GAUGE INVARIANCE AND SECOND CLASS CONSTRAINTS
IN 3-D LINEARIZED MASSIVE GRAVITY

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Abstract
A recently introduced approach for the dynamical analysis and quantization of
field theoretical models with second class constraints is illustrated applied to linearized gravity in 3-D. The canonical structure of two different models of linearized gravity in 3-D, the intermediate and the self dual models, is discussed in detail. It is shown that the first order self dual model whose constraints are all second class may be regarded as a gauge fixed version of the second order gauge invariant intermediate model. In particular it is shown how to construct the gauge invariant hamiltonian of the intermediate model starting from the one of the self dual model. The relation with the topologically massive linearized gravity is also discussed.
I. Introduction

The dynamical analysis and quantization of constrained systems [1] has been actively studied for many years, but still presents unsolved problems which represent serious obstacles to extract the physical content of different models. Some of these problems emerged from the very involved, and sometimes bizarre constraint structure of some of the systems of physical interest. But some others are of a very fundamental nature and relate with the basis of our understanding of, for example, what is a gauge symmetry. This is the case with the very old problem of the quantization of systems with infinitely many degrees of freedom and second class constraints [1]. For these systems at odds with what occurs when there are only a finite number of degrees of freedom the Dirac Bracket construction in the operational approach or the Senjanovic-Fradkin path integral [2] does not represent a practical solution in most of the cases. The reason is that in general, as occurs for example in Superparticle and Superstring models [3], the inversion of the Poisson Bracket matrix of the constraints leads to non-local cumbersome expressions. After the development of the Batalin, Fradkin and Vilkoviski (BFV) method [4] for models with first class constraints, one natural approach to follow for second class systems is to search for an associate canonically equivalent model, with only first class constraints. This idea was explored in some recent papers which addressed this problem [5][6][7][8].

In Ref. [5], a method was presented for the construction of an enlarged phase space where the original second class constraints have first class counterparts. The associated BRST charge in the enlarged phase space, in terms of an auxiliary operator algebra was also discussed. In Ref. [6] some of the ideas of Ref. [5] were developed further leading to a much simpler formulation of the associate first class model in the enlarged phase space. Some aspects of the reduction to the original degrees of freedom were cleared up and in particular the equivalence of the original and enlarged models in the path integral formalism was established. Applied to the Casalbuoni-Brink-Schwarz superparticle this allow a canonical covariant construction of the BRST operator [6].

In Ref. [7] and [8] a somewhat more direct approach for this problem was presented. There, a characterization of the models with second class constraints, which can be viewed as gauge fixed versions of gauge invariant systems was tried. If one can identify, out of the original second class constraints $\theta_i$, a constraint $\varphi$ with vanishing Poisson bracket with itself and with all the other constraints but one (say $\chi$) one observe that the Senjanovic-Fradkin [2] measure split in the form:

$$0 \neq \det\{\theta_i, \theta_i\}^{1/2} = \det\{\varphi, \chi\}\det\{\psi_i, \psi_j\}^{1/2}$$

(1.1)

Here $\psi_i$ stand for the remaining constraints. This is the measure adequate for a gauge theory with gauge transformations generated by $\varphi$, and gauge fixing condition $\chi$ subjected to the original second class constraints $\psi_j$. It remains to find the corresponding gauge invariant hamiltonian. This also can be done but it turns out that at this point non-local terms may reappear.
This approach may be of interest also for second class models for which the Poisson matrix may be handle without much trouble, since it may shed light on their underlying structure. This is the case with self dual and topologically massive electrodynamics in 3-D and with the various models of linearized massive gravity available in 3-D which we will discuss in this paper. All these models and many other field theories in 3-D have been studied intensively in recent years [9][10][11][12][13][14]. The linearized topologically massive model (TMM) [9] which is a gauge invariant model is known to have the same spectrum and in this sense to be equivalent to two other models in 3-D. These are the self-dual model (SDM) and the so called intermediate model (IM) [10][11]. The IM corresponds also to the linearization of the curved Vector Chern-Simons gravity action [12]. Both of them have second class constraints. Moreover they are respectively first order and second order in space-time derivates at the time that (TMM) is third order. Finally the IM presents a reduced gauge symmetry respect to the (TMM) and the (SDM) have no gauge symmetry at all. In Ref. [10] the three actions were shown to be deducible covariantly from a unique master action. Although indicative this fact does not established the canonical equivalence of the systems. There exists examples of systems connected in such fashion which are not equivalent [15]. In this paper we demostrate the canonical equivalence of the (SDM) and the (IM). We show that the (IM) is a gauge theory associated to the (SDM) and we construct explicitly, using the methods presented in Ref. [8], the gauge invariant hamiltonian of the former starting from the hamiltonian of the latter.

Before enter these matters, let us return to our discussion of the determination of the gauge theory associated to a system with second class constraints [8]. To consider a slightly more general case suppose that we have a system subjected to constraints

\begin{align}
\varphi_i(p, q) &= 0 \quad i = 1, \cdots N \\
\chi^i(p, q) &= 0 \quad i = 1, \cdots N \\
\psi_j(p, q) &= 0 \quad j = 1, \cdots 2M
\end{align}

(1.2a)

(1.2b)

(1.2c)

satisfying

\begin{align}
\{\varphi_i, \varphi_j\} &= C_{ij}^k \varphi_k \\
\{\varphi_i, \psi_j\} &= D_{ij}^k \varphi_k \\
\text{det}\{\varphi_i, \chi_j\} &\neq 0
\end{align}

(1.3a)

(1.3b)

(1.3c)

Then (1) generalize in the obvious way with the \(\varphi_i\) the first class constraints, \(\chi^i\) the gauge fixing conditions and \(\psi_j\) the truly second class constraints. Let \(H_0\) be the hamiltonian of the system. The associated gauge invariant hamiltonian must be of the form [8]:

\[\tilde{H} = H_0 + \alpha^i \varphi_i + \beta_i \chi^i + \beta_{jk} \chi^j \chi^k + \beta_{jkl} \chi^j \chi^k \chi^l + \cdots \]

(1.4)
over the manifold defined by (1.2c). The coefficients in this expansions are determined by the condition

\[ \{ \tilde{H}, \varphi_i \} = V_i^j \varphi_j \] (1.5)

and are of the form [8]

\[
\begin{align*}
\beta_j & = -\{ \chi^j, \varphi_i \}^{-1} \{ H_0, \varphi_i \} \\
2\beta_{jk} & = -\{ \chi^k, \varphi_i \}^{-1} \{ \beta_j, \varphi_i \} \\
3\beta_{jkl} & = -\{ \chi^l, \varphi_i \}^{-1} \{ \beta_{jk}, \varphi_i \} \\
& \vdots \\
V_j^i & = -\{ \alpha^j, \varphi_i \} + \alpha^j C_{ki}^j
\end{align*}
\] (1.6)

A similar construction allows to obtain a gauge invariant version of any object in the theory. The appearance of the factor \( \{ \chi^j, \varphi_i \}^{-1} \) in (1.6) implies as mentioned that this approach may also have problems with non-localities.

II. Self dual and topologically massive spin 1 in 3-D

To motivate our presentation to the spin 2 models in section III let us review briefly in this section the case of the spin 1 as discussed in Ref. [8]. In (2 + 1)D massive spin one excitations may be described by the self dual or the topologically massive models defined respectively by

\[
S^{SD} = \frac{m}{2} < mB_{\mu}B^{\mu} - \varepsilon^{\mu\alpha\rho}B_\mu \partial_\alpha B_\rho >
\]

\[
S^{TM} = \frac{1}{2} < -F_\mu F^\mu + mF_\mu A_\mu > ; \quad F_\mu = \varepsilon^{\mu\alpha\rho} \partial_\alpha A_\rho
\]

For the self-dual model \( B_0 \) is a Lagrange multiplier. Replacing the associated constraint one has:

\[
\begin{align*}
H_0^{SD} &= < \frac{m^2}{2} B_i B_i + \frac{1}{2} (\varepsilon_{ij} \partial_i B_j)^2 > \\
P_i &= -\frac{m}{2} \varepsilon_{ik} B_k
\end{align*}
\] (2.3a,b)

The second class constraints (2.3b) may be replaced by the following rotationally invariant ones:

\[
\begin{align*}
\varphi & = \partial_i P_i + \frac{m}{2} \varepsilon_{ik} \partial_i B_k \simeq 0 \\
\chi & = -\varepsilon_{ij} \partial_i P_j + \frac{m}{2} \partial_k B_k \simeq 0
\end{align*}
\] (2.4a,b)

Imposing canonical Poisson brackets for \( P_j \) and \( B_k \) we have

\[
\{ \varphi(x), \chi(y) \} = -m \partial_j^x \partial_j^y \delta^2(x - y)
\]

(2.5)
We identify $\varphi(x)$ as the generator of the gauge transformations and $\chi(x)$ as the gauge fixing condition. The gauge invariant hamiltonian is

$$\tilde{H} = H_0^{SD} + <\beta_1(x,z_1)\chi(z_1)> + <\beta_2(x_1,z_1,z_2)\chi(z_1)\chi(z_2)> + <\alpha(x)\varphi(x)> \quad (2.6)$$

From (1.6) we have

$$\beta_1(x_1,z_1) = mK(x_1-z_1)\partial_iB_i(x_1) \quad (2.7a)$$

$$\beta_2(x_1,z_1,z_2) = -\frac{1}{2}K(x_1-z_2)\delta^2(x_1-z_1) \quad (2.7b)$$

where

$$\Delta^x K(x) = -\delta^2(x) \quad (2.8)$$

Using the transverse + longitudinal (T+L) decomposition

$$B_i(x) = \partial_iB^L + \varepsilon_{ij}\partial_jB^T \quad (2.9)$$

in (2.6) we get the family of gauge invariant hamiltonians

$$\tilde{H} = H_0^{SD} + \frac{3m^2}{4}B^L\Delta B^L + \frac{m}{2}B^L\Delta P^T - \frac{1}{2}P^T\Delta P^T + <\alpha(x)\varphi(x)> \quad (2.10)$$

If one choose

$$\alpha = -\frac{3m}{4}B^T - \frac{1}{2}P^L \quad (2.11)$$

we get finally

$$\tilde{H} = H^{TM} = \frac{1}{2}P_kP_k + \frac{1}{2}\varepsilon_{ij}P_iB_j + \frac{m^2}{8}B_kB_k + \frac{1}{2}(\varepsilon_{ij}\partial_iB_j)^2 \quad (2.12)$$

This hamiltonian is local and is exactly the one of the topologically massive model (2.2) (see [16] for example) and hence, this construction shows the canonical equivalence of the two spin one models.

III. The self dual and the intermediate actions for spin 2 in 3-D

In this section applying the methods described in the previous ones we establish the canonical equivalence between the IM and the SDM. The action for the IM is given by:

$$S^I = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha h_{\beta\rho}\varepsilon^{\rho\sigma\mu\nu} \partial_\mu h_{\alpha\nu} - \frac{1}{4} (\varepsilon_{\alpha\beta\gamma}\partial_\beta h_{\gamma\alpha})^2 + \frac{m}{2} < h_{\rho\sigma}\varepsilon^{\rho\mu\nu} \partial_\mu h_{\sigma} \quad (3.1)$$

where $h_{\alpha\beta}$ is an arbitrary second rank tensor. This action is invariant under the transformations

$$\delta h_{\alpha\beta} = \partial_\alpha \xi_\beta \quad (3.2)$$
Let $\pi_{ij}$ be the conjugate momenta to $h_{ij}$. We introduce the decompositions

$$h_{ij} = \hat{\varepsilon}_{ij} \hat{h} + h_{ij}^s, \quad h_{ij}^s = h_{ji}^s$$  \hspace{1cm} (3.3a)

$$\pi_{ij} = \frac{1}{2} \varepsilon_{ij} \hat{\pi} + \pi_{ij}^s, \quad \pi_{ij}^s = \pi_{ji}^s$$  \hspace{1cm} (3.3b)

(we use $\eta_{ij} = (+ - -)$, $\varepsilon^{012} = 1$, $\varepsilon_{ij} = \varepsilon_{0ij}$). In general a superscript $s$ will denote a spatial symmetric tensor.

Using these variables one can shown easily that the action does not depends on $\dot{h}_{00}$, $\dot{\hat{h}}$ and $\dot{h}_{0j}$ and depends lineary on $\dot{h}_{l0}$. Hence the system is subject to the primary constraints

$$\pi_{00} \approx 0, \pi_{0j} \approx 0,$$

$$\hat{\pi} \approx 0$$  \hspace{1cm} (3.4)

$$\Omega_l \equiv -\varepsilon_{lk} \varepsilon_{ij} \partial_l h_{jk}^s - \frac{m}{2} \varepsilon_{li} h_{ij0} + \pi_{l0} \approx 0$$  \hspace{1cm} (3.5a)

Imposing the conservation of (3.4) and (3.5) since $h_{00}$, $\hat{h}$ and $h_{0j}$ are Lagrange multipliers, generate the secondary constraints

$$\theta_0 \equiv \varepsilon_{ij} \varepsilon_{kl} \partial_i h_{jk}^s - m \varepsilon_{ij} \partial_i h_{j0} \approx 0$$  \hspace{1cm} (3.6a)

$$\theta_k \equiv -\partial_j \pi_{jk}^s - \frac{3m}{4} \varepsilon_{jl} \partial_l h_{jk}^s + \frac{m}{4} \varepsilon_{jk} \partial_l h_{jl}^s + \frac{1}{2} \varepsilon_{lk} \varepsilon_{ij} \partial_i \partial_l h_{j0} \approx 0$$  \hspace{1cm} (3.6b)

$$\psi \equiv 2m \hat{\pi} - \pi_{ll} \approx 0$$  \hspace{1cm} (3.7)

Inspection of the Poisson matrix of the constraints shows that there are three first class constraints given by

$$\Theta_0 = \theta - \partial_l \Omega_l = -\partial_i \pi_{i0} - \frac{m}{2} \varepsilon_{ij} \partial_i h_{j0}$$  \hspace{1cm} (3.8a)

$$\Theta_k = \theta_k + \frac{1}{2} \varepsilon_{kl} \partial_i \hat{\pi}$$  \hspace{1cm} (3.8b)

which we take instead of (3.6). Equations (3.4), (3.5), (3.6b), (3.7) and (3.8) give the complete set of constraints of the systems. The constraints (3.4) are first class and appear, to allow writing the action in a covariant manner. They are analogous to $\pi_0 \approx 0$ in electrodynamics. $\hat{\pi}$, $\Omega_l$ and $\psi$ are second class constraints. $\Theta_A$ ($A = 0, 1, 2$) as stated above are first class and are the truly gauge generators of the model. The 18-dimensional phase space is then reduced by 4 second class constraints, 6 first class constraints are 6 gauge fixing conditions, to the 2 degrees of freedom needed to describe the single excitation of the system.

After substituting the constraints, the hamiltonian for this model is given by:

$$H^I = - \frac{1}{4} \varepsilon_{ij} \partial_i \partial_j h_{l0} \varepsilon_{kl} \partial_k h_{l0} - \frac{m^2}{16} \varepsilon_{ik} \varepsilon_{jl} h_{ij}^s h_{kl}^s + \frac{m^2}{16} h_{ij}^s h_{ij}^s$$

$$- \frac{m}{2} \varepsilon_{ij} h_{ik}^s \pi_{jk}^s - \frac{1}{4} \varepsilon_{ik} \varepsilon_{jl} \pi_{ij}^s \pi_{kl}^s + \frac{1}{4} \pi_{ij}^s \pi_{ij}^s$$  \hspace{1cm} (3.9)
The gauge transformation laws for $h_{j0}$ and $h^s_{ij}$ are easily obtained and take the expected form

$$\delta h_{j0} = \int d^2 x \{ h_{j0}, \xi_A(x) \Theta_A(x) \} = \partial_j \xi_0$$  \hspace{1cm} (3.10a)$$

$$\delta h_{ij} = \int d^2 x \{ h_{ij}, \xi_A(x) \Theta_A(x) \} = \frac{1}{2} (\partial_i \xi_j + \partial_j \xi_i)$$  \hspace{1cm} (3.10b)$$

$$\delta \tilde{h} = \int d^2 x \{ \tilde{h}, \xi_A(x) \Theta_A(x) \} = \frac{1}{2} \varepsilon_{ij} \partial_i \xi_j$$  \hspace{1cm} (3.10c)$$

Let us turn to the SDM. It is defined by the action

$$S^{SD} = \langle - \frac{m^2}{2} \varepsilon^{\alpha\beta\gamma} W^s_{\alpha\rho} \partial_\beta W^s_{\rho \gamma} - \frac{m^2}{2} (W_{\mu\nu} W^{\nu\mu} - W^s_{\rho} W^s_{\sigma}) \rangle$$  \hspace{1cm} (3.11)$$

in terms of the second rank tensor $W_{\mu\alpha}$. Let $P_{\mu\alpha}$ be the conjugate momenta. We define the symmetric and antisymmetric components of $W_{ij}$ and $P_{ij}$ as in (3.3). Since (3.11) is first order in time derivatives it is not possible to resolve the momenta in terms of the velocities. There appear nine primary constraints, one associated to each momenta. We have

$$\varphi^s_{ij} \equiv P^s_{ij} - \frac{m}{4} (\varepsilon_{ik} W^s_{kj} + \varepsilon_{kj} W^s_{ki}) + m \delta_{ij} \tilde{W} \approx 0$$  \hspace{1cm} (3.12a)$$

$$\tilde{P} \approx 0$$  \hspace{1cm} (3.12b)$$

$$\varphi_i \equiv P_{i0} + \frac{m}{2} \varepsilon_{ij} W_{j0} \approx 0$$  \hspace{1cm} (3.12c)$$

$$P_{00} \approx 0 \approx P_{0j}$$  \hspace{1cm} (3.13)$$

The constraints (3.13) are first class and so they can be used to remove six degrees of freedom. Conservation of (3.13) led to the secondary constraints

$$\zeta \equiv m W^s_{il} + \varepsilon_{ij} \partial_i W_{j0} \approx 0$$  \hspace{1cm} (3.14)$$

$$\zeta_k = m W_{k0} + \varepsilon_{ij} \partial_i W^s_{jk} - \partial_k \tilde{W} \approx 0$$  \hspace{1cm} (3.15)$$

This gives twelve constraints and three gauge fixing conditions restricting the 18 dimensional phase space. Looking for tertiary constraints one verifies that conservation of

$$\lambda \equiv \tilde{P} + \zeta - \frac{\partial_i \varphi_i}{m}$$  \hspace{1cm} (3.16)$$

which we take instead of $\zeta$ implies the last constraint

$$\tilde{W} \approx 0$$  \hspace{1cm} (3.17)$$

The hamiltonian for the SDM is simply

$$H^{SD} = \langle - \frac{m^2}{2} W^s_{ij} W^s_{ij} - \frac{m^2}{2} W^s_{ij} W^s_{ij} \rangle$$  \hspace{1cm} (3.18)$$
The momenta $P_{0j}$ and $P_{00}$ and the lagrange multipliers $W_{00}$ and $W_{0j}$ play in this model the same role that $\pi_{0j}$, $\pi_{00}$, $h_{00}$ and $h_{0j}$ in the IM and we have not to be worried further about them. Eq. (3.12b) and (3.17) set the canonically conjugate pair $(\hat{P}, \hat{W})$ to zero. We verify explicitly that $\varphi_{ij}^s$, $\hat{P}$, $\varphi_i$, and $\lambda$ are second class. To show the canonical equivalence of the SDM and the IM our strategy in what follows consists of: 

1) To recognize in the set of constraints of the SDM ((3.12), (3.13), (3.14), (3.15) and (3.17)) the first and second class constraints of the IM ((3.5), (3.6b), (3.7), (3.8)),

2) To relate the remaining constraints of the SDM with gauge fixing conditions in the IM and

3) To show that the gauge invariant hamiltonian of the IM may be written in the form given schematically by (1.4).

Using (3.17) the first class constraints (3.8) of the IM may be recovered in the form

$$\Theta_0|_{h=W; \; \pi=P} = -\partial_i \varphi_i$$  \hspace{1cm} (3.19a)

$$\Theta_k|_{h=W; \; \pi=P} = -\partial_j \varphi_{kj}^s - \frac{1}{2m} \varepsilon_{kj} \partial_j (\partial_l \varphi_l + m\lambda - 2m\hat{P}) + m\partial_i \hat{W}$$  \hspace{1cm} (3.19b)

The second class constraints (3.5) and (3.7) of the IM, in turn, may be reconstructed in the form:

$$\hat{P}|_{h=W; \; \pi=P} = \hat{P}$$  \hspace{1cm} (3.20a)

$$\Omega_i|_{h=W; \; \pi=P} = \varphi_i - \varepsilon_{ij} \partial_j \hat{W}$$  \hspace{1cm} (3.20b)

$$\psi|_{h=W; \; \pi=P} = 4m \hat{W} - \varphi_{ll}$$  \hspace{1cm} (3.20c)

This completes the reconstruction of the constraints of the IM. The physical sub-manifold of the SDM may be then described by (3.19), (3.20) and a set of three other independent constraints also constructed by combinations of $\varphi_{ij}^s$, $\hat{P}$, $\varphi_i$, $\zeta_k$, $\lambda$ and $\hat{W}$ which are to be interpreted as gauge fixing conditions associated to (3.19). We are free to choose any such set with the only requirement of being independent of (3.19) and (3.20). It can be seen that this independent set is expanded by $\lambda$, $\varphi_{ii}$ and $\varepsilon_{ij} \partial_i \varphi_j$. Good choices $\chi_A$ for this gauge fixing conditions are

$$\chi_0 = \varepsilon_{ij} \partial_i \varphi_j + \frac{1}{4m} \Delta \varphi_{ii}$$  \hspace{1cm} (3.21)

$$\chi_i = -\partial_i \lambda + \frac{1}{2} \varepsilon_{ij} \partial_j \varphi_{kk}$$  \hspace{1cm} (3.22)

They satisfy

$$\{\chi_A(x), \theta_B(y)\} = m\delta_A \delta_B \partial_x^i \partial_y^j \delta(x - y)$$  \hspace{1cm} (3.23)

in analogy with the vectorial case (2.5).
Let us discuss now the relation between $H^I$ and $H^{SD}$. An explicit computation shows that on the manifold defined by (3.19), (3.20), (3.21) and (3.22) we have

$$H^I|_{h=W; \pi=p} = H^{SD}$$

(3.24)

Equation (3.24) is a necessary condition for $H^I$ to have the structure given by (1.4). To complete the proof of the canonical equivalence between the SDM and the IM let us compute explicitly the coefficients in the expansion (1.4). which in this case simplify to

$$H^I = H^{SD} + <\alpha_A \Theta_A + \alpha_P \hat{P} + \alpha_\psi \psi + \alpha_i \Omega_i >$$

$$+ <\beta_A \chi_A > + <\beta_{AB} \chi_A \chi_B >$$

(3.25)

Here no additional terms are needed because the hamiltonian is quadratic in the fields. Then, following the general method described in the introduction, we obtain

$$\beta_0(x, z) = 0$$

(3.26)

$$\beta_k(x, z) = -\mu k(x - z)(\varepsilon_{kl} \varepsilon_{ij} \partial_i W_{jl}^{(s)})$$

(3.27)

and the only non vanishing $\beta_{AB}$ is

$$\beta_{kl}(x, z_1, z_2) = -\frac{1}{4} K(x - z_1) \varepsilon_{kl} \partial_i z_2 \varepsilon_{lj} \partial_j z_2 K(z_2 - z_1)$$

(3.28)

From (3.9), (3.18) and (3.25) one can compute the expression for the $\alpha$’s. This can be done for example factorizing the constraints in (3.9) and comparing with (3.18). Then one can see that the $\beta$’s appear naturally and we obtain:

$$<\alpha_A \Theta_A + \alpha_P \hat{P} + \alpha_\psi \psi + \alpha_i \Omega_i > =$$

$$<\Theta_i (-\Delta)^{-1} [\partial_i P_{kk} - \partial_k P_{ki} - \frac{1}{2} \varepsilon_{ik} \partial_k \hat{P} - \frac{m}{4} \varepsilon_{kl} \partial_k W_{li} +$$

$$- \frac{3m}{4} \varepsilon_{ik} \partial_k W_{li} - \frac{1}{2} (\Delta \delta_{ik} - \partial_i \partial_k) W_{k0}] +$$

$$+ \Theta_0 \left[ \frac{1}{2m} \hat{P} + (-\Delta)^{-1} (\delta_{kl} \Delta - \partial_k \partial_l) W_{kl} \right] +$$

$$+ \psi \left[ -\frac{3}{16} P_{ii} - \frac{m}{8} \hat{W} - \frac{m}{2} (-\Delta)^{-1} \varepsilon_{ik} \partial_i \partial_l W_{lk} \right] +$$

$$+ \hat{P} \left[ \frac{1}{2m} \partial_i P_{i0} + \frac{1}{4} \hat{P} - \frac{1}{4} \varepsilon_{ik} \partial_i W_{k0} + 2m (-\Delta)^{-1} (\Delta \delta_{ik} - \partial_i \partial_k) W_{ik} \right] >$$

(3.29)

Introducing (3.29) in (3.25) we have an identity which establishes the canonical equivalence of SDM and IM and our interpretation of the SDM as an explicitly covariant gauge fixed version of the IM.
IV The full gauge invariant extension for $H^I$

In the last section, there were 4 second class constraints, represented by $\psi$, $\hat{\pi}$ and $\Omega_l$ ((3.5), (3.7)), that did not play any role in the proof of the canonical equivalence between the $SDM$ and the $IM$. They could be used, further, to obtain a full gauge invariant form of $H^I$, and expedite the construction of the effective action for the $IM$, and consequently for $SDM$.

Let us rename and redefine this set of constraints as

$$\Theta_3 \equiv \Delta \hat{\pi}$$

$$\Theta_4 \equiv \partial_i \Omega_i = \partial_i \pi_{i0} - \frac{m}{2} \varepsilon_{ij} \partial_i h_{j0} - (\delta_{ij} \Delta - \partial_i \partial_j) h_{ij}$$

$$\chi_3 \equiv \frac{1}{2} \psi - \frac{1}{2m} \varepsilon_{ij} \partial_i \Omega_j = m^2 \hat{\pi} - \frac{1}{2} a_{ii}^{(i)} - \frac{1}{2m} \varepsilon_{ij} \partial_i \pi_j - \frac{1}{2m} \varepsilon_{ij} \partial_i \partial_k h_{kj} - \frac{1}{4} \partial_i h_{i0}$$

$$\chi_4 \equiv -\varepsilon_{ij} \partial_i \omega_j = -\varepsilon_{ij} \partial_i \pi_{j0} - \frac{m}{2} \partial_i h_{i0} - \varepsilon_{ij} \partial_i \partial_k h_{jk}$$

where we are suggesting to take $\Theta_3$ and $\Theta_4$ as first class constraints and the $\chi$’s as gauge fixing conditions. Their Poisson matrix is

$$\{\Theta_{A'}(x), \Theta_{B'}(y)\} = m \delta_{A'B'} \partial_x^x \partial_y^y \delta(x-y)$$

Let $\tilde{H}^I$ be the full gauge invariant form of $H^I$. Like in (1.4) we impose that

$$\tilde{H}^I = H^I + <\alpha_{A'} \Theta_{A'}> + <\beta_{A'} \chi_{A'}> + <\beta_{A'B'} \chi_{A'} \chi_{B'}>$$

where $A' = 3, 4$. Following the same procedure presented, we obtain

$$\beta_3(x, z) = 0$$

$$\beta_4(x, z) = \frac{1}{m} K(x - z) [(\delta_{ij} \Delta - \partial_i \partial_j) \pi_{ij}(x) + \frac{1}{2} \Delta_x \pi_{ii}(x) + \frac{m}{2} \varepsilon_{ij} \partial_i \partial_k h_{kj}(x)]$$

$$\beta_{33}(x_1, z_1, z_2) = \beta_{34}(x_1 z_1, z_2) = \beta_{43}(x_1 z_1, z_2) = 0$$

$$\beta_{44}(x_1, z_1, z_2) = -\frac{1}{4m^2} \delta(x - z_1) \delta(x - z_2)$$

As we pointed out before the $\alpha$’s in (1.4), (3.24) and (4.3) remain unfixed. For a particular choice they may give a canonical connection with the TMM. Nevertheless since this model is written in terms of third derivatives its canonical structure should be analyzed with special care [17] in order to establish such equivalence. This would be done elsewhere [18].
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