DUAL INFINITE WEDGE IS $GL_\infty$-EQUIVARIANTLY NOETHERIAN

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Abstract. We prove the (equivariant) noetherian property for a wide class of varieties generalizing the class of Plücker varieties (Theorem 1). It improves previous results of Draisma–Eggermont who treated the case of bounded Plücker varieties. Key ingredient of our proof is the constructive proof of the equivariant noetherianity for the hyper-Pfaffians (Theorem 36) which implies the equivariant noetherianity of the dual infinite wedge.

1. Introduction

A Plücker variety is a rule $X$ that assigns to a pair of a nonnegative number $p$ and a vector space $V$ an algebraic variety $X_p(V)$. These varieties are required to satisfy some compatibility properties (see Definition 5); for instance, for every $p$ the assignment $X_p(\_)$ is a functor on a category of vector spaces.

Plücker varieties were introduced by Draisma and Eggermont [1]. These varieties give a powerful tool for proving existence of uniform bounds for degrees of equations for varieties $X_p(V)$. The most important implications of the work of Draisma–Eggermont include uniform bounds for degrees of the equations for secant and tangential varieties of Grassmannians.

More generally, their noetherianity result [1, Theorem 1] implies that the degrees of equations for $X_p(V)$ are bounded for all $p$ and $V$ for the so-called bounded Plücker varieties $X$ (Definition 14). The boundedness condition for Plücker varieties originates in the work of Snowden on noetherianity for $\Delta$-modules [2]; $\Delta$-modules are ideals of equations for symmetric power counterparts of Plücker varieties.

The paper [1] raises the natural question whether the same noetherianity result holds for unbounded Plücker varieties. Theorem 1, our first main result, answers affirmatively to this question.

Instead of working with general Plücker varieties, we introduce a notion of a $\wedge$-variety (Definition 6), which generalizes and at the same time simplifies the notion of a Plücker variety. Our main result (Theorem 1) states the topological noetherianity result for the class of $\wedge$-varieties.

The algebraic noetherianity result (a direct analog of the result for $\Delta$-modules) for $\wedge$-varieties does not hold. It fails even for the Grassmannian $\wedge$-variety $Gr$: there are infinitely many (combinatorially different) types of Plücker relations. The paper of R. Laudone [23] follows this direction.

Our work fits in the broader context of noetherianity results for large algebraic structures, e.g., [6, 4, 7, 5]. Namely, the topological noetherianity for algebraic representations of infinite classical groups is proven (originally for $GL_\infty$ in [3], and for other groups in [8]). The dual (unrestricted) infinite wedge is a nontrivial inverse limit of algebraic $GL_\infty$-representations. It is one of the most interesting and used in mathematics literature space among non-algebraic representations of $GL_\infty$. Theorem 2, our second main result, states the $GL_\infty$-equivariant noetherianity of the infinite wedge.

After completing this work, we learned that A. Bik, J. Draisma, and R. Eggermont had obtained similar results in unpublished work. In a forthcoming paper with A. Bik, J. Draisma, and R. Eggermont we consider applications of the main results of this paper to sequences of varieties with contraction morphisms only.

1.1. Main results. The following theorem is our first main result.

Theorem 1. For any $\wedge$-variety $X$ there exists a $p_0 \in \mathbb{Z}_{\geq 0}$ and a finite dimensional vector space $V_0$ such that $X$ is defined set-theoretically by pullbacks of equations for $X_{p_0}(V_0)$. In particular, the degrees of equations defining the varieties $X_p(V)$ are bounded.

The crucial statement for the proof is our second main result:

Theorem 2. The dual unrestricted infinite wedge $(\bigwedge_\infty V_\infty)^*$ is $GL_\infty$-equivariant noetherian. Unfolding, any descending chain of closed $GL_\infty$-invariant subsets of $(\bigwedge_\infty V_\infty)^*$ stabilizes.

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1.2. Plan of the proof. Following the logic of [1], we prove Theorem 1 via the following steps:

1. We define finite dimensional hyper-Pfaffian varieties $\text{HPf}^{(m,l)}$ and their duals $\text{HPf}^{(r,s),*}$. Collecting these varieties, we form the Plücker varieties $\text{HPf}^{(m,l)}$, $\text{HPf}^{(r,s),*}$, and the limiting forms for them $\text{HPf}^{(m,l)}$, $\text{HPf}^{(r,s),*}$. Taking into account both constructions, we build up a two-sided hyper-Pfaffian Plücker variety $\text{HPf}^{(m,k),(r,s)}$ and its limiting form $\text{HPf}^{(m,k),(r,s)}$.

2. We prove (Theorem 31) that for any proper Plücker variety $X$ there exist pairs $(m,l)$, $(r,s)$ such that $X_{\infty} \subseteq \text{HPf}^{(m,l),(r,s)}$.

3. We prove (Theorem 36) that for any pairs $(m,l)$, $(r,s)$ the limiting variety $\text{HPf}^{(m,l),(r,s)}$ is $GL_{\infty}$-noetherian.

4. Finally, Theorem 2 follows from Theorem 31, Proposition 33, and Theorem 36; Theorem 1 follows from this result immediately (Corollary 42).

1.3. Notation and conventions. In what follows, we work over a fixed field $K$ of characteristic 0. Besides algebraic varieties, more often than not we consider affine cones over projective varieties. For instance, by the Grassmannian $\text{Gr}(2,4)$ we mean the affine cone over the actual projective variety inside the vector space $\mathbb{A}^4$.

The vector space generated by set of vectors $\{e_i\}_{i \in I}$ we denote by $\langle e_i \rangle_{i \in I}$ or by $K^I$ when we want to emphasize the generating set $I$ only. Also, for a vector space $V$ we denote by $[V]$ the given set of basis vectors, when this makes sense, e.g., $[K^n] = [n]$ and, more generally, $[K^I] = I$.

By $\mathbb{Z}^*$ we denote the set $\mathbb{Z} \setminus \{0\}$. For pairs of natural numbers we use the following partial order: $(n,p) < (N,P)$ if and only if $n < N$, $p < P$ or $n \leq N$, $p < P$.

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2. $\Lambda$-varieties and equivariant noetherianity

2.1. Infinite wedge. Consider for any $n,p \in \mathbb{Z}_{\geq 0}$ a set $[n,p] = \{-n, \cdots, -1, 1, \cdots, p\}$ and a vector space $V_{n,p} = K[n,p] = \langle e_{-n}, \cdots, e_{-1}, e_1, \cdots, e_p \rangle$ of dimension $n + p$. Then by $V_{\infty}$ we denote the direct limit of $V_{n,p}$ with natural inclusions:

$$V_{\infty} := \lim_{n,p} V_{n,p} = \langle \ldots, e_{-2}, e_{-1}, e_1, e_2, \ldots \rangle_K.$$ 

Let $\langle , \rangle : V_{\infty} \times V_{\infty} \to K$ be a bilinear form on $V_{\infty}$ given by $\langle e_i, e_j \rangle = \delta_{i,j}$. The restrictions of the form $\langle , \rangle$ to the spaces $V_{n,p}$ identify the dual space $V_{n,p}^*$ with $V_{p,n}$.

Next, we consider the exterior powers $\Lambda^k V_{n,p}$. We denote the basis vectors for this space by $e_j = e_{i_1} \wedge \cdots \wedge e_{i_p}$ where $I = \{i_1 < \cdots < i_p\} \subset [n,p]$. The following maps between the exterior powers are of a particular interest for us:

- $i_{n,p} : \Lambda^k V_{n,p} \hookrightarrow \Lambda^k V_{n+1,p}$ is induced by a natural inclusion $V_{n,p} \hookrightarrow V_{n+1,p}$;
- $j_{n,p} : \Lambda^k V_{n,p} \twoheadrightarrow \Lambda^{k+1} V_{n+1,p}$ is multiplication by $e_{p+1}$, i.e., $j_{n,p}(\omega) = \omega \wedge e_{p+1}$;
- $i_{n,p}^\dagger : \Lambda^{k+1} V_{n+1,p} \to \Lambda^k V_{n,p}$ is the dual map to $j_{n,p}$, i.e., $i_{n,p}^\dagger (j_{p,n}(\omega)) = (i_{n,p}^\dagger \omega)$;
- $j_{n,p}^\dagger : \Lambda^{k+1} V_{n+1,p} \to \Lambda^k V_{n,p}$ is the dual map to $i_{n,p}$.

Explicit formulas for the maps $i_{n,p}^\dagger$ and $j_{n,p}^\dagger$ are presented in the proof of Lemma 25.

We note that $j_{n+1,p} \circ i_{n,p} = i_{n,p+1} \circ j_{n,p}$ and $j_{n,p}^\dagger \circ i_{n,p+1}^\dagger = i_{n,p}^\dagger \circ j_{n+1,p}^\dagger$ or, reformulating, the two diagrams – the first one with $i_{n,p}$ and $j_{n,p}$ maps, and the second one with the corresponding $\dagger$-maps – are commutative, see Fig.

Definition 3. The infinite wedge $\Lambda^\infty V_{\infty}$ is the direct limit of the spaces $\Lambda^k V_{n,p}$ with respect to the transition maps $i_{n,p}$ and $j_{n,p}$:

$$\Lambda^\infty V_{\infty} := \lim_{n,p} \Lambda^k V_{n,p}.$$ 

Analogously, we define the (restricted) dual infinite wedge $\Lambda^\infty V_{\infty}^*$:

$$\Lambda^\infty V_{\infty}^* := \lim_{n,p} \Lambda^k V_{n,p}^* = \lim_{n,p} \Lambda^k V_{p,n}.$$

It is isomorphic to the infinite wedge $\Lambda^\infty V_{\infty}$.

The unrestricted dual infinite wedge $(\Lambda^\infty V_{\infty})^*$ is an uncountable dimensional vector space that is defined with the inverse limit of the spaces $\Lambda^k V_{n,p}$ with respect to the transition maps $i_{n,p}^\dagger$ and $j_{n,p}$:

$$(\Lambda^\infty V_{\infty})^* := \lim_{n,p} \Lambda^k V_{n,p}.$$
Definition 5. A Plücker variety is a sequence $(X_p)_{p \in \mathbb{Z}_{\geq 0}}$ of functors from the category $\text{Vect}_K$ to the category of varieties $\text{Var}_K$ satisfying the following axioms:

1. For all vector spaces $V$ and for all $p \in \mathbb{Z}_{\geq 0}$, the variety $X_p(V)$ is a closed subvariety of $\Lambda^p V$.
2. For all $p \in \mathbb{Z}_{\geq 0}$ and for all linear maps $\phi : V \to W$, the map $X_p(\phi) : X_p(V) \to X_p(W)$ coincides with the restriction of $\Lambda^p \phi$.
3. If $V$ is a vector space of dimension $n + p$ with $n, p \in \mathbb{Z}_{\geq 0}$, and $\star : \Lambda^n V \to \Lambda^n V^*$ is the Hodge dual, then the transformation $\star$ maps $X_p(V)$ into $X_n(V^*)$.

Plücker varieties form a category in a natural way; we denote it by $\text{PlVar}_K$.

Next we introduce the more general notion of $\Lambda$-variety.

Definition 6. A $\Lambda$-variety $X$ is a set of closed varieties $X = \{X_{n,p} \subseteq \Lambda^p V_{n,p} \}_{n,p \in \mathbb{Z}_{\geq 0}}$ satisfying the following conditions:

(i) For all $p, n \in \mathbb{Z}_{\geq 0}$, the maps $i_{n,p}$ and $j_{n,p}$ induce injections of $X_{n,p}$ into $X_{n+1,p}$ and $X_{n,p}$ into $X_{n,p+1}$ respectively;
(ii) For all $p, n \in \mathbb{Z}_{\geq 0}$, the maps $i_{n,p}^\dagger$ and $j_{n,p}^\dagger$ induce surjections of $X_{n+1,p}$ onto $X_{n,p}$ and $X_{n,p+1}$ onto $X_{n,p}$ respectively;
(iii) For all $p, n \in \mathbb{Z}_{\geq 0}$, the variety $X_{n,p}$ is $\text{GL}_{n,p}$-invariant.

$\Lambda$-varieties form a category in a natural way; we denote it by $\Lambda\text{Var}_K$. 

Remark 4. The space $\Lambda^\infty V_{\infty}$ was introduced in mathematical physics by Jimbo et al. [14]. We call it the infinite wedge following [1], but this space has many other names: semi-infinite wedge [22], half-infinite wedge [20], charge-0 fermionic Fock space [17] and others.

By $\text{GL}_{n,p}$ and $\text{GL}_\infty$ we denote the groups $\text{GL}(V_{n,p})$ and $\bigcup_{n,p \in \mathbb{Z}_{\geq 0}} \text{GL}(V_{n,p})$ respectively. This group acts on the spaces $V_{\infty}, \Lambda^\infty V_{\infty}, \Lambda^\infty V_{\infty}^*$, and $(\Lambda^\infty V_{\infty})^*$ combining actions of $\text{GL}_{n,p}$ on $V_{n,p}$ for all $n, p$.

From the definition we can see that basis elements of the space $\Lambda^\infty V_{\infty}$ have the form

\[ e_{i_1, i_2, \ldots} := e_{i_1} \wedge e_{i_2} \wedge \ldots \text{ with } i_k = k \text{ for } k \gg 0. \]

The space $\Lambda^\infty V_{\infty}$ is a proper subspace of $(\Lambda^\infty V_{\infty})^*$. Indeed, for any element $w = (w_{n,p})$ of the former space, because of the identity $i_{n,p}^\dagger \circ i_{n,p} = j_{n,p}^\dagger \circ j_{n,p} = \text{id}_{V_{n,p}}$, the same set $(w_{n,p})$ is an element of the latter space. Moreover, due to the dimension count, the spaces are not equal. For instance, the vector $v = (v_{n,p}) \in (\Lambda^\infty V_{\infty})^*$ with $v_{n,p} = (e_1 + e_2) \wedge (e_3 + e_4) \wedge \ldots$ for any $n \gg 0$ do not belong to the space $\Lambda^\infty V_{\infty}$:

\[ v = (e_1 + e_2) \wedge (e_2 + e_3) \wedge (e_3 + e_4) \wedge \ldots \in (\Lambda^\infty V_{\infty})^* \setminus \Lambda^\infty V_{\infty}. \]
Remark 7. In the definition, injectivity and surjectivity conditions are automatically satisfied. Indeed, \(i_{n,p} \) and \(j_{n,p} \) are injective, so are their restrictions to \(X_{n,p} \). Also, \(i_{n,p}^\dagger \circ i_{n,p} = \text{id}_{\bigwedge^p V_{n,p}} \) and \(j_{n,p}^\dagger \circ j_{n,p} = \text{id}_{\bigwedge^p V_{n,p}} \), so the restrictions of \(i_{n,p}^\dagger \) and \(j_{n,p}^\dagger \) to the components of \(\Lambda\)-variety are surjective.

Proposition 8. Every Plücker variety is naturally a \(\Lambda\)-variety, and \(\text{PluVar}_K \) is a subcategory of \(\Lambda\text{Var}_K \):

\[
\text{PluVar}_K \hookrightarrow \Lambda\text{Var}_K.
\]

Proof. For an arbitrary Plücker variety \(X = (X_p)\) we consider the set \(\{X_{n,p}\} \) where \(X_{n,p} := X_p(V_{n,p})\). We prove that this set is a \(\Lambda\)-variety.

The conditions for the maps \(i_{n,p} \) and \(i_{n,p}^\dagger \) follows from condition (2) for Plücker varieties. The conditions for the maps \(j_{n,p} \) and \(j_{n,p}^\dagger \) follows from conditions (2) and (3) for Plücker varieties. In detail, Lemmata 2.3 and 2.4 of [11] and Remark [7] above give the proof. The GL-invariance condition follows from condition (2) for Plücker varieties. \(\square\)

Remark 9. The subcategory \(\text{PluVar}_K \) is not a full subcategory in \(\Lambda\text{Var}_K \).

Pfaffian and hyper-Pfaffian varieties give explicit examples of \(\Lambda\)-varieties which are not Plücker, see Section 5.3.

Definition 10. The limiting variety \(X_\infty \) for a \(\Lambda\)-variety \(X\) is the inverse limit of the varieties \(X_{n,p} \) with respect to the maps \(i_{n,p}^\dagger \) and \(j_{n,p}^\dagger \):

\[
X_\infty := \lim_{n,p} X_{n,p}.
\]

\(X_\infty \) is an inverse limit of affine schemes, so it is a subscheme of the affine space \((\bigwedge^\infty V_\infty)^*\). The \(\text{GL}_\infty\)-action on \(X_\infty \) is inherited from the \(X_{n,p}\)'s; it coincides with the restriction of the \(\text{GL}_\infty\)-action on \((\bigwedge^\infty V_\infty)^*\). The procedure of taking a limiting variety is a functor \((\_)_\infty : \Lambda\text{Var}_K \rightarrow \text{Var}_K \).

2.3. Examples of Plücker and \(\Lambda\)-varieties. We give several examples and constructions of Plücker and \(\Lambda\)-varieties:

1. First examples of Plücker varieties are the trivial ones: \(X_{n,p} = \emptyset\) and \(X_{n,p} = \{0\}\) for all \(n, p \in \mathbb{Z}_{\geq 0}\). An opposite example is the ambient Plücker variety with \(X_{n,p} = N^p V_{n,p}, n, p \in \mathbb{Z}_{\geq 0}\). The limiting variety of the ambient Plücker variety coincides with the unrestricted dual infinite wedge \((\bigwedge^\infty V_\infty)^*\).

2. The most popular example of a Plücker variety is the Grassmann Plücker variety \(\text{Gr} \), defined by the sequence of Grassmannians \(\text{Gr}_{n,p} = \text{Gr}(p, n + p) \subseteq N^p V_{n,p}\), i.e., the sequence of Grassmannians form the Plücker variety \(\text{Gr} \) in a natural manner. The limiting variety \(\text{Gr}_\infty \) is classically known as the Sato Grassmannian \(\text{SGr} \) [13] [14] [15]. We attentively consider this example in Section 2.5.

3. The operations of intersection, union, join, and taking tangential variety are defined in the categories \(\text{PluVar}_K \) and \(\Lambda\text{Var}_K \). So, for example, if two Plücker varieties \(X, Y\) are given, we can consider the union \(X \cup Y\), the intersection \(X \cap Y\), the join \(X + Y\), and the tangential Plücker variety \(\tau X\).

Pfaffian and, more generally, hyper-Pfaffian \(\Lambda\)-varieties, except few exceptions (see Section 5.3), are examples of \(\Lambda\)-varieties which are not Plücker; for definitions see Sections 2.7 and 3 respectively.

2.4. Maximal \(\Lambda\)-varieties. In what follows we will need a notion of the maximal \(\Lambda\)-variety with some fixed component. This section is devoted to the construction of such varieties. In particular, the existence of such varieties follows.

Let \(X\) be a \(\text{GL}_{n,p}\)-invariant subvariety inside \(N^p V_{n,p}\). We can ask for a description of all \(\Lambda\)-varieties \(Y\) such that \(Y_{n,p} = X\).

What are the possibilities for \(Y_{n+1,p}\)? On the one hand, the \((n + 1, p)\)-component should contain the variety \(\text{GL}_{n+1,p} \cdot i_{n, p}(X)\); on the other hand, the component should be contained in the variety \((i_{n,p}^\dagger)^{-1}(X)\). The same logic is applicable for the component \(Y_{n,p+1}\) and \(j\)-maps.

Generalizing, we get the following statement.

Proposition 11. For any natural \(n, p\) and any \(\Lambda\)-variety \(Y\), we have the inclusions

a) \[ \bigcup_i i(Y_{n-1,p}) \subseteq Y_{n,p} \subseteq \bigcap_i (i^\dagger)^{-1}(Y_{n-1,p}), \]

where the right intersection runs over the orbit of the map \(i_{n-1,p}\) under the \(\text{GL}_{n-1,p} \times \text{GL}_{n,p}\)-action and the left union runs over the orbit of \(i_{n-1,p}^\dagger\) under the same group;
Indeed, for the equivalence we note that the case 

\[ \text{GL}_i \]  

where the right intersection runs over the orbit of the map \( j_{n,p-1} \) under the \( \text{GL}_{n,p-1} \times \text{GL}_{n,p} \)-action and the 

left union runs over the orbit of \( j_{1}^{\dagger} \) under the same group.

Explicit examples of situations when the left and right sides do not coincide are given in Section \[ 5 \].

**Definition 12.** We call a \( \Lambda \)-variety \( Y \) maximal with respect to the \( (n, p) \)-coordinate, or just \( (n, p) \)-maximal, if for all pairs \( (N, P) \triangleright (n, p) \) the right inclusions of Proposition \[ 11 \] are equalities, i.e.,

\[ Y_{N,P} = \bigcap_{i^\dagger}(i^\dagger)^{-1}(Y_{N-1,P}) \text{ and } Y_{N,P} = \bigcap_{j^\dagger}(j^\dagger)^{-1}(Y_{N,P-1}). \]

The equalities are consistent due to the commutativity for \( \dagger \)- and \( \dagger \)-maps.

Note that if the \( \Lambda \)-variety \( X \) is \( (n, p) \)-maximal then all components \( X_{N,P} \) with \( N > n \) or \( P > p \) are determined by the component \( X_{n,p} \): informally, all these components are given by (combinatorially) the same equations as the variety \( X_{n,p} \leq \Lambda V_{n,p} \).

The following examples of maximal \( \Lambda \)-varieties clarify this point of view:

1. \( (n, p) \)-maximal variety \( X \) with \( X_{n,p} = \Lambda^k V_{n,p} \) satisfies \( X_{N,P} = \Lambda^k V_{N,P} \) for \( (N, P) \triangleright (n, p) \);
2. \( (2, 2) \)-maximal variety with \( X_{2,2} = \text{Gr}(2, 4) \) coincides with the Grassmannian \( \Lambda \)-variety \( \text{Gr} \);
3. generalizing both previous examples, if the \( \Lambda \)-variety \( X \) is \( (n, p) \)-maximal and the variety \( X_{n,p} \) is defined by the ideal \( I_{n,p} \), then the ideals \( I_{N,P} \) for the varieties \( X_{N,P} \) have the following form

\[ I_{N,P} = \bigcap_{k^\dagger}(k^\dagger)^*(I_{n,p}), \]

where \( k^\dagger \) runs over all compositions of \( i^\dagger \)- and \( j^\dagger \)-maps of the form \( X_{N,P} \to X_{n,p} \).

Further examples are given by Pfaffians (Section \[ 27 \]) and hyper-Pfaffians (Section \[ 3 \]).

Analogously \( (n, p) \)-maximal varieties, we can define the \( (n, p) \)-minimal \( \Lambda \)-variety. However, we emphasize that the existence of such varieties needs a proof (one such the author knows from a private communication with Jan Draisma). Using the existence of minimal varieties, it can be proved that in general the \( j^\dagger \)-image of the \( \bigcap j^\dagger(j^\dagger)^{-1}(Y_{n,p-1}) \) coincides with \( Y_{n,p} \). We do not need these facts, so proofs are omitted.

**2.5. Grassmannian is equivariantly noetherian.** We are interested in \( \Lambda \)-varieties that are “equivariantly noetherian”. Before formulating this property rigorously, we give a motivating example of equivariantly noetherian Plücker variety.

It is known \[ 12 \] that an arbitrary Grassmannian \( \text{Gr}(k, n) \) can be described set-theoretically as an intersection of all possible pullbacks (of linear maps) of the smallest (nontrivial) Grassmannian \( \text{Gr}(2, 4) \):

\[ \text{Gr}(k, n) = \bigcap_{\phi: \mathbb{K}^n \to \mathbb{K}^4} \phi^{-1}(\text{Gr}(2, 4)). \]

So the Plücker variety \( \text{Gr} \) is defined set-theoretically by pullbacks of equations for \( \text{Gr}_{2,2} = \text{Gr}(2, 4) \). In other words, Theorem \[ 1 \] holds in the case \( X = \text{Gr} \) with \( p_0 = 2, V_0 = \mathbb{K}^4 \).

This description can be rephrased geometrically: a (projective) Grassmann variety coincides with a set of decomposable vectors. Equivalently, for any Grassmann variety \( \text{Gr}_{n,p} \), the set of its \( \mathbb{K} \)-points is a union of two \( \text{GL}_{n,p} \)-orbits:

\[ \text{Gr}_{n,p} = \text{GL}_{n,p} \cdot 0 \sqcup \text{GL}_{n,p} \cdot e_1 \wedge \cdots \wedge e_p. \]

Indeed, for the equivalence we note that the case \( (n, p) = (2, 2) \) is the smallest case when the orbit \( \text{GL}_{n,p} \cdot e_1 \wedge \cdots \wedge e_p \) does not coincide with the ambient space \( \Lambda^4 V_{n,p} \setminus \{0\} \). For a detailed proof see \[ 12 \].

The same property can be rephrased in the language of equations as follows. It is classically known \[ 16 \] p. 211 that equations for the Grassmannian \( \text{Gr}(2, n) \) (its \( \mathbb{K} \)-points) are given by pullbacks of the equation for the Grassmannian \( \text{Gr}(2, 4) \). In detail, \( \text{Gr}(2, 4) \) is given by the unique equation

\[ \text{pf}^{(2)}_{(1234)} := x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}, \]

where \( x_{ij} \) are coordinates on the space \( \Lambda^4 \mathbb{K}^4 \). This equation is known as the Pfaffian of degree 2. It spans a 1-dimensional subrepresentation of the \( \text{GL}_4 \)-representation \( \text{Sym}^2(\Lambda^2 \mathbb{K}^4) \) with highest weight \( (1, 1, 1, 1) \). In the case
of an arbitrary $\text{Gr}(2, n)$, the equations form a subrepresentation inside $\text{Sym}^2 (\Lambda^2 \mathbb{K}^n)$ with the same highest weight $(1, 1, 1, 1)$. This subrepresentation has as a basis the following set of polynomials:

$$\text{pf}^{(2)}_{ijkl} = x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} \text{ for all } \{ijkl\} \in \Lambda^4[n].$$

**Remark 13.** The latter property holds for $\mathbb{K}$-points only, it is wrong on an ideal-theoretic level: the decomposition of $\text{Sym}^2 (\Lambda^2 \mathbb{K}_\infty)$ into irreducible has infinitely many non-isomorphic summands, see [24 Example I.8.9(b)]. Related results for ideals of secant varieties of Grassmannians are in [23].

All three reformulations (set-theoretic, orbit-theoretic, and equation-theoretic) can be restated for the single variety $\text{Gr}_\infty$ instead of the set $\{\text{Gr}_{n,p}\}_{n,p \in \mathbb{Z}_{\geq 0}}$. For example, the Sato Grassmannian $\text{Gr}_\infty = \text{SGr}$ coincides with a union of the $\text{GL}_\infty$-orbit of the highest weight vector $e_1, e_2, \ldots, = e_1 \wedge e_2 \wedge e_3 \wedge \cdots \in \Lambda^\infty \mathbb{K}_\infty$ and the zero vector $0 \in \Lambda^\infty \mathbb{K}_\infty$; see [21] for details and connections to the KP hierarchy (originally appeared in [13]).

### 2.6. Bounded Plücker varieties

Describing the class of equivariantly noetherian Plücker and, more generally, $\Lambda$-varieties is a natural problem. Pursuing this question for Plücker varieties, Draisma–Eggermont introduced the following definition.

**Definition 14.** We call a Plücker variety $X$ **bounded** if there exists a finite dimensional vector space $W$ such that the variety $X_2(W)$ does not coincide with $\Lambda^2W$.

The next theorem is a central result of [11].

**Theorem 15.** (Equivariant noetherianity for bounded Plücker varieties) Let $X$ be a bounded Plücker variety. Any closed $\text{GL}_\infty$-stable subset $Z$ of $X_\infty$ is contained in a union of a finite number of $\text{GL}_\infty$-orbits. Reformulating, there exists a $p_0 \in \mathbb{Z}_{\geq 0}$ and a finite dimensional vector space $V_0$ such that $X$ is defined set-theoretically by pullbacks of equations for $X_{p_0}(V_0)$.

### 2.7. Pfaffian $\Lambda$-varieties

The following generalization of the Grassmann Plücker variety is essential for the proof of Draisma–Eggermont.

**Definition 16.** The **Pfaffian form** $\text{pf}^{(l)}_A$ of degree $l$ on a set $A$ of cardinality $2l$ is the polynomial form given by the following formula

$$\text{pf}^{(l)}_A := \sum_{I_1 \sqcup \cdots \sqcup I_l = A} \text{sgn}(I_1, \ldots, I_l)x_{I_1} \cdots x_{I_l},$$

where the summation runs over all unordered partitions $A = I_1 \sqcup \cdots \sqcup I_l$ into $2$-sets $I_i$ and the $\text{sgn}(I_1, \ldots, I_l)$ is a sign of the permutation given in the one-line form by $(I_1, \ldots, I_l)$.

The **Pfaffian variety** $\text{Pf}^{(l)}_B$ of degree $l$ on a set $B$ is the closed subvariety of $\Lambda^2B$ given by the Pfaffians $\text{pf}^{(l)}_A$ for all $A \subseteq B$ of cardinality $2l$:

$$\text{Pf}^{(l)}_B = \text{Pf}^{(l)}(\mathbb{K}^B) := \bigcap_{A \subseteq B, |A| = 2l} \text{pf}^{(l)}_A \cap \bigcap_{A \subseteq B, |A| = 2l} \{\text{pf}^{(l)}_A = 0\}.$$ 

Generally, $\text{Pf}^{(l)}(V)$ is a variety given by all degree-$l$ Pfaffian forms inside the vector space $\Lambda^2V$.

The **Pfaffian $\Lambda$-variety** $\text{Pf}^{(l)}$ of degree $l$ is the $(2l-2, 2)$-maximal $\Lambda$-variety with

$$\text{Pf}^{(l)}_{2l-2, 2} = \text{Pf}^{(l)}(\mathbb{K}^{2l-2, 2}) \subseteq \Lambda^2\mathbb{K}^{2l-2, 2}.$$ 

It will be proven (Theorem [23]) that for every $l$ such defined $\Lambda$-variety exists, and it is uniquely defined by the property above.

In the case $l = 2$, these definitions produce Grassmannians $\text{Pf}^{(2)}_{n,p} = \text{Gr}(p, n + p)$, see Section 2.5. So we have the equality of $\Lambda$-varieties $\text{Pf}^{(2)} = \text{Gr}$; in particular, the $\Lambda$-variety $\text{Pf}^{(2)}$ is Plücker. The general Pfaffian $\text{Pf}^{(l)}$ is also equivariantly noetherian [11], but for $l \geq 3$ they are not Plücker, see Proposition 16.

We note that our notation for Pfaffian varieties $\text{Pf}^{(l)}(V)$ differs from the one of Draisma–Eggermont by the shift of the argument by one: in our convention the Pfaffian forms of degree $l$ are zero on the Pfaffian variety of degree $l$.

As mentioned above, Grassmann variety is the set of all decomposable vectors in the corresponding exterior power of a vector space. Analogously, there exists a geometrical description for all Pfaffian varieties: a vector $v \in \Lambda^2V$ satisfies $v \in \text{Pf}^{(l+1)}(V)$ if and only if $v$ has rank at most $l$. In other words, the rank filtration on the second exterior power $\Lambda^2V$ coincides with the one given by Pfaffians:

$$\{\cdots \subseteq R^l(V) := \{v \in \Lambda^2V : \text{rk}(v) \leq l\} \subseteq R^{l+1}(V) \subseteq \cdots \subseteq \Lambda^2V\} = \{\cdots \subseteq \text{Pf}^{(l)}(V) \subseteq \text{Pf}^{(l+1)}(V) \subseteq \cdots \subseteq \Lambda^2V\}.$$
Using this filtration, we can see that any bounded Plücker variety is contained in some Pfaffian $\mathcal{V}$-variety $\text{Pf}(l)$. Therefore any bounded Plücker variety is equivariantly noetherian.

The goal of this paper is to prove that the Draisma–Eggermont result (equivariant noetherianity) actually holds in its most general form. Conceptually, we describe the analogous filtration on an arbitrary exterior power $\Lambda(V)$ via the so-called hyper-Pfaffian varieties. Description of the filtration in general situation is presented in Section 3.5

3. Hyper-Pfaffians

In this section we recall a natural generalization of Pfaffians, the so-called hyper-Pfaffian varieties [11, Definition 3.4]. We also present here some useful properties of these varieties and give an explicit example of $\Lambda$-variety which is not Plücker.

3.1. Definitions.

Definition 17. The hyper-Pfaffian form $\text{hpf}_{A}^{(m,l)}$ of degree $l$ and width $m = 2m_1$ on a set $A$ of cardinality $|A| = ml$ is the polynomial form in $\text{Sym}(\Lambda^{m} \mathbb{K}^{A})$ given by the formula

$$\text{hpf}_{A}^{(m,l)}(x) := \sum_{I_{1} \sqcup \cdots \sqcup I_{l} = A} \text{sgn}(I_{1}, \ldots, I_{l}) x_{I_{1}} \cdots x_{I_{l}},$$

where the summation runs over all unordered partitions $A = I_{1} \sqcup \cdots \sqcup I_{l}$ into $m$-sets $I_{i}$, the $\text{sgn}(I_{1}, \ldots, I_{l})$ is a sign of the permutation given in the one-line form by $(I_{1}, \ldots, I_{l})$, and $x$ denotes the set of variables $\{x_{I} \mid I \in \Lambda(A)\}$. We will denote the corresponding multilinear form depending on sets of variables $x^{(1)}, \ldots, x^{(l)}$ by $\text{hpf}_{A}^{(m,l)}(x^{(1)}, \ldots, x^{(l)})$.

The hyper-Pfaffian variety $\text{HPf}_{B}^{(m,l)}$ of degree $l$ and width $m$ on a set $B$ is the closed subvariety of $\Lambda^{m} \mathbb{K}^{B}$ given by the hyper-Pfaffians $\text{hpf}_{A}^{(m,l)}$ for all $A \subseteq B$ of cardinality $|A| = ml$:

$$\text{HPf}_{B}^{(m,l)} = \bigcap_{A \subseteq B, |A| = ml} \text{HPf}_{A}^{(m,l)}(\mathbb{K}^{B}) := \bigcap_{A \subseteq B, |A| = ml} \{\text{hpf}_{A}^{(m,l)} = 0\}.$$

We define $\text{HPf}_{B}^{(m,l)}(V) \subseteq \Lambda^{m} V$ as a variety given by all degree-$l$ hyper-Pfaffian forms inside the vector space $\Lambda^{m} V$.

The hyper-Pfaffian $\Lambda$-variety $\text{HPf}_{m,l}^{(m,l)}$ of degree $l$ and width $m$ is the $(m(l-1), m)$-maximal $\Lambda$-variety with $\text{HPf}_{m(l-1),m}^{(m,l)} = \text{HPf}_{m(l-1),m}^{(m,l)}(V_{m(l-1),m}) \subseteq \Lambda^{m} V_{m(l-1),m}$.

Remark 18. We give a couple of remarks about the definitions.

1. In the case $m = 2$, we get the Pfaffians:

$$\text{hpf}_{A}^{(2,l)} = \text{pf}_{A}^{(l)}, \text{HPf}_{B}^{(2,l)}(V) = \text{Pf}(l)(V), \text{and} \quad \text{HPf}_{l}^{(2,l)} = \text{Pf}(l).$$

2. For odd natural number $m$ we can define hyper-Pfaffian forms $\text{hpf}_{A}^{(m,l)}$, but, because of the sign of the monomials, such forms are identically zero (or not $\text{GL}$-invariant if we take only half of the monomials). However, this definition gives correctly defined skew-symmetric forms $\text{hpf}_{A}^{(m,l)} \subseteq \Lambda(V)$ for instance, for $m = 1$ we get the volume form:

$$\text{hpf}_{A}^{(1,n)}((x_{i})_{1 \leq i \leq n}) = x_{1} \wedge \cdots \wedge x_{n}.$$

3. Generalizing the two previous remarks, the hyper-Pfaffian form $\text{hpf}_{A}^{(m,l)}$ stands for an $m$-th root of the determinant. Indeed, on a set $\Lambda^{m} \mathbb{Z}^{ml}$ the following equality holds

$$\left(\text{hpf}_{A}^{(m,l)}\right)^{m} = \det : \Lambda^{m} \mathbb{Z}^{ml} \to \Lambda^{ml} \mathbb{Z}^{ml}.$$

For the main structural result on a general $\Lambda$-variety (Theorem 3.1) we will need the dual notion for a hyper-Pfaffian.

Definition 19. The dual hyper-Pfaffian variety $\text{HPf}_{(r,s),*}^{(r,s)}$ is defined as the Hodge-dual of the $\text{HPf}_{(r,s)}^{(r,s)}$ on a dual vector space:

$$\text{HPf}_{(r,s),*}^{(r,s)}(V) := \ast \text{HPf}_{(r,s)}^{(r,s)}(V^{*}) \subseteq \Lambda^{\dim V - r} V.$$
The following result shows fundamental role of all (not only Pfaffian, but) hyper-Pfaffian forms for the exterior algebra.

**Proposition 20.** Hyper-Pfaffian forms give the structure constants of the exterior algebra. Explicitly, if \( V \) is a finite dimensional \( \mathbb{K} \)-vector space, then for any \( v_1, \ldots, v_l \in \bigwedge^n V \) we have the equality
\[
v_1 \wedge \cdots \wedge v_l = \sum_A \mathsf{hpf}^{(m,l)}_A(v_1, \ldots, v_l)e_A,
\]
where \( \{ e_A := \bigwedge_{i \in A} e_i, A \in \bigwedge^m [V] \} \) is a basis for \( \bigwedge^n V \).

**Proof.** If \( v_i = \sum_{I \in \bigwedge^n [V]} a_{i,I}e_I \), then \( v_1 \wedge \cdots \wedge v_l \) is equal to
\[
\sum_{I_1, \ldots, I_l} a_{1,I_1} \cdots a_{l,I_l}e_{I_1} \wedge \cdots \wedge e_{I_l} = \sum_{A \in \bigwedge^m [V]} \left( \sum_{I_1 \sqcup \cdots \sqcup I_l = A} a_{1,I_1} \cdots a_{l,I_l} \text{sgn}(I_1, \ldots, I_l) \right) e_A.
\]
This expression coincides with the sum \( \sum_A \mathsf{hpf}^{(m,l)}_A(v_1, \ldots, v_l)e_A \). \( \square \)

**Corollary 21.** The hyper-Pfaffian variety \( \mathsf{HPf}^{(m,l)}(V) \) coincides with the set of nilpotency degree-\( l \) vectors in \( \bigwedge^n V \):
\[
\mathsf{HPf}^{(m,l)}(V) = \{ v \in \bigwedge^n V : v^{m-l} = 0 \}.
\]

Analogously, \( \mathsf{HPf}^{(r,s)}(V) = \{ v \in \bigwedge^n V : (\star v)^{n-r} = 0 \} \).

**Proof.** Proposition 20 for \( v_1 = \cdots = v_l = v \) proves the statement. \( \square \)

**Remark 22.** Morally, Proposition 20 is a restatement of the Grassmann–Berezin (fermionic) calculus [18, 19] in a coordinate form with use of the hyper-Pfaffian forms.

### 3.2. Hyper-Pfaffian \( \bigwedge \)-varieties are well-defined.

The following theorem provides us with an explicit description of the components \( \mathsf{HPf}^{(m,l)}_{n,p} \) of hyper-Pfaffian \( \bigwedge \)-varieties. The construction is consistent with definition 17 (as well as definition 13 for Pfaffian \( \bigwedge \)-varieties); the theorem proves the existence and uniqueness of the hyper-Pfaffian \( \bigwedge \)-varieties.

**Theorem 23.** (Set-theoretical description of \( \mathsf{HPf}^{(m,l)}(V) \)) We have the following explicit description of the components \( \mathsf{HPf}^{(m,l)}_{n,p} \subseteq \bigwedge^n V_{n,p} \):

- If \( p < m \), then \( \mathsf{HPf}^{(m,l)}_{n,p} = \bigwedge^p V_{n,p} \).
- If \( p = m \), then \( \mathsf{HPf}^{(m,l)}_{n,m} \) is given by the equations \( \mathsf{hpf}^{(m,l)}_I \) for all \( m \cdot l \)-subsets \( I \) of the set \([n,m]\):

\[
\mathsf{HPf}^{(m,l)}_{n,m} = \begin{cases} 
\bigwedge^n V_{n,m} & \text{for } n < m(l-1), \\
\bigcap_{A \subseteq [n,m], |A| = mL} \{ \mathsf{hpf}^{(m,l)}_A = 0 \} & \text{for } n \geq m(l-1).
\end{cases}
\]

In particular, \( \mathsf{HPf}^{(m,l-1)}_{m(1)-1,m} = \{ \mathsf{hpf}^{(m,l-1)}_{m(1)-1,m} = 0 \} \) is a hyper surface in \( \bigwedge^m V_{m(1)-1,m} \).

- If \( p > m \), then \( \mathsf{HPf}^{(m,l)}_{n,p} = \mathsf{HPf}^{(m,l)}(V_{n,p}) \) is an intersection of pullbacks of the hyper-Pfaffians \( \mathsf{HPf}^{(m,l)}_{n,m} \).

Explicitly,
\[
\mathsf{HPf}^{(m,l)}_{n,p} = \begin{cases} 
\bigwedge^p V_{n,p} & \text{for } n < m(l-1), \\
\bigcap_{A \subseteq [n,p], |A| = mL} \{ \mathsf{hpf}^{(m,l)}_A = 0 \} & \text{for } n \geq m(l-1),
\end{cases}
\]

where
\[ \mathsf{hpf}^{(m,l)}_A = \sum_{I_1 \sqcup \cdots \sqcup I_l = A} \text{sgn}(I_1, \ldots, I_l)x_{I_1} \cdots x_{I_l} \text{ and } J = [n,p] \setminus A. \]

**Remark 24.** Classically known that \( \mathbb{K} \)-points of the Grassmannian \( \mathsf{Pf}^{(2)}_{n,p} = \mathsf{Gr}(p, n + p) \) can be described by Plücker relations of the form
\[
\sum_{j \in J} \text{sgn}(j, I)x_{I_1 \sqcup j}x_{J_1 \setminus j} \text{ for all } I \in \bigwedge^{n-1}[n+p], J \in \bigwedge^{+1}[n+p].
\]

However, the ideal generated by these relations (inside \( \mathsf{Sym}(\bigwedge^p \mathbb{Z}^{2p}) \)) is not radical, i.e., the set of these relations is not sufficient to generate \( \mathsf{Gr}(p, n + p) \) as a scheme over \( \mathsf{Spec}(\mathbb{Z}) \).
The same situation happens for the hyper-Pfaffian varieties. The set of equations in Theorem 23 defines only \( \mathbb{K} \)-points of the varieties. A description of ideals for the hyper-Pfaffians is a non-trivial problem which is related to an explicit description of the plethysms \( \text{Sym}^k \circ \mathbb{L} \).

We recall from Example (3) in Section 2.2.4 that for any \((n, p)\)-maximal \( \mathbb{L} \)-variety \( X \) the ideals defining components \( X_{N, P} \) with \((N, P) \triangleright (n, p)\) are pullbacks of the ideal for \( X_{n, p} \). Therefore the statement of the theorem for \((n, p) \triangleright (m(l - 1), m)\) is clear if we prove the rest.

To get other components of the hyper-Pfaffian variety \( \text{HPf}^{(m,l)} \) we need some computations. Instead of a bulky technical proof in a general case, we exemplify internal strings of the proof technique via the elementary example \( \text{HPf}^{(4,2)} \).

**Lemma 25.** Theorem 23 holds for \( \text{HPf}^{(4,2)} \). Explicitly, for any \((n, p) \not\in (4, 4)\) we have the equality \( \text{HPf}^{(4,2)}_{n, p} = \mathbb{L} V_{n, p} \).

The following elementary observation is extremely useful for the proof of the lemma.

**Observation 26.** Assume that for a vector \( v = \sum v_I e_I \in \mathbb{L}^4 V_{n,4} \) there exists \( j \in [n, 4] \) such that \( v_I = 0 \) if \( j \notin I \). Then \( v \in \text{HPf}^{(4,2)}(V_{n,4}) = \bigcap_{A \subseteq [n,4], |A|=8} \{ \text{hpf}^{(4,2)}_A = 0 \} \).

**Proof.** The observation follows from the fact that for \( \text{hpf}^{(4,2)}_A(v) \) to be nontrivial, we need at least two nonzero coordinates of \( v \) with non-intersecting indices (this is wrong under the assumption for the element \( j \)).

**Proof of Lemma 25.** We begin with the case \( n = 4, p = 3 \). The map \( j_{4,3}^\dagger : \mathbb{L}^4 V_{4,4} \to \mathbb{L}^4 V_{4,3} \) in the coordinate form is given by

\[
e_I \mapsto \begin{cases} \frac{(-1)^{\text{sgn}(I,4)}e_I}{4} & \text{if } 4 \in I; \\ 0 & \text{otherwise}. \end{cases}
\]

For an arbitrary point \((a, j) \in \mathbb{L}^4 V_{4,3} \), the point \((A_I) \in \mathbb{L}^4 V_{4,4} \), where

\[
A_I = \begin{cases} a, & \text{if } I = J \cup 4; \\ 0 & \text{otherwise}, \end{cases}
\]

belongs to the preimage of \((a, j) \) under \( j_{4,3}^\dagger \). Applying the observation for \( n = 4, v = (A_I) \) and \( j = 4 \), we see that \( (A_I) \) belongs to the hyper-Pfaffian \( \text{HPf}^{(4,2)}(V_{4,4}) \).

Therefore we proved that the variety \( \text{HPf}^{(4,2)}(V_{4,4}) \) maps surjectively to \( \mathbb{L}^4 V_{4,3} \), i.e., \( \text{HPf}^{(4,2)}_{4,3} = \mathbb{L}^4 V_{4,3} \). And, more generally, \( \text{HPf}^{(4,2)}_{4,p} = \mathbb{L}^4 V_{4,p} \) for \( p \leq 3 \).

Consider the case of a general \( n \geq 4 \) and \( p = 3 \). Again, for any \((a, j) \in \mathbb{L}^4 V_{n,3} \) we can consider the point \((A_I) \in \mathbb{L}^4 V_{n,4} \) where

\[
A_I = \begin{cases} a, & \text{if } I = J \cup n; \\ 0 & \text{otherwise}. \end{cases}
\]

Applying the observation to the case \( v = (A_I) \) and \( j = n \), we can see that the point \((A_I) \) belongs to the variety \( \text{HPf}^{(4,2)}(V_{n,4}) = \text{HPf}^{(4,2)}_{n,4} \). So the projection \( \text{HPf}^{(4,2)}_{n,4} \to \mathbb{L}^4 V_{n,3} \) is surjective, i.e., \( \text{HPf}^{(4,2)}_{n,p} = \mathbb{L}^4 V_{n,p} \) for any \( n \geq 0 \) and \( p \leq 3 \).

We note that the map \( i_{3,4}^\dagger : \mathbb{L}^4 V_{4,4} \to \mathbb{L}^4 V_{3,4} \) in the coordinate form is given by

\[
e_I \mapsto \begin{cases} e_I & \text{if } -4 \notin I; \\ 0 & \text{otherwise}. \end{cases}
\]

The same technique with the map \( i_{3,4}^\dagger \) as with \( j_{4,3}^\dagger \) gives the equality \( \text{HPf}^{(4,2)}_{3,4} = \mathbb{L}^4 V_{3,4} \). So \( \text{HPf}^{(4,2)}_{n,4} = \mathbb{L}^4 V_{n,4} \) for \( n \leq 3 \). Analogously, for any \((n, p)\) with \( n \leq 3 \) and \( p \geq 4 \) we get the desired equality.

**3.3. Hyper-Pfaffians and GL-orbits in exterior powers.** In this section we talk about GL-orbits in the exterior powers. This description is crucial for Theorem 31.

Consider the exterior power \( \mathbb{L}^\pi V \), where \( V \) is a \( \mathbb{K} \)-vector space of sufficiently high dimension (without loss of generality, we can assume \( V = \mathbb{K}^\infty \)). One can ask about classification of GL(V)-orbits in this space. Theorem 6 of [10] implies that such orbits are related to the decomposition type of tensors inside the exterior power \( \mathbb{L}^\pi V \).

In detail, let \( \pi = (\pi_1 \geq \pi_2 \geq \ldots) \) be a partition of \( p \) and \( k \) be a natural number. We call an element \( \omega \in \mathbb{L}^\pi V \) of type \( (\pi, k) \) if \( \omega \) is equal to a sum of \( k \) elements of the space \( \mathbb{L}^{\pi_1} V \wedge \mathbb{L}^{\pi_2} V \wedge \ldots \) and such \( k \) is minimal:

\[
(\mathbb{L}^\pi V)_{\pi,k} = \left\{ \omega \in \mathbb{L}^\pi V : \omega = \sum_{i=1}^{k} \omega_i, \omega_i \in \mathbb{L}^{\pi_1} V \wedge \mathbb{L}^{\pi_2} V \wedge \ldots, k \text{ is minimal} \right\}.
\]
For example, an element $\omega \in \Lambda^k V$ is of type $((p,0,\ldots),1)$ if and only if $\omega$ is decomposable. So the set of elements of type $((p,0,\ldots),1)$ coincides with the Grassmannian $\text{Gr}(p,V)$, which is $\text{GL}(V)$-invariant. The general statement is the following.

**Proposition 27.** [10] Any $\text{GL}(V)$-invariant algebraic subvariety of $\Lambda^k V$ is contained in a Zariski closure of one of the spaces $(\Lambda^V)_{\alpha,k}$, where $\pi = (\pi_1 \geq \pi_2 \geq \ldots)$ is a partition of $p$ and the latter space consists of all elements that can be represented as a sum of $k$ elements from $\Lambda^1 V \wedge \Lambda^2 V \wedge \ldots$.

**Remark 28.** The result of [10] is far more general: the analogous statement is true for $\text{GL}(V)$-invariant subvarieties of the space $S_\mu V$, where $S_\mu$ is the Schur functor for a partition $\mu$. The analogous result for Proposition 27 in the case $S_\mu = \text{Sym}^k$ is proved in [11].

**Definition 29.** We call a partition $\pi = (\pi_1 \geq \ldots \geq \pi_j)$ even if all of the parts $\pi_i$ are even numbers. Otherwise we call it odd.

**Proposition 30.** For any odd partition $\pi$ and any $k \in \mathbb{N}$, the set $(\Lambda^V)_{\pi,k}$ (as well as its Zariski closure) is contained in the varieties $\text{HPf}((p,k+1)V)$ and $\text{HPf}((\dim V-p,k+1),*V^*)$:

$$(\Lambda^V)_{\pi,k} \subseteq \text{HPf}((p,k+1)V) \text{ and } (\Lambda^V)_{\pi,k} \subseteq \text{HPf}((\dim V-p,k+1),*V^*)$$

**Proof.** We prove the first inclusion only; for the dual variety proof is analogous.

Without loss of generality, we assume that $\pi_1$ is odd. Then for any $\alpha \in (\Lambda^V)_{\pi,k}$ we know that $\alpha \wedge \alpha = 0$ Indeed, $\alpha \wedge \alpha = \alpha_1 \wedge \alpha_1 \wedge \ldots$ and $\alpha_1 \wedge \alpha_1 = 0$ because $\alpha_1$ belongs to $\Lambda^1 V$ with odd $\pi_1$. Therefore for any $\beta = \beta^{(1)} + \cdots + \beta^{(k)} \in (\Lambda^V)_{\pi,k}$ by the pigeonhole principle $\beta^{\wedge(k+1)} = 0$. This equality is equivalent to the desired inclusion due to the geometrical description of the hyper-Pfaffian (Corollary 21).

Hyper-Pfaffian and dual hyper-Pfaffian varieties are closed, so we also have the inclusions $(\Lambda^V)_{\pi,k} \subseteq \text{HPf}((p,k+1)V)$ and $(\Lambda^V)_{\pi,k} \subseteq \text{HPf}((\dim V-p,k+1),*V^*)$ for the Zariski closures.

3.4. Any proper Plücker variety is a subset of a two-sided hyper-Pfaffian.

**Theorem 31.** For any proper $\Lambda$-variety $X$ there exist natural numbers $m,l,r,s$ such that $X_\infty$ is a closed $\text{GL}_\infty$-stable subset of $\text{HPf}((m,l),(r,s))$

**Proof.** Closedness and $\text{GL}_\infty$-invariancy follow from the definition of $X_\infty$, so we prove the inclusion $X_\infty \subseteq \text{HPf}((m,l),(r,s))$ only. The idea for the proof is to combine Proposition 27 and 30.

Namely, let $(N,P)$ be a minimal pair such that $X_{N,P} \not\subseteq \Lambda^PV_{N,P}$ and $N,P$ are even. Then by Proposition 27 the variety $X_{N,P}$ is contained in Zariski closure of $(\Lambda^PV_N)_{(\Lambda,K)}$ for some $\Lambda$ and $K$. Then $X_{N,P+2}$ is contained in the variety $(\Lambda^PV_{N,P+2})_{\Lambda\cup\{1,1\},K}$, where $\Lambda\cup\{1\}$ is the partition formed out of $\Lambda$ and two 1’s:

The partition $\Lambda\cup\{1\}$ is odd (regardless of the $\Lambda$’s parity), so by Proposition 30 we get the chain of inclusions

$$X_{N,P+2} \subseteq (\Lambda^PV_{N,P+2})_{\Lambda\cup\{1\},K} \subseteq \text{HPf}((P,K+1)V_{N,P+2}).$$

This inclusion and the (maximality in the) definition of hyper-Pfaffian $\Lambda$-varieties imply the inclusion of the limiting varieties:

$$X_\infty \subseteq \text{HPf}((P,K+1)).$$

For the dual hyper-Pfaffians, we have $X_\infty \subseteq \text{HPf}((N,K+1),*V)$. Finally, $X_\infty \subseteq \text{HPf}((P,K+1),(N,K+1)).$
3.5. Hyper-Pfaffian filtration. First, we prove an elementary lemma about inclusions for hyper-Pfaffian varieties.

**Lemma 32.** Consider a vector space $V$. Then we have an inclusion
\[ \text{HPf}^{(m,l)}(V) \subseteq \text{HPf}^{(m,l+1)}(V). \]

**Proof.** For any set $A$ of cardinality $m(l + 1)$ we have the equality
\[ \text{hpf}_{A}^{(m,l+1)}(x) = \sum_{I \subset A} \pm x_{I} \cdot \text{hpf}_{A \setminus I}^{(m,l)}(x), \]
where the summation runs over all sets $I \in \bigwedge^{m}(A)$ containing the element 1. This equality proves the desired inclusion of varieties.

**Proposition 33.** For any finite dimensional vector space $V$ we have the exhaustive separable filtration
\[ \{0\} = \text{HPf}^{(m,1)}(V) \subseteq \cdots \subseteq \text{HPf}^{(m,l)}(V) \subseteq \text{HPf}^{(m,l+1)}(V) \subseteq \cdots \subseteq \bigwedge^{m}V. \]

**Proof.** The existence of the filtration follows from Lemma 32.

As mentioned in Remark 33, the hyper-Pfaffian form $\text{hpf}^{(m,l)}$ is an $m$-th root from the determinant $\det[ml]$ on the set of completely antiasymmetric tensors. So for a sufficiently big $L$ the variety $\text{HPf}^{(m,L)}(V)$ coincides with the ambient space $\bigwedge^{m}V$. The number $L$ depends on dimension of the space $V$. □

In light of the proposition, Theorem 31 can be reformulated as follows.

**Corollary 34.** Every proper $\mathbb{K}$-variety has a finite rank, i.e., for any proper $X$ there exists a natural number $N$ such that all $N \times N$ determinants are identically zero on $X_{\infty}$.

**Remark 35.** The filtration of Proposition 33 is exhaustive for a finite dimensional vector spaces $V$ only. Indeed, from Proposition 30 we see that for an odd partition $\pi$ the space $\bigwedge^{m}V_{\pi,k}$ is contained in the hyper-Pfaffian $\text{HPf}^{(m,k+1)}(V)$ as a scheme. However, we do not have such embeddings for even partitions. For instance, even for the simplest case of $\pi = (2, 2)$, none of the hyper-Pfaffian forms $\text{hpf}^{(4,1)}$ belongs to the ideal corresponding to the affine scheme $\bigwedge^{m}V_{(2,2),1}$.

4. Noetherianity proof

**Theorem 36.** For any natural numbers $m, k, r, s$, the variety $\text{HPf}_{\infty}^{(m,l),(r,s)}$ is $\text{GL}_{\infty}$-noetherian.

It’s curious that the proof of Draisma–Eggermont with minor changes is applicable in the general situation. Because of this, here we present a compact version of the proof (keeping the notation of 1).

Following 1, we use the general lemma.

**Lemma 37.** Let $\omega \in (\bigwedge^{m}V)_{\infty}$ and suppose there exist elements $g_{1}, g_{2} \in \text{GL}_{\infty}$ such that $F_{1}(g_{1}\omega) \neq 0$ and $F_{2}(g_{2}\omega) \neq 0$ for some polynomial functions $F_{1,2}$ on $(\bigwedge^{m}V)_{\infty}$. Then there exists an element $g \in \text{GL}_{\infty}$ such that $F_{1}(g\omega) \neq 0$ and $F_{2}(g\omega) \neq 0$. Moreover, the element $g$ can be found in the form $g = \lambda g_{1} + \mu g_{2}$ for some $\lambda, \mu \in K$.

**Proof.** Consider the function $g := g(\lambda, \mu) = \lambda g_{1} + \mu g_{2}$ and the set
\[ F := \{(\lambda, \mu) \in K^{2}: g \notin \text{GL}_{\infty}, F_{1}(g\omega) = 0, \text{ and } F_{2}(g\omega) = 0\}. \]

The set $F$ is Zariski-closed by definition. The polynomial $F_{1}(g\omega)$ in variables $\lambda$ and $\mu$ has a coefficient $F_{1}(g_{1}\omega)$ for a highest degree monomial containing the variable $\lambda$ only. This coefficient in nonzero by the assumption, therefore the polynomial $F_{1}(g\omega)$ is also nonzero. The same logic for the polynomial $F_{2}(g\omega)$ and the variable $\mu$ proves nonzeroness of this polynomial. Therefore $F$ is a proper Zariski-closed subset of $K^{2}$.

Any point $(\lambda, \mu) \notin F$ gives the desired element $g \in \text{GL}_{\infty}$. □

**Proof of Theorem 36.** We fix numbers $m$ and $r$ and proceed by induction on $l$ and $s$.

The base of the induction, the case $l = 1$ or $s = 1$, is clear. Indeed, the $\text{GL}_{\infty}$-orbit of the polynomial $\text{hpf}_{[m]}^{(m,l)}(x) = x_{12\ldots m}$ contains all coordinate variables $x_{I}$, $I \in \bigwedge^{m}N$. The intersection of these polynomials is the point 0.

From now on we assume that $l, s > 1$.

**Step 1: decomposition and the variety Z.** By Lemma 32 we have the decomposition
\[ \text{HPf}_{\infty}^{(m,l+1),(r,s+1)} = \text{HPf}_{\infty}^{(m,l+1),(r,s)} \cup \text{HPf}_{\infty}^{(m,l),(r,s+1)} \cup Z'_{m,r}, \]
where $Z' := Z'_{m,r}$ is the set of all elements $\omega$ such that there exist $g_{1}, g_{2} \in \text{GL}_{\infty}$ with $\text{hpf}_{[m]}^{(m,l)}(g_{1}\omega) \neq 0$ and $\text{hpf}_{[r,s]}^{(r,s)}(g_{2}\omega) \neq 0$.

\[ Z' := \{\omega \in \text{HPf}_{\infty}^{(m,l+1),(r,s+1)}: \text{hpf}_{[m]}^{(m,l)}(g_{1}\omega) \neq 0 \text{ and } \text{hpf}_{[r,s]}^{(r,s)}(g_{2}\omega) \neq 0 \text{ for some } g_{1}, g_{2} \in \text{GL}_{\infty}\}. \]
Applying Lemma 37 for the case $F_1 = \text{hpf}^{(m,l)}$ and $F_2 = \text{hpf}^{(r,s),*}$, we get that $Z' = \text{GL}_\infty \cdot Z$ where

$$Z = \{ \omega \in \text{HPF}_\infty^{(m,l+1),r,s+1} : \text{hpf}^{(m,l)}_\infty(\omega) \neq 0 \text{ and } \text{hpf}^{(r,s),*}_\infty(\omega) \neq 0 \}.$$ 

For the induction step it is enough to show the $GL_\infty$-noetherianity of the set $Z$.

**Step 2: the subgroup $H$.** We will prove that the set $Z$ is noetherian for a certain subgroup $H$ of $GL_\infty$. Let us define this subgroup $H$.

For any $S \subseteq \mathbb{Z}^\times$, let $GL_S$ be the subgroup of $GL_\infty$ that fixes $e_t \in V_\infty$ with $t \notin S$. Then we define the subgroup $H$ as the following product

$$H := GL_{(-\infty,-ml+m)} \times GL_{(rs-r,\infty)},$$

where $(a,b)$ stands for the set $\{ x \in \mathbb{N} : a < x < b \}$.

The group $H$ stabilizes $Z$. Indeed, the coordinates $x_I$ appearing in the forms $\text{hpf}^{(m,l)}$ and $\text{hpf}^{(r,s),*}$ satisfy the condition $I \subseteq [-ml+m, rs-r]$.

**Step 3: A “good” subspace.** To prove that $Z$ is $H$-Noetherian, we embed it into a bigger $H$-Noetherian subspace of $(\wedge \mathcal{V}_\infty)^*$.

**Definition 38.** A subset $I \subseteq \mathbb{Z}^\times$ is good (with respect to $(m,l,r,s)$) if $|I \cap \mathbb{Z}_{<0}|$ and $|I^c \cap \mathbb{Z}_{>0}|$ are finite of the same cardinality, and both $I \cap \mathbb{Z}_{\leq -ml+m-1}$ and $I^c \cap \mathbb{Z}_{>rs-r}$ have cardinality at most 1.

We denote by $(\wedge \mathcal{V}_\infty)_\text{good}^*$ the subspace of $\wedge \mathcal{V}_\infty$ spanned by good coordinates. The dual space $(\wedge \mathcal{V}_\infty)_\text{good}^*$ is naturally $H$-equivariantly embedded into $(\wedge \mathcal{V}_\infty)^*$.

**Lemma 39.** The topological space $(\wedge \mathcal{V}_\infty)_\text{good}^*$ with the Zariski topology is $H$-Noetherian.

The proof of this Lemma is literally the same as for [1] Lemma 6.5: we can embed the space $(\wedge \mathcal{V}_\infty)_\text{good}^*$ into the space $A_{a,b,c,d}$, where

$$A_{a,b,c,d} = (\text{Mat}_{N \times N})^a \times \text{Mat}_{N \times b} \times \text{Mat}_{c \times N} \times \mathbb{R}^d.$$ 

Then the theorem of Draisma–Eggersmont [1] Theorem 1.5] states that the space $A_{a,b,c,d}$ is $GL_\infty \times GL_\infty$-noetherian, so the space $(\wedge \mathcal{V}_\infty)_\text{good}^*$ is.

**Remark 40.** General statement with the subspace of $\wedge \mathcal{V}_\infty$ spanned by all “S-good” coordinates for some finite subset $S \subset N$ (in the considered case $S = [-ml+m, rs-r]$) is true as well. The proof works in the general case.

**Step 4: the injective projection $Z \to (\wedge \mathcal{V}_\infty)_\text{good}^*$**. Finally, we prove that the natural projection from $Z$ to the good subspace $(\wedge \mathcal{V}_\infty)_\text{good}^*$ (the projection forgets non-good coordinates) is injective.

**Claim 41.** On $Z$, each coordinate $x_I$ can be expressed as a rational function of the good coordinates, whose denominator has factors $\text{hpf}^{(m,l)}$ and $\text{hpf}^{(r,s),*}$ only.

It is classically known that coordinates on the space $\wedge \mathcal{V}_\infty$ are in one-to-one correspondence with Young diagrams, the classical reference is [21] Chapter 9, the more contemporary exposition is in [20]. The idea for the proof is the induction on coordinates $\{ x_I, I \in \wedge \mathcal{N} \}$ with respect to the partial order $\prec$ coming from the partial order $\prec$ on the set of Young diagrams. For instance, the relation $\emptyset \prec (2) \prec (2,1)$ for Young diagrams translates to the relation $x_{1234-} < x_{-134-} < x_{-214-}$ for the coordinates.

In detail, consider a non-good coordinate $x_I$ such that all smaller coordinates are good or satisfy the claim.

If $|I \cap \mathbb{Z}_{\leq -ml+m-1}| = 1$, then denote by $I$ any subset of $\mathbb{Z}^\times$ such that the set $I$ is initial subinterval of $I$ (with respect to the natural order on $\mathbb{Z}^\times$). Then the hyper-Pfaffian form $\text{hpf}_I^{(m,l+1)}$ looks as follows

$$\text{hpf}_I^{(m,l+1)} = x_I \cdot \text{hpf}_I^{(m,l)} + Q$$

with all coordinates in $Q$ are strictly smaller than $x_I$. The polynomial $\text{hpf}_I^{(m,l+1)}$ is zero on $Z$ (as a shift of the polynomial $\text{hpf}_I^{(m,l+1)}(x)$ which is identically zero on $Z \subseteq \text{HPF}_\infty^{(m,l+1),r,s+1}$), therefore the following equality holds true on $Z$:

$$x_I = - \frac{Q}{\text{hpf}_I^{(m,l)}}.$$ 

The case $|I \cap \mathbb{Z}_{\geq rs-r}| = 1$ is treated similarly with use of dual hyper-Pfaffian forms.

**Corollary 42.** Theorem 7 holds true.
Theorem 31 says that for any proper

Proposition 44.

\[(j_i \cap j_j)\]

where the right intersection runs over the orbit of the map

\[\nu \in \Lambda^3(V)\] if and only if \(\text{rk}(\nu) \leq 2\), i.e., \(\nu = \nu_1 + \nu_2\) with \(\nu_{1,2} \in (\Lambda^2(V))_{(1,1),1}\). Also, for any trivector \(\nu \wedge v\) with \(\nu \in \Lambda^2(V)\) and \(v \in \Lambda^3(V)\) we can assume that \(\nu \in \Lambda^2(v)\). Therefore any element \(\omega \in \Lambda^2(\Lambda^3(V))\) has the form \(\nu \wedge v\) with \(\text{rk}(\nu) \leq 2\) and \(\nu \in \Lambda^2(v)\), and we have the inclusion \(\Lambda^2(\Lambda^3(V)) \subseteq \Lambda^4(V)\).

Lemma 43. (a) The variety \(Y_{4,3} \subseteq \Lambda^4(\Lambda^3(V))\) coincides with \(\Lambda^2(\Lambda^3(V))\), where

\[\Lambda^2(\Lambda^3(V)) := \{\omega \in \Lambda^3(V) : \omega = (\nu_1 + \nu_2) \wedge v \text{ with } \nu_{1,2} \in (\Lambda^2(V))_{(1,1),1} \text{ and } v \in \Lambda^3(V)\}.

(b) Generally, if \(X_{n,p} = (\Lambda^3(V))_{\pi,k}\), then \(X_{n,p} = \Lambda^2(\Lambda^3(V))\) for any \(P \geq p\).

5. Examples

5.1. Pf(2). We recall from Proposition III that we always have the following inclusions for the \((n,p)\)-component of \(\Lambda\)-variety:

\[Y_{n,p} := \bigcup_j j(Y_{n,p-1}) \subseteq Y_{n,p} \subseteq \bigcap_{j=1}^n (1) =: Y_{n,p},\]

where the right intersection runs over the orbit of the map \(j_{n,p-1}\) under the \(GL_{n,p-1} \times GL_{n,p}\)-action and the left union runs over the orbit of \(j_{n,p-1}\) under the same group.

Let us assume that \(Y_{4,2} = \text{Pf}(2)(V)\). The next lemma describes the minimal bound for the \((4,3)\)-component.

Lemma 43. If \(Y_{4,2} = \text{Pf}(2)(V)\), then \(Y_{4,3} \subseteq Y_{4,3}\). Moreover, \(Y_{4,3} \subseteq (\Lambda^3(V))_{(2,1),2}\) and \(Y_{4,3} \neq (\Lambda^3(V))_{(2,1),2}\).

Proposition 44. If \(Y_{4,2} = \text{Pf}(2)(V)\), then \(Y_{4,3} \subseteq Y_{4,3}\). Moreover, \(Y_{4,3} \subseteq (\Lambda^3(V))_{(2,1),2}\) and \(Y_{4,3} \neq (\Lambda^3(V))_{(2,1),2}\).

Proposition 45. If \(Z_{4,4} = \text{HPf}(4,4)(V)\), then \(Z_{5,4} \neq Z_{5,4}\).

Proposition 5.1. From Proposition III that we always have the following inclusions for the \((n,p)\)-component:

\[Z_{n,p} := \bigcup_i i(Z_{n-1,p}) \subseteq Z_{n,p} \subseteq \bigcap_{i=1}^n (i) =: Z_{n,p},\]

where the right intersection runs over the orbit of the map \(i_{n-1,p}\) under the \(GL_{n-1,p} \times GL_{n,p}\)-action and the left union runs over the orbit of \(i_{n-1,p}\) under the same group.

Let us assume that \(Z_{4,4} = \text{HPf}(4,4)(V)\) and \(Z_{5,4} = \text{HPf}(4,4)(V)\) is the set of all elements \(\omega \in \Lambda^4(V)\) satisfying \(\omega \wedge \omega = 0\). Therefore the variety \(Z_{5,4}\) coincides with \(\text{HPf}(4,2)(V) = \{\omega \in \Lambda^4(V) : \omega \wedge \omega = 0\}\).

Proposition 45. If \(Z_{4,4} = \text{HPf}(4,4)(V)\), then \(Z_{5,4} \neq Z_{5,4}\).

Proof. We can see that dimensions of the varieties are different, or instead just argue that \(e_{-5} \wedge e_{-4} \wedge e_{-3} \wedge e_{-2} + e_{-1} \wedge e_1 \wedge e_2 \wedge e_3 + e_{-5} \wedge e_{-4} \wedge e_{-3} \wedge e_{-1} + e_{-2} \wedge e_1 \wedge e_2 \wedge e_3 + e_{-5} \wedge e_{-2} \wedge e_{-1} \wedge e_4 \neq Z_{5,4}\).
5.3. $\Lambda$-varieties which are not Plücker, and unbounded Plücker varieties. The following proposition explains which of the Pfaffian $\Lambda$-varieties are Plücker.

**Proposition 46.**

1. The $\Lambda$-varieties $\text{Pf}^{(1)}$ and $\text{Pf}^{(2)}$ are Plücker, i.e., these varieties are preserved by the $\star$-symmetry.

2. The $\Lambda$-variety $\text{Pf}^{(r)}$ for $r \geq 3$ are not Plücker. In other words, the $\star$-symmetry does not preserve $\text{Pf}^{(r)}$ for $r \geq 3$.

**Proof.** The first statement follows from a direct computation and Lemma 23(b).

We prove the second statement for $r = 3$ only; the general case is analogous. From Lemma 23(b) we see that $\eta \in \Lambda V_{4,3}$ lies in the $(4,3)$-component of $\text{Pf}^{(3)}$ if and only if $\eta = (w_1 \land w_2 + w_3 \land w_4 + w_5 \land w_6) \land w_7$ for some $w_i \in V_{4,3}$. A direct calculation shows that

$$\star((e_{-4} \land e_{-3} + e_{-2} \land e_{-1} + e_1 \land e_2) \land e_3) = e_{-2} \land e_{-1} \land e_1 \land e_2 + e_{-4} \land e_{-3} \land e_1 \land e_2 + e_{-4} \land e_{-3} \land e_{-2} \land e_{-1}$$

But from the same Lemma we see that $\omega \in \Lambda V_{3,4}$ belongs to the $(3,4)$-component of $\text{Pf}^{(3)}$ if and only if $\omega = (v_1 \land v_2 + v_3 \land v_4) \land v_5 \land v_6$ for some $v_i \in V_{3,4}$. A contradiction.

Generalizing the proposition we can see the following result. We recall that for $l = 1$ and any $m$ the hyper-Pfaffian $\text{HPf}^{(m,1)} = \{0\}$ is trivial.

**Theorem 47.** A nontrivial hyper-Pfaffian $\Lambda$-variety $\text{HPf}^{(m,l)}$ is Plücker if and only if $l = 2$. Moreover, the only nontrivial bounded hyper-Pfaffian Plücker variety is $\text{HPf}^{(2,2)} = \text{Pf}^{(2)} = \text{Gr}$.

**Proof.** We recall that the $\Lambda$-variety $\text{HPf}^{(m,l)}$ is defined as the only $(m(l-1),m)$-maximal variety with $(m(l-1),m)$-component equal to the corresponding hyper-Pfaffian (Theorem 23). Then the statement follows: if the hyper-Pfaffian $\Lambda$-variety is nontrivial ($l \neq 1$), then only for $l = 2$ we have the equality $m(l-1) = m$ which is equivalent to the $\star$-symmetry for $(n,p)$-maximal $\Lambda$-varieties.

From Theorem 23 we see that for $m \geq 4$ the varieties $\text{HPf}_{n,2}^{(m,2)}$ coincide with $\Lambda^2 V_{n,2}$. Therefore the corresponding varieties are unbounded.

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