COMPLEX PRODUCT STRUCTURES ON 6-DIMENSIONAL NILPOTENT LIE ALGEBRAS

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Abstract. We study complex product structures on nilpotent Lie algebras, establishing some of their main properties, and then we restrict ourselves to 6 dimensions, obtaining the classification of 6-dimensional nilpotent Lie algebras admitting such structures. We prove that any complex structure which forms part of a complex product structure on a 6-dimensional nilpotent Lie algebra must be nilpotent in the sense of Cordero-Fernández-Gray-Ugarte. A study is made of the torsion-free connection associated to the complex product structure and we consider also the associated hypercomplex structures on the 12-dimensional nilpotent Lie algebras obtained by complexification.

1. Introduction

The notion of a complex product structure on a (real) Lie algebra was introduced in [4]; it is given by two anticommuting endomorphisms of the Lie algebra, one of them a complex structure and the other one a product structure. Such a structure determines in a unique way a torsion-free connection on the Lie algebra such that the complex structure and the product structure are both parallel, and this connection restricts to flat torsion-free connections on two complementary totally geodesic subalgebras.

In [4] many examples of 4-dimensional Lie algebras admitting complex product structures were exhibited, and the complete classification of such Lie algebras was first given in [1]. This classification, restricted to the solvable case, was also shown in [2]. The next simplest step in order to obtain a better understanding of these structures, would be the classification of the 6-dimensional nilpotent Lie algebras which admit them. The class of 6-dimensional nilpotent Lie algebras has been extensively studied, and there is only a finite number of such Lie algebras, up to isomorphism (see for instance [26]). Furthermore, the family of these Lie algebras admitting a complex structure has also been specified, and their list appears in [26].

Our goal in this work is to obtain the full classification of the 6-dimensional nilpotent Lie algebras admitting a complex product structure, and this classification will be done independently of the results given in [26]. Our approach will make use of properties of simply transitive actions of 3-dimensional nilpotent Lie groups on 3-dimensional Euclidean space or, equivalently, complete left symmetric algebra structures on 3-dimensional nilpotent Lie algebras. The main result that will be used in the classification is Theorem 4.1, which states that any 6-dimensional nilpotent Lie algebra admitting a complex product structure can be decomposed as a direct sum (as vector spaces) of a 4-dimensional subspace and a 2-dimensional central ideal, both of them invariant by the complex and the product structures. As a first corollary of this theorem we find two 6-dimensional nilpotent Lie algebras admitting complex structures (according to [26]) that do not admit any complex product.
structure (see Corollary 4.6). It will turn out later that the 6-dimensional nilpotent Lie algebras which admit complex structures but admit no complex product structures are exactly 3. Another corollary of Theorem 4.1 is the fact that if a complex structure on a 6-dimensional nilpotent Lie algebra forms part of a complex product structure, then the complex structure must be nilpotent in the sense of [13] (see Corollary 4.7).

The outline of the paper is the following: §2 consists only of necessary preliminaries, while in §3 we prove some general results concerning complex product structures on nilpotent Lie algebras. In §4 we restrict ourselves to the 6-dimensional case, and we prove our main result (Theorem 4.1) which describes the structure of such a Lie algebra, and this result will allow us to perform, in §5, the classification mentioned earlier. In §6 we study the associated torsion-free connections, showing that they are always complete and specifying whether these connections are flat or not and, as an application, we use this information together with results in [4] to obtain examples of 12-dimensional nilmanifolds equipped with hypercomplex structures. The corresponding Obata connections are always Ricci flat, but not necessarily flat. Finally, in §7 we consider some simple examples at the Lie group level.

All Lie algebras and Lie groups in this work will be real and finite dimensional, unless otherwise specified.

2. Preliminaries

Let us begin by recalling some basic facts. An almost product structure on any Lie algebra $g$ is a linear endomorphism $E : g \to g$ satisfying $E^2 = 1$ (and not equal to $\pm 1$). It is said to be integrable if

$$E[x, y] = [Ex, y] + [x, Ey] - E[Ex, Ey] \quad \text{for all } x, y \in g.$$  

An integrable almost product structure will be called a product structure. Given an almost product structure $E$ on $g$, we have a decomposition $g = g_+ \oplus g_-$ of $g$ into the direct sum of two linear subspaces, the eigenspaces associated to the eigenvalues $\pm 1$ of $E$. It is easy to verify that $E$ is integrable if and only if $g_+$ and $g_-$ are both Lie subalgebras of $g$; we will write $g = g_+ \otimes g_-$ and will call $(g, g_+, g_-)$ the associated double Lie algebra. If $\dim g_+ = \dim g_-$ then the product structure $E$ is called a paracomplex structure (see [21, 22]).

Next, we recall that an almost complex structure on any Lie algebra $g$ is a linear endomorphism $J : g \to g$ satisfying $J^2 = -1$. If $J$ satisfies the condition

$$J[x, y] = [Jx, y] + [x, Jy] + J[Jx, Jy] \quad \text{for all } x, y \in g,$$

we will say that $J$ is integrable and we will call it a complex structure on $g$. Note that the dimension of a Lie algebra carrying an almost complex structure must be even. A complex structure on $g$ induces a left-invariant complex structure on $G$, the simply connected Lie group with Lie algebra $g$, so that $G$ becomes a complex manifold, although not necessarily a complex Lie group.

If an almost complex structure $J$ satisfies the condition $[Jx, Jy] = [x, y]$ for all $x, y \in g$, then it is automatically integrable, and it is called abelian [7]. This name follows from the fact that the eigenspaces corresponding to the eigenvalues $\pm i$ of the natural extension $J : g^C \to g^C$ are abelian Lie subalgebras of the complex Lie algebra $g^C$. It has been proved in [17] that only solvable Lie algebras admit abelian complex structures.
Finally, a complex product structure on the Lie algebra \( \mathfrak{g} \) is a pair \( \{J, E\} \) of a complex structure \( J \) and a product structure \( E \) satisfying \( JE = -EJ \) (see [4]). If \( (\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-) \) is the double Lie algebra associated to \( E \), then we have \( \mathfrak{g}_- = J\mathfrak{g}_+ \) and hence \( E \) is in fact a paracomplex structure. The endomorphism \( F := JE \) of \( \mathfrak{g} \) is another product structure on \( \mathfrak{g} \) that anticommutes with \( J \); moreover, it can be seen that the operators in \( \{\cos_{\theta} E + \sin_{\theta} F : \theta \in [0, 2\pi)\} \) are all product structures anticommuting with \( J \). Two complex product structures \( \{J, E\} \) and \( \{J', E'\} \) on a Lie algebra \( \mathfrak{g} \) are equivalent if there exists an automorphism \( \varphi \) of \( \mathfrak{g} \) such that \( J' \varphi = \varphi J \) and \( E' \varphi = \varphi E \).

If \( \{J, E\} \) is a complex product structure on \( \mathfrak{g} \) and \( J \) is abelian, then it has been shown in [3] that the Lie subalgebras \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \) determined by \( E \) are both abelian, and the converse also holds. In this case, we will say that \( \{J, E\} \) is an abelian complex product structure.

Let us also recall that a complex product structure \( \{J, E\} \) on \( \mathfrak{g} \) determines uniquely a torsion-free connection \( \nabla^{\text{CP}} \) on \( \mathfrak{g} \) such that \( \nabla^{\text{CP}} J = \nabla^{\text{CP}} E = 0 \), i.e. \( \nabla^{\text{CP}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \) is a \( \mathfrak{g} \)-valued bilinear form on \( \mathfrak{g} \) such that

\[
\nabla^{\text{CP}}_x y - \nabla^{\text{CP}}_y x = [x, y] \quad \text{(torsion-free)},
\]

and also

\[
\nabla^{\text{CP}}_x J y = J \nabla^{\text{CP}}_x y, \quad \nabla^{\text{CP}}_x E y = E \nabla^{\text{CP}}_x y \quad \text{(J and E are \( \nabla^{\text{CP}} \)-parallel)}
\]

for all \( x, y \in \mathfrak{g} \). From this we get that \( \nabla^{\text{CP}}_x y \in \mathfrak{g}_\pm \) whenever \( y \in \mathfrak{g}_\pm \); in particular, \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \) are totally geodesic subalgebras of \( \mathfrak{g} \) with respect to \( \nabla^{\text{CP}} \). The induced connections \( \nabla^+ \) and \( \nabla^- \) on \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \), respectively, are clearly torsion-free and also flat, so that they define left symmetric algebra (LSA) structures on these subalgebras. We recall that a left-symmetric algebra structure on a Lie algebra \( \mathfrak{h} \) is a bilinear product \( \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h} \), \( (x, y) \mapsto x \cdot y \), which satisfies the conditions

\[
(1) \quad x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z,
\]

\[
(2) \quad [x, y] = x \cdot y - y \cdot x
\]

for all \( x, y, z \in \mathfrak{h} \). The correspondence between a flat torsion-free connection \( \nabla \) and an LSA structure on \( \mathfrak{h} \) is given by \( \nabla_x y = x \cdot y \), that is, \( \nabla_x \) represents the left-multiplication by \( x \in \mathfrak{h} \). From [4], it follows that \( \nabla : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}) \) is a representation of \( \mathfrak{h} \). An LSA structure on \( \mathfrak{h} \) is called complete if the corresponding left-invariant connection on \( \mathfrak{h} \), the simply connected Lie group with Lie algebra \( \mathfrak{h} \), is geodesically complete. The completeness of an LSA structure can be characterized in purely algebraic terms by the following condition: the LSA structure is complete if and only if, for all \( x \in \mathfrak{h} \), the right multiplication \( y \mapsto y \cdot x \) of \( \mathfrak{h} \) is nilpotent ([27]).

We move on now to the Lie group level. Let \( G \) be the simply connected Lie group with Lie algebra \( \mathfrak{g} \), where we consider an element of \( \mathfrak{g} \) as a left-invariant vector field on \( G \). A complex product structure \( \{J, E\} \) on \( \mathfrak{g} \) determines a left-invariant complex product structure on \( G \), still denoted by \( \{J, E\} \). This means that \( J \) is a complex structure and \( E \) is a product structure on the manifold \( G \) such that \( JE = -EJ \) and both tensors are invariant by left-translations of \( G \) (see [1] for a study of complex product structures on manifolds). If \( (\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-) \) is the associated double Lie algebra, with \( \mathfrak{g}_- = J\mathfrak{g}_+ \), let \( G_+ \) and \( G_- \) denote the connected Lie subgroups of \( G \) with Lie algebras \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \), respectively. The decomposition \( \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \) determines naturally two complementary distributions on \( G \), both of them involutive and the leaves of the foliations \( \mathcal{F}^+ \) and \( \mathcal{F}^- \) determined by these
distributions are totally real submanifolds of the complex manifold \((G, J)\). Moreover, these leaves are totally geodesic and flat with respect to the canonical torsion-free connection \(\nabla^\text{CP}\) determined by \(\{J, E\}\). It is easy to see that the leaf of \(\mathcal{F}^\pm\) passing through \(x \in G\) is \(xG_\pm\), and it is well known that this leaf is embedded if and only if \(G_\pm\) is closed in \(G\) (see for instance [11]).

3. Complex product structures on nilpotent Lie algebras

Given a Lie algebra \(\mathfrak{g}\), there is the associated lower central series, which is defined recursively by

\[
\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \quad k \geq 1.
\]

The Lie algebra \(\mathfrak{g}\) is called nilpotent if \(\mathfrak{g}^k = \{0\}\) for some \(k\). Equivalently, \(\mathfrak{g}\) is nilpotent if the endomorphisms \(\text{ad} x : \mathfrak{g} \to \mathfrak{g}\) are nilpotent for all \(x \in \mathfrak{g}\), where \((\text{ad} x)y = [x, y]\). If \(\mathfrak{g}^{k-1} \neq \{0\}\), but \(\mathfrak{g}^k = \{0\}\), then \(\mathfrak{g}\) is said to be \(k\)-step nilpotent.

Recall that the Lie bracket on \(\mathfrak{g}\) can be thought of as a linear map \([\cdot, \cdot] : \Lambda^2 \mathfrak{g} \to \mathfrak{g}\). In this way, we can consider its transpose \(d : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*\), defined as follows:

\[
(d f)(x \wedge y) = -f([x, y]) \quad \text{for all } f \in \mathfrak{g}^*, \; x, y \in \mathfrak{g}.
\]

This linear mapping can be extended to \(d : \Lambda^k \mathfrak{g}^* \to \Lambda^{k+1} \mathfrak{g}^*\) for \(k \geq 1\), and the Jacobi identity is equivalent to the vanishing of the composition \(d^2 : \mathfrak{g}^* \to \Lambda^3 \mathfrak{g}^*\). The nilpotency of a Lie algebra \(\mathfrak{g}\) is equivalent to the existence of a basis \(\{e^1, \ldots, e^n\}\) of \(\mathfrak{g}^*\) such that

\[
\text{de}^i = 0, \quad \text{de}^i \in \Lambda^2(\text{span}\{e^1, \ldots, e^{i-1}\}), \quad 2 \leq i \leq n.
\]

Following Salamon [26], we will use this characterization to denote the isomorphism class of a nilpotent Lie algebra: we will identify an \(m\)-dimensional Lie algebra with an \(m\)-tuple, where in the \(k\)th spot we write \(e^k\), further abbreviating \(e^i \wedge e^j\) to \(ij\). For instance, if we write \(\mathfrak{g} = (0, 0, 0, 12, 14 + 23)\), this means that there is a basis \(\{e^1, \ldots, e^6\}\) of \(\mathfrak{g}^*\) such that \(\text{de}^i = 0\) for \(1 \leq i \leq 4\) and \(\text{de}^5 = e^1 \wedge e^2\), \(\text{de}^6 = e^1 \wedge e^3 + e^2 \wedge e^3\). Equivalently, the dual basis \(\{e_1, \ldots, e_6\}\) of \(\mathfrak{g}\) satisfies \([e_1, e_2] = -e_5\), \([e_1, e_4] = [e_2, e_3] = -e_6\).

General properties of nilpotent Lie algebras equipped with complex structures have been given in [26]. Moreover, in that paper the full classification of 6-dimensional nilpotent Lie algebras admitting complex structures was obtained. In forthcoming sections we will classify the 6-dimensional Lie algebras which carry a complex product structure, but this classification will be made independently of the list given by Salamon.

We mention next some properties of LSA structures on nilpotent Lie algebras which will be needed later. We have already mentioned that an LSA structure on any Lie algebra is complete if and only if the right multiplication by any element of the algebra is nilpotent. In the nilpotent case, we have the following equivalent condition, proved by Kim:

**Theorem 3.1** ([21]). An LSA structure on a nilpotent Lie algebra \(\mathfrak{h}\) is complete if and only if the left multiplication \(y \mapsto x \cdot y\) is nilpotent for all \(x \in \mathfrak{h}\).

Many nilpotent Lie algebras admit complete LSA structures; in fact, it was even conjectured that every solvable Lie algebra admits a complete LSA structure, until Benoist [8] gave the first example of a nilpotent Lie algebra not admitting any. However, complete LSA structures on the 3-dimensional Heisenberg Lie algebra \(\mathfrak{h}_3\) are well understood, and we have the following result, proved by Fried and Goldman:

**Theorem 3.2** ([14]). If \(\nabla\) is a complete LSA structure on \(\mathfrak{h}_3\), then \(\nabla_z = 0\) for all \(z \in \mathfrak{z}(\mathfrak{h}_3)\).
It is well known that the existence of a complete LSA structure on a Lie algebra \( g \) (which must be solvable according to a result of Auslander [1]) is equivalent to a simply transitive action of \( G \), the simply connected Lie group with \( \text{Lie}(G) = g \), on \( \mathbb{R}^{\dim G} \) as affine transformations. In this setting, the theorem above implies that if the 3-dimensional Heisenberg group acts simply transitively on \( \mathbb{R}^3 \), then it contains a nontrivial central translation.

Remark. Theorem 3.2 has been generalized in [5] in the following way: if a 2-step nilpotent Lie algebra \( g \) with 1-dimensional commutator ideal carries a complete LSA structure \( \nabla \), then there exists \( z \in \mathfrak{z}(g) \) such that \( \nabla z = 0 \). It is known that a 2-step nilpotent Lie algebra with 1-dimensional commutator ideal is a central extension of a Heisenberg Lie algebra \( \mathfrak{h}_{2n+1} \) (for a proof see [8] or [13]). Recall that \( \mathfrak{h}_{2n+1} = \text{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\} \) with Lie bracket given by \([x_j, y_j] = z\) for all \( j = 1, \ldots, n\).

We begin now the study of complex product structures on nilpotent Lie algebras, focusing in the properties of the torsion-free connection \( \nabla_{\text{CP}} \) associated to this structure. In Theorem 3.3, we prove that the LSA structures defined on the subalgebras determined by the product structure are complete while in Corollary 3.6 we show that \( \nabla_{\text{CP}} \) is Ricci-flat.

Theorem 3.3. Let \( g \) be a nilpotent Lie algebra equipped with a complex product structure \( \{J, E\} \) and let \( \langle g, g_+, g_- \rangle \) be the associated double Lie algebra. Then, the flat torsion-free connections \( \nabla^+ \) and \( \nabla^- \) on \( g_+ \) and \( g_- \), respectively, determined by \( \{J, E\} \) are complete.

In order to prove this theorem, we will need the following general result on product structures on nilpotent Lie algebras.

Lemma 3.4. Let \( E \) be a product structure on the nilpotent Lie algebra \( g \), and let \( \langle g, g_+, g_- \rangle \) be the associated double Lie algebra. Define the representations \( \rho : g_+ \rightarrow \mathfrak{gl}(g_-) \) and \( \mu : g_- \rightarrow \mathfrak{gl}(g_+) \) by

\[
[x, x'] = -\mu(x') x + \rho(x) x',
\]

for \( x \in g_+ \), \( x' \in g_- \). Then \( \rho(x) \) and \( \mu(x') \) are nilpotent endomorphisms of \( g_- \) and \( g_+ \), respectively, for all \( x \in g_+ \) and \( x' \in g_- \).

Proof. Let \( \pi_+ : g \rightarrow g_+ \) and \( \pi_- : g \rightarrow g_- \) denote the projections. We will prove inductively that

\[
\pi_- ((\text{ad} x)^n(x')) = \rho(x)^n(x')
\]

for all \( x \in g_+ \), \( x' \in g_- \). Clearly, (3) holds for \( n = 1 \). Next,

\[
(\text{ad} x)^{n+1}(x') = [x, \pi_+ ((\text{ad} x)^n(x')) + \rho(x)^n(x')] = [x, \pi_+ ((\text{ad} x)^n(x'))] = -\mu(\rho(x)^n(x')) x + \rho(x)^{n+1}(x').
\]

Since the first two terms in the last expression are in \( g_+ \), we get that the \( (g_-) \)-component of \( (\text{ad} x)^{n+1}(x') \) is \( \rho(x)^{n+1}(x') \), and thus (4) is proved.

In the same way one can prove that

\[
\pi_+ ((\text{ad} x')^n(x)) = \mu(x')^n(x)
\]

for all \( x \in g_+ \), \( x' \in g_- \). Since the endomorphisms \( (\text{ad} x) \) and \( (\text{ad} x') \) of \( g \) are nilpotent, then we obtain that \( \rho(x) \in \mathfrak{gl}(g_-) \) and \( \mu(x') \in \mathfrak{gl}(g_+) \) are nilpotent. \( \Box \)
Proof of Theorem 3.3. We recall that $\nabla^+_x = -J \rho(x_+)J$ and $\nabla^-_x = -J \mu(x_-)J$ for $x_+ \in g_+$ and $x_- \in g_-$, with $\rho$ and $\mu$ defined as in (3) (see [1]). From Lemma 3.4, we obtain at once that $\nabla^+_x$ and $\nabla^-_x$ are nilpotent and thus both $\nabla^+$ and $\nabla^-$ are complete, due to Theorem 3.1.

We are going to show that the connection $\nabla^\text{CP}$ associated to a complex product structure on a nilpotent Lie algebra $g$ is always Ricci-flat. We recall that the Ricci tensor is defined by $\text{Ric}(x, y) = \text{tr} (z \mapsto R(z, x)y)$, $x, y, z \in g$, where $R$ is the curvature tensor of $\nabla^\text{CP}$.

Proposition 3.5. Let $\{J, E\}$ be a complex product structure on the nilpotent Lie algebra $g$, and let $\nabla^\text{CP}$ denote the associated torsion-free connection. Then the endomorphisms $\nabla_z^\text{CP}$, $z \in g$, are all traceless.

Proof. As $\nabla^\text{CP}$ is parallel with respect to both $J$ and $E$, we know that each endomorphism $\nabla_z^\text{CP}$ of $g$ must be of the form $\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$, where $X \in \mathfrak{gl}(n, \mathbb{R})$, with respect to suitable bases of $g_+$ and $g_-$. Here $n$ is half the dimension of $g$. If $z = z_+ + z_-$, with $z_+ \in g_+$, then $X$ can be considered as the matrix associated to the endomorphism $(\nabla^+_x - J \nabla^-_x)$ acting on $g_+$, where $J_+ = J|_{g_+} : g_+ \to g_-$ and $J_- = J|_{g_-} : g_- \to g_+$. As $\nabla^+_x$ and $\nabla^-_x$ are both nilpotent, they are traceless, and hence so is $\nabla_z^\text{CP}$ for all $z \in g$.

Corollary 3.6. With the same hypothesis as in Proposition 3.3, the Ricci tensor $\text{Ric}$ of $\nabla^\text{CP}$ vanishes.

Proof. It has been proved in [1] that $\text{Ric}$ is skew-symmetric and that the following relation holds for any $x, y \in g$: $\text{Ric}(x, y) = -\frac{1}{2} \text{tr} R(x, y)$.

Recalling that $R(x, y) = [\nabla_x^\text{CP}, \nabla_y^\text{CP}] - \nabla_{[x, y]}^\text{CP}$, we have that $\text{Ric}(x, y) = \frac{1}{2} \text{tr} \nabla_{[x, y]}^\text{CP}$.

From Proposition 3.3, we obtain that $\text{Ric} = 0$.

Let us move now to the Lie group level. So, let $G$ be the simply connected nilpotent Lie group with Lie algebra $g$, with $\{J, E\}$ its left-invariant complex product structure. In this case, it is known that $G$ is diffeomorphic to $\mathbb{R}^{2n} (2n = \dim G)$ via the exponential map $\exp : g \to G$ and that the connected subgroups $G_+$ and $G_-$ corresponding to $g_+$ and $g_-$, respectively, are closed and simply connected, so that the leaves of the complementary foliations $\mathcal{F}_\pm$ determined by them (the left cosets $xG_\pm$ for $x \in G$) are all simply connected embedded submanifolds of $G$, diffeomorphic to $\mathbb{R}^n$. We already know that these leaves are totally geodesic with respect to $\nabla^\text{CP}$ and flat with respect to the induced connection; furthermore, according to Theorem 3.3, these leaves are also geodesically complete (this fact does not happen in general).

A nilpotent Lie group $G$ admits a lattice (i.e., a cocompact discrete subgroup) $\Gamma$ if and only if there is a basis of its Lie algebra such that the corresponding structure constants are rational ([23]). If $G$ also admits a complex product structure $\{J, E\}$, then it induces a complex product structure $\{\tilde{J}, \tilde{E}\}$ on the nilmanifold $M_\Gamma := \Gamma \backslash G$, and this complex product structure determines naturally two complementary foliations on $M_\Gamma$, denoted by $\mathcal{F}_\pm$. However, since $\Gamma$ is divided out from the left, the foliations $\mathcal{F}_\pm$ on $G$ determined by $\{J, E\}$ induce foliations on $M_\Gamma$, denoted by $\mathcal{F}_\pm^\Gamma$. We will show next that these foliations coincide.
Proposition 3.7. Let $G$ be a Lie group equipped with a left-invariant complex product structure $\{J, E\}$, and let $\mathcal{F}^\pm$ be the associated complementary foliations on $G$. Suppose $\Gamma$ is a lattice in $G$, and let $M_\Gamma := \Gamma \setminus G$. Let $\{\tilde{J}, \tilde{E}\}$ be the complex product structure on $M_\Gamma$ induced by $\{J, E\}$ and $\mathcal{F}^\pm_\Gamma$ be the foliations on $M_\Gamma$ induced by $\mathcal{F}^\pm$. If $\tilde{\mathcal{F}}^\pm$ denote the foliations on $M_\Gamma$ determined by $\{\tilde{J}, \tilde{E}\}$, then $\tilde{\mathcal{F}}^\pm = \mathcal{F}^\pm_\Gamma$.

Proof. This proposition follows from the definition of the induced objects in the quotient $M_\Gamma$, using the fact that the projection $\pi: G \to M_\Gamma$ is a covering map. 

4. The 6-dimensional case

We begin now the study of 6-dimensional nilpotent Lie algebras admitting a complex product structure. In the main result of this section we show that such a Lie algebra can be decomposed as the direct sum (as vector spaces) of two subspaces invariant by the complex product structure, one of which is a 2-dimensional ideal contained in the centre of the algebra. This fact will allow us, in the following section, to obtain the classification of these algebras.

Theorem 4.1. Let $\mathfrak{g}$ be a 6-dimensional nilpotent Lie algebra equipped with a complex product structure $\{J, E\}$. Then there is a decomposition $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{u}$ (direct sum of vector spaces), where $\mathfrak{d}$ is a 4-dimensional subspace and $\mathfrak{u}$ is a 2-dimensional ideal contained in the centre $\mathfrak{z}$ of $\mathfrak{g}$, such that both $\mathfrak{d}$ and $\mathfrak{u}$ are invariant by $J$ and $E$.

Proof. We know that $\mathfrak{g}$ can be written as $\mathfrak{g} = \mathfrak{g}_+ \rtimes \mathfrak{g}_-$, where $\mathfrak{g}_- = J\mathfrak{g}_+$. Since $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are 3-dimensional nilpotent Lie algebras, they are isomorphic to either the abelian Lie algebra $\mathbb{R}^3$ or the Heisenberg Lie algebra $\mathfrak{h}_3$. Recall that this latter Lie algebra has a basis $\{e_1, e_2, e_3\}$ with Lie bracket given by $[e_1, e_2] = e_3$ and $e_3$ is a central element. We will divide the study into two cases: in the first one both subalgebras are abelian, whereas in the second at least one of the subalgebras is isomorphic to $\mathfrak{h}_3$.

(A) First case: both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are abelian, i.e., $\{J, E\}$ is an abelian complex product structure. We will need the following result.

Lemma 4.2. Let $\mathfrak{h}$ be a Lie algebra with an abelian complex product structure and associated double Lie algebra $\langle \mathfrak{h}_+, \mathfrak{h}_- \rangle$, $\mathfrak{h}_\pm \cong \mathbb{R}^n$. If $\mathfrak{h}$ has a non-trivial centre $\mathfrak{z}$, then $\mathfrak{z} \cap \mathfrak{h}_+ \neq \{0\}$, $\mathfrak{z} \cap \mathfrak{h}_- \neq \{0\}$ and $\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{h}_+) \oplus (\mathfrak{z} \cap \mathfrak{h}_-)$. Moreover, $J(\mathfrak{z} \cap \mathfrak{h}_+) = \mathfrak{z} \cap \mathfrak{h}_-$.

Proof. Let us suppose $\mathfrak{z} \cap \mathfrak{h}_+ = \{0\}$, and take $0 \neq z \in \mathfrak{z}$. We can express $z = z_+ + z_-$ with $z_+ \in \mathfrak{h}_+$, $z_- \neq 0$. Thus, $[z_+, x] = -[z_-, x]$ for all $x \in \mathfrak{h}$. Taking $x \in \mathfrak{h}_-$, we have that $[z_+, x] = 0$, and hence $z_+ \in \mathfrak{z} \cap \mathfrak{h}_+$, so that $z_+ = 0$. Therefore, $\mathfrak{z} \subset \mathfrak{h}_-$. However, since $J$ is abelian, it preserves the centre of $\mathfrak{h}$, and thus $J\mathfrak{z} \subset \mathfrak{z} \cap \mathfrak{h}_+ = \{0\}$, which is impossible. Thus, $\mathfrak{z} \cap \mathfrak{h}_+ \neq \{0\}$, and the same holds analogously for $\mathfrak{z} \cap \mathfrak{h}_-$. From the computations above it is clear that the equality $\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{h}_+) \oplus (\mathfrak{z} \cap \mathfrak{h}_-)$ holds.

Take now $z_+ \in \mathfrak{z} \cap \mathfrak{h}_+$; since $J$ is an abelian complex structure we have that, for $x_+ \in \mathfrak{g}_+$, $[Jz_+, x_+] = -[z_+, Jx_+] = 0$.

This relation, together with the fact that $\mathfrak{h}_-$ is abelian, implies that $Jz_+ \in \mathfrak{z} \cap \mathfrak{h}_-$ and therefore $J(\mathfrak{z} \cap \mathfrak{h}_+) \subset \mathfrak{z} \cap \mathfrak{h}_-$. Reversing the roles of $\mathfrak{h}_+$ and $\mathfrak{h}_-$ we obtain the opposite inclusion and hence the proof of the lemma is complete. \qed
Applying Lemma 4.2, we can find a basis \(\{e_1, e_2, e_3\}\) of \(\mathfrak{g}_-\) and a basis \(\{f_1, f_2, f_3\}\) of \(\mathfrak{g}_-\) such that \(f_i = Je_i, i = 1, 2, 3,\) and \(e_3, f_3 \in \mathfrak{z}\). Thus, the subspaces \(\mathfrak{d} = \text{span}\{e_1, e_2, f_1, f_2\}\) and \(u = \text{span}\{e_3, f_3\}\) satisfy the statement of the theorem.

(B) Second case: one of the subalgebras, say \(\mathfrak{g}_+\), is isomorphic to \(\mathfrak{h}_3\). We will prove first that the centre of \(\mathfrak{g}_+\) is contained in the centre of \(\mathfrak{g}\). Let \(\{e_1, e_2, e_3\}\) be a basis of \(\mathfrak{g}_+\) such that \([e_1, e_2] = e_3\) and \([e_3, \mathfrak{g}_+] = 0\).

**Lemma 4.3.** The vector \(e_3 \in \mathfrak{z}(\mathfrak{g}_+)\) belongs to the centre of \(\mathfrak{g}\).

**Proof.** Let \(x \in \mathfrak{g}_-\). Then \([x, e_3] = \nabla^\mathfrak{CP}_x e_3 - \nabla^\mathfrak{CP}_{e_3} x\), where \(\nabla^\mathfrak{CP}\) is the torsion-free connection associated to the complex product structure. Now,

\[
\nabla^\mathfrak{CP}_x x = -J\nabla^\mathfrak{CP}_x e_3 = -J\nabla^\mathfrak{CP}_{e_3} x = 0
\]

due to Theorems 3.2 and 3.3. Thus, \([x, e_3] = \nabla^\mathfrak{CP}_x e_3 \in \mathfrak{g}_+\). So we can write

\[
[x, e_3] = a_1 e_1 + a_2 e_2 + a_3 e_3, \quad a_1, a_2, a_3 \in \mathbb{R}.
\]

Now, on one hand \([x, e_3], e_2\] = \(a_1 e_3\), but on the other hand

\[
[[x, e_3], e_2] = -[[e_3, e_2], x] - [[e_2, x], e_3] = -[[e_2, x], e_3]
\]

using Jacobi’s identity and \(e_3 \in \mathfrak{z}(\mathfrak{g}_+)\). Hence, \([[e_2, x], e_3] = -a_1 e_3\), and since \(\mathfrak{g}\) is nilpotent, we have \(a_1 = 0\). Repeating this argument with \([[x, e_3], e_1]\), we arrive at \(a_2 = 0\) and so we get \([x, e_3] = a_3 e_3\), which implies that \(a_3 = 0\). Therefore \([x, e_3] = 0\) and \(e_3 \in \mathfrak{z}(\mathfrak{g})\). \(\square\)

We will consider now two subcases, according to the isomorphism class of \(\mathfrak{g}_-\).

(i) Let us suppose first that \(\mathfrak{g}_-\) is abelian. If we define \(f_i \in \mathfrak{g}_-\) by \(f_i = Je_i, i = 1, 2, 3,\) then \(\{f_1, f_2, f_3\}\) is a basis of \(\mathfrak{g}_-\). Using Lemma 4.2, we arrive at the following result.

**Lemma 4.4.** The element \(f_3 \in \mathfrak{g}_-\) belongs to the centre of \(\mathfrak{g}\).

**Proof.** Clearly, \([f_3, \mathfrak{g}_-] = 0\). So, let us take now \(x \in \mathfrak{g}_+\) and perform the following computations, using the integrability of \(J\):

\[
J[f_3, x] = -[e_3, x] + [f_3, Jx] - J[e_3, Jx] = 0
\]

since \(e_3\) is central and \(\mathfrak{g}_-\) is abelian. Therefore, \(f_3\) is also central. \(\square\)

The subspaces \(\mathfrak{d} = \text{span}\{e_1, e_2, f_1, f_2\}\) and \(u = \text{span}\{e_3, f_3\}\) satisfy the statement of the theorem.

(ii) Assume now that \(\mathfrak{g}_-\) is isomorphic to \(\mathfrak{h}_3\), and let us consider a basis \(\{f_1, f_2, f_3\}\) of \(\mathfrak{g}_-\) such that \([f_1, f_2] = f_3\) and \([f_3, \mathfrak{g}_-] = 0\). From Lemma 4.3 applied to both \(\mathfrak{g}_+\) and \(\mathfrak{g}_-\), we know that both \(e_3\) and \(f_3\) belong to the centre of \(\mathfrak{g}\) and from this we get the following result.

**Lemma 4.5.** With notation as above, we may suppose that \(Je_1 = f_1, Je_2 = f_2\) and \(Je_3 = cf_3\) for some \(c \neq 0\).

**Proof.** Let the complex structure \(J\) be given by

\[
\begin{align*}
Je_1 &= \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3, \\
Je_2 &= \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3, \\
Je_3 &= \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 f_3,
\end{align*}
\]
with $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$, $i = 1, 2, 3$, and let us suppose that $\gamma_1^2 + \gamma_2^2 \neq 0$. By interchanging $f_1$ with $f_2$, if necessary, we can assume further that $\gamma_1 \neq 0$. By subtracting from $e_1$ and $e_2$ a suitable multiple of $e_3$, we obtain a new ‘$e_1$’ and a new ‘$e_2$’ such that $[e_1, e_2] = e_3$ and

$$\begin{cases}
Je_1 = \alpha_2 f_2 + \alpha_3 f_3,
Je_2 = \beta_2 f_2 + \beta_3 f_3,
Je_3 = \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 f_3,
\end{cases}$$

with $\gamma_1 \neq 0$ and $\Delta := \alpha_2 \beta_3 - \alpha_3 \beta_2 \neq 0$. Note that $Jf_3 = \Delta^{-1}(\beta_2 e_1 - \alpha_2 e_2)$. From the integrability of $J$, we have

$$Je_3 = J[e_1, e_2] = [Je_1, e_2] + [e_1, Je_2] + J[Je_1, Je_2] = -\alpha_2[e_2, f_2] + \beta_2[e_1, f_2]$$

so that

$$Je_3 = \beta_2[e_1, f_2] - \alpha_2[e_2, f_2].$$

Using again the integrability of $J$, we get

$$0 = J[e_1, e_3] = [e_1, Je_3] + J[Je_1, Je_3] = \gamma_1[e_1, f_1] + \gamma_2[e_1, f_2] - \alpha_2 \gamma_1 Jf_3$$

and

$$0 = J[e_2, e_3] = [e_2, Je_3] + J[Je_2, Je_3] = \gamma_1[e_2, f_1] + \gamma_2[e_2, f_2] - \beta_2 \gamma_1 Jf_3$$

so that

$$\begin{cases}
\gamma_1[e_1, f_1] + \gamma_2[e_1, f_2] = \alpha_2 \gamma_1 Jf_3,
\gamma_1[e_2, f_1] + \gamma_2[e_2, f_2] = \beta_2 \gamma_1 Jf_3.
\end{cases}$$

Now, let us compute

$$[[e_1, f_2], Je_3] = [[e_1, f_2], \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 f_3]$$

$$= \gamma_1 [[e_1, f_1], f_2] + \gamma_2 [[e_1, f_2], f_2]$$

$$= \gamma_1 [e_1, [f_1, f_2]] + \gamma_2 [e_1, [f_2, f_2]]$$

$$= \alpha_2 \gamma_1 [Jf_3, f_2]$$

$$= \frac{\alpha_2 \gamma_1}{\Delta} (\beta_2 [e_1, f_2] - \alpha_2 [e_2, f_2])$$

$$= \frac{\alpha_2 \gamma_1}{\Delta} Je_3,$$

and since $g$ is nilpotent and $\gamma_1 \neq 0$, we must have $\alpha_2 = 0$. Let us compute next

$$[[e_2, f_2], Je_3] = [[e_2, f_2], \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 f_3]$$

$$= \gamma_1 [[e_2, f_1], f_2] + \gamma_2 [[e_2, f_2], f_2]$$

$$= \gamma_1 [e_2, [f_1, f_2]] + \gamma_2 [e_2, [f_2, f_2]]$$

$$= \beta_2 \gamma_1 [Jf_3, f_2]$$

$$= \frac{\beta_2 \gamma_1}{\Delta} Je_3,$$

and again since $g$ is nilpotent and $\gamma_1 \neq 0$, we get that $\beta_2 = 0$. Therefore, we have that both $Je_1$ and $Je_2$ are in the subspace spanned by $f_3$, which is impossible since they must be linearly independent. Therefore, the condition $\gamma_1^2 + \gamma_2^2 \neq 0$ can never hold, and as a consequence $Je_3$ is a multiple of $f_3$. Since $[Je_1, Je_2] = c^{-1} f_3$, for some $c \neq 0$, we can take $\{Je_1, Je_2, c^{-1} f_3\}$ as the new basis of $g_-$, which satisfies the conditions stated in the lemma. □
Resuming the proof of the theorem, we see that the subspaces \( \mathfrak{d} = \text{span}\{e_1, e_2, f_1, f_2\} \) and \( \mathfrak{u} = \text{span}\{e_3, f_3\} \) satisfy the conditions in the statement, and therefore the proof is now complete. \( \square \)

As a first consequence of this theorem, together with results in [20], we have the following

**Corollary 4.6.** The Lie algebras \((0, 0, 0, 12, 23, 14-35)\) and \((0, 0, 12, 13, 23, 14+25)\) possess complex structures, but do not admit any complex product structure.

**Proof.** These Lie algebras carry complex structures, as shown in [26]. However, they cannot admit any complex product structure since each centre is one dimensional, and thus there cannot exist a 2-dimensional ideal \( \mathfrak{u} \) contained in the centre. \( \square \)

Let us recall now the definition of a nilpotent complex structure on a nilpotent Lie algebra, which was introduced in [12]. Given a complex structure \( J \) on a nilpotent Lie algebra \( \mathfrak{g} \), there is the ascending series \( \{a_t(J)\} \) associated to \( J \), which is defined inductively by

\[
a_0(J) = \{0\}, \quad a_t(J) = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subset a_{t-1}(J), [Jx, \mathfrak{g}] \subset a_{t-1}(J)\},
\]

for \( t \geq 1 \). \( J \) is called nilpotent if \( a_t(J) = \mathfrak{g} \) for some \( t \geq 1 \). Compact nilmanifolds equipped with nilpotent complex structures (i.e. induced by a nilpotent complex structure on the corresponding Lie algebra) possess many interesting properties (see [13]). As another corollary of Theorem 4.1, we have the following:

**Corollary 4.7.** If \( \{J, E\} \) is a complex product structure on a 6-dimensional nilpotent Lie algebra, then \( J \) is a nilpotent complex structure.

**Proof.** In [12] it was proved that a complex structure \( J \) on a 6-dimensional nilpotent Lie algebra \( \mathfrak{g} \) is nilpotent if and only if \( a_1(J) \neq \{0\} \); note that \( a_1(J) = \{x \in \mathfrak{j}(\mathfrak{g}) : Jx \in \mathfrak{j}(\mathfrak{g})\} \).

Consider now the complex product structure \( \{J, E\} \) on \( \mathfrak{g} \). According to Theorem 4.1, there exists a central ideal \( \mathfrak{u} \) in \( \mathfrak{g} \) which is \( J \)-invariant. Thus, \( a_1(J) \supset \mathfrak{u} \neq \{0\} \) and hence \( J \) is nilpotent. \( \square \)

In general, the dimension of the centre of a nilpotent Lie algebra admitting a complex product structure need not be \( \geq 2 \) and, furthermore, the complex structure need not be nilpotent, as in the 6-dimensional case. The next example will illustrate these facts.

**Example 4.8.** We exhibit next an example of an 8-dimensional nilpotent Lie algebra with 1-dimensional centre which admits a complex product structure. Consider the 4-dimensional nilpotent Lie algebra \( \mathfrak{n}_4 \) with a basis \( \{e_1, \ldots, e_4\} \) such that \( [e_1, e_2] = e_3, [e_1, e_3] = e_4 \), and the complete LSA structure \( \nabla \) on \( \mathfrak{n}_4 \) given by:

\[
\nabla_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \nabla_{e_2} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]
\[
\nabla_{e_3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \nabla_{e_4} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

in the ordered basis \( \{e_1, \ldots, e_4\} \). Observe that \( e_4 \in \mathfrak{j}(\mathfrak{n}_4) \) but \( \nabla_{e_4} \neq 0 \) (compare Theorem 3.2); this LSA structure on \( \mathfrak{n}_4 \) was given by Fried in [18] (see also [20]). Let \( V \) denote the
Remark. Not every nilpotent complex structure on a 6-dimensional nilpotent Lie algebra $J_e = (0, e_1)$ may be part of a complex product structure. For instance, consider the Lie algebra $\mathfrak{g}$ that its associated double Lie algebra is $(\mathfrak{g}, \mathfrak{n}_4, V)$. Note that in this case the central element $e_4$ of $\mathfrak{n}_4$ does not belong to the centre of $\mathfrak{g}$; furthermore, $\mathfrak{z}(\mathfrak{g}) = \text{span}\{(0, e_1)\}$ is 1-dimensional. It can also be seen that the complex structure $J$ satisfies $\alpha_t(J) = \{0\}$ for all $t \geq 0$; therefore, the complex structure $J$ is not nilpotent.

Let $\{J, E\}$ be a complex product structure on the 6-dimensional nilpotent Lie algebra $\mathfrak{g}$. It is known that if $J$ is abelian, then the associated double Lie algebra must be of the form $(\mathfrak{g}, \mathbb{R}^3, \mathbb{R}^3)$. On the other hand, if $J$ is not abelian, then there are essentially two possibilities for the associated double Lie algebra: either $(\mathfrak{g}, \mathfrak{h}_3, \mathbb{R}^3)$ or $(\mathfrak{g}, \mathfrak{h}_3, \mathfrak{h}_3)$. However, in the next result we will show that a Lie algebra admits a complex product structure of the former type if and only if it admits a complex product structure of the latter type. This fact will simplify notoriously the task of classifying the 6-dimensional nilpotent Lie algebras admitting complex product structures, which will be carried out next.

**Proposition 5.1.** Let $\{J, E\}$ be a complex product structure on the 6-dimensional nilpotent Lie algebra $\mathfrak{g}$ with $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ the associated double Lie algebra.

(i) If $\mathfrak{g}_+ \cong \mathfrak{h}_3$, $\mathfrak{g}_- \cong \mathfrak{h}_3$, then there exists a complex product structure $\{J, E'\}$ on $\mathfrak{g}$ such that its associated double Lie algebra is $(\mathfrak{g}, \mathfrak{h}_3, \mathbb{R}^3)$, where $E' = \cos \theta E + \sin \theta JE$ for some $\theta \in [0, 2\pi]$.

(ii) If $\mathfrak{g}_+ \cong \mathfrak{h}_3$, $\mathfrak{g}_- \cong \mathbb{R}^3$, then there exists a complex product structure $\{J, E'\}$ on $\mathfrak{g}$ such that its associated double Lie algebra is $(\mathfrak{g}, \mathfrak{h}_3, \mathfrak{h}_3)$, where $E' = JE$.

**Proof.** (i) According to Lemma 4.2, there exists a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}_+$ and a basis $\{f_1, f_2, f_3\}$ of $\mathfrak{g}_-$ such that $[e_1, e_2] = e_3$, $[f_1, f_2] = f_3$ (with some non zero $[e_i, f_j]$), $Je_i = f_i$ for $i = 1, 2$ and $J e_3 = c f_3$ with $c \neq 0$. Consider the following subspaces of $\mathfrak{g}$:

$$\mathfrak{g}'_+ = \text{span} \{ce_1 + f_1, ce_2 + f_2, e_3 + f_3\},$$

$$\mathfrak{g}'_- = \text{span} \left\{e_1 - cf_1, e_2 - cf_2, \frac{1}{c} e_3 + cf_3 \right\}.$$ 

The complementary subspaces $\mathfrak{g}'_+$ and $\mathfrak{g}'_-$ are in fact Lie subalgebras of $\mathfrak{g}$; moreover, $\mathfrak{g}'_+ \cong \mathfrak{h}_3$, $\mathfrak{g}'_- \cong \mathbb{R}^3$ and $J \mathfrak{g}'_+ = \mathfrak{g}'_-$. The associated product structure $E'$ is given by $E' = \cos \theta E + \sin \theta JE$, where

$$\cos \theta = \frac{c^2 - 1}{c^2 + 1}, \quad \sin \theta = \frac{2c}{c^2 + 1},$$
and (i) is proved.

(ii) There exists a basis \( \{e_1, e_2, e_3\} \) of \( \mathfrak{g}_+ \) and a basis \( \{f_1, f_2, f_3\} \) of \( \mathfrak{g}_- \) such that \([e_1, e_2] = e_3\) (with some non zero \([e_i, f_j]\)) and \(J e_i = f_i\) for \(i = 1, 2, 3\). Consider the following subspaces of \( \mathfrak{g} \):

\[
\mathfrak{g}_+ = \text{span}\{e_1 + f_1, e_2 + f_2, e_3 + f_3\}, \\
\mathfrak{g}_- = \text{span}\{e_1 - f_1, e_2 - f_2, e_3 - f_3\}.
\]

The complementary subspaces \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \) are in fact Lie subalgebras of \( \mathfrak{g} \); moreover, \( \mathfrak{g}_+ \cong \mathfrak{h}_3 \), \( \mathfrak{g}_- \cong \mathfrak{h}_3 \) and \( J \mathfrak{g}_+ = \mathfrak{g}_- \). The associated product structure \( E' \) is given by \( E' = JE \) and hence (ii) is proved. \( \square \)

Let us consider the Lie algebra \( \tilde{\mathfrak{g}} := \mathfrak{g}/\mathfrak{u} \), where we use the notation from Theorem 4.7. It is a 4-dimensional nilpotent Lie algebra, and since \( \mathfrak{u} \) is invariant under \( J \) and \( E \), it carries an induced complex product structure \( \{\tilde{J}, \tilde{E}\} \). There are only two nilpotent Lie algebras of dimension 4 which admit complex product structures, the abelian one \( \mathbb{R}^4 \) and the central extension of the Heisenberg Lie algebra \( \mathfrak{h}_3 \times \mathbb{R} \). We consider next each of these cases separately.

5.1. First case: \( \tilde{\mathfrak{g}} = \mathfrak{h}_3 \times \mathbb{R} \). In [4] there is a classification of complex product structures on \( \mathfrak{h}_3 \times \mathbb{R} \), which is given as follows. Let \( \{v_1, v_2, v_3, v_4\} \) be a basis of \( \mathfrak{h}_3 \times \mathbb{R} \) such that \([v_1, v_2] = v_3\) and \(v_3, v_4\) are central elements. Then every complex product structure on this Lie algebra is equivalent to \( \{J', E_\theta\} \), where the complex structure \( J' \) is \( J' v_1 = v_2 \), \( J' v_3 = v_4 \) and the product structure \( E_\theta \) is given by

\[
E_\theta = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & \sin \theta & -\cos \theta
\end{pmatrix}
\]

for some \( \theta \in [0, 2\pi] \), in the ordered basis \( \{v_1, v_2, v_3, v_4\} \). Any automorphism of \( \tilde{\mathfrak{g}} \) which defines an equivalence between \( \{\tilde{J}, \tilde{E}\} \) and \( \{J', E_\theta\} \) can be lifted to an automorphism of \( \mathfrak{g} \) which is the identity on \( \mathfrak{u} \), and via this latter automorphism we can construct a new complex product structure on \( \mathfrak{g} \) equivalent to the original one, and such that the induced complex structure on \( \tilde{\mathfrak{g}} \) is \( \{J', E_\theta\} \). Therefore, we can suppose without loss of generality that \( \tilde{J} = J' \) and \( \tilde{E} = E_\theta \) for some \( \theta \in [0, 2\pi] \). The subalgebras of \( \mathfrak{h}_3 \times \mathbb{R} \) associated to the eigenvalues \( \pm 1 \) of \( \tilde{E} \) are

\[
\tilde{\mathfrak{g}}_+ = \text{span}\{v_1, \cos \theta/2 v_3 + \sin \theta/2 v_4\},
\]

and

\[
\tilde{\mathfrak{g}}_- = \text{span}\{v_2, -\sin \theta/2 v_3 + \cos \theta/2 v_4\}.
\]

If \( p : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}} \) denotes the canonical projection, then we have \( p(\mathfrak{g}_+) = \tilde{\mathfrak{g}}_+ \) and \( p(\mathfrak{g}_-) = \tilde{\mathfrak{g}}_- \); furthermore, \( p(\mathfrak{d} \cap \mathfrak{g}_+) = \tilde{\mathfrak{g}}_+ \) and \( p(\mathfrak{d} \cap \mathfrak{g}_-) = \tilde{\mathfrak{g}}_- \) as vector spaces. Hence, we can find a basis \( \{e_1, e_2, e_3\} \) of \( \mathfrak{g}_+ \) and a basis \( \{f_1, f_2, f_3\} \) of \( \mathfrak{g}_- \) such that \( \mathfrak{d} = \text{span}\{e_1, e_2, f_2, f_2\} \).
\[ u = \text{span}\{e_3, f_3\}, \; f_i = Je_i \text{ for } i = 1, 2, 3, \text{ and} \]

\[
\begin{align*}
\{e_1, e_2\} &= \alpha e_3 (\alpha \in \{0, 1\}), \\
\{f_1, f_2\} &= \beta f_3 (\beta \in \{0, 1\}), \\
\{e_1, f_1\} &= Ae_2 + Be_3 + Cf_2 + Df_3, \ (A^2 + C^2 \neq 0) \\
\{e_1, f_2\} &\in u, \\
\{e_2, f_1\} &\in u, \\
\{e_2, f_2\} &\in u.
\end{align*}
\]

(5)

Also, from the integrability of \(J\) we get that

\[ J\{e_1, e_2\} = \{f_1, e_2\} + \{e_1, f_2\} + J\{f_1, f_2\}, \]

and so

\[ \{e_1, f_2\} - \{e_2, f_1\} = \beta e_3 + \alpha f_3. \]

(6)

Let us denote \([e_2, f_2] = G e_3 + H f_3\) for some \(G, H \in \mathbb{R}\). By Jacobi’s identity, \([\{e_1, f_1\}, e_2] = -[\{f_1, e_2\}, e_1] - ([e_2, e_1], f_1) = 0\) since \(u \subset \mathfrak{j}\). On the other hand, \([\{e_1, f_1\}, e_2] = -C[e_2, f_2]\), so that \(CG = CH = 0\). In the same way we verify that \(0 = [\{e_1, f_1\}, f_2] = A[e_2, f_2]\), and hence \(AG = AH = 0\). As \(A^2 + C^2 \neq 0\), we must have \(G = H = 0\), so that

\[ \{e_2, f_2\} = 0. \]

We have now two subcases: (i) \(\alpha = \beta = 0\) and (ii) \(\alpha = 1, \beta = 0\). In fact, the case \(\alpha = 0, \beta = 1\) is analogous to the case (ii) above, and the case \(\alpha = 1, \beta = 1\) need not be considered due to Proposition 5.1.

5.1.1. \(\alpha = 0, \beta = 0\). In this case the relations (5), under condition (6), become simply

\[
\begin{align*}
\{e_1, f_1\} &= Ae_2 + Be_3 + Cf_2 + Df_3, \\
\{e_2, f_1\} &= \{e_1, f_2\} = E e_3 + F f_3.
\end{align*}
\]

(7)

If \(E = F = 0\), then \(\mathfrak{g} \cong (0, 0, 0, 0, 12)\). Let us consider now \(E^2 + F^2 \neq 0\). Suppose first \(A \neq 0\). Defining

\[
\begin{align*}
v_1 &:= Ae_1 + Cf_1, \; v_2 := -f_1, \; v_3 := Fe_3 - Ef_3, \\
v_4 &:= Af_2, \; v_5 := A[e_1, f_1], \; v_6 := -A^2[e_2, f_1],
\end{align*}
\]

we have that \(\{v_1, \ldots, v_6\}\) is a basis for \(\mathfrak{g}\) and

\[ [v_1, v_2] = -v_5, \quad [v_2, v_5] = -v_6, \quad [v_1, v_4] = -v_6. \]

So, \(\mathfrak{g} \cong (0, 0, 0, 0, 12, 14 + 25)\). If \(A = 0\), so that \(C \neq 0\), we define

\[
\begin{align*}
v_1 := Cf_1, \; v_2 := e_1, \; v_3 := Fe_3 - Ef_3, \; v_4 := -Ce_2, \; v_5 := C[e_1, f_1], \; v_6 := -C[e_1, f_2],
\end{align*}
\]

and with this basis we see again that \(\mathfrak{g} \cong (0, 0, 0, 0, 12, 14 + 25)\). This concludes this case.
5.1.2. $\alpha = 1$, $\beta = 0$. The relations (8), under condition (f), become in this case

$$\begin{align*}
[e_1, e_2] &= e_3, \\
[e_1, f_1] &= Ae_2 + Be_3 + Cf_2 + Df_3, \quad (A^2 + C^2 \neq 0) \\
[e_2, f_1] &= Ee_3 + Ff_3, \\
[e_1, f_2] &= Ee_3 + (F + 1)f_3.
\end{align*}$$

Changing $f_1$ by $f_1 - Be_2$ we may assume that $B = 0$, so that $[e_1, f_1] = Ae_2 + Cf_2 + Df_3$.

Let us suppose first $E \neq 0$ and $AF - CE \neq 0$. In this case, the vectors $v_1, \ldots, v_6$ defined below form a basis of $g$:

$$\begin{align*}
v_1 &:= E^{-1}f_1, \quad v_2 := Ee_1 + (F + 1)f_1, \quad v_3 := Ee_2 + Ff_2, \\
v_4 &:= [e_1, f_1], \quad v_5 := [e_2, f_1], \quad v_6 := (AF - CE)[e_1, f_2].
\end{align*}$$

These vectors satisfy the relations

$$[v_1, v_2] = -v_4, \quad [v_1, v_3] = -v_5, \quad [v_1, v_4] = -AE^{-1}v_5, \quad [v_2, v_4] = -v_6.$$ 

If $A = 0$, we see immediately that $g \cong (0, 0, 0, 12, 13, 24)$. If $A \neq 0$, multiplying by $AE^{-1}$, we see that $g \cong (0, 0, 0, 12, 13 + 14, 24)$. Let us still suppose that $E \neq 0$, but now with $AF - CE = 0$; note that $A \neq 0$. Consider the vectors

$$\begin{align*}
v_1 &:= -(Ee_1 + (F + 1)f_1), \quad v_2 := E^{-1}f_1, \quad v_3 := E^{-1}f_2, \\
v_4 &:= [e_1, f_1], \quad v_5 := [e_1, f_2], \quad v_6 := AE^{-1}[e_2, f_1].
\end{align*}$$

Then we have

$$[v_1, v_2] = -v_4, \quad [v_1, v_3] = -v_5, \quad [v_2, v_4] = -v_6,$$

so that $g \cong (0, 0, 0, 12, 13, 24)$.

Let us now move to the case $E = 0$. Consider first $F = 0$, and define

$$v_1 := -e_1, \quad v_2 := f_1, \quad v_3 := e_2, \quad v_4 := [e_1, f_1], \quad v_5 := e_3, \quad v_6 := Ae_3 + Cf_3.$$ 

These vectors form a basis of $g$ if $C \neq 0$, and as they satisfy

$$[v_1, v_2] = -v_4, \quad [v_1, v_3] = -v_5, \quad [v_1, v_4] = -v_6,$$

we have that $g \cong (0, 0, 0, 12, 13, 14)$. If $C = 0$ (and hence $A \neq 0$), we consider the basis

$$v_1 := -e_1, \quad v_2 := f_1, \quad v_3 := f_2, \quad v_4 := [e_1, f_1], \quad v_5 := f_3, \quad v_6 := Ae_3,$$

and we have again that $g \cong (0, 0, 0, 12, 13, 14)$. Consider now $F = -1$, so that

$$[e_1, e_2] = e_3, \quad [e_1, f_1] = Ae_2 + Cf_2 + Df_3, \quad [e_2, f_1] = -f_3.$$ 

If $A = 0$, then $[e_1, f_1] \in z$ and it is easily seen that $g \cong (0, 0, 0, 12, 13, 23)$. If $A \neq 0$, define

$$v_1 := -e_1, \quad v_2 := f_1, \quad v_3 := f_2, \quad v_4 := [e_1, f_1], \quad v_5 := Ae_3, \quad v_6 := -Af_3.$$ 

Then, $[v_1, v_2] = -v_4, \quad [v_1, v_4] = -v_5, \quad [v_2, v_4] = -v_6$ and thus, $g \cong (0, 0, 0, 12, 14, 24)$. Let us finally suppose that $F \neq 0$, $F \neq -1$. If $A \neq 0$, consider the basis

$$v_1 := -e_1, \quad v_2 := f_1, \quad v_3 := \frac{AF}{F + 1}f_2, \quad v_4 := [e_1, f_1], \quad v_5 := Ae_3 + C(F + 1)f_3, \quad v_6 := -AFf_3.$$ 

We have then

$$[v_1, v_2] = -v_4, \quad [v_1, v_4] = -v_5, \quad [v_1, v_3] = -v_6, \quad [v_2, v_4] = v_6,$$
and thus \( g \cong (0, 0, 0, 12, 14, 13 + 42) \). If \( A = 0 \) (and hence \( C \neq 0 \)), we consider the basis

\[
v_1 := -e_1, \ v_2 := f_1, \ v_3 := \frac{C(F + 1)}{F} e_2,
\]

\[
v_4 := [e_1, f_1], \ v_5 := \frac{C(F + 1)}{F} e_3, \ v_6 := C(F + 1) f_3,
\]

and we have

\[
[v_1, v_2] = -v_4, \ [v_1, v_3] = -v_5, \ [v_1, v_4] = -v_6, \ [v_2, v_3] = -v_6,
\]

so that \( g \cong (0, 0, 0, 12, 13, 14 + 23) \). This concludes this case.

5.2. **Second case**: \( \tilde{g} = \mathbb{R}^4 \). Any complex product structure on \( \mathbb{R}^4 \) is equivalent to \( \{J_0, E_0\} \), where \( J_0 \) and \( E_0 \) are given by

\[
J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

in some ordered basis of \( \mathbb{R}^4 \), where \( \mathbf{1} \) is the \((2 \times 2)\)-identity matrix. If \( \{\tilde{J}, \tilde{E}\} \) is the complex product structure on \( \tilde{g} \) induced by \( \{J, E\} \), we can suppose without loss of generality that \( \tilde{J} = J_0 \) and \( \tilde{E} = E_0 \). If \( p : g \to \tilde{g} \) denotes the canonical projection, then we have \( p(g_+) = \tilde{g}_+ \) and \( p(g_-) = \tilde{g}_- \); furthermore, \( p(0 \cap g_+) = \tilde{g}_+ \) and \( p(0 \cap g_-) = \tilde{g}_- \) as vector spaces. Hence, we can find a basis \( \{e_1, e_2, e_3\} \) of \( g_+ \) and a basis \( \{f_1, f_2, f_3\} \) of \( g_- \) such that \( \mathfrak{d} = \text{span}\{e_1, e_2, f_1, f_2\} \), \( u = \text{span}\{e_3, f_3\} \), \( f_i = Je_i \) for \( i = 1, 2, 3 \), and

\[
\begin{align*}
[e_1, e_2] &= \alpha e_3 \ (\alpha \in \{0, 1\}), \\
[f_1, f_2] &= \beta f_3 \ (\beta \in \{0, 1\}), \\
[e_i, f_j] &\in u, \quad 1 \leq i, j \leq 2.
\end{align*}
\]

(9)

Also, from the integrability of \( J \) we get that

\[
J[e_1, e_2] = [f_1, e_2] + [e_1, f_2] + J[f_1, f_2],
\]

and so

\[
[e_1, f_2] - [e_2, f_1] = \beta e_3 + \alpha f_3.
\]

(10)

Note that, since \( \tilde{g} \) is abelian, the commutator ideal \( [g, g] \subseteq u \) and therefore \( g \) is 2-step nilpotent with \( \dim[g, g] \leq 2 \).

We have now two subcases: (i) \( \alpha = \beta = 0 \) and (ii) \( \alpha = 1, \beta = 0 \). In fact, the case \( \alpha = 0, \beta = 1 \) is analogous to the case (ii) above, and the case \( \alpha = 1, \beta = 1 \) need not be considered due to Proposition 5.1.

5.2.1. \( \alpha = 0, \beta = 0 \). In this case, the relations (9), under the condition (10), become

\[
\begin{align*}
[e_1, f_1] &= A_1 e_3 + A_2 f_3, \\
[e_1, f_2] &= [e_2, f_1] = B_1 e_3 + B_2 f_3, \\
[e_2, f_2] &= D_1 e_3 + D_2 f_3.
\end{align*}
\]

Performing computations as the ones done in §5.1, which we omit, we arrive at the following result:

\( \diamond \) If \( \dim[g, g] = 1 \), then \( g \) is isomorphic either to \((0, 0, 0, 0, 12)\) or \((0, 0, 0, 0, 12 + 34)\).

\( \diamond \) If \( \dim[g, g] = 2 \), then \( g \) is isomorphic to one of the following Lie algebras:

\[
(0, 0, 0, 12, 34), \ (0, 0, 0, 13 + 42, 14 + 23), \ (0, 0, 0, 12, 14 + 23).
\]
5.2.2. $\alpha = 1$, $\beta = 0$. In this case, the relations (H), under the condition (H0), become

\[
\begin{align*}
[e_1, e_2] &= e_3, \\
[e_1, f_1] &= A_1 e_3 + A_2 f_3, \\
[e_2, f_1] &= C_1 e_3 + C_2 f_3, \\
[e_1, f_2] &= C_1 e_3 + (C_2 + 1) f_3, \\
[e_2, f_2] &= D_1 e_3 + D_2 f_3.
\end{align*}
\]

Note that in this case we have $\dim [g, g] = 2$. Computing again as in §5.1, we can conclude that $g$ must be isomorphic to one of the following Lie algebras:

$(0, 0, 0, 0, 12, 13), (0, 0, 0, 0, 13 + 42, 14 + 23), (0, 0, 0, 12, 14 + 23), (0, 0, 0, 12, 34)$.

5.3. Conclusion. We summarize in the following table the results obtained in the previous subsections.

| $\mathbb{R}^3 \Join \mathbb{R}^3$ | $\mathfrak{h}_3 \Join \mathbb{R}^3$ | $\mathfrak{h}_3 \Join \mathfrak{h}_3$ |
| --- | --- | --- |
| $(0, 0, 0, 0, 0)$ | yes | no | no |
| $(0, 0, 0, 0, 12)$ | yes | no | no |
| $(0, 0, 0, 0, 12 + 34)$ | yes | no | no |
| $(0, 0, 0, 12, 14 + 25)$ | yes | no | no |
| $(0, 0, 0, 12, 13, 13)$ | no | yes | yes |
| $(0, 0, 0, 0, 13 + 42, 14 + 23)$ | yes | yes | yes |
| $(0, 0, 0, 12, 14 + 23)$ | yes | yes | yes |
| $(0, 0, 0, 12, 12)$ | no | yes | yes |
| $(0, 0, 0, 12, 13, 14)$ | no | yes | yes |
| $(0, 0, 0, 12, 13, 23)$ | no | yes | yes |
| $(0, 0, 0, 12, 14, 24)$ | no | yes | yes |
| $(0, 0, 0, 12, 13, 24)$ | no | yes | yes |
| $(0, 0, 0, 12, 13 + 14, 24)$ | no | yes | yes |
| $(0, 0, 0, 12, 13, 14 + 23)$ | no | yes | yes |
| $(0, 0, 0, 12, 14, 13 + 42)$ | no | yes | yes |

Remark. Let us identify some of the Lie algebras appearing in the list above. The Lie algebra $(0, 0, 0, 0, 12)$ is the product $\mathfrak{h}_3 \times \mathbb{R}^3$; the Lie algebra $(0, 0, 0, 0, 12 + 34)$ is the product $\mathfrak{h}_5 \times \mathbb{R}$, while the Lie algebra $(0, 0, 0, 0, 12, 34)$ is the product $\mathfrak{h}_3 \times \mathfrak{h}_3$. The algebra $(0, 0, 0, 13 + 42, 14 + 23)$ is isomorphic to the complex 3-dimensional Heisenberg algebra $\mathfrak{h}_3^c$, considered as a real Lie algebra. Finally, the Lie algebra $(0, 0, 0, 12, 13, 23)$ is isomorphic to the 2-step nilpotent free Lie algebra on 3 generators. Indeed, this Lie algebra is isomorphic to the vector space $\mathbb{R}^3 \oplus \mathbb{R}^3 \wedge \mathbb{R}^3$, with Lie bracket given only by $[u, v] = u \wedge v$ for $u, v \in \mathbb{R}^3$.

We already know that there are two 6-dimensional nilpotent Lie algebras which admit a complex structure but no complex product structure, see Corollary 13. However, comparing with the list in [26], we see that there is another algebra that admits a complex structure but no complex product structure, as it does not appear in the table above; this Lie algebra is the one whose Lie bracket is encoded in $(0, 0, 0, 12, 13 + 42, 14 + 23)$. Next, we verify this fact by direct computations.
Proposition 5.2. The Lie algebra \( g = (0, 0, 0, 12, 13 + 42, 14 + 23) \) does not admit any complex product structure.

Proof. There is a basis \( \{e_1, \ldots, e_6\} \) of \( g \) which satisfies:
\[
[e_1, e_2] = -e_4, \quad [e_1, e_3] = -e_5, \quad [e_2, e_4] = e_5, \quad [e_1, e_4] = [e_2, e_3] = -e_6.
\]
Note that \( \mathfrak{z}(g) = \text{span}\{e_5, e_6\} \).

Let us suppose that \( g \) admits a complex product structure \( \{J, E\} \) with associated double Lie algebra \( (g, g_+, g_-) \). It is known that \( g \) admits both abelian and non abelian complex product structures (see [14, 26]). In the former case we have that the associated double Lie algebra is isomorphic to \( (g, \mathbb{R}^3, \mathbb{R}^3) \), whereas in the latter case it is isomorphic to either \( (g, \mathfrak{h}_3, \mathbb{R}^3) \) or \( (g, \mathfrak{h}_3, \mathfrak{h}_3) \). However, according to Proposition 5.1, we may simply suppose that it is isomorphic to \( (g, \mathfrak{h}_3, \mathbb{R}^3) \). Therefore in any case we have that at least one of the subalgebras \( g_\pm \) is abelian; we may assume hence that \( g_- \) is abelian. From Lemmas 4.2 and 4.4, we can choose a basis \( \{x, y, z\} \) of \( g_- \) with \( z \in \mathfrak{z}(g) \). If we write \( x = \sum x_ie_i, \quad y = \sum y_ie_i \) with \( x_i, y_i \in \mathbb{R}, i = 1, \ldots, 6 \), then from \( [x, y] = 0 \) we obtain the equations
\[
\begin{align*}
x_1y_2 - x_2y_1 &= 0, \\
x_1y_3 - x_3y_1 - x_2y_4 + x_4y_2 &= 0, \\
x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2 &= 0.
\end{align*}
\]
(11)

Suppose first \( x_1 = x_2 = y_1 = y_2 = 0 \); thus \( e_1, e_2 \in g_+ \) and since \( g_+ \) is a subalgebra, we have that \( e_4 = -[e_1, e_2] \in g_+ \). But then \( e_6 = -[e_1, e_4] \) is also in \( g_+ \), which contradicts the fact that \( \dim g_+ = 3 \). Hence, \( x_1^2 + x_2^2 + y_1^2 + y_2^2 \neq 0 \). Now, from the first equation in (11), we have that \( (y_1, y_2) = \alpha(x_1, x_2) \), \( \alpha \in \mathbb{R} \), where we can assume without loss of generality that \( x_1^2 + x_2^2 \neq 0 \). Then, substituting \( y \) by \( y - \alpha x \), we are allowed to consider \( y_1 = y_2 = 0 \).
Using this in (11), we get
\[
\begin{align*}
x_1y_3 - x_2y_4 &= 0, \\
x_1y_4 + x_2y_3 &= 0.
\end{align*}
\]
As \( x_1^2 + x_2^2 \neq 0 \), we have that \( y_3 = y_4 = 0 \), and hence \( y \in \mathfrak{z}(g) \). So, \( \mathfrak{z}(g) \) is contained in \( g_- \), but this contradicts Theorem 4.4. Therefore, there is no complex product structure on \( g \).

Remark. The Lie algebra \( g = (0, 0, 0, 12, 13 + 42, 14 + 23) \) is an example of a Lie algebra that admits both complex and product structures, but does not admit any complex product structure. In fact, the almost complex structure \( J \) on \( g \) given by
\[
J e_1 = e_2, \quad J e_3 = -e_4, \quad J e_5 = e_6,
\]
is integrable, (in fact, it is abelian, see [14]). Also, \( g \) admits paracomplex structures. One example of such a structure is given by the following decomposition of \( g \) as the sum of two 3-dimensional subalgebras \( g = g_+ \oplus g_- \), where
\[
g_+ = \text{span}\{e_1, e_3, e_5\}, \quad g_- = \text{span}\{e_1 + e_2, e_3 + e_4, e_6\}.
\]

6. Associated torsion-free connections

In this section we will determine when the torsion-free connection \( \nabla^{\text{CP}} \) associated to a complex product structure on the Lie algebras classified before are flat, that is, they are LSA structures on these Lie algebras. In what follows, we will use the following notation: as \( \nabla^{\text{CP}} \) is parallel with respect to both \( J \) and \( E \), then each endomorphism \( \nabla_x^{\text{CP}} \) will be of
the form \( \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \), where \( X \in \mathfrak{gl}(3, \mathbb{R}) \), with respect to suitable bases of \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \), so that we will simply write \( \nabla^{\mathbb{C}} = X \). We recall also the definition of \( \nabla^{\mathbb{C}} \), which is given as follows:

\[
\begin{cases}
\nabla_{x_+}^{\mathbb{C}} y_+ = -\pi_+ J [x_+, J y_+], \\
\nabla_{x_-}^{\mathbb{C}} y_- = -\pi_+ J [x_-, J y_-], \\
\nabla_{x_+}^{\mathbb{C}} y_- = \pi_- [x_+, y_-], \\
\nabla_{x_-}^{\mathbb{C}} y_+ = \pi_- [x_-, y_+],
\end{cases}
\]

(12)

for any \( x_+, y_+ \in \mathfrak{g}_+ \), \( x_-, y_- \in \mathfrak{g}_- \), where \( \pi_\pm : \mathfrak{g} \to \mathfrak{g}_\pm \) are the projections (see [1, 4]).

First, let us state a general result on the torsion-free connection associated to a complex product structure \( \{J, E\} \), when the product structure \( E \) is replaced by another product structure in \( \{\cos \theta E + \sin \theta F : \theta \in [0, 2\pi)\} \). This result holds also for complex product structures on manifolds, with exactly the same proof.

**Proposition 6.1.** Let \( \{J, E\} \) be a complex product structure on \( \mathfrak{g} \) and let \( \nabla := \nabla^{\mathbb{C}} \) be its associated torsion-free connection. Let \( E_\theta := \cos \theta E + \sin \theta J E \); if \( \nabla^\theta \) denotes the torsion-free connection associated to the complex product structure \( \{J, E^\theta\} \), then \( \nabla^\theta = \nabla \).

**Proof.** Let us recall that \( \nabla^\theta \) is the only torsion-free connection on \( \mathfrak{g} \) such that \( \nabla^\theta J = \nabla^\theta E_\theta = 0 \). Let us compute now the following:

\[
\nabla_x E_\theta y = \nabla_x (\cos \theta E + \sin \theta J E) y = \cos \theta \nabla_x E y + \sin \theta \nabla_x J E y = \\
= \cos \theta E \nabla_x y + \sin \theta J E \nabla_x y = (\cos \theta E + \sin \theta J E) \nabla_x y = E_\theta \nabla_x y,
\]

so that \( \nabla E_\theta = 0 \). As also \( \nabla J = 0 \), we obtain that both connections coincide. \( \square \)

As a consequence, from this Proposition together with Proposition [5, 1] we obtain that we have to consider only the complex product structures whose associated double Lie algebra is either \( (\mathfrak{g}, \mathbb{R}^3, \mathbb{R}^3) \) or \( (\mathfrak{h}_3, \mathbb{R}^3) \). We will do this in each of the two main cases: when \( \tilde{\mathfrak{g}} = \mathfrak{h}_3 \times \mathbb{R} \) or when \( \tilde{\mathfrak{g}} = \mathbb{R}^4 \). Let us begin now studying each case separately.

6.1. **First case:** \( \tilde{\mathfrak{g}} = \mathfrak{h}_3 \times \mathbb{R} \).

(i) \( \alpha = 0 \), \( \beta = 0 \). In this case we can see that

\[ \nabla_{e_1}^{\mathbb{C}} = \begin{pmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ D & F & 0 \end{pmatrix}, \quad \nabla_{e_2}^{\mathbb{C}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & 0 & 0 \end{pmatrix}, \]

\[ \nabla_{f_1}^{\mathbb{C}} = \begin{pmatrix} 0 & 0 & 0 \\ -A & 0 & 0 \\ -B & -E & 0 \end{pmatrix}, \quad \nabla_{f_2}^{\mathbb{C}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -E & 0 & 0 \end{pmatrix} \]

and \( \nabla_{e_3}^{\mathbb{C}} = \nabla_{f_3}^{\mathbb{C}} = 0 \). Therefore, it is readily verified that

\[ R(e_1, f_1)e_1 = -2(AF - CE)e_3, \quad R(e_1, f_1)f_1 = -2(AF - CE)f_3, \]

and all the other possibilities equal to zero. Thus, \( \nabla^{\mathbb{C}} \) is flat if and only if

\[ AF = CE. \]
(ii) $\alpha = 1, \beta = 0$. In this case, we see that

$$\nabla_{\mathbf{e}_1}^{\text{CP}} = \begin{pmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ D & F + 1 & 0 \end{pmatrix}, \quad \nabla_{\mathbf{e}_2}^{\text{CP}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & 0 & 0 \end{pmatrix},$$

$$\nabla_{\mathbf{f}_1}^{\text{CP}} = \begin{pmatrix} 0 & 0 & 0 \\ -A & 0 & 0 \\ -B & -E & 0 \end{pmatrix}, \quad \nabla_{\mathbf{f}_2}^{\text{CP}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -E & 0 & 0 \end{pmatrix},$$

and $\nabla_{\mathbf{e}_3}^{\text{CP}} = \nabla_{\mathbf{f}_3}^{\text{CP}} = 0$. One can easily see that

$$R(e_1, f_1)e_1 = -(2(AF - CE) + A)e_3, \quad R(e_1, f_1)f_1 = -(2(AF - CE) + A)f_3,$$

so that $\nabla^{\text{CP}}$ is flat if and only if

$$A(2F + 1) = 2CE.$$

6.2. Second case: $\tilde{\mathfrak{g}} = \mathbb{R}^4$. In this case we can prove that $\nabla^{\text{CP}}$ is always flat, irrespective of the values of $\alpha$ and $\beta$.

**Proposition 6.2.** If $\tilde{\mathfrak{g}} \cong \mathbb{R}^4$, then the torsion-free connection $\nabla^{\text{CP}}$ on $\mathfrak{g}$ associated to the complex product structure on $\mathfrak{g}$ is flat.

**Proof.** From the equations (9) and the definition of $\nabla^{\text{CP}}$ (see (12)), we can easily deduce that

$$\nabla^{\text{CP}}_x y \in \mathfrak{u} \quad \text{for all } x, y \in \mathfrak{g}.$$  

Also, it is easy to verify that

$$\nabla^{\text{CP}}_{\mathfrak{u}} x = \nabla^{\text{CP}}_{x} \mathfrak{u} = 0 \quad \text{for all } \mathfrak{u} \in \mathfrak{u}, x \in \mathfrak{g}.$$  

These facts, together with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{u}$, imply that $\nabla^{\text{CP}}$ is flat. \hfill \Box

**Remark.** In the case $\tilde{\mathfrak{g}} = \mathbb{R}^4$, $\alpha = \beta = 0$, we could have proved the flatness of $\nabla^{\text{CP}}$ using the following general result:

**Proposition 6.3.** Any abelian complex product structure on a 2-step nilpotent Lie algebra is flat, i.e. the associated torsion-free connection $\nabla^{\text{CP}}$ is flat.

**Proof.** Although this result might be proved in a straightforward manner, we will give an indirect proof. From results in [4], we know that any complex product structure on a Lie algebra $\mathfrak{g}$ gives rise to a hypercomplex structure on $(\mathfrak{g}^C)_\mathbb{R}$. Furthermore, if the complex product structure is abelian, then the hypercomplex structure is also abelian (meaning that each complex structure is abelian). It is also known that $\nabla^{\text{CP}}$ (the torsion-free connection on $\mathfrak{g}$ associated to the complex product structure) is flat if and only if $\nabla^{\text{HC}}$ (the Obata connection on $(\mathfrak{g}^C)_\mathbb{R}$ associated to the hypercomplex structure) is flat. Now, the proposition follows since in [16] it has been proved that the Obata connection associated to an abelian hypercomplex structure on a 2-step nilpotent Lie algebra is flat. \hfill \Box

Combining the results obtained so far in this section with the results from §5.1 and §5.2, we obtain
Theorem 6.4. (i) The following Lie algebras admit only flat complex product structures:

\[(0, 0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 12, 14 + 23),\]
\[(0, 0, 0, 0, 12), \quad (0, 0, 0, 0, 12, 34),\]
\[(0, 0, 0, 0, 12 + 34), \quad (0, 0, 0, 12, 13, 23),\]
\[(0, 0, 0, 0, 12, 13), \quad (0, 0, 0, 12, 13, 14 + 23).\]

(ii) The following Lie algebras admit only non flat complex product structures:

\[(0, 0, 0, 12, 14, 24), \quad (0, 0, 0, 12, 24).\]

(iii) The following Lie algebras admit both flat and non flat complex product structures:

\[(0, 0, 0, 12, 14 + 25), \quad (0, 0, 0, 12, 13 + 14, 24),\]
\[(0, 0, 0, 12, 13, 14).\]

We shall consider now the question of the completeness of these connections and we will prove that they are always complete. We recall first that a left-invariant connection on a Lie group $H$ is complete if and only if the following differential equation on its Lie algebra $\mathfrak{h}$ admits solutions defined on the whole real line:

\[\dot{x}(t) = -\nabla_{x(t)} x(t),\]

where $\nabla$ denotes the bilinear form on $\mathfrak{h}$ induced by the connection (see for instance [10]).

Proposition 6.5. If $\nabla^{\text{CP}}$ is the torsion-free connection associated to a complex product structure on a 6-dimensional nilpotent Lie algebra, then $\nabla^{\text{CP}}$ is complete.

Proof. We will consider again different cases, according to the ones in §5.1 and §5.2.

First case: $\tilde{\mathfrak{g}} = \mathbb{R} \times \mathbb{R}$.

(i) $\alpha = 0, \beta = 0$. Suppose that the curve $x(t)$ in equation (13) is given by $x(t) = \sum a_i(t) e_i + \sum b_i(t) f_i$, with $a_i, b_i$ real valued functions defined on some interval of the real line. From the brackets given in (8) and the expressions for $\nabla^{\text{CP}}$ given earlier in this section, we obtain that the differential equation (13) yields the system

\[
\begin{align*}
\dot{a}_1 &= 0, \\
\dot{a}_2 &= -Ca_1^2 + Aa_1b_1, \\
\dot{a}_3 &= -Da_1^2 - 2Fa_1a_2 + Ba_1b_1 + Ea_2b_1 + Ea_1b_2, \\
\dot{b}_1 &= 0, \\
\dot{b}_2 &= -Ca_1b_1 + Ab_1^2, \\
\dot{b}_3 &= -Da_1b_1 - Fa_1b_2 - Fa_2b_1 + Bb_1^2 - 2Eb_1b_2.
\end{align*}
\]

From this it follows immediately that $a_1$ and $b_1$ are constant functions, $a_2, b_2$ are linear functions and $a_3, b_3$ are quadratic functions, all of them defined on the whole real line. Thus, the connection $\nabla^{\text{CP}}$ in this case is complete.

(ii) $\alpha = 1, \beta = 0$. From the brackets given in (8) and the expressions for $\nabla^{\text{CP}}$ given early in this section, we obtain that the differential equation (13) yields a system similar to the one above, and it is easy to verify that its solutions are all polynomials of degree $\leq 2$, and hence the connection is complete. We omit the details.

Second case: $\tilde{\mathfrak{g}} = \mathbb{R}^4$. We know from Proposition 5.2 that $\nabla^{\text{CP}}$ is flat, so that it defines an LSA structure on $\mathfrak{g}$. This LSA structure is complete if and only the right multiplications $\rho(x)$ are nilpotent or, equivalently, if $\text{tr} \rho(x) = 0$ for all $x \in \mathfrak{g}$ (see [27]). Since $\nabla^{\text{CP}}$ is torsion-free, we have that $\text{ad}(x) = \nabla^{\text{CP}}_x - \rho(x)$ for all $x \in \mathfrak{g}$, but $\text{ad}(x)$ is
nilpotent and $\nabla^\text{CP}_x$ is traceless, so that $\text{tr} \rho(x) = 0$ for all $x \in \mathfrak{g}$, and therefore $\nabla^\text{CP}$ is complete.

\[ \square \]

6.3. Application: 12-dimensional hypercomplex nilpotent Lie algebras. In \[4\] it was proved that if a Lie algebra $\mathfrak{g}$ carries a complex product structure, then its complexification considered as a real Lie algebra, i.e. $(\mathfrak{g}^\mathbb{C})_\mathbb{R}$, is endowed with a hypercomplex structure. It was shown also that if $\nabla^\text{CP}$ stands for the torsion-free connection on $\mathfrak{g}$ associated to the complex product structure, its natural extension to $(\mathfrak{g}^\mathbb{C})_\mathbb{R}$ coincides with $\nabla^{\text{HC}}$, the Obata connection associated to this hypercomplex structure. Therefore, $\nabla^{\text{HC}}$ is flat if and only $\nabla^\text{CP}$ is flat (see \[4\]).

Applying this to the 6-dimensional nilpotent Lie algebras equipped with a complex product structure, we obtain a list of 12-dimensional hypercomplex nilpotent Lie algebras. The associated Obata connections may be flat or non flat, depending on the flatness of the connection associated to the complex product structure, but they are always Ricci-flat, due to Corollary 3.6.

All these nilpotent Lie algebras have rational structure constants, so that, by a theorem of Malcev, the corresponding simply connected nilpotent Lie groups admit lattices, and hence the associated nilmanifolds admit hypercomplex structures invariant by the action of the group. If we take one of these nilmanifolds with hypercomplex structure such that $\nabla^{\text{HC}}$ is non flat, then we obtain a hypercomplex nilmanifold with holonomy contained in $\text{SL}(n, \mathbb{H})$, since the Ricci tensor vanishes. It was proved in \[28\] that the holonomy of the Obata connection of an abelian hypercomplex structure on a nilmanifold is always contained in $\text{SL}(n, \mathbb{H})$ where $4n$ is the dimension of the nilmanifold. Here we can produce examples of hypercomplex nilmanifolds with holonomy of the Obata connection also contained in $\text{SL}(n, \mathbb{H})$ (since it is Ricci-flat) but with non abelian complex structures. Note that none of these hypercomplex structures underlies a hyperKähler structure on the nilmanifold, since there are no Kähler metrics on nilmanifolds, except for the torus.

Example 6.6. Consider the Lie algebra $\mathfrak{g} = (0, 0, 0, 12, 13, 14)$; it has a basis $\{e_1, \ldots, e_6\}$ such that

$[e_1, e_2] = -e_4, \quad [e_1, e_3] = -e_5, \quad [e_1, e_4] = -e_6.$

According to Theorem 6.4, $\mathfrak{g}$ admits both flat and non flat complex product structures. An example of the former kind is given by $\{J, E_1\}$, with

$J e_1 = -e_2, \quad J e_3 = -e_4, \quad J e_5 = -e_6; \quad E_1|_{\mathfrak{g}_+} = 1, \quad E_1|_{\mathfrak{g}_-} = -1,$

where $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are the Lie subalgebras of $\mathfrak{g}$ given by

$\mathfrak{g}_+ = \text{span}\{e_1, e_3, e_5\}, \quad \mathfrak{g}_- = \text{span}\{e_2, e_4, e_6\},$

while an example of the latter kind is given by $\{J, E_2\}$, with the same $J$ as above and $E_2|_{\mathfrak{g}_+'} = 1, \quad E_2|_{\mathfrak{g}_-'} = -1$, where $\mathfrak{g}_+'$ and $\mathfrak{g}_-'$ are the Lie subalgebras of $\mathfrak{g}$ given by

$\mathfrak{g}_+' = \text{span}\{e_1, e_4, e_6\}, \quad \mathfrak{g}_-' = \text{span}\{e_2, e_3, e_5\}.$

From Theorem 3.3 in \[4\], it follows that each complex product structure $\{J, E_k\}$ gives rise to a hypercomplex structure $\{(J, \mathbb{I}_k)\}$, $k = 1, 2$, on the Lie algebra $\hat{\mathfrak{g}} = (\mathfrak{g}^\mathbb{C})_\mathbb{R}$. This Lie algebra has a basis $\{e_1, \ldots, e_6, \hat{e}_1, \ldots, \hat{e}_6\}$ such that

$[e_1, e_2] = [-\hat{e}_1, \hat{e}_2] = -e_4, \quad [e_1, e_3] = [-\hat{e}_1, \hat{e}_3] = -e_5, \quad [e_1, e_4] = [-\hat{e}_1, \hat{e}_4] = -e_6,$

$[\hat{e}_1, e_2] = [e_1, \hat{e}_2] = -\hat{e}_4, \quad [e_1, e_3] = [e_1, \hat{e}_3] = -\hat{e}_5, \quad [\hat{e}_1, e_4] = [e_1, \hat{e}_4] = -\hat{e}_6.$
The hypercomplex structure $\{J, I_1\}$ on $\mathfrak{g}$ is given by

$$
\begin{align*}
J e_1 &= -e_2, \quad J e_3 = -e_4, \quad J e_5 = -e_6, \quad J \hat{e}_1 = -\hat{e}_2, \quad J \hat{e}_3 = -\hat{e}_4, \quad J \hat{e}_5 = -\hat{e}_6, \\
I_1 e_1 &= \hat{e}_1, \quad I_1 e_3 = \hat{e}_3, \quad I_1 e_5 = \hat{e}_5, \quad I_1 e_2 = -\hat{e}_2, \quad I_1 e_4 = -\hat{e}_4, \quad I_1 e_6 = -\hat{e}_6,
\end{align*}
$$

while, on the other hand, the hypercomplex structure $\{J, I_2\}$ on $\mathfrak{g}$ is given by

$$
\begin{align*}
I_2 e_1 &= \hat{e}_1, \quad I_2 e_4 = \hat{e}_4, \quad I_2 e_6 = \hat{e}_6, \quad I_2 e_2 = -\hat{e}_2, \quad I_2 e_3 = -\hat{e}_3, \quad I_2 e_5 = -\hat{e}_5,
\end{align*}
$$

and $J$ as above. Due to the choice of the complex product structures, the Obata connection associated to $\{J, I_1\}$ is flat, whereas the Obata connection associated to $\{J, I_2\}$ is non-flat but Ricci-flat. Note that these hypercomplex structures are not abelian, since the Lie algebra $\mathfrak{g}$ admits no abelian complex product structure (see [5.3]).

7. Example: The 3-dimensional complex Heisenberg Lie group

We consider the Lie algebra $\mathfrak{g} = \{0, 0, 0, 0, 13 + 42, 14 + 23\}$, which has a basis $\{e_1, \ldots, e_6\}$ such that $[e_1, e_3] = [e_2, e_4] = -e_5, [e_1, e_4] = [e_2, e_3] = -e_6$. This Lie algebra admits a matrix realization as $(3 \times 3)$ complex matrices in the following way:

$$
\mathfrak{g} = \left\{ \begin{pmatrix} 0 & z_1 & z_3 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\},
$$

where we can identify

$$
e_1 = E_{1,2}, \quad e_2 = ie_1, \quad e_3 = E_{2,3}, \quad e_4 = ie_3, \quad e_5 = -E_{1,3}, \quad e_6 = ie_5,
$$

with $E_{r,s}$ the $(3 \times 3)$ complex matrix whose only non-zero element is in the $(r, s)$ position and is equal to 1. The corresponding 6-dimensional simply connected nilpotent Lie group $G$ can be described as a matrix group as follows:

$$
G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\}.
$$

That is, $G$ is the 3-dimensional complex Heisenberg Lie group, considered as a real Lie group.

(i) Let us consider the complex product structure $\{J, E\}$ on $\mathfrak{g}$ given by

$$
J e_1 = e_3, \quad J e_2 = e_4, \quad J e_5 = e_6; \quad E|_{\mathfrak{g}_+} = 1, \quad E|_{\mathfrak{g}_-} = -1,
$$

where $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are the Lie subalgebras of $\mathfrak{g}$ given by

$$
\mathfrak{g}_+ = \text{span}\{e_1, e_2, e_5\}, \quad \mathfrak{g}_- = \text{span}\{e_3, e_4, e_6\}.
$$

Note that both subalgebras are abelian, so that $\{J, E\}$ is an abelian complex product structure on $\mathfrak{g}$. The connected subgroups of $G$ corresponding to $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are given by

$$
G_+ = \left\{ \begin{pmatrix} 1 & z_1 & x_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z_1 \in \mathbb{C}, x_3 \in \mathbb{R} \right\}, \quad G_- = \left\{ \begin{pmatrix} 1 & 0 & iy_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_2 \in \mathbb{C}, y_3 \in \mathbb{R} \right\}.
$$

Since $G$ is nilpotent and simply connected, the subgroups $G_+$ and $G_-$ are also simply connected and closed in $G$. Moreover, it is easy to verify in this case that $G = G_+ \cdot G_-$, i.e., the multiplication on $G$ defines a diffeomorphism $G_+ \times G_- \rightarrow G$, $(g_+, g_-) \mapsto g_+ g_-; \therefore (G, G_+, G_-)$ is a global double Lie group, with $G_+ \cong \mathbb{R}^3 \cong G_-$ (see [28]).
(ii) Let us consider now the complex product structure \( \{ J, E \} \) on \( g \) given by

\[
J e_1 = -(e_4 - e_2), \quad J e_2 = -(e_1 + e_3), \quad J e_5 = e_6; \quad E|_{g_+} = 1, \quad E|_{g_-} = -1,
\]

where \( g_+ \) and \( g_- \) are the Lie subalgebras of \( g \) given by

\[
g_+ = \text{span}\{e_1, e_2, e_5\}, \quad g_- = \text{span}\{e_1 + e_3, e_4 - e_2, e_6\}.
\]

Note that \( g_+ \) is abelian (and is the same subalgebra that appeared in the case (i)), whereas \( g_- \) is isomorphic to \( h_3 \). The connected subgroup of \( G \) corresponding to \( g_- \) appearing in the case (i), while, on the other hand, the subgroup \( G_- \) corresponding to \( g_- \) is

\[
G_- = \left\{ \begin{pmatrix} 1 & \frac{1}{2}z_2 & \frac{i}{2}|z_2|^2 + iy_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_2 \in \mathbb{C}, y_3 \in \mathbb{R} \right\}.
\]

In this case again we have again that \( G = G_+ \cdot G_- \), so that \( (G, G_+, G_-) \) is also a global double Lie group, with \( G_+ \cong \mathbb{R}^3 \), \( G_- \cong H_3 \).

(iii) Let us consider now the complex product structure \( \{ J, E \} \) on \( g \) given by

\[
J e_1 = -(e_4 - e_2), \quad J e_2 = -(e_1 + e_3), \quad J e_5 = e_6; \quad E|_{g_+} = 1, \quad E|_{g_-} = -1,
\]

where \( g_+ \) and \( g_- \) are the Lie subalgebras of \( g \) given by

\[
g_+ = \text{span}\{e_1 + e_2 + e_3, e_1 - e_2 + e_4, e_5 - e_6\}, \quad g_- = \text{span}\{-e_1 + e_2 - e_3, e_1 + e_2 - e_4, e_5 + e_6\}.
\]

Note that both subalgebras are isomorphic to \( h_3 \). The connected subgroups of \( G \) corresponding to \( g_+ \) and \( g_- \) are given by

\[
G_+ = \left\{ \begin{pmatrix} 1 & (1 + i)\overline{z} & \frac{1}{2}|z|^2 + t + i\left(\frac{1}{2}|z|^2 - t\right) \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{C}, t \in \mathbb{R} \right\},
\]

\[
G_- = \left\{ \begin{pmatrix} 1 & (1 - i)\overline{z} & \frac{1}{2}|z|^2 + t - i\left(\frac{1}{2}|z|^2 - t\right) \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{C}, t \in \mathbb{R} \right\}.
\]

In this case we have again that \( G = G_+ \cdot G_- \), so that \( (G, G_+, G_-) \) is also a global double Lie group, with both \( G_+ \) and \( G_- \) isomorphic to \( H_3 \).

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