ABSTRACT. In this paper, as a generalization to content algebras, we introduce amount algebras. Similar to the Anderson-Badawi $\omega_{R[[X]][I[X]]} = \omega_R(I)$ conjecture, we prove that under some conditions, the formula $\omega_B(I^F) = \omega_R(I)$ holds for some amount $R$-algebras $B$ and some ideals $I$ of $R$, where $\omega_B(I)$ is the smallest positive integer $n$ that the ideal $I$ of $R$ is $n$-absorbing. A corollary to the mentioned formula is that if, for example, $R$ is a Prüfer domain or a torsion-free valuation ring and $I$ is a radical ideal of $R$, then $\omega_R([X][I][X])] = \omega_R(I)$.

1. INTRODUCTION

In this paper, all rings are commutative with identity and all algebras are unitary [10]. Let us recall that a proper ideal $I$ of a ring $R$ is an $n$-absorbing ideal of $R$, if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are $n$ of the $x_i$'s whose product is in $I$. Anderson and Badawi [1] conjectured that $\omega_{R[[X]][I[X]]} = \omega_R(I)$ (Anderson-Badawi $\omega$ Conjecture) for each ideal $I$ of an arbitrary ring $R$, where

$$\omega_R(I) = \min\{n: I \text{ is an } n\text{-absorbing ideal of } R\}.$$  

In this direction, the author proved that if $R$ is a Prüfer domain, then for any content $R$-algebra $B$, $\omega_B(IB) = \omega_R(I)$ and since any polynomial ring $R[X]$ is a content $R$-algebra (see Hilfssatz von Dedekind-Mertens on p. 128 in [9]), it is clear that the Anderson-Badawi $\omega$ conjecture is true if $R$ is a Prüfer domain [11, Corollary 11]. The main purpose of this paper is to prove that under some conditions the formula $\omega_{R[[X]][I[X]]]} = \omega_R(I)$ holds as well. In fact, inspired by the recent papers of Epstein and Shapiro [5] and Kang et al. [8], we introduce amount algebras and show that under some conditions - that we are going to report in the upcoming passages - some formulas similar to $\omega_{R[[X]][I[X]]]} = \omega_R(I)$ holds in amount algebras and a corollary to these results is that under some conditions $\omega_{R[[X]][I[X]]]} = \omega_R(I)$ is also true. Here is a brief sketch of the contents of our paper:

In Definition 1 we introduce the concept of amount functions as follows:

Let $R$ be a ring and $B$ an $R$-algebra. We say a function $A$ from $B$ to the set of ideals $\text{Id}(R)$ of $R$ defined by $f \mapsto A_f$ is an amount function if the following properties hold for all $r \in R$ and $f, g \in B$:

1. $A$ preserves 0 and 1, i.e. $A_0 = (0)$ and $A_1 = R$.
2. If $A_f = (0)$ then $f = 0$.

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(3) $A$ is homogeneous, i.e. $A_{rf} = rA_f$.
(4) $A$ is submultiplicative, i.e. $A_{fg} \subseteq A_f A_g$.

A general example for amount functions is the content function $c$ over a faithfully flat $R$-algebra $B$ with this additional property that $B$ as an $R$-module is content (check Theorem $[3]$). Other examples (see Examples $[2]$) include the function $A$ defined on power series rings $R[[X]]$ by $A_f = (r_0, r_1, \ldots, r_n, \ldots)$, where $f = r_0 + r_1 X + \cdots + r_n X^n + \cdots$ is an element of $R[[X]]$ $[6]$. On the other hand, for all $f, g \in R[[X]]$, we have the following amount formulas:

- $A_{f}^{n+1} A_g = A_f^n A_{fg}$, for some $n$, if $R$ is Noetherian and $n \in \mathbb{N}_0$ depending on $g$ is large enough $[5]$ Theorem 2.6).
- $A_f^2 A_g = A_f A_{fg} = A_g A_{gf}$ if $D$ is a valuation ring $[8]$ Theorem 2.8).
- $(A_f A_g)^2 = A_f A_g A_{fg}$ if $D$ is a Prüfer domain $[8]$ Corollary 2.9).

Inspired by the amount formulas mentioned in above, we define amount algebras (check Definition $[6]$) as follows:

Let $R$ be a ring and $B$ an $R$-algebra. We say $B$ is an amount $R$-algebra if the following conditions hold:

- There is an amount function $A$ from $B$ to $\text{Id}(R)$ defined by $f \mapsto A_f$ with this property that for all $f, g \in B$, there are non-negative integers $m, n$ such that
  
  $A_f^m A_g^n A_{fg} = A_f^{m+1} A_g^{n+1}$.

- There is a function $\varepsilon$ from $\text{Id}(R)$ to $\text{Id}(B)$ defined by $I \mapsto I^\varepsilon$ with the following properties:
  1. $A_f \subseteq I$ if and only if $f \in I^\varepsilon$, for all $f \in B$ and $I \in \text{Id}(R)$.
  2. $I^\varepsilon \cap R = I$, for all $I \in \text{Id}(R)$.

Let us recall that an ideal $I$ of a commutative ring $R$ is strongly $n$-absorbing if whenever

$$I_1 \cdots I_{n+1} \subseteq I$$

for some ideals $I_1, \ldots, I_{n+1}$ of $R$, then there are $n$ of the $I_i$’s whose product is a subset of $I$.

In Theorem $[22]$ we prove that if $R$ is a ring such that any $n$-absorbing ideal $I$ of $R$ is strongly $n$-absorbing for any positive integer $n$, also $B$ is an amount $R$-algebra, and $B$ is Gaussian, then $\omega_B(I^\varepsilon) = \omega_R(I)$. A corollary (see Corollary $[23]$) to this is that if $I$ is an ideal of a Dedekind domain $D$, then

$$\omega_{D[[X]]}(I[[X]]) = \omega_{D[X]}(I[X]) = \omega_D(I).$$

Note that an amount $R$-algebra $B$ is Gaussian if $A_{fg} = A_f A_g$ for all $f, g \in B$ (check Definition $[13]$).

Also in Theorem $[24]$ we show that if $R$ is a ring such that any $n$-absorbing ideal $I$ of $R$ is strongly $n$-absorbing for any positive integer $n$, $B$ is an amount $R$-algebra, and $I$ is a radical ideal of $R$, then $\omega_B(I^\varepsilon) = \omega_R(I)$. A corollary (see Corollary $[25]$ and Corollary $[26]$) to this result is that if $I$ is a radical ideal of a ring $R$, and either $R$ is a torsion-free Noetherian ring, or $D$ is a Prüfer domain, or a torsion-free valuation ring, then

$$\omega_{D[[X]]}(I[[X]]) = \omega_{D[X]}(I[X]) = \omega_D(I).$$
We end our paper by conjecturing that if $I$ is an ideal of a ring $R$, then
\[ \omega_R[[x]](I[[x]]) = \omega_R(I). \]

2. AMOUNT ALGEBRAS

We begin this section by introducing the amount functions.

**Definition 1** (Amount functions). Let $R$ be a ring and $B$ an $R$-algebra. We say a function $A$ from $B$ to the set of ideals $\text{Id}(R)$ of $R$ is an amount function if the following properties hold for all $r \in R$ and $f, g \in B$:

1. $A$ preserves 0 and 1, i.e. $A_0 = (0)$ and $A_1 = R$.
2. If $A_f = (0)$ then $f = 0$.
3. $A$ is homogeneous, i.e. $A_{sf} = sA_f$.
4. $A$ is submultiplicative, i.e. $A_{fg} \subseteq A_f A_g$.

**Examples 2.** In the following, we bring two important examples for amount functions:

1. Let $(\Gamma, +, 0, <)$ be a totally ordered commutative additive monoid and $R$ be a ring. Let $f = r_1X^{\alpha_1} + r_2X^{\alpha_2} + \cdots + r_nX^{\alpha_n}$ be an element of the monoid ring $R[\Gamma]$. Define the content of $f$, denoted by $c(f)$, to be an ideal of $R$ generated by the coefficients of $f$, i.e.
   \[ c(f) := (r_1, r_2, \ldots, r_n). \]
   It is easy to verify that $c : R[\Gamma] \rightarrow \text{Id}(R)$ is an amount function. Note that by $\text{Id}(R)$, we mean the set of all ideals of the ring $R$.

2. Let us recall that an element $x$ of a totally ordered semigroup $(\Gamma, +, <)$ is finitely decomposable if there are only finitely many pairs $(y_i, z_i)$ of elements of $\Gamma$ such that $x = y_i + z_i$. Now, let $(\Gamma, +, 0, <)$ be a totally ordered additive commutative monoid. Assume that 0 is the least element of $\Gamma$ and that each element of $\Gamma$ is finitely decomposable (for example, let $\Gamma = \bigoplus \mathbb{N}_0$). Let $R$ be a ring and $R[[\Gamma]]$ be the set of all functions $f : \Gamma \rightarrow R$. Let $f$ and $g$ be arbitrary elements of $R[[\Gamma]]$ and define their addition and multiplication as follows:
   \[ (f + g)(x) = f(x) + g(x), \quad (fg)(x) = \sum_{y+z=x} f(y)g(z). \]
   It is straightforward to see that $R[[\Gamma]]$ is an $R$-algebra [6]. For each $f \in R[[\Gamma]]$, define $A_f$ to be an ideal of $R$ generated by all $f(s)$, i.e. coefficients of $f$. It is easy to see that the function $A$ from $R[[\Gamma]]$ to $\text{Id}(R)$ defined by $A \mapsto A_f$ is an amount function. For instance, for an element $f = s_0 + s_1X + \cdots + s_nX^n + \cdots$ in $R[[X]]$,
   \[ A_f = (s_0, s_1, \ldots, s_n, \ldots). \]
   Let us recall that if $B$ is an $R$-algebra. The content function $c : B \rightarrow \text{Id}(R)$ is defined by
   \[ c(f) = \bigcap \{ I \in \text{Id}(R) : f \in IB \}, \]
   where by $IB$, we mean the extension of the $R$-ideal $I$ in $B$. By definition, $B$ as an $R$-module is content if $f \in c(f)B$ for all $f \in B [12]$.

**Theorem 3.** Let $B$ be an $R$-algebra and a content $R$-module. The content function $c$ is an amount function if and only if $B$ is a faithfully flat $R$-module.
Proof. Let $B$ be an $R$-algebra. It is clear that $c(0) = (0)$. If $B$ is a content $R$-module, then $f \in c(f)B$ and $g \in c(g)B$, for arbitrary elements $f$ and $g$ in $B$ and so, $fg \in c(f)c(g)B$. This implies that $c(fg) \subseteq c(f)c(g)$ (see Proposition 1.1 in [13]). On the other hand, $B$ is flat if and only if $c(rf) = rc(f)$ for all $r \in R$ and $f \in B$ [14, Corollary 1.6]. Also, according to Corollary 1.6 and the Statement 6.1(a) in [14] and Proposition 1.1 in [15], if $B$ is a content and flat $R$-module, then $B$ is faithfully flat if and only if $c(1) = R$ and the proof is complete. □

Remark 4. If an $R$-algebra $B$ as a module is content, then $c(f)$ is finitely generated for all $f \in B$ [14, §1]. Now, let $R[[\Gamma]]$ be as the $R$-algebra defined in Examples 2. It is clear that for $f \in R[[\Gamma]]$, the ideal $A_f$ is not necessarily finitely generated.

The proof of the following is straightforward:

Proposition 5. Let $B$ be an $R$-algebra and $A$ an amount function from $B$ to $\text{Id}(R)$. Then the following statements hold:

1. $A_I = (r)$ for all $r \in R$. In particular in Definition 7 the condition $A_0 = (0)$ is superfluous.
2. The equality $A_fA_g = (0)$ implies $fg = 0$ for all $f, g \in B$.

Now we define amount algebras:

Definition 6. Let $R$ be a ring and $B$ an $R$-algebra. We say $B$ is an amount $R$-algebra if the following conditions hold:

1. There is an amount function $A$ from $B$ to $\text{Id}(R)$ defined by $f \mapsto A_f$ with this property that for all $f, g \in B$, there are non-negative integers $m, n$ such that

$$A_f^mA_g^nA_{fg} = A_f^{m+1}A_g^{n+1}$$

(The Amount Formula).

2. There is a function $\varepsilon$ from $\text{Id}(R)$ to $\text{Id}(B)$ defined by $I \mapsto I^\varepsilon$ with the following properties:

(a) $A_f \subseteq I$ if and only if $f \in I^\varepsilon$, for all $f \in B$ and $I \in \text{Id}(R)$.

(b) $I^\varepsilon \cap R = I$, for all $I \in \text{Id}(R)$.

Proposition 7. Let $B$ be an amount $R$-algebra. Then the following statements hold:

1. $f \in A_f^\varepsilon$ for all $f \in B$.

2. $I \subseteq J$ if and only if $I^\varepsilon \subseteq J^\varepsilon$ for all ideals $I$ and $J$ of $R$.

Proof. (1): Since $A_f \subseteq A_f$, by definition, $f \in A_f^\varepsilon$.

(2): Assume that $I \subseteq J$ and let $f \in I^\varepsilon$. By definition, $A_f \subseteq I$. So, $A_f \subseteq J$. This implies that $f \in J^\varepsilon$. On the other hand, if $I^\varepsilon \subseteq J^\varepsilon$, then $I^\varepsilon \cap R \subseteq J^\varepsilon \cap R$ which is equivalent to say that $I \subseteq J$. □

Let us recall that if $I$ and $J$ are ideals of a ring $R$ then $J$ is a reduction of $I$ if $J \subseteq I$ and $JI^k = I^{k+1}$ for some positive integer $k$ [13, Definition 1].

Lemma 8. Let $B$ be an amount $R$-algebra. Then $A_{fg}$ is a reduction of $A_fA_g$ for all $f, g \in B$.

Proof. Let $f, g \in B$. Then by definition, there are non-negative integers $m, n$ such that

$$A_f^mA_g^nA_{fg} = A_f^{m+1}A_g^{n+1}.$$ 

Let $k = 1 + \max\{m, n\}$. So, $A_{fg}(A_fA_g)^k = (A_fA_g)^{k+1}$. Clearly, $k$
is a positive integer and \(A_{fg} \subseteq A_f A_g\). Hence, \(A_{fg}\) is a reduction of \(A_f A_g\) and the proof is complete.

**Theorem 9.** Let \(B\) be an amount \(R\)-algebra. Then \(A_f A_g \subseteq \sqrt{A_{fg}}\) for all \(f, g \in B\).

**Proof.** Let \(P\) be a prime ideal of \(R\) containing \(A_{fg}\). By Lemma 8, \(A_{fg}\) is a reduction of \(A_f A_g\). So, \(A_{fg} (A_f A_g)^k = (A_f A_g)^{k+1}\) for some positive integer \(k\). This implies that \(P\) contains \(A_f A_g\). Hence, \(A_f A_g \subseteq \bigcap_{P \supseteq A_{fg}} P = \sqrt{A_{fg}}\). This completes the proof.

Let \(B\) be an \(R\)-algebra such that as an \(R\)-module, it is content and faithfully flat. Then, \(B\) is called to be a content \(R\)-algebra [14, §6] if for all \(f, g \in B\), there is a non-negative integer \(n\) such that the Dedekind-Mertens formula \(c(f)^{n+1}c(g) = c(f)^n c(fg)\) holds.

**Theorem 10.** Let \(B\) be a content \(R\)-algebra. Then \(B\) is an amount \(R\)-algebra.

**Proof.** Assume that \(B\) is a content \(R\)-algebra. By Theorem 3, \(c(f)\) is an amount function. Obviously, the Dedekind-Mertens formula is a kind of the amount formula given in Definition [6]. Now, define \(I^n = IB\). Clearly, \(c(f) \subseteq I\) if and only if \(f \in IB\) for all \(f \in B\) and \(I \in \text{Id}(R)\), since \(c(f)\) is the smallest ideal satisfying the condition \(f \in IB\) [14, §1]. Finally, it is clear that \(I \subseteq IB \cap R\). Now, let \(r \in IB \cap R\). So, \(c(r) \subseteq I\). But \(c(r) = (r)\) for all \(r \in R\). Therefore, \(r \in I\). Hence, \(IB \cap R \subseteq I\). From all we said, we conclude that \(B\) is an amount \(R\)-algebra and the proof is complete.

Let \((\Gamma, +, 0, <)\) be a totally ordered commutative additive monoid and \(R\) be a ring. Northcott [12] has proved that \(R[\Gamma]\) is a content \(R\)-algebra. Consequently, we have the following corollary:

**Corollary 11.** If \((\Gamma, +, 0, <)\) is a totally ordered commutative additive monoid and \(R\) is a ring, then the monoid ring \(R[\Gamma]\) is an amount \(R\)-algebra.

**Remark 12** (More examples for amount algebras). Let \(R\) be a ring and \(X\) an indeterminate over \(R\). Define \(A_f\) to be the \(R\)-ideal generated by the coefficients of \(f\) in the power series ring \(R[[X]]\) and set \(I^n = I[[X]]\). Note that \(I[[X]]\) is not in general equal to \(I \cdot R[[X]]\) [7, Proposition 1]). Now, it is easy to verify that all the properties necessary for \(R[[X]]\) to be an amount \(R\)-algebra hold except the possibility of the amount formula given in Definition [6]. However, \(R[[X]]\) is an amount \(R\)-algebra if \(R\) is either Noetherian [5, Theorem 2.6], or a Prüfer domain [8, Corollary 2.9], or a valuation ring [8, Theorem 2.8].

**Definition 13.** We say an amount \(R\)-algebra \(B\) is Armendariz if \(fg = 0\) implies \(A_f A_g = (0)\) for all \(f, g \in B\), where \(A\) is the amount function defined in Definition [1].

Let us recall that a ring \(R\) is reduced if \(r^n = 0\) for some \(n \in \mathbb{N}\) implies \(r = 0\) [10, p. 3].

**Theorem 14.** Let \(R\) be a reduced ring and \(B\) an amount \(R\)-algebra. Then \(B\) is Armendariz. In particular, for all \(f \in B\), we have the following:

\[ f \in Z_B(B) \implies fr = 0 \text{ for some } r \text{ in } R. \quad (\text{McCoy’s property}). \]

**Proof.** Let \(f\) and \(g\) be elements of \(B\) such that \(fg = 0\). By the amount formula in Definition [6] there are non-negative integers \(m\) and \(n\) such that

\[ A_f^{m+1} A_g^{n+1} = (0). \]
Therefore, \( A/R \) is reduced, \( A_fA_g = (0) \). So, we have already proved that \( B \) is Armendariz. Now let \( f \) be a zero-divisor in \( B \). By definition, there is a nonzero element \( g \) in \( B \) such that \( fg = 0 \). Since \( B \) is Armendariz \( A_fA_g = (0) \). Note that \( g \) is nonzero and so \( A_g \) is a nonzero ideal of \( R \). Take \( r \) to be a nonzero element of \( A_g \). Therefore, \( rA_f = (0) \). This implies that \( A_{rf} = (0) \). Hence, \( fr = 0 \), i.e. McCoy’s property holds. This completes the proof. \( \Box \)

**Theorem 15.** Let \( B \) be an amount \( R \)-algebra. Then \( P \) is a prime ideal of \( R \) if and only if \( P^e \) is a prime ideal of \( B \).

**Proof.** Let \( P \) be a prime ideal of \( R \) and \( fg \in P^e \) for arbitrary \( f, g \in B \). It is clear that \( A_{fg} \subseteq P \). On the other hand, by the amount formula in Definition 6, there are non-negative integers \( m \) and \( n \) such that

\[
A_f^mA_g^nA_{fg} = A_f^{m+1}A_g^{n+1}.
\]

Therefore, \( A_f^{m+1}A_g^{n+1} \subseteq P \). Since \( P \) is prime, either \( A_f \subseteq P \) or \( A_g \subseteq P \). This means either \( f \in P^e \) or \( g \in P^e \). Note that \( P^e \neq B \). Therefore, \( P^e \) is a prime ideal of \( B \).

Now let \( P^e \) be a prime ideal of \( B \) and \( r \) and \( s \) be elements of \( R \) such that \( rs \in P \). This implies that \( A_{rs} = (rs) \subseteq P \). So, \( rs \in P^e \). From this, we obtain that either \( r \in P^e \) or \( s \in P^e \) which is equivalent to say that either \( r \in P \) or \( s \in P \) and this completes the proof. \( \Box \)

In the following, we recall the definition of \( n \)-absorbing and strongly \( n \)-absorbing ideals, and also the definition of \( \omega_R(I) \) [1]. For more on \( n \)-absorbing ideals and related topics refer to the recent survey paper [2].

**Definition 16.** Let \( R \) be a ring.

1. A proper ideal \( I \) of \( R \) is an \( n \)-absorbing ideal of \( R \), if whenever \( r_1 \cdots r_{n+1} \in I \) for \( r_1, \ldots, r_{n+1} \in R \), then there are \( n \) of the \( r_i \)’s whose product is in \( I \).
2. If there is a positive integer \( n \) such that \( I \) is an \( n \)-absorbing ideal of \( R \), then

\[
\omega_R(I) = \min \{ n : \text{\( I \) is an \( n \)-absorbing ideal of \( R \)} \}.
\]

Otherwise, \( \omega_R(I) = \infty \).

3. A proper ideal \( I \) of \( R \) is a strongly \( n \)-absorbing ideal of \( R \) if whenever \( I_1 \cdots I_{n+1} \subseteq I \) for some ideals \( I_1, \ldots, I_{n+1} \) of \( R \), then there are \( n \) of the \( I_i \)’s whose product is a subset of \( I \).

The proof of the following statement is straightforward but we bring it only for the sake of reference.

**Proposition 17.** If \( I \) is an ideal of a ring \( R \), then \( \omega_R(I) \leq \omega_R[X](I[X]) \leq \omega_R[[X]](I[[X]]) \).

**Definition 18.** We say an amount \( R \)-algebra \( B \) is Gaussian if \( A_{fg} = A_fA_g \) for all \( f, g \in B \), where \( A \) is the amount function defined in Definition 1.

**Proposition 19.** If an amount \( R \)-algebra \( B \) is Gaussian then it is Armendariz.

**Proof.** Straightforward. \( \Box \)

**Examples 20.**

1. (A general example) Let \( B \) be an amount \( R \)-algebra such that \( A_f \) is a cancellation ideal of \( R \) for all nonzero elements \( f \) in \( B \). Then \( B \) is Gaussian.
(2) Let us recall that a ring $R$ is Gaussian if $c(fg) = c(f)c(g)$ for all $f, g \in R[X]$ \cite{16}. Now it is clear that if $R$ is a Gaussian ring, then the amount $R$-algebra $R[X]$ is Gaussian.

(3) If $D$ is a Dedekind domain, then the amount $D$-algebra $D[[X]]$ is Gaussian (Use Theorem 2.6 in \cite{5} and this fact that each nonzero ideal of a Dedekind domain is a cancellation ideal).

**Lemma 21.** Let $R$ be a ring and $I$ a proper ideal of $R$. Also, let $B$ be an amount $R$-algebra. If $I^e$ is $n$-absorbing, then so is $I$. Moreover, $\omega_R(I) \leq \omega_B(I^e)$.

**Proof.** Let $r_1 \cdots r_{n+1} \in I$. So, $A_{r_1 \cdots r_{n+1}} = (r_1 \cdots r_{n+1}) \subseteq I$. This implies that $r_1 \cdots r_{n+1} \in I^e$. Since $I^e$ is $n$-absorbing, $r_1 \cdots r_{i-1}r_{i+1}r_i$ is in $I^e$ for some index $i$. So,

$$r_1 \cdots r_{i-1}r_{i+1}r_i \in I^e \cap R = I.$$ 

Now, it is clear that $\omega_R(I) \leq \omega_B(I^e)$.

**Theorem 22.** Let $R$ be a ring such that any $n$-absorbing ideal $I$ of $R$ is strongly $n$-absorbing for any positive integer $n$. Let $B$ be an amount $R$-algebra. If $B$ is Gaussian then $\omega_B(I^e) = \omega_R(I)$.

**Proof.** By Lemma 21, $\omega_R(I) \leq \omega_B(I^e)$. Let $I$ be a proper ideal of $R$ such that $\omega_R(I) = n$ for a positive integer $n$. Our claim is that $I^e$ is an $n$-absorbing ideal of $B$. Assume that

$$f_1 \cdots f_{n+1} \in I^e,$$

for arbitrary $f_1, \ldots, f_{n+1} \in B$. It is clear that $A_{f_1 \cdots f_{n+1}} \subseteq I$. Since $B$ Gaussian, $A_{f_1 \cdots f_{n+1}} = A_{f_1} \cdots A_{f_{n+1}}$. By assumption, $I$ is a strongly $n$-absorbing ideal of $R$.

Therefore, $A_{f_1} \cdots A_{f_{i-1}} A_{f_{i+1}} \cdots A_{f_{n+1}} \subseteq I$ for some $i$. This implies that

$$A_{f_1} \cdots f_{i-1} f_{i+1} \cdots f_{n+1} \subseteq I.$$

And this means that

$$f_1 \cdots f_{i-1} f_{i+1} \cdots f_{n+1} \in I^e.$$ 

So, we have already proved that $n = \omega_R(I) \leq \omega_B(I^e) \leq n$. Finally, it is easy to see that $\omega_B(I^e) = \infty$ if and only if $\omega_R(I) = \infty$, and the proof is complete.

**Corollary 23.** Let $D$ be a Prüfer domain. If an amount $D$-algebra $B$ is Gaussian, then $\omega_B(I^e) = \omega_D(I)$ for each ideal $I$ of $D$. In particular, if $I$ is an ideal of a Dedekind domain $D$, then

$$\omega_{D[[X]]}(I[[X]]) = \omega_D(I) = \omega_D(I).$$

**Proof.** Since $D$ is a Prüfer domain, any $n$-absorbing ideal of $D$ is strongly $n$-absorbing for each positive integer $n$ \cite{1] Corollary 6.9}. Now by Theorem 22, $\omega_B(I^e) = \omega_R(I)$. In particular, if $D$ is a Dedekind domain, by Examples 20

$$\omega_D(I)[[X]] = \omega_D(I[[X]]) = \omega_D(I),$$

and this completes the proof.

**Theorem 24.** Let $R$ be a ring such that any $n$-absorbing ideal $I$ of $R$ is strongly $n$-absorbing for any positive integer $n$. Let $B$ be an amount $R$-algebra. If $I$ is a radical ideal of $R$, then $\omega_B(I^e) = \omega_R(I)$. 


Proof. Let \( f_1 \cdots f_{n+1} \in I^e \). Obviously, \( A_{f_1 \cdots f_{n+1}} \subseteq I \). Let \( g = f_2 \cdots f_{n+1} \). By the amount formula in Definition 6, there are non-negative integers \( m, n \) such that
\[
A^m_{f_1} A^n_{g} \subseteq A^m_{f_1} A^n_{g} = A^m_{f_1} A^n_{g} + A^{m+1}A^{n+1}_{g},
\]
and since \( A_{f_1 g} \subseteq I \), we have \( A^{m+1}A^{n+1}_{g} \subseteq I \). Take \( u = \max\{m, n\} \). It is easy to see that
\[
(A_{f_1 A_g})^{u+1} = A^{u+1}A^{u+1}_{g} \subseteq I.
\]
Since \( I \) is a radical ideal of \( R \), we have \( A_{f_1 A_g} \subseteq I \).

Now let \( h = f_3 \cdots f_{n+1} \). It is clear that \( g = f_2 h \) and by the amount formula in Definition 6, there are non-negative integers \( k, l \) such that
\[
A^k_{f_2} A^l_{h} A^m_{g} = A^k_{f_2} A^l_{h} A^m_{g} = A^k_{f_2} A^l_{h} A^m_{g} \subseteq I.
\]
Obviously, we have the following:
\[
A^k_{f_1} A^{k+1}A^l_{h} A^{k+1}_{g} = A^k_{f_1} A^{k+1}A^l_{h} A^{k+1}_{g} = A^k_{f_1} A^{k+1}A^l_{h} A^{k+1}_{g} \subseteq I.
\]
Similarly, since \( I \) is a radical ideal of \( R \), we have \( A_{f_1 A_{f_2} A_h} \subseteq I \). Continuing this process, we obtain that
\[
A_{f_1 \cdots f_{n+1}} \subseteq I.
\]
Now if \( I \) is an \( n \)-absorbing ideal of \( R \), then according to our assumptions, \( I \) is strongly \( n \)-absorbing. Thus,\[
A_{f_1 \cdots f_{i-1} A_{f_{i+1}} \cdots A_{f_{n+1}}} \subseteq I
\]
for some \( i \).

On the other hand, by Definition 1, the amount function \( A \) is submultiplicative. Therefore,
\[
A_{f_1 \cdots f_{i-1} f_{i+1} \cdots f_{n+1}} \subseteq A_{f_1 \cdots f_{i-1} A_{f_{i+1}} \cdots A_{f_{n+1}}}.
\]
This implies that \( f_1 \cdots f_{i-1} f_{i+1} \cdots f_{n+1} \in I^e \) and so \( I^e \) is \( n \)-absorbing.

Now by considering Lemma 21, the rest of the proof is similar to the proof of Theorem 22. This completes the proof. \( \square \)

Let us recall that a ring \((R, +, \cdot)\) is torsion-free if \((R, +)\) is a torsion-free group [3].

Corollary 25. Let \( R \) be a torsion-free Noetherian ring and \( I \) a radical ideal of \( R \). Then
\[
\omega_{R[[X]]}(I[[X]]) = \omega_{R[X]}(I[X]) = \omega_R(I).
\]
Proof. Since \( R \) is Noetherian, by Theorem 2.6 in [5], \( R[[X]] \) is an amount \( R \)-algebra. On the other hand, since \( R \) is torsion-free, by Theorem 4.2 in [4], each \( n \)-absorbing ideal of \( R \) is strongly \( n \)-absorbing for any positive integer \( n \). By using Theorem 24, the proof of this corollary is complete. \( \square \)

Corollary 26. Let \( I \) be a radical ideal of a domain \( D \). If either \( D \) is a Prüfer domain or \( D \) is a torsion-free valuation ring, then
\[
\omega_{D[[X]]}(I[[X]]) = \omega_{D[X]}(I[X]) = \omega_D(I).
\]
Proof. If either \( D \) is a Prüfer domain or \( D \) is a torsion-free valuation ring, then by the Theorem 2.8 and the proof of Corollary 2.9 in [8], in each case, \( D[[X]] \) is an amount \( D \)-algebra. Also, in each of the mentioned cases, any \( n \)-absorbing ideal of \( D \) is strongly \( n \)-absorbing (see Corollary 6.9 in [11] and Theorem 4.2 in [4]). In view of Theorem 24, the proof of this corollary is complete. \( \square \)
Conjecture 27. Let $X$ be an indeterminate over a ring $R$. For any ideal $I$ of $R$,

$$\omega_{R[[X]]}(I[[X]]) = \omega_{R}(I).$$

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Department of Engineering Sciences, Golpayegan University of Technology, Golpayegan, Isfahan Province, Iran

Email address: nasehpour@gut.ac.ir, nasehpour@gmail.com