THE FIVE EXCEPTIONAL SIMPLE LIE SUPERALGEBRAS OF VECTOR FIELDS

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ABSTRACT. The five simple exceptional complex Lie superalgebras of vector fields are described. One of them is new; the other four are explicitly described for the first time. All of the exceptional Lie superalgebras are obtained with the help of the Cartan prolongation or a generalized prolongation.

The description of several of the exceptional Lie superalgebras is associated with the Lie superalgebra sp — the nontrivial central extension of the supertraceless subalgebra spe(4) of the periplectic Lie superalgebra pe(4) that preserves the nondegenerate odd bilinear form on the 4|4-dimensional superspace. (A nontrivial central extension of spe(n) only exists for n = 4.)

INTRODUCTION

V. Kac [K] classified simple finite dimensional Lie superalgebras over C. Kac further conjectured [K] that passing to infinite dimensional simple Lie superalgebras of vector fields with polynomial coefficients we only acquire the straightforward analogs of the four well-known Cartan series vect(n), suct(n), h(2n) and ℓ(2n + 1) (of all, divergence-free, hamiltonian and contact vector fields, respectively, realized on the space of dimension indicated). Since superdimension is a pair of numbers, Kac’s examples of simple vectoral Lie superalgebras double Cartan’s list of simple vectorial Lie algebras.

It soon became clear [L1], [ALSh] that the actual list of simple vectoral Lie superalgebras “doubles” that of Cartan twice, not once (the super counterparts m and sm of ℓ as well as k and sk — the super counterparts of h — were discovered); moreover, even the superalgebras of the 4 well-known series (vect, suct, h and ℓ) have, in addition to the dimension of the superspace on which they are usually realized, one more discrete parameter indicating other, nonstandard realizations. Furthermore, several of these Lie superalgebras have deformations, see [L2], [L3].

Next, three exceptional vectoral algebras were discovered [Sh1], followed by a fourth exception [Sh2]. The purpose of this note is to give a more lucid description of these exceptions, and introduce the most remarkable 5th exception. (For a related construction of Lie superalgebras of string theories see [GLS]; for further study of some of these superalgebras see [LSX] and [LS].)

In this note the ground field is C. First, we recall the background from Linear Algebra in Superspaces. Then we recall the definition of the main tool in the construction of our examples: the notion of Cartan prolongation and its generalization ([Sh1]). We recall the known facts form the classification of simple Lie superalgebras of vector fields, cf. [L2] and [L3].

Our main result is the discovery and a description of the five exceptional simple Lie algebras of vector fields: (spin, as), , (II(T(0)/C · 1), suct(0|3)), , (ab(4), suct(0|3))σ, (hei(8|6), svect3,4(4))κ and k. These names reflect the method of construction of these algebras, rather than their own properties; for private use I call them shortly f1 – f5, respectively. To name them properly, an interpretation is required.

The notions of the generalized and a partial Cartan prolongations are the key tools in the construction.

I divide the description into several sections, according to the method of construction. Boring calculations are gathered in Appendix. The statements on simplicity are proved via Kac’s criteria, cf. [K]. Proof of the fact that the five exceptional Lie superalgebras have only one realization and the classification of their real forms will be given elsewhere.

The reader might wonder if new simple Lie superalgebras of vector fields will continue to appear every several years. We conjecture that there is no such treat/danger any longer: the classification of graded Lie superalgebras of polynomial growth and finite depth is soon to appear, at last [LSh].

Open problems: (1) Give geometric realizations of the exceptional Lie superalgebras (what structures do they preserve?).

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(2) Certain exceptional Lie superalgebra are deformations of (nonsimple) Lie superalgebras whose brackets are easy to describe. In this paper the cocycle is described clumsily, in components of the generating functions. Describe the cocycle (i.e., the bracket itself) in terms of generating functions themselves. (An attempt is made in [PSh].)

(3) Find out what do our exceptional Lie superalgebras add to the list of simple finite dimensional Lie algebras over an algebraically closed field of characteristic 2 via Leites’ conjecture [KL], [L2].

Remark. The results of this paper and the related results on classification of the stringy superalgebras [GLS] were obtained in Stockholm in June 1996 and delivered at the seminar of E. Ivanov, JINR, Dubna (July, 1996) and Voronezh winter school Jan. 12–18, 1997. The interest of V. Kac during October–November 1996 stimulated me to finish editing this paper. I am particularly thankful to P. Grozman, D. Leites and G. Post who helped me.

§0. BACKGROUND

0.1. Linear algebra in superspaces. Generalities. Superization has certain subtleties, often disregarded or expressed as in [L], [L3] or [M]: too briefly. We will dwell on them a bit.

A superspace is a Z/2-graded space; for a superspace $V = V_0 \oplus V_1$ denote by $\Pi(V)$ another copy of the same superspace: with the shifted parity, i.e., $\Pi(V))_i = V_{i+1}$. The superdimension of $V$ is $\dim V = p + q\varepsilon$, where $\varepsilon^2 = 1$ and $p = \dim V_0$, $q = \dim V_1$. Usually $\dim V$ is expressed as a pair $(p, q)$ or $p|q$; this obscures the fact that $\dim V \otimes W = \dim V \cdot \dim W$ which is clear with the use of $\varepsilon$.)

A superspace structure in $V$ induces the superspace structure in the space $\text{End}(V)$. A basis of a superspace is always a basis consisting of homogeneous vectors; let $\text{Par} = (p_1, \ldots, p_{\dim V})$ be an ordered collection of their parities. We call $\text{Par}$ the basis of the basis of $V$. A square supermatrix of format (size) $\text{Par}$ is a $\dim V \times \dim V$ matrix whose $i$th row and $i$th column are of the same parity $p_i$. The matrix unit $E_{ij}$ is supposed to be of parity $p_i + p_j$ and the bracket of supermatrices (of the same format) is defined via Sign Rule: if something of parity $p$ moves past something of parity $q$ the sign $(-1)^{pq}$ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity. For example: setting $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$ we get the notion of the supercommutator and the ensuing notion of the Lie superalgebra (that satisfies the superskew-commutativity and super Jacobi identity).

We do not usually use the sign $\wedge$ for the wedge product of differential forms on supermanifolds: in what follows we assume that the exterior differential is odd and the differential forms constitute a supercommutative superalgebra; we keep using it on manifolds, sometimes, not to diviate too far from conventional notations.

Usually, $\text{Par}$ is of the form $(0, \ldots, 0, \hat{1}, \ldots, \hat{1})$. Such a format is called standard. In this paper we can do without nonstandard formats. But they are vital in the study of systems of simple roots that the reader might be interested in.

The general linear Lie superalgebra of all supermatrices of size $\text{Par}$ is denoted by $\mathfrak{gl}(\text{Par})$; usually, $\mathfrak{gl}(0, \ldots, 0, \hat{1}, \ldots, \hat{1})$ is abbreviated to $\mathfrak{gl}(\dim V_0|\dim V_1)$. Any matrix from $\mathfrak{gl}(\text{Par})$ can be expressed as the sum of its even and odd parts; in the standard format this is the block expression:

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix} + \begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix}, \quad p \begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix} = 0, \quad p \begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix} = 0.
$$

The supertrace is the map $\mathfrak{gl}(\text{Par}) \to \mathbb{C}$, $(A_{ij}) \mapsto \sum (-1)^{p_i} A_{ii}$. Since $\text{str}[x, y] = 0$, the space of supertraceless matrices constitutes the special linear Lie subsuperalgebra $\mathfrak{sl}(\text{Par})$.

Superalgebras that preserve bilinear forms: two types. To the linear map $F$ of superspaces there corresponds the dual map $F^*$ between the dual superspaces; if $A$ is the supermatrix corresponding to $F$ in a basis of the format $\text{Par}$, then to $F^*$ the supertransposed matrix $A^{st}$ corresponds:

$$(A^{st})_{ij} = (-1)^{(p_i + p_j)(p_i + p(A))} A_{ji}.$$ 

The supermatrices $X \in \mathfrak{gl}(\text{Par})$ such that

$$X^{st}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(\text{Par})$$

constitute the Lie superalgebra $\mathfrak{out}(B)$ that preserves the bilinear form on $V$ with matrix $B$. Most popular is the nondegenerate supersymmetric form whose matrix in the standard format is the canonical form $B_{ev}$ or $B_{ev}'$:

$$B_{ev}(m|2n) = \begin{pmatrix}
1_n & 0 \\
0 & J_{2n}
\end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix}
0 & 1_n \\
-1_n & 0
\end{pmatrix}, \quad \text{or } B_{ev}'(m|2n) = \begin{pmatrix}
\text{antidiag}(1, \ldots, 1) & 0 \\
0 & J_{2n}
\end{pmatrix}.$$
The usual notation for $\mathfrak{aut}(B_{ce}(m|2n))$ is $\mathfrak{osp}(m|2n)$ or $\mathfrak{osp}^s(m|2n)$. (Observe that the passage from $V$ to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones, preserved by the “symplectico-orthogonal” Lie superalgebra $\mathfrak{sp}'(2n|m)$ or $\mathfrak{osp}^s(m|2n)$ which is isomorphic to $\mathfrak{osp}^s(m|2n)$ but has a different matrix realization. We never use notation $\mathfrak{sp}'(2n|m)$ in order not to confuse with the special Poisson superalgebra.)

In the standard format the realization of these algebras are:

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & -X^t \\ X & A & B \\ Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^s(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y^t & -X^t & E \end{pmatrix} \right\},$$

where $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n)$, $E \in \mathfrak{o}(m)$ and $^t$ is the usual transposition.

A nondegenerate supersymmetric odd bilinear form $B_{odd}(n|n)$ can be reduced to the canonical form whose matrix in the standard format is $J_{2n}$. A canonical form of the superskew odd nondegenerate form in the standard format is $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. The usual notation for $\mathfrak{aut}(B_{odd}(Par))$ is $\mathfrak{pe}(Par)$. The passage from $V$ to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones and establishes an isomorphism $\mathfrak{pe}^s(Par) \cong \mathfrak{pe}^s(Par)$. This Lie superalgebra is called, as A. Weil suggested, periplectic, i.e., odd-plectic.

The usual notation for $\mathfrak{aut}(B_{odd}(Par))$ is $\mathfrak{pe}(Par)$. The passage from $V$ to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones and establishes an isomorphism $\mathfrak{pe}^s(Par) \cong \mathfrak{pe}^s(Par)$. This Lie superalgebra is called, as A. Weil suggested, periplectic, i.e., odd-plectic. 

The special periplectic superalgebra is $\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) : \text{str}X = 0\}$.

Observe that though the Lie superalgebras $\mathfrak{osp}^s(m|2n)$ and $\mathfrak{pe}^s(2n|m)$, as well as $\mathfrak{pe}^s(n)$ and $\mathfrak{pe}^s(n)$, are isomorphic, the difference between them is sometimes crucial, see Remark 0.6 below.

### 0.2. Vectorial Lie superalgebras. The standard realization.

The elements of $\mathcal{L} = \partial \mathfrak{ter} \mathbb{C}[u]$ are considered as vector fields. The Lie algebra $\mathcal{L}$ has only one maximal subalgebra $\mathcal{L}_0$ of finite codimension (consisting of the fields that vanish at the origin). The subalgebra $\mathcal{L}_0$ determines a filtration of $\mathcal{L}$: set

$$\mathcal{L}_{-1} = \mathcal{L}; \quad \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} : [D, \mathcal{L}] \subseteq \mathcal{L}_{i-1}\}$$

for $i \geq 1$.

The associated graded Lie algebra $L = \bigoplus_{i \geq -1} \mathcal{L}_i$, where $L_i = \mathcal{L}_i/\mathcal{L}_{i+1}$, consists of the vector fields with polynomial coefficients.

Suppose $\mathcal{L}_{\text{sub}} \mathcal{L}$ is a maximal subalgebra of finite codimension and containing no ideals of $\mathcal{L}$. For the Lie superalgebra $\mathcal{L} = \partial \mathfrak{ter} \mathbb{C}[u, \xi]$ the minimal subspace of $\mathcal{L}$ containing $\mathcal{L}_0$ coincides with $\mathcal{L}$. Not all the subalgebras $\mathcal{L}$ of $\partial \mathfrak{ter} \mathbb{C}[u, \xi]$ have this property. Let $\mathcal{L}_{-1}$ be a minimal subspace of $\mathcal{L}$ containing $\mathcal{L}_0$, different from $\mathcal{L}_0$ and $\mathcal{L}_{-1}$-invariant. Construct a filtration of $\mathcal{L}$ by setting

$$\mathcal{L}_{i-1} = [\mathcal{L}_{i-1}, \mathcal{L}_{-1}] + \mathcal{L}_{i-1}, \quad \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} : [D, \mathcal{L}_{-1}] \subseteq \mathcal{L}_{i-1}\}$$

for $i > 0$.

Since the codimension of $\mathcal{L}_0$ is finite, the filtration takes the form

$$\mathcal{L} = \mathcal{L}_{-d} \supset \ldots \mathcal{L}_0 \supset \ldots \quad (\ast)$$

for some $d$. This $d$ is the depth of $\mathcal{L}$ or the associated graded Lie superalgebra. We call all filtered or graded Lie superalgebras of finite depth vectorial, i.e., realizable with vector fields on a finite dimensional supermanifold. Considering the subspaces (\ast) as the basis of a topology, we can complete the graded or filtered Lie superalgebras $L$ or $\mathcal{L}$; the elements of the completion are the vector fields with formal power series as coefficients. Though the structure of the graded algebras is easier to describe, in applications the completed Lie superalgebras are usually needed.

Unlike Lie algebras, simple vectoral superalgebras possess several maximal subalgebras of finite codimension. We describe them, together with the corresponding gradings, in subsect. 0.4.

#### 1) General algebras. Let $x = (u_1, \ldots, u_n, \theta_1, \ldots, \theta_m)$, where the $u_i$ are even indeterminates and the $\theta_j$ are odd ones. The Lie superalgebra $\mathfrak{vect}(n|m)$ is $\partial \mathfrak{ter} \mathbb{C}[x]$; it is called the general vectoral superalgebra.
2) Special algebras. The divergence of the field \( D = \sum_i f_i \frac{\partial}{\partial u_i} + \sum_j g_j \frac{\partial}{\partial \theta_j} \) is the function (in our case: a polynomial, or a series)
\[
\text{div} D = \sum_i \frac{\partial f_i}{\partial u_i} + \sum_j (-1)^p(\theta_j) \frac{\partial g_j}{\partial \theta_j}.
\]

- The Lie superalgebra \( \text{svect}(n|m) = \{ D \in \text{vect}(n|m) : \text{div} D = 0 \} \) is called the special or divergence-free vectoral superalgebra. It is not difficult to see that it is also possible to describe \( \text{svect} \) as \( \{ D \in \text{vect}(n|m) : L_D \text{vol}_x = 0 \} \), where \( \text{vol}_x \) is the volume form with constant coefficients in coordinates \( x \) and \( L_D \) the Lie derivative with respect to \( D \).

- The Lie superalgebra \( \text{svect}_F(0|m) = \{ D \in \text{vect}(0|m) : \text{div}(1 + \lambda \theta_1 \cdots \theta_m)D = 0 \} \) --- the deformation of \( \text{svect}(0|m) \) --- is called the special or divergence-free vectoral superalgebra. Clearly, that \( \text{svect}_F(0|m) \cong \text{svect}(0|m) \) for \( \lambda \mu \neq 0 \). Observe that \( p(\lambda) \equiv m \pmod{2} \), i.e., for odd \( m \) the parameter of deformation \( \lambda \) is odd.

**Remark.** Sometimes we write \( \text{vect}(x) \) or even \( \text{vect}(V) \) if \( V = \text{Span}(x) \) and use similar notations for the subalgebras of \( \text{vect} \) introduced below. Algebraists sometimes abbreviate \( \text{vect}(n) \) and \( \text{svect}(n) \) to \( W_n \) (in honor of Witt) and \( S_n \), respectively.

3) The algebras that preserve Pfaff equations and differential 2-forms.

- Set \( u = (t, p_1, \ldots, p_n, q_1, \ldots, q_n) \); let
\[
\bar{\alpha}_1 = dt + \sum_{1 \leq i \leq n} (p_idq_i - q_idp_i) + \sum_{1 \leq j \leq m} \theta_j d\theta_j \quad \text{and} \quad \omega_0 = d\alpha_1.
\]
The form \( \bar{\alpha}_1 \) is called contact, the form \( \omega_0 \) is called symplectic. Sometimes it is more convenient to redenote the \( \theta \)'s and set
\[
\xi_j = \frac{1}{\sqrt{2}}(\theta_j - i\theta_{r+j}); \quad \eta_j = \frac{1}{\sqrt{2}}(\theta_j + i\theta_{r+j}) \quad \text{for} \ j \leq r = [m/2] \ (\text{here} \ i^2 = -1), \quad \theta = \theta_{2r+1}
\]
and in place of \( \omega_0 \) or \( \bar{\alpha}_1 \) take \( \alpha \) and \( \omega_0 = d\alpha_1 \), respectively, where
\[
\alpha_1 = dt + \sum_{1 \leq i \leq n} (p_idq_i - q_idp_i) + \sum_{1 \leq j \leq r} (\xi_j d\eta_j + \eta_j d\xi_j) \quad \text{if} \ m = 2r
\]
\[
\alpha_1 = dt + \sum_{1 \leq i \leq n} (p_idq_i - q_idp_i) + \sum_{1 \leq j \leq r} (\xi_j d\eta_j + \eta_j d\xi_j) + \theta d\theta \quad \text{if} \ m = 2r + 1.
\]

The Lie superalgebra that preserves the Pfaff equation \( \alpha_1 = 0 \), i.e., the superalgebra
\[
\mathfrak{t}(2n+1|m) = \{ D \in \text{vect}(2n+1|m) : L_D \alpha_1 = f_D \alpha_1 \},
\]
(here \( f_D \in \mathbb{C}[t, p, q, \xi] \) is a polynomial determined by \( D \)) is called the contact superalgebra. The Lie superalgebra that preserves not just the Pfaff equation determined by \( \alpha_1 \) but the form itself, i.e.,
\[
\mathfrak{po}(2n|m) = \{ D \in \mathfrak{t}(2n+1|m) : L_D \alpha_1 = 0 \}
\]
is called the Poisson superalgebra. (A geometric interpretation of the Poisson superalgebra: it is the Lie superalgebra that preserves the connection with form \( \alpha_1 \) in the line bundle over a symplectic supermanifold with the symplectic form \( d\alpha_1 \).

- Similarly, set \( u = q = (q_1, \ldots, q_n) \), let \( \theta = (\xi_1, \ldots, \xi_n; \tau) \) be odd. Set
\[
\alpha_0 = d\tau + \sum_i (\xi_i dq_i + q_id\xi_i), \quad \omega_1 = d\omega_0
\]
and call these forms the odd contact and periplectic, respectively.

The Lie superalgebra that preserves the Pfaff equation \( \alpha_0 = 0 \), i.e., the superalgebra
\[
\mathfrak{m}(n) = \{ D \in \text{vect}(n|n+1) : L_D \alpha_0 = f_D \cdot \alpha_0 \}, \quad \text{where} \ f_D \in \mathbb{C}[q, \xi, \tau],
\]
is called the odd contact superalgebra.

The Lie superalgebra
\[
\mathfrak{b}(n) = \{ D \in \mathfrak{m}(n) : L_D \alpha_0 = 0 \}
\]
is called the Buttin superalgebra ([L3]). (A geometric interpretation of the Buttin superalgebra: it is the Lie superalgebra that preserves the connection with form \( \alpha_1 \) in the line bundle of rank \( \varepsilon \) over a periplectic supermanifold, i.e., the supermanifold with the periplectic form \( d\alpha_0 \).)

The Lie superalgebras
\[
\mathfrak{sm}(n) = \{ D \in \mathfrak{m}(n) : \text{div} D = 0 \}, \quad \mathfrak{sb}(n) = \{ D \in \mathfrak{b}(n) : \text{div} D = 0 \}
\]
are called the divergence-free (or special) odd contact and special Buttin superalgebras, respectively.
Remark. A relation with finite dimensional geometry is as follows. Clearly, \( \ker \alpha_1 = \ker \alpha_0 \). The restriction of \( \omega_0 \) to \( \ker \alpha_1 \) is the orthosymplectic form \( B_{ev}(m|2n) \); the restriction of \( \omega_0 \) to \( \ker \alpha_1 \) is \( B'_{ev}(m|2n) \). Similarly, the restriction of \( \omega_1 \) to \( \ker \alpha_0 \) is the periplectic form \( B_{od}(n|n) \).

0.3. Generating functions. A laconic way to describe the elements of \( \mathfrak{f}, \mathfrak{m} \) and their subalgebras is via generating functions.

- Odd form \( \alpha_1 \). For \( f \in \mathbb{C}[t, p, q, \xi] \) set:
  \[
  K_f = \Delta(f) \frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t} E,
  \]
  where \( E = \sum_i \frac{\partial f}{\partial \theta_i} \) (here the \( y \) are all the coordinates except \( t \)) is the Euler operator (which counts the degree with respect to the \( y \)), \( \Delta(f) = 2f - E(f) \), and \( H_f \) is the hamiltonian field with Hamiltonian \( f \) that preserves \( \delta \alpha_1 \):
  \[
  H_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (1)^{p(f)} \left( \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial}{\partial \theta_j} \right), \quad f \in \mathbb{C}[p, q, \theta].
  \]
  The choice of the form \( \alpha_1 \) instead of \( \hat{\alpha}_1 \) only affects the form of \( H_f \) that we give for \( m = 2k + 1 \):
  \[
  H_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (1)^{p(f)} \sum_{j \leq k} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right), \quad f \in \mathbb{C}[p, q, \xi, \eta, \theta].
  \]

- Even form \( \alpha_0 \). For \( f \in \mathbb{C}[q, \xi, \tau] \) set:
  \[
  M_f = \Delta(f) \frac{\partial}{\partial \tau} - Le_f - (1)^{p(f)} \frac{\partial f}{\partial \tau} E,
  \]
  where \( E = \sum_i \frac{\partial f}{\partial \varphi_i} \) (here the \( y \) are all the coordinates except \( \tau \)) is the Euler operator, \( \Delta(f) = 2f - E(f) \), and
  \[
  Le_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial}{\partial \xi_i} - (1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial q_i} \right), \quad f \in \mathbb{C}[q, \xi].
  \]
  Since
  \[
  L_K_f(\alpha_1) = 2 \frac{\partial f}{\partial t} \alpha_1, \quad L_M_f(\alpha_0) = -(1)^{p(f)} 2 \frac{\partial f}{\partial \tau} \alpha_0,
  \]
  it follows that \( K_f \in \mathfrak{f}(2n + 1|m) \) and \( M_f \in \mathfrak{m}(n) \). Observe that
  \[
  p(Le_f) = p(M_f) = p(f) + \tilde{1}.
  \]

- To the (super)commutators \([K_f, K_g]\) or \([M_f, M_g]\) there correspond contact brackets of the generating functions:
  \[
  [K_f, K_g] = K_{[f,g]_h.b.}, \quad [M_f, M_g] = M_{(f,g)_{l.b.}}.
  \]
  The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on \( t \) (resp. \( \tau \)).

  The Poisson bracket \( \{ \cdot, \cdot \}_{P.b.} \) (in the realization with the form \( \omega_0 \)) is given by the formula
  \[
  \{ f, g \}_{P.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - (1)^{p(f)} \sum_{j \leq m} \left( \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_j} \right),
  \]
  and in the realization with the form \( \omega_0 \) for \( m = 2k + 1 \) it is given by the formula
  \[
  \{ f, g \}_{P.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial \xi_i} - \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial p_i} \right) - (1)^{p(f)} \sum_{j \leq k} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \xi_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right).
  \]

  The Buttin bracket \( \{ \cdot, \cdot \}_{B.b.} \) is given by the formula
  \[
  \{ f, g \}_{B.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial \xi_i} + (1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial q_i} \right).
  \]

Remark. The what we call here Buttin bracket was discovered in pre-super era by Schouten. Buttin was the first to observe that the Schouten bracket determines a Lie superalgebra. The Schouten bracket was originally defined on the superspace of polyvector fields on a manifold, i.e., on the superspace \( \Gamma(A^*(T(M))) \cong A^*(\text{Vect}(M)) \) of sections of the exterior algebra (over the algebra \( F \) of functions) of the tangent bundle. The explicit formula of the Schouten bracket (in which the hatted slot should be ignored, as usual) is
\[
[X_1 \wedge \cdots \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_l, \quad \ast
\]
With the help of Sign Rule we easily superize formula (*) for the case when manifold $M$ is replaced with supermanifold $\mathcal{M}$. Let $x$ and $\xi$ be the even and odd coordinates on $\mathcal{M}$. Setting $\theta_1 = \Pi(\partial x_i) = \dot{x}_i$, $\theta_j = \Pi(\partial \xi_j) = \dot{\xi}_j$ we get an identification of the Schouten bracket of polyvector fields on $\mathcal{M}$ with the Buttin bracket of functions on the supermanifold $\mathcal{M}$; for the definition of $\mathcal{M}$, see [M]. (Physicists call checked variables ghosts [GPS].)

In terms of the Poisson and Buttin brackets, respectively, the contact brackets take the form

$$\{f, g\}_{k.b.} = \Delta(f) \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} \Delta(g) - \{f, g\}_{p.b.}$$

and, respectively,

$$\{f, g\}_{m.b.} = \Delta(f) \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} \Delta(g) - \{f, g\}_{B.b.}$$

The Lie superalgebras of Hamiltonian fields (or Hamiltonian superalgebra) and its special subalgebra (defined only if $n = 0$) are

$$\mathfrak{h}(2n|m) = \{D \in \text{vect}(2n|m) : L_D \omega = 0 \} \quad \text{and} \quad \mathfrak{sh}(m) = \{D \in \mathfrak{h}(0|m) : \text{div} D = 0 \}.$$ 

Its odd analogues are the Lie superalgebra of Leitesian fields introduced in [L1] and its special subalgebra:

$$\mathfrak{le}(n) = \{D \in \text{vect}(n|m) : L_D \omega_1 = 0 \} \quad \text{and} \quad \mathfrak{sle}(n) = \{D \in \mathfrak{le}(n) : \text{div} D = 0 \}.$$ 

It is not difficult to prove the following isomorphisms (as superspaces):

$$\mathfrak{e}(2n+1|m) \cong \text{Span}(K_f : f \in \mathbb{C}[t, p, q, \xi]); \quad \mathfrak{le}(n) \cong \text{Span}(L_{f\xi} : f \in \mathbb{C}[q, \xi]);$$
$$\mathfrak{so}(n) \cong \text{Span}(M_f : f \in \mathbb{C}[\tau, q, \xi]); \quad \mathfrak{h}(2n|m) \cong \text{Span}(H_f : f \in \mathbb{C}[p, q, \xi]).$$

**Remark.**

1) It is obvious that the Lie superalgebras of the series vect, svect, $\mathfrak{h}$ and $\mathfrak{po}$ for $n = 0$ are finite dimensional.

2) A Lie superalgebra of the series $\mathfrak{h}$ is the quotient of the Lie superalgebra $\mathfrak{po}$ modulo the one-dimensional center $\mathfrak{z}$ generated by constant functions. Similarly, $\mathfrak{le}$ and $\mathfrak{sle}$ are the quotients of $\mathfrak{b}$ and $\mathfrak{sb}$, respectively, modulo the one-dimensional (odd) center $\mathfrak{z}$ generated by constant functions.

Set $\mathfrak{spo}(m) = \{K_f \in \mathfrak{po}(0|m) : \int f v_\xi = 0 \};$ clearly, $\mathfrak{sh}(m) = \mathfrak{spo}(m)/\mathfrak{z}$.

Since

$$\text{div} M_f = (-1)^{p(f)} 2 \left( 1 - E \right) \frac{\partial f}{\partial \tau} - \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i},$$

it follows that

$$\text{sm}(n) = \text{Span} \left( M_f \in \mathfrak{m}(n) : (1 - E) \frac{\partial f}{\partial \tau} = \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right).$$

In particular,

$$\text{div} L_{f\xi} = (-1)^{p(f)} 2 \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i}.$$ 

The odd analog of the Laplacian, namely, the operator

$$\Delta = \sum_{i \leq n} \frac{\partial^2}{\partial q_i \partial \xi_i}$$

on a periplectic supermanifold appeared in physics under the name of BRST operator, cf. [GPS]. The divergence-free vector fields from $\mathfrak{sle}(n)$ are generated by harmonic functions, i.e., such that $\Delta(f) = 0$.

Lie superalgebras $\mathfrak{sle}(n)$, $\mathfrak{sb}(n)$ and $\mathfrak{svect}(1)n$ have ideals $\mathfrak{ste}(n)$, $\mathfrak{sbt}(n)$ and $\mathfrak{svect}(n)$ of codimension 1 defined from the exact sequences

$$0 \to \mathfrak{ste}(n) \to \mathfrak{sle}(n) \to \mathbb{C} \cdot L_{\xi_1 \ldots \xi_n} \to 0,$$
$$0 \to \mathfrak{sbt}(n) \to \mathfrak{sb}(n) \to \mathbb{C} \cdot M_{\xi_1 \ldots \xi_n} \to 0,$$
$$0 \to \mathfrak{svect}(n) \to \mathfrak{svect}(1)n \to \mathbb{C} \cdot \xi_1 \ldots \xi_n \to 0.$$. 
0.4. Nonstandard realizations. In [LSh] we proved that the following are all the nonstandard gradings of the Lie superalgebras indicated. Moreover, the gradings in the series $\text{vect}$ induce the gradings in the series $\text{svect}$, and $\text{svect}^\alpha$; the gradings in $m$ induce the gradings in $\text{sm}$, $\text{le}$, $\text{sle}$, $\text{sle}^\alpha$, $b$, $sb$, $sb^\alpha$; the gradings in $t$ induce the gradings in $\text{po}$, $\mathfrak{h}$. In what follows we consider $\mathfrak{t}(2n+1|m)$ as preserving the Pfaff eq. $\alpha = 0$, where $\alpha = dt + \sum i \leq n(p_i d\xi_i - q_i d\delta_i) + \sum j \leq r (\xi_j d\eta_j + \eta_j d\xi_j) + \sum k \leq m-2r \theta_k d\theta_k$.

The standard realizations are marked by $(\ast)$ and in this case indication to $r = 0$ is omitted; note that (bar several exceptions for small $m, n$) it corresponds to the case of the minimal codimension of $L_0$.

| Lie superalgebra | its $\mathbb{Z}$-grading |
|------------------|-------------------------|
| $\text{vect}(n|m;r)$, $0 \leq r \leq m$ | $\deg u_i = \deg \xi_j = 1$ for any $i, j$ $(\ast)$ |
| $\text{m}(n;r)$, $0 \leq r \leq n$ | $\deg \tau = 2$, $\deg q_i = \deg \xi_i = 1$ for any $i$ $(\ast)$ |
| $\mathfrak{t}(2n+1|m;r)$, $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$ | $\deg t = 2$, $\deg p_i = \deg q_i = \deg \xi_j = \deg \eta_j = \deg \theta_k = 1$ for any $i, j, k$ $(\ast)$ |
| $\mathfrak{t}(1|2m;m)$ | $\deg t = \deg \xi_i = 1$, $\deg \eta_i = 0$ for $1 \leq i \leq m$ |

Observe that $\mathfrak{t}(1|2;2) \cong \text{vect}(1|1)$ and $\mathfrak{m}(1;1) \cong \text{vect}(1|1)$.

The exceptional nonstandard gradings. Denote the indeterminates and their respective exceptional degrees as follows (here $\mathfrak{t}(1|2)$ is considered in the realization that preserves the Pfaff eq. $\alpha_1 = 0$):

Denote the nonstandard realizations by indicating the above degrees after a semicolon. We get the following isomorphisms:

$$\text{vect}(1|1;2,1) \cong t(1|2); \quad t(1|2;1,2,-1) \cong \mathfrak{m}(1); \quad \mathfrak{m}(1;1,2,-1) \cong \mathfrak{t}(1|2).$$

Observe that the Lie superalgebras corresponding to different values of $r$ are isomorphic as abstract Lie superalgebras, but as filtered ones they are distinct.

0.5. Cartan prolongs. We will repeatedly use Cartan’s prolongation. So let me recall the definition and generalize it somewhat. Let $\mathfrak{g}$ be a Lie algebra, $V$ a $\mathfrak{g}$-module, $S^i$ the operator of the $i$-th symmetric power. Set $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_0 = \mathfrak{g}$ and define the $i$-th Cartan prolong for $i > 0$ as

$$\mathfrak{g}_i = \{ X \in \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) : X(v)(w, ...) = X(w)(v, ...) \text{ for any } v, w \in \mathfrak{g}_{-1} \}
= (S^i(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0) \cap (S^{i+1}(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}).$$

The Cartan prolong (the result of Cartan’s prolongation) of the pair $(V, \mathfrak{g})$ is $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_\ast = \bigoplus_{i \geq -1} \mathfrak{g}_i$. (In what follows · in superscript denotes, as is now customary, the collection of all degrees, while * is reserved for dualization; in the subscripts we retain the oldfashioned * instead of · to avoid too close a contact with the punctuation marks.)

Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful. Then, clearly,

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_\ast \subset \text{vect}(n) = \text{det} \mathbb{C}[x_1, ..., x_n]$$
where $n = \text{dim} \mathfrak{g}_{-1}$ and

$$\mathfrak{g}_i = \{ D \in \text{vect}(n) : \deg D = i, [D, X] \in \mathfrak{g}_{-1} \text{ for any } X \in \mathfrak{g}_{-1} \}.$$

It is subject to an easy verification that the Lie algebra structure on $\text{vect}(n)$ induces same on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_\ast$.

Of the four simple vectoral Lie algebras, three are Cartan prolongs: $\text{vect}(n) = (\text{id}, \mathfrak{gl}(n))_\ast$, $\text{svect}(n) = (\text{id}, \mathfrak{sl}(n))_\ast$, and $\mathfrak{h}(2n) = (\text{id}, \mathfrak{sp}(n))_\ast$. The fourth one — $\mathfrak{t}(2n+1)$ — is also the prolong under a trifle more general construction described as follows.
A generalization of the Cartan prolongation. Let $\mathfrak{g} = \bigoplus_{i \leq -d} \mathfrak{g}_i$ be a nilpotent $\mathbb{Z}$-graded Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{der}\mathfrak{g}$ a Lie subalgebra of the $\mathbb{Z}$-grading-preserving derivations. For $i > 0$ define the $i$-th prolong of the pair $(\mathfrak{g}, \mathfrak{g}_0)$ to be:

$$\mathfrak{g}_i = ((\mathfrak{g}^\ast(-i)) \otimes \mathfrak{g}_0) \cap ((\mathfrak{g}^\ast(-1)) \otimes \mathfrak{g}_0),$$

where the subscript $i$ in the rhs singles out the component of degree $i$.

Define $\mathfrak{g}_s$, or rather, $(\mathfrak{g}, \mathfrak{g}_s)_0$, to be $\bigoplus_{i \geq -d} \mathfrak{g}_i$; then, as is easy to verify, $(\mathfrak{g}, \mathfrak{g}_s)_0$ is a Lie algebra.

What is the Lie algebra of contact vector fields in these terms? Denote by $\mathfrak{hei}(2n)$ the Heisenberg Lie algebra: its space is $W \oplus \mathbb{C} \cdot z$, where $W$ is a $2n$ dimensional space endowed with a nondegenerate skew-symmetric bilinear form $B$ and the bracket in $\mathfrak{hei}(2n)$ is given by the following conditions: $z$ is in the center and $[v, w] = B(v, w) \cdot z$ for any $v, w \in W$.

Clearly, $\mathfrak{t}(2n + 1)$ is $(\mathfrak{hei}(2n), \mathfrak{osp}(2n))$, where for any $\mathfrak{g}$ we write $\mathfrak{cg} = \mathfrak{g} \oplus \mathbb{C} \cdot z$ or $\mathfrak{c}(\mathfrak{g})$ to denote the trivial central extension with the 1-dimensional even center generated by $z$.

0.6. Lie superalgebras of vector fields as the Cartan prolongs. The superization of the constructions from sec. 0.5 are straightforward: via Sign Rule. We thus get: $\mathfrak{vect}(m|n) = (\mathfrak{id}, \mathfrak{gl}(m|n))_s$; $\mathfrak{vect}(m|n) = (\mathfrak{id}, \mathfrak{sl}(m|n))_s$; $\mathfrak{he}(n) = (\mathfrak{id}, \mathfrak{pe}^{sk}(n))_s$; $\mathfrak{sl}(n) = (\mathfrak{id}, \mathfrak{pe}^{sk}(n))_s$.

Remark. Observe that the Cartan prolongs $(\mathfrak{id}, \mathfrak{osp}^{sk}(m|2n))_s$ and $(\mathfrak{id}, \mathfrak{pe}^{sk}(n))_s$ are finite dimensional.

The generalization of Cartan’s prolongations described in sec. 0.5 has, after superization, two analogs associated with the contact series $\mathfrak{t}$ and $\mathfrak{m}$, respectively.

- First, we define $\mathfrak{hei}(2n|m)$ on the direct sum of a $(2n, m)$-dimensional superspace $W$ endowed with a nondegenerate skew-symmetric odd bilinear form and a $(1, 0)$-dimensional space spanned by $z$.

Clearly, we have $\mathfrak{t}(2n + 1|m) = (\mathfrak{hei}(2n|m), \mathfrak{c}(\mathfrak{osp}^{sk}(m|2n)))_s$ and, given $\mathfrak{hei}(2n|m)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{c}(\mathfrak{osp}^{sk}(m|2n))$, we call $\mathfrak{hei}(2n|m), \mathfrak{g}$, the $k$-prolong of $(W, \mathfrak{g})$, where $W$ is the identity $\mathfrak{c}(\mathfrak{osp}^{sk}(m|2n))$-module.

- The odd analog of $\mathfrak{t}$ is associated with the following odd analog of $\mathfrak{hei}(2n|m)$. Denote by $\mathfrak{ab}(n)$ the antibracket Lie superalgebra: its space is $W \oplus \mathbb{C} \cdot z$, where $W$ is an $n|n$-dimensional superspace endowed with a nondegenerate skew-symmetric odd bilinear form $B$; the bracket in $\mathfrak{ab}(n)$ is given by the following formulas: $z$ is odd and lies in the center; $[v, w] = B(v, w) \cdot z$ for $v, w \in W$.

Set $\mathfrak{m}(n) = (\mathfrak{ab}(n), \mathfrak{c}(\mathfrak{pe}^{sk}(n)))_s$ and, given $\mathfrak{ab}(n)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{c}(\mathfrak{pe}^{sk}(n))$, we call $(\mathfrak{ab}(n), \mathfrak{g})$, the $m$-prolong of $(W, \mathfrak{g})$, where $W$ is the identity $\mathfrak{c}(\mathfrak{pe}^{sk}(n))$-module.

Generally, given a nondegenerate $B$ in a superspace $W$ and a superalgebra $\mathfrak{g}$ that preserves $B$, we refer to the above generalizations as to mk-prolongation of the pair $(W, \mathfrak{g})$.

Partial Cartan prolong. The usual Cartan prolongation starts with nonpositive elements. Define the Cartan prolongation of a positive part. Take a $\mathfrak{g}_0$-submodule $\mathfrak{h}_1$ in $\mathfrak{g}_1$. Suppose that $[\mathfrak{g}_{-1}, \mathfrak{h}_1]$ is the whole $\mathfrak{g}_0$, not a subalgebra of $\mathfrak{g}_0$. Define the 2nd prolongation of $(\bigoplus_{i \leq 0} \mathfrak{g}_i, \mathfrak{h}_1)$ to be $\mathfrak{h}_2 = \{D \in \mathfrak{g}_2 : [D, \mathfrak{g}_{-1}] \in \mathfrak{h}_1\}$.

The terms $\mathfrak{h}_i$ are similarly defined. Set $\mathfrak{h}_i = \mathfrak{g}_i$ for $i < 0$ and $\mathfrak{h}_s = \sum \mathfrak{h}_i$.

Examples. $\mathfrak{vect}(1|n)$ is a subalgebra of $\mathfrak{t}(1|2n)$, the former is obtained as Cartan’s prolong of the same nonpositive part as $\mathfrak{t}(1|2n)$ and a submodule of $\mathfrak{t}(1|2n)$, the simple exceptional superalgebra $\mathfrak{f}$ as introduced in §3 is another example.

0.7. Deformations of the Buttin superalgebras and $\mathfrak{vect}(m|n)$-modules. Here we reproduce a result of Kotchetkoff [Ko1] with corrections from [Ko2], [L3] and [LSh]. To consider the deformations, recall the definition of the $\mathfrak{vect}(m|n)$-module of tensor fields of type $V$, see [BL]. Let $V$ be the $\mathfrak{gl}(m|n) = \mathfrak{vect}_0(m|n)$-module with the lowest weight $\mathbf{L}$. Make $V$ into a $\mathfrak{g}_{\geq 2}$-module, where $\mathfrak{g} = \mathfrak{vect}(m|n)$ and $\mathfrak{g}_{\geq 2} = \bigoplus_{i \geq 0} \mathfrak{g}_i$, setting $\mathfrak{g}_+ \cdot V = 0$ for $\mathfrak{g}_+ = \bigoplus_{i \geq 0} \mathfrak{g}_i$. Let us realize $\mathfrak{g}$ by vector fields on the $m|n$-dimensional linear supermanifold $\mathcal{C}^{m\mid n}$ with coordinates $x$. The superspace $T(V) = \text{Hom}_{\mathfrak{g}_{\geq 2}}(U(\mathfrak{g}), V)$ is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to $\mathbb{C}[x] \otimes V$. Its elements have a natural interpretation as formal tensor fields of type $V$. When $\mathbf{L} = (a, \ldots, a)$ we will simply write $T(\mathbf{a})$ instead of $T(\mathbf{L})$.

Examples. $T(\mathbf{0})$ is the superspace of functions; $\text{Vol}(m|n) = T(1, \ldots, 1; -1, \ldots, -1)$ (the semicolon separates the first $m$ coordinates of the weight with respect to the matrix units $E_{ij}$ of $\mathfrak{gl}(m|n)$) is the superspace of densities or volume forms. We denote the generator of $\text{Vol}(m|n)$ corresponding to the ordered set of coordinates $x$ by $\text{vol}(x)$ or $\text{vol}_x$. The space of $\lambda$-densities is $\text{Vol}^\lambda(m|n) = T(\lambda, \ldots, \lambda; -\lambda, \ldots, -\lambda)$. In particular, $\text{Vol}^\lambda(m|0) = T(\lambda)$ but $\text{Vol}^\lambda(0|n) = T(\lambda, \ldots, \lambda)$. 
As is clear from the definition of the Buttin bracket, there is a regrading (namely, $\mathfrak{b}(n; n)$ given by $\deg \xi_i = 0, \deg q_i = 1$ for all $i$) under which $\mathfrak{b}(n)$, initially of depth 2, takes the form $\mathfrak{g} = \bigoplus_{i \geq -1} \mathfrak{g}_i$ with $\mathfrak{g}_0 = \text{vect}(0|n)$ and $\mathfrak{g}_{-1} \cong \Pi(C[\xi])$.

Let us replace the $\text{vect}(0|n)$-module $\mathfrak{g}_{-1}$ of functions (with inverted parity) with the module of $\lambda$-densities, i.e., set $\mathfrak{g}_{-1} \cong C[\xi](\text{vol}_\lambda)^\lambda$, where

$$L_D(\text{vol}_\lambda)^\lambda = \lambda \text{div} D \cdot \text{vol}_\lambda^\lambda \text{ and } p(\text{vol}_\lambda)^\lambda = 1.$$  

Then the Cartan’s prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ is a deform $\mathfrak{b}_\lambda(n; n)$ of $\mathfrak{b}(n; n)$. The collection of these deformations for various $\lambda \in \mathbb{C}$ constitutes a deformation of $\mathfrak{b}(n; n)$; we called it the \textit{main deformation}. (Though main, this deformation is not the quantization of the Buttin bracket, for the latter see [L3].) The deform $\mathfrak{b}_\lambda(n)$ of $\mathfrak{b}(n)$ is the regrading of $\mathfrak{b}_\lambda(n; n)$ inverse to the regrading of $\mathfrak{b}(n)$ into $\mathfrak{b}(n; n)$.

Another description of the main deformation is as follows. Set

$$\mathfrak{b}_{a, b}(n) = \{M_f \in \mathfrak{m}(n) : a \text{ div } M_f = (-1)^{p(f)}2(a - bn) \frac{\partial f}{\partial r}\}.$$  

It is subject to a direct check that $\mathfrak{b}_{a, b}(n) \cong \mathfrak{b}_\lambda(n)$ for $\lambda = \frac{2a}{m(a - b)}$. This isomorphism shows that $\lambda$ actually runs over $\mathbb{CP}^1$, not $\mathbb{C}$. Observe that for $a = nb$, i.e., for $\lambda = \frac{2}{m + 1}$ we have $\mathfrak{b}_{a, b}(n) \cong \mathfrak{sm}(n)$.

As follows from the description of vectorial superalgebras ([BL]) and the criteria for simplicity of $\mathbb{Z}$-graded Lie superalgebras ([K]), the Lie superalgebras $\mathfrak{b}_\lambda(n)$ are simple for $n \neq 1$ and $\lambda \neq 0, -1, \infty$. It is also clear that the $\mathfrak{b}_\lambda(n)$ are nonisomorphic for distinct $\lambda$’s. (Notice, that at some values of $\lambda$ the Lie superalgebras $\mathfrak{b}_\lambda(n)$ have additional deformations distinct from the above. These deformations are partly described in [L3].)

0.8. \textbf{Several first terms that determine the Cartan and $mk$-prolongations.} To facilitate the comparison of various vectorial superalgebras, consider the following Table. The central element $z \in \mathfrak{g}_0$ is supposed to be chosen so that it acts on $\mathfrak{g}_k$ as $k \cdot \text{id}$. The sign $\oplus$ (resp. $\ominus$) denotes the semidirect sum with the subspace or ideal on the left (right) of it; $\Lambda(r) = C[\xi_1, \ldots, \xi_r]$ is the Grassmann superalgebra of the elements of degree 0.

| $\mathfrak{g}$ | $\mathfrak{g}_{-2}$ | $\mathfrak{g}_{-1}$ | $\mathfrak{g}_0$ |
|----------------|---------------------|---------------------|--------------------|
| $\text{vect}(n|m; r)$ | $-$ | $\text{id} \otimes \Lambda(r)$ | $\mathfrak{gl}(n|m - r) \otimes \Lambda(r) \oplus \text{vect}(0|r)$ |
| $\text{vect}(n|m; r)$ | $-$ | $\text{id} \otimes \Lambda(r)$ | $\mathfrak{sl}(n|m - r) \otimes \Lambda(r) \oplus \text{vect}(0|r)$ |
| $\text{vect}(1|m; m)$ | $-$ | $\Lambda(m)$ | $\Lambda(m) \oplus \text{vect}(0|m)$ |
| $\text{svect}(1|m; m)$ | $-$ | $\text{Vol}(0|m)$ | $\text{vect}(0|m)$ |
| $\text{svect}^\circ(1|m; m)$ | $-$ | $v \in \text{Vol}(0|m) : \int v = 0$ | $\text{vect}(0|m)$ |
| $\text{svect}^\circ(1|2)$ | $-$ | $T^0(0) \cong \Lambda(2)/\mathbb{C} \cdot 1$ | $\text{vect}(0|2) \cong \mathfrak{sl}(1|2)$ |
| $\text{svect}(2|1)$ | $-$ | $\Pi(T^0(0))$ | $\text{vect}(0|2) \cong \mathfrak{sl}(2|1)$ |
| $\mathfrak{e}(2n + 1|m; r)$ | $\Lambda(r)$ | $\text{id} \otimes \Lambda(r)$ | $\mathfrak{cosp}(m - 2r|2n) \otimes \Lambda(r) \oplus \text{vect}(0|r)$ |
| $\mathfrak{h}(2n|m; r)$ | $\Lambda(r)/\mathbb{C} \cdot 1$ | $\text{id} \otimes \Lambda(r)$ | $\mathfrak{osp}(m - 2r|2n) \otimes \Lambda(r) \oplus \text{vect}(0|r)$ |
| $\mathfrak{e}(1|2m; m)$ | $-$ | $\Lambda(m)$ | $\Lambda(m) \oplus \text{vect}(0|m)$ |
| $\mathfrak{e}(1|2m + 1; m)$ | $\Lambda(m)$ | $\Pi(\Lambda(m))$ | $\Lambda(m) \oplus \text{vect}(0|m)$ |
Recall that \( \mathfrak{b}_{a,b}(n) \cong \mathfrak{b}_\lambda(n) \) for \( \lambda = \frac{2a}{n(a-b)} \).

| \( \mathfrak{g} \) | \( \mathfrak{g}_{-2} \) | \( \mathfrak{g}_{-1} \) | \( \mathfrak{g}_0 \) |
|---|---|---|---|
| \( \mathfrak{b}_\lambda(n; \nu) \) | \( \Pi(\Lambda (\nu)) \) | \( \text{id} \otimes \Lambda (\nu) \) | \( (\text{spe}(n - r) \oplus \mathbb{C}(az + bd)) \otimes \Lambda (\nu) \oplus \text{vect}(0|r) \) |
| \( \mathfrak{b}_\lambda(n; n) \) | \( - \) | \( \Pi(V_{\text{od}^A}(0[n])) \) | \( \text{vect}(0|n) \) |
| \( \mathfrak{m}(n; r) \) | \( \Pi(\Lambda (\nu)) \) | \( \text{id} \otimes \Lambda (\nu) \) | \( \text{spe}(n - r) \otimes \Lambda (\nu) \oplus \text{vect}(0|r) \) |
| \( \mathfrak{m}(n; n) \) | \( - \) | \( \Pi(\Lambda (\nu)) \) | \( \Lambda (\nu) \oplus \text{vect}(0|n) \) |
| \( \mathfrak{s} \mathfrak{b}_\nu^E(n; n) \) | \( - \) | \( \Pi(V_{\text{od}^A}(0[n])) \) | \( \text{spe}(n - r) \oplus \text{vect}(0|n) \) |
| \( \text{le}(n; r) \) | \( \Pi(\Lambda (\nu))/\mathbb{C} \cdot 1 \) | \( \text{id} \otimes \Lambda (\nu) \) | \( \text{pe}(n - r) \otimes \Lambda (\nu) \oplus \text{vect}(0|r) \) |
| \( \text{le}(n; n) \) | \( - \) | \( \Pi(\Lambda (\nu))/\mathbb{C} \cdot 1 \) | \( \text{vect}(0|n) \) |
| \( \text{sl}_\nu^E(n; r) \) | \( \Pi(\Lambda (\nu))/\mathbb{C} \cdot 1 \) | \( \text{id} \otimes \Lambda (\nu) \) | \( \text{spe}(n - r) \otimes \Lambda (\nu) \oplus \text{vect}(0|r) \) |
| \( \text{sl}_\nu^E(n; n) \) | \( - \) | \( \Pi(T^0(0)) \) | \( \text{sve}(0|n) \) |

§1. The exceptional Lie superalgebra \((\text{spin}, \mathfrak{as})_s\).

1.1. A. Sergeev’s extension. Let \( \omega \) be a nondegenerate superskew-symmetric bilinear form on an \((n, n)\)-dimensional superspace \( V \). In the standard basis of \( V \) (all the even vectors come first) the canonical matrix of the form \( \omega \) is

\[
\begin{pmatrix}
0 & 1_n \\
1_n & 0
\end{pmatrix}
\]

and the elements of \( \mathfrak{pe}(n) = \mathfrak{aut}(\omega) \) can be represented by supermatrices of the form

\[
\begin{pmatrix}
a & b \\
c & -a^t
\end{pmatrix},
\]

where \( b = b^t, c = -c^t \). The Lie superalgebra \( \mathfrak{spe}(n) \) is singled out by the requirement that \( \text{tr}a = 0 \). Setting

\[
\deg \left( \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \right) = -1, \quad \deg \left( \begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix} \right) = 0, \quad \deg \left( \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right) = 1,
\]

we endow \( \mathfrak{pe}(n) \) with a \( \mathbb{Z} \)-grading. It is known ([K]) that \( \mathfrak{spe}(n) = \mathfrak{pe}(n) \cap \mathfrak{sl}(n|n) \) is a simple Lie superalgebra for \( n \geq 3 \).

A. Sergeev proved that there is just one nontrivial central extensions of \( \mathfrak{spe}(n) \). It exists for \( n = 4 \) and is denoted by \( \mathfrak{as} \). Let us represent an arbitrary element \( A \in \mathfrak{as} \) as a pair \( A = x + d \cdot z \), where \( x \in \mathfrak{spe}(4), d \in \mathbb{C} \) and \( z \) is the central element. In the matrix form the bracket in \( \mathfrak{as} \) is

\[
\begin{pmatrix}
a & b \\
c & -a^t
\end{pmatrix} + d \cdot z, \quad \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + d' \cdot z \cdot \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + \text{tr} cc' \cdot z.
\]

Clearly, \( \deg z = -2 \) with respect to the grading (1.1).

1.2. The Lie superalgebra \( \mathfrak{as} \) can also be described with the help of the spinor representation. Consider \( \mathfrak{po}(0|6) \), the Lie superalgebra whose superspace is the Grassmann superalgebra \( \Lambda (\xi, \eta) \) generated by \( \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \) and the bracket is the Poisson bracket. Recall that \( \mathfrak{h}(0|6) = \text{Span}(H_f : f \in \Lambda (\xi, \eta)) \).

Now, observe that \( \mathfrak{spe}(4) \) can be embedded into \( \mathfrak{h}(0|6) \). Indeed, setting \( \deg \xi_i = \deg \eta_i = 1 \) for all \( i \) we introduce a \( \mathbb{Z} \)-grading on \( \Lambda (\xi, \eta) \) which, in turn, induces a \( \mathbb{Z} \)-grading on \( \mathfrak{h}(0|6) \) of the form \( \mathfrak{h}(0|6) = \otimes_{i \geq 1} \mathfrak{h}(0|6)_i \). Since \( \mathfrak{sl}(4) \cong \mathfrak{o}(6) \), we can identify \( \mathfrak{spe}(4)_0 \) with \( \mathfrak{h}(0|6)_0 \).

It is not difficult to see that the elements of degree \( -1 \) in \( \mathfrak{spe}(4) \) and \( \mathfrak{h}(0|6) \) constitute isomorphic \( \mathfrak{sl}(4) \)-modules. It is subject to a direct verification that it is possible to embed \( \mathfrak{spe}(4)_1 \) into \( \mathfrak{h}(0|6)_1 \).

Sergeev’s extension \( \mathfrak{as} \) is the result of the restriction to \( \mathfrak{spe}(4) \subset \mathfrak{h}(0|6) \) of the cocycle that turns \( \mathfrak{h}(0|6) \) into \( \mathfrak{po}(0|6) \). The quantization deforms \( \mathfrak{po}(0|6) \) into \( \mathfrak{gl}(\Lambda (\xi)) \); the through maps \( T_\lambda : \mathfrak{as} \rightarrow \mathfrak{po}(0|6) \rightarrow \mathfrak{gl}(\Lambda (\xi)) \) are representations of \( \mathfrak{as} \) in the \( 4|4 \)-dimensional modules \( \text{spin}_4 \) isomorphic to each other for all \( \lambda \neq 0 \). (Here \( \lambda \in \mathbb{C} \) plays the role of Planck’s constant.) The explicit form of \( T_\lambda \) is as follows:

\[
T_\lambda : \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + z \rightarrow \begin{pmatrix} a & b - \lambda c \\ c & -a^t \end{pmatrix} + \lambda z \cdot 1_{4|4},
\]

where \( 1_{4|4} \) is the unit matrix and for a skew-symmetric matrix \( c_{ij} = E_{ij} - E_{ji} \) we set \( \tilde{c}_{ij} = c_{kl} \) for the even permutation \((1234) \rightarrow (ijkl)\). Clearly, \( T_\lambda \) is an irreducible representation.

1.3. Theorem. 1) The Cartan prolong \( \mathfrak{f}_\lambda = (\text{spin}_\lambda, \mathfrak{as})_s \) is infinite dimensional and simple for \( \lambda \neq 0 \).

2) \( \mathfrak{f}_\lambda \cong \mathfrak{f}_\mu \) if \( \lambda \cdot \mu \neq 0 \).
**Convention**. For brevity, we denote the isomorphic superalgebras \(f^\lambda = (\text{spin}_\lambda, \text{as})_\ast\) for any \(\lambda \neq 0\) by \((\text{spin}, \text{as})_\ast\).

**Proof.** Heading 1) consists of two statements: a) \((\text{spin}, \text{as})_k \neq 0\) for all \(k > 0\); b) the Lie superalgebra \((\text{spin}_\lambda, \text{as})_\ast\) has no nontrivial \(\mathbb{Z}\)-graded ideals.

a) This follows from the fact that the elements \(u_i^{k+1}\partial_\xi_i\) belong to \((\text{spin}, \text{as})_k\) for any \(k > 0\) and any \(i\) (to prove the statement it suffices to consider only one \(i\)).

b) Assume the contrary: let \(i = \bigoplus i_k\) be a nonzero ideal of \(s = (\text{spin}, \text{as})_\ast\). Let \(x \in i_k\) be a nonzero homogeneous element. Since \(s = \text{vect}(4|4) \supset (\text{spin}, \text{as})_\ast\) is transitive, then the superspace \((k + 1)\) brackets

\[
[h_{-1}, [h_{-1}, \ldots, [h_{-1}, x], \ldots]] = \ldots \ldots \subseteq \ldots \ldots
\]

is a nonzero subspace of \(h_{-1}\). Since \(T_\lambda\) is irreducible, \(i_{-1} = h_{-1}\). The Jacobi identity implies that \([i_{-1}, h_1] \subseteq i_0\) is an ideal of \(h_0\).

But \(h_0 = as\) has only one nontrivial ideal, the center. Since \([i_{-1}, h_1] = [h_{-1}, h_1]\) contains elements of the form \(u_i\partial_\xi_i\) for any \(i\), which do not belong to the center, it follows that \(i_0 = h_0\). In particular, \(i_0\) contains the element \(T_\lambda(z) = -\lambda \sum (u_i\partial_\xi_i + \xi_i\partial_\xi_i)\).

But \([T_\lambda(z), h] = -\lambda \cdot k \cdot h\) for any \(h \in h_k\). Hence, \(i = h\) and \(h\) is simple.

2) is clear.

**1.4.** For \(\lambda = 0\) the representation \(T_0\) is not faithful and \(T_0(\text{as}) = \text{sp}(4)\). The Cartan prolong of the pair \((id, \text{sp}(4))\) is well-known: this is \(\text{sl}(4)\). Recall that we can realize \(\text{le}(n)\) by the generating functions — the elements of \(C[u, \xi]\) — with the Bottin bracket. The subalgebra \(\text{sl}(n)\) is generated by functions \(\Delta\) that satisfy \(\Delta(f) = 0\), where \(\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial u_i \partial \xi_i}\). The exceptional Lie superalgebra \((\text{spin}, \text{as})_\ast\) is a deform of \(\text{sl}(4) \supset C \cdot \sum (u_i\partial_\xi_i + \xi_i\partial_\xi_i)\). An explicit expression of the corresponding cocycle is desirable: it will enable us to express the bracket in \((\text{spin}, \text{as})_\ast\) in terms of harmonic functions (plus one more element).

**§2. An explicit form of the vector fields from \((\text{spin}, \text{as})_\ast \subset \text{vect}(4|4)\)**

Every element \(D \in \text{vect}(4|4)\) is of the form \(D = \sum_{4 \leq i} (P_i \partial_\xi_i + Q_i \partial u_i)\), where \(P_i, Q_i \in C[u, \xi]\).

**2.1. Lemma.** The homogeneous (wrt parity) vector field \(D \in \text{vect}(4|4)\) belongs to \((\text{spin}, \text{as})_\ast\) if and only if it satisfies the following system of equations:

\[
\left\{ \begin{array}{l}
\frac{\partial Q_i}{\partial u_j} + (-1)^{p(D)} \frac{\partial P_j}{\partial \xi_i} = 0 \text{ for any } i \neq j; \\
\frac{\partial Q_i}{\partial u_i} + (-1)^{p(D)} \frac{\partial P_i}{\partial \xi_i} = \frac{1}{2} \sum_{1 \leq j,k \leq 4} \frac{\partial Q_j}{\partial u_k} \text{ for } i = 1, 2, 3, 4; \\
\frac{\partial Q_k}{\partial \xi_j} + \frac{\partial Q_j}{\partial \xi_k} = 0 \text{ for any } i, j; \\
\frac{\partial P_i}{\partial u_j} - \frac{\partial P_j}{\partial u_i} = (-1)^{p(D)} \cdot \lambda \cdot \left( \frac{\partial Q_k}{\partial \xi_i} - \frac{\partial Q_i}{\partial \xi_k} \right) \text{ for any even permutation } \left( \frac{1}{i}, \frac{2}{j}, \frac{3}{k}, \frac{4}{l} \right). 
\end{array} \right.
\]

**2.1.1. Remark.** 1) Observe that the sum of the 4 equations (2.2) yields that \(\text{div} D = 0\), i.e., \((\text{spin}, \text{as})_\ast \subset \text{vect}(4|4)\).

2) For \(\lambda = 0\) the system (2.1)–(2.4) singles out the superalgebra

\[(\text{sl}(4) \supset C \cdot \sum (u_i \partial_\xi_i + \xi_i \partial_\xi_i)) \supset C \cdot \text{Le}_{\xi_i \xi_j \xi_k \xi_l}.
\]

**2.1.2. Remark.** 1) Actually, ANY Cartan prolongation is obtained as a solution of some system of differential equations with constant coefficients. In particular, any \(g_0 \subset \text{gl}(g_{-1})\) is a solution of some homogeneous linear system \(HLS(g)\). Interesting examples are given by Lemmas 2.1 and 5.1. Observe that the number of equations needed to single out \(\text{vect}(V)\) as a subalgebra in \(\text{sl}(\Lambda(V)/C \cdot 1)\) grows with \(\dim V\), since \(\text{codimvect}(V) = (2^n - 2^{n+1}) / n 2^n\). For \(\dim V = 2\) there are no equations; for \(\dim V = 3\) there are 16 equations, etc. Perhaps, these equations can be written in a compact way; e.g., \(\text{div} D = 0\) is a shorthand that eliminates \(n^n\) parameters from the initial \(n^n\) ones.

2) Let \(g_{-1} = V = \text{Span}(\partial_i, i \in I)\). Then any vector field \(D = \sum f_i D_i\) (each \(f_i\) is a function on \(V^\ast\)) generates a linear operator \(L_D : V \rightarrow \text{vect}(V)\) (Lie derivative): \(L_D(D_i) = [D, D_i]\). This operator is a tensor object determined by the matrix \(P(D) = (P_{ij})\), where \(P_{ij} = (-1)^{p(z_j)} \frac{\partial f_i}{\partial x_j}\). If \(D \in \text{vect}(V)_0\),
Define the right inverse \( \Delta \) very simple: the LEFT ad function that for any \( D \), distinct from \( \partial u \), the following formula holds:

\[
\Delta(D) = (-1)^p(D) \frac{\partial P_i}{\partial u_j} + \frac{\partial Q_i}{\partial u_j}, \quad D_{ij} = (-1)^p(D) \frac{\partial P_i}{\partial \xi_j}.
\]

Therefore, equations (2.1) and (2.2) mean that \( a + d^t = (\frac{1}{4} \text{tra}) \cdot 1_4 \), equations (2.3) that \( c + c^t = 0 \), equations (2.4) that \( b - b^t = \lambda (\hat{c} - \hat{c}^t) \).

Then

\[
a_0 = a - \left( \frac{1}{4} \text{tra} \right) \cdot 1_4, \quad d_0 = d - \left( \frac{1}{4} \text{tra} \right) \cdot 1_4.
\]

Comparing formulas (2.5) with (1.2) we see that \( g^0 \) coincides with the image of \( \mathfrak{sle} \) under \( T_\lambda \), i.e., with \( (\mathfrak{spin}, \mathfrak{as})_0 \).

Set

\[
D_{\lambda_i}(D) = \sum_{i \leq 4} \left( \frac{\partial P_i}{\partial u_j} \frac{\partial}{\partial \xi_i} + \frac{\partial Q_i}{\partial u_j} \frac{\partial}{\partial \xi_i} \right) \text{ and } \tilde{D}_{\xi_i}(D) = (-1)^{p(D)} \sum_{i \leq 4} \left( \frac{\partial P_i}{\partial \xi_j} \frac{\partial}{\partial u_i} + \frac{\partial Q_i}{\partial \xi_j} \frac{\partial}{\partial u_i} \right).
\]

The operators \( D_{\lambda_i} \) and \( \tilde{D}_{\xi_i} \), clearly, commute with the \( g^1 \)-action. Observe: the operators commute, not supercommute.

Since equations (2.1)–(2.4) is a linear combination of only these operators, the definition of Cartan prolongation itself ensures an isomorphism of \( (g^1)_n \) with \( (\mathfrak{p}^1)_n \).

**2.2. The right inverse of \( \Delta \).** Let \( f \) be an arbitrary homogeneous wrt the degree in \( u \) and \( \xi \) harmonic function, distinct from \( \xi_1 \ldots \xi_4 \), i.e., an arbitrary generating function for \( \mathfrak{sle}^c(n) \). Then \( f = \Delta(F) \) for some function \( F \) (as follows from the computation of the homology of \( \Delta \) which is an easy exercise; the answer: the homology space \( H(\Delta) \) is spanned by \( \xi_1 \ldots \xi_4 \)). Clearly, \( F \) is determined up to an arbitrary harmonic summand. Set \( \Phi = \sum u_i \xi_i \). Then

\[
\Delta(f) = (\Delta f) f - (-1)^{p(f)} \Phi \Delta f - (-1)^{p(f)} \{ \Phi, f \} = (n + \deg u f - \deg \xi f) f.
\]

Define the right inverse of \( \Delta \) by the formula

\[
\Delta^{-1} f = \frac{1}{\nu(f)} (\Phi f), \quad \text{where } \nu(f) = n + \deg u f - \deg \xi f.
\]

Since the kernel of \( \Delta \) is nonzero, \( \Delta \) has no inverse. Still, \( \Delta^{-1} \) maps \( \mathfrak{sle}^c(n) \) into \( \mathfrak{le}(n) \) without kernel and the following formula holds:

\[
\Delta(\Delta^{-1} f) = f.
\]

**2.3. Theorem.** Any vector field \( D \in g^1 \) can be represented in the form

\[
D = D_f + cZ, \text{ where } c \in \mathbb{C} \text{ and } Z = \sum_{i \leq 4} (u_i \partial u_i + \xi_i \partial \xi_i),
\]

where \( f \in \mathfrak{sle}^c(4) \) and where (recall that \( \mathfrak{A}_n \subset S_n \) denotes the subgroup of even permutations):

\[
D_f = Le_f + \lambda \left( -Le_f + 2 \sum_{1 \leq i \leq 4; (i,j,k,l) \in A_4} \frac{\partial^4 \Delta^{-1}(f)}{\partial \xi_j \partial \xi_k \partial \xi_l} \partial \xi_i \right) \text{ for } \hat{f} = 4 \Delta^{-1} \left( \frac{\partial^4 \Delta^{-1}(f)}{\partial \xi_1 \partial \xi_2 \partial \xi_3 \partial \xi_4} \right).
\]

For the proof see Appendix 1.
Corollary. 1) The Lie superalgebra $\mathfrak{g}^\lambda$ is a deformation of the \Lie superalgebra $\mathfrak{sl}(4) \oplus \mathbb{C} \cdot Z$.

2) If $\deg_\mathfrak{c} f \leq 1$, then $D_f = L_f$, hence, $h = \{e \cdot Z + D_f : \deg_\mathfrak{c} f \leq 1, c \in \mathbb{C}\}$ remains rigid under this deformation.

3) Let $\Omega = du_1 \wedge du_2 \wedge du_3 \wedge du_4$ be the volume element on the underlying manifold of the $\mathbb{C}^{4|1}$. Observe that the volume element $\text{vol}(u, \xi)$ on the whole $\mathbb{C}^{4|1}$ is invariant with respect to the action, but $\Omega$ is not invariant. It is invariant, however, with respect to the deformed subalgebra $h$.

4) Let $D \in \mathfrak{g}^\lambda$; let $L_D$ be the Lie derivative. Denote by $\nabla = \sum \partial u_i \partial \xi_i$ the bivector dual to $\omega$. Observe that the lhs of equations (2.1) – (2.4) determine the coefficients of the 2-form $L_D(\omega)$:

- equations (2.1) determine the coefficients of $d u_i d \xi_i$;
- equations (2.2) determine the coefficients of $d u_i d \xi_i$;
- equations (2.3) determine the coefficients of $d \xi_i d \xi_i$;
- equations (2.4) determine the coefficients of $d u_i d u_i$.

The rhs of eqs. (2.2) determines the nonzero coefficients of the form $\frac{1}{2} (\sum \frac{\partial Q_i}{\partial u_i}) \omega$, while the rhs of eqs. (2.4) determines the nonzero coefficients of the form $\lambda(L_D \Omega) \ast \nabla$, where $\ast$ is the convolution of tensors.

Therefore, eqs. (2.1) – (2.4) can be rewritten in the form

$$L_D \omega = \frac{1}{2} (\sum \frac{\partial Q_i}{\partial u_i}) \omega + \lambda(L_D \Omega) \ast \nabla.$$ 

Besides, if we replace the rhs of eqs. (2.2) with an arbitrary function $\Psi(u, \xi)$ but add the constraint

$$\text{div}(D) = \sum (\frac{\partial Q_i}{\partial u_i} - (-1)^{\rho(D)} \frac{\partial P_i}{\partial \xi_i}) = 0,$$

then the sum of the four eqs. (2.2) with eq. (2.5) automatically yields

$$\left\{ \begin{array}{l}
L_D \omega = \Psi \cdot \omega + \lambda(L_D \Omega) \ast \nabla, \\
\text{div}(D) = 0.
\end{array} \right.$$ 

§3. The exceptional \Lie superalgebra $\mathfrak{f}(1|6)$ of $\mathfrak{f}(1|6)$

If the operator $d$ that determines a $\mathbb{Z}$-grading of the \Lie superalgebra $\mathfrak{g}$ does not belong to $\mathfrak{g}$, we denote the Lie superalgebra $\mathfrak{g} \oplus \mathbb{C} \cdot d$ by $\mathfrak{g} d$. Recall also that $\mathfrak{c}(\mathfrak{g})$ or just $\mathfrak{g}$ denotes the trivial 1-dimensional central extension of $\mathfrak{g}$ with the even center.

3.1. The Lie superalgebra $\mathfrak{g} = \mathfrak{f}(1|2n)$ is generated by the functions from $\mathbb{C}[t, \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n]$. The standard $\mathbb{Z}$-grading of $\mathfrak{g}$ is induced by the $\mathbb{Z}$-grading of $\mathbb{C}[t, \xi, \eta]$ given by $\deg t = 2, \deg \xi = 1, \deg \eta = 1$; namely, $\deg K_f = \deg f - 2$. Clearly, in this grading $\mathfrak{g}$ is of depth 2. Let us consider the functions that generate several first homogeneous components of $\mathfrak{g} = \bigoplus_{i \geq 2} \mathfrak{g}_i$:

| component | $\mathfrak{g}_{-2}$ | $\mathfrak{g}_{-1}$ | $\mathfrak{g}_0$ | $\mathfrak{g}_1$ |
|-----------|----------------------|----------------------|-----------------|-----------------|
| its generators | 1 | $\Lambda^1(\xi, \eta)$ | $\Lambda^2(\xi, \eta) \oplus \mathbb{C} \cdot t$ | $\Lambda^3(\xi, \eta) \oplus t \Lambda^1(\xi, \eta)$ |

As one can prove directly, the component $\mathfrak{g}_1$ generates the whole subalgebra $\mathfrak{g}_+ = \mathfrak{g}$ of elements of positive degree. The component $\mathfrak{g}_1$ splits into two $\mathfrak{g}_0$-modules $\mathfrak{g}_{11} = \Lambda^3$ and $\mathfrak{g}_{12} = t \Lambda^1$. It is obvious that $\mathfrak{g}_{12}$ is irreducible and the component $\mathfrak{g}_{11}$ is trivial for $n = 1$.

The Cartan prolongs of these components are well-known:

$$\begin{align*}
(\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{11})^{mk}_{\ast} & \cong \mathfrak{so}(0|2n) \oplus \mathbb{C} \cdot K_f \cong \mathfrak{so}(0|2n), \\
(\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{12})^{mk}_{\ast} & \cong \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{12} \oplus \mathbb{C} \cdot K_f \cong \mathfrak{osp}(2n|2).
\end{align*}$$

Observe a remarkable property of $\mathfrak{f}(1|6)$. For $n > 1$ and $n \neq 3$ the component $\mathfrak{g}_{11}$ is irreducible. For $n = 3$ it splits into 2 irreducible conjugate modules that we will denote by $\mathfrak{g}_{11}^\xi$ and $\mathfrak{g}_{11}^\eta$. Observe further, that $\mathfrak{g}_0 = \mathfrak{o}(6) \cong \mathfrak{s}(4)$. As $\mathfrak{s}(4)$-modules, $\mathfrak{g}_{11}^\xi$ and $\mathfrak{g}_{11}^\eta$ are the symmetric squares $S^2(\mathfrak{d})$ and $S^2(\mathfrak{d}^*)$ of the identity 4-dimensional representation and its dual, respectively.

3.2. Theorem. 1) The Cartan prolong $$(\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{11}^\xi \oplus \mathfrak{g}_{11}^\eta)^{mk}_{\ast}$$ is infinite dimensional and simple. It is isomorphic to $$(\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{11}^\xi \oplus \mathfrak{g}_{11}^\eta)^{mk}_{\ast}.$$

2) $$(\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{11}^\xi)^{mk}_{\ast} \cong (\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{11}^\eta)^{mk}_{\ast} \cong \mathfrak{s}(\mathfrak{g}_0) \oplus \mathbb{C} \cdot K_f \cong \mathfrak{o}(\mathfrak{g}_0).$$
To clarify the structure of the exceptional Lie superalgebra \( \Pi(T(\bar{0})/\mathbb{C} \cdot 1) \), consider one more construction. Let us describe one wonderful property of \( \mathfrak{sl}(\mathfrak{e}^3(3)) \) which clinches the proof.

\textbf{Lemma.} Denote \( \mathfrak{h} = (\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{11}^\xi \oplus \mathfrak{g}_{12})_{\mathfrak{m}k} \). Consider the \( \mathbb{Z} \)-grading of \( \mathfrak{h} \) induced by the standard grading of \( \mathfrak{g}(1/\mathfrak{t}) \).

For \( k > 1 \) the operator \( T_k = (\text{ad } K_{\xi^2})|_{\mathfrak{h}_k} \) determines an isomorphism of \( \mathfrak{sl}(4) \)-modules \( \mathfrak{h}_k \) and \( \mathfrak{h}_{k+2} \). The operator \( T_1 = (\text{ad } K_{\xi^2})|_{\mathfrak{g}_{11}} \) determines an isomorphism of \( \mathfrak{g}_{11} \) with its image.

Proof. We easily check that \( K_{\xi^2} \in \mathfrak{h} \) and

\[
\text{ad } K_{\xi^2} = 2t(t\partial t + E - 2), \quad \text{where } E = \sum (\xi_i \frac{\partial}{\partial \phi_i} + \eta_i \frac{\partial}{\partial \eta_i}).
\]

Therefore, \( \ker(\text{ad } K_{\xi^2}) \) in \( \mathfrak{g}(1/\mathfrak{t}) \) consists of the fields generated by the functions \( f \) such that \( \deg f + \deg \xi f + \deg \eta f - 2 = 0 \), i.e., \( \ker(\text{ad } K_{\xi^2}) \cong \mathfrak{sl}(4) \oplus \mathfrak{g}_{12} \unrhd \mathfrak{c} K_{\xi^2} \).

This makes it clear that, first, \( \text{ad } K_{\xi^2} \) is \( \mathfrak{sl}(4) \)-invariant; second, the operators \( T_k \) have no kernel for \( k > 0 \).

We will denote the simple exceptional Lie superalgebra described in heading 1) of Theorem 3.2 by \( \mathfrak{e}^3(3) \).

4. Lie superalgebras \( \mathfrak{eg} \). The exceptional Cartan prolong \( \Pi(T(\bar{0})/\mathbb{C} \cdot 1), \mathfrak{vect}(0[3]) \).

In order to inspect these examples we have to recall (see sec. 0.7) that on the supermanifold of purely odd dimension the space of volume forms is \( T(-\bar{1}) \) and the space of half-densities is \( T(-1/2) \) (not \( T(1/2) \) and \( T(1/2) \)) as on manifolds.

4.1. Let us now describe a construction of several exceptional simple Lie superalgebras. Let \( \mathfrak{u} = \mathfrak{vect}(m|n) \), let \( \mathfrak{g} = (\mathfrak{u}_{-1}, \mathfrak{g}_0) \) be a simple Lie subsuperalgebra of \( \mathfrak{u} \). Let, moreover, there exist an element \( D \in \mathfrak{u}_0 \) that determines an exterior derivation of \( \mathfrak{g} \) and has no kernel on \( \mathfrak{u}_+ \). Let us study the prolong \( \tilde{\mathfrak{g}} = (\mathfrak{g}_-, \mathfrak{g}_0 \oplus \mathfrak{C} D) \).

\textbf{Lemma.} Either \( \tilde{\mathfrak{g}} \) is simple or \( \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{C} D \).

Proof. Let \( I \) be a nonzero graded ideal of \( \tilde{\mathfrak{g}} \). The subsuperspace \( \langle \mathfrak{u}_{-1} \rangle^{k+1} a \) of \( \mathfrak{u}_{-1} \) is nonzero for any nonzero homogeneous element \( a \in \mathfrak{u}_k \) and \( k \geq 0 \). Since \( \mathfrak{g}_{-1} = \mathfrak{u}_{-1} \), the ideal \( I \) contains nonzero elements from \( \mathfrak{g}_{-1} \). By simplicity of \( \mathfrak{g} \) the ideal \( I \) contains the whole \( \mathfrak{g} \). If, moreover, \( [\mathfrak{g}_{-1}, \tilde{\mathfrak{g}}_1] = \mathfrak{g}_0 \), then by definition of the Cartan prolongation \( \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{C} D \).

If, instead, \( [\mathfrak{g}_{-1}, \tilde{\mathfrak{g}}_1] = \mathfrak{g}_0 \oplus \mathfrak{C} D \), then \( D \in I \) and since \( [D, \mathfrak{u}_+] = \mathfrak{u}_+ \), we derive that \( I = \mathfrak{g} \). In other words, \( \mathfrak{g} \) is simple.

4.2. Example 1. Take \( \mathfrak{u} = \mathfrak{vect}(2^n-1|2^n-1) \). Consider \( \mathfrak{u}_{-1} \) as \( \Pi(T(\bar{0})/\mathbb{C} \cdot 1) \) and set \( \mathfrak{g}_{-1} = \mathfrak{u}_{-1}, \mathfrak{g}_0 = \mathfrak{vect}(0|n) \). Clearly, \( \mathfrak{g}_{-1} \) is a \( \mathfrak{g}_0 \)-module. Then \( (\mathfrak{g}_{-1}, \mathfrak{g}_0)_+ \) is a simple Lie superalgebra isomorphic to \( \mathfrak{sl}(n|n) \). The isomorphism is established with the help of a regrading. For the operator \( D \) of the exterior derivation of \( \mathfrak{sl}(n|n) \) we take the grading operator \( d \in m(n|n) \subset \mathfrak{u}_0 \), i.e., \( \mathfrak{g}_0 \oplus \mathfrak{C} D \cong \mathfrak{vect}(0|n) \).

In particular, for \( n = 2 \) we have \( \mathfrak{g}_{-1} = \Pi(\xi_1, \xi_2, \xi_3); \mathfrak{g}_0 = \mathfrak{vect}(0|2) \cong \mathfrak{sl}(2|1) \). Then \( \mathfrak{c}(\mathfrak{g}_0) = \mathfrak{gl}(2|1) \) and \( (\Pi(T(\bar{0})/\mathbb{C} \cdot 1), \mathfrak{vect}(0|3))_\ast \cong \mathfrak{vect}(2|1) \).

\textbf{Theorem.} 1) \( (\Pi(T(\bar{0})/\mathbb{C} \cdot 1), \mathfrak{vect}(0|3))_\ast \) is a simple Lie superalgebra.

2) \( (\Pi(T(\bar{0})/\mathbb{C} \cdot 1), \mathfrak{vect}(0|3))_\ast \cong \mathfrak{d}(\mathfrak{sl}(n|n)) \) for \( n > 3 \).

Proof. Thanks to Lemma 4.1.1 heading 1) follows from the fact that \( (\Pi(T(\bar{0})/\mathbb{C} \cdot 1), \mathfrak{vect}(0|3))_\ast \) is a simple Lie superalgebra, is bigger than \( \mathfrak{sl}(\mathfrak{e}^3(3, 3) \oplus \mathfrak{C} D \); we will prove this fact in \( \S 5 \). Heading 2) is proved in Appendix 2.

4.3. To clarify the structure of the exceptional Lie superalgebra \( (\Pi(T(\bar{0})/\mathbb{C} \cdot 1), \mathfrak{vect}(0|3))_\ast \), consider one more construction. Let us describe one wonderful property of \( \mathfrak{sl}(\mathfrak{e}^3(3)) \) that singles it out among the \( \mathfrak{sl}(\mathfrak{e}^3(n)) \).

In the standard grading of \( \mathfrak{g} = \mathfrak{sl}(\mathfrak{e}^3(3)) \) we have: \( \text{dim } \mathfrak{g}_{-1} = (3, 3), \mathfrak{g}_0 \cong \mathfrak{sp}(3) \). For the regraded superalgebra \( \mathfrak{R}_0 = \mathfrak{sl}(\mathfrak{e}^3(3)) \) we have: \( \text{dim } \mathfrak{R}_{-1} = (3, 3), \mathfrak{R}_0 = \mathfrak{vect}(0|3) \cong \mathfrak{sp}(3) \). Therefore, for \( \mathfrak{sl}(\mathfrak{e}^3(3)) \) and only for it among the \( \mathfrak{sl}(\mathfrak{e}^3(n)) \), the regrading \( R \) determines a nontrivial automorphism. In terms of generating functions the regrading \( R \) is given by the formulas:

1) \( \text{deg } f = 0: R(f) = \Delta(f \xi_1 \xi_2 \xi_3) \);
2) \( \text{deg}_x (f) = 1: R(f) = f; \)
3) \( \text{deg}_x (f) = 2: R(f) = \frac{\partial^2 (\Delta^{-1} f)}{\partial \xi_1 \partial \xi_2} \text{ (see (2.6))}. \)

We see that \( R^2 = \text{SIGN} \) for the operator \( \text{SIGN} \) is defined by the formulas \( \text{SIGN}(D) = (-1)^{p(D)} D \) on the vector fields and \( \text{SIGN}(f) = (-1)^{p(f)+1} f \) on the generating functions.

Let now \( g = \text{le}(3, 3) \) and \( i_1: u \rightarrow \text{vect}(3) \) be the embedding that preserves the standard \( \mathbb{Z} \)-grading of \( g \). Let \( h = \text{le}(3) \) and \( \text{ste}^*(3) \subset h \). Then the map \( i_2 = \text{SIGN} \circ i_1 R \) determines an embedding \( h \rightarrow u \) that preserves the grading of \( h \).

Observe that \( h_0 = \text{spe}(3) \) and \( h_0 = \text{pe}(3) \equiv \mathbb{C} \oplus \sum q_i \xi_i \). The action of \( z = -2i_1(\sum q_i \xi_i) + 3d \) on the space \( i_2(h_{-1}) \) coincides with the action of \( \sum q_i \xi_i \) on \( h_{-1} \). Therefore, setting \( i_2(\sum q_i \xi_i) = z \) we get an embedding \( i_2(h_{-1} \oplus h_0) \rightarrow u \) that can be extended to an embedding of \( h \) to \( u \). Under this embedding

\[
i_1(g_1 \oplus g_0) + i_2(h_1 \oplus h_0) = u_{-1} \oplus (g_0 \oplus Cd),
\]
i.e., the nondirect sum of the images of \( i_1 \) and \( i_2 \) covers the whole nonpositive part of \( \Pi(T(\bar{0})/\mathbb{C} \cdot 1), \text{vect}(0|3) \).

Thus, we obtained two distinct embeddings of \( \text{le}(3) \cong \text{le}(3; 3) \) into \( \Pi(T(\bar{0})/\mathbb{C} \cdot 1), \text{vect}(0|3) \):

\[
i_1: \text{le}(3; 3) \rightarrow \Pi(T(\bar{0})/\mathbb{C} \cdot 1), \text{vect}(0|3) \] and \( i_2: \text{le}(3) \rightarrow \Pi(T(\bar{0})/\mathbb{C} \cdot 1), \text{vect}(0|3) \quad (4.1)
\]
such that \( i_1(\text{le}(3; 3)) + i_2(\text{le}(3)) = (\Pi(T(\bar{0})/\mathbb{C} \cdot 1), \text{vect}(0|3)) \) (the sum in the lhs is not a direct one!).

As a linear space, \( \Pi(T(\bar{0})/\mathbb{C} \cdot 1), \text{vect}(0|3) \), is the quotient of \( \text{le}(3; 3) \oplus \text{le}(3) \) modulo the subspace \( V = \{ \text{SIGN} \circ Rg \oplus -g : g \in \text{ste}^*(3) \} \). The map \( \varphi \) defined by the formula

\[
\varphi|_{\text{le}(3; 3)} = \text{SIGN} \circ i_2 i_1^{-1}; \quad \varphi|_{\text{le}(3)} = i_1 i_2^{-1}
\]
determines a nontrivial automorphism of \( \Pi(T(\bar{0})/\mathbb{C} \cdot 1), \text{vect}(0|3) \).

### §5. AN EXPLICIT FORM OF THE VECTOR FIELDS FROM \( \Pi(T(\bar{0})/\mathbb{C} \cdot 1), \text{vect}(0|3) \), \( \subset \text{vect}(4|3) \)

Let us renumber the basis in \( \Pi(\Lambda(\eta_1, \eta_2, \eta_3)) \) by setting:

\[
\Pi(\eta_1\eta_2\eta_3) \mapsto \partial y; \quad \Pi(\eta_i) \mapsto \partial u_i; \quad \Pi(\frac{\partial \eta_1 \eta_2 \eta_3}{\partial \eta_i}) \mapsto \partial \xi_i.
\]

Every element \( D \in \text{vect}(4|3) \) is of the form \( D = \sum_{i \leq 3} (P_i \partial \xi_i + Q_i \partial u_i) + R \partial y \), where \( P_i, Q_i, R \in \mathbb{C}[y, u, \xi] \).

#### 5.1. Lemma

Set \( g_{-1} = \text{Span}(\partial y, \partial u_i, \partial \xi_i \text{ for } i \leq 3), \quad g_0 = \text{vect}(0|3) \). The homogeneous (wrt parity) vector field \( D \in \text{vect}(4|3) \) belongs to \( \Pi(T(\bar{0})/\mathbb{C} \cdot 1), \text{vect}(0|3) \), if and only if it satisfies the following system of equations:

\[
\frac{\partial Q_i}{\partial u_j} + (-1)^{p(D)} \frac{\partial P_i}{\partial \xi_j} = 0 \quad \text{for any } i \neq j; \quad (5.1)
\]

\[
\frac{\partial Q_i}{\partial u_j} + (-1)^{p(D)} \frac{\partial P_i}{\partial \xi_j} = \frac{1}{2} \left( \sum_{1 \leq j \leq 3} \frac{\partial Q_j}{\partial u_j} + \frac{\partial R}{\partial y} \right) \quad \text{for } i = 1, 2, 3; \quad (5.2)
\]

\[
\frac{\partial Q_i}{\partial \xi_j} + \frac{\partial Q_j}{\partial \xi_i} = 0 \quad \text{for any } i, j; \quad \text{in particular } \frac{\partial Q_i}{\partial \xi_i} = 0; \quad (5.3)
\]

\[
\frac{\partial P_i}{\partial u_j} - \frac{\partial P_j}{\partial u_i} = (-1)^{p(D)} \frac{\partial R}{\partial \xi_k} \quad \text{for any } k \text{ and any even permutation } \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}; \quad (5.4)
\]

\[
\frac{\partial Q_i}{\partial y} = 0 \quad \text{for } i = 1, 2, 3; \quad (5.5)
\]

\[
\frac{\partial P_k}{\partial y} = (-1)^{p(D)} \frac{1}{2} \left( \frac{\partial Q_i}{\partial \xi_j} - \frac{\partial Q_j}{\partial \xi_i} \right) \quad \text{for any } k \text{ and any even permutation } \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}. \quad (5.6)
\]

Proof is similar to that of Lemma 2.1.

**Remark** The left hand sides of eqs. (5.1)–(5.6) determine the coefficients of the 2-form \( L_D \omega \), where \( L_D \) is the Lie derivative and \( \omega = \sum_{1 \leq i \leq 3} du_i d\xi_i \). It is interesting to interpret the rhs of these equations.
5.2. Theorem. Every solution of the system (5.1)–(5.6) is of the form:

\[ D = \le f + y A f - (-1)^{p(f)} \left( y \Delta(f) + y^2 \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2 \partial \xi_3} \right) \partial y + \]

\[ A f - (-1)^{p(g)} \left( y \Delta(g) + 2 y \frac{\partial^3 g}{\partial \xi_1 \partial \xi_2 \partial \xi_3} \right) \partial y, \]

where \( f, g \in \mathbb{C}[u, \xi] \) are arbitrary and the operator \( A_f \) is given by the formula:

\[ A_f = \frac{\partial^2 f}{\partial \xi_2 \partial \xi_3 \partial \xi_1} + \frac{\partial^2 f}{\partial \xi_3 \partial \xi_2 \partial \xi_1} + \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2 \partial \xi_3}. \]

(5.7)

Proof. First, let us prove that if \( \tilde{g} \) and the quotient \( g \) are arbitrary and the operator \( A_f \) is given by the formula:

\[ A_f = \frac{\partial^2 f}{\partial \xi_2 \partial \xi_3 \partial \xi_1} + \frac{\partial^2 f}{\partial \xi_3 \partial \xi_2 \partial \xi_1} + \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2 \partial \xi_3}. \]

(5.8)

Proof is similar to that of Theorem 2.3, see Appendix 1 and [PSh].

Formula (5.7) makes it possible to explicitly express the two embeddings (4.1)

\[ i_1, i_2 : \mathfrak{le}(3) \to \Pi(T(\tilde{g})/C \cdot 1), \text{svect}(0|3)). \]

The first embedding \( i_1 \) preserves the grading of \( \mathfrak{le}(3,3) \), cf. 0.4. I do not know any compact general formula for \( i_1 \) and can only determine it component-wise (mind that \( A_3 \) in the first line of the following displayed formula is the group of even permutations, not the operator \( A_f \) for \( f = 3 \)).

\[ i_1(\le f(u)) = \le \sum \frac{\partial^3 f}{\partial \xi_i \partial \xi_j \partial \xi_k} - u f, \text{ where } g \text{ is treated as a parameter} \]

and \( i_1 \) for \( f = 3 \).

The second embedding \( i_2 \) preserves the standard grading of \( \mathfrak{le}(3) \). It is given by the formulas:

\[ i_2(\le f) = \le f + y A f + (-1)^{p(f)} \left( y \Delta(f) + y^2 \frac{\partial^3 f}{\partial \xi_1 \partial \xi_2 \partial \xi_3} \right) \partial y. \]

§6. EXCEPTIONAL SIMPLE LIE SUPERALGEBRAS OF DEPTH 2: \((ab(4), \text{svect}(0|3))^m \) AND \((\text{hei}(8|6), \text{svect}_{3,4}(4))^k \)

Two more examples of exceptional simple Lie superalgebras are obtained with the help of a construction that generalizes the constructions from §4 to Lie superalgebras of depth 2. Let \( u = \bigoplus_{i \geq 2} u_i \) be either \( \mathfrak{m}(n) \) or \( \mathfrak{gl}(2m + 1|n) \); let \( g = (u_-, g_0)_+ \) be a subalgebra of \( u \) such that the subspace \( u_{-2} \) belongs to the center of \( g \) and the quotient \( g/\mathfrak{u}_{-2} \) is simple. Let, moreover, \( D \in u_0 \) determine an exterior derivation of \( g \) without kernel on \( u_{-2} + u_+ \), where \( u_+ = \bigoplus_{i > 0} u_i \).

Let us study the \( mk \)-prolong \( \hat{g} = (g_-, g_0 + CD)^m \). The main result of this section: a description of two simple exceptional Lie superalgebras \((ab(4), \text{svect}(0|3))^m \) (Th. 6.2) and \((\text{hei}(8|6), \text{svect}_{3,4}(4))^k \) (Th. 6.5).

6.1. Lemma. Either \( \hat{g} \) is simple or \( \hat{g} \cong g \oplus CD \).

Proof. First, let us prove that if \( \hat{g} \not\cong g \oplus CD \), then \( u_{-2} \) is not an ideal in \( g \). Indeed, in this case there exist \( g_{-1} \in g_{-1}, g_0 \in g_0 \) and \( g_1 \in g_1 \) such that \( [g_1, g_{-1}] = D + g_0 \).

Let \( u_{-2} = \mathbb{C}_z \). Then \( [g_{-1}, z] = [g_0, z] = 0 \) and we have

\[ [g_{-1}, [z, g_1]] = ([g_{-1}, z], g_1) + (-1)^{p(g_{-1})p(z)} [z, [g_{-1}, g_1]] \]

\[ (-1)^{p(g_{-1})p(z)} [z, D + g_0] = (-1)^{p(g_{-1})p(z)} [z, D] \neq 0. \]

We have taken into account that \( D \) has no kernel on \( u_{-2} \). Hence, \( [z, g_1] \) is a nonzero element of \( g_{-1} \). The rest of the proof mimics that of Lemma 4.1.1.

6.2. Consider \( \hat{g} = b_{-1/2}(n; n) \). We have:

\[ \hat{g}_{-1} = \Pi(T(\overline{-1/2})); \quad \hat{g}_0 = \text{svect}(0|n) \]

and the \( \hat{g}_0 \)-action on \( \hat{g}_{-1} \) preserves the nondegenerate superskew-symmetric form

\[ B(\varphi \sqrt{vol}, \psi \sqrt{vol}) = \int \varphi \psi \cdot vol; \quad p(B) \equiv n \pmod{2}. \]

(6.1)
Now, let $\mathfrak{g} = \hat{\mathfrak{g}}(b_{1/2}(3; 3))$ be the nontrivial central extension (we indicate this fact by tilde over $\mathfrak{g}$) of $\hat{\mathfrak{g}}$ corresponding to (6.1). The depth of $\hat{\mathfrak{g}}(b_{1/2}(3; 3))$ is equal to 2. This central extension is naturally embedded into

$$\mathfrak{u} = \begin{cases} \mathfrak{m}(2^{n-1}) & \text{for } n \text{ odd} \\ \mathfrak{f}(1 + 2^{n-1}2^{n-1}) & \text{for } n \text{ even.} \end{cases}$$

As the operator $D$ described in sec. 6.1 we take the grading operator $d \in \mathfrak{u}_0$, i.e., $\mathfrak{g}_0 \oplus CD \cong \mathfrak{g}_0$.

Example. Let $n = 2$. Then $\mathfrak{g}_{-1} = \Pi(T(-1/2^2))$ and $\mathfrak{g}_- = \mathfrak{hei}(2|2)$. We also have $\mathfrak{g}_0 = \mathfrak{cvect}(0|2) \cong \mathfrak{osp}(2|2)$ and $(\mathfrak{hei}(2|2), \mathfrak{cvect}(0|2))^\times \cong \mathfrak{f}(3|2)$, see sec. 0.6.

Theorem 1. (a) $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$ is a simple Lie superalgebra.

(b) $(\mathfrak{ab}(2n-1), \mathfrak{cvect}(0|2))^{\times n} = 0((\mathfrak{ab}(2n-1), \mathfrak{cvect}(0|2))^{\times n}) \cong \mathfrak{b}(n; n)$ for $n > 3$.

Proof. Thanks to Lemma 6.1 heading 1) follows from the fact that the $m$-prolongation of $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))$ is bigger than $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$. We give explicit formulas in Appendix 3. Heading 2) is proved in Appendix 2.

6.3. Let us clarify the structure of the exceptional Lie superalgebra $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$ with the help of a construction similar to that from §4. To this end, we describe another remarkable property of $\mathfrak{sle}^e(3)$ that singles it out among the $\mathfrak{sle}^e(n)$.

The Lie superalgebra $\mathfrak{g} = \mathfrak{sle}^e(3)$ has a 2z-dimensional nontrivial central extension $\mathfrak{sle}^e(3)$: the element $M_1$ of degree $-2$ with respect to the grading of $\mathfrak{sle}^e(3)$ extends $\mathfrak{sle}^e(3)$ to $\mathfrak{sle}^e(3)$ while $z$ of degree $-2$ with respect to the grading of $\mathfrak{sle}^e(3)$ is associated with the form $B$ on the space $\mathfrak{g}_{-1}$ of halfdensities with shifted parity (see (6.1)) in the realization $\mathfrak{g} = \mathfrak{sle}^e(3; 3)$.

The regrading $R$ interchanges these central elements and establishes a nontrivial automorphism of $\mathfrak{sle}^e(3)$. Let now $\mathfrak{g} = \hat{\mathfrak{g}}(b_{-1/2}(3; 3))$ be the described in sec. 6.2 nontrivial central extension of depth 2 of $\mathfrak{g}_{-1/2}(3; 3)$; clearly, $\mathfrak{g}_{-2} = Cz$. The inverse regrading $R^{-1}$ sends $\mathfrak{g}$ into the nontrivial central extension $\mathfrak{h} = \hat{\mathfrak{h}}(b_{-3,1}(3))$ of $\mathfrak{b}_{-3,1}(3)$, see 0.7, and deg $R^{-1}z = -1$.

From the very beginning we have an embedding $i_1 : \mathfrak{g} \rightarrow \mathfrak{u} = \mathfrak{m}(4)$. Let $\mathfrak{h} = \mathfrak{sle}^e(3) \subset \mathfrak{h}$.

Then the map $i_2 = SIGN \circ i_1 R$ determines an embedding of $\mathfrak{h}$ to $\mathfrak{u}$ preserving the grading of $\mathfrak{h}$.

Observe that $\mathfrak{h}_0 = \mathfrak{sp}(3)$ and $\mathfrak{h}_0 = \mathfrak{pr}(3) \cong \mathfrak{h}_0 \oplus \mathfrak{c} \cdot M_{\sum q_i \xi_i = -3}$. The action of $z = M_{\sum q_i \xi_i = -3}$ on the space $\mathfrak{h}_{-1}$ is of the form:

$$M_q \mapsto 4M_q, \quad M_\xi \mapsto 2M_\xi.$$ 

This coincides with the action of $-3d + \frac{1}{2}i_1(z)$.

Therefore, the embedding $i_2$ of $\mathfrak{h}_{-1}$ can be extended to an embedding of the whole $\mathfrak{h}$. We have

$$i_1(\mathfrak{g}_{-1} \oplus \mathfrak{g}_0) + i_2(\mathfrak{g}_{-1} \oplus \mathfrak{g}_0) = \mathfrak{u}_- \oplus (\mathfrak{g}_0 \oplus CD),$$

i.e., the images of $i_1$ and $i_2$ cover the whole nonpositive part of $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$.

Thus, we have got two different embeddings of $\hat{\mathfrak{h}}(b_{-3,1}(3))$, isomorphic to $\hat{\mathfrak{h}}(b_{-1/2}(3; 3))$ as abstract, but not as graded, Lie superalgebras, into $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$:

$$i_1 : \hat{\mathfrak{h}}(b_{-1/2}(3; 3)) \rightarrow (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$$

and

$$i_2 : \hat{\mathfrak{h}}(b_{-3,1}(3)) \rightarrow (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$$

with the grading of $\hat{\mathfrak{h}}(b_{-1/2}(3; 3))$ preserved and

$$i_2 : \hat{\mathfrak{h}}(b_{-3,1}(3)) \rightarrow (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$$

such that $i_2(\hat{\mathfrak{h}}(b_{-1/2}(3; 3))) + i_2(\hat{\mathfrak{h}}(b_{-3,1}(3))) = (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$ (the sum here is not a direct one!).

As a linear space, $(\mathfrak{cvect}(0|3))^{\times n}$ is the quotient of $\hat{\mathfrak{h}}(b_{-1/2}(3; 3)) \oplus \hat{\mathfrak{h}}(b_{-3,1}(3))$ modulo the subspace $V = \{(SIGN \circ Rg \oplus g) : g \in \mathfrak{sle}^e(3)\}$. The map $\varphi$ defined by the formula

$$\varphi|_{i_1(\hat{\mathfrak{h}}(b_{-1/2}(3; 3)))} = SIGN \circ i_2 i_1^{-1}, \quad \varphi|_{i_2(\hat{\mathfrak{h}}(b_{-3,1}(3)))} = i_1 i_2^{-1}$$

determines a nontrivial automorphism of $(\mathfrak{ab}(4), \mathfrak{cvect}(0|3))^{\times n}$.

6.4. Description of $(\mathfrak{hei}(8|6), \mathfrak{slect}_{a,b}(n))^{\times n}$. Consider the nontrivial central extension $\mathfrak{g} = \hat{\mathfrak{g}}(\mathfrak{sle}e(n; n))$ of $\mathfrak{sle}e(n; n)$ defined as follows. We have: $\mathfrak{g}_0 = \mathfrak{slect}(0|n)$; $\mathfrak{g}_{-1} = \Pi(T^0(\hat{\mathfrak{h}})/\mathfrak{c} \cdot 1)$, where $T^0(\hat{\mathfrak{h}}) = \{f \in T(\hat{\mathfrak{h}}) : \int f \cdot vol(\xi) = 0\}$. Define the central extension with the help of the form $\omega$ on $\mathfrak{g}_{-1}$ given by the formula:

$$\omega(f, g) = \int fg \cdot vol(\xi).$$

The same arguments as in 6.2, show that $(\mathfrak{g}_{-1}, \mathfrak{g}_0)^{\times n}$ can be embedded into $\mathfrak{f}(1 + 2^{n-1}2^{n-1} - 2)$ for $n$ even and into $\mathfrak{m}(2^{n-1} - 1)$ for $n$ odd.
Let $x$ be the operator determining the standard $\mathbb{Z}$-grading of $\mathfrak{svect}(0|n)$ and let $z$ commute with $\mathfrak{svect}(0|n)$; let $a, b \in \mathbb{C}$. For any $a, b$ the element $ax + bz$ determines an outer derivation of $\mathfrak{g}_0$. Set $\mathfrak{svect}_{a,b}(n) = \mathfrak{svect}(0|n) \supset \mathbb{C}(ax + bz)$; set also

$$(g_-, \mathfrak{svect}_{a,b}(n))^n_{mk} = (g_-, \mathfrak{g}_0 \supset \mathbb{C}(ax + bz))^n_{mk}, \text{ where } g_- = \begin{cases} \frac{ab(2n^{-1} - 1)}{\text{hei}(2n^{-1}|2n^{-1} - 2)} & \text{for odd, } n \text{ odd,} \\ \frac{ab(2n^{-1} - 1)}{\text{hei}(2n^{-1}|2n^{-1} - 2)} & \text{for even, } n \text{ even.} \end{cases}$$

**Example.** Let $n = 3$. Then $g_{-1} = \Pi(\xi_1, \xi_2, \xi_3)$ and $g_- \cong (b_\lambda(3))_- = ab(3)$ for any $\lambda$; $g_0 = \mathfrak{svect}(0|3) \cong \mathfrak{sp}(3)$. The operator $x$ becomes $\sum \xi_i \partial \xi_i$ and

$$g_0 \supset \mathbb{C}(ax + bz) \cong \mathfrak{sp}(3) \supset \mathbb{C}(a \sum \xi_i \partial \xi_i + bz) \cong (b_\lambda(3))_0$$

for $\lambda = -\frac{b}{3a}$.

Therefore, $(ab(3), \mathfrak{svect}_{a,b}(3))^n_{mk} \cong b_\lambda(3)$ in particular, $(ab(3), \mathfrak{svect}_{1,3}(3))^n_{mk} \cong \mathfrak{sm}(3)$ and $(ab(3), \mathfrak{svect}_{1,0}(3))^n_{mk} \cong b(3)$.

**6.5. Theorem.** 1) $(\text{hei}(8|6), \mathfrak{svect}_{3,4}(4))^n_{mk}$ is a simple Lie superalgebra.

2) Let $g_- = \begin{cases} \frac{ab(2n^{-1} - 1)}{\text{hei}(2n^{-1}|2n^{-1} - 2)} & \text{for odd, } n \text{ odd,} \\ \frac{ab(2n^{-1} - 1)}{\text{hei}(2n^{-1}|2n^{-1} - 2)} & \text{for even, } n \text{ even.} \end{cases}$ Then

$$(g_-, \mathfrak{svect}_{a,b}(n))^n_{mk} \cong (g_-, \mathfrak{svect}(0|n))^n_{mk} \supset \mathbb{C}(ax + bz) \text{ if } n > 4 \text{ or if } (a,b) \notin \mathbb{C}(3,4) \text{ and } n = 4.$$

**Proof.** As in Theorem 6.1 heading 1) follows from a direct calculations based on Lemma 6.1; for the explicit formulas see Appendix 3. Heading 2) is proved in Appendix 2.

Let us clarify the structure of $(\text{hei}(8|6), \mathfrak{svect}_{3,4}(4))^n_{mk}$. This Lie superalgebra is contained in $u = \mathfrak{t}(9|6)$. In sec. 6.3 we have already described the Lie superalgebra $\mathfrak{g} = \mathfrak{csle}^c(4|4)$ and its embedding $i_1 : \mathfrak{g} \to u$.

Observe that $\mathfrak{g} \supset \mathfrak{as}$ and this embedding preserves the $\mathbb{Z}$-grading described in sec. 2.1:

$$\text{as}_{-2} = g_{-2}; \quad \text{as}_{-1} = g_{-1} = \Pi(\Lambda^2(\xi_1, \xi_2, \xi_3)), \quad \text{as}_0 = \mathfrak{si}(4) \supset \mathfrak{g}_0 = \mathfrak{svect}(0|4); \quad \text{as}_1 = \mathfrak{si}(S^2(q_1, q_2, q_3, q_4)) \supset \mathfrak{g}_1.$$

For the role of $\mathfrak{h}$ (see 4.3 and 6.3) take $\mathfrak{fas}$. It follows from Theorem 3.2 that $\mathfrak{as} \subset \mathfrak{fas}$; set $\mathfrak{h} = \mathfrak{as}$. Let $R : \mathfrak{h} \to \mathfrak{g}$ be the embedding that executes the isomorphism of two copies of $\mathfrak{as}$. (Notice that $R$ preserves the $\mathbb{Z}$-grading (1.1) of $\mathfrak{as}$.) The map $i_2 = i_1 R$ determines an embedding of $\mathfrak{h}$ into $\mathfrak{t}(9|6)$

But $\mathfrak{h}_0 = \mathfrak{h}_0 \oplus \mathbb{C}t$. It turns out that $i_2$ can be extended to an embedding $i_2 : \mathfrak{fas} \to u$.

As in the above examples, we have: $i_1(\mathfrak{g}_0 \supset \mathfrak{g}[0] + i_2(\mathfrak{h}_- \oplus \mathfrak{h}_0) \cong \mathfrak{u}_- \supset \mathbb{C}(3x + 4z)$ (the sum in the lhs here is not a direct one!). As a linear space, $((\text{hei}(8|6), \mathfrak{svect}_{3,4}(0|4))^n_{mk}$ is a nondirect sum of $\mathfrak{g} = \mathfrak{csle}^c(4|4)$ with $\mathfrak{h} = \mathfrak{fas}$ and is the quotient of $\mathfrak{g} \oplus \mathfrak{h}$ modulo the subspace $\mathbb{V} = \{(Rg - g) : g \in \mathfrak{as}\}.

**APPENDIX 1. SOLUTION OF THE SYSTEM (2.1)-(2.4)**

Let $D = \sum_{1 \leq i \leq 4}(P_i \frac{\partial}{\partial \xi_i} + Q_i \frac{\partial}{\partial u_i}) \in \mathfrak{g}^\lambda$ be an homogeneous (wrt parity) vector field. Then by Lemma 2.1 it satisfies the system (2.1)-(2.4).

Equations (2.3) imply that there exists a function $f = f(u, \xi)$ such that $Q_i = (-1)^p(D)\frac{\partial f}{\partial \xi_i}$Equations (2.1) imply further that

$$P_i = \frac{\partial f}{\partial u_i} + \varphi_i(u; \xi) = \frac{\partial f}{\partial u_i} + \varphi_i^c(u) + \varphi_i^t(u) \cdot \xi_i$$

or, in other words, $D = Le_f + \sum(\varphi_t^c(u) + \varphi_t^t(u) \xi_i) \frac{\partial}{\partial \xi_i}$. Equations (2.2) now imply that

$$\varphi_1(u) = \frac{\partial \varphi_t}{\partial u_1} = \frac{1}{2}(-1)^p(D) \sum \frac{\partial Q_i}{\partial u_j} = -\frac{1}{2}\Delta(f), \quad \text{where } \Delta = \sum \frac{\partial^2}{\partial u_i \partial \xi_i},$$

wheras equations (2.4) imply that

$$\frac{\partial \varphi_t}{\partial u_i} - \frac{\partial \varphi_t}{\partial u_i} = -2\lambda \frac{\partial^2 f}{\partial \xi_i \partial \xi_k}.$$ (2.4')

**Remark.** Let $\psi_i = \psi_i(u)$, where $i = 1, \ldots, 4$, be a set of functions such that $\frac{\partial \psi_i}{\partial u_i} - \frac{\partial \varphi_t}{\partial u_i} = 0$. Then there exists a function $\psi(u)$ for which $\psi_i = \frac{\partial \psi}{\partial u_i}$ and $\sum \psi_i(u) \frac{\partial}{\partial u_i} = Le_\psi = D_\psi$ (see Corollary 2 from Theorem 2.3). Thus, for any function $f$ it suffices to find any collection of functions $\varphi_i^o$. 
With the help of the differential forms

$$\alpha = \sum_{i \leq 4} \varphi_i du_i$$
and
$$\omega_0(f) = \sum_{(i,j,k,l) \in A_4} \frac{\partial^2 f}{\partial \xi_i \partial \xi_k} du_j \land du_i$$
equations (2.2') and (2.4') can be expressed in the form

$$d\alpha = \frac{1}{2} \Delta(f) \cdot \omega - 2\lambda \cdot \omega_0(f).$$  \hspace{1cm} (A.1)

Equation (A.1) is solvable if and only if the form in the rhs is exact or, since our considerations are local, equivalently, if and only if it is closed.

Direct calculations show that the condition that the form in the rhs is closed is equivalent to the system

$$\begin{cases}
\frac{\partial \Delta f}{\partial u_i} &= 0 \text{ for all } i = 1, 2, 3, 4 \\
\frac{1}{2} \frac{\partial \Delta f}{\partial \xi_i} - 2\lambda \frac{\partial^3 f}{\partial \xi_j \partial \xi_k \partial \xi_l} &= 0 \text{ for all } i = 1, 2, 3, 4 \text{ and } (i, j, k, l) \in A_4
\end{cases} \hspace{1cm} (A.1.1)$$

Eqs. (A.1.1) imply that $\Delta(f)$ only depends on $u$; eqs. (A.1.2) imply that $\text{deg}_\xi f \leq 3$.

First, suppose that $\text{deg}_\xi f \leq 2$. Then eqs. (A.1.2) imply that

$$\Delta(f) = \text{const.} \hspace{1cm} (A.2)$$

Denote $-\frac{1}{4} \Delta(f)$ by $c$. Thus, $f = -c \sum_{i \leq 4} u_i \xi_i + f_0$, where $\Delta(f_0) = 0$. By (2.2') then, $\varphi_i^1(u) = 2c$ and $D = Le_c \sum u_i \xi_i + f_0 + \sum \varphi_i^\circ \partial \xi_i + 2c \sum \xi_i \partial \xi_i = Le_{f_0} + \sum \varphi_i^\circ \partial \xi_i + cZ$. Replace $f$ with $f_0$. Then $\Delta(f) = 0$ and we have

$$\frac{\partial \varphi_i^2}{\partial u_j} - \frac{\partial \varphi_j^2}{\partial u_i} = -2\lambda \frac{\partial^2 f}{\partial \xi_j \partial \xi_k}. \hspace{1cm} (2.4'')$$

If $\text{deg}_\xi f < 2$, then the rhs of (2.4'') is equal to 0. So due to Remark we can take $\varphi_i^\circ = 0$. In this case

$$D = Le_f + cZ = D_f + cZ. \hspace{1cm} (A.3)$$

Let $\text{deg}_\xi f = 2$. As $\Delta(f) = 0$, we can set $g = \Delta^{-1}(f)$. Then $\text{deg}_\xi g = 3$ and $f = \Delta(g)$. For the role of functions $\varphi_i^\circ$ satisfying equation (2.4'') we can take

$$\varphi_i^\circ = 2\lambda \frac{\partial^2 g}{\partial \xi_j \partial \xi_k \partial \xi_l}, \text{ where } (i, j, k, l) \in A_4.$$ We get:

$$D_f = Le_f + 2\lambda \sum_{(i,j,k,l) \in A_4; 1 \leq i \leq 4} \frac{\partial^2 g}{\partial \xi_j \partial \xi_k \partial \xi_l} \cdot \partial \xi_i + cZ = D_f + cZ. \hspace{1cm} (A.4)$$

Let $\text{deg}_\xi f = 3$. Let us represent $f$ in the form $f = f_3 + f_{<3}$, where $f_3$ is an homogeneous (wrt the degree in $\xi$) polynomial of degree 3, while $f_{<3}$ is a polynomial of lesser degree.

Since $\Delta(f)$ only depends on $u$, we see that $\Delta(f_3) = 0$ and, therefore, we can introduce $H = \Delta^{-1}(f_3) = F(u)\xi_1\xi_2\xi_3\xi_4$ for some function $F(u)$.

From (2.4') we deduce that

$$\frac{\partial \varphi_i^1}{\partial u_j} = 2\lambda \frac{\partial F}{\partial u_j}, \text{ or, with (2.2'), } \varphi_i^1 = 2\lambda F = -\frac{1}{2} \Delta(f).$$

Therefore, $\Delta(f) = \Delta(f_{<3}) = -4\lambda F$. Set $\hat{f} = 4\Delta^{-1}(F)$. We obtain that $f = \Delta(H) - \lambda \hat{f} + g$ for some function $g$ such that $\Delta(g) = 0$ and $\text{deg}_\xi g < 3$. But we have already described the solutions for all such $g$. So now we assume $g = 0$. In this case we can take $\varphi_i^\circ = 0$ (due to Remark). We get:

$$D = Le_f + \lambda(-Le_f + 2F \sum \xi_i \partial \xi_i) =$$

$$Le_f + \lambda \left(-Le_f + 2 \sum_{(i,j,k,l) \in A_4; 1 \leq i \leq 4} \frac{\partial^2 H}{\partial \xi_j \partial \xi_k \partial \xi_l} \cdot \partial \xi_i \right) = D_f. \hspace{1cm} (A.5)$$

Formulas (A.3)-(A.5) prove Theorem 2.3.
Appendix 2. Proof of headings 2 of Theorems 4.2, 6.2 and 6.5

A.2.1. Lemma. Let \((g_{-1}, g_0)_*\) be simple; let the \(g_0\)-module \(g_{-1}\) be irreducible. If \(h = (g_{-1}, c g_0)_*\) is also simple, then for every \(v \in g_{-1}\) there exists an \(F \in g_1\) such that \([v, F] \notin g_0\).

The same applies to \((g_{-1}, g_0)_*\) and \(h = (g_{-1}, c g_0)_*\).

Proof. By simplicity of \((c g_0)_*\), (due to Lemma 4.1.1) we have \([h_{-1}, h_1] = c g_0\), i.e., there exists \(v_0 \in g_{-1}, F_0 \in g_1\) such that \([v_0, F_0] \notin g_0\).

Let
\[
V_1 = \{v_1 \in g_{-1} : [g, v_1] = v_0 \text{ for some } g \in g_0\}.
\]
Then for any \(v_1 \in V_1\) we have
\[
g_0 \not\ni [g, v_1], F_0 = \pm [g, F_0], v_1 \pm [g, v_1], F_0],
\]
where the signs are governed by Sign Rule. Therefore, one of the two cases holds:
1) \([v_1, F_0] \notin g_0\), hence, \(F = F_0\);
2) \([v_1, F_0] \in g_0\).

In case 2) we have \([g, v_1, F_0] \in g_0\); hence,
\[
[F, v_1] \notin g_0 \text{ for } F = [g, F_0].
\]

Similarly, introduce the sequence of spaces
\[
V_2 = \{v_2 \in g_{-1} : [g, v_2] \in V_1 + <v_0> \text{ for some } g \in g_0\},
\]
By irreducibility of \(g_0\)-action on \(g_{-1}\) for every \(v \in g_{-1}\) there exists an \(n\) such that \(v \in V_n\), and, therefore, \(F = F_n\), where \([v, F_n] \notin g_0\).

The arguments for depth 2 are literally the same. \(\square\)

A.2.2. Corollary. Let \(g_{-1} = \Pi(\Lambda(n)/C1), g_0 = \text{vect}(0|n), i.e., g_* = (g_{-1}, g_0)_* = k(n; n).\) Then \((g_{-1}, c g_0)_*\) is not simple for \(n > 3\).

Hereafter in this Appendix we often abuse the notations and denote the elements by their generating functions.

Proof. By Lemma A.2.1 the simplicity of \(g_*\) implies that for any \(v \in g_{-1}\) there exists \(F \in g_{-1}\) such that \([v, F] \notin g_0\).

Take \(v = \xi_1 \ldots \xi_n\); let \(d\) be the central element of \(c(g_0)\) normalized so that \(ad_{|g_{-1}} = - id\). Let \(F \in g_1\) be such that
\[
[v, F] = d + g, \text{ for } g \in g_0.
\]

Then
\[
\pm [v, [F, v_1]] \text{ by Jacobi id. } = ([v, F], v_1) = (d + g)v_1 = -v_1 + g v_1.
\]

In other words, \(g_1 = [F, v_1] maps v to \(-v_1 + g v_1\) up to a sign. But in the \(g_0\)-module considered the element \(v\) can only be mapped into a function of degree \(n - 1\).

Hence, \(g v_1 = v_1 + \varphi(v_1)\), where \(\deg \varphi \geq n - 1\), for any \(v_1\) of degree \(< n - 1\). Consequently, the projection \(g_0\) of \(g\) on the zeroth component of \(\text{vect}(0|n)\) with respect to the standard \(Z\)-grading, i.e., on \(gl(n)\), satisfies the condition
\[
g_0|_{\text{Span}(v_1 : \deg v_1 < n - 1)} = id.
\]

But in \(\text{vect}(0|n)\) the dimension of the maximal torus \(\text{Span}(\xi_i \partial_i : 1 \leq i \leq n)\) is equal to \(n\) and there is no operator whose restrictions to the spaces of homogenous functions in \(\xi\) of at least two distinct degrees are scalar operators.

Since \(n - 2 \geq 2\) for \(n > 3\), the Lie superalgebra \((g_{-1}, c g_0)_*\) is not simple. \(\square\)

A.2.3. Corollary. Let \(g_* = \begin{cases} \mathbb{H}(2n-1) & \text{for } \text{odd } n \\ \mathbb{A}(2n-1) & \text{for } \text{even } n \end{cases}\) The Lie superalgebra \((g_-, c \text{vect}(0|n))_*\) is not simple for \(n > 3\).

Proof follows the lines of the proof of Corollary A.2.1 with the correction that (A.2.0) is now true not for all \(v_1 \in g_{-1}\) but only for those which satisfy \([v_1, v] = 0\). Such elements \(v_1\) are represented by functions \(f \in \Lambda(n)\) such that \(0 < \deg f \leq n - 1\). There are \(\geq 2\) distinct degrees which satisfy this inequality for \(n > 3\). \(\square\)
A.2.4. Corollary. Let \( g_- = \begin{cases} \frac{ab(2n-1) - 1}{k!2^{n-1}} & \text{for } n \text{ odd} \\ \frac{ab(2n-1) - 1}{k!2^{n-1} - 2} & \text{for } n \text{ even} \end{cases} \) Then \( g = (g_-, \text{svect}_{a,b}(0|n))_n^k \) is not a simple Lie superalgebra if either \( n > 4 \) or \( n = 4 \) and \( (a, b) \notin \mathbb{C}(3, 4) \).

Proof is obtained by a slight modification of the proof of Corollary A.2.2. As \( v \) we now take \( \xi_1 \ldots \xi_{n-1} \in g_- \); let \( F \) be such that \([v, F] = ax + bd + g\), where \( g \in \text{svect}(0|n)\). Then

\[
[[F, v_1], v] = \pm [[v, F], v_1] = (ak - b)v_1 + g v_1 \tag{A.2.1}
\]

for any monomial \( v_1 \in g_- \) of degree \( k \) and distinct from \( \xi_n \). Since every element from \( g_0 \) lowers the degree of any monomial not more than by 1, we see that the projection \( g_0 \) of \( g \) on \( \text{svect}(0|n)_0 \) satisfies the relation

\[
g_0 v_1 = (b - ak)v_1 \tag{A.2.2}
\]

for any monomial \( v_1 \in g_- \) of degree \( k < n - 2 \) and distinct from \( \xi_n \). In particular, for \( n > 4 \) this means that \( g_0 \) acts on \( \text{Span}(\xi_1, \ldots, \xi_{n-1}) \) by multiplication by \( b - a \) and on \( \Lambda^2(\xi) \) by multiplication by \( b - 2a \). Hence, \( g_0 = 0 \), i.e., \( a = 0 \).

In (A.2.2) \( k < n - 2 \). So if \( n = 4 \), then \( k = 1 \). The component \( g_0 \) is defined by its action on \( \xi_1, \ldots, \xi_n \). But (A.2.2) gives the action of \( g_0 \) only on \( \xi_1, \ldots, \xi_{n-1} \). Its action on \( \xi_n \) can be arbitrary with only one condition: \( g_0 \in \text{svect}(0|4) \); this is what (A.2.3) means:

\[
g_0(\xi_1\xi_4) = -2(b - a)\xi_1\xi_4 + c_1\xi_1\xi_2 + c_2\xi_1\xi_3. \tag{A.2.3}
\]

Look at formula (A.2.1) with \( v = \xi_1\xi_2\xi_3 \) and \( v_1 = \xi_1\xi_4 \). It means that \( ad[F, v_1] \) (which is an element from \( \text{svect}(0|4) \oplus \mathbb{C}(ax + bd) \)) sends \( \xi_1\xi_2\xi_3 \) to \((2a - b)\xi_1\xi_4 + g(\xi_1\xi_4)\). Since no vector field can send \( v \) to \( v_1 \), we deduce that \( g_0(v_1) \) must compensate \((2a - b)\xi_1\xi_4\). But from formula (A.2.3) we derive that \( b - 2a = -2b + 2a \), implying \( 3b = 4a \).

Due to Lemmas 4.1, 6.1, Corollaries A.2.2–A.2.4 are equivalent to the headings 2) of Theorems 4.2, 6.2 and 6.5, respectively.

APPENDIX 3. PROOF OF SIMPLICITY OF THE LIE SUPERALGEBRAS \( g = (ab(4), \text{svect}(0|3))_n^m \) and \( g = (\text{hei}(8|6), \text{svect}_{3,4}(0|3))_n^k \)

Due to Lemma 6.1, to prove the simplicity of \( g \) it suffices to exhibit an element \( \hat{F} \in g_1 \) such that \([g_-1, \hat{F}] \) is not entirely contained in \( \text{svect}(0|3) \) and \( \text{svect}(0|3) \), respectively.

A.3.1. Simplicity of \( (ab(4), \text{svect}(0|3))_n^m \). First, let us show how to embed \( g = \text{svect}(0|3))_n^m \) into \( m(4) \). We consider \( m(4) \) as preserving the Pfaff equation given by the form \( \alpha_0 = d\tau + \sum_{i=0}^3 (\eta_i du_i + u_i d\eta_i) \). Denote the basis elements of \( g \) as follows

| \( g_- \) | A basis of \( g_- = \Pi(\text{Vol}^2) \) | notations of the corr. functions that generate \( g_- \subset m(4) \) |
|---|---|---|
| \( M_1 \) | \( \xi_1 \xi_2 \xi_3, \xi_2 \xi_3, \xi_3 \xi_1, \xi_1 \xi_2 \) | \( \eta_0 \) |
| | \( \xi_1, \xi_2, \xi_3 \) | \( u_1, u_2, u_3 \) |
| | \( 1 \) | \( \eta_1, \eta_2, \eta_3 \) |
| | | \( u_0 \) |
The following is an explicit realization of the embedding $i: g_0 = \text{vect}(0|3) \rightarrow \mathfrak{e}(3)$. We only indicate the generating functions of the image:

| $\deg D$ | $D \in \text{vect}(0|3)$ | $i(D)$ |
|-----------|-----------------|--------|
| $-1$      | $\partial_1, \partial_2, \partial_3$ | $-u_0 u_1 + \eta_2 \eta_3, -u_0 u_2 + \eta_3 \eta_1, -u_0 u_1 + \eta_1 \eta_2$ |
| $0$       | $\text{div} D = 0: \xi_i \partial_j$ | $-u_i \eta_j$ for $i \neq j; i, j > 0$ |
| $0$       | $\xi_1 \partial_1$ | $\frac{1}{2} (u_0 \eta_0 - u_1 \eta_1 + u_2 \eta_2 + u_3 \eta_3)$ |
| $0$       | $\xi_2 \partial_2$ | $\frac{1}{2} (u_0 \eta_0 + u_1 \eta_1 - u_2 \eta_2 + u_3 \eta_3)$ |
| $0$       | $\xi_3 \partial_3$ | $\frac{1}{2} (u_0 \eta_0 + u_1 \eta_1 + u_2 \eta_2 - u_3 \eta_3)$ |
| $1$       | $\xi_2 \xi_3 \partial_1, \xi_3 \xi_1 \partial_2, \xi_1 \xi_2 \partial_3$ | $-\frac{1}{2} u_1^2, -\frac{1}{2} u_2^2, -\frac{1}{2} u_3^2$ |
| $1$       | $\xi_1 (\xi_2 \partial_2 - \xi_3 \partial_3), \xi_2 (\xi_3 \partial_3 - \xi_1 \partial_1), \xi_3 (\xi_1 \partial_1 - \xi_2 \partial_2)$ | $-u_2 u_3, -u_1 u_3, -u_1 u_2$ |
| $1$       | $\xi_1 (\xi_2 \partial_2 + \xi_3 \partial_3), \xi_2 (\xi_3 \partial_3 + \xi_1 \partial_1), \xi_3 (\xi_1 \partial_1 + \xi_2 \partial_2)$ | $\eta_0 \eta_1, \eta_0 \eta_2, \eta_0 \eta_3$ |
| $2$       | $\xi_2 \xi_3 \xi_4 \partial_1$ | $\eta_1^2, \eta_2^2, \eta_3^2$ |

To check condition (A.3.1), take

$$F = M_F,$$

where $F = 2 \tau u_1 - 2 \eta_0 \eta_2 \eta_3 + u_0^2 \eta_1$.

Then the brackets with $g_{-1}$ are

$$\{F, u_0\}_{\text{m.b.}} = -2u_0 u_1 + 2 \eta_2 \eta_3, \quad \{F, \eta_i\}_{\text{m.b.}} = 2 \tau;$$
$$\{F, u_1\}_{\text{m.b.}} = -3u_1 \eta_1, \quad \{F, \eta_i\}_{\text{m.b.}} = -2u_1 \eta_i (i = 0, 2, 3);$$
$$\{F, u_2\}_{\text{m.b.}} = -2u_1 u_2 - 2 \eta_0 \eta_3; \quad \{F, u_3\}_{\text{m.b.}} = -2u_1 u_3 + 2 \eta_0 \eta_2.$$  (A.3.2)

We get $M_F$, while the remaining elements in the rhs of (A.3.2) lie in vect$(0|3)$.

### A.3.2. Simplicity of $(\text{hei}(8|6), \text{vect}_{3,4}(0|4))^\xi$. First, let us show how to embed vect$_{3,4}(0|4)^\xi$ into $\mathfrak{f}(9|6)$. We realize $\mathfrak{f}(9|6)$ as preserving the Pfaff equation given by the form $\alpha_1 = dt - \sum_{i \leq 4} (p_i dq_i - dq_i p_i) - \sum (\eta_j d\xi_j + \xi_j d\eta_j)$. Let us reenumerate the basis elements of $g_{-1}$:

| A basis of $g_{-1}$ | notations of the corr. functions that generate $g_{-1} \subset \mathfrak{f}(9|6)$ |
|----------------------|----------------------------------------------------------------------------|
| $\xi_1, \xi_2, \xi_3, \xi_4$ | $p_1, p_2, p_3, p_4$ |
| $\xi_1 \xi_2, \xi_1 \xi_3, \xi_1 \xi_4$ | $\eta_1, \eta_2, \eta_3$ |
| $-\xi_3 \xi_4, \xi_2 \xi_4, -\xi_2 \xi_3$ | $\xi_1, \xi_2, \xi_3$ |
| $\xi_2 \xi_3 \xi_4, -\xi_1 \xi_3 \xi_4, \xi_1 \xi_2 \xi_4, -\xi_1 \xi_2 \xi_3$ | $q_1, q_2, q_3, q_4$ |

The following is an explicit realization of the embedding $i: g_0 = \text{vect}(0|3) \rightarrow \mathfrak{h}(8|6)_0$. We only indicate the generating functions of the image. For $D \in \text{vect}(0|3)$ we have

$\deg D = -1$:

$$\partial_{\xi_1} \mapsto \xi_1 p_2 + \xi_2 p_3 + \xi_3 p_4$$
$$\partial_{\xi_2} \mapsto -\xi_1 p_1 + \eta_2 p_4 - \eta_3 p_3$$
$$\partial_{\xi_3} \mapsto -\xi_2 p_1 + \eta_3 p_2 - \eta_1 p_4$$
$$\partial_{\xi_4} \mapsto -\xi_3 p_1 - \eta_2 p_2 + \eta_1 p_3$$

$\deg D = 0$:

$$\xi_1 \partial_2 \mapsto -p_1 q_2 - \eta_2 \eta_3$$
$$\xi_2 \partial_1 \mapsto -p_2 q_1 + \eta_2 \xi_3$$
$$\xi_3 \partial_1 \mapsto -p_1 q_3 + \eta_1 \eta_3$$
$$\xi_3 \partial_2 \mapsto -p_3 q_1 - \eta_1 \xi_3$$
$$\xi_1 \partial_4 \mapsto -p_1 q_4 - \eta_1 \eta_2$$
$$\xi_2 \partial_4 \mapsto -p_2 q_4 + \eta_3 \eta_1$$

$$\xi_1 \partial_1 - \xi_2 \partial_2 \mapsto -p_1 q_1 + p_2 q_2 + \eta_2 \xi_2 + \eta_3 \xi_3$$
$$\xi_2 \partial_3 - \xi_3 \partial_4 \mapsto -p_3 q_3 + p_4 q_4 + \eta_3 \xi_3 - \eta_2 \xi_2$$

$$\sum_{\xi_i \partial_i} \mapsto -\sum p_i q_i - 2t$$  (A.3.3)

$\deg D = 1$:
The image under $i$ is generated by $q_i q_j$ for any $1 \leq i, j \leq 4$.

Now, set

$$x_0 = K - \sum p_i q_i - 2t, \quad x_1 = K - \sum p_i q_i + 3p_4 q_4 + \eta_1 \xi_1 + \eta_2 \xi_2 - \eta_3 \xi_3,$$

see (A.3.3). Set:

$$f = t + \sum_{i \leq 3} p_i q_i + 3p_4 q_4 + \eta_1 \xi_1 + \eta_2 \xi_2 - \eta_3 \xi_3.$$

Then

$$K_f = \frac{1}{2} x_1 + \frac{3}{2} x_2 - \frac{3}{2} x_3 - 2K_1 \in \mathfrak{so}(0,4) \ni \mathbb{C}(3x_0 + 4K_1).$$

To check the condition (A.3), take

$$\hat{F} = K_F,$$

where $F = tp_4 + p_4(\sum_{i \leq 4} p_i q_i + \eta_1 \xi_1 + \eta_2 \xi_2 - \eta_3 \xi_3) - 2\xi_1 \eta_2 p_1 + 2\xi_1 \eta_3 p_2 + 2\xi_2 \eta_3 p_3$.

The commutators of $F$ with $\mathfrak{t}$. are of the form:

$$\{q_i, F\}_{k.b.} = q_i \frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial q_i}, \quad \{\eta_i, F\}_{k.b.} = \eta_i \frac{\partial F}{\partial \eta_i} - \frac{\partial F}{\partial \eta_i};$$

$$\{p_i, F\}_{k.b.} = p_i \frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial q_i}, \quad \{\zeta_i, F\}_{k.b.} = \zeta_i \frac{\partial F}{\partial \eta_i} - \frac{\partial F}{\partial \eta_i}.$$

Hence,

$$\{q_4, F\}_{k.b.} = f; \quad \{\eta_1, F\}_{k.b.} = 2(\eta_1 p_4 + \zeta_2 p_1 - \eta_3 p_3) \mapsto -2\partial_1;$$

$$\{\eta_2, F\}_{k.b.} = 2(\eta_2 p_4 - \zeta_1 p_1 - \eta_3 p_3) \mapsto 2\partial_3;$$

$$\{\eta_3, F\}_{k.b.} = \{\zeta_1, F\}_{k.b.} = \{\zeta_2, F\}_{k.b.} = 0;$$

$$\{\zeta_3, F\}_{k.b.} = 2(\zeta_3 p_4 + \zeta_1 p_2 + \zeta_2 p_3) \mapsto 2\partial_1;$$

$$\{q_1, F\}_{k.b.} = 2(\eta_1 p_4 - \zeta_1 \zeta_2) \mapsto -2\xi_1 \partial_1;$$

$$\{q_2, F\}_{k.b.} = 2(q_4 p_4 + \zeta_1 \eta_3) \mapsto -2\xi_2 \partial_3;$$

$$\{q_3, F\}_{k.b.} = 2(q_3 p_4 + \zeta_2 \eta_3) \mapsto -2\xi_3 \partial_3;$$

$$\{p_i, F\}_{k.b.} = 0 \quad \text{for } i = 1, 2, 3, 4.$$

So we get $K_f$, while the remaining brackets lie in $\mathfrak{so}(0,3)$.

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