On the pure steered states of Einstein-Podolsky-Rosen steering

H. Chau Nguyen\textsuperscript{1,*} and Kimmo Luoma\textsuperscript{2,†}
\textsuperscript{1}Max-Planck-Institut für Physik Komplexer Systeme, Nöthnitzer Straße 38, D-01187 Dresden, Germany
\textsuperscript{2}Institut für Theoretische Physik, Technische Universität Dresden, D-01062 Dresden, Germany

In the Einstein–Podolsky–Rosen experiment, when Alice makes a measurement on her part of a bipartite system, Bob’s part is collapsed to, or steered to, a specific ensemble. Moreover, by reading her measurement outcome, Alice can specify which state in the ensemble Bob’s system is steered to and with which probability. The possible states that Alice can steer Bob’s system to are called steered states. In this work, we study the subset of steered states which are pure after normalisation. We illustrate that these pure steered states, if they exist, often carry interesting information about the shared bipartite state. This information content becomes particularly clear when we study the purification of the shared state. Some applications are discussed. These include a generalisation of the fundamental lemma in the so-called ‘all-versus-nothing proof of steerability’ for systems of arbitrary dimension.

PACS numbers: 03.65.Ud, 03.65.Ta, 03.67.Mn

Introduction

Einstein–Podolsky–Rosen (EPR) steering, or just steering, indicates the ability of Alice to collapse Bob’s system to a specific ensemble of states by measuring locally her own system, provided the two systems are in an appropriate quantum state. This was first discussed in the seminal papers of Einstein, Podolsky and Rosen \cite{EPR35} and Schrödinger \cite{Schr35}. Recently, Wiseman \textit{et al.} gave a precise operational definition of steerability \cite{Wiseman02,Wiseman02b}, which distinguishes it from other classes of nonlocality in quantum mechanics such as nonseparability \cite{Wiseman00} and Bell nonlocality \cite{Wiseman01}.

EPR steering can be seen also as a two-party quantum information processing task where Alice tries to convince Bob that the state they share is entangled. Bob does not trust Alice but he trusts on quantum mechanics and his own measurements \cite{Wiseman01}. This view has motivated a lot of research on how to witness steerability in a specific experimental setting by inequalities \cite{Wiseman05,Wiseman06,Wiseman07a,Wiseman07b,Wiseman08}. Beyond sufficient inequalities, a sufficient and necessary condition for steerability in terms of quantum channels has been developed \cite{Wiseman05,Fewster06}. It has also been shown that the steerability is related to the concept of joint measurability of quantum observables \cite{Wiseman07a,Wiseman07b,Wiseman08}.

On the other hand, some authors concentrate on characterising the steering process in EPR experiments independently from a specific measurement setting. In this line, the EPR steering is characterised by a positive linear map, which maps a measurement outcome operator, or \textit{effect}, on Alice’s system to a conditional state on Bob’s system, which is normalised to a steered state \cite{Wiseman05}. Then an analysis of an EPR experiment is based on all effects on Alice’s system, and all respective conditional states on Bob’s system \cite{Wiseman05,Wiseman06,Wiseman07a,Wiseman07b}. The set of conditional states on Bob’s system and its relative have been proved useful in characterising not only steerability but also more general aspects of bipartite quantum states \cite{Wiseman05,Wiseman06,Wiseman07a,Wiseman07b,Wiseman08}.

In this work we concentrate on a special case of steered states: those that are pure, or rank-1. In general, there may be no such pure steered states. However, we show that if they exist, they often carry interesting information about the shared bipartite state. The special role of pure steered states becomes particularly clear when we study the behavior of the purified system in the EPR experiment. The following simple observation illustrates the idea. Suppose Charlie holds a third system such that the joint system of his with Alice and Bob is in a pure state. When Alice makes a measurement and obtains some outcome, the shared system of Bob and Charlie (BC) is steered to some state, which is nonseparable in general. However, if Alice learns that Bob’s system is in a pure state, she knows immediately the two (BC) are separable and Charlie’s system is also in a pure state. This is a clue that pure steered states imply special structure of the shared state, which becomes visible in its purification. The detailed structure of the purified state is explained in the following sections. Applications in proving steerability are also discussed. Among these, a generalisation of the fundamental lemma of the so-called ‘all-versus-nothing proof of steerability’ \cite{Wiseman07b} is proven in an elementary way.

Effects and steered states

Consider the case where Alice and Bob share a state $\rho$ of a bipartite quantum system over $\mathcal{H}_A \otimes \mathcal{H}_B$ with dimension $d_A \times d_B$. Suppose Alice makes a positive operator valued measurement (POVM) described by effects $\{E_i\}_{i=1}^n$ on her system $A$, where the effects $E_i$ are positive semi-definite operators and $\sum_{i=1}^n E_i = I_A$. Imagine that Alice can make all possible measurements. Then
the set of all possible effects is $M_A = \{ E | 0 \leq E \leq I_A \}$. Regardless of the POVM Alice makes, whenever she gets an effect $E \in M_A$, Bob’s system $B$ is steered to a conditional state $E' = \text{Tr}_A[\rho(E \otimes 1_B)]$. The set of possible conditional states Alice can steer Bob’s system to is $M'_A = \{ E' | E \in M_A \} [20, 23]$. A conditional state $E'$ can also be characterised by the steered state $\sigma = E'/\text{Tr}(E')$ and its steering probability $p = \text{Tr}(E') [20, 21]$.

The set of steered states have been studied recently and a deep connection to the nonlocal properties of the shared bipartite state has been revealed [20–23]. In this work, we consider a special subset of steered states, the pure steered states. For a general bipartite state $\rho$, there may exist no pure steered states. However when they do exist, they often carry interesting information about the shared state $\rho$.

**Pure steered states and non-degenerate projective effects**

To steer Bob’s system to a specific state, we may imagine that Alice tries to design measurements that are efficient in the sense that the steering probability is maximal. We first note that to steer Bob’s system to a pure state, non-degenerate projective measurements are as efficient as general POVMs.

Recall that a POVM $\{ E_i \}_{i=1}^n$ is called a projection valued measurement (PVM), or projective measurement, if the effects $E_i$ are mutual orthogonal projections. A projective measurement is non-degenerate if all the orthogonal effects are rank-1. Effects of a non-degenerate projective measurement, which are of always the form $|\alpha\rangle\langle\alpha|$ for some vector $|\alpha\rangle$, are also called non-degenerate projective effects.

**Lemma 1.** If Alice can steer Bob’s system to a pure state, she can always gain the maximal steering probability for that steered state with a non-degenerate projective measurement.

**Proof.** Suppose Alice can steer Bob’s system to $|\beta\rangle\langle\beta|$ with maximal probability $p$, then there exists an effect $E \in M_A$ such that $E' = p|\beta\rangle\langle\beta|$. Let us consider the spectral decomposition of $E$, $E = \sum_{i=1}^n \lambda_i |\alpha_i\rangle\langle\alpha_i|$, where we kept only $\lambda_i > 0$. Accordingly, $E' = \sum_{i=1}^n \lambda_i (|\alpha_i\rangle\langle\alpha_i|) = p|\beta\rangle\langle\beta|$. But since $|\beta\rangle\langle\beta|$ is a pure state, and pure states are extremal [27], this is only possible if for all $i$, either $(|\alpha_i\rangle\langle\alpha_i|) = 0$ or $(|\alpha_i\rangle\langle\alpha_i|) \propto |\beta\rangle\langle\beta|$. Suppose for $1 \leq i \leq m$, $(|\alpha_i\rangle\langle\alpha_i|) \propto |\beta\rangle\langle\beta|$, and for $m < i \leq n$, $(|\alpha_i\rangle\langle\alpha_i|) = 0$, then $\sum_{i=1}^m \lambda_i (|\alpha_i\rangle\langle\alpha_i|) = p|\beta\rangle\langle\beta|$. This in fact implies that $\lambda_i = 1$ for $1 \leq i \leq m$, otherwise we can always increase $\lambda_i$ to increase $p$. Thus $\sum_{i=1}^m (|\alpha_i\rangle\langle\alpha_i|) = p|\beta\rangle\langle\beta|$. This means that for a non-degenerate projective measurement that contains all $\{ |\alpha_i\rangle\langle\alpha_i| \}_{i=1}^m$, Alice is able to steer Bob’s system to $|\beta\rangle\langle\beta|$ with probability $p$. Note that all effects $\{ |\alpha_i\rangle\langle\alpha_i| \}_{i=1}^m$ are mapped to the same steered state $|\beta\rangle\langle\beta|$, and the probability is accumulated to $p$. \(\square\)

The above lemma implies that Alice can always obtain the maximal steering probability of a pure steered state by accumulating multiple non-degenerate projective effects.\[\text{Behaviour of the purified state in steering}\]

Further insight can be gained by considering the purification of the joint state. Let $C$ be an ancillary system attached to $AB$ such that the whole system is in a pure state $|\Psi\rangle$ and $\rho = \text{Tr}_C(|\Psi\rangle\langle\Psi|)$. Since $|\Psi\rangle$ is pure, after Alice gets an outcome in a non-degenerate projective measurement on $A$, $BC$ is also in a pure state. If further $B$ is in a pure state, so must be $C$. This observation leads directly to the following lemma.

**Lemma 2.** If with a non-degenerate projective effect $|\alpha\rangle\langle\alpha|$, Alice steers Bob’s system to a pure state $|\beta\rangle\langle\beta|$ with probability $p$, then $C$ is also collapsed to a pure state $|\gamma\rangle\langle\gamma|$. Accordingly, the purified state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = c|\alpha\rangle|\beta\rangle + |\tilde{\Phi}\rangle,$$

where $|c|^2 = p$ and the partial projection of $|\tilde{\Phi}\rangle$ on $|\alpha\rangle$ vanishes, i.e., $\langle\alpha|\tilde{\Phi}\rangle = 0$. The tilde indicates that $|\tilde{\Phi}\rangle$ is not normalised.

**Proof.** Suppose Alice makes a non-degenerate projective measurement and gets $|\alpha\rangle\langle\alpha|$ with probability $p$, then the purified state can always be written as

$$|\Psi\rangle = c|\alpha\rangle|\delta\rangle + |\tilde{\Phi}\rangle,$$

where $|c|^2 = p$ and $\langle\alpha|\tilde{\Phi}\rangle = 0$. Here $|\delta\rangle$ is a pure state of $BC$. However if $B$ is in pure state $|\beta\rangle$, we must have $|\delta\rangle = |\beta\rangle|\gamma\rangle$ and (1) follows. \(\square\)

Using Lemma 1, one can easily extend Lemma 2 to the case where the pure conditional state $p|\beta\rangle\langle\beta|$ is obtained by accumulating several non-degenerate projective effects, $p|\beta\rangle\langle\beta| = \sum_{i=1}^m \langle\alpha_i|\langle\alpha_i|'$. In this case, the purified state can be written as

$$|\Psi\rangle = \sum_{i=1}^m c_i|\alpha_i,\beta,\gamma_i\rangle + |\tilde{\Phi}\rangle,$$

where $\sum_{i=1}^m |c_i|^2 = p$ and $\langle\alpha_i|\tilde{\Phi}\rangle = 0$ for all $i$.

More interesting is the extension to the case of several different pure steered states. In this case, we need to consider the relationship between them. The following lemma is an obvious extension when several pure steered states are obtained in a single non-degenerate projective measurement.

**Lemma 3.** If with a non-degenerate projective measurement $\{ |\alpha_i\rangle\langle\alpha_i| \}_{i=1}^d$, Alice can steer Bob’s system to (not necessarily orthogonal) pure states $\{ |\beta_i\rangle\langle\beta_i| \}_{i=1}^d$ with
purified by attaching a third qubit. The purified state is necessarily orthogonal to systems of arbitrary dimension (Theorem 26). In addition, if the states are orthogonal, pure steered states that are obtained because of the orthogonality of \( \{ |\beta_i\rangle \}_{i=1}^n \). Moreover \( \sum_{i=1}^n |c_i|^2 = 1 \). It then follows that

\[
|\Psi\rangle = \sum_{i=1}^n c_i |\alpha_i, \beta_i, \gamma_i\rangle.
\]

with \( |c_i|^2 = p_i \).

We ignore the proof, which is an easy extension of that of Lemma 2.

**Example 1.** Consider the case where Alice and Bob share two qubits in state

\[
\rho = \eta |0, \beta_1\rangle \langle 0, \beta_1| + (1 - \eta) |1, \beta_2\rangle \langle 0, \beta_2| + \sqrt{\eta(1 - \eta)} (z |0, \beta_1\rangle \langle 1, \beta_2| + z^* |1, \beta_2\rangle \langle 0, \beta_1|),
\]

where \( 0 \leq \eta \leq 1 \), \( |z| \leq 1 \) and \( |\beta_1\rangle \) and \( |\beta_2\rangle \) are two arbitrary states. Note that for \( z = 0 \) the state \( \rho \) is separable and for \( \eta = 1 \) or \( \eta = 0 \) it is pure. The state can be purified by attaching a third qubit. The purified state is

\[
|\Psi\rangle = \sqrt{\eta} |0, \beta_1, \gamma_1\rangle + \sqrt{1 - \eta} |1, \beta_2, \gamma_2\rangle.
\]

with \( \langle \gamma_2 | \gamma_1 \rangle = z \). This is of the form (4) and it is obvious that by measuring \( \sigma_z \), Alice can steer Bob’s system to pure states \( |\beta_1\rangle \) or \( |\beta_2\rangle \).

The form (4) of the purified state turns out to be very informative. Later, using Lemma 3, we will generalize the fundamental lemma of the all-versus-nothing proof of steerability, originally designed only for two qubits \( [26] \), to systems of arbitrary dimension (Theorem 2). Before doing so, we consider some other natural questions regarding the general properties of the pure steered states.

The assumption in Lemma 3 requires that the steered states are obtained in a single non-degenerate projective measurement. We ask the question if looking only at the steered states, is it possible to establish that several pure steered states are generated from a single non-degenerate projective measurement. We show that this is possible, albeit with rather restrictive assumptions.

**Lemma 4.** Consider a set of pure steered states

\( \{ |\beta_i\rangle \langle \beta_i| \}_{i=1}^n, \) each obtained from a single non-degenerate projective effect. In addition, if the states are orthogonal and have steering probabilities \( p_i \), summed up to 1, \( \sum_{i=1}^n p_i = 1 \), then they can be obtained from a non-degenerate projective measurement.

Note that the assumption of this lemma is rather restrictive; in fact, pure steered states that are obtained by accumulating several non-degenerate projective effects are excluded from consideration.

**Proof.** Suppose with non-degenerate projective effects \( \{ |\alpha_i\rangle \langle \alpha_i| \}_{i=1}^n, \) Alice can steer Bob’s system to orthogonal states \( \{ |\beta_i\rangle \langle \beta_i| \}_{i=1}^n \) with probabilities \( p_i \), \( \sum_{i=1}^n p_i = 1 \). We can assume that \( p_i \) are non-zero, since outcomes with empty probability can always be added to any measurement. We are going to show that \( \{ |\alpha_i\rangle \langle \alpha_i| \}_{i=1}^n \) are orthogonal. This would imply that we can extend \( \{ |\alpha_i\rangle \langle \alpha_i| \}_{i=1}^n \) to form a non-degenerate projective measurement.

According to Lemma 2, for each effect \( |\alpha_i\rangle \langle \alpha_i| \) we can write

\[
|\Psi\rangle = c_i |\alpha_i, \beta_i, \gamma_i\rangle + |\tilde{\Phi}_i\rangle,
\]

where \( \langle \alpha_i | \tilde{\Phi}_i \rangle = 0 \) with \( |c_i|^2 = p_i \). Note that in the Hilbert space of \( ABC \), the set \( \{ |\alpha_i, \beta_i, \gamma_i\rangle \}_{i=1}^n \) are orthogonal because of the orthogonality of \( \{ |\beta_i\rangle \}_{i=1}^n \). Moreover \( \sum_{i=1}^n |c_i|^2 = 1 \). It then follows that

\[
|\Psi\rangle = \sum_{i=1}^n c_i |\alpha_i, \beta_i, \gamma_i\rangle,
\]

by exhaustive expansion of \( |\Psi\rangle \) with respect to orthonormal states \( \{ |\beta_i\rangle \}_{i=1}^n \). Comparing with (7), we get

\[
|\tilde{\Phi}_i\rangle = \sum_{j=1,j\neq i}^n c_j |\alpha_j, \beta_j, \gamma_j\rangle.
\]

Since \( \langle \alpha_i | \tilde{\Phi}_i \rangle = 0 \), we have

\[
0 = \sum_{j=1,j\neq i}^n c_j \langle \alpha_i | \alpha_j \rangle \langle \beta_j, \gamma_j\rangle.
\]

But since \( \{ |\beta_j, \gamma_j\rangle \}_{j=1,j\neq i}^n \) are orthogonal and \( c_j \neq 0 \), we must have \( \langle \alpha_i | \alpha_j \rangle = 0 \) for \( j \neq i \), i.e., \( \{ |\alpha_i\rangle \langle \alpha_i| \}_{i=1}^n \) are orthogonal.

**Example 2.** One can consider the state (5) over two qubits, now with \( |\beta_1\rangle = |0\rangle \) and \( |\beta_2\rangle = |1\rangle \). The two conditional states (\( |0\rangle |0\rangle \) or \( |1\rangle |1\rangle \) or \( |1\rangle |1\rangle \) or \( |1\rangle |1\rangle \) are each obtained from a single non-degenerate projective effect. Moreover, they are orthogonal and have probabilities summed up to 1. Therefore, they must be obtained from a single non-degenerate projective measurement on \( A \), which is \( \sigma_z \) in this case. It is also clear from this example that the assumption in Lemma 4, which requires \( \langle \beta_1 | \beta_2 \rangle = 0 \), is more restrictive than Lemma 3.

**The emergence of subspaces of pure steered states**

Suppose Alice and Bob share two qubits which are in a pure nonseparable state. When Alice performs a local non-degenerate projective measurement on her system, Bob’s system is steered to a pure state. With all possible non-degenerate projective effects, Alice can steer Bob’s system to all possible pure states with different probabilities. We see that Alice’s pure steered states cover the whole Hilbert space of Bob’s system.

We will see that in general Alice’s steered states may cover only a subspace of Bob’s Hilbert space. This notion
of subspace of pure steered states also appears when the shared bipartite system is in a suitable mixed state. In fact, these mixed states can be viewed as ‘partial pure states’—states that look like pure ones in some restricted subspace. The following theorem allows us to characterise these particular mixed states via their behaviour in EPR experiments.

**Theorem 1.** Suppose the set of steered states on Bob’s side contains a subset of pure steered states \( \{ |\beta_i\rangle \} \), each obtained from a single non-degenerate projective effect. In addition, suppose that these steered states are orthogonal and have steering probabilities \( p_i \) summed up to 1, \( \sum_{i=1}^{n} p_i = 1 \). Now if Alice can steer Bob’s system to another pure state \( |\beta\rangle \) by a non-degenerate projective effect, then she can also steer Bob’s system to any state in \( \text{span}(\{ |\beta_i\rangle \} |\beta\rangle \neq 0 \}) \).

Note that here and from now on we also use vectors such as \( |\beta\rangle \) on behalf of their projections \( |\beta\rangle \langle \beta| \) to indicate pure states.

In order to prove this theorem, we need the following simple lemma.

**Lemma 5.** Let \( \{ |\alpha_i\rangle \} \) be a set of \( n \leq d_A \) orthonormal states of system \( A \), and \( \{ |\beta_i\rangle \} \) be arbitrary states of system \( B \). If there exist \( n \) non-zero numbers \( a_i \) such that the linear combination \( \sum_{i=1}^{n} a_i |\alpha_i\rangle |\beta_i\rangle \) is a product state, then \( |\beta_1\rangle = |\beta_2\rangle = \cdots = |\beta_n\rangle \). \( \square \)

**Proof.** Suppose \( \sum_{i=1}^{n} a_i |\alpha_i\rangle |\beta_i\rangle \) is a product state, i.e., \( \sum_{i=1}^{n} a_i |\alpha_i\rangle |\beta_i\rangle = |\alpha\rangle |\beta\rangle \). Then when tracing over \( A \), one gets

\[
|\beta\rangle \langle \beta| = \sum_{i=1}^{n} a_i a_i^* |\beta_i\rangle \langle \beta_i| ,
\]

where we have used the fact that \( \{ |\alpha_i\rangle \} \) are orthogonal. Since pure states are extremal states [27], this only happens for non-zero \( a_i \) if \( |\beta_1\rangle = |\beta_2\rangle = \cdots = |\beta_n\rangle \). \( \square \)

**Proof of Theorem 1.** Let us attach a system \( C \) to purify \( AB \) to a state \( |\Psi\rangle \). Because of Lemma 4, the purified state takes the form

\[
|\Psi\rangle = \sum_{i=1}^{n} c_i |\alpha_i, \beta_i, \gamma_i\rangle ,
\]

where \( \{ |\alpha_i\rangle \} \) are orthogonal. Now if Alice can steer \( B \) to another state \( |\beta\rangle \) by a non-degenerate projective effect, by Lemma 2, there exists state \( |\alpha\rangle \) in \( \mathcal{H}_A \) and state \( |\gamma\rangle \) in \( \mathcal{H}_C \) such that \( \langle \alpha|\Psi\rangle = c|\beta\rangle |\gamma\rangle \) for some \( c \neq 0 \). Using (12), one has

\[
|\beta, \gamma\rangle = \sum_{i=1}^{n} \frac{c_i}{c} |\alpha_i\rangle |\beta_i, \gamma_i\rangle .
\]

Since \( \{ |\beta_i\rangle \} \) are orthogonal by assumption, one finds \( \langle \beta_i|\beta\rangle = \frac{c_i}{c} \langle \alpha_i|\gamma_i\rangle \). For any \( i \) such that \( \langle \beta_i|\beta\rangle \neq 0 \), one has \( \frac{c_i}{c} \langle \alpha_i|\gamma_i\rangle \neq 0 \). Thus by Lemma 5, (13) implies that for all \( |\beta_i\rangle \) such that \( \langle \beta_i|\beta\rangle \neq 0 \), \( |\gamma_i\rangle = |\gamma\rangle \). Without loss of generality, we assume that \( \langle \beta_i|\beta\rangle \neq 0 \) for \( 1 \leq i \leq m \) and \( \langle \beta_i|\beta\rangle = 0 \) for \( m < i \leq n \). The purified state \( |\Psi\rangle \) can now be written as

\[
|\Psi\rangle = \left( \sum_{i=1}^{m} c_i |\alpha_i, \beta_i\rangle \right) |\gamma\rangle + \sum_{i=m+1}^{n} c_i |\alpha_i, \beta_i, \gamma_i\rangle .
\]

Note that \( \{ |\alpha_i\rangle \} \) are orthogonal due to Lemma 4. We claim that with this form, the bipartite state allows Alice to steer \( B \) to any state in \( \text{span}(\{ |\beta_i\rangle \}) \). To see that we just calculate the (unnormalised) state of Bob’s system given that Alice gets state \( |\alpha\rangle = \sum_{i=1}^{m} a_i |\alpha_i\rangle \) in a non-degenerate projective measurement,

\[
\langle \alpha|\Psi\rangle = \left( \sum_{i=1}^{m} a_i^* c_i |\beta_i\rangle \right) |\gamma\rangle ,
\]

where the separated ancillary state \( |\gamma\rangle \) can be simply ignored. Since \( c_i \neq 0 \) and \( a_i \) are arbitrary, the conditional states (15) cover the whole \( \text{span}(\{ |\beta_i\rangle \}) \).

**Example 3.** When applied to a two-qubit system, this theorem is particularly simple. For a two-qubit system, suppose by non-degenerate projective measurements, Alice can steer Bob’s system to two orthogonal pure states with probabilities summed up to 1, and another additional pure state, then \( AB \) must be in a pure state. This can be an operational way for Alice to prove that Bob’s shared state is a pure entangled one (instead of steering Bob’s system to infinitely many pure states).

**Example 4.** Consider the case where Alice and Bob share two qutrits in state

\[
\rho = \eta |\psi^-\rangle \langle \psi^-| + (1-\eta) |00\rangle \langle 00| + \sqrt{\eta(1-\eta)} (z|\psi^-\rangle \langle 00| + z^*|00\rangle \langle \psi^-| ),
\]

with \( |\psi^-\rangle = \frac{1}{\sqrt{2}}(|11| - |11|, 0 < \eta < 1 \) and \( |z| \leq 1 \). The state can be purified by attaching a qubit. The purified state is

\[
|\Psi\rangle = \sqrt{\eta} |\psi^-\rangle |\gamma_1\rangle + \sqrt{1-\eta} |00\rangle |\gamma_2\rangle ,
\]

with \( \langle \gamma_2|\gamma_1\rangle = z \), which is clearly of the form (14). It is then easy to see that Alice can steer Bob system to any linear combination of \( |\gamma_1\rangle \) and \( |\gamma_2\rangle \).

**Pure steered states and steerability**

In their paper, Wiseman et al. [3] discovered that only for certain states the EPR steering experiment is verifiable to be truly nonlocal, while for other states the steering experiment can actually be locally simulated. The former are regarded as steering states, and the latter are called unsteerable states.

To demonstrate steering, Alice prepares a bipartite quantum state \( \rho \) over \( \mathcal{H}_A \otimes \mathcal{H}_B \). She sends part \( B \)
to Bob and specifies a set of measurements $A$ (called measurement assemblage) she can make. Bob then asks her to make a measurement $A_x$ from $A$, which consists of effects $\{A_{a|x}\}_a$. As we described above, it is expected that upon Alice making the measurement, Bob’s system is steered to unnormalised conditional states $A'_x = \{\text{Tr}_A[\rho(A_{a|z} \otimes I)]\}_a$, which form an ensemble $\{P(a|x), \rho_{a|x}\}_a$ with $P(a|x) = \text{Tr}(A'_a|x)$ and $\rho_{a|x} = A'_{a|x}/P(a|x)$. When receiving the outcome from Alice, Bob can perform state tomography to verify his corresponding conditional state. Being able to perform any specified measurement $x$, Alice intends to convince Bob that she can steer his system to different ensembles in the steering assemblage $A'$ from a distance.

However, when Bob does not trust Alice, then this verification protocol may not be always convincing. Indeed, there may exist a strategy for Alice to cheat Bob. In the cheating strategy, instead of sending Bob halves of entangled systems, Alice sends him random states choosing from an ensemble of Local Hidden States (LHS) $\{P(\xi), \sigma_{\xi|x}\}$ of non-degenerate projective measurements. The problem of determining the steerability inequalities is not explicitly specified, we assume that it consists of all non-degenerate projective measurements.

Following the above consideration, we call $A = \{A_{a|x}\}_a$ the set of effects for the assemblage $A$, which is a subset of $\mathcal{M}_A$. Accordingly, $A' = \{A'_a|x\}_a$ is a subset of $\mathcal{M}'_A$, called the set of conditional states on Bob’s system for the assemblage $A$. The problem of determining the existence of a LHS ensemble is also greatly simplified if $A'$ contains one or several (unnormalised) pure states. At the heart of this simplification is the following lemma.

**Lemma 6.** If Alice can steer Bob’s system to a pure state $|\beta\rangle\langle\beta|$ with probability $p$, then the LHS ensemble, if exists, must contain $|\beta\rangle\langle\beta|$ with probability no less than $p$.

**Proof.** By assumption, in equation (18), $A'_{a|x} = p|\beta\rangle\langle\beta|$ for some $a$ and $x$. But since pure states are extremal points of the set of states of $B$ [27], the decomposition in terms of states of the LHS ensemble (18) is only possible if $\sigma_{\xi} = |\beta\rangle\langle\beta|$ for some $\xi$ and $P(a|x,\xi)$ all vanish except for $\xi = \xi_0$. Moreover $P(a|x,\xi_0) \leq 1$, thus $P(\xi_0) \geq p$. 

Despite its simplicity, Lemma 6 is in fact very useful. As a simple corollary, if Alice can steer Bob to some pure states (not necessarily orthogonal) with probabilities summed up greater than 1, then they cannot be contained in any LHS ensemble and the state of $AB$ is steerable from Alice’s side.

**Example 5.** For a pure nonseparable state, Alice can steer Bob’s system to infinitely many pure states by using different non-degenerate projective measurements. The total probability is infinite, therefore nonseparable pure states are all steerable.

**Example 6.** One can take the state (16) in example 4 with $0 < \eta < 1$. Although the state is mixed, the subspace of pure steered state is infinite. Steerability is therefore also guaranteed. Although we have obtained the results with fairly elementary arguments, it is interesting to note that the steerability of the state is not easily detected by steering inequalities. For example, consider the steering inequality proposed in [28],

$$\langle(S^a_A \otimes S^a_B)^2 - (S_y^a)^2 - (S_y^a)^2\rangle < 0$$

where $\langle K_{AB}\rangle = \text{tr}[K_{AB}\rho_{AB}]$, $S^a_A$, $S^a_B$ with $s_A$, $s_B = \pm 1$ are the spin raising and lowering operators, and $S^y$ with $i = x, y, z$ are the usual self-adjoint spin operators. In the inequality, $S^a_A$ and $S^a_B$ can be chosen independently to optimize the possible violation. When the inequality is true, it witnesses the steerability of the state. For simplicity, we consider a simple case of (16) where $z = 0$,

$$\rho = \eta|\psi^+\rangle\langle\psi^+| + (1 - \eta)|00\rangle\langle00|.$$

For this state, the left hand-side of equation (19) gives 0 and the right hand-side gives $\frac{-4l}{50 - 41\eta}$. Thus the inequality is not violated. Note also that this example can also be easily extended to systems of arbitrary dimension.

Following this line of argument, when the set of conditional states $A'$ contains a set of pure states with steering probabilities summed up to 1, the LHS ensemble, if exists, can only be these pure states with their corresponding steering probabilities. Determining the steerability of the state is therefore reduced to checking if the LHS ensemble can explain all measurements in the assemblage $A$ in the sense of equation (18). The following theorem utilises the idea.

**Theorem 2.** If by a non-degenerate projective measurement Alice can steer Bob’s system $B$ to a set of independent pure states, then the joint state of $AB$ is either separable or steerable.

**Lemma 7.** Let $\{|\beta_i\rangle\}_i$ be independent states of some quantum system, then $\{E_{ij} = |\beta_i\rangle\langle\beta_j|\}_{i,j=1}^n$ are independent operators.

**Proof.** We can suppose that $\{|\beta_i\rangle\}_i$ is a basis of the system; otherwise we can extend it to form a basis. Let
\[\{\beta_j^i\}_{i=1}^n\] be the dual basis, defined by \(\langle \beta_j^i | \beta_j^j \rangle = \delta_{ij}.\] Consider a linear combination \(Q = \sum_{i,j=1}^n \lambda_{ij} E_{ij},\) we observe that \(\langle \beta_j^i | Q | \beta_j^j \rangle = \lambda_{ij}.\) It follows that \(Q = 0\) if and only if \(\lambda_{ij} = 0\) for all \(i\) and \(j,\) or \(\{E_{ij}\}_{i,j=1}^n\) are independent.

**Proof of theorem 2.** Suppose the joint state is unsteerable, we will show that it is separable. Suppose Alice performs the non-degenerate projective measurement \(A_x = \{\langle \alpha_k | \phi_k \rangle \}_{k=1}^d\) locally. After the measurement, Alice's state is described by an ensemble \(\{P(a|x), |\beta_k\rangle |\beta_k\rangle\}_{a=1}^d\) and correspondingly, Bob's system is steered to an ensemble of not necessarily orthogonal pure states \(\{P(a|x), |\beta_k\rangle |\beta_k\rangle\}_{a=1}^d.\) Note that \(\sum_a P(a|x) = 1.\) It follows from our above discussion that the purified state must be \(\{P(a|x), |\beta_k\rangle |\beta_k\rangle\}_{a=1}^d.\) Obviously, we have a very important information.

Let us attach a system \(C\) to purify the state \(\rho\) of \(AB\) to a pure state \(|\Psi\rangle\) of \(ABC\). From Lemma 3, we know that the purified state must be of the form

\[|\Psi\rangle = \sum_{a=1}^{d_A} |\alpha_a| |\beta_a\rangle |\gamma_a\rangle.\]  

(21)

with \(|\alpha_a|^2 = P(a|x).\) Then let Alice make another measurement \(A_{x'}\) in the assemblage \(A\) here being all possible non-degenerate projective measurements, which is a collection of new rank-1 projections \(\{A_{x'}|\alpha_{a'}\rangle = |\alpha_{a'}\rangle |\alpha_{a'}\rangle\}_{a'=1}^{d_A}.\) The new rank-1 projections are related to \(\{\langle \alpha_k | \phi_k \rangle \}_{k=1}^d\) by a \(d_A \times d_A\) unitary matrix \(U,\)

\[|\alpha_{a'}| |\beta_{a'}\rangle = \sum_{a=1}^{d_A} |\alpha_a| U_{aaa'}.\]  

(22)

Now the joint state can be expressed using \(\{\langle \alpha_{a'} | x' \rangle \}_{a'=1}^{d_A}\) as

\[|\Psi\rangle = \sum_{a=1}^{d_A} \sum_{a'=1}^{d_A} c_{a,a'} |\alpha_{a'}| U_{aaa'} |\beta_{a'}\rangle |\gamma_{a'}\rangle.\]  

(23)

If in the measurement \(x'\), Alice gets the outcome \(a',\) \(BC\) is then steered to

\[\sum_{a=1}^{d_A} c_{a,a'} U_{aaa'} |\beta_{a}\rangle |\gamma_{a}\rangle.\]  

(24)

After tracing out the ancillary system \(C,\) one gets the corresponding conditional state of \(B,\)

\[A_{x'}|x'\rangle = \sum_{b,d=1}^{d_A} c_{b,d} |\beta_{b}\rangle U_{bd} |\beta_{d}\rangle |\gamma_{d}\rangle.\]  

(25)

Note that \(\{|\beta_{b}\rangle\}_{b=1}^{d_A}\) are independent, thus so are \(\{|\beta_{d}\rangle\}_{d=1}^{d_A}\) by Lemma 7. The coefficients in the decomposition (25) are therefore unique.

Since the \(\{P(a|x), |\beta_k\rangle |\beta_k\rangle\}_{a=1}^d\) is the LHS ensemble, the conditional states \(A_{x'}|x'\rangle\) must be expanded by rank-1 projections \(\{|\beta_{b}\rangle\}_{b=1}^{d_A}\) for all unitary matrices \(U\) and for all outcomes \(a'.\) This means those operators \(|\beta_{b}\rangle \langle \beta_{d}\rangle\) with \(b \neq d\) must be absent from the decomposition (25), i.e.,

\[c_{b,d}^* U_{bd} |\gamma_{d}\rangle |\gamma_{b}\rangle = 0.\]  

(26)

Now for a particular pair \(b \neq d,\) one can choose some \(a'\) and a unitary matrix \(U\) such that \(U_{bd}^* U_{d'b} = 0.\) For example, one can choose \(a' = b' = d' = 0,\) and for all outcomes \(a'.\) This makes the state of \(AB,\)

\[\rho = \sum_{b,d=1}^{d_A} c_{b,d}^* U_{bd} |\gamma_{d}\rangle |\gamma_{b}\rangle |\beta_{b}\rangle_1 |\beta_{d}\rangle_2.\]  

In fact, from this proof, a more detailed statement can be made.

**Corollary.** Suppose by making a non-degenerate projective measurement consisting of orthonormal states \(\{|\beta_i\rangle\}_{i=1}^{d_A},\) Alice can steer Bob system to independent pure states \(\{|\beta_i\rangle\}_{i=1}^{d_A},\) then either:

(i) the joint state is steerable (thus nonseparable), in which case \(\langle \alpha_i | \beta_i | \rho | \alpha_j | \beta_j \rangle\) must not vanish for some \(j \neq i;\)

(ii) the joint state \(\rho\) is separable of the form \(\rho = \sum_{i=1}^{d_A} p_i |\alpha_i | \beta_i \rangle \langle \alpha_i | \beta_i |.\)

**Proof.** Again applying Lemma 3, the assumption of the corollary implies that the purified state is of the form (21),

\[|\Psi\rangle = \sum_{i=1}^{d_A} c_i |\alpha_i | \beta_i | \gamma_i \rangle.\]  

(27)

By tracing over the ancillary system, we find

\[\rho = \sum_{i=1}^{d_A} c_i c_i^* |\gamma_i \rangle \langle \gamma_i | |\alpha_i | \beta_i \rangle \langle \alpha_j | \beta_j |.\]  

(28)

Note that \(\langle \alpha_i | \beta_i | \rho | \alpha_j | \beta_j \rangle = c_i c_j^* |\gamma_j \rangle \langle \gamma_i |.\) Then it is clear that if \(\langle \alpha_i | \beta_i | \rho | \alpha_j | \beta_j \rangle \neq 0\) for some \(j \neq i\), the state is steerable (thus nonseparable) according to the above proof; else if \(\langle \alpha_i | \beta_i | \rho | \alpha_j | \beta_j \rangle = 0\) for all \(j \neq i,\) \(\rho\) reduces to the separable form stated in (ii) with \(p_i = |c_i|^2.\)

When applied to a two-qubit system, the statement is particularly simple. For a two-qubit system, if by a non-degenerate projective measurement Alice can steer \(B\) to two pure states, then the joint state of \(AB\) is either separable or steerable. This result has been recently obtained by Chen et al. [26]. Experimental applications of this special case were also discussed [29]. Here we report a more general result with a somewhat more systematic proof.
Example 7. Take again the two-qubit state (5) in example 1, which satisfies the assumption of the above corollary. Note that \(0, \beta_1 |\rho | 1, \beta_2 = z \sqrt{\eta(1 - \eta)}\). We then know that it is steerable (or equivalently, nonseparable) if and only if \(|z|^2 \eta(1 - \eta) > 0\).

**Conclusion**

We have shown that the pure steered states in the EPR experiments carry interesting information about the shared state, in particular its nonlocal properties. Thanks to the purification technique, results are obtained in an elementary and systematic way. Although our work only concentrates on the pure steered states, we speculate that analysing the behaviour of the purified system in the EPR experiments might be an interesting approach to quantum nonlocality. Applications of the special case of Theorem 2 for two-qubit systems have been discussed [26, 29]. As experimental research is moving toward higher dimensional systems, we hope that our generalisation will be useful for the future experiments.

**ACKNOWLEDGMENTS**

We thank David Gross, Eric Lutz, Roope Uola and Huangjun Zhu for useful discussions. The authors also acknowledge the Referee of PRA for the useful comments and particularly for correcting our terminology.

[1] A. Einstein, B. Podolsky, and N. Rosen, “Can quantum-mechanical description of physical reality be considered complete?” Phys. Rev. 47, 777–780 (1935).

[2] E. Schrödinger, “Discussion of probability relations between separated systems,” Proc. Cambridge Philos. Soc. 31, 555–563 (1935).

[3] H. M. Wiseman, S. J. Jones, and A. C. Doherty, “Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox,” Phys. Rev. Lett. 98, 140402 (2007).

[4] S. J. Jones, H. M. Wiseman, and A. C. Doherty, “Entanglement, Einstein-Podolsky-Rosen correlations, Bell nonlocality, and steering,” Phys. Rev. A 76, 052116 (2007).

[5] R. F. Werner, “Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model,” Phys. Rev. A 40, 4277–4281 (1989).

[6] J. Bell, “On the Einstein-Podolsky-Rosen paradox,” Physics 1, 195–200 (1964).

[7] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, “Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox,” Phys. Rev. A 80, 032112 (2009).

[8] E. G. Cavalcanti, Q. Y. He, M. D. Reid, and H. M. Wiseman, “Unified criteria for multipartite quantum nonlocality,” Phys. Rev. A 84, 032115 (2011).

[9] J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti, and J. C. Howell, “Einstein-Podolsky-Rosen steering inequalities from entropic uncertainty relations,” Phys. Rev. A 87, 062103 (2013).

[10] A. C. S. Costa and R. M. Angelo, “Quantification of Einstein-Podolsky-Rosen steering for two-qubit states,” Phys. Rev. A 93, 020103 (2016).

[11] H. Zhu, M. Hayashi, and L. Chen, “Universal steering criteria,” Phys. Rev. Lett. 116, 070403 (2016).

[12] D. J. Saunders, S. J. Jones, H. M. Wiseman, and G. J. Pryde, “Experimental EPR-steering using Bell-local states,” Nat. Phys. 6, 845–849 (2010).

[13] J. Schneeloch, P. B. Dixon, G. A. Howland, C. J. Broadbent, and J. C. Howell, “Violation of continuous-variable Einstein-Podolsky-Rosen steering with discrete measurements,” Phys. Rev. Lett. 110, 130407 (2013).

[14] M. Piani, “Channel steering,” J. Opt. Soc. Am. B 32, A1–A7 (2015).

[15] M. Piani and J. Watrous, “Necessary and sufficient quantum information characterization of Einstein-Podolsky-Rosen steering,” Phys. Rev. Lett. 114, 060404 (2015).

[16] R. Uola, T. Moroder, and O. Gühne, “Joint measurability of generalized measurements implies classicality,” Phys. Rev. Lett. 113, 160403 (2014).

[17] M. T. Quintino, T. Vértesi, and N. Brunner, “Joint measurability, Einstein-Podolsky-Rosen steering, and Bell nonlocality,” Phys. Rev. Lett. 113, 160402 (2014).

[18] R. Uola, C. Budroni, O. Gühne, and J.-P. Pellonpää, “One-to-one mapping between steering and joint measurability problems,” Phys. Rev. Lett. 115, 230402 (2015).

[19] R. Uola, K. Luoma, T. Moroder, and T. Heinosaari, “Adaptive strategy for joint measurements,” Phys. Rev. A 94, 022109 (2016).

[20] H. C. Nguyen and T. Vu, “Nonseparability and steerability of two-qubit states from the geometry of steering outcomes,” Phys. Rev. A 94, 012114 (2016).

[21] S. Jevtic, M. Pusey, D. Jennings, and T. Rudolph, “Quantum steering Ellipsoids,” Phys. Rev. Lett. 113, 020402 (2014).

[22] S. Jevtic, M. J. W. Hall, M. R. Anderson, M. Zwierz, and H. M. Wiseman, “Einstein–podolsky–rosen steering and the steering ellipsoid,” J. Opt. Soc. Am. B 32, A40–A49 (2015).

[23] H. Chau Nguyen and T. Vu, “Necessary and sufficient condition for steerability of two-qubit states by the geometry of steering outcomes,” Europhys. Lett. 115, 10003 (2016).

[24] A. Milne, S. Jevtic, D. Jennings, H. Wiseman, and T. Rudolph, “Quantum steering Ellipsoids, extremal physical states and monogamy,” New J. Phys. 16, 083017 (2014).

[25] A. Milne, D. Jennings, and T. Rudolph, “Geometric representation of two-qubit entanglement witnesses,” Phys. Rev. A 92, 012311 (2015).

[26] J.-L. Chen, X.-J. Ye, C. Wu, H.-Y. Su, A. Cabello, L. C. Kwek, and C. H. Oh, “All-versus-nothing proof of Einstein-Podolsky-Rosen steering,” Sci. Rep. 3 (2013), 10.1038/srep02143.

[27] T. Heinosaari and M. Ziman, *The Mathematical Language of Quantum Theory: From Uncertainty* (Cambridge University Press, 2011).
[28] Q. Y. He, P. D. Drummond, and M. D. Reid, “Entanglement, EPR steering, and Bell-nonlocality criteria for multipartite higher-spin systems,” Phys. Rev. A 83, 032120 (2011).

[29] K. Sun, J.-S. Xu, X.-J. Ye, Y.-C. Wu, J.-L. Chen, C.-F. Li, and G.-C. Guo, “Experimental demonstration of the Einstein-Podolsky-Rosen steering game based on the all-versus-nothing proof,” Phys. Rev. Lett. 113, 140402 (2014).