Motivic secondary characteristic classes and the Euler class

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Abstract

We discuss secondary characteristic classes in motivic cohomology for algebraic vector bundles with trivial top Chern class. We then show that vanishing of the Chow-Witt theoretic Euler class implies not only vanishing of the top Chern class, but also vanishing of certain secondary characteristic classes in motivic cohomology.

Contents

1 Introduction 1
2 Secondary characteristic classes in motivic cohomology 3
3 Milnor K-cohomology and motivic cohomology 8
4 Connecting homomorphisms as cohomological operations 10
5 The Euler class and motivic secondary classes 18

1 Introduction

Suppose $k$ is a field having characteristic unequal to 2, $A$ is a $d$-dimensional smooth affine $k$-algebra and $P$ is a projective module of rank $r$ over $A$. There is a well-defined primary obstruction to $P$ splitting off a free rank 1 summand given by “the” Euler class $e(P)$ of $P$ (see [Mor12, Theorem 8.2], [Fas08, Chapitre 13] and [AF13], which shows the two definitions coincide for oriented vector bundles). When $r = d$, Morel shows that this primary obstruction is the only obstruction to splitting off a free rank 1 summand.

Because the Euler class is defined using Chow-Witt theory, which is not part of an oriented cohomology theory (say in the sense of [LM07]), it is difficult to compute in general. The vanishing of the Euler class implies vanishing of $c_r(P)$ in $CH^r(\text{Spec } A)$ [AF12c, Proposition 5.2], though the

*Aravind Asok was partially supported by National Science Foundation Awards DMS-0966589 and DMS-1254892.
converse is not true in general (see, e.g., Example 2.9). It is therefore natural to try to approximate $e(P)$ using structures defined only in terms of oriented cohomology theories.

The Euler class is defined for vector bundles $E$ on arbitrary smooth schemes $X$. In this generality, vanishing of the Euler class of $E$ still implies vanishing of the top Chern class of $E$, but the link with splitting off a free summand of rank 1 is lost. When the top Chern class of $E$ vanishes, it is possible to define secondary characteristic classes associated with the action of the motivic Steenrod algebra on the complement of the zero section $E^c$ of $E$. The main result of this work can be stated as follows.

**Theorem 1** (See Theorems 4.12 and 5.3). If $X$ is a smooth $d$-fold and $\xi : E \to X$ is a rank $r$ vector bundle on $X$ with $e(\xi) = 0$, then $c_r(E) = 0$, and a secondary characteristic class in $H^{2r+1,r+1}(X)/(Sq^2 + c_1(E) \cup)H^{2r-1,r}(X,\mathbb{Z}/2)$ is defined and trivial.

**Remark 2.** Another more conceptual reason for the difference between the top Chern class and the Euler class is as follows. While the top Chern class only depends on the stable isomorphism class of $E$, the secondary characteristic class we discuss above is a not an invariant of the stable isomorphism class of $E$. Indeed, the bundle $E \oplus O_X$ has a nowhere vanishing section, and Proposition 2.5 implies all its (motivic) secondary characteristic classes are defined and trivial. See Remark 2.6 for related discussion along these lines.

**Remark 3.** The secondary class described in the previous theorem lives in a coset that is formally similar to that housing the secondary obstruction to existence of a nowhere vanishing section of a topological vector bundle (see, e.g., [Lia54, §IV] or [PS62, Proposition 4.4]). However, this topological secondary obstruction manifests itself above as part of the primary obstruction to existence of a nowhere vanishing section. The topological secondary obstruction also appears as a “true” secondary obstruction (see, e.g., [AF12c, §5] for a discussion in low dimensions).

Bhatwadekar and Sridharan ask whether the only obstruction to splitting a free rank 1 summand off a rank $(2n+1)$ vector bundle $E$ on a smooth affine $(2n+1)$-fold $X = \text{Spec} \, A$ is vanishing of a variant of the top Chern class living in a group $E_0(A)$ [BS00, Question 7.12]. The group $E_0(A)$ housing their obstruction class is isomorphic to the Chow group of 0-cycles on $\text{Spec} \, A$ in some cases; see, e.g., [BS99, Remark 3.13 and Theorem 5.5]. It is an open problem whether the group $E_0(A)$ is isomorphic to the Chow group of zero cycles in general. A natural byproduct of their question is whether (or, perhaps, when) vanishing of the top Chern class is sufficient to guarantee that $E$ splits off a free rank 1 summand. Using a vanishing theorem for $\mathcal{H}$-cohomology from [AF12b, Proposition 5.6], we can deduce the following result.

**Corollary 4.** Suppose $X$ is a smooth affine $d$-fold over a field $k$ having cohomological dimension $r$ and $E$ is a rank $d$ vector bundle on $X$. Assume that $0 = c_d(E) \in CH^d(X)$.

i) If $r \leq 1$, then $E$ splits off a free rank 1 summand.

ii) If $r \leq 2$, then $E$ splits off a free rank 1 summand if and only if $0 = c_d(E) \in CH^d(X)$ and a secondary characteristic class in $H^{2d+1,d+1}(X)/(Sq^2 + c_1(E) \cup)H^{2d-1,d}(X,\mathbb{Z}/2)$ is trivial.

**Remark 5.** One way to guarantee that the secondary characteristic class in question is always trivial is to establish vanishing of $H^{2d+1,d+1}(X,\mathbb{Z}/2)$ for $X$ a smooth affine scheme. However, there is no general result (or, to the best of our knowledge, even a conjecture) in motivic cohomology that
suggests this for vanishing for smooth affine varieties over fields having cohomological dimension \( \geq 2 \). Thus, if one hopes to construct a counterexample to the conjecture of Bhatwadekar-Sridharan, it seems natural to consider varieties over fields having cohomological dimension \( \geq 2 \).

Remark 6. Over fields of cohomological dimension \( \geq 3 \), the gap between triviality of the \( r \)th Chern class in Chow theory and triviality of \( e(P) \) widens. In these cases, vanishing of the Euler class will imply vanishing of tertiary and higher order characteristic classes. Keeping track of the indeterminacy in these cases becomes even more cumbersome. Moreover, for varieties over fields having finite cohomological dimension, a more careful analysis should show that vanishing of the Euler class can be characterized in terms of vanishing of a finite number of higher order characteristic classes in mod 2 motivic cohomology.

**Preliminaries/Overview of Sections**

Whenever Witt theory is invoked, we assume \( k \) is a field having characteristic unequal to 2. When mentioning motivic cohomology, we will assume \( k \) is perfect. Thus, for simplicity, the reader can assume that \( k \) is perfect and has characteristic unequal to 2 throughout the paper. The proof of Theorem 4.12 in positive characteristic depends on the main result of the preprint \([HKO13]\), which, at the time of writing, depends on several other pieces of work that are still only available in preprint form. We refer the reader to \([Fas08]\) for results regarding Chow-Witt theory, \([MVW06]\) for general properties of motivic cohomology, and \([MV99]\) for results about \( \mathbb{A}^1 \)-homotopy theory.

Section 2 contains a discussion of secondary characteristic classes in motivic cohomology associated with vector bundles with vanishing top Chern class. Section 3 shows how to use the Milnor conjecture (resp. Bloch-Kato conjecture) to identify certain Milnor \( K \)-cohomology groups with motivic cohomology groups. We use these identifications to transport operations in mod \( p \) motivic cohomology to Milnor \( K \)-cohomology. Section 4 uses the results of Section 3 to identify a homomorphism defined using the connecting homomorphism in \( I^j \)-cohomology with a (twisted) cohomology operation involving \( Sq^2 \). The bulk of the section is devoted to establishing non-triviality of this homomorphism and then the identification is made using Voevodsky’s classification of bi-stable operations in motivic cohomology. Finally, Section 5 puts everything together to establish the results stated above.

**Acknowledgements**

We thank Burt Totaro for a discussion related to the proof of Theorem 4.12.

## 2 Secondary characteristic classes in motivic cohomology

In this section, we define secondary characteristic classes for algebraic vector bundles with vanishing top Chern class following the cohomological method of Peterson-Stein (after Massey) \([PS62]\). In outline: suppose \( \mathcal{E} \) is a rank \( r \) vector bundle on a smooth scheme \( X \) with vanishing \( r \)th Chern class (in Chow groups). Write \( \mathcal{E}^\circ \) for the complement of the zero section of \( \mathcal{E} \). By the self-intersection formula, vanishing of the \( c_r(\mathcal{E}) \) implies that the Gysin long exact sequence associated with the inclusion \( \mathcal{E}^\circ \hookrightarrow \mathcal{E} \) splits into short exact sequences. By considering the action of the motivic Steenrod algebra on \( \mathcal{E}^\circ \), one can attach secondary characteristic classes to \( \mathcal{E} \). The situation might differ
somewhat from that in classical topology because we do not know whether all operations in motivic cohomology are bi-stable (this is certainly false in general, but unknown for operations in the range we consider; we include some comments about this later).

**Motivic cohomology of a sphere bundle**

Suppose $B$ is a smooth scheme and $E$ is a rank $n$ vector bundle on $B$. Write $E^\circ$ for the $\mathbb{A}^n \setminus \{0\}$ bundle obtained from $E$ by removing the zero section. Let $p : E^\circ \to B$ be the projection map. The open immersion $E^\circ \to E$ together with homotopy invariance yield a cofiber sequence of the form

$$E^\circ \xrightarrow{p} B \xrightarrow{\psi} Th(E) \xrightarrow{} \cdots,$$

which yields a long exact sequence in motivic cohomology (with arbitrary coefficients, suppressed from the notation):

$$H^{*,*}(Th(E)) \to H^{*,*}(B) \to H^{*,*}(E^\circ) \to H^{*+1,*}(Th(E)).$$

The Thom isomorphism yields an identification $H^{*,*}(Th(E)) \cong H^{*-2n,*-n}(B)$, and there is a canonical class $\gamma_E$ in $H^{2n,n}(Th(E))$ corresponding to $1 \in H^{0,0}(B)$ under this isomorphism. Combining the Thom isomorphism with the long exact sequence above, we obtain an exact sequence of the form

$$H^{*-2n,*-n}(B) \to H^{*,*}(B) \to H^{*,*}(E^\circ) \to H^{*+1-2n,*-n}(B).$$

The self-intersection formula says that the left hand map is precisely cup product with $c_n(E)$.

Thus, if $c_n(E)$ is trivial, it follows that the long exact sequences above break into short exact sequences of the form

$$0 \to H^{i,j}(B) \xrightarrow{p^*} H^{i,j}(E^\circ) \xrightarrow{\psi} H^{i+1-2n,j-n}(B) \to 0.$$

Let $\gamma_E$ be an element of $H^{2n-1,n}(E^\circ)$ whose image in $H^{2n,n}(Th(E))$ is the Thom class.

If $x \in H^{i+1-2n,j-n}(B)$, then we have the formula

$$\psi(\gamma_E \cdot p^* x) = \psi(\gamma_E) x = x.$$

In other words, the above short exact sequence is actually split. It follows that any element $\delta \in H^{i,j}(E^\circ)$ can be written (uniquely) as

$$p^*(\delta_1) + \gamma_E \cdot p^*(\delta_2),$$

where $\delta_2 = \psi(\delta)$.

**The action of the motivic Steenrod algebra**

We refer the reader to [Voe03b, Rio12] for a detailed discussion of the motivic Steenrod algebra and proofs of all the results we use here. Recall that if $k$ is a perfect field, and $\ell$ is a fixed prime number, then there are reduced power operations

$$P^i : \tilde{H}^{p,q}(\cdot, \mathbb{Z}/\ell) \to \tilde{H}^{p+2i(\ell-1),q+i(\ell-1)}(\cdot, \mathbb{Z}/\ell),$$
and the Bockstein homomorphism
\[ \beta : \tilde{H}^{p,q}(\cdot, \mathbb{Z}/\ell) \to \tilde{H}^{p+1,q}(\cdot, \mathbb{Z}/\ell). \]

One then defines operations \( B^i := \beta P^i \); when \( \ell = 2 \), we set \( Sq^{2i} := P^i \) and \( Sq^{2i+1} := B^i \).

Recall that an admissible sequence \( I \) is a sequence \((\epsilon_0, s_1, \epsilon_1, \ldots, s_r, \epsilon_r)\) where \( \epsilon_r \in \{0, 1\} \) and the \( s_i \) are non-negative integers such that \( s_i \geq \ell s_{i+1} + \epsilon_i \). The elements \( P^I \) defined by
\[
P^I := \beta^{\epsilon_0} P^{s_1} \ldots P^{s_r} \beta^{\epsilon_r}
\]
form a basis of the motivic Steenrod algebra \([\text{Voe03b}, \text{Proposition 11.1}]\). We will write \((n(I), w(I))\) for the bidegree by which \( P^I \) increases cohomology.

By the product formula for reduced power operations \([\text{Voe03b}, \text{Proposition 9.7}]\), the discussion of the previous subsection implies that action of the Steenrod algebra on \( \text{mod}\ \ell\)-motivic cohomology of \( \mathcal{E}^0 \) is determined by its action on \( B \) and its action on the chosen element \( \gamma_{\mathcal{E}} \). For an admissible sequence \( I \), we can then use the following formula
\[
P^I(\delta) = p^*(\alpha_I(\delta)) + \gamma_{\mathcal{E}} \cdot p^*(\beta_I(\delta)),
\]
to define classes \( \alpha_I(\delta) \) and \( \beta_I(\delta) \).

**Proposition 2.1.** If \( \gamma'_{\mathcal{E}} \) is another element of \( H^{2n-1,n}(\mathcal{E}^0) \) whose image in \( H^{0,0}_{\mathcal{E}}(B) \) is 1, then \( \gamma'_{\mathcal{E}} = \gamma_{\mathcal{E}} + p^*(b) \) for some \( b \in H^{2n-1,n}(B) \). Moreover,
\[
\alpha_I(\gamma'_{\mathcal{E}}) = \alpha_I(\gamma_{\mathcal{E}}) + P^I(b) + \beta_I(\gamma_{\mathcal{E}}) \cdot b,
\]
and
\[
\beta_I(\gamma'_{\mathcal{E}}) = \beta_I(\gamma_{\mathcal{E}}).
\]

*Proof.* As \( \mathcal{E} \) is fixed, we drop \( \mathcal{E} \) as a subscript and write \( \gamma := \gamma_{\mathcal{E}} \) and \( \gamma' := \gamma'_{\mathcal{E}} \). With that notation, the image of \( \gamma' - \gamma \) in \( H^{0,0}_{\mathcal{E}}(B) \) is 0. Therefore, there is a (necessarily unique) element of \( H^{2n-1,n}(B) \) such that \( \gamma - \gamma = p^*(b) \).

For the second statement, we have
\[
P^I(\gamma') = P^I(\gamma + p^*(b)) = P^I(\gamma) + P^I(p^*b)
\]
\[
= p^*(\alpha_I(\gamma)) + p^*(\beta_I(\gamma)) \cdot \gamma + p^* P^I(b)
\]
\[
= p^*(\alpha_I(\gamma)) + p^*(\beta_I(\gamma)) \cdot (\gamma' - p^*(b)) + p^* P^I(b)
\]
\[
= p^*(\alpha_I(\gamma) + \beta_I(\gamma) \cdot b + P^I(b)) + p^*(\beta_I(\gamma)) \cdot \gamma',
\]
from which the statement follows immediately. \(\square\)

If \( \xi : \mathcal{E} \to B \) is a rank \( r \) vector bundle over a smooth \( k \)-scheme \( B \) such that \( c_r(\mathcal{E}) = 0 \), then we can consider the following expressions:
\[
\alpha_I(\xi) = \{ \alpha_I(\gamma_{\mathcal{E}}) \} \in H^{2n-1+n(I),n+w(I)}(B)/[P^I + \beta_I(\xi) \cup] H^{2n-1,n}(B)
\]
\[
\beta_I(\xi) \in H^{n(I),w(I)}(B).
\]
The next result shows that \( \beta_I(\xi) \) admits a description in terms of Chern classes of \( \xi \).
Lemma 2.2. Given an admissible sequence \( I \), if any \( \epsilon_i \) is non-zero, it follows that \( \beta_I(\xi) = 0 \). On the other hand, if all \( \epsilon_i \) are zero, then the \( \beta_I(\xi) \) can be expressed in terms of polynomials in the Chern classes of \( \xi \).

Proof. It suffices to establish this for the Steenrod squaring operations and the Bockstein. Since we have a cofiber sequence
\[
\mathcal{E}^0 \rightarrow \mathcal{E} \rightarrow Th(\mathcal{E}) \rightarrow \Sigma_1^1 \mathcal{E} \rightarrow \cdots,
\]
the connecting homomorphism in the long exact sequence in motivic cohomology studied above is compatible with the action of Steenrod operations. The map
\[
\delta^* : H^{*,*}(\Sigma_1^1 \mathcal{E}^0) \rightarrow H^{*,*}(Th(\mathcal{E}))
\]
by definition sends \( \gamma_{\mathcal{E}} \) to the Thom class \( t_{\mathcal{E}} \in H^{2n,n}(Th(\mathcal{E})) \). Therefore, to understand \( Sq^I(\gamma_{\mathcal{E}}) \), it suffices to compute \( Sq^I(t_{\mathcal{E}}) \). This computation is precisely [Voe03b, Theorem 14.2].

Definition 2.3. If \( \xi : \mathcal{E} \rightarrow X \) is a rank \( r \) vector bundle on \( X \) with \( c_r(\mathcal{E}) = 0 \), then the motivic secondary characteristic class \( \Phi_I(\mathcal{E}) \) attached to an admissible sequence \( I \) (and a prime \( p \)) is the coset of the class \( \alpha_I(\xi) \) described above.

Example 2.4. Consider the operation \( Sq^2 \), and suppose \( \xi : \mathcal{E} \rightarrow X \) is a rank \( r \) vector bundle on a smooth scheme \( X \) with \( c_r(\mathcal{E}) = 0 \) so that we can define secondary characteristic classes. Let \( t_{\mathcal{E}} \) be the Thom class of \( \mathcal{E} \in H^{2r,r}(Th(\mathcal{E})) \) and let \( \gamma_{\mathcal{E}} \in H^{2r-1,r}(\mathcal{E}^0) \) be a choice of lift of the Thom class. In this case, \( Sq^2(t_{\mathcal{E}}) = c_1(\mathcal{E}) \cup t_{\mathcal{E}} \). It follows that, in this case, \( \beta_I = c_1(\mathcal{E}) \). Thus, the secondary characteristic class associated with \( \mathcal{E} \) and \( I \) corresponding to \( Sq^2 \) takes values in the coset
\[
H^{2r+1,r+1}(X, \mathbb{Z}/2)/(Sq^2 + c_1(\mathcal{E}) \cup) H^{2r-1,r}(X, \mathbb{Z}/2).
\]

The next result is proven, with evident notational changes, in a fashion parallel to [PS62, Proposition 4.3] so we omit the proof.

Proposition 2.5. If a rank \( r \) vector bundle has a nowhere vanishing section, then the motivic secondary characteristic classes are defined and trivial.

Remark 2.6. If \( \mathcal{E}' \) is another vector bundle on \( X \), we expect it is possible to prove a Whitney sum formula expressing the secondary characteristic classes of \( \mathcal{E} \oplus \mathcal{E}' \) in terms of the secondary classes of \( \mathcal{E} \) and the Chern classes of \( \mathcal{E}' \), but we have not pursued this here; see [PS62, §6] for the corresponding topological result. It also seems reasonable to expect that one may prove analogs of the Wu formulae for motivic secondary characteristic classes.

Comments on the universal example

The \( r \)-th Chern class corresponds to an \( \mathbb{A}^1 \)-homotopy class of maps \( c_r : Gr_r \rightarrow K(\mathbb{Z}(r), 2r) \), where \( K(\mathbb{Z}(r), 2r) \) is a motivic Eilenberg-Mac Lane space. Set
\[
K_r := \text{hofib}(c_r).
\]
We therefore have an \( \mathbb{A}^1 \)-fiber sequence of the form
\[
K(\mathbb{Z}(r), 2r - 1) \rightarrow K_r \rightarrow Gr_r.
\]
By construction, if \( X \) is a smooth scheme and \( \xi : \mathcal{E} \to X \) is a vector bundle with vanishing top Chern class, then there is an \( \mathbb{A}^1 \)-homotopy class of maps \( X \to K_r \) determined by \( \xi \).

Now, the motivic cohomology of the base is known: it is a polynomial ring in \( c_1, \ldots, c_r \) over the motivic cohomology of a point. In contrast to the situation in classical topology, the motivic cohomology of the fiber is “unknown,” in the following sense. Voevodsky studied the algebra of bistable operations (i.e., operations that are stable with respect to \( \mathbb{P}^1 \)-suspension) and proved that, over fields having characteristic 0, all bi-stable operations arise from the Bockstein and reduced power operations [Voe10, Theorem 3.49]. Recent work generalizes these computations to positive characteristic [HKØ13]. It seems reasonable to expect that one can compute the “bistable” part of the motivic cohomology of \( K_r \) and that all operations arise from cohomology of the Grassmannian and the secondary characteristic classes discussed above, but we won’t attempt to make this precise here.

However, there are additional operations on motivic cohomology that are “unstable” or, perhaps more appropriately, only stable with respect to simplicial suspension. Such operations were studied by Guillou-Weibel [GW13], but they write (see [GW13, §9]) that they know little about operations on \( H^{2i-1,i} \), which is the range of interest to us. In particular, any “new” unstable operations give rise to motivic secondary characteristic classes that are not of the form studied above.

**Examples: Stiefel varieties and non-trivial motivic secondary classes**

If \( m < n \), the projection map \( SL_n/SL_{m-1} \to SL_n/SL_m \) makes \( SL_n/SL_{m-1} \), up to \( \mathbb{A}^1 \)-homotopy, into the complement of the zero section of a vector bundle of rank \( m \) over \( SL_n/SL_m \). Indeed, this follows from the isomorphism \( SL_n/SL_{m-1} \cong SL_n \times^{SL_m} SL_m/SL_{m-1} \), the fact that the map

\[
SL_n/SL_{m-1} \cong SL_n \times^{SL_m} SL_m/SL_{m-1} \to SL_n \times^{SL_m} \mathbb{A}^m \setminus 0
\]

is an \( \mathbb{A}^1 \)-weak equivalence, and the fact that \( SL_n \times^{SL_m} \mathbb{A}^m \setminus 0 \) is the complement of the zero section of the vector bundle \( SL_n \times^{SL_m} \mathbb{A}^m \), where \( SL_m \) acts on \( \mathbb{A}^m \) by the standard representation.

If \( V_m(\mathbb{A}^n) \) denotes the open subscheme of the affine space of \( m \times n \)-matrices consisting of matrices having rank exactly \( m \), then there is a projection morphism \( SL_n/SL_{m-1} \to V_m(\mathbb{A}^n) \) that is an \( \mathbb{A}^1 \)-weak equivalence; the variety \( V_m(\mathbb{A}^n) \) can be realized as a quotient of \( SL_n \) by an extension of \( SL_m \) by a (split) unipotent group, and the induced map of homogeneous spaces is Zariski locally trivial with affine space fibers.

The motivic cohomology ring of \( V_m(\mathbb{A}^n) \) is determined in [Wil12, Theorem 19]. In particular,

\[
H^{*,*}(V_m(\mathbb{A}^n), \mathbb{Z}/p) = H^{*,*}(\text{Spec } k, \mathbb{Z}/p)[\rho_{n-m+1}, \ldots, \rho_{n-1}, \rho_n]/I
\]

where \( \rho_i \) has bidegree \( (2i-1, i-1) \) and \( I \) is generated by the relations \( \rho_i^2 = 0 \) if \( 2i-1 > n \), and \( \rho_i^2 = \{-1\} \rho_{2i-1} \) if \( 2i-1 \leq n \) (here \( \{-1\} \) is the class of \(-1\) in \( H^{1,1}(\text{Spec } k) \cong k^* \)).

This computation shows that the Chow groups \( V_m(\mathbb{A}^n) \) are always trivial (this also follows from the observation that \( V_m(\mathbb{A}^n) \) is an open subscheme of \( \mathbb{A}^{mn} \)), so the top Chern class of any vector bundle on \( V_m(\mathbb{A}^n) \) is trivial. In particular, it makes sense to speak of the motivic secondary characteristic classes of any vector bundle on \( V_m(\mathbb{A}^n) \).

The action of the motivic Steenrod algebra on the motivic cohomology of \( V_m(\mathbb{A}^n) \) is determined in [Wil12, Theorems 20 and 21]. When \( p = 2 \), the action of the even motivic Steenrod squares is
specified by the formulas:

\[
Sq^{2i}(\rho_j) = \begin{cases} 
(i+1) \rho_{j+i} & \text{if } i + j \leq n, \\
0 & \text{otherwise,}
\end{cases}
\]

while if \( p \) is odd, the action of the reduced power operations \( P^j \) is given by the formula

\[
P^j(\rho_j) = \begin{cases} 
(i+1) \rho_{j+i(p-1)} & \text{if } j + i(p-1) \leq n, \text{ and} \\
0 & \text{otherwise.}
\end{cases}
\]

From these observations, we can compute secondary cohomology operations.

**Lemma 2.7.** The motivic secondary characteristic class for a rank \((n - m)\) vector bundle on \(SL_n/SL_{n-m}\) corresponding to the operation \(Sq^{2(n-m)}\) has no indeterminacy. Similarly, if \((n - m) = i(p - 1)\), then the motivic secondary characteristic class corresponding to \(P^i\) (modulo \(p\)) has no indeterminacy.

**Proof.** This follows from Lemma 2.2, and [Wil12, Theorem 19].

The projection map \(SL_n/SL_{n-m} \to SL_n/SL_{n-m}\) is taken under the weak equivalences above to the “forgetful” map \(V_{m+1}(\mathbb{A}^n) \to V_m(\mathbb{A}^n)\). In particular, the induced map on motivic cohomology sends the generator \(\rho_{n-m}\) in \(H^*(V_{m+1}(\mathbb{A}^n))\) to 0 in \(H^*(V_m(\mathbb{A}^m))\) and acts as the identity on the remaining \(\rho_i\). From this fact and the computations above, we deduce the following result.

**Corollary 2.8.** Consider the canonical rank \((n - m)\) vector bundle on \(SL_n/SL_{n-m}\). Fix a prime \(p\), and assume \(m = i(p - 1)\). The motivic secondary characteristic class associated with \(P^i\) is zero if and only if \(p \mid (n - m - 1)\).

**Example 2.9.** In the case where \(m = 1\), the only \(p\) for which \((p - 1)m\) is 2. In this case, one can consider the motivic secondary class associated with \(Sq^2\). Observe that the motivic secondary class associated with the universal rank \(n \to 1\) vector bundle on \(SL_n/SL_{n-1}\) is zero if and only if \(n\) is even. When \(n = 2m\), the bundle \(SL_{2r}/SL_{2r-2} \to SL_{2r}/SL_{2r-2}\) admits a section given by the composite map \(SL_{2r}/SL_{2r-1} \cong Sp_{2r}/Sp_{2r-2} \to SL_{2r}/SL_{2r-2}\). In contrast, the Euler class of the rank \((n - 1)\) vector bundle on \(SL_n/SL_{n-1}\) is non-zero by [AF12a, Lemma 3.3].

**Example 2.10.** In the case where \(m = 2\), the only \(p\) for which \((p - 1)m\) are 2 and 3. In this case, one can consider the motivic secondary characteristic classes associated with \(Sq^4\) and \(P^1\) (modulo 3). When \(p = 2, i = 2\), and the motivic secondary characteristic class of the statement is zero if and only if \(2 \mid (n - 3)\), i.e., if and only if \(n\) is congruent to 0 or 3 modulo 4. When \(p = 3\), the motivic secondary characteristic class of the statement is zero if and only if \(3 \mid (n - 3)\), i.e., if and only of \(n\) is divisible by 3.

**3 Milnor K-cohomology and motivic cohomology**

The goal of this section is to describe some operations on Milnor K-cohomology in terms of motivic cohomology; we will use these results in the next section when we study the Euler class in Chow-Witt groups. To connect the two sorts of operations, we use the Voevodsky-Rost solution to the Bloch-Kato conjecture.
Mod $\ell$ motivic Eilenberg-Mac Lane spaces

We begin by reviewing some results about the $\mathbb{A}^1$-homotopy theory of mod $\ell$-motivic Eilenberg-Mac Lane spaces. Throughout, we assume that $\ell$ is invertible in the base field.

**Proposition 3.1.** For any prime $\ell$ invertible in $k$, there are canonical isomorphisms of strictly $\mathbb{A}^1$-invariant sheaves with transfers

$$H^i(Z/\ell(n)) \sim \begin{cases} H^i_{\text{ét}}(\mu_\ell^{\otimes n}) & \text{if } 0 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}.$$

**Proof.** Since both sides of the isomorphism are strictly $\mathbb{A}^1$-invariant sheaves with transfers, it suffices by Rost’s theory of cycle modules to construct a map on finitely generated field extensions that is compatible with residue maps and transfers. First, observe that the hypercohomology spectral sequence yields isomorphisms

$$H^i(Z/\ell(n))(\text{Spec } k) \cong H^0(\text{Spec } k, H^i(Z/\ell(n))) \rightarrow H^{i,n}(\text{Spec } k, Z/\ell).$$

Now, for $i > n$, the groups on the right hand side vanish for dimensional reasons. For $i \leq n$, the Beilinson-Lichtenbaum conjecture (see [Voe03a, Voe11]) together with the computation of Lichtenbaum motivic cohomology (e.g., [Voe03a, Theorem 6.1]) yields isomorphisms

$$H^{i,n}(\text{Spec } k, Z/\ell) \rightarrow H^i_{\text{ét}}(\text{Spec } k, \mu_\ell^{\otimes n}).$$

These isomorphisms are compatible with residue maps and transfers and this yields the required isomorphism. \qed

The above result can be reformulated as follows.

**Corollary 3.2.** For any integer $n$ and any prime $\ell$ invertible in $k$, there are identifications

$$\pi_1^A(K(Z/\ell(n), 2n)) \sim \begin{cases} H^{2n-i}_{\text{ét}}(\mu_\ell^{\otimes n}) & \text{if } n \leq i \leq 2n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $K(Z/\ell(n), 2n)$ is $\mathbb{A}^1-(n-1)$-connected.

The above results allow us to say more about the relationship between mod $\ell$-motivic cohomology and cohomology with coefficients in the sheaves $H^i_{\text{ét}}(\mu_\ell^{\otimes n})$. If the base field contains a primitive $\ell$-th root of unity, we can identify the sheaves $H^i_{\text{ét}}(\mu_\ell^{\otimes n})$ with mod $\ell$ unramified Milnor K-theory sheaves.

**Lemma 3.3.** For any smooth scheme $X$, and any $\ell$ invertible in $k$ we have $H^j_{\text{Nis}}(X, H^1_{\text{ét}}(\mu_\ell^{\otimes n})) = 0$ for $j > i$.

**Proof.** This is an immediate consequence of the form of the Gersten resolution for these sheaves. \qed
We can now use these results to relate the motivic cohomology groups to cohomology with coefficients in Milnor K-theory sheaves, this will allow us to compare the groups housing the motivic secondary characteristic classes and the obstruction classes we defined above.

**Corollary 3.4.** For any smooth scheme $X$ over a perfect field $k$, and any prime $\ell$ invertible in $k$, the edge map in the hypercohomology spectral sequence induces a functorial in $X$ isomorphism $\tilde{H}^{2n-1,n}(X, \mathbb{Z}/\ell) \sim H^{n-1}(X, \mathcal{H}_\ell^n(\mu_{\ell^n}^{\otimes n})).$

Furthermore, the sheafified norm-residue isomorphism $K^M_n/\ell \sim H^{n-1}(X, K^M_n/\ell)$ (a.k.a. the Bloch-Kato conjecture) yields a further isomorphism $\tilde{H}^{2n-1,n}(X, \mathbb{Z}/\ell) \sim H^{n-1}(X, K^M_n/\ell)$.

**Proof.** Using the vanishing results stated above, the hypercohomology spectral sequence degenerates to yield the above isomorphism.

---

**Some operations on Milnor $K$-cohomology**

Since the operations $Sq^{2i}$ are morphisms $H^{p-2i,q-i}(X, \mathbb{Z}/2) \to H^{p,q}(X, \mathbb{Z}/2)$, and since the isomorphisms of Corollary 3.4 are functorial in $X$, if $k$ has characteristic unequal to $2$, we can view $Sq^{2i}$ as inducing natural transformations $Sq^{2i}: H^{n-i-1}(_, K_n^{M}/\ell) \to H^{n-i}(_, K_n^{M}/\ell)$.

Similarly, if $\ell$ is invertible in $k$, then the operations $P^i \mod \ell$ can be viewed as natural transformations $P^i: H^{n-i(\ell-1)-1}(_, K_n^{M}/\ell) \to H^{n-i}(_, K_n^{M}/\ell)$.

Using these operations, our next goal is to relate the universal obstruction described above to motivic cohomology.

---

**4 Connecting homomorphisms as cohomological operations**

Suppose $X$ is a smooth $k$-scheme and $\mathcal{L}$ is a line bundle on $X$. The sheaf $\mathcal{I}^j(\mathcal{L})$ comes equipped with a reduction map $\mathcal{I}^j(\mathcal{L}) \to \bar{\mathcal{I}}^j$. There is a canonical homomorphism $\mathcal{K}_j^{M}/\ell \to \bar{\mathcal{I}}^j$. The Milnor conjecture on quadratic forms [OVV07] implies this map is an isomorphism on sections over finitely generated extensions of the base field and since both sheaves in question are strictly $\mathbb{A}^1$-invariant, an isomorphism of sheaves; we use this identification without mention in the sequel. The exact sequence

$$0 \to \mathcal{I}^{j+1}(\mathcal{L}) \to \mathcal{I}^j(\mathcal{L}) \to \bar{\mathcal{I}}^j \to 0$$

yields a connecting homomorphism $H^i(X, \mathcal{I}^{j+1}(\mathcal{L})) \xrightarrow{\partial_{\mathcal{L}}} H^{i+1}(X, \mathcal{I}^{j+1}(\mathcal{L}))$.

The reduction map yields a homomorphism $H^{i+1}(X, \mathcal{I}^{j+1}(\mathcal{L})) \to H^{i+1}(X, \bar{\mathcal{I}}^{j+1})$.

Taking the composite of these two maps yields a homomorphism that we will now study.
Definition 4.1. If $X$ is a smooth scheme, and $L$ is a line bundle on $X$, write

$$
\Phi_{i,j,L} : H^i(X, \bar{I}^j) \to H^{i+1}(X, \bar{I}^{j+1}).
$$

for the composite of the connecting homomorphism $\partial_L$ and the reduction map just described. If $L$ is trivial, suppress it from the notation and write $\Phi_{i,j}$ for the resulting homomorphism.

When $i = j$, via the identification $\bar{I}^j \cong K_M^j / 2$, the map $\Phi_{i,i}$ can be viewed as a morphism $Ch_i(X) \to Ch_{i+1}(X)$, where $Ch^i(X) = CH^i(X) / 2$. In $[\text{Tot03}]$, this homomorphism is identified as $Sq^2$. More generally, via the identification $\bar{I}^j \cong K_M^j / 2$, and the isomorphism of Corollary 3.4, the map $\Phi_{i-1,i,L}$ can be identified as a morphism

$$
\Phi_{i-1,i,L} : H^{2i-1,i}(X, \mathbb{Z}/2) \to H^{2i+i}(X, \mathbb{Z}/2).
$$

The connecting homomorphism is functorial with respect to pullbacks and the goal of this section is to identify it as a cohomology operation.

The operations $\Phi_{i,j}$ and “bi-stability”

We now study bi-stability, i.e., stability with respect to $\mathbb{P}^1$-suspension, of the operations $\Phi_{i,j}$. If $X$ is a smooth scheme, we then need to compare an operation on $X$ and a corresponding operation on $X \times \mathbb{P}^1$. To understand operations on the latter, we first consider $X \times \mathbb{P}^1$ and we use the projective bundle formula in $\bar{I}$-cohomology (see, e.g., $[\text{Fas13}, \S 4]$) to identify this group in terms of cohomology on $X$.

The projective bundle formula yields an identification

$$
H^i(X \times \mathbb{P}^1, \bar{I}^j) \cong H^i(X, \bar{I}^j) \oplus H^{i-1}(X, \bar{I}^{j-1}) \cdot c(O(-1)),
$$

and, unwinding the definitions, this corresponds to a functorial in $X$ isomorphism of the form

$$
H^i(X_+ \wedge \mathbb{P}^1, \bar{I}^j) \cong H^{i-1}(X, \bar{I}^{j-1}).
$$

Using this isomorphism, we can compare the operation $\Phi_{i,j}$ on $H^i(X_+ \wedge \mathbb{P}^1, \bar{I}^j)$ with the operation $\Phi_{i-1,j-1}$ on $H^{i-1}(X, \bar{I}^{j-1})$.

Proposition 4.2. There is a commutative diagram of the form

$$
\begin{array}{ccc}
H^i(X_+ \wedge \mathbb{P}^1, \bar{I}^j) & \xrightarrow{\Phi_{i,j}} & H^{i+1}(X_+ \wedge \mathbb{P}^1, \bar{I}^{j+1}) \\
\downarrow & & \downarrow \\
H^{i-1}(X, \bar{I}^{j-1}) & \xrightarrow{\Phi_{i-1,j-1}} & H^i(X, \bar{I}^j),
\end{array}
$$

where the vertical maps are the isomorphisms described before the statement. Moreover, the sequence of operations $\Phi_{i,j}$ corresponds to a bi-stable operation.
Proof. The operation $\Phi_{i,j}$ is induced by the composite morphism of the connecting homomorphism associated with the short exact sequence

$$0 \rightarrow \mathcal{I}^{j+1} \rightarrow \mathcal{I}^j \rightarrow \bar{\mathcal{I}}^j \rightarrow 0$$

and the reduction map $\mathcal{I}^{j+1} \rightarrow \bar{\mathcal{I}}^{j+1}$. The contractions of $\mathcal{I}^j$ and $\bar{\mathcal{I}}^j$ are computed in [AF12a, Lemma 2.7 and Proposition 2.8] and our result follows immediately from the proofs of those statements. \qed

Remark 4.3. Because of the above result, we will abuse terminology and refer to $\Phi_{i,j}$ as a bi-stable operation.

Non-triviality of the operation $\Phi_{i-1,i,L}$

Our goal in this section is to prove that the operation $\Phi_1$ is non-trivial. It follows from [Wil12, Theorems 19, 20 and 21] that $H^1(SL_3, \mathcal{K}^M_2/2) = \mathbb{Z}/2$, $H^2(SL_3, \mathcal{K}^M_3/2) = \mathbb{Z}/2$ (we use these identifications repeatedly below) and that the Steenrod square operation $Sq^2 : H^1(SL_3, \mathcal{K}^M_2/2) \rightarrow H^2(SL_3, \mathcal{K}^M_3/2)$ is an isomorphism. In order to compute the operation $\Phi_1$, we first need to find explicit generators of the above groups. We will implicitly use the Gersten resolution of the sheaves $\mathcal{K}^M_i/2$ in our computations below.

Consider the “projection onto the first column” map $p : SL_3 \rightarrow \mathbb{A}^3 \setminus 0$. There is an $SL_3$-action on $\mathbb{A}^3 \setminus 0$ by left multiplication making the above map equivariant. If we fix the base-point $(1, 0, 0)^t$ in $\mathbb{A}^3 \setminus 0$, we can identify $\mathbb{A}^3 \setminus 0$ as a homogeneous space of the form $SL_3/P$, where $P$ consists of matrices of the form

$$\left( \begin{array}{cc} 1 & u \\ 0 & M \end{array} \right)$$

with $M \in SL_2$.

The projection map $SL_3 \rightarrow \mathbb{A}^3 \setminus 0$ is Zariski locally trivial, and there is an explicit trivialization over the “standard” open cover of $\mathbb{A}^3 \setminus 0$ by open sets of the form $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^2$ where each such open set corresponds to the locus of points in $\mathbb{A}^3 \setminus 0$ where the $i$-th coordinate is non-zero. Indeed, write $\alpha_i : \mathbb{A}^1 \setminus 0 \times \mathbb{A}^2 \hookrightarrow \mathbb{A}^3 \setminus 0$ for the inclusion morphism. If $x_{ij}$ are the usual matrix coordinates on $SL_3$, then the restriction of $p$ to $\alpha_i$ is $SL_3[x_{11}^{-1}]$. Under the identification of the previous paragraph, the variety $SL_3[x_{11}^{-1}]$ is isomorphic to $(\mathbb{A}^1 \setminus 0 \times \mathbb{A}^2) \times (SL_2 \times \mathbb{A}^2)$. We now make this isomorphism explicit.

Lemma 4.4. The morphism

$$\Psi_1 : (\mathbb{A}^1 \setminus 0 \times \mathbb{A}^2) \times (SL_2 \times \mathbb{A}^2) \rightarrow SL_3[x_{11}^{-1}]$$

defined by

$$\Psi_1(t, w, x, y_{11}, y_{12}, y_{21}, y_{22}, u, v) = \left( \begin{array}{ccc} t & u & v \\ w & t^{-1}(y_{11} + uw) & t^{-1}(y_{12} + vw) \\ tx & y_{21} + uw & y_{22} + vx \end{array} \right)$$

is an isomorphism with inverse

$$\varphi_1 : SL_3[x_{11}^{-1}] \rightarrow \mathbb{A}^1 \setminus 0 \times \mathbb{A}^2 \times (SL_2 \times \mathbb{A}^2)$$
defined by \( \varphi_1(x_{ij}) = (x_{11}, x_{21}, x_{31}/x_{11}, M, x_{12}, x_{13}/x_{11}) \) with

\[
M = \begin{pmatrix}
  x_{11}x_{22} - x_{12}x_{21} & x_{11}x_{23} - x_{13}x_{21} \\
  x_{32} - x_{12}x_{31}/x_{11} & x_{33} - x_{13}x_{31}/x_{11}
\end{pmatrix}.
\]

We now begin our cohomological computations, starting with a general lemma.

**Lemma 4.5.** Let \( X \) be a smooth scheme. Then

\[
H^i(X \times \mathbb{A}^1 \setminus 0, K^M_j/2) \cong H^i(X, K^M_j/2) \oplus H^i(X, K^M_{j-1}/2) \cdot \{t\}
\]

where \( \{t\} \) is the class of a coordinate of \( \mathbb{A}^1 \setminus 0 \) in \( H^0(\mathbb{A}^1 \setminus 0, K^M_1/2) \).

*Proof.* This is a consequence of the localization sequence associated with the embedding \( X \times \mathbb{A}^1 \setminus 0 \subset X \times \mathbb{A}^1 \). \( \square \)

**Lemma 4.6.** If \( y_1 \in SL_3^{(1)} \) is defined by the ideal \( \langle x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21} \rangle \) and \( y_2 \in SL_3^{(1)} \) is defined by the ideal \( \langle x_{11} \rangle \), then a generator for the group \( H^1(SL_3, K^M_2/2) \cong \mathbb{Z}/2 \) is given by the class of the symbol

\[
\xi := \{x_{11}x_{23} - x_{13}x_{21}\} + \{x_{21}\}
\]

in \( K^M_1(k(y_1))/2 \oplus K^M_1(k(y_2))/2 \).

*Proof.* We first check that \( \xi \) is a cycle. The image of \( \{x_{21}\} \) under the boundary map is the generator of \( K^M_0(k(z))/2 \) where \( z \) is the point defined by the ideal \( \langle x_{11}, x_{21} \rangle \).

Consider now the ideal \( I = \langle x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21} \rangle \). Since \( \det(x_{ij}) = 1 \), we see that \( x_{31}(x_{12}x_{23} - x_{22}x_{13}) = 1 \) (mod \( I \)) and it follows that \( (x_{12}x_{23} - x_{22}x_{13}) \) is invertible modulo \( I \). Now

\[
x_{11}(x_{12}x_{23} - x_{22}x_{13}) = x_{12}(x_{11}x_{23} - x_{13}x_{21}) - x_{13}(x_{11}x_{22} - x_{12}x_{21})
\]

shows that \( x_{11} \in I \). The same kind of argument proves that \( x_{21} \in I \) and thus \( I = \langle x_{11}, x_{21} \rangle \). It follows that the boundary of \( \{x_{11}x_{23} - x_{13}x_{21}\} \) in \( K^M_1(k(y_1))/2 \) is also the generator of \( K^M_0(k(z))/2 \) where \( z \) is the point defined by the ideal \( \langle x_{11}, x_{21} \rangle \), and therefore \( \xi \) is a cycle.

We now prove that \( \xi \) is non trivial in \( H^1(SL_3, K^M_2/2) \). To see this, observe that the image of the class of \( \xi \) under the localization map

\[
H^1(SL_3, K^M_2/2) \to H^1(SL_3[x_{11}^{-1}], K^M_2/2)
\]

is the class of the symbol \( \{x_{11}x_{23} - x_{13}x_{21}\} \) in \( K^M_1(k(y_1))/2 \). This class corresponds to the class of the cycle \( \{y_{22}\} \) in \( K^M_1(k(y'_1))/2 \) under the isomorphism of Lemma 4.4, where \( y'_1 \in \mathbb{A}^1 \setminus 0 \times SL_2 \times \mathbb{A}^4 \) is the point corresponding to the ideal \( \langle y_{11} \rangle \). Now the pull-back homomorphism \( H^1(SL_2, K^M_2/2) \to H^1(SL_2 \times \mathbb{A}^1 \setminus 0, K^M_2/2) \) splits because \( \mathbb{A}^1 \setminus 0 \) has a rational point, and the cycle \( \{y_{22}\} \) in \( K^M_1(k(y'_1))/2 \) is a generator of \( H^1(SL_2, K^M_2/2) \) (for instance by [AF12a, Lemma 4.2], or by using the localization sequence associated with \( SL_2 \simeq \mathbb{A}^2 \setminus 0 \subset \mathbb{A}^2 \)). It follows that the class of \( \xi \) is non trivial in \( H^1(SL_3, K^M_2/2) \). \( \square \)
Next, we provide an explicit generator of $H^2(SL_3, K_3^M/2) \cong \mathbb{Z}/2$. To this end, we introduce two other subschemes of $SL_3$ that will aid in our computation. First, let $SL'_3 \subset SL_3$ be defined by $x_{11} = 0$. By construction $SL'_3 \subset SL_3$ is the preimage under $p$ of the closed subscheme of $\mathbb{A}^3 \setminus 0$ corresponding to column vectors with first entry zero; this subscheme is isomorphic to $\mathbb{A}^2 \setminus 0$. Second, consider the variety $SL'_3[x_{21}^{-1}]$, which can be identified with $(\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1) \times (SL_2 \times \mathbb{A}^2)$ by an argument analogous to that in Lemma 4.4 (it is the preimage under $p$ of a subvariety of $\mathbb{A}^3 \setminus 0$ isomorphic to $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1$).

**Lemma 4.7.** There is an isomorphism $H^1(SL'_3, K_2^M/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. If $z_1 \in (SL'_3)^{(1)}$ is the point defined by the ideal $(x_{31})$ and $z_2 \in (SL'_3)^{(1)}$ is the point defined by the ideal $(x_{12})$, then explicit generators for the group $H^1(SL'_3, K_2^M/2)$ are given by the classes of the cycles $\eta_1 := \{x_{31}\}$ in $K_1^M(k(z_1))/2$ and $\eta_2 := \{x_{13}\}$ in $K_1^M(k(z_2))/2$.

**Proof.** By the description of $SL'_3$ given before the theorem statement, it follows by homotopy invariance that $SL'_3$ is the complement of the zero section of a rank 2 vector bundle on $\mathbb{A}^2 \setminus 0$. A straightforward computation with the localization sequence implies that $H^1(SL'_3, K_2^M/2)$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. We now proceed to show that the elements in the theorem statement provide explicit generators of the two factors.

Consider the following portion of the localization sequence associated with the embedding $SL'_3[x_{21}^{-1}] \subset SL_3$:

$$H^0(SL_3[x_{21}^{-1}], K_2^M/2) \longrightarrow H^1(SL'_3, K_2^M/2) \longrightarrow H^1(SL_3, K_2^M/2) \longrightarrow H^1(SL'_3[x_{21}^{-1}], K_2^M/2).$$

Applying homotopy invariance we see that $H^1(SL_3[x_{21}^{-1}], K_2^M/2) \cong H^1(\mathbb{A}^1 \setminus 0 \times SL_2, K_2^M/2)$, and Lemma 4.5 then yields an isomorphism $H^1(\mathbb{A}^1 \setminus 0 \times SL_2, K_2^M/2) \cong H^1(SL_2, K_2^M/2) \oplus H^1(SL_2, K_1^M/2)$. Since $H^1(SL_2, K_1^M/2) \cong Pic(SL_2)/2 = 0$, it follows that $H^1(SL_3[x_{21}^{-1}], K_2^M/2) \cong H^1(SL_2, K_2^M/2) \cong \mathbb{Z}/2$.

We now show that $H^2_{x_{21}}(SL'_3, K_2^M/2) = 0$. To this end, let $SL''_3 \subset SL'_3$ be defined by $x_{21} = 0$. There are isomorphisms $H^2_{x_{21}}(SL'_3, K_2^M/2) \cong H^1(SL''_3, K_1^M/2) \cong H^1(\mathbb{A}^1 \setminus 0 \times SL_2, K_1^M/2) = Pic(\mathbb{A}^1 \setminus 0 \times \mathbb{A}^2 \setminus 0)/2$; this last group is trivial because $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^2 \setminus 0$ is an open subscheme of affine space. It follows that the map $H^1(SL'_3, K_2^M/2) \rightarrow H^1(SL_3[x_{21}^{-1}], K_2^M/2)$ in the localization exact sequence above is surjective. Arguing as in the previous lemma, we see that the class of $\eta_2$ generates $H^1(SL'_3[x_{21}^{-1}], K_2^M/2)$.

There is also an isomorphism $H^1_{x_{12}}(SL'_3, K_2^M/2) \cong H^0(SL''_3, K_1^M/2)$ and the latter group is isomorphic to $K_1^M(k)/2 \oplus \mathbb{Z}/2 \cdot \{x_{31}\}$, with the first factor being given by pull-back from the base and the second factor given by the invertible element $x_{31}$ in $SL''_3$. The same argument shows that $H^0(SL'_3[x_{21}^{-1}], K_2^M/2) = K_2^M(k)/2 \oplus K_1^M(k)/2 \cdot \{x_{21}\}$. A straightforward computation shows that the $K_2^M(k)/2$ factor has trivial image in $H^1_{x_{21}}(SL'_3, K_2^M/2)$ while the factor $K_1^M(k)/2 \cdot \{x_{21}\}$ is mapped isomorphically to the first factor in $H^1_{x_{21}}(SL'_3, K_2^M/2)$. It follows that $\eta_1$ gives the explicit generator of the second factor.

**Corollary 4.8.** Let $z \in SL_3^{(2)}$ be the generic point of the variety defined by the ideal $(x_{11}, x_{21})$. Then $H^2(SL_3, K_3^M/2)$ is generated by the class of the cycle $\{x_{31}\}$ in $K_1^M(k(z))/2$. 

Proof. We use the localization sequence associated with the embedding $SL_3[x_{11}^{-1}] \subset SL_3$; the portion of interest for this corollary is:

$$H^1(SL_3[x_{11}^{-1}], K_3^M/2) \rightarrow H^2_{x_{11}}(SL_3, K_3^M/2) \rightarrow H^2(SL_3, K_3^M/2).$$

There is an isomorphism $H^2_{x_{11}}(SL_3, K_3^M/2) \cong H^1(SL_3, K_3^M/2)$ whose generators are given explicitly in Lemma 4.7. Let $y_1$ be the point corresponding to the ideal $(x_{11}x_{22} - x_{12}x_{21})$, and let $\{x_{11}, x_{11}x_{23} - x_{13}x_{21}\}$ in $K_2^M(k(y_1))/2$. The element of $K_2^M(k(y_1))/2$ just mentioned defines a cycle since $x_{11}$ is invertible and $(x_{11}x_{22} - x_{12}x_{21}, x_{11}, x_{11}x_{23} - x_{13}x_{21}) = (x_{11}, x_{21})$ (see the proof of Lemma 4.6).

We now compute the image of the class described in the previous paragraph under the homomorphism

$$d : H^1(SL_3[x_{11}^{-1}], K_3^M/2) \rightarrow H^2_{x_{11}}(SL_3, K_3^M/2).$$

Observe first that the symbol $\{x_{11}, x_{11}x_{23} - x_{13}x_{21}\}$ in $K_2^M(k(y_1))/2$ is ramified in the ideal $(x_{11}, x_{21}) = (x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21})$ defining $z$. From

$$x_{11}(x_{12}x_{23} - x_{13}x_{21}) = x_{12}(x_{11}x_{23} - x_{13}x_{21}) - x_{13}(x_{11}x_{22} - x_{12}x_{21})$$

we deduce that

$$\{x_{11}, x_{11}x_{23} - x_{13}x_{21}\} = \{x_{12}(x_{11}x_{23} - x_{13}x_{21})/(x_{12}x_{23} - x_{13}x_{22}), x_{11}x_{23} - x_{13}x_{21}\}$$

locally around $z$, and the latter is

$$\{x_{12}/(x_{12}x_{23} - x_{13}x_{22}), x_{11}x_{23} - x_{13}x_{21}\} = \{(x_{11}x_{23} - x_{13}x_{21}), -1\}.$$

Its residue is then the symbol $\{x_{12}/(x_{12}x_{23} - x_{13}x_{22})\} + \{-1\}$ in $K_1^M(k(z))$. At $z$, we have $x_{21}(x_{12}x_{23} - x_{13}x_{22}) = 1$, and we finally find a residue of the form $\{x_{12}\} + \{x_{31}\} + \{-1\}$ in $K_1^M(k(z))$.

Now $\{x_{11}, x_{11}x_{23} - x_{13}x_{21}\}$ is also ramified in the ideal $(x_{11}, x_{12})$. Let $z'$ be the associated point on $SL_3$. It is easy to see that $x_{11}x_{23} - x_{13}x_{21}$ does not belong to $(x_{11}, x_{12})$, and then the residue in $z'$ is of the form $\{-x_{13}x_{21}\} = \{-1\} + \{x_{13}\} + \{x_{21}\}$ in $K_1^M(k(z'))$. So, finally, we see that

$$d(\{x_{11}, x_{11}x_{23} - x_{13}x_{21}\}) = \{x_{12}\} + \{x_{31}\} + \{-1\} + \{-1\} + \{x_{13}\} + \{x_{21}\}$$

in $K_1^M(k(z)) \oplus K_1^M(k(z'))$.

We next observe that $\{x_{12}\} + \{-1\} + \{-1\} + \{x_{21}\}$ is the boundary of the symbol $\{-1, x_{21}\} + \{-1, x_{12}\} + \{x_{12}, x_{21}\}$ on $K_2^M(k(y_2))$ (recall that $y_2$ is the point associated with the ideal $(x_{11})$), and thus vanishes in $H^2_{x_{11}}(SL_3, K_3^M/2)$. It follows that the two generators of $H^2_{x_{11}}(SL_3, K_3^M/2)$ detailed in Lemma 4.7 are identified under the homomorphism

$$d : H^1(SL_3[x_{11}^{-1}], K_3^M/2) \rightarrow H^2_{x_{11}}(SL_3, K_3^M/2).$$

It follows that $H^2(SL_3, K_3^M/2) \cong \mathbb{Z}/2$ is generated by the class of the cycle $\{x_{31}\}$ in $K_1^M(k(z))/2$. 

\[\square\]

Proposition 4.9. The operation $\Phi_{i-1,i}$ is non-trivial.
Proof. We compute the operation $\Phi_{1,2}$ on $H^1(SL_3, K_2^M / 2)$. By definition, it is the composite

$$H^1(SL_3, K_2^M / 2) = H^1(SL_3, I^2) \to H^2(SL_3, I^3) \to H^2(SL_3, I^3) = H^2(SL_3, K_3^M / 2)$$

where the left-hand map is the boundary homomorphism associated with the exact sequence of sheaves

$$0 \to I^3 \to I^2 \to I^2 \to 0$$

and the right-hand map is the projection associated with the morphism of sheaves $I^3 \to I^3$. We will show that $\Phi_{1,2}$ is an isomorphism by showing that the explicit generator of $H^1(SL_3, K_2^M / 2)$ constructed in Lemma 4.6 is mapped to the explicit generator of $H^2(SL_3, K_3^M / 2)$ constructed in Corollary 4.8.

Recall that the sheaves $I^j$ have an explicit Gersten resolution $C(X, I^j)$ ([Fas08, Chapitre 9]) of the form

$$I^j(k(X)) \to \bigoplus_{x \in X^{(i)}} I_{fl}^{j-i}(k(x)) \xrightarrow{d_i} \bigoplus_{x \in X^{(2)}} I_{fl}^{j-2}(k(x)) \to \bigoplus_{x \in X^{(2)}} I_{fl}^{j-2}(k(x)) \to \ldots$$

where $X = SL_3$, and $I_{fl}^{j-1}(k(x)) = I_{fl}^{j-1}(k(x)) \cdot W_{fl}(O_{X,x})$ is an analog of the $(j - 1)$-st power of the fundamental ideal in the Witt group of finite length $O_{X,x}$-modules.

An explicit lift of the generator of $H^1(SL_3, K_2^M / 2)$ given in Lemma 4.6 is of the form

$$\langle -1, x_{11}x_{23} - x_{13}x_{21} \rangle \cdot \rho_1 + \langle -1, x_{21} \rangle \cdot \rho_2$$

where $\rho_1 : k(y_1) \to \text{Ext}^1_{O_{X,y_1}}(k(y_1), O_{X,y_1})$ is defined by mapping 1 to the Koszul complex $Kos(x_{11}x_{22} - x_{12}x_{21})$ associated with the regular sequence $x_{11}x_{22} - x_{12}x_{21}$, and similarly $\rho_1 : k(y_1) \to \text{Ext}^1_{O_{X,y_1}}(k(y_1), O_{X,y_1})$ is defined by $1 \mapsto Kos(x_{11})$. Using [Fas08, Section 3.5], the boundary $d_1$ of the above generator is of the form $\nu_1 + \nu_2$, where

$$\nu_1 : k(z) \to \text{Ext}^2_{O_{X,z}}(k(z), O_{X,z})$$

is defined by $1 \mapsto Kos(x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21})$ and

$$\nu_2 : k(z) \to \text{Ext}^2_{O_{X,z}}(k(z), O_{X,z})$$

is defined by $1 \mapsto Kos(x_{11}, x_{21})$. Now we have

$$\begin{pmatrix} x_{11}x_{22} - x_{12}x_{21} \\ x_{11}x_{23} - x_{13}x_{21} \end{pmatrix} = \begin{pmatrix} x_{22} & -x_{12} \\ x_{23} & -x_{13} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$$

and $(x_{12}x_{23} - x_{13}x_{22})x_{31} = 1$ modulo $\langle x_{11}, x_{21} \rangle$. Thus

$$\nu_1 + \nu_2 = \langle 1, x_{31} \rangle \cdot \nu_1 = ((1, 1) + \langle -1, x_{31} \rangle) \cdot \nu_1$$

A simple computation shows that $(1, 1) \cdot \nu_1$ is the boundary of $(\langle 1, x_{21} \rangle \otimes \langle 1, x_{21} \rangle) \cdot \rho_2$ and therefore vanishes in $H^2(SL_3, I^2)$. Now the class of $\langle -1, x_{31} \rangle \cdot \nu_1$ in $H^2(SL_3, I^3) = H^2(SL_3, K_3^M / 2)$ is precisely a generator as shown by Corollary 4.8. Thus $\Phi_{1,2} : H^1(SL_3, K_2^M / 2) \to H^2(SL_3, K_3^M / 2)$ is an isomorphism. \qed
Corollary 4.10. We have an identification $\Phi_{i-1,i} = Sq^2$.

Proof. The operation $\Phi_{i-1,i}$ is bistable, commutes with pullbacks, and changes bidegree by $(2,1)$ so it is pulled back from a universal class on a motivic Eilenberg-Mac Lane space. On the other hand, the group of bi-stable operations of bidegree $(2,1)$ is isomorphic to $\mathbb{Z}/2$ generated by $Sq^2$: if $k$ has characteristic zero, this is [Voe10, Theorem 3.49], while if $k$ has characteristic unequal to 2, this is [HKØ13, Theorem 1.1]. Since the operation $\Phi_{i-1,i}$ is non-trivial by Proposition 4.9, it follows that it must be equal to $Sq^2$. \hfill \qed

Identification of $\Phi_{i-1,i,L}$

Under the identifications of Corollary 3.4 we have the morphism

$$Sq^2 + c_1(\xi)\cup: H^{i-1}(X, K^M_i)/2 \to H^i(X, K^M_{i+1}/2).$$

We now relate this operation to $\Phi_{i-1,i,L}$.

Theorem 4.11. For any smooth scheme $X$, any $i, j \in \mathbb{N}$ and any line bundle $L$ over $X$, we have

$$\Phi_{i,j,L} = (\Phi_{i,j} + c_1(L))\cup$$

Proof. As in the proof of Proposition 4.9, we denote by $C(X, I^j)$ the Gersten-Witt complex of $X$ (filtered by the powers of the fundamental ideal). When $L = O_X$, we simply drop it from the notation. Let $\alpha \in H^j(X, I^j)$ and let $\alpha' \in C^i(X, I^j)$ be a lift of $\alpha$. Under the equivalences of [BW02, Theorem 6.1, Proposition 7.1], $\alpha'$ can be seen as a complex $P_\bullet$ of finitely generated $O_X$-locally free modules, together with a symmetric morphism (for the $i$-th shifted duality)

$$\psi: P_\bullet \to T^i\text{Hom}(P_\bullet, O_X)$$

whose cone is supported in codimension $\geq i + 1$. By definition, $d_i(\alpha')$ is the localization at the points of codimension $i + 1$ of the symmetric quasi-isomorphism on the cone of $\psi$ (constructed for instance in [BW02, Proposition 1.2]), after dévissage ([BW02, Theorem 6.1, Proposition 7.1]). Let $L$ be the total space of $L$ and let $p: L \to X$ be the projection. The zero section

$$s: O_L \to p^*L$$

can be seen as a symmetric morphism $O_L \to \text{Hom}_{O_X}(O_L, p^*L)$ which is an isomorphism after localization at the generic point of $L$, and whose cone is supported in codimension 1. It follows that $s$ can be seen as an element of $C^0(L, W, p^*L)$. We can consider the class of $d_0^p L(s)$ in $H^1(L, I, p^*L)$, and its projection in $H^1(L, I) = Pic(L)/2$ is precisely the first Chern class of $p^*L$. Using the product structure (say the left one) on derived categories with duality of [GN03], we get a symmetric morphism

$$p^*\psi \otimes s: p^*P_\bullet \otimes p^*L \to T^i\text{Hom}(p^*P_\bullet \otimes p^*L, p^*L).$$
which can be seen as an element of $C^i(L, \mathcal{I}', p^*L)$ lifting $p^*\alpha \in H^i(L, \mathcal{I}')$. The degeneracy locus of $p^*\psi$ in the sense of [Bal05] is by definition the support of its cone, which is of codimension $\geq i + 1$ in $L$. The degeneracy locus of $s$ is of codimension 1 in $L$ and intersects the degeneracy locus of $p^*\psi$ transversally. We can thus use Leibnitz formula [Bal05, Theorem 5.2] in the spirit of [Fas07, Example 2.19] to see that (we won’t bother with signs since we will eventually work with the sheaf $\mathcal{I}'$ whose cohomology groups are 2-torsion)

\begin{align}
(4.1) \quad d^p_{r}\mathcal{E} (p^* \psi \otimes s) = d_i(p^* \psi) \otimes s + p^* \psi \otimes d^r_0 \mathcal{E} (s)
\end{align}

in $C^{i+1}(L, \mathcal{I}', p^*L)$. Since $p^*\alpha \in H^i(L, \mathcal{I}')$ and $p^*\psi \otimes s$ lifts $p^*\alpha$ in $C^i(L, \mathcal{I}', p^*L)$ it follows that $d^p_{r}\mathcal{E} (p^* \psi \otimes s)$ actually belongs to $C^{i+1}(L, \mathcal{I}', p^*L)$. For the same reason, we have $d_i(p^* \psi) \in C^{i+1}(L, \mathcal{I}')$ and then $d_i(p^* \psi) \otimes s \in C^{i+1}(L, \mathcal{I}', p^*L)$. Thus $p^* \psi \otimes d^r_0 \mathcal{E} (s)$ is in $C^{i+1}(L, \mathcal{I}', p^*L)$ as well. It follows that all three terms in $(4.1)$ define classes in $C^{i+1}(L, \mathcal{I}')$. The left term yields a class in $H^{i+1}(L, \mathcal{I}')$ which is $\Phi_{j-i, p^*\alpha} (p^*\alpha)$ by definition. The middle term projects to $\Phi_{j-i} (p^*\alpha)$ and the right-hand term to the class $p^*\alpha \cdot c_1(p^*L)$ in $H^{i+1}(L, \mathcal{I}')$. We can now use the fact that the operation $\Phi_{j-i, \mathcal{E}}$ commutes with pull-backs by [Fas13, Proposition 2.11] together with homotopy invariance to get the result.

As a corollary, we obtain the following theorem.

**Theorem 4.12.** For any smooth scheme $X$, for any (strictly) positive integer $i$, and for any rank $r$ vector bundle $\xi : \mathcal{E} \to X$, the operation $(Sq^2 + c_1(\xi))$ coincides with $\Phi_{1-i, \det \xi}$.

**Proof.** We already know that $\Phi_{1-i, i} = Sq^2$ by Corollary 4.10. The result follows then from Theorem 4.11. 

\section{The Euler class and motivic secondary classes}

The Euler class $e(\xi)$ of a rank $r$ vector bundle $\xi : \mathcal{E} \to X$ is the primary obstruction to splitting off a free rank 1 summand, and it lives in $H^r(X, K^M_{\text{MW}}(\det \mathcal{E}))$. The vanishing of $e(\xi)$ guarantees vanishing of $c_i(\xi)$ in $CH^r(X)$ but, as we saw in Example 2.9, the converse is not true in general. Thus, for vector bundles with vanishing top Chern class but with non-vanishing Euler class, we can define motivic secondary characteristic classes as above. For vector bundles with vanishing top Chern class, the Euler class still has a “quadratic” part, and goal of this section is to relate this “quadratic” part with motivic secondary characteristic classes.

**The quadratic part of the Euler class**

There is a short exact sequence of sheaves

\[ 0 \to \mathcal{I}^{r+1}(\det \mathcal{E}) \to K^M_{\tau}(\det \mathcal{E}) \to K^M_{\tau} \to 0. \]

The long exact sequence in cohomology associated with this short exact sequence takes the form

\[ \cdots \to H^{r-1}(X, K^M_{\tau}) \xrightarrow{\partial_r} H^r(X, \mathcal{I}^{r+1}(\det \mathcal{E})) \to H^r(X, K^M_{\tau}(\det \mathcal{E})) \to H^r(X, K^M_{\tau}) \to \cdots. \]
The connecting homomorphism $\partial_\xi$ can be decomposed as follows. The cartesian square of sheaves
\[
\begin{array}{ccc}
K^M_r(\det E) & \rightarrow & K^M_r \\
\downarrow & & \downarrow \\
\Gamma(\det E) & \rightarrow & \Gamma
\end{array}
\]
yields a diagram of exact sequences
\[
\begin{array}{ccc}
0 & \rightarrow & \Gamma^{r+1}(\det E) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Gamma^{r+1}(\det E) \\
\end{array}
\begin{array}{ccc}
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
K^M_r(\det E) & \rightarrow & K^M_r \\
\downarrow & & \downarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\end{array}
\begin{array}{ccc}
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\Gamma(\det E) & \rightarrow & \Gamma \\
\downarrow & & \downarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\end{array}
\rightarrow 0.
\]

It follows that $\partial_\xi$ fits into the commutative diagram
\[
\begin{array}{ccc}
H^{r-1}(X, K^M_r) & \rightarrow & H^r(X, \Gamma^{r+1}(\det E)) \\
\downarrow & & \downarrow \\
H^{r-1}(X, \Gamma) & \rightarrow & H^r(X, \Gamma^{r+1}(\det E))
\end{array}
\]
where the left vertical arrow is the map induced by the morphism of sheaves $K^M_r \rightarrow K^M_r/2 \simeq \Gamma$ and the bottom horizontal map is the connecting homomorphism associated with the exact sequence of sheaves
\[
\begin{array}{ccc}
0 & \rightarrow & \Gamma^{r+1}(\det E) \\
\rightarrow & \rightarrow & \rightarrow \\
0 & \rightarrow & \Gamma^{r+1}(\det E) \\
\end{array}
\begin{array}{ccc}
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\Gamma(\det E) & \rightarrow & \Gamma \\
\downarrow & & \downarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\end{array}
\rightarrow 0.
\]

Now, the image of the Euler class $e(\xi)$ in $H^r(X, K^M_r) \cong CH^r(X)$ is precisely the top Chern class. Thus, if $\xi$ is a vector bundle with vanishing $r$-th Chern class, then there is an element $e_q(\xi)$ of $H^r(X, \Gamma^{r+1}(\det E))$ that maps to $e(\xi)$, and any two such elements lifting $e(\xi)$ differ by elements in the image of $H^{r-1}(X, K^M_r)$.

There is also a reduction map
\[
H^r(X, \Gamma^{r+1}(\det E)) \rightarrow H^r(X, K^M_{r+1}/2),
\]
and the image of a lift $e_q(\xi)$ determines an element of $H^r(X, K^M_{r+1}/2)$. We also have the composite map
\[
H^{r-1}(X, K^M_r) \rightarrow H^r(X, \Gamma^{r+1}(\det E)) \rightarrow H^r(X, K^M_{r+1}/2).
\]
This composite factors as follows.

**Lemma 5.1.** The composite $H^{r-1}(X, K^M_r) \rightarrow H^r(X, K^M_{r+1}/2)$ factors as
\[
H^{r-1}(X, K^M_r) \rightarrow H^{r-1}(X, K^M_r/2) \rightarrow H^r(X, K^M_{r+1}/2),
\]
where the first map is the reduction map.

**Proof.** This is a straightforward consequence of Diagram (5.1).
Secondary characteristic classes revisited

If \( \xi : E \to X \) is a rank \( r \) vector bundle on \( X \), we can choose a lift \( e_q(\xi) \) of the Euler class to \( H^r(X, \Gamma^{r+1}) \). The choice of such a lift is well-defined up to an element of \( H^{r-1}(X, K^M_r) \), and Theorem 4.12 shows that the image of \( e_q(\xi) \) in \( H^r(X, K^M_{r+1}/2) \) is well-defined up to the image of \((Sq^2 + c_1(\xi)\cup) : H^{r-1}(X, K^M_r/2) \to H^r(X, K^M_{r+1}/2)\).

**Definition 5.2.** If \( \xi : E \to X \) is a rank \( r \) vector bundle on \( X \) with \( c_r(\xi) = 0 \), then the coset of the image of \( e_q(\xi) \in H^r(X, K^M_{r+1}/2)/(Sq^2 + c_1(\xi)\cup)(H^{r-1}(X, K^M_r/2)) \) is independent of the choice of lift \( e_q(\xi) \in H^r(X, \Gamma^{r+1}) \) of \( e(\xi) \) and will be denoted \( \Phi_{Sq^2}(\xi) \).

The next result is implied by Definition 5.2 and the fact that if a vector bundle splits off a free rank 1 summand then the Euler class \( e(\xi) \) is trivial; this result implies Theorem 1 by means of the identifications of Corollary 3.4.

**Theorem 5.3.** If \( \xi : E \to X \) is a rank \( r \) vector bundle on \( X \) with \( c_r(\xi) = 0 \), then \( \xi \) splits off a free rank 1 summand only if, in addition, \( \Phi_{Sq^2}(\xi) = 0 \).

**Remark 5.4.** We expect it is possible to show that the coset \( \Phi_{Sq^2}(\xi) \) of Definition 5.2 coincides with the coset \( \Phi_{I}(\xi) \) of Definition 2.3 when \( I \) corresponds to \( Sq^2 \) (via Example 2.4). It is natural to attempt to check this is in the “universal case” corresponding to the universal vector bundle with trivial top Chern class over the space \( K_r \) discussed in Section 2. However, there are very few techniques for computing motivic cohomology of spaces such as \( K_r \) at the moment.

**Proof of Corollary 4**

Suppose \( k \) has cohomological dimension \( \leq 2 \), \( X \) is a smooth affine \( d \)-fold over \( k \), and \( \xi : E \to X \) is a rank \( d \) vector bundle on \( X \). If \( c_d(E) = 0 \), then the Euler class of \( E \) lifts to an element \( e_q(\xi) \in H^d(X, \Gamma^{d+1}(\det \xi)) \). If \( k \) has cohomological dimension 1, then \( H^d(X, \Gamma^{d+1}(\det \xi)) = 0 \) by [AF12b, Proposition 5.6] (the assumption regarding secondary classes is unnecessary here), and so the Euler class of \( E \) is trivial, i.e., \( E \) splits off a free rank 1 summand.

Suppose \( k \) has cohomological dimension 2. In that case, \( e_q(\xi) \) is well-defined modulo the image of \( H^{d-1}(X, K^M_d/2) \), and the image of \( e_q(\xi) \) in \( H^d(X, \Gamma^{d+1}/\Gamma^{d+2}) \) is well-defined modulo the image of \((Sq^2 + c_1(\xi)\cup)H^{d-1}(X, K^M_d/2)\). By assumption, the secondary characteristic class \( \Phi_{Sq^2}(\xi) \) is trivial. Thus, we can find a representative \( e_q(\xi) \) whose image in \( H^d(X, \Gamma^{d+1}/\Gamma^{d+2}) \) is zero. In that case, \( e_q(\xi) \) lifts to \( H^d(X, \Gamma^{d+2}(\det \xi)) \). Then, again applying [AF12b, Proposition 5.6], we conclude that \( H^d(X, \Gamma^{d+2}(\det \xi)) = 0 \). It follows that \( e_q(\xi) \) is trivial and therefore that \( e(\xi) \) is trivial. Thus, \( \xi \) splits off a free rank 1 summand.

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