ASYMPTOTIC BEHAVIOR OF COMPLEX SCALAR FIELDS IN A FRIEDMAN-LEMAITRE UNIVERSE

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Abstract

We study the coupled Einstein-Klein-Gordon equations for a complex scalar field with and without a quartic self-interaction in a curvatureless Friedman-Lemaître Universe. The equations can be written as a set of four coupled first order nonlinear differential equations, for which we establish the phase portrait for the time evolution of the scalar field. To that purpose we find the singular points of the differential equations lying in the finite region and at infinity of the phase space and study the corresponding asymptotic behavior of the solutions. This knowledge is of relevance, since it provides the initial conditions which are needed to solve numerically the differential equations. For some singular points lying at infinity we recover the expected emergence of an inflationary stage.

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1. INTRODUCTION

The recent developments in particle physics and cosmology suggest that scalar fields may have played an important role in the evolution of the early Universe, for instance in primordial phase transitions, and that they may constitute part of the dark matter. Moreover, scalar fields are predicted by most of the particle physics models based on the unification of the fundamental forces, as for instance in superstring theories. Scalar particles are needed in cosmological models based on inflation, whose relevance is supported by the results of the COBE-DMR measurements that are consistent with an Harrison-Zel’dovich (scale invariant n=1) spectrum [1]. These facts, in particular inflation, motivated the study of the coupled Einstein-scalar field equations to determine the time evolution and also the gravitational equilibrium configurations of the scalar fields. The latter one in particular for massive complex scalar fields, which may form so-called boson stars [2, 3].

A detailed study of the solutions of the Einstein equations for a homogeneous isotropic Friedman-Lemaître Universe with a real scalar field has been done in particular by Belinsky et al. [4, 5, 6], see also ref.[7]. In this paper we extend, following ref.[4, 5, 6], these investigations to a complex scalar field.

This analysis is important in order to see the degree of generality of solutions possessing an inflationary stage and also due to the fact that these solutions constitute the background, starting from which one can study in the early Universe the time evolution of perturbations for the scalar field [8, 4]. A fact this, which is of relevance if scalar fields make up part of the dark matter. If this is the case they may also form compact objects, such as bose stars, or trigger the formation of observed large scale structures in the Universe.

Since we consider a complex scalar field with or without a quartic self-interaction in a curvatureless Friedman-Lemaître Universe, we get for the Einstein-Klein Gordon equations a set of four first order nonlinear differential equations, for which we study the phase portrait for the time evolution of the scalar field. We first determine the singular points of the differential equations and then find analytically the asymptotic behavior for the solutions nearby these points. For the singular point lying in a finite region of the phase space, we can use for $m \neq 0$, in an adapted coordinate system, the averaging method in order to get the asymptotic behavior of the solution, whereas, for the points lying at infinity, we first compactify the phase space on the lower hemisphere of the 3-dimensional sphere. We can then apply the Poincaré-Dulac theorem to get the corresponding asymptotic behavior. From the solution we then see if there is inflation and how long it lasts. For some singular points lying at infinity, we recover the expected emergence of
an inflationary stage.

The paper is organized as follows: in section 2 we present the basic equations which we will use. In section 3 we first study for a massive scalar field the singular point lying in the finite region of the phase space and then the singular points lying at infinity both with and without a quartic self-interaction. Section 4 is devoted to the massless scalar field with a quartic self-interaction and a short summary concludes the paper.

2. BASIC EQUATIONS

We consider a massive complex scalar field with quartic self-interaction in a Friedman-Lemaître Universe with the following action

\[ S = \int \left( -\frac{R}{16\pi G} + e_\mu \varphi e^\mu \varphi^* + m^2 \varphi \varphi^* + \lambda (\varphi \varphi^*)^2 \right) \sqrt{-g} \, d^4x , \]  

(1)

where \( g \) is the determinant of the metric

\[ ds^2 = g_{\mu\nu} \theta^\mu \theta^\nu = -\theta^0 \theta^0 + \delta_{ij} \theta^i \theta^j , \]  

(2)

with

\[ \theta^0 = dt , \quad \theta^i = \frac{a(t) dx^i}{(1 + \frac{k}{4} r^2)} , \quad r^2 = \sum_{i=1}^{3} x_i^2 \]

and \( e_\mu \) is the dual basis of \( \theta^\mu \). By varying the action with respect to \( g^{\mu\nu} \) we get the Einstein field equation

\[ G_{\mu\nu} = 8\pi GT_{\mu\nu} , \]  

(3)

with

\[ T_{\mu\nu} = e_\mu \varphi e^\nu \varphi^* + \sum_{\nu} e_\nu \varphi e_\mu \varphi^* - g_{\mu\nu} (g^{\alpha\beta} e_\alpha \varphi e_\beta \varphi^* + m^2 \varphi \varphi^* + \lambda (\varphi \varphi^*)^2) . \]  

(4)

The (00) component of eq.(3) leads to the constraint equation

\[ H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} (\dot{\varphi} \dot{\varphi}^* + m^2 \varphi \varphi^* + \lambda (\varphi \varphi^*)^2) , \]  

(5)

with \( H = \dot{a}/a \) and dot means derivative with respect to time. For the (ij) component we have

\[ (-2\dot{H} - 3H^2 - \frac{k}{a^2}) \delta_{ij} = 8\pi G (\dot{\varphi} \dot{\varphi}^* - m^2 \varphi \varphi^* - \lambda (\varphi \varphi^*)^2) \delta_{ij} . \]  

(6)

By varying the action with respect to \( \varphi^* \) and \( \varphi \) we get the Klein-Gordon equation

\[ \ddot{\varphi} + 3H \dot{\varphi} + m^2 \varphi + 2\lambda (\varphi \varphi^*) \varphi = 0 , \]  

(7)
and its complex conjugate. The system is fully determined by the independent eqs. (5) and (7). In fact, one can easily show that eq. (6) follows from eqs. (5) and (7). For the massive scalar field case we use the following dimensionless variables

\[
\begin{align*}
t &\to \eta = m t, \\
\lambda &\to \Lambda = \frac{\lambda}{8\pi G m^2}, \\
\varphi &\to x_1 + i x_2 = \sqrt{\frac{8\pi G}{3}} \varphi, \\
\dot{\varphi} &\to y_1 + i y_2 = \sqrt{\frac{8\pi G}{3} \frac{\varphi}{m}}, \\
H &\to z = \frac{H}{m}, \\
k &\to \tilde{k} = \frac{k}{m^2}.
\end{align*}
\]

This way, we get for eqs. (5) and (7) the following set

\[
\begin{align}
\frac{\tilde{k}}{a^2} + z^2 &= y_1^2 + y_2^2 + x_1^2 + x_2^2 + 3\Lambda(x_1^2 + x_2^2)^2, \\
y_1' &= -3zy_1 - x_1 - 6\Lambda x_1(x_1^2 + x_2^2), \\
x_1' &= y_1, \\
y_2' &= -3zy_2 - x_2 - 6\Lambda x_2(x_1^2 + x_2^2), \\
x_2' &= y_2,
\end{align}
\]

where prime means derivative with respect to \( \eta \). The only singular point (defined as the point \((x_1, x_2, y_1, y_2)\) for which the right hand side of eqs. (9) - (13) vanishes), which we denote by \( A \), lying in a finite region of the phase space (defined by \( \varphi, \dot{\varphi} \) or equivalently \( x_1, x_2, y_1, y_2 \)) is the coordinate origin. Eqs. (9) - (13) are invariant under the transformations

\[
\begin{align*}
a) &\quad x_1 \to -x_1 \quad \text{and} \quad y_1 \to -y_1, \\
b) &\quad x_2 \to -x_2 \quad \text{and} \quad y_2 \to -y_2, \\
c) &\quad x_2 \leftrightarrow x_1 \quad \text{and} \quad y_2 \leftrightarrow y_1.
\end{align*}
\]

For every solution which describes an expanding Universe (i.e. \( z > 0 \)) there is a corresponding solution describing a collapsing Universe. This can be seen by performing one of the following transformations on the set of eqs. (9) - (13)

\[
\begin{align*}
a) &\quad \eta \to -\eta, \\
z \to -z, \\
x_1 \to -x_1, \\
x_2 \to -x_2, \\
b) &\quad \eta \to -\eta, \\
z \to -z, \\
x_1 \to -x_1, \\
y_2 \to -y_2, \\
c) &\quad \eta \to -\eta, \\
z \to -z, \\
y_1 \to -y_1, \\
x_2 \to -x_2, \\
d) &\quad \eta \to -\eta, \\
z \to -z, \\
y_1 \to -y_1, \\
y_2 \to -y_2.
\end{align*}
\]
For the massless scalar field case we use the dimensionless variables

\[ t \to \eta_0 = \frac{t}{\sqrt{8\pi G}} , \]
\[ \varphi \to x_{10} + ix_{20} = \frac{8\pi G}{3} \varphi , \]
\[ \dot{\varphi} \to y_{10} + iy_{20} = \frac{8\pi G}{\sqrt{3}} \dot{\varphi} , \]
\[ H \to z_0 = \sqrt{8\pi G}H , \]
\[ k \to \tilde{k}_0 = 8\pi G k , \]

and \( \lambda \) remains unchanged. This way, we get for eqs.(5) and (7) the set

\[ \frac{\tilde{k}_0}{a^2} + z_0^2 = y_{10}^2 + y_{20}^2 + 3\lambda(x_{10}^2 + x_{20}^2)^2 , \]  
(17)
\[ y_{10}' = -3z_0y_{10} - 6\lambda x_{10}(x_{10}^2 + x_{20}^2) , \]  
(18)
\[ x_{10}' = y_{10} , \]  
(19)
\[ y_{20}' = -3z_0y_{20} - 6\lambda x_{20}(x_{10}^2 + x_{20}^2) , \]  
(20)
\[ x_{20}' = y_{20} , \]  
(21)

where prime means here derivative with respect to \( \eta_0 \). These equations are also invariant under the transformations given by eqs.(14) and (15), and the coordinate origin is the only singular point lying in the finite region of the phase space.

3. MASSIVE SCALAR FIELD IN A CURVATURELESS FRIEDMAN-LEMAITRE UNIVERSE

As next we study the asymptotic behavior of the solutions of eqs.(4)-(8) nearby the singular points. Due to the increasing complexity of the analysis involved for a complex scalar field with respect to a real one, we restrict ourselves to the case \( k = 0 \). In section 5 we briefly comment on the extension to \( k = \pm 1 \). With \( \tilde{k} = 0 \) in eq.(3) we can then rewrite eqs.(16)-(19) in spherical coordinates, defined as follows

\[ x_1 = r \cos \vartheta_3 \cos \vartheta_1 , \]
\[ y_1 = r \cos \vartheta_3 \sin \vartheta_1 , \]
\[ x_2 = r \sin \vartheta_3 \cos \vartheta_2 , \]
\[ y_2 = r \sin \vartheta_3 \sin \vartheta_2 , \]

with \( \vartheta_1, \vartheta_2 \in [0, 2\pi) \) and \( \vartheta_3 \in [0, \pi) \). This way we obtain

\[ z^2 = r^2 + 3\Lambda r^4(\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2)^2 , \]  
(23)
\[ \vartheta_1' = -3z \sin \vartheta_1 \cos \vartheta_1 - 1 - 6\Lambda r^2(\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) \cos^2 \vartheta(24) \]
\[ \vartheta_2' = -3z \sin \vartheta_2 \cos \vartheta_2 - 1 - 6\Lambda r^2(\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) \cos^2 \vartheta(25) \]
\[ \vartheta_3' = \sin \vartheta_3 \cos \vartheta_3 [-3z (\sin^2 \vartheta_2 - \sin^2 \vartheta_1) - 6\Lambda r^2 (\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) (\sin \vartheta_2 \cos \vartheta_2 - \sin \vartheta_1 \cos \vartheta_1)] \]

\[ r' = -3rz (\cos^2 \vartheta_3 \sin^2 \vartheta_1 + \sin^2 \vartheta_3 \sin^2 \vartheta_2) - 6\Lambda r^3 (\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) \times (\cos \vartheta_1 \sin \vartheta_1 \cos^2 \vartheta_3 + \cos \vartheta_2 \sin \vartheta_2 \sin^2 \vartheta_3) \, . \] (27)

Due to the transformations given in eq. (14) which leave the equations invariant, we can restrict the domain of definition for \( \vartheta_1, \vartheta_2 \) and \( \vartheta_3 \) respectively to \([0, \pi), [0, \pi) \) and \([\pi/2, \pi) \). At the singular point \( A \) lying at \( r = 0 \) the above equations reduce to \( \vartheta'_1 = -1, \vartheta'_2 = -1, \vartheta'_3 = 0 \) and \( r' = 0 \). To get the asymptotic behavior of the solution nearby \( A \) we apply the averaging method (for details see ref. [10]). Thus, we have to solve the differential equation

\[ r' = -\frac{3}{2} r^2 \, , \] (28)

which is obtained by averaging eq. (27) over the angular variables. We get the asymptotic behavior for the solution nearby \( A \)

\[ x_1 = \frac{2}{3\eta} \cos \vartheta_{30} \cos (\eta - \eta_1) \, , \]
\[ y_1 = -\frac{2}{3\eta} \cos \vartheta_{30} \sin (\eta - \eta_1) \, , \]
\[ x_2 = \frac{2}{3\eta} \sin \vartheta_{30} \cos (\eta - \eta_2) \, , \]
\[ y_2 = -\frac{2}{3\eta} \sin \vartheta_{30} \sin (\eta - \eta_2) \, , \] (29)

where \( \vartheta_{30}, \eta_1 \) and \( \eta_2 \) are integration constants. We see that \( A \) is an asymptotically stable winding point. Setting \( \vartheta_{30} = 0 \) corresponds to consider only a real scalar field and we recover the solution discussed by Belinsky et al. in ref. [4, 5, 6].

We study now all singular points lying at infinity in phase space. First, we consider the case with no quartic self-interaction term.

### 3.1. Properties of the singular points lying at infinity for \( \Lambda = 0 \)

In order to find the singular points lying at infinity we perform a transformation which maps them on the boundary of a unit 3-sphere, defined as follows

\[
\begin{align*}
& r \to \rho = \frac{r}{1+r}, \\
& d\eta \to d\tau = \frac{d\eta}{1-\rho},
\end{align*}
\] (30)

with \( \rho \in [0,1) \). With this transformation eqs. (24) - (27) become

\[ \frac{d\rho}{d\tau} = -3\rho^2 (1-\rho) (\cos^2 \vartheta_3 \sin^2 \vartheta_1 + \sin^2 \vartheta_3 \sin^2 \vartheta_2) \, , \] (31)
\[
\frac{d\vartheta_1}{d\tau} = -3\rho \sin \vartheta_1 \cos \vartheta_1 - (1 - \rho), \quad (32)
\]
\[
\frac{d\vartheta_2}{d\tau} = -3\rho \sin \vartheta_2 \cos \vartheta_2 - (1 - \rho), \quad (33)
\]
\[
\frac{d\vartheta_3}{d\tau} = 3\rho \sin \vartheta_3 \cos \vartheta_3 (\sin^2 \vartheta_1 - \sin^2 \vartheta_2). \quad (34)
\]

Since the system of differential equations is well defined for \( \rho = 1 \), we can extend the domain of definition for \( \rho \) to the boundary. This way, we have now a compactified phase space. At infinity (i.e. setting \( \rho = 1 \) in the above eqs. (31)-(34)) we find two singular curves and two singular points, which are given by

\[
\begin{align*}
  l_1 & : \vartheta_1 = 0, \vartheta_2 = 0, \frac{\pi}{2} \leq \vartheta_3 < \pi, \\
  l_2 & : \vartheta_1 = \frac{\pi}{2}, \vartheta_2 = \frac{\pi}{2}, \frac{3\pi}{2} \leq \vartheta_3 < \pi, \\
  p_1 & : \vartheta_1 = \frac{\pi}{2}, \vartheta_2 = 0, \vartheta_3 = \frac{\pi}{2}, \\
  p_2 & : \vartheta_1 = 0, \vartheta_2 = \frac{\pi}{2}, \vartheta_3 = \frac{\pi}{2}.
\end{align*}
\]

To study the asymptotic behavior of the solutions nearby \( l_1 \) is rather involved, due to the fact that every point \( l_0 = (1, 0, 0, \vartheta_{30}) \in l_1 \) is non hyperbolic (see ref. [11] for details). We first perform a variable shift in eqs. (31)-(34) defined as follows: \( \delta \rho = \rho - 1, \delta \vartheta_1 = \vartheta_1, \delta \vartheta_2 = \vartheta_2 \) and \( \delta \vartheta_3 = \vartheta_3 - \vartheta_{30} \). We then expand the differential equations in the new coordinates up to third order in a sufficiently small neighborhood around the singular point \( l_0 \). Next, we define the linear coordinate transformation:

\[
\begin{align*}
  x &= \delta \rho, \\
  y &= -\frac{1}{3} \delta \rho + \delta \vartheta_1, \\
  v &= -\frac{1}{3} \delta \rho + \delta \vartheta_2, \\
  w &= \delta \vartheta_3,
\end{align*}
\]

such that we get a set of differential equations of the form

\[
(\vec{y})' = D\vec{y} + \vec{p}, \quad (35)
\]

where \( \vec{y} = (x, y, v, w) \). \( D \) is a diagonal matrix and \( \vec{p} \) is a vector whose components are polynomials in \( x, y, v \) and \( w \) containing monomials of degree 2 and 3. We can now use the Poincaré-Dulac theorem [12] to classify the monomials in \( \vec{p} \) into resonant and non-resonant ones. For the non-resonant monomials of degree \( n \), there is a polynomial change of coordinates of degree \( n \), so that they are transformed into polynomials of at least degree \( n + 1 \). This is not the case for the resonant monomials. The polynomial change of coordinates is found by solving the so-called homological equation (see ref. [12] for more details). Performing the polynomial change of coordinates on the differential equations, the non-resonant terms of degree 2 and 3 become of higher order and are thus neglected. Retaining only terms up to third order we obtain the set of
equations
\[
\frac{d\tilde{x}}{d\tau} = \frac{1}{3} \tilde{x}^3, \\
\frac{d\tilde{y}}{d\tau} = -3\tilde{y} - 3\tilde{x}\tilde{y} + \frac{2}{3} \sin^2 \vartheta_{30} (\tilde{x}^2\tilde{y} - \tilde{x}^2\tilde{v}), \\
\frac{d\tilde{v}}{d\tau} = -3\tilde{v} - 3\tilde{x}\tilde{v} - \frac{2}{3} \cos^2 \vartheta_{30} (\tilde{x}^2\tilde{y} - \tilde{x}^2\tilde{v}), \\
\frac{d\tilde{w}}{d\tau} = 0,
\]
where \((x, y, v, w) = (\tilde{x}, \tilde{y}, \tilde{v}, \tilde{w}) + \text{polynomials of degree 2 or higher in} \ (\tilde{x}, \tilde{y}, \tilde{v}, \tilde{w})\). Since the solution of eq.(36) does monotonically increase as a function of \(\tau\), \(l_0\) is a saddle point. The above equations can now be solved analytically and furthermore by transforming back to the original variables one gets for the asymptotic behavior nearby any point \(l_0\) of \(l_1\)

\[
\varphi = \frac{-M_p m t}{\sqrt{3}} e^{i\vartheta_{30}}, \\
\dot{\varphi} = \frac{-M_p m}{\sqrt{3}} e^{i\vartheta_{30}}, \\
H = \frac{-m^2 t}{3},
\]
for \(t \to -\infty\), corresponding to the initial singularity and where \(M_p = \frac{1}{\sqrt{8\pi G}}\). This solution corresponds to an outgoing line, the so-called separatrix. Using eq.(42) and the definition of \(H\) we have

\[
\frac{a(t_f)}{a(t_i)} = \exp \left( \frac{1}{6} m^2 (t_i^2 - t_f^2) \right)
\]
for \(t_f > t_i\).

The effective equation of state near \(l_0\) tends to \(\varepsilon = -p\), where \(\varepsilon = T_{00}\) and \(p = \frac{1}{3} (T_{11} + T_{22} + T_{33})\). The solutions near the separatrix are characterized by the fact that \(\dot{\varphi} \varphi^* \ll m^2 \varphi \varphi^*\) and the phase \(\vartheta_{30}\) being constant. If for a time \(t_f - t_i\) a trajectory \(T\) satisfies these two conditions and, at time \(t_f\), \(T\) is near to the separatrix, then it follows with eq.(3) that \(\varphi \sim \frac{\sqrt{3} M_p}{m} H e^{i\vartheta_{30}}\). Using eq.(7) it is easy to show that

\[
\frac{a(t_f)}{a(t_i)} = \left| \frac{\dot{\varphi}(t_i) + C_1}{\dot{\varphi}(t_f) + C_1} \right|^\frac{1}{3},
\]
for \(t_f > t_i\), with \(C_1 = \frac{M_p m}{\sqrt{3}} e^{i\vartheta_{30}}\). Every trajectory which lies close enough to the separatrix will thus meet the criteria for inflation.
To establish the asymptotic behavior of the solution nearby the singular saddle point $p_1$, the same strategy as for the line $l_1$ has to be applied. We therefore give here only the result, which is

$$\varphi = -\frac{i M_p m t}{\sqrt{3}} , \quad (45)$$

$$\dot{\varphi} = -\frac{i M_p m}{\sqrt{3}} , \quad (46)$$

$$H = \frac{-m^2 t}{3} , \quad (47)$$

for $t \to -\infty$. This solution corresponds also to an outgoing separatrix. A more detailed analysis shows that the singular point can only be reached if $\vartheta_3 \equiv \frac{\pi}{2}$. Otherwise, starting from points nearby $p_1$ with $\vartheta_3 \neq \pi/2$ the phase of the scalar field varies strongly. In this region of the phase space inflation is driven by the imaginary part of the scalar field. Also here the phase of $\varphi$ remains constant along the separatrix and we get the same equation of state as for the previous case. Applying the transformations defined in eq.(14) on eq.(45), we get instead of $p_1$ the singular point $\tilde{p}_1 = (1, 0, \frac{\pi}{2}, 0)$, for which the corresponding solution is now real. Hence, the analysis made in ref.[4, 5, 6] applies here as well.

For all points $b = (1, \frac{\pi}{2}, \frac{\pi}{2}, \vartheta_3)$ in $l_2$ we can directly solve the linearized differential equations. It turns out that $\vartheta_3$ remains constant and that on the plane $\vartheta_3 = \vartheta_{30} = \text{const}$ the solution expands away from $b$. Using the inverse transformations of eqs.(30) and (8), we obtain the asymptotic behavior of the scalar field and of the Hubble parameter

$$\varphi = \frac{M_p}{\sqrt{3}} \ln \left( \frac{t}{t_0} \right) e^{i\vartheta_{30}} , \quad (48)$$

$$\dot{\varphi} = \frac{M_p}{\sqrt{3} t} e^{i\vartheta_{30}} , \quad (49)$$

$$H = \frac{1}{3t} , \quad (50)$$

for $t \to 0^+$ corresponding to the initial cosmological singularity and where $t_0$ is an integration constant. The equation of state nearby these points tends to $\varepsilon = p$ (i.e. stiff matter). From eqs.(50) and (51) we get, as long as $\dot{\varphi} \dot{\varphi}^* \gg m^2 \varphi \varphi^*$, that $\dot{H} = -3H^2$. Solving this differential equations and using the definition of $H$, we obtain as expected

$$\frac{a(t_f)}{a(t_i)} = \left( \frac{t_f}{t_i} \right)^{\frac{1}{3}} . \quad (51)$$

Following the same method used for the points in $l_2$, we get for the saddle point $p_2$

$$\varphi = M_p \sqrt{3} \left[ C_2 + \frac{i}{3} \ln \left( \frac{t}{t_0} \right) \right] , \quad (52)$$
\[ \dot{\varphi} = M_p \sqrt{3} \left[ -\frac{C_2 m^2 t}{2} + \frac{i}{3t} \right], \quad (53) \]

\[ H = \frac{1}{3t}, \quad (54) \]

for \( t \to 0^+ \), with \( C_2 \) and \( t_0 \) being integration constants. The results of the analysis for \( l_2 \) do also apply here.

This completes the study of the phase portraits for \( \Lambda = 0 \), for which we found two singular curves \( l_1, l_2 \) and two singular points \( p_1, p_2 \). All other singular curves and points, due to the transformations given in eq.(14), can be reduced to one of this cases. For the curve \( l_1 \) and the point \( p_1 \) the solutions correspond to an outgoing separatrix where inflation occurs. Along these separatrices the phase of \( \varphi \) remains constant. For the curve \( l_2 \), for which the effective equation of state corresponds to stiff matter, the phase remains also constant. Setting \( \vartheta_{30} = 0 \) in the solutions around \( l_1 \) and \( l_2 \), we recover the results found by Belinsky et al.[4, 5, 6] for a real scalar field. Contrary to the previous ones the solution around \( p_2 \), for which we also get an effective equation of state for stiff matter, can not be obtained by simply adding a constant phase to the corresponding asymptotic solution for the real scalar field. As next we turn to the case where there is a quartic self-interaction term.

### 3.2. Properties of the singular points lying at infinity for \( \Lambda \neq 0 \)

We perform on eqs.(24)-(27) the transformation defined by eq.(30). We then obtain the set of equations

\[ \frac{d\rho}{d\tau} = -3\rho^2 f (\cos^2 \vartheta_3 \sin^2 \vartheta_1 + \sin^2 \vartheta_3 \sin^2 \vartheta_2) \]

\[ - 6\Lambda \rho^3 g (\sin \vartheta_1 \cos \vartheta_1 \cos \vartheta_3 + \sin \vartheta_2 \cos \vartheta_2 \sin^2 \vartheta_3), \quad (55) \]

\[ \frac{d\vartheta_1}{d\tau} = -\frac{1}{1-\rho} \left[ (1-\rho)^2 + 3\rho f \sin \vartheta_1 \cos \vartheta_1 + 6\Lambda \rho^2 g \cos^2 \vartheta_1 \right], \quad (56) \]

\[ \frac{d\vartheta_2}{d\tau} = -\frac{1}{1-\rho} \left[ (1-\rho)^2 + 3\rho f \sin \vartheta_2 \cos \vartheta_2 + 6\Lambda \rho^2 g \cos^2 \vartheta_2 \right], \quad (57) \]

\[ \frac{d\vartheta_3}{d\tau} = -\frac{1}{1-\rho} \left[ 3\rho f \sin \vartheta_3 \cos \vartheta_3 (\sin^2 \vartheta_2 - \sin^2 \vartheta_1) \right. \]

\[ + 6\Lambda \rho^2 g \cos \vartheta_3 \sin \vartheta_3 (\cos \vartheta_2 \sin \vartheta_2 - \cos \vartheta_1 \sin \vartheta_1) \], \quad (58) \]

where

\[ f = \sqrt{(1-\rho)^2 + 3\Lambda \rho^2 g^2} \quad (59) \]

and

\[ g = \cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2. \quad (60) \]
Since the right-hand side of eq.(55) is well defined and continuous for \(\rho = 1\), we can find all sets of angles \((\vartheta_{10}, \vartheta_{20}, \vartheta_{30})\) for which the condition \(\frac{d\rho}{d\tau} = 0\) at \(\rho = 1\) is fulfilled. For each solution \((\vartheta_{10}, \vartheta_{20}, \vartheta_{30})\) we have to check if \(\lim_{\rho \to 1^-} f_i(\rho, \vartheta_{10}, \vartheta_{20}, \vartheta_{30}) = 0\), where \(f_i\) stands for the right-hand side of eqs.(56) - (58). If this is the case, the point \((1, \vartheta_{10}, \vartheta_{20}, \vartheta_{30})\) is a singular point at infinity. Again, we find two singular curves and two singular points

\[
L_1 : \quad \vartheta_1 = \arctan \left( \frac{-2\sqrt{3}\Lambda}{3} \right) + \pi, \quad \vartheta_2 = \arctan \left( \frac{-2\sqrt{3}\Lambda}{3} \right) + \pi, \quad \frac{\pi}{2} \leq \vartheta_3 < \pi, \quad \rho \to 1,
\]

\[
L_2 : \quad \vartheta_1 = \frac{\pi}{2}, \quad \vartheta_2 = \frac{\pi}{2}, \quad \frac{\pi}{2} \leq \vartheta_3 < \pi, \quad \rho \to 1,
\]

\[
P_1 : \quad \vartheta_1 = \frac{\pi}{2}, \quad \vartheta_2 = \arctan \left( \frac{-2\sqrt{3}\Lambda}{3} \right) + \pi, \quad \vartheta_3 = \frac{\pi}{2}, \quad \rho \to 1,
\]

\[
P_2 : \quad \vartheta_1 = 0, \quad \vartheta_2 = \frac{\pi}{2}, \quad \vartheta_3 = \frac{\pi}{2}, \quad \rho \to 1.
\]

Notice that the limit \(\Lambda \to 0\) is rather subtle, since in the differential equations there are terms involving \(\Lambda/(1 - \rho)\), which are not defined when \(\rho \to 1\) and \(\Lambda \to 0\) simultaneously. We expand the differential equations around an arbitrary point \(P_0\) of \(L_1\) defined by the following coordinates

\[
\rho, \vartheta_1 = \arctan \left( \frac{-2\sqrt{3}\Lambda}{3} \right) + \pi, \quad \vartheta_2 = \arctan \left( \frac{-2\sqrt{3}\Lambda}{3} \right) + \pi, \quad \vartheta_3 = \vartheta_{30},
\]

where \(\vartheta_{30}\) is in \([\pi/2, \pi)\). In order to have finite partial derivatives, when \(\rho\) tends to 1, the angular variables \(\vartheta_1\) and \(\vartheta_2\) are kept fixed. When linearizing eq.(58) around \(P_0\), we see that also the angular variable \(\vartheta_3\) has to remain constant. Inserting the values of the angular variables defined in eq.(22), we get for the scalar field

\[
\varphi = \frac{-3M_p m e^{i\vartheta_{30}}}{\sqrt{4\lambda M_p^2 + 3m^2}} r, \quad \rho \to 1,
\]

\[
\dot{\varphi} = -2\sqrt{\frac{\lambda}{3}} M_p \varphi,
\]

for \(r \to \infty\), where \(r\) is a dimensionless parameter defined by eq.(22). Since the phase of \(\varphi\) is constant, we can integrate eq.(66) and thus get \(\varphi\) as a function of \(t\) rather than \(r\). This way we get

\[
\varphi = \varphi_0 e^{i\vartheta_{30}} \exp \left( -2M_p \sqrt{\frac{\lambda}{3}} t \right),
\]

(67)
where \( t \to -\infty \) and \( \varphi_0 \) is a negative integration constant. Using eq. (5), one gets the asymptotic behavior for the Hubble parameter. This solution corresponds to an outgoing separatrix and the equation of state tends to \( \varepsilon = -p \). A trajectory, which lies sufficiently close to the separatrix, can be parametrized by \( \dot{\varphi} \sim \alpha \varphi \), where \( \alpha \sim -2M_p\sqrt{\lambda/3} \). As long as \( H^2 \sim \frac{1}{3M_p^2}\lambda(\varphi\varphi^*)^2 \), with eq. (7) we find along the trajectory

\[
\frac{a(t_f)}{a(t_i)} = \exp\left(\frac{-\alpha^2}{3\alpha + 2\sqrt{3\lambda M_p}}(t_f - t_i)\right),
\]

for \( t_f > t_i \). The factor in the exponential multiplying the time difference is just the Hubble expansion rate, which is positive and tends to infinity as \( \alpha \) reduces to \( -2M_p\sqrt{\lambda/3} \). Thus, any solution which lies sufficiently close to the separatrix will go through an inflationary stage.

Using the same strategy as before for the point \( P_1 \), we obtain

\[
\varphi = i\varphi_0 \exp\left(-2M_p\sqrt{\frac{\lambda}{3}} t\right),
\]

where \( t \to -\infty \) and \( \varphi_0 \) is a negative integration constant. This solution corresponds also to an outgoing separatrix. The singular point can only be reached if \( \vartheta_3 \equiv \frac{\pi}{2} \), thus the real part of the scalar field vanishes. The analysis made for \( L_1 \) applies as well, again there is an inflationary stage if the solution gets close to the separatrix.

For the lines \( L_2 \) we expand the set of differential equations to first order around a point \( B \) defined by the coordinates \( (\rho, \frac{\pi}{2}, \frac{\pi}{2}, \vartheta_{30}) \). For \( \rho \to 1 \) the linearized equations reduce to those obtained for the line \( l_2 \).

Therefore, the solutions are given by eqs. (48) - (50). The inclusion of a quartic term in the lagrangean does not effect the singular line. This is expected, since near \( L_2 \) the potential term is negligible compared to the kinetic term \( \dot{\varphi}\dot{\varphi}^* \). The same remark holds for the singular point \( P_2 \), so that the asymptotic behavior is given by eqs. (52)-(54).

We see that the presence of a quartic self-interaction term does not change substantially the main features of the phase portrait. We find again two singular curves and two singular points. All other solutions can be reduced to these by the transformations given in eq. (14). The solutions which correspond to outgoing separatrices show an inflationary stage. The behavior of the solutions around these separatrices is not much affected by the quartic term. Its main influence is to shift the position of the singular points, where inflation occurs. For the solutions for which no inflation occurs, that is around \( L_2 \) and \( P_2 \), the results are the same as for \( l_1 \) and \( p_2 \) respectively. We obtain the solutions of Belinsky el al. in ref. [4] for the real scalar field by setting \( \vartheta_3 = 0 \) as mentioned in section 3.1.
In Fig.1 and 2 we plot the numerical solutions of the differential equations (10) - (13) for different values of $\Lambda$. For all solutions we take the same initial conditions as given by the asymptotic behavior near the singular point $p_2$. The plots are valid for every value of $m \neq 0$, since eqs.(10) - (13) do not depend on it. The numerical solutions have only a physical meaning in the region where $T_{00} < M_\phi^4$. Hence, depending on the value of $m$, it might be that only a part of the Figure is of physical relevance.

As next, we analyse the asymptotic behavior of the solutions of eqs.(18)-(21) for a massless scalar field with a quartic self-interaction nearby the singular points.

4. MASSLESS SCALAR FIELD IN A CURVATURELESS FRIEDMAN-LE-MAITRE UNIVERSE

Again the coordinate origin is the only singular point lying in the finite region of the phase space. This point is asymptotically stable, since we have the following Liapunov function (see ref.[13] for details)

$$l(x_{10}, x_{20}, y_{10}, y_{20}) = y_{10}^2 + y_{20}^2 + 3\lambda(x_{10}^2 + x_{20}^2)^2,$$

for which the derivative with respect to $\eta_0$ is strictly negative, except at the origin. Applying the transformation defined in eq.(22) on eqs.(17)-(21), we get
\[ z_0^2 = r^2 \left\{ 1 + \left[ 3\lambda r^2 (\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) - 1 \right] \times \right. \\
\left. (\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) \right\} , \tag{70} \]

\[ \vartheta_1' = -3z_0 \sin \vartheta_1 \cos \vartheta_1 - 6\lambda r^2 (\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) \cos^2 \vartheta_1 - \sin^2 (\eta_1) \tag{71} \]

\[ \vartheta_2' = -3z_0 \sin \vartheta_2 \cos \vartheta_2 - 6\lambda r^2 (\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) \cos^2 \vartheta_2 - \sin^2 (\eta_2) \tag{72} \]

\[ \vartheta_3' = \sin \vartheta_3 \cos \vartheta_3 \left[ -3z_0 (\sin^2 \vartheta_2 - \sin^2 \vartheta_1) \right. \\
\left. - 6\lambda r^2 (\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) (\sin \vartheta_2 \cos \vartheta_2 - \sin \vartheta_1 \cos \vartheta_1) \right. \\
\left. + (\sin \vartheta_2 \cos \vartheta_2 - \sin \vartheta_1 \cos \vartheta_1) \right], \tag{73} \]

\[ r' = -3r z_0 (\cos^2 \vartheta_3 \sin^2 \vartheta_1 + \sin^2 \vartheta_3 \sin^2 \vartheta_2) \\
+ r (\cos \vartheta_1 \sin \vartheta_1 \cos^2 \vartheta_3 + \cos \vartheta_2 \sin \vartheta_2 \sin^2 \vartheta_3) \\
- 6\lambda r^3 (\cos^2 \vartheta_3 \cos^2 \vartheta_1 + \sin^2 \vartheta_3 \cos^2 \vartheta_2) \times \\
(\cos \vartheta_1 \sin \vartheta_1 \cos^2 \vartheta_3 + \cos \vartheta_2 \sin \vartheta_2 \sin^2 \vartheta_3). \tag{74} \]

In a sufficiently small neighbourhood around the origin, which we denote by W, we consider the projection of the solutions of eqs.(71)-(74) on the \((x_0, y_0)\)-plane. The angular variable on this plane is \(\vartheta_1\) and its behavior is given by the solution of the equation \(\vartheta_1' = -\sin^2 \vartheta_1\), which is just eq.(71) with \(r = 0\). Therefore, knowing that the coordinate origin is asymptotically stable and that \(\vartheta_1'\) is almost everywhere strictly negative, it follows that the solutions are winding towards the point \(A_0 = (x_0 = 0, y_0 = 0)\). One gets the same behavior when the solutions of eqs.(71)-(74) in W are projected on the \((x_20, y_20)\)-plane.

We now turn to the singular points lying at infinity. One has to apply on eqs.(71)-(74) the transformations given in eq.(30), where \(\eta\) has to be replaced by \(\eta_0\). We obtain the following equations

\[ \frac{d\rho}{d\tau} = -3\rho^2 f (\cos^2 \vartheta_3 \sin^2 \vartheta_1 + \sin^2 \vartheta_3 \sin^2 \vartheta_2) \]
\[ + \left[ -6\lambda \rho^2 g + \rho (1 - \rho)^2 \right] \times \]
\[ \left[ \sin \vartheta_1 \cos \vartheta_1 \cos^2 \vartheta_3 + \sin \vartheta_2 \cos \vartheta_2 \sin^2 \vartheta_3 \right] , \tag{75} \]

\[ \frac{d\vartheta_1}{d\tau} = -\frac{1}{1 - \rho} \left[ 3\rho f \sin \vartheta_1 \cos \vartheta_1 + 6\lambda \rho^2 g \cos^2 \vartheta_1 + (1 - \rho)^2 \sin^2 \vartheta_1 \right] \tag{76} \]

\[ \frac{d\vartheta_2}{d\tau} = -\frac{1}{1 - \rho} \left[ 3\rho f \sin \vartheta_2 \cos \vartheta_2 + 6\lambda \rho^2 g \cos^2 \vartheta_2 + (1 - \rho)^2 \sin^2 \vartheta_2 \right] \tag{77} \]

\[ \frac{d\vartheta_3}{d\tau} = -\frac{\cos \vartheta_3 \sin \vartheta_3}{1 - \rho} \left\{ 3\rho f (\sin^2 \vartheta_2 - \sin^2 \vartheta_1) \right. \\
\left. + \left[ 6\lambda \rho^2 g - (1 - \rho)^2 \right] (\cos \vartheta_2 \sin \vartheta_2 - \cos \vartheta_1 \sin \vartheta_1) \right\} , \tag{78} \]
where from now on

\[ f = \sqrt{(1 - \rho)^2(1 - g) + 3\lambda\rho^2g^2} \quad (79) \]

and \( g \) is still given by eq.(60). The singular points lying at infinity of the phase space are found using the same method as for \( \Lambda \neq 0 \). We get again two singular curves and two singular points, which we denote by \( l_{10}, l_{20}, p_{10} \) and \( p_{20} \) and their coordinates are given by eqs.(51)-(54), but now with \( \Lambda \) replaced by \( \lambda \). To obtain the asymptotic behavior around these singular points we apply the same method used in section 3.2.

The asymptotic behavior around a singular point lying in \( l_{10} \) is given by eqs.(55)-(56), but where \( m \) is now replaced by \( M_p \). The angular variables must be kept fixed in order to have finite partial derivatives, when \( \rho \to 1 \). The analysis made for \( L_1 \) is also valid here and \( l_{10} \) has inflationary stages. Setting \( \vartheta_3 = 0 \) we obtain automatically the asymptotic behavior for a massless real scalar field with a quartic self-interaction for which we recover the inflationary stage. A fact this which was established heuristically by Linde in ref.[14].

One gets the behavior of the solutions around the singular point \( p_{10} \) from the one near \( P_1 \) in the same way as discussed above for \( l_{10} \) from \( L_1 \).

The asymptotic behavior of the solutions around all points \( c = (1, \frac{\pi}{2}, \frac{\pi}{2}, \vartheta_{30}) \) of \( l_{20} \) or around the point \( p_{20} \) is found directly by solving the corresponding linearized differential equations. For the singular points on \( l_{20} \) it turns out that the solutions are given by eqs.(57)-(58), whereas for \( p_{20} \) we get the solutions from eqs.(52)-(54) inserting \( m = 0 \). The form of the solutions around the line \( l_{20} \) and \( p_{20} \) was expected to be similar to that obtained for \( l_2 \) and \( p_2 \), because the potential term is negligible with respect to the kinetic energy term \( \dot{\varphi}\dot{\varphi} \).

5. CONCLUDING REMARKS

The extension of the above analysis to the singular points for \( k = \pm 1 \), although in principle straightforward, is much more involved. One could, for instance, using eq.(4) eliminate the curvature term in eq.(6) and consider this modified equation together with eq.(7). This gives then a set of five non-linear first order differential equations. As a consequence, when performing the transformation to spherical coordinates needed in order to compactify the phase space, one gets one additional angular variable. It turns out that around some singular points the expansion of the differential equations must be done at least up to fourth order. The only singular point lying in the finite region of the phase space is at the coordinate origin. We conjecture that the asymptotic behavior near the singular points lying at infinity with \( k \neq 0 \) will not fulfill the criteria for inflation. This does not imply that inflation can
not occur in an open or a closed universe, but that every trajectory must come close enough to one of the separatrix found for $k = 0$ in order to go through an inflationary stage. This fact has been shown for the real scalar field case (see ref. [4, 5, 6]).

In this paper we have extended to complex scalar fields the analysis of the initial conditions in an homogeneous and isotropic Friedman-Lemaître Universe. The main features found for real scalar fields hold also for complex scalar fields, in particular the existence of inflationary stages. The fact that along the separatrices the phase of $\phi$ remains constant is important and shows that inflation is essentially driven by one component of the complex scalar field. Therefore, the results on inflation valid for a real scalar field (see for instance ref. [2] and references therein) apply also on the component of the complex fields which drives inflation. The behavior around the singular points $p_2, p_{20}$ and $P_2$ is more involved and can not be obtained by just adding a phase to the corresponding solutions for the real scalar field. We also notice that for a massive scalar field the presence of a quartic self-interaction term does not change substantially the main features of the phase portrait.

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Figure Captions

Fig. 1: Plot of the dimensionless variable $y_1$ as function of $x_1$ for different values of $\Lambda$. It corresponds to the phase portrait of the real part of the scalar field. All curves start with the same initial condition given by the asymptotic behavior near $p_2$ (see eq. (22)). The real part of the scalar fields depends on the value of $C_2$, for which we take the value $C_2 = 1$.

Fig. 2: Plot of the phase portrait of the imaginary part of the scalar field in dimensionless variables for the same cases as in Fig.1 The asymptotic behavior of the imaginary part near $p_2$ depends on the value of $t_0$, for which we choose $t_0 = 100$. 
This figure "fig1-1.png" is available in "png" format from:

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