CHARACTERISING THE PATH-INDEPENDENT PROPERTY OF THE GIRSANOV DENSITY FOR DEGENERATED STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we derive a characterisation theorem for the path-independent property of the density of the Girsanov transformation for degenerated stochastic differential equations (SDEs), extending the characterisation theorem of [13] for the non-degenerated SDEs. We further extend our consideration to non-Lipschitz SDEs with jumps and with degenerated diffusion coefficients, which generalises the corresponding characterisation theorem established in [10].

1. INTRODUCTION

Let \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\) be a filtered probability space. Let \(d, m \in \mathbb{N}\) be fixed. We are concerned with the following SDE

\[
\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad t \geq 0,
\]

where

\(b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \otimes m}\)

\((W_t)_{t \geq 0}\) is an \(m\)-dimensional \(\{\mathcal{F}_t\}_{t \geq 0}\)-Brownian motion. Under standard usual conditions, e.g. the two coefficients \(b\) and \(\sigma\) satisfy linear growth and local Lipschitz conditions (for the second variable), there is a unique solution to the above SDE (1) for a given initial data \(X_0\), see, e.g., [3].

The celebrated Girsanov theorem provides a very powerful tool to solve SDEs under the name of the Girsanov transformation or the transformation of the drift. We use \(|\cdot|\) and \(\langle \cdot, \cdot \rangle\) to denote the Euclidean norm and scalar product of vectors in \(\mathbb{R}^m\) or \(\mathbb{R}^d\), respectively. Let \(\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^m\) be a measurable function such that the following exponential integrability along the paths of the solution \((X_t)_{t \geq 0}\) holds (also known as Novikov condition)

\[
\mathbb{E} \left( \exp \left\{ -\int_0^t |\gamma(s, X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s, X_s), \mathrm{d}W_s \rangle \right\} \right) < \infty, \quad t \geq 0.
\]
Then, Girsanov theorem ([2] [3] [4]) says that for any arbitrarily fixed $T > 0$

$$\tilde{W}_t := W_t - \int_0^t \gamma(s, X_s) ds, \quad t \in [0, T]$$

(3)

is an $m$-dimensional \{\mathcal{F}_t\}_{t \in [0, T]}$-Brownian motion under the probability measure

$$Q_T := \exp \left\{ - \int_0^T |\gamma(s, X_s)|^2 ds + \int_0^T \langle \gamma(s, X_s), dW_s \rangle \right\} \cdot \mathbb{P}.\quad (4)$$

Moreover, the solution $(X_t)_{t \in [0, T]}$ fulfills the following SDE

$$dX_t = [b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t)]dt + \sigma(t, X_t)d\tilde{W}_t, \quad t \in [0, T].$$

(5)

Now let us assume that (along the paths of the solution $(X_t)_{t \geq 0}$)

$$b(t, X_t) - \sigma(t, X_t)\gamma(t, X_t) = 0, \quad a.s. \forall \ t \geq 0. \quad (6)$$

Equivalently, $b \in \text{Im}(\sigma)$, where $\text{Im}(\sigma)$ is the image space of $\sigma$. Then

$$dX_t = \sigma(t, X_t)d\tilde{W}_t.$$ 

(7)

We are interested in the path-independent property for the exponent of the Girsanov density of $Q_T$ for any fixed $T > 0$. That is, whether there exists a scalar function

$$v: [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$$

such that

$$Z_t := \frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds + \int_0^t \langle \gamma(s, X_s), dW_s \rangle = v(t, X_t) - v(0, X_0), \quad t \geq 0. \quad (8)$$

This problem arises from a number of studies in economics, finance as well as from stochastic mechanics, just mention a few, see [13] [15] (and references therein).

If $\sigma: [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ (i.e., taking $m = d$) is non-degenerate, that is, the $d \times d$-matrix $\sigma(t, x)$ is invertible for any $(t, x) \in [0, \infty) \times \mathbb{R}^d$, a characterisation of the path-independent property has obtained in [13].

Throughout the article, we assume that there is a unique solution to the above SDE (1) for a given initial data $X_0$. In particular, we allow $\sigma$ be degenerate.

Let $\Lambda_t := \{X_t(\omega) \in \mathbb{R}^d : \omega \in \Omega\} \subset \mathbb{R}^d$, the support of the solution. In particular, for each $t > 0$, we have $\Lambda_t = \Lambda := \bigcap \Lambda_t$ if $b(t, x)$ and $\sigma(t, x)$ satisfy the Hörmander’s conditions for any $t \geq 0$. Then by using Itô’s formula to $v(t, X_t)$ viewing as the composition of $v : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ with the semimartingale $(X_t)_{t \geq 0}$, the utilising the uniqueness of Doob-Meyer decomposition for continuous semimartingales, we can derive for any $t \geq 0$ the following

$$\gamma(t, X_t) = \sigma^*(t, X_t)\nabla v(t, X_t)$$

(9)

and

$$\frac{1}{2} |\gamma(t, X_t)|^2 = \frac{\partial v}{\partial t}(t, X_t) + \langle \nabla v(t, X_t), b(t, X_t) \rangle + \frac{1}{2} Tr[(\sigma \sigma^*(t, X_t) \nabla^2 v(t, X_t))]$$

(10)

where $\sigma^*(t, x)$ stands for the transposed matrix of $\sigma(t, x)$, $\nabla$ and $\nabla^2$ stand for the gradient and Hessian operators with respect to the second variable, respectively. Moreover, we get

$$\left\{ \begin{array}{ll}
\gamma(t, x) = \sigma^*(t, x)\nabla v(t, x) & (I) \\
\frac{1}{2} |\gamma(t, x)|^2 = \frac{\partial v}{\partial x}(t, x) + \langle \nabla v(t, x), b(t, x) \rangle + \frac{1}{2} Tr[(\sigma \sigma^*(t, x) \nabla^2 v(t, x))] & (II) 
\end{array} \right.$$
for any \((t, x) \in [0, \infty) \times \Lambda\). Putting (I) into (II) and (6) yield the following nonlinear parabolic PDE of the (reversible) HJB type

\[
\begin{align*}
\frac{\partial v}{\partial t}(t, x) &= -\frac{1}{2} \left\{ \text{Tr}\left[ (\sigma^*(t, x) \nabla^2 v(t, x)) \right] + |\sigma(t, x) \nabla v(t, x)|^2 \right\}, \\
\sigma(t, x) \sigma^*(t, x) \nabla v(t, x) &= b(t, x), \quad (t, x) \in [0, \infty) \times \Lambda,
\end{align*}
\]

where \(\Lambda := \cap \Lambda_t\).

**Remark 1.1.** All above derivations are reciprocal, namely, that gives a characterisation of path-independence property.

**Theorem 1.2.** Assume that \(\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d\) is a function satisfying (6). Then there exists a unique scalar function \(v \in C^{1,2}((0, \infty) \times \mathbb{R}^d \to \mathbb{R})\) such that

\[
\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds + \int_0^t \left\langle \gamma(s, X_s), dW_s \right\rangle = v(t, X_t) - v(0, X_0)
\]

if and only if (12) holds.

**Proof.** By the previous argument, we only show that the sufficiency. Since

\(v \in C^{1,2}((0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d)\),

we know that \(v(t, X_t)\) is a continuous semimartingale of \(X_t\). Thus we ahve

\[
dv(t, X_t) = \left\{ \frac{\partial v}{\partial t}(t, X_t) + \left\langle \nabla v(t, X_t), b(t, X_t) \right\rangle + \frac{1}{2} \text{Tr}\left[ (\sigma^*(t, X_t) \nabla^2 v(t, X_t)) \right] \right\} dt
\]

\[
+ \left\langle \sigma^*(t, X_t) \nabla v(t, X_t), dW_t \right\rangle.
\]

Combining this with (12), we get

\[
dv(t, X_t)
\]

\[
= \left\{ -\frac{1}{2} |\sigma^*(t, X_t) \nabla v(t, X_t)|^2 + \left\langle \nabla v(t, X_t), b(t, X_t) \right\rangle \right\} + \left\langle \sigma^*(t, X_t) \nabla v(t, X_t), dW_t \right\rangle
\]

\[
= \frac{1}{2} |\gamma(t, X_t)|^2 dt + \left\langle \gamma(t, X_t), dW_t \right\rangle.
\]

This implies (13). The uniqueness of \(v\) is easily obtained by (eq1.13). \(\Box\)

In the following, we give some degenerated examples (see [14] for details) with satisfying Hörmander’s conditions.

**Example 1.3.** [Grushin operator] Let \(b(t, z) = (-xt, -x^k yt)^T\), \(z = (x, y) \in \mathbb{R}^2, t \geq 0\) and \(\sigma(t, z)\) be given by

\[
\sigma(t, z) = \begin{pmatrix} 1 & 0 \\ 0 & x^k \end{pmatrix}, \quad k \in \mathbb{N}, z = (x, y) \in \mathbb{R}^2, t \geq 0.
\]

Then \(b \in \text{Im}(\sigma)\) and the Hörmander’s condition holds for \(\mathcal{H} = \{ \frac{\partial}{\partial x}, x^k \frac{\partial}{\partial y} \}\) with commutators up to order \(k\). Define the subelliptic diffusion operator

\[
L = X^2 + Y^2 + b(t, \cdot).
\]

Let \(\gamma(t, z) = (-xt, -yt)^*\) and \(X_s\) be the associated \(L\)-diffusion process, then \(b(t, z) = \sigma(t, z) \gamma(t, z)\). Assume that \(v \in C^{1,2}((0, \infty) \times \mathbb{R}^2 \to \mathbb{R})\) fulfills the following

\[
\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds + \int_0^t \left\langle \gamma(s, X_s), dW_s \right\rangle = v(t, X_t) - v(0, X_0).
\]
Then, by Theorem 1.2, we know that \( v \) satisfies the equation (\ref{eq:1}).

**Example 1.4. [Kohn operator]** Consider the three-dimensional Heisenberg group realized as \( \mathbb{R}^3 \) equipped with the group multiplication

\[
(x, y, z)(x', y', z') := (x + x', y + y', z + z' + (xy' - x'y)/2),
\]

which is a Lie group with left-invariant orthonormal frame \( \{X, Y, Z\} \), where

\[
X = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad Z = [X, Y] = \frac{\partial}{\partial z}
\]

Then the Kohn-Laplacian is \( \Delta_H := X^2 + Y^2 \). Let

\[
b(t, u) = \left( xt, yt, \frac{z(x - y)}{2} t \right), \quad u = (x, y, z) \in \mathbb{R}^3, \ t \geq 0
\]

and \( \sigma(t, z) \) be given by

\[
\sigma(t, u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{u}{2} & \frac{z}{2} & 0 \end{pmatrix}, \quad u = (x, y, z) \in \mathbb{R}^3, \ t \geq 0. \tag{18}
\]

Define the subelliptic diffusion operator

\[
L = X^2 + Y^2 + b(t, \cdot).
\]

Let \( \gamma(t, z) = (xt, yt, zt)^* \) and \( X_s \) be the associated \( L \)-diffusion process, then \( b \in \text{Im}(\sigma) \) and \( b(t, z) = \sigma(t, z)\gamma(t, z) \). Then, the Hörmander’s condition holds for \( \mathcal{H} = \{X, Y\} \). Assume that \( v \in C^{1,2}((0, \infty) \times \mathbb{R}^2 \to \mathbb{R}) \) fulfills the following

\[
\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds + \int_0^t \langle \gamma(s, X_s), dW_s \rangle = v(t, X_t) - v(0, X_0) \tag{19}
\]

Then, by Theorem 1.2, we know that \( v \) satisfies the equation (\ref{eq:1}).

Next, we present a degenerated example without Hörmander’s conditions.

**Example 1.5.** Let \( f(x) = e^x, x \in \mathbb{R} \). Denote by \( X_t \) the degenerated diffusion process on \( \mathbb{R}^2 \) given by

\[
X_t = (B_t^1, e^{B_t^2})^T,
\]

where \( B_t^i, \ i = 1, 2 \) are independent of one dimensional Brownian motion. For every \( z = (x, y) \in \mathbb{R}^2, \ t \geq 0, \) define

\[
b(t, z) = \left( 0, \frac{1}{2} y \vee 0 \right)^T
\]

\[
\sigma(t, z) = \begin{pmatrix} 1 & 0 \\ 0 & y \vee 0 \end{pmatrix}.
\]

Then \( b \in \text{Im}(\sigma) \) and \( \sigma(t, z), b(t, z) \) does not satisfy Hörmander’s condition. And \( X_s \) be the associated \( L \)-diffusion process for the generator \( L := \sigma\sigma^* \). Let \( \gamma(t, z) = (0, \frac{1}{2} g(\tilde{f}^{-1}(y)))^T \), then \( b(t, z) = \sigma(t, z)\gamma(t, z) \). Assume that \( v \in C^{1,2}((0, \infty) \times \mathbb{R}^2 \to \mathbb{R}) \) fulfills the following

\[
\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds + \int_0^t \langle \gamma(s, X_s), dW_s \rangle = v(t, X_t) - v(0, X_0) \tag{20}
\]

Then, by Theorem 1.2, we know that \( v \) satisfies the equation (\ref{eq:1}).
Theorem 1.6. Assume that \( \gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) is a function satisfying (6). Then there exist a function \( f \in C^2(\mathbb{R} \to \mathbb{R}) \) and a scalar function \( v \in C^{1,2}((0, \infty) \times \mathbb{R}^d \to \mathbb{R}) \) such that

\[
\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds + \int_0^t \langle \gamma(s, X_s), dW_s \rangle = f(t, X_t) - f(0, X_0)
\]  

(21)

if and only if

\[
\begin{align*}
&f'(v)(t, x) \frac{\partial v}{\partial t}(t, x) = -\frac{1}{2} \left\{ f'(v)(t, x) Tr[(\sigma \sigma^*(t, x) \nabla^2 v(t, x))] \\
&+ f''(v)(t, x) |\sigma^*(t, x) \nabla v(t, x)|^2 + |f'(v)(t, x)|^2 |\sigma^*(t, x) \nabla v(t, x)|^2 \right\}, \\
&b(t, x) = f'(v)(t, x) \sigma(t, x) \sigma^*(t, x) \nabla v(t, x), \quad (t, x) \in [0, \infty) \times \Lambda.
\end{align*}
\]

(22)

Proof. According to Theorem 1.2 we know that (21) is equivalent to

\[
\begin{align*}
&\frac{\partial v}{\partial t}(t, x) = -\frac{1}{2} \left\{ Tr[(\sigma \sigma^*(t, x) \nabla^2 f(v)(t, x))] + |\sigma^*(t, x) \nabla f(v)(t, x)|^2 \right\}, \\
&\sigma(t, x) \sigma^*(t, x) \nabla f(v)(t, x) = b(t, x), \quad (t, x) \in [0, \infty) \times \Lambda.
\end{align*}
\]

Since

\[
Tr[(\sigma \sigma^*(t, x) \nabla^2 f(v)(t, x))] = Tr[(\sigma \sigma^*(t, x) \nabla (f'(v)(t, x) \nabla v(t, x)))] = f'(v)(t, x) Tr[(\sigma^*(t, x) \nabla^2 v(t, x))] + f''(v)(t, x) |\sigma^*(t, x) \nabla v(t, x)|^2
\]

and

\[
\sigma^*(t, x) \nabla f(v)(t, x) = f'(v) \sigma^*(t, x) \nabla v(t, x).
\]

Combining all the above equalities, we conclude that (21) is equivalent to (22). \( \square \)

Remark 1.7. Under the conditions of Theorem 1.6, we have the following examples of the function \( f \)

(a) If \( f(x) = x \), then

\[
\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds + \int_0^t \langle \gamma(s, X_s), dW_s \rangle = v(t, X_t) - v(0, X_0)
\]

(23)

if and only if

\[
\sigma(t, x) \sigma^*(t, x) \nabla v(t, x) = b(t, x), \quad (t, x) \in [0, \infty) \times \Lambda
\]

(24)

and \( v \) satisfies the following time-reversed KPZ type equation,

\[
\frac{\partial v}{\partial t}(t, x) = -\frac{1}{2} \left\{ Tr[(\sigma \sigma^*(t, x) \nabla^2 v(t, x))] + |\sigma^*(t, x) \nabla v(t, x)|^2 \right\}, \quad (t, x) \in [0, \infty) \times \Lambda.
\]

(25)

In particular, if \( \sigma \) is invertible, this covers the result obtained in [13].

(b) Assume that \( v \in C^{1,2}((0, \infty) \times \mathbb{R}^d \to \mathbb{R}) \) and \( f(x) = \log x \) for \( x > 0 \), then

\[
\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds + \int_0^t \langle \gamma(s, X_s), dW_s \rangle = \log \frac{v(t, X_t)}{v(0, X_0)}
\]

(26)

if and only if

\[
\sigma(t, x) \sigma^*(t, x) \nabla v(t, x) = v(t)x b(t, x), \quad (t, x) \in [0, \infty) \times \Lambda
\]

(27)

and \( v \) satisfies the following time-reversed heat kernel type equation,

\[
\frac{\partial v}{\partial t}(t, x) = -\frac{1}{2} Tr[(\sigma \sigma^*(t, x) \nabla^2 v(t, x))], \quad (t, x) \in [0, \infty) \times \Lambda.
\]

(28)
In particular, if $\sigma = \text{Id}$, then we have
\[
\frac{1}{2} \int_0^t |b(s, X_s)|^2 \, ds + \int_0^t \langle b(s, X_s), dW_s \rangle = \log \frac{v(t, X_t)}{v(0, X_0)}
\] (29)
if and only if
\[
\nabla v(t, x) = v(t, x)b(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d,
\] (30)
and $v$ satisfies the standard heat kernel equation,
\[
\frac{\partial v}{\partial t}(t, x) = -\frac{1}{2} \Delta v(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d.
\] (31)

(c) If $f(x) = x^{2k+1}$, $k \in \mathbb{N} \cup \{0\}$, or $f(x) = x^{2k+1}$, $k \in \mathbb{Z}$ for $x \neq 0$, then
\[
\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 \, ds + \int_0^t \langle \gamma(s, X_s), dW_s \rangle = v^{2k+1}(t, X_t) - v^{2k+1}(0, X_0)
\] (32)
if and only if
\[
(2k + 1)v^{2k}(t, x)\sigma(t, x)\sigma^*(t, x)\nabla v(t, x) = b(t, x), \quad (t, x) \in [0, \infty) \times \Lambda,
\] (33)
and $v$ satisfies the following time-reversed HJB equation,
\[
\frac{\partial v}{\partial t}(t, x) = -\frac{1}{2} \left\{ \text{Tr}[\sigma \sigma^*(t, x) \nabla^2 v(t, x)] \right\}
\] (34)
\[
+ \frac{(2k + 1)v^{2k+1}(t, x) + 2k}{v(t, x)} |\sigma^*(t, x)\nabla v(t, x)|^2 \right\}.
\]

(d) If $f(x) = \tan(x), x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then
\[
\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 \, ds + \int_0^t \langle \gamma(s, X_s), dW_s \rangle = \tan(v(t, X_t)) - \tan(v(0, X_0))
\] (35)
if and only if
\[
\sigma(t, x)\sigma^*(t, x)\nabla v(t, x) = \cos^2(v(t, x))b(t, x), \quad (t, x) \in [0, \infty) \times \Lambda,
\] (36)
and $v$ satisfies the following time-reversed HJB equation,
\[
\frac{\partial v}{\partial t}(t, x) = -\frac{1}{2} \left\{ \text{Tr}[\sigma \sigma^*(t, x) \nabla^2 v(t, x)] \right\}
\] (37)
\[
+ \frac{[\cos(v(t, x)) + \sin(v(t, x))]|^2}{\cos^2(v(t, x))} |\sigma^*(t, x)\nabla v(t, x)|^2 \right\}.
\]

**Proof.** It is obvious for (a). We only prove (b)((c) may be similarly handed). By Theorem 1.6, we know that (26) is equivalent to
\[
\left\{ \begin{array}{l}
\frac{1}{v(t, x)} \frac{\partial v}{\partial t}(t, x) = -\frac{1}{2} \left\{ \frac{1}{v(t, x)} \text{Tr}[\sigma \sigma^*(t, x) \nabla^2 v(t, x)] \right\}
\] (38)
\[
\quad - \frac{1}{v(t, x)^2} |\sigma^*(t, x)\nabla v(t, x)|^2 + \frac{1}{v(t, x)} |\sigma^*(t, x)\nabla v(t, x)|^2 \right\}
\]
\[
\sigma(t, x)\sigma^*(t, x)\nabla v(t, x) = v(t, x)b(t, x), \quad (t, x) \in [0, \infty) \times \Lambda
\]
which are just (27) and (28), respectively. \qed
2. Non-Lipschitz SDEs with jumps

2.1. The characterisation theorem for SDEs with continuous diffusions on \( \mathbb{R}^d \).

Let \((U, \| \cdot \|_U)\) be a finite dimensional normed space endowed with its Borel \( \sigma \)-algebra \( \mathcal{U} \). Let \( \nu \) be a \( \sigma \)-finite measure defined on \((U, \mathcal{U})\). Let us fix \( U_0 \in \mathcal{U} \) with \( \nu(U \setminus U_0) < \infty \) and \( \int_{U_0} \|u\|_U^2 \nu(du) < \infty \). Furthermore, let \( \lambda : [0, \infty) \times U \to (0,1] \) be a given measurable function. Then, following e.g. [3, 4], there exists a non-negative integer valued \( \{ \mathcal{F}_t \}_{t \geq 0} \)-Poisson random measure \( N_\lambda(dt, du) \) on the given filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}; \{ \mathcal{F}_t \}_{t \geq 0}) \) with intensity \( \mathbb{E}(N_\lambda(dt, du)) = \lambda(t, u)dt\nu(du) \). Set

\[
\tilde{N}_\lambda(dt, du) := N_\lambda(dt, du) - \lambda(t, u)dt\nu(du)
\]

that is, \( \tilde{N}_\lambda(dt, du) \) stands for the compensated \( \{ \mathcal{F}_t \}_{t \geq 0} \)-predictable martingale measure of \( N_\lambda(dt, du) \).

We are concerned with the following SDE on \( \mathbb{R}^d \)

\[
\begin{align*}
\begin{cases}
dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t + \int_{U_0} f(t, X_{t-}, u)\tilde{N}_\lambda(dt, du), & t \in (0, T], \\
X_0 = x_0 \in \mathbb{R}^d,
\end{cases}
\end{align*}
\]  

(39)

for any given \( T > 0 \), where \( b, \sigma \) are Borel measurable as given in the previous section, \( \{ \mathcal{F}_t \}_{t \geq 0} \) is an \( m \)-dimensional \( \{ \mathcal{F}_t \}_{t \geq 0} \)-Brownian motion, \( f : [0, T] \times \mathbb{R}^d \times U_0 \to \mathbb{R}^d \) is Borel measurable, and \( \tilde{N}_\lambda \) is the compensated \( \{ \mathcal{F}_t \}_{t \geq 0} \)-predictable martingale measure of an induced \( \{ \mathcal{F}_t \}_{t \geq 0} \)-Poisson random measure given above which is independent of \( \{ \mathcal{F}_t \}_{t \geq 0} \).

This equation arises in nonlinear filtering and has been considered recently in [11, 8, 9] (see also the monograph [12]).

The characterisation theorem for path-independent property of Girsanov density for the above equation with non-degenerated \( \sigma \) was established in [10]. More precisely, under the following conditions

(\( H_1 \) ) There exists \( \lambda_0 \in \mathbb{R} \) such that for all \( x, y \in \mathbb{R}^d \) and \( t \in [0, T] \)

\[
2\langle x - y, b(t, x) - b(t, y) \rangle + \| \sigma(t, x) - \sigma(t, y) \|^2 \leq \lambda_0|x - y|^2\kappa(|x - y|),
\]

where \( \kappa \) is a positive continuous function, bounded on \([1, \infty)\) and satisfying

\[
\lim_{x \to 0} \frac{\kappa(x)}{\log x} = \delta < \infty.
\]

(\( H_2 \) ) There exists \( \lambda_1 > 0 \) such that for all \( x \in \mathbb{R}^d \) and \( t \in [0, T] \)

\[
|b(t, x)| + \| \sigma(t, x) \|^2 \leq \lambda_1(1 + |x|)^2.
\]

(\( H_3 \) ) \( b(t, x) \) is continuous in \( x \) and there exists \( \lambda_2 > 0 \) such that

\[
\langle \sigma(t, x)h, h \rangle \geq \sqrt{\lambda_2}|h|^2, \quad t \in [0, T], \ x, h \in \mathbb{R}^d.
\]

(\( H_f \) ) For all \( x, y \in \mathbb{R}^d \) and \( t \in [0, T] \),

\[
\int_{U_0} |f(t, x, u) - f(t, y, u)|^2 \nu(du) \leq 2|\lambda_0||x - y|^2\kappa(|x - y|)
\]

and for \( q = 2 \) and \( 4 \)

\[
\int_{U_0} |f(t, x, u)|^q \nu(du) \leq \lambda_1(1 + |x|)^q.
\]
Qiao and Wu in [10] proved a characterisation theorem, where a partial integer-differential equation (PIDE) as the main characterizing equation was derived. We notice that the assumption (H3) on the diffusion coefficient $\sigma$ is too strong. Here we aim to relax this condition. First of all, we let $\sigma$ to be $d \times m$-matrix-valued for $d, m \in \mathbb{N}$, i.e., $\sigma$ is in general not square matrix-valued.

Let $\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^m$ be a measurable function such that the following condition $(H_{\gamma, \lambda})$ holds

$$
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^T |\gamma(s, X_s)|^2 \, ds + \int_0^T \int_{U_0} \left( \frac{1 - \lambda(s, u)}{\lambda(s, u)} \right)^2 \lambda(s, u) \nu(du) \, ds \right\} \right] < \infty.
$$

Set

$$
Z_t : = \exp \left\{ - \int_0^t \langle \gamma(s, X_s), dB_s \rangle - \frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 \, ds 
- \int_0^t \int_{U_0} \log \lambda(s, u) N_\lambda(ds, du) - \int_0^t \int_{U_0} (1 - \lambda(s, u)) \nu(du) \, ds \right\},
$$

$$
M_t : = - \int_0^t \langle \gamma(s, X_s), dB_s \rangle + \int_0^t \int_{U_0} \frac{1 - \lambda(s, u)}{\lambda(s, u)} N_\lambda(ds, du),
$$

and then $(Z_t)$ is the Doléans-Dade exponential of $(M_t)$, see e.g., [2].

Under $(H_1)$, $(H_2)$ and $(H_f)$, it is well known that there exists a unique strong solution to Eq. (39) (cf. [12, Theorem 170, p.140]). This solution will be denoted by $X_t$. In the following, we define the support of a random vector (6) and then present a result about the support of $X_t$ under the above assumptions.

**Definition 2.1.** The support of a random vector $Y$ is defined as

$$
\text{supp}(Y) := \{ x \in \mathbb{R}^d | (\mathbb{P} \circ Y^{-1})(B(x, r)) > 0, \text{ for all } r > 0 \}
$$

where $B(x, r) := \{ y \in \mathbb{R}^d | |y - x| < r \}$, the open ball centered at $x$ with radius $r$.

Under $(H_{\gamma, \lambda})$, $(M_t)$ is a locally square integrable martingale. Moreover, $M_t - M_{t-} > -1$ a.s. and

$$
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} < M^c, M^c >_T + < M^d, M^d >_T \right\} \right] = \mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^T |\gamma(s, X_s)|^2 \, ds + \int_0^T \int_{U_0} \left( \frac{1 - \lambda(s, u)}{\lambda(s, u)} \right)^2 \lambda(s, u) \nu(du) \, ds \right\} \right] < \infty,
$$

where $M^c$ and $M^d$ are continuous and purely discontinuous martingale parts of $(M_t)$, respectively. Thus, it follows from [7] Theorem 6] that $(Z_t)$ is an exponential martingale. Define a measure $\mathbb{P}$ via

$$
\frac{d\mathbb{P}}{d\mathbb{P}} = Z_t.
$$

By the Girsanov theorem for Brownian motions and random measures, one can obtain that under the measure $\mathbb{P}$ the system (39) is transformed into the following

$$
dX_t = [b(t, X_t) + \sigma(t, X_t) \gamma(t, X_t)] \, dt + \sigma(t, X_t) \, dB_t + \int_{U_0} f(t, X_{t-}, u) \, \tilde{N}(dt, du),
$$

Now let us assume that (along th paths of $(X_t)_{t \geq 0}$)

$$
b(t, X_t) + \sigma(t, X_t) \gamma(t, X_t) = 0.
$$
Then we get
\[ dX_t = \sigma(t, X_t)dB_t + \int_{U_0} f(t, X_{t-}, u)\tilde{N}(dt, du), \]
where
\[ \tilde{B}_t := B_t + \int_0^t \gamma(s, X_s)ds, \quad \tilde{N}(dt, du) := N_\lambda(dt, du) - dt\nu(du). \]

Next, we set
\[ Y_t := -\log Z_t = \int_0^t \langle \gamma(s, X_s), dB_s \rangle + \frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds \]
\[ + \int_0^t \int_{U_0} \log \lambda(s, u)N_\lambda(ds, du) + \int_0^t \int_{U_0} (1 - \lambda(s, u))\nu(du)ds. \]

Clearly, \((Y_t)\) is a one-dimensional stochastic process with the following stochastic differential form
\[ dY_t = \langle \gamma(t, X_t), dB_t \rangle + \frac{1}{2} |\gamma(t, X_t)|^2 dt \]
\[ + \int_{U_0} \log \lambda(t, u)N_\lambda(dt, du) + \int_{U_0} (1 - \lambda(t, u))\nu(du)dt. \]

Let \(\Lambda := \text{supp}((t, X_t), t \geq 0)\). Then we have the following.

**Theorem 2.2.** Let \(v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}\) be a scalar function which is \(C^1\) with respect to the first variable and \(C^2\) with respect to the second variable. Then
\[ v(t, X_t) = v(0, x_0) + \int_0^t \langle \gamma(s, X_s), dB_s \rangle + \frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds \]
\[ + \int_0^t \int_{U_0} \log \lambda(s, u)N_\lambda(ds, du) + \int_0^t \int_{U_0} (1 - \lambda(s, u))\nu(du)ds, \quad (41) \]
equivalently,
\[ Y_t = v(t, X_t) - v(0, x_0), \quad t \in [0, T] \]
holds if and only if
\[ \gamma(t, x) = (\sigma \sigma^* \nabla v)(t, x), \quad (t, x) \in \Lambda, \quad (42) \]
\[ \lambda(t, u) = \exp \{v(t, x + f(t, x, u)) - v(t, x)\}, \quad (t, x, u) \in \Lambda \times U_0, \quad (43) \]
and \(v\) satisfies the following time-reversed partial integro-differential equation (PIDE),
\[ \frac{\partial}{\partial t}v(t, x) = -\frac{1}{2} [Tr(\sigma \sigma^*) \nabla^2 v](t, x) - \frac{1}{2} |\sigma^* \nabla v|^2(t, x) - \int_{U_0} \left[ e^{v(t, x + f(t, x, u)) - v(t, x)} - 1 \right. \]
\[ - \langle f(t, x, u), \nabla v(t, x) \rangle e^{v(t, x + f(t, x, u)) - v(t, x)} \nu(du). \quad (44) \]

**Proof.** Following the line of [8]. To the reader’s convenience, we give the detailed proof here. The proof of necessity. By (41),
\[ dv(t, X_t) = \left[ \frac{1}{2} |\gamma(t, X_t)|^2 + \int_{U_0} \left( \lambda(t, u) \log \lambda(t, u) + (1 - \lambda(t, u)) \right) \nu(du) \right] dt \]
\[ + \int_{\mathbb{U}_0} \log \lambda(t, u) \tilde{N}_\lambda(dt, du) + \langle \gamma(t, X_t), dB_t \rangle. \] (45)

It is clear from (45) that \( v(t, X_t) \) is a càdlàg semimartingale with a predictable finite variation part. On the other hand, note that \( X_t \) satisfies Equation (39) and \( v(t, x) \) is a \( C^{1,2} \)-function, by applying the Itô formula to the composition process \( v(t, X_t) \), one could obtain the following

\[
dv(t, X_t) = \frac{\partial}{\partial t} v(t, X_t) dt + \langle b, \nabla v \rangle(t, X_t) dt + \frac{1}{2} [Tr(\sigma^*) \nabla^2 v](t, X_t) dt
\]
\[
+ \int_{\mathbb{U}_0} \left[ v(t, X_{t^-} + f(t, X_{t^-}, u)) - v(t, X_{t^-}) \\
- \langle f(t, X_{t^-}, u), \nabla v(t, X_{t^-}) \rangle \right] \lambda(t, u) \nu(du) dt
\]
\[
+ \int_{\mathbb{U}_0} \left[ v(t, X_{t^-} + f(t, X_{t^-}, u)) - v(t, X_{t^-}) \right] \tilde{N}_\lambda(dt, du)
\]
\[
+ \langle (\sigma^* \nabla v)(t, X_t), dB_t \rangle. \] (46)

Thus, (46) is another decomposition of the semimartingale \( v(t, X_t) \). By uniqueness for decomposition of the semimartingale, it holds that for \( t \in [0, T] \),

\[
\gamma(t, X_t) = (\sigma^* \nabla v)(t, X_t),
\]
\[
\log \lambda(t, u) = v(t, X_{t^-} + f(t, X_{t^-}, u)) - v(t, X_{t^-}), \quad u \in \mathbb{U}_0,
\]

and

\[
\frac{1}{2} |\gamma(t, X_t)|^2 + \int_{\mathbb{U}_0} \left( \lambda(t, u) \log \lambda(t, u) + (1 - \lambda(t, u)) \right) \nu(du)
\]
\[
= \frac{\partial}{\partial t} v(t, X_t) + \langle b, \nabla v \rangle(t, X_t) + \frac{1}{2} [Tr(\sigma^*) \nabla^2 v](t, X_t)
\]
\[
+ \int_{\mathbb{U}_0} \left[ v(t, X_{t^-} + f(t, X_{t^-}, u)) - v(t, X_{t^-}) \\
- \langle f(t, X_{t^-}, u), \nabla v(t, X_{t^-}) \rangle \right] \lambda(t, u) \nu(du), \quad a.s..
\]

Note that \( (t, X_t) \) runs through \( \Lambda \), thus, we have that

\[
\gamma(t, x) = (\sigma^* \nabla v)(t, x), \quad (t, x) \in \Lambda, \quad (47)
\]
\[
\log \lambda(t, u) = v(t, x + f(t, x, u)) - v(t, x), \quad (t, x, u) \in \Lambda \times \mathbb{U}_0, \quad (48)
\]

and

\[
\frac{1}{2} |\gamma(t, x)|^2 + \int_{\mathbb{U}_0} \left( \lambda(t, u) \log \lambda(t, u) + (1 - \lambda(t, u)) \right) \nu(du)
\]
\[
= \frac{\partial}{\partial t} v(t, x) + \langle b, \nabla v \rangle(t, x) + \frac{1}{2} [Tr(\sigma^*) \nabla^2 v](t, x)
\]
\[
+ \int_{\mathbb{U}_0} \left[ v(t, x + f(t, x, u)) - v(t, x) \\
- \langle f(t, x, u), \nabla v(t, x) \rangle \right] \lambda(t, u) \nu(du). \] (49)
It is easy to see that (47) and (48) correspond to (42) and (43), respectively, which together with (49) further yields the PIDE (44).

Next, let us show sufficiency. Assume that there exists a $C_1$ function $v(t, x)$ satisfying (42), (43) and (44). For the composition process $v(t, X_t)$, the Itô formula admits us to get (46). Combining (42), (43) and (44) with (46), we have

$$d v(t, X_t) = \left[ \frac{1}{2} |\gamma(t, X_t)|^2 + \int_{U_0} \left( (\lambda(t, u) \log \lambda(t, u)) \lambda(t, u) + (1 - \lambda(t, u)) \right) \nu(du) \right] dt$$

$$+ \int_{U_0} \log \lambda(t, u) N_\lambda(dt, du) \lambda(t, u) b(t, X_t) dB_t$$

$$= \langle \gamma, dB_t \rangle + \frac{1}{2} |\sigma^{-1}(t, X_t)b(t, X_t)|^2 dt$$

$$+ \int_{U_0} \log \lambda(t, u) N_\lambda(dt, du) + \int_{U_0} (1 - \lambda(t, u)) \nu(du) dt.$$

The proof is completed. □

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