PROPAGATING WEIGHTS OF TORI ALONG FREE RESOLUTIONS

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Abstract. The action of a torus on a graded module over a polynomial ring extends to the entire minimal free resolution of the module. We explain how to determine the action of the torus on the free modules in the resolution, when the resolution can be calculated explicitly. The problem is reduced to analyzing how the weights of a torus propagate along an equivariant map of free modules. The results obtained are used to design algorithms which have been implemented in the software system Macaulay2.

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1. Introduction

Every finitely generated module over a polynomial ring with coefficients in a field has a finite minimal free resolution which is unique up to isomorphism. It is typically used to produce numerical invariants such as projective dimension, regularity, (graded) Betti numbers and the Hilbert series of a module. While there

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are descriptions for certain classes of modules, finding the minimal free resolution of a module is, in general, a very difficult problem. Computational methods offer a solution to this problem in many cases, although they are limited in scope by time and memory constraints. As the matrices of the differentials grow in size, their description is often omitted.

Consider the case of a polynomial ring $A$ endowed with an action of a group $G$ which is compatible with grading and multiplication (see §2.3 for the precise definitions). Let us denote $\text{mod}_{\leq G} A$ the category of finitely generated graded $A$-modules with a compatible action of $G$ and homogeneous $G$-equivariant maps. If $M$ is an object in $\text{mod}_{\leq G} A$, then the action of $G$ extends to the entire minimal free resolution of $M$. A free $A$-module $F$ is isomorphic to $p^n F_{n} A_{n} F_{n-1} A_{n-1} \cdots F_1 A_1 F_{0}$, where $m$ denotes the maximal ideal generated by the variables of $A$. The representation theoretic structure of $F$, i.e. the action of $G$ on $F$, is then controlled by the representation $F/mF_i$. Therefore, if the complex $F_i$:

\[
0 \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_{0}
\]

denotes the minimal free resolution of $M$, we could try to determine the action of $G$ on each representation $F_i/mF_i$.

The representation theoretic structure of $F_i$ may offer some insight into the maps of the complex. Consider the situation of a differential $d_i : F_i \to F_{i-1}$, with $F_i/mF_i$ an irreducible representation of $G$. The map $d_i$ is completely determined by its image on a basis of $F_i$; hence we can reduce to a map of representations $F_i/mF_i \to F_{i-1}/mF_{i-1} \otimes A$. If the decomposition of the tensor product in the codomain contains only one copy of the irreducible $F_i/mF_i$ in the right degree, then Schur’s lemma [Lan02, Ch. XVII, Prop. 1.1] implies that the map is uniquely determined up to multiplication by a constant. In some cases, this information is enough to reconstruct the map completely (see [Gal13a] for a few examples).

The representation theoretic structure of $F_i$ may also be used to determine the class $[M]$, of a module $M$, in $K^G_0(\text{mod}_{\leq G} A)$, the equivariant Grothendieck group of the category $\text{mod}_{\leq G} A$. By construction of the Grothendieck group,

\[
[M] = \sum_{i=0}^{n} (-1)^i [F_i/mF_i],
\]

where the right hand side is the equivariant Euler characteristic of the complex $F_i$.

Motivated by the discussion above, we pose the following question: is it possible to determine the action of a group on the minimal free resolution of a module computationally? The first assumption is that the resolution itself can be computed explicitly in a reasonable amount of time. Secondly, we restrict to a class of groups whose representation theory is well understood and manageable: tori. Every representation of a torus is semisimple, with irreducible representations being one dimensional and parametrized by weights. Moreover, weights can be conveniently represented by integer vectors. More importantly, finite dimensional representations of connected reductive algebraic groups over algebraically closed fields of characteristic zero are uniquely determined by the weights (with multiplicity) of a maximal torus. Therefore successfully developing the case of tori will provide a positive answer to our question for a larger class of groups.
Let \( \varphi : E \rightarrow F \) denote a minimal presentation of a module \( M \). In our experience, the presentation of a module with an action of a reductive group can often be written with respect to bases of weight vectors. Suppose \( \{ \tilde{e}_1, \ldots, \tilde{e}_r \} \) and \( \{ \tilde{f}_1, \ldots, \tilde{f}_s \} \) are bases of weight vectors of \( E \) and \( F \) respectively. For every \( \tilde{e}_j \), there exist polynomials \( a_{i,j} \in A \) such that \( \varphi(\tilde{e}_j) = \sum_{i=1}^{s} a_{i,j} \tilde{f}_i \); moreover, each \( a_{i,j} \) is a weight vector in \( A \) and

\[
\text{weight}(\tilde{e}_j) = \text{weight}(a_{i,j}) + \text{weight}(\tilde{f}_i),
\]

whenever \( a_{i,j} \) is non zero (prop. 3.1.1). If the variables in \( A \) are all weight vectors, then the weights of the \( a_{i,j} \) can be easily obtained (prop. 2.3.6). Then, knowing the weights of the \( \tilde{f}_i \), it is possible to recover the weights of the \( \tilde{e}_j \) thus describing \( E \) as a representation. A minimal presentation is the first differential in a minimal resolution. The weights of the other free modules in the resolution can be found by iterating the process just described, with the only catch that the remaining differentials, as obtained computationally, may not be expressed using bases of weight vectors. This situation is handled by changing to a suitable basis as we indicate in our main result (thm. 3.2.2).

This paper is structured as follows. Section 2 introduces some basic concepts of commutative algebra and the representation theory of tori, and proceeds to describe their natural interactions. In section 3, we analyze how weights of tori propagate along equivariant maps of free modules, first in the easier case of bases of weight vectors (§3.1) and then in a more general setting (§3.2). Our last section is devoted to the design of various algorithms: to propagate weights along an equivariant map from codomain to domain (§4.1), to propagate weights ‘forward’ from domain to codomain (§4.2), for resolutions (§4.3), and, as a bonus, an algorithm to determine the weights of graded components of modules (§4.4). Finally, in §4.5, we discuss the possibility of carrying out all such computations over subfields.

An implementation of the algorithms of this paper for semisimple complex algebraic groups is available for the software system Macaulay2 [GS] and documented in [Gal13b]. The author wishes to thank Jerzy Weyman, for suggesting the project, Claudiu Raicu, for an interesting conversation on the subject, and the entire Macaulay2 community. The author was partially supported through an NSERC grant.

2. Basic notions and notations

We begin by recalling some concepts from commutative algebra (§2.1) and representation theory (§2.2). In sections 2.3 and 2.4, we illustrate some interactions and establish a few basic facts that will be used throughout this work.

2.1. Commutative algebra. Our main source for (computational) commutative algebra are [KR00, KR05]. Most of our notations are lifted from those volumes, with some minor additions that we introduce below, and almost all facts we cite in commutative algebra can be found there.

Let \( K \) be a field. Fix a polynomial ring \( A = \mathbb{K}[x_1, \ldots, x_n] \) with a positive \( \mathbb{Z}^m \)-grading in the sense of [KR05, Defin. 4.2.4]. While we do not need the explicit definition of positive grading, we recall one important consequence of this definition; namely, for every finitely generated graded \( A \)-module \( M \) and \( \forall d \in \mathbb{Z}^m \), the graded component \( M_d \) is a finite dimensional \( \mathbb{K} \)-vector space. All \( A \)-modules considered are finitely generated and graded.
Every free $A$-module $F$ has a basis $\mathcal{F} = \{f_1, \ldots, f_s\}$ consisting (necessarily) of homogeneous elements. In order to distinguish the set $\mathcal{F}$ from the basis of a $\mathbb{K}$-vector subspace of $F$, we refer to it as a ‘homogeneous basis’ or ‘$A$-basis’. Kreuzer and Robbiano denote $\mathbb{T}^n(f_1, \ldots, f_s)$ the set of all terms $tf_i$ in $F$, where $t \in \mathbb{T}^n$ is a term in $A$; we use $\mathbb{T}^n(\mathcal{F})$ to denote the same set. At times, we may need to use more than one homogeneous basis of the same free module $F$; because of this, we denote the support of an element $f \in F$ by $\text{Supp}_F(f)$, with an explicit dependence on the homogeneous basis used.

**Definition 2.1.1.** Let $\sigma$ denote a term ordering on $\mathbb{T}^n$. Let $F$ be a free $A$-module and $\mathcal{F} = \{f_1, \ldots, f_s\}$ a homogeneous basis of $F$. We define some module term orderings on $\mathbb{T}^n(\mathcal{F})$ as follows. For $t_1, t_2 \in \mathbb{T}^n$ and $i, j \in \{1, \ldots, s\}$,

- **term over position up:** $t_1 e_i \succeq t_2 e_j \iff t_1 >_\sigma t_2$ or $t_1 = t_2$ and $i > j$;
- **position up over term:** $t_1 e_i \succeq t_2 e_j \iff i > j$ or $i = j$ and $t_1 >_\sigma t_2$;
- **term over position down:** $t_1 e_i \succeq t_2 e_j \iff t_1 >_\sigma t_2$ or $t_1 = t_2$ and $i < j$;
- **position down over term:** $t_1 e_i \succeq t_2 e_j \iff i < j$ or $i = j$ and $t_1 >_\sigma t_2$.

We refer to the first two orderings together as *position up* module term orderings, and to the last two as *position down* module term orderings.

Any term ordering on $\mathbb{T}^n$ is allowed in what follows. The term ordering will be fixed and so we will drop all references to it. As for module term orderings, we only allow position up/down orderings.

In some parts of this work, we will be using the graded hom functor on the category of graded $A$-modules. The definition can be found in [BH93, p. 33] and the notation is $^\ast\text{Hom}(-,-)$.

### 2.2. Representation theory of tori.

This section contains a brief summary of the representation theory of tori. Our main reference on the subject is [Hum75, §11.4, Ch. 16]. A few facts are presented in the form of propositions so that they can be referenced later. The proofs are standard and typically present in textbooks on the subject, therefore they will be omitted.

An algebraic torus over $\mathbb{K}$ is an algebraic group $T$ which is isomorphic to $\mathbb{K}^\times \times \cdots \times \mathbb{K}^\times$, a finite direct product of copies of the multiplicative group of the field $\mathbb{K}$. The character group of $T$, denoted $X(T)$, is the set of all algebraic group homomorphisms $\chi : T \rightarrow \mathbb{K}^\times$. The set $X(T)$ is an abelian group under point wise multiplication.

A representation of $T$ over $\mathbb{K}$ is a vector space with a $\mathbb{K}$-linear action of $T$. For every finite dimensional representation $V$ of $T$ over $\mathbb{K}$, there exists a unique decomposition $V = \bigoplus_{\chi \in X(T)} V_\chi$, where $V_\chi = \{v \in V \mid \forall \tau \in T, \tau \cdot v = \chi(\tau)v\}$. A character $\chi \in X(T)$ such that $V_\chi \neq 0$ is called a weight of $V$ and $\dim(V_\chi)$ is called the multiplicity of $\chi$ in $V$. The list of weights of $V$, considered with their multiplicity, uniquely determines $V$ as a representation of $T$. Each $V_\chi$ is called a weight space of $V$ and its non zero elements are called weight vectors with weight $\chi$.

If $T \cong (\mathbb{K}^\times)^n$, then $X(T) \cong \mathbb{Z}^n$ as abelian groups. This is important from a computational standpoint because characters can be represented by elements of $\mathbb{Z}^n$, i.e. by lists of integers. If $v \in V$ is a weight vector, we write $\text{weight}(v)$ to identify the weight of $v$ as an element of $\mathbb{Z}^n$. When operating with weights as elements of $\mathbb{Z}^n$, we use an additive notation for the group operation.
If $V, W$ are two finite dimensional representations of $T$ over $\mathbb{K}$, then so is the tensor product $V \otimes_{\mathbb{K}} W$ with the action $\tau \cdot (v \otimes w) = (\tau \cdot v) \otimes (\tau \cdot w)$, $\forall v \in V$, $\forall w \in W$, $\forall \tau \in T$.

**Proposition 2.2.1.** If $v \in V$ and $w \in W$ are weight vectors, then so is $v \otimes w \in V \otimes_{\mathbb{K}} W$ and weight$(v \otimes w) = \text{weight}(v) + \text{weight}(w)$.

**Definition 2.2.2.** Let $V, W$ be two representations of $T$ over $\mathbb{K}$. A $\mathbb{K}$-linear map $\varphi : V \rightarrow W$ is $T$-equivariant, or a map of representations of $T$, if $\forall \tau \in T$, $\forall v \in V$ we have $\varphi(\tau \cdot v) = \tau \cdot \varphi(v)$.

**Proposition 2.2.3.** Let $\varphi : V \rightarrow W$ be $T$-equivariant. If $v \in V$ is a weight vector and $\varphi(v) \neq 0$, then $\varphi(v)$ is a weight vector in $W$ and weight$(\varphi(v)) = \text{weight}(v)$.

If $V$ is a representation of $T$, then the dual (or contragredient) representation is the vector space $V^* := \text{Hom}_\mathbb{K}(V, \mathbb{K})$. An element $\tau \in T$ acts on $V^*$ by $[\tau \cdot f](v) = f(\tau^{-1} \cdot v)$, $\forall f \in V^*$, $\forall v \in V$.

**Proposition 2.2.4.** Let $V$ be a finite dimensional representation of $T$ with a basis of weight vectors $\{v_1, \ldots, v_r\}$. Then the dual basis $\{v_1^*, \ldots, v_r^*\}$ of $V^*$ is a basis of weight vectors and $\text{weight}(v_i^*) = -\text{weight}(v_i)$, $\forall i \in \{1, \ldots, r\}$.

2.3. **Compatible actions of tori on rings and modules.** Let $M$ be an $A$-module.

**Definition 2.3.1.** A $\mathbb{K}$-linear action of $T$ on $M$ is compatible with grading if $\forall \tau \in T$, $\forall d \in \mathbb{Z}^m$ we have $\tau \cdot M_d \subseteq M_d$.

**Definition 2.3.2.** A $\mathbb{K}$-linear action of $T$ on $M$ is compatible with multiplication if $\forall d, d' \in \mathbb{Z}^m$ the map induced by multiplication on $A_d \otimes_{\mathbb{K}} M_{d'} \rightarrow M_{d+d'}$ is $T$-equivariant. In other words $\forall \tau \in T$, $\forall a \in A_d$, $\forall m \in M_{d'}$ we have $\tau \cdot (am) = (\tau \cdot a) (\tau \cdot m)$.

All actions we consider are compatible with grading and multiplication. Indeed we can introduce the category $\text{mod}_{\otimes T} A$ with

- objects: all finitely generated graded $A$-modules with a $\mathbb{K}$-linear action of $T$ compatible with grading and multiplication;
- maps: all homogeneous $A$-linear $T$-equivariant maps,
and work in this category unless otherwise stated.

**Remark 2.3.3.** For any group $G$ with a $\mathbb{K}$-linear action on $A$ that is compatible with grading and multiplication, we could similarly define a category $\text{mod}_{\otimes G} A$.

A homogeneous polynomial $p \in A$ of degree $d \in \mathbb{Z}^m$ is said to be a weight vector if it is a weight vector in $A_d$. We always assume that all variables in $A$ are weight vectors, which can always be obtained up to a linear change of variables in $A$.

**Proposition 2.3.4.** If each $x_i \in A$ is a weight vector, then every term $t \in \mathbb{T}^n$ is a weight vector of $A$. Moreover, if $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, then

\[
\text{weight}(t) = \sum_{i=1}^{n} \alpha_i \text{weight}(x_i).
\]
Proof. Suppose the variable $x_i$ has degree $d_i \in \mathbb{Z}^n$. The degree of $t$ is $d = \sum_{i=1}^{n} \alpha_i d_i$. The map induced by multiplication on

$$A_{d_1}^{\otimes \alpha_1} \otimes \ldots \otimes A_{d_n}^{\otimes \alpha_n} \to A_d$$

is $T$-equivariant because the action of $T$ on $A$ is compatible with multiplication. By proposition 2.2.3, $t$ is a weight vector in $A_d$ because it is the image of a tensor product of weight vectors under the map above. Then the formula for weight($t$) follows from propositions 2.2.3 and 2.2.1.

The previous proposition implies that every graded component $A_d$ of $A$ has a basis of weight vectors consisting of all terms of degree $d$, i.e. the elements of the set $T^n \cap A_d$. Moreover, if $\chi$ is a weight of $A_d$, then the weight space $(A_d)_\chi$ has a basis of weight vectors consisting of all terms of degree $d$ and weight $\chi$, i.e. the elements of the set $T^n \cap (A_d)_\chi$.

Example 2.3.5. We set up an example that will be gradually explored as we proceed through our work.

Let $V = \mathbb{C}^3$ and $G = \text{GL}(V)$. Let $A = \text{Sym}(V)$, the symmetric algebra over $V$; $A$ is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra with graded components $A_d = \text{Sym}^d(V)$ given by the symmetric powers of $V$. The group $G$ has a natural $\mathbb{C}$-linear action on each component $\text{Sym}^d(V)$ which is compatible with multiplication.

By choosing a basis $\{v_1, v_2, v_3\}$ of $V$, we can identify $G$ with $\text{GL}_3(\mathbb{C})$. If $T$ denotes the maximal torus of $G$ corresponding to diagonal matrices in $\text{GL}_3(\mathbb{C})$, then the elements $v_1, v_2, v_3$ form a basis of weight vectors of $V$. Denoting $x_1, x_2, x_3$ the elements $v_1, v_2, v_3 \in \text{Sym}^1(V) = V$, we can also identify $A$ with the standard graded polynomial ring $\mathbb{C}[x_1, x_2, x_3]$. For $g \in G$ and $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ a term in $A$, the action of $G$ is determined by

$$g \cdot x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = (g \cdot x_1)^{\alpha_1} (g \cdot x_2)^{\alpha_2} (g \cdot x_3)^{\alpha_3}.$$

Moreover, the term $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ is a weight vector with weight $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$. In particular, the variables $x_1, x_2, x_3$ have weight $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Proposition 2.3.6. If $p \in A_d$ is a weight vector, then $\forall t \in \text{Supp}(p)$ we have $\text{weight}(t) = \text{weight}(p)$.

Proof. Using the basis of terms of $A_d$, we can write

$$p = \sum_{t \in T^n \cap A_d} c_t t.$$

Suppose $p$ has weight $\chi$, so that $p \in (A_d)_{\chi}$. If $t \in \text{Supp}(p)$, then $c_t \neq 0$; this forces $t \in (A_d)_{\chi}$, otherwise we would have $p \notin (A_d)_{\chi}$.

Let $M$ be an object in mod$_{\mathbb{C}^T}$ $A$. A homogeneous element $m \in M$ of degree $d$ is a weight vector if it is a weight vector in $M_d$. Let $m$ be the maximal ideal of $A$ generated by the variables $x_1, \ldots, x_n$. The quotient $M/mM$ is a finite dimensional graded $\mathbb{K}$-vector space with the action $\tau \cdot (m + mM) = (\tau \cdot m) + mM$, $\forall m \in M$, $\forall \tau \in T$. Moreover, it is an object in mod$_{\mathbb{C}^T}$ $A$. Observe that if $m$ is a weight vector in $M$ and $m + mM$ is non zero, then $m + mM$ is a weight vector in $M/mM$.

Example 2.3.7. In the setting of example 2.3.5, the ideal $m = (x_1, x_2, x_3)$ and the quotient $A/m$ are examples of modules in mod$_{\mathbb{C}^T}$ $A$. From the point of view of representation theory, $m$ is generated by a copy of $V$ in degree 1, whereas $A/m$ is the trivial representation of $G$ in degree 0.
If $M, N$ are objects in $\text{mod}_{\mathcal{T}} A$, there is a natural action of $\mathcal{T}$ on $\text{Hom}_{A}(M, N)$ given by $[\tau \cdot \psi](m) := \psi(\tau^{-1} \cdot m), \forall \psi \in \text{Hom}_{A}(M, N), \forall m \in M$. This action restricts to $\text{Hom}_{A}(M, N)$ as we will illustrate in the next result.

**Proposition 2.3.8.** Let $N$ be an object of $\text{mod}_{\mathcal{T}} A$. The restriction of the functor $\text{Hom}_{A}(-, N)$ to $\text{mod}_{\mathcal{T}} A$ is an endofunctor on $\text{mod}_{\mathcal{T}} A$.

**Proof.** We must show that applying $\text{Hom}_{A}(-, N)$ to an object or morphism in $\text{mod}_{\mathcal{T}} A$ will land us again in $\text{mod}_{\mathcal{T}} A$. We do this in three steps. Let $M, M_1, M_2, N$ be objects in $\text{mod}_{\mathcal{T}} A$ and $\varphi : M_1 \rightarrow M_2$ a morphism in $\text{mod}_{\mathcal{T}} A$. Recall that the graded component of $\text{Hom}_{A}(M, N)$ of degree $d \in \mathbb{Z}^m$ is $\text{Hom}_{d}(M, N) = \{ \psi \in \text{Hom}_{A}(M, N) | \forall d' \in \mathbb{Z}^m, \psi(M_{d'}) \subseteq M_{d+d'} \}$.

**Claim 1:** The action of $\mathcal{T}$ on $\text{Hom}_{A}(M, N)$ restricts to $\text{Hom}_{d}(M, N)$, for all $d \in \mathbb{Z}^m$, i.e. the action of $\mathcal{T}$ on $\text{Hom}_{A}(M, N)$ is compatible with the grading.

For all $d' \in \mathbb{Z}^m$, $\tau \in \mathcal{T}$, and $\psi \in \text{Hom}_{d}(M, N)$, we have that $[\tau \cdot \psi](M_{d'}) = \psi(\tau^{-1} \cdot M_{d'}) \subseteq \psi(M_{d'}) \subseteq M_{d+d'}$ because the action of $\mathcal{T}$ on $M$ is compatible with the grading. This shows $\tau \cdot \psi \in \text{Hom}_{d}(M, N)$.

**Claim 2:** The action of $\mathcal{T}$ on $\text{Hom}_{A}(M, N)$ is compatible with multiplication. Let $d, d' \in \mathbb{Z}^m$ be arbitrary degrees. For all $\tau \in \mathcal{T}, a \in A_d$, $\psi \in \text{Hom}_{d'}(M, N)$ and $m \in M$, we get $[\tau \cdot (a \psi)](m) = a \psi(\tau^{-1} \cdot m) = \psi(a(\tau^{-1} \cdot m)) = \psi(\tau^{-1} \cdot ((\tau \cdot a)m)) = [\tau \cdot \psi][(\tau \cdot a)m] = (\tau \cdot a)[\tau \cdot \psi](m) = [(\tau \cdot a)(\tau \cdot \psi)](m)$ because the action of $\mathcal{T}$ on $M$ is compatible with multiplication. This shows that $\tau \cdot (a \psi) = (\tau \cdot a)(\tau \cdot \psi)$.

**Claim 3:** The map $\text{Hom}_{A}(\varphi, N) : \text{Hom}_{A}(M_2, N) \rightarrow \text{Hom}_{A}(M_1, N)$ is $\mathcal{T}$-equivariant.

For simplicity denote $\text{Hom}_{A}(\varphi, N)$ by $\varphi^*$. For all $\tau \in T, \psi \in \text{Hom}_{A}(M_2, N)$ and $m \in M_1$, we obtain $[(\varphi^* \cdot \psi)](m) = (\tau \cdot \psi)(\varphi(m)) = \psi(\tau^{-1} \cdot \varphi(m)) = \psi(\varphi(\tau^{-1} \cdot m)) = [(\varphi^* \cdot \psi)](\tau^{-1} \cdot m) = [\varphi^* \cdot (\tau \cdot \psi)](m)$ because $\varphi$ is $\mathcal{T}$-equivariant. This shows $\varphi^* \cdot (\tau \cdot \psi) = \tau \cdot \varphi^* \cdot (\psi)$.

We are interested in applying the functor $\text{Hom}_{A}(-, N)$ when $N = A$.

**Definition 2.3.9.** For an object $M$ in $\text{mod}_{\mathcal{T}} A$, we set $M^\vee := \text{Hom}_{A}(M, A)$ and call it the dual of $M$ in $\text{mod}_{\mathcal{T}} A$. For a morphism $\varphi : M_1 \rightarrow M_2$ in $\text{mod}_{\mathcal{T}} A$, we set $\varphi^\vee := \text{Hom}_{A}(\varphi, A)$ and call it the dual of $\varphi$ in $\text{mod}_{\mathcal{T}} A$.

2.4. **Free modules.** In the category of finitely generated $\mathbb{Z}^m$-graded $A$-modules, every free module has the form $\bigoplus_{d \in \mathbb{Z}^m} A(-d)^{\beta_d}$, where $\beta_d \in \mathbb{N}$ for all $d \in \mathbb{Z}^m$, and only finitely many $\beta_d$ are non-zero. In $\text{mod}_{\mathcal{T}} A$, for the same choice of natural numbers $\beta_d$, there may be more than one isomorphism class of free modules depending on how $\mathcal{T}$ acts on a homogeneous basis.
Let $V$ be a finite dimensional $\mathbb{Z}^m$-graded representation of $T$, i.e. a finite dimensional $\mathbb{Z}^m$-graded $\mathbb{K}$-vector space with a $\mathbb{K}$-linear action of $T$ that is compatible with the grading. The $\mathbb{K}$-vector space $V \otimes_\mathbb{K} A$ is naturally graded by

$$(V \otimes_\mathbb{K} A)_d := \bigoplus_{d' + d'' = d} V_{d'} \otimes_\mathbb{K} A_{d''}.$$  

Then $V \otimes_\mathbb{K} A$ becomes a graded $A$-module with multiplication given by $a(v \otimes b) := v \otimes (ab)$, $\forall a, b \in A$, $\forall v \in V$. The usual action of $T$ on the tensor product $V \otimes_\mathbb{K} A$ is compatible with grading and multiplication so $V \otimes_\mathbb{K} A$ is an object in mod$_{\mathbb{K}T} A$.

**Example 2.4.1.** Using the setup of example 2.3.5, for each integer $i \in \mathbb{Z}$ we have a different one dimensional free module in mod$_{\mathbb{K}T} A$ that is generated in degree $d \in \mathbb{Z}$. These modules are given by $(\bigwedge^3 V)^{\otimes i} \otimes_\mathbb{C} A(-d)$, if $i \geq 0$, and by $(\bigwedge^3 V^*)^{\otimes -i} \otimes_\mathbb{C} A(-d)$, if $i < 0$.

**Definition 2.4.2.** A free module in mod$_{\mathbb{K}T} A$ is an object which is isomorphic to $V \otimes_\mathbb{K} A$ for some finite dimensional graded representation $V$ of $T$.

The rank of $V \otimes_\mathbb{K} A$ is $\dim_{\mathbb{K}} V$. If $\dim_{\mathbb{K}} V_d = \beta_d$, then

$$V \otimes_\mathbb{K} A \cong \bigoplus_{d \in \mathbb{Z}^m} A(-d)^{\beta_d},$$

as graded $A$-modules.

**Proposition 2.4.3.** If $F$ is a free module in mod$_{\mathbb{K}T} A$, then $F \cong (F/mF) \otimes_\mathbb{K} A$.

**Proof.** By definition of free module, $\exists V$, finite dimensional graded representation of $T$, such that $F \cong V \otimes_\mathbb{K} A$. Notice that

$$F/mF \cong F \otimes_\mathbb{A} (A/m) \cong (V \otimes_\mathbb{K} A) \otimes_\mathbb{A} (A/m) \cong$$

$$\cong V \otimes_\mathbb{K} (A \otimes_\mathbb{A} (A/m)) \cong V \otimes_\mathbb{K} (A/m) \cong V \otimes_\mathbb{K} \mathbb{K} \cong V$$

where each step holds as isomorphism of graded representations of $T$. Therefore $F \cong (F/mF) \otimes_\mathbb{K} A$. \qed

By the previous proposition, $F_d$ is isomorphic to $\bigoplus_{d' + d'' = d} (F/mF)_{d'} \otimes_\mathbb{K} A_{d''}$ as a representation of $T$. Recall that $A_{d''}$ has a basis of weight vectors consisting of all terms of degree $d''$, so its weights can be recovered using proposition 2.3.4. The weights in a tensor product can be obtained via proposition 2.2.1. Thus the weights of $F_d$ can be described, for any degree $d \in \mathbb{Z}^m$, as long as the weights in $F/mF$ are known.

Let $F$ be a free module of rank $s$ in mod$_{\mathbb{K}T} A$. A homogeneous basis of weight vectors of $F$ is a homogeneous basis $\tilde{F} = \{\tilde{f}_1, \ldots, \tilde{f}_s\}$ of $F$ such that each $\tilde{f}_j$ is a weight vector of $F$. We adopt the convention of decorating homogeneous bases of weight vectors and their elements with a tilde. Notice that the residue classes $\tilde{f}_1 + mF, \ldots, \tilde{f}_s + mF$ form a basis of weight vectors of $F/mF$ and weight($\tilde{f}_j + mF$) = weight($\tilde{f}_j$).

As for graded $A$-modules, we can introduce a graded Hom functor $^\ast\text{Hom}_\mathbb{K}(-, -)$ in the category of graded $\mathbb{K}$-vector spaces (or graded representations of $T$). The dual of a graded $\mathbb{K}$-vector space (or representation of $T$) $V$ is $V^* := ^\ast\text{Hom}_\mathbb{K}(V, \mathbb{K})$. This construction allows us to identify the dual of a free object in mod$_{\mathbb{K}T} A$.

**Remark 2.4.4.** Observe that for every graded $\mathbb{K}$-vector space $V$ and degree $d \in \mathbb{Z}^m$, $(V^*)_d = \text{Hom}_\mathbb{K}(V_{-d}, \mathbb{K})$. 

Proposition 2.4.5. If $F$ is a free module in $\text{mod}_{\mathbb{C}T} A$, then $F^* \cong (F/mF)^* \otimes_{\mathbb{K}} A$.

Proof. The thesis follows from the following chain of isomorphisms in $\text{mod}_{\mathbb{C}T} A$:

$$ F^* \cong *\text{Hom}_{A}(F, A) \cong *\text{Hom}_{A}((F/mF) \otimes_{\mathbb{K}} A, A) \cong *\text{Hom}_{\mathbb{K}}(F/mF, A) \cong (F/mF)^* \otimes_{\mathbb{K}} A. $$

Remark 2.4.6. The proofs of proposition 2.4.3 and 2.4.5 use standard isomorphisms, like associativity of the tensor product or adjunction of $\text{Hom}$ and tensor product. Since those isomorphisms hold in the categories of graded $A$-modules and representations of $T$, they immediately transfer to $\text{mod}_{\mathbb{C}T} A$.

For every free module $V \otimes_{\mathbb{K}} A$ in $\text{mod}_{\mathbb{C}T} A$, there is a natural injection of graded representations of $T$, $i_V : V \to V \otimes_{\mathbb{K}} A$ sending $v$ to $v \otimes 1_A$. Free modules in $\text{mod}_{\mathbb{C}T} A$ satisfy the following universal property (the proof of which is straightforward).

Proposition 2.4.7. Let $V$ be a finite dimensional graded representation of $T$. For every module $M$ in $\text{mod}_{\mathbb{C}T} A$ and any homogeneous map $\psi : V \to M$ of graded representations of $T$, there exists a unique morphism $\psi : V \otimes_{\mathbb{K}} A \to M$ in $\text{mod}_{\mathbb{C}T} A$ such that $\psi = \psi \circ i_V$.

Proposition 2.4.8. Every object $M$ in $\text{mod}_{\mathbb{C}T} A$ admits a finite minimal free resolution in $\text{mod}_{\mathbb{C}T} A$, i.e. an exact complex

$$ 0 \to F_n \xrightarrow{d_n} F_{n-1} \to \ldots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0 $$

of maps and modules in $\text{mod}_{\mathbb{C}T} A$, such that each $F_i$ is free.

Proof. Consider the projection $\pi : M \to M/mM$. As a map of graded representations of $T$, $\pi$ admits a section, i.e. a map $\tilde{d}_0 : M/mM \to M$ such that $\pi \circ \tilde{d}_0 = \text{id}_{M/mM}$. By the universal property of free modules in $\text{mod}_{\mathbb{C}T} A$ (prop. 2.4.7), there exists a unique map $d_0 : (M/mM) \otimes_{\mathbb{K}} A \to M$ such that $d_0 = d_0 \circ i_{M/mM}$.

Set $F_0 := (M/mM) \otimes_{\mathbb{K}} A$. Observe that $d_0$ maps a homogeneous basis of $F_0$ to a minimal generating set of $M$, which makes $d_0$ surjective.

Now suppose that all maps and modules in the complex have been constructed up to a certain index $i \geq 0$. To obtain $d_{i+1} : F_{i+1} \to F_i$, let $M_i := \ker d_i$ and repeat the previous construction using $M_i$ instead of $M$. As a result, $d_{i+1}$ will map a homogeneous basis of $F_{i+1}$ to a minimal generating set of $M_i$, guaranteeing exactness of the complex.

Notice that this construction will produce a complex $F_\bullet$ that is also the minimal free resolution of $M$ in the category of finitely generated graded $A$-modules, hence the process stops when $i > n$.

We can reinterpret the result of proposition 2.4.8 by saying that the usual minimal free resolution of $M$ as an $A$-module carries an action of $T$ that commutes with the differentials.

Remark 2.4.9. Proposition 2.4.8 holds in $\text{mod}_{\mathbb{C}G} A$ for other groups $G$, as long as the category of finite dimensional graded representations of $G$ over $\mathbb{K}$ is semisimple. This guarantees the existence of sections used in the proof.
Example 2.4.10. Continuing example 2.3.5, we look at the complex of free modules in $\text{mod}_{G} A$ given by

$$0 \to \bigwedge^3 V \otimes_{\mathbb{C}} A(-3) \xrightarrow{d_2} \bigwedge^2 V \otimes_{\mathbb{C}} A(-2) \xrightarrow{d_1} \bigwedge^1 V \otimes_{\mathbb{C}} A(-1) \xrightarrow{d_0} A$$

where the maps are defined as follows:

$$d_j: \bigwedge^j V \otimes_{\mathbb{C}} A(-j) \to \bigwedge^{j-1} V \otimes_{\mathbb{C}} A(-j+1)$$

$$v_{i_1} \wedge \ldots \wedge v_{i_j} \mapsto \sum_{k=1}^j (-1)^{k+1} x_k v_{i_1} \wedge \ldots \wedge v_{i_{k-1}} \wedge v_{i_{k+1}} \ldots \wedge v_{i_j}.$$  

This is in fact the Koszul complex $K_n(x_1, x_2, x_3)$ on the variables of $A$, which is the minimal free resolution of $A/\mathfrak{m}$ [BH93, Cor. 1.6.14]. The action of $G$ on each free module is dictated by the exterior powers of $V$. The differential $d_j$ maps the generators of $\bigwedge^j V \otimes_{\mathbb{C}} A(-j)$, which live in degree $j$, into $\bigwedge^{j-1} V \otimes_{\mathbb{C}} A_1$. Notice that $A_1 = V$ so

$$\bigwedge^{j-1} V \otimes_{\mathbb{C}} A_1 \cong \bigwedge^j V \otimes S_{(2,1^{j-2})} V,$$

where $S_{(2,1^{j-2})}$ denotes a Schur functor [Wey03, §2.1] and both summands on the right hand side are irreducible representations of $G$. Such a decomposition can be obtained using Pieri’s formula [Wey03, Cor. 2.3.5]. By Schur’s lemma [Lan02, Ch. XVII, Prop. 1.1], we deduce there is a unique $G$-equivariant map $d_j$ up to multiplication by a scalar. This map can be described as the diagonal map between exterior powers [Wey03, p. 3].

3. Propagating weights

This section contains a detailed description of how the weights for the action of a torus propagate along an equivariant map of free modules. We build gradually towards a fairly general result by first examining the case of maps expressed with respect to bases of weight vectors for the domain and codomain (§3.1). The main theorem is presented in section 3.2.

3.1. Bases of weight vectors. Suppose $\varphi: E \to F$ is a map of free modules in $\text{mod}_{G^T} A$. Assume $E$ has rank $r$ and a homogeneous basis of weight vectors $\mathcal{E} = \{\tilde{e}_1, \ldots, \tilde{e}_r\}$, and that $F$ has rank $s$ and a homogeneous basis of weight vectors $\mathcal{F} = \{\tilde{f}_1, \ldots, \tilde{f}_s\}$. The goal of this section is to explain how to recover weight($\tilde{e}_1$), ..., weight($\tilde{e}_r$) if we assume that weight($\tilde{f}_1$), ..., weight($\tilde{f}_s$) are known. This will provide a complete list of weights of $E$ and hence it will identify $E$ as a representation of $T$.

Proposition 3.1.1. Let $\varphi: E \to F$ be a map of free modules in $\text{mod}_{G^T} A$. Let $\tilde{e} \in E$ be a homogeneous weight vector and assume $\varphi(\tilde{e}) \neq 0$. If $\varphi(\tilde{e}) = \sum_{j=1}^s p_j \tilde{f}_j$ for some homogeneous polynomials $p_1, \ldots, p_s \in A$, then each non zero $p_j$ is a weight vector in $A$ and

$$\text{weight}(\tilde{e}) = \text{weight}(p_j) + \text{weight}(\tilde{f}_j).$$
Proof. By proposition 2.2.3, \( \varphi(\hat{e}) \) is a weight vector and \( \text{weight}(\hat{e}) = \text{weight}(\varphi(\hat{e})) \). Suppose \( \hat{e} \) has weight \( \chi \in X(T) \) and \( \forall j \in \{1, \ldots, s\}, \hat{f}_j \) has weight \( \chi_j \in X(T) \). Then \( \forall \tau \in T \)

\[
\tau \cdot \varphi(\hat{e}) = \varphi(\tau \cdot \hat{e}) = \varphi(\chi(\tau)\hat{e}) = \chi(\tau)\varphi(\hat{e}) = \sum_{j=1}^{s} (\chi(\tau)p_j)\hat{f}_j
\]
and, at the same time,

\[
\tau \cdot \varphi(\hat{e}) = \sum_{j=1}^{s} (\tau \cdot p_j)(\tau \cdot \hat{f}_j) = \sum_{j=1}^{s} (\tau \cdot p_j)(\chi_j(\tau)\hat{f}_j).
\]

Because \( \hat{F} \) is a homogeneous basis of \( F \), we deduce that \( \chi_j(\tau)(\tau \cdot p_j) = \chi(\tau)p_j \) so 

\[
\tau \cdot p_j = \chi(\tau)\chi_j(\tau)^{-1}p_j = \chi\chi_j^{-1}(\tau)p_j
\]
\( \forall j \in \{1, \ldots, s\} \). This implies that each non zero \( p_j \) is a weight vector with weight \( \chi\chi_j^{-1} \). Additively, we may write \( \text{weight}(p_j) = \text{weight}(\hat{e}) - \text{weight}(\hat{f}_j) \) which gives the equality in the thesis. \( \square \)

**Corollary 3.1.2.** If \( \hat{t}\hat{f}_j \in \text{Supp}_{\hat{F}}(\varphi(\hat{e})) \), then

\[
\text{weight}(\hat{e}) = \text{weight}(\hat{t}) + \text{weight}(\hat{f}_j).
\]

**Proof.** If \( \varphi(\hat{e}) = \sum_{j=1}^{s} p_j\hat{f}_j \), the hypothesis implies \( \hat{t} \in \text{Supp}(p_j) \); in particular, \( p_j \neq 0 \) so that \( \varphi(\hat{e}) \neq 0 \) as well. By the previous proposition, \( p_j \) is a weight vector and \( \text{weight}(p_j) = \text{weight}(\hat{t}) \) by proposition 2.3.6. The thesis follows using the weight equality in the previous proposition. \( \square \)

When applying corollary 3.1.2, any term \( \hat{t}\hat{f}_j \in \text{Supp}_{\hat{F}}(\varphi(\hat{e})) \) may be used. In a computational setting, the natural choice of term is \( \text{LT}(\varphi(\hat{e})) \), the leading term of \( \varphi(\hat{e}) \) in a module term ordering on \( \mathbb{T}^n(\hat{F}) \).

If \( \varphi(\hat{e}) = 0 \), then we cannot recover the weight of \( \hat{e} \) using the map \( \varphi \). Indeed, if we expect to use \( \varphi \) to extract information about weight vectors of \( E \), then \( \varphi \) should preserve such information.

**Proposition 3.1.3.** Let \( \varphi: E \to F \) be a map of free modules in \( \text{mod}_{\mathcal{O}_T} A \). Let \( \mathcal{E} = \{e_1, \ldots, e_r\} \) be a homogeneous basis of \( E \) and let \( \langle \mathcal{E} \rangle_K \) denote the \( K \)-vector subspace of \( E \) generated by the elements of \( \mathcal{E} \). The following are equivalent:

1. \( \varphi(e_1), \ldots, \varphi(e_r) \) are minimal generators of \( \text{im} \varphi \);
2. the restriction of \( \varphi \) to \( \langle \mathcal{E} \rangle_K \) is injective.

**Proof.** It is obvious that \( \varphi(e_1), \ldots, \varphi(e_r) \) generate \( \text{im} \varphi \). Then the graded version of Nakayama’s lemma [KR00, Prop. 1.7.15] implies that \( \varphi(e_1), \ldots, \varphi(e_r) \) are minimal generators of \( \text{im} \varphi \) if and only if the residue classes \( \varphi(e_1) + m \text{ im} \varphi, \ldots, \varphi(e_r) + m \text{ im} \varphi \) are linearly independent in \( \text{im} \varphi/m \text{ im} \varphi \). Let \( c_1, \ldots, c_r \in K \). Then:

\[
\varphi \left( \sum_{i=1}^{r} c_i e_i \right) = 0 \iff \sum_{i=1}^{r} c_i \varphi(e_i) = 0 \iff \sum_{i=1}^{r} c_i \varphi(e_i) \in m \text{ im} \varphi \iff \sum_{i=1}^{r} c_i \varphi(e_i) + m \text{ im} \varphi = m \text{ im} \varphi
\]

The left to right direction in the implication \( \iff \) is obvious, while the right to left direction follows from the fact that \( \varphi(e_1), \ldots, \varphi(e_r) \) are homogeneous and \( c_1, \ldots, c_r \).
have degree zero. Then the equivalence of (1) and (2) is a consequence of the comments above and the fact that \( \mathcal{E} \) is a homogeneous basis of \( E \).

Given any two homogeneous bases \( \mathcal{E} \) and \( \mathcal{E}' \) of \( E \), \( \langle \mathcal{E} \rangle_{\mathbb{K}} = \langle \mathcal{E}' \rangle_{\mathbb{K}} \) since any change of basis from \( \mathcal{E} \) to \( \mathcal{E}' \) is \( \mathbb{K} \)-linear. Therefore the second condition in proposition 3.1.3 does not depend on the choice of the homogeneous basis \( \mathcal{E} \).

**Definition 3.1.4.** A map \( \varphi : E \rightarrow F \) of free modules in \( \text{mod}_{\mathbb{K}} A \) that satisfies the equivalent conditions of proposition 3.1.3 is called *minimal*.

**Remark 3.1.5.** The differentials in a minimal free resolution are minimal maps.

The ideas of this subsection are summarized in the following result.

**Proposition 3.1.6.** Let \( \varphi : E \rightarrow F \) be a minimal map of free modules in \( \text{mod}_{\mathbb{K}} A \). Let \( \tilde{\mathcal{F}} = \{ f_1, \ldots, f_s \} \) be a homogeneous basis of weight vectors of \( F \) and assume that \( \mathcal{T}^n(\tilde{\mathcal{F}}) \) is equipped with a module term ordering. If \( \tilde{\mathcal{E}} = \{ \tilde{e}_1, \ldots, \tilde{e}_r \} \) is a homogeneous basis of weight vectors of \( E \), then \( \forall i \in \{1, \ldots, r\} \)

- \( \text{LT}(\varphi(\tilde{e}_i)) \) is a weight vector of \( F \) and \( \text{weight}(\text{LT}(\varphi(\tilde{e}_i))) = \text{weight}(\tilde{e}_i) \);
- if \( \text{LT}(\varphi(\tilde{e}_i)) = \tilde{f}_j \), for some term \( \tilde{f}_j \in \mathcal{T}^n(\mathcal{F}) \), then \( \text{weight}(\tilde{e}_i) = \text{weight}(\tilde{f}_j) \).

If \( \varphi \) is provided as a matrix written with respect to \( \tilde{\mathcal{E}} \) and \( \tilde{\mathcal{F}} \), then this proposition gives a concrete method to obtain a complete list of weights of \( E \) using \( \varphi \).

**Example 3.1.7.** We resume the discussion of our running example from section 2 and we look at the resolution from example 2.4.10. The free module \( F_j = \Lambda^j V \otimes_{\mathbb{C}} A(-j) \) has a homogeneous basis of weight vectors

\[
\tilde{\mathcal{F}}_j = \{ v_{i_1} \wedge \ldots \wedge v_{i_j} \otimes 1_A \mid 1 \leq i_1 < \ldots < i_j \leq 3 \},
\]

where \( \{ v_1, v_2, v_3 \} \) is the coordinate basis of \( V \). Since the weights of \( v_1, v_2, v_3 \) are \( \omega_1 = (1, 0, 0), \omega_2 = (0, 1, 0), \omega_3 = (0, 0, 1) \) respectively, then

\[
\text{weight}(v_{i_1} \wedge \ldots \wedge v_{i_j} \otimes 1_A) = \omega_{i_1} + \ldots + \omega_{i_j}.
\]

We will show how to obtain the same weights using proposition 3.1.6.

Using the homogeneous bases \( \tilde{\mathcal{F}}_j \), the maps of the Koszul complex can be written in matrix form:

\[
0 \rightarrow F_3 \xrightarrow{\begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}} F_2 \xrightarrow{\begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix}} F_1 \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} F_0
\]

Assume \( A \) is endowed with a lexicographic term ordering such that \( x_1 > x_2 > x_3 \) and the modules \( F_i \) are equipped with the term over position up module term ordering. Notice that \( F_0 = A \) and that \( G \) acts trivially on its basis element \( 1_A \). Therefore:

- \( \text{LT}(d_1(v_1 \otimes 1_A)) = x_1 1_A \Rightarrow \text{weight}(v_1 \otimes 1_A) = (1, 0, 0) \),
- \( \text{LT}(d_1(v_2 \otimes 1_A)) = x_2 1_A \Rightarrow \text{weight}(v_2 \otimes 1_A) = (0, 1, 0) \),
- \( \text{LT}(d_1(v_3 \otimes 1_A)) = x_3 1_A \Rightarrow \text{weight}(v_3 \otimes 1_A) = (0, 0, 1) \).
Knowing the weights of the elements in $\tilde{F}_1$, we can proceed with the map $d_2$:

\[
\text{LT}(d_2(v_1 \wedge v_2 \otimes 1_A)) = x_1v_2 \otimes 1_A \implies \\
\Rightarrow \text{weight}(v_1 \wedge v_2 \otimes 1_A) = (1, 0, 0) + (0, 1, 0) = (1, 1, 0), \\
\text{LT}(d_2(v_1 \wedge v_3 \otimes 1_A)) = x_1v_3 \otimes 1_A \implies \\
\Rightarrow \text{weight}(v_1 \wedge v_3 \otimes 1_A) = (1, 0, 0) + (0, 0, 1) = (1, 0, 1), \\
\text{LT}(d_2(v_2 \wedge v_3 \otimes 1_A)) = x_2v_3 \otimes 1_A \implies \\
\Rightarrow \text{weight}(v_2 \wedge v_3 \otimes 1_A) = (0, 1, 0) + (0, 0, 1) = (0, 1, 1).
\]

Finally the map $d_3$:

\[
\text{LT}(d_3(v_1 \wedge v_2 \wedge v_3 \otimes 1_A)) = x_1v_2 \wedge v_3 \otimes 1_A \implies \\
\Rightarrow \text{weight}(v_1 \wedge v_2 \wedge v_3 \otimes 1_A) = (1, 0, 0) + (0, 1, 1) = (1, 1, 1).
\]

The weights we found can be used to identify each module $F_j$ with $\wedge^j V \otimes_C A(-j)$.

### 3.2. The general case.

In general, we cannot expect every map $\varphi : E \to F$ of free modules in $\text{mod}_{C^T} A$ to be written with respect to homogeneous bases of weight vectors. Nevertheless, under reasonable assumptions, it is still possible to recover the weights of $E$ using the weights of $F$ and the map $\varphi$.

#### Example 3.2.1.

To illustrate an issue that can occur, we write the matrix of the map $d_2$ from example 2.4.10 with respect to the following homogeneous basis of $F_2$

\[
\{(v_1 \wedge v_2 - v_1 \wedge v_3) \otimes 1_A, (v_1 \wedge v_2 + v_1 \wedge v_3) \otimes 1_A, (v_1 \wedge v_3 + v_2 \wedge v_3) \otimes 1_A\},
\]

which is created using linear combinations of elements in the homogeneous basis $\tilde{F}_2$ from example 3.1.7. The matrix looks like this:

\[
\begin{pmatrix}
-x_2 + x_3 & -x_2 - x_3 & -x_3 \\
-x_1 & x_1 & -x_3 \\
-x_1 & x_1 & x_1 + x_2
\end{pmatrix}.
\]

Observe that all columns have leading term $x_1v_3 \otimes 1_A$. If we proceed to calculate weights as indicated in proposition 3.1.6, we will obtain the weight $(1, 0, 1)$ three times, which does not fit the representation $\wedge^2 V$.

The situation presented in the example above is somewhat artificial, however it highlights the following fact: unless we are working with homogeneous bases of weight vectors, we are not guaranteed to obtain meaningful lists of weights. The issue is especially relevant in the case of free resolutions constructed using computational techniques. In fact, many algorithms that compute free resolutions express the matrices of the differentials with respect to homogeneous bases that do not, typically, consist of weight vectors.

We adopt the notation of [Lam02, p. 503] for the matrix of a map of free modules. Let $\varphi : E \to F$ be a map of free modules; let $E = \{e_1, \ldots, e_r\}$ and $F = \{f_1, \ldots, f_s\}$ be homogeneous bases of $E$ and $F$ respectively. If $\varphi(e_j) = \sum_{i=1}^s a_{i,j} f_i$, then $M_E^F(\varphi) = (a_{i,j})$ is the matrix of $\varphi$ with respect to $E$ and $F$. If $F$ and $F'$ are homogeneous bases of a free module $F$, then $M_E^{F'}(\text{id}_F)$ is the matrix of the change of basis from $F$ to $F'$.

Here is our main result.

#### Theorem 3.2.2.

Let $\varphi : E \to F$ be a minimal map of free modules in $\text{mod}_{C^T} A$. Suppose that:
H1. \( \mathcal{F} = \{f_1, \ldots, f_s\} \) is a homogeneous basis of \( F \) and \( \mathbb{T}^n(\mathcal{F}) \) is equipped with a position up module term ordering;

H2. \( F \) admits a homogeneous basis of weight vectors \( \tilde{\mathcal{F}} = \{\tilde{f}_1, \ldots, \tilde{f}_s\} \) such that \( \mathcal{M}_{\tilde{\mathcal{F}}}^F(\text{id}_F) \) is upper triangular;

H3. \( E \) admits a homogeneous basis \( \mathcal{E} = \{e_1, \ldots, e_r\} \) such that

\[
\text{LT}(\varphi(e_1)) < \ldots < \text{LT}(\varphi(e_r))
\]

in \( \mathbb{T}^n(\mathcal{F}) \).

Then:

T1. \( E \) admits a homogeneous basis of weight vectors \( \tilde{\mathcal{E}} = \{\tilde{e}_1, \ldots, \tilde{e}_r\} \) such that \( \mathcal{M}_{\tilde{\mathcal{E}}}^E(\text{id}_E) \) is upper triangular;

T2. if \( \text{LT}(\varphi(e_i)) = \hat{t}_f j \), for some \( \hat{t}_f j \in \mathbb{T}^n(\mathcal{F}) \), then

\[
\text{weight}(\tilde{e}_i) = \text{weight}(\hat{t}_f) + \text{weight}(\tilde{f}_j).
\]

In essence, the theorem says the weights of \( E \) can be recovered using the map \( \varphi \) and a homogeneous basis \( \mathcal{E} \) of \( E \) as in H3. This basis is connected to a homogeneous basis of weight vectors of \( E \) by a triangular change of basis, as long as a similar property holds for suitable bases of \( F \). While it may seem that the hypotheses of the theorem are quite restrictive, they are easily met in general enough settings where we wish to apply our algorithm.

We postpone the proof of theorem 3.2.2 to §3.3.

Example 3.2.3. Similarly to example 3.2.1, we write the matrix of the map \( d_2 \) from example 2.4.10 with respect to the following homogeneous basis of \( F_2 \)

\[
\{v_2 \land v_3 \otimes 1_A, v_1 \land v_2 \otimes 1_A, (v_1 \land v_2 + v_1 \land v_3) \otimes 1_A\}.
\]

We get the following matrix:

\[
\begin{pmatrix}
0 & -x_2 & -x_2 - x_3 \\
-x_3 & x_1 & x_1 \\
x_2 & 0 & x_1
\end{pmatrix}.
\]

Now the columns have the same leading terms as the matrix for \( d_2 \) that was written in example 3.1.7, although permuted to form an increasing sequence. Since the homogeneous basis \( \tilde{\mathcal{F}}_1 \) used for \( F_1 \) consists of weight vectors and the module term ordering used is position up, the hypotheses of theorem 3.2.2 are satisfied. The calculation for the weights of \( F_2 \) proceeds as in example 3.1.7, thus we obtain meaningful weights even though the homogeneous basis we are using for \( F_2 \) does not consist of weight vectors. The change of basis from this homogeneous basis to \( \tilde{\mathcal{F}}_2 \) is given by the matrix

\[
\begin{pmatrix}
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

which is upper triangular.

Remark 3.2.4. The module term ordering on \( \mathbb{T}^n(\mathcal{F}) \) in H1 could be taken to be a position down ordering instead of position up. The statement of theorem 3.2.2 would need to be modified as indicated below:

- in H1, \( \mathbb{T}^n(\mathcal{F}) \) is equipped with a position down module term ordering;
- in H2, \( \mathcal{M}_{\tilde{\mathcal{F}}}^F(\text{id}_F) \) is lower triangular;
- in H3, \( \text{LT}(\varphi(e_1)) > \ldots > \text{LT}(\varphi(e_r)) \).
• in $T_1$, $\mathcal{M}_n^F(\text{id}_F)$ is lower triangular.

It might be possible to state a more general version of theorem 3.2.2 that holds with any module term ordering on $\mathbb{T}^n \langle F \rangle$. However, the position up/down module term orderings seem to be the most commonly implemented in software, with one of them often being the default option. Therefore we decided to take a practical approach and limit ourselves to those scenarios that matter for the applications.

3.3. Proof of the main theorem. The first step towards the proof of theorem 3.2.2 is to show how a triangular change of basis can be exploited to calculate the weight of a homogeneous weight vector.

**Proposition 3.3.1.** Let $\varphi: E \to F$ be a map of free modules in $\text{mod}_{\mathbb{Z}T} A$. Suppose hypotheses H1 and H2 of theorem 3.2.2 hold. If $\tilde{e} \in E$ is a homogeneous weight vector with $\text{LT}(\varphi(\tilde{e})) = \hat{t} f_j$, for some $\hat{t} f_j \in \mathbb{T}^n \langle F \rangle$, then:

I. $\hat{t} f_j \in \text{Supp}_F(\varphi(\tilde{e}))$;

II. weight$(\tilde{e}) = \text{weight}(\hat{t}) + \text{weight}(f_j)$.

**Proof.** To prove I, write $\varphi(\tilde{e})$ as a $K$-linear combination of terms in $\mathbb{T}^n \langle F \rangle$:

\[
\varphi(\tilde{e}) = \sum_{t \in \mathbb{T}^n} \sum_{j = 1}^s c_{t,j} \hat{t} f_j.
\]

If $\mathcal{M}_n^F(\text{id}_F) = (u_{i,j})$, then

\[
\varphi(\tilde{e}) = \sum_{t \in \mathbb{T}^n} \sum_{j = 1}^s c_{t,j} \hat{t} \left( \sum_{i = 1}^s u_{i,j} f_i \right) = \sum_{t \in \mathbb{T}^n} \sum_{j = 1}^s \sum_{i = 1}^s c_{t,j} u_{i,j} \hat{t} f_i.
\]

The coefficient of $\hat{t} f_j$ in equation (3.3.2) is

\[
\sum_{j = 1}^s c_{t,j} u_{i,j}.
\]

Since $\text{LT}(\varphi(\tilde{e})) = \hat{t} f_j$ and $\mathbb{T}^n \langle F \rangle$ is equipped with a position up module term ordering (H1), $c_{t,j} = 0$, for all $j > \hat{t}$ and $c_{t,j} \neq 0$. Moreover, $u_{i,j} \neq 0$ and $u_{i,j} = 0$ for all $j < \hat{t}$ because the matrix $\mathcal{M}_n^F(\text{id}_F) = (u_{i,j})$ is upper triangular (H2). Therefore the coefficient of the term $\hat{t} f_j$ is $c_{t,j} u_{i,j}$. Since this coefficient is non zero, we conclude that $\hat{t} f_j \in \text{Supp}_F(\varphi(\tilde{e}))$.

Now II is an immediate consequence of I and corollary 3.1.2.

**Corollary 3.3.2.** Let $\varphi: E \to F$ be a map of free modules in $\text{mod}_{\mathbb{Z}T} A$. Suppose hypotheses H1 and H2 of theorem 3.2.2 hold. If $\tilde{e}_1, \tilde{e}_2 \in E$ are homogeneous weight vectors and $\text{LT}(\varphi(\tilde{e}_1)) = \text{LT}(\varphi(\tilde{e}_2))$, then weight$(\tilde{e}_1) = \text{weight}(\tilde{e}_2)$.

**Proof.** Suppose $\text{LT}(\varphi(\tilde{e}_1)) = \text{LT}(\varphi(\tilde{e}_2)) = \hat{t} f_j$. Then, by the proposition 3.3.1,

\[
\text{weight}(\tilde{e}_1) = \text{weight}(\hat{t}) + \text{weight}(\hat{t} f_j) = \text{weight}(\tilde{e}_2).
\]

Let $M$ be a graded submodule of a graded free $A$-module $F$ endowed with a module term ordering. Recall that the reduced Gröbner basis of $M$ (defined in [KR00, §2.4.C]) consists of homogeneous elements [KR05, Prop. 4.5.1]. Also, given a homogeneous Gröbner basis $G$ of $M$, set $G_{\leq d} = \{ g \in G \mid \deg(g) \leq d \}$, the truncation of $G$ at degree $d$ (see [KR05, §4.5.B] for truncated Gröbner bases).
Proposition 3.3.3. Let \( \varphi: E \to F \) be a minimal map of free modules in \( \text{mod}_{\leq T} A \) with \( E \) generated in a single degree \( d \in \mathbb{Z}^m \). Let \( F \) be a homogeneous basis of \( F \) and assume \( \mathbb{T}^n(\mathcal{F}) \) is equipped with a module term ordering. Let \( Z \) be a \( \mathbb{K} \)-vector subspace of \( E_d \) and let \( M \) be the \( A \)-submodule of \( F \) generated by \( \varphi(Z) \). If \( G \) is the reduced Gröbner basis of \( M \), then \( \varphi^{-1}(G_{\leq d}) \) is a \( \mathbb{K} \)-basis of \( Z \).

**Proof.** Notice that \( M \) is generated in degree \( d \) and \( M_d = \varphi(Z) \). Thus all elements of \( G_{\leq d} \) have degree equal to \( d \). Since \( G \) generates \( M \) as an \( A \)-module, \( G_{\leq d} \) generates \( M_d \) as a \( \mathbb{K} \)-vector space. The elements of \( G \) all have different leading terms because \( G \) is reduced; in particular, the elements of \( G \) are \( \mathbb{K} \)-linearly independent. We conclude that \( G_{\leq d} \) is a \( \mathbb{K} \)-basis of \( M_d \).

To get the thesis, observe that the restriction of \( \varphi \) to \( Z \) is injective, by definition 3.1.4 and the assumption that \( Z \subseteq E_d \). Therefore the preimage of a basis of \( M_d \) is a basis of \( \varphi^{-1}(M_d) = Z \). In particular, \( \varphi^{-1}(G_{\leq d}) \) is a basis of \( Z \). \( \square \)

Corollary 3.3.4. Let \( \varphi: E \to F \) be a minimal map of free modules in \( \text{mod}_{\leq T} A \). If hypotheses H1 and H2 of theorem 3.2.2 hold, then \( E \) admits a homogeneous basis of weight vectors \( \tilde{E} = \{ \tilde{e}_1, \ldots, \tilde{e}_r \} \) such that

\[
\text{LT}(\varphi(\tilde{e}_1)) < \ldots < \text{LT}(\varphi(\tilde{e}_r)).
\]

**Proof.** Let us begin by considering the special case where \( E \) is generated in a single degree \( d \in \mathbb{Z}^m \). Since \( E_d \) is a finite dimensional representation of \( T \), we have a decomposition into weight spaces \( E_d = (E_d)_{\chi_1} \oplus \ldots \oplus (E_d)_{\chi_h} \), for some characters \( \chi_1, \ldots, \chi_h \) of \( T \). By proposition 3.3.3, each weight space \( (E_d)_{\chi_i} \) admits a basis \( \tilde{E}_i \) such that the images of elements in \( \tilde{E}_i \) under \( \varphi \) have different leading terms.

Now consider \( \tilde{E} := \tilde{E}_1 \cup \ldots \cup \tilde{E}_h \). The set \( \tilde{E} \) is a \( \mathbb{K} \)-basis of weight vectors of \( E_d \). Since \( E \) is generated in degree \( d \), \( \tilde{E} \) is also a homogeneous basis of weight vectors \( E \). Moreover if two elements of \( \tilde{E} \) have different weights, their images under \( \varphi \) have different leading terms by corollary 3.3.2. Therefore we can index elements of \( \tilde{E} \) so that \( \tilde{E} = \{ \tilde{e}_1, \ldots, \tilde{e}_r \} \) and \( \text{LT}(\varphi(\tilde{e}_1)) < \ldots < \text{LT}(\varphi(\tilde{e}_r)) \).

For the general case, simply observe that if two homogeneous elements have different degrees, their leading terms must be different. Therefore the conclusion holds for any \( E \). \( \square \)

We are finally ready to prove the main result of this work.

**Proof of theorem 3.2.2.**

**Step 1: reduction to a single degree.**

Without loss of generality we may assume that \( E \) is generated in a single degree \( d \in \mathbb{Z}^m \). If that is not the case, we may write \( E = E_1 \oplus \ldots \oplus E_l \), where each \( E_i \) is a free module in \( \text{mod}_{\leq T} A \) which is generated in a single degree \( d_i \in \mathbb{Z}^m \). Then, by properties of the direct sum, we can define maps \( \varphi_i: E_i \to F \) that satisfy the hypotheses of the theorem, thus reducing to the case where the domain is generated in a single degree.

**Step 2: a special basis of \( E \).**

By corollary 3.3.4, \( E \) admits a homogeneous basis of weight vectors \( \tilde{E} = \{ \tilde{e}_1, \ldots, \tilde{e}_r \} \) such that

\[
\text{LT}(\varphi(\tilde{e}_1)) < \ldots < \text{LT}(\varphi(\tilde{e}_r)).
\]
Step 3: compare leading terms.
Since \( \varphi \) is minimal and \( \tilde{E} \) is a homogeneous basis of \( E \), \( \varphi(e_1), \ldots, \varphi(e_r) \) are minimal generators of \( \text{im} \varphi \). Because \( E \) is generated in degree \( d \), this implies \( \dim_k(\text{im} \varphi)_d = r \). Now let \( \text{LT}(\text{im} \varphi) \) be the leading term module of \( \text{im} \varphi \). Recall that \( \text{LT}(\text{im} \varphi) \) is the monomial submodule of \( F \) generated by the set 
\[
\{ \text{LT}(m) \in F \mid m \in \text{im} \varphi \}.
\]
It is known that \( \text{im} \varphi \) and \( \text{LT}(\text{im} \varphi) \) have the same Hilbert function [KR05, Prop. 5.8.9.f]; in particular, \( \dim_k(\text{LT}(\text{im} \varphi))_d = \dim_k(\text{im} \varphi)_d = r \). By the string of inequalities in \( H_3 \), the terms \( \text{LT}(\varphi(e_1)), \ldots, \text{LT}(\varphi(e_r)) \) are all different and thus \( K \)-linearly independent. We conclude that they form a basis of \( (\text{LT}(\text{im} \varphi))_d \).

Since \( \tilde{E} \) is another homogeneous basis of \( E \), the terms \( \text{LT}(\varphi(\tilde{e}_1)), \ldots, \text{LT}(\varphi(\tilde{e}_r)) \) form another basis of \( (\text{LT}(\text{im} \varphi))_d \), by the same argument. By [KR00, Prop. 1.3.11], the minimal monomial generators of a monomial module are uniquely determined; this gives an equality of sets:
\[
\{ \text{LT}(\varphi(e_1)), \ldots, \text{LT}(\varphi(e_r)) \} = \{ \text{LT}(\varphi(\tilde{e}_1)), \ldots, \text{LT}(\varphi(\tilde{e}_r)) \}.
\]
Finally, the strings of inequalities in \( H_3 \) and (3.3.4) imply that
\[
(3.3.5) \quad \text{LT}(\varphi(e_i)) = \text{LT}(\varphi(\tilde{e}_i)), \ \forall i \in \{1, \ldots, r\}.
\]

Step 4: the triangular change of basis.
Combining the string of inequalities (3.3.4) and the equalities (3.3.5), we get 
\[
\text{LT}(\varphi(\tilde{e}_1)) < \ldots < \text{LT}(\varphi(\tilde{e}_j)) = \text{LT}(\varphi(e_j)),
\]
for all \( j \in \{1, \ldots, r\} \). Thus, for all \( j \in \{1, \ldots, r\} \), \( \exists u_{1,j}, \ldots, u_{j,j} \in K \) such that
\[
(3.3.6) \quad \varphi(e_j) = \sum_{i=1}^{j} u_{i,j} \varphi(\tilde{e}_i) = \varphi \left( \sum_{i=1}^{j} u_{i,j} \tilde{e}_i \right),
\]
and \( u_{i,j} \neq 0 \). The matrix \( (u_{i,j}) \) is then upper triangular and invertible. The map \( \varphi \) being minimal, its restriction to \( E_d \) is injective. Therefore equation (3.3.6) implies
\[
e_j = \sum_{i=1}^{j} u_{i,j} \tilde{e}_i.
\]
In other words, \( M^\varphi_0(\text{id}_E) = (u_{i,j}) \) is upper triangular, which proves \( T_1 \) in the thesis.

Step 5: calculate the weights of \( E \).
For each \( i \in \{1, \ldots, r\} \), \( \text{LT}(\varphi(e_i)) = \text{LT}(\varphi(\tilde{e}_i)) = \hat{t} f_j \), for some \( \hat{t} f_j \in \mathbb{T}^n \langle F \rangle \); then
\[
\text{weight}(\tilde{e}_i) = \text{weight}(\hat{t}) + \text{weight}(f_j),
\]
by proposition 3.3.1. This proves \( T_2 \) in the thesis.

\[ \square \]

4. Algorithms and applications

In this section, we present some applications of theorem 3.2.2. We also design algorithms, in pseudocode, that can be implemented in a software system to carry out the necessary computations. The software system must be able to compute reduced Gröbner bases (truncated at a given degree) over polynomial rings with positive \( \mathbb{Z}^m \)-gradings as well as the change of basis between the reduced Gröbner basis of a module and some user-specified set of generators.
4.1. **Weight propagation along a map.** Our first algorithm can be used to recover the weights of the domain map from those of the codomain. We lay out some assumptions.

(a) \( \varphi: E \rightarrow F \) is a minimal map of free modules in \( \mod \langle T, A \rangle \).

(b) The map \( \varphi \) is provided in matrix form \( \mathcal{M}_E^F(\varphi) \), where \( \mathcal{E} \) is a homogeneous basis of \( E \) and \( \mathcal{F} \) is a homogeneous basis of \( F \).

(c) \( \mathbb{T}_n\langle \mathcal{F} \rangle \) is equipped with a position up (resp. down) module term ordering.

(d) \( F \) admits a homogeneous basis of weight vectors \( \tilde{\mathcal{F}} = \{ \tilde{f}_1, \ldots, \tilde{f}_s \} \) such that \( \mathcal{M}_E^F(\text{id}_F) \) is upper (resp. lower) triangular.

(e) \( W = \{ w_1, \ldots, w_s \} \) is an ordered list with \( w_i = \text{weight}(\tilde{f}_i) \), \( \forall i \in \{ 1, \ldots, s \} \).

First we deal with the case where \( E \) is generated in a single degree \( d \in \mathbb{Z}^m \).

**Algorithm 1:** weight propagation along a map with domain in a single degree

1: function PROPAGATE Singledegree(\( \mathcal{M}_E^F(\varphi), W \))
2: set \( d := \) the degree of the first column of \( \mathcal{M}_E^F(\varphi) \)
3: compute \( \mathcal{G}_{\leq d} \), the \( d \)-truncated reduced Gröbner basis of \( \text{im} \varphi \), using \( \mathcal{M}_E^F(\varphi) \)
4: form \( G \), a matrix with columns the component vectors (in the homogeneous basis \( \mathcal{F} \)) of the elements of \( \mathcal{G}_{\leq d} \)
5: compute \( M \), the matrix of the change of basis in \( E \) such that \( G = \mathcal{M}_E^F(\varphi)M \)
6: if the module term ordering on \( \mathbb{T}_n\langle \mathcal{F} \rangle \) is position up then
7: form \( P \), a permutation matrix such that the columns of \( GP \) are arranged in increasing order of their leading terms
8: else
9: form \( P \), a permutation matrix such that the columns of \( GP \) are arranged in decreasing order of their leading terms
10: end if
11: set \( C := MP \)
12: set \( r := \) number of columns of \( \mathcal{M}_E^F(\varphi) \)
13: for \( i \in \{ 1, \ldots, r \} \) do
14: set \( f := \) the \( i \)-th column of \( GP \)
15: find \( t \in \mathbb{T}_n \) and \( j \in \{ 1, \ldots, s \} \), such that \( \text{LT}(f) = tf_j \)
16: set \( v_i := \text{weight}(t) + w_j \)
17: end for
18: form the ordered list \( V := \{ v_1, \ldots, v_r \} \)
19: return \( (C, V) \)
20: end function

**Proposition 4.1.1.** Under the assumptions of this section, applying algorithm 1 yields a pair \((C, V = \{v_1, \ldots, v_r\})\) that satisfies the following properties:

I. \( E \) admits a homogeneous basis \( \mathcal{E}' = \{ e'_1, \ldots, e'_r \} \) such that \( C = \mathcal{M}_E^F(\text{id}_E) \) and the terms \( \text{LT}(\varphi(e'_1)), \ldots, \text{LT}(\varphi(e'_r)) \) are all different;

II. \( E \) admits a homogeneous basis of weight vectors \( \tilde{\mathcal{E}} = \{ \tilde{e}_1, \ldots, \tilde{e}_r \} \) such that \( \mathcal{M}_E^F(\text{id}_E) \) is upper (resp. lower) triangular and \( \text{weight}(\tilde{e}_i) = v_i \), for all \( i \in \{ 1, \ldots, r \} \).

**Proof.** Using the notation of algorithm 1, \( C = MP \), where both \( M \) and \( P \) correspond to change of bases in \( E \); hence \( C \) is also the matrix of a change of basis in \( E \). Let \( \mathcal{E}' \) be the homogeneous basis of \( E \) defined by \( C \), so that the equality \( C = \mathcal{M}_E^F(\text{id}_E) \) in I follows by construction. Consider the equality \( G = \mathcal{M}_E^F(\varphi)M \)
from line 5 of algorithm 1. Multiplying by $P$ on the right yields $GP = M_{E}^{x}(\varphi)C$. Since $C = M_{E}^{x}(\id_{E})$, we deduce that $GP = M_{E}^{x}(\varphi)$. This implies that the $i$-th column of $GP$ is the component vector of $\varphi(e_{i}')$ in the basis $F$. Recall that $G$ is the matrix of a reduced Gröbner basis, therefore the leading terms of its columns are all different. The same obviously holds for $GP$ which proves $I$.

Now apply theorem 3.2.2, with $E'$ playing the role of the basis $E$ in hypothesis H3 of the theorem. Then, by T1, $E$ admits a homogeneous basis of weight vectors $\tilde{E} = \{\tilde{e}_{1}, \ldots, \tilde{e}_{r}\}$ such that $M_{E}^{x}(\id_{E})$ is upper (resp. lower) triangular. If $\LT(\varphi(\tilde{e}_{i})) = \hat{t}f_{j}$, then

$$\text{weight}(\tilde{e}_{i}) = \text{weight}(\hat{t}) + \text{weight}(f_{j}) = \text{weight}(\hat{t}) + w_{j} = v_{i}.$$  

The first equality follows from T2 in theorem 3.2.2; the second equality comes from the definition of $W$ and the last one descends from the construction of $v_{i}$ on line 16 of algorithm 1. This proves $\Pi$. $\square$

Let $(M_{1}| \ldots | M_{l})$ denote the block matrix obtained by juxtaposing matrices $M_{1}, \ldots, M_{l}$ all with the same number of rows. Also, given matrices $C_{1}, \ldots, C_{l}$, denote $C_{1} \oplus \ldots \oplus C_{l}$ the block diagonal matrix

$$\begin{pmatrix}
C_{1} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & C_{l}
\end{pmatrix}.
$$

We will employ these notations to generalize algorithm 1 to the case where the domain $E$ of $\varphi$ is generated in multiple degrees.

Algorithm 2: weight propagation along a map

1: function PROPAGATE($M_{E}^{x}(\varphi), W$)
2: form $P$, a permutation matrix such that $M_{E}^{x}(\varphi)P = (M_{1}| \ldots | M_{l})$, where, $\forall i \in \{1, \ldots, l\}$, the columns of $M_{i}$ have the same degree $d_{i} \in \mathbb{Z}^{m}$, and the degrees $d_{1}, \ldots, d_{l}$ are all different
3: for $i \in \{1, \ldots, l\}$ do
4: \hspace{1em} set $(C_{i}, V_{i}) := \text{PROPAGATE SINGLE DEGREE}(M_{i}, W)$
5: \hspace{1em} end for
6: set $C := P(C_{1} \oplus \ldots \oplus C_{l})$
7: set $V := V_{1} \cup \ldots \cup V_{l}$
8: return $(C, V)$
9: end function

The symbol $\cup$ on line 7 denotes “ordered union” of ordered lists; in other words the list $V_{2}$ is appended to $V_{1}$, then $V_{3}$ is appended to $V_{1} \cup V_{2}$, etc.

It is enough to trace through the algorithm to realize that the conclusion of proposition 4.1.1 holds for algorithm 2 as well.

Remark 4.1.2. Rather than computing a Gröbner basis for $\text{im} \varphi$, algorithm 2 splits $M_{E}^{x}(\varphi)$ into submatrices with columns of the same degree and then applies algorithm 1 to each one of them. If $M_{E}^{x}(\varphi)$ has columns in degrees $d_{1}, \ldots, d_{l}$, then its reduced Gröbner basis (even a truncated one) could have elements in other degrees which are not part of a minimal generating set of $\text{im} \varphi$. By computing
Gröbner bases in single degrees we can avoid this issue altogether, hence producing another minimal map.

**Remark 4.1.3.** Algorithms 1 and 2 return the matrix $C = \mathcal{M}_E^\nu(id_E)$. The existence of a homogeneous basis of weight vectors $\tilde{E}$ of $E$ such that $\mathcal{M}_E^\nu(id_E)$ is upper (resp. lower) triangular, is guaranteed by theorem 3.2.2. However algorithms 1 and 2 do not provide any means to recover the matrix $\mathcal{M}_E^\nu(id_E)$.

4.2. **Going forward.** For a map $\varphi: E \rightarrow F$ of free modules in mod$_\mathbb{Z}$-$T$, algorithm 2 provides a tool to recover the weights of $E$ from the weights of $F$ by “moving backwards along $\varphi$”. A natural question is whether it is possible to go “forward” instead and recover the weights of $F$ from those of $E$. Here are the assumptions for this section.

(a) $\varphi: E \rightarrow F$ is a map of free modules in mod$_\mathbb{Z}$-$T$. $A$.
(b) The dual map $\varphi^\sim: F^\nu \rightarrow E^\nu$ is minimal.
(c) The map $\varphi$ is provided in matrix form $\mathcal{M}_F^\nu(\varphi)$, where $E$ is a homogeneous basis of $E$ and $F$ is a homogeneous basis of $F$.
(d) $\mathbb{T}^\nu(\tilde{E})$ is equipped with a position up (resp. down) module term ordering.
(e) $E$ admits a homogeneous basis of weight vectors $\tilde{E} = \{\tilde{e}_1, \ldots, \tilde{e}_r\}$ such that $\mathcal{M}_E^\nu(id_E)$ is upper (resp. lower) triangular.
(f) $V = \{v_1, \ldots, v_r\}$ is an ordered list with $v_i = \text{weight}(\tilde{e}_i), \forall i \in \{1, \ldots, r\}$.

For a matrix $M$, let $M^\top$ denote the transpose of $M$. For an ordered list of weights $W = \{w_1, \ldots, w_s\}$, let $-W$ denote the ordered list of weights $\{-w_1, \ldots, -w_s\}$.

**Algorithm 3:** weight propagation going forward along a map

1. function $\text{PROPAGATEFORWARD}(\mathcal{M}_F^\nu(\varphi), V)$
2. if the module term ordering on $\mathbb{T}^\nu(\tilde{E})$ is position up then
3. equip $\mathbb{T}^\nu(\tilde{E})$ with a position down module term ordering
4. else
5. equip $\mathbb{T}^\nu(\tilde{E})$ with a position up module term ordering
6. end if
7. set $(C', W') := \text{PROPAGATE}(\mathcal{M}_F^\nu(\varphi)^\top, -V)$
8. set $C := (C')^\top$
9. set $W := -W'$
10. return $(C, W)$
11. end function

**Proposition 4.2.1.** Under the assumptions of this section, applying algorithm 3 yields a pair $(C, W = \{w_1, \ldots, w_s\})$ that satisfies the following properties:

I. $F$ admits a homogeneous basis $\tilde{F} = \{f'_1, \ldots, f'_s\}$ such that $C = \mathcal{M}_F^{\nu^\top}(id_F)$ and the terms $\text{LT}(\varphi^\sim((f'_1)^\nu)), \ldots, \text{LT}(\varphi^\sim((f'_s)^\nu))$ are all different;

II. $F$ admits a homogeneous basis of weight vectors $\tilde{F} = \{\tilde{f}_1, \ldots, \tilde{f}_s\}$ such that $\mathcal{M}_F^{\nu^\top}(id_E)$ is lower (resp. upper) triangular and $\text{weight}(\tilde{f}_i) = w_i$, for all $i \in \{1, \ldots, s\}$.

**Proof.** Algorithm 3 applies algorithm 2 to the dual map $\varphi^\sim: F^\nu \rightarrow E^\nu$, which is assumed to be minimal. Notice that the matrix $\mathcal{M}_F^{\nu^\top}(\varphi)^\top$ used on line 7 of algorithm 3, is in fact the matrix $\mathcal{M}_F^{\nu^\top}(\varphi^\sim)$, the matrix of $\varphi^\sim$ with respect to the dual bases $E^\nu$ of $E^\nu$ and $F^\nu$ of $F^\nu$. 
If $\mathcal{M}_E^t (\text{id}_E)$ is upper (resp. lower) triangular, then $\mathcal{M}_E^t (\text{id}_E \cdot ) = \mathcal{M}_E^t (\text{id}_E)^\top$ is lower (resp. upper) triangular. Therefore, to use algorithm 2 correctly, $\mathcal{T}^n \langle E^\vee \rangle$ must be equipped with a position down (resp. up) module term ordering, as is done on lines 2-6.

The weights of $E$ are provided in $V$ as part of the input. Thanks to proposition 2.4.5, $E^\vee \cong (E/mE)^* \otimes_K A$, while $E \cong (E/mE) \otimes_K A$ by proposition 2.4.3. Moreover, by proposition 2.2.4, the weights of $(E/mE)^*$ are the opposites of the weights of $E/mE$. Thus weight($\tilde{e}^\vee_i$) = − weight($\tilde{e}_i$), and that is why we apply algorithm 2 with the list $-V$ as input.

After algorithm 2 is applied, the results must be transferred from $F^\vee$ to $F$. For that reason, we transpose $C'$ on line 8 and switch to the opposite weights of $W'$ on line 9.

Proposition 4.2.1 requires that the map $\varphi^\vee: F^\vee \rightarrow E^\vee$ is minimal. Unfortunately, $\varphi: E \rightarrow F$ being minimal does not, in general, imply that $\varphi^\vee: F^\vee \rightarrow E^\vee$ is minimal.

**Example 4.2.2.** Let $A = \mathbb{Q}[x]$. The map of free $A$-modules

$$A(-1) \xrightarrow{\left( \begin{array}{c} x \\ y \end{array} \right)} A^2$$

is minimal. However, the dual map

$$A^2 \xrightarrow{(x \ x)} A(1)$$

is not minimal because the element $(1, -1)^\top$ belongs to the $\mathbb{K}$-vector space generated by the coordinate basis of $A^2$ and is sent to zero (compare with proposition 3.1.3).

The following is a useful criterion for the minimality of a dual map, when computing the weights with respect to a torus contained inside a group $G$.

**Proposition 4.2.3.** Let $G$ be an algebraic group over $\mathbb{K}$ and let $\varphi: E \rightarrow F$ be a non zero map of free modules in $\text{mod}_{\mathbb{Z}G} A$. If $F/mF$ is an irreducible representation of $G$, then the dual map $\varphi^\vee: F^\vee \rightarrow E^\vee$ is minimal.

**Proof.** Because $F/mF$ is an irreducible representation of $G$, the graded vector space $F/mF$ is concentrated in a single degree $d \in \mathbb{Z}^m$. Equivalently, $F$ is generated in a single degree $d$ and $F_d$ is an irreducible representation of $G$. Looking at duals, we have that $F^\vee$ is generated in degree $-d$ and $(F^\vee)_{-d} \cong \text{Hom}_K(F_d, \mathbb{K})$ by proposition 2.4.5. Then $(F^\vee)_{-d}$ is irreducible because $F_d$ is. Notice that $(\ker \varphi^\vee)_{-d}$ is a subrepresentation of $G$ inside $(F^\vee)_{-d}$, hence $(\ker \varphi^\vee)_{-d} = (F^\vee)_{-d}$ or $(\ker \varphi^\vee)_{-d} = 0$ by the irreducibility of $(F^\vee)_{-d}$. The first option would imply $\varphi^\vee$ is the zero map, given that $F^\vee$ is generated in degree $-d$, and this would violate the assumption that $\varphi$ is non zero. Therefore $(\ker \varphi^\vee)_{-d} = 0$ and thus $\varphi^\vee$ is minimal by proposition 3.1.3. □

4.3. **Weight propagation along resolutions.** We can now develop an algorithm to propagate weights along minimal free resolutions. The following will be assumed throughout this section.

(a) The complex of free modules and maps in $\text{mod}_{\mathbb{Z}T} A$

$$F_*: \quad 0 \rightarrow F_m \xrightarrow{d_m} F_{m-1} \rightarrow \ldots \rightarrow F_1 \xrightarrow{d_1} F_{i-1} \rightarrow \ldots \rightarrow F_1 \xrightarrow{d_1} F_0$$

with $m \leq n$, is the minimal free resolution of a module $M$ in $\text{mod}_{\mathbb{Z}T} A$. 
(b) For all \( i \in \{1, \ldots, n\} \), the map \( d_i \) is provided in matrix form \( \mathcal{M}_{\tilde{F}_{i-1}}^{\tilde{F}_i}(d_i) \), where \( \tilde{F}_i \) is a homogeneous basis of \( F_i \) (also for \( i = 0 \)).

(c) There exists \( c \in \{0, \ldots, n\} \) such that \( F_c \) admits a homogeneous basis of weight vectors \( \tilde{F}_c \) such that \( \mathcal{M}_{\tilde{F}_c}^{\tilde{F}_c}(\text{id}_{\tilde{F}_c}) \) is upper (resp. lower) triangular.

(d) \( V_c \) is an ordered list containing the weights of the elements of \( \tilde{F}_c \).

(e) For all \( i \in \{1, \ldots, c\} \), the maps \( d_i^\gamma : F_{i-1}^\gamma \to F_i^\gamma \) are minimal.

(f) For all \( i \in \{0, \ldots, n\} \), \( \mathbb{T}^m(\tilde{F}_i) \) is equipped with a position up (resp. down) module term ordering.

The reason for requiring conditions (c)–(e) is so that the weights can be specified for any one free module in the complex \( F_* \). The idea is then to propagate weights backwards along the maps \( d_{c+1}, \ldots, d_m \) (using algorithm 2) and forward along the maps \( d_1, \ldots, d_1 \) (using algorithm 3).

**Remark 4.3.1.** The minimal free resolution of a module \( M \) is typically obtained, in a computational setting, from a presentation of \( M \). If the presentation is minimal, then it is also the first differential \( d_1 : F_1 \to F_0 \) in the resolution. In our experience, homogeneous bases of weight vectors are often the most natural choice to write a matrix for the presentation \( d_1 \). Using the notation above, this means we have \( \tilde{F}_0 = F_0 \); then \( \mathcal{M}_{\tilde{F}_0}^{\tilde{F}_0}(\text{id}_{\tilde{F}_0}) \) is the identity matrix and assumption (c) is easily met.

**Remark 4.3.2.** Sometimes the minimal free resolution of a module \( M \) can be constructed starting from some differential \( d_{c+1} : F_{c+1} \to F_c \) other than the first one. Our algorithm is designed to deal with this more general setup. For examples of resolutions constructed from the middle (or from the end) we invite the reader to consult [Gal13a].

**Algorithm 4:** weight propagation along a resolution

1: function PropagateResolution(\( \mathcal{M}_{\tilde{F}_0}^{\tilde{F}_1}(d_1), \ldots, \mathcal{M}_{\tilde{F}_{m-1}}^{\tilde{F}_m}(d_m), V_c \))
2: \( C_c := \mathcal{M}_{\tilde{F}_c}^{\tilde{F}_c}(\text{id}_{\tilde{F}_c}) \)
3: for \( i \in \{1, \ldots, m - c\} \) do
4: \( (C_{c+i}, V_{c+i}) := \text{Propagate}(C_{c+i-1}^{-1}\mathcal{M}_{\tilde{F}_{c+i-1}}^{\tilde{F}_{c+i}}(d_{c+i}), V_{c+i-1}) \)
5: end for
6: for \( i \in \{1, \ldots, c\} \) do
7: \( (C_{c-i}, V_{c-i}) := \text{PropagateForward}(\mathcal{M}_{\tilde{F}_{c-i}}^{\tilde{F}_{c-i+1}}(d_{c-i+1})C_{c-i+1}^{-1}, V_{c-i+1}) \)
8: end for
9: return \((V_0, \ldots, V_m)\)
10: end function

**Proposition 4.3.3.** Under the assumptions of this section, algorithm 4 returns a tuple \((V_0, \ldots, V_m)\), where, for all \( i \in \{0, \ldots, m\} \), \( V_i \) is an ordered list containing the weights of the elements of a homogeneous basis of weight vectors \( \tilde{F}_i \) of \( F_i \).

**Proof.** We construct inductively pairs \((C_{c+i}, V_{c+i})\) satisfying the following properties:

- \( \exists \tilde{F}_{c+i} \) homogeneous basis of \( F_{c+i} \) such that \( C_{c+i} = \mathcal{M}_{\tilde{F}_{c+i}}^{\tilde{F}_{c+i}}(\text{id}_{\tilde{F}_{c+i}}) \);
- \( \exists \tilde{F}_{c+i} \) homogeneous basis of weight vectors of \( F_{c+i} \) such that \( \mathcal{M}_{\tilde{F}_{c+i}}^{\tilde{F}_{c+i}}(\text{id}_{\tilde{F}_{c+i}}) \) is upper (resp. lower) triangular, and \( V_{c+i} \) contains the weights of the elements of \( \tilde{F}_{c+i} \).
To start the induction, take \( C_c = \mathcal{M}^{\mathcal{F}_c}(\text{id}_{\mathcal{F}_c}) \), the identity matrix, and \( V_c \) as given in the input. Suppose \((C_{c+i-1}, V_{c+i-1})\) has been constructed for \( i > 0 \). Then
\[
C^{-1}_{c+i-1} \mathcal{M}^{\mathcal{F}_{c+i}}(d_{c+i}) = \mathcal{M}^{\mathcal{F}_{c+i+1}}(d_{c+i})
\]
and the matrix \( \mathcal{M}^{\mathcal{F}_{c+i+1}}(\text{id}_{\mathcal{F}_{c+i+1}}) \) is upper (resp. lower) triangular by the inductive hypothesis. Since all assumptions of algorithm 2 hold, we can apply it as indicated on line 4 to produce the next pair \((C_{c+i}, V_{c+i})\). The properties of the pair are then a consequence of proposition 4.1.1. The process ends when \( c + i > m \).

Similarly, we can produce pairs \((C_{c-i}, V_{c-i})\) for \( i > 0 \), using algorithm 3 as indicated on line 7. The process ends when \( c - i < 0 \). \(\square\)

**Remark 4.3.4.** If \( F_* \) is the minimal free resolution of a Cohen-Macaulay \( A \)-module \( M \) (of grade \( g \)), then \( F'_* \) is a minimal free resolution (of the module \( \text{Ext}_A^1(M, A) \)) (see [BH93, p. 12]). In particular, the duals of all differentials in \( F_* \) are minimal maps.

**Example 4.3.5.** Let \( A = \mathbb{Q}[x] \). The complex
\[
0 \rightarrow A(-1) \xrightarrow{(x)} A^2
\]
is the minimal free resolution of \( A/(x) \oplus A \). The dual of the (only) differential in the resolution is not minimal, as evidenced in example 4.2.2. The module \( A/(x) \oplus A \) has dimension 1 but depth 0, and therefore it is not Cohen-Macaulay.

**Remark 4.3.6.** Suppose \( \mathcal{M}^{\mathcal{F}_c}(\text{id}_{\mathcal{F}_c}) \) is diagonal; this is the case if, for example, \( \mathcal{F}_c = \tilde{\mathcal{F}}_c \). A diagonal matrix is both upper and lower triangular, and so is its transpose. Therefore we can propagate weights forward along \( d_c, \ldots, d_1 \), with a modified algorithm 3 that does not need to switch between position up and down module term orderings. This observation proved useful in our implementation of algorithm 4 [Gal13b] in the software system Macaulay2 [GS], where the module term ordering for all free modules is specified once and for all with the declaration of a ring.

4.4. **Weight propagation for graded components.** Our last algorithm can be used to recover the weights of the graded components of a module starting from a presentation. The following is assumed.

(a) The sequence
\[
F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \rightarrow 0
\]
is a presentation of the module \( M \) in \( \text{mod}_{\text{cyc}} A \).

(b) The map \( d_1 \) is provided in matrix form \( \mathcal{M}^{\mathcal{F}_1}(d_1) \), where \( \mathcal{F}_0 \) is a homogeneous basis of \( F_0 \) and \( \mathcal{F}_1 \) is a homogeneous basis of \( F_1 \).

(c) \( \mathcal{T}^n\langle \mathcal{F}_0 \rangle \) is equipped with a position up (resp. down) module term ordering.

(d) \( F_0 \) admits a homogeneous basis of weight vectors \( \mathcal{F}_0 = \{f_1, \ldots, f_s\} \) such that \( \mathcal{M}^{\mathcal{F}_0}(\text{id}_{\mathcal{F}_0}) \) is upper (resp. lower) triangular.

(e) \( W = \{w_1, \ldots, w_s\} \) is an ordered list with \( w_i = \text{weight}(f_i), \forall i \in \{1, \ldots, s\} \).

**Algorithm 5:** weight propagation for graded components

1: \textbf{function} PROPAGATE GRADED COMPONENTS(\( d, \mathcal{M}^{\mathcal{F}_1}(d_1), W \))
2: \hspace{1cm} compute \( \mathcal{G} \), homogeneous Gröbner basis of \( \text{im} \, d_1 \), using \( \mathcal{M}^{\mathcal{F}_1}(d_1) \)
Let \( t \) be an ordering, we deduce LT\( T \) and consider \( t \) since the terms are elements of \( \tilde{F} \).

Proof. We organize the proof into six separate steps.

**Step 1: a basis of \( M_d \).**

Let \( G \) be a \( \text{Gröbner} \) basis of \( \text{im} \, d_1 \) in the module term ordering on \( T^n \langle F_0 \rangle \). Suppose \( F_0 = \{ f_1, \ldots, f_s \} \). Define the set

\[
B_d := \{ tf_i \in T^n \langle F_0 \rangle \mid \deg(tf_i) = d \text{ and } \forall g \in G, \text{LT}(g) \nmid tf_i \},
\]

consisting of all the degree \( d \) terms of \( T^n \langle F_0 \rangle \) that are not multiples of the leading terms of some element in \( G \). By Macaulay’s basis theorem [KR00, Cor. 2.4.11], the residue classes of elements in \( B_d \) modulo \( \text{im} \, d_1 \) form a \( \mathbb{K} \)-basis of \( M_d \). In particular, \( \dim_{\mathbb{K}}(B_d)_{\mathbb{K}} = \dim_{\mathbb{K}} M_d \), where \( (B_d)_{\mathbb{K}} \) is the \( \mathbb{K} \)-vector subspace of \( (F_0)_d \) generated by the terms in \( B_d \).

**Step 2: a subrepresentation of \((F_0)_d\).**

Define

\[
\tilde{B}_d := \{ t\tilde{f}_i \in F_0 \mid tf_i \in B_d \},
\]

and consider \( \langle \tilde{B}_d \rangle_{\mathbb{K}} \), the \( \mathbb{K} \)-vector subspace of \( (F_0)_d \) generated by \( \tilde{B}_d \). Notice that all elements of \( \tilde{B}_d \) are \( \mathbb{K} \)-linearly independent, hence \( \dim_{\mathbb{K}}(\tilde{B}_d)_{\mathbb{K}} = \dim_{\mathbb{K}}(B_d)_{\mathbb{K}} = \dim_{\mathbb{K}} M_d \). Moreover each element \( t\tilde{f}_i \in \tilde{B}_d \) is a weight vector for the action of \( T \), since the terms \( t \in T^n \) are weight vectors, by proposition 2.3.4, and the \( \tilde{f}_i \) come from a homogeneous basis of \( \mathbb{K} \)-weights of \( F_0 \).

**Step 3: \((F_0)_d = (\text{ker } \pi)_d \oplus \langle \tilde{B}_d \rangle_{\mathbb{K}}\).**

Let \( t\tilde{f}_j \in \tilde{B}_d \). Since \( M_{\tilde{f}_j}^{F_0}(\text{id}_{F_0}) \) is upper (resp. lower) triangular, so is its inverse \( M_{\tilde{f}_j}^{F_0}(\text{id}_{F_0}) \). Suppose \( M_{\tilde{f}_j}^{F_0}(\text{id}_{F_0}) = (u_{i,j}) \), for some \( u_{i,j} \in \mathbb{K} \). Then

\[
t\tilde{f}_j = \sum_{i=1}^{j} u_{i,j} tf_i \quad \text{(resp. } t\tilde{f}_j = \sum_{i=j}^{s} u_{i,j} t\tilde{f}_i \text{)}.
\]

Seeing how \( T^n \langle F_0 \rangle \) is equipped with a position up (resp. down) module term ordering, we deduce LT\( (t\tilde{f}_j) = t\tilde{f}_j \in B_d \).

Now consider any element \( f \in \langle \tilde{B}_d \rangle_{\mathbb{K}} \). Since \( f \) is a \( \mathbb{K} \)-linear combination of elements in \( \tilde{B}_d \), we must have LT\( (f) \in B_d \) by what just observed. This implies that LT\( (f) \) is not divisible by the leading term of any element of \( G \), therefore the remainder of \( f \) upon division by the elements of \( G \) is \( f \) itself.

Suppose \( f \in (\text{ker } \pi)_d \cap \langle \tilde{B}_d \rangle_{\mathbb{K}} \). Since ker \( \pi = \text{im} \, d_1 \) and \( G \) is a \( \text{Gröbner} \) basis of \( \text{im} \, d_1 \), the remainder of \( f \) upon division by elements of \( G \) is zero. This forces \( f = 0 \) and hence the sum of \( (\text{ker } \pi)_d \) and \( \langle \tilde{B}_d \rangle_{\mathbb{K}} \), as subspaces of \((F_0)_d\), is direct. Looking at dimensions we obtain:

\[
\dim_{\mathbb{K}}(\text{ker } \pi)_d + \dim_{\mathbb{K}}(\tilde{B}_d)_{\mathbb{K}} = \dim_{\mathbb{K}}(\text{ker } \pi)_d + \dim_{\mathbb{K}} M_d = \dim_{\mathbb{K}}(F_0)_d.
\]
since $\pi$ is surjective. We conclude $(F_0)_d = \langle \ker \pi \rangle_d \oplus \langle \tilde{B}_d \rangle_\mathbb{K}$. Notice that $(\ker \pi)_d$ is a subrepresentation of $T$ in $(F_0)_d$, since $\pi$ is $T$-equivariant. Therefore the direct sum decomposition holds as a decomposition of representations of $T$.

**Step 4: an explicit section of $\pi$ in degree $d$.**

We will define a $T$-equivariant map $\hat{\varphi} : M_d \to F_0$ such that $\forall m \in M_d$, $\pi(\hat{\varphi}(m)) = m$, a section of $\pi$ in degree $d$. Recall that the elements of $\tilde{B}_d$ are weight vectors in $\langle \tilde{B}_d \rangle_\mathbb{K}$, so that $\pi(\tilde{B}_d)$ is a set of weight vectors in $M_d$. Given the decomposition $(F_0)_d = \langle \ker \pi \rangle_d \oplus \langle \tilde{B}_d \rangle_\mathbb{K}$, we have $\pi(\langle \tilde{B}_d \rangle_\mathbb{K}) = M_d$. Since $\dim_\mathbb{K} \langle \tilde{B}_d \rangle_\mathbb{K} = \dim_\mathbb{K} M_d$, $\pi(\tilde{B}_d)$ is actually a basis of weight vectors of $M_d$. Let us define $\hat{\varphi}$ on $\pi(\tilde{B}_d)$ by setting $\hat{\varphi}(\pi(b)) := \tilde{b}$, $\forall \tilde{b} \in \tilde{B}_d$. Extending linearly gives a $\mathbb{K}$-linear map $\hat{\varphi} : M_d \to F_0$. Since $\pi(\hat{\varphi}(\pi(b))) = \pi(b)$, $\forall \tilde{b} \in \tilde{B}_d$, the map $\hat{\varphi}$ is a section of $\pi$ in degree $d$.

To show that $\hat{\varphi}$ is $T$-equivariant, it is enough to observe that, $\forall \tilde{b} \in \tilde{B}_d$, $\pi(\hat{b})$ is a weight vector with the same weight as $\tilde{b}$, by proposition 2.2.3, because $\pi$ is $T$-equivariant.

**Step 5: the map $\varphi : E \to F_0$.**

By the universal property of free modules in mod-$\mathcal{O}_T A$, $\exists ! \varphi : M_d \otimes \mathbb{K} A \to F_0$ morphism in mod-$\mathcal{O}_T A$ such that $\hat{\varphi} = \varphi \circ i_{M_d}$, where $i_{M_d} : M_d \to M_d \otimes \mathbb{K} A$ sends an element $m \in M_d$ to $m \otimes 1_A$. Set $E := M_d \otimes \mathbb{K} A$. Because $\pi(\tilde{B}_d)$ is a basis of weight vectors of $M_d$, the set $\hat{E} := \{ \pi(b) \otimes 1_A : b \in B_d \}$ is a homogeneous basis of weight vectors of $E$. Moreover, $\forall \tilde{b} \in \tilde{B}_d$, we have

$$\varphi(\pi(\tilde{b}) \otimes 1_A) = \hat{\varphi}(\pi(b)) = \pi(b) \in \langle \tilde{B}_d \rangle_\mathbb{K} \subseteq (F_0)_d.$$ 

Notice that $i_{M_d}$ is an isomorphism in degree $d$, and that $\hat{\varphi}$ is injective because it is a section of $\pi$ in degree $d$; therefore $\varphi$ is injective in degree $d$. Because $\hat{E}$ is generated in degree $d$, we conclude that $\varphi$ is a minimal map (see definition 3.1.4).

**Step 6: an explicit matrix of $\varphi$.**

We will describe the matrix $N := M^E_{\hat{E}_d}(\varphi)$. The elements in $\tilde{B}_d$ are, by definition, of the form $t\tilde{f}_i$, where $t\tilde{f}_i \in B_d$. For $t\tilde{f}_i \in \tilde{B}_d$, we have, by construction,

$$\varphi(\pi(t \tilde{f}_i) \otimes 1_A) = t \tilde{f}_i.$$ 

Therefore the column of $N$ corresponding to the element $\pi(t \tilde{f}_i) \otimes 1_A \in \hat{E}$ has the term $t$ in the $i$-th row and zeros everywhere else. In other words, $N$ is the matrix whose columns are the column vectors of terms in $B_d$ expressed in the homogeneous basis $F_0$ of $F$. Since $B_d$ can be obtained explicitly after computing a Gröbner basis of $\im d_1$, this construction of the matrix $N$ can be carried out explicitly.

Finally we can use algorithm 2, with input the matrix of $\varphi$ just described and the list of weights $W$ of $F_0$, to recover the weights of $E = M_d \otimes \mathbb{K} A$. Since these are the same as the weights of $M_d$, this concludes the proof.

4.5. Computing over subfields. Let $\mathbb{L}$ and $\mathbb{K}$ be fields with $\mathbb{K} \subseteq \mathbb{L}$. Consider the polynomial ring $A_L := \mathbb{L}[x_1, \ldots, x_n]$ with a positive $\mathbb{Z}^n$-grading and identify the polynomial ring $A_K := \mathbb{K}[x_1, \ldots, x_n]$ with a (graded) subring of $A_L$. As usual, all our modules (over $A_L$ or $A_K$) will be finitely generated and graded. Consider a graded $A_L$-submodule $M$ of the free module $F = \bigoplus_{d \in \mathbb{Z}_+^m} A_L(-d)^{\delta_d}$. Following [KR90, Defin. 2.4.14], $M$ is defined over $\mathbb{K}$ if there exist elements $m_1, \ldots, m_l$ in the free $A_K$-module $\bigoplus_{d \in \mathbb{Z}_+^m} A_K(-d)^{\delta_d} \subseteq F$ which generate $M$ as an $A_L$-module.

If $M$ is defined over $\mathbb{K}$, then
• computing the reduced Gröbner basis $G$ of $M$ over $K$ or over $L$, using the elements $m_1, \ldots, m_l$, yields the same result, by [KR00, Prop. 2.4.16.b];
• the leading terms of the elements of $G$ do not depend on the field used for the computation, by [KR00, Prop. 2.4.16.a];
• the matrix of the change of basis between the elements $m_1, \ldots, m_l$ and the elements of $G$ has entries in $A_K$.

The third bullet point is an immediate consequence of the first one.

To explain how this affects our algorithms, let $T$ be a torus over $L$ with an $L$-linear action on $A_L$ that is compatible with grading and multiplication. Let $\varphi: E \to F$ be a minimal map of free modules in the category $\text{mod}_{\Gamma_T} A_L$. Suppose there exist homogeneous bases $E$ of $E$ and $F$ of $F$ such that $M^E_T(\varphi)$ has entries in $K$; equivalently, $\text{im} \, \varphi$ is defined over $K$. Our previous observations imply that the steps used in algorithm 1 do not depend on the field. In practice, using algorithm 1 or 2 to recover the weights of $E$ from the weights of $F$ and the map $\varphi$, will produce the same result whether we carry out our computations over $L$ or over $K$.

Algorithms 3, 4 and 5 are based on algorithm 2 so we expect them to work over subfields as well. Indeed they do, because of the following additional comments. If $\varphi: E \to F$ is a minimal map of free modules in $\text{mod}_{\Gamma_T} A_L$ with $\text{im} \, \varphi$ defined over $K$, then:

• the image of the dual map $\varphi^\vee$ is also defined over $K$ (clearly since a matrix of $\varphi^\vee$ is the transpose of a matrix of $\varphi$);
• the syzygy module of a Gröbner basis of $\text{im} \, \varphi$ is also defined over $K$ (this implies the minimal free resolution of a module can be computed over $K$);
• each graded component of $\text{im} \, \varphi$ has a basis consisting of those terms that are not divisible by the leading terms of the elements in a Gröbner basis of $\text{im} \, \varphi$, in particular this basis does not depend on the field.

Remark 4.5.1. The possibility of performing our algorithms over a subfield $K$ of $L$ is especially useful in the case where computations over $L$ are not feasible. To further illustrate the issue, we discuss the setup for our implementation [Gal13b] in the software system Macaulay2 [GS].

Let $G$ be a complex semisimple algebraic group and let $T$ be a maximal torus in $G$. Every finite dimensional representation $V$ of $G$ is uniquely determined by its weights (counted with multiplicity) for the action of $T$. Moreover $V$ decomposes uniquely into a direct sum of irreducible representations parametrized by so-called highest weights. Given a complete list of weights of $V$ for the action of $T$, the highest weights can be recovered using a recursive formula of Freudenthal [Hum78, §22.3] which holds over $C$.

Our objects of interest are modules in $\text{mod}_{\Gamma_T} A_C$ that are defined over $Q$. While it is not possible to compute over $C$, we can compute over $Q$ in Macaulay2. In particular, we can calculate minimal free resolutions and graded components over the rationals. The implementation of our algorithms provides lists of weights that can then be interpreted and decomposed over the complex numbers.

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