' + n' \). The degeneracy of a \( p \geq q + 1 \)-level is hence \( 2(p^2 - q^2) \).

If \( j = q - 1/2 \), \((2q)\) extra states arise in the \( \rho_3 = 1 \) sector for \( \gamma = -(q + 1) \).

\[
\Psi_0 = u_0 \begin{pmatrix} \Phi_0 \\ 0 \end{pmatrix} \quad \text{for } \rho_3 = 1 \quad \text{and } \gamma = -(q + 1)
\]

(3.20)

where \( u_0 \) solves (3.16) with \( L(\gamma) = q + 1 \). The principal quantum number is now \( p = q \) yielding the \( 2q \)-fold degenerate 0 - energy ground states. Since \( F(0, a, z) = 1 \), and the lowest wave number \( k_0 \) is \( iq \), \( u_0 \) is simply

\[
u_0 = r^{q-1} e^{-qr},
\]

(3.21)

cf. [1, 4]. The situation is shown in Figures 1-2.

### 4 Dyon Helicity and a new SUSY.

Let us now focus our attention to a single \( j = \text{const.} \) sector. The spectra on FIGs. 1-2. are reminiscent of those of a SUSY system except for non-zero ground-state energy: (3.19) with \( n = 0 \) i.e. \( p = p_0 = L_0 + 1 = j - 1/2 \) yields indeed

\[
E^{(j)}_0 = q^2 \left( 1 - \left( \frac{q}{|\gamma|} \right)^2 \right) = q^2 \left( 1 - \left( \frac{q}{j + 1/2} \right)^2 \right),
\]

(4.1)

since \( |\gamma| = j + 1/2 \). Let us subtract therefore the ground-state energy \( E^{(j)}_0 \) from the Hamiltonian and consider rather

\[
\begin{pmatrix} K_1 \\ K_0 \end{pmatrix} = D^2 - E^{(j)}_0 = \begin{pmatrix} H_1 - q^2 + \frac{q^4}{|\gamma|} & \frac{q^4}{|\gamma|} \\ \frac{q^4}{|\gamma|} & H_0 - q^2 + \frac{q^4}{|\gamma|} \end{pmatrix}.
\]

(4.2)

Now we show that the new Hamiltonian (4.2) is indeed supersymmetric: following Biedenharn [7-9], let us define the 'Dyon Helicity' operator \( \mathcal{R} \) as follows: set first

\[
R = w \left[ \left( \partial_r + \text{displaystyle } \frac{1}{r} - \frac{y}{r} \right) y + q^2 \right],
\]

(4.3)

so that

\[
R = \left[ \left( \partial_r + \frac{1}{r} - \frac{x}{r} \right) x + q^2 \right] w,
\]

\[
R^\dagger = w \left[ \left( \partial_r + \frac{1}{r} - \frac{x}{r} \right) x + q^2 \right] w = \left[ \left( \partial_r + \frac{1}{r} - \frac{y}{r} \right) y + q^2 \right] w.
\]

\( \mathcal{R} \) is the operator with \( \mathcal{R} \mathcal{R}^\dagger \) appearing in the section on supersymmetry.
Since \( xR = - Ry \) and \( yR^\dagger = - R^\dagger x \), it is easy to verify that
\[
RR^\dagger = (2H_0 - q^2)x^2 + q^4 \quad \text{and} \quad R^\dagger R = (2H_1 - q^2)y^2 + q^4. \tag{4.5}
\]

Generalizing Bienharn’s approach [7-9], let us now introduce the ‘Dyon Helicity’ or ‘Biedenharn-Johnson-Lippmann’ [11] operator as
\[
\mathcal{R} = \begin{pmatrix} R & R^\dagger \\ R^\dagger & RR \end{pmatrix}. \tag{4.6}
\]
\( \mathcal{R} \) satisfies the (anti)commutation relations,
\[
[\mathcal{R}, J] = 0, \quad \{\mathcal{R}, \Gamma\} = 0, \quad \{\mathcal{R}, \rho_3\} = 0,
\]
and
\[
\mathcal{R}^2 = \begin{pmatrix} R^\dagger R & RR^\dagger \\ RR^\dagger & RR \end{pmatrix} = (D^2 - q^2)\Gamma^2 + q^4. \tag{4.8}
\]

Our Dyon Helicity operator \( \mathcal{R} \) preserves thus the total angular momentum \( j \), changes the sign of \( \Gamma \), and interchanges the chiral eigensectors.

It follows now from (4.6), that, for each \( j = \text{const.} \) sector, we get hence a new supersymmetric system with supercharges
\[
\mathcal{S} = \begin{pmatrix} S & S^\dagger \\ S^\dagger & S \end{pmatrix} = \begin{pmatrix} \omega & \left( \partial_r + \frac{1}{r} - \frac{z}{r} + \frac{q^4}{r^2} \right) \\ \left( \partial_r + \frac{1}{r} - \frac{z}{r} + \frac{q^4}{r^2} \right) & \omega \end{pmatrix}, \tag{4.9}
\]
In fact,
\[
\begin{pmatrix} K_1 \\ K_0 \end{pmatrix} = \mathcal{S}^2, \tag{4.10}
\]
and (4.7) implies the analogous relations
\[
[S, J] = 0, \quad \{S, \Gamma\} = 0, \quad \{S, \rho_3\} = 0. \tag{4.11}
\]

The identities
\[
\sigma \cdot \pi = (\sigma \cdot \hat{r})(\hat{r} \cdot \pi) + (\sigma \times \hat{r})(\hat{r} \times \pi) = (\sigma \cdot \hat{r})(\hat{r} \cdot \pi) + i(\sigma \cdot \hat{r})(\frac{\sigma \cdot \ell}{r})
\]
imply that
\[
i(\sigma \cdot \pi) = w(\partial_r + \frac{1}{r} - \frac{z}{r}), \tag{4.12}
\]
and thus \( \mathcal{S} \) is indeed (1.11).

For positive-energy states we can thus define the new the unitary transformations
\[
V = \mathcal{S} \frac{1}{\sqrt{\mathcal{S}^\dagger \mathcal{S}}}, \quad V^{-1} = V^\dagger = \frac{1}{\sqrt{\mathcal{S}^\dagger \mathcal{S}}} \mathcal{S}^\dagger, \tag{4.13}
\]
which intertwine the eigensectors of the ‘\( \Gamma - \) fermionic operator’
\[
\text{sign} \Gamma = \left| \frac{\Gamma}{\gamma} \right|, \tag{4.14}
\]
\( V^\dagger K_1 V = K_0 \). Plainly, \( x \) (resp. \( y \)) is conserved for the \( K_0 \) (resp. \( K_1 \)) dynamics, \( [x, K_0] = [x, H_0] = 0 \), and \( [y, K_1] = [y, H_1] = 0 \). The new SUSY transformations interchange \( x \) and \( -y \),
\[
V^\dagger y V = - x \quad V x V^\dagger = - y. \tag{4.15}
\]
For $j \geq q + 1/2$, each $j = \text{const.}$ sector has $2(2j + 1)$ ground-states: while the $2j + 1$ counts the angular eigenfunctions $\varphi$ and $\Phi$ constructed in Section 3, the 2 comes from the two single lower-lying dots in each $j$-column in FIG.1-2. These states have wave number $k_0 = iq^2/|\gamma| = iq^2/(j + 1/2)$, and the wave function is
\[
u_0^{(j)} = r^{j+1/2} e^{-r^2/(j+1/2)}.	ag{4.16}
\]
(tim\'es the corresponding angular eigenfunctions $\varphi$ and $\Phi$), which clearly generalizes (3.21) to $j \geq q + 1/2$. For these values of the angular momentum, the Atiyah-Singer index (defined as the difference of the number of solutions of $S\Phi = 0$ in the different supersectors) is 0. For the lowest angular momentum value $j = q - 1/2$ the ground state has vanishing energy and thus $S = Q$, so we recover those ground states (3.21) in the 'upper' sector with Atiyah-Singer index $2q$.

It is amusing to see how these properties are verified in the explicit angular basis described in the previous Section: if $\Psi$ is an $E$-eigenstate of $-D^2$ in the $\gamma$-sector, $\Gamma\Psi = \gamma\Psi$, then $\Gamma(S\Psi) = -S(\Gamma\Psi) = -\gamma(S\Psi)$, so that $V\Psi$ is a state with the same energy in the $(\gamma)$ sector. More precisely, let us consider a $\gamma$-eigenstate $\Psi_{\gamma} = u_{\pm}\Phi_{\pm}$. Since $w\Phi_{\pm} = \varphi_{\pm}$, $S\Psi_{\gamma} = \left[(\partial_r + \frac{1+\gamma}{r} - \frac{q^2}{\gamma})u_{\pm}\right]\varphi_{\mp}$,\tag{4.17}
the action of the Dyon Helicity operator decomposes into radial and angular action, and the angular part ($w$) just switches over the angular eigenfunctions. But using the recurrence relations of the hypergeometric functions one proves \cite{7}, that
\[
\left[(\partial_r + \frac{1+\gamma}{r} - \frac{q^2}{\gamma})u_{\pm}\right]u_{\pm} = \sqrt{k^2 + \frac{q^4}{\gamma^2}} u_{\mp},
\tag{4.18}
\]
so that the two factors combine into
\[
V(\Psi_{\pm|\gamma}) = \psi_{\mp|\gamma} \quad \text{and} \quad V^\dagger(\psi_{\pm|\gamma}) = \Psi_{\mp|\gamma},
\tag{4.19}
\]
i.e. the new SUSY just intertwines the wavefunctions.

It is interesting to note, that both ground states $\Psi_0$ and $\psi_0$ satisfy the first-order relations
\[
S\Psi_0 = 0 \quad \text{and} \quad S^\dagger\psi_0 = 0 \quad \text{for} \quad j \geq q + 1/2
\]
\[
S\psi_0 = 0 \quad \text{for} \quad j = q - 1/2,
\tag{4.20}
\]
where we dropped the upper index $j$ for simplicity. Indeed, for the 'upper' (i.e. $\rho_3 = +1$) sector the ground state corresponds to $y = -\gamma = |\gamma|$ and in in the 'lower' i.e. (i.e. $\rho_3 = -1$) sector $x = -\gamma = |\gamma|$. Therefore the radial parts of $S$ and $S^\dagger$ are, up to an overall sign, identical. But the ground-state wave number is $k_0^2 = -q^2/|\gamma|$ and thus our statement follows from (4.16) and (4.17).

5 The $S$-matrix from SUSY.

For large $r$ the $1/r$ terms can be dropped in the Dyon Helicity operator,
\[
R \rightarrow R_{\text{scatt}} = (-ip_r x - q^2)w \quad \text{and} \quad R^\dagger \rightarrow R^\dagger_{\text{scatt}} = (-ip_r y - q^2)w.
\tag{5.1}
\]
Their actions on the scattering states are therefore

\[ R_{\text{scatt}}^\dagger \Psi_\epsilon = (-i\epsilon k\gamma - q^2)\psi_\epsilon \quad \text{and} \quad R_{\text{scatt}}\psi_\epsilon = (-i\epsilon k\gamma - q^2)\Psi_\epsilon, \quad (5.2) \]

where \( \epsilon = \pm 1 \) for in/out.

The eigenstates \( \Psi_\pm \) and \( \psi_\pm \) can asymptotically be expanded as

\[ \Psi_\pm \sim A_\pm \Psi_\pm^\text{in} + A_\pm \Psi_\pm^\text{out}, \quad \psi_\pm \sim a_\pm \psi_\pm^\text{in} + a_\pm \psi_\pm^\text{out} \quad (5.3) \]

Choose \( A = a \) as in the Coulomb case [9]. Acting on \( \Psi_\pm \) in the r.h.s. of (5.3) by \( R \) produces a state in the opposite sector with shifted index, which is expanded as

\[ R(\Psi_\pm) \sim \sqrt{k^2\gamma^2 + q^4} \left( a_\pm \psi_\pm^\text{in} + a_\pm \psi_\pm^\text{out} \right) \quad (5.4) \]

Acting on the l.h.s. of (5.3) gives in turn, by (5.2),

\[ (-i\epsilon k\gamma - q^2) \left( A_\pm \psi_\pm^\text{in} + A_\pm \psi_\pm^\text{out} \right) \quad (5.5) \]

Equating (5.4) and (5.5) yields the relation

\[ \frac{A_\epsilon}{A_-} = \sqrt{\frac{\epsilon \gamma + iq^2/k}{\epsilon \gamma - iq^2/k}} \quad (5.6) \]

Now remember that the \( \pm \) signs actually mean \( \gamma \) and \( \gamma - 1 \), \( A_+ = A_\gamma \) and \( A_- = A_{\gamma - 1} \) so (5.6) can be viewed as recursion relations whose solutions are

\[ A_\gamma = (i\epsilon)^\gamma \sqrt{\frac{\Gamma(\gamma + iq^2/k)}{\Gamma(\gamma - iq^2/k)}}, \quad (5.7) \]

yielding the phase shift

\[ \frac{A_+}{A_-} = (-1)^\gamma \frac{\Gamma(\gamma + iq^2/k)}{\Gamma(\gamma - iq^2/k)} \times (\text{a } \gamma \text{- independent constant}). \quad (5.8) \]

The S-matrix is hence

\[ S_\gamma = \exp \left( 2i\delta_\gamma \right) C(k), \quad \text{where} \quad \delta_\gamma = \arg \left( \Gamma(\gamma + 1 + iq^2/k) \right). \quad (5.9) \]

The poles of the \( \Gamma \)-function yield once more the positive bound-state spectrum (3.19). The result is consistent with the one obtained in Ref. 2 using the dynamical symmetry.

6 Discussion.

Some aspects of the dyon problem would require further study. One of these concerns the dynamical symmetries: Both Hamiltonians \( H_0 \) and \( H_1 \) have conserved Runge-Lenz vectors as well as extra conserved spin-type vectors, which combine with the angular momentum into an \( o(4) \oplus o(3) \) dynamical symmetry [1, 4]. The spectrum represented on FIGs. 1-2. appears to be consistent with the extension to \( o(4,2) \) [20]. The technique of Barut and Bornzin [21] allows in fact to build an \( o(2,1) \) which should furthermore combine with the \( o(4) \), to provide a spin-dependent realization of \( o(4,2) \), different from the classic one [22]. A related question is to clarify the relation between the Runge-Lenz vector, the spin vectors and the Dyon Helicity operator.
A second remark concern the individual $j = \text{const.}$ and $\rho_3 = \text{fixed sectors in FIGs. 1-2.}$: not only do we get the same patterns in the upper and lower sectors, but, within each sector, the two equations are actually the same up to a shift $j \rightarrow j + 1$. This seems to indicate a shape invariance [23].

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