PRIME AVOIDANCE PROPERTY

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Abstract. Let $R$ be a commutative ring, we say that $A \subseteq \text{Spec}(R)$ has prime avoidance property, if $I \subseteq \bigcup_{P \in A} P$ for an ideal $I$ of $R$, then there exists $P \in A$ such that $I \subseteq P$. We exactly determine when $A \subseteq \text{Spec}(R)$ has prime avoidance property. In particular, if $A$ has prime avoidance property, then $A$ is compact. For certain classical rings we show the converse holds (such as Beazout rings, QR-domains, zero-dimensional rings and $C(X)$). We give an example of a compact $A \subseteq \text{Spec}(R)$ of a Prufer domain $R$ which has not $P,A$-property. Finally, we show that if $V,V_1,\ldots,V_n$ are valuations for a field $K$ and $V[x] \not\subseteq \bigcup_{i=1}^n V_i$ for some $x \in K$, then there exists $v \in V$ such that $v+1 \notin \bigcup_{i=1}^n V_i$.

1. Introduction

Quentel in [21] Proposition 9] proved that the following are equivalent for a commutative reduced ring $R$ (see also [19] Propositions 1.4 and 1.15):

1. $Q(R)$, the classical ring of quotient of $R$, is a VNR.

2. If $I$ is an ideal of $R$ contained in the union of the minimal prime ideals of $R$, then $I$ is contained in one of them.

3. $\text{Min}(R)$ is compact; and if a finitely generated ideal is contained in the union of the minimal prime ideals of $R$, then it is contained in one of them.

In [21], Quentel has produced an example of a reduced ring $R$ where $\text{Min}(R)$ is compact, but $Q(R)$ is not a VNR, which shows the second part of (3) is necessary.

The main aim of this paper is to show that the conditions (2) and (3) of the above are equivalent for each subset $A$ of $\text{Spec}(R)$, for arbitrary commutative ring $R$.

In commutative ideal theory one of the most important covering results is the Prime Avoidance Lemma which state that if an ideal $I$ of a commutative ring $R$ is covered by a finite union of prime ideals $P_1, \ldots, P_n$, then there exists $i$ such that $I \subseteq P_i$, see [15] Theorem 81], [17], [18], [22] and [16]. Let us denote by $V_A(I)$ the set of all prime ideals in $A$ which contains $I$, where $R$ is a commutative ring, $A \subseteq \text{Spec}(R)$ and $I$ is an ideal of $R$. It is well known that the set of all $V_A(I)$, where $I$ ranges over all ideals of $R$, is a topology for closed sets on $A$, which is called hull-kernel or Zariski topology on $A$ (when $A = \text{Spec}(R)$, $A = \text{Max}(R)$ and $A = \text{Min}(R)$ we apply $V(I)$, $V_M(I)$ and $V_m(I)$, respectively). It is well known that $\text{Spec}(R)$ and $\text{Max}(R)$ are compact spaces. In fact one easily find that the fact which implies these two spaces are compact is "each proper ideal of $R$ can be embedded in a maximal ideal" which is well known as Krull Maximal Ideal Theorem; In other words, if $I$ is an ideal of $R$ such that $I \subseteq \bigcup_{M \in \text{Max}(R)} M$, then there exists $M \in \text{Max}(R)$ such that $I \subseteq M$. It seems that there exists a relation between the compactness of a subset $A$ of $\text{Spec}(R)$ and the fact that $A$ has prime avoidance property. Note that by Krull Maximal Ideal Theorem one can easily see that each subset $X$ of $\text{Spec}(R)$ which contains $\text{Max}(R)$ is compact. Moreover, if $R$ is a noetherian ring then each subset of $\text{Spec}(R)$ is compact (i.e., $\text{Spec}(R)$ is noetherian), since each sum of a family of ideals in $R$ reduced to a finite sum of the family. Note that there exist non-noetherian rings for which $\text{Spec}(R)$ is noetherian, in fact $\text{Spec}(R)$ is noetherian if and only if $\text{ACC}$ holds on radical ideals of $R$, i.e., for each ideal $I$ of $R$ there exists $a_1,\ldots,a_n$ such that $\sqrt{I} = \sqrt{(a_1,\ldots,a_n)}$; which also implies that $\text{Min}(I)$ is finite for each ideal $I$.

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of $R$. Note that by [19, Proposition 3.8], for a ring $R$, $\text{Min}(R) = \{P_\alpha\}$ is finite if and only if for each $\beta$, $P_\beta \not\subseteq \bigcup_{\alpha \neq \beta} P_\alpha$, in other words for each $P \in \text{Min}(R)$, the set $\text{Min}(R) \setminus \{P\}$ has prime avoidance property (for another finiteness result for $\text{Min}(R)$ see [2]). Now the following is in order.

**Remark 1.1.** Let $R$ be a ring. Then $\text{Min}(R) = \{P_\alpha\}$ is finite if and only if $\text{Min}(R)$ is compact and for each $\beta$, $Q_\beta := \bigcap_{\alpha \neq \beta} P_\alpha \not\subseteq P_\beta$. In particular, if in addition $R$ is a reduced ring, then each $P_\beta$ is not essential and $P_\beta = \text{ann}(q)$ for some $q \in Q_\beta$. To see this, it is clear that if $\text{Min}(R)$ is finite then $\text{Min}(R)$ is compact and for each $\beta$, $Q_\beta \not\subseteq P_\beta$. Conversely, assume that $\text{Min}(R)$ is compact and for each $\beta$ let $x_\beta \in Q_\beta \setminus P_\beta$. Therefore $V_m(x_\beta)^c = (P_\beta)$. Now, since the collection $\{V_m(x_\beta)^c\}$ is an open cover for $\text{Min}(R)$ and $\text{Min}(R)$ is compact, we immediately conclude that $\text{Min}(R)$ is finite. Finally, if in addition $R$ is a reduced ring then clearly for each $\beta$, $Q_\beta \neq 0$ and $Q_\beta \cap P_\beta = N(R) = 0$. which shows that each $P_\beta$ is not essential and $Q_\beta P_\beta = 0$. Therefore $P_\beta \subseteq \text{ann}(Q_\beta)$ and since $Q_\beta \not\subseteq P_\beta$ we obtain that $\text{ann}(Q_\beta) = P_\beta$. This immediately shows that for each $q \in Q_\beta \setminus P_\beta$ we have $P = \text{ann}(q)$.

By the above remark we give another proof for finiteness of $\text{Min}(R)$ for noetherian rings.

**Corollary 1.2.** Let $R$ be a noetherian ring, then $\text{Min}(R)$ is finite.

**Proof.** First note that $\text{Spec}(R)$ is noetherain, since $R$ is noetherian. Therefore $\text{Min}(R)$ is compact. Without loss of generality we may assume that $R$ is a reduced ring. Let $\text{Min}(R) = \{P_\alpha\}$, hence for each $\beta$ we conclude that there exist $s_\beta \in R \setminus P_\alpha$ such that $P_\beta = \text{ann}(s_\beta)$. Clearly $s_\beta \in Q_\beta$ and therefore by the above remark we are done. □

In this paper all ring are commutative with $1 \neq 0$. All subrings, Modules and ring homomorphisms are unital. A subgroup $S$ of $(R, +)$, which is closed under multiplication of $R$ and $1 \not\in S$ is called Subring-1. Let $S$ be a family of (certain) subset of a ring $R$, then we say that a family $I$ of ideals of $R$ has $A$-property for $S$, if $S \subseteq I$ is covered by $I$, then $S \subseteq I$ for some $I \in I$. In particular, if $\mathcal{S}$ is the family of all ideals of $R$, and $\mathcal{I}$ has $A$-property for $S$, then we say that $\mathcal{I}$ has $A$-property (for $R$). One can easily see that if $C$ is a chain of ideals of $R$, then $C$ has $A$-property for all finitely generated ideals of $R$. When $\mathcal{I} \subseteq \text{Spec}(R)$ and $\mathcal{I}$ has $A$-property for $S$, we say $\mathcal{I}$ has $A$-property for $S$. Clearly, each finite set of prime ideals of a ring $R$ has $P.A$-property for the set of all subrings−1 of $R$ and therefore has $P.A$-property for $R$.

A brief outline of this paper is as follow. In the next section we first proved some basic facts for (certain) $\mathcal{A} \subseteq \text{Spec}(R)$, that has (not) $P.A$-property. In particular, we prove that each compact set of primes of a zero-dimensional rings has $P.A$-property. We give a characterization of a set $A$ of noncomparable prime ideals of a ring $R$ which has $P.A$-property by ring homomorphism; in fact it is shown that if $\mathcal{A}$ has $P.A$-property, then $\mathcal{A}$ is equal to the inverse image of the set of all maximal ideals of a certain ring $T$ under a ring homomorphism from $R$ into $T$. Next, we determine exactly when $\mathcal{A} \subseteq \text{Spec}(R)$ has $P.A$-property. In fact, we prove that $\mathcal{A}$ has $P.A$-property if and only if $\mathcal{A}$ is compact (with Zariski Topology) and $\mathcal{A}$ has $P.A$-property for finitely generated ideals. We show that if $K$ is a formally real field and $X$ is a subset of affine space $K^n$, then $M_X := \{M_P \mid P \in X\}$ has $P.A$-property. We observe that if $\mathcal{A} \subseteq \text{Spec}(R)$, where $R$ is a Bezout ring, zero-dimensional ring or $R = C(X)$ for a topological space $X$, then $\mathcal{A}$ has $P.A$-property if and only if $\mathcal{A}$ is compact. We give an example which shows the previous is not true for Prufer domains, even if $\mathcal{A} \subseteq \text{Spec}(R)$ is compact. We show that if $R$ is a QR-domain, then $\mathcal{A} \subseteq \text{Spec}(R)$ has $P.A$-property for $R$ if and only if $\mathcal{A}$ is compact. If $R$ is a Prufer domain and each $\mathcal{A}$ has $P.A$-property, then we show that $\mathcal{A}$ is a QR-domain. In particular, if $R$ is a Dedekind domain, then each $\mathcal{A} \subseteq \text{Spec}(R)$ has $P.A$-property if and only if $R$ is a QR-domain (i.e., $R$ has torsion class group). It is shown that if $f : R \rightarrow C(X)$ is a ring homomorphism from a ring $R$ to $C(X)$ and $\mathcal{A} \subseteq \text{Spec}(C(X))$ is compact, then $\mathcal{A}^* = \{f^{-1}(P) \mid P \in \mathcal{A}\}$ has $P.A$-property for $R$. In particular, if $I$ is an ideal of $R$ then $I^* = C(X)$ if and only if $f(I)$ contains a unit of $C(X)$. We also prove some corollaries for $A$-property of a finite/countable set of ideals in infinite artinian/noetherian rings with certain cardinality, related to the results of [17], [20] and [22]. In particular, if $R$ is a noetherian integral domain with $|R| > 2^{\aleph_0}$, then $R$ is a $\mathfrak{u}$-ring, i.e., each finite set of ideals of $R$ has $\mathcal{A}$-property. Finally, we prove Davis avoidance Theorem for valuation domains instead of prime ideals. In fact we show that if $K$ is a field, $V, V_1, \ldots, V_n$ be valuations for $K$ and $x \in K$ and $V[x] \not\subseteq \bigcup_{i=1}^n V_i$, then there exists $v \in V$ such that $v + x \not\in \bigcup_{i=1}^n V_i$.

2. Main Result

We begin this section by the following immediate application of Krull Maximal Ideal Theorem.

**Proposition 2.1.** Let $R$ be a ring and $I$ be an ideal of $R$. 
(1) $V(I)$ and $V_M(I)$ have P.A-property for $R$. In particular, if $M$ is a finitely generated $R$-module then $\text{supp}(M)$ has P.A-property.

(2) Let $\mathcal{A} \subseteq \text{Spec}(R)$ has not P.A-property for $R$, then there exists a prime ideal $Q \notin \mathcal{A}$ such that $\bigcap_{P \in \mathcal{A}} P \subseteq Q$.

(3) Let $\mathcal{A} \subseteq \text{Spec}(R)$ and $I$ be an ideal of $R$ such that $I \subseteq \bigcup_{P \in \mathcal{A}} P$. Then either there exists a prime ideal $P$ in $\mathcal{A}$ such that $I \subseteq P$ or $\mathcal{A} \subseteq V(\text{Ann}(I))$. Hence in the latter condition $I + \text{Ann}(I)$ is a proper ideal of $R$.

(4) Let $\mathcal{A} \subseteq \text{Spec}(R)$ and $\mathcal{A}'$ be the set of ideals $I$ of $R$ which are contained in $\bigcup_{P \in \mathcal{A}} P$, but $I \not\subseteq P$ for each $P \in \mathcal{A}$. Then either $\mathcal{A}$ has P.A-property or $\mathcal{A} \subseteq V(I')$ where $I' = \bigcap_{I \in \mathcal{A}} \text{Ann}(I)$.

(5) If $R$ is a zero-dimensional ring and $\mathcal{A} \subseteq \text{Spec}(R)$ is compact. Then $\mathcal{A}$ has P.A-property for $R$.

Proof. The first part of (1) is clear by Krull Maximal Ideal Theorem. For the final part of (1) note that since $M$ is finitely generated we conclude that $\text{supp}(M) = V(\text{Ann}(M))$ and therefore by the first part we are done. For (2), since $\mathcal{A}$ has not P.A-property, we infer that $\mathcal{A} \subseteq V(\bigcap_{P \in \mathcal{A}} P)$, by (1). To see (3) assume that for each $P \in \mathcal{A}$, $I \not\subseteq P$. Therefore $\text{Ann}(I) \subseteq P$, for each $P \in \mathcal{A}$. Thus $\mathcal{A} \subseteq V(\text{Ann}(I))$. Hence by (1) we are done for (3), (4) is clear by (3). Finally for (5), note that in this case $\text{Spec}(R)$ is a Hausdorff space, therefore $\mathcal{A}$ is close. Thus by (1), $\mathcal{A}$ has P.A-property. □

Let $R$ be a ring a subset $S$ of $R$ is called a multiplicatively closed set if $0 \notin X$, $1 \in X$ and $X$ is closed under multiplication of $R$. A well known theorem of I.S. Cohen (which is a generalization of Krull Maximal Ideal Theorem) shows that an ideal $I$ of a ring $R$ disjoint from a multiplicatively closed set $X$ of $R$ if and only if there exists a prime ideal $P$ of $R$ which contains $I$ and disjoin from $X$, see [15, Theorem 1].

The proof of the following proposition which is an immediate consequences of Cohen Theorem and the structure of (prime) ideals of ring of quotient of $R$ respect to a multiplicatively close sets (see [15, Sec. 1-4]) is simple and hence left to the reader.

**Proposition 2.2.** Let $R$ be a ring and $\mathcal{A} \subseteq \text{Spec}(R)$. The following are equivalent:

1. $\mathcal{A}$ has P.A-property for $R$.
2. $\mathcal{A}$ has P.A-property for $\text{Spec}(R)$.
3. $\text{Max}(R_X) \subseteq \{P_X \mid P \in \mathcal{A}\}$, where $X = R \setminus \bigcup_{P \in \mathcal{A}} P$.

We remind that in fact $Q(R)$ is VNR for a reduced ring $R$ if and only if $\text{Max}(Q(R)) = \{P_X \mid P \in \text{Min}(R)\}$, where $X = R \setminus \bigcup_{P \in \text{Min}(R)} P$ is the set of regular (nonzero divisors) of $R$. In the following theorem we generalize the previous fact for arbitrary set of incomparable prime ideals of a ring $R$ and show that the P.A-property is in fact the Krull Maximal Ideal Theorem.

**Theorem 2.3.**

1. Let $R$ be a ring and $\mathcal{A} \subseteq \text{Spec}(R)$ be an incomparable set of primes which has P.A-property for $R$. Then there exists a ring $T$ and a ring homomorphism $f : R \to T$ such that $\mathcal{A} = \{f^{-1}(M) \mid M \in \text{Max}(T)\}$.
2. Let $R$ and $T$ be rings and $f : R \to T$ be a ring homomorphism. Then $\mathcal{A} = \{f^{-1}(M) \mid M \in \text{Max}(T)\}$ has P.A-property for $R$ if and only if for each ideal $I$ of $R$ with $I \cap X = \emptyset$, the ideal $I^c$ is proper in $T$, where $X = R \setminus \bigcup_{P \in \mathcal{A}} P$. Moreover in this case $f$ can be extended to a ring homomorphism from $R_X$ to $T$.

Proof. (1) It suffices to put $T = R_X$, where $X = R \setminus \bigcup_{P \in \mathcal{A}} P$ and $f$ be the natural ring homomorphism from $R$ into $T$. Now note that since $\mathcal{A}$ has P.A-property and elements of $\mathcal{A}$ are incomparable, we immediately infer that $\text{Max}(T) = \{P_X \mid P \in \mathcal{A}\}$, by the structure of prime ideals of $T = R_X$. It is clear that for each $P \in \mathcal{A}$ we have $f^{-1}(P_X) = P$ which complete the proof of (1).

(2) For the if part, let $I$ be an ideal of $R$ which is contained in the union of element of $\mathcal{A}$. Then $I \cap X$ is empty and therefore $I^c$ is a proper ideal of $T$. Hence by Krull Maximal Ideal Theorem $I^c$ is contained in a maximal ideal $M$ of $T$. Thus $I \subseteq f^{-1}(M) \in \mathcal{A}$. Conversely, assume that $\mathcal{A}$ has P.A-property for $R$ and $I$ be an ideal of $R$ with $I \cap X$ is empty. Thus $I$ is contained in the union of element of $\mathcal{A}$ and therefore by assumption $I$ is contained in $f^{-1}(M)$ for some $M \in \text{Max}(T)$. Hence $I^c$ is contained in $M$ and therefore $I^c$ is proper ideal. Finally, for each $t \in X$, we have $f(t) \notin \bigcup_{M \in \text{Max}(T)} M$ and therefore $f(t) \in U(T)$. This immediately shows that $f$ can be extended to a ring homomorphism from $R_X$ to $T$. □

Now we have the following result.

**Proposition 2.4.** Let $f$ be a ring homomorphism from a ring $R$ to a ring $T$ and $\mathcal{A}$ be a set of ideals of $T$ which has A-property for subrings–1 of $T$. Then $\mathcal{A}' := \{Q^c = f^{-1}(Q) \mid Q \in \mathcal{A}\}$ has A-property for subrings–1 of $R$. 

Proof. Assume that $S$ be a subring $-1$ of $R$ which is contained in $\bigcup_{Q \in A} Q^c$. Therefore we infer that $f(S)$ is contained in $\bigcup_{Q \in A} Q$. Now note that $f(S)$ is a subring $-1$ of $T$. Hence by our assumption we infer that there exists a $Q \in A$ such that $f(S) \subseteq Q$. Therefore $S \subseteq Q^c$ and we are done. \hfill \Box

In the next result, we determine exactly when $A \subseteq \text{Spec}(R)$ has $P.A$-property.

**Theorem 2.5.** Let $R$ be a ring and $A \subseteq \text{Spec}(R)$. The following conditions are equivalent:

1. $A$ is compact and has $P.A$-property for all finitely generated ideals of $R$.
2. $A$ has $P.A$-property for $R$.

**Proof.** (1) $\Rightarrow$ (2): Let $I$ be an ideal of $R$ which is contained in the union of $A = \{P_a\}_{a \in \Gamma}$ but for each $\alpha \in \Gamma$, $I$ is not contained in $P_a$. Therefore for each $\alpha \in \Gamma$, there exists $x_a \in I \setminus P_a$. Hence $P_\alpha \cap V_A(x_a)^c$ for each $\alpha \in \Gamma$. Thus the collection $\{V_A(x_a)^c\}_{\alpha \in \Gamma}$ is an open cover for $A$. Now by our assumption $A$ is compact, and therefore there exist finitely many $\alpha_1, \ldots, \alpha_n$ in $\Gamma$ such that $A = V_A(x_{\alpha_1})^c \cup \cdots \cup V_A(x_{\alpha_n})^c$. Thus $V_A(x_{\alpha_1}, \ldots, x_{\alpha_n}) = \emptyset$. But the finitely generated ideal $I = \langle x_{\alpha_1}, \ldots, x_{\alpha_n} \rangle$ of $R$ is contained in $I$ and therefore in the union of $A$, which by (1) immediately implies that there exists $\alpha \in \Gamma$ such that $J \subseteq P_\alpha$, i.e., $P_\alpha \cap V_A(x_{\alpha_1}, \ldots, x_{\alpha_n})$, which is a contradiction.

(2) $\Rightarrow$ (1): It suffices to show that $A$ is compact. Hence assume that $A = \bigcup_{a \in A} V_A(I_a)^c$, where each $I_a$ is an ideal of $R$. Therefore $V_A(\bigcap_{a \in A} I_a) = \emptyset$. Thus by (2) we conclude that $\bigcup_{a \in A} I_a \not\subseteq \bigcup_{P \in \mathcal{A}} P$. This immediately implies that there exist finitely many $\alpha_1, \ldots, \alpha_n$ in $\Gamma$ such that $I_{\alpha_1} + \cdots + I_{\alpha_n} \not\subseteq \bigcup_{P \in \mathcal{A}} P$. Therefore $V_A(I_{\alpha_1} + \cdots + I_{\alpha_n}) = \emptyset$, i.e., $A$ is compact and hence we are done.

Now we give some conclusions of the above theorem.

**Corollary 2.6.** Let $R$ be a ring and $A \subseteq \text{Spec}(R)$ be a chain. Then $A$ is compact if and only if $A$ has $P.A$-property. In particular, if $A$ is compact then $\bigcup_{P \in \mathcal{A}} P \in A$.

**Example 2.7.**

1. Let $K$ be a field and $R = K[x_1, x_2, \ldots]$ be the ring of polynomials of independent variables $x_1, x_2, \ldots$ over $K$. Then clearly $I = (x_1, x_2, \ldots)$ is contained in the union of primes ideals $P_n = (x_1, x_2, \ldots, x_n)$, but $I$ is not contained in $P_n$ for each $n$.

2. Let $V$ be a valuation ring, $A$ a set of prime ideals of $V$ and $I$ an ideal of $V$ which is contained in $\bigcup A$ but is not contained in any element of $A$, then $I = \bigcup A$ and therefore $I$ is prime.

The following is similar to [22] Proposition 2.5.

**Corollary 2.8.** Let $R$ be a ring such that there exists an uncountable family $\{t_a\}_{a \in \Gamma}$ of elements of $R$ such that for each $\alpha \neq \beta$ in $\Gamma$ we have $t_\alpha - t_\beta \in U(R)$. If $A$ is a countable subset of $\text{Spec}(R)$, then $A$ has $P.A$-property for $R$ if and only if $A$ is compact.

**Proof.** By [22] Proposition 2.5, $A$ has $P.A$-property for finitely generated ideals of $R$. Hence by Theorem 2.5 we infer that $A$ has $P.A$-property for $R$ if and only if $A$ is compact. \hfill \Box

**Corollary 2.9.** Let $R$ be a zero-dimensional ring (in particular, if $R$ is VNR) and $A \subseteq \text{Spec}(R)$. Then $A$ has $P.A$-property if and only if $A$ is compact (closed, i.e., $A = V(I)$ for some ideal $I$ of $R$).

**Proof.** The if part is evident by (5) of Proposition 2.1 and the converse holds by Theorem 2.5 (and the fact that Spec($R$) is Hausdorff for zero-dimensional rings). \hfill \Box

**Corollary 2.10.** Let $R$ be a Bezout ring (i.e., every finitely generated ideal of $R$ is principal) and $A \subseteq \text{Spec}(R)$. Then $A$ has $P.A$-property if and only if $A$ is compact.

**Proof.** Assume that $A$ be a set of ideals of $R$ (not necessary prime) and $I$ be a finitely generated ideal of $R$ which is contained in $\bigcup A$. Since $R$ is a Bezout ring we infer that $I = Ra$ for some $a \in R$. Thus there exists $J$ in $A$ such that $a \in J$ and therefore $I \subseteq J$. Hence we are done by Theorem 2.5. \hfill \Box

**Remark 2.11.** The above corollary is still true if $R$ is an almost Bezout domain, see [1]. The proof is similar and need to use [1] Lemma 3.4.

In the next theorem we show that the above corollary does not hold for Prufer domains. We remind the reader that if $R$ is a Prüfer domain with quotient field $K$ and $T$ be an overring of $R$, then each prime ideal $Q$ of $T$ has the form $PT$ where $P = Q \cap T$ and in this case $PR = T_Q$, see [11] Theorem 26.1. Also we remind the reader that an integral domain $D$ is called $QR$-domain if each overring of $D$ is a quotient of $D$ respect to a multiplicatively closed set of $D$. Finally, note that each $QR$-domain is Prüfer, see §27 of [11].
Theorem 2.12. (1) Let \( R \) be a Prufer domain with quotient field \( K \), \( T \) be an overring of \( R \) and \( \mathcal{A} = \{ P \in \text{Spec}(R) \mid PT \in \text{Max}(T) \} \). Then \( \mathcal{A} \) is compact.

(2) If \( R \) is a Prufer domain which is not a QR-domain, then there exists a compact set of prime ideals in \( R \) which has not P.A-property.

(3) If \( R \) is a QR-domain and \( \mathcal{A} \subseteq \text{Spec}(R) \), then \( \mathcal{A} \) has P.A-property if and only if \( \mathcal{A} \) is compact.

Proof. For (1), let \( \mathcal{A} = \bigcup_{\alpha \in \Gamma} V_{\mathcal{A}}(I_\alpha)^c \), where each \( I_\alpha \) is an ideal of \( R \). Hence we infer that \( V_{\mathcal{A}}(\sum_{\alpha \in \Gamma} I_\alpha) = 0 \). Now since each maximal ideal of \( T \) has the form \( P \cdot T \) for some \( P \in \mathcal{A} \), we conclude that \( (\sum_{\alpha \in \Gamma} I_\alpha)T = T \). Therefore there exists \( \alpha_1, \ldots, \alpha_n \in \Gamma \) such that \( (I_{\alpha_1} + \cdots + I_{\alpha_n})T = T \). Thus \( V_{\mathcal{A}}(I_{\alpha_1} + \cdots + I_{\alpha_n}) = 0 \) which immediately implies that \( \mathcal{A} \) is compact.

(2) Assume that \( T \) be an overring of \( R \) which is not a quotient of \( R \) respect to multiplicity closed subsets of \( R \). Suppose \( \text{Max}(T) = \{ Q = PT \mid P \in \mathcal{A} \} \), where \( \mathcal{A} \subseteq \text{Spec}(R) \) and \( X = R \setminus \bigcup_{P \in \mathcal{A}} P \). Hence by the above comment we conclude that

\[
R_X \subseteq T = \bigcap_{Q \in \text{Max}(T)} T_Q = \bigcap_{P \in \mathcal{A}} R_P
\]

Now if \( \mathcal{A} \) has P.A-property then by [11] Proposition 4.8, \( T = R_X \) which is absurd. Hence \( \mathcal{A} \) is compact by (1) and has not P.A-property.

Finally for (3), we first remind that if \( R \) is a QR-domain then for each finitely generated ideal \( I \) of \( R \), there exists a natural number \( n \) and \( a \in I \) such that \( I^n \subseteq (a) \), see [11] Theorem 27.5. Now assume that \( I \) is a finitely generated ideal of \( R \) which is contained in \( \bigcup_{\alpha \in \Gamma} \mathcal{A} \), then by the latter fact we immediately infer that \( I \) is contained in an element of \( \mathcal{A} \) and therefore by Theorem 2.12 we are done. \( \square \)

Remark 2.13. Let \( R \) be a ring. If all subset of \( \text{Spec}(R) \) has P.A-property, then \( \text{Spec}(R) \) is a noetherian space, by Theorem 2.12. In [23], Smith proved a more stronger result: all subset of \( \text{Spec}(R) \) has P.A-property if and only if for each prime ideal \( P \) of \( R \) there exist \( x \) such that \( P = \sqrt{(x)} \). As Gilmer mentioned in [13] Proposition 4, the latter fact is equivalent to: for each ideal \( J \) of \( R \) there exist \( a \in I \) such that \( \sqrt{J} = \sqrt{(a)} \) (which immediately implies that \( \text{Spec}(R) \) is noetherian).

Corollary 2.14. (1) Let \( R \) be a noetherian QR-domain. Then each subset of \( \text{Spec}(R) \) has P.A-property.

(2) Let \( R \) be a Prufer domain. If each \( \mathcal{A} \subseteq \text{Spec}(R) \) has P.A-property, then \( R \) is a QR-domain.

(3) If \( R \) is a Dedekind domain, then each subset of \( \text{Spec}(R) \) has P.A-property if and only if \( R \) is a QR-domain (if and only if \( R \) has torsion class group).

Proof. First note that since \( R \) is noetherian then each subset of \( \text{Spec}(R) \) is compact. Thus if \( R \) is a QR-domain, then by (3) of the previous theorem each subset of \( \text{Spec}(R) \) has P.A-property. Thus (1) holds. For (2), assume that each subset of \( \text{Spec}(R) \) has P.A-property, then by [13] Proposition 4 (or [23]), for each ideal \( I \) of \( R \) there exists \( a \in I \) such that \( \sqrt{I} = \sqrt{(a)} \). Hence if \( I \) is a finitely generated ideal of \( R \) we infer that there exists a natural number \( n \) such that \( I^n \subseteq (a) \) and therefore by [11] Theorem 27.5, we conclude that \( R \) is a QR-domain. (3) is evident by (1) and (2). For the parenthesis fact of (3) see [11] Theorem 40.3. \( \square \)

Remark 2.15. We remind the reader that the following are equivalent for an integral domain \( R \),

(1) \( R \) is a PID.

(2) \( R \) is noetherian and each maximal ideal of \( R \) is principal.

(3) \( R \) has ACC on principal ideals and each maximal ideal of \( R \) is principal.

(4) \( R \) is atomic and each maximal ideal of \( R \) is principal.

To see this first note that clearly (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4). Hence it remains to show that (4) implies (1). We may assume that \( R \) is not a field. Let \( M = (p) \) be an arbitrary maximal ideal of \( R \), thus \( p \) is a prime element of \( R \). We claim that \( J := \bigcap_{n=1}^\infty (p^n) = 0 \). Suppose that \( J \neq 0 \), thus by [15] Exercise 5, P. 7], we infer that \( J \) is a prime ideal and \( J \subseteq M \). Since \( J \) is a nonzero prime ideal of \( R \) and \( R \) is atomic, we conclude that there exists an irreducible element \( q \in J \). Thus \( q \in M = (p) \) and therefore \( q = p \), which is absurd. Hence \( J = 0 \). Again by [15] Exercise 5, P. 7], we deduce that \( M \) contains no properly nonzero prime ideal (we refer the reader to [3] for more interesting results about principal prime ideals in any commutative ring). Hence \( R \) is a PID. \( \square \)

Corollary 2.16. Let \( R \) be an integral domain. Then \( R \) is a PID if and only if \( R \) is a UFD and the family of all principal prime ideals of \( R \) has P.A-property.
Proof. It is clear that if \( R \) is a \( PID \), then each subset of (prime) ideals of \( R \) has \( A \)-property. Conversely, assume that \( R \) is a \( UFD \) which the family of all principal prime ideals of \( R \) has \( P.A \)-property. Let \( \mathcal{I}(R) \) be the set of all prime elements of \( R \) up to associate. Then clearly \( M \subseteq \bigcup_{p \in \mathcal{I}(R)} \{ p \} \), since \( R \) is a \( UFD \). Therefore by assumption we conclude that \( M \subseteq (p) \), for some \( p \in \mathcal{I}(R) \), therefore \( M \) is principal. Thus \( R \) is a \( PID \).

We remind the reader that if \( X \) is a completely regular Hausdorff topological space, then \( C(X) \) denotes the ring of all continuous real functions on \( X \). For each \( x \in X \), let \( M_x = \{ f \in C(X) \mid f(x) = 0 \} \), it is clear that \( C(X)/M_x \cong \mathbb{R} \) as ring and therefore \( M_x \in \text{Max}(C(X)) \), for each \( x \in X \).

**Corollary 2.17.** Let \( X \) be a topological space and \( A \subseteq \text{Spec}(C(X)) \). Then \( A \) has \( P.A \)-property for finitely generated ideals of \( R \), consequently, \( A \) has \( P.A \)-property if and only if \( A \) is compact. In particular, if \( A \) is a subset of \( X \) satisfies at least one of the following conditions:

1. \( X \) is compact and \( A \) is a closed subset of \( X \).
2. \( A \) is a compact subset of \( X \).

Then \( M_A = \{ M_a \mid a \in A \} \) has \( P.A \)-property for \( C(X) \).

Proof. First we remind that for each prime ideal \( P \) of \( C(X) \), the ring \( C(X)/P \) is a totally ordered ring, see [10, Theorem 5.5]. Now let \( I = \langle f_1, \ldots, f_n \rangle \) be a finitely generated ideal of \( C(X) \) which is contained in the union of \( A \). Thus there exists a \( P \in A \) such that \( f_1^2 + \cdots + f_n^2 \in P \), which by the previous fact immediately implies that \( f_i \in P \). Therefore \( I \subseteq P \). Thus by Theorem 2.5 we conclude that \( A \) has \( P.A \)-property if and only if \( A \) is compact. Now assume that (1) or (2) holds, then it is obvious that \( A \) is compact in case (1). Now one can easily see that \( M_A \) is compact and therefore by the previous part has \( P.A \)-property.

**Corollary 2.18.** Let \( R \) be a ring, \( X \) a topological space and \( f : R \to C(X) \) a ring homomorphism. Then the following hold:

1. If \( A \subseteq \text{Spec}(C(X)) \) is compact, then \( A' = \{ f^{-1}(P) \mid P \in A \} \) has \( P.A \)-property for \( R \).
2. If \( I \) is an ideal of \( R \) then \( I^e = \text{C}(X) \) if and only if \( f(I) \) contains a unit of \( C(X) \).

Proof. Let \( I \) be an ideal of \( R \) which is contained in \( \bigcup_{P \in A} f^{-1}(P) \). Thus \( f(I) \) is contained in \( S := \bigcup_{P \in A} P \). We claim that \( I^e = f(I)C(X) \) is contained in \( S \) and therefore is a proper ideal of \( C(X) \). If \( a_1, \ldots, a_n \) are in \( I \), then \( f(a_1^2 + \cdots + a_n^2) \in C(X) \), which shows that \( I^e \subseteq S \) and hence \( I^e \) is proper. Now since \( A \) has \( P.A \)-property, we deduce that there exists a \( P \in A \) such that \( f(a_1^2 + \cdots + a_n^2) \subseteq P \). Since \( C(X)/P \) is a totally ordered ring we infer that \( f(a_i) \in P \), for each \( i \). Therefore we conclude that \( C(X)f(a_1)^2 + \cdots + C(X)f(a_n)^2 \subseteq P \), which shows \( I^e \subseteq S \) and \( I^e \) is proper.

Let \( K \) be a field, \( P \) be a point in the affine space \( K^n \), and \( R = K[x_1, \ldots, x_n] \) be polynomial ring of \( n \) variable over \( K \). Then \( M_P = \{ f \in R \mid f(P) = 0 \} \) is a maximal ideal of \( R \). We remind that \( \text{Spec}(R) \) and \( K^n \) are noetherian spaces (by Zariski topologies). Now the following is in order.

**Corollary 2.19.** Assume that \( K \) be a formally real field and \( X \) a subset of affine space \( K^n \). Then \( M_X = \{ M_P \mid P \in X \} \) has \( P.A \)-property for \( R = K[x_1, \ldots, x_n] \).

Proof. Let \( I = \langle f_1, \ldots, f_n \rangle \) be an ideal of \( R \) which is contained in \( \bigcup_{P \in X} M_P \). Now since \( f := f_1^2 + \cdots + f_n^2 \in I \), we infer that there exists a \( P \in X \) such that \( f \in M_P \). Thus \( f(P) = 0 \) and since \( K \) is formally real we immediately conclude that \( f_i(P) = 0 \) for each \( i \). Thus \( I \subseteq M_P \) and we are done.

**Theorem 2.20.** Let \( R, T \) be rings and \( f \) a ring homomorphism from \( R \) into \( T \). If \( A \subseteq \text{Spec}(T) \) has \( P.A \)-property, then \( A' = \{ f^{-1}(P) \mid P \in A \} \) is compact. Moreover, if \( Im(f) \subseteq T \) has lying-over (in particular, if \( T \) is integral over \( Im(f) \)), then \( M = \{ f^{-1}(M) \mid M \in \text{Max}(T) \} \) has \( P.A \)-property for \( R \).

Proof. Let \( \{ V_A(I_\alpha) \}_{\alpha \in \Gamma} \) be an open cover for \( A' \), where each \( I_\alpha \) is an ideal of \( R \). Hence we infer that \( V_A(I) = \emptyset \), where \( I = \bigcap_{\alpha \in \Gamma} I_\alpha \). Thus for each \( P \in A \) we have \( I \not\subseteq f^{-1}(P) \). Therefore for each \( P \in A \) we conclude that \( I^e \not\subseteq P \). Now since \( A \) has \( P.A \)-property we infer that \( I^e \not\subseteq \bigcup_{P \in A} P \). Hence we deduce that there exist \( a_1, \ldots, a_n \) in \( \Gamma \) such that \( (a_1 + \cdots + a_n)^e \not\subseteq \bigcup_{P \in A} P \). Thus for each \( P \in A \) we have \( (a_1 + \cdots + a_n)^e \not\subseteq P \) and therefore \( I_\alpha + \cdots + I_\alpha \) \( \not\subseteq f^{-1}(P) \) for each \( P \in A \). This shows that
\[ A' = V_{A'}(I_{\alpha}) \cap \cdots \cap V_{A'}(I_{\alpha}) \cap I, \text{ i.e., } A' \text{ is compact.} \]

Now suppose that \( \text{Im}(f) \leq T \) has lying-over and \( I \) be an ideal of \( R \) which is contained in union of \( \mathcal{M} \). Thus \( f(I) \subseteq \bigcup_{N \in \text{Max}\{\text{Im}(f)\}} N \). Therefore by Krull Maximal Ideal Theorem, \( f(I) \) is contained in a maximal ideal \( N \) of \( \text{Im}(f) \). Since lying-over holds, we conclude that there exists a maximal ideal \( M \) of \( T \) over \( N \) and therefore contains \( f(I) \). Thus \( I \) is contained in \( f^{-1}(M) \) and we are done. \( \Box \)

We remind that if \( M \) is a finitely generated \( R \)-module, then \( \text{supp} (M) = V(\text{ann}(M)) \).

**Proposition 2.21.** Let \( R \) be a ring and \( \mathcal{A} = \{ P_{\alpha} \}_{\alpha \in \Gamma} \) be a compact set of prime ideals of \( R \). If \( I \) is an ideal of \( R \) which is contained in \( \bigcup_{\alpha \in \Gamma} P_{\alpha} \), then either \( \mathcal{A} \subseteq \text{supp}(I) \) or \( I \subseteq P_{\alpha} \) for some \( \alpha \in \Gamma \). In particular, if \( I \) is finitely generate and for each \( \alpha \in \Gamma \), \( I \not\subseteq P_{\alpha} \), then \( \mathcal{A} \subseteq \text{supp}(I) \).

**Proof.** Similar to the proof of Theorem 2.20, if for each \( \alpha \), \( I \) is not contained in \( P_{\alpha} \), then there exists a finitely generated ideal \( J \) of \( R \) which is contained in \( I \), but \( J \) is not contained in each \( P_{\alpha} \). Hence we infer that for each \( \alpha \in \Gamma \), \( \text{Ann}(J) \subseteq P_{\alpha} \). Thus by the above comments for each \( \alpha \in \Gamma \), \( P_{\alpha} \in \text{supp}(J) \subseteq \text{supp}(I) \). The final part is evident by Proposition 2.1. \( \Box \)

Let us remind the reader some needed facts from the literature for the next results. In [17], McAdam proved that if \( R \) is a ring such that for each maximal ideal \( M \) of \( R \), the residue ring \( R/M \) is infinite, then each finite set of ideals of \( R \), has \( A \)-property, i.e., if \( I, J, J_{1}, \ldots, J_{n} \) are ideals of \( R \) and \( I \subseteq \bigcup_{k=1}^{n} J_{k} \), then \( I \subseteq J_{k} \) for some \( k \). In [20], Quartararo and Butts, called an ideal \( I \) of \( R \) with the latter property a u-ideal. They proved that each invertible ideal of a ring \( R \) is a u-ideal (see [20] Theorem 1.5) and characterized rings for which each ideal of \( R \) is a u-ideal, and called them u-rings. In fact they proved that a ring \( R \) is a u-ring if and only if for each maximal ideal \( M \) of \( R \) either \( R/M \) is infinite or \( R/M \) is Bezout ring (see [20] Theorem 2.6). Finally, in [22] Corollary 2.6], Sharpe and Vamos proved that if \( (R, M) \) is a local noetherian ring with uncountable residue field and \( I, J_{1}, J_{2}, \ldots \) are ideals of \( R \) such that \( I \subseteq \bigcup_{k=1}^{\infty} J_{k} \), then \( I \subseteq J_{k} \) for some \( k \).

**Corollary 2.22.** Let \((R, M)\) be a local ring which is either an uncountable artinian or a noetherian with \(|R| > 2^{\aleph_0}\). If \( I, J_{1}, J_{2}, \ldots \) be ideals of \( R \) such that \( I \subseteq \bigcup_{n=1}^{\infty} J_{n} \), then \( I \subseteq I_{n} \) for some \( n \geq 1 \).

**Proof.** If \( R \) is an uncountable artinian ring, then by the proof of [5] Proposition 1.4, \( R/M \) is uncountable and therefore we are done by [22] Corollary 2.6]. If \( R \) is a noetherian with \(|R| > 2^{\aleph_0}\), then by the proof of [7] Corollary 2.6], there exist a natural number \( n \) such that \( R/M^{n} \) is uncountable and similar to the proof of the first part \( R/M \) is uncountable and hence we are done. \( \Box \)

**Corollary 2.23.** Let \( R \) be a noetherian integral domain with \(|R| > 2^{\aleph_0}\), then \( R \) is a u-ring.

**Proof.** For each maximal ideal \( M \) of \( R \), by the proof of [4] Corollary 2.7], there exists a natural number \( n \) such that the ring \( R/M^{n} \) is uncountable. Therefore similar to the proof of the first part of the previous corollary we infer that \( R/M \) is uncountable. Thus we are done by [20] Theorem 2.6]. \( \Box \)

Let \( R \) be a ring. A proper subring \( S \) (1 \( R \in S \)) is called a maximal subring if there exists no other subring of \( R \) between \( S \) and \( R \). We refer the reader to [4 – 7] for the existence of maximal subrings in commutative rings.

**Corollary 2.24.** Let \( R \) be noetherian integral domain with nonzero characteristic which is not equal to its prime subring (i.e., \( R \not\cong \mathbb{Z} \), where \( p = \text{Char}(R) \)) then either \( R \) has a maximal subring or \( R \) is a u-ring.

**Proof.** Assume that \( R \) has no maximal subring, then clearly \( R \) is infinite and by [5] Corollary 2.4], we infer that \( R \) is countable. Now by [3] Proposition 3.14], we conclude that for each proper (maximal) ideal \( I \) of \( R \), the residue ring \( R/I \) is infinite. Thus we are done by [20] Theorem 2.6]. \( \Box \)

**Corollary 2.25.** Each infinite artinian local ring is a u-ring.

**Proof.** Let \((R, M)\) be an infinite artian local ring, then by [5] Corollary 1.5], we deduce that \(|R/M| = |R| \). Hence we are done by [20] Theorem 2.6]. \( \Box \)

One of the application of prime avoidance lemma is a theorem which referred in [15] Theorem 124] to E. Davis: Let \( R \) be a commutative ring, \( I \) an ideal of \( R \), \( a \in R \) and \( P_{1}, \ldots, P_{n} \) prime ideals of \( R \); if \( Ra + I \not\subseteq \bigcup_{i=1}^{n} P_{i} \), then \( a + c \not\subseteq \bigcup_{i=1}^{n} P_{i} \), for some \( c \in I \). In particular, if \( R \) is a semilocal ring and \( Ra + I = R \), then there exists \( c \in I \) such that \( a + c \) is a unit of \( R \). The latter result is true for non-commutative semilocal rings as Bass’ Stable Range Theorem. Now the following is in order.
Theorem 2.26. Let $R$ be a ring, $A \subseteq \text{Spec}(R)$, $I$ an ideal of $R$, $a \in R$ and $Q_1, \ldots, Q_n$ be prime ideals of $R$. Assume that $Ra + I \not\subseteq \bigcup_{P \in A} V_P$. If $V_A(a)$ has P.A-property for $R$ (in particular, if $V(a) \subseteq A$ or $A \cap \text{Spec}(R) = V(a)$), then there exists $c \in I$ such that $a + c \not\subseteq \bigcup_{P \in A} V_P \cup Q_1 \cup \cdots \cup Q_n$.

Proof. We may assume that $Q_i \not\subseteq P$, for each $i$ and $P \in V_A(a)$. If $n = 0$, then by assumption there exist $c \in I$ such that $ra + c \not\subseteq \bigcup_{P \in A} V_P$. Since each $P \in V_A(a)$ contains $a$, we infer that $c \not\subseteq \bigcup_{P \in A} V_P$ and hence $a + c \not\subseteq \bigcup_{P \in A} V_P$. Thus the theorem holds for $n = 0$. Hence assume that $n \geq 1$. Since $V_A(a)$ has P.A-property, we conclude that $Q_1 \cap \cdots \cap Q_n \not\subseteq \bigcup_{P \in A} V_P$. Therefore there exists $d \in (Q_1 \cap \cdots \cap Q_n) \setminus \bigcup_{P \in A} V_P$.

Corollary 2.27. Let $R$ be a noetherian QR-domain. Assume that $A \subseteq \text{Spec}(R)$, $I$ an ideal of $R$ and $a \in R$ such that $Ra + I \not\subseteq \bigcup_{P \in A} V_P$. If $A \setminus V_A(a)$ is finite, then there exists $c \in I$ such that $a + c \not\subseteq \bigcup_{P \in A} V_P$.

Proof. By Corollary 2.24, $V_A(a)$ has P.A-property and hence we are done.

Theorem 2.26 might mislead us to generalize the Bass’ Stable Range Theorem for semilocal commutative rings, to commutative rings in which every non-unit element is not contained in only finitely many maximal ideals, but in fact the latter result is the same result, since one can easily see that (by Zariski topology) a ring $R$ is semilocal if and only if each non-unit element is not contained in only finitely many maximal ideals.

Finally in this paper we want to give a valuation version of Davis Theorem. First we need some observation from [3]. Let $V, V_1, \ldots, V_n$ be valuations for a field $K$, if $V \subseteq \bigcup_{i=1}^n V_i$, then $V \subseteq V_i$ for some $i$, see [3] Corollary 3.10 (this fact is called Valuation Avoidance Lemma). More generally, if $W_1, \ldots, W_m$ are also valuation for $K$ and $\bigcap_{i} W_i \subseteq \bigcup_{i=1}^m W_i$, then there exist $i$ and $j$ such that $W_j \subseteq V_i$, see [3] Remark 3.11. We refer the reader to [13] Theorem 6 and Corollary 8 for generalization of these facts. Now the following is in order.

Theorem 2.28. Let $V, V_1, \ldots, V_n$ be valuations for a field $K$ and $x \in K$. If $V[x] \not\subseteq \bigcup_{i=1}^n V_i$, then there exists $v \in V$ such that $v + x \not\subseteq \bigcup_{i=1}^n V_i$.

Proof. We may assume that $V_i, \ldots, V_n$ are incomparable and $x \in V_1 \cap \cdots \cap V_k$ but $x \not\in V_{k+1} \cup \cdots \cup V_n$. If $k = 0$, then it suffices to take $v = 0$; and if $k = n$, then by Valuation Avoidance Lemma we infer that $V \not\subseteq V_1 \cup \cdots \cup V_n$, for otherwise $V \subseteq V_i$ for some $i$ and therefore $V[x] \subseteq V_i$, which is absurd. Thus there exists $v \in V \setminus (V_1 \cup \cdots \cup V_n)$ and clearly $v + x \not\subseteq V_1 \cup \cdots \cup V_n$. Hence suppose that $1 \leq k \leq n - 1$. We claim that $(V \cap V_{k+1} \cap \cdots \cap V_n) \not\subseteq (V \cup V_{k+1} \cup \cdots \cup V_n)$. To see this note that $V \cap V_{k+1} \cap \cdots \cap V_n \subseteq (V \cup V_{k+1} \cup \cdots \cup V_n)$, then by [3] Remark 3.11, either $V \cap V_1 \cup \cdots \cup V_n$ does not contain such $x$, for some $1 \leq i \leq k$, and therefore $V[x] \subseteq V_i$ which is impossible or $V_i \subseteq V_{k+1}$ for some $1 \leq j \leq k$ and $k+1 \leq j \leq n$, which again is impossible since $V_i$’s are incomparable. Thus there exists $v \in (V \cap V_{k+1} \cap \cdots \cap V_n) \not\subseteq (V \cup V_{k+1} \cup \cdots \cup V_n)$ and therefore $v + x \not\subseteq \bigcup_{i=1}^n V_i$.

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