Driven collective instabilities in magneto-optical traps: A fluid-dynamical approach

H. Terc¸as1(a), J. T. Mendonça1,2 and R. Kaiser3

1 CFIF, Instituto Superior Técnico - Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal, EU
2 IPFN, Instituto Superior Técnico - Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal, EU
3 Institut Non Linéaire de Nice, UMR 6618 - 1361 Route des Lucioles, F-06560 Valbonne, France, EU

Received 20 January 2010; accepted in final form 2 March 2010
Published online 26 March 2010

PACS 37.10.Gh – Atom traps and guides
PACS 37.10.Yz – Mechanical effects of lights on atoms, molecules, and ions
PACS 52.35.Dm – Sound waves

Abstract – We present a theoretical model to describe an instability mechanism in ultra-cold gases, where long-range interactions are taken into account. Focusing on the coupling between the collective (plasma-like) and the center-of-mass modes, we show that the resulting dynamics is governed by a parametric equation of the generalized Mathieu type and compute the corresponding stability chart. We apply our model to typical ranges of magneto-optical traps (MOT) parameters and find a good agreement with previous experimental observations.

Introduction. – The celebrated method of magneto-optical cooling of atoms [1] allowed the exploitation of new and very interesting phenomena in atomic physics, and in particular the creation of Bose-Einstein condensates [2]. Cold and ultracold gases produced in a magneto-optical trap (MOT) provide a unique medium to explore the connections between different areas of physics. Due to absorption and re-scattering of light [3–5], collective phenomena of cold gases confined in a MOT became recently an issue of major interest. Under such conditions, the atomic gas in a MOT can be regarded as a fluid with a tunable effective charge, which opens a new area of plasma physics [6–8]. A very interesting manifestation of the latter is the work on the Coulomb explosions of molasses, performed by Pruvost et al. [7]. Moreover, in a recent work, we have proposed the existence of hybrid waves and Tonks-Dattner oscillations in the confined gas of ultra-cold atoms, which turn out to be very similar, but not identical, to those of a plasma medium [8].

It is a well-known fact that cold atomic gases confined in MOTs sustain various types of instabilities. One example is found in the work realized by Kim et al. [9], where a parametric instability is excited by an intensity-modulated laser beam. This instability is based on a nonlinear response of the individual atoms to one of the control parameters of the MOT. Spontaneous instabilities arising for large MOTs have also been investigated [10–12], where the feedback of retroreflected laser beams of the MOT can induce stochastic or deterministic chaos. The latter have been classified as center-of-mass instability. Another type of spontaneous instabilities, due to the large spatial extension of the MOT, has been described in [13]. Here, the instability finds its origin at the edge of the MOT where the Zeeman shift produces a local change of the detuning seen by the atoms with a negative friction for atoms beyond a certain horizon.

In the present work, we theoretically explore the possible existence of a new instability process, due to the coupling between two collective oscillations in a MOT: the “plasma” and the center-of-mass (c.o.m.) modes. We use the formalism developed in our recent work [8], where we have shown that plasma-hybrid waves can propagate in the atomic cloud and have established their dispersion relation. These hybrid waves exhibit features that are common to both electron-plasma and acoustic waves [14–16]. Here, we start with a set of fluid equations describing the dynamics of the atomic medium, where we have retained the main features of the cooling and trapping forces. We then show that these plasma-like oscillations can couple with the center-of-mass of the cloud, giving place to the a possible parametric-instability mechanism. By identifying the relevant parameters and establishing the stability criteria, we are able to show that this mechanism could lead to an instability threshold in the same parameter range as...
the effective plasma mode (also known as breathing mode) MOT can be regarded as a one-component plasma, where field gradient \[8\]. In such a picture, the gas confined in a trap is governed by a set of plasma oscillations. –

et al. the one observed by Labeyrie et al. [13]. New instability regimes are also predicted.

**Coupling between the center-of-mass and the plasma oscillations.** – Following our previous work [8], the collective behavior in the trap is governed by a set of fluid equations:

\[
\begin{align*}
\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) &= 0, \\
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= \frac{\mathbf{F}}{M}, \\
\nabla \cdot \mathbf{F}_c &= Q n,
\end{align*}
\]

where \( n \) and \( \mathbf{v} \) represent the gas density and velocity, respectively, and \( M \) is the atomic mass. The total force is given by \( \mathbf{F} = \mathbf{F}_c + \mathbf{F}_t \), where \( \mathbf{F}_c \) is the collective force and \( Q = (\sigma R - \sigma L) \omega I_0/c \) represents the square of the effective electric charge [7,8], \( c \) is the speed of light and \( I_0 \) the total intensity of the six laser beams. Finally, \( \sigma R \) and \( \sigma L \) represent the emission and absorption cross-sections [3] and \( \mathbf{F}_t = -\nabla U \) stands for the trapping force. Neglecting the multilevel structure of the atom, and thus additional mechanisms such as Sisyphus cooling [1], we approximate the trapping potential by

\[
U(r) = \frac{1}{2} \kappa \ln^2, \quad \kappa = \alpha \frac{\mu_B}{\hbar k} \nabla B.
\]

Here, \( \mu_B \) represents the Bohr magneton, \( \kappa = \kappa(\delta, I_0/I_s) \) is the spring constant, \( \alpha = \alpha(\delta, I_0/I_s) \) is the friction coefficient, \( \delta \) is the laser detuning, \( I_s \) is the atomic saturation intensity and \( \nabla B = |\nabla B| \) represents the magnetic field gradient [8]. In such a picture, the gas confined in a MOT can be regarded as a one-component plasma, where the effective plasma mode (also known as breathing mode) frequency is given by

\[
\omega_p = \sqrt{Q n^0/M}.
\]

The physical origin of this mode is rooted in the collective scattering of light in the trap, which induces a long-range force between the atoms [3–5,7]. Such a plasma description is expected to be valid in the regime where the size of the trap \( L \) increases with the number of atoms \( N_{at} \) (typically, in the range where \( N_{at} > 10^6 \)) [5,17].

We linearize eqs. (1)–(3) by separating each relevant physical quantity into its equilibrium and perturbation components, such that

\[
n = n_0 + n_1 + n_2, \quad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2.
\]

The subscripts 1 and 2 label two different perturbations, corresponding to two distinct collective modes in the MOT. This procedure allows one to include any pair of modes and could be extended to more than two modes, thus broadening the range of applicability of the model. In this work, we consider that 1 and 2 label the perturbations due to center-of-mass (c.o.m.) and plasma oscillations. Accordingly, \( \mathbf{F}_1 = \mathbf{F}_1 \) is the restoring force in the c.o.m. and \( \mathbf{F}_2 = \mathbf{F}_2 \) represents the collective force responsible for the plasma oscillations. For moderate oscillation amplitudes, such that the Zeeman shift always remains negligible compared to the detuning [17], we can use the following ansatz for the center-of-mass velocity \( \mathbf{v}_1 \):

\[
\mathbf{v}_1(t) = \mathbf{u}_1 \sin(\omega_{CM} t + \phi),
\]

where \( \omega_{CM} \) (associated to \( \mathbf{F}_1 \) —for the sake of generality, the details on the calculation of \( \omega_{CM} \) will be discussed later on) represents the c.o.m. mode frequency, \( \mathbf{u}_1 \) its amplitude and \( \phi \) is an arbitrary phase. We will consider that the c.o.m. oscillates as a rigid body, which means that the fluctuations \( n_1 \) and \( n_2 \) can be neglected compared to \( n_0 \) \((n_1 \sim n_2 \ll n_0)\). Combining these approximations with the above fluid equations, one easily obtains

\[
\begin{align*}
\frac{\partial^2 n_2}{\partial t^2} + n_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{v}_2 + \mathbf{u}_1 \cdot \nabla \left[ \frac{\partial n_2}{\partial t} \sin(\omega_{CM} t + \phi) \right. \\
+ n_2 \omega_{CM} \cos(\omega_{CM} t + \phi) \bigg] &= 0. \quad (8)
\end{align*}
\]

Noticing that \( n_0 \partial_t \nabla \cdot \mathbf{v}_2 = \omega_p^2 n_2 \), it follows

\[
\begin{align*}
\frac{\partial^2 n_2}{\partial t^2} + \omega_p^2 n_2 + \mathbf{u}_1 \cdot \nabla \left[ \frac{\partial n_2}{\partial t} \sin(\omega_{CM} t + \phi) \right. \\
+ \omega_{CM} n_2 \cos(\omega_{CM} t + \phi) \bigg] &= 0. \quad (9)
\end{align*}
\]

The third term of this expression describes the coupling between the two collective modes mentioned above. This is one of the main results of the paper. Notice that the plasma mode (5) could be replaced by any of those in the hierarchy \( n_{1,2} \) associated to the Tonks-Dattner (TD) resonances, the standing waves inside the cloud [8]. The reason for this choice is because \( \omega_p \) is the lowest mode in the TD series.

Making use of the ansatz \( n_2(r, t) = B_2(r) A_2(t) \), eq. (9) yields

\[
\frac{d^2 A_2}{d \tau^2} + \left[ \nu + 2 \epsilon \cos(2\tau) \right] A_2 + \epsilon \sin(2\tau) \frac{dA_2}{d\tau} = 0,
\]

where \( \xi = 2\mathbf{u}_1 \cdot \nabla \ln B_2 \) is a free coupling parameter and describes the amplitude of the center-of-mass oscillation. Here, \( 2\tau = \omega_{CM} t + \phi, \nu = 4\omega_p^2/\omega_{CM}^2 \) and \( \epsilon = \xi/\omega_{CM} \) are dimensionless variables. Equation (10) describes the dynamics of a parametrically excited system and belongs to the family of Hill equations. It is formally similar to the Mathieu equation, which is well known in the literature for containing unstable solutions. According to the standard Floquet theory [18], the solutions to eq. (10) can be expanded into Fourier series

\[
A_2(\tau) = \sum_{n=-\infty}^{+\infty} a_n e^{(\gamma+i\nu)\tau},
\]

53001-p2
where $\gamma$ is the characteristic exponent [18]. Plugging it into eq. (10), one obtains a system of equations for the coefficients $a_n$. The non-trivial solution is obtained by computing the roots of the so-called infinite Hill determinant, i.e.,

$$\det (\Delta_{j,k}(n)) = \det \left[ \left[ (\nu + (\gamma + 2jn)^2) \delta_{j,k} + \epsilon \left[ 1 - i \frac{1}{2} (\gamma + 2jn) \right] \delta_{j,k+1} + \epsilon \left[ 1 + i \frac{1}{2} (\gamma + 2jn) \right] \delta_{j,k-1} \right] \right] = 0, \quad (12)$$

where $\delta_{j,k}$ is the Kronecker delta. In order to carry out calculations, only the truncated system $n = -N, \ldots, N$ is considered. This approximation results on the Ince-Strutt diagram [19]. In fig. 1 we plot the stability chart of eq. (10) in the $(\nu, \epsilon)$-plane, truncated at $N = 5$, where the $\pi$ and $2\pi$-periodic marginal curves correspond to $\gamma = 0$ and $\gamma = i$, respectively. The marginal curves $\nu(\epsilon)$ separate the different regions of stability (check ref. [18] for further details on the Floquet theory).

**Experimental evidence of instability.** – For comparison with experiments, it is necessary to express the frequencies $\omega_P$ and $\omega_{CM}$ in terms of real-life parameters. The balance between the trapping and the collective forces is given by the hydrodynamical-equilibrium condition $D_i D_j v \equiv (\partial / \partial t + v \cdot \nabla) v = 0$, which simply corresponds to set

$$\mathbf{F}_t + \mathbf{F}_c = 0, \quad (13)$$

where $F_t = -\nabla U = -\mathbf{x}r$. Taking the divergence of the latter, we obtain the following relation

$$\kappa = (\sigma_R - \sigma_L) \sigma_L I_0 = \frac{Qn_0}{3}, \quad (14)$$

which simply corresponds to the expression for the compression limit of the MOT [5]. This condition establishes a relation between the plasma (5) and the c.o.m. oscillation frequencies: $\omega_{CM} = \omega = \omega P / \sqrt{3}$, where $\omega = \sqrt{E / M}$ is the trapping frequency. The identity $\omega_{CM} = \omega$ is equivalent of setting $F_1 = F_t$ and it is simply a consequence of Kohn’s theorem [20], which basically states that interactions do not affect the c.o.m. motion. The equality $\omega_{CM} = \omega P / \sqrt{3}$ is often recognized in unscreened Coulomb systems [21,22], as it describes the so-called Mie resonance. The analogy between MOTs and Coulomb systems follows from the fact that the effect of the trap is replaced by the background ionic density. For typical experimental conditions, a $^{85}$Rb MOT operating at $\delta = -1.5 \Gamma$ ($\Gamma \approx 6$ MHz is the natural lifetime of the atomic transition), with a magnetic-field gradient $\nabla B = 5 \text{G/cm}$ and $I_0 / I_s \approx 0.3$, the corresponding plasma frequency is $\omega P / 2\pi \approx 120$ Hz. As a consequence of the condition (14), we set the value $\nu = 12$ in eq. (10) and, correspondingly, the instability occurs provided that

$$\epsilon_1 < \epsilon < \epsilon_2 \quad \text{and} \quad \epsilon > \epsilon_3, \quad (15)$$

where $\epsilon_1 = 3.17$, $\epsilon_2 = 3.91$ and $\epsilon_3 = 4.91$ (see fig. 1). Using the definition of $\epsilon$, the marginal curves $\xi = \xi(\epsilon_i)$, with $i = (1, 2, 3)$, representing the critical values of the coupling parameter, are simply given by

$$\left( \frac{\xi}{\epsilon_i} \right)^2 = \omega_{CM}^2 = \frac{8k_L U B I_0}{M} \frac{\left| \delta / \Gamma \right|}{\Gamma (1 + 4\delta^2 / \Gamma^2)^2}, \quad (16)$$

where $k_L$ is the laser wave vector. In fig. 2 we plot the marginal curves $\xi_1$, $\xi_2$ and $\xi_3$ against the relevant MOT parameters. Below the marginal curve $\xi_1$, the oscillations are stable. However, for $\xi_1 < \xi < \xi_2$, the system undergoes large-amplitude density oscillations and becomes unstable. The oscillations become stable between the marginal curves $\xi_2$ and $\xi_3$. Finally for $\xi > \xi_3$ we can observe that the oscillations are again unstable.

This way of coupling different modes of the system, added to an external drive (or sufficient power at the correct frequency in the noise spectrum) and therefore

Fig. 1: (Color online) Stability chart of eq. (10), obtained for a $N = 5$ Hill determinant. The full lines represent $\pi$-period solutions and the dashed lines represent the $2\pi$-period ones. The shadowed (light) regions correspond to stable (unstable) solutions. The red dots represent the marginal values $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ discussed in the text. The negative values of $\nu$ are unphysical.

Fig. 2: (Color online) Marginal curves $\xi_i = \xi(\epsilon_i)$ ($i = 1, 2, 3$) as a function of: a) the detuning $\delta$, at $\nabla B = 3 \text{G/cm}$ and b) the magnetic-field gradient $\nabla B$, at $\delta = -0.75 \Gamma$. In both cases, $I_0 / I_s = 0.3$ and $\xi_1$ (solid line), $\xi_2$ (dashed line) and $\xi_3$ (dotted line) are the marginal curves discussed in the text. The shadowed areas are stable.
Controlling the value $\xi$, could lead to an instability based on mechanisms similar to the ones used in [9]. Applying this idea to a typical range of $(\delta, \nabla B)$ in eq. (16), it is possible to build a stability chart similar to that presented in fig. 3. We note that a stability diagram similar to the one observed by Labeyrie et al. [13] can be obtained for specific values of driving parameters, corresponding here to $\xi = 280$ Hz. In the present paper, however, the basic ingredient for the instability is different from that used in [13], what therefore suggests it is possible to distinguish these two mechanisms: in the latter, the instability is due to a change of sign in the friction coefficient at the edge of the cloud [23], where the driving motion takes place. Instead, we here suggest that an instability in the same parameter range can be obtained by coupling two collective modes, where the energy is transferred from one to the other, as an example of a density-wave mixing phenomenon. Our model also predicts different stability zones, which does not happen in the model of ref. [13]. We believe that the major difference relies on the fact that our model does not depend on the size of the cloud, which is consistent with the linearization of the fluid equations. As a consequence, we do not require the edge to contribute to the dynamics of the system. We also argue that the coupling between the c.o.m. and the different TD modes may occur in real experiments (as they are very close to $\omega_p$) and it should be responsible for the broadening of the intermediate instability zone $U_1$ in fig. 3, which better agrees with the experimental observations of ref. [13]. This is because for $\nu > 12$ the intermediate instability zone $\epsilon_1 < \epsilon < \epsilon_2$ is larger, as one can directly observe in fig. 1.

A short discussion about the experimental features of the present work is in order. First, we notice that the model is valid for instabilities driven both by noise and by an external modulation. In the latter case, where it is possible to better control the relevant experimental parameters, the center-of-mass mode can, e.g., be excited by a controlled external modulation of the trap parameters (detuning, magnetic-field gradient or using an additional pushing laser). The amplitude of plasma mode can be monitored by looking at the frequency spectrum of a local fluorescence signal. Different regimes illustrated in fig. 2) can be studied by tuning the trap parameters. The stable regime may correspond to two different situations: i) either the forced modulation is very weak, so that no coupling between the c.o.m. and plasma modes occurs; ii) or the plasma mode is driven by the c.o.m. oscillation, which may be observed by looking for the emergence of a peak in the spectrum exactly at the breathing mode $\omega_p = \sqrt{3} \omega_{CM}$. On the other hand, the unstable regime corresponds to a strong coupling between the c.o.m. and plasma modes, and the corresponding signature may be the occurrence of higher-order harmonics of $\omega_p$ in the spectrum.

**Conclusion.** – In this work, we have highlighted several aspects of the fluid description of the collective behavior of a cold atomic gas in a magneto-optical trap. We have established the governing equation for the coupling of the center-of-mass and the plasma oscillations in a MOT, which we have shown to be the root of an instability mechanism. It was also shown that the condition of dynamical equilibrium provides a relation between the frequencies of the referred modes, $\omega_{CM} = \omega_p/\sqrt{3}$. Our model yields an instability threshold which can be close to the one observed in [13], even though mechanisms not included in this work have been put forward to explain these self-sustained instabilities. In addition, we predict the existence of new stability regions, compatible with the same coupling parameter, thus giving a new insight to the problem and proposing new areas of experimental research. We concluded that our model can be used to describe, at least qualitatively, a family of instabilities of the collective oscillations in a MOT.

***

One of the authors (HT) would like to thank Dr P. Oliveira, from the Electrical Engineering Department of Instituto Superior Técnico, for helpful discussions during the preparation of a preliminary version of this manuscript. This work was partly supported by Fundação para a Ciência e Tecnologia (FCT-Portugal) through the grant number SFRH/BD/37452/2007, Intercan and ANR CAROL (project ANR-06-BLAN-0096).

**REFERENCES**

[1] CHU S., Rev. Mod. Phys., 70 (1998) 685; COHEN-TANNoudj C., Rev. Mod. Phys., 70 (1998) 707; PHILLIPS W. D., Rev. Mod. Phys., 70 (1998) 721.
[2] CORNELL E. A. and WIEMAN C. E., Rev. Mod. Phys., 74 (2002) 875.
[3] WALKER T., SESKO D. and WIEMAN C., Phys. Rev. Lett., 64 (1990) 408.
[4] DALIBARD J., Opt. Commun., 68 (1988) 203.
[5] SESKO D. W., WALKER T. G. and WIEMAN C. E., J. Opt. Soc. Am. B, 8 (1991) 946.
Driven collective instabilities in magneto-optical traps: A fluid-dynamical approach

[6] Steane A. M., Chowdhury M. and Foot C. J., J. Opt. Soc. Am. B, 9 (1992) 2142.

[7] Pruvost L., Serre I., Hong Tuan Duong and Joshua Jortner, Phys. Rev. A, 61 (2000) 053408.

[8] Mendonca J. T., Kaiser R., Terças H. and Loureiro J., Phys. Rev. A, 78 (2008) 013408.

[9] Kim K., Noh H.-R. and Jhe W., Opt. Commun., 236 (2004) 349.

[10] di Stefano A., Fauquembergue M., Verkerk P. and Hennequin D., Phys. Rev. A, 67 (2003) 033404; di Stefano A., Verkerk P. and Hennequin D., Eur. Phys. J. D, 30 (2004) 243.

[11] Hennequin D., Eur. Phys. J. D, 28 (2004) 135.

[12] Wilkowski D., Ringot J., Hennequin D. and Garreau J. C., Phys. Rev. Lett., 85 (2000) 1839.

[13] Labeyrie G., Michaud F. and Kaiser R., Phys. Rev. Lett., 96 (2006) 023003.

[14] Cheng F. F., Introduction to Plasma Physics and Controlled Fusion, Vol. 1 (Springer) 1984.

[15] Parker J. V., Nickel J. C. and Gould R. W., Phys. Fluids, 7 (1964) 1489.

[16] Guerra R. and Mendonça J. T., Phys. Rev. E, 62 (2000) 1190.

[17] Gattobigio G. L., Pohl T., Labeyrie G. and Kaiser R., Phys. Scr., 81 (2010) 025301.

[18] Nayfeh A. and Mook D., Nonlinear Oscillations (Wiley-VCH) 2008.

[19] van der Pol F. and Strutt M. J. O., Philos. Mag., 5 (1928) 18.

[20] Kohn W., Phys. Rev., 123 (1961) 1242.

[21] Seidl M. and Manninen M., Z. Phys. D, 33 (1995) 163.

[22] Mie G., Ann. Phys. (N.Y.), 25 (1908) 377.

[23] Pohl T., Labeyrie G. and Kaiser R., Phys. Rev. A, 74 (2006) 023409.