Further hardness results on the 
rainbow vertex-connection number of graphs

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Abstract

A vertex-colored graph $G$ is rainbow vertex-connected if any pair of vertices in $G$ are connected by a path whose internal vertices have distinct colors, which was introduced by Krivelevich and Yuster. The rainbow vertex-connection number of a connected graph $G$, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow vertex-connected. In a previous paper we showed that it is NP-Complete to decide whether a given graph $G$ has $rvc(G) = 2$. In this paper we show that for every integer $k \geq 2$, deciding whether $rvc(G) \leq k$ is NP-Hard. We also show that for any fixed integer $k \geq 2$, this problem belongs to NP-class, and so it becomes NP-Complete.

Keywords: vertex-colored graph, rainbow vertex-connection number, NP-Hard, NP-Complete.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Undefined terminology and notation can be found in [2].

Let $G$ be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \cdots , k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path $P$ of $G$ is a rainbow path if no two edges of $P$ are colored the same. The graph $G$ is called rainbow-connected if for any pair of vertices $u$ and $v$ of $G$, there is a rainbow $u-v$ path. The minimum number of colors for which there is an edge-coloring of $G$ such that $G$ is rainbow connected is called the rainbow connection number, denoted by $rc(G)$. Clearly, if a graph is rainbow connected, then it is also connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected, just assign each edge a distinct color. An easy observation is that if $G$ has $n$ vertices then $rc(G) \leq n - 1$, since one may color the edges
of a spanning tree with distinct colors, and color the remaining edges with one of the colors already used. It is easy to see that if $H$ is a connected spanning subgraph of $G$, then $rc(G) \leq rc(H)$. We note the trivial fact that $rc(G) = 1$ if and only if $G$ is a clique, the fact that $rc(G) = n - 1$ if and only if $G$ is a tree, and the easy observation that a cycle with $k \geq 4$ vertices has a rainbow connection number $\lceil k/2 \rceil$. Also notice that $rc(G) \geq diam(G)$, where $diam(G)$ is the diameter of $G$.

Similar to the concept of rainbow connection number, Krivelevich and Yuster \[7\] proposed the concept of rainbow vertex-connection. Let $G$ be a nontrivial connected graph with a vertex-coloring $c : V(G) \rightarrow \{1, 2, \cdots, k\}, k \in \mathbb{N}$. A path $P$ of $G$ is rainbow vertex-connected if its internal vertices have distinct colors. The graph $G$ is rainbow vertex-connected if any pair of vertices are connected by a rainbow vertex-connected path. In particular, if $k$ colors are used, then $G$ is rainbow $k$-vertex-connected. The rainbow vertex-connection number of a connected graph $G$, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow vertex-connected. An easy observation is that if $G$ is of order $n$ then $rvc(G) \leq n - 2$, $rvc(G) = 0$ if and only if $G$ is a complete graph, and $rvc(G) = 1$ if and only if $diam(G) = 2$. Notice that $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2. For the rainbow connection number and the rainbow vertex-connection number, some examples were given to show that there is no upper bound for one of parameters in terms of the other in \[7\]. Krivelevich and Yuster \[7\] proved that if $G$ is a graph with $n$ vertices and minimum degree $\delta$, then $rvc(G) < 11n/\delta$. Li and Shi used a similar proof technique and greatly improved this bound, see \[9\].

The computational complexity of rainbow connection number has been studied extensively. In \[3\], Caro et al. conjectured that computing $rc(G)$ is an NP-Hard problem, and that even deciding whether a graph has $rc(G) = 2$ is NP-Complete. Later, Chakrabory et al. confirmed this conjecture in \[4\]. They also conjectured that for every integer $k \geq 2$, to decide whether $rc(G) \leq k$ is NP-Hard. Recently, Ananth and Nasre confirmed the conjecture in \[1\]. Li and Li \[8\] showed that for any fixed integer $k \geq 2$, to decide whether $rc(G) \leq k$ is actually NP-Complete. For the rainbow vertex-connection number we got a similar complexity result in \[6\].

**Theorem 1** \[6\] Given a graph $G$, deciding whether $rvc(G) = 2$ is NP-Complete. Thus, computing $rvc(G)$ is NP-Hard.

As a generalization of the above result, in this paper we will show the following result:

**Theorem 2** For every integer $k \geq 2$, to decide whether $rvc(G) \leq k$ is NP-Hard. Moreover, for any fixed integer $k \geq 2$, the problem belongs to NP-class, and therefore it is NP-Complete.

In order to prove this theorem, we first show that an intermediate problem called the $k$-subset rainbow vertex-connection problem is NP-Hard by giving a reduction from
the vertex-coloring problem. We then establish the polynomial-time equivalence of the $k$-subset rainbow vertex-connection problem and the problem of deciding whether $rvc(G) \leq k$ for a graph $G$.

2 Proof of Theorem 2

We first describe the problem of $k$-subset rainbow vertex-connection: given a graph $G$ and a set of pairs $P \subseteq V(G) \times V(G)$, decide whether there is a vertex-coloring of $G$ with $k$ colors such that every pair of vertices $(u, v) \in P$ is rainbow vertex-connected. Recall that the $k$-vertex-coloring problem is as follows: given a graph $G$ and an integer $k$, whether there exists an assignment of at most $k$ colors to the vertices of $G$ such that no pair of adjacent vertices are colored the same. It is known that this $k$-vertex-coloring problem is NP-Hard for $k \geq 3$. Now we reduce the $k$-vertex-coloring problem to the $k$-subset rainbow vertex-connection problem, which shows that the problem of $k$-subset rainbow vertex-connection is NP-Hard.

Lemma 1 The problem of $k$-vertex-coloring is polynomially reducible to the problem of $k$-subset rainbow vertex-connection.

Proof. Let $G = (V, E)$ be an instance of the $k$-vertex-coloring problem, we construct a graph $\langle G' = (V', E'), P \rangle$ as follows:

For every vertex $v \in V$ we introduce a new vertex $x_v$. We set

$$V' = V \cup \{x_v : v \in V\} \text{ and } E' = E \cup \{(v, x_v) : v \in V\}.$$

Now we define the set $P$ as follows:

$$P = \{(x_u, x_v) : (u, v) \in E\}.$$

It remains to verify that $G$ is vertex-colorable using $k(\geq 3)$ colors if and only if there is a vertex-coloring of $G'$ with $k$ colors such that every pair of vertices $(x_u, x_v) \in P$ is rainbow vertex-connected.

Let $c$ be the proper $k$-vertex-coloring of $G$. We define the vertex-coloring $c'$ of $G'$ by $c'(x_v) = c(v)$. If $(x_u, x_v) \in P$, then $(u, v) \in E$, $c(u) \neq c(v)$, and so $c'(u) \neq c'(v)$, $x_uuvx_v$ is a rainbow vertex-connected path between $x_u$ and $x_v$.

In the other direction, assume that $c'$ is a $k$-vertex-coloring of $G'$ such that every pair of vertices $(x_u, x_v) \in P$ is rainbow vertex-connected. We define the vertex-coloring $c$ of $G$ by $c(v) = c'(v)$. For every $(u, v) \in E$, $(x_u, x_v) \in P$, since the rainbow vertex-connected
path between \( x_u \) and \( x_v \) must go through \( u \) and \( v \), \( c'(u) \neq c'(v) \), and so \( c(u) \neq c(v) \), thus \( c \) is the proper \( k \)-vertex-coloring of \( G \). 

In the following, we prove that the problem of deciding whether a graph is \( k \)-subset rainbow vertex-connection is polynomial-time equivalent to the problem of deciding whether \( rvc(G) \leq k \) for a graph \( G \).

**Lemma 2** The following problems are polynomial-time equivalent:
1. Given a graph \( G \), decide whether \( rvc(G) \leq k \).
2. Given a graph \( G \) and a set \( P \subseteq V(G) \times V(G) \) of pairs of vertices, decide whether there is a vertex-coloring of \( G \) with \( k \) colors such that every pair of vertices \((u, v) \in P\) is rainbow vertex-connected.

**Proof.** It is sufficient to demonstrate a reduction from Problem 2 to Problem 1. Let \( \langle G = (V, E), P \rangle \) be any instance of Problem 2. We construct a graph \( G_k = (V_k, E_k) \) such that \( G \) is a subgraph of \( G_k \) and \( rvc(G_k) \leq k \) if and only if \( G \) is \( k \)-subset rainbow vertex-connected. We prove the correctness of the reduction by induction on \( k \). For \( k = 2 \) and \( k = 3 \), we give explicit constructions and show that the reduction is valid. Then we show our inductive step to get \( G_k \) and prove the correctness of the reduction.

**Construction of \( G_2 \):** Let \( G_2 = (V_2, E_2) \) where the vertex set \( V_2 \) is defined as follows:

\[
V_2 = \{u\} \cup V_2^{(0)} \cup V_2^{(2)}
\]

\[
V_2^{(0)} = \{v_i^{(2)} : i \in \{1, 2, \ldots, n\}\} \cup \{w_{i,j}^{(2)} : (v_i, v_j) \in (V \times V) \setminus P\}
\]

\[
V_2^{(2)} = \{v_i, 2 : i \in \{1, 2, \ldots, n\}\}
\]

and the edge set \( E_2 \) is defined as:

\[
E_2 = E_2^{(1)} \cup E_2^{(2)} \cup E_2^{(3)} \cup E_2^{(4)} \cup E_2^{(5)} \cup E_2^{(6)}
\]

\[
E_2^{(1)} = \{(u, x) : x \in V_2^{(0)}\}
\]

\[
E_2^{(2)} = \{(v_i^{(2)}, v_i^{(2)}) : i \in \{1, 2, \ldots, n\}\}
\]

\[
E_2^{(3)} = \{(w_{i,j}^{(2)}, w_{i,j}^{(2)}) : (v_i, v_j) \in (V \times V) \setminus P\}
\]

\[
E_2^{(4)} = \{(v_i, 2, v_i, 0), (v_i, 2, v_i, 0) : i \in \{1, 2, \ldots, n\}\}
\]

\[
E_2^{(5)} = \{(v_i, 2, v_i^{(1)}), (v_i, 2, v_i^{(2)}) : (v_i, v_j) \in (V \times V) \setminus P\}
\]

\[
E_2^{(6)} = \{(v_i, 2, v_j, 2) : (v_i, v_j) \in E(G)\}
\]

Denote \( H_2 = G_2[\{v_i, 2 : i \in \{1, 2, \ldots, n\}\}] \). Let \( P_2 = \{(v_i, 2, v_j, 2) : (v_i, v_j) \in P\} \). The graph \( G_2 \) satisfies the property that for all \((v_i, 2, v_j, 2) \in P_2\) there is no path of length \( \leq 3 \) between \( v_i, 2 \) and \( v_j, 2 \) in \( G_2 \setminus E(H_2) \) and also for all \((v_i, 2, v_j, 2) \notin P_2\) the length of the shortest path between \( v_i, 2 \) and \( v_j, 2 \) in \( G_2 \setminus E(H_2) \) is 3.

Let \( c : V \rightarrow \{1, 2\} \) be a 2-vertex-coloring of \( G \) such that every pair of vertices in \( P \) is rainbow vertex-connected. Define the vertex-coloring \( c_2 \) of \( G_2 \) as follows:
• $c_2(u) = 1$.

• $c_2(v^{(1)}_{i,0}) = 1$ and $c_2(v^{(2)}_{i,0}) = 2$ for $i \in \{1, 2, \cdots, n\}$.

• $c_2(w^{(1)}_{i,j}) = 1$ and $c_2(w^{(2)}_{i,j}) = 2$, for all $w^{(a)}_{i,j} \in V^{(0)}_2$, $a \in \{1, 2\}$.

• $c_2(v_{i,2}) = c(v_i)$, for $i \in \{1, 2, \cdots, n\}$.

It can be easily verified that $rvc(G_2) \leq 2$ if and only if $G$ is 2-subset rainbow vertex-connected.

**Construction of $G_3$:** Let $G_3 = (V_3, E_3)$ where the vertex set $V_3$ is defined as follows:

$$V_3 = V_3^{(0)} \cup V_3^{(1)} \cup V_3^{(3)}$$

$$V_3^{(0)} = \{v^{(1)}_{i,0}, v^{(2)}_{i,0} : i \in \{1, 2, \cdots, n\}\} \cup \{u^{(1)}_{i,j}, u^{(2)}_{i,j} : (v_i, v_j) \in (V \times V) \setminus P\}$$

$$V_3^{(1)} = \{v^{(1)}_{i,1}, v^{(2)}_{i,1} : i \in \{1, 2, \cdots, n\}\} \cup \{w^{(1)}_{i,j}, w^{(2)}_{i,j} : (v_i, v_j) \in (V \times V) \setminus P\}$$

$$V_3^{(3)} = \{v_{i,3} : i \in \{1, 2, \cdots, n\}\}$$

and the edge set $E_3$ is defined as:

$$E_3 = E_3^{(1)} \cup E_3^{(2)} \cup E_3^{(3)} \cup E_3^{(4)} \cup E_3^{(5)} \cup E_3^{(6)} \cup E_3^{(7)}$$

$$E_3^{(1)} = \{(x, y) : x, y \in V_3^{(0)}\}$$

$$E_3^{(2)} = \{(v^{(a)}_{i,0}, v^{(b)}_{i,1}) : i \in \{1, 2, \cdots, n\}, \ a, b \in \{1, 2\}\}$$

$$E_3^{(3)} = \{(u^{(a)}_{i,j}, w^{(b)}_{i,j}) : (v_i, v_j) \in (V \times V) \setminus P, \ a, b \in \{1, 2\}\}$$

$$E_3^{(4)} = \{(v^{(1)}_{i,1}, v^{(2)}_{i,1}) : i \in \{1, 2, \cdots, n\}\}$$

$$E_3^{(5)} = \{(v_{i,3}, v^{(1)}_{i,1}), (v_{i,3}, v^{(2)}_{i,1}) : i \in \{1, 2, \cdots, n\}\}$$

$$E_3^{(6)} = \{(v_{i,3}, w^{(1)}_{i,j}), (v_{i,3}, w^{(2)}_{i,j}) : (v_i, v_j) \in (V \times V) \setminus P\}$$

$$E_3^{(7)} = \{(v_{i,3}, v_{j,3}) : (v_i, v_j) \in E(G)\}$$

Denote $H_3 = G_3\{v_{i,3} : i \in \{1, 2, \cdots, n\}\}$. Let $P_3 = \{(v_{i,3}, v_{j,3}) : (v_i, v_j) \in P\}$. The graph $G_3$ satisfies the property that for all $(v_{i,3}, v_{j,3}) \in P_3$ there is no path of length $\leq 4$ between $v_{i,3}$ and $v_{j,3}$ in $G_3 \setminus E(H_3)$ and also for all $(v_{i,3}, v_{j,3}) \notin P_3$ the length of the shortest path between $v_{i,3}$ and $v_{j,3}$ in $G_3 \setminus E(H_3)$ is 4.

Let $c : V \rightarrow \{1, 2, 3\}$ be a 3-vertex-coloring of $G$ such that every pair of vertices in $P$ is rainbow vertex-connected. Define the vertex-coloring $c_3$ of $G_3$ as follows:

• $c_3(v^{(1)}_{i,0}) = 1$ and $c_3(v^{(2)}_{i,0}) = 2$, for $i \in \{1, 2, \cdots, n\}$,

• $c_3(u^{(1)}_{i,j}) = 1$ and $c_3(u^{(2)}_{i,j}) = 2$, for $u^{(1)}_{i,j}, u^{(2)}_{i,j} \in V_3^{(0)}$.

• $c_3(v^{(1)}_{i,1}) = 2$ and $c_3(v^{(2)}_{i,1}) = 3$, for $i \in \{1, 2, \cdots, n\}$,

• $c_3(w^{(1)}_{i,j}) = 2$ and $c_3(w^{(2)}_{i,j}) = 3$, for $w^{(1)}_{i,j}, w^{(2)}_{i,j} \in V_3^{(1)}$. 

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\[ c_3(v_{i,3}) = c(v_i), \text{ for } i \in \{1, 2, \cdots, n\}. \]

It can be easily verified that \( rvc(G_3) \leq 3 \) if and only if \( G \) is 3-subset rainbow vertex-connected.

**Inductive construction of \( G_k \):** Assuming that we have constructed \( G_{k-2} = (V_{k-2}, E_{k-2}) \), the graph \( G_k = (V_k, E_k) \) is then constructed as follows: Each base vertex \( v_{i,k-2} \) in \( V_{k-2} \) is split into the vertices \( v_{i,k-2}^{(1)} \) and \( v_{i,k-2}^{(2)} \), and edges are added between them. Any edge of the form \( (x, v_{i,k-2}) \) is replaced by \( (x, v_{i,k-2}^{(1)}), (x, v_{i,k-2}^{(2)}) \). After doing this, we add the vertices \( v_{i,k} \) and edges \( (v_{i,k}, v_{i,k-2}^{(1)}), (v_{i,k}, v_{i,k-2}^{(2)}) \) for \( i \in \{1, 2, \cdots, n\} \). Formally the graph \( G_k \) is defined as follows:

When \( k \) is even: \( V_k = \{u\} \cup V_k^{(0)} \cup V_k^{(2)} \cup \cdots \cup V_k^{(k)} \), where
\[
V_k^{(i)} = \{v_{i,k-2}^{(i)} : i \in \{1, 2, \cdots, k - 4\}\},
\]
\[
V_k^{(k-2)} = \{v_{i,k-2}^{(1)}, v_{i,k-2}^{(2)} : i \in \{1, 2, \cdots, n\}\},
\]
\[
V_k^{(k)} = \{v_{i,k} : i \in \{1, 2, \cdots, n\}\}.
\]

When \( k \) is odd: \( V_k = V_k^{(0)} \cup V_k^{(1)} \cup V_k^{(3)} \cup \cdots \cup V_k^{(k)} \), where
\[
V_k^{(i)} = \{v_{i,k-2}^{(i)} : i = 0, 1, 3, \cdots, k - 4\},
\]
\[
V_k^{(k-2)} = \{v_{i,k-2}^{(1)}, v_{i,k-2}^{(2)} : i \in \{1, 2, \cdots, n\}\},
\]
\[
V_k^{(k)} = \{v_{i,k} : i \in \{1, 2, \cdots, n\}\}.
\]

For all \( k \geq 4 \), \( E_k \) is defined as follows:
\[
E_k = E_{k-2} \setminus (E_{k-2}(V_{k-2}^{(k-4)}, V_{k-2}^{(k-2)}) \cup E(H_{k-2}))
\]
\[
\cup \{(v_{i,h,k-2}^{(a)}, x) : (v_{i,h,k-2}, x) \in E_{k-2}(V_{k-2}^{(k-4)}, V_{k-2}^{(k-2)}), i \in \{1, 2, \cdots, n\}, a \in \{1, 2\}\}
\]
\[
\cup \{(v_{i,h,k-2}^{(1)}, v_{i,h,k-2}^{(2)}) : i \in \{1, 2, \cdots, n\}\}
\]
\[
\cup \{(v_{i,h}, v_{i,h,k}^{(a)}) : i \in \{1, 2, \cdots, n\}, a \in \{1, 2\}\} \cup E(H_k)
\]
where \( E(H_1) = \{(v_{i,j}, v_{i,j}) : (v_i, v_j) \in E(G)\} \) and \( E_{k-2}(V_{k-2}^{(k-4)}, V_{k-2}^{(k-2)}) = \{(u, v) : u \in V_{k-2}^{(k-4)}, v \in V_{k-2}^{(k-2)}\} \).

Let \( P_k = \{(v_{i,k}, v_{j,k}) : (v_i, v_j) \in P\} \). Then we show that the graph \( G_k \) satisfies the following properties as claims:

**Claim 1** For any \((v_{i,k}, v_{j,k}) \in P_k\), there is no path of length less than \( k + 2 \) between \( v_{i,k} \) and \( v_{j,k} \) in \( G_k \setminus E(H_k) \).

**Proof.** It has been shown that the assertion is true for \( G_2 \) and \( G_3 \). Assume that the assertion is true for \( G_{k-2} \). Let \((v_i, v_j) \in P\), then \((v_{i,k-2}, v_{j,k-2}) \in P_{k-2}\), and hence by
induction, there is no path of length less than \( k \) between \( v_{i,k-2} \) and \( v_{j,k-2} \) in \( G_{k-2} \setminus E(H_{k-2}) \). By the construction of \( G_k \), we do not shorten the paths between any two vertices, so the paths from \( v_{i,k-2}^{(\alpha)} \) to \( v_{j,k-2}^{(\beta)} \) will still be of length at least \( k \) for \( \alpha, \beta \in \{1, 2\} \). Consider the graph \( G_k \setminus E(H_k) \). Since the neighbors of the vertex \( v_{i,k} \) are only \( v_{i,k}^{(1)}, v_{i,k}^{(2)} \), the path between \( v_{i,k} \) and \( v_{j,k} \) must be \( v_{i,k} v_{i,k-2}^{(\alpha)} \cdots v_{j,k-2}^{(\beta)} v_{j,k} \) for \( \alpha = 1 \) or \( 2 \), \( \beta = 1 \) or \( 2 \), thus their lengths are at least \( k + 2 \).

**Claim 2** For any \( (v_{i,k}, v_{j,k}) \notin P_k \), the shortest path between \( v_{i,k} \) and \( v_{j,k} \) is of length \( k + 1 \) in \( G_k \setminus E(H_k) \).

**Proof.** It has been shown that the assertion is true for \( G_2 \) and \( G_3 \). Suppose that the assertion is true for \( G_{k-2} \). Let \( (v_i, v_j) \notin P \) then \( (v_{i,k-2}, v_{j,k-2}) \notin P \), and hence by induction, the shortest path between \( v_{i,k-2} \) and \( v_{j,k-2} \) is of length \( k - 1 \) in \( G_{k-2} \setminus E(H_{k-2}) \). By the construction of \( G_k \), we do not shorten the paths between any two vertices, so the shortest path between \( v_{i,k-2}^{(\alpha)} \) and \( v_{j,k-2}^{(\beta)} \) will still be of length \( k - 1 \) for \( \alpha, \beta \in \{1, 2\} \). Consider the graph \( G_k \setminus E(H_k) \). Since the neighbors of the vertex \( v_{i,k} \) are only \( v_{i,k}^{(1)}, v_{i,k}^{(2)} \), the shortest path between \( v_{i,k} \) and \( v_{i,k} \) must be \( v_{i,k} v_{i,k-2}^{(\alpha)} \cdots v_{j,k-2}^{(\beta)} v_{j,k} \) for \( \alpha = 1 \) or \( 2 \), \( \beta = 1 \) or \( 2 \), thus the length of the path is \( k + 1 \).

**Claim 3** \( G \) is \( k \)-subset rainbow vertex-connected if and only if \( G_k \) is \( k \)-rainbow vertex-connected.

**Proof.** Denote \( H_k = G_k[\{v_{i,k} : i \in \{1, 2, \ldots, n\}\}] \). It can be seen that \( H_k \) is isomorphic to \( G \).

If \( G_k \) is \( k \)-rainbow vertex-connected, let \( c_k : V(G_k) \to \{1, 2, \cdots, k\} \) be a vertex-coloring of \( G_k \) with \( k \) colors such that every pair of vertices in \( G_k \) is rainbow vertex-connected. We define the vertex-coloring \( c \) of \( G \) as follows: \( c(v_i) = c_k(v_{i,k}) \) for \( i \in \{1, 2, \cdots, n\} \). If \( (v_i, v_j) \in P \), then \( (v_{i,k}, v_{j,k}) \in P_k \). By Claim 1 there is no path between \( v_{i,k} \) and \( v_{j,k} \) with length less than \( k + 2 \) in \( G_k \setminus E(H_k) \). Hence the entire rainbow vertex-connected path between \( v_{i,k} \) and \( v_{j,k} \) must lie in \( H_k \) itself. Correspondingly, there is a rainbow vertex-connected path between \( v_i \) and \( v_j \) in \( G \). Thus, \( G \) is \( k \)-subset rainbow vertex-connected.

In the other direction, if \( G \) is \( k \)-subset rainbow vertex-connected, let \( c : V(G) \to \{1, 2, \cdots, k\} \) be a vertex-coloring of \( G \) with \( k \) colors such that every pair of vertices in \( P \) is rainbow vertex-connected. We define the vertex-coloring \( c_k \) of \( G_k \) by induction. We have given the vertex-colorings \( c_2, c_3 \) of \( G_2, G_3 \). Assume that \( c_{k-2} : V(G_{k-2}) \to \{1, 2, \cdots, k-2\} \) is a vertex-coloring of \( G_{k-2} \) such that \( G_{k-2} \) is rainbow vertex-connected. We define the vertex-coloring \( c_k \) of \( G_k \) as follows:

When \( k \) is even:

- \( c_k(u) = k - 1 \).
• $c_k(v) = c_{k-2}(v)$, for $v \in V_k^{(0)} \cup V_k^{(2)} \cup \cdots \cup V_k^{(k-4)}$.  
• $c_k(v_{i,k-2}^{(1)}) = k - 1$, $c_k(v_{i,k-2}^{(2)}) = k$, for $i \in \{1, 2, \cdots, n\}$.  
• $c_k(v_{i,k}) = c(v_i)$, for $i \in \{1, 2, \cdots, n\}$.

When $k$ is odd:

• $c_k(v_{i,0}^{(1)}) = c_{k-2}(v_{i,0}^{(1)})$, $c_k(v_{i,0}^{(2)}) = k - 1$, for $i \in \{1, 2, \cdots, n\}$.  
• $c_k(u_{i,j}) = c_{k-2}(u_{i,j})$, $c_k(u_{i,j}^{(2)}) = k - 1$ for $u_{i,j}, u_{i,j}^{(2)} \in V_k^{(0)}$.

Proposition 1 The vertex-coloring $c_k$ of $G_k$ defined above makes $G_k$ rainbow vertex-connected.

Proof. Let $v, w \in V_k$, we now show that $v, w$ are rainbow vertex-connected in $G_k$.

Case 1. $k$ is even.

By the vertex-coloring $c_k$, we have $c_k(v_{i,j}^{(1)}) = j + 1$, $c_k(v_{i,j}^{(2)}) = j + 2$, $c_k(u) = k - 1$ and $c_k(v_{i,k}) = c(v_i)$ for $i \in \{1, 2, \cdots, n\}$, $j \in \{0, 2, \cdots, k - 2\}$.

Subcase 1.1. $v \in V_k^{(p)}$, $w \in V_k^{(q)}$, where $p, q \in \{0, 2, \cdots, k - 2\}$.

If $v = v_{i,p}^{(a)}$, $w = v_{i,q}^{(b)}$ for $a, b \in \{1, 2\}$, then $v v_{i,p-2}^{(1)} v_{i,p-4}^{(1)} \cdots v_{i,0}^{(1)} u v_{j,0}^{(2)} \cdots v_{j,q-2}^{(2)} w$ is the rainbow vertex-connected path between $v$ and $w$.

Subcase 1.2. $v = v_{i,k}$, $w \in V_k^{(q)}$, where $q \in \{0, 2, \cdots, k - 2\}$.

If $w = v_{j,q}^{(a)}$ for $a \in \{1, 2\}$, then $v v_{i,k-2}^{(2)} v_{i,k-4}^{(1)} \cdots v_{i,0}^{(1)} u v_{j,0}^{(2)} \cdots v_{j,q-2}^{(2)} w$ is the rainbow vertex-connected path between $v$ and $w$.

Subcase 1.3. $v = v_{i,k}$, $w = v_{j,k}$.  

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If \((v_{i,k}, v_{j,k}) \in P_k\), then \((v_i, v_j) \in P\). By the vertex-coloring \(c\) of \(G\), there is a rainbow vertex-connected path between \(v_i\) and \(v_j\) in \(G\). Correspondingly, since \(c_k(v_{i,k}) = c(v_i)\), there is a rainbow vertex-connected path between \(v_{i,k}\) and \(v_{j,k}\) in \(G_k\).

If \((v_{i,k}, v_{j,k}) \notin P_k\), then \(v_{i,k}v_{i,k-2}v_{i,k-4}\cdots v_{i,2}w_{i,j}w_{i,j}v_{j,2}\cdots v_{j,k-2}v_{j,k}\) is the rainbow vertex-connected path between \(v_{i,k}\) and \(v_{j,k}\).

**Case 2.** \(k\) is odd.

By the vertex-coloring \(c_k\), we have

\[
\begin{align*}
c_k(v_{i,j}^{(1)}) &= j + 1, \\
c_k(v_{i,j}^{(2)}) &= j + 2, \\
c_k(v_{i,0}^{(1)}) &= 1, \\
c_k(v_{i,0}^{(2)}) &= k - 1, \\
c_k(u_{i,j}^{(1)}) &= 1, \\
c_k(u_{i,j}^{(2)}) &= k - 1, \\
c_k(w_{i,j}^{(1)}) &= 2, \\
c_k(w_{i,j}^{(2)}) &= 3, \\
c_k(v_{i,k}) &= c(v_i),
\end{align*}
\]

for \(i \in \{1, 2, \cdots, n\}\).

**Subcase 2.1.** \(v \in V_k^{(p)}, w \in V_k^{(q)}\), where \(p, q \in \{1, 3, \cdots, k - 2\}\).

If \(v = v_{i,p}^{(a)}, w = v_{j,q}^{(b)}\) for \(a, b \in \{1, 2\}\), then \(vv_{i,p-2}v_{i,p-4}\cdots v_{i,0}v_{j,0}v_{j,1}\cdots v_{j,q-2}w\) is the rainbow vertex-connected path between \(v\) and \(w\).

If \(v = v_{i,j}^{(a)}, w = w_{i,j}^{(b)}\) for \(a, b \in \{1, 2\}\), then \(vv_{i,p-2}v_{i,p-4}\cdots v_{i,0}w_{i,j}w_{i,j}w\) is the rainbow vertex-connected path between \(v\) and \(w\).

If \(v = w_{i,j}^{(a)}, w = w_{i,j}^{(b)}\) for \(a, b \in \{1, 2\}\), then \(vw_{i,0}^{(a)}w_{i,0}u_{i,j}w\) is the rainbow vertex-connected path between \(v\) and \(w\).

**Subcase 2.2.** \(v = v_{i,k}, w \in V_k^{(q)}\), where \(q \in \{1, 3, \cdots, k - 2\}\).

If \(w = v_{j,q}^{(a)}\) for \(a \in \{1, 2\}\), then \(vv_{j,k}^{(1)}v_{j,k-2}v_{j,k-4}\cdots v_{j,0}v_{i,0}v_{i,1}v_{i,2}\cdots v_{i,q-2}w\) is the rainbow vertex-connected path between \(v\) and \(w\).

If \(w = w_{i,j}^{(a)}\) for \(a \in \{1, 2\}\), then \(vv_{i,k}^{(2)}v_{i,k-2}v_{i,k-4}\cdots v_{i,0}v_{i,1}w_{i,j}w_{i,j}w\) is the rainbow vertex-connected path between \(v\) and \(w\).

**Subcase 2.3.** \(v = v_{i,k}, w = v_{j,k}\).

If \((v_{i,k}, v_{j,k}) \in P_k\), then \(v_i, v_j \in P\). By the vertex-coloring \(c\) of \(G\), there is a rainbow vertex-connected path between \(v_i\) and \(v_j\) in \(G\). Correspondingly, since \(c_k(v_{i,k}) = c(v_i)\), there is a rainbow vertex-connected path between \(v_{i,k}\) and \(v_{j,k}\) in \(G_k\).

If \((v_{i,k}, v_{j,k}) \notin P_k\), then \(v_{i,k}v_{i,k-2}v_{i,k-4}\cdots v_{i,2}u_{i,j}u_{i,j}v_{j,2}v_{j,3}\cdots v_{j,k-2}v_{j,k}\) is the rainbow vertex-connected path between \(v_{i,k}\) and \(v_{j,k}\).

**Proof of Theorem 2:** From the above Lemmas 1 and 2, the first part of Theorem 2 the NP-Hardness, follows immediately.
In the following we will prove the second part of Theorem 2. Recall that a problem belongs to NP-class if given any instance of the problem whose answer is “yes”, there is a certificate validating this fact which can be checked in polynomial time. For any fixed integer \( k \), to prove the problem of deciding whether \( rvc(G) \leq k \) is in NP-class, we can choose a rainbow \( k \)-vertex-coloring of \( G \) as a certificate. For checking a rainbow \( k \)-vertex-coloring, we only need to check that \( k \) colors are used and for any two vertices \( u \) and \( v \) of \( G \), there exists a rainbow vertex-connected path between \( u \) and \( v \). Notice that for any two vertices \( u \) and \( v \) of \( G \), there are at most \( n^{\ell-1} u - v \) paths of length \( \ell \), since if we let \( P = uv_1v_2 \cdots v_{\ell-1}v \), then there are less than \( n \) choices for each \( v_i \) \((i \in \{1, 2, \ldots, \ell - 1\})\). Therefore, \( G \) contains at most \( \sum_{\ell=1}^{k+1} n^{\ell-1} = \frac{n^{k+1} - 1}{n - 1} \leq n^k u - v \) paths of length at most \( k + 1 \). Then, check these paths in turn until one finds one path whose internal vertices have distinct colors. It follows that the time used for checking is at most \( O(n^k \cdot n^2 \cdot n^2) = O(n^{k+4}) \). Since \( k \) is a fixed integer, we conclude that the certificate can be checked in polynomial time, which implies that the problem of deciding whether \( rvc(G) \leq k \) belongs to NP-class, and therefore it is NP-Complete.

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