One common solution to the singularity and perihelion problems

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Abstract

Based on the kinetic energy theorem, as one of the fundamental theorems from the classical mechanics, throughout the first part of the article an attempt has been made to derive the mathematical model of a material point motion in the three-dimensional spatial subspace of the integral four-dimensional space-time continuum and in the field of action of an active force $\mathbf{F}$. Accordingly, with a view to surmounting the singularity problem on the one hand, as well as the moving perihelion problem of the planets on the other, as two acutely vexed questions within Newton’s gravity concept, the paper ends with modification of Newton’s gravity concept itself.
1 Introduction

The space-time mathematical model of motion of a material point in the relativistic (Einstein’s) mechanics, based on the principle of the constancy of the light velocity in vacuum relative to inertial frames [4, 7, 11], is said to be approximately more general with respect to the so called spatial mathematical model derived in the classical (or Newton’s) mechanics [2, 9]. The four-dimensional space-time continuum a priori established by Minkowski is the foundation stone of whole relativistic mechanics. Hence, with a view to pointing out the physical sense of a configurative space of the space-time continuum, throughout the first part of the paper an attempt has been made to derive, on the basis of some fundamental principles of the mechanics, the mathematical model of a material point motion in three-dimensional spatial subspace of an ambient integral four-dimensional space-time continuum.

On the other hand, Newton’s gravity concept, in existence for already three centuries, which describes with sufficiently exactness, in spite of some acutely vexed questions within it, the so called Sun’s planetary system via Kepler’s laws of a planetary motion, is one of the fundamental laws of the classical mechanics, particularly of the celestial mechanics. The first vexed question, based on the purely theoretical basis, is the so-called singularity problem. Namely, on the basis of the mathematical model of two material points motion of the same mass in the field of action of the central Newton’s gravity force, when the direction of material points motion coincides with the assaulted direction of the force [9, 12], it is easy to see that absolute values of all relevant physical variables, such as velocity, force, kinetic and potential energy, in the limit as mutual distance of the material points tends to zero, tend to infinity. The second one, which is cleanly empirical nature, is the perihelion problem. Namely, it has been experimentally stated that the perihelion of Mercury’s orbits moves into the plane of its planetary motion around the Sun. In other words, all planetary motions of Sun’s planetary system depart from elliptical orbits obtained from Newton’s mathematical gravity model [7, 16]. By the strict Schwarzchild-Droste’s solution to the static gravitational field with spherical symmetry, in the general Einstein’s relativity theory [11], the perihelion problem has been approximately solved. However, this solution does not solve the singularity problem. Accordingly, to solve simultaneously these two acutely vexed questions within Newton’s gravity concept the manuscript ends with an approximative modification of Newton’s gravity concept itself.
1.1 Fundamental characteristics of the space-time continuum

By the notion of a material point, introduced for the purpose of an useful idealization, as one of the underlying notions of physics in the general sense, but not only of physics, one means a geometrical point, which is spatially no dimensional on the one hand and exactly fixed mass on the other \[9\]. Closely related to the geometrical point notion is the set of values \([a_1, a_2, ..., a_n]\) of some arbitrary \(n\) variables \([x_1, x_2, ..., x_n]\), such that a set of all geometrical points, and for all real values of the variables, is the real \(n\)-dimensional configurative space \[2\]. The geometrical point defined by a set of zero values \([0, 0, ..., 0]\) is the zero co-ordinate point. If and only if one of \(n\) arbitrary variables \(x_\alpha (\alpha = 1, 2, ..., n)\) is the time variable \(t\), the space aforementioned becomes the space-time continuum (the integral space) \[4, 5, 16\]. The value of \(t\) is called moment or instant \[4\].

The set of all geometrical points of the spatial subspace of the integral space, to which the mass \(m\) can be joined in some strictly monotonous sequence of permitted instants of the time \(t\), makes an odograph, that is a trajectory (a path of motion) of the material point \(M\). The time variable \(t\) is taken for a unique independent variable \[4\], so that all remaining spatial variables \(x_\alpha (t)\) are functional variables. The set of all geometrical points \(x_\alpha (t)\) of the integral space is an integral curve of \(M\) and an odograph, that is a trajectory (a path of motion) of a representative point \(\hat{M} (m = \hat{m})\) of the space-time continuum.

The vectors \(r [x_\alpha (t)]\) and \(\rho [x_\alpha (t)]\) defined with respect to the origin are position vectors of \(M\) and \(\hat{M}\), respectively, in the configurative space of the space-time continuum. The concept of a vector in vector hyper-dimensional spaces \((n > 3)\) should be conditionally comprehended in the sense of its geometrical presentation in a form of segments. Hence it bears a name linear tensor \[11\]. The set of infinitesimal values \(dx_\alpha (\alpha = 1, 2, ..., n)\) corresponds to infinitesimal movements of \(M\) along the trajectory of motion.

Covariant vectors \(e_\alpha = \partial_\alpha \rho (x_\beta)\), where \(\partial_\alpha\) denotes \(\partial/\partial x_\alpha\), form a covariant vector basis \(\{e_\alpha\}_{\alpha = 1}^{n}\) of the configurative space. The vectors \(e_\beta\), such that at any point of the space \(e_\alpha \cdot e_\beta = \delta_\alpha^\beta\), where the second order system of the unit values \(\delta_\alpha^\beta\) is the identity \(n \times n\) matrix (Kronecker’s delta-symbol) \[2, 6, 11\], form a basis \(\{e_\beta\}_{\beta = 1}^{n}\), which is called a dual basis of the covariant vector basis \(\{e_\alpha\}_{\alpha = 1}^{n}\). This is so-called natural isomorphism from \(\{e_\alpha\}_{\alpha = 1}^{n}\) onto \(\{e_\beta\}_{\beta = 1}^{n}\). Accordingly, the differential \(d\rho\) of the position vector \(\rho\) of \(\hat{m}\) is defined by \(d\rho = dx_\alpha e_\alpha = dx_\beta e_\beta\), where the so called Einstein’s convention

\[1\] Greek indices take values 1, 2, ..., \(n\), and Latin ones 1, 2, ..., \(n - 1\).
is applied to a summation with respect to the repetitive indexes (uppers and lowers) \([2, 11]\), herein as well as in the further text of the paper.

2 Main results

2.1 A metric of configurative spaces

On the basis of preceding description of the space-time continuum one can conclude that the space-time continuum is a metrical affine vector space, whose linearly independent basic (fundamental) co-ordinate vectors, reduced to the origin, form a basic \(n\)-hedral \([2]\). As for a metric of metrical space of the space-time continuum it is explicitly related to fundamental properties of a material point motion. Namely, on the one hand the basic kinematical characteristic of a material point motion is the velocity \(\mathbf{v} = \frac{d}{dt} \mathbf{r}\), where \(d_t\) denotes \(d/dt\), while on the other hand the basic dynamical characteristics are the quantity of movement \(\mathbf{K} = mv\) and the kinetic energy \(\mathcal{E} = \frac{m}{2} (\mathbf{v} \cdot \mathbf{v}) = \frac{mv^2}{2}\) \([9, 15]\).

The functional expressions for kinetic energies \(\mathcal{E}\) (of \(\mathcal{M}\)) and \(\mathcal{K}\) (of \(\mathcal{M}^\prime\)) can be stated in more appropriate forms:

\[
\frac{2}{m} \mathcal{E} (dtdt) = d\mathbf{r} \cdot d\mathbf{r} = (ds)^2
\]

and

\[
\frac{2}{m} \mathcal{K} (dtdt) = d\rho \cdot d\rho = (d\sigma)^2,
\]

where \(ds\) and \(d\sigma\) are line elements of affine metrical spaces of the spatial and space-time continuum, respectively \([2]\). On the other hand, according to the differential equation of energy balance of \(\mathcal{M} \) \([1, 13]\) (the differential equation of the kinetic energy theorem \([9]\)):

\[
d(\frac{1}{2}m\mathbf{v} \cdot \mathbf{v}) = d\mathcal{E} = \mathbf{F} \cdot d\mathbf{r} = d\mathcal{A},
\]

derived from the second \(Newton's\) low (principle) of a material point motion in the field of action of an active force \(\mathbf{F}\) \([9, 10]\):

\[
md_t\mathbf{v} = \mathbf{F},
\]

an infinitesimal change of \(\mathcal{E}\) is equal to an infinitesimal work \(d\mathcal{A}\) of \(\mathbf{F}\) during infinitesimal movements of \(\mathcal{M}\) along the path of motion, that is

\[
d (\mathcal{E} - \mathcal{A}) = 0.
\]
If an additive integral constant $\hat{k}$ of (5) is introduced into the analysis:

$$(6) \quad \mathcal{E} - \mathcal{A} = \hat{k} = \frac{1}{2} mc^2,$$

then, by (1) and (6), it follows that

$$(7) \quad dt \mathbf{r} \cdot dt \mathbf{r} - \hat{c}^2 = \frac{2}{m} \mathcal{A},$$

that is

$$(8) \quad (ds)^2 - \hat{c}^2 (dt)^2 = \frac{2}{m} \mathcal{A} (dt)^2.$$

**Assumption 2.1:** An integral curve of $\mathcal{M}$ in the space-time continuum is a curve in the four-dimensional space-time continuum of Minkowski.

On the basis of the preceding assumption and the fact that

$$(9) \quad (ds)^2 - c^2 (dt)^2 = (d\sigma)^2,$$

in the space-time continuum of Minkowski [11, 7], where the constant $c$ is nominally equal to the light velocity in vacuum, it follows that

$$(10) \quad \mathcal{K} = \frac{1}{2} \hat{m} (dt \rho \cdot dt \rho) = \mathcal{A} + \bar{k},$$

where $\bar{k}$ is an additive integral constant of $d (\mathcal{K} - \mathcal{A}) = 0$, that is

$$(11) \quad \mathcal{E} - \mathcal{A} = \bar{k} = \frac{1}{2} mc^2 + \bar{k}$$

taking the relation (6) into account. In addition $\hat{c}^2 = c^2 + 2\bar{k}/m$, since $m = \hat{m}$.

In the event that the Pfaff form $\mathbf{F} \cdot d\mathbf{r}$ of (3) is absolute differential, in other words if there exists a scalar valued function $\mathcal{P}$ depending on $\mathbf{r}$ such that $\mathbf{F} = -\text{grad} \mathcal{P} (\mathbf{r})$, that is $d\mathcal{A} = -d\mathcal{P}$, then a material point motion is that in the field of action of an active potential force $\mathbf{F}$ with the potential $\mathcal{P}$ [2, 9]. Accordingly,

$$(12) \quad \mathcal{U} = \mathcal{E} + \mathcal{P}$$

is a functional of the total mechanical energy of $\mathcal{M}$. Having in view the fact that

$$(13) \quad d (\mathcal{E} + \mathcal{P}) = 0,$$

$\mathcal{U}$ is an integral of a motion of $\mathcal{M}$ too [9].

In the general case of a material point motion in the field of action of an active potential force $\mathbf{F}$, it follows immediately from the differential form of
the energy conservation, see (13), that
\[ \mathcal{E} + \mathcal{P} = \mathcal{U} = k + \tilde{k} = \frac{1}{2}mc^2 + (k + \tilde{k}) = k + \kappa, \]
where \( \tilde{k} \) is an additive integral constant of \( dA = -d\mathcal{P} \) and \( k = mc^2/2 \) as well as \( \kappa = k + \tilde{k} \).

Note that the metric form \( (d\sigma)^2 = d\rho \cdot d\rho \) of the configurative space has been adopted in such a way to represent the kinetic energy theorem on the one hand and Assumption 2.1 on the other. Accordingly, one can say that in that a way it is possible to reduce the analysis of a material point motion to the analysis of a representative point motion in the configurative space. Namely, if one starts with the action \( S \) in the Lagrange sense along the path of \( \mathcal{M} \) in the configurative space [2]:
\[ S = 2 \int_{t_2}^{t_1} K dt, \]
then it follows from the relations (1) and (10) that
\[ K = k + \tilde{k} - \mathcal{P} = \kappa - \mathcal{P} = \frac{1}{2}m (d_t \sigma)^2, \]
that is
\[ S = \sqrt{2m} \int_{\sigma(t_1)}^{\sigma(t_2)} \sqrt{K} d\sigma = \sqrt{2m} \int_{\sigma(t_1)}^{\sigma(t_2)} \sqrt{\kappa - \mathcal{P}} d\sigma. \]

In that case an action line element \( dw \) is introduced [2] as follows
\[ \sqrt{\frac{1}{2}mc^2} dw = \sqrt{\kappa - \mathcal{P}} d\sigma, \]
so that the action metric form of the configurative space is
\[ k (dw)^2 = k a_{\alpha\beta} dx^\alpha dx^\beta = (\kappa - \mathcal{P}) e_{\alpha\beta} dx^\alpha dx^\beta = (\kappa - \mathcal{P}) (d\sigma)^2, \]
where \( e_{\alpha\beta} = e_\alpha \cdot e_\beta \) is the metric tensor of \( (d\sigma)^2 \) [11].

By the well-known Maupertius-Lagrange’s principle [2, 7, 9, 11] a path of motion of \( \mathcal{M} \) in the configurative space is just the path along which the action is stationary, more precisely along which the following condition
\[ \Delta S = \sqrt{2m} \Delta \int_{\sigma(t_1)}^{\sigma(t_2)} \sqrt{\kappa - \mathcal{P}} d\sigma = 0 \]
is satisfied, that is, by the relation (18), the condition
\[ mc \Delta \int_{w(t_1)}^{w(t_2)} dw = 0, \]
where $\Delta$ is the variational operator \[2,9\].

It is well-known from the tensorial analysis \[2\] that curves of the action configurative space, for which the condition (21) is satisfied, are geodesics. In addition, the absolute (Bianchi’s) derivation $d_w x^\beta$ of the unit tangent vector $u = d_w \rho$ of those curves, in the direction of curves, is equal to zero. In other words, the projection of $du/dw$ onto the tangent hyper-plane of the dual vector basis $\{a^\beta\}_{\beta=1}^n$ of the action metric form, is equal to zero. Accordingly, the geodesic equations are

\begin{equation}
\frac{du}{dw} \cdot a^\gamma = d_w (d_w x^\alpha a_\alpha) \cdot a^\gamma = d_w^2 x^\alpha a_\alpha \cdot a^\gamma + d_w x^\alpha d_w a_\alpha \cdot a^\gamma = d_w^2 x^\gamma + \hat{\Gamma}^\gamma_{\alpha\beta} d_w x^\alpha d_w x^\beta = 0,
\end{equation}

where $d_w^2$ denotes $d^2/(dw)^2$, the vectors $a_\alpha = \partial_a \rho (x^\delta)$ are covariant basic vectors forming the covariant vector basis $\{a_\alpha\}_{\alpha=1}^n$ and the mixed system of values $\Gamma^\gamma_{\alpha\beta} = \partial_\beta a_\alpha \cdot a^\gamma = a^\alpha (\partial_\beta a_\alpha + \partial_\alpha a_\beta - \partial_\delta a_{\alpha\beta})/2$ are the second order Christoffel’s symbols \[2,11\].

In the case when $F$ is a potential force, that is $F = -\text{grad} \mathcal{P}$, it follows from the condition $(\kappa - \mathcal{P}) e_{\alpha\beta} = ka_{\alpha\beta}$, see (19), that \[2\]

$$\hat{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + \frac{1}{2(\kappa - \mathcal{P})} \left( \partial_\alpha \mathcal{P} \delta_\beta^\gamma + \partial_\beta \mathcal{P} \delta_\alpha^\gamma - e^{\gamma\delta} e_{\alpha\beta} \partial_\delta \mathcal{P} \right),$$

since $\Gamma^\gamma_{\alpha\beta} = e^{\gamma\delta} (\partial_\beta e_{\alpha\delta} + \partial_\alpha e_{\beta\delta} - \partial_\delta e_{\alpha\beta})/2$. The new form of the geodesic equations, for a constrained material point $\mathcal{M}(F \neq 0 \Leftrightarrow \mathcal{P} \neq \text{const.})$, is

\begin{equation}
d_{ww}^2 x^\gamma + \Gamma^\gamma_{\alpha\beta} d_w x^\alpha d_w x^\beta = \frac{1}{k(\kappa - \mathcal{P})} \partial_\delta \mathcal{P} d_w x^\delta d_w x^\gamma - \frac{k}{2(\kappa - \mathcal{P})^2} \partial_\delta \mathcal{P} e^{\gamma\delta},
\end{equation}

that is

\begin{equation}
m (d_{tt}^2 x^\gamma + \Gamma^\gamma_{\alpha\beta} d_t x^\alpha d_t x^\beta) = -\partial_\delta \mathcal{P} e^{\gamma\delta} = F^\gamma,
\end{equation}

since $d_{ww}^2 x^\gamma (d_t w)^2 + d_t x^\gamma d_w d_t^2 w = d_{tt}^2 x_t x^\gamma$, $d_t w = c(\kappa - \mathcal{P})$ and $d_{tt}^2 w d_w t = -[\partial_\delta \mathcal{P} / (\kappa - \mathcal{P})] d_t x^\delta$.

So, (24) represents Euler-Lagrange’s differential equations of extreme curves in the explicit form \[9\], and at the same time Newton’s low in the field of action of a potential force $F^k = -\partial_l \mathcal{P} e^{kl}$ in the contravariant form \[2,9\]:

\begin{equation}
m (d_{tt}^2 x^k + \Gamma^k_{ij} d_t x^i d_t x^j) = -\partial_l \mathcal{P} e^{kl} = F^k.
\end{equation}

Accordingly, one may conclude that the dynamical (Newton’s) equations (25) are formally derived from the geometric equations (22) representing the kinetic energy theorem on the one hand and Assumption 2.1 on the other.
In the case of a free material point $M$, when $P$ is constant: $P = \text{const.}$, both the basic and action metric form of the configurative space are pseudo-euclidean, while integral curves are straight-lines [2]. To prove these facts one starts with Euler-Lagrange’s differential equations

$$d_w \left( \partial_{dwx^a} W - \partial_\beta W \right) = 0,$$

where

$$W = \frac{\kappa - P}{k} e_{\alpha\beta} dwx^\alpha dwx^\beta,$$

as the condition for the action (21) to be stationary. The geodesic equations (22) are explicitly obtained from it in a known way. If spatial co-ordinates of the configurative space are spherical ones $(r, \theta, \varphi)$, then the components of $e_{\alpha\beta}$ are only functions of the co-ordinates $r$ and $\varphi$ [10], so that it follows from (26) that

$$d_w \left( \frac{\kappa - P}{k} e_{11} cdt \right) = 0$$

and

$$d_w \left( \frac{\kappa - P}{k} e_{33} dw \theta \right) = 0,$$

that is

$$(\kappa - P) c dt = k$$

and

$$(\kappa - P) \left( r^2 \cos^2 \varphi \right) dw \theta = k \alpha.$$

Let the polar extension $r$ and the polar angle $\theta$ be intensities of $r$ and an angle between the position vector $r$ and the polar axis $p$ passing through the origin and the perihelion point, respectively. Then, since $S = r^2 \dot{\theta} = \text{const.}$, where $S = r \times v$ is the sector velocity vector [12], it follows from the condition (31) that a free material point motion is the plane motion $(\varphi = 0)$ and $S = \alpha c$. As $(ds)^2 = (dr)^2 + r^2 (d\theta)^2$ then we obtain finally from (9), (19) and (31) that

$$ (dr)^2 = r^4 \left[ \frac{1}{\alpha^2} - \frac{1}{r^2} + \frac{\kappa - P}{k \alpha^2} \right] (d\theta)^2,$$

that is

$$d_{\theta}^2 \frac{1}{r} + \frac{1}{r} = 0,$$

and what is just Binet’s differential equation [9]. The solution to this differential equation defines a straight-line in the polar co-ordinates: $r_0 = r \cos \theta$, where $r_0$ is the perihelion distance.
If (15) is analyzed anew, in the case when the upper limit of integration is changeable \((t_2 = t)\)

\[ S = 2 \int_{t_1}^{t} K dt, \]

then considering the fact that

\[ d_t S = 2K = me_{\alpha\beta}d_t x^\alpha d_t x^\beta, \]

it is easy to see that from great interest in the analysis is the functional \(S^H\) [9], nominally equal to \(S\), for which the following functional relations hold:

\[ \partial_\alpha S^H = me_{\alpha\beta}d_t x^\beta \]

and

\[ d_t S^H = 2K, \]

as well as the functional \(Z^H\) satisfying the condition

\[ S^H = Z^H - (\frac{1}{2} mc^2 - \kappa)t, \]

that is, the condition

\[ d_t Z^H = L, \]

since \(d_t S^H = 2(\kappa - \mathcal{P})\).

The functional \(\mathcal{L} = \mathcal{E} - \mathcal{P}\) is Lagrange's functional and in the acute case of the standard Lagrange's system it is also Lagrangian of \(\mathcal{M}[9]\).

Since \(\partial_\alpha S^H = -mc^2\) for \(x^4 = ct\), see (36), it follows from the condition (38) that \(\partial_t Z^H = -\mathcal{U}\), that is

\[ \partial_t Z^H + \mathcal{H} = 0. \]

The preceding equation is Hamilton-Jacobi's equation and according to that the functional \(Z^H\) is the principal Hamilton's functional of \(\mathcal{M}[9]\). The Hamiltonian (Hamilton's functional) \(\mathcal{H}\) of \(\mathcal{M}\) is equal to the functional of the total mechanical energy \(\mathcal{U}\), more precisely to an integral of motion, and what in accordance with the fact that the kinetic energy \(\mathcal{E}\) of \(\mathcal{M}\) is a homogenous square function of \(d_t x^\alpha[9]\),

As \(\partial_t Z^H = me_{ij}d_t x^j\), see (36) and (38), that is

\[ e^{kl} d_k Z^H d_l Z^H = m^2 e_{ij}d_t x^i d_t x^j = 2m\mathcal{E}, \]

then it follows that

\[ -\partial_t Z^H - \frac{1}{2m} e^{kl} d_k Z^H d_l Z^H = \mathcal{P}, \]

what is only the second form of Hamilton-Jacobi's equation.
2.2 Principle of invariance and co-ordinate transformations

In Section 1 the mathematical model of a material point motion in the configurative space (in the spatial subspace of the space-time continuum) and in the field of action of an active potential force $\mathbf{F}$ with the potential $P$ has been derived on the basis of the kinetic energy theorem and Assumption 2.1. Note anew that the basic metric form $(ds)^2$ represents the kinetic energy theorem. In addition, $(ds)^2$ is explicitly related to the kinetic energy of $\mathcal{M}$.

**Principle of invariance:** Any form of energy of $\mathcal{M}$ is an invariant in the more expansive sense (a scalar invariant), more precisely a zero order tensor.▼

Co-ordinate systems of configurative spaces are only auxiliary tools for analysis, so that, on the one hand, all fundamental values characterizing inner (natural) dynamical properties of a system, as in this case ether the potential or kinetic energy of $\mathcal{M}$, are invariants with respect to co-ordinate transformations [2]. Accordingly, the fundamental differential functional form of the kinetic energy theorem is also an absolute invariant (an invariant functional form), so that on the basis of these two principles of invariance we can say that the time is also tensor invariant. In other words, in any system of co-ordinate transformations of configurative spaces the time is absolutely only one independent variable.

At the choice of co-ordinate transformations of ether basic or action metric forms of configurative spaces it is necessary in addition to the condition for the so called Jacobian of transformation to be different from zero to take also the principle of invariance 2.2 into consideration. It is different from co-ordinate transformations of ether basic or action metric forms of configurative spaces at the choice of which the condition for Jacobian of transformation to be different from zero has to be satisfied only. Hence, it is clear that the co-ordinate transformations, as Lorentz’s transformations [4, 5, 6, 11], do not meet the established criteria for the choice of co-ordinate transformations of ether basic or action metric forms of configurative spaces, and for reason that the principle of invariance 2.2 is being destroyed by them.
2.3 Example: The mathematical model of two material points motion in the field of action of the central conservative force, as an idealization of two body problem

Let $v_1 = dt r_1$ and $v_2 = dt r_2$ be velocity vectors of material points $M_1$ and $M_2$, respectively, in a spatial subspace of the space-time continuum. Then, a motion analysis of $M_1$ and $M_2$ reduced to that of the compound motion of a virtual material point of the mass $\mu = (m_1 m_2) / M$ ($M = m_1 + m_2$) in the configurative space of the spatial continuum. This is based on defined property of the central conservative force $F$ that its direction of assaulted action coincides with the direction of the relative position vector $r = r_1 - r_2$ of $M_1$ and $M_2$, as well as on Newton's law of motion and the moment law of quantity of movement $[9, 12]$. Having in view the fact that an absolute motion of the mass centre of $M_1$ and $M_2$ is uniformly $[9]$ the mathematical model of motion of $M_1$ and $M_2$ in the configurative space of the spatial continuum and in the field of action of the central conservative force $F$ can be derived as follows. This mathematical model cans from that of motion of a virtual material point $M_\mu$ in an immovable configurative space of the spatial continuum and in the field of action of the central conservative force $F$ whose assaulted direction of action coincides with the direction of the position vector $r$ of $M_\mu$ with respect to the immovable centre of mass as the origin. Clearly, in spite of all that the relativistic principle of the classical mechanics that says that a relative movement in an inertial co-ordinate system is analogous to an absolute movement in an immovable co-ordinate system $[12]$ will be a base for further analysis.

As, in this case too, the sector velocity vector $S = r \times v$ is obviously constant, then it follows from (19):

\[
(43) \quad k (dw)^2 = (\kappa - P) (d\sigma)^2 = (\kappa - P) [ (dr)^2 + r^2 (d\theta)^2 - c^2 (dt)^2 ],
\]

as well as from the geodesic equations, see (28) and (29):

\[
(44) \quad dw (\frac{\kappa - P}{k} dw t) = 0
\]

and

\[
(45) \quad dw (\frac{\kappa - P}{k} r^2 dw \theta) = 0,
\]

more precisely from (32):

\[
(46) \quad (dr)^2 = r^4 \left[ \left( \frac{1}{\alpha} \right)^2 - \left( \frac{1}{r} \right)^2 + \frac{\kappa - P}{k \alpha^2} \right] (d\theta)^2,
\]

that

\[
(47) \quad \frac{d^2_{\theta\theta}}{r} + \frac{1}{r} = - \frac{1}{2 k \alpha^2} \frac{\partial^2 P}{\partial \theta^2},
\]
what is the well-known Binet’s differential equation of the trajectory of motion of $M_\mu$ in an immovable configurative space of the spatial continuum.

2.4 Modified Newton’s gravity concept

For the conservative Newton’s gravity force $F_N = -\nabla \mathcal{P}$ the expression $\kappa - \mathcal{P}$ is of the following functional form $\kappa - \mathcal{P} = -k(\hat{\lambda} - 2\varrho/r)$, where $k = \mu c^2/2$, $\hat{\lambda}$ is an integral constant of initial conditions, and $\varrho$ is a constant of the gravitational radius: $\varrho = \gamma M/c^2$ ($\gamma$ is the well-known gravitational constant), so that (47) is reduced to Binet’s differential equation of motion of a constrained material point $M_\mu$ in an immovable configurative space of the spatial continuum and in the field of action of the central Newton’s gravity force $F_N$:

\begin{equation}
\frac{d^2}{d\theta^2} \frac{1}{r} + \frac{1}{r} = \frac{\varrho}{\alpha^2},
\end{equation}

where $\alpha = S/c$ and $S = r^2 d\theta = \text{const}$.

Since in the limit as $r \to 0^+$ the coefficient of the proportionality between the action $(dw)^2$ and basic $(d\sigma)^2$ metric forms:

\begin{equation}
\hat{\lambda} - \frac{2\varrho}{r} = \lambda + 1 - \frac{2\varrho}{r},
\end{equation}

where $\lambda = \hat{\lambda} - 1$, tends to infinity, it is logical to assume that the last two terms on the right hand side of (49) approximately represent an expansion of the exponential function $e^{-2\varrho/r}$ into Taylor’s functional series. In other words, it is logical to assume that the real functional coefficient of the proportionality between the action $(dw)^2$ and basic $(d\sigma)^2$ metric forms, in the case when $M_\mu$ moves in the field of action of the modified central Newton’s gravity force $F_N$, is as follows

\begin{equation}
(dw)^2 = -(\lambda + e^{\frac{-2\varrho}{r}}) (d\sigma)^2.
\end{equation}

Accordingly, the modified Binet’s differential equation of motion of $M_\mu$ in the field of action of the modified central conservative Newton’s gravity force $F_N$:

\begin{equation}
F_N = -\nabla \mathcal{P}(r) = -\frac{2k\varrho}{r^3} e^{-\frac{2\varrho}{r}} r,
\end{equation}

with the potential $\mathcal{P}(r)$: $\kappa - \mathcal{P} = -k(\lambda + e^{-2\varrho/r})$, is the following differential form

\begin{equation}
\frac{d^2}{d\theta^2} \frac{1}{r} + \frac{1}{r} = \frac{\varrho}{\alpha^2} e^{-\frac{2\varrho}{r}}.
\end{equation}
2.4.1 A material point motion in the field of action of $F_N$

Start with the differential equation of the second *Newton’s law* (principle) of a material point motion \[9, 10\]

\[
\mu \ddot{r} = -\frac{2k_0}{r^3} e^{-2\alpha/r} r.
\]

Multiplying the preceding differential equation on the right by the sector velocity vector $S = r \times v$:

\[
\mu \ddot{r} \times S = -\frac{2k_0}{r^3} e^{-2\alpha/r} r \times S,
\]

we obtain that

\[
d_t (v \times S) = \gamma Me^{-\frac{2\alpha}{r}} d_t \frac{r}{r},
\]

since $d_t S = 0$, that is

\[
d(v \times S - \gamma M r_0) = \gamma M(e^{-\frac{2\alpha}{r}} - 1) dr_0.
\]

By (46):

\[
(d\theta r)^2 = r^4 \left[ (\frac{1}{\alpha})^2 - (\frac{1}{r})^2 - \frac{1}{\alpha^2} (\lambda + e^{-\frac{2\alpha}{r}}) \right],
\]

it follows obviously that

\[
\lambda + e^{-\frac{2\alpha}{r}} = 1 - \alpha^2 [(d\theta r)^2 + (\frac{1}{r})^2] = 1 - \frac{v^2}{c^2}.
\]

Accordingly, based on (56), there holds

\[
dL = \gamma M(e^{-\frac{2\alpha}{r}} - 1) dr_0 = -\gamma M(\lambda + \frac{v^2}{c^2}) dr_0,
\]

where the vector

\[
L = v \times S - \gamma M r_0,
\]

satisfying the relations: $L \cdot S = 0$ and $L \cdot L = L^2 = v^2 S^2 - 2\gamma M/r + \gamma^2 M^2$, in this acute case is not more an element of *Milanković’s constant vector elements*, more precisely is not more *Laplace’s integration vector constant* \[8\].

By multiplying (60) by $r$: $L \cdot r = (v \times S) \cdot r - \gamma M r$, and considering the fact that $(v \times S) \cdot r = S^2$, the equation of the trajectory of motion of $M_\mu$ in the field of action of the modified *Newton’s gravity force* $F_N$ is reduced to

\[
r = \frac{1}{\gamma M} \frac{S^2}{1 + \frac{r}{\gamma M} \cos \varphi}.
\]
where \( \varphi \) is an angle between the vectors \( \mathbf{r} \) and \( \mathbf{L} \).

On the basis of the preceding equalities the following functional equalities are easily obtained
\[
\frac{v^2}{c^2} = \frac{\alpha^2}{\varphi^2} \left[ 1 + \left( \frac{L}{\gamma M} \right)^2 + \frac{2L}{\gamma M} \cos \varphi \right]
\]
and
\[
\frac{2g}{r} = \frac{2g^2}{\alpha^2} (1 + \frac{L}{\gamma M} \cos \varphi),
\]
as well as
\[
d\mathbf{L} = -\gamma M \{ \lambda + \frac{g^2}{\alpha^2} [1 + (\frac{L}{\gamma M})^2 + \frac{2L}{\gamma M} \cos \varphi] \} d\mathbf{r}_0
\]
and
\[
d\mathbf{L} = \gamma M [e^{-\frac{2g^2}{\alpha^2} (1 + \frac{L}{\gamma M} \cos \varphi)} - 1] d\mathbf{r}_0.
\]

As \( \theta = \varphi + \omega \) and \( d\mathbf{r}_0 = d\theta \mathbf{t} \), where \( \omega \) is an angle of deviation of \( \mathbf{L} \) with respect to the perihelion direction: \( d (\mathbf{L}/L) = d\mathbf{L}_0 = d\omega \mathbf{k} \), when \( \varphi = 0 \), while \( \mathbf{t} \) and \( \mathbf{k} \) are unit vectors being orthogonal onto the unit vectors \( \mathbf{r}_0 \) and \( \mathbf{L}_0 \), respectively, multiplying (64) and (65) by \( \mathbf{k} \), we obtain finally that
\[
d\omega = -\frac{\gamma M}{L} \{ \lambda + \frac{g^2}{\alpha^2} [1 + (\frac{L}{\gamma M})^2 + \frac{2L}{\gamma M} \cos \varphi] \} \cos \varphi (d\varphi + d\omega)
\]
and
\[
d\omega = \frac{\gamma M}{L} [e^{-\frac{2g^2}{\alpha^2} (1 + \frac{L}{\gamma M} \cos \varphi)} - 1] \cos \varphi (d\varphi + d\omega).
\]

Accordingly, approximative values of the ratio of angular speeds of \( \mathbf{L} \) and \( \mathbf{r} \), at the perihelion (\( \varphi = 0 \)) and aphelian (\( \varphi = \pi \)), are
\[
\frac{d\omega}{d\varphi} \bigg|_{\varphi=0} \approx -\frac{2g^2}{\alpha^2} (1 + \frac{\gamma M}{L}) \approx -\frac{2g^2 \gamma M}{a(1 - \frac{L}{\gamma M})}
\]
and
\[
\frac{d\omega}{d\varphi} \bigg|_{\varphi=\pi} \approx -\frac{2g^2}{\alpha^2} (1 - \frac{\gamma M}{L}) \approx \frac{2g \gamma M}{a(1 + \frac{L}{\gamma M})},
\]
where a constant \( 2a \) is the major axis of an elliptical orbit of \( \mathcal{M}_\mu \) in the field of action of \( F_N \) [12].
By the middle approximative values of the recessional and precessional angular deviations of the perihelial point vector per revolution ($\varphi = 2\pi$):

\begin{equation}
(\bar{\omega})_r \approx -\frac{\pi \varrho^2}{\alpha^2} (1 + \frac{\gamma M}{L}) \approx -\frac{\pi \varrho \gamma M}{a(1 - \frac{L}{\gamma M})}
\end{equation}

and

\begin{equation}
(\bar{\omega})_p \approx -\frac{\pi \varrho^2}{\alpha^2} (1 - \frac{\gamma M}{L}) \approx \frac{\pi \varrho \gamma M}{a(1 + \frac{L}{\gamma M})},
\end{equation}

the total middle approximative value of the perihelion advance per revolution ($\varphi = 2\pi$), is

\begin{equation}
\bar{\omega} = (\bar{\omega})_r + (\bar{\omega})_p \approx -\frac{2\pi \varrho^2}{\alpha^2} \approx -\frac{2\pi \varrho}{a[1 - (\frac{L}{\gamma M})^2]}.
\end{equation}

Note that the exact value of an angular deviation of the perihelial point vector with respect to the rotation of the position vector $\mathbf{r}$ can be obtained by integrating (66) and (67), and what is very complex problem. Anyway, it is very important that the angular deviation of the perihelial point vector has been just obtained as a result of the analysis of a material point motion in the field of action of the approximately modified Newton’s gravity force $F_N$.

In addition, it is very indicative that curvatures of the Riemannian spaces:

\begin{equation}
dw^2 = e^{\nu(r)} dr^2 + r^2 d\theta^2 + \sin^2 \theta d\varphi^2 + e^{-\nu(r)} dt^2,
\end{equation}

and

\begin{equation}
d\hat{w}^2 = e^{\hat{\nu}(r)} \left( dr^2 + r^2 d\theta^2 + \sin^2 \theta d\varphi^2 + dt^2 \right),
\end{equation}

where $\nu(r) = -\ln(1 - 2\varrho/r)$ and $\hat{\nu}(r) = -2\varrho/r$, are approximately equivalent $\kappa \simeq \hat{\kappa}$, since (see [14])

\begin{equation}
\kappa = 2\frac{\varrho^2}{r^6}
\end{equation}

and

\begin{equation}
\hat{\kappa} = 2\frac{\varrho^2}{r^6} e^{-\frac{\varrho}{2r}} \sqrt{(1 + \frac{\varrho}{2r})(1 + \frac{\varrho}{r})}.
\end{equation}

Clearly, the first space is the strict Schwarzchild-Droste’s solution to the static gravitational field with spherical symmetry, and the second one is its modification based on the approximately modified Newton’s gravity force $F_N$ (see Figure 1.).
3 Conclusion

The mathematical model of a material point motion in the three-dimensional spatial subspace of the four-dimensional integral space-time continuum and in the field of action of a conservative active force $F$ is analogous to Newton’s mathematical model of the classical mechanics. In addition, the configurative integral space of the space-time continuum, whose the metric form $(d\sigma)^2$ represents one of the fundamental theorems of the material point dynamics the kinetic energy theorem as well as Assumption 2.1, is the configurative space of the space-time continuum of Minkowski from Einstein’s relativity theory. Accordingly, it can be said that in the paper a new connection has been established, in contrast to an approximative one, between the classical Newton’s mathematical model and the relativistic Einstein’s mathematical model.

On the other hand the approximately modified Newton’s gravity concept is not, from any point of view, in collision with old Newton’s one. At the same time it solves the acutely vexed questions within old Newton’s gravity concept (the singularity and perihelion problems). Furthermore, analyzing the analytical expression for the modified Newton’s gravity force $F_N$, see

Figure 1. The modified Newton’s gravity force
(51), we can separate the four indicative domains of its field of action (Fig. 1.). The first one is a domain of the weak action on finitely small distances. The second one is a domain of the strong action in a neighborhood of the gravitational radius \( \rho = \frac{\gamma M}{c^2} \) \((\partial r F|_{r=\rho} = 0\) and \(\partial^2_{rr} F|_{r=\rho} < 0\)). The third one is a domain of action on finitely large distances relative to the gravitational radius \( \rho \) and with the relatively small velocities relative to the light velocity, and the fourth on finitely large distances relative to the gravitational radius \( \rho \) and with velocities that are comparable to the light velocity. Previously separated domains of the field of action of the modified Newton’s gravity force \( F_N \) it would be desirable to compare to the fields of action of the four so far non-unified fundamental forces (weak and strong nuclear interactions, gravity and Lorenz’s electromagnetism), and what could be the subject of a separate analysis. Accordingly, note at the end that a correction to Newton’s gravity law in the form of the functional dependence \( r^{-3}e^{-r/rc} \) irresistibly reminding of the modified Newton’s gravity force, see (51), and obviously wrongly called the "fifth force", has been revealed by a reexamination of the old attraction data and careful new force measurements [3].

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