Applications of Effective Lagrangians

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Abstract. Effective Lagrangians were originally used only at the tree level as so-called phenomenological Lagrangians since they were in general non-renormalizable. Today they are treated as effective field theories valid below a characteristic energy scale. Quantum corrections can then be calculated in a consistent way as for any renormalizable theory. A few applications of the Euler-Heisenberg Lagrangian for interacting photons at low energies are presented together with recent developments in the use of QED for non-relativistic systems. Finally, the ingredients of an effective theory for the electroweak sector of the Standard Model are discussed in the case of a non-linear realization of the Higgs mechanism using the Stückelberg formalism.

INTRODUCTION

As a graduate student at Caltech in the early seventies when the Standard Model was established, we were surprised to notice that Richard Feynman did not take very much interest in unified theories of weak and electromagnetic interactions. When he was asked why, he said that these theories were all built on the principle of being renormalizable. And since nobody understood what renormalization really was, one were not allowed to construct new physical theories based on such a principle.

However, at the very same time Ken Wilson was unraveling the physics behind renormalization, although in the somewhat separate field of critical phenomena. His deep insight led to the understanding of any field theory being directly tied to the scale of the phenomena it is supposed to describe. If we have a theory at a certain scale, we can from it construct a new, effective field theory at a larger scale involving new fields and interactions which in principle can be calculated from the theory at the shorter scale. Going in the opposite direction from a large to a smaller scale, there is no reason why the same theory should be applicable. So field theories in general should be non-renormalizable. In the special case where one can go to a much smaller scale with the same theory, the theory is renormalizable. But that is the exceptional case.

In such a short talk it is impossible to say much about the very many applications of effective Lagrangians in modern high energy physics. More detailed reviews have been given by Kaplan [1] and Manohar [2]. I have here chosen first to describe the illustrative example involving the interactions of low energy photons based upon the Euler-Heisenberg effective Lagrangian. Including non-relativistic electrons one

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can then use NRQED to calculate radiative and relativistic corrections in atomic physics to very high accuracy. Finally, I’ll say a few words about the possibility of standard, electroweak interactions also being described by an effective theory without an elementary Higgs boson valid below a characteristic energy scale where new, unknown physics appears.

I will not have time to discuss the surprising and beautiful results coming out of the marriage of quantum theory and gravitation. Usually this is said not to work and to represent one of the remaining big, unsolved problems in modern physics. Again the reason has been that Einstein’s geometric Lagrangian is a non-renormalizable theory. But now John Donoghue [3] has shown that by treating it as an effective theory, quantum corrections can be calculated in a consistent way for all energies below the Planck scale. The lowest order quantum correction to the gravitational attraction between two masses has been calculated and corresponds to Newton’s constant effectively becoming smaller at shorter distances. This is just as to be expected in a non-abelian gauge theory. In almost every thinkable astrophysical situation the quantum effects are completely negligible. At ordinary energies and curvatures this will not change even if we should get a more fundamental theory of gravity valid at shorter scales.

**EULER-HEISENBERG INTERACTION**

QED describes the interactions between the electromagnetic field $A_\mu(x)$ and the electron field $\psi(x)$ by the standard Lagrangian

$$\mathcal{L}(A,\psi) = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi} (\gamma^\mu (i\partial_\mu - eA_\mu) - m) \psi$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. With no matter present and for energies much below the electron mass $m$, the photons can interact only indirectly via virtual electron loops in the vacuum. These interactions are suppressed by inverse powers of the electron mass compared with the Maxwell term and are thus very small. The effective Lagrangian should be Lorentz and gauge invariant and respect parity invariance. It can thus be constructed only from the field strength $F_{\mu\nu}$ and its derivatives. In this way we can write down the form of the effective Lagrangian involving the lowest, dimension six operator,

$$\mathcal{L}_U = -\frac{1}{4} F_{\mu\nu}^2 + C_U \frac{1}{m^2} F_{\mu\nu} F^{\mu\nu}$$

**FIGURE 1.** The low-energy Uehling interaction from the vacuum polarization loop.
where $\Box \equiv \partial_\mu \partial^\mu$. This new term is just the Uehling interaction modifying the photon propagator at low energies [4]. To lowest order in the fine structure constant $\alpha = e^2/4\pi$ the new coupling constant is $C_U = \alpha/60\pi$ resulting from the Feynman diagram in Figure 1.

However, in the absence of matter this new term will not contribute to any physics. This is most easily seen by using the equation of motion for the free field which is simply $\Box F_{\mu\nu} = 0$ and makes the Uehling term go away. Higher order operators in the effective Lagrangian will then contribute instead. Their coefficients must be found by matching to the underlying theory which is here QED. This is equivalent to integrating out the electron field in the Lagrangian [5]. The first non-trivial photon interaction is obtained with dimension eight,

$$\mathcal{L}_{EH} = -\frac{1}{4} F^2_{\mu\nu} + \frac{\alpha^2}{90m^4} \left[ (F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right]$$

(2)

where $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$. This is the Euler-Heisenberg Lagrangian [6] giving a non-linear interaction between photons. At the microscopic scale it is caused by the coupling of the four photons to a virtual electron loop as shown in Figure 2. Higher order terms in the expansion will represent interactions between more photons.

We will here mention just two applications of this effective Lagrangian. It was already used by Euler in 1936 to derive the elastic photon-photon cross section at the tree level. Even if it represents a non-renormalizable theory, Halter [7] showed that a finite result for the one-loop correction to the scattering amplitude can be calculated from the diagram in Figure 3 and its crossed versions. Just from dimensional arguments we know that the correction must be order of magnitude $\alpha^2 (\omega/m)^4$ smaller than the tree-level amplitude where $\omega$ is the photon energy. However, this is not the lowest radiative correction to the Euler cross section. If the effective Lagrangian (2) is derived from QED more accurately, i.e. to two-loop order, the four-photon couplings will be modified by terms of order $\alpha$. These have been calculated by Ritus [8],

$$(FF)^2 \to \left(1 + \frac{40\alpha}{9\pi}\right) (FF)^2$$

(3)

and

$$(F\tilde{F})^2 \to \left(1 + \frac{1315\alpha}{252\pi}\right) (F\tilde{F})^2$$

FIGURE 2. The Euler-Heisenberg interaction between four photons.
and recently confirmed by Reuter, Schmidt and Schubert [9]. Obviously, the resulting corrections to the cross section are much larger than the contribution obtained by Halter. This shows that if one wants to obtain correct results of a fundamental theory using a low-energy effective theory, they have to be properly matched to the appropriate order in the fundamental coupling constant.

Free photons at finite temperature $T$ exert a pressure given by the Stefan-Boltzmann formula $P = (\pi^2/45)T^4$. In full QED they will interact with the virtual electrons in the vacuum. For temperatures $T < m$ the dominant interaction is given by the Euler-Heisenberg Lagrangian (2). Again we see from dimensional reasoning that the correction to the pressure will be of the order $\alpha^2(T/m)^4$. It can be obtained from the Feynman diagram in Figure 4 which gives for the total photon pressure [10]

$$P = \frac{\pi^2}{45}T^4 + \frac{22\pi^4\alpha^2}{3^55^3} \frac{T^8}{m^4}$$

(4)

In full QED it will follow from the evaluation of a difficult three-loop diagram, while using effective field theory it is obtained essentially from a one-loop diagram. The above result has previously been obtained by Barton [11] using semi-classical methods. Since the Uehling term is absent in the effective Lagrangian, we see that there is no term $\propto T^6$ in the result. This is also the case for the pressure of massless pions in chiral theory [12].
NON-RELATIVISTIC QED

When there are also non-relativistic electrons interacting with the photons, covariant and renormalizable QED can be replaced by a non-relativistic and non-renormalizable effective theory called NRQED as first proposed by Caswell and Lepage [13]. Relativistic effects are represented by local operators of higher dimensions. Since the relevant, low-energy degrees of freedom are separated out in the initial, effective Lagrangian, it allows a cleaner and more systematic derivation of higher order radiative corrections. Divergent loop integrations are cut off at a momentum $\Lambda \leq m$. The higher order coupling constants will in general also involve this cutoff and are most conveniently obtained by matching QED and NRQED scattering amplitudes. With the values of the effective couplings established this way, they can then also be used for bound states. The over-all $\Lambda$-dependence will then drop out of the final, physical result. In this way it can be used to calculate radiative shifts of energy levels in simple atoms with high precision [14]. There is no longer any need for the covariant Bethe-Salpeter formalism for bound states. Instead one can use the standard Schrödinger wave functions found in any textbook on quantum mechanics.

The general structure of the NRQED Lagrangian can be derived from the requirement of being invariant under gauge and Galilean transformations. Many of the terms can be obtained directly from the Dirac equation in the non-relativistic limit. The leading part is the Pauli Lagrangian

$$L_{\text{Pauli}} = \bar{\psi} \left[ iD_0 + \frac{1}{2m} D^2 + \frac{e}{2m} \sigma \cdot B \right] \psi$$

for a non-relativistic electron interacting with the photon field $A^\mu = (\Phi, A)$. Here we have introduced the covariant derivatives $D_0 = \partial_0 + ie\Phi$ and $D = \nabla - ieA$ in the Coulomb gauge where $\nabla \cdot A = 0$ and $B = \nabla \times A$ is the magnetic field. A photon with the four-momentum $k^\mu = (\omega, k)$ has a static Coulomb propagator $D_{00}(\omega, k) = 1/k^2$ while the transverse propagator is

$$D_{ij}(\omega, k) = \frac{1}{k^2 + ie} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right)$$

Similarly, the propagator for a non-relativistic electron with energy $E$ and momentum $p$ is $G(E, p) = 1/(E - p^2/2m + ie)$. There are three couplings of the photon in this theory, an electric coupling to the electron charge and a magnetic coupling to the electron spin plus a seagull coupling involving two photons.

FIGURE 5. Loop correction to the electron propagator due to a transverse photon.
The lowest order propagator correction comes from the diagram in Figure 5. When both the vertices are electric, it gives

$$\Sigma = \left(\frac{ie}{2m}\right)^2 \int \frac{d^4k}{(2\pi)^4} (2p_i - k_i)(2p_j - k_j)G(E - k_0, p - k)D_{ij}(k_0, k)$$

Integrating first over $k_0$ gives $k_0 = |k| = \omega$. The remaining integral is linearly divergent. After an angular integration, it then gives the self-energy $\Sigma = -(2\alpha/3\pi m^2)p^2\Lambda$. It can be interpreted as a radiative correction to the electron mass. The same diagram with one electric and one magnetic coupling gives zero, while two magnetic couplings give similarly a cubic divergent result.

Next we consider the simplest vertex correction shown in Figure 6 for an electron scattering off a potential $\Phi(x)$. If $p$ and $q$ are the initial and final electron momenta, the momentum transfer from the potential is $k = q - p$. The linearly divergent corrections on the external electron legs are cancelled by the above self-energy. There is a remaining, logarithmically divergent wave function renormalization which combines with a similar contribution from the non-relativistic vertex correction. They both exhibit an IR divergence which is cured by giving the photon a small mass $\lambda$ so that $\omega^2 = k^2 + \lambda^2$. We are then left with the net contribution

$$V_{qp} = -\frac{\alpha}{3\pi m^2}(p - q)^2 \left(\log \frac{2\Lambda}{\lambda} - \frac{5}{6}\right)e\Phi(q - p)$$

from these three diagrams. The result is thus proportional with $k^2\Phi(k)$. In coordinate space it corresponds to a higher order interaction $\nabla \cdot E$ where $E = -\nabla \Phi$ is the electric field. But this is just the Darwin interaction. It is needed as a counterterm when the Pauli Lagrangian is used at the 1-loop level.

When one considers other scattering amplitudes, additional counterterms are needed. So to this accuracy the non-relativistic Lagrangian (5) has to be extended by including these new interactions and it will take the form

$$\mathcal{L}_{NRQED} = \psi^\dagger \left[iD_0 + \frac{1}{2m}D^2 + C_K \frac{1}{8m^3}D^4 + C_F \frac{e}{2m}\sigma \cdot B + C_D \frac{e}{8m^2}(D \cdot E - E \cdot D) + C_{LS} \frac{ie}{8m^2}\sigma \cdot (D \times E - E \times D)\right] \psi$$

\[\text{(6)}\]
The most general effective Lagrangian to order $1/m^3$ has recently been presented by Manohar [15]. Just as the Darwin term is a relativistic effect, the strength of these new interaction can be obtained from the non-relativistic reduction of the Dirac equation. The coupling constants $C_i$ in the above effective Lagrangian are therefore all equal to one at tree level. To first order in $\alpha$ NRQED must now give the same results as obtained from renormalized QED. Again considering scattering off external potentials, this form of matching gives for the correction to the kinetic energy $C_K = 1$ while [14]

$$C_F = 1 + \frac{\alpha}{2\pi}$$
$$C_D = 1 + \frac{8\alpha}{3\pi} \left[ \log \frac{m}{2\Lambda} - \frac{3}{8} + \frac{5}{6} \right] + \frac{\alpha}{\pi}$$
$$C_{LS} = 1 + \frac{\alpha}{\pi}$$

The magnetic Fermi coupling has been renormalized so that the effective magnetic moment of the electron at low energies now includes the anomalous contribution. Notice that the photon mass $\lambda$ does not appear in the effective coupling constants. This is because the IR behaviour of both NRQED and QED should agree and thus $\lambda$ cancels out in the matching [16]. These renormalized couplings together with the Uehling modification (1) of the photon propagator are now sufficient to give a much more direct derivation of the Lamb shift in hydrogen than is usually found in textbooks [17].

Kinoshita and Nio have used NRQED to calculate the hyperfine splitting in muonium to the highest precision [14]. Similarly, Labelle [18] and collaborators [19] have made applications to radiative shifts of energy levels and lifetimes of positronium. In lowest order one encounters here the four-fermion operator $(\psi^\dagger \sigma \chi) \cdot (\chi^\dagger \sigma \psi)$ describing the annihilation and subsequent creation of an electron $\psi$ and a positron $\chi$ via a photon as shown in Figure 7. Evaluating the diagram, one finds the corresponding coupling constant to be $\alpha\pi/m^2$. Since the virtual photon has the relativistic energy $2m$, the process is thus represented by a local operator in NRQED. Together with the ordinary Fermi interaction it will contribute to the lowest hyperfine splitting $\Delta E$ between ortho- and para-positronium as shown in Figure 8. They give the textbook result $\Delta E = \alpha^4 m (1/3 + 1/4)$.

When considering higher order interactions in positronium, additional electron-positron contact terms will be needed. They will have the general spinor structure
FIGURE 8. Lowest order contributions to the hyperfine splitting in positronium.

\[ \mathcal{L}_{ct} = C_1(\psi^\dagger \chi)(\chi^\dagger \psi) + C_2(\psi^\dagger \sigma \chi) \cdot (\chi^\dagger \sigma \psi) + C_3(\psi^\dagger D \chi) \cdot (\chi^\dagger D \psi) + \ldots \]

when allowing for Fierz transformations. The effective coupling constants will in general be complex and thus contribute both to the radiative shifts of energy levels and lifetimes. Again by matching to QED the full values of \( C_1 \) and \( C_2 \) to lowest order in \( \alpha \) has recently been obtained [20]. They reproduce the Karplus-Klein result for the hyperfine splitting in positronium to order \( \alpha^5 \) [21]. An even more precise matching gives now the splitting to order \( \alpha^6 \) [22].

**ELECTROWEAK VECTOR BOSONS**

In the Standard Model the electroweak bosons get masses from the spontaneous breaking of the underlying \( SU(2)_L \otimes U(1)_Y \) symmetry. When this is realized in a linear way, we will have a renormalizable theory with an elementary Higgs bosons. In principle, we then have a quantum field theory describing essentially all fundamental phenomena ranging over scales varying roughly fifteen orders of magnitude. Something like this we have never seen in physics before and the Standard Model would really have to be a very fundamental theory. But with so many unknown parameters and with a rather ad hoc internal structure, it is more likely that the model is just an effective theory valid up to some unknown energy threshold where new physics appears. The requirement of being renormalizable is then not so pressing anymore, and the electroweak symmetry can be realized in a non-linear way with no elementary Higgs boson. But the would-be Goldstone bosons are still needed in order to give masses to the electroweak vector bosons and allow a perturbative evaluation of their quantum effects.

This was understood already in 1956 by St"uckelberg [23]. Consider the Lagrangian

\[ \mathcal{L}_V = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} m^2 V^2 + a (V_\mu V^\mu)^2 + b (F_{\mu\nu} F^{\mu\nu})^2 + \ldots \]  

(7)

for an interacting, massive vector field \( V_\mu \) where \( F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \). Except for Lorentz and parity invariance, there are no other symmetries. It defines in general a non-renormalizable field theory which we now can treat as an effective theory. But looking at the free propagator
\[
\Delta_{\mu\nu}(k) = \left( \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right) \frac{1}{k^2 - m^2 + i\epsilon}
\]

we see that is goes to a constant at very high momenta \( k \gg m \) which makes the evaluation of higher order loop diagrams impossible.

This technical problem can be cured according to Stueckelberg by replacing the massive vector boson \( V_\mu \) by a massless vector boson \( A_\mu \) plus a massless scalar field \( \theta \). From the covariant derivative \( D_\mu = \partial_\mu - igA_\mu \), we can then replace the massive field with the composite field

\[
V_\mu = i g U^{\dagger} D_\mu U = A_\mu - \frac{1}{m} \partial_\mu \theta
\]

where \( U(x) = \exp \left( i \theta(x)/v \right) \). The mass term in \( \mathcal{L}_V \) then becomes the kinetic term for the \( \theta \)-field with \( m = gv \). Now we have \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and the resulting theory of these two massless fields has developed a new, local \( U(1) \) gauge symmetry under which \( \theta \rightarrow \theta + v x \) and \( A_\mu \rightarrow A_\mu + (1/g) \partial_\mu x \). Via the Stueckelberg construction we thus see that the above massive Lagrangian (7) is just the strongly coupled Abelian Higgs model in the unitary gauge.

This point of view was first introduced in the context of the Standard Model by Cornwall, Levin and Tiktopoulos [24] in 1974. Since then it was advocated for by Chanowitz, Georgi and Golden [25] and others [26]. It has been used extensively by Yuan and collaborators [27] to consistently calculate radiative corrections in the Standard Model without any elementary Higgs boson. Any coupling between the physical vector bosons \( A_\mu, Z_\mu \) and \( W^{\pm}_\mu \) are then allowed as long as it is invariant under Lorentz and local electromagnetic gauge transformations. In this effective theory of electroweak interactions anomalous couplings will therefore appear reflecting the existence of new physics at higher energies. They will offer a much richer world of physical phenomena than with a renormalizable Standard Model and a fundamental Higgs particle.

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