Nets of conics and associated Artinian algebras of length 7

Translation and update of the 1977 version by J. Emsalem and A. Iarrobino

Nancy Abdallah · Jacques Emsalem · Anthony Iarrobino

Received: 22 November 2021 / Revised: 9 August 2022 / Accepted: 8 October 2022 / Published online: 27 March 2023

© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2023

Abstract
We classify the orbits of nets of conics under the action of the projective linear group and we determine the specializations of these orbits, using geometric and algebraic methods. We study related geometric questions, as the parametrization of planar cubics. We show that Artinian algebras of Hilbert function $H = (1, 3, 3, 0)$ determined by nets, can be smoothed—deformed to a direct sum of fields; and that algebras of Hilbert function $H = (1, r, 2, 0)$, determined by pencils of quadrics, can also be smoothed. This portion is a translation and update of a 1977 version, a typescript by the second two authors that was distributed as a preprint of University of Paris VII. In a new Historical Appendix A we describe related work prior to 1977. In an Update Appendix B we survey some developments since 1977 concerning nets of conics, related geometry, and deformations of Artinian algebras of small length.

Keywords Nets of conics · Artinian algebra · Deformation · Discriminant · Hessian · Hilbert function · Isomorphism class · Normal form · Parametrization · Pencils of conics · Planar conics · Planar cubic · Quadric · Smoothable · Ternary cubic

Mathematics Subject Classification 14C21 · 14-02 · 13E10

Réseaux de Coniques et algèbres de longueur sept associées (1977) Univ. Paris VII.

Anthony Iarrobino
a.iarrobino@northeastern.edu
Nancy Abdallah
nancy.abdallah@hb.se

1 University of Borás, Borás, Sweden
2 Paris, France
3 Department of Mathematics, Northeastern University, Boston, MA 02115, USA
Contents

Translators’ note and acknowledgements ........................................... 2
1 Introduction ..................................................................................... 4
2 Space of projective conics ........................................................... 9
3 Pencils of conics ............................................................................ 10
4 Nets of conics ................................................................................ 17
5 Algebraic method for the classification of planes of conics .................. 30
6 The closures of orbits: specializations ............................................. 34
7 Deformations .................................................................................. 48
8 Regular maps of degree 4 from the projective plane into itself ............... 53
9 Note on the Hessian form of a smooth cubic .................................... 55
Appendix A: Historical Note, pre 1977 .............................................. 58
Appendix B: Update on related topics ............................................... 61
References ......................................................................................... 68

Translators’ note and acknowledgements

The original “Réseaux de Coniques et algèbres de longueur sept associées”, was typed with hand-drawn diagrams, and distributed in 1977 as a preprint of Université de Paris VII, where Jacques Emsalem was Assistant Titulaire. It was drafted in 1975–1976 during a one year visit to École Polytechnique, then in Paris, by Tony Iarrobino under an exchange program of Centre Nationale de Recherche Scientifique (CNRS) and the National Science Foundation (NSF). They were grateful to Lê Dung Trang, Monique Lejeune-Jalabert, and Bernard Teissier, who sponsored this visit. In particular, Lê realized that Jacques and Tony were working on the same problem from different viewpoints, and this led to their collaboration.

The original goal of this classification of nets of planar conics was to establish the classification of the related graded commutative Artinian algebras $A$ of length 7, having Hilbert function $H(A) = (1, 3, 3)$. Emsalem had independently determined the classification up to isomorphism of those having length at most 6, and understood that $H(A) = (1, 3, 3)$ or $(1, 4, 2)$ were the smallest with moduli. See the historical note for further comment (Appendix A).

We should like to thank Ivan Cheltsov for the encouragement to write this translation. He was interested in a portion of the classification, relevant to the divisor $X$ in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1, 2)$ (which is the same thing as a net of conics), and showed independently that if $X$ is smooth, then $X$ is in one of the three classes in the first line of Table 1. Here $X$ is a Fano threefold No. 2–24 in [13] and the result helps determine which $X$ in the family are Kähler–Einstein [5, 25]. We should like to thank also Sandro Verra, who had a copy of the preprint, and informed Ivan of it—and we used the pdf file Sandro sent! We thank Aldo Conca, Steve Kleiman, Hal Schenck, Igor Dolgachev, and Joachim Jelisiejew for helpful comments. We appreciate the detailed comments of the referees.

Nancy and Tony made the first draft of the translation, then corrected with Jacques. In making the translation, we have in places updated or clarified the wording; we have introduced main results as theorems in the introduction. We have corrected some mistakes, mostly typos in the original (in particular the top right entry of Table 1). We
have added a few translator notes when we felt it would help the reader. The many figures in the original were hand-drawn by Jacques; we have used tikz and made new figures.

We have added a Historical Appendix A to give some further references to work earlier than 1977 on nets of planar conics and Artinian algebras having small length.

We have added an Appendix B where we discuss selectively some developments since 1977 related to nets of conics, small-length Artinian algebras, and possible directions for future research.

We have retained the 7 Original References for the 1977 preprint after Sect. 9, separate from the References at end.

Nancy Abdallah, Jacques Emsalem¹, and Tony Iarrobino

Notation

Here \( k \) is an algebraically closed field of characteristic zero.² We let \( R = k[[X, Y, Z]] \) be the formal power series over \( k \). Here \( R_i \) denotes the vector subspace of \( R \) constituted by 0 and the homogeneous polynomials of degree \( i \).

One denotes by \( P \) the projective plane whose generic point admits \((X, Y, Z)\) as system of homogeneous coordinates. If \( V \) is the dual vector space to \( R_1 \) it amounts to the same to say that \( P \) is the projective space associated to \( V \).

As \( V \) is a vector space of dimension 3, we denote by \( \text{Gl}(3) \) the group of linear automorphisms of \( V \). Thus, \( \text{Gl}(3) \) has a canonical action on each \( R_i \). We denote by \( \text{Pgl}(3) \) the quotient \( \text{Gl}(3)/k^* \) of \( \text{Gl}(3) \) by the group of non-null homotheties of \( V \) with center the origin (scalings). One denotes by \( P^* \) the projective space associated to the vector space \( R_1 \). It is what we classically call the dual of \( P \) and it can be canonically identified with the set of lines of \( P \).

More generally, one may consider for each \( i \) the projective space associated to \( R_i \), which is canonically identified with the set of projective curves of \( \mathbb{P}^2 \) having degree \( i \). Here, we consider primarily the projective space associated to \( R_2 \), which we will denote by \( C_5 \) (\( C \) for conic, the index 5 indicating the dimension of the space). If \( F \in R_i \) we will denote by \( (F) \) the corresponding curve, which we will term indifferently the curve of equation \( F = 0 \) or the curve \( (F) \).

The group \( \text{Pgl}(3) \) has a natural action on \( P, P^*, C_5 \), and, more generally on the projective space associated to \( R_i \) for each \( i \), and, more generally still, on the \( k \)-th Grassmannian \( \text{Grass}(k, R_i) \) of \( R_i \).

One terms pencil of conics, respectively net of conics a line (respectively, a plane) of \( C_5 \). More generally, one terms pencil of hypersurfaces of degree \( n \) a line in the projective space associated to \( R_n \).

¹ Jacques Emsalem, a friend and colleague, died on July 12, 2022, after submission of the completed MS. Nancy and Tony made final corrections in response to the referee’s comments. We have greatly appreciated the opportunity to work together with Jacques—N.A and A.I.

² Note of Translators: The classification results for pencils of ternary conics, nets of ternary conics, ternary cubics, and length 7 algebras, except those involving Hessians, extend to algebraically closed fields of characteristic \( p \neq 2 \) or 3. For duality results one needs to replace differentiation by the contraction pairing from \( S = k[x, y, z] \) to the divided power ring related to \( R \) [48]. See the page where Theorem B is stated.
1 Introduction

We let $k$ be an algebraically closed field of characteristic zero, recall $R = k[[X, Y, Z]]$ and $R_i$ the vector space of degree- $i$ forms in $R$. The group $\text{Gl}(3)$ acts on $R_i$ and its quotient $\text{Pgl}(3) = \text{Gl}(3)/k^*$ acts on the set of vector subspaces of $R_i$; in particular if $V = \langle v_1, v_2, v_3 \rangle$ is a subspace of $R_3$ of dimension 3 (the case that interests us) and $\sigma \in \text{Gl}(3)$ is a representative of $\sigma' \in \text{Pgl}(3)$ then $\sigma'(V) = \langle \sigma(v_1), \sigma(v_2), \sigma(v_3) \rangle$. Here, we classify the orbits of the dimension-three subspaces under this natural action of $\text{Pgl}(3)$. It is the same as classifying up to isomorphism the “nets of projective plane conics”, or, also, to classify up to isomorphism the commutative local $k$-algebras $A$ of Hilbert function $(1, 3, 3)$, so of the form $A = k[X, Y, Z]/(V, (X, Y, Z)^3)$. We describe also the closures of the orbits, as subvarieties of the Grassmannian Grass$(3, R_2)$, parametrizing the vector spaces in $R_2$ having dimension three. This is the problem of “specialization of orbits”.

The Artinian algebras that we have just considered have dimension 7 as vector spaces over $k$. This is the smallest dimension for which there appear continuous families of orbits.\textsuperscript{4} That such a family occurs in the case of Hilbert function $(1, 3, 3)$ Artinian algebras can be readily seen: the variety Grass$(3, R_2)$ has dimension $3(6 - 3) = 9$, but $\text{Pgl}(3)$ has dimension $3^2 - 1 = 8$. One expects, then, to find a one-parameter family of orbits, that we will describe. An analogous calculation (Sect. 8) shows that there is also a one-parameter family of isomorphism classes of algebras of Hilbert function $(1, 4, 2)$. We have here the two Hilbert functions for length-7 local algebras that are isomorphic to their associated graded algebras, and that have a continuous family of orbits.

Returning to our main goal, the algebras of Hilbert function $(1, 3, 3)$, the one-parameter family is obtained as follows: let $\phi$ be a general enough cubic form in $R$; then the vector space $J_\phi$ spanned by the first-order partial derivatives of $\phi$, has dimension 3, and we will see that “$\phi \cong \phi' (\text{mod Pgl}(3))$” is equivalent to $J_\phi = J'_\phi$ (mod $\text{Pgl}(3)$”). Thus, there is a family of vector spaces $V = J_\phi$ that, like the cubics $\phi$ are parametrized up to isomorphism by the $j$-invariant of $\phi$. We denote by $\Gamma(V)$ the cubic parametrizing the decomposable elements of the vector space $V$; those elements $v$ that can be written as a product of linear forms. This is the discriminant locus, parametrizing the singular conics of the net $V$. We will also denote by $\pi$ the plane of $V$, and by $\Gamma_\pi = \Gamma(V)$ the cubic.

Our first main result concerns the nets $V$ of conics whose discriminant cubic $\Gamma(V)$ is smooth, and their $j$-constant specializations; and as well the nets for which $\Gamma(V)$ is a cubic with a node having distinct tangents. An elemental specialization will be from $A$ to $B$ where there is no intermediate specialization $C$ not equal to $A$ or $B$. We will refer to the orbits in Table 1 by their dimension and their position from the left, thus #7b refers to the second orbit in the dimension seven row.

\textsuperscript{3} The Hilbert function of a local algebra is the sequence $(n_i)$, where $n_i = \text{dim } m^i/m^{i+1}$, $m$ being the maximal ideal. Note of translators: the original article used “type” meaning Hilbert function; we no longer do so, as we reserve “type” for the sense in Definition 4.1).

\textsuperscript{4} For the algebras of dimension at most 6, their classification has been obtained by different persons. The most interesting case is treated implicitly here in Sect. 3. The other cases are simple enough. [Note of translators: see Appendix A].

\textsuperscript{7} Springer
Theorem 1 (Theorem 4.1 (A), Sect. 6.5 (2), Proposition 6.24) The nets of conics corresponding to smooth $\Gamma(V)$ are Jacobian nets $V = J_\phi$ of smooth cubics $\phi$, whose isomorphism classes are determined by the $j$ invariant of $\phi$. There are elementary $j$-constant specializations of the net $J_\phi$ for each permissible $j$ to the net $#7b \langle X^2 + YZ, XY, Z^2 \rangle$ corresponding to a cuspidal cubic $\Gamma(V)$. There are likewise $j$-constant specializations for permissible $j$ to each of the specializations $#6b, 6c, 5a, 5b, 4, 2a, 2b$ of $#7b$. The specializations of the two nets $#8a, 8c$ whose $\Gamma(V)$ is a cubic having a node with two distinct tangents are as shown in Table 1.

We will use two methods to study other orbits, not corresponding to smooth cubics: the geometric method is more useful. Consider for each vector space $V = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^2$, the isomorphism class of the cubic $\Gamma(V)$ with equation $f(a_1, a_2, a_3) = 0$ which characterizes the singular conics: that is, $\Gamma(V)$ parametrizes the elements $v = a_1v_1 + a_2v_2 + a_3v_3 \in V$ that are decomposable: $\exists \ell, b \in \mathbb{R} \setminus 1 | v = \ell \cdot b$. That is, if $P(V)$ is the projective plane associated to $V$, we let $\Gamma(V) = P(V) \cap S_4$ where $S_4$ is the cubic hypersurface of degenerate conics (that can be factored). The geometric method takes into account also the set $D_2$ of “double lines” (singular conics whose equation is the square of a linear form). We will show, for those orbits for which $\Gamma(V)$ is singular,

Theorem 2 (1) (Theorem 4.1 (B)) Each of the orbits $\text{Pgl}(3) \circ V$ for which $\Gamma(V)$ is singular is completely determined by the isomorphism classes of triples of projective algebraic sets

$$[D_2 \cap P(V), \Gamma(V) = S_4 \cap P(V), P(V)],$$

viewed as schemes.
(2) (Sect. 6.2, Sect. 6.6, Proposition 6.24) The orbits, their dimensions, and the specializations of these orbits are as shown in Table 1.
(3) (Sect. 6.1, Proposition 6.6, and Note of Translators 6.7) The equivalence classes and their specializations satisfy a left right symmetry, that arises from a duality between an orbit and its dual.

An alternative method, used to verify this classification, is more algebraic. It is easy to show, according to intersection theory on Grass$(2, \mathbb{R}^2)$, that every 3-dimensional vector space contains a dimension 2 subspace having a particular form, that is, a vector subspace belonging to the closure of the $\text{Pgl}(3)$ orbit of $\langle xy, x^2 + yz \rangle$. This method easily reduces the problem of classification to eliminating repetitions in a finite list (Sect. 5.2).

We give without further delay in Table 1 the table of orbits of nets of conics, and their specializations, reserving for later a detailed explanation. We give for each orbit

1. a vector space belonging to it,
2. a diagram indicating the isomorphism type of of its associated cubic $\Gamma(V)$, where each point of $\Gamma(V) \cap D_2$ is marked by a circle around it,
3. the dimension of the orbit is indicated by its height in the table.
Theorem 3 (Theorem 7.1) The classification up to isomorphism of graded Artinian algebras having Hilbert function $H(A) = (1, 3, 3)$ is that determined implicitly by Table 1. Each such algebra is smoothable, that is, has a flat deformation to $k^7$. Every graded algebra of Hilbert function $H(A) = (1, r, 2)$ is smoothable, that is, has a flat deformation to $k^{r+3}$.

1.1 Outline

We give now the main lines of the paper. In Sect. 2, we recall general facts about algebras of Hilbert function $(r_1, r_2, r_3)$. According to [54, Proposition 2] we may take $j$ satisfying the hypotheses for $\phi$. Remark 4.4 shows that the map of $j$ invariants $j$ to $j(\Gamma(J_\phi))$ is generically a 3 : 1 map.

In Sect. 4.2.3, we will show (Proposition 4.5) that when $\Gamma_\pi$ for the net $\pi$ is a smooth cubic, then $\pi = J_\phi$ for a cubic $\phi$ and $\Gamma$ has a canonical involution $\iota$ without fixed point, such that the three-dimensional vector space $J_\phi$ has the following property:

$\lambda$ Note of translators: A typo in the top right entry in the table has been corrected from the original. The correct entry is $(Z^2, X^2 - YZ, Y^2 - XZ)$. The original top right entry $(XY, X^2 + YZ, Z^2 + X^2)$ is in fact isomorphic to $(XY, X^2 + YZ, Y^2 + XZ)$ at the top left.

6. $j(\lambda)$ is the $j$-invariant of the cubic $\phi$: here $\phi$ is always smooth, not isomorphic to the cubic of equation $X^3 + Y^3 + Z^3 = 0$. A smooth cubic can always be put into this form of Hesse $X^3 + Y^3 + Z^3 - 3\lambda XYZ = 0$, satisfying the hypotheses for $\lambda$ listed in the table under #8b. The symbol $\omega$ denotes a primitive cube root of 1. Note that when $\lambda = -1, -\omega$ or $-\omega^2$ the associated cubic is singular and isomorphic to #6a—then the net has a common point; if $\lambda$ is 0, 2, $2\omega$, $2\omega^2$ then the net is isomorphic to #6d (see the text at the end of Sect. 9). According to [54, Proposition 2] we may take $j(\lambda) = \frac{\lambda^3(\lambda^3 - 8\lambda^3)^3}{27(\lambda + 1)^3(\lambda + \omega)^3(\lambda + \omega^2)^3}$. See Proposition 4.3ff.
Table 1 The orbits of \( \text{Pgl}(3) \) acting on the planes of conics

\[
J_\phi = \left\{ v \in R_2 \mid \forall C \in \Gamma, \ C \text{ and } \iota(C) \text{ are conjugate in relation to } v \right\}.
\]

Then, in Sect. 4.3, we classify the vector spaces \( V \) for which \( \Gamma(V) \) is singular. The idea consists of using a geometric construction to find a basis of a space isomorphic to \( V \), in a canonical form. The algebraic method of classification is presented in Sect. 5: in the present work, we have used it as a verification of the results of the geometric method. But the idea of this method may furnish some simplifications of other problems of algebraic classification.
The specializations, or in other words the closures of orbits, are discussed in Sect. 6. In view of shortening that section, we use the notion of duality among orbits (Sect. 6.1), a calculation of the dimension of each orbit (Sect. 6.2) and an upper semi-continuous invariant, the length of the projective scheme associated to the ideal generated by the vector space. This is useful also for considering the specializations of the orbits of low dimension (Sect. 6.3). In Sect. 6.4, we apply these notions to restrict the set of potential specializations of nets. In Sect. 6.5, we study specializations of cubics. In Sect. 6.5.1, we give an argument from invariant theory about the closure of each orbit in the one parameter family of smooth orbits, or the orbit of the singular cubic with a double point having distinct tangents. We show:

**Theorem 6.12** Let $F$ denote an orbit of cubics of dimension less than 8. Either all cubics of the above family specialize to $F$; or exactly one does.

In Sect. 6.5.2 we give the specializations of singular cubics; we also specify the discriminant cubics of Table 1; in the new Sect. 6.5.3 we for completeness list the polar nets $J_{\phi}$—those determined by the partial derivatives of a cubic form $\phi$. In Sect. 6.6, we establish explicitly the existence of the specializations of nets of conics announced in Table 1.

In Sect. 7, we examine the deformations of Artinian algebras. We show in Sect. 7.1 that the algebras of Hilbert function $(1, 3, 3)$ are smoothable, that is that they deform to algebras isomorphic to $k^7$ (given the product structure of the algebras). In Sect. 7.2, we discuss the form of the algebras of Hilbert function $(1, 4, 2)$ and, more generally, algebras of Hilbert function $(1, r, 2)$. The algebras of Hilbert function $(1, 4, 2)$ constitute the other case of algebras of length 7 presenting a continuous family of distinct isomorphism classes; we describe the $(r - 3)$-parameter family of isomorphism classes of sufficiently general algebras of Hilbert function $(1, r, 2)$. Then, in exhibiting the generators and the relations among these generators, we show that these algebras also have smooth deformations.

Finally, in Sect. 8 we will show that the study of the regular maps of degree 4 from the projective plane into itself is equivalent to the study of nets of conics. We give an interpretation of the invariants of a net of conics in terms of the critical locus and the discriminant of a regular map of degree 4. In Sect. 9 we discuss Hesse pencils of cubics and their connection to nets.

The geometric arguments, including the existence of specializations, in particular the use of the theory of invariants in Sect. 6.5.1, and Sect. 8 are due to Emsalem. Iarrobino has determined algebraically the list of orbits (Sect. 6), the duality (Sect. 6.1) and the deformations of Sect. 7.2. We thank Bernard Teissier for his encouragement and comments. We note that C.T.C. (see [Wall]) has considered independently the same problem and found very succinctly an analogous table (except for the specializations of constant $j$, our Sect. 6.5), both in the complex case that we study, and in the real case that we do not consider; we develop the algebra and the geometry of the situation in more detail (Sects. 4.2, 4.3, 5, etc.). The non-separability of the space of orbits is a phenomenon similar to that described by Nagata in [Nag].

We thank Martine Aeschbacher who promptly deciphered and made readable a handwritten manuscript that was often obscure.
2 Space of projective conics

Introduction Here we describe the main aspects of the geometry of the space of conics in the projective plane.

We denote by $C_5$ the projective space associated to $R_2$ (which has dimension 5), that we identify with the space of planar conics. In this space, the set of singular conics form a remarkable cubic hypersurface that we call $S_4$ in the sequel:

$$S_4 = \left\{ \text{the } \mathbb{C}^* \text{ classes of } F, F \in R_2, \text{ such that } \det_{x,y,z} \left( F'_x, F'_y, F'_z \right) = 0 \right\}.$$

$S_4$ in turn contains $D_2$, the set of “double lines”. $D_2$ is the 2-Veronese embedding of $\mathbb{P}^2$:

$$D_2 = \{ \text{classes of } L^2, L \in R_1 \}.$$

Remark 2.1 It is easy to see that $D_2$ is exactly the singular locus of $S_4$ and that $S_4$ is the union of the planes of $C_5$ tangent to $D_2$. Thus, $S_4$ is also a rational variety. $D_2$ and $S_4$ are evidently stable under the action of $\text{Pgl}(3)$. In the general case where a plane $\pi$ of $C_5$ intersects $S_4$ properly, the scheme intersection with $S_4$ is a cubic $\Gamma_{\pi}$ in $\pi$, and we will try to conclude the maximum information on the class of the plane modulo $\text{Pgl}(3)$ from the pair $(\Gamma_\pi, \pi)$. We note finally that a plane of $C_5$ does not in general meet $D_2$ for the obvious reason of dimension, and that an intersection point of a plane with $D_2$, when it exists, can only be a singular point of the intersection cubic in $S_4$, as it is already a singular point of $S_4$.$^7$

We give now some details on certain families of linear projective subvarieties of $C_5$:

(2.1) The tangent plane to $D_2$ at the point corresponding to a double line is the set of singular conics having as one of its irreducible components the support of that double line.

(2.2) The hyperplane tangent to $S_4$ at a smooth point — that is, a point corresponding to a singular conic of rank two — is the set of all conics passing through the unique singular point of that conic.

(2.3) The Zariski tangent cone to $S_4$ at a singular point of $S_4$, that is, at a double line, is a quadratic cone of dimension 4. This cone is the set of conics tangent to the support of the double line. It contains $D_2$ and the tangent plane to $D_2$ at the point considered. One may regard it also as the set of all classes comprised of a sum of a representative of a point of this tangent plane, and of a representative of a point of $D_2$.

$^7$ We list the discriminant cubics in Sect. 6.5.2, Table 6; see Example 6.20.
(2.4) The maximal linear subvarieties of $S_4$ are the planes comprising the following two disjoint sets:

(2.4a) The set of planes $\pi_x : x \in P$ (the plane corresponding to the dual of $R_1$) where $\pi_x = \{ \gamma : \gamma \in S_4, x \text{ is a singular point of } \gamma \}$. 

(2.4b) The set of planes $\pi_\delta : \delta \in P^*$ where $\pi_\delta = \{ \gamma, \gamma \in S_4, \text{such that } \delta \text{ is an irreducible component of } \gamma \}$. 

One has the following properties where, if $\gamma \in S_4 - D_2$, $m_\gamma$ denotes the unique singular point of $\gamma$ and $T_\gamma(S_4)$ denotes the hyperplane of $C_5$ tangent at $\gamma$ to $S_4$:

(i) For every $\gamma \in S_4 - D_2$, $\pi_{m_\gamma} - D_2 = \{ \gamma' : \gamma' \in S_4 - D_2, T_\gamma S_4 = T_{\gamma'} S_4 \}$. Also, $\pi_{m_\gamma} = \pi_{m_\gamma} - D_2$.

(ii) For every $x, y \in P$, $x \neq y \rightarrow \pi_x \cap \pi_y = \{ u_{xy} \}$, where $u_{xy}$ is the element of $D_2$ whose support contains $x$ and $y$.

(iii) For every $\delta \in P^*$, $\pi_\delta$ is the tangent plane to $D_2$ at the point whose support is $\delta$.

(iv) For every $\delta, \delta' \in P^*, \delta \neq \delta' \Rightarrow \pi_\delta \cap \pi_{\delta'} = \{v_{\delta \delta'}\}$ where $v_{\delta \delta'}$ is the element of $S_4 - D_2$ having as irreducible components $\delta$ and $\delta'$.

(2.5) Consider the action of Pgl$(5)$ on the projective space of conics $C_5$. The preceding considerations show the following assertion:

Let $\sigma \in \text{Pgl}(5) = \text{Aut}(C_5)$, the following conditions are equivalent:

(i) The automorphism $\sigma$ arises from an element of Pgl$(3)$ acting on Proj($R_2$);

(ii) $\sigma$ stabilizes $S_4$ (maps $S_4$ onto $S_4$);

(iii) $\sigma$ stabilizes $D_2$.

For the proofs keep in mind that $D_2$ is the singular locus of $S_4$.

3 Pencils of conics

Before discussing the classification of the planes of $C_5$, we will present the classification of the lines of $C_5$. Table 2 in Sect. 3.1 sets out this classification. The accompanying proof uses as main tools a weak form of Noether’s theorem and Bezout’s theorem. In Sect. 3.2, we follow this classification by determining the specializations, which are listed in Table 3.

3.1 Classification of pencils

The notations are those in [Ful]. We need the following version of Noether’s theorem that can be deduced immediately from [Ful] p. 121, Proposition 1].

**Proposition 3.1** Let $F$ and $G$ be two distinct plane projective curves with the same degree $n$, where $F$ is assumed to be smooth. Let $H$ be a plane projective curve of degree $n$. For $H$ to belong to the pencil generated by $F$ and $G$, it is necessary and sufficient that for every point $M$ in the intersection of the supports of $F$ and $G$, the intersection number $I(M ; F \cap H) \geq I(M ; F \cap G)$. 

In Table 2, $U = \langle F, G \rangle$ denotes a line of $C_5$. The first diagram represents $U$ marked by points of $U \cap S_4$ and, possibly, of $U \cap D_2$. The points of $U \cap D_2$ are represented
by $\bigcirc$. The points of $U \cap S_4$ which do not belong to $D_2$ are represented by $\rightarrow$. When a point of $U$ intervenes without belonging to $S_4$, it is noted $\bigtriangledown$. When a point of $U \cap S_4$ has multiplicity $k$ in this intersection, it is repeated $k$ times.

The second diagram represents the relative situation in the ambient plane $P$ of the conics which correspond to the points in the first diagram. Finally, a system of equations of these conics is provided for a suitably chosen coordinate system. However, in part II of the table, all the points of $U$ belong to $S_4$; we only indicate those which intervene in the chosen presentation. We continue, however, to systematically point out the elements of $U \cap D_2$.

**Exhaustion of the classes of pencils of conics not contained in $S_4$.** We will apply the previous proposition to all the pencils of conics not entirely contained in $S_4$ to find in each case a particular algebraic representation of a system of generators.

Let $U$ be such a pencil and $F$ and $G$ two generators of $U$, $F$ chosen to be smooth. By Bezout’s theorem [Ful, Section 5.3, p. 57], the intersection of $F$ and $G$ can consist of:

1. four distinct points, each of multiplicity one. According to the proposition, the conics of $U$ are those which pass through these four points. Three of these points are never aligned. We can therefore choose a homogeneous coordinate system where these points have the following representations: $(1, 1, 1), (-1, 1, 1), (1, -1, 1), (1, 1, -1)$. Three (and only three) singular conics pass through these four points. They have for equations $S$: $(X^2 - Y^2), (Y^2 - Z^2), (Z^2 - X^2)$ and two of them determine the pencil.

2. three distinct points $J, K$ and $L$ such that $I(J; F \cap G) = I(K; F \cap G) = 1$ and $I(L; F \cap G) = 2$. $J, K, L$ cannot be aligned. Let $T$ be the tangent to $F$ in $J$; $T$ cannot contain either of $J, K$. We can choose a homogeneous coordinate system such that the equation for $T$ is $Z = 0$ and the points $J, K, L$ have coordinates $(0, 1, 0), (1, 0, 1), (-1, 0, 1)$ respectively. We verify that the only two singular conics that intersect $F$ in $J, K, L$ with the suitable multiplicities to belong to $U = (\kappa ZY + \mu(X^2 - Z^2))$ have equations $(ZY = 0)$ and $(X^2 - Z^2 = 0)$. The condition for which this conic is singular is written: $\kappa^2 \mu = 0$. In the intersection between $U$ with $S_4$, the conic $(ZY)$ has multiplicity 1 and $(X^2 - Z^2)$ has multiplicity 2. This explains case (b) in Table 2.

In the following cases, we only give a sketch of proof.

3. two distinct points $J$ and $K$ with multiplicity 2, for example $J = (0, 1, 0)$ and $K = (1, 0, 0)$. Let $R$ and $S$ be the tangents to $F$ in $J$ and $K$ respectively. We choose $Z = 0$ for the equation of $JK$, $X = 0$ for the equation of $R$ and $Y = 0$ for the equation of $S$. The only singular conics of the pencil $(\kappa Z^2 + \mu XY)$ have for equations: $Z^2 = 0$ and $XY = 0$. The condition for which this conic is singular is written $\kappa^2 \mu = 0$, which explains the result in case (c) in Table 2.

4. one point $J$ with multiplicity three and one single point $K$, as $J = (0, 0, 1)$ and $K = (1, 0, 0)$. Let $R$ and $S$ be the tangents to $F$ in $J$ and $K$ respectively. In a suitable homogeneous coordinate system, $R, JK$ and $S$ have equations $X = 0$, $Y = 0$ and $Z = 0$ respectively, and $F$ has equation $XZ - Y^2 = 0$. $(XY)$ is the only singular conic of $U = (\kappa XY + \mu(XZ - Y^2))$, and the condition for which
Table 2  Classification of pencils of conics

I. $U \not\subset S_4$.

(a) \[ \begin{array}{c}
A & B & C \\
\hline
B & & A
\end{array} \]

$(A, B, C) = ((X^2 - Z^2), (Y^2 - Z^2), (X^2 - Y^2))$

(b) \[ \begin{array}{c}
2B & & A \\
\hline
B & & A
\end{array} \]

$(B, A) = ((X^2 - Z^2), (ZY))$

(c) \[ \begin{array}{c}
2B & & C \\
\hline
C & & B
\end{array} \]

$(A, B) = ((XY), (Z^2))$

(d) \[ \begin{array}{c}
3B & & C \\
\hline
C & & B
\end{array} \]

$(B, C) = ((XY), (XZ - Y^2))$

(e) \[ \begin{array}{c}
3B & & C \\
\hline
B & & C
\end{array} \]

$(B, C) = ((X^2), (XZ - Y^2))$

II. $U \subset S_4$.

(f) \[ \begin{array}{c}
A & B \\
\hline
B & A
\end{array} \]

$(A, B) = ((XY), (XZ))$

(g) \[ \begin{array}{c}
A & B \\
\hline
B & A
\end{array} \]

$(A, B) = ((X^2), (XY))$

(h) \[ \begin{array}{c}
B & C \\
\hline
C & B
\end{array} \]

$(A, B) = ((X^2), (Y^2))$
this conic is singular is written $\mu^3 = 0$, which explains the result in case (d) in Table 2.

(5) one point of multiplicity 4. With the conventions of the previous case, the only singular conic of the pencil $U = \kappa X^2 + \mu (XZ - Y^2)$ has equation $(X^2 = 0)$ and we find all the results of case (e) in Table 2.

Exhaustion of the classes of pencils of conics contained in $S_4$. Let $\alpha$ and $\beta$ be lines in $P$. We denote by $v_{\alpha\beta}$ the element of $S_4$ whose irreducible components are $\alpha$ and $\beta$ if $\alpha \neq \beta$ and the element of $D_2$ whose support is $\alpha$ if $\alpha = \beta$.

From the previous examination, and by elimination, we can affirm that two singular conics $v_{\alpha\alpha'}$ and $v_{\beta\beta'}$ determine a pencil contained in $S_4$ if and only if they admit a common irreducible component or a common singular point, these conditions not being exclusive. The conjunction of the two conditions gives the case (g) of Table 2, the first condition without the second gives the case (f) and the second without the first gives case (h).

General classification of pencils of projective plane conics. According to the preceding discussion, the table presented is exhaustive. On the other hand, it is without repetition and therefore constitutes a classification: indeed, according to the remarks given in Sect. 2, it becomes clear that for two pencils $U$ and $U'$ to belong to the same orbit under $\text{Pgl}(3)$,

$U \cap D_2$ and $U' \cap D_2$ on the one hand and $U \cap S_4$ and $U' \cap S_4$ on the other hand, must have the same isomorphism type under linear maps: the types of these intersections described in the different cases are all distinct.

3.2 Specialization of pencils

See Table 3 on the following page.

(A) We begin by calculating the dimension of each orbit. These dimensions are indicated on Table 3. The reasonings leading to the calculation of this dimension are, in fact simple, of the same type as those we will take up again later (Sect. 6.2) for the orbits of planes in $C_5$. So we do not give them here, except for the following example of type (1): To give a pencil of type 1 is equivalent to give four points of the projective plane, none being collinear with two others. There is a variety parametrizing subsets of the plane comprised of four points having this property, which is the quotient of an open set in the product of four copies of the projective plane by a finite group, and that variety thus has dimension 8.

(B) The only specializations that we might envision given the dimension calculation, and the fact that the specialization of a type having an element of $D_2$ should, itself, contain an element of $D_2$, are the specializations indicated by the arrows of the table, and their composites.

(C) One verifies by the following calculations that all the specializations exist.
| Dimension of the orbit |
|------------------------|
| 8                      |
| 7                      |
| 6                      |
| 5                      |
| 4                      |
| 3                      |

**Table 3** Specializations of orbits of pencils in $C_5$
(1) Existence of

\[ [(X^2 - Z^2, Y^2 - X^2)] \quad \rightarrow \quad [(X^2 - Z^2, (ZY))] \]

One considers the family of pencils: \( U_t = [(X - Z)(X + tZ), (Y^2 - X^2)] \). For \( t \neq 0 \), \( U_t \) belongs to the orbit of \( [(X^2 - Z^2), (Y^2 - X^2)] \), and for \( t = 0 \), \( U_0 \) belongs to the orbit of \( [(ZY), (X^2 - Z^2)] \).

(2) Existence of

\[ [(X^2 - Z^2), (ZY)] \quad \rightarrow \quad [(XY), (Z^2)] \]

One considers the family of pencils \( U_t = [(X^2 - tZ^2), (ZY)] \). The desired specialization occurs for \( t = 0 \).

(3) Existence of

\[ [(X^2 - Z^2), (ZY)] \quad \rightarrow \quad [(XY), XZ - Y^2] \]

One considers the family of pencils \( U_t = [(Z(X + tY), (ZY - X^2)] \) which realizes for \( t = 0 \) the looked for specialization.
(4) Existence of

One considers the family \( U_t = [(YZ - X^2 - tZ^2), (Y^2)] \), which for \( t = 0 \) realizes the specialization shown.

(5) Existence of

One considers the family \( U_t = [(Y(tX + Y)), (YZ - X^2)] \) as \( t = 0 \).

(6) Existence of

The family \( U_t = [(X^2), (tXZ - Y^2)] \) realizes for \( t = 0 \) the specialization.

(7) Existence of

Use the family \( U_t = [(XY), (YZ - tX^2)] \) for \( t = 0 \).
is realized by the family $U_t = [(X^2), (XY + tY^2)]$ as $t = 0$.

is realized by the family $U_t = [(XY), (X(tZ + X))]$ as $t = 0$.

4 Nets of conics

4.1 Introduction to Sects. 4.2 and 4.3: classification of planes of conics

Notations If $\pi$ is a plane of $C_5$ we denote by $\Gamma_{\pi}$ the schematic intersection of $\pi$ with $S_4$, the hypersurface of degenerate conics. When that intersection is proper in $S_4$, then $\Gamma_{\pi}$ will be entirely characterized by the class of a cubic form in the projective space associated to $R_3$. We denote by $\delta_{\pi}$ the schematic intersection of $\pi$ with the family of double lines $D_2$.

Definition 4.1 (Definition and Note) One says that the sequences $(\delta_{\pi} \subset \Gamma_{\pi} \subset \pi)$ and $(\delta_{\pi'} \subset \Gamma_{\pi'} \subset \pi')$ are isomorphic if there is a projective linear isomorphism from $\pi$ to $\pi'$ (that is, arising from an element of $\text{Pgl}(3)$ mapping $\mathbb{P}^5 = \mathbb{P}(R_2)$ to itself) that transforms $\Gamma_{\pi}$ to $\Gamma_{\pi'}$ and $\delta_{\pi}$ to $\delta_{\pi'}$; or, equivalently, from an element of $\text{Pgl}(5)$ stabilizing $S_4$ or $D_2$ (as in (2.5) of Sect. 2).

The isomorphism class of a triple $(\delta_{\pi} \subset \Gamma_{\pi} \subset \pi)$ will be called its type. When $\Gamma_{\pi}$ is a cubic, a figure representing $\Gamma_{\pi}$ with the points of $\delta_{\pi}$ indicated, will symbolize the type of $(\delta_{\pi} \subset \Gamma_{\pi} \subset \pi)$. When $\Gamma_{\pi} = \pi$ the points or curve $\delta$ pictured within a hatched plane (that represents $\Gamma_{\pi} = \pi$) will symbolize the type of $(\delta_{\pi} \subset \Gamma_{\pi} \subset \pi)$.

Examples:

On the other hand if $\phi$ is a smooth cubic with defining equation $F$, we will denote by $J_{\phi}$, the plane of $C_5$ containing the conics of equations the partials $F'_X, F'_Y, F'_Z$, and we will term this the Jacobian plane of the cubic $\phi$.

General introduction to the classification of the planes of $C_5$. We show the following result.

Theorem 4.1 There are two disjoint collections of isomorphism classes of planes $\pi \subset C_5$. 
A) The set of classes of planes $\pi$ such that $\Gamma_\pi$ is smooth. For such a plane $\pi$, there is a unique cubic $\phi$ such that $\pi = J_\phi$. This is a smooth cubic, not isomorphic to the cubic $(X^3 + Y^3 + Z^3)$. The isomorphism class of the plane is completely determined by the isomorphism class of $\phi$. If $j$ is the fundamental invariant of $\phi$ we will denote the corresponding class $\pi$ by the symbol $\bigodot_{j}$, representing the smooth cubic indexed by its $j$-invariant.

B) The set of classes of planes $\pi$ such that $\Gamma_\pi$ is not a smooth cubic. Each isomorphism class in this set is completely determined by the type of the sequence $(\delta_\pi \subset \Gamma_\pi \subset \pi)$ with respect to any one of its representatives. The admissible types are those underlying the isomorphism classes that are pictured in Table 1.

Demonstrating these results is the object of Sects. 4.2 and 4.3: Sect. 4.2 treats the case $\Gamma_\pi$ is smooth (Case 1) and Sect. 4.3 treats the case $\Gamma_\pi$ is singular (Case 2–10).

Suppose we are given a plane $\pi$ of $C_5$. We will organize our study in Sect. 4.3 following the situation of $\pi$ with respect to the locus $S_4$ of degenerate conics.

General classification of cases for $\Gamma$.

(I) We consider first the case $\pi \not\subseteq S_4$.

In this case $\pi \cap S_4$, considered as a scheme, is a cubic $\Gamma_\pi$. A priori, following the well-known classification of cubics in the projective plane over an algebraically closed field one has to consider the following cases.

1. $\Gamma$ is smooth (Sect. 4.2): $\bigodot_{j}$; Or $\Gamma$ is singular (Sect. 4.3, #2-10):
2. $\Gamma$ is singular, with singularity a point having two distinct tangents: $\bigodot$;
3. $\Gamma$ is singular at a cusp: $\bigodot$;
4. $\Gamma$ decomposes into a conic union a secant line: $\bigodot$;
5. $\Gamma$ decomposes into a conic union a tangent line: $\bigodot$;
6. $\Gamma$ decomposes into three non-concurrent lines: $\bigodot$;
7. $\Gamma$ decomposes into three concurrent lines: $\bigodot$;
8. $\Gamma$ decomposes into a double line union a simple line: $\bigodot$;
9. $\Gamma$ consists of a triple line: $\bigodot$.

(II) We consider next the case where $\pi \subset S_4$. There are two families of such planes, according to whether

10a) $\delta_\pi$ consists of a parabola $\bigodot$ or
10b) $\delta_\pi$ consists of a point $\bigodot$.

---

8 Establishing that classification is an easy exercise after [Ful, Exercise 5.24].
4.2 The planes of conics whose associated cubic $\Gamma_\pi$ is smooth

**Introduction** In this paragraph we will give first an exhaustion of the cases where $\Gamma_\pi$ is smooth, by examining the cubic $\Gamma_\pi$ geometrically. Then we introduce from the representation of a plane $\pi$, a cubic $\phi$ such that $\pi = J_\phi$. Hence the classification (Sect. 4.2.2). Finally, we develop in Sect. 4.2.3 a new point of view by associating to each plane $J_\phi$ a third cubic $H$ endowed with a regular involution $i$ without fixed point such that:

$$\pi = \{ \gamma : \gamma \in C_5, \forall x \in H, x \text{ and } i(x) \text{ are conjugate with respect to } \gamma \}.$$ 

Here $H$ is the Hessian of $\phi$, and the problem of classification done in Sect. 4.2 comes down to classifying the smooth cubics endowed with a regular involution without a fixed point.

4.2.1 The case where the associated cubic is smooth (Case 1)

Let $\pi$ be a plane in $C_5$ such that $\Gamma_\pi$ is smooth. In what follows, we write $\Gamma$ for $\Gamma_\pi$. We can choose in $\pi$ distinct points $O, A, B, C, D$ (see figure below) verifying the following properties: $A$ is an inflection point of $\Gamma$, $OA$ is the inflectional tangent to $\Gamma$ at $A$; $B, C, D$ are aligned with $O$ and belong to $\Gamma$, $AB, AC, AD$ being the respective tangents to $\Gamma$ in $B, C, D$. (This is the geometric version of putting the cubic equation in Weierstrass form.)

According to Remark 2.1, none of the points of $\pi$ belong to $D_2$, because otherwise it would be a singular point of $\Gamma$. We use the notation of Table 2 to identify the type of pencils which then appear: the pencil OBCD is of type (a); the pencils $AB, AC, AD$ are of type (b); $AO$ is of type (d). According to Table 2, we can choose a coordinate system where: $B = (X^2-Y^2), C = (Y^2-Z^2), D = (Z^2-X^2)$. The bi-ratio $(O, A, B, C)$ then determines the equation of a conic $O$ which is of the form: $\kappa X^2 - (1+\kappa)Y^2 + Z^2 = 0$, with $\kappa$ uniquely determined and $\kappa \neq 0, -1$. The pencils $AB, AC, AD$ being of type (b), the singular points of $B, C, D$ lie on the irreducible components of the conic determined by $A$ outside its singular point. $OA$ being of type (d), an irreducible component of $A$ must be tangent to the conic determined by $O$ at the singular point of $A$. 

\[ Springer\]
Finally, the conic $A$ is determined as follows: one of its irreducible components $A_1$ must pass through two of the singular points $B$, $C$ and $D$. The other component $A_2$ is the tangent to $O$ in one of the two intersection points of $A_1$ and $O$. The last condition ($A_2$ passes through the singular point of the third conic) is also ipso facto satisfied. Then there are six possibilities for $A$, two by two isomorphic:

$$A = \left( Y[k^{1/2}X \pm iZ], (X[Z \pm (1 + \kappa)^{1/2}Y]), (Z[k^{1/2}X \pm (1 + \kappa)^{1/2}Y]) \right).$$

By reordering $B$, $C$, $D$ we therefore have in all cases the following presentation of $\pi$:

$$B = (X^2 - Y^2); \quad C = (Y^2 - Z^2); \quad D = (Z^2 - X^2);$$
$$A = (X[Z + (1 + \kappa)^{1/2}Y]); \quad O = (\kappa X^2 - (1 + \kappa)Y^2 + Z^2).$$

A change of coordinates leads to a new representation of $\pi$:

$$\pi = (YZ - X^2), (Y - Z)(Y - \alpha Z), (XY); \quad \alpha \neq 0, 1.$$

We have then the following proposition.

**Proposition 4.2** Let $\pi$ be a plane in $C_5$ such that $\Gamma_\pi$ is a smooth cubic. There exists a homogeneous coordinate system in $P$ such that:

$$\pi = \left[ (YZ - X^2), (Y - Z)(Y - \alpha Z), (XY) \right] \text{ with } \alpha \neq 0, 1.$$

Conversely, any plane $\pi$ in $C_5$ having in a suitable coordinate system such a presentation has an associated smooth cubic $\Gamma_\pi$.

### 4.2.2 Jacobian plane of a smooth cubic

The above provides an exhaustion of all planes $\pi$ such that $\Gamma_\pi$ is smooth, but is not in itself a classification up to $\text{Pgl}(3)$ isomorphism. To obtain this classification, let us prove the following proposition:

**Proposition 4.3** Let $\pi$ be a plane of $C_5$ such that $\Gamma_\pi$ is smooth.

(a) There exists a unique cubic projective plane curve $\phi$ of $P$ such that $\pi = J_\phi$.

(b) This cubic $\phi$ is smooth, not isomorphic to $X^3 + Y^3 + Z^3$. 
(c) Conversely, for every smooth cubic \( \phi \) not isomorphic to \( X^3 + Y^3 + Z^3 \), \( \Gamma_{J_\phi} \) is smooth.

(d) The fundamental invariant \( j_\phi \) of the cubic \( \phi \) classifies the plane \( \pi \) up to \( \text{Pgl}(3) \) isomorphism.

Let \( \pi \) be a plane of \( C_5 \) and \( V \) the corresponding vector space in \( R_2 \). By Sect. 4.2.1, up to a change of coordinates we can suppose that there exists \( \alpha \neq 0, 1 \) such that \( V \) is generated by \( YZ - X^2, (Y - Z)(Y - \alpha Z), XY \). We want to find an element \( F = \phi \) of \( R_3 \) whose partial derivatives of order 1 also generate \( V \). It is therefore a matter of finding \( a_i, b_i, c_i \) such that:

\[
\exists F \in R_3, \forall i, 1 \leq i \leq 3 \Rightarrow a_i(YZ - X^2) + b_i XY + c_i(Y - Z)(Y - \alpha Z) = F'_{X_i}
\]

where \( X_1 = X, X_2 = Y, X_3 = Z \).

This condition is equivalent to the following condition:

\[
\forall i, \forall j \mid i, j \in \{1, 2, 3\}, i \neq j \text{ we have } \frac{d}{dX_i} \left[ a_j(YZ - X^2) + b_j XY + c_j(Y - Z)(Y - \alpha Z) \right] = \frac{d}{dX_j} \left[ a_i(YZ - X^2) + b_i XY + c_i(Y - Z)(Y - \alpha Z) \right].
\]

By identifying the coefficients of the monomials of the two sides (which are of degree 1, therefore there are three coefficients) of each of these three relations, we obtain a system of 9 linear equations with 9 unknowns (the \( a_i, b_i, c_i \)) easy to solve, which is exactly of rank 8; we find:

\[
\begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3 \\
\end{pmatrix} = \rho
\begin{pmatrix}
0 & 2(\alpha - 1)^2 & 0 \\
0 & 0 & -(\alpha + 1) \\
-(\alpha - 1)^2 & 0 & 2\alpha \\
\end{pmatrix}
\]

So \( F \) therefore exists, unique up to multiplication by a non-zero scalar, and is defined by

\[
3F = XF'_X + YF'_Y + ZF'_Z = \frac{\rho}{3} \left(3(\alpha - 1)^2 X^2 Y + 2\alpha^2 Z^3 - 3\alpha(\alpha + 1)YZ^2 + 6\alpha Y^2 Z - (\alpha + 1)Y^3 \right)
\]

We verify that \( \{F'_X, F'_Y, F'_Z\} \) is of rank 3 and thus indeed constitutes a system of generators of \( V \), and we take \( \phi : F = 0 \). Which ends up proving (a).

Let us show (b). The partial derivatives of \( X^3 + Y^3 + Z^3 \) generate a vector space \( V \) whose associated plane \( \pi \) has a cubic \( \Gamma_{J_\phi} \) constituted by three non-intersecting lines. Therefore \( \phi \) is not isomorphic to \( (X^3 + Y^3 + Z^3) \) since otherwise \( \Gamma_{J_\phi} \) would not be smooth.

Let us show in the same way that \( \phi \) is not singular, that is, \( \phi \) is not a specialization of a family of cubics isomorphic to \( (X^3 + Y^3 + XYZ) = \phi_0 \) (double point cubics...
with distinct tangents, #8a in Table 1). However, \( \Gamma_{J_{\phi_0}} \) is not smooth, and this property holds for a cubic isomorphic to \( \phi_0 \) and for specializations. Therefore, \( \phi \) is not singular.

Let us now show (c). If \( \phi \) is smooth, we can find a system of coordinates in which \( \phi \) has “Hessian form”: \( \phi = (X^3 + Y^3 + Z^3 + 3\lambda XYZ) \), where \( \lambda \neq -1, -\omega, -\omega^2 \) (\( \omega \) and \( \omega^2 \) are the cube roots of 1). That \( \phi \) is not isomorphic to \( (X^3 + Y^3 + Z^3) \) means then that \( \lambda \neq 0, 2, 2\omega, 2\omega^2 \). Saying that \( (aF'_X + bF'_Y + cF'_Z) \) is a singular conic is equivalent to the relation:

\[
(2^3 + 2\lambda^3) abc - 2\lambda^2(a^3 + b^3 + c^3) = 0. \tag{4}
\]

“\( \Gamma_{J_{\phi}} \) is smooth” is equivalent then to \( -\frac{2^3 + 2\lambda^3}{2\lambda^2} \neq -3, -3\omega, -3\omega^2 \) and \( \lambda \neq 0 \). The first condition decomposes into the conditions: \( \lambda \neq -1, 2, -\omega, 2\omega, -\omega^2, 2\omega^2 \) which are all satisfied by hypothesis. \( \Box \)

(d) is clear.

**Presentation in the Table 1.** Considering the cubic \( \phi \) of the form:

\[
(X^3 + Y^3 + Z^3 + 3\lambda XYZ), \quad \lambda \neq -1, -\omega, -\omega^2, 0, 2, 2\omega, 2\omega^2,
\]

we write the plane \( \pi \) in the form \( (X^2 + \lambda YZ, Y^2 + \lambda ZX, Z^2 + \lambda XY) \). In doing so, we do not forget that it is not \( \lambda \) which classifies \( \pi \), but rather the fundamental invariant \( j_{\lambda} \) of \( \phi \), which is the rational function of \( \lambda \) noted in the table \( j_{\lambda} \).

**Remark 4.4** The equation \( \frac{x^3 + 2x^3}{2x^2} = \beta \) generically admits three roots \( \alpha_1, \alpha_2, \alpha_3 \) such that the curves, \( (X^3 + Y^3 + Z^3 + 3\alpha_i XYZ), 1 \leq i \leq 3 \) are two by two non-isomorphic.

Thus, generally, given an isomorphism type \( j_{\Gamma} \) of a smooth cubic curve \( \Gamma \), there exist three types of plane \( \pi \) (that is, isomorphism classes \( j_{\phi} \) of nets), giving rise to the type of curve \( \Gamma \): that is, \( \Gamma_{\phi} \) is in the class \( j_{\Gamma} \).

### 4.2.3 Canonical involution of the Hessian of a smooth cubic

Let \( \pi \) be a plane in \( C_5 \). A conic belongs then to \( \pi \) if three linear conditions on its coefficients hold. Among such conditions, there are some remarkable ones, such as being forced to go through a point, a condition which is obviously very particular. A remarkable condition of a more general nature is that the conic admits two conjugate

---

9 *Note of translators:* this formula is found in [96], as follows:

p. 138, (18): Weierstrass formula \( y^2 = 4x^3 - g_2 x - g_3 \);

p. 138, (15): definition of the discriminant \( \Delta = g_2^2 - 27g_3^3 \);

p. 145, (22): expression for the j-invariant, \( j = 1728g_3^3/\Delta \).

Any homography—rational function of this \( j \) can be used as an invariant, such as \( \Delta/(g_3)^2 \) or \( \Delta/(g_2)^3 \).

As we noted earlier, [54, Proposition 2] identifies \( j_{\phi} = \frac{\lambda^3(\lambda^3 - g_3^3)}{27(\lambda + 1)^3(\lambda + \omega)^3(\lambda + \omega^2)^2} \). The mapping \( \lambda \rightarrow j_{\phi} \) for smooth cubics \( \phi \) is \( 12:1 \), except for the homographic cubic \( \phi_0 = X^3 + Y^3 + Z^3 \), where the mapping is \( 4:1 \) [54, Proposition 3]. We corrected some of the notation to avoid a double use of \( \beta \) in the original.

10 The Hessian cubic \( \Gamma(V) \), \( V = J_{\phi} \) has \( \lambda' = \frac{-4\lambda^3}{3\lambda^2} \) in terms of \( \lambda \) for \( \phi \), when written \( \Gamma(V) : X^3 + Y^3 + Z^3 + 3\lambda' XYZ \) (from [27, Theorem 4.7]).
points. We will therefore wonder if, being given a plane $\pi$ of conics, there are enough pairs of conjugate points with respect to all the conics of $\pi$ to determine $\pi$.

In the case where $\Gamma_\pi$ is smooth, the answer is positive. It is given with other information by the following proposition whose proof is easy.

**Proposition 4.5** Let $\pi$ be a plane in $C_5$ such that $\Gamma_\pi$ is smooth. Let $\phi$ be the cubic such that $\pi = J_\phi$ and let $H$ be the Hessian of $\phi$.

(a) For a point of $P$ to admit a common conjugate with respect to all the conics of $\pi$, it is necessary and sufficient that it belongs to $H$. Its common conjugate is unique, distinct from it, and belongs also to $H$. There is then a canonical and regular involution on $H$ without fixed point, $i : H \to H$ such that

$$\pi = \{ \gamma : \gamma \in C_5 \text{ and } \forall x \in H, \ x \text{ and } i(x) \text{ are conjugate with respect to } \gamma \}.$$  

(b) Conversely, given a smooth cubic $H$, endowed with a regular involution, without fixed point $i$, the set

$$\{ \gamma : \gamma \in C_5 \text{ and } \forall x \in H, \ x \text{ and } i(x) \text{ are conjugate with respect to } \gamma \},$$

is a plane $\pi$ of $C_5$ such that $\Gamma_\pi$ is smooth. If $\phi$ is the cubic such that $\pi = J_\phi$, $H$ is the Hessian of $\phi$ and $i$ is obtained on $H$ by the process described in the direct proposition in (a).

(c) The classification of smooth cubics endowed with a regular involution without a fixed point is therefore equivalent to the classification of planes $\pi$ of $C_5$ such that $\Gamma_\pi$ is smooth.

(d) There are exactly three involutions without fixed points on a smooth cubic which, as soon as we have fixed an algebraic group structure on the cubic (i.e. an origin), can be considered as the translations by the three points of order two of this group.

A smooth cubic is the Hessian of exactly three smooth cubics not isomorphic to $(X^3 + Y^3 + Z^3)$.

The three involutions without a fixed point of a smooth cubic and their three cubics of which the smooth cubic is the Hessian correspond one-to-one by the process described in (a).

We have an equivalent definition of the cubic $H$ thanks to the following proposition, whose proof is left to the reader:

**Proposition 4.6** Under the same hypotheses as the previous proposition, $H$ is the set of all the singular points of the singular conics of $\pi$. Given a singular point $A$ of a singular conic of $\pi$, its polar with respect to $\phi$ is a singular conic of $\pi$, whose unique singular point $B$ is the image of $A$ by the involution $i$ of $H$. The map from $\Gamma_\pi$ to $H$ which sends the singular conic $\gamma$ to its unique singular point is an isomorphism from $\Gamma_\pi$ to $H$.

See also the “Note on the Hessian form of a smooth cubic,” Sect. 9.
4.3 The planes $\pi$ of conics having a singular cubic $\Gamma_\pi$

We will henceforth not give the full demonstration that allows us to arrive at the presentation of each case of this exhaustion. We will indicate only how to choose a system of conics in the plane $\pi$ that is distinguished by its disposition relative to $\Gamma_\pi$, and the conclusions that follow relative to the different pencils of conics to consider. The verifications will be easy. We take again the order of examination adopted in the general Introduction to Sects. 1 and 2. Case 1 (smooth) was treated in Sect. 4.2, so we begin here with Case 2.

Recall the conclusion of this section, which was announced as Theorem 4.1 (B) in Sect. 4.1, the general introduction to Sect. 4: the Pgl(3) class of a plane $\pi$ for which $\Gamma_\pi$ is not a smooth cubic is completely determined by the linear isomorphism class of the triplet of projective algebraic schemes $(\delta_\pi \subset \Gamma_\pi \subset \pi)$. On the figures representing the plane $\pi$, the points associated to double lines are noted: $\circ$. We follow the classification that was introduced in “General classification of cases for $\Gamma$” in the introduction 4.1 to Sect. 4. In I. we consider $\pi \not\subset S_4$ (Case 2–9), and in II. $\pi \subset S_4$ (Case 10a, 10b).

I. $\pi \not\subset S_4$.

Case 2. ($\Gamma$ has singularity a point $C$ having two distinct tangents) Two cases occur:

(a) $C$ has two distinct irreducible components ($\propto$).

Choose then in $\pi$ points $A$, $B$, $C$, $O$ satisfying the following conditions: $C$ is the double point, $A$ is a point of inflection, $AB$ is the tangent at $B$ to $\Gamma$, $O \in BC$ and $AO$ is the tangent at $A$ to $\Gamma$.

One verifies that $BC$, $AB$, $AC$ are of type (b) and $OA$ is of type (d). From this we have a relatively unique disposition of $A$, $B$, $C$, $O$ represented by the figure below:

---

\[ \text{Springer} \]
From this, we have in a convenient system of homogeneous coordinates, the following representation:

\[ A = (XY), \quad B = Z(Y - Z), \quad C = (X^2 - Z^2), \quad O = (YZ - X^2). \]

(b) \( C \) is a double line \( \bigcirc \).

One chooses points \( A, B, C, O \) in the plane \( \pi \) satisfying the same conditions as in (a). \( BC \) and \( AC \) are of type (c), \( AB \) is of type (b), \( OA \) is of type (d). From this we have the following disposition:

\[ \begin{array}{c}
A \\
\bigcirc \\
C
\end{array} \]

and a system of coordinates where \( \pi \) is represented by the following equations:

\[ A = (X(Y + Z)), \quad B = (X^2 - Z^2), \quad C = (Y^2), \quad O = (X^2 + Y^2 - Z^2). \]

Case 3. (\( \Gamma \) is singular at a cusp):

(a) Suppose that the cusp corresponds to a conic having two distinct irreducible components. One verifies the impossibility of this eventuality.

(b) Suppose that the cusp corresponds to a double line \( \bigcirc \).

One chooses in the plane \( \Pi \) points \( A, O, C \) such that: \( C \) is the cusp, \( A \) is an inflection point, \( OC \) is the tangent to \( \Gamma \) at \( C \), \( OA \) is the tangent to \( \Gamma \) at \( A \). One then finds that \( AC \) is of type (c), \( OA \) of type (d), \( OC \) of type (e). From this we have the disposition in the figure below,
and the following equations after a suitable change of coordinates:

\[ A = (XY), \quad C = (Z^2), \quad O = (YZ - X^2). \]

Case 4. (\(\Gamma\) decomposes into a conic union a secant line) Suppose the intersection of these two irreducible components of \(\Gamma\) are two distinct points \(C\) and \(D\). Three cases can be envisioned:

(a) Neither \(C\) nor \(D\) is a double line. Choose \(O\) the intersection point of the tangents at \(C\) and \(D\) to the conic that is one of the irreducible components of \(\Gamma\). \(OC\) and \(OD\) are of type (d). \(CD\) is of type (f).

(b) Only one of the two conics \(C, D\) is a double line. This case never occurs.

(c) \(C\) and \(D\) are each a double line. Choose \(O\) as before: \(OC\) and \(OD\) are of type (e). \(CD\) is of type (h). From these we have the disposition below:
and the following equations in a suitably chosen coordinate system:

\[ C = (X^2), \quad D = (Y^2), \quad O = (XY - Z^2). \]

Case 5. (\( \Gamma \) decomposes into a conic union a tangent line) Let \( \gamma \) be the conic and \( \delta \) the line. Let \( U = \delta \cap \gamma \) be the point of tangency. Two cases may be envisioned:

(a) \( U \) is not a double line. This case does not occur.
(b) \( U \) is a double line. Let \( A \) and \( B \) be two elements of \( \pi \), \( A \) on \( \delta \) and \( B \) on \( \gamma \), both distinct from \( U \), such that \( AB \) is the tangent at \( B \) to \( \gamma \). \( AB \) is of type (b), \( AU \) of type (g), \( UB \) of type (c). From this we have the following disposition:

and the equations in a convenient coordinate system: \( A = (ZY), \quad B = (X^2 + Z^2), \quad U = (Y^2) \). We may change coordinates to have the presentation \( \langle X^2, XY, (Y + Z)Z \rangle \).

Note: The plane \( \pi \) in this case can be defined in the following way. There are two points \( A \) and \( B \) of \( P \) and a line \( \delta \) of \( P \) that contains neither \( A \) nor \( B \) such that the conics of \( \pi \) are exactly the conics passing through \( A, B \) and such that the polar of \( AB \) with respect to each point belongs to \( \delta \).

Case 6. (\( \Gamma \) decomposes into three non-concurrent lines) We should consider a priori the following four cases:
(a) None of $A$, $B$, $C$ are double lines. One obtains the following disposition and the possible representation:

\[ A = (XY), \quad B = (YZ), \quad C = (ZX), \]

Note: The plane $\pi$ is constituted by the conics passing through the three given points, that are not collinear.

(b) One only of the three points $A$, $B$, $C$ is a double line. This case does not occur.

(c) Two of the three points $A$, $B$, $C$ are double lines. This case does not occur.

(d) $A$, $B$ and $C$ are each double lines. One has the following disposition and the possible representation:

\[ A = (X^2), \quad B = (Y^2), \quad C = (Z^2), \]

Note: The plane $\pi$ is constituted in this case by the conics admitting a given autopolar triangle.

(7) $\Gamma$ decomposes into three concurrent lines. This case never occurs, with or without double line.

(8) ($\Gamma$ decomposes into a double line union a single line): Three cases present themselves:

(a) $A = (XZ), \quad C = (XY), \quad B = (Z^2)$.

Note: Here it is a question of conics restricted to pass through two given points, and being tangent at one of the two points to a given double line that does not pass through the other point.
Nets of conics and associated Artinian...

(b) \[ A = (Y^2), \quad B = (Z^2), \quad C = (XY). \]

**Note:** It is a question of conics satisfying the following conditions: There are two given points \( m \) and \( m' \) of \( P \) that are conjugate with respect to each. There is a point \( n \in P, n \notin [m, m'] \), the line determined by \( m \) and \( m' \), and a line \( \delta \) of \( P \), with \( m, n \in \delta \) such that each conic passes through \( n \) and is tangent at \( n \) to \( \delta \).

(c) \[ U = (X^2), \quad V = (Y^2), \quad A = (X^2 - Y^2), \quad C = ((X + Y)(Z)). \]

**Note:** There are \( m, m' \in P, n \in P - [m, m'] \) and \( \delta \) a line of \( P \) with \( n \in \delta \) but \( m, m' \notin \delta \) such that the conics of \( \pi \) are those that admit the pair \( m \) and \( m' \) as conjugates, that also pass through \( n \), and that are tangent at \( n \) to \( \delta \).

**Note:** One can also put the vector space in the form \( \langle ZX, ZY, X^2 + Z^2 \rangle \).

(9) \( \Gamma \) consists of a triple line: \[ O = (YZ - X^2), \quad A = (Y^2), \quad B = (XY). \]

\( OB \) is of type (d), \( OA \) of type (e), \( AB \) of type (g)).

**Note:** The plane \( \pi \) is constituted by the osculating conics at a given point to a given conic.

II. \( \pi \subset S_4 \). Two cases present themselves:

(10a) \( \pi \cap D_2 \) is a parabola.

\[ A = (X^2), \quad B = (Y^2), \quad C = ((X + Y)^2). \]

(10b) \( \pi \cap D_2 \) is reduced to a point.

\[ A = (X^2), \quad B = (XY), \quad C = (XZ). \]

**Conclusion:** Classification of the planes \( \pi \) of \( C_5 \).

The preceding study constitutes not only an exhaustion of all the cases, but also a classification. In effect, there are no isomorphisms between any two of the cases described: for, if there were, the dispositions of \( \pi \cap D_2 \subset \pi \cap S_4 \) would be algebraically isomorphic, which is never the case. The classification of planes \( \pi \) up to \( \text{Pgl}(3) \) action is thus well described by Table 1.
5 Algebraic method for the classification of planes of conics

We prove with an algebraic method the following proposition.

Proposition 5.1 Every three dimensional vector space $V$ either has an associated smooth cubic $\Gamma(V)^{11}$ or belongs to one of the 14 special orbits in Table 1.

There are two stages in this exposition: in Sect. 5.1, we notice with the help of intersection theory on the Grassmannian $G = \text{Grass}(2, R_2)$, that every vector space $V$ of dimension 3 of $R$ contains at least one 2-dimensional subspace that belongs to the closure of the orbit under $\text{Pgl}(3)$ of the vector space $W = \langle XY, X^2 + YZ \rangle$. We know, from Sect. 3, that this closure is the union of the orbits of $W$, $\langle Y^2, X^2 + YZ \rangle$, $\langle XY, XZ \rangle$, $\langle XY, X^2 \rangle$ and of $\langle X^2, Y^2 \rangle$ (see Tables 2 and 3). Then, in Sect. 5.2, we show the possibilities for a third generator of $V$ up to an action of an element of $\text{Pgl}(3)$. We then check the list of special orbits in Table 1. We find that when $\Gamma(V)$ is smooth, $V$ is isomorphic to a vector space $W_3$, $\lambda \neq 0, 1$ of a one-parameter family. However, we need the considerations of Sect. 4.2.2 to make the classification in this case, which then completes the general classification.

5.1 The intersections on Grass$(2, R_2)$

The Grassmannian variety $G = \text{Grass}(2, R_2)$ which parametrizes the 2-dimensional vector subspaces of $R_2$ is of dimension 8. If $V$ is a vector subspace of dimension 3 of $R_2$, there is a plane $P(V)$ in $G$ which parametrizes the 2-dimensional subspaces of $V$. The class of $P(V)$ in the cohomology ring $H^2(\text{Grass}(2, R_2))$ is independent of $V$, and is in fact a Schubert cycle. We would like to find, in a general space $V$, a subspace of dimension 2 of the most particular form to find a canonical expression of $V$; we will find a closed subvariety $\mathcal{Z}$ of $G$ such that $\mathcal{Z} \cap P(V)$ is always non-empty and in general finite. Since the cohomology class of $P(V)$ is independent of $V$, the intersection theory on $G$ affirms that if a closed subvariety $\mathcal{Z}$ of $G$ of dimension 6 intersects $P(V)$ in a non-empty finite set for a given $V$ (that is, the intersection between $\mathcal{Z}$ and $P(V)$ is proper and non-empty) then $\mathcal{Z}$ intersects effectively all the $P(V)$; furthermore $\mathcal{Z}$ will intersect a generic $P(V)$ in a non-empty finite set. In other words, if a 6-dimensional closed subvariety $\mathcal{Z}$ of $G$ contains a finite number of subspaces of dimension 2 of a particular 3-dimensional vector space, then $\mathcal{Z}$ contains at least a 2-dimensional vector subspace of every 3-dimensional vector space in $R_2$.

We prove now that if $W = \langle XY, X^2 + YZ \rangle$, the closure $\mathcal{Z}$ of the orbit of $W$ (which, according to the results in Table 3 of Sect. 3, is a sub-variety of $G$ of dimension 6), only contains as 2-dimensional subspaces of the 3-dimensional vector space $V_0 = \langle XY, X^2 + YZ, Z^2 \rangle$, the space $W$ itself$^{12}$ and the space $\langle X^2 + YZ, Z^2 \rangle$.

Indeed, there is no vector subspace of dimension two of $V_0$ whose elements all have in common a linear factor, and there is no subspace isomorphic to $\langle X^2, Y^2 \rangle$. The only elements of $\mathcal{Z}$ that can belong to $P(V_0)$ belong then either to the orbit of $W$ or to

---

11 If $V$ is a vector space of dimension 3 of $R_2$, let $\pi$ be the correspondent plane of $C_3$. We denote by $\Gamma(V)$ the cubic $\Gamma_{\pi}$.

12 Typo in original corrected.
Nets of conics and associated Artinian...

that of \((X^2 + YZ, Z^2)\). In any case, the elements correspond to lines in the plane \(\pi\) associated to \(V_0\) that meet \(\Gamma_\pi\) in a triple point. However, \(\Gamma_\pi\) is here an irreducible cubic with a cusp, and the plane \(\pi\) contains only two such lines, the tangent in the unique inflection point that corresponds to \(W\) and the tangent at the cusp that corresponds to \((X^2 + YZ, Z^2)\).

Algebraic arguments that are not reproduced here can replace these geometric arguments.

5.2 Algebraic classification

We classify now completely the \(\text{Pgl}(3)\) orbits of the subspace \(V\) of dimension 3 of \(R_2\), such that \(\Gamma(V)\) is singular, and also the \(\text{Pgl}(3)\) orbits of \(V\) such that \(\Gamma(V)\) is smooth: in the case #8b, where each \(\text{Pgl}(3)\) orbit has dimension 8, we are classifying these orbits by \(j\)-invariant up to a finite group of automorphisms. We give a representation for each orbit with an index that corresponds to its dimension and the symbol of the associated cubic \(\Gamma(V)\) as in Table 1.

**Case 1** \(\langle XY, X^2 + YZ \rangle \subset V\). Then

\[ V = \langle XY, X^2 + YZ, aX^2 + bY^2 + cZ^2 + dXZ \rangle. \]

**Case 1.A** \(c \neq 0\) or \(d \neq 0\). By replacing \(Z\) by \(Z - \alpha X\) with \(c\alpha^2 - d\alpha + a = 0\), we eliminate \(a\) (that is, we replace \(V\) by \(V^g\) with \(g \in \text{GL}(3)\), \(g: (X, Y, Z) \rightarrow (X, Y, Z - \alpha X)\)). From now on we will omit such explanations. If now \(b \neq 0\), a “change in scale” leads to a family:

\[ V_c = \langle XY, X^2 + YZ, Y^2 + cZ^2 + XZ \rangle \quad \text{if} \quad d \neq 0; \]

and to

\[ V' = \langle XY, X^2 + YZ, Y^2 + Z^2 \rangle \quad \text{if} \quad d = 0. \]

We study these two cases later.

If \(b = 0\), \(V = \langle XY, X^2 + YZ, Z^2 \rangle = \bigcirc\) or
\[ V = \langle XY, X^2 + YZ, XZ \rangle = \bigcirc \quad \text{or} \]
\[ V = \langle XY, X^2 + YZ, Z^2 + XZ \rangle. \]

In the latter case, the substitution \(Z = (Z' + \frac{1}{2}X)\) and a “change in scale” leads to the form \(V = \langle XY, X^2 + YZ, Z^2 + YZ \rangle\), and the substitution \(X = (Y' - X')/2, \ Y = (Z' - Y' - X')/2, \ Z = (Y' - X')/2\) provides the expression

\[ \langle XY, X^2 + YZ, Y^2 + XZ \rangle_8 = \bigcirc. \]

We see that this is a limited subset of the group of cases previously considered.

We now study the continuous family \(V_c = \langle XY, X^2 + YZ, Y^2 + cZ^2 + XZ \rangle\) and also \(V_0 = \langle XY, X^2 + YZ, X^2 + Y^2 \rangle\). In Sect. 4.2.2 Proposition 4.3ff. we studied
the family $\mathcal{V}_\lambda = \langle XY, X^2 + YZ, (Y - Z)(Y - \lambda Z) \rangle$ with $\lambda \in k$, and we have shown that if $\lambda \neq 0, 1$, $\mathcal{V}_\lambda$ has a non singular associated cubic $\Gamma(\mathcal{V}_\lambda)$. It is clear that $V_0 \simeq \mathcal{V}_0$ and that $\mathcal{V}_{-1} \simeq \mathcal{V}'$. If $c \neq 0$, consider the following substitution in $V_c : Z = rZ' - \frac{1}{rc}X'$, $X = r^{1/2}X'$, $Y = Y'$. Then

$$V_c = \left( XY, X^2 + YZ, Y^2 + \frac{1}{4c}YZ + cr^2Z^2 \right) = \mathcal{V}_\lambda \text{ if } \left( \frac{r}{4c} - 1 \right) = cr^2 = \lambda.$$

Note that:

(a) Unless $c = 0$ (case already considered), we can always choose $r$ such that $\frac{4}{rc} - 1 = cr^2$. Therefore, each $V_c$ belongs to the orbit of an element of the family $\mathcal{V}_\lambda$.

(b) In the map $\{V_c\} \rightarrow \{\mathcal{V}_\lambda\}$, $V_0$ is reached only by $V_0$ and $\mathcal{V}_1$ is reached only when $r = 8c$, and $cr^2 = 64c^3 = 1$ or $c = \sqrt[3]{1/64}$. In this case, $V_c \simeq \mathcal{V}_1 = \langle XY, X^2 + YZ, Y^2 + 2YZ + Z^2 \rangle = \langle XY, X^2 + YZ, (Y + Z)^2 \rangle = \bigcirc$. We conclude that every vector space of the one parameter family $V_c$ has a non singular associated cubic $\Gamma(V_c)$ except for:

$$\langle XY, X^2 + YZ, Y^2 + XZ \rangle_8 = \bigcirc$$
and

$$\langle XY, X^2 + YZ, (X + Z)^2 \rangle_8 = \bigcirc.$$

Case 1.B If $d$ and $c$ are both non zero in Case 1, then:

$$V = \langle XY, X^2 + YZ, X^2 + Y^2 \rangle_6 = \bigcirc \bigcirc$$

or

$$V = \langle XY, YZ, X^2 \rangle \simeq \langle XY, XZ, Z^2 \rangle_5 = \bigcirc \bigcirc$$

or

$$V = \langle XY, X^2 + YZ, Y^2 \rangle_4 = \bigcirc \bigcirc.$$

Case 2 $(Y^2, X^2 + YZ) \subset V$:
We have then $V = \langle Y^2, X^2 + YZ, aZ^2 + bXY + cYZ + dZX \rangle$. Considering the transformations: $X' = X + \lambda Y$, $Z' = Z - 2\lambda X + \mu Y$, composed with the "changes in scale" we get the following five cases:
\( \langle Y^2, X^2 +YZ, Z(Z+X) \rangle \) which is a form of

\( \langle Y^2, X^2 +YZ, XZ \rangle \) which represents

\( \langle Y^2, X^2 +YZ, Z^2 \rangle \) which represents

\( \langle Y^2, X^2 +YZ, YZ \rangle \) which is a form of

\( \langle Y^2, X^2 +YZ, XY \rangle \) which represents

Case 3 \( \langle XY, XZ \rangle \subset V \), so that \( V = \langle XY, XZ, aX^2 + bY^2 + cZ^2 + dYZ \rangle \).

Case 3.A \( d \neq 0 \). Replacing \( Y \) by \( Y + \alpha Z \) with \( ba^2 + d\alpha + c = 0 \), we eliminate \( c \). If \( d \neq O \), we replace \( Z \) by \( Z' - \left( \frac{b}{d} \right) Y \) to eliminate \( b \). After multiplication by a scalar, we get: \( V = \langle XY, XZ, X^2 + YZ \rangle \) already considered in 1.A, or \( V = \langle XY, XZ, YZ \rangle_6 = \triangle \), or \( V = \langle XY, XZ, X^2 \rangle_2 = \square \).

Case 3.B \( d = 0 \). The substitution \( Y \to Y + dZ \) or \( Z \to Z + dY \) will make \( d \) non-zero, unless \( b = c = d = O \), or \( V = \langle XY, XZ, X^2 \rangle \) which has just been considered or \( V = \langle XY, XZ, Z^2 \rangle \) considered in 1.B. Thus the discussion of 3.A applies unless \( V \cong \langle XY, XZ, Y^2 \rangle \) (already considered) or that: \( V \cong \langle XY, XZ, X^2 + Y^2 \rangle \) in which case the substitution used in 3.A returns the initial vector space. This last space \( V = \langle XY, YZ, X^2 + Y^2 \rangle \) is isomorphic to \( \langle XY, X^2 + YZ, X^2 + Y^2 \rangle \) of case 1.B.

Case 4 \( \langle XY, X^2 \rangle \subset V \), so that \( V = \langle XY, X^2, aY^2 + bZ^2 + cXZ + dYZ \rangle \).

Case 4.A \( b \neq 0 \). We eliminate \( a \) and \( c \) by the substitution

\[ Z \to Z' + \sqrt{a/b} Y + \left( \frac{c}{2} \right) X. \]

A change of scale gives:

\[ V \cong \langle XY, X^2, Z^2 + YZ \rangle_6 = \bigcirc \quad \text{or} \quad V = \langle XY, X^2, Z^2 \rangle_5 = \bigcirc \bigcirc \bigcirc . \]

Case 4.B \( b = 0 \). If \( d \neq 0 \), we eliminate \( c \) by \( Y \to Y' - \left( \frac{c}{d} \right) X \) and a change in scale gives: \( V = \langle X^2, XY, Y^2 + YZ \rangle = \langle X^2, XY, XZ \rangle \) or \( V = \langle X^2, XY, YZ \rangle \) of Case 1.B.

If \( d = b = 0 \), \( V = \langle XY, X^2, Y^2 + XZ \rangle = \langle XY, X^2 + YZ, Y^2 \rangle \) of Case 1.B or \( V = \langle XY, XZ, X^2 \rangle \) of Case 3.A or

\[ V = \langle XY, X^2, Y^2 \rangle_2 = \square \square . \]

Case 5 \( \langle X^2, Y^2 \rangle \subset V \). Then, \( V = \langle X^2, Y^2, aXY, bYZ + cXZ + dZ^2 \rangle \).

Case 5.A If \( d \neq 0 \), replace \( Z \) by \( Z' + \frac{a}{2} X + \frac{b}{2} Y \); after a change in scale

\[ V \cong \langle X^2, Y^2, XY + Z^2 \rangle_7 = \bigcirc \bigcirc \quad \text{or} \quad V = \langle X^2, Y^2, Z^2 \rangle_6 = \bigcirc \bigcirc . \]
Case 5.B If \( d = 0 \), \( V \simeq \langle X^2, Y^2, XZ \rangle \) (see 3.A) or \( \langle X^2, Y^2, Z^2 \rangle \) (see 3.B) or \( V \simeq \langle X^2, Y^2, Z(X + Y) \rangle \). This latter space, after substitution \( X \rightarrow X' + Y', Y \rightarrow X' - Y' \), becomes \( V = \langle X'^2 + Y'^2, X'Y', X'Z' \rangle \) which, as we showed in 3.B, is isomorphic to \( \langle XY, X^2 + YZ, X^2 + Y^2 \rangle \) of Case 1.B.

This completes the verification of the 14 special orbits of Table 1 by the algebraic method.

6 The closures of orbits: specializations

Introduction, definitions

Definition 6.1 Let \( U \) and \( V \) be two orbits of planes of \( C \) under the action of \( \text{Pgl}(3) \) operating in \( P \). We say that \( U \) specializes in \( V \) or that \( V \) is a specialization of \( U \) if \( V \subset \overline{U} \).

Remark 6.2 (a) It is the same to say that there exists a family \( F_T \subset \text{Grass}(3, R_2) \times T \) of elements \( F_t, t \in T \) of \( \text{Grass}(3, R_2) \) over a neighborhood \( T \) of 0 in the affine line, whose generic element \( F_t, t \in T \setminus 0 \) belongs to \( U \), and whose element \( F_0 \) corresponding to 0 belongs to \( V \).

(b) The relation “\( V \) is a specialization of \( U \)” is an order relation \( V < U \) on the set of orbits (order of genericity).

We will describe this order in Sects. 6.4–6.6. Beforehand, we study in Sect. 6.1 a duality relation on the set of orbits; in Sect. 6.2, the dimensions of orbits; and in Sect. 6.3, the length of the projective schemes naturally associated to elements of \( \text{Grass}(3, R_2) \).

6.1 Duality

We consider on \( R_2 \) the scalar product which makes the basis

\[
\left( \frac{1}{\sqrt{2}} x^2, \frac{1}{\sqrt{2}} y^2, \frac{1}{\sqrt{2}} z^2, xy, yz, xz \right)
\]

orthonormal. Let \( E \) and \( E' \) be subspaces of \( R_2 \). We can easily verify that if \( E \) and \( E' \) have the same orbit under the action of \( \text{Gl}_3(k) \), so have their respective orthogonals as well. The passage to the orthogonal therefore defines a linear involution (obviously regular) of the Plücker embedding of \( \text{Grass}(3, R_2) \), which passes to the quotient modulo the action of \( \text{Gl}_3(k) \) to give an involution of the set of orbits.

Definition 6.3 The involution thus obtained over the set of orbits of planes of plane projective conics is called duality. If \( X \) is such an orbit, we denote by \( X^\perp \) the dual orbit.

We have the following obvious propositions:

Proposition 6.4 If \( X' \) is a specialization of \( X \), then \( X'^\perp \) is a specialization of \( X^\perp \).
**Proposition 6.5** \[ \dim X^\perp = \dim X. \]

Recall from Table 1 that \( \bigcirc \mathcal{O} \langle j(\lambda) \rangle \) designates the orbit of the plane generated by the partial derivatives of \( X^3 + Y^3 + Z^3 + 3\lambda XYZ \).

**Proposition 6.6** We have the following list of couples of dual orbits (Table 4).

**Note of Translators 6.7** Emsalem and Iarrobino omitted to develop the notion of projective dual of the space of conics, in the dual plane \( P^\vee \) of \( P \). It is a method of treating intrinsically the equivalence between isomorphism class of a net and that of its orthogonal, while, on the other hand, the scalar product they have used does not respect the action of \( \text{Pgl}(3) \). Further, the duality between the two spaces of conics makes obvious the equivalence between

1. Classification of conics, and the classification of hyperplanes of conics;
2. Classification of pencils of conics, and the classification of webs of conics—linear projective subvarieties of dimension 3 of the space of conics.

It becomes clear from this consideration that the classification of all linear projective subvarieties of the space of conics can be deduced simply from the results of the paper.

### 6.2 Dimension of orbits

We will calculate the dimension of most orbits by finding the isotropy group of a 3-dimensional subspace of \( \mathbb{R}^2 \) under the action of \( \text{Pgl}(3) \) or \( \text{Sl}_3 \). \( \text{Sl}_3 \) is a finite covering space of \( \text{Pgl}(3) \) so that

\[
\dim \text{Pgl}(3)(V) = \dim \text{Sl}_3(V) = \dim \text{Sl}_3 - \dim \text{Stab}V = 8 - \dim \text{Stab}V,
\]

where \( \text{Stab}V \) is the isotropy group of \( V \). We start by listing these isotropy groups, orbit by orbit. However, considering the duality, we do not have to perform the calculation for an orbit whose dual already has its known dimension.

(1) **Dimension of** \( \bigcirc \mathcal{O} \langle j(\lambda) \rangle \). By Sect. 4.2, the orbit \( \bigcirc \mathcal{O} \langle j(\lambda) \rangle \) is in a regular one-to-one correspondence with the orbit of a smooth cubic which is not isomorphic to \( (X^3 + Y^3 + Z^3) \) in the space of cubics under the action of \( \text{Pgl}(3) \) operating on \( P \). This last dimension is 8. Then \( \bigcirc \mathcal{O} \langle j(\lambda) \rangle \) is also of dimension 8.

(2) **Dimension of** \( \bigcirc \infty \). We verify that a plane of this class is determined by the partial derivatives of a third degree homogeneous polynomial corresponding to an irreducible nodal cubic, this cubic being unique. The dimension sought is therefore equal to the dimension of the orbit of irreducible nodal cubics within the space of projective plane cubics under the action of \( \text{Pgl}(3) \) on \( P \). This dimension is 8. (See the remarks preceding Theorem 6.12 in Sect. 6.5.)

(2') **Dimension of** \( \bigcirc \infty \). Considering the duality, this dimension is also 8.
(3) Dimension of \( \langle X \rangle \) \( (= \langle X^2 + YZ, XY, XZ \rangle) \). \( X \) is the unique linear form (up to a multiplication by a non-zero scalar) such that \( \dim_k (XR_1 \cap V) = 2 \); we then have \( \langle Y, Z \rangle = V : X \), so that the spaces \( \langle X \rangle \) and \( \langle Y, Z \rangle \) are determined only by \( V \). Let \( G_V \) be the isotropy group of \( V \) in \( SL(3) \). If \( g \in G_V \), \( g(X) = aX \) and \( g(Y), g(Z) \in \langle Y, Z \rangle \). Then \( V \) contains \( g(X^2 + YZ) = a^2 X^2 + g(Y)g(Z) \). Necessarily, by the conditions on \( g(Y) \) and \( g(Z) \), \( a^2 X^2 + g(Y)g(Z) \) is proportional to \( X^2 + YZ \); then either \( g(Y) = aY \) or \( g(Z) = \frac{a^2}{a}Z \), or \( g(Y) = aZ \) and...
$g(Z) = \frac{\alpha^2}{a} Y$. We have $1 = \det g = \pm \alpha^3$ and $G_V \simeq (Z/3Z)^2 \times k$ is of dimension 1. Therefore, $\bigwedge$ is of dimension 7.

(3′) By duality, $\bigwedge$ is also of dimension 7.

(4) **Dimension of** $\langle XY, X^2 + YZ, Z^2 \rangle$. The space $\langle Z \rangle$ is completely determined by $V$. Let $g \in G_V$, the isotropy group, satisfy $g(Z) = aZ$, $g(X) = bX + cY + dZ$, $g(Y) = b'X + c'Y + d'Z$. Then,

$$g(XY) = g(X)g(Y) = (bX + cY + dZ)(b'X + c'Y + d'Z) \in V$$

so that

$$O \equiv (-bb' + c'd)YZ + (bd' + b'd)XZ + cc'Y^2 + (bd' + b'd)XZ \pmod{V}$$

and $g(X^2 + YZ) = (bX + cY + dZ)^2 + aZ(b'X + c'Y + d'Z) \in V$ so that

$$O \equiv (-b^2 + 2cd + ac')YZ + c^2Y^2 + (2bd + ab')XZ \pmod{V}$$

Then $c = 0$ and $1 = \det g = abc'$. Finally, $g(Z) = aZ$, $g(X) = bX$, $g(Y) = c'Y$, with $b^3 = 1$, and $a = b^2/c'$, $c' \in k^*$. $G_V \simeq Z_3 \times k^*$ is of dimension 1 and $\bigwedge$ is of dimension 7.

(5) $\bigtriangleup$ $(= (X^2, Y^2, Z^2))$.

The set of three spaces $\langle X \rangle$, $\langle Y \rangle$, $\langle Z \rangle$ is determined by $V$ and $V$ determines them in turn. If $g \in G_V$, it is the product of a substitution of variables and a transformation of the form $(X, Y, Z) \rightarrow (aX, bY, cZ)$ with $abc = 1$. $\dim G_V = 2$ and

$$\dim \bigtriangleup = 6.$$ (5′) By duality, $\dim \bigtriangleup = 6$.

(6) **Dimension of** $\bigtriangleup$ $(= \langle XY, X^2, Z^2 + YZ \rangle)$.

$\langle X \rangle$ and $\langle X, Y \rangle$ are uniquely determined by $\langle V \rangle$. If $g \in G_V$, $g(X) = aX$, $g(Y) = bX + cY$, $g(Z) = dX + eY + fZ$, and $1 = \det g = aef$ and $g(Z^2 + YZ) = (dX + eY + fZ)(d + b)X + (e + c)Y + fZ \equiv 0 \pmod{V}$, therefore

$$0 \equiv e(fc)Y^2 + f(2d + b)XZ + f(2e + c)YZ + f^2Z^2 \pmod{V},$$

hence, $(c = 0$ or $e + c = 0)$ and $b = -2d$, $2e + c = f$, $a = \frac{1}{cf}$. Finally, the isotropy group $G_V = Z/2Z \times k^* \times k$ is of dimension 2, and $\bigtriangleup$ is of dimension 6.

(6′) By duality, $\bigtriangleup$ is also of dimension 6.
Again, \( (Y), (X, Y) \) and \( (Z) \) are uniquely determined by \( V \). If \( g \in G_V, g(Y) = aY, g(Z) = bZ \) and \( g(X) = cX + dY, \) \( \det g = abc. \) Conversely, such an automorphism is in \( G_V. \) Therefore \( G_V \simeq (k^*)^2 \times k \) has dimension 3 and \( \overset{7'}{\square} \) has dimension 5.

By duality \( \overset{8}{\square} \) has dimension 5.

The vector space determines \( \langle Y \rangle \) and \( \langle X - Y \rangle. \) If \( g \in G_V, g(Y) = aY, g(X) = bX + cY, \) and \( g(Z) = dX + eY + fZ, 1 = \det g = abf \) and \( g(YZ - X^2) \in V \Rightarrow 0 \equiv aY(dX + eY + fZ) - (bX + cY)^2 (mod V) \Rightarrow afYZ - b^2X^2 \in V \Rightarrow af = b^2. \) Finally \( G_V \simeq Z_3 \times k^* \times k^3 \) is of dimension 4 and \( \overset{9}{\square} \) has dimension 4.

It is easy to verify that \( \overset{7}{\square} (= \langle X^2, XY, XZ \rangle) \) and \( \overset{8}{\square} (= \langle Y^2, XY, YZ - X^2 \rangle). \) both have dimension 2.

### 6.3 Scheme length as an invariant of special orbits

**Definition 6.8** Let \( u \in \text{Grass}(3, R_2). \) Let \( P, Q, R \in R_2 \) be such that \( u = \langle P, Q, R \rangle. \) We call “projective scheme associated with \( u \)” the scheme

\[
\text{Proj} k[X, Y, Z]/(P, Q, R).
\]

In general, this scheme is empty, and if \( (P, Q, R) \) does not belong to the orbit of \( \langle X^2, XY, XZ \rangle \), nor to the orbit of \( \langle X^2, XY, Y^2 \rangle, \) then it is finite.

**Definition 6.9** Let \( S \) be a finite scheme. We call *scheme length* of this scheme the integer

\[
\sum_{s \in S} \ell(O_{S,s}) \quad \text{where for } s \in S, \ O_{S,s} \text{ is the local ring of } S \text{ in } s.
\]

In what follows, we give the length of the projective schemes associated to different orbits of \( \text{Grass}(3, R_2), \) an integer in the interval \([0, 3]\). Indeed, two elements of \( \text{Grass}(3, R_2) \) that belong to the same orbit have isomorphic associated schemes, so have the same scheme length. The calculation of this length is easily done by carrying out a generic dehomogenization to obtain \( \text{Proj}_k[X, Y, Z]/(P, Q, R) \) in the form of the spectrum of an Artinian algebra. The length in question is the height \( \ell \) of the infinite string of the Hilbert function \( H(k[X, Y, Z]/(P, Q, R)) = (1, 3, 3, a, \ell). \) When \( (P, Q, R) \) is a complete intersection this is 0.

Table 5 shows this scheme length \( \ell \) immediately after the symbol associated with each orbit. It also summarizes the two previous discussions: each horizontal line of
Table 5. Length of the associated projective scheme to a net of conics

| j(λ) | 1 | 0 | j(−2λ) | 0 | 0 | 8 |
|------|---|---|---------|---|---|---|
|      | 2 | 1 |         | 0 | 0 | 7 |
|      | 3 | 2 |         | 0 | 0 | 6 |
|      | 3 | 2 |         | 0 | 0 | 5 |
|      | ∞ | 3 |         | 0 | 0 | 4 |

the table corresponds to a dimension of an orbit, that is noted at its right end. The axial symmetry corresponds to duality.\(^{13}\)

**Notation** If \( U \) is the class of an element in Grass(3, \( R_2 \)), in the sequel we let \( ℓ(U) \) be the length of the projective schemes associated to its various elements.

### 6.4 Impossibility of some specializations of nets of conics

We call from now on \( \mathcal{C} \) the set of classes of projective planes of conics modulo the natural action of \( \text{GL}(3) \).

**Definition 6.10** Suppose \( A, B ∈ \mathcal{C} \). We say that \( (A, B) \) is an **elementary specialization** if \( B \) is a specialization of \( A \), (if \( A ≠ B \)) and if, whenever \( C \) is a specialization of \( A \) and specializes to \( B \), then \( C \) is equal to \( A \) or \( B \). It will turn out that all elementary specializations of orbits in the nets of conics will be from one of dimension \( c \) to another of dimension \( c − 1 \).

We have the following proposition, whose proof is trivial.\(^{14}\)

\(^{13}\) **Note of translators**: the original table did not have the dimension two entries, Emsalem and Iarrobino perhaps considering them as too special.

\(^{14}\) **Note of translators**: for (b) above, the scheme length \( ℓ(A) \) is \( \dim_k A', A' = A/⟨l⟩ \) where \( l \) is a general enough linear form, and this dimension is upper semicontinuous. The **determinantal** nets of conics, namely,
Proposition 6.11  Let \((A, B)\) be a specialization with \(A \neq B\). Then

(a) \(\dim B < \dim A\).
(b) \(\ell(B) \geq \ell(A)\).
(c) If a representative of \(A\) has a non-empty intersection with \(D_2\), all representatives of \(B\) have a non-empty intersection with \(D_2\).
(d) \((A^\perp, B^\perp)\) is a specialization with \(A^\perp \neq B^\perp\).

Taking into account (a) of the proposition, the description of the order of genericity reduces to the establishing of an exhaustive list of elementary specializations.

In order to establish this list, the proposition permits us to eliminate the possibility of the existence of elementary specializations \((A, B)\) that do not figure in Table 1, with the exception of the following potential families of specializations:

\[
\begin{align*}
\bigcirc \langle \lambda \to \bigcirc \\
\bigcirc \langle \lambda \to \bigtriangleup \\
\bigcirc \langle \lambda \to \bigtriangleup \\
\bigcirc \langle \lambda \to \bigcirc
\end{align*}
\]

which are duals in pairs, and whose elimination is the goal of the following paragraph.

6.5 Specialization of plane cubics and related nets of conics

6.5.1 Specializations of Jacobian planes of a family of cubics \(f\) having constant \(j\)-invariant

In what follows, we will consider the specializations of an orbit \(\bigcirc \langle \lambda\), \(\lambda\) being fixed, which comes down to fixing \(j(\lambda)\).

(1) The potential specializations \(\bigcirc \langle \lambda \to \bigcirc\) and \(\bigcirc \langle \lambda \to \bigtriangleup\) do not exist. It suffices, in considering the duality, to show that the first specialization does not exist. In considering the map which associates to \(\pi\) the cubic \(\Gamma_\pi\), and which is rational, this will imply that, in the space of cubics, the cubic that decomposes into a conic with a secant line will belong to the closure of the orbit of \(\Gamma_\pi\), where \(\pi_\lambda\) is the plane associated to \((X^2 + \lambda Y Z, Y^2 + \lambda X Z, Z^2 + \lambda X Y)\). However, as we have seen in Sect. 4.2.2, \(\Gamma_\pi\) is smooth, which excludes the preceding possibility, according to the Proposition 6.14 below.

In this way, in the space of projective plane cubics, each cubic \(\bigcirc\) would belong to the closure (in the sense of Zariski topology) of at least one orbit of a smooth cubic under the natural action of \(\text{Pgl}(3)\). Thus, the closure of the orbits of these smooth cubics as well as the closure of the orbit of irreducible cubics having a double point with distinct tangents are hypersurfaces (thus, of dimension 8), in the space of cubics.

those arising as the \(2 \times 2\) minors of a \(2 \times 3\) matrix of linear forms, are the four nets of maximum finite scheme length 3. These are seen in the literature (see Appendix B and [47]).

\(\copyright\) Springer
More precisely [Gur, Exercise 14–31, Section 28],\textsuperscript{15} there are two fundamental relative homogeneous invariants $S_4$ and $S_6$ of degrees respectively 4 and 6 in the coefficients of a general cubic, such that a fundamental rational invariant (rationally equivalent to the classic “$j$”) can be written $\frac{S_4^3}{S_6^2}$. Here the closures of the orbits of smooth cubics with a fixed $j$-invariant, and also the closure of the cubic with a double point having two distinct tangents constitute in each case a pencil of hypersurfaces of degree 12, whose general equation is written $\beta S_4^3 + \alpha S_6^2 = 0$, $\beta, \alpha \in \mathbb{k}$.

Since in a pencil of hypersurfaces, if a point belongs to two different hypersurfaces, then it belongs to all the hypersurfaces of the pencil, we thus have the following result.

**Theorem 6.12** For a cubic whose orbit is of dimension strictly smaller than 8 one of the two following occurs concerning its presence among the closures of the dimension eight orbits:

(i) either it belongs to the closure of each of the $j$-constant orbits of the smooth cubics and also to the closure of the orbit of singular cubics having a double point with distinct tangents, or

(ii) it belongs to exactly one of them.

Also, the existence of the specialization of cubics $\infty \rightarrow \infty$, which is shown below, proves

**Proposition 6.13** Every cubic that is comprised of a conic and a secant line belongs to the closure of the orbit of irreducible cubics having a singular point with two distinct tangents.

Thus, we will show that $\infty$ does not belong to the closure of the cubic $(X^3+Y^3+Z^3)$. The conjunction of the two propositions implies also

**Proposition 6.14** Each cubic comprised of a conic and a secant line does not belong to the closure of the orbit of any smooth cubic.

It remains to show

**Lemma 6.15** A cubic comprised of a conic and a secant line does not belong to the closure of the orbit of $(X^3+Y^3+Z^3)$.

To prove the lemma, we will show

**Proposition 6.16** Let $F$ be an element of $R_3$. For the curve $(F)$ $(F = 0)$ to belong to the closure of the orbit of $X^3+Y^3+Z^3$, it is necessary and sufficient that there exists $G \in R_3$ such that $(G) \neq (F)$ and we have

$$\langle G'_X, G'_Y, G'_Z \rangle = \langle F'_X, F'_Y, F'_Z \rangle.$$  

\textsuperscript{15} Exercise 15 of [Gur, Section 28], defines $S = S_4$, $T = S_6$ as invariants, and Exercise 29 is to show if $T^2 - 4S^3 \neq 0$ that each invariant of a canonical cubic form is a rational integral function of $S, T$. Note of translators: Salmon [91, Chapter V, Section V] has a detailed study of invariants of the cubic $f$: Sections 220, 221 give formulas for $S, T$ in terms of the coefficients of $f$. See also Aronhold invariants [81, Section 4.6], [37, Section 10.3] (which classifies cubics over arbitrary algebraically closed fields, and specifies $S, T$), and [38, Section 3.4].
Proof of Proposition 6.16 In effect, if \((F) = (X^3 + Y^3 + Z^3)\), there exists \(G = 2X^3 + Y^3 + Z^3 \neq F\) such that \(⟨G'_X, G'_Y, G'_Z⟩ = ⟨F'_X, F'_Y, F'_Z⟩\). In the reverse direction the Proposition 4.3 shows that if \((F)\) is smooth but not isomorphic to \((X^3 + Y^3 + Z^3)\), then each cubic \((G)\) such that \(⟨G'_X, G'_Y, G'_Z⟩ = ⟨F'_X, F'_Y, F'_Z⟩\) is equal to \((F)\). \(\square\)

One verifies a similar result if \((F)\) is an irreducible cubic having a singular point with two distinct tangents.

Finally, the set of \((F)\) such that there is \((G) \neq (F)\) for which \(⟨G'_X, G'_Y, G'_Z⟩ = ⟨F'_X, F'_Y, F'_Z⟩\) is a closed hypersurface of the space of cubics. In effect, that condition translates itself to the annihilation of a homogeneous polynomial (hypersurface) which is the determinant associated to a system of linear conditions that generalizes equation (1) considered in Sect. 4.2.2.

It results from the last property that this hypersurface is exactly the closure of the orbit of \((X^3 + Y^3 + Z^3)\).

In using the preceding Proposition 6.16 to show Lemma 6.15, it suffices to exhibit an element \(F ∈ R_3\), a cubic comprised of a conic and a secant line, and such that for all \(G ∈ R_3\) the equality \(⟨G'_X, G'_Y, G'_Z⟩ = ⟨F'_X, F'_Y, F'_Z⟩\) implies \(⟨G⟩ = ⟨F⟩\). This is the case for \(F = X(X^2 − YZ)\). \(\square\)

Remark 6.17 The ideal of polynomials in the coefficients of a general cubic that vanish on the orbit of a cubic comprised of a conic and a secant line (as \(F = X(X^2 − YZ)\) above), furnishes the example of an invariant ideal under \(PGl(3)\) that does not have a system of generators that are individually invariant (because they cannot be expressed as polynomials in \(S_4\) and \(S_6\)).

(2) A key specialization: The specialization of nets of conics \(\bigcirc_\lambda \rightarrow \bigcirc\) exists for any \(\lambda\). In effect, given \(\lambda\) there exists \(\mu\) such that for all \(b \neq 0\), the plane associated to the space \((ZX, YZ − X^2, (Y − bZ)(Y − \mu^2bZ))\) belongs to the orbit \(\bigcirc_\lambda\). (Compare the conclusions of Sects. 4.2.1 and of 4.2.2 and make for \(b \neq 0\) a change of variables transforming the expression given at the end of Sect. 4.2.1 to the expression given here.)

For \(b = 0\) the space \((ZX, YZ − X^2, (Y − bZ)(X − \mu^2bZ))\) belongs to the orbit \(\bigcirc\).

(3) The specializations of nets of conics \(\bigcirc_\lambda \rightarrow \bigcirc\) and \(\bigcirc_\lambda \rightarrow \bigcirc\) do not exist. In considering the duality, it suffices to show that the first specialization does not exist. For, if it existed, in the space of cubics \((XYZ)\) would belong to the closure of the smooth cubic \(\Gamma_{πλ}\). However we have

Proposition 6.18 In the space of cubics, \((XYZ)\) belongs to the closure of the orbit of irreducible cubics having a double point with two distinct tangents, and it does not belong to the closure of the orbit of any smooth cubic.

(a) Considering the family \(XYZ + μ(X^3 + Y^3)\) whose elements for \(μ \neq 0\) belong to the closure of the orbit of the irreducible cubic having a double point with distinct tangents and whose element corresponding to \(μ = 0\) is \((XYZ)\), one obtains the first part of the proposition.
(b) By Proposition 6.14 the second part of Proposition 6.18 results from (a) and the fact \((XYZ)\) does not belong to the closure of the orbits of \((X^3 + Y^3 + Z^3)\) (Lemma 6.15). In effect, up to multiplication by a scalar \(XYZ\) is the unique element of \(R_3\) whose partial derivatives generate the vector space \((XY, XZ, YZ)\).

### 6.5.2 Specializations of singular cubics \(\Gamma_\pi\)

**Note of Translators 6.19** The diagrams of Table 1 (except for the last, dimension-two orbits), are those for the cubics that are discriminants \(\Gamma_\pi\) of the nets \(\pi\) of conics: in each case this cubic \(\Gamma_\pi\) is the locus of conics in the net \(\pi\) that have a linear factor: the figure for the cubic is garnished by circles around points corresponding to a conic that is a double line. A specialization of nets \(A\) to \(B\) requires the specialization of the corresponding discriminant cubics; but a specialization of cubics that are discriminants would not necessarily come from a specialization of nets.\(^{16}\) We specify here in Table 6 the cubic discriminants of each of the nets of dimension at least two. We give an example of the calculation for \# 8a.\(^{17}\) We then provide a table of specializations of cubics, Table 7.

The discriminant cubic \(\Gamma_\pi\) in the plane \(\pi\) of the net. If the net is generated by \([f, g, h]\), we let

\[
G = Af + Bg + Ch.
\]

To get a singular curve, we should find the cubic curve \(G = \Gamma_\pi\) in the variables \(A, B, C\) such that \(G_X = G_Y = G_Z = 0\) has a nontrivial solution.

\[
\begin{align*}
G_X &= Af_X + Bg_X + Ch_X = 0, \\
G_Y &= Af_Y + Bg_Y + Ch_Y = 0, \\
G_Z &= Af_Z + Bg_Z + Ch_Z = 0.
\end{align*}
\]

**Example 6.20** (Discriminant \(\Gamma_\pi\) for \# 8a) Here \(\pi = (f, g, h) = (XY, X^2 + YZ, Y^2 + XZ)\).

\[
\begin{align*}
G &= AXY + B(X^2 + YZ) + C(Y^2 + XZ), \\
G_X &= AY + 2BX + CZ = 0, \\
G_Y &= AX + BZ + 2CY = 0, \\
G_Z &= BY + CX = 0.
\end{align*}
\]

\[
\det \begin{pmatrix} 2B & A & C \\ A & 2C & B \\ C & B & 0 \end{pmatrix} = C(AB - 2C^2) - B(2B^2 - AC) = 2ABC - 2(C^3 + B^3)
\]

---

\(^{16}\) For example \# 6a and \# 6d have the same discriminant cubic \(ABC\) before decoration with double points, but there is no specialization of the corresponding nets.

\(^{17}\) Recall the usual notation: \# 8a refers to the orbit of dimension 8, and at the left of Table 1.
Table 6  The equations of the discriminant cubics of Table 1

| π | cubic | \( \Gamma_\pi \) |
|---|---|---|
| 8a | \( B^3 + C^3 - ABC \) | |
| 8b | \( \lambda^2(A^3 + B^3 + C^3) - (\lambda^3 + 4)ABC \) | |
| 8c | \( B^3 + C^3 - ABC \) | |
| 7a | \( A(A^2 - BC) \) | |
| 7b | \( A^3 + B^2C \) | |
| 7c | \( A(A^2 - BC) \) | |
| 6a | \( ABC \) | |
| 6b | \( B(AB + C^2) \) | |
| 6c | \( BC^2 \) | |
| 6d | \( ABC \) | |
| 5a | \( A^2(A + C) \) | |
| 5b | \( BC^2 \) | |
| 4 | \( A^4 \) | |

which is the equation of a node. In Table 6 we give the equations of the discriminant locus, up to a non-zero constant multiple; the diagram is that of the cubic \( \Gamma_\pi \) in Table 1 with double points indicated.

Remark 6.21 Considerations we have not set out in this section and some easy calculations of dimensions of orbits permit us to present the following Table 7 of elementary specializations in the space of cubics. We give some of the specializations\(^{18}\).

(a) Smooth cubic \( j \) fixed, to the cusp. We use a Weierstrass presentation

\[
Y^2 Z = X(X - Z)(X - aZ), \tag{5}
\]

and introduce new variables \( H \) and \( K \) in such a way that: \( Z = b^2 H, \ Y = b^{-1} K \). Then equation (5) becomes \( K^2 H = X(X - b^2 H)(X - ab^2 H) \). When \( b = 0 \), we get \( K^2 H = X^3 \) which is an equation of the cubic with cusp.

(b) Singular cubic with a node \( \bigcirc \) to the cusp \( \bigtriangleup \). We have \( X^3 + Y(Y + aX)Z \) as an equation for the node; letting \( a \to 0 \), one gets \( X^3 + Y^2 Z \), which is an equation of a cusp.

(c) Singular cubic with node to an union of a smooth conic and a transverse line, then to a smooth conic and tangent line. \( ZXY + aX^3 + X^2Y + Y^3 \) gives for \( a = 0 \), \( Y(XX + X^2 + Y^2) \). Then \( X^3 + Y^2 - Z^2(Y - bZ) \) gives for \( b = 1 \), \((X^3 + Y^2 - Z^2)(Y - Z)\).

\(^{18}\) Added in translation

\( \copyright \) Springer
(d) Cusp to a union of a smooth conic and a tangent line, then to a triangle. \( aX^3 + YX^2 + Y^2Z \) gives when \( a = 0, Y(X^2 + YZ) \). Then \( Y(X^2 + aY^2 - Z^2) \) gives for \( a = 0, Y(X^2 - Z^2) \).

(e) The union of a smooth conic and a tangent line to three concurrent lines: \((X^2 - Y^2 - aYZ)Y\) when \( a = 0 \) gives \((X^2 - Y^2)Y\).

(f) The triangle to the union of three concurrent lines to the union of a double line and a transverse line, to a triple line are clear.

The arguments above show there is no \( j \) constant specialization from a family of smooth conics to the cubic \( XYZ \): this and the \( j \)-constant specialization from smooth cubics to a cusp shows the following.

**Lemma 6.22** An algebraic family of cubics isomorphic to the cusp \( X^3 + Y^2Z \), or to the smooth Fermat cubic \( X^3 + Y^3 + Z^3 \) cannot have as limit (specialization) the cubic \( XYZ \).

**Proof** (Second proof)\(^1\) In the Macaulay duality (see [48]), using the contraction pairing from \( S = k[x, y, z] \) to \( R = k[X, Y, Z] \) a graded Gorenstein algebra quotient \( A = S/IF, I_F = \text{Ann} \ F \) of Hilbert function \( H(A) = (1, 3, 3, 1) \) is determined by a degree-three form \( F \in R \). For the Fermat cubic and the cusp one has, respectively \( I_F = (xy, xy, yz, x^3 - y^3, x^3 - z^3), \) or \( I_F = (xy, xz, z^2, y^3, x^3 - y^2z) \), each with five generators. The annihilating ideal for \( XYZ \) is the complete intersection \((x^2, y^2, z^2)\). But the number of generators of an ideal in a finite length \( n \) (here \( n = 8 \)) Artinian quotient of \( S \) is semicontinuous: it is not possible to specialize from an ideal with five generators to a complete intersection. This completes the proof. \( \square \)

### 6.5.3 Polar nets \( J_\Phi \) of singular cubics \( \Phi \)

**Note of Translators 6.23** (See [28]\(^2\)) We give the list of those nets of conics that are polars—Jacobian planes—of cubics. We use the notation for classes of nets in Table 1 where the number is the dimension of the orbit, and the letter is position from the left of the table. The first class is the polars studied above of the family of smooth cubics #8b not isomorphic to \( X^3 + Y^3 + Z^3 \).

There are six discrete classes, the polar nets \( \pi = J_\Phi \) of

1. A cubic having a double point with two distinct tangents, #8a, \( X^3 + Y^3 + Z^3 + 3XYZ \).
2. A cubic comprised of a smooth conic and a transversal line, #7a, \( X^3 + 3XYZ \).
3. A cubic decomposed into three non-concurrent lines, #6a, \( XYZ \).
4. The cubic for #6d, \( X^3 + Y^3 + Z^3 \).
5. The cubic for #5b, \( XY^2 + Z^3 \), a cusp.
6. A smooth conic union a tangent line #4, \( X^2Y + Y^2Z \).

Of the nine discrete classes of cubics (Table 7) two have polars that are pencils, and one a unique conic. See also Appendix B, and [28, p. 1568]. Note that a specialization of a cubic \( F_A \) to \( F_B \) here does imply a specialization \( A \) to \( B \) of the related polar nets.
Table 7 Specializations of planar cubics

| Dimension of the orbit | 8 |
|------------------------|---|
|                         | j |

Table 8 Polar nets

| Case | $F$ | Jacobian $(F_X, F_Y, F_Z)$ | det Hess($F$) | $\Gamma_{\pi}$ |
|------|-----|---------------------------|--------------|---------------|
| 8a   | $X^3 + Y^3 + 3XYZ$ | $(X^2 + YZ, Y^2 + XZ, XY)$ | $X^3 + Y^3 - XYZ$ | $B^3 + C^3 - ABC$ |
| 8b   | $X^3 + Y^3 + Z^3 + 3\lambda XYZ$ | $(X^2 + \lambda YZ, Y^2 + \lambda XZ, Z^2 + \lambda XY)$ | $X^3 + Y^3 + Z^3$ | $\lambda^2(A^3 + B^3 + C^3)$ |
| 7a   | $X^3 + 3XYZ$ | $(X^2 + YZ, XZ, YZ)$ | $3X^3 - XYZ$ | $A^3 - ABC$ |
| 6a   | $XYZ$ | $(YZ, ZX, XY)$ | $XYZ$ | $ABC$ |
| 6d   | $X^3 + Y^3 + Z^3$ | $(X^2, Y^2, Z^2)$ | $XYZ$ | $ABC$ |
| 5b   | $XY^2 + Z^3$ | $(Y^2, XY, Z^2)$ | $Y^2Z$ | $BC^2$ |
| 4    | $X^2Y + Y^2Z$ | $(XY, X^2 + 2YZ, Y^2)$ | $Y^3$ | $A^3$ |
In Table 8 we list $F$, the Jacobian or polar net. The Hessian of $F$ is related to but not the same as the $\Gamma_\pi$ given in the right column. See also A. Conca [28, p. 1568].

### 6.6 Existence of specializations of nets of conics

An elementary specialization will be one from an orbit of dimension $c$ to one of dimension $c - 1$. In this section, we will show

**Proposition 6.24** All the elementary specializations in Table 1 in fact do occur.

We restrict the demonstration here to specialization from orbits not in # 8b, as we have treated the $j$-constant specializations from # 8b in Sect. 6.5.1, Theorem 6.12. We show the Proposition by enumerating the remaining elementary specializations that we have not already considered. In each of the cases (2)–(5) below, the proof of the existence of the corresponding elementary specializations shown in Table 1 reduces to showing the few examples given.

1. Specializations $(A, B)$ where $\dim A = 8$, $\dim B = 7$.
   (a) $\cup \rightarrow \bigcirc$. One considers the # 8a family $(XY, ZY - \mu X^2, ZX - Y^2)$ whose element for $\mu \neq 0$ belongs to $\cup$ and for $\mu = 0$ belongs to $\bigcirc$.
   (b) By duality, one has $\cup \rightarrow \bigcirc$.
   (c) $\bigcirc \xrightarrow{\lambda} \cup$ has already been verified in Sect. 6.5 (2) after Remark 6.17.

2. Specializations $(A, B)$ with $\dim A = 7$, $\dim B = 6$.
   All the specializations envisioned exist: using the duality, it suffices to verify that the following specializations exist:
   
   $\bigcirc \rightarrow \bigtriangleup \quad (XY, ZY, ZX - \mu Y^2) \in \begin{cases} \bigcirc & \text{if } \mu \neq 0 \\ \bigtriangleup & \text{if } \mu = 0 \end{cases}$
   
   $\bigcirc \rightarrow \bigtriangleup \quad (X(X - Z), Y(Y - \mu X), YZ) \in \begin{cases} \bigcirc & \text{if } \mu \neq 0 \\ \bigtriangleup & \text{if } \mu = 0 \end{cases}$
   
   $\bigcirc \rightarrow \bigtriangleup \quad (XY, Z^2, (Y + X)Z - \mu X^2) \in \begin{cases} \bigcirc & \text{if } \mu \neq 0 \\ \bigtriangleup & \text{if } \mu = 0 \end{cases}$
   
   $\bigcirc \rightarrow \bigtriangleup \quad (X(X - Z), (Y - \mu X)^2, YZ) \in \begin{cases} \bigcirc & \text{if } \mu \neq 0 \\ \bigtriangleup & \text{if } \mu = 0 \end{cases}$

3. Specializations $(A, B)$ with $\dim A = 6$, $\dim B = 5$. All the specializations envisioned exist, and the demonstration reduces, by duality, to proving the existence

---

19 Added in translation.
20 Note of translators: we have added this section as a clarification.
of the following specializations:

\[
\begin{align*}
\triangle & \rightarrow \quad (X^2 - aXZ, XY, YZ) \in \begin{cases} \triangle & \text{if } a \neq 0 \\ \circ & \text{if } a = 0 \end{cases} \\
\circ & \rightarrow \quad (Y^2, (X - aZ)(aX - Z), YZ) \in \begin{cases} \circ & \text{if } a \neq 0 \\ \triangle & \text{if } a = 0 \end{cases} \\
\square & \rightarrow \quad (X^2, Y(X - aY), YZ) \in \begin{cases} \square & \text{if } a \neq 0 \\ \circ & \text{if } a = 0 \end{cases}
\end{align*}
\]

(4) Specializations \((A, B)\) with \(\dim A = 5, \dim B = 4\).

All the specializations envisioned exist: using the duality, it suffices to show that the following example of specialization exists:

\[
\begin{align*}
\square & \rightarrow \quad (YZ - X^2, (Y - aX)X, (Y - aX)Y) = \begin{cases} \square & \text{if } a \neq 0 \\ \circ & \text{if } a = 0 \end{cases}
\end{align*}
\]

(5) Specializations \((A, B)\) with \(\dim A = 4, \dim B = 2\). All the specializations envisioned exist: using the duality, it suffices to show that the following specialization exists:

\[
\begin{align*}
\square & \rightarrow \quad (Y^2, YZ + aX^2, XY) = \begin{cases} \square & \text{if } a \neq 0 \\ \circ & \text{if } a = 0 \end{cases}
\end{align*}
\]

Thus, by the examples above, and our classification of the nets in Table 1 (Theorem 4.1, Proposition 5.1) we have the list of all the elementary specializations that occur. All the other specializations that may be envisioned for Table 1 are the products of the elementary specializations, whose existence has been shown.

## 7 Deformations

**Introduction** The graded Artinian local algebras of Hilbert function \(H(A) = (1, 3, 3)\) have been implicitly classified in Theorem 2 (see Introduction 1). We here show

**Theorem 7.1** (1) (Sect. 7.1) Artinian local algebras \(A\) of Hilbert function \(H(A) = (1, 3, 3)\) are smoothable. That is to say, \(A\) can be deformed to a smooth algebra of dimension 7 (isomorphic to the product algebra \(k^7\)).
(2) (Sect. 7.2, Proposition 7.3) Each Artinian algebra of Hilbert function $H(A) = (1, r, 2)$ is smoothable.\(^{21}\)

The second length 7 continuous family of algebras is those of Hilbert function $H(A) = (1, 4, 2)$. To show smoothability, we study the more general case of algebras of type $(1, r, 2)$, for which the proof is not more difficult. To show Theorem 7.1 (2) we consider a general family with $r - 3$ parameters of non-isomorphic algebras of type $(1, r, 2)$; we give generators and the relations between generators; and we show that for $r \geq 3$, such an algebra deforms into an algebra $A' = k \times A''$ where $A''$ is an algebra of type $(1, r - 1, 2)$. From which it follows, by induction on $r$, that $A$ may be deformed to the product algebra of $r + 3$ copies of $k$.

### 7.1 Deformation of algebras of Hilbert function $(1, 3, 3)$ into a smooth algebra

An algebra of Hilbert function $(1, 3, 3)$ has general form $k[X, Y, Z]/(P, Q, R, m^3)$ where $P, Q, R$ are three linearly independent elements in $R_2$. The above and Table 1 proves that, generically, we can find a presentation of such an algebra in the form

$$k[X, Y, Z]/(Y^2 + \lambda XZ, Z^2 + \lambda XY, X^2 + \lambda YZ) + m^3$$

In addition, we have the equality

$$(Y^2 + \lambda XZ, Z^2 + \lambda XY, X^2 + \lambda YZ) + m^3 = (Y^2 + \lambda XZ, Z^2 + \lambda XY, X^2 + \lambda YZ, XYZ),$$

hence the presentation:

$$k[X, Y, Z]/(X^2 + \lambda YZ, Y^2 + \lambda XZ, Z^2 + \lambda XY, XYZ).$$

Consider then the family of algebras:

$$A_t = k[X, Y, Z]/(Y^2 + \lambda XZ, Z^2 + \lambda XY, X^2 + \lambda YZ + tX, XYZ + \lambda^2 t(1 + \lambda^3)^{-1}X^2).$$

For $t \neq 0$ it is the product of a local algebra of dimension 4 and three copies of $k$. Indeed, the zeros of the ideal

$$\left(Y^2 + \lambda XZ, Z^2 + \lambda XY, X^2 + \lambda YZ + tX, XYZ + \lambda^2 t(1 + \lambda^3)^{-1}X^2\right)$$

\(^{21}\) Our convention is that a deformation of an algebra $A(0)$ to $B$ will be a flat family $A \to T$ over a parameter variety $T$, such that the special fiber $A(t_0)$ over the point $t_0$ of $T$ is $A(0)$ and the general fiber $A(t), t \neq t_0$ is isomorphic to $B$ [MS].
are
\[
(0, 0, 0), \quad \left( -\frac{t}{1 + \lambda^3}, \frac{\lambda t}{1 + \lambda^3}, \frac{\lambda t}{1 + \lambda^3} \right),
\]
\[
\left( -\frac{t}{1 + \lambda^3}, \frac{\lambda j t}{1 + \lambda^3}, \frac{\lambda j^2 t}{1 + \lambda^3}, \frac{\lambda j t}{1 + \lambda^3} \right), \quad \left( -\frac{t}{1 + \lambda^3}, \frac{\lambda j^2 t}{1 + \lambda^3}, \frac{\lambda j t}{1 + \lambda^3} \right);
\]
where \( j \) is the cube root of unity; the localization of this algebra at \((0, 0), \) which is isomorphic to \( k[X, Y]/(X^2, Y^2), \) has length 4; and the other localizations have length 1.

We therefore have a flat deformation of \( A_0 \) into a family of algebras isomorphic to \( k[X, Y]/(X^2, Y^2) \times k^3. \) Since all Artinian algebras with two generators are smoothable (see \([\text{Fog}]\)), \( k[X, Y]/(X^2, Y^2) \) may be deformed into \( k^4, \) and \( A_0 \) is finally deformable into \( k^7. \)

**Note of Translators 7.2** That an algebra \( A_0 \) is smoothable, does not answer the question of whether it is alignable: has a deformation to a curvilinear algebra: one isomorphic to \( k[t]/(t^n). \) It is open whether these algebras of Hilbert function \( H = (1, 3, 3, 0) \) are alignable. Furthermore, there is at present no effective deformation theory giving obstructions to being alignable—aside of the dimension of the family. The alignable algebras of length \( n \) in \( r \) variables are a family of dimension \((r - 1)(n - 1), \) so for \( r = 3, n = 7 \) their dimension is 12, while the family of \( H = (1, 3, 3, 0) \) algebras are parametrized by the nets of conics, so have dimension 9: so here dimension is not an obstruction.\(^{22}\)

An algebra of length \( n \) is peelable if it can be smoothed in exactly \( n - 1 \) steps by splitting off a regular point. Since all codimension two algebras are alignable by \( [17] \) (characteristic zero, other authors have shown this for \( k \) algebraically closed in characteristic \( p \)) and alignable algebras are peelable, the above proof shows that all Artinian algebras of Hilbert function \((1, r, 2)\) are peelable over an algebraically closed field of characteristic zero, or characteristic \( p \) not 2.

**7.2 Deformation of algebras of Hilbert function \((1, r, 2)\) into a smooth algebra**

Such an algebra of Hilbert function \((1, r, 2)\) is a quotient of \( k[X_1, \ldots, X_r] \) determined by a subspace of \( R_2 \) having codimension 2. We will first study a generic family with \((r - 3)\) parameters of non-isomorphic vector spaces having dimension 2, where \( r \geq 2.\)\(^{23}\) By duality, we immediately deduce an analogous family of non-isomorphic quotients of \( k[[X_1, \ldots, X_r]] \) of Hilbert function \((1, r, 2)\). We give a presentation of the algebras of this family by specifying generators and the relations between these

\(^{22}\) Codimension three complete intersection (CI) algebras are smoothable but may or may not be alignable. It is open whether general graded \( H = (1, 3, 3, 1) \) algebras defined by a CI net of conics are alignable. For some discussion and still open problems about alignability and peelability (see below) of CI local algebras in codimension three see \([64]\).

\(^{23}\) These are pencils of quadrics, that correspond to Artinian algebras of Hilbert function \((1, r, (\binom{r+1}{2} - 2, 0)\). The defining ideals of these algebras are generated in degrees two and three. The defining ideals of their dual algebras of Hilbert function \((1, r, 2)\) are generated in degree two alone if the corresponding pencil is general enough.
generators. Then we describe a deformation for which all these relations extend, so that this deformation is flat; the deformed algebra splits into a product \( k \times A'' \) where \( A'' \) is an algebra of Hilbert function \((1, r - 1, 2)\). Since all the quotients of \( k[[X_1, X_2]] \) are smoothable, a reasoning by induction proves that the algebras \((1, r, 2)\) are smoothable.

We show that a general enough 2-dimensional subspace of \( R_2 \) (i.e. belonging to an open dense set of \( \text{Grass}(2, R_2) \)) is isomorphic to a pencil \( V[\lambda_4, \lambda_5, \ldots, \lambda_r] = \langle f', g \rangle \) defined as follows:

\[
f = X_1^2 + X_2^3 + \cdots + X_r^2, \quad g = X_2^2 + X_3^2 + \lambda_4 X_4^2 + \cdots + \lambda_r X_r^2.
\]

Furthermore, if the \( \lambda_i \) are all distinct, \( V[\lambda_4, \lambda_5, \ldots, \lambda_r] \simeq V[\lambda_4', \lambda_5', \ldots, \lambda_r'] \) is equivalent to \( \{ \lambda_4, \ldots, \lambda_r \} = \{ \lambda_4', \ldots, \lambda_r' \} \). First, notice that we expect a space of orbits, of dimension

\[
\dim \text{Grass}(2, R_2) - \dim \text{Pgl}(r - 1) = 2 \left[ \frac{(r + 1)}{2} - 2 \right] - [r^2 - 1] = r - 3.
\]

Assume therefore that: \( \langle f', g \rangle \in \text{Grass}(2, R_2) \). Let us look at the singular quadrics of the pencil defined by \( \langle f', g \rangle \). Here \( tf' + (1 - t)g \) corresponds to a singular quadric if its partial derivatives are simultaneously zero. That is, a determinant, which is a polynomial of degree \( r \) in \( t \), must be zero. This polynomial has in general \( r \) distinct roots so that there are \( r \) singular quadrics in the pencil. Let us make a change of coordinates so that singular points of two of them have for homogeneous coordinates \((1, 0, \ldots, 0)\) and \((0, 1, 0, \ldots, 0)\), respectively. So the first quadric has for equation in general

\[
X_1^2 + X_1 h(X_3, \ldots, X_r) + h'(X_3, \ldots, X_r)
\]

and after a change of coordinates only on \( X_1 \)

\[
X_1^2 + f''(X_3, \ldots, X_r).
\]

In a similar way, a change of variable affecting only \( X_2 \) provides for the second quadric an equation \( X_2^2 + g''(X_3, \ldots, X_r) \).

If \( r = 3 \), we get a normal form \( \langle f', g \rangle = \langle X_1^2 + X_3^2, X_3^2 + X_1^2 \rangle \). If \( r > 3 \), we use induction: there is a new basis of the space of the forms generated by \( x_3, \ldots, x_r \), such that \( f'' \) and \( g'' \) have simultaneous diagonalizations. We finally have the announced form where none of the \( \lambda_i \)'s is zero.

It is clear then, when the \( \lambda_i \)'s are distinct, that the only isomorphisms between the \( V(\lambda_4, \ldots, \lambda_r) \) consist in permuting the variables \( X_4, \ldots, X_r \). Thus we have obtained a family of pencils parametrized by the affine space \( A_{r-3} \) such that:

(a) There exists an open in the space of pencils of quadrics such that every pencil of this open is isomorphic to a pencil of this family.

(b) The pencils of the family, isomorphic to a given pencil, form a set of cardinality less than or equal to \((r - 3)\).
The vector space dual to \( V[\lambda_4, \ldots, \lambda_r] \), namely \( V'[\lambda_4, \ldots, \lambda_r] \) (following the definition of Sect. 6.1) contains the vector space \( W \) generated by all the monomials \( X_iX_j \) where \( i \neq j \) and a supplement of dimension \((r - 2)\) of \( W \) generated by \( h, h_4, \ldots, h_r \), where:

\[
\begin{align*}
  h &= X_2^2 - X_1^2 - X_2^2, \\
  h_1 &= X_4^2 - X_1^2 - \lambda_4 X_2^2, \\
  h_i &= X_i^2 - X_1^2 - \lambda_i X_2^2, \\
  h_r &= X_r^2 - X_1^2 - \lambda_r X_2^2.
\end{align*}
\]

Calling \((V')\) the ideal generated by \( V' \), we consider the algebra

\[
A = k[[X_1, \ldots, X_r]]/(V'),
\]

of Hilbert function \( H(A) = (1, r, 2, 0) \). We will show that there exists a flat deformation of such an algebra to the family of algebras

\[
B_t = k[[X_1, \ldots, X_r]]/(h_r + tX_r, h_{r-1}, \ldots, h_4, h).
\]

In order to verify that this deformation is flat, we must list all the non-trivial relations between \( h, h_4, \ldots, h_r \) and show that these relations extend to the generators of the deformed algebra.\(^{24}\)

First, we list the linear relations between the generators whose coefficients are in \( R_1 \). Those are:

\[
\begin{align*}
  e_{1i} : X_1h_i - X_1h - X_i(X_1X_i) + X_3(X_1X_3) + (1 - \lambda_i)X_2(X_1X_2), & \quad 4 \leq i \leq r, \\
  e_{2i} : X_2h_i - \lambda_iX_2h - X_i(X_2X_i) + \lambda_iX_3(X_2X_3) + (1 - \lambda_i)X_1(X_1X_2), & \quad 4 \leq i \leq r, \\
  e_{3i} : X_ih - X_3(X_3X_1) + X_1(X_1X_i) + X_2(X_2X_i), & \quad 4 \leq i \leq r, \\
  e_{si} : X_s h_i + X_1(X_1X_s) + X_2(X_2X_1) - X_i(X_iX_s), & \quad s \neq i, 4 \leq i, 2 \leq s.
\end{align*}
\]

The linear relations between the monomial \( X_iX_j, i \neq j \), are of the form:

\[
e_{kij} : X_k(X_iX_j) - X_j(X_kX_i), \quad i, j, k \text{ distinct.}
\]

These relations correspond to the sets of two adjacent edges of a simplex (whose vertices are the \( X_i \), the edges \( X_iX_j, \ldots \)).

There is a relation between the \( e_{kij} \) for each sub-simplex of dimension 2. Then, there are in total \( \binom{r-1}{2} - \binom{3}{2} \) independent linear relations among the generators of \( W \). There are \( 3(r - 3) + (r - 3)^2 \) linear relations concerning \( h \) and the \( h_i \)'s. Therefore, we have completed a list of \( r\binom{r-1}{2} - \binom{3}{2} + r(r - 3) = \frac{1}{3}(r_3 - 7r) \) linear relations that are clearly linearly independent. It is easy to verify that \( R_3 \subset I = (V') \) and to conclude that there are then \( r \) (minimal number of generators of \( I \))-dim \( R_3 \) linear relations with coefficients in \( R_1 \), which is \( r\binom{r}{2} + r - 2 - \binom{r + 2}{3} = (r^3 - 7r)/3 \). So we have found a basis for these relations. For more complete calculations the reader will refer to \([50]\).

\(^{24}\) Note of translators: the extension of relations is a well-known criterion for flatness of a family, see \([15, \text{ Chapter I.2.11}], [95, \text{ Corollary A.11}]\).
We affirm that these relations and trivial relations span all relations. Suppose indeed that:

\[ e = ah + a_4h_4 + \cdots + a_rh_r + u \]

is a homogeneous relation where \( u \) is formed by a linear combination of monomials of \( W \) having coefficients of degree \( j \) in \( k[X_1, \ldots, X_r] \). According to the previous discussion \( j \geq 2 \). Using multiples of the first relations \( e_{i_1}, e_{i_2}, e_{i_3} \), we can eliminate all the terms in \( h, h_1, \ldots, h_4 \) except for \( X_1h, X_2h, X_3h, X_4h, X_5h_5, \ldots, X_4h_r \). But these terms should be zeroes, since they cannot be compensated by any other term neither of \( u \), nor a multiple of \( h \), nor any of the \( h_i \). Thus, we have reduced the relation to a relation between monomials in \( W \). But it is trivial to verify that the \( e_{kij} \) span all the relations among the monomials of \( W \).

Now we consider the following deformation:

\[ I(t) = (W, h_4, h_5, \ldots, h_{r-1}, h_r + tX_r). \]

The relationships we have listed extend as follows:

These new linear relations are the same as the old ones up to the following modifications:

(a) In \( e_{1r} \), we replace the coefficient \(-X_r\) of \( X_1X_r \) by \(-(X_r + t)\).
(b) In \( e_{2r} \), we replace the coefficient \(-X_r\) of \( X_2X_r \) by \(-(X_r + t)\).
(c) In the other \( e_{sr} \), we replace the coefficient \(-X_r\) of \( X_sX_r \) by \(-(X_r + t)\).

Since all the relations between the generators of \( I \) extend to relations between the generators of \( I(t) \), the given deformation is flat. Its effect at the origin is to add \( X_r \) to the ideal \( I \), thus providing, as localized at the origin, an algebra of Hilbert function \((1, r-1, 2)\) with the same values \((\lambda_4, \ldots, \lambda_r)\) as before. As the deformation is flat, and the dimension of the localization at the origin only decreases by one, a point must be separated from the origin with a localization of length 1. Thus the deformed algebra is isomorphic to a product \( A(t) \simeq k \times A'' \).

We can continue the same procedure with \( A'' \), until \( r = 3 \). Then, \( A'' \) is of Hilbert function \((1, 2, 2)\) and by an old result of Hartshorne and Fogarty (irreducibility of the Hilbert scheme of points in \( \mathbb{P}^2 \), see [Fog]), \( A'' \) is smoothable. Therefore, all the algebras that we have considered are smoothable, and since they form a dense set in the set of algebras of Hilbert function \((1, r, 2)\), we conclude:

\[ \text{Proposition 7.3} \quad \text{All the algebras of Hilbert function } (1, r, 2) \text{ are smoothable.} \]

8 Regular maps of degree 4 from the projective plane into itself

A regular map (morphism) \( \lambda : \mathbb{P}^2 \to \mathbb{P}^2 \) can be written in homogeneous coordinates by a formula:

\[ \lambda((X, Y, Z)) = [P_n[X, Y, Z], Q_n[X, Y, Z], R_n[X, Y, Z]]. \]  

25 Note of translators: this smoothability result has been significantly generalized in [24, Theorem C].
Table 9 Regular maps from $\mathbb{P}^2$ to $\mathbb{P}^2$

\[
\begin{align*}
(X, Y, Z) &\mapsto (X^2 + \lambda YZ, Y^2 + \lambda XZ, Z^2 + \lambda XY) \\
(X, Y, Z) &\mapsto (Z^2, X^2 - YZ, Y^2 - XZ) \\
(X, Y, Z) &\mapsto (X^2 + YZ, Y^2, Z^2) \\
(X, Y, Z) &\mapsto (X^2, Y^2, Z^2)
\end{align*}
\]

where $P_n, Q_n, R_n$ are three homogeneous polynomials of the same degree $n$ without a common zero, and $(X, Y, Z)$ is the point $(X : Y : Z)$ in projective plane associated to the vector $(X, Y, Z)$.

Using Bezout’s Theorem, one sees that $\lambda$ is of degree $n^2$. As a consequence, when $n = 2$ we obtain a map of degree 4. We will say that two regular maps $\lambda_1 : \mathbb{P}^2 \to \mathbb{P}^2$ and $\lambda_2 : \mathbb{P}^2 \to \mathbb{P}^2$ are of the same type if there are two biregular maps

\[
\alpha : \mathbb{P}^2 \to \mathbb{P}^2 \quad \text{and} \quad \beta : \mathbb{P}^2 \to \mathbb{P}^2,
\]

such that $\beta \circ \lambda_2 = \lambda_1 \circ \alpha$. As the biregular maps from $\mathbb{P}^2$ to $\mathbb{P}^2$ are none other than the elements of $\text{Pgl}(3)$, it becomes evident that the classification of the regular applications of degree 4 from $\mathbb{P}^2 \to \mathbb{P}^2$ is equivalent to the classification of nets of planar conics that do not have a base point.

Finally, it is clear that these maps specialize as the corresponding nets. Using the Table 5 of Sect. 6.3 and the specializations of Table 1, we have then the classification of regular maps from $\mathbb{P}^2$ to $\mathbb{P}^2$ (Table 9).

In what follows, we give canonical procedures for passing from a regular map of degree 4, $\lambda : \mathbb{P}^2 \to \mathbb{P}^2$ to a net of conics, and vice versa.

If $\lambda : \mathbb{P}^2_a \to \mathbb{P}^2_b$ is such a map, the net $R$ corresponding to it is obtained in the $\mathbb{P}^2_a$ source of $\lambda$ as inverse image of the net of all lines of $\mathbb{P}^2_b$.

26 Note of Translators: Correction from original: #8a from our Table 1 has the base point $(0, 0, 1)$ so by equation (6)ff. it does not lead to a regular map. See also [87].
The lines of $\mathbb{P}^2_b$ correspond to the points of $R$ and, by duality, the points of $\mathbb{P}^2_b$ to the lines of $R$.

We therefore have the reciprocal canonical process:

Let $P$ be a projective plane together with a net $\pi$ of conics without base point (we change notation to return to the notations of Sect. 4). We may define a map:

$$\lambda : P \to \pi^*$$

in the following manner. If $x \in P$ then the set of conics of $\pi$ that pass through $x$ form a pencil $\delta$. We set $\lambda(x) = \delta$. One verifies immediately that $\lambda$ is of degree 4, and that the inverse image by $\lambda$ of the net of lines of $\pi^*$ is indeed the net of conics of $P$ belonging to $\pi$.

Further, in the case where $\pi$ has the generic form studied in Sect. 4.2.2 we have the following interpretation in terms of $\lambda$ of the cubics $\Gamma_1$ (defined in Sect. 4.1), and $H$ (defined in Sect. 4.2.3).

**Proposition 8.1** (1) $H$ is the singular locus of $\lambda$; (2) $\Gamma$ is the curve dual to the discriminant locus of $\lambda$.

**Proof** To show (1), it suffices to use formula (6) for $\lambda$ where

$$P_n(X, Y, Z) = X^2 + \lambda YZ,$$
$$Q_n(X, Y, Z) = Y^2 + \lambda ZY,$$
$$R_n(X, Y, Z) = Z^2 + \lambda XY.$$

Let us show (2). A point $y$ of $\pi^*$ belongs to the discriminant locus $\Delta$ of $\lambda$ if and only if its inverse image by $\lambda$ has set-wise three points instead of four. The points are the points of the base of the pencil $\delta$ that $y$ represents. Thus, to say that $\delta$ has only three base points, is to say also that $\delta$ has set-wise only two singular conics instead of three. By Bezout’s Theorem, that is to say that $\delta$ is tangent to $\Gamma$. Thus $\Delta$ is the dual curve of $\Gamma$ and, dually, $\Gamma$ is the dual curve of the discriminant of $\lambda$. □

9 Note on the Hessian form of a smooth cubic

Here we review some well known facts developing our earlier discussion of Hesse pencils in Propositions 4.5 and 4.6 in Sect. 4.2.3.27

Let us consider the pencil $P$ generated by the degenerate cubic $XYZ$ (three lines) and the smooth cubic $X^3 + Y^3 + Z^3$.

27 Note of translators: see the 2009 [6], and the 2022 [27, Section 4] for further exposition, development and references.
The base points of the pencil are the points with homogeneous coordinates (here $\omega$ is a primitive cube root of 1)

\[
\begin{align*}
(0, 1, -1) & \quad (0, 1, -\omega) & \quad (0, 1, -\omega^2) \\
(1, 0, -1) & \quad (1, 0, -\omega) & \quad (1, 0, -\omega^2) \\
(1, -1, 0) & \quad (1, -\omega, 0) & \quad (1, -\omega^2, 0) 
\end{align*}
\]

In order for a cubic to belong to the pencil $P$, it is necessary and sufficient that it passes through these nine points. These nine points are the nine inflection points of each smooth cubic of the pencil $P$. In effect, given such a smooth cubic, its Hessian generates $P$.

Further, this set $E$ of nine points has the following property:

(A) Given two of them, there is a third point of $E$ that is collinear with them.

That is to say, $E$ has the structure of an affine plane over $\mathbb{Z}/3\mathbb{Z}$, whose lines are the subsets of three collinear points of $E$.

Conversely, given a set of nine points of $\mathbb{P}^2$ satisfying the property (A), there is a homogeneous system of coordinates of $\mathbb{P}^2$ such that the set of points is identified with $E$.

Thus, given a smooth cubic of $\mathbb{P}^2$, it is easy to see that its nine points of inflection satisfy (A). Consequently, the pencil generated by a smooth cubic and its Hessian is identified with the pencil $P$.

This is to say that for every smooth cubic there is a system of homogeneous coordinates for which it has the Hesse form:

\[
(X^3 + Y^3 + Z^3 + \lambda XYZ).
\]

In $(\mathbb{Z}/3\mathbb{Z})^2$ there are twelve lines that may be partitioned into four directions. The following table gives the lines with this partition:

\[
\begin{align*}
X = 0 & \quad (0, 1, -1) & \quad (0, 1, -\omega) & \quad (0, 1, -\omega^2) \\
Y = 0 & \quad (1, 0, -1) & \quad (1, 0, -\omega) & \quad (1, 0, -\omega^2) \\
Z = 0 & \quad (1, -1, 0) & \quad (1, -\omega, 0) & \quad (1, -\omega^2, 0) \\
X + Y + Z = 0 & \quad (0, 1, -1) & \quad (1, 0, -1) & \quad (1, -1, 0) \\
X + \omega Y + \omega^2 Z = 0 & \quad (0, 1, -\omega) & \quad (1, 0, -\omega) & \quad (1, \omega, 0) \\
X + \omega^2 Y + \omega Z = 0 & \quad (0, 1, -\omega^2) & \quad (1, 0, -\omega^2) & \quad (1, -\omega, 0) \\
\omega X + Y + Z = 0 & \quad (0, 1, -1) & \quad (1, 0, -\omega) & \quad (1, -\omega, 0) \\
X + \omega Y + Z = 0 & \quad (0, 1, -\omega) & \quad (1, 0, -1) & \quad (1, -\omega^2, 0) \\
X + Y + \omega Z = 0 & \quad (0, 1, -\omega^2) & \quad (1, 0, -\omega^2) & \quad (1, -1, 0) \\
\omega^2 X + Y + Z = 0 & \quad (0, 1, -1) & \quad (1, 0, -\omega^2) & \quad (1, -\omega^2, 0) \\
X + \omega^2 Y + Z = 0 & \quad (0, 1, -\omega^2) & \quad (1, 0, -1) & \quad (1, -\omega, 0) \\
X + Y + \omega^2 Z = 0 & \quad (0, 1, -\omega) & \quad (1, 0, -\omega) & \quad (1, -1, 0) 
\end{align*}
\]
The cubics of the pencil decomposed into three lines correspond to the sets of three lines passing through the nine base points. They correspond to the four directions of \((\mathbb{Z}/3\mathbb{Z})^2:\)

\[
(X + Y + Z)(X + \omega Y + \omega^2 Z)(X + \omega^2 Y + \omega Z) = X^3 + Y^3 + Z^3 - 3XYZ = 0,
\]
\[
\frac{1}{\omega} (\omega X + Y + Z)(X + \omega Y + Z)(X + Y + \omega Z) = X^3 + Y^3 + Z^3 - 3\omega XYZ = 0,
\]
\[
\frac{1}{\omega^2} (\omega^2 X + Y + Z)(X + \omega^2 Y + Z)(X + Y + \omega^2 Z) = X^3 + Y^3 + Z^3 - 3\omega^2 XYZ = 0.
\]

These are the only singular cubics of the pencil \(P\). One verifies this property by finding those \(\lambda\)'s such that \(X^3 + Y^3 + Z^3 + 3\lambda XYZ = 0\) has a singular point, that is, that the conics defined by the three partial derivatives have a common zero.\(^{28}\)

Now we find the condition on a cubic of the pencil \(X^3 + Y^3 + Z^3 + 3\lambda XYZ\) for it to be isomorphic to the cubic \(X^3 + Y^3 + Z^3\). (See Proposition 4.3). In this case the vector space spanned by the quadratic forms \(X^2 + \lambda YZ, Y^2 + \lambda ZX, Z^2 + \lambda XY\) is spanned by the partial derivatives of another cubic form \(G\). We should have

\[
G_X' = a[X^2 + \lambda YZ] + b[Y^2 + \lambda ZX] + c[Z^2 + \lambda XY],
\]
\[
G_Y' = a'[X^2 + \lambda YZ] + b'[Y^2 + \lambda ZX] + c'[Z^2 + \lambda XY],
\]
\[
G_Z' = a''[X^2 + \lambda YZ] + b''[Y^2 + \lambda ZX] + c''[Z^2 + \lambda XY]
\]

with

\[
G_{XY}'' = \lambda cX + 2bY + \lambda aZ = 2a'X + \lambda c'Y + \lambda b'Z = G_{XY}',
\]
\[
G_{XZ}'' = \lambda bX + \lambda aY + 2cZ = 2a''X + \lambda c''Y + \lambda b''Z = G_{XZ}',
\]
\[
G_{YZ}'' = \lambda b'X + \lambda a'Y + 2c'Z = \lambda c''X + 2b''Y + \lambda a''Z = G_{YZ}'.
\]

We may suppose \(\lambda \neq 0\). One should have \(b' = a = c''\),

\[
c' = \frac{2b}{\lambda}; \quad a' = \frac{\lambda c}{2}; \quad b'' = \frac{2c}{\lambda}; \quad a'' = \frac{\lambda b}{2}; \quad b'' = \frac{\lambda a'}{2} = \frac{\lambda^2 c}{4}; \quad a'' = \frac{2c'}{\lambda} = \frac{4b}{\lambda^2}.
\]

From which \(\frac{2c}{a} = \frac{\lambda^2 c}{4}\) and \(\frac{\lambda b}{2} = \frac{4b}{\lambda^2}\), that is to say

\[
(\lambda^3 - 8)b = (\lambda^3 - 8)c = 0.
\]

If \(\lambda^3 \neq 8\), one has \(b = c = a'' = b'' = a' = c' = 0, b' = a = c''\), that is to say that \(G\) is proportional to the original cubic.

\(^{28}\) Note of translators: it is straightforward to verify that each of these singular nets have three common zeroes, so by Table 5 correspond to #6a in Fig. 1.
The cubic \((X^3 + Y^3 + Z^3 + 3\lambda XYZ)\) is thus isomorphic to \((X^3 + Y^3 + Z^3)\) if and only if \(\lambda^3 = 0\) or \(8\), so that \(\lambda = 0, 2, 2\omega, 2\omega^2\) (excluded values in \#8b of Table 1).

The four cubics of the pencil isomorphic to \((X^3 + Y^3 + Z^3)\) correspond also to the four directions of the plane \((\mathbb{Z}/3\mathbb{Z})^2\).

For each direction one finds these cubics by taking the sum of the cubes of the linear forms defining each of the lines of the direction.

Here \(X^3 + Y^3 + Z^3\) corresponds to \(XYZ\), \(X^3 + Y^3 + Z^3 + 6XYZ\) corresponds to \(X^3 + Y^3 + Z^3 - 3XYZ\), \(X^3 + Y^3 + Z^3 + 6\omega^2 XYZ\) corresponds to \(X^3 + Y^3 + Z^3 - 3\omega XYZ\), \(X^3 + Y^3 + Z^3 + 6\omega XYZ\) corresponds to \(X^3 + Y^3 + Z^3 - 3\omega^2 XYZ\).

One may describe this correspondence in another way, by saying that the second term of each correspondence is the Hessian of the first term. Finally, there is a third way to describe it.

Associate to each cubic of the pencil, the vector space \(V\) of its partial derivatives. Let us consider also the orthogonal \(V_\perp\) to the space in the space of quadratic forms with scalar product defined in Sect. 6.1. This space \(V_\perp\) may be obtained as the space of partial derivatives of a cubic in the pencil. If \(X^3 + Y^3 + Z^3 + 3\lambda XYZ\) is the first cubic, this correspondence associates to it the cubic \(X^3 + Y^3 + Z^3 - \frac{6}{\lambda} XYZ\) if \(\lambda \neq 0\); this correspondence also associates the cubics isomorphic to \(X^3 + Y^3 + Z^3\) in one-to-one manner to the cubics isomorphic to \(XYZ\).

Original References

[Fog] Fogarty, J.: Algebraic families on an algebraic surface. Amer. J. Math. 90(2), 511–521 (1968)

[Ful] Fulton, W.: Algebraic Curves: An Introduction to Algebraic Geometry. Mathematics Lecture Note Series. Benjamin, New York (1969). Notes written with the collaboration of R. Weiss

[Gur] Gurevich, G.B.: Foundations of the Theory of Algebraic Invariants. Noordhoff, Groningen (1964)

[Iar-Ems] Iarrobino, A., Emsalem, J.: Finite algebras having small tangent space: some zero-dimensional generic singularities, to appear [Later, Compositio Math. 36(2), 145–188 (1978)]

[MS] Mumford, D., Suominen, K.: Introduction to the theory of moduli. In: Oort, F. (ed.) Algebraic Geometry, Oslo, 1970. Proceedings of the 5th Nordic Summer School in Mathematics held in Oslo, pp. 171–222. Wolters-Noordhoff, Groningen (1972)

[Nag] Nagata, M.: Notes on orbit spaces. Osaka Math. J. 14, 21–31 (1962)

[Wall] Wall, C.T.C.: Nets of conics, preprint, Department of Pure Mathematics of University of Liverpool, April, 1976 [Later, Math. Proc. Cambridge Philos. Soc. 81(3), 351–364 (1977)]

Appendix A: Historical Note, pre 1977

Classification of pencils of conics, pre-1977. This is often attributed to Corrado Segre in his 1883 article [94]: he classified the pencils of quadric hypersurfaces (then termed \(n\)-ary conics) over the complexes in any number of variables. The 1884 note [90] on simultaneous invariants of two ternary conics is a letter from Segre to M.J. Rosanes
concerning this classification in response to [88]. However there were earlier works: the book of Igor Dolgachev [38, p. 468] mentions the 1868 article of Karl Weierstrass [115] as giving the classification of pencils of conics up to isomorphism, in work partially dependent on a note by James Sylvester of 1851. The article [14, Theorem 3.1] attributes determining the four simultaneous invariants of two ternary conics to the 1882 article of Paul Gordan [55]. In any case by 1900 there was a substantial literature about pencils of quadric hypersurfaces in any $\mathbb{P}^r$ (see, for example, P. Newstead’s MathSciNet review of [33] and also the 1898–1904 Encyclopedia, which has many detailed historical remarks [77]); Emsalem notes in the Outline 1.1 above that he had learned the classification of pencils in a course of E. Riche.

Nets of planar conics before 1977.29 Unbeknownst to Emsalem and Iarrobino in 1977, the classification up to isomorphism of nets of conics over the complexes had been attempted by Camille Jordan in 1906 [69].30 Jordan’s article in principle considers an $m$-dimensional vector space of $n$-ary forms: after determining the eight ternary pencils, he goes on in Sections 12–24 to handle nets of ternary conics, arriving at a table of 13 classes in [69, Section 25, p. 427], omitting one class (our #6c), and apparently combining the continuous class #8b with #8c. 31 He continues to treat dimension four and five spaces of ternary conics, without using duality. His methods combine geometry and algebra, use the related cubic we called $\Gamma$, but he does not formally use the notion of type of a triple as fundamental invariant, as we have in Theorem 4.1 (B), and in our diagrams. The article of Jordan does not discuss specialization.

In the modular case, the 1914 article of the L.E. Dickson student, Albert Wilson, classifies nets of conics over $\text{GF}(p^n)$ when $p$ is odd, but he leaves the “rank one” case where the net contains a double line incomplete [116].32 The 1928 article of Alan Campbell [19] classifies these nets over the field $\text{GF}(2^n)$, giving 34 classes; however, Zanella in 2012 exhibits classes of nets over any $\text{GF}(q)$ containing only nonsingular conics, contradicting a statement in [19] that these only exist when $p$ is odd [117, Section 6]. Campbell treated the real case in the 1928 article [18].

A 1943 article of Corrado Ciamberlini [26] discusses the simultaneous invariants of three ternary forms; a recent note of Blind [14, Section 5] presents these and other results concerning simultaneous invariants of $k$ ternary forms, giving new proofs using the Jordan algebra structure on ternary forms.

The 1898–1904 Encyclopedia [77, Section 32ff., p. 224ff.] reviews nets of planar conics, beginning with the Hesse form Section 32; however, the results reported are not without errors.

29 We report on these results, but we have not checked the proofs.

30 Until doing the present literature search, with convenient modern tools of MathSciNet, we were still unaware of Jordan’s work. Although MSN does not list Jordan’s article, it does list the 1914 article of Wilson about the modular case [116], and later articles of A. Campbell, which cite it. Dolgachev and Kondo in [40, Example 7.2.9] give the classification over characteristic $k \neq 2$, attributing it to C. Jordan.

31 We note our classes in Table 1 #8a, #8b, #8c, …, #2b where # denotes the dimension and a,b,… the position from left. Then the correspondence of Jordan’s classes I–XIII with ours is #8a III; #8b,c I; #7a XII; #7b II; #7c IV; #6a XIII; #6b VIII; #6c missing; #6d V; # 5a IX, #5b VI; #4 X; #2a XI, #2b, VII.

32 Wilson does not explicitly state that $p$ is odd—apparently considered obvious, as a key method is to complete the square! A careful analysis, correction, and completion of Wilson’s work in this $p$ odd case is made in [72], see Appendix B.
A 1957 article of Václav Alda gives a lucid presentation of the six equivalence classes of nets that arise as Hessians, as well as three degenerate classes of dimension 2 or 1 [3]. He corrects an error in [77, Section 34, line 6, p. 228], which incorrectly states [our translation] “Inversely, given three conics \( f = g = h = 0 \) not part of the same pencil, employing new variables allows us to represent them as partial derivatives of a cubic”; Alda gives the exact conditions on the net for which this representation occurs [3, p. 50].

Classification and specialization of planar cubics. This classification over the reals had been begun by Isaac Newton, and continued by many. Experts assured us that this classification up to algebraic isomorphism, including specializations was known in the 19th century, although we have not found the diagram in our Figure 6 (see also [45, Figure 1.13]). The book of George Salmon [91, Chapter V, Section V, §197ff.] gives a classification, and [71, p. 548–549] has some discussion of specializations. A later section of the same Encyclopedia [77, Chapter III.4, Section 29, p. 316], by Hieronymus Zeuthen and Mario Pieri, includes reference to work of S.N. Maillard and Zeuthen specifying the numbers of singular and degenerate cubics in the one parameter families defined by the conditions to pass through \( n \) general points and to be tangent to \( 8 - n \) general lines for \( 0 \leq n \leq 8 \).

Already in 1902, Alfred Dixon had shown that a general enough degree \( n \) planar curve over \( \mathbb{C} \) can be obtained as a determinant of a symmetric matrix of linear forms [36], a theme not developed in [49]. We discuss this further in Appendix B.

Classification of Artinian algebras of small lengths before 1977. This classification for \( n \leq 6 \) was known to Emsalem and Iarrobino at the time of writing Réseaux, as indicated in footnote 3, p. 5. See the history of early work from 1880 to 1920 by Happel [57]. Tables of isomorphism class and specialization for \( n \leq 6 \) for embedding dimension one and two are in Briançon’s [17] (see especially the figures in the actual dissertation of 1972). Suprunenko had shown in 1956 that the length seven algebras (of Hilbert function \((1, 4, 2)\)) have continuous isomorphism families [101] (not difficult), and Dyment had shown in 1966 that there are a finite number of classes for length six, listing 34 classes [43]. Mazzola’s 1980 article [75] gave a complete classification of isomorphism classes of unitary algebras (not necessarily commutative) for \( n \leq 6 \), over an algebraically closed field of characteristic not 2 or 3, using a variety of methods; Mazzola uses this and Hochschild cocyles to show the irreducibility of the family of local length \( n \) commutative algebras for \( n \leq 7 \), over any such field: that is, he showed that each such algebra may be deformed to \( k[z]/(z^n) \). He accomplished this without classifying up to isomorphism the local algebras of length 7. Mazzola’s result implies

\[ \text{We thank Steve Kleiman for this reference.} \]

\[ \text{We thank a referee for suggesting mention of Dixon’s result. It was reproved by John Grace and Alfred Young in 1903 [56], see the Historical note [38, Section 4, p. 205].} \]

\[ \text{This no longer holds for } n = 8, \text{ as the algebras of Hilbert function } H(A) = (1, 4, 3) \text{ defined by a vector space of quadrics having dimension 7, form a generic point of a component of the Hilbert scheme } \text{Hilb}^8(\mathbb{P}^4); \text{ these correspond by duality to nets of quadrics in } \mathbb{P}^3, \text{ having generically } \dim \text{Grass}(10, 3) - \dim \text{Pgl}(4) = 6 \text{ parameters [50, Section 2.2].} \]
that the punctual Hilbert scheme $\text{Hilb}^n(P^r)$ is irreducible for $n \leq 7$ (see Emsalem’s MSN review of [75] and Iarrobino’s MSN review of [21]).

**Structure, smoothability:** Emsalem’s *Géométrie des points épais* [48] of 1978 concerning the topology of the punctual schemes, and the Macaulay duality, forms part of the background of this paper. Also, an interest in smoothability questions as in Sect. 7 arose from Iarrobino’s [62] where it had been shown by a dimension argument that there are non-smoothable Artinian algebras. The number of irreducible components of the Hilbert scheme parameterizing length-$n$ commutative unitary algebras had been studied in [63]. It took until [50] for a specific non-smoothable scheme to be given, corresponding to graded Artinian algebras of Hilbert function $H = (1, 4, 3, 0)$, defined by seven general enough quadratic forms (see also [58, Section 1.5.8], [20], and the discussion of smoothability below in Appendix B).

### Appendix B: Update on related topics

In this section we give a survey of more recent work, since the original Réseaux of 1977, relating to pencils and nets of planar conics, and their associated algebras.

**Classification of pencils.** This classification was already well known in 1977, as noted in “Appendix A”, even for pencils of quadrics in $n$ variables. The 1983 article of Dimca [33] gives for pencils $\pi$ of quadric hypersurfaces in $n$ variables, a geometric approach to their classification, showing that the class of a pencil is uniquely determined by that of its associated line, and the position of the line with respect to the rank subvarieties of quadrics, and the singularity set of the zeroes of $\pi$. There are connections with Fano varieties: the thesis of Reid [86], articles by X. Wang [114] and others, now include classification of pencils of conics in $\mathbb{P}^2$ in characteristic $p$. Finally we have the very recent survey for pencils of quadric hypersurfaces in $n$ variables by Fevola, Mandelshtam, and Sturmfels [53]: they review what is known about these pencils in $n$ variables, in particular the classification using Segre pencils [94], [60, Section XIII.10]. They also extend the classification over the reals to parametrizing Jordan type loci, and other strata in the appropriate Grassmannian, with examples over three and four variables. They give applications to the maximum likelihood estimation for Gaussian distributions in algebraic statistics.

**Classification of nets of planar conics.** Appearing in 1977 was Wall’s complete and succinct classification [111] of these nets over the complexes and over the reals. Emsalem and Iarrobino saw the Wall preprint only after writing most of Réseaux de Coniques; the appearance of [111] was a factor in leaving [49] in preprint status for so long. Since Wall’s goal was the study of map germs of finite type up to source-image isomorphism (right-left equivalence) there is a different emphasis, and an efficient algebraic approach to classification of nets. By contrast, the Réseaux preprint [49] is more geometric in style.

We also note that the classification in Table 1 of isomorphism classes of nets of planar conics over an algebraically closed field is valid in characteristic $k > 3$; although
Table 10  Concordance of tables for nets of conics from our Table 1 (EI), Wall [Wall]), and Conca [28, p. 1567]

|   | 8a | 8b | 8c | 7a | 7b | 7c | 6a | 6b | 6c | 6d | 5a | 5b | 4 | 2a | 2b |
|---|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|
| EI |     |    |    |    |    |    |    |    |    |    |    |    |   |    |    |
| Wall | B* | A | B | D* | C | D | E* | F* | F | E | G* | G | H | I* | I |
| Conca | 14 | 15 | 13 | 6 | 12 | 5 | 4 | 10 | 9 | 3 | 8 | 7 | 11 | 2 | 1 |

Indeed, this classification over algebraically closed fields of arbitrary characteristic except 2 or 3 is nicely explained in a recent posting by Naoto Onda [79], which uses an efficient analysis of the normal forms of the net having a given isomorphism type of discriminant, to classify the ten orbits for which the discriminant is singular. The context of [79] is of classifying Artinian algebras of Hilbert function $\left(1, 3, 3\right)$ up to isomorphism—which, of course, is equivalent to classifying the equivalence classes of the nets of planar conics, a fact not noted by the author. N. Onda refers to the following result of M. Bhargava and Wei Ho for the nonsingular case. They set $K$ to be an arbitrary field of characteristic not 2 or 3; $K$ need not be algebraically closed.

**Theorem B** ([11, Theorem 5.8]) Let $V_1$ and $V_2$ be 3-dimensional vector spaces over $K$. Then the nondegenerate $\text{Gl}(V_1) \times \text{Gl}(V_2)$ orbits of $V_1 \otimes \text{Sym}^2(V_2)$ are in bijection with the isomorphism classes of triples $(C, L, P)$ where $C$ is a genus one curve over $K$, $L$ is a degree 3 line bundle on $C$ and $P$ is a nonzero 2-torsion point of the Jacobian $\text{Jac}(C)(K)$.

The paper of Domokos, Fehér, Rimányi [41] extends the discussion of nets of conics over the complexes $\mathbb{C}$ to invariants and equivariants, in essence determining cohomology classes of the closures of orbits. The classes of nets of conics of codimension at least two (all but the top row of our Table 1) are the $e^0$ portion of their Table 1 and 3, which include further invariants; the corresponding portion of their Figure 2 has the specialization diagram of our Table 1. They apply this to determine Thom polynomials of contact singularities of type $(3, 3)$. Thus, in the course of their work with equivariant cohomology, they give an independent derivation of the specializations in our Table 1 or of [Wall]) over the complexes — not including the discussion in our Section 6.5 concerning specializations with $\lambda$ constant from the smooth orbits #8b in our Table 1.36 See also [77, Appendix B].

**Concordance of tables of nets of conics.** Conca gave in the 2009 [28, p. 1568] a table of nets of conics with considerable further information, including whether the algebra $A = R/(V)$ is Koszul, whether the net is of gradient type (i.e. polar, or Jacobian), and he gives the Hilbert series for $R/(V)$. He notes that the point $(a : b : c)$ belongs to the locus defined by $V$ if and only if $(ax + by + cz)^2$ is in $V^*$. The duality is indicated in Wall’s notation by an asterisk, as $E$ vs. $E^*$. For a comparison of these classifications with that of our Table 1 see our Concordance Table 10.

36 We for short label the orbits in our Table 1 by the dimension and position from the left in each dimension, thus #6b is the second orbit shown in the dimension 6 row.
Note that Conca in his column \( p \) [28, p. 1568] gives the number of distinct points in the support of the punctual scheme (in all but case 2 where the support is a line), while our Table 5 gives the total scheme length, including multiplicities: thus the entries are different.

Koszul properties of nets are discussed in Conca’s [28] and in D’Ali’s [31] the latter handling characteristic \( p \) cases.

**Nets of conics, and Hessian of smooth cubics.** The theme of our Note on the Hessian form of a smooth cubic (Sect. 9) is greatly developed by Artebani and Dolgachev in [6], who discuss a related elliptic fibration, two sextic curves and a \( K \)-3 surface associated. The article [4] treats the Hessian pencil over all characteristics but three. See also [54] and [27, Section 4].

**Classification of nets of conics over algebraically closed fields of small characteristic, or over finite fields** \( k \). This classification in characteristic 2 for \( k \) closed is discussed in [7, 93], see also [16] and [40, Example 7.13]. We have mentioned in Appendix A as an update of Wilson’s [116] the very recent article by Lavrauw, Popiel, and Sheekey [72] for a finite field of odd order \( q \), finding 15 orbits of rank one, that is, which contain a conic that is a double line. Zanella studies nets of conics over finite fields [117]. Sivatski for an arbitrary field \( k \) of characteristic \( p \) not 2, shows that three planar quadrics \( f_1, f_2, f_3 \) have a common zero if and only if the quadratic form \( u_1 f_1 + u_2 f_2 + u_3 f_3 \) over the field \( K = k(u_1, u_2, u_3) \) is isotropic (has a \( K \)-rational point), and the cubic form \( t_1 f_1 + t_2 f_2 + t_3 f_3 \) is isotropic over \( k \) [99].

**Applications of the classification of nets of planar conics.** Notable applications besides the study of Artinian algebras with Hilbert functions \( H(A) = (1, 3, 3, \ldots) \), include the study of map germs from \( (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0) \) (see [35, 41, 44, 87, 111, 113]). A finite mapping germ \( F = (f_1, \ldots, f_m) \) from \( \mathbb{C}^r \rightarrow \mathbb{C}^m \) is one such that the local algebra \( A = k[x_1, \ldots, x_r]/(f_1, \ldots, f_m) \) has finite vector space dimension; the result of Mather, was that right-left equivalence of stable finite mapping germs is the same as isomorphism of their local algebras [74]. Thus, the classification of local Artinian algebras up to isomorphism is relevant to singularity theory—see also [50, Appendix], [32, 65].

The article [41] applies the study of nets to determining the Thom classes of related maps from \( (\mathbb{C}^3, 0) \) to \( (\mathbb{C}^p, 0) \), \( p \geq 3 \) using their calculation of the \( \text{Gl}(3) \times \text{Aut}(\mathbb{P}^2) \) equivariant cohomology classes of the orbit closures of the nets. The study of the intersections of three conics by Feng and Shen, [52] includes a careful development of the scheme length we studied briefly in Sect. 6.3. There is a connection with Fano threefolds [5]. Verra has used the classification of these nets as a step in characterizing certain Enriques surfaces that correspond to a fibred product of two planes \( \mathbb{P}^2 \) over \( \mathbb{P}^2 \) with the projection maps being two rational maps of degree four, each corresponding to a net of conics as in Sect. 8 [108, Theorem 1.3]: [109] the result uses Proposition 8.1 to characterize the discriminant locus of a regular map, \( \mathbb{P}^2 \) to \( \mathbb{P}^2 \). Ronga [87] studies
topologically over $\mathbb{C}$ the four isomorphism classes of maps $(f_0, f_1, f_2): \mathbb{C}^3 \setminus 0 \to \mathbb{C}^3 \setminus 0$. For reference on Enriques surfaces see [30, 40].

Classification and specialization of planar cubics. Tables of singular cubics and their specializations in [34, p.52, Theorem 5.17] and [45, Figure 1.13] agree with our Table 7, except for an extra arrow from node to triangle in the former, which is proscribed by our Lemma 6.22.3. The latter reference goes on to determine the degrees of the loci [45, Section 1.2.5]. Several authors have discussed the Waring problem for cubics in $\mathbb{P}^2$—decomposing the cubic into the sums of powers, in particular over the reals, some making connections to nets of conics [39, 76, 80, 81]. In particular the Aronhold invariant $S = \text{Ar}(f)$ of a cubic $f$ vanishes if $f$ is equivalent to a sum of cubes (a Fermat cubic); it is written as a certain Pfaffian in [81, Section 4.7]. Dolgachev and Kanev show that $\text{Ar}(f) = 0$ only if $f$ is a Fermat cubic, a cone, or a conic union a non-transversal line [39, Theorem 5.13.2]: this is equivalent to being in the closure of the family of Fermat cubics. As noted earlier in Sect. 6.5.1 the invariants $S, T$ of a ternary cubic were given by Salmon [91]. For a modern derivation of these explicit formulas see [100, Theorem 4.4.6, Proposition 4.4.7, Example 4.6.3]. For further study of cubics and their invariants see Okonek and Van de Ven [78, Examples 6, 8, Section 5.3], [37, Section 10.3] (classifying plane cubics over algebraically closed fields of arbitrary characteristic), Frium [54] and [38, Chapter 3].

Symmetric determinants of linear forms, discriminant, intersections of quadrics. The connection of general enough degree $n$ planar curves with symmetric determinants of linear forms, introduced by Dixon then Grace and Young over the complexes [36, 56], was further developed by Wall for quadrics in $\mathbb{P}^3$ in characteristic zero [112], by Cook and Thomas for quadrics in $\mathbb{P}^r$ over an algebraically closed field [29]; see also Dolgachev’s detailed discussion of determinants of such matrices [38, Section 4] and for references the Historical note [38, Section 4, p. 205]. These determinants were further studied for smooth planar curves in a context of a local-global property over number fields—see Ishitsuka and Ito [66] and the references cited there. A consequence of Dixon’s 1902 article [36] is that the discriminant parametrizing the singular quadrics of a general enough net of quadrics in $\mathbb{P}^{d-1}$ is a degree-$d$ plane curve: for $d = 3$ this is the cubic $\Gamma(V)$ studied here. The corresponding discriminant in higher dimensions/degree has been studied, in particular for $d = 4$; it is connected also to the study of certain rank two vector bundles on $\mathbb{P}^{d-1}$ or related varieties. See, for example [8, 12, 39, 59, 110, 112, 113] and references cited there. Further articles on the intersection of quadrics include the thesis of Reid [86] and Tjurin’s [106, 107].

Determinantal nets of conics and reciprocal nets. Recall that the determinantal nets of conics are those arising as the $2 \times 2$ minors of a $2 \times 3$ matrix of linear forms (footnote p. 40). See also [105] for an exposition obtaining invariants for a compactification of the determinantal nets of conics. Analogous to the article [53] for pencils, Dye, Kohn, Rydell, Sinn have applied the classification of nets of conics over $\mathbb{C}$ to obtain maximum likelihood estimates for a Gaussian distribution [42]. They use the reciprocal surface of an invertible net (one with nonsingular symmetric matrices), a rational variety closure

\footnote{37 The author communicated that this is an error.}
of the inverse images of nonsingular matrices in the net, which they show is an image of the Veronese. Their work uses the intersection of the reciprocal surface with the polar net.

Classification of commutative local algebras for \( n \leq 8 \). Iarrobino wrote in 1985, “There is a virtually complete picture of isomorphism classes of Artinian algebras and their deformations for \( n \leq 8 \), but that picture has never been written down in one place [65].” To explain this comment, for lengths \( n \leq 6 \) we had Mazzola [75], and for earlier authors see [57]. For codimension two and small \( n \leq 5 \) see the actual thesis of Briançon. For the case codimension three and length \( n = 7 \), for \( H = (1, 3, 3) \), see this article in Sect. 7.1; for \( H = (1, 3, 4) \) consider the Macaulay duals of pencils of conics (this article in Sect. 7.2). Various classifications of spaces of quadrics in \( r \) variables were given by Suprunenko and Tyškevič in the 1966 book [102].

A succinct derivation of the isomorphism classes of commutative local algebras over an algebraically closed field in lengths up to six is given by Poonen in [84] [the web-posted update has a correction in the characteristic 2 case]. The case of \( n = 6 \) over \( \mathbb{C} \), corresponding to length 5 nilpotent algebras is also in [70], there obtained by quite different methods involving central extensions of algebras, and given in a different language. See also the 1969 [82, 83] for some other cases specifying socle degree (class) and the number of variables \( n \).

Graded algebras with Hilbert function \( H = (1, 3, 1, 1, 1) \) or \( (1, 4, 1, 1) \) are special and easier to classify because of the tail including \((1, 1)\) which requires that \( \ell R_{j-2} = \ell R_j - 2 \) for some linear \( \ell \) (by Gotzmann’s theorems). The classification of those having Hilbert function \((1, 3, 2, 1)\) will follow from the classification of 4-spaces \( V \subset R_2 \) of conics, and the very restrictive condition that \( \dim R_1 V = 1 \) - allowing an approach using the cubic \( F \) in an inverse system to \( A \). Classifying Gorenstein algebras \( R/I \) of Hilbert function \( H = (1, r, 1) \) is essentially the classification of the conic perpendicular in the Macaulay duality to \( I_2 \).

Thus, the lack of specific writing down of the classifications mentioned in the 1985 [65], has since been somewhat remedied, with relevant articles by Poonen again taking up the classification for \( n \leq 6 \) [84], and with a whole new direction of research showing that Gorenstein Artinian algebras of lengths up to 13 are smoothable (see [21, 24]). The article [85] gives an asymptotic formula for the dimension of the family of length-\( n \) Artinian algebras with a fixed basis.

A subtle issue when the socle degree is at least 3 is whether the classification of local algebras of a given Hilbert function is the same as the classification of graded algebras of that Hilbert function: is each \( A \) canonically graded? Articles by Elias and Rossi [46], Jelisiejew [67], and others [23] go deeply into the connections between

---

38 See [84, Remark 1.1] for some further comparison with the classifications of nilpotent algebras, and historical references. Kaygorodov in a note to the authors pointed out that the algebra \( k^6 \) of [84], corresponding to a nilpotent algebra 0 has no analogue in the classification of length-5 nilpotent algebras [70], thus a difference in the number of classes listed in [70, 84].
local and graded Artinian algebras, for example showing that $H = (1, r, r, 1)$ algebras are canonically graded [46, Theorem 3.3] for $k = 0$, [67, Proposition 1.1].

**Smoothability.** We define elementary components of the punctual Hilbert scheme \( \text{Hilb}^n(\mathbb{P}^r) \), to be those irreducible components that parametrize finite length punctual schemes concentrated at a single point of \( \mathbb{P}^r \); every irreducible component is made up of a concatenation of elementary components, concentrated at different points of \( \mathbb{P}^r \) [63, 68]. The deformation space of a non-smoothable length-\( n \) punctual scheme must contain a scheme of possibly smaller length having an elementary component. As mentioned in Appendix A, the first specific example noticed of an elementary component was the graded algebras of Hilbert function \( H(A) = (1, 4, 3, 0) \) defined by seven general enough quadratic forms [50], shown using a “small tangent space” method, where the available tangent space is matched to elements of the family.\(^{39}\) Shafarevich using new methods greatly generalized this result: he showed that for algebras \( A = k[x_1, \ldots, x_r]/I \) of Hilbert function \( (1, r, n, 0) \), \( r \geq 4 \), that for \( 2 < n \leq \frac{(r-1)(r-2)}{6} + 2 \) (a third of the available values) the generic algebra of Hilbert function \( (1, r, n, 0) \) determines an elementary component of \( \text{Hilb}^{n+r+1}(\mathbb{P}^r) \). For large \( n \), about a third of the available values, all the algebras in \( \text{Hilb}^{n+r+1}(\mathbb{P}^r) \) are smoothable—that is, they are in the closure of the separable algebras, those isomorphic to \( K^{1+n+r} \); for the remaining third of the values the question of smoothability remains open [97].

Further study of the “principal component” of the punctual Hilbert scheme, parametrizing smoothable algebras is in [89]; and the usually reducible scheme \( \mathcal{B} \) parametrizing non-smooth but smoothable algebras is studied in [98]. Distinct approaches to smoothability are given by Erman and Velasco, who introduce a finer, syzygetic “k-vector” obstruction to smoothability in [51], by Casnati and Notari in [21], and by Jelisiejew [68]. Szachniewicz shows that \( \text{Hilb}^{15}(\mathbb{A}^6) \) is non-reduced at a point corresponding to a \( H(A) = (1, 6, 6) \) algebra [103]. A striking and surprising result (counterexamples to [98, Conjecture 1] and to an erroneous [50, Proposition A.5]) was that the length-\( n \) singular schemes have in general codimension only one in the family of smooth schemes in \( \mathbb{P}^r \) when both \( r \geq 11 \), and \( n \geq 15 \) [51, Proposition 1.6]: they construct these by adding smooth points and lifting, from a single basic example of the smoothable local algebras concentrated at a single point \( 0 \in \mathbb{A}^{11} \) and satisfying \( H(A) = (1, 11, 3) \)—a family they show has dimension 153 (so dimension 164 as a family fibered over \( \mathbb{A}^{11} \)).

Substantial work of Casnati and Notari and others concerning Gorenstein algebras of lengths no greater than thirteen have resulted in showing that they are smoothable [21, 22, 24] while the \( H = (1, 6, 6, 1) \) generic Gorensteins form an elementary component [50]. Surprisingly the \( H = (1, 7, 7, 1) \) Gorensteins have been shown

\(^{39}\) We showed that \( A = R/I \) having only trivial negative one tangents in Hom(\( I, R/I \)) implies that all higher order negative tangents are zero, which suffices to show that \( A \) is elementary. We relied on what we Footnote 39 continued

had learned from Schlessinger. Jelisiejew reproves this in an interesting way [68, Theorem 4.5]; his article does survey—and advance—much of the current state of the art of finding elementary components. The simpler example \( A = k[a, b, c, d]/(a^2, ab, b^2, c^2, cd, d^2, ad - bc) \), also of Hilbert function \( H = (1, 4, 3, 0) \) and having no nontrivial negative tangents is reported in [85, Proposition 9.6].
to be smoothable using new methods of marked bases (Bertoni, Cioffi, Roggero [9]), while a dimension calculation shows that generic Gorensteins of Hilbert function \( H = (1, r, r, 1) \) are not smoothable for \( r \geq 8 \) [50, p. 152]. Szafarczyk has recently shown that these generic Gorensteins give elementary components of the Hilbert scheme \( \text{Hilb}^n(\mathbb{P}^{r-1}) \) [104]. The small-tangent space method [50] suggests there are similar elementary components in codimension \( r \) four and larger, determined by suitable-dimension vector spaces of degree-\( j \) forms, \( j \geq 3 \); however, the nub is, given a suitable triple \((r, j, d)\) to verify that the tangent space in at least one example has the conjectured small dimension.

Other elementary components of the punctual Hilbert scheme had been found by Huibregtse [61]: these, strikingly, are comprised of essentially non-homogeneous algebras. Satriano and Staal have recently shown that there is an infinite set of choices \( d \) for which \( \text{Hilb}^d(A^4) \) has an elementary component of very small length, less than \( 3(d - 1) \) [92], answering a question asked in [66, p. 186].

**Classification of isomorphism types of Artinian Gorenstein (AG) algebras having Hilbert function** \( H = (1, 3^k, 1) \), \( k \geq 2 \). This is closely related to the study of length 3 punctual subschemes of \( \mathbb{P}^2 \). The article of Bernardi et al. [10, Section 4.2, Theorem 37] classifies the isomorphism types of length 3 subschemes of \( \mathbb{P}^n \) in characteristic zero; when \( n = 2 \) this is closely related to the classification of nets of ternary conics: the scheme-length 3 cases of Table 5 and classifying also those length-3 schemes that are on a line will give this classification. Casnati, Jelisiejew, and Notari determine for arbitrary characteristic the three isomorphism classes of forms \( F \in k\langle X, Y, Z \rangle \) whose apolar algebra \( A = R/\text{Ann} F \) has Hilbert function \( H(A) = (1, 3, \ldots, 3^{j-1}, 1_j) \) when \( j \geq 4 \) [24, Proposition 4.9]. This can be derived also from the list of scheme-length-3 nets of conics in Table 5.

Jelisiejew [67] determines the isomorphism types of AG algebras \( A \) of Hilbert function \( H(A) = (1, 3, 3, 3, 1) \) including those for non-graded \( A \). In a sequel using the current “Nets of conics” paper, the authors with Yaméogo determine the isomorphism types of graded Artinian Gorenstein algebras \( A \) of Hilbert function \( H(A) = (1, 3^k, 1) \) for all \( k \geq 2 \); and also show that the closure of this family is all the graded algebras \( GT \) of this Hilbert function [2]. This is unlike the case for other Gorenstein sequences, such as \( T' = (1, 3, 5, 3, 1) \) where \( GT \) has several irreducible components, one the closure of \( \text{Gor}(T') \).

**Classification of Jordan types for pairs** \((A, \ell)\), \( A \in \text{Gor}(H) \). The Jordan type \( P_{A,\ell} \) for a pair \((A, \ell)\) is the partition of length \( |A| \) giving the Jordan block decomposition of multiplication \( m_\ell \) on \( A \). For a generic \( \ell \) the graded AG Gorenstein algebras of Hilbert function \( H = (1, 3^k, 1) \) are strong Lefschetz [1]; for an arbitrary linear \( \ell \) the partitions \( P_{A,\ell} \) are unknown.

We remark that the book “Classical Algebraic Geometry” [38] by Dolgachev, and the books “Enriques Surfaces, I” [30] with Cossec and Liedtke, “Enriques Surfaces, II” [40] with Kondo give many further results pertaining to the geometry of nets of conics, nets of quadrics, and have extensive bibliographies including many classical as well as modern results in related areas.
References

1. Abdallah, N., Altafi, N., Iarrobino, A., Secealanu, A., Yaméogo, J.: Lefschetz properties for some codimension three Artinian Gorenstein algebras (2022). arXiv:2203.01258
2. Abdallah, N., Emsalem, J., Iarrobino, A., Yaméogo, J.: Limits of graded Gorenstein algebras of Hilbert function $(1, 3^5, 1)$. arXiv:2302.00287
3. Alda, V.: Les réseaux de coniques. Czechoslovak Math. J. 7(82), 48–56 (1957)
4. Anema, A., Top, J., Tuijp, A.: Hesse pencils and 3-torsion structures. SIGMA Symmetry Integrability Geom. Methods Appl. 14, 102 (2018)
5. Araujo, C., Castravet, A.-M., Cheltsov, I., Fujita, K., Kaloghiros, A.-S., Martinez-Garcia, J., Shramov, C., Süss, H., Viswanathan, N.: The Calabi Problem for Fano Threefolds. MPIM preprint (2021) (to appear, London Mathematical Society Lecture Series, Cambridge University Press). https://www.maths.ed.ac.uk/cheltsov/pdf/Fanos.pdf
6. Artebani, M., Dolgachev, I.: The Hesse pencil of plane cubic curves. Enseign. Math. 55(3–4), 235–273 (2009)
7. Baldisserri, N.: Nets of conics in a plane over a field of characteristic 2. Rend. Mat. 5(3–4), 355–365 (1985)
8. Barth, W.: Moduli of vector bundles on the projective plane. Invent. Math. 42, 63–91 (1977)
9. Bertone, C., Cioffi, F., Roggero, M.: Smoothable Gorenstein points via marked schemes and double-genital initial ideals. Exp. Math. 31(1), 120–137 (2022)
10. Bernardi, A., Gimigliano, A., Idà, M.: Computing symmetric rank for symmetric tensors. J. Symbolic Comput. 46(1), 34–53 (2011)
11. Bhargava, M., Ho, W.: Coregular spaces and genus one curves. Cambridge J. Math. 4(1), 1–116 (2016)
12. Bhosle, U.N.: Nets of quadrics and vector bundles on a double plane. Math. Z. 192(1), 29–43 (1986)
13. Bielmans, P.: Fanography, a tool to visually study the geography of Fano three-folds. https://fanography.pythonanywhere.com
14. Blind, B.: La théorie des invariants des formes quadratiques ternaires revisité (2008). arXiv:0805.4135
15. Bourbaki, N.: Elements of Mathematics, Commutative Algebra. Hermann, Paris (1972)
16. Boughon, P., Nathan, J., Samuel, P.: Courbes planes en caractéristique 2. Bull. Soc. Math. France 83, 275–278 (1955)
17. Briançon, J.: Description de $\text{Hilb}^6 C[x, y]$. Invent. Math. 41(1), 45–89 (1977)
18. Campbell, A.D.: Nets of conics in the real domain. Amer. J. Math. 50(3), 459–466 (1928)
19. Campbell, A.D.: Nets of conics in the Galois fields of order $2^n$. Bull. Amer. Math. Soc. 34(4), 481–489 (1928)
20. Cartwright, D.A., Erman, D., Velasco, M., Viray, B.: Hilbert schemes of 8 points. Algebra Number Theory 3(7), 763–795 (2009)
21. Casnati, G., Notari, R.: On some Gorenstein loci in $\text{Hilb}_6(\mathbb{P}^4)$. J. Algebra 308(2), 493–523 (2007)
22. Casnati, G., Notari, R.: On the Hilbert locus of the punctual Hilbert scheme of degree 11. J. Pure Appl. Algebra 218(9), 1635–1651 (2014)
23. Casnati, G., Elias, J., Notari, R., Rossi, M.E.: Poincaré series and deformations of Gorenstein local algebras. Comm. Algebra 41(3), 1049–1059 (2013)
24. Casnati, G., Jelisiejew, J., Notari, R.: Irreducibility of the Gorenstein loci of Hilbert schemes via ray families. Algebra Number Theory 9(7), 1525–1570 (2015)
25. Cheltsov, I.: Calabi problem for smooth Fano threefolds (2021). Slides for a talk https://www.maths.ed.ac.uk/cheltsov/talk2021x.pdf
26. Ciamberlini, C.: Sulla condizione necessaria e sufficiente affinché un triangolo sia acutangolo, rettangolo o ottusangolo. Boll. Unione Mat. Ital. 5, 37–41 (1943)
27. Ciliberto, C., Ottaviani, G.: The Hessian map. Int. Math. Res. Not. IMRN 2022(8), 5781–5817 (2022)
28. Conca, A.: Gröbner bases for spaces of quadrics of codimension 3. J. Pure Appl. Algebra 213(8), 1564–1568 (2009)
29. Cook, R.J., Thomas, A.D.: Line bundles and homogeneous matrices. Q. J. Math. Oxford Ser. 30(120), 423–429 (1979)
30. Cossec, F., Dolgachev, I., Liedtke, C.: Enriques Surfaces, I (2021). http://www.math.lsa.umich.edu/~idolga/EnriquesOne.pdf
31. D’Ali, A.: The Koszul property for spaces of quadrics of codimension three. J. Algebra 490, 256–282 (2017)
32. Damon, J.: Thom–Mather theory at 50 years of age (2022) (preprint)
33. Dimca, A.: Geometric approach to the classification of pencils. Geom. Dedicata 14(2), 105–111 (1983)
34. Dimca, A.: Topics on Real and Complex Singularities: An Introduction, Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig (1987)
35. Dimca, A., Gibson, C.G.: Classification of equidimensional contact unimodular map germs. Math. Scand. 56(1), 15–28 (1985)
36. Dixon, A.: Note on the reduction of a ternary quantic to a symmetrical determinant. Proc. Cambridge Philos. Soc. 11, 350–351 (1902)
37. Dolgachev, I.: Lectures on Invariant Theory. London Mathematical Society Lecture Note Series, vol. 296. Cambridge University Press, Cambridge (2003)
38. Dolgachev, I.V.: Classical Algebraic Geometry: A Modern View. Cambridge University Press, Cambridge (2012)
39. Dolgachev, I., Kanev, V.: Polar covariants of plane cubics and quartics. Adv. Math. 98(2), 216–301 (1993)
40. Dolgachev, I., Kondo, S.: Enriques Surfaces II (2021). http://www.math.lsa.umich.edu/~idolga/EnriquesTwo.pdf
41. Domokos, M., Fehér, L.M., Rimányi, E.: Equivariant and invariant theory of nets of conics with an application to Thom polynomials. J. Singul. 7, 1–20 (2013). arXiv:1110.5601
42. Dye, S., Kohn, K., Rydell, F., Sinn, R.: Maximum likelihood estimation for nets of conics. Matematicheski (Catania) 76(2), 399–414 (2021). arXiv:2011:08989
43. Dyment, Z.M.: Maximal commutative nilpotent subalgebras of a matrix algebra of the sixth degree. Vesni Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk 1966(3), 53–68 (1966). (in Russian)
44. Edwards, S.A., Wall, C.T.C.: Nets of quadrics and deformations of $\sigma^{3,3}$ singularities. Math. Proc. Cambridge Philos. Soc. 105(1), 109–115 (1989)
45. Eisenbud, D., Harris, J.: 3264 and All That: A Second Course in Algebraic Geometry. Cambridge University Press, Cambridge (2016)
46. Elias, J., Rossi, M.E.: Isomorphism classes of short Gorenstein local rings via Macaulay’s inverse system. Trans. Amer. Math. Soc. 364(9), 4589–4604 (2012)
47. Ellingsrud, G., Peine, R., Strømme, S.A.: On the variety of nets of quadrics defining twisted cubics. In: Ghione, F., et al. (eds.) Space Curves. Lecture Notes in Mathematics, vol. 1266, pp. 84–96. Springer, Berlin (1987)
48. Emsalem, J.: Géométrie des points épais. Bull. Soc. Math. France 106(4), 399–416 (1978)
49. Emsalem, J., Iarrobino, A.: Réseaux de Coniques et algèbres de longueur sept associées. Université de Paris VII, Paris (1977) (preprint)
50. Emsalem, J., Iarrobino, A.: Some zero-dimensional generic singularities: finite algebras having small tangent spaces. Compositio Math. 36(2), 145–188 (1978)
51. Erman, D., Velasco, M.: A syzygetic approach to the smoothability of zero-dimensional schemes. Adv. Math. 224(3), 1143–1166 (2010)
52. Feng, R., Shen, L.-Y.: Computing the intersections of three conics according to their Jacobian curve. J. Symbolic Comput. 73, 175–191 (2016)
53. Fevola, C., Mandelshtam, Y., Sturmfels, B.: Pencils of quadrics: old and new. Matematiche (Catania) 76(2), 319–335 (2021). arXiv:2009.04334
54. Fruin, H.R.: The group law on elliptic curves on Hesse form. In: Mullen, G.L., et al. (eds.) Finite Fields with Applications to Coding Theory, Cryptography and Related Areas, pp. 123–151. Springer, Berlin (2002)
55. Gordan, P.: Ueber Büschel von Kegelschnitten. Math. Ann. 19(4), 529–552 (1882)
56. Grace, J.H., Young, A.: The Algebra of Invariants. Cambridge Library Collection. Cambridge University Press, Cambridge (2010). Reprint of the 1903 original
57. Happel, D.: Klassifikationstheorie endlich-dimensionaler algebren in der Zeit von 1880 bis 1920. Enseign. Math. 26(1–2), 91–102 (1980)
58. Hartshorne, R.: Deformation Theory. Graduate Texts in Mathematics, vol. 257. Springer, New York (2010)
59. Hassett, B., Tschinke, Yu.: Varieties of planes on intersections of three quadrics. Eur. J. Math. 7(2), 613–632 (2021)
60. Hodge, W.V.D., Pedoe, D.: Methods of Algebraic Geometry, Quadrics and Grassmann Varieties, vol. 2. Cambridge University Press, Cambridge (1952)
61. Huibregtse, M.E.: Some elementary components of the Hilbert scheme of points. Rocky Mountain J. Math. 47(4), 1169–1225 (2017)
62. Iarrobino, A.: Reducibility of the families of 0-dimensional schemes on a variety. Invent. Math. 15, 72–77 (1972)
63. Iarrobino, A.: The number of generic singularities. Rice Univ. Stud. 59(1), 49–51 (1973)
64. Iarrobino, A.: Deforming complete intersection Artin algebras. Appendix: Hilbert function of $\mathbb{C}[x, y]/I$. In: Orlik, P. (ed.) Singularities, Part I. Proceedings of Symposia in Pure Mathematics, vol. 40, pp. 593–608. American Mathematical Society, Providence (1983)
65. Iarrobino, A.: Hilbert scheme of points: Overview of last ten years. In: Bloch, S.J. (ed.) Algebraic Geometry: Bowdoin 1985, Part 2. Proceedings of Symposia in Pure Mathematics, vol. 46.2, pp. 297–320. American Mathematical Society, Providence (1987)
66. Ishitsuka, Y., Ito, T.: The local-global principle for symmetric determinantal representations of smooth plane curves. Ramanujan J. 43(1), 141–162 (2017)
67. Jelisiejew, J.: Classifying local Artinian Gorenstein algebras. Collect. Math. 68(1), 101–127 (2017)
68. Jelisiejew, J.: Elementary components of Hilbert schemes of points. J. London Math. Soc. 100(1), 249–272 (2019)
69. Jordan, C.: Réduction d’un réseau de formes quadratiques ou bilinéaire. J. Math. Pures Appl. 6, 403–438 (1906)
70. Kaygorodov, I., Rakhimov, I., Said Husain, Sh.K.: The algebraic classification of nilpotent associative commutative algebras. J. Algebra Appl. 19(11), 2050220 (2020)
71. Kline, M.: Mathematical Thought from Ancient to Modern Times. Oxford University Press, New York (1972)
72. Lavrauw, M., Popiel, T., Sheekey, J.: Nets of conics of rank one in $\text{PG}(2, q)$, $q$ odd. J. Geom. 111(3), 36 (2020)
73. Lee, M., Patel, A., Tseng, D.: Equivariant degenerations of plane curve orbits (2019). arXiv:1903.10069
74. Mather, J.N.: Stability of $C^\infty$ mappings: IV: Classification of stable germs by $R$-algebras. Inst. Hautes Études Sci. Publ. Math. 37, 223–248 (1969)
75. Mazzola, G.: Generic finite schemes and Hochschild cocycles. Comment. Math. Helv. 55(2), 267–293 (1980)
76. Michałek, M., Moon, H., Sturmfels, B., Ventura, E.: Real rank geometry of ternary forms. Ann. Mat. Pura Appl. 196(3), 1025–1054 (2017)
77. Molk, J.: Encyclopédie des sciences mathématiques. III 3. Géométrie algébrique plane. Edition française, Gauthier-Villars, Paris, 1898–1904; Edition Allemande: F. Meyers, ed., Teubner, Leipzig (1914); Reprint (2000) Editions J. Gabay (Paris)
78. Okonek, Ch., Van de Ven, A.: Cubic forms and complex 3-folds. Enseign. Math. 41(3–4), 297–333 (1995)
79. Onda, N.: Isomorphism types of commutative algebras of finite rank 7 over an arbitrary algebraically closed field of characteristic not 2 or 3 (2021). arXiv:2107.04959
80. Ottaviani, G.: An invariant regarding Waring’s problem for cubic polynomials. Nagoya Math. J. 193, 95–110 (2009)
81. Ottaviani, G.: Five lectures on projective invariants. Rend. Semin. Mat. Univ. Politec. Torino 71(1), 119–194 (2013)
82. Pavlov, I.A.: Commutative nilpotent matrix algebras of class $n – 3$: I. Vesci Akad. Navuk BSSR Ser. Fiz.–Mat. Navuk 1968(4), 39–46 (1968). (in Russian)
83. Pavlov, I.A.: Commutative nilpotent matrix algebras of class $n – 3$: II. Vesci Akad. Navuk BSSR Ser. Fiz.–Mat. Navuk 1968(5), 10–16 (1968). (in Russian)
84. Poonen, B.: Isomorphism types of commutative algebras of finite rank over an algebraically closed field. In: Lauter, K.E., Ribet, K.A. (eds.) Computational Arithmetic Geometry. Contemporary Mathematics, vol. 463, pp. 111–120. American Mathematical Society, Providence (2008). (Revised): http://www.math.mit.edu/~poonen/papers/dimension6.pdf
85. Poonen, B.: The moduli space of commutative algebras of finite rank. J. Eur. Math. Soc. (JEMS) 10(3), 817–836 (2008)
86. Reid, M.: The Complete Intersection of Two or More Quadrics. Ph.D. Thesis, Trinity College, Cambridge (1972). https://homepages.warwick.ac.uk/~masda/3folds/qq.pdf
87. Ronga, F.: Applications polynomiales de degré deux du plan projectif complexe dans lui-même. Enseign. Math. 22(1–2), 41–54 (1976)
88. Rosanes, M.J.: Erweiterung eines bekannten Satzes auf Formen von beliebig vielen Veränderlichen. Math. Ann. 23(3), 412–415 (1884)
89. Rydh, D., Skjelnes, R.: An intrinsic construction of the principal component of the Hilbert scheme. J. London Math. Soc. 82(2), 459–481 (2010)
90. Segre, C., Rosanes, M.J.: Sur les invariants simultanés de deux formes quadratiques. Math. Ann. 24(1), 152–156 (1884)
91. Salmon, G.: A Treatise on the Higher Plane Curves: Intended as a Sequel to "A Treatise on Conic Sections". 3rd edn. Chelsea Publishing, New York (1960)
92. Satriano, M., Staal, A.P.: Small elementary components of Hilbert schemes of points (2021) arXiv:2112.01481
93. Segre, B.: Intorno alla geometria sopra un campo di caratteristica due. Rev. Fac. Sci. Univ. Istanbul. Sér. A 21, 97–123 (1956)
94. Segre, C.: Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni. Mem. della R. Acc. delle Scienze di Torino 36(2), 3–86 (1883)
95. Sernesi, E.: Deformations of Algebraic Schemes. Grundlehren der Mathematischen Wissenschaften, vol. 334. Springer, Berlin (2006)
96. Serre, J.P.: Cours d’arithmétique. Collection SUP: “Le Mathématicien”. 2 Presses Universitaires de France, Paris (1970)
97. Shafarevich, I.R.: Deformations of commutative algebras of class 2. Leningrad Math. J. 2(6), 1335–1351 (1991)
98. Shafarevich, I.R.: Degeneration of semisimple algebras. Comm. Algebra 29(9), 3943–3960 (2001)
99. Sivasuki, A.S.: On zeros of a system of quadratic forms in 3 variables. J. Algebra 449, 237–245 (2016)
100. Sturmfels, B.: Algorithms in Invariant Theory: Texts and Monographs in Symbolic Computation, 2nd edn. Springer, Vienna (2008)
101. Suprunenko, D.A.: On maximal commutative subalgebras of the full linear algebra. Uspehi Mat. Nauk (N.S) 11(69), 181–184 (1956). (in Russian)
102. Suprunenko, D.A., Tyškevič, R.I.: Commutative Matrices. Academic Press (1968)
103. Szachniewicz, M.: Non-reducedness of the Hilbert schemes of few points (2021) arXiv:2109.11805
104. Szafarczyk, R.: New elementary components of the Gorenstein locus of the Hilbert scheme of points (2022) arXiv:2206.03732
105. Tjøtta, E.: Rational curves on the space of determinantal nets of conics (1998) arXiv:math/9802037
106. Tjurin, A.N.: An invariant of a net of quadrics Izv. Akad. Nauk SSSR Ser. Mat. 39, 23–27 (1975). (in Russian)
107. Tjurin, A.N.: The intersection of quadrics. Russian Math. Surveys 30(6), 51–105 (1975)
108. Verra, A.: Superfici di Enriques et reti di quadriche. Ann. Mat. Pura Appl. 130, 307–320 (1982)
109. Verra, A.: On Enriques surface as a fourfold cover of $\mathbb{P}^2$. Math. Ann. 266(2), 241–250 (1983)
110. Verra, A.: Edge and Fano on nets of quadrics. Eur. J. Math. 4(3), 1264–1277 (2018)
111. Wall, C.T.C.: Nets of conics. Math. Proc. Cambridge Philos. Soc. 81(3), 351–364 (1977)
112. Wall, C.T.C.: Nets of quadrics, and theta-characteristics of singular curves. Philos. Trans. R. Soc. London Ser. A 289(1357), 229–269 (1978)
113. Wall, C.T.C.: Singularities of nets of quadrics. Compositio Math. 42(2), 187–212 (1980/81)
114. Wang, X.: Maximal linear spaces contained in the based loci of pencils of quadrics. Algebr. Geom. 5(3), 359–397 (2018)
115. Weierstrass, K.: Zur Theorie der bilinearen und quadratischen Formen. Berliner Monatsberichte, 310–338 (1868)
116. Wilson, A.H.: The canonical types of nets of modular conics. Am. J. Math. 36(2), 187–210 (1914)
117. Zanello, C.: On finite Steiner surfaces. Discrete Math. 312(3), 652–656 (2012)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.