Presburger arithmetic with threshold counting quantifiers is easy

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Abstract
We give a quantifier elimination procedure for the extension of Presburger arithmetic with a unary threshold counting quantifier \( \exists \geq c y \) that determines whether the number of different \( y \) satisfying some formula is at least \( c \in \mathbb{N} \), where \( c \) is given in binary. Using a standard quantifier elimination procedure for Presburger arithmetic, the resulting theory is easily seen to be decidable in \( 4 \text{ExpTime} \). Our main contribution is to develop a novel quantifier-elimination procedure for a more general counting quantifier that decides this theory in \( 3 \text{ExpTime} \), meaning that it is no harder to decide than standard Presburger arithmetic. As a side result, we obtain an improved quantifier elimination procedure for Presburger arithmetic with counting quantifiers as studied by Schweikardt [ACM Trans. Comput. Log., 6(3), pp. 634-671, 2005], and a \( 3 \text{ExpTime} \) quantifier-elimination procedure for Presburger arithmetic extended with a generalised modulo counting quantifier.

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1 Introduction

Counting the number of solutions to an equation, or the number of elements in a set subject to constraints, is a fundamental problem in mathematics and computer science. In discrete geometry, a canonical \#P-complete problem is to count the number of integral points in polyhedra. The celebrated algorithm due to Barvinok [2] solves this problem in polynomial time in fixed dimension. This and other powerful insights motivate the study of algorithmic aspects of the more general problem of counting the number of models of formulae in Presburger arithmetic, the first-order theory of the integers with addition and order, and more generally to considering counting extensions of this logic.

It has long been known that the decision problem for Presburger arithmetic itself is recursively solvable [8] and that there is a quantifier elimination procedure for Presburger arithmetic running in \( 3 \text{ExpTime} \) [7]. In this article, we study quantifier elimination procedures for extensions of Presburger arithmetic with counting quantifiers. We are primarily interested in its extension with a unary threshold counting quantifier \( \exists \geq c y \), where \( c \) is given in binary. Given an assignment of integers to the first-order variables \( y, z_1, \ldots, z_n \), a formula \( \exists \geq c y \Psi(y, z_1, \ldots, z_n) \) evaluates to true whenever there are at least \( c \) different values of \( y \) satisfying \( \Psi(y, z_1, \ldots, z_n) \). Note that the number of different \( y \) may depend on the values of the variables \( z_1, \ldots, z_n \). It is easily seen that a formula \( \exists \geq c y \Psi(y, z_1, \ldots, z_n) \) can equivalently be expressed as \( \exists x_1 \cdots \exists x_c \bigwedge_{1 \leq i \leq c} \Psi(x_i, z_1, \ldots, z_n) \land \bigwedge_{1 \leq i < j \leq c} x_i \neq x_j \). However, since \( c \) is
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given in binary, this translation incurs an exponential blow-up, and thus the extension of
Presburger arithmetic with an $\exists^2 y$ counting quantifier can only be decided in $4\text{ExpTime}$
using the standard quantifier elimination procedure for Presburger arithmetic.

The main contribution of this paper is to develop a novel quantifier-elimination procedure
that enables us to solve this decision problem in $3\text{ExpTime}$, i.e., at no additional cost
compared to standard Presburger arithmetic.

The counting quantifiers $\exists^2 x y$ and $\exists^x y$. The unary threshold counting quantifier we
consider is a special instance of a more general unary counting quantifier $\exists^2 x y$, which itself
generalises the counting quantifier $\exists^x y$ for Presburger arithmetic studied by Apelt [1] and
Schweikardt [10]. Given an assignment of integers to the first-order variables $x, z_1, \ldots, z_n$, a
formula $\exists^2 x y \Psi(x; y, z_1, \ldots, z_n)$ evaluates to true whenever the number of different $y$
satisfying $\Psi(x, y, z_1, \ldots, z_n)$ is at least the value of $x$. The semantics of the $\exists^x y$ counting
quantifier is defined analogously, but observe that $\exists^x y$ does not hold when there is an infinite
number of different $y$. Both [1] and [10] show decidability of Presburger arithmetic extended
with the counting quantifier $\exists^2 x y$ by establishing a quantifier-elimination procedure.

For our $3\text{ExpTime}$ algorithm for Presburger arithmetic extended with the $\exists^2 x y$ counting
quantifier, we develop a novel quantifier-elimination procedure for the most general $\exists^2 x y$
counting quantifier. While this procedure a priori runs in non-elementary time, we show
that it can be performed in $3\text{ExpTime}$ when specialised to $\exists^2 x y$ counting quantifiers.

We remark that, crucially, such counting quantifiers are always unary, so as to keep the
logic decidable. Indeed, consider a binary counting quantifier $\exists^{x z}(y_1, y_2)$ counting the number
of different $y_1$ and $y_2$ satisfying a formula. Then the formula $\Phi(x, z) = \exists^{x z}(y_1, y_2)(0 \leq y_1, y_2 < z)$
holds for $x = z^2$, which in turn allows one to define multiplication, leading to undecidability
of the resulting first-order theory. Note that for threshold counting, in contrast, non-unary
quantifiers do not lead to undecidability. Thus, our results lead to the problem of eliminating
such quantifiers in a resource-efficient manner, which we leave open.

The counting quantifier $\exists^{(r,q)} y$. It is wide open whether there is an algorithm with element-
ary running time that decides Presburger arithmetic extended with $\exists^x y$ or $\exists^2 x y$ counting
quantifiers. Moreover, at present, no stronger lower bounds than those established for plain
Presburger arithmetic are known [3]. To shed more light on the complexity of Presburger
arithmetic with an $\exists^x y$ counting quantifier, Habermehl and Kuske [5] gave a quantifier-
elimination procedure for eliminating a unary modulo counting quantifier $\exists^{(r,q)} y$: here
$\exists^{(r,q)} y \Psi(y, z_1, \ldots, z_n)$ holds whenever the number of different $y$ satisfying $\Psi(y, z_1, \ldots, z_n)$
is congruent to $r$ modulo $q$. An analysis of the growth of the constants and coefficients
occurring in their quantifier-elimination procedure then enables them to derive an automata-
based $3\text{ExpTime}$ algorithm for deciding Presburger arithmetic extended with the $\exists^{(r,q)} y$
counting quantifier. As a side result, we show that our quantifier elimination procedure gives
a $3\text{ExpTime}$ upper bound for Presburger arithmetic with a generalised modulo counting
quantifier $\exists^{(x,q)} y \Psi(x, y, z_1, \ldots, z_n)$ that evaluates to true when $x$ is congruent modulo $q$
to the number of different $y$ satisfying $\Psi$, thereby strictly generalising the result of Habermehl
and Kuske.

Key techniques. An advantage of our quantifier elimination procedure for the $\exists^2 x y$ and
$\exists^x y$ counting quantifiers is that it avoids the introduction of additional $\exists$- and $\forall$-quantifiers
when eliminating a counting quantifier on which Schweikardt’s procedure [10] relies. Her
quantifier-elimination procedure replaces a counting quantifier $\exists^2 x y$ with an equivalent
quantified formula of Presburger arithmetic and requires a full transformation into disjunctive
normal form.
One key technique we employ is that, for Presburger arithmetic, it is possible to transform any Boolean combination of inequalities into a “disjoint” disjunctive normal form in polynomial time when the number of variables is fixed, see e.g. [9, 13]. This enables us to make use of a highly desirable disjunctive normal form for counting purposes without the drawback of non-elementary growth that repeated translation into usual disjunctive normal form normally entails. Another crucial ingredient of our quantifier eliminating procedure is evaluation of counting functions on bounded segments during the elimination process. This enables us to circumvent the introduction of additional standard first-order quantifiers that occurs in Scheweikardt’s procedure.

Putting these two ingredients together, we obtain a procedure that, in an analogue of nondeterministic guessing, pre-evaluates atomic predicates in the input formula. When a quantifier is eliminated, our “almost evaluated” formula is reduced to subformulae that are introduced during this “guessing” and evaluation of the counts.

Further related work. The counting quantifiers considered in this paper are derived from the so-called Härtig quantifier that enables reasoning about equicardinality between the sets defined by two formulae of first-order logic [6]. For Presburger arithmetic, the aforementioned undecidability result imposes tight restrictions on how first-order variables counting integral points can be used. Woods [13] studied properties of Presburger counting functions: given a formula \( \Phi(x_1,\ldots,x_m,p_1,\ldots,p_n) \) of Presburger arithmetic, the Presburger counting function associated to \( \Phi \) is the function

\[
g_\Phi(p_1,\ldots,p_n) = \# \{(x_1,\ldots,x_m) \in \mathbb{N}^m : \Phi(x_1,\ldots,x_m,p_1,\ldots,p_n)\}
\]

In [13], Woods shows that \( g_\Phi \) is a so-called piecewise quasi-polynomial. Computing such quasi-polynomials is of high relevance, for instance in numerous compiler optimisation approaches, see e.g. [11] and the references therein. More generally, Bogart et al. [4] have recently studied quantifier elimination procedures and counting problems for parametric Presburger arithmetic.

2 Presburger arithmetic with counting quantifiers

General notation. The symbols \( \mathbb{Z}, \mathbb{N} \) and \( \mathbb{N}_+ \) denote the set of integers, natural numbers including zero, and natural numbers without zero, respectively. We usually use \( a,b,c,\ldots \) for integer numbers, which we assume being encoded in binary. Given \( n \in \mathbb{N} \), we write \([n] \equiv \{0,\ldots,n-1\}\). We write \#A for the cardinality of a set \( A \). If \( A \) is infinite, then \#A = \infty, and we postulate \( n \leq \infty \) for all \( n \in \mathbb{Z} \).

Structure. We consider the structure \( \mathcal{Z} = \langle \mathbb{Z},(c)_{c \in \mathbb{Z}},+,<,(\equiv_q)_{q \in \mathbb{N}_+}\rangle \), where \((c)_{c \in \mathbb{Z}}\) are constant symbols that shall be interpreted as their homographic integer numbers, the binary function symbol + is interpreted as addition on \( \mathbb{Z} \), the binary relation < is interpreted as “less than”, and \( \equiv_q \) is interpreted as the modulo relation, i.e., \( a \equiv_q b \) iff \( q \) divides \( a - b \).

Basic syntax. Let \( X = \{x,y,z,\ldots\} \) be a countable set of first-order variables. Linear terms, usually denoted by \( t, t_1, t_2, \) etc., are expressions of the form \( a_1x_1 + \ldots + a_dx_d + c \) where \( x_1,\ldots,x_d \in X, a_1,\ldots,a_d,c \in \mathbb{Z} \). The integer \( a_i \) is the coefficient of the variable \( x_i \). Variables not appearing in the linear term are tacitly assumed to have a 0 coefficient. A term \( t \) is said to be \( x \)-free if the coefficient of the variable \( x \) in \( t \) is 0. The integer \( c \) is the constant of the linear term. Linear terms with constant 0 are said to be homogeneous.

Given a term \( t \), the lexeme \( t < 0 \) is understood as a linear inequality, and \( t \equiv 0 \mod q \) is a modulo constraint. Syntactically, Presburger arithmetic with counting quantifiers (PAC) is
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the closure of linear inequalities and modulo constraints under the Boolean connectives \( \land \) and \( \neg \) (i.e. conjunction and negation, respectively), the first-order quantifier \( \exists y \) and the unary counting quantifier \( \exists 2^x y \), where \( x, y \in X \). We assume the two variables \( x \) and \( y \) appearing in a counting quantifier to be syntactically different. Formulae of PAC are denoted by \( \Psi, \psi, \varphi, \psi, \gamma, \) etc. We write \( \text{vars}(\varphi) \) and \( \text{fv}(\varphi) \) for the set of variables and free variables of \( \varphi \), respectively. For the counting quantifier, we have \( \text{fv}(\exists 2^x y \varphi) = \{x\} \cup (\text{fv}(\varphi) \setminus \{y\}) \). We say that a formula \( \varphi \) is \( z \)-free if \( z \in X \) does not occur in \( \varphi \). Given linear terms \( t \) and \( t' \), we write \( \varphi[t'/t] \) for the formula obtained from \( \varphi \) by syntactically replacing every occurrence of \( t \) by \( t' \).

**Semantics.** An assignment is a function \( \nu: X \to \mathbb{Z} \) assigning an integer value to every variable. As usual, we extend \( \nu \) in the standard way to a function that maps every term to an element of \( \mathbb{Z} \). For instance, \( \nu(x + 3x + 2) = \nu(x) + 3\nu(y) + 2 \). Given a variable \( x \) and an integer \( n \), we write \( \nu[n/x] \) for the assignment obtained form \( \nu \) by updating the value of the \( x \) to \( n \), i.e. \( \nu[n/x](x) = n \), and for all variables \( y \) distinct from \( x \), \( \nu[n/x](y) = \nu(y) \). Given a formula \( \varphi \) of PAC and an assignment \( \nu \), the satisfaction relation \( \nu \models \varphi \) is defined as usual for linear inequalities, modulo constraints, Boolean connectives and the existential quantifier ranging over \( \mathbb{Z} \). For the counting quantifier, we define

\[
\nu \models \exists 2^x y \varphi \text{ if and only if } \#\{n \in \mathbb{Z} | \nu[n/y] \models \varphi\} \geq \nu(x).
\]

Informally, \( \exists 2^x y \varphi \) is satisfied by \( \nu \) whenever there are at least \( \nu(x) \) distinct values for the variable \( y \) that make the formula \( \varphi \) true. A formula \( \varphi \) of PAC is satisfiable whenever there is an assignment \( \nu \) such that \( \nu \models \varphi \). Two formulae \( \varphi \) and \( \psi \) are equivalent, written \( \varphi \equiv \psi \), whenever they are satisfied by the same set of assignments.

**Syntactic abbreviations.** We define \( \bot \equiv 0 < 0 \) and \( \top \equiv \neg \bot \). The Boolean connectives \( \lor, \rightarrow \) and \( \iff \) and the universal first-order quantifier \( \forall \) are derived as usual, and so are the (in)equalities \( <, \leq, =, \geq, \) and \( > \), between terms. For instance, \( t_1 < t_2 \) corresponds to \( t_1 - t_2 < 0 \), where we tacitly manipulate \( t_1 - t_2 \) with standard operation of linear arithmetic in order to obtain an equivalent term. Given two terms \( t_1 \) and \( t_2 \), and \( q \in \mathbb{N}_+ \), we write \( t_1 \equiv_q t_2 \) for the modulo constraint \( t_1 - t_2 \equiv 0 \mod q \). For a variable \( x \in X \) and \( r \in \{q\} \), we call \( x \equiv_q r \) a simple modulo constraint. All modulo constraints introduced by our quantifier elimination procedure given in Section 3 are simple.

We now introduce the counting quantifiers \( \exists 2^c y \), \( \exists c^x y \) and \( \exists (x,q) y \), where \( x, y \in X \), \( c \in \mathbb{Z} \) and \( q \geq 1 \), and both \( c \) and \( q \) are encoded in binary. Let \( \nu \) be an assignment. Informally, \( \nu \models \exists 2^c y \varphi \) if and only if there are at least \( c \) values for the variable \( y \) that make \( \varphi \) true. Similarly, \( \nu \models \exists c^x y \varphi \) if and only if there are exactly \( \nu(x) \) values for the variable \( y \) that make \( \varphi \) true. Finally, \( \nu \models \exists (x,q) y \varphi \) if and only if the number of values for the variable \( y \) that make \( \varphi \) true is congruent to \( \nu(x) \) modulo \( q \). The formal definition of these three counting quantifiers is given below, where \( z \) is a variable not occurring in \( \text{fv}(\varphi) \),

\[
\exists 2^c y \varphi \equiv \exists z (z = c \land \exists 2^z y \varphi),
\exists c^x y \varphi \equiv (\exists 2^x y \varphi) \land \neg \exists z (z = x + 1 \land \exists 2^z \varphi),
\exists (x,q) y \varphi \equiv \exists z (z \equiv_q x \land \exists 2^z y \varphi).
\]

The counting quantifier \( \exists 2^c y \) is the counting quantifier considered in [10], whereas the modulo counting quantifier \( \exists (x,q) y \) is a generalisation of the quantifier \( \exists (c^x) y \) introduced in [5], where \( r \) is a fixed natural number instead of a variable. More precisely, \( \exists (c^x) y \varphi \) is equivalent to \( \exists x (x \equiv_q c \land \exists (c^x) y \varphi) \). Finally, notice that the formula \( \exists y \varphi \) is equivalent to \( \exists 2^1 y \varphi \). This fact enables us to eliminate a standard first-order existential quantifier by slightly tweaking the quantifier-elimination procedure for \( \exists 2^c y \varphi \).
Parameters of formulae. Following Oppen [7] and Weispfenning [12], we establish bounds
on the absolute value of the variable assignments that suffice for deciding satisfiability of \( \varphi \).
To this end, we introduce a set of parameters of formulae of PAC:

- \(|\varphi|\) denotes the length of the formula \( \varphi \), i.e., the number of symbols to write down \( \varphi \),
  with numbers encoded in binary,
- \(\text{lin}(\varphi)\) is the set of all linear terms \( t \) that appear in a linear inequality \( t < 0 \) of \( \varphi \) (recall that \( t_1 < t_2 \) is syntactic sugar for \( t_1 - t_2 < 0 \)),
- \(\text{hom}(\varphi)\) is the set of homogeneous linear terms obtained from all terms in \(\text{lin}(\varphi)\) by setting
  their constants to \( 0 \), and
- \(\text{mod}(\varphi)\) is the set of all moduli \( q \in \mathbb{N} \) appearing in a modulo constraint \( t_1 \equiv q \ t_2 \) of \( \varphi \). We always assume \( 1 \in \text{mod}(\varphi) \), even if \( \varphi \) has no modulo constraints.

For \( A \subseteq \mathbb{Z} \) finite set, we write \( \|A\| = \max\{|n| \mid n \in A\} \) for the absolute-value norm of \( A \). For a term \( t \), \( \|t\| \) it the maximum coefficient or constant appearing in \( t \), in absolute value. For a set of terms \( T \), \( \|T\| \equiv \max\{|t| \mid t \in T\} \). For a formula \( \varphi \), we define \( \|\varphi\| \equiv \|\text{lin}(\varphi) \cup \text{mod}(\varphi)\| \).

3 A quantifier elimination procedure for unary counting quantifiers

In this section, we develop a quantifier elimination procedure (QE procedure) for the counting
quantifier \( \exists^2 x y \) that allows us to establish the following result.

\( \blacktriangleright \) Theorem 1. Let \( \varphi \) be quantifier-free. Then \( \exists^2 x y \varphi \) is equivalent to a Boolean combination
of linear inequalities and simple modulo constraints.

Our QE procedure perform a series of formula manipulations that we divide into five steps.
At the end of the \( i \)-th step, the procedure produces a formula \( \Psi_i \) equivalent to the original
formula \( \exists^2 x y \varphi \). Ultimately, \( \Psi_5 \) is a Boolean combination of inequalities and simple modulo
constraints allowing us to establish Theorem 1. In this section, we present the procedure and brieﬂy
discuss its correctness, leaving the computational analysis of parameters \( \text{lin}(\Psi_5), \text{hom}(\Psi_5) \) and \( \text{mod}(\Psi_5) \) to subsequent sections.

Step I: Normalise the coefficients of the variable \( y \). Given the input formula \( \Psi_0 = \exists^2 x y \varphi \),
the first step of the procedure is a standard step for QE procedures for Presburger arithmetic.
It produces an equivalent formula \( \Psi_1 \) in which all non-zero coefficients of \( y \) appearing in
a linear term are normalised to \( 1 \) or \(-1 \). For simplicity, we ﬁrst translate every modulo constraint
in \( \varphi \) into simple modulo constraints, by relying on the lemma below.

\( \blacktriangleright \) Lemma 2. Every modulo constraint \( t \equiv q \ 0 \) is equivalent to a disjunction \( \psi \) of simple
modulo constraints such that \( \text{vars}(\psi) \subseteq \text{vars}(t \equiv q \ 0) \) and \( \text{mod}(\psi) = \{q\} \).

Here is the ﬁrst step of the procedure:

- Using Lemma 2 translate every modulo constraint in \( \varphi \) into simple modulo constraints.
- Let \( k \) be the \( \text{lcm} \) of the absolute values of all coefﬁcients of \( y \) appearing in \( \text{hom}(\varphi) \).
- Let \( \varphi' \) be the formula obtained from \( \varphi \) by applying the following three rewrite rules to
  each linear inequality and simple modulo constraint in which \( y \) appears, where \( t \) is a term,
  \( q \geq 1 \) and \( r \in [q] \):
  - \( ay + t < 0 \quad \rightarrow \quad ky + (k/a) \cdot t < 0 \), if \( a > 0 \),
  - \( ay + t < 0 \quad \rightarrow \quad -ky - (k/a) \cdot t < 0 \), if \( a < 0 \), and
  - \( y \equiv q \ r \quad \rightarrow \quad ky \equiv q kr \).
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Let \( \Psi_1 \equiv \exists x y \{ y \equiv_k 0 \wedge \varphi'[y/ky] \} \).

\( \triangleright \) Claim 3. \( \Psi_0 \leftrightarrow \Psi_1 \), and in \( \Psi_1 \), all non-zero coefficients of \( y \) are either 1 or \( -1 \).

**Step II: Subdivide the formula according to term orderings and residue classes.** We define an ordering of \( n \) linear terms to be a formula of the form

\[
(t_1 <_1 t_2) \land (t_2 <_2 t_3) \land \cdots \land (t_{n-1} <_{n-1} t_n),
\]

where \( \{t_1, \ldots, t_n\} \) is the set being ordered and \( \{ <_1, \ldots, <_{n-1} \} \subseteq \{ <, = \} \).

**Lemma 4.** There is an algorithm that, given a set \( T \) of \( n \) linear terms over \( d \) variables, computes in time \( n^{O(d)} \log \| T \|^{O(1)} \) a set \( \{ O_1, \ldots, O_o \} \) of orderings for \( T \) such that (I) \( \bigwedge_{i \in [1, o]} O_i \) is a tautology, (II) for every \( i \neq j \) in \( [1, o], O_i \lor O_j \) is unsatisfiable, and (III) \( o = O(n^{2d}) \).

Our QE procedure manipulates \( \Psi_1 \) as follows:

- Let \( T \) be the set of all \( y \)-free terms \( t \) such that \( t, y - t \) or \( -y + t \) belong to \( \text{lin}(\Psi_1) \).
- Using Lemma 4, build a disjunction of orderings \( \psi_{\text{ord}} \equiv \bigvee_{i \in [1, o]} O_i \) for the terms \( T \cup \{ 0 \} \).
- \( Z = \text{vars}(\varphi) \) and \( m = \text{lcm}(\text{mod}(\Psi_1)) \).
- For each \( i \in [1, o] \) and every \( r : Z \to [m] \), let \( \Gamma_{i, r} \equiv O_i \land (\bigwedge_{z \in Z} z \equiv_m r(z)) \).
- Let \( \Psi_2 \equiv \bigvee_{i \in [1, o]} \bigvee_{r : Z \to [m]} (\Gamma_{i, r} \land \Psi_1) \).

\( \triangleright \) Claim 5. \( \Psi_1 \leftrightarrow \Psi_2 \).

In Steps III to V of the procedure, we focus on each disjunct separately, iterating over all pairs of \( i \in [1, o] \) and \( r : Z \to [m] \).

**Step III: Split the range of \( y \) into segments.** Recall that \( \Psi_1 = \exists x y \psi \), where \( \psi \) is some Boolean combination of inequalities and modulo constraints with variables from \( \text{vars}(\varphi) \), in which the non-zero coefficients of \( y \) are either 1 or \( -1 \) (by Claim 3). Let \( t_1', \ldots, t_n' \) be all of the terms \( T \cup \{ 0 \} \) that the formula \( O_i \) asserts pairwise non-equal, taken in the ascending order. In other words, we obtain \( t_1', \ldots, t_n' \) by removing from the sequence \( t_1, \ldots, t_n \) in Equation 1 all terms \( t_j \) for which \( g_j \) is \( = \). Let \( \text{seg}(y, O_i) \) be the set of formulae

\[
\big\{ y < t_1', \ y = t_1', \ (t_1' < y \land y < t_2'), \ y = t_2', \ \ldots, \ (t_{\ell - 1}' < y \land y < t_\ell'), \ y = t_\ell', \ t_\ell' < y \big\}
\]

We have \( \#(\text{seg}(y, O_i)) = 2\ell + 1 \). Given \( \kappa \in \text{seg}(y, O_i) \), the formula \( O_i \land \kappa \) imparts a linear ordering on the terms \( T \cup \{ 0, y \} \). This enables us to “almost evaluate” the formula \( \psi \):

\( \triangleright \) Claim 6. For every \( \kappa \in \text{seg}(y, O_i) \), there is a Boolean combination \( \psi_{\kappa,r}^{t,r} \) of simple modulo constraints s.t. \( \text{vars}(\psi_{\kappa,r}^{t,r}) = \{ y \} \), \( \text{mod}(\psi_{\kappa,r}^{t,r}) \subseteq \text{mod}(\psi) \) and \( \Gamma_{i, r} \land \kappa \land \psi_{\kappa,r}^{t,r} \leftrightarrow \Gamma_{i, r} \land \kappa \land \psi_{\kappa,r}^{t,r} \).

The procedure continues as follows:

- Let \( \text{seg}(y, O_i) = \{ \kappa_0, \ldots, \kappa_{2\ell} \} \) and, for every \( j \in [0, 2\ell] \), take \( \psi_{\kappa_j}^{t,r} \) from Claim 6
- Let \( \Psi_3' \equiv \exists x y \{ x \leq x_0 + \cdots + x_{2\ell} \land \bigwedge_{j \in [0, 2\ell]} \exists x y (\kappa_j \land \psi_{\kappa_j}^{t,r}) \} \).
- Let \( \Psi_3 \equiv \bigvee_{i \in [1, o]} \bigvee_{r : Z \to [m]} (\Gamma_{i, r} \land \Psi_3') \).

\( \triangleright \) Claim 7. \( \Psi_2 \leftrightarrow \Psi_3 \).
Step IV: Compute the number of solutions for each segment. We next aim to eliminate the counting quantifiers introduced in Step III in the sub-formulae $\exists \exists x: y (\kappa_j \land \psi_{k_i}^{t_j})$. We go over each $\kappa \in \text{seg}(y, O_i)$, and consider three cases depending on whether it specifies (syntactically) an infinite interval, a finite segment, or a single value for $y$.

Notice that $r$ is in fact an assignment to variables, so $r(t) \in \mathbb{Z}$ is well-defined for every term $t$ with free variables $Z$. Compute the following numbers for $j \in [1, \ell]$:

- $c_j$ is 1 if the assignment $y \mapsto r(t'_j)$ satisfies $\psi_{k_i}^{t_j}$ where $\kappa = (y = t'_j)$ and 0 otherwise.

For $j \in [2, \ell]$:

- $p_j \in [0, m]$ is the number of $y \in [m]$ satisfying $\psi_{k_i}^{t_j}$,
- $u_j = (r(t'_j - 1) \mod m)$ and $\pi_j$ is the smallest integer congruent to $r(t'_j) \mod m$ and $> u_j$,
- $r'_j \in [0, m]$ is the number of $y \in [u_j + 1, \pi_j - 1]$ satisfying $\psi_{k_i}^{t_j}$,
- $r_j = -p_j \cdot (\pi_j - u_j) + m \cdot r'_j$.

**Lemma 8.** Given a formula $\psi_{k_i}^{t_j}$ and $m, u_j, \pi_j$, the numbers $p_j$ and $r'_j$ can be computed in $\#P$, or by a deterministic algorithm with running time $O(m \cdot |\psi_{k_i}^{t_j}|)$.

The numbers $c_j, p_j, r_j$ determine, for each formula $\kappa \in \text{seg}(y, O_i)$, how many assignments to the variable $y$ satisfy the formula $\psi_{k_i}^{t_j}$ in the conjunction $\Gamma_{i, r} \land \kappa \land \psi_{k_i}^{t_j}$. Intuitively, this is $c_j$ for $\kappa$ of the form $y = t'_j$, and $(p_j(t'_j - t'_j - 1) + r_j)/m$ for $\kappa$ of the form $t'_j < y \land y < t'_j$.

We say “intuitively” here, because in the latter case the expression above depends on other variables so is not, strictly speaking, a number. The following claims formalise this.

▷ Claim 9. Let $\kappa \in \{y < t'_1, t'_1 < y\}$. If $\exists y (\kappa \land \psi_{k_i}^{t_j})$ is satisfiable, then $\Gamma_{i, r} \land \psi_{k_i}^{t_j} \leftrightarrow \Gamma_{i, r}$.

▷ Claim 10. Let $\kappa$ be the formula $y = t'_j$ for some $j \in [1, \ell]$ and let $z$ be a fresh variable. Then $\Gamma_{i, r} \land \exists y (\kappa \land \psi_{k_i}^{t_j}) \leftrightarrow \Gamma_{i, r} \land z \leq c_j$.

▷ Claim 11. Let $\kappa$ be the formula $t'_j - 1 < y \land y < t'_j$ for some $j \in [2, \ell]$ and let $z$ be a fresh variable. Then $\Gamma_{i, r} \land \exists y (\kappa \land \psi_{k_i}^{t_j}) \leftrightarrow \Gamma_{i, r} \land mz \leq p_j(t'_j - t'_j - 1) + r_j$.

The procedure replaces each disjunct of $\Psi_3$ with a new formula as follows:

- Let $\Psi_4 \equiv \top$ if $\exists y (\kappa \land \psi_{k_i}^{t_j})$ is satisfiable for some $\kappa \in \{y < t'_1, t'_1 < y\}$; otherwise let $\Psi_4 \equiv \exists x_2 \ldots \exists x_\ell \left( x \leq \sum_{j=2}^{\ell} x_j + \sum_{j=1}^{\ell} c_j \land \bigwedge_{j \in [2, \ell]} m x_j \leq p_j(t'_j - t'_j - 1) + r_j \right)$.
- Let $\Psi_4 \equiv \bigvee_{i \in [1, \ell]} \bigvee_{r: z \mapsto [m]} (\Gamma_{i, r} \land \psi_{k_i}^{t_j})$.

▷ Claim 12. $\Psi_3 \leftrightarrow \Psi_4$.

Step V: Sum up the numbers of solutions. It remains to get rid of the variables $x_i$ introduced earlier. For each disjunct $\Gamma_{i, r} \land \psi_{k_i}^{t_j}$ of $\Psi_4$, we use the notation from Step IV.

- Let $\Psi_5 \equiv \top$ if $\forall z: \psi_{k_i}^{t_j}$; otherwise let $\Psi_5 \equiv \exists x_2 \ldots \exists x_{\ell-1} \left( z \leq \sum_{j=2}^{\ell-1} p_j(t'_j - t'_j - 1) + r_j \right)$.
- Let $\Psi_5 \equiv \bigvee_{i \in [1, \ell]} \bigvee_{r: z \mapsto [m]} (\Gamma_{i, r} \land \psi_{k_i}^{t_j})$.

The procedure terminates with $\Psi_5$ as output. The following claim implies Theorem 4.

▷ Claim 13. $\Psi_4 \leftrightarrow \Psi_5$. The formula $\Psi_5$ is quantifier-free.
4 Discussion and summary of results, and roadmap

The QE procedure for a single counting quantifier $\exists \geq x y$ from Section 3 forms the basis of our results. In this section we discuss its use and lay out its applications.

Analysis of the procedure. The next lemma tells us how fast formulae and their parameters grow in ourQE procedure.

Lemma 14. Let the formula $\Psi_5$ be obtained by applying the quantifier elimination procedure from Section 3 to a formula $\exists \geq x y \varphi$, where $\varphi$ is quantifier-free and $\#\text{vars}(\varphi) = d$. Then:

- $\text{mod}(\Psi_5) = \{ m \}$ with $m = k \cdot \text{lcm}(\text{mod}(\varphi))$ and $k \leq \|\text{hom}(\varphi)\|^{\#\text{hom}(\varphi)}$,
- $\#\text{lin}(\Psi_5) \leq N^{O(d)}$, $\|\text{lin}(\Psi_5)\| \leq O(N) \cdot \|\text{lin}(\varphi)\|$, $\#\text{hom}(\Psi_5) \leq N^{O(d)}$, $\|\text{hom}(\Psi_5)\| \leq O(N) \cdot \|\text{hom}(\varphi)\|$, where $N = m^2 \cdot \#\text{lin}(\varphi)$.

A trivial consequence of Lemma 14 is that the QE procedure from Section 3 gives an algorithm for deciding a formula $\varphi$ of Presburger arithmetic with counting quantifiers $\exists \geq z y$ in time $2^{\cdots 2}$, where the height of the tower is at most $O(|\varphi|)$.

Remark 15. With minor changes to the procedure, one can eliminate quantifiers $\exists = x y$ in addition to $\exists \geq x y$, with the same complexity bounds as in Lemma 14. Because of space constraints, this is only briefly described in Section 6 and further details are relegated to Appendix D.

Let us pinpoint where the non-elementary blow-up appears if the procedure is applied multiple times to eliminate all quantifiers from a formula. Putting together the upper bounds and equations given by Lemma 14 for $\#\text{hom}(\Psi_5)$, $N$, $m$, and $k$, we observe that the upper bound for $\#\text{hom}(\Psi_5)$ is exponential in $\#\text{hom}(\varphi)$. This means that more fine-grained bounds are necessary for decision procedures with elementary complexity, i.e., with running time bounded from above by a $k$-fold exponential in the size of the input formula.

Tracing the exponential dependence of $\#\text{hom}(\Psi_5)$ on $\#\text{hom}(\varphi)$ back to the QE procedure, one can see that the quantity $k$ from Lemma 14 stems from computing the least common multiple of the coefficients at $y$ in Step I of the procedure. Each of them can be as big as $\|\text{hom}(\varphi)\|$, and there can be $\#\text{hom}(\varphi)$-many of them. Unfortunately, there does not appear to be a stronger upper bound on the magnitude of their common multiple, even in subsequent rounds of the QE procedure. Indeed, $y$-free terms $t'_{j_1}, \ldots, t'_{j_d}$ in the remaining variables do not only get subtracted from one another in $\Psi_5^{t'}$, but also get multiplied by factors $p_j$ as they are in formulae $\Psi_4^{t'}$. These factors, computed at the beginning of Step IV, represent the limit density of suitable assignments for $y$ in the intervals $t'_{j_{i-1}} < y < t'_{j_i}$ that are long enough. As such, they are model counts of univariate quantifier-free formulae $\psi^{t';r}$, so a priori nothing prevents many different factors $p_j$ from taking different values in the range $[0, m]$ and contributing to a big least common multiple in the next QE round.

3ExpTime decision procedures. We will now explain how this growth of parameters can be countered for more restricted quantifiers, arriving at a 3ExpTime quantifier elimination and decision procedures. This analysis relies on and extends Lemma 14. The following theorem is our main result.

Theorem 16. There is a 3ExpTime quantifier elimination procedure for Presburger arithmetic with threshold counting quantifiers $\exists \geq y$.
In essence, the procedure of Theorem 16 is our main QE procedure from Section 3 that treats the quantifier $\exists^c y$ as if it were $\exists z y$. After substituting $c$ for $z$ at the end, we are able to improve the bound on $\#\text{hom}(\Psi_5)$ from Lemma 14. This results in an elementary decision procedure. We discuss details in Section 5.

Similarly to the case of quantifiers $\exists x y$ mentioned in Remark 15, with very minor changes to the procedure from Section 3, one can eliminate quantifiers $\exists(x^y)$ too, with the same complexity bounds as in Lemma 14.

| Theorem 17. There is a $3\text{ExpTime}$ quantifier elimination procedure for Presburger arithmetic with modulo counting quantifiers $\exists(x^y)$.

A proof outline for Theorem 17 is given in Section 6.

5 Eliminating threshold counting quantifiers

In this section, we extend the quantifier elimination procedure of Section 3 in order to directly deal with the threshold counting quantifiers $\exists^c y$. Afterwards, we provide the complexity analysis of the quantifier elimination procedure.

Consider a formula $\Psi_0 = \exists^c y \varphi$ with $\varphi$ quantifier free. By definition, $\Psi_0$ is equivalent to the formula $\exists z (z = c \wedge \exists^c y \varphi)$ for some variable $z$ not occurring in $\varphi$. In order to eliminate the threshold counting quantifier $\exists^c y$, we first perform the quantifier elimination procedure described in Section 3 on input $\exists^c y \varphi$, obtaining the formula $\Psi_5$. We then eliminate the existential quantifier $\exists z$ from the formula $\exists z (z = c \wedge \Psi_5)$ by relying on the ad hoc procedure we now describe. As explained in Section 3, the prefix $\exists z (z = c \wedge)$ allows us to drastically simplify the set of homogeneous terms in $\Psi_5$, leading to $3\text{ExpTime}$.

Dealing with threshold quantifiers in a single step. With $\Psi_0$ and $\Psi_5$ defined as above, we have $\Psi_0 \iff \exists z (z = c \wedge \Psi_5) \iff \Psi_5[c/z]$. Recall that $\Psi_5$ is defined as

$$\Psi_5 \equiv \bigvee_{i \in [1, o]} \bigvee_{r : Z \rightarrow [m]} (\Gamma_{i, r} \wedge \Psi_{i, r}^{t, r}).$$

Here, $Z = \text{vars}(\varphi)$, $m = \text{lcm(mod($\Psi_1$))}$ (defined as in the Step II of the procedure) and $\Gamma_{i, r}$ is a conjunction of inequalities and simple modulo constraints with variables from $Z$ (hence, $z$-free). Therefore, $\Gamma_{i, r}[c/z] = \Gamma_{i, r}$. Moreover, $\Psi_{i, r}^{t, r}$ is either $T$ or a formula of the form

$$mz \leq p_2(t'_2 - t'_{1}) + p_2(t'_2 - t'_{1} - 1) + r_{\ell} + m(c_1 + \cdots + c_{\ell}). \quad (2)$$

where the terms $t'_1, \ldots, t'_\ell$ are from $T \cup \{0\}$ (with $T$ defined as in Step II of Section 3), and thus written with variables from $Z$. Therefore, the following property holds:

| Claim 18. In $\Psi_5$, $z$ only appears on the left hand side of inequalities of the form $2$. |

We manipulate the disjuncts of $\Psi_5[c/z]$ separately. Fix $i \in [1, o]$ and $r : Z \rightarrow [m]$. We define a formula equivalent to $\Psi_{5, r}^{t, r}[c/z]$ by relying on the following lemma, where $d = \text{vars}(\Psi_{5, r}^{t, r})$.

| Lemma 19. Consider $\Psi_{5, r}^{t, r}$ as in (2). Let $e = m(c - \sum_{j=1}^\ell c_j) - \sum_{j=2}^\ell r_j$. It is possible to compute in time $(e + \ell)^O(d) \log(e \cdot \|\Psi_{5, r}^{t, r}\|^2)O(1)$ a formula $\gamma_{i, r} = \bigvee_{(t'_{j}, \ldots, t'_{j} \in T) : j \in [2, \ell]} t'_j - t'_{j-1} \geq i_j$ such that (1) $I \subseteq [0, e]^\ell$, (2) $\#I \leq O((e + \ell)^{2d})$, and (3) $\Gamma_{i, r} \wedge \Psi_{5, r}^{t, r}[c/z] \iff \Gamma_{i, r} \wedge \gamma_{i, r}$.

To prove Lemma 19 we apply Lemma 4 on the set of terms $\{t'_j - t'_{j-1} : j \in [2, \ell]\} \cup [0, e]$, and manipulate the resulting tautology to filter out orderings that do not satisfy $\Psi_{5, r}^{t, r}[c/z]$.

The QE procedure proceeds as follows.

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For every $i \in [1, o]$ and $r : \mathbb{Z} \to [m]$,

- if $\Psi^{i,r}_5 = \top$, then let $\Psi^{i,r}_6 \equiv \top$,
- else let $\Psi^{i,r}_6 \equiv \gamma_{i,r}$, according to Lemma 19.

Let $\Psi^c_6 \equiv \bigvee_{i \in [1, o]} \bigvee_{r: \mathbb{Z} \to [m]} (\Gamma_{i,r} \land \Psi^{i,r}_6)$.

After defining $\Psi^c_6$, the procedure ends. Notice that the inequalities $t'_{j} - t'_{j-1} \geq i_j$ that replace the inequalities given in (2) are such that $t'_{j-1}, t'_j \in T \cup \{0\}$. This leads to a better bound on the size of the set $\text{hom}(\Psi^c_6)$ (more precisely, quadratic on $\text{#hom}(\varphi)$), which ultimately enables us to establish the 3\text{ExpTime} membership of Presburger arithmetic with threshold counting quantifiers.

▷ Claim 20. $\Psi_0 \leftrightarrow \Psi^c_6$. The formula $\Psi^c_6$ is quantifier-free.

Proof idea for Theorem 16

The key role in the analysis is played by the following lemma.

▶ Lemma 21. $\text{#hom}(\Psi^c_6) = O(\text{#hom}(\varphi)^2)$.

As already stated, this quadratic bound is key in order to obtain an elementary decision procedure. In particular, this improvement over the “baseline” Lemma 14 leads to the following bounds on the elimination of an arbitrary number of threshold counting quantifiers.

▶ Lemma 22. Let $\varphi$ be a formula of Presburger arithmetic with threshold counting quantifiers. There is an equivalent quantifier-free formula $\Psi$ such that

$\text{#lin}(\Psi), \|\text{lin}(\Psi)\|, \|\text{hom}(\Psi)\|, \|\text{mod}(\Psi)\|$ are at most $2^{3^O(|\varphi|^2)}$,

$\text{#hom}(\Psi) \leq 2^{2^O(|\varphi|^2)}$ and $\text{#mod}(\Psi) \leq |\varphi|$.

Proof idea. In a nutshell, elementary upper bounds of Lemma 22 are obtained by first iterating Lemma 21 across all quantifier elimination rounds. This results in a doubly exponential bound on the cardinalities of sets $\text{hom}(\Psi^c_6)$ throughout the entire procedure. With this bound in hand, exponentiation in the right-hand side of the inequalities of Lemma 14 does not blow the parameters above triple exponential.

Theorem 16 follows by combining Lemma 22 with upper bounds on the running time of a single quantifier elimination round. These upper bounds are all subsumed by the size of the obtained formulae, except possibly for the procedures of Lemmas 4 and 19 and the model counting procedure of Lemma 8. For Lemmas 4 and 19, the running time is only exponential in the size of the original formula, and thus runs in polynomial time on the size of the obtained formula, as soon as this formula has size at least exponential. For Lemma 8, observe that the factor $m$ in the running time cannot be more than the product of all elements of the set $\text{hom}(\Psi)$. Hence, the bounds of Lemma 22 suffice for a triply exponential time overall.

6 Eliminating modulo counting quantifiers

Consider a formula $\Psi_0 = \exists(x,y) \varphi$, where $\varphi$ is quantifier-free. By definition, $\Psi_0$ is equivalent to $\exists z (z \equiv x \land \exists^=z \varphi)$, where $z$ is a variable not occurring in $\varphi$. Thus, in order to eliminate a modulo counting quantifier, it makes sense to piggyback on a quantifier elimination procedure for the $\exists^=z$ counting quantifier. Due to space constraints, we only briefly describe the main aspects of eliminating an $\exists^=z$ counting quantifier, further details can be found
We manipulate variables in Steps VII: Eliminate existential quantifiers. We conclude the procedure by manipulating Claim 24. ▷

Let \( \Psi_5^\equiv \) be the formula obtained from performing the quantifier-elimination procedure for the \( \exists^\equiv x y \) counting quantifier on \( \exists^\equiv x y \varphi \), so that \( \Psi_0 \leftrightarrow \exists x (z \equiv x \land \Psi_5^\equiv) \). We have that \( \Psi_5^\equiv \) is defined as
\[
\Psi_5^\equiv \equiv \bigvee_{i \in [1, o]} \bigwedge_{r: Z \rightarrow [m]} (\Gamma_{i,r} \land \Psi_5^{i,r}),
\]
where \( Z = \text{vars}(\varphi), m = \text{lcm}(\text{mod}(\Psi_1)) \) and \( \Gamma_{i,r} = O_i \land (\bigwedge_{w \in Z} w \equiv m, r(w)) \) is a conjunction of an ordering \( O_i \) and simple modulo constraints \( w \equiv m r(w) \) with variables from \( Z \) (and so, \( z \)-free). Moreover, \( \Psi_5^{i,r} \) is either \( \perp \) or a formula of the form
\[
mz = \sum_{j=2}^{\ell} (p_j (t'_j - t'_{j-1}) + r_j) + m \cdot \sum_{j=1}^{\ell} c_j,
\]
where the terms \( t'_1, \ldots, t'_\ell \) are from \( T \cup \{0\} \) (where \( T \) is defined as in Step II of Section 3), and hence \( z \)-free. Analogously to Section 3 Claim 18 the following property holds.
▷ Claim 23. In \( \Psi_5^\equiv \), \( z \) only appears on the left hand side of equalities of the form (3).

We manipulate \( \exists z (z \equiv x \land \Psi_5^\equiv) \) with the following steps, denoted by VI and VII to stress the fact that they are performed after the five steps of the \( \exists^\equiv x y \) quantifier elimination procedure.

**Step VI: Subdivide the formula according the residue classes (again).** To efficiently eliminate the existential quantifier of the formula \( \exists z (z \equiv x \land \Psi_5^\equiv) \), we first guess the residue classes of all variables in \( Z \cup \{x\} \) modulo \(mq\) (instead of just \( m \), as done in \( \Psi_5^r \) for the variables in \( Z \)).

- Let \( \gamma = \bigvee_{s: (Z \cup \{x\}) \rightarrow [mq]} \bigvee_{i \in [1, o]} \bigwedge_{r: Z \rightarrow [m]} ((\bigwedge_{w \in Z \cup \{x\}} w \equiv m q s(w)) \land \Gamma_{i,r} \land \Psi_5^{i,r}) \).

- For every \( s: (Z \cup \{x\}) \rightarrow [mq] \), every \( i \in [1, o] \), and every \( r: Z \rightarrow [m] \), consider the disjunct \((\bigwedge_{w \in Z \cup \{x\}} w \equiv m q s(w)) \land \Gamma_{i,r} \land \Psi_5^{i,r}\) of the formula \( \gamma \) and evaluate every modulo constraint \( w \equiv m r(w) \) in \( \Gamma_{i,r} (w \in Z) \) to \( \perp \) or \( \top \), according to the truth of \( s(w) \equiv m r(w) \).

Since every function \( r: Z \rightarrow [m] \) can be seen as a partial function from \( Z \cup \{x\} \) to \([mq]\), after the two steps above, for every \( s: (Z \cup \{x\}) \rightarrow [mq] \) and \( i \in [1, o] \), all but one disjunct of the subformula \( \bigvee_{r: Z \rightarrow [m]} ((\bigwedge_{w \in Z \cup \{x\}} w \equiv m q s(w)) \land \Gamma_{i,r} \land \Psi_5^{i,r}) \) of \( \gamma \) evaluate \( \perp \). We conclude that \( \gamma \) is equivalent to
\[
\bigvee_{i \in [1, o]} \bigwedge_{r: (Z \cup \{x\}) \rightarrow [mq]} (\Gamma_{i,s} \land \Psi_5^{i,s}),
\]
where \( \Gamma_{i,s} \equiv O_i \land (\bigwedge_{w \in Z \cup \{x\}} w \equiv m q s(w)) \) and \( \Psi_5^{i,s} \) is \( \perp \) or a formula as described in 3.

Let \( \Psi_6^{(x,q)} \equiv \bigvee_{i \in [1, o]} \bigwedge_{r: (Z \cup \{x\}) \rightarrow [mq]} \exists z (z \equiv x \land \Gamma_{i,s} \land \Psi_5^{i,s}) \), the following holds:
▷ Claim 24. \( \exists z (z \equiv x \land \Psi_5^\equiv) \leftrightarrow \Psi_6^{(x,q)} \).

**Step VII: Eliminate existential quantifiers.** We conclude the procedure by manipulating each disjunct of \( \Psi_6^{(x,q)} \) separately. Fix \( i \in [1, o] \) and \( s: (Z \cup \{x\}) \rightarrow [mq] \). We aim at defining a quantifier-free formula \( \gamma_{i,s} \) equivalent to the disjunct \( \exists z (z \equiv x \land \Gamma_{i,s} \land \Psi_5^{i,s}) \) of \( \Psi_6^{(x,q)} \).

- If \( \Psi_5^{i,s} = \perp \) then let \( \gamma_{i,s} = \perp \).
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We developed a QE procedure for Presburger arithmetic extended with the unary threshold quantifiers, 

\[ m \equiv_{mq} \sum_{j=2}^{\infty} (p_j(t_j' - t_j') + r_j) + m \cdot \sum_{j=1}^{\infty} c_j. \]

By Claim 23, this formula has variables from \( Z \cup \{x\} \). Evaluate this formula on \( s \). If it is found to be equivalent to \( \perp \), let \( \gamma_{i,s} \equiv \perp \). Otherwise, let \( \gamma_{i,s} \equiv \Gamma_{i,s} \).

\[ \text{Let } \Psi_7^{(x,q)} \equiv \bigwedge_{i \in [1,\infty]} \bigvee_{s \in (\mathbb{Z} \cup \{x\}) \to \{mq\}} \gamma_{i,s}. \]

After defining \( \Psi_7^{(x,q)} \), the procedure ends. Notice that all the disjuncts of \( \Psi_7^{(x,q)} \) are either \( \perp \) or \( \Gamma_{i,s} = O_i \wedge \bigwedge_{w \in Z \cup \{x\}} \{w \equiv_{mq} s(w)\} \), where \( \text{vars}(O_i) \subseteq Z \cup \{x\} \). The (in)equalities appearing in \( O_i \) are of the form \( t \prec t' \), where \( \prec \in \{<,=\} \) and \( t, t' \in T \cup \{0\} \). Exactly as in the case of threshold quantifiers, this leads to \( \#\text{hom}(\Psi_7^{(x,q)}) \leq (\#T \cup \{0\})^2 \), which ultimately leads to a \( 3\text{ExpTime} \) running time for the QE procedure.

\[ \text{Claim 25. } \Psi_6^{(x,q)} \leftrightarrow \Psi_7^{(x,q)}. \] The formula \( \Psi_7^{(x,q)} \) is quantifier-free.

\[ \textbf{Proof idea for Theorem 17} \]

The key role in the analysis is played by the following lemma.

\[ \textbf{Lemma 26. } \#\text{hom}(\Psi_7^{(x,q)}) = O(\#\text{hom}(\varphi)^2). \]

When eliminating an arbitrary number of \( \exists^{(x,q)} y \), Lemma 26 leads to the following result.

\[ \textbf{Lemma 27. Let } \varphi \text{ be a formula of Presburger arithmetic with threshold quantifiers. There is an equivalent quantifier-free formula } \Psi \text{ such that } \]

\[ \#\text{lin}(\Psi), \#\text{lin}(\Psi), \#\text{hom}(\Psi), \#\text{mod}(\Psi) \text{ are at most } 2^{O(|\varphi|^2)}, \]

\[ \#\text{hom}(\Psi) \leq 2^{O(|\varphi|^2)} \text{ and } \#\text{mod}(\Psi) \leq |\varphi|. \]

Lemma 27 allows us to establish that the QE procedure for Presburger arithmetic with modulo counting quantifiers runs in \( 3\text{ExpTime} \). The proof follows the pattern of Theorem 16.

7 Conclusion

We developed a QE procedure for Presburger arithmetic extended with the unary threshold counting quantifier \( \exists^{(x,q)} y \) that runs in \( 3\text{ExpTime} \), i.e., at no additional cost compared to standard QE procedures for Presburger arithmetic, see e.g. [7]. From the estimation of the growth of the constants occurring in our QE procedure, using standard relativisation arguments, see e.g. [12], we can derive that the decision problem for Presburger extended with the \( \exists^{(x,q)} \) quantifier is in \( 2\text{ExpSpace} \). This matches the complexity of deciding standard Presburger arithmetic closely. Indeed, the latter is complete for the complexity class \( \text{STA}(\ast, 2^{\text{poly}(n)}, O(n)) \) [3]. Fully settling the complexity of Presburger arithmetic extended with \( \exists^{(x,q)} y \) will likely require generalising the STA complexity measure, which we leave as an interesting avenue for further investigation.

Our QE procedure is based on a QE procedure for the more general \( \exists^{(x,q)} y \) counting quantifier that we developed in this paper. While the latter procedure slightly improves the QE procedure given by Schweikardt [10], it still only runs in non-elementary time. We have pinpointed precisely at where the non-elementary growth occurs. It remains to be seen whether our QE procedure can be further improved, or whether, possibly based on the insights obtained from our QE procedure, a non-elementary lower bound for Presburger arithmetic extended with the \( \exists^{(x,q)} y \) quantifier can be established.
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A Missing proofs from Section [3]

Lemma 2. Every modulo constraint \( t \equiv_q 0 \) is equivalent to a disjunction of simple modulo constraints such that \( \text{vars}(\psi) \subseteq \text{vars}(t \equiv_q 0) \) and \( \text{mod}(\psi) = \{q\} \).

Proof. Let \( Z = \text{vars}(t \equiv_q 0) \). We guess the residue classes of the variables in \( Z \), as shown in the right hand side of the following equivalence:

\[
t \equiv_q 0 \iff \bigvee_{r : Z \rightarrow [q]} (t \equiv_q 0 \land \bigwedge_{z \in Z} z \equiv_q r(z)).
\]

Fix \( r : Z \rightarrow [q] \), and consider the disjunct \( (t \equiv_q 0 \land \bigwedge_{z \in Z} z \equiv_q r(z)) \). Let \( x \) be a variable occurring in \( t \). As \( r \) assigns to \( x \) a residue class modulo \( q \), the following equivalence holds:

\[
t \equiv_q 0 \land \bigwedge_{z \in Z} z \equiv_q r(z) \iff (t \equiv_q 0)[r(x)/x] \land \bigwedge_{z \in Z} z \equiv_q r(z).
\]

Therefore, by substituting in \( t \) every variable \( x \) with \( r(x) \), we derive

\[
t \equiv_q 0 \land \bigwedge_{z \in Z} z \equiv_q r(z) \iff r(t) \equiv_q 0 \land \bigwedge_{z \in Z} z \equiv_q r(z).
\]

Since \( r(t) \equiv_q 0 \) does not have free variables (i.e. it is a statement), it is equivalent to \( T \) or \( \bot \). Let \( \psi_r \in \{T, \bot\} \) such that \( r(t) \equiv_q 0 \iff \psi_r \), and let \( \psi = \bigvee_{r : Z \rightarrow [q]} (\psi_r \land \bigwedge_{z \in Z} z \equiv_q r(z)) \).

The formula \( \psi \) satisfies the required properties.

Claim 3. \( \Psi_0 \iff \Psi_1 \), and in \( \Psi_1 \), all non-zero coefficients of \( y \) are either 1 or \(-1\).

Proof. We show the following sequence of equivalences:

\[
\Psi_0 \iff \exists \geq x y \varphi' \quad \text{(4)}
\]

\[
\iff \exists \geq z y \exists (z = ky \land \varphi'[z/ky]) \quad \text{for some fresh variable } z \quad \text{(5)}
\]

\[
\iff \exists \geq z \exists y (z = ky \land \varphi'[z/ky]) \quad \text{(6)}
\]

\[
\iff \exists \geq z (z \equiv_k 0 \land \varphi'[z/ky]) \quad \text{(7)}
\]

\[
\iff \exists \geq y (y \equiv_k 0 \land \varphi'[y/ky]) = \Psi_1 \quad \text{(8)}
\]

The equivalence (4) holds, because the rewrite rules used to produce \( \varphi' \) from \( \varphi \) come from biconditional axioms of (modular) arithmetic, e.g. \( a \equiv_b c \iff ka \equiv_kb \) for all \( k \geq 1 \). In \( \varphi' \), all non-zero coefficients of \( y \) are either \( k \) or \(-k \). This directly establishes equivalence (5).

In the formula \( \varphi'[z/ky] \) (and thus in \( \Psi_1 \)) all non-zero coefficients of \( y \) are either 1 or \(-1\). The equivalence (6) holds as the expression \( z = ky \) induces a bijection between all possible values of \( y \) and \( z \). The equivalence (7) holds as \( y \) does not occur in \( \varphi'[z/ky] \). Notice that \( z \equiv_k 0 \iff \exists y z = ky \). The equivalence (8) follows as we rename \( z \) by \( y \).

Lemma 4. There is an algorithm that, given a set \( T \) of \( n \) linear terms over \( d \) variables, computes in time \( n^{O(d)} \log \|T\|^{O(1)} \) a set \( \{O_1, \ldots, O_n\} \) of orderings for \( T \) such that (I) \( \bigvee_{i \in [1 \ldots n]} O_i \) is a tautology, (II) for every \( i \neq j \) in \( [1 \ldots n] \), \( O_i \cap O_j \) is unsatisfiable, and (III) \( o = O(n^{2d}) \).

Proof. We first show the existence of a family of orderings with required properties. This part of the proof relies on the insight that \( n \) hyperplanes split \( \mathbb{R}^d \) into \( O(n^d) \) regions. This is the basis of multiple “geometric” decision procedures and algorithms; see, e.g., [11, 13].

Claim 28. Given \( s_1, \ldots, s_m \) linear terms over \( d \) variables, there are at most \( O(m^d) \) conjunctions of the form

\[
(s_1 R_1 0) \land (s_2 R_2 0) \land \ldots \land (s_m R_m 0),
\]

where \( R_i \in \{<, =, >\} \), that are satisfiable over the reals \( \mathbb{R} \).
Proof. This is a small variation of the classic proof, using double induction. For \( d = 1 \), the number of such conjunctions is clearly at most \( 2m + 1 \), because \( m \) points can split the line into (at most) \( m + 1 \) finite or infinite open intervals and \( m \) points themselves.

For larger \( d \), we proceed as follows. We assume an \( \mathcal{O}(m^{d-1}) \) bound for dimension \( d - 1 \). For \( m = 1 \), the number of satisfiable conjunctions is at most 3. Let us deal with larger \( m \) now. Suppose we have already computed (or, rather, bounded from above) the number of satisfiable conjunctions of terms \( s_1, \ldots, s_{j-1} \), and suppose this number is \( N \). Consider what happens when the term \( s_j \) is added to them. Each region of \( \mathbb{R} \) that corresponds to one of the \( N \) satisfiable conjunctions of \( s_1, \ldots, s_{j-1} \) can be “cut” by the new term into at most 3 regions, according to whether \( s_j < 0 \), \( s_j = 0 \), or \( s_j > 0 \) (and thus adding 2 new regions). This is the only way new regions, and thus satisfiable conjunctions of the form \((s_1 R_1 0) \land \ldots \land (s_j R_j 0)\) are composed. However, we can now observe that the number of regions that are “cut” is not, in general, as big as \( N \). Indeed, the number of regions that are “cut” is bounded from above by the number of regions inside the set \( \{ x \in \mathbb{R}^d \mid s_j = 0 \} \) formed by the terms \( s_1, \ldots, s_{j-1} \). But this number is \( \mathcal{O}((j-1)^{d-1}) \) by the bound for dimension \( d - 1 \). Therefore, for dimension \( d \) we obtain an overall bound of

\[
\mathcal{O}(1) + \sum_{j=2}^{m} 2 \cdot \mathcal{O}((j-1)^{d-1}) = \mathcal{O}(m^d).
\]

\( \triangleright \) Claim 29. Given \( t_1, \ldots, t_n \) linear terms over \( d \) variables, there are at most \( \mathcal{O}(n^{2d}) \) orderings that satisfy properties (I) and (II).

Proof. This is a consequence of Claim 28. Indeed, we can form \( m = \binom{n}{d} \) terms of the form \( t_i - t_j \), \( i \neq j \). For any valuation to the \( d \) variables, the signs of these \( m \) terms determine an ordering of \( t_1, \ldots, t_n \), satisfiable over the reals. Therefore, we obtain \( \mathcal{O}(m^d) = \mathcal{O}(n^{2d}) \) orderings in total.

Given Claim 29, let us now proceed to the second, algorithmic part of the proof. The idea can be seen as dynamic programming.

Our algorithm runs as follows. Let \( t_1, \ldots, t_n \) be the terms from the statement of the lemma. We construct several families of orderings, incrementally: family \( F_j \) is the required family for the the first \( j \) terms, \( t_1, \ldots, t_j \).

For \( j = 1 \), the family \( F_1 \) is trivial. For \( j = 2, \ldots, n \), we compute \( F_j \) from \( F_{j-1} \) as follows. Start from \( F_j = \emptyset \). For each ordering \( o \) from \( F_{j-1} \), enumerate all possible positions to insert \( t_j \) into it. There are at most \( 2j - 1 \) possible options here; their precise number depends on the number of equalities (as opposed to inequalities) among the relations on \( t_1, \ldots, t_j \). For each of these options, check if the resulting ordering is satisfiable when the variables are interpreted over \( \mathbb{R} \), using any polynomial-time algorithm for linear programming. If so, add it to \( F_j \), otherwise just skip it. In the end, \( F_n \) is a family of orderings with the required properties.

Let us analyse this algorithm. From the (non-algorithmic) part of the lemma, already proved above as Claim 29, we know that there are \( \mathcal{O}(j^{2d}) \) orderings for \( j \) terms. For each of these orderings, we will try to insert \( t_{j+1} \) at \( \mathcal{O}(j) \) possible positions. So there are \( \mathcal{O}(j^{2d+1}) \) satisfiability checks to run. Over all \( j \), this is \( \mathcal{O}(n^{2d+2}) \) checks.

Notice that it is sufficient for us to look at satisfiability over the reals or rationals here, as long as the number of orderings we get is not too high. Indeed, if some ordering is satisfiable over \( \mathbb{R} \) (or, equivalently, over \( \mathbb{Q} \)) but not satisfiable over \( \mathbb{Z} \), then we may still include it. This means that we can rely on polynomial-time algorithms for linear programming. Each instance has at most \( n \) constraints over \( d \) variables. The bit size of each coefficient be bounded by
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\[ b \triangleq \log \max_j \| t_j \|. \] Therefore, a satisfiability check for a system of constraints of this form can be run in time \( \text{poly}(b, n, d) \). The overall running time for the entire algorithm is

\[ O((2d+2)(bnd)^{O(1)}) = n^{O(d)}(nd)^{O(1)}b^{O(1)} = n^{O(d)} \log \max_j \| t_j \|^{O(1)}. \] 

\[ \triangleright \text{Claim 5. } \Psi_1 \leftrightarrow \Psi_2. \]

Proof. Let \( \psi_{\text{rem}} \) be the formula \( \bigvee_r: Z \rightarrow [m] \bigwedge_{z \in \mathbb{Z}} z \equiv_m r(z) \) whose disjuncts represent a combination of residue classes modulo \( m \) for the variables in \( Z \). We have

\[
\Psi_1 \leftrightarrow \psi_{\text{ord}} \land \psi_{\text{rem}} \land \Psi_1 \\
\leftrightarrow \Psi_2.
\]

The equivalence \( \triangleright \) follows from the fact that both the formulae \( \psi_{\text{ord}} \) and \( \psi_{\text{rem}} \) are tautologies.

\[ \triangleright \text{Claim 6. For every } \kappa \in \text{seg}(y, O_1), \text{there is a Boolean combination } \psi_{\kappa}^{y,r} \text{ of simple modulo constraints s.t. } \text{vars}(\psi_{\kappa}^{y,r}) = \{ y \}, \text{mod}(\psi_{\kappa}^{y,r}) \subseteq \text{mod}(\psi) \text{ and } \Gamma_{i,r} \land \kappa \land \psi \leftrightarrow \Gamma_{i,r} \land \kappa \land \psi_{\kappa}^{y,r}. \]

Proof. Let \( \kappa \) be fixed. We recall that the formulae \( O_i \) were constructed based on the set of terms that includes \( 0 \). This means, in particular, that for all assignments that satisfy the conjunction \( \Gamma_{i,r} \land \kappa \) (if any exist) the truth value of all inequalities that occur in the formula \( \psi \) is the same. Indeed, for inequalities not involving the variable \( y \) this is because the formula \( O_i \) asserts or implies the sign of every linear term. For inequalities involving \( y \), this is due to our choice of the set \( \text{seg}(y, O_1) \).

We now consider modulo constraints that occur in \( \psi \). Those of them where the variable \( y \) does not appear also evaluate to just true or false on all assignments satisfying \( \Gamma_{i,r} \land \kappa \), because \( r \) specifies residue classes modulo \( m = \text{lcm}(\text{mod}(\psi)) \) for all variables except \( y \). Since \( y \) can only occur with coefficient \( 1 \) or \( -1 \), all the remaining modulo constraints become simple, i.e., take the form \( y \equiv_q r \) for some \( q \in \text{mod}(\psi) \).

To sum up, replacing all constraints in the \( \psi \) part of the formula \( \Gamma_{i,r} \land \kappa \land \psi \) with their truth values or their simplified form, as described above, we obtain an equivalent formula \( \Gamma_{i,r} \land \kappa \land \psi_{\kappa}^{y,r} \), as required.

\[ \triangleright \text{Claim 7. } \Psi_2 \leftrightarrow \Psi_3. \]

Proof. Let \( i \in [1, o_1], r: Z \rightarrow [m] \). Establishing \( \Gamma_{i,r} \land \Psi_1 \leftrightarrow \Gamma_{i,r} \land \Psi_3^{i,r} \) suffices, where \( \Psi_1 = \exists \mathbb{Z}^y \psi \) and \( \Psi_3^{i,r} = \exists x_0 \ldots \exists x_{2r} \left( x \leq x_0 + \cdots + x_{2r} \land \bigwedge_{j \in [0,2r]} \exists \mathbb{Z}^y (y_j \land \psi_{i,j}^{y,r}) \right) \). Directly from Claim 6. The formula \( \Gamma_{i,r} \land \Psi_3^{i,r} \) is equivalent to

\[
\gamma \triangleq \Gamma_{i,r} \land \exists x_0 \ldots \exists x_{2r} \left( x \leq x_0 + \cdots + x_{2r} \land \bigwedge_{j \in [0,2r]} \exists \mathbb{Z}^y (y_j \land \psi_j) \right)
\]

where we notice that all the formulae of the form \( \psi_{i,j}^{y,r} \) are substituted with \( \psi \). Proving the equivalence \( \Gamma_{i,r} \land \Psi_1 \leftrightarrow \gamma \) is rather straightforward. \((\Rightarrow):\) Let \( \nu \) be an assignment such that \( \nu \models \Psi_1 \land \gamma \). Therefore, there are values \( x_0, \ldots, x_{2r} \) for the variables \( x_0, \ldots, x_{2r} \) such that \( \nu[v_0/x_0, \ldots, v_{2r}/x_{2r}] \models x \leq x_0 + \cdots + x_{2r} \land \bigwedge_{j \in [0,2r]} \exists \mathbb{Z}^y (y_j \land \psi_j) \). Since the variables \( x_0, \ldots, x_{2r} \) do not appear in \( \kappa_j \land \psi \), we have \( \nu \models x \leq v_0 + \cdots + v_{2r} \land \bigwedge_{j \in [0,2r]} \exists \mathbb{Z}^y (y_j \land \psi_j) \).

Lastly, in view of the definition of the set \( \text{seg}(y, O_1) \), given \( \kappa, \kappa' \in \text{seg}(y, O_1) \) there is no value \( v \) for \( y \) such that \( \nu[v/y] \models \kappa \land \psi \) and \( \nu[v/y] \models \kappa' \land \psi \). We conclude that there are at least \( \sum_{j=0}^{2r} v_j \) distinct values \( v \) for \( y \) such that \( \nu[v/y] \models \psi \), and thus \( \nu \models \Gamma_{i,r} \land \Psi_1 \) directly from \( \nu \models x \leq v_0 + \cdots + v_{2r} \).
(⇒): Suppose \( \nu \models \Gamma_{i,r} \land \Psi_1 \). So, there are at least \( \nu(x) \) distinct values \( y \) for \( x \) such that \( \nu[y/y] \models \psi \). Given \( j \in [0, 2^\ell] \), let \( v_j \) be the number distinct values for \( y \) such that \( \nu[v_j/y] \models \kappa_j \land \psi \). Again from the fact that, given \( \kappa, \kappa' \in \text{seg}(y, O_i) \) there is no value \( y \) for \( x \) such that \( \nu[y/y] \models \kappa \land \psi \) and \( \nu[v_j/y] \models \kappa' \land \psi \), we conclude that \( \nu(x) \leq \sum_j^2 v_j \). Thus, \( \nu[v_0/x_0, \ldots, v_{2^\ell}/x_{2^\ell}] \models x \leq x_0 + \cdots + x_{2^\ell} \land \bigwedge_{j \in [0, 2^\ell]} \exists^2 y \kappa_j \land \psi \), and so \( \nu \models \gamma \).
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make the formula $\kappa \land \psi_{\kappa}^{i,r}$ satisfied, in which case $z$ can be set to all non-positive integers (and only to them).

\[ \triangleright \text{Claim 11. Let } \kappa \text{ be the formula } t_j^* - t_j - 1 < y < y < t_j^* \text{ for some } j \in [2, t] \text{ and let } z \text{ be a fresh variable. Then } \Gamma_{i,r} \land \exists z(y (\kappa \land \psi_{\kappa}^{i,r})) \iff \Gamma_{i,r} \land m z \leq p_j (t_j^* - t_j - 1) + r_j. \]

Proof of Claim 11 For every assignment $\nu$ to variables other than $y$, take $L = \nu(t_{j-1})^* + 1$ and $U = \nu(t_j^*)$ and consider the segment of integers $[L, U - 1]$, using the convention $[a, b] = \emptyset$ if $a > b$. These are all the potential values of $y$ that satisfy the formula $\kappa$, which under the conditions of the Claim has the form $t_j^* - t_j - 1 < y < t_j^*$. We need to determine how many of these values actually satisfy the larger formula $\kappa \land \psi_{\kappa}^{i,r}$, and this number will be the maximum possible value attained by the variable $z$.

As in the proof of Claim 9 recall from Step III (and Claim 9) that the formula $\psi_{\kappa}^{i,r}$ is a Boolean combination of simple modulo constraints with $\text{vars}(\psi_{\kappa}^{i,r}) = \{y\}$ and that $\text{mod}(\psi_{\kappa}^{i,r}) \subseteq \text{mod}(\psi)$. So our previous choice of $m = \text{lcm}(\text{mod}(\Psi_1))$ is a multiple of $\text{lcm}(\text{mod}(\psi_{\kappa}^{i,r}))$, and thus the set of all assignments (for $y$) that satisfy the formula $\psi_{\kappa}^{i,r}$ is periodic with period $m$. Formally, denote $S = \{n \in \mathbb{Z} \mid (y \mapsto n) \models \psi_{\kappa}^{i,r}\}$ and note that $n \in S$ iff $n + m \in S$.

The technical hurdle we need to overcome in this proof is that the values of $L$ and $U$ depend on the assignment $\nu$. Importantly, it suffices to consider assignments that satisfy $\Gamma_{i,r}$, i.e., the formula $O_i \land (\wedge_{z \in Z} z \equiv_m r(z))$. We focus on the modulo constraints in this formula and, from now on, we assume that $\nu$ satisfies all of them. Our goal is to compute the cardinality of the set $[L, U - 1] \cap S$, which by the arguments above is exactly the number of variable assignments to $y$ that satisfy $\kappa \land \psi_{\kappa}^{i,r}$, if all other variables have already been assigned values by $\nu$.

Let $L' = r(t_{j-1}^*) + 1$ and $U' = \min\{r(t_j^*) + m \cdot h \mid h \in \mathbb{Z}\} \cap [L', +\infty)$. These two numbers are almost the same as $y_j$ and $\overline{y}_j$, respectively, but we will use the capital letter notation to keep symbols for different segment endpoints uniform. Now $L' \leq U'$, $\#([L', U' - 1] = U' - L' \in [0, m - 1] \text{ and, because of our assumption about } \nu, L \equiv L' \mod m \text{ and } U \equiv U' \mod m$. (For the proof of these congruences, observe that, firstly, $r(z) \equiv \nu(z) \mod m$ because $\nu$ satisfies $\Gamma_{i,r}$. This implies that $ar(z) \equiv \nu(r(z)) \mod m$ for all $a \in \mathbb{Z}$. Summing up several congruences of this kind results in another congruence, of the form $r(t) \equiv \nu(t) \mod m$. Setting $t = t_{j-1}^* + 1$ and $t = t_j^*$ concludes the proof.)

We are now ready to compute the cardinality of $[L, U - 1] \cap S$. As $S$ is periodic with period $m$, we will split $[L, U - 1]$ into two disjoint parts: $[L, U - 1] = [L, L^* - 1] \cup [L^*, U - 1]$, where $L^*$ is the largest integer not exceeding $U$ and congruent to $L$ modulo $m$. We consider each part separately:

- As $L^*$ is congruent to $L$ modulo $m$, it is clear that the integer segment $[L, L^* - 1]$ consists of zero, one, two or more copies of a full period of $S$. Therefore,

\[ \#([L, L^* - 1] \cap S) = \#([0, m - 1] \cap S) \cdot \frac{L^* - L}{m} = p_j \cdot \frac{L^* - L}{m}. \]

- For the second part, observe that all three numbers $L$, $L'$, and $L^*$ are congruent modulo $m$; similarly, $U'$ and $U$ are congruent modulo $m$. By definition of $L^*$, we have $\#([L^*, U - 1]) = U - L^* \in [0; m - 1]$. Therefore, the following two constraints hold:

\[ \#([L^*, U - 1]) = \#([L', U' - 1]) \text{ and } L^* \equiv L' \mod m. \]
By periodicity of $S$, for all $v \in [L^*, U - 1]$ we have
\[ v \in S \text{ if and only if } v + (L' - L^*) \in S \]
and therefore
\[ \#([L^*, U - 1] \cap S) = \#([L', U' - 1] \cap S) = r'_j. \]
Let us sum up the results above. Due to the semantics of the quantifier $\exists \exists^+ y$, the constraint on the variable $z$ is equivalent to the following one:
\[ z \leq p_j \cdot \frac{L^* - L}{m} + r'_j, \]
which is the same as $m \cdot z \leq p_j \cdot (L^* - L) + m \cdot r'_j$. It remains to return to the original terms, “undoing” the variable assignment $\nu$. Observe that
\[ L^* - L = (U - L) - (U - L^*) = (U - L) - (U' - L'). \]
In the constraint, instead of $U - L$ we write $t'_j - (t'_{j-1} + 1)$, and the value of $U' - L'$ can be computed from $r(t'_{j-1})$ and $r(t'_j)$ using simple arithmetic. Putting everything together, we obtain
\[ m \cdot z \leq p_j \cdot (t'_j - t'_{j-1} - 1) - (U' - L')) + m \cdot r'_j. \]
Since $U' - L' \in [0, m - 1]$ and $r'_j \in [0, p_j]$, the following bounds hold:
\[ p_j \cdot (-1 - (U' - L')) + m \cdot r'_j \leq m \cdot r'_j \leq m^2, \]
\[ p_j \cdot (-1 - (U' - L')) + m \cdot r'_j \geq -p_j \cdot m \geq -m^2. \]
This completes the proof. 

> Claim 12. $\Psi_3 \leftrightarrow \Psi_4$.

Proof: Follows directly from Claim 9 Claim 10 and Claim 11 together with simple formulae manipulations.

> Claim 13. $\Psi_4 \leftrightarrow \Psi_5$. The formula $\Psi_5$ is quantifier-free.

Proof: Let $i \in [1, o], r: Z \to [m]$. Establishing $\Gamma_i, r \land \Psi_4^{i,r} \leftrightarrow \Gamma_i, r \land \Psi_5^{i,r}$ suffices. If $\Psi_4^{i,r} = \top$ then $\Psi_5^{i,r}$ is defined as $\top$ and the equivalence holds. Otherwise, we have
\[ \Psi_4^{i,r} \equiv \exists x_2 \ldots \exists x_\ell \left( x \leq \sum_{j=2}^\ell x_j + \sum_{j=1}^\ell c_j \land \bigwedge_{j \in [2, \ell]} mx_j \leq p_j(t'_j - t'_{j-1} + r_j) \right) \]
\[ \Psi_5^{i,r} \equiv mx \leq \sum_{j=2}^\ell (p_j(t'_j - t'_{j-1} + r_j) + m \cdot \sum_{j=1}^\ell c_j. \]

($\Rightarrow$): It is easy to see that $\Psi_5^{i,r}$ is obtained from $\Psi_4^{i,r}$ by first multiplying both sides of the inequality $x \leq \sum_{j=2}^\ell x_j + \sum_{j=1}^\ell c_j$ by $m$, and then substituting $mx_j$ with $p_j(t'_j - t'_{j-1} + r_j) + r_j$. 

($\Leftarrow$): Let $\nu$ be an assignment such that $\nu \models \Gamma_i, r \land \Psi_5^{i,r}$. We show that $\nu \models \Psi_4^{i,r}$. First of all, we consider $j \in [2, \ell]$, and aim at showing that $\nu(p_j(t'_j - t'_{j-1} + r_j))$ is a multiple of $m$. We recall the definition of $r_j, y_j$ and $\overline{y}_j$, as introduced in Step IV:
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By definition of \( r_j \), the term \( p_j(t'_j - t'_{j-1}) + r_j \) is equivalent to \( p_j(t'_j - t'_{j-1} - \bar{\gamma}_j + \bar{\gamma}_j) + m \cdot r'_j \).

Since \( \nu \models \Gamma_i \), we have \( \bar{\gamma}_j \equiv_m \nu(t'_j - 1) \) and \( \bar{\gamma}_j \equiv_m \nu(t'_j) \). From axioms of modular arithmetic, \( \nu(t'_j) - \nu(t'_{j-1}) = \bar{\gamma}_j + \bar{\gamma}_j \equiv_m 0 \), and thus \( p_j(\nu(t'_j) - \nu(t'_{j-1}) - \bar{\gamma}_j + \bar{\gamma}_j) + m \cdot r'_j \equiv_m 0 \), which allows us to conclude that \( \nu(p_j(t'_j - t'_{j-1}) + r_j) \) is a multiple of \( m \). Therefore, for every \( j \in [2, \ell] \), there is \( v_j \in \mathbb{Z} \) such that \( m \cdot v_j = \nu(p_j(t'_j - t'_{j-1}) + r_j) \). Let \( x_2, \ldots, x_\ell \) be fresh variables. We consider the assignment \( \nu(v_2/x_2, \ldots, v_\ell/x_\ell) \) that updates \( v \) by assigning \( v_j \) to the variable \( x_j \), for every \( j \in [2, \ell] \). We have,

\[
\nu(v_2/x_2, \ldots, v_\ell/x_\ell) \models mx \leq \sum_{j=2}^\ell mx_j + m \cdot \sum_{j=1}^\ell c_j \land \bigwedge_{j \in [2, \ell]} mx_j = p_j(t'_j - t'_{j-1}) + r_j.
\]

Divide both side of the leftmost inequality by \( m \in \mathbb{N} \), and weaken the equalities of the form \( mx_j = p_j(t'_j - t'_{j-1}) + r_j \) to inequalities of the form \( mx_j \leq p_j(t'_j - t'_{j-1}) + r_j \). We obtain

\[
\nu(v_2/x_2, \ldots, v_\ell/x_\ell) \models x \leq \sum_{j=2}^\ell x_j + \sum_{j=1}^\ell c_j \land \bigwedge_{j \in [2, \ell]} mx_j \leq p_j(t'_j - t'_{j-1}) + r_j.
\]

By definition of the existential quantifier,

\[
\nu \models \exists x_2 \ldots \exists x_\ell \left( x \leq \sum_{j=2}^\ell x_j + \sum_{j=1}^\ell c_j \land \bigwedge_{j \in [2, \ell]} mx_j \leq p_j(t'_j - t'_{j-1}) + r_j \right).
\]

That is, \( \nu \models \Psi_4^{t,r} \).

# Missing proofs from Section 4

In this appendix, we provide the computational analysis on the parameters \( \text{lin}(\cdot) \), \( \text{hom}(\cdot) \) and \( \text{mod}(\cdot) \), of the formula obtained form the elimination of the quantifier \( \exists \exists^x y \) via the procedure of Section 3.

Let \( \varphi \) be a quantifier-free formula, and let \( d = \#\text{fv}(\varphi) \). Consider the formula \( \Psi_5 \) obtained by performing the quantifier-elimination procedure of Section 3 on the formula \( \exists \exists^x y \varphi \).

The following lemma restates Lemma 14 by expressing the bounds on \( \Psi_5 \) explicitly.

**Lemma 30.** The following bounds are established for \( \Psi_5 \):

\[
\#\text{mod}(\Psi_5) = \{ m \} \text{ with } m = k \cdot \text{lcm}(\text{mod}(\varphi)) \text{ and } k \leq O(\|\text{hom}(\varphi)\| \cdot \#\text{hom}(\varphi)).
\]

\[
\#\text{lin}(\Psi_5) \text{ and } \#\text{hom}(\Psi_5) \text{ are bounded by } (m \cdot \#\text{lin}(\varphi))^O(d),
\]

\[
\|\text{lin}(\Psi_5)\| \leq O(m^2 \cdot \#\text{lin}(\varphi) \cdot \|\text{lin}(\varphi)\|),
\]

\[
\|\text{hom}(\Psi_5)\| \leq O(m^2 \cdot \#\text{lin}(\varphi) \cdot \|\text{hom}(\varphi)\|).
\]

**Proof.** First of all, from Lemma 2, we notice that translating every modulo constraint appearing in \( \varphi \) into simple modulo constraints does not change the sets \( \text{lin}(\varphi), \text{hom}(\varphi) \) and \( \text{mod}(\varphi) \). Therefore, assume \( \varphi \) to be a Boolean combination of linear inequalities and simple modulo constraints. Let \( k \) be the \( \text{lcm} \) of the absolute values of all coefficients in \( y \) appearing in \( \text{hom}(\varphi) \). We have \( k \leq \|\text{hom}(\varphi)\| \cdot \#\text{hom}(\varphi) \). The first step essentially multiplies every term in \( \varphi \) by \( k \), producing the formula \( \Psi_1 \) with bounds

\[
\#\text{lin}(\Psi_1) = \#\text{lin}(\varphi) \text{ and } \|\text{lin}(\Psi_1)\| \leq k \|\text{lin}(\varphi)\|,
\]

\[
\#\text{hom}(\Psi_1) = \#\text{hom}(\varphi) \text{ and } \|\text{hom}(\Psi_1)\| \leq k \|\text{hom}(\varphi)\|,
\]

\[
\text{mod}(\Psi_1) = \{ kq \mid q \in \text{mod}(\varphi) \}.
\]
Let $T$ be the set of all $y$-free terms $t$ such that $t$, $y - t$ or $-y + t$ belong to $\text{lin}(\Psi_1)$. So, $#T \leq \#\text{lin}(\Psi_1)$, $|T| \leq \|\text{lin}(\Psi_1)\|$ and all coefficients of variables in terms of $T$ are bounded by $\|\text{hom}(\Psi_1)\|$. Let $m = \text{lcm}(\text{mod}(\Psi_1)) = k \cdot \text{lcm}(\text{mod}(\varphi))$. In the second step of the procedure, the orderings introduce terms $t < t'$, where $< \in \{\leq, =\}$ and $t, t' \in T \cup \{0\}$. So, at most $(#T \cup \{0\})^2$ new terms are introduced, increasing $\text{lin}(\cdot)$ and $\text{hom}(\cdot)$ quadratically in cardinality. The magnitude of coefficients and constants doubles. Simple modulo constraints of the form $x \equiv_m r$ are also introduced. Because of this, the formula $\Psi_2$ produced in the second step of the procedure has the following bounds:

- $\#\text{lin}(\Psi_2) \leq (\#\text{lin}(\Psi_1) + 1)^2 + \#\text{lin}(\Psi_1)$ and $\|\text{lin}(\Psi_2)\| \leq 2\|\text{lin}(\Psi_1)\|$, 
- $\#\text{hom}(\Psi_2) = (\#\text{hom}(\Psi_1) + 1)^2 + \#\text{hom}(\Psi_1)$ and $\|\text{hom}(\Psi_2)\| \leq 2\|\text{hom}(\Psi_1)\|$, 
- $\text{mod}(\Psi_2) = \{m\} \cup \text{mod}(\Psi_1)$.

To study the bounds on the formula $\Psi_5$, analysing the bounds obtained from the third and fourth steps of the procedure is unnecessary. Indeed, we recall that $\Psi_5$ is defined as

$$\Psi_5 = \bigvee_{i \in [1, \ell]} \bigvee_{r: z \rightarrow [m]} (\Gamma_{i,r} \land \Psi_{i,r}^{k,r}),$$

where every $\Gamma_{i,r}$ is a conjunction of simple modulo constraints of the form $z \equiv_m r$ and linear inequalities from $\Psi_2$, and every $\Psi_{i,r}^{k,r}$ is either $T$ or a formula of the form

$$mx \leq \sum_{j=2}^\ell (p_j(t'_j - t'_{j-1}) + r_j) + m \cdot \sum_{j=1}^\ell c_j$$

where $\ell \leq #T + 1 \leq \#\text{lin}(\Psi_1) + 1$, for every $j \in [1, \ell]$, $c_j \in \{0, 1\}$, and for every $j \in [2, \ell]$, $p_j \in [0, m]$ and $|r_j| \in [-m^2, m^2]$ (see Claim 11) and the terms $t'_j$ and $t'_{j-1}$ belong to $T$. This implies that variable coefficients in $\Psi_{i,r}^{k,r}$ are bounded (in absolute value) by $m \cdot (2 \cdot \ell \cdot \|\text{lin}(\Psi_1)\| + 1)$, whereas the constant term is bounded by $2 \cdot \ell \cdot m \cdot \|\text{lin}(\Psi_1)\| + \ell \cdot m^2 + m$, again in absolute values. By recalling that the number of disjunctions of $\Psi_5$ is $m^4\alpha \leq m^d(2(#T \cup \{0\})^2 + 1)^d$ (see Lemma 4 for the bound on $\alpha$), we derive

- $\text{mod}(\Psi_5) = \{m\}$, and so $\|\text{mod}(\Psi_5)\| \leq \text{lcm}(\text{mod}(\varphi)) \cdot \|\text{hom}(\varphi)\|^{\#\text{hom}(\varphi)}$, 
- $\#\text{lin}(\Psi_5) \leq (\#\text{lin}(\Psi_1) + 1)^2 + (2(\#\text{lin}(\Psi_1) + 1)^2 + 1)^d m^d \leq O(m^d \cdot \#\text{lin}(\psi)^{2d})$, 
- $\|\text{lin}(\Psi_5)\| \leq 2 \cdot \ell \cdot m \cdot \|\text{lin}(\Psi_1)\| + \ell \cdot m^2 + m \leq O(m^2 \cdot \#\text{lin}(\psi) \cdot \|\text{lin}(\varphi)\|)$, 
- $\#\text{hom}(\Psi_5) \leq (\#\text{hom}(\Psi_1) + 1)^2 + (2(\#\text{lin}(\Psi_1) + 1)^2 + 1)^d m^d \leq O(m^d \cdot \#\text{lin}(\psi)^{2d})$, 
- $\|\text{hom}(\Psi_5)\| \leq m \cdot (2 \cdot \ell \cdot \|\text{lin}(\Psi_1)\| + 1) \leq O(m^2 \cdot \#\text{lin}(\psi) \cdot \|\text{hom}(\varphi)\|)$.

**C Missing proofs from Section 5**

In the lemma below, we recall that $d = \text{vars}(\Psi_5^{k,r})$, and that $\Psi_5^{k,r}$ has the following form (see Equation 2 in the body of the paper):

$$mx \leq \sum_{j=2}^\ell (p_j(t'_j - t'_{j-1}) + r_j) + m \cdot \sum_{j=1}^\ell c_j$$

**Lemma 19.** Consider $\Psi_5^{k,r}$ as in [2]. Let $e = m(c - \sum_{j=1}^\ell c_j) - \sum_{j=2}^\ell r_j$. It is possible to compute in time $(e + \ell)^O(d) \log(c \cdot \#\Psi_5^{k,r})^O(1)$ a formula $\gamma_{i,r} = \bigvee_{(i_1, \ldots, i_d) \in I} \bigvee_{j \in [2, \ell]} t'_j - t'_{j-1} \geq i_j$ such that (1) $I \subseteq [0, e]^d$, (2) $#I \leq O((e + \ell)^{2d})$, and (3) $\Gamma_{i,r} \land \Psi_5^{k,r}[c/z] \leftrightarrow \Gamma_{i,r} \land \gamma_{i,r}$. 


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Proof. Consider the set of $e + \ell$ terms $T' = \{ t'_j - t'_{j-1} \mid j \in [2, \ell] \} \cup \{0, e\}$. Note that $e \leq 2 \cdot c \cdot \|\Psi_5^t\|$, and so $\|T'\| \leq O(c \cdot \|\Psi_5^t\|)$. Applying Lemma 4, we compute a set $\{O'_1, \ldots, O'_n\}$ of orderings for $T'$ such that $\psi \equiv \bigvee_{k \in [1, \ell]} O'_k$ is a tautology and $\ell' = O((e+\ell)^{2d})$. Since $O'_k$ is an ordering for $T'$, for all $k \in [1, \ell']$ and $t', t'' \in T'$, exactly one of the entailments $O'_k \models t' < t''$, $O'_k \models t' = t''$ or $O'_k \models t' > t''$ holds.

We iterate over all $k \in [1, \ell']$, at each step generating a formula $\psi_k$ that satisfies

$$\Gamma_{i,r} \land \Psi_{b}^t[c/z] \leftrightarrow (\Gamma_{i,r} \land \psi_k).$$

(11)

At the end of the process, the formula $\psi_o'$ is the formula $\gamma_{i,r}$ required by the lemma. Let $\psi_0 = \bigvee_{k \in [1, \ell]} (\Psi_{b}^t[c/z] \land O'_k)$. Since $\psi$ is a tautology and $\psi_0 \equiv \Psi_{b}^t[c/z] \land \psi$, the formula $\psi_0$ satisfies the equivalence in (11). Let $n = \#T' = e + \ell$. For all $k \in [1, \ell']$, suppose

$$O'_k = b_1 \land b_2 \land \cdots \land b_{n-1} \land b_n$$

where $\{b_1, \ldots, b_n\} = T'$ and $\{<, 1, \ldots, \leq n - 1\} \subseteq \{<, =\}$. We inductively assume that $\psi_{k-1}$ satisfies the equivalence (11), and we compute $\psi_k$ following the cases below. Notice that checking which of the cases is satisfied by $O'_k$ can be done in linear time with respect to $|O'_k|$, by simply scanning the ordering.

case: $O'_k$ does not respect the order $0 < 1 \cdots < e$. Then, $O'_k$ is unsatisfiable and $\psi_k$ is obtained from $\psi_{k-1}$ by removing the disjunct $\Psi_{b}^t[c/z] \land O'_k$. Since $\psi_{k-1}$ satisfies the equivalence (11), so does $\psi_k$.

case: $O'_k \models t'_j - t'_{j-1} < 0$, for some $j \in [2, \ell]$. Since $\Psi_{b}^t[c/z] \land O'_k$ is unsatisfiable. Again, $\psi_k$ is obtained from $\psi_{k-1}$ by removing the disjunct $\Psi_{b}^t[c/z] \land O'_k$, and $\psi_k$ satisfies the equivalence (11).

otherwise, for every $j \in [2, \ell]$ there is $i_j \in [0, e]$ such that either $O'_k \models i_j = t'_j - t'_{j-1}$ or $i_j = e$ and $O'_k \models i_j < t'_j - t'_{j-1}$. By simply parsing of the ordering, can find all the $i_j$ in time $O(|O'_k|)$. Now, if $e \leq \sum_{j=2}^{\ell} p_j \cdot i_j$ does not hold, then the formula $\Psi_{b}^t[c/z] \land O'_k$ is unsatisfiable and, as in the previous cases, we define $\psi_k$ from $\psi_{k-1}$ by removing this disjunct. We obtain a formula that satisfies the equivalence (11). Otherwise, let $\gamma = \bigwedge_{j=2}^{\ell} t'_j - t'_{j-1} \geq i_j$. By definition, $O'_k \models \gamma$ and $\gamma \models \Psi_{b}^t[c/z]$. Let $\psi_k$ be the formula obtained from $\psi_{k-1}$ by replacing the disjunct $\Psi_{b}^t[c/z] \land O'_k$ by the formula $\gamma$. Notice that $\psi_{k-1} \models \psi_k$ directly from $O'_k \models \gamma$. We show that $\psi_k$ satisfies the equivalence (11).

$\Rightarrow$: Let $\nu$ be an assignment such that $\nu \models \Gamma_{i,r} \land \Psi_{b}^t[c/z]$. Since $\psi_{k-1}$ satisfies the equivalence (11), we have $\nu \models \rho_{i,r} \land \psi_{k-1}$. By $\psi_{k-1} \models \psi_k$, we derive $\nu \models \Gamma_{i,r} \land \psi_k$.

$\Leftarrow$: Let $\nu$ be an assignment such that $\nu \models \Gamma_{i,r} \land \psi_k$. If $\nu$ satisfies a disjunct of $\psi_k$ that is different from $\gamma$, then $\nu \models \psi_{k-1}$ and, since $\psi_{k-1}$ satisfies the equivalence (11), $\nu \models \Gamma_{i,r} \land \Psi_{b}^t[c/z]$. Otherwise, $\nu \models \gamma$ and, by $\gamma \models \Psi_{b}^t[c/z]$ we deduce that $\nu \models \Gamma_{i,r} \land \Psi_{b}^t[c/z]$.

As already said, the formula $\gamma_{i,r} \equiv \psi_o'$. The formula $\gamma_{i,r}$ satisfies all the expected properties. In particular, $I \subseteq [0, e]$ holds by definition of the various $i_j$ in the third case of the procedure, and $\#I \leq O((e + \ell)^{2d})$ holds from the bound on the number $\ell'$ of disjuncts of $\psi$. By Lemma 4 computing the initial formula $\psi$ can be done in time $(e + \ell)^{O(d)} \log \|T'\|^{O(1)}$.

Similarly, the case analysis on the disjuncts of $\psi$ has a overall running time that is linear in $|\psi| \leq (e + \ell)^{O(d)} \log \|T'\|^{O(1)}$.

\begin{claim}
\label{claim:psi0}
$\Psi_0 \leftrightarrow \Psi_0^t$. The formula $\Psi_0^t$ is quantifier-free.
\end{claim}

Proof. By definition of $\Psi_0^t$ and Lemma 19, $\Psi_5 \leftrightarrow \Psi_0^t$. Then, the claim follows from the chain of claims “$\Psi_i \leftrightarrow \Psi_{i,t+1}$" starting from Claim 3 and ending with Claim 19. 

\begin{flushright}
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\end{flushright}
The following bounds are established for $\Psi_6^i$:

- $\#\text{mod}(\Psi_6^i) = \{m\}$ with $m = k \cdot \text{lcm}(\varphi)$ and $k \leq \|\text{hom}(\varphi)\| \cdot \#\text{hom}(\varphi)$,

- $\#\text{hom}(\Psi_6^i) \leq O(\#\text{hom}(\varphi)^3)$ and $\|\text{hom}(\Psi_6^i)\| \leq O(k \cdot \|\text{hom}(\varphi)\|)$,

- $\#\text{lin}(\Psi_6^i) \leq O(c \cdot m^2 \cdot \#\text{lin}(\varphi)^2)$ and $\|\text{lin}(\Psi_6^i)\| \leq O(k \cdot c \cdot m^2 \cdot \#\text{lin}(\varphi) \cdot \|\text{lin}(\varphi)\|)$.

Proof. Without loss of generality, we assume $\#\text{hom}(\varphi)$, $\#\text{lin}(\varphi)$, $\|\text{hom}(\varphi)\|$ and $\|\text{lin}(\varphi)\|$ to be at least 1. We also assume $c$ to be at least 1, as otherwise the formula $\exists^2 c y \varphi$ is trivially true. These assumptions hide constant factors in the exponent. We recall the bounds (as in Lemma 30) on the formula $\Psi_1$ obtained after performing the normalisation of the coefficients of $y$, as described in Step I of Section 3. We have

- $\#\text{lin}(\Psi_1) = \#\text{lin}(\varphi)$ and $\|\text{lin}(\Psi_1)\| \leq k \cdot \|\text{lin}(\varphi)\|$, 

- $\#\text{hom}(\Psi_1) = \#\text{hom}(\varphi)$ and $\|\text{hom}(\Psi_1)\| \leq k \cdot \|\text{hom}(\varphi)\|$.

where $k \leq \|\text{hom}(\varphi)\| \cdot \#\text{hom}(\varphi)$ is the lcm of all coefficients of $y$ appearing in linear inequalities.

We recall that the formula $\Psi_6^i$ is defined as $\bigvee_{w \in [1,m]} \bigwedge_{(I_{i,r})^{c}} \bigwedge_{z \in [1,c]} (T_{i,r} \land \Psi_6^{i,r} \land \Psi_6^{i,w} \equiv m r(w))$, where $O_{i}$ is an ordering on the set of terms $T \cup \{0\}$, and $\Psi_6^{i,r}$ is either $\top$ or of the form (see Lemma 19)

$V_{(i_2, \ldots, i_{\ell})} \land \bigwedge_{j \in [2,\ell]} t'_j - t'_{j-1} \geq i_j$.

Here, $\ell \leq \#T + 1$ and, for all $j \in [2,\ell]$, $t'_j, t'_{j-1} \in T$ and $i_j \in [0,e]$ where $e \leq m \cdot (c + (m + 1) \cdot \ell)$. Moreover, $\#T \leq \#\text{lin}(\Psi_1)$, $\|T\| \leq \|\text{lin}(\Psi_1)\|$ and all coefficients of variables in terms of $T$ are bounded by $\|\text{hom}(\Psi_1)\|$. So, when accounting for all orderings (O_i)_{i \in [1,o]} and all inequalities of the form $t'_j - t'_{j-1} \geq i_j$, the formula $\Psi_6^i$ contains $(e + 1) \cdot (\#T + 1)^2$ inequalities. However, the set hom($\Psi_6^i$) is only quadratic on the size of the set of homogeneous terms built from pairs of terms in $T \cup \{0\}$, as we do not account for the natural numbers $i_j$. Since the terms in $T$ are constructed by removing $y$ from terms in $\text{lin}(\Psi_1)$, #hom($\Psi_6^i$) is quadratic on #hom($\Psi_1$). The magnitude of the coefficients of the variables in linear inequalities of $\Psi_6^i$ doubles with respect to $\|T\|$, whereas the magnitude of the constants is bounded by $2\|T\| + e$. Lastly, every modulo constraint in $\Psi_6^i$ is of the form $w \equiv_m r(w)$, and thus mod($\Psi_6^i$) = $\{m\}$. Overall, the following bounds are derived:

- $\#\text{mod}(\Psi_6^i) = \{m\}$ with $m = k \cdot \text{lcm}(\varphi)$ and $k \leq \|\text{hom}(\varphi)\| \cdot \#\text{hom}(\varphi)$,

- $\#\text{hom}(\Psi_6^i) \leq (\#\text{hom}(\varphi) + 1)^2 \leq 4 \cdot \#\text{hom}(\varphi)^2$, 

- $\|\text{hom}(\Psi_6^i)\| \leq 2 \cdot k \cdot \|\text{hom}(\varphi)\|$, 

- $\#\text{lin}(\Psi_6^i) \leq (e + 1) \cdot (\#\text{lin}(\varphi) + 1)^2 \leq 20 \cdot c \cdot m^2 \cdot \#\text{lin}(\varphi)^3$, 

- $\|\text{lin}(\Psi_6^i)\| \leq 2 \cdot k \cdot \|\text{lin}(\varphi)\| + e \leq 6 \cdot k \cdot c \cdot m^2 \cdot \#\text{lin}(\varphi) \cdot \|\text{lin}(\varphi)\|$, 

where we recall that we are assuming $\#\text{hom}(\varphi)$, $\#\text{lin}(\varphi)$, $\|\text{hom}(\varphi)\|$, $\|\text{lin}(\varphi)\|$, $e \geq 1$.

Proof. This is a simple consequence of Lemma 31.

Lemma 21. #hom($\Psi_6^i$) = $O(\#\text{hom}(\varphi)^2)$.

Proof. This is a simple consequence of Lemma 31.

Lemma 22. Let $\varphi$ be a formula of Presburger arithmetic with threshold quantifiers. There is an equivalent quantifier-free formula $\Psi$ such that

- $\#\text{lin}(\Psi), \|\text{lin}(\Psi)\|, \|\text{hom}(\Psi)\|$ and $\|\text{mod}(\Psi)\|$ are at most $2^{2^{O(|\varphi|^2)}}$,

- $\#\text{hom}(\Psi) \leq 2^{2^{O(|\varphi|^2)}}$ and $\#\text{mod}(\Psi) \leq |\varphi|$. 

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**Proof.** Recall that the standard first-order quantifier \( \exists y \) is equivalent to \( \exists^1 y \). Therefore, without loss of generality, we can assume \( \varphi \) to only contain threshold counting quantifiers. For simplicity, we also assume \( \# \text{hom}(\varphi), \# \text{lin}(\varphi), \# \text{mod}(\varphi) \) and \( \| \text{lin}(\varphi) \| \) to be at least 1. This hides constant factors in the exponent of the bounds that we derive. Let us introduce some shortcuts.

- Let \( d \) be the the quantifier-depth of \( \varphi \),
- let \( B \) be 2 plus \( \# \text{mod}(\varphi) \), plus the number of Boolean connectives in \( \varphi \), and
- let \( \bar{c} \) be the maximal integer such that \( \exists^{\geq \bar{c}} y \) occurs in \( \varphi \).

We show the following bounds for \( \Psi \), sharpening the ones in the statement of the lemma.

\[
\# \text{hom}(\Psi) \leq A_d \equiv (4 \cdot B)^{2^d - 1} \cdot \# \text{hom}(\varphi)^{2^d},
\]

\[
\# \text{mod}(\Psi) \leq B,
\]

\[
\| \text{hom}(\Psi) \| \leq C_d \equiv 2(2A_d)^{d-1}\# \text{hom}(\varphi)^{2^d},
\]

\[
\| \text{mod}(\Psi) \| \leq D_d \equiv (C_d)(B^{d-1}) \cdot \text{lcm}(\text{mod}(\varphi))^B^d,\]

\[
\# \text{lin}(\Psi) \leq E_d \equiv (20 \cdot \bar{c} \cdot B \cdot D_d^2)^{3^d - 1} \cdot \# \text{lin}(\varphi)^{3^d},
\]

\[
\| \text{lin}(\Psi) \| \leq F_d \equiv (6 \cdot \bar{c} \cdot D_d^2) \cdot \# \text{lin}(\varphi)^{2^d}.\]

Notice that \( A_d, C_d, D_d, E_d \) and \( F_d \) are monotonic in \( d \). Moreover, notice that \( B \leq O(\| \varphi \|) \). The proof is by induction on the quantifier-depth of \( \varphi \). The base case for \( d = 0 \), i.e. \( \varphi \) quantifier-free, is trivial. For the induction step, let \( S = \{ \exists^{\geq c_i} y_1 \psi_1, \ldots, \exists^{\geq c_k} y_n \psi_n \} \) be a minimal family of formulae such that \( \varphi \) is a Boolean combination of formulae from \( S \). Notice that \( n \leq B \). Let \( j \in [1, n] \). The quantifier-depth of \( \psi_j \) is at most \( d - 1 \). We apply the quantifier elimination procedure on \( \psi_j \), obtaining the formula \( \Psi_j \). By induction hypothesis, we have

\[
\# \text{hom}(\Psi_j) \leq A_{d-1} = (4 \cdot B)^{2^{d-1} - 1} \cdot \# \text{hom}(\varphi)^{2^{d-1}},
\]

\[
\# \text{mod}(\Psi_j) \leq B,
\]

\[
\| \text{hom}(\Psi_j) \| \leq C_{d-1} = 2(2A_{d-1})^{d-1} \# \text{hom}(\varphi)^{(2A_{d-1})^{d-1}},
\]

\[
\| \text{mod}(\Psi_j) \| \leq D_{d-1} = (C_{d-1})(B^{d-1}) \cdot \text{lcm}(\text{mod}(\varphi))^B^{d-1},\]

\[
\# \text{lin}(\Psi_j) \leq E_{d-1} = (20 \cdot \bar{c} \cdot B \cdot D_{d-1}^2)^{(3^{d-1} - 1)} \cdot \# \text{lin}(\varphi)^{3^{d-1}},
\]

\[
\| \text{lin}(\Psi_j) \| \leq F_{d-1} = (6 \cdot \bar{c} \cdot D_{d-1}^2) \cdot \# \text{lin}(\varphi)^{2^{d-1}}.\]

For \( j \in [1, k] \), we consider every formula \( \exists^{\geq c_j} y_j \Psi_j \) and perform the quantifier elimination procedure for threshold counting quantifiers, obtaining a formula \( \tilde{\Psi}_j \). From Lemma 31 (see the proof of this lemma for the exact bounds) we have

\[
\# \text{mod}(\tilde{\Psi}_j) = \{ n \} \text{ with } n = k \cdot \text{lcm}(\text{mod}(\Psi_j)) \text{ and } k \leq \| \text{hom}(\Psi_j) \| \cdot \# \text{hom}(\Psi_j),
\]

\[
\# \text{hom}(\tilde{\Psi}_j) \leq 4 \cdot \# \text{hom}(\Psi_j)^2,
\]

\[
\| \text{hom}(\tilde{\Psi}_j) \| \leq 2 \cdot k \cdot \| \text{hom}(\Psi_j) \| \leq 2 \cdot \# \text{hom}(\Psi_j)^2 \cdot \# \text{hom}(\Psi_j),
\]

\[
\# \text{lin}(\tilde{\Psi}_j) \leq 20 \cdot \bar{c} \cdot m^2 \cdot \# \text{lin}(\Psi_j) \cdot \| \text{lin}(\Psi_j) \| \leq 20 \cdot \bar{c} \cdot m^3 \cdot \# \text{lin}(\Psi_j) \cdot \| \text{lin}(\Psi_j) \|,
\]

We derive:

\[
\# \text{hom}(\tilde{\Psi}_j) \leq 4 \cdot (4 \cdot B)^{2^{d-1} - 1} \cdot \# \text{hom}(\varphi)^{2^{d-1}} \leq B^{2^d - 2} A_d \cdot \# \text{hom}(\varphi)^{2^d} = A_d^2.
\]

\[
\| \text{hom}(\tilde{\Psi}_j) \| \leq 2(2A_{d-1})^{d-1} \# \text{hom}(\varphi)^{(2A_{d-1})^{d-1}}\]

\[
\leq 2(2A_{d-1})^{d-1} \# \text{hom}(\varphi)^{(2A_{d-1})^{d-1}} \leq 2(2A_d)^{d-1} \# \text{hom}(\varphi)^{(2A_d)^d} = C_d.\]
Notice that \( k \leq \|\text{hom}(\Psi_j)\|^{\#\text{hom}(\Psi_j)} \leq C_d \) and that \( \text{lcm}(\text{mod}(\Psi_j)) \leq \|\text{mod}(\Psi_j)\|^B \).

- \( m \leq \text{lcm}(\text{mod}(\Psi_j)) \cdot C_d \leq C_d \cdot \|\text{mod}(\Psi_j)\|^B \)
  \[ \leq C_d \cdot ((C_d-1)^{B^d-1} \cdot \text{lcm}(\text{mod}(\varphi))^{B^d-1})^B \]
  \[ \leq (C_d)^{B^d-B+1} \cdot \text{lcm}(\text{mod}(\varphi))^{B^d} \leq (C_d)^{B^d-1} \cdot \text{lcm}(\text{mod}(\varphi))^{B^d} = D_d, \]
  where we recall that we assume \( B \geq 2 \). Hence, \( \|\text{mod}(\Psi_j)\| \leq D_d. \)

- \( \#\text{lin}(\Psi_j) \leq 20 \cdot \bar{e} \cdot m^2 \cdot \#\text{lin}(\Psi_j)^3 \leq (20 \cdot \bar{e} \cdot m \cdot ((20 \cdot \bar{e} \cdot B \cdot D_d)^2)^{(3^d-1)} \cdot \#\text{lin}(\varphi)^{3^d-1})^3 \)
  \[ \leq B^{3^d-3}(20 \cdot \bar{e} \cdot D_d^2)^{3^d-2} \cdot \#\text{lin}(\varphi)^3 \leq E_d. \]

- \( \|\text{lin}(\Psi_j)\| \leq 6 \cdot \bar{e} \cdot m^3 \cdot \#\text{lin}(\Psi_j) \cdot \|\text{lin}(\Psi_j)\| \leq (6 \cdot \bar{e} \cdot D_d^3 \cdot E_d) \cdot (6 \cdot \bar{e} \cdot D_d^3 \cdot E_d)^{d-1} \|\text{lin}(\varphi)\| = F_d. \)

For every \( j \in [1, n] \), we replace in \( \varphi \) each occurrence of the formula \( \exists y \in x_j \psi_j \) not in the scope of a quantification with the formula \( \tilde{\Psi}_j \). We obtain the formula \( \Psi \) that is a Boolean combination of at most \( B \) formulae from \( \{\Psi_1, \ldots, \Psi_n\} \). So, \( \Psi \) is quantifier-free. We have:

- \( \#\text{hom}(\Psi) = \sum_{j=1}^n \#\text{hom}(\tilde{\Psi}_j) \leq A_d, \)

- \( \#\text{mod}(\Psi) = \sum_{j=1}^n \#\text{mod}(\tilde{\Psi}_j) \leq B, \)

- \( \|\text{hom}(\Psi)\| \leq \max\{\|\text{hom}(\tilde{\Psi}_j)\| \mid j \in [1, n]\} \leq C_d, \)

- \( \|\text{mod}(\Psi)\| \leq \max\{\|\text{mod}(\tilde{\Psi}_j)\| \mid j \in [1, n]\} \leq D_d, \)

- \( \#\text{lin}(\Psi) = \sum_{j=1}^n \#\text{lin}(\tilde{\Psi}_j) \leq E_d, \)

- \( \|\text{lin}(\Psi)\| \leq \max\{\|\text{lin}(\tilde{\Psi}_j)\| \mid j \in [1, n]\} \leq F_d. \)

\[ \text{D Eliminating the counting quantifier } \exists^x y \]

We adapt the quantifier-elimination procedure of Section 3 in order to directly deal with the counting quantifier \( \exists^x y \). To shorten the presentation, we only provide the (minor) changes to the procedure of Section 3. Consider \( \Psi_0 = \exists^x y \varphi \), where \( \varphi \) is quantifier-free.

**Steps I–III.** The first two steps of the procedure follow in the same way as described in Section 3, the only difference being that the counting quantifier \( \exists^x y \) is substituted with \( \exists^x y \). The third step is also analogous, but instead of defining \( \Psi_3^{x,r} \) as

\[ \exists x_0 \ldots \exists x_{2\ell} \left( x \leq x_0 + \cdots + x_{2\ell} \land \bigwedge_{j \in [0,2\ell]} \exists^x y(\kappa_j \land \psi^{i,r}_j) \right), \]

we define it as

\[ \exists x_0 \ldots \exists x_{2\ell} \left( x = x_0 + \cdots + x_{2\ell} \land \bigwedge_{j \in [0,2\ell]} \exists^x y(\kappa_j \land \psi^{i,r}_j) \right). \]

This change of the procedure is to be expected, in view of the difference between the two forms of quantification.

**Step IV.** The fourth step of the procedure is updated to deal with the different semantics that \( \exists^x y \psi \) and \( \exists^x y \psi \) assume when infinitely many values of \( y \) satisfy \( \psi \). In the first case, the formula \( \exists^x y \psi \) is equivalent to \( \top \), this follows from Claim 39. As already stated, the formula \( \exists^x y \psi \) instead evaluates to \( \bot \), which lead us to update Claim 9 as follows.

**Claim 32.** Let \( \kappa \in \{y < t'_1, t'_2 < y\} \). If \( \exists y (\kappa \land \psi^{i,r}_\kappa) \) is satisfiable, then \( \Gamma_{i,r} \land \Psi_3^{x,r} \leftrightarrow \bot \).

The procedure then follows as described in Section 3 using Claim 32 instead of Claim 9 to set \( \Psi_4^{i,r} \) to \( \bot \) when necessary. Whenever \( \Psi_4^{i,r} \) is different form \( \bot \), instead of having the form
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\[ \exists x_0 \ldots \exists x_{\ell} \left( x \leq x_0 + \cdots + x_{\ell} + c_1 + \cdots + c_{\ell} \land \bigwedge_{j \in \{2, \ldots, \ell\}} m x_j \leq p_j (t_j' - t_{j-1}') + r_j \right) \]

(as defined in Section 3), it is of the form

\[ \exists x_0 \ldots \exists x_{\ell} \left( x = x_0 + \cdots + x_{\ell} + c_1 + \cdots + c_{\ell} \land \bigwedge_{j \in \{2, \ldots, \ell\}} m x_j = p_j (t_j' - t_{j-1}') + r_j \right). \]

Again, this update to the procedure only reflects the differences between

\[ \text{Step V.} \quad \text{The last step of the procedure is updated following the changes done to } \Psi_{4,r}. \text{ In particular, for every } i \in [1, o] \text{ and } r : Z \to [m], \text{ if } \Psi_{i,r} = \perp \text{ then } \Psi_{i,r} = \perp, \text{ otherwise} \]

\[ \Psi_{i,r}^5 \equiv \Psi_{i,r} = \bigwedge_{i \in [1, o]} r \bigwedge_{Z \to [m]} (\Gamma_{i,r} \land \Psi_{i,r}) \]

Let \( \Psi_{5}^\equiv \equiv \bigwedge_{s \in [1, a]} \bigwedge_{r : Z \to [m]} \Gamma_{i,s} \land \Psi_{i,s}^5 \) be the formula obtained by performing the QE procedure described in this section, on input \( \Psi_0 \). Recall that the formula \( \Gamma_{i,r} \) defined in the second step of the procedure is a conjunction of inequalities with variables from \( \text{vars}(\varphi) \) together with simple modulo constraints.

One can show the following claim with minor adaptation to the proof of correctness of the QE procedure of Section 3.

▷ Claim 25. \( \Psi_0 \leftrightarrow \Psi_{5}^\equiv \). The formula \( \Psi_{5}^\equiv \) is a Boolean combination of linear inequalities and simple modulo constraints.

E Missing proofs from Section 6

▷ Claim 24. \( \exists z \left( z \equiv_q x \land \Psi_{5}^\equiv \right) \leftrightarrow \Psi_{6}^{(x,q)}. \)

Proof. Let \( \varphi_{\text{res}} \) be the formula \( \bigvee_{S : (Z \cup \{x\}) \to [mq]} \bigwedge_{w \in Z \cup \{x\}} w = mq \, s(w) \). We have

\[ \Psi_{5}^\equiv \leftrightarrow \varphi_{\text{res}} \land \Psi_{5}^\equiv \]

\[ \leftrightarrow \gamma \]

\[ \leftrightarrow \bigvee_{i \in [1, o]} \bigwedge_{s : (Z \cup \{x\}) \to [mq]} (\Gamma_{i,s} \land \Psi_{i,s}^5) \]

\[ \text{Indeed, the equivalence (12) holds since } \varphi_{\text{res}} \text{ is a tautology. The equivalence (13) holds as } \gamma \text{ is obtained from } \varphi_{\text{res}} \land \Psi_{5}^\equiv \text{ by distributing the atomic formulæ } w = mq \, s(w) \text{ of } \varphi_{\text{res}} \text{ over the disjunctions given by } \bigvee_{i \in [1, o]} \text{ and } \bigvee_{r : Z \to [m]}. \text{ The nature of (14) is already explained during the procedure: since every function } r : Z \to [m] \text{ can be seen as a partial function from } Z \cup \{x\} \text{ to } [mq], \text{ after the two steps above, for every } s : (Z \cup \{x\}) \to [mq] \text{ and } i \in [1, o], \text{ all but one disjunct of the subformula } \bigvee_{r : Z \to [m]} \bigvee_{w \in Z \cup \{x\}} w = mq \, s(w) \land \Gamma_{i,r} \land \Psi_{i,r}^5 \text{ of } \gamma \text{ evaluate } \perp. \text{ Thanks to (12)–(14), we conclude:} \]

\[ \exists z \left( z \equiv_q x \land \Psi_{5}^\equiv \right) \leftrightarrow \exists z \left( z \equiv_q x \land \left( \bigvee_{i \in [1, o]} \bigwedge_{s : (Z \cup \{x\}) \to [mq]} (\Gamma_{i,s} \land \Psi_{i,s}^5) \right) \right) \]

\[ \leftrightarrow \Psi_{6}^{(x,q)} \]

The equivalence (15) follows from (12)–(14), whereas for the equivalence (16) it is sufficient to distribute the existential quantifier \( \exists z \) and the formula \( z \equiv_q x \) over all the disjunctions given by \( \bigvee_{i \in [1, o]} \) and \( \bigvee_{s : (Z \cup \{x\}) \to [mq]} \).

▷ Claim 25. \( \Psi_{6}^{(x,q)} \leftrightarrow \Psi_{4}^{(x,q)}. \) The formula \( \Psi_{4}^{(x,q)} \) is quantifier-free.
Proof. We show that, given $i \in [1, o]$ and $s: (Z \cup \{x\}) \rightarrow \{mq\}$, $\exists z (z \equiv_{q} x \land \Gamma_{i,s} \land \Psi_{5}^{s}) \leftrightarrow \gamma_{i,s}$. If $\Psi_{5}^{i,s} = \perp$ then $\gamma_{i,s}$ is defined as $\perp$ and the equivalence holds. Otherwise we have $\Psi_{5}^{i,s} \equiv m\gamma_{z} = p_{2}(t_{2} - t_{1}') + r_{2} + \ldots + p_{t}(t_{t} - t_{t-1}') + r_{t} + m(c_{1} + \ldots + c_{t})$. In this case, $\gamma_{i,s} = \Gamma_{i,s}$ if $s(m\gamma)$ is congruent to $S \equiv s(p_{2}(t_{2} - t_{1}') + r_{2} + \ldots + p_{t}(t_{t} - t_{t-1}')) + r_{t} + m(c_{1} + \ldots + c_{t})$ modulo $mq$, and otherwise $\gamma_{i,s} = \perp$. We do a case split following these to cases:

1. Suppose that $s(m\gamma)$ and $S$ are not congruent modulo $mq$. In order to prove that the equivalence $\exists z (z \equiv_{q} x \land \Gamma_{i,s} \land \Psi_{5}^{i,s}) \leftrightarrow \gamma_{i,s}$ holds, it is sufficient to show that the formula $\exists z (z \equiv_{q} x \land \Gamma_{i,s} \land \Psi_{5}^{i,s})$ is unsatisfiable (since $\gamma_{i,s}$ is defined as $\perp$). From $m\gamma = p_{2}(t_{2} - t_{1}') + r_{2} + \ldots + p_{t}(t_{t} - t_{t-1}') + r_{t} + m(c_{1} + \ldots + c_{t}).$

and $z \equiv_{q} x \leftrightarrow m\gamma \equiv_{mq} m\gamma$ we derive $m\gamma \equiv_{mq} p_{2}(t_{2} - t_{1}') + r_{2} + \ldots + p_{t}(t_{t} - t_{t-1}') + r_{t} + m(c_{1} + \ldots + c_{t}).$

Ad absurdum, suppose that $\nu \models \exists z (z \equiv_{q} x \land \Gamma_{i,s} \land \Psi_{5}^{i,s})$, for some assignment $\nu$. As $\nu \models \Gamma_{i,s}$, for every variable $w \in Z \cup \{x\}$ we have $\nu(w) \equiv_{mq} s(w)$. However, this is contradictory, as $\nu \models m\gamma \equiv_{mq} p_{2}(t_{2} - t_{1}') + r_{2} + \ldots + p_{t}(t_{t} - t_{t-1}') + r_{t} + m(c_{1} + \ldots + c_{t})$ implies $s(m\gamma) \equiv_{mq} S$.

2. Assume that $s(m\gamma)$ and $S$ are congruent modulo $mq$. By definition, $\gamma_{i,s} = \Gamma_{i,s}$. The left-to-right direction of the equivalence $\exists z (z \equiv_{q} x \land \Gamma_{i,s} \land \Psi_{5}^{i,s}) \leftrightarrow \gamma_{i,s}$ is thus trivial: if an assignment $\nu$ satisfies the left hand side of this equivalence, then $\nu \models \Gamma_{i,s}$. For the right-to-left direction, assume $\nu \models \Gamma_{i,s}$. Hence, for every variable $w \in Z \cup \{x\}$, $\nu(w)$ is equivalent to $s(w)$ modulo $mq$. This implies that $\nu(m\gamma)$ and $V = \nu(p_{2}(t_{2} - t_{1}') + r_{2} + \ldots + p_{t}(t_{t} - t_{t-1}') + r_{t} + m(c_{1} + \ldots + c_{t}))$ are congruent modulo $mq$. Since moreover $\nu(m\gamma) = m\gamma(x)$ is a multiple of $m$, we conclude that there is $v \in Z$ such that $m \cdot v = V$. Consider the assignment $\nu[v/z]$, with $z$ fresh. Since $z \notin \{Z \cup \{x\}$, we have $\nu[v/z] \models \Gamma_{i,s}$. Moreover, from $\nu(m\gamma) \equiv_{mq} m\gamma$ we conclude that $\nu[v/z] \models m\gamma \equiv_{mq} m\gamma$. Equivalently, $\nu[v/z] \models x \equiv_{q} z$. Lastly, by definition of $v$, $\nu[v/z] \models m\gamma = p_{2}(t_{2} - t_{1}') + r_{2} + \ldots + p_{t}(t_{t} - t_{t-1}') + r_{t} + m(c_{1} + \ldots + c_{t}).$

Therefore, $\nu[v/z] \models z \equiv_{q} x \land \Gamma_{i,s} \land \Psi_{5}^{i,s}$, and thus $\nu \models \exists z (z \equiv_{q} x \land \Gamma_{i,s} \land \Psi_{5}^{i,s})$. ▶

**Lemma 34.** The following bounds are established for $\Psi_{7}^{(x,q)}$:

- $\text{mod}(\Psi_{7}^{(x,q)}) = \{m \cdot q\}$ with $m = k \cdot \text{lcm}(\text{mod}(\varphi))$ and $k \leq \|\text{hom}(\varphi)\|^{\#\text{hom}(\varphi)},$
- $\#\text{hom}(\Psi_{7}^{(x,q)}) \leq O(\#\text{hom}(\varphi)^{2})$ and $\|\text{hom}(\Psi_{7}^{(x,q)})\| \leq O(k \cdot \|\text{hom}(\varphi)\|),$
- $\#\text{lin}(\Psi_{7}^{(x,q)}) \leq O(\#\text{lin}(\varphi)^{2})$ and $\|\text{lin}(\Psi_{7}^{(x,q)})\| \leq O(k \cdot \|\text{lin}(\varphi)\|).$

**Proof.** The proof follows similarly to the one of Lemma 31. Without loss of generality, we assume $\#\text{hom}(\varphi), \#\text{lin}(\varphi), \|\text{hom}(\varphi)\|$ and $\|\text{lin}(\varphi)\|$ to be at least 1. This hides constant factors in the exponent. The quantifier-elimination procedure for $\exists z^{(x,q)} \varphi z$ starts by performing the normalisation of the coefficients of $y_{i}$, as described in Step I of Section 3. Let $\Psi_{1}$ be the resulting formula. As already discussed in the proof of Lemma 30, we have

- $\#\text{lin}(\Psi_{1}) = \#\text{lin}(\varphi)$ and $\|\text{lin}(\Psi_{1})\| \leq k \cdot \|\text{lin}(\varphi)\|,$
- $\#\text{hom}(\Psi_{1}) = \#\text{hom}(\varphi)$ and $\|\text{hom}(\Psi_{1})\| \leq k \cdot \|\text{hom}(\varphi)\|.$
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where \( k \leq \|\hom(\varphi)\|^{\#\hom(\varphi)} \) is the \( \text{lcm} \) of all coefficients of \( y \) appearing in linear inequalities.

We recall that the formula \( \Psi_7^{(x,q)} \) is defined as \( \bigwedge_{i \in [1,q]} \bigvee \mathbb{Z} \cup \{0\} \gamma_{i,s} \), where \( Z = \mathsf{fv}(\varphi) \), \( m = \text{lcm}(\text{mod}(\varphi)) \cdot \|\hom(\varphi)\|^{\#\hom(\varphi)} \), \( \gamma_{i,s} \) is \( 1 \) or \( O_i \land (\wedge_{w \in \mathbb{Z} \cup \{0\}} w \equiv_{mq} s(w)) \). Here, the sub-formula \( O_i \) is an ordering on the set of terms \( T \cup \{0\} \), as defined in Step II of Section 3.

In particular, \( \#T \leq \#\mathsf{lin}(\Psi_1) \), \( ||T|| \leq ||\mathsf{lin}(\Psi_1)|| \) and all coefficients of variables in terms of \( T \) are bounded by \( ||\hom(\Psi_1)|| \). So, even when accounting for all orderings \( (O_i)_{i \in [1,q]} \), the formula \( \Psi_7^{(x,q)} \) only contains at most \( (\#T \cup \{0\})^2 \) inequalities, whose magnitude of coefficients and constants doubles with respect to the one of the terms in \( \Psi_1 \). Lastly, every modulo constraint in \( \Psi_7^{(x,q)} \) is of the form \( w \equiv_{mq} s(w) \). Therefore, we have the following bounds:

\[
\begin{align*}
\#\mathsf{lin}(\Psi_7^{(x,q)}) &\leq (\#\mathsf{lin}(\varphi) + 1)^2 \leq 4 \cdot \#\mathsf{lin}(\varphi)^2, \\
\|\mathsf{lin}(\Psi_7^{(x,q)})\| &\leq 2 \cdot k \cdot \|\mathsf{lin}(\varphi)\|, \\
\#\hom(\Psi_7^{(x,q)}) &\leq (\#\hom(\varphi) + 1)^2 \leq 4 \cdot \#\hom(\varphi)^2, \\
\|\hom(\Psi_7^{(x,q)})\| &\leq 2 \cdot k \cdot \|\hom(\varphi)\|, \\
\text{mod}(\Psi_7^{(x,q)}) &\equiv \{mq\}, \text{ where } m = k \cdot \text{lcm}(	ext{mod}(\varphi)),
\end{align*}
\]

where we recall that we are assuming \( \#\hom(\varphi) \), \( \#\mathsf{lin}(\varphi) \), \( \|\hom(\varphi)\| \) and \( \|\mathsf{lin}(\varphi)\| \) to be at least 1.

\begin{lemma}
\label{lem:hom-2}
\( \#\hom(\Psi_7^{(x,q)}) = \mathcal{O}(\#\hom(\varphi)^2) \).
\end{lemma}

\begin{proof}
This is a simple consequence of Lemma \[24\].
\end{proof}

\begin{lemma}
\label{lem:linear-2}
Let \( \varphi \) be a formula of Presburger arithmetic with threshold quantifiers. There is an equivalent quantifier-free formula \( \Psi \) such that

\[
\begin{align*}
\#\mathsf{lin}(\Psi), \|\mathsf{lin}(\Psi)\|, \|\hom(\Psi)\| \text{ and } \|\text{mod}(\Psi)\| \text{ are at most } 2^{2^{O(\|\varphi\|^2)}}, \\
\#\hom(\Psi) &\leq 2^{2^{O(\|\varphi\|^2)}} \text{ and } \#\text{mod}(\Psi) \leq \|\varphi\|. \\
\end{align*}
\end{lemma}

\begin{proof}
Without loss of generality, we assume \( \#\hom(\varphi) \), \( \#\mathsf{lin}(\varphi) \), \( \|\hom(\varphi)\| \) and \( \|\mathsf{lin}(\varphi)\| \) to be at least 1. This hides constant factors in the exponent. The proof follows very closely the one of Lemma \[22\] with no surprises.

- Let \( d \) be the the quantifier-depth of \( \varphi \),
- let \( B \) be 2 plus \( \#\text{mod}(\varphi) \), plus the number of Boolean connectives in \( \varphi \), and
- let \( \bar{q} \) be the maximal integer such that \( \exists (x,q) y \) occurs in \( \varphi \).

We show the following bounds for \( \Psi \), sharpening the ones in the statement of the lemma.

\[
\begin{align*}
\#\hom(\Psi) &\leq A_d \equiv (4 \cdot B)^{2^d - 1} \cdot \#\hom(\varphi)^{2^d}, \\
\#\text{mod}(\Psi) &\leq B, \\
\|\hom(\Psi)\| &\leq C_d \equiv 2^{(2A_d)^{d - 1} \#\hom(\varphi)(2A_d)^d}, \\
\|\text{mod}(\Psi)\| &\leq D_d \equiv (\bar{q} \cdot C_d)(B^d - 1) \cdot \text{lcm}(	ext{mod}(\varphi)) B^d, \\
\#\mathsf{lin}(\Psi) &\leq E_d \equiv (20 \cdot B \cdot D_d^2)^{3^d - 1} \cdot \#\mathsf{lin}(\varphi)^{3^d}, \\
\|\mathsf{lin}(\Psi)\| &\leq F_d \equiv (6 \cdot D_d^3 \cdot E_d)^d \cdot \|\mathsf{lin}(\varphi)\|.
\end{align*}
\]
Recall that the logic features both modulo counting quantifiers and standard first-order quantifiers. To this end, notice that $A_d$, $B$, $C_d$, $E_d$ and $F_d$ overapproximate the homonymous bounds given in Lemma 22 for the threshold quantifiers, for the case where all thresholds $c$ in $\exists^{c^d}y$ equal 1 (since $\exists^{\leq 1}y \equiv \exists y$). Therefore, we deal with standard first-order quantifiers exactly as in Lemma 22. Below, let us focus uniquely on modulo counting quantifiers.

Notice that $A_d$, $C_d$, $D_d$, $E_d$ and $F_d$ are monotonous in $d$. Moreover, notice that $B \leq O(|\varphi|)$. The proof is by induction on the quantifier-depth of $\varphi$ ($d$ takes into account both modulo counting quantifiers and first-order quantifiers). The base case for $d = 0$, i.e. $\varphi$ quantifier-free, is trivial. For the induction step, let $S = \{\exists^{x_1, \ldots, x_n}y_1 \psi_1, \ldots, \exists^{x_1, \ldots, x_n}y_n \psi_n\}$ be a minimal family of formulae such that $\varphi$ is a Boolean combination of formulae from $S$. Notice that $n \leq B$. Let $j \in [1, n]$. The quantifier-depth of $\psi_j$ is at most $d - 1$. We apply the quantifier elimination procedure on $\psi_j$, obtaining the formula $\Psi_j$. By induction hypothesis,

- $\#\text{hom}(\Psi_j) \leq A_{d-1} = (4 \cdot B)^{d-1} \cdot \#\text{hom}(\varphi)^{2^{d-1}}$,
- $\#\text{mod}(\Psi_j) \leq B$,
- $\|\text{hom}(\Psi_j)\| \leq C_{d-1} = 2^{(2A_{d-1})^{d-1}-1} \|\text{hom}(\varphi)\|^{2^{A_{d-1}}-1}$,
- $\|\text{mod}(\Psi_j)\| \leq D_{d-1} = (\tilde{q} \cdot C_{d-1})^{B^{d-1}-1} \cdot \text{lcm}(\text{mod}(\varphi))^{B^d-1}$,
- $\#\text{lin}(\Psi_j) \leq E_{d-1} = (20 \cdot B \cdot D_{d-1}^{-2})^{(2^{d-1}-1)} \cdot \#\text{lin}(\varphi)^{2^{d-1}}$,
- $\|\text{lin}(\Psi_j)\| \leq F_{d-1} = (6 \cdot D_{d-1}^{-3} \cdot E_{d-1})^{d-1} \cdot \|\text{lin}(\varphi)\|$.

For $j \in [1, k]$, we consider every formula $\exists^{(x_j, q_j)}y_j \Psi_j$ and perform the quantifier elimination procedure for threshold counting quantifiers, obtaining a formula $\tilde{\Psi}_j$. Lemma 34 (see the proof of this lemma for the exact bounds) we have

- $\#\text{mod}(\tilde{\Psi}_j) = \{q_j \cdot m\}$ with $m = k \cdot \text{lcm}(\text{mod}(\Psi_j))$ and $k \leq \|\text{hom}(\Psi_j)\|^{\#\text{hom}(\Psi_j)}$.
- $\#\text{hom}(\tilde{\Psi}_j) \leq 4 \cdot \#\text{hom}(\Psi_j)^2$,
- $\|\text{lin}(\tilde{\Psi}_j)\| \leq 2 \cdot k \cdot \|\text{lin}(\Psi_j)\|$,
- $\#\text{mod}(\tilde{\Psi}_j) \leq 4 \cdot \#\text{mod}(\Psi_j)^2$,
- $\#\text{hom}(\tilde{\Psi}_j) \leq 2 \cdot k \cdot \|\text{hom}(\Psi_j)\| \leq 2 \cdot \|\text{hom}(\Psi_j)\|^{2\#\text{hom}(\Psi_j)}$,

We derive:

- $\#\text{hom}(\tilde{\Psi}_j) \leq 4 \cdot ((4 \cdot B)^{2d-1} \cdot \#\text{hom}(\varphi))^{2d-1} \leq B^{2d-2} \cdot 4^{2d-1} \cdot \#\text{hom}(\varphi) = \frac{4^d}{B}$,
- $\|\text{hom}(\tilde{\Psi}_j)\| \leq 2^{(2A_{d-1})^{d-1}-1} \|\text{hom}(\varphi)\|^{(2A_{d-1})^{d-1}} \leq 2^{(2A_{d-1})^{d-1}} \|\text{hom}(\varphi)\|^{2^{A_{d-1}}-1} \leq 2^{(2A_{d-1})^{d-1}} \|\text{hom}(\varphi)\|^{2^{A_{d-1}}} = C_d$.

Notice that $k \leq \|\text{hom}(\Psi_j)\|^{\#\text{hom}(\Psi_j)} \leq C_d$ and that $\text{lcm}(\text{mod}(\Psi_j)) \leq \|\text{mod}(\Psi_j)\|^{B}$.

- $m \leq q_j \cdot C_d \cdot \text{lcm}(\text{mod}(\Psi_j)) \leq \tilde{q} \cdot C_d \cdot \|\text{mod}(\Psi_j)\|^{B}$
- $\leq \tilde{q} \cdot C_d \cdot (\tilde{q} \cdot C_{d-1})^{(B^{d-1}-1)} \cdot \text{lcm}(\text{mod}(\varphi))^{B^{d-1}}$
- $\leq (\tilde{q} \cdot C_d)^{B^{d-1}-1} \cdot \text{lcm}(\text{mod}(\varphi))^{B^{d-1}} \leq (\tilde{q} \cdot C_d)^{B^{d-1}} \cdot \text{lcm}(\text{mod}(\varphi))^{B^d} = D_d$,

where we recall that we assume $B \geq 2$. Hence, $\#\text{mod}(\tilde{\Psi}_j) = 1$ and $\|\text{mod}(\tilde{\Psi}_j)\| \leq D_d$.

- $\#\text{lin}(\tilde{\Psi}_j) \leq 4 \cdot (20 \cdot B \cdot D_{d-1}^{-2})^{(2d-1)-1} \#\text{lin}(\varphi)^{2d-1} \leq B^{3d-2} \cdot (20 \cdot D_{d-1}^{-2})^{(2d-1)} \#\text{lin}(\varphi)^{2d-1} \leq \frac{E_d}{B}$.
- $\|\text{lin}(\tilde{\Psi}_j)\| \leq 2 \cdot C_d \cdot \|\text{lin}(\Psi_j)\| \leq (2 \cdot C_d) \cdot (6 \cdot D_{d-1}^{-3} \cdot E_{d-1})^{d-1} \cdot \|\text{lin}(\varphi)\| \leq F_d$,

where we notice that $C_d \leq D_d$. 


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For every $j \in [1, n]$, we replace in $\varphi$ each occurrence of the formula $\exists^{(x_{j}, q_{j})} x_{j} \psi_{j}$ not in the scope of a quantification with the formula $\tilde{\psi}_{j}$. We obtain the formula $\Psi$ that is a Boolean combination of at most $B$ formulae from $\{\tilde{\psi}_{1}, \ldots, \tilde{\psi}_{n}\}$. So, $\Psi$ is quantifier-free. We have:

- $\#hom(\Psi) = \sum_{j=1}^{n} \#hom(\tilde{\psi}_{j}) \leq A_{d}$,
- $\#mod(\Psi) = \sum_{j=1}^{n} \#mod(\tilde{\psi}_{j}) \leq B$,
- $\|hom(\Psi)\| \leq \max\{\|hom(\tilde{\psi}_{j})\| \mid j \in [1, n]\} \leq C_{d}$,
- $\|mod(\Psi)\| \leq \max\{\|mod(\tilde{\psi}_{j})\| \mid j \in [1, n]\} \leq D_{d}$,
- $\#lin(\Psi) = \sum_{j=1}^{n} \#lin(\tilde{\psi}_{j}) \leq E_{d}$,
- $\|lin(\Psi)\| \leq \max\{\|lin(\tilde{\psi}_{j})\| \mid j \in [1, n]\} \leq F_{d}$. ◀