FOURIER TRANSFORM AND REGULARITY OF CHARACTERISTIC FUNCTIONS

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Abstract. Let \( E \) be a bounded domain in \( \mathbb{R}^d \). We study regularity property of \( \chi_E \) and integrability of \( \hat{\chi}_E \) when its boundary \( \partial E \) satisfies some conditions. At the critical case these properties are generally known to fail. By making use of Lorentz and Lorentz-Sobolev spaces we obtain the endpoint cases of the previous known results. Our results are based on a refined version of Littlewood-Paley inequality, which makes it possible to exploit cancellation effectively.

1. Introduction

Let \( E \) be a bounded measurable subset in \( \mathbb{R}^d \). Sobolev regularity property of \( \chi_E \) and \( L^p \)-integrability of \( \hat{\chi}_E \) have long been of interest in connection with various problems and studied extensively by many authors [6, 10, 13]. The sharp range of regularity and integrability are relatively well known and these properties generally fail at the critical exponent. In extended functions spaces validity of such properties at the critical exponent is not clearly understood. In this short note investigate this issue with Lorentz and Lorentz-Sobolev spaces.

Integrability of \( \hat{\chi}_E \). If \( 2 \leq p \leq \infty \), by the Hausdorff-Young inequality it follows that \( \hat{\chi}_E \in L^p \). This holds without any dependence on the geometric structure of \( E \). However, for \( p < 2 \), it becomes no longer trivial to determine \( L^p \)-boundedness of \( \hat{\chi}_E \) and the geometric information of \( E \) comes into play, especially the geometry of the boundary of \( E \).

In order to describe the boundary, we set

\[
(\partial E)_\delta = \{ x : \text{dist} (x, \partial E) < \delta \}
\]

and consider the condition that, for \( 0 < \gamma \leq d \),

\[
| (\partial E)_\delta | \lesssim \delta^{d-\gamma}.
\]

(1.1)

This is satisfied if \( \partial E \) is a \( \gamma \) set (see [12, Definition 3.1] for definition of \( \gamma \)-set). As remarked in (p.5-6 in [12]), the Minkowski content is equivalent to the Hausdorff measure for \( \gamma \)-set.

\( L^p \)-integrability of \( \hat{\chi}_E \) and Sobolev regularity of \( \chi_E \) are closely related. In fact, the first on some range can be deduced from the latter. We denote by \( L^q_s(\mathbb{R}^d) \),

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for $0 < s < \infty$, the Bessel potential spaces consisting of all tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with the norm
\[
\|f\|_{L^q} = \left\| \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{f}(\xi) \right\|_q,
\]
where $\hat{\cdot}$ denotes the inverse Fourier transform and by $\Lambda_s^{q,r}(\mathbb{R}^d)$ the Nikol’skij-Besov spaces endowed with the norm
\[
\|f\|_{\Lambda_s^{q,r}} = \|f\|_q + \left( \int_{\mathbb{R}^d} \frac{\|f(x + t) - f(x)\|_q^r}{|t|^{d+rs}} \, dt \right)^{1/r}
\]
for $0 < s < 1$ and $q \in [1, \infty]$ and $r \in [1, \infty)$. Then we see that $\Lambda_s^{q,q} = W_s^q$ where $W_s^q$ denotes the fractional Sobolev spaces. Then we have the following characterization of $W_s^q$ due to Sickel \cite{10} Proposition 3.6]. The converse direction is also true if $E$ is a quasiball \cite[Theorem 1.3]{3].

**Theorem 1.1.** Let $0 < s < 1$ and $E$ be a measurable subset of $\mathbb{R}^d$ with $|E| < \infty$. Suppose
\[
\int_0^1 \delta^{-qs} \left| (\partial E)_{\delta} \right| \frac{d\delta}{\delta} < \infty,
\]
then $\chi_E \in W_s^q(\mathbb{R}^d)$ for $1 \leq q < \infty$.

If $\frac{2(d-\alpha)}{d} < p \leq 2$, by Hölder’s inequality and Plancherel’s theorem we have
\[
\|\hat{\chi}_{E}\|_p \lesssim \|\chi_{E} (1 + |\cdot|^2)^{\frac{\alpha}{2p}}\|_2 \sim \|\chi_{E}\|_{W^{2/p}_{\alpha/p}(\mathbb{R}^d)}.
\]
We here use the fact that $W_s^2(\mathbb{R}^d)$ and $L^2_s(\mathbb{R}^d)$ are equivalent for $0 < s < 1$ (see \cite{11}). Now, by Theorem \ref{1.1} $\chi_{E} \in W^{2/p}_{\alpha/p}(\mathbb{R}^d)$ if $0 < \frac{\alpha}{p} < \frac{d-\gamma}{2}$. Hence, $\hat{\chi}_{E} \in L^p(\mathbb{R}^d)$ for $p > \frac{2d}{d+\gamma}$ whenever the condition (1.1) holds and $\gamma \geq d-2$ (see \cite{7} for a slightly different argument).

Especially for a bounded domain with $C^1$-boundary, this (with $\gamma = d-1$ in (1.1)) recovers the classical result due to Herz \cite{6} who showed that $\hat{\chi}_{E} \in L^p(\mathbb{R}^d)$ for $p > \frac{2d}{d+1}$ under the assumption that $E$ is compact and convex with smoothness condition.

If we assume that $E$ is a bounded domain, an improved characterization is possible in terms of $(\partial E)_{\delta}$. The following is our first result.

**Proposition 1.2.** Let $1 \leq p \leq 2$ and $E$ be a bounded domain. Then,
\[
\|\hat{\chi}_{E}\|_p \lesssim |E| + \left( \int_0^1 \delta^{-d(1-\frac{\gamma}{2})} \left| (\partial E)_{\delta} \right| \frac{d\delta}{\delta} \right)^{1/p}.
\]

However, at the critical $p = \frac{2d}{d+\gamma}$, the condition (1.1) doesn’t generally imply $\hat{\chi}_{E} \in L^p(\mathbb{R}^d)$. For example, if $E$ is a ball $B$, (1.1) is satisfied with $\gamma = d-1$, but $\hat{\chi}_{B} \notin L^{\frac{2d}{d+1}}$ because, for $|\xi| \gg 1$,
\[
(1.2) \quad |\hat{\chi}_{B}(\xi)| \geq C|\xi|^{-\frac{d+1}{2}} \sin(2\pi|\xi|) - C'||\xi|^{-\frac{d+3}{2}}.
\]
Lebedev \cite{7} showed that $\hat{\chi}_{E} \notin L^p(\mathbb{R}^d)$ for $p \leq \frac{2d}{d+1}$ if $E$ has $C^2$- boundary.
Though $\hat{\chi}_E$ generally fails to be in $L^{\frac{2d}{d-\gamma}}(\mathbb{R}^d)$ under the assumption (1.1), from the above example it seems natural to expect that $\hat{\chi}_E$ is contained in the weaker $L^{\frac{2d}{d-\gamma},\infty}$. Here $L^{p,\infty}$ denotes the weak $L^p$ space. At the critical $p = \frac{2d}{d-\gamma}$, we have the following.

**Theorem 1.3.** Let $E$ be a bounded domain in $\mathbb{R}^d$. Assume that, for some $0 < \gamma < d$, (1.1) holds for $0 < \delta \ll 1$. Then $\hat{\chi}_E \in L^{\frac{2d}{d-\gamma},\infty}(\mathbb{R}^d)$.

When $\gamma = d - 1$, Theorem 1.3 is optimal by (1.2) in that $\hat{\chi}_E \notin L^{p,\infty}(\mathbb{R}^d)$ for $p < \frac{2d}{d+1}$. For $\gamma$ other than $d - 1$ the same seems to be true but we are not able to construct an example at this moment. In particular, if $E$ has Lipschitz boundary, then $|\langle \partial E\rangle_\delta| \lesssim \delta$. Hence we get the following.

**Corollary 1.4.** Let $E$ be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. Then $\hat{\chi}_E \in L^{p,\infty}(\mathbb{R}^d)$ for $p = \frac{2d}{d+1}$.

**Regularity property of $\chi_E$.** By Theorem 1.1 $\chi_E \in W^q_s(\mathbb{R}^d)$ if $s < \frac{d-\gamma}{q}$ but $\chi_E$ generally fails to be in $W^q_s(\mathbb{R}^d)$ at the critical exponent $s = (d-\gamma)/q$. In [13, Theorem 3] it is shown that, for $1 \leq d - 1 \leq \gamma < d$ and for $1 \leq q < \infty$, there is a bounded star-like domain $E$ such that the boundary $\partial E$ is a $\gamma$-set and $\chi_E \notin W^q_{(d-\gamma)/q}(\mathbb{R}^d)$. Another result is that if $E$ is a $K$-quasiball such as Koch snowflake whose boundary has nonzero $\gamma$-dimensional lower Minkowski content, then $\chi_E \notin W^q_{(d-\gamma)/q}(\mathbb{R}^d)$ for $1 \leq q < \infty$ (Theorem 1.3 in [3]). This can be shown by observing that if the lower Minkowski content is nonzero, then the opposite direction of (1.1) holds.

We can also characterize the regularity of $\chi_E$ by using the Bessel potential spaces $L^q_s$. When $q \neq 2$, $L^q_s(\mathbb{R}^d)$ and $W^q_s(\mathbb{R}^d)$ do not coincide in general. But, for $1 < q < \infty$ and $0 < s < 1$, there are the well-known embeddings

$$\Lambda^q_s \subset L^q_s \quad \text{for} \quad q \leq 2, \quad \Lambda^{q,2}_s \subset L^q_s \quad \text{for} \quad q \geq 2,$$

and

$$L^q_s \subset \Lambda^{q,2}_s \quad \text{for} \quad q \leq 2, \quad L^q_s \subset \Lambda^{q,q}_s \quad \text{for} \quad q \geq 2,$$

(see [11]). As a consequence of embedding (1.3), by the analogous argument in the proof of Theorem 1.1 (see [10], also see [3]) it is easy to see that, for $q \geq 2$,

$$\|\chi_E\|_{L^1_s} \lesssim |E|^{1/q} + \left( \int_0^1 \delta^{-2s}|\langle \partial E\rangle_\delta|^{\frac{q}{2}} \frac{d\delta}{\delta} \right)^{1/2},$$

and, for $q \leq 2$,

$$\|\chi_E\|_{L^q_s} \lesssim |E|^{1/q} + \left( \int_0^1 \delta^{-q}|\langle \partial E\rangle_\delta| \frac{d\delta}{\delta} \right)^{1/q}.$$
for $1 \leq q < \infty$ and $1 \leq r \leq \infty$ where $L^{q,r}$ denote the Lorentz spaces. By the Chebyshev inequality, the Bessel potential spaces $L^{q}(\mathbb{R}^{d})$ are embedded in $L^{q,\infty}(\mathbb{R}^{d})$. For more detail on Lorentz-Sobolev spaces we refer the reader to the recent literature [5, 8, 14, 15].

In what follows we show that $\chi_{E} \in L^{q,\infty}(\mathbb{R}^{d})$ at the critical $s = (d - \gamma)/q$.

**Theorem 1.5.** Let $E$ be a bounded domain in $\mathbb{R}^{d}$ satisfies (1.1) for $0 < \delta \ll 1$. If $0 < \gamma < d$ and $1 < q < \infty$, then $\chi_{E} \in L^{q,\infty}(\mathbb{R}^{d})$ for $s = (d - \gamma)/q$.

The paper is organized as follows: In Section 2, we prove a refined Littlewood-Paley inequality in which the projection operators have preferable cancellation property. Theorem 1.3 and Proposition 1.2 are proved in Section 3. The proof of Theorem 1.5 is given in Section 4.

2. Preliminaries

In this section we prove a version of Littlewood-Paley inequality which plays a crucial role for the proof of our results. Most important feature is that the associated projection operators have a cancellation property when they are applied to the characteristic functions of open sets. For this purpose we need to find a smooth function $\phi \in \mathcal{S}(\mathbb{R}^{d})$ which satisfies special properties.

We denote by $B_{r}(a)$ the open ball of radius $r$ which is centered at the point $a$.

**Lemma 2.1.** There exists a Schwartz function $\phi$ such that $\phi^{\vee}$ is supported on $B_{1}(0)$ and $\phi$ satisfies

\begin{equation}
\int_{\mathbb{R}^{d}} \phi^{\vee}(x) \, dx = 0,
\end{equation}

and, for some constants $C_{1}, C_{2} > 0$,

\begin{equation}
C_{1} \leq \sum_{k=-\infty}^{\infty} \phi^{2}(2^{-k}\xi) \leq C_{2}.
\end{equation}

Moreover, for any positive integer $N$,

\begin{equation}
\int_{\mathbb{R}^{d}} x^{\beta} \phi^{\vee}(x) \, dx = 0 \quad \text{if} \quad |\beta| < 2^{N}.
\end{equation}

**Proof.** Choose a radial function $\psi_{0} \in \mathcal{S}(\mathbb{R}^{d})$ such that $\psi_{0}$ is supported on $B_{1/2}(0)$ and $\int \psi_{0}(x) \, dx = \hat{\psi}_{0}(0) \neq 0$. Then we select $\phi$ by setting

$\phi^{\vee}(x) = \psi_{0}(x) - 2^{-d}\psi_{0}(2^{-1}x)$

and after a change of variables (2.1) follows.

We now prove the estimate (2.2). For scaling it suffices to prove (2.2) for $1 \leq |\xi| \leq 2$. Since $\phi^{2}(0) = 0$, we have $|\phi^{2}(2^{-k}\xi)| \leq C2^{-k}|\xi|$. Hence,

$$
\sum_{k=-\infty}^{\infty} \phi^{2}(2^{-k}\xi) \leq \sum_{k=-\infty}^{\infty} \min\left(2^{-k}|\xi|, (2^{-k}|\xi|)^{-1}\right) \leq C_{2}.
$$
This gives the upper bound of (2.2). For the lower bound note that \( \phi(\xi) = \widehat{\psi_0}(\xi) - \widehat{\psi_0}(2\xi) \). Since \( \sum_{k=-\infty}^{\infty} \phi(2^{-k}\xi) = \widehat{\psi_0}(0) \neq 0 \) converges uniformly for \( 1 \leq |\xi| \leq 2 \), there exists \( i_0 \in \mathbb{Z}_+ \) such that \( \frac{1}{2} |\widehat{\psi}(0)| \leq |\sum_{k=-i_0}^{i_0} \phi(2^{-k}\xi)| \). By the Cauchy-Schwarz inequality, we see

\[
\begin{align*}
(2.4) \quad \frac{1}{2} |\widehat{\psi}(0)| & \leq (2i_0 + 1)^{\frac{1}{2}} \left( \sum_{k=-i_0}^{i_0} \phi^2(2^{-k}\xi) \right)^{\frac{1}{2}} \leq (2i_0 + 1)^{\frac{1}{2}} \left( \sum_{k=-\infty}^{\infty} \phi^2(2^{-k}\xi) \right)^{\frac{1}{2}},
\end{align*}
\]

which gives the desired uniform lower bound of (2.2) for \( 1 \leq |\xi| \leq 2 \).

We now have (2.3) with \( \beta = 0 \). We may assume that \( \phi^\beta(x) \) is supported in \( B_{2^{-N}}(0) \) by replacing \( \phi \) with \( \phi(2^{-N}\cdot) \). In order to have (2.3) with bigger \( |\beta| < 2^N \) we need only to consider \( \phi^{2^N}(\xi) \) instead of \( \phi(\xi) \). Then \( \phi^{2^N}(x) \) is supported on \( B_1 \) and \( \partial^\beta(\phi^{2^N})(0) = 0 \) for \( |\beta| < 2^N \) which gives (2.3). As before, the estimate (2.2) follows by applying the Cauchy-Schwarz inequality \( N + 1 \) times.

We now prove the Littlewood-Paley inequality in which the projection operators are defined by the Schwartz function in Lemma 2.1 and give a characterization of the Lorentz-Sobolev spaces. The associated projection operators \( P_{\leq 0} \) and \( P_k \) for \( k \geq 1 \) are similar to the classical Littlewood-Paley operators but they are different in that the multiplier does not have compact support. However, the standard argument works without much of modification. For the reader’s convenience we include a proof.

Let \( \phi \) be given as in Lemma 2.1 and define

\[
(2.5) \quad \overline{P_k}f(\xi) = \phi^2(2^{-k}\xi) \hat{f}(\xi), \quad \overline{P_{\leq 0}}f(\xi) = \Phi_0(\xi) \hat{f}(\xi)
\]

where \( \Phi_0(\xi) = (1 + |\xi|^2)^{s/2} \sum_{k=-\infty}^{0} \phi^4(2^{-k}\xi) \). In what follows we prove Littlewood-Paley inequality in Lorentz-Sobolev spaces.

**Lemma 2.2.** Let \( s > 0 \) and \( 1 < q < \infty \), \( 1 \leq r \leq \infty \). Then there exists a constant \( C = C(d, q, s) \) such that, for \( f \in L^{q,r}_s(\mathbb{R}^d) \),

\[
(2.6) \quad \|P_{\leq 0}f\|_{q,r} + \left\| \left( \sum_{k=1}^{\infty} 2^{ks} |P_kf| \right)^{\frac{1}{2}} \right\|_{q,r} \leq C \|f\|_{L^{q,r}_s}
\]

and, for \( f \in S'(\mathbb{R}^d) \),

\[
(2.7) \quad C^{-1} \|f\|_{L^{q,r}_s} \leq \|P_{\leq 0}f\|_{q,r} + \left\| \left( \sum_{k=1}^{\infty} 2^{ks} |P_kf| \right)^{\frac{1}{2}} \right\|_{q,r}.
\]

Before proving this lemma, we need to prove the following estimates which allow us to use the Mikhlin multiplier theorem.

**Lemma 2.3.** Let \( s > 0 \) and \( \phi \) be given as in Lemma 2.1. Also, let \( m_k \) and \( m'_k \) be given by

\[
(2.8) \quad m_k(\xi) = 2^{ks} \phi^2(2^{-k}\xi)(1 + |\xi|^2)^{-\frac{s}{2}},
\]

\[
(2.9) \quad m'_k(\xi) = 2^{-ks} \phi^2(2^{-k}\xi)(1 + |\xi|^2)^{\frac{s}{2}} \left( \sum_{j=-\infty}^{\infty} \phi^4(2^{-j}\xi) \right)^{-1}.
\]
Let \( \{\omega_k\} \) be a sequence of constants having values \( \pm 1 \). Then

\[
\left| \partial_\xi^\alpha \left( \sum_{k=1}^L \omega_k m_k(\xi) \right) \right| \lesssim |\xi|^{-|\alpha|}
\]

for all \( |\alpha| \leq \frac{d}{2} + 1 \) and an analogue of (2.10) also holds if \( m_k \) is replaced with \( m'_k \).

Here we remark that \( \sum_{j=-\infty}^\infty \phi^4(2^{-j} \xi) \sim 1 \) so that \( (\sum_{j=-\infty}^\infty \phi^4(2^{-j} \xi))^{-1} \) is well defined. In fact, this can be shown by applying the Cauchy-Schwarz inequality to (2.3).

**Proof.** Let \( \mu \) be a multi-index. If we choose a sufficiently large \( N \) in Lemma 2.1 then (2.3) guarantees that \( \partial^\mu (\partial^\phi (0) = 0 \) for \( |\beta| \leq \frac{d}{2} + 1 \) and \( |\mu| \leq N \). Hence, for \( |\beta| \leq \frac{d}{2} + 1 \) and \( M > 0 \),

\[
|\partial^\beta (\phi^2(2^{-k} \xi))| \lesssim 2^{-k|\beta|} \min \left\{ \left( 2^{-k}|\xi| \right)^N, (2^{-k}|\xi|)^{-M} \right\}.
\]

Using (2.11) with large enough \( N, M \), we see that, for \( |\alpha| \leq \frac{d}{2} + 1 \),

\[
\left| \partial^\alpha \left( \sum_{k=1}^L \omega_k m_k(\xi) \right) \right| \lesssim \sum_{k=1}^L \sum_{\beta+\gamma=\alpha} 2^{ks} \left| \partial^\beta \left( \phi^2(2^{-k} \xi) \right) \right| \left( \left( 1 + |\xi|^2 \right)^{-\frac{1}{2}} \right) \left( \sum_{j=-\infty}^\infty \phi^4(2^{-j} \xi) \right)^{-1} \]

\[
\lesssim \sum_{k=1}^\infty \sum_{\beta+\gamma=\alpha} 2^{k(s-|\beta|)} \min \left\{ \left( 2^{-k}|\xi| \right)^N, (2^{-k}|\xi|)^{-M} \right\} |\xi|^{-s-|\gamma|} \lesssim |\xi|^{-|\alpha|}.
\]

This gives the desired inequality (2.10). Similarly, \( |\partial^\alpha \left( \sum_{k=1}^L \omega_k m'_k(\xi) \right) | \) is bounded by

\[
\sum_{k=1}^L \sum_{\beta+\gamma+\delta=\alpha} 2^{-ks} \left| \partial^\beta \left( \phi^2(2^{-k} \xi) \right) \right| \left( \left( 1 + |\xi|^2 \right)^{-\frac{1}{2}} \right) \left( \sum_{j=-\infty}^\infty \phi^4(2^{-j} \xi) \right)^{-1} \]

\[
\lesssim \sum_{k=1}^\infty \sum_{\beta+\gamma+\delta=\alpha} 2^{-k(s+|\beta|)} \min \left\{ \left( 2^{-k}|\xi| \right)^N, (2^{-k}|\xi|)^{-M} \right\} (1 + |\xi|^2)^{s/2} |\xi|^{-|\gamma|-|\delta|}.
\]

by (2.11) with sufficiently large \( N, M \) and is bounded by \( |\xi|^{-|\alpha|} \). This completes the proof. \( \square \)

Now we prove Lemma 2.2.

**Proof.** To prove the first part of Lemma 2.2, we make use of the Mikhlin multiplier theorem (cf. [4] Theorem 5.2.7) and Khintchine’s inequality. By the standard density argument we may assume that \( f \) is contained in the Schwartz class.

As for the operator \( P_{\leq 0} \), by following the argument in the proof of Lemma 2.3 it easy to see that the multiplier \( \sum_{k=-\infty}^0 \phi^4(2^{-k} \xi) \) satisfies \( |\partial^\alpha (\sum_{k=-\infty}^0 \phi^4(2^{-k} \xi) )| \lesssim |\xi|^{-|\alpha|} \). Hence, by the Mikhlin multiplier theorem we have \( \| P_{\leq 0} f \|_q \lesssim \| f \|_q \) for \( 1 < q < \infty \). Then the Lorentz bound

\[
\| P_{\leq 0} f \|_{q,r} \lesssim \| f \|_{q,r}
\]

follows from the Marcinkiewicz interpolation theorem between the \( L^q \) estimates.
To bound the square function of (2.6), we consider the multiplier \( m_k(\xi) \) defined by (2.8). Let \( \{\omega_k\} \) be independent random variables taking values \( \pm 1 \) with equal probability. Since \((m_k\hat{f}_s)^\vee = 2^{ks}P_kf\), Khintchine’s inequality gives, for \( 0 < q < \infty \),

\[
(2.13) \quad \left( \sum_{k=1}^{L} \left( 2^{ks}|P_kf| \right)^2 \right)^\frac{q}{2} \approx \mathbb{E} \left( \left| \sum_{k=1}^{L} (\omega_k m_k \hat{f}_s)^\vee \right|^q \right)
\]

with the implicit constants independent of \( L \). Thanks to (2.10) in Lemma 2.3 we can apply the Mikhlin multiplier theorem to the right-hand side of (2.13). Taking integral both side of (2.13), passing to the limit \( L \to \infty \) and using Fatou’s lemma give

\[
(2.14) \quad \left\| \left( \sum_{k=1}^{\infty} \left( 2^{ks}|P_kf| \right)^2 \right)^\frac{q}{2} \right\|_q \lesssim \| f_s \|_q^q
\]

for all \( 1 < q < \infty \). Then (2.6) is an immediate consequence of the Marcinkiewicz interpolation theorem and this proves the first part (2.6) of Lemma 2.2.

Now we show the inequality (2.7) by using the duality argument. Let \( f, g \) be Schwartz functions. By the Plancherel identity we have

\[
\int f_s(x)\overline{g}(x) \, dx = \int \hat{f}_s(\xi)\overline{\hat{g}(\xi)} \, d\xi.
\]

Using the identity \( 1 = (\sum_{k=-\infty}^{0} \phi^4(2^{-k}\xi) + \sum_{k=1}^{\infty} \phi^4(2^{-k}\xi)) \left( \sum_{j=-\infty}^{\infty} \phi^4(2^{-j}\xi) \right)^{-1} \), we decompose \( \hat{f}_s\hat{g} \) so that

\[
\hat{f}_s\hat{g} = \hat{P}_{\leq 0}f \hat{m}_0\hat{g} + \sum_{k=1}^{\infty} 2^{ks} \hat{P}_k f \hat{m}_k\hat{g},
\]

where \( m_0'(\xi) = \left( \sum_{j=-\infty}^{\infty} \phi^4(2^{-j}) \right)^{-1} \) and \( m_k' \) is defined in (2.9). By repeated use of the Plancherel identity, we have

\[
(2.15) \quad \int f_s(x)\overline{g}(x) \, dx = \int \hat{P}_{\leq 0}f(x)\overline{(m_0'\hat{g})^\vee}(x) \, dx + \sum_{k=1}^{\infty} \int 2^{ks} P_k f(x)\overline{(m_k'\hat{g})^\vee}(x) \, dx.
\]

Let \( 1 < q < \infty \) and \( 1 \leq r \leq \infty \) satisfying \( \frac{1}{q} + \frac{1}{r} = 1 \) and \( \frac{1}{q'} + \frac{1}{r'} = 1 \). We may apply the Cauchy-Schwarz inequality and the Hölder-type inequality for Lorentz spaces to obtain

\[
\left| \int f_s(x)\overline{g}(x) \, dx \right| \leq \| P_{\leq 0}f \|_{q,r} \| (m_0'\hat{g})^\vee \|_{q',r'}
\]

\[
+ \left\| \left( \sum_{k=1}^{\infty} \left( 2^{ks}|P_kf| \right)^2 \right)^\frac{1}{2} \right\|_{q,r} \left\| \left( \sum_{k=1}^{\infty} \left| (m_k'\hat{g})^\vee \right|^2 \right)^\frac{1}{2} \right\|_{q',r'}.\]

Then it is easy to see that \( \| (m_0'\hat{g})^\vee \|_{q',r'} \) is bounded by \( \| g \|_{q',r'} \) by following the same argument which shows the boundedness of \( P_{\leq 0} \) (see (2.12)). Since \( m_k' \) also satisfies
(2.10) in the place of $m_k$, by repeating the argument for (2.6) it follows that
\[
\left\| \left( \sum_{k=1}^{\infty} |(m_k g) |^{2} \right)^{\frac{1}{2}} \right\|_{q',r'} \lesssim \| g \|_{q',r'}.
\]
Hence, combining these two estimates we have
\[
\left| \int f(x) \overline{g}(x) \, dx \right| \lesssim \| P_{\leq 0} f \|_{q,r} \| g \|_{q',r'} + \left\| \left( \sum_{k=1}^{\infty} \left( 2^{ks} |P_k f| \right)^{2} \right)^{\frac{1}{2}} \right\|_{q,r} \| g \|_{q',r'}.
\]
Finally, taking supremum over Schwartz function $g$ with $\| g \|_{q',r'} \leq 1$ gives the desired inequality (2.17).

By making use of the function $\phi$ in Lemma 2.1 we prove the following which relates the $L^p$-norm of $P_k \chi_E$ and the measure of $(\partial E)_{2^{-k}}$.

**Lemma 2.4.** Let $E$ be a bounded domain in $\mathbb{R}^d$ and $1 \leq p < \infty$. If a Schwartz function $\phi^\vee$ is supported on $B_1(0)$ and satisfies (2.1), then
\[
(2.16) \quad \left\| \left( \phi(2^{-k} \xi) \chi_E(\xi) \right)^\vee \right\|_p \lesssim |(\partial E)_{2^{-k}}|^{1/p}.
\]
Additionally, if we use $\phi^2$ instead of $\phi$,
\[
(2.17) \quad \| P_k \chi_E \|_p = \left\| \left( \phi^2(2^{-k} \xi) \chi_E(\xi) \right)^\vee \right\|_p \lesssim |(\partial E)_{2^{-k}}|^{1/p}.
\]
**Proof.** Fix $p$, $1 \leq p < \infty$, and consider
\[
\left\| \left( \phi(2^{-k} \xi) \chi_E(\xi) \right)^\vee \right\|_p = \left( \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \phi^\vee(y) \chi_E(\xi - 2^{-k} y) \, dy \right|^p \, d\xi \right)^{1/p}.
\]
The crucial observation is that for any fixed $k$, the inner integral above survives only for $\xi$ such that $\text{dist} (\xi, \partial E) \leq 2^{-k}$. This is because $\phi^\vee$ is supported on $B_1$ and then the integral vanishes for $\xi \in E$ such that $\text{dist} (\xi, \partial E) > 2^{-k}$ due to (2.1). As a consequence, the $L^p$-norm must be smaller than $|(\partial E)_{2^{-k}}|^{1/p}$ as desired.

For the second inequality we need only to observe that $\int (\phi^2)^\vee = 0$ and $(\phi^2)^\vee$ is supported in $B_2(0)$.

3. **Proof of Theorem 1.3 and Proposition 1.2**

In this section we prove Theorem 1.3 and Proposition 1.2.

**Proof of Theorem 1.3.** Let $\phi$ be given as in Lemma 2.1. Then, we have
\[
(3.1) \quad |\widehat{\chi}_E(\xi)| \lesssim \sum_{k=\infty}^{\infty} \phi^2(2^{-k} \xi) |\widehat{\chi}_E(\xi)|.
\]
To get boundedness in the weak $L^p$ spaces, we separately handle the $L^p$-norm of $\phi(2^{-k} \cdot) \widehat{\chi}_E$. Using Lemma 2.4 with $p = 2$ and the condition (1.1), we have $\| \phi(2^{-k} \xi) \widehat{\chi}_E(\xi) \|_2 \lesssim 2^{-k(d-2)}$. Hence, using the Cauchy-Schwarz inequality, we get
\[
(3.2) \quad \| \phi^2(2^{-k} \xi) \widehat{\chi}_E(\xi) \|_1 \lesssim 2^{k/2}.
\]
and by (2.17)

\[ \| \phi^2(2^{-k}\xi)\widehat{\chi_E}(\xi) \|_2 \lesssim 2^{-\frac{k(d-\gamma)}{2}}. \]

Let \( N \) be an integer to be chosen later. We consider the distribution function of \( (3.1) \) and apply Chebyshev’s inequality so that

\[
\left| \left\{ \xi : \sum_{k=-\infty}^{\infty} \phi^2(2^{-k}\xi)|\widehat{\chi_E}(\xi)| > \lambda \right\} \right| \\
\leq \left| \left\{ \xi : \sum_{k=-\infty}^{N} \phi^2(2^{-k}\xi)|\widehat{\chi_E}(\xi)| > \frac{\lambda}{2} \right\} \right| + \left| \left\{ \xi : \sum_{k=N+1}^{\infty} \phi^2(2^{-k}\xi)|\widehat{\chi_E}(\xi)| > \frac{\lambda}{2} \right\} \right| \\
\lesssim \lambda^{-1} \left\| \sum_{k=-\infty}^{N} \phi^2(2^{-k}\xi)|\widehat{\chi_E}(\xi)| \right\|_1 + \lambda^{-2} \left\| \sum_{k=N+1}^{\infty} \phi^2(2^{-k}\xi)|\widehat{\chi_E}(\xi)| \right\|_2^2.
\]

The last term is bounded by

\[
\lambda^{-1} \sum_{k=-\infty}^{N} \| \phi^2(2^{-k}\xi)\widehat{\chi_E}(\xi) \|_1 + \lambda^{-2} \left( \sum_{k=N+1}^{\infty} \| \phi^2(2^{-k}\xi)\widehat{\chi_E}(\xi) \|_2^2 \right)^{1/2}
\]

by the Minkowski inequality. By application of the estimates (3.2) and (3.3) and summation along \( k \), we get

\[
\left| \left\{ \xi : \sum_{k=-\infty}^{\infty} \phi^2(2^{-k}\xi)|\widehat{\chi_E}(\xi)| > \lambda \right\} \right| \lesssim \lambda^{-1} 2^{\frac{d-\gamma}{2}} + \lambda^{-2} 2^{-N(d-\gamma)}.
\]

If we take \( N \) such that \( 2^N \approx \lambda^{-2/(2d-\gamma)} \), the right hand side is bounded by \( \lambda^{-2d/(2d-\gamma)} \). Hence, \( \widehat{\chi_E} \in L^{\frac{2d}{d-\gamma}}(\mathbb{R}^d) \) and this concludes the proof. \( \square \)

In what follows we prove Proposition 1.2. This is done by relating the \( L^p \)-norm of \( \widehat{\chi_E} \) to the integral of \( |(\partial E)_g| \).

**Proof of Proposition 1.2**. From (2.16) with \( p = 2 \), \( \| \phi(2^{-k}\xi)\widehat{\chi_E}(\xi) \|_2 \lesssim |(\partial E)_{2-k}|^{1/2} \). For \( 1 \leq p \leq 2 \), Hölder’s inequality yields

\[ \| \phi^2(2^{-k}\xi)\widehat{\chi_E}(\xi) \|_p \lesssim 2^{kd(\frac{1}{p} - \frac{1}{2})}|(\partial E)_{2-k}|^{1/2}. \]

Since \( \sum_{k=-\infty}^{\infty} \phi^4(2^{-k}\xi) \) is bounded below as indicated in Lemma 2.3, we get

\[
\| \widehat{\chi_E} \|_p \lesssim \left\| \sum_{k=-\infty}^{0} \phi^4(2^{-k}\xi)|\widehat{\chi_E}(\xi)| \right\|_p + \left\| \sum_{k=1}^{\infty} \phi^4(2^{-k}\xi)|\widehat{\chi_E}(\xi)| \right\|_p.
\]

Clearly, the first term in the right-hand side is bounded by \( C\| \widehat{\chi_E} \|_{\infty} \), which is in turn bounded by \( |E| \). For the second term it follows by Hölder’s inequality and (3.4) that

\[
\left\| \sum_{k=1}^{\infty} \phi^4(2^{-k}\xi)|\widehat{\chi_E}(\xi)| \right\|_p \leq \left( \sum_{k=1}^{\infty} \| \phi^2(2^{-k}\xi)|\chi_E(\xi)| \|_p \right)^{1/p} \left\| \sum_{k=1}^{\infty} 2^{kd(1-\frac{1}{p})}|(\partial E)_{2-k}|^{\frac{1}{p}} \right\|_p \lesssim \sum_{k=1}^{\infty} 2^{kd(1-\frac{1}{p})}|(\partial E)_{2-k}|^{\frac{1}{p}}.
\]
Note that the last sum is bounded by \( \int_0^1 \delta^{-d(1-\frac{p}{2})}\vert (\partial E) \delta \vert^\frac{p}{2} d\delta \) because \( \vert (\partial E) \delta \vert \) is increasing in \( \delta \). Therefore, combining these estimates gives
\[
\|\hat{\chi}_E\|_p \lesssim |E| + \left( \int_0^1 \delta^{-d(1-\frac{p}{2})}\vert (\partial E) \delta \vert^\frac{p}{2} d\delta \right)^{1/p}.
\]
This completes the proof. \( \square \)

4. Proof of Theorem 1.5

In this section we prove Theorem 1.5 by using the Littlewood-Paley inequality in the Lorentz-Sobolev spaces \( L^{q,\infty}_s \) which we have proved in Lemma 2.2

**Proof of Theorem 1.5.** Let \( 1 < q < \infty \), and \( s = (d-\gamma)/q \). For the proof of Theorem 1.5 it is sufficient to show that the right-hand side of (2.7) is finite while \( f = \chi_E \) and \( r = \infty \).

To estimate the first term, we note that the multiplier \( \Phi_0 \) of the operator \( P_{\leq 0} \) satisfies the Mikhlin multiplier condition by an analogous proof of (2.12). In particular, we have
\[
(4.1) \quad \| P_{\leq 0} \chi_E \|_{q,\infty} \lesssim \| \chi_E \|_{q,\infty} \sim |E|^\frac{1}{\gamma}.
\]

We now examine the square function which appears in (2.7). Let us choose \( p_0 \) and \( p_1 \) such that \( 1 < 2p_1 < q < 2p_0 < \infty \), and as before let \( N \) be an integer to be chosen later. By a simple manipulation and Chebyshev’s inequality we see
\[
\left| \left\{ x : \left( \sum_{k=1}^{\infty} \left( 2^k |P_k \chi_E(x)| \right)^2 \right)^{1/2} > \lambda \right\} \right| 
\leq \left| \left\{ x : \sum_{k=1}^{N} 2^{2k} |P_k \chi_E(x)|^2 > \frac{\lambda^2}{2} \right\} \right| + \left| \left\{ x : \sum_{k=N+1}^{\infty} 2^{2k} |P_k \chi_E(x)|^2 > \frac{\lambda^2}{2} \right\} \right| 
\lesssim \lambda^{-2p_0} \left\| \sum_{k=1}^{N} 2^{2k} |P_k \chi_E(x)|^2 \right\|_{p_0}^p + \lambda^{-2p_1} \left\| \sum_{k=N+1}^{\infty} 2^{2k} |P_k \chi_E(x)|^2 \right\|_{p_1}^p.
\]
By Minkowski’s inequality, the above sum is bounded by
\[
\lambda^{-2p_0} \left( \sum_{k=1}^{N} 2^{2k} \| (P_k \chi_E(x))^2 \|_{p_0}^p \right)^{1/p} + \lambda^{-2p_1} \left( \sum_{k=N+1}^{\infty} 2^{2k} \| (P_k \chi_E(x))^2 \|_{p_1}^p \right)^{1/p}.
\]
Applying (2.17) with \( p = 2p_0 \) and \( p = 2p_1 \) respectively and (1.1) with \( d - \gamma = sq \), we get
\[
\left| \left\{ x : \left( \sum_{k=1}^{\infty} \left( 2^k |P_k \chi_E(x)| \right)^2 \right)^{1/2} > \lambda \right\} \right| 
\lesssim \lambda^{-2p_0} \left( \sum_{k=1}^{N} 2^{2k} 2^{-ksq/p_0} \right)^{p_0} + \lambda^{-2p_1} \left( \sum_{k=N+1}^{\infty} 2^{2k} 2^{-ksq/p_1} \right)^{p_1} 
\lesssim \lambda^{-2p_0} 2^{Ns(2p_0 - q)} + \lambda^{-2p_1} 2^{Ns(2p_1 - q)}.
\]
Choosing $N$ to be $2^{Ns} \approx \lambda$, the right hand side is bounded by $C\lambda^{-q}$. Hence,

$$
\left\| \left( \sum_{k=1}^{\infty} \left| 2^{ks} P_k \chi_E(x) \right| \right)^{1/2} \right\|_{q,\infty} \lesssim 1.
$$

Combining this with (4.1) and using (2.7) we conclude that $\| \chi_E \|_{L^q,\infty(\mathbb{R}^d)} < \infty$. This completes the proof. □

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