Analytical and numerical modelling of the programmable percolation route formation when planning two-phase operations

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Abstract. A swarm of robots, as a system of relatively simple interconnected managed objects, performs a common task simultaneously and in a distributed manner. When planning swarm operations associated with the creation in the service area through the front strip of the zones of the trust instrument – payload objects swarm, there is a problem of transmitting a signal from one robot to another for operational rearrangement of the swarm, as at the time of planning the exact purpose of the swarm operation has not yet been determined, or is a secret, or is determined by a number of random circumstances. The execution of the swarm operation is advisable to carry out in two phases, and the first phase to start even before the resolution of these uncertainties by creating a basic random network with a small concentration of robots in it. In the second phase of the operation, a programmable percolation route is formed by local rearrangement of the robots, which provides target coverage of the target equipment of the swarm objects of a certain service zone. In this case, you can significantly reduce the time of the operation. The corresponding analytical dependences were obtained.

1. Introduction

The application a large network of relatively simple mobile robots (swarm) allows to solve a large number of important practical problems in a distributed manner. For example, the systems of: surveillance, laying a safe path, firefighting, intelligence, communication, remote sensing tasks [1], [2], [5], [9]-[11], [20]. Many of these tasks are related to the formation in the operating area of a linear topology for placing swarm objects close to the front line.

In this paper, the planning of swarm operations is considered under conditions of uncertainty, when the place and time of the operation in an operating area is not defined or is a secret. Meanwhile, after making a decision and becoming certain, it may take too much time for the formation of the front line in the operating area. The solution to this problem is proposed to be carried out in two phases. In the first phase, a stochastic basis is created in advance in the operating area from generally randomly distributed swarm objects. In the second phase, the intervals between the formed clusters of swarm objects are filled with objects from the immediate environment of the planned path with the formation of a through target frontal path the operating area in the minimum time.

A large network of relatively simple interconnected swarm objects has several advantages over a single more complex and effective robot (or several such robots):
• Greater efficiency of such a system comes due to the ability to distribute the network (swarm) throughout the operating area and their simultaneous operation.

• There is a smooth decrease in system efficiency when individual objects fail or are lost, as opposed to a sharp and complete loss of efficiency of a centralized system when a more complex platform fails.

In this paper, the problem of planning swarm operations under uncertainty is being investigated and analysed using methods of the theory of programmable percolation. This approach has a scientific novelty and is an alternative and unique in relation to other existing methods of solving this problem.

In particular, the similar interpretation of swarm robotics task fits well with percolation theory tasks. The percolation theory finds a through “conductive” route, i.e. the percolation route formed in a random environment when the concentration of “conductive objects” increases and reaches the critical value.

The classical percolation theory considers the matrix with random filling as the model of random operating environment model in direct geometrical interpretation [3], [12], [15], [21]-[24]. In this square matrix with the number of lines \(L\), the random part of cells is “black”. It conducts a liquid or gas flow, transport or information flows. Other cells are “white”, which do not conduct the flow. As the concentration (probability of occurrence) of the black cells increases, some of their ribs start osculating in a random manner, which can be interpreted as the occurrence of close interaction, and merge. The osculating ribs of the “black” cells form random conductive clusters, which are generated and grow as the concentration of the “black” cells increases [15], [24].

The classical percolation theory [15], [21]-[24] finds the concentration of the swarm objects in the service zone – “black” cells \(p_b\) – stochastic percolation threshold when the through random route is formed over the “black” cells via the entire matrix in the set direction, i.e. the stochastic percolation cluster. However, the stochastic percolation cluster has a loose structure, multiple “dead branches” and fairly excess quantity of the swarm objects in terms of solving many practical tasks. In connection with this fact, the well studied classical percolation theory for flat grids is impractical for our task, notwithstanding the fact that the task under consideration is related to flat geometrical arrangement of the swarm objects.

So, the authors develop the programmable or artificial percolation theory (non-stochastic one as in the tasks of percolation via the porous environment). We do not increase the concentration of “conductive black cells” until the stochastic breakdown concentration known for the flat grid appears, but we organize actively the through percolation route occurrence by filling intervals between small clusters occurring on the matrix and supplementing the planned through route with active swarm objects. All these things take place at the concentration much less than the stochastic breakdown concentration (i.e. with less quantity of the swarm objects) [7], [8].

Therefore, the authors develop the theory of programmable or artificial percolation (not stochastic as in problems of percolation through a porous medium).

When implementing the programmable percolation [6]-[8], [16]-[19], the stochastic base is created at the first phase using randomly distributed objects, with their concentration values much lower than the stochastic percolation threshold; the through percolation route is built at the second phase by introducing (installing) additional objects into the available inter-cluster intervals. In this case the stochastic base concentration is selected so that total costs for a two-phase operation are minimum.

For resource-efficient signal transmission between swarm members via network repeaters, it is proposed to use methods of two-phase operations in percolation theory in its direct geometric interpretation.

Using the method of programmable percolation in a two-phase operation allowed to strongly reduce the number of swarm objects that need to be used in a given operating area. In this article, using statistical modeling and analytical research, the concentration of swarm objects is found, which minimizes the relative cost of performing a two-phase operation. The revealed optimal concentration value is almost three times less than if stochastic percolation methods were used to solve this problem.
The considered approach with two-phase operations for percolation of the operating area is effective not only for a swarm of mobile objects, but also for a large network of stationary objects: video surveillance cameras, nodes of computer network, etc. In this case, this approach can be implemented when related objects with special properties must be placed along a linear topology, forming a frontal structure – a through target programmable percolation path the operating area. Examples of such properties: security, changes in the frequency of activation, etc. The approach with two-phase operations for percolation of the operating area in this case allows determining the optimal concentration of objects with special properties.

The solution of these and the previously mentioned practical problems by the method of programmable percolation is based on the dependencies obtained by statistical modeling on a large number of matrices with randomly filled objects with different concentrations. Translation of these dependencies obtained statistically into analytical form is highly desirable for speeding up the work of the corresponding software for analyzing two-phase swarm operations.

This problem can be solved by selecting appropriate approximating analytical expressions. But, the authors considered that the choice of approximating functions and their parameters through analytical conclusion of the equations of cluster formation and growth, filling of inter-cluster intervals with objects forming a through percolation path, will simultaneously confirm the correctness of the results of statistical modeling.

Of course, at the same time, the inevitable reconciliation of the results of a mathematical experiment and the results of analytical analysis is possible and should be simple. That allow us to associate the parameters of two-phase operations for constructing a path of programmable percolation with the concentration of swarm objects in the operating area.

2. Expression for the average "width" of clusters formed when the concentration of objects in the operating area changes

Let's designate the average length of the programmable percolation route from the concentration as $L(p)$. It is evident that this average length is the sum of two other functions -- $N(p)$ and $R(p)$, namely the average quantity of the black cells and the average quantity of the added cells (hereinafter: red cells) from the concentration, respectively.

The quantity of the cells in the percolation matrix is equal to $N$, and the concentration of the objects on the matrix amounts to $p$. Before this [7], the dependence of the average quantity of the object clusters on the concentration $M(p) = \sum_{s=1}^{N} \sum_{t=t_{min}}^{t_{max}} g_{stN} p^s (1 - p)^t$ was obtained, where $p$ is the concentration; $s$ is the quantity of the cells included into the cluster; $t$ is the required quantity of empty (white) cells around the cluster; $g_{stN}$ is the quantity of different clusters which can be obtained from $s$ cells and with $t$ empty neighboring cells. Since $L(p) = N(p) + R(p)$, it is proposed to find separately $N(p)$ and $R(p)$.

Let's consider a certain percolation matrix used to build the artificial percolation route (see Figure 1).

The black cells in the figure, which are engaged in the percolation route, are shown in dark red color; the bright red cells are the cells added to the programmable percolation route of the cell.

To calculate the average quantity of the black cells engaged in the percolation route, let's imagine a certain percolation matrix composed of the same clusters. To do this, the notion “cluster influence zone” as a certain areal characteristic on the percolation matrix will be introduced. The closest analogue for the “cluster influence zone” will be Thiessen-Voronoi polygons [27] where object clusters are used instead of points.

$$\frac{N}{NM(p)} = \frac{1}{M(p)}$$  \hspace{1cm} (1)

where $NM(p)$ is the average quantity of the clusters on the matrix, with the number of cells $N$. Expression 1 will be the “cluster influence zone”. Similarly, the average area will be found, namely the average quantity of the cells in the cluster on such matrix:
\[ \frac{pN}{NM(p)} = \frac{p}{M(p)} \]  

(2)

**Figure 1.** Percolation matrix $30 \times 30$ with the concentration $p = 0.4$ and built artificial percolation route

This means that the matrix composed of the same clusters will be the matrix composed of some “cluster influence zones”, each of which has a cluster inside with the average area being the same for all the clusters in this matrix (expression 2).

The average quantity of the black cells in this matrix, which are engaged in the percolation route, can be considered as the sum of all the black cells from the programmable percolation route or as the product of the average quantity of the black cells over a certain average “width” of the cluster by the length $l$, along which the percolation route passes. The average “width” means the average number of the cells for all the lines in the cluster line, which is perpendicular to the percolation direction.

The length, $l$, can be taken as the average quantity of the black cells in the matrix, $N$, which is located along the direction of the percolation distribution. In this case this is the matrix length, $L = \sqrt{N}$, multiplied by the concentration.

Now let’s find this average “width”. To do this, let’s consider our cluster. In general, the cluster has a random form but [7] shows that this form for most clusters tends to the form of circle (and the Gaussian circle for final small matrices). The radius of such circle will be as follows:

\[ \frac{p}{M(p)} = \pi r^2(p) \Rightarrow r(p) = \sqrt{\frac{p}{M(p)\pi}} \]  

(3)

Here the evident average height of the cluster is given as $2r(p)$ – the maximum length in the cluster along the percolation direction. For a circular cluster, the above height coincides with the diameter of this cluster. When the average area of the cluster and its average height are known, it is easy to find the average “width” of the cluster:

\[ S_{cl}(p) = \frac{p}{M(p)2r(p)} = \frac{p\sqrt{M(p)\pi}}{2M(p)\sqrt{p}} = \frac{1}{2} \sqrt{\frac{p\pi}{M(p)}} \]  

(4)

3. The range of deviations of the percolation path from the original direction of percolation

When the expression for the average “width” of the cluster is known, the average quantity of the black cells engaged in the percolation route and distributed over this “width” of the cluster can be found.

The behaviour of the “black” cells on the “width” is random oscillatory deviations from the direction of percolation movement. This behaviour is well described by the Rayleigh-Rice random variable distribution for a two-dimensional random vector whose orthogonal coordinates can be the implementation of a random walk on a straight line [25], [26].
The consideration of the random walk process results in the range of deviations of associated black cells from the initial vector of percolation movement. For example, Figure 2 shows the deviation of the percolation route on some lines from the initial percolation vector (movement in the straight line, from top to bottom). Higher the concentration, greater this deviation, down to the maximum value at the stochastic percolation breakdown concentration. Then the range of these deviations starts decreasing and the percolation route straightens.

When considering the behaviour of the black cells engaged in the programmable percolation route, the randomness of their distribution along the average “width” of the cluster can be seen. Such behaviour of the associated black cells is well described with the methods of random walks in the straight line [25]. In fact, the percolation route is sufficiently tortuous at high concentrations $p$ due to the considerable looseness of the clusters (see Figure 2), which causes the variation in the quantity of the black cells engaged in the route, inside this “width” of the cluster (see Figure 2 b).

The Rayleigh-Rice distribution is used to describe the length of the two-dimensional random vector, which rectangular coordinates are independent normal random values with equal dispersions and, supposedly, different mathematical expectations [26].

The programmable percolation route can be considered as the random two-dimensional vector due to the presence of random deviations from the initial vector of percolation movement. The deviation of the associated black cells from the percolation movement direction can be described with the random walk methods. As it follows from solving the random walk tasks, the associated black cells are distributed by the normal distribution law [25]. Since the maximum range of these deviations cannot go beyond the cluster boundaries (i.e. exceed the average “width” of the cluster), the normal law of distribution of deviations from the percolation movement direction becomes the law of two-way truncated normal distribution [25], [26] described in expression 6. We shall consider this range of deviations for $p_{fix}$ – any fixed concentration is such that $p_{fix} \in [0;1]$. This means that we shall obtain a set of expressions with similar structure, which describe this range, but have different values of the average “width” of the cluster.

So, the range of the deviations from the set direction of the programmable percolation route is well described with the help of the distribution which is similar in structure with the Rayleigh-Rice distribution and has a floating scale parameter; therefore, it is proposed to use it to describe:

$$f(x) = \frac{x}{\sigma_{cl}^2} e^{-\left(\frac{x^2+m^2}{2\sigma_{cl}^2}\right)} I_0\left(\frac{xm}{\sigma_{cl}^2}\right)$$

where $f(x)$ is the function which is similar in structure to the Rayleigh-Rice distribution density and has the floating scale parameter; $\sigma_{cl}^2$, $\sigma_{cl}^2$ are the scale parameters (same dispersions of the vector
coordinates being random values), $m$ is the form parameter, which is equal to $m = \sqrt{\mu_1^2 + \mu_2^2}$ ($\mu_1$ and $\mu_2$ are the mathematical expectations of the vector coordinates), $I_0(x)$ is the modified zero-order Bessel function.

Let’s consider the cluster behaviour depending on the concentration. In accordance with the expressions of the average area of the cluster $\frac{p}{M(p)}$ and average quantity of the clusters on the percolation matrix $M(p) = \sum_{s=1}^{N} \sum_{t=\text{max}_s}^{\text{min}_s} g_{stN} p^s(1-p)^t$ [7], it’s easy to see that the cluster area increases with the concentration increase $p$. It is evident that the average “width” of the cluster also increases with the concentration increase. As the average “width” of the cluster increases, the quantity of the black cells engaged in the route also increases. Since the value (black or associated black) in each such cell is considered as the implementation of independent random values with the same dispersions and, generally speaking, the same mathematical expectations, the actual space should be determined along the average “width” of the cluster along which the two-way truncated normal distribution acts. All the cells in the average “width” of the cluster can be potentially engaged in building the programmable percolation route and the boundaries of such truncated normal distribution will be the boundaries of the average “width” of the cluster:

$$f_{\text{nor}}(x) = \frac{1}{\gamma \sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \alpha \leq x \leq \beta$$

where $f_{\text{nor}}(x)$ is the two-way truncated normal distribution, $\gamma = \Phi_0\left(\frac{\beta-\mu}{\sigma}\right) - \Phi_0\left(\frac{\alpha-\mu}{\sigma}\right)$; $\alpha$ and $\beta$ are the left and right boundaries of the truncation, respectively; $\sigma$ is the root-mean-square error of the deviation of the associated black cells from the percolation movement direction distributed by the two-way truncated normal law, which is received from the large-scale mathematical experiment; $\mu$ is the mathematical expectation of normal distribution.

Based on the rule $3\sigma$ and the fact that the truncated normal distribution acts along all the cells of the average “width”, the dispersion of the truncated normal distribution being the scale parameter for the distribution which is similar in structure with the Rayleigh-Rice distribution and the floating scale parameter can be given by:

$$S_{\text{cl}}(p) = 2C_1(p)3\sigma_{\text{cl}}(p) \Rightarrow \sigma_{\text{cl}}(p) = \frac{S_{\text{cl}}(p)}{6C_1(p)}$$

where $S_{\text{cl}}(p)$ is the average “width” of the cluster; $\sigma_{\text{cl}}(p)$ is the scale parameter of the distribution law which is similar in structure with the Rayleigh-Rice distribution and has the floating scale parameter being the dispersion of the two-way truncated normal distribution; $C_1(p)$ is the consistent factor which is, in general, the function from the concentration that can be found by the following equation for each fixed value of the concentration $p_{\text{fix}}$ so that $\forall p_{\text{fix}} \in [0; 1]$:

$$\sigma_{\text{cl}}(p | p = p_{\text{fix}})^2 = \left(\frac{S_{\text{cl}}(p | p = p_{\text{fix}})}{6C_1(p)}\right)^2 = \sigma^2 \left(1 + \frac{1}{\gamma} (u_1 \varphi(u_1) - u_2 \varphi(u_2))\right) - (\bar{x} - \mu)^2$$

$$\Rightarrow C_1(p) = \frac{1}{6} \sqrt{\sigma^2 \left(1 + \frac{1}{\gamma} (u_1 \varphi(u_1) - u_2 \varphi(u_2))\right) - (\bar{x} - \mu)^2}$$

where $p_{\text{fix}}$ is the specific fixed value of the concentration, $u_1 = \frac{\beta-\mu}{\sigma}$, $u_2 = \frac{\alpha-\mu}{\sigma}$ ($\alpha$ and $\beta$ are found in expression 6); $\varphi(u)$ is the density of the standard normal distribution probability; $\sigma(p)$ is the root-mean-square error of the deviation of the associated black cells from the percolation movement direction defined by means of statistic modelling. In the concentration range $p$ which is important for practical applications ($p = 0.15 - 0.4$) the values $C_1(p)$ are close to 1 (in Table 1).

The necessity to fix the concentration is explained by the fact that each concentration value has its own value of the average “width” of the cluster being the boundaries of the truncated normal
distribution describing random deviation from the initial vector of the percolation movement. Consequently, the dispersion of the truncated normal distribution will change in expressions 7 and 8.

| Table 1. Values $C_1(p)$ from some $p$ |
|---|
| $p$ | 0 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.45 | 0.5 | 0.55 | 0.6 | 0.65 |
| $C_1(p)$ | 1.083 | 0.975 | 0.94 | 0.95 | 0.96 | 1.077 | 1.068 | 0.998 | 0.912 | 0.875 | 0.858 | 0.835 | 0.817 |

Then our function in expression 5 which is similar in structure with the Rayleigh-Rice distribution and has the floating scale parameter will be the function of probability [4] of the black cell occurrence in the programmable percolation route from the concentration (in specific infinitely small vicinity of the specific concentration) and will look like as follows:

$$f(p) = \frac{p}{S_{cl}(p)} \sqrt{2} e^{-\left(\frac{p^2+m^2}{2(S_{cl}(p))^2}\right)} I_0\left(\frac{pm}{S_{cl}(p)}\right)$$

(9)

The average quantity of the black cells $\mathfrak{R}(p)$ engaged in the programmable percolation route and distributed over the average “width” is as follows:

$$\mathfrak{R}(p) = \frac{p}{\frac{1}{12S_{cl}(p)} M(p)} e^{-\left(\frac{p^2+m^2}{2\left(\frac{1}{12S_{cl}(p)} M(p)\right)}\right)} I_0\left(\frac{pm}{\left(\frac{1}{12S_{cl}(p)} M(p)\right)}\right) \frac{\sqrt{M(p)}}{2}$$

(10)

To find the average quantity of the black cells engaged in the programmable percolation route, it is necessary to multiply the last expression by the average quantity of the black cells along the percolation direction:

$$N(p) = \mathfrak{R}(p) p L = \frac{p}{\frac{1}{12S_{cl}(p)} M(p)} e^{-\left(\frac{p^2+m^2}{2\left(\frac{1}{12S_{cl}(p)} M(p)\right)}\right)} I_0\left(\frac{pm}{\left(\frac{1}{12S_{cl}(p)} M(p)\right)}\right) \frac{\sqrt{M(p)}}{2} p L$$

(11)

where $pL$ is the average quantity of the black cells along the percolation direction, $L = 1$ for the single matrix $N$ (matrix which length is equal to 1).

The average integral value $C_1(p)$ is equal to 0.92; the proximity to 1 proves the good compliance of the used model of proximity of the process of random deviation of associated black cells from the percolation movement direction to the statistic research results.

Or this can be summarized as follows from expression 11:

$$N(p) = f(p, m, \sigma_{cl}(p)) S_{cl}(p) p$$

(12)

where $N(p)$ is the average quantity of the black cells engaged in the programmable percolation route; $f(p, m, \sigma_{cl}(p))$ is the function which is similar in structure with the Rayleigh-Rice distribution density and has the floating scale parameter; $\sigma_{cl}(p)$ is the scale parameter; $m$ is the form parameter ($m \to 0$); $S_{cl}(p)$ is the average “width”; $p$ is the concentration.

4. Analytical expression for the average length of the path of the programmable percolation and the average number of objects added to the cluster intervals

The resulting expression 12 can be strengthened based on the following reasoning considerations:

As the concentration increases, the object clusters start expanding not only due to the addition of new objects to them but also due to the merging (consolidation) of the neighbouring clusters. Their total average quantity will start decreasing already from the concentration $p = 0.26$ just due to the merging of the neighbouring clusters. It is evident that starting from the specific high concentration,
one giant cluster will be formed and the percolation route will be built along it without adding red cells. As the concentration continues increasing, such route will be only straightened and its tortuosity and length will decrease.

Let’s fix \( p = p_{\text{fix}} \) so that the average diameter of the cluster at such concentration will be equal to the length of the matrix along which the percolation will be directed. Then at such concentration \( p_{\text{fix}} \), the function is \( M(p = p_{\text{fix}}) \to \frac{1}{N} \). And consequently:

\[
\frac{p}{\frac{1}{N} \pi} = \frac{N}{4} \Rightarrow p = \frac{\pi}{4}
\]

The reinforced function \( N(p) \) can be represented in the form of the piecewise set function where \( \xi \left( \frac{\pi}{4} \right) = f \left( \frac{\pi}{4}, \mu, \sigma_{\text{cl}} \left( \frac{\pi}{4} \right) \right) S_{\text{cl}} \left( \frac{\pi}{4} \right) \frac{\pi}{4} = 1 \), which tends linearly to zero with further increase in the concentration \( \xi(p) \):

\[
N(p) = \begin{cases} 
  f \left( p, \mu, \sigma_{\text{cl}}(p) \right) S_{\text{cl}}(p)p, & \text{if } p \leq \frac{\pi}{4} \\
  1 + \xi(p), & \text{else } p > \frac{\pi}{4}
\end{cases}
\]

\( 14 \)

Figure 3 and 4 shows the diagram of this expression.

![Figure 3](image)

**Figure 3.** The function of the average quantity of swarm objects from the stochastic base which are engaged in the programmable percolation route depending on the concentration.

The resulting expression for the average number of black cells in the path of a programmable percolation is pointless to use without an expression for the average number of red cells, since only their sum determines the average length of the path of a programmable percolation.

Similarly, we shall find the function of the average quantity of objects added to the inter-cluster intervals of the programmable percolation route depending on the concentration, namely the cells added \( R(p) \).

Let’s consider the area of the cluster operation. The percolation route built via this cluster can be divided into two areas: the area belonging to the cluster and consisting only of the black cells, respectively, and the area belonging to the toroidal structure of the area limited by the “cluster influence zone” and composing only of the (red) cells added. Since this toroidal structure is sufficiently complex for a separate analysis, let's consider it as the result of the cluster removal from the “cluster influence zone”. In other words, let’s consider the percolation route behaviour separately in the “cluster influence zone” and inside the cluster itself by representing that the cluster is composed of the white cells. The difference between these two expressions is the function \( R(p) \).
The “cluster influence zone” is equal to $\frac{1}{M(p)}$. Then the average minimum possible radius of the cluster is equal to $r_{\min}(p) = \frac{1}{\sqrt{M(p)} \pi}$ and the maximum possible one is equal to $r_{\max}(p) = \frac{1}{2M(p)}$. The length of the $n$-dimensional random vector which can be considered as cluster radii is described by the distribution of the module of the $n$-dimensional random vector which coincides with the Rayleigh distribution at $n = 2$ [26]. In this case, the average radius can be considered as the mathematical expectation of the Rayleigh’s two-way truncated distribution which boundaries are $r_{\min}$ and $r_{\max}$:

$$r_{\text{moyen}} = \frac{r_{\max}}{r_{\min}} \int_{r_{\min}}^{r_{\max}} \frac{(x - r_{\min})^2}{\sigma^2} e^{-\frac{(x-r_{\min})^2}{2\sigma^2}} dx =$$

$$= \frac{1}{\sqrt{2}} \text{erf}\left(\frac{r_{\max} - r_{\min}}{\sigma \sqrt{2}}\right) - (r_{\max} - r_{\min}) e^{\frac{(r_{\max}-r_{\min})^2}{2\sigma^2}}$$

(15)

If $r_{\max}(p) = \frac{1}{M(p)}$ is reinforced, the last summand in the difference will be negligible, i.e. $e^{\frac{(r_{\max}-r_{\min})^2}{2\sigma^2}} \to 0 \ \forall \ p \in (0; 1)$. This means:

$$r_{\text{moyen}} \approx \sigma \sqrt{\frac{\pi}{2}} \text{erf}\left(\frac{r_{\max} - r_{\min}}{\sigma \sqrt{2}}\right)$$

(16)

Let’s fix $p = p_{\text{fix}} \in (0; 1)$, then:

$$e^{-\frac{(r_{\max}(p_{\text{fix}}) - r_{\min}(p_{\text{fix}}))^2}{2\sigma^2}} = e^{-\frac{(r_{\max}(p_{\text{fix}}) - r_{\min}(p_{\text{fix}}))^2}{2\sigma^2}} = \xi \to 0$$

$$\Rightarrow \left(\frac{r_{\max}(p_{\text{fix}}) - r_{\min}(p_{\text{fix}})}{\sigma}\right)^2 = -\ln \xi \Rightarrow \sigma = \frac{r_{\max}(p_{\text{fix}}) - r_{\min}(p_{\text{fix}})}{\sqrt{-2 \ln \xi}}$$

(17)

If $\xi \to 0$, then $\text{erf}\left(\frac{r_{\max}(p_{\text{fix}}) - r_{\min}(p_{\text{fix}})}{\sigma \sqrt{2}}\right) \to 1$. Then:

$$r_{\text{moyen}}(p) = \frac{(r_{\max}(p) - r_{\min}(p)) \sqrt{\pi}}{2 \sqrt{-\ln \xi}}$$

(18)

Now, when the average radius of the “cluster influence zone” is known, we shall find the average “width” both for the “influence zone” and the area occupied by the cluster.

The average “width” will be as follows for the average “cluster influence zone”:

$$S_1(p) = \frac{1}{M(p) 2r_{\text{moyen}}(p)}$$

(19)

We shall do the same for the area occupied by the cluster:

$$S_2(p) = \frac{p}{M(p) 2r_{\text{moyen}}(p)}$$

(20)

The average “width” of the cluster itself is the difference between two last expressions.

It was earlier stated that the average “width” of the cluster is used to find the floating scale parameter in the distribution which is similar in structure with the Rayleigh-Rice distribution and has the floating scale parameter. It should be noted that the normal distribution truncated on both sides in this case shall be limited greater than the boundaries of the average “width”, since the quantity of the cells added to the programmable percolation route under the optimization condition shall be minimum and the minimum quantity of the cells added is possible only when moving in the straight line along the percolation movement vector almost with zero random deviations observed on actual matrices. To make such limitation, we shall consider the height of cluster’s “influence zone”, $h(p) = \frac{2}{\sqrt{\pi M(p) \pi}}$, as the average height of the cluster. In this case, the distribution scale parameter which is similar in structure with the Rayleigh-Rice distribution and has the floating scale parameter can be assessed similarly to the previous case as follows:
\[ \sigma_{pol}(p) = \frac{p}{M(p)h(p)6C_2(p)} \]  

(21)

where \( \frac{p}{M(p)} \) is the cluster volume; \( h(p) \) is the height of the “cluster influence zone”; \( C_2(p) \) is the function of consistent factors found similarly to \( C_1(p) \). On the analogy of the previous case, we shall write the expression for the function of the average quantity of the cells added to the programmable percolation route, \( R(p) \), for the single matrix:

\[ R(p) = f \left( p, m, \sigma_{pol}(p) \right) S_{pol}(p)(1 - p) \]  

(22)

where \( R(p) \) is the average quantity of the cells added to the programmable percolation route; \( f \left( p, m, \sigma_{pol}(p) \right) \) is the function which is similar in structure with the Rayleigh-Rice distribution density and has the floating scale parameter; \( \sigma_{pol}(p) \) is the scale parameter; \( m \) is the form parameter \((m \to 0)\); \( S_{pol}(p) \) is the average “width” equal to \( S_{pol}(p) = S_1(p) - S_2(p) \); \( p \) is the concentration.

Let’s consider the function \( R(p) \) with the extremely low values of the concentration \((p \to 0)\). It is evident that at such \( p \), the artificial percolation route will be almost the straight line composed of almost the red cells. It is evident that the length of such route in the single matrix will be equal to 1. However, using the original expression for \( \sigma_{pol}(p) \), the function \( R(p) \) will not be defined because \( \sigma_{pol}(p|p \to 0) \to 0 \). So, it is necessary to reinforce the expression of the scale parameter from the function similar in the Rayleigh-Rice distribution and has the floating scale parameter for such critical values of the concentration. To do this, the following equation should be solved:

\[ R(p|p \to 0) = f \left( p, m, \sigma_{pol} \right) S_{pol}(p)(1 - p) \]  

(23)

By solving this equation in case of random walks in the straight line \((m = 0)\), we shall receive the following:

\[ \sigma_{pol}(p|p \to 0) = \frac{1}{\sqrt{-\frac{2}{p^2}W \left( -\frac{p}{2(1-p)S_{pol}(p)} \right)}} \]  

(24)

where \( W(x) \) is the Lambert W function \([14], [25]\), \( R(p|p \to 0) = 1, p \to 0 \).

Then the expression for the scale parameter will be as follows:

\[ \sigma_{pol}(p) = \begin{cases} \frac{1}{\sqrt{-\frac{2}{p^2}W \left( -\frac{p}{2(1-p)S_{pol}(p)} \right)}}, & p \to 0 \\ \frac{p}{M(p)h(p)6C_2} & 0 \ll p < 1 \end{cases} \]  

(25)

Figure 5 shows the graphic of the function \( R(p) \) of the average quantity of the cells added to the programmable percolation route.

When analysing the expression \( R(p) \), it may be concluded that the quantity of the agents added at the concentration \( p = 0.59 \) becomes negligible, so the natural percolation breakdown occurs at this concentration.

5. Statistic modelling of the task of network signal transmission between the swarm members along the programmable percolation route

An extensive mathematical experiment was conducted to study the two-phase operation on the percolation matrix of final size. This experiment was carried out as follows: first the objects with different concentrations were inoculated on a set of percolation matrices. Then the Hoshen-Kopelman algorithm \([4], [13]\) was used to define the clusters; this was followed by building an optimal (with the minimum quantity of the cells added) programmable percolation route (the experiment was conducted for the matrices \( 50 \times 50, 100 \times 100, \) and \( 200 \times 200 \) in size). The obtained results rated by the matrix length are given in Figure 4 and 5 and almost completely coincide with the obtained analytical results.
Figure 4. Diagram 1 – statistic average length of the programmable percolation route; diagram 2 – analytical average length of the programmable percolation route.

Figure 5. Diagram 1 – statistic average quantity of the cells $R(p)$ rated by the matrix size $L$, which are added to the percolation route; diagram 2 – analytical average quantity of the cells added.

6. Conclusions
As a result of the completed work, the following conclusions can be made:

1. Rapid construction of the frontal structure of swarm objects for solving applied problems in a given part of the operating area is possible using the theory of programmable percolation. When realising a programmable percolation, the first phase creates a base of randomly distributed swarm objects with their concentration values much lower than the stochastic percolation threshold, and the second phase builds a through percolation path by introducing (installing) complementary objects in the available inter-cluster intervals. In this case, the concentration of the stochastic basis can be selected so that the total cost of a two-phase operation is minimal.

2. Analytical expressions are revealed for the average length of the path of programmable percolation and the average number of objects added to the inter-cluster intervals to form a through shortest path, as a function of the concentration of objects in the operating area. The choice of approximating functions and their parameters is made through analytical modeling of the processes of formation and growth of clusters of objects, filling of inter-cluster intervals with swarm objects forming a through percolation path.
3. The results of statistical modeling to determine the same characteristics of the path of programmable percolation, allowing to optimize the implementation of two-phase operations, coincide with acceptable accuracy with the results of analytical modeling. In this case, the agreement of the results of a mathematical experiment and the results of analytical analysis is possible using a single matching coefficient, variable in concentration, but little different from 1.

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