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Homodyne monitoring of postselected decay

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We use homodyne detection to monitor the radiative decay of a superconducting qubit. According to the classical theory of conditional probabilities, the excited-state population differs from an exponential decay law if it is conditioned upon a later projective qubit measurement. Quantum trajectory theory accounts for the expectation values of general observables, and we use experimental data to show how a homodyne detection signal is conditioned upon both the initial state and the finally projected state of a decaying qubit. We observe, in particular, how anomalous weak values occur in continuous weak measurement for certain pre- and postselected states. Subject to homodyne detection, the density matrix evolves in a stochastic manner, but it is restricted to a specific surface in the Bloch sphere. We show that a similar restriction applies to the information associated with the postselection, and thus bounds the predictions of the theory.

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I. INTRODUCTION

Exponential decay is a fundamental process in classical and quantum physics \cite{1,2}. While the fraction of a large ensemble of systems surviving decay with a rate $\gamma$ until any given time $t$ is represented by an exponential law, $\exp(-\gamma t)$, if the radiative decay of a single system is monitored as a function of time, its actual state evolves in a conditional manner and differs in general from the exponential behavior \cite{3–5}. In a similar way to how the state of a quantum system evolves in time subject to information retrieval from measurements, our probabilistic description of a system at a given time in the past is also influenced by information retrieved after that time. To illustrate this, consider how the exponential decay law is modified if we observe the time evolution of a single quantum (or classical) system for which we know the state at a given final time $T$. If an initially excited two-level system, decaying with a rate $\gamma$, is observed to be still in its excited state at time $T$, a previous measurement could not possibly have found the system in its ground state, i.e., the exponential decay is replaced by a constant unit excitation probability as we have found the system in its ground state, i.e., the exponential law, $\exp(-\gamma T)$, if the system is found in state $\langle e| g, T \rangle$ at time $T$. These joint probabilities can be written in terms of the conditional probabilities, $P(e, t; g, T) = P(e, t) P(g, T|e, t) = e^{-\gamma T}(1 - e^{-\gamma(T-t)})$ and $P(g, t; g, T) = P(g, t) P(g, T|g, t) = (1 - e^{-\gamma T}) \times 1$. The excited-state probability at time $t$, conditioned on the initial excited and final ground state at time $T$, is thus given by the ratio

$$P(e, t|g, T) = \frac{e^{-\gamma T}(1 - e^{-\gamma(T-t)})}{e^{-\gamma T}(1 - e^{-\gamma(T-t)}) + (1 - e^{-\gamma T})},$$

(1)

which, as shown in Fig. 1, interpolates smoothly between unity at $t = 0$ and zero at $t = T$. Equation (1) reflects the predictions we can make about the system state, i.e., the measurement at time $T$ does not impose a physical interaction with the system at time $t$; it merely updates our (present) knowledge about it.

In this article we consider measurements by homodyne detection of the field emitted by a quantum system prepared in an initial state and eventually measured in a given final state. Measurements on quantum systems subject to pre- and postselection have been subject to theoretical and experimental analysis \cite{7–14} and can be generally described with the past quantum state (PQS) formalism \cite{15}. Here we use this formalism to analyze the outcome of homodyne detection of the signal emitted during spontaneous decay by a qubit system, conditioned on both its initial preparation and on a later projective detection, and we show how anomalous weak values in the continuous measurement signal emerge for certain pre- and postselected states. We then examine how the initial and final states, together with the continuous measurement record, combine to describe the probability distribution of different qubit observables at any given time: supplementing the stochastic evolution of the quantum system conditioned on the measurement record obtained before a given time $t$ with the information accumulated after time $t$, we observe that the PQS predictions are at any time confined to certain regions in the Bloch vector picture.

This article is organized as follows. In Sec. II, we first describe the experimental setup and present the past quantum state theory. We then compare our experimental signal arising from homodyne detection of the radiation emitted by the qubit, conditioned on its initial preparation and final projective measurement, to the predictions of PQS theory. In Sec. III, we assess the back action on the quantum emitter due to homodyne detection of the radiated field, and we determine effective Bloch vector components yielding predictions for projective qubit measurements. Section IV concludes the article.

II. PREDICTION AND RETRODICTION OF THE HOMODYNE SIGNAL FROM A DECAYING TWO-LEVEL EMITTER

Our experiment is realized in a hybrid two-level system which behaves as a quantum emitter that when initially
prepared in the excited state $|e\rangle$ radiates at its resonant frequency $\omega_e/2\pi = 6.541$ GHz. The emitter is comprised of a transmon qubit embedded in a 3D aluminum cavity [16,17] connected to a 50Ω transmission line. The interaction between the emitter and the transmission line is described by a Hamiltonian $H_{\text{int}} \propto a\sigma_- + a^\dagger\sigma_+$, where $a$ (a) is the creation (annihilation) operator for a photon in the transmission line and $\sigma_\pm$ is the pseudospin raising (lowering) operator. The strength of this interaction is given by the Purcell enhanced [1] radiative decay rate $\gamma = 1.628 \mu s^{-1}$ into the transmission line. We use a near-quantum-limited Josephson parametric amplifier to perform homodyne detection of the fluorescence from the excited-$|e\rangle$ to ground-state $|g\rangle$ transition. The homodyne measurement signal is proportional to the amplitude of a specific field quadrature, $a^\dagger e^{i\phi} + a e^{-i\phi}$, and by virtue of the interaction Hamiltonian is a measurement of the corresponding emitter dipole $\sigma_- e^{i\phi} + \sigma_+ e^{-i\phi}$. By adjusting the homodyne phase $\phi = 0$, the resulting homodyne signal conveys information about the $\sigma_\pm$ dipole observable of the qubit [3].

Using a classical drive, we may prepare the emitter in an arbitrary initial superposition state. As the qubit decays, the master equation for the density matrix yields the mean dipole and, hence, predicts the mean value of the time-dependent emitted signal, as measured by homodyne detection. If we also know the outcome of a later, projective measurement on the system, the expected outcome of the homodyne measurement changes and is given by the past quantum state theory, which generalizes our introductory classical analysis of joint probabilities to quantum systems.

In quantum mechanics, a general measurement is described by POVMs, i.e., a set of operators $M_m$, satisfying $\sum_m M_m^\dagger M_m = 1$. If the system at time $t$ is described by the density matrix $\rho_t$, the probability for outcome $m$ is $P(m) = \text{Tr}(M_m^\dagger M_m \rho_t)$, which coincides with Born’s rule in the case of projective measurements.

The POVM operators associated with a homodyne fluorescence detection signal $V$, obtained with a detector efficiency $\eta$, are given by [18,19]

$$M_V = \left(\frac{1}{2\pi \gamma dt}\right)^{\frac{i}{2}} e^{\frac{i \eta}{\gamma} \frac{\gamma dt}{2} (\sigma_+ + \sqrt{\eta} \gamma V)}.$$

and satisfy $\int M_m^\dagger M_m dV = 1$. The probability for the measurement to yield a value $V$ is

$$P(V) = \text{Tr}(M_V^\dagger M_V \rho_t) = \frac{1}{\sqrt{2\pi \gamma dt}} \exp\left(-\frac{V^2}{2\gamma dt}\right) \left(1 + \sqrt{\eta} \gamma V\right)$$

leading to the expected average value, $\bar{V} = \int V P(V) dV = \sqrt{\eta} \gamma V$, with characteristic Gaussian fluctuations.

If the outcome of later measurements on the system are available, they contribute to our knowledge about the system at time $t$ and the resulting modification of the outcome probabilities for the earlier measurement can be written [15]:

$$P_p(m, t) = \frac{\text{Tr}(M_m^\dagger M_m E_t)}{\int \text{d} V \text{Tr}(M_v^\dagger M_v E_t)},$$

where the positive, Hermitian effect matrix $E_t$ depends on the information accumulated from time $t$ to a final time $T$.

Equation (4) reduces to the classical example offered in the Introduction [Eq. (1)] when the $M_m$ are taken to be the projection operators on the excited and ground states of the emitter, while for homodyne detection, it yields the probability for the measurement signal $V$ conditioned on both prior and posterior measurements,

$$P_p(V, t) = \frac{\text{Tr}(M_v^\dagger M_v E_t)}{\int \text{d} V \text{Tr}(M_v^\dagger M_v E_t)}.$$

The density matrix of a decaying quantum system obeys the master equation $d\rho_t = \gamma dt [D|\sigma_-|\rho_t = \gamma dt \sigma_- \rho_t \sigma_+ - \frac{i}{2} \gamma dt \{\sigma_+ \sigma_- \rho_t\}$ with the time-dependent solution expressed in terms of the matrix elements of $\rho_t$,

$$\rho_{ee}^t = e^{\gamma t}, \quad \rho_{gg}^t = e^{-\gamma t}, \quad \rho_{ge}^t = 1 - e^{\gamma t}.$$

Similarly, the matrix $E$ solves the adjoint equation $dE_t = \gamma dt [D|\sigma_-|E_t = \gamma dt \sigma_- E_t \sigma_+ - \frac{i}{2} \gamma dt \{\sigma_+ \sigma_- E_t\}$, where we apply the convention $dE_t = E_t - E_t^\dagger$, because we shall solve the equation backwards in time. Equation (6) does not conserve the trace, but this is not a formal problem, since Eqs. (4) and (5) are explicitly renormalized. The (backwards) evolution of $E_t$ from time $T$ yields the solution

$$E_{ee}^t = e^{\gamma (T-t)}, \quad E_{gg}^t = e^{-\gamma (T-t)}, \quad E_{ge}^t = E_{ee}^t + (E_{ee}^t - E_{gg}^t) e^{-\gamma (T-t)},$$

where $E_T$ is the projection operator on the state of the final heralding measurement (postselection). In the absence of postselection, $E_T$ is the identity matrix, and (7) also yields the identity matrix for all earlier times. In this case $P_p(m, t)$, Eqs. (4) and (5) reduce to the usual Born rule for quantum expectation values.

From Eqs. (5), (6), and (7), we can express the retrodicted mean value $\bar{V}_p(t) = \int V P_p(V, t) dV$ of the homodyne signal...
to first order in the infinitesimal time interval \( dt \) by the matrix elements of \( \rho_1 \) and \( E_i \),

\[
\mathcal{V}_p(t) = 2\sqrt{\eta\gamma} dt \text{Re} \left[ E_i^{\ast} \rho_1^{\ast} E_i + \rho_1 E_i^{\ast} E_i^{\ast} \right],
\]

We shall compare Eq. (8) with the experimental homodyne detection signal averaged over many experiments, for different choices of the initial state \( \rho_0 \) and final projection \( E_T \).

We first examine the experimental average homodyne signal that is obtained without postselection \( \mathcal{V} = \sqrt{\eta\gamma} (\sigma_x) dt \). We prepare the emitter in a state \( |\theta\rangle = \cos \frac{\theta}{2} |g\rangle + \sin \frac{\theta}{2} |e\rangle \) by a rotation pulse \( R_\theta \) and obtain the average homodyne signal \( \tilde{V} \) right after the preparation pulse by integrating 60 ns of recorded homodyne signal as depicted in Fig. 2(c). In Fig. 2(c), we display our experimental results \( \tilde{V} \), testing the predicted average signal \( \tilde{V} \) for different \( \theta \). \( \tilde{V} \) oscillates as a function of \( \theta \) and reaches a maximum (minimum) at \( \theta = \frac{\pi}{2} \) (\( \theta = -\frac{\pi}{2} \)) as expected [3]. The experimental and theoretical curves are in good agreement and show that the average homodyne signal \( \tilde{V} \) without postselection never exceeds the maximum value \( \sqrt{\eta\gamma} \) \( \gamma dt \) (dashed horizontal lines).

To confirm the theory prediction for the mean signal with postselection, \( \mathcal{V}_p \), we conduct the experimental sequence illustrated in Fig. 2(d). We first initialize the qubit in state \( |\theta\rangle \), then record 0.5 \( \mu s \) of homodyne signal, and finally postselect the state \( |\theta - \frac{\pi}{2}\rangle \). The average, postselected signal \( \mathcal{V}_p \) is obtained by averaging 60 ns of homodyne signal right after the initial-state preparation pulse from the experimental runs which successfully preselect state \( |\theta\rangle \) and postselect state \( |\theta - \frac{\pi}{2}\rangle \). After correcting for the postselection fidelity (see Appendix D) the experimental results \( \mathcal{V}_p \) are in good agreement with the theory prediction \( \mathcal{V}_p \), calculated from Eqs. (6), (7), and (8) with \( \rho_0 = |\theta\rangle \langle \theta | \) and \( E_T = |\theta - \frac{\pi}{2}\rangle \langle \theta - \frac{\pi}{2} | \). Furthermore, we observe anomalous weak values [9] where \( |\mathcal{V}_p| \) exceeds \( \sqrt{\gamma} \gamma dt \). This is due to the low overlap between the pre- and postselected states when \( \theta = \{ -\pi, \pi, \frac{3\pi}{2}, \pi \} \) as displayed in Fig. 2(d). Note that ideally we could obtain the average \( \mathcal{V}_p \) by postselecting state \( |\theta - \frac{\pi}{2}\rangle \) immediately after the 60 ns signal integration, but transient behavior associated with the rotations and readout affects the homodyne signal. Therefore, as indicated in Fig. 2(d), we wait for 0.5 \( \mu s \) before making the postselection measurement.

### III. EVOLUTION DYNAMICS SUBJECT TO HOMODYNE DETECTION

In our experiment, the emitter state is continuously monitored with the homodyne signal which is sensitive to the \( \sigma_x \) component of the two-level system. If we, rather than averaging over many experiments as we have in the previous section, consider a single run of the experiment, the state of the system evolves in time as a quantum trajectory which can be inferred from the record of the detected homodyne signal. Homodyne detection with efficiency \( \eta \) gives rise to a signal \( V = \sqrt{\eta}\gamma \text{Tr}[\sigma_x \rho] dt + \sqrt{\eta} dW_t \), with a stochastic Wiener increment \( dW_t \), with zero mean and variance \( dt \) [20], and the density matrix of the emitter solves the stochastic master equation (SME) [18],

\[
d\rho_t = \gamma dt \text{D}[\sigma_-] \rho_t + \sqrt{\eta\gamma} \text{Tr}[\sigma_x \rho] dt \times \mathcal{H}[\sigma_-],
\]

where the term proportional to \( \mathcal{H}[\sigma_-] \rho_t = \sigma_- \rho_t + \rho_t \sigma_+ - \text{Tr}[\sigma_- \rho_t] \rho_t \) is added to the unobserved master equation to account for the stochastic measurement back action. The trajectory followed by a monitored quantum emitter is well described by the stochastic master equation (9), and quantum trajectories for the density matrix \( \rho_t \) have been studied, e.g., in [3,5,21–24].

The homodyne signal \( V \) is scaled to have a variance \( \sigma^2 = \gamma dt \), and by recording two histograms for \( V \) separated by \( \Delta V = 2\sqrt{\gamma\gamma} dt \), the quantum efficiency of our experimental setup is found to be \( \eta = 0.3 \) (see Appendix B). Note that this scaling yields a dimensionless signal \( V \), whereas under other conventions it has units of (time)\( ^2 \) [5,25].

Similar to the density matrix \( \rho_t \), being now conditioned on the initial state and the homodyne detection record until time \( t \), the matrix \( E_t \) at time \( t \) is conditioned on the homodyne signal recorded after \( t \). It solves the adjoint counterpart of the (SME)
Eq. (9) backwards from the final time $T$ [15],
\[
dE_t = \gamma dt \left[ \sigma_\pm E_t + \sqrt{\eta} (V - \sqrt{\eta} \gamma \text{Tr}[\sigma_\pm E_t]) dt \right] \\
\times \mathcal{H}(\sigma_\pm E_t).
\]
(10)

A. Bloch representation of $\rho_t$ and $E_t$

To graphically present the results, the density matrix of a two-level quantum system may be represented by a real Bloch vector $(x_\rho, y_\rho, z_\rho)$,
\[
\rho_t = \frac{1}{2} (1 + x_\rho \sigma_x + y_\rho \sigma_y + z_\rho \sigma_z),
\]
(11)
where $u_\rho = \text{Tr}(\sigma_u \rho_t)$ for $u = x, y, z$. The stochastic master equation (9) describes how the evolution of a decaying qubit monitored by homodyne detection is conditioned on the measurement signal. For perfect detection ($\eta = 1$), an initially pure state remains pure and the Bloch vector explores the surface of the unit Bloch sphere, while imperfect detection ($0 \leq \eta < 1$) leads to a mixed state inside the Bloch sphere. In our experiments, the system is prepared (and postselected) in states with vanishing $\langle \sigma_z \rangle$, and as the homodyne detection effectively probes the $\sigma_z$ operator, the (conditional) $y_\rho$ component of the Bloch vector remains zero at all times. The SME is, hence, equivalent to the following coupled stochastic equations for the $x_\rho$ and $z_\rho$ Bloch vector components:
\[
dx_\rho = -\frac{\gamma}{2} x_\rho dt + \sqrt{\eta} (1 - z_\rho - x_\rho^2) (V - \sqrt{\eta} \gamma x_\rho dt),
\]
\[
dz_\rho = \gamma (1 - z_\rho) dt - \sqrt{\eta} (1 - z_\rho)x_\rho (V - \sqrt{\eta} \gamma x_\rho dt).
\]
(12)

Similarly, we wish to introduce a Bloch sphere representation to illustrate the conditional evolution of $E_t$. While the role of $E_t$ in predicting measurement outcomes does not require unit trace due to the normalization factor in Eq. (4), the Bloch sphere representation assumes a normalized state matrix. The term $D^\dagger [\sigma_-] E_t$ in the SME Eq. (10) is not trace preserving, but since $\gamma dt \text{Tr}(D^\dagger [\sigma_-] E_t) = \gamma dt \text{Tr}(\sigma_\pm E_t)$, we may introduce the following normalized version of the SME:
\[
dE_t = \gamma dt \left[ \sigma_- E_t - \gamma dt \text{Tr}(\sigma_\pm E_t) E_t \right] \\
+ \sqrt{\eta} (V - \sqrt{\eta} \gamma \text{Tr}[\sigma_\pm E_t]) dt \mathcal{H}(\sigma_\pm E_t),
\]
(13)
and an associated Bloch vector
\[
E_t = \frac{1}{2} (1 + x_E \sigma_x + y_E \sigma_y + z_E \sigma_z),
\]
(14)
where $u_E = \text{Tr}(\sigma_u E_t)$ for $u = x, y, z$. Just like (12), we can obtain a set of stochastic Bloch equations for $E_t$,
\[
dx_E = -\frac{\gamma}{2} x_E [1 + 2 z_E + \eta (1 + z_E - x_E^2)] dt \\
+ \sqrt{\eta} (1 - z_E - x_E^2) V,
\]
\[
dz_E = -\gamma (z_E + z_E^2 - \eta (1 + z_E) x_E^2) dt \\
- \sqrt{\eta} (x_E + 1) x_E V.
\]
(15)

In Fig. 3(a), we show schematically how we prepare the emitter in the state $|\psi_i\rangle$, then digitize the detected homodyne signal $V(t)$, accumulated for a time interval of 1.68 $\mu s$, and finally measure the emitter in the $|\psi_f\rangle$ state by a high fidelity projective measurement. Using Eqs. (12) and (15), we determine the conditional Bloch vectors for $\rho_t$ and $E_t$ and in Fig. 3 we show the resulting trajectories with the colors red, green, cyan, and blue, corresponding to different time intervals $[0.42n, 0.42(n + 1)] \mu s (n = 0, 1, 2, 3)$. Here the panels (b)–(d) represent different choices for the initial and final states. The trajectories for $\rho_t$, shown in the first column in Figs. 3(b)–3(d) diffuse through the Bloch sphere, but are confined to different deterministic curves for different evolution times (blue dashed lines in Fig. 3) [3,5]. In a similar way, the trajectories for $E_t$ diffuse backwards in time through the Bloch sphere from the postselected state and they are also, for different evolution times, confined to different deterministic curves. Analytic expressions for these curves are provided in the following subsection.

The Bloch vector components of $\rho_t$ are the expectation values of the Pauli operators, but can also be written as the weighted mean value of their eigenvalues, e.g., $(\sigma_z) =$
\[ \rho_{SS}^{\text{eq}} + \rho_{T}^{\text{eq}} \times (-1) = 2 \rho_{SS}^{\text{eq}} - 1. \]

Since the past quantum state answers the question “what is the probability that a measurement of an observable gave a certain outcome a time \( t \) later?” we can use Eq. (4) to obtain such probabilities for projection operators on the eigenstates of \( \sigma_x \), \( \sigma_y \), and \( \sigma_z \), and subsequently display the weighted eigenvalues as Bloch vector components, e.g.,

\[ \langle \sigma_x \rangle_p(t) = P_{\rho}(\sigma_x = +1, t) - P_{\rho}(\sigma_x = -1, t). \] (16)

Similar equations apply for \( \sigma_y \) and \( \sigma_z \), measurements and, using the Bloch vector representation of \( \rho_t \) and \( E_t \), the retrodict expectation values of the three spin components acquire the elegant form

\[
\begin{align*}
\langle \sigma_x \rangle_p &= \frac{x_p + x_E}{1 + x_p x_E}, \\
\langle \sigma_y \rangle_p &= \frac{y_p + y_E}{1 + y_p y_E}, \\
\langle \sigma_z \rangle_p &= \frac{z_p + z_E}{1 + z_p z_E}.
\end{align*}
\] (17)

These expressions are used together with the solutions of Eqs. (12) and (15) to plot the trajectories of the retrodict expectation values in the third columns in Figs. 3(b)–3(d). These trajectories diffuse through the state space, and notably assume values that are outside of the Bloch sphere. This is as expected, since, e.g., the prediction for the outcome of a measurement of the ground-state population at late times is unity with almost certainty, while postselection upon a final measurement of \( \sigma_z \) certifies an immediately foregoing measurement of \( \sigma_z \) would have to yield the same result. Note that this is not at variance with Heisenberg’s uncertainty relation which concerns only predictions of future measurements and does not apply for the combined prediction and retrodiction of observations. We emphasize that, while the mean values and probabilities for the outcome of measurements along any rotated spin direction simply follow from the projection of the Bloch vector along those directions, due to the nonlinear expressions in Eq. (17) the same reasoning does not apply for the vector plotted in the third columns in Figs. 3(b)–3(d). Prediction of the spin measurement along a 45° direction between the \( x \) and \( z \) axes would require a separate calculation, using the \( \rho_t \) and \( E_t \) Bloch vector components along that direction.

**B. Deterministic properties of \( \rho_t \) and \( E_t \)**

In this section, we examine the character of the stochastic trajectories in more detail. In Ref. [5] it is derived how the stochastic evolution of a decaying qubit subject to homodyne detection is at all times confined to the surface of a deterministic spheroid inside the Bloch sphere. In the case of homodyne detection, only one component of the pseudospin is probed, and the three-dimensional spheroid is replaced by a two-dimensional ellipse. In previous work, the deterministic properties of this ellipse have been used for state preparation [3] and analyzing the emergence of multiple most-likely paths in the quantum trajectories [26]. Here we wish to extend these results to the Bloch representation Eq. (14) of the matrix \( E_t \).

For completeness, we first rederive the expressions for the ellipse pertaining to the density matrix. The quest is to identify a function \( \alpha(x_p, z_p) \) of the stochastically evolving Bloch components, for which the equation of motion is deterministic. We shall see that such a function exists and that it indeed describes an ellipse in \((x_p, z_p)\). For a generic function, the equation of motion is derived from the stochastic Bloch equations (12),

\[
d\alpha = \frac{\partial}{\partial x_p} dx_p + \frac{\partial}{\partial z_p} dz_p
\]

\[
+ \frac{1}{2} \left[ \frac{\partial^2}{\partial x_p^2} (d x_p)^2 + \frac{\partial^2}{\partial z_p^2} (d z_p)^2 + 2 \frac{\partial}{\partial x_p} \frac{\partial}{\partial z_p} d x_p d z_p \right].
\] (18)

where the second-order terms yield contributions from the noise terms in Eq. (12) of the same order in \( dt \) as the first-order deterministic terms. The evolution of \( \alpha(x_p, z_p) \) is deterministic if all terms proportional to \( d W_t \) in \( d\alpha \) cancel. After applying Eq. (12) in Eq. (18) this requirement dictates the following form of \( \alpha(x_p, z_p) \):

\[
\alpha(x_p, z_p) = \frac{2}{1 - z_p} - \frac{x_p^2}{(1 - z_p)^2}.
\] (19)

This can be rewritten

\[
\alpha^2 (1 - z_p - 1/\alpha)^2 + \alpha x_p^2 = 1,
\] (20)

which shows that the Bloch components of \( \rho_t \) are at all points in time restricted to an ellipse centered at \((x, z) = (0, 1 - 1/\alpha)\) and with major axis \( 1/\sqrt{\alpha} \) (\( x_p \) direction) and minor axis \( 1/\alpha \) (\( z_p \) direction). Furthermore, applying Eq. (19) on the right-hand side of Eq. (18) yields an ordinary differential equation for the time evolution of the parametrizing function \( \alpha(x_p, z_p) \),

\[
\frac{d\alpha}{dt} = \gamma (\alpha - \eta),
\] (21)

with the solution

\[
\alpha(t) = \eta + [\alpha(t = 0) - \eta] e^{\gamma t},
\] (22)

where \( \alpha(t = 0) \) follows from Eq. (19) with the initial Bloch components at time \( t = 0 \). For any pure initial state \( \alpha(t = 0) = 1 \), and the ellipse Eq. (20) is the full Bloch sphere. The time evolution of \( \alpha \) for an initial pure state is shown in Fig. 4(a) for different values of the detection efficiency \( \eta \). \( \alpha(t) \) increases, and hence the center \( z \) coordinate of ellipse increases with a rate \( \gamma \) in accordance with the decay of the qubit. As a signature of the loss of information associated with nonperfect monitoring, the axes of the ellipse reduce faster for smaller values of the detector efficiency \( \eta \) and the qubit explores a range of mixed states. At large times, both axes of the ellipse diminish and the (pseudo)spin is certain to be found in the ground state.

To derive a similar result for \( E_t \), we define a generic function \( \beta(x_E, z_E) \) of the Bloch components in Eq. (14), and we seek a form of this function evolving in a deterministic manner. The equation of motion for \( \beta(x_E, z_E) \) follows from the stochastic Bloch equations (15) in a way equivalent to that for \( \alpha(x_p, z_p) \) in Eq. (18), and the requirement that all terms proportional to \( V \) cancel yields

\[
\beta(x_E, z_E) = -\frac{2}{z_E + 1} + \frac{x_E^2}{(z_E + 1)^2}.
\] (23)
Rewriting and noting that $\beta < -1$ reveals that $\beta$ parametrizes an ellipse in $(x_E, z_E)$.

$$1 = \beta^2(z_E + 1/\beta)^2 - \beta(x_E^2 + y_E^2), \quad \text{(24)}$$

centered at $(x, z) = [0, -(1 + 1/\beta)]$ and with major axis $1/\sqrt{-\beta}$ (x_E direction) and minor axis $1/\beta$ (z_E direction).

In addition, one finds that the time evolution of $\beta(x_E, z_E)$ fulfills the differential equation,

$$\frac{d\beta}{dt} = \gamma(-\beta + \eta - 2), \quad \text{(25)}$$

which must be solved backwards in time from the final value at the time of postselection $\beta(t = T)$. This gives the following evolution for $\beta(x_E, z_E)$:

$$\beta(t) = \eta - 2 + [\beta(T) - \eta + 2]e^{\gamma(t - T)}. \quad \text{(26)}$$

Equation (23) provides the value of $\beta(t = T)$ from the Bloch components of the postselected state. For any pure state $\beta(t = T) = -1$ and the ellipse is the full Bloch sphere. Without postselection $E_T = 1/2$ so $\beta(t = T) = -2$ and the final ellipse is smaller and includes the origin $(x, z) = (0, 0)$. The time evolution of $\beta$ for a final postselection in a pure state at time $T = 4\gamma^{-1}$ is shown in Fig. 4(b) for different values of the detection efficiency $\eta$. Similar to the case of the density matrix, lower efficiency causes a faster (backwards) decay of the ellipse towards that corresponding to a fully mixed effect matrix.

The retrodicted expectation values of the spin components at any point in time during an experiment follows in Eq. (17) from the Bloch components of the density and effect matrices at that point in time. Combining the restriction of $\rho_i$ and $E_i$ to deterministic curves in the Bloch sphere plot, the retrodiction for $\sigma_x$ and $\sigma_z$ measurements becomes confined to time-dependent $(x, z)$ domains. For different realizations of the homodyne signals, the retrodicted outcomes of measurements of the two spin components explore this area, as illustrated in the third columns in Figs. 3(b)–3(d).

**APPENDIX B: CALIBRATION OF HOMODYNE SIGNAL**

We conduct a simple experiment illustrated in Fig. 5(a) to calibrate our measurement homodyne signal by applying a $R_x^\pi(R_z^\pi)^*$ pulse to prepare the emitter in $|+x\rangle$ or $|-x\rangle$. We collect 20 ns of homodyne signal immediately after the state
sequences to prepare the emitter in the $|+\rangle$ and $|-\rangle$ state. When postselecting on rare events with for example states, we simply apply a qubit rotation before the readout.

The postselection fidelity is the ratio of correct postselections to the total number of detection events. The number of incorrect postselections is the product of the error rate and the number of trials.

APPENDIX C: HIGH FIDELITY POSTSELECTION MEASUREMENTS

Postselection experiments often look for rare events, and in experiments with modest measurement fidelity, postselection errors can easily contaminate the measurement results. Here, we focus on maximizing the postselection fidelity at the expense of the postselection efficiency. In our experiment, we realize high fidelity postselections by adjusting the readout power to the extent that minimizes the error occurrence while maintaining a modest success rate. In the language of photodetection we want to minimize the dark counts (the postselection errors) even at the expense of low photodetection efficiency. We define the postselection fidelity as the fraction of correct postselections. We test the postselection error rate by preparing the qubit in the ground state and then performing a readout measurement. On average, by choosing an appropriate threshold (arrows in Fig. 6), we found two error occurrences out of 5000 runs of the experiment as shown in the blue region in Fig. 6(a). If we prepare the qubit in the excited state, we have 314 occurrences from the same number of runs with the same threshold. At this point, we know the postselection error rate is below 1%. To apply this postselection technique to other states, we simply apply a qubit rotation before the readout.

While it is possible to reduce the postselection error rate below 1%, when postselecting on rare events with for example an expected occurrence of one in $10^6$, these postselection errors will dominate the experimental results. This limits the types of postselections that can be reliably made, and we focus on postselections where successes rate (the ratio of number of successful runs to total experiment runs) greatly outweighs the error rate. Figure 6(d) characterizes the postselection fidelity for different pre- and postselected states. To test the postselection fidelity, we conduct the experimental sequence as illustrated in Fig. 6(d); we first apply a rotation to prepare the qubit in the state $|\theta\rangle$ in the $x-z$ plane of the Bloch sphere. After 0.5 $\mu$s we then postselect the $|\theta - \frac{\pi}{2}\rangle$ state by applying a corresponding rotation $R_x^{\theta}$ and a projective measurement $\Pi_{\pm z}$. The postselection fidelity is the ratio of correct postselections to the total number of detection events. The number of incorrect postselections is the product of the error rate and the number of trials.

APPENDIX D: CORRECTION OF THE PREDICTED MEAN VALUE $V_p$ DUE TO POSTSELECTION FIDELITY

In the experiment, we prepare the emitter in the state $|\theta\rangle = \cos(\theta/2)\langle g| + \sin(\theta/2)e|e\rangle$ at $t = 0$ and postselect state $|\theta - \frac{\pi}{2}\rangle = \cos(\frac{\theta}{2} - \frac{\pi}{4})\langle g| + \sin(\frac{\theta}{2} - \frac{\pi}{4})\langle e|$ at $t = T$. Ideally, we have the density matrix $\rho_{\text{post}}(\theta) = |\theta\rangle\langle\theta|$ and the effect matrix $E_{\text{post}}(\theta) = |\theta - \frac{\pi}{2}\rangle\langle\theta - \frac{\pi}{2}|$. In the experiment, however, the postselection fidelity $\eta_p$ is subunity as shown in Fig. 6(d). To account for this in the analysis, the effect matrix $E_T$ at time $T$ for calculating $V_p$ is corrected in the following way:

$$E_T^{\text{eff}}(\theta) = (1 - \eta_p)\cos^2\left(\frac{\theta - \frac{\pi}{2} - \pi}{2}\right) + \eta_p\cos^2\left(\frac{\theta - \frac{\pi}{2}}{2}\right),$$

$$E_T^{\text{eff}}(\theta) = \frac{1}{2}(1 - \eta_p)\sin\left(\frac{\theta - \frac{\pi}{2} - \pi}{2}\right) + \frac{1}{2}\eta_p\sin\left(\frac{\theta - \frac{\pi}{2}}{2}\right),$$

$$E_T^{\text{eff}}(\theta) = (1 - \eta_p)\sin^2\left(\frac{\theta - \frac{\pi}{2} - \pi}{2}\right) + \eta_p\sin^2\left(\frac{\theta - \frac{\pi}{2}}{2}\right).$$
After taking the postselection fidelity $\eta_p$ into consideration, the experimental and theoretical curves agree well as displayed in Fig. 2(d) of the main text.

**APPENDIX E: DISTRIBUTION OF BLOCH VECTOR COMPONENTS**

In this appendix, we revisit the deterministic ellipses and allowed areas of the Bloch components for the trajectories introduced in Sec. III B. While the deterministic ellipses pose outer boundaries for the Bloch components and retrodicted expectation values, they do not hold information on the actual distribution of trajectories realized over many experimental runs.

In Fig. 7 we show results of 10,000 Monte Carlo simulations of the SMEs (9) and (13) which allow sampling of the time-dependent distribution of trajectories of the density and effect matrices of a monitored, decaying (pseudo)spin as well as of the effective retrodicted Bloch vector components. As the $\rho$ and $E$ Bloch vector trajectories are confined to quite localized segments along the ellipses and they may be correlated with each other, the distribution of retrodicted Bloch vectors is restricted to more narrow regions than allowed by full deterministic ellipses.

The dashed, black lines in Fig. 7 track the unconditional or ensemble-averaged state, and it is seen that the deterministic ellipses of the conditional Bloch components of $\rho$ and $E$ does not include this state. This leads to a discrepancy between the most likely state represented by the bright, yellow areas in the color plots and the average state. This feature of the density matrix of a monitored quantum system is well
known; see, e.g., Ref. [23]. Similar results apply for the conditional trajectories of the effect matrix and, as seen from Figs. 7(b) and 7(c), when the qubit is postselected in $|\pm x\rangle$, the most likely retrodicted set of expectation values differ from the unconditional retrodiction.

The experiments similarly allow an analysis of the distribution of trajectories. In Fig. 8, we display separate histograms of the $x$ and $z$ Bloch components of $\rho$, $E$, and of the ($\rho$, $E$) retrodicted expectation values, corresponding to the different pre- and postselections that were studied in Fig. 3. These distributions agree with the theoretical simulations, and they confirm that the Bloch vector coordinates are restricted to finite intervals, and sometimes very well localized within even tighter regions.

Both the simulations and the experiments provide distributions for the $(\rho, E)$ trajectories that extend beyond the Bloch sphere and, e.g., approach the $x, z = \pm 1$ corner of the “Bloch square.” We recall that the two coordinates of the retrodicted Bloch vector provide the probabilities of separate measurements of the $x$ and the $z$ pseudospin components of the qubit. Close to $x, z = \pm 1$ we are thus able to make a confident, joint prediction for the outcome of a measurement of any of the two spin components. While this is normally forbidden by Heisenberg’s uncertainty relation, we recall that we are not predicting the outcome of a future measurement, but rather retrodicting the outcome of a past one. If the state prior to such a past measurement is close to a $\sigma_z$ eigenstate (e.g., the ground-state long time after preparation of the initial excited state), one could not have obtained the excited state in a $\sigma_z$ measurement. At the same time, if a subsequent final measurement yields $\sigma_z = 1$, one could not possibly have measured $\sigma_z = -1$ just prior to that. Hence the majority of Bloch components based on the $(\rho, E)$ retrodiction may fall outside of the Bloch sphere as seen in the third column of Figs. 7(b) and 7(c), and indicated by the curves in the third column of Figs. 8(c) and 8(d).

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