Teleportation via generalized measurements, and conclusive teleportation

Tal Mor and Pawel Horodecki
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In this work we show that teleportation \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\] is a special case of a generalized Einstein, Podolsky, Rosen (EPR) non-locality. Based on the connection between teleportation and generalized measurements we define conclusive teleportation. We show that perfect conclusive teleportation can be obtained with any pure entangled state, and it can be arbitrarily approached with a particular mixed state.

KEY WORDS: Quantum information processing, Entanglement, Teleportation, Nonlocality, Generalized measurements, Conclusive teleportation, Distillation;

I. INTRODUCTION

Quantum information processing (QIP) \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\] discusses information processing in which the basic units are two-level quantum systems (e.g., spin-half particles, the polarization of individual photons, etc.) known as quantum bits or shortly, qubits. The classical states, 0 and 1, of a classical bit are generalized to quantum states of a qubit, \(|0\rangle \equiv (1/\sqrt{2})|0\rangle + |1\rangle \equiv (\beta/\sqrt{2})|0\rangle + (\alpha/\sqrt{2})|1\rangle\). The nonclassical aspect of a qubit is that it can also be in a superposition \(|\alpha|^{2} + |\beta|^{2} = 1\), and two or more qubits can be in a superposition which cannot be written as tensor product, and is known as an entangled state. The minimal resources required for teleportation are first noted by Einstein-Podolsky-Rosen (EPR) \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\], and a proof for the special nonclassicality was first obtained by Bell \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\]. The EPR-Bohm singlet state, \(|\Psi^{-}\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)\), of pair of qubits is the most important example of entanglement. [We refer, for simplicity, to use these “braket” notations for two-particle states while using vector notations for one-particle states.] The singlet state can be complimented to a basis \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\] (known now as the “Bell basis”) by adding the three states \(|\Psi^{+}\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle)\), \(|\Phi^{-}\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle)\), and \(|\Phi^{+}\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)\), the Bell-states (or the Braunstein, Mann, Revzen (BMR) states). We shall usually refer here to two qubits in any one of the Bell-BMR states as an EPR-pair, and to the EPR-Bloch state as the singlet state. Entanglement—the quantum feature visualized by such states—is an origin of fascinating quantum phenomena in quantum information theory: quantum computation \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\], entanglement-based quantum cryptography \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\], quantum error correction \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\], and more.

One of the most fascinating discoveries is quantum teleportation \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\] which lies in the heart of quantum information theory (see \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\]), and has been recently realized experimentally \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\]. Quantum teleportation is a process of transmission of an unknown quantum state \(|\phi\rangle\) = \((\alpha/\sqrt{2})|0\rangle + (\beta/\sqrt{2})|1\rangle\) via a previously shared EPR pair with the help of only two classical bits transmitted via a classical channel (usually visualized by phone): Alice (the sender) has a qubit in an unknown quantum state which she wishes to transmit to Bob (the receiver) using additional EPR pair shared by her and Bob. To do this she performs joint measurement on the two particles which are in her hands, then she sends (via phone) her two-bit result to Bob, who performs some unitary operation on his particle “transferring” his particle to the (still unknown) original state \(|\phi\rangle\). The initial state of Alice’s unknown state, and the EPR-pair (say, in a singlet state) is \((\alpha/\sqrt{2})|\Psi^{-}\rangle\). The teleportation is based on the fact that this initial state can also be written as \[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\]:
bits neither Alice nor Bob can learn anything about the unknown parameters of the state $|\phi\rangle$.

The alternative approach presented in this paper somewhat clarifies the mystery. Namely, we interpret the teleportation in the light of the paper of Hughston, Jozsa and Wootters (HJW) \cite{27}, and we present the teleportation process as a unique case of generalized EPR-nonlocality (we use the language of generalized measurement to express the ideas of \cite{27}).

A positive operator valued measure (POVM) provides the most general physically realizable measurement in quantum mechanics \cite{26,262,2622}, and we also call these measurements “generalized measurements”. Formally, a POVM is a collection of positive operators $A_i$ on a Hilbert space $\mathcal{H}_n$ of dimension $n$ which sum up to the identity, $A_1 + \ldots + A_r = I_n$. [When viewed as matrices, these are matrices which can be diagonalized and have only non-negative eigenvalues.] Standard measurements (which are usually described by some Hermitian operator in quantum mechanics books) arise as a special case where $A_i = |\psi_i\rangle \langle \psi_i|$ and $A_iA_j = \delta_{ij}$. We discuss here only pure POVMs in which each of the $A_i$ is proportional to a projection $A_i = q_i|\psi_i\rangle \langle \psi_i|$, but the operators $A_i$ are not necessarily orthogonal to each other, so that $r \geq n$. Any POVM can be implemented (at least in principle) by adding an ancilla in a known state, and performing a standard measurement in the enlarged Hilbert space $\mathcal{H}_n$.

To describe the EPR-nonlocality and its generalization, let us first define the notion of $\rho$-ensembles \cite{27}. An ensemble of quantum state is defined by a collection of normalized states $|\psi_1\rangle, \ldots, |\psi_m\rangle$ taken with a-priori probabilities $p_1, \ldots, p_m$ respectively. To any such ensemble one can associate its density matrix:

$$\rho = \sum_{i=1}^{m} p_i |\psi_i\rangle \langle \psi_i| ,$$

and the term $\rho$-ensemble refers to an ensemble with a density matrix $\rho$. For instance, for the completely mixed state in 2-dimensions, $\rho = I/2$, the following are all legitimate $\frac{1}{2}$-ensembles:

$$E_1 = \{|\psi_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : p_1 = p_2 = 1/2\}$$

$$E_2 = \{|\psi_1\rangle = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, |\psi_2\rangle = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} : p_1 = p_2 = 1/2\}$$

$$E_3 = \{|\psi_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |\psi_3\rangle = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, |\psi_4\rangle = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} : p_i = 1/4, 1 \leq i \leq 4\}$$

$$E_4 = \{|\psi_1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, |\psi_2\rangle = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}, |\psi_3\rangle = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}, |\psi_4\rangle = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} : p_i = 1/4, 1 \leq i \leq 4\} .$$

When a classical system is subjected to a measurement of any of its properties a definite outcome exists (at least in principle). However, when a quantum particle (say a qubit) is in a state which is well defined in one bases, say $(\frac{1}{\sqrt{2}})$ in the rectilinear basis $(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})$, the state is undefined in any other basis, and a measurement, say, in the diagonal basis $(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})$, does not have a definite outcome which can be predicted, and only the probabilities (of the possible outcomes) can be calculated. This is the well known uncertainty principle.

The EPR paradox \cite{21} is as follows: If Alice and Bob share a singlet state, the state of Bob’s particle is undefined (if we trace-out Alice’s particle, then Bob’s particle is in a completely mixed state $I/2$, but without tracing out Alice’s particle, the state of Bob’s particle by itself is not defined). However, if Alice measures in any basis she chooses to, say the rectilinear or the diagonal, she fully “learns” the state of Bob’s particle. Assuming that a quantum state is “real” (as the state of a classical object) and assuming that the state cannot be changed instantaneously (immediately after Alice’s measurement) when Alice and Bob are far apart, EPR concluded that the state of Bob’s particle must have been previously defined in both bases, in contradiction with the uncertainty principle. They further concluded that this is a paradox (the EPR paradox) and thus that quantum mechanics is incomplete. Today we know, due to \cite{11}, that indeed quantum mechanics is not described by a realistic-local model, and thus the EPR-paradox is resolved.

We refer to the following fact as the EPR nonlocality: the state of Bob’s particle, previously undefined, become completely specified by Alice nonlocal operation. Thus, the EPR nonlocality is not a nonlocality in the sense of \cite{11}, but the profound feature which allows to “create” quantum states from different ensembles as it was discussed in the original EPR analysis.

Using the language of $\rho$-ensembles, the EPR nonlocality is described as follows: Alice can choose whether Bob’s state will be in a $\rho$-ensemble $E_1$ or $E_2$ by choosing an appropriate measurement on her member of the EPR pair. Thus, while Bob holds the mixed state $\rho$, Alice has an additional information regarding his state.

The EPR nonlocality is further generalized by HJW in \cite{27}, by allowing Alice to perform generalized measurements (POVMs), hence enabling her to create any $\rho$-ensemble in Bob’s site, and also knowing precisely the state he has. Note that she cannot chose $\rho$, and also she cannot chose the resulting state in Bob’s hands, but she can choose the $\rho$ ensemble, and learn the state. Generating $\rho$-ensembles at a distance is the generalization of the EPR nonlocality in which only standard (projection) measurements are used. We shall refer to this generalized EPR nonlocality as the EPR-HJW nonlocality.

In particular Alice can create the $\rho$-ensemble $E_4$, and we shall show in Section III, that creating this ensemble
corresponds to the teleportation process, once we add the transmission of classical information from Alice to Bob (she transmits the outcome of her measurement). Thus, teleportation is a special case of generating \( \rho \)-ensembles at a distance, when Alice uses a special POVM and where the operations done by Alice and Bob are independent of the parameters of the (unknown) state. We call this view of the teleportation process “telePOVM” (see the acknowledgement), or teleportation via generalized measurements.

The next natural step is to use this approach to generalize the concept of teleportation, by removing the demand that the transmitted state can always be recovered. In Section IV, we define the concept of conclusive teleportation. The term “conclusive” is taken from quantum information theory, when one asks the following question (see, for instance, [1]): what is the optimal mutual information which can be extracted from two nonorthogonal quantum states each sent with probability half? One can obtain a definite (correct) answer (regarding the given state) sometimes for the price of nothing in other occasions [2]. Here we adapt this term presenting the conclusive teleportation in which the teleportation process is sometimes successful, and the sender knows if it is successful or not. When Alice and Bob use an entangled pure state which is not fully entangled the conclusive teleportation scheme allows them to teleport a quantum state with fidelity one. This is done for the price of occasional failures, and the sender knows whether it is successful or it is not. For many purposes (e.g., for quantum cryptography [26,1,24,25]), one would prefer performing this conclusive teleportation rather than the original one which leads to a transfer fidelity which is smaller than one [31], and yields fidelity one only when the shared state is maximally entangled. [The fidelity of a state \( \rho \) relative to a pure state \( \psi \) is given by \( \langle \psi | \rho | \psi \rangle \); for other properties of the fidelity, see [32].] For instance, if Alice has an unknown qubit which she wishes to teleport to Bob, while they only share partially-entangled states, she can first create a fully entangled state, and try to teleport a qubit from such a pair to Bob via a conclusive teleportation. If she fails, she can try again (using another shared pair) till she succeeds. Once she succeeds to teleport an EPR-pair member, she can teleport the unknown qubit with fidelity one.

A further generalization is to let Bob also perform a conclusive measurement that sometimes succeeds (this requires, a 2-way classical communication). Surprisingly, we shall show in Section V, that this type of teleportation can allow for a conclusive teleportation even when the shared entangled state is mixed. The conclusive teleportation obtained in this case is with arbitrarily high fidelity, but for the price of a probability of success decreasing to zero as the fidelity increases. We refer to it as a quasi-conclusive teleportation. The questions of quasi-conclusive teleportation with fixed probability of success or with only one-way classical communication allowed will be discussed elsewhere.

II. TELEPOVM

Suppose that Alice and Bob share any two-particle entangled pure state in any dimension, such that the reduced density matrix in Bob’s hands is \( \rho \). Then, according to Hughston, Jozsa and Wootters [27], any measurement at Alice side, performed on her part of the entangled state, creates a specific \( \rho \)-ensemble in Bob’s hands. All \( \rho \)-ensembles are indistinguishable (recall that a quantum system is fully described by its density matrices) unless there exist an additional information somewhere. For example, in the Bennett-Brassard-84 (BB84) cryptographic scheme [28] Bob receives the same density matrix \( \rho \) whether Alice uses the rectilinear basis or the diagonal basis, but he receives different \( \rho \)-ensembles. He cannot distinguish between the two ensembles and between the states in each particular occasion, unless he receives more information from Alice. When receiving additional information (the basis) he is told which \( \rho \)-ensemble he has, and (in this particular case) can find which state.

In the same sense, the EPR-scheme [17], provides a simple example of the HJW meaning of \( \rho \)-ensembles: when Alice chooses to measure her member of the singlet state in the rectilinear basis or in the diagonal basis, she “creates” a different \( \rho \)-ensemble in Bob’s hands, \( E_1 \) or \( E_2 \) respectively. Bob can distinguish the two states to find Alice’s bit after receiving additional information from Alice who tells him the basis (hence her choice of a \( \rho \)-ensemble). Alice’s choice of measurement determines the \( \rho \)-ensemble, and furthermore, her result in each occasion, tells her which of the states is in Bob’s hands. If the measurement is chosen in advance, and Alice tells Bob the outcome of the measurement (by sending one bit of information) he can know precisely the state of the qubit in his hands.

The generalization done by HJW replaces the standard, projection measurement by a generalized measurement [26] (POVM), so the number of results can be larger than the dimension of the Hilbert space in Alice’s site or in Bob’s site. Thus, the HJW-EPR nonlocality argument implies that the set of Bob states contains nonorthogonal states. Furthermore, if Alice sends him an additional information (her measurement’s result) Bob can recognize in which of these states his particle is now. This is a very interesting result of [27] and we now show that teleportation provides a fascinating usage of it.

Let Alice and Bob share an EPR pair (say, the singlet state). Consider the following POVM \( \mathcal{A} \):

\[
A_1 = \frac{1}{2} \begin{pmatrix} \alpha^2 & \beta \alpha^* \\ \beta \alpha & \beta^2 \end{pmatrix} ; \quad A_2 = \frac{1}{2} \begin{pmatrix} \beta^2 & -\beta^* \alpha \\ -\beta \alpha & \alpha^2 \end{pmatrix} ;
\]
with complex parameters $\alpha, \beta$, such that $|\alpha|^2 + |\beta|^2 = 1$.

It should be stressed that Alice’s measurements do not depend on the parameters $\alpha, \beta$, thus these need not be known to her. Moreover she can learn nothing about the latter as all four results of her generalized measurement corresponding to operations (6) can happen with equal probabilities. In the case of starting with the singlet state, all four Alice’s results occur with equal probabilities and the initial state of Bob’s particle is the maximally mixed state $\frac{I}{2}$ (reduced state of a maximally entangled state). Thus, it is clear that the teleportation is equivalent to the creation of a specific $\rho = \frac{I}{2}$ ensemble at a distance, where the specific $\frac{I}{2}$-ensemble is $E_4$. This can be done even if Alice and Bob do not know the state of the ancilla, $(\beta \rangle$ chosen by someone else, and this is exactly the process of teleportation of an unknown state.

This process will also teleport a density matrix (a mixed state) or a particle entangled with others. It can also easily be generalized to fully entangled states in higher $(N^2)$ dimensions discussed in [4].

### III. Generating $\rho$-Ensembles in Quantum Key Distribution

To see one application of the ideas described above, let us view a different scenario (taken from quantum key distribution): Suppose that Alice has in mind a set of states and their probabilities, say, $E_3$, which is used in the BB84 [29] quantum key distribution scheme. This describes a particular $\rho$-ensemble (the $\frac{I}{2}$-ensemble in the BB84 case) sent to Bob. If Alice doesn’t care which of the states is sent in each experiment, but only that it belongs to that set, she does not need to send the states. Instead of sending Bob the states, she sends him a member of some entangled state such that the reduced density matrix in Bob’s hands is $\rho$. Then she applies a specific POVM which creates the desired ensemble in Bob’s hands. The relevant example is the EPR scheme [17], in which an EPR-pair is shared by Alice and Bob. As we have seen before, Alice creates either the $\frac{I}{2}$-ensemble $E_1$ or $E_2$, when she apply a measurement in the rectilinear or the diagonal bases respectively. However, since the probability of each basis is 1/2, Alice’s full operation, including the choice of the basis, can also be described by a POVM which leads to the ensemble $E_3$.

Let us present a less trivial example. Let the state

$$|\chi_{23}\rangle = a|00\rangle + b|11\rangle$$

(9)

(with $a, b$ real, and $a^2 + b^2 = 1$) be prepared by Alice and let one particle be sent from Alice to Bob. Then let Alice measure her particle using a standard measurement in the computation (the rectilinear) basis. As result, the following $\rho$-ensemble is generated in Bob’s hands:

$$\left(\frac{1}{\sqrt{2}} \frac{a+a^*}{a-b} \frac{1}{\sqrt{2}} \frac{a-b}{a+b}\right); \quad p_1 = p_2 = 1/2.$$
IV. CONCLUSIVE TELEPORTATION WITH ANY PURE ENTANGLED STATE

We first present the use of an additional one-way classical communication to modify the teleportation process: if Alice wishes to teleport to Bob a quantum state of which she can make more copies (e.g., to teleport a member of an EPR-pair) or if she wishes to teleport an arbitrary state from a set (e.g., a BB84 state), she can improve the teleportation process very much by using conclusive teleportation: a teleportation process which is sometimes successful. After performing her measurement, Alice uses the classical channel to tell Bob if the teleportation succeeded, and he uses the received state only if the teleportation succeed.

For instance, one can use conclusive teleportation to save time or classical bits. Let Alice and Bob share a fully entangled state, and use it to perform a conclusive teleportation: Alice performs a measurement which distinguishes the singlet state from the other three (triplet) states. Instead of sending 2 bits she sends Bob only one bit telling him whether she received a singlet state or not. Bob doesn’t need to do any operation on his particle. In a 4 of the occasions she receives this result (the singlet state), hence performs a successful teleportation. This process makes sense when the classical bits are as expansive as the shared quantum states, or when a fast teleportation of arbitrary states (e.g. BB84 states) is required. Also, it allows teleportation when Bob is technologically limited and cannot perform the required rotations.

The process of conclusive teleportation makes more sense when Alice and Bob share a pure entangled state which is not fully entangled.

Let Alice and Bob share the state \(|\phi_1\rangle\) (any pure state can be written in that form called the Schmidt decomposition \(|\begin{array}{c} \alpha \\ \beta \end{array}\rangle\)), which they use to teleport a quantum state \(\phi_1 = |a\rangle^1\).

Following the method of \(|\begin{array}{c} \alpha \\ \beta \end{array}\rangle\), the state of the three particles is written using the Bell-BMR states as:

\[
\begin{align*}
|\Psi_{123}\rangle & = |\phi_1\rangle|\chi_{23}\rangle = \frac{1}{\sqrt{2}} \left[ |\Phi_{12}\rangle_{3} \left( a\alpha + b\beta \right) + |\Phi_{12}\rangle_{3} \left( a\alpha - b\beta \right) + |\Psi_{+}\rangle_{3} \left( a\beta \right) + |\Psi^{-}\rangle_{3} \left( -a\beta \right) \right].
\end{align*}
\]

If Alice and Bob were to use the standard teleportation process, a Bell measurement still creates the same POVM as before. But, unlike the case of using a fully entangled state, the states created in Bob’s hands depend also on \(a\) and \(b\), and not only on the state of the ancilla. The fidelity is clearly less than one (e.g., if Alice received a state \(\Phi^{+}\) in her measurement (which happen with probability \(p_{\Phi^{+}} = (|a|^2a^2 + |\beta|^2b^2)/2\), the fidelity \(|\langle \phi_1 | \phi_1\rangle|^2\) of the output state is \((|a|^2a^2 + |\beta|^2b^2)/(|a|^2a^2 + |\beta|^2b^2)\), which depends on \(a\) and \(b\), and on the teleported state.

The POVM that reproduce the four desired states can be found. It is not performed by a Bell measurement and will depend on the state of the ancilla which is supposed to be unknown to both sides. So perfect teleportation will not take place this time.

We present a different measurement which generates the desired states in Bob’s hands with perfect fidelity. The price we pay for the perfect state obtained, is that the process cannot be done with 100% probability of success, therefore it is a conclusive teleportation. To explain how it works, let us return to the case of fully entangled state (standard teleportation, with initial EPR-pair \(\Phi^{+}\)) and separate the Bell measurement into two measurements (one follows the other):

1. A measurement which checks whether the state is in the subspace spanned by \(|00\rangle\) and \(|11\rangle\), or in the subspace spanned by \(|01\rangle\) and \(|10\rangle\).

2. A measurement in the appropriate subspace (according to the result of the previous step), which projects the state on one of the two possible Bell states in that subspace, \(\Psi^{\pm}\) respectively.

When |\(\Psi_{23}\rangle\) is not fully entangled we still repeat the first step of that two-steps process. To see the outcome, note that the state of the three particles can also be written as

\[
|\Psi_{123}\rangle = \frac{1}{2} \left[ (a|00\rangle + b|11\rangle) \left( a\beta \right)_{3} + (a|00\rangle - b|11\rangle) \left( -a\beta \right)_{3} + (b|01\rangle + a|10\rangle) \left( \beta \right)_{3} + (b|01\rangle - a|10\rangle) \left( -\beta \right)_{3} \right].
\]

The first step projects |\(\Psi_{123}\rangle\) on either the first two possibilities or the last two with equal probabilities. In the second step, let us assume that the result of the first step was the subspace spanned by the states \(|00\rangle \equiv (1)_{00;01}\) and \(|11\rangle \equiv (0)_{00;11}\). A similar analysis can easily be done for the other case where the result of the first step is the subspace spanned by the states \(|01\rangle \equiv (1)_{01;10}\) and \(|10\rangle \equiv (0)_{01;10}\).

In this \(|00;11\rangle\) subspace, Alice now performs a second measurement, but not in the Bell-BMR basis which is now the states \((1/\sqrt{2})_{00;11}\), as in the ideal case. Instead, Alice performs a POVM which conclusively distinguishes the two states \((\frac{\alpha}{b})_{00;11}\) and \((-\frac{a}{b})_{00;11}\) (which are the first two states in the above expression). Assuming (without loss of generality) that \(a^2 \geq b^2\) the POVM elements in that subspace are:

\[
\begin{align*}
A_1 & = \begin{pmatrix} b^2 & ba \\ ba & a^2 \end{pmatrix}; \\
A_2 & = \begin{pmatrix} b^2 & -ba \\ -ba & a^2 \end{pmatrix}; \\
A_3 & = \begin{pmatrix} 1 - (b^2/a^2) & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]
Such a POVM can never give a wrong result, and it gives an inconclusive result when the outcome is $A_3$. This POVM was found in Ref. [35], in the context of distinguishing the two states of $(|b\rangle\langle 0|)$. It is the optimal process for obtaining a perfect conclusive outcome, and a conclusive result is obtained with probability $1 - (|a|^2 - |b|^2)$. In our case, this is the probability of a successful teleportation.

Alice tells Bob whether she succeeded in teleporting the state by sending him one bit, and in addition to this bit, Alice still has to send Bob the two bits for distinguishing the four possible states (so he can perform the required rotation). Alternatively, she can send him only one bit telling him whether he received the state or not (as we explained for the case of fully entangled state) loosing $\frac{3}{4}$ of the successful teleportations.

When used for distinguishing non-orthogonal states, this POVM allows to get the optimal deterministic information from two non-orthogonal states, although, on average, it yields less mutual information than the optimal projection measurement. In the same sense, on average, the conclusive teleportation does not yield the optimal average fidelity, but when it is successful – the fidelity is one.

The conclusive teleportation process proves that any (pure) entangled state presents quantum non-locality. This fact can also be seen using the filtering method [35] when applied to pure states.

V. ARBITRARY GOOD CONCLUSIVE BILOCAL TELEPORTATION VIA MIXED STATES

In a perfect conclusive teleportation Alice performs a teleportation process which is sometime successful, and when it is successful, the fidelity of the teleported state is one. In an imperfect conclusive teleportation, Alice performs a teleportation process which is sometime successful, and when it is successful, the fidelity of the teleported state is less than one but better than could be achieved with a standard teleportation.

The original idea of teleportation involves only one way classical communication from Alice to Bob. We shall now extend it, allowing Bob to call Alice as well so that bilocal protocol is used. Note that here we do not consider the most general bilocal protocol (the so called ping-pong protocol) but only allow Bob and Alice to operate independently of the operation of the other. A ping-pong protocol could improve the probability of successful projection (e.g., increase the $p'(p)$ described below), by allowing several “paths” of successful distillation depending on the outcomes of the measurements in each step of the protocol. The communication (in our example) is just used to verify that the state was teleported. This generalization of teleportation makes sense, as in many cases the classical communication is treated as a free resource.

We have shown previously that a perfectly reliable conclusive teleportation can be achieved when pure entangled states are shared. We now show that it is possible to perform arbitrary good bilocal conclusive teleportation when certain mixed states are used (however, see the remark in the acknowledgements). The arbitrary good conclusive teleportation (which we call “quasi-conclusive teleportation”) is not described by a particular POVM, $A = \{A_1, \ldots A_m\}$, but by a series of POVMs $A^n = \{A_1(n), \ldots A_m(n)\}$, where $n$ is the index of this series. For any $\epsilon$ we can find $n$ such that the POVM $A^n$ yields fidelity better than $1 - \epsilon$ for teleportation. Yet, perfect fidelity cannot be achieved since the probability of success goes also to zero when $\epsilon$ goes to zero. Thus, we show that quasi-conclusive teleportation is successfully done via mixed states.

We first purify the mixed state, and then use it for teleportation.

Consider the state

$$\rho_p = p|\Psi^-\rangle\langle \Psi^-| + (1 - p)|00\rangle\langle 00|, \quad 0 < p < 1 \quad (13)$$

which is a mixture of a singlet (with probability $p$) and a $|00\rangle$ state (with probability $1 - p$). Let the bilocal Alice and Bob action be described in the following way:

$$\rho_p \to \rho' \equiv \frac{V_1 \otimes W_1(\varphi)V_1^\dagger \otimes W_1^\dagger}{Tr(V_1 \otimes W_1(\varphi)V_1^\dagger \otimes W_1^\dagger)} \quad . \quad (14)$$

It can be realized by performing generalized measurements by Alice and Bob independently, i.e., Alice performs the measurement defined by the pair of operators $\{V_1, V_2 \equiv \sqrt{I - V_1V_1^\dagger}\}$, and Bob performs the measurement defined by the pair of operators $\{W_1, W_2 \equiv \sqrt{I - W_1W_1^\dagger}\}$. Alice’s POVM is the set $\mathcal{A} = \{A_1 = V_1^\dagger V_1, A_2 = V_2^\dagger V_2\}$, and Bob’s POVM is the set $\mathcal{B} = \{B_1 = W_1^\dagger W_1, B_2 = W_2^\dagger W_2\}$. When the outcomes of both Alice and Bob is 1, which correspond to the first operator in each lab ($V_1$ and $W_1$ respectively) the above transformation is successfully done.

After getting the results of their measurements Alice and Bob communicate via phone to keep only those particles for which both results correspond to the successful case. To show that quasi conclusive teleportation can be performed, we define the sequence of POVM operators (in basis $\{|0\rangle, |1\rangle\}$):

$$V_1(n) = \begin{pmatrix} (1/n) & 0 \\ 0 & 1 \end{pmatrix}; W_1(n) = \begin{pmatrix} (1/n) & 0 \\ 0 & 1 \end{pmatrix} \quad . \quad (15)$$
After the action of the corresponding POVM the new state is $\rho' = \rho_{p'} \equiv \rho' (\Psi^-)(\Psi^-) + (1 - p')(00)(00)$ with the parameter $p'$ depending on the input parameter $p$ as follows

$$p'(p) = \frac{1}{1 + \frac{2p}{n_p}}. \quad (16)$$

The probability of successful transition from $\rho_p$ to $\rho_{p'}$ is

$$P_{p \rightarrow p'} = \frac{1}{n^2} [1 + (n-1)p]. \quad (17)$$

Thus one can produce the state which has arbitrary good singlet fraction (a fidelity with a singlet) $F(\rho_p) = \langle \Psi^- | \rho_p \Psi^- \rangle$, which obviously allows for arbitrary good conclusive teleportation. The key point is that the probability of successful teleportation decreases to zero with fidelity of teleportation (or equivalently singlet fraction) going to unity. But it is nonzero for any required fidelity arbitrary close to perfect one.

One natural question is whether it is possible to make teleportation arbitrary good via other mixed states. In general the answer is negative. In the case of Werner states (states in which a fully entangled state is mixed with the completely mixed state), for instance, this is a consequence of the fact that arbitrary good conclusive distillation is impossible [37]. In fact for those states the entanglement fidelity cannot be increased. Its best value $F_{\text{max}}$ is the initial (before the conclusive process) value $F_0$. Thus following [34], the maximal teleportation fidelity is equal to $\frac{2F_{\text{max}}^{(1)}}{3}$ and is less than 1 apart from the trivial case where the initial state is fully entangled.

Another interesting question is whether it is possible to perform quasi-conclusive teleportation with only one way classical communication.

This represents a more complicated issue which requires a more complicated technical analysis, and will be analyzed elsewhere.

VI. SUMMARY

In this paper we presented a new way of viewing the teleportation of an unknown quantum state. We showed that teleportation is a special and particular case of generating $p$-ensembles at a distance, hence, a special case of generalized EPR nonlocality (the HJW-EPR nonlocality). We believe that this view of teleportation reduces some of the mystery of that process, and in particular, explains why two classical bits can be sufficient for the teleportation of a qubit. This work also showed the usefulness of the HJW generalized EPR nonlocality, and their understanding that any $p$-ensemble can be generated nonlocally.

We feel that understanding the connection between these two important forms of nonlocality improves much the understanding of entanglement.

Based on the connection between teleportation and generalized measurements, we presented the process of conclusive teleportation, a teleportation which is sometime successful. We showed that any pure entangled state can be used to perform conclusive teleportation with fidelity one, and more surprising, certain mixed states can also be used to achieve conclusive teleportation with fidelity as close to one as we like.

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The first sections (teleportation via POVMs and conclusive teleportation) were presented before (but not published) [38]. The idea of bilocal teleportation and the ability to perform quasi-conclusive teleportation with mixed states were presented in [10], but the example we provide here is much simpler, and involves only two qubits. The connection between conclusive teleportation and conclusive purification (or distillation) from a single pair was obtained in [36] and the idea of conclusive distillation appears earlier in previous works [35].

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