Accuracy and efficiency of three stress integration schemes for the SANISAND-04 model

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ABSTRACT

In this paper, three types of stress integration algorithms are evaluated for the SANISAND-04 model based on Runge-Kutta method (RK4), cutting plane method (CPM) and closest point projection method (CPPM). Performance of the three integration schemes is first assessed in monotonic drained and undrained triaxial compression test simulations, and then in cyclic undrained triaxial test simulations. These simulations show that the errors of CPPM and CPM are similar and predictable, generally increasing with increasing strain step size. The error of the RK4 is highly dependent on the loading conditions and material parameters. Under undrained cyclic loading, the overall performance of CPM is the best in terms of both accuracy and convergence. As for the computational cost, CPM is the most efficient integration scheme, especially for small strain increments.

Keywords: SANISAND-04 model, integration schemes, Runge-Kutta method, cutting plane method, closest point projection method

1 INTRODUCTION

Complex elastoplastic constitutive models have been developed to provide high fidelity description of the mechanical behavior of soils. These complex models require accurate and efficient numerical implementation when used to solve boundary value problems. The numerical implementation of such models requires the integration of stress-strain rate equations. Therefore, the choice of an appropriate integration scheme is utmost important. Fully implicit integration schemes are often considered to be the most accurate, but are generally difficult to derive and may suffer from low efficiency due to its iterative approach, e.g. Runesson (1987) and Manzari (2011). In comparison, explicit and semi-explicit integration schemes are much simpler in formulation, but may suffer from non-strict compliance with consistency conditions, like Sloan (1987) and Sloan (2001). Implicit, explicit, and semi-explicit integration schemes such as the closest point projection method (CPPM), the substepping method, and the cutting plane method (CPM) have been developed, e.g. Simo and Ortiz (1985), and have been applied to several constitutive models, such as Manzari (2001) and Manzari (2011).

In this study, three different integrations schemes are implemented for a very widely used plasticity model, SANISAND-04 by Dafalias and Manzari (2004), and compared in terms of accuracy and computation efficiency. The integration schemes include the explicit Runge-Kutta method (RK4), the cutting plane method (CPM), and the fully implicit closest point projection method (CPPM). The performance of the three integration schemes are first evaluated in undrained and drained compression simulations, then in undrained cyclic simulations.

2 INTEGRATION SCHEMES

2.1 SANISAND-04 model

The SANISAND-04 model is a state dependent bounding surface plasticity model proposed by Dafalias and Manzari (2004). The framework of the model is:

\[ d\varepsilon^e = \frac{ds}{2G}; \quad d\varepsilon^p = \frac{dp}{K} \]  
\[ d\varepsilon^p = \langle L \rangle \mathbf{R}; \quad d\varepsilon^p = \langle L \rangle D \]  
\[ G = G_0 p_{at} (\frac{2.97 - e}{1 + e})^2 - (\frac{p}{p_{at}})^{1/2}; \quad K = \frac{2(1 + \nu)}{3(1 - 2\nu)} G \]  
\[ f = [(s - \rho a) : (s - \rho a)]^{1/2} - \sqrt{2/3} pm = 0 \]
Here \( \alpha \) is the fabric-dilatancy tensor; \( p_{at} \) represents atmospheric pressure. Fig. 1 shows the yield surface, bounding surface, critical surface and dilatancy surface in the deviatoric stress ratio space.

### 2.2 Runge-Kutta method

The fourth order Runge-Kutta formula can be used for explicit stress integration of the model:

\[
\begin{align*}
\sigma_1^{(m+1)} &= h(\sigma^{(m)}, \varepsilon^{(m+1)}) \\
\sigma_2^{(m+1)} &= h(\sigma^{(m)} + \frac{1}{2} \Delta \sigma_1^{(m)}, \varepsilon^{(m+1)}) \\
\sigma_3^{(m+1)} &= h(\sigma^{(m)} + \frac{1}{2} \Delta \sigma_2^{(m)}, \varepsilon^{(m+1)}) \\
\sigma_4^{(m+1)} &= h(\sigma^{(m)} + \Delta \sigma_3^{(m)}, \varepsilon^{(m+1)}) \\
\sigma^{(m+1)} &= \sigma^{(m)} + \frac{1}{6} (\Delta \sigma_1^{(m)} + 2\Delta \sigma_2^{(m)} + 2\Delta \sigma_3^{(m)} + \Delta \sigma_4^{(m)})
\end{align*}
\]

where \( h \) is the function to calculate \( \Delta \sigma \) based on SANISAND-04 model formulation. It should be noted that the Runge-Kutta method does not guarantee the compliance of the consistency equation, and may cause unpredictable errors.

The method involves two steps. In the first step, the strain increments are assumed to be elastic and the stress increments are calculated correspondingly. The yield function \( f \) is then checked for plastic loading. If \( f \leq 0 \), the calculation is complete. If not, Eq. (17) is used to obtain \( \Delta \sigma \).

### 2.3 Cutting plane method

The semi-implicit cutting plane method can also be applied to implement the SANISAND-04 model. The cutting plane method was developed by Simo and Ortiz (1985), and has been used in the numerical integration of several elasto-plastic constitutive models (e.g. Manzari and Prachathananukit, 2001; Wang et al., 2014). The main steps of the method are as follows.

(I) Initialize trial step by assuming elastic response

\[
k = 0, \Delta \varepsilon^{(0)} = 0, \Delta \sigma^{(0)} = 0, L^{(0)} = 0
\]

(II) Compute the stress for the new strain increment

\[
p_{n+1}(k) = p_n + K_n(k)(\Delta \varepsilon_{v,n+1} - \Delta \varepsilon^{(k)})
\]

\[
s^{(k)}_{n+1} = s_n + 2G_n^{(k)}(\Delta \varepsilon_{v,n+1} - \Delta \varepsilon^{(k)})
\]

Here \( G_{n+1}^{(k)} \) and \( K_{n+1}^{(k)} \) are calculated by Eq. (3) with \( p_n \) and \( \varepsilon_{v,n+1} \) (\( e \) stands for the void ratio).

Calculate the yield function

\[
f^{(k)}_{n+1} = \left[ (a_{n+1}^{(k)} - a_n^{(k)}) : (\sigma^{(k)}_{n+1} - \sigma_n^{(k)}) \right]^2 - \sqrt{2/3m}
\]

Here \( a_n^{(0)} = a_n \).
IF \( f_{n+1}^{(k)} \leq \text{Tol}_1 \), THEN: EXIT.

ELSE:

(III) Compute the increment of \( L \) using:

\[
dL_{n+1}^{(k)} = - f_{n+1}^{(k)} \frac{\begin{pmatrix} -2G_{n+1}^{(k)} & R^{(k)} + DK_{n+1}^{(k)} & a_n \\
2 & 3 & \rho_h \left( a_{n+1}^{(k)} - a_{n+1}^{(k)} \right) \\
\rho_h \left( a_{n+1}^{(k)} - a_{n+1}^{(k)} \right) & \sqrt{(r_{n+1}^{(k)} - a_{n+1}^{(k)})^2} & \left( r_{n+1}^{(k)} - a_{n+1}^{(k)} \right)
\end{pmatrix}}{\sqrt{2/3DK_{n+1}^{(k)}}}
\]

\( \text{R}', \text{D}, h \) and \( a_n^{(k)} \) are only calculated once.

(IV) Update state variables

\[
L_{n+1}^{(k+1)} = L_{n+1}^{(k)} + dL_{n+1}^{(k)}
\]

\[
de_{v,n+1}^{(k+1)} = \left( L_{n+1}^{(k+1)} \right) D
\]

\[
de_{p,n+1}^{(k+1)} = \left( L_{n+1}^{(k+1)} \right) R'
\]

\[
z_{n+1}^{(k)} = z_{n+1}^{(k)} + c_z \left( -de_{v,n+1}^{(k+1)} \right)
\]

\[
\tilde{a}_{n+1}^{(k)} = \tilde{a}_{n+1}^{(k)} + \left( 2/3 \right) \left( b_0 \left( a_n^{(k)} - a_{n+1}^{(k)} \right) / \left( a_{n+1}^{(k)} - a_{n+1}^{(k)} \right) \right)
\]

\( n \) and \( b_0 \) are only calculated once.

Set \( k \rightarrow k+1 \) and Go To II.

As is observed from above algorithm, the cutting plane method requires iteration during the calculation for plastic correction. However, several internal variables are computed only once for efficiency during the iteration, thus the consistency condition is not strictly guaranteed.

2.4 Closest point projection method

An implicit integration scheme based on the closest point projection method is implemented for better compliance of the consistency condition (Manzari, 2012). The main steps of the method are as follows:

(I) Initialize trial step by assuming elastic response

\[
k = 0, de_{p}^{(0)} = 0, \text{de}_{p}^{(0)} = 0, L_{n+1}^{(0)} = 0
\]

(II) Calculate the stress state

\[
P_{n+1}^{(k)} = p_n + K_{n+1}^{(k)} (de_{v,n+1} - de_{v,n+1})
\]

\[
\tilde{a}_{n+1}^{(k)} = \tilde{a}_{n+1}^{(k)} + 2G_{n+1}^{(k)} (de_{p,n+1} - de_{p,n+1})
\]

Calculate the yield function

\[
f^{(k)}_{n+1} = \left( \left( \tilde{a}_{n+1}^{(k)} - \tilde{a}_{n+1}^{(k)} \right) \cdot \tilde{a}_{n+1}^{(k)} \right)^{1/2} - \sqrt{3m}
\]

IF \( f_{n+1}^{(k)} < 0 \), THEN: EXIT.

ELSE:

(III) Calculate the following residuals:

\[
R_1 = dp_{n+1}^{(k)} - e_{v,n+1}^{(k)} (de_{v,n+1} - de_{v,n+1})
\]

\[
R_2 = de_{v,n+1} - L_{n+1}^{(k)}
\]

\[
R_3 = -\gamma_{n+1}^{(k)} - 2G_{n+1}^{(k)} (de_{p,n+1} - de_{p,n+1})
\]

\[
R_8 = -\theta_{n+1}^{(k)} - f_{n+1}^{(k)} - R'
\]

\[
R_{13} = -17 - da_{n+1}^{(k)} + 2/3 \left( a_n^{(k)} - a_{n+1}^{(k)} \right)
\]

\[
R_{18} = -22 - dz_{n+1}^{(k)} + \alpha^{(k)} \left( z_{n+1}^{(k)} - 2/3 \alpha^{(k)} \right)
\]

\[\text{D}, \text{R}', \text{a}_n^{(k)} \text{ and n are only calculated once, which has been shown to have negligible influence on the accuracy of the method. IF } ||f_{n+1}^{(k)}||_{2} \leq \text{Tol}_2, \text{THEN: EXIT.}\]

ELSE: obtain incremental plastic strains and internal variables by solving the following linear system of equations

\[
T_{n+1}^{(k)} \Delta U_{n+1}^{(k)} = -R_{n+1}^{(k)}
\]

for \( U_{n+1}^{(k)} \) that contains 23 unknowns: \( dp_{n+1}, de_{v,n+1}, de_{p,n+1}, da_{n+1}, dz_{n+1} \) and \( L_{n+1}^{(k)} \).

The matrix \( T_{n+1}^{(k)} \) is defined as \( T_{n+1}^{(k)} = \partial R_{n+1}^{(k)} / \partial U_{n+1}^{(k)} \).

For efficiency, \( T_{n+1}^{(k)} \) is only calculated once.

(IV) Update stresses, plastic strains and internal variables

\[
dp_{n+1}^{(k+1)} = dp_{n+1}^{(k)} + \Delta U_{n+1}^{(k+1)}
\]

\[
de_{v,n+1}^{(k+1)} = de_{v,n+1}^{(k)} + \Delta U_{n+1}^{(k+1)}
\]

\[
de_{p,n+1}^{(k+1)} = de_{p,n+1}^{(k)} + \Delta U_{n+1}^{(k+1)}
\]

where \( \Delta U_{n+1}^{(k+1)} \) is the first number of the \( \Delta U_{n+1}^{(k+1)} \) vector, and so on.

Set \( k \rightarrow k+1 \) and GO TO step III.

To simplify calculation and improve efficiency, the Jacobian matrix is only calculated once. This approach is observed to improve efficiency without sacrificing accuracy.
The performance of the integration schemes described in the previous sections is evaluated by simulating monotonic triaxial undrained and drained compression tests, and undrained cyclic triaxial tests. In these simulations, two sets of model parameters are used for all the simulations (Table 1). The step size is defined as the absolute value of the increment of the \( \varepsilon_{11} \) component in triaxial tests, with \( \varepsilon_{11} \) being the major principal strain in triaxial compression. The tolerance values, \( \text{Tol}_1 \) and \( \text{Tol}_2 \), are defined as \( \text{Tol}_1 = \text{Tol}_2 = 10^{-4} \). The relative errors in monotonic undrained and drained triaxial compression test simulations, \( \delta \), is calculated as:

\[
\delta = \left( \frac{\int [q - q^*]dy}{\int |q^*|dy} \right) \times 100\% \quad (47)
\]

where \( q^* \) is the “exact solution” and \( q \) is the calculated deviatoric stress based on numerical integration. The “exact solution” is obtained by using a very small strain increment with the CPPM method.

| Parameter | Set 1 | Set 2 |
|-----------|-------|-------|
| \( G_0 \)  | 125   | 130   |
| \( v \)    | 0.05  | 0.05  |
| \( M \)    | 1.25  | 1.27  |
| \( c \)    | 0.712 | 0.712 |
| \( \lambda_c \) | 0.019 | 0.02  |
| \( \omega \) | 0.934 | 0.858 |
| \( \xi \)  | 0.7   | 0.69  |
| \( m \)    | 0.01  | 0.02  |
| \( h_0 \)  | 7.05  | 8.5   |
| \( c_h \)  | 0.968 | 0.968 |
| \( n^b \)  | 1.1   | 1.05  |
| \( A_0 \)  | 0.704 | 0.6   |
| \( n^d \)  | 3.5   | 2.5   |
| \( z_{\max} \) | 4     | 4     |
| \( \varepsilon \) | 600   | 50    |

The relative error-step size plots for the three integration schemes and two sets of parameters in monotonic undrained and drained conditions are shown in Fig. 2 and Fig. 3, respectively. The simulations are continued till axial strain \( \varepsilon_q = 0.25 \). The markers on the dashed line NA stand indicate failure of convergence of the corresponding method at the particular step size.

First, we assess the Runge-Kutta method (RK4). The relative error of RK4 which is expressed in green circles is almost step size independent in undrained simulations, while in drained conditions the error increases with the increase of step size. The results indicate that although the RK4 method can have smaller errors than the other two methods in certain cases, the method often yield unpredictable errors, sometimes failing to converge, and its performance is very much stress path and model parameter dependent.

These characteristics are often undesirable in a numerical scheme targeting general loading paths.
using parameter Set 1; (b) simulations using parameter Set 2 ($p_{in} = 500$ kPa; $e_{in} = 0.810$).

Fig. 4 Simulations of cyclic undrained triaxial tests using RK4, CPM and CPPM (parameter Set 1; step size = $10^{-5}$).

Fig. 5 Comparison of errors in cyclic undrained triaxial test simulations using RK4, CPM and CPPM: (a) simulations using parameter Set 1; (b) simulations using parameter Set 2 ($e_{in} = 0.735$).

Fig. 6 Comparison of efficiency in (a) the undrained triaxial compression test simulation and (b) drained triaxial compression test simulation using RK4, CPM and CPPM (data from the simulation of Fig. 2(a) and Fig. 3(a)).

The errors of the cutting plane method (CPM) and closest point projection method (CPPM) are also assessed, which are expressed by blue “X”s and red triangles, respectively. The relative errors of these two integration schemes are similar in all the monotonic test simulation conditions regardless the step sizes and model parameters. They both show a predictable increase in relative error with increase in step size. In undrained compression tests, the errors of CPM and CPPM are smaller than RK4’s at small step size, and at large step sizes, the relative error of these two methods may even surpass that of RK4. Under drained conditions, the CPM and CPPM always exhibit smaller errors than RK4.

The stress path and stress-strain results of three methods in undrained cyclic triaxial simulations are shown in Fig. 4. The error-step size plots for the three algorithms are shown in Fig. 5. The relative error of these conditions is calculated by using

$$\delta = \left( \frac{\tilde{S} - \tilde{S}^*}{\tilde{S}^*} \right) \times 100\% ; \quad \tilde{S} = \left\lfloor \frac{q_{dy}}{n_{st}} \right\rfloor 2$$

(48)

where $n_{st}$ = number of stress reversals; the floor brackets $\left\lfloor \cdots \right\rfloor$ operate as $\left\lfloor 2.5 \right\rfloor = 2$; $\tilde{S}$ is the calculated value and $\tilde{S}^*$ is the “exact value”. All the simulations in Fig. 5 start with an initial isotropic stress of 300 kPa, deviatoric stress amplitude $q = 100$ kPa, and
are terminated at \( \varepsilon_q = 0.04 \).

The relative errors of cyclic tests are much greater compared to that of the monotonic triaxial compression tests, due to the accumulation of errors. This indicate that when conducting dynamic analysis, smaller step size should be used. It can be found that the red and blue lines almost coincide in Fig. 4, showing that the errors of the CPM and CPPM are almost the same in Fig. 5. When step size is small, the relative error of RK4 is greater than CPM and CPPM evident from Fig. 4 and Fig. 5. At larger step sizes, the results become more unstable and the error of RK4 may occasionally be smaller. RK4 and CPPM are prone fail at the low effective stress state approaching liquefaction, especially at large time steps. This problem arises in the simulation process in Fig. 5(a).

The calculation time-step size plots for the three integration schemes in monotonic drained and undrained conditions are shown in Fig. 6. It can be found that CPM is the fastest method of the three, and RK4 is the slowest one. There is a negative correlation between time and step size. Combining considerations for error, convergence, and efficiency, the CPM method appears to be the best numerical integration scheme for this particular constitutive model.

4 CONCLUSIONS

Numerical integration of the SANISAND-04 model proposed by Dafalias and Manzari (2004) is conducted in this study using the Runge-Kutta method, cutting plane method, and closest point projection method. The performance of the three integration schemes is compared in a series of monotonic drained and undrained triaxial test, and undrained cyclic triaxial test simulations. The accuracy of the explicit RK4 method is found to be highly dependent on the loading conditions and material parameters, and is the most unstable of the three methods. The CPM is the most efficient scheme, and has the capability to handle low effective stress conditions better than the other two methods. CPPM is only slightly more accurate than CPM, with the tradeoff of lower efficiency and worse convergence when the stress path evolves towards liquefaction. To establish a globally more accurate and efficient integration scheme, combination of the various methods may be explored.

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REFERENCES

1) Dafalias Y., Manzari M. (2004): Simple plasticity sand model accounting for fabric change effects, Journal of Engineering Mechanics, 130(6), 622-634
2) Manzari M., Prachathamukit R. (2001): On integration of a cyclic soil plasticity model, International Journal for Numerical and Analytical Methods in Geomechanics, 25(6), 525-549.
3) Manzari M., Yonten K. (2011): Comparison of two integration schemes for a micropolar plasticity model, Computational Methods in Civil Engineering, 2(1), 21-42.
4) Simo S., Ortiz M. (1985): A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive equations, Compute Methods Appl Mech Eng, 49(2), 221-45.
5) Sloan S. (1987): Substepping Schemes for The Numerical Integration of Elastoplastic Stress-strain Relations, International Journal for Numerical Methods in Engineering, 24:893-911.
6) Sloan S., Abbo A., Sheng D. (2001): Refined explicit integration of elastoplastic models with automatic error control, Engineering Computations, Vol. 18, 1/2, 121-154.
7) Ramirez J., Barrero A., Chen, Dashihi S., Ghofrani A., Taiebat M., Arduino P. (2018): Site Response in a Layered Liquefiable Deposit: Evaluation of Different Numerical Tools and Methodologies with Centrifuge Experimental Result, Journal of Geotechnical and Geoenvironmental Engineering, 144(10):04018073.
8) Runesson K. (1987): Implicit integration of elastoplastic relations with reference to silts. Short Communication, International Journal for Numerical and Analytical Methods in Geomechanics, Vol. 11, 315-321.
9) Wang R., Zhang J.M., Wang G. (2014). A unified plasticity model for large post-liquefaction shear deformation of sand, Computers and Geotechnics, 59, 54-66.