Comment on “Casimir effect in a weak gravitational field: Schwinger’s approach”

A.P.C.M. Lima¹, G. Alencar and R.R. Landim⁰

¹Departamento de Física, Universidade Federal do Ceará, Caixa Postal 6030, Campus do Pici, 60455-760, Fortaleza, Ceará, Brazil.

July 20, 2020

Abstract

We show that the statement in F. Sorge [Class. Quant. Grav. 36, no. 23, 235006 (2019)] that the Casimir effect receives second order corrections due to gravity is not consistent. We remark especially on the tracing of the proper time Hamiltonian, where the correct procedure is to use the eigenfunctions and eigenvalues of the covariant D’Alembertian. After some cancellations we find that the value of the functional $W[0]$ is the same as obtained by Sorge. However, we argue that the proper vacuum energy density carries extra space-time volume terms that cancel over the gravitational correction, returning to the same expression as in Minkowski space-time.

1 On the tracing procedure for the proper time Hamiltonian

As shown by Schwinger in [1, 2], the Casimir energy [3] can be obtained from the functional $W[0] \equiv i \ln(Z[0])$, which can be written as

$$W[0] = \frac{i}{2} \int \frac{ds}{s} Tr[\exp(-is\hat{H})],$$

(1)

where $\hat{H}$ is the proper time Hamiltonian. The trace contained in (1) can be simply expressed in condensed notation as

$$Tr[\exp(-is\hat{H})] = \sum_n e^{-is\lambda_n},$$

(2)

where $\lambda$ are the eigenvalues of $\hat{H}$ in any given representation. The proper Hamiltonian can be interpreted as an differential operator acting on the space of scalar fields obeying Dirichlet conditions $\psi(z = 0) = \psi(z = L) = 0$. Given an complete set of eigenfunctions $\psi_n$ with normalization

$$\int d\nu \psi_n^* \psi_m = \delta_{nm},$$

(3)
where $dv_x$ is the invariant space-time volume element, the trace \ref{eq:trace} can be expanded as

\[ Tr[\exp(-is\hat{H})] = \int dv_x \sum_n e^{-is\lambda_n} |\psi_n(x)|^2. \tag{4} \]

For flat space-time, we have $\hat{H} \equiv \Box$, and we can use the well known normalized eigenfunctions

\[ \psi_n = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{2}{L}} \sin\left(n\pi z/L\right) \exp\left[i(\omega t - k_\perp x_\perp)\right]. \tag{5} \]

Thus, we have

\[ W[0] = i \frac{2}{\sqrt{-g}} \sum_n \int d^4x \omega d^2k_\perp ds |\psi_n(x)|^2 \exp[-is(\omega^2 - k_\perp^2 - n^2\pi^2/L^2)], \tag{6} \]

this is equivalent to the expression presented in \ref{eq:W0}. In curved space-time, the same expression \ref{eq:W0} is valid, we just have to adjust the elements in \ref{eq:trace}(e.g. space-time volume element, eigenfunctions and eigenvalues of $\hat{H}$).

With the above arguments in mind, next we would like to make a few remarks in the calculation from \ref{eq:W0}. The author splits the total value of $W[0]$ in its flat space-time value plus a gravitational correction coming from the modified proper time Hamiltonian

\[ W[0] = W_{flat}[0] + \delta W[0]. \tag{7} \]

Even though the operator whose trace is to be evaluated in the first term is the same as in Minkowski case, in the second term

\[ \delta W = -i \int \frac{ds}{s} Tr[4i\gamma_s(z + is\partial_z)\Box e^{-is\hat{H}_0}], \tag{8} \]

the field modes \ref{eq:psi_n} do not represent eigenfunctions of the operator in square brackets, and the subsequently $z$ dependent values to be integrated do not correspond to formal eigenvalues. Metric terms are missing in the space-time integration, although this is canceled out by the use of the flat modes. Notice also that the proper time Hamiltonian used is simplified by a factor of $(1 - 2\gamma z)$, which needs to be accounted for in the eigenvalue equation.

In the next section we will perform an explicit calculation of the $W[0]$ functional value given by

\[ W[0] = i \frac{2}{\sqrt{-g}} \sum_\lambda \int \frac{ds}{s} e^{-is\lambda} |\psi_\lambda(x)|^2, \tag{9} \]

where the $\lambda$ and $\psi_\lambda$ are to obtained from the eigenvalue equation

\[ \Box \psi_\lambda(x) = \lambda \psi_\lambda(x), \tag{10} \]

and $\Box$ is the generalized D’Lambertian.
2 Casimir energy density

We begin with the eigenvalue equation (10), which for the metric
\[ds^2 = (1 + 2\gamma z)dt^2 - (1 - 2\gamma z)(dx^2 + dy^2 + dz^2),\]
(11)
can be written as
\[((1 - 2\gamma z)\partial_t^2 - (1 + 2\gamma z)\nabla^2)\psi_\lambda = \lambda\psi_\lambda.\]
(12)

As in [6], we expand the solutions as
\[\psi_{n,k,\omega}(x) = A_{n,k,\omega}\chi_n(z)e^{i(\omega t - k_{\perp}x_{\perp})},\]
(13)
leading to
\[((1 - 2\gamma z)\omega^2 + (1 + 2\gamma z)(k^2 - \partial_z^2))\chi_n(z) = -\lambda_{n,k,\omega}\chi_n(z).\]
(14)

This equation can be rearranged into
\[\partial_z^2\chi - a\chi + b\chi = 0,\]
(15)
where \(a = (4\omega - 2\lambda)\gamma\) and \(b = (\omega^2 - k_{\perp}^2 + \lambda)\). Solutions to the above equation can be found in analogy to the mode solutions in [6] (in [7] an explicit expression of these modes as perturbations to the flat case solutions is shown). Also from [6], imposing the boundary conditions on the resulting modes leads to
\[b - aL^2 \simeq n^2\pi^2 L^2,\]
(16)
which can be used to obtain the eigenvalues
\[
\lambda = \left( k_{\perp}^2 + \frac{n^2\pi^2}{L^2} \right) (1 + \gamma L) - \omega^2(1 - \gamma L),
\]
(17)
for \(\lambda = 0\), we recover the mode frequencies originally presented in [6]. Notice that if we had simplified the proper-time Hamiltonian as in [5], we would have obtained different values.

Then, with the insertion of [17] and the regularization parameter from [8], equation (11) becomes
\[W^{(\nu)}[0] = \frac{i}{2} \sum_n \int d^4x d^2k_{\perp} d\omega dss^{-1} \sqrt{-g} |\psi_{n,k,\omega}|^2 \times \exp \left[ -is \left( k_{\perp}^2 + \frac{n^2\pi^2}{L^2} \right) (1 + \gamma L) + i\omega^2(1 - \gamma L) \right].\]
(18)

Proceeding with the calculation as usual (use condition (3) to perform space-time integrations), we find
\[W^{(0)}[0] = (1 + \gamma L)^{-1+3/2}(1 - \gamma L)^{-1/2} W_{\text{flat}}^{(0)}[0] = -(1 + \gamma L) \frac{AT\pi^2}{1440L^3}.\]
(19)

This coincides with the result in [5].
It is important to notice that the functional $W$ is directly related to the vacuum to vacuum transition rates as measured in the world line of the static observer and is constructed as an scalar quantity. Remember the similarly invariant mean proper Casimir energy density constructed in [6], defined as

$$\bar{\epsilon} = \frac{1}{V_p} \int d^3 x \sqrt{h} u^\mu u^\nu \langle 0 | T_{\mu \nu} | 0 \rangle,$$

(21)

where $h$ is the induced metric determinant and $V_p$ the proper volume of the cavity. In the present approach this quantity can be obtained by factoring out the space-time 4-volume (remember $W[0]$ also contains the time factor), so:

$$\bar{\epsilon} = \frac{W[0]}{V_p^{(4)}},$$

(22)

where $V_p^{(4)} \equiv \int d^4 x \sqrt{-g}$. This is in contrast with the original proposal from [5], where the denominator in the above equation is just $V^{(4)} = ALT$, using coordinate rather than proper parameters. Thus

$$\bar{\epsilon} = -(1 + 2\gamma L) \frac{\pi^2}{1440L^4} = -\frac{\pi^2}{1440L_p^4},$$

(23)

where $L_p$ is the proper length $L_p = \int_0^L dz \sqrt{-g_{33}}$. This recovers the result from [7], indicating no correction to the Casimir energy to order of $\gamma$.

### 3 Conclusion

We reviewed a few points in the calculation by [5], addressing some formal aspects that we believe to be non appropriately applied. Although our suggested changes lead to the same value for $W[0]$ (20), we state that these changes must be considered in future works to avoid possibly inconsistent results. We also argued, based on invariance grounds, that the relation between the functional obtained and the mean Casimir energy as defined in [6], needed to be corrected by considering space-time proper volume factors (22), which canceled out the previously presented gravitational correction. Thus being in agreement with the result from [7].

It was claimed in [5] that the calculations from [7] used an non exactly normalized mode expansion. We have checked on the modes normalization but did not find any problems. As it was also not explicitly shown in [5] and for conciseness reasons due to the format of the manuscript, we did not include this check here. A short remark on that can be found instead in [9], where a more general background metric is considered. We do however reaffirm the null correction result that is re-obtained through the present formalism culminating in expression (23).

We also reiterate that the setup and mathematical model considered in this discussion are very simplified. Thus the results are nowhere definitive, there are other factors to be considered, as mentioned previously [5][7]. Nevertheless, the seemingly oddness of the result (23) can be a good motivation for further investigation. For this purpose, the Schwinger’s approach may be an valuable tool as it presents an, although less intuitive, simplified alternative to possibly very
complicated curved space-time field solutions. Given the space-time metrics, it is only required to find the explicit form of the eigenvalues in equation (10) to find the Casimir energy.

Acknowledgements

The authors would like to thank Alexandra Elbakyan and sci-hub, for removing all barriers in the way of science.

We acknowledge the financial support by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Fundação Cearense de Apoio ao Desenvolvimento Científico e Tecnológico (FUNCAP) through PRONEM PNE0112-00085.01.00/16.

References

[1] J. Schwinger, “Casimir effect in source theory,” Letters in Mathematical Physics, 1(1), 43(1975).

[2] J. Schwinger, “Casimir effect in source theory II. Letters in Mathematical Physics,” 24(1), 59(1992).

[3] H. Casimir, Proc. K. Ned. Akad. Wet., 51, 793(1948)

[4] L. E. Parker and D. Toms, “Quantum Field Theory in Curved Spacetime,” doi:10.1017/CBO9780511813924

[5] F. Sorge, “Casimir effect in a weak gravitational field: Schwinger’s approach,” Class. Quant. Grav. 36, no.23, 235006 (2019) doi:10.1088/1361-6382/ab4def

[6] F. Sorge, “Casimir effect in a weak gravitational field,” Class. Quant. Grav. 22, 5109(2005) doi:10.1088/0264-9381/22/23/012

[7] A. Lima, G. Alencar, C. Muniz and R. Landim, “Null Second Order Corrections to Casimir Energy in Weak Gravitational Field,” JCAP 07, 011 (2019) doi:10.1088/1475-7516/2019/07/011 [arXiv:1903.05512 [hep-th]].

[8] M. V. Cougo-Pinto, C. Farina, A. J. Seguí-Santonja, “On Schwinger’s method for obtaining the Casimir effect,” Letters in Mathematical Physics, 30(2), 169(1994).

[9] A. P. C. M. Lima, G. Alencar and R. R. Landim, “Null second order corrections to Casimir energy in weak gravitational field: The Schwinger’s approach,” arXiv:2007.07163 [hep-th].