ON A MEAN FIELD THEORY OF TOPOLOGICAL 2D GRAVITY

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Abstract. We present a one-dimensional mean field theory for topological 2D gravity. We discuss possible generalizations to other topological field theories, in particular those related to semisimple Frobenius manifolds.

1. Introduction

The mathematical theory of the topological 2D gravity studies the following intersection numbers on the Deligne-Mumford moduli spaces:

\[ \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g := \int_{\mathcal{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}. \]

As is well-known, instead of considering these numbers individually, it is more effective to study them collectively by considering their formal generating series:

\[ F(t; \lambda) = \sum_{g \geq 0} \lambda^{2g-2} F_g(t), \]

where

\[ F_g(t) := \sum \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \frac{t_{a_1} \cdots t_{a_n}}{n!}. \]

Apparently \( F \) involves integrations on infinitely many spaces and infinitely many parameters. In this paper, we will present a mean field theory for this theory which is one-dimensional, more precisely, a model that depends on formal integration of a formal field on a one-dimensional space, depending on infinitely parameters. This is very striking because the original theory involves all sorts of topology of Riemann surfaces and their suitable compactifications, it is very hard to expect that all the information can be essentially encoded in a one-dimensional theory. The secret is that we hide all the complexities in the interactions of the mean field theory.

The idea of a mean field theory is a very well-known old idea in statistical physics where one considers a system with a very large degree of freedom. One approximates the system by finding a field theory
for suitably chosen finitely many order parameters so that the Euler-Lagrange equations approximate the equations of states of the systems. Usually quantum corrections are needed to improve the approximation. Renormalization flow was developed to provide a scheme to canonically improve the model.

The application of the approach of mean field theory to topological string theory was initiated in [3]. The authors focus on the genus zero case based on topological recursion relations in genus zero (see also [10] for mean field theory in genus zero for topological 2D gravity). They also discussed the cases of genus one and higher genera based on integrable hierarchies. Their discussions are mostly concerned with the derivation of the equations of states. In this paper we will supplement their work by providing the suitable action functional that leads to these equations. We will focus on the case of pure topological gravity. We will speculate on the generalizations to the general case in the final Section 5 and leave the details to be worked out in subsequent work.

For the order parameter in the case of topological 2D gravity, we take the specific heat defined as the second derivative of the free energy:

$$ u(t; \lambda) = \sum_{g \geq 0} \lambda^{2g} u_g(t) = \sum_{g \geq 0} \lambda^{2g} \frac{\partial^2 F_g(t)}{\partial t_0^2}. $$

It is customary to take $t_0$ as the space variable and denote it by $x$, $u$ then can be thought of as field on the space $\mathbb{R}$ with coordinate $x$, parameterized by time variables $t_1, t_2, \ldots$. By Witten Conjecture/Kontsevich Theorem [15, 13], $u$ satisfies the KdV hierarchy

$$ \frac{\partial u}{\partial t_n} = \partial_x R_n[u], $$

for the sequence of Gelfand-Dickey differential polynomials $R_n[u]$. It is known that there is a differential polynomial $T_n[u]$ such that

$$ \partial_x T_n[u] = \partial_x u \cdot R_n[u]. $$

The action we find for our mean field theory is

$$ S[u] = \frac{1}{\lambda^2} \sum_{n \geq 0} (t_n - \delta_{n1}) \cdot \int_{-\infty}^{\infty} T_n[u(x)] dx + F(t)|_{t_0=0}. $$

We will show that its Euler-Lagrange equation is equivalent to the string equation:

$$ u = \sum_{k=1}^{\infty} t_n R_n[u] + x. $$
Furthermore, if \( u(t) \) is determined by this equation, then we have

\[
F(t) = \frac{1}{\lambda^2} \sum_{n \geq 0} (t_n - \delta_{n0}) \cdot \int T_n[u(t)] dx + F(t)|_{t_0=0}.
\]

This means our mean field theory is exact semiclassically without any quantum corrections.

Note in the above we have not specified the bounds for integration. This is because we are working in a formal setting without considering the issue of convergence. In fact, we will show that we cannot expect the convergence in this case. Since \( F_g(t) \) involves infinitely many variables, to make sense of the convergence problem one can restrict to the subspaces where only fixed numbers of finitely many variables are possibly nonvanishing. In fact we will consider the restriction to the space where all variables except for \( t_0 \) and \( t_2 \) vanish. Denote by \( F_g(t_0, t_2) \) the restriction of \( F_g(t) \) on this space. We will establish the following result:

\[
\sum_{g \geq 0} u_g(t_0, t_2) \lambda^{2g} = \frac{1}{t_2} + \sum_{g \geq 0} \frac{2a_g}{24g} t_2^{3g-1}(1 - 2t_0t_2)^{-(5g-1)/2}\lambda^{2g},
\]

where \( \{a_g\}_{g \geq 0} \) is a sequence of integers studied in probability theory of graphs \([11, 12]\), see also \([7]\). By the asymptotic formula for \( a_n \), we have

\[
\lim_{n \to \infty} \frac{1}{a_n^{1/n}} = 0.
\]

It follows that when \( u_2 \neq 0 \), the radius of convergence of the series

\[
\sum_{g \geq 0} F_g(t_0, t_2) \lambda^{2g}
\]

as a power series in \( \lambda \) is zero.

The rest of the paper is arranged as follows. In Section 2 we recall the Gelfand-Dickey polynomials \( R_n[u] \) and some related differential polynomials \( T_n[u] \) and establish some variational properties of \( T_n \). The mean field theory of the topological 2D gravity is derived in Section 3 based on the string equation. The convergence problem in Section 4 will be addressed in Section 3. In the final Section 5 we present some speculations about possible generalizations.
2. Gelfand-Dickey Polynomials and Their Properties

2.1. Gelfand-Dickey polynomials. Following [8], define a sequence \( \{ R_n \} \) of differential polynomials in \( u \) by the Lenard recursion relations:

\[
R_0 = 1, \quad \partial_x R_{n+1} = \frac{1}{2n+1} \left( \partial_x u \cdot R_n + 2u \cdot \partial_x R_n + \frac{\lambda^2}{4} \partial^3_x R_n \right). \tag{13}
\]

For example,

\[
\partial_x R_1 = \partial_x u, \quad R_1 = u, \\
\partial_x R_2 = u \cdot \partial_x u + \frac{\lambda^2}{12} \partial^3_x u, \quad R_2 = \frac{1}{2} u^2 + \frac{\lambda^2}{12} \partial^2_x u, \\
\partial_x R_3 = \frac{1}{2} u^2 \cdot \partial_x u + \frac{\lambda^2}{12} u \cdot \partial^3_x u + \frac{\lambda^2}{6} \partial_x u \cdot \partial^2_x u + \frac{\lambda^4}{240} \partial^5_x u, \quad R_3 = \frac{1}{6} u^3 + \frac{\lambda^2}{12} u \cdot \partial^2_x u + \frac{\lambda^2}{24} (\partial_x u)^2 + \frac{\lambda^4}{240} \partial^4_x u.
\]

Rewrite (13) as follows:

\[
R_{n+1} = \frac{1}{2n+1} \left( u \cdot \partial_x R_n + \frac{\lambda^2}{4} \partial^2_x R_n \right) R_n + \frac{1}{2n+1} \partial_x^{-1}(u \cdot \partial_x R_n), \tag{14}
\]

or alternative as

\[
R_{n+1} = \frac{1}{2n+1} \left( 2u \cdot \frac{\lambda^2}{4} \partial^2_x \right) R_n - \frac{1}{2n+1} \partial_x^{-1}(\partial_x u \cdot R_n), \tag{15}
\]

To find \( R_{n+1} \), one needs to show that \( u \cdot \partial_x R_n \) or \( \partial_x u \cdot R_n \) is a total derivative and finds the corresponding antiderivative. More generally, in [8] it was proved that for \( k, l \geq 0 \), there exists a differential polynomial \( P_{k,l} \) such that

\[
R_k \cdot \partial_x R_l = \partial_x P_{k,l}. \tag{16}
\]

In particular, there are differential polynomials \( T_n \) such that

\[
\partial_x u \cdot R_n = \partial_x T_n. \tag{17}
\]
The following are the first few terms:

\[
\begin{align*}
T_0 &= u, \\
T_1 &= \frac{1}{2} u^2, \\
T_2 &= \frac{1}{6} u^3 + \frac{1}{24} u_x^2 \lambda^2, \\
T_3 &= \frac{1}{24} u^4 + \frac{1}{24} u u_x^2 \lambda^2 - \frac{1}{480} u_{2x}^2 \lambda^4 + \frac{1}{240} u_x u_{3x} \lambda^4, \\
T_4 &= \frac{1}{120} u^5 + \frac{1}{48} u^2 u_x^2 \lambda^2 + \frac{1}{240} u_x^2 u_{2x} \lambda^4 - \frac{1}{480} u u_x^2 \lambda^4 \\
&\quad + \frac{1}{240} u u_x u_{3x} \lambda^4 + \frac{1}{13440} u_{3x}^2 \lambda^6 - \frac{1}{6720} u_{2x} u_{4x} \lambda^6 + \frac{1}{6720} u_x u_{5x} \lambda^6.
\end{align*}
\]

2.2. **Variational derivative.** Denote by \( A = \oplus_{n \geq 0} A_n \) the space of differential polynomials, i.e., polynomials in \( u_0 = u(x), u_1 = \partial_x u(x), \ldots, u_n = \partial^n_x u(x), \ldots \). By \( A_n \) we denote the space of homogeneous differential polynomials of degree \( n \). On the space \( A \) the operators \( \frac{\partial}{\partial u_k} \) naturally act, so do the operator

\[
\partial_x = \sum_{k \geq 0} u_{k+1} \frac{\partial}{\partial u_k}
\]

and the operator \( \delta \) defined as follows:

\[
\delta = \sum_{k \geq 0} (-1)^k \partial_x^k \frac{\partial}{\partial u_k}.
\]

For \( f \in A \), \( \delta f \) will be called the variational derivative of \( f \). Since

\[
\left\{ \frac{\partial}{\partial u_k}, \partial_x \right\} = \begin{cases} 0, & \text{if } k = 0, \\ \frac{\partial}{\partial u_{k-1}}, & \text{if } k \geq 1, \end{cases}
\]

one can easily see that

\[
\delta \partial_x = 0.
\]

In [8] the following sequence

\[
0 \to \mathbb{R} \to A \xrightarrow{\delta} A \xrightarrow{\delta} A
\]

is proved to be exact. Furthermore, the following identity is established by symbolic computations:

\[
\delta(u_1 \cdot \delta f) = 0,
\]

for \( f \in A \). By slightly modifying the proof, one can also prove the following identity:

\[
\delta(u \cdot \delta f) = n \cdot \delta f.
\]
for \( f \in A_n \).

2.3. Variational property of \( T_n \). In [8], the following relation is proved:

\[
\delta R_{n+1} = R_n.
\]

By (23) one then has

\[
\delta (\partial_x u \cdot R_n) = 0.
\]

Lemma 2.1. For \( n \geq 0 \),

\[
\delta (uR_n) = (n + 1)R_n - \frac{1}{2} \lambda \frac{\partial}{\partial \lambda} R_n.
\]

Proof. Write \( R_n = \sum_{g \geq 0} \lambda^{2g} R_n^{(g)} \), where \( R_n^{(g)} \in A_n \). By (13) it is not hard to see that

\[
R_n^{(g)} \in A_{n-g}.
\]

So we have

\[
\delta (uR_n) = \delta (u \cdot R_n) = \delta (u \cdot \delta R_{n+1})
\]

\[
= \sum_{g \geq 0} \lambda^{2g} \delta (u \cdot \delta R_{n+1}^{(g)})
\]

\[
= \sum_{g \geq 0} \lambda^{2g} (n + 1 - g) \delta R_{n+1}^{(g)}
\]

\[
= (n + 1 - \frac{1}{2} \lambda \frac{\partial}{\partial \lambda}) R_n.
\]

The following result plays a key role in the next Section:

Proposition 2.2. For \( n \geq 0 \), the following holds:

\[
\delta T_n = (1 - \lambda \frac{\partial}{\partial \lambda}) R_n.
\]

Proof. Rewrite (13) as follows:

\[
T_n = 2u \cdot R_n + \frac{\lambda^2}{4} \partial_x^2 R_n - (2n + 1)R_{n+1},
\]

Apply \( \delta \) on both sides:

\[
\delta T_n = 2\delta (u \cdot R_n) + \frac{\lambda^2}{4} \delta \partial_x^2 R_n - (2n + 1) \cdot \delta R_{n+1}
\]

\[
= 2(n + 1 - \frac{1}{2} \lambda \frac{\partial}{\partial \lambda}) R_n - (2n + 1) R_n
\]

\[
= (1 - \lambda \frac{\partial}{\partial \lambda}) R_n.
\]
3. Mean Field Theory of Topological 2D Gravity

3.1. The KdV equations. The KdV hierarchy is the following sequence of partial differential equations:

\[ \partial_t u = \partial_t R_{n+1}, \]  

where \( t_0 = x. \)

3.2. The string equation. By the puncture equation we mean the following equation:

\[ \frac{\partial F}{\partial t_0} = \sum_{k=1}^{\infty} t_k \frac{\partial F}{\partial t_{k-1}} + \frac{t_0^2}{2\lambda^2}. \]

In the mathematical literature this is referred to as the string equation. Following physicists, we will reserve this name for the equation \( \text{(35)} \) below. Take \( \lambda^2 \partial^2 u_0 \) on both sides of \( \text{(32)} \):

\[ \frac{\partial u}{\partial t_0} = \sum_{k=1}^{\infty} t_k \frac{\partial u}{\partial t_{k-1}} + 1. \]

And so by \( \text{(31)} \),

\[ \partial_x u = \sum_{k=1}^{\infty} t_k \partial_x R_k + 1. \]

It can be shown that when \( u = \partial^2 F \) satisfies the KdV hierarchy and \( F \) satisfies the puncture equation, the following equation holds \( \text{[2, 9, 14]} \):

\[ u = \sum_{k=1}^{\infty} t_k R_k + x. \]

This is called the string equation in the physics literature.

3.3. Landau-Ginzburg equation. Expanding both sides of the string equation as series in \( \lambda \), one gets by comparing the leading terms the following equation:

\[ u_0 = \sum_{k=1}^{\infty} \frac{1}{k!} t_k \cdot u_0^k + x. \]

We will call this equation the Landau-Ginzburg equation.
3.4. Derivation of the mean field theory. Multiply both sides of (35) by $u_x$ and integrate with respect to $x$:

$$\int uu_x dx = \sum_{k=1}^{\infty} t_k \int u_x R_k dx + \int xu_x dx. \tag{37}$$

The last term on the right-hand side can be found by integration by parts:

$$\int xu_x dx = \int xdu = xu - \int u dx = xu - \lambda^2 \frac{\partial F}{\partial x}.$$

So we get

$$\frac{1}{2} u^2 = \sum_{k=1}^{\infty} t_k T_k + xu - \lambda^2 \frac{\partial F}{\partial x}. \tag{38}$$

Rewrite it as follows:

$$\frac{\partial}{\partial t_0} F = \frac{1}{\lambda^2} \sum_{n \geq 0} (t_n - \delta_{n1}) \cdot T_n.$$

Integrate once more, we obtain the following

**Proposition 3.1.** Suppose that $u = u(t)$ is determined by (35), then one has:

$$F(t) = F(t)|_{t_0=0} + \frac{1}{2} \frac{1}{\lambda^2} \sum_{n \geq 0} (t_n - \delta_{n1}) \cdot \int T_n[u(t)] dx, \tag{39}$$

**Theorem 3.2.** Define the following action for a formal field $u = u(x)$ on $\mathbb{R}^1$:

$$S[u] = F(t)|_{t_0=0} + \frac{1}{2} \frac{1}{\lambda^2} \sum_{n \geq 0} (t_n - \delta_{n1}) \cdot \int T_n[u(x)] dx. \tag{40}$$

Then the Euler-Lagrange equation for this action is equivalent to the string equation (35).

**Proof.** This is because

$$\frac{\delta}{\delta u} S[u] = \frac{1}{\lambda^2} \sum_{n \geq 0} (t_n - \delta_{n1}) \cdot \delta T_n[u(x)]$$

$$= \frac{1}{\lambda^2} \sum_{n \geq 0} (t_n - \delta_{n1}) \cdot (1 - \lambda \frac{\partial}{\partial \lambda}) R_n[u(x)]$$

$$= \sum_{n \geq 0} (t_n - \delta_{n1}) \cdot \sum_{g \geq 0} \lambda^{2g-2} (1 - 2g) R_n^{(g)}[u(x)].$$
Therefore, if \( \frac{\delta}{\delta u} S[u] = 0 \), then one has for all \( g \geq 0 \),
\[
(1 - 2g) \cdot \sum_{g \geq 0} \sum_{n \geq 0} (t_n - \delta_{n1}) \cdot R_n^{(g)} [u(x)] = 0,
\]
and so
\[
\sum_{g \geq 0} \sum_{n \geq 0} (t_n - \delta_{n1}) \cdot R_n^{(g)} [u(x)] = 0.
\]
It follows that (35) holds. Conversely, if \( u \) satisfies (35), then it also satisfies the Euler-Lagrange equation.

By the explicit expression for the first few \( T_n \)'s given earlier, we see that the first few terms of the Lagrangian for our action functional is:
\[
L = t_0 u + t_1 \frac{1}{2} u^2 + t_2 \left( \frac{1}{6} u^3 + \frac{1}{24} u_x^2 \lambda^2 \right) + t_3 \left( \frac{1}{24} u^4 + \frac{1}{48} u u_x^2 \lambda^2 - \frac{1}{480} u_x^2 \lambda^4 + \frac{1}{240} u x u_3 \lambda^4 \right) + t_4 \left( \frac{1}{120} u^5 + \frac{1}{48} u^2 u_x^2 \lambda^2 + \frac{1}{240} u_x^2 u_x \lambda^4 - \frac{1}{480} u u_x^2 \lambda^4 + \frac{1}{240} u u_x u_3 \lambda^4 \right) + \cdots.
\]
The genus zero part of the Lagrangian is:
\[
L_0 = \sum_{n \geq 0} (t_n - \delta_{n1}) \frac{u^{n+1}}{(n+1)!}.
\]
This is what we used in [18] for the mean field theory of the topological 1D gravity. Taking all \( t_n = 0 \), one gets a plane algebraic curve:
\[
L_0 = -\frac{1}{2} u^2.
\]
This is the Airy curve [1, 5, 16] that determines the topological 2D gravity by Eynard-Orantin topological recursion [6].

4. Convergence Problem

Now we restrict to the \((t_0, t_2)\)-plane. The string equation (35) becomes:
\[
u(t_0, t_1) = t_0 + t_2 \left( \frac{1}{2} u^2(t_0, t_1) + \frac{\lambda^2}{12} \partial_{t_0}^2 u(t_0, t_1) \right).
\]
We solve the above equation recursively by rewriting it as follows:
\[
u_0(t_0, t_1) = t_0 + \frac{t_2}{2} u_0^2(t_0, t_1).
\]
and for $g \geq 1$,

$$u_g(t_0, t_2) = \frac{t_2}{2} \sum_{g_1+g_2=g} u_{g_1}(t_0, t_2) \cdot u_{g_2}(t_0, t_2) + \frac{t_2}{12} \frac{\partial^2}{\partial t_0^2} u_{g-1}(t_0, t_2).$$

One can rewrite (45) in the following form:

$$u_g(t_0, t_2) = \frac{t_2}{1-t_2u_0(t_0, t_2)} \left( \frac{1}{2} \sum_{g_1=1}^{g-1} u_{g_1}(t_0, t_2) \cdot u_{g-g_1}(t_0, t_2) + \frac{1}{12} \frac{\partial^2}{\partial t_0^2} u_{g-1}(t_0, t_2) \right).$$

From (44) one can get [17]:

$$u_0(t_0, t_2) = \frac{1-(1-2t_0t_2)^{1/2}}{t_2}.$$  

After using (45) to get

$$u_1(t_0, t_2) = \frac{1}{12} t_2^2 (1-2t_0t_2)^{-2},$$

$$u_2(t_0, t_2) = \frac{49}{288} t_2^5 (1-2t_0t_2)^{-9/2},$$

we make the following ansatz:

$$u_g = c_g t_2^{3g-1} (1-2t_0t_2)^{- (5g-1)/2} + \delta_g t_2^{-1}$$

and get the following recursion relations for the coefficients $c_g$:

$$c_g = \frac{1}{2} \sum_{g_1=1}^{g-1} c_{g_1} c_{g-g_1} + \frac{1}{12} (5g-4)(5g-6) c_{g-1}. $$

Set $c_g = \frac{2}{24} a_g$, then one has

$$a_0 = -\frac{1}{2},$$

and for $n > 0$ the recursion relation:

$$a_n = \sum_{k=1}^{n-1} a_k a_{n-k} + 2(5n-4)(5n-6) a_{n-1}.$$ 

The sequence $a_n$ is the sequence A094199 on Sloane’s The On-Line Encyclopedia of Integer Sequences. This sequence appeared in [11, 12], see also [17]. By [12, Theorem 4.2], as $n \to \infty$,

$$a_n \sim \beta \cdot 50^{n-1}(n-1)!$$

for some constant $\beta$. The constant $\beta$ has been determined by Kotesovec to be

$$\beta = \frac{5\sqrt{15}}{2\pi^2}. $$
By Stirling’s formula,
\( a_n \sim \sqrt{32^{n-1}5^{2n-1/2}n^{2n-1}}/(\pi \exp(2n)). \)

It follows that
\( c_n \sim \sqrt{35^{2n-1/2}n^{2n-1}}/(\pi 12^n \exp(2n)). \)

Therefore, as \( n \to \infty \),
\( c_1^n/n \sim 25^{12}e^{-n^2}. \)

Therefore, when \( t_2 \neq 0 \),
\( u(t_0, t_2) = \sum_{g \geq 0} \lambda^g c_g t_2^{3g-1} (1 - 2t_0 t_2)^{-(5g-1)/2} + t_2^{-1} \)
has radius of convergence equal to zero.

5. DISCUSSIONS

We expect to extend our construction of the mean field theory to other topological field theories. The following discussions are based on speculative assumptions. We will check these assumptions in subsequent work. The free energy of such a theory in two dimensions depends on infinitely many parameters \( \{t^{a,n}\}_{0 \leq a \leq m, n \geq 0} \) for some fixed \( m \), where \( m + 1 \) is the number of primary operators. Suppose that
\( \frac{\partial^3 F_0(t)}{\partial t^{0,0} \partial t^{a,0} \partial t^{b,0}} = \eta_{ab}, \)
where \( (\eta_{ab}) \) is a nondegenerate symmetric matrix. Denote its inverse matrix by \( (\eta^{ab}) \). For the order parameters, as in [3] we take
\( u_a := \lambda^2 \frac{\partial^2 F}{\partial t^{0,0} \partial t^{a,0}}. \)
Suppose that they satisfy the integrable hierarchy
\( \frac{\partial u_a}{\partial t^{b,n}} = \partial_x R_{a,b,n}[u], \)
where \( R_{a,b,n}[u] \) are some differential polynomials. Furthermore, assume that the free energy \( F \) satisfies the puncture equation of the form:
\( \frac{\partial F}{\partial t^{0,0}} = \sum_{a=0}^m \sum_{n=1}^{\infty} t^{a,n} \frac{\partial F}{\partial t^{a,n-1}} + \frac{1}{2\lambda^2} \eta_{ab} t^{a,0} t^{b,0}, \)
such that one can derive from it the string equations:
\( u_b = \sum_{n \geq 1} \sum_{c=0}^m t^{c,n} R_{b,c,n}[u] + \eta_{bo} x. \)
Multiply both sides by $\eta^{ab}\partial_x u_a$ and sum over repeated indices:

\begin{equation}
\eta^{ab}\partial_x u_a \cdot u_b = \sum_{n \geq 1} \sum_{c=0}^{m} \eta^{ab}t^{c,n}_b \partial_x u_a \cdot R_{b,c,n}[u] + x\partial_x u_0.
\end{equation}

Suppose that after integration one has

\begin{equation}
\frac{1}{2}\eta^{ab}u_au_b = \sum_{n \geq 1} \sum_{c=0}^{m} \eta^{ab}t^{c,n}_b T_{a,b,c,n}[u] + xu_0 - \frac{\lambda^2}{2} \frac{\partial F}{\partial x}.
\end{equation}

Then one gets:

\[
F = \lambda^{-2} \int \left( \sum_{n \geq 1} \sum_{c=0}^{m} \eta^{ab}t^{c,n}_b T_{a,b,c,n}[u] + xu_0 - \frac{1}{2}\eta^{ab}u_au_b \right) dx + F|_{t=0,0=0},
\]

where $u$ satisfies the string equations (63). Next we define the action functional to be

\[
S = \lambda^{-2} \int \left( \sum_{n \geq 1} \sum_{c=0}^{m} \eta^{ab}t^{c,n}_b T_{a,b,c,n}[u] + xu_0 - \frac{1}{2}\eta^{ab}u_au_b \right) dx + F|_{t=0,0=0},
\]

and our final assumption is that the system of Euler-Lagrange equations

\begin{equation}
\delta S \over \delta u_a = 0, \quad a = 0, \ldots, m,
\end{equation}

is equivalent to the system of string equations (63). When all our assumptions are met, we then arrive at a one-dimensional mean field theory for the original theory. We conjecture this is the case for the theories arising from semisimple Frobenius manifolds [4].

Another direction for possible generalizations is to find $(n-1)$-dimensional mean field theory for topological $n$-dimensional gravity. In [18] we have studied topological 1D gravity by a 0-dimensional mean field theory. We have seen in Section 3 that the genus zero part of the Lagrangian density for the topological 2D gravity is the Lagrangian density for the topological 1D gravity. Furthermore, in both cases the system of equations of motion has only one formal solution. It will be very interesting to generalize these to higher dimensions.

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