Note on the First Law with $p$-form potentials

Geoffrey Compère

Physique Théorique et Mathématique,
Université Libre de Bruxelles
and
International Solvay Institutes
Campus Plaine C.P. 231,
B-1050 Bruxelles, Belgium

Abstract

The conserved charges for $p$-form gauge fields coupled to gravity are defined using Lagrangian methods. Our expression for the surface charges is compared with an earlier expression derived using covariant phase space methods. Additional properties of the surfaces charges are discussed. The proof of the first law for gauge fields that are regular when pulled-back on the future horizon is detailed and is shown to be valid on the bifurcation surface as well. The formalism is applied to black rings with dipole charges and is also used to provide a definition of energy in plane wave backgrounds.

PACS numbers: 04.65.+e, 04.70.-s, 11.30.-j, 12.10.-g
Remarkably, the first law of black hole mechanics has been demonstrated for arbitrary perturbations around a stationary black hole with bifurcation Killing horizon in any diffeomorphism invariant theory of gravity \cite{1}. Also, this law has been shown to hold when gravity is coupled to Maxwell or Yang-Mills fields as a consequence of conservation laws and of geometric properties of the horizon \cite{2,3}.

Recently, black rings with gauge charge along the ring, the so-called dipole charge, have been found in five dimensional supergravity \cite{4}. As shown in \cite{5}, the black ring solutions with dipole charge have a potential which diverges at the bifurcation surface. This implies that the computations of \cite{1,2} are not directly applicable to that case.

Hamiltonian methods were applied to gravity coupled to a $p$-form and a scalar field in order to explain the occurrence of dipole charges in the first law \cite{5}. Quasilocal formalism \cite{6} as well as covariant phase space methods \cite{7,8} have also been developed. The first aim of this paper is to improve the covariant analysis \cite{7,8} by deriving an expression for the conserved charges taking better care of the form factors. Following the Lagrangian methods based on cohomological results \cite{9,10}, our expression for the surface charges will moreover get round the usual ambiguities of covariant phase methods. Several properties of these surface charges will be discussed.

It was observed in \cite{3,8} that a consistent thermodynamics can be done on the future event horizon with gauge potentials that may be irregular on the bifurcation surface if, nevertheless, the potential is regular when pulled-back on the future horizon. We will extend the analysis of \cite{7,8} by detailing how this regularity hypothesis allows for proving the first law in that context. We point out that the proof of the first law is valid on the bifurcation surface as well. We will then show that the potential for the black rings \cite{4} admits a regular pull-back on the future event horizon and can thus be treated by this method. Note that this analysis covers only electric-type charges and not magnetic charges where the potential is necessarily singular on the future event horizon.

Conservations laws have been defined in asymptotically flat and anti-de Sitter backgrounds, see e.g. the seminal works \cite{11,12,13}. A natural question, raised in \cite{14,15,16}, is how mass can be defined in asymptotic plane wave geometries. We show in the last section that the conserved charges defined in this paper can be used in this context and lead to the correct first law.
In what follows, we will consider the action

\[ S[g, A, \phi] = \frac{1}{16\pi G} \int \left[ \ast 1 \, R - \ast 1 \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{1}{2} e^{-\alpha \chi} H \wedge \ast H \right] , \quad (1) \]

where \( \chi \) is a dilaton and \( H = dA \) is the field strength of a \( p \)-form \( A \), \( p \geq 1 \). The fields of the theory are collectively denoted by \( \phi^i \equiv (g_{\mu\nu}, A, \chi) \). We will set \( 16\pi G = 1 \) for convenience.

1. Conservation Laws

A very convenient mathematical setting to handle with \( n-1 \) or \( n-2 \)-form conservation laws or more generally \( (n-q) \)-form conservation laws \( (0 \leq q < n) \) is the study of local cohomology in field theories \[17, 18\], see also \[19\] for an introduction. A conservation law consists in the existence of a \( (n-q) \)-form \( k \) which is conserved on-shell \( dk^{(n-q)} \approx 0 \) and which is non-trivial, i.e. not the differential of another form on-shell, \( k^{(n-q)} \neq d(\cdot) \).

In Minkowski spacetime \( g_{\mu\nu} = \eta_{\mu\nu} \), \( \chi = 0 \) and for a trivial bundle \( A \), all these lower degree conserved forms are classified by the characteristic cohomology of \( p \)-form gauge theories \[20\]. These laws are generated in the exterior product by the forms \( \ast H \) dual to the field strength \[36\]. More precisely, for odd \( n-p-1 \), one can construct the conserved \( (n-p-1) \)-form \( \ast H \). For even \( n-p-1 \), factors \( \ast H \) mutually commute and one may construct the conserved forms \( l(n-p-1) \underline{\ast H \wedge \cdots \wedge \ast H} \) for any integer \( l \) such that \( l(n-p-1) < n-1 \).

When gravity and the scalar field are present, the charges

\[ Q^{(n-p-1)} = e^{-\alpha \chi} \ast H, \quad n-p-1 \text{ odd} \quad (2) \]
\[ Q^{(n-p-1)} = e^{-\alpha \chi} \underline{\ast H \wedge \cdots \wedge \ast H}, \quad n-p-1 \text{ even} \quad (3) \]

still enumerate the non-trivial conservation laws \[20, 21, 37\].

In order to investigate the first law of thermodynamics, where variations around a solution are involved, we now extend the analysis to the linearized theory.

In linearized gravity, only \( (n-2) \)-form conservation laws are allowed \[22\]. The classification of non-trivial conserved \( (n-2) \)-forms was described in \[9\] and is straightforward to specialize in our case. The equivalence classes of conserved \( (n-2) \)-forms of the linearized theory for the variables \( \delta \phi^i \) around a fixed reference solution \( \phi^i \) are in correspondence with
equivalence classes of gauge parameters $\xi^\mu(x), \Lambda(x)$ satisfying the reducibility equations
\[
\delta_{\xi,\Lambda} \phi^i = 0 \quad [38],
\]
i.e.
\[
\begin{align*}
\mathcal{L}_\xi g_{\mu\nu} &= 0, \\
\mathcal{L}_\xi A + d\Lambda &= 0, \\
\mathcal{L}_\xi \chi &= 0.
\end{align*}
\]
(4)

In this paper, we construct a $(n-2)$-form $k_{\xi,\Lambda}$ enjoying the following properties. First, for each generalized Killing vector $(\xi, \Lambda)$ satisfying the reducibility equations (4), the surface form $k_{\xi,\Lambda}$ will be closed on-shell. As a result, the infinitesimal charge difference between solutions $\phi^i$ and $\phi^i + \delta \phi^i$ associated with any parameter $(\xi, \Lambda)$ satisfying (4),
\[
\delta Q_{\xi,\Lambda} \hat{=} \oint_S k_{\xi,\Lambda}[\delta \phi; \phi],
\]
will only depend on the homology class of $S$. Second, since the $(n-2)$-form will be built from the weakly vanishing Noether current, the usual ambiguities that should be treated with care in covariant phase space methods [1] will be avoided here [39]. For additional properties of these surface charges, as the representation theorem of the Lie algebra of reducibility parameters, the reader is referred to the original work [9, 10].

2. SURFACE FORMS

Following the lines of [9, 10], one can construct the weakly vanishing Noether currents associated with the couple $(\xi, \Lambda)$ by integrating by parts the expression $\delta_{\xi,\Lambda} \phi^i \frac{\delta L}{\delta \phi^i}$ and using the Noether identities. We obtain
\[
S_{\xi,\Lambda} = \star \left( (-2G_{\mu}^\nu + T^\nu_{A\mu} + T^\nu_{\chi\mu}) \xi_\nu dx^\mu \right. \\
- \left. \frac{1}{(p-1)!} D_\beta (e^{-\alpha \chi} H_{\beta \mu^1 \cdots \mu^{p-1}}^\nu (\xi^\rho A_{\rho \mu^1 \cdots \mu^{p-1}} + \Lambda_{\mu^1 \cdots \mu^{p-1}}) dx^\mu) \right),
\]
where the stress tensors are given by
\[
T^\mu_{A\nu} = e^{-\alpha \chi} \left( \frac{1}{p!} H^{\mu^1 \cdots \mu^p}_\beta H_{\nu^1 \cdots \nu^p}^{\mu^1 \cdots \mu^p} - \frac{1}{2(p+1)!} g^{\mu\nu} H^2 \right),
\]
(7)
\[
T^\mu_{\chi\nu} = (\partial^\mu \chi \partial^\nu \chi - \frac{1}{2} g^{\mu\nu} \partial^\alpha \chi \partial_\alpha \chi).
\]
(8)
The surface form $k_{\xi,\Lambda}[\delta \phi; \phi] = k_{\xi,\Lambda}^{[\mu]} (d^{n-2}x)_{\mu\nu}$ can be obtained as a result of a contracting homotopy $I_{\delta \phi}^{n-1}$ acting on the current $S_{\xi,\Lambda}$, see e.g. [10, 17]. Using the following property of
the homotopy operators,
\[
d^{q-1}_{\delta \phi}(\omega^{(q-1)}) + I^{q}_{\delta \phi}d\omega^{(q-1)} = \delta \omega^{(q-1)}, \quad \forall \omega^{(q-1)}, \quad q \leq n,
\]
one has
\[
dk_{\xi, \Lambda} = \delta S_{\xi, \Lambda} - I^{n-2}_{\delta \phi} \left( \delta_{\xi, \Lambda} \phi^j \frac{\delta L}{\delta \phi^j} \right).
\]
The closure \(dk_{\xi, \Lambda}[\delta \phi; \phi] \approx 0\) then hold whenever \(\phi^j\) satisfies the equations of motion, \(\delta \phi^j\) the linearized equations of motion and \((\xi, \Lambda)\) the system \([4]\).

Let us now split the current into different contributions, \(S_{\xi, \Lambda} = S^g_{\xi} + S^\chi_{\xi} + S^A_{\xi, \Lambda}\) with
\[
S^g_{\xi} = \star(-2G_{\mu \nu} \xi_{\nu} dx^\mu),
\]
\[
S^\chi_{\xi} = \star(T_{\chi \mu} \xi_{\nu} dx^\mu),
\]
and \(S^A_{\xi, \Lambda}\) being the remaining expression. Since the homotopy \(I^{n-1}_{\delta \phi}\) is linear in its argument, the surface form can be decomposed as \(k_{\xi, \Lambda} = k^g_{\xi} + k^\chi_{\xi} + k^A_{\xi, \Lambda}\).

The gravitational contribution \(k^g_{\xi}\), which depends only on the metric and its deviations, coincides with the Abbott-Deser expression \([12]\) and, for Killing vectors, with the expression derived in the Hamiltonian approach of Regge-Teitelboim \([13]\). It can be written as
\[
k^g_{\xi}[\delta g; g] = -\delta Q^g_{\xi} + Q^g_{\delta \xi} - i_\xi \Theta[\delta g] - E_{\xi}[\mathcal{L}_{\delta g}],
\]
where
\[
Q^g_{\xi} = \star \left( \frac{1}{2} (D_{\mu} \xi_{\nu} - D_{\nu} \xi_{\mu}) dx^\mu \wedge dx^\nu \right),
\]
is the Komar \(n-2\) form and
\[
\Theta[\delta g] = \star \left( (D^{\alpha} \delta g_{\mu \sigma} - g^{\alpha \beta} D_{\mu} \delta g_{\alpha \beta}) dx^\mu \right).
\]
\[
E_{\xi}[\mathcal{L}_{\delta g}, \delta g] = \star \left( \frac{1}{2} \delta g_{\mu \alpha} (D^{\alpha} \xi_{\nu} + D_{\nu} \xi^{\alpha}) dx^\mu \wedge dx^\nu \right).
\]
The supplementary term, \(E_{\xi}\), with respect to the Iyer-Wald form \([1]\) vanishes for Killing vectors.

The scalar contribution is easily found to be \(k^\chi_{\xi}[\delta g, \delta \chi; g, \chi] = i_\xi \Theta_{\chi}[23]\) with
\[
\Theta_{\chi} = \star (d\chi \delta \chi).
\]
Let us now compute the contribution \( k_{\xi,A} \) from the \( p \)-form. After some algebra, one can rewrite the current \( S_{\xi,A} \) as

\[
S_{\xi,A} = -dQ_{\xi,A} + e^{-\alpha x}(L_{\xi}A + dA) \wedge H - \frac{1}{2} e^{-\alpha x}i_{\xi}(H \wedge *H)
\]  

(18)

with

\[
Q_{\xi,A} = e^{-\alpha x}(i_{\xi}A + \Lambda) \wedge *H.
\]

(19)

Using the property (9), the surface form \( k_{\xi,A} \) reduces to

\[
k_{\xi,A} = -\delta Q_{\xi,A} + Q_{\xi,\delta A} + dI_{\delta A}^{p-2}Q_{\xi,A} + I_{\delta \phi}^{p-1}(e^{-\alpha x}(L_{\xi}A + dA) \wedge *H - \frac{1}{2} e^{-\alpha x}i_{\xi}(H \wedge *H)),
\]

(20)

where the exact term \( dI_{\delta A}^{p-2}Q_{\xi,A} \) is trivial and can be dropped. The last term can then be computed easily since it admits only first derivatives of the gauge potential. The homotopy thus reduces in that case to \( I_{\delta A}^{p-1} = \frac{1}{2} \delta A \frac{\partial}{\partial H} \). We eventually get

\[
k_{\xi,A}[\delta g, \delta A, \delta \chi, g, A, \chi] = -\delta Q_{\xi,A} + Q_{\xi,\delta A} + i_{\xi} \Theta_{A} - E_{A}^{L}[L_{\xi}A + dA, \delta A]
\]

(21)

with

\[
\Theta_{A} = e^{-\alpha x}\delta A \wedge *H,
\]

(22)

\[
E_{A}^{L}[L_{\xi}A + dA, \delta A] = e^{-\alpha x} \ast \left( \frac{1}{2} \frac{1}{(p-1)!} \delta A_{\mu_{a_{1}}\cdots a_{p-1}} \right.
\]

\[
(L_{\xi}A + dA)_{\mu_{a_{1}}\cdots a_{p-1}} dx^{\mu} \wedge dx^{\nu}
\]

(23)

which has a very similar structure as the gravitational field contribution (13). For reducibility parameters (4), the term involving \( L_{\xi}A + dA \) vanishes. The form (19) will be referred to as a Komar term, in analogy with the gravitational Komar term (14).

For \( p = 1 \) and reducibility parameters, the surface form (21) reduces to the well-known expression for electromagnetism, see e.g. [24]. Expression (21) and the one derived in [7, 8] have a similar structure but differ in two respects. First, our surface form contains the additional term \( E_{A}^{L}[L_{\xi}A + dA, \delta A] \). Nevertheless, since this term vanishes for reducibility parameters, it will not be relevant for exact conservation laws. Second, the form factors in the Komar term \( Q_{\xi,A} \) differ from [7, 8]. The results of [7, 8] agree with ours when the right-hand side of equation (10) of [7] and equation (4) of [8] are multiplied by \(-\frac{p+1}{2}\).

Let us assume that (4) holds for a field configuration \((g, A, \chi)\). As a consistency check, note that the surface form (21) satisfies the equality on-shell \( k_{\xi,A}^{A}[\delta g = 0, \delta A = d\omega^{(p-1)}, \delta \chi = 0] \).
0; g, A, χ] ≈ d(·). The charge difference (5) between two configurations differing by a gauge transformation $δA = dω^{p-1}$, is thus zero on-shell.

Besides generalized Killing vectors ($ξ, Λ$) which are also symmetries of the gauge field and of the scalar $χ$, there may be charges associated with non-trivial gauge parameters ($ξ = 0, Λ ≠ d(·)$). For $p = 1$, in electromagnetism, $Λ = constant ≠ 0$ is such a parameter and the associated charge is the electric charge (2). For $p > 1$, non-exact forms $Λ$ may exist if the topology of the manifold is non-trivial. The charges with a non-trivial closed form $Λ$ which does not vary along solutions is given by

$$Q_{0,-Λ} = \oint_S e^{-αχ}Λ \wedge *H = \oint_T e^{-αχ} *H,$$

where $S$ is a $n-2$ surface enclosing the non-trivial cycle $T$ dual to the form $Λ$. It is simply the integral of (2) on the non-trivial cycle. The charges (24) are thus the generalization for $p$-forms of electric charges.

The properties of the surface form (21) under transformations of the potential $A$ are worth mentioning. The transformation $A \rightarrow A + dε$ preserves the reducibility equations (1) if $dL_ξ ε = 0$. In that case, $L_ξ ε$ can be written as the sum of an exact form and a harmonic form that we denote as $f(ε, ξ)Λ'$ with $Λ'$ not varying along solutions, $δΛ' = 0$ and $f(ε, ξ)$ constant. In Einstein-Maxwell theory, one has $Λ' = 1$ and $f(ε, ξ) = L_ξ ε$. Under the transformation $A \rightarrow A + dε$, the surface form (21) changes according to

$$k^A_ξ,A \rightarrow k^A_ξ,A - f(ε, ξ)δ(Λ' \wedge e^{-αχ} *H) + d(·) + t_ξ, \quad t_ξ \approx 0.$$  

Defining the charge associated to $Λ'$ as (24), one sees that the infinitesimal charge (5) varies on-shell as

$$δQ_{ξ,A} \rightarrow δQ_{ξ,A} - f(ε, ξ)δQ_{0,-Λ'}.$$  

As a consequence, a transformation $A \rightarrow A + dε$ admitting a non-vanishing function $f(ε, ξ)$ cannot be considered as a gauge transformation because such a transformation does not leave the conserved charges of the solution invariant.

3. FIRST LAW

We now assume that $φ^i$ and $φ^i + δφ^i$ are stationary black hole solutions with Killing horizon. The generator of the Killing horizon of $φ^i$, $ξ = \partial_t + Ω^a \partial_φ^a$ is a combination of the
Killing vectors $\partial_t$ and $\partial_{\phi^a}$, $a = 1 \ldots \left\lfloor (n-1)/2 \right\rfloor$. The variation of energy $\delta E$ and angular momenta $\delta J^a$ are defined as the charges associated with the Killing vectors $\partial_t$ and $-\partial_{\phi^a}$, respectively [40]. Remark that this definition of energy is more natural than the one used in [7, 8], where a factor $\alpha = \frac{n^2 - 3}{n - 2}$ was artificially added in equation (16) of [7] and in equation (8) of [8].

We assume that $\xi$ is a solution of (4) with $\Lambda = 0$. We also require that $\xi + \delta \xi$ is a symmetry of the perturbed black hole $\phi^i + \delta \phi^i$.

The first law is then a consequence of the equality [41]

$$\int_{S^\infty} \kappa_{\xi,0}[\delta \phi; \phi] = \int_H \kappa_{\xi,0}[\delta \phi; \phi],$$

where $S^\infty$ is a $(n-2)$-sphere at infinity and $H$ is any cross-section of the Killing horizon.

Using the linearity of $k_{\xi,0}$ with respect to $\xi$, the left-hand side is simply given by $\delta E - \Omega^a \delta J_a$. Splitting the right-hand side, we get

$$\delta E - \Omega^a \delta J_a = \int_H k_{\xi,0}^\theta[\delta \phi; \phi] + \int_H k_{\xi,0}^\chi[\delta \phi; \phi] + \int_H k_{\xi,0}^A[\delta \phi; \phi].$$ (28)

The geometric properties of the Killing horizon then allow one to express the pure gravitational contribution into the form [1, 26, 27, 28]

$$\int_H k_{\xi,0}^\theta[\delta \phi; \phi] = \frac{\kappa}{8\pi G} \delta A,$$ (29)

where $\kappa$ is the surface gravity and $A$ the area of the black hole and where $G$ factors have been restored. Here, the cross-section of the horizon could be chosen to lie on the future horizon or, when it exists, to be the bifurcation surface $H_B$. See also [29] for a derivation of the first law (29) for stationary perturbations on the future event horizon without assumption on the way to perform the variation.

It is now convenient for the rest of the computation to choose a cross-section lying on the future horizon. The integration measure for the $(n - 2)$-forms then becomes

$$\sqrt{|g|}(d^{n-2}x)_{\mu \nu} = \frac{1}{2}(\xi_\mu n_\nu - n_\mu \xi_\nu) dA,$$ (30)

where $dA$ is the angular measure and $n^\mu$ is an arbitrary null vector transverse to the horizon normalized with $n^\mu \xi_\mu = -1$, see e.g. equations (6.14) and (6.70) of [25] for details.

Using (17), the scalar contribution can be written as

$$\int_H k_{\xi,0}^\chi[\delta \phi; \phi] = - \int_H dA \delta \chi (\mathcal{L}_{\xi \chi} + \xi^2 \mathcal{L}_{n \chi}) = 0,$$ (31)
which vanishes thanks to the reducibility equations (4), assuming the regularity of the scalar field on the horizon. By continuity, this result is also valid on the bifurcation surface $H_B$.

The contribution of the $p$-form can be computed using the arguments of [5, 30]. The Raychaudhuri equation gives $R_{\mu\nu}{\xi}^\mu{\xi}^\nu = 0$ on the horizon. It follows by Einstein’s equations and by the identity $L_{\xi}\phi = 0$ that $i_\xi H$ has vanishing norm on the horizon. But as $i_\xi(i_\xi H) = 0$, $i_\xi H$ is tangent to the horizon. $i_\xi H$ has thus the form $\xi \wedge \cdots \wedge \xi$ by antisymmetry of $H$ and its pullback to the horizon vanishes. The equation $L_{\xi}A = 0$ can be written as $di_\xi A = -i_\xi H$. Therefore, the pull-back of $i_\xi A$ on the horizon is a closed form.

For $p = 1$, $-i_\xi A = \Phi$ is simply the scalar electric potential at the horizon. When $p > 1$, the quantity $-i_\xi A$ pulled-back on the horizon is the sum of an exact form $de$ and an harmonic form $h$. If the horizon has non-trivial $n - p - 1$ cycles $T_a$, one can define the harmonic forms dual to $T_a$ by duality between homology and cohomology as

$$\int_{T_a} \sigma = \int_H \Omega_a \wedge \sigma, \quad \forall \sigma.$$  \hspace{1cm} (32)

The harmonic form $h$ is then a sum of terms $h = \Phi^a \Omega_a$ with $\Phi^a$ constant over the non-trivial cycles.

The contribution from the potential contains three terms (21). The Komar term (19) can be written as

$$\oint_H Q^A_{\xi,0} = -\Phi^a \oint_{T_a} e^{-a\chi} \star H,$$  \hspace{1cm} (33)

where the exact form $de$ do not contribute on-shell. We recognize on the right-hand side the conserved form written in (24). Let us denote by $Q_a$ the integral $\oint_{T_a} e^{-a\chi} \star H$.

Using (33), the contribution $\int_H i_\xi \Theta_A[\delta\phi, \phi]$ reads as

$$\int_H i_\xi \Theta_A[\delta\phi, \phi] = \int_H e^{-a\chi}(i_\xi \delta A) \wedge \star H - \int_H dA \xi^2 \star (\delta A \wedge \star (i_\xi H)).$$  \hspace{1cm} (34)

The first term of (34) nicely combines with the second term of (21) into $-\oint_{T_a} \delta\Phi^a e^{-a\chi} \star H = -\delta\Phi^a Q_a$ because $\delta\Phi^a$ is constant as a consequence of the hypotheses on the variation. In the second term of (34), one can replace $\delta A$ by its pull-back $\phi_* \delta A$ on the future horizon. Indeed, decomposing $\delta A = n \wedge \omega^{(1)} + \phi_* \delta A$, one sees that the term involving $n$ do not contribute because of the antisymmetry of $H$. Therefore, the second term in (34) will vanish if $H$ is regular and if the pull-back $\phi_* \delta A$ on the future horizon is regular.
Finally, the contribution from the potential on the horizon reduces to

$$\oint_{H} k^A_{\xi}[\delta \phi; \dot{\phi}] = \Phi^a \delta Q_a, \quad (35)$$

as it should to give the first law

$$\delta \mathcal{E} - \Omega^a \delta J_a = \frac{\kappa}{8\pi G} \delta A + \Phi^a \delta Q_a. \quad (36)$$

Since the computation can be done entirely on the future horizon, this first law is valid in the extremal case, with $\kappa = 0$. The relations (29) and (31) hold on any cross-section of the horizon. Since the surface charges (5) only depend on the homology class of the surface $S$, the third term in the right-hand side of (28) has to be equal to (35) for any cross-section of the horizon as well. Therefore, when the bifurcation surface exists and when the regularity hypotheses are fulfilled, the first law (36) also holds there.

4. APPLICATION TO BLACK RINGS

Let us consider the black ring with dipole charge described in [4]. This black ring is a solution to the action (1) in five dimensions for a two-form $A$. The solution admits three independent parameters: the mass, the angular momentum and a dipole charge $\oint_{S^2} e^{-\alpha \chi} \star H$ where $S^2$ is a two-sphere section of the black ring whose topology is $S^2 \times S^1$.

The thermodynamics of this solution was worked out in the original paper [4]. The role of dipole charges in the formalism of Sudarsky and Wald [2] was elucidated in [5]. The metric, the scalar field and the gauge potential are written in equations (3.2)-(3.3)-(3.4) of [5]. There, the gauge potential

$$A = B_{t\psi} dt \wedge d\psi, \quad (37)$$

was shown to be singular on the bifurcation surface in order to avoid a delta function in the field strength on the black ring axis. Here, we point out that this singularity in the potential does not prevent one from studying thermodynamics on the future event horizon along the lines above since the pull-back of the potential is regular there.

Indeed, following [31], one can introduce ingoing Eddington-Finkelstein coordinates near
the horizon of the black ring as

\[ d\psi = d\psi' + \frac{dy}{G(y)}\sqrt{-F(y)H^N(y)}, \quad (38) \]
\[ dt = dv - CDR(1 + y)\sqrt{-F(y)H^N(y)}\frac{dy}{F(y)G(y)}d\gamma. \quad (39) \]

The metric is regular in these coordinates and the gauge potential can be written as

\[ A = B_{t\psi}dv \wedge d\psi' + dy \wedge \omega^{(1)}, \quad (40) \]

for some \( \omega^{(1)} \). The pull-back of the gauge potential to the future horizon \( y = -1/\nu \) is explicitly regular because \( B_{t\psi} \) is finite and \( v \) and \( \psi' \) are good coordinates.

The first law for black rings may then be seen as a consequence of (36).

5. APPLICATION TO BLACK STRINGS IN PLANE WAVES

We now turn to the definition of mass in asymptotic plane wave geometries. Here, we show that the integration of the surface form \( k_{\theta,0}[\delta\phi, \phi] \) along a path \( \gamma \) in solution space [10, 32],

\[ \mathcal{E} = \int_{\gamma} \oint_{S^\infty} k_{\theta,0}[\delta\phi, \phi] \quad (41) \]

provides a natural definition of mass, satisfying the first law of thermodynamics.

The action of the NS-NS sector of bosonic supergravity in \( n \)-dimensions in string frame reads

\[ S[G, B, \phi_s] = \frac{1}{16\pi G} \int d^n x \sqrt{-G} e^{-2\phi_s} \left[ R_G + 4\partial_\mu \phi_s \partial^\mu \phi_s - \frac{1}{12} H^2 \right], \]

when all fields in the \( D - n \) compactified dimensions vanish. In Einstein frame, \( g_{\mu\nu} = e^{-4\phi/(n-2)}G_{\mu\nu}, \phi = \alpha\phi_s \), the action can be written as [11] with \( \alpha = \sqrt{8/(n-2)} \) and \( A = B \).

Neutral black string in the \( n \)-dimensional maximally symmetric plane wave background \( \mathcal{P}_n \), with \( n > 4 \), are given by [14, 15, 16]

\[ ds^2_s = -f_n(r)(1 + \beta^2 r^2)dt^2 - \frac{2\beta^2 r^2 f_n(r)}{k_n(r)}dtdy + r^2d\Omega_{n-3}^2 \]
\[ + \left( 1 - \frac{\beta^2 r^2}{k_n(r)} \right) dy^2 + \frac{dr^2}{f_n(r)} - \frac{r^4 \beta^2 (1 - f_n(r))}{4k_n(r)} d\sigma^2, \]
\[ e^{\phi_s} = \frac{1}{\sqrt{k_n(r)}}, \quad B = \frac{\beta^2 r^2}{k_n(r)}(f_n(r)dt + dy) \wedge \sigma_n \quad (42) \]
where
\[ f_n(r) = 1 - \frac{M}{r^{n-4}}, \quad k_n(r) = 1 + \frac{\beta^2 M}{r^{n-6}}. \] (43)

The black strings have horizon area per unit length given by
\[ A = M^{\frac{n-2}{2}} A_{n-3} \]

is the area of the \( n - 3 \) sphere. Choosing the normalization of the horizon generator as \( \xi = \partial_t \), the surface gravity is given by
\[ \kappa = \sqrt{-1/2(D\mu \xi \nu D\nu \xi)} = \frac{n-4}{2} M^{-\frac{1}{n-1}}. \]

Using the surface forms defined above, the charge difference associated with \( \frac{\partial}{\partial t} \) between two infinitesimally close black string solutions \( \phi, \phi + \delta \phi \) is given by
\[ \delta Q_{\partial_t} = \oint k_{\partial_t,0}[\delta \phi, \phi] = \frac{n-3}{16\pi G} A_{n-3} \delta M, \] (45)

which reproduces the expectations of [14, 15, 16]. This quantity is integrable and allows one to define \( Q_{\partial_t} = \frac{n-3}{16\pi G} A_{n-3} M \) where the normalization of the background has been set to zero. It is easy to check that the first law is satisfied.

Note that one freely can choose a different normalization for the generator \( \xi' = N \partial_t \). In that case, the surface gravity changes according to \( \kappa' = N \kappa \), the charge associated to \( \xi' \) becomes \( \delta Q_{\xi'} = \frac{n-3}{16\pi G} A_{n-3} N \delta M \) and the first law is also satisfied. However, \( N \) cannot be a function of \( \beta \). Otherwise, the charge \( Q_{\xi'} \) would not be defined.

Acknowledgments

The author thanks G. Barnich, K. Copsey and M. Henneaux for their valuable comments. The author is Research Fellow at the National Fund for Scientific Research (FNRS Belgium). This work is also supported in part by a “Pôle d’Attraction Interuniversitaire” (Belgium), by IISN-Belgium, convention 4.4505.86, by Proyectos FONDECYT 1970151 and 7960001 (Chile) and by the European Commission program MRTN-CT-2004-005104, in which the author is associated to V.U. Brussel.

[1] V. Iyer and R. M. Wald, Phys. Rev. D50, 846 (1994), gr-qc/9403028.
[2] D. Sudarsky and R. M. Wald, Phys. Rev. D46, 1453 (1992).
[3] S. Gao, Phys. Rev. D68, 044016 (2003), gr-qc/0304094.
[4] R. Emparan, JHEP 03, 064 (2004), hep-th/0402149.
[5] K. Copsey and G. T. Horowitz, Phys. Rev. D73, 024015 (2006), hep-th/0505278.
[6] D. Astefanesei and E. Radu, Phys. Rev. D73, 044014 (2006), hep-th/0509144.
[7] M. Rogatko, Phys. Rev. D72, 074008 (2005), hep-th/0509150.
[8] M. Rogatko, Phys. Rev. D73, 024022 (2006), hep-th/0601055.
[9] G. Barnich and F. Brandt, Nucl. Phys. B633, 3 (2002), hep-th/0111246.
[10] G. Barnich, Class. Quant. Grav. 20, 3685 (2003), hep-th/0301039.
[11] R. Arnowitt, S. Deser, and C. Misner, Gravitation, an Introduction to Current Research (Wiley, New York, 1962), chap. 7. The Dynamics of General Relativity, pp. 227–265.
[12] L. F. Abbott and S. Deser, Nucl. Phys. B195, 76 (1982).
[13] T. Regge and C. Teitelboim, Ann. Phys. 88, 286 (1974).
[14] E. G. Gimon, A. Hashimoto, V. E. Hubeny, O. Lunin, and M. Rangamani, JHEP 08, 035 (2003), hep-th/0306131.
[15] V. E. Hubeny and M. Rangamani, Mod. Phys. Lett. A18, 2699 (2003), hep-th/0311053.
[16] A. Hashimoto and L. Pando Zayas, JHEP 03, 014 (2004), hep-th/0401197.
[17] I. Anderson, Tech. Rep., Formal Geometry and Mathematical Physics, Department of Mathematics, Utah State University (1989).
[18] G. Barnich, F. Brandt, and M. Henneaux, Phys. Rept. 338, 439 (2000), hep-th/0002245.
[19] C. G. Torre (1-7 Dec 1996), lectures given at 2nd Mexican School on Gravitation and Mathematical Physics, Tlaxcala, Mexico, hep-th/9706092.
[20] M. Henneaux, B. Knaepen, and C. Schomblond, Commun. Math. Phys. 186, 137 (1997), hep-th/9606181.
[21] G. Barnich, F. Brandt, and M. Henneaux, Nucl. Phys. B455, 357 (1995), hep-th/9505173.
[22] G. Barnich, F. Brandt, and M. Henneaux, Commun. Math. Phys. 174, 57 (1995), hep-th/9405109.
[23] G. Barnich, proceedings of 3rd International Sakharov Conference on Physics, Moscow, Russia, 24-29 Jun 2002 (2002), gr-qc/0211031.
[24] S. Gao and R. M. Wald, Phys. Rev. D64, 084020 (2001), gr-qc/0106071.
[25] P. K. Townsend (1997), gr-qc/9707012.
[26] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).
Here, all forms are written with bold letters, $A = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$. The convention for the Hodge dual of a $p$-form $\omega^p$ is $\star \omega^p = \sqrt{|g|} \omega_{\mu_1 \cdots \mu_p} (d^n - p) x^{\mu_1} \cdots dx^{\mu_p}$. Hence, if $\alpha(p)$ and $\beta(q)$ are $p$ and $q$ forms with $q \leq p \leq n$, one has $\beta(q) \wedge \star \alpha(p) = \frac{1}{q!} \alpha(p)_{\mu_1 \cdots \mu_p-q} \beta(q)_{\mu_1 \cdots \mu_q} (d^n - (p-q)) x^{\mu_1} \cdots dx^{\mu_{p-q}}$. The inner product $i \xi \omega^{(n-1)}$ can be written explicitly as $(\xi^\nu \omega^{(n-1)} \mu - \xi^\mu \omega^{(n-1)} \nu) (d^{n-2} x)_{\mu \nu}$.

When magnetic charges are allowed, there are additional conserved quantities as $\oint H \neq 0$. However, the field strength $H$ cannot be written as the derivative of a potential $B$ and the action principle has to be modified. This case will not be treated below.

The conservations laws that we consider here are called dynamical because they explicitly involve the equations of motion. It exists also specific topological conservation laws, see e.g. [33].

This correspondence is one-to-one for gauge parameters that may depend on the linearized fields $\varphi^i$ and that satisfy $\delta_{\xi(x, \varphi^i)} A(x, \varphi^i) \delta^i \approx_{lin} 0$, i.e. zero for solutions $\varphi^i$ of the linearized equations of motion. However, it has been proven in [34] that this $\varphi$-dependence is not relevant in the case of Einstein gravity. Such a dependence will not be considered here.

Indeed, by construction, the weakly vanishing Noether current does not depend on boundary terms that may be added to the Lagrangian. Moreover, if one adds a weakly vanishing exact $(n-1)$-form $dl^{(n-2)}$ to the Noether current, the resulting $(n-2)$-form will be supplemented by an irrelevant exact term and by $\delta l^{(n-2)}$ which vanishes on-shell.

The relative sign difference between the definitions of $\delta E$ and $\delta J^a$ trace its origin to the Lorentz signature of the metric [1].

The first law can be straightforwardly generalized to reducibility parameters satisfying $L_\xi A +

[27] T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D49, 6587 (1994), gr-qc/9312023.
[28] R. M. Wald, Phys. Rev. D48, 3427 (1993), gr-qc/9307038.
[29] G. Compere, Proceedings of the second Modave Summer School in Mathematical Physics (2006), gr-qc/0611129.
[30] J. P. Gauntlett, R. C. Myers, and P. K. Townsend, Class. Quant. Grav. 16, 1 (1999), hep-th/9810204.
[31] R. Emparan and H. S. Reall, Class. Quant. Grav. 23, R169 (2006), hep-th/0608012.
[32] R. M. Wald and A. Zoupas, Phys. Rev. D61, 084027 (2000), gr-qc/9911095.
[33] C. G. Torre, Class. Quant. Grav. 12, L43 (1995), gr-qc/9411014.
[34] G. Barnich, S. Leclercq, and P. Spindel, Lett. Math. Phys. 68, 175 (2004), gr-qc/0404006.

[35] Here, all forms are written with bold letters, $A = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$. The convention for the Hodge dual of a $p$-form $\omega^p$ is $\star \omega^p = \sqrt{|g|} \omega_{\mu_1 \cdots \mu_p} (d^n - p) x^{\mu_1} \cdots dx^{\mu_p}$ with $(d^n - p) x_{\mu_1 \cdots \mu_p} = \frac{1}{p!(n-p)!} \epsilon_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_n} dx^{\mu_{p+1}} \cdots dx^{\mu_n}$. Hence, if $\alpha(p)$ and $\beta(q)$ are $p$ and $q$ forms with $q \leq p \leq n$, one has $\beta(q) \wedge \star \alpha(p) = \frac{1}{q!} \alpha(p)_{\mu_1 \cdots \mu_p-q \rho_1 \cdots \rho_q} \beta(q)_{\mu_1 \cdots \mu_q} (d^n - (p-q)) x^{\mu_1} \cdots dx^{\mu_{p-q}}$. The inner product $i \xi \omega^{(n-1)}$ can be written explicitly as $(\xi^\nu \omega^{(n-1)} \mu - \xi^\mu \omega^{(n-1)} \nu) (d^{n-2} x)_{\mu \nu}$.

[36] When magnetic charges are allowed, there are additional conserved quantities as $\oint H \neq 0$. However, the field strength $H$ cannot be written as the derivative of a potential $B$ and the action principle has to be modified. This case will not be treated below.

[37] The conservations laws that we consider here are called dynamical because they explicitly involve the equations of motion. It exists also specific topological conservation laws, see e.g. [33].

[38] This correspondence is one-to-one for gauge parameters that may depend on the linearized fields $\varphi^i$ and that satisfy $\delta_{\xi(x, \varphi^i)} A(x, \varphi^i) \delta^i \approx_{lin} 0$, i.e. zero for solutions $\varphi^i$ of the linearized equations of motion. However, it has been proven in [34] that this $\varphi$-dependence is not relevant in the case of Einstein gravity. Such a dependence will not be considered here.

[39] Indeed, by construction, the weakly vanishing Noether current does not depend on boundary terms that may be added to the Lagrangian. Moreover, if one adds a weakly vanishing exact $(n-1)$-form $dl^{(n-2)}$ to the Noether current, the resulting $(n-2)$-form will be supplemented by an irrelevant exact term and by $\delta l^{(n-2)}$ which vanishes on-shell.

[40] The relative sign difference between the definitions of $\delta E$ and $\delta J^a$ trace its origin to the Lorentz signature of the metric [1].

[41] The first law can be straightforwardly generalized to reducibility parameters satisfying $L_\xi A +
\[ \Lambda = 0 \text{ with } \Lambda \neq d(\cdot). \] This simply amounts to add a contribution at infinity and at the horizon.