ON C*-ALGEBRAS WHICH CANNOT BE DECOMPOSED INTO TENSOR PRODUCTS WITH BOTH FACTORS INFINITE-DIMENSIONAL

TOMASZ KANIA

ABSTRACT. We prove that C*-algebras which, as Banach spaces, are Grothendieck cannot be decomposed into a tensor product of two infinite-dimensional C*-algebras. By a result of Pfitzner, this class contains all von Neumann algebras and their norm-quotients. We thus complement a recent result of Ghasemi who established a similar conclusion for the class of SAW*-algebras.

1. Introduction and statement of the main result

During the London Mathematical Society Meeting held in Nottingham on 6th September 2010, Simon Wassermann asked a question of whether the Calkin algebra can be decomposed into a C*-tensor product of two infinite-dimensional C*-algebras. This question stems from the study of the elusive nature of the automorphism group of the Calkin algebra whose structure is independent of the usual axioms of Set Theory ([7, 18]). Ghasemi ([9]) studied tensorial decompositions of SAW*-algebras answering the above-mentioned question in the negative—one cannot thus expect to build automorphisms of such algebras out of automorphisms of non-trivial tensorial factors. Let us remark that the commutative version of Ghasemi’s result was known to experts as it follows directly from the conjunction of [24, Theorem B] with the main theorem of [4].

The aim of this note is to prove that C*-algebras which satisfy a certain Banach-space property cannot be decomposed into a tensor product of C*-algebras. More specifically, we prove that C*-algebras which, as Banach spaces, are Grothendieck (i.e., weak*-null sequences in the dual space converge weakly) do not allow such a tensorial decomposition. In particular, we give a new solution to the problem of Wassermann as the Calkin algebra falls into the class of Grothendieck spaces.

Theorem 1.1. Let $A$ be a C*-algebra which, as a Banach space, is a Grothendieck space. Suppose that $E$ and $F$ are C*-algebras such that

$$A \cong E \otimes, F$$

for some C*-norm $\gamma$. Then either $E$ or $F$ (or both) are finite-dimensional.

In other words, a C*-algebra, which is also a Grothendieck space, cannot be decomposed into a tensor product of two infinite-dimensional C*-algebras.
Let us list examples of classes of C*-algebras which meet the assumptions of Theorem 1.1.

**Proposition 1.2.** C*-algebras in each of the following classes are Grothendieck spaces:

(i) von Neumann algebras (or more generally, AW*-algebras) and their norm-quotients; in particular \(\mathcal{B}(H)\) and the Calkin algebra,

(ii) ultraproducts of C*-algebras over countably incomplete ultrafilters,

(iii) unital C*-algebras with the countable Riesz interpolation property,

**Proof.** By Corollary 2.4, von Neumann algebras and hence their continuous, linear images (such as the Calkin algebra) satisfy the hypothesis of Theorem 1.1. The assertion for AW*-algebras follows from Proposition 2.5 as every maximal abelian self-adjoint subalgebra of an AW*-algebra is Grothendieck by the main results of [22] and [23].

Avilés et al. ([2, Proposition 3.3]) proved that ultraproducts of Banach spaces over countably incomplete ultrafilters cannot contain complemented copies of \(c_0\). This, combined with Theorem 2.2 and Proposition 2.1(iii), yields that ultraproducts of C*-algebras are Grothendieck spaces.

The assertion (iii) follows from applying [19, Theorem 9] to the real Banach space \(A_{sa}\) of all self-adjoint elements of a unital C*-algebra \(A\) with the countable Riesz interpolation property and noticing that the Grothendieck property passes from \(A_{sa}\) to the complex Banach space \(A = A_{sa} \oplus iA_{sa}\). \(\square\)

It is perhaps worthwhile to mention that even at the abelian level there exist many Grothendieck \(C(X)\)-spaces that are not SAW* (i.e., for which \(X\) is not sub-Stonean); an example of such is a space constructed by Haydon ([12]).

Each unital C*-algebra with the countable Riesz interpolation property is an SAW*-algebra in the sense of Pedersen ([25, Proposition 2.7]); see [15] for the definition of an SAW*-algebra. Conjecturally all unital SAW*-algebras have the countable Riesz interpolation property ([25, p. 117]), hence our result covers all known examples of SAW*-algebras—thus, we extend the main result of [9] to the class of ultraproducts of C*-algebras and other C*-algebras created, for instance, out of Grothendieck abelian C*-algebras that are not SAW*.

To the best of our knowledge, it is not known whether a maximal abelian self-adjoint subalgebra of an SAW*-algebra is SAW* too. If this were the case, Proposition 2.5 would immediately imply that SAW*-algebras are Grothendieck spaces, because abelian SAW*-algebras are of the form \(C_0(X)\) for some locally compact sub-Stonean space \(X\), hence Grothendieck by [1] or [24].

### 2. Preliminaries

**Grothendieck spaces.** A series \(\sum_{n=1}^{\infty} y_n\) in a Banach space \(E\) is *weakly unconditionally convergent* if the scalar series \(\sum_{n=1}^{\infty} |\langle f, y_n \rangle|\) converges for each \(f \in E^*\). An operator between Banach spaces is *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent series. A Banach space \(E\) has property (V)
if for each Banach space $F$ the class of unconditionally converging operators $T: E \to F$ coincides with the class of weakly compact operators. It is a result of Pełczyński that $C(X)$-spaces (abelian C*-algebras) have property (V) ([16]).

A Banach space $E$ is Grothendieck if every weak*-null sequence in $E^*$ converges weakly. The name Grothendieck space stems from a result of Grothendieck ([11]) who identified $\ell_\infty$ as a space having this property. It is a trivial remark that reflexive spaces have this property too. By the Hahn–Banach theorem, the class of Grothendieck spaces is closed under surjective linear images, i.e., whenever $E$ and $F$ are Banach spaces, $T: E \to F$ is a surjective bounded linear operator, then if $E$ is Grothendieck, so is $F$.

Let us record a proposition which links property (V) with the Grothendieck property.

**Proposition 2.1.** Let $X$ be a Banach space. Then the following are equivalent.

(i) $X$ is a Grothendieck space,

(ii) each bounded linear operator $T: X \to c_0$ is weakly compact.

(iii) $X$ has property (V) and no subspace of $X$ isomorphic to $c_0$ is complemented.

**Proof.** For the proof of equivalences (i) $\iff$ (ii) see [5, Corollary 5 on p. 150]. Equivalence (i) $\iff$ (iii) is due to Räbiger ([20]); see also [10, Theorem 28] (this argument is also implicit in the proof of [4, Corollary 2]).

We require the following theorem of Pfitzner ([17, Theorem 1]), which can be thought as a non-commutative generalisation of the above-mentioned result of Pełczyński.

**Theorem 2.2 (Pfitzner).** Let $A$ be a C*-algebra and let $K \subset A^*$ be a bounded set. Then $K$ is not relatively weakly compact if and only if there are a sequence $(x_n)_{n=1}^\infty$ of pairwise orthogonal, norm-one self-adjoint elements of $A$ and $\delta > 0$ such that

$$\sup_{f \in K} |\langle f, x_n \rangle| > \delta.$$ 

In particular, C*-algebras have property (V).

The original proof was highly sophisticated and relied on numerous deep facts from Banach space theory. Fortunately, Fernández-Polo and Peralta ([8]) supplied a short and elementary proof of Pfitzner’s theorem.

By virtue of Proposition 2.1(iii), we arrive at the following corollary.

**Corollary 2.3.** A C*-algebra is a Grothendieck space if and only if it does not contain complemented subspaces isomorphic to $c_0$.

The special case of Corollary 2.3 where the C*-algebra is also a von Neumann algebra was noted by Pfitzner ([17, Corollary 7]):

**Corollary 2.4.** Von Neumann algebras are Grothendieck spaces.

Let us take this opportunity to record the following easy corollary to Theorem 2.2.

**Proposition 2.5.** Let $A$ be a C*-algebra with the property that each maximal abelian self-adjoint subalgebra $B$ of $A$ is a Grothendieck space. Then $A$ is a Grothendieck space.
Proof. Let \( T : A \to c_0 \) be a bounded linear operator. By Proposition 2.1(ii) it is enough to show that \( T \) is weakly compact.

Assume contrapositively that \( T \) is not weakly compact. By Gantmacher’s theorem, \( T \) is weakly compact if and only if \( T^* \) is, so the set \( \mathcal{K} = T^*(B) \) is not relatively weakly compact, where \( B \) is the unit ball of \( c_0^* \). By Theorem 2.2, there exist \( \delta > 0 \) and a sequence \( (x_n)_n \) of pairwise orthogonal, norm-one self-adjoint elements in \( A \) such that

\[
\sup_{f \in \mathcal{K}} |\langle f, x_n \rangle| = \sup_{y \in B} |\langle T^*y, x_n \rangle| = \sup_{y \in B} |\langle y, Tx_n \rangle| > \delta. \tag{2.1}
\]

Let \( B_0 \subseteq A \) be the C*-algebra generated by \( \{x_n : n \in \mathbb{N}\} \). Since the \( x_n \) (\( n \in \mathbb{N} \)) are pairwise orthogonal, \( B_0 \) is abelian. Let \( B \) be a maximal abelian subalgebra of \( A \) containing \( B_0 \).

Consider the restriction \( T|_B : B \to c_0 \). It is not weakly compact by (2.1), so \( B \) is not a Grothendieck space. \( \square \)

3. Proof of Theorem 1.1

We are now in a position to prove our main result. The general strategy of the proof was inspired by the path taken by Cembranos in [4].

Proof of Theorem 1.1. Let \( A \) be a C*-algebra and suppose that it is a Grothendieck space.

Assume towards a contradiction that \( A \cong E \otimes F \) for some infinite-dimensional C*-algebras \( E, F \) and a C*-norm \( \gamma \).

Since *-homomorphisms between C*-algebras have closed range, there is a surjective *-homomorphism \( Q : E \otimes F \to E \otimes_{\min} F \) that extends the identity map on \( E \odot F \). (Here \( E \otimes_{\min} F \) denotes the minimal C*-tensor product of \( E \) and \( F \).)

Let \( B_1 \subseteq E \) and \( B_2 \subseteq F \) be infinite-dimensional abelian C*-algebras. (Such subalgebras exist because infinite-dimensional C*-algebras contain self-adjoint elements with infinite spectrum ([13, Ex. 4.6.12]), hence the assertion follows from the spectral theorem.) We may thus identify \( B_1 \otimes_{\min} B_2 \) with a subalgebra of \( E \otimes_{\min} F \) (cf. [3, II.9.6.2]). However, the (minimal) tensor product of abelian C*-algebras is the same as the Banach-space injective tensor product, i.e.,

\[
B_1 \otimes_{\min} B_2 = B_1 \tilde{\otimes} B_2.
\]

Let \( (e_n)_n \) be a sequence of pairwise orthogonal, positive, norm-one elements in \( B_1 \). In particular, \( E_0 = \text{span}\{e_n : n \in \mathbb{N}\} \) is isometric to \( c_0 \) and \( (e_n)_n \) is equivalent to the canonical basis for \( c_0 \). Choose norm-one functionals \( e_n^* \in E^* \) such that \( \langle e_n^*, e_m \rangle = \delta_{n,m} \) \( (n, m \in \mathbb{N}) \). Moreover, let \( (x_n)_n \) be a sequence of unit vectors in \( F^* \) which converges to 0 in the weak* topology. (Such a sequence exists by the Josefson–Nissenzweig theorem (see [6, Chapter XII].) Let \( (x_n)_n \subseteq F \) be a sequence such that \( \langle x_n, x_n^* \rangle = 1 \). Without loss of generality we may suppose that \( \|x_n\| \leq 2 \) for all \( n \).

Define a map \( T : E \otimes_{\min} F \to \ell_\infty \) by the formula

\[
T\xi = ((e_n^* \otimes x_n^*)\xi)_n \quad (\xi \in E \otimes_{\min} F).
\]
This is a well-defined bounded linear operator because \((e^*_n \otimes x^*_n)_{n=1}^\infty\) is a bounded sequence of functionals on \(E \otimes_{\text{min}} F\). We can thus extend \(T\) to the whole of \(E \otimes_{\text{min}} F\). Moreover, for all \(f \in E\) and \(x \in F\) we have
\[
|\langle e^*_n, f \rangle \cdot \langle x, x^*_n \rangle| \leq \|f\| \cdot |\langle x, x^*_n \rangle|
\]
so \(T\) takes values in \(c_0\) as \((x^*_n)_{n=1}^\infty\) is a weak*-null sequence. Since the injective tensor product ‘respects subspaces’ (see [21, p. 49]), \(E_0 \otimes B_2\) can be identified with a subspace of \(B_1 \otimes B_2\) and the latter is a subspace of \(E \otimes_{\text{min}} F\).

As \(E_0\) and \(c_0\) are isometrically isomorphic, so are \(E_0 \otimes B_2\) and \(c_0 \otimes B_2 \cong c_0(B_2)\) (cf. [21, Example 3.3]). Let \(n \in \mathbb{N}\) and let \((a_k)_{k=1}^\infty\) be a scalar sequence with only finitely many non-zero entries. We have
\[
\left\| \sum_{k=1}^n a_k e_k \otimes x_k \right\| = \left\| \sum_{k=1}^n e_k \otimes (a_k x_k) \right\| \leq \sup_{1 \leq k \leq n} \|a_k x_k\| \leq 2 \max\{|a_k| : 1 \leq k \leq n\}.
\]
By [14, Proposition 4.3.9], \(\sum_{n=1}^\infty e_n \otimes x_n\) is a weakly unconditionally convergent series in \(E \otimes_{\text{min}} F\). On the other hand, for all \(k, n \in \mathbb{N}\) we have \((T(e_n \otimes x_n))(k) = \delta_{k,n}\) so
\[
\sum_{n=1}^\infty T(e_n \otimes x_n)
\]
fails to converge in \(c_0\). Consequently, \(E \otimes_{\text{min}} F\) is not a Grothendieck space as we proved that the \(c_0\)-valued operator \(T\) is not unconditionally converging. Indeed, if \(E \otimes_{\text{min}} F\) were Grothendieck, \(T\) would be weakly compact (Proposition 2.1(ii)), hence also unconditionally converging (Proposition 2.1(iii)).

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Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster LA1 4YF, United Kingdom.

E-mail address: tomasz.marcin.kania@gmail.com