Asymptotically locally AdS and flat black holes in the presence of an electric field in the Horndeski scenario

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Abstract

Asymptotically locally AdS and asymptotically flat black hole solutions are found for a particular case of the Horndeski action. The action contains the Einstein-Hilbert term with a cosmological constant, a real scalar field with a non minimal kinetic coupling given by the Einstein tensor, the minimal kinetic coupling and the Maxwell term. There is no scalar potential. The solution has two integration constants related with the mass and the electric charge. The solution is given for all dimensions. A new class of asymptotically locally flat spherically symmetric black holes is found when the minimal kinetic coupling vanishes and the cosmological constant is present. In this case we get a solution which represents an electric Universe. The electric field at infinity is only supported by $\Lambda$. When the cosmological constant vanishes the black hole is asymptotically flat.
I. INTRODUCTION

Scalar fields have a prominent role in high energy physics. At subatomic scales they are an essential part of the quantum description of the electroweak interaction. Indeed, a fundamental scalar field excitation is given by the well known Brout-Englert-Higgs particle, which allows a consistent mathematical description of the the short range of the weak force and lepton masses.

At the galactic and cosmological scales, scalar fields arise as the simplest candidate to the explanation of many phenomena. At these scales, the general theory of relativity successfully describes the gravitational interaction. However, despite the great success of the theory, it cannot give a satisfactory description of certain cosmological phenomena, such as the origin of the early Universe and its late time accelerated expansion, as well as the presence of dark matter and dark energy. The properties of these phenomena make the scalar field a suitable candidate able to solve such unknowns, giving rise to a wide variety of theories such as Brans-Dicke theory, inflation theories and several cosmological models.

Moreover scalar fields appear naturally in theories like Kaluza-Klein compactifications and in theories that intend to give a natural description of gravity at the quantum level, such as string theory, which includes the dilaton scalar field.

While it is true that the study of scalar-tensor theories is not a new topic, currently a great interest resurfaced due to the study of galileon theories and their applications. This have revived the study of the most general scalar-tensor theory which has second order field equations and second order energy-momentum tensor, problem that was solved by Horndeski four decades ago. Horndeski theory along with a big amount of interesting properties also includes galileon gravity and massive gravity.

If we focus our attention in a four dimensional curved spacetime, the most general Lagrangian which can be constructed with the above properties is given by

\[ L = \lambda_1 \phi R_{ef} R_{hi} + \lambda_2 \phi \nabla_a \phi \nabla^a R_{ef} + \lambda_3 \phi R_{ab} R_{cd} + \Theta + B \epsilon_{abcd} R_{qab} R_{pcd} \]  

where \( B \) is a constant, \( \lambda_i \) are arbitrary functions of the scalar \( \phi \) and \( \Theta \) is and arbitrary function of the scalar field and its squared gradient, i.e. \( \Theta = \Theta(\nabla_a \phi \nabla^a \phi, \phi) \).

At this point, we can see that obtaining scalar field Lagrangians, whose kinetic term has non-minimal couplings with the curvature, is possible. In a cosmological context, theories
where this non-minimal derivative coupling is given by the Einstein tensor, provides an expansion of the Universe without a scalar potential \[8\]. Accelerating behaviors were observed as well in the case of a coupling given by the Ricci tensor \[9\]. Many models appeared in this context \[10\][11][12].

Let us focus our attention on kinetic terms \(S\) which are quadratic in the derivatives of the field in arbitrary dimension \(n\). Requiring second order energy-momentum tensor, as well as field equations for the field, single out \(S\) as a linear combination of the following terms

\[
S^{(p)} = E^{(p)}_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi ,
\]

where \(E^{(p)}_{\mu\nu}\) is \(p\)-th order Lovelock tensors \[13\]

\[
E^{(p)}_{\nu \mu} = \delta^{\nu \alpha_1 \ldots \alpha_{2p}}_{\mu \beta_1 \ldots \beta_{2p}} R_{\alpha_1 \alpha_2} \ldots R_{\beta_{2p-1} \beta_{2p}}.
\]

By setting \(p = 0\), the standard kinetic term is therefore obtained. Since \(E^{(1)}_{\mu\nu}\) is proportional to the Einstein tensor, the first non-standard term in \(S\) already includes a non-minimal kinetic coupling of the scalar field and the curvature.

In this paper we shall focus on the study of black hole solutions and their properties that emerge from this theory. The action principle is given by

\[
\mathcal{I}[g_{\mu\nu}, \phi] = \int \sqrt{-g} d^n x \left[ \kappa (R - 2\Lambda) - \frac{1}{2} (\alpha g_{\mu\nu} - \eta G_{\mu\nu}) \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right].
\]

The strength of the non minimal kinetic coupling is controlled by \(\eta\). Here \(\kappa := \frac{1}{16\pi G}\). The possible values of the dimensionfull parameters \(\alpha\) and \(\eta\) will be determined below requiring the positivity of the energy density of the matter field.

The first exact black hole solution to this system was found by Rinaldi in \[14\] for the case of vanishing cosmological constant \(\Lambda\) and without the Maxwell term. In that solution the scalar field becomes imaginary in the domain of outer communications, and the weak energy condition is violated outside of the horizon.

A great interest has been generated by spacetimes which are asymptotically of constant curvature, particularly asymptotically AdS spacetimes. This interest is largely motivated by the AdS/CFT correspondence \[15\] which relates the observables in a gauged supergravity theory with those of a conformal field theory in one dimension less. In this way, black

\[1\] The most general symmetric tensors which are divergency-free and contain up to second order derivatives of the metric.
hole solutions with a negative cosmological constant are important because in principle they could provide the possibility of studying the phase diagram of a CFT theory. As we know, a black hole in an asymptotically flat spacetime is thermodynamically unstable. In order to solve this problem it is possible to put the black hole inside a cavity of finite size. However, there is an alternative method to stabilize such a black hole. It consists in adding a negative cosmological constant. The properties of the AdS spacetime stabilize the black hole simulating a reflecting cavity.

Therefore, it seems natural to study the case where a negative cosmological constant is present. This was done in [16], where a real scalar field outside the horizon was found and where the positivity of the energy density is given by this reality condition. Recently in reference [17] it has been shown that allowing the scalar to depend on time permits to construct a black hole solution in which the scalar field is analytic at the future or at the past horizon. In a similar context exact solutions were found in [18].

The aim of this work is to continue in this line and generalize the results in reference [16] by adding a Maxwell term given by a spherically symmetric gauge field \( A = A_0(r)dt \).

A numerical solution in this case was found in [19], where phase transitions to charged black hole with complex anisotropic scalar hair were explored. We also extend the solution to the topological case in arbitrary dimension \( n \geq 4 \) and show that it is also possible to obtain a non-trivial solution when \( \alpha = 0 \). In this later case, when the black hole is spherically symmetric, we obtain an asymptotically locally flat black hole with \( \Lambda \neq 0 \) and an asymptotically flat black hole (i.e the metric is Minkowski at spatial infinity)\(^2\) when \( \Lambda = 0 \).

The variation of the action (4) with respect to the metric tensor, the scalar field and the gauge field yields

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{\alpha}{2\kappa} T^{(1)}_{\mu\nu} + \frac{\eta}{2\kappa} T^{(2)}_{\mu\nu} + \frac{1}{2\kappa} T^{em}_{\mu\nu} ,
\]

\[
\nabla_\mu \left[ (\alpha g^{\mu\nu} - \eta G^{\mu\nu}) \nabla_\nu \phi \right] = 0 ,
\]

\[
\nabla_\mu F^{\mu\nu} = 0 ,
\]

\(^2\) This is the difference with the asymptotically locally flat solution. However, both solutions have curvatures which vanishes at spatial infinity.
respectively. Here we have defined

\[ T^{(1)}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \phi \nabla^\lambda \phi , \]

\[ T^{(2)}_{\mu\nu} = \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi R - 2 \nabla_\lambda \phi \nabla_\mu \phi R^\lambda_{\nu \rho} - \nabla^\lambda \phi \nabla^\rho \phi R_{\mu \lambda \rho} - \nabla_\mu \nabla_\nu \phi + \frac{1}{2} G_{\mu\nu} (\nabla \phi)^2 \]

\[ - g_{\mu\nu} \left[ - \frac{1}{2} (\nabla^\lambda \nabla^\rho \phi)(\nabla_\lambda \nabla_\rho \phi) + \frac{1}{2} (\nabla \phi)^2 + \nabla_\lambda \phi \nabla_\rho \phi R^\lambda_{\rho} \right] , \]

\[ T^{em}_{\mu\nu} = F^{\lambda}_{\mu} F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F^2 . \]

We will consider the family of spacetimes

\[ ds^2 = -F(r)dt^2 + G(r)dr^2 + r^2 d\Sigma_K^2 , \]

where \( d\Sigma_K \) is the line element of a closed, \((n-2)\)-dimensional Euclidean space of constant curvature \( K = 0, \pm 1 \). The metric \( \Sigma_K \) corresponds to the most general static spacetime compatible with the possible local isometries of \( \Sigma_K \) acting on a spacelike section. For \( K = 1 \), the space \( \Sigma_K \) is locally a sphere, for \( K = 0 \) it is locally flat, while for \( K = -1 \) it locally reduces to the hyperbolic space. Hereafter we will consider a static and isotropic scalar field, i.e. \( \phi = \phi(r) \).

The outline of the paper is as follows: in section 2 the four-dimensional solution is given for arbitrary \( K \), and the energy density is computed. In section 3, the spherically symmetric solution is described in detail and the constraints in the couplings parameters are described in order to obtain a real scalar field and positive energy density. We comment as well on some of the thermodynamical properties of the solution. In section 4, the solution in arbitrary dimension \( n \) is given. Finally in section 5 the solution in the special case when \( \alpha = 0 \) is analyzed. In this paper we use the “mostly plus signature” and Greek indices stand for indices in the coordinate basis.

II. FOUR DIMENSIONAL SOLUTION

Using the ansatz \( \Sigma_K \) the equation of motion for the scalar field \( \phi \) admits a first integral, which implies the equation

\[ r \frac{F'(r)}{F(r)} = \left[ K + \frac{\alpha}{\eta} r^2 - \frac{C_0}{\eta} \frac{G(r)}{\psi(r) \sqrt{F(r)G(r)}} \right] G(r) - 1 , \]  

\[ \text{for} \quad \alpha \neq 0 , \]

\[ \frac{F'(r)}{F(r)} = \frac{1}{r} \left[ \frac{C_0}{\eta} \frac{G(r)}{\psi(r) \sqrt{F(r)G(r)}} \right] G(r) - 1 , \]

\[ \text{for} \quad \alpha = 0 . \]

\[ \text{We use a normalized symmetrization} \quad A_{(\mu\nu)} := \frac{1}{2} (A_{\mu\nu} + A_{\nu\mu}) . \]
where $C_0$ is an integration constant, $\psi(r) := \phi'(r)$, and $(')$ stands for derivation with respect to $r$. As it was done in reference [14], and then in [16] we (arbitrarily) set $C_0 = 0$, which allows to find a simple relation between the metric functions $F(r)$ and $G(r)$

$$G(r) = \frac{\eta}{F(r)} \left( \frac{r F'(r) + F(r)}{r^2 \alpha + \eta K} \right).$$

(10)

The Maxwell equation admits a first integral as well, providing the following relation

$$G(r) = \frac{r^4}{q^2 F(r)} (A_0'(r))^2,$$

(11)

where $\frac{1}{q^2}$ is an integration constant. These two last equations allow us to find an expression for the first radial derivative of the electric potential

$$(A_0'(r))^2 = \frac{q^2 \eta}{r^4} \left( \frac{r F'(r) + F(r)}{r^2 \alpha + \eta K} \right).$$

(12)

In this way, equations (10) and (12) together with the $tt$ and $rr$ components of (5), provide a consistent system which for $K = \pm 1$ and $\eta \Lambda \neq \alpha$, has the following solution

$$F(r) = \frac{r^2}{l^2} + \frac{K}{\alpha} \sqrt{\alpha \eta K} \left( \frac{\alpha + \Lambda \eta + \frac{\alpha^2 q^2}{4 \eta K}}{\alpha - \Lambda \eta} \right)^2 \arctan \left( \frac{\sqrt{\alpha \eta K} r}{r} \right) - \mu + \frac{\alpha^2}{\kappa (\alpha - \Lambda \eta)^2} \frac{q^2}{r^2} + \frac{\alpha^2 q^4}{16 \eta^2 K^2 (\alpha - \Lambda \eta)^2} - \frac{\alpha^2 q^4}{48 \eta^2 K^2 (\alpha - \Lambda \eta)^2} + \frac{3 \alpha + \Lambda \eta K}{\alpha - \Lambda \eta},$$

$$G(r) = \frac{1}{16} \frac{\alpha^2 (4 \kappa \alpha + \eta \Lambda) r^4 + 8 \eta K r^2 - \eta q^2)^2}{r^4 \kappa^2 (\alpha - \eta \Lambda)^2 (\alpha r^2 + \eta K)^2 F(r)},$$

$$\psi^2(r) = -\frac{1}{32} \frac{\alpha^2 (4 \kappa \alpha + \eta \Lambda) r^4 + \eta q^2)^2 (4 \alpha - \eta \Lambda) r^4 + 8 \eta K r^2 - \eta q^2)^2}{r^6 \eta^2 K^2 (\alpha - \eta \Lambda)^2 (\alpha r^2 + \eta K)^3 F(r)},$$

$$A_0(r) = \frac{q \sqrt{\alpha}}{4 \eta^2 \kappa^2} \frac{(4 \beta \kappa K^2 (\alpha + \eta \Lambda) + \alpha q)}{(\alpha - \eta \Lambda)} \arctan \left( \frac{\sqrt{\alpha \eta K}}{\eta K} \right) + \alpha \left( \frac{8 \eta K^2 + \alpha q}{4 \eta \kappa K^2 (\alpha - \eta \Lambda)} \right) - \frac{\alpha q^3}{12 \kappa K (\alpha - \eta \Lambda)}.$$

Here we have defined the effective (A)dS radius $l$ by $l^{-2} := \frac{\alpha}{3 \eta}$. In the case of a locally flat transverse section ($K = 0$) the system integrates in a different manner and the solution
takes the form

\[
F(r) = \frac{r^2}{l^2} - \frac{\mu}{r} + \frac{\alpha}{2\kappa(\alpha - \eta \Lambda)} \frac{q^2}{r^2} + \frac{\alpha \eta}{80\kappa^2(\alpha - \eta \Lambda)^2} \frac{q^4}{r^6},
\]

\[
G(r) = \frac{1}{16} \frac{(4\kappa(\alpha - \eta \Lambda) r^4 - \eta q^2)^2}{\kappa^2(\alpha - \eta \Lambda) r^8 F(r)},
\]

\[
\psi(r)^2 = -\frac{1}{32} \frac{(4\kappa(\alpha + \eta \Lambda) r^4 + \eta q^2)(4\kappa(\alpha - \eta \Lambda) r^4 + \eta q^2)^2}{\alpha \eta r^{12} \kappa^2(\alpha - \Lambda \eta) r^8 F(r)},
\]

\[
A_0(r) = -\left(\frac{20\kappa(\alpha - \eta \Lambda) r^4 - \eta q^2}{20\kappa(\alpha - \eta \Lambda) r^5}\right) q.
\]

In the case when we set \( q \to 0 \) we recover the result obtained in [16] for the cases \( K = \pm 1 \) as well as for the case \( K = 0 \). The later case reduces to topological Schwarzschild solution with locally flat horizon [20].

It can be seen that this solution is asymptotically locally dS or AdS for \( \alpha/\eta < 0 \) or \( \alpha/\eta > 0 \), respectively, since when \( r \to \infty \) the components of the Riemann tensor go to

\[
R^{ab}_{\ cd} \to -\frac{1}{3\eta} \delta^{ab}_{\ cd} = -\frac{1}{l^2} \delta^{ab}_{\ cd},
\]

justifying our previous definition of the effective (A)dS radius. The asymptotic expansion \((r \to \infty)\) of the metric functions and of the gauge field reads

\[
g_{tt} \to \frac{r^2}{l^2} + \frac{3\alpha + \eta \Lambda}{\alpha - \eta \Lambda} K + \frac{K}{2\alpha} \sqrt{\alpha \eta K} \left(\frac{(\alpha + \eta \Lambda) + \frac{\alpha^2 q^2}{4\eta \kappa K^2}}{\alpha - \eta \Lambda}\right)^2 \frac{\pi \sigma - 2\mu}{r} + O\left(r^{-2}\right),
\]

\[
g^{rr} \to \frac{r^2}{l^2} + \frac{7\alpha + \eta \Lambda}{3(\alpha - \eta \Lambda)} K + \frac{K}{2\alpha} \sqrt{\alpha \eta K} \left(\frac{(\alpha + \eta \Lambda) + \frac{\alpha^2 q^2}{4\eta \kappa K^2}}{\alpha - \eta \Lambda}\right)^2 \frac{\pi \sigma - 2\mu}{r} + O\left(r^{-2}\right),
\]

\[
A_0(r) \to a_0 - \frac{q}{r} + O(r^{-2}),
\]

where \( \sigma \) is the sign of \( \eta K \) and \( a_0 \) is a constant. From here it is possible to see that our electric potential reproduces the Coulomb potential at infinity. There is a curvature singularity at \( r = 0 \) since for example the Ricci scalar diverges as

\[
R = \frac{4K}{r^2} + O(1). \tag{13}
\]

If \( \rho(r) \) is the energy density, then the total energy \( \mathcal{E} \) is given by

\[
\mathcal{E} = V(\Sigma) \int dr \rho(r), \tag{14}
\]
where $V(\Sigma)$ stands for the volume of $\Sigma$. Therefore

$$\rho(r) := r^2 \sqrt{G(r)} F(r)^{-1} T_{tt}. \quad (15)$$

Now, the $tt$ component of the energy momentum tensor reads

$$T_{tt} = -\frac{(\alpha + \Lambda \eta)}{\eta \kappa^2} F(r) \left[1 - H(r) F(r)\right], \quad (16)$$

where $H(r)$ is given by the expression

$$H(r) = \frac{64 \eta^2 r^2 (\alpha - \Lambda \eta)^2 (r^2 \alpha + \eta K)}{\alpha^2 \kappa^2 (\alpha + \Lambda \eta)} \left(\frac{q^2 \kappa (2r^2 \alpha + \eta K) - 4K (\alpha + \Lambda \eta) r^4}{4(\alpha - \Lambda \eta) r^4 + 8^2 \eta K - \eta \kappa q^2}\right).$$

If we take the limit $q \to 0$ we recover the $T_{tt}$ component of the uncharged case.

### III. SPHERICALLY SYMMETRIC CASE

Now we study the particular case with a spherically symmetric transverse section $K = 1$. The solution for the metric components and for the square of the derivative of the scalar field reduces to

$$F(r) = \frac{r^2}{l^2} + \frac{1}{\alpha} \sqrt{\alpha \eta} \left(\frac{\alpha + \Lambda \eta + \frac{\alpha^2}{4 \eta \kappa} q^2}{\alpha - \Lambda \eta}\right) \arctan \left(\frac{\sqrt{\alpha \eta}}{\eta} r\right) - \mu \frac{\alpha}{r^2} + \frac{\alpha^2}{\kappa (\alpha - \Lambda \eta)^2} \frac{q^2}{r^2} - \frac{\alpha^3}{16 \kappa^2 (\alpha - \Lambda \eta)^2} \frac{q^4}{r^2} + \frac{3 \alpha + \Lambda \eta}{\alpha - \Lambda \eta},$$

$$G(r) = \frac{1}{16} \frac{\alpha^2 (4 \kappa (\alpha - \eta \Lambda) r^4 + 8 \eta \kappa r^2 - \eta q^2)^2}{r^4 \kappa^2 (\alpha - \eta \Lambda)^2 (\alpha r^2 + \eta)^2 F(r)},$$

$$\psi^2(r) = -\frac{1}{32} \frac{\alpha^2 (4 \kappa (\alpha - \eta \Lambda) r^4 + \eta q^2)(4 \kappa (\alpha - \eta \Lambda) r^4 + 8 \eta \kappa r^2 - \eta q^2)^2}{r^6 \kappa^2 (\alpha - \eta \Lambda)^2 (\alpha r^2 + \eta)^3 F(r)},$$

$$A_0(r) = \frac{1}{4} q \sqrt{\alpha} \left(\frac{4 \beta \kappa^2 (\alpha + \eta \Lambda) + \alpha^2 q}{\alpha - \eta \Lambda}\right) \arctan \left(\frac{\sqrt{\alpha \eta}}{\eta} r\right) + \alpha \left(\frac{8 \eta \kappa + \alpha q}{4 \eta \kappa (\alpha - \eta \Lambda)}\right) \frac{q}{r} - \frac{\alpha}{12 \kappa (\alpha - \eta \Lambda)} \frac{q^3}{r}.$$

In order to analyze the features proper of a black hole in our solution we need to analyze the lapse function $F(r)$. As we approach the origin, the lapse function goes to minus infinity. On the other hand, as we go to infinity along coordinate $r$, $F(r)$ tends to plus infinity. Therefore, it is clear that this function being continuous has at least one zero. We can
prove that this function has more than one cero. Since we know the existence of at least one cero \( r_H \), we can parametrize the function with \( r_H \) as parameter. From the equation \( F(r_H) = 0 \) we get \( \mu \equiv \mu(r_H) \) which can be used to express the lapse function as \( F(r, \mu(r_H)) \).

To prove the existence of the second event horizon, we can do the same as before but with the electric charge. We propose the existence of \( r_h \), then \( F(r_h) = 0 \), and using this we get \( q^2 \equiv q^2(r_h, r_H) \). It is possible to find two roots for \( F(r_h) = 0 \) or in other words, two suitable values of \( q^2 \) for a possible \( r_h \). This values in some cases are both negatives, both positive or one positive and the second negative, but at least the existence of one positive root is enough to prove the existence of \( r_h \). As we said, due to the shape near the origin and at infinity of the lapse function, the existence of two zeros of the function implies the existence of a third zero for some range of parameters. Therefore \( F(r) \) can have just one zero, two zeros\(^4\) or three zeros. Each of these cases exist for a specific set of values of the coupling and cosmological constants. From hereafter and for simplicity, we will focus in the case when the lapse function has just one zero.

Reality condition of the lapse function requires \( \alpha \eta > 0 \). Therefore \( l^{-2} := \frac{\alpha}{3 \eta} \) is positive defined and the spacetime is asymptotically AdS. As it was noted in the uncharged case without loss of generality it is possible to choose both parameters positive, since the solution with both \( \alpha \) and \( \eta \) negative is equivalent to the former by changing \( \mu \rightarrow -\mu \).

In order to obtain a real scalar field in the domain of outer communications and satisfy the positivity of the energy, we need to impose some constraints in our parameters. In fact, the value of the cosmological constant is restricted to be

\[
\Lambda < -\frac{q^2}{4v_H^2} - \frac{\alpha}{\eta} .
\] (17)

It is important to note that we cannot switch off the scalar field. This implies that our solution is not continuously connected with the maximally symmetric background. Despite of this, setting \( \mu = 0 \) and \( q = 0 \) we observe that the spacetime is regular, actually is the only regular spacetime that can be found within this family. Such a case describes an asymptotically AdS gravitational soliton. Close to \( r = 0 \) and after a proper reescaling on the time coordinate the spacetime metric takes the following form

\[
ds_{\text{soliton}}^2 = -\left(1 - \frac{\Lambda}{3} r^2 + O(r^4)\right) dt^2 + \left(1 - \frac{3\alpha + 2\Lambda \eta}{3\eta} r^2 + O(r^4)\right) dr^2 + r^2 d\Omega^2.
\] (18)

\(^4\) This case is an special case in the sense that contains a zero which is a local minimum. When that local minimum is the outer horizon this corresponds to an extremal black hole.
The thermal version of this spacetime can be used as the background metric for obtain a regularized euclidean action which could be used to obtain the thermodynamical properties of the black holes in the Hawking-Page approach.

IV. N-DIMENSIONAL CASE

In this section we analyze the $n-$dimensional solution to the action principle defined by (4). For doing this, we take the variation of our Lagrangian with respect to all the functions involved $F(r), G(r), \phi(r)$ and $A_0(r)$. This procedure gives us the equations of motion of the system.

Therefore, following the same strategy than in four dimensions, the equation of motion for the scalar field admits a first integral. Setting to zero the integration constant of this equation we obtain a relation between the metric coefficients, but now in arbitrary dimension

$$G_n(r) = \eta(n-2) \frac{F_n(r) + F_n(r)(n-3)}{2r^2 + \eta K(n-2)(n-3)} . \quad (19)$$

The equation coming from the variation with respect to the electric field gives us the following relation

$$\left(A_n(r)\right)^2 = q^2 F_n(r) G_n(r)r^{(4-2n)} .$$

In the same spirit, and using the last result, it is possible to obtain a relation for $\psi(r)^2$. Then

$$\psi_n(r)^2 = -\frac{1}{2}(n-2) \left(\frac{\Xi_1^2 + \Xi_2^2}{\Xi_3^2}\right) ,$$

where we have defined

$$\Xi_1^2 = (n-3)^2(4\kappa \Lambda \eta r^2 + 4\kappa r^2 \alpha + q^2 r^{(-2n+6)} \eta) F_n(r)^2$$

$$+ 2(n-3)(q^2 r^{(-2n+7)} \eta + 4\kappa \Lambda \eta r^3 + 4\alpha r^3 \kappa) F'_n(r) F_n(r) ,$$

$$\Xi_2^2 = (4\kappa \Lambda \eta r^4 + q^2 r^{(-2n+8)} \eta + 4\alpha r^4 \kappa) F'_n(r)^2 ,$$

$$\Xi_3^3 = F_n(r)(2r^2 \alpha \eta K n^2 - 5\eta K n + 6\eta K)^2 ((n-3) F_n(r) + F'_n(r)r) .$$

Using these expressions and the equation resulting from the variation with respect to the function $F_n(r)$, we can obtain a relation which allows to obtain the explicit form of $F_n(r)$ for an arbitrary value of the dimension $n$, and in this way, the complete solution to our system. We checked the result from $n = 4$ to $n = 10$. 

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V. ASYMPTOTICALLY LOCALLY FLAT BLACK HOLES WITH CHARGE SUPPORTED BY THE EINSTEIN-KINETIC COUPLING

In this section we will study the particular case where the scalar field is coupled to the background only with the Einstein tensor. It is possible to do this by setting $\alpha = 0$. Under the presence of an electric field, we obtain asymptotically locally flat black hole solutions in the case where the cosmological constant is present. Therefore, the action principle is given by

$$I[g_{\mu\nu}, \phi] = \int \sqrt{-g} d^4x \left[ \kappa (R - 2\Lambda) + \frac{\eta}{2} G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right].$$ \hspace{1cm} (20)

Following the same procedure (with $\alpha \neq 0$ and $K = 1$)\footnote{In the case where $K = 0$, the system integrate in a different manner. In fact, $\Lambda$ and $q$ have to vanish in order to fulfil the field equations. Then, we obtain the same degenerated system found in [16].} we obtain

$$ds^2 = -F(r) dt^2 + \frac{15[4\kappa r^2(2 - \Lambda r^2) - q^2]^2}{r^4} \frac{dr^2}{F(r)} + r^2 d\Omega^2,$$ \hspace{1cm} (21)

where

$$F(r) = 48\kappa^2 \Lambda^2 r^4 - 320\kappa^2 \Lambda r^2 + 120\kappa(8\kappa + \Lambda q^2) - \frac{\mu}{r} + 240\kappa \frac{q^2}{r^2} - 5\frac{q^4}{r^4},$$

$$\psi(r)^2 = -\frac{15(4\kappa \Lambda r^4 + q^2)(4\kappa r^2(2 - \Lambda r^2) - q^2)^2}{r^6 \eta} - \frac{1}{F(r)},$$

$$A_0(r) = \sqrt{15} \left( \frac{q^3}{3r^3} - 8\kappa \frac{q}{r} - 4\kappa \Lambda r q \right).$$

This solution shows the following features:

- The solution is asymptotically locally flat, namely we have

$$\lim_{r \to \infty} R^{\mu\nu}_{\lambda\rho} \to 0.$$

- For a non degenerated horizon $r = r_H$ we have $F(r_H) = 0$, then the scalar field vanishes at the horizon and is not analytic there.

- In order to obtain a real scalar field outside of the horizon we can impose two different conditions:

  1. $\Lambda > 0$ and $\eta < 0$ or
  2. $\Lambda < -\frac{q^2}{4\kappa r_H^4}$ and $\eta > 0$. 
• For any value of the integration constant $\mu$ we have the curvature singularities

$$r_0 = 0,$$

$$r_{1,2} = \sqrt{\frac{2\kappa\Lambda(2\kappa \pm \sqrt{4\kappa^2 - \kappa\Lambda q^2})}{2\kappa\Lambda}}.$$

Then for $\Lambda < 0$ the only singularity is located at the origin of coordinates. If the cosmological constant is positive, in order to rule out the existence of singularities different than $r = 0$, we need to impose the following constraint in the value of $\Lambda$

$$\Lambda > \frac{4\kappa}{q^2}. \quad (22)$$

• We point out that in the limit $r \to \infty$ our electric potential represents a constant electric field at that point supported by the cosmological constant, and in this way we obtain an asymptotically electric Universe.

• Finally the limit $q \to 0$ we recover the results obtained in [16].

Let us put $\Lambda = 0$, then the solution takes the form

$$ds^2 = -F(r)dt^2 + \frac{3(8\kappa r^2 - q^2)^2}{r^4} \frac{dr^2}{F(r)} + r^2d\Omega^2,$$

where

$$F(r) = 192\kappa^2 - \frac{\mu}{r} + 48\kappa q^2 r^2 - \frac{q^4}{r^4},$$

$$\psi(r)^2 = -\frac{15}{2} \frac{(8\kappa r^2 - q^2)^2}{r^6} \frac{q^2}{F(r)},$$

$$A_0(r) = \sqrt{15} \left( \frac{q^3}{3r^3} - 8\kappa \frac{q}{r} \right).$$

In this case we have:

• The solution is asymptotically flat

$$ds^2 = - \left( 1 - \frac{\mu}{r} + O(r^{-2}) \right) dt^2 + \left( 1 + \frac{\mu}{r} + O(r^{-2}) \right) dr^2 + r^2d\Omega^2,$$

which is reasonable because when we have $\Lambda = 0$, the electric field at infinity vanishes.

• For a non degenerated horizon $r = r_H$ we have $F(r_H) = 0$, then the scalar field vanish at the horizon, as in the previous cases, is not analytic there.
• In order to obtain a real scalar field outside of the horizon we impose

\[ \eta < 0 . \]

• For any value of the integration constant \( \mu \) we have the curvature singularities

\[ r_0 = 0 , \]
\[ r_1 = \sqrt{\frac{1}{8\kappa}} |q| . \]

• The electric field goes to zero at infinity.

• Taking the limit when \( q \to 0 \) we obtain a trivial scalar field and then we recover the Schwarzschild solution.

VI. DISCUSSION

In this work a particular sector of the Horndeski theory was considered where the gravity part is given by the Einstein-Hilbert term, and where the matter source is represented by a scalar field which has a non minimal kinetic coupling constructed with the Einstein tensor. The main novelty of this work is the inclusion of the Maxwell field. We found exact solutions to this system for a spherically symmetric and topological horizons in all dimensions. The solution gives a new class of asymptotically locally AdS and asymptotically locally flat black hole solutions.

These solutions are obtained using two important observations. The first one, is the fact that the equation of motion for the scalar field admits a first integral, which after setting the integration constant to zero (arbitrarily) gives a simple relation between the two metric functions. The second one, is that the Maxwell equations are easily integrated for our ansatz and symmetry conditions, given a simple relation between the electric potential term and the metric functions. Mixing these two results we obtain a complete description of the system, obtaining in that way the exact solution for the topological case in \( n \geq 4 \) dimensions.

We observe and point out that in the case of the asymptotically locally AdS solution, the cosmological constant at infinity is not given by the cosmological \( \Lambda \) term in the action but rather in terms of the coupling constants \( \alpha \) and \( \eta \) that appear in the kinetic coefficients of the field. The electric field is well behaved and goes to the Coulombian one at infinity.
The solutions are not continuously connected with the maximally symmetric AdS or flat backgrounds since the scalar field cannot be turned off. Nevertheless, since our family of metrics contains a further integration constant, it is possible to show that within such a family there is a unique regular spacetime. Such spacetime is a gravitational soliton and it is useful in the four dimensional spherically symmetric case to define a regularized Euclidean action and to explore the thermodynamics of the black hole solution. A similar situation occurs with the AdS soliton, which can be considered as the background for the planar AdS black holes, as well as in gravity in 2+1 with scalar fields, where the gravitational solitons are the right backgrounds to give a microscopic description of the black hole entropies [22][23][24].

In the particular case when the scalar field is only coupled to the metric through the Einstein tensor, namely, $\alpha = 0$ we obtain an asymptotically locally flat black hole solution. When $\Lambda \neq 0$ this solution presents some interesting properties. The solution exist in both cases, where the cosmological constant is positive and when is negative, given a real scalar field configuration depending on constraints imposed on the electric charge and on the coupling constant $\eta$. In any of these cases we obtain a constant electric field at infinity, representing in this way our solution a electric Universe. This constant electric field at infinity is just supported by the cosmological constant.

In the case where $\Lambda = 0$ we obtain a real scalar configuration just in case where the coupling constant is negative. The solution is asymptotically flat and the electric field vanishes at infinity when $\Lambda = 0$. If we switch off the electric field setting $q = 0$, we get a trivial scalar field and then we recover the Schwarzschild solution.

It is important to note that Horndeski theory offers the possibility of exploring its solutions in many different ways. In another context, using the same action principle, but without the Maxwell term an asymptotically Lifshitz solution was recently found in [25]. Moreover, even if it is not possible to obtain an analytic solution to the most general case of the Horndeski theory for the general static black hole solution, it would be interesting to study the cases where the non minimal coupling is given by more general tensors than the Einstein one, namely the Lovelock tensors.
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