Effects of spacetime curvature on spin-1/2 particle
zitterbewegung

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Abstract
This paper investigates the properties of spin-1/2 particle zitterbewegung in the presence of a general curved spacetime background described in terms of Fermi normal coordinates, where the spatial part is expressed using general curvilinear coordinates. Adopting the approach first introduced by Barut and Bracken for zitterbewegung in the local rest frame of the particle, it is shown that non-trivial gravitational contributions to the relative position and momentum operators appear due to the coupling of zitterbewegung frequency terms with the Ricci curvature tensor in the Fermi frame, indicating a formal violation of the weak equivalence principle. Explicit expressions for these contributions are shown for the case of quasi-circular orbital motion of a spin-1/2 particle in a Vaidya background. Formal expressions also appear for the time derivative of the Pauli–Lubanski vector due to spacetime curvature effects coupled to the zitterbewegung frequency. Also, the choice of curvilinear coordinates results in non-inertial contributions in the time evolution of the canonical momentum for the spin-1/2 particle, where zitterbewegung effects lead to stability considerations for its propagation, based on the Floquet theory of differential equations.

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1. Introduction

The concept of zitterbewegung or ‘trembling motion’ in German due to the rapid oscillation of a spin-1/2 particle about its classical worldline is one that still generates great interest about foundational issues in quantum mechanics, even after over 70 years since its initial discovery by Schrödinger [1]. When investigating the free-particle motion of an electron via the Dirac equation, he showed that its instantaneous velocity \( v_{\text{inst}} \) equals the speed of light \( c \), which...
clearly disagrees with the fact that a massive particle’s observed speed will always be \( v_{\text{obs}} < c \). However, Dirac convincingly emphasized [2] that this apparent contradiction is resolved by noting that \( v_{\text{obs}} \) is really an average velocity determined by measuring the electron’s position over a sufficiently small time interval \( \Delta t \). When \( c\Delta t \gg \hbar/mc \), the Compton wavelength for the electron of mass \( m \), it follows that the practical measurement of \( v_{\text{obs}} \) always leads to a preservation of causality, and without any complications due to \textit{zitterbewegung}. Nevertheless, this highly non-intuitive oscillatory behaviour identified to exist for massive spin-1/2 particles is a conceptually rich feature of relativistic quantum mechanics that still offers both thought-provoking challenges and opportunities for deeper exploration [3–5], especially when applied to more complicated situations.

While past and recent investigations of \textit{zitterbewegung} since Schrödinger’s and Dirac’s studies have been largely motivated by attempts to understand the intrinsic structure of the electron [6], there currently exists an intensive practical study of the concept within the context of condensed-matter theory and experiment. Recent explorations of this phenomenon involving two-dimensional carbon sheets known as \textit{graphene} have been proposed [7, 8], where \textit{zitterbewegung} effects are potentially observable in the non-relativistic limit of the Dirac equation, while a second paper claims a direct observation for photons in a two-dimensional photonic crystal [9], and a third paper proposes detection of \textit{zitterbewegung} using ultracold neutral atoms [10]. These studies are especially interesting given the largely accepted understanding that free-particle \textit{zitterbewegung} arises only in situations where positive- and negative-energy solutions to the Dirac equation can interact, allowing for quantum interference effects to manifest the rapid oscillation terms that would otherwise remain decoupled and unobservable.

This leads to the following consideration where it concerns interactions in a gravitational background. Given that external fields can introduce non-trivial effects that lead to observable consequences for electron \textit{zitterbewegung}, and given that gravitation is best described (for now) by Einstein’s theory of general relativity in terms of the dynamical curvature of spacetime due to external matter, it is reasonable to envision curved spacetime as something like a medium with an energy density content not unlike that of an electromagnetic background. Of course, there is the fundamental distinction that Einstein’s equivalence principle is an integral part of general relativity, reflected mathematically as diffeomorphism invariance under coordinate transformations. This is certainly reasonable when considering purely classical descriptions of matter, where it is meaningful to speak about its localization within a compact spatial region. However, conceptual difficulties arise when considering quantum matter in a curved spacetime background, where issues of non-locality due to the wave-particle duality of quantum objects overlap with classical descriptions of spacetime curvature. This is a particularly relevant issue considering that gravitational effects are manifested locally from observing tidal forces via the geodesic deviation equation, which requires the existence of a neighbouring geodesic in relation to the reference worldline. It is very unclear how to disentangle the competing effects of quantum non-locality and the locality of classical gravitation in a way that results in meaningful statements about quantum matter effects in a curved spacetime background, including \textit{zitterbewegung}.

The purpose of this paper is to explore the physical properties of spin-1/2 particle \textit{zitterbewegung} in a curved spacetime background described locally in terms of Fermi normal coordinates. Its motivation is based on understanding whether spatial fluctuations about the particle’s worldline lead to non-trivial gravitational corrections that influence its propagation in spacetime. A recent innovation in the standard formalism of Fermi normal coordinates is to describe the spatial fluctuations in terms of curvilinear spatial coordinates [11–13]. This paper begins with a brief review of the existing treatment of \textit{zitterbewegung} in a flat spacetime
background, as given in section 2. Because the Fermi normal coordinate system is defined locally about the particle’s worldline, it is important to make use of a local perspective for the occurrence of zitterbewegung and explore its physical consequences. It so happens that the approach introduced by Barut and Bracken [14] is particularly well suited for this purpose, and so this perspective is adopted for consideration in a curved spacetime background. This leads to a brief introduction of the covariant Dirac equation in Fermi normal coordinates in section 3, which includes the curvilinear coordinate generalization. A presentation of the corresponding Dirac Hamiltonian derived within this framework is given in section 4, where contributions due to zitterbewegung reveal a formal violation of the weak equivalence principle due to direct coupling of the gravitational field with the particle’s mass. From this Hamiltonian, it becomes possible to compute the time evolution of quantum operators in section 5 and show how zitterbewegung formally appears in the expressions. Particular attention is given to the time evolution of the position and momentum operators, and also the covariant spin operator described by the Pauli–Lubanski vector. A brief conclusion follows in section 6.

For this paper, geometric units of $G = c = 1$ are adopted, while the curvature tensor definitions follow the conventions given by Misner, Thorne and Wheeler [15], but with metric signature $-2$. Also, the flat spacetime gamma matrices follow the conventions of Itzykson and Zuber [16].

2. Review of free-particle spin-1/2 zitterbewegung in flat spacetime

2.1. General formalism

Before introducing zitterbewegung in curved spacetime, it is useful to first review the original treatment of the problem in flat spacetime by Schrödinger [1, 2, 14]. Assuming spacetime coordinates in Cartesian form $X^\hat{\mu} = (T, X)$, the Schrödinger equation is

$$i\hbar \frac{\partial}{\partial T} \psi(X) = H_0 \psi(X),$$

(1)

is the free-particle Dirac Hamiltonian defined in terms of spin-1/2 particle mass $m$, canonical momentum $P_{\hat{\mu}} = i\hbar \partial / \partial X^\hat{\mu}$, and $4 \times 4$ matrices $\alpha_{\hat{\mu}} = \gamma^0 \gamma_{\hat{\mu}}$ and $\beta_{\hat{\mu}} = \gamma^0$ with $[\gamma^\hat{\alpha}, \gamma^\hat{\beta}] = 2\eta^{\hat{\alpha}\hat{\beta}}$ [16]. As expected, the canonical momentum operators satisfy

$$[P_{\hat{\mu}}^{(0)}, P_{\hat{\nu}}^{(0)}] = 0,$$

(2)

while the corresponding position operators $X_{\hat{\mu}}^{(0)}$ satisfy

$$[X_{\hat{\mu}}^{(0)}, X_{\hat{\nu}}^{(0)}] = 0, \quad [X_{\hat{\mu}}^{(0)}, P_{\hat{\nu}}^{(0)}] = -i\hbar \delta_{\mu}^{\nu}.$$  

(3)

Adopting the Heisenberg picture, it follows that the time derivative of some operator $A$ is

$$\frac{dA(T)}{dT} = i \frac{\hbar}{\hbar} [H_0, A(T)].$$

(4)

Clearly, $dP_{\hat{\mu}}^{(0)}(T)/dT = dH_0/dT = 0$, while the instantaneous velocity and acceleration operators are

$$\frac{dX_{\hat{\mu}}^{(0)}(T)}{dT} = \alpha_{\hat{\mu}}^{(0)}(T),$$

$$\frac{d^2X_{\hat{\mu}}^{(0)}(T)}{dT^2} = \frac{d\alpha_{\hat{\mu}}^{(0)}(T)}{dT} = -\frac{2i}{\hbar} \eta_{\hat{\mu}\nu} H_0,$$

(5)

(6)
where
\[ \eta^j_{(0)}(T) = \alpha^j_{(0)}(T) - P^j_{(0)}(T)H_0^{-1}. \] (7)
Because \( P^j_{(0)}(T) \) and \( H_0 \) are constants of the motion, it is also true that
\[ \frac{d\eta^j_{(0)}(T)}{dT} = \frac{d}{dT} [\alpha^j_{(0)}(T) - P^j_{(0)}(T)H_0^{-1}] = \frac{d\alpha^j_{(0)}(T)}{dT} = -\frac{2i}{\hbar}\eta^j_{(0)}(T)H_0. \] (8)
An immediate solution follows for \( \eta^j_{(0)}(T) \), such that
\[ \eta^j_{(0)}(T) = \eta^j_{(0)}(0)e^{-2i\hbar T/\hbar}, \] (9)
or
\[ \alpha^j_{(0)}(T) = P^j_{(0)}(0)H_0^{-1} + \eta^j_{(0)}(0)e^{-2i\hbar T/\hbar}. \] (10)
Direct integration of (10) then leads to the free-particle solution for \( X^j_{(0)}(T) \), such that the time evolution for the position and canonical momentum operators are
\[ X^j_{(0)}(T) = X^j_{(0)}(0) + (P^j_{(0)}(0)H_0^{-1})T + \frac{\hbar}{2}\eta^j_{(0)}(0)H_0^{-1}[\sin(2H_0 T/\hbar) - 2i\sin^2(H_0 T/\hbar)], \] (11)
\[ P^j_{(0)}(T) = P^j_{(0)}(0). \] (12)
The first two terms of (11) denote the expected mean position as a function of time \( T \), while the remaining terms show the rapid oscillation effect superimposed about the spin-1/2 particle’s worldline due to zitterbewegung. For future reference, it is straightforward to show that zitterbewegung also applies to \( \beta_{(0)}(T) \) and the Pauli spin matrix \( \sigma^{j}_{(0)}(T) \), such that
\[ \beta_{(0)}(T) = \beta_0 e^{-2i\hbar T/\hbar} + mH_0^{-1}[2\sin^2(H_0 T/\hbar) + i\sin(2H_0 T/\hbar)], \] (13)
\[ \sigma^{j}_{(0)}(T) = \sigma^{j}_{(0)}(0) + i(\alpha_0 \times P^{j}_{(0)}(0))H_0^{-1}[2\sin^2(H_0 T/\hbar) + i\sin(2H_0 T/\hbar)], \] (14)

where
\[ \alpha^j_0 \equiv \alpha^j_{(0)}(0), \quad \beta_0 \equiv \beta_{(0)}(0), \quad \sigma^j_0 \equiv \sigma^{j}_{(0)}(0). \] (15)

Before continuing, it is worthwhile to explore the mathematical details of the operators which describe the zitterbewegung phenomenon. Based on a precisely formulated description of the problem [17], it is possible to establish that the domain \( \mathcal{D} \) corresponding to the position operator \( X^j_{(0)}(0) \) is invariant under time evolution, such that
\[ \mathcal{D}(X^j_{(0)}(T)) = e^{-i\hbar H_0 T/\hbar} \mathcal{D}(X^j_{(0)}(0)) = \mathcal{D}(X^j_{(0)}(0)). \] (16)
As such, (16) implies that \( X^j_{(0)}(T) \) remains bounded for all finite \( T \). In addition, the expectation value for the time-evolved position operator is well behaved, in that the component of (11) exhibiting zitterbewegung integrates to zero in the expectation value as \( T \to \infty \). Furthermore, it is well known [17–19] that the Hilbert space identified with solutions of the free-particle Dirac equation decouples into invariant subspaces defined by positive and negative energy states, respectively. This implies that the expectation values of \( X^j_{(0)}(T) \) formed by exclusively positive energy or negative energy states never reveal any zitterbewegung-induced effects. Not only does it explain why zitterbewegung is not an observable for a propagating free particle, it also suggests the existence of a breakdown in the single-particle treatment of quantum matter, justifying the need for the standard quantum field theory description.
2.2. The Barut–Bracken approach to zitterbewegung

The main expressions above can apply to spin-1/2 particle motion in a general reference frame with $P^j_{(0)}(T) \neq 0$. Schrödinger’s original view of zitterbewegung was to search for ‘microscopic’ degrees of freedom for position, momentum and angular momentum to explain the electron’s rapid oscillatory behaviour as a separable effect superimposed on its observable motion [1, 14]. To this end, he proposed the existence of a microscopic position operator $\hat{\xi}^j(T)$, according to

$$X^j_{(0)}(T) = X^j_{(A)}(0) + \hat{\xi}^j(T), \quad (17)$$

where

$$X^j_{(A)}(T) = X^j_{(0)}(0) + \left( P^j_{(0)}(0)H_0^{-1} \right) T - \frac{i\hbar}{2} \eta^j_{(0)}(0)H_0^{-1}, \quad (18)$$

$$\hat{\xi}^j(T) = \frac{i\hbar}{2} \eta^j_{(0)}(0)H_0^{-1} e^{-2iH_0T/\hbar}, \quad (19)$$

along with a microscopic momentum associated with $\hat{\xi}^j(T)$, in the form

$$\eta^j_{(0)}(T)H_0 + P^j_{(0)}(T) = \frac{dX^j_{(0)}(T)}{dT}H_0, \quad (20)$$

In re-examining zitterbewegung with the motive to understand the electron’s intrinsic structure for addressing self-energy and renormalization issues in QED, Barut and Bracken [14] saw no difficulty with Schrödinger’s formulation of microscopic position, according to (17). However, they had serious issues with his definition of microscopic momentum (20), primarily motivated by the claim that an equally attractive—but incompatible—definition could be proposed within his formalism, such that

$$H_0\hat{\eta}^j_{(0)}(T) + P^j_{(0)}(T) = \frac{dX^j_{(0)}(T)}{dT}H_0. \quad (21)$$

Another major deficiency is that (20) and (21) are not Hermitian operators, and that it is impossible to construct a viable Hermitian combination of these two momentum definitions. Also, because the canonical momentum $P^j_{(0)}(T)$ is a constant of the motion, they argue that it should represent the total overall momentum of the system, and that the microscopic nature of (20) should not have any dependence on an inherently macroscopic description of momentum.

To deal with these conceptual deficiencies identified by Barut and Bracken in Schrödinger’s treatment of zitterbewegung, they proposed the existence of a relative momentum operator $P^j_{(B)}(T)$ that results from performing the computations in the local rest frame$^2$. This amounts to letting the operators $A_{(0)}(T) \rightarrow A_{(B)}(T)$, where

$$P^j_{(B)}(T) = 0. \quad (22)$$

By also setting the initial position to $X^j_{(B)}(0) = 0$, a relative position operator in the local frame can be identified, such that

$$X^j_{(B)}(T) \approx \frac{\hbar}{2m} \left[ \sin(\omega_{zitt} T) + 2i \sin^2(\omega_{zitt} T/2) \beta_0 \right] \alpha^j_{(0)} \quad (23)$$

while the remaining operators (10), (13) and (14) become

$$\alpha^j_{(B)}(T) \approx \alpha^j_{(0)} \left[ \cos(\omega_{zitt} T) - i \sin(\omega_{zitt} T) \beta_0 \right]. \quad (24)$$

$^2$ The main advantage of Barut’s and Bracken’s approach over Schrödinger’s is that the derived relative position and momentum operators form a harmonic oscillator relationship with frequency $\omega_{zitt}$ and are part of a basis set satisfying an SO(5) Lie algebra structure [14].
\[ \beta(B)(T) \approx \beta_0, \]  
(25)

\[ \sigma_j(B)(T) \approx \sigma_j^0, \]  
(26)

where \( \omega_{zitt} = 2m/\hbar \) is the zitterbewegung frequency derived from

\[ H_0 = m\beta(B)(T) \approx m\beta_0, \]  
(27)

\[ H_0^{-1} \approx m^{-1}\beta_0. \]  
(28)

Barut and Bracken subsequently propose that the relative momentum is described by

\[ \hat{P}_j(B)(T) = m\hat{\alpha}_j(B)(T), \]  
(29)

with respect to the spin-1/2 particle’s classical worldline.

It should be noted that \( \hat{X}_j(B)(T) \) differs slightly from \( \hat{\xi}_j(T) \) incorporated in the local rest frame by Barut and Bracken [14]. This is due to a minor change in definition, such that the third term of (18) is absorbed by (19) of form (23), instead of expressing (11) as given by (17)–(19) in the local rest frame, where

\[ \hat{X}_j(A)(T) = \frac{i\hbar}{2m}\beta_0\alpha_0^j, \]  
(30)

\[ \hat{\xi}_j(T) = -\frac{i\hbar}{2m}\beta_0\alpha_0^j e^{-i\omega_{zitt} T\beta_0} = \frac{\hbar}{2m} [\sin(\omega_{zitt} T) - i \cos(\omega_{zitt} T) \beta_0] \alpha_0^j = \frac{\hbar}{2m} e^{i\omega_{zitt}(T-\pi/2)\beta_0} \alpha_0^j. \]  
(31)

With the definitions for relative position and momentum now given, it is possible to derive the uncertainty in their measurements, to compare with the Heisenberg uncertainty relation. To begin, consider the time-averaged operator over a complete cycle in terms of \( \omega_{zitt} \), such that

\[ \langle A(T) \rangle = \frac{\omega_{zitt}}{2\pi} \int_0^{2\pi/\omega_{zitt}} A(T) \, dT. \]  
(32)

It follows that the zitterbewegung time average of (23) is

\[ \langle \hat{X}_j(B)(T) \rangle = \frac{i\hbar}{2m}\beta_0\alpha_0^j, \]  
(33)

and subsequently leads to

\[ |\langle \hat{X}_j(B)(T) \rangle | = \left[ -\eta_j \langle \hat{X}_j(B)(T) \rangle |\hat{X}_j(B)(T)\rangle \right]^{1/2} = \frac{\sqrt{3}}{2} \left( \frac{\hbar}{m} \right), \]  
(34)

in agreement with an earlier computation derived in a similar context [20]. However, the squared magnitude of \( \hat{X}_j(B)(T) \) prior to time averaging is

\[ |\hat{X}_j(B)(T)|^2 = -\eta_j \hat{X}_j(B)(T) \hat{X}_j(B)(T) = \frac{12}{\omega_{zitt}^2} \sin^2(\omega_{zitt} T/2), \]  
(35)

which leads to

\[ \langle |\hat{X}_j(B)(T)|^2 \rangle^{1/2} = \sqrt{\frac{3}{2}} \left( \frac{\hbar}{m} \right). \]  
(36)

Therefore, the local uncertainty in relative position is

\[ \Delta X(B) = \sqrt{\langle |\hat{X}_j(B)(T)|^2 \rangle - |\hat{X}_j(B)(T)|^2} = \sqrt{\frac{3}{2}} \left( \frac{\hbar}{m} \right). \]  
(37)
To compute the corresponding local uncertainty in the relative momentum, it is self-evident that

\[
\langle \hat{P}_{j(B)}(T) \rangle = 0, \tag{38}
\]

while

\[
\langle |\hat{P}_{j(B)}(T)|^2 \rangle^{1/2} = \sqrt{3}m. \tag{39}
\]

This results in \( \Delta \hat{P}_{j(B)} = \sqrt{3}m \), leading to the expression

\[
(\Delta X_{j(B)})(\Delta \hat{P}_{j(B)}) = 3 \left( \frac{\hbar}{2} \right), \tag{40}
\]

well in agreement with the Heisenberg position-momentum uncertainty relation \( (\Delta X_{j(B)})(\Delta \hat{P}_{j(B)}) \geq \hbar/2 \).

3. An approach to the curved spacetime generalization

3.1. The hypothesis of locality

Fundamental to the requirements of this paper is consideration of the implicitly accepted hypothesis of locality [21] for classical phenomena and how it relates to the intersection of quantum mechanical behaviour in curved spacetime. This hypothesis essentially states that, at any instantaneous moment of proper time on a classical worldline, it is possible to identify an instantaneous comoving inertial frame tangent to the worldline. A smooth one-to-one correspondence follows from this identification, along with an intrinsic length and time scale correlated with the comoving observer.

When considering purely classical matter, this hypothesis is undeniably successful. Issues become much more ambiguous, however, when applying this hypothesis to single-particle quantum states—let alone quantum fields, since the wave-particle duality implies the need for a region of spacetime to establish the location of quantum matter that is consistent with Heisenberg’s uncertainty principle. This especially applies to quantum matter with long wavelengths when compared to the characteristic length of the background gravitational source, where quantum interference effects become most relevant to disentangle conceptually. If a measurement applied to quantum matter is somehow performed on a sufficiently small time scale compared to the comoving observer’s intrinsic time scale, it is reasonable to surmise that the hypothesis of locality still applies. However, this claim is still a tenuous one, at best, and should be subjected to deeper analysis.

3.2. Covariant Dirac equation in fermi normal coordinates

Accepting the hypothesis of locality as a viable starting point, the approach to zitterbewegung in curved spacetime is to adopt Fermi normal coordinates \( X^\mu \) with respect to a local comoving frame [11], with \( X^0 = T \) identified as the spin-1/2 particle’s proper time and \( X^j \) as the Cartesian spatial coordinates orthogonal to the worldline. Again, for purely classical phenomena, this choice for \( X^j \) is perfectly reasonable. However, when considering the propagation of quantum matter, it may be more appropriate to use general curvilinear coordinates \( U^\mu = (T, u^j) \) to better reflect the symmetries associated with the particle’s trajectory in space for general motion between neighbouring intervals of proper time, where \( X^j = X^j(u) \) [11]. Eventually, the spatial Fermi normal coordinates will be identified with quantum fluctuations about the classical worldline, such that \( X^j \rightarrow \hat{X}^j_{j(B)}(T) \) when deriving the Dirac Hamiltonian. This is
the central hypothesis that results in the description of curved spacetime zitterbewegung to follow.

Starting with a worldline $C$ defined in a general spacetime background and parametrized by proper time $\tau$, the Fermi frame [22–25] is determined at some event $P_0$ on $C$ by constructing a local orthonormal vierbein set $\{\lambda^\alpha_\mu\}$ and inverse set $\{\lambda^\mu_\alpha\}$. Then the local spatial axes are defined by $\lambda^\mu_0 = \frac{dx^\mu}{d\tau}$ and $\lambda^\mu_a$. If $\xi^\mu = (dx^\mu/d\sigma)_0$ denotes the unit spatial tangent vector from $P_0$ to a neighbouring event $P$, where a unique space-like geodesic orthogonal to $C$ exists along proper length $\sigma$, then $T = \tau$ and $X^i = \sigma \xi^\mu \lambda^\mu_0$ become the Fermi normal coordinates at $P$. The corresponding spacetime metric is then described by

$$ds^2 = Fg_{\mu\nu}(X)\, dX^\mu dX^\nu,$$

where

$$Fg_{00}(X) = 1 + \frac{1}{2}FR_{0ij}(T)X^iX^j + \cdots,$$  \hspace{1cm} (42a)

$$Fg_{0j}(X) = \frac{2}{3}FR_{0ij}(T)X^iX^k + \cdots,$$  \hspace{1cm} (42b)

$$Fg_{ij}(X) = \eta_{ij} + \frac{1}{3}FR_{ijk}(T)X^kX^l + \cdots,$$  \hspace{1cm} (42c)

and

$$FR_{\alpha\beta\gamma\delta}(T) = R_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}\lambda^\alpha_a\lambda^\beta_\bar{\beta}\lambda^\gamma_\bar{\gamma}\lambda^\delta_\bar{\delta}.$$  \hspace{1cm} (43)

The covariant Dirac equation for a spin-1/2 particle with mass $m$ can be written in terms of (42a)–(42c) as

$$[i\gamma^\mu(\partial_\mu + i\Gamma_\mu(X)) - m/\hbar]\psi(X) = 0,$$  \hspace{1cm} (44)

where $\partial_\mu = \partial/\partial X^\mu$ and $\Gamma_\mu(X)$ is the spin connection defined with respect to $\{\gamma^\mu(X)\}$, the set of gamma matrices satisfying $[\gamma^\mu(X), \gamma^\nu(X)] = 2g^\mu\nu(X)$. A local Lorentz frame [26] can be determined according to $\tilde{e}_\alpha^\alpha(X) = \eta_{\bar{\alpha}\bar{\beta}}\tilde{e}^\alpha_m(X)\tilde{e}_\beta^n(X)$, where $\{\tilde{e}^\alpha_m(X)\}$ and $\{\tilde{e}_\alpha^m(X)\}$ form a respective orthonormal vierbein and inverse vierbein set, such that

$$\tilde{e}_\alpha^0(X) = 1 + \frac{1}{2}FR_{00ij}(T)X^iX^j,$$  \hspace{1cm} (45a)

$$\tilde{e}_0^i(X) = \frac{1}{2}FR_{0ijk}(T)X^iX^k,$$  \hspace{1cm} (45b)

$$\tilde{e}^i_\alpha(X) = -\frac{1}{2}FR_{\bar{i}\bar{0}jk}(T)X^iX^k,$$  \hspace{1cm} (45c)

$$\tilde{e}^i_j(X) = \delta^i_j - \frac{1}{3}FR_{\bar{i}\bar{j}kl}(T)X^kX^l,$$  \hspace{1cm} (45d)

and

$$\tilde{e}^0_{\bar{i}}(X) = 1 - \frac{1}{2}FR_{\bar{i}\bar{0}0j}(T)X^iX^j,$$  \hspace{1cm} (46a)

$$\tilde{e}^0_i(X) = \frac{1}{2}FR_{\bar{i}\bar{0}jk}(T)X^iX^k,$$  \hspace{1cm} (46b)

$$\tilde{e}^i_\bar{0}(X) = -\frac{1}{2}FR_{\bar{i}\bar{0}0j}(T)X^iX^j,$$  \hspace{1cm} (46c)

$$\tilde{e}^i_j(X) = \delta^i_j + \frac{1}{3}FR_{\bar{i}\bar{j}kl}(T)X^kX^l.$$

The spin connection is then determined to be

$$\Gamma_\mu(X) = \frac{i}{4}\gamma^\mu(\nabla_\mu\gamma_\alpha(X)) = -\frac{1}{4}\sigma^{\bar{a}\bar{b}}\eta_{\bar{a}\bar{b}}\tilde{e}^\alpha_m(\nabla_\mu\tilde{e}^\beta_n).$$  \hspace{1cm} (47)
where $\nabla_\mu$ is the covariant derivative operator and $\sigma^\hat{\alpha}\hat{\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta]$. To first order in the Riemann tensor, it is shown from (47) that

$$\Gamma_0(X) = iy^0\gamma^j \left[ \frac{1}{2} \mathcal{F}R_{00k}(T) + \frac{1}{2} \mathcal{F}R_{j0k,0}(T)X^j \right] X^k,$$

(48)

$$\Gamma_i(X) = iy^0\gamma^j \left[ \frac{1}{2} \mathcal{F}R_{k0j}(T) + \mathcal{F}R_{i0k}(T) - \frac{1}{12} \mathcal{F}R_{ijkl}(T) \right] X^k,$$

(49)

where $\mathcal{F}R_{ijkl}(T) = \frac{1}{2} [\mathcal{F}R_{ikl0}(T) - \mathcal{F}R_{i0k,l}(T)]$ denotes antisymmetrization of the middle two indices.

3.3. Conversion to curvilinear coordinates

It is straightforward to introduce a conversion of Fermi normal coordinates from locally Cartesian to general curvilinear coordinates, in terms of a new set of orthonormal vierbeins described by

$$e^\beta_\alpha(U) = \frac{\partial U^\beta}{\partial X^\alpha} \bar{e}^\alpha_\alpha(X),$$

(50a)

$$e^\beta_\mu(U) = \frac{\partial X^\alpha}{\partial U^\mu} \bar{e}^\alpha_\alpha(X).$$

(50b)

The corresponding spin connection in curvilinear coordinates becomes

$$\Gamma_\mu(U) = \frac{i}{4} \gamma^\alpha(U)[\nabla_\mu \gamma_\alpha(U)] = -\frac{1}{4} \sigma^\hat{\alpha}\hat{\beta}\hat{\gamma} \bar{e}^\alpha_\alpha(\nabla_\mu \gamma^\beta)$$

$$= \frac{\partial X^\alpha}{\partial U^\mu} \Gamma_\alpha(X).$$

(51)

It follows that the covariant Dirac equation (44), expressed in curvilinear coordinates and projected onto the local Lorentz frame, is

$$[i\gamma^\mu(\hat{P}_\mu - \hbar \hat{\Gamma}_\mu(U)) - m/\hbar] \bar{\psi}(U) = 0,$$

(52)

where $\hat{\nabla}_\mu = \nabla_\mu + i \hat{\Gamma}_\mu(U)$ is the flat spacetime covariant derivative operator in curvilinear coordinate form. The corresponding line element is represented by

$$ds^2 = dT^2 + \eta_{ij}(\lambda^1(u) \bar{\lambda}^1(u) \bar{\lambda}^2(u) \bar{\lambda}^2(u) du^i),$$

(53)

where $\lambda^i(u)$ are dimensional scale functions [27], and

$$\nabla_0 \equiv \frac{\partial}{\partial T}, \quad \nabla_j \equiv \frac{1}{\lambda^{(j)}(u)} \frac{\partial}{\partial u^j}.$$  

(54)

A straightforward computation from (52) leads to the covariant Dirac equation in the form

$$[\gamma^\mu (P_\mu - \hbar \Gamma_\mu(U)) - m] \bar{\psi}(U) = 0,$$

(55)

where

$$P_\mu = p_\mu + \Omega_\mu$$

(56)

is the canonical momentum operator in curvilinear coordinates, with

$$p_\mu = i\hbar \nabla_\mu,$$

(57)

$$\Omega_\mu = i\hbar [\nabla_\mu \ln(\lambda^1(u)\bar{\lambda}^2(u)\bar{\lambda}^3(u)])^{1/2}],$$

(58)

$$i\Gamma_\mu = \Gamma^{(S)}_\mu + \gamma^\nu \gamma^{\hat{\alpha}} \Gamma^{(T)}_{\hat{\alpha}\hat{\beta}\hat{\mu}} \delta^\beta_\nu,$$

(59)
\[ \bar{\Gamma}_0^{(S)} = \frac{1}{12} F_{mjk,0}(T) X^j X^k, \]
\[ \bar{\Gamma}_j^{(S)} = -\left[ \frac{1}{2} F_{00m}(T) + \frac{1}{2} F_{j0m,0}(T) X^j \right] X^m, \]
\[ \bar{\Gamma}_{[ij],0}^{(T)} = \frac{1}{2} F_{lmk,0}(T) X^l. \]

For example, it is straightforward to verify in spherical coordinates \( u^i = (r, \theta, \phi) \) that \[ P_r = -\frac{i\hbar}{r} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right), \]
\[ P_\theta = -\frac{i\hbar}{r \cot \theta} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \right), \]
\[ P_\phi = -\frac{i\hbar}{r \sin \theta} \frac{\partial}{\partial \phi}, \]
where \( \lambda^{(1)}(u) = 1, \lambda^{(2)}(u) = r \) and \( \lambda^{(3)}(u) = r \sin \theta \).

The final step follows by recalling the identity \[ \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho = \eta^\nu^\rho \eta^\mu^\nu - 2 \gamma^\nu \eta^\rho^\mu - i \gamma^\rho \epsilon^{\mu \nu \rho \sigma} \epsilon, \]
where \( \epsilon^{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \) is the Levi-Civita symbol with \( \epsilon^{\hat{0} \hat{1} \hat{2} \hat{3}} = 1 \). Use of (64) leads to a new expression for the spin connection, such that
\[ \Gamma_\mu = \gamma^5 \bar{\Gamma}_\mu^{(C)} - i\bar{\Gamma}_\mu^{(S)}, \]
\[ \bar{\Gamma}_\mu^{(C)} = \epsilon^{\hat{0}\hat{\mu}\hat{\nu}} \Gamma_{[ij],0}^{(T)} \]
where the ‘C’ in (66) is the chiral-dependent part of the spin connection, while the ‘S’ in (60)–(61) denotes the symmetric part under chiral symmetry.

### 4. Spin-1/2 particle Dirac Hamiltonian for general motion in curved spacetime and zitterbewegung effects

Having obtained the required computations for the covariant Dirac equation in Fermi normal coordinates, it is straightforward to determine from the Schrödinger equation \( \hbar (\partial/\partial T) \psi(U) = H \psi(U) \) that
\[ H = m \beta + (\alpha \cdot P) + \hbar \left[ \gamma^j \mathcal{B}_j(T, X) - \frac{i}{2} \alpha^j \mathcal{E}_j(T, X) \right] \]
\[ = \frac{i\hbar}{3} \frac{1}{R_{jm0,0}(T)} \alpha^j X^m \alpha^i X^i - \frac{i\hbar}{12} \frac{1}{R_{jm0,0}(T)} X^m \]

is the Dirac Hamiltonian expressed in terms of the ‘electric’ and ‘magnetic’ field components \[ \mathcal{E}_j(T, X) = -\frac{1}{2} R_{jm0,0}(T) X^k, \]
\[ \mathcal{B}_j(T, X) = \frac{1}{2} \epsilon^{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \frac{1}{R_{jm0,0}(T)} X^k, \]
where \( \partial_j \mathcal{B}_j(T, X) = 0 \) and \( \partial_j \mathcal{E}_j(T, X) = -\frac{1}{2} R_{00}(T) \), the analogue of Poisson’s equation in electromagnetism, whose matter field source generates the Ricci curvature tensor in the Fermi frame. The Dirac Hamiltonian (67) is comprised of operators which are defined in the local Lorentz frame and are related to corresponding operators in general curvilinear coordinates in the standard way through the vierbein projection functions (50a) and (50b). As such, the mathematical properties of these operators locally correspond precisely to their counterparts in strictly flat spacetime, with the understanding that the range...
of curvature deviation satisfies the condition that $|F g_{\mu\nu}(X) - \eta_{\mu\nu}| \ll 1$, particularly when $X^j \rightarrow X^j_{(B)}(T)$. Apart from the spin-dependent term coupled to (69), it is evident that the gravitational contributions to (67) in its present form are anti-Hermitian. If the time variations of the gravitational field are deemed small, then the last two terms in (67) can be neglected. However, for this paper all terms in the Dirac Hamiltonian are retained.

At this point, it is important to note that the Dirac Hamiltonian $H \rightarrow H_0 = m\beta + (\alpha \cdot P)$ as $X^j \rightarrow 0$, satisfying the weak equivalence principle for purely classical conceptions of spacetime. However, it so happens that if $X^j \rightarrow X^j_{(B)}(T)$ in the Fermi frame, then a quantum violation of the weak equivalence principle emerges due to *zitterbewegung*. This can be shown explicitly by the following process, where for the sake of notational convenience the local Lorentz frame indices will be unhatted for all subsequent computations. Suppose the Dirac Hamiltonian is expressed as

$$H = H_0 + H_G,$$

(70)

where $H_0$ is given by (1) and $H_G$ is the interaction Hamiltonian involving spacetime curvature. Given that $|X^j| \sim \hbar/m$ and the curvature tensors in the Fermi frame are defined on the classical worldline to satisfy $|F g_{\mu\nu}(X) - \eta_{\mu\nu}| \ll 1$, it is reasonable to surmise that $H_G \ll H_0$. If $X^j \rightarrow X^j_{(B)}(T)$ in the Fermi frame as operators, then the $X^j X^m$ within (67) become

$$X^j_{(B)}(T) X^m_{(B)}(T) = \frac{1}{i} \{ X^j_{(B)}(T), X^m_{(B)}(T) \} + \frac{i}{2} [ X^j_{(B)}(T), X^m_{(B)}(T) ],$$

(71)

the sum of anticommutator and commutator expressions, respectively, involving $X^j_{(B)}(T)$ and $X^m_{(B)}(T)$. This leads to

$$H_G = \frac{\hbar}{2} \sum_{l, m} \epsilon_{\alpha\beta\gamma\delta} F R_{\alpha \beta \gamma \delta}(T) \sigma^j_{(B)}(T) X^k_{(B)}(T) - \frac{i\hbar}{2} F R_{0j0k}(T) \alpha^j_{(B)}(T) X^k_{(B)}(T)$$

$$- \frac{i\hbar}{6} F R_{jlm0,0}(T) \alpha^j_{(B)}(T) \{ X^l_{(B)}(T), X^m_{(B)}(T) \}$$

$$+ \frac{i\hbar}{12} F R_{jlm0,0}(T) \alpha^j_{(B)}(T) [ X^l_{(B)}(T), X^m_{(B)}(T) ]$$

$$- \frac{i\hbar}{24} F R_{\alpha \beta \gamma \delta}(T) \{ X^j_{(B)}(T), X^m_{(B)}(T) \}$$

(72)

from substitution of (71) into the interaction Hamiltonian, with $F R_{jlm0,0}(T) = \frac{1}{2} \{ F R_{jlm0}(T) + F R_{jlm0}(T) \}$ denoting symmetrization of the middle two indices, and using the cyclic permutation property of indices for the Riemann tensor in the fourth term of (72). Upon evaluation in terms of

$$\{ X^j_{(B)}(T), X^m_{(B)}(T) \} = -\frac{8}{\omega_{zitt}} \sin^2(\omega_{zitt} T/2) \eta_{lm},$$

(73)

$$[ X^j_{(B)}(T), X^m_{(B)}(T) ] = -\frac{8i}{\omega_{zitt}} \sin(\omega_{zitt} T/2) e^{i\omega_{zitt}} \sigma^0_{lm},$$

(74)

and with substitutions of (23), (24) and (26) into (72), it follows that

$$H_G = -\frac{\hbar}{\omega_{zitt}} \sin^2(\omega_{zitt} T/2) \left\{ F R_{00}(T) + 2 \left[ F R_{0j0k}(T) + \frac{4}{3\omega_{zitt}} \sin(\omega_{zitt} T) F R_{0j0k}(T) \right] \alpha^j_0 \right\} \beta_0$$

$$+ \frac{i\hbar}{\omega_{zitt}} \sin(\omega_{zitt} T) F R_{00}(T) + \frac{1}{3\omega_{zitt}} \sin^2(\omega_{zitt} T/2) F R_{jkl,0}(T)$$

(75)
\[
- \left[ \sin(\omega_{\text{zitt}} T) \mathcal{F}R_{0j}(T) \\
+ \frac{8}{3\omega_{\text{zitt}}} \sin^2(\omega_{\text{zitt}} T/2) \cos(\omega_{\text{zitt}} T) \mathcal{F}R_{0j,0}(T) \right] \alpha_0^j, \tag{75}
\]

where the quantum violation of the weak equivalence principle formally appears due to the explicit coupling of spacetime curvature to the spin-1/2 particle’s rest mass via the \(\omega_{\text{zitt}}\). This is by no means the first time that a suggestion of weak equivalence violation has appeared in the literature. For example, it was suggested years ago by Greenberger that, for a bound-state quantum mechanical particle in an external gravitational potential, various violations of the weak equivalence principle emerge due to the direct coupling of particle mass to the gravitational field that subsequently do not appear in the classical limit [30].

Clearly, (75) vanishes as \(\bar{\hbar} \to 0\) or \(m \to \infty\), while the Hermitian part of \(H_G\) is regular as \(m \to 0\). In fact, it is shown that

\[
\lim_{m \to 0} H_G = i\bar{\hbar} \left[ FR_{00}(T) + \frac{1}{13} FR_{jk}(T)T \right] - \left[ FR_{0j}(T) + \frac{2}{3} FR_{0j,0}(T)T \right] \alpha_0^j \beta_0, \tag{76}
\]

which is relevant for (almost) massless neutrinos. It is particularly interesting to note that the first two terms in (75) coupled to \(\beta_0\) form a gravitationally induced effective mass, where \(FR_{00}(T)\) is likened to a potential energy term, while the remaining terms serve as energy flux contributions with instantaneous velocity \(v_{\text{inst}} \equiv |\alpha_0| = 1\). A similar identification exists for the anti-Hermitian terms not coupled to \(\beta_0\), which yield decay width contributions due to \(\text{zitterbewegung}\) in curved spacetime.

It is worthwhile to make a general comment about the anti-Hermitian contributions to (75). Since it is practically true that the time scale associated with the Fermi frame curvature tensor will be very much longer than the time scale corresponding to the \(\text{zitterbewegung}\) frequency, it is possible to regard \(FR_{\mu\nu\alpha\beta}(T)\) as effectively constant in time over a complete cycle defined by \(\omega_{\text{zitt}}\). When this is taken into account, (75) reduces to

\[
H_G \approx - \frac{\bar{\hbar}}{\omega_{\text{zitt}}} \sin^2(\omega_{\text{zitt}} T/2) \left[ FR_{00}(T) \right. \\
+ 2 \left. \left[ FR_{0j}(T) + \frac{4}{3\omega_{\text{zitt}}} \sin(\omega_{\text{zitt}} T) FR_{0j,0}(T) \right] \alpha_0^j \beta_0 \right. \\
+ \left. \frac{\bar{\hbar}}{3\omega_{\text{zitt}}} \sin^2(\omega_{\text{zitt}} T/2) \left[ FR_{jk}(T)T \right. \right. \\
- \left. \left. 8 \cos(\omega_{\text{zitt}} T) FR_{0j,0}(T) \alpha_0^j \right] \beta_0 \right], \tag{77}
\]

If it so happens that the curvature tensors time evolve adiabatically [31] and \(\alpha_0^j \beta_0 \to \frac{1}{2} \{\alpha_0^j, \beta_0\} = 0\), then the sole contribution to \(H_G\) is Hermitian. Therefore, in a practical sense the anti-Hermitian terms in (75) are of little consequence in determining the dynamics of a massive spin-1/2 particle in curved spacetime due to \(\text{zitterbewegung}\).

5. Time evolution of quantum operators

Having now determined the Dirac Hamiltonian (70) with \(\text{zitterbewegung}\) contributions according to (1) and (75), it is possible to compute the time evolution of quantum operators via the Heisenberg equations of motion. This can be done in perturbative form by solving for \(A \approx A_0 + A_1\), where

\[
\frac{dA_0(T)}{dT} \approx \frac{i}{\hbar} [H_0, A(T)], \tag{78}
\]
\[ \frac{dA_{(T)}}{dT} \approx \frac{i}{\hbar} [H_G, A(T)]. \]  

By this approach, explicit computations for the time evolution of the relative position and momentum operators can be determined in a gravitational background, where the specific case of quasi-circular motion around a spherical black hole described by the Vaidya metric is considered in detail [32]. This eventually leads to additional contributions to the position-momentum measurement uncertainties obtained in (40) due to spacetime curvature and zitterbewegung. A separate computation for the time evolution of the canonical momentum \( P^j(T) \approx P_{(0)}^j(T) + P_{(1)}^j(T) \) can also be determined, noting that in curvilinear coordinates the zeroth-order terms \( P_{(0)}^j(T) \) are not constants of the motion. When coupled with the periodic nature of the resulting vector differential equation, it follows that solutions to \( P^j(T) \) take the form given by Floquet’s theorem [33, 34]. Finally, a formal presentation of the time derivative for the Pauli–Lubanski vector, the operator that describes particle spin in covariant form, is given with respect to the perturbation Hamiltonian \( H_G \) and briefly discussed.

5.1. Time evolution of the relative position and momentum in curved spacetime due to zitterbewegung

5.1.1. Relative position \( X^j(T) \)

In the local rest frame, the relative position operator is

\[ X^j(T) \approx X^j(B)(T) + X^j(T), \]  

(80)

where \( X^j(B)(T) \) is given by (23) and

\[ \frac{dX^j(T)}{dT} = \frac{4}{\omega_{\text{zitt}}^2} \sin^2(\omega_{\text{zitt}} T/2) \]

\[ \times \left[ F_{R00}(T) \left[ \sin^2(\omega_{\text{zitt}} T/2) \alpha^j_0 + \frac{i}{2} \sin(\omega_{\text{zitt}} T) (i\alpha^j_0 \beta_0) \right] \right. \]

\[ - i \left[ \frac{8}{3\omega_{\text{zitt}}} F^0 j, 0(T) \sin^2(\omega_{\text{zitt}} T/2) \beta_0 \right. \]

\[ - 2 \left[ F_{R0k}(T) + \frac{2}{3\omega_{\text{zitt}}} F_{R0k, 0}(T) \sin(\omega_{\text{zitt}} T) \right] e^{i/2 k} \sigma^0 \].

(81)

Retaining only the Hermitian part of (81) and integrating then lead to

\[ X^j(T) = \frac{4}{\omega_{\text{zitt}}^2} \int_0^T F_{R00}(T') \sin^2(\omega_{\text{zitt}} T'/2) \]

\[ \times \left[ \sin^2(\omega_{\text{zitt}} T'/2) \alpha^j_0 + \frac{i}{2} \sin(\omega_{\text{zitt}} T) (i\alpha^j_0 \beta_0) \right] dT'. \]

(82)

Diagonalization of (82) then results in

\[ X^j(T) \bigg|_{\text{diag}} = \frac{4}{\omega_{\text{zitt}}^2} \sigma^0 \int_0^T F_{R00}(T') \sin^3(\omega_{\text{zitt}} T'/2) dT', \]

(83)

where the squared magnitude of \( X^j(T) \) before time averaging yields

\[ |X^j(T)|^2 = -\eta_{ij} X^i(T) X^j(T) \]

\[ = \left( 4\sqrt{3} \omega_{\text{zitt}}^2 \right) \int_0^T F_{R00}(T') \sin^3(\omega_{\text{zitt}} T'/2) dT'. \]

(84)
5.1.2. Relative momentum $\mathcal{P}^{j}(T)$. With the relative momentum described by

$$\mathcal{P}^{j}(T) = m \vec{v}^{j}(T) \approx \mathcal{P}^{j}_{(1)}(T) + \mathcal{P}^{j}_{(2)}(T),$$  

(85)

the time derivative of the first-order perturbation is

$$\frac{d\mathcal{P}^{j}_{(1)}(T)}{dT} = -\frac{2m}{\omega_{zitt}} \mathcal{F}(T) \left[ \sin(\omega_{zitt}T)\alpha^{j}_{0} + i\cos(\omega_{zitt}T)\alpha^{j}_{0}\beta_{0} \right]$$

$$+ \frac{2im}{\omega_{zitt}} [\cos(\omega_{zitt}T)\mathcal{C}^{j}(T) - \sin(\omega_{zitt}T)\mathcal{D}^{j}(T)]\beta_{0}$$

$$- \frac{2im}{\omega_{zitt}} e^{\delta^{j}_{kl}} \left[ \sin(\omega_{zitt}T)\mathcal{C}^{k}(T) + \cos(\omega_{zitt}T)\mathcal{D}^{k}(T) \right] \sigma^{l}_{0},$$  

(86)

where

$$\mathcal{F}(T) = -\sin^{2}(\omega_{zitt}T/2)F_{00}(T),$$  

(87)

$$\mathcal{C}^{j}(T) = -2\sin^{2}(\omega_{zitt}T/2) \left[ F_{0j}(T) + \frac{4}{3\omega_{zitt}} \sin(\omega_{zitt}T)F_{0j,0}(T) \right],$$  

(88)

$$\mathcal{D}^{j}(T) = - \left[ \sin(\omega_{zitt}T)F_{0j}(T) + \frac{8}{3\omega_{zitt}} \sin^{2}(\omega_{zitt}T/2) \cos(\omega_{zitt}T)F_{0j,0}(T) \right].$$  

(89)

By also retaining only the Hermitian part of (86) and integrating, it is shown that

$$\mathcal{P}^{j}_{(1)}(T) = -\frac{2m}{\omega_{zitt}} \int_{0}^{T} \mathcal{F}(T') \left[ \sin(\omega_{zitt}T')\alpha^{j}_{0} + \cos(\omega_{zitt}T')i\alpha^{j}_{0}\beta_{0} \right] dT'.$$  

(90)

where in diagonalized form (90) becomes

$$\mathcal{P}^{j}_{(1)}(T) \mid_{\text{diag}} = \frac{2m}{\omega_{zitt}} \sigma^{j}_{0} \int_{0}^{T} F_{00}(T') \sin^{2}(\omega_{zitt}T'/2) dT'.$$  

(91)

A corresponding computation for the squared magnitude of $\mathcal{P}^{j}_{(1)}(T)$ prior to time averaging then results in

$$|\mathcal{P}^{j}_{(1)}(T)|^2 = -\eta_{ij} \mathcal{P}^{i}_{(1)}(T)\mathcal{P}^{j}_{(1)}(T)$$

$$= \left| \frac{2\sqrt{3}m}{\omega_{zitt}} \int_{0}^{T} F_{00}(T') \sin^{2}(\omega_{zitt}T'/2) dT' \right|^2.$$  

(92)

5.1.3. Position-momentum measurement uncertainties in the Vaidya background. To illustrate the contribution of zitterbewegung in curved spacetime to the position-momentum uncertainties in measurement, consider the quasi-circular motion of a spin-$1/2$ particle around a spherically symmetric black hole in the presence of radiation, as described by the Vaidya metric [32]. For this computation, it is assumed that the background source $M$ monotonically changes in time according to

$$M = M_{0} + \Delta M(T),$$  

(93)

where $M_{0}$ is the (static) Schwarzschild mass,

$$\Delta M(T) = \frac{a}{A} \left| \frac{d(\Delta M)}{dT} \right| T,$$  

(94)

$$A^2 = 1 - \frac{2M_{0}}{r},$$  

(95)
\( r \) is the particle’s orbital radius and \(|\Delta M|/\xi|\) is the source’s rate of change along \( \xi \), a radial null coordinate. If \( \alpha = +1 \), then \( \xi \) is the advanced null coordinate corresponding to infalling radiation, while \( \alpha = -1 \) implies that \( \xi \) is the retarded null coordinate for outgoing radiation. Then it is determined that [32]

\[
FR_{00}(T) \approx \frac{2\alpha}{N^2r^2} \left| \frac{d(\Delta M)}{d\xi} \right| \left( 1 + 2\alpha \frac{d(\Delta M)}{d\xi} \right) \left( \frac{r\Omega_K}{2N^2} C (r, \Omega_K T) - \frac{2}{N^2A^2} \sin^2 (\Omega_K T) \right),
\]

where \( \Omega_K = \sqrt{M_0/r^3} \) is the Keplerian frequency of the orbit,

\[
N^2 = 1 - \frac{3M_0}{r} \quad (r > 3M_0)
\]

and

\[
C(r, \Omega_K T) = 2 \sin(2\Omega_K T) + \frac{N}{r\Omega_K} [(1 - 2r\Omega_K) \sin(2\Omega_K T) - 2\Omega_K T].
\]

To leading order in \(|d(\Delta M)/d\xi|\), it is straightforward to show that

\[
\langle X_1(T) \rangle_{\text{Vaidya}} \approx \frac{16\alpha}{3N^2r^2\alpha_{\text{int}}^2} \left| \frac{d(\Delta M)}{d\xi} \right| \left( \frac{\hbar}{m} \right) \sigma_0.
\]

\[
\langle |X(T)|^2 \rangle_{\text{Vaidya}} \approx \langle |X_{(B)}(T)|^2 \rangle + \frac{16\alpha}{N^2r^2\alpha_{\text{int}}^2} \left| \frac{d(\Delta M)}{d\xi} \right| \left( \frac{\hbar}{m} \right)^2 \left( -\frac{8}{5\pi} \gamma_0^5 + i\beta_0 \gamma_0^5 \right),
\]

\[
\langle |X(T)|^2 \rangle_{\text{Vaidya}} = \langle |X_{(B)}(T)|^2 \rangle + \frac{16\alpha}{N^2r^2\alpha_{\text{int}}^2} \left| \frac{d(\Delta M)}{d\xi} \right| \left( \frac{\hbar}{m} \right)^2 (i\beta_0 \gamma_0^5),
\]

which results in

\[
\langle \Delta X \rangle_{\text{Vaidya}} \approx \left( 1 - \frac{256\alpha}{15\pi N^2r^2\alpha_{\text{int}}^2} \left| \frac{d(\Delta M)}{d\xi} \right| \gamma_0^5 \right) \langle \Delta X_{(B)} \rangle
\]

for the uncertainty in local position measurement. Similarly, it becomes evident that

\[
\langle |P(T)|^2 \rangle_{\text{Vaidya}} \approx \langle |P_{(B)}(T)|^2 \rangle - \frac{18\gamma_0}{N^2r^2\alpha_{\text{int}}^2} \left| \frac{d(\Delta M)}{d\xi} \right| (i\beta_0 \gamma_0^5) m^2,
\]

\[
\langle |P(T)|^2 \rangle_{\text{Vaidya}} = \langle |P_{(1)}(T)|^2 \rangle_{\text{Vaidya}} \approx \left( \frac{2\sqrt{3}\gamma_0}{N^2r^2\alpha_{\text{int}}^2} \left| \frac{d(\Delta M)}{d\xi} \right| m \right)^2,
\]

leading to

\[
\langle \Delta P \rangle_{\text{Vaidya}} \approx \left( 1 - \frac{3\gamma_0}{N^2r^2\alpha_{\text{int}}^2} \left| \frac{d(\Delta M)}{d\xi} \right| (i\beta_0 \gamma_0^5) \right) \langle \Delta P_{(B)} \rangle
\]

for the corresponding uncertainty for relative momentum.

Therefore, the combination of (102) and (105) leads to the gravitationally modified relative position-momentum uncertainty relation

\[
\langle \Delta X \rangle \langle \Delta P \rangle_{\text{Vaidya}} \approx \left[ 1 - \frac{\alpha}{N^2r^2\alpha_{\text{int}}^2} \left| \frac{d(\Delta M)}{d\xi} \right| \left( \frac{256\gamma_0}{15\pi} + 3i\beta_0 \right) \gamma_0^5 \right] \langle \Delta X_{(B)} \rangle \langle \Delta P_{(B)} \rangle.
\]
whose diagonalized form, in terms of the chiral representation for the gamma matrices [16], is

\[
(\Delta X)(\Delta P)_{\text{Vaidya}}|_{\text{diag}} \approx \left[1 \mp \sqrt{9 + \frac{256}{15\pi}} \frac{\alpha}{N^2r^2\omega_{\text{zitt.}}} \frac{d(\Delta M)}{d\xi} \right] (\Delta X(B))(\Delta P(B)).
\]

(107)

where the upper sign in (107) refers to the right-handed spinor, while the lower sign refers to the left-handed spinor. It is interesting to note that when \(\omega_{\text{zitt.}} \to 0\), the absolute magnitude of \((\Delta X)(\Delta P)_{\text{Vaidya}}\) becomes infinitely large, while \(\omega_{\text{zitt.}} \to \infty\) reduces (107) to \((\Delta X(B))(\Delta P(B))\).

Assuming the upper sign in (107), the choice of \(\alpha = +1\) shows that infalling radiation serves to reduce the uncertainty relation computed in flat spacetime, though \(|\frac{d(\Delta M)}{d\xi}| \ll 1\) for realistic spherical mass accretion rates involving astrophysical black holes due to the Eddington luminosity limit [32]. As such, the gravitational contribution to the position-momentum uncertainty in measurement is negligibly small for this circumstance. However, it is unclear if this condition still applies where it concerns microscopic black holes, especially if \(r\) reduces to the Compton wavelength scale or smaller.

Finally, to ensure that the Heisenberg uncertainty relation is satisfied, the prefactor in front of \((\Delta X(B))(\Delta P(B))\) in (107) must remain nonzero. Given (97), this condition implies that

\[
r > 3M_0 \left[1 \pm \sqrt{9 + \frac{256}{15\pi}} \frac{\alpha}{9M_0^2\omega_{\text{zitt.}}} \frac{d(\Delta M)}{d\xi} \right].
\]

(108)

Again, assuming the upper sign and infalling radiation for (108), it is interesting to observe a growth in the lower bound for \(r\) due to gravitational and zitterbewegung effects. Since \(r = 3M_0\) is the (unstable) photon orbit in Schwarzschild spacetime, it is not surprising that the lower bound will grow due to mass accretion from infalling radiation. However, its explicit dependence on the zitterbewegung frequency is unusual, a byproduct of the quantum violation of the weak equivalence principle.

5.2. Time evolution of the canonical momentum due to zitterbewegung

When (78) and (79) are applied to the canonical momentum, the result is

\[
\frac{dP_j^{(0)}}{dT} = -\Omega(T)\frac{k}{k}P_k^{(0)}(T),
\]

(109)

\[
\frac{dP_j^{(1)}}{dT} = -\Omega(T)\frac{k}{k}P_k^{(1)}(T) + \nabla_j H_G(T),
\]

(110)

where

\[
\Omega(T)\frac{k}{k} = [(\nabla_j \ln \lambda^{(k)})\delta_j^k - (\nabla_j \ln \lambda^{(k)})\delta_j^k]\alpha_{(0)}^{(j)}(T).
\]

(111)

There are two very important points to note concerning (111). First, the fact that \(\Omega(T)\frac{k}{k} \neq 0\) in curvilinear coordinates indicates that the standard free-particle approach only applies for Cartesian coordinates, which implicitly requires strictly inertial or rectilinear motion for the spin-1/2 particle. For non-inertial motion with rotation, a more complicated time evolution applies. The second point is that (111) is periodic due to zitterbewegung contributions from \(\alpha_{(0)}^{(j)}(T)\). As a consequence, it follows that

\[
\Omega(T + 2\pi/\omega_{\text{zitt.}})\frac{k}{k} = \Omega(T)\frac{k}{k}.
\]

(112)
which satisfies the conditions for Floquet’s theorem to apply \[33, 34\]. This leads to the free-particle solution of the form
\[
P_j^{(0)}(T) \approx e^{i\lambda \omega_{\text{zitt}} T} \Pi_j^{(0)}(T),
\]
where
\[
\Pi_j^{(0)}(T + 2\pi/\omega_{\text{zitt}}) = \Pi_j^{(0)}(T).
\]
With explicit initial conditions, the free-particle solution for \( P_j^{(0)}(T) \) is
\[
P_j^{(0)}(0) \approx P_j^{(0)}(0) + e^{i\lambda \omega_{\text{zitt}} T} \Pi_j^{(0)}(0),
\]
where \( \text{Re} \lambda > 0 \) in (115) indicates an instability in the solution space for certain values of \( T \) and/or regions of the curved spacetime background where this condition may be satisfied.

To solve for \( P_j^{(1)}(T) \) from (110), it is possible to multiply from the left by a time-dependent invertible matrix \( \mu(T) j k \), such that
\[
\mu(T) j k \left( \nabla k H G(T) \right) = \frac{d}{dT} \left[ \mu(T) j k P_k^{(1)}(T) \right] - \left[ \frac{d\mu(T)}{dT} j k - \mu(T) j l \Omega(T) l k \right] P_k^{(1)}(T).
\]
Then so long as a solution exists for the matrix differential equation
\[
\frac{d\mu(T)}{dT} j k - \mu(T) j l \Omega(T) l k = 0,
\]
which is possible to obtain using Floquet theory, it becomes evident that \( \mu(T) j k \) is an integrating factor for (116), yielding the general solution
\[
P_j^{(1)}(T) = \left[ \mu(T)^{-1} j l \int_0^T \mu(T) l i k (\nabla i H G(T')) dT' + \mu(0) l i k P_k^{(1)}(0) \right]
\]
for the first-order perturbation due to the interaction Hamiltonian \( H_G \).

### 5.3. Time evolution of covariant spin due to zitterbewegung

The final set of computations concern the time evolution of the Pauli–Lubanski vector, the operator describing covariant spin, involving zitterbewegung in a curved spacetime background. For spin-1/2 particles, the Pauli–Lubanski vector in a local Lorentz frame is \[11, 12\]
\[
W^\mu(T) = -\frac{1}{2} \epsilon^{\mu \alpha \beta \gamma} \sigma_{\alpha \beta}(T) P^\gamma(T),
\]
where \( \sigma^{\alpha \beta}(T) \) can be expressed as
\[
\sigma_{\alpha \beta}(T) = 2i\delta^{[\alpha} \delta^{\beta]} j \alpha^j(T) - \epsilon^{\alpha \beta j} \sigma^i(T).
\]

Applying (78) and (79) to (119) eventually leads to
\[
\frac{dW^\mu_{(0)}}{dT} = -\frac{m}{\hbar} \epsilon^{\alpha j(\omega \delta \beta)} j \alpha^j(T) P^\beta_0(T) + \eta^{\mu[i]} \sigma^k_0(T)(\nabla_i \ln \lambda^{j(0)}) P^0_j(T)
+ \frac{1}{\hbar} \delta^{[\alpha} j \left[ P^\beta_0(T) P^{\beta}_j(T) - \delta^j_\alpha P^\beta_j(T) \right] \sigma^k_0(T)
+ \hbar R^\mu_{(0)}(T),
\]
\[17\]
\[
\frac{dW_μ^{(1)}(T)}{dT} = -\frac{1}{4} \varepsilon_{μβγ} σ^αβ(T) (\nabla^γ H_G(T)) - \frac{1}{ω_zitt} \varepsilon_{μ0}^0 \left[ \cos(ω_zitt \cdot T) \left[ F(T) \alpha^0_0 - C^j(T) \right] + \sin(ω_zitt \cdot T) D^j(T) \right] \beta_0
\]

\[
+ \frac{4}{ω_zitt} \left[ \cos(ω_zitt \cdot T) + \sin(ω_zitt \cdot T) \right] δ^μ_μ δ_β_0 \left[ C^k(T) σ^0_0 \right] P^0_0(T) \alpha_0^j P^0_0(T)(i + β_0)
\]

(122)

for the zeroth-order and first-order time derivatives of \(W_μ(T)\) \(≈ W^{(0)}_μ(T) + W^{(1)}_μ(T)\), where

\(R^L_{(0)}(T) = \frac{i}{2\hbar} \varepsilon^{0μμ0} \left[ P^0_μ(T), P^0_μ(T) \right] = \varepsilon^{0μμ0} (\nabla^L) \alpha_0^μ P^0_μ(T)\)

in (121) is the non-inertial dipole operator [11–13], which identically vanishes for momentum operators in Cartesian coordinates. Because of the explicit and widespread dependence of \(P^0_μ(T)\) in both (121) and (122), it is possible to foresee instabilities emerging in the time evolution of \(W_μ(T)\) due to instabilities in (115).

6. Conclusion

This paper explores the consequences of spin-1/2 particle zitterbewegung in a general curved spacetime background using Fermi normal coordinates, where the spatial vectors \(X_j\) orthogonal to the classical worldline are treated as quantum fluctuations \(X_j(B)(T)\) in a local frame, according to the formalism first introduced by Barut and Bracken [14]. An overarching conclusion is that the associated Dirac Hamiltonian \(H_G\) reveals a quantum violation of the weak equivalence principle due to zitterbewegung, confined to a worldtube of radius \(ω_{−1}zitt\). Apart from one anti-Hermitian term in (75) that is coupled to \(F R^0_0(T)\), all the gravitational contributions in \(H_G\) are dependent on the Ricci curvature tensor and its time derivative in the Fermi frame. In the classical limit as \(\hbar \to 0\), it follows that \(H_G \to 0\), as expected. It should also be noted that, while the definition of quantum operators employed in this paper is sufficient to demonstrate the possible existence of zitterbewegung effects in a curved spacetime background, a more mathematically rigorous definition may be required to illustrate the full extent of these effects. This is a worthwhile consideration for future research on this topic.

When applied to the special case of quasi-circular orbital motion in the presence of a spherical black hole in radiation, as described by the Vaidya metric, it is shown that uncertainties in the simultaneous measurement of the particle’s locally determined position and momentum are modified due to the coupling of \(ω_zitt\) with the black hole’s rate of change of mass \(\frac{d(\Delta M)}{d\xi}\), with interesting physical consequences. By further allowing a description of the problem in terms of general curvilinear coordinates to accommodate non-inertial motion along the worldline, the purely flat spacetime treatment of this problem becomes generalized by showing that the canonical momentum is subject to the Floquet theory of differential equations involving oscillation frequency \(ω_zitt\), with potential stability implications due to zitterbewegung.

There are numerous potentially significant consequences and future directions worth exploring from the main expressions presented in this paper. For example, the fact that \(H_G\)
is explicitly time dependent due to *zitterbewegung* indicates the possibility of identifying transition rates for local energy states about the spin-1/2 particle’s classical worldline [34]. While the gravitational and *zitterbewegung* contributions do not survive for a *classical* vacuum spacetime background, where $\tilde{F}_{\mu \nu} (T) = 0$ identically, this may not necessarily be true if zero-point fluctuations and vacuum polarization effects [35] due to the Casimir effect, for example, appear as a consequence of virtual pair production. Given that QED predicts the existence of vacuum bubbles that have no observable effect in purely flat spacetime, it is reasonable to consider whether this property holds true in the local presence of spacetime curvature. Finally, the results presented here may have potential cosmological implications in terms of the very small value for the (observed) cosmological constant, with the perceived global acceleration of the Universe as an observable effect. These and other possibilities may be explored at a later date.

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