Research Article
A Composite Algorithm for Numerical Solutions of Two-Dimensional Coupled Burgers’ Equations

Vikas Kumar, Sukhveer Singh, and Mehmet Emir Koksal

1Department of Mathematics, D. A. V. College Pundri, Kaithal 136026, Haryana, India
2Department of Mathematics, Indian Institute of Technology, Roorkee 247667, India
3Department of Mathematics, Ondokuz Mayas University, Atakum, Samsun 55139, Turkey

Correspondence should be addressed to Mehmet Emir Koksal; mekoksal@omu.edu.tr

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1. Introduction

In this paper, the authors considered the following dimensionless form of two-dimensional (2D) coupled Burgers’ equation:

\[
\begin{align*}
\frac{1}{\text{Re}} \nabla^2 u & = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}, \\
\frac{1}{\text{Re}} \nabla^2 v & = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}
\end{align*}
\]  

(1a, 1b)

with initial conditions (ICs),

\[
\begin{align*}
u(x, y, 0) & = \psi_1(x, y), \quad (x, y) \in [\alpha, \beta] \times [\gamma, \delta], \\
v(x, y, 0) & = \psi_2(x, y), \quad (x, y) \in [\alpha, \beta] \times [\gamma, \delta],
\end{align*}
\]  

(2a, 2b)

and Dirichlet boundary conditions (BCs),

\[
\begin{align*}
u(\alpha, y, t) & = h_1(y, t), \\
u(\beta, y, t) & = h_2(y, t), \\
u(x, \gamma, t) & = h_3(x, t), \\
u(x, \delta, t) & = h_4(x, t),
\end{align*}
\]  

(3)

where \(\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)\) is Laplace operator, \(u(x, y, t)\) and \(v(x, y, t)\) are velocity components to be determined. Also, \(h_i, i = 1, 2, \ldots, 4\) are known smooth functions and \(\text{Re} = (\rho \overline{u} |u| L/\mu)\) is the Reynolds number with density \(\rho\), viscosity \(\mu\), characteristic length \(L\), and \(\overline{u} = [u, v]^T\).

The nonlinear convection-diffusion model is simply represented by Burgers’ equation [1]. This famous equation describes the flow theory through a shockwave moving in viscous liquid [2], phenomena of turbulence [3], and various other kinds of phenomena in aerodynamics.

Due to its extensive scope of applicability, various numerical schemes have been constructed to study its...
Recently, DQMs have become popular for solving nonlinear phenomena. DQMs discretize the first and second derivatives over 1D domain $\Omega = [\alpha, \beta]$ as follows:

$$u_x(x_i, t) = \sum_{j=1}^{N} a_{ij}^{(1)} u(x_j, t), \quad i = 1, 2, \ldots, N, \quad (4)$$

$$u_{xx}(x_i, t) = \sum_{j=1}^{N} a_{ij}^{(2)} u(x_j, t), \quad j = 1, 2, \ldots, N, \quad (5)$$

where $a_{ij}^{(1)}$ and $a_{ij}^{(2)}$ are unknown coefficients weighting the first and second derivatives, respectively, and $x_i, t = 1, 2, \ldots, N$, are uniform grids as well as nonuniform grids that exist in the domain. Bellman et al. [28] introduced two approaches to calculate WCs. Furthermore, to modify Bellman’s approaches for finding WCs, many efforts have been carried out such as Lagrange interpolated cosine functions, spline functions, Legendre polynomials, Lagrange interpolation polynomials, and radial basis functions (see [19, 29–35] and the references therein) to determine these coefficients. In this study, we determine WCs with the use of CTBS functions after some modifications.

### 2.1. Cubic Trigonometric B-Spline Functions

In this section, we mesh the solution domain $\alpha \leq x \leq \beta$ into $N$ subintervals $[x_i, x_{i+1}], i = 0, 1, \ldots, N-1$ with the help of knots $x_i$ such that $x_0 < x_1, \ldots, < x_N = \beta$ is a uniform partition with step length $a = x_{i+1} - x_i = (\beta - \alpha)/N, i = 0, 1, \ldots, N - 1$.

Now, the piecewise CTBS basis functions $rB_j(x)$ over the uniform mesh are defined as follows [36, 37]:

$$B_j(x) = \frac{1}{\omega} \begin{cases} r^3(x_i), & x \in [x_{i-2}, x_{i-1}], \\ r(x_i)[r(x_i)s(x_{i+2}) + s(x_{i+3})r(x_{i+1})] + s(x_{i+4})r^2(x_{i+1}), & x \in [x_{i-1}, x_i], \\ s(x_{i+4})[r(x_{i+1})s(x_{i+2}) + s(x_{i+3})r(x_{i+2})] + r(x_i)s^2(x_{i+3}), & x \in [x_i, x_{i+1}], \\ s^3(x_{i+4}), & x \in [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise,} \end{cases}$$

### Table 1: Coefficients of CTBSs and their derivatives at knots $x_j$.

| $x$ | $x_{j-2}$ | $x_{j-1}$ | $x_j$ | $x_{j+1}$ | $x_{j+2}$ |
|-----|-----------|-----------|-------|-----------|-----------|
| $B_j(x)$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
| $B_j'(x)$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_5$ |
| $B_j''(x)$ | $a_5$ | $a_6$ | $a_5$ | $a_5$ | $a_5$ |

In this section, a new numerical algorithm is developed based on the finite difference and the modified cubic trigonometric B-spline (CTBS) DQMs for approximate solutions of coupled two-dimensional Burgers’ equations’ weighting coefficients (WCs) of DQM are calculated by using the modified CTBS functions as test functions which are different from the conventional technique of Lagrange interpolation [27]. Some well-known test problems are worked out to inspect the correctness and competence of the planned approach. The techniques lead to correct results with insignificant $L_{\infty}$, RMS and $L_2$ errors.

### 2. Differential Quadrature Method

Recently, DQMs have become popular for solving nonlinear partial differential equations (PDEs) arising in nonlinear
where

\[ r(x_i) = \sin\left(\frac{x - x_i}{2}\right), \]
\[ s(x_i) = \sin\left(\frac{x_i - x}{2}\right), \]
\[ \omega = \sin\left(\frac{a}{2}\right)\sin\left(\frac{3a}{2}\right). \]

The basis over the region \( a \leq x \leq \beta \) is formed by the set
\[ \{B_{-1}(x), B_0(x), \ldots, B_N(x), B_{N+1}(x)\}. \]

Every CTBS covers four elements. Now, with the help of Table 1, we have tabulated the values of \( B_j(x) \) and its derivatives as follows:

\[ a_1 = \frac{\sin^2(a/2)}{\sin(a)\sin(3a/2)}, \]
\[ a_2 = \frac{2}{1 + 2\cos(a)}, \]
\[ a_3 = \frac{-3}{4\sin(3a/2)}, \]
\[ a_4 = \frac{3}{4\sin(3a/2)}, \]
\[ a_5 = \frac{3[1 + 3\cos(a)]}{16\sin^2(a/2)[2\cos(a/2) + \cos(3a/2)]}, \]
\[ a_6 = \frac{3\cos^2(a/2)}{2\sin^2(a/2)[1 + 2\cos(a)]}. \]

### 2.2. Modified Cubic Trigonometric B-Spline Functions

In this work, we compute WCs of DQM with the help of modified CTBS function defined in (6) as follows:

\[
\begin{cases}
\bar{B}_0(x) = B_0(x) + 2B_{-1}(x), & j = 0, \\
\bar{B}_1(x) = B_1(x) - B_{-1}(x), & j = 1, \\
\bar{B}_j(x) = B_j(x), & j = 2, 3, \ldots, N - 2, \\
\bar{B}_{N-1}(x) = B_{N-1}(x) - B_{N+1}(x), & j = N - 1, \\
\bar{B}_N(x) = B_N(x) + 2B_{N+1}(x), & j = N.
\end{cases}
\]

It is worth mentioning that the modified functions \( \{\bar{r}B_j(x)\}, j = 0, 1, \ldots, N \) are linearly independent. On the solution domain \([a, \beta]\), these functions create a family of basis functions.

### 2.3. Weighting Coefficients for Modified Cubic Trigonometric B-Spline Differential Quadrature Method

Now, substitute the modified functions \( \{\bar{r}B_j(x)\} \), for \( j = 0, 1, \ldots, N \), into equation (4). The matrix form of the equation is as follows:

\[
\begin{bmatrix}
Au_{10} = B_0 \\
Au_{11} = B_1 \\
\vdots \\
Au_{N-1} = B_{N-1} \\
Au_N = B_N
\end{bmatrix}
\]

where \( A \) is \((N + 1) \times (N + 1)\) coefficient matrix:

\[
A = \begin{bmatrix}
a_1 + 2a_1 & a_1 & 0 & 0 & \cdots & 0 \\
a_2 & a_2 & a_1 & 0 & \cdots & 0 \\
0 & a_1 & a_2 & a_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_1 + 2a_1
\end{bmatrix}
\]

Furthermore, with the help of Thomas algorithm WCs, \( \alpha_{ij}^{(1)} \) are achieved as solutions of tridiagonal systems of equation (11). Similarly, with the help of the above method, it is easy to calculate second-order WCs \( \beta_{ij}^{(2)} \).

### 2.4. Two-Dimensional Modified Cubic Trigonometric B-Spline Differential Quadrature Method

In order to apply this method to 2D nonlinear problems, first of all, decompose the domain \( \Omega = \{(x, y): a_1 \leq x \leq \beta_1; a_2 \leq y \leq \beta_2\} \) as \( \Omega = \{(x_i, y_j), i = 1, 2, \ldots, N; j = 1, 2, \ldots, M\} \) by adopting step length \( \Delta x = x_i - x_{i-1} \) and \( \Delta y = y_j - y_{j-1} \) in \( x \) and \( y \) direction, respectively. This modified technique helps to estimate the 1st order partial derivatives of \( u(x, y, t) \) at a point as follows:

\[
u_x(x_i, y_j, t) = \sum_{k=1}^{N} a_{ik}^{(1)} u(x_k, y_j, t), \quad i = 1, 2, \ldots, N, \quad (14)
\]
\[
u_y(x_i, y_j, t) = \sum_{k=1}^{M} \beta_{jk}^{(1)} u(x_i, y_k, t), \quad j = 1, 2, \ldots, M, \quad (15)
\]
where \( \alpha_{ij}^{(1)} \) is WCs for the 1st order derivatives w.r.t. \( x \). Similarly, \( \beta_{ij}^{(1)} \) are coefficients w.r.t. \( y \).

In order to compute the 2D WCs, we can define the functions \( \{ rB_j(y) \}, j = 0, 1, \ldots, N \), as in equation (10). Furthermore, take the test functions as \( T_{ij}(x, y) = rB_j(x) rB_j(y) \). Now, with the help of the axioms of vector space and substituting the value of \( T_{ij}(x, y) \) into equations (14) and (15), we have

\[
\begin{align*}
B_j'(x_i) &= \sum_{k=1}^{N} \alpha_{kj}^{(1)} B_j(x_k), \quad j, i = 1, 2, \ldots, N, \\
B_j'(y_i) &= \sum_{k=1}^{M} \beta_{kj}^{(1)} B_j(y_k), \quad j, i = 1, 2, \ldots, M.
\end{align*}
\] (16)

Furthermore, applying the well-known algorithm “Thomas algorithm” and proceeding with the same methods as in the case of equation (11), the solutions of the systems give the value of \( \alpha_{ij}^{(1)} \) and \( \beta_{ij}^{(1)} \). In 2D case, the WCs in higher-order derivatives can be considered as follows:

\[
\begin{align*}
\alpha_{ij}^{(r)} &= r \left[ \alpha_{ij}^{(1)} \alpha_{ij}^{(r-1)} - \frac{\alpha_{ij}^{(r-1)}}{x_i - x_j} \right], \quad \text{for } i, j = 1, 2, \ldots, N; i \neq j; r = 2, 3, \ldots, N - 1, \\
\alpha_{ii}^{(r)} &= - \sum_{j=1, j \neq i}^{N} \alpha_{ij}^{(r)}, \quad \text{for } i = j, \\
\beta_{ij}^{(r)} &= r \left[ \beta_{ij}^{(1)} \beta_{ij}^{(r-1)} - \frac{\beta_{ij}^{(r-1)}}{y_i - y_j} \right], \quad \text{for } i, j = 1, 2, \ldots, N; i \neq j; r = 2, 3, \ldots, N - 1, \\
\beta_{ii}^{(r)} &= - \sum_{j=1, j \neq i}^{M} \beta_{ij}^{(r)}, \quad \text{for } i = j,
\end{align*}
\] (17)

where \( \alpha_{ij}^{(r)} \) and \( \beta_{ij}^{(r)} \) are WCs for \( r \)th order partial derivatives w.r.t. \( x \) and \( y \), respectively.

### 3. Numerical Algorithm for Two-Dimensional Coupled Burgers’ Equation

In this section, the numerical algorithm is developed in the following sections.

\[
\begin{align*}
\frac{u(x, y, t^{n+1}) - u(x, y, t^n)}{\Delta t} &= \frac{1}{\text{Re}} \left[ \theta \nabla^2 u(x, y, t^{n+1}) + (1 + \theta) \nabla^2 u(x, y, t^n) \right] - (uu_x)^{n+1} - (uu_y)^{n+1}, \quad n = 0, 1, \ldots, K, \quad (18a) \\
\frac{v(x, y, t^{n+1}) - v(x, y, t^n)}{\Delta t} &= \frac{1}{\text{Re}} \left[ \theta \nabla^2 v(x, y, t^{n+1}) + (1 + \theta) \nabla^2 v(x, y, t^n) \right] - (vv_x)^{n+1} - (vv_y)^{n+1}, \quad n = 0, 1, \ldots, K, \quad (18b)
\end{align*}
\]
where \( u(x, y, t^{n+1}) = u(x, y, t + n\Delta t), \quad v(x, y, t^{n+1}) = v(x, y, t + n\Delta t), \Delta t \) step length in time direction, and \( 0 \leq \theta \leq 1 \). The nonlinear term is linearized in the following manner:

\[
\begin{align*}
(uu_x)^{n+1} &= uu_x^n, \\
(vu_x)^{n+1} &= vu_x^n, \\
(uu_y)^{n+1} &= uu_y^n, \\
(vu_y)^{n+1} &= vu_y^n,
\end{align*}
\]

(19a)

and prescribed BCs (3).

After simplification, equations (18a) and (18b) can be written as follows:

\[
\begin{align*}
u^{n+1} - \frac{\Delta t}{Re} \partial^2_y u^{n+1} &= (1 - \Delta t\theta )u^n + \Delta t (1 - \theta )\Delta^2 u^n - \Delta t \nu^n u_y^n, \\
v^{n+1} - \frac{\Delta t}{Re} \partial^2_y v^{n+1} &= (1 - \Delta t\theta )v^n + \Delta t (1 - \theta )\Delta^2 v^n - \Delta t \nu^n v_y^n,
\end{align*}
\]

(21a, 21b)

which is a system of second-order differential equations, where \( u^{n+1}(x, y) = u(x, y, t^{n+1}) \) and equations (21a) and (21b) are a system of second-order differential equations.

3.2 Fully Discretization in Space. In this section, spatial derivatives that occur in equations (21a) and (21b) are discretized by modified CTBS DQM over the given domain. After spatial discretization, equations (21a) and (21b) convert into a system of linear equations for each \( n \) in the following form:

\[
\begin{align*}
u_{ij}^{n+1} - \frac{\Delta t\theta}{Re} \left( \sum_{k=1}^{N} a_{jk}^{(2)} u_{kj}^{n+1} + \sum_{k=1}^{M} b_{jk}^{(2)} u_{ik}^{n+1} \right) &= \left( 1 - \Delta t \sum_{k=1}^{N} b_{jk}^{(1)} u_{kj}^n \right) u_{ij}^n + \Delta t (1 - \theta ) \left( \sum_{k=1}^{N} a_{jk}^{(2)} u_{kj}^n + \sum_{k=1}^{M} b_{jk}^{(2)} u_{ik}^n \right) - \Delta t u_{ij}^n \sum_{k=1}^{M} a_{jk}^{(1)} v_{ik}^n, \\
v_{ij}^{n+1} - \frac{\Delta t\theta}{Re} \left( \sum_{k=1}^{N} a_{jk}^{(2)} v_{kj}^{n+1} + \sum_{k=1}^{M} b_{jk}^{(2)} v_{ik}^{n+1} \right) &= \left( 1 - \Delta t \sum_{k=1}^{N} a_{jk}^{(1)} v_{kj}^n \right) v_{ij}^n + \Delta t (1 - \theta ) \left( \sum_{k=1}^{N} a_{jk}^{(2)} v_{kj}^n + \sum_{k=1}^{M} b_{jk}^{(2)} v_{ik}^n \right) - \Delta t v_{ij}^n \sum_{k=1}^{M} a_{jk}^{(1)} u_{ik}^n,
\end{align*}
\]

(22a, 22b)

where \( u_{ij} = u^n(x_i, y_j) \) and \( a_{jk}^{(2)} \) and \( b_{jk}^{(2)} \) are WCs of 2nd order partial derivatives w.r.t. \( x \) and \( y \).

3.3. Implementation of Dirichlet Boundary Conditions. The Dirichlet BCs given in equation (3) as \( u(\beta, y, t) = h_1(y, t), u(\beta, y, t) = h_3(y, t), u(x, y, t) = h_1(x, t), \) and \( u(x, \delta, t) = h_4(x, t) \) can be implemented directly as follows:

\[
\begin{align*}
u_{ij}^{n+1} - \frac{\Delta t\theta}{Re} \left( \sum_{k=1}^{N} a_{jk}^{(2)} u_{kj}^{n+1} + \sum_{k=1}^{M} b_{jk}^{(2)} u_{ik}^{n+1} \right) &= S_{ij}, \\
v_{ij}^{n+1} - \frac{\Delta t\theta}{Re} \left( \sum_{k=1}^{N} a_{jk}^{(2)} v_{kj}^{n+1} + \sum_{k=1}^{M} b_{jk}^{(2)} v_{ik}^{n+1} \right) &= T_{ij}, \\
S_{ij} &= \left( 1 - \Delta t \sum_{k=1}^{N} a_{jk}^{(1)} u_{kj}^n \right) u_{ij}^n + \Delta t (1 - \theta ) \left( \sum_{k=1}^{N} a_{jk}^{(2)} u_{kj}^n + \sum_{k=1}^{M} b_{jk}^{(2)} u_{ik}^n \right) - \Delta t u_{ij}^n \sum_{k=1}^{M} a_{jk}^{(1)} v_{ik}^n, \\
T_{ij} &= \left( 1 - \Delta t \sum_{k=1}^{M} b_{jk}^{(1)} v_{ij}^n \right) v_{ij}^n + \Delta t (1 - \theta ) \left( \sum_{k=1}^{N} a_{jk}^{(2)} v_{kj}^n + \sum_{k=1}^{M} b_{jk}^{(2)} v_{ik}^n \right) - \Delta t v_{ij}^n \sum_{k=1}^{N} a_{jk}^{(1)} u_{ik}^n.
\end{align*}
\]

(24a, 24b)

As a result of applying the BCs on systems (23a) and (23b), the system can be written as follows:
The system of equations (18a) and (18b) is a Lyapunov system of equations of the form

\[
\begin{align*}
[A_1][U] + [U][B_1] + [C_1] &= 0, \\
[A_2][U] + [U][B_2] + [C_2] &= 0,
\end{align*}
\]

where

\[
[A_1] = \begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}_{(N-2)\times(N-2)}
\]

\[
[B_1] = -\frac{\Delta t \theta}{\text{Re}} \begin{bmatrix}
\alpha_{22}^{(2)} & \alpha_{23}^{(2)} & \cdots & \alpha_{2(N-2)}^{(2)} & \alpha_{2(N-1)}^{(2)} \\
\alpha_{32}^{(2)} & \alpha_{33}^{(2)} & \cdots & \alpha_{3(N-2)}^{(2)} & \alpha_{3(N-1)}^{(2)} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\alpha_{(N-2)2}^{(2)} & \alpha_{(N-2)3}^{(2)} & \cdots & \alpha_{(N-2)(N-2)}^{(2)} & \alpha_{(N-2)(N-1)}^{(2)} \\
\alpha_{(N-1)2}^{(2)} & \alpha_{(N-1)3}^{(2)} & \cdots & \alpha_{(N-1)(N-2)}^{(2)} & \alpha_{(N-1)(N-1)}^{(2)}
\end{bmatrix}_{(N-2)\times(N-2)}
\]

\[
[C_1] = \begin{bmatrix}
S_{22} & S_{23} & \cdots & S_{2(M-2)} & S_{2(M-1)} \\
S_{32} & S_{33} & \cdots & S_{3(M-2)} & S_{3(M-1)} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
S_{(N-2)2} & S_{(N-2)3} & \cdots & S_{(N-2)(M-2)} & S_{(N-2)(M-1)} \\
S_{(N-1)2} & S_{(N-1)3} & \cdots & S_{(N-1)(M-2)} & S_{(N-1)(M-1)}
\end{bmatrix}_{(N-2)\times(M-2)}
\]
Also,

$$[A_2] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{(N-2) \times (N-2)}$$

$$\frac{\Delta t \theta}{\text{Re}}
\begin{bmatrix}
\alpha_{22}^{(2)} & \alpha_{23}^{(2)} & \cdots & \alpha_{2(N-2)}^{(2)} & \alpha_{2(N-1)}^{(2)} \\
\alpha_{32}^{(2)} & \alpha_{33}^{(2)} & \cdots & \alpha_{3(N-2)}^{(2)} & \alpha_{3(N-1)}^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{(N-2)^2}^{(2)} & \alpha_{(N-2)^3}^{(2)} & \cdots & \alpha_{(N-2)(N-2)}^{(2)} & \alpha_{(N-2)(N-1)}^{(2)} \\
\alpha_{(N-1)^2}^{(2)} & \alpha_{(N-1)^3}^{(2)} & \cdots & \alpha_{(N-1)(N-2)}^{(2)} & \alpha_{(N-1)(N-1)}^{(2)} \\
\end{bmatrix}_{(N-2) \times (N-2)}$$

$$[V] =
\begin{bmatrix}
v_{22}^{\nu+1} & v_{23}^{\nu+1} & \cdots & v_{2(M-2)}^{\nu+1} & v_{2(M-1)}^{\nu+1} \\
v_{32}^{\nu+1} & v_{33}^{\nu+1} & \cdots & v_{3(M-2)}^{\nu+1} & v_{3(M-1)}^{\nu+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
v_{(N-2)^2}^{\nu+1} & v_{(N-2)^3}^{\nu+1} & \cdots & v_{(N-2)(M-2)}^{\nu+1} & v_{(N-2)(M-1)}^{\nu+1} \\
v_{(N-1)^2}^{\nu+1} & v_{(N-1)^3}^{\nu+1} & \cdots & v_{(N-1)(M-2)}^{\nu+1} & v_{(N-1)(M-1)}^{\nu+1} \\
\end{bmatrix}_{(N-2) \times (M-2)}$$

$$[B_2] = \frac{\Delta t \theta}{R}
\begin{bmatrix}
\beta_{22}^{(2)} & \beta_{23}^{(2)} & \cdots & \beta_{2(M-2)}^{(2)} & \beta_{2(M-1)}^{(2)} \\
\beta_{32}^{(2)} & \beta_{33}^{(2)} & \cdots & \beta_{3(M-2)}^{(2)} & \beta_{3(M-1)}^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{(M-2)^2}^{(2)} & \beta_{(M-2)^3}^{(2)} & \cdots & \beta_{(M-2)(M-2)}^{(2)} & \beta_{(M-2)(M-1)}^{(2)} \\
\beta_{(M-1)^2}^{(2)} & \beta_{(M-1)^3}^{(2)} & \cdots & \beta_{(M-1)(M-2)}^{(2)} & \beta_{(M-1)(M-1)}^{(2)} \\
\end{bmatrix}_{(M-2) \times (M-2)}$$

$$[C_2] =
\begin{bmatrix}
T_{22} & T_{23} & \cdots & T_{2(M-2)} & T_{2(M-1)} \\
T_{32} & T_{33} & \cdots & T_{3(M-2)} & T_{3(M-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
T_{(N-2)^2} & T_{(N-2)^3} & \cdots & T_{(N-2)(M-2)} & T_{(N-2)(M-1)} \\
T_{(N-1)^2} & T_{(N-1)^3} & \cdots & T_{(N-1)(M-2)} & T_{(N-1)(M-1)} \\
\end{bmatrix}_{(N-2) \times (M-2)}.$$
Table 2: $L_\infty$, RMS, and $L_2$ errors of Problem 1 at different times and Reynolds number for $u(x, y, t)$ with $N = M = 40$.

| $T$  | $L_\infty$ Re = 50 | $L_\infty$ Re = 100 |
|------|-------------------|---------------------|
|      | RMS               | RMS                 |
|      | $L_2$             | $L_2$               |
| 0.1  | $7.064E-04$       | $2.287E-04$         |
| 0.5  | $6.864E-04$       | $2.393E-04$         |
| 1.0  | $6.386E-04$       | $2.424E-04$         |
| 1.5  | $6.158E-04$       | $2.403E-04$         |
| 2.0  | $6.028E-04$       | $2.348E-04$         |

Table 3: $L_\infty$, RMS, and $L_2$ errors of Problem 1 at different times and Reynolds number for $u(x, y, t)$ with $N = M = 40$.

| $T$  | $L_\infty$ Re = 200 | $L_\infty$ Re = 400 |
|------|-------------------|---------------------|
|      | RMS               | RMS                 |
|      | $L_2$             | $L_2$               |
| 0.1  | $9.269E-04$       | $3.269E-03$         |
| 0.5  | $9.948E-04$       | $3.742E-03$         |
| 1.0  | $9.167E-04$       | $3.910E-03$         |
| 1.5  | $9.020E-04$       | $4.481E-03$         |
| 2.0  | $9.037E-04$       | $4.891E-03$         |

Figure 1: NSs and ESs of $u(x, y, t)$ in 3D form for $T = 1.0$ of Problem 1.

Figure 2: NSs and ESs of $u(x, y, t)$ in 3D form for $T = 2.0$ of Problem 1.
Equations (19a) and (19b) are the coupled Lyapunov system, first solved for \( n \geq 0 \) and then solved simultaneously for \( n \geq 1, 2, \ldots, K \) by developing code in MATLAB 7.

4. Numerical Experiments and Discussion

Under this heading, to check the correctness and competence of the algorithm modified CTBS DQM, two test problems have been considered, which are available in the literature. All the computation work is conducted by using MATLAB 7.0. The following formulas are used for computing maximum absolute error \( L_\infty \), root mean square (RMS) error, and \( L_2 \) error, respectively:

\[
L_\infty = \max_{1 \leq i \leq N} \left| e_{ij} \right|, \\
\text{RMS} = \left[ \frac{1}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} \left| e_{ij} \right|^2 \right]^{1/2}, \\
L_2 = \left[ \sum_{i=1}^{N} \sum_{j=1}^{M} \left| e_{ij} \right|^2 \right]^{1/2},
\]

(28)

Table 4: Maximum absolute error \( L_\infty \) of the problems at different nodes for \( u(x, y, t) \).

| Problem 2 | Problem 1 |
|-----------|-----------|
| Re = 100  | Re = 100  |
| \( N = M = 10 \) | \( N = M = 10 \) | \( N = M = 40 \) | \( N = M = 40 \) | \( N = M = 40 \) | \( N = M = 40 \) |
| 0.1 | 2.923E-04 | 5.569E-05 | 2.950E05 | 0.747E-03 | 0.326E-03 | 2.287E-04 |
| 0.5 | 4.958E-04 | 7.518E-05 | 2.421E-05 | 0.381E-03 | 0.103E-03 | 2.393E-04 |
| 1.0 | 3.932E-04 | 8.167E-05 | 1.387E-05 | 0.444E-03 | 9.792E-04 | 2.424E-04 |

Equations (19a) and (19b) are the coupled Lyapunov system, first solved for \( n = 0 \) and then solved simultaneously for \( n = 1, 2, \ldots, K \) by developing code in MATLAB 7.
where \( u_{ij} \) and \( \overline{u}_{ij} \) are approximate and exact solutions, respectively, and \( e_{ij} = u_{ij} - \overline{u}_{ij} \).

**Problem 1.** As the first problem, consider 2D Burgers’ equations (1a) and (1b). The exact solutions over the domain \( D = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1 \} \) is generated by the Hopf-Cole transformation \([12, 15, 17]\) and obtained as

\[
\begin{align*}
  u(x, y, t) &= \frac{3}{4} \frac{1}{1 + e^{Re(4y - 4x - t)/32}} \\
  v(x, y, t) &= \frac{3}{4} \frac{1}{1 + e^{Re(4y - 4x - t)/32}}
\end{align*}
\]  

(29)  

(30)

ICs and BCs are taken from exact solutions (29) and (30). The numerical results are shown with the help of Tables 2 and 3 and Figures 1–4 in form of errors, three-dimensional, and contour plots. Convection prevails the flow which causes the errors become larger and larger as we increase the value of Re. \( L_{\infty} \) is smaller than [15] for \( T = 2.0, Re = 100 \) with less grid points \( N = M = 40 \). The figures show that exact solutions and numerical solutions are well consistent in three-dimensional and contour form. Table 4 shows that, as we increase the values of \( M \) and \( N \), the absolute errors decrease which shows the convergence of the method.

**Problem 2.** Consider 2D Burgers’ equations (1a) and (1b) over the computational domain \( D = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1 \} \) with the ICs [12],

\[
\begin{align*}
  u(x, y, 0) &= \frac{-4\pi \cos(2\pi x) \sin(\pi y)}{Re[2 + \sin(2\pi x) \sin(\pi y)]} \quad (x, y) \in D, \\
  v(x, y, 0) &= \frac{-2\pi \sin(2\pi x) \cos(\pi y)}{Re[2 + \sin(2\pi x) \sin(\pi y)]} \quad (x, y) \in D,
\end{align*}
\]

(31)
Figure 6: NSs and ESs of $u(x, y, t)$ in contour form for $T = 2.0$ of Problem 2.

Figure 7: NSs and ESs of $v(x, y, t)$ in 3D form for $T = 2.0$ of Problem 2.

Figure 8: NSs and ESs of $v(x, y, t)$ in contour form for $T = 2.0$ of Problem 2.
The main results of this study are summarized in the proposed algorithm to reveal the computational modeling of 2D coupled Burgers’ equations. In this study, a modified CTBS DQM and a new algorithm are developed. The proposed algorithm is tested on two benchmark problems appearing in the literature. The main results of this study are summarized as follows:

(i) A different technique using modified CTBS functions is presented to determine the WGs of 2D DQM than Lagrange interpolation traditional technique [22].

(ii) CTBS DQ algorithm proposed in [33] has extended for 2D problems in different forms, and it has concluded the algorithm worked nicely for the same problems.

(iii) The developed algorithm is better than the DQ algorithms proposed in [31, 32, 34] due to more smoothness of CTBS functions.

(iv) The presented method leads to quite similar results to those treated in [12, 15, 17, 18] and good accuracy in the case of a small number of grid points.

(v) After some modifications, the presented method can be extended to solve 2D or higher-dimensional equations. In this way, it can be used to analyze many other biological, mechanical or physical events, such as reaction, linear diffusion, dispersion, and nonlinear convection.

The exact solutions of the problem are given by

\[ u(x, y, t) = -2\pi \exp\left(\frac{-5\pi^2 t}{Re}\right) \sin\left(\frac{\pi y}{Re}\right), \]

\[ v(x, y, t) = -2\pi \exp\left(\frac{-5\pi^2 t}{Re}\right) \sin\left(\frac{2\pi x}{Re}\right). \]

The exact solutions for three-dimensional form are well consistent in three-dimensional and contour form. Figure 5–8 show the convergence of the method.

5. Conclusion

In this study, a modified CTBS DQM and a new algorithm to reveal the computational modeling of 2D coupled Burgers’ equations are developed. The proposed algorithm is tested on two benchmark problems appearing in the literature. The main results of this study are summarized as follows:

The authors declare that they have no conflicts of interest.

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