Scale-deviating operators of Riesz type and the spaces of variable dimensions

Vladimir Kobelev
Department of Natural Sciences, University of Siegen, Siegen, Germany

Correspondence
Vladimir Kobelev, Department of Natural Sciences, University of Siegen, D-57076, Siegen, Germany.
Email: vladimir.kobelev@uni-siegen.de

Funding information
Alexander von Humboldt-Stiftung

Abstract
The article introduces the scale-deviating operator. The scale-deviating differential operator comprises the parameters to designate the operator order and the parameters to define the dimension of space. The operator order depends on the characteristic length $\kappa$. There are two types of linear scale-deviating operators. For the distances $r$, which are much less than $\kappa$, the scale-deviating operator of the first type $A$ reduces to the common operators. For the distances, which exceed the length $\kappa$, this operator reduces to the fractional Riesz operator. The second type of the scale-deviating operator $B$ behaves oppositely. For the distances $r$, which are much higher than $\kappa$, the scale-deviating operator of the second type reduces to the common operators. Finally, for the distances, which below the length $\kappa$, this operator lessens to the fractional Riesz operator. These linear, isotropic operators possess the order, less than two. The solutions of new scale-deviating equations and the shell theorem for these operators are provided closed form.

KEYWORDS
fractional scalable operators, Riesz potential, scalable operators

1 | INTRODUCTION

The linear isotropic operator (“scaling operator”) extends the common differential operators of mathematical physics. The common differential operators of mathematical physics possess the same distance dependence in the entire space, from zero distance to infinity. This means, that space has everywhere the same dimensionality. The scaling operator allows straightforwardly and continually vary the distance dependence. In the near regions to the source, the distance dependence differs arbitrary from the distance dependence in the far regions of space. Two constants determine the alternation radius and the distance dependence. Well known, that the distance dependence uniquely linked to the dimensionality of space. Namely, the distance dependence is the dimensionality of space minus one. The scaling operator allows to study without difficulty the mathematical physics in the spaces of continuously variable dimensions. The scaling operator uses the standard mathematical technique.

The seminal paper presented the mathematically concrete realization of spaces with non-integer dimensions. In the cited paper, five axioms were proposed for the definition of dimension, which is not restricted to the positive integers. Four topological the axioms and one integration measure method were proposed. When the dimension is a positive integer, the corresponding space acts like a conventional Euclidean vector space. In the opposite case, a space with some non-vector...
character occurs. This Stillinger space conforms to informal usage of continuously variable dimension in several physical contexts. The number of mutually perpendicular lines in the Stillinger space can exceed the formal dimensions’ number.

To discuss the variable dimensions in context of the fractional operators, the scaling method was proposed in Reference [2] using the volume and the surface area of a generalized hypersphere in a fractional dimensional space. The cited paper demonstrated using the scaling method, which that the total dimension of the fractional space could be obtained by summing the dimension of the fractional line element along each axis.

The actual study applies the standard procedure for solving of elliptical equations via Fourier transform and the Green functions to study the equations of mathematical physics in the spaces of the variable dimensions. The proposed operator introduces the continuous, isotropic variability of the dimension. The scaling operator $S_{(n)}$ is the linear pseudodifferential operator, which is based on the Riesz potential. For this purpose, we introduce the scalable differential operators of Riesz type for the common operators of mathematical physics. The definition of the fractional dimensional space reflects the variability of the radial dependence of the Green’s functions of the scaled operators.

The structure of the article is the following.

The Section 2 introduces the scale-deviating operators of the order two, which obey the definite symmetry conditions. Based on these definitions, we demonstrate the exact solutions for Green’s functions of the linear scale-deviating operators. Notable, that the variable dimensions are formally equivalent to the certain order of the scale-deviation. Other saying, the definite scale-deviation manifests the variable dimensionality of the embedding space.

The solutions are derived using the Fourier method in the Section 3 for the first scale-deviating Helmholtz equation, in the Section 4 for the first scale-deviating Laplace equation. In the Section 5, the Green functions are derived for the first scale-deviating Helmholtz equation, in the Section 6 for the second scale-deviating Laplace equation.

The Section 7 exposes the necessary mathematical preliminaries for further generalizations of Newton equations. Newtonian mechanics will be presented using the Galilean Relativity Principle. For this purpose, Einstein’s theory of Special and General Relativity in the Newtonian limit is briefly summarized. After that, the scale-deviating Newtonian mechanics is debated, applying the mathematical formalism.

The Section 8 deliberates the gravitation flux on the surface of homogeneous sphere and derives the potential of the homogeneous spherical source.

2 | SCALE-IN Variant ANd SCALE-DEVIATING OPERATORS

In this manuscript, we consider the partial differential equations for the scalar function $\phi(r)$:

$$\mathcal{L} \phi(r) = q(r).$$

In this equation $\mathcal{L}$ is the linear differential operator, $q$ is the source density, and $r$ is vector in $D$-dimensional unbounded space. The equations with common D'Alembert, Laplace and Helmholtz equations are respectively:

$$\Box \phi = q, \quad \Delta \phi = q, \quad H \phi = q. \quad (1)$$

The formal description is essentially based on the representation of the left side of the equation $\mathcal{L} \phi = q$ as the integral Fourier operator:

$$\langle \mathcal{L} \phi \rangle(r) \equiv \mathcal{F}_{k \rightarrow r}[s(k) \mathcal{F}_{r \rightarrow k} \phi(r)]. \quad (2)$$

We use in Equation (2) the direct and inverse Fourier transforms:

$$\tilde{\phi}(k) = \mathcal{F}_{r \rightarrow k} \phi(r), \quad \phi(r) = \mathcal{F}_{k \rightarrow r} \tilde{\phi}(k). \quad (3)$$

The appropriate functions $\phi(r), \tilde{\phi}(k)$ are defined in $k \in R^D, x \in R^D$. The scalar function $s(k)$ in (2) is a symbol of operator $\mathcal{L}$, such that

$$s(k) = \text{symbol}(\mathcal{L}). \quad (4)$$

Generally saying, the equations of mathematical physics are scale deviating, if the equations contain the derivatives of different order. There are various scale-deviating equations. For example, the equations of shells and plates are scale deviating, because these equations contain the partial derivatives of the second and fourth order. The D'Alembert and
Laplace equations are scale-invariant. However, it is impossible to construct the partial differential equations of the order less than two, which do not contradict the definite symmetry conditions. The ground is the violence of the rotation isotropy of the differential equations of the first order.

In this manuscript, we consider the scale-deviating isotropic equations. We study primarily the half-integer orders of derivative. The orders are the integer multiples of half-integer values \( \alpha = \pm 1/2 \). Fractional derivative, being applied twice, results in the common integer derivative. Particularly, the Fourier transform of the fractional derivative is the product of the Fourier transform of the operator and the Fourier transform of the function, such that

\[
\partial^{(\alpha)} \phi(r) = \mathcal{F}_k \cdot r^{-|\alpha|} \mathcal{F}_r \cdot k^\alpha \phi(r), \quad \text{symbol}(\partial^{(\alpha)}) = (-i k)^\alpha, \quad k = |k| > 0.
\]

The application of the scaling technique is demonstrated below for the common D’Alembert, Laplace and Helmholtz partial differential equations.

We introduce two types of the scale-deviating equations for Laplace-type operators:

\[
\mathcal{A}_\Delta \phi = q, \quad \mathcal{B}_\Delta \phi = q.
\]

The scale-deviating operators \( \mathcal{A}_\Delta, \mathcal{B}_\Delta \) are defined as (Table 1):

\[
\mathcal{A}_\Delta \equiv \mathcal{S}_{(1)} \Delta, \quad \mathcal{B}_\Delta \equiv \mathcal{S}_{(-1)} \Delta.
\]

The scaling operator \( \mathcal{S}_{(\alpha)} \) of the Laplace operator \( \Delta \) is defined with the Fourier integrals as:

\[
\mathcal{S}_{(\alpha)} \phi(r) = \left( \frac{1}{2\pi} \right)^D \int_{K_0} \int_{K_0} e^{i k r (1 + \kappa^{\alpha} k^{\alpha}) n} d^D k \int_{R_0} \phi(\rho) e^{-i k \rho} d^D \rho, \quad |n| \leq 1, \quad \lim_{k \to 0} \mathcal{S}_{(\alpha)} = 1,
\]

where \( \rho = |\rho| \).

The symbol of the scaling operator (8) is:

\[
\text{symbol}(\mathcal{S}_{(\alpha)}) = (1 + \kappa^{\alpha} k^{\alpha})^n.
\]

If \( \text{Re} \ \alpha < 0 \), the operator (8) does not alter the character of solutions on short distances, but change the behavior on the long distances. Otherwise, if \( \text{Re} \ \alpha > 0 \), the operator (8) does not alter the character of solutions on long distances, but change the behavior on the short distances. The presence in Equation (9) of a fixed length scale \( \kappa \) specifies the scale deviation. The parameter \( \kappa \) will be reffered to as the “scaling dimension”.

From (8) follows that the operators \( \mathcal{S}_{(1)}, \mathcal{S}_{(-1)} \) are:

\[
\begin{align*}
\mathcal{S}_{(1)} \phi(r) &= \left( \frac{1}{2\pi} \right)^D \int_{K_0} \int_{K_0} e^{i k r (1 + \kappa^{n} k^{n})} d^D k \int_{R_0} \phi(\rho) e^{-i k \rho} d^D \rho, \quad n = 1, \\
\mathcal{S}_{(-1)} \phi(r) &= \left( \frac{1}{2\pi} \right)^D \int_{K_0} \int_{K_0} e^{i k r (1 - \kappa^{n} k^{n})} d^D k \int_{R_0} \phi(\rho) e^{-i k \rho} d^D \rho, \quad n = -1.
\end{align*}
\]

The scaling operators \( \mathcal{S}_{(1)}, \mathcal{S}_{(-1)} \) are applicable for the operator \( \mathcal{H} \) as well (Table 2). The scale-deviating operators of Helmholtz-type \( \mathcal{A}_\mathcal{H}, \mathcal{B}_\mathcal{H} \) are defined as (Table 2):

\[
\mathcal{A}_\mathcal{H} \equiv \mathcal{S}_{(1)} \mathcal{H}, \quad \mathcal{B}_\mathcal{H} \equiv \mathcal{S}_{(-1)} \mathcal{H}.
\]
For the D'Alembert equation, the scaling operator \( \mathcal{T}_1(n) \) could be defined for \( n = -1, n = 1 \) as well. The scaling operator \( \mathcal{T}_1(n) \) results from (8) after the replacement of \( k \) by \( l \), as it requires the symmetry of the operator \( \square \):

\[
\mathcal{T}_1(n) \phi(r) = \left( \frac{1}{2 \pi} \right)^D \int_{k^0} e^{i k r (1 + \kappa n \rho^\alpha)} a d^D k \int_{\rho^0} \phi(\rho) e^{-i k \rho} d^D \rho, \quad |n| \leq 1, \lim_{\kappa \to 0} \mathcal{T}_1(n) = 1.
\]

The symbol of the scaling operator (12) of the order \(-1 < Re \alpha < 1\) is:

\[
symbol(\mathcal{T}_1(n)) = (1 + \kappa n \rho^\alpha)^n.
\]

The wave-type scale-deviating operators \( A_D, B_D \) are defined as (Table 3):

\[
A_D \equiv \mathcal{T}_1(1) \square, B_D \equiv \mathcal{T}_1(-1) \square.
\]

In Equation (13) the scaling operators are:

\[
\begin{align*}
\mathcal{T}_1(1) \phi(r) &= \left( \frac{1}{2 \pi} \right)^D \int_{k^0} e^{i k r (1 + \kappa n \rho^\alpha)} a d^D k \int_{\rho^0} \phi(\rho) e^{-i k \rho} d^D \rho, & n = 1, \\
\mathcal{T}_1(-1) \phi(r) &= \left( \frac{1}{2 \pi} \right)^D \int_{k^0} e^{i k r (1 + \kappa n \rho^\alpha)} a d^D k \int_{\rho^0} \phi(\rho) e^{-i k \rho} d^D \rho, & n = -1, \kappa > 0.
\end{align*}
\]

According to the above definition, the scale-deviating D'Alembert, Laplace and Helmholtz equations for \( n = -1 \) or \( n = 1 \) respectively read:

\[
\begin{align*}
\mathcal{T}_1(n) \square \phi &= q, \quad S_1 \Delta \phi &= q, \quad S_1 \mathcal{H} \phi &= q.
\end{align*}
\]

The scale-deviating equations possessing special invariant properties are summarized in Tables 1, 2, 3. As we discuss later, there are two types of linear scale-deviating operators. For the distances \( r \), which are much less than \( \kappa \), the scale-deviating operators of the first type \( (A_L, A_3, A_D) \) reduce to the operators \( (\Delta, \mathcal{H}, \square) \) correspondingly. For the distances \( r \), which exceed the length \( \kappa \), these operators act as the fractional Riesz operator. The second type of the scale-deviating operators \( (B_L, B_3, B_D) \) perform oppositely. For the distances \( r \), which are much higher than the scale-deviating operators of the second type act as the consistent operators \( (\Delta, \mathcal{H}, \square) \). For the distances, which below the length \( \kappa \), this operator asymptotically match the certain fractional Riesz operator.
3 | FIRST SCALE-DEViating HELmholtz EQUATION

The first Equation (11) is resolvable in closed form. For the scalar function \( \phi = \phi(r) \), the first scale-deviating fractional Poisson’s equation reads:

\[
\mathcal{A}_{\mathcal{F}} \phi(r) = q, \quad (16)
\]

\[
\lim_{\kappa \to 0} \mathcal{A}_{\mathcal{F}} \phi(r) = \mathcal{H} \phi . \quad (17)
\]

The operator \( \mathcal{A}_{\mathcal{F}} \) in Equation (16) expresses as:

\[
\mathcal{A}_{\mathcal{F}} \phi(r) = -\left( \frac{1}{2\pi} \right)^D \int_{K^D} e^{i k \cdot r} (k^2 + \Omega^2) (1 + \kappa^\alpha k^\alpha) d^D k \int_{\mathbb{R}^D} \phi(\rho) e^{-i k \cdot \rho} d^D \rho, \quad (18)
\]

To solve the scale-deviating fractional Helmholtz Equation (16) with Riesz fractional derivative of order \( \alpha \), the Fourier transform method, is applied. The Fourier transforms are used for solution of (16):

\[
\left[ \mathcal{F}_{\mathcal{F}} \phi(\rho) \right] = \psi(k) = \int_{K^D} \phi(\rho) e^{i k \cdot \rho} d^D \rho, \quad [\mathcal{F}_{\mathcal{F}} q(\rho)] = \tau(k) = \int_{K^D} q(\rho) e^{i k \cdot \rho} d^D \rho. \quad (19)
\]

The inverse Fourier transforms are:

\[
[\mathcal{F}_{\mathcal{F}}^{-1} \psi(k) ] = \mathcal{F}_{\mathcal{F}}^{-1} \tau(k) = \int_{K^D} \tau(k) e^{-i k \cdot \rho} d^D \rho. \quad (20)
\]

The application of the Fourier transform for Equation (16) with respect to the spatial variable leads to:

\[
\psi(k) = \frac{\tau(k)}{(k^2 + \Omega^2) \cdot (1 + \kappa^\alpha k^\alpha)}. \quad (21)
\]

To solve the linear inhomogeneous pseudodifferential equation, we first solve the equation for Green’s function, subject to the appropriately selected boundary conditions. The Green’s function satisfies the scaled Helmholtz equation:

\[
\mathcal{S} \mathcal{F} g_{\mathcal{A}} = \delta(r). \quad (22)
\]

With its solution \( g_{\mathcal{A}} \), we immediately obtain the solution to our differential equation:

\[
\phi(r) = \int_{\mathbb{R}^D} T(\rho) g_{\mathcal{A}}(r - \rho) d^D \rho. \quad (23)
\]

With the inverse Fourier transforms, the Green’s function is given by:

\[
g_{\mathcal{A}}(r) = \left( \frac{1}{2\pi} \right)^D \int_{K^D} \frac{d^D k}{e^{i k \cdot r} (k^2 + \Omega^2) \cdot (1 + \kappa^\alpha k^\alpha)}. \quad (24)
\]

For the further transformation of Equation (23), we use the identity:

\[
\int_{K^D} \phi(k) \exp(i k \cdot \rho) d^D k = \int_{0}^{\infty} \phi(\rho) \rho^{D-1} d \rho \int_{\Phi^{-1}} e^{i \rho \theta} d \theta. \quad (25)
\]

The substitution of Equation (24) in (25) results the following identities:

\[
\int_{K^D} \frac{\exp(i k \cdot \rho)}{(k^2 + \Omega^2) \cdot (1 + \kappa^\alpha k^\alpha)} d^D k = \int_{0}^{\infty} \frac{\rho^{D-1}}{(\rho^2 + \Omega^2) \cdot (1 + \kappa^\alpha \rho^2)} \rho \int_{\Phi^{-1}} e^{i \rho \theta} d \theta. \quad (26)
\]
The integral in right term of (26) could be represented with the Bessel function $J_\nu(r)$ of the first kind:

$$J_\nu(r) = \frac{(r/2)\nu}{\sqrt{\pi} \Gamma \left( \frac{\nu}{2} + \frac{1}{2} \right)} \int_{-1}^{1} e^{i\xi r (1 - \xi^2)^{-\frac{1}{2}}} d\xi,$$

With Equation (28), the Equation (23) reduces to the integral form for the potential:

$$g(r) = \left( \frac{1}{2\pi} \right)^{D/2} \int_{0}^{\infty} \frac{\rho^{D-1} J_{\frac{D}{2}-1}(\rho r)}{(\rho^2 + \Omega^2)^{\frac{D}{2}}} \cdot \frac{d\rho}{1 + \kappa^a \rho^a}.$$  

For $D = 3$, the potential of a point mass the integral (29) reduces to:

$$g(r) = \left( \frac{1}{2\pi^2} \right) \int_{0}^{\infty} \frac{\rho \sin(\rho r)}{\rho^2 + \Omega^2} \cdot \frac{d\rho}{1 + \kappa^a \rho^a}.$$  

For certain values of $\alpha$, Equation (29) expresses the potential of the point source in terms of the higher functions:

$$g(r) = \frac{1}{4\pi r} \cdot \left\{ \begin{array}{ll}
\frac{\theta(r) + \pi e^{-r\Omega}}{\pi (1 + k^2\Omega^2)} & \text{for } \alpha = 1, \\
\frac{-\theta(r) + \pi e^{r\Omega}}{-\pi (1 + k^2\Omega^2)} & \text{for } \alpha = -1, \\
\frac{\pi (1 + k^2\Omega^2)}{1 + k^2\Omega^2 e^{-r\Omega}} & \text{for } \alpha = -2, \\
\frac{e^{\Omega^2} - 1}{1 - k^2\Omega^2 e^{-r\Omega}} & \text{for } \alpha = 2.
\end{array} \right.$$  

The auxiliary function in Equation (31) is:

$$\theta(r) = 2 \cos \left( \frac{r}{k} \right) \cdot \text{Si} \left( \frac{r}{k} \right) - 2 \sin \left( \frac{r}{k} \right) \cdot \text{Ci} \left( \frac{r}{k} \right) + \kappa \Omega [e^{-r\Omega} \text{Ei}(r\Omega) - e^{r\Omega} \text{Ei}(-r\Omega)].$$

In Equation (31), we use the following functions:

- $\text{Si}(x) = \text{Si}(x) - \pi/2$ is the shifted sine integral,
- $\text{Si}(x)$ is the sine integral,
- $\text{Ci}(x)$ is the cosine integral,
- $\text{Ei}(x)$ is the exponential integral.

### 4 | FIRST SCALE-DEVIATING POISSON’S EQUATION

Using the similar techniques, Equation (6) is solvable in closed form as well. For the scalar function $\phi = \phi(r)$, the scale-deviating fractional Poisson’s equation reads:

$$\mathcal{A}_\xi \phi(r) = q,$$  

$$\lim_{\kappa \to 0} \mathcal{A}_\xi \phi(r) = \Delta \phi.$$  

The operator $\mathcal{A}_\xi$ in Equation (16) expresses as:

$$\mathcal{A}_\xi \phi(r) = - \left( \frac{1}{2\pi} \right)^D \int_{k^0} e^{i k \cdot r} (1 + \kappa^a k^a) d^D k \int_{R^D} \phi(\rho)e^{-i k \cdot \rho} d^D \rho.$$
The Green’s function \( g_A \) of Equation (32) follows from the solution (29) as the limit case \( \Omega = 0 \):

\[
g_A(r) = \left( \frac{1}{2\pi} \right)^{D/2} \int_0^\infty \frac{\rho^{D-3} J_{\frac{D}{2}-1}(\rho \tau)}{(\rho r)^{\frac{D}{2}-1}} \cdot \frac{d\rho}{1 + \kappa^2 \rho^2}.
\]  

(35)

The Green’s function of an ordinary Poisson’s equation follows from (35) for \( \kappa = 0 \). The integral (35) expresses for a positive \( D \) as:

\[
g_N(r) = \left( \frac{1}{2\pi} \right)^{D/2} \int_0^\infty \frac{J_{\frac{D}{2}-1}(\rho \tau)}{(\tau r)^{\frac{D}{2}-1}} \cdot \frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{D/2} r^{D-2}} = O\left(\frac{1}{r^{D-2}}\right).
\]  

(36)

For \( \alpha = 0 \), the Green’s function (35) is \( g_N(r)/2 \).

For \( D = 3 \), the integral (35) reduces to:

\[
g_A(r) = \frac{1}{4\pi^2 r} \int_0^\infty \frac{\sin(\rho \tau)}{\rho} \cdot \frac{d\rho}{1 + \kappa^2 \rho^2}.
\]  

(37)

For the certain values of \( \alpha \), Equation (37) expresses the Green’s function in terms of the higher functions:

\[
g_A(r) \equiv \frac{1}{4\pi^2 r} \cdot \begin{cases} 
\frac{2}{\pi^2} \cdot \left[ \sin\left(\frac{\tau}{\kappa}\right) \cdot \text{Ci}\left(\frac{\tau}{\kappa}\right) - \cos\left(\frac{\tau}{\kappa}\right) \cdot \text{Ssi}\left(\frac{\tau}{\kappa}\right) \right], & \text{for } \alpha = -1, \\
\exp\left(-\frac{\tau}{\kappa}\right), & \text{for } \alpha = -2, \\
1 - \exp\left(-\frac{\tau}{\kappa}\right), & \text{for } \alpha = 2,
\end{cases}
\]  

(38)

In Equation (38), \( m([\cdot], [\cdot], x) \) is the Meijer G-function.

The expression (38) for \( \alpha = -2 \) corresponds to the Yukawa potential. There are also closed form solutions in terms of hypergeometric or Meijer functions for some other values of parameter \( \alpha \).

The asymptotic behavior of the functions in (38) could be evaluated without difficulties. In particular, for \( \alpha = -1 \):

\[
g_A(r) \equiv \begin{cases} 
\frac{1}{4\pi^2} + \frac{2\tau - 2\ln \pi - \ln r}{2\pi^2} + O(r), & \text{as } r \to 0, \\
\frac{\kappa}{\pi^2 r^2} - \frac{2\tau - 2\ln \pi + \ln r}{2\pi^2 r} + O\left(\frac{1}{r^3}\right), & \text{as } r \to \infty.
\end{cases}
\]  

(39)

5 | SECOND SCALE-DEVIATING HELMHOLTZ EQUATION

For a scalar function \( \phi = \phi(r) \), the second scale-deviating Helmholtz Equation (11) reads:

\[
B_{2\gamma} \phi = q, \quad B_{\gamma} = S_{(-1)} H \quad \Rightarrow \quad B_{2\gamma} \phi = -\left( \frac{1}{2\pi} \right)^D \int_{\mathbb{R}^D} \frac{1 + |\mathbf{k}|^2}{|\mathbf{k}|^2 + \Omega^2} e^{i k \cdot r} d^D k \int_{\mathbb{R}^D} \phi(r) e^{-i k \cdot r} d^D k.
\]  

(40)

The application of the Fourier transform (19) (2.7) with respect to the spatial variable leads to the relation between the transforms \( \psi(k) \) and \( r(k) \):

\[
\psi(k) = \frac{1 + \kappa^{-\alpha} k^{-\alpha}}{k^2 + \Omega^2} r(k).
\]  

(41)
The solution of (32) reads:

\[ \phi(r) = \int_{\mathbb{R}^3} q(\rho) g_B(r - \rho) d^3\rho. \]  

(42)

The Green’s function in Equation (34) is given by:

\[ g_B(r) = \left( \frac{1}{2\pi} \right)^D \int_{|k|}^{1} \frac{1 + k^{-\alpha}k^{-\alpha}}{k^2 + \Omega^2} e^{i k \cdot r} d^3k. \]  

(43)

The following identity follows from (43) with (24):

\[ \int_{|k|}^{1} \frac{1 + k^{-\alpha}k^{-\alpha}}{k^2 + \Omega^2} e^{i k \cdot r} d^3k = \int_{0}^{\infty} \frac{1 + \rho^{-\alpha}k^{-\alpha}}{\rho^2 + \Omega^2} \rho^{D-1} d\rho \int_{\Theta^{D-1}} e^{i \rho \theta} d\theta, \]  

(44)

Finally, the Equation (43) reduces for an arbitrary integer dimension \( D \) to the integral:

\[ g_B(r) = \left( \frac{1}{2\pi} \right)^{D/2} \int_{0}^{\infty} \rho^{D-1} J_{\frac{D}{2}-1}(\rho r) \cdot \frac{1 + \rho^{-\alpha}k^{-\alpha}}{\rho^2 + \Omega^2} \cdot d\rho. \]  

(45)

For \( D = 3 \), the integral (45) reduces to the expression:

\[ g_B(r) = \frac{1}{2\pi^2 r} \int_{0}^{\infty} \rho \sin(\rho r) \cdot \frac{1 + \rho^{-\alpha}k^{-\alpha}}{\rho^2 + \Omega^2} \cdot d\rho. \]  

(46)

For \( \Omega > 0, \kappa > 0 \), the integral (46) expresses in terms of the hypergeometric function:

\[ g_B(r) = \frac{1}{4\pi r} \cdot \left\{ e^{-\Omega} - \frac{\sinh(\Omega r)}{(\Omega \kappa)^\alpha} \cos\left(\frac{\pi \alpha}{2}\right) - \left(\frac{r}{2}\right)^\alpha \cdot \frac{2\Gamma\left(\frac{1+\alpha}{2}\right)}{\alpha \sqrt{\pi} \Gamma\left(\frac{\alpha}{2}\right)} \cdot {}_2F_2\left(\left[1, \left[1 + \frac{\alpha}{2}, \frac{1}{2}\right]\right], \left(\frac{(\Omega r)^2}{4}\right)\right)\right\}. \]  

(47)

Particularly, for the certain values of \( \alpha \), Equation (47) resolves:

\[ g_B(r) \equiv \frac{1}{4\pi r} \cdot \left\{ \begin{array}{ll}
\frac{1}{k^2 \Omega^\alpha} [(k^2 \Omega^2 - 1)e^{-\Omega^2} + 1], & \text{for } \alpha = 2, \\
e^{-\Omega^2} \cdot \frac{1}{\sqrt{2\pi} \Omega^\alpha} \cdot [2 \sinh(\Omega r) - e^{\Omega} \text{erf}(\sqrt{\Omega r}) - e^{-\Omega} \text{erf}(\sqrt{r \Omega})], & \text{for } \alpha = \frac{1}{2}, \\
e^{-\Omega^2} + \sqrt{\frac{2\pi}{\Omega}} - \sqrt{\frac{2\pi r}{\Omega}} \cdot [2 \sinh(\Omega r) - e^{\Omega} \text{erf}(\sqrt{\Omega r}) + e^{-\Omega} \text{erf}(\sqrt{r \Omega})], & \text{for } \alpha = -\frac{1}{2}.
\end{array} \right. \]  

(48)

The functions in Equation (3.9) are:\n\( {}_2F_2([ ], [ ], \alpha) \) is the hypergeometric function,\nerf(\alpha) is the error function,\nerf(\alpha) is the imaginary error function.

6 \hspace{1em} SECONDScale-deviating Poisson’s Equation

The second scale-deviating fractional Poisson’s Equation (6) reads for a scalar function \( \phi = \phi(r) \):

\[ B_{\lambda} \phi = q, \hspace{0.5em} B_{\lambda} = S(-1)\Delta, B_{\lambda} \phi = -\left( \frac{1}{2\pi} \right)^D \int_{|k|}^{1} \frac{1 + |k|^{\alpha}}{k^2} e^{i k \cdot r} d^3k \int_{\mathbb{R}^3} \phi(\rho) e^{-i k \cdot \rho} d^3\rho. \]  

(49)
The application of the Fourier transform (19)–(20) with respect to the spatial variable leads to the relation between the transforms \( \psi(\mathbf{k}) \) and \( r(\mathbf{k}) \):

\[
\psi(\mathbf{k}) = \frac{1 + \kappa^{-a}k^{-a}}{k^2} r(\mathbf{k}). \tag{50}
\]

The solution of (49) delivers:

\[
\phi(r) = \int_{R^2} g(\rho) \, g_B(\mathbf{r} - \mathbf{r}) d^D \rho. \tag{51}
\]

The Green’s function in Equation (51) yields:

\[
\psi_B(\mathbf{r}) = \left( \frac{1}{2\pi} \right)^D \int_{\mathbb{R}^D} \frac{1 + \kappa^{-a}k^{-a}}{k^2} e^{i \mathbf{k} \cdot \mathbf{r}} d^D \mathbf{k}. \tag{52}
\]

The identity results from (52) and (24):

\[
\int_{\mathbb{R}^D} \frac{1 + \kappa^{-a}k^{-a}}{k^2} e^{i \mathbf{k} \cdot \mathbf{r}} d^D \mathbf{k} = \int_0^\infty \frac{1 + \rho^{-a}k^{-a}}{\rho^2} \rho^{D-1} d\rho \int_0^{\theta_0} e^{i \rho \theta} d\theta. \tag{53}
\]

After the substitution of Equation (53), the Equation (52) reduces to the formula:

\[
\psi_B(\mathbf{r}) = \left( \frac{1}{2\pi} \right)^D \int_0^\infty \frac{J_{\frac{D}{2}-1}(\rho r) \cdot \frac{1 + \rho^{-a}k^{-a}}{\rho^{\frac{D}{2}-1}} \cdot d\rho = I_5 + I_6. \tag{54}
\]

The integral (54) solves with the expressions \( I_5, I_6 \) from Equation (A14), (A15). For \( \kappa = 0 \), the Green’s function (54) reads (Newtonian potential):

\[
\psi_N(\mathbf{r}) = \left( \frac{1}{2\pi} \right)^D \int_0^\infty \frac{J_{\frac{D}{2}-1}(\rho r) \cdot \frac{1}{\rho^{\frac{D}{2}-1}} \cdot d\rho = \frac{1}{4\pi r} \right. \tag{55}
\]

For \( a = 0 \), the Green’s function (54) is \( 2\psi_N(\mathbf{r}) \).

Using for \( D = 3, a \neq 1 \) and \( \kappa > 0 \) the Equation (A13), the integral (54) condenses to the closed form formula:

\[
\psi_B(\mathbf{r}) = \frac{1}{2\pi^2 r} \int_0^\infty \frac{\sin(\rho r)}{\rho} \cdot (1 + \rho^{-a}k^{-a}) d\rho = \frac{1}{4\pi r} \left\{ 1 + \frac{2}{a} \sqrt{\frac{r}{2\kappa}} \right\}. \tag{56}
\]

The Green’s function follows for certain values of \( a \) from Equation (56):

\[
\psi_B(\mathbf{r}) \equiv \frac{1}{4\pi r} \cdot \begin{cases} 
1 + 2 \cdot \sqrt{\frac{r}{2\kappa}}, & \text{for } a = \frac{1}{2}, \\
1 + \frac{2a}{\sqrt{\kappa}}, & \text{for } a = -1, \\
1, & \text{for } a = -2, \\
1 - \frac{r}{2\kappa}, & \text{for } a = 2, \\
1 + \sqrt{\frac{r}{2\kappa}}, & \text{for } a = -\frac{1}{2}.
\end{cases} \tag{57}
\]

Figure 1 demonstrates the potentials \( g_A(\mathbf{r}), g_B(\mathbf{r}) \) and their short- and long-range asymptotes for \( a = -1 \). According to (39), the function \( g_A(\mathbf{r}) \) behaves as \( g_N(\mathbf{r}) \) on the short distance. Otherwise, the function \( g_B(\mathbf{r}) \) coincides with \( g_A(\mathbf{r}) \) on the long distance.

For all admissible values of parameters \( D \) and \( \kappa > 0 \), the following inequalities are obviously satisfied:

\[
g_B(\mathbf{r}) > g_A(\mathbf{r}). \tag{58}
\]

This behavior displays Figure 2 with the plots of the Green’s functions \( g_A(\mathbf{r}) \) for \( a = -1 \) and \( g_B(\mathbf{r}) \) for \( a = -\frac{1}{2} \). For both graphs the value was set to \( \kappa = 1 \).
**FIGURE 1** Green's functions $\mathcal{G}_A$, $\mathcal{G}_B$ and their short- and long-range asymptotes

\[ \mathcal{G}_A = \frac{1}{2\pi r^2} \left[ \sin \left( \frac{r}{\kappa} \right) \text{Ci} \left( \frac{r}{\kappa} \right) - \cos \left( \frac{r}{\kappa} \right) \text{Ssi} \left( \frac{r}{\kappa} \right) \right] \]

$\alpha = -1$, Eq. (3.7)

\[ \mathcal{G}_N = \frac{1}{4\pi r} \]

\[ \mathcal{G}_B = \frac{1}{4\pi r} + \frac{\kappa}{2\pi^2 r^2} \]

$\alpha = -1$, Eq. (5.10)

**FIGURE 2** Green's functions $\mathcal{G}_A$, $\mathcal{G}_B$ and their short- and long-range asymptotes

\[ \mathcal{G}_A = \frac{\kappa}{2\pi^2 r^2} \]

\[ \mathcal{G}_N = \frac{1}{4\pi r} \]

\[ \mathcal{G}_B = \frac{1}{4\pi r} + \frac{1}{\sqrt{2\pi^3} r^3} \]

$\alpha = -1/2$, Eq. (5.10)

\[ \mathcal{G}_A = \frac{1}{2\pi^2 r^2} \left[ \sin \left( \frac{r}{\kappa} \right) \text{Ci} \left( \frac{r}{\kappa} \right) - \cos \left( \frac{r}{\kappa} \right) \text{Ssi} \left( \frac{r}{\kappa} \right) \right] \]

$\alpha = -1$, Eq. (3.7)
From (56) and (37) follows the asymptotic behavior of Green’s functions:

\[ g_A(r) \sim g_B(r) \sim \frac{\Gamma\left(\frac{1-a}{2}\right)}{4\pi^{\frac{1}{2}}(2\kappa)^a} \frac{1}{r^{1-a}} = O\left(\frac{1}{r^{1-a}}\right) \quad \text{for } D = 3 \text{ as } r \to \infty. \]  

(59)

For an arbitrary dimension \( D \), the long-range asymptotic expansions are\(^{11}\):

\[ g_A(r) \sim g_B(r) \sim \frac{\Gamma\left(\frac{D-2-a}{2}\right)}{4\pi^{\frac{1}{2}}(2\kappa)^a} \frac{1}{r^{D-2-a}} = O\left(\frac{1}{r^{D-2-a}}\right), \quad \text{as } r \to \infty. \]  

(60)

\[ \nabla g_A \sim \nabla g_B \sim \frac{\Gamma\left(\frac{D-2-a}{2}\right)}{4\pi^{\frac{1}{2}}(2\kappa)^a} \frac{D-2-a}{r^{D-1-a}} = O\left(\frac{1}{r^{D-1-a}}\right) \quad \text{as } r \to \infty. \]  

(61)

Comparing the asymptotic behaviors (60) to (59) one can formally conclude, that \( a \) leads the fluctuation of the long-range dimension of the space from \( D \) to \( D - a \).

Another argument provides the shell theorem.\(^{12}\) Consider the sphere \( \Omega \) of radius \( R_S \) in three-dimensional space. The density of the sphere is homogeneous and equals in each point to \( \rho \). The gravitational flux over the surface \( \Omega \), that encapsulates \( \Omega \), is \( \Phi_3[g_A] \).\(^{13}\) According to the shell theorem,\(^{12}\) the Newtonian flux of the point or sphere of mass \( M = 4\pi \rho R_S^3/3 \) does not depend on radius and is equal to:

\[ \Phi_3[\nabla g_N] = 4G\pi \rho. \]

On the large distances, \( R_S \ll r \), the flux \( \Phi_3[g_A] \) is, however, the decreasing function of radius \( r \). The flux \( \Phi_3[g_B] \) is the growing function of radius \( r \). The ratios of fluxes for both operators to the Newtonian flux \( \Phi_3[\nabla g_N] \) are shown on the Figure 3.

We search now the dimension of space, for which the flux remains constant for the scale-deviating Laplace-type operators (6). For this purpose, the \( D \)-ball of radius \( r \) is studied. The surface of the \((D - 1)\)-dimensional sphere \( \Omega \), embedded in the \( D \)-dimensional space, is equal to\(^{14}\):
\[
S_{D-1} = \frac{2 \pi^{\frac{D}{2}} r^{D-1}}{\Gamma\left(\frac{D}{2}\right)}.
\]

There is the point source \( q = \delta(r) \) in the center of the \( D \)-ball, which induces the field \( g_A(r) \). According to (60), the flux \( \Phi_D \) over the surface area \( \Gamma_{D-1} \) of the \((D-1)\)-sphere depends on radius:

\[
\Phi_D[g_A] = \int_{\mathcal{O}=\Gamma_{D-1}} \nabla g_A(r) \cdot d\Gamma_{D-1} = \frac{\Gamma\left(\frac{D-d}{2}\right)}{(2\pi)^{\frac{D}{2}}} \Gamma\left(1 + \frac{d}{2}\right) \Gamma\left(\frac{D}{2}\right) r^d.
\]

Remarkably, that the flux \( \Phi_{D-d} \) over the surface area of the \((D - 1 - d)\)-dimensional sphere remains constant:

\[
\Phi_{D-d}[g_A] = \int_{\mathcal{O}=\Gamma_{D-1-d}} \nabla g_A(r) \cdot d\Gamma_{D-1-d} = \frac{1}{(2\pi)^d} \Gamma\left(1 + \frac{d}{2}\right) \Gamma\left(\frac{D}{2}\right).
\]

This remark extends the shell theorem to the surfaces of the dimension \( D - d \). Namely, if the dimension of the space varies from \( D \) to \( D - d \), the fluxes of the scale-deviating operators \( \mathcal{A}_L, \mathcal{B}_L \) remain constant.

7 \hspace{1em} SCALE-DEVIATING LORENZ INVARIANT EQUATIONS

The fractional derivatives in the dynamic equations are considered as partial fractional derivatives with respect to space coordinates \( x^\nu \) and momentum coordinates \( k_\nu, \nu = 0, 1, 2, 3 \). When a Greek index variable appears twice in a single term, it indicates summation of that term over the values from 0 to \( D = 3 \). In Cartesian coordinates, the metric for Lorentz-Minkowski space is defined and ordered as \( \eta_{\mu\nu} = diag(-1, 1, 1, 1) \). Eliminating the first row and first column of \( \eta_{\mu\nu} \), we are left with the metric tensor of a Euclidean \( 3 \) – space. The index \( i \) of space coordinates \( x^i \) and moment coordinates \( k_i \) runs in the static equations of mathematical physics from one to \( D \). When a latin index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index from 1 to \( D \). Such that \( k \) is the Pythagorean norm of moment coordinates. The Lorentz norm \( l \) reads:

\[
l^2 = \eta_{\mu\nu} k^\mu k^\nu = k_\mu k^\mu.
\]

We use the Lorentz-Minkowski metric to solve the wave equations. For formal derivations with D’Alembert wave equations, one can use the imaginary time in space \( \hat{x}^\nu = \{ i k^0, x^1, x^2, x^3 \} \) and moment coordinates \( \hat{k}_\nu = \{ i k_0, k_1, k_2, k_3 \} \). The metric in this case remains the Euclidean \( \tilde{\eta}_{\mu\nu} = diag(1, 1, 1, 1) \) and the Lorentz norm reads:

\[
l^2 = \tilde{\eta}_{\mu\nu} \hat{k}_\mu \hat{k}^\nu.
\]

We will use the imaginary coordinate for derivation of the solutions of 3-dimensional wave equations from the 4-dimensional Laplace equations. The Lorentz norm and Pythagorean norm related to each other as \( l = \sqrt{-k_0^2 + k^2} \).

If the governing equations satisfy some invariance condition, the same is valid for the corresponding symbol. For the equations of mathematical physics, the symbols

\[
s = s(k), \quad s = s(l)
\]

must satisfy the certain conditions.

There are some restrictions, imposed on the symbols of the operators of mathematical physics. The physically admissible symbols conform several symmetry conditions:

1. Uniformity of space. The symbol space must be invariant to shift of spatial coordinates. The symbol does not depend explicitly on spatial coordinates. This symmetry guarantees the momentum conservation.
2. Uniformity of time. The symbol does not depend explicitly upon time shift, what assurances energy conservation.
3. Rotational symmetry of symbol reassures angular momentum conservation.
4. Isometric invariance, or Lorenz invariance, the relativistic conformity.
The common character of the equations of mathematical physics is also their scale invariance. The scale invariance is a feature of the equations of mathematical physics that do not alter, if the scales of the independent variables are multiplied by a mutual factor. For example, if length or time are multiplied by a certain constant, the physical laws do not vary. Scale invariance symbolizes a certain universality. For example, the classical field theory is commonly pronounced by a function, which depends on coordinates. If a theory is scale-invariant, its field equations should be invariant under a rescaling of the coordinates, combined with some specified rescaling of the fields. Scale invariance will typically hold if no fixed length scale appears in the theory. The scale invariance obey some restrictions on the symbol of the governing equation. By definition, a homogeneous polynomial is a polynomial whose nonzero terms all have the same degree. If the symbol is the homogeneous polynomial of second order with respect to $l$ or $k$, namely

$$s = -l^2, \quad s = -k^2,$$

then the governing equations are scale-invariant. The D’Alembert and Laplace Equations (1) possess this feature.

As an example of the application of the above technique, we discuss briefly the scale-deviating Newtonian potential. In the classical Newtonian mechanics, space and time are decoupled. Newtonian space is a flat manifold and time $x^0 = ct$ is absolute. The D-dimensional Newtonian space–time is foliated by the absolute time function $t$. Each foliation is virtually flat space. The spatial coordinates $x^\mu$ and time coordinate $t$ of the inertial observers are connected via the Galilei group. Newtonian gravity acts as an instantaneous force between gravitating masses. Consider two particles separated a distance $r$ with gravitational masses $M$ and $m$ in $D$ spatial dimensions. The Newtonian gravitational force $F(r)$ between two particles is given by:

$$F = \frac{GMm}{r^D} \cdot r = |r|.$$

(64)

The correspondence principle dictates that under definite conditions the theory of General Relativity should replicate Newton’s theory of gravity. These conditions are known as the Newtonian limit. In the Minkowski background of spatial dimension $D = 3$, consider the point particle with embedding coordinates:

$$\{x^\mu(\tau)\} = \{x^0(\tau) = ct, x^i(\tau)\}$$

(65)

For such particle, the Newtonian limit is defined by four conditions:

I. $x^0 \gg x^i$,

II. $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon f_{\mu\nu}$,

III. $g_{0j} = 0$,

IV. $g_{\mu\nu} = g_{\mu\nu}(x^0)$.

The condition (I) states that the longitudinal “velocity” is much larger than the transverse velocity. This means, that the speed $|v|$ of the particle is much less than the speed of light $c$:

$$c = \frac{v^2}{c^2} \ll 1.$$

The order of approximation in Bachmann–Landau notation of physically significant quantities is of $O(\epsilon)$.

The requirement (II) means that gravity is weak. Consequently, we can expand the metric $g_{\mu\nu}$ around the Minkowski vacuum metric $\eta_{\mu\nu}$. For this expansion, the metric $g_{\mu\nu}$ uses the approximation of the first order in the perturbation $f_{\mu\nu}$.

The condition (III) demands that the line element is invariant under the transformation $x^0 \to -x^0$. The transformation (IV) leads to a static gravitational field.

The weakness of the curvature leads to:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon f_{\mu\nu}, g'^{\mu\nu} = \eta'^{\mu\nu} - \epsilon f'^{\mu\nu}.$$

(66)
From Equation (66) follows:
\[ g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha + O(\varepsilon^2). \]

In the Newtonian approximation, the Christoffel symbols and the contacted Ricci tensor are:
\[ 2\Gamma^\alpha_{\mu\nu} = \eta^\alpha^\beta (\partial_\mu f_{\beta\nu} + \partial_\nu f_{\beta\mu} - \partial_\beta f_{\mu\nu}). \]
\[ R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha}. \]  
(67)

For the reduction of formulas, we use the deviatoric perturbation of the metric tensor:
\[ F_{\mu\nu} = f_{\mu\nu} - \frac{1}{2} f_{\eta_{\mu\nu}}. \]
\[ f = Sp f_{\mu\nu}. \]

With the deviatoric perturbation, appears the standard linearized expression of Einstein tensor:
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R_{\eta_{\mu\nu}} = -\frac{1}{2} \Delta F_{\mu\nu} + \frac{1}{2} \partial^\alpha \partial_\mu F_{\nu\alpha} + \frac{1}{2} \partial^\alpha \partial_\nu F_{\mu\alpha} - \frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta F_{\alpha\beta} = k^2 T_{\mu\nu}. \]  
(68)

The harmonic coordinates \( x^\alpha \rightarrow X^\alpha = x^\alpha - \zeta^\alpha(x) \)

satisfy the equations:
\[ \partial^2 \zeta_\alpha + \partial^\nu F_{\nu\alpha} = 0. \]  
(69)

In the harmonic coordinates, the Einstein tensor (68) simplifies to:
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R_{\eta_{\mu\nu}} = -\frac{1}{2} \partial^2 F_{\mu\nu}. \]  
(70)

Finally, the standard form for the linearized Einstein equations in harmonic coordinates reduces to the Laplace equation for each of components of metric perturbation.\( \text{18} \) The Newtonian equations in the covariant notation read:
\[ \partial^2 F_{\mu\nu} = -2k^2 T_{\mu\nu}. \]  
(71)

For the static field, one component of perturbation tensor is substantial: \( F_{00} = 2k^2 \phi. \) The source density reads \( =2k^2 T_{00}. \) 

Apparently, the symbol is the homogeneous polynomial of second order, such that the governing equations are scale-invariant.

If the symbol is not a homogeneous polynomial of second order, the governing equations are not scale-invariant. The symbols and corresponding equations will be referred to as scale-deviating.

Remarkably, there are scale-deviating symbols of the order less or equal than two, which obey the symmetry conditions. We study the scale-deviating equations using the correction operators, which do not alter the character of solutions on short distances, but change the behavior on the long distances. The order of the operators must less than one. There are oppositely acting correction operators, which alter the character of solutions on short distances, but preserve the behavior on the long distances.

Using Equation (71), the two types of Riesz corrections of the linearized Einstein Equations (71) are:
\[ AF_{\mu\nu} = -2k^2 T_{\mu\nu}, BF_{\mu\nu} = -2k^2 T_{\mu\nu}. \]  
(72)

For (72), the scale-deviating operators of Riesz type \( A, B \) are the scale-deviating fractional Poisson’s equations for each component of metric perturbation \( F_{\mu\nu}. \)

8 | EXAMPLE: SCALE-DEVIATING FRACTIONAL POISSON’S EQUATIONS

We calculate the potential of the sphere with the radius \( R_S. \) Let \( \rho \) be the homogeneous density (mass per unit area) of the shell. Accordingly the mass of the shell reads:
\[ M = \frac{4\pi \rho}{3} R_S^3. \]
The mass of a small test object a distance \( r \) from the center of the shell is \( m \). The integrals of the Green’s functions are evaluated in the observation point (Figure 4).

In the cylindrical coordinates the distance from the point on the shell and the observation point is:

\[
P = \sqrt{R^2 + r^2 - 2rR \cos \varphi}.
\] (73)

The relation between the angles \( \theta, \varphi \) read:

\[
\theta = \arcsin \left( \frac{R}{P} \sin \varphi \right).
\] (74)

At first, the integrals of the Green’s functions are evaluated for the thin spherical shell with radius \( R \) and then integrated over the radius of the shell (\( 0 < R \leq R_S \)):

\[
\Psi_A(R) = 2\pi \int_0^{R_S} \left[ \int_0^\pi \frac{1}{2\pi^2 P} \left( \sin \left( \frac{P}{K} \right) Cl \left( \frac{P}{K} \right) - \cos \left( \frac{P}{K} \right) Ssi \left( \frac{P}{K} \right) \right) \sin \varphi \, d\varphi \right] R^2 dR.
\] (75)

Further, the calculation of the gradient leads to the acceleration in the examined point:

\[
a_A = \nabla \Psi_A(R).
\] (76)

The substitution of (76) into equation for flux leads to the exact expressions for the flux:

\[
\Phi_D[g_A] = 4\pi R_S^2 |a_A|, \quad \Phi_D[g_B] = 4\pi R_S^2 |a_B|
\] (77)

The integrals are evaluated in the closed form for several cases. Recall that \( R_S \) is the radius of the sphere and \( r \) is the distance of the observation point from the center of the sphere. We introduce the dimensionless parameters:

\[
\zeta = \frac{R_S}{K}, \quad \xi = \frac{r}{K}, \quad \epsilon = \frac{R_S}{r} \geq 1.
\]

The calculation of the integrals (75) could be performed in the closed form in some special cases.

1°. According to Equation (75), potential Equation (28a) reads:

\[
\Psi_A(r) \equiv 2\pi \int_0^{R_S} \left\{ \int_0^\pi \left[ \sin \left( \frac{P}{K} \right) Cl \left( \frac{P}{K} \right) - \cos \left( \frac{P}{K} \right) Ssi \left( \frac{P}{K} \right) \right] \sin \varphi \, d\varphi \right\} R^2 dR.
\] (78)
For the calculations the Equations (A1)–(A3) from Appendix A are applied:

\[ \int_{0}^{\pi} \frac{1}{2\pi^2 P} \left[ \sin \left( \frac{P}{\kappa} \right) Ci \left( \frac{P}{\kappa} \right) - \cos \left( \frac{P}{\kappa} \right) Ssi \left( \frac{P}{\kappa} \right) \right] \sin \varphi \, d\varphi = I_1 + I_2 + I_3. \]  

(79)

With the expressions (A4)-(A11) from Appendix A, the integral (78) reads:

\[ \Psi_A(r) \equiv \frac{4\rho \kappa^2}{\xi} \left[ C_{c1} Ci((1 + \epsilon)\zeta) + C_{c2} Ci((1 - \epsilon)\zeta) + C_{s1} Ssi((1 + \epsilon)\zeta) + C_{s2} Ssi((1 - \epsilon)\zeta) + C_1 \ln \left( \frac{1 + \epsilon}{1 - \epsilon} \zeta \right) + C_0 \right]. \]  

(80)

The coefficients in Equation (80) are:

\[ C_{c1} = -\sin((1 + \epsilon)\zeta) + \epsilon \xi \cos((1 + \epsilon)\zeta), \]
\[ C_{c2} = -\sin((1 - \epsilon)\zeta) - \epsilon \xi \cos((1 - \epsilon)\zeta), \]
\[ C_{s1} = -\cos((1 + \epsilon)\zeta) - \epsilon \xi \sin((1 + \epsilon)\zeta), \]
\[ C_{s2} = \cos((1 - \epsilon)\zeta) + \epsilon \xi \sin((1 - \epsilon)\zeta), \]
\[ C_1 = 1 + \xi^2 (\rho^2 + 1), \]
\[ C_0 = \xi^2 \epsilon - \pi \epsilon \xi \cos((1 - \epsilon)\zeta) - \pi \sin((1 - \epsilon)\zeta). \]  

(81)

2°. Integral (75) for the potential Equation (29) calculates to:

\[ \Psi_A(r) \equiv 2\pi \int_{0}^{R_S} \left\{ \int_{0}^{\pi} \frac{\exp \left( -\frac{P}{\kappa} \right)}{4\pi P} \sin \varphi \, d\varphi \right\} R^2 \, dR = \frac{6\pi \rho \kappa (1 + \zeta)}{\xi^2} \left[ (1 + \zeta) e^{-2\varphi} + 1 - \zeta \right]. \]  

(82)

3°. The flux of the Newtonian potential \( \sim 1/P \):

\[ \Psi_N(r) = 2\pi \int_{0}^{R_S} \left\{ \int_{0}^{\pi} \frac{\sin \varphi}{4\pi P} \, d\varphi \right\} R^2 \, dR = \frac{4\pi \rho}{3} R_S^3 = \frac{4\pi \rho \kappa^2 \xi^3}{3 \xi} = \frac{M}{r}. \]  

(83)

As already deliberated in Section 6, for the potential (83) the flux remains constant. Consequently, the classical shell theorem is valid.12

9 | CONCLUSIONS

The mathematical model introduces the scale-deviating versions for the equations of mathematical physics. As an example, we study the covariant Newtonian equations. The scale-deviating equations are isotropic equations of the lower order, than the Laplacian equations. The equations contain two additional parameters. One parameter \( \kappa \) portrays the character length of deviation. The second parameter \( \alpha \) provides the order of deviation. The asymptotic behavior on the short distances matches the behavior of the Newtonian equations. The solutions of the scale-deviating equations could be used for the portraying the deviations at infinity. These features could provide the simple mathematical apparatus for the fitting of the experimentally observed flux functions on the long distances.

ACKNOWLEDGMENTS

The research was partially supported by Alexander von Humboldt Foundation, Bonn, Germany.

CONFLICT OF INTEREST

Authors declare no competing interests. The manuscript was not be submitted to more than one journal for simultaneous consideration. The submitted work is original and not have been published elsewhere in any form or language (partially or in full), unless the new work concerns an expansion of previous work. A single study was not split up into several parts to increase the quantity of submissions and submitted to various journals or to one journal over time. Results are presented clearly, honestly, and without fabrication, falsification or inappropriate data manipulation. Authors adhere to discipline-specific rules for acquiring, selecting and processing data.
AUTHOR CONTRIBUTION
All authors whose names appear on the submission (1) made substantial contributions to the conception or design of the work; or the acquisition, analysis, or interpretation of data; or the creation of new software used in the work; (2) drafted the work or revised it critically for important intellectual content; (3) approved the version to be published; and (4) agree to be accountable for all aspects of the work in ensuring that questions related to the accuracy or integrity of any part of the work are appropriately investigated and resolved.

LIST OF SYMBOLS

- $S_{(n)}$: Scaling isotropic operator
- $T_{(n)}$: Scaling isometric operator
- $\mathcal{H} = \Delta + \Omega^2$: Helmholtz operator
- $\alpha$: Scalarization order
- $\partial^2 \equiv \Delta$: Laplace operator
- $A_L \equiv S_{(1)} \partial^2$: First scale-deviating Laplace operator
- $B_L \equiv S_{(-1)} \partial^2$: Second scale-deviating Laplace operator
- $A_H \equiv S_{(1)} \mathcal{H}$: First scale-deviating Helmholtz operator
- $B_H \equiv S_{(-1)} \mathcal{H}$: Second scale-deviating Helmholtz operator
- $\kappa$: scaling parameter dimension of an inverse time or inverse length
- $k^2 = \frac{8 \pi G}{c^4}$: Coupling constant
- $n$: Type parameter, $n = 1$ or $n = -1$
- $\zeta = \frac{r}{R_S}$: Ratio of the distance from the center of the sphere to the scaling length (Dimensionless parameter)
- $\xi = \frac{R_S}{\kappa}$: Ratio of the sphere to the scaling length (Dimensionless parameter)
- $M = \frac{4 \pi \rho R_S^3}{3}$: Mass of the sphere the 3-dimensional space
- $\rho$: Homogeneous density of the sphere
- $R_S$: Radius of the sphere the 3-dimensional space
- $r$: Distance from the center of the sphere to the observation point
- $\Box = -\frac{\partial^2}{\partial t^2} + \Delta$: D’Alembert operator
- $A_D \equiv T_{(1)} \Box$: First scale-deviating D’Alembert operator
- $B_D \equiv T_{(-1)} \Box$: Second scale-deviating D’Alembert operator
- $\Phi_D[\phi]$: Flux of field $\phi$ in space of dimension $D$
- $S_{D-1}$: Surface area of the $(D - 1)$-sphere, embedded in the $D$-dimensional space

ORCID

Vladimir Kobelev ⓒ https://orcid.org/0000-0002-2653-6853

REFERENCES

1. Stillinger FH. Axiomatic basis for spaces with non integer dimension. J Math Phys. 1977;18(6):1224-1234.
2. Muslih SI, Agrawal OP. Riesz fractional derivatives and fractional dimensional space. Int J Theoret Phys. 2010;49:270-275.
3. Reed M, Simon B. Methods of Modern Mathematical Physics. Fourier Analysis Self-Adjointness. Vol 2. London, UK: Academic Press; 1992.
4. Treves F. Introduction to Pseudodifferential and Fourier Integral Operators. New York, NY: Plenum Press; 1982.
5. Samko SG, Kilbas AA, Marichev OI. Fractional Integrals and Derivatives: Theory and Applications. New York, NY: Gordon and Beach; 2006.
6. Vladimirov VS. Equations of Mathematical Physics. New York, NY: Marcel Dekker; 1971.
7. Ohtaka K. Green’s functions. In: Trigg GL, ed. Mathematical Tools for Physicists. Weinheim, Wiley; 2005.
8. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Amsterdam, Netherlands: Elsevier; 2006.
9. NIST NIST Digital Library of Mathematical Functions. Ver. 1.0.28; 2020. https://dlmf.nist.gov/.
10. Olver FW, Lozier DW, Boisvert RF, Clark CW. NIST Handbook of Mathematical Functions. Cambridge, MA: Cambridge University Press; 2010.
11. de Bruijn NG. Asymptotic Methods in Analysis. North-Holland Publ. Co, Amsterdam; 1958.
12. Arens R. Newton’s observations about the field of a uniform thin spherical shell. Note di Matematica. 1990;X(Suppl. 1):39-45.
13. Misner CW, Thorne KS, Wheeler JA. Gravitation. San Francisco: W. H. Freeman; 1973.
14. Huber G. Gamma function derivation of n-sphere volumes. Amer Math Monthly. 1982;89(5):301-302. https://doi.org/10.2307/2321716.
APPENDIX A

\[ I_1 = \int_0^\pi \frac{\cos(\sqrt{\eta^2 + \zeta^2 - 2\eta \zeta \cos \varphi})}{\sqrt{\eta^2 + \zeta^2 - 2\eta \zeta \cos \varphi}} \sin \varphi \, d\varphi = \frac{\eta \kappa}{4\xi^2 \pi} [\sin(\zeta + \eta) + \sin(\eta - \zeta)]. \]  
\[ I_2 = -2 \int_0^\pi \frac{\sin(\sqrt{\eta^2 + \zeta^2 - 2\eta \zeta \cos \varphi}) \cdot \cos(\sqrt{\eta^2 + \zeta^2 - 2\eta \zeta \cos \varphi})}{\sqrt{\eta^2 + \zeta^2 - 2\eta \zeta \cos \varphi}} \sin \varphi \, d\varphi = \frac{\eta \kappa}{4\xi^2 \pi} (\theta_1 + \theta_3 + \theta_4). \]  
\[ I_3 = 2 \int_0^\pi \frac{\cos(\sqrt{\eta^2 + \zeta^2 - 2\eta \zeta \cos \varphi}) \cdot \cos(\sqrt{\eta^2 + \zeta^2 - 2\eta \zeta \cos \varphi})}{\sqrt{\eta^2 + \zeta^2 - 2\eta \zeta \cos \varphi}} \sin \varphi \, d\varphi = \frac{\eta \kappa}{4\xi^2 \pi} (\theta_2 - \theta_3 + \theta_4). \]  
\[ \theta_1 = 2 \text{Si}(\eta - \zeta) \cdot \sin(\eta - \zeta) - 2 \text{Si}(\eta + \zeta) \cdot \sin(\eta + \zeta). \]  
\[ \theta_2 = 2 \text{Ci}(\eta - \zeta) \cdot \cos(\eta - \zeta) - 2 \text{Ci}(\eta + \zeta) \cdot \cos(\eta + \zeta). \]  
\[ \theta_3 = \text{Ci}(2\eta - 2\zeta) - \text{Ci}(2\eta + 2\zeta). \]  
\[ \theta_4 = \ln \frac{\eta + \zeta}{\zeta - \eta}. \]  
\[ \int_0^\zeta \theta_3 \, d\eta = (\zeta - \xi) \text{Ci}(2\xi - 2\zeta) - (\zeta + \xi) \text{Ci}(2\xi + 2\zeta) + 2\xi \text{Ci}(2\xi) - 2\sin^2 \zeta \sin 2\xi. \]  
\[ \int_0^\zeta \text{Ci}(\eta + \zeta) \cdot \cos(\eta + \zeta) \, d\eta = \frac{1}{2} \text{Si}(2\zeta) - \frac{1}{2} \text{Si}(2\zeta + 2\xi) + \text{Ci}(\zeta + \xi) \sin(\zeta + \xi) - \text{Ci}(\xi) \sin(\xi). \]  
\[ \int_0^\zeta \text{Si}(\eta + \zeta) \cdot \sin(\eta + \zeta) \, d\eta = -\frac{1}{2} \text{Si}(2\zeta) + \frac{1}{2} \text{Si}(2\zeta + 2\xi) - \text{Si}(\zeta + \xi) \cos(\zeta + \xi) + \text{Si}(\xi) \cos(\xi). \]
\[ \int_0^\zeta \text{Si}(\eta - \xi) \cdot \sin(\eta - \xi) d\eta = \frac{1}{2} \text{Si}(2\xi) + \frac{1}{2} \text{Si}(2\zeta - 2\xi) - \text{Si}(\zeta - \xi) \cos(\zeta - \xi) - \text{Si}(\xi) \cos(\xi). \quad (A11) \]

\[ \int_0^\zeta \text{Ci}(\eta - \xi) \cdot \cos(\eta - \xi) d\eta = -\frac{1}{2} \text{Si}(2\xi) - \frac{1}{2} \text{Si}(2\zeta - 2\xi) + \text{Ci}(\zeta - \xi) \sin(\zeta - \xi) + \text{Ci}(\xi) \sin(\xi). \quad (A12) \]

\[ \int_0^\infty \frac{\sin(\rho r)}{\rho} \frac{d\rho}{\rho^{\alpha+\kappa-\alpha}} = \frac{\sqrt{\pi}}{2} \left( \frac{1}{\kappa} \right)^\alpha \Gamma \left( \frac{1-\alpha}{2} \right) \cdot \left( \frac{\rho}{\kappa} \right)^\alpha. \quad (A13) \]

\[ I_5 = \left( \frac{1}{2\pi} \right)^{D/2} \int_0^\infty \rho^{D-3} J_{\frac{D}{2}-1}(\rho r) \cdot \frac{1}{(\rho r)^{\alpha-1}} \cdot \frac{1}{\Gamma \left( \frac{D}{2} - 1 \right)} \cdot \frac{1}{4\pi^{D/2}} \cdot \frac{1}{r^{D-2}}. \quad (A14) \]

\[ I_6 = \left( \frac{1}{2\pi} \right)^{D/2} \int_0^\infty \rho^{D-3} J_{\frac{D}{2}-1}(\rho r) \cdot \frac{\rho^{-\alpha-\kappa+\alpha}}{(\rho r)^{\alpha-1}} \cdot \frac{1}{\Gamma \left( \frac{D}{2} - 1 \right)} \cdot \frac{1}{2^{\alpha+2\pi^{D/2}k^2}} \cdot \frac{1}{r^{D-2}}. \quad (A15) \]