ON GLOBAL DYNAMICS OF REACTION–DIFFUSION SYSTEMS AT RESONANCE

PIOTR KOKOCKI

Abstract. In this paper we use the homotopy invariants methods to study the global dynamics of the reaction-diffusion systems that are at resonance at infinity. Considering degrees of the resonance for the nonlinear perturbation we establish Landesman-Lazer type conditions and use them to express the Rybakowski-Conley index of the invariant set consisting of all bounded solutions. Obtained results are applied to study the existence of solutions connecting stationary points for the system of nonlinear heat equations.

1. Introduction

We are concerned with the following system of autonomous differential equations

\[
\begin{cases}
\dot{u}_k(t) = -A_k u_k(t) + \lambda_k u_k(t) + f_k(x, u(t), \nabla u(t)), & 1 \leq k \leq m, \ x \in \Omega, \\
\quad u_k(t) = 0, & 1 \leq k \leq m, \ x \in \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^n \) is an open bounded set with the smooth boundary and \( \lambda_1, \ldots, \lambda_m \) are real parameters. Given \( 1 \leq k \leq m \), we assume that \( f_k : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R} \) is a continuous function and

\[ A_k u := -D_i(a^{ij}_k D_j u), \quad u \in C^2(\Omega; \mathbb{R}^m) \]

is symmetric uniformly elliptic differential operator, that is, \( a^{ij}_k = a^{ji}_k \in C^1(\Omega; \mathbb{R}^m) \) for \( 1 \leq i, j \leq n \) and there is \( c > 0 \) such that the following inequality holds

\[ a^{ij}_k(x)\xi_i \xi_j \geq c|\xi|^2, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, \ x \in \Omega. \]

Throughout this paper we assume that the each of the linear operators \( A_k \) is considered on the domain

\[ D(A_k) := cl_W^2(\Omega) \{ u \in C^2(\Omega; \mathbb{R}^m) \mid u(x) = 0 \text{ for } x \in \partial \Omega \}, \]

where \( p \geq 2 \) is the exponent that will be precisely chosen later. We are interested in the existence and topological properties of the set consisting of all bounded solutions of the system \([1.1]\) in the case of the resonance at infinity, that is,

\[ \text{Ker}(\lambda_k I - A_k) \neq \{0\} \quad \text{and} \quad f_k \text{ is a bounded map for } 1 \leq k \leq m. \]  

(1.2)

It is known that under the assumption \([1.2]\), there are examples of the nonlinear perturbations \((f_1, \ldots, f_m)\) such that the semiflow associated with the system \([1.1]\) does not admit bounded solutions (see Remark \([1.3]\)). In particular, it can not have even stationary points. During last years many effort has been made to study the influence of the resonance phenomena on the existence of solutions for partial differential equations. In the fundamental paper \([29]\) the Landesman-Lazer conditions were introduced to establish the existence results for the following problem

\[
\begin{cases}
D_i(a^{ij} D_j u) + \lambda u + f(x, u) = 0, & x \in \Omega, \\
\quad u = 0, & x \in \partial \Omega,
\end{cases}
\]

(1.3)

Key words and phrases. semiflow, invariant set, Rybakowski-Conley index, resonance.
where \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a bounded map, \( D_j (a^{ij} D_j ) \) is a symmetric elliptic differential operator with the Dirichlet boundary conditions, which is considered on the space \( H^2(\Omega) \cap H^1_0(\Omega) \) and \( \lambda \in \mathbb{R} \) is its simple eigenvalue. Assuming that the limits \( \hat{f}^\pm (x) := \lim _{s \to \pm \infty} f(x, s) \) exist for all \( x \in \Omega \), we say that the Landesman-Lazer conditions are satisfied provided the following inequality

\[
\int _{\Omega_+} \hat{f}^+ \, dx + \int _{\Omega_-} \hat{f}^- \, dx > 0, \quad \text{(resp.} \int _{\Omega_+} \hat{f}^+ \, dx + \int _{\Omega_-} \hat{f}^- \, dx < 0) \quad (1.4)
\]

holds for any \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) satisfying the equation \( \lambda u + D_j (a^{ij} D_j u) = 0 \), where \( \Omega_+ := \{ x \in \Omega \mid \pm u(x) > 0 \} \). The results of [29] were improved in [12] and [31] by dropping the assumption concerning the simplicity of the eigenvalue \( \lambda \), whereas in [2] and [3] the effect of the conditions (1.4) on the existence of multiple equilibrium points of (1.3) were studied. The problem concerning the smoothness of the solutions of this equation obtained under the Landesman-Lazer conditions was considered in [27], where the results stating the regularity in the Besov and Triebel-Lizorkin spaces were derived. We also refer the reader to the papers [7], [18], and [22] for the analogous bifurcation problem for the \( p \)-Laplace counterpart of the equation (1.3) using variational methods and linking type argument. Surprisingly, it appears that the Landesman-Lazer conditions can be applied to the study of the global dynamics of partial differential equations. For example, in [9] the results concerning the existence of global attractors for the heat equation with nonlinear boundary conditions were obtained.

Recently, in [16] the resonance conditions (1.4) were used to study the existence of bifurcations from infinity for the solutions of the semilinear Schrödinger equation on \( \mathbb{R}^n \). We also refer the reader to [13] and [22] for the analogous bifurcation problem for \( p \)-Laplace and Hamilton-Jacobi-Bellman equations, respectively. Clearly the conditions (1.4) do not work if the equation (1.3) is at strong resonance at infinity, which means that \( f(x, s) \to 0 \) as \( |s| \to \infty \). To handle with this case, topological and variational methods were applied in [5], [6], [8], [11], [21], [41], [43], [44], [46] to prove the existence of solutions for the equation (1.3) under various assumptions imposed on the perturbation \( f \). These studies were continued in [1] for the \( p \)-Laplace version of the problem (1.3). As a result, there were obtained criteria on the existence of positive solutions for the \( p \)-Laplace equation in the terms of the sign of the limit \( c := \lim _{|s| \to \infty} s f(x, s) \), which is assumed to be independent from \( x \in \Omega \). As far as we know there are not many results in the literature concerning the resonance phenomena in case of the systems of partial differential equations. In the paper [4], the topological degree methods were applied to study the existence of periodic solutions for the following quasilinear system

\[
\begin{align*}
[\phi_k(w(x))]' &= g_k(x, w(x), w'(x)), & x \in (0, T), & 1 \leq k \leq m, \\
w_k(0) &= w_k(T), & w_k'(0) &= w_k'(T), & 1 \leq k \leq m,
\end{align*}
\tag{1.5}
\]

where the mapping \( \phi : \mathbb{R}^n \to \mathbb{R}^n \) satisfies appropriate monotonicity and growth assumptions and \( g_k : (0, T) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) for \( 1 \leq k \leq m \), are bounded continuous functions. There was shown that the system (1.5) has a solution provided the following Landesman-Lazer type conditions hold

\[
\int _0^T g_k^+ (x) \, dx < 0 < \int _0^T g_k^- (x) \, dx, \quad 1 \leq k \leq m.
\]

Here, for any \( 1 \leq k \leq m \), the function \( g_k^\pm : (0, T) \to \mathbb{R} \) is given by

\[
g_k^\pm (x) := \lim _{s \to \pm 0} g_k(x, u + s e_k, y), \quad x \in \Omega,
\]
where \( \{e_j \mid 1 \leq j \leq m\} \) is the standard Euclidean basis and the limit is assumed to be uniform with respect to \( u \in \text{span}\{e_j \mid j \neq k\} \) and \( y \in \mathbb{R}^n \). Motivated by the above results we intend to consider more general Landesman-Lazer type resonance conditions for the system of differential equations \((1.1)\). To this end, we assume that \( \sigma_1, \ldots, \sigma_m \in [0, 1] \) are given numbers and \( f_k^\pm : \Omega \to \mathbb{R} \) for \( 1 \leq k \leq m \), are continuous functions such that

\[
f_k^\pm(x) := \lim_{s \to \pm\infty} |s|^{\sigma_k} f_k(x, u + se_k, y), \quad x \in \Omega, \tag{1.6}
\]

where we assume that the limit \((1.6)\) is uniform with respect to \( u \in \text{span}\{e_j \mid j \neq k\} \) and \( y \in \mathbb{R}^m \). Let us define \( J_1 \) (resp. \( J_2 \)) to be the collection of indexes \( 1 \leq j \leq l \) (resp. \( l + 1 \leq j \leq m \)) such that the element \( \sigma_j \) is minimal in the set \( \{\sigma_1, \ldots, \sigma_l\} \) (resp. \( \{\sigma_{l+1}, \ldots, \sigma_m\} \)). Then we consider the following resonance conditions

\[
(LL1)_\pm \left\{ \begin{aligned}
&\sum_{k \in J_1} \left( \pm \int_{\{u_k > 0\}} f_k^+(x)|u_k(x)|^{1-\sigma_k} dx \mp \int_{\{u_k < 0\}} f_k^+(x)|u_k(x)|^{1-\sigma_k} dx \right) > 0 \\
&\text{for} \ (u_1, \ldots, u_l) \in \text{Ker} (\lambda_1 I - A_1) \times \ldots \times \text{Ker} (\lambda_l I - A_l),
\end{aligned} \right.
\]

and

\[
(\text{LL}2)_\pm \left\{ \begin{aligned}
&\sum_{k \in J_2} \left( \pm \int_{\{u_k > 0\}} f_k^-(x)|u_k(x)|^{1-\sigma_k} dx \mp \int_{\{u_k < 0\}} f_k^-(x)|u_k(x)|^{1-\sigma_k} dx \right) > 0 \\
&\text{for} \ (u_{l+1}, \ldots, u_m) \in \text{Ker} (\lambda_{l+1} I - A_{l+1}) \times \ldots \times \text{Ker} (\lambda_m I - A_m).
\end{aligned} \right.
\]

Let us observe that in the assumption \((1.6)\) we use the parameter \( \sigma_k \) to measure the strength of the resonance for the component \( f_k \) of the nonlinear perturbation. In particular, the cases \( \sigma_k = 1 \) and \( \sigma_k = 0 \) correspond to the known situations that were studied for a single differential equation in \([11]\) and \([20]\), respectively. To the best of our knowledge, the intermediate case \( \sigma_k \in (0, 1) \) seems to be not considered in the literature so far. The main results of this paper are Theorems \([4.1]\) and \([4.2]\) that express the Rybakowski-Conley index of the set consisting of all bounded solutions of the system \((1.1)\) in the terms of the resonance conditions \((LL1)_\pm\) and \((\text{LL}2)_\pm\). The theorems extend the earlier result of \([37]\), where the index formula were obtained for the parabolic differential equation defined on a bounded domain, with the assumption that the resonance at infinity does not occur. The homotopy invariant that we use in our studies was developed in \([30]\) and \([38]\), as an infinite dimensional generalization of the classical Conley index for the flows defined on finite dimensional spaces (see \([15]\), \([40]\) and \([42]\) for more details). The main advantage coming from the application of the homotopy invariant is that we do not require the system of equations \((1.1)\) to have a gradient form, which in turn is a crucial assumption in the variational approach.

In the proof of Theorems \([4.1]\) and \([4.2]\) we exploit the spectral theorem for the operator \( A \) to obtain a direct sum decomposition of the space \( L^p(\Omega; \mathbb{R}^m) \) into three components, among which we have the kernel of the operator \( A \) and two other spaces corresponding to the positive and negative part of the spectrum of \( A \). Using the homotopy invariance of the Rybakowski-Conley index we can deform the semiflow associated with the system \((1.1)\) to the product of semiflows defined on the spaces coming from the spectral decomposition. One of them is the \( C_0 \) semiflow generated by a restriction of the operator \(-A\), while the other one is the semiflow associated with the vector field obtained by the projection of the nonlinear perturbation \((f_1, \ldots, f_m)\) onto the kernel \( \text{Ker} A \). Then the crucial point of our argument is to determine the contribution to the homotopy index coming from the later semiflow, which appears to be dependent from the resonance conditions.
The paper is organized as follows. In Section 2 we set the abstract framework to define the semiflow $\Phi$ associated with the reaction-diffusion system \([1,1]\) and furthermore, we recall the definition and properties of the Rybakowski-Conley homotopy index. Section 3 is devoted to the spectral decomposition of the operator $A$ on the space $L^p(\Omega; \mathbb{R}^m)$. In Section 4 we state the main results of the paper and construction of the family of semiflows \(\{\Psi^s\}_{s \in [0,1]}\) that will be used as the homotopy deformation of $\Phi$. In Section 5 we apply the resonance conditions $(LL1)_\pm$ and $(LL2)_\pm$ to obtain the guiding function estimates for the nonlinear perturbation $(f_1, \ldots, f_m)$, whereas in Section 6 we establish a priori estimates for the bounded full solutions of the family \(\{\Psi^s\}_{s \in [0,1]}\). Then, in Section 7, we provide the proof of Theorems 4.1 and 5.2 and finally, in Section 8 we provide applications of the obtained results to study the existence of solutions connecting stationary points for the system of nonlinear heat equations.

2. Abstract framework and homotopy index

Let us write $X := [L^p(\Omega)]^m$ for the real vector space equipped with the norm

$$
\|u\|^p := \sum_{k=1}^m \int_\Omega |u_k(x)|^p \, dx, \quad u = (u_1, \ldots, u_m) \in X
$$

and assume that $A := (A_1 - \lambda_1 I) \times \ldots \times (A_m - \lambda_m I)$ is the product operator defined on the space $X$. It is known (see e.g. \([14, 25, 34, 45]\)) that the operator $A$ is sectorial, that is, there are $\gamma \in (0, \pi/2)$, $C_1 \geq 1$ and $a \in \mathbb{R}$, such that the sector

$$
\Sigma_{a, \gamma} := \{\lambda \in \mathbb{C} \mid \gamma \leq |\arg(\lambda - a)| \leq \pi, \ \lambda \neq a\}
$$

is contained in the resolvent set $\rho(A)$ of the operator $A$ and the inequality holds

$$
\| (\lambda I - A)^{-1} \| \leq C_1/|\lambda - a|, \quad \lambda \in \Sigma_{a, \gamma}.
$$

Furthermore $-A$ is a generator of a compact analytic $C_0$ semigroup \(\{S_A(t)\}_{t \geq 0}\) of bounded linear operators on $X$. Let us observe that from \([26\text{ Theorem 16.7.2}]\) we have the following useful kernel relation

$$
\ker A = \ker (I - S_A(t)), \quad t > 0. \quad (2.1)
$$

If we write $\delta := 1 + \max\{\lambda_k \mid 1 \leq k \leq m\}$, then the operator $A_\delta := A + \delta I$ is positively defined, that is, $\Re \mu > 0$ for any element $\mu$ from the spectrum $\sigma(A + \delta I)$. Hence, given $\alpha \geq 0$, we can define the fractional space $X^\alpha$ as the domain of the fractional power $(\delta I + A)^\alpha$ (see \([34\text{ Section 2.6}]\)), endowed with the graph norm

$$
\|u\|_\alpha := \|(A + \delta I)^\alpha u\|, \quad u \in X^\alpha.
$$

It is known that $X^\alpha$ is a Banach space, continuously embedded in $X$, that is, there is a constant $C_2 > 0$ such that the following inequality holds

$$
\|u\| \leq C_2\|u\|_\alpha, \quad u \in X^\alpha. \quad (2.2)
$$

From now on we assume additionally that

$$
\alpha \in (3/4, 1), \quad p \geq 2n \quad (2.3)
$$

and furthermore we require that, for any $1 \leq k \leq m$, the map $f_k : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ satisfies the following conditions:

(F1) given $R > 0$, there exists a constant $L_R > 0$ such that

$$
|f_k(x, s_1, y_1) - f_k(x, s_2, y_2)| \leq L_R (|s_1 - s_2| + |y_1 - y_2|),
$$

for $x \in \Omega$, $s_1, s_2 \in \mathbb{R}^m$ and $y_1, y_2 \in \mathbb{R}^m$ with $|s_1|, |s_2|, |y_1|, |y_2| \leq R$;
Theorem 1.6.1, gives the inclusion $X \subset C^1(\mathbb{R})$ together with the inequality

$$\|u\|_{C^1(\mathbb{R})} \leq C_4\|u\|_\alpha, \quad u \in X^\alpha,$$

where $C_4 > 0$ is a constant. Therefore, we can use [14, Theorem 3.2.1] to deduce that any bounded set is well-defined and straightforward calculations show that it is bounded and satisfies the Lipschitz condition on the bounded subsets of $X^\alpha$. Consequently the system (1.1) can be written in the following abstract form

$$\dot{u}(t) = -Au(t) + F(u(t)), \quad t > 0. \quad (2.4)$$

Definition 2.1. Given the interval $I \subset \mathbb{R}$, we say that the function $u : I \to X^\alpha$ is a mild solution of the equation (2.4), provided

$$u(t) = S_A(t-t')u(t') + \int_s^t S_A(t-\tau)F(u(\tau))d\tau \quad \text{for} \quad t, t' \in I \quad \text{with} \quad t' < t.$$

From [25, Theorem 3.3.3], [25, Corollary 3.3.5] and Remark 4.6 it follows that, for any $u_0 \in X^\alpha$, equation (2.4) admits a unique global mild solution $u(\cdot; u_0) : [0, +\infty) \to X^\alpha$ starting at $u_0$. Hence we can define a semiflow $\Phi : [0, +\infty) \times X^\alpha \to X^\alpha$ associated with the equation (2.4) by the following formula

$$\Phi(t, u_0) := u(t; u_0), \quad t \geq 0, \quad u_0 \in X^\alpha.$$

From [14, Proposition 2.3.2], we infer that the semiflow is continuous, that is, for any sequence $(u_n)$ in $X^\alpha$ such that $u_n \to u_0$ as $n \to \infty$, we have

$$\Phi(t; u_n) \to \Phi(t; u_0) \quad \text{for} \quad t \geq 0, \quad \text{as} \quad n \to \infty$$

and the convergence is uniform for the time $t$ from bounded subsets of the interval $[0, +\infty)$. Furthermore it is known that the operator $A$ has compact resolvents and therefore, we can use [14, Theorem 3.2.1] to deduce that any bounded set $M \subset X^\alpha$ is admissible with respect to $\Phi$, which means that, for every sequences $(u_n)$ in $X^\alpha$ and $(t_n)$ in $[0, +\infty)$, if $t_n \to +\infty$ as $n \to \infty$ and

$$\Phi([0, t_n] \times \{u_n\}) \subset M, \quad n \geq 1,$$

then the set $\{\Phi(t_n, u_n) \mid n \geq 1\}$ is relatively compact in the space $X^\alpha$.

Definition 2.2. Let us assume that $u : [-\delta_1, \delta_2) \to X^\alpha$, where $\delta_1 \geq 0$ and $\delta_2 > 0$, is a continuous map. We say that $u$ is a solution of the semiflow $\Phi$, provided

$$\Phi(t, u(s)) = u(t + s) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad s \in [-\delta_1, \delta_2]$$

such that $t + s \in [-\delta_1, \delta_2)$. In particular, if the map $u$ is defined on the whole real line, then $u$ is called the full solution of the semiflow $\Phi$. \hfill \Box

We recall that the set $K \subset X^\alpha$ invariant with respect to $\Phi$ provided, for every $u_0 \in K$ there is a full solution $u$ of the semiflow $\Phi$ such that $u(0) = u_0$ and $u(\mathbb{R}) \subset K$. Therefore, given $M \subset X^\alpha$, we define its maximal invariant subset $\text{Inv} M = \text{Inv} (M, \Phi)$ as the set of points $u_0 \in N$ with the property that there is a full solution $u$ of the semiflow $\Phi$ such that $u(0) = u_0$ and $u(\mathbb{R}) \subset M$. In particular, we call $K$ an isolated invariant set, if there is a closed set $M \subset X^\alpha$ such that

$$K = \text{Inv} M \subset \text{int} M.$$

Then we say that $M$ is an isolating neighborhood for $K$. 

(F2) there exists a constant $C_3 > 0$ such that, for any $1 \leq k \leq m$, we have

$$|f_k(x, s, y)| \leq C_3, \quad x \in \Omega, \quad s \in \mathbb{R}^m, \quad y \in \mathbb{R}^{mn}.$$
Definition 2.3. Assume that $B \subset X^\alpha$ is a closed set and let $u_0 \in \partial B$. We say that $u_0$ is a strict egress point (resp. strict ingress point, resp. bounce off point), if for any solution $u : [-\delta_1, \delta_2) \to X^\alpha$, where $\delta_1 \geq 0$ and $\delta_2 > 0$, of the semiflow $\Phi$ such that $u(0) = u_0$ the following conditions are satisfied:

(a) there is $\varepsilon_2 \in (0, \delta_2)$ such that $u(t) \notin B$ (resp. $u(t) \in \text{int} B$, resp. $u(t) \notin B$) for $t \in (0, \varepsilon_2]$;

(b) if $\delta_1 > 0$ then there is $\varepsilon_1 \in (0, \delta_1)$ such that $u(t) \in \text{int} B$ (resp. $u(t) \notin B$, resp. $u(t) \notin B$) for $t \in [-\varepsilon_1, 0)$.

Then we write $B^e$ (resp. $B^i$, resp. $B^b$) for the set of strict egress points (resp. strict ingress points, resp. strict bounce off points) and furthermore, we set $B^- := B^e \cup B^b$. We say that the set $B \subset X^\alpha$ is an isolating block, provided $\partial B = B^e \cup B^i \cup B^b$ and $B^-$ is a closed set in $X^\alpha$.

Definition 2.4. We write $[Y, y_0]$ for the homotopy type of pointed topological space $(Y, y_0)$. In particular, if $Y = \{y_0\}$ then we say that the homotopy type $[\{y_0\}, y_0]$ is trivial and we denote it by $0$. Furthermore, given $k \geq 0$, we set $\Sigma^k := [S^k, s_0]$, where $S^k$ is $k$-dimensional unit sphere and $s_0 \in S^k$ is an arbitrary point.

From [38, Theorem I.5.1] we know that, for any isolated invariant set $K$, which admits an admissible isolating neighborhood, we can construct an isolating block $B$ such that $K = \text{Inv} B$. Then we define the homotopy index of $K$ as

$$h(\Phi, K) := \begin{cases} [B/B^-, [B^-]], & \text{if } B^- \neq \emptyset, \\ [B \cup \{c\}, c], & \text{if } B^- = \emptyset, \end{cases}$$

where $B/B^-$ is the quotient topological space and $B \cup \{c\}$ is a disjoint sum of $B$ and the one point space $\{c\}$. It is known that the homotopy index is independent from the choice of isolating block $B$ for the set $K$ and has the following properties.

(H1) If $M \subset X^\alpha$ is an admissible isolating neighborhood and the homotopy index of $K := \text{Inv} M$ is nontrivial, then the set $K$ is non-empty.

(H2) Let $\varphi_j : [0, +\infty) \times A_j \to A_j$, for $j = 1, 2$, be semiflows defined on the closed components of the direct sum decomposition $X^\alpha = A_1 \oplus A_2$. Assume that, for any $j = 1, 2$, the set $M_j \subset A_j$ is an admissible isolating neighborhood for $K_j := \text{Inv} (M_j, \varphi_j)$. Then the set $M_1 \oplus M_2$ is an admissible isolating neighborhood with respect to the product semiflow $\varphi_1 \oplus \varphi_2$ and

$$h(\varphi_1 \oplus \varphi_2, K) = h(\varphi_1, K_1) \wedge h(\varphi_2, K_2),$$

where $K := \text{Inv} (M_1 \oplus M_2, \varphi_1 \oplus \varphi_2)$.

(H3) Assume that the closed set $M \subset X^\alpha$ is admissible with respect to the family of semiflows $\{\Psi^s\}_{s \in [0, 1]}$, that is, for every sequences $(u_n)$ in $[0, 1]$ and $(t_n)$ in $[0, +\infty)$, if $t_n \to +\infty$ as $n \to \infty$ and

$$\Psi^{s_n}([0, t_n] \times \{u_n\}) \subset M, \quad n \geq 1,$$

then the set $\{\Psi^{s_n}(t_n, u_n) \mid n \geq 1\}$ is relatively compact in $X^\alpha$. If the set $M$ is an isolating neighborhood of $K_s := \text{Inv} (\Psi^s, M)$ for all $s \in [0, 1]$, then

$$h(\Psi^0, K_0) = h(\Psi^1, K_1).$$

Let us assume that $u$ is a full solution of the semiflow $\Phi$. We define the limit set $\alpha(u)$ (resp. $\omega(u)$) to be the collection of points $u' \in X^\alpha$ such that $u(t_n) \to u'$ for some $t_n \to -\infty$ (resp. $t_n \to +\infty$). The following proposition (see [38, Theorem 11.5]) provides a tool to study the existence of solutions connecting invariant sets by the use of the homotopy index.
Proposition 2.5. Let us assume that $K$ and $K_0$ are isolated invariant sets such that each of them admits an admissible isolating neighborhood and furthermore
\[ h(\varphi, K_0) \neq \emptyset \text{ and } h(\varphi, K) = \Sigma_k^\infty \text{ for some } k \geq 0. \]
If $h(\varphi, K_0) \neq h(\varphi, K)$ then there is a full solution $u$ of the semiflow $\Phi$ such that either $a(u) \subset K_0$ or $\omega(u) \subset K_0$.

3. Spectral decomposition of the product operator

In this section we assume that $\lambda := (\lambda_1, \ldots, \lambda_m)$ is a vector consisting of real numbers. Let us observe that, for any $1 \leq k \leq m$, the spectrum $\sigma(\lambda_k)$ consists of a bounded below sequence of real eigenvalues, which is either finite or divergent to infinity. Since the respective eigenspaces of the operator $\lambda_k$ are finite dimensional, we can define the number $d_k(\lambda) := 0$ if $\lambda_k$ is the first eigenvalue of $\lambda_k$ and otherwise
\[ d_k(\lambda) := \sum_{\lambda<\lambda_k} \dim \ker (\lambda I - A_k) \]
where in the above summation the parameter $\lambda$ ranges over the set $\sigma_p(A_k)$ of eigenvalues of the operator $A_k$. Then we put
\[ d_\infty(\lambda) := d_1(\lambda) + \ldots + d_m(\lambda). \tag{3.1} \]
Let us consider the auxiliary product operator $\hat{A} := (\hat{A}_1 - \lambda_1 I) \times \ldots \times (\hat{A}_m - \lambda_m I)$, defined on the space $X := L^2(\Omega; \mathbb{R}^m)$, where, for any $1 \leq k \leq m$, we assume that $\hat{A}_k$ is a linear operator on $L^2(\Omega)$ given by
\[ \begin{aligned}
D(\hat{A}_k) := & C^2(\Omega) | u(x) = 0 \text{ for } x \in \partial \Omega, \\
\hat{A}_k u := & D_1(a_{ij}^k D_j u), \quad u \in D(\hat{A}_k).
\end{aligned} \tag{3.2} \]
\[ \sigma(\hat{A}_k) \subset \mathbb{R} \]
Remark 3.1. We claim that $\sigma_p(A) = \sigma_p(\hat{A}) \subset \mathbb{R}$. Indeed, since $\hat{A}$ is a symmetric operator and $A \subseteq \hat{A}$, its spectrum consists of real eigenvalues. Furthermore, the fact that $A \subseteq \hat{A}$ implies $\sigma_p(A) \subset \sigma_p(\hat{A})$. To check the opposite inclusion let us observe that, by the regularity properties of the elliptic operators (see e.g. [45]), the eigenvalues of the operator $\hat{A}$ can be considered as smooth functions on the set $\Omega$. Therefore, if $\mu \in \sigma_p(\hat{A})$ and $u \in \ker (\mu I - \hat{A})$, then $u \in D(A)$ and $(\mu I - A)u = 0$. Hence $\mu \in \sigma_p(A)$ and $\sigma_p(\hat{A}) \subset \sigma_p(A)$ as desired. \[ \square \]
Since the operator $A$ has compact resolvents, its spectrum $\sigma(A)$ can be represented as the sequence of complex isolated eigenvalues $(\mu_k)_{k \geq 1}$, which is finite or $|\mu_k| \to \infty$ as $k \to \infty$ (see e.g. [14], [45]). On the other hand, from Remark 3.2 we know that $\mu_k \in \mathbb{R}$ for $k \geq 1$. On the other hand, the resonance assumption (3.1), gives $\ker A \neq \{0\}$, and hence we can choose $r \geq 1$ such that
\[ 0 < \mu_k - \mu_{k+1}, \quad k \geq r + 1. \]
By the spectral theorem for the symmetric operators with compact resolvents (see [25] Theorem 1.5.2) we obtain the direct sum decomposition $X = \hat{X}_0 \oplus \hat{X}_1 \oplus \hat{X}_2$ on the closed and mutually orthogonal in $X$ spaces $\hat{X}_0$, $\hat{X}_1$ and $\hat{X}_2$, where
\[ \hat{X}_0 = \ker (\mu I - \hat{A}) \quad \text{and} \quad \hat{X}_1 = \ker (\mu_1 I - \hat{A}) \oplus \ldots \oplus \ker (\mu_{r-1} I - \hat{A}). \tag{3.3} \]
Furthermore we have the following inclusions
\[ \hat{A}(D(\hat{A}) \cap \hat{X}_k) \subset \hat{X}_k, \quad k = 0, 1, 2 \tag{3.4} \]
and, if $\hat{A}_k$ is a part of the operator $\hat{A}$ in the space $\hat{X}_k$, then
\[ \sigma(\hat{A}_1) = \{ \mu_1, \ldots, \mu_{r-1} \} \quad \text{and} \quad \sigma(\hat{A}_2) = \{ \mu_k | k \geq r + 1 \}. \tag{3.5} \]
Let us define $X_k := X \cap \tilde{X}_k$ for $k = 0, 1, 2$ and write

$$N_1 := \text{Ker} \,(A_1 - \lambda_1 I) \times \ldots \times \text{Ker} \,(A_l - \lambda_l I),$$

$$N_2 := \text{Ker} \,(A_{i+1} - \lambda_{i+1} I) \times \ldots \times \text{Ker} \,(A_m - \lambda_m I).$$

**Remark 3.2.** We claim that

$$\dim X_1 = d_\infty(\lambda).$$

Indeed, as an immediate consequence of (3.3), we obtain

$$X_1 = \tilde{X}_1 = \text{Ker} \,(\mu_1 I - \hat{A}) \oplus \ldots \oplus \text{Ker} \,(\mu_l I - \hat{A}). \quad (3.6)$$

Furthermore, for any $1 \leq k \leq r - 1$, we have

$$\text{Ker} \,(\mu_k I - \hat{A}) = \text{Ker} \,(\mu_k + \lambda_j)I - \hat{A}_j). \quad (3.7)$$

Let us observe that, given $1 \leq j \leq m$, we have

$$\{\mu_k + \lambda_j \mid 1 \leq k \leq r - 1 \text{ and } \text{Ker} \,(\mu_k + \lambda_j)I - \hat{A}_j) \neq \{0\}\} = \left\{\lambda < \lambda_j \mid \text{Ker} \,(\lambda I - \hat{A}_j) \neq \{0\}\right\},$$

which together with (3.4) and (3.7) give

$$\dim X_1 = \sum_{k=1}^{r-1} \dim \text{Ker} \,(\mu_k I - \hat{A}) = \sum_{k=1}^{r-1} \sum_{j=1}^{m} \dim \text{Ker} \,(\mu_k + \lambda_j)I - \hat{A}_j)$$

$$= \sum_{j=1}^{m} \sum_{\lambda < \lambda_j} \dim \text{Ker} \,(\lambda I - \hat{A}_j) = \sum_{j=1}^{m} d_j(\lambda) = d_\infty(\lambda),$$

as claimed.

Let us observe that $X_k = \tilde{X}_k \subset D(A)$ for $k = 0, 1$ and the space $X$ can be represented as the direct sum $X = N_1 \oplus N_2 \oplus X_1 \oplus X_2$, where the component spaces are closed in $X$ and mutually orthogonal in $\tilde{X} = L^2(\Omega; \mathbb{R}^m)$. Given $k = 1, 2$, we denote by $P_k$ and $Q_k$ the projections on the spaces $N_k$ and $X_k$, respectively, that are determined by this decomposition. We also write $Q_0 := P_1 + P_2$. Let us observe that, using the continuity of the inclusion $X_\alpha \subset X$, we obtain

$$X_\alpha = N_1 \oplus N_2 \oplus X_1^\alpha \oplus X_2^\alpha,$$

where $X_k^\alpha := X_\alpha \cap X_k$ for $k = 1, 2$, are closed subspaces of $X_\alpha$. Therefore the linear operator $Q_k$ can be restricted to the bounded map $Q_k : X_\alpha \to X_\alpha$ for $k = 1, 2$.

**Remark 3.3.** Given $k = 1, 2$, we denote by $A_k$ the part of the operator $A$ in the space $X_k$. We claim that $\rho(A) \subset \rho(A_k)$ and

$$(\mu I - A)^{-1} v_k = (\mu I - A_k)^{-1} v_k, \quad v_k \in X_k. \quad (3.8)$$

Indeed, let us assume that $k = 1, 2$ is fixed and take $\mu \in \rho(A)$. Then have $\text{Ker} \,(\mu I - A_k) = \{0\}$ because $A_k \subset A$. On the other hand, if we take arbitrary $v \in X_k$ then $(\mu I - A)u = v$ for some $u \in D(A)$. Let us write $u = u_0 + u_1 + u_2$, where $u_i \in X_i$ for $i \in \{0, 1, 2\}$. Since $u, u_0, u_1 \in D(A)$, it follows also that $u_2 \in D(A)$ and therefore

$$v = (\mu I - A) u = (\mu I - A) u_0 + (\mu I - A) u_1 + (\mu I - A) u_2.$$

Combining this with (3.4) and the fact that $v \in X_k$, we have $v = (\mu I - A) u_k = (\mu I - A) u_k$, which gives $u_k \in D(A_k)$ and $(\mu I - A_k) u_k = v$. Hence the operator $\mu I - A_k$ is invertible on $X_k$ and (3.3) holds. Observe that the operator $A_k$ is closed as a part of the closed operator $A$ in the closed subspace $X_k \subset X$. Consequently the inverse operator $(\mu I - A_k)^{-1}$ is bounded on $X_k$ and $\mu \in \rho(A_k)$ as claimed. \qed
that given $k$, the opposite inclusion we take arbitrary $c, C$ of bounded linear operators and there are constants $\sigma_p(A_k)$.

Remark 3.4. We claim that $\sigma_p(A_k) = \sigma_p(\hat{A}_k)$ for $k = 1, 2$. Indeed, let us observe that given $k = 1, 2$, the relation $A_k \subset \hat{A}_k$ implies that $\sigma_p(A_k) \subset \sigma_p(\hat{A}_k)$. To prove the opposite inclusion we take arbitrary $\mu \in \sigma_p(\hat{A}_k)$. Then there is a non-zero $u \in D(\hat{A}_k) \subset \hat{X}_k$ such that $0 = (\mu I - \hat{A}_k)u = (\mu I - \hat{A})u$. Using the regularity properties of the elliptic operators once again (see e.g. [45]), we infer that $u \in C^\infty(M)$. Consequently $u \in D(A) \cap X_k$ and $(\mu I - A)u = (\mu I - \hat{A})u = 0$. Hence $\mu \in \sigma_p(A_k)$ and $\sigma_p(\hat{A}_k) \subset \sigma_p(A_k)$ as claimed.

Since the operator $A$ has compact resolvents, by Remark 3.3 it follows that the operator $A_2$ also has the property. Combining this with Remark 3.4 gives $\sigma(A_2) = \sigma_p(A_2) = \sigma_p(\hat{A}_2)$, which together with (3.5) and the fact that $A_1 = \hat{A}_1$ provide

$$\sigma(A_1) = \{\mu_1, \ldots, \mu_{r-1}\} \quad \text{and} \quad \sigma(A_2) = \{\mu_j \mid j \geq r + 1\}. \quad (3.9)$$

Let us take arbitrary $\mu \in \rho(A)$ and observe that from the definition of the operator $A$ and equality (3.8), we have

$$Q_k(\mu I - A)^{-1}u = (\mu I - A)^{-1}Q_ku = (\mu I - A_k)^{-1}Q_ku, \quad u \in X,$$

$$P_k(\mu I - A)^{-1}u = (\mu I - A)^{-1}P_ku = (\mu I - A_k)^{-1}P_ku, \quad u \in X,$$

which by the Euler formula for the $C_0$ semigroups (see [34, Theorem 8.3]) yields

$$S_{A_k}(t)Q_ku = S_A(t)Q_ku = Q_kS_A(t)u, \quad t \geq 0, \quad u \in X, \quad (3.10)$$

$$S_A(t)P_ku = P_kS_A(t)u, \quad t \geq 0, \quad u \in X. \quad (3.11)$$

Remark 3.5. The semigroup $\{S_A(t)\}_{t \geq 0}$ extends on the space $X_1$ to a $C_0$ group of bounded linear operators and there are constants $c, C_5 > 0$ such that

$$\|S_A(t)u\| \leq C_5e^{ct}\|u\|, \quad t \leq 0, \quad u \in X_1, \quad (3.12)$$

$$\|S_A(t)u\| \leq C_5e^{ct}\|u\|, \quad t \geq 0, \quad u \in X_2, \quad (3.13)$$

$$\|S_A(t)u\| \leq C_5e^{ct\alpha}\|u\|, \quad t > 0, \quad u \in X_2. \quad (3.14)$$

Indeed, observe that the equality (3.10) implies that

$$S_A(t)u = S_{A_k}(t)u, \quad t \geq 0, \quad u \in X_1 \quad (3.15)$$

Since $X_1$ is a finite dimensional space, the operator $A_1$ generates the $C_0$ group of bounded linear operators on $X_1$, which by the equality (3.15) is the desired extension of $\{S_A(t)\}_{t \geq 0}$ on $X_1$. Furthermore, by (3.9), the operator $-A_1$ is positively definite which implies that

$$\|S_A(t)u\| = \|S_{A_k}(t)u\| \leq Ce^{ct}\|u\|, \quad t \leq 0, \quad u \in X_1,$$  

where $c, C > 0$ are constants and consequently the inequality (3.12) follows. As a direct consequence of Remark 3.3 and (3.9) we find that $A_2$ is positively defined sectorial operator on $X_2$, which by [25, Theorem 6.13] allows us to modify the constants $c, C > 0$ if necessary, to obtain the following estimates

$$\|S_{A_k}(t)u\| \leq Ce^{-ct}\|u\|, \quad t \geq 0, \quad u \in X_2, \quad (3.16)$$

$$\|A_2^\alpha S_{A_2}(t)u\| \leq Ce^{-ct\alpha}\|u\|, \quad t > 0, \quad u \in X_2. \quad (3.17)$$

On the other hand, (3.10) implies that $S_A(t)u = S_{A_2}(t)u$ for $t \geq 0$ and $u \in X_2$, which together with the inequality (3.10) gives

$$\|S_A(t)u\| = \|S_{A_2}(t)u\| \leq Ce^{-ct}\|u\|, \quad t \geq 0, \quad u \in X_2,$$

and (3.13) follows. Observe that, by [25, Theorem 1.4.6] we have the equality of domains $D(A_2^\alpha) = D((\delta + A_2)^\alpha)$ and the equivalence of the corresponding norms

$$C''\|A_2^\alpha u\| \leq \|(\delta + A_2)^\alpha u\| \leq C'\|A_2^\alpha u\|, \quad u \in D(A_2^\alpha), \quad (3.18)$$
where \( C', C'' > 0 \). On the other hand, by the definition of the fractional power of the positive sectorial operator \((\delta I + A_2)^\alpha \subset (\delta I + A)^\alpha\), which together with (3.17) and (3.18), for any \( t > 0 \) and \( u \in X_2 \) yields
\[
\|S_A(t)u\|_\alpha = \|\delta I + A)^\alpha S_A(t)u\| = \|(\delta I + A)^\alpha S_{A_2}(t)u\| \\
= \|(\delta I + A_2)^\alpha S_{A_2}(t)u\| \leq C\|A_2^2 S_{A_2}(t)u\| \leq C' e^{-ct}\alpha n \|u\|,
\]
and the estimate (3.14) follows. \( \square \)

### 4. Statement of the main results

Given \( \lambda = (\lambda_1, \ldots, \lambda_m) \) we impose the following standing resonance assumption
\[
\text{Ker} (\lambda_kI - A_k) \neq \{0\}, \quad 1 \leq k \leq m
\] (4.1)
and define the following numbers
\[
n_1(\lambda) := \sum_{i=1}^l \dim \text{Ker} (\lambda_iI - A_i), \quad n_2(\lambda) := \sum_{i=l+1}^m \dim \text{Ker} (\lambda_iI - A_i).
\] (4.2)

The main results of this paper are the following theorems.

**Theorem 4.1.** Suppose that \( \{h_k\}_{k=1}^n \) is a family of \( L^2(\Omega) \) functions such that
\[
(C1)_\pm \quad \left\{ \begin{array}{l}
\pm f_k(x, u, y)|u_k|^{\alpha} \text{sgn } u_k \geq h_k(x), \quad x \in \Omega, \ u \in \mathbb{R}^m, \ y \in \mathbb{R}^{nm},
\end{array} \right.
\]
for any \( 1 \leq k \leq l \) the following inequality holds
and furthermore
\[
(C2)_\pm \quad \left\{ \begin{array}{l}
\pm f_k(x, u, y)|u_k|^{\alpha} \text{sgn } u_k \geq h_k(x), \quad x \in \Omega, \ u \in \mathbb{R}^m, \ y \in \mathbb{R}^{nm}
\end{array} \right.
\]
for any \( l + 1 \leq k \leq m \) the following inequality holds
If conditions \((LL1)_\pm \) and \((LL2)_\pm \) are satisfied, then the set \( K_\infty \) consisting of all bounded full solutions of the semiflow \( \Phi \) admits an admissible isolating neighborhood. Furthermore the homotopy index of \( K_\infty \) is given by
\[
h(\Phi, K_\infty) = \Sigma^{d_{\infty}(\lambda) + n_1(\lambda) + n_2(\lambda)},
\]
(4.3)
if the conditions \((C1)_+, \ (C2)_+, \ (LL1)_+, \ (LL2)_+ \) hold and
\[
h(\Phi, K_\infty) = \Sigma^{d_{\infty}(\lambda)},
\]
(4.4)
if the conditions \((C1)_-, \ (C2)_-, \ (LL1)_-, \ (LL2)_- \) are satisfied.

In the subsequent result we prove analogous homotopy index formula in the case of the resonance conditions \((LL1)_\pm \) and \((LL2)_\pm \).

**Theorem 4.2.** Suppose that \( \{h_k\}_{k=1}^n \) is a family of \( L^2(\Omega) \) functions such that the inequalities \((C1)_\pm \) and \((C2)_\pm \) hold. If conditions \((LL1)_\pm \) and \((LL2)_\pm \) are satisfied, then the set \( K_\infty \) consisting of all bounded full solutions of the semiflow \( \Phi \) admits an admissible isolating neighborhood and the homotopy index of \( K_\infty \) is given by
\[
h(\Phi, K_\infty) = \Sigma^{d_{\infty}(\lambda) + n_1(\lambda)},
\]
(4.5)
if the conditions \((C1)_+, \ (C2)_-, \ (LL1)_+, \ (LL2)_- \) hold and
\[
h(\Phi, K_\infty) = \Sigma^{d_{\infty}(\lambda) + n_2(\lambda)},
\]
(4.6)
if the conditions \((C1)_-, \ (C2)_+, \ (LL1)_-, \ (LL2)_+ \) are satisfied.
Remark 4.3. Let us consider the nonlinear perturbation given by $F(u) = v_0$ for $u \in X^*$, where $v_0 \in \text{Ker} A \setminus \{0\}$. We claim that the set $K_\infty$ is empty. Indeed, if $u$ would be a bounded full solution for the semiflow $\Phi$, then

$$u(t) = S_A(t)u(0) + \int_0^t S_A(t-\tau)v_0 \, d\tau, \quad t \geq 0,$$

which by the equality (2.1), gives

$$u(t) = S_A(t)u(0) + tv_0, \quad t \geq 0.$$ 

Acting on this equation by the projection $Q_0 = P_1 + P_2$ and using (3.11), we obtain

$$Q_0u(t) = S_A(t)(Q_0u(0)) + tQ_0v_0 = Q_0u(0) + tv_0, \quad t \geq 0,$$

which contradicts the assumption that the solution $u$ is bounded, because $v_0 \neq 0$, and the claim follows. \qed

Remark 4.4. Let us observe that if $\sigma_k = 0$ for some $1 \leq k \leq m$, then the existence of the function $h_k \in L^2(\Omega)$ satisfying the corresponding inequality

$$\pm f_k(x, u, y) \text{sgn } u_k \geq h_k(x), \quad x \in \Omega, \ u \in \mathbb{R}^m, \ y \in \mathbb{R}^{nm},$$

is an obvious consequence of the assumption (F2). Clearly it is enough to take $h_k := \mp C_3$, where $C_3$ is the bounding constant of the maps $f_k$ for $1 \leq k \leq m$. \qed

Remark 4.5. Let us assume that $I_1, \ldots, I_r$ are mutually disjoint sets of natural numbers such that $I_1 \cup \ldots \cup I_r = \{1, \ldots, m\}$. For any $1 \leq k \leq r$, we define $J_k$ as the collection of all the indexes $i \in I_k$ such that the element $\sigma_i$ is minimal in the set $\{\sigma_j \mid j \in I_k\}$. Given $\epsilon_1, \ldots, \epsilon_r \in \{+,-\}$, we assume that the following resonance conditions are satisfied

\begin{align*}
(LL 1)_{\epsilon_1} \quad & \left\{ \sum_{k \in J_1} \epsilon_1 \left( \int_{(u_k > 0)} f_k^+(x)|u_k(x)|^{1-\sigma_k} \, dx - \int_{(u_k < 0)} f_k^-(x)|u_k(x)|^{1-\sigma_k} \, dx \right) > 0 \right. \\
& \left. \text{for all } u_k \in \text{Ker} (\lambda_k I - A_k), \text{ where } k \in J_1, \right.
\end{align*}

$$\vdots$$

\begin{align*}
(LL r)_{\epsilon_r} \quad & \left\{ \sum_{k \in J_r} \epsilon_r \left( \int_{(u_k > 0)} f_k^+(x)|u_k(x)|^{1-\sigma_k} \, dx - \int_{(u_k < 0)} f_k^-(x)|u_k(x)|^{1-\sigma_k} \, dx \right) > 0 \right. \\
& \left. \text{for all } u_k \in \text{Ker} (\lambda_k I - A_k), \text{ where } k \in J_r. \right.
\end{align*}

Analyzing the proof of Theorems 4.1 and 4.2 we can check that, under the resonance conditions $(LL 1)_{\epsilon_1} - (LL r)_{\epsilon_r}$, the set $K_\infty$ consisting of all bounded full solutions of the semiflow $\Phi$ has an admissible isolating neighborhood and the homotopy index of $K_\infty$ is given by

$$h(\Phi, K_\infty) = \Sigma^d_{\infty}(\lambda)+\beta_1 \tilde{n}_1(\lambda)+\ldots+\beta_r \tilde{n}_r(\lambda),$$

where, given $1 \leq k \leq r$, we define

$$\tilde{n}_k(\lambda) := \sum_{j \in I_k} \dim \text{Ker} (\lambda_j I - A_j)$$

and furthermore we write $\beta_k := 1$ if $\epsilon_k$ has the plus sign and $\beta_k := 0$ otherwise. \qed

In the proof of Theorems 4.1 and 4.2 we use the homotopy invariance property (H3) of the Rybakowski-Conley index and we deform the semiflow $\Phi$ using the following family of the differential equations

$$\dot{u}(t) = -Au(t) + H(s, u(t)), \quad t > 0,$$ 

(4.7)
where \( H : [0, 1] \times X^\alpha \to X \) is a map given by
\[
H(s, u) := Q_0 F(sQ_1 u + sQ_2 u + Q_0 u) + sQ_1 F(u) + sQ_2 F(u)
\] (4.8)
for \( s \in [0, 1] \) and \( u \in X^\alpha \).

**Remark 4.6.** We claim that, for any \( s \in [0, 1] \), there is a constant \( \tilde{L}_R > 0 \) such that
\[
\|H(s, u_1) - H(s, u_2)\| \leq \tilde{L}_R \|u_1 - u_2\|, \quad s \in [0, 1], \quad \|u_1\|_\alpha, \|u_2\|_\alpha \leq R. \quad (4.9)
\]
To check this, let us write \( R' := (\|Q_0\|_\alpha + \|Q_1 + Q_2\|_\alpha + 1)R \). By the condition \((F1)\), there is a constant \( L_R' > 0 \) such that
\[
\|F(u_1) - F(u_2)\| \leq L_R' \|u_1 - u_2\|_\alpha, \quad \text{if} \quad \|u_1\|_\alpha, \|u_2\|_\alpha \leq R'.
\]
Let us take \( u_1, u_2 \in X^\alpha \) such that \( \|u_1\|_\alpha, \|u_2\|_\alpha \leq R \). Then, for \( k = 1, 2 \), we have
\[
\|sQ_1 u_k + sQ_2 u_k + Q_0 u_k\|_\alpha \leq (\|Q_0\|_\alpha + \|Q_1 + Q_2\|_\alpha)R = R', \quad s \in [0, 1].
\]
Therefore, for any \( s \in [0, 1] \), we obtain
\[
\|H(s, u_1) - H(s, u_2)\| \leq L_R' (\|Q_0\|_\alpha \|sQ_1 u + sQ_2 u + Q_0 u\|_\alpha + \|Q_1 + Q_2\|_\alpha \|u_1 - u_2\|_\alpha)
\]
\[
\leq L_R' (\|Q_0\|_\alpha + \|Q_1 + Q_2\|_\alpha) (\|Q_0\|_\alpha + \|Q_1 + Q_2\|_\alpha + 1) \|u_1 - u_2\|_\alpha,
\]
which gives the inequality \((4.9)\), as desired. Observe that, by condition \((F2)\), we easily deduce that the set \( \{F(v) \mid v \in X^\alpha\} \) is bounded in \( X \). This implies that
\[
\|H(s, u)\| \leq \|Q_0\|_\alpha \|sQ_1 u + sQ_2 u + Q_0 u\| + \|Q_1 + Q_2\|_\alpha \|F(u)\|
\]
\[
\leq \sup \{\|F(v)\| \mid v \in X^\alpha\} (\|Q_0\|_\alpha + \|Q_1 + Q_2\|_\alpha) := C_0,
\]
for \( s \in [0, 1] \) and \( u \in X^\alpha \), which shows that \( H \) is a bounded map. \( \Box \)

Arguing similarly as in Section 2, we can verify that, for any \( s \in [0, 1] \) and \( u_0 \in X^\alpha \), the equation \((4.7)\) admits a unique mild solution \( u(t; s, u_0) : [0, +\infty) \to X^\alpha \) starting at \( u_0 \). In fact, it is enough to use \([25, \text{Theorem 3.3.3}],[25, \text{Corollary 3.3.5}]\) and Remark 4.3. Therefore we are able to define the semiflow associated with the equation \((4.7)\) by the following formula
\[
\Psi^s(t, u_0) := u(t; s, u_0), \quad t \in [0, +\infty), \quad s \in [0, 1], \quad u_0 \in X^\alpha.
\]
Furthermore, applying \([14, \text{Proposition 2.3.2}]\) once again, we deduce that the family of semiflows is continuous, that is, for any sequence \( (u_n) \) in \( X^\alpha \) and \( (s_n) \) in \( [0, 1] \) such that \( u_n \to u_0 \) and \( s_n \to s_0 \) as \( n \to \infty \), we have
\[
\Psi^{s_n}(t; u_0) \to \Psi^{s_0}(t; u_0) \quad \text{for} \quad t \geq 0, \quad \text{as} \quad n \to \infty
\]
and the convergence is uniform for the time \( t \) bounded subsets of \([0, +\infty)\).

Taking into account the fact that the operator \( A \) has compact resolvents, we can use \([11, \text{Theorem 3.2.1}]\) to find that any bounded set \( M \subset X^\alpha \) is admissible with respect to the family \( \{\Psi^s\}_{s \in [0, 1]} \).

5. Estimates for the nonlinear perturbation

In this section we use the resonance conditions \((LL1)_\pm \) and \((LL2)_\pm \) to obtain guiding function type estimates on the nonlinear perturbation \( F \). Let us observe that, due to the inclusion \( L^p(\Omega; \mathbb{R}^m) \subset L^2(\Omega; \mathbb{R}^m) \), we can define the bilinear forms
\[
\langle u, v \rangle_1 := \sum_{k=1}^l \int_{\Omega} u_k(x)v_k(x) \, dx, \quad \langle u, v \rangle_2 := \sum_{k=1}^m \int_{\Omega} u_k(x)v_k(x) \, dx, \quad u, v \in X,
\]
that determine the functions \( \|u\|_k^2 := \langle u, u \rangle_k \) for \( u \in X \) and \( k = 1, 2 \). We intend to prove the following proposition.
Proposition 5.1. Let us assume that \( \{h_k\}_{k=1}^\infty \) is a family of \( L^2(\Omega) \) functions such that the inequalities \((C1)_1^+\) hold. If condition \((LL1)_1^+\) is satisfied, then, for any bounded set \( W \subset X_1^1 \oplus X_2^1 \), there are \( r > 0 \) and \( R > 0 \) such that
\[
\pm(F(u+v+w),u)_1 > r
\]
for \((u,v,w) \in N_1 \times N_2 \times W\) such that \( \|u\|_1 \geq R \).

Proof. Arguing by contradiction, we can suppose that there are sequences \((r_n)\) of positive numbers, \((u_n,v_n)\) in \( N_1 \times N_2 \) and \((w_n)\) in \( W \) such that \( r_n \to 0 \) and \( \|u_n\|_1 \to \infty \) as \( n \to \infty \) and furthermore
\[
\langle F(u_n + v_n + w_n), u_n \rangle \leq r_n, \quad n \geq 1.
\]
For any \( n \geq 1 \), we write \( z_n := u_n/\|u_n\|_1 \). Since \((z_n)\) is a bounded sequence of the finite dimensional space \( N_1 \) and the embedding \( X_1^1 \subset X \) is compact, without loss of generality we can assume that there are \( z_0 \in N_1 \) with \( \|z_0\|_1 = 1 \) and \( w_0 \in X_1^1 \) such that \( \|z_n - z_0\| \to 0 \) and \( \|w_n - w_0\| \to 0 \) as \( n \to \infty \). Let us write \( p_n := u_n + v_n + w_n \) for \( n \geq 1 \). Given \( 1 \leq k \leq l \) we define the sets
\[
\Omega_k^+: = \{ x \in \Omega \mid z_k^+(x) > 0 \}, \quad \Omega_k^- := \{ x \in \Omega \mid z_k^-(x) < 0 \}, \quad 1 \leq k \leq l.
\]
Let us observe that, for any \( 1 \leq k \leq l \), we have \( v_k^+ = 0 \) and
\[
p_n^k/\|u_n\|_1 = z_n^+ + v_k^+/\|u_n\|_1 + w_k^+/\|u_n\|_1 = z_n^+ + w_k^+/\|u_n\|_1 \to z_0^+, \quad n \to \infty
\]
in the space \( L^2(\Omega) \). Therefore, passing to a subsequence if necessary, we obtain
\[
w_k^+(x) \to w_0^+(x) \quad \text{and} \quad p_n^k/\|u_n\|_1 \to z_0^+(x) \quad \text{as} \quad n \to \infty \quad \text{for a.a.} \ x \in \Omega.
\]
Furthermore there are functions \( a_k,b_k \in L^2(\Omega) \) such that
\[
|w_k^+(x)| \leq a_k(x) \quad \text{and} \quad |p_n^k(x)/\|u_n\|_1| \leq b_k(x), \quad \text{for a.a.} \ x \in \Omega, \ n \geq 1.
\]
If we write \( \tilde{\sigma} := \min\{\sigma_1,\ldots,\sigma_l\} \), then, by the inequality \((5.2)\), we have the estimates
\[
\|u_n\|_1^{\tilde{\sigma}-1}r_n \geq \|u_n\|_1^{\tilde{\sigma}-1}(F(u_n + v_n + w_n),u_n)_1 = \|u_n\|_1^{\tilde{\sigma}-1}(F(p_n),u_n)_1
\]
\[
= \sum_{k=1}^l \int_\Omega \|u_n\|_1^{\tilde{\sigma}-1}f_k(x,p_n(x),\nabla p_n(x))w_k^+(x) \, dx.
\]
Let us assume that the inequalities \((C1)_1^+\) and resonance condition \((LL1)_1^+\) are satisfied. If we take arbitrary \( 1 \leq k \leq l \), then, by \((5.4)\), we have
\[
\|u_n\|_1^{\tilde{\sigma}-1}f_k(x,p_n(x),\nabla p_n(x))p_n^k(x)
\]
\[
= \|u_n\|_1^{\tilde{\sigma}-1}f_k(x,p_n(x),\nabla p_n(x))|p_n^k(x)|^{\sigma_k} |p_n^k(x)/\|u_n\|_1|^{1-\sigma_k}
\]
\[
\geq h_k(x)|p_n^k(x)/\|u_n\|_1|^{1-\sigma_k} \geq -|h_k(x)|b_k(x)^{1-\sigma_k}
\]
for \( x \in \Omega \) and \( n \geq 1 \). On the other hand, if we set \( c_0 := \sup\{\|u_n\|_1^{-1} \mid n \geq 1\} \), then, by the condition \((F2)\) and inequality \((5.4)\), we obtain
\[
-\|u_n\|_1^{\tilde{\sigma}-1}f_k(x,p_n(x),\nabla p_n(x))w_n^k(x) \geq -C_3 c_0^{-1-\sigma_k}a_k(x), \quad \text{for a.a.} \ x \in \Omega, \ n \geq 1,
\]
which together with \((5.3)\) gives the following estimate
\[
\|u_n\|_1^{\tilde{\sigma}-1}f_k(x,p_n(x),\nabla p_n(x))w_n^k(x)
\]
\[
\geq -|h_k(x)|b_k(x)^{1-\sigma_k} - C_3 c_0^{-1-\sigma_k}a_k(x)
\]
for \( x \in \Omega \) and \( n \geq 1 \). Let us observe that in the case \( \sigma_k = 1 \) we have
\[
|h_k|b_k^{1-\sigma_k} + C_3 c_0^{1-\sigma_k}a_k = |h_k| + C_3 a_k \in L^2(\Omega) \subset L^1(\Omega).
Furthermore, if $\sigma_k \in [0, 1)$ then $b_k^{1-\sigma_k} \in L^2(\Omega)$. Since $2/(1 - \sigma_k) \geq 2$ and the domain $\Omega \subset \mathbb{R}^n$ is bounded, it follows that $b_k^{1-\sigma_k} \in L^2(\Omega)$ and consequently

$$
\int_\Omega |h_k(x)||b_k(x)|^{1-\sigma_k} \, dx \leq \|h_k\|_{L^2(\Omega)} \|b_k^{1-\sigma_k}\|_{L^2(\Omega)} < \infty.
$$

This in turn, implies that the function $|h_k|b_k^{1-\sigma_k} + C_3c_0^{1-\sigma_k}a_k$ is integrable. Let us observe that, for any $1 \leq k \leq l$, we have

$$
||u_n||_{L^1(\Omega)}^{\sigma_k-1} f_k(x, p_n(x), \nabla p_n(x)) u_n^k(x) = [\text{sgn } p_n^k(x)] f_k(x, p_n(x), \nabla p_n(x))|p_n^k(x)|^{\sigma_k}|p_n^k(x)|^{-1} / \|u_n\|_1^{1-\sigma_k} \quad (5.8)
$$

and furthermore, for any $x \in \Omega_k^\pm$, the following limit hold

$$
\lim_{n \to \infty} f_k(x, p_n(x), \nabla p_n(x))|p_n^k(x)|^{\sigma_k}|p_n^k(x)|^{-1} / \|u_n\|_1^{1-\sigma_k} = f_k^\pm(x)|z_0^k(x)|^{1-\sigma_k}. \quad (5.11)
$$

It is not difficult to check that for $\sigma_k \in (0, 1]$ the limit $1.6$ implies that

$$
\lim_{s\to \pm \infty} f_k(x, u + se_k, y) = 0 \quad \text{for} \quad x \in \Omega, \ u \in \text{span}\{e_j \mid j \neq k\} \quad \text{and} \quad y \in \mathbb{R}^m,
$$

which together with (5.8) and (5.9) give

$$
\lim_{n \to \infty} ||u_n||_{L^1(\Omega)}^{\sigma_k-1} f_k(x, p_n(x), \nabla p_n(x)) u_n^k(x) = 0, \quad x \in \Omega_k^\pm \cup \Omega_k^\pm. \quad (5.13)
$$

Combining (5.8), (5.11), (5.12) and (5.13) we obtain

$$
\lim_{n \to \infty} ||u_n||_{L^1(\Omega)}^{\sigma_k-1} f_k(x, p_n(x), \nabla p_n(x)) u_n^k(x) = f_k^+(x)|z_0^k(x)|^{1-\sigma_k}, \quad x \in \Omega_k^+, \quad (5.14)
$$

and

$$
\lim_{n \to \infty} ||u_n||_{L^1(\Omega)}^{\sigma_k-1} f_k(x, p_n(x), \nabla p_n(x)) u_n^k(x) = -f_k^-(x)|z_0^k(x)|^{1-\sigma_k}, \quad x \in \Omega_k^-. \quad (5.15)
$$

Since the function $|h_k|b_k^{1-\sigma_k} + C_3c_0^{1-\sigma_k}a_k$ is integrable, we can apply (5.7), (5.14), (5.13) and the Fatou lemma to obtain

$$
\liminf_{n \to \infty} \int_{\Omega_k^+} ||u_n||_{L^1(\Omega)}^{\sigma_k-1} f_k(x, p_n(x), \nabla p_n(x)) u_n^k(x) \, dx \geq \int_{\Omega_k^+} f_k^+(x)|z_0^k(x)|^{1-\sigma_k} \, dx \quad (5.16)
$$

and furthermore

$$
\liminf_{n \to \infty} \int_{\Omega_k^-} ||u_n||_{L^1(\Omega)}^{\sigma_k-1} f_k(x, p_n(x), \nabla p_n(x)) u_n^k(x) \, dx \geq -\int_{\Omega_k^-} f_k^-(x)|z_0^k(x)|^{1-\sigma_k} \, dx. \quad (5.17)
$$

As a consequence of the unique continuation property for elliptic operators (see [24, Theorem 1.1] and [17, Proposition 3]), we infer that the set $\Omega_0^- := \{x \in \Omega \mid z_0^k(x) = 0\}$ has the Lebesgue measure equal to zero for all $1 \leq k \leq m$. Furthermore, we
observe that, for $k \not\in J_1$, we have $\|u_n\|_1^{\nu} \rightarrow 0$ as $n \rightarrow \infty$. Hence, by (5.5), (5.16) and (5.17), we find that

$$0 \geq \sum_{k=1}^{n} \liminf_{n \rightarrow \infty} \left( \|u_n\|_1^{\nu} \int_{\Omega} \|u_n\|_1^{\nu} f_k(x, p_n(x), \nabla p_n(x)) u_n(x) \, dx \right)$$

$$= \sum_{k \in J_1} \liminf_{n \rightarrow \infty} \int_{\Omega} \|u_n\|_1^{\nu} f_k(x, p_n(x), \nabla p_n(x)) u_n(x) \, dx$$

$$\geq \sum_{k \in J_1} \left( \int_{\Omega_k^+} f_k^+(x)|\xi_k^+(x)|^{1-\nu} \, dx - \int_{\Omega_k^-} f_k^-(x)|\xi_k^-(x)|^{1-\nu} \, dx \right),$$

which contradicts (LL1) and proves the plus sign case of the inequality (5.1). Analogous argument, the details of which we leave to the reader, shows that the inequalities (C1) and resonance condition (LL1) imply the minus sign form of (5.1) and thus the proof of the proposition is completed.

Following the lines of the above proof, we can show the following proposition concerning the guiding function type estimates for the nonlinear perturbation $F$, with respect to the space $N_2$.

**Proposition 5.2.** Suppose that $\{h_k\}_{k=1}^{\infty}$ is a family of $L^2(\Omega)$ functions such that the inequalities (C2) hold. If condition (LL2) is satisfied, then, for any bounded set $W \subset X_1^+ \oplus X_2^+$, there is $r > 0$ and $R > 0$ such that

$$\pm(F(u + v + w), v)_2 > r$$

for $(u, v, w) \in N_1 \times N_2 \times W$ such that $\|v\|_2 \geq R$.

6. Estimates for bounded solutions of the homotopy flow

We begin with the following lemma, which provides some a priori bounds for the projections of solutions of the parametrized equation (6.1) onto the space $X_1^+ \oplus X_2^+$. 

**Proposition 6.1.** There is $R_0 > 0$ such that for any $s \in [0, 1]$ and for any bounded full solution $u = u_s$ for the semiflow \( \Psi^s \), the following inequality holds

$$\|Q_k u(t)\|_s \leq R_0 \quad \text{for } t \in \mathbb{R} \text{ and } k = 1, 2.$$  \hspace{1cm} (6.1)

**Proof.** Let us assume that $u = u_s$ is a full solution for the equation (6.1) for some $s \in [0, 1]$. From the continuity of the projections $Q_1, Q_2 : X^k \rightarrow X^k$ we deduce the boundedness of the sets $\{Q_1 u(t) \mid t \geq 0\}$ and $\{Q_2 u(t) \mid t \leq 0\}$ in the space $X^k$. Since $u$ is a solutions of the semiflow $\Psi^s$, it follows that the equality $\Psi^s(t-t', u(t')) = u(t)$ holds for all $t, t' \in \mathbb{R}$ such that $t \geq t'$. This in turn can be written in the following integral form

$$u(t) = S_A(t-t')u(t') + \int_{t'}^{t} S_A(t-\tau)H(s, u(\tau)) \, d\tau, \quad t > t'. \hspace{1cm} (6.2)$$

Acting on (6.2) with the operator $Q_k$, where $k = 1, 2$, and using (3.10), we obtain

$$Q_k u(t) = S_A(t-t')Q_k u(t') + \int_{t'}^{t} S_A(t-\tau)Q_k H(s, u(\tau)) \, d\tau, \quad t > t'. \hspace{1cm} (6.3)$$

Since the semigroup $\{S_A(t)\}_{t \geq 0}$ extends on the space $X_1$ to the $C_0$ group of bounded linear operators, we can apply $S_A(t-t')$ on the formula (6.3) to derive

$$S_A(t-t')Q_1 u(t) = Q_1 u(t') + \int_{t'}^{t} S_A(t-\tau)Q_1 H(s, u(\tau)) \, d\tau, \quad t \geq t'. \hspace{1cm} (6.4)$$
Then, the inequalities (2.2) and (3.12) imply that
\[ \|S_A(t' - t)Q_1u(t)\| \leq C_5e^c(t' - t)\|Q_1u(t)\| \leq C_2C_5e^c(t' - t)\|Q_1u(t)\|_\alpha \]
and hence, using the boundedness of the set \( \{Q_1u(t) \mid t \geq 0\} \) in \( X^\alpha \), we find that
\[ \|S_A(t' - t)Q_1u(t)\| \to 0, \quad t \to +\infty. \tag{6.5} \]
Combining this inequality (6.5) with (3.12) and (4.10), we obtain
\[ \|Q_1u(t')\| \leq \|S_A(t' - t)Q_1u(t)\| + C_5 \int_{t'}^{t} e^{c(t' - \tau)}\|Q_1H(s, u(\tau))\| d\tau \]
\[ \leq \|S_A(t' - t)Q_1u(t)\| + C_5C_6\|Q_1\| \int_{t'}^{t} e^{c(t' - \tau)} d\tau \]
\[ \leq \|S_A(t' - t)Q_1u(t)\| + C_5C_6c^{-1}\|Q_1\|. \tag{6.6} \]
Since \( X_1 \) is a finite dimensional space there is constant \( C_7 > 0 \) such that
\[ \|u\|_\alpha \leq C_7\|u\|, \quad u \in X_1. \tag{6.7} \]
Passing in (6.6) to the limit with \( t \to +\infty \) and using (6.5) with (6.7) we obtain
\[ \|Q_1u(t')\|_\alpha \leq C_5C_6C_7\|Q_1\|c^{-1}, \quad t' \in \mathbb{R}, \tag{6.8} \]
which gives the desired estimate (6.1) for \( k = 1 \). Let us observe that, by the inequalities (2.2) and (3.14), we have
\[ \|S_A(t' - t)Q_2u(t')\|_\alpha \leq C_5e^{-c(t' - t)}(t - t')^{-\alpha}\|Q_2u(t')\| \]
\[ \leq C_2C_5e^{-c(t' - t)}(t - t')^{-\alpha}\|Q_2u(t')\|_\alpha, \]
which together with the boundedness of the set \( \{Q_2u(t) \mid t \leq 0\} \) in \( X^\alpha \) give
\[ \|S_A(t' - t)Q_2u(t')\|_\alpha \to 0, \quad t' \to -\infty. \tag{6.9} \]
Combining the formula (6.3) with the inequalities (3.14) and (4.10), we obtain
\[ \|Q_2u(t)\|_\alpha \leq \|S_A(t' - t)Q_2u(t')\|_\alpha + C_5 \int_{t'}^{t} \frac{e^{-c(t' - \tau)}}{(t - \tau)\alpha}\|Q_2H(s, u(\tau))\| d\tau \]
\[ \leq \|S_A(t' - t)Q_2u(t')\|_\alpha + C_5C_6\|Q_2\| \int_{t'}^{t} \frac{e^{-c(t' - \tau)}}{(t - \tau)\alpha} d\tau. \tag{6.10} \]
If we take \( t, t' \in \mathbb{R} \) with \( t' + 1 < t \), then we have the following estimates
\[ \int_{t'}^{t} \frac{e^{-c(t' - \tau)}}{(t - \tau)\alpha} d\tau = \int_{t'}^{t-1} \frac{e^{-c(t' - \tau)}}{(t - \tau)\alpha} d\tau + \int_{t-1}^{t} \frac{e^{-c(t' - \tau)}}{(t - \tau)\alpha} d\tau \]
\[ \leq \int_{t'}^{t-1} e^{-c(t' - \tau)} d\tau + \int_{t-1}^{t} (t - \tau)^{-\alpha} d\tau \leq e^{-c}/c + 1/(1 - \alpha) \]
that together with the inequality (6.10), provide
\[ \|Q_2u(t)\|_\alpha \leq \|S_A(t' - t)Q_2u(t')\|_\alpha + C_5C_6\|Q_2\|((e^{-c}/c + 1/(1 - \alpha))). \tag{6.11} \]
Using (6.9) and passing in (6.11) to limit with \( t' \to -\infty \) we infer that
\[ \|Q_2u(t)\|_\alpha \leq C_5C_6\|Q_2\|((e^{-c}/c + 1/(1 - \alpha)) \], \quad t \in \mathbb{R}. \tag{6.12} \]
Consequently we obtain the estimate (6.1) for \( k = 2 \) and the proof of the proposition is completed. \( \square \)

We proceed to the following proposition, which provides the estimates for the projections of the solutions of the equation (4.7) onto the space \( N_1 \).
Proposition 6.2. Let us assume that \( W \subset X_1^0 \oplus X_2^0 \) is a ball centered at the origin and \( r, R > 0 \) are such that either
\[
\{F(u + v + w), u\}_1 > r \quad \text{for} \quad (u, v, w) \in N_1 \times N_2 \times W \quad \text{with} \quad \|u\|_1 \geq R
\] (6.13)
or
\[
\{F(u + v + w), u\}_1 < -r \quad \text{for} \quad (u, v, w) \in N_1 \times N_2 \times W \quad \text{with} \quad \|u\|_1 \geq R.
\] (6.14)
Then, for any \( s \in [0, 1] \) and any bounded full solution \( u \) of the semiflow \( \Psi^s \) such that \( Q_1 u(t) + Q_2 u(t) \in W \) for \( t \in \mathbb{R} \), the following inequality holds
\[
\|P_1 u(t)\|_1 \leq R, \quad t \in \mathbb{R}.
\] (6.15)

Proof. Since \( N_1 \) is finite dimensional space, the functions \( \| \cdot \|_1 \) and \( \| \cdot \| \) are equivalent norms on \( N_1 \). Hence the boundedness of the solution \( u \) in the space \( X \) gives
\[
\sup_{t \in \mathbb{R}} \|P_1 u(t)\|_1 < +\infty.
\] (6.16)

We argue by a contradiction and assume that there is \( s \in [0, 1] \) and a full solution \( u \) of the semiflow \( \Psi^s \) such that \( \|P_1 u(t_0)\|_1 > R \) for some \( t_0 \in \mathbb{R} \). Acting by the operator \( P_1 \) on the integral formula
\[
u(t) = S_A(t - t')u(t') + \int_{t'}^t S_A(t - \tau)H(s, u(\tau)) \, d\tau, \quad t \geq t'
\] and using (6.11) we obtain
\[
P_1 u(t) = S_A(t - t')P_1 u(t') + \int_{t'}^t S_A(t - \tau)P_1 H(s, u(\tau)) \, d\tau,
\] which by the kernel equality (2.1), takes the following form
\[
P_1 u(t) = P_1 u(t') + \int_{t'}^t P_1 F(sQ_1 u(\tau) + sQ_2 u(\tau) + P_1 u(\tau) + P_2 u(\tau)) \, d\tau.
\]
Let us assume that \( \langle \cdot , \cdot \rangle \) is the standard scalar product on \( L^2(\Omega, \mathbb{R}^m) \). In view of the fact that the spaces \( N_1, N_2, X_1 \) and \( X_2 \) are mutually orthogonal, for any \( u \in N_1, v \in N_2 \) and \( w \in X_1^0 \oplus X_2^0 \), we have
\[
\langle P_1 F(u + v + w), u \rangle_1 = \langle P_1 F(u + v + w), u \rangle = \langle (P_1 + P_2)F(u + v + w), u \rangle = \langle F(u + v + w), u \rangle = \langle F(u + v + w), u \rangle_1,
\] which in turn, for any \( t \in \mathbb{R} \), yields
\[
\frac{d}{dt} \|P_1 u(t)\|_1^2 = 2\langle \frac{d}{dt} P_1 u(t), P_1 u(t) \rangle_1
\] (6.17)
\[
= 2\langle P_1 F(sQ_1 u(t) + sQ_2 u(t) + P_1 u(t) + P_2 u(t)), P_1 u(t) \rangle_1
\]
\[
= 2\langle F(sQ_1 u(t) + sQ_2 u(t) + P_1 u(t) + P_2 u(t)), P_1 u(t) \rangle_1.
\]
Let us assume that the condition (6.13) is satisfied. If we define
\[
t_0^+ := \sup \{ t \geq t_0 \mid \|u(t')\|_1 \geq R \ \text{for} \ t' \in [t_0, t] \},
\]
then \( t_0^+ = +\infty \), because otherwise, the fact that \( \|u(t_0^+)\|_1 \geq R \) together with (6.17) and the inequality (6.13) would yield
\[
\frac{d}{dt} \|P_1 u(t)\|_1^2 \bigg|_{t=t_0^+} = 2\langle F(s(Q_1 + Q_2)u(t_0^+) + P_1 u(t_0^+) + P_2 u(t_0^+) \rangle, P_1 u(t_0^+) \rangle_1 > 2r.
\]
Consequently, there would exists \( \delta > 0 \) such that
\[
\|P_1 u(t)\|_1 > \|P_1 u(t_0^+)\|_1 \geq R, \quad t \in [t_0^+, t_0^+ + \delta],
\]
contrary to the definition of the number \( t_0^+ \). This implies that \( \| P_1 u(t) \|_1 \geq R \) for \( t \geq t_0 \) and hence, using (6.14) once again, we obtain
\[
\frac{d}{dt} \| P_1 u(t) \|_1^2 = 2(F(s(Q_1 + Q_2) u(t) + P_1 u(t) + P_2 u(t)), P_2 u(t))_1 > 2r, \quad t \geq t_0.
\]
Therefore the following inequality is satisfied
\[
\| P_1 u(t) \|_1^2 \geq \| P_1 u(t_0) \|_1^2 + 2(t - t_0)r, \quad t \geq t_0,
\]
which contradicts (6.10) and proves the estimate (6.15). On the other hand, if the condition (6.14) holds, then we write
\[
t_0^- := \inf \{ t \leq t_0 \mid \| u(t') \|_1 \geq R \text{ for } t' \in [t, t_0] \}.
\]
If \( t_0^- \) would be a finite real number, then the inequality \( \| u(t_0^-) \|_1 \geq R \) combined with (6.14) and (6.11) would give
\[
\frac{d}{dt} \| P_1 u(t) \|_1^2 \big|_{t=t_0^-} = 2(F(s(Q_1 + Q_2) u(t_0^-) + P_1 u(t_0^-) + P_2 u(t_0^-)), P_1 u(t_0^-))_1 < -2r.
\]
This in turn would imply the existence of \( \delta > 0 \) such that
\[
\| P_1 u(t) \|_1 > \| P_1 u(t_0^-) \|_1 \geq R, \quad t \in [t_0^- - \delta, t_0^-],
\]
which is impossible due to the definition of \( t_0^- \). Therefore \( t_0^- = -\infty \) and consequently \( \| P_1 u(t) \|_1 \geq R \) for \( t \leq t_0 \). Combining this inequality with (6.14) we obtain
\[
\frac{d}{dt} \| P_1 u(t) \|_1^2 = 2(F(s(Q_1 + Q_2) u(t) + P_1 u(t) + P_2 u(t)), P_1 u(t))_1 < -2r, \quad t \leq t_0,
\]
which after integration gives
\[
\| P_1 u(t) \|_1^2 + 2(t - t_0)r < \| P_1 u(t_0) \|_1^2, \quad t \leq t_0.
\]
This again contradicts (6.10) and shows the estimate (6.15). Thus the proof of the proposition is completed.

In the similar way, we can show the following proposition concerning the estimates of the solutions of the equation (4.7) after projection onto the space \( N_2 \).

**Proposition 6.3.** Let us assume that \( W \subset X_1^1 \oplus X_2^2 \) is a ball centered at the origin and \( r, R > 0 \) are such that either
\[
(F(u + v + w), u)_2 > r \quad \text{for} \quad (u, v, w) \in N_1 \times N_2 \times W \quad \text{with} \quad \| u \|_2 \geq R
\]
or
\[
(F(u + v + w), u)_2 < -r \quad \text{for} \quad (u, v, w) \in N_1 \times N_2 \times W \quad \text{with} \quad \| u \|_2 \geq R.
\]

Then, for any \( s \in [0, 1] \) and any bounded full solution \( u \) of the semiflow \( \Psi^s \) such that \( Q_1 u(t) + Q_2 u(t) \in W \) for \( t \in \mathbb{R} \), the following inequality holds
\[
\| P_2 u(t) \|_2 \leq R, \quad t \in \mathbb{R}.
\]

7. **Proof of Theorems 4.1 and 4.2**

**Step 1.** Proposition 6.1 says that there is a constant \( R_0 > 0 \) such that, for any \( s \in [0, 1] \) and for any bounded full solution \( u \) of the semiflow \( \Psi^s \), we have
\[
\| Q_1 u(t) + Q_2 u(t) \|_\alpha \leq R_0, \quad t \in \mathbb{R}.
\]

Let us define \( W := \{ u \in X_1^1 \oplus X_2^2 \mid \| u \|_\alpha \leq R_0 \} \). If condition (LL1) is satisfied, then Proposition 6.1 asserts the existence of \( r_1 > 0 \) and \( R_1 > 0 \) such that
\[
\pm (F(u + v + w), u)_1 > r_1 \quad \text{for} \quad (u, v, w) \in N_1 \times N_2 \times B \quad \text{with} \quad \| u \|_1 \geq R_1.
\]
Consequently, by the inequality (7.1) and Proposition 6.2, we infer that
\[
\| P_1 u(t) \|_1 \leq R_1, \quad t \in \mathbb{R}.
\]
On the other hand, if condition \((LL2)\) hold, then Proposition \([5,2]\) says that, there are \(r_2 > 0\) and \(R_2 > 0\) such that

\[ \pm (F(u + v + w), v)_{L^2} > r_2 \quad \text{for} \quad (u, v, w) \in N_1 \times N_2 \times B \quad \text{with} \quad \|u\|_2 \geq R_2. \]  

\((7.4)\)

Therefore, using the inequality \((7.1)\) and Proposition \([5,3]\) we deduce that

\[ \|P_2u(t)\|_2 \leq R_2, \quad t \in \mathbb{R}. \]  

\((7.5)\)

Let us define the following sets

\[ M_0 := \{ u \in X_1^0 \oplus X_2^0 \mid \|u\|_\alpha \leq R_0 + 1 \}, \]

\[ M_1 := \{ u \in N_1 \mid \|u\|_1 \leq R_1 + 1 \}, \]

\[ M_2 := \{ v \in N_2 \mid \|v\|_2 \leq R_2 + 1 \} \]

and write \(M := M_0 \oplus M_1 \oplus M_2\). By the estimates \((7.1), (7.3)\) and \((7.5)\), we deduce that, if \(u\) is a bounded full solution of \(\Psi\), where \(s \in [0,1]\), then it is contained in the interior of the set \(M\), which in particular, is an admissible isolating neighborhood for the family of the semiflows \(\{\Psi^s\}_{s \in [0,1]}\) and Inv \(M = K_{\infty}\). Hence, by the homotopy invariance of the Rybakowski-Conley index (see property \((H3)\)), we obtain

\[ h(\Phi, K_{\infty}) = h(\Psi^1, K_1) = h(\Psi^0, K_0), \]  

\((7.6)\)

where we denote \(K_s := \text{Inv} (M, \Psi^s)\) for \(s \in [0,1]\).

**Step 2.** Let \(\psi_1 : [0, +\infty) \times X_0 \to X_0\) be the semiflow associated with the equation

\[ \dot{u}(t) = Q_0F(u(t)), \quad t > 0. \]

We show that \(M_1 \oplus M_2\) is an isolating block for \(\psi_1\) and its exit set is such that

\[ (M_1 \oplus M_2)^{-} := \begin{cases} \partial X_0(M_1 \oplus M_2) & \text{if (C1)\+, (C2)\+, (LL1)\+, (LL2)\+ hold,} \\ (\partial N_1M_1) \oplus M_2 & \text{if (C1)\-, (C2)\-, (LL1)\-, (LL2)\- hold,} \\ M_1 \oplus (\partial N_2M_2) & \text{if (C1)\-, (C2)\+, (LL1)\-, (LL2)\+ hold,} \\ \emptyset & \text{if (C1)\-, (C2)\-, (LL1)\-, (LL2)\- hold.} \end{cases} \]  

\((7.7)\)

To this end, let us assume that \(u : [-\delta_2, \delta_1) \to X_0\), where \(\delta_1 > 0, \delta_2 \geq 0\), is a solution for \(\psi_1\) such that \(u_0 := u(0) \in \partial X_0(M_1 \oplus M_2)\). Then we have

\[ u(t) = u(0) + \int_{0}^{t} Q_0F(u(\tau)) \, d\tau, \quad t \in [-\delta_2, \delta_1), \]

which implies that, for any \(t \in (-\delta_2, \delta_1)\) and \(k = 1, 2\), we have

\[ \frac{1}{2} \frac{d}{dt} \|P_k u(t)\|^2_k = \frac{d}{dt} \langle P_k u(t), P_k u(t) \rangle_k = \langle P_k Q_0 F(u(t)), P_k u(t) \rangle_k = \langle Q_0 F(u(t)), P_k u(t) \rangle_k = \langle F(u(t)), P_k u(t) \rangle = \langle F(P_1 u(t) + P_2 u(t)), P_k u(t) \rangle_k. \]  

\((7.8)\)

Let us assume that the conditions \((LL1)\) and \((LL2)\) are satisfied. Since \(u_0 \in \partial X_0(M_1 \oplus M_2)\), we have either \(u_0 \in (\partial N_1M_1) \oplus M_2\) or \(u_0 \in M_1 \oplus (\partial N_2M_2)\). In the former case we have \(\|P_1 u_0\|_1 = R_1 + 1\) and \(\|P_2 u_0\|_2 \leq R_2 + 1\), which together with the equation \((7.8)\) and inequality \((7.2)\), implies that

\[ \pm \frac{d}{dt} \|P_1 u(t)\|^2_1 \big|_{t=0} > 0. \]  

\((7.9)\)

On the other hand, if \(u_0 \in M_1 \oplus (\partial N_2M_2)\), then \(\|P_1 u_0\|_1 \leq R_1 + 1\) and \(\|P_2 u_0\|_2 = R_2 + 1\). Applying the equation \((7.8)\) together with the inequality \((7.4)\), we obtain

\[ \pm \frac{d}{dt} \|P_2 u(t)\|^2_2 \big|_{t=0} > 0. \]  

\((7.10)\)
Combining (7.9) and (7.10) we infer that the exit set of \( M_1 \oplus M_2 \) takes the form

\[
(M_1 \oplus M_2)^- := \begin{cases} \partial X_0(M_1 \oplus M_2) & \text{if } (C1)_+, (C2)_+, (LL1)_+, (LL2)_+ \text{ hold,} \\ \emptyset & \text{if } (C1)_-, (C2)_-, (LL1)_-, (LL2)_- \text{ hold.} \end{cases} \tag{7.11}
\]

On the other hand, if we assume that the conditions \((LL1)_{\pm}\) and \((LL2)_{\pm}\) are satisfied, then, proceeding in the same way, we obtain

\[
\pm \frac{d}{dt} \|P_1 u(t)\|^2 \bigg|_{t=0} > 0, \quad \text{if } u_0 \in (\partial N_1 M_1) \oplus M_2,
\]

and furthermore

\[
\pm \frac{d}{dt} \|P_2 u(t)\|^2 \bigg|_{t=0} > 0, \quad \text{if } u_0 \in M_1 \oplus (\partial N_2 M_2).
\]

This in turn implies that

\[
(M_1 \oplus M_2)^- := \begin{cases} (\partial N_1 M_1) \oplus M_2 & \text{if } (C1)_+, (C2)_-, (LL1)_+, (LL2)_- \text{ hold,} \\ M_1 \oplus (\partial N_2 M_2) & \text{if } (C1)_-, (C2)_+, (LL1)_-, (LL2)_+ \text{ hold.} \end{cases}
\]

which together with (7.11) provide (7.12) as desired. In particular, we infer that \( M_1 \oplus M_2 \) is an isolated neighborhood for the invariant set \( K_0^1 := \text{Inv}(\psi_1, M_1 \oplus M_2) \) and its homotopy index is given by

\[
h(\psi_1, K_0^1) = \begin{cases} \Sigma^{n_1(\lambda) + n_2(\lambda)} & \text{if } (C1)_+, (C2)_+, (LL1)_+, (LL2)_+ \text{ hold,} \\ \Sigma^{n_1(\lambda)} & \text{if } (C1)_+, (C2)_-, (LL1)_+, (LL2)_- \text{ hold,} \\ \Sigma^{n_2(\lambda)} & \text{if } (C1)_-, (C2)_+, (LL1)_-, (LL2)_+ \text{ hold,} \\ \Sigma^0 & \text{if } (C1)_-, (C2)_-, (LL1)_-, (LL2)_- \text{ hold.} \end{cases} \tag{7.12}
\]

**Step 3.** Let us assume that \( \psi_2 \) is a semiflow obtained by the restriction of the semigroup \( \{S_A(t)\}_{t \geq 0} \) to the space \( X_0^\alpha \oplus X_2^\alpha \), that is,

\[
\psi_2(t, u) := S_A(t)u, \quad t \geq 0, \quad u \in X_1^\alpha \oplus X_2^\alpha.
\]

Combining the estimates (3.12) and (3.13) with the following commutative property

\[
(\delta I + A)^a S_A(t)u = S_A(t)(\delta I + A)^a u, \quad u \in X_0^\alpha,
\]

we deduce that

\[
\|S_A(t)u\|_\alpha \leq C_5 e^{ct} \|u\|_\alpha, \quad t \leq 0, \quad u \in X_1,
\]

\[
\|S_A(t)u\|_\alpha \leq C_5 e^{-ct} \|u\|_\alpha, \quad t \geq 0, \quad u \in X_2.
\]

Hence [38, Theorem 11.1] shows that \( M_0 \) is an admissible isolating neighborhood, \( K_0^2 := \text{Inv}(\psi_2, M_0) = \{0\} \) and the homotopy index of \( K_0^2 \) satisfies

\[
h(\psi_2, K_0^2) = \Sigma^{\dim X_1} = \Sigma^{d_\infty(\lambda)}. \tag{7.13}
\]

where the last equality is a consequence of Remark 3.2.

**Step 4.** Let us observe that the semiflow \( \Psi^0 \) corresponding to the equation

\[
\dot{u}(t) = -Au(t) + Q_0 F(Q_0 u(t)), \quad t > 0,
\]

satisfies the following equality

\[
\Psi^0(t, u + v) = \psi_1(t, u) + \psi_2(t, v), \quad u \in X_0, \quad v \in X_0^\alpha \oplus X_2^\alpha.
\]

Therefore, by the multiplication property of the homotopy index (H2), we have

\[
h(\Psi^0, K_0) = h(\psi_1, K_0^1) \land h(\psi_2, K_0^2). \tag{7.14}
\]

Combining (7.4), (7.13) and (7.14) we deduce that

\[
h(\Phi, K_\infty) = h(\psi_1, K_0^1) \land \Sigma^{d_\infty(\lambda)},
\]
which together with (7.12) provides the homotopy index formulas (4.3), (4.4), (4.5) and (4.6). Thus the proof of Theorems 4.1 and 4.2 is completed. □

8. APPLICATIONS TO THE EXISTENCE OF CONNECTING SOLUTIONS

In this section we provide applications of Theorems 4.1 and 4.2 to the study of the existence of solutions connecting stationary points for the following system of nonlinear heat equations with the Dirichlet boundary conditions

\[
\begin{aligned}
\dot{u}_k(t) &= \Delta u_k(t) + \lambda_k u_k(t) + f_k(x, u(t), \nabla u(t)), \quad 1 \leq k \leq m, \ x \in \Omega, \\
u_k(t) &= 0, \quad 1 \leq k \leq m, \ x \in \partial \Omega,
\end{aligned}
\]

where, for any \(1 \leq k \leq m\), the parameter \(\lambda_k\) is a real number and \(f_k : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m} \to \mathbb{R}\) is a bounded continuously differentiable map with the property that

\[
f(x, 0, 0) = 0, \quad D_x f(x, 0, 0) = G, \quad D_u f(x, 0, 0) = 0, \quad x \in \Omega,
\]

where \(G\) is a symmetric \(m \times m\) matrix and we define \(f := (f_1, \ldots, f_m)\). Let us consider the operator \(A_0\) given by

\[
\begin{aligned}
\{ D(A_0) := \text{cl}_W \{ u \in C^2[\Omega; \mathbb{R}^m] \mid u(x) = 0 \ for \ x \in \partial \Omega \}, \\
A_0u := -\Delta u, \quad u \in D(A_0).
\end{aligned}
\]

Since \(A_0\) is symmetric and has compact resolvents its spectrum \(\sigma(A_0)\) consists of a sequence of real positive eigenvalues, which is either finite or divergent to the infinity. Throughout this section we assume that the system (8.1) is at resonance at infinity, that is,

\[\text{Ker}(\lambda_k I - A_0) \neq \{0\}, \quad 1 \leq k \leq m.\]

It is not difficult to check that the regularity assumption of \(f\) implies that \(F : X^\sigma \to X\) is a \(C^1\) map satisfying conditions (F1) and (F2). Furthermore, from the assumption (8.2) we infer that \(F(0) = 0\) and the derivative of \(F\) at the origin is a linear map \(DF(0) : X^\sigma \to X\) given by the following formula

\[
DF(0)[w](x) = Gw(x) \quad \text{for a.a.} \ x \in \Omega,
\]

where \(w = (w_1, \ldots, w_m) \in X^\sigma\) is a vector. If we define \(\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)\), then the matrix \(G + \Lambda\) is symmetric and consequently its spectrum \(\sigma(G + \Lambda)\) consists of a finite sequence of real eigenvalues \(\theta_1 \leq \theta_2 \leq \ldots \leq \theta_m\). Let us define

\[d_0(\lambda) := \sum_{k=1}^{m} \sum_{\nu < \theta_k} \dim \text{Ker}(\nu I - A_0),\]

where we write \(\lambda := (\lambda_1, \ldots, \lambda_m)\) and, in the above summation, the parameter \(\nu\) is taken from the set of eigenvalues of the operator \(A_0\) that are contained in the set \((-\infty, \theta_k)\) for \(1 \leq k \leq m\). In this section we intend to prove the following theorems that provide sufficient conditions for the existence of compact full solutions connecting stationary points for the system (8.1).

**Theorem 8.1.** Let us assume that \(\{h_k\}_{k=1}^{m}\) is a family of \(L^2(\Omega)\) functions such that the inequalities (C1)\(\pm\) and (C2)\(\pm\) are fulfilled and suppose that the following non-resonance condition at the origin holds

\[\sigma(A_0) \cap \sigma(G + \Lambda) = \emptyset.\]

If the resonance conditions (LL1)\(\pm\) and (LL2)\(\pm\) are satisfied, then there is a non-trivial bounded full solution \(u\) of the system (8.1) such that either \(u(t) \to 0\) as \(t \to +\infty\) or \(u(t) \to 0\) as \(t \to -\infty\), provided

\[d_0(\lambda) \neq d_\infty(\lambda) + (n_1(\lambda) + n_2(\lambda))/2 \pm (n_1(\lambda) + n_2(\lambda))/2. \quad (8.4)\]
Theorem 8.2. Let us assume that \( \{b_k\}_{k=1}^n \) is a family of \( L^2(\Omega) \) functions such that the inequalities \( (C1)_{\pm} \) and \( (C2)_{\mp} \) are fulfilled and suppose that the following non-resonance condition at the origin holds

\[
\sigma(A_0) \cap \sigma(G + \Lambda) = \emptyset. \tag{8.5}
\]

If the resonance conditions \( (LL1)_{\pm} \) and \( (LL2)_{\mp} \) are satisfied, then there is a non-trivial bounded full solution \( u \) of the system \( \Phi(t)u = \xi \) such that either \( u(t) \to 0 \) as \( t \to +\infty \) or \( u(t) \to 0 \) as \( t \to -\infty \), provided

\[
d_0(\lambda) \neq d_\infty(\lambda) + (n_1(\lambda) + n_2(\lambda))/2 \pm (n_1(\lambda) - n_2(\lambda))/2. \tag{8.6}
\]

Proof of Theorems 8.1 and 8.2. Observe that the operator \( L \), given by the formula

\[
L := (A_0 - \lambda_1 I) \times \cdots \times (A_0 - \lambda_m I) - DF(0),
\]

is sectorial and has compact resolvents as [34, Proposition 3.1.4] and [34, Theorem 3.2.1] say. Then the spectrum of \( L \) consists of a sequence of real eigenvalues, which is either finite or diverges to infinity. Furthermore the respective eigenspaces are finite dimensional. We claim that

\[
d_0(\lambda) = \sum_{\mu < 0} \dim \ker (\mu I - L), \tag{8.7}
\]

where, in the above summation, the parameter \( \mu \) is taken from the set of all negative eigenvalues of the operator \( L \). Indeed, in view of [83], we have

\[
Lu = (A_0 u_1, \ldots, A_0 u_m) - (G + \Lambda)u, \quad u \in D(L).
\]

If we take \( O \) to be the \( m \times m \) orthogonal matrix such that

\[
O^2(G + \Lambda)O = \text{diag}(\theta_1, \theta_2, \ldots, \theta_m),
\]

then, for any \( u \in D(L) \), the following holds

\[
L(Ou) = (O(A_0 u_1, \ldots, A_0 (Ou)_m) - (G + \Lambda)(Ou)
\]

\[
= O(A_0 u_1, \ldots, A_0 u_m) - O(\theta_1 u_1, \ldots, \theta_m u_m), \tag{8.8}
\]

\[
= O((A_0 - \theta_1 I) u_1, \ldots, (A_0 - \theta_m I) u_m),
\]

which implies that \( \sigma(L) = \sigma((A_0 - \theta_1 I) \times \cdots \times (A_0 - \theta_m I)) \). Consequently

\[
\sum_{\mu < 0} \dim \ker (\mu I - L) = \sum_{\mu < 0} \sum_{k=1}^m \dim \ker ((\mu + \theta_k) I - A_0), \tag{8.9}
\]

where in the above summations the parameters \( \mu \) ranges over the negative elements of the set \( \sigma(L) \). Let us observe that, for any \( 1 \leq k \leq m \), we have

\[
\{\mu + \theta_k | \mu \in \sigma(L), \mu < 0 \text{ and } \ker ((\mu + \theta_k) I - A_0) \neq \{0\} \}
\]

\[
= \{\nu < \theta_k | \ker (\nu I - A_0) \neq \{0\} \}.
\]

Combining this with (8.5), we obtain

\[
\sum_{\mu < 0} \dim \ker (\mu I - L) = \sum_{k=1}^m \sum_{\mu < 0} \dim \ker ((\mu + \theta_k) I - A_0)
\]

\[
= \sum_{k=1}^m \sum_{\mu < \theta_k} \dim \ker (\nu I - A_0) = d_0(\lambda),
\]

where in the above summations the parameters \( \mu \) and \( \nu \) ranges over the sets \( \sigma(L) \) and \( \sigma(A_0) \), respectively. This gives (8.7) as desired. Let us observe that the condition (8.5) implies that \( \ker [(A_0 - \theta_1 I) \times \cdots \times (A_0 - \theta_m I)] = \{0\} \), which together with (8.8) yield \( \ker L = \{0\} \). Therefore, from [34] Theorem (3.5) it follows that the invariant set \( K_0 := \{0\} \) admits an admissible isolating neighborhood and

\[
h(\Phi, K_0) = \sum_{\mu = 0} d_0(\lambda),
\]

where \( \Phi \) is teh semiflow associated with the system (8.1). If

\[
\tau_h \in \mathcal{D}(\Phi), \quad \Phi(t)\tau_h \to 0, \quad t \to +\infty.
\]
the inequalities \((C1)\pm, (C2)\pm\) and conditions \((LL1)\pm, (LL2)\pm\) are satisfied, then, by Theorem 8.1, we infer that the set \(K_\infty\) consisting of all bounded full solutions of the semiflow \(\Phi\) also has an admissible isolating neighborhood and its homotopy index is given by

\[
h(\Phi, K_\infty) = \begin{cases} 
\sum d_0(\lambda) + n_1(\lambda) + n_2(\lambda) & \text{if } (C1)_+, (C2)_+, (LL1)_+, (LL2)_+ \text{ hold}, \\
\sum d_0(\lambda) & \text{if } (C1)_-, (C2)_-, (LL1)_-, (LL2)_- \text{ hold}.
\end{cases}
\]

Clearly \(K_0 \subset K_\infty\) and \(h(\Phi, K_0) \neq 0\). By the condition \(8.4\), we have also that \(h(\Phi, K_0) \neq h(\Phi, K_\infty)\). Hence Proposition 2.3 gives the existence of a non-trivial full solution \(u\) of the semiflow \(\Phi\) such that either \(u(t) \to 0\) as \(t \to +\infty\) or \(u(t) \to 0\) as \(t \to -\infty\). Thus the proof of Theorems 8.1 is completed. On the other hand, if the inequalities \((C1)_\pm, (C2)_\pm\) together with the conditions \((LL1)_\pm, (LL2)_\pm\) are satisfied, then Theorem 4.2 says that the set \(K_\infty\) admits an admissible isolating neighborhood and its homotopy index is such that

\[
h(\Phi, K_\infty) = \begin{cases} 
\sum d_0(\lambda) + n_1(\lambda) & \text{if } (C1)_+, (C2)_-, (LL1)_+, (LL2)_- \text{ hold}, \\
\sum d_0(\lambda) + n_2(\lambda) & \text{if } (C1)_-, (C2)_+, (LL1)_-, (LL2)_+ \text{ hold}.
\end{cases}
\]

Consequently the condition \(8.6\), implies that again \(h(\Phi, K_0) \neq h(\Phi, K_\infty)\) and therefore Proposition 2.3 provides the existence of a non-trivial full solution \(u\) of the semiflow \(\Phi\) such that either \(u(t) \to 0\) as \(t \to +\infty\) or \(u(t) \to 0\) as \(t \to -\infty\). This in turn completes the proof of Theorem 8.2. \(\square\)

In Theorems 8.1 and 8.2 we employed the resonance conditions \((LL1)_\pm\) and \((LL2)_\pm\) to derive the existence of a full solution \(u\) of the semiflow \(\Phi\) with the relatively compact image \(u(\mathbb{R}) \subset X^\alpha\) such that \(0 \in \alpha(u) \cup \omega(u)\). To deduce that \(u\) connects the origin with a nontrivial stationary point we make an additional assumption on the semiflow \(\Phi\).

**Definition 8.3.** We say that the semiflow \(\Phi\) is gradient-like with respect to the functional \(V : X^\alpha \to \mathbb{R}\) provided

\[
V(\Phi(u_0, t_1)) \geq V(\Phi(u_0, t_2)), \quad u_0 \in X^\alpha, \ t_1 > t_2 \geq 0
\]

and, for any non-constant full solution \(u\) of \(\Phi\), the value \(V(u(t))\) is not constant for \(t \in \mathbb{R}\). Then \(V\) is called the Liapunov function for the semiflow \(\Phi\). \(\square\)

**Remark 8.4.** The usual assumption which makes \(\Phi\) a gradient-like semiflow is the existence of a smooth potential function \(f : \Omega \times \mathbb{R}^m \to \mathbb{R}\) such that

\[
f_k(x, s, y) = \partial_{s_k} \tilde{f}(x, s), \quad x \in \Omega, \ s \in \mathbb{R}^m, \ y \in \mathbb{R}^m, \ 1 \leq k \leq m.
\]

Then the energy functional \(E : X^\alpha \to \mathbb{R}\) given, for any \(u = (u_1, \ldots, u_m) \in X^\alpha\), by

\[
E(u) := \sum_{k=1}^m \int_{\Omega} |\nabla u_k(x)|^2 \, dx + \int_{\Omega} \tilde{f}(u(x)) \, dx
\]

is the Liapunov function for the semiflow \(\Phi\) (see [14] for more details). \(\square\)

The following corollaries are simple consequences of Theorems 8.1 and 8.2.

**Corollary 8.5.** Let us assume that the semiflow \(\Phi\) is gradient-like and

\[
\sigma(A_0) \cap \sigma(G + \Lambda) = \emptyset.
\]

Suppose that \(|h_k|_{k=1}^m\) is a family of \(L^2(\Omega)\) functions satisfying the inequalities \((C1)_\pm\) and \((C2)_\pm\). If the resonance conditions \((LL1)_\pm\) and \((LL2)_\pm\) hold and

\[
d_0(\lambda) \neq d_\infty(\lambda) + (n_1(\lambda) + n_2(\lambda))/2 \pm (n_1(\lambda) + n_2(\lambda))/2,
\]

then...
then there are non-zero stationary point \( u_0 \in X^\alpha \) and full solution \( u \) of the system (1.1) such that either \( u(t_n) \to u_0 \) for some \( t_n \to +\infty \) and \( u(t) \to 0 \) as \( t \to -\infty \) or \( u(t_n) \to u_0 \) for some \( t_n \to -\infty \) and \( u(t) \to 0 \) as \( t \to +\infty \).

**Corollary 8.6.** Let us assume that the semiflow \( \Phi \) is gradient-like and

\[
\sigma(A_0) \cap \sigma(G + \Lambda) = \emptyset.
\]

Suppose that \( \{h_k\}_{k=1}^{\infty} \) is a family of \( L^2(\Omega) \) functions satisfying the inequalities

\[
(C1)_\pm \quad \text{and} \quad (C2)_\pm.
\]

If the resonance conditions \((LL1)_\pm \) and \((LL2)_\pm\) hold and

\[
d_0(\lambda) \neq d_{\infty}(\lambda) + (n_1(\lambda) + n_2(\lambda))/2 \pm (n_1(\lambda) - n_2(\lambda))/2,
\]

then there are non-zero stationary point \( u_0 \in X^\alpha \) and full solution \( u \) of the system (1.1) such that either \( u(t_n) \to u_0 \) for some \( t_n \to +\infty \) and \( u(t) \to 0 \) as \( t \to -\infty \) or \( u(t_n) \to u_0 \) for some \( t_n \to -\infty \) and \( u(t) \to 0 \) as \( t \to +\infty \).

**Proof of Corollaries 8.5 and 8.6.** In view of Theorems 8.1 and 8.2 we infer that there is a non-trivial full solution \( u \) of the system (8.1) such that either \( u(t) \to 0 \) as \( t \to +\infty \) or \( u(t) \to 0 \) as \( t \to -\infty \). Since the semiflow \( \Phi \) is gradient-like, from [18] Theorem II.5.4 we know that the the limit sets \( \alpha(u) \) and \( \omega(u) \) are non-empty, disjoint and consist of the stationary points of the semiflow \( \Phi \). Consequently, if \( u(t) \to 0 \) as \( t \to +\infty \), then we can choose a non-zero element \( u_0 \in \alpha(u) \) such that \( u(t_k) \to u_0 \) for some \( t_k \to -\infty \). On the other hand, if \( u(t) \to 0 \) as \( t \to -\infty \), then we can take \( u_0 \in \omega(u) \) such that \( u_0 \neq 0 \) and \( u(t_k) \to u_0 \) for some \( t_k \to +\infty \). Thus the proof of the corollaries is completed.

\( \square \)

**References**

1. A. Ambrosetti, D. Arcoya, *On a quasilinear problem at strong resonance*, Topol. Methods Nonlinear Anal. 6 (1995), no. 2, 255–264.
2. A. Ambrosetti, G. Mancini, *Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance. The case of the simple eigenvalue*, J. Differential Equations 28 (1978), no. 2, 229–245.
3. A. Ambrosetti, G. Mancini, *Theorems of existence and multiplicity for nonlinear elliptic problems with noninvertible linear part*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 5 (1978), no. 1, 15–28.
4. P. Amster, P. De Nápoli, Pablo . Landesman-Lazer type conditions for a system of p-Laplacian like operators, J. Math. Anal. Appl. 326 (2007), no. 2, 1236–1243.
5. D. Arcoya, A. Cañada, *Critical point theorems and applications to nonlinear boundary value problems*, Nonlinear Anal. 14 (1990), no. 5, 393–411.
6. D. Arcoya, D.G. Costa, *Nontrivial solutions for a strongly resonant problem*, Differential Integral Equations 8 (1995), no. 1, 151–159.
7. D. Arcoya, L. Orsina, *Landesman-Lazer conditions and quasilinear elliptic equations*, Nonlinear Anal. 28 (1997), no. 10, 1623–1632.
8. D.G. Costa, E.A. Silva, *Existence of solutions for a class of resonant elliptic problems*, J. Math. Anal. Appl. 175 (1993), no. 2, 411–424.
9. J. Arrieta, R. Pardo, A. Rodríguez-Bernal, *Equilibria and global dynamics of a problem with bifurcation from infinity*, J. Differential Equations 246 (2009), 2055–2080.
10. J. Bouchala, P. Drábek, *Strong Resonance for some quasilinear elliptic equations*, J. Math. Anal. Appl. 245 (2000), no. 1, 7–19.
11. P. Bartolo, V. Benci, D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with ‘‘strong’’ resonance at infinity*, Nonlinear Anal. 7 (1983), no. 9, 981–1012.
12. H. Brézis, L. Nirenberg, *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 5 (1978), no. 2, 225–326.
13. L. Cesari, R. Kannan, *An abstract existence theorem at resonance*, Proc. Amer. Math. Soc. 63 (1977), no. 2, 221–225.
14. J. W. Cholewa, T. Dlotko, *Global attractors in abstract parabolic problems*, London Mathematical Society Lecture Note Series, vol. 278, Cambridge University Press, Cambridge, 2000.
15. C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics, vol. 38, American Mathematical Society, Providence, R.I., 1978.
