GEODESICS ASSOCIATED TO THE BALANCED METRIC ON THE SIEGEL-JACOBI BALL

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ABSTRACT. We determine the Christoffel’s symbols for the Siegel-Jacobi ball endowed with the balanced metric. We study the equations of geodesics on the Siegel-Jacobi ball. We calculate the covariant derivative of one-forms in the variables in which is expressed the balanced metric on the Siegel-Jacobi ball.

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1. INTRODUCTION

The Jacobi group of index \( n \), \( G^J_n \), defined as the semidirect product of the symplectic group with Heisenberg group of appropriate dimension, is an interesting object in several branches of Mathematics and Physics, see references in \([7, 9, 10, 11, 16]\). \( G^J_n \) acts transitively on the homogeneous manifold associated with it, the Siegel-Jacobi ball \( \mathcal{D}^J_n \), a partially bounded domain, whose points are in \( \mathbb{C}^n \times \mathcal{D}_n \), where \( \mathcal{D}_n \) denotes the Siegel ball. In fact, there is a real and a complex version of the Jacobi group, with factor group \( \text{Sp}(n, \mathbb{R}) \), respectively \( \text{Sp}(n, \mathbb{R})_{\mathbb{C}} \), while the associated manifold in the real case is the Siegel-Jacobi upper half plane \( X^J_n \), whose points are in \( \mathbb{C}^n \times X_n \), where \( X_n \) is the Siegel upper half plane. The case of the Jacobi group of index \( n = 1 \) was considered in \([26, 21]\).

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There are generalizations of the Jacobi group and their associated homogeneous spaces, where instead $\mathbb{C}^n$ are considered spaces $\mathbb{C}^{mn}$, $m, n \in \mathbb{N}$. There is an isomorphism between the complex and real version of the Jacobi group and the homogenous spaces associated to them [10, 12]. The present paper is mainly devoted to the complex version of the Jacobi group.

In [9, 12] we have determined the $G_n^J$-invariant metric on $D_n^J$ and we have shown that it corresponds to the metric determined in the papers of J.-H. Yang [40], generalizing the results of E. Kähler [30] and R. Berndt [26] in the case $n = 1$. We mention that the starting point of our approach in [9, 12] was the construction of coherent states (CS) [35] attached to the Jacobi group $G_n^J$, with support in $D_n^J$. The case $n = 1$ was studied in [7]. What we have find starting from the construction of CS attached to $G_n^J$ was the Kähler potential on $D_n^J$. In [16] we have underlined that the metric determined in [9, 12] is the balanced metric [28, 1] and we have determined the metric matrix $h(z, W)$ associated to the Kähler two-form $\omega_{D_n^J}(z, W)$, where $z \in \mathbb{C}^n$ and $W \in M(n, \mathbb{C})$ is a symmetric matrix describing points in $D_n$. We have introduced [12] a useful variable $\eta \in \mathbb{C}^n$, such that if we make a change of variables $(z, W) = FC(\eta, W)$, $z = \eta - W\bar{\eta}$, the metric on the Siegel-Jacobi ball expressed in $(\eta, W)$ separates as sum of the metric on $\mathbb{C}^n$ and $D_n$, i.e. the $FC$-transform is a Kähler homogeneous diffeomorphism and realizes the fundamental conjecture for the homogeneous Kähler manifold $D_n^J$ [38, 25]. In [16] we have also determined the inverse matrix $h^{-1}$ in the same variables.

In the present paper we use the results of [16] and calculate the Christoffel symbols on $D_n^J$. This allows us to write down explicitly the equations of geodesics on $D_n^J$. We are not able to integrate these equations, but we make some comments related to the geometric and physical meaning of the variables used. In order to clarify the physical meaning of the $FC$-transform, we use some notions concerning the CS, even that the present work is mainly a simple matrix calculation, where the CS does not appear explicitly. As a byproduct of the calculation of the $\Gamma$-s on $D_n^J$, we also calculate the covariant derivatives $D(dz)$ and $D(dW)$ (called in [41] holomorphic $\Gamma$-module connections associated to the Christoffel symbols) - this kind of calculation was done also in [42] in order to determine derivations of the Jacobi forms on $X_{mn} = X_n \times \mathbb{C}^{mn}$. The reason to present the calculation of $D(dz)$ and $D(dW)$ is to underline again the utility of the variables introduced in [9, 12].

The paper is divided in two parts. In §2 are recalled some notations and concepts introduced in pervious works and used in the present paper. In §2.1 we present what we mean by Jacobi group in this paper, see details in [9, 12]. §2.2 recalls some simple facts about the CS defined by Perelomov [35], in order to understand how the balanced metric was obtained in [16]. The formulae used for calculation of geodesics are recalled in §2.3. The geometry of $D_1^J$ [7, 13], summarized in §2.4, is a guide for the calculation to be done on $D_n^J$, especially Proposition 1 which gives equations of geodesics on the Siegel-Jacobi disk. §2.5 reproduces the results from [16] concerning the matrix of the balanced metric on $D_n^J$ and is its inverse, the starting point of the present investigation - see Theorem 1. In Proposition 3 we extract from [16] some facts on the geometry of the Jacobi group and its action on the Siegel-Jacobi ball. The new results of the present paper are contained in §3. The equations of geodesics in the variable $z \in \mathbb{C}^n$, $W \in D_n$
are given in §3.1 which also contains the calculation of Γ-s. Equations of geodesics are collected in Proposition 5 of §3.2, followed by a discussion about the significance of the result in the context of the FC-transform on \( D^J \), which is shown to be not a geodesic mapping. It was proved \( \text{(1)} \) that for symmetric manifolds the FC-transform gives geodesics, but the studied Siegel-Jacobi ball is not a symmetric space. Section 3.3 summarizes our calculation of the covariant derivative of \( d z \) and \( d W \). The appendix in §4 is dedicated to the equations of geodesics on \( D_n \) deduced from equations of geodesics on \( X_n \). The new results of these paper are contained in Propositions 4, 5, 6 and 7.

**Notation.** We denote by \( \mathbb{R}, \mathbb{C}, \mathbb{Z}, \) and \( \mathbb{N} \) the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. We denote the imaginary unit \( \sqrt{-1} \) by \( i \), and the Real and Imaginary part of a complex number by \( \Re \) and \( \Im \), i.e. we have for \( z \in \mathbb{C} \), \( z = \Re z + i \Im z \), and \( \bar{z} = \Re z - i \Im z \). \( M(m \times n, \mathbb{F}) \cong \mathbb{F}^{mn} \) denotes the set of all \( m \times n \) matrices with entries in the field \( \mathbb{F} \). \( \mathbb{M}(n \times 1, \mathbb{F}) \) is identified with \( \mathbb{F}^n \). Set \( \mathbb{M}(n, \mathbb{F}) = \mathbb{M}(n \times n, \mathbb{F}) \). For any \( A \in \mathbb{M}_n(\mathbb{C}) \), \( A^t \) denotes the transpose matrix of \( A \). For \( A \in \mathbb{M}_n(\mathbb{C}) \), \( \bar{A} \) denotes the conjugate matrix of \( A \) and \( A^* = \bar{A}^t \). For \( A \in \mathbb{M}_n(\mathbb{C}) \), the inequality \( A > 0 \) means that \( A \) is positive definite. The identity matrix of degree \( n \) is denoted by \( \mathbb{I}_n \) and \( \mathbb{D}_n \) denotes the \( (n, \mathbb{F}) \)-matrix with all entries 0. We consider a complex separable Hilbert space \( \mathcal{H} \) endowed with a scalar product which is antilinear in the first argument, \( (\lambda x, y) = \bar{\lambda}(x, y) \), \( x, y \in \mathcal{H} \), \( \lambda \in \mathbb{C} \setminus \{0\} \). If \( A \) is a linear operator, we denote by \( A^\dagger \) its adjoint. If \( \pi \) is a representation of a Lie group \( G \) on the Hilbert space \( \mathcal{H} \) and \( \mathfrak{g} \) is the Lie algebra of \( G \), we denote \( X := d\pi(X) \) for \( X \in \mathfrak{g} \). We use Einstein convention that repeated indices are implicitly summed over.

2. Preliminaries

2.1. The Jacobi group. The (complex) Jacobi group of index \( n \) is defined as the semidirect product \( G^J_n = H_n \ltimes \text{Sp}(n, \mathbb{R})_{\mathbb{C}} \), where \( H_n \) denotes the \((2n+1)\)-dimensional Heisenberg group \([39, 9, 12]\), endowed with the composition law

\[(g_1, \alpha_1, t_1)(g_2, \alpha_2, t_2) = (g_1 g_2, g_2^{-1} \times \alpha_1 + \alpha_2, t_1 + t_2 + \Im(g_2^{-1} \times \alpha_1 \bar{\alpha}_2)).\]

\( \alpha_i \in \mathbb{C}^n, t_i \in \mathbb{R} \) and \( g_i \in \text{Sp}(n, \mathbb{R})_{\mathbb{C}}, i = 1, 2 \) have the form (2.2)

\[g = \begin{pmatrix} p & q \cr \bar{q} & \bar{p} \end{pmatrix}, \quad p, q \in \mathbb{M}(n, \mathbb{C}),\]

and \( g = p + q \alpha \) and \( g^{-1} = p^* - q^* \alpha \). The matrices \( p, q \in \mathbb{M}(n, \mathbb{C}) \) have the properties

\[(2.3a) \quad pp^* - qq^* = \mathbb{I}_n, \quad pq^t = qp^t;\]

\[(2.3b) \quad p^* p - q^* \bar{q} = \mathbb{I}_n, \quad \bar{p} \bar{q} = q^* p.\]

To the Jacobi group \( G^J_n \) it is associated a homogeneous manifold - the Siegel-Jacobi ball \( \mathbb{D}^J_n [9] \) whose points are in \( \mathbb{C}^n \times \mathbb{D}_n \), i.e. a partially-bounded space. \( \mathbb{D}_n \) denotes the Siegel (open) ball of index \( n \). The non-compact hermitian symmetric space \( \text{Sp}(n, \mathbb{R})_{\mathbb{C}}/ U(n) \) admits a matrix realization as a homogeneous bounded domain:

\[(2.4) \quad \mathbb{D}_n := \{ W \in \mathbb{M}(n, \mathbb{C}) : W = W^t, N > 0, N := \mathbb{I}_n - WW^t \}.\]
For \( g \in \text{Sp}(n, \mathbb{R}) \mathcal{C} \) of the form (2.2), (2.3) and \( \alpha, z \in \mathbb{C}^n \), the transitive action \((g, \alpha) \times (W, z) = (W_1, z_1)\) of the Jacobi group \( G_n^{1} \) on the Siegel-Jacobi ball \( \mathcal{D}_n^{1} \) is given by the formulae [9]

\[
\begin{align*}
(2.5a) & \quad W_1 = (pW + q)(qW + p)^{-1} = (Wq^* + p^*)^{-1}(q^* + Wp^*), \\
(2.5b) & \quad z_1 = (Wq^* + p^*)^{-1}(z + \alpha - W\alpha).
\end{align*}
\]

2.2. Balanced metric and coherent states. We consider a Kähler manifold \( M \) endowed with a Kähler-two form, locally written as

\[
\omega_M(z) = i \sum_{\alpha, \beta=1}^{n} h_{\alpha \beta}(z) \, \text{d} z_\alpha \wedge \text{d} \bar{z}_\beta, \quad h_{\alpha \beta} = \bar{h}_{\beta \alpha} = h_{\beta \alpha},
\]

We consider a \( G \)-invariant Kähler two-form \( \omega_M \) (2.6) on the \( 2n \)-dimensional homogeneous manifold \( M = G/H \) derived from the Kähler potential \( f(z, \bar{z}) \) (2.7)

\[
h_{\alpha \beta} = \frac{\partial^2 f}{\partial z_\alpha \partial \bar{z}_\beta}.
\]

The condition of the metric to be a Kählerian one is (cf. (6) p. 156 in [32])

\[
\frac{\partial h_{\alpha \beta}}{\partial z_\gamma} = \frac{\partial h_{\alpha \bar{\beta}}}{\partial \bar{z}_\gamma}, \quad \alpha, \beta, \gamma = 1, \ldots, n.
\]

As was underlined in [13] for \( \mathcal{D}_1^{1} \) and proved in [16] for \( \mathcal{D}_n^{1} \), the homogeneous hermitian metric determined in [7, 9, 12] is in fact a balanced metric, because it corresponds to the Kähler potential calculated as the scalar product of two CS-vectors

\[
f(z, \bar{z}) = \ln K_M(z, \bar{z}), \quad K_M(z, \bar{z}) = (e_z, e_{\bar{z}}).
\]

We recall that in the approach of Perelomov [35] to CS it is supposed that there exists a continuous, unitary, irreducible representation \( \pi \) of a Lie group \( G \) on the separable complex Hilbert space \( \mathfrak{H} \). The coherent vector mapping \( \varphi \) is defined locally (cf. [5, 6]) \( \varphi : M \to \mathfrak{H} \), \( \varphi(z) = e_z \), where \( \mathfrak{H} \) denotes the Hilbert space conjugate to \( \mathfrak{H} \).

We can introduce the normalized (unnormalized) vectors \( \xi_x \) (respectively, \( e_z \)) defined on \( G/H \)

\[
\xi_x = \exp\left( \sum_{\phi \in \Delta_+} x_\phi \frac{X_\phi^+}{\phi} - \bar{x}_\phi \frac{X_\phi^-}{\phi} \right) e_0, \quad e_z = \exp\left( \sum_{\phi \in \Delta_+} z_\phi \frac{X_\phi^+}{\phi} \right) e_0,
\]

where \( e_0 \) is the extremal weight vector of the representation \( \pi, \Delta_+ \) are the positive roots of the Lie algebra \( \mathfrak{g} \) of \( G \), and \( X_\phi, \phi \in \Delta_+ \), are the generators. \( \frac{X_\phi^+}{\phi} (\frac{X_\phi^-}{\phi}) \) corresponds to the positive (respectively, negative) generators. See details in [35, 6].

Let us denote by \( FC \) the change of variables \( x \to z \) in formula (2.10) such that

\[
\xi_x = \bar{e}_z, \quad \bar{e}_z := (e_z, e_z)^{-\frac{1}{2}} e_z, \quad z = FC(x).
\]

The reason for calling the transform \( \xi_x = FC \) (fundamental conjecture) is explained later, see Proposition 2. We also recall that we have proved that for symmetric spaces the dependence \( z(t) = FC(tX) \) from (2.11) gives geodesics in \( M \) with the property that \( z(0) = p \) and \( \dot{z}(0) = X \).
Using Perelomov’s CS vectors $e_z \in \mathcal{F}_z$, $z \in M$ we have considered Berezin’s approach to quantization on Kähler manifolds, with the supercomplete set of vectors verifying the Parceval overcompletness identity.

Berezin’s approach to quantization on Kähler manifolds, with the supercomplete set of vectors verifying the Parceval overcompletness identity

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{F}_K} = \int_M \langle \psi_1, e_z \rangle \langle e_z, \psi_2 \rangle d\nu_M(z, \bar{z}), \quad \psi_1, \psi_2 \in \mathcal{F}_z,$$

$$d\nu_M(z, \bar{z}) = \Omega_M(z, \bar{z} \rangle \langle e_z, e_{\bar{z}} \rangle; \quad \Omega_M := \frac{1}{n!} \omega \wedge \ldots \wedge \omega \text{ } n \text{ times.}$$

Let us denote by $\mathcal{H}_f$ the weighted Hilbert space of square integrable holomorphic functions on $M$, with weight $e^{-f}$.

$$\mathcal{H}_f = \{ \phi \in \text{hol}(M) | \int_M e^{-f} |\phi|^2 \Omega_M < \infty \}.$$

In order to identify the Hilbert space $\mathcal{H}_f$ defined by (2.14) with the Hilbert space with scalar product (2.12), it was introduced the so called $\epsilon$-function.

$$\epsilon(z) = e^{-f(z)} K_M(z, \bar{z}).$$

If the Kähler metric on the complex manifold $M$ is such that $\epsilon(z)$ is a positive constant, then the metric is called balanced. Donaldson used this denomination for compact manifolds, then it was used in [1] for noncompact manifolds and later in the context of Berezin quantization on homogeneous bounded domains.

The balanced hermitian metric of $M$ in local coordinates is

$$d^2s^2_M(z, \bar{z}) = \sum_{\alpha, \beta=1}^{n} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \ln(K_M(z, \bar{z})) d \bar{z}_\beta \otimes dz_\alpha, \quad K_M(z, \bar{z}) = \langle e_z, e_{\bar{z}} \rangle.$$

**2.3. Geodesics on Kähler manifolds.** In terms of a local coordinate system $x^1, \ldots, x^n$ the equations of geodesics on a manifold $M$ with linear connection with components of the linear connections $\Gamma$ are (see e.g. Proposition 7.8 p. 145 in [31])

$$\frac{d^2 x^i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad i = 1, \ldots, n.$$

The components $\Gamma^i_{jk}$ of a Riemannian (Levi-Civitta) connection are obtained by solving the linear system (see e.g. [31], p 160)

$$h_{ik} \Gamma^l_{ji} = \frac{1}{2} \left( \frac{\partial h_{ki}}{\partial x_j} + \frac{\partial h_{jk}}{\partial x_i} - \frac{\partial h_{ij}}{\partial x_k} \right).$$

The connection matrix (form) $\theta = \partial \ln h$ (see §6 in [24] or §3.2 in [2]) has the matrix elements:

$$\theta^i_j = \Gamma^j_{ik} dx^k.$$

The covariant derivative of a contravariant vector (one-form) is given by

$$Du_i = -\theta^k_j u_j.$$
If we take into account the hermiticity condition in (2.7) on the metric and the Kählerian restrictions (2.8), the non-zero Christoffel’s symbols $\Gamma$ of the Chern connection (cf. e.g. §3.2 in [2]; in the case of Kähler manifolds, also Levi-Civita connection, cf. e.g. Theorem 4.17 in [2]) on Kähler manifolds which appear in (2.18) are determined by the equations (see also e.g. (12) at p. 156 in [32])

\begin{equation}
(2.21)
\frac{\partial h_{\alpha\beta}}{\partial z_\gamma} = \frac{\partial h_{\gamma\beta}}{\partial z_\alpha} = \frac{\partial h_{\alpha\gamma}}{\partial z_\beta}.
\end{equation}

We introduce the convention

\begin{equation}
(2.22)
\tilde{h}^{\alpha\beta} := (h_{\alpha\beta})^{-1}, \text{ i.e. } h_{\alpha\beta} \tilde{h}^{\beta\gamma} = \delta_{\alpha\gamma}.
\end{equation}

From (2.21), we have:

\begin{equation}
(2.23)
\Gamma^\gamma_{\alpha\beta} = \bar{h}^{\gamma\zeta} \frac{\partial h_{\zeta\beta}}{\partial z_\alpha} = h^{\epsilon\gamma} \frac{\partial h_{\epsilon\beta}}{\partial z_\alpha}.
\end{equation}

2.4. Geodesics on the Siegel-Jacobi disk. We recall some formulae describing the the Jacobi group $G^J_1$ and the geometry of the Siegel-Jacobi disk $D^J_1$ [7, 13, 15].

The transitive action of the group $G^J_1 = H_1 \rtimes SU(1, 1) \ni (g, \alpha) \times (z, w) \rightarrow (z_1, w_1) \in D^J_1$ on the Siegel disk is given by the formulae:

\begin{equation}
(2.24)
w_1 = \frac{aw + b}{\delta}, \quad \delta = b\bar{w} + \bar{a}, \text{SU(1, 1)} \ni g = \left(\begin{array}{cc}a & b \\ \bar{b} & \bar{a} \end{array}\right), \text{where } |a|^2 - |b|^2 = 1,
\end{equation}

\begin{equation}
(2.25)z_1 = \gamma \delta, \quad \gamma = z + \alpha - \bar{\alpha} w.
\end{equation}

The balanced Kähler two-form $\omega_{k\mu}$ on $D^J_1$, $G^J_1$-invariant to the action (2.24), (2.25), can be written as:

\begin{equation}
(2.26)\quad - i \omega_{k\mu}(z, w) = 2k \frac{d w \wedge d \bar{w}}{P^2} + \mu \frac{A \wedge \bar{A}}{P}, \quad A = \frac{d z + d w \bar{\eta}(z, w)}{P} = 1 - w\bar{w},
\end{equation}

\begin{equation}
(2.27)\quad z = \eta - w\bar{\eta}, \quad \text{and } \eta = \eta(z, w) := \frac{z + \bar{z} w}{P}.
\end{equation}

$k$ indexes the positive discrete series of $SU(1, 1)$ ($2k \in \mathbb{N}$), while $\mu > 0$ indexes the representations of the Heisenberg group.

The matrix of the balanced metric (2.16) $h = h(\varsigma), \varsigma := (z, w) \in \mathbb{C} \times D_1$ is

\begin{equation}
(2.28)\quad h(\varsigma) = \begin{pmatrix} \frac{\mu}{P} & \frac{\mu \bar{\eta}}{P} \\ \frac{\mu \eta}{P} & \frac{2k |\eta|^2}{P^2} + \mu \frac{|\eta|^2}{P^2} \end{pmatrix}.
\end{equation}

The inverse of the matrix (2.28) reads

\begin{equation}
(2.29)\quad h^{-1}(\varsigma) = \begin{pmatrix} \frac{P}{\mu} + \frac{P^2 |\eta|^2}{2k} & \frac{P \bar{\eta}}{2k} \\ \frac{P \eta}{2k} & \frac{P^2 |\eta|^2}{2k} + \frac{P^2 |\eta|^2}{2k} \end{pmatrix}.
\end{equation}
If we introduce the notation
\[(2.30) \quad \mathcal{G}_M(z) := \det(h_{\alpha \beta})_{\alpha, \beta = 1, \ldots, n}, \]
then we find
\[(2.31) \quad \mathcal{G}_D(z, w) = \frac{2k\mu}{(1 - w\bar{w})^3}, \quad z \in \mathbb{C}, \quad |w| < 1. \]

In the variables \((z, w) \in (\mathbb{C}, D_1)\) the equations of geodesics \((2.17)\) read
\[(2.32) \quad \begin{cases} \frac{d^2\bar{z}}{dt^2} + \Gamma_{\bar{z}z}^z \left(\frac{dz}{dt}\right)^2 + 2\Gamma_{\bar{z}w}^z \frac{dz}{dt} \frac{dw}{dt} + \Gamma_{\bar{w}w}^z \left(\frac{dw}{dt}\right)^2 = 0; \\ \frac{d^2\bar{w}}{dt^2} + \Gamma_{\bar{z}z}^w \left(\frac{dz}{dt}\right)^2 + 2\Gamma_{\bar{z}w}^w \frac{dz}{dt} \frac{dw}{dt} + \Gamma_{\bar{w}w}^w \left(\frac{dw}{dt}\right)^2 = 0. \end{cases} \]

The equations \((2.21)\) which determine the \(\Gamma\)-symbols for the Siegel-Jacobi disk are
\[(2.33) \quad \begin{cases} h_{zz} \Gamma_{zz}^z + h_{wz} \Gamma_{zw}^w = \frac{\partial h_{zz}}{\partial z}, \\ h_{z\bar{w}} \Gamma_{z\bar{w}}^z + h_{w\bar{w}} \Gamma_{w\bar{w}}^w = \frac{\partial h_{z\bar{w}}}{\partial z}. \end{cases} \]
\[(2.34) \quad \begin{cases} h_{zz} \Gamma_{zz}^w + h_{wz} \Gamma_{zw}^w = \frac{\partial h_{zz}}{\partial z}, \\ h_{z\bar{w}} \Gamma_{z\bar{w}}^w + h_{w\bar{w}} \Gamma_{w\bar{w}}^w = \frac{\partial h_{z\bar{w}}}{\partial z}. \end{cases} \]
\[(2.35) \quad \begin{cases} h_{zz} \Gamma_{zz}^w + h_{wz} \Gamma_{zw}^w = \frac{\partial h_{zz}}{\partial z}, \\ h_{z\bar{w}} \Gamma_{z\bar{w}}^w + h_{w\bar{w}} \Gamma_{w\bar{w}}^w = \frac{\partial h_{z\bar{w}}}{\partial z}. \end{cases} \]

From \((2.33)-(2.35)\) or from \((2.23)\), we get
\[(2.36) \quad \begin{align*} \Gamma_{zz}^z &= h_{zz} \frac{\partial h_{zz}}{\partial z} + h_{wz} \frac{\partial h_{z\bar{w}}}{\partial z}, \\ \Gamma_{zw}^w &= h_{zz} \frac{\partial h_{zw}}{\partial w} + h_{wz} \frac{\partial h_{w\bar{w}}}{\partial w}. \end{align*} \]
\[(2.37) \quad \begin{align*} \Gamma_{zz}^w &= h_{zz} \frac{\partial h_{z\bar{w}}}{\partial z} + h_{wz} \frac{\partial h_{w\bar{w}}}{\partial z}, \\ \Gamma_{zw}^w &= h_{zz} \frac{\partial h_{zw}}{\partial w} + h_{wz} \frac{\partial h_{w\bar{w}}}{\partial w}. \end{align*} \]

With \((2.28)\), we calculate easily the derivatives
\[(2.37) \quad \begin{align*} \frac{\partial h_{zz}}{\partial z} &= 0; \\ \frac{\partial h_{z\bar{w}}}{\partial z} &= \mu \frac{\bar{w}}{P^2}; \\ \frac{\partial h_{w\bar{w}}}{\partial z} &= \mu \frac{\bar{w}}{P^2}; \\ \frac{\partial h_{zw}}{\partial w} &= \frac{\bar{z}}{P^2} \frac{\partial \bar{z}}{\partial w} + 3\mu \frac{\bar{w}|\eta|^2}{P^2} + 4k \frac{\bar{w}}{P^3}. \end{align*} \]

Introducing \((2.37)\) into \((2.33)-(2.35)\), we find for the Christoffel’s symbols \(\Gamma\)-s the expressions
\[(2.38) \quad \begin{align*} \Gamma_{zz}^z &= -\lambda \bar{\eta}; \\ \Gamma_{zw}^z &= -\lambda \bar{\eta}^2 + \frac{\bar{w}}{P}; \\ \Gamma_{zw}^w &= \lambda \bar{\eta}; \\ \Gamma_{wz}^w &= -\lambda \eta^2; \\ \Gamma_{ww}^w &= \lambda \eta^2 + 2 \frac{\bar{w}}{P}; \\ \lambda &= \frac{\mu}{2k}. \end{align*} \]
Proposition 1. The equations of geodesics on the Siegel-Jacobi corresponding to metric defined by $\omega_{\mu\nu}$ are

\begin{align}
(2.39a) \quad \mu \eta G_1^2 &= 2kG_3, \quad G_1 = \frac{dz}{dt} + \frac{\eta}{P} \frac{dw}{dt}, \quad G_3 = \frac{d^2 z}{dt^2} + 2\frac{\bar{w}}{P} \frac{dz}{dt} \frac{dw}{dt}, \quad P = 1 - w\bar{w}; \\
(2.39b) \quad \mu G_2^2 &= -2kG_2, \quad G_2 = \frac{d^2 w}{dt^2} + 2\frac{\bar{w}}{P} \left(\frac{d w}{dt}\right)^2.
\end{align}

The connection matrix $w_{\mathcal{D}_1}$ on $\mathcal{D}_1$ has the value

\begin{equation}
\theta_{\mathcal{D}_1} := \begin{pmatrix}
\theta_z^z & \theta_z^w \\
\theta_z^w & \theta_w^w
\end{pmatrix} = \begin{pmatrix}
\Gamma_z^{zz} dz + \Gamma_z^{zw} dw & \Gamma_z^{wz} dw + \Gamma_z^{ww} dw \\
\Gamma_z^{zw} dz + \Gamma_z^{ww} dw & \Gamma_z^{ww} dz + \Gamma_z^{ww} dw
\end{pmatrix}
\end{equation}

has the value

\begin{equation}
\theta_{\mathcal{D}_1} = \begin{pmatrix}
-\lambda \eta A + \frac{\bar{w}}{P} dw & -\lambda \bar{w}^2 A + \frac{\bar{w}}{P} dz \\
\lambda \bar{w} A & \lambda \bar{w} A + 2\frac{\bar{w}}{P} dw
\end{pmatrix}.
\end{equation}

The covariant derivative of $dz$ on $\mathcal{D}_1$ has the expression

\begin{equation}
D(dz) = \begin{pmatrix}
dz \\
dw
\end{pmatrix} = \begin{pmatrix}
\lambda \eta & \lambda \bar{w}^2 - \frac{\bar{w}}{P} \\
\lambda \bar{w} & \lambda \bar{w} A + 2\frac{\bar{w}}{P}
\end{pmatrix} \begin{pmatrix}
dz \\
dw
\end{pmatrix}
\end{equation}

The covariante derivative of $dw$ has the expression

\begin{align}
(2.42a) \quad -D(dw) &= \begin{pmatrix}
dz \\
dw
\end{pmatrix} = \begin{pmatrix}
\lambda & \lambda \eta \\
0 & \lambda A + 2\frac{\bar{w}}{P}
\end{pmatrix} \begin{pmatrix}
dz \\
dw
\end{pmatrix} \\
(2.42b) \quad &= \begin{pmatrix}
\lambda & 0 \\
0 & \lambda A + 2\frac{\bar{w}}{P}
\end{pmatrix} \begin{pmatrix}
dz \\
dw
\end{pmatrix}.
\end{align}

2.5. Balanced metric on the Siegel-Jacobi ball. The Jacobi algebra is the the semi-direct sum $\mathfrak{g}_n^I := \mathfrak{h}_n \rtimes \mathfrak{sp}(n, \mathbb{R})_C$ \cite{[8] [9] [12]}. The Heisenberg algebra $\mathfrak{h}_n$ is generated by the boson creation (respectively, annihilation) operators $a^I$, $(a)$, and

\begin{equation}
[a_i, a_j^\dagger] = \delta_{ij}; \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0.
\end{equation}

$K_{ij}^{\pm,0}$ are the generators of the $\mathfrak{sp}(\mathbb{R})_C$ algebra

\begin{align}
(2.44a) \quad &[K_{ij}^-, K_{kl}^+] = [K_{ij}^+, K_{kl}^-] = 0, \quad 2[K_{ij}^-, K_{kl}^0] = K_{ij}^- \delta_{kj} + K_{ji}^- \delta_{ki}, \\
(2.44b) \quad &2[K_{ij}^-, K_{kl}^0] = K_{kj}^0 \delta_{ij} + K_{ij}^0 \delta_{ki}, \quad K_{ij}^0 \delta_{ij} + K_{ji}^0 \delta_{ki}, \\
(2.44c) \quad &2[K_{ij}^+, K_{kl}^0] = -K_{ik}^+ \delta_{jl} - K_{jk}^+ \delta_{li}, \quad 2[K_{ij}^+, K_{kl}^0] = K_{ij}^0 \delta_{ki} - K_{kl}^0 \delta_{ij}.
\end{align}

$\mathfrak{h}_n$ is an ideal in $\mathfrak{g}_n^I$, and

\begin{align}
(2.45a) \quad &[a_k, K_{ij}^+] = [a_k, K_{ij}^-] = 0, \\
(2.45b) \quad &[a_i, K_{kj}^+] = \frac{1}{2} \delta_{ik} a_j^\dagger + \frac{1}{2} \delta_{ij} a_k^\dagger, \quad [K_{kj}, a_i^\dagger] = \frac{1}{2} \delta_{ik} a_j + \frac{1}{2} \delta_{ij} a_k, \\
(2.45c) \quad &[K_{ij}^0, a_k^\dagger] = \frac{1}{2} \delta_{jk} a_i^\dagger, \quad [a_k, K_{ij}^0] = \frac{1}{2} \delta_{ik} a_j.
\end{align}
Perelomov’s CS vectors [35] associated to the group $G^d_n$ with Lie algebra the Jacobi
algebra $g^d_n$ based the Siegel-Jacobi ball $D^d_n$ have been defined as [35] [35]

$$e_{z,W} = \exp(X)e_0, \quad X := \sqrt{\mu} \sum_{i=1}^{n} z_i a_i + \sum_{i,j=1}^{n} w_{ij} K^+_{ij}, \quad z \in \mathbb{C}^n; W \in D_n.$$  (2.46)

The vector $e_0$ appearing in (2.46) verifies the relations

$$a_i e_0 = 0, \quad i = 1, \cdots, n; \quad K^+_{ij} e_0 \neq 0, \quad K^-_{ij} e_0 = 0, \quad K^0_{ij} e_0 = \frac{k}{4} \delta_{ij} e_0, \quad i, j = 1, \cdots, n.$$  (2.47)

The reproducing kernel $K(z,W) = (e_{z,W} e_{z,W})_{k\mu}, \quad z \in \mathbb{C}^n, W \in D_n$ is

$$K(z, W) = \det(M) \frac{k^4}{2} \exp \mu F, \quad M = (\mathbb{I}_n - W\bar{W})^{-1},$$  (2.48)

$$2F = 2\bar{z}^t Mz + z^t \bar{W} Mz + \bar{z}^t MWz,$$  (2.49a)

$$2F = 2\bar{\eta}^t \eta - \eta^t \bar{W} \eta - \bar{\eta}^t W\bar{\eta},$$  (2.49b)

$$\eta = M(z + W\bar{z}); \quad z = \eta - W\bar{\eta}.$$  (2.50)

The manifold $D^d_n$ has the Kähler potential (2.51), $f = \log K$, with $K$ given by (2.43),

$$f = -\frac{1}{2} \log \det(\mathbb{I}_n - W\bar{W})$$  (2.51)

$$+ \mu \left\{ \bar{z}^t (\mathbb{I}_n - W\bar{W})^{-1} z + \frac{1}{2} \bar{z}^t [\bar{W}(\mathbb{I}_n - W\bar{W})^{-1}] z + \frac{1}{2} \bar{z}^t [(\mathbb{I}_n - W\bar{W})^{-1} W\bar{z}] \right\}.$$  (2.52)

We use the following notation for the matrix of the metric:

$$h = \left( \begin{array}{cc} h_1 & h_2 \\ h_3 & h_4 \end{array} \right) := \left( \begin{array}{cc} h_{ij} & h_{ipq} \\ h_{pq} & h_{pqmn} \end{array} \right) \in M(n(n + 3)/2, \mathbb{C}), \quad p \leq q, m \leq n. \quad h = h^*,$$  (2.53)

We use the following convention: indices of $z \in M(n \times 1, \mathbb{C})$ are denoted with: $i, j, k, l$; indices of $W = W^t, W \in M(n, \mathbb{C})$ are denoted with: $m, n, p, q, r, s, t, u, v$.

We have determined the “inverse” $h^{-1}$ of (2.52) such that

$$\Delta_{pq}^i := \frac{\partial w_{ij}}{\partial w_{pq}} = \delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp} - \delta_{ij} \delta_{pq} \delta_{ip} = (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}) f_{ij}, \quad w_{ij} = w_{ji}.$$  (2.54)

We use the notation

$$f_{pq} := 1 - \frac{1}{2} \delta_{pq}; \quad d_{pq} := 1 - \delta_{pq}.$$  (2.55)

In [16] we have proved

**Theorem 1.** The Kähler two-form $\omega_{D^d_n}$, associated with the balanced metric of the
Siegel-Jacobi ball $D^d_n$, $G^d_n$-invariant to the action (2.5a), (2.5b) has the expression

$$-i \omega_{D^d_n}(z,W) = \frac{k}{2} \text{Tr}(\mathbb{B} \wedge \bar{\mathbb{B}}) + \mu \text{Tr}(\mathbb{A} \wedge \bar{\mathbb{M}}), \quad A = dz + dW\bar{\eta},$$  (2.56)

$$\mathbb{B} = M dW, \quad M = (\mathbb{I}_n - W\bar{W})^{-1}.$$
The matrix (2.52) of the metric on $D_n^1$ has the matrix elements (2.57):

(2.57a) \[ h_{ij} = \mu M_{ij}, \]
(2.57b) \[ h_{ipq} = \mu (\eta_q \bar{M}_{ip} + \eta_p \bar{M}_{iq}) f_{pq}, \]
(2.57c) \[ h_{pq} = \mu (\eta_q \bar{M}_{pi} + \eta_p \bar{M}_{qi}) f_{pq}, \]
(2.57d) \[ h_{pqmn} = \frac{k}{2} h_{pqmn} + \mu h_{pqmn}^\mu, \]
(2.57e) \[ h_{pqmn}^\mu = 2 (M_{mp} M_{nq} + M_{mq} M_{np}) - 2 M_{mp} (M_{np} \delta_{pq} + M_{mq} \delta_{mn}) + M^2_{mp} \delta_{pq} \delta_{mn} = 2 M_{mp} M_{nq} d_{pq} + 2 M_{mq} M_{np} d_{mn} + M^2_{mp} \delta_{pq} \delta_{mn}, \]
(2.57f) \[ h_{pqmn}^{\mu} = \bar{\eta}_p (\eta_m M_{nq} + \eta_m M_{mq}) d_{pq} + \bar{\eta}_q (\eta_n M_{mp} + \eta_n M_{np}) - \eta_m (\bar{\eta}_p M_{nq} + \bar{\eta}_q M_{mq}) \delta_{mn} + \bar{\eta}_n (\eta_m M_{mp} + \eta_m M_{np}) \delta_{mn} = [\bar{\eta}_p (\eta_m M_{nq} + \eta_m M_{mq}) + \bar{\eta}_q (\eta_n M_{mp} + \eta_n M_{np})] f_{pq} f_{mn}. \]

The “inverse” of the matrix with elements $h_{pqmn}^\mu$ given by (2.57e) has the matrix elements

(2.58) \[ k_{mn\bar{\nu}} = \frac{1}{2} (N_{en} N_{mu} + N_{vm} N_{nu}), \quad N = \mathbb{I}_n - WW, \]
and we have

(2.59) \[ \sum_{m \leq n} h_{pqmn}^\mu k_{mn\bar{\nu}} = \Delta_{pq}^uv. \]

The “inverse” $h^{-1}$ of the metric matrix $h$ which verifies (2.53), has the elements $h_{1}^{1} - h_{4}^{4}$ given by

(2.60a) \[ (h_{1})_{ij} = \sigma M_{ij}^{-1}, \quad \sigma = \frac{1}{\mu} + \alpha \frac{2}{k}, \quad \alpha = \eta^t \bar{M}^{-1} \bar{\eta} = \bar{\eta} S_n, \quad S_n = \sum \eta_q \bar{M}_{qn}^{-1}; \]
(2.60b) \[ (h_{2})_{im\bar{n}} = -\frac{1}{k} (S_n \bar{M}_{in}^{-1} + S_m \bar{M}_{in}^{-1}); \]
(2.60c) \[ (h_{3})_{mn\bar{i}} = -\frac{1}{k} (S_n \bar{M}_{ni}^{-1} + S_m \bar{M}_{ni}^{-1}); \]
(2.60d) \[ (h_{4})_{pqmn} = \frac{2}{k} (h_{1})_{pqmn} = \frac{1}{k} (\bar{M}_{mn}^{-1} \bar{M}_{pm}^{-1} + \bar{M}_{mn}^{-1} \bar{M}_{qm}^{-1}). \]

The determinant of the metric matrix $h$ is

(2.61) \[ \mathcal{D}_n(z, W) = \det h_{D_n^1}(z, W) = \left( \frac{k}{2} \right)^{\frac{n(n+1)}{2}} \mu^n \det (\mathbb{I}_n - WW)^{-n-2}. \]

In the case $n = 1$ formulae (2.57), (2.60), (2.61) became the formulae (2.28), (2.29), respectively (2.31), with $2k \leftrightarrow \frac{k}{2}$.

Now we recall (see Proposition 4 in [12], Lemma 2 and Proposition 3 in [13], and Propositions 1 and 2 in [14]) the significance of the change of coordinates (2.50) as a FC-transform in the language of coherent states and also in the context of fundamental conjecture for homogeneous Kähler manifolds of Gindikin-Vinberg [38] (see the proof in [25]).
Proposition 2. a) The change of coordinates  
\[(2.50) \quad (z,W) = FC(\eta,W), z = \eta - W\bar{\eta},\]
is a FC-transform in the meaning of  
\[(2.11).\]

b) The FC-transform  
\[(2.50) \quad \C^n \times \D_n \ni (\eta,W) \overset{FC}{\to} (z,W) \in \D_n^J\]
is Kähler homogeneous diffeomorphism, i.e.
\[
\omega_{\C^n \times \D_n}(z,\eta) = FC^*[\omega_{\D_n^J}(z,W)] = \omega_{\D_n}(W) + \omega_{\C^n}(\eta),
\]
where
\[
(2.62) \quad -i\omega_{\D_n}(W) = \frac{k}{2}\text{Tr}(\mathcal{B} \wedge \bar{\mathcal{B}}), \quad -i\omega_{\C^n} = \mu\text{Tr}(d\eta^i \wedge d\bar{\eta}).
\]
The Kähler two-form  
\[
\omega_{\C^n \times \D_n}(z,\eta)
\]
is invariant to the  
\[G_n^J\]-action on  
\[\D_n \times \C^n: (g,\alpha) \times (\eta,W) \to (\eta_1,W_1),\]
where  
\[g\] has the expression  
\[(2.2),\]
\[W_1\] is given by  
\[(2.5a),\]
and  
\[\eta_1 = p(\eta + \alpha) + q(\bar{\eta} + \bar{\alpha}).\]

In the following proposition we collect several geometric properties of the Siegel-Jacobi group and its action on the Siegel-Jacobi ball (see Remark 4, Proposition 3 and Proposition 4 from [16], where all the terms appearing in the enunciation below are explained).

Proposition 3. i) The Jacobi group  
\[G_n^J\] is an unimodular, non-reductive, algebraic group of Harish-Chandra type.

ii) The Siegel-Jacobi domain  
\[\D_n^J\] is a homogeneous reductive, non-symmetric manifold associated to the Jacobi group  
\[G_n^J\] by the generalized Harish-Chandra embedding.

iii) The homogeneous Kähler manifold  
\[\D_n^J\] is contractible.

iv) The Kähler potential of the Siegel-Jacobi ball is global.  
\[\D_n^J\] is a Lu Qi-Keng manifold, with nowhere vanishing diastasis.

v) The manifold  
\[\D_n^J\] is a quantizable manifold.

vi) The  
\[\D_n^J\] is projectively induced, and the Jacobi group  
\[G_n^J\] is a CS-type group.

vii) The Siegel-Jacobi ball  
\[\D_n^J\] is not an Einstein metric with respect to the balanced metric attached to the Kähler two-form  
\[(2.56),\]
but it is one with respect to the Bergman Kähler two-form.

 ix) The scalar curvature is constant and negative.

3. Geodesics on the Siegel-Jacobi ball

3.1. Calculation of  \(\Gamma\)-s. In the case of the Siegel-Jacobi ball, the equations of geodesics  
\[(2.17)\] read
\[
(3.1a) \quad \frac{d^2 z_i}{dt^2} + \Gamma^i_{jk} \frac{dz_j}{dt} \frac{dz_k}{dt} + 2 \sum_{p \leq q} \Gamma^i_{jpq} \frac{dz_j}{dt} \frac{dw_{pq}}{dt} + \sum_{p \leq q, m \leq n} \Gamma^i_{pqmn} \frac{dw_{pq}}{dt} \frac{dw_{mn}}{dt} = 0,
\]
\[
(3.1b) \quad \frac{d^2 w_{pq}}{dt^2} + \Gamma^p_{jk} \frac{dz_j}{dt} \frac{dz_k}{dt} + 2 \sum_{m \leq n} \Gamma^p_{imn} \frac{dz_i}{dt} \frac{w_{mn}}{dt} + \sum_{m \leq n, u \leq v} \Gamma^p_{mnuv} \frac{w_{mn}}{dt} \frac{w_{uv}}{dt} = 0.
\]
Equations  
\[(2.32)\] are a particular case of  
\[(3.1).\]
The equations (2.21) which determine the $\Gamma$-symbols for the Siegel-Jacobi ball $\mathcal{D}_n^J$ are

\[
\begin{align*}
\{ h_{ij} \Gamma_{kl}^i + & \sum_{m \leq n} h_{mnj} \Gamma_{kl}^{mn} = \frac{\partial h_{ij}}{\partial z_k} = \frac{\partial h_{ij}}{\partial \bar{z}_l}, \\
h_{ipq} \Gamma_{kl}^i + & \sum_{m \leq n} h_{mnj} \Gamma_{kl}^{mn} = \frac{\partial h_{ipq}}{\partial z_k} ,
\end{align*}
\]

(3.2)

\[
\begin{align*}
\{ h_{ij} \Gamma_{kpq}^i + & \sum_{m \leq n} h_{mnj} \Gamma_{kpq}^{mn} = \frac{\partial h_{ipq}}{\partial z_k} , \\
h_{ipq} \Gamma_{krs}^i + & \sum_{m \leq n} h_{mnj} \Gamma_{krs}^{mn} = \frac{\partial h_{ipq}}{\partial w_k} .
\end{align*}
\]

(3.3)

\[
\begin{align*}
\{ h_{ij} \Gamma_{pquv}^i + & \sum_{m \leq n} h_{mnj} \Gamma_{pquv}^{mn} = \frac{\partial h_{uv1}}{\partial z_k} , \\
h_{ipq} \Gamma_{pquv}^i + & \sum_{m \leq n} h_{mnj} \Gamma_{pquv}^{mn} = \frac{\partial h_{uv1}}{\partial w_k} .
\end{align*}
\]

(3.4)

Equations (3.2)-(3.4) are a generalization on $\mathcal{D}_n^J$ of the corresponding ones (2.33)-(2.35) on $\mathcal{D}_1^J$.

With formula (2.23), we have

\[
\begin{align*}
\Gamma_{jk}^i &= h_{it} \frac{\partial h_{kl}}{\partial z_j} + \sum_{p \leq q} h_{pqk} \frac{\partial h_{kpl}}{\partial z_j}, \\
\Gamma_{ijpq}^i &= h_{ik} \frac{\partial h_{pqk}}{\partial z_j} + \sum_{r \leq s} h_{rsi} \frac{\partial h_{pqrs}}{\partial z_j}, \\
\Gamma_{jqk}^p &= h_{ipq} \frac{\partial h_{kij}}{\partial z_j} + \sum_{m \leq n} h_{mnq} \frac{\partial h_{knm}}{\partial z_j}, \\
\Gamma_{pmn}^{qij} &= h_{ij} \frac{\partial h_{mnj}}{\partial z_i} + \sum_{u \leq v} h_{wpq} \frac{\partial h_{mnv}}{\partial z_i}, \\
\Gamma_{pqmn}^{ijk} &= h_{ij} \frac{\partial h_{mnj}}{\partial w_{pq}} + \sum_{r \leq s} h_{rsi} \frac{\partial h_{mnrs}}{\partial w_{pq}}, \\
\Gamma_{mnuv}^{pq} &= h_{ipq} \frac{\partial h_{uv1}}{\partial w_{mn}} + \sum_{s \leq t} h_{stq} \frac{\partial h_{uvst}}{\partial w_{mn}}.
\end{align*}
\]

(3.5a)-(3.5f)

Equations (3.5) for $\mathcal{D}_n^J$ generalize equations (2.36) for $\mathcal{D}_1^J$.

We shall use the formulae given in Theorem 1 for the matrix elements of the metric and its inverse and also some formulae (see (3.28) and (3.29) in [16])

\[
\begin{align*}
\frac{\partial M_{ab}}{\partial w_{ik}} &= (M_{ai}X_{bk} + M_{ak}X_{ib})f_{ik}, \text{ where } X = X^t = \bar{W}M = \bar{M}W, \\
\frac{\partial X_{ab}}{\partial w_{ik}} &= (X_{ai}X_{bk} + X_{ak}X_{ib})f_{ik}, \\
\frac{\partial X_{ab}}{\partial w_{ik}} &= (M_{ai}M_{bk} + M_{ak}M_{bi})f_{ik}.
\end{align*}
\]

(3.6a)-(3.6c)
\[
\frac{\partial \eta_k}{\partial z_i} = M_{qk}, \quad \frac{\partial \bar{\eta}_k}{\partial z_j} = X_{qj},
\]
(3.7)
\[
\frac{\partial \eta_k}{\partial w_{pq}} = (M_{tp} \bar{\eta}_q + M_{tq} \bar{\eta}_p) f_{pq}, \quad \frac{\partial \bar{\eta}_l}{\partial w_{pq}} = (\bar{\eta}_p X_{ql} + \bar{\eta}_q X_{pl}) f_{pq}.
\]

Now we want to calculate the partial derivatives which appear in (3.5), a generalization of the partial derivatives (2.37). We use (3.9) and (3.7) to calculate the partial derivatives which appear in the equations (3.5) of $\Gamma$-s and we get
\[
\frac{\partial h_{ki}}{\partial z_j} = 0, \quad \frac{\partial h_{kpq}}{\partial z_j} = \mu (M_{pk} M_{qj} + M_{pq} M_{kj}) f_{pq},
\]
\[
h_{pqk} = \mu (X_{qj} M_{kp} + X_{pj} M_{kq}) f_{pq}, \quad \frac{\partial h_{pqrs}}{\partial z_j} = \mu M_{rp} [X_{aq} \eta_s + \bar{\eta}_q M_{sj}],
\]
\[
\frac{1}{\mu} \frac{\partial h_{pqrs}}{\partial z_j} = \left[ X_{pq} (\eta_s M_{qr} + \eta_r M_{qs}) + X_{aq} (\eta_s M_{pr} + \eta_r M_{ps}) \right] f_{pq} f_{rs},
\]
\[
+ \left[ \bar{\eta}_p (M_{sj} \bar{M}_{qr} + M_{rj} \bar{M}_{qs}) + \bar{\eta}_q (M_{sj} \bar{M}_{pr} + M_{rj} \bar{M}_{ps}) \right] f_{pq} f_{rs},
\]
\[
\frac{1}{\mu} \frac{\partial h_{mnij}}{\partial w_{pq}} = \left[ \Lambda_{pq} M_{mij} + \bar{\eta}_n (M_{jp} X_{qm} + M_{jq} X_{pm}) \right] f_{mn} f_{pq},
\]
\[
+ \left[ \Lambda_{pq} M_{mij} + \bar{\eta}_m (M_{jp} X_{qn} + M_{jq} X_{pn}) \right] f_{mn} f_{pq},
\]
\[
\frac{\partial h_{mnij}}{\partial w_{pq}} = 2 \left[ \frac{1}{\mu} M_{rp} X_{qm} + M_{rq} X_{pm} \right] d_{mn} f_{pq}
\]
\[
+ 2 \left[ \frac{1}{\mu} M_{rp} X_{qn} + M_{rq} X_{pn} \right] d_{mn} f_{pq}
\]
\[
+ 2 \frac{1}{\mu} M_{rn} (M_{sp} X_{qm} + M_{sq} X_{pn}) \delta_{mn} \delta_{rs} f_{pq},
\]
where
(3.9)
\[
\Lambda_{pq} := \bar{\eta}_p X_{qm} + \bar{\eta}_q X_{pn}.
\]

We have also
(3.10)
\[
\frac{\partial h_{mnij}}{\partial w_{pq}} = \Upsilon_{pq} f_{mn} f_{rs} f_{pq},
\]
where
\[
\Upsilon_{pq} = \Lambda_{pq} (\eta_s M_{nr} + \eta_r M_{ns}) + \Lambda_{pq} (\bar{\eta}_s \bar{M}_{nr} + \bar{\eta}_r \bar{M}_{ns})
\]
\[
+ \bar{\eta}_n (X_{sp} M_{rq} + \bar{\eta}_p M_{sq} \bar{M}_{rp} + \bar{\eta}_q M_{sp} X_{rq} + \bar{\eta}_p M_{sq} X_{rn} + \eta_r M_{ps} X_{qn} + \eta_s M_{qs} X_{pn})
\]
\[
+ \bar{\eta}_n (X_{sp} M_{rq} + \bar{\eta}_p M_{sq} \bar{M}_{rp} + \bar{\eta}_q M_{sp} X_{rq} + \bar{\eta}_p M_{sq} X_{rn} + \eta_r M_{ps} X_{qn} + \eta_s M_{qs} X_{pn})
\]
\[
+ \bar{\eta}_n (X_{sp} M_{rq} + \bar{\eta}_p M_{sq} \bar{M}_{rp} + \bar{\eta}_q M_{sp} X_{rq} + \bar{\eta}_p M_{sq} X_{rn} + \eta_r M_{ps} X_{qn} + \eta_s M_{qs} X_{pn}).
\]

In order to calculate $\Gamma_{pqmn}$ with (3.5c), we calculate the first term
\[
\frac{1}{h^{ij}} \frac{\partial h_{mnij}}{\partial w_{pq}} = (1 + 2 \epsilon \alpha) X_{pqmn}^i f_{mn}, \quad \epsilon = \frac{\mu}{k}.
\]
where

\[
X_{pqmn}^i = (\delta_{mi} \Lambda_m^{pq} + \delta_{ni} \Lambda_n^{pq} + \bar{\eta}_m \Omega_{pqmn}^i + \bar{\eta}_n \Omega_{pqmn}^i) f_{pq},
\]

(3.12)

\[
\Omega_{pqmn}^i := \delta_{qp} X_{qm} + \delta_{iq} X_{pm}.
\]

(3.13)

Now we calculate

\[
2 \Gamma^i_{pqmn} := \sum_{r \leq s} h^{rs}_{it} \frac{\partial h_{mnrs}}{\partial w_{pq}} = 2k \Gamma^i_{pqmn} + 2\mu \Gamma^i_{pqmn},
\]

where

\[
2k \Gamma^i_{pqmn} = -\sum_{s \leq r} (\bar{S}_s \bar{M}_{rs}^{-1} + \bar{S}_r \bar{M}_{si}^{-1}) \frac{\partial h_{kmfp}}{\partial w_{pq}}.
\]

We find

\[
2k \Gamma^i_{pqmn} = -\frac{1}{2} X^i_{pqmn},
\]

while

\[
2k \Gamma^i_{pqmn} = -X^i_{pqmn}, m \neq n,
\]

i.e.

\[
2k \Gamma^i_{pqmn} = -X^i_{pqmn} f_{mn}.
\]

For

\[
2\mu \Gamma^i_{pqmn} := \sum_{r \leq s} h^{stpq} \frac{\partial h_{munf}}{\partial w_{pq}},
\]

we find

\[
2\mu \Gamma^i_{pqmn} := -\epsilon f_{mn} f_{pq} Y^i_{pqmn},
\]

where

\[
Y^i_{pqmn} = \alpha (\delta_{mi} \Lambda_m^{pq} + \delta_{ni} \Lambda_n^{pq} + \bar{\eta}_m \Omega_{pqmn}^i + \bar{\eta}_n \Omega_{pqmn}^i) + 2S_i (\bar{\eta}_n \Lambda_m^{pq} + \bar{\eta}_m \Lambda_n^{pq}) + 2\bar{\eta}_p \bar{\eta}_q (\bar{\eta}_m \delta_{in} + \bar{\eta}_n \delta_{im}) + 2\bar{\eta}_n \bar{\eta}_m (\bar{\eta}_q \delta_{ip} + \bar{\eta}_p \delta_{iq}),
\]

(3.14)

We find

\[
\Gamma^i_{pqmn} = \epsilon f_{mn} f_{pq} Z^i_{pqmn},
\]

where

\[
Z^i_{pqmn} = \alpha (\delta_{mi} \Lambda_m^{pq} + \delta_{ni} \Lambda_n^{pq} + \bar{\eta}_m \Omega_{pqmn}^i + \bar{\eta}_n \Omega_{pqmn}^i) - 2S_i (\bar{\eta}_n \Lambda_m^{pq} + \bar{\eta}_m \Lambda_n^{pq}) + 2\bar{\eta}_p \bar{\eta}_q (\bar{\eta}_m \delta_{in} + \bar{\eta}_n \delta_{im}) - 2\bar{\eta}_n \bar{\eta}_m (\bar{\eta}_q \delta_{ip} + \bar{\eta}_p \delta_{iq}),
\]

(3.15)

In (3.5f) we calculate firstly

\[
1 \Gamma^p_{mnuv} := h^{tq}_{iu} \partial h_{uv}^{mi} \partial w_{mn},
\]

(3.16)

\[
= -\epsilon f_{uv} f_{mn} S_q (\delta_{pu} \Lambda_v^{mn} + \bar{\eta}_v \Omega_{rpmnu}^{p} + \delta_{pw} \Lambda_u^{mn} + \bar{\eta}_u \Omega_{rpmnu}^{p}) - \epsilon f_{uv} f_{mn} S_p (\delta_{qu} \Lambda_v^{mn} + \bar{\eta}_v \Omega_{nmup}^{q} + \delta_{q} \Lambda_u^{mn} + \bar{\eta}_u \Omega_{nmup}^{q}).
\]
Now we calculate in (3.15)

\[ 2k \Gamma^q_{mnw} := h^{rst} \frac{\partial h^k_{mnw}}{\partial w_{pq}} = f_{uv} f_{mn}(\delta_{vq} \Omega^p_{mnw} + \delta_{wp} \Omega^q_{mnv} + \delta_{up} \Omega^q_{mnv} + \delta_{vp} \Omega^q_{mnw}). \]

where

\[ U^s_{mnw} = \Lambda^m_u (S_q \delta_{pv} + S_p \delta_{qv}) + \Lambda^m_n (S_q \delta_{pu} + S_p \delta_{qu}) + (\bar{\eta}_m \delta_{mp} + \bar{\eta}_m \delta_{np}) \]

\[ + \eta_n (S_q \Omega^p_{mnv} + S_p \Omega^q_{mnv}) + \eta_v (S_q \Omega^p_{mnw} + S_p \Omega^q_{mnw}). \]

Now we calculate

\[ \Gamma^q_{mnw} := 2k \Gamma^q_{mnw} + 2 \mu \Gamma^q_{mnw}. \]

We find

\[ V^p_{mnw} = \delta_{vq} \Omega^p_{mnw} + \delta_{wp} \Omega^q_{mnv} + \delta_{up} \Omega^q_{mnv} + \delta_{vp} \Omega^q_{mnw} + \epsilon W^p_{mnw}; \]

\[ (3.19) \]

\[ W^p_{mnw} = (\bar{\eta}_m \delta_{mp} + \bar{\eta}_m \delta_{np}) (\delta_{vp} \eta_u + \delta_{up} \bar{\eta}_w) + (\bar{\eta}_n \delta_{mp} + \bar{\eta}_n \delta_{np}) (\delta_{vp} \eta_u + \delta_{up} \bar{\eta}_w). \]

We have proved

**Proposition 4.** The Christoffel’s symbols of the Siegel-Jacobi ball \( \mathcal{D}_n^J \) endowed with the balanced metric attached to the Kähler two-form (2.56) are

\[ (3.20a) \Gamma^i_{jk} = -\epsilon (\bar{\eta}_j \delta_{ik} + \bar{\eta}_k \delta_{ij}), \quad \text{where} \quad \epsilon = \frac{\mu}{k}, \]

\[ (3.20b) \Gamma^i_{jq} = \left( 1 + \epsilon \alpha \right) (X_{qj} \delta_{pi} + X_{pj} \delta_{qi}) - \epsilon S_i (\bar{\eta}_p X_{qj} + \bar{\eta}_q X_{pj})], f_{pq}, \]

\[ (3.20c) - \epsilon (\bar{\eta}_j (\bar{\eta}_q \delta_{pi} + \bar{\eta}_p \delta_{qi}) + 2 \bar{\eta}_q \bar{\eta}_p \delta_{ij}], f_{pq}, \]

\[ (3.20d) \Gamma^i_{jk} = \epsilon (\delta_{kp} \delta_{jq} + \delta_{jq} \delta_{kp}); \]

\[ (3.20e) \Gamma^i_{mn} = [\bar{\eta}_m (\delta_{mp} \delta_{iq} + \delta_{iq} \delta_{mp}) + \bar{\eta}_n (\delta_{mp} \delta_{iq} + \delta_{iq} \delta_{mp})], f_{mn}; \]

\[ (3.20f) \Gamma^i_{pqmn} = \epsilon f_{mn} f_{pq} Z^i_{pqmn}; \]

\[ (3.20g) \Gamma^i_{pqmn} = f_{uv} f_{mn} V^p_{pqmn}. \]

\( S_i \) was defined in (2.60a), \( X \) was defined in (3.6a), \( Z^i_{pqmn} \) is given by (3.13), \( V^p_{mnw} \) is given by (3.19), \( \Lambda^p_{mn} \) is defined in (3.9), \( \Omega^m_{pqmn} \) is given by (3.13).

Equations (3.20) of the Christoffel’s symbols for \( \mathcal{D}_n^J \) generalize the corresponding ones (2.38) on \( \mathcal{D}_1^J \).

For the proof of the last assertion in Proposition 4, we have in the case case \( n = 1 \) the following relations:

\[ \alpha = |\eta|^2 P, \quad S = \eta P, \quad M = \bar{M} = P^{-1}, \quad X := \bar{W} M \rightarrow \bar{w} P^{-1}, \quad \Lambda = \bar{\eta} \bar{w} / P, \quad \Omega = X, \]

\[ Y = \frac{\bar{\eta} \bar{w}}{P^2} (\bar{\eta} + 2 \bar{\eta} \bar{w}), \quad X^i_{pqmn} \rightarrow 2 (\Lambda + \bar{\eta} \Omega) = 4 \bar{\eta} \Omega, \quad Y = 4 \bar{\eta}^2 (\bar{\eta} + 2 \bar{\eta} \bar{w}), \quad Z = -4 \bar{\eta}^3. \]

\( \frac{1}{2k} \) corresponds in the case \( n = 1 \) to \( \frac{1}{2}, \) i.e. \( \epsilon = \frac{1}{2} \lambda. \)
3.2. Equations of geodesics. Now we calculate the equations of geodesics (3.1). In order to calculate (3.1a), we find successively:

\[ \Gamma^i_{jk} \frac{dz_j}{dt} \frac{dz_k}{dt} = -2\epsilon L \frac{dz_i}{dt}, \]

\[ \sum_{p \leq q} \Gamma^i_{jpq} \frac{dz_j}{dt} \frac{dw_{pq}}{dt} = Q_i + \epsilon[\alpha Q_i - S_i \eta^i Q - L\lambda_i - \eta^i \frac{dW}{dt} \frac{\bar{\eta} dz_i}{dt}], \]

\[ \sum_{p \leq q, m \leq n} \Gamma^i_{pqmn} \frac{dw_{pq}}{dt} \frac{dw_{mn}}{dt} = 2\epsilon[\alpha (\lambda \frac{dW}{dt})_i - S_i (\lambda^i X \frac{dW}{dt}) \bar{\eta} - \eta^i \frac{dW}{dt} \eta \lambda_i], \]

where

\[ L = \bar{\eta} \frac{dz_k}{dt}, \lambda = \eta^i \frac{dW}{dt}, Q = \frac{dW}{dt} X \frac{dz}{dt}. \]

We write down (3.1a) as

\[ G^3 = 2\epsilon G^1, \]

where

\[ G^1_i = (\eta^i Y)_i + (\eta^i \frac{dW}{dt} XY) S_i - \alpha (\frac{dW}{dt} X) Y_i, \ Y = \frac{dZ}{dt} + \frac{dW}{dt} \bar{\eta}, \]

\[ G^3 = \frac{d^2 z}{dt^2} + 2Q. \]

Now we calculate (3.1b). We find successively

\[ \Gamma^{pq}_{jk} \frac{dz_j}{dt} \frac{dz_k}{dt} = 2\epsilon \frac{dz_p}{dt} \frac{dz_q}{dt}, \]

\[ \sum_{m \leq n} \Gamma^{pq}_{imn} \frac{dz_i}{dt} \frac{dw_{mn}}{dt} = \epsilon(\lambda_p \frac{dz_q}{dt} + \lambda_q \frac{dz_p}{dt}), \]

\[ \sum_{m \leq n, u \leq v} \Gamma^{pq}_{mnuv} \frac{dw_{mn}}{dt} \frac{dw_{uv}}{dt} = 2\epsilon \lambda_p \lambda_q + 2(\frac{dW}{dt} X \frac{dW}{dt})_{pq}. \]

We write (3.1b) as

\[ G^2 = -2\epsilon G^4, \]

where

\[ G^2 = \frac{d^2 W}{dt^2} + 2\frac{dW}{dt} X \frac{dW}{dt}; \ G^4 = Y \otimes Y \]

We rewrite equations (3.21), (3.24) as (3.26)

\[ (3.26a) \quad kG^3 = 2\mu G^1, \]

\[ (3.26b) \quad kG^2 = -2\mu G^4, \]

and we have proved
Proposition 5. The equations of geodesics (3.1) on the Siegel-Jacobi ball \( D^I_n \) corresponding to the balanced metric attached to the Kähler two-form (2.56) are given in (3.26), where \( G^1 \), \( G^2 \) and \( G^4 \), \( G^3 \), are given by (3.22), (3.25), respectively (3.23).

Equations (3.26) in the case of \( D^I_1 \) are those given in Proposition 1.

Now we discuss the solution of the system of differential equations (3.26).

(a) If we consider \( \mu = 0 \), then in (3.26b) we get \( G^2 = 0 \), i.e. the equation of geodesics on the Siegel ball \( D_n \) (see the Appendix)

\[
G^2 := \frac{d^2 W}{dt^2} + 2 \frac{dW}{dt} X \frac{dW}{dt} = 0.
\]

The solution of the equation (3.27) with \( W(0) = \emptyset \) and \( \dot{W}(0) = B \) is

\[
W(t) = B \frac{\tanh(t\sqrt{BB})}{\sqrt{BB}}.
\]

Note that the change of coordinates (3.28) is a FC-transform in the sense of (2.10) on the Siegel ball \( D_n \) (see the comment after Remark 7 in [13] for \( D_1 \) and formula (2.25) for \( D_n \) in [9]).

(b) If \( \mu \neq 0 \), a particular solution \((z, W)\) of the equations of geodesics on Siegel-ball Jacobi \( D^J_n \) is given by \( z(t) = \eta_0 - W(t) \bar{\eta}_0 \), where \( W(t) \) has the expression (3.28) and \( \eta_0 \) is independent of \( t \). This is a particular case of the solution corresponding to \( \eta = \eta_0 + t \eta_1 \) of the equations of geodesics \( \frac{d^2 \eta}{dt^2} = 0 \) on the flat manifold \( \mathbb{C}^n \) corresponding to the separation of variables by the FC-transform. We can formulate the

Proposition 6. The FC-transform (2.50) is not a geodesic mapping, but it gives geodesics \((z(t), W(t)) = FC(\eta_0, W(t))\) on the nonsymmetric space \( D^J_n \), with \( W(t) \) given by (3.28).

For more details, see Remark 8 and appendix A in [13], where the notion of geodesic mapping (cf. Definition 5.1 p. 127 in [31]) is explained.

3.3. Covariant derivative of one-forms. We calculate the connection matrix (2.19) on the Siegel-Jacobi ball using the Christoffel’s symbols obtained in Proposition 4

\[
\theta = \begin{pmatrix}
\theta^i_j \\
\theta^i_{jq} \\
\theta^i_{pq} \\
\theta^i_{mn}
\end{pmatrix}.
\]

We have

\[
\theta^i_j := \Gamma^i_{jk} dz_k + \sum_{r<q} \Gamma^i_{jq} d w_{pq}.
\]

We get

\[
\theta^i_j = (1 + \epsilon \alpha) \Xi^i_j - \epsilon [\bar{\eta}_j A_i + \delta_{ij} \bar{\eta}^l A + S_i (\bar{\Xi} \bar{\eta})]_j,
\]
where we have introduced the notation
\[(3.32)\quad \Xi = X \, dW.\]

We calculate
\[(3.33)\quad \theta^i_{pq} := \Gamma^i_{pqj} \, dz_j + \sum_{m \leq n} \Gamma^i_{pqn} \, dw_{mn},\]
and we obtain
\[(3.34)\quad \theta^i_{pq} = f_{pq} [\delta_{ip}(X \, dz)_q + \delta_{iq}(X \, dz)_p + \epsilon T^i_{pq}],
\]
\[T^i_{pq} = \alpha[\delta_{ip}(X \mathcal{A})_q + \delta_{iq}(X \mathcal{A})_p + \bar{\eta}_p \Xi_q + \bar{\eta}_q \Xi_p] - S_i \{\bar{\eta}_p [X(A + dW\bar{\eta})]_q + \bar{\eta}_q [X(A + dW\bar{\eta})]_p\}
- [(\bar{\eta}_q \delta_{ip} + \bar{\eta}_p \delta_{iq})(\bar{\eta}_t \mathcal{A}) + 2\bar{\eta}_p \bar{\eta}_q A_i].\]

For
\[(3.35)\quad \theta^pq := \Gamma^pq_{ij} \, dz_j + \sum_{m \leq n} \Gamma^pq_{imn} \, dw_{mn},\]
we get
\[(3.36)\quad \theta^pq = \epsilon(\delta_{iq}A_p + \delta_{ip}A_q).\]

We calculate
\[(3.37)\quad \theta^pq_{mn} := \Gamma^pq_{imn} \, dz_i + \sum_{u \leq v} \Gamma^pq_{mnuv} \, dw_{uv},\]
as
\[(3.38)\quad \Gamma^pq_{imn} \, dz_i = \epsilon[\bar{\eta}_m (\delta_{np} \, dz_q + \delta_{nq} \, dz_p) + \bar{\eta}_n (\delta_{mp} \, dz_q + \delta_{mq} \, dz_p)] f_{mn},
\]
\[\sum_{u \leq v} \Gamma^pq_{mnuv} \, dw_{uv} = [\delta_{pm}\Xi_{mq} + \delta_{pm}\Xi_{mq} + \delta_{qm}\Xi_{np} + \delta_{qm}\Xi_{np}] f_{mn}
+ \epsilon[(\bar{\eta}_m \delta_{mq} + \bar{\eta}_n \delta_{nq})(\tilde{\eta}^t \, dW)_p + (\bar{\eta}_m \delta_{mp} + \bar{\eta}_n \delta_{np})(\tilde{\eta}^t \, dW)_q] f_{mn}\]
For \[(3.37)\] we obtain
\[(3.39)\quad \theta^pq_{mn} = [\delta_{pm}\Xi_{mq} + \delta_{pm}\Xi_{mq} + \delta_{qm}\Xi_{np} + \delta_{qm}\Xi_{np}] f_{mn}
+ \epsilon[(\bar{\eta}_m \delta_{mq} + \bar{\eta}_n \delta_{nq})A_p + (\bar{\eta}_m \delta_{mp} + \bar{\eta}_n \delta_{np})A_q] f_{mn}.\]

We calculate the covariant derivative of the one form \(d \, z_i\) with \[(2.20)\] as
\[(3.40)\quad -D(d \, z_i) = w^j_i \, dz_j + \sum_{p \leq q} \theta^j_{pq} \, dw_{pq}.\]

We get successively
\[(3.41)\quad \theta^j_i \, dz_j = (1 + \epsilon \alpha)(\Xi^t \, d \, z)_i - \epsilon[(\tilde{\eta}^t \, d \, z)A_i + (\tilde{\eta}^t \mathcal{A}) \, d \, z_i + (\tilde{\eta}^t \Xi^t \, d \, z)S_i],
\]
\[\sum_{p \leq q} \theta^j_{pq} \, dw_{pq} = (\Xi^t \, d \, z)_i + \epsilon\{\alpha[\Xi^t(A + dW\bar{\eta})]_i + [\tilde{\eta}^t\Xi^t(A + dW\bar{\eta})]S_i\}
- \epsilon[(\tilde{\eta}^t \mathcal{A})(dW\bar{\eta})_i + (\tilde{\eta}^t \, dW\bar{\eta})A_i].\]
We get for (3.40) the expression

\[(3.42) \quad - \frac{1}{2} D(dz_i) = \{ \Xi^t[(1 + \epsilon\alpha)A - dW\bar{\eta}] \}_i - \epsilon[(\bar{\eta}^tA)A_i + (\bar{\eta}^i\Xi^tA)S_i].\]

Now we calculate the covariant derivative of \(dwp_{pq}\):

\[(3.43) \quad - D dwp_{pq} = \theta^pq_i d z_i + \sum_{m \leq n} \theta^pq_{mn} d w_{mn}.\]

We obtain

\[\theta^pq_i d z_i = \epsilon(A_p d z_q + A_q d z_p),\]

\[\sum_{m \leq n} \theta^pq_{mn} d w_{mn} = (dW\Xi)pq + (dW\Xi)qp + \epsilon[A_p(dW\bar{\eta})q + A_q(dW\bar{\eta})p],\]

which introduced in (3.43) gives

\[- D(dwp_{pq}) = (dW\Xi)pq + (dW\Xi)qp + 2\epsilon A_p A_q,\]

which can be written as

\[(3.44) \quad - D(dW) = 2 \left( \begin{array}{cccc} \epsilon I_n & 0 \\ 0 & A \end{array} \right) \left( \begin{array}{c} dW \end{array} \right).\]

We have proved

**Proposition 7.** The connections matrix (3.29) on \(D_n^J\), whose matrix elements are calculated with formulae (3.30), (3.33), (3.35), and (3.37), are given by (3.31), (3.34), (3.36), respectively (3.39).

The covariant derivative \(D(dz_i)\) (3.40) and \(D(dwp_{pq})\) (3.43) on the Siegel-Jacobi ball \(D_n^J\) are given by formulae (3.42), respectively (3.44), which generalize (2.41), respectively (2.42) on the Siegel-Jacobi disk \(D_1^J\).

4. Appendix

The real Jacobi group of index \(n\) is defined as \(G_n^J(\mathbb{R}) = H_n(\mathbb{R}) \rtimes \text{Sp}(n, \mathbb{R})\), where \(H_n(\mathbb{R})\) is the real Heisenberg group of real dimension \((2n + 1)\) [39, 12].

To the Jacobi group \(G_n^J(\mathbb{R})\) it is associated the homogeneous manifold - the Siegel-Jacobi upper half-plane \(- \mathcal{X}_n^J \approx \mathbb{C}^n \times \mathcal{X}_n\), where the Siegel upper half-plane \(\mathcal{X}_n = \text{Sp}(n, \mathbb{R})/U(n)\) is realized as

\[\mathcal{X}_n := \{ V \in M(n, \mathbb{C}) | V = S + iR, S, R \in M(n, \mathbb{R}), R > 0, S^t = S; R^t = R \}.\]

Siegel has determined the metric on \(\mathcal{X}_n, \text{Sp}(n, \mathbb{R})\)-invariant to the action (1.1)

\[(4.1) \quad V_1 = (aV + b)(cV + d)^{-1} = (Vc^t + d^t)^{-1}(Va^t + b^t); \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{Sp}(n, \mathbb{R}),\]

(see equation (2) in [37] or Theorem 3 at p. 644 in [29]):

\[(4.2) \quad d s_{\mathcal{X}_n}^2(R) = \text{Tr}(R^{-1} dV R^{-1} d\bar{V}).\]

With the Cayley transform (4.3)

\[(4.3) \quad V = i(I_n - W)^{-1}(I_n + W),\]
and the relations

\begin{align}
\dot{V} &= 2iU\dot{W}U, \quad U = (I_n - W)^{-1}, \\
\ddot{V} &= 2iU(\dot{W} + 2\dot{W}U\dot{W})U, \\
2R &= (I_n + W)U + (I_n + \dot{W})U, 
\end{align}

introduced in (4.2), it is obtained the metric on $\mathcal{D}_n$, $\text{Sp}(n, \mathbb{R})_c$-invariant to the action (2.5a):

\begin{equation}
zs^2_{2\mathcal{D}_n}(W) = 4\text{Tr}(M dW\tilde{M} d\tilde{W}), \quad W \in \mathcal{D}_n, \quad M = (I_n - \dot{W}\tilde{W})^{-1}.
\end{equation}

(4.4b) is equation (3.2) in [40] and corresponds to $\omega_{\mathcal{D}_n}$ in (2.62).

Equations of geodesics on the Siegel-upper half plane $\mathcal{X}_n$ (see e.g. equation (39) in [37] or Theorem 11 in at p. 478 in [29]) are

\begin{equation}
\ddot{V} = -iV R^{-1}\dot{V}.
\end{equation}

The relation (4.4c) can be written as

\begin{equation}
R = (I_n - \tilde{W})^{-1}(I_n - \tilde{W}W)(I_n - W)^{-1} = (I_n - W)^{-1}(I_n - W\tilde{W}')(I_n - \tilde{W})^{-1}.
\end{equation}

Equation (4.6) on $\mathcal{X}_n$ becomes on $\mathcal{D}_n$

\begin{equation}
\ddot{W} + 2\dot{W}\Sigma\dot{W} = 0
\end{equation}

where

\begin{equation}
\Sigma = \tau U, \\
\tau = I_n - [v(I_n - W)]^{-1} = 2\tilde{U}(I_n - W\tilde{W}),
\end{equation}

and we have

\begin{equation}
\Sigma = \tilde{W}(I_n - W\tilde{W})^{-1},
\end{equation}

i.e. $\Sigma$ is our previous $X$ defined in (3.6a) and equation (3.27) is retrieved. Note that the equation of geodesics on the Siegel ball $\mathcal{D}_n$ has the same expression as on the noncompact Grassmannian $\text{SU}(n,n)/\text{S}(\text{U}(n) \times \text{U}(n))$, see equation (6.13) in [3].

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