A simple approach to the angular momentum distribution in the ground states of many-body systems

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Abstract

We propose a simple approach to predict the angular momentum $I$ ground state ($I_{g.s.}$) probabilities of many-body systems that does not require the diagonalization of hamiltonians with random interactions. This method is found to be applicable to all cases that have been discussed: even and odd fermion systems (both in single-$j$ and many-$j$ shells), and boson (both $sd$ and $sdg$) systems. A simple relation for the highest angular momentum g.s. probability is found. Furthermore, it is suggested for the first time that the 0g.s. dominance in boson systems and in even-fermion systems is given by two-body interactions with specific features.

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The low-lying spectra of many-body systems with even numbers of fermions were recently examined by Johnson et al. [1–3] using the two-body random ensemble (TBRE), and the results showed a preponderance of $I^g = 0^+$ ground states (0g.s.). The 0g.s. dominance was soon confirmed in sd-boson systems [4,5]. Therefore, the 0g.s. dominance in even fermion systems and boson systems is robust and insensitive to the detailed statistical properties of the random hamiltonian, suggesting that the pairing features arise from a very large ensemble of two-body interactions other than a simple monopole pairing interaction and might be independent of the specific character of the force. Very recently, several interesting studies were performed to check whether the spectroscopy based random and/or displaced random ensembles can simulate that of realistic systems [6–8].

An understanding of the 0g.s. dominance in many-body systems is extremely important, since this observation seems to be contrary to what is traditionally assumed. For example, in nuclear physics the 0g.s. dominance in even-even nuclei is usually explained as a reflection of the strong pairing that results for a short-range attraction between identical nucleons. There have been several efforts to understand this observation. Work has been reported on its connection with the distribution widths of eigenvalues [8] or the distribution of the lowest eigenvalues of each angular momentum [4], its connection with geometric chaoticity of the angular momentum coupling [10], its connection with the largest and smallest diagonal matrix elements [11], its connection with random polynomials [12] or a Hartree-Bose mean field analysis [13], and its connection with deviations from the random theory expectations [14]. These studies are interesting and important, and have potentially impacted our understanding on the origin of one of the most characteristic features of nuclear spectra. All of these approaches, however, address only simple or very specific (sp and sd bosons, or fermions in small and single-$j$ shells) cases. It is desirable, therefore, to construct a simple and universal approach to understand the 0g.s. dominance of even-fermion systems and sd-boson systems, which can also be applied when discussing the probability that the g.s. has angular momentum $I$ ($I$g.s. probability) of very different systems (boson systems, even and odd fermions in single-$j$ or many-$j$ shells).

Towards that goal, we present in this Letter a new view on the origin of the 0g.s. dominance in even-fermion and boson systems. For the first time, the $I$g.s. probabilities of both fermions (in single-$j$ and many-$j$ shells) and bosons are studied on the same footing. In the process, some previously unrecognized features of the highest angular momentum $I = I_{\text{max},g.s.}$ probabilities of fermions in single-$j$ and many-$j$ shells, and of sd and sdg bosons will be discerned and explained, thereby providing further new insights on the problem. We shall show that the 0g.s. dominance in even-fermion and boson systems comes from two-body interactions with specific features.

We now very succinctly describe our method and then discuss its application to the various possible cases. The basic idea is as follows. We first set one of the two-body matrix elements of the problem to -1 and all the rest to zero and then see which angular momentum $I$ gives the lowest eigenvalue among all of the eigenvalues of this many-body system. If the
number of independent two-body interaction matrix elements is $N$, the above procedure is iterated $N$ times, with each of the matrix elements assuming the privileged role of being set to -1. After all $N$ calculations have been done, we simply count how many times (denoted as $N_I$) the angular momentum $I$ gives the lowest eigenvalue \([13]\). Finally, the $I$g.s. probability is given by $N_I/N$.

We will now show the universality of this simple method by comparing its predictions with the results obtained by diagonalizing random hamiltonians for a variety of very different systems. In all cases, we will use the TBRE \([1,2,4,5,11,12]\) to define the random two-body interactions, $G_J(j_1 j_2 j_3 j_4)$. Namely, $G_J(j_1 j_2 j_3 j_4)$ will be chosen to be random numbers with a distribution

$$
\rho(G_J(j_1 j_2 j_3 j_4)) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(G_J(j_1 j_2 j_3 j_4))^2}{2}\right).
$$

For single-$j$ shells, the labels $(j_1 j_2 j_3 j_4)$ are unnecessary and will thus be suppressed.

Let us start by considering fermions in single-$j$ shells. TABLE I shows the angular momenta $I$ which produce the lowest eigenvalues for the different possible choices for which $G_J$ to be set to -1. The results are shown for $j$ values ranging from $7/2$ to $31/2$ and for systems of $n = 4$. The number $N_{I=0}$ of 0g.s staggers with $j$ with an interval $\delta j=3$. FIG. 1 gives a comparison between the predicted 0g.s. probabilities (open diamond symbols) and those obtained by diagonalizing the TBRE hamiltonian (solid square symbols). The agreement is quite good, both for small-$j$ and large-$j$ shells. The predicted 0g.s. probabilities show the same staggering character as those obtained using the TBRE hamiltonian. A similar staggering behavior of 0g.s. probabilities for single-$j$ shells with $n = 6$ was observed earlier \([7]\) and it can be explained in the same way.

In single-$j$ shells, the highest angular momentum state (denoted as $I_{max}$) was found to have a sizable probability to be the g.s. \([10,11]\). This can be understood from the observation that $N_{I_{max}} = 1$ always. It is easy to confirm that the eigenvalue of the $I = I_{max}$ state is the lowest when $G_{I_{max}} = -1$ and all other parameters are switched off. A detailed study of the eigenvalue of the state with $I = I_{max}$ in single-$j$ shells was given in Ref. \([7]\). Because $N_{I_{max}} = 1$, the predicted $I_{max}$g.s. probabilities of fermions in single-$j$ shells are $\frac{1}{N} = \frac{1}{j+1/2} \times 100\%$, a formula which is valid for all particle numbers (even or odd). It is predicted that the $I = I_{max}$g.s. probability decreases gradually with $j$ and vanishes in the large-$j$ limit.

In Ref. \([4]\) the authors found that the $I_{max}$g.s. probability is also large for $sd$-boson systems. This can be explained in the same way. Among the two-body interactions, the interaction $-(d^\dagger d^\dagger)^4 (dd)^4$ always gives the lowest eigenvalue for the $I_{max} = 2n$ state, independent of the boson number. In the $sd$-boson model, the predicted $I = 2n$ g.s. probability is $1/7 = 14.3\%$, well consistent \([10]\) with the previous results ($\sim 15\%$) \([4,11]\).

FIG. 2a) shows the $I_{max}$ probabilities for different fermion numbers in single-$j$ shells. The $I_{max}$g.s. probabilities obtained by diagonalizing the TBRE hamiltonian and those based on our simple $\frac{1}{N} \times 100\%$ ($N = (2j + 1)/2$) rule are in good agreement. For the first time, the
The $I_{max}$ g.s. probabilities of fermions in single-$j$ shells are shown to be independent of particle number $n$ and to follow a simple analytic relation.

Our argument on the $I_{max}$ g.s. probabilities for single-$j$ shells can be readily generalized to many-$j$ shells. Consider, for example, two shells with angular momenta $j_1$ and $j_2$. Following the same logic as was used for a single-$j$ shell, we predict that the two angular momenta $I'_{max} = I_{max}(j_1^a)$ and $I_{max}(j_2^a)$ have g.s. probabilities which are at least as large as $1/N \times 100\%$. Here, $I_{max}(j^a)$ refers to the highest angular momentum of a state constructed from the $j^a$ configuration. In other words, we can predict in this way the lower limit on these $I'_{max}$ g.s. probabilities. This is analogous to the Schmidt diagram of magnetic moments of odd-$A$ nuclei in nuclear structure theory.

FIG. 2b presents the $I'_{max} = I_{max}(j_1^a)$ and $I_{max}(j_2^a)$ g.s. probabilities. They are compared with a simple $1/N$ plot. Here $N$ is the number of independent two-body interactions of a $(j_1, j_2)$ shell. Indeed, the $1/N$ predicted lower limit on the $I'_{max}$ g.s. probabilities works quite well. It should also be noted that the $I$ g.s. probabilities with $I$ very near $I'_{max}$ are extremely small (less than 1%) in these examples.

FIGS. 3a) and 3b) present a comparison between the predicted $I$ g.s. probabilities and those obtained by diagonalizing the TBRE hamiltonian for a $j = \frac{9}{2}$ shell. We present two cases: the case of $n = 4$ (even) is shown in 3a) and the case of $n = 5$ (odd) is shown in 3b). In both cases, the agreement is good. Note that the agreement does not deteriorate when we go to larger single-$j$ shells or to many-$j$ shells where there are more interaction parameters.

FIGS. 3c) and 3d) show the $I$ g.s. probabilities for a system of 7 fermions in two-$j$ shells ($j_1 = \frac{7}{2}$, $j_2 = \frac{5}{2}$), and an sd-boson system (with 10 bosons), respectively. The predicted $I$ g.s. probabilities are reasonably consistent with those obtained by diagonalizing the TBRE hamiltonian. We checked many cases, such as (a) single-$j$ shells with $j$ up to $\frac{31}{2}$ and with $n$=4, 5, 6, (b) two-$j$ shells with $(2j_1, 2j_2)$=(5, 7), (5, 9), (11, 3), (11, 5), (11, 9) and (13, 9) and with $n = 4, 5, 6$, (c) sd-boson systems with $n$ up to 17, and (d) sdg-boson systems with $n = 4, 5, 6$, and in all cases the agreement is reasonably good.

One may ask why we study the $I$ g.s. probabilities in the way proposed in this Letter. A rationale can be seen from the following analysis. The eigenvalues, though non-linear in principle, are linear in terms of each interaction in a local space in which we set the one non-zero matrix element to -1 and looked for the state with the lowest eigenvalue. Therefore, instead of studying the effects of all two-body matrix elements simultaneously, we decompose the problem into $N$ parts. In each part, we focus on only one interaction matrix element. As a specific example, we shall consider a single-$j$ shell with $j = \frac{31}{2}$. We set $G_0 = -1$ and all other matrix elements to their TBRE values multiplied by a factor $\epsilon$, with $\epsilon$ running from 0 to 10. As expected, almost all of the ground states have $I = 0$ when $\epsilon$ is small. Interestingly, the 0g.s. probability remains large even when $\epsilon$ becomes large (say, 1.5). If we switch off all of the interaction matrix elements for which the lowest eigenvalue corresponds to an $I = 0$ state, then the 0g.s. probabilities become small. Similar results can be noticed in other
Previously, it was noticed that the 0g.s. dominance is not dependent on having monopole pairing for fermions in the sd-shell [1–3]. It was not known, however, which interactions are crucial in order to have 0g.s. dominance, and it was assumed by many authors that 0g.s. dominance is thus an intrinsic property of the model space. Using the method proposed in this Letter, we are able to readily tell which interactions (not only monopole pairing interactions) are crucial for 0g.s. dominance.

To summarize, we have presented in this Letter a simple approach to predict the Ig.s. probabilities for many-body systems. The agreement between the predicted Ig.s. probabilities and those obtained by diagonalizing the TBRE random hamiltonian is good. This method is applicable to both (even and odd) fermion systems (with both single-j and multi-j shells) and to boson systems. It predicts the 0g.s. probability and addresses the other Ig.s. probabilities as well. We believe, therefore, that we have provided in this Letter a universal approach for studying the Ig.s. probabilities.

Using this method, we have been able to address several important issues regarding the spectra of random hamiltonians. We have shown, for the first time, that the $I_{\text{max}}$g.s. probabilities of fermions in single-j shells are determined solely by the number of two-body matrix elements and are independent of particle number, and furthermore that they follow a simple $1/N$ relation. A generalization of such a regularity to fermion systems in many-j shells and to (sd and sdg) boson systems has been shown to work well too. In addition, we discovered which interactions (not only the monopole pairing interaction) are essential for producing 0g.s. dominance in boson and even-fermion systems. Our results suggest that the 0g.s. dominance (and other Ig.s. dominance in odd-A valence fermion systems) in boson systems and even-fermion systems is not a reflection of an intrinsic property of the model space, as previously assumed, but is related to the two-body matrix elements that give $I = 0$ ground states when they are set to -1 and all others are set to 0. The analogous remark applies to the dominance of a given Ig.s in odd-A fermion systems.

What is not yet understood at a more microscopic level is why $N_0$ is so large for even-fermion systems, nor why there is a staggering of the $N_0$ values for an even number of fermions in single-j shells. Further consideration of these issues is warranted.

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[15] The lowest eigenvalues should be equivalent to the largest eigenvalues in our discussion. But, these largest eigenvalues are usually (exactly or nearly) zero for many $I$ matrices, especially for many-$j$ shells and large single-$j$ shells. To have the “rule” as simple as possible, we use only the lowest eigenvalues with one of $G_J = -1$ and others switched off.

[16] The term $(s^\dagger d^\dagger)(sd)$ gives degenerate eigenvalues for many $I$ states. Therefore, we may also use 6 (instead of 7) as the number of independent two-body interactions. The difference due to this minor modification is very small, though.
Caption:

FIG. 1 The $I=0$ g.s. probabilities of 4 fermions in different single-$j$ shells. Solid squares are obtained from 1000 runs of the TBRE, and the open diamonds are from our simple prescription.

FIG. 2 (a) Calculated $I_{\text{max}}$g.s. probabilities for single-$j$ shells. The solid line plots the results from our simple prescription whereas the other results were obtained by diagonalizing the TBRE hamiltonian. (b). The $I'_{\text{max}} = I_{\text{max}}(j_1^n)$ and $I_{\text{max}}(j_2^n)$ g.s. probabilities for two-$j$ shells. The lower limit of the $I'_{\text{max}}$ g.s. probabilities is predicted by our simple formula $\frac{1}{N} \times 100\%$ (solid line). The other results were obtained by diagonalizing the TBRE hamiltonian.

FIG. 3 The predicted $I$g.s. probabilities of four very different systems. Solid squares are obtained by diagonalizing the TBRE hamiltonian and open squares are the predicted $I$g.s. probabilities of this Letter. a) $j = \frac{9}{2}$ with 4 fermions; b) $j = \frac{9}{2}$ with 5 fermions; c) a 7 fermions in a two-$j$ shell ($j_1 = 7/2, j_2 = 5/2$); d) an sd-boson system with 10 bosons.
TABLE I. The angular momenta which give the lowest eigenvalues when $G_J = -1$ and all other parameters are 0 for 4 fermions in single-$j$ shells.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{18}$ | $G_{20}$ | $G_{22}$ | $G_{24}$ | $G_{26}$ | $G_{28}$ | $G_{30}$ |
|------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 7    | 0     | 4     | 2     | 8     |       |         |         |         |         |         |         |         |         |         |         |         |
| 9    | 0     | 4     | 0     | 0     | 12    |         |         |         |         |         |         |         |         |         |         |         |
| 11   | 0     | 4     | 0     | 4     | 8     | 16      |         |         |         |         |         |         |         |         |         |         |
| 13   | 0     | 4     | 0     | 2     | 2     | 12      | 20      |         |         |         |         |         |         |         |         |         |
| 15   | 0     | 4     | 0     | 2     | 0     | 0       | 16      | 24      |         |         |         |         |         |         |         |         |
| 17   | 0     | 4     | 6     | 0     | 4     | 2       | 0       | 20      | 28      |         |         |         |         |         |         |         |
| 19   | 0     | 4     | 8     | 0     | 2     | 8       | 2       | 16      | 24      | 32      |         |         |         |         |         |         |
| 21   | 0     | 4     | 8     | 0     | 2     | 0       | 0       | 20      | 28      | 36      |         |         |         |         |         |         |
| 23   | 0     | 4     | 8     | 0     | 2     | 0       | 10      | 2       | 0       | 24      | 32      | 40      |         |         |         |         |
| 25   | 0     | 4     | 8     | 0     | 2     | 4       | 8       | 10      | 6       | 0       | 28      | 36      | 44      |         |         |         |
| 27   | 0     | 4     | 8     | 0     | 2     | 4       | 2       | 0       | 0       | 4       | 20      | 32      | 40      | 48      |         |         |
| 29   | 0     | 4     | 8     | 0     | 0     | 2       | 6       | 8       | 12      | 8       | 0       | 24      | 36      | 44      | 52      |         |
| 31   | 0     | 4     | 8     | 0     | 0     | 2       | 0       | 8       | 14      | 16      | 6       | 0       | 32      | 40      | 48      | 56      |
FIG. 1

Pred.

TBRE, Pred.

0g.s. Probabilities (in %)

j

2 4 6 8 10 12 14 16
**FIG. 2**

(a) $l_{\text{max}}$ g.s. probabilities (in %)

- pred., $n=3$
- $n=4$, $\triangledown$
- $n=5$, $\triangle$
- $n=6$, $\square$
- $n=7$

(b) $l_{\text{max}}/N$ g.s. probabilities (in %)

- $j=1, 11/2, 3/2$
- $j=1, 11/2, 5/2$
- $j=1, 11/2, 9/2$
- $j=1, 13/2, 9/2$
- $j=1, 9/2, 5/2$
- $j=1, 9/2, 7/2$
- $j=1, 7/2, 5/2$
- $j=1, 11/2, 7/2$
FIG. 3

\( I^G \) S probabilities (in %)

\( a) \) TBR E, \( j=9/2 \) shell with 4 fermions

\( b) \) TBR E, \( j=9/2 \) shell with 5 fermions

\( c) \) 7 fermions in the \( j_1=7/2, j_2=5/2 \) orbits

\( d) \) 10 sd bosons system

angular momenta \( l \)