Expanding Visibility Polygons by Mirrors

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Abstract

We consider extending visibility polygon (VP) of a given point q (VP(q)), inside a simple polygon P by converting some edges of P to mirrors. We will show that several variations of the problem of finding mirror-edges to add at least k units of area to VP(q) are NP-complete, or NP-hard. Which k is a given value. We deal with both single and multiple reflecting mirrors, and also specular or diffuse types of reflections.

In specular reflection, a single incoming direction is reflected into a single outgoing direction. In this paper diffuse reflection is regarded as reflecting lights at all possible angles from a given surface.

The paper deals with finding mirror-edges to add at least k units of area to VP(q). In the case of specular type of reflections we only consider single reflections, and the multiple case is still open.

Specular case of the problem is more tricky. We construct a simple polygon for every given instance of a 3-SAT problem. There are some specific spikes which are visible only by some particular mirror-edges. Consequently, to have minimum number of mirror-edges it is required to choose only one of these mirrors to see a particular spike. There is a reduction polygon which contains a clause-gadget corresponding to every clause, and a variable-gadget corresponding to every variable.

3-SAT formula has n variables and m clauses, so the minimum number of mirrors required to add an area of at least k to VP(q) is l = 3m + n + 1 if and only if the 3-SAT formula is satisfiable. This reduction works in these two cases: adding at least k vertex of P to VP(q), and expanding VP(q) at least k units of area.

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1 Introduction

Many variations of visibility polygons have been studied so far. Generally, we have a simple polygon P and a viewer such as a point or a segment inside it. The goal is to find the
maximal sub-polygon of $P$ visible to the viewer ($VP(q)$ for a point $q$). There are linear time algorithms to compute $VP(q)$ \cite{8} or when the viewer is a segment \cite{5}.

Visibility in the presence of mirrors was first introduced by Klee in 1969 \cite{6}. He asked whether every polygon whose edges are all mirrors is illuminable from every interior point. In 1995 Tokarsky constructed an all-mirror polygon inside which there exists a dark point \cite{10}.

Some papers have considered visibility with mirrors previously \cite{2}. Two different types of reflection were discussed; Diffuse-reflection and Specular-reflection.

Diffuse-reflection is the reflection of light from a surface such that an incident ray is reflected at many angles. Here we consider diffuse-reflection to reflect light at all possible angles from a given surface.

Specular-reflection is the mirror-like reflection of light from a surface, in which light from a single incoming direction is reflected into a single outgoing direction. The direction in which light is reflected is defined by the law-of-reflection, which states that the incident, surface-normal and reflected directions are coplanar, and the incident ray and the reflected ray make the same angle with the surface normal.

In each type of reflection, we may count on a single reflection or multiple reflections. Which in the former case the viewer can only see articles by just one mirror in the middle, but in the latter case a target may be visible through more than one mirror. In some papers they specified a max number for the middling mirrors \cite{11}.

We only discuss finding the edges to be mirrors, but it is shown that having two mirrors, the resulting visibility polygon, may not be a simple polygon \cite{8}. Also, having $h$ mirrors, the number of vertices of the resulting visibility polygon, can be $O(n + h^2)$, and for $h$ mirrors, each projection, and its relative visibility polygon can be computed in $O(n)$ time, which leads to overall time complexity of $O(hn)$.

we deal with adding at least $k$ units of area to $VP(q)$ with minimizing $m$.

This paper is organized as follows: In Section 2, notations are described. Next, we present two reductions in Section 3 In Subsection 3.1 we deal with the specular type, and in Subsection 3.2 we consider the diffuse case. Section 4 covers the discussion part of the paper.

2 Notations

Suppose $P$ is a simple polygon and $int(P)$ denotes its interior. Two points $x$ and $y$ are visible to each other, if and only if the open line segment $xy$ lies completely in $int(P)$. The visibility polygon of the point $q$ in $P$, denoted as $VP(q)$, consists of all points of $P$ visible to $q$.

Edges of $VP(q)$ that are not edges of $P$ are called windows. Weak visibility polygon of a segment $s$, denoted as $WVP(s)$, is the maximal sub-polygon of $P$ visible to at least one point (not endpoints) of $s$.

Suppose an edge $e$ of $P$ is a mirror. Two points $x$ and $y$ inside $P$ can see each other using $e$, if and only if they are directly visible with one specular-reflection.

$VP(q)$ with a mirror-edge $e$, is the maximal sub-polygon of $P$ visible to $q$ either directly or via $e$.

Two points or segments are “mirror-visible” if and only if they can see each other with an edge middling as a mirror. When we refer to an edge as “mirror-edge”, we will temporary count it as a mirror (convert it to a mirror), and check a property of that as a mirror. These mirror-edges may have the potential of being mirrors in the final solution.

We consider the whole edge as mirror, thus two points can be mirror-visible by just a
part of an edge. Also, if a point can see a part of a segment through a mirror, they are mirror-visible.

There are some regions in the polygon the area of which is not visible to \( q \) directly. We refer to each of them as a “spike”.

**3-SAT problem:** Given a boolean formula in conjunctive normal form, with each clause having exactly three variables, can any assignment of the variables satisfy the formula? In our reduction we assume that clauses with a variable and its complement have already been removed.

**The art gallery problem** is to determine the number of guards that are sufficient to see every point in the interior of an art gallery room. The art gallery can be viewed as a polygon \( P \) of \( n \) vertices and the guards are stationary points in \( P \). If guards are placed at vertices of \( P \), they are called vertex guards [4].

In 1986 Lee and Lin proved that the vertex guard version of the art gallery problem is NP-hard, they reduced the 3-SAT problem to that, by constructing a simple polygon for every given instance of a 3-SAT problem [9]. Their goal in this paper is similar to ours, however, we present a completely different ideas. The functionality of our reduction polygon and its gadgets is different. In this polygon there some especial spikes and edges. And each spike is only mirror-visible through some predetermined mirror-edges.

Also, we have a construction algorithm that provides a polynomial way to construct a reduction polygon which reduces the 3-SAT problem to ours.

**Paper overall:**

We discuss the problem of finding the minimum number of edges that need to be turned into mirrors to increase the area of the visibility polygon by at least a given quantity \( k \).

### 3 Expanding at least \( k \) units of area

In this Section in order to add at least \( k \) units of area to the visibility polygon of a point \( q \), we assume that all edges of the polygon are mirrors. It is shown by Aronov [2] in 1998 that in such a polygon which all of its edges are mirrors the visibility polygon of a point can contains hole. And also, when we consider at most \( r \) specular reflections for every ray, we can compute the visibility polygon of a point inside that within \( O(n^{2r}\log n) \) of time complexity and \( O(n^{2r}) \) of space complexity [1].

#### 3.1 Diffuse type of reflections

▶ **Theorem 1.** Given a simple polygon without holes, and a query point \( q \) inside the polygon, can \( l \) of the edges of the polygon be turned to mirrors, so that the area added to \( VP(q) \) through a single diffuse reflection is at least \( k \) units?

**Claim**: The problem is NP-hard and so is its multiple reflection case.

**Proof.** The construction uses the method of reduction of the 3-SAT problem to the art-gallery problem with vertex guards, by Lee et al [9]. In the original reduction, for a given 3-SAT formula with \( n \) variables and \( m \) clauses, a simple polygon without holes was constructed, which would require a minimum of exactly \( n + 3m + 1 \) vertex-guards to be completely seen if and only if the given 3-SAT formula was satisfiable.

In the original construction, there are several spikes, for which it would be necessary to turn certain vertices into vertex-guards if the whole polygon is to be made visible from only \( n + 3m + 1 \) vertices. We call these the candidate-vertices. In our construction we replace the candidate vertices for vertex-guards with small edges that we call candidate-edges. These
are marked in red in Figure 1. Also, we have the vertex guard at the top-left corner of the

![Figure 1 Modified art-gallery polygon.](image)

polygon, required to see the main rectangular region, and the query point \( q \) so that it sees the whole rectangular region as well as each candidate edge for mirrors. We set \( k \) to be the difference between the area of the polygon and the area of \( VP(q) \). This makes the polygon to require \( l = n + 3m + 1 \) mirrors if and only if the given 3-SAT formula is satisfiable.

### 3.1.1 Multiple reflections

**Conjecture 1.** We strongly believe that, since there are some areas which we can make visible just through the specified mirror-edges, which we should choose to expand the visibility polygon at least \( k \) unit squares—no matter how many reflections are allowed—these minimum number of mirror-edges are required to get to at least \( k \) unit squares. Therefore, the previous reduction works in the case of multiple reflections, too.

### 3.2 Specular type of reflections

**Theorem 2.** Given a simple polygon without holes, and a query point \( q \) inside the polygon, can \( l \) of the edges of the polygon be turned to mirrors, so that the area added to \( VP(q) \) through a single specular reflection is at least \( k \) units?

**Claim:** The problem is NP-hard.

**Proof.** In specular reflection, the angle of incidence is equal to the angle of reflection. This reduction requires a significant modification of the reduction for diffuse reflection. The gadgets for the construction are explained below. In our reduction in the 3-SAT instance we assume that clauses with a variable and its complement have already been removed.

### 3.2.1 Description of the polygon

A scheme of the polygon is shown in Figure 2. It contains a variable-gadget for each variable and a clause-gadget for each clause. Each variable \( x_i \) is represented by a variable pattern, which have wells labeled \( w(x_i) \) and \( w(\overline{x_i}) \), representing the literals \( x_i \) and \( \overline{x_i} \) respectively.
The query point \( q \) is placed near to the top left corner of the polygon, and the variable-gadgets are drawn such that \( q \) can see the inside of each well. Each well \( w(x_i) \) contains a blue edge on its right boundary, labeled \( e(x_i) \). Similarly, \( w(\overline{x_i}) \) contains \( e(\overline{x_i}) \) on its right boundary.

The polygon contains several spikes that point \( q \) can see fully only through some predetermined mirror-edges via a single specular reflection. If those edges that either one or two of them are supposed to see a specified spike completely when the construction is complete are, say, \( e(a) \) and \( e(b) \), then the spike is labeled \( s(e(a), e(b)) \).

The clause-gadget for any clause \( c_j \) contains six orange edges which represent the literals in \( c_j \) and their complements. In fact in a clause-gadget for every literal, say \( x_i \), there is two mirror-edges one represents \( x_i \) and the other one represents \( \overline{x_i} \). So, for the clause \( c_j \) having \( x_i \), its clause-gadget contains two orange mirror-edges labeled \( oc(e_j(x_i)) \) and \( oc(e_j(\overline{x_i})) \).

The left boundary of each \( w(\overline{x_i}) \) has a blue spike labeled \( s(e(\overline{x_i}), e(x_i)) \). So, \( q \) can only see this spike via \( e(\overline{x_i}) \) or \( e(x_i) \) blue edges in the \( w(x_i) \) and \( w(\overline{x_i}) \).

For every clause \( c_j \) in which either \( x_i \) or \( \overline{x_i} \) occur, \( w(\overline{x_i}) \) and \( w(x_i) \) each has one purple spike on their left boundaries, labeled \( s(e(c_j(x_i))) \) and \( s(e(c_j(\overline{x_i}))) \) respectively.

Between \( w(x_i) \) and \( w(\overline{x_i}) \), the boundary of the polygon contains red spikes labeled \( s(e(c_j(x_i))), e(c_j(\overline{x_i}))) \), for every clause \( c_j \) where \( \overline{x_i} \) or \( x_i \) occur.

Hence, if a literal \( x_i \) or its negation \( \overline{x_i} \) occur in a total of \( l \) clauses, their corresponding wells \( w(\overline{x_i}) \) and \( w(x_i) \) will have \( l \) purple spikes each, and another \( l \) red spikes between \( w(\overline{x_i}) \) and \( w(x_i) \). To the right of each clause-gadget, there is a green spike. For example, if the clause \( c_j \) is \( (x_1 \lor x_2 \lor x_3) \), then the green spike in its gadget is labeled \( s(e(c_j(x_1))), e(c_j(\overline{x_2})), e(c_j(x_3))) \).

**Analysis**

3-SAT formula has \( n \) variables and \( m \) clauses, then the minimum number of mirrors required to add an area of at least \( k \) to \( V P(q) \) is \( l = 3m + n \) if and only if the 3-SAT formula is satisfiable.

For example, if the clause concerned in Figure 3(a) is \( (x_1 \lor \overline{x_2} \lor x_3) \), then the edges correspond to literals in the following way: \( e_1 \rightarrow x_3, e_2 \rightarrow \overline{x_2}, e_3 \rightarrow x_1, e_4 \rightarrow x_3, e_5 \rightarrow \overline{x_2} \) and \( e_6 \rightarrow \overline{x_1} \).

From the above, it is clear that to see each blue spike \( s(e(\overline{x_i}), e(x_i)) \), at least one blue mirror-edge \( e(\overline{x_i}) \) or \( e(x_i) \) have to be turned into a mirror.

To see the equivalent of red spike \( s(\text{ome}(e_j(x_i))), \text{ome}(e_j(\overline{x_i})) \), at least one of the edges \( \text{ome}(e_j(\overline{x_i})) \) or \( \text{ome}(e_j(x_i)) \) have to be turned into a mirror. So, turning both \( e(\overline{x_i}) \) and \( e(x_i) \) into mirrors to see all the spikes in \( w(\overline{x_i}) \) and \( w(x_i) \) would not give the minimum number of mirrors required, since the number of edges required to see the purple spikes in one of \( w(x_i) \) and \( w(\overline{x_i}) \) would have to be turned into mirrors anyway. This would force us to consistently turn edges corresponding to a literal to a mirror in whichever clause it is present. Note that by doing this for all variables, we have \( 3m \) mirrors in all the \( c_i \)'s together, and \( n \) mirrors for all the \( x_i \)'s together. Now, if and only if the 3-SAT formula is satisfiable, there will be a way to assign 1 to some variables and 0 to others, such that each clause contains a literal which is assigned 1. This is equivalent to at least one of the latter three edges in every \( c_i \) being turned into a mirror, which means that the small green spike shown in Figure 3(b) is visible to \( q \) for each \( c_i \), and hence the whole polygon is visible to \( q \). Figure 3(b) reveals details of the green projection. A green spike forces at least one of the three orange mirror-edges to be a mirror. This shows a close up of the concave region under the edges \( e_4, e_5 \) and \( e_6 \). Each \( r_i \) and \( r'_i \) are the rightmost and leftmost unblocked reflected rays of \( q \) from edge \( e_i \), respectively, if it is turned into a mirror. To see the green spike in the concave region, at
Figure 2 This shows a scheme of the whole polygon for the reduction. The regions that represent clauses and variables are labeled as $c_i$ and $x_i$ respectively. The query point is labeled as $q$. There are two wells for each variable which contain blue and purple spikes. Each blue spike can be seen by $q$ only through a reflection via each of the two subsequent blue edges. The purple spikes can be seen by $q$ only through reflections via one blue and one orange edges. The red spikes can be seen by $q$ only through reflections via two orange edges. The green regions contain a spike each that can be seen by $q$ only via a reflection through three orange edges above it. The various parts of the polygon are described below with the subsequent figures.
Figure 3 (a) These gadgets stand for a clause each. Edges $e_1$, $e_2$ and $e_3$ correspond to the complement of the literals in the clause, while edges $e_4$, $e_5$ and $e_6$ correspond to the literals in the clause. (b) The green inward spike separating each such gadget from the next one has a spike inside it which is fully visible only by a single reflection through any of the mirrors $e_4$, $e_5$ and $e_6$. $IP$ is a point which illustrates the beginning of the green spike. We will use it later to construct the green spike. (c) The reflected rays from each orange edge go to the bottom of the polygon and see a purple and a red spike each. Furthermore, reflected rays from two edges representing a literal and its negation see the same red spike. (d) The green spike obstructs the reflected rays significantly from only one of the edges. (e) A full variable-gadget is shown. (f) The orange edges after introduction of the regulator-gadget.
least one of \( e_4, e_5 \) or \( e_6 \) have to be turned into a mirror. This is equivalent to having at least one literal in the clause to be assigned 1.

Thus, the minimum number of mirrors required to add an area of at least \( k \) to \( VP(q) \) is \( 3m + n \) if and only if the 3-SAT formula is satisfiable.

Figure 3(c) exhibits the gadget for each variable. The two consecutive wells \( w(\overline{x}_i) \) and \( w(x_i) \) correspond to a variable \( x_i \) and its negation, and contain many spikes.

The reflected rays coming from the orange mirror-edges \( ome(c_j(x_i)) \) and \( ome(c_j(\overline{x}_i)) \) in each clause gadget, corresponding to \( \overline{x}_i \) and \( x_i \) respectively in a given clause \( c_j \), are the only mirror-edges that allow \( q \) to see the red spike \( s(e(c_j(x_i)), e(c_j(\overline{x}_i))) \) entirely through a single reflection. So, in the final solution one of them must be turned into a mirror.

For \( q \) to see the blue spike \( s(e(\overline{x}_i), e(x_i)) \), at least one of the blue edges \( e(\overline{x}_i) \) and \( e(x_i) \) must be turned into a mirror.

Every purple spike in \( w(\overline{x}_i) \) (\( w(x_i) \)) is visible to \( q \) by reflected rays from \( e(\overline{x}_i) \) (\( e(x_i) \)). Also, the complete area of each purple spike individually can be seen from the corresponding orange edge in the clause gadget which either \( x_i \) or \( \overline{x}_i \) appeared in the pattern of that clause in the given instance of the 3-SAT problem. However, the orange edge \( ome(c_j(x_i)) \) can make a purple spike in \( w(\overline{x}_i) \) mirror-visible to \( q \). There are two candidate orange edges for seeing each spike. For example, if we turn edge \( e(\overline{x}_i) \) into a mirror, then we will have to turn the edges corresponding to \( \overline{x}_i \) in every clause gadget into a mirror as well, to see the spikes, say \( s(e(c_j(\overline{x}_i)), e(x_i)), s(e(c_j(\overline{x}_i)), e(x_i)) \) and \( s(e(c_j(\overline{x}_i)), e(x_i)) \). And so, on the subject of minimum number of mirror-edges the mirror-edge \( e(x_i) \) should not be converted to a mirror.

As is shown in Figure 3(d), the green spike can be drawn so that it partially blocks the reflected rays coming from any one of the three orange edges.

Purple spikes can be seen from their corresponding orange edges, because of the span of each of the reflected rays. They may be fully seen from some other orange edge as well. We introduce the regulator-gadgets shown in Figure 3(f) to partially obstruct the orange edges so that for each orange edge, its reflected rays do not see any spike in \( w(\overline{x}_i) \) above the purple spike it corresponds to. This makes each purple spike visible only from a single orange edge and the blue edge of \( w(\overline{x}_i) \). The inside of these regulator-gadgets are completely visible from point \( q \).

### 3.3 Construction algorithm

The main goal of this section is to show that it is possible to construct such a polygon mentioned above in polynomial time. As a result, the problem is NP-hard.

We first assemble the bottom of the polygon, or gadgets of the variables. In our construction we first determine the positions of spikes, and at the end we put appropriate spikes there. Dotted segments may be used to locate the spikes.

#### 3.3.1 Gadgets of variables

To do this, several vertical and horizontal lines are used (see Figure 4), which help organize a simple construction. These lines are assumed to be apart one unit (it does not matter what is our measurement) both vertically and horizontally. Let \( hf_i \) be the \( i \)th half-line starting form \( q \). During the construction of each gadget, it is recommended to have equal angles between these half-lines. Assume that the lower endpoint of the \( i \)th blue segment (from the left side of the polygon) is denoted by \( lep_i \), and the upper endpoint by \( uep_i \).

Appendix A shows how to draw the spikes at the specified locations. At the end, we can draw appropriate spikes with right directions so that in each spike there would be an
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Figure 4 The start point of the construction. We can use the law-of-reflection to specify the right angle for a mirror.

area which is only mirror-visible by only some predetermined mirror-edges according to the polygon we had in Section 3.1.

To build the first variable-gadget, we draw a dotted blue segment on a few (say a hundred) horizontal lines below q. This blue spike is denoted by $s(be(x_1), be(x_1))$, has one unit length, and is on the same vertical line as $q(hf_1)$.

Let $hl_i$ illustrate the $i$th horizontal line, starting on the line crossing $uep_1$ we have $hl_1$. As Figure 4 illustrates, $be(x_1)$ starts at the point where $hf_3$ intersects $hl_2$ ($uep_{be}(x_1)$). We use an hypothetical orange line from the upper endpoint of the blue dotted segment to this intersection point to specify the correct angle for $be(x_1)$. $be(x_1)$ starts from this intersection point, and ends at $hf_2$, and reflects $hf_3$ to the top endpoint of $s(be(x_1), be(x_1))$.

The intersection point of $hf_2$ and $be(x_1)$ is denoted as $lep_{be}(x_1)$. A ray from $q$ to $lep_{be}(x_1)$, and reflected from $lep_{be}(x_1)$ to $hf_1$ indicates the $be(x_1)$-mirror-visible interval on $hf_1$ (The orange arrow in Figure 4 illustrates this visible interval). Later, we will utilize this visible interval up to $lep_1$ to place purple spikes.

Next, we draw a red segment starting from $uep_{be}(x_1)$, on $hl_2$, and ending at $hf_5$ (see Figure 5). We will use this segment to place red spikes. The right endpoint of this red segment in the $i$th variable gadget is denoted as $rep_i$.

Constructing $be(x_1)$: A hypothetical orange segment from $uep_1$ to $hf_8$, on $hl_1$, indicates the upper endpoint of $be(x_1)$ ($uep_{be}(x_1)$) In order for the mirror-edge $e(x_1)$ to see the blue spike $s(be(x_1), be(x_1))$, $uep_{be}(x_1)$ should reflect $hf_8$ on $uep_1$ this way we determine the correct angle for the blue mirror-edge $be(x_1)$. We protract $be(x_1)$ to $hf_6$. Consequently, $be(x_1)$-mirror-visible interval on $hf_6$ provides enough space so that later we can place purple spikes properly. See Figure 5, the orange arrow starting from $lep_{be}(x_1)$ reveals $be(x_1)$-mirror-visible interval on $hf_5$.

Note that, we can use a larger angle between $hf$ half-lines, or more than 8 halflines we can provide more space to place the purple spikes.

The second variable-gadget is constructed like the first one, except that $s(be(x_1), be(x_2))$ should lie on $hf_6$, for $w(x_2)$ to be completely visible to $q$. The distance between the two gadgets can be equal to the distance of two consecutive half-lines. And also, it is sufficient for
the red segment (a locate to place red spikes) to be only between two consecutive half-lines (see Figure 5). This is because, from the second gadget onward, half-lines have increasingly lower slopes and provide more space to construct necessary spikes or mirror-edges. Other gadgets are constructed the same as the second one.

We split each purple segment into three equal segments which indicate places to put three purple spikes.

3.4 Constructing clause gadgets

The purple spikes are separated with some points which are indicated by \( p_{1(i)}, p_{2(i)}, \ldots \) starting from the lower point in the \( i \)th well (see Figure 6). Appendix A shows how to construct purple spikes.

As is shown in Figure 6, after we construct all the variable-gadgets, we draw a hypothetical big circle, whose center is the center of the patterns of the variable (The point in the middle of \( uep_1 \) and \( uep_{be}(x_n) \)). For now assume that this circle crosses \( q \). As we will see later, we may use a bigger circle with the same center.

Such a circle will help us to construct orange mirror-edges for clause-gadgets. And, it works as a roof over all the variable gadgets.

Without lost of generality, we sort clause patterns according to the variables appeared in them increasingly. A literal has priority over its complement. For example, the order of \((\overline{x}_1, x_2, x_3) \lor (\overline{x}_1, \overline{x}_2, x_3) \lor (x_1, x_2, x_3)\) clauses is \((x_1, \overline{x}_2, \overline{x}_3) \lor (\overline{x}_1, x_2, x_3) \lor (\overline{x}_1, \overline{x}_2, \overline{x}_3)\). We put the clause gadgets in the polygon in this order from left to right. By doing this we have an easier construction.

To build the upper section of the polygon, again, we use some half-lines or rays from \( q \) (\( uhf \)). These half-lines help ensure that all the mirrors are visible by \( q \). Also, we use \( uhf \) rays to determine correct angles for orange mirror-edges. The circle plus these half-lines will help us to determine edges of the polygon between the clause gadgets and the mirror-edges.

Every orange mirror-edge corresponding to a literal \((x_i)\) should see an interval which starts from a predetermined purple spike (in \( w(\overline{x_i}) \)) and ends at the endpoint of a prede-
Figure 6 The second variable-gadget is constructed. $ome(x_1)$ are illustrated in every clause-gadget.

termined red spike. For the orange mirror-edge corresponds to the complement of the same literal ($\overline{x_i}$) this interval starts from the start point of the same red spike and ends at a purple spike in $w(x_i)$.

Thus, for every orange mirror-edge we have a fix mirror-visibility interval. As we will see later in the paper, for every orange mirror-edge we fix a point on the circle, as the start point of that mirror-edge, which ensures us that $q$ sees the start point of the corresponding mirror-visibility interval without any obstacle. Also, using this point and the law-of-reflection we can fix the correct angle for this orange mirror-edge. So, the mirror-visibility area, the start point and the angle of every orange mirror-edge can be determined. In order for $q$ to see the endpoint of the mirror-visibility interval from the endpoint of this mirror-edge, Appendix B shows how we can find the endpoint of an orange mirror-edge. The start point of the mirror-edge is either on the left or right side of the mirror-edge depending on the fact that the edge is a literal or a complement of a literal.

As everything is fixed, while constructing any orange mirror-edge, we can compute the exact size of the arc it uses on the circle, and if the edge needs more space, Appendix C shows how we can shift other previous constructed orange mirror-edges. In the worst case
we can use a bigger circle with the same center, which can accommodate an arc with the size we need from the construction of the first orange mirror-edge till the mirror-edge we are working on in the current step.

3.4.1 Details of the construction of orange mirror-edges

To construct the first orange mirror-edge in the first clause \((\text{ome}(c_1(x_1)))\), we use a hypothetical arrow which starts from \(p_{1(1)}\) in Figure 6, and crosses \(\text{uep}_1\). This arrow intersects the circle, we set \(uhf_6\) to cross this intersection point on the circle. We draw another parallel arrow from \(p_{2(1)}\) to the circle, and we set \(uhf_5\) to cross the intersection of the this latter arrow with the circle. This intersection point on the circle is the start point of \(\text{ome}(c_1(x_1))\).

We set the angle of \(\text{ome}(c_1(x_1))\) so that the mirror reflects \(uhf_5\) to \(p_{2(1)}\) (The start point of a purple spike). \(\text{ome}(c_1(x_1))\) starts from \(uhf_5\), and ends at a point we can find easily through Appendix B. We change \(uhf_6\) and set it to cross this endpoint of \(\text{ome}(c_1(x_1))\). This endpoint of \(\text{ome}(c_1(x_1))\) reflects \(uhf_6\) to \(r_1\) which is a point on the red segment between \(w(x_1)\) and \(w(x_1)\) (see Figure 7).

Note that every orange mirror-edge and its complement in a clause-gadget can see a common interval on the corresponding red segment.

Similarly, using \(p_{2(1)}, p_{3(1)}, p_{4(1)}, \ldots\), we can construct all \(\text{ome}(x_1)\) orange mirror-edges in every corresponding clause-gadget on the circle.

To construct \(\text{ome}(x_1)\) mirror-edges we act the same way as \(\text{ome}(x_1)\) except that \(\text{uwp}(x_1)\) will be used instead of \(\text{uwp}(x_1)\). Also, we fix the right endpoint of the mirror-edge on the circle and the correct angle then we use Appendix B to find the left endpoint of the mirror so that it can see the left endpoint of the corresponding red segment (see Figure 7).

For clause-gadgets we need to know the exact 3-SAT statement to construct the polygon. Because, if a clause pattern contains \(x_1\), then the clause gadget contains five more orange mirror-edges on the left side of \(\text{ome}(x_1)\). Otherwise, there must be two orange mirror-edges on the left side, and three edges on the right side of \(\text{ome}(x_1)\).

There are quite a few 3-SAT statements with different polygons. We can construct a polygon equivalent to every possible 3-SAT instance using the algorithm proposed in this paper. Therefore, without loss of generality, suppose we have only three clauses, and our 3-SAT statement is \((x_1, \overline{x_2}, x_3) \lor (\overline{x_1}, x_2, x_1) \lor (\overline{x_1}, \overline{x_2}, x_3)\).

Each mirror-edge, at least one endpoint of which lies on the circle, is between two halflines from \(q(\text{uhf}_s)\). We indicate the left endpoint of the \(i\)th orange mirror-edge with \(\text{lepome}_i\), and the right one with \(\text{repome}_i\) \((1 \leq i \leq m)\).

However, when \(\text{ome}\) has to see a purple spike in a \(w(x)\), we need to set the right endpoint of \(\text{ome}\) to lie on the circle, then we act in a similar way mentioned above. And also, we have to protract \(\text{ome}\) from its left side.

To do this, more space on the circle is needed to accommodate all the orange mirror-edges, so we need to use a bigger circle with the same center.

To construct red spikes, in every clause-gadget we find the common visible interval of each orange mirror-edge and its complement on the corresponding red segment. A whole red segment may be visible to all the corresponding orange mirror-edges, in this case we can simply split the red segment into three equal intervals. Appendix A reveals how we construct the spikes.

To construct green gadgets and green spikes, we need to focus on a particular intersection point in each clause. We call such an intersection point in the \(i\)th clause-gadget–from the
The third variable-gadget is constructed. Every orange mirror-edge needs to see a proper interval of its corresponding red segment. We can split the red segment in each variable-gadget into three subsegments and set the left endpoint of the corresponding orange mirror-edge to see an appropriate red point. For example the left endpoint of $ome(c_1(x_1))$ sees $uepbe(x_1)$. And the right endpoint of $ome(c_1(x_1))$ sees $r_1$. So, a red spike from $uepbe(x_1)$ to $r_1$ works correctly for these two orange mirror-edges.

To have a more precise definition for this point, without loss of generality, suppose we have $x_1$ in clause $c_j$. After the construction of the corresponding green-gadget, $ome(x_1(c_j))$-mirror-visibility will be blocked the most among the other two orange mirror-edges in $cg_j$. This is because, we need a little area of the first three orange mirror-edges in clause-gadget $cg_j$ to construct the green spike of $cg_j$. As Figure 7 illustrates, we can construct a green-gadget that obstructs the mirror-visibility of the two upper orange mirror-edges (rather than $x_1(c_j)$ just a little bit so that their functionality to see a red spike and a purple spike will not suffer. However, it is not possible to do that for $ome(x_1(c_j))$, because $IP_j$ blocks $ome(x_1(c_j))$-mirror-visible area.

Suppose $whf_k$ is on $repome_j$ of $ome(x_1(c_j))$. To construct a green spike for $cg_j$, we draw a green line from $repome_j$ of $ome(x_1(c_j))$ on the circle to $IP_j$. The green spike starts from $IP_j$. We draw two connected green lines from the last endpoint of the green-spike to a proper point below $IP_j$ (similar to Figure 3(b)). These last two green lines and point should be in a way that we can accommodate the green-spike properly (see Figure 3(b)).

As the last endpoint of the green-gadget is really close to $IP$, in our discussion we consider $IP$ as the blocker.

Before the construction of green-gadget, every orange mirror-edge is constructed in a way that it can see its corresponding red spike and purple spike. After constructing the green-gadget, we check whether $ome(x_1(c_j))$ can see an interval on the left wall of $w(x_1)$ to place the purple spike $s(c_j(x_1), x_1)$. If $ome(x_1(c_j))$ cannot see such kind of interval, then
we need to rotate \( ome(\overline{x}_2(c_j)) \) in counterclockwise direction. The rotation of \( ome(\overline{x}_2(c_j)) \) moves \( IP_j \) so that \( IP_j \) becomes closer to the ray reflected from \( repome_j \) on \( \overline{x}_2(c_j) \). We continue the rotation process till \( ome(\overline{x}_2(c_j)) \) see an interval on the left wall of \( w(x_1) \).

Note that in another case, an orange mirror-edge \( ome(\overline{x}_1) \) instead of \( ome(\overline{x}_2) \) is in clause \( c_j \). And, the most obstructed orange edge is \( ome(\overline{x}_1) \). Here, the above procedure works, except that we need to consider an appropriate red interval of the corresponding red segment instead of a place for the corresponding purple spike.

We need to check \( ome(\overline{x}_2(c_j))-mirror-visibility \) to provide an enough spacious mirror-visibility area. If \( ome(\overline{x}_2(c_j)) \) cannot see a common red interval with \( ome(x_2(c_j)) \) on the red segment. Then, we need to make a longer orange mirror-edge and stretch it form its left side. We can use a bigger circle, and use the same process that we mentioned before when we explained how to make a longer orange mirror-edge to see a corresponding red spike.

Again, in the case we have \( x_2 \) in our clause, then we can simply use the same process to make a bigger mirror-edge for \( ome(x_2) \) to see the equivalent purple spike.

Appendix B covers the analysis of the algorithm.

\section*{Discussion}

In this paper, we proved to find the mirror edges which can extend the visibility polygon of a given point is a hard problem. Specifically we proved to add a space with a minimum space of \( k \) unit squares, the problem was discussed in 1998 \cite{1}. We can check the polygon considering all of its edges as mirrors. We can investigate to see if they can add \( k \) units, or more than that to the surface area of the visibility polygon in \( O(n^{2r}\log n) \) time, which at most \( r \) specular reflections are allowed. Nonetheless, we proved in the paper that the minimum version the problem is NP-Hard.

This result works for:

- Specular version considering single reflections.
- Diffuse version considering single and multiple reflections.

\textbf{Open Problem:} However, the specular case considering multiple reflections, of the at least \( k \) version is still open.

The problem can be extended as: put mirrors inside the polygon, have more than one point inside it, a point with a limited visibility area, find some edges which can give the point a specific vision to name but a few.
A  Finding mirror endpoints

Generally, for every orange mirror-edge $e$ we first fix a point on the circle from which $e$ should make a point at the bottom of the polygon $e$-mirror-visible to $q$. There is another point at bottom of the polygon that the endpoint of $e$ needs to make it mirror-visible. This appendix shows how we can find the mirror-endpoint.

Lemma 3. If we fix a start point and an angle for a mirror-edge ($e$) in a way that $q$ (the viewer) can see a predetermined point, say $a$, on a segment $ab$. Then, we can find the exact point on the mirror, say $d$, that can make $b$ $e$-mirror-visible to $q$.

Proof. See Figure 11. We only need to protract $ca$ through the back of $e$, and with the size of $|cq|$. The new point we reach is denoted by $q'$ which is the projection of $q$ on $e$. The intersection of $q'b$ and the half-line containing $e$, which starts from $c$, reveals the position of $d$ (the end point of the mirror-edge). In another word, $q$ can see $b$ from $d$ on the mirror-edge $e$.

![Figure 9](Looking for the end point of a mirror-edge. The position of $d$ is unknown first and we find it.)

B  Changing the position of mirror-edges

In order for the circle to accommodate all clause gadgets and orange mirror-edges, we can shift orange mirror-edges to the left.

See the following Lemma;
Lemma 4. We can change the position of an orange mirror-edge \((e)\) in a way that the mirror-visibility interval does not change. Mirror-visibility interval is a segment we fixed previously and a mirror can make it mirror-visible to the viewer.

Proof. See Figure 10, the orthogonal projection of \(a\) on the circle is denoted as \(op(e)\). We can easily shift the position of \(c\) (the starting point of mirror-edge, which reflects rays from \(q\) (the viewer) to \(a\)) from anyplace on the right of \(op(e)\) to \(op(e)\).

![Figure 10](image.png) Changing a mirror-edge position.

According to the law-of-reflection we can find the right angle for \(e\) in the new position. And we can find the exact endpoint of the mirror-edge which reflects rays exactly to \(b\) using Lemma 3.

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