A SHAPE-TOPOLOGICAL CONTROL PROBLEM FOR NONLINEAR CRACK - DEFECT INTERACTION: THE ANTI-PLANE VARIATIONAL MODEL

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Abstract. We consider the shape-topological control of a singularly perturbed variational inequality. As the reference geometry-dependent state problem the paper addresses a heterogeneous medium with a micro-object (defect) and a macro-object (crack) modeled in 2d and illustrated analytically in 1d.

The corresponding nonlinear optimization problem subject to inequality constraints at the crack is considered within a general variational framework. For the reason of asymptotic analysis, singular perturbation theory is applied resulting in topological sensitivity of an objective function representing the release rate of the strain energy. In the vicinity of the nonlinear crack, the anti-plane strain energy release rate is expressed by means of the mode-III stress intensity factor, that is examined with respect to small defects like micro-cracks, holes, and inclusions of varying stiffness. The result of shape-topological control is useful either for arrests or rise of crack growth.

1. INTRODUCTION

The paper aims at shape-topological control of geometry-dependent variational inequalities, which are motivated by application to non-linear cracking phenomena.

From a physical point of view, both cracks and defects appears in heterogeneous media and composites in the context of fracture. We refer to [28] for phenomenological approach to fracture with and without defects. Particular cases for the linear model of a stress-free crack interacting with inhomogeneities and micro-defects were considered in [10, 27, 29]. In the present paper we investigate sensitivity of a nonlinear crack in respect to a small object (called defect) of arbitrary physical and geometric nature.

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While the classic model of a crack is assumed linear, the physical consistency needs nonlinear modeling. Nonlinear crack models subject to non-penetration (contact) conditions have been developed in [7, 14, 19, 22] and other works by the authors. Recently, nonlinear cracks were bridged with thin inclusions under non-ideal contact, see [13, 17, 18]. In the present paper we confine ourselves to the anti-plane model simplification, in which case the inequality type constraints at the plane crack are argued in [15, 16]. The linear crack is included here as the particular case.

From a mathematical point of view, a topology perturbation problem is considered by varying defects posed in a cracked domain. For shape and topology optimization of cracks we refer to [2, 4, 8]. As the size of the defects tends to zero, we have to employ singular perturbation theory. The respective asymptotic methods were developed in [1, 12, 26] mostly for linear partial differential equations (PDE) stated in singularly perturbed domains. Nevertheless, nonlinear boundary conditions are admissible to impose at those boundaries which are separated from the varying object, as it is described in [5, 9].

From the point of view of shape and topology optimization, we investigate a novel setting of interaction problems between dilute geometric objects. In a broad scope, we consider a new class of geometry-dependent objective functions $J$ which are perturbed by at least two interacting objects $\Gamma$ and $\omega$ such that

$$J : \{\Gamma\} \times \{\omega\} \mapsto \mathbb{R}, \quad J = J(\Gamma, \omega).$$

In particular, we look how a perturbation of one geometric object, say $\omega$, will affect a topology sensitivity, here the derivative of $J$ with respect to another geometric object $\Gamma$. Our particular setting of the interaction problem refer $\Gamma$ to a crack, and $\omega$ to an inhomogeneity (defect) in a heterogeneous medium.

The principal difficulty is that $\Gamma$ and $\omega$ enter the objective $J$ in a fully implicit way through a solution of a state (PDE) geometry-dependent problem. Therefore, to get explicit formula, we rely on asymptotic modeling of small $\omega$. Moreover, we generalize the state problem by allowing it to be a variational inequality. In fact, the variational approach to the perturbation problem allows us to incorporate nonlinear boundary conditions stated at the crack $\Gamma$.

Outline of the paper is as follows.

To get an insight into the mathematical problem, in Section 2 we start with a general concept of shape-topological control for singular perturbations of abstract variational inequalities. We illustrate it with a one dimensional (1d) example. In Sections 3 and 4 this technique is generalized to 2d for the nonlinear dipole problem of crack-defect interaction.

For the anti-plane model introduced in Section 3 further in Section 4 we provide the topological sensitivity of an objective function expressing the strain energy release rate $J_{\text{ERR}}$ by means of the mode-III stress intensity factor $J_{\text{SIF}}$ which is of primary importance for engineers. The first order
asymptotic term implies the so-called topological derivative of the objective function with respect to diminishing defects like holes and inclusions of varying stiffness. We prove its semi-analytic expression by using a dipole representation of the crack tip - the defect center with the help of a Green type (weight) function. The respective dipole matrix is related inherently to polarization and virtual mass matrices, see [30].

Within an equivalent ellipse concept, see, for example, [6, 21, 29], further we derive explicit formulas of the topological derivative for the particular cases of the ellipse shaped defects. Holes and rigid inclusions are accounted here as the two limit cases of the stiffness parameter $\delta \downarrow +0$ and $\delta \uparrow \infty$, respectively (see Appendix A).

The asymptotic result of shape-topological control is useful to force either shielding or amplification of an incipient crack by posing trial inhomogeneities (defects) in the test medium.

2. Shape-topological control

In the abstract context of shape-topological differentiability, see e.g. [24, 25], our construction can be outlined as follows.

We deal with variational inequalities of the type: Find $u^0 \in K$ such that

$$\langle Au^0 - g, v - u^0 \rangle \geq 0 \quad \text{for all } v \in K \quad (1)$$

with a linear strongly monotone operator $A : H \mapsto H^*$, fixed $g \in H^*$, and

a polyhedric cone $K \subset H$, which are defined in a Hilbert space $H$ and its dual space $H^*$. The solution of variational inequality (1) implies a metric projection $P_K : H^* \mapsto K, g \mapsto u^0$. Its differentiability properties are useful in control theory, see [24, 25].

For control in the right-hand side of (1), one employs regular perturbations of $g$ with a small parameter $\varepsilon > 0$ in the direction of $h \in H^*$: Find $u^\varepsilon \in K$ such that

$$\langle Au^\varepsilon - (g + \varepsilon h), v - u^\varepsilon \rangle \geq 0 \quad \text{for all } v \in K \quad (2)$$

Then the directional differentiability of $P_K(g + \varepsilon h)$ from the right as $\varepsilon = +0$ implies the following linear asymptotic expansion

$$u^\varepsilon = u^0 + \varepsilon q + o(\varepsilon) \quad \text{in } H \quad \text{as } \varepsilon \searrow +0 \quad (3)$$

with $q \in S(u^0)$ determined uniquely on a proper convex cone $S(u^0), K \subset S(u^0) \subset H$, and depending on $u^0$ and $h$, see [24, 25] for details.

In contrast, our underlying problem implies singular perturbations and the control of the operator $A$ of (1), namely: Find $u^\varepsilon \in K$ such that

$$\langle A_\varepsilon u^\varepsilon - g, v - u^\varepsilon \rangle \geq 0 \quad \text{for all } v \in K \quad (4)$$

where $A_\varepsilon = A + \varepsilon F_\varepsilon$, with a bounded linear operator $F_\varepsilon : H \mapsto H^*$ such that $A_\varepsilon$ is strongly monotone, uniformly in $\varepsilon$, and $\varepsilon \| F_\varepsilon \| = O(\varepsilon)$. In this
In case, we arrive at the nonlinear representation in $\varepsilon \downarrow 0$

$$u^\varepsilon = u^0 + \varepsilon \tilde{q}^\varepsilon + O(f(\varepsilon)) \text{ in } H, \quad \varepsilon \|\tilde{q}^\varepsilon\| = O(\varepsilon).$$

In (5) $\tilde{q}^\varepsilon$ depends on $u^0$ and $F_\varepsilon$. A typical example, $\tilde{q}^\varepsilon(x) = \tilde{q}(\varepsilon x)$, implies the existence of a boundary layer, e.g. in homogenization theory. In contrast to the differential $\varepsilon q$ in (3), a representative $\varepsilon \tilde{q}^\varepsilon$ is defined not uniquely with respect to $\varepsilon$, but up to $o(f(\varepsilon))$-terms. The example are slant derivatives.

The asymptotic behavior $f(\varepsilon)$ of the residual in (5) may differ for concrete problems. Thus, in the subsequent analysis $f(\varepsilon) = \varepsilon^{1+d/2}$ with respect to the spatial dimension $d = 1, 2$.

In order to find the representative $\tilde{q}^\varepsilon$ in (5), we suggest sufficient conditions (6)–(9) below.

**Proposition 1.** If the following relations hold:

1. $u^0 + \varepsilon \tilde{q}^\varepsilon \in K$,
2. $u^\varepsilon - \varepsilon \tilde{q}^\varepsilon \in K$,
3. $\langle A_\varepsilon \tilde{q}^\varepsilon + F_\varepsilon u^0 - R_\varepsilon, v \rangle = 0$ for all $v \in H$,
4. $\varepsilon \|R_\varepsilon\| = O(f(\varepsilon))$,

then (5) holds for the solutions of variational inequalities (1) and (4).

**Proof.** Indeed, plugging test functions $v = u^\varepsilon - \varepsilon \tilde{q}^\varepsilon \in K$ in (1) due to (7) and $v = u^0 + \varepsilon \tilde{q}^\varepsilon \in K$ in (4) due to (6), after summation

$$\langle A_\varepsilon (u^\varepsilon - u^0) + \varepsilon F_\varepsilon u^0, u^\varepsilon - u^0 - \varepsilon \tilde{q}^\varepsilon \rangle \leq 0,$$

and substituting $v = u^\varepsilon - u^0 - \varepsilon \tilde{q}^\varepsilon$ in (8) multiplied by $-\varepsilon$, this yields

$$\langle A_\varepsilon (u^\varepsilon - u^0 - \varepsilon \tilde{q}^\varepsilon) + \varepsilon R_\varepsilon, u^\varepsilon - u^0 - \varepsilon \tilde{q}^\varepsilon \rangle \leq 0.$$

Applying here the Cauchy–Schwarz inequality together with (9) it follows (5) and completes the proof. \qed

Our consideration aims at shape-topological control by means of mathematical programs with equilibrium constraints (MPEC): Find optimal parameters $p \in P$ from a feasible set $P$ such that

$$\min_{p \in P} J(u^{(\varepsilon,p)}) \text{ subject to } \Pi(u^{(\varepsilon,p)}) = \min_{v \in K_p} \Pi(v).$$

In (10) the functional $\Pi : H \mapsto \mathbb{R}$, $\Pi(v) := \frac{1}{2} A_\varepsilon v + g, v \rangle$ associates the strain energy (SE) of the state problem, such that variational inequality (1) implies the first order optimality condition for the minimization of $\Pi(v)$ over $K_p \subset H$. The multi-parameter $p$ may include the right-hand side $g$, geometric variables, and other data of the problem. The optimal value function $J$ in (10) is motivated by underlying physics, which we will specify in examples below.

The main difficulty of the shape-topological control is that geometric parameters are involved in MPEC in fully implicit way. In this respect, relying
on asymptotic models under small variations $\varepsilon$ of geometry is helpful to linearize the optimal value function. See e.g. the application of topological sensitivity to inverse scattering problems in [23].

In order to expand (10) in $\varepsilon \searrow +0$, the uniform asymptotic expansion (5) is useful which, however, is varied by $f(\varepsilon)$. The variability of $f(\varepsilon)$ is inherent here due to non-uniqueness of a representative $\varepsilon \tilde{g}$ defined up to the order $f(\varepsilon)$-terms. As an alternative, developing variational technique based on Green functions and Fourier series, in Section 2.3 we employ local asymptotic expansions in the near-field, which are unique ones.

In the following Sections 2.1 and 2.2 we illustrate our construction analytically for 1d problems which obey exact solutions.

2.1. One dimensional nonlinear 'dipole' problem. We start with an example configuration of the multiple domain $x \in (0, \varepsilon^-) \cup (\varepsilon^+, r + t)$ joining two segments, with $0 < r_0 < r < r_1$, $r_0 - r < t < r_1 - r$, and $0 < \varepsilon < \varepsilon_0 < r_0$. One pole $x = 0$ marks the position of the 'defect' $\omega = (0, \varepsilon^-)$, and the other pole $x = r + t$ defines the position of the nonlinear crack $\Gamma$ under an inequality constraint. The defect is represented with the help of the characteristic function such that $\chi^{\delta}_0(x) = \delta$ for $x < \varepsilon$, otherwise $\chi^{\delta}_0(x) = 1$ for $x > \varepsilon$, where $\delta > 0$ stands for the stiffness parameter.

The function space here is $v \in H_1 := H^1(0, r + t)$, the polyhedral cone $K_1 \subset H_1$ is represented by the equality $v(0) = g$ and the inequality $v(r + t) \geq 0$ constraints. For fixed $g \in \mathbb{R}$, the variational inequality (11) takes the form: Find $u^{(\varepsilon, t)} \in K_1$ such that

$$
\int_0^{r+t} \chi^{\delta}_{0,\varepsilon}(u^{(\varepsilon, t)})'(v - u^{(\varepsilon, t)})' \, dx \geq 0 \quad \text{for all } v \in K_1.
$$

Here and in what follows we mark the dependence of functions on two variables $\varepsilon$ and $t$ (the geometric parameters of length) according to the dipole model.

The variational inequality (11) implies the boundary value problem:

$$
-(u^{(\varepsilon, t)})''(x) = 0 \quad \text{for } x \in (0, \varepsilon^-) \cup (\varepsilon^+, r + t),
$$

$$
u^{(\varepsilon, t)}(0) = g,
$$

$$
u^{(\varepsilon, t)}(\varepsilon^+) - u^{(\varepsilon, t)}(\varepsilon^-) = 0, \quad (u^{(\varepsilon, t)})'(\varepsilon^+) - \delta \cdot (u^{(\varepsilon, t)})'(\varepsilon^-) = 0,
$$

$$
u^{(\varepsilon, t)}(r + t) \geq 0, \quad (u^{(\varepsilon, t)})'(r + t) \geq 0,
$$

$$
(u^{(\varepsilon, t)})'(r + t) \cdot u^{(\varepsilon, t)}(r + t) = 0,
$$

which is derived from (11) in the standard way by applying integration by parts for all $v \in K_1$:

$$
- \int_0^{r+t} \chi^{\delta}_{0,\varepsilon}(u^{(\varepsilon, t)})''(v - u^{(\varepsilon, t)}) \, dx + (u^{(\varepsilon, t)})'(r + t) \cdot (v(r + t) - u^{(\varepsilon, t)}(r + t))
- (u^{(\varepsilon, t)})'(\varepsilon^+) - \delta \cdot (u^{(\varepsilon, t)})'(\varepsilon^-) \cdot (v(\varepsilon) - u^{(\varepsilon, t)}(\varepsilon)) \geq 0.
$$
We construct the solution to (12)–(16) explicitly. Indeed, for arbitrary \( c(\varepsilon,t) \in \mathbb{R} \) the relations (12)–(14) can be solved by

\[
\begin{align*}
    u^{(\varepsilon,t)}(x) &= g + \frac{c(\varepsilon,t)}{\delta} x, \\
    (u^{(\varepsilon,t)})' &= \frac{c(\varepsilon,t)}{\delta},
\end{align*}
\]

implying the piecewise linear continuous function

\[
(17) \quad u^{(\varepsilon,t)}(x) = g + c(\varepsilon,t) \left( x + \frac{1-\delta}{\delta} \min\{\varepsilon, x\} \right).
\]

With (17) the complementarity condition (16) takes the form

\[
(18) \quad c(\varepsilon,t) \cdot (g + c(\varepsilon,t) (r + t + \frac{1-\delta}{\delta} \varepsilon)) = 0.
\]

Hence, due to (15) the nonnegative constant \( c(\varepsilon,t) \) can be found uniquely

\[
(19) \quad c(\varepsilon,t) = \max \left\{ 0, -g(r + t + \frac{1-\delta}{\delta} \varepsilon)^{-1} \right\}.
\]

As \( \varepsilon \searrow 0 \), from (17) and (18) we have the reference state

\[
(20) \quad u^{(0,t)}(x) = g + c(0,t)x,
\]

which solves the reference variational inequality (corresponding to (1)): Find \( u^{(0,t)} \in K_t \) such that

\[
(21) \quad \int_0^{r+t} (u^{(0,t)})'(v - u^{(0,t)})' dx \geq 0 \quad \text{for all } v \in K_t.
\]

Alluding to the asymptotic expansion in Proposition 2 below, we need to consider a layer near the defect boundary \( x = \varepsilon \). It is obtained after mapping \( (0,\varepsilon) \mapsto (0,1), x \mapsto \varepsilon y \) by solving the auxiliary transmission problem (see Lemma 1 for generalization in 2d): Find \( w \in H^1(\mathbb{R}_+) \) such that

\[
(22) \quad \int_0^\infty \chi_{(0,1)} \delta^\delta w'(y)v'(y) dy = (1-\delta)v(1) \quad \text{for all } v \in H^1(\mathbb{R}_+).
\]

Using integration by parts, variational equation (22) implies the boundary value problem:

\[
\begin{align*}
    -w''(y) &= 0 \quad \text{for } y \in (0,1^-) \cup (1^+,\infty), \\
    w(x) &\to 0 \quad \text{as } x \nearrow \infty, \\
    w(1^+) - w(1^-) &= 0, \quad w'(1^+) - \delta \cdot w'(1^-) = -(1 - \delta),
\end{align*}
\]

the unique solution of which is given by the piecewise linear continuous function

\[
(23) \quad w(y) = \frac{1-\delta}{\delta} \min\{0, y-1\}.
\]

After stretching the coordinates \( y = \frac{z}{\varepsilon} \) in (23), we get the boundary layer

\[
(24) \quad \varepsilon w\left(\frac{z}{\varepsilon}\right) = \frac{1-\delta}{\delta} \min\{0, x - \varepsilon\}, \quad \|\varepsilon w\left(\frac{z}{\varepsilon}\right)\| = O(\varepsilon^{1/2}) \quad \text{in } H_t,
\]
where the square root asymptotic order is due to the seminorm estimate
\[ \sqrt{\int_0^{r+t} (\varepsilon w(\xi'))^2 \, dx} = \sqrt{\int_0^{\varepsilon} \left(\frac{1-\varepsilon}{\delta}\right)^2 \, dx} = O(\sqrt{\varepsilon}). \]

In this case we justify the asymptotic formula \((\mathbf{5})\) as follows.

**Proposition 2.** The solutions \(u^{(\varepsilon,t)}\) and \(u^{(0,t)}\) of variational inequalities \((\mathbf{11})\) and \((\mathbf{21})\) admit the following residual estimate as \(\varepsilon \searrow 0\):
\[
(\mathbf{25}) \quad u^{(\varepsilon,t)} = u^{(0,t)} + \varepsilon q^{(\varepsilon,t)} + O(\varepsilon^{3/2}) \quad \text{in } H_t,
\]
with the principal asymptotic term
\[
(\mathbf{26}) \quad \varepsilon q^{(\varepsilon,t)}(x) := (u^{(0,t)})'(0) \cdot \left[ \varepsilon w\left(\frac{x}{\varepsilon}\right) + \varepsilon^{1-\delta} \left(1 - \frac{x}{r+t}\right)\right].
\]

**Proof.** Indeed, for sufficiently small \(\varepsilon\) we have \((r+t)(r+t+\frac{1-\delta}{\delta})^{-1} > 0\), hence from \((\mathbf{18})\) it follows that
\[
c^{(\varepsilon,t)} = \left(1 + \frac{1-\delta}{\delta(r+t)}\right)^{-1} \cdot \max\{0, -g(r+t)^{-1}\},
\]
and together with \((\mathbf{20})\) this results in the expansion
\[
(\mathbf{27}) \quad c^{(\varepsilon,t)} = c^{(0,t)}\left(1 - \frac{1-\delta}{\delta(r+t)}\varepsilon + O(\varepsilon^2)\right).
\]
Substituting \((\mathbf{27}), (\mathbf{24}),\) and \((\mathbf{19})\) in \((\mathbf{17})\) we get
\[
u^{(\varepsilon,t)}(x) = u^{(0,t)}(x) - c^{(0,t)}x + c^{(0,t)}(1 - \frac{1-\delta}{\delta(r+t)}\varepsilon)\left[x + \frac{1-\delta}{\delta} \varepsilon + \varepsilon w\left(\frac{x}{\varepsilon}\right)\right] + O(\varepsilon^2)
\]
and derive iteratively the following uniform estimates in \(H_t:\)
\[
u^{(\varepsilon,t)}(x) = u^{(0,t)}(x) + O(\varepsilon^{1/2}),
\]
\[
u^{(\varepsilon,t)}(x) = u^{(0,t)}(x) + (u^{(0,t)})'(0) \cdot \varepsilon w\left(\frac{x}{\varepsilon}\right) + O(\varepsilon),
\]
\[
u^{(\varepsilon,t)}(x) = u^{(0,t)}(x) + (u^{(0,t)})'(0) \cdot \left[ \varepsilon w\left(\frac{x}{\varepsilon}\right) + \varepsilon^{1-\delta} \left(1 - \frac{x}{r+t}\right)\right] + O(\varepsilon^{3/2}),
\]
where we have used \(c^{(0,t)} = (u^{(0,t)})'(0)\). The latter equality enforces \((\mathbf{25})\) with notation \((\mathbf{26})\), thus completing the proof. \(\square\)

We remark that \(\varepsilon q^{(\varepsilon,t)}(x)\) from Proposition \(2\) satisfies relations \((\mathbf{6}) - (\mathbf{9})\) in Proposition \(1\) with \(f(\varepsilon) = \varepsilon^{3/2}\), which can be checked straightforwardly. For 2d-generalization of Proposition \(2\) see Theorem \(1\) in Section \(4\).

2.2. Shape-topological control in the one dimensional nonlinear dipole problem. Here we discuss 1d-examples of MPEC problem \((\mathbf{10})\) for various objectives \(J(u^{(\varepsilon,t)})\) subject to the optimal state \(u^{(\varepsilon,t)} \in K_t\) from Section \(2.1\). MPEC \((\mathbf{10})\) here is of the form:
\[
(\mathbf{28}) \quad \min_{(\varepsilon,t) \in (0,\varepsilon_0) \times (R_0 - r, r_1 - r)} J(u^{(\varepsilon,t)}) \quad \text{subject to } \Pi(u^{(\varepsilon,t)}) = \min_{v \in K_t} \Pi(v),
\]
where the strain energy (SE) functional \(\Pi : H_t \mapsto \mathbb{R}\), \(\Pi(v) := \frac{1}{2} \int_0^{r+t} \chi\left(0,\varepsilon\right)(v'(x))^2 \, dx.\)
The variational inequality (11) implies the first order optimality condition for the constrained minimization \( \min_{v \in K} \Pi(v) \).

It is important to comment that, for fixed \( \varepsilon \in (0, \varepsilon_0) \), variations of the parameter \( t \in (r_0 - r, r_1 - r) \) describe regular perturbations of the boundary of the domain \( (0, \varepsilon^-) \cup (\varepsilon^+, r + t) \), thus \textit{shape variation}. The limiting procedure \( \varepsilon \to 0 \) implies diminishing of the defect \( (0, \varepsilon^-) \), and, hence, a \textit{topology change} from the disconnected set to the 1-connected set \( (0, r + t) \).

First, we control the optimal value function \( J_{SE} = \Pi \) of the strain energy (29) with respect to the topology change as \( \varepsilon \to 0 \). Relying on small \( \varepsilon \), we substitute the optimal state \( u^{(\varepsilon,t)} \) with its asymptotic model (25) and (26), thus calculating the approximation of the optimal value function

\[
\Pi(u^{(\varepsilon,t)}) = \Pi(u^{(0,t)}) + c_{(0,t)} [\varepsilon w(\frac{\varepsilon}{\varepsilon}) + \varepsilon^{1/3} \delta (1 - \frac{x}{\varepsilon + t})] + O(\varepsilon^{3/2})
\]

\[
= \frac{1}{2} \int_0^{r+t} \chi_{(0,t)} \left[ (u^{(0,t)})' + c_{(0,t)} \left[ \varepsilon w(\frac{\varepsilon}{\varepsilon}) - \varepsilon^{1/3} \delta (1 - \frac{x}{\varepsilon + t}) \right] \right]^2 dx + O(\varepsilon^{3/2})
\]

\[
= \frac{c_{(0,t)}^2}{2} \left[ \int_0^\varepsilon \delta (1 - \frac{2\varepsilon(1-\delta)}{\delta + \varepsilon^2}) + \frac{1}{\varepsilon^2} \int_{\varepsilon}^{r+t} \left( 1 - \frac{2\varepsilon(1-\delta)}{\delta + \varepsilon^2} \right) dx \right] + o(\varepsilon)
\]

\[
= \frac{c_{(0,t)}^2}{2} (r + t - \frac{c_{(0,t)}(1-\delta)}{\varepsilon}) + o(\varepsilon), \quad \Pi(u^{(0,t)}) = \frac{c_{(0,t)}^2}{2} (r + t),
\]

due to (19), (24), and (29). From (30) it follows that \( (0, \varepsilon_0) \to \mathbb{R}, \varepsilon \to \Pi(u^{(\varepsilon,t)}) \) is differentiable at \( \varepsilon = 0 \), and we get the topological derivative

\[
\frac{d}{d\varepsilon} \Pi(u^{(\varepsilon,t)})|_{\varepsilon=0} = -c_{(0,t)}^2 (1-\delta) \frac{1}{25} = -\Pi(u^{(0,t)}) \frac{1-\delta}{\sigma_r(t)}.
\]

Second, we control the objective function \( J_{SERR} = -\frac{d}{d\varepsilon} \Pi \) of the strain energy release rate, which implies shape variation and associates a Griffith functional used in fracture mechanics.

To calculate \( -\frac{d}{d\varepsilon} \Pi \) from (29), we apply the constitutive formula proven in (11). Indeed, let a cut-off function \( \eta \) be such that \( \eta(x) = 0 \) as \( x < \varepsilon \) and \( \eta(x) = 1 \) as \( x > \varepsilon + \beta \), with some \( \beta \) such that \( \varepsilon + \beta < r_0 \). For small \( s \in (r_0 - r - t, r_1 - r - t) \), the translation \( \Phi_s : (0, r + t) \to (0, r + t + s) \), \( z = x + s\eta(x) \) yields the representation of \( \Pi(u^{(\varepsilon,t+s)}) \) as

\[
\frac{1}{2} \int_0^{r+t+s} \chi_{(0,t)} \left[ (u^{(\varepsilon,t+s)})' \right]^2 dz = \frac{1}{2} \int_0^{r+t} \chi_{(0,t)} \left[ \left( \frac{(u^{(\varepsilon,t+s)}_t + \Phi_s)'_t}{1 + s\eta'} \right)^2 (1 + s\eta') \right] dx
\]

\[
= \Pi(u^{(\varepsilon,t+s)} - \frac{s}{2} \int_0^{r+t} \chi_{(0,t)} \left[ (u^{(\varepsilon,t+s)}_t + \Phi_s)'_t \right]^2 \eta' dx + o(s).
\]

Since \( u^{(\varepsilon,t+s)} + \Phi_s \in K_t \), we infer \( u^{(\varepsilon,t+s)} + \Phi_s \to u^{(\varepsilon,t)} \) strongly in \( H_t \) as \( s \to 0 \), and conclude, see (11) for detail, with the asymptotic expansion

\[
\Pi(u^{(\varepsilon,t+s)}) = \Pi(u^{(\varepsilon,t)}) - \frac{s}{2} \int_0^{r+t} \chi_{(0,t)} \left[ (u^{(\varepsilon,t)}_t) \right]^2 \eta' dx + o(s).
\]

From (32) it follows directly the explicit formula of shape derivative

\[
J_{SERR}(u^{(\varepsilon,t)}) := -\frac{d}{d\varepsilon} \Pi(u^{(\varepsilon,t)}) = \frac{1}{2} \int_0^{r+t} \chi_{(0,t)} \left[ (u^{(\varepsilon,t)}_t) \right]^2 \eta' dx.
\]
We observe that $J_{SERR}$ depends on $u^{(ε,t)}$, but not on $εq^{(ε,t)}$ in expansion (25). The latter fact is in accordance with [24, 25].

For the shape-topological control, now we insert (25) in (33), which implies the asymptotic model

$$J_{SERR}(u^{(0,t)} + c(0,t)\left[εw(\frac{ε}{x}) + ε\frac{1−δ}{δ}\left(1 − \frac{x}{r+t}\right)\right] + O(ε^{3/2}))$$

$$= \frac{1}{2} \int_{r+t}^{ε+t} \chi(0,ε)(u^{(0,t)})^t + c(0,t)\left[εw'(\frac{ε}{x}) - \frac{ε(1−δ)}{δ(r+t)}\right] E' dx + O(ε^{3/2})$$

$$= \frac{1}{2} \int_{ε}^{ε+β} \left(1 − \frac{2ε(1−δ)}{δ(r+t)}\right) E' dx + o(ε) = \frac{c^2(0,t)}{2}\left(1 − \frac{2ε(1−δ)}{δ(r+t)}\right) + o(ε)$$

$$= J_{SERR}(u^{(0,t)} - εc^2(0,t)\frac{1−δ}{δ(r+t)}) + o(ε), \quad J_{SERR}(u^{(0,t)}) = \frac{c^2(0,t)}{2}.$$  

In particular, from (34) the formula for the topological derivative follows

$$\frac{d}{dε}J_{SERR}(u^{(ε,t)})|_{ε=0} = -c^2(0,t)\frac{1−δ}{δ(r+t)}.$$  

Moreover, in view of the definition (33), it implies the mixed second derivative $−\frac{d^2}{dε dt}\Pi(u^{(ε,t)})|_{ε=0}$, which is symmetric: $\frac{d^2}{dε dt}\Pi(u^{(ε,t)})|_{ε=0} = −\frac{d^2}{dt dε}\Pi(u^{(ε,t)})|_{ε=0}.

Thus, we have proved the following.

**Proposition 3.** For the solutions $u^{(ε,t)}$ and $u^{(0,t)}$ of variational inequalities (11) and (21), there exists the shape-topological derivative

$$\frac{d}{dε}J_{SERR}(u^{(ε,t)})|_{ε=0} = -\frac{d^2}{dε dt}\Pi(u^{(ε,t)})|_{ε=0} = -\frac{d^2}{dt dε}\Pi(u^{(ε,t)})|_{ε=0}$$

$$= -c^2(0,t)\frac{1−δ}{δ(r+t)}.$$  

### 2.3. Local asymptotic expansion in the one dimensional nonlinear dipole problem.

We recall that Proposition 3 is derived based on the uniform asymptotic formula (25) which, however, is not unique. The representation (25) which is uniform over domain matches the near-field (the boundary layer near defect) and the far-field (extendable to infinity) asymptotic representations, which both are unique. This is the reason of our alternative approach to the shape-topological control. Since in 1d the far-field is trivial (zero), here we employ only the near-field for the 1d nonlinear dipole problem from Section 2.1.

In the near-field of pole $x = r + t$, any solution $u^{(ε,t)}$ of the homogeneous equation (12) can be written as a linear function

$$u^{(ε,t)}(x) = u^{(ε,t)}(r + t) + (u^{(ε,t)})'(r + t) \cdot [x − (r + t)]$$ for $x > ε.$

In this sense, the factor of the principal term $x − (r + t)$ in (37) is called stress intensity factor (SIF) in crack mechanics. We associate it with the objective

$$J_{SIF}(u^{(ε,t)}) = (u^{(ε,t)})'(r + t) =: c(ε,t),$$

and we aim at proper formula for its calculation without knowledge of the analytic solution (17) and (18) from Section 2.2.
For this reason, we construct a Green function $\zeta_t$ (called the weight function in crack mechanics) obeying the bounded singularity $\zeta_t(r + t) \neq 0$ and $\zeta'_t(r + t) \neq 0$ at the pole $x = r + t$ and solving the homogeneous problem:

$$-\zeta''_t(x) = 0 \quad \text{for} \quad x \in (0, r + t),$$

$$\zeta_t(0) = 0.$$  

All solutions of \(39\) and \(40\) are given by straight lines $\alpha x$ and defined up to arbitrary factor $\alpha \neq 0$. If we set the normalization condition

$$1 = \int_0^{r + t} (\zeta'_t(x))^2 \, dx = \zeta'_t(r + t) \cdot \zeta_t(r + t)$$

due to \(39\) and \(40\), then the unique $\alpha x$ satisfying \(41\) is

$$\zeta_t(x) = \frac{x}{\sqrt{r + t}}.$$  

Using \((12)-(14)\) and \((39)-(40)\), the second Green formula yields

$$0 = \int_0^{r + t} \left[(u^{(\varepsilon,t)})'' \zeta_t - u^{(\varepsilon,t)} \zeta''_t\right] \, dx = -\|\,(u^{(\varepsilon,t)})'\|_2 \zeta_t(\varepsilon) + g_\zeta'(0)$$

$$+ \left((u^{(\varepsilon,t)})'(r + t) \cdot \zeta_t(r + t) - u^{(\varepsilon,t)}(r + t) \cdot \zeta'_t(r + t),
$$

denotation a jump $\left[\left((u^{(\varepsilon,t)})'(\varepsilon)\right] := (u^{(\varepsilon,t)})'(\varepsilon^+) - (u^{(\varepsilon,t)})'(\varepsilon^-)$. Multiplying \(43\) either with $(u^{(\varepsilon,t)})'(r + t)$ or $u^{(\varepsilon,t)}(r + t)$ and using complementarity conditions \((15), (16)\), we derive the representations

$$\left((u^{(\varepsilon,t)})'(r + t)\right) = \max\{0, \zeta'_t(r + t) \left[\left((u^{(\varepsilon,t)})'(\varepsilon)\right] \zeta_t(\varepsilon) - g_\zeta'(0)\right]\},$$

$$u^{(\varepsilon,t)}(r + t) = \max\{0, \frac{1}{\zeta_t'(r + t)} \left[\left((u^{(\varepsilon,t)})'(\varepsilon)\right] \zeta_t(\varepsilon) + g_\zeta'(0)\right]\},$$

where we have used normalization \((41)\) to get \(44\). In comparison to the explicit formula \((18)\) of $c_{(\varepsilon,t)}$, expressions \((44)\) and \((45)\) are implicit ones. We plug in \((44)\) expansion \((25)\) and infer the asymptotic model

$$c_{(\varepsilon,t)} := (u^{(\varepsilon,t)})'(r + t) = \max\{0, \zeta'_t(r + t) \left[\left((u^{(\varepsilon,t)})'(\varepsilon)\right] \zeta_t(\varepsilon) + g_\zeta'(0)\right]\},$$

Moreover, we apply to \((46)\) the local representation $\zeta_t(x) = \zeta'_t(0) x$ following from \((39)\) and \((40)\), hence $\zeta_t(\varepsilon) = \zeta'_t(0) \varepsilon$. In this way we have proved the following.

**Proposition 4.** For the solutions $u^{(\varepsilon,t)}$ and $u^{(0,t)}$ of variational inequalities \((11)\) and \((21)\), the following asymptotic representation of SIF holds:

$$J_{\text{SIF}}(u^{(\varepsilon,t)}) = c_{(\varepsilon,t)}$$

$$= \max\{0, \zeta'_t(r + t) \zeta'_t(0) \left[\left((u^{(0,t)})'(0)[w'(1)] - O(\varepsilon^2)\right)\right],$$

$$J_{\text{SIF}}(u^{(0,t)}) = c_{(0,t)} = \max\{0, -g_\zeta'_t(r + t) \zeta'_t(0)\}. $$
We note that the max-function in (47) is, generally, nondifferentiable in $\varepsilon$ when $g = 0$. Nevertheless, further we need the square of the max-function which is differentiable with respect to its argument. Indeed, the square of (47) constitutes the form:

$$\left( c_{\varepsilon, t}^2 = c_{(0,t)}^2 + 2\varepsilon c_{(0,t)}(u(0, t))'(0) w'(1) \bigl[ w'(1) \bigr] \zeta(r + t) \zeta'(0) + O(\varepsilon^2). \right)$$

As the corollary of Proposition 4 we restate the asymptotic result on shape-topological control of $J_{\text{SERR}}$ and $J_{\text{SE}}$ from Section 2.2.

Inserting the exact solution (17) in (33), $J_{\text{SERR}}$ is given by

$$J_{\text{SERR}}(u^{(\varepsilon, t)}) = -\frac{d}{d\varepsilon} \Pi(u^{(\varepsilon, t)}) = \frac{1}{2} c_{\varepsilon, t}^2.$$

With the help of (48), from (49) we immediately obtain the shape-topological derivative $-\frac{\partial^2}{\partial\varepsilon\partial t} \Pi(u^{(\varepsilon, t)})|_{\varepsilon = 0}$ as

$$\frac{d}{d\varepsilon} J_{\text{SERR}}(u^{(\varepsilon, t)})|_{\varepsilon = 0} = c_{(0,t)}(u(0, t))'(0) w'(1) \bigl[ w'(1) \bigr] \zeta(r + t) \zeta'(0).$$

In order to validate (50), after substitution the exact analytic expressions (19), (24), and (42) of solutions $u(0, t)$, $w$, and $\zeta$, respectively, this results in $\frac{d}{d\varepsilon} J_{\text{SERR}}(u^{(\varepsilon, t)})|_{\varepsilon = 0} = -c_{(0,t)}^2(1 - \frac{\varepsilon}{\delta})^2 + O(\varepsilon^2)$ thus coinciding with expression (36) derived in Proposition 3.

Similarly, substituting (17) in $\Pi(u^{(\varepsilon, t)})$ given in (29), straightforward calculation provides equivalent expression of SE-optimal value function

$$J_{\text{SE}}(u^{(\varepsilon, t)}) = \Pi(u^{(\varepsilon, t)}) = \frac{1}{2} c_{\varepsilon, t}^2 (r + t + \frac{1-\delta}{\delta} \varepsilon)$$

$$= \left[ \frac{c_{(0,t)}^2}{2} + \varepsilon c_{(0,t)}(u(0, t))'(0) w'(1) \bigl[ w'(1) \bigr] \zeta(r + t) \zeta'(0) + O(\varepsilon^2) \right] (r + t + \frac{1-\delta}{\delta} \varepsilon)$$

$$= \frac{c_{(0,t)}^2}{2} \left( 1 + \frac{2(1-\delta)}{\delta(r + t)} + O(\varepsilon^2) \right) (r + t + \frac{1-\delta}{\delta} \varepsilon) = \frac{c_{(0,t)}^2}{2} (r + t + \frac{1-\delta}{\delta} \varepsilon) + O(\varepsilon^2),$$

where we have used here the expansion (48) of SIF $c_{\varepsilon, t}^2$. Thus, we arrive again at formula (30).

In further Sections 3 and 4 we extend our technique of shape-topological control to the nonlinear problem of crack-defect interaction in 2d, where no analytic solutions but only variational formulations are available. Nevertheless, we will prove semi-analytic expressions for the topological derivatives of $J_{\text{SIF}}^2$ and $J_{\text{SERR}}$. 

3. NONLINEAR PROBLEM OF CRACK-DEFECT INTERACTION IN 2D

We start with the 2d-geometry description.

3.1. Geometric configuration. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary $\partial\Omega$ and the normal vector $n = (n_1, n_2)^\top$ at $\partial\Omega$. For $x = (x_1, x_2)^\top \in \mathbb{R}^2$ we set the semi-infinite straight crack $\Gamma_\infty = \{ x \in \mathbb{R}^2 : x_1 < 0, x_2 = 0 \}$ and associate its tip with the origin 0. Let $n = (0, 1)^\top$ be the unit normal vector at $\Gamma_\infty$. We assume that 0 $\in \Omega$ and assign it to the tip of a finite crack $\Gamma := \Gamma_\infty \cap \Omega$. An example geometric configuration is drawn in Figure 1.
Let $x^0$ be an arbitrarily fixed point in the cracked domain $\Omega \setminus \Gamma$. We associate the poles $0$ and $x^0$ with two polar coordinate systems $x = \rho (\cos \theta, \sin \theta)^T$, $\rho > 0$, $\theta \in [-\pi, \pi]$, and $x - x^0 = \rho_0 (\cos \theta_0, \sin \theta_0)^T$, $\rho_0 > 0$, $\theta_0 \in (-\pi, \pi]$. Here $x^0 = r (\cos \phi, \sin \phi)^T$ is given by $r > 0$ and $\phi \in (-\pi, \pi)$ as it is depicted in Figure 1 (a). We refer $x^0$ to the center of a defect $\omega_{\varepsilon}(x^0)$ posed in $\Omega$ as illustrated in Figure 1 (b).

More precisely, let a trial geometric object be given by the compact set $\omega_{\varepsilon}(x^0) = \{x \in \mathbb{R}^2 : \frac{x-x^0}{\varepsilon} \in \omega\}$ which is parametrized by an admissible triple of the shape $\omega \in \Theta$, center $x^0 \in \Omega \setminus \Gamma$, and size $\varepsilon > 0$. By admissible shapes $\Theta$ we mean those compact sets in $\mathbb{R}^2$ which exhibit a minimum enclosing ball $B_1(0)$ of the radius one centered at the origin $0$, and $0 \in \omega$. Thus, the shapes are invariant to translations and isotropic scaling, so that we express them with the equivalent notation $\omega = \omega_1(0)$. Admissible geometric parameters $(\omega, \varepsilon, x^0) \in \Theta \times \mathbb{R}_+ \times (\Omega \setminus \Gamma)$ should satisfy the consistency condition $\omega_{\varepsilon}(x^0) \subset B_\varepsilon(x^0) \subset \Omega \setminus \Gamma$, where $B_\varepsilon(x^0)$ stands for the ball of radius $\varepsilon$ centered at $x^0$.

In the following we assume that $\text{meas}_2(\omega) > 0$, the boundary $\partial \omega_{\varepsilon}(x^0)$ is Lipschitz continuous and assign $n$ to the unit normal vector at $\partial \omega_{\varepsilon}(x^0)$ which points outward to $\omega_{\varepsilon}(x^0)$. In a particular situation, our consideration admits also the degenerate case when $\omega_{\varepsilon}(x^0)$ shrinks to a 1d Lipschitz manifold of co-dimension one in $\mathbb{R}^2$, thus, allowing for defects like curvilinear inclusions. The degenerate case will appear in more detail when shrinking ellipses to line segments as described in Appendix A.

3.2. **Variational problem.** In the reference configuration of the cracked domain $\Omega \setminus \Gamma$ with the fixed inclusion $\omega_{\varepsilon}(x^0)$ we state a constrained minimization problem related to PDE, here, a model problem with the scalar Laplace operator. Motivated by 3d-fracture problems with possible contact...
between crack faces, as described in [15], in the anti-plane framework of linear elasticity, we look for admissible displacements \( u(x) \) in \( \Omega \setminus \Gamma \) which are restricted along the crack by the inequality constraint

\[
\begin{align*}
[u] &= u|_{\Gamma^+} - u|_{\Gamma^-} \geq 0 \quad \text{on } \Gamma.
\end{align*}
\]

The positive \( \Gamma^+_\infty \) (hence, its part \( \Gamma^+ = \Gamma^+_\infty \cap \Omega \)) and the negative \( \Gamma^-_\infty \) (hence, \( \Gamma^- = \Gamma^-_\infty \cap \Omega \)) crack faces are distinguished as the limit of points \((x_1, x_2)\) for \( x_1 < 0 \) and \( x_2 \to 0 \), when \( x_2 > 0 \) and \( x_2 < 0 \), respectively, see Figure 1.

Now we get a variational formulation of a state problem due to the unilateral constraint (51).

Let the external boundary \( \partial \Omega \) consist of two disjoint parts \( \Gamma_N \) and \( \Gamma_D \). We assume that the Dirichlet part has the positive measure \( |\Gamma_D| > 0 \), otherwise we should exclude the nontrivial kernel (the rigid displacements) for coercivity of the objective functional \( \Pi \) in (53) below. The admissible displacements \( u(x) = 0 \) for \( x \in \Gamma_D \) from the Sobolev space

\[
H(\Omega \setminus \Gamma) = \{ u \in H^1(\Omega \setminus \Gamma) : u = 0 \quad \text{on } \Gamma_D \}
\]

and (51) yield together the admissible set

\[
K(\Omega \setminus \Gamma) = \{ u \in H(\Omega \setminus \Gamma) : [u] \geq 0 \quad \text{on } \Gamma \}
\]

which is a convex cone in \( H(\Omega \setminus \Gamma) \). We note that the jump of traces at \( \Gamma \) is defined well in the Lions–Magenes space \([u] \in H^{1/2}_0(\Gamma)\), see [14].

Let \( \mu > 0 \) be a fixed material parameter (the Lame constant) in the reference homogeneous domain \( \Omega \setminus \Gamma \). We distinguish the inhomogeneity with the help of a variable parameter \( \delta > 0 \), such that the characteristic function

\[
\chi^\delta_{\omega\varepsilon(x^0)}(x) := 1 - (1 - \delta)1_{\omega\varepsilon(x^0)} = \begin{cases} 1, & x \in \Omega \setminus \omega\varepsilon(x^0) \\ \delta, & x \in \omega\varepsilon(x^0) \end{cases}
\]

In the following we use the notation \( \mu \chi^\delta_{\omega\varepsilon(x^0)} \), which implies, due to (52), the material parameter \( \mu \) in the homogeneous domain \( \Omega \setminus \omega\varepsilon(x^0) \), and the material parameter \( \mu \delta \) in \( \omega\varepsilon(x^0) \) characterizing stiffness of the inhomogeneity. Given \( \delta \) accounts for three physical situations: inclusions of varying stiffness for finite \( 0 < \delta < \infty \), holes for \( \delta \nearrow +0 \), and rigid inclusions for \( \delta \searrow +\infty \).

For given boundary traction \( g \in L^2(\Gamma_N) \), the heterogeneous medium obeys the strain energy given by the functional \( \Pi : H(\Omega \setminus \Gamma) \to \mathbb{R} \),

\[
\Pi(u; \Gamma, \omega\varepsilon(x^0)) := \frac{1}{2} \int_{\Omega \setminus \Gamma} \mu \chi^\delta_{\omega\varepsilon(x^0)} |\nabla u|^2 \, dx - \int_{\Gamma_N} gu \, dS_x,
\]

which is quadratic and strongly coercive over \( H(\Omega \setminus \Gamma) \). Henceforth, the Babuska–Lax–Milgram theorem guarantees the unique solvability of the constrained minimization of \( \Pi \) over \( K(\Omega \setminus \Gamma) \), which implies the variational formulation of the heterogeneous problem: Find \( u^{(\omega, \varepsilon, x^0, \delta)} \in K(\Omega \setminus \Gamma) \) such
that
\[
\int_{\Omega \setminus \Gamma} \mu \chi_{\omega_{\varepsilon}^0}^\delta \left( \nabla u^{(\omega_{\varepsilon},x^0,\delta)} \right)^\top \nabla (v - u^{(\omega_{\varepsilon},x^0,\delta)}) \, dx \\
geq \int_{\Gamma_N} g(v - u^{(\omega_{\varepsilon},x^0,\delta)}) \, dS_x \quad \text{for all } v \in K(\Omega \setminus \Gamma).
\] (54)

The variational inequality (54) describes the weak solution of the following boundary value problem:

\begin{align}
- \Delta u^{(\omega_{\varepsilon},x^0,\delta)} &= 0 \quad \text{in } \Omega \setminus \Gamma, \\
v^{(\omega_{\varepsilon},x^0,\delta)} &= 0 \quad \text{on } \Gamma_D, \quad \mu \frac{\partial u^{(\omega_{\varepsilon},x^0,\delta)}}{\partial n} = g \quad \text{on } \Gamma_N, \\
\left[ \frac{\partial u^{(\omega_{\varepsilon},x^0,\delta)}}{\partial n} \right] = 0, \quad \left[ u^{(\omega_{\varepsilon},x^0,\delta)} \right] &\geq 0, \quad \frac{\partial u^{(\omega_{\varepsilon},x^0,\delta)}}{\partial n} \leq 0, \\
\frac{\partial u^{(\omega_{\varepsilon},x^0,\delta)}}{\partial n} |_{\omega_{\varepsilon}(x^0)^+} - \delta \frac{\partial u^{(\omega_{\varepsilon},x^0,\delta)}}{\partial n} |_{\omega_{\varepsilon}(x^0)^-} &= 0, \\
\left[ u^{(\omega_{\varepsilon},x^0,\delta)} \right] &= 0 \quad \text{on } \partial \omega_{\varepsilon}(x^0). \tag{55}
\end{align}

In (55d) the jump across the defect boundary is defined as
\[
[u] = u^{\partial \omega_{\varepsilon}(x^0)^+} - u^{\partial \omega_{\varepsilon}(x^0)^-} \quad \text{on } \partial \omega_{\varepsilon}(x^0),
\]
where + and − correspond to the chosen direction of the normal \( n \), which is outward to \( \omega_{\varepsilon}(x^0) \), see Figure 1(b). We remark that the \( L^2 \)-regularity of the normal derivatives at the boundaries \( \Gamma_N, \Gamma \), and \( \partial \omega_{\varepsilon}(x^0) \) is needed in order to have strong solutions in (55). The exact sense to the boundary conditions (55) can be given for the traction \( \frac{\partial u^{(\omega_{\varepsilon},x^0,\delta)}}{\partial n} \) in the dual space of \( H_0^{1/2}(\Gamma_1) \), to (55b) for \( \frac{\partial u^{(\omega_{\varepsilon},x^0,\delta)}}{\partial n} \) in the dual space of \( H_0^{1/2}(\Gamma_N) \), and to (55d) for \( \frac{\partial u^{(\omega_{\varepsilon},x^0,\delta)}}{\partial n} \) on \( \partial \omega_{\varepsilon}(x^0) \) is \( C^\infty \)-smooth away from the crack tip, boundary of defect, and possible irregular points of external boundary, for detail see [14].

If \( \varepsilon \downarrow +0 \), similarly to (54) there exists the unique solution of the homogeneous problem: Find \( u^0 \in K(\Omega \setminus \Gamma) \) such that for all \( v \in K(\Omega \setminus \Gamma) \)

\[
\int_{\Omega \setminus \Gamma} \mu (\nabla u^0)^\top \nabla (v - u^0) \, dx \geq \int_{\Gamma_N} g(v - u^0) \, dS_x,
\] (57)

which implies the boundary value problem:

\begin{align}
- \Delta u^0 &= 0 \quad \text{in } \Omega \setminus \Gamma, \\
u^0 &= 0 \quad \text{on } \Gamma_D, \quad \mu \frac{\partial u^0}{\partial n} = g \quad \text{on } \Gamma_N, \\
\left[ \frac{\partial u^0}{\partial n} \right] &= 0, \quad \left[ u^0 \right] \geq 0, \quad \frac{\partial u^0}{\partial n} \leq 0, \quad \frac{\partial u^0}{\partial n} [u^0] = 0 \quad \text{on } \Gamma, \\
\left[ \frac{\partial u^0}{\partial n} \right] &= 0, \quad \left[ u^0 \right] = 0 \quad \text{on } \partial \omega_{\varepsilon}(x^0). \tag{58}
\end{align}
We note that (58) is written here for comparison with (55), and it implies that the homogeneous solution $u^0$ is $C^\infty$-smooth in $B_\varepsilon(x^0) \supset \omega_\varepsilon(x^0)$ compared to $u^{(\omega,\varepsilon,x^0,\delta)}$.

With the help of Green formulas written in $(\Omega \setminus \Gamma) \setminus \omega_\varepsilon(x^0)$ and in $\omega_\varepsilon(x^0)$, from (57) and (58) we can write the equivalent variational inequality with the heterogeneous material parameter:

$$
\int_{\Omega \setminus \Gamma} \mu \chi_\delta \omega_\varepsilon(x^0) (\nabla u^0)^\top \nabla (v - u^0) \, dx \geq \int_{\Gamma_N} g(v - u^0) \, dS
- (1 - \delta) \int_{\partial \omega_\varepsilon(x^0)} \mu \frac{\partial u^0}{\partial n} (v - u^0) \, dS
$$

for all $v \in K(\Omega \setminus \Gamma)$.

$$
(59)
$$

The left hand side of (59) obeys the same operator as (54), this fact will be used in Section 4 for asymptotic analysis of the solution $u^{(\omega,\varepsilon,x^0,\delta)}$.

### 4. Topology asymptotic analysis

To examine the heterogeneous state (54) in comparison with the homogeneous one (57) in an explicit way, we rely on small defects, thus passing $\varepsilon \downarrow +0$ leads to the first order asymptotic analysis. First, for the solution of the state problem we obtain a two-scale asymptotic expansion, which is related to Green functions. For this reason we apply the singular perturbation theory and endow it with variational arguments. With its help, second, we provide topology sensitivity of geometry dependent objective functions representing the mode-III stress intensity factor (SIF) and the strain energy release rate (SERR) which are the primary physical characteristics of fracture.

#### 4.1. Asymptotic analysis of the solution

We start with the Fourier series of the homogeneous solution $u^0$ of the variational inequality (57), which is $C^\infty$-smooth in the ball $B_R(x^0)$ of the radius $R < \min\{r, \text{dist}(x^0, \partial \Omega)\}$. We remind that $r$ is the distance of the defect center $x^0$ from the crack tip at the origin 0. Due to (58a), we have the representation

$$
u^0(x) = u^0(x^0) + \nabla u^0(x^0)^\top (x - x^0) + U_{x^0}(x) \quad \text{for } x \in B_R(x^0),$$

(60)

$$
\int_{-\pi}^{\pi} U_{x^0} \, d\theta_0 = \int_{-\pi}^{\pi} U_{x^0} \frac{x - x^0}{\rho_0} \, d\theta_0 = 0, \quad U_{x^0} = O(\rho_0^2).
$$

From (60) we infer the expansion of the traction

$$
\frac{\partial u^0}{\partial n} = \nabla u_0(x^0)^\top n + \frac{\partial U_{x^0}}{\partial n} = O(\varepsilon) \quad \text{on } \partial \omega_\varepsilon(x^0)
$$

which will be used further for expansion of the right hand side in (59).

Moreover, to compensate the $O(1)$-asymptotic term in (61), we will need to construct a boundary layer near $\partial \omega_\varepsilon(x^0)$. For this task, we stretch the coordinates as $y = \frac{x - x^0}{\varepsilon}$ which implies the diffeomorphic map $\omega_\varepsilon(x^0) \mapsto \omega_1(0) \subset B_1(0)$. In the following, the stretched coordinates $y = (y_1, y_2)^\top \in$
$\mathbb{R}^2$ refer always to the infinite domain. In the whole $\mathbb{R}^2$ we introduce the weighted Sobolev space

$$H^1_v(\mathbb{R}^2) = \{ v : \nu v, \nabla v \in L^2(\mathbb{R}^2) \}, \quad \nu(y) = \frac{1}{|y| \ln |y|} \text{ in } \mathbb{R}^2 \setminus B_1(0),$$

with the weight $\nu \in L^\infty(\mathbb{R}^2)$ due to the weighted Poincare inequality in exterior domains, see [3]. In this space we state the following auxiliary result (cf. [22]).

**Lemma 1.** There exists the unique solution of the following variational problem: Find $w \in (H^1_v(\mathbb{R}^2) \setminus \mathbb{P}_0)^2$, $w = (w_1, w_2)^\top (y)$, such that

$$\int_{\mathbb{R}^2} \chi_\omega \nabla w_i^\top \nabla v \, dy = (1 - \delta) \int_{\partial \omega_1(0)} n_i v \, dS_y \text{ for all } v \in H^1_v(\mathbb{R}^2),$$

for $i = 1, 2$, which satisfies the Laplace equation in $\mathbb{R}^2$ and the following transmission boundary conditions across $\partial \omega_1(0)$:

$$\frac{\partial w}{\partial n} \big|_{\partial \omega_1(0)^+} - \delta \frac{\partial w}{\partial n} \big|_{\partial \omega_1(0)^-} = -(1 - \delta)n, \quad w \big|_{\partial \omega_1(0)^+} - w \big|_{\partial \omega_1(0)^-} = 0.$$

After rescaling, the far-field representation by the Fourier series holds

$$w\left(\frac{x - x^0}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} A(\omega, \delta) \frac{x - x^0}{\rho_0} + W(x) \quad \text{for } x \in \mathbb{R}^2 \setminus B_\varepsilon(x^0),$$

$$\int_{-\pi}^{\pi} W \, d\theta = 0, \quad W = O\left(\left(\frac{\varepsilon}{\rho_0}\right)^2\right),$$

where the dipole matrix $A(\omega, \delta) \in \text{Sym}(\mathbb{R}^{2 \times 2})$ has entries $(i, j = 1, 2)$:

$$(A(\omega, \delta))_{ij} = (1 - \delta) \left\{ \delta_{ij} \text{meas}_2(\omega_1(0)) + \int_{\omega_1(0)} w_i n_j \, dS_y \right\}.$$

Moreover, $A(\omega, \delta) \in \text{Spd}(\mathbb{R}^{2 \times 2})$ if $\delta \in [0, 1)$ and $\text{meas}_2(\omega_1(0)) > 0$.

**Proof.** The existence of a solution to (62) up to a free constant follows from the results of [3]. Following [8, Lemma 3.2], below we prove the far-field pattern (65) in representation (64).

For this reason, we split $\mathbb{R}^2$ in the far-field $\mathbb{R}^2 \setminus B_1(0)$ and the near-field $B_1(0)$. Since $w$ from (62) solves the Laplace equation, in the far-field it exhibits the Fourier series

$$w(y) = \frac{1}{2\pi} A(\omega, \delta) \frac{y}{|y|^2} + W(y) \quad \text{for } y \in \mathbb{R}^2 \setminus B_1(0),$$

$$\int_{-\pi}^{\pi} W \, d\theta = 0, \quad W = O\left(\left(\frac{1}{|y|}\right)^2\right),$$

which implies (64) after rescaling $y = \frac{x - x^0}{\varepsilon}$.

In the near-field, we apply the second Green formula for $i, j = 1, 2$,

$$0 = \int_{B_1(0)} \chi_\omega \left\{ \Delta w_i y_j - w_i \Delta y_j \right\} \, dy = \int_{\partial B_1(0)} \left\{ \frac{\partial w_i}{\partial |y|} y_j - w_i \frac{\partial y_j}{\partial |y|} \right\} \, dS_y$$

$$- \int_{\partial \omega_1(0)} \left\{ \frac{\partial w_i}{\partial n} \big|_{\partial \omega_1(0)^+} - \delta \frac{\partial w_i}{\partial n} \big|_{\partial \omega_1(0)^-} \right\} y_j - (1 - \delta) w_i \frac{\partial y_j}{\partial |y|} \right\} \, dS_y,$$
and substitute here the transmission conditions (63) to derive that
\[(67) \quad - \int_{\partial B_1(0)^{+}} \{ \frac{\partial w_i}{\partial y_j} - w_i \} \frac{y_j}{|y_j|} \, dS_y = (1 - \delta) \int_{\partial \omega_1(0)^{+}} \{ n_i y_j + w_i \frac{\partial y_j}{\partial y_i} \} \, dS_y. \]
We apply to (67) the divergence theorem
\[
\int_{\partial \omega_1(0)} n_i y_j \, dS_y = \int_{\omega_1(0)} y_{j,i} \, dy = \delta_{ij} \text{ meas}_2(\omega_1(0))
\]
and substitute (66) to calculate the integral over \(\partial B_1(0)\) as
\[
- \int_{\partial B_1(0)} \{ \frac{\partial w_i}{\partial y_j} - w_i \} \frac{y_j}{|y_j|} \, dS_y = \frac{1}{\pi} \int_{-\pi}^{\pi} (A(\omega,\delta))_{ij} \frac{\partial y_i}{\partial y_j} \, d\theta = (A(\omega,\delta))_{ij},
\]
which together give (65).

Now we prove the symmetry and positive definiteness properties of \(A(\omega,\delta)\). Inserting \(v = w_j\), \(j = 1, 2\), into (62) it holds
\[
\int_{\mathbb{R}^2} \chi_{\omega_1(0)}^\delta \nabla w_i^\top \nabla w_j \, dy = (1 - \delta) \int_{\partial \omega_1(0)} n_i w_j \, dS_y = (1 - \delta) \int_{\omega_1(0)} n_j w_i \, dS_y,
\]
hence, the symmetry \((A(\omega,\delta))_{ij} = (A(\omega,\delta))_{ji}\) for \(i, j = 1, 2\) in (65). For arbitrary \(z \in \mathbb{R}^2\), from (62) we have
\[
0 \leq \int_{\mathbb{R}^2} \chi_{\omega_1(0)}^\delta |\nabla (z_1 w_1 + z_2 w_2)|^2 \, dy = (1 - \delta) \sum_{i,j=1}^{2} w_i z_i n_j z_j \, dS_y.
\]
Henceforth, multiplying (65) with \(z_i z_j\) and summing the result over \(i, j = 1, 2\), it follows that
\[
\sum_{i,j=1}^{2} (A(\omega,\delta))_{ij} z_i z_j = (1 - \delta) \left\{ |z|^2 \text{ meas}_2(\omega_1(0)) + \sum_{i,j=1}^{2} w_i z_i n_j z_j \, dS_y \right\}
\][
\[\geq (1 - \delta)|z|^2 \text{ meas}_2(\omega_1(0)) > 0,
\]
if \(1 - \delta > 0\) and \(\text{ meas}_2(\omega_1(0)) > 0\). This completes the proof. \(\square\)

It is important to comment on the transmission conditions (63) in relation to the stiffness parameter \(\delta > 0\). On the one hand, for \(\delta < +0\) implying the hole \(\omega_1(0)\), conditions (63) split as
\[(68) \quad w^- = w^+ \text{ on } \partial \omega_1(0)^-, \quad \frac{\partial w}{\partial n}^+ = -n \text{ on } \partial \omega_1(0)^+,
\]
where the indexes \(\pm\) mark the traces of the functions in (68) at \(\partial \omega_1(0)^\pm\), respectively. Henceforth, to find \(A(\omega,\delta)\) in (65) instead of (62), it suffices to solve the exterior problem under the Neumann condition (68): Find \(w \in \left( H^1_{\nu}(B^2 \setminus \omega_1(0)) \setminus \mathbb{P}_0 \right)^2 \) such that for \(i = 1, 2\)
\[
\int_{\mathbb{R}^2 \setminus \omega_1(0)} \nabla w_i^\top \nabla v \, dy = -\int_{\partial \omega_1(0)} n_i v \, dS_y \quad \text{for all } v \in H^1_{\nu}(B^2 \setminus \omega_1(0)).
\]
In this case, \(A(\omega,\delta)\) is called the virtual mass, or added mass matrix according to (30).
On the other hand, for $\delta \nearrow +\infty$ implying the rigid inclusion $\omega_1(0)$, conditions (63) read
\begin{equation}
\frac{\partial w^-}{\partial n} = -n \quad \text{on } \partial\omega_1(0)^-, \quad w^+ = w^- \quad \text{on } \partial\omega_1(0)^+.
\end{equation}
In this case, (62) is split in the interior Neumann problem in $\omega_1(0)$, and the exterior Dirichlet problem in $\mathbb{R}^2 \setminus \omega_1(0)$. The respective $A(\omega, \delta)$ is called the polarization matrix in [30].

Thus, we have the following.

Corollary 1. The auxiliary problem (62) under the transmission boundary conditions (63) describes the general case of inclusions of varying stiffness, and it accounts for holes (hard obstacles in acoustics) under the Neumann condition (68) as well as rigid inclusions (soft obstacles in acoustics) under the Dirichlet condition (69) as the limit cases of the stiffness parameter $\delta \searrow 0$ and $\delta \nearrow +\infty$, respectively.

With the boundary layer $w$ constructed in Lemma 1 we can represent the first order asymptotic term in expansion of the perturbed solution $u(\omega, \varepsilon, x_0, \delta)$ as $\varepsilon \searrow 0$ in the following theorem (cf. Proposition 2).

Theorem 1. The solution $u(\omega, \varepsilon, x_0, \delta) \in K(\Omega \setminus \Gamma)$ of the heterogeneous problem (54) and the solution $u^0$ of the homogeneous problem (57) admit the uniform asymptotic representation for $x \in \Omega \setminus \Gamma$,
\begin{equation}
(70) \quad u(\omega, \varepsilon, x_0, \delta)(x) = u^0(x) + \varepsilon \nabla u^0(x_0)^T w^0(x_0) w(x_0) \eta_{\Gamma_D}(x) + Q(x),
\end{equation}
where $\eta_{\Gamma_D}$ is a smooth cut-off function which is equal to one except of a neighborhood of the Dirichlet boundary $\Gamma_D$ on which $\eta_{\Gamma_D} = 0$. The residual term $Q \in H(\Omega \setminus \Gamma)$ is such that
\begin{equation}
(71) \quad \|Q\|_{H^1(\Omega \setminus \Gamma)} = O(\varepsilon^2).
\end{equation}

Proof. Since $\|w^\varepsilon\| = 0$ on $\Gamma_\infty$, we can substitute $v = u^0 + \varepsilon \nabla u^0(x_0)^T w^\varepsilon \eta_{\Gamma_D} \in K(\Omega \setminus \Gamma)$ in (54), and $v = u(\omega, \varepsilon, x_0, \delta) - \varepsilon \nabla u^0(x_0)^T w^\varepsilon \eta_{\Gamma_D} \in K(\Omega \setminus \Gamma)$ in (59) as the test functions, which yields two inequalities. Summing them together in the standard way, we get
\begin{equation}
(72) \quad \int_{\Omega \setminus \omega_0(x_0)^{\delta}} (u(\omega, \varepsilon, x_0, \delta) - u^0)^T \nabla Q dx \leq (1 - \delta) \int_{\partial\omega_0(x_0)^{\delta}} \frac{\partial w^0}{\partial n} Q dS_x,
\end{equation}
where $Q := u(\omega, \varepsilon, x_0, \delta) - u^0(x) - \varepsilon \nabla u^0(x_0)^T w^\varepsilon \eta_{\Gamma_D} \in H(\Omega \setminus \Gamma)$ is defined according to (70).

After rescaling $y = \frac{x_0 - x_0}{\varepsilon}$, with the help of the Green formula in $\Omega \setminus \Gamma$, from (62) we obtain the following variational equation for $w^\varepsilon(x) := w^0(\frac{x_0 - x_0}{\varepsilon})$, \[ \int_{\Gamma_0} (\frac{\partial w^\varepsilon}{\partial n})^T \nabla Q dS_y \leq (1 - \delta) \int_{\partial\omega_0(x_0)^{\delta}} \frac{\partial w^0}{\partial n} Q dS_x, \]
$i = 1, 2$, in the bounded domain
\begin{equation}
\int_{\Omega \setminus \Gamma} \chi_\omega^{\delta} \varepsilon (\nabla w^\varepsilon)^\top \nabla v \, dx = \frac{1-\delta}{\varepsilon} \int_{\partial\omega^{x_0}} \nu_i v \, dS_x \\
+ \int_{\Gamma_N} \frac{\partial w^\varepsilon}{\partial n} v \, dS_x - \int_{\Gamma} \frac{\partial w^\varepsilon}{\partial n} [v] \, dS_x \quad \text{for all } v \in H(\Omega \setminus \Gamma).
\end{equation}

Inserting $v = Q$ into (73) multiplied element-wisely with the vector $\varepsilon \nabla u^0(\omega^{x_0})$ and subtracting it from (72) results in the following residual estimate
\begin{equation}
\int_{\Omega \setminus \Gamma} \chi_\omega^{\delta} \varepsilon |\nabla Q|^2 \, dx \leq (1-\delta) \int_{\partial\omega^{x_0}} \left( \frac{\partial u^0}{\partial n} - \nabla u^0(\omega^{x_0})^\top n \right) \|Q\| \, dS_x \\
- \varepsilon \int_{\Gamma_N} \frac{\partial}{\partial n} (\nabla u^0(\omega^{x_0})^\top w^\varepsilon) Q \, dS_x + \varepsilon \int_{\Gamma} \frac{\partial}{\partial n} (\nabla u^0(\omega^{x_0})^\top w^\varepsilon) \|Q\| \, dS_x \\
+ \varepsilon \int_{\text{supp}(1-\eta_{D})} \chi_\omega^{\delta} \varepsilon \nabla (\nabla u^0(\omega^{x_0})^\top w^\varepsilon (1-\eta_{D}))^\top \nabla Q \, dx.
\end{equation}

We apply here the expansion (61) at $\partial\omega^{x_0}$ and estimate pointwisely $w^\varepsilon = O(\varepsilon)$ far from $\omega^{x_0}$ due to (64), which follows that $\|\nabla Q\|_{L^2(\Omega \setminus \Gamma)} = O(\varepsilon^2)$, hence (71). The proof is complete. $\square$

In the following sections we apply Theorem 1 for topology sensitivity of objective functions which depend on the both crack $\Gamma$ and defect $\omega^{x_0}$.

4.2. Topology sensitivity of SIF-function. We start with the notation of stress intensity factor (SIF). At the crack tip $0$, where the stress is concentrated, from (55a) and (55c) we infer the Fourier series (compare to (60)) for $x \in B_R(0)$ with $R = \min \{ r, \text{dist}(0, \partial\Omega) \}$:
\begin{equation}
\begin{aligned}
u^{(\omega,\varepsilon,x_0,\delta)}(0) &+ \frac{1}{\mu} \sqrt{\frac{2}{\pi}} \eta \sqrt{\rho} \sin \frac{\theta}{2} + U(x), \\
\int_{-\pi}^{\pi} U \, d\theta &\equiv \int_{-\pi}^{\pi} U(\cos \frac{\theta}{2}, \sin \frac{\theta}{2})^\top \, d\theta = 0, \quad U = O(\rho).
\end{aligned}
\end{equation}

In the fracture literature, the factor $c^{(\omega,\varepsilon,x_0,\delta)}_G$ in (74) is called SIF, here due to the mode-III crack in the anti-plane setting of the spatial fracture problem. The SIF characterizes the main singularity at the crack tip. Moreover, the inequality conditions (55c) require necessary
\begin{equation}
c^{(\omega,\varepsilon,x_0,\delta)}_G \geq 0.
\end{equation}

For justification of (74) and (75) see [15,16]. From a mathematical viewpoint, the factor in (74) can be determined in the dual space of $H(\Omega \setminus \Gamma)$ through a Green function (the so-called weight function), which we introduce next.

Let $\eta(\rho)$ be a smooth cut-off function supported in $B_{2R}(0) \subset \Omega$, $\eta \equiv 1$ in $B_R(0)$, and $R > r$, where $r > 0$ stands always for the distance to the defect.
With the cut-off function we extend in $\Omega$ the tangential vector $\tau$ from the crack $\Gamma$ by the velocity vector
\[ V(x) := \tau \eta(\rho), \quad \tau = (1,0)^\top. \] (76)

Using the notation of matrices for
\[ D(V) := \text{div}(V) \text{Id} - \frac{\partial V}{\partial x} - \frac{\partial V}{\partial x}^\top \in \text{Sym}(\mathbb{R}^{2\times 2}), \] (77)
where Id means the identity, we formulate the auxiliary variational problem: Find $\xi \in H(\Omega \setminus \Gamma)$ such that for all $v \in H(\Omega \setminus \Gamma)$
\[ \int_{\Omega \setminus \Gamma} \nabla \xi^\top \nabla v \, dx = -\int_{\Omega \setminus \Gamma} \nabla \left( \sqrt{\frac{2}{\pi}} \sqrt{\rho} \sin \frac{\theta}{2} \right)^\top D(V) \nabla v \, dx. \] (78)

In order to get the strong formulation we use the following identities for the 'square-root' function $S(x) := \sqrt{\frac{2}{\pi}} \sqrt{\rho} \sin \frac{\theta}{2}$ in the right-hand side of (78):
\[ \text{div}\left( \nabla S^\top D(V) \right) = \text{div}(V) \Delta S - \Delta V^\top \nabla S \]
\[ -2(\nabla V_1^\top \nabla S_1 + \nabla V_2^\top \nabla S_2) = -\Delta(V^\top \nabla S) \]
where we have applied $\Delta S = 0$ in $\Omega \setminus \Gamma$, and
\[ (\nabla S^\top D(V)) n = 0 = -(V^\top \nabla S)n \quad \text{on } \Gamma^\pm \]
due to $V_2 = 0$, $\frac{\partial V}{\partial n} = 0$, and $\frac{\partial S}{\partial n} = 0$, recalling that $\frac{\partial}{\partial n} = -\frac{1}{\rho} \frac{\partial}{\partial \rho}$ at $\Gamma^\pm$, as $\theta = \pm \pi$. We compute $V^\top \nabla S = -\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho}} \sin \frac{\theta}{2} \eta$, which implies the directional derivative of $S$ with respect to $V$. Henceforth, after integration of (78) by parts, the unique solution of (78) satisfies the boundary value problem:
\[ -\Delta \xi = \Delta(\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho}} \sin \frac{\theta}{2} \eta) \quad \text{in } \Omega \setminus \Gamma; \] (79a)
\[ \xi = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial \xi}{\partial n} = 0 \quad \text{on } \Gamma_N; \] (79b)
\[ \frac{\partial \xi}{\partial n} = -\frac{\partial}{\partial n} \left( \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho}} \sin \frac{\theta}{2} \eta \right) \quad \text{on } \Gamma^\pm. \] (79c)

From (78) and (79) we define the weight function (here $t > 0$ small)
\[ \zeta := \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho}} \sin \frac{\theta}{2} \eta + \xi \in L^2(\Omega \setminus \Gamma) \cap H^1((\Omega \setminus \Gamma) \setminus B_t(0)), \] (80)
which is a non-trivial singular solution of the homogeneous problem
\[ -\Delta \zeta = 0 \quad \text{in } \Omega \setminus \Gamma; \] (81a)
\[ \zeta = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial \zeta}{\partial n} = 0 \quad \text{on } \Gamma_N; \] (81b)
\[ \frac{\partial \zeta}{\partial n} = 0 \quad \text{on } \Gamma^\pm. \] (81c)

From (80) it follows that
\[ \zeta(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho}} \sin \frac{\theta}{2} + \xi(x) \quad \text{for } x \in B_R(0) \setminus \{0\}, \] (82)
which is useful in the following.
Lemma 2. The stress intensity factor $c_{T}^{(\omega,\varepsilon,x^{0},\delta)}$ from (74) and (75) is determined by the following integral formula

\begin{equation}
(83) \quad c_{T}^{(\omega,\varepsilon,x^{0},\delta)} = \max\{0, \int_{\Gamma_{N}} g_{\zeta} \, dS_{x} - \mu \int_{\partial\omega_{c}(x^{0})} \left[ \frac{\partial u^{(\omega,\varepsilon,x^{0},\delta)}}{\partial n_{0}} \right] \zeta \, dS_{x} \}
\end{equation}

with the weight function $\zeta$ defined in (78) and (80) and obeying the properties (81) and (82).

Proof. Using the second Green formula in $(\Omega \setminus \Gamma) \setminus B_{t}(0)$ with small $t > 0$, from (55) and (79) we derive that

\begin{align*}
0 &= \int_{(\Omega \setminus \Gamma) \setminus B_{t}(0)} \{ \Delta \zeta u^{(\omega,\varepsilon,x^{0},\delta)} - \zeta \Delta u^{(\omega,\varepsilon,x^{0},\delta)} \} \, dx \\
&\quad + \int_{\partial\omega_{c}(x^{0})} \left[ \frac{\partial u^{(\omega,\varepsilon,x^{0},\delta)}}{\partial n_{0}} \right] \zeta \, dS_{x} - \int_{\partial B_{t}(0)} \left\{ \frac{\partial \zeta}{\partial \rho} u^{(\omega,\varepsilon,x^{0},\delta)} - \zeta \frac{\partial u^{(\omega,\varepsilon,x^{0},\delta)}}{\partial \rho} \right\} \, dS_{x} \\
&\quad + \int_{\Gamma \setminus B_{t}(0)} \left\{ \frac{\partial \zeta}{\partial \rho} u^{(\omega,\varepsilon,x^{0},\delta)} - \zeta u^{(\omega,\varepsilon,x^{0},\delta)} \right\} \, dS_{x}.
\end{align*}

We note that the latter integral over $\Gamma \setminus B_{t}(0)$ is zero here due to the complementarity conditions (55c) implying either $[u^{(\omega,\varepsilon,x^{0},\delta)}] = 0$ or $\frac{\partial u^{(\omega,\varepsilon,x^{0},\delta)}}{\partial n_{0}} = 0$ at the crack. The former integral over $\partial B_{t}(0)$ can be calculated by plugging the representations (74) and (82) here

\begin{align*}
&\quad - \int_{\partial B_{t}(0)} \left\{ \frac{\partial \zeta}{\partial \rho} u^{(\omega,\varepsilon,x^{0},\delta)} - \zeta u^{(\omega,\varepsilon,x^{0},\delta)} \right\} \, dS_{x} = \frac{c_{T}^{(\omega,\varepsilon,x^{0},\delta)}}{\mu_{T}} \int_{-\pi}^{\pi} \sin^{2}(\frac{\theta}{2}) \, d\theta \\
&\quad - \int_{-\pi}^{\pi} \left\{ \frac{\partial \zeta}{\partial \rho} \left( u^{(\omega,\varepsilon,x^{0},\delta)}(0) + \frac{c_{T}^{(\omega,\varepsilon,x^{0},\delta)}}{\mu_{T}} \sqrt{\frac{2t}{\pi}} \sin \frac{\theta}{2} + U \right) \right\} \, d\theta = \frac{1}{\mu_{T}} c_{T}^{(\omega,\varepsilon,x^{0},\delta)} + O(\sqrt{t})
\end{align*}

which holds true due to $\xi = O(1)$, $\frac{\partial \zeta}{\partial \rho} = O(\frac{1}{\sqrt{t}})$ (similarly to (74)) and using the properties in (74) of $U(\rho, \theta)$ as $\rho = t$ and $\theta \in (-\pi, \pi)$. Therefore, passing $t \searrow 0$ and accounting for (75) we have proven formula (83).

Next, using (70) we expand the right hand side of (83) in $\varepsilon \searrow 0$ and derive the main result of this section (cf. Proposition 4).

Theorem 2. The SIF $c_{T}^{(\omega,\varepsilon,x^{0},\delta)}$ of the heterogeneous problem (54) given in (83) admits the following asymptotic representation

\begin{equation}
(84) \quad c_{T}^{(\omega,\varepsilon,x^{0},\delta)} = \max\{0, \int_{\Gamma_{N}} g_{\zeta} \, dS_{x} - \varepsilon^{2} \mu \nabla u^{0}(x^{0})^{T} A_{(\omega,\delta)} \nabla \zeta(x^{0}) \}
\end{equation}

\begin{align*}
&\quad + \text{Res}, \quad \text{Res} = O(\varepsilon^{3}),
\end{align*}

where $A_{(\omega,\delta)}$ is the dipole matrix and $\nabla \zeta(x^{0}) = \frac{1}{2\sqrt{2\pi}} r^{-3/2} \left( -\sin \frac{3\phi}{2}, \cos \frac{3\phi}{2} \right)^{T} + O(r^{-1/2})$ at the defect center $x^{0} = r(\cos \phi, \sin \phi)^{T}$.
Proof. To expand the integral in the right hand side of (83) as \( \varepsilon \searrow +0 \) we substitute here the expansion (70) of the solution \( u(\omega, \varepsilon, x^0, \delta) \) which implies

\[
\int_{\partial \omega_\varepsilon(x^0)} \left[ \frac{\partial u(\omega, \varepsilon, x^0, \delta)}{\partial n} \right] \zeta \, ds = \varepsilon \nabla u(0) \, \int_{\partial \omega_\varepsilon(x^0)} \left[ \frac{\partial w}{\partial n} \right] \zeta \, ds + O(\varepsilon^4).
\]

Below we apply to the right hand side of (85) the expansion (64) of the boundary layer \( u^e \) and the Fourier series of \( \zeta \), which is a \( C^\infty \)-function in the near field of \( x^0 \), written similarly to (60) as

\[
\zeta(x) = \zeta(x^0) + \nabla \zeta(x^0) \cdot (x-x^0) + Z(x) \quad \text{for} \quad x \in B_R(x^0),
\]

\[
\int_{-\pi}^{\pi} Z \, d\theta_0 = \int_{-\pi}^{\pi} Z \frac{x-x^0}{\rho_0} \, d\theta_0 = 0, \quad Z = O(\rho_0^2).
\]

Next inserting (64) and (86) into the second Green formula in \( B_\varepsilon(x^0) \),

\[
\int_{\partial \omega_\varepsilon(x^0)} \left\{ \frac{\partial u^e}{\partial n} \zeta + (1-\delta)w^e \frac{\partial \zeta}{\partial n} \right\} \, ds = \int_{\partial B_\varepsilon(x^0)} \left\{ \frac{\partial \zeta}{\partial n} w^e - \zeta \frac{\partial w^e}{\partial n} \right\} \, ds,
\]

we estimate its terms as follows. The divergence theorem provides

\[
\int_{\partial \omega_\varepsilon(x^0)} w^e \frac{\partial \zeta}{\partial n} \, ds = \int_{\partial \omega_\varepsilon(x^0)} n \cdot \nabla \zeta(x^0) \, ds + O(\varepsilon^2)
\]

\[
= \int_{\omega_\varepsilon(x^0)} (\nabla w^e) \cdot \nabla \zeta(x^0) \, dx + O(\varepsilon^2) = O(\varepsilon^2),
\]

and we calculate analytically the integral over \( \partial B_\varepsilon(x^0) \) as

\[
\int_{\partial B_\varepsilon(x^0)} \left\{ \frac{\partial \zeta}{\partial \rho_0} w^e - \zeta \frac{\partial w^e}{\partial \rho_0} \right\} \, ds = \frac{\varepsilon}{\pi} \int_{-\pi}^{\pi} \nabla \zeta(x^0) \cdot \frac{x-x^0}{\rho_0} A(\omega, \delta) \frac{x-x^0}{\rho_0} \, d\theta_0 + O(\varepsilon^2)
\]

\[
= \varepsilon A(\omega, \delta) \nabla \zeta(x^0) + O(\varepsilon^2).
\]

Therefore, we obtain the asymptotic expansion

\[
\int_{\partial \omega_\varepsilon(x^0)} \left[ \frac{\partial w^e}{\partial n} \right] \zeta \, ds = \varepsilon A(\omega, \delta) \nabla \zeta(x^0) + O(\varepsilon^2).
\]

Inserting (85) and (87) into (83) it yields (84). Finally, the value of \( \nabla \zeta(x^0) \) can be estimated analytically from (82), while \( \xi \) has the \( O(\rho^{1/2}) \)-singularity similar to (74), hence \( \nabla \zeta(x^0) = O(\rho^{-1/2}) \). This completes the proof. \( \Box \)

As the corollary of Lemma 2 and Theorem 2 we find the SIF of the solution \( u^0 \in K(\Omega \setminus \Gamma) \) of the homogeneous problem (57), which is the limit case of the heterogeneous problem as \( \varepsilon \searrow +0 \). Namely, similar to (74) and (75) we have the Fourier series

\[
u^0(x) = u^0(0) + \frac{1}{\mu} \sqrt{\frac{2}{\pi}} \sqrt{\rho} \sin \frac{\theta}{2} + U^0(x) \quad \text{for} \quad x \in B_R(0),
\]

\[
\int_{-\pi}^{\pi} U^0 \, d\theta = \int_{-\pi}^{\pi} U^0(\cos \frac{\theta}{2}, \sin \frac{\theta}{2}) \, d\theta = 0, \quad U^0 = O(\rho),
\]
with the reference SIF $c_0^0 \geq 0$ determined by the formula

\[(89)\]

\[c_0^0 = \max \{0, \int_{\Gamma_N} g \zeta \, dS_x\}.\]

In the following we get interpretation of Theorem 2 from the point of view of shape-topological control.

We parametrize the crack growth by means of the position of the crack tip along the fixed path $x_2 = 0$ as

\[\Gamma_\infty(t) := \{x \in \mathbb{R}^2 : x_1 < t, x_2 = 0\}, \quad \Gamma(t) := \Gamma_\infty(t) \cap \Omega,\]

such that $\Gamma = \Gamma(0)$ in this notation. Formula (83) defines the optimal value function depending on both $\Gamma(t)$ and $\omega_\varepsilon(x^0)$

\[(90)\]

\[J_{\text{SIF}} : \mathbb{R} \times \Theta \times \mathbb{R}_+ \times (\Omega \setminus \Gamma) \times \mathbb{R}_+ \to \mathbb{R}_+, \quad (t, \omega, \varepsilon, x^0, \delta) \mapsto J_{\text{SIF}}(\Gamma(t), \omega_\varepsilon(x^0)) := c_{\Gamma(t)}^{(\omega, \varepsilon, x^0, \delta)}\]

and satisfying the consistency condition $\omega_\varepsilon(x^0) \subset B_\varepsilon(x^0) \subset \Omega \setminus \Gamma(t)$. From the physical point of view, the reason of (90) is to control SIF of the crack $\Gamma(t)$ by means of the defect $\omega_\varepsilon(x^0)$. The reference homogeneous state implies

\[(91)\]

\[J_{\text{SIF}}(\Gamma(t), \emptyset) = c_0^0_{\Gamma(t)}.\]

For fixed $\Gamma(0) = \Gamma$, formula (84) proves the topology sensitivity of $J_{\text{SIF}}$ from (90) and (91) with respect to diminishing the defect $\omega_\varepsilon(x^0)$ as $\varepsilon \searrow +0$.

We note that, in comparison to the linear crack, see [10, 27, 29], $J_{\text{SIF}}$ for the nonlinear crack subject to inequality constraint (75) is a non-smooth function of $\varepsilon$ due to the presence of max-operator in (84).

In the following section we introduce another geometry dependent objective function inherently related to fracture, namely, the strain energy release rate (SERR). It is smooth with respect to $\varepsilon$ and we lead its first order topology sensitivity analysis using the result of Theorem 2. The first order asymptotic term provides us with the respective topological derivative, see in [11] a generalized concept of topological derivatives suitable for fracture due to cracks.

4.3. Topological derivative of SERR-function. The widely used Griffith criterion of fracture declares that a crack starts to grow when its SERR attains a critical value (the material parameter of fracture resistance). Therefore, decreasing SERR would arrest the incipient crack growth, while increasing SERR, conversely, will affect its rise. This gives us practical motivation of the topological derivative of the SERR objective function, which we construct below.
Indeed, from the local asymptotic expansion (74) written at the crack tip (95), the derivative of $\Pi$ in (92) with respect to $\varepsilon$ (95) proves directly the asymptotic model of SERR as (96).

The derivative of $\Pi$ in (92) with respect to $t$, taken with the minus sign, is called strain energy release rate (SERR) and defines the optimal value function similar to (90) as

$$J_{\text{SERR}} : \mathbb{R} \times \mathcal{Q} \times \mathbb{R}_+ \times (\Omega \setminus \Gamma) \times \mathbb{R}_+ \mapsto \mathbb{R}_+,$$

$$(t, \omega, \varepsilon, x^0, \delta) \mapsto J_{\text{SERR}}(\Gamma(t), \omega_\varepsilon(x^0)) := -\frac{d}{dt} \Pi(\Gamma(t), \omega_\varepsilon(x^0)).$$

It obeys the equivalent representations (see [11, 14, 20] for detail) generalizing (33) from Section 2.2

$$J_{\text{SERR}} = \frac{1}{2} \int_{\Omega \setminus \Gamma(t)} \mu \chi_{\omega_\varepsilon(x^0)} \left( \nabla u^{(\omega, \varepsilon, x^0, \delta)} \right)^\top D(V) \nabla u^{(\omega, \varepsilon, x^0, \delta)} dx$$

$$= \lim_{R \to +0} I_R,$$

where $I_R := \mu \int_{\partial B_R((t, 0))} \left\{ \frac{1}{2} \left( V^\top \frac{z}{\rho} \right) (\nabla u^{(\omega, \varepsilon, x^0, \delta)}) \nabla u^{(\omega, \varepsilon, x^0, \delta)} \right\} dS.$

The key issue is that from (94) we derive the following expression (cf. (49))

$$J_{\text{SERR}}(\Gamma(t), \omega_\varepsilon(x^0)) = \frac{1}{2\mu} (c^{(\omega, \varepsilon, x^0, \delta)}_{\Gamma(t)})^2 \geq 0.$$

Indeed, from the local asymptotic expansion (74) written at the crack tip $t^0$ it follows

$$\nabla u^{(\omega, \varepsilon, x^0, \delta)} = \frac{1}{\mu \sqrt{2\pi R}} (c^{(\omega, \varepsilon, x^0, \delta)}_{\Gamma(t)}) \left( -\sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right) + \nabla U \quad \text{on} \ \partial B_R((t, 0)).$$

Plugging this expression into the invariant integral $I_R$ in (94), due to $|\nabla U| = O(1)$, $V = (1, 0)$ and $\frac{z}{\rho} = (\cos \theta, \sin \theta)$ at $\partial B_R((t, 0))$, we calculate

$$I_R = \mu \int_{-\pi}^\pi \left\{ \frac{1}{2} \cos \theta \frac{1}{2\pi R \mu^2} (c^{(\omega, \varepsilon, x^0, \delta)}_{\Gamma(t)})^2 + \sin^2 \left( \frac{\theta}{2} \right) \frac{1}{2\pi R \mu^2} (c^{(\omega, \varepsilon, x^0, \delta)}_{\Gamma(t)})^2 \right\} Rd\theta = \frac{1}{2\mu} (c^{(\omega, \varepsilon, x^0, \delta)}_{\Gamma(t)})^2 + O(\sqrt{R}).$$

Passing $R \to +0$ it follows (95). Now, the substitution of expansion (84) in (95) proves directly the asymptotic model of SERR as $\varepsilon \to +0$ given next.

**Theorem 3.** The strain energy release rate at the tip of the crack $\Gamma = \Gamma(0)$ admits the following asymptotic representation

$$J_{\text{SERR}}(\Gamma, \omega_\varepsilon(x^0)) = \frac{1}{2\mu} (c^0_{\Gamma})^2 - \varepsilon^2 c^0_{\Gamma} \nabla u^0(x^0)^\top A_{(\omega, \delta)} \nabla \zeta(x^0) + \text{Res}, \quad \text{Res} = O(\varepsilon^3) \text{ and } \text{Res} \geq 0 \text{ if } c^0_{\Gamma} = 0,$$
due to diminishing the defect $\omega_\varepsilon(x^0)$. The reference $J_{SERR}(\Gamma, 0) = \frac{1}{2\pi} (c_0^0)^2$ implies SERR for the homogeneous state $u^0$ without defect, $A_{(\omega, \delta)}$ is the dipole matrix, and $\nabla \zeta(x^0) = \frac{1}{2\sqrt{2\pi}} r^{-3/2} (-\sin \frac{3\phi}{2}, \cos \frac{3\phi}{2})^\top + O(r^{-1/2})$ at the defect center $x^0 = r(\cos \phi, \sin \phi)^\top$. Moreover, the first asymptotic term in (96) provides the topological derivative

$$\lim_{\varepsilon \searrow 0} \frac{J_{SERR}(\Gamma, \omega_\varepsilon(x^0)) - J_{SERR}(\Gamma, 0)}{\varepsilon^2} = -c_0^0 \nabla u^0(x^0)^\top A_{(\omega, \delta)} \nabla \zeta(x^0).$$

5. Discussion

In the context of fracture, from Theorem 3 we can discuss the following. The Griffith fracture criterion suggests that the crack $\Gamma$ starts to grow when $J_{SERR} = G$ attains the fracture resistance threshold $G > 0$. For incipient growth of the nonlinear crack subject to inequality $c_0^0 > 0$, its arrest needs necessary the negative topological derivative to decrease $J_{SERR}$, hence positive sign of $\nabla u^0(x^0)^\top A_{(\omega, \delta)} \nabla \zeta(x^0)$ in (96).

The sign and value of the topological derivative depends in semi-analytic implicit way on the homogeneous solution $u^0$, trial center $x^0$, shape $\omega$ and stiffness $\delta$ of the defect. The latter two parameters enter the topological derivative through the dipole matrix $A_{(\omega, \delta)}$. In Appendix A we present explicit values of the dipole matrix for the specific cases of the ellipse shaped holes and inclusions. This describes also the degenerate case of cracks and thin rigid inclusions called anti-cracks.

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Appendix A. Ellipse and Crack Shaped Defects

Let the shape $\omega$ of a defect be ellipsoidal. Namely, we consider the ellipse $\omega$ enclosed in the ball $B_1(0)$, which has the major one and the minor $b \in (0, 1]$ semi-axes, where the major axis has an angle of $\alpha \in [0, 2\pi)$ with the $x_1$-axis counted in the anti-clockwise direction.

With the rotation matrix $Q(\alpha)$, the dipole matrix for the elliptic defect has the form (see [6, 21, 29])

$$A(\omega, \delta) = Q(\alpha)A(\omega', \delta)Q(\alpha)^\top, \quad Q(\alpha) := \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix},$$

(98a) $$A(\omega', \delta) = \pi(1 + b) \begin{pmatrix} \frac{(1-\delta)b}{1+\delta b} & 0 \\ 0 & \frac{(1-\delta)b}{\delta+b} \end{pmatrix}.$$

(98b)

Further we consider the limit cases of (98b) when the stiffness parameter $\delta \searrow +0$ and $\delta \nearrow +\infty$, which correspond to the ellipse shaped holes and rigid inclusions according to Corollary 1.

On the one hand, for the elliptic hole $\omega$, passing $\delta \searrow +0$ in (98b) we obtain the virtual mass, or added mass matrix

$$A(\omega', \delta) = \pi(1 + b) \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix},$$

(99) which is positive definite. In particular, for the straight crack $\omega$ as $b \searrow +0$, (99) turns in the singular matrix

$$A(\omega', \delta) = \pi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(100)

On the other hand, for the rigid ellipse $\omega$, passing $\delta \nearrow +\infty$ in (98b) we obtain the polarization matrix

$$A(\omega', \delta) = \pi(1 + b) \begin{pmatrix} -1 & 0 \\ 0 & -b \end{pmatrix},$$

(101) which is negative definite. In particular, for the rigid segment $\omega$ as $b \searrow +0$, (101) turns in the singular matrix

$$A(\omega', \delta) = \pi \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(102)