On the quenched CLT for stationary random fields under projective criteria

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Abstract

Motivated by random evolutions which do not start from equilibrium, in a recent work, Peligrad and Volný (2018) showed that the quenched CLT (central limit theorem) holds for ortho-martingale random fields. In this paper, we study the quenched CLT for a class of random fields larger than the ortho-martingales. To get the results, we impose sufficient conditions in terms of projective criteria under which the partial sums of a stationary random field admit an ortho-martingale approximation. More precisely, the sufficient conditions are of the Hannan’s projective type. As applications, we establish quenched CLT’s for linear and nonlinear random fields with independent innovations.

Key words: random fields, quenched central limit theorem, ortho-martingale approximation, projective criteria.

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1 Introduction

An interesting problem with many practical applications is to study limit theorems for processes conditioned to start from a fixed past trajectory. This problem is difficult, as the stationary processes started from a fixed past trajectory are no longer stationary. This type of convergence is also known under the name of almost sure conditional limit theorem or the quenched limit theorem. The issue of the quenched CLT for stationary processes has been widely explored for the last few decades. Among many others, we mention papers by Derriennic and Lin (2001), Cuny and Peligrad (2012), Cuny and Volný (2013), Cuny and Merlevède (2014), Volný and Woodroofe (2014), Barrera et al. (2016). Some of these results were surveyed in Peligrad (2015).

As far as we know, this type of convergence was rarely investigated for stationary random fields. Let $d$ be a positive integer. A random field consists of multi-indexed random variables $(X_u)_{u \in \mathbb{Z}^d}$. The main difficulty when analyzing the asymptotic properties of random fields, is the fact that the future and the past do not have a unique interpretation. To compensate for the lack of ordering of the filtration, mathematicians utilize the notion of commuting filtrations. Traditionally, this kind of filtration is constructed based on random fields which are functions of independent and identically distributed random variables. Alternatively, commuting filtrations can be induced by stationary random fields with independent columns or rows. See for example, El Machkouri et al. (2013) and Peligrad and Zhang (2018a). As in the case of random processes, a fruitful approach for proving the limit theorems for random fields is via the martingale approximation method, which was started by Rosenblatt (1972) and its development is still in progress. Recently, the interest

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is in the approximation by ortho-martingales which were introduced by Cairoli (1969). We would like to mention several important recent contributions in this direction by Gordin (2009), Volný and Wang (2014), Volný (2015), Cuny et al. (2015), Peligrad and Zhang (2018a), Giraudo (2017) and Peligrad and Zhang (2018b). However the corresponding quenched version of these results have rarely been explored. To the best of our knowledge, the only quenched invariance principle so far for random fields is due to Peligrad and Volný (2018), in which they studied a quenched functional CLT for ortho-martingales and a quenched functional CLT for random fields via co-boundary decomposition. By constructing an example of an ortho-martingale which satisfies the CLT but not its quenched form, Peligrad and Volný (2018) showed that the finite second moment condition is not enough for the quenched CLT and they provided a minimal moment condition, that is, $EX_0^2 \log(1 + |X_{0,0}|) < \infty$, for the validity of this type of results.

Here, we aim to establish sufficient conditions in terms of projective criteria such that a quenched CLT holds. The main result of this paper is a natural extension of the quenched CLT for ortho-martingales in Peligrad and Volný (2018) to more general random fields under the generalized Hannan projective condition (1973). Our result is also a quenched version of the main theorem in Peligrad and Zhang (2018a). The tools for proving these results consist of ortho-martingale approximations, projective decompositions and ergodic theorems for Dunford-Schwartz operators.

Our paper is organized as follows. In the first section, we introduce the notion of stationary random fields, commuting filtrations, projection operators, martingale differences, martingale approximations and the main results for double-indexed random fields. In Section 3, we prove our main results for double-indexed random fields. The extension to general indexed random fields and its proof are given in Section 4. In Section 5, we apply our results to linear and Volterra random fields with independent innovations, which are often encountered in economics. Our results could also be formulated in the language of dynamical systems, leading to new results in this field. For the convenience of the reader, in the Appendix, we provide one well-known inequality for martingales and an important theorem in decoupling theory which will be of great importance for the proof of our main results.

2 Preliminaries and Results

For the sake of clarity, especially due to the complicated notation, in this section, we shall only talk about the double-indexed random fields. After obtaining results for double-indexed random fields, we will extend them to random fields indexed by $\mathbb{Z}^d$, $d > 2$. We shall introduce a stationary random field adapted to a stationary filtration. In order to construct a flexible filtration it is customary to start with a stationary real valued random field $(\xi_{n,m})_{n,m \in \mathbb{Z}}$ defined on a probability space $(\Omega, \mathcal{K}, P)$ and define the filtrations

$$\mathcal{F}_{k,\ell} = \sigma(\xi_{j,u} : j \leq k, u \leq \ell).$$

For all $i, j \in \mathbb{Z}$, we also define the following sigma algebras generated by the union of sigma algebras: $\mathcal{F}_{\infty,j} = \vee_{n \in \mathbb{Z}} \mathcal{F}_{n,j}$, $\mathcal{F}_{i,\infty} = \vee_{m \in \mathbb{Z}} \mathcal{F}_{i,m}$ and $\mathcal{F}_{\infty,\infty} = \vee_{i,j \in \mathbb{Z}} \mathcal{F}_{i,j}$.

To ease the notation, sometimes the conditional expectation will be denoted by $E_{a,b}X = E(X|\mathcal{F}_{a,b})$.

In addition we consider that the filtration is commuting in the sense that

$$E_{a,b}E_{a,b}X = E_{a \land a, b \land b}X,$$

where the symbol $a \land b$ stands for the minimum between $a$ and $b$. As we mentioned before, this type of filtration is induced, for instance, by an initial random field $(\xi_{n,m})_{n,m \in \mathbb{Z}}$ of independent
random variables or more generally can be induced by stationary random fields \((\xi_{n,m})_{n,m\in \mathbb{Z}}\) where only the columns are independent, i.e. \(\eta_m = (\xi_{n,m})_{n\in \mathbb{Z}}\) are independent. This model often appears in statistical applications when one deals with repeated realizations of a stationary sequence. It is interesting to point out that commuting filtrations can be described by the equivalent formulation: for \(a \geq u\) we have
\[
E_{u,v}E_{a,b}X = E_{u,b\wedge v}X.
\] (3)

This follows from the Markovian-type property (see for instance Problem 34.11 in Billingsley, 1995).

Without restricting the generality we shall define \((\xi_u)_{u\in \mathbb{Z}^2}\) in a canonical way on the probability space \(\Omega = R^{\mathbb{Z}^2}\), endowed with the \(\sigma\)-field, \(\mathcal{B}(\Omega)\), generated by cylinders. Now on \(R^{\mathbb{Z}^2}\) we shall introduce the operators
\[
T^u((x_v)_{v\in \mathbb{Z}^2}) = (x_v + u)_{v\in \mathbb{Z}^2}.
\]
Two of them will play an important role in our paper namely, when \(u = (1,0)\) and when \(u = (0,1)\).

By interpreting the indexes as notations for the lines and columns of a matrix, we shall call
\[
T((x_{u,v})_{(u,v)\in \mathbb{Z}^2}) = (x_{u+1,v})_{(u,v)\in \mathbb{Z}^2}
\]
the vertical shift and
\[
S((x_{u,v})_{(u,v)\in \mathbb{Z}^2}) = (x_{u,v+1})_{(u,v)\in \mathbb{Z}^2}
\]
the horizontal shift.

Now we introduce the stationary random field \((X_m)_{m\in \mathbb{Z}^2}\) in the following way. For a real-valued measurable function \(f\) on \(R^{\mathbb{Z}^2}\), we define
\[
X_{j,k} = f(T^jS^k(\xi_{a,b}))_{a\leq 0, b\leq 0}).
\] (4)

The variable \(X_{0,0}\) will be assumed to be square integrable (in \(L^2\)) and with mean 0. We notice that the variables \((X_{n,m})_{n,m\in \mathbb{Z}}\) are adapted to the filtration \((\mathcal{F}_{n,m})_{n,m\in \mathbb{Z}}\).

Let \(\phi : [0, \infty) \rightarrow [0, \infty)\) be a Young function, that is, a convex, even function satisfying
\[
\lim_{x \rightarrow 0} \frac{\phi(x)}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\phi(x)}{x} = \infty.
\]

We shall define the Luxemburg norm associated with \(\phi\) which will be needed in the sequel. For any measurable function \(f\) from \(\Omega\) to \(\mathbb{R}\), the Luxemburg norm of \(f\) is define by
\[
||f||_\phi = \inf\{k \in (0, \infty) : E\phi(|f|/k) \leq 1\}.
\] (5)

In the sequel, we use the notations
\[
S_{k,j} = \sum_{u,v=1}^{k,j} X_{u,v}, \quad P^\omega(\cdot) = P(\cdot|\mathcal{F}_{0,0})(\omega) \text{ for any } \omega \in \Omega.
\]

Also, we shall denote by \(E^\omega\) the expectation corresponding to \(P^\omega\) and \(\Rightarrow\) the convergence in distribution.

For an integrable random variable \(X\), we introduce the projection operators defined by
\[
P_{0,0}(X) := (E_{0,0} - E_{-1,0})(X)
\]
\[
P_{0,0}(X) := (E_{0,0} - E_{0,-1})(X).
\]
Note that, by (3), we have
\[ P_0(X) := P_{0,0}P_{0,0}(X) = P_{0,0}P_{0,0}(X) = (E_{0,0} - E_{0,-1} - E_{-1,0} + E_{-1,-1})(X). \]

Then for \((u, v) \in \mathbb{Z}^2\), we can define the projections \(P_{u,v}\) as follows
\[ P_{u,v}(\cdot) := (E_{u,v} - E_{u,v-1} - E_{u-1,v} + E_{u-1,v-1})(\cdot) \]

We shall introduce the definition of an ortho-martingale, which will be referred to as a martingale with multiple indexes or simply martingale.

**Definition 2.1** Let \(d\) be a function and define
\[ D_{n,m} = d(\xi_{i,j}, i \leq n, j \leq m). \]

Assume integrability. We say that \((D_{n,m})_{n,m \in \mathbb{Z}}\) is a martingale differences field if \(E_{a,b}(D_{n,m}) = 0\) for either \(a < n\) or \(b < m\).

Set
\[ M_{k,j} = \sum_{u,v=1}^{k,j} D_{u,v}. \]

**Definition 2.2** We say that a random field \((X_{n,m})_{n,m \in \mathbb{Z}}\) defined by (4) admits a martingale approximation if there is a field of martingale differences \((D_{n,m})_{n,m \in \mathbb{Z}}\) defined by (6) such that
\[ \lim_{n \wedge m \to \infty} \frac{1}{nm} E^{\omega}(S_{n,m} - M_{n,m})^2 = 0 \text{ for almost all } \omega \in \Omega. \]

Our main result is the following theorem, which is an extension of the quenched CLT for ortho-martingales in Peligrad and Volný (2018) to stationary random fields satisfying the generalized Hannan condition (1973).

**Theorem 2.3** Assume that \((X_{n,m})_{n,m \in \mathbb{Z}}\) is defined by (4) and the filtrations are commuting. Also assume that \(T\) (or \(S\)) is ergodic and in addition
\[ \sum_{u,v \geq 0} ||P_{0,0}(X_{u,v})||_2 < \infty. \]

Then, for almost all \(\omega \in \Omega,
\[ \frac{1}{n}(S_{n,n} - R_{n,n}) \Rightarrow N(0, \sigma^2) \text{ under } P^{\omega} \text{ when } n \to \infty, \]
where \(R_{n,n} = E_{n,0}(S_{n,n}) + E_{0,n}(S_{n,n}) - E_{0,0}(S_{n,n}).\)

It should be noted that, for a stationary orthomartingale, the existence of finite second moment is not enough for the validity of a quenched CLT when the summation in taken on rectangles (see Peligrad and Peligrad and Volný (2018)). In order to assure the validity of a martingale approximation with a suitable moment condition we shall reinforce condition (8) when dealing with indexes \(n\) and \(m\) which converge independently to infinite.

**Theorem 2.4** Assume now that (8) is reinforced to
\[ \sum_{u,v \geq 0} ||P_{0,0}(X_{u,v})||_\phi < \infty, \]
where \(\phi(x) = x^2 \log(1 + |x|)\) and \(|| \cdot ||_\phi\) is defined by (5). Then, for almost all \(\omega \in \Omega,
\[ \frac{1}{(nm)^{1/2}}(S_{n,m} - R_{n,m}) \Rightarrow N(0, \sigma^2) \text{ under } P^{\omega} \text{ when } n \wedge m \to \infty, \]
where \(R_{n,m} = E_{n,0}(S_{n,m}) + E_{0,m}(S_{n,m}) - E_{0,0}(S_{n,m}).\)
The random centering is not needed if we impose two regularity conditions.

**Corollary 2.5** Assume that the conditions of Theorem 2.4 hold. If

\[ E_{0,0} \left( \frac{E_{0,m}^2(S_{n,m})}{nm} \right) \to 0 \text{ a.s. and } \frac{E_{0,0} \left( E_{n,0}^2(S_{n,m}) \right)}{nm} \to 0 \text{ a.s. when } n \wedge m \to \infty, \]

then for almost all \( \omega \in \Omega \),

\[ \frac{1}{(nm)^{1/2}} S_{n,m} \Rightarrow N(0, \sigma^2) \text{ under } P^\omega \text{ when } n \wedge m \to \infty. \]  

(12)

If the conditions of Theorem 2.3 hold and (11) holds with \( m = n \), then for almost all \( \omega \in \Omega \),

\[ \frac{1}{n} S_{n,n} \Rightarrow N(0, \sigma^2) \text{ under } P^\omega \text{ when } n \to \infty. \]

(13)

For the sake of applications, we provide a sufficient condition which will take care of both (9) and also of regularity assumptions (11).

**Corollary 2.6** Assume that \((X_{n,m})_{n,m} \in \mathbb{Z}\) is defined by (4) and the filtrations are commuting. Also assume that \( T \) (or \( S \)) is ergodic and in addition for \( \delta \geq 0 \)

\[ \sum_{u,v \geq 1} \frac{\|E_{1,1}(X_{u,v})\|_{2+\delta}}{(uv)^{1/(2+\delta)}} < \infty. \]

(14)

(a) If \( \delta = 0 \), then the quenched convergence (13) holds.
(b) If \( \delta > 0 \), then the quenched convergence (13) holds.

**Remark 2.7** Assume that \((X_{n,m})_{n,m} \in \mathbb{Z}\) is defined by (4) and the filtrations are commuting. Also assume that \( T \) (or \( S \)) is ergodic, (9) holds for \( \phi(x) = x^2 \log(1 + |x|) \) and (14) holds for \( \delta = 0 \). Then, the quenched convergence (13) holds.

3 Proofs

Let us point out the main idea of the proof. Since Peligrad and Volný (2018) proved a quenched CLT for orthomartingales, the proof can be reduced to prove the existence of an almost sure orthomartingale approximation for the random field we consider. We start with the proof of Theorem 2.4 since the proof of Theorem 2.3 is similar with the exception that we use different ergodic theorems.

Let us denote by \( \hat{T} \) and \( \hat{S} \) the operators on \( L^2 \) defined by \( \hat{T} f = f \circ T \), \( \hat{S} f = f \circ S \).

**Proof of Theorem 2.4.**

Starting from condition (9), by triangle inequality we have that

\[ f_0 := \sum_{u,v \geq 0} |P_{0,0}(X_{u,v})| < \infty \text{ a.s.} \]  

(15)

and

\[ ||f_0||_\phi \leq \sum_{u,v \geq 0} ||P_{0,0}(X_{u,v})||_\phi < \infty, \]

which clearly implies that \( E(f_0^2 \log^+ |f_0|) < \infty \).
Note that by [15], \( \mathcal{P}_{1,1}(S_{n,m}) \) is convergent almost surely. Denote the pointwise limit by

\[
D_{1,1} = \lim_{n \wedge m \to \infty} \mathcal{P}_{1,1}(S_{n,m}) = \sum_{u,v \geq 1} \mathcal{P}_{1,1}(X_{u,v}).
\]

Meanwhile, by the triangle inequality and [9], we obtain

\[
\sup_{n,m \geq 1} |\mathcal{P}_{1,1}(S_{n,m})| \leq \sum_{u,v \geq 1} |\mathcal{P}_{1,1}(X_{u,v})|
\]

and

\[
E \left( \sum_{u,v \geq 1} |\mathcal{P}_{1,1}(X_{u,v})| \right)^2 \leq \left( \sum_{u,v \geq 1} \|\mathcal{P}_{1,1}(X_{u,v})\|_2^2 \right)^2 < \infty.
\]

Thus by the dominated convergence theorem, \( \mathcal{P}_{1,1}(S_{n,m}) \) converges to \( D_{1,1} \) a.s. and in \( L^2(P) \) as \( n \wedge m \to \infty. \)

Since \( E_{0,1}(\mathcal{P}_{1,1}(S_{n,m})) = 0 \) a.s. and \( E_{1,0}(\mathcal{P}_{1,1}(S_{n,m})) = 0 \) a.s., by defining for every \( i, j \in \mathbb{Z} \), \( D_{i,j} = \hat{T}^{i-1} \hat{S}^{j-1} D_{1,1} \), we conclude that \( (D_{i,j})_{i,j \in \mathbb{Z}} \) is a martingale differences field. By the expression of \( D_{1,1} \) above,

\[
D_{i,j} = \sum_{(u,v) \geq (i,j)} \mathcal{P}_{i,j}(X_{u,v}).
\]

Now we look at the decomposition of \( S_{n,m} \) (See Peligrad and Zhang (2018b) for details):

\[
S_{n,m} - R_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathcal{P}_{i,j}(\sum_{u=i}^{n} \sum_{v=j}^{m} X_{u,v})
\]

where

\[
R_{n,m} = E_{n,0}(S_{n,m}) + E_{0,m}(S_{n,m}) - E_{0,0}(S_{n,m}).
\]

Therefore

\[
\frac{S_{n,m} - R_{n,m} - M_{n,m}}{\sqrt{nm}} = \frac{1}{\sqrt{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \mathcal{P}_{i,j} \left( \sum_{u=i}^{n} \sum_{v=j}^{m} X_{u,v} \right) - D_{i,j} \right).
\]

By the orthogonality of the martingale differences field \( (\mathcal{P}_{i,j} - D_{i,j})_{i,j \in \mathbb{Z}} \) and the assumption that the filtration is commuting, we have

\[
\frac{1}{nm} E_{0,0} \left( S_{n,m} - R_{n,m} - M_{n,m} \right)^2 = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} E_{0,0} \left( \mathcal{P}_{i,j} \left( \sum_{u=i}^{n} \sum_{v=j}^{m} X_{u,v} \right) - D_{i,j} \right)^2.
\]

From the main results in Peligrad and Volný (2018), we know that the quenched CLT holds for \( M_{n,m}/\sqrt{nm} \). Therefore by Theorem 25.4 in Billingsley (1995), in order to prove the conclusion of this theorem, it is enough to show that

\[
\lim_{n \wedge m \to \infty} \frac{1}{nm} E_{0,0} \left( S_{n,m} - R_{n,m} - M_{n,m} \right)^2 = 0.
\]

Define the operators

\[
Q_1(f) = E_{0,\infty}(\hat{T} f); \quad Q_2(f) = E_{\infty,0}(\hat{S} f).
\]

Then we can write

\[
E_{0,0} \left( \mathcal{P}_{i,j}(X_{u,v}) \right)^2 = Q_1^i Q_2^j \left( \mathcal{P}_{0,0}(X_{u-i,v-j}) \right)^2.
\]
By simple algebra we obtain

\[ E_{0,0} \left( \mathcal{P}_{i,j} \left( \sum_{u=i}^{n} \sum_{v=j}^{m} X_{u,v} - D_{i,j} \right) \right)^2 \]

\[ = E_{0,0} \left( \sum_{u=n+1}^{\infty} \sum_{v=j}^{m} \mathcal{P}_{i,j}(X_{u,v}) + \sum_{u=i}^{n} \sum_{v=m+1}^{\infty} \mathcal{P}_{i,j}(X_{u,v}) \right)^2. \]

Therefore, by elementary inequalities we have the following bound

\[ \frac{1}{nm} E_{0,0} \left( S_{n,m} - R_{n,m} - M_{n,m} \right)^2 = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} E_{0,0} \left( \mathcal{P}_{i,j} \left( \sum_{u=i}^{n} \sum_{v=j}^{m} X_{u,v} - D_{i,j} \right) \right)^2 \leq 2(I_{n,m} + II_{n,m}), \]

where we have used the notations

\[ I_{n,m} = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} Q_{1}^{i} Q_{2}^{j} \left( \sum_{u=n+1-i}^{\infty} \sum_{v=0}^{\infty} \left| \mathcal{P}_{0,0}(X_{u,v}) \right| \right)^2 \]

and

\[ II_{n,m} = \frac{1}{nm} \sum_{i=n-c+1}^{n} \sum_{j=1}^{m} Q_{1}^{i} Q_{2}^{j} \left( \sum_{u=n+1-i}^{\infty} \sum_{v=m+1-j}^{\infty} \left| \mathcal{P}_{0,0}(X_{u,v}) \right| \right)^2. \]

The task is now to show the almost sure negligibility of each term. By symmetry we treat only one of them.

Let \( c \) be a fixed integer satisfying \( c < n \). We decompose \( I_{n,m} \) into two parts

\[ \frac{1}{nm} \sum_{i=1}^{n-c} \sum_{j=1}^{m} Q_{1}^{i} Q_{2}^{j} \left( \sum_{u=n+1-i}^{\infty} \sum_{v=0}^{\infty} \left| \mathcal{P}_{0,0}(X_{u,v}) \right| \right)^2 := A_{n,m}(c) \quad (17) \]

and

\[ \frac{1}{nm} \sum_{i=n-c+1}^{n} \sum_{j=1}^{m} Q_{1}^{i} Q_{2}^{j} \left( \sum_{u=n+1-i}^{\infty} \sum_{v=m+1-j}^{\infty} \left| \mathcal{P}_{0,0}(X_{u,v}) \right| \right)^2 := B_{n,m}(c). \quad (18) \]

Note that

\[ B_{n,m}(c) \leq \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} Q_{1}^{i} Q_{2}^{j} f_{0}^{2} \]

\[ = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} Q_{1}^{i} Q_{2}^{j} f_{0}^{2} - \frac{1}{nm} \sum_{i=1}^{n-c} \sum_{j=1}^{m} Q_{1}^{i} Q_{2}^{j} f_{0}^{2}, \]

where \( f_{0} \) is given by [15].

By the ergodic theorem for Dunford-Schwartz operators (Krengel (1985), Theorem 1.1, ch.6), for each \( c \) fixed

\[ \lim_{n \land m \to \infty} \frac{1}{nm} \sum_{i=1}^{n-c} \sum_{j=1}^{m} Q_{1}^{i} Q_{2}^{j} f_{0}^{2} = E(f_{0}^{2}) \quad \text{a.s.} \quad (19) \]

Therefore, for all \( c > 0 \)

\[ \lim_{n \land m \to \infty} B_{n,m}(c) = 0 \quad \text{a.s.} \]
In order to treat the first term in the decomposition of $I_{n,m}$, note that
\[
A_{n,m}(c) \leq \frac{1}{nm} \sum_{i=1}^{n-c} \sum_{j=1}^{m} Q_i^j f_0(c) \quad \text{where} \quad f_0(c) = \sum_{u=c}^{\infty} \sum_{v=0}^{\infty} |P_{0,0}(X_{u,v})|.
\]

Again, by the ergodic theorem for Dunford-Schwartz operators (Krengel (1985), Theorem 1.1, Ch. 6), for each $c$ fixed
\[
\lim_{n \land m \to \infty} \frac{1}{nm} \sum_{i=1}^{n-c} \sum_{j=1}^{m} Q_i^j f_0(c) = E(f_0^2(c)) \quad \text{a.s.} \quad (20)
\]

In addition, by (15), we know that $\lim_{c \to \infty} |f_0(c)| = 0$. So, by the dominated convergence theorem, we have
\[
\lim_{c \to \infty} \lim_{n \land m \to \infty} A_{n,m}(c) \leq \lim_{c \to \infty} E(f_0^2(c)) = 0 \quad \text{a.s.}
\]

The proof of the theorem is now complete. ■

The proof of Theorem 2.3 requires only a slight modification of the proof of Theorem 2.4. Indeed instead of Theorem 1.1 in Ch. 6 in Krengel (1985), we shall use Theorem 2.8 in Ch. 6 in the same book.

**Proof of Corollary 2.5**

By Theorem 2.4 together with Theorem 25.4 in Billingsley (1995), it suffices to show that (11) implies that
\[
\lim_{n \land m \to \infty} \frac{1}{nm} E_{0,0}(R_{n,m}^2) = 0 \quad \text{a.s.} \quad (21)
\]

Simple computations involving the fact that the filtration is commuting gives that
\[
E_{0,0}(R_{n,m}^2) = E_{0,0} \left( E_{n,0}(S_{n,m}) \right) + E_{0,0} \left( E_{0,m}(S_{n,m}) \right) - E_{0,0}(S_{n,m}) \quad (22)
\]

and since
\[
E_{0,0}(S_{n,m}) \leq E_{0,0} \left( E_{0,m}(S_{n,m}) \right), \quad \text{we have} \quad \lim_{n \land m \to \infty} \frac{1}{nm} E_{0,0}(R_{n,m}^2) = 0 \quad \text{a.s.} \quad \text{by condition (11).}
\]

■

**Proof of Corollary 2.6**

Throughout the proof, denote by $C_5 > 0$ a generic constant depending on $\delta$ which may take different values from line to line.

Before we prove the corollary, we shall first establish a preparatory fact, namely that (14) implies
\[
\sum_{u \geq 1} \frac{1}{u^{1/(2+\delta)}} \sum_{v \geq 0} \|P_{0,0}(X_{u,v})\|_{2+\delta} < \infty. \quad (23)
\]

By the Hölder’s inequality and the Rosenthal inequality for martingales (see Theorem 6.1 in the Appendix), we have
\[
\sum_{v \geq 1} \|P_{0,0}(X_{u,v})\|_{2+\delta} = \sum_{v \geq 1} \|P_{u,v}(X_{0,0})\|_{2+\delta} \leq \sum_{n \geq 0} (2^n)^{\frac{1+\delta}{2+\delta}} \left( \sum_{v = 2^n}^{2^{n+1}-1} \|P_{u,v}(X_{0,0})\|_{2+\delta} \right)^{\frac{1}{2+\delta}}
\]
\[
\leq C_5 \sum_{n \geq 0} (2^n)^{\frac{1+\delta}{2+\delta}} \left( \sum_{v = 2^n}^{2^{n+1}-1} P_{u,v}(X_{0,0}) \right)_{2+\delta} \leq 2C_5 \sum_{n \geq 0} (2^n)^{\frac{1+\delta}{2+\delta}} \|E_{u,2^n}(X_{0,0})\|_{2+\delta}.
\]

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Since the sequence \( (\|E_{-u,-n}(X_{0,0})\|_2)_{n \geq 1} \) is non-increasing in \( n \), it follows that
\[
(2^n)^{1+\delta} \|E_{-u,-2^n}(X_{0,0})\|_2 \leq 2 \sum_{k=2^{n-1}}^{2^n-1} \frac{\|E_{-u,-k}(X_{0,0})\|_2}{k^{1/(2+\delta)}}.
\]
So
\[
\sum_{v=1}^{\infty} \|P_{0,0}(X_{u,v})\|_2 \leq C_\delta \sum_{k=1}^{\infty} \frac{\|E_{-u,-k}(X_{0,0})\|_2}{k^{1/(2+\delta)}}.
\]  
(24)
Thus relation (23) holds by (14), (24) and the stationarity.

In addition we also have for any \( u \geq 0 \)
\[
\sum_{v=1}^{\infty} \|P_{0,0}(X_{u,v})\|_2 < \infty.
\]  
(25)
By symmetric roles of \( m \) and \( n \), we have for any \( v \geq 0 \)
\[
\sum_{u=1}^{\infty} \|P_{0,0}(X_{u,v})\|_2 < \infty.
\]  
(26)

Now we will proceed to prove Corollary 2.6 in two steps:

- **Step 1.** Condition (14) implies
\[
\lim_{n \wedge m \to \infty} \frac{1}{nm} \frac{E_{0,0}(S_{n,m})}{n m} = 0 \text{ a.s.}
\]
First we show that (14) implies that
\[
\frac{E_{0,0}^2(S_{n,m})}{nm} \to 0 \text{ a.s. when } n \wedge m \to \infty.
\]
We bound this term in the following way
\[
\left|\frac{E_{0,0}(S_{n,m})}{\sqrt{nm}}\right| \leq \frac{1}{\sqrt{nm}} \sum_{u=1}^{n} \sum_{v=1}^{m} \left|E_{0,0}(X_{u,v})\right|
\]
\[
\leq \frac{1}{\sqrt{n}} \sum_{u=1}^{c} \sum_{v=1}^{\infty} \left|E_{0,0}(X_{u,v})\right| + \sum_{u=c+1}^{\infty} \sum_{v=1}^{\infty} \left|E_{0,0}(X_{u,v})\right| \frac{1}{\sqrt{uv}}
\]
\[
\leq \frac{c}{\sqrt{n}} \sup_{1 \leq u \leq c} \sum_{v=1}^{\infty} \left|E_{0,0}(X_{u,v})\right| + \sum_{u=c+1}^{\infty} \sum_{v=1}^{\infty} \left|E_{0,0}(X_{u,v})\right| \frac{1}{\sqrt{uv}}.
\]
Now, (14) implies that
\[
\sum_{u=1}^{\infty} \sum_{v=1}^{\infty} \left|E_{0,0}(X_{u,v})\right| \frac{1}{\sqrt{uv}} < \infty \text{ a.s.}
\]
Therefore,
\[
\frac{E_{0,0}(S_{n,m})}{\sqrt{nm}} \to 0 \text{ a.s.}
\]  
(27)
by letting \( n \to \infty \) followed by \( c \to \infty \).
By \((22)\) and the symmetric roles of \(m\) and \(n\), the corollary will follow if we can show that
\[
E_{0,0} \left( \frac{E_{0,m}(S_{n,m})}{nm} \right) \to 0 \text{ a.s. when } n \wedge m \to \infty.
\]

By \((27)\) this is equivalent to showing that
\[
\frac{1}{nm} E_{0,0} \left( E_{0,m}(S_{n,m}) - E_{0,0}(S_{n,m}) \right)^2 \to 0 \text{ a.s. when } n \wedge m \to \infty.
\]

We start from the representation
\[
E_{0,0} \left( E_{0,m}(S_{n,m}) - E_{0,0}(S_{n,m}) \right)^2 = \sum_{j=1}^{m} E_{0,0} \left[ P_{0,j} \left( \sum_{u=1}^{n} \sum_{v=j}^{m} X_{u,v} \right) \right]^2
\]
\[
= \sum_{j=1}^{m} E_{0,0} \left[ \hat{S}^j \left( \sum_{u=1}^{n} \sum_{v=0}^{m-j} P_{0,0}(X_{u,v}) \right) \right]^2.
\]

So,
\[
\frac{1}{nm} E_{0,0} \left( E_{0,m}(S_{n,m}) - E_{0,0}(S_{n,m}) \right)^2 \leq \frac{2}{mn} \sum_{j=1}^{m} E_{0,0} \left[ \hat{S}^j \left( \sum_{u=1}^{c} \sum_{v=0}^{m-j} |P_{0,0}(X_{u,v})| \right) \right]^2
\]
\[
+ \frac{2}{m} \sum_{j=1}^{m} E_{0,0} \left[ \hat{S}^j \left( \sum_{u=c+1}^{n} \sum_{v=0}^{m-j} |P_{0,0}(X_{u,v})| \right) \right]^2
\]
\[
= I_{n,m,c} + II_{n,m,c}.
\]

Let us introduce the operator
\[
Q_{0}(f) = E_{0,0}(\hat{S} f).
\]

We treat first the term \(I_{n,m,c}\). For \(c\) fixed
\[
I_{n,m,c} \leq \frac{2c^2}{mn} \sup_{1 \leq u \leq c} \sum_{j=1}^{m} E_{0,0} \left[ \hat{S}^j \left( \sum_{v=0}^{\infty} |P_{0,0}(X_{u,v})| \right) \right]^2
\]
\[
= \frac{2c^2}{mn} \sup_{1 \leq u \leq c} \sum_{j=1}^{m} Q_{0}^j \left[ \left( \sum_{v=0}^{\infty} |P_{0,0}(X_{u,v})| \right) \right]^2.
\]

By \((25)\), the function
\[
g(u) = \sum_{v=0}^{\infty} |P_{0,0}(X_{u,v})|
\]

is square integrable. By the ergodic theorem for Dunford-Schwartz operators (see Theorem 11.4 in Eisner et al., 2015 or Corollary 3.8 in Ch.3, Krengel, 1985)
\[
\frac{1}{m} \sum_{j=1}^{m} Q_{0}^j \left[ g^2(u) \right] \to E(g^2(u)) \text{ a.s.}
\]

and therefore, since \(c\) is fixed,
\[
\lim_{n \wedge m \to \infty} I_{n,m,c} = 0 \text{ a.s.}
\]
In order to treat the second term, note that

$$H_{n,m,c} \leq \frac{2}{m} \sum_{j=1}^{m} Q_{0}^j \left[ \left( \sum_{u=e}^{\infty} \frac{1}{\sqrt{u}} \sum_{v=0}^{\infty} \|P_{0,0}(X_{u,v})\| \right)^2 \right].$$

Denote

$$h(c) = \sum_{u=e}^{\infty} \frac{1}{\sqrt{u}} \sum_{v=0}^{\infty} \|P_{0,0}(X_{u,v})\|.$$ 

By [23], we know that

$$\sum_{u=1}^{\infty} \frac{1}{\sqrt{u}} \sum_{v=0}^{\infty} \|P_{0,0}(X_{u,v})\|_{2+\delta} < \infty. \quad (28)$$

So, \(E(h^2(c)) < \infty\). Again, by the ergodic theorem for the Dunford-Schwartz operators (see Theorem 11.4 in Eisner et al., 2015 or Corollary 3.8 in Ch.3, Krengel, 1985)

$$\frac{1}{m} \sum_{j=1}^{m} Q_{0}^j(h^2(c)) \to E(h^2(c)) \leq \left( \sum_{u=e}^{\infty} \frac{1}{\sqrt{u}} \sum_{v=0}^{\infty} \|P_{0,1}(X_{u,v})\|_2 \right)^2.$$ 

So, by (28)

$$\lim_{c \to \infty} \lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} Q_{0}^j(h^2(c)) = 0 \text{ a.s.}$$

- **Step 2.** Condition (14) implies

$$\sum_{u,v \geq 0} \|P_{0,0}(X_{u,v})\|_{2+\delta} < \infty, \quad (29)$$

which clearly implies (11).

In fact, by applying twice the Rosenthal inequality for martingales (see Theorem 6.1 in the Appendix), for any integers \(a \leq b\) and \(c \leq d\), we have

$$\sum_{k=a}^{b} \sum_{k'=c}^{d} \|P_{k,-k'}(X_{0,0})\|_{2+\delta}^{2+\delta} \leq C_{\delta} \left( \sum_{k=a}^{b} \sum_{k'=c}^{d} \|P_{k,-k'}(X_{0,0})\|_{2+\delta} \right)^{2+\delta}. \quad (30)$$

In addition, note that for any integers \(a \leq b\) and \(c \leq d\), we have

$$\|\sum_{k=a}^{b} \sum_{k'=c}^{d} P_{k,-k'}(X_{0,0})\|_{2+\delta}^{2+\delta} \leq 4^{2+\delta} \|E_{-u,-c}(X_{0,0})\|_{2+\delta}^{2+\delta}. \quad (31)$$

Then by the Hölder’s inequality together with (30) and (31), we obtain

$$\sum_{n,m \geq 0} \|P_{u,v}(X_{0,0})\|_{2+\delta} \leq \sum_{n,m \geq 0} (2^n 2^m)^{\frac{1}{2+\delta}} \left( \sum_{k=2^n}^{2^{n+1} - 1} \sum_{k'=2^m}^{2^{m+1} - 1} \|P_{k,-k'}(X_{0,0})\|_{2+\delta} \right)^{\frac{1}{2+\delta}}$$

$$\leq 4C_{\delta} \sum_{n,m \geq 0} (2^n 2^m)^{\frac{1}{2+\delta}} \|E_{2^n,-2^m}(X_{0,0})\|_{2+\delta}.$$ 

Since \(\|E_{-2^n,-2^m}(X_{0,0})\|\) is non-increasing in \(n\) and \(m\), it follows that

$$\left(2^n 2^m\right)^{\frac{1}{2+\delta}} \|E_{-2^n,-2^m}(X_{0,0})\|_{2+\delta} \leq 4 \sum_{u=2^{n-1}}^{2^n} \sum_{v=2^{m-1}}^{2^m} \frac{\|E_{u,v}(X_{0,0})\|_{2+\delta}}{(uv)^{1/(2+\delta)}}.$$
Therefore, by the relations above, we have proved that (13) implies
\[ \sum_{u,v \geq 1} \| P_{u,v}(X_{0,0}) \|_{2+\delta} < \infty. \]

Similarly we have
\[ \sum_{u=1}^{\infty} \| P_{u,0}(X_{0,0}) \|_{2+\delta} < \infty \text{ and } \sum_{v=1}^{\infty} \| P_{0,v}(X_{0,0}) \|_{2+\delta} < \infty. \]

Thus by stationarity (29) holds.

The proof of the corollary is now complete by a combination of Theorem 2.4 for \( \delta > 0 \) and Theorem 2.3 for \( \delta = 0 \) via Theorem 25.4 in Billingsley (1995).

**Proof of Remark 2.7**
The remark follows by the proof of Step 1 of Corollary 2.6 and Theorem 2.4 via Theorem 25.4 in Billingsley (1995).

### 4 Random fields with multi-dimensional index sets

In this section we extend our results to random fields indexed by \( Z^d, d > 2 \). By \( u \leq n \) we understand \( u = (u_1, ..., u_d) \), \( n = (n_1, ..., n_d) \) and \( 1 \leq u_1 \leq n_1, ..., 1 \leq u_d \leq n_d \). We shall start with a strictly stationary real-valued random field \( \xi = (\xi_u)_{u \in Z^d} \), defined on the canonical probability space \( R^{2d} \) and define the filtrations \( F_u = \sigma(\xi_j : j \leq u) \). We shall assume that the filtration is commuting if \( E_u E_a(X) = E_{u\wedge a}(X) \), where the minimum is taken coordinate-wise and we used notation \( E_u(X) = E(X|F_u) \). We define
\[ X_m = f((\xi_j)_{j \leq m}) \text{ and set } S_k = \sum_{u=1}^{k} X_u. \] (32)

The variable \( X_0 \) is assumed to be square integrable (in \( L^2 \)) and with mean 0. We also define \( T_i \) the coordinate-wise translations and then
\[ X_k = f(T_{i}^{1} \circ ... \circ T_{d}^{k}(\xi_u)_{u \leq 0}). \]

Let \( d \) be a function and define
\[ D_m = d((\xi_j)_{j \leq m}) \text{ and set } M_k = \sum_{u=1}^{k} D_u. \] (33)

Assume integrability. We say that \( (D_m)_{m \in Z^d} \) is a martingale differences field if \( E_u(D_m) = 0 \) if at least one coordinate of \( a \) is strictly smaller than the corresponding coordinate of \( m \). Now we introduce the \( d \)-dimensional projection operator. By using the fact that the filtration is commuting, it is convenient to define projections \( P_u \) in the following way
\[ P_u(X) := P_{u(1)} \circ P_{u(2)} \circ ... \circ P_{u(d)}(X), \]
where
\[ P_{u(j)}(Y) := E(Y|F_u) - E(Y|F_{u(j)}). \] (34)

where \( u(j) \) has all the coordinates of \( u \) with the exception of the \( j \)-th coordinate, which is \( u_j - 1 \). For instance when \( d = 3 \), \( P_{u(2)}(Y) = E(Y|F_{u_1,u_2,u_3}) - E(Y|F_{u_1,u_2,u_3-1}). \)

We say that a random field \( (X_n)_{n \in Z^d} \) admits a martingale approximation if there is a field of martingale differences \( (D_m)_{m \in Z^d} \) such that for almost all \( \omega \in \Omega \)
\[ \frac{1}{|n|} E^\omega (S_n - M_n)^2 \to 0 \text{ when } \min_{1 \leq i \leq d} n_i \to \infty, \] (35)
where $|\mathbf{n}| = n_1 \ldots n_d$.

Let $R_n$ be the remainder term of the decomposition of $S_n$ such that

$$S_n = \sum_{u=1}^{n} P_u(S_n) + R_n.$$ 

In this context we have:

**Theorem 4.1** Assume that $(X_n)_{n \in \mathbb{Z}^d}$ is defined by (32) and there is an integer $i$, $1 \leq i \leq d$, such that $T_i$ is ergodic and the filtrations are commuting. In addition assume that

$$\sum_{u \geq 0} ||P_0(X_u)||_2 < \infty. \quad (36)$$

Then, for almost all $\omega \in \Omega$,

$$(S_{n,\ldots,n} - R_{n,\ldots,n})/n^{d/2} \Rightarrow N(0, \sigma^2) \text{ under } P^\omega \text{ when } n \to \infty.$$ 

**Theorem 4.2** Furthermore, assume now condition (36) is reinforced to

$$\sum_{u \geq 0} ||P_0(X_u)||_\varphi < \infty, \quad (37)$$

where $\varphi(x) = x^2 \log^{d-1}(1 + |x|)$ and $|| \cdot ||_\varphi$ is defined by (3).

Then, for almost all $\omega \in \Omega$,

$$\frac{1}{\sqrt{|\mathbf{n}|}}(S_n - R_n) \Rightarrow N(0, \sigma^2) \text{ under } P^\omega \text{ when } \min_{1 \leq i \leq d} n_i \to \infty.$$ 

**Corollary 4.3** Assume that the conditions of Theorem 4.2 hold and for all $j$, $1 \leq j \leq d$ we have

$$\frac{1}{|\mathbf{n}|}E_{\mathbf{n}_j}(E^2_{\mathbf{n}_j}(S_n)) \to 0 \text{ a.s. when } \min_{1 \leq i \leq d} n_i \to \infty. \quad (38)$$

where $\mathbf{n}_j \in \mathbb{Z}^d$ has the $j$-th coordinate 0 and the other coordinates equal to the coordinates of $\mathbf{n}$. Then, for almost all $\omega \in \Omega$,

$$S_n/\sqrt{|\mathbf{n}|} \Rightarrow N(0, \sigma^2) \text{ under } P^\omega \text{ when } \min_{1 \leq i \leq d} n_i \to \infty. \quad (39)$$

If the conditions of Theorem 4.1 hold and (38) holds with $\mathbf{n} = (n, n, \ldots, n)$, then for almost all $\omega \in \Omega$,

$$\frac{1}{n^{d/2}}S_{n,\ldots,n} \Rightarrow N(0, \sigma^2) \text{ under } P^\omega \text{ when } n \to \infty. \quad (40)$$

**Corollary 4.4** Assume that $(X_n)_{n \in \mathbb{Z}^d}$ is defined by (32) and the filtrations are commuting. Also assume that there is an integer $i$, $1 \leq i \leq d$, such that $T_i$ is ergodic and in addition for $\delta > 0$,

$$\sum_{u \geq 1} \frac{||E_1(X_u)||_{2+\delta}}{|u|^{1/(2+\delta)}} < \infty. \quad (41)$$

(a) If $\delta = 0$, then the quenched CLT (41) holds.

(b) If $\delta > 0$, then the quenched convergence (32) holds.
As for the case of random fields with two indexes, we start with the proof of Theorem 4.2 since the proof of Theorem 4.1 is similar with the exception that we use different ergodic theorems.

Proof of Theorem 4.2.
The proof of this theorem is straightforward following the same lines of proofs as for a double-indexed random field. It is easy to see that, by using the commutativity property of the filtration, the martingale approximation argument in the proof of Theorem 2.4 remains unchanged if we replace $Z^2$ with $Z^d$ for $d \geq 3$. The definition of the approximating martingale is also clear. The only difference in the proof is that for the validation of the limit in (19) and (20) when $\min_{1 \leq i \leq d} n_i \to \infty$, in order to apply the ergodic theorem for Dunford-Schwartz operators, conform to Theorem 1.1 in Ch. 6 in Krengel (1985), we have to assume $E \left[ f_0^2 \log^{-1}(1 + |f_0|) \right] < \infty$, which is implied by (37).

More precisely, let us denote by $\tilde{T}_i$, $1 \leq i \leq d$, the operators defined by $\tilde{T}_i f = f \circ T_i$. Then for $i = (i_1, \cdots, i_d) \in Z^d$, we define $Q^i = \prod_{k=1}^d Q_{i_k}^k$ where $(Q_i)_{1 \leq i \leq d}$ are operators associated with coordinate-wise translations $(T_i)_{1 \leq i \leq d}$ defined as follows

$$Q_1(f) = E_{0, \infty, \cdots, \infty}(\tilde{T}_1 f), \ Q_2(f) = E_{\infty, 0, \infty, \cdots, \infty}(\tilde{T}_2 f), \cdots, Q_d(f) = E_{\infty, \cdots, \infty, 0}(\tilde{T}_d f).$$

Then bound the following quantity

$$\frac{1}{|n|} E_0 \left[ |S_n - R_n - M_n|^2 \right]$$

by the sum of $d$ terms with the first term of them in the form

$$I_n = \frac{1}{|n|} \sum_{i=1}^n Q^i \left( \sum_{u=n_1+1-i_1}^\infty \sum_{v \geq 0} |P_0(X_{u,v})| \right)^2 \text{ where } v \in Z^{d-1}.$$

By symmetry, we only need to deal with this one. Let $c$ be a fixed integer satisfying $c < n_1$, we decompose $I_n$ into two parts:

$$A_n(c) := \frac{1}{|n|} \sum_{i_1=1}^{n_1-c} \sum_{i'_{1}=1}^{n'} Q^i \left( \sum_{u=n_1+1-i_1}^\infty \sum_{v \geq 0} |P_0(X_{u,v})| \right)^2$$

and

$$B_n(c) := \frac{1}{|n|} \sum_{i_1=1}^{n_1} \sum_{i'_{1}=1}^{n'} Q^i \left( \sum_{u=n_1+1-i_1}^\infty \sum_{v \geq 0} |P_0(X_{u,v})| \right)^2$$

with $i' = (i_2, \cdots, i_d)$ and $n' = (n_2, \cdots, n_d)$. Afterwards, we just proceed by following step by step the proof for negligibility of $A_{n,m}(c)$ and $B_{n,m}$ (see (17) and (18) from the proof of Theorem 2.4).

The proof of Theorem 4.3 follows by just replacing Theorem 1.1 in Ch. 6 in Krengel (1985) in the proof by Theorem 2.8 in Ch. 6 in the same book.

Proof of Corollary 4.3. The negligibility of the reminder $R_n$ can be shown exactly the same way as the negligibility of the term $R_{n,m}$ in the proof of Corollary 2.5.

Proof of Corollary 4.4. By the same way we proved (43) and (44) in Corollary 2.6, we can show that (41) implies the following facts:

$$\sum_{u \geq 1} \frac{1}{\sqrt{|u|}} \sum_{v \geq 0} \|P_0_{(d)}(X_{u,v})\|_{2+\delta} < \infty,$$

(42)
\[ \sum_{v \geq 0} \| P_0(X_{u,v}) \|_{2+\delta} < \infty \]  

(43)

and

\[ \sum_{u \geq 1} \frac{1}{\sqrt{u}} \sum_{v \geq 0} \| P_0(X_{u,v}) \|_{2+\delta} < \infty, \]

(44)

where \( 0 = (0, \cdots, 0) \in \mathbb{Z}^d, u, v \in \mathbb{Z}^{d-1} \) and \( P_0 = P_0(2) \circ P_0(3) \circ \cdots \circ P_0(d) \) with \( P_0(j) \) defined by (41).

To prove the corollary, we need to show that

\[ \frac{1}{|n|} E_0 \left( E_{n,0}^{2}(S_n) \right) \rightarrow 0 \text{ a.s. when } \min_{1 \leq i \leq d} n_i \rightarrow \infty, \]

(45)

where \( n^{(k)} \in \mathbb{Z}^d \) has \( k \) coordinates equal to the corresponding coordinates of \( n \) and the other \( n - k \) coordinates zero for all \( 0 \leq k \leq d - 1 \). We will proceed by induction.

First, we shall show that

\[ \frac{1}{|n|} E_0 \left( E_{n,0}^{2}(S_n) \right) \rightarrow 0 \text{ a.s. and } \frac{1}{|n|} E_0 \left( E_{n,0,0,\cdots,0}^{2}(S_n) \right) \rightarrow 0 \text{ a.s. when } \min_{1 \leq i \leq d} n_i \rightarrow \infty, \]

(46)

which are easy to establish by similar arguments in the proof of Corollary 2.6 using (41) and (42).

That is, (45) holds for \( k = 0 \) and \( k = 1 \). Now assume for \( k < d - 1 \) the result holds. We need to show the result for \( k = d - 1 \) which follows straightforward using (43) and (44). The proof of this corollary is complete now. \( \square \)

5 Examples

We shall give examples providing new results for linear and Volterra random fields. Let \( d \) be an integer greater than 1 and \( \delta > 0 \). Throughout this section, as before by \( C_\delta > 0 \), we denote a generic constant depending on \( \delta \) which may be different from line to line.

Example 5.1 (Linear field) Let \( (\xi_n)_{n \in \mathbb{Z}^d} \) be a random field of independent, identically distributed random variables which are centered and \( E \left( |\xi_0|^{2+\delta} \right) < \infty \). For \( k \geq 0 \) define

\[ X_k = \sum_{j \geq 0} a_j \xi_{k-j}. \]

Assume that

\[ \sum_{k \geq 1} \frac{1}{|k|^{1/(2+\delta)}} \left( \sum_{j \geq k-1} a_j^2 \right)^{\frac{1}{2}} < \infty. \]  

(46)

Then the results of Corollary 4.4 hold.

Proof. Since

\[ E_1(X_k) = \sum_{j \geq k-1} a_j \xi_{k-j}, \]

by the independence of \( \xi_n \) and the Rosenthal inequality (see Theorem 6.1 given in the Appendix), we obtain
\[ \|E_1(X_k)\|_{2+\delta}^{2+\delta} = \left\| \sum_{j \geq k-1} a_j \xi_{k-j} \right\|_{2+\delta}^{2+\delta} \leq C_\delta \left[ \left( \sum_{j \geq k-1} a_j^2 E(\xi_{k-j}^2) \right)^{\frac{2+\delta}{2}} + \sum_{j \geq k-1} |a_j|^2 \right] \]

where \( \delta \) is a small constant. By the monotonicity of norms in \( \ell_p \), we have

\[ \left( \sum_{j \geq k-1} |a_j|^{2+\delta} \right)^{\frac{1}{2+\delta}} \leq \left( \sum_{j \geq k-1} a_j^2 \right)^{\frac{1}{2}}, \]

therefore

\[ \|E_1(X_k)\|_{2+\delta} \leq C_\delta \left( \sum_{j \geq k-1} a_j^2 \right)^{\frac{1}{2}}. \]

So condition (41) is implied by (40). Whence the results of Corollary 4.4 holds. \( \blacksquare \)

**Example 5.2 (Volterra field)** Let \((\xi_n)_{n \in \mathbb{Z}^d}\) be a random field of independent random variables identically distributed centered and \(E(|\xi_0|^{2+\delta}) < \infty \). For \( k \geq 0 \), define

\[ X_k = \sum_{(u,v) \geq (0,0)} a_{u,v} \xi_{k-u} \xi_{k-v}. \]

where \( a_{u,v} \) are real coefficients with \( a_{u,u} = 0 \) and \( \sum_{u,v \geq 0} a_{u,v}^2 < \infty \). In addition, assume that

\[ \sum_{k \geq 1} \frac{1}{|k|^{1/(2+\delta)}} \left( \sum_{(u,v) \geq (k-1,k-1)} a_{u,v}^2 \right)^{1/2} < \infty. \]  

(47)

Then the results of Corollary 4.4 hold.

**Proof.** Note that

\[ E_1(X_k) = \sum_{(u,v) \geq (k-1,k-1)} a_{u,v} \xi_{k-u} \xi_{k-v}. \]

Let \((\xi'_n)_{n \in \mathbb{Z}^d}\) and \((\xi''_n)_{n \in \mathbb{Z}^d}\) be two independent copies of \((\xi_n)_{n \in \mathbb{Z}^d}\). By independence and \( a_{k,k} = 0 \), applying the decoupling inequality together with the Rosenthal inequality, both of which are given for convenience in the Appendix, (see Theorem 6.2 and Theorem 6.1 from the Appendix), we obtain

\[ \|E_1(X_k)\|_{2+\delta}^{2+\delta} = \left\| \sum_{(u,v) \geq (k-1,k-1)} a_{u,v} \xi_{k-u} \xi_{k-v} \right\|_{2+\delta}^{2+\delta} \leq C_2 \left( \sum_{(u,v) \geq (k-1,k-1)} a_{u,v}^2 \xi'_{k-u} \xi''_{k-v} \right)^{\frac{2+\delta}{2}} \]

\begin{align*}
&\leq C_\delta \left[ \left( \sum_{(u,v) \geq (k-1,k-1)} a_{u,v}^2 E(\xi_{k-u}^2) \xi''_{k-v}^2 \right)^{\frac{2+\delta}{2}} + \sum_{(u,v) \geq (k-1,k-1)} |a_{u,v}|^{2+\delta} E\left(|\xi'_u \xi''_v|^{2+\delta}\right) \right] \\
&\leq C_\delta \left[ \left( \sum_{(u,v) \geq (k-1,k-1)} a_{u,v}^2 E(\xi_{k-u}^2 \xi''_{k-v}^2) \right)^{\frac{2+\delta}{2}} + \sum_{(u,v) \geq (k-1,k-1)} |a_{u,v}|^{2+\delta} E\left(|\xi_0|^{2+\delta}\right)^2 \right]
\end{align*}
Above, the first inequality holds by Theorem 6.2 while the second one is implied by Theorem 6.1.
Again by the monotonicity of norms in $\ell_p$, we have

$$\|E_1(X_k)\|_{2,5} \leq C_6\left(\sum_{u,v \geq k-1} a_{u,v}^2\right)^{\frac{1}{2}}.$$  

Thus the results of Corollary 4.4 hold.

\[\blacksquare\]

6 Appendix

For convenience, we mention one classical inequality for martingales, see Theorem 2.11, p. 23, Hall and Heyde (1980) and also Theorem 6.6.7 ch. 6, p. 322, de la Peña and Giné (1999).

**Theorem 6.1 (Rosenthal’s Inequality)** Let $p \geq 2$. Let $M_n = \sum_{k=1}^{n} X_k$ where $\{M_n, \mathcal{F}_n\}$ is a martingale with martingale differences $X_i$, then there are constants $0 < c_p, C_p < \infty$ such that

$$c_p \left\{ \sum_{k=1}^{n} E|X_k|^p \right\} + E\left[ \left( \sum_{k=1}^{n} E(X_k^2 | \mathcal{F}_{k-1}) \right)^{p/2} \right] \leq \|M_n\|_p \leq C_p \left\{ E\left[ \left( \sum_{k=1}^{n} E(X_k^2 | \mathcal{F}_{k-1}) \right)^{p/2} \right] + \sum_{k=1}^{n} E|X_k|^p \right\}.$$

The following is a decoupling result for U-statistics, which can be found on p. 99, Theorem 3.1.1, de la Peña and Giné (1999).

**Theorem 6.2 (Decoupling inequality)** Let $(X_i)_{1 \leq i \leq n}$ be $n$ independent random variables and let $(X_i^k)_{1 \leq i \leq n}$, $k = 1, \ldots, m$, be $m$ independent copies of this sequences. For each $(i_1, i_2, \ldots, i_m) \in I_n^m$, let $h_{i_1, \ldots, i_m} : R^m \to R$ be a measurable function such $E|h_{i_1, \ldots, i_m}(X_{i_1}, \ldots, X_{i_m})| \leq \infty$. Let $f : [0, \infty) \to [0, \infty)$ be a convex non-decreasing function such that $Ef(|h_{i_1, \ldots, i_m}(X_{i_1}, \ldots, X_{i_m})|) < \infty$ for all $(i_1, i_2, \ldots, i_m) \in I_n^m$ where $I_n^m = \{(i_1, \ldots, i_m) : i_j \in \mathbb{N}, 1 \leq i_j \leq n, i_j \neq i_k, j \neq k\}$. Then there exists $C_m > 0$ such that

$$Ef\left( \sum_{I_n^m} |h_{i_1, \ldots, i_m}(X_{i_1}, \ldots, X_{i_m})| \right) \leq C_mEf\left( \sum_{I_n^m} |h_{i_1, \ldots, i_m}(X_{i_1}^1, \ldots, X_{i_m}^m)| \right).$$

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