Mathematical modelling and design of isolation diaphragms for pressure gauges

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Abstract. The paper discusses some questions of equilibrium, stability and shape optimization of circular membranes with arbitrary profile along the meridian. Such membranes can be used as a flexible barrier to isolate the media on one side of a diaphragm from media on the other side, whilst allowing the transmission of pressure with minimal loss. The main goal of the analysis is to propose techniques for designing such shape of corrugated membranes that satisfy the requirements of so call “slow growing linear or almost linear” behavior of the pressure-volume diagram. The applied Kirchhoff-Love theory for non-shallow shells was used as a base for deriving nonlinear as well as linearized boundary value problems of equilibrium. A genetic algorithm was used to evaluate and optimize the characteristics of corrugated membranes. The presented analysis of the membrane with the maximum linear stroke showed its efficiency and reliability. At the same time, special attention should be paid to the basic parametrization used in modelling, since the mathematically optimal solution is not always suitable from the point of view of membrane production technology.

1. Introduction

Sensors with the open membrane, including high-temperature sensors, are often required for pressure measurements in the food industry, in the manufacture of plastic articles, building components, for measuring the pressure and the level of liquids. Profiled membranes (also known as corrugated membranes, micro-structured membranes, patterned membranes, membranes with designed topography or notched membranes) are gaining increasing academic and industrial attention and recognition as a viable alternative to flat membranes.

A large number of corrugated membranes are used as elastic elements in the devices of precision mechanics [1, 2]. The understanding of deflection, stress and strain of thin diaphragms with clamped edges under various loads is of great importance for designing and fabricating sensors and actuators [3]. A feasible approach to reduce the residual stress is to incorporate a corrugated diaphragm in the microphone structure. This appears to be a more convenient way to increase the mechanical sensitivity of the condenser microphones than the approach of changing the deposition process during the fabrication [4]. The corrugated composite bipolar plate can improve the performance of the vanadium redox flow battery by reducing pumping and ohmic energy losses [5]. The feasible method to compensate for the high-order effect during bunch length compression, thereby enhancing the peak
current of a high-repetition-rate X-ray free-electron laser source is to insert the corrugated structure downstream of the high-order harmonic cavities. This structure functions as a passive linearizer and enhances the longitudinal profile of the electron beam [6]. The profiled ion exchange membranes have shown to significantly improve the performance of reverse electrodialysis by eliminating the spacer shadow effect and by inducing hydrodynamic changes, leading to ion transport rate enhancement [7]. In all the mentioned cases problem of the effective choose of the profile or corrugation is of great importance.

Shape optimization problems are usually solved numerically, by using iterative methods [8]. That is, one starts with an initial guess for a shape, and then gradually evolves it, until it morphs into the optimal shape. Another strategy in the optimization process, based on discretization and linearization techniques, is introduced by reliable finite element formulation and used in [9] and [10] for the membrane form-finding. A survey of shape parametrization techniques for the effective high fidelity shape optimization is presented in [11].

The next alternative in dealing with the shape optimization problems is the genetic algorithm (GA), developed by Holland in 1975 [12], a programming technique that mimics biological evolution as a problem-solving strategy. In [13] it was used to optimize membrane separation modules, while in [14] it was applied to problems of designing axisymmetric shells of minimal weight. A GA optimizer and evolutionary search algorithms have been applied in [15] in order to solve the optimal design of mufflers. A shape optimization system for fluid dynamic problems, which used the genetic code, was presented in [16] and applied to the optimization of bidimensional aerodynamic profiles with geometric constraints.

The present paper starts from the description of the used model of the corrugated circular shell. To describe the large elastic strains of the membrane under hydrostatic pressure two-dimensional nonlinear equations [17, 18], based on the Kirchhoff’s hypotheses, were used. Some aspects of the numerical analysis of the equilibrium and stability of such shells are discussed in [19]. The modified genetic algorithm is used to design corrugation providing a sufficiently large value of the linear section length of the membrane loading diagram, i.e. plot of applied pressure versus volume under the membrane surface. Some practical aspects of the parametrizations used in modeling discussed to guarantee that the mathematically optimal solution is suitable for the real-life membrane production technology. In the conclusions, some directions of the future work are presented.

2. Membranes modelling

We consider the general problem for a circular shell of revolution of arbitrary profile and assume that the profile of such shell having a thickness \( h \) and radius \( a \gg h \) is given by the function \( z = f(r) \) in cylindrical coordinates. Shape optimization problems were solved by analysis of multi-parameter expressions of \( f(r) \), where all parameters were considered as unknowns.

To describe the large elastic strains of the membrane under hydrostatic pressure \( p \) we use two-dimensional nonlinear equations [17, 18], based on the Kirchhoff’s hypotheses. In addition to the detailed description of non-axisymmetric equations of the shell equilibrium, the boundary value problem for the ODE system describing the axisymmetric behavior of the shell is provided in [19]. The main method of numerical analysis of this two-point nonlinear boundary value problem was shooting method; the scheme of its application to the nonlinear boundary value problems in consideration was presented in [20]. The essential feature of its realization here is associated with the impossibility to select the one loading parameter for the whole cycle of plotting the loading diagram, i.e. the dependence between force (say, applied pressure \( p \)) and geometric (e.g., deflection at the membrane center point, \( w_0 \)) characteristics. Traditionally, either pressure or deflection at the center of the membrane is chosen as such parameter, but in the case of a shape close to a spherical dome one can encounter the situation where there is no functional relationship between these parameters: for non-shallow profiles the loading diagram on the plane \( p - w_0 \) is a complex curve with
self-intersections. To solve this problem a special algorithm was realized to automatically change and select the loading parameter. This algorithm is in some sense close to that of developed in [21].

The scheme for investigating of stability of the constructed axisymmetric solution is based on the bifurcation approach completely described in [19]. Strictly speaking, such an analysis was not the goal of this work since the characteristic of the optimal membrane should have been linear, and therefore not contains extremum points, indicating a possible loss of stability. However, in the process of finding the optimal form, it was important to ensure such stability. The scheme of introducing restrictions on the range of permissible values of parameters that determine the membrane geometry, which was based upon the information on the stability region of a spherical dome, will be described below.

3. The optimization task
The main goal of the research was elaboration the technique of the designing the shape of the corrugated membrane that satisfies the requirements of so call “slow growing linear or almost linear” behavior of the pressure-volume diagram. More precisely these requirements can be formulated by means of the figure 1 and a set of notations presented below.

![Figure 1. Definition of the “linear” behavior](image)

Let’s introduce the following quantities:
- the pressure range, $\Delta p = p_{\text{max}} - p_{\text{min}}$ ;
- the volume range, $\Delta V = V_{\text{max}} - V_{\text{min}}$ ;
- stiffness of the corrugated membrane, $s = \Delta p / \Delta V$ ;
- maximal admissible stiffness, $s_{\text{adm}}$ ;
- nonlinearity of the membrane, $\varepsilon$ ;
- maximal admissible nonlinearity, $\varepsilon_{\text{adm}}$ ;
- the required operation range of the volume, $V_{\text{req}}$ .

Then the condition of slow growth can be reflected by the relation

$$s \leq s_{\text{adm}} \quad (1)$$

that should be valid within the sufficiently large region of the volume

$$\Delta V \geq V_{\text{req}} \quad (2)$$

The condition of linearity means, firstly, that the nonlinearity parameter $\varepsilon$ should be small enough
\[ \varepsilon \leq \varepsilon_{\text{adm}} \]  

(3)

and, secondly, that \( p - V \) curve, or the characteristic curve of the corrugated diaphragm, can intersect the straight line (its linear approximation) no more than at the three points.

One extra requirement is connected with the boundedness of the mechanical stresses: within the region \( p_{\text{min}} < p < p_{\text{max}} \) all mechanical stresses should be limited by the elastic limit of the membrane material.

One can see that conditions (1)–(3) can be easily checked by numerical analysis of the loading diagram (i.e. plot of \( p - V \) dependence) though the construction of this diagram is not so easy. The calculations were based upon the nonlinear equilibrium equations for the thin membrane of arbitrary shape within the limits of axisymmetric deformations mentioned in the previous section. This process requires the solution of a series of nonlinearly boundary value problems.

4. The optimization technique

The optimization process was divided into two steps. At the first one some shape of the membrane profile was chosen and \( N \)-parametrical description of this shape was constructed. Possible example of such shape and its description by the set of \( 2n+5 \) parameters \((r_1, r_2, \ldots, r_n, R_0, R_1, \ldots, R_{n+2}, H_1, H_2)\) is presented at the figure 2. Actually some of them are more important while others are not so significant so usually from 5 to 10 parameters were used for the optimization purposes.

![Figure 2. Example of the shape parametrization](image)

The optimization step, or the process of finding the optimal values of the chosen set of parameters, was performed by means of the genetic algorithm (GA). This algorithm has been inspired by evolutionary biology and incorporates techniques such as inheritance, mutation, inverse, selection and crossover to find a better solution. The used algorithm can be called modified as it compares different techniques, such as three-stepwise single-point crossover, elitism strategy and truncation method in order to achieve certain results in an acceptable time. The detailed description of the used GA was presented in [22].

Very important part of this algorithm is the choosing of the fitness function \( F \), i.e. the function we want to minimize. In our case it was chosen to reflect requirements (1) and (3) so roughly speaking it could be written as

\[ F = K_1 \frac{s}{3} + K_2 \frac{\varepsilon}{10} \]  

(4)

The role of coefficients \( K_i \) in (4) was to reflect the importance of linearity (so we set \( K_2 \gg K_1 \)) or the importance of the slow growth (so we set \( K_2 \gg K_1 \)). These coefficients were also used as penalty functions depending upon the volume \( V \) with quick growth in the area \( \Delta V < V_{\text{req}} \) to guarantee condition (2) fulfillment.
To narrow the search range for the optimal values of the geometry parameters, the following approach was used, based on the requirement of the membrane operation without loss of stability. Such check for a given membrane shape is rather time-consuming, so it is inexpediency to use it every time when calculating the objective function $F$. Therefore, for each checking geometry, before calculating the objective function, a rough empirical assessment of possible problems with stability was performed. To do this, in the vicinity of the membrane surface, two spherical domes were built – the inner and the outer (see figure 3). The membrane was considered suitable for consideration if both were stable in the required range of applied pressure $p < p_{\text{max}}$. The classical expression from [23] was used as a criterion for the stability of the dome. Stability tests were realized regardless of the GA program by analyzing only optimal forms.

![Figure 3](image.png)

**Figure 3.** Dome surfaces used to roughly estimate stability

5. **Numerical results**

To demonstrate the used approach the example of the membrane with the radius $\rho = 25$ mm and the thickness $h = 0.1$ mm was presented in [24] based on a set of profile heights $z_i$ ($i = 1...6$) in six points with fixed radii $r_i$ ($i = 1...6$), and the profile itself is built according to a given set based on the spline interpolation. In the process of optimization, a set of parameters constituting the membrane profile with the maximum linear stroke was found. The constructed membrane differs very little from a circular plate; however the difference in loading diagrams between the corrugated membrane and the plate is not only significant, but also qualitative. The example in [24] demonstrates the good mathematical possibilities of the method, however, the chosen formal parametrization led to an unsatisfactory result in terms of technology: the production of a thin biconvex membrane (the obtained values for $z_1, z_2, ..., z_6$ are positive though $z_6$ is negative) is very laborious, which immediately reduces the efficiency of the resulting solutions.

Therefore, as a numerical example, consider a more realistic problem. As a parametrization, we will use the scheme shown in figure 2, where the number of rollers is three, all of which have the same radii. The idea to use the flat area in the center of membrane was borrowed from [25]. We assume that this flat area has a radius of 5 mm, and the horizontal zone of fixing the edge of the membrane has a width of 2 mm. Thus, we obtain seven parameters for optimization: $r_1, R_1, R_2, R_3, H_1, H_2, H_3$.

The profile and three-dimensional view of the optimal membrane obtained by GA are shown in figure 4, its loading diagram is presented in figure 5. Please note that different scales used for visibility in vertical and horizontal directions at figure 4.
Figure 4. Schematic view of the optimized membrane: a) profile, b) 3D view

Figure 5. Loading diagram for the membrane presented at figure 4

It turned out that the characteristics (nonlinearity and stiffness) of this membrane are slightly (5-10%) worse than those obtained earlier in [24], but the technology of its manufacturing is not so difficult.

To check whether the designed diaphragm is stable under the applied pressure the bifurcation equations were analyzed. In particular it has been shown that for pressure within limits from zero to 200 MPa no bifurcations with mode from 0 to 10 can happen. This proves the fact of mechanical stability of the proposed construction.

To guarantee that proposed design can be realized technologically it is necessary to show that it is stable with respect to possible technological imperfections, that is, to check how minor changes in the shape will influence upon the performance of the membrane. To check this necessary condition a series of numerical experiments were performed and a lot of data on the dependence of the “pressure-volume” graph upon the parameters of the geometry was obtained. Analysis of these distortions of the membrane shape shows that in some cases they can slightly change the shape of the plot but it still will satisfy the requirements (1)–(3).

6. Conclusions
A genetic algorithm was used to evaluate and optimize the characteristics of corrugated membranes. The presented analysis of the membrane with the maximum linear stroke showed its efficiency and reliability. The future research in this field will be related with increasing the speed of the proposed
algorithm using parallel computations and its application to the optimization problems for the membranes, taking into account more significant limitations, in particular, on the allowable stresses level.

References

[1] Giovanni M Di 1982 *Flat and Corrugated Diaphragm Design Handbook* (New York: Marcel Dekker, Inc)

[2] Charcosset C 2012 *Membrane Processes in Biotechnology and Pharmaceutics* (Amsterdam: Elsevier)

[3] Wang Q and Ko W H 1999 Modeling of touch mode capacitive sensors and diaphragms *Sensors and Actuators* 75 230–41

[4] Chen C-H, Kann H-C, Yang P-H and Wang Y-T 2007 Modeling of corrugated diaphragms for condenser microphones *Proc. Int. Microsystems, Packaging, Assembly and Circuits Technology (Taipei)* 161–4

[5] Choe J, Kim J and Lee D G 2016 Shape optimization of the corrugated composite bipolar plate (CCBP) for vanadium redox flow batteries (VRFBs) *Composite Structures* 158 333–9

[6] Wang Z, Feng C, Huang D-Z, Qiang Gu and Zhang M 2018 Nonlinear energy chirp compensation with corrugated structures *Nuclear Science and Tech.* 29 175

[7] Pawlowski S, Crespo J G and Svetlozar V 2019 Profiled ion exchange membranes: a comprehensible review *Int. J. Mol. Sci.* 20 165

[8] Delfour M C and Zol J-P 2011 *Shapes and Geometries: Analysis, Differential Calculus, and Optimization* (Philadelphia: SIAM)

[9] Bletzinger K-U 1998 *Form Finding and Optimization of Membranes and Minimal Surfaces* (Lyngby: Tech. Univ. of Denmark)

[10] Gosling P D and Lewis W J 1996 Form-finding of prestressed membranes using a curved quadrilateral finite element for surface definition *Comput. Structs* 61 871–83

[11] Jamshid A S 2001 Survey of shape parameterization techniques for high-fidelity multidisciplinary shape optimization *AIAA Journal* 39 877–84

[12] Holland J H 1975 *Adaptation in Natural and Artificial Systems* (Michigan: Univ. of Michigan Press)

[13] Yuen C C, Aatmeeyata, Gupta S K and Ray A K. 2000 Multi-objective optimization of membrane separation modules using genetic algorithm *J of Membrane Sci* 176 177–96

[14] Banichuk N V, Ivanova S Yu and Makeev E V 2007 Some problems of optimizing shell shape and thickness distribution on the basis of a genetic algorithm *Mech Solids* 42 (6) 956–64

[15] Yeh L J, Chang Y C and Chiu M.C. 2004 Application of genetic algorithm to the shape optimization of a constrained double-chamber muffler with extended tubes *J of Marine Science and Technology* 12 (3) 189–99

[16] López D, Angulo C, Fernández de Bustos I, and García V 2013 Framework for the shape optimization of aerodynamic profiles using genetic algorithms *Math Problems in Engineering* 2013 275091

[17] Libai A and Simmonds J G 1998 *The Nonlinear Theory of Elastic Shells* (Cambridge: Cambridge University Press)

[18] Vorovich I 1999 *Nonlinear Theory of Shallow Shells* (New York: Springer)

[19] Getman I P, Karyakin M I and Ustinov U A 2010 Nonlinear behavior analysis of circular membranes with arbitrary radius profile *J Applied Mathematics and Mechanics* 74 917–27

[20] Gavrilvachenko T V, Karyakin M I and Sukhov D Y 2008 Designing of the interface for nonlinear boundary value problem solver using Maple *Proc of the International Conference on Computational Sciences and its Applications (Perugia)* (Los Alamitos: IEEE) 284–91

[21] Grigolyuk E I and Lopanitsyn Ye A 2002 The axisymmetric postbuckling behaviour of shallow spherical domes *J Applied Mathematics and Mechanics* 66 605–16

[22] Karyakin M and Sigaeva T 2011 Application of genetic algorithms to the shape optimization of...
the nonlinearly elastic corrugated membranes Adv. Struct. Materials 15 297–306
[23] Singer J, Arbocz J and Weller T 2002 Buckling Experiments: Experimental Methods in Buckling of Thin-Walled Structures: Shells, Built-Up Structures Composites and Additional Topics (New York: John Wiley & Sons, Inc)
[24] Karyakin M, Pustovalova O and Ustinov U 2018 Mathematical Modelling of Circular Corrugated Membranes Proc 5th International Conference on Mathematics and Computers in Sciences and Industry – MCSI 2018 8769777
[25] Andrianov I V, Diskovsky A A and Syerko E 2017 Optimal design of a circular corrugated diaphragm using the homogenization approach Mathematics and Mechanics of Solids 22 283–303