INVARIANT RADON MEASURES AND MINIMAL SETS FOR COMMUTATIVE SUBGROUPS OF Homeo$_+$(\mathbb{R})

HUI XU & ENHUI SHI & YIRUO WANG

Abstract. For every $\alpha > 0$, we show that every $C^{1+\alpha}$ commutative subgroup of Homeo$_+$(\mathbb{R}) has an invariant Radon measure and has a minimal closed invariant set. A counterexample of $C^1$ commutative subgroup of Homeo$_+$(\mathbb{R}) is constructed.

1. Introduction

In the theory of dynamical systems, the following two facts are well known:

(1) if $G$ is a group consisting of homeomorphisms on a compact metric space $X$, then $G$ has a minimal set $K$ in $X$, that is $K$ is minimal among all nonempty $G$-invariant closed subsets with respect to the inclusion relations on sets;

(2) if $G$ is an amenable group consisting of homeomorphisms on a compact metric space $X$, then $G$ has an invariant Borel probability measure on $X$.

In general, these two results do not hold if $X$ is not compact. However, if the topology of $X$ is very constrained and the acting group $G$ possesses some specified structures, then the existence of invariant Radon measures (Borel measures which are finite on every compact set) or minimal sets can still be true, even if $X$ is noncompact.

When $X$ is the real line $\mathbb{R}$ and $\Gamma$ is a finitely generated virtually nilpotent subgroup, Plante obtained the following theorem in [3].

Theorem 1.1. If $\Gamma$ is a finitely generated virtually nilpotent subgroup of Homeo$_+$(\mathbb{R}), then $\Gamma$ preserves a Radon measure on the line.

Here, Homeo$_+$(\mathbb{R}) means the orientation preserving homeomorphism group on $\mathbb{R}$. The following theorem appears in A. Navas’ book (see Prop. 2.1.12 in [1]).

Theorem 1.2. Every finitely generated subgroup of Homeo$_+$(\mathbb{R}) admits a nonempty minimal invariant closed set.

Date: November 5, 2019.
In this paper, we are interested in the commutative subgroup of Homeo⁺(ℝ), which consists of \( C^{1+\alpha} \) homeomorphisms with \( \alpha \geq 0 \), and get the following theorem.

**Theorem 1.3.** For every \( \alpha > 0 \) and every \( C^{1+\alpha} \) commutative subgroup \( H \) of Homeo⁺(ℝ), \( H \) has an invariant Radon measure and has a minimal closed invariant set.

We should note that there is no requirement of finite generation or even countability for \( H \) appearing in Theorem 1.3. This is the key point that differs from Theorem 1.1 and Theorem 1.2.

The strategy to prove Theorem 1.3 is to establish a combinatorial lemma in Section 3 which shows the equivalence between the existence of invariant measures, the existence of minimal sets, and the nonexistence of an infinite tower covering the whole line (see Section 2 for the definition). This together with a generalized version of Kopell’s lemma due to A. Navas implies the conclusion.

As a supplement of Theorem 1.3, we construct an abelian subgroup of Homeo⁺(ℝ) consisting of \( C^1 \) diffeomorphisms in Section 4, which has neither invariant Radon measure nor minimal set. Certainly, such groups can’t be finitely generated by Theorem 1.1 and Theorem 1.2.

2. **Notions and Auxiliary Lemmas**

In this section, we give some definitions and lemmas which will be used in the proof of the main theorem.

Let \( G \) be a subgroup of Homeo⁺(ℝ). For \( x \in ℝ \), we denote the orbit of \( x \) by \( Gx = \{ g(x) : g \in G \} \). For \( g \in G \), we denote by \( \text{Fix}(g) \) the set of fixed points of \( g \) and denote by \( \text{Fix}(G) \) the set of global fixed points of \( G \). For commuting homeomorphisms, we recall the following facts that we will used frequently.

**Fact 2.1.** Let \( f \) and \( g \) be two commuting homeomorphisms on a space \( X \). Then \( g(\text{Fix}(f)) \subseteq \text{Fix}(f) \).

**Proof.** For any \( x \in \text{Fix}(f) \), \( f(g(x)) = g(f(x)) = g(x) \). Thus \( g(x) \in \text{Fix}(f) \). Hence \( g(\text{Fix}(f)) \subseteq \text{Fix}(f) \). \( \square \)

From Fact 2.1 we immediately get the following

**Fact 2.2.** Let \( f \) and \( g \) be two commuting homeomorphisms in Homeo⁺(ℝ) with \( \text{Fix}(f) \neq \emptyset \), and let \( (\alpha, \beta) \) be a connected component of \( ℝ \setminus \text{Fix}(f) \). Then either \( g((\alpha, \beta)) = (\alpha, \beta) \) or \( g((\alpha, \beta)) \cap (\alpha, \beta) = \emptyset \).
Fact 2.3. Let $G$ be a subgroup of $\text{Homeo}_+ (\mathbb{R})$. For any $x \in \mathbb{R}$, set

\[ \alpha := \inf \{ Gx \}, \quad \beta := \sup \{ Gx \}. \]

Then either $\alpha = -\infty$ (resp. $\beta = +\infty$) or $\alpha \in \text{Fix}(G)$ (resp. $\beta \in \text{Fix}(G)$).

Proof. We may assume that $\alpha \neq -\infty$. Then for any $g \in G$,

\[ g(\alpha) \geq \alpha, \quad \text{and} \quad g^{-1}(\alpha) \geq \alpha \implies g(\alpha) \leq \alpha. \]

Hence $g(\alpha) = \alpha$. It is similar for $\beta$. \hfill \Box

Definition 2.4. If $\{I_i\}_{i=1}^{\infty}$ is a sequence of closed intervals such that $I_1 \subsetneq I_2 \subsetneq \cdots$, and $\{f_i\}_{i=1}^{\infty}$ is a sequence of orientation preserving homeomorphisms on $\mathbb{R}$ such that $\text{Fix}(f_i) \cap I_i = \text{End}(I_i)$ for each $i$, where $\text{End}(I_i)$ denotes the endpoint set of interval $I_i$, then we call the sequence of pairs $\{(I_i, f_i)\}_{i=1}^{\infty}$ an infinite tower.

Lemma 2.5. Let $H$ be a commutative subgroup of $\text{Homeo}_+ (\mathbb{R})$. If for every $f \in H$, $\text{Fix}(f) \neq \emptyset$ and there is no global fixed point for $H$, then there exists an infinite tower $\{(I_i, f_i)\}_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} I_i = \mathbb{R}$. 

Proof. Since $\text{Fix}(H) = \emptyset$, we can choose $f_1 \in H, f_1 \neq \text{id}$. Since $f_1$ is nontrivial, we have $\text{Fix}(f_1) \subsetneq \mathbb{R}$. Then $\mathbb{R} \setminus \text{Fix}(f_1)$ is a nonempty open set. Take a connected component $(\alpha_1, \beta_1)$ of $\mathbb{R} \setminus \text{Fix}(f_1)$.

We claim that $-\infty < \alpha_1 < \beta_1 < +\infty$. In fact, since $\text{Fix}(f_1) \neq \emptyset$, at least one of $\alpha_1, \beta_1$ is finite. We may assume that $\alpha_1 \in \mathbb{R}$. Since $H$ has no global fixed point, by Fact 2.3, there exists $f_2 \in H \setminus \{f_1\}$ such that $f_2(\alpha_1) > \max \{\alpha_1, 2\}$. Since $f_1$ commutes with $f_2$, $f_2((\alpha_1, \beta_1)) \cap (\alpha_1, \beta_1) = \emptyset$ by Fact 2.2. Therefore, $\beta_1 < f_2(\alpha_1) < +\infty$.

Set $\alpha_2 = \inf \{f_2^i(\alpha_1) : i \in \mathbb{Z}\}$ and $\beta_2 = \sup \{f_2^i(\alpha_1) : i \in \mathbb{Z}\}$. Then either $\alpha_2 \neq -\infty$ or $\beta_2 \neq +\infty$ by the assumption that $\text{Fix}(f) \neq \emptyset$ for every $f \in H$. Similar to the argument of the previous claim, we have $\alpha_2 \in \mathbb{R}$ and $\beta_2 \in \mathbb{R}$. Then $\alpha_2 < \alpha_1 < \beta_1 < \beta_2$ and $\text{Fix}(f_2) \cap [\alpha_2, \beta_2] = \{\alpha_2, \beta_2\}$ and $\beta_2 > 2$.

Similar to the above arguments, we can get $\alpha_3, \beta_3 \in \mathbb{R}$ and $f_3 \in H$ such that $\alpha_3 < \alpha_2 < \beta_2 < \beta_3$, and $\text{Fix}(f_3) \cap [\alpha_3, \beta_3] = \{\alpha_3, \beta_3\}$, and $\alpha_3 < -3$.

Continuing this process, we obtain a nested closed intervals $[\alpha_1, \beta_1] \subsetneq [\alpha_2, \beta_2] \subsetneq \cdots$ and a sequence $f_1, f_2, \cdots \in H$ such that

$\text{Fix}(f_i) \cap [\alpha_i, \beta_i] = \{\alpha_i, \beta_i\}, i = 1, 2, \ldots,$

and $\alpha_{2i-1} < -(2i-1)$ and $\beta_{2i} > 2i$ for each $i > 0$. Set $I_i = [\alpha_i, \beta_i]$. Then $\{(I_i, f_i)\}_{i=1}^{\infty}$ is the desired infinite tower. \hfill \Box
Lemma 2.6. Let $f$ be an orientation preserving homeomorphism of $\mathbb{R}$. If $f$ has no fixed point, then $f$ or $f^{-1}$ is topologically conjugate to the unit translation $L_1 : x \mapsto x + 1$, by an orientation preserving homeomorphism.

Proof. We may assume that $f(0) > 0$, otherwise replace $f$ by $f^{-1}$. We need to construct an orientation preserving homeomorphism $\varphi$ of $\mathbb{R}$ such that $\varphi f = L_1 \varphi$. Since $f$ has no fixed point, we have $(f^n(0), f^{n+1}(0))$ are pairwise disjoint and $\bigcup_{n \in \mathbb{Z}} [f^n(0), f^{n+1}(0)] = \mathbb{R}$. Let $\varphi$ be the homeomorphism on $\mathbb{R}$ such that $\varphi | [0, f(0)] : [0, f(0)] \to [0, 1]$ is affine and $\varphi(x) = \varphi(f^{-n}x) + n$ for $x \in [f^n(0), f^{n+1}(0)]$, $n \in \mathbb{Z}$. Then $\varphi$ satisfies the requirement. □

Lemma 2.7. Let $X$ be a compact metric space and let $G$ be a commutative subgroup of $\text{Homeo}(X)$. Then there is a $G$ invariant Borel probability measure on $X$.

Proof. Let $\mathcal{M}(X)$ be the set of all Borel probability measures on $X$. Then $\mathcal{M}(X)$ is compact under the weak-$*$ topology. For any finite subset $F$ of $G$, since the group $\langle F \rangle$ generated by $F$ is a countable commutative group, we have that the set of common invariant measures of $F$, $\text{Inv}(F) \neq \emptyset$, by the amenability of $\langle F \rangle$. Thus the family of closed sets $\{\text{Inv}(g) : g \in G\}$ has the finite intersection property. By the compactness of $\mathcal{M}(X)$, we have $\text{Inv}(G) = \cap_{g \in G} \text{Inv}(g) \neq \emptyset$. □

Lemma 2.8. Let $G$ be a commutative subgroup of $\text{Homeo}_+(\mathbb{R})$. If $G$ commutes with the unit translation $L_1$ of the real line, then $G$ naturally induces an action on the circle $S^1$. Moreover, every invariant Borel probability measure and every minimal subset of $S^1$ under the induced action can be lifted to an invariant Radon measure and a minimal set of $\mathbb{R}$ under the $G$ action respectively.

Proof. Let $\pi : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ be the standard quotient map. Since $G$ commutes with $L_1$, we can define a $G$ action on $S^1$ by

$$(2.1) \quad g \cdot (x + \mathbb{Z}) := g(x) + \mathbb{Z}, \ x \in \mathbb{R}.$$

By Lemma 2.7 there exists a $G$ invariant Borel probability measure $\nu$ on $S^1$. We need to show that $\nu$ can be lifted to a $G$ invariant Radon measure $\mu$ on $\mathbb{R}$. For any Borel measurable set $E$ of $\mathbb{R}$, define

$$(2.2) \quad \mu(E) = \sum_{k \in \mathbb{Z}} \nu \left( \pi \left( E \cap [k, k + 1) \right) \right).$$

From the definition, we immediately have that for any $t \in \mathbb{R}$,

$$(2.3) \quad \mu(E) = \sum_{k \in \mathbb{Z}} \nu \left( \pi \left( L^k_t(E) \cap [t, t + 1) \right) \right).$$
Since $G$ commutes with $L_1$, for any $g \in G$ and $x \in \mathbb{R}$,
\begin{equation}
(2.4) \quad g(x + 1) - g(x) = gL_1(x) - g(x) = L_1g(x) - g(x) = 1.
\end{equation}
Thus, for any $g \in G$ and any Borel measurable set of $\mathbb{R}$,
\begin{align*}
\mu(gE) &= \sum_{k \in \mathbb{Z}} \nu\left(\pi\left(L_1^k(gE) \cap [0, 1]\right)\right) \text{ (by 2.3)} \\
&= \sum_{k \in \mathbb{Z}} \nu\left(\pi\left(gL_1^kE \cap [0, 1]\right)\right) \text{ (by commutativity)} \\
&= \sum_{k \in \mathbb{Z}} \nu\left(\pi\left(L_1^k\pi gE \cap g^{-1}([0, 1])\right)\right) \\
&= \sum_{k \in \mathbb{Z}} \nu\left(\pi\left(L_1^kE \cap g^{-1}(0), g^{-1}(0) + 1\right)\right) \text{ (by 2.4)} \\
&= \sum_{k \in \mathbb{Z}} \nu\left(\pi\left(L_1^kE \cap [g^{-1}(0), g^{-1}(0) + 1]\right)\right) \text{ (by the invariance of $\nu$)} \\
&= \mu(E) \text{ (by 2.3)}.
\end{align*}
Hence $\mu$ is $G$ invariant. On the other hand, if $E$ is compact, then the summation in 2.2 is finite. Since each summand in 2.2 is no greater than 1, we have $\mu(E)$ is bounded. Therefore, $\mu$ is a $G$ invariant Radon measure on $\mathbb{R}$.

Now let $\Lambda$ be a nonempty closed minimal subset of $\mathbb{S}^1$ for the induced $G$ action on $\mathbb{S}^1$. Define
\[ \tilde{\Lambda} = \pi^{-1}(\Lambda). \]
We claim that $\tilde{\Lambda}$ is a nonempty closed minimal subset of $\mathbb{R}$ for the $G$ action. By the minimality of $\Lambda$, for any $x \in \tilde{\Lambda}$,
\[ \Lambda = \{g(\pi(x)) : g \in G\} = \{\pi g(x) : g \in G\}. \]
This implies
\[ \tilde{\Lambda} = \pi^{-1}(\Lambda) = \{g(x) : g \in G\}. \]
Therefore, $\tilde{\Lambda}$ is a nonempty closed minimal subset of $\mathbb{R}$. \hfill \Box

3. Proof of the main theorem

In this section, we first establish a combinatorial lemma which is key for the proof of the main theorem.

**Lemma 3.1.** Let $G$ be a commutative subgroup of $\text{Homeo}_+(\mathbb{R})$. Then the following items are equivalent:
(1) there exists a $G$-invariant Radon measure;
(2) there exists a nonempty closed minimal set;
(3) there does not exist an infinite tower $\{(I_i, f_i)\}_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} I_i = \mathbb{R}$.

Proof. We show the lemma by proving $(1) \iff (3)$ and $(2) \iff (3)$.

$(1) \implies (3)$: Let $\mu$ be a $G$-invariant Radon measure on $\mathbb{R}$. If there exists an infinite tower $\{(I_i, f_i)\}_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} I_i = \mathbb{R}$, then there is $N \in \mathbb{N}^+$ such that $\mu(\text{int}(I_N)) > 0$. Let $B = \text{int}(I_N)$. By the definition of infinite tower, we see that $B, f_{N+1}(B), f_{N+1}^2(B), ...$ are pairwise disjoint and are all contained in $I_{N+1}$. Since $\mu$ is $G$-invariant, we have
$$
\mu(B) = \mu(f_{N+1}(B)) = \mu(f_{N+1}^2(B)) = \cdots
$$
and then
$$
\mu(I_{N+1}) \geq \sum_{i=0}^{\infty} \mu(f_{N+1}^i(B)) = \infty,
$$
which contradicts the assumption that $\mu$ is a Radon measure.

$(3) \implies (1)$:

**Case1** $\text{Fix}(G) \neq \emptyset$. Then take any fixed point $x \in \text{Fix}(G)$, the Dirac measure $\delta_x$ is a $G$-invariant Radon measure.

**Case2** $\text{Fix}(G) = \emptyset$. By Lemma 2.5, there exists $f \in G$ such that $\text{Fix}(f) = \emptyset$. By Lemma 2.6, $f$ (or $f^{-1}$) is topologically conjugate to the unit translation on the real line, i.e., there exists $\varphi \in \text{Homeo}_+(\mathbb{R})$, such that $\varphi f = L_1 \varphi$. Hence the group $\varphi G \varphi^{-1}$ commutes with $\varphi f \varphi^{-1} = L_1$. By Lemma 2.8, there exists a $\varphi G \varphi^{-1}$ invariant Radon measure $\mu$ on $\mathbb{R}$.

We claim that $\mu^* := \varphi^{-1}_{*\mu}$ is a $G$-invariant Radon measure on $\mathbb{R}$, where $\varphi^{-1}_{*\mu}(E) = \mu(\varphi(E))$, for any Borel measurable set $E$. This can be seen as follows
$$
\mu^*(gE) = \mu(\varphi g E) = \mu(\varphi g \varphi^{-1}(\varphi E)) = \mu(\varphi E) = \mu^*(E),
$$
for any $g \in G$ and any Borel measurable set $E$.

$(2) \implies (3)$: Assume that $\Lambda$ is a nonempty closed minimal subset of $\mathbb{R}$. Fix a point $x \in \Lambda$. If there exists an infinite tower $\{(I_i, f_i)\}_{i=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n = \mathbb{R}$, then there exists $N \in \mathbb{N}^+$ such that $x \in \text{int}(I_N)$. Write $I_N = [a, b]$. We may assume that $f_N(x) > x$, otherwise replace $f_N$ by $f_N^{-1}$. Then $\lim_{n \to +\infty} f_N^n(x) = b$. Then $b \in \text{Fix}(f_N)$, and $Gb \subseteq \text{Fix}(f_N)$ by Fact 2.1. Since $\text{Fix}(f_N) \cap (a, b) = \emptyset$, $x \notin Gb$, which contradicts the minimality of $\Lambda$. 

6
We may assume that $\text{Fix}(G) = \emptyset$, otherwise any point in $\text{Fix}(G)$ is a nonempty closed minimal set. By Lemma 2.5 there exists $f \in G$ such that $\text{Fix}(f) = \emptyset$. By Lemma 2.6 we may assume that $f$ is topologically conjugate to the unit translation on the real line. As showing in Case 2 in the proof of $(3) \implies (1)$, the group $\varphi G \varphi^{-1}$ commutes with $\varphi f \varphi^{-1} = L_1$. By Lemma 2.8 there is a nonempty closed minimal subset $\Lambda$ of $\mathbb{R}$ for $\varphi G \varphi^{-1}$. Then $\varphi^{-1}(\Lambda)$ is a nonempty closed minimal subset of $\mathbb{R}$ for $G$, since

$$\varphi^{-1}(\Lambda) = \varphi^{-1}\{\varphi g \varphi^{-1}(\varphi x) : g \in G\} = \{gx : g \in G\},$$

for any $x \in \varphi^{-1}(\Lambda)$.

The following theorem is due to A. Navas, which is a generalized version of Kopell’s lemma.

**Theorem 3.2.** ([2, Theorem B]) Let $\{I_{i_1,\cdots,i_{d+1}} : (i_1,\cdots,i_{d+1}) \in \mathbb{Z}^{d+1}\}$ be a family of subintervals of $[0,1]$ that are disposed respecting the lexicographic order. Assume that $f_1,\cdots,f_{d+1}$ are diffeomorphisms of class $C^{1+\alpha_1},\cdots,C^{1+\alpha_{d+1}}$ such that

$$f_j(I_{i_1,\cdots,i_{j-1},i_j,i_{j+1},\cdots,i_{d+1}}) = I_{i_1,\cdots,i_{j-1},i_j+1,i_{j+1},\cdots,i_{d+1}},$$

for all $1 \leq j \leq d+1$. Then $\alpha_1 + \cdots + \alpha_d < 1$.

**Proof of Theorem 1.3.** By Lemma 3.1 it suffices to show that there does not exist any infinite tower $\{(I_i,g_i)\}_{i=1}^\infty$ with $\bigcup_{i=1}^\infty I_i = \mathbb{R}$. Assume to the contrary that there exists such an infinite tower. For each $d > 0$, let

$$f_1 = g_{d+2}, f_2 = g_{d+1}, \cdots, f_{d+1} = g_2,$$

and

$$I_{i_1,\cdots,i_{d+1}} = f_1^{i_1} f_2^{i_2} \cdots f_{d+1}^{i_{d+1}}(I_1), \quad (i_1,\cdots,i_{d+1}) \in \mathbb{Z}^{d+1}.$$

Then $\{I_{i_1,\cdots,i_{d+1}} : (i_1,\cdots,i_{d+1}) \in \mathbb{Z}^{d+1}\}$ is a family of subintervals of $I := I_{d+3}$ and satisfies

$$f_j(I_{i_1,\cdots,i_{j-1},i_j,i_{j+1},\cdots,i_{d+1}}) = I_{i_1,\cdots,i_{j-1},i_j+1,i_{j+1},\cdots,i_{d+1}},$$

for all $1 \leq j \leq d+1$, by the definition of infinite tower. Applying Theorem 3.2 we have $d\alpha < 1$. This contradicts $\alpha > 0$, since $d$ is arbitrary. □
4. A Counterexample of $C^1$ Subgroup

In this section, we construct an example which shows that Theorem 1.3 does not hold for $C^1$ commutative subgroups of $\text{Homeo}_+(\mathbb{R})$. The following construction is due to Yoccoz ([4, Lemma 2.1]).

**Lemma 4.1.** For any closed intervals $I = [a, b], J = [c, d]$ there exists a $C^1$ orientation preserving diffeomorphism $\phi_{I,J} : I \rightarrow J$ with the following properties:

1. $\phi_{I,J}'(a) = \phi_{I,J}'(b) = 1$;
2. Given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in [a, b],
   \left| \phi_{I,J}(x) - 1 \right| < \varepsilon$, whenever $\left| \frac{d - c}{b - a} - 1 \right| < \delta$;
3. For any closed interval $K$ and for any $x \in I$,
   
   $\phi_{I,K}(x) = \phi_{J,K}(\phi_{I,J}(x))$.

**Theorem 4.2.** There exists a non-finitely generated abelian group $G$ consisting of $C^1$ orientation preserving diffeomorphisms of $\mathbb{R}$ such that there exists an infinite tower $\{(I_j, f_j)\}_{j=1}^{\infty}$ with $f_j \in G, j = 1, 2, \cdots$, such that $\bigcup_{j=1}^{\infty} I_j = \mathbb{R}$.

(Then, by Lemma 3.1 there exists neither $G$ invariant Radon measure nor nonempty closed minimal set.)

**Proof.** Firstly, we define $f_1 : [-1, 1] \rightarrow [-1, 1]$ by

$$
   f_1(x) = \begin{cases} 
   \exp \left( \frac{1}{x-1} - \frac{1}{x+1} \right) + x, & x \in (-1, 1) \\
   -1, & x = -1 \\
   1, & x = 1.
   \end{cases}
$$

Then $f_1$ satisfies

- $f_1$ is a $C^1$ orientation preserving diffeomorphism of $[-1, 1]$;
- $f_1(\pm 1) = \pm 1$ and $f_1(x) > x$ for any $x \in (-1, 1)$;
- $f_1'(1) = 1$.

Next, choose two infinite sequences $-2 < \cdots < a_2 < a_1 < a_0 = -1$ and $1 = b_0 < b_1 < b_2 < \cdots < 2$ such that

$$
   \lim_{n \rightarrow \infty} a_n = -2, \quad \lim_{n \rightarrow \infty} b_n = 2,
$$

and

$$
   \lim_{n \rightarrow \infty} \frac{a_{n-1} - a_n}{a_n - a_{n+1}} = 1, \quad \lim_{n \rightarrow \infty} \frac{b_{n+1} - b_n}{b_n - b_{n-1}} = 1.
$$

For example, we can take

$$
   a_n = -2 + \frac{1}{n+1}, \quad b_n = 2 - \frac{1}{n+1}, \quad n = 1, 2, \cdots.
$$
Define
\[
\begin{align*}
 f_2(x) &= \begin{cases} 
 \phi_{[a_{n+1}, a_n], [a_n, a_{n-1}]}(x), & x \in [a_{n+1}, a_n], n = 1, 2, \ldots \\
 \phi_{[a_1, a_0], [-1, 1]}(x), & x \in [a_1, a_0] \\
 \phi_{[-1, 1], [b_n, b_1]}(x), & x \in [-1, 1] \\
 \phi_{[b_n, b_{n+1}], [b_{n+1}, b_{n+2}]}(x), & x \in [b_n, b_{n+1}], n = 0, 1, 2, \ldots \\
 \pm 2, & x = \pm 2,
\end{cases}
\end{align*}
\]

Then, by Lemma 4.1 and the choices of \(\{a_n\}\) and \(\{b_n\}\), \(f_2\) satisfies
- \(f_2\) is a \(C^1\) orientation preserving diffeomorphism of \([-2, 2]\);
- \(f_2(\pm 2) = \pm 2\) and \(f_2(x) > x\) for any \(x \in (-2, 2)\);
- \(f_2'(2) = 1\).

Then we extend \(f_1\) to a diffeomorphism \(\tilde{f}_1\) of \([-2, 2]\):
\[
\tilde{f}_1(x) = \begin{cases} 
 f_2^{-(n+1)} f_1 f_2^{n+1}(x), & x \in [a_{n+1}, a_n], n = 1, 2, \ldots \\
 f_1(x), & x \in [-1, 1] \\
 f_2^{n+1} f_1 f_2^{-(n+1)}(x), & x \in [b_n, b_{n+1}] \\
 \pm 2, & x = \pm 2.
\end{cases}
\]

We denote \(\tilde{f}_1\) by \(f_1\) for \(x \in [-2, 2]\). Then
\[
f_1 f_2(x) = f_2 f_1(x), \quad \forall x \in [-2, 2].
\]

Continuing the above process, we can construct a sequence of commuting \(C^1\) orientation preserving diffeomorphisms \(f_1, f_2, \cdots\) of \(\mathbb{R}\). More precisely, assume that we have constructed pairwise commuting \(C^1\) orientation preserving diffeomorphisms \(f_1, \cdots, f_k\) of \([-k, k]\) for \(k \in \mathbb{N}^+\) with the following properties:
1. \(f_i(\pm i) = \pm i\) and \(\forall x \in (-i, i), f_i(x) > x\), for \(i = 1, 2, \cdots, k\);
2. \(f_i((-i)) = f_i'(i) = 1\), for \(i = 1, 2, \cdots, k\);
3. \(f_i f_j(x) = f_j f_i(x)\) for all \(x \in [-k, k]\) and \(1 \leq i, j \leq k\).

Then choose two infinite sequences \(- (k+1) < \cdots < c_2 < c_1 < c_0 = -k\) and \(k = d_0 < d_1 < d_2 < \cdots < k + 1\) such that
\[
\lim_{n \to \infty} c_n = -(k + 1), \quad \lim_{n \to \infty} d_n = k + 1,
\]
and
\[
\lim_{n \to \infty} \frac{c_{n-1} - c_n}{c_n - c_{n+1}} = 1, \quad \lim_{n \to \infty} \frac{d_{n+1} - d_n}{d_n - d_{n-1}} = 1.
\]

For example, we can take
\[
c_n = -(k + 1) + \frac{1}{n + 1}, \quad d_n = k + 1 - \frac{1}{n + 1}, \quad n = 1, 2, \cdots.
\]
Define

$$f_{k+1}(x) = \begin{cases} 
\phi_{[c_n, c_{n-1}]}[c_n, c_{n-1}](x), & x \in [c_{n+1}, c_n], n = 1, 2, \cdots \\
\phi_{[c_1, c_0]}[-k,k](x), & x \in [c_1, c_0] \\
\phi_{[-k,k]}[d_0, d_1](x), & x \in [-k, k] \\
\phi_{[d_n, d_{n+1}]}[d_{n+1}, d_{n+2}](x), & x \in [d_n, d_{n+1}], n = 0, 1, 2, \cdots \\
\pm(k+1), & x = \pm(k+1).
\end{cases}$$

Then by Lemma 4.1 and the choices of \{c_n\} and \{d_n\}, \(f_{k+1}\) satisfies

- \(f_{k+1}\) is a \(C^1\) orientation preserving diffeomorphism of \([-k - 1, k + 1]\);
- \(f_{k+1}(\pm(k+1)) = \pm(k+1)\) and \(f_{k+1}(x) > x\) for any \(x \in (-(k+1), k + 1)\);
- \(f_{k+1}'(k-1) = f_{k+1}'(k + 1) = 1\).

We extend \(f_1, \cdots, f_k\) to diffeomorphisms \(\tilde{f}_1, \cdots, \tilde{f}_k\) of \([-k+1, k + 1]$: for \(i = 1, \cdots, k,$

$$\tilde{f}_i(x) = \begin{cases} 
\tilde{f}_{k+1}^{-n+1} f_i f_{k+1}^{n+1}(x), & x \in [c_{n+1}, c_n], n = 1, 2, \cdots \\
\tilde{f}_k(x), & x \in [-k, k], \\
\tilde{f}_{k+1}^{-1} f_i f_{k+1}^{n+1}(x), & x \in [d_n, d_{n+1}], \\
\pm(k+1), & x = \pm(k+1).
\end{cases}$$

Denote \(\tilde{f}_i\) by \(f_i\) for \(x \in [-k + 1, k + 1]\). Then \(f_1, \cdots, f_{k+1}\) are commuting orientation preserving \(C^1\) diffeomorphisms of \([-k + 1, k + 1]\).

From the constructing process, we see that \([-1, 1] \subset [-2, 2] \subset \cdots\) and \(f_1, f_2, \cdots\) form an infinite tower, and the group \(G\) generated by \(f_1, f_2, \cdots\) is a non-finitely generated abelian group consisting of \(C^1\) orientation preserving diffeomorphisms of \(\mathbb{R}\). This completes the proof. \(\square\)

**Acknowledgements.** The work is supported by NSFC (No. 11771318, No. 11790274).

**References**

[1] A. Navas, Groups of circle diffeomorphisms. Chicago Lectures in Math. (2011).

[2] A. Navas, On centralizers of interval diffeomorphisms in critical (intermediate) regularity. Journal d’Analyse Math. 121 (2013), 21-30.
[3] J. Plante, On solvable groups acting on the real line. Trans. AMS 278 (1983), 401-414.

[4] B. Farb & J. Franks, Groups of homeomorphisms of one-manifolds III: nilpotent subgroups. Ergodic Theory and Dynamical Systems 23 (2003), 1467-1484.

Enhui Shi, School of mathematical and sciences, Soochow University, Suzhou, 215006, P.R. China (ehshi@suda.edu.cn)

Yiruo Wang, School of mathematical and sciences, Soochow University, Suzhou, 215006, P.R. China (632804847@qq.com)

Hui Xu, School of mathematical and sciences, Soochow University, Suzhou, 215006, P.R. China (20184007001@stu.suda.edu.cn)