Abstract—A problem of electromagnetic wave radiation by narrow slots cut in an end wall of a semi-infinite waveguide section into space above a perfectly conducting sphere is solved in a strict self-consistent formulation by the generalized method of induced magnetomotive forces (MMF). Inside the waveguide section, a reentrant cavity formed by the volume between a slotted diaphragm and the waveguide end wall is located. The waveguide is operating in the frequency range of a single-mode regime. The electrodynamic characteristics of this radiating system with the spherical screen of resonant dimensions are investigated numerically and experimentally. The possibility to develop the spherical antennas with a narrow-band frequency, energy, and spatial characteristics is substantiated.

1. INTRODUCTION

In all sectors of the economy and other spheres of modern society life, microelectronic robotics, information technologies, and radio communications are being applied at an increasing rate. Most of the devices used in these spheres are functioning in close proximity to each other both in the physical space and in frequency ranges. Therefore, a practically important problem concerning electromagnetic compatibility (EMC) of electronic systems exposed to interference from other electronic devices and natural phenomena arises. If the levels of the intrasystem and intersystem EMC do not reach the required values for all permissible variations of the equipment parameters, additional technical measures, for example, appropriate frequency filters in the transceiver paths should be used. In particular, such filters for slot antennas with waveguide excitation that do not impair their mass and size characteristics can be realized by waveguide resonant diaphragms [1].

As known in [2], non-protruding slotted radiators are widely used on mobile objects, since slot antennas do not significantly change their aerodynamic properties. The application of such antennas ranges from space crafts [3] to autonomous microprobes [4]. Quite often, a body of a mobile objects or their structural part can be approximately modelled by spherical surfaces, whose radii are comparable with the antenna operating wavelength. For example, spherical antennas are used for microwave hyperthermia allowing deep controlled heating of various tissues and organs of a living organism [5]. Therefore, interest in slotted spherical antennas with resonant spheres dimensions radiating into complex media has not disappeared for several decades.

The characteristics of spherical antennas with resonant rectangular slot radiator were studied in [6–12]. The external electrodynamic characteristics of slot antennas were considered in [6–8] under the assumption that the magnetic currents along the narrow half-wave slot radiator are defined by using a predefined cosine distribution. In the monograph [13], the authors of this article have proposed a numerical-analytical method known as the generalized method of induced MMF for solving the
diffraction problems concerning slotted coupling elements between electrodynamic volumes. This method was applied for solving the problem of electromagnetic wave radiations by waveguide-slot structures into the external space above a perfectly conducting sphere [9–12]. The problem of electromagnetic wave radiation into space outside the perfectly conducting sphere through the slot cut in the end wall of the semi-infinite rectangular waveguide, known as the waveguide-slot spherical antenna (WSSA), was presented in a strict electrodynamic formulation in [9]. It was shown that the spherical screen shape with corresponding design parameters does not significantly limit the operating frequency band obtained for the waveguide-slot radiator placed over the infinite flat screen.

As known, a combination of internal resonators and slot radiators with pronounced frequency-selective properties allows us to obtained for such antennas required narrow-band frequency-energy and spatial characteristics [14–16]. Note that the results in [14, 15] were obtained for the slot excitation by a voltage generator. This article is aimed at investigation of electrodynamic characteristics of the WSSA [9] with the reentrant resonator formed by the one or two slot resonant diaphragm and waveguide end wall. Since simulation of multielement waveguide structures by using existing commercial programs are low-efficient, the boundary-value problem is proposed to solve in a rigorous, self-consistent formulation by the generalized method of induced MMF.

2. THE PROBLEM FORMULATION AND SOLUTION

Geometry of the WSSA and adopted notations are shown in Fig. 1. Let us consider three electrodynamic volumes separated by perfectly conducting wall: a semi-infinite rectangular waveguide, rectangular resonator, and space over a sphere marked by the indices Wg, R, and Sp, respectively, coupling with each other through rectilinear slots S1, S2, S3 cut in common walls. The waveguide cross-section is \( \{a \times b\} \), the resonator dimensions are \( \{a \times b \times H\} \) and the sphere radius is \( R \). A fundamental \( H_{10} \)-wave propagates in the waveguide from the region \( z = \infty \). Let us introduce a Cartesian coordinate system associated with the waveguide and a spherical coordinate system associated with the sphere (Fig. 1). Cartesian coordinates of geometric centers of the internal slot aperture are \( (a/2, y_0, 0) \), and the spherical coordinates of the external slot aperture are \( (R, \pi/2, 0) \). The length of the external slot aperture along the sphere is \( 2L_\varepsilon \). The diaphragm thickness of the with the slots \( S_1 \) and \( S_2 \) is equal to \( h \). The slot center coordinates are \( (a/2, y_{0m}, H) \). The slot \( S_3 \) radiates into space with material parameters \( \varepsilon_1, \mu_1 \).

![Figure 1. Geometry of the WSSA and corresponding notations.](image)

The slot dimensions satisfy the following inequalities

\[
\frac{d_m}{2L_m} \ll 1, \quad \frac{d_m}{\lambda} \ll 1, \quad m = 1, 2, 3,
\]

where \( 2L_m \) and \( d_m \) are the length and width of the slots, and \( \lambda \) is the wavelength in free space. The
equivalent magnetic currents in the slots can be represented in the form

$$\vec{J}_m(s_m) = \vec{e}_{s_m} \cdot J_{0m} f_m(s_m) \chi_m(\xi_m),$$  

(2)

where $\vec{e}_{s_m}$ are unit vectors; $s_m$ and $\xi_m$ are local coordinates associated with the slots; $J_{0m}$ is the current amplitude, and the axes $\{0\xi_m\}$ are located in the plane $x = a/2$. The functions $f_m(s_m)$ must satisfy the boundary conditions $f_m(\pm L_m) = 0$. The functions $\chi_m(\xi_m)$ take into account the electric field behavior on the edges of the slots [13, 17] and satisfy the normalization conditions $\int_0^{\xi_m} \chi_m(\xi_m) d\xi_m = 1$. For the infinitely thin diaphragm $h = 0$, these functions can be presented as

$$\chi_m(\xi_m) = \frac{1/\pi}{\sqrt{(d_m/2)^2 - \xi_m^2}}.$$  

(3)

If the diaphragm thickness is finite ($h \neq 0$), and the slot edges are perfectly conducting rectangular wedges, the function $\chi_m(\xi_m)$ can be written as

$$\chi_m(\xi_m) = \frac{\Gamma(7/6)/\Gamma(2/3)}{\sqrt{\pi(d_m/2)^3}} \left(1 - (2\xi_m/d_m)^2\right).$$  

(4)

where $\Gamma(x)$ is the gamma function.

The functions $f_{1,2}(s_{1,2})$ and $\{f_3^e(\varphi) (f_3(s_3))\}$ were obtained as approximate solutions of the integral equation for the magnetic currents in the resonant slots cut in the diaphragm in the waveguide excited by the $H_{10}$-wave. The functions defining magnetic currents in the slot cut in the perfectly conducting sphere excited by a plane electromagnetic wave, whose vector $\vec{H}$ is parallel to the vector $\vec{e}_{s_3}$ [13], can be written as

$$f_{1,2}(s_{1,2}) = \cos ks_{1,2} \cos \frac{\pi L_{1,2}}{a} - \cos kL_{1,2} \cos \frac{\pi s_{1,2}}{a},$$

$$f_3^e(\varphi) = (\cos(k_1 R \varphi) \cos(\pi L_3^e/a) - \cos k_1 L_3^e \cos(\pi R \varphi/a)), \quad f_3(s_3) = \cos ks_3 - \cos kL_3.$$  

(5)

Using continuity conditions for the tangential components of the magnetic field on the slot surfaces and the generalized method of induced MMF for the multi-slot structure, we obtain a system of algebraic equations relative to the unknown current amplitudes $J_{0m}$

$$\begin{cases}
J_{01} \left(Y_{11}^W + Y_{11}^R\right) + J_{02} \left(Y_{12}^W + Y_{12}^R\right) + J_{03} Y_{13}^R = -\frac{i\omega}{2k} \int_{-L_1}^{L_1} f_1(s_1) H_{0s_1}(s_1) ds_1, \\
J_{02} \left(Y_{22}^W + Y_{22}^R\right) + J_{01} \left(Y_{21}^W + Y_{21}^R\right) + J_{03} Y_{25}^R = -\frac{i\omega}{2k} \int_{-L_2}^{L_2} f_2(s_2) H_{0s_2}(s_2) ds_2, \\
J_{03} \left(Y_{33}^W + Y_{33}^{Sp}\right) + J_{01} Y_{31}^R + J_{02} Y_{32}^R = 0,
\end{cases}$$  

(6)

where

$$Y_{mm}^{Wg,R,(Sp)} = \frac{1}{2k(k_1)} \int_{-L_m}^{L_m} f_m(s_m) \left( \frac{d^2}{ds_m^2} + k^2(k_1^2) \right) \int_{-L_m}^{L_m} f_m(s'_m) G_{s_m}^{Wg,R,(Sp)}(s_m, s'_m) ds'_m ds_m$$  

(7)

are intrinsic slot conductivities and

$$Y_{mm}^{Wg,R} = \frac{1}{2k} \int_{-L_m,n}^{L_m,n} f_{m,n}(s_{m,n}) \left( \frac{d^2}{ds_{m,n}^2} + k^2 \right) \int_{-L_m,n}^{L_m,n} f_{m,n}(s'_{m,n}) G_{s_{m,n}}^{Wg,R}(s_{m,n}, s'_{m,n}) ds'_{m,n} ds_{m,n}$$  

(8)

are mutual slot conductivities; $G_{s}^{Wg,R,(Sp)}$ are $s$-components of quasi-one-dimensional $(|\xi_m - \xi'_m| \approx d_m/4)$ Green’s functions for the vector potentials of the corresponding volumes (see Appendix A); $H_{0s_1,2}(s_1,2)$ are projections of fields of external sources on the axis of the slots, $k = 2\pi/\lambda$, and $k_1 = k\sqrt{\varepsilon_{1}\mu_{1}}$. 

The magnetic Green’s functions of a semi-infinite rectangular waveguide, rectangular resonator, and space outside the perfectly conducting sphere (see Appendix A) allow us to derive, based on Eqs. (5), (7), (8), the expressions for the intrinsic and mutual slot conductivities

\[ Y_{mn}^{Wg}(kL_m, kL_n) = \frac{4\pi n}{ab} \sum_{m=1,3\ldots}^{\infty} \sum_{n=0}^{\infty} \varepsilon_n \left( k^2 - k_x^2 \right) \frac{\cos k_y y_{0m} \cos k_y \left( y_{0m} + \frac{d_{en}}{4} \right)}{kk_z} I_1(kL_m)I_1(kL_n), \]

\[ Y_{mn}^{R}(kL_m, kL_n) = \frac{4\pi n}{ab} \sum_{m=1,3\ldots}^{\infty} \sum_{n=0}^{\infty} \varepsilon_n \left( k^2 - k_x^2 \right) \frac{\coth k_z H}{kk_z} \cos k_y \left( y_{0m} + \frac{d_{en}}{4} \right) I_1(kL_m)I_1(kL_n), \]

\[ Y_{m3}^{R}(kL_m, kL_3) = Y_{3m}^{R}(kL, kL_m) = \frac{4\pi n}{ab} \sum_{m=1,3\ldots}^{\infty} \sum_{n=0}^{\infty} \varepsilon_n \left( k^2 - k_x^2 \right) \frac{k_z \coth k_z H}{kk_z} \cos k_y \left( y_{03} + \frac{d_{e3}}{4} \right) I_1(kL_m)I_2(kL_3), \]

\[ Y_{33}^{Sp}(kL_3', kL_1) = -\frac{4}{k_1R_{h1}} \sum_{n=0}^{\infty} \frac{1}{n(n+1)} \left[ (n+1) - k_1R_{h_{n+1}}(k_1R)/h_{n+1}^{(2)}(k_1R) \right] \cos k_y \left( y_{03} + \frac{d_{e3}}{4} \right) I_1(kL_3)I_2(kL_3), \]

where

\[ I_1(kL_m) = 2 \left\{ \frac{k \sin kL_m \cos k_x L_m - k_x \cos kL_m \sin k_x L_m \cos \frac{\pi L_m}{a}}{k^2 - k_x^2} \right\}, \]

\[ I_2(kL_3) = 2 \frac{k_x \sin kL_3 \cos k_x L_3 - k \cos kL_3 \sin k_x L_3}{k_x}, \]

\[ A_n^m = \sqrt{\frac{\pi}{C_{nm}}} \Phi_n^m, \quad \tilde{A}_n^m = \sin \theta \left| \frac{dP_n^m(\cos \theta)}{d\theta} \right|_{\theta = \frac{\pi}{2} + \frac{d\theta}{4R}}, \]

\[ B_n^m = \sqrt{\frac{\pi}{C_{nm}}} \Phi_n^m, \quad \tilde{B}_n^m = \tilde{P}_n^m \left( \cos \left( \frac{\pi}{2} + \frac{d\theta}{4R} \right) \right), \]

\[ C_m = \frac{\cos(\pi L^e/\alpha)}{m^2 - (kR)^2} \left[ \frac{mL^e R}{R} \cos k_1 L_3 - k_1 R \cos \frac{mL^e R}{R} \sin k_1 L_3^e \right] - \frac{mL^e R}{R} \cos k_1 L_3^e - \frac{mL^e R}{R} \sin k_1 L_3^e \]

\[ = C_m^I - C_m^II, \]

\[ \tilde{P}_n^m(\cos \theta) = \sqrt{\frac{n}{n_m} P_n^m(\cos \theta)} \text{ are the normalized associated Legendre functions, } k_x = \frac{m\pi}{a}, k_y = \frac{n\pi}{b}, k_z = \sqrt{k_x^2 + k_y^2 - k^2}, m \text{ and } n \text{ are integers, } \varepsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \neq 0 \end{cases}, \text{ and } y_{0m} \text{ are coordinates of the slot centers.} \]
The slot currents can be found by solving the equations system in Eq. (6). If the $H_{10}$-wave with the amplitude $H_0$ propagates in the semi-infinite rectangular waveguide, the fields can be presented as $H_{012} = 2H_0 \cos \frac{\pi z}{a}$. Then the field reflection coefficient $S_{11}$ and the power radiation coefficient $|S_{21}|^2$ can be written as

$$S_{11} = \left\{ 1 - \frac{8\pi k_g}{i \omega k a^3} \left[ \tilde{J}_{01} F(kL_1) + \tilde{J}_{02} F(kL_2) \right] \right\} e^{-i2k_g z},$$

$$|S_{21}|^2 = 1 - |S_{11}|^2,$$

where $\tilde{J}_{0m} = J_{0m} (-i \omega \mu_0 H_0)^{-1}$ are the normalized currents amplitude in the slots, and $k_g = \sqrt{k^2 - (\pi/a)^2}$ is the propagation constant, and

$$F(kL_m) = 2 \cos \frac{\pi L_m}{a} \sin \frac{k L_m \cos \frac{\pi L_m}{a}}{1 - (\pi/ka)^2} - \cos \frac{k L_m}{a} + \frac{2\pi L_m}{a}.$$

The equivalent width $d_{e3}$ of the slot was calculated by the formula $d_{e3} \approx d_3 \exp(-\pi h_e/2d_3)$ under the condition $(h_3d_3/\lambda^2) \ll 1$, where $h_e = V^v/S_3$, $V^v$ is the volume of the slot cavity, $S_i$ the internal slot aperture area, and $h_3$ the maximum dimension of the slot tunnel cavity in the radial direction [9]. The thickness of the diaphragm wall was taken into account using a similar formula under condition $h_e = h$.

### 3. RADIATION FIELDS OF THE WSSA

The formula for determining the magnetic current on the slot aperture in space outside the sphere according to Eqs. (5) and (9) is

$$J(s) = \frac{\text{i} \omega}{k^2} H_0 F(kL_3) \left[ \cos(k_1 s) \cos(\pi L_3^2/a) - \cos k_1 L_3^2 \cos(\pi s/a) \right].$$

The distribution of the slot magnetic current in Eq. (12) allows us to calculate the electrodynamic characteristics of the spherical antenna in a space outside the sphere. The total radiation field of the spherical antenna (Fig. 1(c)) can be determined by two components of the electric Hertz vector, $\Pi^e_m(r, \theta, \varphi)$ and $\Pi^m_m(r, \theta, \varphi)$ [9], after substituting the current distribution $J(s) = J(R \varphi')$ in Eq. (12). Then the components of the total radiation field of the spherical antenna can be presented as:

$$E_{ex}(r, \theta, \varphi) = \frac{1}{r} \sum_{n=0}^{\infty} Q_n(r) \left( \frac{FC_0(\varphi)}{2C_{n0}} P_n(\cos \theta) \frac{dP_n(\cos \theta)}{d\theta} \right)_{\theta=\pi/2},$$

$$E_{e\theta}(r, \theta, \varphi) = \frac{1}{r} \sum_{n=0}^{\infty} \frac{\varepsilon_n}{2(n+1)C_{nm}} \left[ \frac{m^2 P_n(\cos \theta)}{\sin \theta} F_n + \frac{dP_n(\cos \theta)}{d\theta} \Phi_n^m \right],$$

$$E_{e\varphi}(r, \theta, \varphi) = \frac{1}{r} \sum_{n=0}^{\infty} \frac{mQ_n^m(r) F_{Sm}(\varphi)}{n(n+1)C_{nm}} \left[ \frac{dP_n(\cos \theta)}{d\theta} F_n + \frac{P_n(\cos \theta)}{\sin \theta} \Phi_n^m \right],$$

$$H_{ex}(r, \theta, \varphi) = \frac{1}{i \omega \mu_1 r^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} Q_n(r) \frac{mFS_m(\varphi)}{C_{nm}} \left( Q_n^m(r) - 2Q_n(r) \right) P_n^m(\cos \theta) F_n^m,$$

$$H_{e\theta}(r, \theta, \varphi) = \frac{k_2}{i \omega \mu_1} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{Q_n(r) mFS_m(\varphi)}{n(n+1)C_{nm}} \left[ \frac{P_n(\cos \theta)}{\sin \theta} \Phi_n^m + \left( 1 - \frac{n(n+1)}{(k_1 r)^2} \right) \frac{dP_n(\cos \theta)}{d\theta} F_n^m \right],$$

$$H_{e\varphi}(r, \theta, \varphi) = \frac{k_2}{i \omega \mu_1} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{Q_n(r) FC_m(\varphi)}{2n(n+1)C_{nm}} \left[ \frac{P_n(\cos \theta)}{\sin \theta} \Phi_n^m + 2m^2 \left( 1 - \frac{n(n+1)}{r^2} \right) \frac{P_n(\cos \theta)}{\sin \theta} F_n^m \right].$$
where

\[ Q_n(r) = \frac{h_n^{(2)}(k_1 r)}{(n+1) h_n^{(2)}(k_1 R) - k_1 R h_{n+1}^{(2)}(k_1 R)}, \quad F_m^n = P_m^n (\cos \theta') \bigg|_{\theta' = \pi/2}, \quad \Phi_m^n = \frac{dP_m^n (\cos \theta')}{d\theta'} \bigg|_{\theta' = \pi/2} \]

\[ FC_m(\varphi) = \int_{-L_c/R}^{L_c/R} f(\varphi') \cos (m(\varphi - \varphi')) \, d\varphi', \quad FS_m(\varphi) = \int_{-L_c/R}^{L_c/R} f(\varphi') \sin (m(\varphi - \varphi')) \, d\varphi', \]

\[ C_{nm} = \frac{2\pi (n + m)!}{(2n + 1)(n - m)!}, \quad Q_n^*(r) = \frac{\partial}{\partial r} \left( rQ_n(r) \right) = \frac{(n+1) h_n^{(2)}(k_1 r) - k_1 r h_{n+1}^{(2)}(k_1 r)}{(n+1) h_n^{(2)}(k_1 R) - k_1 R h_{n+1}^{(2)}(k_1 R)}, \]

\[ \varepsilon_m = \begin{cases} 1, & m = 0, \\ 2, & m \neq 0, \end{cases} \]

In the second slot in the diaphragm, the total reflection of the incident wave occurs. The radiation coefficient increases at shorter wavelengths, and the resonance curve has a large steepness, and its shape approaches a rectangular form. The plots show that the reentrant cavity placed in the waveguide substantially increases the system Q-factor. The resonance curve has a large steepness, and its shape approaches to a rectangular form. The numerical and experimental results are presented for various numbers of slots marked as: curve 1 — slot 2, curve 2 — slots 1 and 3, 3 — slots 1, 2, and 3. The WSSA parameters are as follows: \( R = 80/\pi \) mm, \( 2L_1 = 2L_3 = 16 \) mm, \( 2L_2 = 14 \) mm, \( d_1 = d_2 = d_3 = 0.8 \) mm, \( y_{01} = b/8, \ y_{02} = b/2, \ y_{03} = b/2, \ h = 1.0 \) mm, \( H = a/2. \)

The plots show that the reentrant cavity placed in the waveguide substantially increases the system Q-factor. The resonance curve has a large steepness, and its shape approaches to a rectangular form. The numerical and experimental results are presented for various numbers of slots marked as: curve 1 — slot 2, curve 2 — slots 1 and 3, 3 — slots 1, 2, and 3. The WSSA parameters are as follows: \( R = 80/\pi \) mm, \( 2L_1 = 2L_3 = 16 \) mm, \( 2L_2 = 14 \) mm, \( d_1 = d_2 = d_3 = 0.8 \) mm, \( y_{01} = b/8, \ y_{02} = b/2, \ y_{03} = b/2, \ h = 1.0 \) mm, \( H = a/2. \)
Figure 2. The power radiation coefficient $|S_{\Sigma}|^2$ of the WSSA versus the wavelength.

Figure 3. The power radiation coefficient versus the wavelength for the WSSA with one and two slot reentrant cavity.

where $\lambda_c$ is a critical wavelength of the $H_{10}$-wave, and $\alpha = 1/\{8 \ln[d_2/(8L_2)]\}$ is the small parameter,

$$F(a, b, L_2, d_{e2}, y_{02}) \approx 2\pi \left\{ \frac{4\cos^2 \frac{\pi L_2}{a}}{\gamma_{10}^2 L_2} \left[ \frac{2\pi \cos^2 \frac{\pi y_{02}}{b}}{k_{11}^2 b} - 2\cos^2 \frac{\pi y_{02}}{b} \right] \right. $$

$$+ \left. \left( \frac{\gamma_{10} b}{9} \right)^2 \cos^2 \frac{2\pi y_{02}}{b} - \ln \left( \frac{\pi d_{e2}}{2b} \sin \frac{\pi y_{02}}{b} \right) \right\} $$

$$- \frac{4\cos^2 \frac{3\pi L_2}{a}}{k_{30}^2 a L_2} \left[ K_0 \left( \frac{d_{e2}}{4k_{30}} \right) + K_0(2k_{30}y_{02}) \right] + \left( \ln \frac{16L_2}{d_{e2}} - 1 \right)$$

$$- \ln \frac{1 - L_2/a}{1 + L_2/a} + \ln \frac{1 - 2L_2/a}{1 + 2L_2/a} + \ln \frac{1 - 2L_2/(3a)}{1 + 2L_2/(3a)}$$

$$- \frac{a}{2L_2} \left[ \ln \left( 1 - \left( \frac{2L_2}{a} \right)^2 \right) - 2\ln \left( 1 - \left( \frac{L_2}{a} \right)^2 \right) + 3\ln \left( 1 - \left( \frac{2L_2}{3a} \right)^2 \right) - \left( \frac{L_2}{a} \right)^2 \right]$$
\[-\frac{4a}{\pi^2 L_2} \left( K_0 \left( \frac{\pi d_2}{4a} \right) \sin^2 \frac{\pi L_2}{a} + \frac{1}{9} K_0 \left( \frac{3\pi d_2}{4a} \right) \sin^2 \frac{3\pi L_2}{a} + \sum_{m=5,7, \ldots}^{\infty} K_0 \left( \frac{m\pi d_2}{4a} \right) \right) \right] \right\}. \quad (16)

Here \( \gamma_{10} = \sqrt{(\pi/(2L_2)) - (\pi/a)^2} \), \( k_{11} = \sqrt{(\pi/a)^2 + (\pi/b)^2 - (\pi/(2L_2))^2} \), \( k_{30} = \sqrt{(3\pi/a)^2 - (\pi/(2L_2))^2} \), \( K_0(x) \) is Macdonald function. Since the function \( K_0(x) \) decreases rapidly with increasing argument, only the first few terms should be taken into account in the series of Eq. (16).

By varying the length \( H \) of the reentrant cavity, maximum radiation can be achieved at one or several wavelengths (Fig. 4: \( 2L_2 = 14.0 \text{ mm}, \ y_02 = 7b/8 \)), while the value \( \lambda_{res}/\lambda_c \) is practically independent of the reentrant cavity dimensions.

![Figure 4](image1.png)

**Figure 4.** The WSSA power radiation coefficient versus the wavelength for various length \( H \) of the reentrant cavity.

The typical normalized RPs of the WSSA shown in Fig. 1, \( c \) are shown in Fig. 5. The plots were obtained with the following parameters: \( \lambda = 32 \text{ mm}, \ R = \lambda \text{ mm}, \ 2L_3 = 0.5\lambda, \) and \( d_3 = 0.05\lambda. \) Note that in \( E \)-plane RP of the slot radiator placed over the infinite flat screen is presented by the constant function, while the \( E \)-plane RP of WSSA is significantly non-constant. The degree of the difference between the two cases increases when the sphere radius is decreased.

The validity of the numerical simulation was confirmed by comparing the calculated and experimental data. A photograph of a prototype WSSA model used for experimental studies is shown.

![Figure 5](image2.png)

**Figure 5.** The RPs of the WSSA.
Figure 6. The prototype WSSA model.

Figure 7. The power radiation coefficient of the WSSA versus the wavelength: $R = 50$ mm, $a = 23$ mm, $b = 10$ mm, $2L_3^c = 18$ mm, $d_m = 1.5$ mm, $h_m = 2.0$ mm, $2L_1 = 16.0$ mm, $2L_2 = 14.0$ mm, $H = a/2$, $y_{01} = y_{03} = b/4$, $y_{02} = 3b/4$.

in Fig. 6. The WSSA geometrical parameters and waveguide wall thickness for the standard rectangular waveguide are given in the caption of Fig. 7. The plots show that the theoretical curves and experimental data are in good agreement with each other.

5. CONCLUSION

The problem concerning radiation of electromagnetic waves into space outside the perfectly conducting sphere through a narrow slot cut in the end wall of the semi-infinite rectangular waveguide with the reentrant cavity in it is solved by the generalized method of induced MMF. The problem was solved by using the basis functions obtained as the analytical solutions of the integral equations for the slot currents by the asymptotic averaging method. The concept of equivalent slot width was introduce allowing us to eliminate the need to determine the fields in the internal slot cavities. The proposed mathematical model was verified by comparing the numerical results and experimental data. The results can be used in the design and development of WSSAs.
APPENDIX A. MAGNETIC GREEN’S FUNCTIONS OF THE CONSIDERED ELECTRODYNAMIC VOLUMES

1. The hollow half-infinite rectangular waveguide of the cross-section \( \{a \times b\} \) with the perfectly conducting walls:

\[
\hat{G}^m(\vec{r}, \vec{r}') = \frac{2\pi}{ab} \sum_{m,n} \frac{\varepsilon_m \varepsilon_n}{k_z} \left\{ (\vec{e}_x \otimes \vec{e}_x)^m(x, y; x', y') \left[ e^{-k_z|z-z'|} + e^{-k_z(z+z')} \right] \right.
\]
\[
+ (\vec{e}_y \otimes \vec{e}_y)^m(x, y; x', y') \left[ e^{-k_z|z-z'|} + e^{-k_z(z+z')} \right] \\
+ (\vec{e}_z \otimes \vec{e}_z)^m(x, y; x', y') \left[ e^{-k_z|z-z'|} - e^{-k_z(z+z')} \right] \right\}.
\]

The following symbols are accepted in Eq. (A1):

\[
\Phi_x(x, y; x', y') = \cos k_x x \cos k_y y \sin k_y y', \\
\Phi_y(x, y; x', y') = \sin k_x x \cos k_y y \cos k_y y', \\
\Phi_z(x, y; x', y') = \sin k_x x \sin k_y y \sin k_y y',
\]

\[
\varepsilon_{m,n} = \begin{cases} 1, & m, n = 0 \\
2, & m, n \neq 0 \end{cases}, \quad Rk_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \sqrt{k_x^2 + k_y^2 - k^2}, \quad m \text{ and } n \text{ are the integral numbers; } \vec{e}_x, \\
\vec{e}_y \text{ and } \vec{e}_z \text{ are the unit vectors of the rectangular coordinate system; } \vec{I} = (\vec{e}_x \otimes \vec{e}_x) + (\vec{e}_y \otimes \vec{e}_y) + (\vec{e}_z \otimes \vec{e}_z) \text{ is the unit dyadic, and } \otimes \text{ stands for dyadic product.}
\]

2. The hollow rectangular resonator \( \{a_R \times b_R \times H\} \) with perfectly conducting walls:

\[
\hat{G}^m(\vec{r}, \vec{r}') = \frac{2\pi}{a_R b_R} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\varepsilon_m \varepsilon_n}{k_z} \\
\times \left\{ (\vec{e}_x \otimes \vec{e}_x)^m(x, y; x', y') \left[ \text{chk}_z(H - |z-z'|) + \text{chk}_z(H - |z+z'|) \right] \right.
\]
\[
+ (\vec{e}_y \otimes \vec{e}_y)^m(x, y; x', y') \left[ \text{chk}_z(H - |z-z'|) + \text{chk}_z(H - |z+z'|) \right] \\
+ (\vec{e}_z \otimes \vec{e}_z)^m(x, y; x', y') \left[ \text{chk}_z(H - |z-z'|) - \text{chk}_z(H - |z+z'|) \right] \right\}.
\]

The following notations are adopted in Eq. (A2):

\[
\Phi_x^m(x, y; x', y') = \sin k_x x \sin k_y y \cos k_y y', \\
\Phi_y^m(x, y; x', y') = \cos k_x x \sin k_y y \cos k_y y', \\
\Phi_z^m(x, y; x', y') = \cos k_x x \cos k_y y \cos k_y y'.
\]

The remaining notations coincide with that in Eq. (A1).

3. Space outside the perfectly conducting sphere of the radius \( \bar{R} = R \) with the permittivity \( \varepsilon_1 \) and the permeability \( \mu_1 \) of the medium (Fig. A.1):

\[
\hat{G}^m(\rho, \theta; \rho', \theta') = \begin{bmatrix} G_{\rho\rho}^m & 0 & 0 \\
0 & G_{\theta\theta}^m & G_{\theta\varphi}^m \\
0 & G_{\varphi\theta}^m & G_{\varphi\varphi}^m \end{bmatrix},
\]

\[
G_{\rho\rho}^m(\rho, \theta; \rho', \theta') = -\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\varepsilon_m}{2C_{nm}} \frac{P_n^m(\cos \theta) P_n^m(\cos \theta')}{n m \cos m(\varphi - \varphi')},
\]

\[
G_{\theta\theta}^m(\rho, \theta; \rho', \theta') = -\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\varepsilon_m}{2n(n+1)} \frac{C_{nm} \sin \theta \sin \theta'}{n m \cos m(\varphi - \varphi')},
\]

\[
G_{\varphi\varphi}^m(\rho, \theta; \rho', \theta') = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\varepsilon_m}{2n} \frac{C_{nm} \sin \theta \sin \theta'}{n m \cos m(\varphi - \varphi')}.
\]
Here $P_n^m(\cos \theta)$ is the associated Legendre functions of the first sort,

$$C_{nm} = \frac{2\pi (n + m)!}{(2n + 1)(n - m)!}.$$ 

$$h_n^m(\rho, \rho') = \begin{cases} 
4\pi k_1 h_n^{(2)}(k_1\rho') & \left[ j_n(k_1\rho) Q_n\left(y_n\left(k_1\tilde{R}\right)\right) - y_n(k_1\rho) j_n\left(k_1\tilde{R}\right) \right], \quad \tilde{R} \leq \rho < \rho', \\
4\pi k_1 h_n^{(2)}(k_1\rho) & \left[ j_n(k_1\rho') Q_n\left(y_n\left(k_1\tilde{R}\right)\right) - y_n(k_1\rho') j_n\left(k_1\tilde{R}\right) \right], \quad \rho > \rho', 
\end{cases}$$

$$Q_n(f_n(k_1\tilde{R})) = \frac{n f_n\left(k_1\tilde{R}\right) - k_1\tilde{R} f_{n+1}\left(k_1\tilde{R}\right)}{n h_n^{(2)}(k_1\tilde{R}) - k_1\tilde{R} h_{n+1}^{(2)}(k_1\tilde{R})},$$

$$u_n^m(\rho, \rho') = \begin{cases} 
4\pi k_1 \frac{h_n^{(2)}(k_1\rho)}{h_n^{(2)}(k_1\rho')} & \left[ j_n(k_1\rho) y_n\left(k_1\tilde{R}\right) - y_n(k_1\rho) j_n\left(k_1\tilde{R}\right) \right], \quad \tilde{R} \leq \rho < \rho', \\
4\pi k_1 \frac{h_n^{(2)}(k_1\rho)}{h_n^{(2)}(k_1\rho')} & \left[ j_n(k_1\rho') y_n\left(k_1\tilde{R}\right) - y_n(k_1\rho') j_n\left(k_1\tilde{R}\right) \right], \quad \rho > \rho', 
\end{cases}$$

$$h_n^{(2)}(k_1\rho) = j_n(k_1\rho) - iy_n(k_1\rho) = \sqrt{\frac{\pi}{2k_1\rho}} H_n^{(2)}_{n+1/2}(k_1\rho)$$ is the Hankel spherical function of the second sort; $j_n(k_1\rho) = \sqrt{\frac{\pi}{2k_1\rho}} J_{n+1/2}(k_1\rho)$ and $y_n(k_1\rho) = \sqrt{\frac{\pi}{2k_1\rho}} N_{n+1/2}(k_1\rho)$ are the Bessel spherical function and the Neumann one, correspondingly; $J_{n+1/2}(k_1\rho)$ is the Bessel function; $N_{n+1/2}(k_1\rho)$ is the Neumann function, and $H_n^{(2)}_{n+1/2}(k_1\rho)$ is the Hankel function of the second sort with the half-integral index.

$$G_{\theta\varphi'}^{m, n}(\rho, \varphi; \rho', \varphi') = \sum_{n=0}^{\infty} \frac{m m_n(\rho, \rho') \sin(m(\varphi - \varphi'))}{n(n + 1) C_{nm}} \times \left[ \frac{dP_n^m(\cos \theta) P_n^m(\cos \theta')}{\sin \theta'} \right] + \frac{P_n^m(\cos \theta)}{\sin \theta} \frac{dP_n^m(\cos \theta')}{d\theta'} \right], \quad \rho > \rho', \rho < \rho'. $$

\section*{Figure A1.}
REFERENCES

1. Lewin, L., *Advanced Theory of Waveguides*, Iliffe & Sons, London, 1951.
2. Mittra, R., *Computer Techniques for Electromagnetics*, Pergamon Press, NY, 1973.
3. Reznikov, G. B., *Antennas of Flying Vehicles*, Soviet Radio, Moscow, 1967 (in Russian).
4. Schantz, H., “Nanoantennas: A concept for efficient electrically small UWB devices,” *IEEE International Conference ICU 2005*, 264–268, Sept. 2005.
5. Vorst, A. V., A. Rosen, and Yu. Kotsuka, *RF/Microwave Interaction with Biological Tissues*, Willey-IEEE Press, NY, 2006.
6. Mushiake, Y. and R. E. Webster, “Radiation characteristics with power gain for slots on a sphere,” *IRE Trans. Antennas Propagat.*, Vol. 5, 47–55, 1957.
7. Leung, K. W., “Theory and experiment of a rectangular slot on a sphere,” *IEEE Trans. Microwave Theory Tech.*, Vol. 46, 2117–2123, 1998.
8. Kwok, W. L., “Rectangular and zonal slots on a sphere with a backing shell: Theory and experiment,” *IEEE Trans. Antennas Propagat.*, Vol. 51, 1434–1442, 2003.
9. Berdnik, S. L., Yu. M. Penkin, V. A. Katrich, M. V. Nesterenko, and V. I. Kijko, “Electromagnetic waves radiation into the space over a sphere by a slot in the end-wall of a semi-infinite rectangular waveguide,” *Progress In Electromagnetics Research B*, Vol. 46, 139–158, 2013.
10. Berdnik, S. L., V. A. Katrich, M. V. Nesterenko, and Yu. M. Penkin, “Spherical antenna excited by a slot in an impedance end-wall of a rectangular waveguide,” *Proc. of the XVIII-th International Seminar/Workshop on Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory*, 111–114, Lviv, Ukraine, 2013.
11. Berdnik, S. L., V. A. Katrich, Yu. M. Penkin, M. V. Nesterenko, and S. V. Pshenichnaya, “Energy characteristics of a slot cut in an impedance end-wall of a rectangular and radiating into the space over a perfectly conducting sphere,” *Progress In Electromagnetics Research M*, Vol. 34, 89–97, 2014.
12. Berdnik, S. L., V. S. Vasylkovskyi, M. V. Nesterenko, and Yu. M. Penkin, “Radiation fields of the spherical slot antenna in a material medium,” *Proc. of the X-th Anniversary International Conference on Antenna Theory and Techniques*, Kharkiv, Ukraine, 282–284, 2015.
13. Nesterenko, M. V., V. A. Katrich, Yu. M. Penkin, and S. L. Berdnik, *Analytical and Hybrid Methods in Theory of Slot-Hole Coupling of Electrodynamic Volumes*, Springer Science+Business Media, New York, 2008.
14. Long, S. A., “Experimental study of the impedance of cavity-backed slot antennas,” *IEEE Trans. Antennas Propag.*, Vol. 23, 1–7, 1975.
15. Lee, J. Y., T. Sh. Horng, and N. G. Alexopoulos, “Analysis of cavity-backed aperture antennas with a dielectric overlay,” *IEEE Trans. Antennas Propag.*, Vol. 42, 1556–1562, 1994.
16. Nesterenko, M. V. and V. A. Katrich, “The method of induced magnetomotive forces for cavity-backed slot radiators and coupling slots,” *Radioelectronics and Communications Systems*, Vol. 47, 8–13, 2004.
17. Mittra, R. and S. W. Lee, *Analytical Techniques in the Theory of Guided Waves*, Collier-Macmillan Limited, London, 1971.