Canonical almost-Kähler metrics dual to general plane-fronted wave Lorentzian metrics

Mehdi Lejmi$^1$ · Xi Sisi Shen$^2$

Received: 22 August 2022 / Accepted: 14 February 2023 / Published online: 18 March 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
In the compact setting, Aazami and Ream (Lett Math Phys 112(4):17, 2002) proved that Riemannian metrics dual to a class of Lorentzian metrics, called (compact) general plane-fronted waves, are almost-Kähler. In this note, we explain how to construct extremal and second-Chern–Einstein non-Kähler almost-Kähler metrics dual to those general plane-fronted waves.

Keywords
Extremal almost-Kähler Metrics · Second-Chern–Einstein almost-Kähler metrics · Lorentzian metrics · pp-wave spacetimes

Mathematics Subject Classification Primary 53C55; Secondary 53B35

1 Introduction

Let $(M, g)$ be a Riemannian manifold of dimension $2n$. The Riemannian metric $g$ is called almost-Kähler if there is a $g$-orthogonal almost-complex structure $J$ on $M$ such that the induced 2-form $\omega$ defined by $\omega(\cdot, \cdot) := g(J \cdot, \cdot)$ is a symplectic form. When $J$ is integrable, $g$ is a Kähler metric. On an almost-Kähler manifold $(M, g, \omega, J)$, the 1-parameter family of canonical Hermitian connections introduced by Gauduchon in [21] all coincide and are equal to the canonical Hermitian connection $\nabla$ which is the unique connection preserving the almost-Kähler structure $(g, \omega, J)$ and whose torsion is the Nijenhuis tensor of $J$ [21, 29].

The authors are warmly grateful to Amir Babak Aazami for giving his insights on pp-wave spacetimes and his comments on the note. The authors are also thankful to Daniele Angella, Giuseppe Barbaro, Abdellah Lahdili and Ali Maalaoui for very useful discussions. The first author is supported by the Simons Foundation Grant #636075.

Mehdi Lejmi
mehdi.lejmi@bcc.cuny.edu

Xi Sisi Shen
xss@math.columbia.edu

1 Department of Mathematics, Bronx Community College of CUNY, Bronx, NY 10453, USA

2 Department of Mathematics, Columbia University, New York, NY 10027, USA
(in particular $\nabla$ is the Chern connection on $(M, g, \omega, J)$). More explicitly,

$$\nabla_X Y = D^g_X Y - \frac{1}{2} J \left( D^g_X J \right) Y,$$

where $D^g$ is the Levi–Civita connection with respect to $g$ and $X, Y$ are vector fields on $M$. The Hermitian scalar curvature $s^H$ of the almost-Kähler metric $g$ is obtained by taking twice the trace of $R^\nabla$ the curvature of $\nabla$ with respect to $\omega$.

Donaldson [18] showed how one can use a formal framework of the geometric invariant theory [31] in order to define natural representatives of almost-Kähler metrics called extremal almost-Kähler metrics [8, 26, 27]. An almost-Kähler metric $g$ is extremal if the symplectic gradient of $s^H$ is an infinitesimal isometry of $J$ (or equivalently a Killing vector field with respect to $g$). Extremal almost-Kähler metrics include almost-Kähler metrics with constant Hermitian scalar curvature and Calabi extremal Kähler metrics [14]. For more results about extremal non-Kähler almost-Kähler metrics, we refer the reader to [2, 13, 16, 17, 23, 25–28, 40, 41]. In addition to extremal almost-Kähler metrics, other possible canonical almost-Kähler metrics are the so-called second-Chern–Einstein metrics. Namely, an almost-Kähler metric is second-Chern–Einstein if $R^\nabla(\omega)$ is proportional to $\omega$ (here $s^H$ is not necessarily constant). When $J$ is integrable, second-Chern–Einstein almost-Kähler metrics are Kähler–Einstein. On complex manifolds, Hermitian metrics satisfying the second-Chern–Einstein condition were studied for instance in [3, 4, 10, 17, 20, 22, 30, 35, 39]. We remark that very few examples of compact Hermitian non-Kähler second-Chern–Einstein manifolds are known. For instance, by [22, Theorem 2], the only 4-dimensional compact Hermitian non-Kähler second-Chern–Einstein manifold is the Hopf surface.

Recently, Aazami and Ream [1] demonstrate a one-to-one correspondence between certain almost-Kähler metrics and a class of Lorentzian metrics called pp-wave spacetimes (see, e.g., [11, Chapter 13] and [1] and the references therein). In the compact setting, a generalization of pp-wave spacetimes is given by general plane-fronted wave Lorentzian metrics [15]. More precisely, given a compact Riemannian manifold $(M, g_M)$, a (compact) general plane-fronted wave Lorentzian metric $h$ is defined on $S^1 \times S^1 \times M$ by

$$h := 2 d\varphi d\theta + H d\theta^2 + g_M,$$

where $\varphi, \theta$ denote the standard angular coordinates on $S^1 \times S^1$ and $H$ a function on $S^1 \times M$ independent of the first angular coordinate $\varphi$. If we suppose that $g_M$ is almost-Kähler then it turns out that the Riemannian metric $g$ dual to general plane-fronted wave Lorentzian metric $h$ with respect to the vector field $T := \frac{1}{2} (H + 1) \partial_\varphi - \partial_\theta$, is almost-Kähler [1, Theorem 2]. More explicitly, the metric $g$ has the form

$$g := h + 2 T^b \otimes T^b,$$  \hspace{1cm} (1)

where $\flat$ the $h$-Lorentzian dual of $T$. In this paper, we first construct extremal almost-Kähler metrics of the form (1).

**Theorem** (See Theorem 8) Let $(M, g_M)$ be a closed extremal almost-Kähler manifold with a non-constant Hermitian scalar curvature $s^H_M$. Suppose that $H = s^H_M$. Then, on $S^1 \times S^1 \times M$, the metric $g$, of the form (1) dual to a general plane-fronted wave Lorentzian metric, is an extremal almost-Kähler metric.

Then, we focus on second-Chern–Einstein almost-Kähler metrics and we obtain the following:
**Theorem** (see Theorem 11) Let $\Sigma$ be a compact Riemann surface equipped with a Riemannian metric $g_\Sigma$. Suppose that $H$ is a non-constant function on $\Sigma$. Then, the metric $g$, of the form (1) dual to a general plane-fronted wave Lorentzian metric, is a second-Chern–Einstein non-Kähler almost-Kähler metric on $\mathbb{S}^1 \times \mathbb{S}^1 \times \Sigma$ if and only if
\[
\|\text{grad}^g H\|^2 = 2s^H_\Sigma,
\]
where $\text{grad}^g H$ is the gradient of $H$ with respect to the metric $g$ and $s^H_\Sigma$ is twice the Gaussian curvature of $(\Sigma, g_\Sigma)$. In particular, $\Sigma$ is isomorphic to the sphere $\mathbb{S}^2$. Moreover, with a suitable normalization of $H$, a metric $g_\Sigma$ satisfying Eq. (2) always exists.

In the present paper, after Preliminaries, we explain in Sect. 3 how to construct extremal almost-Kähler metrics of the form (1) dual to a general plane-fronted wave Lorentzian metric. We also remark that when $(M, g_M)$ has a constant Hermitian scalar curvature then the metric $g$, of the form (1) dual to a general plane-fronted wave Lorentzian metric, has also a constant Hermitian scalar curvature (see Theorem 6). In Sect. 4, first we study the second-Chern–Einstein condition in dimension 4 and we obtain the existence of second-Chern–Einstein non-Kähler almost-Kähler metrics on $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^2$ of the form (1) dual to a general plane-fronted waves. Then, we construct an example in dimension 6 (see Theorem 14).

## 2 Preliminaries

Let $(M, g)$ be a Riemannian manifold of dimension $2n$. The metric $g$ is almost-Kähler if there is a $g$-orthogonal almost-complex structure $J$ i.e. $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ such that the 2-form $\omega(\cdot, \cdot) := g(J\cdot, \cdot)$ is $d$-closed i.e. $d\omega = 0$ where $d$ is the exterior derivative. The almost-complex structure $J$ is integrable if and only if the Nijenhuis tensor $N$ defined by
\[
4N(X, Y) := [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY],
\]
vanishes [32] (here $X, Y$ are vector fields on $M$). For an almost-Kähler metric, we have the following
\[
g \left(D^g_X JY, Z\right) = 2g \left(JX, N(Y, Z)\right),
\]
where $D^g$ is the Levi–Civita connection of $g$. In particular, $D^g$ preserves $J$ if and only if $g$ is Kähler. The canonical Hermitian connection $\nabla$ is defined by
\[
\nabla_X Y = D^g_X Y - \frac{1}{2} J \left( D^g_X J \right) Y.
\]
The connection $\nabla$ preserves the almost-Kähler structure $(g, \omega, J)$ and its torsion is given by the Nijenhuis tensor $N$. We denote by $R^\nabla$ the curvature of $\nabla$ and we use the convention $R^\nabla_{X, Y} = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$. The first-Chern–Ricci form $\rho^\nabla$ is defined by
\[
\rho^\nabla(X, Y) = \frac{1}{2} \sum_{i=1}^{2n} g \left(R^\nabla_{X, Y} e_i, J e_i\right),
\]
where $\{e_1, e_2 = Je_1, \ldots, e_{2n-1}, e_{2n} = Je_{2n-1}\}$ is a $J$-adapted $g$-orthogonal local frame of the tangent bundle. The 2-form $\rho^\nabla$ is $d$-closed and it is a representative of $2\pi c_1(TM, J)$ the first Chern class of $M$. On the other hand, the second-Chern–Ricci form $r$ is defined by
\[
r(X, Y) = \frac{1}{2} \sum_{i=1}^{2n} g \left(R^\nabla_{e_i, Je_i} X, Y\right).
Equivalently, \( r = R^V(\omega) \). The 2-form \( r \) is \( J \)-invariant but not closed in general.

**Definition 1** An almost-Kähler metric is second-Chern–Einstein if

\[
r = \lambda \omega,
\]

for some function \( \lambda \).

We would like to express the difference between \( \rho V \) and \( r \) in terms of the Nijenhuis tensor \( N \). Denote by \( \rho^* \) the \( \ast \)-Ricci form defined as

\[
\rho^*(X,Y) = R^g(\omega)(X,Y) = \frac{1}{2} \sum_{i=1}^{2n} g \left( R^g_{e_i,J e_i} X, Y \right),
\]

where \( R^g \) is the curvature of the Levi–Civita connection \( D^g \). Then, from [7], we have

\[
\rho^V(X,Y) = \rho^*(X,Y) - \frac{1}{4} \sum_{i=1}^{2n} g \left( (D^g_J J)^2 e_i, e_i \right)
\]

On the other hand from [10, Proposition 1] and Eq. (3), it is easy to see that

\[
r(X,Y) = (\rho^*)^{J,+}(X,Y) + \frac{1}{4} \sum_{i=1}^{2n} g \left( (D^g_J J)^2 e_i, e_i \right),
\]

where \((\cdot)^{J,+}\) denotes the \( J \)-invariant part. We obtain then

\[
r(X,Y) = \left( \rho^V \right)^{J,+}(X,Y)
- \frac{1}{4} \sum_{i=1}^{2n} g \left( (D^g_J J)^2 e_i, e_i \right) + \frac{1}{4} \sum_{i=1}^{2n} g \left( (D^g_J J)^2 e_i, e_i \right).
\]

From Eq. (3), we get the following formula

**Corollary 2** Let \((M,g)\) be an almost-Kähler manifold. Then,

\[
r(X,Y) = \left( \rho^V \right)^{J,+}(X,Y) + \sum_{i,k=1}^{2n} g \left( J X, N(e_i, e_k) \right) g \left( Y, N(e_i, e_k) \right)
+ \sum_{i,k=1}^{2n} g \left( N(X, e_k), e_i \right) g \left( N(Y, e_k), J e_i \right),
\]

where \( \{e_1, e_2 = J e_1, \ldots, e_{2n-1}, e_{2n} = J e_{2n-1} \} \) is a \( J \)-adapted \( g \)-orthonormal local frame of the tangent bundle.

The trace of \( \rho^V \) and \( r \) with respect to \( \omega \) is equal to the Hermitian scalar curvature

\[
s^H = \sum_{i,k=1}^{2n} \rho^V(e_i, J e_i) = \sum_{i=1}^{2n} r(e_i, J e_i).
\]

On the other hand the Riemannian scalar curvature \( s^g \) is defined as

\[
s^g = \sum_{i,j=1}^{n} g \left( R^g_{e_i,e_j} e_i, e_j \right).
\]
and coincides with $s^H$ when $g$ is Kähler.

An almost-Kähler metric $g$ is extremal [26] if the symplectic gradient $\nabla_{\omega}s^H = J\nabla^g s^H$ of the Hermitian scalar curvature $s^H$ is a Killing vector field with respect to $g$ (here $\nabla^g$ denotes the $g$-Riemannian gradient).

Given a Lorentzian metric $h$, if $T$ is a nowhere vanishing timelike vector field (i.e. $h(T, T) < 0$) then $h$ has a Riemannian dual given by

$$ g := h + 2 T^b \otimes T^b, $$

where $b$ the $h$-Lorentzian dual of $T$ (we assume here that $h(T, T) = -1$). Aazami and Ream [1] constructed then almost-Kähler metrics dual to a certain class of Lorentzian metrics.

**Theorem 3** [1, Theorem 2] Let $(M, g_M)$ be a closed almost-Kähler manifold. Let $\varphi, \theta$ denote the standard angular coordinates on $S^1 \times S^1$ and $H$ a function on $S^1 \times M$ independent of the first angular coordinate $\varphi$. Consider the (compact) general plane-fronted wave Lorentzian metric $h$ defined on $S^1 \times S^1 \times M$:

$$ h := 2 d\varphi \, d\theta + H \, d\theta^2 + g_M. $$

With respect to the vector field

$$ T := \frac{1}{2} (H + 1) \partial_\varphi - \partial_\theta, $$

let $g$ denote the Riemannian metric dual to $h$:

$$ g := h + 2 T^b \otimes T^b, $$

where $b$ the $h$-Lorentzian dual of $T$. Then $g$ is an almost-Kähler metric on $S^1 \times S^1 \times M$ which is not a warped product, and which is Kähler if and only if $H$ is constant on $M$ and $(M, g_M)$ is Kähler. Furthermore, the fundamental 2-form of $g$ is also co-closed with respect to the Lorentzian metric $h$.

### 3 Extremal almost-Kähler metrics dual to general plane-fronted wave Lorentzian metrics

Let $(M, g_M, \omega_M, J_M)$ be a closed almost-Kähler manifold of dimension $2n$. On the manifold $M$, we use the Darboux coordinates $\{z_1, \ldots, z_n, t_1, \ldots, t_n\}$ defined on an open set. The symplectic form $\omega_M$ has the form

$$ \omega_M = \sum_{i=1}^n dz_i \wedge dt_i. $$

Let $\varphi, \theta$ denote the standard angular coordinates on $S^1 \times S^1$. Let $H$ be an arbitrary smooth function on $S^1 \times M$ that is independent of the angular coordinate $\varphi$. Then, on $S^1 \times S^1 \times M$, we consider the almost-Kähler metric $g$ of the form (1) dual to a (compact) general plane-fronted wave Lorentzian metric. Then, the metric $g$ can be expressed as

$$ g = g_M + H (d\varphi \otimes d\theta + d\theta \otimes d\varphi) + \frac{1}{2} (1 + H^2) \, d\theta^2 + 2 \, d\varphi^2. $$ (4)
The almost-Kähler metric \( g \) is compatible with the following symplectic form \( \omega \) (i.e. \((g, \omega)\) is an almost-Kähler structure)
\[
\omega = \omega_M + d\theta \wedge d\phi. \tag{5}
\]
The metric \( g \) is Kähler if and only if \( H \) is constant on \( M \) and \( g_M \) is Kähler \cite[Theorem 2]{article1}.

We would like to compute the first-Chern–Ricci form \( \rho \) and the Hermitian scalar curvature \( s_H \) on the almost-Kähler manifold \((S^1 \times S^1 \times M, g, \omega)\). We apply the formula computed in \cite[Equation (4.2)]{article22} using Darboux coordinates.

**Proposition 4** Let \( \rho^V \) be the first-Chern–Ricci form of the almost-Kähler manifold \((S^1 \times S^1 \times M, g, \omega)\). Then,
\[
\rho^V = \rho^V_M + \frac{1}{2} \sum_{i=1}^n H_{\theta z_i} d\theta \wedge dz_i + \frac{1}{2} \sum_{i=1}^n H_{\theta t_i} d\theta \wedge dt_i,
\]
where \( \rho^V_M \) is the first-Chern–Ricci form of \((M, g_M, \omega_M, J_M)\). Here we use the notation \( H_{\theta z_i} = \frac{\partial^2 H}{\partial \theta \partial z_i} \), etc.

**Corollary 5** Let \( s^H \) be the Hermitian scalar curvature of the almost-Kähler manifold \((S^1 \times S^1 \times M, g, \omega)\). Then,
\[
s^H = s^H_M,
\]
where \( s^H_M \) is the Hermitian scalar curvature of \((M, g_M, \omega_M, J_M)\).

**Proof** This is a direct consequence of Proposition 4. \(\square\)

As a consequence of Corollary 5, we obtain the following

**Theorem 6** Let \((M, g_M, \omega_M, J_M)\) be a closed almost-Kähler manifold of constant Hermitian scalar curvature. Then, on \(S^1 \times S^1 \times M\), the metric \( g \), of the form (1) dual to a general plane-fronted wave Lorentzian metric, has a constant Hermitian scalar curvature.

We also get the following as a consequence of Proposition 4.

**Corollary 7** Let \((M, g_M, \omega_M, J_M)\) be a closed non-Kähler almost-Kähler manifold with a zero first-Chern–Ricci form \( \rho^V_M = 0 \). Suppose that \( H \) is a function on \( M \). Then, the metric \( g \), of the form (1) dual to a general plane-fronted wave Lorentzian metric, defines on \(S^1 \times S^1 \times M\) a non-Kähler almost-Kähler metric with vanishing first-Chern–Ricci form \( \rho^V = 0 \).

Now, we would like to discuss the construction on \(S^1 \times S^1 \times M\) of extremal almost-Kähler metrics with non-constant Hermitian scalar curvature.

**Theorem 8** Let \((M, g_M, \omega_M, J_M)\) be a closed extremal almost-Kähler manifold with a non-constant Hermitian scalar curvature \( s^H_M \). Suppose that \( H = s^H_M \). Then, on \(S^1 \times S^1 \times M\), the metric \( g \), of the form (1) dual to a general plane-fronted wave Lorentzian metric, is an extremal almost-Kähler metric.

**Proof** We need to prove that \( \text{grad}_\omega s^H \) is a Killing vector field of \( g \), where \( \omega \) is given by (5). From Corollary 5, it follows that \( s^H = s^H_M \) and so \( \text{grad}_\omega s^H = \text{grad}_\omega s^H_M \). Consider the \( J_M \)-adapted \( g_M \)-orthonormal local frame of the tangent bundle of \( M \) \( \{X_1, J MX_1, \ldots, X_n, J MX_n\} \). We define as in \cite[Theorem 2]{article1} \( T = \frac{1}{2} (H + 1) \partial \psi - \partial \theta \).
Now, we prove that the almost-complex structure \( J_M \) is then extended to an almost-complex structure \( J \) on \( S^1 \times S^1 \times M \) by defining
\[
JT = \frac{1}{2}(H - 1)\partial \varphi - \partial \theta.
\]
The frame \( \{X_1, JX_1, \ldots, X_n, JX_n, T, JT\} \) defines on \( S^1 \times S^1 \times M \) a \( J \)-adapted orthonormal local frame. The \( g \)-Riemannian duals of \( T \) and \( JT \) are the 1-forms
\[
T^b = \frac{1}{2}(H - 1)d\theta + d\varphi, \quad (JT)^b = -\frac{1}{2}(H + 1)d\theta - d\varphi.
\]
Then, the Riemannian metric \( g \) is given by
\[
g = g_M + T^b \otimes T^b + (JT)^b \otimes (JT)^b.
\]
Now, we prove that \( \mathcal{L}_{\text{grad}_\omega s_M^H}g = 0 \), where \( \mathcal{L} \) is the Lie derivative. First, for any two vectors \( X_i, X_j \) in \( \{X_1, JX_1, \ldots, X_n, JX_n\} \), we have
\[
\left( \mathcal{L}_{\text{grad}_\omega s_M^H}g \right)(X_i, X_j) = \left( \left( \mathcal{L}_{\text{grad}_\omega s_M^H}T^b \right) \otimes T^b + \mathcal{L}_{\text{grad}_\omega s_M^H}(JT)^b \otimes (JT)^b \right)(X_i, X_j) = 0,
\]
using (8) and because \( \text{grad}_\omega s_M^H \) is a Killing vector field of \( g_M \). For the same reasons
\[
\left( \mathcal{L}_{\text{grad}_\omega s_M^H}g \right)(T, X_i) = \left( \mathcal{L}_{\text{grad}_\omega s_M^H}g \right)(JT, X_i) = 0.
\]
Moreover using Cartan formula and the fact that \( g(\text{grad}_\omega s_M^H, T) = g(\text{grad}_\omega s_M^H, JT) = 0 \) we get
\[
\left( \mathcal{L}_{\text{grad}_\omega s_M^H}g \right)(T, T) = \left( \left( \mathcal{L}_{\text{grad}_\omega s_M^H}T^b \right) \otimes T^b + \mathcal{L}_{\text{grad}_\omega s_M^H}(JT)^b \otimes (JT)^b \right)(T, T),
\]
\[
= 2 \left( \mathcal{L}_{\text{grad}_\omega s_M^H}T^b \right)(T),
\]
\[
= 2 dT^b(\text{grad}_\omega s_M^H, T),
\]
\[
= (dH \wedge d\theta)(\text{grad}_\omega s_M^H, T),
\]
\[
= - dH \wedge (T^b + (JT)^b) (\text{grad}_\omega s_M^H, T),
\]
\[
= - ds_M^H(\text{grad}_\omega s_M^H) = 0.
\]
Here we use the assumption that \( H = s_M^H \) and the fact that \( -d\theta = T^b + (JT)^b \) which follows from (6) and (7). In a similar way,
\[
\left( \mathcal{L}_{\text{grad}_\omega s_M^H}g \right)(JT, JT) = \left( \left( \mathcal{L}_{\text{grad}_\omega s_M^H}T^b \right) \otimes T^b + \mathcal{L}_{\text{grad}_\omega s_M^H}(JT)^b \otimes (JT)^b \right)(JT, JT),
\]
\[
= 2 \left( \mathcal{L}_{\text{grad}_\omega s_M^H}(JT)^b \right)(JT),
\]
\[
= 2 d(JT)^b(\text{grad}_\omega s_M^H, JT),
\]
\[
= - (dH \wedge d\theta) (\text{grad}_\omega s_M^H, JT),
\]
\[
dH \wedge (T^b + (JT)^b) (\text{grad}_\omega s_M^H, JT),
\]
\[
= ds_M^H(\text{grad}_\omega s_M^H) = 0.
\]
\[
\left( \mathcal{L}_{\text{grad} \omega}^s H_M \right)^b (T, JT) = \left( \mathcal{L}_{\text{grad} \omega}^s H_M T^b \right) \odot T^b + \mathcal{L}_{\text{grad} \omega}^s H_M (JT)^b \odot (JT)^b (T, JT),
\]

\[
= \left( \mathcal{L}_{\text{grad} \omega}^s H_M T^b \right) (JT) + \left( \mathcal{L}_{\text{grad} \omega}^s H_M (JT)^b \right) (T),
\]

\[
= dT^b (\text{grad} \omega H_M, JT) + d(JT)^b (\text{grad} \omega H_M, T),
\]

\[
= \frac{1}{2} (dH \wedge d\theta) \left( \text{grad} \omega H_M, JT \right) - \frac{1}{2} (dH \wedge d\theta) \left( \text{grad} \omega H_M, T \right),
\]

\[
= -\frac{1}{2} dH \wedge (T^b + (JT)^b) (\text{grad} \omega H_M, JT)
\]

\[
+ \frac{1}{2} dH \wedge (T^b + (JT)^b) \left( \text{grad} \omega H_M, T \right) = 0.
\]

The theorem follows. \hfill \Box

4 Second-Chern–Einstein almost-Kähler metrics dual to general plane-fronted wave Lorentzian metrics

4.1 The four-dimensional case

Let \( \Sigma \) be a compact Riemann surface with a Riemannian metric \( g_\Sigma \) and a complex structure \( J_\Sigma \). In this section, we would like to investigate the existence of second-Chern–Einstein almost-Kähler metrics of the form (1) dual to a general plane-fronted wave Lorentzian metric on \( S^1 \times S^1 \times \Sigma \). On \( \Sigma \), we use the isothermal coordinates \( \{ x, y \} \) so that

\[
g_\Sigma = e^{2u} (dx \otimes dx + dy \otimes dy),
\]

for some function \( u = u(x, y) \).

We have the \( g_\Sigma \)-unit vectors

\[
X = e^{-u} \partial_x, \quad J_\Sigma X = e^{-u} \partial_y.
\]

The complex structure \( J_\Sigma \) is extended to an almost-complex structure \( J \) on \( S^1 \times S^1 \times \Sigma \) by considering the vector fields

\[
T = \frac{1}{2} (H + 1) \partial_\varphi - \partial_\theta,
\]

\[
JT = \frac{1}{2} (H - 1) \partial_\varphi - \partial_\theta.
\]

On the almost-Kähler manifold \( (S^1 \times S^1 \times \Sigma, g, J) \), where \( g \) is of the form (1), we can express the second-Chern-Ricci form \( r \) in terms of the first-Chern–Ricci form \( \rho^\nabla \) using Corollary 2 or [10, Proposition 24]

\[
r = \left( \rho^\nabla \right)^{J,+} - \frac{1}{4} \| N \|^2 \omega + 4 (N(X, T))^b \wedge (J N(X, T))^b,
\]

(9)

where \( (\cdot)^{J,+} \) denotes the \( J \)-invariant part, \( N \) is the Nijenhuis tensor of \( J \), \( \| N \|^2 = 8 \| N(X, T) \|^2 \) and \( \flat \) denotes the \( g \)-Riemannian dual.

Now, we compute \( r \) using (9). First, we have

\[
[X, T] = [X, JT] = \frac{X(H)}{2} (T - JT),
\]

\( \Box \) Springer
\[ [JX, T] = [JX, JT] = \frac{JX(H)}{2} (T - JT). \]

Hence,
\[
4N(X, T) = [JX, JT] - [X, T] - J[X, JT] - J[JX, T],
\]
\[
= \frac{JX(H)}{2} (T - JT) - \frac{X(H)}{2} (T - JT) - \frac{X(H)}{2} (JT + T) - \frac{JX(H)}{2} (JT + T),
\]
\[
= -X(H)T - JX(H)JT.
\]

We obtain that
\[
4N(X, T) = \frac{(X(H))^2 + (JX(H))^2}{4} (T^b \wedge JT^b),
\]
\[
= \frac{(X(H))^2 + (JX(H))^2}{4} d\theta \wedge d\varphi. \quad (10)
\]

On the other hand, we compute \( \rho^\nabla \) using isothermal coordinates on \( \Sigma \) and we get the following

**Proposition 9** On \((\mathbb{S}^1 \times \mathbb{S}^1 \times \Sigma, g, J)\), we have
\[
\rho^\nabla = -dJdu + \frac{1}{2}H_x \theta \ d\theta \wedge dx + \frac{1}{2}H_y \theta \ d\theta \wedge dy,
\]

where \( H_x \theta = \frac{\partial^2 H}{\partial x \partial \theta} \), etc.

**Proof** We follow computations done in [5, 6]. Two independent \( J \)-anti-invariant forms \( \phi, J\phi = -\phi(J\cdot, \cdot) \) of square norms 2 are given by
\[
\phi = X^b \wedge T^b - (JX)^b \wedge (JT)^b,
\]
\[
= e^u \left( dx \wedge \left( \frac{1}{2} (H - 1)d\theta + d\varphi \right) + dy \wedge \left( \frac{1}{2} (H + 1)d\theta + d\varphi \right) \right),
\]
and
\[
J\phi = X^b \wedge (JT)^b + (JX)^b \wedge T^b,
\]
\[
= e^u \left( dy \wedge \left( \frac{1}{2} (H - 1)d\theta + d\varphi \right) - dx \wedge \left( \frac{1}{2} (H + 1)d\theta + d\varphi \right) \right).
\]

We then have
\[
d\phi = \tau_\phi \wedge \phi, \quad d(J\phi) = \tau_{J\phi} \wedge J\phi,
\]
for some 1-forms \( \tau_\phi, \tau_{J\phi} \). We obtain that
\[
\tau_\phi = \left( u_x + \frac{H_x}{2} - \frac{H_y}{2} \right) dx + \left( u_y + \frac{H_x}{2} - \frac{H_y}{2} \right) dy = du + \left( \frac{H_x}{2} - \frac{H_y}{2} \right) (dx + dy),
\]
and
\[
\tau_{J\phi} = \left( u_x - \frac{H_x}{2} - \frac{H_y}{2} \right) dx + \left( u_y + \frac{H_x}{2} + \frac{H_y}{2} \right) dy = du - \left( \frac{H_x}{2} + \frac{H_y}{2} \right) (dx - dy).
\]

The first-Chern–Ricci form is given by
\[
\rho^\nabla = -\frac{1}{2} d \left( J\tau_\phi + J\tau_{J\phi} \right).
\]
The proposition follows. □

Combining (9) and (10) and Proposition 9, we get the following

**Corollary 10** On \((S^1 \times S^1 \times \Sigma, g, J)\), the second-Chern-Ricci form is given by

\[
r = -dJ du + \frac{H_x}{4} d\theta \wedge dx + \frac{H_y}{4} d\theta \wedge dy - \frac{H_{xy}}{2} d\phi \wedge dx
\]

As a consequence of Corollary 10, we get the following theorem

**Theorem 11** Let \(\Sigma\) be a compact Riemann surface equipped with a Riemannian metric \(g_\Sigma\). Suppose that \(H\) is a non-constant function on \(\Sigma\). Then, the metric \(g\), of the form (1) dual to a general plane-fronted wave Lorentzian metric, is a second-Chern–Einstein non-Kähler almost-Kähler metric on \(S^1 \times S^1 \times \Sigma\) if and only if

\[
\|\text{grad}^g H\|^2 = 2 s_H^\Sigma,
\]

where \(\text{grad}^g H\) is the gradient of \(H\) with respect to the metric \(g\) and \(s_H^\Sigma\) is twice the Gaussian curvature of \((\Sigma, g_\Sigma)\). In particular, \(\Sigma\) is isomorphic to the sphere \(S^2\). Moreover, for a suitable normalization of \(H\), a metric \(g_\Sigma\) satisfying Eq. (11) always exists.

**Proof** Equation (11) follows from Corollary 10. To prove the last statement, let \(g_0\) be the metric on \(\Sigma\) with Gaussian curvature equal to 1. Suppose that \(g_\Sigma = e^{2f} g_0\), for some function \(f\). Then, Eq. (11) is equivalent to

\[
\|\text{grad}^{g_0} H\|_{g_0}^2 = 2 e^{2f} s_H^\Sigma,
\]

where the gradient and the norm are with respect to \(g_0\). So given a function \(H\) on \(\Sigma\), we need to find a solution \(f\) to

\[
\|\text{grad}^{g_0} H\|_{g_0}^2 = 4 + 4 \Delta^{g_0} f,
\]

where \(\Delta^{g_0}\) is the Laplacian with respect to \(g_0\). We conclude that given a non-constant \(H\), there exists a metric \(g_\Sigma\) satisfying Eq. (11) if \(H\) is normalized so that

\[
\int_{\Sigma} \|\text{grad}^{g_0} H\|_{g_0}^2 = 4 V_{g_0},
\]

where \(V_{g_0}\) is the total volume with respect to \(g_0\). □

**Remark 12** If \(H\) is a constant function and \(s_H^\Sigma = 0\), in particular \(\Sigma\) is isomorphic to the torus \(T^2\), then we get a Kähler Ricci flat metric on \(S^1 \times S^1 \times \Sigma\).

Theorem 11 provides examples of non-Kähler almost-Kähler metrics \(g\) that are second-Chern–Einstein and weakly \(*)\-Einstein (i.e. the \(*)\-Ricci form \(\rho^*\) is proportional to \(\omega\)) with a \(J\)-anti-invariant Riemannian Ricci tensor (in particular the Riemannian scalar curvature \(s^g = 0\)) and a \(J\)-invariant first-Chern–Ricci form. We remark that \(s^* = 2 s^H = \|\text{grad}^g H\|^2\) and \(g\) can not be Riemannian Einstein (more precisely \(g\) can not be Riemannian Ricci flat) unless the metric \(g\) is Kähler [36, 37] (see also [9, 19, 33, 34, 38]) (here \(s^*\) is twice the trace of the \(*)\-Ricci form \(\rho^*\) with respect to \(\omega\)).
4.2 Six-dimensional example

We would like to construct here a six-dimensional example of second-Chern–Einstein non-Kähler almost-Kähler metric of the form (1) dual to a general plane-fronted wave Lorentzian metric. We denote by \( M = \mathbb{P}(O_{\mathbb{C}P^1} \oplus O_{\mathbb{C}P^1}(1)) \) the first-Hirzebruch surface. On \( M \), we consider the \( U(2) \)-invariant Kähler metric \( g_M \) given by [12]

\[
g_M = h^2 (e^1 \otimes e^1 + e^2 \otimes e^2) + h^2 h'^2 e^3 \otimes e^3 + dt^2,
\]

where \( e^1, e^2, e^3 \) are the invariant 1-forms on \( S^3 \) dual to the vectors \( X, Y, Z \) satisfying \([X, Y] = 2V, [Y, V] = 2X, [V, X] = 2Y \) and \( t \) is a coordinate transverse to the \( U(2) \)-orbits. The function \( h = h(t) \) is a positive function on the interval \((0, l)\) and satisfy the boundary conditions

\[
h(0) > 0, h(l) > 0, h''(0) = \frac{1}{h(0)}, h''(l) = -\frac{1}{h(l)}, h^{(2p+1)}(0) = h^{(2p+1)}(l) = 0, \forall p \geq 0.
\]

Let \( \{E_1 = \frac{1}{h}X, E_2 = \frac{1}{h}Y, E_3 = \frac{1}{h'}V, E_4 = \frac{\partial}{\partial t}\} \) be a local \( g_M \)-orthonormal basis of the tangent bundle of \( M \) and we consider the \( g_M \)-orthogonal complex structure \( J_M \) defined by

\[
E_2 = J_M E_1, \quad E_4 = J_M E_3.
\]

On \( S^1 \times S^1 \times M \), we consider the almost-Kähler metric \( g \) of the form (1) dual to a general plane-fronted wave Lorentzian metric. The complex structure \( J_M \) is extended to an almost-complex structure \( J \) on \( S^1 \times S^1 \times M \) by introducing \( T, JT \) defined by

\[
T = \frac{1}{2}(H + 1)\partial_{\varphi} - \partial_{\theta},
\]

\[
JT = \frac{1}{2}(H - 1)\partial_{\varphi} - \partial_{\theta}.
\]

Suppose that \( H \) is a function on \( M \) invariant under the action of \( U(2) \) so \( H = H(t) \). Since \( H \) is a function on \( M \) then it follows from Proposition 4 that on \( (S^1 \times S^1 \times M, g, J) \)

\[
\rho^\nabla = \rho^\nabla_M,
\]

where \( \rho^\nabla_M \) is the first-Chern–Ricci form of \((M, g_M, J_M)\).

**Proposition 13** On \((S^1 \times S^1 \times M, g, J)\), we have

\[
r = \rho^\nabla_M - \frac{1}{8}H'^2 E_3^b \wedge E_4^b + \frac{1}{8}H'^2 d\theta \wedge d\varphi,
\]

where \( b \) denotes the \( g \)-Riemannian dual and \( H' = H'(t) \).

**Proof** We consider the \( J \)-adapted local \( g \)-orthonormal frame \( \{e_1, \ldots, e_6\} = \{E_1, E_2, E_3, E_4, T, JT\} \). We have

\[
[E_4, T] = [E_4, JT] = \frac{H'}{2}(T - JT).
\]

Hence, \( 4N(E_3, T) = -H' JT \) and \( 4N(E_4, T) = -H' T \), where \( N \) is the Nijenhuis tensor of \( J \). Using Corollary 2, we have the following

\[
r(E_i, E_j) - \rho^\nabla_M(E_i, E_j) = \sum_{l,k=1}^{6} g(J E_i, N(e_l, e_k)) g(E_j, N(e_l, e_k))
\]
We introduce the function \( y \) on \( M \) where the function \( h \). In particular,

\[
\left( r - \rho^V_M \right) \text{ applied to the pairs } (E_1, E_2), (E_1, E_3), (E_1, E_4), (E_2, E_3) \text{ and } (E_2, E_4) \text{ vanishes. However,}
\]

\[
r(E_3, E_4) - \rho^V_M(E_3, E_4) = -2g(N(E_3, T), JT)g(N(E_4, T), T),
\]

\[
= -\frac{1}{8}H^2.
\]

Similarly, using Corollary 2, \( (r - \rho^V_M) \) applied to \( (E_i, T) \) and \( (E_i, JT) \) vanishes. Finally,

\[
r(T, JT) = \sum_{l,k=1}^{6} g(JT, N(e_l, e_k))g(JT, N(e_l, e_k)) + \sum_{l,k=1}^{6} g(N(T, e_k), e_l)g(N(JT, e_k), J e_l),
\]

\[
= \sum_{l,k=1}^{6} g(JT, N(e_l, e_k))^2 - \sum_{l,k=1}^{6} g(N(T, e_k), e_l)^2,
\]

\[
= \frac{1}{8}H^2.
\]

The form \( \rho^V_M \) was computed in [24] (see also [4]). Hence, from Proposition 13 we get that on \( (\mathbb{S}^1 \times \mathbb{S}^1 \times M, g, J) \) the second-Chern–Ricci form is

\[
r = \frac{4 - 4h'^2 - 2hh''}{h^2}E_1^b \wedge E_2^b - \left( \frac{5h'' + hh'''}{h'} + \frac{1}{8}H^2 \right)E_3^b \wedge E_4^b + \frac{1}{8}H^2 d\theta \wedge d\varphi. \tag{13}
\]

We would like to find second-Chern–Einstein almost-Kähler metrics on \( (\mathbb{S}^1 \times \mathbb{S}^1 \times M, g, J, \omega) \) where

\[
\omega = E_1^b \wedge E_2^b + E_3^b \wedge E_4^b + d\theta \wedge d\varphi.
\]

Then, it follows from (13) that we need to solve

\[
- \frac{h'''}{h'} - \frac{h''}{h} + 8\frac{h'^2}{h^2} - \frac{8}{h^2} = 0, \tag{14}
\]

where the function \( h = h(t) \) satisfies the boundary conditions (12) with

\[
\frac{4 - 4h'^2 - 2hh''}{h^2} = - \frac{5h'' + hh'''}{2h'} = \frac{1}{8}H^2 \geq 0. \tag{15}
\]

We introduce the function \( y = y(h) \) defined by \( h' = \sqrt{y(h)} \). Then, \( h'' = \frac{1}{2}y'(h) \) and \( h''' = \frac{1}{2}y''(h)h' \) (here \( y'(h) \) is the derivative with respect to \( h \) etc). Then, Eq. (14) becomes

\[
- \frac{1}{2}y'' - \frac{1}{2h}y' + \frac{8}{h^2}y = \frac{8}{h^2}.
\]
The solution is given by
\[ y(h) = c_1 h^4 + \frac{c_2}{h^4} + 1, \]
for some constants \( c_1, c_2 \). Denote by \( h(0) = h_0 > 0 \) and \( h(l) = h_l > 0 \). To satisfy the boundary conditions (12), we need a solution \( y(h) \) such that
\[ y(h_0) = y(h_l) = 0, \quad y'(h_0) = \frac{2}{h_0}, \quad y'(h_l) = -\frac{2}{h_l}. \]
Then,
\[ c_1 = -\frac{1}{h_0^4 + h_l^4}, \]
\[ c_2 = -\frac{h_0^4 h_l^4}{h_0^4 + h_l^4}, \]
\[ h_0^4 + h_l^4 = -2(h_0^4 - h_l^4). \]  
(16)
Define \( x = \frac{h_l}{h_0} \). Then, Eq. (16) is equivalent to
\[ P(x) = -x^4 + 3 = 0. \]
Remark that \( P(1) = 2 > 0 \), and so we get the existence of a solution of \( P(x) = 0 \) with \( x = \frac{h_l}{h_0} > 1 \). Thus, there are solutions of (16) making \( h \) a positive and an increasing function on the interval \((0, l)\) and satisfying the boundary conditions (12). We also remark that the function \( y = y(h) \) is a positive on the interval \((h_0, h_l)\).
Explicitly,
\[ y(h) = -\frac{1}{4h_0^4} h^4 - \frac{3h_0^4}{4} h^{-4} + 1, \]
where \( h_0 > 0 \).
Moreover, the condition (15) is also satisfied for any \( h_0 > 0 \) (actually \( H'(t) \) is nowhere zero on the interval \([0, l]\)).
We can then define the function \( t(h) \) on the interval \((0, l)\) to be
\[ t(h) = \int_{h_0}^{h} \frac{dh}{\sqrt{y(h)}}. \]
The function \( h(t) \) is then the inverse of \( t(h) \). We deduce then the following

**Theorem 14** Let \( M \) be the first-Hirzebruch surface. Then, there exists on \( S^1 \times S^1 \times M \) an infinite family of non-Kähler almost-Kähler second-Chern–Einstein metrics of the form (1) dual to a general plane-fronted wave Lorentzian metric, of positive (non-constant) Hermitian scalar curvature.

**Remark 15** We would like to mention that we tried the same procedure for the \( n \)-Hirzebruch surface \( M = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(n)) \) with \( n > 1 \) but with no success.

**References**

1. Aazami, A.B., Ream, R.: Almost Kähler metrics and pp-wave spacetimes. Lett. Math. Phys. 112(4), 17 (2022)
2. Alekseevsky, D.V., Podestà, F.: Homogeneous almost-Kähler manifolds and the Chern-Einstein equation. Math. Z. 296(1–2), 831–846 (2020)
3. Angella, D., Calamai, S., Spotti, C.: Remarks on Chern-Einstein Hermitian metrics. Math. Z. 295(3–4), 1707–1722 (2020)
4. Angella, D., Pediconi, F.: On cohomogeneity one Hermitian non-Kähler metrics. arXiv:2010.08475v3 (2022)
5. Apostolov, V., Armstrong, J., Drăghici, T.: Local models and integrability of certain almost Kähler 4-manifolds. Math. Ann. 323(4), 633–666 (2002)
6. Apostolov, V., Drăghici, T.: On some 4-dimensional almost Kähler manifolds. Kodai Math. J. 18(1), 156–168 (1995)
7. Apostolov, V., Drăghici, T.: Almost Kähler 4-manifolds with $J$-invariant Ricci tensor and special Weyl tensor. Q. J. Math. 51(3), 275–294 (2000)
8. Apostolov, V., Drăghici, T.: The curvature and the integrability of almost-Kähler manifolds: a survey. In: Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), volume 35 of Fields Inst. Commun., pages 25–53. Amer. Math. Soc., Providence, RI (2003)
9. Armstrong, J.: On four-dimensional almost Kähler manifolds. Quart. J. Math. Oxford Ser. (2) 48(192), 405–415 (1997)
10. Barbaro, G., Lejmi, M.: Second-Chern-Einstein metrics on 4-dimensional almost-Hermitian manifolds. arXiv:2205.03452v1 (2022)
11. Beem, J.K., Ehrlich, P.E.: Global Lorentzian geometry. In: Monographs and Textbooks in Pure and Applied Mathematics, vol. 67. Marcel Dekker Inc, New York (1981)
12. Bérard-Bergery, L.: Sur de nouvelles variétés riemanniennes d’Einstein. In: Institut Élie Cartan, 6, volume 6 of Inst. Élie Cartan, pages 1–60. Univ. Nancy, Nancy (1982)
13. Cahen, M., Gutt, S., Hayyani, M., Raouyane, M.: Some pseudo-Kähler Einstein 4-symmetric spaces with a twin special almost complex structure. Differ. Geom. Appl. 86, 101958 (2023)
14. Calabi, E.: Extremal Kähler metrics. In: Seminar on Differential Geometry, volume 102 of Ann. of Math. Stud., pp. 259–290. Princeton University Press, Princeton (1982)
15. Candela, A.M., Flores, J.L., Sánchez, M.: On general plane fronted waves. Geodesics. Gen. Relativ. Gravit. 35(4), 631–649 (2003)
16. Della Vedova, A.: Special homogeneous almost complex structures on symplectic manifolds. J. Symplectic Geom. 17(5), 1251–1295 (2019)
17. Di Scala, A.J., Vezzoni, L.: Chern-flat and Ricci-flat invariant almost Hermitian structures. Ann. Glob. Anal. Geom. 40(1), 21–45 (2011)
18. Donaldson, S.: Fields Medalists’ lectures, volume 5 of World Scientific Series in 20th Century Mathematics. World Scientific Publishing Co., Inc., River Edge, NJ; Singapore University Press, Singapore (1997)
19. Drăghici, T.C.: On some 4-dimensional almost Kähler manifolds. Kodai Math. J. 18(1), 156–168 (1995)
20. Gauduchon, P.: La topologie d’une surface hermitienne d’Einstein. C. R. Acad. Sci. Paris Sér. A-B 290(11), A509–A512 (1980)
21. Gauduchon, P.: Hermitian connections and Dirac operators. Boll. Un. Mat. Ital. B (7) 11(2, suppl.), 257–288 (1997)
22. Gauduchon, P., Ivanov, S.: Einstein-Hermitian surfaces and Hermitian Einstein-Weyl structures in dimension 4. Math. Z. 226(2), 317–326 (1997)
23. Keller, J., Lejmi, M.: On the lower bounds of the $L^2$-norm of the Hermitian scalar curvature. J. Symplectic Geom. 18(2), 537–558 (2020)
24. Koca, C., Lejmi, M.: Hermitian metrics of constant Chern scalar curvature on ruled surfaces. Kodai Math. J. 43(3), 409–430 (2020)
25. Legendre, E.: A note on extremal toric almost Kähler metrics. In: Moduli of K-stable varieties, volume 31 of Springer INdAM Ser., pp. 53–74. Springer, Cham (2019)
26. Lejmi, M.: Extremal almost-Kähler metrics. Internat. J. Math. 21(12), 1639–1662 (2010)
27. Lejmi, M.: Stability under deformations of extremal almost-Kähler metrics in dimension 4. Math. Res. Lett. 17(4), 601–612 (2010)
28. Lejmi, M.: Stability under deformations of Hermitian-Einstein almost Kähler metrics. Ann. Inst. Fourier (Grenoble) 64(6), 2251–2263 (2014)
29. Libermann, P.: Sur les connexions hermitiennes. C. R. Acad. Sci. Paris 239, 1579–1581 (1954)
30. Liu, K.-F., Yang, X.-K.: Geometry of Hermitian manifolds. Internat. J. Math. 23(6), 1250055 (2012)
31. Mumford, D., Fogarty, J., Kirwan, F.: Geometric invariant theory, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer, Berlin, 3rd edn (1994)
32. Newlander, A., Nirenberg, L.: Complex analytic coordinates in almost complex manifolds. Ann. Math. 2(65), 391–404 (1957)
33. Oguro, T., Sekigawa, K.: Four-dimensional almost Kähler Einstein and ∗-Einstein manifolds. Geom. Dedicata 69(1), 91–112 (1998)
34. Oguro, T., Sekigawa, K., Yamada, A.: Four-dimensional almost Kähler Einstein and weakly ∗-Einstein manifolds. Yokohama Math. J. 47(1), 75–92 (1999)
35. Podestà, F.: Homogeneous Hermitian manifolds and special metrics. Transform. Groups 23(4), 1129–1147 (2018)
36. Sekigawa, K.: On some 4-dimensional compact Einstein almost Kähler manifolds. Math. Ann. 271(3), 333–337 (1985)
37. Sekigawa, K.: On some compact Einstein almost Kähler manifolds. J. Math. Soc. Jpn. 39(4), 677–684 (1987)
38. Sekigawa, K., Vanhecke, L.: Four-dimensional almost Kähler Einstein manifolds. Ann. Mat. Pura Appl. 4(157), 149–160 (1990)
39. Streets, J., Tian, G.: Hermitian curvature flow. J. Eur. Math. Soc. (JEMS) 13(3), 601–634 (2011)
40. Vedova, A.D., Gatti, A.: Almost Kähler geometry of adjoint orbits of semisimple Lie groups. Math. Z. 301(3), 3141–3183 (2022)
41. Vezzoni, L.: A note on canonical Ricci forms on 2-step nilmanifolds. Proc. Am. Math. Soc. 141(1), 325–333 (2013)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.