E-INFINITY COALGEBRA STRUCTURE ON CHAIN COMPLEXES WITH COEFFICIENTS IN Z

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Abstract. The aim of this paper is to construct an $E_\infty$-operad $\mathcal{R}$ and prove that this operad induces an $E_\infty$-coalgebra structure on chain complexes with coefficients in $\mathbb{Z}$. The operad $\mathcal{R}$ is an alternative to the description of the $E_\infty$-coalgebra structure on chain complexes by the Barrat-Eccles operad.

1. Introduction

In [10], Smith describes an $E_\infty$-coalgebra structure on the chain complex of a simplicial set when the coefficients ring is $\mathbb{Z}$. In order to do this, he uses an $E_\infty$-operad, denoted $\Theta$, with components $\Sigma n$, the $\Sigma n$-free bar resolution of $\mathbb{Z}$. The morphisms $f_n : \Sigma n \otimes C_\ast(X) \to C_\ast(X)^{\otimes n}$ determined by the $E_\infty$-coalgebra structure contains a family of higher diagonals on $C_\ast(X)$, starting with an homotopic version of the iterated Alexander-Whitney diagonal (given by $x \mapsto f_n(\otimes n \otimes x)$). The construction made by Smith can be seen as a version of the Barratt-Eccles operad (see [1]). Moreover, Berger and Fresse (see [2]) construct a explicit coaction over the normalized chain complex associated to a simplicial set by the Barrat-Eccles operad that extend the structure given by the Alexander-Whitney diagonal.

In this article, it is constructed an $E_\infty$-operad $\mathcal{R}$ which is used to give an alternative description of the $E_\infty$-structure on the chain complex of a simplicial set. The method used to construct $\mathcal{R}$ gives an simply way to produce $E_\infty$-operads.

The operad $\mathcal{R}$ presents similarities with the bar-cobar resolution of Ginzburg-Kapranov (see [6]). Berger and Moerdijk (see [3]) show that this resolution can identified with the $W$-construction of Boardman and Vogt (see [4]), given as a result that applied to the Barratt-Eccles operad, the $W$-construction gives a cofibrant resolution of it. Then, the construction of $\mathcal{R}$ can be seen as a middle point between the Barratt-Eccles operad and its $W$-construction.

The results in this article are based in the Phd thesis of the author [9], where the construction of $E_\infty$-operads is needed to study homotopy properties, described by Alain Prout in [7] and [8], of structures associated to chain complexes determinated by the Eilenberg-Mac lane transformation.

2. Preliminaries

2.1. Differential graded modules. A $\mathbb{Z}$-module $M$ is graded if there is a collection $\{M_i\}_{i \in \mathbb{Z}}$ of submodules of $M$ such that $M = \bigoplus_{i \in \mathbb{Z}} M_i$. A differential graded module with augmentation and coefficients in $\mathbb{Z}$, or $DGA$-module for short, is a graded module $M$ together with an application $\partial : M \to M$ of degree $-1$ such that $\partial^2 = 0$, an applications $\epsilon : M \to \mathbb{Z}$, $\eta : \mathbb{Z} \to M$ of degree $0$, called the augmentation.

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and coaugmentation of $M$, respectively, such that $\epsilon \circ \eta = 1_Z$. The category of DGA-modules is denoted $\text{DGA-Mod}$.

2.2. Operads. An operad $P$ on the monoidal category $\text{DGA-Mod}$ is a collection of $\text{DGA}$-modules $\{P(n)\}_{n \geq 1}$ together with right actions of the symmetric group $\Sigma_n$ on each component $P(n)$, and morphisms of the form $\gamma : P(r) \otimes P(i_1) \otimes P(i_r) \to P(i_1 + \cdots + i_r)$, which satisfies the usual conditions of existence of an unit, associativity and equivariance. The morphisms $\gamma$ will be called composition morphisms or simply the composition of the operad. A morphism between operads $f : P \to Q$, is a collection of $\text{DGA}$-morphisms $f_n : P(n) \to Q(n)$ of degree 0, respecting the units, composition and equivariance. The category of operads is denoted $\text{OP}$.

If we forget the composition morphism of an operad $P$, the collections with the right actions by the symmetrics groups are called $\text{S}$-modules. They form a category denoted $\text{S-Mod}$. The forgetful functor $U : \text{OP} \to \text{S-Mod}$ has a right adjoint denoted $F : \text{S-Mod} \to \text{OP}$, called the free operad functor.

**Definition 2.1.** Let $P$ be an operad on the category of $\text{DGA}$-$\mathbb{Z}$-modules, with composition $\gamma$. A sub $\text{S}$-module $I$ of $U(P)$ is called an operadic ideal of $P$ if it satisfies $\gamma(x \otimes y_1 \otimes \cdots \otimes y_k) \in I$, whenever some of the elements $x, y_1, \ldots, y_k$ belongs to $I$.

**Definition 2.2.** Let $P$ be an operad and $I$ an operadic ideal of $P$. We define the quotient operad $P/I$ as the operad with components given by $(P/I)(k) = P(k)/I(k)$ for every $k \geq 1$, and composition induced by the composition of $P$.

**Remark 2.3.** Clearly the operad structure $P/I$ is well defined by the properties of the ideal, which allows the pass to the quotient of the composition in $P$.

2.3. The Bar Resolution. $\Sigma_n$ will denote the symmetric group on of the set $[n] = \{1, \ldots, n\}$. The chain complex with coefficients in $\mathbb{Z}$ given by the $\Sigma_n$-free bar resolution of $\mathbb{Z}$ is denoted $R\Sigma_n$. Recall that degree $m$ elements of $R\Sigma_n$ are $\mathbb{Z}$-linear combinations of elements of the form $\sigma[\sigma_1/\cdots/\sigma_m]$, where $\sigma, \sigma_1, \ldots, \sigma_m \in \Sigma_n$ and their border is determinated by the equations $\partial = \sum_{i=0}^{m} (-1)^i \partial_i$, where $\partial_0[\sigma_1/\cdots/\sigma_m] = \sigma_1[\sigma_2/\cdots/\sigma_m]$, for $0 < i < m$ $\partial_i[\sigma_1/\cdots/\sigma_m] = [\sigma_1/\cdots/\sigma_i, \sigma_{i+1}/\cdots/\sigma_m]$, and $\partial_m[\sigma_1/\cdots/\sigma_m] = [\sigma_1/\cdots/\sigma_{m-1}]$. In degree zero, the $\mathbb{Z}[\Sigma_n]$-module is generated by the element writed $[\cdot]$.

The contracting chain homotopy for the chain complex $R\Sigma_n$ is the application $\psi_n : R\Sigma_n \to R\Sigma_n$ of degree 1 defined by the relations $\psi_n[\sigma_1/\cdots/\sigma_m] = 0$ and $\psi_n[\sigma_1/\cdots/\sigma_m] = [\sigma/\sigma_1/\cdots/\sigma_m]$.

2.4. $E_\infty$-Operads.

**Definition 2.4.** An operad $P$ on the category $\text{DGA-Mod}$ is called $E_\infty$-operad if each component $P(n)$ is a $\Sigma_n$-free resolution of $\mathbb{Z}$.

**Definition 2.5.** We call $E_\infty$-algebra any $P$-algebra with $P$ an $E_\infty$-operad. And in the same way, an $E_\infty$-coalgebra is an $P$-coalgebra where the operad $P$ is an $E_\infty$-operad.

We introduce a notion of morphism between $E_\infty$-coalgebras which is well suited for our purpose.

**Definition 2.6.** Let $P$ be an $E_\infty$-operad on the category $\text{DGA-Mod}$, and let $A, B$ $P$-coalgebras. A morphism $f : A \to B$ of $P$-coalgebras is a morphism of $\text{DGA-Mod}$.
which preserves the $\mathcal{P}$-coalgebra structure up to homotopy, that is, the following diagram

$$
\begin{array}{ccc}
\mathcal{P}(n) \otimes A & \xrightarrow{\varphi_n^A} & A^\otimes n \\
1 \otimes f & \downarrow & f^\otimes n \\
\mathcal{P}(n) \otimes B & \xrightarrow{\varphi_n^B} & B^\otimes n
\end{array}
$$

is commutative up to homotopy for every $n > 0$, where $\varphi_n^A$ and $\varphi_n^B$ are the associated morphisms of the $\mathcal{P}$-coalgebra structure of $A$ and $B$, respectively. The category of coalgebras on the operad $\mathcal{P}$ is denoted $\mathcal{P}$-Coalg.

3. The Operad $\mathcal{R}$

In this section, it is constructed an $E_\infty$-operad $\mathcal{R}$ which is used to describe $C_*(X)$ as an $E_\infty$-coalgebra. Roughly speaking, to construct the operad $\mathcal{R}$, first take the $\mathcal{S}$-module with components the $\mathbb{Z}[\Sigma_n]$-free bar resolutions of $\mathbb{Z}$, and then make the quotient of the free operad on this $\mathcal{S}$-module by a suitable operad ideal $\mathcal{I}$ (see [6] §2.1), such that our operad will have only one generator of degree 0 in each component.

Definition 3.1. Let $S$ be the $\mathcal{S}$-module on the category $DGA$-$\text{Mod}$, with components $S(n) = R\Sigma_n$, the $\mathbb{Z}[\Sigma_n]$-free bar resolution of $\mathbb{Z}$. Define the operad $\mathcal{R}$ as the quotient operad $F(S)/\mathcal{I}$, where $\mathcal{I}$ is the operadic ideal of the free operad $F(S)$ generating by the elements of degree zero of $F(S)$ of the form $x - y$, where $x$ and $y$ are not null.

Theorem 3.2. The operad $\mathcal{R}$ is an $E_\infty$-operad and induces an $E_\infty$-coalgebra structure on $C_*(X)$.

Proof. It suffices to exhibit in each arity an contracting chain homotopy. In arity $n$, the contracting chain homotopy $\Phi_n : R(n) \to R(n)$ is obtained by extending on $R(n)$ the contracting chain homotopy $\psi_n$ from the component $R\Sigma_n$ of $S$ as follows.

$R(2)$ is isomorphic to $S(2)$, so the contracting chain homotopy remains the same. When $n > 2$, $R(n)$ has two types of elements: the elements from the injection $S(n) \to R(n)$ and the elements of the form $\gamma(x; y_1, \ldots, y_r)$, where $x \in S(r)$ and $y_j \in R(i_j)$. In the first case $\Phi_n$ will behaves as the contracting chain homotopy in $S(n)$, and for the second case, we define $\Phi_n \gamma(x; y_1, \ldots, y_r) = \gamma(\Phi_n(x); y_1, \ldots, y_r)$.

To check that $\partial \Phi_n + \Phi_n \partial = 1$, let $x$ of the form $[\sigma_1] \cdots [\sigma_l]$, with $\sigma_j \in \Sigma_r$. Now $\partial \Phi_n \gamma(x; y_1, \ldots, y_r) = \partial \gamma(\Phi_n(x); y_1, \ldots, y_r) = 0$. On the other hand,

\begin{align*}
(3.1) & \quad \Phi_n \partial \gamma(x; y_1, \ldots, y_r) \\
(3.2) & \quad = \Phi_n \gamma(\partial x; y_1, \ldots, y_r) + \text{(sign)} \sum \Phi_n \gamma(x; y_1, \ldots, y_j, \ldots, y_r) \\
(3.3) & \quad = \gamma(\Phi_n \partial x; y_1, \ldots, y_r) + \text{(sign)} \sum \gamma(\Phi_n x; y_1, \ldots, y_j, \ldots, y_r) \\
(3.4) & \quad = \gamma(x - \partial \Phi_n x; y_1, \ldots, y_r) \\
(3.5) & \quad = \gamma(x; y_1, \ldots, y_r)
\end{align*}
When \( x \) has the form \( \sigma|\sigma_1|\cdots|\sigma_l \) the verification is similar, because the compositions \( \gamma \) satisfy the following equivariance relation:

\[
\gamma(\sigma|\sigma_1|\cdots|\sigma_l; y_1,\ldots,y_r) = \gamma((\sigma_1|\cdots|\sigma_l); y_{\sigma^{-1}(1)},\ldots,y_{\sigma^{-1}(l)}).
\]

Now, the universal property of the coaugmentation \( \iota \) of the adjunction \( F \vdash U \), gives the commutative diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{\iota} & F(S) \\
\downarrow{\iota} & & \downarrow{p} \\
\mathcal{S} & & \text{CoEnd}(C_\ast(X))
\end{array}
\]

(3.6)

Where the morphism \( \iota \) is the identity of \( \mathcal{S} \)-modules. It is easy to see that \( p \) respect the ideal \( \mathcal{J} \) because, when the free operad construction is interpreted by rooted trees, \( p \) is essentially the contraction of vertices of trees. Thus \( p \) pass to the quotient and we obtain a morphism of operads \( \overline{p} : \mathcal{R} \to \mathcal{S} \), which implies that every \( \mathcal{S} \)-coalgebra is an \( \mathcal{R} \)-coalgebra.

**Corollary 3.3.** The construction in theorem 3.2 is functorial.

**Proof.** The functoriality of the \( \mathcal{S} \)-coalgebra structure is hereditary by the \( \mathcal{R} \)-coalgebra structure by the operad morphism \( \overline{p} : \mathcal{R} \to \mathcal{S} \) in the proof of theorem 3.2 as shows the following commutative diagram for every morphism \( f : X \to Y \):

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{\overline{p}} & \mathcal{S} \\
\downarrow{f_*} & & \downarrow{f_*} \\
\text{CoEnd}(C_\ast(X)) & & \text{CoEnd}(C_\ast(Y))
\end{array}
\]

(3.7)

\[ \square \]

We can understand the relation between the operad \( \mathcal{R} \) and the operad \( \mathcal{S} \) by the following proposition.

**Corollary 3.4.** There is an operad ideal \( \mathcal{I} \) such that \( \mathcal{S} \cong \mathcal{R}/\mathcal{I} \).

**Proof.** This is because the underlying \( \mathcal{S} \)-module of \( \mathcal{S} \) is \( S \), and a direct consequence of the definition of compositions \( \gamma \) of \( \mathcal{S} \) (see [10]), in the sense that, the operadic ideal \( \mathcal{I} \) is defined by the identification needed for \( \gamma \).

In [5] Vallette and Dehling describe an operad similar to \( \mathcal{R} \) and they show that this operad can be used to explicitly state (by the use relations) the definition of \( E_\infty \)-algebras, as it is already possible for \( A_\infty \)-algebras.

**Corollary 3.5.** Let \( A \) be a DGA-module together with:

1. For every integer \( m \geq 1 \), \( n \geq 1 \) and \( \sigma,\sigma_1,\ldots,\sigma_n \in \Sigma_m \), morphisms of degree \( n \):

   \[
   \mu_{\sigma|\sigma_1|\cdots|\sigma_n} : A \to A^\otimes n.
   \]

2. For every integer \( m \geq 1 \) and \( \sigma \in \Sigma_m \), applications of degree 0:

   \[
   \mu_{|\sigma} : A \to A^\otimes n.
   \]

Suppose these morphisms satisfy the following relations:

1. \( \mu_{\sigma x} = \mu_{x \sigma} \), where \( \sigma \) is the right action on \( n \) factors.
\(\mu_{x+y} = \mu_x + \mu y \) and \(\partial \mu_x = \mu \partial x\).

(3) \( (\mu_1 \otimes \cdots \otimes \mu_n) = \mu_{1+\cdots+n} \).

Then, \(A\) is an \(R\)-coalgebra if and only if \(A\) has an structure of this type.

Proof. This is directly implied by the operad morphism \(R \to \text{Coend}(A)\). \(\square\)

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