A new look at anomaly cancellation in heterotic $M$-theory

Ian G. Moss

School of Mathematics and Statistics, University of Newcastle Upon Tyne, NE1 7RU, UK

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This paper considers anomaly cancellation for eleven-dimensional supergravity on a manifold with boundary and theories related to heterotic $M$-theory. The Green-Schwarz mechanism is implemented without introducing distributions. The importance of the supersymmetry anomaly in constructing the low energy action is discussed and it is argued that a recently proposed action for heterotic $M$-theory gives a supersymmetric theory to all orders in the gravitational coupling $\kappa$.

I. INTRODUCTION

Horava and Witten have argued that the strong coupling limit of the ten-dimensional heterotic string is eleven-dimensional supergravity with gauge multiplets confined to two ten-dimensional hypersurfaces forming the boundary of the eleven-dimensional spacetime manifold [1, 2]. This theory is a very promising starting point for phenomenological models based on compactifications to four dimensions (see, for example, [3, 4, 5, 6, 7]). In applications such as these, it is important to know the action in as much detail as possible.

The form of the action originally put forward was found by relying partly on anomaly cancellation and supersymmetry. Gauge and gravitational anomalies in the theory cancel via a novel modification of the Green-Schwarz mechanism involving the supergravity 3-form. The cancellation, which involves some remarkable algebraic coincidences, requires that the matter action contains a factor of order $\kappa^{2/3}$ compared to the supergravity action, where $\kappa$ is the eleven-dimensional gravitational coupling strength.

Imposing local supersymmetry on the action fixes all of the terms at order $\kappa^{2/3}$. However, when the same procedure is applied to order $\kappa^{4/3}$, singular terms depending on the square of the delta-function start to arise. This problem has recently been overcome by modifying the boundary conditions on the gravitino and the supergravity 3-form, so that now an action can be constructed which is non-singular and supersymmetric to higher orders [8, 9]. The effect of these boundary conditions on anomaly cancellation is one of the issues to be addressed in this paper.

As we extend the theory to higher orders in the gravitational coupling, we have to take account of the supersymmetry anomaly which appears at order $\kappa^2$ (i.e. $\kappa^2$ times the gravitational action). The existence of a supersymmetry anomaly implies that the classical action should not be supersymmetric at this order. However, it is reasonable to suppose that the supersymmetry anomaly, like the gauge anomaly, is cancelled by the Green-Schwarz mechanism, and the action should therefore be supersymmetric up to the variation of the Green-Schwarz terms [10, 11]. This was not appreciated in [8], where it was shown that the supersymmetric variation of the new action for heterotic $M$-theory reduced to a single term of precisely this type. Now, taking into account the supersymmetry anomaly, this theory appears to be supersymmetric to all orders in $\kappa$, at least when truncated to terms up to first order in the Riemann tensor.

A heuristic argument for cancellation of the supersymmetry anomaly by the Green-Schwarz terms can be made from the Wess-Zumino consistency conditions, which relate the supersymmetry anomaly to the gauge anomaly [10, 11]. When the Green-Schwarz terms are added to the effective action, the total has vanishing gauge variation. Therefore, provided that the consistency conditions have a unique solution, the variation of the Green-Schwarz terms should cancel the supersymmetry anomaly as well.

We shall consider the gauge and supersymmetry anomaly cancellation in more detail. We work throughout on the ‘downstairs picture’ of a manifold with boundary, rather than lifting to the covering space $R^{10} \times S^1$. For the present, we truncate the action to first order in the Riemann tensor. The gauge anomaly from the chiral fermion on one of the boundary components can be described by a formal 12-form $I_{12}(F)$ [12]. To generate the anomaly, we introduce the notation $T$, such that locally $dT \omega = \omega$ for a closed form $\omega$. The anomalous variation of the chiral fermion effective action under gauge transformations $\delta_a$ is given by integrating a 10-form $I^A_{10}$, defined by

$$I^A_{10} = T \delta_a T I_{12}$$

The anomalous variation under supersymmetry variations is given by the sum of two other 10-forms, $I^S_{10} + I'^{S'}_{10}$, where

*Electronic address: ian.moss@ncl.ac.uk*
according to \[10, 11\],

$$I^{S}_{10} = l_\eta TI_{12}$$  \hspace{1cm} (2)

and \(I^{S'}_{10}\) is gauge invariant. The anti-derivative \(l_\eta\) is defined by \(l_\eta A = 0\) and \(l_\eta F = \delta_\eta A\).

In the case of the gauge group \(E_8\), \((4\pi)^2 I_{12} = (\text{tr}F^2)^3/12\) and we have

$$I^A_{10} = \frac{1}{12(4\pi)^2} \text{tr}(\alpha F)(\text{tr}F^2)^2$$  \hspace{1cm} (3)

$$I^C_{10} = \frac{1}{12(4\pi)^2} \text{tr}(\delta_\eta AA)(\text{tr}F^2)^2 + \frac{1}{3(4\pi)^2} \text{tr}(\delta_\eta AF)(\text{tr}F^2)^2.$$  \hspace{1cm} (4)

Now the observation of Horava and Witten was that \(I^A_{10}\) can be cancelled by a variation of the \(CGG\) term in the supergravity action \[12\]. This can be done by requiring \(G \sim \text{tr}F^2\) on the boundary and \(\delta_\alpha C \sim \delta(x^{11})\text{tr}(\alpha F)\). If we follow this route further, we are eventually lead to the theory with \(\delta(x^{11})^2\) terms \[2\].

An alternative way to arrange the Green-Schwarz cancellation was first described in \[14\]. If we let \(\delta_\alpha C \sim da\), where \(a\) is any 2-form which satisfies \(a = \text{tr}(\alpha F)\) on the boundary, and require that \(G \sim \text{tr}F^2\) on the boundary, then the variation of the Green-Schwarz term is a total derivative,

$$\delta CGG \sim d(\alpha GG).$$  \hspace{1cm} (5)

This integrates to give a term which can cancel the anomaly \[8\]. Similarly, if we add an extra supersymmetry variation \(\delta_\eta C \sim df\) to the 3-form, where \(f = \text{tr}(\delta_\eta AA)\) on the boundary, then part of the supersymmetry anomaly \(I^C_{10}\) is cancelled \[8\]. (The rest of the anomaly is cancelled by the usual transformation of \(C\), as we shall see shortly).

The gauge and supersymmetry variations of \(C\) are precisely those which are required to maintain the gauge and supersymmetry invariance of the 3-form boundary condition given in \[8\], namely

$$C = \frac{\sqrt{2}}{12} \epsilon (\omega_{3Y} + \omega_\chi)$$  \hspace{1cm} (6)

where \(\omega_{3Y}\) is the Chern-Simons form \(T\text{tr}F^2\) and

$$\omega_\chi = \frac{1}{4} \chi^a \Gamma_{ABC} \chi^a.$$  \hspace{1cm} (7)

The constant \(\epsilon\) is fixed by supersymmetry to the normalisation of the matter action. It is related to the gauge coupling \(\lambda\) by \(\epsilon = \kappa^2/2\lambda^2\). Since the Chern-Simons form has a gauge transformation \(\delta_\alpha \omega_{3Y} = d(\text{tr}\alpha F)\), the boundary condition remains valid if the variation of \(C\) is given by

$$\delta_\alpha C = \frac{\sqrt{2}}{12} \epsilon da$$  \hspace{1cm} (8)

where \(a = \text{tr}(\alpha F)\) on the boundary. (Some details of the use of \(p\)-form boundary conditions in quantum field theory can be found in \[13, 10\]. A more careful treatment would consider the Abelian BRST variations of the boundary condition, but these are similar in form to the Abelian gauge variations.)

The fermion term \(7\) in the boundary condition is required to make the boundary condition supersymmetric. It also plays an important role in obtaining the correct ten-dimensional reduction. Unfortunately, the gaugino enters into the variation of the \(CGG\) term through the value of \(G = 6dC\) on the boundary,

$$G = \frac{\epsilon}{\sqrt{2}} (\text{tr} F^2 + d\omega_\chi).$$  \hspace{1cm} (9)

In order to avoid spoiling the anomaly cancellation, we have to add extra boundary corrections to the Green-Schwarz terms. The \(CGG\) term is taken from the usual supergravity action (with gravitational coupling \(\kappa^2/2\) \[17\]),

$$S_C = \frac{2\sqrt{2}}{\kappa^2} \int_{\mathcal{M}} CGG.$$  \hspace{1cm} (10)

The boundary terms can only involve \(\omega_{3Y}, \omega_\chi\) and \(F\) and they must vanish when \(\omega_\chi = 0\). The unique combination which has the desired effect is

$$S_3 = -\frac{1}{6\kappa^2} \oint_{\partial\mathcal{M}} \omega_{3Y} \omega_\chi (2\text{tr}F^2 + d\omega_\chi).$$  \hspace{1cm} (11)
The variation of \( S_C \) and \( S_3 \) using (8) is

\[
\delta_\alpha S_C + \delta_\alpha S_3 = -\frac{e^3}{6\kappa^2} \int_{\partial M} \text{tr}(\alpha F)(\text{tr}F^2)^2.
\]

(12)

The variation cancels the gauge anomaly (8) and fixes the value of \( \epsilon \),

\[
\epsilon = \frac{1}{4\pi} \left( \frac{\kappa}{4\sqrt{2}\pi} \right)^{2/3}.
\]

(13)

This agrees with [17], which corrected a factor of 2 in [2]. The result differs by a factor of 3 from the one obtained on the covering space in [18]. The difference is possibly due to the way in which the theory is lifted to the covering space.

If our assumptions are correct, then the supersymmetric variation of the Green-Schwarz terms should now cancel the supersymmetry anomaly,

\[
\delta_\eta S_C + \delta_\eta S_3 + \int (I_{10}^S + I_{10}^\omega) = 0.
\]

(14)

A supersymmetry variation of \( S_C \) and \( S_3 \) allows us to read off the non-gauge-invariant part of the anomaly

\[
I_{10}^S = \frac{e^3}{\kappa^2} \text{tr}(\delta_\eta A A)(\text{tr}F^2)^2 + \frac{2e^3}{3\kappa^2} \text{tr}(\delta_\eta AF)\omega_3 Y \text{tr}F^2.
\]

(15)

This is in complete agreement with (11), proving that this part of the supersymmetry anomaly does indeed cancel. We also generate the gauge invariant part of the supersymmetry anomaly,

\[
I_{10}^{\omega'} = \frac{e^3}{\kappa^2} \text{tr}(\delta_\eta AF)\omega_\chi (2 \text{tr}F^2 + d\omega_\chi) + \frac{e^3}{6\kappa^2} (\delta_\eta \omega_\chi) \omega_\chi (3 \text{tr}F^2 + 2d\omega_\chi).
\]

(16)

In these expressions, we have local supersymmetry transformations

\[
\delta_\eta A = \frac{3}{2} \eta \Gamma_\chi
\]

\[
\delta_\eta \omega_\chi = \frac{1}{8} \eta \Gamma_{ABC} \Gamma_{DE} \chi^a \hat{F}_{DE}^a + \frac{3}{8} \eta \Gamma_D \psi [A \chi^a \Gamma^D BC] \chi^a,
\]

(17)

(18)

where \( \hat{F}_{AB} = F_{AB} - \hat{\psi}_B \Gamma_B \chi \). The supersymmetry anomaly in ten dimensions has been calculated previously up to four fermi terms [19, 20]. Our result has a similar form, although a direct comparison is not worthwhile because our result is specific to the gauge group \( E_8 \) and contains contributions from the eleventh dimension (indicated by the presence of the gravitino field \( \psi_\chi \)).

We can make use of the anomaly (15) in connection with the action of heterotic \( M \)-theory. The action \( S \) proposed in [9] consisted of usual supergravity action with boundary terms \( S_0 \) and a boundary matter action

\[
S_1 = -\frac{2e}{\kappa^2} \int_{\partial M} dv \left( \frac{1}{4} F_{AB} F^{AB} + \frac{1}{2\sqrt{2}} \chi^a \Gamma^A \Omega^* \chi^a + \frac{1}{4} \hat{\psi}_A \Gamma^A \Gamma^B \Gamma^C \Gamma^D \chi^a \right),
\]

(19)

where \( F^* = (F + \hat{F})/2 \), \( \Omega \) is the supergravity spin connection and \( \Omega^{*\ ABC} = \Omega^{ABC} + \frac{1}{2\sqrt{2}} \psi^D \Gamma_{ABC} \psi^E \). We have discovered a new result that we must also add the term \( S_3 \) for the anomaly cancellation to work properly. In [8], it was shown that the supersymmetric variation of the action was

\[
\delta_\eta S = \frac{2\sqrt{3}}{\kappa^2} \int_{\partial M} \delta_\eta C C G,
\]

(20)

up to one possible four fermi term and all orders in \( \kappa \). We now recognise this as the variation of the Green-Schwarz term, and therefore cancels with the supersymmetry anomaly. The extra four fermi terms in (10) explain also why there was a four-fermi term left in the variation. Up to the limitations of truncating out the higher order curvature terms, the action \( S = S_0 + S_1 + S_3 \) describes a theory which is supersymmetric to all orders in \( \kappa \).

The treatment of gauge and gravitational anomalies in the original Horava-Witten model included terms which are higher order in the Riemann tensor [2]. The 12-form which generates the anomalies was obtained from the gauginos and boundary effects on the gravitino [1],

\[
I_{12} = \frac{1}{12(2\pi)^3} (I_4^3 - 4I_4 X_8),
\]

(21)
where \( I_4 = \text{tr} F^2 - \frac{1}{3} \text{tr} R^2 \) and \( X_8 = -\frac{1}{4} \text{tr} R^4 + \frac{1}{2} (\text{tr} R^2)^2 \). The gauge, gravitational and supersymmetry anomalies due to the first term in \( I_{12} \) can be removed by the \( CGG \) term as described above, provided that the boundary condition on \( C \) is modified,

\[
C = \frac{\sqrt{2}}{12} \left( \omega_3 Y - \frac{1}{2} \omega_3 L + \omega_\chi - \frac{1}{2} \omega_\psi \right),
\]

where \( \omega_3 = T \text{tr} R^2 \) and \( \omega_\psi \), according to dimensional analysis, is bilinear in the derivative of the gravitino. The construction of a fully supersymmetric theory with this boundary condition has not yet been done, but it seems inevitable that \( R^2 \) terms will also appear in the action \[21, 22\]. These terms would be needed to ascertain the precise form of \( \omega_\psi \).

Similarly, the \( X_8 \) terms in the gauge anomaly should be cancelled by a Green-Schwarz term \( CX_8 \) in eleven dimensions \[2, 21\]. This can be done with \( \delta \omega C \sim da \) as above, but the cancellation is not exact, because there are extra terms in the eleven dimensional curvature appearing in the Green-Schwartz term which are not present in the ten dimensional curvature part of the anomaly. These would be removed by adding additional boundary terms, or possibly by finding new contributions to the anomaly.

In conclusion, it is possible to cancel the gauge anomalies in eleven dimensional supergravity with boundaries without introducing singular gauge transformations. The \( CGG \) term in the supergravity action acts as a Green-Schwarz term, but with fermions present in the boundary conditions it is necessary to introduce an extra boundary term depending on the gaugino field. It is interesting that the boundary conditions and action appear to be well-determined from gauge and supersymmetry invariance without making any use of the covering space. This agrees with recent work by van Nieuwenhuizen and Vassilevich \[23\], who have found that supersymmetry severely restricts the boundary conditions for pure supergravity. Given also that eleven dimensional supergravity with more than two boundaries can now be consistently formulated (at least as \( \kappa \to 0 \)) \[24\], it looks increasingly likely that the manifold with boundary picture is the more fundamental way of formulating heterotic \( M \)-theory.

We have seen the supersymmetry anomaly has to be taken into account when constructing the action and, at least in the limit of small curvature, there is an action for supergravity with boundary matter which gives a fully supersymmetric quantum field theory. However, dimensional reduction to four dimensions involves curvature terms in the internal dimensions which are not small, but comparable in size to the gauge field strength \[8, 11, 14, 17\]. It would be very desirable to find a supersymmetric action which includes the \( R^2 \) terms suggested by the gauge and gravity anomalies, and then we would have confidence in using the theory as a basis for particle physics phenomenology.

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