MUSINGS ON MAGNUS

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Abstract. The object of this paper is to describe a simple method for proving that certain groups are residually torsion-free nilpotent, to describe some new parafree groups and to raise some new problems in honour of the memory of Wilhelm Magnus.

1. Introduction

I first heard of Wilhelm Magnus in 1956, when I was attending some lectures by B.H. Neumann on amalgamated products. At some point during the course of these lectures, Neumann remarked that Magnus was the first mathematician to recognize the value of amalgamated products and had shown just how effective a tool they were, in his work on groups defined by a single relation. I was working, at that time, on an universal algebra variation of free groups, involving groups with unique roots, which I called $D$-groups [1]. Consequently, in an attempt to find analogues of various theorems about free groups for $D$-groups, I found myself reading a beautiful paper of Magnus, in which he proved the residual torsion-free nilpotence of free groups. Much of my talk today will be concerned with these two topics, one-relator groups and residual nilpotence. Let me begin by reminding you of some of the definitions involved. Let $P$ be a property of groups.

Definition. We say that a group $G$ is residually a $P$-group if for each $g \in G$, $g \neq 1$, there exists a normal subgroup $N$ of $G$, such that $g \notin N$ and $G/N$ has $P$.

The properties that I will be mainly concerned with here are freeness, nilpotence and torsion-free nilpotence. I will make use of the usual commutator notation. Thus if $H$ and $K$ are subgroups of a group $G$ then

$$[H, K] = gp(h^{-1}k^{-1}hk \mid h \in H, k \in K)$$

is the subgroup generated by all the commutators $h^{-1}k^{-1}hk$. The lower central series of $G$ is defined to be the series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_n(G) \geq \cdots,$$

where $\gamma_{n+1}(G) = [\gamma_n(G), G]$. $G$ is termed nilpotent if $\gamma_{c+1}(G) = 1$ for some $c$. I am now in a position to formulate the theorem of Magnus [12] that I alluded to before.

1991 Mathematics Subject Classification. Primary 20F05, 20F14.
Key words and phrases. Residually torsion-free nilpotent groups, one-relator groups, $D$-groups.

The author was supported in part by NSF Grant #9103098
Theorem 1. Free groups are residually torsion-free nilpotent.

The basic idea involved in the proof of Theorem 1 is beautifully simple. Magnus concocts a faithful representation of a given free group \( F \) in the group of units of a carefully chosen ring \( R \) with 1. Each of the elements \( f \in F \) takes the form \( 1 + \phi \), where \( \phi \) lies in an ideal \( R^+ \) of \( R \). \( R^+ \) carries with it the structure of a metric space designed so as to ensure that if \( f \in \gamma_n(F) \), then \( d(\phi, 0) \leq 2^{-n} \), where here \( d(\phi, 0) \) denotes the distance between \( \phi \) and 0. This suffices to ensure that

\[
\bigcap_{n=1}^{\infty} \gamma_n(F) = 1.
\]

The sketch of the proof, below, amplifies these remarks.

Proof. Let \( F \) be free on \( x_1, \ldots, x_q \). Consider the ring \( R \) of power series in the non-commuting indeterminates \( \xi_1, \ldots, \xi_q \) with rational coefficients. Each element \( r \in R \) can be thought of as an infinite sum which takes the form

\[
r = r_0 + r_1 + \cdots + r_n + \ldots,
\]

where \( r_0 \in \mathbb{Q} \) and the *homogeneous component* \( r_n \) of \( r \) of degree \( n > 0 \) is a finite sum of rational multiples \( c\xi_{i_1} \cdots \xi_{i_n} \) of monomials \( \xi_{i_1} \cdots \xi_{i_n} \) of degree \( n \), \( c \in \mathbb{Q}, i_1, \ldots, i_n \in \{1, \ldots, q\} \). Here \( \xi_{i_1} \cdots \xi_{i_n} = \xi_{j_1} \cdots \xi_{j_m} \) only if \( n = m \) and \( i_r = j_r \) for each \( r = 1, \ldots, n \). Each of the elements \( a_i = 1 + \xi_i \) is invertible in \( R \), with inverse \( a_i^{-1} = 1 - \xi_i + \xi_i^2 - \xi_i^3 + \ldots \). It then turns out that the elements \( a_1, \ldots, a_q \) freely generate a free subgroup of rank \( n \) of the multiplicative group of units of \( R \). We define \( R^+ \) to be the ideal of \( R \) consisting of those elements \( r \in R \) such that \( r_0 = 0 \). We define a metric \( d \) on \( R^+ \) by setting \( d(r, 0) = 0 \) if \( r = 0 \) and, if \( r \neq 0 \),

\[
d(r, 0) = 2^{-n}
\]

where \( n \) is the degree of the first non-zero homogeneous component of \( r \). Magnus proves that if \( f = 1 + \phi \in \gamma_n(F) \), then all of the homogeneous components \( f_1 = f_2 = \cdots = f_{n-1} = 0 \), i.e., \( d(\phi, 0) \leq 2^{-n} \). It is not hard then to deduce that \( F \) is residually torsion-free nilpotent.

One of the many consequences of this theorem of Magnus is the following characterisation of finitely generated free groups, which Magnus proved a few years later in [13]

**Theorem 2.** If the group \( G \) can be be generated by \( q \) elements, where \( q \) is finite, and if \( G/\gamma_n(G) \cong F/\gamma_n(F) \), where \( F \) is a free group of rank \( q \), then \( G \cong F \).

Theorem 2 will play a role here in due course.

2. One-relator groups

My next encounter with Magnus came some years later. After spending a post-doctoral year in Manchester, as a Special Lecturer, I came to Princeton in 1959 as an instructor. Some time towards the end of that year, Trueman MacHenry, a doctoral student of Magnus whom I had met in Manchester, came to visit me in Princeton. He was accompanied by Bruce Chandler, another of Magnus’ doctoral
students. MacHenry brought greetings from Magnus and an implicit offer of a job at the Courant Institute. I was very happy at the prospect of working with Magnus, and came up to New York to give a talk early in 1960. This was the first time that I had actually met Magnus and I was very impressed with his quickness and his vast knowledge, not only of mathematics but of almost everything else as well. He must have been amused by my talk, which was to an audience which consisted mainly of analysts, because I talked about an extremely esoteric theorem that Norman Blackburn and I [4] had proved about 9 months earlier, a theorem that only the most special of specialists would have found interesting. Nonetheless, the analysts were apparently convinced by Magnus that I should be offered a position and the offer was made explicit soon after. I immediately accepted the position and came to the Courant in the late summer of 1960. I spent part of that summer in Pasadena, where I met Danny Gorenstein and Roger Lyndon. Roger Lyndon had been working on groups with parametric exponents [9]. There were other things on his mind as well and he asked me a number of questions about one-relator groups while we were in Pasadena. I had not read Magnus’s papers on one-relator groups, and so I was forced to spend part of that summer trying to understand Magnus’s work. There are two main theorems that I would like to describe. The first of these is Magnus’ celebrated ”Freiheitssatz” [10].

**Theorem 3.** Let

\[ G = \langle x_1, \ldots, x_q; r = 1 \rangle \]

be a group defined by a single relator \( r \). If the first and last letters of \( r \) are not inverses and if \( x_1 \) appears in \( r \), then the subgroup of \( G \) generated by \( x_2, \ldots, x_q \) is a free group, freely generated by \( x_2, \ldots, x_q \).

One of the byproducts of the proof of the Freiheitssatz was an extraordinary unravelling of the structure of these groups, which allowed Magnus to deduce, in due course, that one-relator groups have solvable word problem [11]:

**Theorem 4.** Let \( G \) be a group defined by a single relation. Then \( G \) has a solvable word problem.

### 3. Surface groups

As I remarked earlier, I came to the Courant Institute in the late summer of 1960. Magnus was part of the electromagnetic group of Morris Kline and used to go to a seminar on electromagnetism. I remember one occasion, after tea, when Morris Kline and Magnus and I were in the elevator, in the old hat factory which had become the Courant Institute. Kline was talking to Magnus and addressed him as Bill. I saw Magnus shudder at the prospect of being called Bill. Magnus was much too polite to say anything, but I did not feel at all constrained to be silent. So I said something like this to Kline: ”Morris, Wilhelm is a Geheimrat and so he simply cannot be called Bill. Wilhelm is more appropriate”. Morris Kline smiled and indicated that this was okay with him. To which Magnus responded by muttering under his breath: ”Thank God”. He later told me that he had hated being addressed as Bill, and was grateful to me for telling Kline to call him Wilhelm. Magnus had a large group of Ph.D. students in 1961. They included Karen Fredericks, Bruce Chandler, Seymour Lipschutz and Trueman MacHenry, all of whom worked with Magnus on Combinatorial Group Theory. Martin Greendlinger had already
completed his beautiful work on small cancellation theory, but was still around. In addition, Magnus had some other students who worked with him on Hill’s equation. There was always a line of students outside his office and I offered to take on some of them to relieve him of some of the burden. Magnus ran a seminar on Combinatorial Group Theory at the time. The seminar was a mixture of pure research and reports by students on papers that they were reading. Shortly after I arrived, the research part of the seminar was broadened and the role of students was reduced essentially to zero. One of Magnus’ habits was to propose a number of problems in the seminar. He was always interested in purely algebraic proofs of theorems that had been proved by other means. In particular, he asked late in 1961, whether there was a direct proof of the residual finiteness of the fundamental groups of two-dimensional orientable surfaces. Both Karen Frederick and I began to work on this problem and we both eventually, independently, came up with a solution. Her solution [7] was very closely tied to the actual presentation of these surface groups. I took a somewhat more general approach which yielded somewhat more [2]. The net result was the following

**Theorem 5.** Let $F$ be a free group on $X$, $A$ a free abelian group on $Y$, $f$ an element of $F$ which is not a proper power in $F$, and $a$ an element of $A$ which is not a proper power in $A$. Then the group

$$G = \langle X \cup Y; [y, y'] = 1(y, y' \in Y), f = a \rangle$$

is residually free.

In particular, it follows from this theorem that surface groups are residually free and hence residually torsion-free nilpotent (and also residually finite). The residual nilpotence of one-relator groups was itself a topic of some interest to Magnus. He later put Bruce Chandler to work on this topic. Chandler [6] eventually found an alternative proof of the residual torsion-freeness of surface groups by making use of the ring $R$ that I described at the outset. Some years later I found yet another means of proving the residual nilpotence of surface groups. I never discussed this method with Magnus, and subsequently forgot about it. A few months ago, J. Lewin told me that he had also found a very simple argument to prove the residual nilpotence of surface groups. This prompted me to rethink some of my old ideas and it is to these thoughts that I want to turn next.

4. **Residually nilpotent groups**

The fundamental groups of two dimensional orientable surfaces all contain a free subgroup with infinite cyclic factor group. Thus they have the same form as the groups covered by the following theorem.

**Theorem 6.** Let $G$ be a finitely generated group. Suppose that $G$ contains a free, normal subgroup $N$ such that $G/N$ is infinite cyclic. If $G/\gamma_2(N)$ is residually torsion-free nilpotent, then so is $G$.

This theorem is similar in spirit to a theorem of P. Hall [8], who proved that if $G$ is a group with a normal, nilpotent subgroup $N$, then $G$ is nilpotent if $G/\gamma_2(N)$ is nilpotent. Theorem 6 leads to a host of new examples of residually nilpotent groups. In particular, it can be used to give yet another proof of the residual torsion-free nilpotence of surface groups. There is a further use of Theorem 6 that I want to describe here. To this end, let me recall the following definition.
Definition. A group $G$ is termed parafree if $G$ is residually nilpotent and there exists a free group $F$ such that

$$G/\gamma_n(G) \cong F/\gamma_n(F) \text{ for all } n.$$ 

Parafree groups, which can be likened to free groups, exist in profusion, see, e.g., [3]. It should be pointed out that it follows from Magnus’ Theorem 2 that a non-free, finitely generated parafree group $G$ with the same nilpotent factor groups $G/\gamma_n(G)$ as a free group of rank $q$ cannot be generated by $q$ elements. It is not known how closely a parafree group can resemble a free group. I want to describe next some new, non-free parafree groups, which very closely resemble free groups. The proof that these groups are parafree is an easy application of Theorem 5 (see 6).

**Theorem 7.** Let $F$ be the free group on $s, t, a_1, \ldots, a_q$ and let $w$ be an element of $F$ which involves $a_1$ and does not involve $s$. In addition suppose that $w$ lies in the $k$-th term $F^{(k)}$ of the derived series of $F$. Then the one-relator group

$$G_w = \langle s, t, a_1, \ldots, a_q; a_1 = ws^{-1}t^{-1}st \rangle$$

is parafree and not free. Moreover

$$G/G^{(k)}_w \cong H/H^{(k)},$$

where $H$ is a free group of rank $q + 1$.

I will sketch the proofs of Theorems 6 and 7 in section 6.

5. SOME PROBLEMS ON $D$-GROUPS

Let $p_1 = 2, p_2 = 3, \ldots$ be the set of all primes in ascending order of magnitude.

Definition. A group $G$ is called a $D$-group if it admits a set of unary operators

$$\pi_1, \pi_2, \ldots$$

such that for all $g \in G$

$$g^{n_i} \pi_i = g = (g \pi_i)^{n_i}.$$ 

It is not hard to verify that these $D$-groups consist precisely of those groups $G$ in which extraction of $n$-th roots is uniquely possible, for every positive integer $n$. The class of all $D$-groups form a variety of (universal) algebras. The precise technical description of these terms does not matter. It suffices only to say that in such a variety one has the notion of a free $D$-group, as well as all of the other notions that one makes use of in group theory. There are a number of properties of such free $D$-groups that are similar to properties of free groups. For example one has the following
**Theorem 8.** D-subgroups of free D-groups are free.

I was told about this theorem by Tekla Taylor-Lewin, but I have not been able to locate a reference. Magnus was fond of these D-groups, and discussed them in his book with Karrass and Solitar [14]. It seems appropriate, therefore, to raise here some new problems about free D-groups, in his memory. To this end, let $F$ be the free D-group on $x_1, \ldots, x_q$.

It is not hard to see that the mapping

$$x_i \mapsto 1 + \xi_i \ (i = 1, \ldots, q)$$

defines a homomorphism $\phi$ of $F$ into the group of units of $R$.

**Problem 1.** Is $\phi$ a monomorphism?

**Problem 2.** Let

$$G = \langle x_1, \ldots, x_q; r = 1 \rangle.$$

If extraction of $n$-th roots in $G$ is unique, whenever such roots exist, can $G$ be embedded in a D-group?

**Problem 3.** Suppose that the one-relator group

$$G = \langle x_1, \ldots, x_q; r = 1 \rangle$$

can be embedded in a D-group. If $H$ is the one-relator D-group generated, as a D-group, by $x_1, \ldots, x_q$ and defined, as a D-group by the single relation $r = 1$, is the word problem solvable for $H$? In general, is the word problem solvable for one-relator D-groups?

**Problem 4.** Is there a freiheitssatz for one-relator D-groups?

**Problem 5.** Can free groups be characterised by a length function?

**Problem 6.** Does a free D-group act freely on a $\Lambda$-tree, for a suitable choice of $\Lambda$?

6. Proofs

I want to sketch here the proofs of Theorem 6 and Theorem 7. I would like to begin with the proof of Theorem 6. I will adopt the notation used in the statement of the theorem. Since $G/N$ is infinite cyclic, we can choose an element $t \in G$, such that $G = gp(N, t)$. So $t$ is of infinite order modulo $N$. Suppose that $g \in G, g \neq 1$. We want to find a normal subgroup $K$ of $G$ such that $G/K$ is torsion-free nilpotent and $g \notin K$. It is clear that it suffices to consider the case where $g \in N$. Since $N$ is free there exists an integer $k$ such that $g \notin \gamma_k(G)$. Notice that $H = N/\gamma_k(N)$ is a torsion-free nilpotent group and so we can form the Malcev completion $\overline{H}$ of $H$ (see [15]). This group $\overline{H}$ is a minimal D-group containing $H$. It is again nilpotent of class $k - 1$ and torsion-free. Moreover if we denote by $\tau$ the automorphism that $t$ induces on $\overline{H}$, then $\tau$ extends uniquely to an automorphism $\overline{\tau}$ of $\overline{H}$. Let $\overline{G}$ be the semidirect product of $\overline{H}$ with the infinite cyclic group generated by an element $\overline{t}$, where $\overline{t}$ acts on $\overline{H}$ as $\overline{\tau}$. Then $M = \overline{H}/\gamma_2(\overline{H})$ is a direct sum of copies of the additive group of $\mathbb{Q}$ (see [1]). Moreover, it is not hard to deduce from the fact
that $G/\gamma_2(N)$ is residually torsion-free nilpotent, that $\overline{\Gamma}/\gamma_2(H)$ is also residually torsion-free nilpotent. We now view $M$ as a module over the rational group algebra $\Gamma$ of the infinite cyclic group on $t$. Since $\Gamma$ is a principal ideal domain and the group $G$ is finitely generated, the $\Gamma$-module $M$ is finitely generated and hence a direct sum of cyclic modules. This decomposition of $M$ makes it possible to understand the action of $t$ on $\overline{T}$ and, using the fact that $H = F/\gamma_k(F)$, to deduce that $\overline{T}$ is residually torsion-free nilpotent. It is easy to deduce that $G$ is itself residually torsion-free nilpotent. There are three steps in the proof of Theorem 7. The first makes use of Magnus’ method of unravelling the structure of one-relator groups. This method allows to prove, easily, that the normal closure $N$ of $s, a_1, \ldots, a_q$ in $G_w$ is free. The second step involves the verification that $G/\gamma_2(N)$ is residually torsion-free nilpotent. So, by Theorem 6, $G_w$ is residually torsion-free nilpotent. The other properties of $G_w$ follow directly from the form of the defining relation of $G_w$. The final step in the proof of Theorem 7 is the verification that $G_w$ is not free. This is accomplished by invoking an algorithm of J.H.C. Whitehead [15]. I have only talked here about a few of Magnus’ theorems. All of his work is filled with beautiful, new ideas, giving joy to all of us. Wilhelm Magnus is sorely missed, but his work will be with us always.

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