Quiver Gauge Models in F-Theory on
Local Tetrahedron

Lalla Btissam Drissi∗, †, Leila Medari‡, El Hassan Saidi§

1. Lab/UFR- Physique des Hautes Energies, Faculté des Sciences, Rabat, Morocco,
2. GNPHE, focal point, Faculté des Sciences Rabat, Morocco,
3. LPHEA, Physics Department, Faculty of Science Semlalia, Marrakesh, Morocco

August 4, 2009

Abstract

We study a class of 4D $\mathcal{N} = 1$ supersymmetric GUT-type models in the framework of the Beasley-Heckman-Vafa theory. We first review general results on MSSM and supersymmetric GUT; and we describe useful tools on 4D quiver gauge theories in F-theory set up. Then we study the effective supersymmetric gauge theory in the 7-brane wrapping 4-cycles in F-theory on local elliptic CY4s based on a complex tetrahedral surface $T$ and its blown ups $T_n$. The complex 2d geometries $T$ and $T_n$ are non planar projective surfaces that extend the projective plane $\mathbb{P}^2$ and the del Pezzos. Using the power of toric geometry encoding the toric data of the base of the local CY4, we build a class of 4D $\mathcal{N} = 1$ non minimal GUT-type models based on $T$ and $T_n$. An explicit construction is given for the SU(5) GUT-type model.

Key Words: MSSM, GUT, BHV model, tetrahedron, Intersecting Branes.

1 Introduction

In the last few years an increasing interest has been given to linking superstring theory to the low energy elementary particle physics phenomenology [1]-[4]. Several attempts have been particulary focusing on type II superstrings and M-theory to engineer extensions of the Minimal Supersymmetric Standard Model (MSSM) of elementary particles at TeV-
This interest in physics beyond standard model is also motivated by the Large Hadron Collider (LHC) event whose ATLAS and CMS detectors are expected to capture new signals beyond the theory of electroweak interactions. Recall that the energy magnitude used in the LHC and the power of the grid computing constitute the beginning of a new era for testing several ideas and proposals such as supersymmetry and extra dimensions. The access to the TeV energy band will allow to check early phenomenological prototypes beyond the Standard Model such as the SU(3) × SU(2) × U(1) model and the SU(3) × SU(2) × SU(2) Pati-Salam model treating quarks and leptons on equal footing, and the SU(3) tri-unification models. The TeV band allows as well to shed more light on grand unified theory (GUT) proposals; especially those based on gauge symmetry groups like SU(5), flipped SU(5), SO(10) and E6 GUT models.

Recently a model has been proposed to linking quantum physics at TeV energies to twelve dimensional F-theory compactified on a local Calabi-Yau four-folds in the limit of decoupled supergravity. In this proposal, to which we shall refer to as the BHV theory, and which has been further developed in a series of seminal papers, the visible N = 1 supersymmetric local GUT models in the 4D space time is given by an effective non abelian gauge theory living on a seven brane wrapping 4-cycles in F-theory on local elliptically K3 fibered Calabi-Yau four-folds X4,

\[ Y \rightarrow X_4 \]
\[ \downarrow \pi_s \]
\[ S \]

where, to fix the ideas, the base surface S is thought of as the del Pezzo complex base surface dP8. Together with the nature of the singularity in the fiber Y (type A, type D or type E) which engineer the gauge invariance that we see in the 4D space time, the del Pezzo surface dP8 and its dPn sisters with n ≥ 5 are used to engineer chiral matter and Yukawa couplings of the 4D space time standard model and beyond. Notice that besides \( N = 1 \) supersymmetry in 4D space time, the dPn base surfaces play as well a central role in the BHV theory due to their special features; in particular the two following:

1. the dPn’s are in some sense artificial surfaces engineered by performing blow ups of the complex projective plane \( \mathbb{P}^2 \) at n isolated points (n ≤ 8). In addition to the hyperline class H of projective plane \( \mathbb{P}^2 \), the blow ups are generated by n exceptional curves \( E_n \), which altogether with H, generate the \((1+n)\) dimensional homology group \( H_2(dP_n, \mathbb{Z}) \) of real 2-cycles in the complex dPn surfaces.

2. the dPn’s are also remarkably linked to the "exceptional" E6 Lie algebras, which are known to exist in the non perturbative regime of type IIB superstrings
as F-theory. The real 2-cycle homology group $H_2(dP_n, \mathbb{Z})$ decomposes as the direct sum,

$$H_2(dP_n, \mathbb{Z}) = \Omega_n \oplus L_n,$$

with $\Omega_n$ being the anticanonical class of $dP_n$ and the orthogonal class $L_n$ is a $n$ dimensional sublattice that is isomorphic to the root space of the exceptional Lie algebras $E_n$. These two properties make the $dP_n$s very special complex surfaces which allow an explicit geometric engineering of:

(a) chiral matter localizing on complex curves $\Sigma_i$ at the intersections of seven branes wrapping $dP_n$,

(b) the MSSM and GUT tri-fields Yukawa couplings localizing at isolated points in the del Pezzo surface $dP_n$ with $n \geq 5$ where matter curves intersect and where the bulk gauge invariance gets enhanced [32].

The aim of this paper is to contribute to the efforts for the study of embedding the $\text{MSSM}$ and $\mathcal{N} = 1$ supersymmetric GUT models in F-theory compactification on local Calabi-Yau four-folds (CY4). More precisely, we focus on 4D $\mathcal{N} = 1$ supersymmetric GUT- type models along the line of the BHV theory; but by considering a seven brane wrapping 4-cycles in F-theory local CY4-folds based on a tetrahedral surface $T$ of the figure (1) and its toric blown ups $T_n$. Using these backgrounds, we first engineer unrealistic $\mathcal{N} = 1$ supersymmetric GUT- type models based on the tetrahedron $T$. Then we consider extensions based on a particular class of blow ups of the tetrahedron namely the sub-family $T_n^{\text{toric}}$ of toric blow ups of $T$. These extensions, which involve exotic matter, constitute a step towards engineering non minimal quasi-realistic 4D $\mathcal{N} = 1$ supersymmetric GUT on $T$ and $T_n$. To fix the ideas, we shall mainly focus on the engineering of supersymmetric $SU(5)$ GUT- type models based on $T$ and $T_n$; but the method works as well for the other GUT gauge groups.

Figure 1: Toric graph of the tetrahedral surface $T = \bigcup_{a=1}^4 S_a$ with $S_a \cap S_b = \Sigma_{ab}$. The toric fibration of $T$ degenerate once on the six edges $\Sigma_{ab}$ and twice at the vertices $P_{abc} = S_a \cap S_b \cap S_c$. 

3
Before proceeding it is interesting to say few words about the motivations behind our interest into the tetrahedral surface $T$ and its blown ups $T_n$ as a base surface of the local CY4-folds. Besides the fact of being a particular non planar complex surface, our interest into the tetrahedron and the cousin geometries has been motivated by the two following features:

(i) the complex tetrahedral surface $T$, viewed as a toric surface, has a natural toric fibration given by a 2- torus $\mathbb{T}^2$ fibered over a real two dimension base $B_2$,

$$\mathbb{T}^2 \rightarrow T \downarrow \pi_B \quad B_2 \quad (1.3)$$

The complex surface $T$ is nicely represented by a toric graph $\Delta_T$ which is precisely the usual real tetrahedron given by the figure (1). The polytope $\Delta_T$ encodes the toric data on the shrinking cycles of the toric fibration $\pi_B$.

(ii) the toric geometry of the tetrahedral surface $T$ has a set of remarkable properties that have an interpretation in F-theory GUT models building. Below, we describe three of these features:

($\alpha$) the 2-torus fibration $\pi_B$ has an inherent $U(1) \times U(1)$ gauge symmetry which may be interpreted in F-theory compactifications in terms of abelian gauge symmetries. Each $U(1)$ factor describes gauge translation along compact 1-cycles in $\mathbb{T}^2$.

($\beta$) the $\mathbb{T}^2$ fiber has two shrinking properties: first down to 1- cycles on the six edges of the tetrahedron and second down to 0-cycles at its four vertices.

These properties capture in a remarkable way the enhancement of gauge symmetry used in the engineering of the F-theory GUT-models à la BHV.

Notice moreover that the non planar tetrahedral surface $T$ involves:

- four intersecting planar faces $S_a$, $a = 1, 2, 3, 4$, with different 2- torus fibers $\mathbb{T}^2_a$,
- six intersecting edges $\Sigma_{ab} = S_a \cap S_b$ having different 1- cycle fibers $S^1_{ab}$,
- four vertices $P_{abc}$ given by the curves intersection $\Sigma_{ab} \cap \Sigma_{bc} \cap \Sigma_{ca}$. At these special points, the 2- torus fiber eq(1.3) shrinks to zero.

From the view of the F-theory- GUT models building, the faces $S_a$ of the tetrahedron correspond roughly to 4- cycles wrapped by seven branes. These faces intersect mutually along six edges $\Sigma_{ab}$ on which the fibers $\mathbb{T}^2_a$ and $\mathbb{T}^2_b$ shrink down to $S^1_{ab}$. Along these curves seven branes intersect and give rise to bi-fundamental matter. Moreover, three faces $S_a$, $S_b$ and $S_c$ intersect at a point $P_{abc}$ corresponding to a vertex of the figure (1). From this
picture, it follows that the vertices of the tetrahedron are good candidates to host the tri-fields Yukawa couplings such as those of the 4D supersymmetric SU(5)-GUT model namely,

\[
\begin{align*}
H_u Q_10 & \rightarrow 5_H \otimes 10_M \otimes 10_M \rightarrow P_1, \\
\mathcal{F} H_u H_d & \rightarrow 1_E \otimes 5_H \otimes 5_H \rightarrow P_2, \\
N_R H_u Q_5 & \rightarrow 1_M \otimes 5_H \otimes 5_M \rightarrow P_3, \\
H_d Q_5 Q_{10} & \rightarrow 5_H \otimes 5_M \otimes 10_M \rightarrow P_4.
\end{align*}
\]

(1.4)

In these relations, $5_H$ refers to Higgs fields and $5_M, 10_M$ to matter. The vertex $P_1$ stands for $P_{(234)}$ and similarly for the others.

(γ) The third feature behind the study of this local Calabi-Yau four-folds geometry is that the tetrahedral surface $\mathcal{T}$ shares also some basic properties with the del Pezzo surfaces $dP_n$ used in the BHV theory. The point is that each one of the four faces $S_a$ of the tetrahedron is in one to one with the four projective plane $\mathbb{P}^2_a$ in the complex three dimension projective space $\mathbb{P}^3$,

\[
S_a \leftrightarrow \mathbb{P}^2_a, \quad a = 1, \ldots, 4.
\]

(1.5)

On each of these $\mathbb{P}^2_a$, one may a priori perform blow ups leading to a $\mathcal{T}_n$ family of blown tetrahedrons. The number of blow ups of the tetrahedron are obviously richer than the ones encountered in the del Pezzo surfaces since the tetrahedron involves four kinds of projective planes; for more details see [10, 11]. From this view blown ups of the tetrahedral surface may be thought of as given by intersections of del Pezzo surfaces and thereby F-theory GUT models based on blown ups tetrahedron could incorporate the BHV ones based on del Pezzo surfaces.

The presentation of this paper is as follows: In section 2, we review briefly the main lines of MSSM and supersymmetric GUT models in 4D space time. Comments using quiver gauge theory ideas and intersecting brane realizations are also given. In section 3, we review general results on F-theory and we study the engineering of the non abelian gauge symmetries in the frame work of F-theory on local CY4-folds. An heuristic classification of pure and hybrid colliding singularities in CY4s is also made. In section 4, we first review $\mathcal{N} = 1$ supergravity theory coupled to super Yang-Mills. Then, we focus on the gauge theory in the seven branes wrapping 4-cycles and study the engineering of the effective $\mathcal{N} = 1$ supersymmetric gauge theory in 4D obtained by using topological twisted ideas. In section 5, we study the engineering of F-theory GUT-model along the line of the BHV approach. We take this opportunity to give a brane realization of SU(5) GUT model by using five stacks of intersecting seven branes. In section 6, we study F-theory on local CY4-folds based on the complex tetrahedral surface $\mathcal{T}$ and its $\mathcal{T}_n$ blown
ups and we develop a first class of F-theory GUT-type models based on the $T$. In section 7, we build a second class of F-theory GUT-type models based on $T_n$ blown ups and fractional bundle ideas. In section 8, we give our conclusion and in section 9, we give an appendix on the engineering of bi-fundamental matter in F-theory GUT-models building.

2 General on MSSM and GUT

In this section we review briefly some useful tools on the MSSM and the $4D\ N = 1$ Supersymmetric Grand Unified Theories (SGUT) as well as general links to superstrings. These tools are helpful to fix the ideas on: (1) how fundamental matter and gauge particles get unified into group representations method and (2) how the geometric tri-fields Yukawa couplings (1.4) are handled in the $4D\ N = 1$ superfield theory set up. These materials are also needed for later use when we consider the embedding SGUT-type models into the effective non abelian twisted gauge theory \[32\] on the seven brane wrapping 4-cycles in the twelve dimensional F-theory on local Calabi-Yau four-folds.

2.1 MSSM

We start by recalling some general aspects on Standard Model of electroweak interactions. The basic elements in this model are as follows:

(a) the elementary particles namely: quarks, leptons, gauge bosons and Higgs particles,
(b) the $SU_C (3) \times SU_L (2) \times U_Y (1)$ gauge symmetry to be denoted as $G_{str}$,
(c) the $G_{str}$ representations unifying the particles into gauge group multiplets.

In the Cartan basis, the Lie algebra of the Standard Model group $G_{str}$ is generated by the following matrices,

\[
\begin{array}{c|c|c}
         & SU_C (3)                     & SU_L (2) & U_Y (1) \\
\hline
\text{Cartan operators} & H_{su(3)}^1 & H_{su(3)}^2 & H_{su(2)}^0 & Y_{u(1)} & - \\
\text{step operators}   & E_{su(3)}^\pm & E_{su(2)}^\pm & & & \\
\end{array}
\]

where $H_{su(3)}^1, H_{su(3)}^2, H_{su(2)}^0, Y_{u(1)}$ are commuting Cartan generators and $E_{su(3)}^\pm, E_{su(2)}^\pm$ are step operators with $\alpha$ being a generic positive root of the $SU_C (3)$ root system $\Delta_{su(3)}$.

The fundamental particles of the Standard Model are of two kinds:

(i) Elementary fermions forming three hierarchical families $F(e), F(\mu)$ and $F(\tau)$; each one containing quarks and leptons in different representations of the $G_{str}$ group. For the family $F(e)$ of the electron $e^- \equiv e$, the sixteen left-handed fermions are packaged into

---

\[2\]
smaller representations $R_{suC(3)} \times R_{suL(2)} \times R_{uy(1)}$ of the $G_{str}$ gauge symmetry as given below,

| Quarks | Leptons |
|--------|---------|
| $q = \begin{pmatrix} u \\ d \end{pmatrix}$, $u^c$, $d^c$ | $l = \begin{pmatrix} \nu_e \\ e \end{pmatrix}$, $\nu^c$, $e^c$ |

Using the conventional notation $(n, m)_y$ with $m = \dim R_{suC(3)}$, $n = \dim R_{suL(2)}$ and $y$ being the eigenvalue of the hypercharge charge $R_Y$, the group theoretical description of the $F(e)$ family is as follows:

$$q, u^c, d^c, l, \nu^c, e^c$$

Using the conventional notation $(n, m)_y$ with $m = \dim R_{suC(3)}$, $n = \dim R_{suL(2)}$ and $y$ being the eigenvalue of the hypercharge charge $R_Y$, the group theoretical description of the $F(e)$ family is as follows:

$$q, u^c, d^c, l, \nu^c, e^c$$

The usual $U_{em}(1)$ electric charge operator is given by $Q_{U_{em(1)}} = H_{suL(2)}^0 + \frac{Y}{2}$.

Later on (see section 5), we find as well that these matter fields and their group theoretical configurations get a nice geometric interpretation in the framework of F-theory on Calabi-Yau four-folds and intersecting seven branes wrapping 4-cycles and filling the non compact space time directions.

For completeness, notice that implementation of the two other generations of flavors $F_i$ with

| family | Quarks | Leptons |
|--------|--------|---------|
| $F_2 = F(\mu)$ : $\begin{pmatrix} c \\ s \end{pmatrix}$, $c^c$, $s^c$ | $\begin{pmatrix} \nu^c \\ \mu \\ e \end{pmatrix}$, $\nu^c$, $\mu^c$ |
| $F_3 = F(\tau)$ : $\begin{pmatrix} t \\ b \end{pmatrix}$, $t^c$, $b^c$ | $\begin{pmatrix} \nu^c \\ \tau \\ \tau \end{pmatrix}$, $\nu^c$, $\tau^c$ |

is achieved by help of inserting a flavor index $i$ running as $i = 1, 2, 3$ with $F_1 = F(e)$. As such, the full set of $3 \times 16$ elementary fermionic fields will be denoted as $q_i$, $u^c_i$, $d^c_i$, $l_i$, $\nu^c_i$ and $e^c_i$.

In the MSSM, the $F_i$ families get promoted to super- families $\mathcal{F}_i$ where the above $3 \times 16$
elementary fermionic fields are now promoted to $3 \times 16$ chiral superfields

$$Q_i, \quad U_i^c, \quad D_i^c, \quad L_i, \quad N_i^c, \quad E_i^c,$$

with same gauge quantum numbers as in the non supersymmetric case.

Below we denote collectively these chiral superfields by $\Psi (y, \theta)$ living on the chiral super-space $(y, \theta)$ with 4D space time coordinates $x^\mu$ shifted $y^\mu = x^\mu - i\theta \sigma^\mu \bar{\theta}$ and Grassmann odd variable given by a $SO(1, 3)$ Weyl spinor. Since $\theta$ is nilpotent ($\theta^3 = 0$), the $\Psi (y, \theta)$ admits then the following finite $\theta$- expansion

$$\Psi (y, \theta) = \tilde{\phi} (y) + \sqrt{2} \theta^\alpha \psi_\alpha (y) + \theta^2 F (x),$$

where the left handed fermion $\psi_\alpha$ is one of the fields in eq(2.3), $\tilde{\phi}$ the corresponding sparticle and $F$ the usual auxiliary field which, amongst others, plays a central role in the study of supersymmetry breaking and in the geometric interpretation of supersymmetric quiver gauge theories embedded in type II superstrings.

(ii) Bosons are of two types namely Higgs scalars and vector particles. In the MSSM, we need two space time Higgs scalars $h_u = (h^+, h^0)$ and $h_d = (\tilde{h}^0, \tilde{h}^-)$ together with their superpartners. These fields form chiral multiplet denoted by $H_u$ and $H_d$ with $\theta$-expansion as in eq(2.6). Regarding the vector particles, we have in addition to the twelve space time 4- vector potentials

$$A_\mu^{su(3)}, \quad A_\mu^{su(2)} \equiv (W_\mu^\pm, Z_\mu^0), \quad A_\mu^Y \equiv B_\mu,$$

the gaugino partners described by four dimensional space time Majorana spinors.

In 4D $\mathcal{N} = 1$ superspace, the Higgs sector is described by two doublets of chiral Higgs superfield,

| Higgs | $G_{str}$ group |
|--------------------|----------------|
| $H_u = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$ | $(1, 2)_{+1}$ |
| $H_d = \begin{pmatrix} \tilde{H}^0 \\ \tilde{H}^- \end{pmatrix}$ | $(1, 2)_{-1}$ |

These chiral superfields are needed to break the $G_{str}$ gauge symmetry down to $SU_C (3) \times U_{em} (1)$. The gauge fields involve in addition to the space time gauge bosons

$$A_\mu^{su(3)} \oplus A_\mu^{su(2)} \oplus \frac{Y}{2} B_\mu,$$

the gauginos

$$\tilde{\lambda}_{su(3)} \oplus \tilde{\lambda}_{su(2)} \oplus \frac{Y}{2} \tilde{\lambda}_{uy(1)}.$$
These 4D space time fields are combined altogether in 4D $\mathcal{N} = 1$ real superspace $(x, \theta, \bar{\theta})$ to form real 4D superfield

$$V = V_{su(3)} \oplus V_{su(2)} \oplus \frac{Y}{2} V_Y,$$  \hspace{1cm} (2.11)

valued in the Lie algebra $su_C(3) \oplus su_L(2) \oplus u_Y(1)$. The real superfields $V_{su(3)}$, $V_{su(2)}$ and $V_Y$ mediate the gauge interactions with superspace dynamics described by the following lagrangian density

$$\mathcal{L}_{MSSM} = + \int d^4 \theta \sum_{\text{superfields } \Psi} \bar{\Psi} \left( e^{-2[g_{su(3)} V_{su(3)} + g_{su(2)} V_{su(2)} + g_Y \frac{Y}{2} V_Y]} \right) \Psi$$

$$+ \int d^2 \theta \left( \frac{1}{8g_{su(3)}} Tr W^2_{su(3)} + \frac{1}{8g_{su(2)}} Tr W^2_{su(2)} + \frac{1}{8g_Y} W_Y^2 \right) + hc$$

$$+ \int d^2 \theta W + \int d^2 \bar{\theta} \bar{W},$$  \hspace{1cm} (2.12)

where $W$ is the chiral superpotential. This is a gauge invariant superfunction depending on the matter chiral superfields and describing mass terms and Yukawa tri-couplings as shown below,

$$W = -\mu H_u H_d - \sum_{i,j=1}^{3} \frac{m^{ij}}{2} N_i^c N_j^c$$

$$+ \sum_{i,j=1}^{3} \frac{\lambda^e_{ij}}{3} L_i E^c_j H_d + \sum_{i,j=1}^{3} \frac{\lambda^{ij}}{3} L_i N_j^c H_u$$

$$+ \sum_{i,j=1}^{3} \frac{\lambda^d_{ij}}{3} Q_i D^c_j H_d + \sum_{i,j=1}^{3} \frac{\lambda^u_{ij}}{3} Q_i U^c_j H_u,$$  \hspace{1cm} (2.13)

where the numbers $\mu$ and $m^{ij}$ scale as mass and where the dimensionless complex numbers $\lambda^e_{ij}$, $\lambda^{ij}$, $\lambda^d_{ij}$ and $\lambda^u_{ij}$ are complex Yukawa couplings.

### 2.2 Beyond MSSM

In the MSSM, quarks and leptons, together with their superpartners, belong to several irreducible representations of the $G_{str}$ gauge symmetry involving three gauge coupling constants $g_{su_C(3)}$, $g_{su_L(2)}$ and $g_Y$. A true unification model requires however packaging all the fundamental particles in a unique irreducible representation of a simple gauge symmetry group. This is the basic idea behind grand unified theories (GUT) of strong and electroweak interactions using the real 24 dimensional unitary $SU(5)$, the 45 dimensional orthogonal $SO(10)$ and the 78 dimensional exceptional $E_6$. As a first step towards this goal, we distinguish below two main ways in getting the GUT gauge groups, either
by using physical imagination à la Pati-Salam; or by using group theoretical methods à la Georgi-Glashow by looking for the smallest simple gauge group containing $G_{str}$ as a maximal gauge subgroup. Let us describe briefly these two ways.

### 2.2.1 Pati-Salam model and $SO(10)$ GUT

In the Pati-Salam model, the gauge symmetry is given by $SU_C(4) \times SU_L(2) \times SU_R(2)$. There, the quarks and leptons supermultiplets of each one of the three super-families $F_i$ are packaged in two irreducible representations $Q$ and $Q^c$ of this group. The basic idea behind this packaging is to think about the lepton number as the fourth color so that the previous $SU_C(3)$ color gauge symmetry gets promoted to a $SU_C(4)$ gauge invariance. In this way, the quarks and the leptons of the standard model family (2.2) are now put into two $SU_C(4)$ quartets as follows,

| quarks and leptons | $SU_C(4) \times SU_L(2) \times SU_R(2)$ |
|-------------------|--------------------------------------|
| $Q = (q, l)$      | $(4, 2, 1)$                           |
| $Q^c = (q^c, l^c)$| $(4, 1, 2)$                           |

(2.14)

with $q$ and $l$ as in eqs(2.2) and

$$q^c = (u^c, d^c), \quad l^c = (\nu^c, e^c).$$

(2.15)

The baryon number $B$ minus the lepton number $L$ and the electric charge operator $Q_{em}$ act on the representation 4 of $SU(4)$ as follows,

$$B - L = \frac{1}{3} \text{diag}(1, 1, 1, -3), \quad Tr_4(B - L) = 0,$$

$$Q_{em} = H^0_{su_L(2)} + H^0_{su_R(2)} + \frac{(B - L)}{2}, \quad Tr_4(Q_{em}) = 0.$$  

(2.16)

Regarding bosons, we have a quite similar picture. Besides the gauge particles transforming in the adjoint of the gauge symmetry, the two Higgs doublets $H_u$ and $H_d$ of eq(2.8) are also combined into one irreducible quartet Higgs multiplet

$$\mathcal{H} = (H_u, H_d),$$

(2.17)

transforming under the $SU_C(4) \times SU_L(2) \times SU_R(2)$ Pati-Salam group like $(1, 2, \bar{2})$ and so allowing the following unique gauge invariant trilinear Yukawa coupling term

$$\mathcal{L}_{Yukawa} = \lambda_{qdh} \int d^2 \theta Q^c HQ + h.c,$$

(2.18)

where $\lambda_{qdh}$ is the Yukawa coupling constant. Notice that Pati Salam group $SU_C(4) \times SU_L(2) \times SU_R(2)$ which is homomorphic to $SO(6) \times SO(4)$ is not a grand unified gauge
symmetry as it still involves three gauge coupling constants \( g_{SU_C(4)}, g_{SU_L(2)} \) and \( g_{SU_R(2)} \). However, this gauge symmetry can be embedded into the simple \( SO(10) \) group. In this larger simple group, the reducible 16-dimensional matter representation \( (4, 2, 1) \oplus (\bar{4}, 1, \bar{2}) \) of the Pati-Salam group gets interpreted as the left handed \( SO(10) \) spinor representation

\[
SO(10) \rightarrow SO(6) \times SO(4) \rightarrow SU_C(3) \times SU(2) \times U_Y(1)
\]

\[
16_+ \rightarrow (4, 2, 1) \oplus (\bar{4}, 1, \bar{2}) \rightarrow (3, 2)_\frac{1}{3} \oplus (\bar{3}, 1)_{-\frac{1}{3}} \oplus (3, 1)_\frac{1}{3} \oplus (1, 2)_{-1} \oplus (1, 1)_0 \oplus (1, 1)_2
\]

(2.19)

In this embedding, we have a lepton-quark unification as well as a gauge coupling unification. This feature makes the \( SO(10) \) gauge symmetry as one of the most attractant GUT models for gauge unification of strong and electroweak interactions.

### 2.2.2 Georgi Glashow model

In the Georgi-Glashow model based on group theory analysis, the GUT symmetry is given by the simple rank four unitary group \( SU(5) \). There, the leptons and quarks of each sixteen dimensional family of the standard model are packaged into three \( SU(5) \) irreducible representations: the singlet, the anti-fundamental \( \bar{5} \) and the antisymmetric \( 10 = [5 \otimes 5]_A \) representations. This property follows from the decomposition

\[
SO(10) \rightarrow SU(5)
\]

\[
16_M \rightarrow 1_M \oplus \bar{5}_M \oplus 10_M
\]

(2.20)

where the sub-index \( M \) refers for matter. To get more insight in the field content of these elementary particles unification, we recall that the singlet stands for the anti-neutrino, \( 1_M \sim \nu^c \) while the \( \bar{5}_M \) and \( 10_M \) correspond to

\[
\bar{5}_M \sim \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ \nu \end{pmatrix}, \quad 10_M \sim \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1^c & d_1 \\ -u_3^c & 0 & u_1^c & u_2^c & d_2 \\ u_2^c & -u_1^c & 0 & u_3^c & d_3 \\ -u_1^c & -u_2^c & -u_3^c & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{pmatrix}
\]

(2.21)

Similar representations are valid for the two other families and their supersymmetric extensions. Later on, we will see that, along with this group theoretic representation, these matter fields have a nice geometric representation in terms of Riemann surfaces \( \Sigma \) (complex curves) inside the internal space used in the F-theory compactification on real eight dimensional Calabi-Yau four-folds; for illustration, think about this feature as “corresponding” to the matter localized on the edges of the figure (1),

\[
\bar{5}_M \rightarrow \Sigma_M^{(5)}, \quad 10_M \rightarrow \Sigma_M^{(10)}
\]

(2.22)
Notice moreover that altogether with these chiral matter representations, which are promoted to chiral superfield in $SU(5)$ SGUT model, we have moreover:

1. two Higgs multiplets $H_u$ and $H_d$ transforming respectively in the $5_H$ and $\bar{5}_H$ representations,
2. twenty four $4D$ $\mathcal{N} = 1$ gauge multiplets $V^a$ transforming the adjoint representation of the $SU(5)$ gauge symmetry.

Furthermore, the $SU(5)$ gauge invariant chiral superpotential $W$ between two matter superfields and one Higgs superfield has the following structure,

$$W_{Yukawa} = + \frac{\lambda_1}{3} (5_H \otimes 10_M \otimes 10_M) + \frac{\lambda_2}{3} (\bar{5}_H \otimes \bar{5}_M \otimes 10_M) + \frac{\lambda_3}{3} (5_H \otimes \bar{5}_M \otimes 1_M) + \mu (5_H \otimes \bar{5}_H),$$

(2.23)

where $\mu$ is a mass constant and the $\lambda_i$’s are Yukawa coupling constants. This chiral superpotential involves three kinds of chiral superfield vertices as depicted in the figure 2.

Figure 2: Yukawa couplings in supersymmetric SU(5) GUT model: three kind of chiral superfield tri- vertices namely $5 \times 10 \times 10$, $\bar{5} \times \bar{5} \times 10$ and $5 \times \bar{5} \times 1$. Tri-coupling involving Higgs superfield in the 24 adjoint, which is also allowed, is not reported.

### 2.3 MSSM as a quiver gauge theory

In the MSSM with $SU_C(3) \times SU_L(2) \times U_Y(1)$ gauge invariance, the matter fields are generally charged under representations of the groups factors; that is under $SU_C(3)$, $SU_L(2)$ and $U_Y(1)$. This property suggests that MSSM might be thought of as quiver gauge theory that can be embedded in superstrings compactifications. Recall that $4D$ $\mathcal{N} = 1$ supersymmetric quiver gauge theories have been subject to an intensive interest during last decade [42]-[45]. These theories, which may be engineered in different, but dual, ways appear as low energy effective field theory of $10D$ superstrings on CY3-folds, $11D$ M-theory compactification on G2 manifolds and $12D$ F-theory on CY4-folds preserving four supersymmetries [46]-[49].

In this subsection, we explore rapidly what kind of quiver diagram one gets in the engineering of MSSM as a supersymmetric quiver gauge theory.
2.3.1 Engineering the MSQSM

One of the main actors in the Minimal Supersymmetric Quiver Standard Model (MSQSM) is that supersymmetric chiral matter in the three families of elementary particles transform in specific representations of the $SU_C(3) \times SU_L(2) \times U_Y(1)$ gauge symmetry. These representations are mainly given by:

(1) the hermitian adjoint representation of each factor of the MSSM gauge symmetry where transform the twelve MSSM gauge superfields $V^a_{\text{MSSM}}$, that is:

$$V^a_{\text{MSSM}} \sim (8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0.$$  \hfill (2.24)

These hermitian representations have a nice interpretation in terms of massless excitations of open superstrings ending on stacks of D-branes of 10D closed type II superstrings. In this regards, it is interesting to note that in the D-brane setting, a stack of $N$ coincident D-branes of type II superstrings involves $U(N) = U(1) \times SU(N)$ gauge invariance in 4D space time \cite{1,2}. As such the gauge symmetry in the MSQSM is, instead of $G_{\text{str}}$, is rather given by,

$$U_a(3) \times U_b(2) \times U_c(1),$$  \hfill (2.25)

involving two extra undesired $U(1)$ gauge factors namely

$$U_a(1) = U_a(3)/SU_C(3),$$

$$U_b(1) = U_b(2)/SU_L(2),$$  \hfill (2.26)

which may be interpreted as baryon and lepton numbers. The $U_Y(1)$ hypercharge in the $G_{\text{str}}$ group should be then given by the non anomalous combination of the three $U_i(1)$s with a massless gauge field. The two other combinations are anomalous; but following \cite{1,2}, these anomalies may be canceled by a generalized Green-Schwarz mechanism which at the same time gives large masses to the corresponding gauge bosons. As such these abelian symmetries remain as global symmetries in the effective Lagrangian of the theory. If forgetting for a while about the right handed leptons that are charged under the $U_Y(1)$ hypercharge, the quiver graph that would describe this supersymmetric quiver gauge theory without fundamental matter would involve three separated nodes as depicted in the figure (3). Each node\footnote{In fact it is the requirement that the lepton doublets remain charged under the $SU_L(2)$ factor but transform as singlets under $SU_C(3)$ which implies that any minimal embedding will possess at least three quiver nodes.} refers to a gauge group factor and represents the world volume of the branes at some fix points under some given orbifold action on the internal manifold.

(2) Matter of the MSQSM is in several complex representations of the gauge invariance
as shown below,

| Multiplet          | Representation | $U(3)$ | $U(2)$ |
|--------------------|----------------|--------|--------|
| Quark multiplet    | $Q = (3, 2)_{\frac{1}{3}}$ | $U^c = (3, 1)_{-\frac{1}{3}}$ | $D^c = (3, 1)_{\frac{2}{3}}$ |
| Lepton multiplet   | $L = (1, 2)_{-1}$ | $N = (1, 1)_0$ | $E = (1, 1)_2$ |
| Higgs multiplet    | $H_u = (1, 2)_{-1}$ | $H_d = (1, 2)_{+1}$ |

In the brane set up, matter fields in the bi-fundamental representations live at the brane intersections. This is the case for the superfields $Q$, $U^c$, $D^c$, $L$, $H_u$ and $H_d$; but not for the two superfields $N$ and $E$ of eqs (2.27). For these superfields, the corresponding quiver gauge graph requires rather four nodes; for more details see [?], see also figure (4) for a brane representation.

Figure 4: Quiver graph of MQSM: directed lines denote three generations of left-handed chiral fermions. Lines with two arrows determine fermions charged under $U(3) \times U(1)$. Dashed line refers to the SM Higgs doublet. In the supersymmetric version MSQSM, oriented line denotes a chiral superfield and dashed line a vector-like pair of fields.

The gauge group of the MSQSM is $U(3) \times USp(2) \times U(1)$ with non anomalous hyper-

\footnote{Implementation of the right handed leptons that are charged under the $U_Y(1)$ requires adding a fourth brane stack [2].}
In [33], a supersymmetric version of the minimal quiver standard model has been constructed in F-theory on local CY4-folds by partially Higgsing the brane probe theory of a del Pezzo $dP_5$ Calabi-Yau singularity. This extension, which will be implicitly described in section 5, involves orientifolding ideas as a way to solve the problem of engineering the leptonic right handed sector and anomaly cancelation in the hypercharge sector.

We end this section by describing rapidly the four nodes quiver gauge model extending the three nodes one of figure (4). In the language of intersecting D5-branes in type IIB superstrings on local Calabi-Yau threefold orbifolds, the quiver gauge theory involves four stack of D5- branes and an orientifold as depicted in the figure (5) and table (2.30).

\[
Q_Y = \frac{1}{2} Q_{U_a(1)} - \frac{1}{3} Q_{U_b(1)}
\]  \hspace{1cm} (2.28)

The embedding of the MSSM in type IIB superstring may be achieved in this unoriented quiver gauge theories of at least four stacks of intersecting D- branes leaving at the fix points of the orbifold action in the type II superstring compactification to 4D space time. Under this orbifolding, the particles content of the MSSM is engineered by using both

\[
(N_a, \bar{N}_b), \quad (N_a, N_b)
\]  \hspace{1cm} (2.29)

bi-fundamental representations of the gauge group. This possibility is familiar from type II orientifold models in which the world sheet of the string is modded by some operation.
ΩR with Ω being the world sheet parity operation and R some geometrical action. Bi-
fundamental representations of type \((N_a, \bar{N}_b)\) appear from open strings stretched between
branes \(a\) and \(b\) whereas those of type \((N_a, N_b)\) appear from those going between the branes
\(a\) to the branes \(b^*\); the mirror of the branes \(b\) under \(\Omega R\). Inclusion of these representations
in the string theoretic realization is crucial for tadpole cancelation. Following [1][2], the
spectrum of the unoriented quiver gauge theory is given by

| intersection | matter | repres | \(Q_a\) | \(Q_b\) | \(Q_c\) | \(Q_d\) | \(\frac{Y}{2}\) |
|-------------|--------|--------|---------|---------|---------|---------|------------|
| \(ab\)      | \(Q_L\) | (3, 2)  | +1      | -1      | 0       | 0       | +1/6       |
| \(ab^*\)    | \(q_L\) | 2 (3, 2) | +1      | +1      | 0       | 0       | +1/6       |
| \(ac\)      | \(U_R\) | \(3 (\overline{3}, 1)\) | -1    | 0    | +1    | 0       | -2/3       |
| \(ac^*\)    | \(D_R\) | \(3 (\overline{3}, 1)\) | -1    | 0    | -1    | 0       | +1/3       |
| \(bd^*\)    | \(L\)   | \(3 (1, 2)\) | 0    | -1    | 0    | -1    | -1/2       |
| \(cd\)      | \(E_R\) | \(3 (1, 1)\) | 0    | 0    | -1    | -1    | +1         |
| \(cd^*\)    | \(N_R\) | \(3 (1, 1)\) | 0    | 0    | -1    | +1    | 0          |

where the hypercharge \(\frac{Y}{2} = \frac{1}{6}Q_a - \frac{1}{2}Q_c - \frac{1}{2}Q_d\).

3 Non abelian gauge theory on 7- brane

In this section, we describe some basic tools on brane physics to be used later on when
we study the \(4D \mathcal{N} = 1\) supersymmetric GUT- type models along the line of the BHV
proposal [32][33][34].

In the first subsection, we review briefly Vafa’s twelve dimensional F-theory as it is the
framework for building the \(4D \mathcal{N} = 1\) supersymmetric GUT models. In the second
subsection, we consider some useful aspects on the geometry of the elliptic Calabi-Yau
4-folds,

\[
\mathcal{E} \rightarrow X_4
\]
\[
\downarrow \pi
\]
\[
B_3
\]

where \(\mathcal{E}\) stands for the elliptic curve fibered on the complex three dimension base \(B_3\).

For later use, we will particularly focus on the following local geometry of the base

\[
\Sigma_0 \rightarrow B_3
\]
\[
\downarrow \pi
\]
\[
S
\]

where the base \(S\) is a complex surface and \(\Sigma_0 \sim \mathbb{P}^1\) a genus zero complex curves which
locally may be thought of as the complex line \(\mathbb{C}\). So, the resulting local Calabi-Yau four-
folds $X_4$ reduces to the form $(1.1)$. Notice that although the complex base $S$ could be a generic surface, we shall think about it as:

(i) a del Pezzo surface $dP_n$ with $H_2$ homology as in eq(1.2),
(ii) a complex tetrahedral surface $T$ of fig(11) or its $T_n$ blown ups studied in [40].

The second issue constitutes the basis of our contribution in the embedding of GUT-like models building in the F-theory set up.

With this picture in mind, we study the engineering of ADE gauge symmetries in the fiber $Y$ of eq(1.1) with locus in the complex two codimension surface $S$.

In the third subsection, we study the colliding of the singularities in the fiber as well as the enhancement of the gauge invariance at specific loci in the complex surface $S$. As we will see later on, these collisions have a nice realization in the complex tetrahedral geometry where gauge invariance in the bulk gets enhanced once on the edges and twice at the vertices of the tetrahedron of the figure (11); thanks to toric geometry.

To make direct contact with the usual 4D formulation of gauge theory in SGUT models building, we shall often use field theoretical method to interpret geometric quantities in the compact real eight dimensional manifold $X_4$.

### 3.1 F-theory on elliptic CY manifolds

We begin by noting that there are two main related approaches to introduce Vafa’s twelve dimensional F-theory:

(1) in terms of strongly coupled 10D type IIB superstring, or
(2) by using superstrings dualities in lower space time dimensions.

Besides its merit to incorporate F-theory as a part of a unifying picture including the five superstring theories, the duality based manner for defining F-theory has also the advantage to offer a way to engineer non abelian gauge symmetries in terms of geometric singularities in the internal manifolds. Before going into technical details, let us start by reviewing rapidly these two constructions.

#### 3.1.1 10D Type IIB set up

In type II superstring set up, the existence of twelve dimensional F-theory underlying 10D type IIB superstring may be motivated by looking for a link similar to the one existing between Witten’s eleven dimensional M-theory and 10D type IIA superstring [52]. In this view, it has been observed in a seminal work by Vafa [53] that the 10D type IIB superstring theory has indeed a remarkable underlying 12D F-theory description with non constant dilaton and axion. Recall that type IIB has, amongst others, the following features
(a) a constant profile coupling constant $g_s$ over the entire $10D$ spacetime,
(b) a strong/weak self duality captured by the $SL(2,\mathbb{Z})$ symmetry, and
(c) a NS-NS and R-R massless bosonic spectrum

\[
\begin{align*}
\text{NS-NS} : & \quad G_{MN} , \quad B_{MN} , \quad \varphi , \\
\text{R-R} : & \quad \tilde{B}_{MN} , \quad \tilde{D}_{MNPQ}^+ , \quad \tilde{\varphi} . 
\end{align*}
\]  

(3.3)

Moreover the dilaton and the axion vevs $\varphi$ and $\tilde{\varphi}$ of eq (3.3) are interpreted in terms of the complex structure modulus $\tau_{IIB} \equiv \tau = \varphi + i e^{-\varphi}$ of an elliptic curve $E$ with the modular transformation

\[\tau \to \tau' = \frac{n_1 \tau + n_2}{n_3 \tau + n_4} , \quad \left( \begin{array}{cc}
n_1 & n_2 \\
n_3 & n_4 \end{array} \right) \in SL(2,\mathbb{Z}) . \]  

(3.4)

By thinking about this 2-torus $\mathbb{T}^2$ as a universe geometric entity with coordinates $x^{11} \equiv x^{11} + R_1$ and $x^{12} \equiv x^{12} + R_2$, one ends with a $(10+2)$ dimensional space time.

To practically handle this complex elliptic curve $E \sim \mathbb{T}^2$, it is useful to embed it in the complex space $\mathbb{C}^2$ with a local holomorphic coordinates $(u,v)$. In this embedding, the complex elliptic curve $E$ may be naively defined by the typical complex algebraic cubic,

\[E : \quad v^2 = du^3 + eu^2 + fu + g , \quad d \neq 0 , \]  

(3.5)

where $d$, $e$, $f$ and $g$ are some complex constants introduced for later use.

Recall in passing that in the Weierstrass form of the complex elliptic curve $E$, we have $d = 1$ and $e = 0$; but here we have used the equivalent form (3.5) since later on the coefficients $d$, $e$, $f$ and $g$ will be promoted to holomorphic sections of some canonical bundle in the base of the CY4-folds. This promotion is needed in the engineering of elliptic fibrations of CY4-folds and in the implementation of gauge ADE symmetries in the game.

Twelve dimensional F-theory defines then a non perturbative vacua of type IIB superstring theory with non constant dilaton and axion and may be thought of as its strong string coupling limit ($\tau_{IIB} \to \infty$); but with no local on shell dynamics along the two extra compact directions ($x^{11}, x^{12}$). From this view, $10D$ type IIB superstring theory may be seen as the compactification of F-theory on $\mathbb{T}^2$

\[\text{F-Theory}/\mathbb{T}^2 \leftrightarrow 10D \text{ Type IIB} . \]  

(3.6)

---

3 Generally, a complex elliptic curve is a nonsingular cubic curve in the $(u,v)$-complex plane with algebraic equation $\sum_{n,m=0}^{3} a_{nm} u^n v^m = 0$ where $a_{nm}$ are some constants. This complex cubic can be simplified however, by an appropriate change of variables and brought to the usual Weierstrass form $v^2 = u^3 + au + b$ with discriminant $\Delta = -16 (4a^3 + 27b^3)$. In our formulation, we have kept the expression of the cubic quite general in order to give a unified description of the ADE geometries.
Notice in passing that in ten dimensions, we also have a duality between F-theory on a cylinder and $SO(32)$ type I/Heterotic superstrings. There, the modulus of the cylinder $S^1 \times S^1 / \mathbb{Z}_2$ is identified with the type I/Heterotic coupling constants $[51]$.

### 3.1.2 Duality in lower dimensions

Twelve dimensional F-theory may be nicely defined in terms of superstrings dualities at various space time dimensions where more physical features are expected. In eight space time dimensions, F-theory on elliptic K3 is dual to the 10D heterotic superstring on 2-torus $\mathbb{T}^2$,

$$10D \text{ Heterotic superstring}/\mathbb{T}^2 \leftrightarrow \text{ F-theory on K3} \ , \quad (3.7)$$

where topologically $K3 \sim E \times \mathbb{P}^1$.

This duality relation can be used to build other dualities in lower space time dimensions by using the adiabatic argument. By further compactifying (3.7) on a real two sphere $S^2 \sim \mathbb{P}^1$ reducing then the space time dimension to six, we get a duality between F-theory on Calabi-Yau three-folds with elliptic K3 fibration and the Heterotic superstring on elliptic K3,

$$10D \text{ Heterotic superstring/K3} \leftrightarrow \text{ F-theory on CY3} \ , \quad (3.8)$$

where topologically $CY3 \sim K3 \times \mathbb{P}^1$ or more explicitly $E \times \mathbb{P}^1 \times \mathbb{P}^1$.

In $4D$ space time, F-theory on Calabi-Yau four-folds is dual to the Heterotic superstring on Calabi-Yau three-folds,

$$10D \text{ Heterotic superstring/CY3} \leftrightarrow \text{ F-theory on CY}_4 \ . \quad (3.9)$$

As we see, these duality based definitions of F-theory are related and they can be used to build other dualities in various space dimensions by implementing type I, type II superstrings, eleven dimension M- and twelve dimension F-theories. These dualities turn out be crucial in the engineering of non abelian gauge symmetries, the bi-fundamental matter and Yukawa couplings.

### 3.2 Engineering non abelian gauge symmetries

Before coming to the engineering non abelian gauge symmetries in F-Theory on CY4-folds, let us start by recalling basic results that are helpful for the understanding the links between the geometry of CY4-folds and $4D$ space time non abelian gauge invariance.

---

Notice that the geometry of the compactification must be of a special type for this duality to hold. In F-theory GUT models, it is precisely those models that are not dual to heterotic superstring that are important as they allow gauge breaking of the GUT group through the so called hyperflux method.
(1) **Gauge fields in heterotic superstring**

In building GUT models extending MSSM, one needs, amongst others 4D non abelian gauge fields \( A_\mu \); that is operator fields with the non commutativity property \([A_\mu, A_\nu] \neq 0\). As it is well known, this non commutativity feature is solved by taking the hermitian gauge field \( A_\mu \) in the adjoint representation of an ADE gauge group G as given below,

\[
A_\mu = \sum_{a=1}^{\dim G} T_a A^a_\mu, \quad F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu],
\]

with

\[
[A_\mu, A_\nu] = \sum_{a,b=1}^{\dim G} A^a_\mu A^b_\nu [T_a, T_b] = \sum_{a,b=1}^{\dim G} C^{c}_{ab} A^a_\mu A^b_\nu T_c,
\]

where the \( T_a \)'s are the generators of G and \( C^{c}_{ab} \) its constant structures,

\[
[T_a, T_b] = \sum_{c=1}^{\dim G} C^{c}_{ab} T_c.
\]

These 4D massless gauge fields \( A^a_\mu \) appear naturally in the massless spectrum. Compactification down to lower space time dimensions, with some Wilson lines switched on to break partially gauge invariance, still have non abelian gauge fields \( A^a_\mu \). It is this property which made first heterotic superstring much popular and was behind the early days in building superstring inspired semi-realistic MSSM and GUT models \([56, 55]\) by using heterotic superstring compactifications down to 4D.

(2) **Non abelian gauge fields in F-theory**

In the 10D type IIB closed superstring with chiral \( \mathcal{N} = 2 \) supersymmetry, which according to (3.6) may be also viewed as the perturbative regime of 12D F-theory on \( T^2 \), the massless bosonic fields are as in eq(3.3). As we see, there is no non abelian gauge fields \( A^a_\mu \) in the massless spectrum of the theory. But this is not a problem since 10D type IIB closed superstring has Dp-branes with \( p = 1, 3, 5, 7 \). On a stack of \( r \) Dp-branes live...
abelian \((p+1)\) dimensional gauge fields \(A^I_M\) belonging to the spectrum of the quantized open superstrings that end on these branes. These fields can be put altogether like \(A^{(\text{abel})}_M = \sum_{I=1}^r A^I_M H_I\) with the property

\[
\left[ A^{(\text{abel})}_M, A^{(\text{abel})}_N \right] = \sum_{I,J=1}^r A^I_M A^J_N [H_I, H_J] = 0,
\]

(3.14)

In fact, one should think about \(\sum_{I=1}^r A^I_M H_I\) as the commuting part of a more general non abelian expansion involving as well the gauge fields associated with step operators of Lie algebras \(A^{\pm\alpha}_M\) of the strings stretching between the D- branes. Indeed for coincident branes, the gauge fields \(A^{\pm\alpha}_M\) become massless and one is left with a massless non abelian gauge field \(A^\alpha_M \equiv (A^I_M, A^{\pm\alpha}_M)\) in the spectrum. Thanks to the extended solitonic objects and open superstrings; these are exactly what is needed for engineering \textit{non abelian} gauge symmetries in type II superstrings.

By using the Heterotic string/F-theory duality (3.9), it is now clear that gauge symmetries \(G\) of the heterotic superstring on three- folds; with

\[
G \subset E_8 \times E_8 \quad \text{or} \quad G \subset SO(32),
\]

(3.15)

have a counterpart in the F-theory compactification on elliptically fibered CY4- folds \(X_4\). The origin of non abelian gauge fields in F-theory gauge on CY4- folds is then due to the 7- brane wrapping 4-cycles in CY4- folds:

| Heterotic string/CY3 | F-theory on CY4 |
|----------------------|----------------|
| gauge symmetry \(G\) | singularity |
| gauge fields \(A_M = (A^I_M, A^{\pm\alpha}_M)\) | coincident branes |

(3.16)

In this table, the gauge fields \(A^I_M\) and \(A^{\pm\alpha}_M\) are respectively associated with the Cartan Weyl basis generators \((H_I, E_{\pm\alpha})\) the of the Lie algebra \(g\) of the gauge symmetry \(G\); i.e

\[
A_M = \sum_{\alpha \in \Delta} E_{\pm\alpha} A^{\pm\alpha}_M + \sum_{I=1}^r H_I A^I_M,
\]

(3.17)

with \(\Delta = \Delta (g)\) being the root system of \(g\) and \(r = r (g)\) is its rank. To complete the table (3.16) for the case of F-theory GUT models, we still need to study:

(a) the engineering of non abelian gauge symmetry through the implementation of the ADE geometric singularities in the elliptically K3 fiber of the local \(CY4 \sim K3 \times S\),

(b) the 4D effective gauge theory on the seven brane wrapping compact 4- cycles in the base \(S\) of the local CY4-fold.

Below we study these two issues with some details.
3.2.1 4D gauge invariance in F-theory on CY4s

In the F-theory setup of the duality (3.9), the 4D space-time gauge symmetry $G$ has a very nice geometric interpretation. This invariance is in fact captured by a Weierstrass ADE singularity living in the local CY4-fold which may roughly be thought of as,

\[
\begin{align*}
\mathbb{E} \times \mathbb{P}^1 & \rightarrow X_4 \\
\downarrow \pi & \\
S
\end{align*}
\]

and described by the vanishing condition of the discriminant $\Delta_E$ of the elliptic curve $v^2 = u^3 + f(z)u + g(z)$. This condition reads as

\[
\Delta_E = (16f^3 + 27g^2) = 0,
\]

and its solutions depend on the nature of the sections $f(z)$ and $g(z)$.

To get more insight into the ways one deals with this condition for generic elliptically fibered CY4-folds $X_4$, we start by recalling that elliptic $X_4$ may be defined by the following complex four dimension elliptically fibered hypersurface in $\mathbb{C}^5$,

\[
v^2 = D(w_1, w_2, w_3)u^3 + E(w_1, w_2, w_3)u^2 + F(w_1, w_2, w_3)u + G(w_1, w_2, w_3).
\]

In this relation inspired from eq.(3.5), the holomorphic functions $D(w), E(w), F(w)$ and $G(w)$ are special sections on the base manifold $B_3$ of the elliptic Calabi-Yau 4-fold

\[
\begin{align*}
\mathbb{E} & \rightarrow X_4 \\
\downarrow \pi & \\
B_3
\end{align*}
\]

The complex variables $w = (w_1, w_2, w_3)$ are the holomorphic coordinates of the base manifold $B_3$ and the $w$-dependence in the holomorphic sections $D(w), E(w), F(w)$ and $G(w)$ are such that the various monomials in eq.3.20 transform homogeneously under coordinates transformations in the base.

To explicitly exhibit the Weierstrass ADE singularity on the complex 3- dimension base $B_3$ of the CY4-fold, it is interesting to factorize these holomorphic sections $D(w), E(w), F(w)$ and $G(w)$ like

\[
\begin{align*}
D(w) & = \vartheta(s_1, s_2) \times d(z), \\
E(w) & = \vartheta(s_1, s_2) \times e(z), \\
F(w) & = \vartheta(s_1, s_2) \times f(z), \\
G(w) & = \vartheta(s_1, s_2) \times g(z),
\end{align*}
\]
where we have supposed $\dot{\theta}(s_1, s_2) \neq 0$ and where $(s_1, s_2; z)$ are new local holomorphic coordinates of $B_3$ related to the old complex coordinates $(w_1, w_2, w_3)$ by some local analytic coordinate change,

\begin{align*}
    s_1 &= s_1(w_1, w_2, w_3), \\
    s_2 &= s_2(w_1, w_2, w_3), \\
    z &= z(w_1, w_2, w_3).
\end{align*}

(3.23)

In doing so, we have broken the $U(3)$ structure group of the tangent bundle $TB_3$ (with $B_3 \sim \Sigma_0 \times S$) down to $U_R(1) \times U(2)$ with $U_R(1)$ and $U(2)$ being respectively the structure group of the tangent sub-bundles $T\Sigma_0$ and $TS$. These complex structure groups are contained in the R-symmetry groups of the compactification of 12D F-theory down to 4D,

\begin{align*}
    U(3) &\subset SU(4) \simeq SO_R(6), \\
    U_R(1) &\simeq SO_R(2), \\
    U(2) &\subset SU(2) \times SU(2) \simeq SO_R(4).
\end{align*}

(3.24)

Notice also the following features:

(1) the complex holomorphic functions $d(z), e(z), f(z)$ and $g(z)$ are particular sections of the canonical bundle $K_{\Sigma_0}$ on the curve $\Sigma_0$ in $B_3$. These holomorphic functions, which will be specified later on for Weierstrass ADE singularities; see table (3.29), transform homogenously under the change $z \to \rho z$ as follows,

\begin{align*}
    d(z) &\to \rho^{n_d}d(z), \\
    e(z) &\to \rho^{n_e}e(z), \\
    f(z) &\to \rho^{n_f}f(z), \\
    g(z) &\to \rho^{n_g}g(z),
\end{align*}

(3.25)

where $n_d, n_e, n_f$ and $n_g$ are some positive integers.

(2) In the coordinate frame $(u, v, z; s_1, s_2)$, the local CY4-fold is thought of as

\begin{equation}
    Y_2 \to X_4 \\
    \downarrow \pi \\
    S
\end{equation}

(3.26)

where the local surface $Y_2$ is given by the elliptic curve $E$ fibered on the complex line $\Sigma_0$ with coordinate $z (Y_2 \sim E \times \Sigma_0)$. In this realization, the Calabi-Yau condition requires the two following:

(a) the local coordinates $(u, v, z)$ have to transform as sections of the canonical bundle over $S$. Under the “scaling” $s_i \to \sqrt{\lambda} s_i$, the local coordinates $(u, v, z)$ transform like

\begin{equation}
    (u, v, z) \to (\lambda^a u, \lambda^b v, \lambda^c z),
\end{equation}

(3.27)
where $\lambda$ is a non zero complex number and the degrees $a$, $b$ and $c$ are some integers to determine.

(b) the holomorphic 2- form $\Omega^{(2,0)} = \frac{du \wedge dz}{v^2}$ over the local surface $Y_2$ should scale like $\Omega^{(2,0)} \to \lambda \Omega^{(2,0)}$. This condition requires that the degrees $a$, $b$ and $c$ should be constrained as $a - b + c = 1$.

Substituting (3.23) back into (3.20) and setting $v^2 = \vartheta \tilde{v}^2$, we can factorize it as follows

$$\tilde{v}^2 = d(z) u^3 + e(z) u^2 + f(z) u + g(z),$$
$$\vartheta = \vartheta(s_1, s_2). \quad (3.28)$$

From these relations and following the analysis of [32], one can immediately read the Weierstrass form of the standard ADE singularities by specifying the holomorphic sections $d(z)$, $e(z)$, $f(z)$ and $g(z)$ as given below,

| singularity | $d$ | $e$ | $f$ | $g$ |
|-------------|-----|-----|-----|-----|
| $A_n$       | 0   | 1   | 0   | $z^{n+1}$ |
| $D_n$       | 0   | $z$ | 0   | $z^{n-1}$ |
| $E_6$       | 1   | 0   | 0   | $z^4$   |
| $E_7$       | 1   | 0   | $z^3$ | 0     |
| $E_8$       | 1   | 0   | 0   | $z^5$   |

(3.29)

For later use we mainly need the engineering of the following singularities,

$$A_n : \quad \tilde{v}^2 = u^2 + z^{n+1}, \quad n = 4, 5, 6,$$
$$D_n : \quad \tilde{v}^2 = zu^2 + \alpha z^{n-1}, \quad n = 5, 6,$$
$$E_6 : \quad \tilde{v}^2 = u^3 + z^4,$$  \quad (3.30)

in particular the $A_4$ geometry; its one - fold enhancements $A_5$ and $D_5$ and the two- folds enhancements $D_6$ and $E_6$. These enhancements, which are related to switching off Higgs vevs, have a geometric realization in terms of colliding singularities. Below we give some specific examples.

### 3.2.2 Examples

To fix the ideas on colliding singularities, we give below some illustrating examples on the engineering of enhanced non abelian gauge symmetries by colliding geometric singularities in the local CY4- folds. These examples will be used later on when we consider gauge/brane interpretation as well as the engineering of matter and Yukawa couplings.

**SU (2) gauge invariance**

In the description we have been using so far, the Weierstrass singularity capturing the
SU(2) gauge invariance of the low energy quantum field theory embedded in F-theory compactified on K3 fibered CY4- folds, termed in Kodaira classification as $A_1$ $[51]$, is given by,

$$v^2 = \vartheta (u^2 + z^2) ,$$

$$\vartheta = \vartheta (s_1, s_2) ,$$

(3.31)

where $\vartheta (s_1, s_2)$ is a non zero holomorphic function that live on the complex surface $S$ of eq(3.26). The locus of this $A_1$ geometric singularity lives at,

$$\{P_0\} \times S,$$

(3.32)

with $P_0 = (u, v, z) = (0, 0, 0)$ is the singular point in the K3 fiber where the elliptic curve $E$ degenerate. This means that the local CY4- folds $[3, 26]$ is given by the local $A_1$ space fibered on $S$. The fibration is captured by the relation $v^2 = -\vartheta (s_1, s_2) \tilde{v}^2$ and extends directly to higher order $A_n$ geometries fibered on $S$.

$SU(n) \times SU(m)$ gauge invariance

From the preceding example, it is not difficult to see that the Weierstrass singularity capturing the semi simple $SU(n) \times SU(m) \times U(1)$ gauge invariance of the supersymmetric QFT$_4$ embedded in F- theory on the CY4- folds is then given by the following algebraic relation

$$\frac{x^2}{\sigma} = v^2 + (z + t)^n (z - t)^m ,$$

$$\vartheta = \vartheta (s_1, s_2) .$$

(3.34)

The complex modulus $t$ is a section on $K_S$ with same degree as the variable $z$. This modulus may be physically thought of as a vevs of a matter field $\phi$ in the adjoint representation of $SU(n + m)$ with the following Cartan subalgebra value,

$$\langle \phi \rangle = t \sum_{I=1}^{n-1} H_I - t \sum_{I=1}^{m-1} H_{n-1+I} .$$

(3.35)

Notice that the singularity $A_{n-1}$ lives at $\{P_1\} \times S$ with the point $P_1 = (0, 0, -t)$ while the singularity $A_{m-1}$ lives at $\{P_2\} \times S$ with $P_2 = (0, 0, +t)$; see also the figure [5] for illustration.

The locus of the $A_{n-1}$ and $A_{m-1}$ singularity is then given by the set $\{P_1, P_2\} \times S$. Notice moreover that in the case where $t \to 0$, the two $A_{n-1}$ and $A_{m-1}$ singularities collide and the gauge symmetry gets enhanced to $SU(n + m)$ with singular geometry algebraic equation

$$v^2 = \vartheta [u^2 + z^{n+m}] ,$$

(3.36)

25
with $\vartheta = \vartheta(s_1, s_2)$ as before.

*SO(2n) and E_6 gauge invariances*

The elliptic singularity capturing the SO(2n) gauge symmetry is as follows

$$v^2 = (\alpha^2 z^{n-1} - zu^2) \vartheta, \quad n \geq 4, \quad \vartheta = \vartheta(s_1, s_2), \quad (3.37)$$

where the modulus $\alpha$ is trick to handle gauge enhancements [32]. This singularity lives at $(u, v, z) = (0, 0, 0)$ whatever are the complex coordinates $(s_1, s_2)$. Then, the locus of the D_n singularity is $\{P_0\} \times S$ with $P_0 = (0, 0, 0)$. Notice that for $z \neq 0$; say $z = 1$, the above relation describes an A_1 singularity at $(u, v, \alpha) = (0, 0, 0)$.

Similarly, the elliptic singularity capturing the E_6 gauge symmetry is

$$v^2 = (u^3 + z^4) \vartheta, \quad (3.38)$$

with $\vartheta = \vartheta(s_1, s_2)$. This singularity lives at $\{(0, 0, 0)\} \times S$. Aspects of colliding of such kind of singularities will be detailed in the following subsection.

### 3.3 Colliding singularities: pure and hybrids

Colliding singularities in the CY4- folds yields an enhancement of the gauge invariance of the supersymmetric QFT_4 embedded in the F-theory compactified on CY4. Generally speaking, we distinguish the following rough classification:

1. **Colliding singularities of same nature: pure colliding,**
2. **Colliding singularities of different types: hybrids.**

Let us comment briefly this classification through some selected examples.
3.3.1 Pure colliding

This kind of gauge invariance enhancement concerns are the colliding of two or several singularities of same type. Restricting the classification to the ADE case, we distinguish for a given integer \( l \geq 2 \):

(a) Unitary symmetry: \( SU(n_1) \times ... \times SU(n_l) \) with \( n_i \geq 2 \),

(b) Orthogonal symmetry: \( SO(2n_1) \times ... \times SO(2n_l) \) with \( n_i \geq 4 \),

(c) Exceptional symmetry \( E_s^\otimes l \) with \( s = 6, 7, 8 \).

These collisions do not give necessary a singularity of same type as we will show on the following examples.

For the unitary series, the simplest example concerns colliding the \( A_n \) and \( A_m \) singularities which lead to the enhancement,

\[ A_n \times A_m \rightarrow A_{n+m}, \quad (3.39) \]

whose algebraic relation is given by eq(3.34). In the case of three singularities \( A_{n_1}, A_{n_2} \) and \( A_{n_3} \), the collision can be achieved in various ways and leads to the following enhancement,

\[ A_{n_1} \times A_{n_2} \times A_{n_3} \rightarrow \left\{ \begin{array}{c} A_{n_1+n_2} \times A_{n_3} \\
A_{n_1} \times A_{n_2+n_3} \rightarrow A_{n_1+n_2+n_3} \\
A_{n_1+n_3} \times A_{n_2} \end{array} \right. \quad (3.40) \]

The colliding of \( A_n \) singularities is a commutative and associative product. These collisions extend straightforwardly to the case of colliding \( l \) singular components \( A_{n_i} \). We have several ways to do these collisions; but with same result at the end:

\[ A_{n_1} \times A_{n_2} \times ... \times A_{n_l} \rightarrow \left\{ \begin{array}{c} A_{n_1+n_2} \times \cdots \times A_{n_l} \\
\vdots \rightarrow \cdots \rightarrow A_{n_1+n_2+\cdots+n_l} \end{array} \right. \quad (3.41) \]

In the case where all singularities as well as their collision have all of them the complex surface \( S \) as a locus in the local CY4-folds, then the algebraic relation describing the colliding of these singularities reads as follows

\[ \begin{align*}
\frac{\nu^2}{\vartheta} &= x^2 + \prod_{i=1}^{l} (z - t_i)^{n_i} \\
\vartheta &= \vartheta (s_1, s_2)
\end{align*} \quad (3.42) \]

where the complex moduli \( t_i \) are sections on the canonical bundle \( K_S \) with same degree property as \( z \).

Regarding the orthogonal D- singularities, the colliding of two singularities \( D_n \) and \( D_m \) gives an exotic singularity which is beyond the scope of the present study. These singularities may be associated with the indefinite sector in the classification of Lie algebras [57, 58, 59, 60]. The same thing is valid for the colliding of the exceptional singularities.
3.3.2 Hybrids

Given several isolated singularities of type ADE, which are associated with a semi simple gauge group invariance in the QFT4 limit of F-theory on CY4-folds, we can engineer enhanced singularities by performing collisions. In addition to the pure colliding considered above, we also have, amongst others, the following hybrids:

\(\alpha\) case \(A_n \times D_m \to D_{n+m+1}\)
\(\beta\) case \(A_n \times E_6 \to \text{Exotic singularity}, n > 2\)
\(\gamma\) case \(D_n \times E_6 \to \text{Exotic singularity}\).

This analysis extends to the case of colliding more than two singularities. For the case of three singularities, we have

\(\alpha\) case \(A_n \times A_m \times D_k \to D_{n+m+k+1}\)
\(\beta\) case \(A_n \times A_m \times E_6 \to \text{Exotic singularity},\)
\(\gamma\) case \(A_n \times D_m \times E_6 \to \text{Exotic singularity}\).

More hybrids such as the triangular geometries \(T_{n,m,r}\) as those considered in the geometric engineering of superconformal QFT4 embedded in type II compactification on CY3-folds [44, 60] as technical details regarding these hybrids will be reported elsewhere.

4 Seven brane wrapping 4-cycles

In the 4D \(\mathcal{N} = 1\) supersymmetric QFT limit of F-theory on local Calabi Yau four-folds \(X_4\), there is a close relation between the degeneracy loci of the elliptic curve \(E\) in \(X_4\) and the seven brane wrapping 4-cycles. The space time region

\[(x^0, x^1, x^2, x^3; s_1, s_2, s_1, \bar{s}_2; z, \bar{z}, x^{11}, x^{12})\]  

of the 12-dimensional F-theory where the elliptic curve \(E\) (3.28) degenerates, corresponds precisely to the world volume \(V_8\) of the seven brane,

\[x^M = (x^0, x^1, x^2, x^3; s_1, s_2, s_1, \bar{s}_2)\]  

In this region, the \((u, v, z)\) coordinates of the K3 fiber of \(X_4\), with \(u = u(x^{11}, x^{12})\), \(v = v(x^{11}, x^{12})\), take particular values and one is left with the local coordinates (4.2) which parameterize the world volume of the seven brane. Notice that in addition to the non compact 4D space time \(\mathbb{R}^{1,3}\) with the usual real coordinates \((x^0, x^1, x^2, x^3)\), the holomorphic coordinates \((s_1, s_2)\) of (4.2) parameterize the compact complex surface \(S\) sitting in the complex three dimensional base \(B_3\). The compact real four dimensional

\(^5\)In (4.1) the variables \((x^{11}, x^{12})\) are the two real compact coordinates of the extra 2-torus used in F-Theory and which as been realized in terms of the algebraic curve \(v = du^3 + eu^2 + fu + g\).
manifold $S$ is just the locus of the elliptic singularity in the Calabi Yau 4-folds.

In this section, we first consider the $\mathcal{N} = 1$ supergravity in 8D space time \[64\]; then we analyze its reduction to 4D supergravity with four supersymmetric charges by borrowing ideas and results from the twisted topological field analysis of \[32\], in particular the solutions of BPS equations. After that, we use these results to study the 4D $\mathcal{N} = 1$ nonabelian gauge invariance in the seven brane wrapping $S$.

### 4.1 General on $\mathcal{N} = 1$ supergravity in 8D

We start by recalling that in curved eight dimensional space time lives a 8D $\mathcal{N} = 1$ supergravity theory describing the interacting dynamics of the supergravity multiplet $\mathcal{G}_{sugra}^{(8D)}$ coupled to superYang Mills $\mathcal{V}_{SYM}^{(8D)}$. This supersymmetric gauge theory may be viewed as the field theory limit of compactified superstrings theory at Planck scale; in particular as the limit of F-theory on $K3$ which is dual to 10D heterotic superstring on $\mathbb{T}^2$. In this subsection, we first review briefly general results on this supersymmetric gauge theory having sixteen conserved supercharges. Then we consider the super Yang-Mills theory in the limit of decoupled supergravity in connection with the philosophy of the F-theory GUT models building and the gauge theory on the seven brane wrapping 4-cycles of the Calabi Yau 4-folds.

#### 4.1.1 Fields spectrum

The massless spectrum of the $\mathcal{N} = 1$ supergravity in 8D involves two super multiplets: the supergravity multiplet $\mathcal{G}_{sugra}^{(8D)}$ and the Maxwell (super-Yang-Mills) multiplet $\mathcal{V}_{Max}^{(8D)}$. The 8D $\mathcal{N} = 1$ supergravity multiplet $\mathcal{G}_{sugra}^{(8D)}$ has the following fields content:

| Bosonic fields | Fermions |
|----------------|----------|
| $e^A_M$, $B_{MN}$, $G^1_M$, $G^2_M$, $\sigma$ | $\psi^M$, $\chi$ |

The bosonic sector consists of the graviton (eightbein) $e^A_M$ with space time metric $G_{MN} = e^A_M e^A_N$, the antisymmetric field $B_{MN}$, two gravi-photons $G^1_M$, $G^2_M$; and a scalar field $\sigma$: the 8D dilaton. The fermionic sector consists of the 8D Rarita-Schwinger field $\psi^M$ and a 8D pseudo Majorana fermion $\chi$. This supermultiplet contains $48 + 48$ on shell propagating degrees of freedom capturing the pure supergravity dynamics.

The 8D $\mathcal{N} = 1$ Maxwell supermultiplet $\mathcal{V}_{Max}^{(8D)}$ has the following fields content:

| superfield | Bosonic fields | Fermions |
|------------|---------------|----------|
| $\mathcal{V}_{Max}^{(8D)}$ | $A_{M}$, $\phi_1$, $\phi_2$ | $\lambda_\alpha$ |
where $A_M$ (to be some times denoted as $A_M^{(8D)}$) is a $8D$ Maxwell field. The spinor field $\lambda_a$ is the $8D$ gaugino having real 8 propagating degrees of freedom and the fields $(\phi_1, \phi_2)$ are two real $8D$ scalars parameterizing the $SO(1, 2)/SO(2)$ coset manifold

$$SO(1, 2)/SO(2) \sim SU(1, 1)/U(1)$$

(4.5)

defining the interactions of these scalar fields.

Notice that the two scalar fields $\phi_m = (\phi_1, \phi_2)$ are charged under the $U_R(1) \simeq SO_R(2)$ symmetry of eq(3.24) with the $SO_R(2)$ appearing in the breaking of the $10D$ space time group

$$SO(1, 9) \supset SO(1, 7) \times SO_R(2).$$

(4.6)

This property can be immediately viewed by thinking about $(A^{(10D)}_M, \phi_1, \phi_2)$ as following from the reduction of the dimensional field $A^{(10D)}_M$. Reducing the flat space time dimension $\mathbb{R}^{1,9}$ down to $\mathbb{R}^{1,9} \times \mathbb{C}$, the real 1-form gauge connexion $A^{(10D)} = \sum_{M=0}^9 A^{(10D)}_M dx^M$ splits as

$$A^{(10D)} = \left( \sum_{M=0}^7 A^{(8D)}_M dx^M \right) + (\phi dz + \bar{\phi} d\bar{z}),$$

(4.7)

where we have set

$$\phi = \phi_1 + i\phi_2 = \frac{1}{2} \left( A^{(10D)}_8 + iA^{(10D)}_9 \right)$$

(4.8)

and where $z = x^8 + ix^9$ stands for the coordinate of the complex line $\mathbb{C}$. Under the change $z \rightarrow e^{i\theta} z$, then we should have

$$\phi \rightarrow e^{-i\theta} \phi$$

(4.9)

showing that $\phi$ carries a $U_R(1)$ charge $q_\phi = -1$.

**Supergravity action**

To write down the more general supergravity action, let us consider the case of $n$ Maxwell multiplets $\mathcal{V}^{\text{max}}_a = (A^a_M, \phi^a_1, \phi^a_2; \lambda^a_8)$ which, in the case of F- theory on K3, may be thought of as dealing with the gauge theory of $n$ separated 7-branes. Combining these Maxwell multiplets $\mathcal{V}^{\text{max}}_a$ with the gravity multiplet $\mathcal{G}^{(sugra)}$, we get, in addition to the fermions $\psi_M, \chi, A^a_8$, the following bosonic fields

$$e^A_M, \ B_{MN}, \ A^A_M, \ \phi^a, \ \sigma,$$

(4.10)

where we have set $\phi^a = (\phi^a_1, \phi^a_2)$. In this relation, the scalars $\phi^a$ with $a = 1, ..., n$ parameterize the Kahler manifold

$$SO(n, 2)/SO(n) \times SO(2)$$

(4.11)
fixing the interactions of the scalars. Following [61, 62], this coset manifold is conveniently parameterized by the following typical representative \((n + 2) \times (n + 2)\) orthogonal matrix,

\[
L^\Lambda_\Sigma = \begin{pmatrix}
0_{2 \times 2} & (\phi^a)_{2 \times n} \\
(\phi_a)_{n \times 2} & 0_{2 \times 2}
\end{pmatrix}_{(n+2) \times (n+2)}, \quad L^\Lambda_\Sigma L^\Sigma_\Gamma \eta_{\Gamma \Sigma} = \eta_{\Lambda \Sigma},
\] (4.12)

where the metric \(\eta_{\Lambda \Sigma} = \text{diag}(+ + ... + - -)\) of the \(\mathbb{R}^{n,2}\) real space.

Regarding the 8D gauge vector fields \((A^a_M, G^1_M, G^2_M)\) involved in this theory, they may be combined as \(A^\Lambda_M\), with \(\Lambda = 1, ..., n + 2\). These Maxwell type gauge fields transform as a vector of \(SO(n,2)\) while the gaugino partners transform as a vector under \(SO(n)\). To describe the interactions of the scalar fields, we also need the gauge connection \(L^{-1} \partial_M L\) which may be split as follows

\[
L^{-1} \partial_M L = \begin{pmatrix}
Q^b_M a & P^i_M a \\
P^b_M i & Q^i_M b
\end{pmatrix}_{(n+2) \times (n+2)},
\] (4.13)

where \(Q^b_M a\) and \(Q^i_M b\) are respectively the \(SO(n)\) and \(SO(2)\) gauge connections and where \(P^i_M a\) are the Cartan-Maurer Form transforming homogeneously under the \(SO(n) \times SO(2)\) gauge symmetry.

Following [64], the component field action of the \(N = 1\) supergravity in 8D describing the interacting dynamics of the \(G_{\text{sugra}}^{(8D)}\) and the \(n\) vector multiplets \(V_{\text{Max}}^{(8D)}\) reads as

\[
\frac{L_0}{\det e} \simeq \frac{1}{4} \mathcal{R} - \frac{1}{4} e^\sigma \tau_{\Lambda \Sigma} \mathcal{F}^\Lambda_{MN} \mathcal{F}^{MN \Sigma} - \frac{1}{12} e^{2 \sigma} G_{MNQ} G^{MNQ} + \frac{3}{8} \partial_M \sigma \partial^M \sigma + \frac{1}{4} P^i_{Ma} P^M a \quad \text{(4.14)}
\]

with \(\mathcal{F}^\Lambda_{MN}, G_{MNQ}\) and \(\tau_{\Lambda \Sigma}\) given by

\[
\mathcal{F}^\Lambda_{MN} = \partial_M A^\Lambda_N - \partial_N A^\Lambda_M, \quad G_{MNQ} = \partial_M B_{NQ} - \eta_{\Lambda \Sigma} F^\Lambda_{MN} A^\Sigma_Q + \text{cyclic permutation}, \quad \tau_{\Lambda \Sigma} = L^a_\Lambda L^a_\Sigma + L^\Lambda a L^a_\Sigma.
\] (4.15)

### 4.1.2 SYM\(_8\) in decoupling gravity limit

The 8D \(N = 1\) supergravity multiplet \(G_{8D}\) may also couple non abelian superYang-Mills multiplets. The field content of these \(N = 1\) non abelian gauge supermultiplets is given by,

| Bosons | Fermions |
|--------|----------|
| \(A_M = \sum_{a=1}^{\dim G} T_a A^a_M\), \(\phi = \sum_{a=1}^{\dim G} T_a \phi_a\) | \(\lambda = \sum_{a=1}^{\dim G} T_a \lambda^a\) |

(4.16)
where now the $8D$ gauge fields are valued in the Lie algebra of the gauge symmetry with matrix generators $\{T_a\}$ as in eq (3.11).

The component fields action describing the classical interacting dynamics may be constructed perturbatively by using Noether method. This field action reads as

$$S^{(8D)}_{\text{sugra}} = \int_{R^{1,7}} d^8x \mathcal{L}^{(8D)}_{\text{sugra}}(x),$$

(4.17)

with $\mathcal{L}^{(8D)}_{\text{sugra}}$ describing the lagrangian density of the $8D$ supergravity fields

$$\mathcal{L}^{(8D)}_{\text{sugra}} = \mathcal{L} (e^A_M, B_{MN}, G^1_M, G^2_M, \varphi, \psi, \chi; A_M, \phi^1, \phi^2, \lambda).$$

(4.18)

It is given by the sum of the $8D$ Hilbert-Einstein supergravity term $\mathcal{L}_{HE}$ plus $\mathcal{L}_{SYM-E}$ the $8D$ superYang-Mills term coupled to supergravity.

In the limit of decoupled supergravity, the dynamics of the gravity supermultiplet (4.3) is freezed and the above action $S^{(8D)}_{\text{sugra}}$ reduces to the usual supersymmetric Yang Mills theory $S^{(8D)}_{\text{SYM}} = \int_{R^{1,7}} d^8x \mathcal{L}^{(8D)}_{\text{SYM}}$ with,

$$S^{(8D)}_{\text{SYM}} = \int_{R^{1,7}} d^8x Tr \left(-\frac{1}{8} F_{MN} F^{MN} + i \bar{\lambda} \Gamma^M D_M \lambda + D_M \bar{\phi} D^M \phi \right) + \ldots,$$

(4.19)

where $F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$ is the $8D$ field strength valued in the Lie algebra of the gauge group.

4.1.3 Reduction to 4D $\mathcal{N} = 1$ supersymmetry

In the supersymmetric $QFT$ set up of the F-theory GUT models building, the starting point is precisely the $\mathcal{N} = 1$ supersymmetric Yang-Mills lagrangian density $\mathcal{L}^{(8D)}_{\text{SYM}}$ (4.19). Since the seven brane wraps 4-cycles $S$ in the CY4- folds, the $8D$ fields $\Phi(x; s, \bar{s})$ of the 7- brane bulk theory may be thought of as a collection of $4D$ space time fields

$$\Phi_{\{s_1, s_2\}} = \Phi_{\{s_1, s_2\}}(x)$$

(4.20)

labeled by points $s_m = (s_1, s_2) \in S$. Then, to reduce the above $\mathcal{N} = 1$ 8D SYM down to a $\mathcal{N} = 1$ supersymmetry in $4D$ as required by the compactification of F-theory on CY4-folds, one needs to compactify $\mathbb{R}^{1,7}$ as $\mathbb{R}^{1,3} \times S$. Under this compactification, the action $S^{(8D)}_{\text{SYM}}$ (4.19) gets reduced down to $S^{(4D)}$ as follows,

$$S^{(4D)}_{\text{SYM}} = \int_{R^{1,3}} d^4x \mathcal{L}^{(4D)}_{\text{SYM}},$$

(4.21)

32
where $\mathcal{L}_{SYM}^{(4D)}$ is a priori given by,

$$
\mathcal{L}_{SYM}^{(4D)} = \int_S d^2s d^2\bar{s}\ L_{SYM}^{(8D)}[\phi(x; s, \bar{s})].
$$

(4.22)

Notice that in performing the reduction from $8D$ down to $4D$, one should worry about two main things: (1) supersymmetry and (2) chiral matter representations.

**Supersymmetry**

In the flat $8D \mathcal{N} = 1$ supersymmetric Yang-Mills (4.19), there are sixteen conserved supersymmetries. This is too much since compactification of F-theory on CY4-folds has only four conserved supersymmetries. Thus the reduction from $8D$ down to $4D$ should preserve $\frac{1}{4}$ of the original sixteen. Using the compactification $\mathbb{R}^{1,7} \to \mathbb{R}^{1,3} \times S$, the $SO(1,7)$ structure group gets broken down like

$$
SO(1,7) \to SO(1,3) \times U(2) = SO(1,3) \times SU(2) \times U(1).
$$

(4.23)

The mechanism to perform the reduction preserving four supersymmetric charges has been studied in [32] and is based on the mapping

$$
U_R(1) \times U_J(1) \to U_{\text{top}}(1),
$$

(4.24)

with twist charge,

$$
J_{\text{top}} = J + 2R \equiv T,
$$

(4.25)

borrowed from topological field theory ideas.

**Chiral matter**

In $4D \mathcal{N} = 1$ supersymmetric gauge theory, chiral matter is described by chiral superfields transforming in complex representations of the gauge group. The reduction of $\mathcal{N} = 1$ SYM$_{8D}$ down to a $4D \mathcal{N} = 1$ supersymmetric theory gives indeed chiral matter; but only in the *adjoint* representation. This property is immediately seen by decomposing the $\mathcal{N} = 1$ super Yang-Mills multiplets $\mathcal{V}_{SYM}^{(8D)}$. This supermultiplet belongs to the adjoint representation of the gauge group and decomposes as follows:

(a) $\mathcal{N} = 1$ super Yang-Mills multiplets $\mathcal{V}_{SYM}^{(4D)}$,

(b) three massless chiral multiplets $\Phi_0, \Phi_1, \Phi_2$ in the adjoint representation,

(c) an infinite tower of massive KK type modes which may be denoted as $\mathcal{V}_{\pm n}, \Phi_0[n], \Phi_1[n]$ and $\Phi_2[n]$; see also next subsection for more details.

As we see, there is no chiral superfield in complex representations of the gauge group. This difficulty has been solved in wonderful manner in [32], by considering local Calabi-Yau four-folds,

$$
Y_2 \to X_4
$$

$$
\downarrow \pi
$$

$$
\to S,
$$

(4.26)
where now the base surface $S = \cup_a C_a$ with non trivial 4- cycles intersections
\[ C_a \cap C_a = \Sigma_{ab}. \] (4.27)
where $\Sigma_{ab}$ are real 2- cycles inside the CY4- folds. Each real 2- cycle $\Sigma_{ab}$ defines the locus of intersecting seven branes where precisely live chiral matter. In the next subsection, we give some explicit details.

4.2 $SU(N)$ invariance in seven brane

In the F-theory set up, non abelian gauge invariance has a remarkable geometric engineering in terms of seven branes wrapping compact 4- cycles $C_a$ in the CY4- folds. On each 4-cycle the world volume of the seven brane splits into two blocks: 

(1) the four non compact real $(1 + 3)$ space time dimensions viewed as the 4D space where lives the $\mathcal{N} = 1$ supersymmetric GUT.

(2) four compact directions wrapping the 4-cycle $C_a$ a number of times; say $N_a$ times.

In the case of $SU(N_a)$ gauge invariance, the fiber $Y$ of the Calabi-Yau 4- folds (4.26) has a $\mathcal{A}_{N_a - 1}$ singularity described by the following algebraic equation
\[ \frac{v^2}{\vartheta} = u^2 + z^{N_a}, \quad \vartheta = \vartheta(s_1, s_2), \] (4.28)
where the integer $N_a$ in the monomial $z^{N_a}$ captures the number of times the seven brane wraps $C_a$.

4.2.1 QFT$_{8D}$ set up

In the supersymmetric field theory analysis, the non abelian gauge theory in the seven brane involves the following:

First, a non abelian 8D $\mathcal{N} = 1$ supersymmetric $SU(N_a)$ Yang Mills multiplet
\[ \left( A_M^{(8D)}, \lambda_{\dot{a}}^{(8D)}, \phi^{(8D)} \right), \] (4.29)
with flat space time gauge dynamics given by the action $S_{SYM}^{(8D)}$ (4.19). Since $SU(N_a)$ is an exact gauge symmetry for the gauge theory engineered from
\[ Y_2 \rightarrow X_4 \]
\[ \downarrow \pi \]
\[ C_a, \] (4.30)
the vev of the scalar $\phi^{(8D)}$ in the adjoint representation of the $SU(N_a)$ gauge symmetry should vanish, i.e
\[ \left\langle \phi^{(8D)} \right\rangle = 0, \] (4.31)
otherwise $SU(N_a)$ gauge invariance would be broken down to a subsymmetry.

Second, being $SU(N_a)$ matrices in the adjoint representation, the 8D gauge fields can be expanded as follows:

$$
\phi^{(8D)} = \sum_{\alpha \in \Delta} E_{\pm \alpha} \phi^{\pm \alpha} + \sum_{I=1}^{N_a-1} H_I \phi_I,
$$

$$
\mathcal{A}_M^{(8D)} = \sum_{\alpha \in \Delta} E_{\pm \alpha} \mathcal{A}_M^{\pm \alpha} + \sum_{I=1}^{N_a-1} H_I \mathcal{A}_M^I,
$$

$$
\lambda_{\hat{a}}^{(8D)} = \sum_{\alpha \in \Delta} E_{\pm \alpha} \lambda_{\hat{a}}^{\pm \alpha} + \sum_{I=1}^{N_a-1} H_I \lambda_{\hat{a}}^I,
$$

(4.32)

where $\{H_I, E_{\pm \alpha}\}$ is the Cartan Weyl basis of $su(N_a)$ and $\Delta$ its root system. Putting the expansion of $\phi^{(8D)}$ back into (4.32) and setting $\langle \phi^{\pm \alpha} \rangle = 0$, we have

$$
\langle \phi^{(8D)} \rangle = \sum_{I=1}^{N_a-1} H_I \langle \phi_I \rangle.
$$

(4.33)

Giving a non zero vev to some of the $\phi_I$; say $\langle \phi_I \rangle = t_I \neq 0$ with $I = 1, ..., N_0$, the gauge symmetry gets broken down to $SU(N_a - N_0) \times U^{N_0} (1)$.

On the level of the geometry of the local Calabi-Yau 4- folds, this breaking corresponds to performing a complex deformation of the singularity which reduce the degree of the $A_{N_a-1}$ singularity (4.28) down to

$$
\frac{v^2}{\partial} = u^2 + (z - t_1)(z - t_2) \ldots (z - t_{N_0})^{N_a - N_0},
$$

(4.34)

where $\partial = \partial (s_1, s_2)$.

Next, seen that the seven brane wraps the compact 4-cycle $C_a$, the above 8D gauge fields depend on the local coordinates $(x^0, x^1, x^2, x^3, s_1, s_2, \bar{s}_1, \bar{s}_2)$; that is:

$$
\phi_a^{(8D)} = \phi_a^{(8D)} (x^0, x^1, x^2, x^3, s_1, s_2, \bar{s}_1, \bar{s}_2) \equiv \phi_a^{(8D)} (x; s, \bar{s}),
$$

$$
\mathcal{A}_M^{(8D)} = \mathcal{A}_M^{(8D)} (x^0, x^1, x^2, x^3, s_1, s_2, \bar{s}_1, \bar{s}_2) \equiv \mathcal{A}_M^{(8D)} (x; s, \bar{s}),
$$

$$
\lambda_{\hat{a}}^{(8D)} = \lambda_{\hat{a}}^{(8D)} (x^0, x^1, x^2, x^3, s_1, s_2, \bar{s}_1, \bar{s}_2) \equiv \lambda_{\hat{a}}^{(8D)} (x; s, \bar{s}).
$$

(4.35)

However, since $C_a$ is a compact Kahler manifold\footnote{The 4D fields are determined by the zero modes of the Dirac operator on the complex base surface. The chiral and antichiral spectrum is determined by the bundle valued cohomology groups $H^0_\mathcal{A} (S, R^\nu)^v \oplus H_1^\mathcal{A} (S, R) \oplus H_2^\mathcal{A} (S, R^\nu)^v$ and $H^0_\mathcal{A} (S, R) \oplus H_1^\mathcal{A} (S, R^\nu)^v \oplus H_2^\mathcal{A} (S, R)$ where $R$ is the vector bundle on the base surface whose sections transform in the representation $R$ of the structure group.}, these fields may be expanded in harmonic series in terms of the representations of the $U(2)$ structure group of $TC_a$. For the
case of the scalar fields $\phi^{(8D)}$ and $\bar{\phi}^{(8D)}$, we have a priori the following expansion:

$$
\phi^{(8D)} = \phi_0^{(4D)} (x) + \sum_{n>0} \phi_n^{(4D)} (x) R_n ,
$$
$$
\bar{\phi}^{(8D)} = \bar{\phi}_0^{(4D)} (x) + \sum_{n>0} \bar{\phi}_n^{(4D)} (x) \bar{R}_n ,
$$

(4.36)

where the zero mode of the expansion,

$$
\phi_0^{(4D)} (x) \equiv \phi (x) , \quad \bar{\phi}_0^{(4D)} (x) \equiv \bar{\phi} (x) ,
$$

(4.37)

stand for the 4D scalar fields and $\phi_n^{(4D)} (x)$ and $\bar{\phi}_n^{(4D)} (x)$ for the non zero modes associated with the non trivial $U (2)$ representations $R_n = R_n [s, \bar{s}]$ and $R_{[-n]} = R_n [s, \bar{s}]$.

Moreover, following [32, 33] the BPS conditions require the field to be holomorphic on the $S$ so that the representations $R_n$ are holomorphic and may be taken as,

$$
R_n = \sum_{k=-n}^n s_1^{n-k} s_2^k .
$$

(4.38)

Similarly, we have for the 8D vector gauge field,

$$
A^{(8D)}_\mu = \left( A^{(8D)}_\mu , A^{(8D)}_i , A^{(8D)}_{\bar{i}} \right) ,
$$

(4.39)

the following mode expansion

$$
A^{(8D)}_\mu (x; s, \bar{s}) = A_\mu (x) + \sum_{n>0} \left( R_{[n]} A^{[n]}_\mu (x) + \bar{R}_{[n]} A^{[-n]}_\mu (x) \right) ,
$$

(4.40)

where the zero mode $A_\mu (x) \equiv A^{[0]}_\mu (x)$ stands for the massless 4D gauge field and $A^{[\pm n]}_\mu (x)$ for the higher modes. Analogous expansions are valid as well for the four other components on the compact manifold namely

$$
A^{(8D)}_i = A_i (x) + \sum_{n>0} R_{[n]} A^{[n]}_i (x) ,
$$
$$
A^{(8D)}_{\bar{i}} = A_{\bar{i}} (x) + \sum_{n>0} \bar{R}_{[n]} A^{[n]}_{\bar{i}} (x) ,
$$

(4.41)

where the zero modes $A_i (x)$ and $A_{\bar{i}} (x)$ are two $U (2)$ doublets of 4D scalars while $A^{[n]}_i$ and $A^{[n]}_{\bar{i}}$ describe massive excitations.

Regarding the 8D fermionic field $\lambda^{(8D)}_{\hat{a}}$, the reduction is a little bit more technical as it requires splitting this $SO (1, 7)$ spinor in terms of representations of $SO (1, 3) \times U (2)$. Let us treat this decomposition separately as it is interesting as well for the reduction of the sixteen original supersymmetries down to the four conserved supercharges in $\mathcal{N} = 1$ supersymmetric theory in 4D space time.

\footnote{BPS conditions \cite{32} require furthermore that these expansions to be holomorphic in the complex coordinates of the complex surface $C$.}
4.2.2 Twisted gauge theory

We begin by recalling that the $SO(1,7)$ space time group of the $8D$ flat space time $\mathbb{R}^{1,7}$ decomposes in the case of the seven-brane wrapping a 4-cycle $C_a$ in the Calabi-Yau 4-folds like,

$$SO(1,7) \times U_R(1) \supset SO(1,3) \times SO(4) \times U_R(1) \enspace ,$$

$$\supset SO(1,3) \times U(2) \times U_R(1) \enspace ,$$

where $U(2) = U_J(1) \times SU(2)$ is just the structure group of the tangent bundle of $C_a$ and where $U_R(1)$ is as in eq(3.24).

To twist the gauge theory in the seven brane, we combine the $U_R(1)$ charge and the $U_J(1)$ as in eq(4.24) and then think about the compact symmetry group as

$$U_R(1) \times U(2) = U_R(1) \times U_J(1) \times SU(2) \enspace ,$$

$$\supset SU_T(1) \times SU(2) = SU_T(2) \enspace .$$

with $T = J + 2R$ as in the relation (4.25). In other words, we have the following chain of breakings of space time groups

$$SO(1,9) \supset SO(1,7) \times U_R(1) \enspace ,$$

$$\supset SO(1,3) \times U_R(1) \times SO(4) \enspace ,$$

$$\supset SO(1,3) \times U_R(1) \times U_J(1) \times SU(2) \enspace ,$$

$$\supset SO(1,3) \times U_T(1) \times SU(2) \enspace .$$

The sixteen components of the $SO(1,7)$ spinor decomposes in terms of the representations of $SO(1,3) \times SU(2) \times U_T(1)$ as follows:

$$16 = (2,1) \otimes 1_0 \oplus (1,2) \otimes 1_0$$

$$\oplus (1,2) \otimes 2_- \oplus (1,2) \otimes 2_+$$

$$\oplus (1,2) \otimes 1_- \oplus (2,1) \otimes 1_+ \enspace .$$

Thus the gaugino $\lambda_{\dot{a}}^{(8D)}$ decomposes into two $U(2)$ singlets $\eta_\alpha$ and $\chi_{\alpha[mn]}$ of 4D Weyl spinors as well as a doublet $\bar{\psi}_{\dot{\alpha}m}$:

$$(2,1) \otimes 1_0 \equiv \eta_\alpha \enspace , \enspace (1,2) \otimes 1_0 \equiv \bar{\eta}_{\dot{\alpha}} \enspace ,$$

$$(1,2) \otimes 2_- \equiv \bar{\psi}_{\dot{\alpha}m} \enspace , \enspace (2,1) \otimes 2_+ \equiv \psi_{\alpha m} \enspace ,$$

$$(2,1) \otimes 1_- \equiv \chi_{\alpha[mn]} \enspace , \enspace (2,1) \otimes 1_+ \equiv \bar{\chi}_{\dot{\alpha}[mn]} \enspace .$$

Each of these 4D Weyl spinor fields has a harmonic expansion as in (4.36,4.40) and combine with the bosonic fields (4.36,4.40,4.41) to form $\mathcal{N} = 1$ supermultiplets in 4D space time. The bosonic modes $\phi_{[\pm]}$, $A_{[n]}$, $A_{[n]}^i$, $A_{[-n]}^i$ and the fermionic ones $\eta_\alpha^{[n]}$, $\bar{\eta}_{\dot{\alpha}}^{[n]}$, $\chi_{\alpha[n]}$, $\bar{\chi}_{\dot{\alpha}[n]}$, $\psi_{\alpha[n]}$, $\bar{\psi}_{\dot{\alpha}[n]}$, $\xi_{\alpha[n]}$, $\bar{\xi}_{\dot{\alpha}[n]}$.
\(\bar{\psi}^{[n]}_{\alpha m}, \chi^{[n]}_{\alpha[ij]} = \varepsilon_{ij} \chi^{[n]}_{\alpha} \) together with their complex conjugates combine to form \(N = 1\) supermultiplets valued in the \(su (N_a)\) Lie algebra. For the zero modes, we have

\[
gauge \text{ multiplets} : \quad V = (A_\mu, \eta_\alpha, \bar{\eta}_{\bar{\alpha}})
\]

\[
\text{chiral matter multiplets} : \quad \begin{cases} 
\Phi_{ij}^- = \varepsilon_{ij} (\phi^{--}, \chi_{\alpha}^-) \\
\Upsilon_i^+ = (A_i^+, \psi_{\alpha i}^+)
\end{cases}
\]

where the upper charges refer to the \(U_T (1)\) twisted charge \(T = J + 2R\). Similar superfield relations are valid for each excitation level.

5 Engineering F-theory GUT model

In the engineering of supersymmetric GUT models in the framework of F-theory compactification on local CY4-folds, one has to specify, amongst others, the base surface \(S\). A priori, one may imagine several kinds of compact complex surfaces by considering hypersurfaces in higher dimensional complex Kahler manifolds. Typical examples of compact complex surfaces \(S\) which have been considered in F-Theory GUT literature are given by the del Pezzo surfaces \(dP_n\) with \(n = 0, 1, ..., 8\) obtained by preforming up to eight blow ups in the projective plane \(\mathbb{P}^2\). \[37, 32, 38, 39\].

Later on we develop a class of models based on toric manifold involving the complex tetrahedral surface of figure (11) and its blown ups \[40\]. But before that, we want to discuss here the \(dP_n\) based GUT model; as a front matter towards the study of the local tetrahedron model.

We take this opportunity to study a realization of \(SU (5)\) GUT model by using five intersecting 7-branes wrapping 4- cycles in the del Pezzo \(dP_8\) as illustrated by figure (8).

5.1 Del Pezzo surfaces \(dP_k\)

Here, we give some useful tools on del Pezzo surfaces; these are needed for the engineering of the corresponding \(SU (5)\) GUT model based on \(dP_k\) with \(5 \leq k \leq 8\).

5.1.1 2- cycle homology of \(dP_k\)

The \(dP_k\) del Pezzo surfaces with \(k \leq 8\) are defined as blow ups of the complex projective space \(\mathbb{P}^2\) at \(k\) points. Taking into account the overall size \(r_0\) of the \(\mathbb{P}^2\), a surface \(dP_k\) has then real \((k + 1)\) dimensional Kahler moduli,

\[
r_0, \quad r_1, \quad \cdots, \quad r_k
\]

(5.1)
corresponding to the volume of each blown up cycle \[32, 39, 40\]. The \(dP_k\)s possess as well a moduli space of complex structures with complex dimension \((2k - 8)\) where the eight gauge fixed parameters are associated with the \(GL(3)\) symmetry of \(\mathbb{P}^2\). As such, only surfaces with \(5 \leq k \leq 8\) admit a moduli space of complex structures.

The real 2-cycle homology group \(H_2(dP_k, \mathbb{Z})\) is \((k + 1)\) dimensional and is generated by \(\{H, E_1, ..., E_k\}\) where \(H\) denotes the hyperplane class inherited from \(\mathbb{P}^2\) and the \(E_i\) denote the exceptional divisors associated with the blow ups. These generators have the intersection pairing

\[
H^2 = 1, \quad H.E_i = 0, \quad E_i.E_j = -\delta_{ij}, \quad i, j = 1, ..., k, \quad (5.2)
\]

so that the signature \(\eta\) of the \(H_2(dP_k, \mathbb{Z})\) group is given by \(\text{diag}(+ - ... -)\).

The first three blow ups giving \(dP_1, dP_2\) and \(dP_3\) complex surfaces are of toric types while the remaining five others namely \(dP_4, ..., dP_8\) are non toric. These projective surfaces have the typical toric fibration

\[
dP_k = T^2 \times B_2^{(k)}, \quad k = 1, 2, 3,
\]

with real base \(B_2^{(k)}\) nicely represented by toric diagrams \(\Delta_2^{(k)}\) encoding the toric data of the fibration

| surface \(S\) | \(dP_0 = P^2\) | \(dP_1\) | \(dP_2\) | \(dP_3\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| blow ups | \(k = 0\) | \(k = 1\) | \(k = 2\) | \(k = 3\) |
| toric graph \(\Delta_2^{(k)}\) | triangle | square | pentagon | hexagon |
| generators | \(H\) | \(H, E_1\) | \(H, E_1, E_2\) | \(H, E_1, E_2, E_2\) |

(5.3)

In terms of these basic classes of curves, one defines all the tools needed for the present study; in particular the three following:

(1) the generic classes \([\Sigma_a]\) of holomorphic curves in \(dP_k\) given by the following linear combinations,

\[
\Sigma_a = n_a H - \sum_{i=1}^{k} m_{ai} E_i, \quad (5.4)
\]

with \(n_a\) and \(m_a\) are integers. The self- intersection numbers \(\Sigma_a^2 \equiv \Sigma_a \cdot \Sigma_a\) following from eqs (5.4) and (5.2) are then given by

\[
\Sigma_a^2 = n_a^2 - \sum_{i=1}^{k} m_{ai}^2, \quad (5.5)
\]

(2) The canonical class \(\Omega_k\) of the projective \(dP_k\) surface, which is given by \minu s the first Chern class \(c_1(dP_k)\) of the tangent bundle, reads as,

\[
\Omega_k = - \left(3H - \sum_{i=1}^{k} E_i\right), \quad (5.6)
\]

39
and has a self intersection number $\Omega_k^2 = 9 - k$ whose positivity requires $k < 9$. Obviously $k = 0$ corresponds just to the case where there is no blow up; i.e $dP_0 = \mathbb{P}^2$ the complex projective plane.

(3) the degree $d_\Sigma$ of a generic complex curve class $\Sigma = nH - \sum_{i=1}^{k} m_i E_i$ in $dP_k$ is given by the intersection number between the class $\Sigma$ with the anticanonical class ($-\Omega_k$),

$$d_\Sigma = - (\Sigma \cdot \Omega_k) = 3n - \sum_{i=1}^{k} m_i.$$  \hfill (5.7)

Positivity of this integer $d_\Sigma$ puts a constraint equation on the allowed values of the $n$ and $m_i$ integers which should be like,

$$\sum_{i=1}^{k} m_i \leq 3n.$$ \hfill (5.8)

Notice that there is a remarkable relation between the self intersection number $\Sigma^2$ (5.5) of the classes of holomorphic curves and their degrees $d_\Sigma$. This relation, which is known as the adjunction formula, is given by

$$\Sigma^2 = 2g - 2 + d_\Sigma,$$ \hfill (5.9)

and allows to define the genus $g$ of the curve class $\Sigma$ as

$$g = 1 + \frac{n(n - 3)}{2} - \sum_{i=1}^{k} m_i (m_i - 1).$$ \hfill (5.10)

For instance, taking $\Sigma = 3H$; that is $n = 3$ and $m_i = 0$, then the genus $g_{3H}$ of this curve is equal to 1 and so the curve $3H$ is in the same class of the real 2- torus. In general, fixing the genus $g$ to a given positive integer puts then a second constraint equation on $n$ and $m_i$ integers; the first constraint is as in (5.8). For the example of rational curves with $g = 0$, we have

$$\Sigma^2 = d_\Sigma - 2$$ \hfill (5.11)

giving a relation between the degree $d_\Sigma$ of the curve $\Sigma$ and its self intersection. For $d_\Sigma = 0$, we have a rational curve with self intersection $\Sigma^2 = -2$ while for $d_\Sigma = 1$ we have a self intersection $\Sigma^2 = -1$. To get the general expression of genus $g = 0$ curves, one has to solve the constraint equation

$$\sum_{i=1}^{k} m_i (m_i - 1) = 2 + n(n - 3),$$ \hfill (5.12)

by taking into account the condition (5.8). For $k = 1$, this relation reduces to $m (m - 1) = 2 + n(n - 3)$, its leading solutions $n = 1$, $m = 0$ and $n = 0$, $m = -1$ give just the classes $H$ and $E$ respectively with degrees $d_H = 3$ and $d_E = 1$. Typical solutions for this constraint equation are given by the generic class $\Sigma_{n,n-1} = nH - (n-1) E$ which is more convenient to rewrite it as follows $\Sigma_{n,n-1} = H + (n-1) (H - E)$. 

40
5.1.2 Link with exceptional Lie algebras

Del Pezzo surfaces $dP_k$ have also a remarkable link with the exceptional Lie algebras. Decomposing the $\mathbb{H}_2$ homology group as,

$$\mathbb{H}_2(dP_k, Z)_{k \geq 3} = \langle \Omega_k \rangle \oplus \mathcal{L}_k,$$

$$\Omega_k = -3H + E_i + \ldots + E_k,$$

$$\mathcal{L}_k = \langle \Omega_k \rangle \perp,$$ (5.13)

the sublattice $\mathcal{L}_k = \langle \alpha_1, \ldots, \alpha_k \rangle$, orthogonal to $\Omega_k$, is identified with the root space of the corresponding Lie algebra $E_k$. The generators $\alpha_i$ of the lattice $\mathcal{L}_k$ are:

$$\begin{align*}
\alpha_1 &= E_1 - E_2, \\
\vdots \\
\alpha_{k-1} &= E_{k-1} - E_k, \\
\alpha_k &= H - E_1 - E_2 - E_3,
\end{align*}$$ (5.14)

with product $\alpha_i \alpha_j$ equal to minus the Cartan matrix $C_{ij}(E_k)$ of the Lie algebra $E_k$. For the particular case of $dP_2$, the corresponding Lie algebra is $su(2)$. The mapping between the exceptional curves and the roots of the exceptional Lie algebras is given in the following table

| $dP_k$ surfaces | exceptional curves | Lie algebras | simple roots |
|-----------------|--------------------|--------------|--------------|
| $dP_1$          | $E_1$              | -            | -            |
| $dP_2$          | $E_1, E_2$         | $su(2)$      | $\alpha_1$  |
| $dP_3$          | $E_1, E_2, E_3$    | $su(3) \times su(2)$ | $\alpha_1, \alpha_2, \alpha_3$ |
| $dP_4$          | $E_1, E_2, E_3, E_4$ | $su(5)$     | $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ |
| $dP_5$          | $E_1, E_2, E_3, E_4, E_5$ | $so(10)$    | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ |
| $dP_6, dP_7, dP_8$ | $E_1, E_2, \ldots, E_k$ | $E_6, E_7, E_8$ | $\alpha_1, \ldots, \alpha_k$, $k = 6, 7, 8$ |

Notice that one can also use eqs(5.13,5.14) to express the generators $H$ and $\langle E_i \rangle_{1 \leq i \leq k}$ in terms of the anticanonical class $\Omega_k$ and the roots of the exceptional Lie algebra. For the case of the del Pezzo $dP_5$, we have the following useful relations

$$\begin{pmatrix}
H \\
E_1 \\
E_2 \\
E_3 \\
E_4 \\
E_5
\end{pmatrix} = \frac{1}{4} 
\begin{pmatrix}
3 & 2 & 4 & 6 & 3 & 5 \\
1 & -2 & 0 & 2 & 1 & 3 \\
1 & 2 & 0 & 2 & 1 & 3 \\
1 & 2 & 4 & 2 & 1 & 3 \\
1 & 2 & 4 & 6 & 1 & 3 \\
1 & 2 & 4 & 6 & 5 & 3
\end{pmatrix}
\begin{pmatrix}
\Omega_5 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5
\end{pmatrix},$$ (5.16)

\footnote{Here $E_3$, $E_4$, and $E_5$ denote respectively $SU(3) \times SU(2)$, $SU(5)$ and $SO(10)$.}
from which we read the following classes of 2- cycles curves:

\[ H = -\frac{1}{4} (3\Omega_5 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4 + 5\alpha_5) , \]

\[ H - E_1 - E_3 = -\frac{1}{4} (\Omega_5 + 2\alpha_1 + 4\alpha_2 + 2\alpha_3 + 2\alpha_4 - \alpha_5) , \]

\[ 2H - E_1 - E_5 = -\Omega_5 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_5 \quad . \] (5.17)

## 5.2 GUT model based on \(dP_8\)

In [32, 33], a semi-realistic supersymmetric F- theory GUT model based on del Pezzo surfaces \(dP_k\) surfaces, \(k \geq 5\), has been constructed. The bulk gauge symmetry in the F- theory GUT model is broken down to \(SU_C(3) \times SU_L(2) \times U_Y(1)\) via an internal hypercharge flux in one to one correspondence with the roots of underlying exceptional Lie algebras (5.15). Following these seminal works, the chiral matter of the MSSM localize on complex curves \(\Sigma_M\) in the base surface \(S\) of the CY4- folds while Yukawa couplings localize at specific points \(P_\gamma\) in \(S\). On the matter curves \(\Sigma_M\), the bulk gauge invariance \(G_r\) gets enhanced to a rank \(r+1\) symmetry \(G_{r+1}\) while at the points \(P_\gamma\) it gets enhanced to a rank \((r+2)\) invariance \(G_{r+2}\). A typical example is given by the figure (7).

![Figure 7](https://via.placeholder.com/150)

**Figure 7:** This figure is taken from ref [33]: It represents the various matter curves and Higgs ones in the \(SU(5)\) GUT model based on del Pezzo surface \(dP_5\). 4D Yukawa couplings live at the intersection of the curves.

In this subsection, we use this example to develop an explicit realisation of seven brane wrapping cycles of the BHV theory for the case of the \(SU(5)\) GUT model based on del Pezzo \(dP_8\).
5.2.1 BHV- SU(5) GUT model versus seven branes

In the SU(5) model, the chiral matter and Higgs superfields as well as their Yukawa couplings localize on different curves in the base of the local Calabi-Yau 4-folds. Matter and Higgs superfields in the 5 and \( \bar{5} \) representations of SU(5) localize on complex curves \( \Sigma_5 \) and \( \Sigma_{\bar{5}} \) where the bulk SU(5) singularity enhances to SU(6) while those in the 10 and \( \overline{10} \) representations localize on curves \( \Sigma_{10} \) and \( \Sigma_{\overline{10}} \) where the bulk SU(5) gets enhanced to SO(10). Yukawa couplings localize at four isolated points

\[ P_1, P_2, P_3, P_4, \]

in the base where the gauge symmetry gets enhanced either to SU(7), or SO(12) or \( E_6 \). To engineer the above typical SU(5) GUT model within the framework of the BHV theory by using intersecting seven branes, we propose the following:

(1) We consider F-theory compactified on the local Calabi-Yau four-folds along the lines of BHV approach,

\[ Y \to X_4 \]
\[ \downarrow \pi_8 \]
\[ dP_8 \]

with Kodaira type degenerating fiber Y \([32, 36]\).

(2) We assume moreover that there are several singularities in the fiber Y with different degeneracy types and different loci in \( dP_8 \). At these loci live stacks of seven branes wrapping del Pezzo surfaces. These seven brane stacks are as follows:

(a) A bulk seven brane wrapping \( dP_4 \subset dP_8 \) where the fiber Y has an SU(5) singularity. We refer to this bulk seven brane like \( (7B_{GUT})_{SU(5)} \equiv (7B_{GUT})_5 \); it is given by the horizontal 7-brane depicted in the figure (8).

(b) Together with this GUT seven brane, we have four more seven branes intersecting the GUT brane along curves as shown on the figure (8).

To engineer these seven branes, we use the fact that \( dP_8 \) may be obtained from the surface \( dP_4 \) by performing up to four more blow ups at generic points in \( dP_4 \). These blow ups generated by the exceptional curves,

\[ E_5, E_7, E_8, E_9 \]

together with the complex curves \( \Sigma_5, \Sigma_{\bar{5}} \) and \( \Sigma_{10} \) and \( \Sigma_{\overline{10}} \) of the figure (8) allow to determine the wrapping properties of the seven branes. We have:
Figure 8: Brane representation of $SU(5)$ GUT model. The horizontal brane is the GUT brane; it intersects four other branes along matter curves describing chiral matter.

(i) a first seven brane wrapping the complex surface blown up of the curve,

$$
E_5 \rightarrow C_a \\
\downarrow \pi_a \\
\Sigma_M^{(1)}
$$

with base $\Sigma_M^{(1)}$ given by the following matter curve\(^{10}\) in $dP_4$,

$$
\Sigma_M^{(1)} = 2H - E_1 - E_4,
$$

and where the fiber $Y$ has a type I$_1$ geometry on $C_a$. On this seven brane lives a Maxwell gauge supermultiplet with $U_a(1)$ gauge invariance. We will refer below to this seven brane as $(7B)_a$; see also the figure [\textit{S}]. The non compact direction of the $(7B)_a$ brane fill the 4D space time while the four compact ones wraps $C_a$.

(ii) a second seven brane $(7B)_b$ wrapping the local 4- cycle

$$
(E_5 - nE_6) \rightarrow C_b \\
\downarrow \pi_b \\
\Sigma_{H_d}
$$

with $n$ being an integer and the base $\Sigma_{H_d}$ same as the BHV SU(5) model,

$$
\Sigma_{H_d} = \langle H - E_1 - E_3 \rangle
$$

\(^{10}\)Notice that in [33], the curve $\Sigma_M^{(1)}$ has been taken as $2H - E_1 - E_5$. By performing the change $E_4 \leftrightarrow E_5$, we get the same result.
and where $Y$ has as well a type I$_1$ geometry.

(iii) a third seven brane $(7B)_c$ with a $U_c(1)$ gauge symmetry wrapping

$$(m_1 E_5 + m_2 E_6 - m_3 E_7) \rightarrow C_c$$

$$\downarrow \pi_c$$

$$\Sigma_{H_a}$$

where $\Sigma_{H_a} = \langle H - E_1 - E_3 \rangle$ and where $m_i$ are integers which can be determined by solving the brane intersection condition.

(iv) a fourth seven brane $(7B)_d$ with a $U_d(1)$ gauge invariance wrapping

$$(k_1 E_5 + k_2 E_6 - k_2 E_7 - k_3 E_8) \rightarrow C_d$$

$$\downarrow \pi_d$$

$$\Sigma_{(2)}^M$$

with $\Sigma_{(2)}^M = H$ and where the $k_i$s are integers.

These branes intersect with the GUT branes along matter curves where the gauge singularity gets enhanced either to $SU(6)$ or $SO(10)$. But, there are also branes intersections at four isolated points $P_\gamma$ in the GUT branes as shown on the figure [8]. At these points, the gauge symmetry gets enhanced to one of the following rank six gauge groups,

$$SU(7) , \quad SO(12) , \quad E_6 ,$$

with the following typical breakings,

$$SU(7) \rightarrow SU(6) \times U_1(1) \rightarrow SU(5) \times U_1(1) \times U_2(1) ,$$

$$SO(12) \rightarrow SO(10) \times U_1'(1) \rightarrow SU(5) \times U_1'(1) \times U_2'(1) ,$$

$$E_6 \rightarrow SO(10) \times U''_1(1) \rightarrow SU(5) \times U''_1(1) \times U''_2(1) .$$

The decomposition of the adjoint representations of these groups namely the 48 of the $SU(7)$ group, the 66 of the $SO(12)$ symmetry and the 78 for $E_6$, give the bi-fundamental matters that localize on the curves $\Sigma_M^{(1)}$, $\Sigma_M^{(2)}$ and $\Sigma_M^{(3)}$ for each group $G_6$. Below, we give some details on the Yukawa tri-couplings that are invariant under these groups.

Yukawa couplings at $SU(7)$ point

The $SU(7)$ point is an isolated singular point in the surface $S$ where three matter curves $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ meet. The geometric engineering of the $SU(7)$ point in $S$ is obtained by starting from a $SU(7)$ singularity in the fiber $Y$ of the Calabi-Yau four-folds and switching on a $U_1(1) \times U_2(1)$ bundle. The $U_1(1) \times U_2(1)$ fluxes give vevs to adjoint matter in the bulk theory

$$\langle \phi \rangle = t_1 H_1 + t_2 H_2$$

(5.28)
with $H_1$ and $H_2$ being two Cartan generators of $SU(7)/SU(5)$ and induces a geometric deformation in the fiber,

$$v^2 = u^2 + z^5(z - t_1)(z - t_2)$$

(5.29)

where $t_1$ and $t_2$ are two complex moduli. This geometric deformation induces as well a deformation in the base surface leading to rotation of the branes.

Notice that for $t_1 = 0$; but $t_2 \neq 0$ and $t_2 = 0$; but $t_1 \neq 0$ the $SU(5)$ singularity (5.29) gets enhanced to $SU(6)$ while for $t_1 = t_2 \neq 0$, it gets enhanced to $SU(5) \times SU(2)$. Notice also that for the particular case $t_1 = t_2 = 0$; that is when the $U_1(1) \times U_2(1)$ fluxes are switched off; these singularities gets further enhanced to the $SU(7)$ singularity $v^2 = u^2 + z^7$.

To get matters at brane intersections, we decompose the adjoint representation 48 of $SU(7)$ in terms of representations of $SU(5) \times U_1(1) \times U_2(1)$ namely

$$48 = 1_{0,0} \oplus 1_{0,0} \oplus 24_{0,0}$$

$$\oplus (5_{0,-6} \oplus \bar{5}_{0,6}) \oplus (5_{-7,-1} \oplus \bar{5}_{7,1}) \oplus (1_{7,-5} \oplus \bar{1}_{-7,5})$$

(5.30)

In addition to the usual uncharged adjoints, we have moreover the following bi-fundamentals:

(a) Four matter fields in the fundamental representations of $SU(5)$:

(i) two matter fields with charges $(0, \mp 6)$; one in the $5_{0,-6}$ and the other in the conjugate representation $\bar{5}_{0,6}$. Matter in these representations localize on the curve $\Sigma_1$ associated to $\pm 6t_2 = 0$.

(ii) two more matter fields with charges $\pm (7, 1)$; one in the $5_{-7,-1}$ representation and the other in the conjugate $\bar{5}_{7,1}$. They localize on the curve $\Sigma_2$ associated to $\pm (7t_1 + t_2) = 0$.

(b) Two $SU(5)$ matter singlets with charges $(7, -5)$ and $(-7, 5)$ localizing on the curve $\Sigma_3$ associated to $\pm (7t_1 - 5t_2) = 0$.

The $SU(5) \times U_1(1) \times U_2(1)$ gauge invariant Yukawa tri-couplings is given by the following fields overlapping:

$$5_{-7,-1} \otimes \bar{5}_{0,6} \otimes 1_{7,-5} \quad ,$$

$$5_{0,-6} \otimes \bar{5}_{7,1} \otimes \bar{1}_{-7,5} \quad .$$

(5.31)

Upon using the following fields identification

$$5_{H_u} = 5_{-7,-1} \quad , \quad 5_M = \bar{5}_{0,6} \quad , \quad 1_X = 1_{7,-5} \quad ,$$

(5.32)

the three fields overlapping engineer the Yukawa coupling term $5_{H_u} \times 5_M \times 1_X$ originating then from points $P_{SU(7)}$ in the base surface $S$ where the $SU(5)$ singularity gets enhanced to a $SU(7)$ singularity.

This analysis extends directly to the $SO(12)$ and $E_6$ gauge symmetries. Let us give some brief details.

\[11\] To get the decomposition (5.30), we have solved the traceless condition of the fundamental representation of $SU(7)$ in terms of $SU(5) \times U^2(1)$ as $7 = 5_{-1,-1} + 1_{6,0} + 1_{-1,5}$
**Yukawa couplings at SO(12) point**

First, recall that the SO(12) singularity \( v^2 = u^2 z + \alpha z^5 \) may be broken down to SU(5) by using two non-zero vevs \( t'_1 \) and \( t'_2 \) like,

\[
\langle \phi' \rangle = t'_1 H'_1 + t'_2 H'_2
\]  

(5.33)

with \( t'_1 \) and \( t'_2 \) captured by two local Cartan \( H'_1 \) and \( H'_2 \) generators of \( so(12) \) Lie algebra. Under a one parameter deformation by \( \langle \phi' \rangle = t'_1 H'_1 \) \( (t'_2 = 0) \), we can either break the \( SO(12) \) singularity down to \( SO(10) \) or down to \( SU(5) \times SU(2) \). By switching on the second deformation \( (t'_2 \neq 0) \), we can break further the above singularity down to \( SU(5) \) described by the following relation

\[
v^2 = (u - t'_1)(u - t'_2) z + \alpha z^5.
\]  

(5.34)

Under the \( SO(12) \) gauge symmetry breaking down to \( SO(10) \times U'(1) \), the adjoint representation \( 66 \) decomposes as \( 12 \) as \( 1_0 + 45_0 + 10_2 + 10_{-2} \) and by switching on the second flux, the \( SO(10) \times U'(1) \) representation break further down to representations of \( SU(5) \times U'_1(1) \times U'_2(1) \) as given below,

\[
66 = 1_{0,0} + 1_{0,0} + 24_{0,0}
\]

\[
(5_{2,2} + \bar{5}_{-2,-2}) + (5_{-2,2} + \bar{5}_{2,-2}) + 10_{0,4} + 10_{0,-4}.
\]  

(5.35)

This decomposition involves two kinds of bi-fundamental matters. (a) Matter in \( 5_{2,2} \) and \( 5_{-2,2} \) representations which localize on the curves in the \( 2 (t'_1 \pm t'_2) = 0 \) and (b) matter in the \( 10_{4,0} \) localizes on \( \pm 4t'_1 = 0 \).

The \( SU(5) \times U'_1(1) \times U'_2(1) \) gauge invariant Yukawa couplings one can write down by the combination of three matter fields is as follows:

\[
\bar{5}_{+2,-2} \otimes 5_{-2,-2} \otimes 10_{0,+4},
\]

\[
\bar{5}_{-2,+2} \otimes 5_{+2,+2} \otimes 10_{0,-4}.
\]  

(5.36)

Upon using the following fields identification

\[
\bar{5}_{H_d} = 5_{2,-2}, \quad \bar{5}_{M} = 5_{-2,-2}, \quad 10_{M} = 10_{0,+4},
\]

the three fields overlapping engineer the Yukawa coupling term \( \bar{5}_{H_d} \times 5_{M} \times 10_{M} \) originating then from points \( P_{SO(12)} \) in the base surface \( S \) where the \( SU(5) \) singularity gets enhanced to a \( SO(12) \) singularity.

---

\(^{12}\)To get the decomposition of the adjoint of \( SO(12) \) in terms of representations of \( SO(10) \times U(1) \), we have used the splitting \( 12 = 10_0 + 1_2 + \bar{T}_{-2}. \) To get the decomposition in terms of \( SU(5) \times U^2(1) \) representations, we have used as well the splitting \( 12 = 5_{0,2} + \bar{5}_{0,-2} + 1_{2,0} + \bar{T}_{-2,0}. \)
**Yukawa couplings at \(E_6\) point**

In the same manner, under the breaking of the \(E_6\) gauge symmetry down to \(SO(10) \times U(1)\), the adjoint representation \(78\) decomposes as \(1_0 + 45_0 + 16_{-3} + \overline{10}_3\) and by a further breaking down to \(SU(5) \times U'_1(1) \times U''_2(1)\) we get:

\[
78 = 1_{0,0} + 1_{0,0} + 24_{0,0} + 1_{5,3} + 1_{-5,-3} + 5_{-3,3} + 5_{3,-3} + 10_{-1,-3} + \overline{10}_{1,3} + 10_{4,0} + \overline{10}_{-4,0},
\]

(5.37)

where matter in the \(5_{3,-3}\) and \(\overline{5}_{3,-3}\) localizes on the curve \((t''_2 - t''_1) = 0\) and matter in the \(10_{-1,-3}\) and \(10_{4,0}\) as well as their conjugates \(\overline{10}_{1,3}\) and \(\overline{10}_{-4,0}\) localize on the curves \((t'_1 + 3t''_2) = 0\) and \(t'_2 = 0\).

The \(SU(5) \times U'_1(1) \times U''_2(1)\) gauge invariant Yukawa couplings at the \(E_6\) point is given by the following three matter fields interactions:

\[
\begin{align*}
5_{-3,3} \otimes 10_{-1,-3} & \otimes 10_{4,0}, \\
\overline{5}_{3,-3} \otimes \overline{10}_{1,3} & \otimes \overline{10}_{-4,0}.
\end{align*}
\]

(5.38)

By using the fields identification

\[
5_{H_u} = 5_{-3,3}, \quad 10_M = 10_{-1,-3}, \quad 10_M = 10_{4,0},
\]

the three overlapping (5.38) engineer the Yukawa coupling term \(5_{H_u} \times 5_M \times 10_M\) originating then from points in the base surface \(S\) where the \(SU(5)\) singularity gets enhanced to a \(E_6\).

We end this study by giving more explicit expressions of the complex curves on which matter localize. Following [32, 33] and using fractional bundle idea, the configuration of the matter curves that engineer a quasi-realistic F-theory \(SU(5)\) GUT model based on \(dP_8\) are as follows:

(i) the Higgs up \(5_{H_u}\) and the Higgs down \(\overline{5}_{H_d}\) are placed on two distinct matter curves \(\Sigma^{(u)}_{H}\) and \(\Sigma^{(d)}_{H}\) which intersect at a point in \(dP_8\).

(ii) the three generations of the fields in the \(10_M\) are placed on one self-intersecting \(\mathbb{P}^1\)

(iii) the three generations of the fields in the \(\overline{5}_M\) are placed on one smooth \(\mathbb{P}^1\) which does not self-intersect.

The matter content of this supersymmetric \(SU(5)\) model and the corresponding fractional bundle assignments are collected in the following table [33], see also footnote 9:

[33] In [33], this matter field configuration in terms of curves in \(dP_8\) was named Model II.
where $g_\Sigma$ stands for the genus of the matter curves. The geometrical figure representing the various matter curves in this $SU(5)$ model are depicted in figure (7).

The $N = 1$ chiral superpotential $W_{SU(5)}$ capturing the intersections of the various matter and Higgs curves is given by

$$W_{SU(5)} = \sum_{i,j} \lambda_{ij}^{(d)} 5_H \otimes \bar{5}_M^{(i)} \otimes 10_M^{(j)} + \sum_{i,j} \lambda_{ij}^{(u)} 5_H \otimes \bar{5}_M^{(i)} \otimes N_R^{(a)} + \lambda_{ud}^{(\phi)} \Phi \otimes 5_H \otimes \bar{5}_H,$$

where the moduli $\lambda_{2y}^{(z)}$ stand for Yukawa coupling constants. Notice that the interaction term $5_H \otimes \bar{5}_M^{(i)} \otimes N_R^{(a)}$ leads to a two-fold enhancement in rank to an $SU(7)$ singularity so that the singlet $N_R^{(a)}$ may be identified with the right-handed neutrinos. The interaction term $\Phi \otimes 5_H \otimes \bar{5}_H$ with vev $\langle \Phi \rangle$ determines the supersymmetric $\mu$- term [33, 35].

6 Quiver GUT models on tetrahedron

In this section, we set up the basis for constructing a class of quiver GUT like models embedded in F- theory on CY4- folds by using the toric geometry of the complex tetrahedral base surface. The key idea behind this construction stems from thinking about the abelian gauge factors appearing in eqs(5.27) as given by the toric symmetry

$$U(1) \times U(1)$$

of the complex tetrahedral base surface $T$. Denoting by $(s_1, s_2)$ the local holomorphic coordinates of $T$, this toric symmetry is given by,

$$s_1 \rightarrow e^{i\theta_1} s_1,$$

$$s_2 \rightarrow e^{i\theta_2} s_2,$$

with $\theta_1$ and $\theta_2$ being the gauge parameters. This abelian group action has degeneracy loci on the edges $\Sigma_{ab}$ and at the vertices $P_{abc}$ of the tetrahedron (1).

In this section is organized into three parts, we study in the two first ones the geometry of local Calabi-Yau four-folds based on $T$ as a matter to get a more insight of such a particular geometry. In the third subsection, we construct three quiver $SU(5)$ GUT-like models embedded in F- theory on the tetrahedron based CY4s. GUT-like models building using blown ups of tetrahedron will be considered in the next section.

| Model II | curve | class | $g_\Sigma$ | $\mathcal{L}_\Sigma$ | $\mathcal{L}^a_\Sigma$ |
|----------|-------|-------|-----------|-------------------|-----------------|
| $1 \times 5_H$ | $\Sigma_H^{(a)}$ | $H - E_1 - E_3$ | 0 | $\mathcal{L}^{1/5}_{\Sigma_H^{(a)}}$ (1) | $\mathcal{L}^{2/5}_{\Sigma_H^{(a)}}$ (1) |
| $1 \times 5_H$ | $\Sigma_H^{(d)}$ | $H - E_1 - E_3$ | 0 | $\mathcal{L}^{1/5}_{\Sigma_H^{(d)}}$ (1) | $\mathcal{L}^{2/5}_{\Sigma_H^{(d)}}$ (1) |
| $3 \times 10_M$ | $\Sigma_M^{(1)}$ | $2H - E_1 - E_4$ | 0 | $\mathcal{L}_{\Sigma_M^{(1)}}$ | $\mathcal{L}_{\Sigma_M^{(1)}}$ (3) |
| $3 \times 5_M$ | $\Sigma_M^{(2)}$ | $H$ | 0 | $\mathcal{L}_{\Sigma_M^{(2)}}$ | $\mathcal{L}_{\Sigma_M^{(2)}}$ (3) |
6.1 4-cycles in CY4- folds

We begin by recalling that real 4- cycles in Calabi-Yau 4- folds play an important role in the engineering of F-theory GUT models. The seven brane living at the elliptic singularity of the Calabi-Yau four folds has four non compact directions filling the $4D$ Minkowski space time and four compact directions that wrap compact real 4-cycles in the base of $X_4$. Generally speaking, the Calabi-Yau 4- folds has an elliptic curve $E$ fibered on a complex three dimension base $B_3$,

$$
\begin{align*}
\mathbb{E} & \rightarrow X_4 \\
\downarrow & \pi_B \\
B_3 
\end{align*}
$$

but it is locally handled as a ADE geometry fibered on a complex surface $S$. Indeed, defining the elliptic fiber $\mathbb{E}$ by a cubic in the complex plane with coordinates as usual like $v^2 = du^3 + e u^2 + f u + g$ , an explicit expression of $X_4$ is obtained by fibering the cubic on the base $B_3$; i.e,

$$
v^2 = D(w_1, w_2, w_3)u^3 + E(w_1, w_2, w_3)u^2 + F(w_1, w_2, w_3)u + G(w_1, w_2, w_3) .
$$

The complex variables $(w_1, w_2, w_3)$ are local holomorphic coordinates parameterizing the complex three dimension base $B_3$ while $D(w), E(w), F(w)$ and $H(w)$ are tri- holomorphic functions whose explicit expressions depend on the type of the ADE singularity living in the CY4- folds.

6.1.1 Factorization

By breaking the $U(3)$ group structure of the tangent bundle of the complex three dimension base $TB_3$ down to the subgroup $U(2) \times U(1)$, we can locally split $TB_3$ like,

$$
TB_3 \rightarrow TS \oplus (TS)^\perp ,
$$

where $TS$ the tangent bundle of $S$ with group structure $U(2)$ and where $(TS)^\perp$ is the normal codimension one bundle in $TB_3$. Under this decomposition, the fibration (6.3) can be reduced down to the simple form

$$
\frac{\tau^2}{2} = d(z)u^3 + e(z)u^2 + f(z)u + g(z) ,
$$

---

$^{14}$In the Weierstrass form of the elliptic curve, we have $d = 1$ and $e = 0$. 

---
where \( \vartheta = \vartheta (s_1, s_2) \neq 0 \) is a holomorphic function on the complex surface \( S \). In this case, the local CY4- folds is thought of as

\[
Y \longrightarrow X_4 \\
\downarrow \pi_s \\
S
\]

where \( Y \) is an elliptic local K3 surface with a given ADE geometry; i.e \( Y \sim E \times \Gamma \) with \( \Gamma \) being a projective line \( P^1 \) or a collection of intersecting \( P^1 \)'s. Comparing Eq. (6.4) to its equivalent form (6.6), we get the following relations,

\[
D (w_1, w_2, w_3) = \vartheta (s_1, s_2) \times d (z) , \\
E (w_1, w_2, w_3) = \vartheta (s_1, s_2) \times e (z), \\
F (w_1, w_2, w_3) = \vartheta (s_1, s_2) \times f (z) , \\
G (w_1, w_2, w_3) = \vartheta (s_1, s_2) \times g (z) ,
\]

where the holomorphic functions \( D (w) \), \( E (w) \), \( F (w) \) and \( G (w) \) get factorized in terms of products of the holomorphic functions \( \vartheta (s) \) on the complex surface \( S \) and the holomorphic functions \( d (z) \), \( e (z) \), \( f (z) \) and \( g (z) \) on the normal line to the surface \( S \) in the complex base \( B_3 \).

In the case where the complex surface \( S \) in the local CY4- folds has several irreducible compact components \( S_a \) like,

\[
C_4 = \bigcup_{a=1}^M S_a, \tag{6.9}
\]

the factorizations (6.8) apply to each component \( S_a \).

Notice that the irreducible 4- cycle components \( S_a \), describe as well compact complex surfaces in the Calabi Yau 4- folds that are locally parameterized by the complex coordinates \( (s_{ma})_{1 \leq a \leq n} \), i.e:

\[
S_1 = \{ s_{11}, s_{21} \} , \\
S_2 = \{ s_{12}, s_{22} \} , \\
: = : , \\
S_M = \{ s_{1M}, s_{1M} \} . \tag{6.10}
\]

Extending the factorizations (6.8) to each component \( S_a \), we can write,

\[
D = \vartheta (s_{1a}, s_{2a}) \times d (z_a) , \\
E = \vartheta (s_{1a}, s_{2a}) \times e (z_a) , \\
F = \vartheta (s_{1a}, s_{2a}) \times f_i (z_a) , \\
G = \vartheta (s_{1a}, s_{2a}) \times h (z_a) , \tag{6.11}
\]

\(^{15}\)In the case where the base surface has several irreducible 4- cycles \( S_a \), one has to specify the intersections \( S_a \cap S_b \) as well as the fibration of the ADE geometry; see below.
with \( z_a \) parameterizing the normal direction to \( S_a \) in \( B_3 \). Notice in passing that the geometry of the \( B_3 \) base of the Calabi-Yau 4-folds is a little bit complicated. Because of cycles intersections, the splitting (6.5) is not trivial.

Focusing on the 4-cycles in the complex surface (6.9), the irreducible compact components \( S_a \) have intersections captured by the following typical relations,

\[
S_a \cap S_b = \bigcup_{\alpha=1}^{M'} I_{\alpha}^{ab} \Sigma_{\alpha} , \quad I_{ab}^a = I_{ba}^a ,
\]

\[
\Sigma_\alpha \cap \Sigma_\beta = \bigcup_{A=1}^{M''} J_{A}^{\alpha \beta} P_A , \quad J_{\alpha \beta}^{\alpha} = J_{\beta \alpha}^{\alpha} ,
\]

where \( \Sigma_\alpha \) and \( P_A \) stand respectively for 2- and 0-cycles in the local Calabi Yau 4-folds. The intersection numbers \( I_{\alpha}^{ab} \) and \( J_{A}^{\alpha \beta} \) fix also the manner in which the \( S_a \)'s are glued together. Moreover, the complex coordinates \((s_{1a}, s_{2a})\) and \((s_{1b}, s_{2b})\) of any two intersecting cycles \( S_a \) and \( S_b \) are obviously related by holomorphic transition functions as usual.

### 6.1.2 Toric surfaces and blown ups

So far, we have been describing general geometric features of the base surface of the local Calabi Yau 4-folds. A particular class of these surfaces have been considered in the previous section; these are the del Pezzo surfaces \( dP_n \) with their remarkable links with:

1. the projective plane and its blown ups,
2. the finite dimensional exceptional Lie algebras \( E_n \).

Here, we want to contribute to this direction by studying a particular class of complex surfaces that may play the role of the base \( S \) of the local Calabi Yau 4-folds. This class of complex surfaces share basic features of the projective plane

\[
P^2 = dP_0
\]

and the del Pezzos \( dP_n \), but has also the property to allow more possibilities. We will distinguish two kinds of surfaces:

- **a) complex tetrahedral surface** \( T \) and its toric blown ups \( T_n^{toric} \),
- **b) non toric blown ups** \( T_n^{non toric} \) of the tetrahedral surface \( T \).

Below, we focus our attention on CY4-folds based on the complex tetrahedral surface and its toric blow ups \( T_n^{toric} \).

**Toric surfaces**
Toric surfaces $S$, which can be thought of as the fibration,
\[
\mathbb{T}^2 \rightarrow S \\
\downarrow \pi_S \\
B_S
\]
with real two dimensional base $B_S$ and fiber $\mathbb{T}^2$, form a particular generalization of the projective plane $dP_0$. These surfaces have special features that are nicely engineered by using toric geometry property encoded in a toric graph $\Delta_S$. The simplest toric surface is obviously given by the compact $dP_0$; its toric graph is
\[
\Delta_{dP_0} = \text{triangle } [ABC].
\]
Recall that $dP_0$ is defined as the projective plane in the non compact complex three dimension space $\mathbb{C}^3$ like,
\[
dP_0 = \left\{ (x_1, x_2, x_3) \equiv (\lambda x_1, \lambda x_2, \lambda x_3) \right\} \quad (x_1, x_2, x_3) \neq (0, 0, 0)
\]
with $\lambda$ a non zero complex constant. This compact surface has also a nice supersymmetric linear sigma model representation given by
\[
|x_1|^2 + |x_2|^2 + |x_3|^2 = r
\]
with the gauge identification $x_i \equiv e^{i\theta} x_i$ and where $r$ is the Kahler parameter. Other complex surfaces directly related to $dP_0$ are given by the toric blown ups $dP_1$, $dP_2$ and $dP_3$ whose toric graphs $\Delta_{dP_1}$, $\Delta_{dP_2}$ and $\Delta_{dP_3}$ are depicted in the figure (9).

![Toric graphs for $dP_0$, $dP_1$ and $dP_2$.](image)

Complex two dimension toric surfaces may be also engineered by using embedding in complex higher dimensional projective spaces $\mathbb{P}^n$ with $n \geq 3$ thought of as given by the fibration
\[
\mathbb{T}^n \rightarrow \mathbb{P}^n \\
\downarrow \pi_{\mathbb{P}^n} \\
\Delta_{\mathbb{P}^n}
\]
An interesting class of toric surfaces that we are interested in here is given by the complex tetrahedral surface $T$ and its toric blown ups $T_{n}^{toric}$. Let us consider first the non planar toric surface $T$ with fibration

$$\mathbb{T}^2 \rightarrow T \downarrow \pi_{\mathcal{T}} \Delta_{\mathcal{T}} (6.18)$$

A nice way to define complex tetrahedral surface $T$ is in terms of divisors of the complex three dimensions projective space $\mathbb{P}^3$,

$$\left\{ (x_1, x_2, x_3, x_4) \equiv (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4) \right\}$$

$$(x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0) \quad (6.19)$$

with $\lambda \in \mathbb{C}^*$. Being a toric three- folds, the complex space $\mathbb{P}^3$ may be also defined in terms of the supersymmetric linear sigma model D- equation like,

$$\mathbb{P}^3 : \sum_{i=1}^{4} |x_i|^2 = R \quad , \quad x_i \equiv e^{i\theta} x_i \quad , \quad (6.20)$$

where $R$ is the Kahler parameter of $\mathbb{P}^3$. Irreducible divisors $S_a$ in the space $\mathbb{P}^3$ are complex surfaces generated by the equation $x_a = 0$. There are four such divisors in $\mathbb{P}^3$ which form altogether the complex tetrahedron depicted in figure (10).

Figure 10: A toric complex surface given by the union of four intersecting projective planes forming a toric tetrahedron. Each face of the tetrahedron corresponds to an irreducible divisor $S_i$. For instance $S_4$ corresponds to the toric triangle $[P_1 P_2 P_3]$ corresponding as well to a toric representation of the projective plane.

The complex tetrahedral surface $T$ has some particular features which we describe below.

(a) \textit{Link between $T$ and $\mathbb{P}^2$}

The complex tetrahedral surface $T$ has the tetrahedron $\Delta_T$ as a toric graph; it is then a natural extension of the projective plane $\mathbb{P}^2 = dP_0$ whose toric graph is a triangle (6.14).
Since the tetrahedron $\Delta_T$ has four intersecting triangle faces; the non planar surface $T$ involves then four intersecting projective planes

$$\mathbb{P}^2_a = dP^{(a)}_0, \quad a = 1, ..., 4.$$  

(6.21)

Using the link between the projective plane and the del Pezzo surfaces, we may refer to the complex tetrahedral surface $T$ as follows

$$T_0 = dP_{k_1,k_2,k_3,k_4}, \quad (k_1, k_2, k_3, k_4) = (0, 0, 0, 0),$$  

(6.22)

where the four integers $(k_1, k_2, k_3, k_4)$ refer to the number of blow ups of the faces of the non surface $T$. Notice that these blow ups form in fact just a particular family of a larger one. A way to see this feature is to focus on toric singularities where tetrahedron involves both at its edges and its vertices; for useful details see below but for an explicit study regarding these blow ups see [40].

(b) Tetrahedron and gauge enhancements

As a toric surface, the tetrahedron $T \sim \Delta_T \times \mathbb{T}^2$ has a natural $U^2(1)$ symmetry on $\mathbb{T}^2$ with fix points on the following loci:

(i) the six edges of the toric surface $T$ where a 1-cycle of $\mathbb{T}^2$ shrinks to zero,

(ii) its four vertices where 2- cycles shrink to zero.

The $U(1) \times U(1)$ toric gauge symmetry of the fiber of the toric surface $T$ may be:

- interpreted in terms of two wrapped seven branes $(7B)_1$ and $(7B)'_1$,

- used to engineer the enhancement of gauge symmetry along the edges and at the vertices of the tetrahedral surface.

(c) Blown ups of the tetrahedron

Mimicking the relation between the projective plane $dP_0$ and the del Pezzo surfaces $dP_k$, and using the relation between the complex tetrahedral surface $T$ and the complex projective plane, we can perform blow ups of the toric surface $T$. Generally, we distinguish two kinds of blow ups: toric blow ups and non toric ones [40]. Regarding the toric blow ups, one has to distinguish as well two classes of blow ups:

(i) blow ups by projective lines of the edges $\Sigma_{ab}$ of the tetrahedron,

(ii) blow ups of the vertices $P_{abc}$ by projective planes.

Regarding the edges, the bow up at each point on a edge $\Sigma$ is done in terms of projective line $\mathbb{P}^1$. As such the blow up of the full edge $\Sigma \sim \mathbb{P}^1$ is given by a del Pezzo surface $dP1$:

$$\Sigma \rightarrow dP_1 \sim \mathbb{P}^1 \times \mathbb{P}^1.$$  

(6.23)

Concerning, the blow up of each vertex of the tetrahedron, it is done by a projective plane $\mathbb{P}^2$; for illustration see figure [14].
To avoid technicalities, it is enough to notice that for each plane $\mathbb{P}_a^2$ associated with a given face of the complex tetrahedral surface $T$, one may perform up to eight blow ups as given below,

$$\mathbb{P}_a^2 = dP^{(a)}_0 \rightarrow dP^{(a)}_{k_a}, \quad k_a = 1, \ldots, 8, \quad a = 1, 2, 3, 4. \quad (6.24)$$

In doing so, we a priori get the following blown up surfaces of the tetrahedron,

$$T_n = dP_{k_1,k_2,k_3,k_4}, \quad n = k_1 + k_2 + k_3 + k_4, \quad k_a = 1, \ldots, 8. \quad (6.25)$$

Clearly, these complex surfaces give generalizations of the del Pezzo ones which are recovered by setting three of these integers to zero to get $dP_{k_1,0,0,0}$ by taking $k_1 = k_2 = k_3 = 0$. Explicit examples will be given in section 7.

### 6.2 More on tetrahedron geometry

We first describe subspaces in the complex tetrahedral geometry where the bulk gauge invariance in the GUT seven brane undergoes transitions. Then we build explicitly the local Calabi Yau 4-folds based on the tetrahedral surface $T$.

#### 6.2.1 subspaces in tetrahedron

In the complex tetrahedral geometry, the 4-cycle $C_4$ is given by the union of four intersecting components $S_1$, $S_2$, $S_3$ and $S_4$

$$C_4 = S_1 \bigcup S_2 \bigcup S_3 \bigcup S_4, \quad (6.26)$$

where the compact toric surfaces $(S_n)_{1 \leq n \leq 4}$ are four intersecting complex projective surfaces $\mathbb{P}_{a}^2$ belonging to four different planes of the complex three dimension space $\mathbb{P}^3$. We have the relations:

$$S_1 \cap S_2 = \Sigma_{(12)}, \quad S_2 \cap S_3 = \Sigma_{(23)}, \quad S_1 \cap S_3 = \Sigma_{(13)}, \quad S_2 \cap S_4 = \Sigma_{(24)}, \quad S_1 \cap S_4 = \Sigma_{(14)}, \quad S_3 \cap S_4 = \Sigma_{(34)}. \quad (6.27)$$

Moreover, since the complex tetrahedral surface is toric, all the edges $\Sigma_{(ab)}$ are precisely given by projective lines $\mathbb{P}_1$. Furthermore, seen that the tetrahedron is compact, these projective lines $\Sigma_{(ab)}$ intersect mutually at four points $P_A$ in the base of the local Calabi-Yau 4-folds,

$$\Sigma_{(23)} \cap \Sigma_{(24)} = \Sigma_{(23)} \cap \Sigma_{(34)} = \Sigma_{(24)} \cap \Sigma_{(34)} = P_1, \quad (6.28)$$

$$\Sigma_{(14)} \cap \Sigma_{(34)} = \Sigma_{(13)} \cap \Sigma_{(34)} = \Sigma_{(13)} \cap \Sigma_{(14)} = P_2, \quad \Sigma_{(12)} \cap \Sigma_{(24)} = \Sigma_{(14)} \cap \Sigma_{(24)} = \Sigma_{(12)} \cap \Sigma_{(14)} = P_3,$$

$$\Sigma_{(12)} \cap \Sigma_{(23)} = \Sigma_{(13)} \cap \Sigma_{(23)} = \Sigma_{(12)} \cap \Sigma_{(13)} = P_4.$$
Using eq. (6.27), these intersecting points may be also viewed as the intersection of three faces as shown below

\[
S_2 \cap S_3 \cap S_4 = P_1, \\
S_1 \cap S_3 \cap S_4 = P_2, \\
S_1 \cap S_2 \cap S_4 = P_3, \\
S_1 \cap S_2 \cap S_4 = P_4.
\]  

(6.29)

Tetrahedron geometry has other remarkable properties; in particular each face \( S_a \) of the tetrahedron has a toric fibration

\[
\mathbb{T}_a^2 \to S_a \\
\downarrow \pi_{S_a} \\
\Delta_a
\]  

(6.30)

with real two dimension base \( \Delta_S \) represented by a triangular toric graph and a fiber \( \mathbb{T}_a^2 = S_a^1 \times S_a^1 \), \( a = 1, 2, 3, 4 \), where the \( S_a^1 \)'s are associated with the \( U_a (1) \) toric actions. Notice that these toric fibers \( \mathbb{T}_a^2 \) are not the same for all the faces \( S_a \); they change from a \( S_a \) to an other \( S_b \); but intersect along a 1- cycle \( S_{ab}^1 \). Thus, given two faces \( S_a \) and \( S_b \) with intersecting curve,

\[
\Sigma_{(ab)} = S_a \cap S_b,
\]  

(6.31)

we have the following,

| subspace | 2d- base | 2d-fiber | toric action |
|----------|----------|----------|--------------|
| \( S_a \) | \( \Delta_a \) | \( S_a^1 \times S_{ab}^1 \) | \( U_a (1) \times U_{ab} (1) \) |
| \( S_b \) | \( \Delta_b \) | \( S_b^1 \times S_{ab}^1 \) | \( U_b (1) \times U_{ab} (1) \) |
| \( \Sigma_{(ab)} \) | \( \Delta_{(ab)} \) | \( S_{ab}^1 \) | \( U_{ab} (1) \) |

(6.32)

The toric fibers \( S_a^1 \times S_{ab}^1 \) degenerate \textit{once} on the projective edges \( \Sigma_{(ab)} \) and degenerate \textit{twice} at the four vertices \( P_A \). Notice that the 1-cycles \( S_a^1 \) and \( S_b^1 \) shrink to zero on \( \Sigma_{(ab)} \)

\[
S_{ab}^1 \to \Sigma_{ab} \\
\downarrow \pi_{\Sigma} \\
\Delta_{\Sigma}
\]  

(6.33)

Furthermore, the cycle \( S_{ab}^1 \) shrinks down to zero at the meeting point of the two curves \( \Sigma_{(ab)} \) and \( \Sigma_{(ac)} \).

With these tools at hand, we turn now to build the explicit expression of the algebraic equation of the local elliptic Calabi-Yau 4- folds based on the tetrahedron.
6.2.2 Local tetrahedron

In toric language, one may directly read the intersections in the base of the elliptically K3 fibered Calabi-Yau four-folds,

$$ Y \longrightarrow X_4 $$

$$ \downarrow \pi_T $$

$$ T $$

(6.34)

Using the irreducible $S_a$s, the complex tetrahedral surface may be defined as

$$ T = S_1 \cup S_2 \cup S_3 \cup S_4, $$

(6.35)

with the following intersections,

$$ S_a \cap S_b = \Sigma_{(ab)} \quad , \quad a < b = 1, ..., 4 $$

$$ S_a \cap S_b \cap S_c = P_{(abc)} \quad , \quad a < b < c $$

(6.36)

Being a toric surface, the toric fibration of the tetrahedral surface $T \sim \Delta_T \times \mathbb{T}^2_T$ is not homogeneous; it decomposes in terms of the toric fibrations,

$$ T \sim \bigcup_a (\Delta_{S_a} \times \mathbb{T}^2_T) $$

(6.37)

with toric graph given by the figure [10]. From this toric graph, one can directly read the toric data of each component $S_a$ and then those of $T$.

In the toric graph picture of the complex base, the local Calabi-Yau four-folds $X_4$ may be thought of as fibering on each point of $\Delta_T$ a complex three dimension fiber $Z$ given by the 2-torus $\mathbb{T}^2_T$ times the complex two dimension fiber $Y$. Roughly, we have

$$ X_4 \sim T \times Y \sim \Delta_T \times Z $$

(6.38)

with $Z \sim \mathbb{T}^2_T \times Y$.

Below, we construct the explicit expression for $X_4$ as a complex 4 dimension hypersurface in the complex space $\mathbb{C}^5$. First, we give the algebraic equation of the complex base tetrahedron $T$. Then, we study the fiber singularity on the edges $\Sigma_{(ab)}$ and at the vertices $P_{(abc)}$ of the tetrahedron.

**Base surface $T$**

Since the four irreducible components $S_a$ of the complex tetrahedron are given by different projective planes in $\mathbb{C}^4$, we start by introducing the projective coordinates of the complex three dimension projective space $\mathbb{P}^3$,

$$ (x_1, x_2, x_3, x_4) \equiv (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4), $$

(6.39)
with projective parameter \( \lambda \in \mathbb{C}^* \) and \((x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0, 0)\). The tetrahedron surface is engineered by thinking about the compact surfaces \( S_a \) as the planar divisors of \( \mathbb{P}^3 \),

\[
x_a = |x_a| e^{i\phi_a} = 0 \quad , \quad a = 1, 2, 3, 4.
\]

(6.40)

As noticed earlier, this representation has an equivalent description in the supersymmetric linear sigma model set up of toric manifolds. There, the divisors \( S_a \) are given by the standard D-term equations

\[
S_a : \left( \sum_{b=1}^{4} |x_b|^2 \right)_{x_a=0} = R \quad , \quad x_b \equiv e^{i\theta} x_b \quad ,
\]

(6.41)

where \( R \) is the Kahler parameter of \( \mathbb{P}^3 \) and \( e^{i\theta} \) is the \( U(1) \) compact part of the gauge transformation (6.39).

In the mirror complex holomorphic description, it is not difficult to see that the complex algebraic equation describing the base manifold \( T \) of the local Calabi-Yau 4-folds is given by the following complex two dimension surface in the projective space \( \mathbb{P}^3 \),

\[
\mu \left( \prod_{a=1}^{4} x_a \right) = \mu x_1 x_2 x_3 x_4 = 0.
\]

(6.42)

In this relation, \( \mu \) is a complex number and the divisors \( S_a \) are precisely given by the solutions of this relation. This equation may be viewed as well as the large complex structure limit \((\mu \to \infty)\) of the quartic

\[
\sum_{i=1}^{4} A_i x_i^4 + \sum_{i=1}^{4} \left[ x_i^3 \left( \sum_{j \neq i} B_{ij} x_j \right) \right] + \sum_{i=1}^{4} \left[ x_i^2 \left( \sum_{j \neq l \neq m \neq i} C_{ijkl} x_j x_l x_m \right) \right] + \mu x_1 x_2 x_3 x_4 = 0
\]

(6.43)

where the \( A_i \)'s, \( B_{ij} \)'s, \( C_{ijk} \)'s and \( D_{ijlm} \)'s are complex structures. In mirror geometry, the divisors \( S_a \) are explicitly given by,

\[
S_1 : \{ (x_2, x_3, x_4) \equiv (\lambda x_2, \lambda x_3, \lambda x_4) \} \equiv \mathbb{P}^2_1 \quad ,
\]

\[
S_2 : \{ (x_1, x_3, x_4) \equiv (\lambda x_1, \lambda x_3, \lambda x_4) \} \equiv \mathbb{P}^2_2 \quad ,
\]

\[
S_3 : \{ (x_1, x_2, x_4) \equiv (\lambda x_1, \lambda x_2, \lambda x_4) \} \equiv \mathbb{P}^2_3 \quad ,
\]

\[
S_4 : \{ (x_1, x_2, x_3) \equiv (\lambda x_1, \lambda x_2, \lambda x_3) \} \equiv \mathbb{P}^2_4 \quad ,
\]

(6.44)

with the \( \mathbb{C}^* \) action generated by the complex parameter \( \lambda \) inherited from the projective action of the \( \mathbb{P}^3 \) space. Similarly, the intersections \( S_a \cap S_b = \Sigma_{(ab)} \) can be determined...
explicitly from above relations. These are given by the following projective lines in \( \mathbb{P}^3 \),

\[
\begin{align*}
\Sigma_{(12)} &= \{(x_3, x_4) \equiv (\lambda x_3, \lambda x_4)\} \equiv \mathbb{P}_1^1, \\
\Sigma_{(13)} &= \{(x_2, x_4) \equiv (\lambda x_2, \lambda x_4)\} \equiv \mathbb{P}_1^2, \\
\Sigma_{(14)} &= \{(x_2, x_3) \equiv (\lambda x_2, \lambda x_3)\} \equiv \mathbb{P}_1^3, \\
\Sigma_{(23)} &= \{(x_1, x_4) \equiv (\lambda x_1, \lambda x_4)\} \equiv \mathbb{P}_4^1, \\
\Sigma_{(24)} &= \{(x_1, x_3) \equiv (\lambda x_1, \lambda x_3)\} \equiv \mathbb{P}_5^1, \\
\Sigma_{(34)} &= \{(x_1, x_2) \equiv (\lambda x_1, \lambda x_2)\} \equiv \mathbb{P}_6^1.
\end{align*}
\]

Their supersymmetric linear sigma model description may be directly deduced from eqs (6.41) by putting to zero two of the four variables.

Finally, the intersections of these curves are given by points in \( \mathcal{T} \). Up on making an appropriate choice of the \( \mathbb{C}^* \) action, these points may be taken as,

\[
\begin{align*}
P_1 &= (1, 0, 0, 0), \\
P_2 &= (0, 1, 0, 0), \\
P_3 &= (0, 0, 1, 0), \\
P_4 &= (0, 0, 0, 1).
\end{align*}
\]

Using the above analysis, we can now write down the explicit algebraic relation defining the local Calabi-Yau four-fold based on the tetrahedron \( \mathcal{T} \). In the large complex structure limit \( \mu \to \infty \), the local elliptic Calabi-Yau four-folds may defined by the following algebraic relations,

\[
v^2 = \left( \prod_{l=1}^{4} x_l \right) \times \bar{v}^2, \quad \left( \prod_{l=1}^{4} x_l \right) = 0,
\]

with the singular term \( \bar{v}^2 \) given by,

\[
\begin{align*}
\bar{v}^2 &= \sum_{i=1}^{4} \frac{1}{x_i} \left[ d_i (z) u^3 + e_i (z) u^2 + f_i (z) u + h_i (z) \right] \\
+ \sum_{i>j=1}^{4} \frac{1}{x_i x_j} \left[ d_{ij} (z) u^3 + e_{ij} (z) u^2 + f_{ij} (z) u + h_{ij} (z) \right] \\
+ \sum_{i>j>k=1}^{4} \frac{1}{x_i x_j x_k} \left[ d_{ijk} (z) u^3 + e_{ijk} (z) u^2 + f_{ijk} (z) u + h_{ijk} (z) \right].
\end{align*}
\]

These holomorphic relations involve several terms which deserve some comments.

(1) Since \( \Pi_{l=1}^{4} x_l = 0 \) as required by the defining relation of the tetrahedron, then eq (6.47)
is non trivial unless if $\tilde{v}^2$ has poles so that the product $\tilde{v}^2 \times \prod_{l=1}^{4} x_l$ make sense, that is

$$\tilde{v}^2 \times \prod_{l=1}^{4} x_l \rightarrow \text{finite}, \quad (6.49)$$

in the limit $x_k \rightarrow 0$.

(2) Since $\prod_{l=1}^{4} x_l = 0$ has simple, double and triple zeros, then the poles in $\tilde{v}^2$ should be of three kinds: simple, double and triple,

$$\tilde{v}^2 \sim \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x} + \ldots, \quad (6.50)$$

where the dots stand for irrelevant regular terms.

(3) the simple poles are located at $x_a = 0$ and so are associated with the divisors $S_a$. These simple poles correspond to the first terms in eq(6.48). Upon multiplication by $\prod_{l=1}^{4} x_l$, we get cubic monomials $x_i x_j x_k$. More explicitly, this term reads as,

$$+ x_2 x_3 x_4 \left[ d_1 (z) u^3 + e_1 (z) u^2 + f_1 (z) u + h_1 (z) \right] \delta (x_1)$$

$$+ x_1 x_3 x_4 \left[ d_2 (z) u^3 + e_2 (z) u^2 + f_2 (z) u + h_2 (z) \right] \delta (x_2)$$

$$+ x_1 x_2 x_4 \left[ d_3 (z) u^3 + e_3 (z) u^2 + f_3 (z) u + h_3 (z) \right] \delta (x_3)$$

$$+ x_1 x_2 x_3 \left[ d_4 (z) u^3 + e_4 (z) u^2 + f_4 (z) u + h_4 (z) \right] \delta (x_4), \quad (6.51)$$

where we have added the Dirac delta function $\delta (x_a)$ to refer to the divisor $S_a$ in question. Furthermore, the extra term between brackets, namely

$$\tilde{v}^2_a = d_a (z) u^3 + e_a (z) u^2 + f_a (z) u + h_a (z) \quad (6.52)$$

where we have set

$$\tilde{v}^2_a = \frac{(\tilde{v}^2 x_a)}{(\prod_{b=1}^{4} x_b)},$$

captures the way the fiber Y degenerates on $S_a$ as a locus. In the $SU(5)$ GUT type model, eq(6.52) takes the form

$$\tilde{v}^2_a = u^2 + z^5_a (z_a - t_{a1}) (z - t_{a2}), \quad a = 1, 2, 3, 4, \quad (6.53)$$

where $t_{a1}$ and $t_{a2}$ are vevs as in eq(5.28).

(4) the double poles are located at $x_a = x_b = 0$ and are associated with the complex curves $\Sigma_{(ab)} = S_a \cap S_b$. These double poles correspond to the second term in eqs(6.48). Up on multiplying by $(\prod_{b=1}^{4} x_b)$, one ends with quadratic monomial $x_a x_b$ associated with the six matter curves $\Sigma_{(ab)}$.

Moreover, the elliptic curves fibered on the matter curves $\Sigma_{(ab)}$ namely

$$\tilde{v}^2_{ab} = d_{(ab)} (z) u^3 + e_{(ab)} (z) u^2 + f_{(ab)} (z) u + h_{(ab)} (z), \quad (6.54)$$

61
with $\tilde{v}_{ab}^2 = (\tilde{v}^2 x_a x_b) / (\Pi_{c=1}^4 x_c)$, capture the one-fold enhanced gauge symmetry. (5) the triple poles located at $x_a = x_b = x_c = 0$ are associated with the vertices of the tetrahedron. Furthermore, the elliptic curves fibered on the vertices of the tetrahedron

$$\tilde{v}_{abc}^2 = d_{(abc)} (z) u^3 + e_{(abc)} (z) u^2 + f_{(abc)} (z) u + h_{(abc)} (z),$$

(6.55)

with $\tilde{v}_{abc}^2 = (\tilde{v}^2 x_a x_b x_c) / (\Pi_{l=1}^4 x_l)$, capture the two-fold enhanced gauge symmetry namely $SO (12)$, $E_6$ and $SU (7)$.

6.3 $SU (5)$ Quiver models

In this subsection, we consider F-theory on local Calabi-Yau 4-folds based on tetrahedron and we construct a class of three kinds of 4D $\mathcal{N} = 1$ supersymmetric $SU (5)$ quiver GUT-type models. By using the $SU (5)$ group as a gauge invariance on the surfaces $S_a$ of the tetrahedron, we distinguish three models according to the gauge enhanced symmetry that live at the vertices of the tetrahedron. These unrealistic models have respectively a $SU (7)$, a $SO (12)$ or a $E_6$ enhanced invariance.

6.3.1 $SU (7)$ vertex

The quiver gauge diagram of the 4D $\mathcal{N} = 1$ supersymmetric $SU (5)$ GUT-type model with a $SU (7)$ enhanced gauge symmetry is depicted in figure 11.

Figure 11: Quiver gauge diagram for $SU (5)$ GUT-like model with $SU (7)$ enhanced gauge symmetry at the vertices.
The $SU(7)$ symmetry at the vertices of the tetrahedron breaks down to subgroups on the edges and the surfaces. The simplest 4D $\mathcal{N} = 1$ supersymmetric $SU(5)$ GUT-type model one engineers from the $SU(7)$ singularity involves the Yukawa couplings (5.31). The chiral superfields configuration of the model reads as:

| chiral superfields | $SU(5) \times U^2(1)$ | number |
|--------------------|------------------------|--------|
| Matter like $\Phi_M$ | $\tilde{5}_{7,1}$ | 4 |
| Higgs like $\Phi_H$ | $5_{0,-6}$ | 4 |
| Neutino like $\Phi_N$ | $1_{-7,5}$ | 4 |

These superfields follow from the decomposition of the adjoint representation $48$ of the enhanced gauge symmetry $SU(7)$ living at the vertices of the tetrahedron in terms of representations $SU(5) \times U^2(1)$ group as shown below,

$$48 = 1_{0,0} \oplus 1_{0,0} \oplus 24_{0,0} \oplus (5_{0,-6} \oplus \tilde{5}_{0,6}) \oplus (5_{-7,-1} \oplus \tilde{5}_{7,1}) \oplus (1_{7,-5} \oplus \tilde{1}_{-7,5}) \, .$$

(6.57)

From this decomposition, we see that one can build several tri-coupling gauge invariant terms; These are given by the following tri-couplings

$$W_1 = 5_{0,-6} \times 1_{0,0} \times \tilde{5}_{0,6} \quad , \quad W_2 = 5_{-7,-1} \times 1_{0,0} \times \tilde{5}_{7,1} \quad , \quad W_3 = 5_{0,-6} \times \tilde{1}_{-7,5} \times \tilde{5}_{7,1} \quad , \quad W_4 = 5_{0,6} \times 1_{7,-5} \times 5_{-7,-1} \, .$$

(6.58)

Notice that the superpotentials $W_1$ and $W_2$ involve, in addition to two chiral superfields transforming into conjugates bi-fundamentals, an adjoint bulk matter singlet. The superpotentials $W_3$ and $W_4$ involve however only chiral matter in the bi-fundamentals.

A typical $\mathcal{N} = 1$ chiral superpotential that involve a Higgs like superfield $H_u$, matter in the $\tilde{5}$ and neutrino like superfields reads as follows

$$\int d^2\theta \ W_3 = \sum_{a=1}^{4} \lambda_a \int d^2\theta \ \Phi_H^a \Phi_M^a \Phi_N^a \, .$$

(6.59)

where the $\lambda_a$s are coupling constants. Notice that along the matter curve in the $5_{0,-6}$ and $\tilde{5}_{7,1}$ representations, the bulk $SU(5) \times U^2(1)$ gauge symmetry gets enhanced to $SU(6) \times U(1)$ which gets further enhanced to $SU(7)$ at the vertices. Along the matter curve associated with $\tilde{1}_{-7,5}$, the $SU(5)$ singularity on the surface gets enhanced to $SU(5) \times SU(2)$.

### 6.3.2 $SO(12)$ enhanced singularity

The quiver gauge diagram of the supersymmetric $SU(5)$ GUT-type model with an $SO(12)$ enhanced singularity is depicted in figure [12].
The chiral superfield configuration of this model reads as,

| chiral superfields | $SU(5) \times U^2(1)$ | number |
|-------------------|-----------------------|--------|
| Matter like $\Phi_5$ | $5_{-2,-2}$ | 4 |
| Matter like $\Phi_{10}$ | $10_{0,4}$ | 4 |
| Higgs like $\Phi_H$ | $5_{-2,-2}$ | 4 |

where now, we have both matter in the $\overline{5}$ and $10$ representations as well as the Higgs $H_d$.

These complex superfields follow from the decomposition of the adjoint representation $66$ of the two fold enhanced $SO(12)$ symmetry, living at the vertices of the tetrahedron, in terms of representations $SU(5) \times U^2(1)$

$$66 = 1_{0,0} + 1_{0,0} + 24_{0,0} + (5_{2,2} + 5_{-2,-2}) + (5_{-2,2} + 5_{2,-2}) + 10_{0,4} + \overline{10}_{0,-4}.$$ (6.61)

From this decomposition, we see that one can build several tri-coupling gauge invariant terms; These are given by

$$W'_1 = 5_{2,2} \times 1_{0,0} \times 5_{-2,-2}, \quad W'_3 = 5_{-2,-2} \times 5_{-2,-2} \times 10_{0,4},$$

$$W'_2 = 10_{0,4} \times 1_{0,0} \times 10_{0,-4}, \quad W'_4 = 5_{2,2} \times 5_{2,2} \times \overline{10}_{0,-4}.$$ (6.62)

Like in the $SU(7)$ case, here also the $SO(12)$ gauge symmetry gets broken down to subgroups on the edges and the faces of the tetrahedron. Moreover, the superpotentials...
$W'_1$ and $W'_2$ involve, in addition to two bi-fundamentals, an adjoint singlet while $W'_3$ and $W'_4$ involve only matter in the bi-fundamentals which is used to describe Yukawa couplings of GUT- like models. The $\mathcal{N}=1$ chiral superpotential reads as follows

$$\int d^2\theta \ W'_3 = \sum_{a=1}^{4} \lambda'_a \int d^2\theta \ \Phi^a_H \Phi^a_5 \Phi^a_{10}$$

(6.63)

where the $\lambda'_a$ s are coupling constants. Notice that, along the matter curves represented by the edges, the $SU(5) \times U(1)$ gauge symmetry on the surface of the tetrahedron gets enhanced to $SO(10) \times U(1)$ which in turns gets further enhanced to $SO(12)$ at the vertices.

### 6.3.3 $E_6$ enhanced singularity

The quiver gauge diagram of the supersymmetric $SU(5)$ GUT-type model with an $E_6$ enhanced singularity at the vertices of the tetrahedron is depicted in the figure (13),

![Figure 13: Quiver gauge diagram for SU(5) GUT- like model with $E_6$ enhanced gauge symmetry at the vertices and a $SU(5)$ model involving $5 \times 10 \times 10$ tri-couplings.](image)

The quiver gauge model has the following chiral superfield spectrum:

| chiral superfields | $SU(5) \times U^2(1)$ | number |
|-------------------|-----------------------|--------|
| Matter like $\Phi_5$ | $5_{-3,3}$ | 4 |
| Matter like $\Phi_{10}$ | $10_{-1,-3}$ | 4 |
| Matter like $\Phi_H$ | $10_{4,0}$ | 4 |

(6.64)
These chiral superfields follow from the decomposition of the adjoint representation 78 of the enhanced \( E_6 \) in terms of representations \( SU(5) \times U^2(1) \) namely,

\[
78 = 1_{0,0} + 24_{0,0} + 1_{5,3} + 1_{-5,-3} + 5_{-3,3} + 5_{3,-3} + 10_{-1,-3} + \overline{10}_{1,3} + 10_{4,0} + \overline{10}_{-4,0}.
\]  

(6.65)

From this decomposition, we see that we can build several tri-coupling gauge invariant terms; these are:

- \( W''_1 = 5_{-3,3} \times 1_{0,0} \times \overline{5}_{3,-3} \)
- \( W''_2 = 10_{-1,-3} \times 1_{0,0} \times \overline{10}_{1,3} \)
- \( W''_3 = 10_{4,0} \times 1_{0,0} \times \overline{10}_{-4,0} \)
- \( W''_4 = 5_{-3,3} \times 10_{-1,-3} \times 10_{4,0} \)
- \( W''_5 = 5_{3,-3} \times \overline{10}_{1,3} \times \overline{10}_{-4,0} \)

(6.66)

Similarly as before, the superpotentials \( W''_1, W''_2 \) and \( W''_3 \) involve, besides two bi-fundamentals, an adjoint singlet while \( W''_4 \) and \( W''_5 \) involve only matter in the bi-fundamentals.

The \( \mathcal{N} = 1 \) chiral superpotential describing the tri-coupling of the matter in the bi-fundamentals is given by \( W''_4 \). Moreover, along the matter curves in the tetrahedron, the \( SU(5) \times U^2(1) \) gauge symmetry on the surface of the tetrahedron gets enhanced to \( SO(10) \times U(1) \) which gets further enhanced to \( E_6 \) at the vertices.

In the end of this section, notice that in these SU(5) GUT-type models based on tetrahedron, the gauge symmetry at the vertices is of same nature. In what follows, we study other configurations where different gauge symmetries live at the vertices of the tetrahedron. This kind of quiver gauge models requires however performing blown ups of the tetrahedron surface.

7  GUT-like models on blown up Tetrahedron

We start by giving further details on the blown up \( T_n \) on the complex tetrahedral geometry \( T \); in particular the blown ups by projective planes \( \mathbb{P}^2 \) at one and two vertices of \( \Delta_T \). Then, we consider the building of \( SU(5) \) GUT-type models that are embedded in F-theory on Calabi-Yau four-folds based on these geometries.

7.1 More on blown ups of tetrahedron

Starting from the non planar tetrahedral surface \( T \) with its four projective planar faces \( S_a \), its six projective line edges \( \Sigma_{(ab)} \) and the four vertices \( P_{(abc)} \), we can perform blown ups of the tetrahedral surface \( T \) at a finite set of points. Roughly, we distinguish:

(1) blown ups at the four vertices of the tetrahedron \( \Delta_T \),

66
(2) blown ups at the edges of the tetrahedron  
(3) blown ups at a finite number of generic points of the tetrahedron.

In what follows, we will consider the first case of these blown ups and illustrate the main idea by studying \( SU(5) \) GUT-type models based on \( T_1 \) and \( T_2 \) geometries.

### 7.1.1 Blown up at a vertex

Recall that the toric graph of the tetrahedron \( \Delta_T \) has four vertices \( P_{(abc)} \) where meet simultaneously \(^{16}\) three projective lines \( \Sigma_{(ab)} \), \( \Sigma_{(ac)} \) and \( \Sigma_{(bc)} \). Starting from such a graph and focusing on the fourth vertex \( P_4 \) of the figure \((10)\), the blown up of this vertex \( P_4 \) by a projective plane amounts to replacing \( P_4 \) by a projective plane,

\[
\text{point } P_4 \rightarrow \text{projective plane } \mathbb{P}^2. \quad (7.1)
\]

Since in toric geometry, a projective plane is described by a triangle, the blown up of the vertex \( P_4 \) amounts to substitute this point by a triangle \([Q_1Q_2Q_3]\) as depicted in the figure \((14)\).

![Figure 14: The toric graph representing a blown up of the tetrahedral geometry. The vertex \( P_4 \) has been replaced by a projective plane \([Q_1Q_2Q_3]\).](image)

The resulting toric geometry of the blown up tetrahedron at a vertex by a projective plane, to which we shall refer below to as \( T_1 \), has five intersecting faces namely:

\(1\) two complex projective planes with toric graphs given by the triangles

\[
[P_1P_2P_3], \quad [Q_1Q_2Q_3], \quad \text{ (7.2)}
\]

\(^{16}\)Three projective planes meet as well at each vertex of the tetrahedron.
of the figure (14). As we see from this figure, these triangles have no edge intersection. 

(2) three del Pezzo surfaces $dP_1$ with toric graphs given by the quadrilaterals,

$$ [P_1P_2Q_1Q_2] , [P_2P_3Q_2Q_3] , [P_1P_3Q_1Q_3] $$

(7.3)

Thinking about the three edges $[Q_1Q_2], [Q_2Q_3], [Q_1Q_3]$ of the exceptional triangles $[Q_1Q_2Q_3]$ that generate the blown up of the vertex $P_4$ as describing complex projective lines with the Kahler parameters,

$$ [Q_1Q_2] \rightarrow a , $$

$$ [Q_2Q_3] \rightarrow b , $$

$$ [Q_1Q_3] \rightarrow c , $$

(7.4)

and considering the singular limit of the geometry (14) where one or two of these parameters are sent to zero, one recovers new "singular" topologies of blown up of the tetrahedron $\Delta_T$. For instance, putting $a = 0$, and $b = c \neq 0$, the points $Q_1$ and $Q_2$ get identified,

$$ Q_1 = Q_2 \equiv Q_0 , $$

(7.5)

and so the triangle $[Q_1Q_2Q_3]$ gets reduced to a singular line

$$ [Q_1Q_2Q_3] \rightarrow [Q_0Q_3] . $$

(7.6)

Consequently, we get a degenerating blown up of the tetrahedron where the vertex $P_4$ is replaced by the projective line $[Q_0Q_3]$. The resulting geometry has three intersecting projective planes $dP_0$; intersecting as well two del Pezzo surfaces $dP_1$. Notice that in the special case where $a = b = c = 0$, we recover obviously the standard tetrahedron $\Delta_T$.  

### 7.1.2 Blown up at two vertices

The blown up of the tetrahedron $\Delta_T$ at two vertices, say $P_3$ and $P_4$ of the figure (10), is achieved by replacing these two points by projective planes. In toric graph language, this amounts to replace $P_3$ and $P_4$ by the triangles,

$$ P_3 \rightarrow [R_1R_2R_3] , \quad P_4 \rightarrow [Q_1Q_2Q_3] . $$

(7.7)

The toric graph of the two blown up tetrahedron is depicted in the figure (15).

The obtained surface, denoted as $T_2$, has six intersecting faces namely:

(1) two projective planes with toric graphs given by the triangles of the figure (14)

---

\[^{17}\text{Notice that the projective plane has one Kahler parameter; it should not be confused with the auxiliary parameters } a, b \text{ and } c.\]

68
Figure 15: blown up of the tetrahedron at the two points $P_3$ and $P_4$ which have been replaced by the projective planes with toric graphs given by the triangles $[R_1R_2R_3]$ and $[Q_1Q_2Q_3]$ respectively.

namely $[R_1R_2R_3]$ and $[Q_1Q_2Q_3]$.

(2) two del Pezzo surfaces $dP_1$ with toric graphs given by the quadrilaterals $[P_1P_2R_1R_2]$ and $[P_1P_2Q_1Q_2]$.

(3) two del Pezzo surfaces $dP_2$ with toric graphs given by the pentagons $[P_1R_1R_3Q_3Q_1]$ and $[P_2R_2R_3Q_3Q_2]$.

Similarly as in the previous case, one can recover new singular topologies of the blown tetrahedron (15) by taking singular limits of the Kahler parameters $a$, $b$, $c$, $e$, $f$ and $g$. The case where $e = f = g = 0$ leads to the figure (14) and the case where all these parameters are set to zero gives the standard tetrahedron.

7.2 $SU(5)$ GUT model on $T_1$ and $T_2$

In this subsection, we engineer various unrealistic $SU(5)$ GUT-type models that are embedded in consider F-theory on local elliptic K3 fibered Calabi Yau four-folds based on the surfaces $T_1$ and $T_2$. We first construct GUT-type models based on $T_1$ and then we build other models based on $T_2$.

7.2.1 $SU(5)$ GUT type models on $T_1$

The toric graph of the complex surface $T_1$ is given by the figure (14); the fix points of the toric action encode data on the seven brane intersections with the following features:

(1) $T_1$ has five faces where live the $4D \mathcal{N} = 1$ supersymmetric gauge theory with bulk
gauge symmetry \( SU(5) \times U(1) \times U(1) \) where the extra factor \( U^2(1) \) is the toric symmetry in the 2-torus in the toric surface \( T_1 \).

(2) \( T_1 \) has nine edges where localize matter in the fundamental and antisymmetric representations of the \( SU(5) \) gauge symmetry. On these curves, the rank of the gauge invariance gets enhanced by one.

(3) \( T_1 \) has six vertices where live tri-fields Yukawa couplings and where the gauge symmetry gets enhanced to \( SU(7) \), or \( SO(12) \) or also \( E_6 \).

Now using the fact that at the vertices of the surface \( T_1 \), the tri- fields interactions should be gauge invariant under the gauge group \( SU(5) \times U^2(1) \), one can engineer various gauge invariant configurations; in particular the ones depicted in the figures (16),

![Figure 16: tetrahedron models](image)

To engineer \( SU(5) \) GUT- type gauge invariant models with different Yukawa couplings, we use the following relations,

| Yukawa tri-fields couplings | enhanced singularity at vertices |
|----------------------------|---------------------------------|
| \( 1 \otimes 5 \otimes \bar{5} \) | \( \rightarrow \) \( SU(7) \) |
| \( 5 \otimes \bar{5} \otimes 10 \) | \( \rightarrow \) \( SO(12) \) |
| \( 5 \otimes \bar{5} \otimes 10 \) | \( \rightarrow \) \( E_6 \) |

(7.8)

to choose the kind of the ADE singularity one has to put in the fiber over each vertex.
of the base surface $T_1$. Let us illustrate the idea by describing the examples depicted in the figures (16).

The six toric vertices of $T_1$ of the figure (16-a) involves the tri-coupling $1 \otimes 5 \otimes \overline{5}$ and so have a SU(7) enhanced singularity. In the figure (16-b), four toric vertices have a SU(7) singularity and the two others have a SO(12) one since the tri-couplings are given by

$$5 \otimes \overline{5} \otimes 10.$$  (7.9)

The six toric vertices of the figure (16-c) have all of them an SO(12) enhanced singularity. Using the same philosophy, four toric vertices of the figure (16-d) have an SO(12) enhanced singularity and the two others are of type

$$5 \otimes 5 \otimes 10$$  (7.10)

and so are associated with an E$_6$ singularity. Finally, all the six toric vertices of the figure (16-f) are of E$_6$ type while the tri-fields couplings given by the figure (16-e) are equivalent to those of the figure (16-c).

### 7.2.2 SU(5) GUT type models on $T_2$

The toric graph of the surface $T_2$ is given by the figure (15); it has:

1. Six toric faces where localize a 4D $\mathcal{N} = 1$ supersymmetric gauge theory with $SU(5) \times U^2(1)$ gauge symmetry. These faces are given by del Pezzo surfaces of different types:
   - (a) two isolated $dP_0$’s; each one intersects a surface $dP_1$ and two surfaces $dP_2$,
   - (b) two intersecting $dP_1$’s, each one of these $dP_1$’s intersects a $dP_1$ and two $dP_2$,
   - (c) two intersecting $dP_2$’s, each one of these $dP_2$’s intersects the two $dP_0$’s and the two $dP_1$’s.

2. Thirteen toric edges describing the intersections of the del Pezzo surfaces. On these curves localize matter in the singlet, the fundamentals and the antisymmetric representations of $SU(5)$. For the last representations, the gauge symmetry gets enhanced either to $SU(6) \times U(1)$ or to $SO(10) \times U(1)$.

3. Eight toric vertices where live Yukawa couplings and the enhanced gauge singularity. Each of these vertices is associated with the intersection of three edges and it localizes tri-fields Yukawa coupling.

**Model I**

Now using the same approach as for the surface $T_1$, we can engineer various gauge invariant configurations.

One of these configurations, depicted in the figure (17), involves eight Yukawa couplings: six Yukawa couplings of type $5 \otimes 5 \otimes 10$ and two Yukawa couplings of type $5 \otimes 5 \otimes \overline{10}$. 

71
Figure 17: $SU(5)$ GUT-type model based on the $T_2$ geometry. This model has six vertices with an $E_6$ singularity and two others with an $SO(12)$ one.

| Yukawa couplings | Singularity | number of vertices |
|------------------|-------------|--------------------|
| $5 \otimes 5 \otimes 10$ | $E_6$ | 6 |
| $5 \otimes 5 \otimes \overline{10}$ | $E_6$ | 0 |
| $5 \otimes 5 \otimes 10$ | $SO(12)$ | 2 |
| $5 \otimes 5 \otimes 10$ | $SU(7)$ | 0 |

An equivalent configuration is also given by the conjugate representations.

**Model II**

This $SU(5)$ GUT-type model is the dual of the previous one. This duality is in the sense that six of the eight vertices have an $SO(12)$ singularity while the two remaining others have an $E_6$ one. We distinguish two class of models depending on the intersecting surfaces and intersecting edges. We will refer to these models as IIa and IIb:

**Model IIa:** In this model, the Yukawa couplings are depicted in the figure (18)
Figure 18: $SU(5)$ GUT-like model based on the $\mathbb{T}_2$ geometry; it has five $E_6$ vertices and two $SO(12)$ vertices.

From the figure (18), we see that the vertices are as in the following table,

| Yukawa couplings | Singularity | number of vertices |
|------------------|-------------|--------------------|
| $5 \otimes 5 \otimes 10$ | $E_6$        | 2                  |
| $\bar{5} \otimes \bar{5} \otimes \overline{10}$ |             | 0                  |
| $5 \otimes 5 \otimes \overline{10}$ | $SO(12)$    | 6                  |
| $\bar{5} \otimes \bar{5} \otimes 10$ |             | 0                  |
| $5 \otimes 5 \otimes 1$ | $SU(7)$    | 0                  |

We learn also that the two $E_6$ and the six $SO(12)$ vertices are given by the intersections of three del Pezzo surfaces as given below,

\[
\begin{align*}
6 \times E_6 & : dP_1^{(1)} \cap dP_1^{(2)} \cap dP_2, \\
2 \times SO(12) & : dP_0 \cap dP_1 \cap dP_2,
\end{align*}
\]  

(7.13)

where $dP_1^{(1)}$ and $dP_1^{(2)}$ are the two del Pezzo surfaces of the blown up surface $\mathbb{T}_2$.

Model IIb. In this model, the configuration of the Yukawa couplings, depicted in the figure (19), are as in the table (7.12).

The two vertices with $E_6$ singularity and the other six vertices with $SO(12)$ singularity
are given by the following intersections,

\[ 2 \times E_6 : dP_0 \cap dP^{(1)}_2 \cap dP^{(2)}_2, \]
\[ 6 \times SO(12) : dP_0 \cap dP_1 \cap dP_2, \quad (7.14) \]

where \( dP^{(1)}_2 \) and \( dP^{(2)}_2 \) stand for the two del Pezzo surfaces involved in the blown up tetrahedron \( T_2 \).

**Model III**

In this model, the Yukawa couplings are depicted in the figure (20),

| Yukawa couplings | Singularity | number of vertices |
|------------------|-------------|-------------------|
| \( 5 \otimes 5 \otimes 10 \) | \( E_6 \) | 5 |
| \( 5 \otimes 5 \otimes \overline{10} \) | \( E_6 \) | 0 |
| \( 5 \otimes 5 \otimes \overline{10} \) | \( SO(12) \) | 1 |
| \( 5 \otimes 5 \otimes 10 \) | \( SU(7) \) | 0 |
| \( 5 \otimes 5 \otimes 10 \) | \( SU(7) \) | 2 |

The five vertices with an \( E_6 \) singularity is given by two kinds of tri-intersection of the del
Figure 20: $SU(5)$ GUT-like model based on the $T_2$ geometry with five $E_6$ vertices, two $SU(7)$ and one $SO(12)$.

Pezzo surfaces. One vertex is given by the tri-intersection of a projective plane with two del Pezzo surfaces $dP_2$ while the five others are given by the intersection of a projective line and the del Pezzo surfaces $dP_1$ and $dP_2$:

$$1 \times E_6 : dP_0 \cap dP_2^{(1)} \cap dP_2^{(2)} , \quad 5 \times E_6 : dP_0 \cap dP_1 \cap dP_2 ,$$

(7.16)

Regarding the vertex with an $SO(12)$ singularity, we have:

$$1 \times SO(12) : dP_0 \cap dP_1 \cap dP_2 ,$$

(7.17)

and for the two vertices with a $SU(7)$ singularity, the tri-intersections are as follows:

$$1 \times SU(7) : dP_0 \cap dP_1 \cap dP_2 , \quad 1 \times SU(7) : dP_0 \cap dP_2^{(1)} \cap dP_2^{(2)} .$$

(7.18)

8 Conclusion

In this paper we have studied a class of 4D $\mathcal{N} = 1$ supersymmetric quiver gauge models that describe gauge theory limits of 12D F-theory compactification on local tetrahedron. In these supersymmetric models; we have mainly focused on GUT-type gauge
symmetries in particular on the $SU(5)$ symmetry with the $SO(12)$, $SU(7)$ and $E_6$ gauge enhancements. These 4D gauge models should be thought of as a first step for building non minimal supersymmetric GUT-type models along the line of the BHV theory. The other steps are to require the conditions for realistic supersymmetric GUT models building such as GUT breaking and doublet/triplet splitting via hypercharge flux, the absence of bare $\mu$ term and dangerous dimension 4 proton decay operators.

Like in BHV theory, our quiver gauge models are based on using seven branes wrapping 4-cycles in the framework of twelve dimensional F-theory compactified on local elliptic K3 fibered Calabi Yau four-folds

$$ Y \rightarrow X_4$$

where now the base surface $S$ is given by the complex tetrahedral surface $T$ and its blowups $T_n$. The relation between $T_n$ and $T$ should be thought of in the same manner the del Pezzo surfaces $dP_n$ are linked to the complex projective plane. In fact, the complex surfaces $dP_n$ are particular sub-geometries of $T_m$s; a property which make these CY4-manifolds somehow extending the local geometry used in the BHV theory.

The engineering of the non abelian gauge symmetry that is visible at the level of the 4D $\mathcal{N} = 1$ supersymmetric effective GUT model is achieved through singularities in the K3 fiber of the local CY4-folds $X_4$. In the complex base surface $S$, it generally lives a non abelian rank $r$ bulk gauge symmetry $G_r$. This gauge invariance gets enhanced on the matter curves along which seven branes intersect to $G_{r+1} \supset G_r$. It gets further enhanced to $G_{r+2} \supset G_{r+1}$ at isolated points of $S$ where matter curves meet and where live tri-fields Yukawa couplings. The three gauge groups satisfy the embedding property

$$ G_{r+2} \supset G_{r+1} \times U(1) \supset G_r \times U(1) \times U(1) .$$

In the 4D $\mathcal{N} = 1$ supersymmetric $SU(5)$ GUT-like models, these gauge symmetries should be thought of as follows

| $G_{r+2}$ | $G_{r+1}$ | $G_r$ |
|-----------|-----------|-------|
| $E_6$, $SO(12)$, $SU(7)$ | $SO(10)$, $SU(6)$ | $SU(5)$ |

The decomposition of the adjoint representation of $G_{r+2}$ in terms of representations of the $G_r \times U(1) \times U(1)$ allows to generate chiral matter in representations other than the adjoint ones. In our construction, the extra abelian $U(1) \times U(1)$ gauge invariance is interpreted in terms of symmetries of the toric fiber of the complex base surface. Recall
that complex surfaces $S$ exhibits a natural toric fibration,

$$
\begin{array}{c}
\mathbb{T}^2 \\
\downarrow \pi_S \\
B_S
\end{array} \rightarrow S
$$

(8.4)

where $\mathbb{T}^2$ is the usual fiber of the toric geometry and $B_S$ is a real two dimension base which is nicely represented by a toric graph $\Delta_S$. In the case of the complex tetrahedral surface $T$, the corresponding toric graph $\Delta_T$ is given by the tetrahedron of the figure (1). The toric fibration has remarkable shrinking features on the edges of the tetrahedron and at the vertices.

Using the power of toric geometry in the complex base $T$ and the degeneracy of its torus fibration, we have engineered a class of $SU(5)$ GUT-like models based on local tetrahedron $T$ and the two first elements of its blow ups family $T_n$. These $SU(5)$ GUT-type models building extend naturally for generic 4D $\mathcal{N} = 1$ supersymmetric quiver gauge theories that are embedded in F-theory on local CY4- folds based on blown ups of the tetrahedron; in particular for the interesting class of GUT-type models using flipped $SU(5)$ and $SO(10)$ gauge symmetries.

In the end of this conclusion, we would like to emphasize that our interest in GUT-type models buildings based on the complex tetrahedral surface and its toric blown ups has been motivated by a number of remarkable features; in particular the two following:

(1) there is an intimate link between the complex tetrahedral surface and the projective plane $\mathbb{P}^2$. The tetrahedron is precisely given by the four projective planes $dP_0^{(1)}$, $dP_0^{(2)}$, $dP_0^{(3)}$ and $dP_0^{(4)}$ describing the basic divisors of the complex three dimension projective space $\mathbb{P}^3$ while its blow ups are given by a union of the del Pezzo surfaces.

For the case of the blow up of $T$ at a vertex by a projective plane, we have

$$
\begin{align*}
T_1 & = dP_0^{(1)} \cup dP_0^{(2)} \cup dP_0^{(3)} \cup dP_0^{(4)} \cup dP_0^{(5)}, \\
\Sigma_{ij} & = dP_0^{(i)} \cap dP_0^{(j)}, \\
P_{ijk} & = dP_0^{(i)} \cap dP_0^{(j)} \cap dP_0^{(k)},
\end{align*}
$$

(8.5)

where the complex curves $\Sigma_{ij}$ stand for the nine edges of $T_1$ and the six isolated points $P_{ijk}$ for its vertices. These intersections can be read from eqs (6.26-6.29) or directly from their toric graphs given by the respective figures (10) and (14). From this view, the blown ups of the tetrahedron contain several copies of del Pezzo surfaces $dP_n^{(i)}$ as special components on which may be engineered the BHV theory. Recall that the del Pezzo complex surfaces $dP_n$ play a central role in the BHV theory for F-theory GUT models building. These complex $dP_n$s are strongly linked to the projective plane $\mathbb{P}^2$ since they are precisely given by its blown ups at eight isolated points,

$$
dP_0 = \mathbb{P}^2, \quad dP_n, \quad n = 1, \ldots, 8
$$

(8.6)
the special degeneracy properties of the fiber $T^2$ of the toric fibration of the tetrahedron $T \sim \Delta_T \times T^2$. The 1-cycle shrinking loci of the 2-torus fiber down $S^1_\Sigma$ allows to host in a natural way the engineering of the seven branes intersections along the edges. Moreover, the shrinking of the $T^2$ fiber down to zero allows to engineer Yukawa couplings at the vertices.

In a future occasion, we give further refinements of this construction and seek for non minimal quasi-realistic F-theory-GUT models building based on local tetrahedral geometry.

Acknowledgement 1

This research work is supported by the program Protars III D12/25.

9 Appendix

To engineer chiral matter transforming in representations of the gauge invariance $G_S$ other than the $\text{adj} G_S$, we have two ways: either by switching on a gauge bundle $E$ with structure group $H_S$ that breaks the gauge symmetry like $G_S \rightarrow H_S \times G$, or modify the base geometry $S$ of the local Calabi-Yau 4-folds into a larger surface containing at least two intersecting 4-cycles $S_a$ and $S_b$ like,

$$S = S_a \cup S_b \quad \text{,} \quad S_a \cap S_b = \Sigma_{ab} \neq \emptyset,$$

with $\Sigma_{ab}$ standing for a intersecting complex curve where localize bi-fundamental matter. In this way the bulk gauge symmetry $G_S$ gets broken down like $G_S \rightarrow G_{S_a} \times G_{S_b}$. To make an idea on how these deformations work, we review below the key idea behind these methods.

In the case of deformation by switching on fluxes, the adjoint representation $\text{ad} (G_S)$ decomposes as

$$\text{ad} G_S = (\text{ad} H_S, 1) \bigoplus (1, \text{ad} G) \bigoplus \left( \bigoplus \left( \rho_i, U_i \right) \right),$$

where $\rho_i$ and $U_i$ stand respectively for representations of $H_S$ and $G$ respectively and where

$$\dim [\text{ad} G_S] = \dim [\text{ad} H_S] + \dim [\text{ad} G] + \sum_i (\dim \rho_i) \times (\dim U_i).$$

The switching of the bundle $E$ induces then a deformation in the complex surface $S$ and may be interpreted as splitting the winding of the bulk seven branes wrapping $S$ into two intersecting stacks; one stack, to which we refer as matter brane, with gauge symmetry $H_S$ and the second stack with gauge invariance $G$. Along the intersection of the two
stacks of seven branes (matter and bulk), which corresponds geometrically to a curve \( \Sigma \) in \( S \), the gauge symmetry is obviously given by \( G_S \); but outside \( \Sigma \), the symmetry is \( H_S \times G \).

The number \( N_i \) of chiral fields \( \phi_i \) (resp. \( N^*_i \) of anti-chiral fields \( \phi^*_i \)) transforming in the representation \( U_i \) (resp. \( U^*_i \)) of the subgroup \( G \) is determined by the bundle-valued Euler characteristics,

\[
N_i = \chi_S (\mathcal{R}_i) \quad , \quad N^*_i = \chi_S (\mathcal{R}^*_i) \tag{9.4}
\]

where \( \mathcal{R}_i \) and \( \mathcal{R}^*_i \) denote the bundles transforming in \( U_i \) and \( U^*_i \) respectively. On the del Pezzo surface \( dP_8 \), the numbers \( N_i \) and \( N^*_i \) are easily computed by help of the relation

\[
\chi_S (\mathcal{R}) = 1 - \frac{1}{2} \Omega_S . c_1 (\mathcal{R}) + \frac{1}{2} c_1 (\mathcal{R}) . c_1 (\mathcal{R}) \tag{9.5}
\]

where \( \Omega_S \) denotes the canonical class of \( S \).

Notice that this analysis is particularly interesting when the gauge subgroup \( H_S \) is abelian; that is an \( U^{r_0} (1) \) abelian subgroup of the Cartan subalgebra of \( G_S \). In this case, the deformation by fluxes has a nice geometric description in terms of deformation of of the ADE singularity. For instance, by taking as a bulk gauge symmetry \( G_S = SU (N+1) \) which is described by the local geometry of the fiber

\[
u^2 + v^2 + z^{N+1} = 0 \tag{9.6}
\]

and which represents a bulk brane wrapping the surface \( N \) times, the switching of a \( U (1) \) gauge bundle yields the deformation

\[
u^2 + v^2 + z^N (z - t) = 0. \tag{9.7}
\]

Here \( t \) is a non zero complex number behaving as \( z \) and represents a non zero vev of a scalar Higgs field in the adjoint. Under this deformation, the original bulk stacks of wrapped seven branes at \( z = 0 \) gets split to two stacks: one at \( z = 0 \), with gauge symmetry \( SU (N) \) and the other with gauge invariance \( U (1) \) at

\[
z = t, \quad t \in \mathbb{C}. \tag{9.8}
\]

These two stacks intersect along the curve \( \{z = 0\} \cap \{z = t\} \) where gauge symmetry gets enhanced to \( SU (N+1) \).

On this particular example, which applies as well to D7 seven branes of type IIB superstring, one can also read the bi-fundamental matter by decomposing the adjoint of \( U (N+1) = U_S (1) \times SU (N+1) \), describing the gauge symmetry of \( (N+1) \) parallel D7 branes, with respect to \( U (N) \times U_{\Sigma} (1) \),

\[
(N+1)^2 = N^2_0 \oplus 1_0 \oplus N_{q} \oplus N_{-q} \tag{9.9}
\]
where the charge $q = N + 1$. The charged components in this decomposition namely $N_q$ and $\overline{N}_{-q}$ describe precisely charged matter in bi-fundamentals. Notice that after the rotation of one D7-brane; say the $a$-th brane with with gauge group $U_a(1)$, the bi-fundamentals $N_q$ and $\overline{N}_{-q}$ carry the charge $(N,-1)$ and $(-N,+1)$ under the abelian group $U_S(1) \times U_a(1)$. Comparing with eq(9.9), we find that $U_{\Sigma}(1)$ should be identified with the specific linear combination

$$q_{\Sigma} = q_S - q_a \quad (9.10)$$

In the second case, we consider F-theory on a local CY four-folds with base surface (9.1) consisting of at least two components surfaces $S_a$ and $S_b$ with non trivial intersection along a complex curve $S_a \cap S_b = \Sigma_{ab}$. So the seven branes wrapping the respective surfaces $S_a$ and $S_b$ intersect in a six-dimensional space $\mathbb{R}^{1,3} \times \Sigma_{ab}$. Along $\Sigma_{ab}$, the singularity in the fiber gets enhanced to $G_{\Sigma_{ab}}$ with new bi-fundamental matter localized on the curve $\Sigma_{ab}$ determined by decomposing $adG_{\Sigma_{ab}}$ with respect to the representation of the bulk gauge symmetries $G_{S_a} \times G_{S_b}$, that is

$$adG_{\Sigma} = (adG_{S_a}, 1) \bigoplus (1, adG_{S_b}) \bigoplus \bigoplus_i \left( U^a_i, U^b_i \right) \quad (9.11)$$

where $(U^a_i, U^b_i)$ determine the bi-fundamentals under which matter on $\Sigma_{ab}$ transform. Notice the two following features:

(a) there is a strong link between the flux deformation of the base geometry of the Calabi-Yau four-fold and the use of intersecting 4-cycles. Indeed, by setting $G_{S_a} = G_S$ and $G_{S_b} = H_S$, the description using intersecting 4-cycles $S_a$ and $S_b$ may be viewed as having a complex surface $S_a = S$ together with a gauge bundle $E$ with structure group $H_S$.

(b) the intersecting 4-cycles construction and the deformation by fluxes may be combined altogether. By switching on $U(1)$- gauge bundles $\mathcal{L}_a$ and $\mathcal{L}_b$ on the surfaces $S_a$ and $S_b$, the respective gauge symmetries $G_{S_a}$ and $G_{S_b}$ get broken down to subgroups as shown below

$$G_{S_a} \rightarrow G_a \times U_a(1) \quad , \quad G_{S_b} \rightarrow G_b \times U_b(1) \quad . \quad (9.12)$$

This breaking leads to a further decomposition of the bi-fundamental representations $(U^a_i, U^b_i)$. For a given representation $(U^a_i, U^b_i)$ in eq(9.11), we have the typical decomposition

$$(U^a_i, U^b_i) = \bigoplus_j \left( r^a_j, r^b_j \right)_{q_j^a, q_j^b} \equiv \bigoplus_j \left( r_j, \tilde{r}_j \right)_{q_j, p_j} \quad (9.13)$$

where $(q^a_j, q^b_j)$ are $U_a(1) \times U_b(1)$ charges while $r^a_j$ and $r^b_j$ are representations of $G_a$ and $G_b$ respectively. Moreover, following [32, 33, 35], the number $N(r_j, \tilde{r}_j)$ of zero modes.
transforming in the representation \((r_j, \tilde{r}_j)_{q_j, p_j}\) is given by the bundle cohomology

\[
N_{(r_j, \tilde{r}_j)_{q_j, p_j}} = h^0 \left( \Sigma, K^{1/2}_\Sigma \otimes L^a|_{\Sigma} \otimes L^b|_{\Sigma} \right),
\]

(9.14)

where \(L^a|_{\Sigma}\) and \(L^b|_{\Sigma}\) are the restriction of the of the bundles \(L^a\) and \(L^b\) to the curve \(\Sigma\).

References

[1] L.E. Ibanez, F. Marchesano, R. Rabadan, *Getting just the Standard Model at Intersecting Branes*, JHEP 0111 (2001) 002, arXiv:hep-th/0105155.

[2] D. Cremades, L.E. Ibanez, F. Marchesano, *SUSY Quivers, Intersecting Branes and the Modest Hierarchy Problem*, JHEP 0207 (2002) 009, arXiv:hep-th/0201205. *Standard Model at Intersecting D5-branes: Lowering the String Scale*, Nucl.Phys. B643 (2002) 93-130, arXiv:hep-th/0205074. *Intersecting Brane Models of Particle Physics and the Higgs Mechanism*, JHEP 0207 (2002) 022, arXiv:hep-th/0203160.

[3] L.E. Ibanez, *Standard Model Engineering with Intersecting Branes*, Contribution to the proceedings of SUSY-01, Dubna (Russia), June 2001, arXiv:hep-ph/0109082.

[4] L. Aparicio, D.G. Cerdeno, L.E. Ibanez, *Modulus-dominated SUSY-breaking soft terms in F-theory and their test at LHC*, JHEP 0807:099, 2008, arXiv:0805.2943.

[5] Bobby S. Acharya, Konstantin Bobkov, Gordon L. Kane, Piyush Kumar, Jing Shao, *The G_2-MSSM – An M- Theory motivated model of Particle Physics*, Phys.Rev.D78:065038,2008, arXiv:0801.0478.

[6] Piyush Kumar, *Connecting String/M Theory to the Electroweak Scale and to LHC Data*, Fortsch.Phys.55:1123-1280,2008, arXiv:0706.1571.

[7] Bobby S. Acharya, Konstantin Bobkov, Gordon L. Kane, Piyush Kumar, Jing Shao, *Explaining the Electroweak Scale and Stabilizing Moduli in M Theory*, Phys.Rev.D76:126010,2007, arXiv:hep-th/0701034.

[8] Baris Altunkaynak, Phillip Grajek, Michael Holmes, Gordon Kane, Brent D. Nelson, *Studying Gaugino Mass Unification at the LHC*, arXiv:0901.1145.

[9] P. Anastasopoulos, T.P.T. Dijkstra, E. Kiritsis, A.N. Schellekens, *Orientifolds, hypercharge embeddings and the Standard Model*, Nucl.Phys.B759:83-146,2006, arXiv:hep-th/0605226.
[10] I. Antoniadis, E. Kiritsis and T. N. Tomaras, *A D-brane alternative to unification*, Phys. Lett. B 486 (2000) 186, ArXiv:hep-ph/0004214. *D-brane Standard Model*, Fortsch. Phys. 49 (2001) 573, ArXiv:hep-th/0111269.

[11] G. Aldazabal, L. E. Ibáñez, F. Quevedo and A. M. Uranga, *D-branes at singularities: A bottom-up approach to the string embedding of the standard model*, JHEP 0008 (2000) 002, ArXiv:hep-th/0005067.

[12] Z. Kakushadze, Phys. Lett. B434 (1998) 269, hep-th/9804110. Phys. Rev. D58 (1998) 101901, hep-th/9806044; Z. Kakushadze, S.H. Tye, Phys. Rev. D58 (1998) 126001, hep-th/9806143; M. Cvetic, M. Plümercher, J. Wang, JHEP 0004(2000)004, hep-th/9911021.

[13] B. Holdom, *t’ at the LHC: the physics of discovery*, JHEP0703:063,2007, arXiv:hep-ph/0702037.

[14] F. Sandin, J. Hansson, *The observational legacy of preon stars - probing new physics beyond the LHC*, Phys.Rev.D76:125006,2007, arXiv:astro-ph/0701768.

[15] Radovan Dermisek, Ian Low, *Probing the Stop Sector of the MSSM with the Higgs Boson at the LHC*, Phys.Rev.D77:035012,2008, arXiv:hep-ph/0701235.

[16] Ben Lillie, Lisa Randall, Lian-Tao Wang, *The Bulk RS KK-gluon at the LHC*, JHEP0709:074,2007, arXiv:hep-ph/0701166.

[17] D0 Collaboration: V. Abazov, et al, *Search for Large extra spatial dimensions in the dielectron and diphoton channels in p\bar{p} collisions at \sqrt{s} = 1.96 TeV*, Phys.Rev.Lett.102:051601,2009, arXiv:0809.2813.

[18] S.N. Gninenko, N.V. Krasnikov, V.A. Matveev, *Invisible Z’ as a probe of extra dimensions at the CERN LHC*, Phys.Rev.D78:097701,2008, arXiv:0811.0974.

[19] Xing-Gang Wu, Zhen-Yun Fang, *Revisiting the Real Graviton Effects at CERN LHC within the Quantum Gravity Theory with Large Extra Dimensions*, Phys.Rev.D78:094002,2008, arXiv:0810.3314.

[20] Zhi-qiang Guo, Bo-Qiang Ma, *Fermion Families from Two Layer Warped Extra Dimensions*, JHEP0808:065,2008, arXiv:0808.2136.

[21] Vyacheslav Krutelyov (for the CDF and D0 Collaborations), *Searches for Large Extra Dimensions at the Tevatron*, Proceedings of the XVI International Workshop on Deep-Inelastic Scattering and Related Subjects, DIS 2008, 7-11 April 2008, University College London, arXiv:0807.0645.
[22] J.C. Pati and A. Salam, Phys. Rev. D10, 275 (1974).

[23] T. Blazek, S. F. King and J. K. Parry, JHEP 0305, 016 (2003). hep-ph/0303192.

[24] S.L. Glashow, in Proceedings of the Fifth Workshop on Grand Unification, Editors: K.Kang, H. Fried and P.H. Frampton, World Scientific (1984). pages 88-94.

[25] K. S. Babu, E. Ma and S. Willenbrock, Phys. Rev. D 69, 051301 (2004) hep-ph/0307380. S. L. Chen and E. Ma, Mod. Phys. Lett. A 19, 1267 (2004) hep-ph/0403105. A. Demaria, C. I. Low and R. R. Volkas, Phys. Rev. D 72, 075007 (2005) [Erratum-ibid. D 73, 079902 (2006)] hep-ph/0508160. A. Demaria, C. I. Low and R. R. Volkas, Phys. Rev. D 74, 033005 (2006). hep-ph/0603152.

[26] A. Demaria and K. L. McDonald, Phys. Rev. D 75, 056006 (2007) hep-ph/0610346. K. S. Babu, T. W. Kephart and H. Pas, arXiv:0709.0765 [hep-ph].

[27] Stuart Raby, SUSY GUT Model Building, Eur.Phys.J.C59:223-247,2009, arXiv:0807.4921.

[28] D. Emmanuel-Costa, S. Wiesenfeldt, Proton Decay in a Consistent Supersymmetric SU(5) GUT Model, Nucl.Phys.B661:62-82,2003, arXiv:hep-ph/0302272. Hitoshi Nishino, Subhash Rajpoot, Standard Model and SU(5) GUT with Local Scale Invariance and the Weylon, AIP Conf.Proc.881:82-93,2007, arXiv:0805.0613.

[29] Yi Cai, Hai-Bo Yu, An SO(10) GUT Model with S4 Flavor Symmetry, Phys.Rev.D74:115005,2006, arXiv:hep-ph/0608022. J. Sayre, S. Wiesenfeldt, Higher-dimensional operators in SUSY SO(10) GUT models, Phys.Lett. B637 (2006) 295-297, arXiv:hep-ph/0603056.

[30] Howard Baer, Javier Ferrandis, Supersymmetric SO(10) GUT Models with Yukawa Unification and a Positive Mu Term, Phys.Rev.Lett. 87 (2001) 211803, arXiv:hep-ph/0106352.

[31] Paul Langacker, Jing Wang, U(1)’ Symmetry Breaking in Supersymmetric E6 Models, Phys.Rev. D58 (1998) 115010, arXiv:hep-ph/9804428. M.M. Boyce, M.A. Doncheski, H. König, Charged Heavy Lepton Production In Superstring Inspired E6 Models, Phys.Rev. D55 (1997) 68-97, arXiv:hep-ph/9607376.

[32] Chris Beasley, Jonathan J. Heckman, Cumrun Vafa, GUTs and Exceptional Branes in F-theory - I, JHEP 0901:058,2009, arXiv:0802.3391.
[33] Chris Beasley, Jonathan J. Heckman, Cumrun Vafa, *GUTs and Exceptional Branes in F-theory - II: Experimental Predictions*, arXiv:0806.0102

[34] Jonathan J. Heckman, Cumrun Vafa, *From F-theory GUTs to the LHC*, arXiv:0809.3452

[35] Joseph Marsano, Natalia Saulina, Sakura Schafer-Nameki, *Gauge Mediation in F-Theory GUT Models*, arXiv:0808.1571, *An Instanton Toolbox for F-Theory Model Building*, arXiv:0808.2450, *F-theory Compactifications for Supersymmetric GUTs*, arXiv:0904.3932, *Monodromies, Fluxes, and Compact Three-Generation F-theory GUTs*, arXiv:0906.4672

[36] Martijn Wijnholt, *F-Theory, GUTs and Chiral Matter*, arXiv:0809.3878

[37] M.R. Douglas, S. Katz, C. Vafa, *Small Instantons, del Pezzo Surfaces and Type I’ theory*, Nucl.Phys. B497 (1997) 155-172, arXiv:hep-th/9609071

[38] Duiliu-Emanuel Diaconescu, Bogdan Florea, Antonella Grassi, *Geometric Transitions, del Pezzo Surfaces and Open String Instantons*, Adv.Theor.Math.Phys. 6 (2003), 643-702, arXiv:hep-th/0206163

[39] R.Abounasr, M.Ait Ben Haddou, A.El Rhalami, E.H.Saidi, *Algebraic Geometry Realization of Quantum Hall Soliton*, J.Math.Phys. 46 (2005) 022302, arXiv:hep-th/0406036

[40] El Hassan Saidi, *Tetrahedron in F-theory Compactification*, Lab/UFR-HEP/0901, GNPHE/0901, arXiv:0907.2655

[41] Lalla Btissam Drissi, Houda Jehjouh, El Hassan Saidi, *Topological String on Toric CY3s in Large Complex Structure Limit*, Nucl.Phys.B813:315-348,2009, arXiv:0812.0526, *Non Planar Topological 3-Vertex Formalism*, Nucl.Phys.B804:307-341,2008, arXiv:0712.4249

[42] Freddy Cachazo, Nathan Seiberg, Edward Witten, *Phases of N=1 Supersymmetric Gauge Theories and Matrices*, JHEP 0302 (2003) 042, hep-th/0301006

[43] R.Dijkgraaf, C.Vafa, *A Perturbative Window into Non-Perturbative Physics*, hep-th/0208048, *Matrix Models, Topological Strings, and Supersymmetric Gauge Theories*, Nucl.Phys. B644 (2002) 3-20, hep-th/0206255

[44] S. Katz, P. Mayr, C. Vafa, *Mirror symmetry and Exact Solution of 4D N=2 Gauge Theories I*, Adv.Theor.Math.Phys. 1 (1998) 53-114, hep-th/9706110

84
[45] Malika Ait Benhaddou, El Hassan Saidi, *Explicit Analysis of Kahler Deformations in 4D N=1 Supersymmetric Quiver Theories*, Physics Letters B575(2003)100-110, [arXiv:hep-th/0307103](http://arxiv.org/abs/hep-th/0307103)

R. Ahl Laamara, M. Ait Ben Haddou, A Belhaj, L.B Drissi, E.H Saidi, *RG Cascades in Hyperbolic Quiver Gauge Theories*, Nucl.Phys. B702 (2004) 163-188, [arXiv:hep-th/0405222](http://arxiv.org/abs/hep-th/0405222)

Belhaj, A. Elfallah, E.H. Saidi, Class.Quant.Grav.16:3297-3306,1999.

[46] C.Vafa, *On N=1 Yang-Mills in 4 Dimensions*, Adv.Theor.Math.Phys. 2 (1998) 497, [hep-th/9801139](http://arxiv.org/abs/hep-th/9801139)

[47] F. Cachazo, S. Katz, C. Vafa, *Geometric Transitions and N=1 Quiver Theories*, [hep-th/0108120](http://arxiv.org/abs/hep-th/0108120)

[48] B. S. Acharya, *On Realising N=1 Super Yang-Mills in M theory*, [hep-th/0011089](http://arxiv.org/abs/hep-th/0011089)

[49] L. B Drissi, *Theorie-M sur des Varietes d’holonomie G2*, African Journal Of Mathematical Physics. 1. (2004)1-19,

[50] Jonathan J. Heckman, Cumrun Vafa, Herman Verlinde, Martijn Wijnholt, *Cascading to the MSSM*, JHEP0806:016,2008, [arXiv:0711.0387](http://arxiv.org/abs/0711.0387)

[51] M. Berkooz, M. R. Douglas, and R. G. Leigh, *Branes Intersecting at Angles*, Nucl.Phys. B 480 (1996) 265–278, [hep-th/9606139](http://arxiv.org/abs/hep-th/9606139)

[52] E. Witten, *Some Comments on String Dynamics*, hepth/9507121, *Phase Transitions In M-Theory And F-Theory*, Nucl.Phys. B471 (1996) 195-216, [arXiv:hep-th/9603150](http://arxiv.org/abs/hep-th/9603150)

[53] C. Vafa, *Evidence For F-Theory*, hepth/9602022.

[54] D. Morrison and C. Vafa, *Compactification Of F-Theory On Calabi-Yau Threefolds*, hepth/9602114.

[55] Oleg Lebedev, Hans Peter Nilles, Stuart Raby, Saul Ramos-Sanchez, Michael Ratz, Patrick K. S. Vaudrevange, Akin Wingerter, *The Heterotic Road to the MSSM with R parity*, Phys.Rev.D77:046013,2008, [arXiv:0708.2691v1](http://arxiv.org/abs/0708.2691v1)

[56] Pierre Binetruy, Sally Dawson, Ian Hinchcliffe, Marc She, *Phenomenologically viable models from Superstrings*, Nucl.Phys.B273:501,1986.

[57] R. Ahl Laamara, M. Ait Ben Haddou, A Belhaj, L.B Drissi, E.H Saidi, *RG Cascades in Hyperbolic Quiver Gauge Theories*, Nucl.Phys. B702 (2004) 163-188, [arXiv:hep-th/0405222](http://arxiv.org/abs/hep-th/0405222)
[58] M. Ait Ben Haddou, A. Belhaj, E.H. Saidi, *Classification of N=2 supersymmetric CFTs: Indefinite Series*, J.Phys. A38 (2005) 1793-1806, arXiv:hep-th/0308005
M. Ait Ben Haddou, A. Belhaj, E.H. Saidi, *Geometric Engineering of N=2 CFTs based on Indefinite Singularities: Hyperbolic Case*, Nucl.Phys. B674 (2003) 593-614, arXiv:hep-th/0307244

[59] El Hassan Saidi, *Hyperbolic Invariance in Type II Superstrings*, Talk given at IPM String School and Workshop, ISS2005, January 5-14, 2005, Qeshm Island, IRAN, arXiv:hep-th/0502176
Malika Ait Ben Haddou, El Hassan Saidi, *Hyperbolic Invariance*, arXiv:hep-th/0405251

[60] A. Belhaj, E.H. Saidi, *Non simply laced quiver gauge theories in superstrings compactifications*, Afr.J.Math.Phys.1:29-51,2004,

[61] L. Andrianopoli, R. D’Auria, S. Ferrara, *U- Duality and Central Charges in Various Dimensions, Revisited*, Int.J.Mod.Phys. A13 (1998) 431-490, arXiv:hep-th/9612105
S. Ferrara, A. Marrani, *Black Hole Attractors in Extended Supergravity*, arXiv:0708.1268
S. Ferrara, R. Kallosh and A. Strominger, *N = 2 Extremal Black Holes*, Phys. Rev. D52, 5412 (1995), hep-th/9508072.

[62] El Hassan Saidi, *Computing the Scalar Field Couplings in 6D Supergravity*, Nucl.Phys.B803:323-362,2008, arXiv:0806.3207
El Hassan Saidi, *On Black Hole Effective Potential in 6D/7D N=2 Supergravity*, Nucl.Phys.B803:235-276,2008, arXiv:0803.0827
El Hassan Saidi, *BPS and non BPS 7D Black Attractors in M-Theory on K3*, arXiv:0802.0583

[63] A. Belhaj, L. B. Drissi, E. H. Saidi, A. Segui, *N=2 Supersymmetric Black Attractors in Six and Seven Dimensions*, Nucl.Phys.B796:521-580,2008, arXiv:0709.0398
E.H Saidi, A. Segui, *Entropy of Pairs of Dual Attractors in 6D/7D*, arXiv:0803.2945

[64] A. Salam, E. Sezgin, *Supergravities in diverse dimensions*, North-Holland, World Scientific 1989