Chaos expansion of 2D parabolic Anderson model

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Abstract

We prove a chaos expansion for the 2D parabolic Anderson Model in small time, with the expansion coefficients expressed in terms of the annealed density function of the polymer in a white noise environment.

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1 Introduction and main result

Consider the continuous parabolic Anderson model in $d = 2$ formally written as

$$\partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + u \cdot (\dot{W}(x) - \infty), \quad t \geq 0, x \in \mathbb{R}^2,$$

(1.1)

where $\dot{W}(x)$ is a spatial white noise defined on the probability space $(\Omega, \mathcal{F}, P)$ formally satisfying

$$E[\dot{W}(x)\dot{W}(y)] = \delta(x - y).$$

The equation (1.1) was analyzed in [7, 8, 9] by different approaches including the theory of regularity structures, para-controlled calculus, and the method of correctors and two-scale expansions. The main results in these references showed that a smoothed version of (1.1) converges to some limit that is independent of the mollification.

More precisely, let $\varphi: \mathbb{R}^2 \to \mathbb{R}_+$ be a smooth and compactly supported function on $\mathbb{R}^2$ satisfying $\varphi(x) = \varphi(-x)$ and $\int \varphi = 1$. Define $\varphi_\epsilon(\cdot) = \epsilon^2 \varphi(\cdot/\epsilon)$ and

$$\dot{W}_\epsilon(x) = \int_{\mathbb{R}^2} \varphi_\epsilon(x - y)dW(y)$$

(1.2)

as the mollification of $\dot{W}$. The covariance function of $\dot{W}_\epsilon$ is

$$R_\epsilon(x - y) := E[\dot{W}_\epsilon(x)\dot{W}_\epsilon(y)] = \varphi_\epsilon \ast \varphi_\epsilon(x - y).$$

(1.3)

Let $u_\epsilon$ be the solution to the equation with smooth coefficients

$$\partial_t u_\epsilon(t, x) = \frac{1}{2} \Delta u_\epsilon(t, x) + u_\epsilon(\dot{W}_\epsilon(x) - C_\epsilon),$$

(1.4)

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with the diverging constant
\[ C_\epsilon = \frac{1}{\pi} \log \epsilon^{-1}. \] (1.5)

Then \( u_\epsilon \) converges in some weighted Hölder space to a limit \( u \) that is defined to be the solution to (1.1), see [9, Theorem 4.1].

While the solution to (1.1) is well-defined, its statistical property remains a challenge. We refer to \([1, 2, 5, 6, 14]\) for some relevant discussions. The goal of this note is to provide a Wiener chaos expansion of the solution \( u \), in the short time regime. We assume \( u_\epsilon(0, x) = u_0(x) \) for some bounded function \( u_0 \). Theorem 1.1 below shows that for small \( t \), \( u_\epsilon(t, x) \to u(t, x) \) in \( L^2(\Omega) \) as \( \epsilon \to 0 \), and \( u(t, x) \) is written explicitly as a Wiener chaos expansion in terms of the probability density of a polymer in a white noise environment, see (1.17). We hope that the explicit chaos expansion will provide another way of proving the convergence to (1.1), e.g. from a discrete system using the general criteria proved in [3, 14]. The tool we use is a combination of the Feynman-Kac representation and Malliavin calculus. By writing \( u_\epsilon(t, x) \) in terms of a chaos expansion, it suffices to pass to the limit in each chaos.

### 1.1 Elements of Malliavin calculus

We give a brief introduction to Malliavin calculus and refer to [15] for more details. For any function \( \phi \in L^2(\mathbb{R}^2) \), we define \( W(\phi) = \int \phi \ dW \). Let \( F \) be a smooth and cylindrical random variable of the form

\[ F = f(W(\phi_1), \ldots, W(\phi_n)), \]

with \( \phi_i \in L^2(\mathbb{R}^2), f \in C^\infty(\mathbb{R}^n) \) (namely \( f \) and all its partial derivatives have polynomial growth), then the Malliavin derivative of \( F \), denoted by \( DF \), is the \( L^2(\mathbb{R}^2) \)-valued random variable defined by

\[ DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(\phi_1), \ldots, W(\phi_n))\phi_j. \] (1.6)

For each positive integer \( k \), \( D^k F \) is defined to be the \( k \)-th iterated derivative of \( F \), which is a random variable taking values in \( L^2(\mathbb{R}^2)^\otimes k \), the \( k \)-th tensor product of \( L^2(\mathbb{R}^2) \). The operator \( D^k \) is closable from \( L^2(\Omega) \) into \( L^2(\Omega; L^2(\mathbb{R}^2)^\otimes k) \) and we define the Sobolev space \( \mathbb{D}^{k,2} \) as the closure of the space of smooth and cylindrical random variables under the norm

\[ \|D^k F\|_{k,2} = \left\| \mathbb{E} \left[ F^2 + \sum_{j=1}^{k} \|D^j F\|_{L^2(\mathbb{R}^2)^\otimes j}^2 \right] \right\|. \] (1.7)

Define \( \mathbb{D}^{\infty,2} = \bigcap_{k=1}^{\infty} \mathbb{D}^{k,2} \), and \( L^2(\mathbb{R}^2)^\otimes k \) as the \( k \)-th symmetric tensor product of \( L^2(\mathbb{R}^2) \).

For any integer \( n \geq 0 \), we denote by \( \mathbb{H}_n \) the \( n \)-th Wiener chaos of \( W \). We recall that \( \mathbb{H}_0 \) is simply \( \mathbb{R} \), and for \( n \geq 1 \), \( \mathbb{H}_n \) is the closed linear subspace of \( L^2(\Omega) \) generated by the random variables

\[ \{H_n(W(h)) : h \in L^2(\mathbb{R}^2), \|h\|_{L^2(\mathbb{R}^2)} = 1\}, \]

where \( H_n \) is the \( n \)-th order Hermite polynomials. For any \( n \geq 1 \), the mapping

\[ I_n(h^{\otimes n}) := H_n(W(h)) \]

can be extended to a linear isometry between \( L^2(\mathbb{R}^2)^{\otimes n} \) and \( \mathbb{H}_n \), with the isometric relation

\[ \mathbb{E}[I_n(h^{\otimes n})^2] = n!\|h^{\otimes n}\|_{L^2(\mathbb{R}^2)^{\otimes n}}^2 \] (1.8)
Consider now a random variable $F \in L^2(\Omega)$, it can be written as

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$  \hspace{1cm} (1.9)

where the series converges in $L^2(\Omega)$, and the coefficients $f_n \in L^2(\mathbb{R}^\circ)^n$ are determined by $F$. This identity is called the Wiener-chaos expansion of $F$.

When the above $F \in D_{-2}$, the $n-$th coefficient $f_n$ in the Wiener chaos expansion of $F$ can be explicitly written as [16, Page 3, equation (7)]

$$f_n = \frac{\mathbb{E}[D^n F]}{n!}. \hspace{1cm} (1.10)$$

1.2 Brownian self-intersection local time and polymer in white noise

The self-intersection local time of the planar Brownian motion is a classical subject in probability theory [4, 12, 13, 17, 18]. In the following, we discuss its connections to the parabolic Anderson model.

Using the Feynman-Kac formula, we write the solution to (1.4) as

$$u_c(t, x) = \mathbb{E}_B \left[ u_0(x + B_t) \exp \left( \int_0^t \tilde{W}_c(x + B_s) ds - C_t \right) \right], \hspace{1cm} (1.11)$$

where $B$ is a standard Brownian motion starting from the origin which is independent from $\tilde{W}$, and $\mathbb{E}_B$ denotes the expectation with respect to $B$. Taking expectation with respect to $\tilde{W}_c$ and using the fact that the exponent inside the expectation in (1.11) is of Gaussian distribution for each realization of the Brownian motion, we obtain

$$\mathbb{E}[u_c(t, x)] = \mathbb{E}_B \left[ u_0(x + B_t) \exp \left( \int_0^t \int_0^s R_c(B_s - B_u) duds - C_t \right) \right],$$

where we recall that $R_c$ is the covariance function of $\tilde{W}_c$. It is well-known that

$$\gamma_c(t, B) := \int_0^t \int_0^s R_c(B_s - B_u) duds - \int_0^t \int_0^s \mathbb{E}_B[R_c(B_s - B_u)] duds \to \gamma(t, B) \hspace{1cm} (1.12)$$

almost surely, and $\gamma(t, B)$ is the so-called renormalized self-intersection local time of the planar Brownian motion formally written as

$$\gamma(t, B) = \int_0^t \int_0^s \delta(B_s - B_u) duds - \int_0^t \int_0^s \mathbb{E}_B[\delta(B_s - B_u)] duds. \hspace{1cm} (1.13)$$

In addition, there exists some critical $t_c > 0$ such that

$$\mathbb{E}_B[\exp(\gamma(t, B))] \begin{cases} < \infty & t < t_c, \\ = \infty & t > t_c. \end{cases} \hspace{1cm} (1.14)$$

The renormalization constant in (1.5) matches the expectation in (1.12) up to an $O(1)$ correction, and a calculation as in [6, Lemma 1.1] shows that there exists constants $\mu_1, \mu_2$ such that

$$\int_0^t \int_0^s \mathbb{E}_B[R_c(B_s - B_u)] duds - C_t \to t(\mu_1 + \mu_2 \log t)$$

as $\epsilon \to 0$. For small $t$, it was shown in [6] that

$$\mathbb{E}[u(t, x)] = \lim_{\epsilon \to 0} \mathbb{E}[u_c(t, x)] = e^{t(\mu_1 + \mu_2 \log t)} \mathbb{E}_B[u_0(x + B_t)e^{\gamma(t, B)}]. \hspace{1cm} (1.16)$$
This motivates us to define

$$F(t) := \log \mathbb{E}_B[e^{\gamma(t,B)}],$$

so we can write

$$\mathbb{E}[u(t,x)] = e^{t(\mu_1 + \mu_2 \log t) + F(t)} \mathbb{E}_t,B[u_0(x + B_t)],$$

where $\mathbb{E}_t,B$ denotes the expectation with respect to the Wiener measure tilted by the factor $e^{\gamma(t,B)}$, i.e.,

$$\mathbb{E}_t,B[X] = \frac{\mathbb{E}_B[Xe^{\gamma(t,B)}]}{\mathbb{E}_B[e^{\gamma(t,B)}]} = \mathbb{E}_B[Xe^{\gamma(t,B)}]e^{-F(t)}$$

for any bounded $X$. By the formal expression in (1.13), we can view $\mathbb{E}_t,B$ as the expectation with respect to the annealed measure of a polymer in a white noise environment. By (1.14), it is clear that the measure is absolutely continuous with respect to the Wiener measure for small $t$. Applying the Radon-Nikodym theorem, for any $n \in \mathbb{Z}_+$ and $0 < s_1 < \ldots < s_n \leq t < t_\epsilon$, there exists a non-negative measurable function, denoted by

$$\mathcal{F}_{s_1,\ldots,s_n} : \mathbb{R}^{2n} \to \mathbb{R},$$

such that

$$\mathbb{E}_{t,B}[1_A(B_{s_1},\ldots,B_{s_n})] = \int_A \mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)dx$$

for all $A \subset \mathbb{R}^{2n}$. In other words, $\mathcal{F}_{s_1,\ldots,s_n}$ is the joint spatial density function of the polymer path at $s_1 < \ldots < s_n$. We note that $\mathcal{F}$ actually depends on $t$ since the tilted measure depends on $t$. For our purpose, we use the simplified notion since $t$ is fixed. It is an elementary exercise to show that $\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)$ is jointly measurable in $(s_1,\ldots,s_n,x_1,\ldots,x_n)$. For the convenience of the reader, we present a proof in the appendix.

Denote $[0,t]_\epsilon^s := \{0 \leq s_1 < \ldots < s_n \leq t\}$, the following is our main result.

**Theorem 1.1.** There exists $t_0 > 0$ such that for each $t \in (0,t_0)$, $x \in \mathbb{R}^2$, the random variable $u_\epsilon(t,x)$ converges in $L^2(\Omega)$ to

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)).$$

(1.17)

The coefficient $f_n(\cdot,t,x)$ is given by

$$f_n(y_1,\ldots,y_n; t, x)
= e^{t(\mu_1 + \mu_2 \log t) + F(t)} \int_{\mathbb{R}^2} \int_{[0,t]_\epsilon^s} u_0(x + z) \mathcal{F}_{s_1,\ldots,s_n}(y_1 - x, \ldots, y_n - x, z) dsdz.$$

(1.18)

**Remark 1.2.** The small time constraint in Theorem 1.1 seems necessary. It was shown in [6] that $\mathbb{E}[u(t,x)^2]$ is finite and admits a Feynman-Kac representation for small $t$, and we expect that $\mathbb{E}[u(t,x)^2] = \infty$ when $t$ is large, in light of (1.14).

**Remark 1.3.** Since the formal product $u \cdot \dot{W}$ in (1.1) comes from the classical physical polymer $u_\epsilon \dot{W}_\epsilon$ in (1.4), we may interpret it in the Stratonovich’s sense. If it is replaced by the Wick product:

$$\partial_t u(t,x) = \frac{1}{2} \Delta u(t,x) + u(t,x) \circ \dot{W}(x),$$

(1.19)

a different chaos expansion was proved in [10]. Compared with (1.17), the only difference is the lack of the weight $e^{\gamma(t,B)}$ in the definition of $\mathcal{F}$. This reduces the polymer measure to the original Wiener measure, in which case we have

$$\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n) = q_{s_1}(x_1)q_{s_2-s_1}(x_2 - x_1)\ldots q_{s_n-s_{n-1}}(x_n - x_{n-1}),$$

(1.20)
where \( q_t(x) := (2\pi t)^{-1} e^{-|x|^2 / 2t} \) is the standard heat kernel. With \( \mu_1 = \mu_2 = F = 0 \), the expansion coefficient is given by

\[
\begin{align*}
\int_R \int_{[0,t]} u_0(x + z) \mathcal{F}_{s_1,\ldots,s_n,t}(y_1 - x, \ldots, y_n - x, z) ds dz \\
= \int_{[0,t]} \prod_{j=0}^{n} q_{s_j - s_{j+1}}(y_j - y_{j+1}) u_0(z) dz ds,
\end{align*}
\]

(1.21)

with the convention that \( y_0 = x, y_{n+1} = z, s_{n+1} = 0 \). Thus, the resulting chaos expansion is obtained by iterating the mild formulation of (1.19). The missing exponential weight \( e^{\gamma(t,B)} \) favors self-attracting of the polymer paths, which prevails in the intermittency behaviors of parabolic Anderson model. We refer to the recent monograph [11] for more details.

**Remark 1.4.** The same proof works in the one dimensional case, where the small time constraint can be removed, and there is no need to renormalize. A similar expansion coefficient as (1.18) holds.

2 Proof of the main result

For fixed \( t > 0, x \in \mathbb{R}^2, \epsilon > 0 \) and each realization of the Brownian motion, we write the exponent in (1.11) as

\[
\begin{align*}
\int_0^t W_\epsilon(x + B_s) ds &= \int_0^t \int_{\mathbb{R}^2} \varphi_\epsilon(x + B_s - y) dW(y) ds \\
&= \int_{\mathbb{R}^2} \left( \int_0^t \varphi_\epsilon(x + B_s - y) ds \right) dW(y) \\
&= \int_{\mathbb{R}^2} \Phi_{t,x,B}(y) dW(y),
\end{align*}
\]

with

\[
\Phi_{t,x,B}(y) := \int_0^t \varphi_\epsilon(x + B_s - y) ds.
\]

Then it is easy to see that \( u_\epsilon(t,x) \in D^{\infty,2} \), and

\[
D^n u_\epsilon(t,x) := E_B \left[ u_0(x + B_t) D^n \exp \left( \int_{\mathbb{R}^2} \Phi_{t,x,B}(y) dW(y) - C_\epsilon t \right) \right] \\
= E_B \left[ u_0(x + B_t) \exp \left( \int_{\mathbb{R}^2} \Phi_{t,x,B}(y) dW(y) - C_\epsilon t \right) (\Phi_{t,x,B}(\cdot))^{\otimes n} \right].
\]

(2.1)

By the Stroock’s formula (1.10), we can write the Wiener chaos expansion of \( u_\epsilon(t,x) \) as

\[
u_\epsilon(t,x) = \sum_{n=0}^{\infty} I_n(f_{\epsilon,n}(\cdot; t,x)),
\]

(2.2)

with

\[
f_{\epsilon,n}(\cdot; t,x) = \frac{1}{n!} E[D^n u_\epsilon(t,x)] \\
= \frac{1}{n!} E_B \left[ u_0(x + B_t) \exp \left( \int_0^t \int_0^s R_\epsilon(B_u - B_s) du ds - C_\epsilon t \right) (\Phi_{t,x,B}(\cdot))^{\otimes n} \right].
\]

(2.3)
Chaos expansion 2D PAM

By (1.15), we define

\[ r_\epsilon := \int_0^t \int_0^s E_B [R_\epsilon(B_s - B_u)] du ds - C \epsilon t - t(\mu_1 + \mu_2 \log t), \tag{2.4} \]

which goes to zero as \( \epsilon \to 0 \), and rewrite

\[ f_{\epsilon,n}(\cdot; t, x) = \frac{e^{t(\mu_1 + \mu_2 \log t) + r_\epsilon}}{n!} E_B \left[ u_0(x + B_t) \exp(\gamma(t, B)) \left( \Phi_{t,x,B}(\cdot) \right)^n \right]. \tag{2.5} \]

To prove Theorem 1.1, it suffices to show that as \( \epsilon \to 0 \),

\[ \sum_{n=0}^\infty n! \| f_{\epsilon,n}(\cdot; t, x) - f_n(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n}^2 \to 0. \tag{2.6} \]

Define

\[ \tilde{f}_{\epsilon,n}(\cdot; t, x) := \frac{e^{t(\mu_1 + \mu_2 \log t) + r_\epsilon}}{n!} E_B \left[ u_0(x + B_t) \exp(\gamma(t, B)) \left( \Phi_{t,x,B}(\cdot) \right)^n \right]. \tag{2.7} \]

Since

\[
\| f_{\epsilon,n}(\cdot; t, x) - f_n(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n}^2 \\
\leq 2\| f_{\epsilon,n}(\cdot; t, x) - \tilde{f}_{\epsilon,n}(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n}^2 + 2\| \tilde{f}_{\epsilon,n}(\cdot; t, x) - f_n(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n}^2,
\]

the proof of (2.6) reduces to the following three lemmas.

**Lemma 2.1.** There exists \( t_0, C > 0 \) independent of \( \epsilon, n \) such that if \( t < t_0 \),

\[
\| f_{\epsilon,n}(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n} + \| \tilde{f}_{\epsilon,n}(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n} + \| f_n(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n}^2 \leq \frac{(Ct)^n}{n!}.
\]

**Lemma 2.2.** There exists \( t_0 > 0 \) such that if \( t < t_0 \),

\[
\| f_{\epsilon,n}(\cdot; t, x) - \tilde{f}_{\epsilon,n}(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n} \to 0, \quad \text{as} \ \epsilon \to 0.
\]

**Lemma 2.3.** There exists \( t_0 > 0 \) such that if \( t < t_0 \),

\[
\| \tilde{f}_{\epsilon,n}(\cdot; t, x) - f_n(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n} \to 0, \quad \text{as} \ \epsilon \to 0.
\]

In the following, we use the notation \( a \lesssim b \) when \( a \leq Cb \) for some constant \( C > 0 \) independent of \( \epsilon, n \).

**Proof of Lemma 2.1.** The proof of \( f_{\epsilon,n} \) and \( \tilde{f}_{\epsilon,n} \) is the same. Take \( f_{\epsilon,n} \) for example:

\[
\| f_{\epsilon,n}(\cdot; t, x) \|_{L^2(\mathbb{R}^2)^\otimes n}^2 \\
\leq \frac{1}{(n!)^2} \int_{\mathbb{R}^2} E_{B^1, B^2} \left[ \prod_{j=1}^2 \left( e^{\gamma_j(t,B^j)} \prod_{k=1}^n \Phi_{t,x,B^j}(y_k) \right) \right] dy,
\]

where \( B^1, B^2 \) stand for independent Brownian motions. Performing the integral in the \( y \) variable, the r.h.s. of the above display is bounded by

\[
\frac{1}{(n!)^2} E_{B^1, B^2} \left[ e^{\gamma_1(t,B^1) + \gamma_2(t,B^2)} \left( \int_{[0,t]^2} R_t(B^1_s - B^2_s)dsdu \right)^n \right].
\]
An application of Stirling’s approximation yields the desired result. By Lemma 2.3, the same estimate holds for \( L \). Thus, the r.h.s. of the above display goes to zero as \( n \to \infty \).

Proof of Lemma 2.3. First, we claim that \( \tilde{f}_{\epsilon,n}(t,x) \) is a Cauchy sequence in \( L^2(\mathbb{R}^2)^\otimes n \). It suffices to prove the convergence of

\[
\lim_{\epsilon_1,\epsilon_2 \to 0} \langle \tilde{f}_{\epsilon_1,n}(t,x), \tilde{f}_{\epsilon_2,n}(t,x) \rangle_{L^2(\mathbb{R}^2)^\otimes n}.
\]  

(2.8)

By applying Lemma A.1, we have

\[
E_{B_1,B_2} \left[ \prod_{j=1}^{2} u_0(x + B_i^t t) e^{\gamma(t,B_i)} \left( \int_{0}^{t} \int_{0}^{t} R_{\epsilon_1,\epsilon_2}(B_i^t - B_i^s) ds du \right)^n \right]
\]

converges as \( \epsilon_1,\epsilon_2 \to 0 \), where \( R_{\epsilon_1,\epsilon_2} := \varphi_{\epsilon_1} \ast \varphi_{\epsilon_2} \). This proves (2.8).

Next, we show that \( \tilde{f}_{\epsilon,n}(t,x) \to f_n(t,x) \) in \( L^1(\mathbb{R}^2)^\otimes n \) which implies that \( f_n(t,x) \in L^2(\mathbb{R}^2)^\otimes n \) and completes the proof. We have

\[
\tilde{f}_{\epsilon,n}(y_1,\ldots,y_n; t, x) = e^{\mu_1 + \mu_2 \log t + \nu} \frac{1}{n!} E_{B} \left[ u_0(x + B_t t) e^{\gamma(t,B)} \prod_{k=1}^{n} \Phi_{t,x,B}(y_k) \right]
\]

(2.9)

with

\[
\varphi_{\epsilon}^n \ast f \to f \text{ in } L^1(\mathbb{R}^2)^\otimes n.
\]

Since \( \varphi_{\epsilon}^n \) is an approximation to identity, by the classical convolution theorem, the desired result follows.

Thus,

\[
\tilde{f}_{\epsilon,n}(y_1,\ldots,y_n; t, x) = e^{\mu_1 + \mu_2 \log t + \nu} \varphi_{\epsilon}(y_1 - x, \ldots, y_n - x)
\]

in \( L^1(\mathbb{R}^2)^\otimes n \).
A Technical lemmas

A.1 Measurability of $\mathcal{F}$

We show that $\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)$ is jointly measurable in the $(s,x)$ variable. Fix any $0 < s_1 < \ldots < s_n \leq t$, consider

$$\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n) := \mathcal{E}_{t,B} \left[ \prod_{j=1}^{n} \varphi_{s_j}(B_{s_j} - x_j) \right]$$

$$= \int_{\mathbb{R}^2n} \prod_{j=1}^{n} \varphi_{s_j}(y_j - x_j) \mathcal{F}_{s_1,\ldots,s_n}(y_1,\ldots,y_n) dy.$$

The last integral converges in $L^1(\mathbb{R}^{2n})$ to $\mathcal{F}_{s_1,\ldots,s_n}$. It is clear that $\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)$ is continuous in both $s$ and $x$ variable, hence it is measurable. If we can show $\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)$ converges in $L^1([0,t]_\mathbb{Z} \times \mathbb{R}^{2n})$ to some $g_{s_1,\ldots,s_n}(x_1,\ldots,x_n)$, then $g = \mathcal{F}$ almost everywhere in $[0,t]_\mathbb{Z} \times \mathbb{R}^{2n}$, which implies $\mathcal{F}$ is measurable.

For fixed $s_1,\ldots,s_n$, we have

$$\int_{\mathbb{R}^2n} |\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n) - \mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)| dx \to 0$$

as $\epsilon, \delta \to 0$. In addition,

$$\int_{\mathbb{R}^2n} |\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n) - \mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)| dx$$

$$\leq \int_{\mathbb{R}^2n} (\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n) + \mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)) dx = 2.$$

Thus, by the dominated convergence theorem, we have

$$\int_{[0,t]_\mathbb{Z}} \int_{\mathbb{R}^2n} |\mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n) - \mathcal{F}_{s_1,\ldots,s_n}(x_1,\ldots,x_n)| dx ds \to 0$$

as $\epsilon, \delta \to 0$. This completes the proof.

A.2 Estimates on intersection local time

We collect some standard estimates on the intersection local time of planar Brownian motion. Recall that $R_{\epsilon_1,\epsilon_2} = \varphi_{\epsilon_1} \ast \varphi_{\epsilon_2}$, and assume that the Brownian motion is built on the probability space $(\Sigma, \mathcal{A}, P_B)$.

**Lemma A.1.** For any $\lambda > 0$, there exist constants $C, \ell_0 > 0$ such that

$$\sup_{\epsilon \in (0,1], t \in [0,\ell_0]} E_B[e^{\lambda \gamma_{t,\ell_0}(s,B)}] \leq C,$$

(A.1)

and for all $n \in \mathbb{N}$,

$$\sup_{\epsilon_1,\epsilon_2 \in (0,1]} E_B^{1, B^2} \left[ \left( \int_{[0,\ell]^2} R_{\epsilon_1,\epsilon_2}(B_s^1 - B_u^2) ds du \right)^n \right] \leq n!(Ct)^n.$$

(A.2)

In addition,

$$\int_{[0,\ell]^2} R_{\epsilon_1,\epsilon_2}(B_s^1 - B_u^2) ds du \to \int_{[0,\ell]^2} \delta(B_s^1 - B_u^2) ds du$$

(A.3)

in $L^2(\Sigma)$ as $\epsilon_1, \epsilon_2 \to 0$, where the r.h.s. is the so-called mutual intersection local time of planar Brownian motions.
Proof. The uniform exponential integrability (A.1) is shown in [6, Lemma A.1]. It also contains a moment estimate of the form
\[
\sup_{\epsilon \in (0, 1]} E_{B^1, B^2} \left[ \left( \int_{[0,t]^2} R_t(B_s^1 - B_u^2) ds du \right)^n \right] \\
\leq E_{B^1, B^2} \left[ \left( \int_{[0,t]^2} \delta(B_s^1 - B_u^2) ds du \right)^n \right] \leq n!(Ct)^n.
\]
The same proof leads to (A.2).

Since
\[
\int_{[0,t]^2} R_t(B_s^1 - B_u^2) ds du \rightarrow \int_{[0,t]^2} \delta(B_s^1 - B_u^2) ds du
\]
in \(L^2(\Sigma)\), to prove (A.3), it suffices to show that as \(\epsilon_1, \epsilon_2 \rightarrow 0\),
\[
\int_{[0,t]^2} R_{\epsilon_1, \epsilon_2}(B_s^1 - B_u^2) ds du - \int_{[0,t]^2} R_{\epsilon_1}(B_s^1 - B_u^2) ds du \rightarrow 0
\]
in \(L^2(\Sigma)\), which reduces to the convergence of
\[
E_{B^1, B^2} \left[ \int_{[0,t]^4} R_{\epsilon_1, \epsilon_2}(B_s^1 - B_u^2) R_{\epsilon_3, \epsilon_4}(B_s^1 - B_u^2) ds du \right]
\]
as \(\epsilon_j \rightarrow 0\), \(j = 1, 2, 3, 4\). We write \(R_{\epsilon_i, \epsilon_j}\) in the Fourier domain so that the above expectation equals to
\[
\frac{1}{(2\pi)^2} \int_{[0,t]^4} \hat{\varphi}(\epsilon_1 \xi) \hat{\varphi}(\epsilon_2 \xi) \hat{\varphi}(\epsilon_3 \eta) \hat{\varphi}(\epsilon_4 \eta) E_{B^1, B^2} [e^{i\xi(B_s^1 - B_u^2)} e^{i\eta(B_s^1 - B_u^2)}] d\xi d\eta ds du.
\]
It suffices to use the bound
\[
\int_{[0,t]^4} \int_{R^4} E_{B^1, B^2} [e^{i\xi(B_s^1 - B_u^2)} e^{i\eta(B_s^1 - B_u^2)}] d\xi d\eta ds du < \infty
\]
and the dominated convergence theorem to complete the proof.

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