ON THE EDGE OF THE STABLE RANGE

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Abstract. We prove a general homological stability theorem for certain families of groups equipped with product maps, followed by two theorems of a new kind that give information about the last two homology groups outside the stable range. (These last two unstable groups are the ‘edge’ in our title.) Applying our results to automorphism groups of free groups yields a new proof of homological stability with an improved stable range, a description of the last unstable group up to a single ambiguity, and a lower bound on the rank of the penultimate unstable group. We give similar applications to the general linear groups of the integers and of the field of order 2, this time recovering the known stability range. The results can also be applied to general linear groups of arbitrary principal ideal domains, symmetric groups, and braid groups. Our methods require us to use field coefficients throughout.

1. Introduction

A sequence of groups and inclusions \( G_1 \hookrightarrow G_1 \hookrightarrow G_3 \hookrightarrow \cdots \) is said to satisfy homological stability if in each degree \( d \) there is an integer \( n_d \) such that the induced map \( H_d(G_{n-1}) \to H_d(G_n) \) is an isomorphism for \( n > n_d \). Homological stability is known to hold for many families of groups, including symmetric groups \([16]\), general linear groups \([17, 3, 22]\), mapping class groups of surfaces and 3-manifolds \([8, 19, 23, 12]\), diffeomorphism groups of highly connected manifolds \([6]\), and automorphism groups of free groups \([11, 10]\). Homological stability statements often also specify that the last map outside the range \( n > n_d \) is a surjection, so that the situation can be pictured as follows.

\[
\cdots \to H_d(G_{n_d-3}) \to H_d(G_{n_d-2}) \to H_d(G_{n_d-1}) \to H_d(G_{n_d}) \cong H_d(G_{n_d+1}) \cong \cdots
\]

The groups \( H_d(G_{n_d}), H_d(G_{n_d+1}), \ldots \), which are all isomorphic, are said to form the stable range. This paper studies what happens at the edge of the stable range, by which we mean the last two unstable groups \( H_d(G_{n_d-2}) \) and \( H_d(G_{n_d-1}) \). We prove a new and rather general homological stability result that gives exactly the picture above with \( n_d = 2d + 1 \). Then we prove two theorems of an entirely new kind. The first describes the kernel of the surjection \( H_d(G_{n_d-1}) \to H_d(G_{n_d}) \).

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and the second explains how to make the map $H_d(G_{n_d-2}) \to H_d(G_{n_d-1})$ into a surjection by adding a new summand to its domain. These general results hold for homology with coefficients in an arbitrary field.

We apply our general results to general linear groups of principal ideal domains (PIDs) and automorphism groups of free groups. In both cases we obtain new proofs of homological stability, recovering the known stable range for the general linear groups, and improving upon the known stable range for $\text{Aut}(F_n)$. We also obtain new information on the last two unstable homology groups for $\text{Aut}(F_n)$, $\text{GL}_n(\mathbb{Z})$ and $\text{GL}_n(\mathbb{F}_2)$, in each case identifying the last unstable group up to a single ambiguity.

Our proofs follow an overall pattern that is familiar in homological stability. We define a sequence of complexes acted on by the groups in our family, and we assume that they satisfy a connectivity condition. Then we use an algebraic argument, based on spectral sequences obtained from the actions on the complexes, to deduce the result. The connectivity condition has to be verified separately for each example, but it turns out that in our examples the proof is already in the literature, or can be deduced from it. The real novelty in our paper is the algebraic argument. To the best of our knowledge it has not been used before, either in the present generality or in any specific instances. Even in the case of general linear groups of PIDs, where our complexes are exactly the ones used by Charney in the original proof of homological stability [3] for Dedekind domains, we are able to improve the stable range obtained, matching the best known.

1.1. General results. Let us state our main results, after first establishing some necessary terminology. From this point onwards homology is to be taken with coefficients in an arbitrary field $\mathbb{F}$, unless stated otherwise.

A family of groups with multiplication $(G_p)_{p \geq 0}$ consists of a sequence of groups $G_0, G_1, G_2, \ldots$ equipped with product maps $G_p \times G_q \to G_{p+q}$ for $p, q \geq 0$, subject to some simple axioms. See section 2 for the precise definition. The axioms imply in particular that $\bigoplus_{p \geq 0} H_*(G_p)$ is a graded commutative ring. Examples include the symmetric groups, braid groups, the general linear groups of a PID, and automorphism groups of free groups.

To each family of groups with multiplication $(G_p)_{p \geq 0}$ we associate the splitting posets $SP_n$ for $n \geq 2$. If we think of $G_n$ as the group of symmetries of an ‘object of size $n$', then an element of $SP_n$ is a splitting of that object into two ordered nontrivial pieces. See section 3 for the precise definition. The stabilisation map $s_* : H_*(G_{n-1}) \to H_*(G_n)$ is the map induced by the homomorphism $G_{n-1} \to G_n$ that takes the product on the left with the neutral element of $G_1$. Our first main result is the following homological stability theorem.

**Theorem A.** Let $(G_p)_{p \geq 0}$ be a family of groups with multiplication, and assume that $|SP_n|$ is $(n-3)$-connected for all $n \geq 2$. Then the stabilisation map $s_* : H_*(G_{n-1}) \to H_*(G_n)$
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is an isomorphism for \( * \leq \frac{n-2}{2} \) and a surjection for \( * \leq \frac{n-1}{2} \). Here homology is taken with coefficients in an arbitrary field.

Theorem A overlaps with work in progress of Søren Galatius, Alexander Kupers and Oscar Randal-Williams. Indeed, if we were to add the assumption that \( \bigsqcup_{p \geq 0} G_p \) is a braided monoidal groupoid, then it would follow from the work of Galatius, Kupers and Randal-Williams. (The definition of family of groups with multiplication ensures that \( \bigsqcup_{p \geq 0} G_p \) is a monoidal groupoid; the braiding assumption holds in all of our examples.) We will mention other points of overlap as they occur.

In a given degree \( m \), Theorem A gives us the surjection and isomorphisms in the following sequence.

\[
\cdots \to H_m(G_{2m-2}) \to H_m(G_{2m-1}) \to H_m(G_{2m}) \to H_m(G_{2m+1}) \xrightarrow{\cong} H_m(G_{2m+2}) \xrightarrow{\cong} \cdots
\]

Our next two theorems extend into the edge of the stable range.

**Theorem B.** Let \( (G_p)_{p \geq 0} \) be a family of groups with multiplication, and assume that \( |SP_n| \) is \((n - 3)\)-connected for all \( n \geq 2 \). Then the kernel of the map

\[
s_* : H_m(G_{2m}) \to H_m(G_{2m+1})
\]

is the image of the product map

\[
H_1(G_2) \otimes H_1(G_3) \to H_m(G_{2m}),
\]

Here homology is taken with coefficients in an arbitrary field.

**Theorem C.** Let \( (G_p)_{p \geq 0} \) be a family of groups with multiplication, and assume that \( |SP_n| \) is \((n - 3)\)-connected for all \( n \geq 2 \). Then the map

\[
H_m(G_{2m-1}) \oplus H_1(G_2) \otimes H_m(G_{2m}) \to H_m(G_{2m+1})
\]

is surjective. Here homology is taken with coefficients in an arbitrary field.

Homological stability results like Theorem A are often combined with theorems computing the stable homology \( \lim_{n \to \infty} H_*(G_n) \) to deduce the value of \( H_*(G_n) \) in the stable range. In a similar vein, Theorems B and C allow us to bound the last two unstable groups \( H_m(G_{2m}) \) and \( H_m(G_{2m-1}) \) in terms of \( \lim_{n \to \infty} H_*(G_n) \). In the following subsections we will see how this works for automorphism groups of free groups and general linear groups of PIDs. Note that our results do not rule out the possibility of a larger stable range than the one provided by Theorem A. Nevertheless, in what follows we will refer to \( H_m(G_{2m}) \) and \( H_m(G_{2m-1}) \) as the ‘last two unstable groups’.
1.2. Applications to automorphism groups of free groups. The automorphism groups of free groups form a family of groups with multiplication \((\text{Aut}(F_n))_{n \geq 0}\). In this case the splitting poset \(SP_n\) consists of pairs \((A, B)\) of proper subgroups of \(F_n\) satisfying \(A \ast B = F_n\). By relating the splitting poset to the poset of free factorisations studied by Hatcher and Vogtmann in [9], we are able to show that \(|SP_n|\) is \((n - 3)\)-connected, so that Theorems \(A, B\) and \(C\) can be applied. Our first new result is obtained using Theorem \(A\) in arbitrary characteristic, and Theorems \(A, B\) and \(C\) in characteristic other than 2.

**Theorem D.** Let \(\mathbb{F}\) be a field. Then the stabilisation map

\[
s_*: H_*(\text{Aut}(F_{n-1}); \mathbb{F}) \to H_*(\text{Aut}(F_n); \mathbb{F})
\]

is an isomorphism for \(* \leq \frac{n-2}{2}\) and a surjection for \(* \leq \frac{n-1}{2}\). Moreover, if \(\text{char}(\mathbb{F}) \neq 2\), then \(s_*\) is an isomorphism for \(* \leq \frac{n-1}{2}\) and a surjection for \(* \leq \frac{n}{2}\).

Hatcher and Vogtmann showed in [11] that \(s_*: H_*(\text{Aut}(F_{n-1})) \to H_*(\text{Aut}(F_n))\) is an isomorphism for \(* \leq \frac{n-3}{2}\) and a surjection for \(* \leq \frac{n-2}{2}\), where homology is taken with arbitrary coefficients. Theorem \(D\) increases this stable range one step to the left in each degree when coefficients are taken in a field, and two steps to the left in each degree when coefficients are taken in a field of characteristic other than 2. (In characteristic 0 this falls far short of the best known result [10].) In particular we learn for the first time that the groups \(H_m(\text{Aut}(F_{2m+1}); \mathbb{F})\) are stable.

By applying Theorems \(B\) and \(C\) when \(\mathbb{F} = \mathbb{F}_2\), we are able to learn the following about the last two unstable groups \(H_m(\text{Aut}(F_{2m}); \mathbb{F}_2)\) and \(H_m(\text{Aut}(F_{2m-1}); \mathbb{F}_2)\).

**Theorem E.** Let \(t \in H_1(\text{Aut}(F_2); \mathbb{F}_2)\) denote the element determined by the transformation \(x_1 \mapsto x_1, x_2 \mapsto x_1x_2\), and let \(m \geq 1\). Then the kernel of the stabilisation map

\[
s_*: H_m(\text{Aut}(F_{2m}); \mathbb{F}_2) \to H_m(\text{Aut}(F_{2m+1}); \mathbb{F}_2)
\]

is the span of \(t^m\), and the map

\[
H_m(\text{Aut}(F_{2m-1}); \mathbb{F}_2) \oplus \mathbb{F}_2 \to H_m(\text{Aut}(F_{2m}); \mathbb{F}_2), \quad (x, y) \mapsto s_*(x) + y \cdot t^m
\]

is surjective.

This theorem shows that the last unstable group \(H_m(\text{Aut}(F_{2m}); \mathbb{F}_2)\) is either isomorphic to the stable homology \(\lim_{n \to \infty} H_m(\text{Aut}(F_n); \mathbb{F}_2)\), or is an extension of it by a copy of \(\mathbb{F}_2\) generated by \(t^m\). It does not state which possibility holds. Galatius [5] identified the stable homology \(\lim_{n \to \infty} H_*(\text{Aut}(F_n))\) with \(H_*(\Omega_0^\infty S^\infty)\), where \(\Omega_0^\infty S^\infty\) denotes a path-component of \(\Omega^\infty S^\infty = \text{colim}_{n \to \infty} \Omega^n S^n\). Thus we are able to place the following bounds on the dimensions of the last two unstable groups for \(m \geq 1\), where \(\epsilon\) is either 0 or 1.

\[
\dim(H_m(\text{Aut}(F_{2m}); \mathbb{F}_2)) = \dim(H_m(\Omega_0^\infty S^\infty; \mathbb{F}_2)) + \epsilon
\]
\[
\dim(H_m(\text{Aut}(F_{2m-1}); \mathbb{F}_2)) \geq \dim(H_m(\Omega_0^\infty S^\infty; \mathbb{F}_2))
\]
1.3. Applications to general linear groups of PIDs. The general linear groups of a commutative ring $R$ form a family of groups with multiplication $(GL_n(R))_{n \geq 0}$. When $R$ is a PID, the realisation $|SP_n|$ of the splitting poset is precisely the split building $[R^n]$ studied by Charney, who showed that it is $(n - 3)$-connected [3]. Theorems [A] [B] and [C] can therefore be applied in this setting.

Theorem [A] shows that $H_*(GL_{n-1}(R)) \to H_*(GL_n(R))$ is onto for $* \leq \frac{n-2}{2}$ and an isomorphism for $* \leq \frac{n-2}{2}$, where homology is taken with field coefficients. This exactly recovers homological stability with the range due to van der Kallen [22], but only with field coefficients. Theorems [B] and [C] then allow us to learn about the last two unstable groups $H_m(GL_{2m-1}(R))$ and $H_m(GL_{2m}(R))$, where little seems to be known in general. In order to illustrate this we specialise to the cases $R = \mathbb{Z}$ and $R = F_2$ and take coefficients in $F_2$; this is the content of our next two subsections.

1.4. Applications to the general linear groups of $\mathbb{Z}$. We now specialise to the groups $GL_n(\mathbb{Z})$ and take coefficients in $F_2$. Theorems [B] and [C] give us the following information about the final two unstable groups $H_m(GL_{2m}(\mathbb{Z}); F_2)$ and $H_m(GL_{2m-1}(\mathbb{Z}); F_2)$.

**Theorem F.** Let $t$ denote the element of $H_1(GL_2(\mathbb{Z}); F_2)$ determined by the matrix
$$
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
$$
and let $m \geq 1$. Then the kernel of the stabilisation map
$$s_* : H_m(GL_{2m}(\mathbb{Z}); F_2) \to H_m(GL_{2m+1}(\mathbb{Z}); F_2)
$$
is the span of $t^m$, and the map
$$
H_m(GL_{2m-1}(\mathbb{Z}); F_2) \oplus F_2 \to H_m(GL_{2m}(\mathbb{Z}); F_2), \quad (x, y) \mapsto s_*(x) + y \cdot t^m
$$
is surjective.

This theorem shows that the last unstable group $H_m(GL_{2m}(\mathbb{Z}); F_2)$ is either isomorphic to the stable homology $\lim_{n \to \infty} H_m(GL_n(\mathbb{Z}); F_2)$, or is an extension of it by a copy of $F_2$ generated by $t^m$. It does not guarantee that $t^m \neq 0$, and so does not specify which possibility occurs. The theorem also gives us the following lower bounds on the dimensions of the last two unstable groups in terms of $\dim(\lim_{n \to \infty} H_m(GL_n(\mathbb{Z}); F_2))$, and in particular shows that they are highly non-trivial.

$$
\dim(H_m(GL_{2m}(\mathbb{Z}); F_2)) = \dim \left( \lim_{n \to \infty} H_m(GL_n(\mathbb{Z}); F_2) \right) + \epsilon
$$
$$
\dim(H_m(GL_{2m-1}(\mathbb{Z}); F_2)) \geq \dim \left( \lim_{n \to \infty} H_m(GL_n(\mathbb{Z}); F_2) \right)
$$
Here $\epsilon$ is either 0 or 1.

1.5. Applications to the general linear groups of $F_2$. Now let us specialise to the groups $GL_n(F_2)$. Quillen showed that in this case the stable homology $\lim_{n \to \infty} H_*(GL_n(F_2); F_2)$ vanishes [17, Section 11]. Combining this with Maazen’s stability result shows that $H_m(GL_n(F_2); F_2) = 0$ for $n \geq 2m + 1$. It is natural to ask for a description of the final unstable homology groups $H_m(GL_{2m}(F_2); F_2)$. 

These are known to be nontrivial for \( m = 1 \) and \( m = 2 \), the latter case being due to Milgram and Priddy (Example 2.6 and Theorem 6.5 of [15]), but to the best of our knowledge nothing further is known. By applying Theorem [15] we obtain the following result, which determines each of the groups \( H_m(GL_{2m}(F_2); F_2) \) up to a single ambiguity.

**Theorem G.** Let \( t \) denote the element of \( H_1(GL_2(F_2); F_2) \) determined by the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Then \( H_m(GL_{2m}(F_2); F_2) \) is either trivial, or is a copy of \( F_2 \) generated by the class \( t^m \).

We hope that by extending the techniques of the present paper we will be able in future to prove the following conjecture. We anticipate that the known non-vanishing of \( t \) and \( t^2 \) will be an essential ingredient in its proof.

**Conjecture.** For every \( m \geq 1 \) the group \( H_m(GL_{2m}(F_2); F_2) \) is a single copy of \( F_2 \) generated by the class \( t^m \), where \( t \in H_1(GL_2(F_2); F_2) \) is the element determined by the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

A proof of this conjecture would, via the homomorphisms \( Aut(F_n) \rightarrow GL_n(Z) \rightarrow GL_n(F_2) \), also resolve the ambiguities in Theorems [15] and [14] showing that the final unstable homology groups \( H_m(Aut(F_{2m}), F_2) \) and \( H_m(GL_{2m}(Z); F_2) \) are extensions by \( F_2 \) of \( \lim_{n \rightarrow \infty} H_m(Aut(F_n); F_2) \) and \( \lim_{n \rightarrow \infty} H_m(GL_n(Z); F_2) \) respectively. In particular this would confirm that the known homological stability ranges are sharp.

Theorem [15] is relevant to questions about the groups \( H^m(GL_{2m}(F_2); F_2) \) raised by Milgram and Priddy in [15, p.301], and posed explicitly by Priddy in [2, section 5]. Let \( M_{mm} \) denote the subgroup of \( GL_{2m}(F_2) \) consisting of matrices of the form

\[
\begin{pmatrix}
I_m & * \\
0 & I_m
\end{pmatrix}.
\]

Milgram and Priddy describe an element \( \det_m \in H^m(M_{mm}; F_2) \) that is invariant under the action of \( N_{GL_{2m}(F_2)}(M_{mm})/M_{mm} = GL_m(F_2) \times GL_m(F_2) \), and so potentially lifts to an element of \( H^m(GL_{2m}(F_2); F_2) \). Priddy asks whether \( \det_m \) lifts to \( H^m(GL_{2m}(F_2); F_2) \), and if so, whether it spans \( H^m(GL_{2m}(F_2); F_2) \). As explained to us by David Sprehn, \( t^m \) is the image of a class in \( H_m(M_{mm}; F_2) \), and \( \det_m \) spans the invariants \( H^m(M_{mm}; F_2)^{GL_m(F_2) \times GL_m(F_2)} \). Theorem [15] therefore shows that \( H^m(GL_{2m}(F_2); F_2) \) is either trivial, or is a single copy of \( F_2 \) generated by a lift of \( \det_m \).

1.6. **Decomposability beyond the stable range.** Let \( (G_p)_{p \geq 0} \) be a family of groups with multiplication, and consider the bigraded commutative ring \( A = \bigoplus_{p \geq 0} H_*(G_p) \). Homological stability tells us that any element of \( H_*(G_p) \) that lies in the stable range decomposes as a product of elements in the augmentation ideal of \( A \). (In fact it tells us that such an element decomposes as a product with the generator of \( H_0(G_1) \).) We believe that connectivity bounds on the splitting
complex can yield decomposability results far beyond the stable range. The following conjecture was formulated after studying explicit computations for symmetric groups and braid groups [4], in which cases it holds.

**Conjecture.** Let \((G_p)_{p \geq 0}\) be a family of groups with multiplication. Suppose that \(|SP_n|\) is \((n - 3)\)-connected for all \(n \geq 2\). Then the map

\[
\mu: \bigoplus_{p+q=n \atop p,q \geq 1} H_*(G_p) \otimes H_*(G_q) \rightarrow H_*(G_n)
\]

is surjective in degrees \(* \leq (n - 2)\), and its kernel is the image of

\[
\alpha: \bigoplus_{p+q+r=n \atop p,q,r \geq 1} H_*(G_p) \otimes H_*(G_q) \otimes H_*(G_r) \rightarrow \bigoplus_{p+q=n \atop p,q \geq 1} H_*(G_p) \otimes H_*(G_q)
\]

in degrees \(* \leq (n - 3)\). Here \(\mu\) and \(\alpha\) are defined by \(\mu(x \otimes y) = x \cdot y\) and \(\alpha(x \otimes y \otimes z) = (x \cdot y) \otimes z - x \otimes (y \cdot z)\).

We are able to prove the surjectivity statement in degrees \(* \leq \frac{n}{2}\) and the injectivity statement in degrees \(* \leq \frac{n-1}{2}\), both of which are half a degree better than the stable range (Lemmas 11.3 and 11.4), and Theorems B and C are the ‘practical’ versions of these facts. We hope that in future work we will be able to obtain information further beyond the stable range.

### 1.7. Organisation of the paper

In the first half of the paper we introduce the concepts required to understand the statements of Theorems A, B and C and then, assuming these theorems for the time being, we give the proofs of the applications stated earlier in this introduction. Section 2 introduces families of groups with multiplication, and introduces four main examples: the symmetric groups, general linear groups of PIDs, automorphism groups of free groups, and braid groups. Section 3 introduces the splitting posets \(SP_n\) associated to a family of groups with multiplication, and identifies them in the four examples. In section 4 we show that for these four examples, the realisation \(|SP_n|\) of the splitting poset is \((n - 3)\)-connected. Finally, in section 5 we give the proofs of Theorems F, G, D and E.

In the second half of the paper we give the proofs of our three general results, Theorems A, B and C. Section 6 introduces the splitting complex, an alternative to the splitting poset that features in the rest of the argument. Section 7 introduces a graded chain complex \(B_n\) obtained from a family of groups with multiplication. In section 8 we show that, under the hypotheses of Theorems A, B and C there is a spectral sequence with \(E^1\)-term \(B_n\) and converging to 0 in total degrees \(\leq (n - 2)\). Section 9 introduces and studies a filtration on \(B_n\). The filtration allows us to understand the homology of \(B_n\) inductively within a range of degrees. Then sections 10, 11 and 12 give the proofs of the three theorems.
1.8. **Acknowledgements.** My thanks to Rachael Boyd, Anssi Lahtinen, Martin Palmer, Oscar Randal-Williams, David Sprehn and Nathalie Wahl for useful discussions.

2. **Families of groups with multiplication**

In this section we define the families of groups with multiplication to which our methods will apply, and we provide a series of examples.

**Definition 2.1.** A family of groups with multiplication \((G_p)_{p \geq 0}\) is a sequence of discrete groups \(G_0, G_1, G_2, \ldots\) equipped with a multiplication map

\[ G_p \times G_q \rightarrow G_{p+q}, \quad (g, h) \mapsto g \oplus h \]

for each \(p, q \geq 0\). We assume that the following axioms hold:

1. **Unit:** The group \(G_0\) is the trivial group, and its unique element \(e_0\) acts as a unit for left and right multiplication. In other words \(e_0 \oplus g = g = g \oplus e_0\) for all \(p \geq 0\) and all \(g \in G_p\).
2. **Associativity:** The associative law

\[ (g \oplus h) \oplus k = g \oplus (h \oplus k) \]

holds for all \(p, q, r \geq 0\) and all \(g \in G_p, h \in G_q\) and \(k \in G_r\). Consequently, for any sequence \(p_1, \ldots, p_r \geq 0\) there is a well-defined iterated multiplication map

\[ G_{p_1} \times \cdots \times G_{p_r} \rightarrow G_{p_1+\cdots+p_r}. \]

3. **Commutativity:** The product maps are commutative up to conjugation, in the sense that there exists an element \(\tau_{pq} \in G_{p+q}\) such that the squares

\[
\begin{array}{ccc}
G_p & \times & G_q \\
\approx & & \downarrow \ c_{pq} \\
G_q & \times & G_p \\
\end{array}
\]

commute, where \(c_{pq}\) denotes conjugation by \(\tau_{pq}\). (We do not impose any further conditions upon the \(\tau_{pq}\).)

4. **Injectivity:** The multiplication maps are all injective. It follows that the iterated multiplication maps are also injective. Using this, we henceforth regard \(G_{p_1} \times \cdots \times G_{p_r}\) as a subgroup of \(G_{p_1+\cdots+p_r}\) for each \(p_1, \ldots, p_r \geq 0\).

5. **Intersection:** We have

\[ (G_{p+q} \times G_r) \cap (G_p \times G_{q+r}) = G_p \times G_q \times G_r, \]

for all \(p, q, r \geq 0\), where \(G_{p+q} \times G_r, G_p \times G_{q+r}\) and \(G_p \times G_q \times G_r\) are all regarded as subgroups of \(G_{p+q+r}\).

We denote the neutral element of \(G_p\) by \(e_p\).
Remark 2.2. We could delete the intersection axiom from Definition 2.1 at the expense of working with the splitting complex of section 6 instead of the splitting poset. See Remark 6.5 for further discussion.

Example 2.3 (Symmetric groups). For \( p \geq 0 \) we let \( \Sigma_p \) denote the symmetric group on \( n \) letters. Then we may form the family of groups with multiplication \((\Sigma_p)_{p \geq 0}\), equipped with the product maps

\[
\Sigma_p \times \Sigma_q \to \Sigma_{p+q}, \quad (f, g) \mapsto f \uplus g
\]

where \( f \uplus g \) is the automorphism of \( \{1, \ldots, p+q\} \cong \{1, \ldots, p\} \cup \{1, \ldots, q\} \) given by \( f \) on the first summand and by \( g \) on the second. Then the axioms of a multiplicative family are all immediately verified. In the case of commutativity, the element \( \tau_{pq} \) is the permutation that interchanges the first \( p \) and last \( q \) letters while preserving their ordering.

Example 2.4 (General linear groups of PIDs). Let \( R \) be a PID. For \( n \geq 0 \), let \( GL_n(R) \) denote the general linear group of \( n \times n \) invertible matrices over \( R \). Then we may form the family of groups with multiplication \((GL_p(R))_{p \geq 0}\), equipped with the product maps

\[
GL_p(R) \times GL_q(R) \to GL_{p+q}(R), \quad (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}
\]

given by the block sum of matrices. The unit, associativity, commutativity, injectivity and intersection axioms all hold by inspection. In the case of commutativity, the element \( \tau_{pq} \) is the permutation matrix \( \begin{pmatrix} 0 & I_q \\ I_p & 0 \end{pmatrix} \). (It would have been enough to assume that \( R \) is a commutative ring here. However, as we will see later, we will only be able to apply our results when \( R \) is a PID.)

Example 2.5 (Automorphism groups of free groups). For \( p \geq 0 \) we let \( F_p \) denote the free group on \( p \) letters, and we let \( \text{Aut}(F_p) \) denote the group of automorphisms of \( F_p \). Then we may form the family of groups with multiplication \((\text{Aut}(F_p))_{p \geq 0}\), equipped with the product maps

\[
\text{Aut}(F_p) \times \text{Aut}(F_q) \to \text{Aut}(F_{p+q}), \quad (f, g) \mapsto f \ast g.
\]

Here \( f \ast g \) is the automorphism of \( F_{p+q} \cong F_p \ast F_q \) given by \( f \) on the first free factor and by \( g \) on the second. Then the unit, associativity and connectivity axioms all hold by inspection. In the case of commutativity, the element \( \tau_{pq} \) is the automorphism that interchanges the first \( p \) generators with the last \( q \) generators. The injectivity axiom is also clear. We prove the intersection axiom as follows. Suppose that \( f_p \ast f_{q+r} = f_{p+q} \ast f_r \) where each \( f_{\alpha} \) lies in \( \text{Aut}(F_{\alpha}) \). We would like to show that \( f_{q+r} = f_q \ast f_r \) for some \( f_q \in \text{Aut}(F_q) \). Let \( x_i \) be one of the middle \( q \) generators. Then \( f_{q+r} \) sends \( x_i \) to a reduced word in the first \( p+q \) generators and to a reduced word in the last \( q + r \) generators. Since an element of a free group has a unique reduced expression, it follows that \( x_i \) is sent to a word in the middle
generators. Thus \( f_{q+r} = f_q \ast f_r \) for some \( f_q : F_q \to F_q \). By inverting the original equation we see that in fact \( f_q \in \text{Aut}(F_q) \).

**Example 2.6** (Braid groups). Given \( p \geq 0 \), let \( B_p \) denote the braid group on \( p \) strands. This is defined to be the group of diffeomorphisms of the disk \( D^2 \) that preserve the boundary pointwise and that preserve (not necessarily pointwise) a set \( X_p \subset D^2 \) of \( p \) points in the interior of \( D^2 \), arranged from left to right, all taken modulo isotopies relative to \( \partial D^2 \) and \( X_p \).

\[
\begin{array}{c}
D^2 \\
\bullet \bullet \bullet \bullet \bullet \\
X_5
\end{array}
\]

The product maps are

\[
B_p \times B_q \to B_{p+q}, \quad (\beta, \gamma) \mapsto \beta \sqcup \gamma
\]

where \( \beta \sqcup \gamma \) denotes the braid obtained by juxtaposing \( \beta \) and \( \gamma \). More precisely, we choose an embedding \( D^2 \sqcup D^2 \to D^2 \) that embeds two copies of \( D^2 \) `side by side' in \( D^2 \), in such a way that \( X_p \sqcup X_q \) is sent into \( X_{p+q} \) preserving the left-to-right order.

\[
\begin{array}{c}
\bullet \bullet \\
\bullet \bullet \\
\end{array}
\]

Then \( \beta \sqcup \gamma \) is defined to be the map given by \( \beta \) and \( \gamma \) on the respective embedded punctured discs, and by the identity elsewhere. Then the unit, associativity and injectivity axioms are immediate. The commutativity axiom holds when we take \( \tau_{pq} \) to be the class of a diffeomorphism that interchanges the two embedded discs, passing the left one above the right. The intersection axiom follows from the fact that we may identify the subgroup \( B_p \times B_{q+r} \subseteq B_{p+q+r} \) with the set of isotopy classes of diffeomorphisms that fix an arc that cuts the disc in two, separating the first \( p \) punctures from the last \( q + r \) punctures, and similarly for \( B_{p+q} \times B_r \) and \( B_p \times B_q \times B_r \).

### 3. The splitting poset

In this section we define the splitting posets associated to a family of groups with multiplication, and identify them in the case of symmetric groups, braid groups, general linear groups of PIDs, and automorphism groups of free groups.
Conditions on the connectivity of these posets are the key assumptions in all of our main theorems.

**Definition 3.1** (The splitting poset). Let \((G_p)_{p \geq 1}\) be a family of groups with multiplication. Then for \(n \geq 2\), the \(n\)-th splitting poset \(SP_n\) of \((G_p)_{p \geq 1}\) is defined to be the set

\[
SP_n = \frac{G_n}{G_1 \times G_{n-1}} ▴ \frac{G_n}{G_2 \times G_{n-2}} ▴ \cdots ▴ \frac{G_n}{G_{n-2} \times G_2} ▴ \frac{G_n}{G_{n-1} \times G_1}
\]

equipped with the partial ordering \(\leq\) with respect to which

\[g(G_p \times G_{n-p}) \leq h(G_q \times G_{n-q})\]

if and only if \(p \leq q\) and there is \(k \in G_n\) such that

\[g(G_p \times G_{n-p}) = k(G_p \times G_{n-p}) \quad \text{and} \quad h(G_q \times G_{n-q}) = k(G_q \times G_{n-q}).\]

Lemma 3.2 below verifies that the relation \(\leq\) is transitive.

**Lemma 3.2.** Given an arbitrary chain

\[g_0(G_{p_0} \times G_{n-p_0}) \leq g_1(G_{p_1} \times G_{n-p_1}) \leq \cdots \leq g_r(G_{p_r} \times G_{n-p_r})\]

in \(SP_n\) we may assume, after possibly choosing new coset representatives, that \(g_0 = \cdots = g_r\). It follows that \(g_i(G_{p_i} \times G_{n-p_i}) \leq g_j(G_{p_j} \times G_{n-p_j})\) for any \(i \leq j\).

**Proof.** We prove by induction on \(s = 1, 2, \ldots, r\) that given an arbitrary chain \(\{g_0 = \cdots = g_s = g\}\) in \(SP_n\), we may assume, after choosing new coset representatives, that \(g_0 = \cdots = g_s = g\) for some \(g \in G_n\), the case \(s = r\) being our desired result.

When \(s = 1\), the claim is immediate from the definition of \(\leq\).

For the induction step, suppose that the claim holds for \(s\). Take an arbitrary chain \(\{g_0 = \cdots = g_s = g\}\) and use the induction hypothesis to choose new coset representatives so that \(g_0 = \cdots = g_s = g\). Since \(g(G_{p_s} \times G_{n-p_s}) \leq g_{s+1}(G_{p_{s+1}} \times G_{n-p_{s+1}})\) we may assume, after replacing \(g_{s+1}\) if necessary, that \(g(G_{p_s} \times G_{n-p_s}) = g_{s+1}(G_{p_s} \times G_{n-p_s})\).

Then there are \(\gamma \in G_{p_s}\) and \(\delta \in G_{n-p_s}\) such that \(g^{-1}g_{s+1} = \gamma \oplus \delta\). Since \(e_{p_s} \oplus \delta\) lies in \(G_{p_t} \times G_{n-p_t}\) for \(t \leq s\), we may replace \(g\) with \(g(e_{p_s} \oplus \delta)\). And since \(\gamma \oplus e_{n-p_s}\) lies in \(G_{p_{s+1}} \times G_{n-p_{s+1}}\), we may replace \(g_{s+1}\) with \(g_{s+1}(\gamma^{-1} \oplus e_{n-p_s})\). But then \(g_{s+1} = g\). So \(g_0 = \cdots = g_{s+1}\) as required. \(\square\)

Now we will identify the splitting posets associated to the symmetric groups, general linear groups of PIDs, automorphism groups of free groups, and braid groups.

**Proposition 3.3** (Splitting posets for symmetric groups). For the family of groups with multiplication \((\Sigma_p)_{p \geq 0}\), the \(n\)-th splitting poset \(SP_n\) is isomorphic to the poset of proper subsets of \(\{1, \ldots, n\}\) under inclusion.

**Proof.** We define a bijection \(\phi\) from \(SP_n\) to the poset of proper subsets of \(\{1, \ldots, n\}\) by the rule

\[\phi(g(\Sigma_p \times \Sigma_{n-p})) = \{g(1), \ldots, g(p)\}.\]
This $\phi$ is a well-defined bijection, and we must show that
\[ g(\Sigma_p \times \Sigma_{n-p}) \leq h(\Sigma_q \times \Sigma_{n-q}) \iff \{g(1), \ldots, g(p)\} \subseteq \{h(1), \ldots, h(q)\}. \]
If the first condition holds then $p \leq q$ and we may assume that $g = h$, so that the second condition follows immediately. If the second condition holds then $p \leq q$ and, replacing $h$ by $h \circ (k \times \text{Id})$ and $g$ by $g \circ (\text{Id} \times l)$ for an appropriate $k \in \Sigma_q$ and $l \in \Sigma_{n-p}$, we may assume that $g = h$, so that the first condition holds. \(\square\)

Let $R$ be a PID. To identify the splitting posets associated to the family $(\text{GL}_n(R))_{n \geq 0}$, recall that Charney in [3] defined $S_R(R^n)$ to be the poset of ordered pairs $(P, Q)$ of proper submodules of $R^n$ satisfying $P \oplus Q = R^n$, equipped with the partial order $\leq$ defined by
\[ (P, Q) \leq (P', Q') \iff P \subseteq P' \text{ and } Q \supseteq Q'. \]
Charney then defined the split building of $R^n$, denoted by $[R^n]$, to be the realisation $|S_R(R^n)|$. (Note that Charney worked with arbitrary Dedekind domains.)

**Proposition 3.4** (Splitting posets for general linear groups of PIDs). Let $R$ be a PID. For the family of groups with multiplication $(\text{GL}_n(R))_{n \geq 0}$, the splitting poset $SP_n$ is isomorphic to $S_R(R^n)$, so that $|SP_n|$ is isomorphic to the split building $[R^n]$. 

**Proof.** Define $s_1, \ldots, s_{n-1} \in SP_n$ and $t_1, \ldots, t_{n-1} \in S_R(R^n)$ by
\[ s_p = e_n(\text{GL}_p(R) \times \text{GL}_{n-p}(R)), \quad t_p = (\text{span}(x_1, \ldots, x_p), \text{span}(x_{p+1}, \ldots, x_n)), \]
where $e_n \in \text{GL}_n(R)$ denotes the identity element and $x_1, \ldots, x_n$ is the standard basis of $R^n$. Then the following three properties hold for the elements $s_i \in SP_n$, and their analogues hold for the $t_i \in S_R(R^n)$.

1. $s_1, \ldots, s_{n-1}$ are a complete set of orbit representatives for the $\text{GL}_n(R)$ action on $SP_n$.
2. The stabiliser of $s_p$ is $\text{GL}_p(R) \times \text{GL}_{n-p}(R)$.
3. $x \leq y$ if and only if there is $g \in \text{GL}_n(R)$ such that $x = g \cdot s_p$ and $y = g \cdot s_q$ where $p \leq q$.

It follows immediately that there is a unique isomorphism of posets $SP_n \to S_R(R^n)$ satisfying $s_i \mapsto t_i$ for all $i$.

The three properties hold for $s_i \in SP_n$ by definition. We prove them for $t_i \in S_R(R^n)$ as follows. For (1), the fact that $R$ is a PID guarantees that if $(P, Q) \in S_R(R^n)$ then $P$ and $Q$ are free, of ranks $p$ and $q$ say, such that $p + q = n$. If we choose bases of $P$ and $Q$ and concatenate them to form an element $A \in \text{GL}_n(R)$, then $A \cdot t_p = (P, Q)$ as required. Property (2) is immediate. For (3), suppose that $(P, Q) \leq (P', Q')$ and let $p = \text{rank}(P)$ and $p' = \text{rank}(P')$, so that $p \leq p'$. Then
\[ R^n = P \oplus (P' \cap Q) \oplus Q', \quad P \oplus (P' \cap Q) = P', \quad (P' \cap Q) \oplus Q' = Q. \]
Let $g$ denote the element of $\text{GL}_n(R)$ whose columns are given by a basis of $P$, followed by a basis of $(P' \cap Q)$, followed by a basis of $Q'$. Again this is possible
since $R$ is a PID. Then $(P, Q) = g \cdot t_p$ and $(P', Q') = g \cdot t_{p'}$ where $p \leq p'$, as required.

Let us now identify the splitting posets for automorphism groups of free groups. The situation is closely analogous to that for general linear groups. Define $S(F_n)$, for each $n \geq 2$, to be the poset of ordered pairs $(P, Q)$ of proper subgroups of $F_n$ satisfying $P \ast Q = F_n$. It is equipped with the partial order under which $(P, Q) \leq (P', Q')$ if and only if $(P, Q) = (J_0, J_1 \ast J_2) \ast (J_0 \ast J_1, J_2)$ for some proper subgroups $J_0, J_1, J_2$ of $F_n$ satisfying $J_0 \ast J_1 \ast J_2 = F_n$. (Note that the condition in the definition of $\leq$ is stronger than assuming that $P \subseteq P'$ and $Q' \supseteq Q$). The proof of the following proposition is similar to that of Proposition 3.4, and we leave the details to the reader.

**Proposition 3.5** (Splitting posets for automorphism groups of free groups). For the family of groups with multiplication $(\text{Aut}(F_n))_{n \geq 0}$, the splitting poset $S P_n$ is isomorphic to $S(F_n)$.

Let us now identify the splitting posets associated to the family $(B_p)_{p \geq 0}$ of braid groups. See Example 2.6 for the relevant notation. Given $n \geq 2$, let us define a poset $A_n$ as follows. The elements of $A_n$ are the arcs embedded in $D^2 \setminus X_n$, starting at the ‘north pole’ of the disc and ending at the ‘south pole’, such that $X_n$ meets both components of their complement, all taken modulo isotopies in $D^2 \setminus X_n$ that preserve the endpoints.

Given $\alpha, \beta \in A_n$, we say that $\alpha \leq \beta$ if $\alpha$ and $\beta$ have representatives $a$ and $b$ that meet only at their endpoints, and such that $a$ lies ‘to the left’ of $b$. (More precisely, $a$ and $b$ must meet the north pole in anticlockwise order and the south pole in clockwise order.)

Again, the proof of the following is similar to that of Proposition 3.4, and we leave it to the reader to provide details if they wish.
Proposition 3.6 (Splitting posets for braid groups). For the family of groups with multiplication \((B_p)_{p \geq 0}\), we have \(SP_n \cong A_n\).

4. Examples of connectivity of \(|SP_n|\)

Our Theorems \[A\] \[B\] and \[C\] apply to a family of groups with multiplication only when the associated splitting posets satisfy the connectivity condition that each \(|SP_n|\) is \((n - 3)\)-connected. In this section we verify this condition for our main examples: symmetric groups, where the result is elementary; general linear groups of PIDs, where the result was proved by Charney in \[3\]; automorphism groups of free groups, where we make use of Hatcher and Vogtmann’s result on the connectivity of the poset of free factorisations of \(F_n\) in \[9\]; and for braid groups, where the claim is a variant of known results on arc complexes.

Let us fix our definitions and notation for realisations of posets. If \(P\) is a poset, then its order complex (or flag complex or derived complex) \(\Delta(P)\) is the abstract simplicial complex whose vertices are the elements of \(P\), and in which vertices \(p_0, \ldots, p_r\) span an \(r\)-simplex if they form a chain \(p_0 < \cdots < p_r\) after possibly reordering. The realisation \(|P|\) of \(P\) is then defined to be the realisation \(|\Delta(P)|\) of \(\Delta(P)\). We will usually not distinguish between a simplicial complex and its realisation. So if \(P\) is a poset, then the simplicial complex \(|P|\) and topological space \(|(\Delta(P))|\) will both be denoted by \(|P|\). When we discuss topological properties of a poset or of a simplicial complex, we are referring to the topological properties of its realisation as a topological space.

4.1. Symmetric groups. The result for symmetric groups is elementary.

Proposition 4.1 (Connectivity of \(|SP_n|\) for symmetric groups). For the family of groups with multiplication \((\Sigma_p)_{p \geq 0}\) we have \(|SP_n| \cong S^{n-2}\), and in particular \(|SP_n|\) is \((n - 3)\)-connected.

Proof. Let \(\partial\Delta^{n-1}\) denote the simplicial complex given by the boundary of the simplex with vertices \(1, \ldots, n\). Then the face poset \(F(\partial\Delta^{n-1})\) of \(\partial\Delta^{n-1}\) is exactly the poset of proper subsets of \(\{1, \ldots, n\}\) ordered by inclusion. But we saw in Proposition 3.3 that the latter is isomorphic to \(SP_n\). Thus \(|SP_n| \cong |F(\partial\Delta^{n-1})| \cong |\partial\Delta^{n-1}| \cong S^{n-2}\) as required. \(\square\)

4.2. General linear groups of PIDs. Let \(R\) be a PID. In Proposition \[3.4\] we saw that for the family of groups with multiplication \((GL_p(R))_{p \geq 0}\) there is an isomorphism \(SP_n \cong S_R(R^n)\), where \(S_R(R^n)\) is the poset whose realisation is the split building \([R^n]\). Since \(R\) is in particular a Dedekind domain, Theorem 1.1 of \[3\] shows that \([R^n]\) has the homotopy type of a wedge of \((n - 2)\)-spheres. So we immediately obtain the following.

Proposition 4.2 (Connectivity of \(|SP_n|\) for general linear groups of PIDs). Let \(R\) be a PID. For the family of groups with multiplication \((GL_p(R))_{p \geq 0}\), and for any
\( n \geq 2, \ |SP_n| \) has the homotopy type of a wedge of \((n-2)\)-spheres, and in particular is \((n-3)\)-connected.

4.3. Automorphism groups of free groups. Now we give the proof of the connectivity condition on the splitting posets for automorphism groups of free groups. This is the most involved of our connectivity proofs.

**Definition 4.3.** Let \( F \) be a free group of finite rank. Define \( P(F) \) to be the poset of ordered tuples \( H = (H_0, \ldots, H_r) \) of proper subgroups of \( F \) such that \( r \geq 1 \) and \( H_0 \ast \cdots \ast H_r = F \). It is equipped with the partial order in which \( H \geq K \) if \( K \) can be obtained by repeatedly amalgamating adjacent entries of \( H \).

**Theorem 4.4.** If \( F \) has rank \( n \), then \( |P(F)| \) has the homotopy type of a wedge of \( S^{n-2} \)-spheres.

**Corollary 4.5** (Connectivity of \( |SP_n| \) for automorphism groups of free groups). For the family of groups with multiplication \( \text{Aut}(F_p)_{p \geq 0} \), the splitting poset \( |SP_n| \) has the homotopy type of a wedge of \((n-2)\)-spheres, and in particular is \((n-3)\)-connected.

This result has been obtained independently, and with the same proof, as part of work in progress by Kupers, Galatius and Randal-Williams. (See also the remarks after Theorem \([43]\))

**Proof of Corollary 4.3.** If \( P \) is a poset then we denote by \( P' \) the derived poset of chains \( p_0 < \cdots < p_r \) in \( P \) ordered by inclusion. Its realisation satisfies \( |P'| \cong |P| \).

Recall from Proposition \([35]\) that \( SP_n \) is isomorphic to the poset \( S(F_n) \) defined there. So it will suffice to show that \( P(F_n) \) is isomorphic to \( S(F_n)' \), for then \( |SP_n| \cong |S(F_n)| \cong |S(F_n)'| \cong |P(F_n)| \) and the result follows from Theorem \([14]\).

Consider the maps
\[
\lambda: P(F_n) \rightarrow S(F_n)', \quad \mu: S(F_n)' \rightarrow P(F_n)
\]
defined by
\[
\lambda\left( H_0, \ldots, H_{r+1} \right) = \left( H_0, H_1 \ast \cdots \ast H_{r+1} \right) < \cdots < \left( H_0 \ast \cdots \ast H_r, H_{r+1} \right)
\]
and
\[
\mu\left( A_0, B_0 \right) < \cdots < \left( A_r, B_r \right) = \left( A_0, A_1 \cap B_0, A_2 \cap B_1, \ldots, A_r \cap B_{r-1}, B_r \right).
\]
Then one can verify that \( \lambda \) and \( \mu \) are mutually inverse maps of posets. The verification requires one to use the fact that if \( (X_1, Y_1) < (X_2, Y_2) < (X_3, Y_3) \), then \( X_1 \ast (X_2 \cap Y_1) = X_2, Y_2 \ast (Y_1 \cap X_2) = Y_1 \) and \( (X_2 \cap Y_1) \ast (X_3 \cap Y_2) = X_3 \cap Y_1 \), which follow from the definition of the partial ordering on \( S(F_n) \).

We now move towards the proof of Theorem \([13]\). In order to do so we require another definition.
Definition 4.6. Let $F$ be a free group of finite rank. Define $Q(F)$ to be the poset of unordered tuples $H = (H_0, \ldots, H_r)$ of proper subgroups of $F$ such that $r \geq 1$ and $H_0 \ast \cdots \ast H_r = F$. Give it the partial order in which $H \succeq K$ if $K$ can be obtained by repeatedly amalgamating entries of $H$, adjacent or otherwise. Let $f: P(F) \to Q(F)$ denote the map that sends an ordered tuple to the same tuple, now unordered.

The poset $Q(F_n)$ is exactly the opposite of the poset of free factorisations of $F_n$. This poset was introduced and studied by Hatcher and Vogtmann in section 6 of [9], where it was shown that its realisation has the homotopy type of a wedge of $(n-2)$-spheres. It follows that if $F$ is a free group of rank $m$ then $|Q(F)|$ has the homotopy type of a wedge of $(m-2)$-spheres.

We will now prove Theorem 4.4 by deducing the connectivity of $|P(F)|$ from the known connectivity of $|Q(F)|$. In order to do this we will use a poset fibre theorem due to Björner, Wachs and Welker [1]. Let us recall some necessary notation. Given a poset $P$ and an element $p \in P$, we define $P_{<p}$ to be the poset $\{q \in P \mid q < p\}$. We define $P_{<p}$, $P_{>p}$ and $P_{=}p$ similarly. The length $\ell(P)$ of a poset $P$ is defined to be the maximum $\ell$ such that there is a chain $p_0 < p_1 < \cdots < p_\ell$ in $P$; the length of the empty poset is defined to be $-1$. Theorem 1.1 of [1] states that if $f: P \to Q$ is a map of posets such that for all $q \in Q$ the fibre $|f^{-1}Q_{<q}|$ is $\ell(f^{-1}Q_{<q})$-connected, then so long as $|Q|$ is connected, we have

$$|P| \simeq |Q| \vee \bigvee_{q \in Q} |f^{-1}Q_{<q}| \ast |Q_{>q}|$$

where $\ast$ denotes the join. See the introduction to [1] for further details.

Proof of Theorem 4.4. The proof is by induction on the rank of $F$. When rank$(F) = 2$ we need only observe that $P(F)$ is an infinite set with trivial partial order, so that $|P(F)|$ is an infinite discrete set, and in particular is a wedge of 0-spheres. Suppose now that rank$(F) \geq 3$ and that the claim holds for all free groups of smaller rank than $F$. Since rank$(F) \geq 3$, $|Q(F)|$ is connected. Suppose that $H = (H_0, \ldots, H_{r_H}) \in Q(F)$. Then Lemmas 4.7, 4.8, 4.9 and 4.10 below tell us the following.

- $\ell(f^{-1}(Q(F)_{<H})) = r_H - 2$
- $|f^{-1}(Q(F)_{<H})| \simeq S^{r_H - 1}$
- $|Q(F)_{>H}| \simeq \bigvee S^{n-r_H-2}$
Since \( S^{r-1} \) is \((r - 2)\)-connected, we may apply the theorem of Björner, Wachs and Welker, which tells us that

\[
|P(F)| \simeq |Q(F)| \lor \bigvee_{H \in Q(F)} \left( |f^{-1}(Q(F)_{< H})| \ast |Q(F)_{> H}| \right).
\]

\[
\simeq \bigvee S^{n-2} \lor \bigvee_{H \in Q(F)} \left( \left( \bigvee S^{n-r-2} \ast S^{r-1} \right) \right)
\]

\[
\simeq \bigvee S^{n-2} \lor \bigvee_{H \in Q(F)} \left( S^{n-r-2} \ast S^{r-1} \right)
\]

\[
\simeq \bigvee S^{n-2} \lor \bigvee_{H \in Q(F)} S^{n-2}
\]

\[
\simeq \bigvee S^{n-2}
\]

as required. \( \square \)

**Lemma 4.7.** Let \( F \) be a free group of finite rank and let \( H = (H_0, \ldots, H_r) \in Q(F) \). Then \( \ell(f^{-1}(Q(F)_{< H})) = r - 2 \).

**Proof.** After fixing an ordering of the tuple \( H \), one can amalgamate \((r - 1)\) adjacent entries before obtaining a 2-tuple. This shows that \( \ell(f^{-1}(Q(F)_{< H})) = r - 1 \). Since any maximal chain must include \( H \) itself (with some ordering) it follows that \( \ell(f^{-1}(Q(F)_{< H})) = r - 2 \). \( \square \)

**Lemma 4.8.** Let \( F \) be a free group of finite rank and let \( H = (H_0, \ldots, H_r) \in Q(F) \). Then \( |f^{-1}(Q(F)_{< H})| \simeq S^{r-1} \).

**Proof.** The poset \( f^{-1}(Q(F))_{< H} \) is the subposet of \( P(F) \) consisting of tuples \( K = (K_0, \ldots, K_s) \) where each \( K_j \) is an amalgamation of some of the \( H_i \). It is isomorphic to the poset \( X_r \) of sequences \( F = (F_0 \subset F_1 \subset \cdots \subset F_{s-1}) \) of proper subsets of \( \{0, \ldots, r\} \), where \( F' \preceq F \) if \( F' \) can be obtained from \( F \) by forgetting terms of the sequence. The isomorphism

\[
X_r \xrightarrow{\cong} f^{-1}(Q(F))_{< H}
\]

sends \( F = (F_0 \subset \cdots \subset F_{s-1}) \) to \( K = (K_0, \ldots, K_s) \) where for \( i \leq s - 1 \), \( K_j \) is the subgroup generated by the \( H_i \) for \( i \in F_j \setminus F_{j-1} \), and where \( K_s \) is the subgroup generated by the \( H_j \) for \( j \notin \{F_{s-1}\} \). Now \( X_n \) is isomorphic to the poset of faces of the barycentric subdivision of \( \partial \Delta^{r} \), as we see by identifying \( F_0 \subset \cdots \subset F_{s-1} \) with the face whose vertices are the barycentres of the simplices spanned by the \( F_i \). So \( |X_n| \simeq \partial \Delta^r \simeq S^{r-1} \) as claimed. \( \square \)

Before stating the next lemma we introduce some notation. Given a poset \( P \), let \( CP \) denote the poset obtained by adding a new minimal element \(-\).
Lemma 4.9. Let $F$ be a free group of finite rank and let $H = (H_0, \ldots, H_r) \in Q(F)$. Then

\[ \left| Q(F)_{> H} \right| \cong \left| Q(H_0) \right| \ast \cdots \ast \left| Q(H_r) \right|. \]

Proof. There is an isomorphism

\[ Q(F)_{> H} \cong CQ(H_0) \times \cdots \times CQ(H_r). \]

It simply takes a tuple $K = (K_0, \ldots, K_s)$ and sends it to the element of $CQ(H_0) \times \cdots \times CQ(H_r)$ whose $CQ(H_i)$-component is the tuple consisting of those $K_j$ which are contained in $H_i$ if there are more than one such, and which is otherwise, in which case $H_i$ itself appears as one of the $K_j$. This isomorphism identifies $H$ itself with the tuple $(-, \ldots, -)$, so that we obtain a restricted isomorphism

\[ Q(F)_{> H} \cong CQ(H_0) \times \cdots \times CQ(H_r) \setminus (-, \ldots, -). \]

Now the realisation of the right hand side is exactly $|Q(H_0)| \ast \cdots \ast |Q(H_r)|$, so the result follows. \(\square\)

Lemma 4.10. Let $F$ be a free group of finite rank and let $H = (H_0, \ldots, H_r) \in Q(F)$. Then $|Q(H_0)| \ast \cdots \ast |Q(H_r)|$ has the homotopy type of a wedge of $(n-r-2)$-spheres.

Proof. Write $s_i$ for the rank of $H_i$, so that $|Q(H_i)|$ has the homotopy type of a wedge of $(s_i - 2)$-spheres. Since wedge sums commute with joins up to homotopy equivalence, it follows that $|Q(H_0)| \ast \cdots \ast |Q(H_r)|$ has the homotopy type of a wedge of copies of $S^{s_0 - 2} \ast \cdots \ast S^{s_r - 2}$. But then

\[ S^{s_0 - 2} \ast \cdots \ast S^{s_r - 2} \cong S^{(s_0 - 2) + \cdots + (s_r - 2) + r} = S^{(s_0 + \cdots + s_r) - 2(r + 1) + r} = S^{n - r - 2} \]

as required. \(\square\)

4.4. Braid groups. Now we investigate the connectivity of the realisations of the splitting posets for braid groups. In this case we will appeal to well-known connectivity results for complexes of arcs.

Proposition 4.11 (Connectivity of $|SP_n|$ for braid groups). For the family of groups with multiplication $(B_n)_{n \geq 0}$, and for any $n \geq 2$, $|SP_n|$ has the homotopy type of a wedge of $(n-2)$-spheres, and in particular is $(n-3)$-connected.

Proof. Recall from Proposition [3.5] that we identified $SP_n$ with the poset of arcs $A_n$ defined there. Thus $|A_n|$ is (the realisation of) the simplicial complex with vertices the elements of $A_n$, in which vertices $\alpha_0, \ldots, \alpha_r$ span a simplex if and only if, after possibly reordering, $\alpha_0 < \cdots < \alpha_r$. Now $\alpha_0 < \cdots < \alpha_r$ holds if and only if the $\alpha_i$ have representatives $a_i$ that are disjoint except at their endpoints, and such that $a_0, \ldots, a_r$ meet the north pole in anticlockwise order. Thus $|A_n|$ is the realisation of the simplicial complex whose vertices are isotopy classes of nontrivial (they do not separate a disc from the remainder of the surface) arcs in $D^2 \setminus X_n$ from the north pole to the south, where a collection of vertices form a simplex if they have
representing arcs that can be embedded disjointly except at their endpoints. In the notation of section 4 of [23], this is exactly the complex $\mathcal{B}(S, \Delta_0, \Delta_1)$ where $S = D^2 \setminus X_n, \Delta_0 \subset \partial D^2$ is the set containing just the north pole, and $\Delta_1 \subset \partial D^2$ is the set containing just the south pole. Now, replacing $S$ with the complement of $n$ open discs in $D^2$ does not change the isomorphism type of the complex. But in that case, Lemma 4.7 of [23] applies to show that $|A_n|$ has connectivity $(n - 2)$ greater than that of $|A_2|$, which is $(-1)$-connected since it is a simply a nonempty set.

5. Proofs of the applications

In this section we will assume that Theorems A, B and C hold, and we will prove the remaining theorems stated in the introduction. We begin with three closely analogous lemmas about the groups $GL_n(\mathbb{Z}), GL_n(\mathbb{F}_2)$ and Aut($F_n$).

Lemma 5.1. Define elements of $GL_n(\mathbb{Z}), n = 1, 2, 3$ as follows

$$s_1 = (-1), \quad s_2 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

Use the same symbols to denote the corresponding elements of $H_1(GL_1(\mathbb{Z}); \mathbb{Z}) = GL_n(\mathbb{Z})_{ab}$. Then the $H_1(GL_n(\mathbb{Z}); \mathbb{Z})$ for $n = 1, 2, 3$ are elementary abelian 2-groups with generators $s_1 \in H_1(GL_1(\mathbb{Z}); \mathbb{Z}), s_2, t \in H_1(GL_2(\mathbb{Z}); \mathbb{Z})$ and $s_3 \in H_1(GL_3(\mathbb{Z}); \mathbb{Z})$, and the stabilisation maps have the following effect.

$$H_1(GL_1(\mathbb{Z}); \mathbb{Z}) \xrightarrow{s^*} H_1(GL_2(\mathbb{Z}); \mathbb{Z}) \xrightarrow{s^*} H_1(GL_3(\mathbb{Z}); \mathbb{Z})$$

Proof. There are split extensions

$$SL_n(\mathbb{Z}) \longrightarrow GL_n(\mathbb{Z}) \longrightarrow \{\pm 1\}$$

with section determined by $-1 \mapsto s_n$, so that we have isomorphisms

$$H_1(GL_n(\mathbb{Z}); \mathbb{Z}) \cong H_1(SL_n(\mathbb{Z}); \mathbb{Z})_{\{\pm 1\}} \oplus \mathbb{Z}/2\mathbb{Z},$$

where $\mathbb{Z}/2\mathbb{Z}$ is generated by the class of $s_n$. This isomorphism respects the stabilisation maps. Now $H_1(SL_1(\mathbb{Z}); \mathbb{Z})$ obviously vanishes, and $H_1(SL_3(\mathbb{Z}); \mathbb{Z})$ vanishes since $SL_n(\mathbb{Z})$ is perfect for $n \geq 3$. So it suffices to show that $H_1(SL_2(\mathbb{Z}); \mathbb{Z})_{\{\pm 1\}}$ is a group of order 2 generated by $t$.

Let us write

$$u = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Then $H_1(SL_2(\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/12\mathbb{Z}$, where $v \leftrightarrow 3$ and $u \leftrightarrow 2$ [13 p.91]. One can verify that $s_2vs_2^{-1} = v^{-1}$ and $s_2us_2^{-1} = v^{-1}u^{-1}v$, so that $\{\pm 1\}$ acts on $H_1(SL_2(\mathbb{Z}))$
by negation. Consequently $H_1(SL_2(\mathbb{Z}); \mathbb{Z})_{\{\pm 1\}} = (\mathbb{Z}/12\mathbb{Z})_{\{\pm 1\}}$ has order 2 with generator $t = vu$ as required. □

**Lemma 5.2.** $H_1(GL_n(\mathbb{F}_2)) = GL_n(\mathbb{F}_2)_{ab}$ is trivial for $n = 1, 3$, and is generated by the element $t$ determined by the matrix $(0 \ 1 \ \ 1 \ 0)$ for $n = 2$.

**Proof.** For $n = 1$ this is trivial, and for $n = 3$ it follows from the fact that $GL_3(\mathbb{F}_2) = SL_3(\mathbb{F}_2)$ is perfect. For $n = 2$, we simply observe that $GL_2(\mathbb{F}_2)$ is a dihedral group of order 6 generated by the involutions

$$
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
$$

so that the abelianization is a group of order 2 generated by either of the involutions. □

**Lemma 5.3.** Define elements of $Aut(F_n)$, $n = 1, 2, 3$ as follows. For $n = 1, 2, 3$ let $s_i$ denote the transformation that inverts the first letter and fixes the others. And let $t \in Aut(F_2)$ denote the transformation $x_1 \mapsto x_1$, $x_2 \mapsto x_1x_2$. Use the same symbols to denote the corresponding elements of $H_1(Aut(F_n); \mathbb{Z}) = Aut(F_n)_{ab}$. Then the $H_1(Aut(F_n); \mathbb{Z})$ for $n = 1, 2, 3$ are elementary abelian 2-groups with generators $s_1 \in H_1(Aut(F_1); \mathbb{Z})$, $s_2, t \in H_1(Aut(F_2); \mathbb{Z})$ and $s_3 \in H_1(Aut(F_3); \mathbb{Z})$, and the stabilisation maps have the following effect.

$$
H_1(Aut(F_1); \mathbb{Z}) \xrightarrow{s_1} H_1(Aut(F_2); \mathbb{Z}) \xrightarrow{s_2} H_1(Aut(F_3); \mathbb{Z})
$$

$\shortleftarrow t \quad \shortrightarrow 0$

**Proof.** The linearisation map $Aut(F_n) \to GL_n(\mathbb{Z})$ is an isomorphism on abelianisations for all $n$. In the case $n = 1$ this is because the map itself is an isomorphism. In the case $n = 2$ this is because the map $Out(F_2) \to GL_2(\mathbb{Z})$ is an isomorphism, so there is an extension $F_2 \to Aut(F_2) \to GL_2(\mathbb{Z})$ in which the action of $GL_2(\mathbb{Z})$ on $(F_2)_{ab} = \mathbb{Z}^2$ is the tautological one, so that the coinvariants $(F_2)_{ab}^{GL_2(\mathbb{Z})}$ vanish, and the claim follows. And for $n \geq 3$ this is because $SL_n(\mathbb{Z})$ is perfect, as is the subgroup $SA_n$ of $Aut(F_n)$ consisting of automorphisms with determinant one. (For the last claim we refer to the presentation of $SA_n$ given in Theorem 2.8 of [1].) The linearisation map sends the generators $s_1, s_2, s_3, t$ listed here to the corresponding generators from Lemma 5.1, so the claim follows. □

**Proof of Theorem [7]** Let $\mathbb{F}$ be a field of characteristic 2. We will use the Künneth isomorphism $H_1(\_\_ ; \mathbb{F}) \cong H_1(\_\_ ; \mathbb{Z}) \otimes \mathbb{F}$ without further mention. Theorem [3] states that the kernel of the map

$$s_*: H_m(GL_{2m}(\mathbb{Z}); \mathbb{F}) \to H_m(GL_{2m+1}(\mathbb{Z}); \mathbb{F})$$

is the image of the product map

$$H_1(GL_2(\mathbb{Z}); \mathbb{F})^{\otimes m-1} \otimes \ker[H_1(GL_2(\mathbb{Z}); \mathbb{F}) \to H_1(GL_3(\mathbb{Z}); \mathbb{F})] \to H_m(GL_{2m}(\mathbb{Z}); \mathbb{F}).$$
By Lemma 5.1, \( H_1(GL_2(\mathbb{Z}); \mathbb{F}) \) is spanned by the classes \( s_2 \) and \( t \), and \( \ker[H_1(GL_2(\mathbb{Z}); \mathbb{F}) \xrightarrow{\sim} H_1(GL_2(\mathbb{Z}); \mathbb{F})] \) is spanned by \( t \). Any product involving both \( s_2 \) and \( t \) vanishes, since \( s_2 \cdot t = s_3(s_1) \cdot t = s_2 \cdot s_4(t) = 0 \). So it follows that the image of the given product map is precisely the span of \( t^m \), which gives us the claimed description of kernel of (2). Next, Theorem C states that the map

\[
H_m(GL_{2m-1}(\mathbb{Z}); \mathbb{F}) \oplus H_1(GL_2(\mathbb{Z}); \mathbb{F})^\otimes m \twoheadrightarrow H_m(GL_{2m}(\mathbb{Z}); \mathbb{F})
\]

is surjective. The second summand of the domain is spanned by the words in \( s_2 \) and \( t \), but the image of any word involving \( s_2 = s_3(s_1) \) lies in the image of \( H_m(GL_{2m-1}(\mathbb{Z}); \mathbb{F}) \). Thus the image of the given map is in fact spanned by the image of \( H_m(GL_{2m-1}(\mathbb{Z}); \mathbb{F}) \) and of \( t^m \), as required. \( \square \)

**Proof of Theorem E.** Since \( H_m(GL_{2m+1}(\mathbb{F}_2); \mathbb{F}_2) \) vanishes, Theorem E shows that \( H_m(GL_{2m}(\mathbb{F}_2); \mathbb{F}_2) \) is spanned by the image of

\[
H_1(GL_2(\mathbb{F}_2); \mathbb{F}_2)^\otimes (m-1) \otimes \ker[s_*: H_1(GL_2(\mathbb{F}_2); \mathbb{F}) \rightarrow H_1(GL_2(\mathbb{F}_2); \mathbb{F})].
\]

But by Lemma 5.2, this image is precisely the span of \( t^m \). \( \square \)

**Proof of Theorem C.** The first claim is immediate from Theorem A. For the second claim, when \( \text{char}(\mathbb{F}) \neq 2 \) we have \( H_1(\text{Aut}(\mathbb{F}_2); \mathbb{F}) = 0 \) by Lemma 5.3, so that Theorem B shows that \( s_*: H_*(G_{n-1}) \rightarrow H_*(G_n) \) is injective for \( * = \frac{n-1}{2} \), and Theorem C shows that \( s_*: H_*(G_{n-1}) \rightarrow H_*(G_n) \) is surjective for \( * = \frac{n}{2} \). \( \square \)

**Proof of Theorem D.** This is entirely analogous to the proof of Theorem E, this time making use of Lemma 5.3. \( \square \)

6. The splitting complex

In this section we identify the realisation of the splitting poset \( SP_n \) with the realisation of a semisimplicial set that we call the ‘splitting complex’. It is the splitting complex, rather than the splitting poset, that will feature in our arguments from this section onwards. In this section we will make use of semisimplicial sets; see section 2 of [18] for a general discussion of semisimplicial sets (and spaces) and their realisations.

We have borrowed the name ‘splitting complex’ from work in progress of Galatius, Kupers and Randal-Williams. See also the remarks after Theorem A.

**Definition 6.1 (The splitting complex).** Let \( n \geq 2 \). The **\( n \)-th splitting complex** of a family of groups with multiplication \( (G_p)_{p \geq 0} \) is the semisimplicial set \( SC_n \) defined as follows. Its set of \( r \)-simplices is

\[
(SC_n)_r = \bigcup_{q_0+\cdots+q_{r+1}=n} G_{q_0} \times \cdots \times G_{q_{r+1}}
\]

if \( r \leq n-2 \), and is empty otherwise. And the \( i \)-th face map

\[
d_i: (SC_n)_r \rightarrow (SC_n)_{r-1},
\]
Figure 1. The splitting complex $SC_4$

is defined by

$$d_i(g(G_{q_0} \times \cdots \times G_{q_{r+1}})) = g(G_{q_0} \times \cdots \times G_{q_i+1} \times \cdots \times G_{q_{r+1}})$$

for $g \in G_n$.

**Example 6.2.** Figure 1 illustrates the splitting complex $SC_4$. Taking the disjoint union of the terms in each column gives the 0-, 1- and 2-simplices. And the arrows leaving each term represent the face maps on that term, ordered from top to bottom.

**Remark 6.3.** In the expression $G_{q_0} \times \cdots \times G_{q_{r+1}}$ appearing in Definition 6.1, we can imagine the symbols $\times$ as being labelled from $0, \ldots, r$, so that the $i$-th face map $d_i$ simply ‘erases the $i$-th $\times$’.

Let $P$ be a poset. The **semisimplicial nerve** $NP$ of $P$ is defined to be the semisimplicial set whose $r$-simplices are the chains $p_0 < \cdots < p_{r}$ of length $(r + 1)$ in $P$, and whose face maps are defined by $d_i(p_0 < \cdots < p_{r}) = p_0 < \cdots \hat{p_i} \cdots < p_{r}$. The realisation $\|NP\|$ of the semisimplicial nerve is naturally homeomorphic to the realisation $|P|$ of the poset.

**Proposition 6.4.** Let $(G_p)_{p \geq 0}$ be a family of groups with multiplication and let $n \geq 2$. Then $SC_n \cong N(SP_n)$. In particular $|SP_n| \cong \|SC_n\|$.

**Proof.** Let $\phi: SC_n \to N(SP_n)$ denote the map that sends an $r$-simplex $g(G_{q_0} \times \cdots \times G_{q_{r+1}})$ of $SC_n$ to the $r$-simplex

$$g(G_{q_0} \times G_{q_1+\cdots+q_{r+1}}) < g(G_{q_0+q_1} \times G_{q_2+\cdots+q_{r+1}}) < \cdots < g(G_{q_0+\cdots+q_r} \times G_{q_{r+1}})$$
of $N(SP_n)$. One can verify that $\phi$ is indeed a semi-simplicial map. Surjectivity follows from Lemma 3.2. Injectivity follows from the fact that
\[
\bigcap_{i=0}^{r} G_{q_0+\ldots+q_i} \times G_{q_{i+1}+\ldots+q_{r+1}} = G_{q_0} \times \ldots \times G_{q_{r+1}},
\]
which follows by induction from the intersection axiom. □

Remark 6.5 (Splitting posets or splitting complexes?). The results of this section show that if we wish we could replace $|SP_n|$ with $\|SC_n\|$ in the statements of Theorems A, B and C. In doing so, we could jettison the intersection axiom from Definition 2.1 possibly admitting more examples in the process. However, it is arguably simpler to work with the splitting poset, and that was certainly the case in sections 3 and 4 where we studied specific examples. Moreover, the examples of interest to us here all satisfy the intersection axiom. We therefore decided to write our paper with splitting posets at the forefront.

7. A bar construction

In this section we introduce a variant of the bar construction which takes as its input an algebra like $\bigoplus_{p \geq 0} H_*(G_p)$ and produces a graded chain complex (that is, a chain complex of graded vector spaces) called $B_n$. We will see in the next section that $B_n$ is the $E_1$-term of the spectral sequence around which all of our proofs revolve. We fix a field $F$ throughout.

For the purposes of this section we fix a field $F$ and a commutative graded $F$-algebra $A$ equipped with an additional grading that we call the charge. Thus
\[
A = \bigoplus_{p \geq 0} A_p
\]
where $A_p$ is the part of $A$ with charge $p$. We will call the natural grading of $A$ the topological grading, and we will suppress it from the notation wherever possible. We require that the multiplication on $A$ respects the charge grading, and that each charge-graded piece $A_p$ is concentrated in non-negative degrees. We further require that $A_0$ is a copy of $F$ concentrated in topological degree 0 and (necessarily) generated by the unit element 1. In particular, $A$ is augmented. Finally we assume that $(A_1)_0$, the part of $A$ of charge 1 and topological degree 0, is a copy of $F$ generated by an element $\sigma$.

Example 7.1. Our only examples of such algebras will be
\[
A = \bigoplus_{p \geq 0} H_*(G_p)
\]
where $(G_p)_{p \geq 0}$ is a family of groups with multiplication. Here the topological grading is the grading of homology, and the charge grading is obtained from the multiplicative family. The element $\sigma \in (A_1)_0 = H_0(G_1)$ is defined to be the standard generator.
Definition 7.2 (The chain complex $\mathcal{B}_n$). Let $A$ be an $F$-algebra as described at the start of the section. For $n \geq 0$ we define $\mathcal{B}_n$ to be the chain complex of graded abelian groups whose $b$-th term is

$$(\mathcal{B}_n)_b = \bigoplus_{q_0 + \cdots + q_b = n, q_1, \ldots, q_b \geq 1} A_{q_0} \otimes \cdots \otimes A_{q_b}$$

and whose differential is defined by

$$d_b(x_0 \otimes \cdots \otimes x_b) = \sum_{i=0}^{b-1} (-1)^i x_0 \otimes \cdots \otimes x_i \cdot x_{i+1} \otimes \cdots \otimes x_b.$$}

For $n = 0$ we define $\mathcal{B}_0$ by letting all groups vanish except for $(\mathcal{B}_0)_0$, which consists of a single copy of $F$.

Note that $\mathcal{B}_n$ is bigraded. Its homological grading is the grading that is explicit in the definition, and which is reduced by the differential $d_b$. Its topological grading is the grading obtained from the topological grading of $A$, and is preserved by the differential $d_b$. We say that the part of $\mathcal{B}_n$ with homological grading $b$ and topological grading $d$ lies in bidegree $(b, d)$. We reserve the notation $\mathcal{B}_n^b$ for the part of $\mathcal{B}_n$ that lies in homological degree $b$.

Remark 7.3 (\(\mathcal{B}_n\) and the bar complex). Regarding $F$ as a left and right $A$-module via the projection $A \rightarrow (A_0)_0 = F$, we may form the two-sided normalised bar complex $B(F, \bar{A}, F)$

$$F \otimes F \leftarrow F \otimes \bar{A} \otimes F \leftarrow F \otimes \bar{A} \otimes \bar{A} \otimes F \leftarrow F \otimes \bar{A} \otimes \bar{A} \otimes \bar{A} \otimes F \leftarrow \cdots$$

or, more simply,

$$F \leftarrow \bar{A} \leftarrow \bar{A} \otimes \bar{A} \leftarrow \bar{A} \otimes \bar{A} \otimes \bar{A} \leftarrow \cdots$$

where all tensor products are over $F$. This is naturally trigraded: there is the homological grading explicit in the the expressions above, together with charge and topological gradings inherited from $A$. Writing $[B(F, \bar{A}, F)]_{\text{charge}=n}$ for the homogeneous piece with charge grading $n$ inherited from $A$, then we have the following:

$$(\mathcal{B}_n)_b = [B(F, \bar{A}, F)_{b+1}]_{\text{charge}=n}.$$
Example 7.4. Here is a diagram of $\mathcal{B}_4$.

![Diagram of $\mathcal{B}_4$]

The first column of the diagram represents $(\mathcal{B}_4)_0$, the direct sum of the terms in the next column represent $(\mathcal{B}_4)_1$, and so on. The effect of the differential $d_b$ on an element of one of the summands is the alternating sum (taken from top to bottom) of its images under the arrows exiting that summand. The arrows are all constructed using the product of $A$ in the evident way.

8. The spectral sequence

The complex $\mathcal{B}_n$ is our main tool in proving the theorems stated in the introduction. The aim of the present section is to prove the following result, which demonstrates the connection between $\mathcal{B}_n$ and the splitting poset. Throughout this section we fix a family of groups with multiplication $(G_p)_{p \geq 0}$ and the algebra $A = \bigoplus H_*(G_p)$, which is of the kind described at the start of section 7. Throughout this section homology is to be taken with coefficients in an arbitrary field $F$.

**Theorem 8.1.** Let $(G_p)_{p \geq 0}$ be a family of groups with multiplication that satisfies the connectivity axiom, and let $A = \bigoplus_{p \geq 0} H_*(G_p)$. Then there is a first quadrant spectral sequence with $E^1$-term

$$(E^1, d^1) = (\mathcal{B}_n, d_b)$$

for which $E^\infty$ vanishes in bidegrees $(b,d)$ satisfying $b + d \leq (n - 2)$.

**Remark 8.2** (The spectral sequence and Tor). In Remark 7.3, we identified $\mathcal{B}_n$ in terms of a two-sided bar complex. It follows that we may therefore identify the $E^2$-term of the above spectral sequence in terms of a Tor group:

$$E^2_{i,j} = \text{Tor}^A_{i+1}(F,F)_{\text{charge}=n, \text{topological}=j}$$

This observation potentially allows us to use the machinery of derived functors to understand the $E^2$-term of our spectral sequence. We do not do this in the present version of this paper. Instead, our arguments are all done explicitly on the level of $\mathcal{B}_n$ itself. We hope that in a future version of this paper we will rephrase our arguments in terms of Tor wherever possible.
The rest of the section is devoted to the proof of Theorem 8.1. To begin, we introduce a topological analogue of $B_n$. Observe that the multiplication map $G_a \times G_b \to G_{a+b}$ induces a map of classifying spaces $BG_a \times BG_b \to BG_{a+b}$. We call it the *product map on classifying spaces* and denote it by $(x, y) \mapsto x \cdot y$. We will use the product maps on classifying spaces to create an augmented semisimplicial space from which we can recover $B_n$. See section 2 of [18] for conventions about semisimplicial spaces, augmented semisimplicial spaces, and their realisations.

**Definition 8.3** (The augmented semisimplicial space $tB_n$). Given a family of groups with multiplication $(G_p)_{p \geq 0}$, and given $n \geq 2$, we let $tB_n$ denote the augmented semisimplicial set whose set of $r$-simplices is given by

$$(tB_n)_r = \bigsqcup_{\begin{subarray}{c}q_0, \ldots, q_{r+1} \geq 1 \\ q_0 + \cdots + q_{r+1} = n \end{subarray}} BG_{q_0} \times \cdots \times BG_{q_{r+1}}$$

for $r = -1, \ldots, (n-2)$, and which is empty otherwise. The face map $d_i : (tB_n)_r \to (tB_n)_{r-1}$ is defined by

$$d_i(x_0, \ldots, x_{r+1}) = (x_0, \ldots, x_i \cdot x_{i+1}, \ldots, x_{r+1}),$$

where $\cdot$ denotes the product map on classifying spaces.

**Example 8.4.** Here is a diagram of $tB_4$.

The four columns correspond to the $r$-simplices of $tB_4$ for $r = -1, 0, 1, 2$ respectively, the disjoint union of the terms in a column being the space of simplices of the relevant dimension.

The next proposition shows the sense in which $tB_n$ is a topological analogue of $B_n$.

**Proposition 8.5** (From $tB_n$ to $B_n$). There is a spectral sequence with $E_1$-term

$$(E^1, d^1) = (B_n, d_b)$$

and converging to $H_*(||tB_n||)$. 

\[ \text{Diagram here with BG's and product maps} \]
Proof. As in section 2.3 of [18], but with a shift of grading, the augmented semisimplicial set $t\mathcal{B}_n$ gives rise to a spectral sequence, converging to $H_\ast(||t\mathcal{B}_n||)$, and whose $E^1$-term is given by

$$E^1_{s,t} = H_t((t\mathcal{B}_n)_{s-1}),$$

with $d^1$ given by the alternating sum of the maps induced by the face maps of $t\mathcal{B}_n$. Writing each $(t\mathcal{B}_n)_{s-1}$ as a product of spaces and applying the Künneth isomorphism (which applies because homology is taken with coefficients in the field $\mathbb{F}$) we see that this is isomorphic to $\mathcal{B}_n$ equipped with the differential $d_b$. □

**Proposition 8.6.** Suppose that the realisation of the $n$-th splitting poset $SP_n$ is $(n-3)$-connected. Then the realisation $||t\mathcal{B}_n||$ is $(n-2)$-connected.

**Proof.** In order to give this proof, we must be precise about our construction of classifying spaces. Given a group $G$, we define $EG$ to be the realisation of the category obtained from the action of $G$ on itself by right multiplication. (So it is $BG$ in the notation of [21].) Then we define $BG = EG/G$. The map $EG \to BG$ is a locally trivial principal $G$-fibration, and $EG$ is itself contractible. The assignment $G \to EG$ is functorial, and respects products in the sense that if $G$ and $H$ are groups then the map $E(G \times H) \to EG \times EH$ obtained from the projections is an isomorphism. We can therefore construct a homotopy equivalence as follows.

$$BG_{q_0} \times \cdots \times BG_{q_{r+1}} = \frac{EG_{q_0} \times \cdots \times EG_{q_{r+1}}}{G_{q_0} \times \cdots \times G_{q_{r+1}}}$$
$$\approx \frac{E(G_{q_0} \times \cdots \times G_{q_{r+1}})}{G_{q_0} \times \cdots \times G_{q_{r+1}}}$$
$$\approx \frac{EG_n}{G_{q_0} \times \cdots \times G_{q_{r+1}}}$$

Here the first arrow comes from the compatibility with products. The second map comes from the iterated product map $G_{q_0} \times \cdots \times G_{q_{r+1}} \to G_n$, and it is a homotopy equivalence because it lifts to a map of principal $(G_{q_0} \times \cdots \times G_{q_{r+1}})$-bundles whose total spaces are both contractible. There is an isomorphism

$$\frac{EG_n}{G_{q_0} \times \cdots \times G_{q_{r+1}}} \approx \frac{EG_n \times G_n}{G_{q_0} \times \cdots \times G_{q_{r+1}}}$$

sending the orbit of an element $x$ to the orbit of $(x, e_n(G_{q_0} \times \cdots \times G_{q_{r+1}}))$. Combining the two maps just constructed gives us a homotopy equivalence:

$$BG_{q_0} \times \cdots \times BG_{q_{r+1}} \approx EG_n \times G_n \left( \frac{G_n}{G_{q_0} \times \cdots \times G_{q_{r+1}}} \right)$$

(3)
Now let $SC_n^+$ denote the augmented semisimplicial set obtained from $SC_n$ by adding a single point as a $-1$-simplex. The maps (3) then form the components of a homotopy equivalence

$$(t\mathcal{B}_n)_r \xrightarrow{\simeq} EG_n \times_{G_n} (SC_n^+)_r.$$ 

These equivalences in turn assemble to a levelwise homotopy equivalence

$$t\mathcal{B}_n \xrightarrow{\simeq} EG_n \times_{G_n} SC_n^+$$

and consequently induce a homotopy equivalence

$$\|t\mathcal{B}_n\| \xrightarrow{\simeq} \|EG_n \times_{G_n} SC_n^+\|.$$ 

By assumption, $|SP_n|$ is $(n-3)$-connected, so that $\|SC_n\|$ (to which it is isomorphic by Proposition 6.4) is also $(n-3)$-connected. Consequently $\|SC_n^+\|$, which is just the suspension of $\|SC_n\|$, is $(n-2)$-connected. Equivalently, the inclusion of the basepoint $* \hookrightarrow \|SC_n^+\|$ is an $(n-2)$-equivalence. It follows that the map $EG_n \times_{G_n} * \to EG_n \times_{G_n} \|SC_n^+\|$ is also an $(n-2)$-equivalence, so that the quotient

$$\frac{EG_n \times_{G_n} \|SC_n^+\|}{EG_n \times_{G_n} *}$$

is $(n-2)$-connected. But then

$$\|t\mathcal{B}_n\| \cong \|EG_n \times_{G_n} SC_n^+\| \cong \frac{EG_n \times_{G_n} \|SC_n^+\|}{EG_n \times_{G_n} *}$$

is $(n-2)$-connected as required. \hfill \square

9. RELATING $\mathcal{B}_n$ TO THE STABILISATION MAPS

Let $A$ be an $\mathbb{F}$-algebra of the kind described at the start of section 7. Thus $A$ has a natural topological grading with respect to which it is commutative, it has an additional charge grading $A = \bigoplus_{p \geq 0} A_p$, $A_0$ consists of a single copy of $\mathbb{F}$ in topological degree 0, $(A_1)_0$ is a copy of $\mathbb{F}$ generated by an element $\sigma$, and each piece $A_p$ is concentrated in non-negative topological degrees.

**Definition 9.1** (The stabilisation map.). The *stabilisation map* $s: A_{n-1} \to A_n$ is defined by $s(a) = \sigma \cdot a$.

**Example 9.2.** In the case $A = \bigoplus_{p \geq 0} H_* (G_p)$ where $(G_p)_{p \geq 0}$ is a family of groups with multiplication, we take $\sigma$ to be the standard generator of $(A_1)_0 = H_0(G_1)$, and then $s: A_{n-1} \to A_n$ is nothing other than the stabilisation map $s*: H_* (G_{n-1}) \to H_* (G_n)$ defined in the introduction.

The aim of this section is to relate the complex $\mathcal{B}_n$ to the stabilisation maps. In order to do so, we introduce complexes $s_n$ whose homology quantifies the injectivity and surjectivity of the stabilisation maps.
**Definition 9.3** (The complex $S_n$). For $n \geq 1$, let $S_n$ denote the graded chain complex defined as follows. If $n \geq 2$, then $S_n$ is the complex.

\[ \xymatrix{ (S_n)_0 \ar[r]^{d_1} & (S_n)_1 \ar[d]^{s} \ar[l]^{A_n} \ar[r] & A_{n-1} } \]

concentrated in homological degrees 0 and 1. And for $n = 1$, $S_1$ is the complex concentrated in homological degree 0, where it is given by the part of $A_1$ lying in positive degrees, which we denote by $A_{1,>0}$.

In the case where $A = \bigoplus_{p \geq 0} H_*(G_p)$ comes from a family of groups with multiplication $(G_n)_{n \geq 0}$, the complex $S_n$ for $n \geq 2$ is simply

\[ H_*(G_n) \xleftarrow{s} H_*(G_{n-1}) , \]

so that injectivity and surjectivity of the stabilisation map $s_*$ in certain ranges of degrees can be expressed as the vanishing of the homology of $S_n$ in certain ranges of bidegrees. All of our results on the stabilisation map are proved from this point of view.

Our aim now is to relate the stabilisation maps, via the complexes $S_n$, to the complex $B_n$. We do this using the following filtration.

**Definition 9.4.** Given $n \geq 2$, define a filtration

\[ F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} = B_n \]

of $B_n$ by defining $F_{n-1} = B_n$, and by defining $F_r$ for $r \leq (n - 2)$ to be the subcomplex of $B_n$ spanned by summands of the form $A_{n-s} \otimes -$ and $A_{1,0} \otimes A_{n-s-1} \otimes -$ for $s \leq r$. As usual $A_{1,0}$ denotes the part of $A$ lying in bidegree $(1,0)$. Here it is considered as a graded submodule of $A_1$.

**Example 9.5.** Let us illustrate the above definition in the case $n = 3$, i.e. for the filtration $F_0 \subseteq F_1 \subseteq F_2 = B_3$. 

\[ \begin{array}{ccc}
A_3 & F_0 & A_3 \\
\downarrow & \downarrow & \downarrow \\
A_{1,0} \otimes A_2 & F_1 & A_1 \otimes A_1 \otimes A_1 \\
\downarrow & \downarrow & \downarrow \\
A_{1,0} \otimes A_2 & A_1 \otimes A_2 & A_1 \otimes A_1 \otimes A_1 \\
\end{array} \]
Example 9.6. In the case $n = 4$, we can depict $\mathcal{B}_4$ as follows.

Then we can depict the filtration

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 = \mathcal{B}_4$$

symbolically in the form

where a bullet $\bullet$ indicates that the relevant summand of $\mathcal{B}_4$ is included in that term of the filtration, a circle $\circ$ indicates a summand $A_1 \otimes -$ of $\mathcal{B}_4$ that has been replaced by $A_{1,0} \otimes -$, and a dot $\cdot$ indicates an omitted summand.

The next proposition will describe the filtration quotients of the filtration we have just defined. In order to state it we need the following definition.

Definition 9.7. Let $\mathcal{C}$ be a chain complex of graded $\mathbb{F}$-vector spaces (such as $\mathcal{B}_n$ or $S_n$). The homological suspension of $\mathcal{C}$, denoted $\Sigma_b \mathcal{C}$, is defined to be the chain complex of graded $\mathbb{F}$-vector spaces obtained by increasing the homological grading of each term by 1. In other words

$$(\Sigma_b \mathcal{C})_{b,d} = \mathcal{C}_{b-1,d}$$

for $b, d \geq 0$.

Proposition 9.8. For $r \geq 1$ there is an isomorphism

$$F_r/F_{r-1} \cong \Sigma_b [S_{n-r} \otimes \mathcal{B}_r],$$

while

$$F_0 \cong S_n.$$
Example 9.9. Let us illustrate the result of Proposition 9.8 in the case $n = 4$ and $r = 2$. Following on from Example 9.6, we see that $F_2/F_1$ can be depicted like this:

![Diagram](image)

The signs on the arrows indicate whether the arrow is the one obtained from the obvious multiplication map, or is the negative of that map. Observing now that

$$S_2 = (A_2 \leftarrow A_1) \cong (A_2 \leftarrow A_{1,0} \otimes A_1)$$

and that

$$B_2 = (A_2 \leftarrow A_1 \otimes A_1),$$

where the unmarked arrows are obtained from multiplication maps, we see that $F_2/F_1$ is isomorphic to the complex depicted as follows:

![Diagram](image)

The signs on the arrows now indicate whether the arrow is equal to the tensor product of a differential from $S_2$ or $B_2$ with an identity map, or to the negative of such. On the other hand, $\Sigma_0[S_2 \otimes B_2]$ is exactly the same, but where now the
signs are governed by the Koszul sign convention.

\[
\begin{array}{c}
(\mathcal{S}_2)_0 \otimes (B_2)_1 \\
(\mathcal{S}_2)_0 \otimes (B_2)_0 \\
(\mathcal{S}_2)_1 \otimes (B_2)_1 \\
(\mathcal{S}_2)_1 \otimes (B_2)_0
\end{array}
\]

\[
+ \\
- \\
\]

The last two complexes are isomorphic \textit{via} the identity map on the summands \((\mathcal{S}_2)_0 \otimes (B_2)_0\) and \((\mathcal{S}_2)_1 \otimes (B_2)_0\), and \textit{via} the negative of the identity map on the summands \((\mathcal{S}_2)_0 \otimes (B_2)_1\) and \((\mathcal{S}_2)_1 \otimes (B_2)_1\), as claimed in Proposition 9.3.

\textbf{Proof of Proposition 9.8.} For the purposes of the proof, for \(m \geq 1\) we define a chain complex of graded \(F\)-modules \(\bar{S}_m\) as follows. For \(m \geq 2\), \(\bar{S}_m\) is \(\bar{S}_0 \leftarrow \bar{S}_1 \leftarrow \cdots \leftarrow A_m\), concentrated in homological degrees 0 and 1. For \(m = 1\), we define \(\bar{S}_1\) to be the graded submodule \(A_{1,0} \otimes A_{m-1}\) of \(A_1\) consisting of the terms in positive degree. Observe that \(\bar{S}_m\) is isomorphic to \(S_m\) \textit{via} the identity map \(A_m \to A_m\) in homological degree 0, and \textit{via} the isomorphism \(A_{1,0} \otimes A_{m-1} \xrightarrow{\cong} A_{m-1}, \quad \sigma \otimes x \mapsto x\) in homological degree 1. We will prove the result with \(\bar{S}_m\) in place of \(S_m\).

We begin with the case \(r \leq n - 2\). By definition, \((F_r/F_{r-1})_b\) is the direct sum of the terms

\[A_{q_0} \otimes \cdots \otimes A_{q_b}\]

where \(q_0 + \cdots + q_b = n, \ q_1, \ldots, q_b \geq 1, \ q_0 = n - r\), together with the terms

\[A_{1,0} \otimes A_{q_1} \otimes \cdots \otimes A_{q_b}\]

where \(1 + q_1 + \cdots + q_b = n, \ q_1, \ldots, q_b \geq 1, \) and \(q_1 = n - r - 1\). In other words, \((F_r/F_{r-1})_b\) is the direct sum of the terms

\[A_{n-r} \otimes [A_{q_0} \otimes \cdots \otimes A_{q_b-1}]\]

where \(q_0 + \cdots + q_{b-1} = r, \ q_0, \ldots, q_{b-1} \geq 1, \) which is exactly \((\bar{S}_{n-r})_0 \otimes (B_r)_{b-1}\), together with the direct sum of the terms

\[[A_{1,0} \otimes A_{n-r-1}] \otimes [A_{q_0} \otimes \cdots \otimes A_{q_{b-2}}]\]
Lemma 10.2. Then the composite \( b \) is a surjection on homology in the range \( n \) for all integers smaller than \( n \).

Proof. For \( r \) in the range \( n - 1 \geq r \geq 2 \), the inductive hypothesis tells us that \( \mathcal{S}_{n-r} \) and \( \mathcal{B}_r \) are acyclic in the ranges \( b \leq (n-r) - 2d - 1 \) and \( b \leq r - 2d - 1 \).
respectively. Consequently $S_{n-r} \otimes B_r$ is acyclic in the range $b \leq n - 2d - 1$, so that $F_r/F_{r-1} \cong \Sigma_0(S_{n-r} \otimes B_r)$ is acyclic in the range $b \leq n - 2d$. It follows that $F_{r-1} \to F_r$ is a surjection on homology in the range $b \leq n - 2d$ and an isomorphism in the range $b \leq n - 2d - 1$.

(The estimate for the acyclic range of $S_{n-r} \otimes B_r$ is seen as follows. The Künneth Theorem tells us that the homology of $S_{n-r} \otimes B_r$ is the tensor product of the homologies of $S_{n-r}$ and $B_r$. Nonzero elements $x$ and $y$ of these respective homologies must lie in bidegrees $(b_1, d_1)$ and $(b_2, d_2)$ satisfying $b_1 \geq (n-r) - 2d_1$ and $b_2 \geq r - 2d_2$, so that $x \otimes y$ lies in bidegree $(b_1 + b_2, d_1 + d_2)$ satisfying $(b_1 + b_2) \geq n - 2(d_1 + d_2)$, so that $S_{n-r} \otimes B_r$ is acyclic in the range $b \leq n - 2d - 1$, as claimed.)

**Lemma 10.3.** The inclusion $F_0 \hookrightarrow F_1$ is an isomorphism in homology in the range $b \leq n - 2d - 1$.

**Proof.** Consider the chain complex corresponding to the square

$$
\begin{array}{ccc}
A_{n-1} \otimes A_{1,0} & & \\
\downarrow & & \\
A_n & & A_{1,0} \otimes A_{n-2} \otimes A_{1,0} \\
\downarrow & & \\
A_{1,0} \otimes A_{n-1} & & \\
\end{array}
$$

in which the arrows are induced by the multiplication maps of $A$. This is a subcomplex $S_q$ of $B_n$, and indeed of $F_1$. Restricting the filtration $F_0 \subset F_1$ of $F_1$ to $S_q$, gives a filtration $\bar{F}_0 \subset \bar{F}_1$ of $S_q$ for which $\bar{F}_0 = F_0$. There results a commutative diagram with short exact rows and left column an isomorphism.

```
0 \to \bar{F}_0 \to S_q \to \bar{F}_1/\bar{F}_0 \to 0
```

The right-hand vertical map is an injection with cokernel

$\Sigma[S_{n-1} \otimes H_{s \geq 1}(G_1)]$.

Since $S_{n-1}$ is acyclic in the range $b \leq (n-1) - 2d - 1$, this cokernel is acyclic in the range $b \leq [(n-1) - 2(d-1) - 1] + 1 = n - 2d + 1$, so that the right-hand map in the diagram is a surjection in homology in the same range. The connecting homomorphism for the top row is zero, since $S_q$ is isomorphic to the
chain complex obtained from the square

\[
\begin{array}{c}
A_{n-1} \\
\downarrow \\
A_n \\
\downarrow \\
A_{n-2} \\
\downarrow \\
A_{n-1}
\end{array}
\]

in which each map is multiplication by \( \sigma \in A_{1,0} \), where triviality of the connecting homomorphism is evident. The connecting homomorphism for the bottom sequence is therefore zero in the range (of bidegrees for its domain) \( b \leq n - 2d + 1 \). It follows that in the range \( b \leq n - 2d \) we have short exact sequences

\[
0 \rightarrow H_*(F_0) \rightarrow H_*(F_1) \rightarrow H_*(F_1/F_0) \rightarrow 0.
\]

In the smaller range \( b \leq n - 2d - 1 \) the third term vanishes, so that \( H_*(F_0) \rightarrow H_*(F_1) \) is an isomorphism as claimed.

We can now complete the proof of Theorem 10.1. It follows from the last two lemmas that in the range \( b \leq n - 2d - 1 \) the inclusion \( S_n = F_0 \hookrightarrow B_n \) is an isomorphism in homology. The homology of \( S_n \) is concentrated in the range \( b \leq 1 \), so that the homology of \( B_n \) vanishes in the range \( 2 \leq b \leq n - 2d - 1 \). It remains to prove that \( H_*(S_n) = H_*(B_n) = 0 \) in the range where \( b \leq n - 2d - 1 \) and \( b \leq 1 \) both hold.

In order to proceed we use the spectral sequence of Theorem 8.1 which has \( H_*(B_n) = E_2^{s,s} \). No nonzero differentials \( d^r, r \geq 2 \), of the spectral sequence affect terms in the range \( b \leq n - 2d - 1, b \leq 1 \). This is because any differential with source in this range has target outside the first quadrant. And any differential \( d^r \) with target in this range has source \( E_{b+r,d-r+1}^r \), where

\[
b + r \leq n - 2d - 1 + r \leq n - 2(d - r + 1) - 1,
\]

so that \( E_{b+r,d-r+1}^r = 0 \). Thus \( H_*(S_n) = H_*(B_n) = E_\infty^{s,s} \) in the range \( b \leq n - 2d - 1, b \leq 1 \). Recall that \( E_\infty^{s,s} = 0 \) in the range \( d \leq n - 2 - b \). Now for \( n \geq 3 \) and \( b = 0, 1 \) we have

\[
d \leq \frac{n - b - 1}{2} \implies d \leq n - 2 - b.
\]

(The case \( d \geq 1 \) must be treated separately from the case \( d = 0 \), which is vacuous.) Thus \( H_*(S_n) = H_*(B_n) = E_\infty^{s,s} = 0 \) as required.
11. Proof of Theorem C

For the purposes of this section, we let \((G_p)_{p \geq 0}\) be a family of groups with multiplication satisfying the hypotheses of Theorems A, B and C and we define \(A = \bigoplus_{n \geq 0} H_\ast(G_n)\). In this section we will prove Theorem C essentially by extracting a little extra data from the proof of Theorem A, and then exploiting a cheap trick. Throughout the section we will write \(A_{i,j}\) for the part of \(A\) with charge \(i\) and topological degree \(j\). In other words, \(A_{i,j} = (A_i)_j\).

**Lemma 11.1.** For \(m \geq 1\), the graded chain complex \(\mathcal{B}_{2m+1}\) is acyclic in the range \(3 \leq b \leq (2m + 1) - 2d\).

**Proof.** Lemma 10.2 shows that the inclusion \(F_1 \hookrightarrow \mathcal{B}_n\) is a surjection on homology in the range \(b \leq n - 2d\). However, \(F_1\) is concentrated in homological degrees \(b = 0, 1, 2\), and so is acyclic in the range \(b \geq 3\). Combining the two facts gives the result. \(\square\)

**Lemma 11.2.** In the spectral sequence of Theorem 8.1, for \(n = 2m + 1\), there are no differentials affecting the term in bidegree \((1, m)\) from the \(E^2\) page onwards.

**Proof.** Certainly there are no such differentials with source in this bidegree, since the spectral sequence is concentrated in the first quadrant. Since \(E^1 = \mathcal{B}_{2m+1}\), Lemma 11.1 shows that \(E^2\) vanishes in the range
\[
3 \leq b \leq (2m + 1) - 2d.
\]
If \(r \geq 2\), then any differential \(d^r\) with target in bidegree \((1, m)\) has source in bidegree \((b, d) = (1 + r, m - r + 1)\), so that
\[
b = (2m + 1) - 2d - [r - 2] \leq (2m + 1) - 2d,
\]
and consequently the source term vanishes. \(\square\)

**Lemma 11.3.** Let \(m \geq 2\). Then the complex \(\mathcal{B}_{2m+1}\) is acyclic in bidegree \((1, m)\).

**Proof.** In the spectral sequence of Theorem 8.1 for \(n = 2m + 1\), we know that \(E^2_{1,m} = E^\infty_{1,m}\) by Lemma 11.2 and that \(E^\infty_{1,m} = 0\) since \(m \geq 2\) guarantees that \(1 + m \leq (2m + 1) - 2\). So \(E^2_{1,m} = 0\), but this is simply the homology of \(\mathcal{B}_{2m+1}\) in bidegree \((1, m)\). \(\square\)

**Lemma 11.4.** Let \(m \geq 2\). Then \(\mathcal{B}_{2m}\) is acyclic in bidegree \((0, m)\).

**Proof.** Consider the following composite.
\[
\Sigma_b A_{2m} \xrightarrow{\theta} \mathcal{B}_{2m+1} \xrightarrow{\phi} \Sigma_b \mathcal{B}_{2m} \otimes \mathcal{B}_1 \xrightarrow{\psi} \Sigma_b \mathcal{B}_{2m}
\]
Here \(\theta\) is the map that sends \(x \in A_{2m}\) to the element \(x \otimes \sigma - \sigma \otimes x \in (\mathcal{B}_{2m+1})_1\). To check that \(\theta\) is a chain map, we need only check that the differential vanishes on its image, which holds because
\[
d(x \otimes \sigma - \sigma \otimes x) = x \cdot \sigma - \sigma \cdot x = 0.
\]
Next, $\Sigma_b(\mathcal{B}_{2m} \otimes \mathcal{B}_1)$ can be identified with the submodule of $\mathcal{B}_{2m+1}$ consisting of summands of the form $- \otimes A_1$, and $\phi$ is the projection onto these summands. It is a chain map. Finally, $\psi$ is the map that projects $\mathcal{B}_1 = A_1$ onto its degree 0 part $A_{1,0} \cong F$. In homology in bidegree $(1, m)$ this map is zero since it factors through the homology of $\mathcal{B}_{2m+1}$, which vanishes in that bidegree. On the other hand, in this bidegree the composite is simply the suspension of the map $A_{2m} \to \mathcal{B}_{2m}$, which is a surjection in homological degree $b = 0$. It follows that the target of this map, which is the homology of $\mathcal{B}_{2m}$ in bidegree $(0, m)$, is zero.

Proof of Theorem C We have seen that $\mathcal{B}_{2m}$ is acyclic in bidegree $(0, m)$. This means that the map

$$\bigoplus_{p+q=2m} \bigoplus_{p \geq 1 \atop q \geq 0} A_{p,p'} \otimes A_{q,q'} \to A_{2m,m}$$

is surjective. Now, suppose that $p, q, p', q'$ are as in the summation above, with $p' \leq \frac{p-1}{2}$. Then we have the commutative diagram

$$\begin{array}{ccc}
A_{p,p'} \otimes A_{q,q'} & \to & A_{2m,m} \\
\text{id} & \uparrow & \\
A_{p-1,p'} \otimes A_{q,q'} & \to & A_{2m-1,m}
\end{array}$$

in which the left-hand map is surjective by Theorem A so that the image of $A_{p,p'} \otimes A_{q,q'}$ is contained in the image of $s$. Similarly, if $q' \leq \frac{q-1}{2}$ then the image of $A_{p,p'} \otimes A_{q,q'}$ is contained in the image of $s$. The only summands to which these observations do not apply are those indexed by $p, q, p', q'$ as in the summation, satisfying also that

$$p' > \frac{p-1}{2}, \quad q' > \frac{q-1}{2}.$$ 

Adding these inequalities shows that we have

$$m = p' + q' > m - 1.$$ 

Thus the only possibility it that $p'$ is greater than $\frac{p-1}{2}$ by exactly $1/2$, and similarly for $q'$. In other words, we must have $p = 2p'$ and $q = 2q'$. So we have shown that the map

$$\begin{array}{ccc}
A_{2m-1,m} \oplus \bigoplus_{p'+q'=m \atop p',q' \geq 1} A_{2p',p'} \otimes A_{2q',q'} & \to & A_{2m,m}
\end{array}$$

is surjective. In the case $m = 2$ this proves the claim, and for $m > 2$ the claim now follows by induction. \qed
12. Proof of Theorem \[B\]

For the purposes of this section, we let \((G_p)_{p \geq 0}\) be a family of groups with multiplication satisfying the hypotheses of Theorems \([A]\) and \([C]\) and we define \(A = \bigoplus_{n \geq 0} H_n(G_n)\). The aim of this section is to prove Theorem \([B]\) which is an immediate consequence of Theorem \([C]\) and the following.

The section will deal with complexes like \(\mathcal{B}_n\) which have a homological and topological grading. Given such a complex \(\mathcal{C}\), we will write \(H_{i,j}(\mathcal{C})\) for the part of \(H_i(\mathcal{C})\) that lies in topological grading \(j\), in other words \(H_{i,j}(\mathcal{C}) = H_i(\mathcal{C}) \_j\).

**Theorem 12.1.** Let \(m \geq 1\). Then the images of the maps

\[
\ker \left[ s_\ast : H_{m-1}(G_{2m-2}) \to H_{m-1}(G_{2m-1}) \right] \otimes H_1(G_2) \longrightarrow \ker \left[ s_\ast : H_m(G_{2m}) \to H_m(G_{2m+1}) \right]
\]

(4)

\[
H_{m-1}(G_{2m-2}) \otimes \ker \left[ s_\ast : H_1(G_2) \to H_1(G_3) \right] \longrightarrow \ker \left[ s_\ast : H_m(G_{2m}) \to H_m(G_{2m+1}) \right]
\]

(5)

together span \(\ker \left[ s_\ast : H_m(G_{2m}) \to H_m(G_{2m+1}) \right]\).

The main ingredient in the proof of Theorem \([12.1]\) is Lemma \([11.3]\) which states that \(H_1,m(\mathcal{B}_{2m+1}) = 0\) for \(m \geq 1\), and of which it is an entirely algebraic consequence. However our argument is significantly more unpleasant than we would like. Here is the general outline: Theorem \([12.1]\) is a statement about \(H_1,m(\mathcal{S}_{2m+1})\), which is by definition the kernel \(\ker \left[ s_\ast : H_m(G_{2m}) \to H_m(G_{2m+1}) \right]\). We will use the filtration

\[
\mathcal{S}_{2m+1} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{2m} = \mathcal{B}_{2m+1}
\]

from Definition \([9.4]\) to get from what we know about \(H_1,m(\mathcal{B}_{2m+1})\) to what we need to know about \(H_1,m(\mathcal{S}_{2m+1})\). We will do this by using the spectral sequence arising from the filtration in topological degree \(m\).

\[
E_{i,j}^1 = H_{i+j,m}(F_i/F_{i-1}) \implies H_{i+j,m}(\mathcal{B}_{2m+1})
\]

The point is to identify the differentials affecting the term \(E_{0,1}^1 = H_1,m(\mathcal{S}_{2m+1})\) with the maps \([4]\) and \([5]\).

Let us begin the proof in detail. We are interested in the values of \(H_{r,m}(F_i/F_{i-1})\) in the cases \(r = 0, 1, 2\). Recall from Proposition \([9.8]\) that for \(i \geq 1\) we have

\[
F_i/F_{i-1} \cong \Sigma_b [\mathcal{S}_{2m+1-i} \otimes \mathcal{B}_i]
\]

so that

\[
H_{r,m}(F_i/F_{i-1}) \cong H_{r-1,m}[\mathcal{S}_{2m+1-i} \otimes \mathcal{B}_i] \cong \bigoplus_{r_1 + r_2 = r-1, \ m_1 + m_2 = m} H_{r_1,m_1}(\mathcal{S}_{2m+1-i}) \otimes H_{r_2,m_2}(\mathcal{B}_i).
\]

We have the following.
Lemma 12.2. For $r = 0, 1, 2$ and $i = 0, \ldots, 2m$, the only nonzero groups $H_{r,m}(F_i/F_{i-1})$ are as follows.

\[
\begin{align*}
H_{1,m}(F_0) &\cong H_{1,m}(S_{2m+1}) \\
H_{2,m}(F_0) &\cong H_{2,m}(S_{2m+1}) \\
H_{1,m}(F_1/F_0) &\cong H_{0,m}(S_{2m}) \otimes H_{0,0}(B_1) \\
H_{2,m}(F_1/F_0) &\cong H_{1,m}(S_{2m}) \otimes H_{0,0}(B_1) \\
H_{2,m}(F_2/F_1) &\cong H_{1,m-1}(S_{2m-1}) \otimes H_{0,1}(B_2) \\
H_{2,m}(F_3/F_2) &\cong H_{0,m-1}(S_{2m-2}) \otimes H_{1,1}(B_3)
\end{align*}
\]

Proof. Case $i = 0$. In this case we have $H_{r,m}(F_0) = H_{r,m}(S_{2m+1})$, and by Theorem [10.1] this is nonzero only for $r \geq 1$.

Case $i = 1$. In this case we have

\[
H_{r,m}(F_1/F_0) \cong H_{r,m}(\Sigma_b[S_{2m} \otimes B_1]) \cong H_{r-1,m}(S_{2m} \otimes B_1) \cong \bigoplus_{m_1+m_2=m} H_{r-1,m_1}(S_{2m}) \otimes H_{0,m_2}(B_1)
\]

since $B_1$ is concentrated in homological degree $b = 0$. Now by Theorem [10.1] the term $H_{r-1,m_1}(S_{2m})$ vanishes for $m_1 \leq m - r/2$. So for $r = 0$ we require $m_1 > m$, which is impossible, and for $r = 1, 2$ the only possibility is $m_1 = m$, $m_2 = 0$. So the possible terms are

\[
H_{1,m}(F_1/F_0) \cong H_{0,m}(S_{2m}) \otimes H_{0,0}(B_1)
\]

and

\[
H_{2,m}(F_1/F_0) \cong H_{1,m}(S_{2m}) \otimes H_{0,0}(B_1)
\]

Case $2 \leq i \leq 2m$. In this case we have

\[
H_{r,m}(F_i/F_{i-1}) \cong H_{r,m}(\Sigma_b[S_{2m+1-i} \otimes B_i]) \\
\cong H_{r-1,m}(S_{2m+1-i} \otimes B_i) \\
\cong \bigoplus_{r_1 + r_2 = r-1 \atop m_1 + m_2 = m} H_{r_1,m_1}(S_{2m+1-i}) \otimes H_{r_2,m_2}(B_i).
\]

Now from Theorem [10.1] we know that $H_{r_2,m_2}(B_i) = 0$ for $r_2 \leq 2i - 2m_2 - 1$ while $H_{r_1,m_1}(S_{2m+1-i}) = 0$ for $r_1 \leq 2m + 1 - i - 2m_1 - 1$. Thus a nonzero group appearing in the direct sum above must have

\[
r_1 = 2m - i - 2m_1 + \delta \quad \text{and} \quad r_2 = i - 2m_2 - 1 + \epsilon
\]

for $\delta, \epsilon > 0$. Then the constraints $r_1 + r_2 = r - 1$ and $m_1 + m_2 = m$ give us $r = \delta + \epsilon$. Thus, to find a nonzero group when $i \geq 2$ and $r = 0, 1, 2$, the only possibility is that $r = 2$ and $\delta = \epsilon = 1$. But then $(r_1, r_2) = (1, 0)$ or $(r_1, r_2) = (0, 1)$, in which case we have two possible summands, only one of which is possible at a given time,
namely

\[ H_{2,m}(F_i/F_{i-1}) = \begin{cases} H_{0,m-(i-1)/2}(S_{2m+1}) \otimes H_{1,(i-1)/2}(B_{i}) & \text{for } i \text{ odd}, \\ H_{1,-i/2}(S_{2m+1-i}) \otimes H_{0,i/2}(B_{i}) & \text{for } i \text{ even}. \end{cases} \]

However, Lemmas 11.3 and 11.4 guarantee that the second factors vanish for \( i \geq 4 \). Thus the only contributing factors are

\[ H_{2,m}(F_3/F_2) = H_{0,m-1}(S_{2m-2}) \otimes H_{1,1}(B_3) \]

and

\[ H_{2,m}(F_2/F_1) = H_{1,m-1}(S_{2m}) \otimes H_{0,1}(B_2). \]

This completes the proof. \( \square \)

Thus the spectral sequence is as follows.

\[
\begin{array}{ccccccc}
H_{2,m}(S_{2m+1}) & \bullet & \bullet & \bullet & \bullet & \bullet \\
H_{1,m}(S_{2m+1}) & H_{1,m}(S_{2m}) \otimes H_{0,0}(B_1) & \bullet & \bullet \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

We will now investigate the differentials affecting \( H_{1,m}(S_{2m+1}) \). We will need the following preliminary result.

**Lemma 12.3.** An arbitrary element of \( H_{1,1}(B_3) \) has a representative of the form

\[ (x \otimes \sigma - \sigma \otimes x) + q \otimes \sigma \]

where \( x \in A_{2,1} \) and \( q \in \ker(s: A_{2,1} \rightarrow A_{3,1}) \), and \( \sigma \in A_{1,0} \) is the stabilising element.

**Proof.** \( A_1 \) and \( A_2 \) are concentrated in non-negative degrees, and in degree 0 they are spanned by \( \sigma \) and \( \sigma^2 \) respectively. Thus an arbitrary cycle of \( B_3 \) in bidegree \((1, 1)\) has form \( j \otimes \sigma + k \otimes \sigma^2 + \sigma \otimes l + \sigma^2 \otimes m \) for \( j, l \in A_{2,1} \) and \( k, m \in A_{1,1} \). By adding \( d(k \otimes \sigma \otimes \sigma - \sigma \otimes \sigma \otimes m) \), we may assume that \( k = m = 0 \), so that our cycle has the form \( j \otimes \sigma + \sigma \otimes l \). This can be rewritten in the required form with \( x = -l \) and \( q = j + l \). \( \square \)

**Lemma 12.4.** The span of the images of the differentials with target \( H_{1,m}(S_{2m+1}) \) is precisely the span of the maps (4) and (5).
Proof. There are just three positions in the spectral sequence supporting differentials with the given target. We will compute the differentials case by case.

**Case 1: Differentials with domain** $H_{1,m}(S_{2m}) \otimes H_{0,0}(B_1)$. An element $l$ of the domain can be represented by a cycle $l_1 = x \otimes \sigma$ in $S_{2m} \otimes B_1$, where $x \in \ker(s: A_{2m,m} \to A_{2m+1,m})$ and $\sigma \in A_{1,0}$ is the stabilising element. Then under the isomorphism of Proposition 9.8, $l_1$ corresponds to the element $l_2 = \sigma \otimes x \otimes \sigma$ of $F_1/F_0$. We lift this to the element $l_3 = \sigma \otimes x \otimes \sigma$ of $F_1$. Then $d(l_3) = \sigma \cdot x \otimes \sigma - \sigma \otimes x \cdot \sigma = 0$. Thus all differentials $d^r$ vanish on $l$. (In fact there is only one possibility, $d^1$.)

**Case 2: Differentials with domain** $H_{1,m-1}(S_{2m-1}) \otimes H_{0,1}(B_2)$. An element $l$ of the domain can be represented by a linear combination of cycles of the form $x \otimes y$ in $S_{2m-1} \otimes B_2$, where $x \in \ker(s: A_{2m-2,m-1} \to A_{2m-1,m-1})$ and $y \in A_{2,1}$. Let us assume without loss that $l$ is in fact represented by $l_1 = x \otimes y$. Then under the isomorphism of Proposition 9.8, $l_1$ corresponds to the element $l_2 = \sigma \otimes x \otimes y$ of $F_2/F_1$, which we lift to the element $l_3 = \sigma \otimes x \otimes y$ of $F_2$. Now $d(l_3) = \sigma \cdot x \otimes y - \sigma \otimes x \cdot y = -\sigma \otimes x \cdot y$, which lies in $F_0$. Thus $d^1(l) = 0$, while $d^2(l)$ is the class represented by $-\sigma \otimes x \cdot y$, under which the isomorphism of Proposition 9.8 corresponds to the element $-x \cdot y$ of $A_{2m,m} = H_m(G_{2m})$. This is precisely the image of $-x \cdot y$ under the map $[1]$ above. Thus the image of $d^2$ is precisely the image of $[1]$.

**Case 3: Differentials with domain** $H_{0,m-1}(S_{2m-2}) \otimes H_{1,1}(B_3)$. By Lemma 12.3, an element $l$ of the domain has a representative of the form

$$l_1 = \sum_\alpha x_\alpha \otimes (y_\alpha \otimes \sigma - \sigma \otimes y_\alpha) + \sum_\beta p_\beta \otimes (q_\beta \otimes \sigma)$$

where $x_\alpha, p_\beta \in A_{2m-2,m-1}$, $y_\alpha \in A_{2,1}$ and $q_\beta \in \ker(s: A_{2,1} \to A_{3,1})$. Under the isomorphism of Proposition 9.8, $l_1$ corresponds to the element

$$l_2 = \sum_\alpha (x_\alpha \otimes y_\alpha \otimes \sigma - x_\alpha \otimes \sigma \otimes y_\alpha) + \sum_\beta p_\beta \otimes q_\beta \otimes \sigma$$

of $F_3/F_2$. We lift this to the element

$$l_3 = \sum_\alpha (x_\alpha \otimes y_\alpha \otimes \sigma - x_\alpha \otimes \sigma \otimes y_\alpha + \sigma \otimes x_\alpha \otimes y_\alpha) + \sum_\beta p_\beta \otimes q_\beta \otimes \sigma$$

of $F_3$. (The apparently new terms lie in $F_2$.) Then

$$d(l_3) = \sum_\alpha (x_\alpha \cdot y_\alpha \otimes \sigma - \sigma \otimes x_\alpha \cdot y_\alpha) + \sum_\beta p_\beta \cdot q_\beta \otimes \sigma.$$

This lies in $F_1$, so that $d^1(l) = 0$, and its image in $F_1/F_0$ is

$$\sum_\alpha x_\alpha \cdot y_\alpha \otimes \sigma + \sum_\beta p_\beta \cdot q_\beta \otimes \sigma.$$
so that applying the isomorphism of Proposition 9.8 shows that
\[ d^2(l) = \left[ \sum_{\alpha} x_{\alpha} \cdot y_{\alpha} + \sum_{\beta} p_{\beta} \cdot q_{\beta} \right] \otimes [\sigma] \in H_{0,m}(S_{2m}) \otimes H_{0,0}(B_1). \]

Thus \( l \) lies in the kernel of \( d^2 \) if and only if
\[ \left[ \sum_{\alpha} x_{\alpha} \cdot y_{\alpha} + \sum_{\beta} p_{\beta} \cdot q_{\beta} \right] \]
is zero in \( H_{0,m}(S_{2m}) \), or in other words if and only if there is \( w \in A_{2m-1,m} \) such that
\[ \sum_{\alpha} x_{\alpha} \cdot y_{\alpha} + \sum_{\beta} p_{\beta} \cdot q_{\beta} = \sigma \cdot w. \]
In this case, we may again represent \( l \) by \( l_1 \), which again corresponds to the element \( l_2 \) of \( F_3/F_2 \), but which we now lift to the element \( l_3 = \sigma \otimes w \otimes \sigma \) of \( F_3 \). (The additional term lies in \( F_1 \).) But then \( d(l_3 - \sigma \otimes w \otimes \sigma) \) is precisely the element
\[ \sum_{\beta} \sigma \otimes p_{\beta} \cdot q_{\beta} \]
of \( F_0 \). Applying the isomorphism of Proposition 9.8, we find that
\[ d^3(l) = \left[ \sum_{\beta} p_{\beta} \cdot q_{\beta} \right] \in H_{1,m}(S_{2m+1}). \]

But then it follows that the image of \( d^3 \) is precisely the span of the map \([5]\). \( \square \)

We may now complete the proof. Since \( H_{1,m}(B_{2m+1}) = 0 \), it follows that the infinity-page of the spectral sequence must vanish in total degree 1. So then in particular we must have \( E_{1,0}^\infty = 0 \), or in other words, the differentials with target \( H_{1,m}(S_{2m+1}) \) must span. But we have identified the (nonzero) differentials with the maps \([1]\) and \([5]\). Thus it follows that together, the images of these two maps must span. This completes the proof of Theorem 12.1.

REFERENCES

[1] Anders Björner, Michelle L. Wachs, and Volkmar Welker. Poset fiber theorems. Trans. Amer. Math. Soc., 357(5):1877–1899, 2005.
[2] Carles Broto, Nguyen H. V. Hu'ng, Nicholas J. Kuhn, John H. Palmieri, Stewart Priddy, and Nobuaki Yagita. The problem session. In Proceedings of the School and Conference in Algebraic Topology, volume 11 of Geom. Topol. Monogr., pages 435–441. Geom. Topol. Publ., Coventry, 2007.
[3] Ruth M. Charney. Homology stability for GLn of a Dedekind domain. Invent. Math., 56(1):1–17, 1980.
[4] Frederick R. Cohen, Thomas J. Lada, and J. Peter May. The homology of iterated loop spaces. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.
[5] Søren Galatius. Stable homology of automorphism groups of free groups. Ann. of Math. (2), 173(2):705–768, 2011.
[6] Søren Galatius and Oscar Randal-Williams. Homological stability for moduli spaces of high dimensional manifolds. I. arXiv:1403.2334v1, 2014.

[7] S. M. Gersten. A presentation for the special automorphism group of a free group. J. Pure Appl. Algebra, 33(3):269–279, 1984.

[8] John L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. Ann. of Math. (2), 121(2):215–249, 1985.

[9] Allen Hatcher and Karen Vogtmann. Cerf theory for graphs. J. London Math. Soc. (2), 58(3):633–655, 1998.

[10] Allen Hatcher and Karen Vogtmann. Rational homology of $\text{Aut}(F_n)$. Math. Res. Lett., 5(6):759–780, 1998.

[11] Allen Hatcher and Karen Vogtmann. Homology stability for outer automorphism groups of free groups. Algebraic Topology, 4:1253–1272, 2004.

[12] Allen Hatcher and Nathalie Wahl. Stabilization for mapping class groups of 3-manifolds. Duke Math. J., 155(2):205–269, 2010.

[13] Kevin P. Knudson. Homology of linear groups, volume 193 of Progress in Mathematics. Birkhäuser Verlag, Basel, 2001.

[14] Hendrik Maazen. Homology stability for the general linear group. 1979. Thesis (Ph.D.)–Utrecht University.

[15] R. James Milgram and Stewart B. Priddy. Invariant theory and $H^*(\text{GL}_n(\mathbb{F}_p);\mathbb{F}_p)$. In Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985), volume 44, pages 291–302, 1987.

[16] Minoru Nakaoka. Decomposition theorem for homology groups of symmetric groups. Ann. of Math. (2), 71:16–42, 1960.

[17] Daniel Quillen. Finite generation of the groups $K_i$ of rings of algebraic integers. In Algebraic $K$-theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pages 179–198. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[18] Oscar Randal-Williams. Homological stability for unordered configuration spaces. Q. J. Math., 64(1):303–326, 2013.

[19] Oscar Randal-Williams. Resolutions of moduli spaces and homological stability. J. Eur. Math. Soc. (JEMS), 18(1):1–81, 2016.

[20] Oscar Randal-Williams and Nathalie Wahl. Homological stability for automorphism groups. arXiv:1409.3541, 2014.

[21] Graeme Segal. Classifying spaces and spectral sequences. Inst. Hautes Études Sci. Publ. Math., (34):105–112, 1968.

[22] Wilberd van der Kallen. Homology stability for linear groups. Invent. Math., 60(3):269–295, 1980.

[23] Nathalie Wahl. Homological stability for mapping class groups of surfaces. In Handbook of moduli. Vol. III, volume 26 of Adv. Lect. Math. (ALM), pages 547–583. Int. Press, Somerville, MA, 2013.

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