THE GENERALIZED MOYAL-NAHM AND CONTINUOUS MOYAL TODA EQUATIONS

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ABSTRACT

We present in detail a class of solutions to the 4D $SU(\infty)$ Moyal Anti Self Dual Yang Mills equations that are related to reductions of the generalized Moyal Nahm equations using the Ivanova-Popov ansatz. The former yields solutions to the ASDYM/SDYM equations for arbitrary gauge groups. A further dimensional reduction yields solutions to the Moyal Anti Self Dual Gravitational equations. The Self Dual Yang Mills /Self Dual Gravity case requires a separate study. The $SU(2)$ and $SU(\infty)$ (continuous) Moyal Toda equations are derived and solutions to the latter equations in implicit form are proposed via the Lax-Brockett double commutator formalism. An explicit map taking the Moyal heavenly form (after a rotational Killing symmetry reduction) into the $SU(2)$ Moyal Toda field is found. Finally, the generalized Moyal Nahm equations are conjectured that contain the continuous $SU(\infty)$ Moyal Toda equation after a suitable reduction. Three different embeddings of the three different types of Moyal Toda equations into the Moyal Nahm equations are proposed.

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I. Introduction

The quantization program of the 3D continuous Toda theory (2D Toda molecule) is a challenging enterprise that we believe would enable us to understand many of the features of the quantum dynamics and spectra of the quantum self dual membrane [1]. This is based on the observation that the light-cone-gauge (spherical) supermembrane moving in a $D$ dimensional flat spacetime background has a correspondence with a $D-1$ $SU(\infty)$ Super Yang-Mills theory, dimensionally reduced to one temporal dimension; i.e. with a $SU(\infty)$ supersymmetric gauge quantum mechanical model (matrix model) [2,6].

It was shown in [1] that exact (particular) solutions of the $D=11$ light-cone (spherical) supermembrane, related to the $D=10$ $SU(\infty)$ SYM theory, could be constructed based on a particular class of reductions of the SYM equations from higher dimensions to four dimensions [3]. In particular, solutions of the $D=10$ YM equations given by the $D=10$ YM potentials, $A_\mu$, can be obtained in terms of the 4D YM potentials, $A_i$, that obey the $D=4$ Self Dual YM equations. Dimensional reductions of the latter $SU(\infty)$ SDYM equations to one temporal dimension are equivalent to the $SU(\infty)$ Nahm equations in the temporal gauge $A_0 = 0$.

Finally, the embedding of the continuous $SU(\infty)$ Toda equation into the $SU(\infty)$ Nahm equations was performed in [1] based on the connection between the $D=5$ self dual membrane and the $SU(2)$ Toda molecule/chain equations [4]. A continuous Toda theory in connection to self dual gravity was also found by Chapline and Yamagishi [5] in
the description of a three-dimensional version of anyon superconductivity. Based on the theory of gravitational instantons a 3D model describing the condensation of quasiparticles (chirons) with properties related to fractional statistics was found.

The classical Toda theory can be obtained also from a rotational Killing symmetry reduction [15] of the 4D Self Dual Gravitational (SDG) equations expressed in terms of first heavenly form that furnish (complexified) self dual metrics of the form: \( ds^2 = (\partial_x, \partial_{\tilde{x}}, \Omega) dx^i d\tilde{x}^j \) for \( x^i = y, z; \tilde{x}^j = \tilde{y}, \tilde{z} \) and \( \Omega \) is the first heavenly form. The latter equations can, in turn, be obtained from a dimensional reduction of the 4D \( SU(\infty) \) Self Dual Yang Mills equations (SDYM), an effective 6D theory [7,8] and references therein. The Lie algebra \( su(\infty) \) was shown to be isomorphic (in a basis dependent limit) to the Lie algebra of area preserving diffeomorphisms of a 2D surface, \( sdiff(\Sigma) \) by Hoppe [2]. It is for this reason that a WWM quantization of the reductions of the first heavenly equation will be used in this letter.

Using our results of [9] based on [10] we have shown that a Weyl Wigner Groenwold Moyal (WWGM) [11] quantization approach yields a straightforward quantization scheme for the 3D continuous Toda theory (2D Toda molecule). Supersymmetric extensions can be carried out following [7] where we wrote down the supersymmetric analog of the heavenly equations for SD Supergravity.

There are some differences between our results and those which in general have appeared in the literature. Among these are (i) One is not taking the limit of \( \hbar \to 0 \) while having \( N = \infty \) in the classical \( SU(N) \). Recently, Fairlie [23] has written solutions to Moyal-Nahm equations, with \( \hbar \neq 0 \) for the eight tranverse membrane coordinates in \( D = 11 \) in terms of spinors using the WWM formulation. (ii) We are working with the generalized Moyal-Nahm equations involving a Moyal bracket w.r.t an enlarged phase space, \( q, p, q', p' \) and not with the standard \( SU(2) \) Moyal-Nahm equations involving a Moyal bracket w.r.t \( q, p \) only. We have \( \hbar \neq 0; N = \infty \) simultaneously. (iii) The connection with the self dual membrane and \( W_\infty \) algebras was proposed in [1]. The results of [10] become very useful in the implementation of the WWM quantization program and in the embedding of the \( SU(2) \) Moyal-Nahm solutions into the generalized Moyal-Nahm equations studied in the present work.

In the next section we present in detail a class of solutions to the 4D \( SU(\infty) \) Moyal Anti Self Dual Yang Mills equations that are related to reductions of the generalized Moyal Nahm quations using the Ivanova-Popov ansatz. The former yields solutions to the ASDYM/SDYM equations for arbitrary gauge groups. A further dimensional reduction yields solutions to the Moyal Anti Self Dual Gravitational equations. The Self Dual Yang Mills /Self Dual Gravity case requires a separate study.

In \( \text{III} \) the \( SU(2) \) and \( SU(\infty) \) Moyal Toda (continuous) equations are derived and solutions to the latter equations in implicit form are proposed via the Lax-Brockett double commutator formalism of the continuous Toda equation [13]. By \( SU(\infty) \) Moyal one means the Moyal deformations of the symplectic diffeomorphisms in 4D instead of 2D. This is explained in detail in the text. The master Legendre transform between solutions to the rotational Killing symmetry reductions of the Moyal heavenly equations and the Moyal Toda equations is studied. In particular, the explicit map taking the Moyal heavenly form into the \( SU(2) \) Moyal Toda field is found. Finally, in \( \text{IV} \), the generalized Moyal
Nahm equations are provided that contain the continuous $SU(\infty)$ Moyal Toda equation after a suitable reduction. Three different embeddings of the different types of Moyal Toda equations into the Moyal Nahm equations are provided. At the end of IV we show explicitly how to connect the Moyal heavenly form with the continuous Toda field via the Moyal-Lax pair formalism. The remaining obstacle is to write down the explicit Moyal quantization of all the equations involved in the Ivanova-Popov construction [3]. This is necessary in order to write down the equation governing the deformations of the scalar field involved in the construction of [3], and which is mapped into the Moyal deformed continuous Toda field via deformations of the twistor transform. Deformations of twistor surfaces have yet to be constructed. For comments in that direction we refer to the work of Strachan and Takasaki [16,29] in their study of higher dimensional integrable models.

The most salient feature of the generalized Moyal-Nahm equation is that it involves an effective 8$D$ theory associated with symplectic diffeomorphisms of a 4$D$ manifold. This 8$D$ effective theory may have an important role in understanding the quantum dynamics of the 11$D$ supermembrane based on the the M(atrix) Membrane models and their integrability properties [6,23].

In the conclusion we discuss (among other things) briefly how deformation quantization techniques for higher dimensional volume forms, the Zariski quantization [22], may be used to quantize $p$ branes. Other types of deformations are possible, like those which give up the associative character of the Moyal product, the so-called $q$ deformations, and which can be used to construct deformations of the self dual membrane we refer to [24]. We expect that a $q$-Moyal deformation program of the self dual membrane might yield important information about how to quantize the full membrane theory beyond the self dual exactly integrable sector and that the particle/soliton spectrum of the underlying Conformal Affine Toda models will shed some light into the particle content of the more general theory [1,9].

Note:

Sometime after this work was completed we were informed [27] based on Strachan’s work [16] that a map from the master equation involving the Moyal deformations of the Self Dual Yang-Mills equations to Strachan’s $SU(2)$ Moyal Toda equation (which in reality is a Toda lattice model whose discrete spacing is given by multiples of $\hbar$) could be obtained by a suitable dimensional reduction. However, the latter is not a rotational Killing symmetry reduction; i.e the particular dimensional reduction of the effective 6$D$ master equation is not a Moyal deformation of the heavenly equation (although the heavenly equation can be obtained from the master equation). The results of [27] are based on a two step process where after performing the WWGM map and explicit introduction of $\hbar$ is put in by hand. More on this shall be explained in the text.

II. The $SU(\infty)$ MOYAL ASDYM/SDYM equations

We will study in this section the Moyal 4$D$ $SU(\infty)$ ASDYM/SDYM in connection to the ASDG/SDG equations and the continuous 2$D$ Toda molecule. To begin with, the Moyal bracket of two YM potentials $A_y, \tilde{A}_\tilde{y}$, for example, can be expanded in powers of $\hbar$ as [16]:

\[ \]
\[ \{A_y, A_{\bar{y}}\}_{q,p} \equiv \sum_{s=0}^{\infty} \frac{(-1)^{2s}}{(2s+1)!} \sum_{l=0}^{2s+1} (-1)^l \left( C^2_{s+l+1} \right) [\partial_q^{2s+1-l} \partial_p^l A_y][\partial_p^{2s+1-l} \partial_q^l A_{\bar{y}}]. \] (1)

where \( C^2_{s+l+1} \) are the binomial coefficients.

The crucial difference between the solutions of the \( SU(2) \) Moyal-Nahm eqs [10] and the generalized Moyal-Nahm case is that one must have an extra explicit dependence on another set of phase space variables, \( q', p' \). In particular, those reductions of the generalized Moyal Nahm equations that are linked to the Moyal Toda equations must have an extra \( t \) dependence for the YM potentials. The continuous Toda molecule equation as well as the usual Toda system may be written in the double commutator form of the Brockett equation [13] :

\[ \frac{\partial L(\tau, t)}{\partial \tau} = [L, [L, H]]. \] (2)

\( L \) has the form

\[ L \equiv A_+ + A_- = X_0(-iu) + X_{+1}(e^{\rho/2}) + X_{-1}(e^{\rho/2}). \] (3)

with the connections \( A_\pm \) taking values in the subspaces \( \mathcal{G}_o \oplus \mathcal{G}_{\pm 1} \) of some \( \mathbb{Z} \)-graded continuum Lie algebra \( \mathcal{G} = \oplus_m \mathcal{G}_m \) of a novel class. \( H = X_o(\kappa) \) is a continuous limit of the Cartan element of the principal \( sl(2) \) subalgebra of \( \mathcal{G} \). The functions \( \kappa(\tau, t), u(\tau, t), \rho(\tau, t) \) satisfied certain equations given in [13].

Upon the elimination of \( u \) one obtains the Toda equation. A naive look at (2) might beg the question: Where is the \( t \) dependence? The \( t \) dependence is encoded in the continuum algebra commutation relations of the generators that are parametrized by a family of functions depending on both \( \tau, t \). It is in this fashion that the \( t \) dependence makes its presence in (2).

To implement the Weyl Wigner Moyal (WWM) prescription, one may consider the case when \( G \) is a group of unitary operators acting in the Hilbert space of square integrable functions on the line, \( L^2(R^1) \). Then, \( \mathcal{G} \) is now the associated (continuum) Lie algebra of self-adjoint operators acting in the Hilbert space, \( L^2(R^1) \). The following operator-valued quantities depend on the two coordinates, \( \tau, t \) and obey the operator version of the Brockett equation and after the WWM map reads:

\[ \frac{\partial \hat{L}(\tau, t)}{\partial \tau} = \frac{1}{i\hbar} [\hat{L}, \frac{1}{i\hbar} [\hat{L}, \hat{H}]]. \leftrightarrow \frac{\partial \mathcal{L}}{\partial \tau} = \{\mathcal{L}, \{\mathcal{L}, \mathcal{H}\}\}. \] (4)

where \( \mathcal{L}(\tau, t, q, \rho; \hbar), \mathcal{H}(\tau, t, q, \rho; \hbar) \) are the corresponding elements in the phase space after performing the WWM map. The main problem with this approach is that we do not have representations of the continuum \( \mathbb{Z} \)-graded Lie algebras in the Hilbert space, \( L^2(R^1) \) and, consequently, we cannot evaluate the matrix elements \( <q - \frac{\xi}{2} | \hat{L}(r, \tau)|q + \frac{\xi}{2}>; <\hat{H}> \). For this reason we have to recur to another methods to solve this problem.

The quantities \( \mathcal{L}, \mathcal{H} \) are defined as:
\[ \mathcal{L}(\tau, t, q, p; h) \equiv \int_{-\infty}^{\infty} d\xi <q - \frac{\xi}{2} \hat{L}(\tau, t)|q + \frac{\xi}{2} > \exp\left[ i\frac{\xi p}{h} \right]. \]  

(5)

\[ \mathcal{H}(\tau, t, q, p; h) \equiv \int_{-\infty}^{\infty} d\xi <q - \frac{\xi}{2} \hat{H}(\tau, t)|q + \frac{\xi}{2} > \exp\left[ i\frac{\xi p}{h} \right]. \]  

(6)

The latter matrix elements, if known, suffice to determine the quantities \( \mathcal{L} \) and \( \mathcal{H} \) associated with the 2D continuous Toda molecule equation.

Despite not knowing the explicit operator form of \( \hat{L}(\tau, t) \) and \( \hat{H}(\tau, t) \) acting in the Hilbert space, \( L^2(R^1) \), one may still write down solutions for the continuous Moyal-Toda equation. This can be achieved starting from the original Moyal SDYM/ASDYM equations associated with the \( SU(\infty) \) group in \( D = 4 \) and looking for solutions. Ivanova and Popov [3], in a summary of YM equations in \( D \geq 4 \), have discussed that solutions to the ASDYM/SDYM equations in \( D = 4 \) for an arbitrary Lie group, \( G \), which are linked to the Nahm equations may be obtained from the ansatz:

\[ A_\mu = -\eta^\alpha_{\mu\nu} T_\alpha(\phi(x^\mu)) \partial_\nu \phi. \quad \eta^\alpha_{\mu\nu} = \epsilon^\alpha_{\beta\gamma}, \quad \eta^\alpha_{\mu0} = -\delta^\alpha_\mu. \]  

(7)

The 'T Hooft matrices obey the quaternionic algebra:

\[ \eta^\alpha_{\mu\lambda} \eta^\beta_{\lambda\nu} = -\delta^\beta_\mu \eta^\gamma_{\mu\nu} - \epsilon^\alpha_{\beta\gamma} \eta^\gamma_{\mu\nu}. \]  

The function \( \phi \) obeys: \( \partial_\mu \partial^\mu \phi = 0 \) (ASDYM) and \( \phi = x_\mu x^\mu \) (SDYM) and the Lie algebra valued functions \( T_\alpha(\phi) = T^A_\alpha(\phi)L_A \), for \( \alpha = 1, 2, 3 \) satisfy the Nahm equations w.r.t the \( \phi \) function:

\[ \epsilon^\alpha_{\beta\gamma} \frac{dT_\gamma}{d\phi} = \pm \{ T_\alpha, T_\beta \}_{\text{Moyal}}. \]  

(8)

where the \( \pm \) corresponds to the SDYM/ASDYM case. Notice that the simple reflection: \( T_\alpha \rightarrow -T_\alpha \) converts the SDYM to the ASDYM solutions with the proviso that now \( \phi \) obeys the 4D Laplace equation. It is very important to emphasize that Ivanova and Popov are using an Euclidean spacetime signature. This will become important later on when we discuss other results obtained in a \((++, --)\) signature.

The Ivanova-Popov ansatz, for Euclidean signatures, will yield solutions to the Moyal deformations of the Anti-Self Dual Gravitational equations in 4D from dimensional reductions of the \( SU(\infty) \) ASDYM equations. However, this will not be the case for the self-dual gravitational equations that can be obtained from reductions of the \( SU(\infty) \) SDYM equations. The Ivanova-Popov ansatz will not yield solutions to the Moyal deformations of the SDG equations. Another type of solutions will be required. For signatures, \( 2+2 \) the situation is reversed. More on this issue will be clarified in the next sections.

A WWM quantization requires writing down the symbol map of the operators acting in the Hilbert space, \( L^2(R^1) \), associated with the three Lie algebra valued functions, \( \hat{T}_\alpha \) so the Moyal Nahm equations are:

\[ \epsilon^\alpha_{\beta\gamma} \frac{dT_\gamma}{d\phi} = \pm \{ T_\alpha, T_\beta \}_{\text{Moyal}}. \]  

(9)

with \( \mathcal{T}_\alpha(\phi(x^\mu; q, p, h); q, p, h) = \text{symbol} [\hat{T}_\alpha] \) where \( \hat{T}_\alpha \) is a representation of the Lie algebra valued operators in \( L^2(R^1) \).
Rigorously speaking one should write $T_{\alpha}[\mathcal{G}]$ to include the explicit dependence on the Lie algebra $\mathcal{G}$ involved initially in the construction. In the case that $G = SU(\infty)$ one is required then to extend the symplectic diffs in 2D to 4D. The Moyal bracket involves now a generalized phase space $q_i, p_i$ for $i = 1, 2$. For $G = SU(2)$ the fact that the dual of the $SU(2)$ Lie algebra is $\mathbb{R}^3$ allows to view the three scalars $T_\alpha$ as the three components $X, Y, Z$ of a four vector, after fixing the gauge $A_0 = 0$, replacing $\phi$ by $\tau$ and making the correspondence $A_x \leftrightarrow X, \ldots$.

Furthermore, if Moyal deformations of the Toda equations are related to the Moyal Nahm equations via the Lax pair formalism, the scalar $\phi(x^\mu)$ will require in general a deformation of the type: $\phi(x^\mu; q_i, p_i, \hbar)$ obeying a deformation of the Laplace equation. The latter equation is in general modified to include $\hbar$ corrections and derivatives acting on the phase space coordinates as well. This can be explicitly seen when one performs a direct Moyal quantization program of all the equations involved in the Ivanova-Popov construction. Operators are mapped into functions of phase space via the symbol map. Products of operators are mapped into the Moyal star product of their corresponding symbols and this involves derivatives of arbitrary order w.r.t $q, p$. Furthermore, a suitable ordering prescription must be specified a priori. Therefore, a Moyal quantization of the Ivanov-Popov construction induces a $q, p, \hbar$ dependence on the scalar $\phi$ and it deforms the original Laplace equation. We shall go back to this issue when we discuss the twistor transform mapping the nonlinear 3D Toda equation into the 3D Laplace equation for the scalar $\phi$.

A particular class of solutions of the $SU(2)$ Moyal-Nahm equations in terms of the Jacobi elliptic functions w.r.t the undeformed $\phi$ has been given by [10]:

\[
T_1 = sn[\phi] \left( \frac{i}{2} p(q^2 - 1) - \hbar(\beta + \frac{1}{2})q \right), \quad T_2 = dn[\phi] \left( -\frac{1}{2} p(q^2 + 1) - i\hbar(\beta + \frac{1}{2})q \right).
\]

\[
T_3 = cn[\phi] \left( -ipq - \hbar(\beta + \frac{1}{2}) \right), \quad \beta = \text{constant}
\]

where the (undeformed) scalar function $\phi$ has a correspondence with one, and only one, temporal parameter, $\phi \rightarrow \tau$ (which is clearly a solution of Laplace equation in the ASDYM case but does not correspond to the $\phi = x^\mu x_\mu$ required in the SDYM case). When $\phi \rightarrow \tau$ the ansatz of (7) gives that $A_0 = 0, A_i \sim T_i$ and as expected the $SU(2)$ Moyal Nahm equations involve the three components of a four vector and the derivatives are taken w.r.t the temporal variable $\tau$ that does not transform as a scalar like $\phi$. It is important also to remark that if $\phi$ acquires a deformation $\phi(x^\mu; q, p, \hbar)$ the solution given in (10) will no longer hold!

As stated earlier, by $SU(2)$ Moyal Nahm it is meant that the solutions (10) in [10] were obtained by performing the WWM map which takes $su(2)$ Lie algebra-valued operators belonging to the Hilbert space, $L^2(\mathbb{R}^1)$, into functions of $q, p$. Beforehand, the Wolf representation of the $su(2)$ Lie-algebra valued generators, in matrix form using the Pauli spin matrices, $A^a_i \tau_a$, needs to be used prior to evaluating the WWM map. i.e; it was essential to use a representation which takes the three Pauli spin $SU(2)$ matrices into
three known operators in $\hat{q}, \hat{p}$. It is in this sense that one may speak of the solutions (10) to the $SU(2)$ Moyal Nahm equations.

Since representations of $SU(\infty)$ in terms of operators in the Hilbert space $L^2(R)$ are not known (as far as we know) one cannot evaluate explicitly the WWM map. In addition, $SU(\infty)$ requires to use the extended phase space which implies that the Moyal bracket to be used in (9) will be the one w.r.t the $q_i, p_i$ phase space coordinates rather than to $q, p$. We have symplectic diffs in four dimensions instead of two-dimensions. Therefore, the generalized Moyal Nahm equations require an extra set of variables: $q', p'$ that must be introduced to account for the area-preserving diffs algebra associated with a 4D manifold instead of a two-dim surface (sphere, torus) so now $T_\alpha(\phi; q, p, \hbar; q', p')$ The Moyal brackets are then computed w.r.t the enlarged phase space involving the $q, p$ and $q', p'$. We shall discuss this in detail in the last section.

Nevertheless, the generalized "$SU(\infty)$" Moyal-Nahm equations admit reductions to the continuous Moyal Toda equations and equations directly linked to the 4D $SU(\infty)$ Moyal ASDYM/SDYM equations. We refrain from using the term $SU(\infty)$ Moyal Nahm because it is not really a WWM quantization of the classical $SU(\infty)$ Nahm equations but, instead, one has Moyal deformations of the algebra of symplectic diffs in 4D. Such algebra is an infinite dimensional extension of the area-preserving diffs of a two-dim surface \cite{20}.

The reduction one is referring can be attained simply by imposing the dimensional reduction conditions: $q = q', p = p'$ which will reduce the phase space in half and the Moyal bracket will involve one pair of phase space variables only and not two copies of the same pair. In this fashion one has now a set of functions depending on six variables instead of eight:

$$T_\alpha(\phi(x^0, x^1, x^2, x^3); q, p, \hbar). \quad (11a)$$

From now on, by reduction of the generalized Moyal Nahm equations we mean that reduction implied by eq-(11a). A further dimensional reduction/restriction on the $x^\mu$ co-ordinates is needed in order to match the $t, \tau$ variables required in the original classical continuous Toda equation, $\phi$ must be after a dimensional reduction $\phi = \phi(\tau, t)$. We are omitting for convenience the $q, p, \hbar$ dependence on the scalar $\phi$.

Firstly, this may obtained by a dimensional reduction of the six variables to four in (11a); i.e from 6D to 4D, which is how in the first place the Moyal SDG equations are obtained from a dimensional reduction of 4D $SU(\infty)$ SDYM \cite{7,8}. A further rotational Killing symmetry reduction is needed from the Moyal heavenly equation to the 3D continuous Toda theory. First of all, in the ASDYM case, the undeformed $\phi$ obeys the 4D Laplace equation $\partial_\mu \partial^\mu \phi = 0$. Upon complexification one may choose the variables $x^\mu$ to be related to four out of the six variables:

$y, z, \bar{y}, \bar{z}; q, p$, and the dimensional reduction is chosen for the (undeformed) scalar $\phi$ as follows:

$$\phi(x^\mu(y, z, \bar{y}, \bar{z})) \equiv \phi(w, \bar{w}). \quad w = z + \bar{y}; \quad \bar{w} = \bar{z} - y. \quad (11b)$$

so the functions are of the form $T_\alpha(\phi(w, \bar{w}); q, p, \hbar)$.

The dimensional reduction conditions in \cite{8,9} were of the type:
\[ w = z + \tilde{y}. \quad \partial_w = \partial_z = \partial_{\tilde{y}}. \quad \tilde{w} = \tilde{z} - y. \quad \partial_{\tilde{w}} = \partial_{\tilde{z}} = -\partial_y. \] (12a)

with
\[ \tilde{y} = x^2 - ix^3. \quad \tilde{z} = x^0 + ix^1. \quad y = x^2 + ix^3. \quad z = x^0 - ix^1. \] (12b)

The ASDYM equations in 4D Euclidean spacetime read:
\[ F_{y\tilde{y}} + F_{z\tilde{z}} = 0 \quad \text{and} \quad F_{yz} = 0, \]
\[ F_{\tilde{y}\tilde{z}} = 0. \] (13a)

Under \( T \rightarrow -T \), then \( A \rightarrow -A \) and the ASDYM case is retrieved:
\[ A_{\tilde{y}} = -\partial_w \Omega - \frac{1}{2} \tilde{z}. \quad A_{\tilde{z}} = -\partial_{\tilde{w}} \Omega + \frac{1}{2} \tilde{y}. \] SDYM. \quad (++) signature. \quad \text{ASDYM}. \quad (++,--) signature. \quad (13b)

so that the Moyal heavenly equations are equivalent to a zero curvature condition:
\[ \{ \Omega, w, \Omega, \tilde{w} \} = \pm 1 \leftrightarrow \{ \partial_{\tilde{z}} + A_{\tilde{z}}, \partial_{\tilde{y}} + A_{\tilde{y}} \} = 0. \] (14)

The minus sign , \(-1\) is assigned to the ASDYM case related to Anti Self Dual Gravity in 4D with 2 + 2 signature. Whereas, the +1 sign is the one corresponding to the SDYM case with 2 + 2 signature related to Self Dual Gravity. In the Euclidean signature case, the signs appearing in front of the 1 coefficient are reversed. The important thing is that the (undeformed) scalar obeying Laplace equation is the solution that must be used for Self Dual Gravity in 2 + 2. This corresponds precisely to the ASDYM equations in (4, 0) given by Ivanova and Popov.

Therefore, one learns from eqs-(7, 13) that there is a direct relation between the three functions \( T_\alpha \) appearing in (7) and the \( \Omega \) obeying (14) using the reductions (12). Before this can be achieved it is essential to perform a gauge transformation from the YM potentials in (13) to the new ones:
\[ A_\mu \rightarrow A_\mu + \partial_\mu \Lambda + \{ A_\mu, \Lambda \}. \quad \delta F_{\mu\nu} = \{ F_{\mu\nu}, \Lambda \}. \]

Choosing a \( \Lambda \) depending on \( y, z, \tilde{y}, \tilde{z} \) only and not on \( q, p \) gives:
\[ \hat{A}_\tilde{y} = A_{\tilde{y}} + \partial_{\tilde{y}} \Lambda(y, \tilde{y}, z, \tilde{z}). \quad \hat{A}_{\tilde{z}} = A_{\tilde{z}} + \partial_{\tilde{z}} \Lambda(y, \tilde{y}, z, \tilde{z}). \quad F'_{\mu\nu} = F_{\mu\nu} = 0 \] (15a)

The choice \( \Lambda = \frac{1}{2}(y\tilde{y} + z\tilde{z}) + f(y, z) \) is the adequate one that yields \( \hat{A}_\tilde{y}, \hat{A}_{\tilde{z}} \) as functions solely of \( w, \tilde{w}, q, p \). It is now that one can equate in the ASDYM case:
\[
\partial_z \hat{A}_y = \partial_w \hat{A}_y = \partial_w \partial_w \Omega + \frac{1}{2},
\]
\[
\partial_y \hat{A}_z = \partial_w \hat{A}_z = \partial_w \partial_w \Omega - \frac{1}{2}.
\]

(15b)

where the YM potentials \( \hat{A}_y = \hat{A}_2 - i \hat{A}_3 \), and \( \hat{A}_z = \hat{A}_0 + i \hat{A}_1 \) obtained from the ansatz (7), are :
\[
\hat{A}_y = \eta^{\alpha}_2 \mu T_{\alpha} \partial_\mu \phi - i \eta^{\alpha}_3 \mu T_{\alpha} \partial_\mu \phi.
\]

(16a)

\[
\hat{A}_z = \eta^{\alpha}_0 \mu T_{\alpha} \partial_\mu \phi + i \eta^{\alpha}_1 \mu T_{\alpha} \partial_\mu \phi.
\]

(16b)

Concluding, plugging the values of the YM potentials (16) directly into (15) yields the solutions of the Moyal ASDG equations (14) related to the reductions of the generalized Moyal Nahm equations (when the phase space is reduced in half, \( q = q', p = p' \)) that are encoded in the ansatz (7) which solves the ASDYM equations in 4D for the \( SU(\infty) \). The Moyal heavenly form is then, up to integration "constants", \( f(q, p) \):
\[
\Omega(w, \bar{w}, q, p, \bar{h}) = \int (\hat{A}_y - \frac{1}{2} \bar{w}) dw + \int (\hat{A}_z + \frac{1}{2} w) d\bar{w}.
\]

(16c)

where the YM potentials are explicitly given by (16a,16b). Eq-(16c) is the main result of this section. It will be shown later on that eqs-(16), via the Lax pair formalism described in eq-(19), contain the map taking the Moyal Heavenly form into the \( SU(2) \) Moyal Toda field obeying the Moyal Toda equations described in section III.

It remains to determine the (undeformed) function \( \phi \), whereas the \( T_\alpha \) may be solved as follows, for example, by using the solutions of the \( SU(2) \) Moyal-Nahm equations given by (10) that require the (undeformed) scalar \( \phi \) without explicit \( q, p, \bar{h} \) dependence.
\[
- \mathcal{T}_1 = sn[\phi(\tau, t)][(\frac{i}{2} p q^2 - 1) - \hbar (\beta + \frac{1}{2}) q], \quad - \mathcal{T}_2 = dn[\phi(\tau, t)][(\frac{1}{2} p q^2 + 1) - i\hbar (\beta + \frac{1}{2}) q].
\]
\[
- \mathcal{T}_3 = cn[\phi(\tau, t)][-i p q - \hbar (\beta + \frac{1}{2})].
\]

(17)

The minus signs are due to the fact that the ASDYM equations are the ones related to the solutions of the Laplace equation and where it is essential that this embedding restricts the space of solutions \( \phi(x^\mu) \) to depend solely on \( \tau, t \). One must use the undeformed \( \phi \).

An explicit example will be given shortly. In this fashion, the direct relation between the Moyal deformed ASDYM potentials appearing in (16,17) and the Moyal heavenly form obeying (14) may be obtained from eqs-(15).

These reductions are also compatible with the fact that the Toda equations for \( SU(N) \) are obtained from particular reductions of Nahm equations which, in turn, can be represented in a Lax pair form:
\[ L = T_1(\phi) + iT_2(\phi). \quad iT_3(\phi) = M. \quad \frac{dL}{d\phi} = [L, M]. \] (18)

Consequently, in the \( N = \infty \) limit, one can recast the continuous Moyal-Toda equations in the double commutator form after establishing the following correspondence (which are not identifications), see (4),:

\[ \mathcal{L}(u, \rho) \leftrightarrow T_1 + iT_2. \quad iT_3 = \mathcal{M} \leftrightarrow \{\mathcal{L}, \mathcal{H}\}. \quad \frac{\partial \mathcal{L}}{\partial \tau} = \{\mathcal{L}, \{\mathcal{L}, \mathcal{H}\}\} \] (19)

where the \( T_\alpha \) obey (9). Setting aside at the moment the differences between the SDYM and ASDYM cases (the undeformed \( \phi \) differs in each case) one can see that eqs-(16,19) contain the relationship between the Moyal heavenly form, \( \Omega \) and the Toda field \( \rho \) via the YM potentials which, in turn, are expressed in terms of the Moyal-Nahm functions \( T_\alpha \) by using eqs-(16a,16b).

Therefore, barring the differences between the SDYM and ASDYM cases, eqs-(16,19) express the indirect mapping between the Moyal heavenly form and the Toda field via the Lax pair formalism. The main problem is to find an explicit realization of the \( \mathcal{L}, \mathcal{M} \) functions which amounts, again, to finding a representation of the continuum \( \mathbb{Z} \) graded Lie algebras in the Hilbert space \( L^2(R) \).

If one makes a strict identification in eqs-(19), instead of a correspondence, from eqs-(9,18,19) one learns that (the undeformed) \( \phi \) must be restricted further. It then obeys the additional condition:

\[ \frac{\partial \dot{\phi}}{\partial \tau} = \frac{\partial \phi}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tau} = \frac{\partial \dot{\phi}}{\partial w} \frac{\partial w}{\partial \tau} + \frac{\partial \dot{\phi}}{\partial \tilde{w}} \frac{\partial \tilde{w}}{\partial \tau} = 1. \] (20)

with \( w, \tilde{w} \) functions of \( \tau, t \). Eq-(20) amounts to constraining one out of the four coordinates \( x^\mu \).

We will show at the end of this section that this additional restriction on (the undeformed) \( \phi \) would correspond to a very special class of solutions to the \( \rho \) where the \( \tau, t \) dependence of the \( \rho \) is required to be of the form: \( \rho(\tau, t) = \rho(\tau \pm it) \). Therefore, identifying the l.h.s of (19) with the r.h.s furnishes a restricted class of solutions to the Moyal deformations of the Lax-Brockett equations (4) given in terms of special solutions to the Moyal-Nahm equations. The restriction is due to the fact that \( \phi \) is constrained to be \( \tau \pm it \). This restriction can be avoided by establishing the correspondence given by eq-(19).

The reason that one can make the correspondences (which are not identifications) given by (19) is because there are continuum Lie algebras that are isomorphic to Poisson bracket algebras \( \sim su(\infty) \), which correspond to the Lie algebras of area-preserving diffs of the sphere, torus [13]. It is in this sense that the correspondence of eq-(19) is implemented. There two ways to retrieve Moyal Toda equations: one way is to use the Lax-Brockett double commutator form and another to use the Lax representations for the \( SU(\infty) \) Nahm equations. The correspondence between these two constructions of the Moyal equations originates from the fact that there is a Legendre-like transform that maps solutions of the \( 2 + 1 \) continuous Toda equation to those of the three-dim Laplace equation, i.e. \( \rho \rightarrow \phi \).
The 2 + 1 continuous Toda equation occurs in the theory of self-dual Einstein spaces and has a well known Eguchi-Hanson solution \[26\]. Prasad \[25\] and discussed by Chapline in \[5\]) has shown that by a change of variables one can transform the Toda equation into a three dimensional Laplace equation for a certain function \(\phi = \phi(\rho)\) related to the two-center gravitational instanton of Gibbons and Hawking \[26\]. So the correspondence dictated by eq-(19) is a reflection of the Legendre-like map which takes the Toda field \(\rho\) to the \(\phi\) obeying the Laplace equation after a dimensional reduction from 2 + 1 to 2 dimensions. The Legendre-like transform determines the correspondence given in (19), once the suitable maps from the remaining \(u, \kappa\) functions to \(\phi\) are found.

As stated earlier, a subtlety will now arise. Due to the Moyal deformations of the Toda equations it is expected then that an accompanying deformation of the Laplace equation for the scalar field \(\phi\) follows. The Prasad map taking \(\phi \to \rho\) must be deformed as well. Therefore one should be forced to include \(\hbar\) corrections to the scalar \(\phi(x; \hbar)\). Similar considerations have been found by \[27\] after employing the WWGM map: an explicit introduction of \(\hbar\) was made afterwards. Nevertheless, the correspondence given by eq-(19) still holds once the deformed map from \(\phi\) (obeying deformations of Laplace equation) to \(\rho\) (obeying deformations of the Toda equation) is found. Similar arguments apply to remaining functions: \(u, \kappa\). These also acquire an \(\hbar\) explicit dependence.

In general, there must be an explicit dependence on the \(q,p\) phase space coordinates for the continuous Moyal Toda field \(\rho(\tau,t;q,p,\hbar)\). This is implies that the scalar \(\phi = \phi(x^\mu; q,p,\hbar)\) as mentioned earlier. However, in the case of the \(SU(2)\) Moyal Toda field, which is really a Toda lattice theory whose discrete spacing is a multiple of \(\hbar\), one has instead that \(\psi(\tau,t;\hbar)\) in agreement with \[16,27\]. The latter case implies then that the scalar \(\phi = \phi(x^\mu; \hbar)\) only. This will become more transparent in the course of the text.

There is also another reduction from the generalized Moyal Nahm equations to Moyal Toda equations which shall be studied in IV that differs from the reduction obtained from eq-(11a). Such reduction involves using the enlarged phase space variables \(q,p,q',p'\); (using the full symplectic diffs in four dimensions). It is obtained by setting (the undeformed) \(\phi \to \tau, t = q'\) and by integrating out the \(p'\) variable after finding the three functions \(T_\alpha\) which solve the generalized Moyal-Nahm equations. Whereas the reduction obtained from (11a) involves incorporating the \(\tau,t\) variables through the (undeformed) scalar function \(\phi(\tau,t;\hbar)\) and establishing the correspondence with Lax-Brockett formalism. Both types of reductions lead to Moyal Toda equations.

The very special case we are studying in this section requires that one performs a series of coordinate redefinitions and dimensional reductions from 6D to 4D:

\[
\{x^0, x^1, x^2, x^3; q,p\} \to \{y, z, \tilde{y}, \tilde{z}; q,p\} \to \{w, \tilde{w}; q,p\} \to \{\tau, t; q,p\}. \tag{21a}
\]

The Laplace equation fixes the family of (undeformed) functions \(\phi\). Due to the dimensional reduction, the 4D Laplace operator \textit{degenerates} to zero, this can be verified by simple inspection:

\[
w = x^0 - ix^1 + x^2 - ix^3, \quad \tilde{w} = x^0 + ix^1 - x^2 - ix^3, \quad w^* \neq \tilde{w}. \tag{21b}
\]

\[
\partial_0 = \partial_w + \partial_{\tilde{w}}, \quad \partial_1 = -i\partial_w + i\partial_{\tilde{w}}, \quad \partial_2 = \partial_w - \partial_{\tilde{w}}, \quad \partial_3 = -i\partial_w - i\partial_{\tilde{w}}. \tag{21b}
\]
One can verify that the Laplace operator acting on (the undeformed) $\phi$:

$$\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 = \partial_0 \partial_\bar{\mu} + \partial_0 \partial_\mu = \partial_0 \partial_\bar{\nu} - \partial_0 \partial_\nu \equiv 0 \quad (21c)$$

hence, as a result of the dimensional reduction, $\phi(w, \bar{w})$, the 4D Laplace operator acting on (the undeformed) $\phi$ degenerates to zero. i.e. Any function of the form $\phi = \phi(w, \bar{w})$ obeys automatically the 4D Laplace equation.

If one wishes one may restrict the solutions of the 4D Laplace equation, $\partial_\mu \partial^\mu \phi(w, \bar{w}) = 0$, for arbitrary functions $\phi$ to those obeying the 2D Laplace equation, instead,:

$$\partial_\tau^2 \phi + \partial_t^2 \phi = \partial_\tau \partial_\mu \phi = 0 \Rightarrow \phi = f(w) + g(\bar{w}). \quad w \equiv \tau + it. \quad \bar{w} = \tau - it. \quad (21d)$$

In this fashion one can remove the arbitrariness of $\phi$.

It is important to emphasize that $\bar{w} \neq w^*$ and that $\tau, t$ are complex valued since the Moyal heavenly equations are defined in complexified 4D spacetime. A real slice may be taken by choosing $\bar{w} = w^*$ which implies that $\tau, t$ must be real. The general solution to the 2D (complexified in general) (undeformed) Laplace equation for the (undeformed) $\phi$ is:

$$\phi = f(\tau + it) + g(\tau - it) \text{ for } f, g \text{ arbitrary.}$$

Hence, one may choose without loss of generality the following solutions to the reductions of the generalized Moyal-Nahm equations related to the $SU(\infty)$ ASDYM equations in 4D:

$$-T_1 = sn[f + g][(\frac{i}{2}p(q^2 - 1) - \hbar(\beta + \frac{1}{2})q]. \quad -T_2 = dn[f + g][(-\frac{1}{2}p(q^2 + 1) - i\hbar(\beta + \frac{1}{2})q].$$

$$-T_3 = cn[f + g][(-ipq - \hbar(\beta + \frac{1}{2})]. \quad \phi = f(\tau + it) + g(\tau - it). \quad (22a)$$

Inserting the above solutions into eqs-(16a-16b) give automatically through eqs-(15b) an explicit solution, eq-(16c), to the Moyal deformations of the ASDG equations in 4D given in eq-(14). The above solutions to the reductions of the generalized Moyal-Nahm equations are parametrized by an arbitrary family of functions of right/left movers, $f(\tau + it), g(\tau - it)$.

At this point one is ready to examine the meaning of the correspondence of eq-(19). If an strict identification is made, eq- (20) will restrict these family of right/left movers $f, g$ to satisfy the following additional constraint in addition to the two-dim Laplace equation:

$$\frac{\partial \phi}{\partial \tau} = 1 \text{ and } \partial_\tau^2 \phi + \partial_t^2 \phi = 0 \Rightarrow \phi = \tau + at + b = f + g.$$

There are three simple cases to consider, take $b = 0$:

$$(i) f + g = \tau \Rightarrow a = b = 0. \quad (ii) f + g = \tau + it \Rightarrow g = 0. \quad (iii) f + g = \tau - it \Rightarrow f = 0. \quad (22b)$$

case (i) belongs to $f(\tau + it) \equiv \frac{1}{2}(\tau + it), g(\tau - it) \equiv \frac{1}{2}(\tau - it)$ so $f + g = \tau$. This case bears a direct connection to the $SU(2)$ Moyal Nahm equation found by Strachan [16], eq-(29) in
the next section. Cases (ii), (iii), are typical of the above mentioned restricted solutions of the type \( \rho(\tau, t) = \rho(\tau \pm it) \) that bear a connection to the Moyal Toda equation of the type given by eq-(26) in the next section. These are very special cases. In general, the scalar field \( \phi = f + g \) for arbitrary \( f, g \) so one should focus on the most general solution for \( \phi \) instead of a very restricted special case.

The main point of these results is that there must be an additional dependence on \( t \) (besides the original \( \tau \) dependence) in the formulation of the reductions of the generalized Moyal-Nahm equations. Having found a particular choice for (the undeformed) \( \phi = f(\tau + it) + g(\tau - it) \), which solely depends on \( \tau, t \), a solution to eqs-(9) like eqs-(17,22a) will finally yield the sought after solution to the 4D \( SU(\infty) \) ASDYM equations via the ansatz (7), as it has been explicitly provided above by eqs-(14,15,16,17). Once a solution has been found other solutions may be attained by the action of the infinite dimensional group of symmetries of the anti/self dual \( SU(\infty) \) Yang-Mills and heavenly equations [17]. The symmetry group is an infinite dimensional affine Kac-Moody algebra associated with the algebra of area-preserving diffs.

The SDYM equations require a separate study since in this case (the undeformed) \( \phi = y\tilde{y} + z\tilde{z} \) is fixed and does not obey the Laplace equation. A different kind of reduction other than the Ivanova-Popov ansatz, via the Moyal Nahm equations, is necessary to obtain the Moyal SDG equations from the 4D \( SU(\infty) \) SDYM equations. See [7,10]. By setting:

\[
\phi = x^\mu x_\mu = y\tilde{y} + z\tilde{z} = u + v. \quad u \equiv y\tilde{y}, \quad v \equiv z\tilde{z}.
\] (22c)

one has another set of variables besides the \( w, \tilde{w} \) used in the ASDYM case. But whereas the latter could furnish solutions to the Moyal deformations of 4D ASDG, the Ivanova-Popov ansatz in the former SDYM case cannot be used to obtain solutions to Moyal deformations of 4D SDG although it does yield solutions to the 4D \( SU(\infty) \) SDYM equations. Notice that one cannot equate the \( SO(4) \) invariant quantity \( x^\mu x_\mu \) with a non-invariant one like \( \tau \) (a coordinate). Whereas the quantity \( \phi \) in the ASDYM case transforms as a scalar, however, this does not imply that it is \( SO(4) \) invariant: \( \phi(x) = \phi'(x') \neq \phi(x') \).

In eq-(14) the transformation \( \Omega \to i\Omega \) changes the signs of the r.h.s and naively reverses the role of SDYM and ASDYM. However eq-(16c) is not invariant under \( \Omega \to i\Omega \): one cannot absorb the \( i \) factors in the \( d\omega, d\omega \) measure pieces despite being able to absorb them in each single term inside the integrands. Therefore the naive analytical continuation will not convert a SDYM potential into an ASDYM one. Furthermore, the scalar \( \phi \) is different in each situation. A mere \( i \) multiplication will not suffice. This fact will become important when we study the master Legendre transform from rotationally Killing symmetry reductions of the Moyal deformations of the 4D Self Dual Gravitational equations into the Moyal Toda equations.

To finalize this section we point out, once again, that the Moyal Toda equations obtained from the Moyal Nahm equations associated with the quantities \( T_\alpha[\phi = f + g; q, p, \hbar] \) are those related to the \( SU(2) \) Moyal Toda equations derived by Srachan [16] (given by eq-(29)) and to the continuous Moyal Toda equation given by eq-(26); these equations will be studied next.

Concluding, to a first order approximation, neglecting the deformations to the \( \phi \) scalar, the most general solution to the Moyal ASDG equations related to the reductions of the
generalized Moyal Nahm equations \((q = q', p = p')\) may be parametrized by a family of undeformed functions obeying the 2\(D\) Laplace equation for the \textit{undeformed} function \(\phi(w, \bar{w})\) and whose solutions depend on two \textit{arbitrary} functions, \(f(\tau + it), g(\tau - it)\) as prescribed in (22a).

When deformations are included, we can still use eqs-(16,18,19) as the most important equations of this section establishing the indirect correspondence between the Moyal heavenly \(\Omega\) form and the Moyal Toda field \(\rho\) using the Lax pair formalism. In this case \(\phi\) obeys deformations of the Laplace equation obtained by an explicit Moyal quantization of the Ivanova-Popov construction. Hence, it contains an additional \(q,p\) dependence as it should for the continuous \(SU(\infty)\) Moyal Toda field.

In the case of the \(SU(2)\) Moyal Toda field (a spatially-discrete but temporal continuous lattice theory) there is no \(q,p\) dependence as it has been shown in [16] and discussed by [27] in the constructions of a master integrable equation that contains the KP, KdV hierarchies. The explicit relation between \(\Omega\) and the \(SU(2)\) Moyal Toda field will be studied further in 3.3. We proceed now to study the Moyal Toda equations in the next section.

**III. The Moyal Toda Equations**

**3.1. A Continuous Moyal Toda Equation**

In this section we shall display the different forms of the Moyal Toda equations that are related to the the quantities \(\mathcal{L}(u, \rho), \mathcal{H}(\kappa)\) in eqs-(5,6,18,19) and the \(T_\alpha[\phi,q,p,\hbar]\). We return now to this discussion. After the WWM map is performed, \(u, \rho, \kappa\) acquire and additional dependence on \(q,p,\hbar\). To illustrate this, let us look at the operator form of the original continuous Toda equation:

\[ \frac{\partial^2 \hat{\rho}}{\partial \tau^2} = \frac{\partial^2 e^{\hat{\rho}}}{\partial t^2}. \] (23)

Given an operator, \(\hat{\rho}(\tau,t)\), acting in the Hilbert space of square integrable functions on the line, of the form:

\[ \hat{\rho} = \sum_{mn} \rho_{mn}(\tau,t)(\hat{q}^m \hat{p}^n + \hat{p}^m \hat{q}^n + \ldots). \] (24a)

with a Weyl ordering prescription imposed on the monomials in \(\hat{q}^m \hat{p}^n\):

\[ \hat{q}\hat{p} \to \hat{q}\hat{p} + \hat{p}\hat{q}, \quad \hat{q}\hat{p}^2 \to \hat{q}\hat{p}^2 + \hat{p}^2 \hat{q} + 2\hat{p}\hat{q}\hat{p} \ldots. \] (24b)

More complicated operators are also possible that are not necessary sums of monomials. The WWM map converting operators, \(\hat{\rho}(\tau,t)\) into functions in phase space (making use of the symbol map) yields:

\[ \text{symbol }[\hat{\rho}] = \rho(t,\tau,q,p,\hbar), \quad \text{symbol }[e^{\hat{\rho}}] = e^{*\rho} = 1 + \rho + \frac{\rho \ast \rho}{2!} + \frac{\rho \ast \rho \ast \rho}{3!} + \ldots. \] (25)

Hence the putative Moyal continuous Toda molecule equations reads:
\[
\frac{\partial^2 \rho}{\partial \tau^2} = \frac{\partial^2 e^* \rho}{\partial t^2}. \tag{26}
\]

The Moyal star product of two functions of phase space of dimension \(2n\) whose symplectic form has the inverse \(\omega^{IJ}\) is defined:

\[
f \star g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(i \frac{\hbar}{2}\right)^n \omega^{i_1 j_1} \omega^{i_2 j_2} \cdots \omega^{i_n j_n} (\partial_{i_1} \partial_{i_2} \cdots \partial_{i_n} f)(\partial_{j_1} \partial_{j_2} \cdots \partial_{j_n} g). \tag{27}
\]

When \(\omega^{ij}\) is the inverse of the symplectic form in two-dimensions, the derivatives are taken w.r.t the \(q, p\) variables only. To recover the \(2 + 1\) continuous Toda requires replacing the l.h.s of (26) by \((\partial_\tau)^2\) by \(\partial_{z_+} \partial_{z_-}\) and setting \(\rho(z_+, z_-, t, q, p, \hbar).\) Eq-(26) is one of the Moyal Toda equations equation to be studied in this section III.

The above equation was obtained from the operator form of the original continuous Toda equation. In general, at the quantum level, the form of the operator equations of motion are not the same as those of the original classical field. A modification of (26) will be presented shortly where the r.h.s is modified ; i.e. there will be derivatives of infinite order w.r.t the \(\tau, t\) variable that originate from deformations of the continuum graded Lie algebras as a result of replacing ordinary Poisson bracket by Moyal brackets.

The \(\hbar = 0\) limit of (26) yields in the r.h.s the ordinary exponential, \(e^\rho\) because in the classical limit the Moyal star product becomes the ordinary pointwise product of functions. Since in the classical limit, eq-(26) involves a differential equation w.r.t the variables \(\tau, t\), only, the classical limit of (26) does not determine the \(q, p\) dependence of \(\rho(\tau, t, q, p, \hbar = 0)\) which may be completely arbitrary. Assuming that \(\rho\) admits an expansion in powers of \(\hbar\):

\[
\rho = \sum \hbar^n \rho_n(\tau, t, q, p) \quad \text{one can impose the condition that the zeroth-order term does not depend on } q, p : \rho_0(\tau, t, q, p) \equiv \rho_{\text{class}}(\tau, t).\]  

This condition can also be derived from the master Legendre transform (studied in 3.3) that maps the rotational Killing symmetry reductions of the Moyal SDG equations into the continuous Moyal Toda equations [9,21].

Another way to obtain the continuous Moyal Toda equations directly should be to perform the master Legendre transform mapping, if indeed it exists, between \(\Omega(y', \tilde{y}', z', \tilde{z}', \hbar)\) obeying the rotationally Killing symmetry reductions of the Moyal SDG equations [9,21] to \(\rho(\tau, t, q, p, \hbar)\) [1,9] obeying the continuous Moyal Toda equations. Strachan [16] has shown that the \(SU(2)\) Moyal-Nahm equations admit a reduction to the classical continuous Toda chain in the \(\hbar = 0\) limit. Therefore, reductions of the \(SU(2)\) generalized Moyal Nahm equations should yield the continuous Moyal Toda equation. This shall be studied in IV.

We shall continue shortly with the Strachan ansatz and write down a more general equation than (26) which contains derivatives of infinite order w.r.t the \(t\) variable; i.e. the operator equations of motion for the quantized Toda field differ from the classical counterpart. The master Legendre transform will be discussed also. The study of the geometry associated with these Moyal deformations has been given by [14].

3.2 Strachan’s Reduction of the \(SU(2)\) Moyal Nahm Equations to the \(SU(2)\) Moyal Toda Equations

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It is known that the ordinary continuous Toda equation may be obtained from axial-symmetry reductions of the $SU(\infty)$ classical Nahm equations. This fact permitted Strachan to construct Moyal deformations of the Toda equation by replacing the Poisson bracket by the Moyal bracket. [16]. If one writes the Moyal Nahm equations of the type given (with the plus sign) by eq-(8), with $\phi = f + g = \tau$ and imposing the axial-symmetry reductions of the form:

$$T_1 = X_1 = h(\tau, t = q, h) \cos(p). \quad T_2 = X_2 = h(\tau, t = q, h) \sin(p). \quad T_3 = X_3 = z(\tau, t, h). \quad (28)$$

allows the decoupling of the $\cos(p), \sin(p)$ terms after computing the Moyal bracket and, after eliminating the function $z(\tau, t)$. Strachan [16] arrived at:

$$\frac{\partial^2 \psi}{\partial \tau^2} = \frac{1}{4} \left[ \frac{\Delta - \Delta^{-1}}{\hbar} \right]^2 e^\psi = \frac{\partial^2}{\partial t^2} e^\psi + \frac{\hbar^2}{3} \frac{\partial^4}{\partial t^4} e^\psi + O(h^{2n})..... \quad (29)$$

the shift operators, $\Delta, \Delta^{-1}$ and $\psi(\tau, t, h)$ are defined:

$$\Delta \psi = \psi(t + h). \quad \Delta^{-1} \psi = \psi(t - h). \quad e^{\psi/2} \equiv h(\tau, t). \quad (30)$$

Equation (29) is the $SU(2)$ Moyal Toda equation. It involves one field only, $\psi$, in the same way that the Liouville equation is tantamount of a $sl(2)$ Toda field equation. In the classical limit, $\hbar = 0$ one recovers the classical continuous $SU(\infty)$ Toda molecule equation as expected. This can be seen by expanding: $\psi(\tau, t, h) = \psi_0 + \hbar^2 \psi_2 + .....$ and plugging this value into (29). The $\hbar = 0$ limit reproduces again the classical continuous Toda equation for the field $\psi_0(\tau, t)$ as eq-(26) did for $\rho(\tau, t, q, p, \hbar = 0) = \rho_{class}(\tau, t)$.

It is an interesting question (although not the right question to ask) if one could find a representation of the $su(2)$ Lie algebra in terms of $\hat{q}, \hat{p}$ operators that would yield Strachan’s solutions after performing the WWM map of the operator equations associated with the $su(2)$ Nahm’s equations. The solutions (28) clearly differ from those presented in eq-(10); not only in the $\tau$ functional dependence implicit in the elliptic functions but also in the dependence of the phase space variables.

The reason that one should not view Strachan’s construction as a direct WWGM quantization of the $SU(2)$ Toda field, a Liouville theory, is because eqs-(29,30) represent really a $SU(2)/SL(2)$ Toda lattice theory with discrete-spacial spacings in multiples of $h$: the field $\psi$ is evaluated at discrete jumps $t, t \pm h, \ldots$. However, there is continuous temporal dynamics represented by the $\partial_\tau$ derivatives. See the important work of Dimakis et al [28] on this respect. The r.h.s of (29) is a spatially-discrete-difference operator which can be expanded into an infinite number of derivatives, a nonlocal expression. The Moyal product is also nonlocal due to the infinite number of derivatives. Therefore, loosely speaking one may refer to Strachan’s equation as the $SU(2)$ Moyal Toda equation. It has been speculated by Reuter [14] that this maybe the source of nonlocality in Quantum Mechanics.

From the form of (29) one can notice that a WWGM quantization program alters the operator form of the quantum field equations of motion from the original classical field equations. The operator equations read instead:
\[
\frac{\partial^2 \hat{\rho}}{\partial \tau^2} = \frac{1}{4} \left( \frac{\Delta - \Delta^{-1}}{\hbar} \right)^2 e^{\hat{\rho}} = \frac{\partial^2}{\partial t^2} e^{\hat{\rho}} + O(h^{2n}).....
\]
the r.h.s involves derivatives of *infinite* order of the operator \( \hat{\rho}(\tau, t) \) w.r.t the \( t \) variable . This implies that a WWGM quantization program related to an operator of the type given by eq-(24) in terms of monomials of \( \hat{q}, \hat{p} \), is directly linked with Moyal deformations of continuum Lie algebras. Upon quantization the structure of the Lie algebra is itself modified as well. This occurs in the study of quantum groups and quantum Lie algebras where quantum integrability requires deformations of the classical Lie algebraic structures.

In view of this, the correct Moyal continuous Toda equation ( it is a further deformation of the Strachan’s Toda lattice equations by introducing the star exponential ) must be:

\[
\frac{\partial^2 \rho}{\partial \tau^2} = \frac{1}{4} \left( \frac{\Delta - \Delta^{-1}}{\hbar} \right)^2 e^{\rho} = \frac{\partial^2}{\partial t^2} e^{\rho} + O(h^{2n}).....
\]

and is obtained after performing the WWM map of eq-(31). The Moyal star product is taken w.r.t the \( q, p \) variables only. The extra dependence on the two phase space variables, \( q, p \) is due to the WWM symbol map taking operators into functions in phase space. The \( t \) parameter is the one that encodes the continuum Lie algebra generators and commutation relations. Upon quantization, the latter algebra is deformed and hence the operator appearing in the r.h.s of the original Toda equation, \( \partial^2_t \), acquires a deformation in powers of \( h \) and an infinite number of derivative terms is induced. The deformed continuum algebra is the algebra associated with deformations of the Poisson bracket algebra of the sphere, \( \sim SU(\infty) \).

We have learnt that the \( SU(2) \) Moyal Toda equation is *contained* in the \( SU(2) \) Moyal Nahm equations and, similarly, the \( SU(\infty) \) Moyal Toda , eq-(32), should be *contained* in the *generalized* Moyal Nahm equations . It is precisely for this reason that the generalized Moyal Nahm equations must depend on an *extra* set of phase space variables as argued earlier prior to eq-(11a). This will be study in IV where in particular we shall provide a plausible embedding of (32) into the generalized Moyal Nahm equations.

It becomes important now to study the subtleties due to the choices of different signatures taken by the authors in [3,8]. As we stated earlier, the Ivanova Popov ASDYM equations in Euclidean spacetime required the use of the ( undeformed) scalar field \( \phi \) obeying Laplace equation. However, the ASDYM equations in Euclidean space *correspond* to the SDYM equations in spaces of signature \( 2 + 2 \) studied by [8]. In both cases one has : \( F_{\tilde{y}y} + F_{\tilde{z}z} = 0 \). Reductions of the latter equations are the ones which yield the \( 2 + 2 \) Self Dual Gravitational equations, and from these, the Toda equations are obtained after a further rotational Killing symmetry reduction. Therefore, in view of this signature subtlety, Strachan solutions (28,29), as expected, do fit into functions of the type : \( \mathcal{T}_\alpha[\phi(w(\tau, t), \tilde{\omega}(\tau, t)); q, p, \hbar] \). This can be seen by imposing the reduction condition \( t = q \), taking a particular solution for \( \phi = f + g \) where \( f(\tau + it) = \frac{1}{2}(\tau + it) \), \( g(\tau - it) = \frac{1}{2}(\tau - it) \) whose sum yields \( \phi = f + g = \tau \). And, finally , the \( \mathcal{T}_\alpha \) functions have the form described in eq-(28). Concluding, after these steps are taken : Strachan’s ansatz is a very special case of the most general Ivanova-Popov ansatz for the three functions, \( \mathcal{T}_\alpha \).
Solutions to (29) may be obtained through iterations after expanding in even powers of $\hbar$: $\psi_0 + \hbar^2 \psi_2 + \ldots$. In this way an infinite, but known, number of differential equations yields iteratively the solutions for $\psi_{2n}$. To solve this system is another matter. The most general solution to the ordinary classical continuous Toda equations has been found by Saveliev [13]. It is well known by now that the ordinary Liouville equation can be embedded into the continuous Toda equations as follows. Given:

$$\frac{\partial^2 \psi_0}{\partial \tau^2} = \frac{\partial^2 e^{\psi_0}}{\partial t^2}. \quad (33)$$

One may plug in the ansatz which will automatically reproduce the Liouville equation:

$$e^{\psi_0} = \left( \frac{1}{2} t^2 + bt + c \right) e^{\phi_L} \Rightarrow \partial_{z_+} \partial_{z_-} \phi_L(z_+, z_-) = e^{\phi_L}. \quad (34)$$

after performing the dimensional reduction $\tau = z_+ + z_-$. So, in the $\hbar = 0$ limit eqs-(29) reduces to the ordinary classical continuous Toda molecule (chain) equations. Again, the $2+1$ Toda equations are retrieved by inserting $z_+, z_-$ for $\tau$ and taking $(\partial_{\tau})^2$ for $\partial_{z_+} \partial_{z_-}$. This ansatz (34) is the one which reproduces the Eguchi-Hanson gravitational instanton [26].

### 3.3 The Master Legendre Transform

Before we begin this section we must emphasize that the results of [16,27] are very different from the results of this section. Mainly because the results of [27] do not involve a rotational Killing symmetry reduction of the Moyal heavenly equations and also because these involve a particular class of dimensional reductions of the Moyal SDYM master equations that do not lead to the Moyal heavenly equations. The authors [27] based their work on the results by Strachan [16] on the Toda/KP hierarchies and obtained the SU(2) Moyal Toda equations from a different route than the one described here. The equations in [16,17] are the SU(2) Toda lattice equations whose discrete spacing is a multiple of $\hbar$. When $\hbar = 0$ one recovers the continuous Toda equations, or the so-called Boyer-Finley equations obtained originally from Killing symmetry reductions of the heavenly equations.

As promised earlier, we can conclude that eqs-(16,19,28) already contain the required map from the $\Omega$ heavenly form obeying the Moyal heavenly equation, eq-(14), to the SU(2) Moyal Toda field, $\psi$, obeying Strachan equation (29). This can be achieved via the Lax pair formalism given by eq-(19) after using Starchan ansatz for the three Moyal-Nahm functions given by eq-(28). This automatically solves the problem of finding the Legendre transform from $\Omega$ to the SU(2) Moyal Toda field $\psi$.

Hence, eqs-(16) yield $\Omega = \Omega[A_i]$. The Strachan solution corresponds to the particular case of $\phi = \tau$ which impies that $A_i = X_i$ and, finally, eqs-(28,29) give the explicit relations $X_i = X_i(\psi(\tau, t, \hbar), z(\tau, t, \hbar))$ with $e^{\psi/2} \equiv h(\tau, t, \hbar)$. To study the map from $\Omega$ to the continuous Toda field $\rho(\tau, t, q, p, \hbar)$ is another matter. We proceed to study this case below and show that it can be solved in principle by embedding the SU(2) solutions into the continuous SU($\infty$) ones.

It has been known for sometime that rotational Killing symmetry reductions of the Self Dual Gravitational or heavenly equations lead to the continuous Toda equations [15]. The
master Legendre transform, if it exists, between rotational Killing symmetry reductions of $\Omega(y', \tilde{y}', z', \tilde{z}', h)$ obeying the 4D Moyal SDG equations and $\rho(z_+, z_-, t, q, p, \hbar)$ obeying the 2 + 1 continuous Moyal Toda equation, eq-(32), should be defined as the map [9,21]:

$$\Omega(y', \tilde{y}', z', \tilde{z}', h) \equiv \sum_{n=0}^{\infty} \left[ \frac{\hbar}{y'(q, p)} \right]^n \Omega_n(r \equiv y'y'; z', \tilde{z}') \rightarrow \rho(z_+, z_-, t, q, p, \hbar) =$$

$$n=\infty \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hbar^n \rho_n^{lm}(z_+, z_-, t)Y_{lm}(\theta, \varphi). \quad q = \cot(\theta/2)\cos \varphi, \quad p = \cot(\theta/2)\sin \varphi. \quad (35)$$

the limits in $l$ are defined with the proviso that in the limit $\hbar = 0$ the zeroth order terms will survive only giving

$$\rho_0^{00}(\tau, t)Y_{00}(\theta, \varphi) \equiv \rho_{class}(\tau, t). \quad \lim_{\hbar \rightarrow 0} e^{*\rho} = e^{\rho_{class}(\tau, t)}. \quad \lim_{\hbar \rightarrow 0} \rho = \rho_{class}(\tau, t). \quad (36)$$

and, as expected, the zeroth-order terms do not depend on $q, p$. We have performed also the stereographic projection mapping the sphere into the complex plane (phase space of $q, p$). This implies that the $n^{th}$ order Legendre map, if it exists, must establish the correspondence:

$$\left[ \frac{1}{y'(q, p)} \right]^n \Omega_n(r \equiv y'y'; z', \tilde{z}') \rightarrow \sum_{l=0}^{n} \sum_{m=-l}^{l} \rho_n^{lm}(z_+, z_-, t)Y_{lm}(\theta, \varphi). \quad (37)$$

an additional dimensional reduction in (33) $z_+, z_-$ to $\tau$ is of course needed to go from the 2 + 1 continuous Toda to the 2D Toda molecule equation.

It is essential not to confuse the prime variables $y', z'$... with the variables $y, z$.... A detailed discussion of the maps between the primed and unprimed variables was given in [9] based on [21]:

$$\{ \Omega, z', \Omega, y' \} \tilde{z}' y' = 1 \leftrightarrow \{ \Omega, w, \Omega, \tilde{w} \} w, q, p = 1. \quad (38a)$$

The inverse symbol map takes functions of phase space into operators in $L^2(R)$ and Moyal brackets into commutators leading to the operator equations:

$$\frac{1}{i\hbar} [\hat{\Omega}, z', \hat{\Omega}, y'] = \hat{1} \leftrightarrow \frac{1}{i\hbar} [\hat{\Omega}, w, \hat{\Omega}, \tilde{w}] = \hat{1}. \quad (38b)$$

where

$$\Omega(w, \tilde{w}, q, p, \hbar) = \sum (\hbar)^n \Omega_n(w, \tilde{w}, q, p). \quad (38c)$$

Extreme care must be taken not to set $\Omega_n$ as a function of $r = w\tilde{w}$ and $q, p$. If this is wrongly assumed then the r.h.s. of (38) will be zero instead of 1. This corrects an errata in [9]. An example of the coordinate transformation (a field dependent coordinate
transformation up to zeroth order) between the primed and unprimed variable associated with the particular solution $\Omega = \Omega_0 = z' \bar{z}' + y' \bar{y}'$ is [9]:

$$\bar{z}' = q, \quad \bar{y}' = p, \quad z' = w + \frac{\lambda}{q}, \quad y' = \bar{w} - \frac{\lambda}{p}.$$  \hspace{1cm} (39)

with $\lambda$ a complex constant. As $n$ runs the coordinate transformation varies and one speaks of a field dependent transformation to the $n$-th order. This is what eqs-(38a,38b) represent.

At first sight matters seem to indicate that one can transform the infinite number of differential equations (38) for the $\Omega_n(r \equiv y' \bar{y}', z', \bar{z}')$ functions given in [9,21] into an infinite number of differential equations for the $\rho_{lm}^{(n)}(z_+, z_-, t)$ functions after inserting $\rho$ into eq-(32) (the angular dependence has been factored out). However, there appears to be a discrepancy: the differential equations for the $\Omega_n$ involve higher derivatives w.r.t the $r, \bar{z}'$ variables as the integer $n$ runs from 0 to $\infty$. Whereas the differential equations for the $\rho_{lm}^{(n)}$ involve higher derivatives w.r.t the $t$ variable only with solely quadratic derivatives in $\partial_{z_+} \partial_{z_-}$ appearing in the l.h.s.

Identical objections arise if one wishes to perform a transformation from the $\Omega_n$ to the $\psi_n(z_+, z_-, t)$ appearing in Strachan’s $SU(2)$ Moyal Toda equation (29) for $\psi$ after expanding the latter in powers of $\hbar$. This may suggest that the l.h.s of eqs-(29,32) should have higher derivatives in $z_+, z_-$ or $\tau$ variables if a master Legendre transform exists from the infinite number of quantities $\Omega_n$ to the $\rho_{lm}^{(n)}$ or $\psi_n$.

Nevertheless this does not represent a problem for the following reason. At any given level $n = N$, in powers of $\hbar$, the number of differential equations involve solely $\Omega_0, \Omega_1, ..., \Omega_N$. The number of derivatives w.r.t the $r, \bar{z}'$ is higher for the $\Omega_0$ function than for the $\Omega_N$. The number of derivatives w.r.t the $r, \bar{z}'$ at level $n = N$ is the lowest for the $\Omega_N$ function because $\Omega_N$ appears already with a power of $\hbar^N$. Hence, the number of derivatives of $\Omega_N$ w.r.t the $r, \bar{z}'$ variables precisely matches the number of $\tau$ or $z_+, z_-$ derivatives acting on $\rho_{lm}^{(n)}$ [9,21] at that given order $n = N$. Starting from the zeroth order term, by iteration, one can span the whole range of $\Omega_n$ and $\rho_{lm}^{(n)}$ with the right number of derivatives. The same argument applies to the $\psi_n$.

To find the master Legendre transform which takes the infinite number of differential equations [9,21] for the $\Omega_n$ into an infinite number of differential equations for the $\rho_{lm}^{(n)}$ is a very difficult problem. All we can do is outline some of the most important features and propose to embed the $SU(2)$ solutions into the $SU(\infty)$ ones as a special simpler case.

Another discrepancy which at first sight might be troublesome is that the number of variables of $\Omega$ and $\rho$ does not match. The former has four variables whereas the latter has five. It is only after the $q, p$ dependence is factorized as shown in (35) that one can match the variables in $\Omega_n(r, z', \bar{z}')$ with those in $\rho_{lm}^{(n)}(t, z_+, z_-)$. However, the $\psi_n(z_+, z_-, t)$ belonging to the $SU(2)$ Moyal Toda equation (29) admit a perfect match with the $\Omega_n(r, z', \bar{z}')$.

For this reason, when one speaks of the master Legendre transform from the $\Omega$, obeying the rotational Killing symmetry reductions of the Moyal heavenly equation, (38), to the Moyal Toda equation one must refer to Strachan’s $SU(2)$ Moyal Toda equation (29) as discussed at the beginning of this subsection.

Despite these problems, the expansion in spherical harmonics given by eq-(35) is mathematically sound. Since derivatives of spherical harmonics w.r.t. the angle variables
can be reexpressed in terms of sums of spherical harmonics, plugging the value of $\rho$ given by the r.h.s of (35) into eq-(32) yields an infinite number of differential equations for the functions $\rho_{lm}^{n}(\tau,t)$ that in principle could be solved by iterations. A knowledge of the solutions $\rho(\tau,t,q,p,\hbar)$ to eq-(32) will allow to compute the operator $\hat{\rho}(\tau,t)$, obeying the operator equations of motion (31), by recurrring to the inverse symbol map which converts real functions in phase space into self adjoint operators acting in the Hilbert space of square integrable functions on the line. This is precisely what is meant by a WWGM quantization of the continuous Toda field. To solve (32) using (35) is clearly a very complicated matter.

An interesting question, which simplifies matters a little bit, is whether or not one can find a particular subset of solutions to the $\rho_{lm}^{n}$ which solve the continuous Moyal Toda equation in such a way that the following factorization condition occurs:

$$\rho_{lm}^{n}(z_{+},z_{-},t) \sim \psi_{n}(z_{+},z_{-},t) \int d\varphi d(\cos \theta) Y_{lm}(\varphi,\theta) f_{n}(q,p). \quad (41)$$

where $f_{n}(q,p)$ are functions which in principle can be determined from (32) after inserting eq-(40) into eq-(32). This is clearly a very difficult task.

To conclude this subsection, to establish a master Legendre transform from eq-(38) to eq-(32) runs into the discrepancy in the number of derivatives w.r.t the $z_{+}, z_{-}$ or $\tau$ variable appearing in the l.h.s of eq-(32) compared to the arbitrarily high number of derivatives w.r.t the $\tilde{z}'$ variable appearing in eq-(38) [9,21]. After counting derivatives at any given order $n = N$, acting on the quantity $\Omega_{N}$, that is determined iteratively in terms of the previous $\Omega_{0}, ..., \Omega_{N-1}$ ones, reveals that this discrepancy does not seem to represent a major problem. One could forsee the possibility of modifying the number of derivatives appearing in the l.h.s of eq-(32) that includes derivatives of arbitrary order in $\tau$ or $z_{+}, z_{-}$. At the moment this seems unnecessary. The explicit master Legendre transform can be constructed in the very special case when one embeds the $SU(2)$ Moyal Toda into the continuous Moyal Toda. Solving for $\rho_{lm}^{n}$ in (40) yields:

$$\rho_{lm}^{n}(z_{+},z_{-},t) = \psi_{n}(z_{+},z_{-},t) \int d\varphi d(\cos \theta) Y_{lm}(\varphi,\theta) f_{n}(q,p). \quad (41)$$

where $f_{n}(q,p)$ are functions which in principle can be determined from (32) after inserting eq-(40) into eq-(32). This is clearly a very difficult task.

To conclude this subsection, to establish a master Legendre transform from eq-(38) to eq-(32) runs into the discrepancy in the number of derivatives w.r.t the $z_{+}, z_{-}$ or $\tau$ variable appearing in the l.h.s of eq-(32) compared to the arbitrarily high number of derivatives w.r.t the $\tilde{z}'$ variable appearing in eq-(38) [9,21]. After counting derivatives at any given order $n = N$, acting on the quantity $\Omega_{N}$, that is determined iteratively in terms of the previous $\Omega_{0}, ..., \Omega_{N-1}$ ones, reveals that this discrepancy does not seem to represent a major problem. One could forsee the possibility of modifying the number of derivatives appearing in the l.h.s of eq-(32) that includes derivatives of arbitrary order in $\tau$ or $z_{+}, z_{-}$. At the moment this seems unnecessary. The explicit master Legendre transform can be constructed in the very special case when one embeds the $SU(2)$ solutions ( of eq-(29) ) into the $SU(\infty)$ case ( eq-(32)).

Concluding, by sorting out the signature subtelties used by different authors [3,8], the transformation from $\Omega$, after a rotational Killing symmetry reduction, to the $SU(2)$ Moyal Toda field $\psi$ is attained in three step-process by using eqs-(16,19,28). In this fashion one establishes the sought-after $\Omega = \Omega[X_{i}(\psi(\tau,t,\hbar),z(\tau,t,\hbar))]$ relation due to the fact that in this case the YM potentials $A_{i} = X_{i}$. This is probably one of the most relevant result of this work.

**IV. The Generalized Moyal Nahm Equations**

Moyal Deformations of Loop Algebras, Symplectic Diffs in 4D And Continuum Lie Algebras.
We have been studying reductions of the generalized Moyal Nahm equations related to the Moyal deformations of the $SU(\infty)$ ASDYM/SDYM equations in 4D, an effective 6D theory. The most general Moyal Nahm equations require, at least, an extra set of $q', p'$ coordinates and hence one has an effective 8D theory where the Moyal bracket is taken w.r.t the enlarged phase space. i.e; the Weyl Wigner Moyal formalism involves mapping operator valued quantities living in a 4D spacetime ( belonging to a Hilbert space of $L^2(R^2)$, instead of $L^2(R^1)$) into functions of the enlarged phase space, $q, p, q', p'$. Instead of dealing with Moyal deformed symplectic diffs of a two-dim surface one is now dealing with Moyal deformed symplectic diffs in 4D. Therefore the effective theory is now 8D instead of 6D!

We define the generalized Moyal Nahm equations as:

$$\epsilon_{\alpha\beta\gamma} \frac{dT_{\gamma}}{d\phi} = \{\{T_{\alpha}, T_{\beta}\}\}. \quad T_{\alpha}[\phi(x^\mu; q, p, q', p', h); q, p, q', p'; h].$$

where now the Moyal bracket must be taken w.r.t an enlarged phase space $q, p, q', p'$. There is an explicit and implicit $q, p, t, h$ dependence in the Moyal-Nahm functions. The Moyal bracket of $f, g$ w.r.t the enlarged phase space is compactly written as:

$$\{\{f, g\}\} \equiv \frac{1}{\hbar} f(sin[h(\overrightarrow{\partial}_q \overrightarrow{\partial}_p - q \leftrightarrow p + \overrightarrow{\partial}_q \overrightarrow{\partial}_{p'} - q' \leftrightarrow p')])g.$$  

(43)

expanding the sine function in powers of $\hbar$ one retrieves the infinite derivative terms. The WWM map takes self-adjoint operator-valued quantities, living in the Hilbert space $L^2(R^2)$, $\hat{A}(x^\mu)$, into real valued functions in phase space $A(x^\mu; q, p, q', p'; h)$

$$A(x^\mu; q, p, q', p'; h) \equiv \int_{-\infty}^{\infty} d^2\xi \cdot \frac{\xi}{2} |\hat{A}(x^\mu)| q + \frac{\xi}{2} > exp[i\frac{\bar{\xi}\vec{p}}{\hbar}]. \quad q = (q, q'), \quad \bar{p} = (p, p').$$

(44a)

$$|\bar{q} + \frac{\xi}{2} >= |q_1 + \frac{\xi_1}{2} > \otimes |q_2 + \frac{\xi_2}{2} > .......$$

(44b)

Imagine representing now the $SU(2)$ Lie-algebra valued YM potentials (matrices) in terms of operators in $\hat{q}, \hat{p}, \hat{q}, \hat{p}'$ and performing afterwards the WWM map (44a).

The relevant algebra is now the Moyal deformations of symplectic diffs in 4D instead of 2D. For this reason it is incorrect to say that one has "$SU(\infty)$" Moyal Nahm equations. For example, the generators are labelled as $V_{m}^{l, \vec{k}}$ where $\vec{k} = (k_1, k_2)$ $[20]$ and obey the infinite dimensional generalization of the $w_\infty$ algebra (area-preserving diffs of the plane):

$$[V_{m}^{l, \vec{k}}, V_{n}^{j, \vec{l}}] = [(j + 1)m - (l + 1)n] V_{m + n}^{l + j, \vec{k} + \vec{l}} + \vec{k} \times \vec{l} V_{m + n}^{l + j + 1, \vec{k} + \vec{l}}.$$  

(45)

This algebra of symplectic diffs in 4D has a realization in terms of ordinary Poisson brackets w.r.t the $q, p, q', p'$ enlarged phase space variables. The Moyal deformations are obtained by replacing ordinary Poisson brackets by Moyal ones. If one had a representation of the $SU(2)$ algebra as linear operators in $L^2(R^2)$, instead of the known representations in
\[ L^2(R^1) \] [10], one could then evaluate the WWM map (44a) and obtain solutions to the generalized Moyal Nahm equations as it was done in [10].

Another type of generalized Moyal Nahm equations one could write is such where instead of having a Moyal bracket w.r.t the enlarged phase space one has a partial Moyal bracket w.r.t one set of \( q, p \) variables:

\[ \epsilon_{\alpha\beta\gamma} \frac{\partial T_\gamma}{\partial \phi} = \{ T_\alpha, T_\beta \}_{q,p}. \quad T_\alpha[\phi(x^\mu; q, p, q', p', h); q, p, q', p'; h]. \tag{46} \]

The problem with eq-(39) as such is that it does not determine the functional dependence on the \( q', p' \) variables since there is no differential operators which involve now the \( q', p' \) variables. For this reason we should disregard (46) as a valid equation.

We shall study the reductions of (42) with the goal of obtaining the continuous Moyal Toda equation (32). Firstly, one chooses \( \phi = \tau \) then the ansatz (7) gives \( A_0 = 0, A_i \sim T_i \). Secondly, imposing the reduction condition \( t = q' \) while recurring to an ansatz that allows one to decouple the \( \cos p', \sin p' \) terms, after computing the Moyal bracket in (42), giving, finally, an equation involving only the \( \tau, t, q, p \) variables. If we set:

\[ \begin{align*}
T_1 &= R(\tau, t = q', q, p, h)\cos p'. \\
T_2 &= R(\tau, t = q', q, p, h)\sin p'. \\
T_3 &= z(\tau, t = q', h). \tag{47}
\end{align*} \]

After plugging (47) into (42), the terms in \( \cos p', \sin p' \) decouple and eliminating \( z \) one obtains the following highly nontrivial equation for the function \( R \), after computing the Moyal bracket w.r.t the enlarged phase space variables:

\[ \frac{\partial^2 \ln R}{\partial \tau^2} = \alpha \left( \frac{\Delta - \Delta^{-1}}{h} \right)^2 R^2 + \beta \left( \frac{\Delta - \Delta^{-1}}{h} \right) \sum_{n=1}^{\infty} (h)^n C_{i_1 \ldots i_n} (\partial_{i_1} \ldots \partial_{i_{n-1}} R)(\partial_{j_1} \ldots \partial_{j_n} R). \tag{48} \]

The second terms of the r.h.s of (48) contain mixed derivatives of infinite order w.r.t the \( t, q, p \) variables. \( \alpha, \beta \) and \( C_{i_1 \ldots i_n} \) are constants. Strachan’ \( SU(2) \) Moyal Nahm equation is recovered automatically by dropping the extra \( q, p \) dependence on \( R \) (so that the second term in the r.h.s of (48) becomes zero) and by equating \( R^2 = e^\rho \). The l.h.s is then equal to the l.h.s of (29).

Due to the extra \( q, p \) dependence the situation changes drastically. In order to obtain an equation like (32) it is required to establish the new functional relation between the \( R \) and \( \rho \) functions. The previous one \( R^2 = e^\rho \) does not work. The situation now is more complex if one wishes to perform the one-to-one WWM map taking the operator form of the equations given by eq- (31) into the Moyal Toda equation (32). For example, if one sets:

\[ R * R = e^{*\rho} \Rightarrow \frac{\rho}{2} \neq \ln R. \tag{49a} \]

the l.h.s of eq-(48) will no longer be:

\[ \frac{1}{2} \frac{\partial^2 \rho}{\partial \tau^2}. \tag{49b} \]
Furthermore, if (49a) were the correct relation between $R$ and $\rho$, the r.h.s of (48) would not equal the r.h.s of (32). Although the r.h.s of (48) contains the same type of infinite derivative terms as eq-(32) does, after using the relationship (49a), the coefficients differ. Therefore, if (49a) were satisfied the r.h.s of (48) would not equal that of (32).

Assuming that (49a) were the correct relation between $R$ and $\rho$, another obstacle is that the l.h.s of (48) should have been of the form:

$$\frac{\partial^2 \ln_* R}{\partial \tau^2} = \frac{1}{2} \frac{\partial^2 \rho}{\partial \tau^2}. \quad (50a)$$

Where the star logarithm and star square root are defined:

$$\ln_*(e^{*\rho}) \equiv \rho. \quad R * R = e^{*\rho} \Rightarrow R = [e^{*\rho}]_1^{1/2}. \quad (50b)$$

In this fashion the inverse symbol map corresponding to (48) would have had the required operator form. It is not difficult to see that both sides of eq- (48) do not have the adequate form to match eq-(32). Therefore, the anstaz proposed in (47) does not work if one wishes to to find a direct relation between $R$ and $\rho$. What is required is to introduce two auxiliary functions discussed below. The source of the problem is due to the new functional relationship between $R$ and $\rho$ that will render both sides of eq-(32) correctly. Nevertheless matters are not final. Eq-(48) is per se satisfactory in the sense that it yields a well defined differential equation for the $R$ function with an infinite number of derivative terms. Imposing the condition:

$$\frac{1}{2} \frac{\partial^2 \rho}{\partial \tau^2} = \frac{\partial^2 \ln R}{\partial \tau^2} \quad (51)$$

implies that $\rho/2$ and $\ln R$ now differ by:

$$\frac{1}{2} \rho = \ln R + F(t,q,p,\hbar) \tau + G(t,q,p,\hbar). \quad (52)$$

This fixes $\rho$ in terms of $R$ and two auxiliary functions, $F,G$. Solving for $R$ yields:

$$R = e^{\rho/2 - F \tau - G} = e^{\rho/2} e^{-F(t,q,p,\hbar) \tau - G(t,q,p,\hbar)}. \quad (53)$$

Inserting this new value for $R$ into the r.h.s of (48) and equating it to the r.h.s of (32) determines another differential equation for the $F,G$ functions in conjunction with the original Moyal Toda equation. Three coupled differential equations for $\rho$ and $F,G$ are obtained in this fashion given by the three eqs- (32,48,51). It would be erroneous to equate the r.h.s of (48) to the r.h.s of (26) because the number of derivatives w.r.t the $t$ variable does not match. In (48) there is an infinite number whereas in (26) they are only quadratic.

Therefore, one can in principle obtain the continuous Moyal Toda equation from axial symmetry reductions of the Moyal Nahm equation (42,47) if, and only if, one introduces two auxiliary functions, $F,G$ that are determined, in conjunction with $\rho$, by three coupled differential equations (32,48,51). Whether this system of three coupled differential equations is compatible and consistent is another matter. This very difficult question remains
to be answered in addition to the uniqueness and existence of solutions. This implies that for every solution \( \rho \) to eq-(32) the remaining two differential equation (48,51) yield a well defined nontrivial \( F(t,q,p,h) \) in terms of \( \rho \). Similar arguments applied to the remaining auxiliary function \( G(t,q,p,h) \).

It is also reasonable to look for other ways of embedding the Moyal-Toda equations into the Moyal-Nahm equation that do not require a direct decoupling of the \( \cos p', \sin p' \) functions, for example. In other words, integrating out the \( p' \) variable without the need to reduce the generalized Moyal-Nahm equations. A plausible embedding (it is problematic) could have been as follows:

\[
\int_{-\infty}^{+\infty} dq' \int_{-\infty}^{+\infty} dp' \frac{\partial T_3}{\partial \phi} = \frac{\partial^2 \rho(\tau,t,q,p,h)}{\partial \tau^2}.
\]

\[
\int_{-\infty}^{+\infty} dq' \int_{-\infty}^{+\infty} dp' \{T_+,T_-\} = \frac{1}{4} \left( \frac{\Delta - \Delta^{-1}}{h} \right)^2 e^{*\rho} = \frac{1}{3}(h)^2 \frac{\partial}{\partial t^4} e^{*\rho(\tau,t,q,p,h)} + \ldots
\]

with \( T_\pm = T_1 \pm iT_2 \) and \( \phi = f(\tau + it) + g(\tau - it) \) contains implicitly the \( \tau, t \) dependence of \( \rho \). The three functions \( T_\alpha \) are required to obey the generalized Moyal-Nahm equations (42). One has some vanishing boundary terms due to total derivative terms but not all of the terms are total derivatives due to the integration w.r.t half of the phase space variables. However there is a major problem with the above embedding: there are derivatives of infinite order w.r.t the \( t \) variable in the r.h.s of eq-(54b) whereas the derivatives w.r.t the function \( \phi \) are of first order only. Since \( \partial_1 = (\partial \phi / \partial t) \frac{\partial}{\partial \phi} \) the above equations are inconsistent. There is no place where derivatives of infinite order w.r.t the function \( \phi \) appear. Therefore we must disregard the above embedding.

The use of the \( T_\alpha \) was successful in rendering the ordinary \( SU(2) \) Moyal Toda equation (29) from the Moyal Nahm equations as discussed earlier when one imposed the reduction \( q = q', p = p' \) conditions and set \( \phi = \tau, t = q \). Nevertheless, despite the above problems, the putative continuous Moyal Toda equation (26) admits the following embedding into the reductions of the Moyal Nahm equations, after setting for example \( \phi = f + g = \tau \pm it \),

\[
\frac{\partial T_3}{\partial \phi} = \frac{\partial^2 \rho}{\partial \phi^2} \Rightarrow \rho(\phi,q,p,h) = \int T_3 d\phi + F(q,p,h)\phi + G(q,p,h).
\]

\[
\{T_+,T_-\} = -\frac{\partial^2}{\partial \phi^2} e^{*\rho(\phi,q,p,h)}. \quad \rho = \rho(\phi,q,p,h). \quad q = q', p = p'.
\]

Following similar arguments as above yields a system of three differential equations, eqs-(26,55a,55b) to solve for the three functions \( \rho(\phi,q,p,h), F, G \).

Finally, the embedding of the continuous Moyal Toda equation (32) into the generalized Moyal Nahm equations (42) which is more directly linked to eq- (47) is to set \( q' = t \) and to fix the function \( \phi = \tau \) so that the ansatz in (7) yields \( A_0 = 0 \) and \( A_i(\tau, q' = t, p', q,p,h) = T_i \) for \( i = 1, 2, 3 \). Given a solution to the generalized \( SU(2) \)
Moyal Nahm equations (using Moyal brackets w.r.t the extended phase space variables) for the three potentials $A_i$, one selects the particular equation:

$$\frac{\partial A_3}{\partial \tau} = \{\{A_+, A_\pm\}\}. \quad A_\pm = \frac{1}{\sqrt{2}}(A_1 \pm iA_2). \quad (56a)$$

A partial integration taken w.r.t the $p'$ variable only yields the embedding relations:

$$\int_{-\infty}^{+\infty} dp' \frac{\partial A_3}{\partial \tau} = \frac{\partial^2 \rho(\tau, t, q, p, \bar{h})}{\partial \tau^2}. \quad (56b)$$

$$\int_{-\infty}^{+\infty} dp' \{\{A_+, A_\pm\}\} = \frac{1}{4} \left(\frac{\Delta - \Delta^{-1}}{\bar{h}}\right)^2 e^{*\rho} = \partial_t^2 e^{*\rho(\tau, t, q, p, \bar{h})} + \frac{1}{3} (\bar{h})^2 \partial_t^4 e^{*\rho} + \ldots \quad (56c)$$

Both sides of eq-(56c) now contain derivatives of infinite order w.r.t the $q, p, q'$ = $t$ variables and no inconsistency arises. There are some total derivative terms w.r.t $p'$ which vanish after integration but not all of the terms appearing in the l.h.s of (56c) are total derivatives. Eqs-(56) are consistent in their structure. From eq-(56b) one learns that:

$$\rho = \int_{-\infty}^{+\infty} dp' \int_{\tau'=0}^{\tau=\tau} A_3 d\tau' + F(t, q, p, \bar{h})\tau + G(t, q, p, \bar{h}). \quad (57)$$

The functions $F, G$ are not arbitrary but are part of the system of three coupled differential equations given by eqs-(32,56b,56c) ; i.e $F, G$ are given in terms of the $A_i$ satisfying the generalized Moyal Nahm equations (42).

For instance, inserting (57) into (32) and (56c) determines the differential equations for $F(t, q, p, \bar{h})$ and $G(t, q, p, \bar{h})$ in terms of the fields $A_i(\tau, q'= t, p', q, p, \bar{h})$ which are solutions to eq-(42) . The embedding is characterized by integrating out the variable $p'$ without imposing a reduction on the eqs- (42) which would decouple the $p'$ variable directly . In all these embeddings , two auxiliary functions are required.

It is warranted to see whether it is possible to find a Killing symmetry reduction of the generalized Moyal Nahm equations (42) directly to the Moyal Toda equations without the need to recur to auxiliary functions and avoid the complicated set of three coupled differential equations (32,56b,56c). The essence of the problem lies in the fact that the $t$ variable plays two different roles. In one case, like in the rotational Killing symmetry reduction of the heavenly equation, it behaves like an ordinary spacetime variable and in another it behaves like an internal phase space variable associated with the Lie algebra of the area-preserving diffs of the sphere. It seems very difficult to reconcile both roles within the WWGM formalism without recurring to the coupled system of differential equations.

Finally, we shall discuss the ansatz that succeeds in rendering reductions of the generalized Moyal-Nahm equations in the form prescribed by eq- (32) without the introduction of auxiliary functions. We believe that there maybe a reduction ( other than the axial symmetry reduction proposed above ) of the generalized Moyal-Nahm equations, (42), that sucessfully decouples the $p'$ variable and that reproduces eq-(32) without the auxiliary functions.
Essentially one has the data $\{T_\alpha\}$ for three Moyal-Nahm functions and three functions $\{L, H, M\}$ required in the Lax-Brockeett formalism of continuum $\mathbf{Z}$ graded Lie algebras [13]. Eq-(19) establishes the correspondence among these data. However, this is only possible if one can construct the three twistor-like transformations which map the now-deformed scalar field $\phi(x^\mu; q_i, p_i, \hbar)$ into the now-deformed three functions appearing in eqs-(2,3,4) : $\rho, u, \kappa$ depending on $\tau, t, q_i, p_i, h$. One of them is the Moyal deformed continuous Toda field : $\rho$. Presumably this should be related to the problem of deformations of twistor surfaces and Kodaira-Spencer deformation theory [16]. Unfortunately, we cannot say more on this matter. A start will be in constructing Moyal deformations of continuum Lie algebras. We are unaware if this has ever been done.

Roughly speaking, an example of the analog of the twistor-like transformation is to introduce two functionals $F_+, F_3$ so that the correspondence in (19) implies:

$$F_+[T_1(\phi(\tau, t; q_i, p_i h); q_i, p_i, h)) + i T_2((\phi(\tau, t; q_i, p_i h); q_i, p_i, h))] = L[u, \rho; q_i, p_i, h].$$

(58)

Let's assume for simplicity that the functional in (58) is linear, $F_+ [A + B] = F_+ [A] + F_+ [B]$. The other expression is :

$$F_3[i T_3(\phi(\tau, t; q_i, p_i h); q_i, p_i, h))] = M[\kappa; q_i, p_i, h].$$

(59)

with $u, \rho, \kappa$ depending on $\tau, t, q_i, p_i, h$ and the deformed $\phi$ obeys a deformed Laplace equation obtained by the Moyal quantization of the Ivanova-Popov construction. If, and only if, the representations of the $\mathbf{Z}$-graded continuum Lie algebras were known, then one would have a knowledge of the explicit functional relation of $L, M$ in terms of $u, \rho, \kappa$ that can be obtained from eqs-(2-6) after performing the WWGM map. Similarly, the three functions $T_\alpha$ are solutions to the generalized Moyal-Nahm equations (42). Therefore, eqs-(58,59) would have been the defining expression for the two functionals, $F_+, F_3$. Unfortunately we do not know the explicit form of $L, M$ because we lack the knowledge of the representations. All we can do to find out the expressions for $F_+, F_3$ is the following:

Upon establishing the correspondence between the Moyal-Nahm eqs-(42) and the Lax-Brockeett equations (18,19) ( with the crucial difference that now one must use the Moyal bracket w.r.t the enlarged phase space coordinates) gives after the chain rule and due to the fact that we have assumed for simplicity that the functional $F_+$ was linear in its arguments :

$$\frac{\partial L}{\partial \tau} = (\frac{\partial \phi}{\partial \tau})\frac{\partial T_1}{\partial \phi}\frac{\delta F_+}{\delta T_1} + i \frac{\partial T_2}{\partial \phi}\frac{\delta F_+}{\delta T_2} =$$

$$\left(\frac{\partial \phi}{\partial \tau}\right)\frac{\delta F_+}{\delta T_1}\{T_2, T_3\} + i \frac{\delta F_+}{\delta T_2}\{\{T_3, T_1\}\} = \{\{L, M\}\}. $$

(60)

If one chooses the identity functionals for both $F_+, F_3$ then as expected one arrives at the constraint in eq-(20). The correspondence turns into an strict identification. Avoiding this trivial and restrictive special case, we see then that in general the $L, M$ expressions in terms of $\phi$ are given by the functional map in eqs-(58,59). Hence, the last eqs-(58-60) are the defining highly nontrivial differential equations for the two functionals $F_+, F_3$, 27
once the defining equation for the deformed \( \phi \) is known and solutions of the generalized Moyal-Nahm equations (42) for the three Moyal-Nahm functions, \( T_\alpha \) have been found. Clearly, this is a extremely complicated system of equations to solve. Instead of having two auxiliary functions we have in this case two functionals. For this reason, it is of tantamount importance to construct representations of continuum Lie algebras and for that matter representations of \( SU(\infty) \) as well.

To sum up: we have shown that axial symmetry reductions of the generalized Moyal Nahm equations (47) in principle yield the continuous Moyal Toda equation (32) with the provision that a coupled system of three differential equations for \( \rho, F, G \) is solved, eqs-(32,48,51). An embedding of (32) into (42) is also possible after integrating out one of the phase space variables. One is also required to solve three coupled system of complicated differential equations, eqs-(32,56b,56c), for \( \rho \) and the two new auxiliary functions \( F, G \). Eqs-(58-60) depict another way of obtaining Moyal Toda equations directly from the Moyal Lax formalism (19) and the Moyal Nahm equations (42). Again, two functionals must be introduced which are determined by a very complicated system of differential equations.

Other Moyal deformations are those related to the infinite dimensional loop algebras associated with \( w_\infty \) algebras. For example, the loop algebra of \( sdiff(R^2) \), the algebra of maps of the circle into \( w_\infty \), in the basis of functions \( x^{s+m}y^{s-m} \) is:

\[
[v^{s}_m(\sigma), v^{t}_n(\sigma')] = [(t-n)(s+m) - (s-m)(t+n)]v^{s+t-1}_{m+n}(\sigma)\frac{\partial}{\partial \sigma}\delta(\sigma - \sigma').
\]

These loop algebras may admit Moyal deformations as well since the Moyal deformation of the centerless \( w_\infty \) algebra is the centerless \( W_\infty \) algebra [17,18,19,20]. Central extensions can be added as well [19]. Hence, the Moyal deformations of the algebra (42) will be just the infinite dimensional loop algebra associated with \( W_\infty \).

Finally, Moyal deformations of the \( \mathbf{Z} \) graded continuum Lie algebras [13] ought to be very relevant in the Moyal quantization program of the continuous Toda theory. Especially in regards to determining the differential equations for the \( u(\tau,t,q,p,\bar{h}) \) and \( \kappa(\tau,t,q,p,\bar{h}) \) appearing in (4,18,19) and in obtaining eq-(32) directly from the Lax-Brockett double commutator formalism. More on this shall be said in a forthcoming publication.

V. Conclusion

We have explictly presented a class of solutions to the Moyal \( SU(\infty) \) ASDYM equations in four dimensions that are related to the reductions of the generalized Moyal Nahm quations via the Ivanova-Popov ansatz. A dimensional reduction yields solutions to the Moyal deformations of the ASDG equations. The SDYM and SDG case requires a separate study.

Since the ASDYM equations studied by Ivanova and Popov [3] in Euclidean 4D correspond to the SDYM equations in 2 + 2 dimensions studied by [8], one can write down the master Legendre transform that maps the rotational Killing symmetry reductions of the Moyal heavenly equations given by eq-(14) into the \( SU(2) \) Moyal Toda equations given by eq-(29). A three step process is required to attained such a map and it is explicitly given by eqs-(16,19,28). This is one of the most relevant results of this work. Solutions to the
continuous Moyal Toda equation may be obtained by embedding the $SU(2)$ solutions into the $SU(\infty)$ case. In this fashion, a Legendre map from $\Omega$ to the continuous Toda field $\rho$ can be attained.

Three different types of Toda equations have been studied. The $SU(2)$ and continuous Moyal Toda equations, eqs-(26,29,32), have been explicitly derived. To a first order approximation, neglecting the deformations of the scalar $\phi$, the $SU(2)$ Moyal Toda equations can be simply obtained from the $SU(2)$ Moyal Nahm equations when (the undeformed scalar) $\phi = f + g = \tau$ and $t = q$. Finally, the generalized Moyal Nahm equations (42) have been provided that contain the continuous Moyal Toda equations (32) after a suitable reduction, similar to the one performed by Strachan [16] which yields the $SU(2)$ Moyal Toda from the $SU(2)$ Moyal Nahm equations. This reduction requires the introduction of two auxiliary fields. Further details of this reduction is currently under investigation. Embeddings of the various forms of the Moyal Toda equations into the Moyal Nahm equation were also provided and, again, the introduction of two auxiliary fields was required.

The project for the future is to study eqs-(16,19) and the generalized Moyal-Nahm equations (42) as shown in eqs-(58-60). But now one should use the appropriate deformed scalar field $\phi(x^\mu; q_i, p_i, \hbar)$ satisfying deformations of Laplace equation. This will provide the indirect map between $\Omega$ and $\rho$ via the Moyal-Nahm-Lax pair formalism. Unfortunately we lack an explicit knowledge of the form of the $\mathcal{L}, \mathcal{M}$ expressions (given by eq-(19)) in terms of the deformed $\rho, u, \kappa$ fields. If we did, then one could establish the sought-after correspondence after using eq-(19) and eqs-(16).

The main obstacle is the construction of the defining equation for the full deformed scalar $\phi(x^\mu; q_i, p_i, \hbar)$ that must be obtained by an explicit Moyal quantization program of the Ivanova-Popov construction. In the $\hbar = 0$ limit the $\phi(x^\mu, q_i, p_i, \hbar) \rightarrow \phi(x^\mu)$ obeying the original (undeformed) Laplace equation. Presumably, this could be a realization of deformations of twistor surfaces. The connection to Kodaira-Spencer deformation theory [16] is unknown at the moment.

Other Moyal deformations applied to higher extended objects, $p$-branes, remain to be studied: the so-called Moyal-Nambu-Poisson Algebras related to deformations of the volume forms. The natural deformation quantization technique is the Zariski product [22] which generalizes the Moyal product to $p$-branes. Octonionic [30] and Quaternionic Moyal Nahm equations can be constructed as well using the octonionic/quaternionic structure constants instead of the $\epsilon_{\alpha\beta\gamma}$ tensor density. The fact that the generalized Moyal Nahm equations require 8D may have an important role in understanding the quantum dynamics of the 11D membrane [23] and the role of $W_\infty$ algebras [21].

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