Exchange symmetry and global entanglement and full separability

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(Dated: November 9, 2018)

Ichikawa et al. [Phys. Rev. A 78, 052105 (2008)] showed that exchange symmetry gives rise to simple characterization of whether multipartite pure quantum states being either globally entangled or fully separable. In this Brief Report, we provide a simple alternative approach and some extension to their conclusions.

PACS numbers: 03.65.Ud, 03.67.Mn

Entanglement is one of many important properties for quantum systems. It is identified as a resource for many quantum information processing tasks [1]. The characterization and quantification of entanglement in both bipartite and multipartite settings [2, 3] has uncovered many interesting aspects of it, and this has in turn enabled and prompted investigation of entanglement in many-body systems and connection to quantum phase transitions [4, 5].

Symmetry usually makes simpler the classification of properties of a system [6]. Recently, Ichikawa et al. employed group-theoretical method and showed that exchange symmetry gives rise to simple characterization of multipartite quantum states being either globally entangled or fully separable [7]. Here, we provide a simple alternative approach to their conclusions.

Let us start with a multipartite system comprising $n$ parts, each of which can have a distinct Hilbert space. Consider a general $n$-partite pure state (expanded in the local bases $\{|e^{(i)}\rangle\}$):

$$|\psi\rangle = \sum_{p_1,...,p_n} \chi_{p_1p_2...p_n} |e^{(1)}_{p_1}\rangle \otimes |e^{(2)}_{p_2}\rangle \otimes \cdots \otimes |e^{(n)}_{p_n}\rangle. \quad (1)$$

Let us also give precise definitions of global entanglement and full separability.

**Definitions.** A state is globally entangled if it remains entangled across any bi-partition. A state is fully separable if it remains separable across all bi-partitions. A fully separable pure state is also called a completely product state and can be expressed in the form $|\psi\rangle \equiv \otimes_{i=1}^n |\phi^{(i)}\rangle$.

We note that in the mixed-state scenario separable states and product states can mean different things whereas they are the same in the pure-state scenario. Let us clarify this. A product mixed state across bi-partition $A:B$ is of the form $\rho = \rho^A \otimes \rho^B$, where $\rho$’s are the corresponding density matrices. This includes product pure states as a special case. A completely product mixed state is then of the form $\rho = \rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(n)}$. A separable mixed state across bi-partition $A:B$ is a state that can be written as $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$, where $p_i \geq 0$ and $\sum_i p_i = 1$. A fully separable mixed state is a state that can be written as $\rho = \sum_i p_i \rho^{(1)} \otimes \rho_i^{(2)} \otimes \cdots \otimes \rho_i^{(n)}$. Similar to the pure-state case, we will call a mixed state being globally entangled, if it is not separable across any bi-partition. For a more refined classification of mixed states, we refer the readers to Ref. [8]. The structure of mixed-state entanglement is much richer, and exotic states, such as bound entangled states can occur [8, 9]. We will be mainly concerned with pure states, but will make a brief comment on generalization of both results (below) to mixed states at the end.

More often than not, the existence of symmetry helps to reduce the difficulty of the problem and make the solution simpler. For example, when $|\psi\rangle$ is invariant under permutations of parties (namely when the coefficients $\chi$’s in Eq. (1) are invariant under permuting their indices), simplification arises as to the quantification of their entanglement [10, 11], via, e.g., the relative entropy of the entanglement [12], the geometric measure [13, 14], or the Majorana representation [15], and the characterization of “exotic” bound entangled states [21], as well as proposals for direct measurements of entanglement [22]. Below, we analyze the global entanglement and full separability of symmetric states (states that possess symmetry under permutations), considered earlier by Ichikawa et al. [7]. They used group theoretical arguments to deduce the two following main results:

**Result 1:** Symmetric states are either globally entangled or fully separable with all the constituent systems having identical states, whereas antisymmetric states are globally entangled.

**Result 2:** No completely product states can be orthogonal to all symmetric states and symmetrization of a completely product state gives rise to a globally entangled state unless the original product state is symmetric (namely, with all the constituent systems having identical states).

We shall provide an alternative approach to these results.

**Proof of Result 1:** Let us, for the sake of argument, imagine that the parties are arranged on a circle and consider that the state has the symmetry

$$T|\psi\rangle = e^{i\theta} |\psi\rangle, \quad (2)$$

where $T$ is the periodic translation on party labels: $1 \rightarrow 2$, $2 \rightarrow 3$, ..., $n \rightarrow 1$. We claim that if $|\psi\rangle$ is separable under any bi-partition of the form $\{1, 2, ..., k : k + 1, k + 2, ..., n\}$ then $|\psi\rangle$ must be fully separable. The proof is as
The bi-separability implies that
\[ \langle \psi | = | \phi^A_1 \otimes | \phi^B_1 \rangle_{k+1, k+2, \ldots, n} \]  \tag{3}

The translation symmetry implies that \[ | \phi \rangle_{1, 2, \ldots, k} \otimes | \phi \rangle_{k+1, k+2, \ldots, n} \] is also separable under the partition \[ 2, \ldots, k+1 : k+2, k+3, \ldots, n \] and hence
\[ | \psi \rangle = | \phi^A_1 \otimes | \phi^A_2 \rangle_{2, \ldots, k} \otimes | \phi^B_1 \rangle_{k+1} \otimes | \phi^B_2 \rangle_{k+2, \ldots, n} \] \tag{4}

The argument holds regardless of the value \( \theta \), as what matters is the separability. Continuing this argument, we arrive at that \( | \psi \rangle \) must be fully separable. We have thus shown that states that possess translational symmetry are either globally entangled or fully separable, extending results of Ichikawa et al.

A periodic translation operator is also an element in the permutation group and hence, if a permutation invariant state is separable under any bi-partition (and by permutation the partition can be made to be \( \{1, 2, \ldots, k : k+1, k+2, \ldots, n\} \)), it must be fully separable. When a permutation invariant state is separable, its constituent systems must possess identical states (up to irrelevant global phases). Therefore, we conclude that a permutation invariant state (as a special case of translation invariant states) is either globally entangled (i.e., it cannot be separable across any bi-partition) or fully separable.

We note that the argument can be applied to the totally antisymmetric state as well, as it also satisfies \( T | \psi \rangle = \pm | \psi \rangle \). But it cannot be separable, as this implies complete separability, which cannot induce a sign change under any permutation. Thus, an entirely antisymmetric state is entangled, and hence globally entangled. Furthermore, it can also be applied to the case of braid group \( \mathcal{B}_n \) when the state \( | \psi \rangle \) satisfies
\[ \pi_\sigma | \psi \rangle = e^{i \theta} | \psi \rangle, \quad \forall \pi_\sigma \in \mathcal{B}_n, \] \tag{5}
as a periodic translation operator can be constructed from elements in the braid group. For all states \( | \psi \rangle \) satisfying the above symmetry with \( e^{i \theta} \neq 1 \), they are necessarily globally entangled. Hence, we have provide alternate proof of Result 1, which was originally proven in Ref. 7 using group-theoretical arguments. \( \square \)

**Proof of Result 2:**
Next, we shall derive Result 2. We first show that for any product state \( | \Phi \rangle = | \phi_1 \rangle \otimes | \phi_2 \rangle \otimes \cdots \otimes | \phi_n \rangle \), the symmetrization yields a nonzero state
\[ | \Phi_S \rangle = \frac{c}{n!} \sum_{\sigma \in \mathcal{S}_n} \sigma | \Phi \rangle, \] \tag{6}
i.e., \( \langle \Phi_S | \Phi_S \rangle > 0 \), where \( c \neq 0 \) is a normalization constant and \( \mathcal{S}_n \) denotes the permutation group for \( n \) objects. It is straightforward to see that
\[ \langle \Phi_S | \Phi_S \rangle = \frac{|c|^2}{n!} \text{Perm}(\langle \phi_1 | \phi_2 \rangle), \] \tag{7}
where \( \text{Perm} \) denotes the permanent of a matrix. As the matrix \( a_{ij} \equiv \langle \phi_i | \phi_j \rangle \) is positive semi-definite and \( a_{ii} = 1 \), from the results of Marcus, we see that the permanent of matrix \( a_{ij} \) is nonzero, and in particular
\[ 1 \leq \text{Perm}(a_{ij}) \leq n! \] \tag{8}
where the first inequality becomes equality when \( a_{ij} = \delta_{ij} \) and the second inequality becomes equality when \( a_{ij} \) is rank-one, i.e., \( | \phi_i \rangle \)'s are identical up to a phase factor. What we have just shown is that the symmetrization of a state composed of direct product of single-particle states always yields a valid state, which is an implicit assumption in discussing a bosonic state. Conversely, if \( \langle \Phi_S | \Phi_S \rangle = 0 \), this implies that \( \langle \Phi | \Phi \rangle = 0 \) as well.

Now, suppose there exists a completely product state \( | \Phi \rangle \) that is orthogonal to all permutation invariant states. As the inner product of \( | \Phi \rangle \) with any such symmetric state \( | \dot{\chi} \rangle \) is invariant under permuting parties in \( | \Phi \rangle \), this means that its symmetrized state \( | \Phi_S \rangle \) must be orthogonal to all permutation invariant states, hence including itself! This leads to contradiction. Therefore, there cannot exist a completely product state that is orthogonal to all permutation invariant states.

Furthermore, from Result 1 the symmetrized state \( | \Phi_S \rangle \) must be either globally entangled or fully separable. In the latter case, \( | \Phi_S \rangle = | \phi \rangle \otimes (n \) up to normalization and a global phase), we want to show that this implies that the original state \( | \Phi \rangle \) must be uniquely of the form \( | \Phi \rangle = | \phi \rangle \otimes n \) up to a global phase factor. A consequence of \( | \Phi_S \rangle \) being a product state is that its reduced density matrix after tracing over \( k = 1, \ldots, n - 1 \) is still pure. We shall consider tracing over \( (n - 1) \) parties. Now we can rewrite \( | \Phi_S \rangle \)
\[ | \Phi_S \rangle \equiv \frac{c}{n!} \sum_{i=1}^{n} | \phi_i \rangle_A | \psi_i \rangle_B, \] \tag{9}
where the \( (n - 1) \)-partite state \( | \psi_k \rangle \) that is associated with one-partite state \( | \phi_k \rangle \) is of the form:
\[ | \psi_k \rangle = \sum_{\tilde{\sigma} \in \mathcal{S}_{n-1}} \tilde{\sigma} | \phi_{k+1} \rangle \cdots | \phi_n \rangle | \phi_1 \rangle \cdots | \phi_{k-1} \rangle. \] \tag{10}

Tracing over \( (n - 1) \) parties (namely \( B \)), we obtain
\[ \rho = \text{Tr}_{2 \ldots n}(| \Phi_S \rangle \langle \Phi_S |) = \frac{|c|^2}{(n!)^2} \sum_{i,j} | \phi_i \rangle \langle \phi_j | \langle \psi_j | \psi_i \rangle. \] \tag{11}
Since \( \langle \psi_j | \psi_i \rangle \) is positive semi-definite we can diagonalize it:
\[ \langle \psi_j | \psi_i \rangle = \sum_k \lambda_k U_{jk} U^*_{ik}, \] \tag{12}
where \( U \) is unitary and \( \lambda_k \geq 0 \). This means that
\[ \rho = \frac{|c|^2}{(n!)^2} \sum_k \lambda_k | \alpha_k \rangle \langle \alpha_k |, \] \tag{13}
Hence, the corresponding mixed state is fully separable. This is indeed the case. If such a symmetric mixed state is separable with respect to some bi-partition, this implies that for certain decomposition all the pure symmetric states in the mixture are in a product form with respect to this bi-partition. Then by reasoning in Result 1, this means that they are completely product states. As a fully separable state \( \rho \) is composed of completely product states \( |\phi_i\rangle \): \( \rho = \sum_i p_i |\phi_i\rangle \langle \phi_i| \), the inner product with a symmetric state \( |\psi_S\rangle \) becomes \( \sum_i p_i |\langle \phi_i| \psi_S\rangle|^2 \). The expression is zero if and only if \( \langle \phi_i| \psi_S\rangle = 0 \) for all \( i \) (with \( p_i > 0 \)). If this holds for all symmetric states \( |\psi_S\rangle \), it will lead to the same contradiction that \( |\phi_i\rangle = 0 \) in Result 2. Hence, the generalization to mixed states is also correct. However, a generalization of the second part regarding symmetrization of fully separable mixed states does not lead to any interesting outcome, namely, it will not make fully separable mixed states become globally entangled. Because we define the symmetrization (sum over all permutations) at the level of density matrices, a fully separable state remains fully separable under symmetrization.

Acknowledgment. This work was supported by NSERC and MITACS.

where
\[
|\alpha_k\rangle = \sum_{i=1}^{n} U_{ik} |\phi_i\rangle.
\] (14)

But we also have that \( \rho = |\phi\rangle \langle \phi| \), this means that \( |\alpha_k\rangle \sim |\phi\rangle \), which in turn (by inverting the above equation) gives
\[
|\phi_j\rangle = \sum_{k=1}^{n} U_{jk} |\alpha_k\rangle \sim |\phi\rangle.
\] (15)

Thus, we have \( |\phi_j\rangle = |\phi\rangle \) up to a global phase and \( |\Phi\rangle = |\phi^{\otimes n}\rangle \). Hence, Result 2 is proved.

Let us conclude with some comments on mixed states. A generalization of Result 1 would be: mixed states that are composed of mixture of permutation invariant pure states are either globally entangled or fully separable. This is indeed the case. If such a symmetric mixed state is separable with respect to some bi-partition, this implies that for certain decomposition all the pure symmetric states in the mixture are in a product form with respect to this bi-partition. Then by reasoning in Result 1, this means that they are completely product states. Hence, the corresponding mixed state is fully separable.

A partial generalization of Result 2 would be: No fully separable mixed states can be orthogonal to all symmetric mixed states (orthogonal in the sense of this “inner product”: \( \text{Tr}(\rho_1^{\dagger} \rho_2) = 0 \)). This is also correct, as all symmetric mixed states are composed of mixture of pure symmetric states, this reduces to showing no fully separable mixed states can be orthogonal to all symmetric pure states. As a fully separable state \( \rho \) is composed of completely product states \( |\phi_i\rangle \); \( \rho = \sum_i p_i |\phi_i\rangle \langle \phi_i| \), the expression is zero if and only if \( \langle \phi_i| \psi_S\rangle = 0 \) for all \( i \) (with \( p_i > 0 \)). If this holds for all symmetric states \( |\psi_S\rangle \), it will lead to the same contradiction that \( |\phi_i\rangle = 0 \) in Result 2. Hence, the generalization to mixed states is also correct. However, a generalization of the second part regarding symmetrization of fully separable mixed states does not lead to any interesting outcome, namely, it will not make fully separable mixed states become globally entangled. Because we define the symmetrization (sum over all permutations) at the level of density matrices, a fully separable state remains fully separable under symmetrization.

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