Composable security against collective attacks of a modified BB84 QKD protocol with information only in one basis☆

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Abstract
Quantum Cryptography uses the counter-intuitive properties of Quantum Mechanics for performing cryptographic tasks in a secure and reliable way. The Quantum Key Distribution (QKD) protocol BB84 has been proven secure against several important types of attacks: collective attacks and joint attacks. Here we analyze the security of a modified BB84 protocol, for which information is sent only in the z basis while testing is done in both the z and the x bases, against collective attacks. The proof follows the framework of a previous paper [1], but it avoids a classical information-theoretical analysis and proves a fully composable security. We show that this modified BB84 protocol is as secure against collective attacks as the original BB84 protocol, and that it requires more bits for testing.

Keywords: Collective Attacks, Quantum Key Distribution, Cryptography, Error Rate, Test Bits, Information Bits.

1. Introduction
Cryptography is the science of protecting the security and correctness of data against adversaries. One of the most important cryptographic problems is the problem of encryption – namely, of transmitting a secret message from a sender to a receiver. Two main encryption methods are used today:

1. In symmetric-key cryptography, the same secret key is used for both the sender and the receiver: the sender uses the secret key for encrypting his or her message, and the receiver uses the same secret key for decrypting the message. Examples of symmetric-key ciphers include the Advanced Encryption Standard (AES) [3], the older Data Encryption Standard (DES), and one-time pad (“Vernam cipher”).

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In public-key cryptography [4], a public key (known to everyone) and a secret key (known only to the receiver) are used: the sender uses the public key for encrypting his or her message, and the receiver uses the secret key for decrypting the message. Examples of public-key ciphers include RSA [5] and elliptic curve cryptography.

One of the main problems with current public-key cryptography is that its security is usually not formally proved. Moreover, its security relies on the computational hardness of specific computational problems, such as integer factorization and discrete logarithm (that can both be efficiently solved on a quantum computer, by using Shor’s factorization algorithm [6]; therefore, if a scalable quantum computer is successfully built in the future, the security of many public-key ciphers, including RSA and elliptic curve cryptography, will be broken). Symmetric-key cryptography requires a secret key to be shared in advance between the sender and the receiver (in other words, if the sender and the receiver want to share a secret message, they must share a secret key beforehand). Moreover, no security proofs for many current symmetric-key ciphers, such as AES and DES, are known (even if one is allowed to rely on the computational hardness of problems), and unconditional security proofs against computationally-unlimited adversaries are impossible unless the secret key is used only once and is at least as long as the secret message [7].

The one-time pad (symmetric-key) cipher, that, given a message \(M\) and a secret key \(K\) of the same length, defines the encrypted message \(C\) to be \(C = M \oplus K\) (and then decryption can be performed by computing \(M = C \oplus K\)), is fully and unconditionally secure against any adversary [7]: namely, even if the adversary Eve intercepts the encrypted message \(C\), she gains no information about the original message \(M\) (assuming that she has no information about the secret key \(K\)). This means that, for obtaining perfect secrecy, all that is needed is an efficient way for sharing a random secret key between the sender and the receiver; unfortunately, “classical key distribution” cannot be achieved in a fully secure way if the adversary can listen to all the communication between the sender and the receiver.

Quantum key distribution (QKD) protocols take advantage of the laws of quantum mechanics for achieving fully and unconditionally secure key distribution, so that their resulting final key can later be used by other cryptographic primitives (e.g., one-time pad encryption). Most of the QKD protocols have security proofs applicable even against adversaries whose only limitations are the laws of nature (and who are otherwise capable of solving any computational problem and of performing any physically-allowed operation). The two parties (the first party is usually named “Alice”, and the second party is usually named “Bob”) want to create a shared random key, and they use an insecure quantum channel and an unjammable classical channel (to which the adversary may listen, but not interfere). The adversary (eavesdropper), Eve, tries to get as much information as she can on the final shared key. The first and most important QKD protocol is BB84 [8], that uses four possible quantum states (see details below), and it has been proven fully and unconditionally secure.

Boyer, Gelles, and Mor [1] discussed the security of BB84 against collective attacks. The class of the “collective attacks” [9, 10, 11] is an important and powerful subclass of the joint attacks; the class of the “joint attacks” includes all the theoretical
attacks allowed by quantum physics. [1] improved the security proof of Biham, Boyer, Brassard, van de Graaf, and Mor [11] against collective attacks, by using some techniques of Biham, Boyer, Boykin, Mor, and Roychowdhury [12] (that proved security against joint attacks). In this paper, too, we restrict the analysis to collective attacks, because security against collective attacks is conjectured (and, in some security notions, proved [13, 14]) to imply security against joint attacks. In addition, proving security against collective attacks is much simpler than proving security against joint attacks.

Other QKD protocols, either similar to BB84 or ones that use different approaches, have also been suggested, and in some cases have also been proven fully secure. In particular, the “three-state protocol” [15] uses only three quantum states, and it has been proven secure [16, 17, 18]; the “classical Bob” protocol [19] is a two-way protocol such that only Alice has quantum capabilities and Bob has only classical capabilities, and it has been proven robust [19] and secure [20]; and the “classical Alice” protocol [21] is similar to “classical Bob” with Alice being the classical participant instead of Bob, and it has been proven robust [22].

The above QKD protocols are all “Discrete-Variable” protocols. Two other classes of QKD protocols, “Continuous-Variable” protocols and “Distributed-Phase-Reference” protocols, have also been suggested; their security proofs are still weaker than the security proofs of “Discrete-Variable” protocols (see [23] for details).

QKD protocols can be used as a subroutine (secure key distribution) of more complicated cryptographic protocols. In other words, they can be integrated into a system in order to improve its security. See [24] for more details about this integration and about the practical usability of QKD compared to other methods.

In many QKD protocols, including BB84, Alice and Bob exchange several types of bits (encoded as quantum systems, usually qubits): INFO bits, that are secret bits shared by Alice and Bob and are used for generating the final key (via classical processes of error correction and privacy amplification); and TEST bits, that are publicly exposed by Alice and Bob (by using the classical channel) and are used for estimating the error rate. In BB84, each bit is sent from Alice to Bob in a random basis (the \(z\) basis or the \(x\) basis).

In this paper, we extend the analysis of BB84 done in [1] and prove the security of a QKD protocol we shall name \(BB84-INFO-z\). This protocol is almost identical to BB84, except that all its INFO bits are in the \(z\) basis. In other words, the \(x\) basis is used only for testing. The bits are thus partitioned into three disjoint sets: INFO, TEST-Z, and TEST-X. The sizes of these sets are arbitrary (\(n\) INFO bits, \(n_z\) TEST-Z bits, and \(n_x\) TEST-X bits).

We note that, while this paper follows a line of research that mainly discusses a specific approach of security proof for BB84 and similar protocols (this approach, notably, considers finite-key effects and not only the asymptotic error rate), many other approaches have also been suggested: see for example [25, 26, 13, 27]. For comparison, see Section 4.

In the other papers ([9, 10, 11, 12, 1]) that discussed the same approach of security proofs as discussed here, the classical mutual information between Eve and the final key was calculated and bounded, which caused problems with composability (see definition in [13] and in Subsection 1.1). In contrast to those papers, in this paper we suggest a method to prove a fully composable security – namely, to calculate and
bound the trace distance between the quantum state at the end of the real protocol and the quantum state at the end of an ideal protocol. This method is fully composable, because it bounds the distance between quantum states instead of bounding the classical information Eve has (bounding the classical information means, in particular, that we assume that Eve measures at the end of the protocol, while in reality she is not required to measure then, but is allowed to wait until Alice and Bob use the final key). This method is implemented in this paper for proving the fully composable security of BB84-INFO-\(_z\) against collective attacks; it also directly applies to the BB84 security proof in [1] against collective attacks, proving the fully composable security of BB84 against collective attacks. It may be extended in the future to show that the BB84 security proof of [12] proves the fully composable security of BB84 against joint attacks.

(We note that in the conference version of this paper [2], we used a weaker security definition: it was not sufficient for proving fully composable security, but it was more composable than in the previous papers.)

The “qubit space”, \(\mathcal{H}_2\), is a 2-dimensional Hilbert space. The states \(|0^0\rangle\) and \(|1^0\rangle\) form an orthonormal basis of \(\mathcal{H}_2\), called “the computational basis” or “the \(z\) basis”. The states \(|0^1\rangle \triangleq \frac{(|0^0\rangle+|1^0\rangle)}{\sqrt{2}}\) and \(|1^1\rangle \triangleq \frac{(|0^0\rangle-|1^0\rangle)}{\sqrt{2}}\) form another orthonormal basis of \(\mathcal{H}_2\), called “the \(x\) basis”. Those two bases are said to be conjugate bases.

In this paper, we denote bit strings (of \(t\) bits, with \(t \geq 0\) being some integer) by a bold letter (e.g., \(i = i_1 \ldots i_t\) with \(i_1, \ldots, i_t \in \{0, 1\}\)); and we refer to those bit strings as elements of \(\mathbb{F}_2^t\) – that is, as elements of a \(t\)-dimensional vector space over the field \(\mathbb{F}_2 = \{0, 1\}\), where addition of two vectors corresponds to a XOR operation between them. The number of 1-bits in a bit string \(s\) is denoted by \(|s|\), and the Hamming distance between two strings \(s\) and \(s'\) is \(d_H(s, s') = |s + s'|\).

1.1. Security Definitions of Quantum Key Distribution

Originally, a QKD protocol was defined to be secure if the (classical) mutual information between Eve’s information and the final key, maximized over all the possible attack strategies and measurements by Eve, is exponentially small in the number of qubits, \(N\). Examples of security proofs of BB84 that use this security definition are [25, 12, 26]. Those security proofs used the observation that one cannot analyze the classical data held by Eve before privacy amplification (as done in [28]), but must analyze the quantum state held by Eve [29]. In other words, they assumed that Eve could keep her quantum state until the end of the protocol, and only then choose the optimal measurement (based on all the data she observed) and perform the measurement.

Later, it was noticed that this security definition may not be “composable”. In other words, the final key is secure if Eve measures the quantum state she holds at the end of the QKD protocol, but the proof does not apply to cryptographic applications (e.g., encryption) of the final key: Eve might gain non-negligible information after the key is used, even though her information on the key itself was negligible. This means that the proof is not sufficient for practical purposes. In particular, those applications may be insecure if Eve keeps her quantum state until Alice and Bob use the key (thus giving Eve some new information) and only then measures.
Therefore, a new notion of “(composable) full security” was defined \([30, 27, 13]\) by using the trace distance between quantum states, following universal composability definitions for non-quantum cryptography \([31, 32]\). Intuitively, this notion means that the final joint quantum state of Alice, Bob, and Eve at the end of the protocol is very close (namely, the trace distance is exponentially small in \(N\)) to their final state at the end of an ideal key distribution protocol, that distributes a completely random and secret final key to both Alice and Bob. In other words, if a QKD protocol is secure, then except with an exponentially small probability, one of the two following events happens: the protocol is aborted, or the secret key generated by the protocol is the same as a perfect key that is uniformly distributed (i.e., each possible key having the same probability), is the same for both parties, and is independent of the adversary’s information.

Formally, \(\rho_{ABE}\) is defined as the final quantum state of Alice, Bob, and Eve at the end of the protocol (with Alice’s and Bob’s states being simply the “classical” states \(|k_A\rangle_A\) and \(|k_B\rangle_B\), where \(k_A\) and \(k_B\) are bit strings that are the final keys held by Alice and Bob, respectively (note that usually \(k_A = k_B\)); and with Eve’s state including both her quantum probe and the classical information published in the unjammable classical channel); \(\rho_U\) is defined as the complete mixture of all the possible keys that are the same for Alice and Bob (namely, if the set of possible final keys is \(K\), then \(\rho_U = \frac{1}{|K|} \sum_{k \in K} |k\rangle_A \langle k|_A \langle k|_B\); and \(\rho_E\) is defined as the partial trace of \(\rho_{ABE}\) over the system \(AB\). For the QKD protocol to be fully (and compositably) secure, it is required that

\[
\frac{1}{2} \text{tr} |\rho_{ABE} - \rho_U \otimes \rho_E| \leq \epsilon,
\]

where \(\epsilon\) is exponentially small in \(N\). Intuitively, \(\rho_{ABE}\) is the actual joint state of Alice, Bob, and Eve at the end of the QKD protocol; \(\rho_U\) is the ideal final state of Alice and Bob (an equal mixture of all the possible final keys, that is completely uncorrelated with Eve and is the same for Alice and Bob); and \(\rho_E\) is the state of Eve, uncorrelated with the states of Alice and Bob. Note that cases in which the protocol is aborted are represented by the zero operator: see [13, Subsection 6.1.2] for details.

Composable security of many QKD protocols, including BB84, has been proved [30, 27, 13].

2. Full Definition of the “BB84-INFO-\(z\)” Protocol

Below we formally define all the steps of the BB84-INFO-\(z\) protocol, as used in this paper.

1. Before the protocol, Alice and Bob choose some shared (and public) parameters: numbers \(n, n_z,\) and \(n_x\) (we denote \(N = n + n_z + n_x\)), error thresholds \(p_a, p_z,\) an \(r \times n\) parity check matrix \(P_C\) (corresponding to a linear error-correcting code \(C\)), and an \(m \times n\) privacy amplification matrix \(P_K\) (representing a linear key-generation function). It is required that all the \(r + m\) rows of the matrices \(P_C\) and \(P_K\) put together are linearly independent.

2. Alice randomly chooses a partition \(P = (s, z, b)\) of the \(N\) bits by randomly choosing three \(N\)-bit strings \(s, z, b \in \mathbf{F}_2^N\) that satisfy \(|s| = n, |z| = n_z, |b| = n_x,\) and
$|s + z + b| = N$. Thus, $P$ partitions the set of indexes $\{1, 2, \ldots, N\}$ into three disjoint sets:

- $I$ (INFO bits, where $s_j = 1$) of size $n$;
- $T_z$ (TEST-Z bits, where $z_j = 1$) of size $n_z$; and
- $T_x$ (TEST-X bits, where $b_j = 1$) of size $n_x$.

3. Alice randomly chooses an $N$-bit string $i \in \mathbb{F}_2^N$ and sends the $N$ qubit states $|i_{b_1}\rangle, |i_{b_2}\rangle, \ldots, |i_{b_N}\rangle$, one after the other, to Bob using the quantum channel. Notice that Alice uses the $z$ basis for sending the INFO and TEST-Z bits, and that she uses the $x$ basis for sending the TEST-X bits. Bob keeps each received qubit in quantum memory, not measuring it yet\(^1\).

4. Alice sends to Bob over the classical channel the bit string $b = b_1 \ldots b_N$. Bob measures each of the qubits he saved in the correct basis (namely, when measuring the $i$-th qubit, he measures it in the $z$ basis if $b_i = 0$, and he measures it in the $x$ basis if $b_i = 1$).

The bit string measured by Bob is denoted by $i^B$. If there is no noise and no eavesdropping, then $i^B = i$.

5. Alice sends to Bob over the classical channel the bit string $s$. The INFO bits (that will be used for creating the final key) are the $n$ bits with $s_j = 1$, while the TEST-Z and TEST-X bits (that will be used for testing) are the $n_z + n_x$ bits with $s_j = 0$. We denote the substrings of $i, b$ that correspond to the INFO bits by $i_s$ and $b_s$, respectively.

6. Alice and Bob both publish the bit values they have for all the TEST-Z and TEST-X bits, and they compare the bit values. If more than $n_z \cdot p_a, z$ TEST-Z bits are different between Alice and Bob, or more than $n_x \cdot p_a, x$ TEST-X bits are different between them, they abort the protocol. We note that $p_a, z$ and $p_a, x$ (the pre-agreed error thresholds) are the maximal allowed error rates on the TEST-Z and TEST-X bits, respectively – namely, in each basis ($z$ and $x$) separately.

7. The values of the remaining $n$ bits (the INFO bits, with $s_j = 1$) are kept in secret by Alice and Bob. The bit string of Alice is denoted $x = i_s$, and the bit string of Bob is denoted $x^B$.

8. Alice sends to Bob the syndrome of $x$ (with respect to the error-correcting code $C$ and to its corresponding parity check matrix $P_C$), that consists of $r$ bits and is defined as $\xi = xP_C^T$. By using $\xi$, Bob corrects the errors in his $x^B$ string (so that it is the same as $x$).

9. The final key consists of $m$ bits and is defined as $k = xP_K^T$. Both Alice and Bob compute it.

\(^1\) Here we assume that Bob has a quantum memory and can delay his measurement. In practical implementations, Bob usually cannot do that, but is assumed to measure in a randomly-chosen basis ($z$ or $x$), so that Alice and Bob later discard the qubits measured in the wrong basis. In that case, we need to assume that Alice sends more than $N$ qubits, so that $N$ qubits are finally detected by Bob and measured in the correct basis. In the original scheme, the probability of choosing each basis ($z$ or $x$) was $\frac{1}{2}$, which caused half of the sent qubits to be lost; in the improved scheme suggested by [33], the probability of choosing the $z$ basis can be much higher, which means that much less qubits get lost.
The protocol is defined similarly to BB84 (and to its description in [1]), except that it uses the generalized bit numbers \( n, n_z, \) and \( n_x \) (numbers of INFO, TEST-Z, and TEST-X bits, respectively); that it uses the partition \( \mathcal{P} = (s, z, b) \) for dividing the \( N \)-bit string \( i \) into three disjoint sets of indexes \((I, T_Z, T_X)\); and that it uses two separate thresholds \((p_{a,z} \text{ and } p_{a,x})\) instead of one \((p_a)\).

3. **Proof of Security for the BB84-INFO-\( z \) Protocol Against Collective Attacks**

3.1. **The General Collective Attack of Eve**

Before the QKD protocol is performed (and, thus, independently of \( i \) and \( \mathcal{P} \)), Eve chooses some collective attack to perform. A collective attack is bitwise: it attacks each qubit separately, by using a separate probe (ancillary state). Each probe is attached to the compound system \( (0^E)_E \), and applies some unitary operator \( U_j \) of her choice to the compound system \( (0^E)_E \). Then, Eve keeps to herself (in a quantum memory) the subsystem \( E_j \), which is her probe state; and sends to Bob the subsystem \( T_j \), which is the qubit sent from Alice to Bob (which may have been modified by her attack \( U_j \)).

The most general collective attack \( U_j \) of Eve on the \( j \)-th qubit, represented in the orthonormal basis \((|0^b \rangle_{T_j}, |1^b \rangle_{T_j})\), is

\[
U_j |0^E_E \rangle |0^b \rangle_{T_j} = |E_{00}^b_E \rangle |0^b \rangle_{T_j} + |E_{01}^b_E \rangle |1^b \rangle_{T_j} \tag{2}
\]

\[
U_j |0^E_E \rangle |1^b \rangle_{T_j} = |E_{10}^b_E \rangle |0^b \rangle_{T_j} + |E_{11}^b_E \rangle |1^b \rangle_{T_j} \tag{3}
\]

where \( |E_{00}^b_E \rangle, |E_{01}^b_E \rangle, |E_{10}^b_E \rangle, \) and \( |E_{11}^b_E \rangle \) are non-normalized states in Eve’s probe system \( E_j \) attached to the \( j \)-th qubit.

We thus notice that Eve can modify the original product state of the compound system, \(|0^E_E \rangle |i^b \rangle_{T_j}\), into an entangled state (e.g., \(|E_{00}^b_E \rangle |0^b \rangle_{T_j} + |E_{10}^b_E \rangle |1^b \rangle_{T_j}\)). Eve’s attack may thus cause Bob’s state to become entangled with her probe. On the one hand, this may give Eve some information on Bob’s state; on the other hand, this causes disturbance that may be detected by Bob. The security proof shows that the information obtained by Eve and the disturbance caused by Eve are inherently correlated: this is the basic reason QKD protocols are secure.

3.2. **Results from [1]**

The security proof of BB84-INFO-\( z \) against collective attacks is very similar to the security proof of BB84 itself against collective attacks, that was detailed in [1]. Most parts of the proof are not affected at all by the changes made to BB84 to get the BB84-INFO-\( z \) protocol (changes detailed in Section 2 of the current paper), because those
parts assume fixed strings $s$ and $b$, and because the attack is collective (so the analysis is restricted to the INFO bits).

Therefore, the reader is referred to the proof in Section 2 and Subsections 3.1 to 3.5 of [1], that applies to BB84-INFO-Z without any changes (except changing the total number of bits, $2n$, to $N$, which does not affect the proof at all), and that will not be repeated here.

We denote the rows of the error-correction parity check matrix $P_c$ as the vectors $v_1, \ldots, v_r$ in $F_2^n$, and the rows of the privacy amplification matrix $P_k$ as the vectors $v_{r+1}, \ldots, v_{r+m}$. We also define, for every $r'$, $V_{r'} \triangleq \text{Span}\{v_1, \ldots, v_{r'}\}$; and we define

$$d_{r,m} \triangleq \min_{r \leq r' < r+m} d_H(v_{r+1}, V_{r'}) = \min_{r \leq r' < r+m} d_{r',1}. \quad (4)$$

For a 1-bit final key $k \in \{0, 1\}$, we define $\hat{\rho}_k$ to be the state of Eve corresponding to the final key $k$, given that she knows $\xi$. Thus,

$$\hat{\rho}_k = \frac{1}{2^{n-r-1}} \sum_{x | k = \xi, x_{v_{r+1}} = 1} \rho^b_x, \quad (5)$$

where $\rho^b_x$ is Eve’s state after the attack, given that Alice sent the INFO bit string $x$ encoded in the bases $b' = b$. In [1], the state $\hat{\rho}_k$ was also defined: it is a lift-up of $\hat{\rho}_k$ (which means that $\hat{\rho}_k$ is a partial trace of $\hat{\rho}_k$), in which the states $\rho^b_x$ appearing in $\hat{\rho}_k$ are replaced by their purifications (see full definition in Subsection 3.4 of [1]).

In the end of Subsection 3.5 of [1], it was found that (in the case of a 1-bit final key, i.e., $m = 1$)

$$\frac{1}{2} \text{tr} |\hat{\rho}_0 - \hat{\rho}_1| \leq 2 \sqrt{P \left[ |C_l| \geq \frac{d_{l,1}}{2} \mid B_l = \text{flip}(b^l) \right]}, \quad (6)$$

where $C_l$ is a random variable whose value is the $n$-bit string of errors on the $n$ INFO bits; $B_l$ is a random variable whose value is the $n$-bit string of bases of the $n$ INFO bits; $\text{flip}(b^l)$ is the bit-flipped string of $b^l = b_l$; and $d_{l,1}$ (and, in general, $d_{r,m}$) was defined above.

Now, according to [34, Theorem 9.2 and page 407], and using the fact that $\hat{\rho}_k$ is a partial trace of $\hat{\rho}_k$, we find that $\frac{1}{2} \text{tr} |\hat{\rho}_0 - \hat{\rho}_1| \leq \frac{1}{2} \text{tr} |\hat{\rho}_0 - \hat{\rho}_1|$. From this result and from inequality (6) we deduce that

$$\frac{1}{2} \text{tr} |\hat{\rho}_0 - \hat{\rho}_1| \leq 2 \sqrt{P \left[ |C_l| \geq \frac{d_{l,1}}{2} \mid B_l = \text{flip}(b^l) \right]}, \quad (7)$$

### 3.3. Bounding the Differences Between Eve’s States

We define $c \triangleq i + i^2$: namely, $c$ is the XOR of the $N$-bit string $i$ sent by Alice and of the $N$-bit string $i^2$ measured by Bob. For all indexes $1 \leq l \leq N$, $c_l = 1$ if and only if Bob’s $l$-th bit value is different from the $l$-th bit sent by Alice. The partition $\mathcal{P}$ divides the $N$ bits into $n$ INFO bits, $n_z$ TEST-Z bits, and $n_x$ TEST-X bits. The corresponding substrings of the error string $c$ are $c_s$ (the string of errors on the INFO bits), $c_z$ (the string of errors on the TEST-Z bits), and $c_b$ (the string of errors on the TEST-X bits).
The random variables that correspond to \( e_s, e_z, \) and \( e_b \) are denoted by \( C_I, C_{T_z}, \) and \( C_{T_k}, \) respectively.

We define \( C_I \) to be a random variable whose value is the string of errors on the INFO bits if Alice had encoded and sent the INFO bits in the x basis (instead of the z basis dictated by the protocol). In those notations, inequality (7) reads as

\[
\frac{1}{2} \text{tr} |\hat{\rho}_0 - \hat{\rho}_1| \leq 2 \sqrt{P \left[ |\hat{C}_I| \geq \frac{d_{r,m}}{2} \mid \mathcal{P} \right]} = 2 \sqrt{P \left[ |\hat{C}_I| \geq \frac{d_{r,m}}{2} \mid e_z, e_b, \mathcal{P} \right]},
\]

(8)

using the fact that Eve’s attack is collective, so the qubits are attacked independently, and therefore, the errors on the INFO bits are independent of the errors on the TEST-Z and TEST-X bits (namely, of \( e_z \) and \( e_b \)).

As explained in [1], inequality (8) was not derived for the actual attack \( U = U_1 \otimes \ldots \otimes U_N \) applied by Eve, but for a virtual flat attack (that depends on \( b \) and therefore could not have been applied by Eve). That flat attack gives the same states \( \hat{\rho}_0 \) and \( \hat{\rho}_1 \) as given by the original attack \( U \), and it gives a lower (or the same) error rate in the conjugate basis. Therefore, inequality (8) holds for the original attack \( U \), too. This means that, starting from this point, all our results apply to the original attack \( U \) rather than to the flat attack.

So far, we have discussed a 1-bit key. We will now discuss a general \( m \)-bit key \( k \). We define \( \hat{\rho}_k \) to be the state of Eve corresponding to the final key \( k \), given that she knows \( \xi \):

\[
\hat{\rho}_k = \frac{1}{2^{m-r-m}} \sum_{x \mid x^T \xi = k} \rho_x^{b^{T_j}}
\]

(9)

**Proposition 1.** For any two keys \( k, k' \) of \( m \) bits,

\[
\frac{1}{2} \text{tr} |\hat{\rho}_k - \hat{\rho}_{k'}| \leq 2m \sqrt{P \left[ |\hat{C}_I| \geq \frac{d_{r,m}}{2} \mid e_z, e_b, \mathcal{P} \right]}.
\]

(10)

**Proof.** We define the key \( k_j \), for \( 0 \leq j \leq m \), to consist of the first \( j \) bits of \( k' \) and the last \( m - j \) bits of \( k \). This means that \( k_0 = k, k_m = k' \), and \( k_{j-1} \) differs from \( k_j \) at most on a single bit (the \( j \)-th bit).

First, we find a bound on \( \frac{1}{2} \text{tr} |\hat{\rho}_{k_{j-1}} - \hat{\rho}_{k_j}| \): since \( k_{j-1} \) differs from \( k_j \) at most on a single bit (the \( j \)-th bit, given by the formula \( x \cdot v_{r+j} \)), we can use the same proof that gave us inequality (8), attaching the other (identical) key bits to \( \xi \) of the original proof; and we find that

\[
\frac{1}{2} \text{tr} |\hat{\rho}_{k_{j-1}} - \hat{\rho}_{k_j}| \leq 2 \sqrt{P \left[ |\hat{C}_I| \geq \frac{d_j}{2} \mid e_z, e_b, \mathcal{P} \right]},
\]

(11)

where we define \( d_j \) as \( d_j = d_{p_j}(v_{r+j}, V'_j) \), and \( V'_j \triangleq \text{Span}\{v_1, v_2, \ldots, v_{r+j-1}, v_{r+j+1}, \ldots, v_{r+m}\} \).

Now we notice that \( d_j \) is the Hamming distance between \( v_{r+j} \) and some vector in \( V'_j \), which means that \( d_j = \left| \sum_{i=1}^{r+j} a_i v_i \right| \) with \( a_i \in \mathbb{F}_2 \) and \( a_{r+j} \neq 0 \). The properties of
Hamming distance assure us that $d_j$ is at least $d_H(v_{r'}, V_{r'})$ for some $r \leq r' < r + m$. Therefore, we find that $d_{r_m} = \min_{r < r' < r + m} d_H(v_{r'}, V_{r'}) \leq d_j.$

The result $d_{r_m} \leq d_j$ implies that if $|C_i| \geq \frac{d_j}{2}$ then $|C_i| \geq \frac{d_{r_m}}{2}$. Therefore, inequality (11) implies

$$\frac{1}{2} \tr |\hat{\rho}_{k_{j-1}} - \hat{\rho}_{k_j}| \leq 2 \sqrt{P \left[ |C_i| \geq \frac{d_{r_m}}{2} \mid c_x, c_b, P \right]}.$$  \hspace{1cm} (12)

Now we use the triangle inequality for norms to find

$$\frac{1}{2} \tr |\hat{\rho}_k - \hat{\rho}_{k'}| = \frac{1}{2} \tr |\hat{\rho}_{k_0} - \hat{\rho}_{k_m}| \leq \sum_{j=1}^m \frac{1}{2} \tr |\hat{\rho}_{k_{j-1}} - \hat{\rho}_{k_j}|$$

$$\leq 2m \sqrt{P \left[ |C_i| \geq \frac{d_{r_m}}{2} \mid c_x, c_b, P \right]},$$ \hspace{1cm} (13)

as we wanted. \hfill \Box

We would now like to bound the expected value (namely, the average value) of the trace distance between two states of Eve corresponding to two final keys. However, we should take into account that if the test fails, no final key is generated, in which case we define the distance to be 0. We thus define the random variable $\Delta_{Eve}^{(p_a,z,p_u,x)}(k, k')$ for any two final keys $k, k'$:

$$\Delta_{Eve}^{(p_a,z,p_u,x)}(k, k'|P, \xi, c_x, c_b) \triangleq \begin{cases} \frac{1}{2} \tr |\hat{\rho}_k - \hat{\rho}_{k'}| & \text{if } \frac{|c_a|}{n_z} \leq p_{a,z} \text{ and } \frac{|c_b|}{n_x} \leq p_{a,x} \\ 0 & \text{otherwise} \end{cases} \hspace{1cm} (14)$$

We need to bound the expected value $\langle \Delta_{Eve}^{(p_a,z,p_u,x)}(k, k') \rangle$, that is given by:

$$\langle \Delta_{Eve}^{(p_a,z,p_u,x)}(k, k') \rangle = \sum_{P, \xi, c_x, c_b} \Delta_{Eve}^{(p_a,z,p_u,x)}(k, k'|P, \xi, c_x, c_b) \cdot P(P, \xi, c_x, c_b) \hspace{1cm} (15)$$

(In Subsection 3.6 we prove that this expected value is indeed the quantity we need to bound for proving fully composable security, defined in Subsection 1.1.)

Theorem 2.

$$\langle \Delta_{Eve}^{(p_a,z,p_u,x)}(k, k') \rangle \leq 2m \sqrt{P \left[ \left( \frac{|C_i|}{n} \geq \frac{d_{r_m}}{2m} \right) \land \left( \frac{|C_{r_x}|}{n_x} \leq p_{a,x} \right) \land \left( \frac{|C_{r_y}|}{n_y} \leq p_{a,y} \right) \right]} \hspace{1cm} (16)$$

where $\frac{|C_i|}{n}$ is a random variable whose value is the error rate on the INFO bits if they had been encoded in the x basis, $\frac{|C_{r_x}|}{n_x}$ is a random variable whose value is the error rate on the TEST-Z bits, and $\frac{|C_{r_y}|}{n_y}$ is a random variable whose value is the error rate on the TEST-X bits.
Proof. We use the convexity of $x^2$, namely, the fact that for all $\{p_i\}_i$ satisfying $p_i \geq 0$ and $\sum p_i = 1$, it holds that $(\sum p_i x_i)^2 \leq \sum p_i x_i^2$. We find that:

\[
(\Delta_{\text{Eve}}^{(p_{a,z}; p_{a,x})}(k, k'))^2
\]

\[
= \left[ \sum_{\mathcal{P}, \xi, c_z, c_b} \Delta_{\text{Eve}}^{(p_{a,z}; p_{a,x})}(k, k'|\mathcal{P}, \xi, c_z, c_b) \cdot p(\mathcal{P}, \xi, c_z, c_b) \right]^2 \quad \text{(by (15))}
\]

\[
\leq \sum_{\mathcal{P}, \xi, c_z, c_b} \left( \frac{1}{2} \text{tr} |\hat{\rho}_k - \tilde{\rho}_k| \right)^2 \cdot p(\mathcal{P}, \xi, c_z, c_b) \quad \text{(by convexity of $x^2$)}
\]

\[
= \sum_{\mathcal{P}, \xi, \{a\}_n \leq p_{a,z}, \{b\}_n \leq p_{a,x}} P \left[ \left| \hat{c}_l \right| \geq \frac{d_{w,m}}{2} \mid c_z, c_b, \mathcal{P} \right] \cdot p(\mathcal{P}, \xi, c_z, c_b) \quad \text{(by (14))}
\]

\[
\leq 4m^2 \cdot \sum_{\mathcal{P}, \xi, \{a\}_n \leq p_{a,z}, \{b\}_n \leq p_{a,x}} P \left[ \left| \hat{c}_l \right| \geq \frac{d_{w,m}}{2} \mid c_z, c_b, \mathcal{P} \right] \cdot p(\mathcal{P}, c_z, c_b)
\]

\[
= 4m^2 \cdot \sum_{\mathcal{P}, \xi, \{a\}_n \leq p_{a,z}, \{b\}_n \leq p_{a,x}} P \left[ \left| \hat{c}_l \right| \geq \frac{d_{w,m}}{2} \right] \wedge \left( \left| \frac{c_{r_k}}{n_z} \right| \leq p_{a,z} \right) \wedge \left( \left| \frac{c_{r_k}}{n_x} \right| \leq p_{a,x} \right) \mid \mathcal{P} \right] \cdot p(\mathcal{P}) \quad \text{(by (10))}
\]

\[
= 4m^2 \cdot P \left[ \left| \frac{c_{r_k}}{n_z} \right| \leq p_{a,z} \wedge \left| \frac{c_{r_k}}{n_x} \right| \leq p_{a,x} \right] \quad \text{(17)}
\]

as we wanted. \hfill \Box

3.4. Proof of Security

Following [1] and [12], we choose matrices $R_c$ and $R_k$ such that the inequality $\frac{d_{w,m}}{2r} > p_{a,x} + \epsilon$ is satisfied for some $\epsilon$ (we will explain in Subsection 3.7 why this is possible). This means that

\[
P \left[ \left| \frac{c_{r_k}}{n_z} \right| \geq \frac{d_{w,m}}{2n} \right] \wedge \left( \left| \frac{c_{r_k}}{n_z} \right| \leq p_{a,z} \right) \wedge \left( \left| \frac{c_{r_k}}{n_x} \right| \leq p_{a,x} \right)
\]

\[
\leq P \left[ \left| \frac{c_{r_k}}{n_z} \right| > p_{a,x} + \epsilon \right] \wedge \left( \left| \frac{c_{r_k}}{n_x} \right| \leq p_{a,x} \right) \quad \text{(18)}
\]

We will now prove the right-hand-side of (18) to be exponentially small in $n$.

As said earlier, the random variable $\tilde{c}_l$ corresponds to the bit string of errors on the INFO bits if they had been encoded in the $x$ basis. The TEST-X bits are also encoded in the $x$ basis, and the random variable $c_{r_k}$ corresponds to the bit string of errors on those bits. Therefore, we can treat the selection of the indexes of the $n$ INFO bits and the $n_x$ TEST-X bits as a random sampling (after the numbers $n, n_z$, and $n_x$ and the indexes of the TEST-Z bits have all already been chosen) and use Hoeffding’s theorem (that is described in Appendix A of [1]).

Therefore, for each bit string $c_1 \ldots c_{n+n_x}$ that consists of the errors in the $n + n_x$ INFO and TEST-X bits if the INFO bits had been encoded in the $x$ basis, we apply Hoeffding’s theorem: namely, we take a sample of size $n$ without replacement from
the population \( c_1, \ldots, c_{n+k} \) (this corresponds to the random selection of the indexes of the INFO bits and the TEST-X bits, as defined above, given that the indexes of the TEST-Z bits have already been chosen). Let \( \bar{X} = \frac{C_X}{n} \) be the average of the sample (this is exactly the error rate on the INFO bits, assuming, again, that the INFO bits had been encoded in the x basis); and let \( \mu = \frac{|C_X| + |C_{X}^\perp|}{n + n_k} \) be the expectancy of \( \bar{X} \) (this is exactly the error rate on the INFO bits and TEST-X bits together). Then \( \frac{|C_X|}{n_k} \leq p_{a,x} \) is equivalent to \((n + n_k) \mu - n \bar{X} \leq n_k \cdot p_{a,x} \), and, therefore, to \( n \cdot (\bar{X} - \mu) \geq n_k \cdot (\mu - p_{a,x}) \). This means that the conditions \( \frac{|C_X|}{n} > p_{a,x} + \varepsilon \) and \( \frac{|C_X|}{n_k} \leq p_{a,x} \) rewrite to

\[
\left( \bar{X} - \mu > \varepsilon + p_{a,x} - \mu \right) \land \left( \frac{n}{n_k} \cdot (\bar{X} - \mu) \geq \mu - p_{a,x} \right),
\]

which implies \( \left( 1 + \frac{n}{n_k} \right) (\bar{X} - \mu) > \varepsilon \), which is equivalent to \( \bar{X} - \mu > \frac{n}{n + n_k} \varepsilon \). Using Hoeffding’s theorem (from Appendix A of \cite{1}), we get:

\[
P \left[ \left( \frac{|C_X|}{n} > p_{a,x} + \varepsilon \right) \land \left( \frac{|C_X|}{n_k} \leq p_{a,x} \right) \right] \leq P \left[ \bar{X} - \mu > \frac{n}{n_k} \varepsilon \right] \leq e^{-n \left( \frac{n}{n + n_k} \varepsilon \right)^2}
\]

(20)

In the above discussion, we have actually proved the following Theorem:

**Theorem 3.** Let us be given \( \delta > 0 \), \( R > 0 \), and, for infinitely many values of \( n \), a family \( \{v_1, \ldots, v_{n+k}\} \) of linearly independent vectors in \( \mathbb{F}_2^n \) such that \( \delta < \frac{d_{a,x}}{n} \) and \( \frac{d_{a,x}}{n} \leq R \). Then for any \( p_{a,z}, p_{a,x} > 0 \) and \( \varepsilon_{sec} > 0 \) such that \( p_{a,z} + \varepsilon_{sec} \leq \frac{R}{2} \), and for any \( n, n_z, n_k > 0 \) and two \( m_{a} \)-bit final keys \( k, k' \), the distance between Eve’s states corresponding to \( k \) and \( k' \) satisfies the following bound:

\[
\langle \Delta_{Eve}^{(p_{a,z}, p_{a,x})} (k, k') \rangle \leq 2Rn e^{-n \left( \frac{n}{n + n_k} \varepsilon \right)^2}
\]

(21)

In Subsection 3.7 we explain why the vectors required by this Theorem exist.

We note that the quantity \( \langle \Delta_{Eve}^{(p_{a,z}, p_{a,x})} (k, k') \rangle \) bounds the expected values of the Shannon Distinguishability and of the mutual information between Eve and the final key, as done in \cite{1} and \cite{12}, which is sufficient for proving non-composable security; but it also avoids composability problems: Eve is not required to measure immediately after the protocol ends, but she is allowed to wait until she gets more information. In Subsection 3.6 we use this bound for proving a fully composable security.

3.5. Reliability

Security itself is not sufficient; we also need the key to be reliable (namely, to be the same for Alice and Bob). This means that we should make sure that the number of errors on the INFO bits is less than the maximal number of errors that can be corrected by the error-correcting code. We demand that our error-correcting code can correct \( n(p_{a,z} + \varepsilon_{sec}) \) errors (we explain in Subsection 3.7 why this demand is satisfied). Therefore, reliability of the final key with exponentially small probability of failure is
guaranteed by the following inequality: (as said, $C_i$ corresponds to the actual bit string of errors on the INFO bits in the protocol, when they are encoded in the $z$ basis)

$$P \left[ \left( \frac{|C_i|}{n} > p_{a,z} + e_{\text{rel}} \right) \land \left( \frac{|C_T|}{n_z} \leq p_{a,z} \right) \right] \leq e^{-2 \left( \frac{n}{n + n_z} \right)^2 n_{\text{rel}}} \quad (22)$$

This inequality is proved by an argument similar to the one used in Subsection 3.4: the selection of the indexes of the INFO bits and the TEST-Z bits is a random partition of $n + n_z$ bits into two subsets of sizes $n$ and $n_z$, respectively (assuming that the indexes of the TEST-X bits have already been chosen), and thus it corresponds to Hoeffding’s sampling.

### 3.6. Proof of Fully Composable Security

We now prove that the BB84-INFO-$z$ protocol satisfies the definition of composable security for a QKD protocol: namely, that it satisfies equation (1) presented in Subsection 1.1. We prove that the expression $\frac{1}{4} \text{Tr} |\rho_{\text{ABE}} - \rho_{\text{U}} \otimes \rho_{\text{E}}|^{2}$ is exponentially small in $n$, with $\rho_{\text{ABE}}$ being the actual joint state of Alice, Bob, and Eve; $\rho_{\text{U}}$ being an ideal (random, secret, and shared) key distributed to Alice and Bob; and $\rho_{\text{E}}$ being the partial trace of $\rho_{\text{ABE}}$ over the system $AB$.

To make reading easier, we use the following notations, where $i$ is the bit string sent by Alice, $i^B$ is the bit string received by Bob, and $e = i \oplus i^B$ is the string of errors:

$$i^A_T \triangleq (i_1, i_2, i^B_3, i^B_4) \quad (23)$$

$$T \triangleq \begin{cases} 1 & \text{if } \frac{|e_k|}{n_c} \leq p_{a,z} \text{ and } \frac{|e_k|}{n_z} \leq p_{a,x} \\ 0 & \text{otherwise} \end{cases} \quad (24)$$

In other words, $i^A_T$ consists of all the TEST-Z and TEST-X bits of Alice and Bob; and $T$ is the random variable representing the result of the test.

According to the above definitions, the states $\rho_{\text{ABE}}$ and $\rho_{\text{U}}$ are

$$\rho_{\text{ABE}} = \sum_{i^B_T, P, T} P (i^B_T, P) \cdot |i_A^T \rangle \langle i_A | \otimes |k^i_B \rangle \langle k^i_B|$$

$$\otimes \left( \rho_{\text{E}}^{b^T} \right)_E \otimes |i^A_T \rangle \langle i^A_T| \otimes |k^i_B \rangle \langle k^i_B| \otimes |\xi \rangle \langle \xi|_C \quad (25)$$

$$\rho_{\text{U}} = \frac{1}{2^m} \sum_k |k_A \rangle \langle k_A| \otimes |k_B \rangle \langle k_B|_B, \quad (26)$$

where $\left( \rho_{\text{E}}^{b^T} \right)_E$ is defined to be Eve’s quantum state if Alice sends the INFO string $x = i_k$ in the bases $b^T = b_k$ and Bob gets the INFO string $x^B = i^B_k$. All the other states actually represent classical information: subsystems $A$ and $B$ represent the final keys held by Alice ($k = x^A_k$) and Bob ($k^i_B$, that is obtained from $x^B$, $\xi = x^B_{P_{\xi}}$, and $P_{\xi}$), and subsystem $C$ represents the information published in the unjammmable classical channel during the protocol (this information is known to Alice, Bob, and Eve) – namely, $i^A_T$ (all the test bits), $P$ (the partition), and $\xi$ (the syndrome).

We note that in the definition of $\rho_{\text{ABE}}$, we sum only over the events in which the test is passed (namely, in which the protocol is not aborted by Alice and Bob): in
Proposition 4.
Under the same conditions as Theorem 3, it holds that
\[ \rho \] means, by definition, that \( r \), all the probability can be reformulated as:
\[ \text{not only on} \]
\[ \text{Proof.} \]
We notice that in
\[ \text{for} \]
\[ \text{This state is identical to } \rho_{ABE}, \text{ except that Bob holds the Alice’s final key (} k \text{) instead of his own calculated final key (} k’ \text{). In particular, the similarity between } \rho_{ABE} \text{ and } \rho'_{ABE} \text{ means, by definition, that } \rho_E \triangleq \text{tr}_{AB}(\rho_{ABE}) \text{ and } \rho'_E \triangleq \text{tr}_{AB}(\rho'_{ABE}) \text{ are the same state: namely, } \rho_E = \rho'_E. \]

**Proposition 4.** Under the same conditions as Theorem 3, it holds that
\[ \frac{1}{2} \text{tr}|\rho'_{ABE} - \rho_U \otimes \rho_E| \leq 2Rn_{e}^{-}\left(\frac{n}{\sqrt{m}}\right)^{2}n_{e}^{-}, \]
for \( \rho'_{ABE} \) and \( \rho_U \) defined above and for the partial trace \( \rho_E \triangleq \text{tr}_{AB}(\rho_{ABE}). \)

**Proof.** We notice that in \( \rho'_{ABE}, \) the only factors depending directly on \( x \) and \( x^b \) (and not only on \( k \) and \( \xi \)) are the probability \( P(i, i^b, \mathcal{P}) \) and Eve’s state \( \left( \rho_{x, x^b}^b \right)_E. \) The probability can be reformulated as:
\[ P(i, i^b, \mathcal{P}) = P(i^b, \mathcal{P}, \xi) \cdot P(k | i^b, \mathcal{P}, \xi) \]
\[ \cdot P(x | k, i^b, \mathcal{P}, \xi) \cdot P(x^b | x, k, i^b, \mathcal{P}, \xi) \]
\[ = P(i^b, \mathcal{P}, \xi) \cdot \frac{1}{2^m} \cdot \frac{1}{2^{n-r-m}} \cdot P(x^b | x, k, i^b, \mathcal{P}, \xi) \]
(Because all the possible \( n \)-bit values of \( x \) have the same probability, \( \frac{1}{2^m}; \) and because all the \( r + m \) rows of the matrices \( P_C \) and \( P_K \) are linearly independent, so there are exactly \( 2^{n-r-m} \) values of \( x \) corresponding to each specific pair (\( \xi, k \)).

Therefore, the state \( \rho'_{ABE} \) takes the following form:
\[ \rho'_{ABE} = \frac{1}{2^m} \sum_{k, i^b, \mathcal{P}, \xi | T=1} P(i^b, \mathcal{P}, \xi) \cdot |k\rangle_A \langle k|_A \otimes |k\rangle_B \langle k|_B \]
\[ \otimes \left[ \frac{1}{2^{n-r-m}} \sum_{x, x^b | x^b = e, x^b = k} P(x^b | x, k, i^b, \mathcal{P}, \xi) \cdot \left( \rho_{x, x^b}^b \right)_E \right] \]
\[ \otimes |i^b, \mathcal{P}, \xi\rangle_C \langle i^b, \mathcal{P}, \xi|_C \]
\[
\rho_E = \rho'_E = \frac{1}{2^m} \sum_{k,k',p,q,r,T=1} P(i^{AB}, p, \xi) \cdot |k\rangle_A \langle k|_A \otimes |k\rangle_B \langle k|_B
\]
\[
\otimes (\hat{\rho}_k)_E \otimes |\bar{i}^{AB}_T, p, \xi\rangle c(\bar{i}^{AB}_T, p, \xi|_C)
\]
(\hat{\rho}_k was defined in equation (9).)

The partial trace \(\rho'_E = tr_{AB}(\rho'_{ABE})\), that (as proved above) is the same as \(\rho_E\), is
\[
\rho_U \otimes \rho_E = \frac{1}{2^m} \sum_{k,k',p,q,r,T=1} P(i^{AB}, p, \xi) \cdot |k\rangle_A \langle k|_A \otimes |k\rangle_B \langle k|_B
\]
\[
\otimes (\hat{\rho}_k')_E \otimes |\bar{i}^{AB}_T, p, \xi\rangle c(\bar{i}^{AB}_T, p, \xi|_C).
\]

By using the triangle inequality for norms, since \(\rho'_{ABE}\) and \(\rho_U \otimes \rho_E\) are the same (except the difference between Eve's states, \((\hat{\rho}_k)_E\) and \((\hat{\rho}_k')_E\)), we get, by using the definition of \(\langle \Delta_{\text{Eve}}^{(p_{A;B;x},p_{A;B})}(k,k')\rangle\) (equation (15)) and Theorem 3:
\[
\frac{1}{2} \text{tr} |\rho'_{ABE} - \rho_U \otimes \rho_E| \leq \frac{1}{2^m} \sum_{k,k',p,q,r,T=1} P(i^{AB}, p, \xi) \cdot \frac{1}{2} \text{tr} |(\hat{\rho}_k)_E - (\hat{\rho}_k')_E|
\]
\[
= \frac{1}{2^m} \sum_{k,k'} \langle \Delta_{\text{Eve}}^{(p_{A;B;x},p_{A;B})}(k,k')\rangle
\]
\[
\leq 2Rn\epsilon \left( \frac{n_1}{n+n_2} \right)^2 n_2 \epsilon
\]
(33)
as we wanted. \(\square\)

We still have to bound the following difference:
\[
\rho_{ABE} - \rho'_{ABE} = \sum_{i^{AB}, p} P(i^{AB}, p)
\]
\[
\cdot |k\rangle_A \langle k|_A \otimes [k']_B |B - |k\rangle_B |B
\]
\[
\otimes \left( \rho_{k;k}^0 \right)_{EB} \otimes |\bar{i}^{AB}_T, p, \xi\rangle c(\bar{i}^{AB}_T, p, \xi|_C)
\]
\[
= P((k \neq k') \land (T = 1))
\]
\[
\cdot \sum_{i^{AB}, p} P(i^{AB}, p \mid (k \neq k') \land (T = 1))
\]
\[
\cdot |k\rangle_A \langle k|_A \otimes [k']_B |B - |k\rangle_B |B
\]
\[
\otimes \left( \rho_{k;k}^0 \right)_{EB} \otimes |\bar{i}^{AB}_T, p, \xi\rangle c(\bar{i}^{AB}_T, p, \xi|_C)
\]
(34)
Because the trace distance between every two normalized states is bounded by 1, and because of the reliability proof in Subsection 3.5, we get:

$$\frac{1}{2} \text{tr} |\rho_{ABE} - \rho_{ABE}'| \leq P \left( (k \neq k') \land (T = 1) \right) \leq e^{-2\left(\frac{n^2z}{n + m} \right) n^2 \varepsilon_{\text{rel}}} \quad (35)$$

(Because if $k \neq k'$, Alice and Bob have different final keys, and this means that the error correction stage did not succeed. According to the discussion in Subsection 3.5, this can happen only if there are too many errors in the information string – namely, if $\frac{|C|}{n} > p_{a,z} + \varepsilon_{\text{rel}}$.)

To sum up, we get the following bound:

$$\frac{1}{2} \text{tr} |\rho_{ABE} - \rho_U \otimes \rho_E| \leq \frac{1}{2} \text{tr} |\rho_{ABE} - \rho_{ABE}'| + \frac{1}{2} \text{tr} |\rho_{ABE}' - \rho_U \otimes \rho_E| \leq e^{-2\left(\frac{n^2z}{n + m} \right) n^2 \varepsilon_{\text{rel}}} + 2Re^{-\left(\frac{n^2z}{n + m} \right) n^2 \varepsilon_{\text{sec}}}$$

This bound is exponentially small in $n$. Thus, we have proved the composable security of BB84-INFO-z.

### 3.7. Security, Reliability, and Error Rate Threshold

According to Theorem 3 and to the discussion in Subsection 3.5, to get both security and reliability we only need vectors $\{v^n_1, \ldots, v^n_{m+n+m_x}\}$ satisfying both the conditions of the Theorem (distance $\frac{d_{\text{min}}}{2n} > \frac{\delta}{2} \geq p_{a,x} + \varepsilon_{\text{sec}}$) and the reliability condition (the ability to correct $n(p_{a,z} + \varepsilon_{\text{rel}})$ errors). Such families were proven to exist in Appendix E of [12], giving the following upper bound on the bit-rate:

$$R_{\text{secret}} = \frac{m}{n} - H_2(2p_{a,x} + 2\varepsilon_{\text{sec}}) - H_2\left(p_{a,z} + \varepsilon_{\text{rel}} + \frac{1}{n}\right)$$

(37)

where $H_2(x) \triangleq -x \log_2(x) - (1 - x) \log_2(1 - x)$.

Note that we use here the error thresholds $p_{a,x}$ for security and $p_{a,z}$ for reliability. This is possible, because in [12] those conditions (security and reliability) on the codes are discussed separately.

To get the asymptotic error rate thresholds, we require $R_{\text{secret}} > 0$, and we get the condition:

$$H_2(2p_{a,x} + 2\varepsilon_{\text{sec}}) + H_2\left(p_{a,z} + \varepsilon_{\text{rel}} + \frac{1}{n}\right) < 1$$

(38)

The secure asymptotic error rate thresholds zone is shown in Figure 1 (it is below the curve), assuming that $\frac{1}{n}$ is negligible. Note the trade-off between the error rates $p_{a,z}$ and $p_{a,x}$. Also note that in the case $p_{a,z} = p_{a,x}$, we get the same threshold as BB84 ([12] and [1]), which is 7.56%.

### 4. Discussion

In the current paper, we have proved the BB84-INFO-z protocol to be fully secure against collective attacks. We have discovered that the results of BB84 hold very similarly for BB84-INFO-z, with only two exceptions:
1. The error rates must be separately checked to be below the thresholds $p_{a,z}$ and $p_{a,x}$ for the TEST-Z and TEST-X bits, respectively, while in BB84 the error rate threshold $p_a$ applies to all the TEST bits together.

2. The exponents of Eve’s information (security) and of the failure probability of the error-correcting code (reliability) are different than in [1], because different numbers of test bits are now allowed ($n_z$ and $n_x$ are arbitrary). This implies that the exponents may decrease more slowly (or more quickly) as a function of $n$. However, if we choose $n_z = n_x = n$ (thus sending $N = 3n$ qubits from Alice to Bob), then we get exactly the same exponents as in [1].

The asymptotic error rate thresholds found in this paper allow us to tolerate a higher threshold for a specific basis (say, the $x$ basis) if we demand a lower threshold for the other basis ($z$). If we choose the same error rate threshold for both bases, then the asymptotic bound is 7.56%, exactly the bound found for BB84 in [12] and [1].

We conclude that even if we change the BB84 protocol to have INFO bits only in the $z$ basis, this does not harm its security and reliability (at least against collective attacks). This does not even change the asymptotic error rate threshold. The only drawbacks of this change are the need to check the error rate for the two bases separately, and the need to either send more qubits ($3n$ qubits in total, rather than $2n$) or get a slower exponential decrease of the exponents required for security and reliability.

We thus find that the feature of BB84, that both bases are used for information, is not very important for security and reliability, and that BB84-INFO-$z$ (that lacks this feature) is almost as useful as BB84. This may have important implications on the security and reliability of other protocols that, too, use only one basis for information qubits, such as [15] and some two-way protocols [19, 21].

We also present a better approach for the proof, that uses the quantum distance
between two states rather than the classical information. In [1], [11], and [12], the classical mutual information between Eve’s information (after an optimal measurement) and the final key was calculated (by using the trace distance between two quantum states); although we should note that in [1] and [12], the trace distance was used for the proof of security of a single bit of the final key even when all other bits are given to Eve, and only the last stages of the proof discussed bounding the classical mutual information. In the current paper, on the other hand, we use the trace distance between the two quantum states until the end of the proof, which allows us to prove fully composable security.

Therefore, our proof shows the fully composable security of BB84-INFO-z against collective attacks (and, in particular, security even if Eve keeps her quantum states until she gets more information when Alice and Bob use the key, rather than measuring them at the end of the protocol); and a very similar approach can be applied to [1], immediately proving the composable security of BB84 against collective attacks. Our proof also makes a step towards making the security proof in [12] (security proof of BB84 against joint attacks) prove the composable security of BB84 against joint attacks.

Our results show that the BB84-INFO-z protocol can be securely used for distributing a secret key; the security is of an ideal implementation and against an adversary limited to collective attacks (it may be possible to generalize the proof, so that it applies to the most general attacks (joint attacks), by using the methods of [12], but such generalization is beyond the scope of the current paper). Moreover, the security of the final key is universally composable, which means that the key may be used for any cryptographic purpose without harming the security, even if Eve keeps her quantum states and uses all the information she gets in the future in an optimal way.

The techniques described in our proof may be applied in the future for proving the security of other protocols by using similar methods, and, in particular, for proving the security of other QKD protocols that use only one basis for the information bits, such as [15, 19, 21] mentioned above.

We note that this paper strengthens the security proofs described in [1, 11, 12], both because it slightly generalizes them (from security of BB84 to security of BB84-INFO-z) and because it makes them composable. Those security proofs have various advantages over other methods to prove security: first of all, they are mostly self-contained, while other security proofs require many results from other areas of quantum information (such as various notions of entropy needed for the security proof of [13, 27], and entanglement purification and quantum error correction needed for the security proof of [26]); second, they give tight finite-key bounds, unlike several other methods (see details below); and finally, at least in some sense, they are simpler than other proof techniques. On the other hand, their generality and their asymptotic error-rate threshold (7.56%, rather than 11% given by [27, 26]) are yet to be improved by future research.

Our method for proving security gives explicit and tight finite-key bounds. In contrast to this, the security proof of [26] gives only asymptotic results (for infinitely long keys). For the security proof of [13, 27], it is proved today that for some protocols (including BB84), one can get tight finite-key bounds [35] that are the same as the ones found by our method; but at first that security method gave very pessimistic bounds (by using the de Finetti theorem [13, 36]), and later, the bounds were improved for several protocols (including BB84) [37], but were still not tight (see [35] for comparison).
We also note that the existence of many different proof techniques is important, because some proofs may be more adjustable to various QKD protocols or to practical scenarios; some proofs may be clearer to different readers with different backgrounds; analyzing the differences between the proofs and between their obtained results may lead to important insights on the strengths and weaknesses of various techniques; and the existence of many proofs makes the security result more certain and less prone to errors.

We note that our security proof, similarly to many other full security proofs of QKD, assumes an ideal implementation (of ideal quantum systems consisting of exactly one qubit) and theoretical attacks. Practical implementations of QKD, almost always using photons, exist (see [38, 23] for details); their security analysis is much more complicated, because both Alice’s photon source and Bob’s detector devices have weaknesses and deviations from the theoretical protocol (especially when more than one photon is emitted by Alice or is sent by the eavesdropper). Those imperfections give rise to various practical attacks, such as the “Photon-Number Splitting” attack [39] (in which the eavesdropper takes advantage of emissions of two or more photons by Alice and gets full information) and the “Bright Illumination” attack [40] (in which the eavesdropper takes advantage of a weakness of specific detectors used by Bob and gets full information).

Possible solutions to those problems of actual physical realizations (see [38, 23] for more details) include a much more careful analysis of the practical devices and of practical implementations; “Measurement-Device Independent” QKD protocols [41, 42, 43, 44], that may be secure even if the measurement devices are controlled by the adversary; and “Device Independent” QKD protocols [45, 46, 47], that may be secure even if all the quantum devices are controlled by the adversary (under certain assumptions).

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