Sivers Function in the Quasi-Classical Approximation

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The Role of Spin in Hadronic Physics

- Motivation: transverse spin asymmetries and the proton spin crisis
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- Transverse-Momentum-Dependent Parton Distributions and Factorization
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- The importance of the glue: initial and final state interactions

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Spin in the Quasi-Classical Bjorken Limit

- A common regime: a heavy nucleus at Bjorken kinematics
- Quasi-classical factorization of the SIDIS cross-section and TMD's
- The Sivers function in the classical limit and orbital angular momentum
- Nuclear shadowing: the new importance of initial and final state interactions

Outlook
The Role of Spin in Hadronic Physics
Inaugurating an Era of Spin

**1988: The Proton Spin Crisis**

Measurements of the valence quark polarization of the proton by the European Muon Collaboration showed that much of the proton's total spin is “missing”:

\[ \frac{1}{2} = \frac{1}{2} \Sigma + G + L \]

"Hence (14\pm9\pm21)\% of the proton spin is carried by the spin of the quarks. The remaining spin must be carried by gluons or orbital angular momentum."

Phys. Lett. B 206 (1988) 364
arXiv: 1212.1701

**1991: Large Transverse Spin Asymmetries**

Measurements of pion production in polarized proton collisions at FNAL showed large transverse spin asymmetries and hence strong spin-orbit effects.

Phys. Lett. B 264 (1991) 462-466
Semi-Inclusive Deep Inelastic Scattering (SIDIS)

- Lepton-hadron scattering producing a specific tagged particle.
- Leptonic part can be factorized out from the hadronic part, reducing to the scattering of a spacelike virtual photon: $\gamma^* + N \rightarrow q + X$
- A colorless initial state produces a colored final state: sensitive to final state interactions.

The Drell-Yan Process (DY)

- Hadron-hadron scattering producing a hard virtual photon, which decays into a tagged dilepton pair.
- The dileptons can be integrated out, reducing to the production of a timelike virtual photon: $\overline{q} + N \rightarrow \gamma^* + X$
- A colored initial state produces a colorless final state: sensitive to initial state interactions.
For these processes, transverse-momentum-dependent factorization has been proven when the hard scale is large:

\[
d\sigma^{A+B\rightarrow C+X} \sim \sum_{abc} f_{a/A} \otimes f_{b/B} \otimes H^{a+b\rightarrow c+x} \otimes D_{C/c} \quad Q^2 \gg \Lambda^2
\]

(Convolutions over longitudinal and transverse momenta)

The parton distribution functions are now nonlocal in transverse coordinate space and require a nontrivial gauge link:

\[
\Phi_{ij}(x, \vec{k}_\perp) \equiv \int \frac{d^2r \ d(p \cdot r)}{(2\pi)^3} e^{i(xp - r - \vec{k}_\perp \cdot \vec{r}_\perp)} \langle p, S | \overline{\psi}_j(0) U[r] \psi_i(r) | p, S \rangle
\]

Transverse momentum permits many new types of parton distribution that couple spin and transverse momentum:

\[
\Phi(x, \vec{k}_\perp) = \frac{1}{2} f_1(x, k_T) \frac{p^\mu}{p^+} + \frac{1}{2} f_{1T}(x, k_T) \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \frac{p^\nu}{p^+} \frac{k_\perp^\rho}{m} S_\perp^\sigma - \frac{1}{2} g_{1s}(x, \vec{k}_\perp) \frac{p^\perp}{p^+} \gamma^5
\]

\[
- \frac{1}{2} h_{1s}(x, k_T) i\sigma_{\mu\nu} \gamma^5 S_\perp^{\mu} \frac{p^\nu}{p^+} - \frac{1}{2} h_{1T}(x, \vec{k}_\perp) i\sigma_{\mu\nu} \gamma^5 \frac{k_\perp^\mu}{m} \frac{p^\nu}{p^+} + \frac{1}{2} h_1(x, k_T) \sigma_{\mu\nu} \frac{k_\perp^\mu}{m} \frac{p^\nu}{p^+}
\]
The Sivers Function: A Lesson In Time Reversal

• One of these TMD's is the Sivers function, which measures the intrinsic STSA in a transversely-polarized hadron:

\[ \frac{(\vec{S}_\perp \times \vec{k}_\perp) \cdot \hat{P}}{m} f_{1T}^{\perp}(x, k_T) = \frac{1}{2} \text{Tr} [\gamma^+ \Phi(x, \vec{k}_\perp; \vec{S})] - (\vec{k}_\perp \rightarrow -\vec{k}_\perp) \]

• But the correlation \((\vec{S}_\perp \times \vec{k}_\perp) \cdot \hat{P}\) is odd under time-reversal, and the hadron is an eigenstate of the T-Even QCD Hamiltonian....

• Collins showed that, when one neglects the presence of the gauge link, time-reversal invariance requires the Sivers function to vanish.

Nucl.Phys. B396 (1993) 161-182

• But the gauge link reflects more than just gauge invariance; it contains the physics associated with initial / final state interactions.

• The gauge link also transforms under time reversal: initial state interactions become final state interactions, so Collins' derivation instead predicts an exact sign reversal between SIDIS and DY:

\[ f_{1T}^{\perp}|_{\text{SIDIS}} = - f_{1T}^{\perp}|_{\text{DY}} \]

Phys. Lett. B536 (2002) 43-48
The Importance of the Glue

- TMD's are **not** solely properties of the hadronic wave functions; they encapsulate aspects of the scattering dynamics as well.

- Brodsky *et al.* showed that final state interactions of SIDIS permit a T-odd imaginary phase that occurs when an intermediate state goes on-shell:

\[
\frac{1}{\ell^2 - m^2_\ell + i\epsilon} = \text{P.V.} \left[ \frac{1}{\ell^2 - m^2_\ell} \right] - i\pi\delta(\ell^2 - m^2_\ell)
\]

(Phys. Lett. B530 (2002) 99-107)

- This “lensing” mechanism generates a preferred direction through the coherent attractive force the quark experiences as it escapes the hadron.

- Crossing symmetry relates the SIDIS and DY amplitudes, and to leading twist accuracy, their T-odd phases satisfy the sign flip:

\[
f_{1T}^{1T}|_{\text{SIDIS}} = -f_{1T}^{1T}|_{\text{DY}} + O\left(\frac{1}{Q^2}\right)
\]

(Phys. Rev. D88 (2013) 1)

**SIDIS (FSI)**

- Pointlike proton (colorless)
- Pointlike scalar (antiquark)
- Yukawa coupling: proton/quark/scalar

**DY (ISI)**

+ C.C.
The Emergence of Parton Saturation At High Energy
Regge Kinematics and Eikonal Scattering

- The high-energy Regge limit: kinematic simplifications occur when the scattering particles travel nearly along the light cone:

\[ s \gg t \gg \Lambda^2 \]
\[ Q^2 \sim t = \text{const.} \quad x \approx \frac{Q^2}{s} \ll 1 \]

- These high-energy particles undergo eikonal scattering, in which a transverse momentum kick is exchanged, but the longitudinal momenta are unaffected.

\[ A(x, y; x', y') \equiv \mathcal{F} \cdot \mathcal{T} \left[ \frac{M(p_1, p_2, k)}{s} \right] \sim g^2 \delta^4(x - x')\delta^4(y - y')\delta(x^+ - y^+)\delta(x^- - y^-) \ln \frac{1}{|x - y|_T} \]

- The interactions occur instantaneously and are naturally ordered along the light cone. This eikonal propagation of a projectile through the field of the target is re-summed into a Wilson line:

\[ V_x \equiv \mathcal{P} \exp \left[ ig \int dx^- A^+(0, x^-, x) \right] \]
Quantum Evolution: the Small-x Gluon Cascade

- When the energy becomes so large that its logarithm can compete with the coupling, it reorders the perturbation series:

\[ s \sim \frac{1}{2} e^{1/\alpha_s} \gg \frac{1}{2} \gg \Lambda^2, \quad \alpha_s \ln \frac{s}{\Lambda^2} \sim 1 \]

- Emission of an extra longitudinally-soft (small-x) gluon is suppressed by the coupling, but contributes a factor of its phase space.

- These emissions must be re-summed to give the small-x gluon cascade which dominates the physics of high energies.

- This increase in gluon bremsstrahlung with energy gives the BFKL equation, which drives up the gluon density at small-x.
Unitarity and Gluon Saturation

- BFKL gives a cross-section that grows too quickly with energy and would violate unitarity if it continued unabated.

- At high enough densities, nonlinear gluon fusion begins to compete with bremsstrahlung, saturating the gluon density to a parametrically large value.

\[ \rho_{g,\text{max}} \sim \frac{Q_s^2}{\alpha_s} \]

- The proliferation of incoherent color sources generates a correlation length, described by the saturation scale, which cuts off the gluon distributions in the IR.

\[ \phi \sim \frac{1}{r_T^2} \left( 1 - e^{-\frac{1}{4}r_T^2Q_s^2} \right) \]

- Including these recombination processes yields a nonlinear evolution equation (ie, BK, JIMWLK):

\[ \frac{\partial}{\partial \ln s} N(x_{10}, s) = \frac{\alpha_sN_c}{2\pi^2} \int d^2x_2 \frac{x_{10}^2}{x_{12}x_{20}} \left[ N(x_{12}, s) + N(x_{20}, s) - N(x_{10}, s) - N(x_{12}, s)N(x_{20}, s) \right] \]

- Nonlinear evolution yields a saturation scale that grows with energy, so eventually, it cuts off the IR while still in the perturbative domain.

\[ Q_s(x) \sim Q_{s0} \left( \frac{1}{x} \right)^\# \]
• Nature provides us with another way to approach the limit of dense color charges: using a heavy nucleus with a large number $A$ of nucleons.

• The number of nucleons at a given transverse position, $A^{1/3}$, provides a parameter to control the density of color charges and define the saturation scale:

$$Q_s^2 \sim \alpha_s^2 A^{1/3}$$

$$\alpha_s \ll 1, \quad A \gg 1, \quad \alpha_s^2 A^{1/3} \sim 1$$

• The nucleus provides a well-defined regime to re-sum the high-density corrections, without the need for quantum evolution.

• **DIS in the Regge Limit**: the virtual photon has a long coherence length and fluctuates into a quark/antiquark pair, undergoing eikonal rescattering from the $A^{1/3}$ nucleons.

$$\ell_{coh} \sim \frac{1}{m x}$$

$$\text{Tr} \left[ V_{x} V_{y}^{\dagger} \right]$$

• The leading order effect in $A^{1/3}$ is a combinatoric enhancement that prefers each rescattering to occur on a different nucleon.

• Integrating over the momentum transfer $k$ between nucleons puts the intermediate propagators on-shell and factorizes into a product of scattering on independent nucleons.

$$\frac{1}{N_c} \text{Tr} \left[ V_{x} V_{y}^{\dagger} \right] \approx \exp \left[ -\frac{1}{4} |x - y|^2 Q_s^2 \ln \frac{1}{|x - y| T \Lambda} \right]$$
The Dense Limit is the Quasi-Classical Limit

- The 2-gluon/nucleon resummation parameter $\alpha_s^2 A^{1/3} \sim 1$ corresponds to interacting with the classical Weizsacker-Williams gluon field of the target.

\[
\phi_{WW}(x, k_T^2) \equiv \frac{\pi}{(2\pi)^3} \sum_{\lambda, a} \langle A| \hat{a}_{k\lambda}^{a\dagger} a_{k\lambda}^a |A\rangle
\]

- The gluon fields of the nucleus are characterized by high occupation numbers, reducing to their classical limit.

\[
\phi_{WW}(x, k_T^2) = \frac{C_F}{\pi^3 \alpha_s} \int d^2 b \, d^2 r \, e^{i k \cdot t} \frac{1}{r_T^2} \left[ 1 - \exp \left( -\frac{1}{4} r_T^2 Q_s^2(b) \ln \frac{1}{r_T \Lambda} \right) \right]
\]

- Equivalently, one can solve the classical Yang-Mills equations for a heavy nucleus moving along the light-cone and recover the same formulas (McLerran – Venugopalan model).

\[
\mathcal{D}_\mu F^{\mu\nu} = \delta^{\nu+} \rho(x^-, \vec{x})
\]

- The high energy limit of QCD is the limit of high-density classical gluon fields.
- The resummation parameter $\alpha_s^2 A^{1/3} \sim 1$ embeds the perturbation series in a classical background field.
Spin in the Quasi-Classical Bjorken Limit

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There is a common limit that employs both the kinematics necessary for spin physics and the high densities necessary for saturation: **Bjorken kinematics in a heavy nucleus**

In SIDIS, the virtual photon has a short coherence length and interacts via a local “knockout” process due to the “large-x” Bjorken kinematics.

But the struck quark has a long coherence length and can undergo eikonal final-state rescattering on the spectator nucleons.

Since $Q^2 \gg \Lambda^2$, TMD factorization holds, and we can express the nuclear TMD's in terms of nucleons and Wilson lines.

Since $Q_s^2 \gg \Lambda^2$, we can do the calculations perturbatively, using the machinery of saturation physics.
The SIDIS Cross Section (1)

\[
\frac{d\sigma^{\gamma^*+A\rightarrow q+X}}{d^2k\,dy} = \sum_p \langle \phi_N | \frac{\hat{g}_I^{\gamma^*}}{\hat{g}_I^{\gamma^*}} | \phi_N \rangle \langle \phi_N | \frac{\hat{g}_I^{\gamma^*}}{\hat{g}_I^{\gamma^*}} | \phi_N \rangle \langle \phi_N | \frac{\hat{g}_I^{\gamma^*}}{\hat{g}_I^{\gamma^*}} | \phi_N \rangle
\]

- We can relate the scattering amplitude on the nucleus to the light-cone wave functions of the nucleons and the rest of the amplitude (knockout + rescattering).

\[
M_{\text{tot}} \sim \int d^2p_1 d^2p_2 \psi(p_1)\psi(p_2) \, M(p_1, p_2)
\]

\[
M_{\text{tot}}^* \sim \int d^2p'_1 d^2p'_2 \psi^*(p'_1)\psi^*(p'_2) \, M^*(p'_1, p'_2)
\]

- By Fourier transforming the momentum difference between the amplitude and C.C., we obtain the Wigner distribution of nucleons in the nucleus:

\[
W(p, b) \equiv \int \frac{d^2(\delta p)}{(2\pi)^3} e^{-i(\delta p)\cdot b} \psi(p + \frac{1}{2}\delta p) \psi^*(p - \frac{1}{2}\delta p)
\]

- After introducing the Wigner functions, the knockout + rescattering amplitudes are still off-forward.

- But, using the fact that \(W(p,b)\) varies with impact parameter only over macroscopic scales \(\sim A^{1/3}\), we can neglect the off-forwardness in the transverse momentum.
But, as we did in the Regge limit, we can integrate over the longitudinal momentum carried between nucleons to put the intermediate propagator on shell.

At this point, the knockout + rescattering amplitudes are still off-forward in the longitudinal momenta.

This enforces path ordering and fully factorizes the knockout of a quark from a nucleon from the rescattering of that quark on all the other spectators. The result is a “quasi-classical factorization” of the SIDIS cross-section:
TMD's in the Quasi-Classical Limit

- We can perform the same analysis using the definition of the TMD distribution functions.

\[ \Phi_{ij}^A(x, k; P, S) \equiv \frac{P^+}{(2\pi)^3} \int d^2\, (x - y) \, e^{ik \cdot (x - y)} \, \langle A(P, S) | \bar{\psi}_j(x) \, U[x, y] \, \psi_i(y) | A(P, S) \rangle \]

- Decompose the nuclear state into a superposition of nucleons using the light-cone wave functions:

\[ |A(P, S)\rangle = \int d^2\, p_1 d^2\, p_2 \cdots \, \Psi(p_1) \Psi(p_2) \cdots \, |N(p_1)\rangle \otimes |N(p_2)\rangle \otimes \cdots \]

- The kinematics of the knockout and rescattering proceed in the same way, reducing the matrix element of the nucleus down to the matrix element of the nucleon:

\[ \langle A | \bar{\psi} U \psi | A \rangle \sim W(p, b) \otimes \langle N | \bar{\psi} U \psi | N \rangle \otimes D_{xy} \]

- This expresses the TMD's of the nucleus in terms of the TMD's of the nucleons, their Wigner distributions, and the eikonal rescattering factor.

\[ \Phi_A(x, k; P, J) = A \int \frac{dp^+ \, d^2 p \, db^-}{2(2\pi)^3} \, d^2 x \, d^2 y \sum_{\sigma} W_N^\sigma \left( p, b^-, \frac{x + y}{2} \right) \int \frac{d^2 k'_{-}}{(2\pi)^2} e^{-i(k-k') \cdot (x-y)} \times \phi_N(x, k'_{-} - x \, p; p, \sigma) \, D_{xy}[+\infty, b^-] + O\left(\frac{1}{Q^2}\right) + O(A^{-1/3}) \]
Using our quasi-classical factorization formula, we can directly compute the Sivers function of the nucleus:

\[
\frac{\mathbf{S}_\perp \times \mathbf{k}_\perp}{m} \cdot \hat{P}_{1T}^A(x, k_T) = \frac{1}{2} \text{Tr}[\gamma^+ \Phi(x, \mathbf{k}_\perp; \mathbf{S})] - (\mathbf{k}_\perp \rightarrow -\mathbf{k}_\perp)
\]

By splitting each factor into symmetric and antisymmetric parts, we can use the symmetry properties of the Sivers function to identify which channels can contribute:

\[
\hat{z} \cdot (J \times k) f_{1T}^A(x, k_T) = M_A \int \frac{dp^+ d^2p db^-}{2(2\pi)^3} d^2x d^2y \frac{d^2k'}{(2\pi)^2} e^{-i(k-k') \cdot (x-y)}
\]

\[
\times \left\{ i x \frac{p \cdot (x-y)}{2} A W^{OAM}_{\text{unp}} \left( p^+, p, b^-, \frac{x+y}{2} \right) f^N_1(x, k_T')
\right.
\]

\[
+ \frac{1}{m_N} \hat{z} \cdot (S \times k') W^{symm}_{\text{trans}} \left( p^+, p, b^-, \frac{x+y}{2} \right) \underline{f_{1T}^N(x, k_T')}
\left\} S_{xy}[+\infty, b^-]
\]

**Orbital Angular Momentum:**

\[
W^{OAM}(p, b) \equiv \frac{1}{2} \left[ W(p, b) - (\mathbf{p}_\perp \rightarrow -\mathbf{p}_\perp) \right]
\]

**Intrinsic Sivers function of the nucleons:**

\[
f_{1T}^N(x, k_T')
\]

**The Odderon (T-odd rescattering):**

\[
iO_{xy} \equiv \frac{1}{2} \text{Tr} \left[ V_x V_y^\dagger - V_y V_x^\dagger \right]
\]

\[\mathcal{O}(A^{-1/3})\]
We find a new mechanism that can generate the Sivers function: (orbital angular momentum of nucleons) x (their quark TMD's) x (symmetric rescattering).

The essential spin-orbit effect is the presence of OAM, but OAM alone is not enough to generate the Sivers function. When we impose PT symmetry on the Wigner distribution, we find that the OAM part integrates to zero:

$$W^{OAM}(p, b_\perp, b_z) = -W^{OAM}(p, b_\perp, -b_z)$$

$$\int db_z W^{OAM}(p, b) = 0$$

If we neglect the role of multiple rescattering, then it is equally likely to eject the quark from the front of the nucleus as from the back, and the net asymmetry integrates to zero.

Final state interactions break this front-back symmetry through nuclear shadowing, making the quark more likely to be produced near the back of the nucleus.

This mechanism is quite different from the "lensing" mechanism of the Sivers function, in which the rescattering is color-correlated and generates the preferred direction.
Comparing the Channels

- The OAM channel has a particularly simple interpretation of the SIDIS / DY sign flip: shadowing from the front vs. back of the nucleus.

- The OAM channel uses the unpolarized quark TMD, which enters at leading order in $\alpha_s$. But it requires at least one rescattering to be nonzero, bringing in a factor of $Q_s^2 / k_T^2$ at large $k_T$.

- The transversity channel uses the Sivers function, which enters at order $\alpha_s^2$, but does not rely on nuclear shadowing.

- Thus OAM dominates for much of the range, with the transversity channel only taking over at very large $k_T$. 

$$f_{1T}^{\perp A}|_{OAM} \sim W_{unp}^{OAM} \otimes f_1^N \otimes S_{xy}$$

$$f_{1T}^{\perp A}|_{\text{trans}} \sim W_{\text{symm}}^{\text{trans}} \otimes f_{1T}^N \otimes S_{xy}$$
Outlook and Conclusion
The machinery we have developed is a robust method of calculating TMD's in the dense limit and can be applied to any of the interesting spin correlations.

The one key modeling assumption about the target is the existence of a large parameter $A^{1/3}$ that controls the charge density.

The system under consideration need not be a real nucleus; our approach is valid for any composite particle being decomposed into a large number of constituents (i.e., partons in a high-energy proton).

Our approach allows a clear separation into the “wave function part” which must be P, T even, and the “interaction part” which may contain a T-odd part.

We can now proceed to evaluate other TMD's in the quasi-classical limit. A good baseline for comparison would be $f_1, g_1, f_{1T}$ for quarks and gluons.

Full calculations of quark target TMD's exist in the literature, so we can in principle use these to do detailed calculations of the nuclear TMD's.

Maybe this can shed some light on the zoo of spin correlations and on the distribution of the proton spin.
Toward the Regge Limit and Quantum Evolution

- TMD factorization seems to only require that $Q^2 \gg \Lambda^2$, so this approach should also be valid if we consider the small-x limit $s \gg Q^2 \gg \perp^2 \gg \Lambda^2$.

- Such a limit may be useful for studying the overlap and transition between the TMD formalism, small-x saturation formalism, and twist-3 collinear formalism.

- What do the relevant evolution equations look like from this perspective? The nonlocal, semi-infinite Wilson lines are unusual quantities; will their small-x evolution be linear or nonlinear? Can we see the connections to DGLAP evolution, CSS evolution, and BK/JIMWLK evolution?

- Formulating, understanding, and solving the relevant evolution equations will be essential for linking these low-order calculations to future EIC phenomenology.
Summary

- Working in the **quasi-classical large-x, large-A regime**, we have derived a relation between the TMD's of a heavy nucleus and the TMD's of its nucleons.

- We analyzed the T-odd **Sivers function**, identifying two contributing channels: an orbital angular momentum channel and a transversity channel.

- The OAM channel is a **new mechanism**, distinct from the usual “lensing” mechanism. It uses the inherent **orbital motion**, together with nuclear shadowing, to generate the preferred direction.

- This machinery is quite general and can be used to **calculate the nuclear TMD's directly from first principles**. This is an important step as we move toward the intersection of spin and saturation in the coming EIC era.
Toy Model: A Rigid Rotator

- Assumptions: Rigid Rotator

\[ W_{unp}(p, b) \approx \frac{2(2\pi)^3}{A} \rho(b, b^-) \delta^2 \left( p - \hat{y} p_{\max}(b_x) \frac{b^-}{R^-(b_x)} \right) \delta \left( p^+ - \frac{p^+}{A} \right) \]

\[ W_{trans}(p, b) = \beta W_{unp}(p, b) \]

- Result:

\[ f_{1T}^{\perp A}(\bar{x}, k_T) = \frac{M_A N_c}{4\pi \alpha_s J C_F} \frac{1}{k_T^2} \int d^2b \left\{ 4 \bar{x} p_{\max}(b) C_1 \left[ e^{-k_T^2/Q_s^2(b)} + 2 \frac{k_T^2}{Q_s^2(b)} Ei \left( -\frac{k_T^2}{Q_s^2(b)} \right) \right] \right. \]

\[ + \alpha_s \beta m_N C_2 e^{-k_T^2/Q_s^2(b)} \left. \right\} \]

- Asymptotics:

\[ f_{1T}^{\perp A}(\bar{x}, k_T) \bigg|_{k_T \gg Q_s} = \frac{S}{J} \left[ -\frac{4\alpha_s m_N \bar{x} C_1}{3\beta k_T^6} \ln \frac{k_T^2}{\Lambda^2} \int d^2b T(b) p_{\max}(b) Q_s^2(b) + A f_{1T}^{\perp N}(\bar{x}, k_T) \right] \]