Analytical Studies on Holographic Insulator/Superconductor Phase Transitions

Rong-Gen Cai 1,* Huai-Fan Li 1,2,3,† and Hai-Qing Zhang 1‡

1Key Laboratory of Frontiers in Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100190, China

2Department of Physics and Institute of Theoretical Physics, Shanxi Datong University, Datong 037009, China

3Department of Applied Physics, Xi’an Jiaotong University, Xi’an 710049, China

We investigate the analytical properties of the s-wave and p-wave holographic insulator/superconductor phase transitions at zero temperature. In the probe limit, we analytically calculate the critical chemical potentials at which the insulator/superconductor phase transition occurs. Those resulting analytical values perfectly match the previous numerical values. We also study the relations between the condensation values and the chemical potentials near the critical point. We find that the critical exponent for condensation operator is $1/2$ for both models. The linear relations between the charge density and the chemical potential near the critical point are also deduced in this paper, which are qualitatively consistent with the previous numerical results.

I. INTRODUCTION

The AdS/CFT correspondence [1] provides a powerful theoretical method to understand the strongly coupled field theories. Recently, it has been proposed that the AdS/CFT correspondence can also be used to describe superconductor phase transitions [2, 3]. Since the condensed matter physics deals with the systems at finite charge and finite temperature, from the AdS/CFT correspondence the dual gravity should be described by a charged black hole.

The phase transition studied in [2, 3] is indeed a holographic superconductor/metal phase transition. The simplest model for holographic superconductors can be constructed by an Einstein-Maxwell theory coupled to a complex scalar field. In particular, when the

*Electronic address: cairg@itp.ac.cn
†Electronic address: huaifan.li@stu.xjtu.edu.cn
‡Electronic address: hqzhang@itp.ac.cn
temperature of the black hole is below a critical temperature, the black hole will become unstable to develop a scalar hair near the horizon. And this scalar hair will break the U(1) symmetry of the system. From the AdS/CFT correspondence, the complex scalar field is dual to a charged operator in the boundary field theory. And the breaking of the U(1) symmetry in gravity will cause a global U(1) symmetry breaking in the dual boundary theory. This induces a superfluid (superconductor) phase transition [4].

The holographic insulator/superconductor phase transition was first studied in [5]. In particular, they used a five-dimensional AdS soliton background [6] coupled to a Maxwell and scalar field to model the holographic insulator/superconductor phase transition at zero temperature. The normal phase in the AdS soliton is dual to a confined gauge theory with a mass gap which resembles an insulator phase [7]. When the chemical potential is sufficiently large, the AdS soliton becomes unstable to forming scalar hair which is dual to a superconducting phase in the boundary field theory. The holographic insulator/superconductor phase transition was also studied in [8–10].

In this paper, using the variational method for the Sturm-Liouville eigenvalue problems [11], we analytically studied the s-wave and p-wave holographic insulator/superconductor phase transitions in probe limit at zero temperature. We constructed the s-wave model with the Einstein-Maxwell-scalar field theory in an AdS soliton background. The order parameter for s-wave is the scalar operator. On the other hand, we built the p-wave model with the Einstein-Yang-Mills theory coupled to an AdS soliton background with the order parameter to be the vector operator. The condensation of the order parameter represents the onset of the superconducting. We analytically calculated the critical chemical potentials in both s-wave and p-wave models and found that they were in perfect agreement with the previous numerical values in [5, 12]. We also analytically obtained the relations between the condensation values of the operators and the chemical potentials near the critical point $\mu_c$, we found that the general critical exponent 1/2 would always appear, i.e. $\langle O \rangle \propto \sqrt{\mu - \mu_c}$ which was qualitatively consistent with the former numerical results [5, 12]. The linear relation of the charge density $\rho$ and $(\mu - \mu_c)$ near the critical chemical potential was obtained analytically which was also qualitatively consistent with the previous numerical results. Other analytical studies on holographic superconductors can be found in [13–18].

The paper is organized as follows. We studied the holographic s-wave insulator/superconductor phase transition in Sec.II. In particular we calculated the critical chemical potential for operators of various conformal dimensions. We also calculated the relations of condensed values of operators and the charge density with respect to $(\mu - \mu_c)$. The same procedure was also employed in Sec.III for the p-wave case. At last, we draw our conclusions in Sec.IV.
II. S-WAVE HOLOGRAPHIC INSULATOR/SUPERCONDUCTOR PHASE TRANSITION

We construct the model of holographic insulator/superconductor phase transition with the Einstein-Maxwell-scalar action in five-dimensional spacetime:

\[ S = \int d^5x \sqrt{-g} (R + \frac{12}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |\nabla_\mu \psi - iq A_\mu \psi|^2 - m^2 |\psi|^2). \] (1)

where \( L \) is the radius of AdS spacetime.

Following Ref.[5], in the probe limit, we setup this model in the AdS soliton background [6]:

\[ ds^2 = L^2 \frac{dr^2}{f(r)} + r^2 (-dt^2 + dx^2 + dy^2) + f(r) d\chi^2. \] (2)

where, \( f(r) = r^2 - r_0^2/r^2 \). In fact, this soliton solution can be obtained from a five-dimensional AdS Schwarzschild black hole by making use of two Wick rotations. The asymptotical AdS space-time approaches to a \( R^{1,2} \times S^1 \) topology near the boundary. And the Scherk-Schwarz circle \( \chi \sim \chi + \pi L/r_0 \) is needed in order to have a smooth geometry. The geometry looks like a cigar whose tip is at \( r = r_0 \) if we extract the coordinates \((r, \chi)\). Because of the compactified direction \( \chi \), this background provides a gravity description of a three-dimensional field theory with a mass gap, which resembles an insulator in the condensed matter physics. The temperature in this background is zero.

For simplicity, we will make ansatz of the matter fields as \( A_t = \phi(r), \quad \psi = \psi(r) \), which is consistent with the equations of motions (EoMs):

\[ \partial_t^2 \psi + \left( \frac{\partial_r f}{f} + \frac{3}{r} \right) \partial_r \psi + \left( -\frac{m^2}{f} + \frac{q^2 \phi^2}{r^2 f} \right) \psi = 0, \] (3)

\[ \partial_t^2 \phi + \left( \frac{\partial_r f}{f} + \frac{1}{r} \right) \partial_r \phi - \frac{2q^2 \psi^2}{f} \phi = 0. \] (4)

The boundary conditions near the infinity \( r \to \infty \) are:

\[ \psi = \psi^{(1)} r^{-2+\sqrt{4+m^2}} + \psi^{(2)} r^{-2-\sqrt{4+m^2}} + \cdots, \] (5)

\[ \phi = \mu - \frac{\rho}{r^2} + \cdots. \] (6)

where, \( \psi^{(i)} = \langle O^{(i)} \rangle, \quad i = 1, 2, \quad O^{(i)} \) are the corresponding dual operators of \( \psi^{(i)} \) in the boundary field theory. The conformal dimensions of the operators are \( \Delta_\pm = -2 \pm \sqrt{4 + m^2} \). There are two alternative quantizations for the scalar field in AdS_5, i.e. the operators \( O^{(i)} \) are all normalizable [19], if

\[ 0 < \sqrt{4 + m^2} < 1 \Rightarrow -4 < m^2 < -3. \] (7)

In this paper, we will always set \( m^2 = -15/4 \) except in subsection (II A 2), in order to compare with the numerical results in Ref.[5]. \( \mu \) and \( \rho \) are the corresponding chemical potential and charge density in the boundary field theory.
The boundary conditions at the tip \( r = r_0 \) are:

\[
\psi = a + b \log(r - r_0) + c(r - r_0) + \cdots, \\
\phi = A + B \log(r - r_0) + C(r - r_0) + \cdots.
\]

In order to take the Neumann-like boundary conditions, we impose \( b = B = 0 \) to render the physical quantities finite, see Ref.[5]. It is worth noting that unlike in the AdS black holes, the gauge field here is finite at \( r = r_0 \), i.e. \( A_t \neq 0 \).

Following Ref.[5], we will scale \( r_0 = 1, \ q = 1 \) in the subsequent calculations. Let \( z = 1/r \), the EoMs (3) and (4) become

\[
\psi'' + \left( \frac{f'}{f} - \frac{1}{z} \right) \psi' + \frac{(-m^2 + \phi^2 z^2)}{z^4 f} \psi = 0, \\
\phi'' + \left( \frac{f'}{f} + \frac{1}{z} \right) \phi' - \frac{2\psi^2}{z^4 f} \phi = 0.
\]

where \( ' \) denotes the derivative with respect to \( z \).

**A. The critical chemical potential \( \mu_c \)**

From the numerical analysis in [5], we can see that when the chemical potential \( \mu \) exceeds a critical chemical potential \( \mu_c \), the condensations of the operators will turn out. This can be viewed as a superconductor (superfluid) phase. However, when \( \mu < \mu_c \), the scalar field is zero and this can be interpreted as the insulator phase because this system has a mass gap, which is due to the confinement in the (2+1)-dimensional gauge theory via the Scherk-Schwarz compactification. Therefore, the critical chemical potential \( \mu_c \) is the turning point of this holographic insulator/superconductor phase transition.

When \( \mu \leq \mu_c \), the scalar field in nearly zero viz. \( \psi \approx 0 \), therefore we can analytically solve the gauge field equation (11)

\[
\phi'' + \left( \frac{f'}{f} + \frac{1}{z} \right) \phi' \approx 0,
\]

The general solution is

\[
\phi = C_2 + C_1 \log\left( \frac{1 - z^2}{1 + z^2} \right),
\]

where \( C_2 \) and \( C_1 \) are the integration constants. The Neumann boundary condition (9) near \( z = 1 \) imposes \( C_1 = 0 \), so \( \phi(z) \equiv C_2 = \mu \). This means \( \phi(z) \) is a constant if \( \psi(z) = 0 \), and from the boundary condition (6) near \( z = 0 \) we can get that \( \rho = 0 \). This analytical results are consistent with the numerical results in Figure 2 in Ref.[5], where \( \rho = 0 \) when \( \mu < \mu_c \).

1. **Operators of dimension \( \Delta = 3/2 \)**

Now, we take \( m^2 = -15/4 \) like in [5]. For this \( m^2 \), the operators \( O_{(1)} \) and \( O_{(2)} \) are all normalizable as we have mentioned above. In this subsection, we will focus on the
operator $O_{(1)}$ of conformal dimension $\Delta = 3/2$. We will discuss the operator $O_{(2)}$ in the next subsection.

When $\mu \to \mu_c$, the scalar EoM (10) becomes

$$
\psi'' + \left(\frac{f'}{f} - \frac{1}{z}\right)\psi' + \frac{1}{z^4}f\left(\frac{15}{4} + \mu^2z^2\right)\psi = 0.
$$

(14)

To solve this equation, we can introduce a trial function $F(z)$ for $\psi(z)$ near $z = 0$ as [11]

$$
\psi|_{z \to 0} \approx \langle O_{(1)} \rangle z^{3/2}F(z),
$$

(15)

The boundary condition for $F(z)$ is $F(0) = 1$, $F'(0) = 0$. In this case, it is easy to deduce the EoM of $F(z)$ as

$$
F'' + \frac{4z^3}{z^4 - 1}F' + \frac{-9z^4 + 4\mu^2z^2}{4z^2(1 - z^4)}F = 0.
$$

(16)

Multiply $T(z)$ to both sides of the above equation, where

$$
T(z) = z^4 - 1,
$$

(17)

we have the EoM of $F(z)$

$$
\frac{d}{dz}[(z^4 - 1)F'] + \frac{9}{4}z^2F - \mu^2F = 0.
$$

(18)

We can define the following parameters

$$
k = z^4 - 1, \quad P = -\frac{9}{4}z^2, \quad Q = -1.
$$

(19)

Thus, the minimum eigenvalues of $\mu^2$ can be obtained from taking variations with the following functional [20]

$$
\mu^2 = \frac{\int_0^1 dz(kF'^2 + PF^2)}{\int_0^1 dz QF^2}
$$

(20)

In order to estimate the minimum value of $\mu^2$, we use the trial function, $F(z) = 1 - \alpha z^2$, where $\alpha$ is a constant. The minimum appears as

$$
\mu_{\text{min}}^2 = \frac{3}{20} \times \frac{863\sqrt{230} - 14950}{\sqrt{230} - 414} \Rightarrow \mu_{\text{min}} \approx 0.837,
$$

(21)

when $\alpha = 3(35 - 2\sqrt{230})/61 \approx 0.230$. Therefore, $\mu_c = \mu_{\text{min}} \approx 0.837$, which is in perfect agreement with the numerical value $\mu_c \approx 0.84$ in Figure 2 of Ref. [5].
2. Operators of general dimensions and $\Delta = 5/2$

The operator $O(2)$ is normalizable when $m^2 > m_{BF}^2 = -4$, where $m_{BF}^2$ is the Breitenlohner-Freedman (BF) bound of the mass square of scalar field in the AdS space-time. In this subsection, we will calculate the critical chemical potential $\mu_c$ for the general case $-4 < m^2 < 0$.

Following the the steps in the preceding subsection, we introduce a trial function $F(z)$ into $\psi(z)$ near $z = 0$,

$$\psi|_{z \to 0} \approx \langle O(2) \rangle z^{2 + \sqrt{4 + m^2}} F(z),$$

(22)

The boundary conditions also impose $F(0) = 1$, $F'(0) = 0$. It is easy to obtain the EoM of $F(z)$ as

$$F'' + \frac{1 + 2\sqrt{4 + m^2} - z^4(5 + 2\sqrt{4 + m^2})}{z - z^5} F' + \frac{(8 + m^2 + 4\sqrt{4 + m^2})z^2}{z^4 - 1} F + \frac{\mu^2}{1 - z^4} F = 0.$$

(23)

In this case, we multiply to both sides of the above equation with $T(z)$:

$$T(z) = z^{1 + 2\sqrt{4 + m^2}}(z^4 - 1),$$

(24)

Thus the EoM of $F(z)$ reduces to

$$\frac{d}{dz} \left[ z^{1 + 2\sqrt{4 + m^2}}(z^4 - 1) F' \right] + z^{3 + 2\sqrt{4 + m^2}}(8 + m^2 + 4\sqrt{4 + m^2}) F - z^{1 + 2\sqrt{4 + m^2}} \frac{\mu^2}{Q} F = 0.$$

(25)

And the minimum eigenvalue of $\mu^2$ can be obtained by varying the following functional:

$$\mu^2 = \frac{\int_0^1 dz (kF'^2 + PF^2)}{\int_0^1 dz QF^2}$$

$$= \left[ (15\alpha^2 - 32\alpha + 17) m^6 + (517\alpha^2 - 1160\alpha + 671) m^4 + (4456\alpha^2 - 9792\alpha + 5940) m^2 \right.
+ 720 (15\alpha^2 - 32\alpha + 20) + \sqrt{m^2 + 4} ((\alpha - 1)^2 m^6 + (107\alpha^2 - 238\alpha + 133) m^4
+ 2 (776\alpha^2 - 1728\alpha + 1035) m^2 + 360 (15\alpha^2 - 32\alpha + 20)) \right] / \left[ \left( \sqrt{m^2 + 4} m^2 + 9 m^2 + 30 \sqrt{m^2 + 4} + 60 \right) \left( \alpha^2 m^2 - 2 \alpha m^2 + m^2 + 3 \sqrt{m^2 + 4} \alpha^2 + 6 \alpha^2 \right. \
- 8 \sqrt{m^2 + 4} \alpha - 14 \alpha + 5 \sqrt{m^2 + 4} + 10 \right] \right] \] (26)

In order to estimate it, we have set $F(z) = 1 - \alpha z^2$.

We plot the function $\mu^2 = \mu^2(\alpha, m^2)$ in the part A of Figure.(1). It can be seen that there indeed exist the minimum values of $\mu^2$ and are very close to $\alpha \sim 0$. In part B of Figure.(1),
FIG. 1: (A.) The square of chemical potential as a function of $\alpha$ and $m^2$ where $-4 < m^2 < 0$. (B.) The analytical relations between the critical chemical potential $\mu_c$ and the mass square of scalar field $m^2$. The black point represents the particular value for $m^2 = -15/4$.

we plot the relations of $\mu_c$ and $m^2$. It can be seen that when the mass square of the scalar field grows, the critical chemical potential grows too.

In particular, for $m^2 = -15/4$ in Ref.[5], we get

\[ \mu^2_{\text{min}} = \frac{113190 - 1405\sqrt{462}}{15400 + 364\sqrt{462}} \Rightarrow \mu_{\text{min}} \approx 1.890, \]  

(27)

when $\alpha = 5(63 - 2\sqrt{462})/303 \approx 0.330$. This critical value $\mu_c = \mu_{\text{min}} \approx 1.890$ is in well agreement with the numerical values $\mu_c \approx 1.88$ in Figure 2 of Ref.[5]. We have denoted this particular value in part B of Figure.(1) as the black point.

B. Relations of $\langle \mathcal{O} \rangle - (\mu - \mu_c)$ and $\rho - (\mu - \mu_c)$

1. Operators of dimension $\Delta = 3/2$

When $\mu$ is away from (but very close to) $\mu_c$, we can substitute (15) into the EoM of $\phi(z)$ (11) as

\[ \phi'' + \left( \frac{f'}{f} - \frac{1}{z} \right) \phi' = \frac{2\langle O(1) \rangle^2 F^2}{zf} \phi = 0 \]  

(28)

Because near the critical chemical, the condensation of the operator is very small, we can expand $\phi(z)$ in $\langle O(1) \rangle$ as

\[ \phi(z) \sim \mu_c + \langle O(1) \rangle \chi(z) + \cdots, \]  

(29)

The boundary condition at the tip imposes the correction function $\chi(z)$ to be $\chi(1) = 0$. It is easy to deduce the EoM of $\chi(z)$ as

\[ \chi'' - \frac{1 + 3z^4}{z - z^5} \chi' = \frac{2\langle O(1) \rangle \mu_c F^2}{zf}. \]  

(30)
Multiplying $T(z)$ to both sides of the above equation, where

$$T(z) = \frac{z^4 - 1}{z}$$ (31)

The EoM of $\chi(z)$ is reduced to

$$\frac{d}{dz} \left[ \frac{z^4 - 1}{z} \chi' \right] = -2\langle O_{(1)} \rangle \mu_c F^2$$ (32)

Making integration of both sides, we get

$$\left. \frac{z^4 - 1}{z} \chi' \right|_0^1 = \left. \frac{\chi'(z)}{z} \right|_{z=0} = -2\langle O_{(1)} \rangle \mu_c \int_0^1 dz (1 - \alpha z^2)^2,$$ (33)

where we have used the trial function $F(z) = 1 - \alpha z^2$.

Near $z = 0$, $\phi(z)$ can be expanded as

$$\phi(z) \sim \mu - \rho z^2 \approx \mu_c + \langle O_{(1)} \rangle \left( \chi(0) + \chi'(0)z + \frac{1}{2} \chi''(0)z^2 + \cdots \right)$$ (34)

Comparing the coefficients of $z^0$ term in both sides of the above formula, we get

$$\mu - \mu_c \approx \langle O_{(1)} \rangle \chi(0).$$ (35)

Besides, from the $z^1$ term in (34), we obtain that $\chi'(0) = 0$. Therefore, from the EoM (30) and the boundary conditions of $\chi(z)$, we can solve $\chi(z)$ to be

$$\chi(z) = \frac{\langle O_{(1)} \rangle \mu_c}{60} \left( 8\alpha (-\alpha z^3 + 10z + \alpha - 10) + \pi (3\alpha^2 + 10\alpha + 15) \ight.$$

$$-4 (3\alpha^2 + 10\alpha + 15) \arctan(z) + (3\alpha^2 - 10\alpha + 15) (4 \log(z+1) - 2 \log(z^2+1))$$

$$\left. + (-6\alpha^2 + 20\alpha - 30) \log 2 \right).$$ (36)

And then,

$$\chi(0) = \frac{\langle O_{(1)} \rangle \mu_c}{60} \left( 8(\alpha - 10)\alpha + \pi (3\alpha^2 + 10\alpha + 15) + (-6\alpha^2 + 20\alpha - 30) \log 2 \right).$$ (37)

Therefore, from (35) we can deduce that

$$\mu - \mu_c \approx \frac{\langle O_{(1)} \rangle^2 \mu_c}{60} \left( 8(\alpha - 10)\alpha + \pi (3\alpha^2 + 10\alpha + 15) + (-6\alpha^2 + 20\alpha - 30) \log 2 \right)$$ (38)

And further we have

$$\langle O_{(1)} \rangle \approx 1.940 \sqrt{\mu - \mu_c},$$ (39)

when $\alpha = 0.230$. This critical exponent $1/2$ for the condensation value and $(\mu - \mu_c)$ qualitatively match the numerical curves in Figure.1 of Ref.[5].
Comparing the coefficients of the $z^2$ term in (34), we get

$$\rho = -\frac{1}{2} \langle O_{(1)} \rangle \chi''(0).$$

(40)

From the EoM (30) and the formula (33), we can obtain

$$\chi''(0) = \frac{1 + 3z^4}{z - z^5} \chi'(z) \bigg|_{z \to 0} = \frac{\chi'(z)}{z} \bigg|_{z \to 0} = -2\langle O_{(1)} \rangle \mu_c \int_0^1 dz (1 - \alpha z^2)^2.$$  

(41)

Thus, using the relation (39), it is easy to deduce that

$$\rho \approx 2.700(\mu - \mu_c),$$

(42)

when $\alpha = 0.230$. This linear relation between the charge density $\rho$ and the chemical potential difference $(\mu - \mu_c)$ qualitatively matches the numerical curves in Figure.2 of Ref.[5]. Moreover, this linear relation between $\rho$ and $(\mu - \mu_c)$ can also be frequently seen in the numerical analysis in Ref.[8, 12].

2. Operators of dimension $\Delta = 5/2$

For the operators of dimension $\Delta = 5/2$, we can follow the preceding steps to get that

$$\langle O_{(2)} \rangle \approx 1.801\sqrt{\mu - \mu_c}$$

(43)

and

$$\rho \approx 1.329(\mu - \mu_c),$$

(44)

when $\alpha = 0.330$. Eqs. (43) and (44) are qualitatively consistent with the numerical curves in Figure.1 and Figure.2 of Ref.[5].

III. P-WAVE HOLOGRAPHIC INSULATOR/SUPERCONDUCTOR PHASE TRANSITION

Following [21], in this section we consider a five-dimensional SU(2) Einstein-Yang-Mills theory with a negative cosmological constant. The action is

$$S = \int d^5x \sqrt{-g} \left[ \frac{1}{2} (R - \Lambda) - \frac{1}{4} F^{\alpha}_{\mu \nu} F^{\alpha \mu \nu} \right],$$

(45)

where $F^{\alpha}_{\mu \nu}$ is the field strength of the SU(2) gauge theory and $F^{\alpha}_{\mu \nu} = \partial_\mu A^{a}_\nu - \partial_\nu A^{a}_\mu + \epsilon^{abc} A^{b}_\mu A^{c}_\nu$. $a, b, c = (1, 2, 3)$ are the indices of the SU(2) Lie algebra generator. $A^{a}_\mu$ are the components of the mixed-valued gauge fields $A = A^{a}_\mu \tau^a dx^\mu$, where $\tau^a$ are the generators of the SU(2) Lie algebra with commutation relation $[\tau^a, \tau^b] = \epsilon^{abc} \tau^c$. And $\epsilon^{abc}$ is a totally antisymmetric tensor with $\epsilon^{123} = +1$. 
As a consistent solution of the system, in the probe limit, the background of the metric can also be an AdS soliton solution like (2) (We have scaled $L \equiv 1, r_0 \equiv 1$),

$$ds^2 = \frac{dr^2}{r^2 g(r)} + r^2(-dt^2 + dx^2 + dy^2) + r^2 g(r) d\chi^2,$$  \hspace{1cm} (46)

where we have set $f(r) = r^2 g(r) = r^2(1 - 1/r^4)$.

We adopt the ansatz for the gauge field as [22]

$$A(r) = \phi(r) \tau^3 dt + \psi(r) \tau^1 dx.$$  \hspace{1cm} (47)

In this ansatz, the gauge boson with nonzero component $\psi(r)$ along $x$-direction is charged under $A^3_t = \phi(r)$. According to AdS/CFT dictionary, $\phi(r)$ is dual to the chemical potential in the boundary field theory while $\psi(r)$ is dual to the $x$-component of some charged vector operator $O$. The condensation of $\psi(r)$ will spontaneously break the $U(1)_3$ gauge symmetry and induce the phenomena of superconducting on the boundary field theory.

Let $z = 1/r$, the EoMs for $\phi(z)$ and $\psi(z)$ in the $z$ coordinate are

$$\phi'' + \left(\frac{g'}{g} - \frac{1}{z}\right)\phi' - \frac{\psi^2}{g} \phi = 0,$$

$$\psi'' + \left(\frac{g'}{g} - \frac{1}{z}\right)\psi' + \frac{\phi^2}{g} \psi = 0.$$  \hspace{1cm} (48, 49)

The boundary conditions at infinity, i.e. $z \to 0$ are

$$\phi = \mu - \rho z^2,$$

$$\psi = \psi^{(0)} + \psi^{(2)} z^2,$$  \hspace{1cm} (50, 51)

where, $\mu$ and $\rho$ can be interpreted as chemical potential and charge density on the boundary field, respectively. $\psi^{(0)}$ and $\psi^{(2)}$ represent the source and vacuum expectation value (vev) of the dual operator on the boundary. We always set $\psi^{(0)} = 0$ since we are interested in the case where the dual operator is not sourced.

On the tip $z = 1$, the boundary conditions are like the former ones (8) and (9):

$$\psi = \alpha_0 + \alpha_1 (1 - z) + \cdots,$$

$$\phi = \beta_0 + \beta_1 (1 - z) + \cdots.$$  \hspace{1cm} (52, 53)

A. The critical chemical potential $\mu_c$

Following the analysis in the previous section, when $\mu \leq \mu_c$, $\psi$ is nearly zero, i.e. $\psi \sim 0$. Solving the equations (48) we can get $\rho = 0$ and $\phi(z) = \text{Cons.} = \mu$ when $\mu < \mu_c$. This is consistent with the numerical values in Figure.1 in Ref.[12].

We can also introduce a trial function $F(z)$ into $\psi^{(2)}$ near $z = 0$,

$$\psi|_{z \to 0} \sim \psi^{(2)} z^2 \approx \langle O_{(2)} \rangle z^2 F(z)$$  \hspace{1cm} (54)
where we have set $\psi^{(2)} = \langle O^{(2)} \rangle$. And the boundary condition for $F(z)$ is $F(0) = 1$, $F'(0) = 0$. Therefore, EoM of $F(z)$ is

$$F'' + \frac{3 - 7z^4}{z - z^5} F' - \frac{8z^2}{1 - z^4} F + \frac{\mu^2}{g} F = 0.$$  \hspace{1cm} (55)

Multiplying on both sides of the above equation with $T(z) \equiv z^7 - z^3$, we have the EoM of $F(z)$ as

$$\frac{d}{dz} \left[ (z^7 - z^3) F' \right] + 8z^5 F - \mu^2 z^3 F = 0.$$  \hspace{1cm} (57)

We define three parameters as follows

$$k = z^7 - z^3, \quad P = -8z^5, \quad Q = -z^3.$$  \hspace{1cm} (58)

The minimum eigenvalues of $\mu^2$ can be obtained by varying the following functional

$$\mu^2 = \int_0^1 dz (kF'^2 + PF^2) \int_0^1 dz QF^2.$$  \hspace{1cm} (59)

It turns out that the minimum value is

$$\mu_{\text{min}}^2 = \frac{16}{5} \times \frac{10 - 15\alpha + 8\alpha^2}{6 - 8\alpha + 3\alpha^2} \Rightarrow \mu_{\text{min}} \approx 2.267$$  \hspace{1cm} (60)

when $\alpha = (18 - \sqrt{134})/19 \approx 0.338$. The critical value $\mu_c = \mu_{\text{min}} \approx 2.267$ is in great agreement with the numerical values $\mu_c \approx 2.26$ in Figure.1 of Ref.[12].

**B. Relations of $\langle O \rangle$-(\(\mu - \mu_c\)) and $\rho$-(\(\mu - \mu_c\))**

When $\mu \rightarrow \mu_c$, the condensation value of $\psi(z)$ is very small, we can expand $\phi(z)$ in $\langle O^{(2)} \rangle$ as

$$\phi \sim \mu_c + \langle O^{(2)} \rangle \chi(z) + \cdots.$$  \hspace{1cm} (61)

where the boundary condition imposes $\chi(1) = 0$. Substituting (61) into (48), we get the EoM of $\chi(z)$ as

$$\chi'' - \frac{1 + 3z^4}{z - z^5} \chi' - \langle O^{(2)} \rangle \mu_c \frac{z^4}{1 - z^4} F^2 = 0.$$  \hspace{1cm} (62)

Near $z = 0$, we can expand $\phi(z)$ as

$$\phi = \mu - \rho z^2 \approx \mu_c + \langle O^{(2)} \rangle \chi(0) + \chi'(0) z + \frac{1}{2} \chi''(0) z^2 + \cdots.$$  \hspace{1cm} (63)
Comparing the coefficients of the $z^0$ term, we obtain
\[ \mu = \mu_c + \langle O_{(2)} \rangle \chi(0). \] (64)

And from the $z^1$ term, we know that $\chi'(0) = 0$. We can solve $\chi(z)$ via the EoM (62) and the boundary conditions of $\chi(z)$ to be
\[ \chi(z) = -\frac{\langle O_{(2)} \rangle \mu_c}{48} \left[ z^6 + 3z^2 - 4 \right] \alpha^2 + \left( -4z^4 + 8 \log \left( \frac{2}{z^2 + 1} \right) + 4 \right) \alpha + 6 \left( z^2 - 1 \right), \] (65)

which gives
\[ \chi(0) = \frac{\langle O_{(2)} \rangle \mu_c}{24} \left( 3 + 2\alpha^2 - 2\alpha(1 + 2 \log 2) \right). \] (66)

Further we deduce from (64) that
\[ \langle O_{(2)} \rangle \approx 2.560\sqrt{\mu - \mu_c} = 3.855 \sqrt{\frac{\mu}{\mu_c} - 1}, \] (67)

when $\alpha = 0.338$. Once again, this critical exponent $1/2$ is qualitatively consistent with the numerical curves in Figure.1 of Ref.[12].

From the equation (62), we can have
\[ \chi''(0) = \frac{1 + 3z^4}{z - z^3} \chi'(z) \bigg|_{z \to 0} = \frac{\chi'(z)}{z} \bigg|_{z \to 0}. \] (68)

Multiplying $T(z)$ on both sides of (62), where
\[ T(z) = \frac{z^4 - 1}{z} \] (69)

we have
\[ \frac{d}{dz} \left[ \frac{z^4 - 1}{z} \chi \right] = -\langle O_{(2)} \rangle \mu_c z^3 F^2, \] (70)

which gives us with
\[ \frac{z^4 - 1}{z} \chi'(z) \bigg|_0^1 = \frac{\chi'(z)}{z} \bigg|_{z \to 0} = -\langle O_{(2)} \rangle \mu_c \int_0^1 dz z^3 F^2. \] (71)

From (68) we can obtain with $F(z) = 1 - \alpha z^2$ that
\[ \chi''(0) = -\langle O_{(2)} \rangle \mu_c \left( \frac{1}{4} - \frac{\alpha}{3} + \frac{\alpha^2}{8} \right). \] (72)

Comparing the coefficients of the $z^2$ term of (63), we reach
\[ \rho = \frac{1}{2} \langle O_{(2)} \rangle^2 \mu_c \left( \frac{1}{4} - \frac{\alpha}{3} + \frac{\alpha^2}{8} \right) \approx 1.126 (\mu - \mu_c), \] (73)

when $\alpha = 0.338$. This linear relation between $\rho$ and $\mu - \mu_c$ is qualitatively consistent with the numerical curves in Figure.1 of Ref.[12].
IV. CONCLUSIONS

In this paper, we have studied the analytical properties of the s-wave and p-wave holographic insulator/superconductor phase transitions at zero temperature. In particular, we have ignored the back-reaction of the gauge field and the scalar field to the AdS soliton background. When the chemical potential $\mu$ is lower than the critical chemical potential $\mu_c$, the AdS soliton background is stable and the dual field theory can be interpreted as an insulator; However, when $\mu > \mu_c$ the AdS soliton background will be unstable to form the condensations of the scalar field (in s-wave) or the vector field (in p-wave). And the dual field theory is in a superconducting phase.

Actually, we have analytically obtained the critical chemical potentials in both s-wave and p-wave models. We found that the critical chemical potential $\mu_c$ we obtained are perfectly in agreement with the previous numerical values. In particular, for the scalar operator of conformal dimension $2 + \sqrt{4 + m^2}$, we found that when $m^2$ grows, the critical chemical potential $\mu_c$ grows as well. Besides, we calculated the relations between the condensation values of the dual operator and the chemical potential near $\mu_c$, and found that the critical exponent of condensation operator is always $1/2$ in both models, i.e. $\langle O \rangle \propto \sqrt{\mu - \mu_c}$. In addition, we also obtained the linear relations between the charge density and the chemical potential near $\mu_c$, which is also qualitatively consistent with the previous numerical results. Our results combining with others in the literature [11, 16, 17] show that the analytic method is quite powerful to study the holographic superconducting phase transition near the critical point.

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