Abstract

Sparse subspace clustering (SSC) is an elegant approach for unsupervised segmentation if the data points of each cluster are located in linear subspaces. This model applies, for instance, in motion segmentation if some restrictions on the camera model hold. SSC requires that problems based on the $l_1$-norm are solved to infer which points belong to the same subspace. If these unknown subspaces are well-separated this algorithm is guaranteed to succeed.

The algorithm rests upon the assumption that points on the same subspace are well spread. The question what happens if this condition is violated has not yet been investigated. In this work, the effect of particular distributions on the same subspace will be analyzed. It will be shown that SSC fails to infer correct labels if points on the same subspace fall into more than one cluster.

1 Introduction

This paper considers unsupervised classification of data of points which reside on multiple unknown and low-dimensional subspaces. The problem is to decide which points belong to the same subspace. This subspace clustering arises in problems such as motion segmentation (Kanatani 2001; Vidal, Ma, and Sastry 2005; Elhamifar and Vidal 2013), hand written digit clustering (Zhang et al. 2012), face clustering (Ho et al. 2003) and compression (Hong et al. 2005).

Sparse subspace clustering (SSC) (Elhamifar and Vidal 2013) estimates the self-expressiveness within the data: which points can be used to linearly approximate a point in $\mathbb{R}^d$. This approach was given in which the graph was disconnected.

In this paper we focus on the same question. Different prior works, we analyze the case that points on the same subspace are not evenly distributed as required in (Soltanolkotabi and Candès 2012), but fall into different clusters. It will be shown that a few edges indeed connect vertices of different clusters. However, it will also be shown that the weights of such edges are negligible small compared with edges connecting vertices of the same cluster. In other words, the relative connectivity is too low to be significant. The reason is a bias of $l_1$-norm based estimators.

The graph constructed from the sparse coefficients then consists of more disconnected subgraphs than the correct yet unknown number of subspaces. Since the subspace model is often motivated by some underlying physical model, for instance each rigidly moving body induces a 4-dimensional subspace in motion segmentation, over-segmenting the data can be difficult to correct. On the other hand, because most algorithms of this class (Elhamifar and Vidal 2013; Patel, Nguyen, and Vidal 2013; Li and Vidal 2015; Liu, Lin, and Yu 2010) require the number of clusters to be known in advance, cluster centers will be incorrect which causes the labelling to be wrong. It is therefore more desirable to address the original problem in the first place.

This paper is structured as follows: In Sec. 2, we summarize the notation used in this paper. In Sec. 3, the original sparse subspace clustering algorithm is shortly explained. An example of the effect of non-uniformly distributed points is given in Sec. 4. A formal analysis on the magnitude of the sparse coefficients in given in Sec. 5. The proposed algorithm is introduced in Sec. 6. Experimental results are shown in Sec. 7. The paper concludes with a summary in Sec. 8.

2 Notation

Bold lower-case letters $\mathbf{x}$ indicate vectors, normal capital letters matrices, e.g. $X = [x_1 \cdots x_N]$, or constants, e.g. $N$. By $x(j)$, we mean the $j$th element of vector $x$.
Sparse subspace clustering rests upon the assumption that there are \( L \) subspaces \( S^{(l)} \subseteq \mathbb{R}^m \), \( l = 1, \ldots, L \) with \( N_l \) points \( X^{(l)} = [x^{(l)}_1, \ldots, x^{(l)}_{N_l}] \) on it. Let \( d^{(l)} \) denote the dimension of \( S^{(l)} \). Let \( X = [X^{(1)} \ldots X^{(L)}] \) denote the matrix of all \( N_1 + \cdots + N_L = N \) data points. Assume without loss of generality that \( X \) is ordered. The points \( x_j \) are supposed to have unit length \( \|x_j\|_2 = 1 \).

Let \( T^{(l)} \) indicate the set of indices of all the points \( x^{(l)}_j \in S^{(l)} \). Let \( S_j \subseteq T^{(l)} \) indicate the support set of point \( x_j \), \( j = \{1, \ldots, N\} \). The cardinality of a set is denoted by \( |\cdot| \).

The \( \ell_k \)th point of the support set \( S_j \) of \( x^{(l)}_j \) is denoted by \( x^{(l)}_{j,k} \). Let \( y \) be any of the points of \( X \). The matrix \( X - y \) indicates the matrix \( X \) without the column corresponding to \( y \).

By \( \|\cdot\|_{0,1,2,F} \) we indicate the \( 0 \)-pseudo-norm, or the \( l_{11}, l_2, \) or Frobenius-norm of the argument.

### 3 Sparse Subspace Clustering

Sparse subspace clustering (SSC) (Elhamifar and Vidal 2013) is based on the fact that any point \( x_j \in S^{(l)} \) can be expressed by a linear combination of \( d^{(l)} \) other points in the same subspace. Since such linear combinations do not use points of other classes, they can be used to infer the unknown class labels.

The problem is to compute the least number of points \( x_k \), \( k \neq j \) such that a linear combination of the \( x_k \) yields \( x_j \). Denote by \( c_j \) the vector of mixing coefficients with its \( j \)th entry being equal to zero such that \( x_j = X c_j \) holds true. The task to estimate the vector \( c_j \) can be solved by minimizing

\[
\min_{\|c_j\|_0} \quad s.t. \quad \|X c_j - x_j\|_2^2 = 0 \quad \text{and} \quad c_{jj} = 0. \quad (1)
\]

Since optimizing the \( l_0 \)-norm is difficult, it is often approximated by using the \( l_1 \)-norm. Allowing for data points \( x_j \) contaminated by noise, it is possible to instead optimize

\[
\min_{\|c_j\|_1} \quad s.t. \quad \|X c_j - x_j\|_2^2 \leq \lambda \quad \text{and} \quad c_{jj} = 0. \quad (2)
\]

for a scalar \( \lambda > 0 \).

Given the matrix \( C = [c_1 \cdots c_N] \), class labels can be inferred by means of spectral clustering of the graph \( G = (V, E) \) with \( |V| = N \) vertices corresponding to the points \( x_j \) and edge weights \( E_{ij} \) defined by the matrix \( |C| + |C^T| \). These steps are the basis of the so-called sparse subspace clustering (SSC) algorithm proposed in (Elhamifar and Vidal 2013).

### 4 Subspace Connectivity in \( \mathbb{R}^2 \)

Since only few edge weights are known, the authors of (Nasihatokht and Hartley 2011) raised the question whether connectivity between vertices of the same cluster is always guaranteed. In other words, is the subgraph consisting of the vertices of one particular cluster connected? Their answer was that connectivity is guaranteed – and thus a correct result of SSC – if the subspace dimension is 2. However, at an angle of 4°, the two sets almost form two disconnected clusters.
resulting graph consists of two almost disconnected clusters. Apparent- ly, the question raised in (Nashatmak and Hartley 2011), namely whether there is connectivity, i.e. are there non-zero entries in the affinity matrix, is not sufficient. The affinities shown on the anti-diagonal of the rightmost affinity are > 0 yet their magnitude is negligible as compared to the other entries.

This example shows that spurious clusters can emerge if sparse coefficients are used as affinities. The example is equivalent to points on an affine line (d = 2) normalized to unit length. It therefore readily generalizes to higher-dimensional structures in \( \mathbb{R}^m \), \( m > 2 \). The remainder of this paper focuses on the question how gaps between data on the same subspace influence the solutions \( c_j \) of Eq. (2) and thus the relative connectivity. The relative connectivities then determine if the corresponding vertices of the graph form more or less disconnected clusters.

5 The Bias of \( l_1 \)-Norm Estimators

The analysis in this section solely concentrates on points on a single subspace \( S^{(l)} \). For the sake of simplicity, the superscript (l) is omitted in the following.

Let the angle between two points \( x_{j_1} \) and \( x_{j_2} \), \( j_1 \neq j_2 \) be defined by the inverse cosine of the scalar product of two unit length vectors, \( \langle x_{j_1}, x_{j_2} \rangle = \cos(x_{j_1}^T \cdot x_{j_2}) \). Assume without loss of generality that the points \( x_{j,k} \) are ordered such that \( \langle x_{j_1}, x_{j_2} \rangle \geq \cdots \geq \langle x_{j_j}, x_{j,k} \rangle \).

Noiseless \( l_1 \)-Norm: \( \lambda = 0 \)

Given a point \( y \), a matrix \( X_{-y} \) and the \( K \) points \( x_{y,k}, k = 1, \ldots, K \) of a support set \( S_y \) of \( y \), and let \( c_y \) be a solution to \( \| X_{-y} c_y - y \|_2 \leq \lambda \) so that all entries but those corresponding to the points \( x_{y,k} \) in \( X_{-y} \) are zero. The question then is how does \( \| c_y \|_1 \) change if a different support set \( S_y' \subseteq T_y \) is selected.

The geometry of this case is shown in the left plot of Fig. 2. The intersection of the red dash-dotted line with the line through the point \( x_{y,1} \) indicates the point \( c_y(1)x_{y,1} \). The point \( s \) indicates the intersection of this line with the subspace \( \text{span}(Q) \) spanned by the remaining points \( Q = [x_{y,2} \cdots] \). It can be seen that \( \| s \|_1 \) grows as the angle \( \theta \) between \( \text{span}(Q) \) and \( \text{span}(x - y) \) decreases. This intuition motivates the following proposition:

Proposition 1. The \( l_1 \)-norm of \( c_y \) increases with the angle \( \angle (y, x_{y,1}) \) between \( y \) and \( x_{y,1} \)

\[
\| c_y \|_1 \geq \cot \left( \frac{\| \langle y, x_{y,1} \rangle \| + \rho - \frac{\pi}{2}}{\eta} \right).
\] (3)

Proof. Let \( v_1 = (c_{y,1}x_{y,1} - y)/\| c_{y,1}x_{y,1} - y \|_2 \), \( v_2 = (P_Q y - y)/\| P_Q y - y \|_2 \) where \( P_Q \) is the orthogonal projector onto \( \text{span}(Q) \), and \( f(\langle y, x_{y,1} \rangle) = \cos^{-1} v_1 \). Let \( \alpha = \cos^{-1} v_1 \), \( \beta \) be the angle between the line connecting \( y \) and \( c_{y,1}x_{y,1} \). Assuming that \( \| T_y \| > \| S_y \| \), the support set \( x_{y,2} \cdots \) is selected such that \( \text{span}(Q) \perp \text{span}(y - x_{y,1}) \) in order to minimize \( \| x \|_2 \). Then, we can define the angles \( \alpha_1 = \cos^{-1} -y^T v_2 \), \( \alpha_2 = \pi - \alpha_1 - \pi/2 \), and \( \theta = \alpha + \alpha_1 - \pi/2 \) which motivates

\[
\| x - P_Q y \|_2 = \cot \theta \cdot \| (I - P_Q) y \|_2.
\] (4)

If we now define \( \nu = (I - P_Q) y \), \( \eta = \| \nu \|_2 \), and \( \rho = \cos^{-1}(y^T \nu)/\eta \) we arrive at the claim since \( \| y \|_1 \geq \| y \|_2 \).

Given \( y \) and \( d \) points \( x_{y,1}, \ldots, x_{y,d} \) of a support set \( S_y \) and the corresponding sparse solution \( c_{y} \) to \( \| X_{-y} c_y - y \|_2 = 0 \) let \( \mathcal{H}(y, Q) \) be the plane through \( y \) with orthonormal basis \( Q \). Assume that there is a point \( x_n \in X_T \setminus S_y \) with \( \langle y, x_n \rangle < \langle y, x_{y,1} \rangle \) on the same side of \( \mathcal{H}(y, Q) \) as \( x_{y,1} \). Let \( S'_y \) be the set of indices of the points \( x_{n,1}, x_{y,2}, \ldots, x_{y,d} \) and \( c_{y'} \) be the solution to \( \| X_{-y} c_{y'} - y \|_2 = 0 \).

Proposition 2. The solution \( c_{y'} \) of \( \| X_{-y} c_{y'} - y \|_2 = 0 \) satisfies

\[
\| c_{y'} \|_1 < \| c_y \|_1.
\] (5)

Proof. The proof follows from Prop. 1.

Corollary 1. The \( d \) points \( x_{y,1}, \ldots, x_{y,d} \) in \( X_T \) closest to \( y \) minimize \( \| c \|_1 \) such that \( \| X_{-y} c - y \|_2 = 0 \).

The implication is as follows: The usual assumption is that the data uniformly distributed. In the context of sparse subspace clustering, this implies that data on the same subspace need be uniformly distributed (cf. (Soltanolkotabi and Candes 2012)). If this condition is violated, for instance because points fall into two well separated clusters, the support of the points of one cluster will not include points of the other. This does not change the results in (Soltanolkotabi and Candes 2012) since the central Theorem 2.5 there rests upon the assumption that points on the same subspace are more or less evenly distributed. The idea in this work considers a particular violation of this assumption.

Robust \( l_1 \)-Norm: \( \lambda > 0 \)

Define \( \lambda_1 \) and \( \lambda_2 \) such that \( \lambda_1 + \lambda_2 = \lambda \). The idea here is to define an auxiliary line parallel to \( [x_{y,1}] \) with distance \( \lambda_1 \) into direction of \( y \), and an auxiliary subspace parallel to \( [x_{y,2} \cdots] \) with distance \( \lambda_2 \) also into direction of \( y \). The geometry of this configuration is shown in the middle of Fig. 2.

Proposition 3. The \( l_1 \)-norm of \( c \) increases with the angle \( \langle y, g(\langle y_{y,1}, \lambda_1 \rangle) \rangle \) between \( y \) and \( g(\langle y_{y,1}, \lambda_1 \rangle) \)

\[
\| c \|_1 \geq h_2(\eta, \lambda_2) \cdot \| f(\langle y, g(\langle y_{y,1}, \lambda_1 \rangle) \rangle + h_1(\rho, \lambda_2) - \frac{\pi}{2}) \.
\] (6)

Proof. The idea for the robust \( l_1 \)-norm estimator is to create auxiliary variables \( \lambda_1 \) and \( \lambda_2 \) so that \( \lambda = \lambda_1 + \lambda_2 \). These two are used to define two affine spaces parallel to \( \text{span}(x_{y,1}) \) and \( \text{span}(Q) \) with distances \( \lambda_1 \) and \( \lambda_2 \) such that the distances to \( y \) are reduced by \( \lambda_1 \) and \( \lambda_2 \). The function \( g \) constructs the affine space parallel to \( \text{span}(x_{y,1}) \). The functions \( h_1 \) and \( h_2 \) modify the projection of \( y \) not onto \( \text{span}(Q) \) but its parallel affine space, and correct \( \rho \), respectively.
Lasso
The configuration for the lasso estimator
\[
\min \|c\|_1 + \lambda \cdot \|X_{-y}c - y\|_2
\]
(7)
is very similar to the one shown for the robust l1-norm and is
shown in the right plot of Fig. 2. We can now define a variable
\(\epsilon = \epsilon_1 + \epsilon_2\) that absorbs the \(l_2\)-error. Here \(\epsilon_1\) indicates the
distance of an auxiliary line parallel to span \([y]\) with distance
\(\epsilon_1\) and an auxiliary subspace parallel to span \([x_{y,2}, \ldots]\)
with distance \(\epsilon_2\). The difference to the robust \(l_1\)-norm is that
\(\epsilon\) is now variable.

Due to the close similarity between the two estimators,
Proposition 3 can easily be adapted to the lasso estimator:

**Proposition 4.** The \(l_1\)-norm of \(c\) is increases with the angle
\(\angle (y, g(x_{y,1}, \lambda_1))\) between \(y\) and \(g(x_{y,1}, \lambda_1)\)
\[
\|c\|_1 \geq h_2(\eta, \epsilon_2) \cdot \cot \left( f(\angle (y, g(x_{y,1}, \epsilon_1)) + h_1(\rho, \epsilon_2) - \frac{\pi}{2} \right).
\]
(8)

**Proof.** The proof of Proposition 4 is identical as the one of
Prop. 3 except that the admissible error \(\epsilon = \epsilon_1 + \epsilon_2\) is now variable.

### 6 Selective Pursuit

#### Selective Dantzig Selector
Inspired by the Dantzig selector (Candes and Tao 2007; Qu
and Xu 2015), \(X^* = X^\top\), we notice that the product \(X^*X\) measures correlations to the subspace if we define a modified
Dantzig selector
\[
X^* = X^\top_{S_y^c}.
\]
(9)
The matrix \(X_{S_y^c}\) consists of those columns of \(X\) that are
the support of \(y\). Since points on the same subspace have
smaller distances to it than points from different subspaces,
we can expect that the products \(X^*x_j, j = 1, \ldots, N\ \{\mathcal{I}(y), \supp(y)\}\)
to be large if \(j \in T_y\).

The idea proposed here is as follows: The support set \(S_y\)
of \(y\) usually consists of points from the set of neighbors of
\(y\). Rather than stopping at this point, it is possible to select
additional points, if their coherences with \(\text{span}(X_{S_y})\) is are
large.

Let \(S_y^c\) be the extended support set of \(y\) which consists
of the original support set, and the additionally selected
points. If the modified Dantzig selector is now taken to
be \(X^* = X^\top_{S_y^c}\), the selection process becomes more and more
influenced by noise. Because noise causes spurious singular
values in \(X^*\), we reduce their effect by scaling with
\[
\rho = \text{trace}(X^*)^\top X^*.
\]
(10)

Since the trace of the square of a matrix equals the sum of
its singular values, this reduces the effect of the spurious
singular values. Finally, new extended support vectors are
chosen if for a threshold \(\delta\)
\[
\arg \max_j \frac{\|X^*x_j\|_2^2}{\rho} > \delta, \quad j \in \{1, \ldots, N\} \ \{\mathcal{I}(y), S_y^c\}.
\]
(11)

#### Subspace Selector
To further reduce the effect of spurious singular values, it is possible to fit an \(d^{(l)} \approx \text{length}(S_y)\) dimensional orthonormal
basis \(B\) to \(X^*\)
\[
B = \arg \max_j \|BX_j\|_2^2, \quad j = 1, \ldots, \text{length}(S_y^c).
\]
(12)

We can now choose additional points based on the Eucli-
dean distance to the subspace \(\text{span}(B)\) by
\[
\arg \max_j \|P_Bx_j\|_2^2 > \delta, \quad j \in \{1, \ldots, N\} \ \{\mathcal{I}(y), S_y^c\}.
\]
(13)

for a threshold \(\delta\). Here, \(P_B\) is the orthogonal projector onto
\(\text{span}(B)\).

### 7 Experiments
To experimentally demonstrate the effect of disconnectivity
between clusters of points on the same subspace, we created
two normally distributed point clouds on the unit sphere in

The topic of this work is sparse subspace clustering. The principle idea behind this class of algorithms is to approximate each data point by a linear combination of as few other data points as possible. This problem is often approached by solving $l_1$-norm problems. The sparse coefficients are used to define edge weights of a graph. Computing a minimum cut through this graph reveals which points are located on the same subspace.

Since there are few edges per vertex, it could be possible that vertices of a graph form disconnected subgraphs although the corresponding points lie on the same subspace. In a previous work, the possibility for this to happen was answered negatively if the subspace dimension is or 3. Four 4-dimensional subspaces, an example was given when disconnected subgraphs at an angle of 45° (left column in Fig. 4).

8 Summary and Conclusions

The two algorithms proposed in Sec. 6 are compared against a lasso-estimator

$$\|c_z\|_1 + \lambda \|Y_jc_z - y\|_2$$

which is optimized using CVX (CVX). We used only Eq. (16) because the results of Eq. (2) are almost identical. Further, an algorithm to estimate the $l_0$-norm using a variational approach (Logsdon et al. 2012) was included into the comparison. This is motivated by the idea that the $l_0$-norm is not supposed to be effected by the bias that the $l_1$-norm estimators have.

The results are shown in the plots in Fig. 3. The left plot corresponds to no noise, the middle one to a medium noise ($\sigma = 1\%$), and the right plot to strong noise ($\sigma = 2\%$). The $x$-axes correspond to angles between the clusters (cf. the explanation in Sec. 7), the $y$-axes to the relative connectivity $\xi$ defined by Eq. (15). The dotted green line indicates the lasso-method (16), the dash-dotted red line the $l_0$-norm optimization, the dashed blue line the proposed Selective Dantzig Selector, and the solid black line the proposed Subspace Selector.

As can be seen, the two proposed algorithms do not suffer as much from the gap between the two clusters as the other algorithms. Very surprisingly, the $l_0$-norm estimator is also strongly affected.

Average affinity matrices computed by three different methods are shown in Fig. 4. Apparently, for the lasso estimator the two clusters cause almost disconnected subgraphs at an angle of 45° (left column in Fig. 4).

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Since there are few edges per vertex, it could be possible that vertices of a graph form disconnected subgraph although the corresponding points lie on the same subspace. In a previous work, the possibility for this to happen was answered negatively if the subspace dimension is 2 or 3. Four 4-dimensional subspaces, an example was given when dis-
connectivity occurs.

This work investigates the relaxed definition of not exact disconnectivity but relative disconnectivity. This means that all edge weights between two subgraphs are so low that the minimum cut either separates the two subgraphs, or – if the number of clusters is fixed – estimates a very erroneous result.

It is shown that this problem is caused by a gap between the points corresponding to the vertices of the subgraphs. In other words, if points on the same subspace fall into separate clusters, then the two subgraphs necessarily have a so low connectivity that a successful clustering is no longer possible.

This problem does not invalidate the result published in (Soltanolkotabi and Candès 2012), since the authors explicitly state that their Theorem 2.5 requires well distributed points. The idea in this work is to consider the case that this assumption is not given.

Two different algorithms are proposed which are not as susceptible to the shown effect. Surprisingly, even an estimator of the l0-norm suffers from biased distributions of points.

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