INTRODUCTION TO QUANTUM RADAR

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Abstract. After a brief introduction to the notion of quantum entanglement and quantum correlations, several schemes for a quantum radar based upon the quantum illumination and others protocols are discussed. We review different concepts that have been introduced to overcome several of the inherent difficulties in the implementation of quantum generation and/or detection of quantum sensing protocols for RADAR applications. Our review is an up-to-date critical presentation of the state of the art, with emphasis in the case by case assessment of the feasibility of the different concepts. We also aim that the review is accessible to non-experts in the field. Hence several appendixes and a technical glossary are included.

Keywords: Quantum entanglement, Quantum interferometric radar, Quantum Radar, Quantum Illumination

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1. Introduction

2. Quantum entanglement, non-classical correlations and discrimination of quantum operations and channels

2.1. Quantum entanglement

2.2. Entanglement in pure states

2.3. Entanglement in mixed states

2.4. Entanglement of states described by continuous variables

2.5. Discrimination of quantum operations and detection of targets

3. General approaches to quantum radar

4. Quantum radar protocols

4.1. Theoretical protocols and experiments on interferometric quantum radar

5. Quantum illumination: general features

5.1. Lloyd’s proposal on quantum illumination

5.2. The work of Lloyd and Shapiro on coherent state and quantum illumination

5.3. Gaussian quantum illumination

5.4. Detection problem, detectors protocols and experimental demonstrations

5.5. Impracticability of gaussian optical illumination for radar applications

5.6. Experiments demonstrating Lloyd’s quantum illumination

5.7. Experimental demonstration of optical gaussian quantum illumination

5.8. Microwave quantum illumination

6. Problems in the implementation of quantum illumination for radar applications

6.1. Reception and process of the correlated signal/idler beams

6.2. Generation of the entangled beams at microwave frequencies

6.3. Losses introduced by the storage of the idler beam

6.4. The time band-width problem

7. Hybrid quantum illumination protocols

7.1. Quantum radar prototype of Chang et. al. and experiments

7.2. Experimental results

7.3. Covariant matrix for classical correlated noise radar and quantum correlated noise radar

7.4. Quantum radar prototype of Barzahjeh et al. and experiments

7.5. Comparison of quantum noise radar performance with coherent illumination

7.6. Discussion of the scope of Shapiro’s analysis

8. The quantum radar protocol of Maccone and Ren

8.1. Maccone-Ren protocol for quantum radar

8.2. Practical issues implementing Maccone-Ren’s quantum radar protocol

9. Quantum illumination with Maccone-Ren’s quantum radar protocol

9.1. Lloyd’s quantum illumination using Maccone-Ren’s quantum radar protocol

9.2. Determination of the range and transverse position using quantum illumination with Maccone-Ren protocol
1. INTRODUCTION

Building on the work of M. Sacchi on entanglement enhancement in entanglement breaking quantum channels [68, 69], S. Lloyd started quantum illumination [52]. It was shown that quantum illumination was an example of quantum protocol where the enhancement in sensitivity derived by the use of quantum entangled states survives the effects of environmental noise. Indeed, the advantage of entanglement respect to non-entangled sources is, surprisingly and counter-intuitively, larger when the system is more noisy.

This theoretical result ignited several investigations and demonstrations on the topic of quantum illumination, at the theoretical and experimental level. One of the goals in such research was and still remains, the realization of a quantum radar, a radar that explodes quantum illumination to range detection with sensitivity beyond classical resources. As a result, it became clear that the enhancement of quantum illumination over classical illumination was limited by theoretical and technological considerations and that the goal of the realization of a quantum radar will need to come after the resolution of several issues at the practical and theoretical levels and also, that the enhancement will finally will not be as strong as initially was thought. Nevertheless, even the more modest theoretical advantages that quantum illumination has over classical illumination for target detection remains a strong motivation to develop the concept and more generally, to study quantum radar protocols and its practical and technological implementations. The maximal goal is the realization of a quantum radar, a goal which is not completed yet.

The present review aims to introduce for the non-expert in quantum radar the main achievements, concepts, methods and results developed in the field, but also the limitations and problems currently appearing in the theory and applications. The pre-requisites for understanding the material presented here is to have a basic acquaintance with the fundamental mathematical tools of quantum mechanics (notion of Hilbert space and linear operators, scalar products,...). We expect that the review serves as a bridge between two communities. First, the quantum radar
community, as a research activity located mainly in the fields of quantum optics, quantum metrology and quantum sensing. Second, the classical radar community with interest to initiate research in the area of quantum radar. We have approached the review with a minimal of formalism and we have developed several notions of quantum mechanics, quantum information and quantum optics in the appendix section and the Glossary. In this way the reader familiar with the more technical notions can be dispensed from following their reading in the manuscript.

The structure of this review is the following. In section 2, a succinct introduction to quantum entanglement and several of the aspects of entanglement for quantum radar applications is provided. Section 3 is a general description of the existing approaches to quantum radar. Section 4 describes quantum interferometric radar. Several aspects of quantum illumination are discussed in sections 5 and 6. Section 7 discuss a form of quantum illumination that we have called hybrid quantum radar. Section 8 is an introduction to a novel approach to quantum radar developed by L. Maccone and C. Ren [59]. Section 9 describes an innovative approach to quantum radar based in a combination of the methods from Maccone-Ren and Lloyd’s quantum illumination. This new idea will be developed further in a shortcoming separate paper. Section 10 describes other new approaches to quantum radar, especially a recent proposal by Durak et al. [22]. We observe that several of the developments in the appendix are difficult to be found directly in the literature. This applies to exposition of several details of Lloyd’s quantum illumination that we have presented in Appendix C. The Appendix introduce most of the relevant states used in quantum radar. The Glossary introduces several technical terms that are of relevance for quantum illumination and quantum radar.

We are intellectually in debt with several reviews on quantum illumination and quantum radar that have appeared recently in the literature. Especially, the review on quantum illumination from J. Shapiro [71] and the book from M. Lanzagorta [49]. From the first, we depart mainly in the way of addressing the presentation and in the scope of the topics considered, since [71] only discusses quantum illumination protocols. From the second, we depart in an more up to date account, considering developments up to now. Finally, a review of this nature can hardly be encomprehensive. Most of the topics are force by the main research lines, but others are emphasized from a more authors perspective. Moreover, the field is continuously changing. Hence we apologize first if some relevant research is not appropriately considered in this review. On the other hand, we hope that the review is useful for the researchers willing to enter this new are.

2. Quantum Entanglement, Non-Classical Correlations and Discrimination of Quantum Operations and Channels

Quantum radar is builds on the use of the properties of quantum entanglement and other quantum properties of electromagnetic radiation field. According with the characteristics and use of entanglement, quantum sensors are adequately classified in three types [38, 49]:

- **Type 1 quantum sensors**: The quantum sensor transmits un-entangled quantum states of light. This includes single photons quantum radars and classical LIDARS (Light Detection and And Ranging). An example of a Type-1 sensor is the LIDAR proposed in [26].
• **Type 2 quantum sensors**: The quantum sensors transmit un-entangled states of light, but use quantum photo sensors to increase performance. References [38, 49] offer different scope in the treatment of quantum LIDARS. Specially extensive treatment is [38].

• **Type 3 quantum sensors**: The quantum sensors transmit quantum states of light which are originally entangled. This includes quantum illumination protocols, Maccone-Ren protocol, Durak et al. protocol. We will discuss these protocols in this review.

In this review we will concentrate our attention in protocols for type 3 sensors. Quantum illumination and quantum radar refers to quantum sensing protocols based on the use of entangled sources of photon beams for radar applications and other quantum sensing and metrological applications. In such protocols, the use of entanglement varies from the application of a strong notion of entanglement, when the entanglement is preserved up to the reception of the scattered signal, to the use of entangled breaking discrimination protocols, where although entanglement is not present in the reception because environmental incoherence totally degrades the quantum coherence before the signal arrives the receptor, but still the protocols present enhanced sensitivity benefits due to the inherited non-classical correlation properties of the initially entangled states.

Since these notions are necessary to understand quantum radar protocols and quantum illumination protocols, a brief introduction to the notion of entanglement and quantum correlation is in order. Note that although the concepts and applications are counter-intuitive at some point, a pragmatic point of view has to be adopted, if one does not desires to enter in the discussion of the deepest points of the foundations of quantum theory. Hence the concepts that we will discuss related with quantum entanglement and quantum correlations pertain to the ambit of standard quantum mechanics, quantum optics and quantum information.

2.1. **Quantum entanglement.** An introduction to the concept of entanglement is necessary to understand the applications in quantum illumination. A good introduction for this purpose can be found in chapter 6 of the monograph of Garrison and Chiao [31]. A comprehensive treatment of quantum entanglement for continuous states of entanglement can be found in the review paper [1]. Reference [31] also offers an introduction to the notions of quantum optics need for quantum illumination and quantum radar. References for quantum sensing are [36, 62].

Quantum entanglement is a consequence of the fundamental principle of linear superposition of states in quantum mechanics applied to compound systems. The most elementary notion of quantum entanglement, describing systems composed by two distinguishable parts (here denoted 1 and 2 is the following. If the Hilbert space describing the system is a product of the form $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the separable elements of $\mathcal{H}$ are the ones compatible with the **classical principle of separability**.

**Complete knowledge of the state of a compound systems yields complete knowledge of the individual states of the parts.**

The origin of this principle goes back to the discussion of the Einstein-Podolsky-Rosen paradox [24] by Schrödinger [73]. The negation of this postulate leads to the notion of entanglement. Hence an entangled state is the one where the knowledge of the component states does not imply a complete knowledge of the parts.
Entanglement is a property on the basis of many practical and theoretical developments. The theory of quantum computation is based upon the quantum mechanical notion of entanglement, for instance. Also, all the quantum radar protocols that we will consider in this review are based on the concept of entanglement. Thus the quantum theory of entanglement is without doubt, among the key notions in modern applications of quantum physics in technology. On the other hand, is probably (al-together with the notion of quantum non-locality), the most difficult concept to grasp. Attempts to understand the concept have been explored recently. Two examples are the $EP = EPR$ conjecture [57, 79] and two-dimensional time dynamics (see for instance [28] and chapter 9 in [29]). Although advance on quantum foundations could bring changes in the description and understanding of quantum entanglement and quantum correlations, in this review we will adopt a conventional point of view on quantum entanglement and quantum correlations, with the view on applications of the theory in quantum radar.

In the next paragraphs we will discuss several forms of entanglement and also, several associated notions of quantum correlations associated to entanglement which are of relevance in different quantum protocols.

### 2.2. Entanglement in pure states.

Separable pure states in quantum mechanics are described by elements of $\mathcal{H}$ that are compatible with the classical principle of separability. They are elements of the form $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ that can be described as product of the form $\psi = \psi_1 \otimes \psi_2$, with $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$. For these notion of separable state, the statistical properties associated to the subsystem 1 and the statistical properties associated to the subsystem 2 are statistically independent from each other. This can be seen as a consequence from the fact that for separable states, the joint density probability function is the product

$$|\psi(x_1, x_2)|^2 = |\psi_1(x_1)|^2 |\psi_2(x_2)|^2,$$

where $x_1$ labels the components of $\psi_1$ and $x_2$ the components of $\psi_2$. Remarkably, according to quantum mechanics, $\psi$, $\psi_1$ and $\psi_2$ are states from which we can extract the maximal information (all the statistical information about measurements) respect to the total system $1 \cup 2$, the system 1 and the system 2, respectively. Also, the information for 1 (respectively, respect to 2) is obtained from the state $\psi$ by the process of integrating respect to 2 (respectively, integrating respect to 1) or summing up, if the indices are discrete, the distribution function $|\psi|^2(x_1, x_2)$ to obtain marginal distribution functions.

An entangled pure state is a pure state of a composed system which is not separable, namely, the classical principle of separability does not hold. In the quantum mechanical setting above, the pure entangled states are elements of the product Hilbert space $\psi \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ which are not product states. A simple case of entangled state is the *Bell’ state* $^4$

$$|\text{spin } = 0\rangle = \sqrt{\frac{1}{2}} (|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2),$$

where 1 and 2 refers to two separate regions of spacetime. The labels $\uparrow$, $\downarrow$ refers to the possible states of the spin at 1 and 2. A Bell’ state describes a system composed by two quantum particles with zero spin. Bell’ states are of fundamental relevance

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$^4$The Einstein Podolski Rosen paradox discussed in terms of 1/2-spin states appears in D. Bohm’s book [8], section 22.17.
for applications and theory of quantum entanglement. Also, in the investigation of quantum non-locality. Note that, however, spacelike separation or timelike separation between 1 and 2 is not of fundamental in the construction of Bell’ states. However, the paradoxical conclusions of the analysis of this system appear when 1 and 2 are spacelike separated [24], when, according with the theory of relativity, it is not possible to send a physical signal between the system at 1 and the system at 2, but, according to quantum mechanics, correlations between 1 and 2 occur [24, 7].

For pure states, as for instance Bell’ states and related states, there are several criteria and measures of entanglement. One of them is builds on the Schmidt’s decomposition in finite dimensional Hilbert spaces in terms of maximally projected product states. This is a type of decomposition in terms of product states. If the state is separable, then the Schmidt’s decomposition is trivial, since coincides with the state itself. Non-trivial decompositions reveal entanglement. If the state is not separable, there are several elements that appear in the Schmidt’s decomposition, with weights less than 1. The state is entangled iff the maximal projection coefficient of the Schmidt’s decomposition is less than 1 [31].

2.3. Entanglement in mixed states. Mixed states describe physical systems where the a priori knowledge that we have is partial. Mixed states are usually describing ensembles of individual states. In particular, an entangled mixed state is such that in the ensemble \((\{\psi_i, p_i\}\) that determines the statistical ensemble of the mixture contains at least one entangled state.

For mixed states, described by density matrices, the concept of entangled states is reduced to the concept of entanglement in the case of pure states.

For mixed states, it is more difficult to provide a measurement of entanglement in terms of measurement of correlations, since the correlations due to the mixing and the quantum correlations are undistinguishable. That is, it is not possible to specify if an experimental correlation between observables is due to quantum fluctuations or statistical fluctuation of the mixture (see for instance [31], section 6.4). A necessary condition for separability is the condition of positive partial transpose of the density matrix [63], which in the present form applies to identical systems with \(\mathcal{H}_1 \cong \mathcal{H}_2\). This condition of separability is described as follows:

**Positive partial transpose criterion.** The density matrix of a bi-partite separable system is of the form

\[
\rho = \sum_A \omega_A \rho_{1A} \otimes \rho_{2A},
\]

then the partial transpose

\[
\sigma = \sum_A \omega_A \rho_{1A}^{\top} \otimes \rho_{2A}
\]

must also describe a separable state.

From the properties of the density matrix, one has that the transpose matrix \(\rho_{1A}^{\top} = \rho_{1A}^*\) is also non-negative and has unit trace. Therefore, \(\rho_{1A}^{\top}\) can also be a legitimate density matrix for the subsystem 1. This implies that all the eigenvalues of the partial transposed matrix \(\sigma\) must be non-negative. This algebraic requirement is the positive partial transpose criteria [63, 43]. If the criteria is not met for a particular density matrix, then the system is non-separable. This can be the case
even if Bell’s inequalities are satisfied \[82\]. Hence the partial transpose criteria is stronger than Bell’s inequalities as a necessary criteria for separability.

2.4. **Entanglement of states described by continuous variables.** A class of entanglement of particular relevance for quantum illumination and quantum radar protocols is the class of entangled states parameterized by continuous variable systems \[1\]. In quantum mechanics, a continuous variable system with \(N\) modes is described by the cartesian product of \(N\) Fock spaces. Hence these systems are used to describe states of the electromagnetic quantum field. In particular, the Gaussian states that appear in quantum illumination are continuous variable states.

A relevant first example of this type of entanglement is realized by EPR-states,

\[
\psi(x_1, x_2) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} F(k) \exp(ik(x_1 - x_2)).
\]

These states are null eigenvectors of the total linear momentum operator,

\[
(\hat{p}_1 + \hat{p}_2) \psi(x_1, x_2) = 0,
\]

no matter how spacetime points \(x_1, x_2\) are spacelike, timelike or lightlike separated. This is the basis for the Einstein Podolsky Rosen paradox. This particular class of states provides an example of quantum entanglement, namely, the anti-correlation among the linear moment of the two systems.

**Continuous mixed entangled states.** For applications to quantum illumination and quantum radar, it is important to have a separability criteria for continuous entangled states for mixed states, since several relevant protocols are based on continuous states. Several results have been obtained, among them the Peres-Horodecki partial transpose criteria for gaussian states discussed in \[74\] and the necessary and sufficient criteria for non-separability discussed in \[21\], which are applied in different protocols of quantum illumination.

2.5. **Discrimination of quantum operations and detection of targets.** The main difficulty in the practical use of entanglement to enhance sensitivity in quantum measurements is that entanglement is a quantum property easy to be spoiled by interaction of the quantum system with the environment. Such processes, the loss of quantum coherence (quantum superpositions) is called decoherence\[5\]. Indeed, some of the protocols for quantum radar make use of a property of entangled states: entangled states are used as a signal may produce an enhancement of distinguishably of entanglement-breaking channels \[68\] \[69\]. Quantum illumination protocols are based on this property.

Let us consider two quantum channels, represented by the operators \(\mathcal{C}_1\) and \(\mathcal{C}_2\). If \(\rho\) is the initial state matrix density, the problem is to find the state \(\rho\) such that the probability of discrimination for the output states \(\mathcal{C}_1(\rho)\) and \(\mathcal{C}_2(\rho)\) provides the minimal error. In the case of bipartite entangled states, the initial state is a mixed state whose associated pure states are in a product space \(\mathcal{H} \otimes \mathcal{K}\). If one of the parts is transmitted (signal) and the other is retained (idler), the entanglement breaking channels produce outputs of the form \((\mathcal{C}_1 \otimes I_{\mathcal{K}})[\rho]\) and \((\mathcal{C}_2 \otimes I_{\mathcal{K}})[\rho]\) respectively, where \(I_{\mathcal{K}}\) is the identity on the Hilbert space \(\mathcal{K}\).

\[5\]Indeed, this notion, either implies distinguishing special basis in the Hilbert space or is basis dependent concept.
In the space of finite rank operators on $\mathcal{H} \otimes \mathcal{K}$ (operators whose trace is well defined and finite; see for instance [12], section 2.7), the trace norm is defined by the expression $\| A \|_1 := Tr \sqrt{A^\dagger A}$. If the channel 1 has assigned a probability $p_1$ and the channel 2 has assigned a probability $p_2$. For non-entangled states, the minimal error probability is of the form

$$p'_E = \frac{1}{2} \left( 1 - \max_{\rho \in \mathcal{H}} \| p_1 C_1(\rho) - p_2 C_2(\rho) \|_1 \right).$$

Instead, if one uses quantum entangled states, we have

$$p_E = \frac{1}{2} \left( 1 - \max_{\rho \in \mathcal{H} \otimes \mathcal{K}} \| p_1 (C_1 \otimes I_\mathcal{K})(\rho) - p_2 (C_2 \otimes I_\mathcal{K})(\rho) \|_1 \right).$$

Convexity properties of the space of states $\rho$ constructed as mixed states from $\mathcal{H}$ and from $\mathcal{H} \otimes \mathcal{K}$, linearity and convexity property

$$\| aA + (1-a)B \|_1 \leq a \| A \|_1 + b \| B \|_1$$

implies that the maximum probability for distinguish $C_1$ and $C_2$, that is, the maximum for the probabilities $p'_E$ and $p_E$ is achieved for pure states. And among them, $p'_E \leq p_E$, that is, entanglement enhance discrimination sensitivity in entanglement-breaking channels. This is the counter-intuitive result, since the space $\mathcal{H} \otimes \mathcal{K}$ is larger than $\mathcal{H}$. Furthermore, the advantage of using entanglement is more evident for large dimensions of the Hilbert space [69].

**Quantum entanglement or quantum correlations?** The concept of quantum non-locality or quantum correlation differs from the notion of quantum entanglement. Indeed, particular quantum systems can show correlations beyond the allowed by classical distributions even in the case where there is no entanglement present [6]. It is in this context that the notion of quantum discord is, besides the discussion on quantum channel discrimination, be of interest in quantifying quantum correlations in a more general context than entanglement [41, 65], which is a notion of relevance for quantum illumination [84].

### 3. General approaches to quantum radar

Quantum radar is the application of entangled light for radar use. Several approaches has been proposed in the literature:

- **Interferometric quantum radar.** In this protocol, interference phases of an idler (retained) and a signal (send to explore) beams are measured. After the signal beam probes the region of interest, there is a joint measurement performed at the receiver location where the phase difference between the two interfered beams is measured. Quantum interference with beams composed by entangled NOON-state allows to reach the Heisenberg limit in sensitivity in the difference in phase between the two paths followed by each beam, instead of reaching the standard quantum limit in the form of the shot noise limit as it appears in Mach-Zehnder interferometer [34, 36, 49]. Other forms of photon beams, like those composed by photons in squeeze states, also beat the shot noise limit (see [36] and references there in).

  Interferometric quantum entanglement requires to keep the entanglement alive from the initial entangled beams through the whole process of interference. Such a constraint limits the applicability of the protocol very
• **Quantum radar based on quantum illumination protocol.** Quantum illumination uses an entangled source of light for signal and idler beams to detect or track a possible space region where a target could be located. The theory of quantum illumination shows an enhancement in domain and sensitivity in several detection observables, for a theoretical low reflective, low signal intensity and bright noise background environment at optical frequencies [52,80] and at microwave frequencies [4]. These gains in detection sensitivity are resilient to the lost of quantum entanglement during the round trip. Indeed, after entanglement is lost by the process of decoherence and losses, the residual correlations between the received beam (when the target is there) and idler beam can be higher than for protocols working with entangled photon beams than for protocols based upon non-entangled light sources. How this happens, how the enhancement of the correlations persists, it is a counterintuitive fact. A derivation of how correlations are preserved even after the lost of entanglement in quantum illumination will be discussed in the Appendix B, but only in a formal way.

• **Hybrid quantum radar systems.** In these protocols, entangled light is prepared to be used as a pair of signal/idler system as in a quantum radar illumination protocol. Again, the protocol relies in the enhancement of the residual correlations heritage from the original correlations due to the quantum entanglement. However, the detection and storage processes for both, the idler and the received beam, are achieved using classical digitalization procedures and storage methods. This has certain advantage (it can capture and storage for long time information), but also the handicap of huge losses in sensitivity and enhancement respect to quantum radar based in full quantum illumination protocols. These methods have been discussed and experimentally demonstrated recently by two different groups [5,54].

• **Maccone-Ren theoretical quantum-radar protocol and further developments.** L. Maccone and C. Ren have developed a theoretical protocol for a quantum radar, able to determine the range and transverse displacement of a target. The fundamental idea is to exploit the advantage on the quantum sensing capability of EPR-type multiple entangled photons respect to illumination using non-entangled light states. In particular, it is shown that using beams composed by \(N\) photons as signal beams, the standard error in the localization of the target is reduced in a factor \(\sqrt{N^3}\) by state detected compared with the protocol of using \(N\) non-entangled states.

Although the Maccone-Ren’s protocol effectively is a quantum radio detection and ranging protocol, it presents special handicaps. Among them is the sensitivity to environmental noise. We will discuss a combined strategy between Maccone-Ren’s protocol and quantum illumination to solve this problem and the so called range problem in quantum illumination all together.
Interferometric quantum radar can find applications at least in quantum metrology and quantum scanner. While the first protocol works in a regime of entanglement, the other forms of quantum radar (quantum illumination and hybrid quantum illumination) are applicable even under complete loss of entanglement. This makes those protocols particularly useful for radar applications, where entanglement is easily lost in the microwave frequency. But this potential application, for realization, has to overcome several technical difficulties. Indeed, all known quantum radar protocols present handicaps of theoretical and/or technological nature, that eventually lead to the lost totally or partially of the aforementioned benefits.

4. Quantum radar protocols

4.1. Theoretical protocols and experiments on interferometric quantum radar. Interferometric quantum radar is based upon the analogy of the general protocol of radar detection with Mach-Zender interferometry. In Mach-Zender interferometry with coherent light, a light beam pass through a beam splitter, that divides the beam in two beams and let pass each of the beams through different evolutions. Then the beams are recombined by a second beam splitter. The difference in the optical paths that the beams follow implies a difference on phase $\varphi$ between the quantum states, via path integral interpretation of amplitude transitions. Information on the characteristics of one of the paths can be extracted from $\varphi$. But $\varphi$ can be obtained from the measurements of the intensity of the beams at the detection point, that is, from the measurement of individual photons arriving from both paths. It is found that when using coherent light as a source beam composed by $N$ independent photons, the error in the estimation of $\varphi$ goes asymptotically with the number of probes $N$ or with the number of photons of the beam, as $1/\sqrt{N}$. This is a direct consequence of the statistical independent character of the $N$ individual photons composing the light beam [36] (see also the glossary in the appendix section).

The analogy between interferometric radar detection and Mach-Zender interferometry relies in the following protocol for the interferometric radar. Let us assume first that the source beam is coherent light. In the interferometric radar system, an initial source light beam is split in two beams. One of them is sent to explore an optical path (signal beam), while the second beam is keep alive in the detector during the whole experiment. After, the received beam is recombined with the idler beam and an interferometric experiment performed. As for Mach-Zender interferometry, one can recover $\varphi$ from measuring intensities of the received beam and the idler beam. Quantum metrology theory [34, 36] establishes that the error on $\varphi$ implies an error in the estimation of the target range,

$$\delta R = \mathcal{O}\left(\frac{1}{\Delta \omega \sqrt{N}}\right),$$

where $\Delta \omega$ is the signal band-width. An interferometric radar works as the case of a Mach-Zender interferometer, but where one of the arms of the interferometer (corresponding to the signal beam) is much larger than the other [49], section 5.2.

The analogy between quantum interferometry and radar sensing can be generalized to the case when beams are composed by entangled quantum states of light. NOON-states are among the type of quantum entangled states which are used in
quantum sensing applications \[10, 49\]. A NOON-quantum state is of the form

\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left( (a_1|N\rangle_{1} \otimes \text{Id}_2 + \text{Id}_1 \otimes (a_2|N\rangle_{2} \right) |0\rangle_1 |0\rangle_2 \\
\equiv \frac{1}{\sqrt{2}} (|N0\rangle_1 + |0N\rangle_2),
\]

where $|0\rangle_1 |0\rangle_2$ is the vacuum state and $a_i$, $i = 1, 2$ are the annihilation operators for the photons associated to pass by arm 1 or arm 2. The effect of a phase shift $\Phi$ along one of the arms implies that the quantum state is of the form

\[
|\psi\rangle \equiv \frac{1}{\sqrt{2}} (|N0\rangle_1 + e^{iN\Phi} |0N\rangle_2)
\]

(4.2)

It can be shown that the error in the estimation of the phase $\Phi$ when using a particular observable for this NOON-estate is of order $\delta R \sim 1/N \[32, 49\]$. This implies the same order of precision for the estimation of the range $R$ of the target, that contrast with the asymptotic precision of order $1/\sqrt{N}$ when using classical states. This precision in the estimation range is generically known as the Heisenberg limit \[58\].

However, the Heisenberg limit cannot be reached using NOON-states in presence of attenuation due to absorption and scattering of the beam with propagation media \[49, 32\]. The effect of attenuation in the determination of range sensitivity has been discussed in \[49, 32\], including the case of atmospheric attenuation in \[33\]. In the model used in the literature, the quantum state in an attenuated media is of the form

\[
|\psi\rangle \equiv \frac{1}{\sqrt{2}} e^{-\left(\eta_1 (\omega c / 2) + \kappa_1 L_1 / 2\right)} (\hat{a}_1^\dagger)^N |0\rangle_1 |0\rangle_2 \\
+ \frac{1}{\sqrt{2}} e^{-\left(\eta_2 (\omega c / 2) + \kappa_2 L_2 / 2\right)} (\hat{a}_2^\dagger)^N |0\rangle_1 |0\rangle_2,
\]

(4.3)

where $\eta_i$, $i = 1, 2$ are refractive indices of the paths $i = 1, 2$, $\kappa_i (\omega)$ $i = 1, 2$ are attenuation indices, $\omega$ is the frequency of the radiation and $c$ is the speed of light in vacuum. The model implies for $L_2 \approx L_1$ and $\kappa_1 << \kappa_2$ an exponential attenuation along the path 2 respect the path 1,

\[
|\psi\rangle \rightarrow \frac{1}{\sqrt{2}} e^{-\left(\eta_1 (\omega c / 2) + \kappa_1 L_1 / 2\right)} (\hat{a}_1^\dagger)^N |0\rangle_1 |0\rangle_2 + \delta|\psi\rangle,
\]

where $\delta|\psi\rangle$ is an exponentially attenuated state respect to the first term.

It can be shown as consequences of the attenuated state model \[49, 32\] that:

- Quantum interference using NOON-states does not surpass the standard quantum limit \[43, 49\] except in very low attenuations levels. This is a potential obstacle for the use of quantum interferometry using NOON-states for radar applications \[32\].
- The effects in the error on the estimation of the range present periodic divergences in the azimuth angle \[49, 76\].

Note that this process of attenuation is not fully identified with decoherence, but with a related dissipative process (see for instance \[53\] or the description of a model for decoherence in the Glossary in the Appendixes section). A full treatment of atmosphere effects on quantum entanglement needs to consider decoherence. It is reported that decoherence will destroy quantum superpositions able to be use for quantum radar applications (microwave frequencies).
On the use of adaptive optics correction in quantum interferometric radar. In order to overcome the problem of the attenuation and the effect of the atmosphere on entanglement, it has been propose the use of adaptive techniques [76, 49]. The analysis presented in [76] shows that a maximal range over maximal range over 1000 km can be achieved. Without adaptive techniques, a maximal range of 60 km can be achieved. There is several idealizations on these estimates, as the use of ideal detectors.

However, the use of adaptive techniques requires the forehand knowledge of several relevant parameters of the target, specifically of the target range. This limitation precludes the use of this techniques in building a quantum interferometric radar for large distances. This problem will be indeed a recurrent issue in quantum radar protocols. Despite this, the use of adaptive optics methods can imply benefits when using quantum interferometric radar as scanning system.

Other possible approaches to quantum interferometric radar. Quantum interferometric radar could also be formalized by means of entangled states as described in [87, 20, 36]. These states are of the form

\[ |\Psi\rangle = \frac{1}{2} (|N_+\rangle_A |N_-\rangle_B + |N_-\rangle_A |N_+\rangle_B) . \]

(4.4)

Similarly as for NOON-states, the error in the measurement of \( \phi \) is of the form \( 1/N \), a reduction of the estimation error of order \( \sqrt{N} \) respect to coherent light. The use of such states as models for quantum interferometric radar seems not being pursued in the literature.

Another option for enhancement in interferometric quantum radar could be the use of squeezed states that interfere with coherent states. In quantum metrology squeezed state provide an enhancement that, although it is not optimal, still surpass the shot noise level [12, 3]. The use of squeezed states, by injecting them in port 1 of the interferometer and a coherent state in the port 2, allows to achieve a sensitivity in \( \phi \) of order \( 1/N^{3/4} \). This is an improvement compared with que standard quantum limit \( 1/\sqrt{N} \) that one reaches in the case the port 1 is injected a vacuum state. [3].

5. QUANTUM ILLUMINATION: GENERAL FEATURES

The fundamental criteria for enhancement in sensitivity detection of quantum illumination protocols respect to the analogous classical illumination protocols are based upon the theory of quantum detection and estimation theory [40], that appears as a generalization of Chernov’s theory [17], ultimately in the form of quantum Chernov’s theorem [2] in the case of a particular form of entanglement breaking channels.

In quantum illumination, two beams of entangled photons are generated. One of the generated beams will be used as a signal, while the other will be stored as idler in the detector for a while. The signal is sent to probe the region where the possible target could be and then it is detected back. The received beam is compared with the idler beam, usually by means of a joint measurement of the in-phase and quadrature voltages of the received and idler beams. This procedure leads to a theoretical enhancement in sensitivity and signal to noise ratio respect to the use of classical light beams with the same characteristics of low brightness, and in situations of low brightness beams, low reflectivity target and noise environment [52, 80, 4].
The generation of the entangled beams is achieved by means of spontaneous parametric down conversion methods \cite{85, 13} (in short SPDC or PDC) and modifications of these mechanisms. In a nutshell, the method consists in the generation of two beams composed by pairs of entangled photons at frequencies $\omega_s, \omega_i$ by pumping with a frequency $\omega_p$ a crystal with a second order non-linear susceptibility.

As a consequence of the method of production of the beams, the idler and signal beams are correlated by the following characteristics \cite{52, 49, 71}:

- **Correlation in frequency.** The idler and signal systems are correlated by conservation of energy, during the generation by the SPDC. The energy conservation implies the correlation for the frequencies of the photons

$$\omega_p = \omega_s + \omega_i,$$

with $\omega_s$ the signal frequency and $\omega_i$ the idler frequency, implying energy and momentum conservation at the photon level,

$$\hbar \omega_p = \hbar \omega_s + \hbar \omega_i, \quad \hbar k_p = \hbar k_s + \hbar k_i.$$

- **Correlation in arrival times.** They are exactly correlated on the instant where both the idler and the signal are created by SPDC process. Hence they must exactly meet in space and time when a joint measurement is performed, in order to exploit the profit of the correlations of the quantum state heritage from the entanglement.

- **The intensity of the idler and signal beams are the same, because for each of the photons in the idler beam, there is an entangled photon in the signal beam.**

Quantum illumination is resilient to noise and loss of entanglement due to decoherence \cite{93}. This robustness is the main difference with respect to other quantum sensing protocols \cite{36} and makes it potentially useful in the development of radar and metrology technology. However, some non-idealities of the mechanism of detection, combined with the form in which it is generated, make the potential enhanced sensitivity to drop to lower figures. On the other hand, the enhancement is almost independent from the loss of entanglement when the signal beam arrives at the receiver. This enhancement of heritage correlations with respect to the non-entangled sources is surprising, but can be shown that this is the case generically by direct computation using techniques from quantum mechanics.

In the following, we review several proposal for quantum illumination, illustrating the enhancement benefits and the problematic points that they represent from a practical point of view. A recent critical review on quantum illumination can be found in \cite{71}.

### 5.1. Lloyd’s proposal on quantum illumination.

Lloyd’s original theoretical proposal compared the detection capability of two different optical transmitters. In the first, the beam is composed of un-entangled photons. It is sent to probe a spatial region and received back after exploration. In the second, two entangled beams (the idler and the signal beam) are generated. The signal beam is sent to explore the region, while the idler is retained in the detector. After scattering with the target and being received the signal beam, the detector makes a joint measurement of the idler and received beams. Entanglement is assumed to be completely lost during the round trip.
The characteristics of Lloyd’s quantum illumination protocol are the following:

- $N$ pulses with a single photon per pulse.
- High time-bandwidth product $M = TW \gg 1$, where $W$ is the bandwidth and $T$ is the temporal detecting window.
- Low reflectivity index $0 < \eta \ll 1$ in presence of target. In absence of target, $\eta = 0$.
- The background light’s average photon number per mode, $N_B$, satisfies the low-brightness condition $N_B \ll 1$.
- For each transmitted signal pulse, at most one photon is detected by time, implying the condition $MN_B \ll 1$.

Under these assumptions, there are two differentiated regimes of interest. For single photon beams, the good regime happens when $\eta/N_B \gg 1$, while for beams formed by entangled photons, the good regime happens when $\eta > N_B/M$. For non-entangled light, namely, single photon states, the probability of error $Pr^+(e)_{SP}$ is bounded as

$$Pr^+(e)_{SP} = e^{-N\eta}/2, \quad \text{for } \eta \gg N_B,$$

for single photon beams, while for quantum illumination,

$$Pr^+(e)_{QI} = e^{-N\eta}/2, \quad \text{for } \eta \gg N_B/M,$$

showing an enhancement of the region of validity of this good regime (were the probability of error is very small) in the case of quantum illumination, despite there is not an enhancement in the probability of false positive.

In the so called bad regimes (the probabilities of error are in some sense large), we have the following probabilities of false detection,

$$Pr^-(e)_{SP} = e^{-N\eta^2/8N_B}/2, \quad \text{for } \eta \ll N_B,$$

and

$$Pr^-(e)_{QI} = e^{-N\eta^2 M/8N_B}/2, \quad \text{for } \eta \ll N_B/M.$$

Thus for very non-reflective systems, the probability of error in quantum illumination is reduced drastically with $M \gg 1$ respect to single photon beams. Furthermore, there is an enhancement of the region of validity of this result, from $\eta \ll N_B$ to $\eta \ll N_B/M$. To give a figure of this advantage, let us consider an optical frequency of 300 THz and a pulse of 1 $\mu$s with a 0.3 percent of bandwidth yields $M \sim 10^6$. This reduces notably $Pr^-(e)_{QI}$ respect to $Pr^-(e)_{SP}$, that corresponds to $60dB$ higher signal to noise ratio from quantum illumination respect to single photon illumination.

The condition of low-brightness background noise $N_B \ll 1$ is full-filled in the optical regime for the sky normal conditions [70]. However, Lloyd’s quantum illumination could be extended to bright backgrounds $N_B \gg 1$, as the second model considered by Lloyd itself suggests [52]. Although, this is not a particularly physical condition at optical frequencies, since for sky day light in the optical regime $N_B \ll 1$, the condition $N_B \gg 1$, it can be full-filled in presence of bright jamming.

Lloyd’s analysis presupposes the following technical assumptions:

- There is at disposition a source of high-TW entangled photons,
- There is not losses in the storage system,
The receiver performs optimally: it detects individual pair of correlated photons.

Experience with other relevant protocols of quantum illumination has shown that such assumptions could be unrealistic and that non-ideal physical conditions could reduce considerably the theoretical benefits of quantum illumination. Specially difficult to implement are detectors with low or negligible losses.

5.2. The work of Lloyd and Shapiro on coherent state and quantum illumination. Despite the significant advantage of quantum illumination at the optical regime, coherent beams outperforms quantum illumination at the optical regime, as an analysis of Lloyd and Shapiro showed [72]. Under the assumptions that

- An ideal laser produces \( N \) pulses, each of which has unity average photon number,
- For all reflectivity \( 0 < \eta \leq 1 \),
- For low-brightness background, \( 1 \gg N_B \geq 0 \),

it can be proved that the probability of false positive for beams composed by coherent state is such that the Chernov’s type bound

\[
Pr(e)_{CI} \leq e^{-N\eta}/2.
\]

(5.5)

holds good. Performance of coherent light equals to the performance of quantum illumination in the good regime, but outperforms when quantum illumination operates in the bad regime.

The fact that an ideal coherent light illumination protocol theoretically outperform Lloyd’s quantum illumination protocol in enhanced sensitivity and in signal to noise ratio could be seen as a strong limitation to the applications of quantum illumination, except in practical circumstances where the use of coherent light is disregarded.

5.3. Gaussian quantum illumination. The work from Lloyd and Shapiro [72] showed that quantum illumination based upon Lloyd’s protocol [52] did not outperform a generic detection system based upon non-entangled single beams operating at the same energy and frequency characteristics, and indeed, quantum illumination could be substantial less sensitive than illumination protocols using coherent state light.

However, quantum illumination stimulated further research on the enhancement on sensitivity of quantum entangled states. Short after the work from Lloyd and Shapiro on coherent and quantum illumination, Tan et al. [80] showed that quantum illumination based upon gaussian light packets theoretically outperforms any classical system, including coherent light illumination protocols. Apart from the gaussian nature of the quantum states of the light, the main difference with the previous protocol is that Tan et al. considered bright noise scenarios, where \( N_B \gg 1 \). Indeed, the theory of gaussian quantum illumination assumes the following conditions:

- The wave packets are composed by a number of photons per mode very small, \( N_S \ll 1 \).
- High time-bandwidth product \( M = TW \gg 1 \).
- Low reflectivity index \( 0 < \eta \ll 1 \) in presence of target. In absence of target, \( \eta = 0 \).
The background light average photon number per mode, $N_B$, satisfies the high-brightness condition $N_B \gg 1$.

Note that the last condition in Lloyd’s quantum illumination, namely, at most one photon is detected by time, implying the condition $MN_B \ll 1$, is dropped from the conditions in gaussian quantum illumination as developed in [80]. In this protocol of quantum gaussian illumination, the quantum Chernov bounds for coherent and gaussian quantum illumination are rather different than in Lloyd’s quantum illumination. For the coherent light illumination, it was found that

$$ Pr(e)_CI \leq e^{-M\eta N_S/4N_B/2}, \quad (5.6) $$

while for gaussian quantum illumination is

$$ Pr(e)_QI \leq e^{-M\eta N_S/N_B/2}, \quad (5.7) $$

when $N_B \gg 1$, $0 < \eta \ll 1$ and $N_S \ll 1$. In this operating bad regime and for $M \sim 10^6$, gaussian illumination offers a theoretical 6 dB improvement respect to coherent light illumination working on the same conditions. This is a drastic reduction from the original assessment in Lloyd’s analysis of quantum illumination, that claimed an improvement in sensitivity around 60 dB, but still concedes a theoretical ample advantage of quantum illumination respect to protocols based upon classical light illumination.

5.4. Detection problem, detectors protocols and experimental demonstrations. The main difficulty in the reception protocol in Tan et al. quantum illumination is that the observable of the theory where quantum advantage is shown, is impossible to observe directly. This observable are the phase sensitive cross-correlations

$$ \langle \hat{a}_{Rm} \hat{a}_{Im} \rangle_{H_i}, \quad H_i = 0, 1, \quad (5.6) $$

where

$$ \hat{a}_{Rm} = \sqrt{\eta} \hat{a}_{Sm} + \sqrt{1-\eta} \hat{a}_{Bm} $$

is the annihilation operators of the received mode $m$. $\langle \hat{a}_{Im} \rangle$ is the annihilations operator of the idler mode $m$, $\hat{a}_{Bm}$ is the annihilation operator for the back-ground field and $\hat{a}_{Sm}$ is the corresponding operator for the signal mode. $H_0$ denotes the hypothesis that there is no target present, while $H_1$ is the hypothesis that there is target present. This correlation cannot be measure by measuring quadratures of idler and received signal, because it is necessary to know the al-together four quadratures and Heisenberg principle precludes to do it so. This is independent of being in the low-bright regime ($N_S << 1$) or the bright regime ($N_S >> 1$). Furthermore, at the time of such developments, direct detection was not enough precise to provide quantum illumination any type of advantage over coherent illumination.

An alternative detection procedure consists on measuring the phase insensitive cross correlations

$$ \langle \hat{a}^\dagger_{Rm} \hat{a}_{Im} \rangle_{H_i}, \quad H_i = 0, 1. \quad (5.7) $$

But for Tan et al. theory, this cross-correlation is zero for quantum illumination, for both $H_0$ and $H_1$ hypothesis.

A partial solution to the above situation was found by Guha and Erkmen’s optical parameter amplifier (OPA) [37]. The theory of OPA is essentially based in an inverse type process than spontaneous parametric down converted generation.
of entangled modes (SPDC), which are both non-linear optic processes (see also the short introduction provided in Appendix D). Using OPA method of detection, Zhang et al. \cite{89} reported the first experimental demonstration of gaussian quantum illumination. The reported improvement of quantum illumination respect to coherent light illumination is rather modest, of order 20% in signal to noise ratio. The reduction on the efficiency between experiments and the theory elaborated by Tan et al. is explained by the multiple non-idealities of the experimental scheme and detector procedure.

5.5. **Impracticability of gaussian optical illumination for radar applications.** Apart from the range problem, the problem of how to determine the range of the target, the main problem that precludes the use of gaussian illumination for realistic radar applications in the optical regime is related with the conditions of brightness $N_B \gg 1$ that gaussian quantum illumination assumes. In the optical regime (were sources of entangled beams by means of parametric down converted methods are relatively easy to find) do not hold for normal sky light conditions, where $N_B(\text{opticalsky}) \ll 1$. Henceforth one arrives to the conclusion that for long range radar purposes, gaussian illumination will be impractical, except in contra-jamming measures applications.

5.6. **Experiments demonstrating Lloyd’s quantum illumination.** Experiment demonstrating Lloyd’s quantum illumination theory have been performed. The first one was the experiment performed by Lopaeva et al. \cite{53}. In such experiment, the generation of the entangled states of photons was achieved by means of spontaneous parametric down conversion. The target was a highly reflective object located at a fixed known position. The mechanism of SPDC generates a pair signal/idler beams at optical wavelengths appropriate for gaussian quantum illumination. After scattering, the signal beam is received in a highly efficient coincidence-counting receiver CDC camera, while the idler beam was also detected in a CDC camera (the same CDC camera than it is used for the detection of the signal beam). The classical illumination is generated by first stopping one of the beams generated in spontaneous parametric down conversion. The results shows a clear enhancement of quantum illumination respect to classical illumination, given by the factor ratio between the signal to noise ratio of the quantum and classical light radars,

$$QE = \frac{SNR_Q}{SNR_C},$$

which was demonstrated in the experiment from Lopaeva et al. to be larger than 1. Indeed, for low intensity beams, with average number of photons per mode $N_S \ll 1$, the enhancement parameter can be of several orders of magnitude. In the experiment, the average number of photons per mode was $N_S \approx 0.075$.

A recent experiment on quantum illumination has been reported by England et al. \cite{25}. The general idea of the experiment follows the previous experiment by Lopaeva et al. \cite{53}, but there are significant differences between both experiments. In the quantum illumination experiment of England et al., the source of quantum entangled states is an spontaneous four wave mixing generator (SFWM). This procedure to generate entangled states is based upon an effective four photon interaction \cite{18, 77}, where the interaction of a powerful pump with a non-linear
birefringent media produces two entangled photon state [77] (spontaneous parametric down conversion based on an effective three photon interaction [48], [31], section 13.3). Also, the target used in the experiment is different than for Lopaeva et al. experiment, since in the experiment from England et al. a diffusive target situated at a constant distance of 32 cm from the detector (and source). The experiment compares the performance of an standard radar system that works using classical light with a quantum illumination system. For the quantum illumination scheme, two beams are generated. One is send to explore the presence of an object, while the second is retained in the detector system. The intensity of the classical light and quantum light illuminations are the same. After the signal is received, there is a joint measurement of both beams. This is achieve by using the same signal beam as when the system operates in the quantum regime but disregarding the idler beam.

The detection system is by single photo counting, in the case of classical illumination, and by the use of coincidence event, in the case of quantum illumination. Therefore, the signal to noise ratio was defined by the phenomenological expression

\[ SNR = \frac{N_{in} - N_{out}}{N_{out}} \]

for both, the classical and the quantum illumination systems. In this expression, \(N_{in}\) is the number of detected photons where the target is there and \(N_{out}\) is the number of photons where the target is not there. Furthermore, the theory of spontaneous four wave mixing generation provides close expressions for the SNR in the case of classical and quantum illumination. Also, for light generated by the SFWM, it can be proved that the enhancement, measured as the ratio between the quantum illumination \(SNR_Q\) and the classical illumination \(SNR_C\) is independent (in first approximation) of the losses in the detector [77] and also of the intensity the intensity of the laser jamming. Furthermore, it can be shown that the non-classicality of the signal received is equivalent to the condition the bound

\[ QE = \frac{SNR_Q}{SNR_C} \geq 2 \]

and that this ratio of quantum enhancement is larger when the power of the signal beam is low, that is, when the average number of photons per mode \(N_B \ll 1\). This enhancement is a distinctive character of the mechanism of generation SFWM of the pair of entangled photons. The frequency of the jamming laser determines the time bin. Also, it is assumed that there is maximum of one photon per bin. In the case of low noise environment, although there is a decrease respect to the theoretical gaussian quantum illumination 6 dB benchmark, there is still benefit respect to classical illumination when the target is not there. However, this part of the experiment operates in the low-bright regime \(N_B \ll 1\), a regime where quantum entangled states do not provide theoretical advantage over coherent light illumination. Indeed, the experiment also shows that in such conditions classical illumination works well enough for detection.

However, the main advantage of quantum illumination appears when there is a bright background noise. The experiment also investigated quantum illumination in presence of jamming, showing a clear benefit of quantum illumination over classical illumination. This is shown by the signal to noise ratio, using quantum illumination and using classical illumination.
5.7. **Experimental demonstration of optical gaussian quantum illumination.** The experiment from Lopaeva et al. on quantum enhancement of quantum illumination respect to classical raised several criticisms. As it was discussed in [89], Lopaeva’s experiment compares a non-optimal source of classical light (thermal states) with quantum illumination. But coherent states are the benchmark for classical illumination sensitivity, instead of thermal states. Furthermore, the CDC camera used in Lopaeva’s experiments are far from being the most efficient, that it turns out to be homodyne-detection receiver for coherent light.

Such deficiencies were partially emended in Zhang et al. experiment, where the use OPA detectors and proving enhancement respect to coherent light homodyne-detection scheme was demonstrated [89]. In Zhang et al. experiment, a laser pump at $\lambda_p = 780 \text{nm}$ is used in a SPDC process to generate two entangled beams at wavelengths $\lambda_s = 1590 \text{nm}$ and $\lambda_i = 1530 \text{nm}$. Further, noise is added at the same wavelength than the signal beam. The recombination of the idler and signal-noise beam are detected at a OPA detector (instead than a CDC-camera), which is theoretically the best detector for Gaussian quantum illumination [37]. In theory, the Guha et al. OPA detector can enhance up to $3 \text{dB}$ in signal-to-noise ratio (SNR) when using quantum illumination respect to coherent light illumination detection. In practices, it is much less, as mention above, 20% enhancement on SNR respect to homodyne-detection for coherent light of the same characteristics.

**Remark 5.1. Two general remarks on experiments on quantum illumination.** In all experiments [53, 89, 25], the enhancement is higher for low average number of photons per mode, that is, when the condition $N_S << 1$.

In the above discussed experiments, the target range must be known. Furthermore, the design of the experiments of Lopaeva et al. [53], Zhang et al. [89] and England et al. [25] are on the optical wavelength, which is unpractical for long range radar applications, although it can be applied as non-intrusive short range quantum LIDAR or scanning system applications.

5.8. **Microwave quantum illumination.** The main limitation of optical gaussian quantum illumination for radar applications relies on the inadequate implementation under which optical quantum illumination works for daily sky normal conditions. This problem is overcome in microwave quantum illumination by the use of microwave gaussian beams as signals. At microwave wavelength, daily sky conditions implies $N_B \gg 1$. It is a high noisy environment, which is one of the premisses to exploit the benefits of quantum illumination and also it was demonstrated by the experiments discussed above [53, 25].

In this context, the proposal for quantum illumination in the microwave regime of Barzanjeh et al. [4] from 2015 was the following. Two beams of gaussian states entangled photons are generated at optical frequencies using a SDPC system. The two beams have the same intensity. One of the beams is retained in the receiver as the idler and keep alive for a later joint measurement; the other beam pass through an electro-optomechanical (EOM) cavity that converts the optical to microwave signal. After collimation, the microwave signal is send to probe the region and later it is detected and converted back to another optical received signal using a second EOM cavity. After this, a joint measurement is performed of the detected/converted signal and the retained idler beam as described by Tan et al. gaussian quantum illumination theory.
The theoretical probability of error in detection of a false positive $Pr(e)_{QI}$ in the operating regime of gaussian quantum illumination ($0 < \eta \ll 1$, $N_S \ll 1$, $N_B \gg 1$) depends upon the characteristics of receiver, implying an overall gain from 3 dB to 6 dB when using entangled light states respect to use coherent light [71]. The original analysis employed receivers with theoretical gains in $Pr(e)_{QI}$ equivalent to 3 dB [4]. Furthermore, since the receiver is not ideal, there are additional losses at the receiver. Specifically, to retain the idler beam using optical storage methods implies the generation of noise at a rate around 0.2 dB/km (noise induced/fiber propagation). This implies that, in order to keep the 3 dB advantage of microwave quantum illumination over coherent microwave illumination, the range of the maximal radar range should be restricted to approximately 11 km [4,71].

A possible resolution of the idler storage problem is the use of quantum memories [75,93] for the storage of the information concerning the idler mode [4]. Although the implementation of quantum memory technology for quantum illumination is technologically demanding and still to be achieved, it can potentially solve the storage problem in microwave quantum illumination, allowing the 3 to 6 dB theoretical increase performance for longer range detection, as indicated initially in [4].

There is also the concern of the loses in the optical to microwave signal conversion.

6. Problems in the implementation of quantum illumination for radar applications

Quantum illumination has stimulated and continuous stimulating the research on the sensitivity enhancement of quantum entanglement in detection protocols. However, it is also apparent that non-optimal technological implementations and subversive problems appear in the experimental and in the possible practical implementation of the protocols. In this section we discuss the most relevant of such issues.

6.1. Reception and process of the correlated signal/idler beams. Quantum illumination requires the identification of the correlated photons of the signal received beam and idler beam at the moment of detection. This requires previous knowledge of the target range [52,19,15,74,6]. Because of this constrain, it has been suggested the use of quantum illumination once the target range is known, as a scanner system or in bio-medical applications [19,5,46]. The advantage of quantum illumination over classical illumination can be exploded for having better resolution of the structure of possible targets, once the target has been located by other means.

Besides from new versions of the quantum illumination protocol that could effectively resolve this fundamental issue, there are other theoretical protocols of quantum radar that provide a resolution to this problem (see below sections 8,9,10).

6.2. Generation of the entangled beams at microwave frequencies. Quantum illumination requires the generation of two beams composed by entangled pairs of photons. In the optical regime, the generation has been successfully accomplished by the use of spontaneous parametric down conversion techniques [48]. One problem to implement quantum illumination for radar applications is the generation of
beams with the required frequency characteristics adequate to radar applications, namely, signal beams in the microwave frequency. A resolution of this difficulty was achieve by the use of EOM cavities [4, 89]. Currently, two different techniques has been used that remain fidel to the original quantum illumination/radar protocol:

(1) **Frequency conversion.** The use of electro-optomechanical converters, to transfer from optical frequencies to microwave frequencies and vice versa, in the generation the detection of the signal beams of the signal beam [4]. Using this method it was demonstrated the reliability of microwave quantum illumination at the laboratory level (under cryogenic conditions).

The major weak point of this technique of microwave entangled pairs generation is the low rate of the conversion from optical to microwave frequencies and vice versa. Although the advantage of quantum illumination respect to coherent illumination is greater at low power signal, the signal achieved by these techniques could be too low for practical applications.

(2) **Josephson amplification.** Recently, non-degenerate Josephson parameter amplifier (JPA) [88, 14] has been used as a quantum microwave source in quantum illumination and quantum radar [15, 54, 5]. JPA has been studied extensively in recent years because they work as supersensitive microwave amplifiers [14]. JPA generators have been used in the new proposal of *quantum-enhanced noise radar* [15] and in the experimental verification of the advantages of quantum microwave illumination at macroscopic distances and room temperature [5].

The major weak point of these methods is the extremely low temperatures that the JPA requires for operation.

The problem of generating entangled signal/idler states of pairs of photon beams in the microwave regime suitable for radar operation remains an important factor for the further development of quantum radar. Indeed, to have other forms of generation would be an interesting development.

6.3. **Losses introduced by the storage of the idler beam.** One source of losses comes from the storage of the idler signal. Although this is not an important concern for short-range applications, for long range applications, those losses possess a limit on the maximal target range. The use of optical fiber delay line implies a limitation of the range of around 11 km, assuming a fiber loss of 0.2 dB and fiber propagation speed around 2/3 c.

The proposal of using a quantum memory for storage of the idler beam could potentially improve this range limit, since the efficient in some quantum memories reach up to order 89% [14]. However, as mentioned before, the implementation of quantum memories in quantum radar is still a theoretical concept and significantly depends on the quality of the quantum memory.

**Losses due to atmospheric absorption.** This is another important source for losses. This difficulty has been already discussed before, in the case of quantum interferometric radar. It could have a similar treatment by means of adaptive optics correction [76], with the same limitations and constrains in practical implementation.
6.4. The time band-width problem. High time-bandwidth product is essential for the theoretical advantage of quantum illumination respect to classical illumination. But in the microwave regime this is difficult to achieve [71]. To start with, the mechanisms proposed in [4] of optical conversion to microwave implies a narrow band generation. Even using broadband amplifiers [56], the time-bandwidth is very small in the microwave compared with what it could be easily available at optical frequencies. For example, a 1/3-percent of fractional bandwidth at 1 μm wavelength that gives 1μs pulse duration at 10⁶ time-bandwidth product, only gives 10² bandwidth at 1 cm wavelength. But pushing to the mm-wave operation these figures will imply a decrease the efficiency on the single pulse bin interrogation efficiency of quantum illumination, spoiling the eventual advantage [71, 62].

7. Hybrid quantum illumination protocols

Hybrids quantum radar protocols described below employ entangled light for the protocols of the illumination, but the mechanisms for the reception the signal beam are based on digital classical methods and classical matching filter techniques. In this area, we will describe the research work of two groups where these new proposal methods have been investigated and demonstrated [54, 5]. For both, the following guidelines describe the procedure, that we present in closer form to the one discussed in [54].

7.1. Quantum radar prototype of Chang et. al. and experiments. The radar prototype described in [15–54] develops and demonstrates experimentally the concept of quantum-enhanced noise radar (also named quantum two-mode squeezing radar in [54]). It is a demonstration of how the use of quantum entangled signals/idler beams can outperform a classical equivalent (two squeezed noise radar system) working under the same conditions.

The radar prototype experiment has steps [54]. For entangled light illumination system, the protocol is the following:

1. Produce two correlated noise signals at the JPA and amplification.
2. One of the signals (the idler beam) is amplified and measured immediately after generation using classical digital techniques. The result is stored using classical digital techniques. In particular, the in-phase and quadrature voltages for the idler signal are measured and stored.
3. The signal beam is amplified and send to explore a spacetime region where the target could be located. In the experiment, the signals are send through free space.
4. Receive and measure the signal using classical digital techniques as for previously was done for the idler beam.
5. Declare a detection if the detector output, based in matched filtering techniques, exceeds a given value threshold.

The methodology is analogous for the case when the signal/idler beams are not entangled (two mode noise radar, in short TMN radar) than for the entangled generated sources (quantum two mode squeezing radar, in short QTMS radar), making the standards for the protocol comparison the same.

The difference between the TMN radar and the QTMS radar arises in the generation of the corresponding sources. In the case of QTMS-radar, the sources are generated by a Josephson parametric amplifier (JPA), that generates two entangled
photons microwave frequencies $\omega_1 = 7.5376 \, GHz$ (corresponding to a wave length of approximately 43 cm), $\omega_2 = 6.1445 \, GHz$ (corresponding to a wave length of approximately 18 cm). After the generation, the signals are amplified. Note that the amplification process is such that introduce noise in the system and also is a cause of the lose of entanglement\(^6\). The idler (with $\omega_2 = 6.1445 \, GHz$) is digitalized and stored as a classical record immediately after generation. The amplified signal (with $\omega_1 = 7.5376 \, GHz$) is sent to probe the spacetime region. The signal at the JPA as it fed at the transmit horn antenna is $-82 \, dBm$. The signal coming directly from the JPA is $-145.43 \, dBm$, after which it suffers an amplification to the power $-82 \, dBm$. The received signal is measured using heterodyne methods, digitalized after being received. Both are stored and compared as classical records using filtering techniques.

For the TMN radar, the radar prototype has the following characteristics. The generation of the signal is as follows. A carrier signal at 6.84105 GHz is generated and mixed with Gaussian noise centered at 069655 GHz and band-limited width of 5 MHz. This will produce two sidebands at frequencies $\omega_1 = 7.5376 \, GHz$, $\omega_2 = 6.1445 \, GHz$, as in the case of QTMS radar protocol. The signals are then treated through the same amplifiers chain than for the QTMS radar signals. After the signals pass through an splitter, the signal $\omega_2$ is then detected by heterodyne methods and the results digitalized, while the signal $\omega_1$ reaches an X-band antenna horn and is then send it through free space.

The reception mechanism consists of the same kind of X-band horn antenna connected to an amplifier at 25 dB. Then the amplified signal is connected to a digitizer performing measurements at $\omega_1 = 7.5376 \, GHz$.

The free space that separates the two horns is $R = 0.5m$. In the experiment, there is no target intersected. In this sense it is also not this experiment a quantum illumination experiment.

The total power injected in the transmit horn is $-63 \, dB$; The power of the Gaussian noise generator, after discounting the noise of the amplification is $-82 \, dBm$. Hence the signal to noise ratio is of order $-19 \, dB$.

The implementation of the above procedures ensures that the QTMS radar and the TMN radar prototypes operate under the same conditions and characteristics.

### 7.2. Experimental results.

Experimental demonstration of the above prototype has been performed. Instead of evaluating the signal to noise ratio, the receiver operating characteristics curves (ROC curves)\(^7\) are plotted and analyzed for different samples consisting different number of photon pairs detected. The result is a remarkable factor of up to 10 enhanced in the probability of detection respect to

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\(^6\)Let us remark that there is a bit of confusion at this point. The notion of entangled sources used in \([24]\) is a pragmatical one: two beams are entangled if the corresponding correlation matrix show higher correlations than the corresponding correlation matrix for classically correlated beams operating at the same specific characteristics and preparations. One needs to understand this as a pragmatical notion of entanglement, not in concurrence with the fact that the pair of photons generated at the JPA are entangled in a quantum mechanical sense. Indeed, the notion in Luong et al. of quantum entanglement is called non-classical or quantum correlation in quantum mechanical terms.

\(^7\)In this case, these are curves characterized by the number of photon pairs detected and correlated, in the plane probability of false alarm versus probability of detection. The precise way the probability of false positive and probability of detection are analyzed can be found in \([55]\).
the radar protocol based upon classical correlated sources, especially as the number of photon pairs increase and for low probability of false alarm \[15, 54\]. Furthermore, the operation time of the QTMS radar can be reduced in a factor up to eight respect to the operation time of the TMN radar.

However, by the nature of the technique used, further degradation of the theoretical maximal enhancement of quantum illumination respect to classical illumination of \(6 \text{ dB} \) \[80\] is introduced, it is observed by analysis of the still performing better than the two mode noise radar.

Note that although in these experiments the JPA is settle in a cryogenic environment, the target or the trip of the signal beam can be located at room temperature, without target. On the other hand, the JPA itself is very sensitive to noise. It must be keep at very low temperature of \(7 \text{ mK} \), in order to produce a vacuum respect to frequencies above \(4 \text{ GHz} \).

7.3. Covariant matrix for classical correlated noise radar and quantum correlated noise radar. The detector functions that Luong et al. considered are constructed from the in-phase and quadrature voltages of the signals 1 and 2. The signal 1 is send to probe the region of the space; the signal 2 is measured right after generation and stored using conventional methods. Let us consider the four dimensional vector

\[
x^\top := (I_1, Q_1, I_2, Q_2)
\]

When the two signals are generated, the covariant matrix has the form \[54\]

\[
E[x x^\top](0) = \begin{bmatrix} R_{11} & R_{12}(0) \\ R_{21}(0) & R_{22} \end{bmatrix}
\]

where each \(R_{ij}\) is a \(2 \times 2\) matrix. Assuming stationary signals, the block matrices \(R_{11}\) and \(R_{22}\) does not depend upon time. Hence after a time of evolution, the covariant matrix will be of the form

\[
E[x x^\top](t) = \begin{bmatrix} R_{11} & R_{12}(t) \\ R_{21}(t) & R_{22} \end{bmatrix}
\]

It is on the off-diagonal blocks \(R_{12}(t)\) and \(R_{21}(t)\) where the information about the absence or presence of the target is encoded.

This formalism is applied to the evolution under the 0-hypothesis and under the 1-hypothesis and it is applicable to analyse the QTMS-radar and for the TMN-radar. The details can be found in reference \[54\], but a suitable form of them will be provided below for further discussion.

7.4. Quantum radar prototype of Barzaranjeh et al. and experiments. The the general concept and technics behind the prototype of quantum radar discussed in \[5\] are related to the hybrid radar discussed in \[54\], specially in the generation of the quantum entangled source. The protocol of Barzaranjeh et al. has the following steps:

1. Two entangled microwave beams are generated directly from a JPA source.
2. The idler beam is measured using heterodyne detection and recorded right after its amplification. Meanwhile, after amplification the signal beam is send to probe the spacetime region where the target could be located.
(3) Classical digital filtering techniques are used in the detection and storage of the received beam. Both, heterodyne and homodyne detections were performed.

(4) Matched Filtering to compare the both signals.

For the classical light illumination, the procedure is the same, with the same conditions of temperature and coherent light illumination of the energy and power for the idler/signal generation than in the case of the quantum illumination. Several artificial sources of noise are added to simulate the noise introduced in the amplification process of the quantum illumination radar (introduced during amplification).

In this experiment, the target is located up to a 1 m fixed distance from the sending antenna at room temperature. The reflected signal is detected using also heterodyne detection. Then the two measurements are post-processed and used to calculate the signal to noise radio.

The JPA and amplification is done in cryogenic conditions. Indeed, before amplification, the generation of the microwave modes by the JPA is in a cryogenic container at \(7 \text{ mK}\). The two beams generated are at frequencies \(\omega_1 = 10.09 \text{ GHz}\) and \(\omega_2 = 6.8 \text{ GHz}\), the experiments are done at room temperature, in the sense that the signal is send to detect the target which is at room temperature.

In the experiment of Barzanjeh et at. [5], the demonstration was performed comparing quantum illumination against coherent state illumination with the same specifications and prepared under the same characteristics. This aspect differs from Luong et al. experiment, where the comparison was done with two mode squeezed noise radar prototype [54]. Also, in the experiment performed by Barzanjeh et at. there was a possible target in propagation space (the target was absent in [54]).

**Experimental results.** The experiment showed an enhancement in sensitivity when using the quantum entangled microwave radar respect to the coherent light of up to 3 dB in the SNR in the low intensity regime (less than 0.5 average photons per mode) respect to coherent light using heterodyne measurement. The enhancement is of 1 dB respect to classical coherent illumination when using homodyne measurements.

**Advantages respect to previous experiments.** In the experiment, the target is located at a fixed, known distance. However, the JPA allows for a modulation of the signal and idler frequencies within a narrow range, that could be used to provide a range variable using such a modulation in the detection. Using such a technique The location of the target has to be known only approximately. The analysis of this concept can be found in [16].

Since the enhancement of signal to noise ratio is higher for quantum radar than for coherent light, with an improvement up to a factor 4 dB, it was suggested in reference [5] the use of this protocol of quantum illumination in situations where the target range is approximately known and where non-invasive techniques are fundamental, for example, in biological and medical applications and as a short-range radar for security applications.

**7.5. Comparison of quantum noise radar performance with coherent illumination.** J. H. Shapiro has analyzed the theoretical performance of the protocol discussed in Luong et al. and Barzanjeh et al. and compared with coherent light illumination [71]. The key point in Shapiro’s analysis is the following. As we mentioned above, one difference between quantum illumination protocols and coherent
illumination is on the way the intensity can be distributed between the signal/idler pair of beams. While for quantum illumination, the beams need to have the same intensity (because the way they are generated as composed by entangled beams), in coherent illumination the relative intensities of the idler signal beams can be arbitrarily distributed. As a result of this relative intensity variable, one can have a weak signal beam classically correlated with a strong idler beam.

Let us consider the hybrid quantum radar protocol as the ones discussed in [54] or in [5]. Such hybrid radar systems are named by Shapiro quantum correlated radar, while the equivalents protocols at the classical level were called classical correlated radar. The correlation matrices showed an interplay between the intensities of the idler/signal beams that implies the possibility of outperform the sensitivity conditions achieved by a quantum noise radar by the classical noise radar. It is the above mentioned difference in the signal/idler beam intensity on how this can happen[71], since it is possible to construct coherent signals of the same characteristic than the signal beam of the quantum radar counterpart, while keep a stronger signal idler, due to the remain effect of the classical correlations in the coherent case when the intensity of the idler beam is large enough.

The argument is based on the analysis of the correlation matrices for QCR and CCR in [54]. The $M$ modes of the measurement (heterodyne measure) are a set of independent, identically distributed, complex valued random columns vectors, whose quadrature components have zero mean Gaussian distributions with covariant matrices as follow. For the correlated noise radar in the case of stationary signals, the correlation matrices are $4 \times 4$ symmetric matrices that do not depend on time parameter and such that act on four vectors, formally as

$$x^\top \equiv (a_{Rm}, \sqrt{N_S + 1}/\sqrt{G_A} a_{Im}),$$

where $G_A$ is the pre-amplifiers gain coefficient [71]. The correlation matrices can be re-written in the form (compare with the form discussed in [54]),

$$(7.3)$$

$$E_{\eta, \theta}^{QCN} [xx^\top] = \begin{bmatrix}
N_R + N_F & 0 & \sqrt{\eta N_S \cos(\theta)} & -\sqrt{\eta N_S \sin(\theta)} \\
0 & N_R + N_F & \sqrt{\eta N_S \sin(\theta)} & \sqrt{\eta N_S \cos(\theta)} \\
-\sqrt{\eta N_S \sin(\theta)} & \sqrt{\eta N_S \cos(\theta)} & 1 + \frac{N_F - 1}{N_S + 1} & 0 \\
\sqrt{\eta N_S \cos(\theta)} & \sqrt{\eta N_S \sin(\theta)} & 0 & 1 + \frac{N_F - 1}{N_S + 1}
\end{bmatrix}$$

for the quantum correlated noise radar, while for the classical correlated radar, the correlation matrix is of the form

$$(7.4)$$

$$E_{\eta, \theta}^{CCN} [xx^\top] = \begin{bmatrix}
N_R + N_F & 0 & \sqrt{\eta N_S \cos(\theta)} & -\sqrt{\eta N_S \sin(\theta)} \\
0 & N_R + N_F & \sqrt{\eta N_S \sin(\theta)} & \sqrt{\eta N_S \cos(\theta)} \\
-\sqrt{\eta N_S \sin(\theta)} & \sqrt{\eta N_S \cos(\theta)} & 1 + \frac{N_F}{N_I} & 0 \\
\sqrt{\eta N_S \cos(\theta)} & \sqrt{\eta N_S \sin(\theta)} & 0 & 1 + \frac{N_F}{N_I}
\end{bmatrix}$$

In these expressions, $\eta$ is the reflectivity coefficient. Its value is in the interval $0 \leq \eta \ll 1$. For the null hypothesis, the correlation matrices are determined by the condition $\eta = 0$ (absence of target). The phase $\theta$ is on the interval $0 \leq \theta \leq 2\pi$. $N_S$ is the average number of photons per mode in the signal beam; $N_I$ is defined analogously for the idler beam. $N_F$ is the noise figure. $N_R$ is given by the expression $N_R = \eta N_S + N_B$. It is such that $N_F \geq 1$ and it is equal to 1 in the ideal case.
As we remark before, \( N_I \) and \( N_S \) are independent variables. Also, it is \( N_F \) too. Because of this independence, there are several regimes where classical light outperforms quantum light:

- For an ideal detector with \( N_F = 1 \), in the limit \( N_I \to \infty \), the covariant matrices are identical and hence, the performance of the classical correlated noise radar and quantum correlated noise radar is identical.
- If \( N_F > 1 \), the classical correlation radar outperforms the quantum analogue when the condition

\[
1 + \frac{N_F - 1}{N_S + 1} > 1 + \frac{N_F}{N_I} \quad \text{meets.}
\]

This happens when the idler beam is of such intensity such that

\[
N_I > N_F \frac{N_S + 1}{N_F - 1}.
\]  

\((7.5)\)

7.6. **Discussion of the scope of Shapiro’s analysis.** The direct consequence from Shapiro’s analysis is that, under the conditions investigated, the scheme of hybrid quantum radar with heterodyne detection, cannot universally outperform a coherent light radar working with a signal of the same intensity and energy characteristics and under the same detection capabilities. The trick in Shapiro’s argument is to retain a bright enough idler, while the signal has the required properties.

This claim partially contrasts with the results of the experiments performed by [5] and in [54] demonstrating a modest improvement using entangled light respect to coherent light. Partially explaining such a discrepancy is that [54] did not compare the two model squeezed radar model directly with coherent light. On the other hand, in the experiment described in case of [5] they did such a comparison. They investigated both heterodyne and homodyne detection, but they used a low intensity idler. Hence, Shapiro’s analysis is not in contradiction with the outcome of such experiments. In addition, note that Shapiro’s analysis applies to heterodyne detection, while in [5] reported also experiment with homodyne detection, with a relatively modest increase in sensitive of 1 dB in the signal to noise ratio when using quantum entangled light respect to the strongest classical benchmark as is using coherent light as signal and homodyne detection.

Besides the discrepancies between the experimental results discussed in [5] and in [54] and the criticism raised by Shapiro in [71] an even if we accept the argument from Shapiro’s in the conditions that currently works, for specific tasks, hybrid quantum illumination working with heterodyne detection could bring technical advantage over coherent light protocols. Namely, the hybrid protocols are of practical interest when the use of coherent light for illumination or ranging is impractical. This could be the situation in security applications and in non-invasive scanning application. On the other hand, it can happen that the power of the idler is not possible to satisfy the condition \((7.5)\).

Furthermore, the use of other protocols for quantum illumination could avoid directly Shapiro’s argument.

8. **The quantum radar protocol of Maccone and Ren**

From the preceding discussion, it is clear that one of the main problems in the application of quantum illumination to radar is what we can call the range target problem. In particular, the experimental realizations described in [5] and in [54], but also in the case of Gaussian quantum illumination [53, 89, 25] require exact or approximate knowledge of the target location. Recently, Maccone and Ren have
suggested a different protocol to apply quantum entangled light to radar purposes in a way that corresponds to a quantum radar. We describe briefly below the Maccone-Ren theoretical protocol.

8.1. Maccone-Ren protocol for quantum radar. In Maccone-Ren’s protocol, quantum states with \( N \) entangled photons are prepared. For each individual state, all the \( N \) photons are sent to explore a region of spacetime possibly containing a non-cooperative point-like target. Thus the difference with quantum illumination protocols and hybrid protocols \([22, 71, 80, 4, 5, 54, 54]\) is that all the entangled photons are sent to explore the target and none is preserved as idler.

In order to introduce systematically these ideas, we use an analogous analysis as in Lloyd’s quantum illumination scheme.

A. Protocol for the radar using entangled light. The entangled state in Maccone-Ren’s protocol is an EPR-like state of the form

\[
|\psi_N \rangle \equiv \int d\omega d\vec{k} \psi(\omega, \vec{k}) \left( a^\dagger(\omega, \vec{k}) \right)^N |0\rangle,
\]

where \( a^\dagger(\omega, \vec{k}) \) is the creation operator of a photon with frequency \( \omega \) and transverse moment \( \vec{k} = (k_x, k_y) \); the propagation of the photons is along the \( z \)-direction. The function \( \psi(\omega, \vec{k}) \) is the biphoton structure function.

In what will follow, one assumes that the far-field approximation

\[
|\vec{k}_3|^2 = (k_x^2 + k_y^2 + k_z^2)^2 \gg (k_x^2 + k_y^2)^2 = |\vec{k}|^2
\]

holds good. After scattered by the point object, the joint probability to detect the \( N \) photons at times and transverse locations \( \{(t_j, \vec{r}_j)\}, j = 1, ..., N \) is given by an expression of the form

\[
p(\{(t_j, \vec{r}_j)\}_{j=1,...,N}) \propto \left| \langle 0 | \prod_j E^+(t_j, \vec{r}_j) |\psi_N \rangle \right|^2,
\]

where \( E^+(t, \vec{r}) \) is the electric field operator in the Heisenberg picture of dynamics at the transverse location \( \vec{r} \) and instant \( t \) where the photon is detected at the receiver and the scattering is due to a point object located at a transverse distance \( \vec{r}_p \). In the far field approximation, \( E^+(t, \vec{r}) \) is of the form

\[
E^+(t, \vec{r}) \propto \int d\omega d\vec{k} \int d\vec{r}_0 \delta(\vec{r}_0 - \vec{r}) a(\omega, \vec{k}) e^{i(\omega t - \omega \Delta t_0)} e^{i\vec{k} \cdot (\vec{r}_0 - \vec{r})},
\]

where \( a(\omega, \vec{k}) \) is the annihilation operator of the mode. The joint probability is then given by the expression

\[
p(\{(t_j, \vec{r}_j)\}_{j=1,...,N}) \propto \left| \tilde{\psi} \left( \sum_{j=1}^N t_j - Nt_0, \sum_{j=1}^N \vec{r}_j - N\vec{r}_p \right) \right|^2,
\]

where

\[
\tilde{\psi}(t, \vec{r}) = \int d\omega d\vec{k} \psi(\omega, \vec{k}) e^{-\omega t + \vec{k} \cdot \vec{r}}.
\]
Therefore, the expression for the joint probability of detecting the \( N \) photons at times \( t_j \) and transverse locations \( \vec{r}_j \) can be re-written in the form

\[
(8.3) \quad p(\{t_j, \vec{r}_j\}_{j=1,\ldots,N}) \propto \left| \psi\left( N \left( \frac{\sum_{j=1}^{N} t_j}{N} - t_0 \right) , N \left( \frac{\sum_{j=1}^{N} \vec{r}_j}{N} - \vec{r}_p \right) \right) \right|^2.
\]

This expression has several relevant consequences. First, it provides a method to determine the target range. From the detection times one can extract the target range,

\[
(8.4) \quad r_z = \frac{\sum_{j=1}^{N} (t_j - t_0)}{2 N c},
\]

where \( c \) is the speed of light. The transverse location of the target is similarly estimated to be given by the average transverse displacement relation,

\[
(8.5) \quad \vec{r} = \frac{\sum_{j=1}^{N} \vec{r}_j}{N}.
\]

These methodology is a generalization of the method of radar distance when using non-entangled light \((N = 1)\).

**B. Protocol for the radar that uses non-entangled light.** For the experiment where \( N \) individual, non-entangled photons are sent to explore and detect the target, the probability of detection a single photon at time \( t \) and transverse position \( \vec{r} \) is given by the expression of the form

\[
(8.5) \quad p(t, \vec{r}) \propto \left| \tilde{\psi}(t, \vec{r}) \right|^N.
\]

The time of detection transverse location are established by the expectation value of the corresponding coordinates.

**Enhancement of sensitivity using quantum entanglement respect to non-entangled light.** Let us assume that the distribution \( \tilde{\psi}(t, \vec{r}) \) is Gaussian. Then the comparison of the expressions \((8.3)\) and \((8.5)\) shows a reduction on the standard deviation by a factor of \( \sqrt{N} \) on the expected arrival time \( t \equiv \sum_{j=1}^{N} t_j \) and transversal displacement detection vector \( \vec{r} \equiv \sum_{j=1}^{N} \vec{r}_j \). Since the range of the object can be defined by the expression \((8.4)\), one can say that entanglement imprints a reduction in the range detection error of order \( \sqrt{N}/2 \) per measurement. Similar conclusion is established for the expected transverse displacement detection position \( \vec{r} \equiv \sum_{j=1}^{N} \vec{r}_j/N \).

When repeated many times the experiment, this advantage becomes enhanced by statistical independence.

**8.2. Practical issues implementing Maccone-Ren’s quantum radar protocol.** There are three relevant concerns in the practical implementation of this protocol. The first is the inherent difficulty in the generation of the entangled states required \((8.1)\).

Second, the randomness in the distribution and arrival in time for the \( N \) entangled photons implies an infinite time detection and infinite size of the detector. These two issues are considered in \([59]\) and cured by using partially entangled states. It is shown that, although the enhancement in precision for the measurement of the range and transverse displacement is not as high as when using the maximally
entangled states (8.1), the use of partially entangled states of the form

\[
|\phi\rangle := \int d\omega d\vec{k} \prod_j d\omega_j d\vec{k}_j \psi(\omega, \vec{k}) \gamma(\omega_j) \xi(\vec{k}_j) a^\dagger(\omega + \omega_j, \vec{k} + \vec{k}_j) |0\rangle.
\]

allows for a finite time of detection, finite size of the detection screen. Furthermore, the states (8.6) are easier to produce than the maximally entangled states (8.1) using spontaneous parametric down conversion methods, at least for states associated with two entangled photons [51, 64, 86]. Even if the gain is not as large as using entangled states of the form (8.1), the use of partially entangled states for quantum radar still has an advantage over classical photon state protocols [59].

The third issue in Maccone-Ren’s quantum radar protocol is related with the effect of thermal noise, since the entangled states used in the protocol are very sensitive to noise. Indeed, usually protocols in quantum metrology are very sensitive to noise. In the case of Maccone-Ren quantum radar protocol, the lost of one of the states associated with two entangled photons renders the other useless, since their detection is produced at random times and transverse locations. It has been suggested several strategies to solve this problem. Two of such suggestions are discussed in [59]. The first is to use the partially entangled states (8.6). Such a states are more robust against noise. The second is suggested by the protocols discussed in [35] and involve nested systems of entangled states. Both strategies reduce the effect of noise at the price of a reduction on the enhancement in precision by using quantum illumination.

9. Quantum illumination with Maccone-Ren’s quantum radar protocol

We describe below a new protocol for a quantum radar which is resilient to thermal noise. The protocol scheme is a combination of Lloyd’s quantum illumination protocol and Maccone-Ren’s quantum radar protocol. Further details will be developed in [30].

9.1. Lloyd’s quantum illumination using Maccone-Ren’s quantum radar protocol. Let us consider an EPR-like state of the form (8.1) with \(N = 3\). One of the photons will be keep as idler, while the remaining two are send to explore the region of spacetime where a possible target could be present. The claim is that quantum radar protocols based on this form of illumination/ancillary systems will have the combined advantages of being resilient to noise, have a noise enhancement effect in sensitivity as in quantum illumination and take advantage from Maccone-Ren’s protocol to obtain a procedure to evaluate the range and transverse displacement of the target. Hence the protocol is potentially the theoretical foundation for more realistic quantum radar schemes.

The initial entangled state of Maccone-Ren type that we will consider first is of the form

\[
|\psi_3\rangle \equiv \int d\omega d\vec{k} \psi(\omega, \vec{k}) \left( a^\dagger(\omega, \vec{k}) \right)^3 |0\rangle.
\]

This state is split into two, one state describing an idler photon and the other state describing two signal photons. The signal photon states are sent to explore a given region in space. Repeating this procedure, we obtain from an initial beam
two beams (idler and signal beams). The quantum state describing the system composed by the two different beams is the statistical mixture

\[ \rho = |\alpha_1|^2 \rho_1 + |\alpha_2|^2 \rho_2, \]

where \( \rho_1 \) is the density matrix of the pure state

\[ |\psi_1\rangle \equiv \int d\omega d\vec{k} \psi(\omega, \vec{k}) a^\dagger(\omega, \vec{k})|0\rangle \]

and \( \rho_2 \) is the density matrix of the pure state

\[ |\psi_2\rangle \equiv \int d\omega d\vec{k} \psi(\omega, \vec{k}) \left(a^\dagger(\omega, \vec{k})\right)^2 |0\rangle; \]

\( \alpha_1 \) and \( \alpha_2 \) are in general complex coefficients satisfying \(|\alpha_1|^2 + |\alpha_2|^2 = 1\).

The first that we note here is that the biphoton wave function structure \( \psi(\omega, \vec{k}) \) is modeled to simulate the noise environment. One can assume first that noise is described by the state \( \rho_0 \) given by the relation (C.1) as in Lloyd’s theory. Consequently, in order to apply an analogous protocol as Lloyd’s quantum illumination (see Appendix C for a detailed treatment of Lloyd’s argument) and to simplify the treatment of our protocol, we consider instead the following states for the idler and signal states. For the full signal-idler system, the state is of the form

\[ |\psi_3\rangle \equiv \frac{1}{\sqrt{M}} \left(a^\dagger(\omega, \vec{k})\right)^3 |0\rangle \]

after decoherence has happened. Once the decoherence between the idler and the signal is lost, the states are the following. For the idler beam, the photons are described by the same class of states than in Lloyd’s theory, namely,

\[ |\psi_1\rangle \equiv \frac{1}{\sqrt{M}} \sum_{k=1}^{M} a^\dagger(\omega, \vec{k})|0\rangle. \]

For the signal system, the state \( |\psi_2\rangle \) is the pure state

\[ |\psi_2\rangle \equiv \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \left(a^\dagger(\omega, \vec{k})\right)^2 |0\rangle. \]

Now we can easily provide estimates of the probability of false positive and probability of right detection for Lloyd’s protocol of quantum illumination using Maccone-Ren’s like states (9.2)-(9.3). Let us assume direct photo detection. Then the criteria that we can follow for a positive detection is formulated as follows:

*Criterion for positive detection:* We declare that the target is present if two photons in the spectrum range of the signal are detected back within an established time window at the same time than a idler photon is detected, in a joint measurement.

The notion of time window detection must be adapted to the particular situations. It cannot be too large to confuse uncorrelated photons, but also not to short to miss entangled photons.

In a noise environment, the detection of two noise photons with the same frequency or the detection of only one of the photons of the initial entangled pair signal can mislead us. For the first case, one can estimate the probability of a false positive for protocols based in classical illumination or based upon Maccone-Ren. This is estimated by the probability of a false positive and will be discussed below and it shows, in a similar way as in Lloyd’s quantum illumination, enhancement of...
quantum illumination respect to non-entangled illumination.

**A. Illumination with non-entangled light.** Similarly as in LLoyd’s theory (see Appendix C), when the target is not there and the illumination is done with non-entangled light, the quantum states is described by a density matrix of the form

\[ \rho_0 \approx \left\{ (1 - MN_B)|0\rangle \langle 0| + N_B \sum_{k=1}^{M} |a^\dagger(\omega, \vec{k})|0\rangle \langle 0|a(\omega, \vec{k})| \right\}, \]

This is the noise state used in Lloyd’s theory (see the expression (C.1) in Appendix C). The probability of false positive can be read directly from the structure of the state and, by the criteria of detection discussed above, it is the probability of detecting two photons in the same time window. Therefore, it is given by

\[ p_0(+) = (N_B)^2. \]

On the other hand, when the target is there, and under the same assumptions, the state is given by the density matrix

\[ \rho_1 = (1 - \eta)\rho_0 + \eta \tilde{\rho} \]

\[ \approx (1 - \eta) \left\{ (1 - MN_B)|0\rangle \langle 0| + N_B \sum_{k=1}^{M} |a^\dagger(\omega, \vec{k})|0\rangle \langle 0|a(\omega, \vec{k})| \right\} + \eta \tilde{\rho}, \]

where \( \tilde{\rho} \) stands for the state describing the signal. In our case, it is the pure state \( |\psi\rangle_2 \). The probability of simultaneous detection of two photons in presence of a target is

\[ p_1(+) = ((1 - \eta)N_B + \eta)^2 \]

The signal to noise ratio in quantum illumination with Maccone-Ren’s protocol when the illumination is performed with non-entangled light is given by the expression

\[ SNR_{CIMR} = \frac{p_1(+)}{p_0(+)} = \frac{((1 - \eta)N_B + \eta)^2}{(N_B)^2}. \]

One observes that this signal to noise ratio is given by the square of the signal to noise ratio \( SNR_{QI} \) in Lloyd’s theory (expression (C.8) in Appendix C). Therefore, using the above criteria for detection and using classical illumination, reduces considerably the SNR respect to the usual criteria of positive only if a photon is detected.

**B. Illumination with entangled light.** The idler state is a mixed state whose density matrix \( \tilde{\rho}_1 \) is of the form

\[ \tilde{\rho}_1 = |\alpha|_1^2 \frac{1}{M} \sum_{k=1}^{M} |a^\dagger(\omega, \vec{k})|0\rangle \langle 0|a(\omega, \vec{k})|. \]

The state of the signal, after decoherence and after a possible interaction with the target, will be denoted by \( \tilde{\rho}_2 \) and will be a non-entangled system.

When there is not target there, the state noise-idler is described by the density matrix \( \tilde{\rho}_0^\dagger \) and it has the form

\[ \tilde{\rho}_0^\dagger \approx \left\{ (1 - MN_B)|0\rangle \langle 0| + N_B \sum_{k=1}^{M} |a^\dagger(\omega, \vec{k})|0\rangle \langle 0|a(\omega, \vec{k})| \right\} \otimes \tilde{\rho}_1. \]
The probability of a false positive is the probability to attribute to the presence of the target the detection of two simultaneous returned photons. Within the scope of the approximations that we are considering, such a probability is independent of the details of the signal state and given by the expression

\[ p_e^0(+) = |\alpha|_1^4 \left( \frac{N_B}{M} \right)^2. \]  

(9.8)

This relation shows an enhancement respect to the analogous relation in Lloyd’s quantum illumination protocol (eq. (C.10) in Appendix B.9), by a factor \(|\alpha|_1^4\) (which depends on the number of photons entangled, in this case \(N = 3\)) and on the equivalent probability of false positive when using non-entangled light, given by expression (9.4), where in this case, the enhancement is given by the factor \(|\alpha|_2^4\).

When the target is there, the state after decoherence and interaction signal-target is of the form

\[ \tilde{\rho}_1 = (1 - \eta) \cdot \tilde{\rho}_0^\ast + \eta \tilde{\rho}_2, \]

where here \(\tilde{\rho}_2\) is the two photon signal state after decoherence happens, assuming that decoherence has totally annihilate the quantum entanglement. By a similar argument as in Lloyd’s theory, the probability of detection using entangled light signal states when the target is there for one trial is

\[ p_e^1(+) = \left( (1 - \eta) |\alpha|_1^2 \frac{N_B}{M} + \eta |\alpha|_2^2 \right)^2. \]  

(9.9)

To evaluate the signal to noise ratio, we take the values \(|\alpha|_1^2 = |\alpha|_2^2 = 1/2\), in which case we have

\[ SNR_{QIMR} = \frac{p_e^1(+)}{p_e^0(+)} = \left( \frac{M}{N_B} \right)^2 \left( (1 - \eta) \frac{N_B}{M} + \eta \right)^2, \]

which is the square of the signal to noise ratio obtained for Lloyd’s quantum illumination in the analogous case, equation (C.15). Expression (9.10) reflects two different enhancements: the first from the use of quantum entangled states respect to non-entangled states; the second by using Maccone-Ren respect to Lloyd’s quantum illumination.

**Remark 9.1.** The above evaluation of the probabilities of detection assumes joint measurements between the signal and idler photons, as in the case of Lloyd’s quantum illumination. The direct implementation of this mechanism is to keep the idler beam alive. Other techniques are related with matched filtering as in hybrids systems discussed before.

### 9.2. Determination of the range and transverse position using quantum illumination with Maccone-Ren protocol.

If the target is small enough, the criterion for the detection of a stealth target is that, as discussed before, the target is declared detected if two individual photons with the same frequency and momenta are detected within the same detection time window. Under the further assumption that there is only a pair of photons on fly, the detection of a pair of correlated photons provides a measure also of \(t - t_0\) in an analogous way to Maccone-Ren’s theory and hence, it determines the range by the expression

\[ \vec{r} \equiv \frac{\sum_j \vec{r}_j}{2}. \]  

(9.11)
The measurement of the location of the two photons determines the transverse location of the target as the average location of the photons arrivals.

The above strategy to determine the range could also be applied to Lloyd’s quantum illumination protocol, but the precision of the method is smaller than when it is applied to the protocol of quantum illumination using Maccone-Ren type states. The idea of applying the method to Lloyd’s quantum illumination complemented with other methods, has been discussed in detail by Durak, Jam and Dindar [22] and is in concordance with the discussion in the work of Maccone-Ren [59] and in concordance with the conjecture presented in [23] on a general hierarchy on quantum strategies in quantum enhancement in presence of noise. We will discuss Durak-Jam-Dindar protocol briefly below. Note that the net effect of the two photon signal states is to mark with an extra-reinforcement in the correlations, the signals coming from the scattering with the target. Furthermore, at least two photon signal states are necessary to determine the transverse position.

Current experimental demonstrations for quantum radar [54, 5] are based in hybrid schemes, where the light source is entangled, but the detection is based upon classical protocols. For these protocols, the methodology discussed above for the quantum illumination with Maccone-Ren’s quantum states can also be implemented.

9.3. Short discussion of the method. The main problem in quantum illumination combined with Maccone-Ren protocol is the generation of the required entangled states in the form (9.1) or the states proposed in Maccone-Ren with \( N = 3 \). Currently, there are techniques to create two-photon positively momentum correlated entangled states [51, 64, 86], but for the protocol in question, states with three entangled photons, not necessarily momentum correlated, are required. The essential point of the method presented here, as in Lloyd’s quantum illumination and that differs from Maccone-Ren protocol, is that all the three photons are correlated in energy and in time. The generation of three photon states for quantum radar is technical problem, specially in the microwave regime required for target detection in open atmosphere.

A different possibility for the generation of the desired states could be to quantum states where two photons are positively correlated in momentum (they will be part of the signal beam), but the third photon is not positively correlated (these states will constitute the idler beam). One way to achieve this is by using a sequence of two spontaneous parametric down conversions. The first one is of the form \( \gamma \rightarrow \gamma_1 + \gamma_0 \), where \( \gamma_1 \) will determine the photons of the idler beam. Then \( \gamma_0 \) is used as the initial state in a second spontaneous parametric down conversion \( \gamma_0 \rightarrow \gamma_2 + \gamma_3 \) to generate the two positive momentum entangled photons \( \gamma_2, \gamma_3 \) that will serve for the signal. The use two spontaneous parametric down conversions will introduce a reduction in the efficiency. Note that although the second parameter down conversion will break the entanglement \( \gamma_1 - \gamma_0 \), this is a fact that is in any case un-avoidable in quantum illumination. We do not observe this will limit the procedure of using two Spontaneous parametric down conversions to generate the states \( (\gamma_1, \gamma_2 - \gamma_3) \), where \( (\gamma_2, \gamma_3) \) are entangled. The pair \( (\gamma_2, \gamma_3) \) and the photon \( \gamma_1 \) are correlated in time of generation and in frequency. In the case that the generation of the Maccone-Ren type states is by means of double parametric down conversion, then one has the possibility to reduce around \( 1/2 \) the final frequency of the signal photons respect
to usual optical parametric down conversion generation. This could benefit the
application of the techniques discussed above. But will not be enough to generate
microwave entangled photons. The possibility of generation of such photon states
is discussed in the Appendix D.

To use quantum illumination with Maccone-Ren protocol for radar purposes, the
signal beam must be generated in the microwave regime, in order to be applicable
in radar technology. However, the original entangled state $|\psi_3\rangle$ does not need
to correspond to a in the microwave regime, because existing current frequency
conversion methods (either electro-optomechanical converter [4] can be applied.
The application of such methods will reduce the efficiency drastically. On the other
hand, advances in quantum sensing in the terahertz spectrum [47] are potentially
could potentially be extended to the microwave regime. How these methods reduce
the quantum-illumination with Maccone-Ren state should be analyzed in detail. Also,
the use of JPA in this protocol appears very interesting and deserves attention.

Another technical problem in the implementation of the protocol that we are
proposing, starting with the criteria of detection stated above, the idler photon
needs in principle to be stored and keep alive the idler beam, in order to perform a
quantum measurement when the two pairs of beams arrive, since it is fundamental
to keep track from the time correlation. To have a detection, the idler photon
must be such that at time $t-t_0$ is located in a position equivalent to the distance
$2\vec{r} = 2 \sum_{j} \vec{r}_j$. This is because at that time, the two photon signal scattered with
the target arrives to the detector. This is the original proposal. However, one
can envisage hybrid methods, where matched filtering techniques are using and the
idler is not need to be keep alive from the beginning. As for quantum illumination,
these methods, discussed in section 4 will reduce the enhancement of quantum illumination.

10. OTHER PROPOSALS AND DEMONSTRATIONS OF QUANTUM RADAR CONCEPT

The above protocols and prototypes for quantum radar do no exhaust all the
proposals for quantum radar discussed in the last years. In this section we will
discuss in some detail another proposal for quantum radar that address the range
problem. Other proposals that have recently appeared and that also offer solutions
for the range problem are discussed in [27], where the explode the properties of
mixed squeezed states to enhance and in [39], although the second of them only
address the presence/non-presence of the target.

10.1. The proposal for quantum radar of Durak, Jam and Dindar. In the
work of Durak, Jam and Dindar [22] it has been discussed a protocol for quantum
radar which is in principle capable to provide the range of a target without need
of previous approximate knowledge of the position of the target. The protocol is a
realization of Lloyd’s quantum illumination protocol, where the signal and the idler
photons are correlated in frequency, polarization and detection time. Each signal
photon of the entangled pair is sent to explore the region of interest. The scattered
signal is received first through a telescope, then an avalanche single photon detection
is used to detect the photon and the event is registered. The distribution of timing of
detection is $D_R(t+\tau)$. The idler series of photons are detected using single avalanche
detector and time recorded just after generation with a time distribution of the
form $D_I(t)$. The protocol assumes than only entangled photons in polarization will
The cross correlation is of the form
\[
ccf(\tau, \gamma, f_d) = K \int_0^T D_I(t) D_R(t + \tau, \gamma, f_d) dt,
\]
where \(\gamma = \frac{1}{\sqrt{1 - \beta^2}}\) is the relativistic \(\gamma\)-factor, and \(f_d\) is the Doppler shift of the received signal. By using a cross-correlation technique between the timing of arrival and the Doppler frequency shift, the authors discuss a methodology to obtain the location of the target and its speed. The target is located by the time of the pick of the cross-correlation, while the velocity is found by measuring the Doppler frequency shift at the time \(f_d\) of the correlations.

One of the theoretical benefits is that the protocol of Durak, Jam Dindar can be used at any signal frequency, provided the detection requirements are available. This is especially significative and difficult in the case of microwave detection on earth ground.

Durak et al. reported an experimental demonstration of the protocol in the form of a quantum radar prototype. The signal/idler entangled photons are obtained by parametric down conversion such that the pump beam is at \(\lambda_0 = 402\, \text{nm}\) and the down conversion is to wavelengths \(\lambda_1 = 780\, \text{nm}\) and \(\lambda_2 = 842\, \text{nm}\). After passing through a filtering and collimation processes, the idler and the signal are also correlated in polarization. Then the signal is sent to a telescope of aperture 50 mm to explore the location (in the experiment the range of the target is fixed) of an object composed by black anodized Aluminum. The scattered photons are detected back by the telescope, filtered and the arrival time registered using single photon counting detector. The photo detector used allowed to count \(5 \times 10^5\) pairs of photons per second.

In the reported experiment, the object is situated at a fixed distance from the telescope. The time delay in the arrival of the signal photon respect to the idler is theoretically modeled by a curve of the form \(f(r) = b + a R^{-2}\), where \(R\) is the target range. This model is based upon the principle that the power detected by the telescope is of the form \(P \sim D^2/R^2\), where \(D\) is the aperture of the telescope. The fit parameters \(a, b\) depends on the telescope aperture. The paper of Durek et al. reports a good performance of the experimental quantum radar prototype up to a range of 200 mm. This maximal range highly depends upon the telescope aperture, the detector time jitter, the power of the source of entangled photons. Improvement in these parameters will imply an extension of the range. In the report, there is no a direct comparison with an equivalent radar based on non-entangled illumination.

10.2. Critical view of Durak et al. protocol. Several comments are in order.

First, we would like to remark that in the experimental demonstration described in the paper there was no comparison with the equivalent classical prototype. Hence one cannot claim enhancement from such demonstration. Indeed, it is not shown also theoretically, the origin of the enhancement due to the use of entangled light. The argument reported by Durak et al. relies on theoretical considerations on quantum illumination. But we have learn that there is a huge gap between ideal benefits of quantum illumination and the real gain. Indeed, the protocol used is purely classical, since it could be applied in a very close way to a classical source of illumination.

The proposal from Durak et al. is based in a weak notion of entanglement, similar to the he working definition of entanglement used by Chang et al. [13].
These notions are related with the concept of quantum discord, briefly discussed in the Glossary and enhancement in entangled breaking channels, according to Sacchi [68, 69]. Also, in order to implement Durak et al. protocol for a realistic radar, the signal must work on the microwave regime, where the difficulties of detection of individual photons is higher. Electro-optomechanical conversion could be a method to be implemented, but at the expenses of reducing the efficiency and introducing further noise.

Increasing the number of detection pairs per second implies that the window for photo detection must be smaller. Available avalanche photodiode detectors (APD) with a coincidence window (that we can take as the time jitter of the detector) is 82 ps. This scale allows for detection of $10^{10}$ photon pairs per second. The amount of pairs detected can be increase if the APD can have a shorter time jitter. Increasing the capability to detect more pairs per second implies that powerful sources can be used for the illumination and hence, the range of the radar can be expanded. Currently, APD with a time jitter of approximately 10 ps are being investigated [67], which will imply an increase in the performance of the quantum radar protocol. For APD detectors with a time jitter of order 10 ps, the range will correspond up to 300 m. However, these figures are still far from the upper bound on the correlation time $t_c \sim 10^{-1} ps$ in the spontaneous down conversion [11].

In conclusion, although the method proposed by Durak et al. is very interesting from a practical point of view, it is necessary of further studies to determine the existence of enhancement respect to classical illumination. We think that such experimental studies are feasible with un-expensive equipment.

11. CONCLUSION AND OUTLOOK: QUANTUM RADAR, FROM THEORY TO REALISTIC POTENTIAL APPLICATIONS

Quantum radar, in an ample sense, refers to several protocols and prototypes that, either theoretically or in a preliminary experimental phase, aims to explode quantum entanglement properties for enhancement in target detection sensitivity. Quantum radar is a very active area of research. However, the realization of a realistic quantum radar prototype remains elusive. Indeed, the acknowledged large gap between the theoretical expectations and the preliminary experimental results could be a source of critics and skepticism on the possibilities of real enhancement of quantum radar respect to conventional radar in practical situations. This criticism has been raised especially in the case of quantum illumination [71, 78], since quantum illumination raised very high expectations that in the following years have amply relaxed. Indeed, it is remarkable that in the scientific literature, there is no reference to a real genuine radio detector and ranging prototype project based on quantum illumination. But the benefits that a quantum radar technology could have over conventional technologies is still and with justice, a great motivation to pursue and investigate such avenues.

There is a clear classification of theoretical quantum radar protocols. In first instance, there are the protocols where quantum entanglement has been lost during the round trip. These class of protocols and experiments include quantum illumination and hybrid protocols [52, 72, 80, 4, 5, 54, 4] and the protocols and prototypes discussed by Maccone and Ren [59, 22] and the combined idea discussed in section 9 and that will be developed elsewhere in a forthcoming paper [30].
The physical realization of the protocols poses new problems at several levels. The main problems are the following:

- The target range problem. As we have discussed, this problem affects to any quantum illumination protocol based solely in Lloyd’s quantum illumination and its variants and also to other protocols (example, to quantum interferometric radar). We think that this is the most pre-eminent problem to prevents the realization of a realistic quantum radar based on quantum illumination protocols. In principle, the protocols and prototypes discussed by Maccone and Ren [59], Durak et al. [22] and the author [30] discussed briefly in section 9 provide different solutions for the range problem.
- Loss of intensity in the signal beam by attenuation processes. This is a problem that affects all types of quantum radar protocols. In the context of quantum illumination has been examined recently by Sorelli et al. [78]. The conclusion reached by their analysis is that, at the microwave regime, the power used in signalling is not enough for target detection, in the regime $N_s \ll 1$ of quantum advantage.
- Idler storage. In order to make joint measurements or in general, to correlate the idler with the received signal beam, the information provided by the idler must be handle. Two ways have been discussed: 1. The use of very efficient idler storages, including quantum memories [4], 2. The use of classical digital methods and matched filtering [5, 54, 22]. Currently, these methods introduce a radical reduction in maximal target range and in the enhancement.
- Generation of entangled beams. The usual processes for generation of quantum entanglement is spontaneous parametric down conversion. While this is a procedure quite useful for optical frequencies, in the range of microwave frequencies presents several problems. Direct methods like JPA generation require a very low cryogenic temperature [5, 13, 14, 54], while frequency conversion [4] is currently highly inefficient. This is a point that can be greatly improve if new methods to generate quantum entangled states are discussed briefly in Appendix D.
- Detection. For quantum illumination, there is an ideal detector, with a maximal possible gain of $6 \, dB$, but real efficiencies are drastically reduced respect to the ideal case [59]. Single photo counting are used in several proposals and experiments [53, 25, 22] [50]. But the photo counting rate limits the range of the detection significatively [22].

The problems pointed out above currently prevent the realization of an ultimate, universal quantum radar. However, different protocols can potentially have advantages respect to classical illumination protocols and prototypes in specific applications, specially in noise, entanglement breaking channels, non-invasive applications and in space target detection.

In particular, the target range problem is specially relevant in quantum illumination and related protocols and needs to be handle specially. On the other hand, when the range is approximately known, quantum illumination can be used in noise resilient, non-invasive scanning systems. The protocols where the target range problem is solved, as in Maccone-Ren protocol [59], Durak et al. [22] and the protocol discussed in section 5; the main problems are related with losses...
and with detection. A way to overcome the detection limitations could be use of classical digital techniques as in hybrid protocol and prototypes in [54][5]. Other related protocols that have recently appear are [27] and [39].

The second general form of quantum radar that has been investigated are protocols where quantum entanglement is preserved during the round trip, specifically in quantum interferometric radar protocols [49][10][33][32][76]. The problems discussed above for entanglement breaking quantum channels protocols also affect quantum interferometric quantum protocols. Losses and noise are even more dramatic, because the phenomena of quantum decoherence and attenuation. Based on these considerations, quantum decoherence effects will provide a very short maximal target range detection, as it is discussed for instance in the Glossary. This contrast with the claims in [49], section 5.2 and in [76].

The difficulties in the practical implementation of a quantum radar can be seen as an opportunity to apply and develop quantum technology in the field. Both, the generation of required signal/idler beam and the difficulties in the detection of the incoming signal are research areas that require advances and new developments to satisfy the demanded specifications of quantum radar, in any of the forms discussed in this review. Hence, the development of quantum radar implies for the future the continuation of a strong synergy with parallel quantum technological developments.

APPENDIX A. GLOSSARY

In this Glossary we have collected and discuss in some detail several theoretical notions of quantum mechanics, quantum optics and quantum metrology that we hope will facilitate to the non-expert the reading of the main part of the manuscript. The collection cannot be in any instance comprehensive and complete and depends on the taste and interest of the authors, but we hope that contribute in the understanding of the text. Most of the instance are compiled from other sources, except the discussion of the application of the model of decoherence for localization of the wave function, which we have follow an original contribution to the topic.

1. Pure and mixed states in quantum mechanics. In the standard formulation of quantum mechanics [19], the state of a physical system $S$ is completely specified when a normalized element $\psi$ of the Hilbert space ($\mathcal{H}$ is given. In Dirac bra-ket notation, normalized means that $\langle \psi | \psi \rangle = 1$. According to the Copenhagen interpretation of quantum mechanics, a pure state provides the most complete description of the system: no further experiments with identically prepared states can provide a more precise description.

A mixed state of a physical system is an state that is not pure. This means that, given the description of the state, one can at least ideally conceive an experiment that, together with the current description of the state, the new experiment will provide a more complete information (hence, description) of the state. Since the complete description of systems is in terms of normalized elements of the Hilbert space, the description of a mixed state will be given in terms of them, but cannot be associated to a particular normalized element of $\mathcal{H}$. Indeed, the description of a mixed state is associated to a collection $\{(\psi_i, p_i)\}$, where $\psi_i \in \mathcal{H}$ such that

$$p_i > 0, \quad \sum_i p_i = 1.$$  

(A.1)
$p_i$ is the probability of the state $\psi_i$. The collection $\{(\psi, p_i)\}$ is usually interpreted as describing an statistical ensemble, where some elements are described by $\psi_1$, some elements by $\psi_2$, etc... with corresponding probabilities $p_1, p_2, \ldots$. Then the expected value of an observable is

$$\langle X \rangle = \sum_i p_i \langle \psi_i | X | \psi_i \rangle = Tr[\hat{\rho} X],$$

(A.2)

$$\hat{\rho} = \sum_i p_i | \psi_i \rangle \langle \psi_i |.$$

(A.3)

This relation introduce the density matrix. A pure state can also be described by a density matrix of the form $\hat{\rho}_\psi = | \psi \rangle \langle \psi |$.

In occasions, it is more convenient to interpret the density matrix as the primary object provided. In this case, the characterization of a pure state $\psi \in \mathcal{H}$ by means of a density matrix $\hat{\rho}_\psi$ is that

$$Tr[\hat{\rho}_\psi^2] = 1.$$  

(A.4)

For a pure state, the purity is maximal and equal to $\mathcal{P}[\hat{\rho}] = 1$. The maximally mixed state is found for ensembles where either $p(\psi) = 0$ or $\pi(\psi) = a$, a constant independent of $\psi$ in the decomposition of $\hat{\rho}$.

A second measure of mixing is von Neumann entropy,

$$\mathcal{S} = -Tr[\hat{\rho} \ln \hat{\rho}].$$

(A.5)

For a state of the form (A.3) subjected to the condition (A.4), Von Neumann entropy is zero for a pure state and is maximal for a maximally mixed state.

The basic properties of the density matrix can be found for instance [31]. General treatments can be found in [42, 81].

2. Schmidt’s decomposition. Given an entangled state of the form

$$| \psi \rangle = \sum_{i,j} \psi_{ij} | \Phi_i \rangle \otimes | \eta_j \rangle,$$

where $\{| \Phi_i \rangle \otimes | \eta_j \rangle \}$ determines a basis of the product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then there is an alternative decomposition of $\hat{\rho}$ in terms of product spaces such that

$$| \psi \rangle = Y_1 | \xi_1 \rangle \otimes | \vartheta_1 \rangle + | \psi_1 \rangle$$

such that the coefficient $| Y_1 \rangle$ is maximal. Following this procedure, for finite dimensional Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ there is a decomposition of $| \psi \rangle$ in product states of the form

$$| \psi \rangle = \sum_{n=1}^r Y_n | \xi_n \rangle \otimes | \vartheta_n \rangle,$$

(A.6)

where $r \leq min(\dim(\mathcal{H}_1), \dim(\mathcal{H}_2))$. The minimal value $r = 1$ occurs when $| \psi \rangle$ is a product space. Note that the set $\{| \xi_n \rangle \otimes | \vartheta_n \rangle \}$ in the Schmidt’s decomposition depends upon the initial state $| \psi \rangle$.

3. Classical and Quantum Chernov’s bounds. The classical result of H. Chernov concerns the problem of given a set of measurements $\{X_i, i = 1, 2, \ldots, N\}$ in the measure of $N$ identically distributed random variables, to determine which is the distribution between two possible options $p_1$ and $p_2$ with the minimal possible
error. The probability of error $p_e$ is the probability to obtain the hypothesis 0 under the condition that the correct hypothesis is 1 plus the probability of choosing hypothesis 1 under the condition that the correct hypothesis is 0.

Chernov solved this problem in the case of asymptotically large trials $N \to \infty$ \cite{17}, that showed that the probability of error $p_e$ in discriminating among $p_1$ and $p_2$ decrease exponentially with the number of trials $N$,

$$P_e \sim \exp(-CN).$$

The exponent $C$ is known as the Chernov’s distance, Chernov’s entropy or Chernov bound. It is determined in Chernov’s theory (given by the form of Theorem 1 in the original Chernov’s work \cite{17}).

Chernov’s theorem is a seminal result in classical decision theory. This type of problems appears also in quantum mechanical issues. Specially, the discrimination problem between quantum states or quantum canals, is an example. Therefore, it is interesting to have a quantum version of the above result from Chernov theorem. There are many different generalizations at the quantum level. The formulation of the quantum Chernov bound makes uses of a positive operator valued measure, that consists of two operators $E_0, E_1$ such that $E_0 + E_1 = I$ and $E_i \geq 0$. Given two possible quantum states with corresponding a priori assigned probabilities $\pi_1, \pi_2$, the error probability is given by the expression

(A.7) $$p_e = \pi_0 Tr[E_1 \hat{\rho}_0] + \pi_1 Tr[E_0 \hat{\rho}_1].$$

The minimum of this error probability is denoted by $p_{e,min}$. The basic problem to be solved is to understand how the error behaves for $N$ experiments where the state can be either in state $\hat{\rho}_0$ (hypothesis $H_0$) or in the state $\hat{\rho}_1$ (hypothesis $H_1$). The resolution determines the quantum Chernov bound, which states that \cite{2}

(A.8) $$p_{e,min,N} \sim \exp(-N C_q),$$

where the exponent is given by the expression

(A.9) $$C_q = \lim_{N \to \infty} -\frac{\log(P_{e,min,N})}{N}.$$ 

This expression of the quantum Chernov bound reduces to the classical result \cite{2}.

The quantum Chernov bound is in general difficult to be determined. Other related notions are used in the literature, specifically in the theory of quantum illumination \cite{80}. Also related, is the notion of quantum channel discrimination \cite{68, 69}, which is on the basis for the initial studies on quantum illumination \cite{52}.

4. Quantum decoherence. Quantum decoherence is the process by which quantum coherence is lost by interaction between the quantum initially coherent system and the environment. Originally motivated by the problem of measurement in quantum mechanics \cite{91, 93}, the theory of quantum coherence has become of fundamental relevance for quantum computation and in general, the implementation of quantum technologies, due to the limitations on the stability of quantum computers that decoherence poses (see for instance \cite{60}, chapter 8).

The simplest way to introduce quantum decoherence is by look at von Neumann’s projection postulate when applied to spin 1/2-spin system coupled to a detector coupled to a environment \cite{93}. If the system is measured to be in state $|+\rangle$, then the detector will in state $|d+\rangle$, while if the system is in state $|−\rangle$, then the detector
will be in state $|d-\rangle$. Therefore, a generic state of the system-detector will be of the form

$$|\psi\rangle = \alpha |+\rangle \otimes |d+\rangle + \beta |-\rangle \otimes |d-\rangle.$$  

This system is entangled. It must also be always coupled to an environment, represented by $|En\rangle$. Thus the total system-detector-environment is of the form

$$|\Psi\rangle = |\psi\rangle \otimes |En\rangle = (\alpha |+\rangle \otimes |d+\rangle \otimes |En\rangle + \beta |-\rangle \otimes |d-\rangle \otimes |En\rangle).$$

This state is also entangled. Tracing out respect to the environment state, an operation that can be read as passing from an individual to an ensemble system and ignoring the details of the environment state, the density matrix corresponding to the pure state $\hat{\rho}_c = |\Psi\rangle\langle\Psi|$ passes to a density matrix of the form

$$\hat{\rho}_r = Tr_{En} |\Psi\rangle\langle\Psi| = |\alpha|^2 |+\rangle\langle+| \otimes |d+\rangle\langle d+| + |\beta|^2 |-\rangle\langle-| \otimes |d-\rangle\langle d-|,$$

which is the reduced state in von Neumann postulate and is not entangled. The main idea behind this example is that the above type of transitions $\hat{\rho}_c \Rightarrow \hat{\rho}_r$ can be described by models of interaction between the system-detector state and the environment. Indeed, the idea of decoherence generalizes the above processes to the general case of suppression of quantum interference phenomena by interaction with the environment.

Several models for quantum decoherence have been investigated in the literature \[51\]. We consider briefly here the model of a point particle (harmonic oscillator) with coordinate position $x$ interacting with an environmental scalar field $\phi$. Although such a dynamical system does not directly describes the problem of decoherence effects and noise interaction in the propagation of free photons in atmosphere, it illustrates the structure of the environment-system interactions. The interaction Hamiltonian is of the form

$$H_{int} = \epsilon x \frac{d\phi}{dt}.$$  

The effective equation of motion, found after solving the exact Schrödinger equations for the field and the point particle corresponding to a particle in superposition at $x$ and at $x'$ is of the form

$$\dot{\hat{\rho}}(x, x') = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(x, x')]$$

$$- \gamma (x - x') \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \hat{\rho}(x, x')$$

$$- \frac{2m\gamma k_BT}{\hbar^2} ((x - x')) \hat{\rho}(x, x'),$$

(A.10)

where $\hat{H}$ is the Hamiltonian of the particle, $\gamma = \frac{\epsilon^2}{4m}$ is the relaxation rate, $k_B$ the Boltzmann constant, $T$ the temperature and $\Delta x = x - x'$ is the displacement between the two position superposition locations at $x$ and at $x'$. In equation (A.10), the term

$$-\frac{i}{\hbar} [\hat{H}, \hat{\rho}(x, x')]$$

is an unitary evolution equation, the term

$$\gamma (x - x') \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \hat{\rho}(x, x')$$

is a dissipative term, and

$$- \frac{2m\gamma k_BT}{\hbar^2} ((x - x')) \hat{\rho}(x, x')$$

is a diffusion term.
is a relaxation term, while the term
\[ \frac{2m\gamma k_BT}{\hbar^2} ((x - x')) \hat{\rho}(x, x') \]
is the responsible for the decoherence.

The ratio between the relaxation time \( \tau_R = \gamma^{-1} \) and the decoherence time \( \tau_D \) provides a direct comparison between the characteristic time scales of the processes of relaxation and decoherence. For the above model the ratio is
\[ \tau_D/\tau_R = \left( \frac{\hbar}{\Delta x \sqrt{2mk_BT}} \right)^2. \]  
(A.11)

The comparison for a point electron provides a ratio \( \tau_D/\tau_R \sim 10^{-13} \) when the distance on the localization is of order \( \Delta x = 1 \text{cm} \), indicating that the effects of quantum decoherence acts much faster than the effects of relaxation (attenuation) at macroscopic distance scales.

Even if the above model describes a massive particle interacting with a scalar field, one could expect that the general features shown by it will be repeated for entangled photon systems. In particular, the decoherence effects are notoriously larger as the separation \( \Delta x \) among the two photons increases, indicated by the inverse square dependence \( 1/(\Delta x)^2 \) in the expression (A.11). Indeed, we can estimate the decoherence time using the model above in the following way. The relaxation time has the form
\[ \tau_R = \frac{\varrho}{2m^2}, \]
for a massive harmonic oscillator, where \( \varrho = \epsilon^2/2 \) is a constant (viscosity) given by the interaction strength \( \epsilon \) in the Hamiltonian \( H_{\text{int}} \). We first generalize this expression to general systems and assume that provides a first order approximation, obtaining the expression
\[ \tau_R = \frac{\varrho \epsilon^2}{2E}, \]
where \( E \) stand for the energy. Then the relation (A.11) can be re-written as
\[ \tau_D = \frac{1}{\varrho k_BT} \frac{\hbar^2}{\Delta x^2}. \]  
(A.12)

In this expression and as first approximation, we observe that for any system with energy \( E \), in an environment of a thermal bath with temperature \( T \) (by bosonic scalar field) and with distance on the localization of order \( \Delta x \), then:

- The decoherence time \( \tau_D \) reduces with the inverse of the viscosity coefficient \( \varrho \) (inverse square of the interaction coupling \( \epsilon \)).
- \( \tau_D \) is independent of the energy. Assuming that the harmonic oscillator system describes a pair of photon state (assuming that the expressions for the relaxation time and decoherence time can be generalized for photons of energy \( E \)), then in the above approximation the decoherence time is independent of the frequencies.
- \( \tau_D \) reduces with the inverse of the square of the distance localization \( \Delta x \).
- \( \tau_D \) reduces with the inverse of the temperature of the thermal bath \( T \).

The dynamical systems of pairs entangled photons propagating in the atmosphere also suffer from quantum decoherence, associated with the entanglement in
quantum number. Indeed, it is natural to think that for entangled systems, decoherence effects appear faster than for non-entangled systems. Thus the above analysis is in support of the view expressed in quantum illumination and quantum radar studies, that the idler-signal systems loses the quantum entanglement due to coherence very fast. However, note that decoherence do not affect, however, correlation in polarization, as long range entanglement photon experiments demonstrate [90].

5. Quantum discord. Quantum discord is a measurement of how much a density matrix corresponds to a classical state [65, 93]. A general formulation of the notion of quantum discord can be found in [65], but for the purposes of this paper, the following considerations requires to introduce less quantum mechanical formalism.

The mutual information of two systems $A_1$ and $A_2$ is given by the expression of the entropies associated to the corresponding density matrices,

$$I(A_1, A_2) := \mathcal{S}(A_1) + \mathcal{S}(A_2) - \mathcal{S}(A_1, A_2).$$

$\mathcal{S}(A_1, A_2)$ is the joint entropy of the two systems. For a classical system,

$$\mathcal{S}(A_1, A_2) = \mathcal{S}(A_2) + \mathcal{S}(A_2|A_1),$$

where $\mathcal{S}(A_2|A_1)$ is the conditional entropy. One can define then the classical mutual information

$$J(A_1, A_2) = \mathcal{S}(A_1) + \mathcal{S}(A_2) - (\mathcal{S}(A_2) + \mathcal{S}(A_2|A_1)).$$

One then defines the difference

$$\delta(A_1|A_2) = I - J = (\mathcal{S}(A_2) - \mathcal{S}(A_2|A_1)) - \mathcal{S}(A_1, A_2).$$

In quantum physics, the state collapse to one of the eigenstates of the measured observable. Hence in order that $I - J$ describes a measure for quantum correlation, it is necessary to maximize respect all possible quantum projective measurements on the system $A_2$. Then the quantum discord respect to a basis of eigenvectors associated to all the eigenvalues of measuring the system $A_2$ is given by [93]

$$\delta\langle k_2 \rangle A_1|A_2) = \mathcal{S}(A_2) - \mathcal{S}(A_1, A_2) + \min_{\{k_2\}} \mathcal{S}(A_2|\{k_2\}). \tag{A.13}$$

For an entangled bipartite system, the quantum discord is positive in all basis $\{k_2\}$; for the reduced matrix $\hat{\rho}_r$, the quantum discord is zero in an appropriate basis [93].

The relevance of the notion of quantum discord for quantum radar is the following. As we have discussed in the main text, quantum illumination provides a quantum entangled enhancement protocol for quantum sensing where quantum entanglement is necessarily loss by the action of the noise environment. Usually, this behavior is understood as an example of entanglement can enhance the distinguishably of entanglement-breaking channels [52], in the context of Sacchi’s theory [68] and in practical terms, encoded in the properties of the correlation matrix in the case of Gaussian illumination [?]. But it turns out that the enhancement when the quantum entanglement has been destroy can be seen as a consequence of an underlying quantum correlation, present as a residual quantum correlation. Indeed, there is a quantitative correlation between entangled enhancement of sensitivity in quantum illumination protocol and quantum discord, as discussed in [84].

6. Quantum Heisenberg limit and standard shot limit in quantum interferometry. We will follow the review of Giovannetti, Lloyd and Maccone [36].
A typical scheme for quantum interferometry is based upon Mach-Zehnder apparatus. A light of beam is divided by a beam splitter into a reflected $B$-beam and a transmitted $A$-beam. Both beams pass through different path. If there is no phase difference between the paths $\varphi = 0$, all the photons are collected at path parallel to $A$ (port $D$); if there is a phase of $\varphi = \pi$ among the path, all the photons are detected at port $C$. In the intermediate situation, a proportion of $\cos^2(\varphi/2)$ of the photons will pass through the port $D$ and a proportion $\sin^2(\varphi/2)$ will pass through the port $C$. The quantity $\cos^2(\varphi/2)$ is obtained as the statistical average

$$\frac{\sum_{j=1}^{N} x_j}{N}$$

of the independent stochastic variables $\{x_j\}$, where each $x_j$ takes values at $\{0, 1\}$. There is independence since the photons are correlated between each other. Because each $x_j$ is independent, the error of the average is the average of the errors,

$$\Delta \left( \frac{\sum_{j=1}^{N} x_j}{N} \right) = \sqrt{\frac{\sum_{j=1}^{N} \Delta^2 x_j}{N}}$$

and since all the distributions are identical $\Delta x_j = \Delta x$. Hence one has

$$\sum_{j=1}^{N} \Delta x_j = \Delta x \sqrt{N}, \quad \text{(A.14)}$$

The dependence $1/\sqrt{N}$ on the precision of the phase $\varphi$ is known as the shot noise limit. The same precision is obtained if the distributions are applied to $N$ individual identical photons, instead of ensembles.

Careful designed quantum procedures can surpass the precision imposed by the shot noise limit. If one use the states

$$|\Psi\rangle = \frac{1}{2} (|N_+\rangle_A |N_-\rangle_B + |N_-\rangle_A |N_+\rangle_B),$$

where $A, B$ indicate the ports and $N_{\pm} \equiv (N \pm 1)/2$. If $a, b, c, d$ are the annihilation operators at the ports $A, B, C, D$, measuring the observable

$$M \equiv a^\dagger a - c^\dagger c = (a^\dagger a - b^\dagger b) \cos(\varphi) + (a^\dagger b + b^\dagger a) \cos(\varphi)$$

provides a method to calculate $\Delta \varphi$. First, for the states \text{(A.15)} it holds that

$$\langle M \rangle = -N_+ \sin^2(\varphi),$$

and its variance is

$$\Delta^2 M = \cos 2(\varphi) + N_+^2 \sin^2(\varphi).$$

For the phase variance one has

$$\Delta \varphi = \Delta M / \partial \langle M \rangle / \partial \varphi.$$

For $\varphi \approx 0$ this results with a scaling $\Delta \varphi \sim 1/N$. Other quantum procedures avoid the constrain $\varphi \sim 1/N$. This precision $1/N$ is known as the Heisenberg limit. It can be shown from fundamental principles, namely, the Heisenberg uncertainty relations, that the Heisenberg limit is an absolute limit in quantum mechanical systems [9, 66].

Precision in the parameter estimation is similarly enhanced by the use of entangled states. An excellent introduction to this topic is again [36].
Appendix B. Quantum states that appear in quantum radar protocols

In this appendix, we discuss several types of entangled states that are of relevance for quantum radar protocols.

1. Coherent states. For the quantum mechanical oscillator, a system whose algebra is determined by the relation $[\hat{a}, \hat{a}^\dagger] = 1$ and by the Hamiltonian
$$\hat{H} = \hbar \omega \hat{a}^\dagger \hat{a},$$
the coherent state $|\alpha\rangle$ is defined to be the eigenstate of the annihilation operator $\hat{a}$ with eigenvalue $\alpha$,
$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle.$$
(B.1)
The fundamental characteristic of the state $|\alpha\rangle$ is that the corresponding expectation value of the Hamiltonian coincides with the value of the energy for the classical mechanical oscillator,
$$\langle \alpha | \hat{H} | \alpha \rangle = E_{cl} = \hbar \omega |\alpha|^2.$$ In another equivalent characterization, the coherent state $|\alpha\rangle$ is such that the operators $\hat{a}$ and $\hat{a}^\dagger$ are statistically independent. Namely, for a coherent state, it holds that
$$\langle \alpha | (\hat{a}^\dagger)^n \hat{a}^m | \alpha \rangle = (\langle \alpha | \hat{a}^\dagger | \alpha \rangle)^n (\langle \alpha | \hat{a} | \alpha \rangle)^m.$$ (B.2)
This relation is a characterization of a coherent state.

For quantum states describing the state of a radiation oscillation, the same characterization (B.2) can be applied. However, some changes in notation and interpretation are in order. First, the characterization (B.1) is substituted by
$$\hat{a}_k |\alpha_k\rangle = \delta_{kk'} \alpha_k |\alpha_k\rangle.$$ (B.3)
It implies the characterization
$$\langle \alpha_k | (\hat{a}_k^\dagger)^n \hat{a}_k^m | \alpha_k \rangle = (\langle \alpha_k | \hat{a}_k^\dagger | \alpha_k \rangle)^n (\langle \alpha_k | \hat{a}_k | \alpha_k \rangle)^m.$$ (B.4)
Second, the $k$-coherent states live in the subspace expanded by the number states of the $k$-mode of the electromagnetic field. This implies that
$$|\alpha_k\rangle = \sum_{n=0}^{\infty} b_n |n\rangle_k.$$ The algebra of the harmonic oscillator implies that $b_n = b_0 \alpha^n / \sqrt{n!}$. The constant $b_0$ is fixed by normalization, obtaining the expression of a coherent state for a $k$-mode as
$$|\alpha_k\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_k.$$ (B.5)
$k$ indicates the mode of the electromagnetic radiation.

If $n$ is the outcome of measuring the photon number operator, then the distribution of probability $P(n)$ is a Poisson distribution, namely, $P_k(n) = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$, where $\bar{n} = \langle \alpha_k | \hat{a}_k^\dagger \hat{a}_k | \alpha_k \rangle$.

Coherent states is commonly associated to the state of light in lasers. This is discussed in section 5.3 of the book of Garrison-Chiao [31].

2. Continuous variable Gaussian states. A continuous quantum variable system is a system described by a Hilbert space such that any generation system is labeled by a continuous variable. In quantum radar, special roles is played by
bosonic Gaussian states whose states are described by continuous variables (see for instance [11] [83] and chapter 5 in [42] for different reviews of Gaussian states in quantum information theory). The Hilbert space of $N$ bosonic identical modes is a product space of the form $\mathcal{H} = \prod_{k=1}^{N} \mathcal{H}_k$. The annihilation and creation operators of the modes

$$\hat{b} := (\hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2, \hat{a}_2^\dagger, \ldots, \hat{a}_N, \hat{a}_N^\dagger)^\top$$

satisfy the commutation relations

$$[\hat{b}_i, \hat{b}_j] = \Omega_{ij}, \quad i, j = 1, \ldots, 2N. \quad (B.6)$$

The symplectic matrix $\Omega_{ij}$ is the $2N \times 2N$ skew-symmetric matrix

$$\Omega := \sum_{k=1}^{N} \omega = \begin{pmatrix} \omega \\ \vdots \\ \omega \end{pmatrix}, \quad (B.7)$$

where

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (B.7)$$

The quadrature operators are defined by the expressions

$$\hat{q}_k = \hat{a}_k + \hat{a}_k^\dagger, \quad \text{quad} \quad \hat{p}_k = 2\imath \left( \hat{a}_k^\dagger - \hat{a}_k \right), \quad k = 1, \ldots, N. \quad (B.8)$$

Denoting by

$$\hat{x} := (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_N, \hat{p}_N)^\top,$$

one has the commutations relations

$$[\hat{x}_i, \hat{x}_j] = 2\imath \Omega_{ij}, \quad i, j = 1, \ldots, 2N. \quad (B.8)$$

Given a quantum system described by a density matrix $\rho$, the first momenta of the distribution are defined by the expression

$$\bar{x} := \langle \hat{x} \rangle = \text{Tr} (\hat{x} \hat{\rho}),$$

while the variance matrix is the $2N \times 2N$ matrix defined by

$$V_{ij} := \frac{1}{2} \langle [\Delta \hat{x}_i] [\Delta \hat{x}_j] \rangle \quad (B.8)$$

Gaussian states are a type quantum state that are fully characterized by the first momenta and the covariance matrix. Examples of Gaussian states are coherent states discussed above and two mode squeezed quantum states discussed later, as well as a type of EPR states [83].

3. States for Lloyd’s quantum illumination. In Lloyd’s theory of quantum illumination, each of the entangled quantum states for idler-signal system is of the form

$$|\psi\rangle_{sa} = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} |k\rangle_s \otimes |k\rangle_a, \quad (B.9)$$

where the number of modes of the state $M = TW > 1$ and the state $|k\rangle_s (|k\rangle_a)$ represents the either vacuum or one photon state for the signal(idler) in the $k$-mode.
One observes the entanglement mode by mode. A fundamental aspect of the theory is that $M \gg 1$. This technical aspect is on the basis of the entanglement.

In contrast, the states for coherent quantum illumination in Lloyd’s theory are of the form

$$|\psi\rangle_s = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} |k\rangle_s.$$

4. Entangled states for Gaussian quantum illumination. In Tan et al. [80], the entangled quantum states for quantum illumination are Gaussian states. Each $T$ seconds long transmission comprises $M = WT > 1$ signal-idler modes, where $W$ is the SPDC phase-matching bandwidth. For each of the $m$ temporal modes, the state is of the form

$$|\psi_m\rangle_{sa} = \sum_{n=0}^{+\infty} \sqrt{\frac{N_S^n}{(1+N_S)^{n+1}}} |n\rangle_{sm} \otimes |n\rangle_{am},$$

where, differently from Lloyd’s quantum illumination, $|n\rangle_{sm}$ (resp. $|n\rangle_{am}$) represent the state containing $n$ photons in the mode $m$ of the electromagnetic field. The states (B.10) are zero first momenta Gaussian states, whose covariant matrix is of the form

$$V_{SI} = \begin{pmatrix} NS + 1 & 0 & \sqrt{NS(NS + 1)} \\ 0 & NS + 1 & 0 \\ \sqrt{NS(NS + 1)} & 0 & NS \end{pmatrix}.$$

For comparison, the $m$th temporal mode of Tan et al. theory is in the coherent state of the form

$$|\psi_m\rangle_s = \sum_{n=0}^{\infty} \sqrt{\frac{N_S^n}{n!}} e^{-N_S} |n\rangle_{sm}.$$

5. Squeezed states. Several forms of squeezed states appear in different protocols of interferometric and quantum radar and quantum illumination. Therefore, a brief introduction to the general setting of squeezed states is in order. With this aim, we follow here the exposition in [31], chapter 15. For one mode with creation operator $\hat{a}^\dagger$ and annihilation operator $\hat{a}$, the quadrature operators associated to are of the form

$$\hat{X}_0 = \frac{1}{2} (\hat{a}^\dagger + \hat{a}), \quad \hat{Y}_0 = \frac{1}{2} (\hat{a}^\dagger - \hat{a}).$$

From the relation $[\hat{a}, \hat{a}^\dagger] = I$, it follows the commutation relation

$$[X_0, Y_0] = \frac{i}{2}.$$

This corresponds to an uncertainty relation of the form

$$\Delta X_0 \Delta Y_0 \geq \frac{1}{4}.$$
Physically, $\hat{X}_0$ represents an electric field, while $\hat{Y}_0$ is the electric field. However, arbitrary combinations
\begin{equation}
\hat{X} = \hat{X}_0 \cos \beta + \hat{Y}_0 \sin \beta, \quad \hat{Y} = -\hat{X}_0 \sin \beta + \hat{Y}_0 \cos \beta,
\end{equation}
for real $\beta$. For the pair $\{X, Y\}$, the relations (B.13), (B.14) also hold. For particular combinations, the phase $\beta$ can be fixed relative to a local oscillator, in what it is called an homodyne detection. The operators $\{X, Y\}$ are the quadrature operators.

For a coherent state, the variance of the quadrature operators
\begin{align*}
\Delta^2 \hat{X} = \Delta^2 \hat{Y} &= 1/4
\end{align*}
and the product of uncertainties is $\Delta \hat{X} \Delta \hat{Y} = 1/4$.

A state $\rho$ is said to be squeezed along the quadrature $\hat{X}$, if the variance respect to $\rho$,
\begin{align*}
\Delta^2_\rho \hat{X} := \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2
\end{align*}
satisfies $\Delta^2_\rho \hat{X} < 1/4$. Similarly, a squeezed state along the quadrature $\hat{Y}$ is defined.

Therefore, one mode squeezed states are such that the uncertainty in a given quadrature is lower than for the coherent state. A price to pay is that along the complementary quadrature, the uncertainty is higher, in a way that Heisenberg uncertainty principle (B.14) is full-filled.

Usually, squeezed states are creating either by four wave mixing generation in a non-linear optical medium with a $\chi^{(3)}$ generation, or by a strongly by down conversion in a $\chi^{(2)}$ crystal. In both cases, the generator Hamiltonian is of the form
\begin{align*}
H_{\text{gen}} = \hbar \Omega_p \left( (\hat{a}^\dagger)^2 - H_c \right).
\end{align*}
This suggest a method to parameterize squeezed states by introducing the squeezed operator. For one mode states, the squeezed operator is of the form
\begin{align*}
S(\zeta) = e^{\frac{1}{2}(\zeta^* \hat{a}^\dagger - \zeta \hat{a})^2},
\end{align*}
$\zeta = r \exp(2i\phi)$ is the complex squeezing parameter. The single mode squeezed vacuum is then
\begin{align*}
|s\rangle = S(\zeta) |0\rangle.
\end{align*}
Squeezed coherent states are of the form
\begin{align*}
|\zeta; \alpha\rangle = S(\zeta) D(\alpha) |0\rangle,
\end{align*}
where $D(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$.

The formalism can be efficiently generalize to multi-mode states. The multi-mode squeezing operator is of the form
\begin{align*}
S(\tilde{\zeta}) = \prod_k \exp \left( \frac{1}{2} \sum_k \left( \zeta_k \hat{a}_k^2 - \zeta_k^* (\hat{a}_k^\dagger)^2 \right) \right)
\end{align*}
the two mode squeezed coherent vacuum states are of the form
\begin{align*}
|\zeta; \alpha\rangle = S(\tilde{\zeta}) D(\tilde{\alpha}) |0\rangle,
\end{align*}
where $D(\tilde{\alpha})$ is the multi-mode version of the displacement operator.

The advantage in squeezed states for quantum metrology rest on the fact that one of the quadratures can be measured with a priori lower sensitivity than coherent light. This is useful in beating the short noise quantum limit and in quantum lithography.
APPENDIX C. ENHANCEMENT OF SENSITIVITY IN LLOYD’S QUANTUM ILLUMINATION: AN ILLUSTRATIVE EXAMPLE

In the following lines we discuss in detail Lloyd’s theoretical protocol [52]. We partially follow the exposition described in [49]. As before, situation 0 means that the target is not there, while when the target is there, the situation is labeled by 1. We use here the notation introduced in sub-section 5.1.

A. Non-entangled light illumination. When the light used for experiments is described by non-entangled photons, the density matrix of the system idler-signal-noise, when the target is not there (hypothesis 0) is

$$\rho_0 \approx \{ (1 - MN_B)|0\rangle\langle 0| + N_B \sum_{k=1}^{M} |k\rangle_n\langle k|_n\}$$

where $|k\rangle_n$ stands for a noise photon mode. Hence the probability of a false positive is

$$p_0(+) = N_B,$$

while the probability to be correct in the forecast that the target is not there is

$$p_0(−) = 1 - p_0(+) = 1 - N_B.$$

If we repeat the experiment $m$ times, the probability of a false positive is

$$p_0(+, M) = (N_B)^M.$$

If the target is there (hypothesis 1), then the density matrix is given by

$$\rho_1 = (1 - \eta)\rho_0 + \eta \tilde{\rho}$$

$$\approx (1 - \eta) \left\{ (1 - MN_B)|0\rangle\langle 0| + N_B \sum_{k=1}^{M} |k\rangle_n\langle k|_n\right\} + \eta |\psi\rangle_s\langle \psi|_s,$$

where $|\psi\rangle_s$ stands for the state describing the signal, that one can assume first is a pure state and $\eta$ is the reflective index. It follows that the probability to measure the arrival of photon is

$$p_1(+) = (1 - \eta)N_B + \eta$$

and that consequently, the probability of false negative is

$$p_1(−) = 1 - p_1(+) = 1 - ((1 - \eta)N_B + \eta) = (1 - \eta)(1 - N_B).$$

The signal to noise ratio is given by the expression

$$SNR_{QI} = \frac{p_1(+)}{p_0(+)} = \frac{(1 - \eta)N_B + \eta}{(N_B)}.$$

B. Entangled light illumination. Let us now consider that the illumination is made using entangled states. For the case when there is no target there, the density matrix is given by the expression

$$\rho_0 \approx \left\{ (1 - MN_B)|0\rangle\langle 0| + N_B \sum_{k=1}^{M} |k\rangle_n\langle k|_n\right\} \otimes \left( \frac{1}{M} \sum_{k=1}^{M} |k\rangle_A\langle k|_A\right).$$
where $\sum_{k=1}^{M} |k\rangle_{A} \langle k|_{A}$ is the state of the idler. The state
\[
\rho_0 = \left\{ (1 - MN_B)|0\rangle\langle 0| + N_B \sum_{k=1}^{M} |k\rangle_n \langle k|_n \right\}
\]
is the state that will describe the absence of the target. It determines the probability distributions to detect one photon due to noise only. The modes determining the idler $k = 1, \ldots, M$ are selected to coincide with the modes of the noise. In this context, it is remarkable that the false positive probability for one individual detection,
\[
 p_0^c(+) = \frac{N_B}{M},
\]
is dramatically reduced with the number of modes $M$. This was first highlighted by S. Lloyd in his seminal work \[52\]. The probability of forecasting correctly the absence of the target is given by the probability of the complement,
\[
 p_0^c(-) = 1 - \frac{N_B}{M}.
\]
Note that when the experiment is repeated a number $m$ of times in an independent way, the probability of a false positive after detecting $m$ independent photons is
\[
 p_0^c(+, m) = \left( \frac{N_B}{M} \right)^m.
\]

In the case that the target is there, for entangled states, the system idler-noise-signal is described by a density matrix of the form
\[
 \rho_1^e = (1 - \eta) \cdot \rho_0^c + \eta \rho_s,
\]
where $\rho_s$ is the density matrix of the signal photon system. From this expression, one can extract the probability of detecting the target is
\[
 p_1^e(+) = (1 - \eta) \frac{N_B}{M} + \eta.
\]
The probability of no detection (interpreted as a false negative) is of the form
\[
 p_1^e(-) = 1 - p_1^e(+) = (1 - \frac{N_B}{M}) (1 - \eta).
\]
When applied $m$ independent experiments, the probability of right detection is
\[
 p_1^e(+, m) = \left( (1 - \eta) \frac{N_B}{M} + \eta \right)^m.
\]

For the case of false negative,
\[
 p_1^e(-, m) = 1 - p_1^e(+) = (1 - \frac{N_B}{M})^m (1 - \eta)^m.
\]
\[
 \text{SNR}^e_{QI} = \frac{p_1^e(+) / p_0^c(+)}{p_0^c(+)} = \left( \frac{M}{N_B} \right)^2 \left( 1 - \eta \frac{N_B}{M} + \eta \right)^2,
\]

Further details of the enhancement of sensitivity using Lloyd’s protocol can be found summarized in \[52\] and in \[19\], section 5.5.3.

Remark C.1. It is remarkable that the expressions \[C.13\] and \[C.14\] are independent of the details of the state $\rho_s$. The initial treatment in Lloyd’s work was to consider $\rho_s$ to correspond to the pure entangled state idler-signal \[B.9\]. However, due to a rapid decoherence process by interaction with the noisy media, this is unrealistic. Instead, the state to be considered for $\rho_s$ is the reduced matrix obtained
by decoherence. Nevertheless, the results for the probabilities \( C.14 \) - \( C.13 \) remain the same.

From the above formulae and comparing the probabilities of false positive and detection using quantum enhancement respect to classical light, one observes a clear enhancement in sensitive when using quantum illumination, as advanced by Lloyd [52].

Appendix D. Non-linear optical processes relevant for quantum radar protocols

The theory of non-linear optics necessary for quantum radar concepts is related with both, the generation of entangled states and in the theory of receivers for quantum illumination protocols. The notions that follow are extensively treated in the quantum optics in the literature, for instance in the book from Garrison and Chiao [31]. However, here we squeeze the required treatment to the minimum necessary for the applications involved in quantum radar.

Our starting point is the relation between the displacement vector \( \vec{D} \) and the macroscopic electric field \( \vec{E}_i \) in classical electrodynamics,

\[
\vec{D}_i = \varepsilon_0 \vec{E}_i + \mathcal{P}_i, \quad i = 1, 2, 3,
\]

where \( \vec{E}_i \) are the components of electric field and \( \mathcal{P}_i \) is the polarization of the media. Averaging the charge density distribution, for non-dispersive media, the polarization vector is of the form

\[
\mathcal{P}_i = \varepsilon_0 \left[ \chi^{(1)}_{ij} \mathcal{E}_j + \chi^{(2)}_{ijk} \mathcal{E}_j \mathcal{E}_k + \chi^{(3)}_{ijkl} \mathcal{E}_j \mathcal{E}_k \mathcal{E}_l + \ldots \right].
\]

The constants \( \chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \ldots \) are the non-linear susceptibilities tensors. When the media is dispersive, the susceptibilities depend upon the frequencies.

One approach to the quantum theory of electrodynamics with non-linear susceptibility media is to formulate the general Hamiltonian corresponding to each of the above terms. In this way, the Hamiltonian of the electric in a non-isotropic media (that consists of a large voxel of volume \( V \)) is of the form

\[
H_{\text{em}} = H^{(2)} + H^{NL} = H^{(2)} + H^{(3)} + H^{(4)} + \ldots
\]

where each of the Hamiltonian terms are of the form

\[
H^{(2)} = \sum_{k,s} \hbar \omega_{k,s} \hat{a}_{k,s}^\dagger \hat{a}_{k,s},
\]

\[
H^{(3)} = \frac{1}{V^{3/2}} \sum_{k_0,x_0,k_1,s_1,k_2,s_2} C(k_0 - k_1 - k_2) \delta_{\omega_0,\omega_1+\omega_2} g^{(3)}_{n_0,s_1,s_2} (\omega_1, \omega_2) \left[ \hat{a}_{k_1,s_1}^\dagger \hat{a}_{k_2,s_2}^\dagger \hat{a}_{k_0,s_0} - H.C. \right],
\]
\[ H^{(4)} = \frac{1}{\sqrt{2}} \sum_{k_{0}k_{1},k_{2},k_{3}} \mathcal{C}(k_{0} - k_{1} - k_{2} - k_{3}) \delta_{\omega_{0},\omega_{1} + \omega_{2} + \omega_{3}}, \]
\[ \cdot g^{(4)}_{0k_{0}k_{1}k_{2}k_{3}}(\omega_{1},\omega_{2},\omega_{3}) \left[ \hat{a}^\dagger_{k_{1}s_{1}} \hat{a}^\dagger_{k_{2}s_{2}} \hat{a}_{k_{3}s_{3}} \hat{a}_{k_{0}0} + H.C. \right] \]
\[ + \frac{1}{\sqrt{2}} \sum_{k_{0}k_{1},k_{2},k_{3}} \mathcal{C}(k_{0} + k_{1} - k_{2} - k_{3}) \delta_{\omega_{0} + \omega_{1} + \omega_{2} + \omega_{3}}, \]
\[ \cdot f^{(4)}_{0k_{0}k_{1}k_{2}k_{3}}(\omega_{1},\omega_{2},\omega_{3}) \left[ \hat{a}^\dagger_{k_{1}s_{1}} \hat{a}^\dagger_{k_{2}s_{2}} \hat{a}_{k_{1}s_{1}} \hat{a}_{k_{0}0} + H.C. \right], \]

and so on. The \( k \)-variables are the wave number vectors of the waves and \( s \)-variable the polarizations. In these expressions, \( g^{(3)}_{0k_{0}k_{1}k_{2}k_{3}}(\omega_{1},\omega_{2},\omega_{3}), g^{(4)}_{0k_{0}k_{1}k_{2}k_{3}}(\omega_{1},\omega_{2},\omega_{3}), f^{(4)}_{0k_{0}k_{1}k_{2}k_{3}} \) are the third order and four order coupling strength and are proportional to the non-linear \( \chi^{(2)} \) and \( \chi^{(3)} \) susceptibilities of the classical theories, respectively. Note that we have considered the case of dispersive media, where susceptibilities could depend upon the frequencies.

The quantum effective Hamiltonian corresponds to the quantum version of the average classical description. In such average description, the matching conditions

\[
\begin{align*}
\omega_{0} &= \omega_{1} + \omega_{2}, \\
\omega_{0} &= \omega_{1} + \omega_{2} + \omega_{3}, \\
\omega_{0} + \omega_{1} &= \omega_{2} + \omega_{3},
\end{align*}
\]

appear as a consistent requirement for the slow-varying enveloping fields, after a long time interaction between the electromagnetic field and the crystal. The requirement that one needs for these constraints to hold in the quantum theory is that the Hamiltonian must be invariant under time translations. Furthermore, for large crystals, also under spatial translations, which implies the constrains

\[ C(k) \sim V \delta(k) \]

Then the general rules of quantum mechanics leads to energy-momentum conservations, in concordance with the classical complete phase matching conditions,

\[
\begin{align*}
(D.1) & \quad \omega_{0} = \omega_{1} + \omega_{2}, \quad k_{0} = k_{1} + k_{2} \\
(D.2) & \quad \omega_{0} = \omega_{1} + \omega_{2} + \omega_{3}, \quad k_{0} = k_{1} + k_{2} + k_{3} \\
(D.3) & \quad \omega_{0} + \omega_{1} = \omega_{2} + \omega_{3}, \quad k_{0} + k_{1} = k_{2} + k_{3}.
\end{align*}
\]

In practice, the coupling constants are obtained experimentally for a given phenomenological process. Also, for each process, the effective Hamiltonian is a restriction from \( H \) to the corresponding piece.

D.1. **Three photon interactions.** The Hamiltonian piece \( H^{(3)} \) is responsible for the processes as spontaneous parametric down conversion and up frequency conversion. In spontaneous parametric down conversion, the phase matching relation is of the form \( \omega_{p} = \omega_{1} + \omega_{2} \). Typically, a laser beam pumps an anisotropic crystal with a non-vanishing second order susceptibility \( \chi^{(2)} \). Examples of such a crystals are lithium niobate (LiNbO\(_{3}\)), potassium titanyl phosphate (KTiOP\(_{4}\)) or ammonium dihydrogen phosphate (NH\(_{4}\))(H\(_{2}\)PO\(_{4}\)). Generically, the properties of the beams depend upon the details of the cut of the crystal and the initial conditions of the beams.
The three-photon Hamiltonian is of the form

\[ H^{(3)} = \frac{1}{\sqrt{3}/2} \sum_{k_{0s0}, k_{1s1}, k_{2s2}} g^{(3)} \mathcal{C}(k_0 - k_1 - k_2) \hat{a}^{\dagger}_{k_1s1} \hat{a}^{\dagger}_{k_2s2} \hat{a}_{k_0s0} + H.C., \]

where the matching conditions \[(D.1)\] are understood to hold.

The Hamiltonian \(H^{(3)}\) is time reversible. The Hamiltonian \[(D.4)\] leads to two second order non-linear optics processes. The first one is the so-called spontaneous down conversion (or spontaneous parametric down conversion), where a high frequency photon beam heats the crystal and far away from the system, two photon beams emerge, such that the phase matching conditions \[(D.1)\] holds good. The pair of photons created by such methods are correlated in time of creation and in polarization. Furthermore, in SPDC the initial pump field is typically a coherent state, described by a continuous momenta variable. Hence one speaks of continuous wave Spontaneous Parametric Down Conversion (cw SPDC).

Spontaneous down conversion constitutes one of the most common methods to generated entangled beams at optical frequencies and easier to implement experimentally. The reasons for this is that it does not require vacuum conditions and is a highly directional generator of pairs, where the two photons produce are emitted in opposite sides of a thin cone rainbow surrounding the initial photon beam. Furthermore, there is no need of cryogenic conditions for the production of the entangled pairs. However, because the parametric down conversion happens typically at the nm scale, it is not possible to use the technique to produce quantum entangled pairs at microwave wavelength for use in radar applications.

The Hamiltonian piece \(H^{(3)}\) also leads to sum a second type of second order processes, namely, sum frequency conversion. In this process, a pair of photons are combined in one third photon in such a way that the conditions \[(D.1)\] hold. In the Guha Erkmen optical parametric amplifier uses a second order non-linear susceptibility crystal operating at a very low gain. The idler and receiver light are combined as the output idler mode,

\[ \hat{a}_{I_m}^{\text{out}} = \sqrt{G} \hat{a}_{I_m} + \sqrt{G-1} \hat{a}_{R_m}^{\dagger}, \]

where \(\hat{a}_{R_m}^{\dagger}\) is the bosonic operator for the received mode and \(G\) is the gain associated to the interaction with the crystal. It turns out that the associated number operators

\[ \hat{N}_T = \sum_{m=1}^{N} \hat{a}_{I_m}^{\dagger} \hat{a}_{I_m} \]

can be measure by direct photo-counting during the window time \(T\) of the OPA duration time. This leads in Guha and Erkman theory to the analysis of expectation values for \(\hat{N}_T\) and also to specific Chernov’s bounds, showing an increase of 3 dB respect to coherent light \[37, 71\].

**D.2. Four photons interactions.** The piece \(H^{(4)}\) of the Hamiltonian leads to four photon interactions. The Hamiltonian piece proportional to the coupling \(f^{(4)}\) is responsible of the photon-photon interaction \(\gamma_1 + \gamma_2 \rightarrow \gamma_3 + \gamma_4\). In this case, conservation of energy is required,

\[ h\omega_1 + h\omega_2 = h\omega_3 + h\omega_4 \]
This is the process on which spontaneous four wave mixing generation (SFWM) relies on. TSFWM generation of entangled photon pairs has been recently used in quantum illumination experiments [25]. The scheme used for photon pair production is based on a birefringent optical fiber. The mechanism avoids the Raman noise effects that are common in four wave mixing generation and the need of cooling the system. Furthermore, it can be adapted to the generation of entangled photons of arbitrary wavelength [77]. Note that the complete matching conditions of the form (D.3) are not fulfilled for this particular mechanism [77].

The piece proportional to coupling $g^{(4)}$ is responsible of the frequency tripling (sum frequency generation) and down conversion of one photon to three photons. In both cases, the complete phase matching conditions are of the form (D.2). This second process is of relevance for Maccone-Ren quantum illumination discussed in Sections 8-9. Recently, three photons down conversion generation has been demonstrated experimentally under cryogenic conditions.

D.3. Methods of generation of entangled states for quantum radar. From the above elementary discussion on non-linear quantum optics process, we recollect here the main applications in the generation of quantum entangled photon states.

- **Spontaneous down conversion and related techniques.** The most common method use is spontaneous down conversion. This is a process of the form $\omega_p \rightarrow \omega_I + \omega_S$. It generates a pair of entangled photons which are correlated on polarization and in energy. SPDC generation does not require cryogenic conditions.

  However, currently, only provides photons in the nano-meter regime. To remedy this situation, optomechanical converters to microwave were used to generate indirectly entangled photons in the microwave regime [4]. The price is a drastic reduction of the intensity at microwave wavelength.

- **Josephson parametric amplification.** This mechanism is based in down conversion of a frequency pump directly to microwave regime. The fundamental process in the JPC is the reduction from $\omega_p$ to $\omega_I$, the amplification and then the mixing from $\omega_I \rightarrow \omega_I + \omega_S$. This happens in the JPA by the coupling of two microwave resonators (at $\omega_I$ and $\omega_S$ to the core of the JPA, which is a Josephson ring modulator) These processes need very low temperature (of order 7 mK).

  A practical advantage is that a JPA generator allows partial modulation of the generated frequencies by modulation of the pump frequency. This is a mechanism that allows to partially determine the range of the target for short distance radar or for scanning applications [46]. The main problem is the practical implementation of the cryogenic conditions.

- **Four wave mixing.** In photon-photon interaction within a birefringent crystal, two entangled photons are generated. In this case, no cryogenic conditions are required and, by appropriate preparation of the incident angles, incident frequency and cutting of the crystal [77]. For these reasons, it can be adapted for generation of pairs of entangled photons at microwave.

We would like to emphasize that nobel techniques for generation of entangled photons is still one of the most relevant problems in the area, despite the above already existing mechanisms. Higher intensity generation of entangled light is one of the main concerns.
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References
[1] G. Adesso and F. Illuminati, Entanglement in continuous variable systems: Recent advances and current perspectives, J. Phys. A 40, 7821 (2007).
[2] K.M.R. Audenaert et al., The Quantum Chernoff Bound, Phys. Rev. Lett., 98, 160501 (2007).
[3] S. M. Barnett, C. Fabre and A. Maitre, Ultimate quantum limits for resolution of beam displacements, Eur. Phys. J. D 22, 513 (2003).
[4] S. Barzanjeh, S. Guha, C. Weedbrook, D. Vitale, J. H. Shapiro and S. Pirandola, Microwave quantum illumination, Phys. Rev. Lett. 114, 080503 (2015).
[5] S. Barzanjeh, S. Pirandola, D. Vitali and J. M. Fink, Microwave Quantum Illumination using a digital receiver, Science Advances Vol. 6, no. 19, eabb0451 (2020).
[6] C. Bennett et al., Quantum Nonlocality without Entanglement, Phys. Rev. A 59, 1070 (1999).
[7] J. S. Bell, On the Einstein-Podolski-Rosen paradox, Physica 1, 195 (1964).
[8] D. Bohm, Quantum Theory, Dover (1999).
[9] J. J. Bollinger et al., Optimal frequency measurements with maximally correlated states, Phys. Rev. A 54, R(4649 (R) (1996).
[10] A. N. Boto et al., Quantum interferometric optical lithography: Exploiting Entanglement to beat the diffraction limit, Phys. Rev. Lett. 85, 2733 (2000).
[11] David C. Burnham and Donald L. Weinberg, Observation of Simultaneity in Parametric Production of Optical Photon Pairs Phys. Rev. Lett. 25, 84 (1970).
[12] C. M. Caves Quantum-mechanical noise in an interferometer, Phys. Rev. D 23, 1693 (1981).
[13] C. M. Caves and B. L. Schumaker, New formalism for two-photon quantum optics I. Quadratic phases and squeezed states, Phys. Rev. A 31, 3068 (1985).
[14] C. W. Sandbo Chang et al., Generating Multimode Entangled Microwaves with a Superconducting Parametric Cavity, Phys. Rev. Applied 10, 044019 (2018).
[15] C. W. Sandbo Chang, A. M. Vadiraj, J. Bourassa, B. Balaji, and C. M. Wilson, Quantum enhanced noise radar, Appl. Phys. Lett. 114, 112601 (2019).
[16] C. W. Sandbo Chang et al., Observation of Three-Photon Spontaneous Parametric Down-Conversion in a Superconducting Parametric Cavity Phys. Rev. X 10, 011011 (2020).
[17] H. Chernov, A measure of the asymptotic efficiency for tests of a hypothesis based on the sum of observations, Ann. Math. Stat., 23, 493 (1952).
[18] J. Chen, X. Li, and P. Kumar, Two-photon-state generation via four-wave mixing in optical fibers, Phys. Rev. A 72, 033801 (2005).
[19] P. A. M. Dirac, The Principles of Quantum Mechanics, Fourth Edition, Oxford University Press (1958).
[20] J. P. Dowling, Correlated input-port, matter-wave interferometer: Quantum-noise limits to the atom-laser gyroscope, Phys. Rev. A 57, 4736 (1998).
[21] L. M. Duan, G. Giedke, J. I. Cirac and P. Zoller, Inseparability criterion for continuous variable systems, Phys. Rev. Lett. 84, 2722 (2000).
[22] K. Durak, N. Jam and . Dindar, Object Tracking and Identification by Quantum Radar, arXiv:1908.06890.
[23] R. Demkowicz-Dobrzański L. Maccone, Using entanglement against noise in quantum metrology, Phys. Rev. Lett. 113, 250801 (2014).
[24] A. Einstein, B. Podolsky and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. 47, 777 (1935).
[25] D. G. England, B. Balaji, and B. J. Sussman1, Quantum-enhanced standoff detection using correlated photon pairs, Phys. Rev. A 99, 023828 (2019).
[26] Feng Fei et al., Quantum LIDAR based on squeezed states of light. Acta Photonica SINICA 46, 0527001 (2017).
[27] S. Frick, A. McMillan, J Rarity, Quantum Rangefinding, arXiv:2006.03875v1 [quant-ph].
[28] R. Gallego Torromé, Emergent Quantum Mechanics and the Origin of Quantum Non-local Correlations, Int. Jour. Theo. Phys. 56, 3323(2017).
[29] R. Gallego Torromé, *Foundations for a theory of emergent quantum mechanics and emergent classical gravity*, arXiv:1402.5070v16 [math-ph].

[30] R. Gallego Torromé, *Quantum illumination with multiple entangled photons*, to appear.

[31] J. C. Garrison and R. Y. Chiao, *Quantum Optics*, Oxford University Press (2007).

[32] G. Gilbert and Y. S. Weinstein, *Aspects of practical remote quantum sensing*, J. of Mod. Optics 55 10 (2008).

[33] G. Gilbert and M. Hamrick, MITRE Technical Report, (2000), *Practical Quantum Cryptography: A Comprehensive Analysis (Part One)*, [http://arXiv.org/abs/quant-ph/0009027](http://arXiv.org/abs/quant-ph/0009027).

[34] V. Giovannetti, S. Lloyd and L. Maccone, *Quantum enhanced positioning and clock synchronization*, Nature 412, 417 (2001).

[35] V. Giovannetti, S. Lloyd and L. Maccone, *Positioning and clock synchronization through entanglement*, Phys. Rev. A 65, 022309 (2002).

[36] V. Giovannetti, S. Lloyd and L. Maccone, *Quantum-Enhanced Measurements: Beating the Standard Quantum Limit*, Science 306, 1330 (2004).

[37] S. Guha and B. I. Erkman, *Gaussian-state quantum illumination receivers for target detection*, Phys. Rev. A 80, 052310 (2009).

[38] Harris Corporation, *Quantum sensors program*, AFRL-RI-RS-TR-2009-208 Final Technical Report, August (2009).

[39] H. He et al., *Non-classical Semiconductor Photon Sources Enhancing the Performance of Classical Target Detection Systems*, Journal of Lightwave Technology, VOL. XX, NO. X, JULY (2019), [arXiv:2004.06785](http://arXiv.org/abs/quant-ph/0009027).

[40] C. W. Helstrom, *Quantum detection and Estimation Theory*, New York, NY : Academic Press (1976).

[41] L. Henderson and V. Vedral, *Classical, quantum and total correlations*, Journal of Physics A 34, 6899 (2001).

[42] A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, Edizioni della Normale (2011).

[43] M. Horodecki, P. Horodecki and R. Horodecki, *Separability of mixed states: necessary and sufficient conditions*, quant-ph/9605038.

[44] S. F. Huelga et al., *Improvement of frequency standards with quantum entanglement*, Phys. Rev. Lett., 79, 3865 (1997).

[45] K. Heshami et al., *Quantum memories: emerging applications and recent advances*, Journal of Modern Optics., 63 (20): 20052028 (2016).

[46] A. Karsa, G. Spedalieri, Q. Zhuang and S. Pirandola, *Quantum Illumination with a generic Gaussian source*, arXiv:2005.07733v3 [quant-ph].

[47] M. Kutas, B. Haase, P. Bickert, F. Rießinger, D. Molter and G. von Freymann, *Terahertz Quantum Sensing*, arXiv:1909.06855.

[48] Paul G. Kwiat, Klaus Mattle, Harald Weinfurter, Anton Zeilinger, Alexander V. Sergienko and Yanhua Shih, *New High-Intensity Source of Polarization-Entangled Photon Pairs*, Phys. Rev. Lett., 75, 4337 (1995).

[49] M. Lanzagorta, *Quantum Radar*, Synthesis Lectures on Quantum Computing n. 5, Morgan and Claypool publishers (2011).

[50] L. B. Levitin and Y. Toffoli, *Fundamental limit on the rate of quantum dynamics: The unfiedbound is tight*, Phys. Rev. Lett., 103, 160502 (2009).

[51] W. Liu, P-x. Chen, C. Z. Li and J-M. Yuan, *Preparation and identification of two-photon positively momentum correlated entangled states*, Phys. Rev. A 79, 061802 (2009).

[52] S. Lloyd, *Enhanced Sensitivity of Photodetection via Quantum Illumination*, Science 321, 1463 (2008).

[53] Lopaeva et al., *Experimental realization of quantum illumination*, Phys. Rev. Lett. 111, 010501 (2013).

[54] D. Luong, C. W. Sandbo Chang, A. M. Vadiraj, A. Damini, C. M. Wilson and B. Balaji, *Receiver Operating Characteristics for a Prototype Quantum Two-Mode Squeezing Radar*, https://arxiv.org/abs/1903.0101.

[55] D. Luong, S. Rajan and B. Balaji *Quantum Two-Mode Squeezing Radar and Noise Radar: Correlation Coefficients for Target Detection*, IEEE Sensors Journal, Vol: 20, Issue: 10, May15, 15 (2020).
[56] C. Macklin et al. *A near quantum limited Josephson travelling-wave parameter amplifier*, Science 350, 307-310 (2015).

[57] J. Maldacena and L. Susskind, *Cool horizons for entangled black holes*, Progress of Physics, Vol. 61, 781 (2013).

[58] N. Margulus and L. B. Levitin, *The maximal speed of dynamical evolution*, Physica D 120, 188 (1998).

[59] L. Maccone and C. Ren, *Quantum Radar*, Phys. Rev. Lett. 124, 200503 (2020).

[60] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2010).

[61] J. P. Paz and W. H. Zurek, *Environment-induced decoherence and the transition from quantum to classical*, Lectures given by both authors at the 72nd Les Houches Summer School on Coherent Matter Waves”, JulyAugust (1999).

[62] S. Pirandola, B. Roy Bardham, T. Gehring, C. Weedbrook and S. Lloyd, *Advances in photonic quantum sensing*, Nature Photonics 12, 724(2018).

[63] A. Peres, *Separability Criterion for Density Matrix*, Phys. Rev. Lett. 77, 1413 (1996).

[64] *Observation of near field correlations in spontaneous parametric down conversion*, Phys. Rev. A 79, 041801 (2009).

[65] H. Ollivier and W. H. Zurek, *Quantum Discord: A Measure of the Quantumness of Correlations*, Phys. Rev. Lett. 88, 017901 (2001).

[66] Z. Y. Ou, *Fundamental quantum limit in precision phase measurement*, Phys. Rev. A 55, 2598 (1997).

[67] J. Rothman et al., *HgCdTe APDs detector developments at CEA/Leti for atmospheric lidar and free space optical communications*, International Conference on Space Optics-ICSO 2018, International Society for Optics and Photonics (2019).

[68] M. F. Sacchi, *Entanglement can enhance the distinguishability of entanglement-breaking channels*, Phys. Rev. A 72, 014305 (2005).

[69] M. F. Sacchi, *Optimal distribution of quantum operations*, Phys. Rev. A 71, 062340 (2005).

[70] J. H. Shapiro, S. Guha and B. I, Erkman, *Ultimate channel capacity of free-space optical communications*, J. Opt. Netw. 4, 501 (2005).

[71] J. H. Shapiro, *The Quantum Illumination Story*, IEEE Aerospace and Electronic Systems Magazine35, Issue: 4 , April 1 (2020).

[72] J. H. Shapiro and S. Lloyd, *Quantum illumination versus coherent-state target detection*, New Journal of Physics 11 063045 (2009).

[73] E. Schrödinger, *Die gegenwärtige Situation in der Quantenmechanik*, Naturwissenschaften, 23, 807, 823, 844 (1935).

[74] R. Simon, *Peres-Horodecki separability criterion for continuous variable systems*, Phys. Rev. Lett., 84, 2726 (2000).

[75] C Simon et al., *Quantum memories*, European Physical Journal D 58, 1 (2010).

[76] J. F. Smith III, *Quantum entanglement radar theory and a correction method for the effects of the atmosphere on entanglement*, Proceedings of the SPIE Quantum Information and Computation VII conference, 2009.

[77] B. J. Smith et al., *Photon pair generation in birefringent optical fibers*, Opt. Express 17, 23589 (2009).

[78] G. Sorelli, N. Treps, F. Grosshans and F. Boust, *Detecting a target with quantum entanglement*, [arXiv:2005.07110] [quant-ph].

[79] L. Susskind, *ER=EPR, GHZ, and the Consistency of Quantum Measurements*, [arXiv:1412.8483v1] [hep-th].

[80] S.-H. Tan et al., *Quantum Illumination with Gaussian States*, Phys. Rev. Lett. 101, 253601 (2008).

[81] J. von Neumann, *Mathematical foundations of quantum mechanics*, Princeton Landmarks in Mathematics, Princeton University Press (1955).

[82] R. F. Werner, *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys. Rev. A 40,4277 (1989).

[83] C. Weedbrook et al., *Gaussian Quantum Information*, Reviews of Modern Physics 84, 621 (2012).

[84] C. Weedbrook et al., *How discord underlies the noise resilience of quantum illumination*, New J. Phys. 18, 043027 (2016).
F. N. C. Wong, J. H. Shapiro and T. Kim, Efficient generation method of polarized-entangled photons in a non-linear crystal, Laser Phys. 16, 1517 (2006).

S. Yun et al., Generation of positively momentum correlated biphotos from spontaneous parametric down conversion, Phys. Rev. A 86, 023852 (2012).

B. Yurke, S. L. McCall and J R. Klauder, SU(2) and SU(1,1) interferometers Phys. Rev. A 33, 4033 (1986).

B. Yurke et al., Observation of parametric amplification and deamplification in a Josephson parametric amplifier, Phys. Rev. A 39, 2519 (1989).

Z. Zhang et al., Entanglement-Enhanced Sensing in a lossy and Noisy Environment, Phys. Rev. Lett. 114, 110506 (2015).

A. Zeilinger, Long-distance quantum cryptography with entangled photons, Proc. SPIE 6780, Quantum Communications Realized, 67800B (2007).

H. D. Zeh, On the interpretation of quantum measurement in quantum mechanics, Found. Phys., 1, 69 (1970).

M. Zhong et al. Optically addressable nuclear spins in a solid with a six-hour coherence time, Nature (London) 517, 177 (2015).

W. H. Zurek, Decoherence and the Transition from Quantum to Classical-Revisited, Los Alamos Science 27, 2002.