A Reconstruction theorem for homeomorphism groups without small sets and non-shrinking functions of a normed space

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1 Introduction

Let \( X \) be a topological space and \( G \) be a subgroup of the group \( H(X) \) of all auto-homeomorphisms of \( X \). The pair \( (X, G) \) is then called a space-group pair. Let \( K \) be a class of space-group pairs. \( K \) is called a faithfull class if for every \( (X, G), (Y, H) \in K \) and an isomorphism \( \varphi \) between the groups \( G \) and \( H \) there is a homeomorphism \( \tau \) between \( X \) and \( Y \) such that \( \varphi(g) = \tau \circ g \circ \tau^{-1} \) for every \( g \in G \).

The first important theorem on faithfulness is due to J. Whittaker [W] (1963). He proved that the class of homeomorphism groups of Euclidean manifolds is faithful. That is, \( \{(X, H(X)) \mid X \text{ is a Euclidean manifold}\} \) is faithful.

Other faithfulness theorems were proved in R. McCoy [McC] 1972, W. Ling [Lg1] 1980, M. Rubin [Ru1] 1989, M. Rubin [Ru2] 1989, K. Kawamura [Ka] 1995, M. Brin [Br1] 1996, M. Rubin [Ru3] 1996, A. Banyaga [Ba1] 1997, A. Leiderman and R. Rubin [LR] 1999 and J. Borzellino and V. Brunsden 2000.
Among the classes shown to be faithful in [Ru2] is the class of manifolds over normed spaces. In [RY] 2000, M. Rubin and Y. Yomdin obtained various strengthenings and continuations of the this result from [Ru2]. One central result from [RY] is the following theorem.

**Theorem A** Let $K_{LLIP}$ be the class of all space-group pairs $(X, G)$ such that $X$ is an open subset of a normed space and $G$ is a subgroup of $H(X)$ which contains all locally bilipschitz homeomorphisms of $X$. Then $K_{LLIP}$ is faithful.

Let $(X, G)$ be a space-group pair and $\emptyset \neq U \subseteq X$ be open. We say that $U$ is a small set with respect to $(X, G)$, if for every open nonempty $V \subseteq U$ there is $g \in G$ such that $g(U) \subseteq V$.

It is easy to show that if $X$ is an open subset of a normed space, and $LIP(X)$ is the group of all bilipschitz homeomorphisms of $X$, then the family of subsets of $X$ which are small with respect to $(X, LIP(X))$ is an open cover of $X$. The same is obviously true for any group of homeomorphisms containing $LIP(X)$. The existence of a cover consisting of small sets is indeed used in the proof of Theorem A. This fact is also used in all previous faithfulness results applicable to infinite dimensional normed spaces.

This leads to the question of discovering subgroups of $H(X)$ which are rich enough to allow the recovery of $X$, but which are not sufficiently big to admit small sets.

This work addresses this question. We prove a new faithfulness result (Theorem 2.3) which does not assume the existence of small sets. And we also construct a large class of groups which do not have small sets, and which are covered by this new faithfulness result.

Theorem 2.3 deals with a general class of first countable spaces. We apply it to the class of open subsets of normed spaces. It is also applicable outside the class of normed spaces. One such application - to the class of metrizable locally convex spaces, is proved in Theorem 4.9.

The following statement is a special case of Theorem 2.3. Let $(X, G)$ be
a space-group pair and $S \subseteq X$ be open. $S$ is *strongly flexible*, if for every infinite $A \subseteq S$ without accumulation points in $X$, there is a nonempty open set $V \subseteq X$ such that for every nonempty open set $W \subseteq V$ there is $g \in G$ such that the sets $\{a \in A \mid g(a) \in W\}$ and $\{a \in A \mid \text{for some neighborhood } U \text{ of } a, \ g \mid U = Id\}$ are infinite.

**Theorem B** Let $K_F$ be the class of all space-group pairs $(X, G)$ such that
1. $X$ is regular, first countable and has no isolated points.
2. For every $x \in X$ and an open neighborhood $U$ of $x$ the set
   $$\{g(x) \mid g \in G \text{ and } g \upharpoonright (X - U) = Id\}$$
   is somewhere dense.
3. The family of strongly flexible sets is a cover of $X$.
Then $K_F$ is faithful.

**Normed spaces**

We next describe our results for normed spaces. Let $LLIP(X)$ denote the group of locally bilipschitz homeomorphisms of a metric space $X$. For every nonempty open subset $X$ of an infinite dimensional normed space we shall define a certain subgroup $G_X \subseteq LLIP(X)$. We shall prove the following theorem concerning the $G_X$’s.

**Theorem C** (a) If $X$ is a nonempty open subset an infinite dimensional normed space, then $X$ does not have small sets with respect to $(X, G_X)$.
(b) Let $K_{NSML}$ be the class of all space-group pairs $(X, H)$ such that $G_X \subseteq H \subseteq H(X)$. Then $K_{NSML}$ is faithful.

The group $G_X$ is defined in Definition 4.2. Part (a) of Theorem B is a corollary of Theorem 3.1(a). Part (b) of Theorem B follows from Theorems 2.3 and 4.3.

Note that $K_{LLIP} \subseteq K_{NSML}$. Recall that for every $(X, H) \in K_{LLIP}$, the family of small sets is a cover of $X$. On the other hand, $(X, G_X)$ has no small sets at all, and it belongs to $K_{NSML}$. So $K_{LLIP} \subsetneq K_{NSML}$. Hence Theorem C(b) strengthens Theorem A.
It needs to be mentioned that if $E$ is an infinite dimensional normed space and $X \subseteq E$ is open and nonempty, then $G_X$ is obtained from $G_E$ in the following way.

$$G_X = \{ g|X \mid g \in G_E \text{ and } g|(E \setminus X) = Id \}.$$ 

So it suffices to show that $(E, G_E)$ does not have small sets.

The nonexistence of small sets follows from a stronger property which is called here the non-shrinking property. For a metric space $X$, $x \in X$ and $r > 0$ let $B(x, r)$ denote the closed ball with center at $x$ and radius $r$. Let $X$ be a metric space and $\Gamma$ be a semigroup of functions from $X$ to $X$. The operation in $\Gamma$ is composition of functions. $\Gamma$ is non-shrinking if there are no $g \in \Gamma$, $x \in X$ and $r > 0$ such that $g(B(x, r))$ is contained in a finite union of closed balls with radius $< r$.

The final result of Section 3 is Theorem 3.1. For a normed space $E$ we shall define two semigroups of functions from $E$ to $E$: $\Gamma_1(E)$ and $\Gamma_2(E)$. $\Gamma_1(E)$ is a subsemigroup of $\Gamma_2(E)$. Theorem 3.1(a) states that for every infinite dimensional normed space $E$, $\Gamma_1(E)$ is non-shrinking.

Theorem 3.1(b) says that if $E$ has an infinite dimensional subspace $F$ such that $c_0$ (the space of real sequences converging to 0), is not isomorphically embeddable in the completion of $F$, then $\Gamma_2(E)$ is non-shrinking.

Also observed in Section 3 is that $\Gamma_2(c_0)$ is not non-shrinking.

The group $G_E$ from Theorem C is contained in $\Gamma_1(E)$. So the fact that $G_E$ and hence $G_X$ has no small sets follows from Theorem 3.1(a).

Metrizable locally convex spaces

Theorem D is another application of Theorem 2.3. It is an analogue of Theorem A to the class of metrizable locally convex spaces.

**Theorem D**  Let $K_M$ be the class of all space-group pairs $(X, G)$ in which $X$ is an open subset of a locally convex metrizable topological vector space, and $G$ is a group containing all locally bi-uniformly continuous homeomorphisms of $X$. Then $K_M$ is faithful.
Theorem C is restated in 4.9(a).

Acknowledgements Lemma 3.5 which is used in the proof of Theorem 3.1(a) was found by Michael Levin. We thank him for his kind permission to include it. We had another proof that did not rely on Lemma 3.5. But the proof which uses Levin’s Lemma is simpler.

We also thank Arkady Leiderman for his help in the proof of Proposition 4.12.

2 The reconstruction theorem

Notations 2.1. Let $X$ be a topological space. If $A \subseteq X$. Then the closure and the interior of $A$ in $X$ are denoted respectively by $cl_X(A)$ and $int_X(A)$. Also, $acc_X(A)$ denotes the set of accumulation points of $A$ in $X$. If $x \in X$ then $Nbr_X(x)$ denotes the set of open neighborhoods of $x$ in $X$. The subscript $X$ is sometimes omitted.

Definition 2.2. Let $X$ be a topological space.

(a) A subset $A \subseteq X$ is discrete if $acc(A) = \emptyset$.

(b) Let $(X, G)$ be a space-group pair. The group $G$ is called a locally moving group of $X$, if for every open nonempty set $U \subseteq X$ there is $g \in G \setminus \{Id\}$ such that $g|(X \setminus U) = Id$.

(c) Let $(X, G)$ be a space-group pair, $A \subseteq X$ be infinite and $V \subseteq X$ be open and nonempty. We say that $V$ dissect $A$, if for for every open nonempty $W \subseteq V$ there is $g \in G$ such that $\{a \in A \mid g(a) \in W\}$ is infinite, and $\{a \in A \mid \text{there is } S \in Nbr(a) \text{ such that } g \upharpoonright S = Id\}$ is infinite. $A$ is dissectable if there is $V$ such that $V$ dissects $A$.

(d) Let $(X, G)$ be a space-group pair and $U \subseteq X$ be open. $U$ is flexible if there is a dense subset $D \subseteq U$ such that every infinite discrete subset of $D$ is dissectable.

(e) Let $(X, G)$ be a space-group pair.

(1) The set $D(X, G)$ is defined as follows. A point $x$ is in $D(X, G)$ iff
\{g(x) \mid g \in G\} is somewhere dense. That is, if \(\text{int}\left(\text{cl}\left(\{g(x) \mid g \in G\}\right)\right) \neq \emptyset\).

(2) Define the relation \(DF_{X,G}(x,y)\) as follows. \(DF_{X,G}(x,y)\) holds if the set \(\{g(x) \mid g \in G \text{ and there is } U \in \text{Nbr}(y) \text{ such that } g\upharpoonright U = \text{Id}\}\) is somewhere dense.

(f) Define the class \(K_1\) of space-group pairs as follows. Let \((X,G)\) be a space-group pair. \((X,G) \in K_1\) if the following hold:

(P1) \(X\) is regular and first countable.

(P2) \(G\) is a locally moving group of \(X\).

(P3) For every distinct \(x,y \in X\), \(DF_{X,G}(x,y)\) holds.

(P4) The set of flexible subsets of \(X\) is a cover of \(X\).

Define the class \(K\) of space-group pairs as follows. Let \((X,G)\) be a space-group pair. \((X,G) \in K\) if the following hold:

(P1) \(X\) is regular and first countable.

(P2) \(G\) is a locally moving group of \(X\).

(Q1) \(D(X,G)\) is a dense subset of \(X\).

(Q2) For every distinct \(x,y \in D(X,G)\), \(DF_{X,G}(x,y)\) holds or \(DF_{X,G}(y,x)\) holds.

(P4) The set of flexible subsets of \(X\) is a cover of \(X\).

Note that \(K_1 \subseteq K\). Note also that \(D(X,G)\) is invariant under \(G\).

**Theorem 2.3.** (a) \(K_1\) is faithful.

(b) For every \((X_1,G_1),(X_2,G_2) \in K\) and \(\varphi : G_1 \cong G_2\) there is \(\tau : D(X_1,G_1) \cong D(X_2,G_2)\) such that \(\tau\) induces \(\varphi\). That is, for every \(g \in G\), \(\varphi(g) \upharpoonright D(Y,H) = \tau \circ (g \upharpoonright D(X,G)) \circ \tau^{-1}\).

**Remark** Part (b) of Theorem 2.3 implies Part (a). In this work only Part (a) is used. But there are concrete classes of space-group pairs, which require the use of Part (b). An example of such a class, is the class of all space-group pairs \((X,G)\), in which \(X\) is the closure of an open subset of a normed space, and \(G\) is the group of all homeomorphisms of \(X\) which take the boundary of \(X\) to itself.

We shall use a theorem from [Ru3]. It is quoted here as Theorem 2.4. The set of regular open subsets of \(X\) is denoted by \(Ro(X)\). Recall that
U is a regular open set if \( \text{int}(\text{cl}(U)) = U \). For \( U, V \in \text{Ro}(X) \) define \( U + V = \text{int}(\text{cl}(U \cup V)) \) and \( \sim U = \text{int}(X - U) \). Then \( (\text{Ro}(X), +, \cap, \sim) \) is a complete Boolean algebra, which we denote by \( \text{Ro}(X) \). The partial ordering of the Boolean algebra \( \text{Ro}(X) \) is \( \subseteq \). Note that every \( g \in H(X) \) induces an automorphism of \( \text{Ro}(X) \) which we also denote by \( g \).

**Theorem 2.4.** Let \( \langle X, G \rangle \) and \( \langle Y, H \rangle \) be space-group pairs. Assume that \( G \) and \( H \) are locally moving groups of \( X \) and \( Y \) respectively. Let \( \varphi : G \cong H \). Then there is a unique \( \eta : \text{Ro}(X) \cong \text{Ro}(Y) \) such that for every \( g \in G \), \( \varphi(g) = \eta \circ g \circ \eta^{-1} \).

**Proof** See [Ru3] Definition 1.2, Corollary 1.4 or Corollary 2.10 and Proposition 1.8.

**Definition 2.5.** (a) Let \( B \) be a set of sets. \( B \) is called a pairwise disjoint family if for every distinct \( A, B \in B \), \( A \cap B = \emptyset \). \( B \) is called a regular open family, if \( B \subseteq \text{Ro}(X) \). Let \( A \) be an infinite set of subsets of \( X \) and \( x \in X \). \( \lim \mathcal{A} = x \), if for every \( S \in \text{Nbr}(x) \), \( \{ A \in \mathcal{A} \mid A \not\subseteq S \} \) is finite. \( A \) is convergent if for some \( x \in X \), \( \lim \mathcal{A} = x \).

(b) Let \( \mathcal{A} \) be a set of subsets of \( X \). \( \text{acc}(\mathcal{A}) \) is defined as follows: \( x \in \text{acc}(\mathcal{A}) \) iff there is \( B \subseteq \bigcup \mathcal{A} \) such that for every \( A \in \mathcal{A} \), \( |B \cap A| \leq 1 \) and \( x \in \text{acc}(B) \).

(c) Let \( \mathcal{U} \) be an infinite pairwise disjoint regular open family and \( V \in \text{Ro}(X) \setminus \{ \emptyset \} \). \( V \) absorbs \( \mathcal{U} \), if for every \( W \in \text{Ro}(X) \setminus \{ \emptyset \} \): if \( W \subseteq V \), then there is \( g \in G \) such that \( \{ U \in \mathcal{U} \mid g(U) \not\subseteq W \} \) is finite.

\( \mathcal{U} \) is absorbable if there is \( V \in \text{Ro}(X) \setminus \{ \emptyset \} \) such that \( V \) absorbs \( \mathcal{U} \).

(d) Let \( \mathcal{U} \) be an infinite pairwise disjoint regular open family and \( V \in \text{Ro}(X) \setminus \{ \emptyset \} \). \( V \) splits \( \mathcal{U} \) if for every \( W \in \text{Ro}(X) \setminus \{ \emptyset \} \): if \( W \subseteq V \), then there is \( g \in G \) such that \( \{ U \in \mathcal{U} \mid g(U) \cap W \neq \emptyset \} \) is infinite and \( \{ U \in \mathcal{U} \mid \text{there is } U' \in \text{Ro}(X) \setminus \{ \emptyset \} \text{ such that } U' \subseteq U \text{ and } g \upharpoonright U' = \text{Id} \} \) is infinite.

\( \mathcal{U} \) is splittable if there is \( V \in \text{Ro}(X) \setminus \{ \emptyset \} \) such that \( V \) splits \( \mathcal{U} \).
Proposition 2.6. Let \( U \) be an infinite pairwise disjoint regular open family and \( A \subseteq \bigcup U \). Suppose that for every \( U \in U \), \( |A \cap U| \leq 1 \), and that \( A \) dissectable. Then \( U \) is splittable.

**Proof** Trivial. \( \square \)

Proposition 2.7. Let \((X, G)\) be a space-group pair, and let \( U \) denote a regular open family in \( X \). We define \( \psi_{\text{cnvrg}}(U) \) to be the following property of \( U \).

(\( C_1 \)) \( U \) is infinite and pairwise disjoint.

(\( C_2 \)) \( U \) is absorbable.

(\( C_3 \)) \( U \) is not splittable.

Let \((X, G) \in K \) and \( U \subseteq \text{Ro}(X) \).

The following are equivalent.

(1) \( U \) is pairwise disjoint and convergent, and \( \lim U \in D(X, G) \).

(2) \( U \) satisfies \( \psi_{\text{cnvrg}} \).

**Proof** (1) \( \Rightarrow \) (2) Let \( U \) be pairwise disjoint and convergent, and \( \lim U \in D(X, G) \). Clearly, \( U \) fulfills (\( C_1 \)).

Let \( x = \lim U \). There is \( V \in \text{Ro}(X) \setminus \{\emptyset\} \) such that \( \{g(x) \mid g \in G\} \cap V \) is dense in \( V \). Let \( W \subseteq V \) and \( W \in \text{Ro}(X) \setminus \{\emptyset\} \). Then there is \( g \in G \) such that \( g(x) \in W \). So \( \{U \in U \mid g(U) \not\subseteq W\} \) is finite. Hence \( U \) satisfies (\( C_2 \)).

Let \( S \in \text{Ro}(X) \setminus \{\emptyset\} \). We show that \( S \) does not split \( U \). There is \( W' \subseteq S \) such that \( W'' \in \text{Ro}(X) \setminus \{\emptyset\} \) and \( |\{U \in U \mid U \cap W' \neq \emptyset\}| \leq 1 \). Let \( W \in \text{Ro}(X) \setminus \{\emptyset\} \) be such that \( \text{cl}(W) \subseteq W' \). Let \( g \in G \), and suppose that \( V := \{V' \in U \mid g(V') \cap W \neq \emptyset\} \) is infinite. Since \( x = \lim V \), \( g(x) = \lim g(V) \). Hence \( g(x) \in \text{cl}(W) \subseteq W' \). So \( \{V' \in U \mid g(V') \not\subseteq W'\} \) is finite. Thus \( \{V' \in U \mid \text{there is } V'' \in \text{Ro}(X) \setminus \{\emptyset\} \text{ such that } V'' \subseteq V' \text{ and } g \upharpoonright V'' = Id\} \) is finite. Hence \( S \) does not split \( U \). So (\( C_3 \)) holds.

(2) \( \Rightarrow \) (1) Suppose that \( U \) satisfies \( \psi_{\text{cnvrg}} \). So there is \( V \in \text{Ro}(X) \setminus \{\emptyset\} \) such that \( V \) absorbs \( U \). Let \( S' \) be an open nonempty flexible set which intersects \( V \) and \( S = S' \cap V \). Then \( \emptyset \neq S \subseteq V \) and \( S \) is flexible. Let \( g \in G \) be such that \( \{U \in U \mid g(U) \not\subseteq S\} \) is finite. Let \( U' = \{g(U) \mid U \in \text{Ro}(X) \setminus \{\emptyset\}\} \).
Proposition 2.8. Let $\psi^{1}_{eq}(U,V)$ be the following property.
There does not exist a nonempty regular open set $S$ such that: For every $T \in Ro(X) \setminus \{\emptyset\}$: if $T \subseteq S$, then there is $g \in G$ such that $\{\ U \in U \mid g(U) \not\subseteq T \}$ is finite, and $\{\ V \in V \mid g \upharpoonright V \neq Id \}$ is finite. Let

$$\psi_{eq}(U, V) \equiv \psi_{eq}^1(U, V) \land \psi_{eq}^1(V, U).$$

Let $U, V$ be convergent pairwise disjoint regular open families such that $\lim U, \lim V \in D(X, G)$. Then $\psi_{eq}(U, V)$ holds iff $\lim U = \lim V$.

**Proof** Suppose that $\lim U = \lim V$. Let $S \in Ro(X) \setminus \{\emptyset\}$. Let $x = \lim U$. Choose $T \in Ro(X) \setminus \{\emptyset\}$ such that $T \subseteq S$ and $x \not\in \text{cl}(T)$. Let $g \in G$ be such that $A := \{\ U \in U \mid g(U) \not\subseteq T \}$ is finite. Since $x = \lim (U \setminus A)$, $g(x) \in \text{cl}(T)$. So $g(x) \neq x$. Let $Q, R$ be pairwise disjoint neighborhoods of $x$ and $g(x)$ respectively. Since $x = \lim V$, $g(x) = \lim g(V)$. Hence $\{V \in V \mid g(V) \not\subseteq R \}$ is finite. Since $\lim V = x$ and $Q \in Nbr(x)$, $\{V \in V \mid V \not\subseteq Q \}$ is finite. Hence $g(V) \cap V = \emptyset$ for all but finitely many members of $V$. So $\{V \in V \mid g \upharpoonright V \neq Id \}$ is infinite. Hence $(U, V)$ fulfills $\psi_{eq}^1$. Similarly, $(V, U)$ fulfills $\psi_{eq}^1$. So $(U, V)$ fulfills $\psi_{eq}$.

Let $U, V$ be convergent pairwise disjoint regular open families such that $\lim U, \lim V \in D(X, G)$. Suppose that $x = \lim U \neq \lim V = y$. By (Q2) of Definition 2.2(f), $DF_{X,G}(x, y)$ holds or $DF_{X,G}(y, x)$ holds. We may assume that $DF_{X,G}(x, y)$ holds. Let $S$ be a regular open set such that $\{g(x) \mid g \in G$ and for some $R \in Nbr(y)$, $g \upharpoonright R = Id \}$ is dense in $S$. Let $T \subseteq S$ be regular open and nonempty. There is $g \in G$ such that $g(x) \in T$ and for some $R \in Nbr(y)$, $g \upharpoonright R = Id$. Then $\{U \in U \mid g(U) \not\subseteq T \}$ is finite. Let $V' = \{V \in V \mid V \subseteq R \}$. Then $V \setminus V'$ is finite, and for every $V \in V'$, $g \upharpoonright V = Id$. That is, $\{V \in V \mid g \upharpoonright V \neq Id \}$ is finite. Hence $(U, V)$ does not fulfill $\psi_{eq}$.

**Proposition 2.9.** Let $\psi_{eq}(U, V)$ be the following property.

For every $W$: if $\psi_{eq}(W)$ and $\psi_{eq}(U, W)$, then $\{W \in W \mid W \not\subseteq V \}$ is finite.

Then for every $U$ satisfying $\psi_{eq}$ and $V \in Ro(X)$: $\psi_{eq}(U, V)$ holds iff $\lim U \in V$.

**Proof** It is obvious that if $\lim U \in V$, then $\psi_{eq}(U, V)$ holds. Suppose that $\lim U \not\in V$. Let $x = \lim U$. Since $V \in Ro(X)$, $x \not\in \text{int}(\text{cl}(V))$. So
\(x \in \text{cl}(X \setminus \text{cl}(V))\). Hence there is a regular open pairwise disjoint family \(\mathcal{W}\) such that for every \(W \in \mathcal{W}\), \(W \subseteq X \setminus \text{cl}(V)\) and \(\lim \mathcal{W} = x\). So \(\psi_{eq}(\mathcal{U}, \mathcal{W})\) holds and \(\{W \in \mathcal{W} | W \not\subseteq V\}\) is not finite. So \(\psi_{eq}(\mathcal{U}, V)\) does not hold.

\[\square\]

**Proof of Theorem 2.3** Let \((X, G), (Y, H) \in K\) and \(\varphi : G \cong H\). By Theorem 2.4, there is \(\eta : \text{Ro}(X) \cong \text{Ro}(X)\) such that \(\varphi(g) = \eta \circ g \circ \eta^{-1}\) for every \(g \in G\). That is, for every \(g \in G\) and \(U \in \text{Ro}(X)\), \(\varphi(g)(\eta(U)) = \eta(g(U))\).

We define \(\tau : D(X, G) \rightarrow D(Y, H)\). Let \(x \in D(X, G)\). Let \(\mathcal{U}\) be a pairwise disjoint regular open family such that \(\lim \mathcal{U} = x\). By Proposition 2.7, \(\mathcal{U}\) satisfies \(\psi_{\text{cnvrg}}\). So \(\eta(\mathcal{U})\) satisfies \(\psi_{\text{cnvrg}}\). By Proposition 2.7, \(\eta(\mathcal{U})\) converges to a member of \(D(Y, H)\). Define \(\tau(x) = \lim \eta(\mathcal{U})\). If \(V\) is another pairwise disjoint regular open family converging to \(x\), then by Proposition 2.8, \(\psi_{eq}(\mathcal{U}, V)\) holds. So \(\psi_{eq}(\eta(\mathcal{U}), \eta(V))\) holds. Hence by Proposition 2.8, \(\lim \eta(\mathcal{U}) = \lim \eta(\mathcal{V})\). So the definition of \(\tau\) is independent of the choice of \(\mathcal{U}\). It follows easily that \(\tau\) is a bijection between \(D(X, G)\) and \(D(Y, H)\).

We show that \(\tau : D(X, G) \cong D(Y, H)\). It suffices to show that for every \(x \in D(X, G)\) and \(V \in \text{Ro}(X)\), \(x \in V\) iff \(\tau(x) \in \eta(V)\). Let \(\mathcal{U}\) be a pairwise disjoint regular open family converging to \(x\). By Proposition 2.9, \(\psi_{eq}(\mathcal{U}, V)\) holds iff \(x \in V\). Also, \(\psi_{eq}(\mathcal{U}, V)\) holds iff \(\psi_{eq}(\eta(\mathcal{U}), \eta(V))\) holds iff \(\lim \eta(\mathcal{U}) \in \eta(V)\). So \(x \in V\) iff \(\lim \eta(\mathcal{U}) \in \eta(V)\). That is, \(x \in V\) iff \(\tau(x) \in \eta(V)\).

It is left to the reader to check that for every \(g \in G\),

\[\varphi(g) \upharpoonright D(Y, H) = \tau \circ (g \upharpoonright D(X, G)) \circ \tau^{-1}.
\]

3 The non-shrinking property of the semigroups \(\Gamma_i(E)\)

In this section we consider two subsemigroups of \(\{f \upharpoonright D(X, G) \cong D(Y, H)\}\), where \(X\) is an infinite dimensional normed space: \(\Gamma_1(X)\) and \(\Gamma_2(X)\). By their definition, \(\Gamma_1(X) \subseteq \Gamma_2(X)\).
We prove that (1) for every infinite dimensional normed space $X$, $\Gamma_1(X)$ is non-shrinking. We also show that (2) if $X$ is an infinite dimensional normed space, and $X$ has an infinite-dimensional subspace $Y$, such that the completion of $Y$ does not have a subspace isomorphic to the space $c_0$ (of real sequences converging to 0), then $\Gamma_2(X)$ is non-shrinking.

Finally we prove that (3) $\Gamma_2(c_0)$ is not non-shrinking.

The proofs of (1) and (2) have a very similar structure. In both proofs we need to show that the unit sphere of $X$ does not have a cover with certain properties. But for spaces which do not embed $c_0$ a stronger statement of this kind can be proved. The proof of (2) relies on the fact (Lemma 3.9) that if $X$ is an infinite dimensional Banach space, and does not have a subspace isomorphic to $c_0$ then $X$ has the property:

(*) The sphere of $X$ does not have a locally finite cover consisting of weakly closed sets which do not contain 0.

This fact is not true for a general infinite dimensional Banach space, and in fact, $c_0$ does not have this property.

The following fact (Corollary 3.7) replaces (*) when proving (1). Every infinite dimensional normed space $X$ has the property:

(**) The sphere of $X$ does not have a locally finite cover with finite order consisting of closed convex sets which do not contain 0.

A set $\mathcal{S}$ of subsets of a set $X$ has finite order $k \in \mathbb{N}$, if $\max_{x \in X} |\{S \in \mathcal{S} \mid x \in S\}| = k$.

Let $E$ be a normed space. Denote by $\overline{E}$ the completion of $E$. We define a set of functions $\hat{\Gamma}_2 = \hat{\Gamma}_2(E)$ from $E$ to $E$. A function $h : E \to E$ belongs to $\hat{\Gamma}_2$ if there is a set of pairs

$\{(\hat{S}_j, E_j) \mid j \in J\}$ such that:

(1) Every $\hat{S}_j$ is a weakly closed subset of $\overline{E}$.
(2) $\{\hat{S}_j \mid j \in J\}$ is locally finite in $\overline{E}$.
(3) $\bigcup\{\hat{S}_j \mid j \in J\}$ is bounded.
(4) For every $j \in J$ and $x \in \hat{S}_j \cap E$, 

A set $\mathcal{S}$ of subsets of a set $X$ has finite order $k \in \mathbb{N}$, if $\max_{x \in X} |\{S \in \mathcal{S} \mid x \in S\}| = k$. Let $E$ be a normed space. Denote by $\overline{E}$ the completion of $E$. We define a set of functions $\hat{\Gamma}_2 = \hat{\Gamma}_2(E)$ from $E$ to $E$. A function $h : E \to E$ belongs to $\hat{\Gamma}_2$ if there is a set of pairs

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(2) $\{\hat{S}_j \mid j \in J\}$ is locally finite in $\overline{E}$.
(3) $\bigcup\{\hat{S}_j \mid j \in J\}$ is bounded.
(4) For every $j \in J$ and $x \in \hat{S}_j \cap E$,
\[ h(x) - x \in E_j, \]
and for every \( x \in E \setminus \bigcup \{ \hat{S}_j \mid j \in J \} \),
\[ h(x) = x. \]

For every \( h \in \hat{\Gamma}_2 \) we pick a family \( \{ (\hat{S}_j, E_j) \mid j \in J \} \) satisfying the above and in which \( \hat{S}_i \neq \hat{S}_j \) for every distinct \( i, j \in J \). We then denote \( J, \hat{S}_j, E_j \) by \( J^h, \hat{S}_j^h \) and \( E_j^h \) respectively. If \( X \) is a metric space, \( x \in X \) and \( r > 0 \), define \( B_X(x, r) = \{ y \in X \mid d(x, y) \leq r \} \). For a normed space \( E \) denote \( B_E(0, 1) \) by \( B_E \).

Let \( \Gamma_2 = \Gamma_2(E) \) be the semigroup generated by \( \hat{\Gamma}_2 \), that is, the closure of \( \hat{\Gamma}_2 \) under composition.

Note that \( \hat{\Gamma}_2 \) contains discontinuous functions.

Let \( S \) be a set of subsets of a metric space \( (X, d) \) and \( r > 0 \). \( S \) is \textit{r-separated} if for every distinct \( S, T \in S \), \( d(S, T) > r \). We say that \( S \) is \textit{separated} if for some \( r > 0 \), \( S \) is \( r \)-separated.

We define \( \hat{\Gamma}_1 = \hat{\Gamma}_1(E) \). A function \( h : E \to E \) belongs to \( \hat{\Gamma}_1 \) if there is a set of pairs \( \{ (S_j, E_j) \mid j \in J \} \) such that:
(1) Every \( S_j \) is a closed convex subset of \( E \).
(2) \( \{ S_j \mid j \in J \} \) is separated.
(3) \( \bigcup \{ S_j \mid j \in J \} \) is bounded.
(4) For every \( j \in J \) and \( x \in S_j \),
\[ h(x) - x \in E_j, \]
and for every \( x \in E \setminus \bigcup \{ S_j \mid j \in J \} \),
\[ h(x) = x. \]

Let \( \Gamma_1 = \Gamma_1(E) \) be the semigroup generated by \( \hat{\Gamma}_1 \).

Note that \( \hat{\Gamma}_1 \subseteq \hat{\Gamma}_2 \). For \( h \in \hat{\Gamma}_1 \), define \( \hat{S}_j^h = \text{cl}_E(S_j^h) \). Then \( \{ (\hat{S}_j^h, E_j^h) \mid j \in J^h \} \) fulfills Clauses (1) - (4) in the definition of \( \Gamma_2 \).

The theorem below is the final goal of this section.

**Theorem 3.1.** (a) For every infinite dimensional normed space \( E \), \( \Gamma_1(E) \) is non-shrinking.

(b) Let \( E \) be an infinite dimensional normed space. Suppose that \( E \) con-
tains an infinite dimensional subspace $F$ such that $c_0$ is not isomorphic to a subspace of the completion of $F$. Then $\Gamma_2(E)$ is non-shrinking.

For a function $g$, $\text{supp}(g)$ is defined as $\{x \in \text{Dom}(g) \mid g(x) \neq x\}$. Clearly, $\text{supp}(h_n \circ \ldots \circ h_1) \subseteq \bigcup_{i=1}^{n} \text{supp}(h_i)$. So for every $g \in \Gamma_2$, $\text{supp}(g)$ is bounded.

The definition of the non-shrinking property relies on the choice of the metric on $X$. We observe that the non-shrinkingness of $\Gamma_1$ and $\Gamma_2$ persists when one norm on $E$ is replaced by an equivalent norm.

**Proposition 3.2.**

(a) If $\Gamma_2$ is not non-shrinking, then for every $s \in (0,1)$ there are $g \in \Gamma_2$ and a finite subset $\sigma \subseteq E$ such that $g(B_E) \subseteq \bigcup_{x \in \sigma} (x + s \cdot B_E)$.

(b) The same holds for $\Gamma_1$. That is, if $\Gamma_1$ is not non-shrinking, then for every $s \in (0,1)$ there are $g \in \Gamma_1$ and a finite subset $\sigma \subseteq E$ such that $g(B_E) \subseteq \bigcup_{x \in \sigma} (x + s \cdot B_E)$.

(c) Let $(E, \|\|)$ be a normed space, and $\|\||$ be an equivalent norm on $E$. If $\Gamma_2$ is non-shrinking with respect to $\|\||$, then it is non-shrinking with respect to $\|\||$.

(d) The same holds for $\Gamma_1$.

**Proof**

(a) Suppose that $g \in \Gamma_2$, $r \in (0,1)$ and $\sigma \subseteq E$ are such that $\sigma$ is finite and $g(B_E) \subseteq \bigcup_{x \in \sigma} (x + r \cdot B_E)$.

For $v \in E$ and $A \subseteq E$ let $\text{tr}_v$ be the function $x \mapsto x + v$ and $\text{tr}_{v,A} = \text{tr}_v \upharpoonright A \cup \text{Id} \upharpoonright (E \setminus A)$. So if $A$ is bounded, then $\text{tr}_{v,A} \in \Gamma_1$.

For $v \in E$ and $\lambda > 0$ let $g_{\lambda,v}$ be the function acting on $v + \lambda \cdot B_E$ in the same way that $g$ acts on $B_E$. That is, $g_{\lambda,v} = \text{af}_{\lambda,v} \circ g \circ (\text{af}_{\lambda,v})^{-1}$, where $\text{af}_{\lambda,v}$ is the affine function: $x \mapsto \lambda x + v$. Clearly, $g_{\lambda,v} \in \Gamma_2$ and $g_{\lambda,v}(v + \lambda \cdot B_E) \subseteq \bigcup_{x \in \sigma} (x + v + r \cdot B_E)$.

Let $\sigma = \{x_1, \ldots, x_k\}$. Choose $\{v_i \mid i \leq k\}$ such that for every $i < j \leq k$,

1. $\text{supp}(g_{v_i,v_j} \cap (v_j + r \cdot B_E)) = \emptyset$;
2. $g_{v_i,v_j}(v_i + r \cdot B_E) \cap \text{supp}(g_{v_j,v}) = \emptyset$;
3. $v_i + r \cdot B_E \cap \bigcup_{\ell=1}^{k} (x_{\ell} + r \cdot B_E) = \emptyset$.

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Let \( A_i = x_i + r \cdot B_E \setminus \bigcup_{\ell < i}(x_{\ell} + r \cdot B_E) \), \( f_i = \text{tr}_{v_i - x_i,A_i} \) and \( f = f_k \circ \ldots \circ f_1 \).

Relying on (3),

\[(3.1) \text{ For every } i \leq k, \quad f(x_i + r \cdot B_E \setminus \bigcup_{\ell < i}(x_{\ell} + r \cdot B_E)) \subseteq v_i + r \cdot B_E.\]

Let

\[h = g_{v_k, r} \circ \ldots \circ g_{v_1, r} \quad \text{and} \quad \hat{h} = h \circ f.\]

We check that

\[\hat{h} \circ g(B_E) \subseteq \bigcup \{x_i + v_j + r^2 \cdot B \mid 1 \leq i \leq k, 1 \leq j \leq k\}.\]

For every \( i \leq k \), \( g_{v_i, r}(v_i + r \cdot B_E) \subseteq \bigcup_{j \leq k}(x_j + v_i + r^2 \cdot B_E) \).

For every \( \ell < i \), \( \left. g_{v_i, r} \right|_{(v_i + r \cdot B_E)} = Id \). This follows from (1). So

\[(3.2) \quad g_{v_i, r} \circ \ldots \circ g_{v_1, r}(v_i + r \cdot B_E) = g_{v_i, r}(v_i + r \cdot B_E) \subseteq \bigcup_{j \leq k}(x_j + v_i + r^2 \cdot B_E).\]

Clause (2) together with the equality in (3.2) imply that

\[(3.3) \text{ For every } i \leq k, \quad \hat{h}(v_i + r \cdot B_E) = g_{v_i, r} \circ \ldots \circ g_{v_1, r}(v_i + r \cdot B_E) \subseteq \bigcup_{j \leq k}(x_j + v_i + r^2 \cdot B_E).\]

By (3.1) and (3.3),

\[(3.4) \text{ For every } i \leq k, \quad \hat{h}(A_i) \subseteq \bigcup_{j \leq k}(x_j + v_i + r^2 \cdot B_E).\]

Since \( \bigcup_{i \leq k} A_i = \bigcup_{i \leq k}(x_i + r \cdot B_E) \),

\[(3.5) \quad \hat{h}(\bigcup_{i \leq k}(x_i + r \cdot B_E)) \subseteq \bigcup_{j \leq k}(x_j + v_i + r^2 \cdot B_E).\]

It follows that \( \hat{h} \circ g(B_E) \subseteq \bigcup_{j \leq k}(x_j + v_i + r^2 \cdot B_E) \). Hence it is contained in the union of finitely many balls with radius \( r^2 \).

Let \( s \in (0, 1) \). Iterating the above construction sufficiently many times, one obtains \( g_s \in \Gamma_2 \) such that \( g_s(B_E) \) is contained in the union of finitely many balls with radius \( \leq s \).

(b) Note that if in the above construction \( g \in \Gamma_1 \), then \( \hat{h} \in \Gamma_1 \).

(c) Let \( B' \) be the unit ball of \((E, \|\cdot\|)\). Let \( K > 1 \) be such that \( B' \subseteq K \cdot B_E \) and \( B_E \subseteq K \cdot B' \). Let \( S > K^2 \). By Part (a), there is \( g \in \Gamma_2 \) such that \( g(K \cdot B_E) \) is contained in a finite union \( \bigcup_{i \leq k} D_i \) of \( \|\cdot\|\)-balls with radius \( \frac{K}{S} \).

So \( g(K \cdot B_E) \subseteq \bigcup_{i \leq k} D_i' \), where each \( D_i' \) is a \( \|\cdot\|\)-ball with radius \( \frac{K^2}{S} \). So \( g(B') \subseteq \bigcup_{i \leq k} D_i' \), and the \( D_i'\)'s are \( \|\cdot\|\)-balls with radius \( < 1 \).

(d) Part (d) follows from Part (b) in the same way that Part (c) follows from Part (a). \( \square \)
Suppose that $L, M$ are linear subspaces of a vector space $X$ and $L \cap M = \{0\}$. The function $P : M + L \to L$ defined by
\[
P(v + u) = u, \quad v \in M, \; u \in L
\]
is called the projection of $(L, M)$.

**Lemma 3.3.** Let $X$ be a separable normed space. Then there is an equivalent norm $\|\cdot\|$ on $X$ such that

(*) for any finite-dimensional subspace $M \subseteq X$ and any $\varepsilon > 0$ there is a finite-codimensional subspace $L \subseteq X$ such that $M \cap L = \{0\}$ and if $P : M + L \to L$ is the projection of $(L, M)$, then $\|P\| < 1 + \varepsilon$.

**Proof.** Let $Y$ be any separable Banach space with a basis which contains $X$ isomorphically. For instance, take $Y = C([0,1])$. Let $\{e_i \mid i \in \mathbb{N}\}$ be a basis for $Y$ such that for every $i \in \mathbb{N}$, $\|e_i\| = 1$. Let $\{e_i^* \mid i \in \mathbb{N}\} \subseteq Y^*$ be the biorthogonal sequence for $\{e_i \mid i \in \mathbb{N}\}$. For $n = 0, 1, \ldots$ define the operators from $Y$ to $Y$
\[
S_n x = \sum_{i=1}^n e_i^*(x)e_i \quad \text{and} \quad R_n x = x - S_n x,
\]
and put
\[
\|x\| = \sup\{\|S_n x\|, \|R_n x\| \mid n = 0, 1, \ldots\}.
\]
So $S_0$ is the 0-operator and $R_0 = Id$. The norm $\|\cdot\|$ has the following properties.

(P1) $\|\cdot\|$ is equivalent to the original norm on $Y$.

(P2) $\|S_n\| = \|R_n\| = 1$, $n = 1, 2, \ldots$

**Proof of P1** For every $x \in X$, $\|x\| \geq \|R_0 x\| = \|x\|$.

The fact that $\{x_n \mid n \in \mathbb{N}\}$ is a basic sequence is equivalent to the existence of $C > 0$ such that for every $x \in X$ and $n \in \mathbb{N}$, $\|S_n x\| \leq C\|x\|$. Hence for every $n \in \mathbb{N}$, $\|R_n x\| = \|x - S_n x\| \leq (1 + C)\|x\|$. Since $\|\cdot\|$ is the supremum of the above, $\|x\| \leq (1 + C)\|x\|$.

**Proof of P2** Since for $n = 1, 2, \ldots$, $S_n$ and $R_n$ are nonzero projections, their norm must be $\geq 1$. It follows trivially from the definition of $\|\cdot\|$ that $\|R_n\|, \|S_n\| \leq 1$. 

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We claim that the norm ||.|| restricted to $X$ has Property ($\ast$).

Let $Y_m = [e_i]_{i=1}^m$ and $Y^m = [e_i]_{i=m+1}^\infty$. Then $S_m : Y^m + Y_m \rightarrow Y_m$ is the projection of $(Y_m, Y^m)$ and $R_m : Y_m + Y^m \rightarrow Y^m$ is the projection of $(Y^m, Y_m)$. Since $\|S_m\| = \|R_m\| = 1$, for every $u \in Y_m$ and $v \in Y^m$, $\|u\|, \|v\| \leq \|u+v\|$.

Let $M \subseteq X$ be a finite dimensional subspace of $X$ and $\varepsilon > 0$. We may assume that $\varepsilon < 1$. Let $m \in \mathbb{N}$ be so large that for every $x \in S_M$, $\inf(\{|\|x-y\|| \mid y \in Y_m\}) \leq \frac{\varepsilon}{3}$. Such an $m$ exists, since $S_M$ is compact and $\bigcup_{i=1}^\infty Y_m$ is dense in $Y$. We check that $M \cap Y^m = \{0\}$. If this is not so, there is $x \in S_M \cap Y^m$. Let $y \in Y_m$ be such that $\|x-y\| \leq \frac{\varepsilon}{3}$. Hence $\|y\| \geq 1 - \frac{\varepsilon}{3}$. So for every $z \in Y^m$, $\|y-z\| \geq 1 - \frac{\varepsilon}{3}$. In particular, $\|y-x\| \geq 1 - \frac{\varepsilon}{3}$. This is impossible since $\frac{\varepsilon}{3} \leq \frac{\varepsilon}{3}$.

Let $Q : M + Y^m \rightarrow Y^m$ be the projection of $(Y^m, M)$. We show that $\|Q\| \leq 1 + \varepsilon$. Let $x \in M + Y^m$ and $\|x\| = 1$. So for some $u \in M$ and $v \in Y^m$, $x = u + v$. Let $w \in Y^m$ be such that $\|u-w\| \leq \frac{\varepsilon}{3} \cdot \|u\|$. So

$$\|v\| \leq \|v+w\| \leq \|v+u\| + \|u-w\| \leq \|v+u\| + \frac{\varepsilon}{3} \cdot \|u\|. \quad (1)$$

$$\|u\| \leq \|u+v\| + \|v-w\| = 1 + \|v\| \leq 1 + \|v+w\| \leq \|v+u\| + \frac{\varepsilon}{3} \cdot \|u\|. \quad (2)$$

Hence

$$\text{ }(1 - \frac{\varepsilon}{3}) \cdot \|u\| \leq \|v+u\|. \quad (3)$$

That is,

$$\|u\| \leq \frac{3}{3-\varepsilon} \cdot \|v+u\|. \quad (4)$$

By (1) and (4),

$$\|v\| \leq \|v+u\| + \frac{\varepsilon}{3} \cdot \|v+u\| = (1 + \frac{\varepsilon}{3-\varepsilon}) \cdot \|v+u\| \leq (1 + \varepsilon) \cdot \|v+u\|. \quad (5)$$

So $\|Q\| \leq 1 + \varepsilon$. Let $L = Y^m \cap X$. Since $Y^m$ has finite codimension in $Y$, $L$ has finite codimension in $X$. Clearly, $M \cap L = \{0\}$. The projection $P$ of $(L, M)$ is the restriction of $Q$ to $M + L$. So $\|P\| \leq 1 + \varepsilon$. \hfill \Box
Let $S$ be a set of sets and $k \in \mathbb{N}$. $S$ has finite order, if there is $k \in \mathbb{N}$ such that for every $x$, $|\{S \in S \mid x \in S\}| \leq k$. The order of $S$ is the minimal such $k$.

**Lemma 3.4.** Let $M \subseteq X$ be a finite dimensional subspace of a normed space $X$ and $L \subseteq X$ be a closed subspace which is a complement of $M$ in $X$. Let $P : X \to L$ be the projection of $(L, M)$.

(a) Let $A \subseteq X$ be a bounded closed set. Then $P(A)$ is closed.

(b) Let $S$ be a locally finite family of subsets of $X$ such that $\bigcup_{S \in S} S$ is bounded. Then $\{P(S) \mid S \in S\}$ is a locally finite family.

(c) Let $S$ be a separated family of subsets of $X$ such that $\bigcup_{S \in S} S$ is bounded. Then $\{P(S) \mid S \in S\}$ has finite order.

**Proof** (a) Let $x = \lim x_n$, $x_n \in P(A)$. Then for every $n$ there is $z_n \in A$ such that $Pz_n = x_n$. Clearly, $z_n = x_n + y_n$, where $y_n \in M$. Since $A$ is bounded, $\{z_n \mid n \in \mathbb{N}\}$ is a bounded sequence. $x_n$ is convergent and thus it is bounded. So $y_n = z_n - x_n$ is a bounded sequence. Also, $\{y_n \mid n \in \mathbb{N}\} \subseteq M$, and $M$ is finite-dimensional. So $\{y_n \mid n \in \mathbb{N}\}$ has a convergent subsequence $\{y_{n_k} \mid k \in \mathbb{N}\}$. Denote $y = \lim y_{n_k}$.

Then we have $\lim z_{n_k} = \lim (x_{n_k} + y_{n_k}) = x + y$. Since $A$ is closed and $z_{n_k} \in A$, it follows that $x + y \in A$. Clearly, $P(x + y) = x \in P(A)$. This proves (a).

(b) Let $S$ be as in Part (b), and assume by contradiction that $\{P(S) \mid S \in S\}$ is not locally finite. Let $x$ be an accumulation point of $P(S)$. That is, there is a 1–1 sequence $\{S_i \mid i \in \mathbb{N}\} \subseteq S$, $x_i \in P(S_i)$ such that $\lim x_i = x$. Let $y_i \in M$ be such that $y_i + x_i \in S_i$. Since $\bigcup_{S \in S} S$ is bounded and the projection of $(M, L)$ is bounded, $\{y_i \mid i \in \mathbb{N}\}$ is bounded. Since $M$ is finite dimensional $\{y_i \mid i \in \mathbb{N}\}$ has a convergent subsequence. Denote it by $\{y_{i_k} \mid k \in \mathbb{N}\}$. Both $\{y_{i_k}\}$ and $\{x_{i_k}\}$ are convergent. So $\{x_{i_k} + y_{i_k}\}$ is convergent. But $x_{i_k} + y_{i_k} \in S_{i_k}$, and the $S_{i_k}$'s are distinct. So $S$ is not locally finite. A contradiction. This proves Part (b).
(c) A subset $A$ of a metric space is $\varepsilon$-separated, if the distance between every two distinct points of $A$ is $\geq \varepsilon$. There is an integer $\ell = \ell(m, R, \varepsilon)$ such that for every $m$-dimensional normed space $E$ and an $\varepsilon$-separated subset $A \subseteq B_E(0, R)$, $|A| \leq \ell$. For spaces $E$ isometric to $\mathbb{R}^m_\infty$ the number $\ell' = \left(\left[\frac{m^2 R}{\varepsilon}\right] + 1\right)^m$ fulfills the requirement.

Let $E$ be an $m$-dimensional normed space. There is a norm $\|\|\|$ on $E$ such that $(E, \|\|\|)$ is isometric to $\mathbb{R}^m_\infty$ and for every $x \in E$, $\frac{m^2 R}{\varepsilon} \leq \|x\| \leq m\|x\|$.

Let $A \subseteq B_E(0, R)$ be $\varepsilon$-separated. Then $A \subseteq B_{E^\|\|}(0, mR)$ and $A$ is $\frac{\varepsilon}{m}$-separated with respect to $\|\|\|$. So $|A| \leq \left(\left[\frac{m^2 R}{\varepsilon}\right] + 1\right)^m$.

Suppose that $\dim(M) = m$ and let $S$ be as in Part (c). Let $\delta > 0$ be such that $S$ is $\delta$-separated. Suppose that $\bigcup_{S \in S} S \subseteq B_X(0, R)$. Let $Q$ be the projection of $(M, L)$. So $Q(\bigcup_{S \in S} S) \subseteq B_M(0, \|Q\| \cdot R)$. Denote $R' = \|Q\| \cdot R$. Let $k \in \mathbb{N}$ be such that there are distinct $S_1, \ldots, S_k \in S$ and $x$ with $x \in \bigcap_{i=1}^k P(S_i)$. So there are $y_1, \ldots, y_k \in M$ such that for every $i = 1, \ldots, k$, $y_i + x \in S_i$. It follows that $\{y_1, \ldots, y_k\} \subseteq B_M(0, R')$. Also, for every $i < j \leq k$, $\|y_i - y_j\| = \|(x + y_i) - (x + y_j)\| \geq \delta$. So $k \leq \ell(m, R', \delta)$. It follows that the order of $S$ is $\leq \ell(m, R', \delta)$. \hfill \Box

For a normed space $E$, $S_E$ denotes the unit sphere of $E$.

The next lemma is due to Michael Levin.

**Lemma 3.5.** (M. Levin) Let $X$ be an $n$-dimensional normed space. Suppose that $\mathcal{F}$ is a finite set of closed subsets of $X$ such that

1. $S_X = \bigcup_{F \in \mathcal{F}} F$.
2. For every $F \in \mathcal{F}$ and $x \in F$, $-x \notin F$.

Then the order of $\mathcal{F}$ is $\geq \frac{n}{2} + 1$.

**Proof** Let $k$ be the order of $\mathcal{F}$ and assume by contradiction that $k < \frac{n}{2} + 1$. There is $\varepsilon > 0$ such that the order of $\mathcal{U} := \{B_X(F, \varepsilon) \mid F \in \mathcal{F}\}$ is $k$, and for every $x \in U \in \mathcal{U}$, $-x \notin U$. Let $\{\psi_U \mid U \in \mathcal{U}\}$ be a partition of unity for $\mathcal{U}$. That is,

1. For every $U \in \mathcal{U}$, $\text{Dom}(\psi_U) = \bigcup_{V \in \mathcal{U}} V$. Denote $W = \bigcup_{V \in \mathcal{U}} V$.
2. For every $U \in \mathcal{U}$ and $x \in U$, $\psi_U(x) > 0$. 

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(3) For every \( U \in \mathcal{U} \) and \( x \in W \setminus U \), \( \psi_U(x) = 0 \).
(4) For every \( x \in W \), \( \sum_{U \in \mathcal{U}} \psi_U(x) = 1 \).

Let \( K \) be the simplex whose vertices are the members of \( \mathcal{U} \), and define \( f : S_X \to K \).

\[
 f(x) = (\psi_U(x))_{U \in \mathcal{U}}, \quad x \in S_E.
\]

Let \( L = \text{Rng}(f) \), and denote by \( m \) the topological dimension of \( L \). If \( z \in L \), then the number of coordinates of \( z \) which are different from 0 is \( \leq k \). That is, \( L \) is contained in the \((k-1)\)-dimensional skeleton of \( K \). So \( m \leq k - 1 \).

\[
2(k - 1) + 1 < 2\left(\frac{n}{2} + 1 - 1\right) + 1 = n + 1.
\]

That is, \( 2m + 1 \leq n \). By [E] Theorem 1.11.4 p.95, there is an embedding \( g : L \to \mathbb{R}^n \). It is obvious that if \( x \in S_X \), then \( g \circ f(x) \neq g \circ f(-x) \). This contradicts the theorem of Borsuk and Ulam which says that for every continuous function \( h : S_{\mathbb{R}^n} \to \mathbb{R}^{n-1} \) there is \( x \in S_{\mathbb{R}^n} \) such that \( h(x) = h(-x) \). \( \square \)

**Corollary 3.6.** Let \( X \) be an infinite dimensional normed space. Then \( S_X \) does not have a cover \( \mathcal{S} \) such that \( \mathcal{S} \) has finite order, \( \mathcal{S} \) is locally finite, and for every \( S \in \mathcal{S}, \) \( S \) is a closed and convex and \( 0 \notin S \).

**Proof** Let \( k \) be the order of \( \mathcal{S} \). Let \( n \) be such that \( \frac{n}{2} + 1 \geq k \), and \( L \) be an \( n \)-dimensional subspace of \( X \). Let \( \mathcal{F} = \{ S \cap S_L \mid S \in \mathcal{S}\} \). \( L \) and \( \mathcal{F} \) fulfill the conditions of Lemma 3.5. That is,

(Q1) Every member of \( \mathcal{F} \) is closed, and does not contain antipodal points.
(Q2) \( \bigcup_{F \in \mathcal{F}} F = S_L \).
(Q3) \( \mathcal{F} \) is finite.

The argument that (Q3) holds is as follows. \( \mathcal{S} \) is locally finite. This implies that \( \mathcal{F} \) is locally finite. But a locally finite family of subsets of a compact space must be finite.

By Lemma 3.5, the order of \( \mathcal{F} \) is \( > \frac{n}{2} + 1 \). This contradicts the fact that the order of \( \mathcal{S} \) is \( k \). \( \square \)

**Proposition 3.7.** (a) Let \( X \) be a normed space, \( Y \) be a separable subspace of \( X \) and \( g \in \Gamma_2(X) \). Then there is a separable subspace \( Z \) of \( X \) such that \( Y \subseteq Z \) and \( g(Z) \subseteq Z \).
(b) (i) Let $X$ be a normed space, $Z$ be a subspace of $X$ and $g \in \Gamma_2(X)$. Suppose that $g(Z) \subseteq Z$. Then $g \restriction Z \in \Gamma_2(Z)$.

(ii) The same holds for $\Gamma_1$.

**Proof** (a) Suppose first that $g \in \hat{\Gamma}_1(X)$. Then $S_1 := \{S^g_j \cap Y \mid j \in J^g\}$ is locally finite. $Y$ is second countable, and in a second countable Hausdorff space every locally finite family has cardinality $\leq \aleph_0$. So $|S_1| \leq \aleph_0$.

For every $S \in S_1$, let $j_S \in J^g$ be such that $S^g_j \cap Y = S$ and let $J_1 = \{j_S \mid S \in S_1\}$ and $E_1 = \bigcup_{j \in J_1} E^g_j$. So $|J_1| \leq \aleph_0$ and hence $dim(E_1) \leq \aleph_0$. Let $Y_1 = Y + E_1$. Clearly, $Y \cup g(Y) \subseteq Y_1$. Since $Y$ is separable and $dim(E_1) \leq \aleph_0$, $Y_1$ is separable.

Applying the same process to $Y_1$ instead of $Y$ one obtains a subspace $Y_2$ such that $(\ast)$ $Y_1 \cup g(Y_1) \subseteq Y_2$ and $Y_2$ is separable.

Repeating this procedure countably many times we get a sequence of separable subspaces

$$Y = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$$

Such that for $i = 0, 1, \ldots$, $g(Y_i) \subseteq Y_{i+1}$. Let $Z = \bigcup_{i=0}^{\infty} Y_i$. Then $Y \subseteq Z$, $Z$ is separable and $g(Z) \subseteq Z$.

Assume now that $g \in \Gamma_1(X)$. Let $g = h_n \circ \cdots \circ h_1$, where $h_1, \ldots, h_n \in \hat{\Gamma}_1(X)$. Let \{\{f_i \mid i \in \mathbb{N}\} be an enumeration of \{h_1, \ldots, h_n\} such that for every $m \leq n$ \{i \mid f_i = h_m\} is infinite. Let $Z_0 = Y$ and define by induction a chain of separable subspaces $Z_0 \subseteq Z_1 \subseteq \ldots$ such that for every $i$, $f_i(Z_i) \subseteq Z_{i+1}$. Let $Z = \bigcup_{i=0}^{\infty} Z_i$. Then $Y \subseteq Z$, $g(Z) \subseteq Z$ and $Z$ is separable.

(b) Part (b) is Trivial. \hfill \Box

**Proof of Theorem 3.1(a)** We first prove that Theorem 3.1(a) is true for separable spaces. So let $(X, ||.||)$ be an infinite dimensional separable normed space.

Assume to the contrary that $\Gamma_1(X)$ is not non-shrinking. There is a norm $||.||$ on $X$ such that $||.||$ is equivalent to $||.||$, and $||.||$ fulfills the conclusion of Lemma 3.3. By Proposition 3.2(d), $\Gamma_1(X)$ is not non-shrinking with respect to $(X, ||.||)$. We may thus assume that $||.||$ fulfills the conclusion of Lemma 3.3.
Let $g \in \Gamma_1(X)$, $r \in (0, 1)$ and $x_1, \ldots, x_m \in X$ be such that
\[
g(B_X) \subseteq \bigcup_{k=1}^{m} (x_k + rB_X) \quad (1)
\]

Let $g = h_n \circ h_{n-1} \circ \ldots \circ h_1$, where $h_i \in \widehat{\Gamma}_1(X)$, $i = 1, \ldots, n$.

Every separable metric space is second countable, and in a second countable Hausdorff space every locally finite family has cardinality $\leq \aleph_0$. So $|J^h| \leq \aleph_0$.

We may assume that for every $i$, $|J^h_i| = \aleph_0$. This can be achieved by adding to $\{(S^h_j, E^h_j) \mid j \in J^h\}$ additional pairs of the form $(S, \{0\})$. So denote the set of pairs associated with $h_i$ by $\{(S^i_j, E^i_j) \mid j \in \mathbb{N}\}$.

Let $\sigma_1 = \{j \in \mathbb{N} \mid 0 \in P_1(S^1_j)\}$. Since $\{S^i_j \mid j \in \mathbb{N}\}$ is separated, $\sigma_1$ is finite. (In fact, $|\sigma_1| \leq 1$). Let $M_1$ be the following finite-dimensional subspace of $X$.
\[
M_1 = \{x_1, \ldots, x_m\} \cup \bigcup_{j \in \sigma_1} E^1_j
\]

(The points $x_1, \ldots, x_m$ were defined in (1)). Let $L_1 \subseteq X$ be a complement of $M_1$ in $X$ and $P_1 : X \to L_1$ be the projection of $(L_1, M_1)$. (That is, $P_1 \upharpoonright M_1 = 0$ and $P_1 \upharpoonright L_1 = Id$).

By Lemma 3.4(c), $\{P_1(S^2_j) \mid j \in \mathbb{N}\}$ has finite order. So $\sigma_2 := \{j \in \mathbb{N} \mid 0 \in P_1(S^2_j)\}$ is finite. Put $M_2 = [M_1 \cup \bigcup_{j \in \sigma_2} E^2_j]$, and let $L_2 \subseteq X$ be a complement of $M_2$ in $X$. Let $P_2 : X \to L_2$ be the projection of $(L_2, M_2)$.

Denote $M_0 = \{\{x_1, \ldots, x_m\}\}$ and $P_0 = Id_X$. Proceeding in the above way $n$ times we construct:
A chain of finite-dimensional subspaces of $X$
\[
M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n,
\]
A sequence of complements of the $M_i$’s with respect to $X$
\[
L_1, L_2, \ldots, L_n.
\]
The projections
\[
P_i : X \to L_i \quad \text{of} \quad (L_i, M_i), \ i = 1, \ldots, n.
\]
And finite sets
\[
\sigma_i = \{j \in \mathbb{N} \mid 0 \in P_{i-1}(S^i_j)\}, \ i = 1, \ldots, n
\]
such that
\[ M_i = [M_{i-1} \cup \bigcup_{j \in \sigma_i} E_j^i], \quad i = 1, \ldots, n. \]
(In fact, \( L_n \) and \( P_n \) will not be used).

Choose \( \varepsilon > 0 \) such that \( r(1 + \varepsilon) < 1 \). Recall that \( \| \cdot \| \) has the property of Lemma 3.3. Let \( L \) be a subspace of \( X \) as guaranteed by Lemma 3.3 for \( M_n \) and \( \varepsilon \). That is,

1. \( L \) has finite codimension in \( X \) and \( L \cap M_n = \{0\} \).
2. Let \( P : M_n + L \to L \) be the projection of \((L, M_n)\). Then \( \|P\| < 1 + \varepsilon \).

We may also assume that
3. \( L \subseteq \bigcap_{i=1}^{n-1} L_i \).

This is so, since \( L \cap \bigcap_{i=1}^{n-1} L_i \) too, fulfills (P1) and (P2).

Let
\[ C = S_L \setminus \bigcup_{i=1}^{n} \bigcup_{j \notin \sigma_i} P_{i-1}(S_j^i). \quad (2) \]

We show that \( C \neq \emptyset \). Suppose otherwise. So \( S_L \subseteq \bigcup_{i=1}^{n} \bigcup_{j \notin \sigma_i} P_{i-1}(S_j^i) \).

By Lemma 3.4(a), for every \( i, j \), \( P_{i-1}(S_j^i) \) is closed. If \( j \notin \sigma_i \), then \( 0 \notin P_{i-1}(S_j^i) \). Hence since \( P_{i-1}(S_j^i) \) is convex, it does not contain antipodal points.

We use the facts that for every \( i = 1, \ldots, n \), \( \{S_j^i \mid j \in \mathbb{N}\} \) is separated and bounded. By Lemma 3.4(c), \( \{P_{i-1}(S_j^i) \mid j \in \mathbb{N} \setminus \sigma_i\} \) has finite order. So \( S := \{P_{i-1}(S_j^i) \mid i = 1, \ldots, n, \ j \notin \sigma_i\} \) has finite order. By Lemma 3.4(b), \( S \) is locally finite. The facts:

(i) \( S \) covers \( S_L \);
(ii) For every \( S \in S \), \( S \) is closed and convex and \( 0 \notin S \);
(iii) \( S \) is locally finite;
(iv) \( S \) has finite order;

contradict Corollary 3.7. So \( C \neq \emptyset \).

Recall that \( P \) is the projection of \((L, M_n)\). We claim that

(*) For every \( x \in C \), \( g(x) = x + v \) for some \( v \in M_n \).

Let \( x \in C \).

**Step 1:** By (2), either \( x \notin \bigcup_{j \in \mathbb{N}} S_j^1 \), or \( x \in \bigcup_{j \in \mathbb{N}} S_j^1 \). In either case,
\[ h_1(x) = x + v_1, \quad v_1 \in M_1. \]
From the facts that \( x \in L \subseteq L_1 \) and \( v_1 \in M_1 \), it follows that \( x = P_1(x + v_1) \).

**Step 2:** Assume by contradiction that \( x + v_1 \in \bigcup_{j \notin \sigma_2} S_j^2 \). Then

\[
x = P_1(x + v_1) \in \bigcup_{j \notin \sigma_2} P_1(S_j^2),
\]
contradicting the fact that \( x \in C \). So \( x + v_1 \notin \bigcup_{j \notin \sigma_2} S_j^2 \). Hence either \( x + v_1 \notin \bigcup_{j \in \mathbb{N}} S_j^1 \), or \( x + v_1 \in \bigcup_{j \notin \sigma_2} S_j^2 \). In either case,

\[
h_2 \circ h_1(x) = h_2(x + v_1) = x + v_1 + v_2, \quad v_2 \in M_2.
\]

Since \( M_1 \subseteq M_2 \), \( v_1 + v_2 \in M_2 \). Also, \( x \in L \subseteq L_2 \). So \( P_2(x + v_1 + v_2) = x \).

In Step 3 of this argument one concludes that

\[
g(x) = x + \sum_{i=1}^n v_i, \quad v_i \in M_i, \; i = 1, \ldots, n.
\]

For every \( i \leq n \), \( M_i \subseteq M_n \). So \( v = \sum_{i=1}^n v_i \in M_n \). That is, \( g(x) = x + v \), where \( v \in M_n \). So \((*)\) holds.

It follows from \((*)\) that

\[
(**) \quad \text{For every } x \in C, \; Pg(x) = x.
\]

We check that

\[
(***) \quad \text{For every } x \in C, \; g(x) \in \bigcup_{k=1}^m x_k + rB_{M_n + L}.
\]

It is given that \( g(x) \in \bigcup_{k=1}^m x_k + rB_X \). Write \( g(x) = x_k + ru \), where \( u \in B_X \).

By \((*)\), \( g(x) \in M_n + L \), and from the definition of \( M_1 \) follows that

\( x_k \in M_1 \subseteq M_n + L \). So \( u = \frac{g(x) - x_k}{r} \in M_n + L \). So \( u \in B_{M_n + L} \).

By \((***)\),

\[
g(C) \subseteq \bigcup_{k=1}^m (x_k + rB_{M_n + L}).
\]

By \((***)\),

\[
C = P(g(C)) \subseteq \bigcup_{k=1}^m P(x_k + rB_{M_n + L}).
\]

Since \( x_k \in M_n \), it follows that \( P(x_k + rB_{M_n + L}) = rP(B_{M_n + L}) \). Hence

\[
C \subseteq \bigcup_{k=1}^m rP(B_{M_n + L}).
\]

We now use the facts that \( \|P\| < 1 + \varepsilon \) and \( (1 + \varepsilon)r < 1 \). So

\[
C \subseteq \bigcup_{k=1}^m rP(B_{M_n + L}) \subseteq \bigcup_{k=1}^m r(1 + \varepsilon) \cdot B_{M_n + L}.
\]

We know that \( \emptyset \neq C \subseteq S_L \). But all the points in \( \bigcup_{k=1}^m r(1 + \varepsilon) \cdot B_{M_n + L} \) have norm \( < 1 \). A contradiction.
We have shown that the claim of Theorem 3.1(a) is true for separable spaces.

Let $X$ be any normed space. Suppose by contradiction that $\Gamma_1(X)$ is not non-shrinking. Let $g \in \Gamma_1(X)$, $r \in (0, 1)$ and $x_1, \ldots, x_m \in X$ be such that $g(B_X) \subseteq \bigcup_{k=1}^m (x_k + rB_X)$. Let $Z$ be a separable subspace of $X$ such that $\{x_1, \ldots, x_k\} \subseteq Z$ and $g(Z) \subseteq Z$. Such $Z$ exists by Proposition 3.7. Relying on the fact that $x_1, \ldots, x_m \in Z$, we conclude that $g(B_Z) \subseteq \bigcup_{k=1}^m (x_k + rB_Z)$. Also, $g \upharpoonright Z \in \Gamma_1(Z)$. So $\Gamma_1(Z)$ is not non-shrinking. This contradicts the first part of the proof. So Part (a) of the theorem is proved.

The following additional facts are needed in the proof of Theorem 3.1 (b).

**Lemma 3.8.** ([FL] Crollary 3, [FL] Theorem 2.3) Assume that an infinite dimensional Banach space $E$ contains a nonempty bounded open subset which is a weak $G_\delta$-set. Then $E$ contains $c_0$ isomorphically.

**Lemma 3.9.** Let $L$ be an infinite dimensional Banach space that does not contain $c_0$ isomorphically, and $A$ be a locally finite family of $w$-closed subsets of $L$ which do not contain 0. Then $S_L \setminus \bigcup_{A \in A} A \neq \emptyset$.

**Proof** Assume to the contrary that

$$S_L \subseteq \bigcup_{A \in A} A.$$

Let $E \subseteq L$ be any separable closed infinite dimensional subspace of $L$. Then $B = \{A \cap E \mid A \in A\}$ is a locally finite family of $w$-closed subsets of $E$ and $S_E \subseteq \bigcup_{A \in B} A$.

Since $E$ is second countable $|B| \leq \aleph_0$. Define

$$G = B_E \setminus \bigcup_{A \in B} A.$$

Clearly, $0 \in G$. Since $E$ is separable it follows that $B_E$ is a weak $G_\delta$ set, and hence $G$ is a weak $G_\delta$ set too. Clearly $G$ is bounded. We check that $G$ is open. Let $x \in G$. Since $S_E \subseteq \bigcup_{A \in B} A$, $x \in \text{int}(B_E)$. Since $B$ is a locally finite family and $x \not\in \bigcup_{A \in B} A$, it follows that there is $r > 0$ such that $(x + rB_E) \cap \bigcup_{A \in B} A = \emptyset$. Put $\alpha = \min(d(x, S_E), r)$. Then $x + \alpha B_E \subseteq G$. By Lemma 3.8, $c_0$ is embeddable in $E$. A contradiction. 

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Lemma 3.10. Let $M \subseteq X$ be a finite-dimensional subspace of a Banach space $X$, $L \subseteq X$ be a closed complement of $M$ in $X$ and $P : X \to L$ be the projection of $(L, M)$. Then for every $w$-closed bounded subset $A \subseteq X$, $P(A)$ is $w$-closed.

Proof Denote the weak limit of a net $N$ by $w \text{-} \lim N$. We prove that $P(A)$ is closed under convergent nets. Let $(D, \leq_D)$ be a directed poset and $\{x_d \mid d \in D\} \subseteq P(A)$ be a net in $A$ such that $\{x_d \mid d \in D\}$ is $w$-convergent in $L$, and let $x = w \text{-} \lim \{x_d \mid d \in D\}$. Let $z_d \in A$ be such that $P(z_d) = x_d$. So $z_d = x_d + y_d$, where $y_d \in M$. The set $\{y_d \mid d \in D\}$ is bounded, since it is the image of the bounded set $\{z_d \mid d \in D\}$ under a bounded operator. So its closure is compact in $M$, and hence the net $\{y_d \mid d \in D\}$ has a convergent subnet. Let $\{y_{dc} \mid c \in C\}$ be a convergent subnet of $\{y_d \mid d \in D\}$ and $y = w \text{-} \lim \{y_{dc} \mid c \in C\}$. So $x = w \text{-} \lim \{x_{dc} \mid c \in C\}$. It follows that $x + y = w \text{-} \lim \{x_{dc} + y_{dc} \mid c \in C\}$. Since $x_{dc} + y_{dc} = z_{dc} \in A$ and $A$ is $w$-closed, $x + y \in A$. Clearly, $x = P(x + y) \in P(A)$.

We have shown that $P(A)$ is $w$-closed.

Proof of Theorem 3.1(b) We first prove that Theorem 3.1(b) is true for separable spaces. So let $(X, \|\cdot\|)$ be an infinite dimensional separable normed space.

Assume to the contrary that $\Gamma_2(X)$ is not non-shrinking. There is a norm $\|\cdot\|$ on $X$ such that $\|\cdot\|$ is equivalent to $\|\cdot\|$, and $\|\cdot\|$ fulfills the conclusion of Lemma 3.3. By Proposition 3.2(c), $\Gamma_2(X)$ is not non-shrinking with respect to $(X, \|\cdot\|)$. We may thus assume that $\|\cdot\|$ fulfills the conclusion of Lemma 3.3.

Let $g \in \Gamma_2(X)$, $r \in (0, 1)$ and $x_1, \ldots, x_m \in X$ be such that

$$g(B_X) \subseteq \bigcup_{k=1}^{m} (x_k + rB_X)$$

Let $g = h_n \circ h_{n-1} \circ \ldots \circ h_1$, where $h_i \in \Gamma_2(X)$, $i = 1, \ldots, n$.

Every separable metric space is second countable, and in a second countable Hausdorff space every locally finite family has cardinality $\leq \aleph_0$. So $|J^{h_i}| \leq \aleph_0$. 26
We may assume that for every $i$, $|J_i^h| = \aleph_0$. This can be achieved by adding to $\{(\hat{S}^h_j, E^h_j) | j \in J^h\}$ additional pairs of the form $(\hat{S}^i, \{0\})$. So denote the set of pairs associated with $h_i$ by $\{(\hat{S}^i_j, E^i_j) | j \in \mathbb{N}\}$.

Let $\sigma_1 = \{j \in \mathbb{N} | 0 \in \hat{S}^1_j\}$. Since $\{\hat{S}^1_j | j \in \mathbb{N}\}$ is locally finite in $\overline{X}$, $\sigma_1$ is finite. Let $M_1$ be the following finite-dimensional subspace of $X$.

\[ M_1 = \{(x_1, \ldots, x_m) \cup \bigcup_{j \in \sigma_1} E^1_j \}. \]

(The points $x_1, \ldots, x_m$ were defined in (1)). Let $L_1 \subseteq X$ be a complement of $M_1$ in $\overline{X}$ and $P_1 : \overline{X} \rightarrow L_1$ be the projection of $(L_1, M_1)$. (That is, $P_1 \upharpoonright M_1 = 0$ and $P_1 \upharpoonright L_1 = Id$).

By Lemma 3.4(b), $\{P_1(\hat{S}^2_j) | j \in \mathbb{N}\}$ is locally finite. So $\sigma_2 := \{j \in \mathbb{N} | 0 \in P_1(\hat{S}^2_j)\}$ is finite. Put $M_2 = [M_1 \cup \bigcup_{j \in \sigma_2} E^2_j]$, and let $L_2 \subseteq \overline{X}$ be a complement of $M_2$ in $\overline{X}$. Let $P_2 : X \rightarrow L_2$ be the projection of $(L_2, M_2)$.

Denote $M_0 = [\{x_1, \ldots, x_m\}]$ and $P_0 = Id_\overline{X}$. Proceeding in the above way $n$ times we construct:

A chain of finite-dimensional subspaces of $X$:

\[ M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n. \]

A sequence of complements of the $M_i$'s with respect to $\overline{X}$:

\[ L_1, L_2, \ldots, L_n. \]

The projections

\[ P_i : \overline{X} \rightarrow L_i \quad \text{of} \quad (L_i, M_i), \quad i = 1, \ldots, n. \]

And finite sets

\[ \sigma_i = \{j \in \mathbb{N} | 0 \in P_{i-1}(\hat{S}^i_j)\}, \quad i = 1, \ldots, n \]

such that

\[ M_i = [M_{i-1} \cup \bigcup_{j \in \sigma_i} E^i_j], \quad i = 1, \ldots, n. \]

(In fact, $L_n$ and $P_n$ will not be used).

Choose $\varepsilon > 0$ such that $r(1 + \varepsilon) < 1$. Recall that $\|\cdot\|$ has the property of Lemma 3.3. Let $L$ be a subspace of $X$ as guaranteed by Lemma 3.3 for $M_n$ and $\varepsilon$. That is,

(P1) $L$ has finite codimension in $X$ and $L \cap M_n = \{0\}$.

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(P2) Let $P : M_n + L \to L$ be the projection of $(L, M_n)$. Then $\|P\| < 1 + \varepsilon$.

We may also assume that

(P3) $L \subseteq \bigcap_{i=1}^{n-1} L_i$.

This is so, since $L \cap \bigcap_{i=1}^{n-1} L_i$ too, fulfills (P1) and (P2). Then by the assumptions of Part (b), $X$ has an infinite dimensional subspace $Y$ such that $c_0$ is not isomorphically embeddable in $\overline{Y}$. Let $Z = Y \cap L$. Since $L$ has finite codimension in $X$, $Z$ is infinite dimensional. Clearly,

(i) $c_0$ is not isomorphic to a subspace of $Z$.

Let

$$C = S_Z \setminus \bigcup_{i=1}^n \bigcup_{j \notin \sigma_i} P_{i-1}(\hat{S}_j^i).$$

(2)

We show that $C \neq \emptyset$. Suppose otherwise. So

(ii) $S_Z \subseteq \bigcup_{i=1}^n \bigcup_{j \notin \sigma_i} P_{i-1}(\hat{S}_j^i)$.

Recall that $\bigcup_{j \in \mathbb{N}} \hat{S}_j^i$ is bounded. By Lemma 3.10 for every $i, j$, $P_{i-1}(\hat{S}_j^i)$ is w-closed in $X$. So

(iii) $P_{i-1}(\hat{S}_j^i) \cap Z$ is w-closed in $Z$.

By Lemma 3.24(b), for every $i$, $\{P_{i-1}(\hat{S}_j^i) : j \in \mathbb{N} \setminus \sigma_i\}$ is locally finite. So

(iv) $\{P_{i-1}(\hat{S}_j^i) \cap Z : i = 1, \ldots, n, j \in \mathbb{N} \setminus \sigma_i\}$ is locally finite.

By the definition of the $\sigma_i$’s,

(v) $0 \notin P_{i-1}(\hat{S}_j^i)$ whenever $j \notin \sigma_i$.

(i) - (v) contradict Lemma 3.9. So $C \neq \emptyset$.

We check that $C$ is open in $S_Z$. This is so, since $C$ is the complement in $S_Z$ of the union of the locally finite family of closed sets

$$\{P_{i-1}(\hat{S}_j^i) \cap Z : i = 1, \ldots, n, j \in \mathbb{N} \setminus \sigma_i\}.$$ 

Since $Z \cap S_Z$ is dense in $S_Z$, $C \cap Z \neq \emptyset$. In particular, $C \cap L \neq \emptyset$.

Recall that $P$ is the projection of $(L, M_n)$. We claim that

(*) For every $x \in C \cap L$, $g(x) = x + v$ for some $v \in M_n$.

Let $x \in C \cap L$.

Step 1: By (2), either $x \notin \bigcup_{j \in \mathbb{N}} \hat{S}_1^i$, or $x \in \bigcup_{j \in \sigma_1} \hat{S}_1^i$. In either case,

$$h_1(x) = x + v_1, \quad v_1 \in M_1.$$ 

From the facts that $x \in L \subseteq L_1$ and $v_1 \in M_1$, it follows that $x = P_1(x + v_1)$. 28
Step 2: Assume by contradiction that \( x + v_1 \in \bigcup_{j \in \sigma^2} \hat{S}_j^2 \). Then

\[
x = P_1(x + v_1) = \bigcup_{j \in \sigma^2} P_1(\hat{S}_j^2),
\]

contradicting the fact that \( x \in C \). So \( x + v_1 \not\in \bigcup_{j \in \sigma^2} \hat{S}_j^2 \). Hence either \( x + v_1 \not\in \bigcup_{j \in \sigma^2} \hat{S}_j^2 \), or \( x + v_1 \not\in \bigcup_{j \in \sigma^2} \hat{S}_j^2 \). In either case,

\[
h_2 \circ h_1(x) = h_2(x + v_1) = x + v_1 + v_2, \quad v_2 \in M_2.
\]

Since \( M_1 \subseteq M_2 \), \( v_1 + v_2 \in M_2 \). Recalling that \( x \in L \subseteq L_2 \), we conclude that \( P_2(x + v_1 + v_2) = x \).

In Step \( n \) of this argument one concludes that

\[
g(x) = x + \sum_{i=1}^{n} v_i, \quad v_i \in M_i, \quad i = 1, \ldots, n.
\]

For every \( i \leq n \), \( M_i \subseteq M_n \). So \( v = \sum_{i=1}^{n} v_i \in M_n \). That is, \( g(x) = x + v \), where \( v \in M_n \). So (*) holds.

It follows from (*) that

(**) For every \( x \in C \cap L \), \( Pg(x) = x \).

We check that

(***) For every \( x \in C \cap L \), \( g(x) \in \bigcup_{k=1}^{\mu} x_k + rB_{M_n+L} \).

It is given that \( g(x) \in \bigcup_{k=1}^{\mu} x_k + rB_{X} \). Write \( g(x) = x_k + ru \), where \( u \in B_{X} \).

By (*), \( g(x) \in M_n + L \), and from the definition of \( M_1 \) follows that \( x_k \in M_1 \subseteq M_n + L \). So \( u = \frac{g(x)-x_k}{r} \in M_n + L \). So \( u \in B_{M_n+L} \).

By (***)

\[
g(C \cap L) \subseteq \bigcup_{k=1}^{\mu} (x_k + rB_{M_n+L}).
\]

By (**),

\[
C \cap L = P(g(C \cap L)) \subseteq \bigcup_{k=1}^{\mu} P(x_k + rB_{M_n+L}).
\]

Since \( x_k \in M_n \), it follows that \( P(x_k + rB_{M_n+L}) = rP(B_{M_n+L}) \). Hence

\[
C \cap L \subseteq \bigcup_{k=1}^{\mu} rP(B_{M_n+L}).
\]

We now use the facts that \( \|P\| < 1 + \varepsilon \) and \( (1 + \varepsilon)r < 1 \). So

\[
C \cap L \subseteq \bigcup_{k=1}^{\mu} rP(B_{M_n+L}) \subseteq \bigcup_{k=1}^{\mu} r(1 + \varepsilon)B_{M_n+L}.
\]

We know that \( \emptyset \neq C \cap L \subseteq S_L \). But all the points in \( \bigcup_{k=1}^{\mu} r(1 + \varepsilon)B_{M_n+L} \) have norm \( < 1 \). A contradiction.
We have shown that the claim of Theorem 3.1(b) is true for separable spaces.

Let $X$ be any normed space. Suppose by contradiction that $\Gamma_2(X)$ is not non-shrinking. Let $g \in \Gamma_2(X)$, $r \in (0,1)$ and $x_1, \ldots, x_m \in X$ be such that $g(B_X) \subseteq \bigcup_{k=1}^{m} (x_k + rB_X)$. Let $Z$ be a separable subspace of $X$ such that $[\{x_1, \ldots, x_k\}] \subseteq Z$ and $g(Z) \subseteq Z$. Such $Z$ exists by Proposition 3.7. Relying on the fact that $x_1, \ldots, x_m \in Z$, we conclude that $g(B_Z) \subseteq \bigcup_{k=1}^{m} (x_k + rB_Z)$. Also, $g \upharpoonright Z \in \Gamma_2(Z)$. So $\Gamma_2(Z)$ is not non-shrinking. This contradicts the first part of the proof. So Part (b) of the theorem is proved.

Proposition 3.11. Theorem 3.1(b) does not hold for $X = c_0$.

Proof Let $V$ be the set of all the sequence in $c_0$ which have only finitely many nonzero coordinates and in which every coordinate is 0, 2 or $-2$. For $v \in V$ let $S_v = v + B_{c_0}$.

It is not difficult to check that $\{S_v \mid v \in V\}$ is a locally finite family and that $\bigcup_{v \in V} S_v = 3B_{c_0}$. For every $x \in 3B_{c_0}$, let $v_x$ be such that $x \in S_{v_x}$. Define a map $h : c_0 \rightarrow c_0$ as follows. If $x \in 3B_{c_0}$, then $h(x) = x - v_x$, and if $x \in c_0 \setminus 3B_{c_0}$, then $h(x) = x$. Clearly, $h \in \Gamma_2(c_0)$ and $h(3B_{c_0}) = B_{c_0}$.

Question 3.12. Is $H(c_0) \cap \Gamma_2(c_0)$ non-shrinking?
4 Faithfulness in normed spaces and in metrizable locally convex spaces

This section deals with two faithfulness theorems: the first concerns with open subsets of normed spaces, and the second with open subsets of metrizable locally convex spaces. Recall that $K_1$ is the faithful class of Theorem 2.3(a).

For every normed space $E$ we shall define a subgroup $G_E$ of $H(E)$ such that $G_E \subseteq \Gamma_1(E) \cap LLIP(E)$, and for every nonempty open subset $X \subseteq E$ we define $G_X = \{g \mid X : g \in G_E \text{ and } \text{supp}(g) \subseteq E\}$. In the first theorem, Theorem 4.3, we prove that $(X, G_X) \in K_1$. Also, since $G_E \subseteq \Gamma_1(E)$, $(X, G_X)$ does not have small sets.

Every metrizable topological vector space has a metric which is invariant under $+$. We shall deal only with such metrics. We next define the group $G_X$ mentioned above.

**Definition 4.1.** (a) Let $(E, \tau)$ be a metrizable topological vector space and $d$ be metric on $E$ whose topology is $\tau$ and which is invariant under $+$. Then $(E, d)$ is called a metric vector space. In particular a metric locally convex space is a metric vector space which is locally convex.

(b) Let $X, Y$ be metric spaces and $f : X \to Y$ be 1–1. The function $f$ is bilipschitz if $f$ and $f^{-1}$ are Lipschitz functions. $f$ is locally bilipschitz if for every $x \in X$ there is $U \in Nbr(x)$ such that $f \restriction U$ is bilipschitz.

(c) For a metric space $X$, $A \subseteq X$ and $r > 0$, denote $B_X(A, r) = \bigcup_{x \in A} B_X(x, r)$. For a vector space $E$ and $x, y \in E$ denote $[x, y] = \{tx + (1-t)y \mid t \in [0, 1]\}$.

(d) Let $E$ be a metric locally convex space and $h \in H(E)$. We say that $h$ is a basic homeomorphism if there are $x, v \in E$ and $r > 0$ such that

1. $\text{supp}(h) \subseteq B_E([x, x + v], r)$;
2. For every $z \in E$, there is $\lambda(z) \in \mathbb{R}$ such that $h(z) = z + \lambda(z) \cdot v$.
3. $h(x) = x + v$. 

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Denote $x = x^h$, $x + v = y^h$, $v = v^h$, $r = r^h$ and $B([x, x + v], r) = S^h$.

Note that $h = Id$ is a basic homeomorphism with $v^h = 0$.

**Faithfulness in Normed spaces**

**Definition 4.2.** Let $E$ be a normed space. We define $\hat{G}_E \subseteq H(E)$. A homeomorphism $h$ belongs to $\hat{G}_E$ if there is a sequence of basic homeomorphisms $\{h_i | i \in \mathbb{N}\}$ such that

1. $\{S^{h_i} | i \in \mathbb{N}\}$ is separated.
2. $h| (E \setminus \bigcup_{i \in \mathbb{N}} S^{h_i}) = Id$, and for every $i \in \mathbb{N}$, $h| S^{h_i} = h_i| S^{h_i}$.
3. $\text{supp}(h)$ is a bounded set.
4. For every $i,j \in \mathbb{N}$, $r^{h_i} = r^{h_j}$.
5. $h$ is locally bilipschitz.

Let $\hat{G}_{E}^{\text{Lip}} = \{h \in \hat{G}_E | h$ is bilipschitz$\}$. Define $G_E$ and $G_E^{\text{Lip}}$ to be the subgroups of $H(E)$ generated by $\hat{G}_E$ and $\hat{G}_E^{\text{Lip}}$ respectively. Suppose that $X \subseteq E$ is open and nonempty, and define

$$G_X = \{g|X | g \in G_E \text{ and } g|(E \setminus X) = Id\}$$

and

$$G_X^{\text{Lip}} = \{g|X | g \in G_E^{\text{Lip}} \text{ and } g|(E \setminus X) = Id\}.$$

Note that $\hat{G}_E \subseteq \hat{T}_1(E)$ and $(\hat{G}_E)^{-1} = \hat{G}_E$. Hence $G_E \subseteq \Gamma_1(E)$.

**Theorem 4.3.** (a) For every normed space $E$ and an open nonempty subset $X \subseteq E$, $(X, G_X) \in K_1$.

(b) For every Banach space $E$ and an open nonempty subset $X \subseteq E$, $(X, G_X^{\text{Lip}}) \in K_1$.

**Remark** The definition of $K_1$ implies that $K_1$ is closed upwards. That is, if $(X, G) \in K_1$ and $G \subseteq H \subseteq H(X)$, then $(X, H) \in K_1$. So the fact that $(X, G_X^{\text{Lip}}) \in K_1$ implies that $(X, G_X) \in K_1$.

We need the following proposition. It appears in [RY] as Proposition 2.14(c).
Proposition 4.4. There is a function $M(\ell, t)$ increasing in $\ell$ and decreasing in $t$ such that for every normed space $E$, $x, y \in E$ and $r > 0$, there is $h \in H(E)$ such that:

1. $\text{supp}(h) \subseteq B([x, y], r)$;
2. $h(z) - z \in \text{span}\{y - x\}$, for every $z \in B([x, y], r)$;
3. $h(x) = y$;
4. $h|B(x, \frac{3r}{4}) = \text{tr}_{y-x}|B(x, \frac{3r}{4})$.

(Recall that $\text{tr}_v$ denotes the function $x \mapsto x + v$).

Note that in the above proposition Clauses (1) - (3) imply that $h$ is a basic homeomorphism.

For a metric space $Y$, $x \in Y$ and $r > 0$ let $B_Y(x, r)$ denote the open ball of $Y$ with center at $x$ and radius $r$.

Proposition 4.5. Let $E$ be a normed space and $X \subseteq E$ be an open nonempty set. Let $x \in X$ and $r > 0$ be such that $B_E(x, 4r) \subseteq X$. Then every infinite separated subset $A \subseteq B_E(x, r)$ is dissectable with respect to $G^{LIP}_X$.

Proof If a set $C$ has a dissectable subset, then $C$ itself is dissectable. So we may assume that $A$ is countable. Let $A = \{x_n \mid n \in \mathbb{N}\}$. There is $e > 0$ and a subsequence $\{x'_{n}\}$ of $\{x_n\}$ such that for every $\varepsilon > 0$ there is $n_0$ such that for every $m, n > n_0$, $e - \varepsilon < \|x'_m - x'_n\| < e + \varepsilon$. We may thus assume that for every distinct $m, n$, $\frac{7e}{8} < \|x_m - x_n\| < \frac{9e}{8}$. Denote $x_1 = u$. We show that $B_E(u, \frac{e}{2})$ dissects $A$. By removing $x_1$ from $A$ we may assume that for every $n$, $\frac{7e}{8} < \|x_n - u\| < \frac{9e}{8}$.

Claim 1 For every $0 < s < \frac{e}{2}$ there are $h \in G^{LIP}_E$ and $a > 0$ such that

(i) $\text{supp}(h) \subseteq X$,

(ii) for every $n$, $h(x_{2n}) \in B_E(u, s)$ and $h|B_E(x_{2n+1}, a) = Id$.

Proof For simplicity assume that $u = 0$. Let $y_n = \frac{e}{4} \cdot \frac{x_n}{\|x_n\|}$ and $L_n = [x_n, y_n]$. We leave to the reader to check that there are $b, c > 0$ such that:

1. $B_E(L_n, b) \subseteq X$,

2. For every distinct $m, n$,
For every even $n \in \mathbb{N}$ let $h_n$ be as assured by Proposition 4.4. That is,

1. $\text{supp}(h_n) \subseteq B_E(L_n, b)$;
2. $h_n(x_n) = y_n$;
3. $h_n$ is $M(\|x_n - y_n\|, b)$-bilipschitz.

Note that because of the increasingness of $M(\ell, t)$ in $\ell$ there is $K$ such that for every $n \in 2\mathbb{N}$, $h_n$ is $K$-bilipschitz. Let $h = \bigcup_{n \in 2\mathbb{N}} h_n \restriction B_E(n, b) \cup \text{Id} \restriction (E \setminus \bigcup_{n \in 2\mathbb{N}} B_E(n, b))$.

It is easy to see that $h$ is as required in Claim 1.

We now show that $\overline{B}_E(x, \frac{c}{2})$ dissects $A$. Let $U$ be a nonempty open subset of $\overline{B}_E(u, \frac{c}{2})$ and $y \in U$. There is $t > 0$ such that $B_E([u, y] \cap \ell, t) \subseteq B_E(u, \frac{c}{2})$.

By Proposition 4.4 there is $g \in G^{LIP}_E$ such that $g(u) = y$ and $\text{supp}(g) \subseteq B_E(u, \frac{c}{2})$. Let $s > 0$ be such that $g(B_E(u, s)) \subseteq U$. By Claim 1, there is $h \in G^{LIP}_E$ such that $\text{supp}(h) \subseteq X$, for every $n \in 2\mathbb{N}$, $h(x_n) \in \overline{B}(u, s)$ and for every $n \in 2\mathbb{N} + 1$, there is $T \in \text{Nbr}(x_n)$ such that $h \restriction T = \text{Id}$. Clearly, $\text{supp}(g \circ h) \subseteq X$. Let $f = (g \circ h) \restriction X$. Then

1. $f \in G^{LIP}_X$;
2. $f(x_n) \in U$, for every $n \in 2\mathbb{N}$;
3. If $n \in 2\mathbb{N} + 1$, then there is $T \in \text{Nbr}(x_n)$ such that $f \restriction T = \text{Id}$. We have shown that $\overline{B}_E(x, \frac{c}{2})$ dissects $A$.

Let $E$ be a normed space and $g \in LIP(E)$. Then $g$ can be extended uniquely to a homeomorphism $\overline{g}$ of $\overline{E}$. For simplicity, if $x \in E$ we denote $\overline{g}(x)$ by $g(x)$.

**Proposition 4.6.** Let $E$ be a normed space, $u \in E$ and $x \in \overline{B}_E(u, r) \setminus E$. Let $\emptyset \neq V \subseteq \overline{B}_E(u, r)$ be open. Then there is $h \in G^{LIP}_E$ such that $\text{supp}(h) \subseteq \overline{B}_E(u, r)$ and $h(x) \in V$.

**Proof** Let $v \in V \cap E$. Choose $w \in E$ sufficiently close to $x$ so that $B_E([w, v], 3\|w - x\|) \subseteq B_E(u, r)$ and $\|w - x\| < d(v, \overline{E} \setminus V)$. Denote $s = \|w - x\|$. We now use Proposition 4.4. Let $h \in H(E)$ be such that

1. $\text{supp}(h) \subseteq B([w, v], 3s)$;

$$d(B_E(L_m, b), B_E(L_n, b)) > c.$$
(2) \( h(z) - z \in \text{span}(\{v - w\}) \), for every \( z \in B([w, v], 3s) \);
(3) \( h(w) = v \);
(4) \( h \) is \( M(\|w - v\|, 3s) \)-bilipschitz;
(5) \( h|B(w, 2s) = \text{tr}_{v-w}|B(w, 2s) \).

Hence \( h \in G_E^{\text{lip}} \) and \( \text{supp}(h) \subseteq B(u, r) \). Also, 
\[ h(x) = h(w) + (x - w) = v + (x - w). \]
So \( \|h(x) - v\| = \|x - w\| < d(v, \overline{E} \setminus V). \)
So \( h(x) \in V. \)

Let \( Y \) be a metric space and \( \overline{Y} \) be its completion. Let \( U \subseteq Y \) be open. Denote \( \text{cmpl}(U) = \{x \in \overline{E} \mid \text{there is } V \in \text{Nbr}_{\overline{E}}(x) \text{ such that } V \cap Y \subseteq U\} \). Note that \( \text{cmpl}(U) \) is open in \( \overline{E} \) and that \( \text{cmpl}(U) \cap E = U \).

**Proposition 4.7.** Let \( E \) be a normed space and \( X \subseteq E \) be open. Let \( \{x_n \mid n \in \mathbb{N}\} \subseteq X \) be a Cauchy sequence such that \( \lim_n x_n \in \text{cmpl}(X) \setminus X \). Then \( \{x_n \mid n \in \mathbb{N}\} \) is dissectable with respect to \( G_X \).

**Proof** Denote \( x = \lim_n x_n \). Let \( u \in X \) and \( r > 0 \) be such that \( \overline{B}_E(u, 4r) \subseteq X \) and \( \|x - u\| < \frac{r}{10} \). We may assume that the \( x_n \)'s are pairwise distinct and that for every \( n \), \( \|x_n - x\| < \frac{r}{10} \). Let \( v \in E \) be such that \( \|v\| = r \), \( L = [x, x + v] \) and \( L_n = [x_n, x_n + v] \). Then \( L \subseteq B_E(u, 2r) \setminus E \) and \( L_n \subseteq B_E(u, 2r) \). There are \( t > s > 0 \) such that \( B_E(L_1, t) \subseteq B_E(u, 4r) \) and for every \( n \), \( B_E(L_n, s) \subseteq B_E(L_1, t) \), (for example, take \( t \) and \( s \) to be \( \frac{3r}{10} \) and \( \frac{r}{10} \)). Let \( \{s_n\} \) be a sequence of positive numbers such that for every \( n \neq m \), \( B_E(L_n, s_n) \cap B_E(L_m, s_m) = \emptyset \) and \( B_E(L_n, s_n) \subseteq B_E(L_1, t) \). For every \( n \in 2\mathbb{N} \) let \( h_n \) be as assured by Proposition 4.3 and such that \( h_n(x_n) = x_n + v \) and \( \text{supp}(h_n) \subseteq B_E(L_n, s_n) \).

Let \( h = \bigcup_{n \in 2\mathbb{N}} h_n \upharpoonright B_E(L_n, s_n) \cup \text{Id} \mid (E \setminus \bigcup_{n \in 2\mathbb{N}} B_E(L(n, s_n))). \)
Every accumulation point of \( \{B_E(L_n, s_n) \mid n \in 2\mathbb{N}\} \) in \( \overline{E} \) belongs to \( L \), so since \( L \cap E = \emptyset \), \( \{B_E(L_n, s_n) \mid n \in 2\mathbb{N}\} \) is discrete in \( E \). It follows that \( h \in H(E) \).

Since for every \( n \in 2\mathbb{N} \), \( h_n \) is bilipschitz, \( h \) is locally bilipschitz. Also, \( \text{supp}(h) \subseteq B_E([x_1, x_1 + v], t) \) and \( h(x_1) = x_1 + v \). Finally, for every \( w \in E \), \( h(w) - w \in \text{span}(\{v\}) \). So \( h \) is a basic homeomorphism. Hence \( h \in G_E \).
Note that \( \text{supp}(h) \subseteq B_E(L_1, t) \subseteq X \). Also, for every \( n \in 2 \mathbb{N} + 1 \), \( B_E(L_n, s_n) \in \text{Nbr}(x_n) \) and \( h \upharpoonright B_E(L_n, s_n) = \text{Id} \).

Let \( w \in E \) be such that \( \| (x + v) - w \| < \frac{r}{4} \). We show that \( \overline{B}_E(w, \frac{r}{2}) \) dissects \( \{ x_n \mid n \in \mathbb{N} \} \).

Clearly, \( x + v \in \overline{B}_E(w, \frac{r}{2}) \), \( d(x, B_E(w, \frac{r}{2}) > \frac{3r}{4} \) and \( B_E(w, \frac{r}{2}) \subseteq X \).

Let \( \emptyset \neq V \subseteq \overline{B}_E(w, \frac{r}{2}) \) be open. By Proposition 4.6 there is \( g \in G_E \) such that \( g(x + v) \in V \) and \( \text{supp}(g) \subseteq \overline{B}_E(w, \frac{r}{2}) \). So \( \text{supp}(g) \subseteq X \). It follows that \( (g \circ h) \mid X \in G_X \).

Since \( \lim_{n \in \mathbb{N}} h(x_{2n}) = x + v \) and \( g(x + v) \in V \), for all but finitely many \( n \)'s, \( g \circ h(x_{2n}) \in V \).

Let \( n \) be odd. Then there is \( T \in \text{Nbr}(x_n) \) such that \( h \upharpoonright T = \text{Id} \). Also, \( \| x_n - x \| < \frac{r}{4} \). So \( d(x, B_E(w, \frac{r}{2})) > 0 \). Hence \( d(x_n, \text{supp}(g)) > 0 \). It follows that there is \( S \in \text{Nbr}(x_n) \) such that \( g \circ h \upharpoonright S = \text{Id} \).

Let \( f = (g \circ h) \upharpoonright X \). We have shown that \( f \in G_X \), and that the sets \( \{ n \mid f(x_n) \in V \} \) and \( \{ n \mid \text{there is } T \in \text{Nbr}(x_n) \text{ such that } f \upharpoonright T = \text{Id} \} \) are infinite. So \( B_E(w, \frac{r}{2}) \) dissects \( \{ x_n \mid n \in \mathbb{N} \} \).

**Proof of Theorem 4.3** We prove Parts (a) and (b) together.

Let \( X \) be an open subset of a normed space \( E \). Then \( X \) is regular and first countable. That is, (P1) of Definition 2.2(f) holds.

By Proposition 4.4, \( G_X^{\text{LIP}} \) is locally moving. Hence every subgroup of \( H(X) \) containing \( G_X^{\text{LIP}} \) is locally moving. That is, (P2) holds for every \( H \) containing \( G_X^{\text{LIP}} \).

Similarly, Proposition 4.4 implies that \( DF(X, G_X^{\text{LIP}}) = X \). Hence the same is true for every \( H \) containing \( G_X^{\text{LIP}} \). That is, (P3) holds for every \( H \) containing \( G_X^{\text{LIP}} \).

We show that (P4) holds. Assume that \( E \) is a Banach space. Let \( x \in X \) and \( r > 0 \) be such that \( \overline{B}(x, 4r) \subseteq X \). We show that \( \overline{B}(x, r) \) is flexible with respect to \( G_X^{\text{LIP}} \). Let \( A \subseteq \overline{B}(x, r) \) be an infinite set which is discrete in \( X \). So \( A \) is discrete in \( E \). Then \( A \) contains an infinite separated subset \( A' \). By Proposition 4.5, \( A' \) is dissectable with respect to \( G_X^{\text{LIP}} \). So \( A \) is dissectable
with respect to $G^L_X$, and hence $\overline{B}(x,r)$ is flexible. Since $\{\overline{B}(x,r) | \overline{B}(x,4r) \subseteq X\}$ is a cover of $X$, it follows that $(X, G^L_X)$ satisfies (P4), and the same is true for every $H$ containing $G^L_X$. We have proved Part (b) of Theorem 4.3.

Assume next that $E$ is a normed space. Let $x \in X$ and $r > 0$ be such that $\overline{B}(x,4r) \subseteq X$. We show that $\overline{B}(x,r)$ is flexible with respect to $G_X$. Let $A \subseteq \overline{B}(x,r)$ be an infinite set which is discrete in $X$. Then $A$ is discrete in $E$. Then either (i) $A$ contains an infinite separated subset $A'$, or (ii) $A$ contains a Cauchy sequence $\{x_n | n \in \mathbb{N}\}$ converging to a point in $E \setminus E$. If (i) happens, then by Proposition 4.5, $A'$ is dissectable with respect to $G^L_X$. Hence $A$ is dissectable with respect to $G_X$.

If (ii) happens, then by Proposition 4.7, $\{x_n | n \in \mathbb{N}\}$ is dissectable with respect to $G_X$. Hence $A$ is dissectable with respect to $G_X$. So $\overline{B}(x,r)$ is flexible with respect to $G_X$. Hence $(X, G_X)$ satisfies (P4). We have proved Part (a) of Theorem 4.3. □

**Faithfulness in metrizable locally convex spaces**

We now turn to general locally convex spaces. The question whether the class of locally convex topological vector spaces is faithful is open. That is, it is unknown whether $\{(X,H(X)) | X \text{ is a locally convex space}\}$ is faithful.

In [LR] it was shown that if $K_2$ is the class of space-group pairs $(X, G)$ such that $X$ is an open subset of a normal locally convex space $E$, and $E$ has a nonempty open set which intersects every line in a bounded set. Then $K_2$ is faithful.

So $K_2$ includes spaces which are not first countable. On the other hand, spaces which are a countable product of normed spaces, and in particular, $\mathbb{R}^{\aleph_0}$ do not belong to $K_2$. These spaces do belong to the faithful class considered below.

Theorem D in the introduction states that the class of all space-group pairs $(X, G)$ in which $X$ is an open subset of a locally convex metrizable topological vector space, and $G$ is a group containing all locally bi-uniformly continuous homeomorphisms of $X$ is faithful. Below we define the group $P_X$
and prove the \((X, P_X) \in K_1\). This implies Theorem D.

**Definition 4.8.** Let \(E\) be a metric locally convex space.

(a) Let \(h \in H(E)\). Suppose that there are \(x_0, \ldots, x_n, k_1, \ldots, k_n\) and \(r > 0\) such that \(h = k_n \circ \ldots \circ k_1\), and for every \(i\), \(k_i\) is a basic homeomorphism with \(x^h = x_{i-1}\), \(y^k = x_i\) and \(r^k = r\). Then \(h\) is called a *polygonal homeomorphism*. Denote \(S^h = \bigcup_{i=1}^{n} S^{k_i} \) and \(r^h = r\).

(b) We define \(\hat{P}_E \subseteq H(X)\). A member \(h \in H(E)\) belongs to \(\hat{P}_E\) if there is a set of polygonal homeomorphisms \(\{h_i \mid i \in \mathbb{N}\}\) such that

1. \(h|(E \setminus \bigcup_{i \in \mathbb{N}} S^{h_i}) = Id\), and for every \(i \in \mathbb{N}\), \(h|S^{h_i} = h_i|S^{h_i}\).
2. \(h\) is locally bi-uniformly continuous.
3. \(\text{supp}(h)\) is a bounded set.
4. \(\{S^{h_i} \mid i \in \mathbb{N}\}\) is discrete.

Let \(P_E\) be the subgroup of \(H(E)\) generated by \(\hat{P}_E\)

Let \(X \subseteq E\) be open and nonempty. Define

\[P_X = \{g|X \mid g \in P_E\text{ and }g|(E \setminus X) = Id\} \]

**Theorem 4.9.** (a) Let \(K_M\) be the class of all space-group pairs \((X,G)\) in which \(X\) is a nonempty open subset of a metrizable locally convex topological vector space \(E\), and \(G \subseteq H(X)\) is a group containing all locally bi-uniformly continuous homeomorphisms of \(X\).

Then \(K_M\) is faithful.

(b) For every metric locally convex space \(E\) and a nonempty open subset \(X \subseteq E\), \((X,P_X) \in K_1\).

(c) Let \(K_P = \{(X,G) \mid X\text{ is a nonempty open subset of a metric locally convex space and }P_X \subseteq G \subseteq H(X)\}\).

Then \(K_P\) is faithful.

**Remarks** (a) In 4.9(a) the “uniform continuity” is with respect to the uniformity of the topological group \((E, +)\). However, for every metric \(d\) on \(E\), if \(d\) is invariant under + and \(d\) induces on \(E\) the original topology, then the uniformity of \((E, d)\) is equal to the uniformity of \((E, +)\).
(b) Because $K_1$ is closed upwards, §4.9(b) implies §4.9(c). Since $K_M \subseteq K_P$, §4.9(c) implies §4.9(a).

(c) The proof of §4.9(b) is analogous to the proof of Theorem §4.3(a).

For a metric locally convex space $(E,d)$ and $x \in E$, denote $\|x\| = d(0,x)$.

**Proposition 4.10.** Let $(E,d)$ be a metric locally convex space. Let $x,y \in E$ and $r > 0$. Then there is a bi-uniformly continuous basic homeomorphism $h$ such that $h(x) = y$ and $\text{supp}(h) \subseteq B([x,y],r)$.

**Proof** We may assume that $x = 0$. Let $L = \text{span}(\{y\})$. Since $E$ is locally convex, there is a complement $F$ of $L$ such that the projection $P$ of $(F,L)$ is continuous. In a metric vector space with an invariant metric every continuous linear operator is uniformly continuous. In particular, $P$ is uniformly continuous.

By the triangle inequality and the fact that the metric $d$ is $+$-invariant, $B_L(0,\frac{r}{2}) + B_F(0,\frac{r}{2}) \subseteq B_E(0,r)$. So $B_L([0,y],\frac{r}{2}) + B_F(0,\frac{r}{2}) \subseteq B_E([0,y],r)$.

Let $p = \min(\frac{r}{2}, \frac{\|y\|}{2})$. Choose $y_1 \in [0,y]$ such that $\|y_1\| = p$, and let $q$ be such that $qy = y_1$. We define a function $g : [0,p] \times \mathbb{R} \to \mathbb{R}$ as follows. Let $t \in [0,p]$. We first define a function $g_t : \mathbb{R} \to \mathbb{R}$. The function $g_t$ is the unique piecewise linear function satisfying:

1. The breakpoints of $g_t$ are $-q, 0$ and $1 + q$,
2. $g_t(s) = 0$ for every $s \in [-\infty, -q] \cup [1 + q, \infty]$,
3. $g_t(0) = \frac{p - t}{p}$.

Define $g$ by $g(t,s) = g_t(s)$. Clearly, $g$ is uniformly continuous.

Let $z \in E$. Suppose that $z = \lambda y + v$, where $v \in F$. So $\lambda y = P(z)$. Denote $\varphi(z) = \lambda$ and $v = Q(z)$. Since $Q = \text{Id} - P$, $Q$ is uniformly continuous. $\varphi$ is a continuous homomorphism from $(E,+)$ to $(\mathbb{R},+)$. So $\varphi$ is uniformly continuous. Define

$$h(z) = \begin{cases} g_{\|v\|}(\lambda) \cdot y + v & \|v\| \leq p \\ z & \|v\| > p \end{cases}$$

Note that if $\|v\| = p$, then $h(z) = z$. Now, if $\|v\| \leq p$, then $h(z) = g(\|Q(v)\|, \varphi(z)) \cdot y + Q(z)$. Hence $h$ is uniformly continuous. Clearly,
\[
    h^{-1}(z) = \begin{cases} 
    (g_{\|Q(z)\|})^{-1}(\phi(z)) \cdot y + Q(z) & \|Q(z)\| \leq p \\
    z & \|Q(z)\| \geq p
    \end{cases}
\]

Since the function \((t, s) \mapsto (g_t)^{-1}(s)\) is uniformly continuous, \(h^{-1}\) is uniformly continuous.

Let \(z \in E\). Suppose that \(z = \lambda y + v\), where \(v \in F\). If \(\|v\| \geq p\), then \(h(z) = z\), and if \(\lambda \not\in [-q, 1+q]\), then \(h(z) = z\). It follows that 
\[supp(h) \subseteq B_L([0, y], p) + B_F(0, p) \subseteq B_E([0, y], r)\]. So \(h\) is as required.

**Definition 4.11.** Let \(E\) be a metrizable infinite dimensional locally convex space. Denote the completion of \(E\) by \(\overline{E}\). A nonempty open subset \(U \subseteq E\) is called *polygonally flexible*, if the following holds. For every infinite \(A \subseteq U\), if \(A\) is discrete in \(\overline{E}\), then there are an infinite set \(B \subseteq A\) and \(x \in U\) such that \(P(x, B)\) holds, where \(P(x, B)\) is the following statement.

For every \(W \in Nbr(x)\), there are a set \(\{y_b \mid b \in B\} \subseteq W\) and a family of polygonal lines \(\{L_b \mid b \in B\}\) such that:

1. For every \(b \in B\), \(L_b \subseteq cl(U)\) and the endpoints of \(L_b\) are \(b\) and \(y_b\).
2. \(\{L_b \mid b \in B\}\) is discrete in \(\overline{E}\).

**Proposition 4.12.** (a) The separable Hilbert space \(\ell_2\) is polygonally flexible. Moreover, for every infinite discrete \(A \subseteq \ell_2\) there is an infinite \(B \subseteq A\) and a nonempty open set \(D\) such that for every \(x \in D\), \(P(x, B)\) holds.

(b) Let \(E\) be an infinite dimensional metric locally convex space and \(U \subseteq E\) be a convex open set. Then \(U\) is polygonaly flexible.

**Proof**  
(a) The proof of Part (a) is easy and is left to the reader.

(b) Let \(A \subseteq U\) be a countably infinite set discrete in \(\overline{E}\). Choose a separable infinite dimensional closed subspace \(E_1\) of \(E\) such that \(E_1\) contains the linear span of \(A\). Let \(U_1 = U \cap E_1\) and \(F_1 = cl_{E_1}(U_1)\).

Clearly, \(F_1\) is the closure of an open convex subset of a separable Fréchet space. By [BP] Corollary 6.1 p.191, such a set is homeomorphic to \(\ell_2\). Choose \(h : F_1 \cong \ell_2\). Let \(A' = h(A)\). Then \(A'\) is discrete in \(\ell_2\). So there is an infinite \(B' \subseteq A'\) and a ball \(D'\) of \(\ell_2\) such that for every \(x' \in D'\), \(P(x', B')\) holds.

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Let \( x' \in D' \cap h(U_1) \). Let \( x = h^{-1}(x') \) and \( B = h^{-1}(B') \). We show that \( P(x, B) \) holds. Let \( W \in Nbr_E(x) \). Denote \( W_1 = cmpl(W \cap E_1) \). Then \( h(W_1) \in Nbr_{E_2}(x') \). Let \( \{L_{b'} \mid b' \in B'\} \) be as assured by \( P(x', B') \) for \( h(W) \), and \( M_b = h^{-1}(L_{h(b)}) \). Replace \( M_b \) by polygonal line \( L_b \) which is sufficiently close to \( M_b \) and whose vertices are in \( E \). Then \( L_b \subseteq E \). Also, \( \{L_b \mid b \in B\} \) is discrete in \( F_1 \). Since \( F_1 \) is closed in \( E \), \( \{L_b \mid b \in B\} \) is discrete in \( E \). \( \square \)

For \( x \in E \) and \( r > 0 \) let \( \overline{B}_E(x, r) \) denote the open ball with center at \( x \) and radius \( r \).

**Proposition 4.13.** Let \( X \) be an open set in an infinite dimensional metric locally convex space \( E \). Suppose that \( U \) is an open nonempty bounded convex subset of \( E \) such that \( cl_E(U) \subseteq X \). Let \( A \subseteq U \) be infinite, and suppose that \( A \) is discrete in \( E \). Then \( A \) is dissectable in \( (X, P_X) \).

**Proof** Let \( B \subseteq A \) and \( x \in U \) be such that \( P(x, B) \) holds. We may assume that \( |B| = \aleph_0 \) and that \( x \not\in B \).

**Claim 1** For every \( V \in Nbr(x) \) and \( C \subseteq B \) there is \( h \in P_X \) such that for every \( c \in C \), \( h(c) \in V \), and for every \( b \in B \setminus C \) there is \( W \in Nbr(b) \) such that \( h \upharpoonright W = Id \).

**Proof** For every \( b \in B \) let \( L_b \subseteq cl_E(U) \) be a polygonal line and \( y_b \in V \) be such that (i) the endpoints of \( L_b \) are \( b \) and \( y_b \), (ii) \( \{L_b \mid b \in B\} \) is discrete in \( E \). Let \( d_b = d(L_b, (E \setminus X) \cup \{L_{b'} \mid b' \in B \setminus \{b\}\}) \). So \( d_b > 0 \). Let \( \{r_b \mid b \in B\} \) be a sequence of positive numbers converging to 0 such that \( r_b < \frac{d_b}{2} \). There is a bi-uniformly continuous polygonal homeomorphism \( h_b \) such that \( h_b(b) = y_b \) and \( supp(h_b) \subseteq B(L_b, r_b) \). The homeomorphism \( h_b \) is obtained by applying Proposition [4.10] to every edge of \( L_b \). So for every distinct \( b, b' \in B \), \( supp(h_b) \cap supp(h_{b'}) = \emptyset \) and \( supp(h_b) \subseteq X \). Also, since \( \{r_b \mid b \in B\} \) converges to 0, \( \{B(L_b, r_b) \mid b \in B\} \) is discrete in \( E \). Let \( h = Id \upharpoonright (X \setminus \bigcup_{c \in C} B(L_c, r_c)) \cup \bigcup_{c \in C} h_c \). Then \( h \) is as required.

**Claim 2** Let \( d = \min(d(x, E \setminus U), \frac{d(x, B)}{2}) \) and \( T \) be an open convex set contained in \( B(x, d) \) and containing \( x \). Then \( T \) dissects \( B \).

**Proof** Let \( W \subseteq T \) be open and nonempty. Let \( y \in W \). So \( [x, y] \subseteq U \).
Hence there is \( r > 0 \) such that \( B([x, y], r) \subseteq \bar{B}(x, d) \). There is a bi-uniformly continuous basic homeomorphism \( g' \) such that \( \text{supp}(g') \subseteq B([x, y], r) \) and \( g'(x) = y \). This follows from Proposition 4.10. Let \( g = g' \mid X \). Then \( g \in P_X \).

Denote \( V = g^{-1}(W) \). Then \( V \in \text{Nbr}(x) \). Let \( C \subseteq B \) be such that \( C \) and \( B \setminus C \) are infinite. By Claim 1, there is \( h \in P_X \) such that \( h(C) \subseteq V \) and for every \( b \in B \setminus C \), there is \( S \in \text{Nbr}(b) \) such that \( h \upharpoonright S = \text{Id} \).

It follows that for every \( c \in C \), \( g \circ h(c) \in W \). Also, \( g \mid (X \setminus B([x, y], r)) = \text{Id} \) and \( X \setminus B([x, y], r) \in \text{Nbr}(b) \) for every \( b \in B \). So for every \( b \in B \setminus C \) there is \( S \in \text{Nbr}(b) \) such that \( g \circ h \upharpoonright S = \text{Id} \). This shows that \( T \) dissects \( B \). This proves Claim 2. Since \( B \subseteq A \), \( T \) dissects \( A \) in \((X, P_X)\).

If \( g \in H(E) \) and \( g \) is bi-uniformly continuous, then \( g \) has a unique extension to \( \overline{E} \). So \( g(x) \) is defined for any \( x \in \overline{E} \).

**Proposition 4.14.** Let \( E \) be a metric locally convex space. Let \( x \in \overline{E} \setminus E \). Let \( U \) be an open convex subset of \( E \) such that \( x \in \text{cmpl}(U) \). Then for every nonempty open subset \( V \subseteq U \) there is a bi-uniformly continuous basic homeomorphism \( g \) of \( E \) such that \( \text{supp}(g) \subseteq U \) and \( g(x) \in \text{cmpl}(V) \).

**Proof** Let \( z \in \text{cmpl}(V) \) be such that \( y := z - x \in E \). Let \( r > 0 \) be such that \( B_{\overline{E}}([x, z], 2r) \subseteq \text{cmpl}(U) \). So if \( x' \in \overline{E} \) is such that \( \|x' - x\| < r \), then \( B_{\overline{E}}([x', x' + y], r) \subseteq \text{cmpl}(U) \). By Proposition 4.10 there is a bi-uniformly continuous basic homeomorphism \( h \) of \( E \) such that \( \text{supp}(h) \subseteq B([0, y], r) \) and \( h(0) = y \). Let \( \bar{h} \) be the extension of \( h \) to \( \overline{E} \). Let \( e > 0 \) be such that \( B_{\overline{E}}(z, 2e) \subseteq \text{cmpl}(V) \). There is \( \varepsilon > 0 \) such that for every \( u \in \overline{E} \): if \( \|u\| < \varepsilon \), then \( \|\bar{h}(u) - y\| < e \). Let \( x' \in E \) be such that \( \|x' - x\| < \min(r, \varepsilon, e) \). Let \( g = \text{tr}_{x' \circ h \circ \text{tr}_{x'}^{-1}} \). So \( g \) is a bi-uniformly continuous basic homeomorphism. Clearly, \( \text{supp}(g) \subseteq B([x', x' + y], r) \subseteq U \). Since \( \|x' - x\| < \varepsilon \), \( \|g(x) - g(x')\| < e \). Also, \( g(x') = x' + z - x \). So \( \|g(x) - x' - z + x\| < e \). Hence \( \|g(x) - z\| \leq \|g(x) - x' - z + x\| + \|x' - x\| < 2e \).

So \( g(x) \in \text{cmpl}(V) \).

**Proposition 4.15.** Let \( E \) be a metric locally convex space. Let \( X \subseteq E \) be open, and \( \{x_n \mid n \in \mathbb{N}\} \subseteq X \) be a Cauchy sequence such that
\[ \lim_{n} x_n \in \text{cmpl}(X) \setminus X. \] Then \( \{x_n \mid n \in \mathbb{N}\} \) is dissectable with respect to \( P_X \).

**Proof** The proof is analogous to the proof of Proposition 4.7. It relies on Propositions 4.10 and 4.14 in the same way that the proof of 4.7 relies on Propositions 4.4 and 4.6.

**Proof of Theorem 4.9** Part (c) of 4.9 follows from Part (b) and Part (a) follows from Part (c). We prove Part (b).

Let \( X \) be a nonempty open subset of a metric locally convex space \( E \). Then \( X \) is regular and first countable. That is, (P1) of Definition 2.2(f) holds. By Proposition 4.10, \( P_X \) is locally moving. That is, (P2) holds for \( P_X \). Proposition 4.10 also implies that \( DF_{X,P_X}(x,y) \) holds for any distinct \( x,y \in X \). Indeed, choose \( r > 0 \) such that \( y \not\in B_E(x,r) \subseteq X \). Then by Proposition 4.10, for every \( z \in B_E(x,r) \) there is \( g \in P_X \) such that \( g(x) = z \), and \( g\restriction U = Id \) for some \( U \in \text{Nbr}(y) \). Hence (P3) holds.

We show that (P4) holds. Let \( U \) be a bounded convex open set such that \( cl_E(U) \subseteq \text{cmpl}(X) \). Let \( A \subseteq U \) be an infinite set, and assume that \( A \) is discrete in \( X \). Either \( A \) is discrete in \( E \) or \( A \) contains a 1–1 Cauchy sequence converging to a member of \( E - E \). If \( A \) is discrete in \( E \), then by Proposition 4.13, \( A \) is dissectable in \( (X,P_X) \).

Suppose that \( B = \{x_n \mid n \in \mathbb{N}\} \subseteq A \) is a 1–1 Cauchy sequence converging to \( x \) and \( x \in E - E \). Then \( x \in \text{cmpl}(X) - X \). By Proposition 4.15, \( B \) is dissectable in \( (X,P_X) \). We have shown that \( U \) is flexible in \( (X,P_X) \). So \( X \) has an open cover consisting of flexible sets. That is, (P4) holds. 

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