Locally compact groups with every isometric action bounded or proper

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(with an appendix by Corina CIOBOTARU)

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Abstract
A locally compact group $G$ has property PL if every isometric $G$-action either has bounded orbits or is (metrically) proper. For $p > 1$, say that $G$ has property $BP_{L^p}$ if the same alternative holds for the smaller class of affine isometric actions on $L^p$-spaces. We explore properties PL and $BP_{L^p}$ and prove that they are equivalent for some interesting classes of groups: abelian groups, amenable almost connected Lie groups, amenable linear algebraic groups over a local field of characteristic 0.

The appendix provides new examples of groups with property PL, including non-linear ones.

1 Introduction
Let the locally compact group $G$ act by isometries on a metric space $(X,d)$. The action is locally bounded if $Kx$ is bounded for every $x \in X$ and every compact set $K \subset G$; the action is bounded if every orbit is bounded. On the other hand, the action is (metrically) proper if $\lim_{g \to \infty} d(gx, x) = +\infty$ for every $x \in X$.

A length function on $G$ is a non-negative function $L$ on $G$ which is bounded on compact subsets, is symmetric ($L(g) = L(g^{-1})$ for every $g \in G$), and is sub-additive: $L(gh) \leq L(g) + L(h)$ for every $g, h \in G$. Clearly if $G$ admits a locally bounded action by isometries on $(X, d)$, then for every $x \in X$ the function $L : G \to \mathbb{R}^+ : g \mapsto d(gx, x)$ is a length function on $G$. It is known that $G$ admits a proper length function if and only if $G$ is $\sigma$-compact (see Section 2 in [Co2]). In the next definition, the equivalence of the two statements is Proposition 1.2 in [Co2].

Definition 1.1. (see [Co2]) A locally compact group $G$ has property PL if every locally bounded action of $G$ by isometries is either bounded or proper; equivalently, every length function on $G$ is either bounded or proper.
For $p \geq 1$, a length function $L$ on $G$ is a $L^p$-type length function if it comes from a continuous affine isometric action $\alpha$ of $G$ on some $L^p$-space $L^p(X, \mu)$, i.e. $L(g) = \|\alpha(g)x - x\|$ for some $x \in L^p(X, \mu)$. In the terminology of Definition 6.5 in [CDH], the invariant kernel $(g, h) \mapsto L(g^{-1}h)^p$ has type $p$ on $G$.

**Definition 1.2.** For $p \geq 1$, a locally compact group $G$ has property $BP_{L^p}$ if every affine isometric action of $G$ on a $L^p$-space is either bounded or proper; equivalently every $L^p$-type length function on $G$ is either bounded or proper.

Recall from [BFGM] that $G$ has property $FL^p$ if every continuous affine isometric action of $G$ on a $L^p$-space, has a fixed point. Obviously property $BP_{L^p}$ is implied both by property PL and by property $FL^p$.

A surprising fact, discovered by Y. Shalom ([Sh1], Theorem 3.4), is that simple Lie groups with finite center have property $BP_{L^2}$. Since those have property $FH$ except when locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$, this is really a statement about the latter two classes of groups. A stronger result was proved by Y. Cornulier ([Co2], Theorem 1.4): property PL holds for all simple algebraic groups over a local field. In the same paper, it is also proved that certain semi-direct products have property PL, e.g. $\mathbb{R}^d \rtimes K$, where $K$ is a closed subgroup of the orthogonal group $O(d)$ acting transitively on the unit sphere (see Proposition 1.8 in [Co2]); or $F \rtimes A^\times$, where $F$ is a non-Archimedean local field and $A^\times$ is the invertible group of its ring of integers (Proposition 1.9 in [Co2]).

The aim of the present paper is to investigate the relation between properties PL and $BP_{L^p}$. It was a surprise for us that, for some interesting classes of groups, they turn out to be equivalent. For example, for abelian groups, both are equivalent to compactness:

**Theorem 1.3.** Let $A$ be a locally compact abelian (LCA) group. The following are equivalent:

a) $A$ has property PL;

b) for some (resp. every) $p \geq 1$ the group $A$ has property $BP_{L^p}$;

c) $A$ is compact.

For amenable locally compact groups, we have:

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1Property $FL^2$ is more commonly denoted by FH and, for $\sigma$-compact locally compact groups, property FH is equivalent to Kazhdan’s property (T); see [BHV] for all this.

2For isometric actions which are continuous, not just locally bounded, an even stronger result holds for simple algebraic groups over local field: a continuous isometric action either is proper or has a globally fixed point, see Theorem 6.1 in [BG].
Theorem 1.4. Let $G$ be a locally compact group admitting a closed co-compact normal subgroup $V$ such that $V \cong F^d$ with $F$ a local field of characteristic 0 and $d > 0$ (so that the compact group $G/V$ acts on $V$). The following are equivalent:

a) $G$ has property PL;

b) for some (resp. every) $p \geq 1$ the group $G$ has property $BP_{L^p}$;

c) $G/V$ is infinite and it acts irreducibly on $V$.

Notation: We denote by $\mathcal{G}_{LA}$ the union of the class of almost connected Lie groups and the class of groups of the form $G(F)$, the group of $F$-rational points of a linear algebraic group $G$ defined over a non-Archimedean local field $F$ with characteristic 0.

Theorem 1.5. Let $G$ be an amenable non-compact group in $\mathcal{G}_{LA}$. The following are equivalent:

a) $G$ has property PL;

b) for some (resp. every) $p \geq 1$ the group $G$ has property $BP_{L^p}$;

c) there exists a compact normal subgroup $W$ of $G$ such that $H := G/W$ has a closed co-compact subgroup $V$ isomorphic to $F^d$ for some $d \geq 1$, with $H/V$ infinite and acting irreducibly on $V$.

For a group $G$, we denote by $Ad(G)$ the image of $G$ in its group of inner automorphisms. For non-amenable groups we have:

Theorem 1.6. Let $G$ be a non-amenable locally compact group which is either an almost connected Lie group or a linear algebraic group over a local field of any characteristic. The following are equivalent:

a) $G$ has property PL;

b) every closed normal subgroup of $G$ is either compact or co-compact;

c) there exists a compact normal subgroup $W$ of $G$ such that $G/W$ admits a closed, co-compact, normal subgroup $N$ which is isomorphic to a direct product $S_1 \times \ldots \times S_n$ of simple groups, and the simple factors $S_1, \ldots, S_n$ are permuted transitively under $Ad(G)$.

The previous result actually holds under weaker assumptions on $G$, see Theorem 5.1 for the precise statement.

The linear algebraic groups with property FH have been characterized by S.P. Wang [Wan]. So to classify linear algebraic groups with property $BP_{L^2}$, we may assume that they do not have property FH.
Theorem 1.7. Let $G$ be a group in $G_{LA}$. Assume that $G$ does NOT have property FH and is non-amenable. The following are equivalent:

a) The group $G$ has property $BP_{L^2}$.

b) $G$ admits a finite normal subgroup $W$ such that $G/W$ admits a closed, co-compact, normal subgroup $N$ which is isomorphic to a direct product $S_1 \times ... \times S_n$ of simple groups, and the simple factors $S_1, ..., S_n$ are permuted transitively under $\text{Ad}(G)$. Moreover, if $G = G(F)$ with $F$ is non-Archimedean, each simple factor of $N$ is a simple algebraic group of rank 1 over $F$; if $G$ is Lie almost connected, each simple factor of $N$ is locally isomorphic to $SO(n,1)$ or $SU(m,1)$.

Finally, we prove a general result about centers of $BP_{L^p}$-groups.

Theorem 1.8. Fix $p > 1$. Let $G$ be a compactly generated locally compact group satisfying property $BP_{L^p}$ but not property $FL^p$. Then the center of $G$ is compact.

In a previous paper [CTV], property $BP_0$ was introduced for a locally compact group $G$: it means that $G$ satisfies the bounded/proper alternative for affine isometric actions on Hilbert spaces, such that the linear part is a $C_0$, or mixing, representation. The class of groups with $BP_0$ is significantly larger than the class of groups with $BP_{L^2}$. For example, it was proved in [CTV] that every group with non-compact center (in particular every abelian group) has $BP_0$.

The paper is structured as follows. Section 2 contains generalities on property $BP_{L^p}$. In particular we prove that, for $1 \leq p \leq 2$ and $G$ locally compact separable, property $BP_{L^p}$ is equivalent to every action of $G$ on a space with measured walls being bounded or proper (Proposition 2.8). Section 3 contains generalities on property $PL$. Theorems 1.3, 1.4, 1.5 and 1.8 are proved in section 4, which is the core of the paper. Theorem 1.6 is proved in section 5. Section 6 deals specifically with property $BP_{L^2}$: we prove Theorem 1.7 and make in Proposition 6.2 the connection with the Howe-Moore property, by proving that it implies property $BP_{L^2}$. This provides a new proof of the already mentioned Theorem 3.4 in [Sh1], stating that $SO(n,1)$ and $SU(n,1)$ have property $BP_{L^2}$ (the original proof used the Mautner phenomenon). Since property $BP_{L^2}$ is implied both by property PL and the Howe-Moore property, it is natural to ask for any relationship between PL and Howe-Moore, and this is an interesting open question. In the appendix, Corina Ciobotaru shows that a closed non-compact subgroup of the automorphism group of the $d$-regular tree ($d \geq 3$) that acts 2-transitively on the boundary, satisfies property PL. As a consequence of her result, all known examples of groups with the Howe-Moore property (see [Cio]) have property PL.
This paper is a natural continuation of [CTV, CCLTV], but can be read independently.

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2 Generalities on property $BP_{L^p}$

The two next results follow immediately from definitions.

**Proposition 2.1.** Let $G$ be a locally compact group, and $N$ a closed normal subgroup.

1) If $G$ has property $BP_{L^p}$, then so does $G/N$.

2) If $G$ has property $BP_{L^p}$, and $N$ is not compact, then $G/N$ has property $FL_{L^p}$.

3) If $G/N$ has property $BP_{L^p}$, and $N$ is compact, then $G$ has property $BP_{L^p}$.

□

**Proposition 2.2.** Let $H$ be a closed co-compact subgroup in $G$. If $H$ has property $BP_{L^p}$, then so does $G$.

□

**Example 2.3.** Let $N = SL_2(R) \times SL_2(R)$, and let $\mathbb{Z}/2\mathbb{Z}$ act on $N$ by exchanging factors. Form the semi-direct product $G = N \rtimes \mathbb{Z}/2\mathbb{Z}$. Clearly $N$ does not have Property $BP_{L^2}$, but $G$ has Property $BP_{L^2}$ by Theorem 1.4. This example shows that Property $BP_{L^2}$ is not inherited by finite index subgroups.

**Example 2.4.** Let $G$ be the universal covering group of $SU(n,1)$ ($n \geq 1$). For every $p \geq 1$, the group $G$ does not have property $BP_{L^p}$, by Proposition 2.1 (since the quotient $G/Z(G)$ of $G$ by the non-compact normal subgroup $Z(G)$, does not have property $FL_p$). This shows that property $BP_{L^p}$ is not inherited by non-trivial central extensions.

**Remark 2.5.** There are plenty of discrete groups with Property $BP_{L^2}$ provided by discrete groups with Property FH. But we do not know any example of a discrete group with Property $BP_{L^2}$ but without Property FH. It is a result by
Peterson-Thom (Theorem 2.6 in [PT]) that, if $G$ is a countable group with non-zero first $L^2$-Betti number, containing some infinite amenable subgroup (e.g. $\mathbb{Z}$), then there exists a 1-cocycle with respect to the regular representation, which is neither bounded or proper; so such a group does not have property $BP_{L^2}$, nor the weaker property $BP_0$.

Since a locally compact group admitting a proper isometric action on some metric space must be $\sigma$-compact, we have in particular:

**Proposition 2.6.** A group with property $BP_{L^p}$ but without property $FL^p$, is $\sigma$-compact. □

Recall that a locally compact group $G$ is *locally elliptic* if every compact subset is contained in a compact subgroup. Observe that an locally elliptic group is amenable, as a direct limit of compact groups. For an arbitrary locally compact group, the *locally elliptic radical* $R_{\text{ell}}(G)$ is the unique maximal locally elliptic closed normal subgroup of $G$; see Example 4.D.7(7) in [CH].

**Proposition 2.7.** Let $G$ be a $\sigma$-compact group with property $BP_{L^p}$. If $G$ is not compactly generated, then $G$ is locally elliptic and not almost connected.

**Proof:** Let $K$ be a compact set in $G$, and let $U$ be the closed subgroup generated by $K$. Upon replacing $K$ by its union with a compact neighborhood of the identity, we may assume that $U$ is an open subgroup. Let $(K_n)_{n \geq 0}$ be an increasing sequence of compact subsets of $G$, with $K = K_0$ and $G = \bigcup_{n \geq 0} K_n$, and let $U_n$ be the subgroup generated by $K_n$. By assumption $U_n \neq G$. As explained e.g. in the proof of Proposition 2.4.1 of [BHV], the set $T = \bigsqcup_{n \geq 0} G/U_n$ carries a natural $G$-invariant tree structure such that, for the $G$-representation $\pi$ on $\ell^p$ of the set of oriented edges, there is an unbounded 1-cocycle $b \in Z^1(G, \pi)$. Actually $\|b(g)\|_p = d(gx_0, x_0)$, where $d(.,.)$ is the distance in $T$ and $x_0$ is the trivial coset in $G/U$ (see Proposition 2.3.3 in [BHV]). By property $BP_{L^p}$, this unbounded cocycle $b$ must be proper, in particular vertex stabilizers in $T$ must be compact. So $U = \{ g \in G : gx_0 = x_0 \}$ is compact, i.e. $G$ is locally elliptic. It then follows from Proposition 4.D.3 in [CH], that the connected component of identity in $G$ is compact. □

Recall from [CMV] that a *space with measured walls* is a 4-tuple $(X, W, B, \mu)$ where $W$ is a set of walls on $X$ (i.e. partitions of $X$ into 2 classes), $B$ is a $\sigma$-algebra of subsets of $W$ and $\mu$ is a measure on $B$ such that, for any $x, y \in X$, the set $W(x|y)$ of walls separating $x$ from $y$ is in $B$ and has finite measure.

The kernel $(x, y) \mapsto \mu(W(x|y))$ is then a pseudo-metric on $X$, called the *wall distance*. 
Proposition 2.8. For a locally compact group $G$ and $p \geq 1$, consider the following statements:

1) $G$ has property $BP_{L^p}$.

2) Every action of $G$ on a space with measured walls $X$ is either bounded or proper (when $X$ is endowed with the wall distance).

Then $(1) \Rightarrow (2)$. The converse holds if $1 \leq p \leq 2$ and $G$ is separable.

Proof: $(1) \Rightarrow (2)$ follows essentially from the proof of Proposition 3.1 in [CTV] and the remark following it. We recall the main features. If $G$ acts on a space with measured walls $X$, and $x_0$ is some base-point in $X$, there is an affine isometric action $\alpha_X$ of $G$ on $L^p$ of the space of half-spaces in $X$, such that $\|\alpha_X(g)(0)\|_p = \mu(W(gx_0|x_0))$, the measure of the set of walls separating $gx_0$ from $x_0$. So the $G$-action on $X$ is proper (resp. bounded) if and only if $\alpha_X$ is proper (resp. bounded).

$(2) \Rightarrow (1)$. This is a combination of results from [CDH]. Assume $1 \leq p \leq 2$ and $G$ separable, and let $\alpha$ be an affine isometric action of $G$ on $L^p(\Omega, \nu)$. Fix $v \in L^p(\Omega, \nu)$ and set $\psi(g) = \|\alpha(g)v - v\|_p^p$. In the terminology of Definition 6.5 in [CDH], the invariant kernel $(g, h) \mapsto \psi(g^{-1}h)$ has type $p$. By Corollary 6.11(1) in [CDH], the function $\psi$ is conditionally negative definite on $G$ (because $1 \leq p \leq 2$). By Theorem 6.25(2) in [CDH], since $G$ is separable there exists a median space $(X, d)$, a point $x_0 \in X$ and a continuous isometric $G$-action on $X$ such that $\sqrt{\psi(g)} = d(gx_0, x_0)$ for every $g \in G$. Finally, by Theorem 5.1 in [CDH], because $X$ is median it carries a structure of space with measured walls $(X, W, B, \mu)$ such that $d(x, y) = \mu(W(x|y))$ for every $x, y \in X$, and every isometry of $X$ is an automorphism of $(X, W, B, \mu)$. So $\psi$ is bounded (resp. proper) if and only if the $G$-action on $(X, W, B, \mu)$ is bounded (resp. proper). \qed

3 Generalities on property PL

Let $G$ be a locally compact $\sigma$-compact group. Observe that, if $G$ has property PL, then every closed normal subgroup is either compact or co-compact. If $G$ is amenable with property $BP_{L^p}$ for some $p \geq 1$, and $N$ is a closed non-compact normal subgroup, then $G/N$ is both amenable and property $FL_p$, so $G/N$ is compact. So also in this case every closed normal subgroup of $G$ is either compact or co-compact.

In Theorem E of [CM], Caprace and Monod obtained structural results for compactly generated, locally compact groups $G$ with the property that every
non-trivial closed normal subgroup is co-compact. If $G$ is not compact, then $G$ falls into one of the following three cases:

1. $G$ is isomorphic to a semi-direct product $\mathbb{R}^d \rtimes K$ where $K$ is a compact subgroup of $\text{GL}_d(\mathbb{R})$ acting irreducibly on $\mathbb{R}^d$;

2. $G$ is a compact extension of a quasi-product of finitely many non-compact, pairwise isomorphic, topologically simple groups, permuted transitively by $\text{Ad}(G)$;

3. $G$ is discrete and residually finite.

**Lemma 3.1.** Let $G$ be a non-compact, locally compact group. Assume either that $G$ has property PL, or that $G$ is amenable with property $\text{BP}_{L^p}$ for some $p \geq 1$.

a) If $G$ is not compactly generated, then $G$ is locally elliptic.

b) If $G$ is compactly generated, then $R_{\text{ell}}(G)$ is compact and every non-trivial closed normal subgroup of $G/R_{\text{ell}}(G)$ is co-compact.

**Proof:** If $G$ is not compactly generated, the result follows from Proposition [2.7](as PL implies $\text{BP}_{L^p}$). If $G$ is compactly generated, then every closed normal subgroup of $G$ is either compact or co-compact: this is obvious if $G$ has property PL; if $G$ is amenable with property $\text{BP}_{L^p}$, this follows from the fact that any non-compact compactly generated amenable group admits a proper action on $L^p$. In particular $R_{\text{ell}}(G)$ is compact, and $G/R_{\text{ell}}(G)$ is a non-compact group, without non-trivial compact normal subgroup, and every non-trivial closed normal subgroup co-compact. \(\square\)

For almost connected Lie groups, Lemma 3.1 cleans things up, as those are compactly generated. An immediate consequence of Lemma 3.1 and the Caprace-Monod theorem, is:

**Proposition 3.2.** Let $G$ be a non-compact almost connected real Lie group. Assume that $G$ either has property PL, or that $G$ is amenable with property $\text{BP}_{L^p}$ for some $p \geq 1$. There is a compact normal subgroup $W$ of $G$ such that:

a) if $G$ is amenable, then $G/W$ is isomorphic to a semi-direct product $\mathbb{R}^d \rtimes K$ where $K$ is a compact subgroup of $\text{GL}_d(\mathbb{R})$ acting irreducibly on $\mathbb{R}^d$;

b) if $G$ is non-amenable, then $G/W$ is a compact extension of a product of finitely many non-compact, pairwise isomorphic, simple Lie groups, permuted transitively by $\text{Ad}(G)$. \(\square\)
4 Amenable groups

4.1 Semi-direct products

For $A$ a LCA group, we denote by $\hat{A}$ its Pontryagin dual, and by $1_A \in \hat{A}$ the trivial character. Set $\hat{A}^* = \hat{A} \setminus \{1_A\}$.

**Proposition 4.1.** Fix $p \geq 1$. Let $A$ be a LCA group, and let $K$ be a compact group of automorphisms. Let $\mu$ be an infinite $K$-invariant Radon measure on $\hat{A}^*$. Assume that, for every compact subset $C \subset A$, we have

$$\sup_{a \in C} \int_{\hat{A}^*} |\chi(a) - 1|^p \, d\mu(\chi) < +\infty. \quad (1)$$

For $(a, k) \in A \rtimes K$, set:

$$L(a, k) = \left(\int_{\hat{A}^*} |\chi(a) - 1|^p \, d\mu(\chi)\right)^{1/p}.$$

Then $L$ is an unbounded $L^p$-type length function on $A \rtimes K$.

**Proof:** Let $F_\mu$ be the space of $\mu$-measurable functions on $\hat{A}^*$, modulo equality $\mu$-almost everywhere. We define a linear representation $\pi_\mu$ of $A \rtimes K$ on $F_\mu$ by

$$(\pi_\mu(a, k)f)(\chi) = \chi(a)f(k^{-1} \cdot \chi)$$

for $(a, k) \in A \rtimes K, f \in F_\mu, \chi \in \hat{A}^*$. Observe that $\pi_\mu$ has no non-zero fixed vector. View the space $L^p(\hat{A}^*, \mu)$ as a subspace of $F_\mu$: it is invariant under $\pi_\mu$, that induces an isometric representation of $A \rtimes K$ on $L^p(\hat{A}^*, \mu)$. Let $t$ be the translation by the constant function $1$ on $F_\mu$, so that $t(f) = f + 1$ for $f \in F_\mu$. Define an affine action $\alpha_\mu$ of $A \rtimes K$ on $F_\mu$ by:

$$\alpha_\mu = t^{-1} \circ \pi_\mu \circ t.$$

More precisely, for $(a, k) \in A \rtimes K, f \in F_\mu, \chi \in \hat{A}^*$:

$$(\alpha_\mu(a, k)f)(\chi) = (\pi_\mu(a, k)f)(\chi) + \chi(a) - 1.$$

Observe that the constant function $-1$ is the only fixed point of $\alpha_\mu$ in $F_\mu$.

By assumption $\chi \mapsto \chi(a) - 1$ is in $L^p(\hat{A}^*, \mu)$ for every $a \in A$, so $L^p(\hat{A}^*, \mu)$ is $\alpha_\mu$-invariant, and $\alpha_\mu$ defines a continuous affine isometric action of $A \rtimes K$ on $L^p(\hat{A}^*, \mu)$. Then $L(a, k) = \|\alpha_\mu(a, k)(0)\|$ is indeed a $L^p$-type length function on $A \rtimes K$. Since $-1 \notin L^p(\hat{A}^*, \mu)$, the action $\alpha_\mu$ has no fixed point in $L^p(\hat{A}^*, \mu)$, so that $L$ is unbounded. Indeed, this follows from the fact that an isometric action on $L^p$ with bounded orbits fixes a point: this is a consequence of the center lemma for $p > 1$, and of [BGM] for $p = 1$.

□
Remark 4.2. Suppose that $A = \mathbb{R}^d$; every continuous character on $\mathbb{R}^d$ is of the form $x \mapsto \exp(2\pi i \langle x | y \rangle)$, for some $y \in \mathbb{R}^d$ (where $\langle . | . \rangle$ denotes the usual scalar product), so for $(x, k) \in \mathbb{R}^d \rtimes K$ we have

$$L(x, k)^p = \int_{(\mathbb{R}^d)^*} \left| \exp(2\pi i \langle x | y \rangle) - 1 \right|^p d\mu(y) = 2^{p/2} \int_{(\mathbb{R}^d)^*} (1 - \cos(2\pi \langle x | y \rangle))^p/2 d\mu(y)$$

$$= 2^p \int_{(\mathbb{R}^d)^*} \left| \sin^p(\pi \langle x | y \rangle) \right| d\mu(y).$$

Using $|\sin t| \leq |t|$ and the Cauchy-Schwarz inequality, we see that, to ensure the finiteness condition (1), it is sufficient that $\mu$ has a $p$-th moment, i.e. $\int_{(\mathbb{R}^d)^*} \|y\|^p d\mu(y) < +\infty$.

We will apply Proposition 4.1 to semi-direct products $A \rtimes F$, with $F$ finite. In the case $A = \mathbb{Z}$ and $F$ trivial, the next result is due to Edelstein (Theorem 2.1 in [Ede]); for $A = \mathbb{R}$ and $F$ trivial, see Corollary 5.3 in [CTV].

Theorem 4.3. Let $A$ be a non-compact, $\sigma$-compact LCA group, and let $F$ be a finite group of automorphisms of $A$. For every $p \geq 1$, the semi-direct product $A \rtimes F$ does not have property $BP_{L^p}$.

Proof: We will use Proposition 4.1 to construct a specific unbounded $L^p$-type length function on $A \rtimes F$, which will turn out to be not proper. Let $(K_n)_{n>0}$ be a strictly increasing sequence of compact subsets of $A$, with $A = \bigcup_{n>0} K_n$. Clearly we may assume that each $K_n$ is $F$-invariant. For $a \in A$, let $|a|$ be the unique integer $n > 0$ such that $a \in K_n \setminus K_{n-1}$, so that $K_n = \{a \in A : |a| \leq n\}.$

Claim: There exists sequences $(a_n)_{n>0}$ in $A$, and $(\chi_n)_{n>0}$ in $\hat{A}^*$ such that:

- $|a_n| > |a_{n-1}|$;
- for $f \in F, k \leq n$ we have: $|(f \cdot \chi_k)(a_n) - 1| \leq 2^{-n};$
- $\max_{|a| \leq |a_n|} |\chi_n(a) - 1| \leq 2^{-n}$ for every $n > 0$.

Taking the Claim for granted, we define the measure $\mu$ on $\hat{A}^*$ as a sum of Dirac masses:

$$\mu = \sum_{f \in F} \sum_{k>0} \delta_{(f \cdot \chi_k)}.$$  

Then $\mu$ is an infinite $F$-invariant Radon measure on $\hat{A}^*$. Moreover for $a \in K_n$ we have a uniform bound:

$$\int_{\hat{A}^*} |\chi(a) - 1|^p d\mu(\chi) = \sum_{f \in F} \sum_{k=1}^n |(f \cdot \chi_k)(a) - 1|^p + \sum_{f \in F} \sum_{k>n} |(f \cdot \chi_k)(a) - 1|^p$$

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\[ \leq 2^n |F| + |F| \sum_{k>n} 2^{-pk} \]

where the inequality follows from the Claim and \( K_n \subset K[a_n] \subset K[a_{k-1}] \) for \( k > n \).

Then by Proposition 4.1,

\[ L(a, g) = \left\| \sum_{f \in F} \sum_{k=0}^n (|f \cdot \chi_k|^2 - 1|p|^{1/p} \right\| \]

((a, g) \in A \times F) defines an unbounded \( L^p \)-type length function on \( A \times F \). To show that it is not proper, we show that \( L \) remains bounded along the unbounded sequence \( (a_n, Id_A)_{n>0} \) in \( A \times F \). But by the Claim:

\[ L(a_n, Id_A)^p = \left(\sum_{f \in F} \sum_{k=0}^n (|f \cdot \chi_k|^2 - 1|p|^{1/p} \sum_{f \in F} \sum_{k>n} (|f \cdot \chi_k|^2 - 1|p|^{1/p} \leq |F| \cdot n \cdot 2^{-m} + |F| \sum_{k>n} 2^{-pk} \]

as \( |a_n| \leq |a_{k-1}| \) for \( k > n \). So \( \lim_{n \to \infty} L(a_n, Id_A) = 0 \), so the sequence \( (L(a_n, Id_A))_{n>0} \) is bounded.

It remains to prove the Claim. Assume that \( a_1, \ldots, a_{n-1} \in A \) and \( \chi_1, \ldots, \chi_{n-1} \in \hat{A}^* \) have been constructed. As \( A \) is not compact, so that \( \hat{A} \) is not discrete, we find \( \chi_n \in \hat{A}^* \) such that \( \max_{|\alpha| \leq m} |\chi_n(\alpha) - 1| \leq 2^{-n} \). Let \( H \) be the subgroup of \( \hat{A} \) generated by the \( f \cdot \chi_k \)'s, with \( f \in F, k \leq n \). Let \( \iota \) denote the inclusion of \( H \) into \( \hat{A} \). We then have the dual homomorphism \( \iota^* : \hat{A} = A \to \hat{H} \) with \( \hat{H} \) compact. Since \( \iota^* \) has dense image (see Corollaire 6 in Chap II.1.7 of [Bou]), and \( A \) is not compact, in any complement of a compact set in \( A \) we can find \( a \) with \( \iota^*(a) \) arbitrarily close to the trivial element in \( \hat{H} \). In particular we can find \( a_n \) with \( |a_n| > |a_{n-1}| \) with \( |(\iota^*(a_n))(f \cdot \chi_k) - 1| \leq 2^{-m} \), i.e. \( |(f \cdot \chi_k)(a_n) - 1| \leq 2^{-m} \) for \( f \in F, k \leq n \).

**Remark 4.4.** Say that \( A = \mathbb{R}^d \) in Theorem [4.3] and assume that the finite group \( F \) stabilizes some proper, closed, unbounded subgroup \( B \) of \( \mathbb{R}^d \) (think of \( B \) as either a proper linear subspace, or a lattice). Then the proof of Theorem [4.3] becomes much simpler. Indeed let \( y_0 \in \mathbb{R}^d \) be a non-zero vector such that the character \( x \mapsto \exp(2\pi i \langle x, y_0 \rangle) \) is in the annihilator \( B^\perp = \{ \chi \in \hat{A} : \chi|_B \equiv 1 \} \). Set then \( \chi_k(x) = \exp(\frac{2\pi i}{k t^\perp} \langle x, y_0 \rangle) \) Then the measure \( \mu = \sum_{f \in F} \sum_{k>0} \delta(f \cdot \chi_k) \) on \( (\mathbb{R}^d)^* \) has finite \( p \)-th moment, so by remark [4.2] and Proposition [4.1] the function \( L(a, g) =: \sum_{f \in F} \sum_{k>0} |(f \cdot \chi_k(a)) - 1|^{p/2} \) defines an unbounded \( L^p \)-type length function on \( A \times F \). On the other hand, pick any non-zero vector \( a_0 \in B \), and
set $a_n = n! \cdot a_0$. We claim that $\lim_{n \to \infty} L(a_n, Id_A) = 0$. Indeed observe that $(f \cdot \chi_k)(a_n) = 1$ for $f \in F, k \leq n$, so that as in remark 4.2:

$$L(a_n, Id_A)^p = \sum_{f \in F} \sum_{k > n} |(f \cdot \chi_k)(a_n) - 1|^p = 2^p \sum_{f \in F} \sum_{k > n} |\sin^p(\pi \frac{n!}{k!}(a_0|y_0) )|$$

$$\leq (2\pi)^p \|a_0\|^p \|y_0\|^p \|F\| \sum_{k > n} \left(\frac{n!}{k!}\right)^p \leq (2\pi)^p \|a_0\|^p \|y_0\|^p \|F\| \sum_{k > n} \left(\frac{1}{k}\right)^p (n-k)$$

using the bound $\frac{n!}{k!} < \frac{1}{n-k}$ for $k > n$. So we have

$$L(a_n, Id_A)^p \leq (2\pi)^p \|a_0\|^p \|y_0\|^p \|F\| \cdot \frac{1}{n^p - 1},$$

establishing the claim. The choice of the weights in defining $\mu$ and the sequence $(a_n)_{n>0}$, is inspired by the proof of Theorem 2.1 in [Ede]. We will come back to finite groups stabilizing a lattice in $\mathbb{R}^d$, in Corollary 6.3 below.

Here is a noteworthy consequence of Theorem 4.3.

**Corollary 4.5.** Let $\Gamma$ be an infinite, finitely generated, virtually abelian group. Then $\Gamma$ does not have property $BP_{LP}$, for every $p \geq 1$.

**Proof:** Write $\Gamma$ as the central term of a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} F \to 1$$

(2)

with $F$ finite. We claim that $\Gamma$ embeds as a co-compact lattice in a semi-direct product $G = \mathbb{R}^n \rtimes F$. Since $G$ does not have property $BP_{LP}$ (by Theorem 4.3, the Corollary follows from Proposition 2.2).

To prove the claim, let $c \in Z^2(F; \mathbb{Z}^n)$ be the 2-cocycle on $F$ describing the extension (2). The group $\mathbb{Z}^n$ becomes an $F$-module through the conjugation action of $F$, and $c(g,g) = s(g)s(g')s(gg')^{-1}$ for some section $s : F \to \Gamma$ of the map $p$. Now the $F$-action on $\mathbb{Z}^n$ canonically extends to $\mathbb{R}^n$, and the law

$$(v, g)(v', g') = (v + g \cdot v' + c(g,g'), gg') \quad (v \in \mathbb{R}^n, g \in F)$$

defines on $G = \mathbb{R}^n \rtimes F$ the structure of an almost connected Lie group in which $\Gamma$ embeds as a co-compact lattice. Since $H^2(F; \mathbb{R}^n) = 0$, the extension

$$0 \to \mathbb{R}^n \to G \to F \to 1$$

splits, so that $G = \mathbb{R}^n \rtimes F$. This proof was inspired by the proof of Theorem 1 in [AK].
4.2 Abelian groups

The following Lemma can be deduced from the proof of Proposition 2.5.9 in [BHV] (where it is proved for solvable groups and $p = 2$). We include the simple proof for locally compact abelian (LCA) groups.

**Lemma 4.6.** Fix $p \geq 1$. A LCA group $A$ has property $FL^p$ if and only if $A$ is compact.

**Proof:** One implication is trivial. For the non-trivial one, let $A$ be a LCA group with property $FL^p$. We consider two cases:

- $A$ is discrete. Assume by contradiction that $A$ is infinite. Since every infinite abelian group has a countably infinite quotient (see Theorem 2.5.2 in [Rud]), we may assume that $A$ is countably infinite with property $FL^p$. But a countable group with property $FL^p$ is finitely generated (by Corollary 2.4.2 in [BHV]), hence $A$ is isomorphic to $\mathbb{Z}^n \oplus F$, with $n > 0$ and $F$ finite abelian. But such a group does not have property $FL^p$, so a contradiction is reached.

- $A$ is arbitrary. By structure theory for LCA groups (see Theorem 2.4.1 in [Rud]), $A$ admits an open subgroup $U$ of the form $U = K \times \mathbb{R}^m$, for some $m \geq 0$ and some compact group $K$. The group $A/U$ is discrete with property $FL^p$, so it is finite by the first case of the proof, i.e. $U$ has finite index in $A$. So $U$ has property $FL^p$ too (by Proposition 2.5.7 in [BHV]). This clearly forces $m = 0$, so $A$ is compact. □

**Proof of Theorem 1.3.** Implications $(c) \Rightarrow (a) \Rightarrow (b)$ are clear. We prove $(b) \Rightarrow (c)$ by contradiction. So suppose there is a non-compact LCA group $A$ with property $BP_{LP}$, for some $p \geq 1$. As $A$ is not compact, $A$ does not have property $FL^p$, by Lemma 4.6. Since $A$ has property $BP_{LP}$, the group $A$ must be $\sigma$-compact, by Proposition 2.6. But this contradicts Theorem 1.3. □

**Corollary 4.7.** Let $G$ be a locally compact group with property $BP_{LP}$, for some $p \geq 1$. Then $G/[G,G]$ is compact. □

4.3 Centers: proof of Theorem 1.8

Let $S$ be a compact generating subset of $G$. Since $G$ does not have property $FL^p$, it admits an affine isometric action $\sigma$ on some $L^p$-space $E$ with non-zero
displacement:

$$\inf_{x \in E} \sup_{s \in S} \|\sigma(s)x - x\| > 0.$$  

Indeed, this follows from [Gro 3.8.D], taking the “energy” $E(x)$ to be $\sup_{s \in S} \|\sigma(s)x - x\|$. In other words, letting $\pi$ and $b$ be respectively the linear part and cocycle part of $\sigma$, we have that $b$ is non-trivial in reduced first cohomology. Let $Z(G)$ denote as usual the center of $G$. By [BFGM Proposition 2.6], we have a $\pi(G)$-invariant continuous decomposition $E = E_1 \oplus E_2 \oplus E_3$, where $E_1$ is the space of $\pi(G)$-invariant vectors, and $E_1 \oplus E_2$ the space of $Z(G)$-invariant vectors. The projection $b_1$ of $b$ on $E_1$ is a group homomorphism $G \to E_1$, which by Corollary [4.7] is zero. Observe then that the projection $b_2$ of $b$ on $E_2$ vanishes on $Z(G)$. Indeed, the cocycle relation shows that $b_2(z)$ is a $\pi(G)$-invariant vector for all $z \in Z(G)$. So assuming by contradiction that $Z(G)$ is not compact, by property $BP_{LP}$ the affine action corresponding to $b_2$ is bounded, so it has a fixed point and $b_2$ is a coboundary. Finally, as the center $Z(G)$ is non-compact, the projection $b_3$ is trivial in $\mathcal{H}^1(\pi, E)$ by [BRS Corollary 5], finally implying that $b$ itself is an almost coboundary: this is a contradiction. \[\square\]

Note that the above proof really needs $p > 1$ to appeal to Proposition 2.6 of [BFGM] and to Corollary 5 of [BRS].

4.4 Proof of Theorem 1.4

Lemma 4.8. Let $F$ be a local field of characteristic 0, $V = F^d$, and $K$ an infinite compact subgroup of $GL(V)$, acting irreducibly on $V$. Denote by $\mathfrak{k}$ the Lie algebra of $K$. For every non-zero $x \in V$, there exists $X \in \mathfrak{k}$ such that $Xx \neq 0$.

Proof: Contraposing, we assume that there is a non-zero vector $x \in V$ such that $Xx = 0$ for every $X \in \mathfrak{k}$, and will show that $K$ is finite. Let $W$ be the space of vectors $v \in V$ such that $Xv = 0$ for every $X \in \mathfrak{k}$: this is clearly a $K$-invariant subspace, so by irreducibility we have $W = V$. This implies $\mathfrak{k} = 0$ and hence $K$ is finite. \[\square\]

The next lemma says that, if $G$ is as in Theorem 1.4, it is close to being a semi-direct product.

Lemma 4.9. Let $G$ be as in Theorem 1.4. There exists a compact subgroup $C$ of $G$ such that $G = VC$.

Proof: If $F = \mathbb{R}$, it is a classical fact (see Theorem 2.3 in Chapter III of [Hoc]) that any extension of a finite-dimensional real vector space by a compact group, is split. For $F$ non-Archimedean, we observe that $V$ is locally elliptic and
appeal to the fact that local ellipticity is preserved by extensions (see Proposition 4.D.4(2) in [CH]): hence $G$ is locally elliptic. So if $K$ is a compact set that surjects onto $G/V$ by the quotient map $G \to G/V$, the set $K$ is contained in a compact subgroup $C$ of $G$, and clearly $G = VC$.

**Example 4.10.** Let $G$ be the following closed subgroup of the Heisenberg group over $\mathbb{Q}_p$:

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{Z}_p, z \in \mathbb{Q}_p \right\}. $$

Then $G$ is a central extension of $V = \mathbb{Q}_p$ by $\mathbb{Z}_p^2$. The extension is not split as $G$ is not abelian. However we have $G = VC$ where $C$ is the Heisenberg group over $\mathbb{Z}_p$.

**Proof of Theorem 1.4:** The implication $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (c)$ Assume that $G$ has property $BP_{l,p}$. By Proposition 2.8 we may assume $p > 1$. By Lemma 4.9 we can write $G = VC$ for some compact subgroup $C$ of $G$. Observe that $V \cap C$ is normal in $G$. Denote by $\alpha : V \to V/(V \cap C)$ and $\beta : C \to C/(V \cap C)$ the quotient maps. Then the map

$$G \to V/(V \cap C) \times C/(V \cap C) : g = vc \mapsto (\alpha(v), \beta(c))$$

is well-defined and is a continuous surjective homomorphism. So the semi-direct product $V/(V \cap C) \times C/(V \cap C)$ has property $BP_{l,p}$ and Theorem 4.3 implies that $G/V \simeq C/(V \cap C)$ is infinite. If $W$ is a non-zero $G/V$-invariant linear subspace in $V$, then $W$ is a normal subgroup in $G$. By Proposition 2.11 the quotient $G/W$ has property $FL^p$, so it is compact as $G/W$ is also amenable. Hence $W = V$, i.e. $G/V$ acts irreducibly on $V$.

$(c) \Rightarrow (a)$ Set $K = G/V$. Assume that $K$ acts irreducibly on $V$ and is infinite, so that its Lie algebra $\mathfrak{k}$ is non-zero. We proceed in several steps.

- For $x \in V \setminus \{0\}$, set $\phi^x : K^d \to V : (g_1, \ldots, g_d) \mapsto g_1x + \ldots + g_dx$. We claim that the image of $\phi^x$ contains some open set in $V$. For this, it is enough to show that the differential $D\phi^x_{(g_1, \ldots, g_d)}$ has rank $d$ for some $(g_1, \ldots, g_d) \in K^d$.
  But for $(X_1, \ldots, X_d) \in \mathfrak{k}^d$ we have:

$$D\phi^x_{(g_1, \ldots, g_d)}(X_1, \ldots, X_d) = g_1X_1x + \ldots + g_dX_dx. $$

By Lemma 4.8 we find $X \in \mathfrak{k}$ such that $Xx \neq 0$. As $Xx$ is a cyclic vector for $K$ (because $K$ acts irreducibly), we find $g_1, \ldots, g_d \in K$ such that \{g_1Xx, \ldots, g_dXx\} is a basis of $V$. This means that $D\phi^x_{(g_1, \ldots, g_d)}$ has rank $d$. 

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• Endow $V = F^d$ with the $\ell^\infty$ norm $\|x\| = \max_{1 \leq i \leq d} |x_i|$. For $x \in V \setminus \{0\}$, set $\psi_x : K^{2d} \to V : (g_1, ..., g_d, h_1, ..., h_d) \mapsto \phi^x(g_1, ..., g_d) + \phi^{-x}(h_1, ..., h_d)$. By the previous point, the image of $\psi_x$ contains some open set around 0. Let $\epsilon_x$ denote the radius of the largest open ball centered at 0 and contained in the image of $\psi_x$. Since $\psi_x$ depends smoothly on $x$, the function $x \mapsto \epsilon_x$ on the unit sphere of $V$, is bounded below by some positive $\epsilon > 0$.

• As in Lemma 4.9, write $G = VC$. Let $L$ be a length function on $G$, let $N > 0$ be such that $L|_C \leq N$. For $x \in V, g \in C$, the relation $gxg^{-1} = g(x)$ in $G$ implies:

$$L(g(x)) \leq L(x) + 2N.$$ 

Assume that $L$ is not proper, so that there is a sequence $(x_n)_{n>0}$ in $V$, with $\|x_n\| \to \infty$, and a constant $M > 0$ such that $L(x_n) \leq M$ for every $n > 0$. Fix $y \in V$, and choose $n$ large enough so that $\|y\| < \epsilon \|x_n\|$. This implies that $y$ is in the image of $\psi^{x_n}$, say $y = \sum_{i=1}^{d} g_i(x_n) + \sum_{i=1}^{d} h_i(-x_n)$ for suitable $g_1, ..., g_d, h_1, ..., h_d \in K$. Then

$$L(y) \leq \sum_{i=1}^{d} L(g_i(x_n)) + \sum_{i=1}^{d} L(h_i(-x_n)) \leq 2d(L(x_n) + 2N)$$

$$\leq 2d(M + 2N).$$

Finally for $y \in V, c \in C$ we have: $L(yc) \leq L(y) + L(c) \leq 2d(M + 2N) + N$, meaning that $L$ is bounded.

\[\square\]

4.5 Proof of Theorem 1.5

The implication $(a) \Rightarrow (b)$ is trivial while $(c) \Rightarrow (a)$ follows from Theorem 1.4 together with the fact that property PL is stable under extensions with compact kernels (see Lemma 3.1 in [Co2]).

To prove $(b) \Rightarrow (c)$, let $G$ be non-compact amenable in $G_{LA}$, with property $BP_{L^p}$. We will repeatedly appeal to what Proposition 2.1 says for amenable groups: if a locally compact amenable group has property $BP_{L^p}$, then every closed normal subgroup is either compact or co-compact. The case of almost connected Lie groups follows from Proposition 3.2(a) and Theorem 4.3, so we may focus on the algebraic case, i.e. $G = G(F)$ with $F$ a non-Archimedean local field $F$ of characteristic 0. Let $\mathcal{O}$ be the valuation ring of $F$ and $\pi$ be a uniformizer, so that $F^x = \mathcal{O}^x \pi \mathbb{Z}$. 

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Let $G^0$ be the Zariski-connected component of identity, and $R_u(G^0)$ its unipotent radical. We proceed in two steps:

- We claim that the unipotent radical $R_u(G^0)$ is non-trivial. Suppose by contradiction that it is trivial, i.e. $G^0$ is reductive. Consider the Levi decomposition $G^0 = R(G^0)S$; as $G^0$ is non-compact amenable, $S$ is compact/anisotropic and the radical $R(G^0)$ is a non-compact torus, say $R(G^0) \simeq (F^\times)^r$ with $r > 0$. Let $T \simeq (O^\times)^r$ be the unique maximal compact subgroup of $R(G^0)$: then $TS$ is the unique maximal compact subgroup of $G^0$, so it is normal in $G$. The quotient $G/TS$ contains $Z_r$ with finite index. Because of the assumption $G/TS$ has property $BP_{L^p}$, contradicting Corollary 4.5.

- $[R_u(G^0), R_u(G^0)] = \{1\}$, i.e. $R_u(G^0)$ is abelian (otherwise $[R_u(G^0), R^i(G^0)]$ is a non-compact and non-co-compact closed normal subgroup in $G$). So $R_u(G^0) \simeq F^d$ for some $d \geq 1$. Since $R_u(G^0)$ is normal in $G$, it is co-compact. The result then follows from $(b) \Rightarrow (c)$ in Theorem 1.4.

5 Non-amenable groups: proof of Theorem 1.6

It is actually possible to weaken the assumption of Theorem 1.6 rather drastically. For this we need two more definitions.

A locally compact group is **locally linear** if it admits an open subgroup which is linear over some local field. We also define the class of **elementary groups** as the smallest class of locally compact totally disconnected groups containing all discrete groups, all profinite groups, and closed under group extensions and directed unions of open subgroups.

**Theorem 5.1.** Let $G$ be a non-amenable locally compact group. Assume moreover that $G$ is either an almost connected Lie group, or a non-elementary locally linear totally disconnected group. The following are equivalent:

a) $G$ has property PL;

b) every closed normal subgroup of $G$ is either compact or co-compact;

c) there exists a compact normal subgroup $W$ and a closed co-compact normal subgroup $N$ of $G$, with $W \trianglelefteq N$, such that $N/W$ is isomorphic to a direct product $S_1 \times \ldots \times S_n$ of simple algebraic groups over some local field $F$, and the simple factors $S_1, \ldots, S_n$ are permuted transitively under $Ad(G)$. 

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In particular Theorem 5.1 applies to any non-amenable group of the form $G(F)$, the group of $F$-rational points of a linear algebraic group $G$ defined over a non-Archimedean local field $F$ of any characteristic.

**Proof of Theorem 5.1:**

$(c) \Rightarrow (a)$ Let $G, W$ be as in $c)$. By Lemma 3.1 in [Co2], it is enough to show that $G/W$ has property PL. So let $L$ be an unbounded length function on $G/W$, we show that $L$ is proper. As $M := N/W$ is co-compact, $L|_M$ is unbounded. So there exists some index $i$ such that $L|_{S_i}$ is unbounded. Say $i = 1$. By assumption, for $j = 2, ..., r$, there exists $g_j \in G$ such that $Ad(g_j)(S_j) = S_1$. Then, for $s_j \in S_j$ we have by the triangle inequality: $L(g_j s_j g_j^{-1}) \leq L(s_j) + 2L(g_j)$, so that $L|_{S_j}$ is unbounded too. By Theorem 1.4 in [Co2], $L|_{S_i}$ is proper for every $i = 1, ..., r$. By Lemma 1.7 in [Co2], $L|_M$ is proper. So $L$ is proper.

$(a) \Rightarrow (b)$ We already observed that, in a locally compact $\sigma$-compact group with property PL, every closed normal subgroup is either compact or co-compact.

$(b) \Rightarrow (c)$ The Lie group case follows immediately from the already quoted Caprace-Monod theorem (Theorem E in [CM]) and the discussion preceding Proposition 3.2.

For the totally disconnected case, we appeal to a structural result by Caprace and Stulemeijer (Corollary 1.2 in [CS]): if $G$ is totally disconnected and locally linear, there exists closed characteristic subgroups $W \triangleleft N \triangleleft G$ such that $W$ is elementary, $N/W$ (if non-trivial) is a product $S_1 \times ... \times S_n$ of topologically simple algebraic groups over local fields $F_1, ..., F_n$ (in particular $S_i$ is compactly generated and abstractly simple), and $G/N$ is elementary. In view of the assumption that $G$ is non-elementary, $N/W$ is non-trivial, hence non-compact, in our case. If we assume that all closed normal subgroups of $G$ are either compact or co-compact, we get that $W$ is compact and $N$ is co-compact. Finally $Ad(G)$ acts transitively on the simple factors of $N/W$, since a proper orbit would allow to construct a closed normal subgroup of $G$ that is neither compact nor co-compact. \[\Box\]

6 **Property $BP_{L^2}$ in particular**

6.1 **Proof of Theorem 1.7**

The implication $(b) \Rightarrow (a)$ is clear: if $G$ has the described form, then by Theorem 1.6 the group $G$ has property PL, *a fortiori* it has property $BP_{L^2}$.

The proof of $(a) \Rightarrow (b)$ is very similar in spirit to the proof of $(b) \Rightarrow (c)$ in Theorem 1.6. Let $G$ be either an almost connected Lie group, or $G = G(F)$, a linear algebraic group over a local field $F$ of characteristic 0. Let $g$ be the
Lie algebra of $G$, let $\mathfrak{g} = \mathfrak{r} \times \mathfrak{s}$ be a Levi decomposition, write $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$, where $\mathfrak{s}_c$ stands for the compact/anisotropic factors, and $\mathfrak{s}_{nc}$ stands for the non-compact/isotropic factors. Assume $G$ non-amenable, so that $\mathfrak{s}_{nc} \neq 0$.

Suppose that $G$ has property $BP_{PL}^2$ but not FH. Then $G$ admits a proper isometric action on a Hilbert space, i.e. $G$ has the Haagerup property. By Theorem 1.10 in [Co1], this implies $[\mathfrak{r}, \mathfrak{s}_{nc}] = 0$, and all simple factors of $\mathfrak{s}_{nc}$ have $F$-rank 1, and are locally isomorphic to $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$ ($n \geq 2$) if $F = \mathbb{R}$ or if $G$ is Lie and almost connected. By property $BP_{PL}^2$, any quotient of $G$ by a closed non-compact normal subgroup, must have property FH (see Proposition 2.1). We now distinguish the two cases.

6.1.1 The algebraic case

Let $G^0$ be the Zariski-connected component of identity of $G$. The radical $R(G^0)$ is compact. Suppose not: as $R(G^0)$ is characteristic in $G^0$, it is a closed non-compact subgroup of $G$, and the quotient $G/R(G^0)$ does not have property FH, contradicting property $BP_{PL}^2$ of $G$.

Let $S_c$ (resp. $S_{nc}$) be the Zariski-connected subgroup of $G^0$ corresponding to $\mathfrak{s}_c$ (resp. $\mathfrak{s}_{nc}$). Since $\mathfrak{s}_{nc}$ is a characteristic ideal in $\mathfrak{g}$, the subgroup $S_{nc}$ is characteristic in $G^0$, hence normal and co-compact in $G$. Finally $Ad(G)$ acts transitively on the simple factors of $S_{nc}$, since a proper orbit would allow to construct a quotient of $G$ by a closed non-compact normal subgroup, not having property FH. Let $W = Z(S_{nc})$ be the center of $S_{nc}$; then the subgroup $N := S_{nc}/W$ is the desired subgroup of $G/W$.

6.1.2 The Lie case

Let $G$ be a non-amenable, almost connected Lie group with $BP_{PL}^2$ and without FH. Let $G^0 = RS$ be a Levi decomposition of the connected component of identity $G^0$ of $G$. Then $R$ is compact (otherwise, as above, there is a quotient by a non-compact normal subgroup, not having FH), so $R$ is a compact torus, and $G^0$ is reductive.

Let $S$ and $S_{nc}$ be the analytic subgroups corresponding to $\mathfrak{s}_c$ and $\mathfrak{s}_{nc}$ respectively. Note that $S_{nc}$ is closed in $G$ because $R$ is compact. The subgroup $S_{nc}$ is characteristic in $G^0$ hence normal in $G$, so $S_{nc}$ is co-compact in $G$. As above, $Ad(G)$ permutes the simple factors of $S_{nc}$ transitively.

Set $W := Z(S_{nc})$, the center of $S_{nc}$. Since $W$ is normal in $G$ and $G/W$ does not have property FH, the subgroup $W$ must be finite. As in the algebraic case, we set $N := S_{nc}/W$, and it is the desired subgroup of $G/W$. 

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6.2 Link with the Howe-Moore property

Let $H$ be a closed subgroup of the locally compact group $G$. We recall Definition 1.3 of [CCLTV]:

**Definition 6.1.** The pair $(G, H)$ has the relative Howe-Moore property if every unitary representation $\pi$ of $G$, either has $H$-invariant vectors, or is such that $\pi|_H$ is a $C_0$-representation. The group $G$ is a Howe-Moore group if the pair $(G, G)$ has the relative Howe-Moore property.

**Proposition 6.2.** Let $N$ be a closed, co-compact normal subgroup of $G$. If the pair $(G, N)$ has the relative Howe-Moore property, then $G$ has property $BP_{L^2}$. In particular every Howe-Moore group has property $BP_{L^2}$.

**Proof:** Let $\pi$ be a unitary representation of $G$, and let $b$ be a 1-cocycle with respect to $\pi$. Set $\psi(g) = \|b(g)\|^2$. Assuming that $(G, N)$ has the Howe-Moore property, we must prove that $\psi|_N$ is either bounded or proper. So suppose that $\psi|_N$ is unbounded.

By Schöenberg’s theorem, for $t > 0$, the function $\phi_t(g) = e^{-t\psi(g)}$ is positive definite on $G$. So there exists a Hilbert space $\mathcal{H}_t$ and a unitary representation $\pi_t$ of $G$ on $\mathcal{H}_t$, with a cyclic vector $\xi_t \in \mathcal{H}_t$, such that:

$$\phi_t(g) = \langle \pi_t(g)\xi_t|\xi_t \rangle$$

for every $g \in G$.

**Claim:** $\pi_t$ has no non-zero $N$-fixed vector.

To see this, let $\mathcal{H}^0$ be the space of $N$-fixed vectors in $\mathcal{H}_t$, and $\mathcal{H}^\perp$ be its orthogonal complement. We must show that $\mathcal{H}^0 = 0$. Observe that $\mathcal{H}^0$ and $\mathcal{H}^\perp$ are $G$-invariant, as $N$ is normal in $G$. For $\xi \in \mathcal{H}$, write $\xi = \xi^0 + \xi^\perp$ in the decomposition $\mathcal{H}_t = \mathcal{H}^0 \oplus \mathcal{H}^\perp$. As $\xi_t$ is cyclic, it is enough to show that $\xi^0_t = 0$. But, for $h \in N$:

$$\phi_t(h) = \langle \pi_t(h)\xi^\perp_t|\xi^\perp_t \rangle + \|\xi^0_t\|^2.$$ 

As $\psi|_N$ is unbounded, we can find a sequence $(h_n)_{n>0}$ in $N$ such that $\lim_{n \to \infty} \psi(h_n) = +\infty$. Then $\lim_{n \to \infty} \phi_t(h_n) = 0$. On the other hand, coefficients of $\pi_t|_N$ on $\mathcal{H}^\perp$ are $C_0$, by the Howe-Moore property for $(G, N)$. So $\lim_{n \to \infty} \langle \pi_t(h_n)\xi^\perp_t|\xi^\perp_t \rangle = 0$. Hence $\|\xi^0_t\| = 0$, proving the claim.

From the claim, plus the fact that $(G, N)$ has the Howe-Moore property, we deduce that $\phi_t|_N$ is a $C_0$-function. This is equivalent to saying that $\psi|_N$ is proper. □
We will see in Example 6.4 below that the converse of Proposition 6.2 does not hold in general.

We revisit semi-direct products of the form \( V \rtimes K \), where \( V = \mathbb{R}^d \) (\( d \geq 2 \)) and \( K \) is a closed subgroup of the unitary group \( U(d) \). Denote by \( K^0 \) the connected component of identity of \( K \).

**Corollary 6.3.** Consider the following statements:

\begin{itemize}
  \item [a)] \( K^0 \) acts irreducibly on \( V \).
  \item [a’)] The pair \( (V \rtimes K, V) \) has the relative Howe-Moore property.
  \item [b)] \( K \) is infinite and acts irreducibly on \( V \).
  \item [b’)] \( V \rtimes K \) has property \( BP_{L^2} \).
  \item [c)] \( K \) stabilizes no proper closed unbounded subgroup of \( V \).
  \item [d)] \( K \) acts irreducibly on \( V \).
\end{itemize}

Then \((a) \iff (a’) \implies (b) \iff (b’) \implies (c) \implies (d)\). If \( K \) is connected, all those statements are equivalent.

**Proof:** \((a) \iff (a’)\) follows immediately from Theorem 4.5 in [CCLTV].

\((a) \implies (b)\) follows by observing that, if \( K^0 \) acts irreducibly, then \( K^0 \) is non-trivial, hence infinite.

\((b) \iff (b’)\) is Theorem 1.4 above.

\((b’) \implies (c)\) If the semi-direct product \( V \rtimes K \) has property \( BP_{L^2} \) then every closed normal subgroup of \( V \rtimes K \) is either compact or co-compact. This rules out any proper closed unbounded \( K \)-invariant subgroup of \( W \).

\((c) \implies (d)\) is trivial, and so is \((d) \implies (a)\) when \( K = K^0 \).

In general the implications \((a) \implies (b), (b) \implies (c), (c) \implies (d)\) cannot be reversed, as the following examples show.

**Example 6.4.** 1. Let \( K \) be the semi-direct product \( K = (SO(2) \times SO(2)) \rtimes C_2 \), where \( C_2 \) acts by flipping the two factors. Then \( K \) acts on \( \mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2 \), with the first (resp. second) copy of \( SO(2) \) acting by rotations on the first (resp. second) copy of \( \mathbb{R}^2 \), and \( C_2 \) flipping the two copies of \( \mathbb{R}^2 \). Then \( K \) acts irreducibly on \( \mathbb{R}^4 \) but \( K^0 = SO(2) \times SO(2) \) acts reducibly. So \((b) \implies (a)\) does not hold in general. Since \( \mathbb{R}^4 \rtimes K \) has property \( BP_{L^2} \) but the pair \( (\mathbb{R}^4 \rtimes K, \mathbb{R}^4) \) does not have the relative Howe-Moore property, the same example shows that the converse of Proposition 6.2 fails in general.
2. Let $C_n$ denote the cyclic group of order $n \geq 2$. Let $C_n$ act on $\mathbb{R}^2$ by rotations of angles a multiple of $2\pi/n$. For $n \neq 2, 3, 4, 6$, the group $K = C_n$ stabilizes no proper closed unbounded subgroup of $\mathbb{R}^2$. So $(c) \Rightarrow (b)$ does not hold in general.

3. Consider the same action of $C_n$ by rotations on $\mathbb{R}^2$, this time with $n = 3, 4, 6$: the action is irreducible but stabilizes a lattice in $\mathbb{R}^2$. So $(d) \Rightarrow (c)$ does not hold in general.

7 Groups acting on trees with property PL

Appendix by Corina CIOBOTARU

In this appendix we prove that closed non-compact subgroups of $\text{Aut}(T_d)^+$ that act 2-transitively on $\partial T_d$ have property PL. Beside linear examples as $\text{SL}(2, \mathbb{Q}_p)$, the latter family of groups contains examples of non-compact locally compact groups that are non-linear, at least in characteristic 0: those are the universal groups $U(F)^+$ introduced by Burger–Mozes in [BM, Section 3].

We denote by $T_d$ a $d$-regular tree, with $d \geq 3$, and by $\text{Aut}(T_d)$ its group of automorphisms, which is a locally compact group with respect to the compact-open topology. Let $\text{Aut}(T_d)^+$ be the group of all type-preserving automorphisms of $T_d$. By a type-preserving automorphism of $T_d$ we mean one that preserves an orientation of $T_d$ that is fixed in advance; this is the same as saying that the automorphism acts without inversion. We denote by $\partial T_d$ the set of endpoints of $T_d$ (they are also called the ideal points of $T_d$) and we call $\partial T_d$ the boundary of $T_d$. For every two points $x, y \in T_d \cup \partial T_d$ we denote by $[x, y]$ the unique geodesic between $x$ and $y$ in $T_d \cup \partial T_d$.

For $G \leq \text{Aut}(T_d)$ and $x, y \in T_d \cup \partial T_d$ we define

$$G_{[x,y]} := \{ g \in G \mid g \text{ fixes pointwise the geodesic } [x, y] \}.$$ 

In particular, $G_x = \{ g \in G \mid g(x) = x \}$. For $\xi \in \partial T_d$ we define $G_\xi := \{ g \in G \mid g(\xi) = \xi \}$ and $G^0_\xi := \{ g \in G \mid g(\xi) = \xi \text{ and } g \text{ fixes at least one vertex of } T_d \}$. Notice that $G_\xi$ can contain hyperbolic elements; if this is the case then $G^0_\xi \leq G_\xi$.

For the remaining of the appendix we consider $G$ to be a closed non-compact subgroup of $\text{Aut}(T_d)^+$ that acts 2-transitively on $\partial T_d$. One easily sees [Tits] that $G$ contains at least one hyperbolic element. Typical examples of such subgroups $G$ are $\text{SL}(2, \mathbb{Q}_p)$ and the universal groups introduced by Burger–Mozes in [BM, Section 3]. Those groups are moreover topologically simple. We will see below that the universal groups are not linear.
**Definition 7.1.** Let $a$ be a hyperbolic element of $G$. Corresponding to $a$ we define the set
\[ U_a^+ := \{ g \in G \mid \lim_{n \to \infty} a^{-n}ga^n = e \} \]

Notice that $U_a^+$ is a subgroup of $G$. It is called the contraction group corresponding to $a$, and in general it is not a closed subgroup of $G$. In the same way, but using $a^n ga^{-n}$ we define $U_a^-$. For example $U_a^\pm$ are closed when $G = \text{SL}(2, \mathbb{Q}_p)$ and not closed when $G$ is the universal group $U(F)^+$ of Burger–Mozes.

Let us recall some important properties of $G$, when $G$ is 2–transitive on $\partial T_d$. Let us fix for what follows a hyperbolic element $a$ of $G$ and denote by $\mathbf{\xi}^-, \mathbf{\xi}^+ \subset T$ the translation axis of $a$, where $\mathbf{\xi}^-, \mathbf{\xi}^+ \in \partial T_d$ are the repelling and respectively, the attracting endpoints of $a$. Without loss of generality, we can assume from now on that $a$ is of translation length 2 (see [Cio, Example 4.10] where 2-transitivity is explicitly used). We fix a vertex $x \in (\mathbf{\xi}^-, \mathbf{\xi}^+)$. One has the following Cartan decomposition (see e.g. Ciobotaru [Cio, Example 4.10]): $G = KA_k$, where $K := G_x = \{ g \in G \mid g(x) = x \}$ and $A := G_{\mathbf{\xi}^-, \mathbf{\xi}^+} = \{ g \in G \mid g(\mathbf{\xi}^-) = \mathbf{\xi}^-, g(\mathbf{\xi}^+) = \mathbf{\xi}^+ \}$.

Note that $A$ is a closed subgroup of $G$ containing $a$. Notice also that each element of $A$ either is elliptic, thus fixing pointwise the axis $(\mathbf{\xi}^-, \mathbf{\xi}^+)$, or it is hyperbolic and thus translating along the axis $(\mathbf{\xi}^-, \mathbf{\xi}^+)$. Moreover, every element $g \in G$ is of the form $g = k_1a^n k_2$, for some $k_1, k_2 \in K$ and some $n \in \mathbb{Z}$. For the latter decomposition we used the fact that $a$ has translation length 2.

By [Cio] Proposition 4.11], we have that
\[ G = \langle G_\mathbf{x}^+, G_\mathbf{x}^- \rangle. \]

Moreover, by [Cio] Proposition 4.15 and Corollary 4.17] we have
\[ G_{\mathbf{\xi}^-} = U_a^- A, \ G_{\mathbf{\xi}^+} = U_a^+ A, \ G_{\mathbf{\xi}^-}^0 = U_a^- (A \cap G_{\mathbf{\xi}^-}^0) \text{ and } G_{\mathbf{\xi}^+}^0 = U_a^+ (A \cap G_{\mathbf{\xi}^+}^0). \]

Notice that $G_{\mathbf{\xi}^-}$, $G_{\mathbf{\xi}^+}$, $G_{\mathbf{\xi}^-}^0$ and $G_{\mathbf{\xi}^+}^0$ are closed subgroups of $G$, and that $A \cap G_{\mathbf{\xi}^-}^0 = A \cap G_{\mathbf{\xi}^+}^0 = G_{\mathbf{\xi}^-} \cap G_{\mathbf{\xi}^+}$.

The following lemma says that hyperbolic elements in $A$ are boundedly generated by $G_{\mathbf{\xi}^-}^0 \cup G_{\mathbf{\xi}^+}^0$.

**Lemma 7.2.** (See the proof of [CC, Lemma 3.5]) For every hyperbolic element $\gamma \in A$ there exist $\gamma_1 \in A$ hyperbolic and $u \in (G_{\mathbf{\xi}^+}^0 \cap G_{\mathbf{\xi}^-}^0)$ such that $\gamma = \gamma_1 u$, $\gamma_1$ has same translation length as $\gamma$ and $\gamma_1$ is a product of 6 elements from $G_{\mathbf{\xi}^-}$ and $G_{\mathbf{\xi}^+}$. 23
Proof: As in the proof of [CC] Lemma 3.5 we start with a basic observation.

Let \((\eta, \xi)\) be a bi-infinite geodesic line in \(T_d\), with \(\eta \in \partial T_d \setminus \{\xi, \xi\} \). Then the intersection \((\eta, \xi) \cap (\xi, \xi)\) is a geodesic ray \([y, \xi]\), with \(y\) a vertex in \((\xi, \xi)\). We claim that if there is some \(u \in G^0_{\xi, \xi}\), fixing \((\eta, \xi)\) to \((\xi, \xi)\) and fixing \([y, \xi]\) pointwise. Indeed, because \(\eta\) and \(\xi\) are opposite to \(\xi\) in \(\partial T_d\) and \(G\) is 2-transitive on \(\partial T_d\) we obtain that \(G^0_{\xi, \xi}\) is transitive on \(\partial T_d \setminus \{\xi, \xi\}\) by [CC] Lemma 3.5; there exists an element \(u \in G^0_{\xi, \xi}\) with the desired property. By the same argument applied to the pair of bi-infinite geodesic lines \((\xi, \eta)\) and \((\xi, \xi)\), we deduce the existence of \(u \in G^0_{\xi, \xi}\) mapping \((\xi, \eta)\) to \((\xi, \xi)\) and fixing \((\xi, \eta) \cap (\xi, \xi)\) pointwise.

Next we claim that for every vertex \(y \in (\xi, \xi)\) there exists \(r \in \langle G^0_{\xi, \xi}, G^0_{\xi, \xi}\rangle\), product of three elements from \(G^0_{\xi, \xi}\) and \(G^0_{\xi, \xi}\), such that \(r\) fixes \(x\) and swaps \(\xi\) to \(\xi\); i.e. \(r(y) = y, r(\xi) = \xi\) and \(r(\xi) = \xi\). Indeed, fix a vertex \(y \in (\xi, \xi)\). Because \(T_d\) is \(d\)-regular with \(d \geq 3\), there exists \(\eta \in \partial T_d\) with \((\eta, \xi) \cap (\xi, \xi)\) and \(\eta\). Moreover, we also have that \((\xi, \eta) \cap (\xi, \xi)\) and \((\xi, \eta) \cap (\xi, \xi)\) pointwise. By the above claim, we can find an element \(u \in G^0_{\xi, \xi}\) fixing \([y, \xi]\) pointwise and mapping \([y, \xi]\) to \([y, \eta]\). Similarly, there are elements \(v, w \in G^0_{\xi, \xi}\) both fixing \([y, \xi]\) pointwise and such that \(v([y, \eta]) = [y, \xi]\) and \(w([y, \xi]) = u^{-1}([y, \xi])\). Now we set \(r := vwu\). By construction \(r\) fixes the vertex \(y\), \(r \in \langle G^0_{\xi, \xi}, G^0_{\xi, \xi}\rangle\) and \(r\) is the product of three elements from \(G^0_{\xi, \xi}\) and \(G^0_{\xi, \xi}\). Moreover we have

\[
r([y, \xi]) = vwu^{-1}([y, \xi]) = v([y, \xi]) = [y, \xi]
\]

and

\[
r([y, \xi]) = vu([y, \xi]) = v([y, \eta]) = [y, \xi],
\]

so that \(r\) swaps \([y, \xi]\) and \([y, \xi]\).

Let now \(\gamma \in A\) be a hyperbolic element, thus of translation length \(2n\), for some \(n \in \mathbb{N}\) (recall that \(G\) is type-preserving). Fix \(x \in (\xi, \xi)\) and let \(y \in (\xi, \xi)\) be the midpoint of the segment \([x, \gamma(x)]\). Because \(\gamma\) has even translation length, \(y\) is a vertex of \(T_d\). By our second claim above, there exist \(r_1, r_2 \in \langle G^0_{\xi, \xi}, G^0_{\xi, \xi}\rangle\), each being product of three elements from \(G^0_{\xi, \xi}\) and \(G^0_{\xi, \xi}\), such that \(r_1(x) = x, r_2(y) = y\) and \(r_1(\xi) = \xi, r_i(\xi) = \xi, \) for \(i \in \{1, 2\}\). We claim that \(r_2 r_1 \in A\) is a hyperbolic element of translation length \(2n\) and it is a product of six elements from \(G^0_{\xi, \xi}\) and \(G^0_{\xi, \xi}\). Indeed, \(r_2 r_1(\xi) = r_2(r_1(\xi)) = r_2(\xi) = \xi\) and \(r_2 r_1(\xi) = r_2(r_1(\xi)) = r_2(\xi) = \xi\), so \(r_2 r_1 \in A\). Moreover, \(r_2 r_1(x) = r_2(x) = \gamma(x)\) and thus \(r_2 r_1\) is hyperbolic as desired. In particular, we obtain that \(u^{-1} := \gamma^{-1} r_2 r_1\) fixes pointwise the bi-infinite geodesic line \((\xi, \xi)\), thus \(u \in G^0_{\xi, \xi} \cap G^0_{\xi, \xi}\). By taking \(\gamma_1 := r_2 r_1\) the conclusion follows. □
Proposition 7.3. (cf. Cornulier [Co2, Proposition 4.1]) Let $L$ be a length function on $U^+_aA$. If $L$ is non-proper on $A$, then $L$ is bounded on $U^+_a$ and also on $G^0_{\xi^+}$.

Proof: Let $W$ be a compact neighborhood of the identity element $e$ in $G_{\xi^+}$, so that $L$ is bounded by a constant $M$ on $W$.

Suppose that the length function $L$ is not proper on $A$. Then there exists an unbounded sequence $\{a_n\}_{n \in \mathbb{N}} \subset A$ such that $\{L(a_n)\}_{n \in \mathbb{N}}$ is bounded by a constant $M'$. Then, for every $n \in \mathbb{N}$, we can write $a_n = k_na^{m_n}k'_n$, for some $k_n, k'_n \in K$ and some $m_n \in \mathbb{Z}$. We obtain that $\{L(a^{m_n})\}_{n \in \mathbb{N}}$ is bounded by a constant $M''$ and that $m_n \to \infty$ (by extracting a subsequence), when $n \to \infty$. By replacing $m_n$ with $-m_n$ (as $L$ is symmetric) and by taking $u \in U^+_a$ one can suppose that $\lim_{n \to \infty} a^{-m_n}ua^{m_n} = e$. Therefore, for $m_n$ large enough we have that $a^{-m_n}ua^{m_n} \in W$. We obtain that $L(u) \leq M + 2M''$, for every $u \in U^+_a$. As $G^0_{\xi^+} = U^+_a(A \cap G^0_{\xi^+})$ and because $A \cap G^0_{\xi^+}$ is compact as a closed subgroup of $K$, the function $L$ is also bounded on $G^0_{\xi^+}$. The conclusion follows. □

Corollary 7.4. Let $G$ be a closed non-compact subgroup of $\text{Aut}(T_d)^+$ that acts 2-transitively on $\partial T_d$. Then $G$ has property PL.

Proof: Let $L$ be a length function on $G$. If $L$ is non-proper, then by Proposition 7.3 we have that $L$ is bounded on $G^0_{\xi^+}$ and $G^0_{\xi^-}$. As $G = KAK$ and $L$ is bounded on the compact subgroups $K$, $A \cap G^0_{\xi^+}$ and $A \cap G^0_{\xi^-}$, it is enough to prove that $L$ remains bounded on the set of all hyperbolic elements in $A$. This follows by applying the bounded generation result of Lemma 7.2 to every hyperbolic element of $A$. Thus $L$ is bounded on $G$ as desired. □

As we have mentioned above, beside $\text{SL}(2,\mathbb{Q}_p)$, examples of closed non-compact subgroups of $\text{Aut}(T_d)^+$ that are topologically simple and act 2-transitively on $\partial T_d$ are the universal groups $U(F)^+$ introduced by Burger–Mozes in [BM, Section 3]. These groups are defined as follows.

Definition 7.5. Let $E(T_d)$ be the set of unoriented edges of the tree $T_d$. Let $\iota : E(T_d) \to \{1, \ldots, d\}$ be a function whose restriction to the star $E(x)$ of every vertex $x \in T_d$ is a bijection. Such a function $\iota$ is called a legal coloring of the tree $T_d$.

Definition 7.6. Let $F$ be a subgroup of permutations of the set $\{1, \ldots, d\}$ and let $\iota$ be a legal coloring of $T_d$. The universal group, with respect to $F$ and $\iota$, is defined as

$$U(F) := \{g \in \text{Aut}(T_d) \mid \iota \circ g \circ (\iota|_{E(x)})^{-1} \in F, \text{ for every } x \in T_d\}.$$
By $U(F)^+$ one denotes the subgroup generated by the edge-stabilizing elements of $U(F)$, and $U(F)^+ \leq \text{Aut}(T_d)^+$. Moreover, Amann [Amm, Proposition 52] tells us that the group $U(F)$ is independent of the legal coloring $\iota$ of $T_d$.

Immediately from the definition one deduces that $U(F)$ and $U(F)^+$ are closed subgroups of $\text{Aut}(T_d)$. Notice that, when $F$ is the full permutation group $\text{Sym}(d)$, then $U(F) = \text{Aut}(T_d)$ and $U(F)^+ = \text{Aut}(T_d)^+$, the latter group being an index 2, simple subgroup of $\text{Aut}(T_d)$ (for this see Tits [Tit]).

An important property of these groups is that $U(F)$ and $U(F)^+$ act 2–transitively on the boundary $\partial T_d$ if and only if $F$ is 2–transitive. Moreover, $U(F)^+$ is either trivial or it is a topologically simple group (see [BM, Amm]).

Moreover, the group $U(F)^+$ is not linear. This is because $U(F)^+$ has Tits’ independence property (see Amann [Amm]); this implies by Caprace-De Medts [CD, Section 2.6] that the contraction groups $U_a^\pm$ corresponding to hyperbolic elements $a \in U(F)^+$ are not closed. By Wang [Wan, Theorem 3.5(ii)] we know that the contraction groups corresponding to a $p$-adic Lie group are closed, thus one obtains the non-linearity of $U(F)^+$ in characteristic 0. In particular, when $F$ is 2-transitive, we conclude that the universal group $U(F)^+$ is non-linear (in characteristic 0) and has property $PL$.

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