EXISTENCE RESULTS OF SOLITARY WAVE SOLUTIONS FOR
A DELAYED CAMASSA-HOLM-KP EQUATION

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Abstract. This paper is concerned with the Camassa-Holm-KP equation, which is a model for shallow water waves. By using the analysis of the phase space, we obtain some qualitative properties of equilibrium points and existence results of solitary wave solutions for the Camassa-Holm-KP equation without delay. Furthermore we show the existence of solitary wave solutions for the equation with a special local delay convolution kernel by combining the geometric singular perturbation theory and invariant manifold theory. In addition, we discuss the existence of solitary wave solutions for the Camassa-Holm-KP equation with strength 1 of nonlinearity, and prove the monotonicity of the wave speed by analyzing the ratio of the Abelian integral.

1. Introduction. The classical Korteweg-de Vries-type equations and Camassa-Holm-type equations are fundamental models for shallow water waves. Physical structures of such equations have been extensively studied by many authors [35, 10, 6, 20, 38, 32, 39, 42, 23, 24]. In the past few decades remarkable progresses have been made in understanding the Korteweg-de Vries (KdV) equation, it can be considered as a paradigm in nonlinear science and has many applications in weakly nonlinear and weakly dispersive physical systems. The KdV equation

\[ u_t + \alpha uu_x + u_{xxx} = 0, \]

which is first suggested by Korteweg and de Vries in 1895 [18] is a typical soliton equation where the balance between the nonlinear convection term \( uu_x \) and the dispersion effect term \( u_{xxx} \) gives rise to solitons, while dissipation effects are small enough to be neglected in the lowest order approximation. The KdV equation is a model that governs the one-dimensional propagation of small amplitude, weakly dispersive waves. The two best known models for the two-dimensional generalizations

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of the KdV equation are the Kadomtsev-Petviashivili (KP) equation \[17\]

\[(u_t + auu_x + u_{xxx})_x + u_{yy} = 0,\]  \(1\)

and the Zakharov-Kuznetsov (ZK) equation \[40\]

\[u_t + auu_x + (\nabla^2 u)_x = 0,\]

where \(\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2\) is the isotropic Laplacian. In mathematics and physics, the KP equation introduced by Kadomtsev and Petviashivili is a partial differential equation to describe nonlinear wave motion. It has been studied in many papers, because it commonly appears in different physical applications, and it is integrable by means of the inverse scattering transform method. In addition to being used as a model for the evolution of surface waves \[1\], the KP equation has also been studied as a model for ion-acoustic wave propagation in isotropic media \[31\]. Saut and Tzvetkov \[33\] considered the Cauchy problems of KP equations in \(\mathbb{R}^d\) \((d = 2, 3)\) with

\[\begin{aligned}
& (u_t + \alpha uu_{xxx} + \beta u_{xxxx} + uu_x)_x + u_{yy} = 0, \\
& u(0, x, y) = \phi(x, y),
\end{aligned}\]

in the two dimensional case, and

\[\begin{aligned}
& (u_t + \alpha uu_{xxx} + \beta u_{xxxx} + uu_x)_x + u_{yy} + u_{zz} = 0, \\
& u(0, x, y, z) = \phi(x, y, z),
\end{aligned}\]

in the three dimensional case. The usual KP equations correspond to \(\beta = 0\) and \(\alpha = -1\) (KP-I) or \(\alpha = +1\) (KP-II \[15, 14, 28\]). Molinet et al. \[28\] studied the initial value problem for the KP-II equation and they proved the global well-posedness of the Cauchy problem.

Another shallow water wave equation different from the KP equation is the Camassa-Holm (CH) equation, which is an integrable, dimensionless and nonlinear partial differential equation

\[u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \ x \in \mathbb{R}, \ t > 0,\]

where \(u(x, t)\) denotes the fluid velocity, or can also be interpreted as the height of the water’s free surface above a flat bottom, and the constant \(k\) is related to the critical shallow water wave speed. In the special case that \(k\) is equal to zero, the CH equation has peakon solutions: solitons with a sharp peak, so with a discontinuity at the peak in the wave slope. This equation was first derived by Fuchssteiner and Fokas \[10\] as an abstract bi-Hamiltonian equation with infinitely many conservation laws, and later rederived by Camassa and Holm \[6\] from physical principles. Traveling wave solutions are basic patterns of different nonlinear differential equations. Lenells \[20\] provided a complete classification of all traveling wave solutions of the CH equation \(4\). Qu et al. \[32\] introduced a \(\mu\)-version of the modified CH equation and investigated its integrability, well-posedness, wave breaking, and existence of peaked soliton and multi-peakon solutions. Novruzov and Haguaev \[29\] considered Cauchy problem for the CH equation with dissipative term and compactly supported initial data. They found a simple condition guaranteeing blow-up of the solution by using some properties of the solution generated by initial data.

Wazwaz \[36\] considered the following CH-KP equations given by

\[\begin{aligned}
& (u_t - u_{xxt} + 2ku_x - au^n u_x)_x + u_{yy} = 0, \\
& \end{aligned}\]  \(5\)

and

\[\begin{aligned}
& (u_t - u_{xxt} + 2ku_x - au^n (u^n)_x)_x + u_{yy} = 0, \\
& \end{aligned}\]  \(6\)
where \( a > 0 \), \( k \in \mathbb{R} \) and \( n \) is called the strength of the nonlinearity. The aforementioned variants (5) and (6) are developed similarly to the KP equation (1), derived from the modified CH equations

\[
\begin{align*}
    u_t - u_{xxt} + 2k u_x - au^n u_x &= 0, \\
    u_t - u_{xxt} + 2k u_x - au^n (u^n)_x &= 0.
\end{align*}
\]

Therefore, equations (5) and (6) are called CH-KP equations. By using the sine-cosine method and the tanh technique, the solitons, compactons, solitary patterns and periodic solutions for equations (5) and (6) were obtained and expressed analytically. Lai et al. [19] studied the generalized forms of equations (5) and (6), which are written by

\[
\begin{align*}
    (u_t + 2k u_x - (u^m)_{xxt} - au^n u_x)_x + u_{yy} &= 0, \\
    (u_t + 2k u_x - (u^m)_{xxt} - au^n (u^n)_x)_x + u_{yy} &= 0,
\end{align*}
\]

and derived families of exact traveling wave solutions of equations (5) and (6) by means of the mathematical method. Wei et al. [37] investigated the single peak solutions of the CH-KP equation for \( m = 2, n = 1, k = 1, a = 1 \)

\[
(u_t + 2u_x - (u^2)_{xxt} - uu_x)_x + u_{yy} = 0,
\]

under the boundary condition \( \lim_{x,y \to \pm \infty} u(x, y) = A \). Recently, Biswas [5] obtained an exact 1-solitons solution of the generalized CH-KP equation by the solitary wave ansatze. By using the bifurcation theory of planar dynamical systems to the generalized CH-KP equations, Zhang et al. [41] proved the existence of smooth and nonsmooth traveling wave solutions.

Traveling wave solutions correspond to heteroclinic or homoclinic orbits of related ordinary differential equations. In general, such orbits can be found by means of geometric singular perturbation theory [9], which has been used by many researchers to obtain the existence of traveling waves for different nonlinear differential equations including generalized KdV equations [16], Fisher equations [2], perturbed BBM equation [7], FitzHugh-Nagumo equation [13, 25], reaction-diffusion equations [30], tissue interaction model [3], predator-prey models and epidemiology [22, 11], etc. The method has also received a great deal of interests in studying semilinear elliptic equations [4], slow-fast dynamic systems [26, 27], Liénard equations [21], etc. Recently, Du et al. [8] discussed the existence of solitary wave solution for the delayed CH equation

\[
u_t - u_{xxt} + 2k u_x + 3(f \ast u)u_x + \tau u_{xx} = 2u_x u_{xx} + uu_{xxx},\]

by using the method of dynamical system, especially the geometric singular perturbation theory and invariant manifold theory. Here \( f \ast u \) is the spatial-temporal convolution representing distributed delay and \( \tau \) is a small constant.

In fact, geometric singular perturbation theory uses invariant manifolds in phase space to better understand the global structure of the phase space or to construct orbits with desired properties. The essential idea behind geometric singular perturbation theory is that this persistence can be established by showing that these singular structures correspond to transversal intersections of a pair of stable and unstable manifolds.
Motivated by above papers, this paper is to establish the existence of solitary wave solutions for the following CH-KP equation

\[(u_t - u_{xxt} + 2ku_x - a(f \ast u)u^{n-1}u_x + \tau u_{xx})_x + u_{yy} = 0, \tag{10}\]

where the convolution \(f \ast u\) is defined by

\[f \ast u(x, y, t) = \int_{-\infty}^{t} f(t - s)u(x, y, s)\,ds,\]

and the kernel \(f : [0, +\infty) \to [0, +\infty)\) satisfies the normalization assumption \(f(t) \geq 0\) for all \(t \geq 0\), and \(\int_{0}^{\infty} f(t)\,dt = 1\). It is notable that the normalization assumption on \(f\) ensures that the uniform non-negative steady-state solutions are unaffected by the delay. The kernel \(f(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}\) and \(f(t) = \frac{t}{\tau^2}e^{-\frac{t}{\tau}}\), are frequently used in the literature on delay differential equations. The first one is called the weak generic delay kernel and the second one is called the strong general delay kernel. In this paper, we discuss the strong general delay kernel, i.e.,

\[f \ast u(x, y, t) = \int_{0}^{\infty} f(s)u(x, y, t - s)\,ds = \int_{0}^{\infty} \frac{s}{\tau^2}e^{-\frac{s}{\tau}}u(x, y, t - s)\,ds.\]

The remaining part of this paper is organized as follows. In Section 2, we present geometric singular perturbation theory which is important to obtain our main results. In Section 3, We prove the existence of solitary wave solutions for the the CH-KP equation (10) without perturbation and delay. In Section 4, we investigate the existence of solitary wave solutions for the CH-KP equation (10) with a special local delay convolution kernel by using the method of dynamical system, especially the geometric singular perturbation theory and invariant manifold theory. In Section 5, we discuss the existence of solitary wave solutions for equation (10) with \(n = 1\), and prove the monotonicity of the wave speed by analyzing the ratio of the Abelian integral.

2. Preliminaries. In this section, to present our results on the existence of traveling wave, we first introduce the geometric singular perturbation theory ([9, 16, 12, 34]).

Consider the system

\[
\begin{cases}
u'(t) = f(u, v, \epsilon), \\
v'(t) = \epsilon g(u, v, \epsilon),
\end{cases}
\tag{11}
\]

where \(\frac{d}{dt}\), \(u \in \mathbb{R}^k\), and \(v \in \mathbb{R}^l\) with \(k, l \geq 1\). The parameter \(\epsilon\) is a small parameter, which gives the system a singular character. The function \(f\) and \(g\) are assumed to be sufficiently smooth. Here ‘sufficiently smooth’ means at least \(C^1\) in \(u, v\) and \(\epsilon\). In general, to obtain \(C^r\) invariant manifolds, \(f\) and \(g\) should be \(C^{r+1}\) functions of \(u, v, \epsilon\), and the subset \(M_0 = \{ f(u, v, 0) = 0 \}\) we consider below should be a \(C^{r+1}\) submanifold of the phase space \(\mathbb{R}^{k+l}\).

System (11) can be reformulated with a change of time-scale as

\[
\begin{cases}
u'(t) = f(u, v, \epsilon), \\
v'(t) = \epsilon g(u, v, \epsilon),
\end{cases}
\tag{12}
\]

where \(\frac{d}{dt}\) and \(\tau = \epsilon t\). The time scale given by \(\tau\) is said to be slow whereas that for \(t\) is fast. Thus we call (11) the fast system and (12) the slow system. As long as
$\epsilon \neq 0$ the two systems are equivalent. Each of the scalings is naturally associated with a limit as $\epsilon \to 0$. These limits are respectively given by

$$\begin{align*}
\left\{\begin{array}{l}
u'(t) = f(u,v,0), \\
\nu'(t) = 0,
\end{array}\right.
\end{align*}$$

(13)

and

$$\begin{align*}
\left\{\begin{array}{l}
0 = f(u,v,0), \\
\dot{v}(t) = g(u,v,0).
\end{array}\right.
\end{align*}$$

(14)

The former is called the layer problem and the latter is called the reduced system. The problem (13) and (14) are lower dimensional and can often be analysed in sufficient detail. By ‘gluing’ together fast and slow pieces of orbits, respectively obtained in the fast and slow limits, one can formally construct global singular structures, such as singular periodic orbits and singular homoclinic orbits. Moreover, the flow under (13) on the $l$-dimensional set $f(u,v,0) = 0$ is trivial. On the other hand, (14) does prescribe a nontrivial flow on $f(u,v,0) = 0$, but at the same time its validity is limited to only this set. The goal of geometric singular perturbation theory is to analyse the dynamics of system (11) with $\epsilon$ nonzero but small by suitably combining the dynamics of these two limits.

**Definition 2.1.** ([16]) A manifold (with corner) $M_0$ on which $f(u,v,0) = 0$ is called a critical manifold. A critical manifold $M_0$ is said to be normally hyperbolic if the linearization of system (13) at each point in $M_0$ has exactly $l$ eigenvalues on the imaginary axis $\Re(\lambda) = 0$.

If $M_0$ is an $l$-dimensional manifold contained in $f(u,v,0) = 0$, and $M_0$ is normally hyperbolic, then Fenichel’s first theorem is as follows.

**Lemma 2.2** (Fenichel’s first theorem [12]). Suppose $M_0 \subset \{f(u,v,0) = 0\}$ is compact, possibly with boundary, and normally hyperbolic, that is, the eigenvalues $\lambda$ of the Jacobian $\frac{\partial f}{\partial u}(u,v,0)|_{M_0}$ all satisfy $\Re(\lambda) \neq 0$. Suppose $f$ and $g$ are smooth. Then for $\epsilon > 0$ and sufficiently small, there exists a manifold $M_\epsilon$, $O(\epsilon)$ close and diffeomorphic to $M_0$, that is locally invariant under the flow of the full problem (11).

Consider the equation (11). Suppose that, for $\epsilon = 0$, the normally hyperbolic critical manifold $M_0 \subset \{f(u,v,0) = 0\}$ has an $(l + m)$-dimensional stable manifold $W^s(M_0)$ and an $(l + n)$-dimensional unstable manifold $W^u(M_0)$, with $m + n = k$. In other words, suppose that the Jacobian $\frac{\partial f}{\partial u}(u,v,0)|_{M_0}$ has $m$ eigenvalues $\lambda$ with $\Re(\lambda) < 0$ and $n$ eigenvalues $\lambda$ with $\Re(\lambda) > 0$. Then the following theorem holds.

**Lemma 2.3** (Fenichel’s second theorem [12]). Suppose $M_0 \subset \{f(u,v,0) = 0\}$ is compact, possibly with boundary, and normally hyperbolic, and suppose $f$ and $g$ are smooth. Then for $\epsilon > 0$ and sufficiently small, there exist manifold $W^s(M_\epsilon)$ and $W^u(M_\epsilon)$, that are $O(\epsilon)$ close and diffeomorphic to $W^s(M_0)$ and $W^u(M_0)$, respectively, and that are locally invariant under the flow of (11).

**Definition 2.4.** A traveling wave solution $u(x,y,t) = \phi(x+y-ct) =: \phi(\xi)$ of the equation (11) below is called a solitary wave solution if $\lim_{\xi \to \pm \infty} \phi(\xi) = 0$. Here $c > 0$ is the wave speed.

3. **Solitary wave solutions for CH-KP equation without delay.** In this section, we will prove the existence of solitary wave solutions for (10) without delay,
i.e.,

\[ (u_t - u_{xx} + 2ku_x - au^n u_x)_x + u_{yy} = 0. \]  

(15)

Let

\[ u(x, y, t) = \phi(\xi), \quad \xi = x + y - ct, \]  

(16)

where \( c \neq 0 \). Substituting (16) into (15), then we obtain the following solitary wave equation

\[ (2k - c + 1)\phi'' + c\phi''' - n a \phi^{n-1}(\phi')^2 - a \phi^n \phi'' = 0, \]  

(17)

where \( ' = \frac{d}{d\xi} \). It can be integrated twice to yield the equation

\[ (2k - c + 1)\phi + c\phi'' - \frac{a}{n+1} \phi^{n+1} = 0, \]  

(18)

which is equivalent to the following system of first-order equation

\[
\begin{align*}
\phi' &= \psi, \\
\psi' &= \frac{1}{c} \left( -(2k - c + 1)\phi + \frac{a}{n+1} \phi^{n+1} \right).
\end{align*}
\]  

(19)

It has first integral

\[ H(\phi, \psi) = \frac{1}{2} \left( \psi^2 + \frac{(2k - c + 1)}{c} \phi^2 \right) - \frac{a}{c(n+1)(n+2)} \phi^{n+2}. \]  

(20)

By (20) and the analysis of the phase space, we obtain the following two theorems when \( n \) is odd and even respectively.

**Theorem 3.1.** When \( n \) is odd, then system (19) has two equilibrium points \( E_1(0,0) \) and \( E_2(\left(\frac{(2k+1-c)(n+1)}{a}\right)^{\frac{1}{n}}, 0) \) in the \((\phi, \psi)\) phase plane and the following results hold.

(i) If \( a > 0, c < 2k + 1 \), then \( E_1 \) is a center and \( E_2 \) is a saddle, and system (19) has a homoclinic orbit to the equilibrium point \( E_2 \) (Figure 1(a)).

(ii) If \( a < 0, c > 2k + 1 \), then \( E_1 \) is a saddle and \( E_2 \) is a center, and system (19) has a homoclinic orbit to the equilibrium point \( E_1 \) (Figure 1(b)).

(iii) If \( a > 0, c > 2k + 1 \), then \( E_1 \) is a saddle and \( E_2 \) is a center, and system (19) has a homoclinic orbit to the equilibrium point \( E_1 \) (Figure 1(c)).

(iv) If \( a < 0, c < 2k + 1 \), then \( E_1 \) is a center and \( E_2 \) is a saddle, and system (19) has a homoclinic orbit to the equilibrium point \( E_2 \) (Figure 1(d)).

**Theorem 3.2.** When \( n \) is even, the following results hold in the \((\phi, \psi)\) phase plane.

(i) If \( a > 0, c < 2k + 1 \), then system (19) has three equilibrium points \( E_1(0,0) \), \( E_2(\left(\frac{(2k+1-c)(n+1)}{a}\right)^{\frac{1}{n}}, 0) \) and \( E_3(-\left(\frac{(2k+1-c)(n+1)}{a}\right)^{\frac{1}{n}}, 0) \). \( E_1 \) is a center, \( E_2 \) and \( E_3 \) are saddles, and system (19) has two heteroclinic orbits which is connected by two equilibria \( E_2 \) and \( E_3 \) (Figure 2(a)).

(ii) If \( a < 0, c > 2k + 1 \), then system (19) has three equilibrium points \( E_1(0,0) \), \( E_2(\left(\frac{(2k+1-c)(n+1)}{a}\right)^{\frac{1}{n}}, 0) \) and \( E_3(-\left(\frac{(2k+1-c)(n+1)}{a}\right)^{\frac{1}{n}}, 0) \). \( E_1 \) is a saddle, \( E_2 \) and \( E_3 \) are center, and system (19) has two homoclinic orbits to the equilibrium point \( E_1 \) which is connected by two homoclinic orbits (Figure 2(b)).
Figure 1. The phase portraits of system (19) for $n$ is odd

Figure 2. The phase portraits of system (19) for $n$ is even
4. Solitary wave solution for CH-KP equation with local delay and viscosity. In this section, we establish the existence of solitary wave solutions for equation (10) with the condition \( a < 0, c > 2k + 1 \) and \( n \) is odd.

Consider the case of small positive averaging delay \( \tau > 0 \). We search for solutions of (10) in the form \( u(x, y, t) = \phi(\xi) \) with \( \xi = x + y - ct \), then (10) is changed to
\[
(2k - c + 1)\phi'' + c\phi''' - (a\eta\phi^{n-1}\phi')' + \tau\phi''' = 0,
\]
where
\[
\eta(\xi) = \int_0^\infty \frac{t}{\tau^2} e^{-\frac{t}{\tau}} \phi(\xi + ct) dt.
\]
Then differentiating \( \eta(\xi) \) with respect to \( \xi \), we obtain
\[
\frac{d\eta}{d\xi} = \frac{1}{c\tau}(\eta - \zeta),
\]
where
\[
\zeta = \int_0^\infty \frac{1}{\tau} e^{-\frac{t}{\tau}} \phi(\xi + ct) dt.
\]
Differentiating \( \zeta \) with respect to \( \xi \), we obtain
\[
\frac{d\zeta}{d\xi} = \frac{1}{c\tau}(\zeta - \phi).
\]
Using the boundary condition at \(-\infty\), the equation (21) can be integrated once to yield the equation
\[
(2k - c + 1)\phi' + c\phi''' - a\eta\phi^{n-1}\phi' + \tau\phi'' = 0.
\]
Therefore the solitary wave equation (24) is equivalent to the following system
\[
\begin{aligned}
\dot{\phi}' &= \psi, \\
\psi' &= \nu, \\
c\nu' &= (c - 2k - 1)\psi + a\eta\phi^{n-1}\psi - \tau\nu, \\
c\tau\eta' &= \eta - \zeta, \\
c\tau\zeta' &= \zeta - \phi.
\end{aligned}
\]
Note that when \( \tau \to 0 \), we get \( \eta \to \phi \) and arrive at the non-delay equation.

We want to investigate the persistence of the homoclinic orbit for small \( \tau > 0 \). By setting \( \xi = \tau s \), the system (25) becomes
\[
\begin{aligned}
\dot{\phi}' &= \tau\psi, \\
\psi' &= \tau\nu, \\
c\nu' &= \tau((c - 2k - 1)\psi + a\eta\phi^{n-1}\psi - \tau\nu), \\
c\eta' &= \eta - \zeta, \\
c\zeta' &= \zeta - \phi,
\end{aligned}
\]
where \( \dot{} = \frac{d}{ds} \). We refer to (25) as the slow system and (26) as the fast system. The above two systems are equivalent when \( \tau > 0 \). If we set \( \tau = 0 \) in (25), then the flow of system (25) is restricted to the set
\[
M_0 = \{(\phi, \psi, \nu, \eta, \zeta) \in \mathbb{R}^5 : \eta = \zeta = \phi\},
\]
which is a three-dimensional manifold of equilibrium for (26) with \( \tau = 0 \). Note that we have restricted the discussion in a neighborhood of the unperturbed homoclinic
orbit. The linearization of (26) with $\tau = 0$ is
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & c^{-1} \\
-\frac{1}{c} & 0 & 0 & 0 & -\frac{1}{c}
\end{pmatrix}.
\]
It can be easily seen that the matrix has five eigenvalues: 0, 0, 0, $\frac{1}{c}$, $\frac{1}{c}$, and thus we can conclude that $M_0$ is normally hyperbolic with two unstable normal directions. According to Lemma 2.2, there exists $M_\tau$ for $0 < \tau \ll 1$:
\[
M_\tau = \{(\phi, \psi, \nu, \eta, \zeta) \in \mathbb{R}^5 : \eta = \phi + g(\phi, \psi, \nu, \tau), \zeta = \phi + h(\phi, \psi, \nu, \tau)\},
\]
where the functions $g, h$ are smooth functions defined on a compact domain, and satisfy
\[
g(\phi, \psi, \nu, 0) = 0, h(\phi, \psi, \nu, 0) = 0.
\]
Thus the function $g, h$ can be expanded into the form of a Taylor series about $\tau$
\[
g(\phi, \psi, \nu, \tau) = \tau g_1(\phi, \psi, \nu) + \tau^2 g_2(\phi, \psi, \nu) + \cdots,
\]
\[
h(\phi, \psi, \nu, \tau) = \tau h_1(\phi, \psi, \nu) + \tau^2 h_2(\phi, \psi, \nu) + \cdots.
\]
Substituting $\eta = \phi + g(\phi, \psi, \nu, \tau), \zeta = \phi + h(\phi, \psi, \nu, \tau)$ into the slow system (25), we have
\[
ct \left(1 + \frac{\partial g}{\partial \phi} \psi + \frac{\partial g}{\partial \psi} \nu + \frac{\partial g}{\partial \nu} \frac{1}{c} \left((c - 2k - 1)\psi + a\phi^n \psi - \tau \nu\right)\right) = g - h,
\]
\[
ct \left(1 + \frac{\partial h}{\partial \phi} \psi + \frac{\partial h}{\partial \psi} \nu + \frac{\partial h}{\partial \nu} \frac{1}{c} \left((c - 2k - 1)\psi + a\phi^n \psi - \tau \nu\right)\right) = h.
\]
An easy calculation shows the following asymptotic expansion:
\[
h = c\tau \psi + O(\tau^2), \quad g = 2c\tau \psi + O(\tau^2).
\]
Therefore, restricted to $M_\tau$, (25) becomes the following system
\[
\begin{cases}
\phi' = \psi, \\
\psi' = \nu, \\
c\nu' = (c - 2k - 1)\psi + a(\phi + 2c\tau \psi)\phi^n - \tau \nu + O(\tau^2).
\end{cases}
\]
Because both normal directions are unstable, the persistent homoclinic orbit (if ever exists) must be contained totally in $M_\tau$. In addition, it is obvious that system (27) can be rewritten as the following form
\[
\begin{cases}
\phi' = \psi, \\
\psi' = \frac{1}{c} \left((c - 2k - 1)\phi + \frac{a}{n + 1} \phi^{n+1}\right) + \tau \nu, \\
\nu' = 2a\phi^n \psi^2 - \frac{1}{c} \left(\frac{1}{c} \left((c - 2k - 1)\phi + \frac{a}{n + 1} \phi^{n+1}\right) + \tau \nu\right) + O(\tau).
\end{cases}
\]
Now we consider the homoclinic orbit of system (28).

**Theorem 4.1.** For sufficiently small $\tau > 0$, there exists a unique speed $c$ such that equation (10) with the strong generic local delay kernel has a generalized solitary wave solution in the sense that the corresponding singular perturbation equation (28) has a homoclinic orbit to the equilibrium point $(0, 0)$. 
Proof. For $\tau = 0$, the system (28) contains two manifolds of equilibria $\bar{M}_0 = \{ \phi = \psi = 0 \}$ and $\bar{N}_0 = \{ \phi = (\frac{2k+1-c(n+1)}{c})\tilde{z}, \psi = 0 \}$. We focus on $\bar{M}_0$, which is normally hyperbolic with one stable and one unstable normal direction. From Lemma 2.2, there exists $\bar{M}_\tau$ for $0 < \tau \ll 1$, which is $O(\tau)$ close to $\bar{M}_0$. Lemma 2.3 indicates the existence of two-dimensional stable and unstable manifold $W^s_\tau$ and $W^u_\tau$ of $\bar{M}_\tau$, being $O(\tau)$ close to corresponding stable and unstable manifold $W^s_0$ and $W^u_0$ of $\bar{M}_0$ respectively.

Viewing $\tilde{\nu}$ as parameter, (28) is a Hamiltonian system, with non-transversal intersection of $W^s_0$ and $W^u_0$ and a two-dimensional homoclinic manifold. We need to study the separation of $W^s_\tau$ and $W^u_\tau$ along the unperturbed homoclinic orbit $\Gamma$ which satisfies

$$\begin{cases}
\phi' = \psi, \\
\psi' = \frac{1}{c} \left( (c - 2k - 1)\phi + \frac{a}{n + 1}\phi^{n+1} \right),
\end{cases} \ (29)$$

with

$$\frac{1}{2} \left( \psi^2 + \frac{(2k - c + 1)\phi^2}{c} \right) - \frac{a\phi^{n+2}}{c(n + 1)(n + 2)} = 0.$$

Since the bounded parts of $W^s_0$ and $W^u_0$ coincide, the distance between the perturbed manifolds $W^s_\tau$ and $W^u_\tau$ is of order $O(\tau)$ and may thus be computed via the adiabatic Melnikov integral

$$Q(c, \tau) = \tau M + O(\tau^2), \quad M = \int_{-\infty}^{\infty} \psi \tilde{\nu} d\xi, \quad (30)$$

where $\phi, \psi$ are evaluated along $\Gamma$, and $\tilde{\nu}$ satisfies

$$\begin{cases}
\frac{d\tilde{\nu}}{d\xi} = 2a\phi^{n-1}\psi^2 - \frac{1}{c^2} \left( (c - 2k - 1)\phi + \frac{a}{n + 1}\phi^{n+1} \right), \\
\tilde{\nu} = 0 \text{ at } \xi = 0,
\end{cases} \ (31)$$

with $\phi, \psi$ evaluated along $\Gamma$. Obviously, solutions of the equation $Q(c, \tau) = 0$ correspond to homoclinic orbits of (28).

To compute $M$, we use integration by parts. By (29), one has

$$M = \phi\psi \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi \left( 2a\phi^{n-1}\psi^2 - \frac{1}{c^2} \left( -(2k - c + 1)\phi + \frac{a}{n + 1}\phi^{n+1} \right) \right) d\xi,$$

i.e.,

$$M = -\int_{-\infty}^{\infty} \phi \left( 2a\phi^{n-1}\psi^2 - \frac{1}{c^2} \left( -(2k - c + 1)\phi + \frac{a}{n + 1}\phi^{n+1} \right) \right) d\xi.$$

From (29) and by using integration by parts again, we have

$$M = -\int_{-\infty}^{\infty} 2a\phi^n\psi^2 d\xi - \frac{1}{c} \int_{-\infty}^{\infty} \psi^2 d\xi$$

$$= -2 \int_{-\infty}^{\infty} (a\phi^n - (2k - c + 1))(\phi')^2 d\xi - 2(2k - c + 1) \int_{-\infty}^{\infty} (\phi')^2 d\xi$$

$$= -\frac{1}{c} \int_{-\infty}^{\infty} \psi^2 d\xi$$

$$= 2c \int_{-\infty}^{\infty} (\phi')^2 d\xi - 2(2k - c + 1) \int_{-\infty}^{\infty} (\phi')^2 d\xi - \frac{1}{c} \int_{-\infty}^{\infty} (\phi')^2 d\xi$$

$$= 2c \int_{-\infty}^{\infty} (\phi')^2 d\xi - \frac{1}{c}(1 + 2c(2k - c + 1)) \int_{-\infty}^{\infty} (\phi')^2 d\xi.$$
We obtain
\[ M = \frac{1}{c} \left( 2(A + B)c^2 - 2(2k + 1)Bc - B \right). \]
Let
\[ G(c) = 2(A + B)c^2 - 2(2k + 1)Bc - B. \]
A simple calculation yields that
\[ \frac{(2k + 1)B}{2(A + B)} > 0, \quad -\frac{B}{2(A + B)} < 0, \]
and
\[ \Delta = 4(2k + 1)^2B^2 + 8B(A + B) > 0. \]
It follows that there exists a unique \( c_0 > 0 \) such that
\[ M = M(c_0) = 0, \quad \text{and} \quad \frac{\partial M}{\partial c} \bigg|_{c_0} \neq 0. \]
Thus, we can solve the equation \( Q(c, \tau) = 0 \) locally for \( c = c_0 \) by the implicit function theorem. This indicates \( W^* \) and \( W^u \) intersect transversally and a homoclinic orbit to \( M_\tau \) persists.

\[ \blacksquare \]

**Remark 1.** This paper only discusses the existence of solitary wave solution for the case that \( a < 0, c > 2k + 1, \) and \( n \) is odd. In fact, we could obtain the existence of solitary (traveling) wave solution for other cases by similar discussions, and we omit the details here.

5. **Monotonicity of wave speed for CH-KP equation with** \( n = 1. \) In this section, we will study the monotonicity of the wave speed for CH-KP equation with \( n = 1. \) To do this, we will use another convenient way to obtain the existence of solitary wave solutions for (10) with \( n = 1. \) In fact, this verifies again the result for \( n = 1 \) in previous section. Then, by analyzing the ratio of Abelian integrals, we obtain the monotonicity of the wave speed and give the upper and lower bounds of the limit wave speed. When \( n = 1, \) equation (10) becomes
\[ (u_t - u_{xxt} + 2ku_x - a(f * u)u_x + \tau u_{xx})_x + u_{yy} = 0. \] (32)

First, we use the Hamiltonian function to discuss the equation (32) without delay, i.e.,
\[ (u_t - u_{xxt} + 2ku_x - auu_x)_x + u_{yy} = 0. \] (33)
Substituting \( u(x, y, t) = \phi(\xi), \xi = x + y - ct \) into (33), then we obtain
\[ (2k - c + 1)\phi'' + c\phi''' - a(\phi')^2 - a\phi\phi'' = 0, \] (34)
where \( \xi = \frac{d}{dt}. \) It can be integrated twice to yield the equation
\[ (2k - c + 1)\phi + c\phi'' - \frac{a}{2}\phi^2 = 0. \] (35)
Taking the transformations \( \phi_1 = \frac{\phi}{e^{2k-1}}, \) and \( z = \sqrt{\frac{c-2k-1}{c}}\xi, \) we obtain
\[ \phi_1 - \phi_1^2 + \frac{a}{2}\phi_1^2 = 0, \] (36)
which is equivalent to the following system of first-order equation

\[
\begin{align*}
\dot{\phi}_1 &= \psi_1, \\
\dot{\psi}_1 &= \frac{a}{2} \phi_1^2 + \phi_1,
\end{align*}
\] (37)

where \(\dot{\cdot} = \frac{d}{dz}\). It has first integral

\[
H(\phi_1, \psi_1) = -\frac{1}{2} \psi_1^2 + \frac{a}{2} \phi_1^2 + \frac{a}{6} \phi_1^3.
\] (38)

From (38) and easy analysis of the phase space, we obtain the following theorem.

**Theorem 5.1.** If \(a < 0, c > 2k + 1\), then system (37) has two equilibrium points: \(E_1(0, 0)\) is a saddle and \(E_2(-\frac{a}{2}, 0)\) is a center, and system (37) has a homoclinic orbit to the equilibrium point \(E_1\).

Consider the case of small positive averaging delay \(\tau > 0\). We search for solutions of (32) in the form

\[u(x, y, t) = \phi(\xi), \quad \xi = x + y - ct,\]

then (32) is changed to

\[
(2k - c + 1)\phi'' + c\phi''' - (a\eta\phi')' + \tau\phi'' = 0,
\] (39)

where

\[
\eta(\xi) = \int_0^\infty \frac{t}{\tau^2} e^{-\frac{t}{\tau}} \phi(\xi + ct) dt.
\]

Using the boundary condition at \(-\infty\), equation (39) can be integrated once to yield the equation

\[
(2k - c + 1)\phi' + c\phi''' - a\eta\phi' + \tau\phi'' = 0.
\] (40)

Taking the transformations \(\phi_1 = \frac{\phi}{\sqrt{c-2k-1}}\) and \(z = \sqrt{\frac{c-2k-1}{c}} \xi\), we obtain

\[
\dot{\phi}_1 - \frac{\tau}{\sqrt{c(c-2k-1)}\psi_1} \dot{\psi}_1 + a\eta_1 \dot{\phi}_1 - \frac{\tau}{\sqrt{c(c-2k-1)}} \ddot{\phi}_1 = 0,
\] (41)

where

\[
\eta_1(z) = \int_0^\infty \frac{t}{\tau^2} e^{-\frac{t}{\tau}} \phi_1\left(\sqrt{\frac{c}{c-2k-1}} z + ct\right) dt.
\]

Therefore the solitary wave equation (41) is equivalent to the following system

\[
\begin{align*}
\dot{\phi}_1 &= \psi_1, \\
\dot{\psi}_1 &= \nu_1, \\
\dot{\nu}_1 &= \psi_1 + a\eta_1 \psi_1 - \frac{\tau}{\sqrt{c(c-2k-1)}} \nu_1, \\
\sqrt{c(c-2k-1)}\tau\psi_1 &= \eta_1 - \zeta_1, \\
\sqrt{c(c-2k-1)}\tau\zeta_1 &= \zeta_1 - \phi_1,
\end{align*}
\] (42)

where

\[
\zeta_1 = \int_0^\infty \frac{1}{\tau} e^{-\frac{t}{\tau}} \phi_1\left(\sqrt{\frac{c}{c-2k-1}} z + ct\right) dt.
\]
By setting $z = \tau s$, the system (42) becomes

$$
\begin{align*}
\frac{d\phi_1}{ds} &= \tau \psi_1, \\
\frac{d\psi_1}{ds} &= \tau \nu_1, \\
\frac{d\nu_1}{ds} &= \tau \left( \psi_1 + a\eta_1 \psi_1 - \frac{\tau}{\sqrt{c(c - 2k - 1)}} \nu_1 \right), \\
\sqrt{c(c - 2k - 1)} \frac{d\eta_1}{ds} &= \eta_1 - \xi_1, \\
\sqrt{c(c - 2k - 1)} \frac{d\xi_1}{ds} &= \xi_1 - \phi_1.
\end{align*}
$$

(43)

If $\tau$ is set to zero in (42), then the flow of that system is restricted to the set

$$M_0 = \{ (\phi_1, \psi_1, \nu_1, \eta_1, \xi_1) \in \mathbb{R}^5 : \eta_1 = \xi_1 = \phi_1 \},$$

which is a three-dimensional manifold of equilibrium for (43) with $\tau = 0$. The linearization of (43) with $\tau = 0$ is

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{c(c - 2k - 1)}} & - \frac{1}{\sqrt{c(c - 2k - 1)}} \\
- \frac{1}{\sqrt{c(c - 2k - 1)}} & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

It can be easily seen that the matrix has five eigenvalues: 0, 0, 0, $\frac{1}{\sqrt{c(c - 2k - 1)}}$, and thus we can conclude that $M_0$ is normally hyperbolic with two unstable normal directions. According to Lemma 2.2, there exists $M_\tau$ for $0 < \tau \ll 1$: $M_\tau = \{ (\phi_1, \psi_1, \nu_1, \eta_1, \xi_1) \in \mathbb{R}^5 : \eta_1 = \phi_1 + g(\phi_1, \psi_1, \nu_1, \tau), \xi_1 = \phi_1 + h(\phi_1, \psi_1, \nu_1, \tau) \}$, where the functions $g, h$ are smooth functions defined on a compact domain, and satisfy

$$g(\phi_1, \psi_1, \nu_1, 0) = 0, h(\phi_1, \psi_1, \nu_1, 0) = 0.$$

Easy calculation shows the following asymptotic expansion:

$$h = \sqrt{c(c - 2k - 1)} \tau \psi_1 + O(\tau^2), \quad g = 2 \sqrt{c(c - 2k - 1)} \tau \psi_1 + O(\tau^2).$$

Therefore restricted to $M_\tau$, (42) is the following system

$$
\begin{align*}
\dot{\phi}_1 &= \psi_1, \\
\dot{\psi}_1 &= \nu_1, \\
\dot{\nu}_1 &= \psi_1 + a(\phi_1 + 2\tau \psi_1 \sqrt{c(c - 2k - 1)}) \psi_1 - \frac{\tau}{\sqrt{c(c - 2k - 1)}} \nu_1 + O(\tau^2). \quad (44)
\end{align*}
$$

Because both normal directions are unstable, the persistent homoclinic orbit (if ever exists) must be contained totally in $M_\tau$. In addition, it is obvious that system (44) can be rewritten as the following form

$$
\begin{align*}
\phi_1 &= \psi_1, \\
\dot{\psi}_1 &= \phi_1 + a \phi_1^2 + \tau \nu_1, \\
\dot{\nu}_1 &= 2a \sqrt{c(c - 2k - 1)} \psi_1^2 - \frac{1}{\sqrt{c(c - 2k - 1)}} (\phi_1 + a \phi_1^2 + \tau \nu_1) + O(\tau). \quad (45)
\end{align*}
$$

Now we consider the homoclinic orbit of system (45).
Theorem 5.2. For sufficiently small \( \tau > 0 \), there exists a unique speed \( c \) such that equation (32) with the strong generic local delay kernel has a generalized solitary wave solution in the sense that the correspond singular perturbation equation (45) has a homoclinic orbit to the equilibrium point \((0, 0)\).

Proof. Similar to the proof of Theorem 4.1, we have

\[
M = \phi_1 \dot{v}_1 \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi_1 \left( 2a \sqrt{c(c-2k-1)} \gamma_1 - \frac{1}{\sqrt{c(c-2k-1)}} (\phi_1 + \frac{a}{2} \phi_1^2) \right) dz
\]

\[
= 2 \sqrt{c(c-2k-1)} \int_{-\infty}^{\infty} (\phi_1)^2 dz
\]

\[
+ \frac{1}{\sqrt{c(c-2k-1)}} (2c(c-2k-1) - 1) \int_{-\infty}^{\infty} (\phi_1)^2 dz
\]

\[
= \sqrt{c(c-2k-1)} \left( 2 \int_{-\infty}^{\infty} (\phi_1)^2 dz \right)
\]

Thus, we can solve the Melnikov distance \( Q(c, \tau) = 0 \) locally for \( c = c_0 \) by the implicit function theorem. This indicates \( W_s^{\tau} \) and \( W_u^{\tau} \) intersect transversally and a homoclinic orbit to \( M^{\tau} \) persists. \( \square \)

In the following, we discuss the monotonicity of the wave speed \( c \) and the upper and lower bounds of the wave speed by analyzing the ratio of the Abelian integral. First of all, let \( Q \) and \( R \) be

\[ Q = \frac{1}{2} \int_{z_2}^{z_1} \phi_1^2 dz, \quad R = \frac{1}{2} \int_{z_2}^{z_1} \phi_1^2 dz. \]

Denote

\[ f(\phi_1) := \frac{a}{3} \phi_1^3 + \phi_1^2. \]

Then, we have

\[ f(\phi_1) - M = \frac{a}{3} \phi_1^3 + \phi_1^2 - M. \]

Denote the two non-negative roots of \( \frac{a}{3} \phi_1^3 + \phi_1^2 - M = 0 \) by \( \alpha(M) \) and \( \beta(M) \) with \( \alpha(M) < \beta(M) \), and \( 0 < M < \frac{4}{3a^2} \). The orbit \((\phi_1(z), \psi_1(z))\) is on the level curve \( H = \frac{M}{2} \), therefore, by (37) we have

\[ Q = \int_{\alpha}^{\beta} \frac{(\frac{a}{3} \phi_1^3 + \phi_1^2)^2}{E(\phi_1)} d\phi_1, \quad R = \int_{\alpha}^{\beta} E(\phi_1) d\phi_1, \]

where

\[ E(\phi_1) = \sqrt{\frac{a}{3} \phi_1^3 + \phi_1^2 - M}. \]

Now \( Q \) and \( R \) are the functions of \( M \) only. The purpose of this section is to prove the following proposition which will assert the monotonicity of the speed on \( M \).

Proposition 1. Let \( X(M) = \frac{Q}{R} \), then we have

\[ X'(M) > 0 \text{ and } \frac{5}{7} < X(M) < 1, \text{ for } 0 < M < \frac{4}{3a^2}. \]

Moreover

\[ \lim_{M \to 0} X(M) = \frac{5}{7} \text{ and } \lim_{M \to \frac{4}{3a^2}} X(M) = 1. \]
In order to prove the proposition, it is convenient to represent $Q$ and $R$ by the following integrals

$$J_n(M) = \int_\alpha^\beta \phi_1^n E(\phi_1) d\phi_1, \ n = 0, 1, 2, \ldots$$

Then it holds

$$\int_\alpha^\beta \frac{\phi_1^n}{E(\phi_1)} d\phi_1 = -2J'_n(M).$$

Therefore, $Q$ and $R$ are represented as follows:

$$Q = -2J'_2 - 2aJ'_3 - \frac{a^2}{2}J'_4,$$

and

$$R = J_0.$$

First, let us study the basic properties of $J_0$ and $J_1$ by the following four lemmas.

**Lemma 5.3.** $\lim_{M \to 0} J_0 = \frac{12}{5a^2}, \lim_{M \to 0} J_1 = -\frac{144}{35a^3}$.

*Proof.* The conclusion can be obtained by a direct calculation of $J_0$ and $J_1$ with $M = 0$.

**Lemma 5.4.** $\lim_{M \to \frac{4}{5a^2}} \frac{J_1(M)}{J_0(M)} = -\frac{2}{a}$.

**Lemma 5.5.**

$$\begin{pmatrix} J_0 \\ J_1 \end{pmatrix} = \Lambda(M) \begin{pmatrix} J'_0 \\ J'_1 \end{pmatrix},$$

where

$$\Lambda(M) = \begin{pmatrix} 6M & 4 \\ -12M & 3 \left(2M - \frac{16}{5a^2}\right) \end{pmatrix}.$$

*Proof.* From the relation

$$E^2 = \frac{a}{3} \phi_1^3 + \phi_1^2 - M,$$

we have

$$2E \frac{dE}{du} = 2\phi_1 + a\phi_1^2.$$

$J_0$ can be calculated as

$$J_0 = \int_\alpha^\beta Ed\phi_1 = \int_\alpha^\beta E^2 \frac{d\phi_1}{E} = \int_\alpha^\beta \left(\frac{a}{3} \phi_1^3 + \phi_1^2 - M\right) \frac{d\phi_1}{E}$$

$$= \int_\alpha^\beta \left(\frac{a}{3} \phi_1 + 1\right) \phi_1^2 - M \frac{d\phi_1}{E} = \int_\alpha^\beta \left(\frac{a}{3} \phi_1 + 1\right) \left(\frac{2E}{a} \frac{dE}{d\phi_1} - \frac{2}{a} \phi_1\right) - M \frac{d\phi_1}{E}$$

$$= 2MJ'_0 + \frac{4}{3a} J'_1 - \frac{2}{3}J_0.$$

Thus

$$J_0 = \frac{6}{5} M J'_0 + \frac{4}{5a} J'_1.$$
\[
\int_\alpha^\beta \left( \phi_1^2 \left( \frac{a}{3} \phi_2^2 + \phi_1 \right) - \phi_1 M \right) \frac{d\phi_1}{E} = -\frac{4}{a} J_0 - \frac{4}{3} J_1 + \frac{4M}{a} J_0' + 2M J_1'.
\]
Thus
\[
J_1 = \frac{3}{7} \left( -\frac{4}{a} J_0 + \frac{4M}{a} J_0' + 2M J_1' \right) = -\frac{12}{35a} M J_0' + \frac{3}{7} \left( 2M - \frac{16}{5a^2} \right) J_1'.
\]
So we get
\[
\begin{pmatrix}
J_0 \\
J_1
\end{pmatrix} = \Lambda(M) \begin{pmatrix}
J_0' \\
J_1'
\end{pmatrix}.
\]

**Lemma 5.6.**
\[
\begin{pmatrix}
J_0'' \\
J_1''
\end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix}
-\frac{1}{2} \frac{M}{a} & -\frac{1}{2} \frac{M}{a} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix} \begin{pmatrix}
J_0' \\
J_1'
\end{pmatrix},
\]
where \( \Delta = 3M^2 - \frac{4M}{a^2} \).

**Proof.** By Lemma 5.5, \( J'' = \Lambda^{-1}(I - \Lambda')J' \) holds, where \( I \) denotes the identity matrix. We have
\[
\Lambda' = \begin{pmatrix}
6 & 0 \\
\frac{12}{35a} & \frac{1}{7}
\end{pmatrix}.
\]
Therefore, we can get
\[
I - \Lambda' = \begin{pmatrix}
-\frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{7}
\end{pmatrix},
\]
and
\[
\Lambda^{-1}(k) = \frac{35}{12\Delta} \begin{pmatrix}
\frac{3}{7} \left( 2M - \frac{16}{5a^2} \right) & -\frac{4}{5a} \\
\frac{12}{35a} & \frac{6}{5} M
\end{pmatrix}.
\]
Hence,
\[
\Lambda^{-1}(I - \Lambda')J' = \frac{1}{\Delta} \begin{pmatrix}
-\frac{1}{2} \frac{M}{a} & -\frac{1}{2} \frac{M}{a} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix} \begin{pmatrix}
J_0' \\
J_1'
\end{pmatrix}.
\]
This implies the lemma.

After a series of calculation, we note that all \( J_n \) can be represented only by \( J_0 \) and \( J_1 \). Therefore, the following lemma holds.

**Lemma 5.7.**
\[
\begin{aligned}
J_2 &= -\frac{2}{a} J_1, \\
J_3 &= \frac{6M}{11a} J_0 + \frac{48}{11a^2} J_1, \\
J_4 &= \left( \frac{12M}{13a} - \frac{1440}{143a^3} \right) J_1 - \frac{180M}{143a^2} J_0.
\end{aligned}
\]
Proof.

\[ J_2 = \int_\alpha^\beta \phi_1^2 E d\phi_1 = \int_\alpha^\beta \left( \frac{2E}{a} \frac{dE}{d\phi_1} - \frac{2}{a} \phi_1 \right) E d\phi_1 = -\frac{1}{a} \int_\alpha^\beta \phi_1 E d\phi_1 = -\frac{2}{a} J_1, \]

\[ J_3 = \int_\alpha^\beta \phi_1^3 E d\phi_1 = \int_\alpha^\beta \left( \frac{2E}{a} \frac{dE}{d\phi_1} - \frac{2}{a} \phi_1 \right) \phi_1 E d\phi_1 = -\frac{8}{3} J_3 + \frac{16}{a^2} J_1 + \frac{2M}{a} J_0, \]

then, we can obtain

\[ J_3 = \frac{6M}{11a} J_0 + \frac{48}{11a^2} J_1, \]

\[ J_4 = \int_\alpha^\beta \phi_1^4 E d\phi_1 = \int_\alpha^\beta \phi_1^2 \left( \frac{2E}{a} \frac{dE}{d\phi_1} - \frac{2}{a} \phi_1 \right) E d\phi_1 = \frac{4M}{a} J_1 - \frac{10}{a} J_3 - \frac{10}{3} J_4. \]

Therefore we get

\[ \frac{13}{3} J_4 = \frac{4M}{a} J_1 - \frac{10}{a} J_3 = \left( \frac{4M}{a} - \frac{480}{11a^2} \right) J_1 - \frac{60M}{11a^2} J_0. \]

This completes the proof. \(\square\)

To analyze \(Q\) and \(R\), let us represent them by \(J_0\) and \(J_1\). Applying Lemmas 5.5–5.7, we get

\[ Q = -2J_2^2 - 2aJ_3 - a^2 \frac{J_4'}{2} - \frac{6}{7} MJ_0' + \left( \frac{4}{7a} - \frac{6a}{7} M \right) J_4' = -J_0 - aJ_1, \quad R = J_0. \quad (47) \]

Denote \(X := \frac{Q}{R}\), then we will investigate the monotonicity of \(X\). By the above relation, we have

\[ X = \frac{Q}{R} = -1 - \frac{a}{J_0} J_1. \]

To simplify the presentation in the rest of this section, we denote \(\tilde{X} := \frac{J_1}{J_0}\) and \(Z := \frac{J_1'}{J_0'}\). Then by Lemma 5.6

\[ Z' = \frac{1}{(J_0')^2} (J_1'' J_0' - J_1' J_0'') = \frac{1}{3a\Delta} (Z^2 + 3aMZ + 3M) \]

\[ = \frac{1}{3a\Delta} \left( \left( Z + \frac{3aM}{2} \right)^2 - \frac{3}{4} a^2 M \Delta \right) > 0. \]

Lemma 5.8. If \(\tilde{X}'(M_0) = 0\) for some \(0 < M_0 < \frac{4}{3a^2}\), then \(\tilde{X}''(M_0) < 0\).

Proof. By the definitions of \(\tilde{X}\) and \(Z\), we have

\[ J_0' \tilde{X} + J_0 \tilde{X}' = J_1', \]

\[ J_0'' \tilde{X} + 2J_0' \tilde{X}' + J_0 \tilde{X}'' = J_1'', \]

\[ J_0' Z = J_1', \]

\[ J_0'' Z + J_0' Z' = J_1''. \]

Since \(\tilde{X}'(M_0) = 0\), we have

\[ \tilde{X}''(M_0) = \frac{J_1''(M_0) - J_0''(M_0) \tilde{X}(M_0)}{J_0} = \frac{J_1'(M_0) Z'(M_0)}{J_0(M_0)}. \]
According to $J_0(M) > 0$, $J'_0(M) < 0$ and $Z'(M) > 0$, we can conclude $\tilde{X}''(M_0) < 0$.

**Lemma 5.9.** If $\tilde{X}'(M_0) = 0$ for some $0 < M_0 < \frac{4}{3a^2}$, then $-\frac{12}{7a} < \tilde{X}(M_0) < -\frac{2}{a}$.

**Proof.** By the equations

$$J_0 = \frac{6}{5}MJ'_0 + \frac{4}{5a}J'_1,$$
and

$$J_1 = -\frac{12}{35a}MJ'_0 + \frac{3}{7}(2M - \frac{16}{3a^2})J'_1,$$

we can get

$$12J_0 + 7aJ_1 = 6M(2J'_0 + aJ'_1),$$
and

$$\frac{12}{7} + aJ_0 = \frac{6M}{7J_0}J'_0(2 + a\tilde{X}).$$

Since $\tilde{X}'(M_0) = 0$, we have

$$\frac{J'_0(M_0)J_0(M_0) - J'_0(M_0)J_1(M_0)}{J'_0(M_0)} = 0.$$ 

Therefore, we can get

$$Z(M_0) = \tilde{X}(M_0).$$

So, we have

$$\frac{12}{7} + a\tilde{X} = \frac{6M}{7J_0}J'_0(2 + a\tilde{X}).$$

Hence,

$$\left(\frac{12}{7} + a\tilde{X}\right)(2 + a\tilde{X}) < 0,$$

this completes the proof.

Therefore combining these facts with Lemmas 5.8 and 5.9, we can get the following lemma.

**Lemma 5.10.** For $0 < M < \frac{4}{3a^2}$, $\tilde{X}'(M) > 0$.

Now, we can complete the proof of Proposition 1. As mentioned above, the relationship between $X$ and $\tilde{X}$ is

$$X = -1 - a\tilde{X}.$$ 

Therefore,

$$X' = -a\tilde{X}'.$$

From Lemma 5.10, the proof of monotonicity of $X$ in the Proposition 1 is obvious. Moreover, from Lemma 5.3 and 5.4, we have

$$\lim_{M \to 0} \frac{J_1(M)}{J_0(M)} = -\frac{12}{7a}, \quad \lim_{M \to \frac{4}{3a^2}} J_1(M) = -\frac{2}{a}.$$ 

That is

$$\lim_{M \to 0} \tilde{X}(M) = -\frac{12}{7a}, \quad \lim_{M \to \frac{4}{3a^2}} \tilde{X}(M) = -\frac{2}{a}.$$ 

Therefore,

$$\lim_{M \to 0} X(M) = \frac{5}{7}, \quad \lim_{M \to \frac{4}{3a^2}} X(M) = 1.$$
So far, we have completed the proof of Proposition 1. By equation (46), we know that the limit speed $c_0$ satisfies

$$2c_0^2 - 2(2k + 1)c_0 - \frac{1}{X(M) + 1} = 0,$$

(48)

Therefore we get the following theorem.

**Theorem 5.11.** The limit wave speed $c_0$ is a smooth decreasing function for $M \in (0, \frac{4}{3a^2})$, moreover,

$$\lim_{M \to 0} c_0(M) = \frac{(2k + 1) + \sqrt{(2k + 1)^2 + \frac{7}{6}}}{2},$$

and

$$\lim_{M \to \frac{4}{3a^2}} c_0(M) = \frac{(2k + 1) + \sqrt{(2k + 1)^2 + 1}}{2}.$$

**Proof.** Equation (48) of the limit speed $c_0$ can be solved as

$$c_0 = \frac{(2k + 1) + \sqrt{(2k + 1)^2 + \frac{2}{X(M) + 1}}}{2}.$$

for $M \in (0, \frac{4}{3a^2})$, then we have

$$c'_0(M) = -\frac{X'(M)}{2(X(M) + 1)^2 \sqrt{(2k + 1)^2 + \frac{2}{X(M) + 1}}}. $$

According to Proposition 1, we can obtain that $c'_0(M) < 0$ for $M \in (0, \frac{4}{3a^2})$, so we have

$$\lim_{M \to 0} c_0(M) = \frac{(2k + 1) + \sqrt{(2k + 1)^2 + \frac{7}{6}}}{2},$$

and

$$\lim_{M \to \frac{4}{3a^2}} c_0(M) = \frac{(2k + 1) + \sqrt{(2k + 1)^2 + 1}}{2}.$$

**Remark 2.** In this paper, we consider a CH-KP equation with a special local delay convolution kernel for describing real wave propagation. In fact, if the kernel $f$ is replaced by the weak generic delay, the approach is still applicable. And we only discuss the monotonicity of the wave speed for the case $n = 1$, and $J_n$ can be represented by $J_1$ and $J_2$. However, for the general $n$, it is difficult to express $J_n$ exactly, and the proof for the monotonicity of the wave speed is more complicated. We will study it in the future.

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REFERENCES

[1] M. Ablowitz and P. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge Univ. Press, Cambridge, 1991.
[2] S. Ai, Traveling wave fronts for generalized Fisher equations with spatio-temporal delays, J. Differential Equations, 232 (2007), 104–133.
[3] S. Ai, S. Chow and Y. Yi, Travelling wave solutions in a tissue interaction model for skin pattern formation, J. Dynam. Differential Equations, 15 (2003), 517–534.
[4] W. Bates and J. Shi, Existence and instability of spike layer solutions to singular perturbation problems, J. Funct. Anal., 196 (2002), 211–264.
[5] A. Biswas, 1-Soliton solution of the generalized Camassa-Holm Kadomtsev-Petviashvili equation, Commun. Nonlinear Sci. Numer. Simulat., 14 (2009), 2524–2527.
[6] R. Camassa and D. Holm, An integrable shallow water equation with peaked soliton, Phys. Rev. Lett., 71 (1993), 1661–1664.
[7] A. Chen, L. Guo and X. Deng, Existence of solitary waves and periodic waves for a perturbed generalized BBM equation, J. Differential Equations, 261 (2016), 5324–5349.
[8] Z. Du, J. Li and X. Li, The existence of solitary wave solutions of delayed Camassa-Holm equation via a geometric approach, J. Funct. Anal., 275 (2018), 988–1007.
[9] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, J. Differential Equations, 31 (1979), 53–98.
[10] B. Fuchsteiner and A. Fokas, Symplectic structures, their backland transformations and hereditary symmetries, Phys. D, 4 (1981), 47–66.
[11] A. Gourley and G. Ruan, Convergence and traveling wave fronts in functional differential equations with nonlocal terms: A competition model, SIAM. J. Math. Anal., 35 (2003), 806–822.
[12] G. Hek, Geometrical singular perturbation theory in biological practice, J. Math. Biol., 60 (2010), 347–386.
[13] C. Hsu, T. Yang and C. Yang, Diversity of traveling wave solutions in FitzHugh-Nagumo type equations, J. Differential Equations, 247 (2009), 1185–1205.
[14] P. Isaza and J. Mejia, On the support of solutions to the Kadomtsev-Petviashvili (KP-II) equation, Commun. Pure Appl. Anal., 10 (2011), 1239–1255.
[15] J. Isaza, L. Mejia and N. Tzvetkov, A smoothing effect and polynomial growth of the Sobolev norms for the KP-II equation, J. Differential Equations, 220 (2006), 1–17.
[16] C. Jones, Geometrical singular perturbation theory, In Dynamical Systems, Lecture Notes in Mathematics (R. Johnson ed.), vol. 1609, Springer, New York, 1995.
[17] B. Kadomtsev and V. Petviashvili, On the stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl., 15 (1970), 539–541.
[18] D. Korteweg and G. Vries, On the change of form of long waves advancing in a rectangular channel and on a new type of long stationary waves, Philos. Mag. R Soc. London, 39 (1895), 422–443.
[19] S. Lai and Y. Xu, The compact and noncompact structures for two types of generalized Camassa-Holm-KP equations, Math. Comput. Model., 28 (2008), 1089–1098.
[20] J. Lenells, Traveling wave solutions of the Camassa-Holm equation, J. Differential Equations, 271 (2005), 393–430.
[21] C. Li and K. Lu, Slow divergence integral and its application to classical Liénard equations of degree 5, J. Differential Equations, 257 (2014), 4437–4469.
[22] C. Li and H. Zhu, Canard cycles for predator-prey systems with Holling types of functional response, J. Differential Equations, 254 (2013), 879–910.
[23] X. Liu, Orbital stability of peakons for a modified Camassa-Holm equation with higher-order nonlinearity, Discrete. Contin. Dyn. Syst., 38 (2018), 5505–5521.
[24] X. Liu, Z. Qiao and Z. Yin, On the Cauchy problem for a generalized Camassa-Holm equation with both quadratic and cubic nonlinearity, Commun. Pure Appl. Anal., 13 (2014), 1283–1304.
[25] W. Liu and E. Vleck, Turning points and traveling waves in FitzHugh-Nagumo type equations, J. Differential Equations, 225 (2006), 381–410.
[26] N. Lu and C. Zeng, Normally elliptic singular perturbations and persistence of homoclinic orbits, J. Differential Equations, 250 (2011), 4124–4176.
[27] P. Maesschalck and F. Dumortier, Canard solutions at non-generic turning points, Trans. Amer. Math. Soc., 358 (2006), 2291–2334.
[28] L. Molinet, J. Saut and N. Tzvetkov, Global well-posedness for the KP-II equation on the background of a non-localized solution, Ann. I. H. Poincare-AN, 28 (2011), 653–676.

[29] E. Novruzov and A. Hagverdiyev, On the behavior of the solution of the dissipative Camassa-Holm equation with the arbitrary dispersion coefficient, J. Differential Equations, 257 (2014), 4525–4541.

[30] C. Ou and J. Wu, Persistence of wave fronts in delayed nonlocal reaction-diffusion equations, J. Differential Equations, 238 (2007), 219–261.

[31] V. Petviashvili and V. Yan'kov, Solitons and turbulence, Rev. Plasma Phys., XIV (1989), 1–62.

[32] C. Qu, Y. Fu and Y. Liu, Well-posedness, wave breaking and peakons for a modified μ-Camassa-Holm equation, J. Funct. Anal., 266 (2014), 433–477.

[33] J. Saut and N. Tzvetkov, The cauchy problem for higher-order KP equations, J. Differential Equations, 153 (1999), 196–222.

[34] P. Smolyan, Transversal heteroclinic and homoclinic orbits in singular perturbation problems, J. Differential Equations, 92 (1991), 252–281.

[35] A. Tovbis, Breaking of symmetrical periodic solutions in a singularly perturbed KdV model, SIAM J. Math. Anal., 40 (2008), 1516–1549.

[36] A. Wazwaz, The Camassa-Holm-KP equations with compact and noncompact travelling wave solutions, Appl Math Comput, 170 (2005), 347–360.

[37] M. Wei, X. Sun and S. Tang, Single peak solitary wave solutions for the CH-KP(2,1) equation under boundary condition, J. Differential Equations, 259 (2015), 628–641.

[38] L. Yang, Z. Rong, S. Zhou and C. Mu, Uniqueness of conservative solutions to the generalized Camassa-Holm equation via characteristics, Discrete. Contin. Dyn. Syst., 38 (2018), 5205–5220.

[39] S. Yang and T. Xu, Symmetry analysis, persistence properties and unique continuation for the cross-coupled Camassa-Holm system, Discrete. Contin. Dyn. Syst., 38 (2018), 329–341.

[40] V. Zakharov and E. Kuznetsov, On three-dimensional solitons, Sov. Phys., 39 (1974), 285–288.

[41] K. Zhang, S. Tang and Z. Wang, Bifurcation of travelling wave solutions for the generalized Camassa-Holm-KP equations, Commun. Nonlinear Sci. Numer. Simulat., 15 (2010), 564–572.

[42] L. Zhang and B. Liu, Well-posedness, blow-up criteria and gevrey regularity for a rotation-two-component camassa-holm system, Discrete. Contin. Dyn. Syst., 38 (2018), 2655–2685.

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