ON THE LUSTERNIK-SCHNIRELMANN CATEGORY
OF SPACES
WITH 2-DIMENSIONAL FUNDAMENTAL GROUP

ALEXANDER N. DRANISHNIKOV

(Communicated by Daniel Ruberman)

Abstract. The following inequality
\[ \text{cat}_{LS} X \leq \text{cat}_{LS} Y + \left\lceil \frac{hd(X) - r}{r + 1} \right\rceil \]
holds for every locally trivial fibration \( f : X \to Y \) between ANE spaces which
admits a section and has the \( r \)-connected fiber, where \( hd(X) \) is the homotopical
dimension of \( X \). We apply this inequality to prove that
\[ \text{cat}_{LS} X \leq \text{cd}(\pi_1(X)) + \left\lceil \frac{\dim X - 1}{2} \right\rceil \]
for every complex \( X \) with \( \text{cd}(\pi_1(X)) \leq 2 \), where \( \text{cd}(\pi_1(X)) \) denotes the coho-

1. Introduction

In \cite{DKR} we proved that if the Lusternik-Schnirelmann category of a closed \( n \)-

manifold, \( n \geq 3 \), equals 2, then the fundamental group of \( M \) is free. In the opposite
direction we proved that if the fundamental group of an \( n \)-manifold is free, then
\( \text{cat}_{LS} M \leq n - 2 \). J. Strom proved that \( \text{cat}_{LS} X \leq \frac{2}{3}n \) for every \( n \)-complex, \( n > 4 \),
with free fundamental group \cite{St}. Yu. Rudyak conjectured that the coefficient
\( 2/3 \) in Strom’s result could be improved to \( 1/2 \). Precisely, he conjectured that the

function \( f \) defined as
\[ f(n) = \max\{\text{cat}_{LS} M^n \} \]
is asymptotically \( \frac{1}{3}n \), where the maximum is taken over all closed \( n \)-manifolds with free fundamental group.

In this paper we prove Rudyak’s conjecture. Our method gives the same estimate

for \( n \)-complexes. Moreover, we give the same asymptotic upper bound for \( \text{cat}_{LS} \) of

\( n \)-complexes with the fundamental group of cohomological dimension \( \leq 2 \). In view

of this, the following generalization of Rudyak’s conjecture seems to be natural.

Conjecture 1.1. For every \( k \) the function \( f_k \) defined as
\[ f_k(n) = \max\{\text{cat}_{LS} M^n \mid \text{cd}(\pi_1(M^n) \leq k) \} \]
is asymptotically \( \frac{1}{k}n \).

The smallest \( k \) when it is unknown is 3.
This paper is organized as follows. Section 2 is an introduction to the Lusternik-Schnirelmann category based on an analogy with dimension theory. Section 3 contains a fibration theorem for $\text{cat}_{LS}$. In Section 4 this fibration theorem is applied for the proof of Rudyak’s conjecture.

2. Kolmogorov-Ostrand’s approach to the Lusternik-Schnirelmann category

A subset $A \subset X$ of a topological space $X$ is called $X$-contractible if it can be contracted to a point in $X$. A cover $U$ of a topological space $X$ by $X$-contractible sets is called $X$-contractible. By definition, $\text{cat}_{LS} X \leq n$ if there is an $X$-contractible open cover $U = \{U_0, \ldots, U_n\}$ of $X$ that consists of $n+1$ sets.

We recall [CLOT] that a sequence $\emptyset = O_0 \subset O_1 \subset \cdots \subset O_{n+1} = X$ is called categorical of length $n+1$ if each difference $O_{i+1} \setminus O_i$ is contained in an $X$-contractible open set. It was proven in [CLOT] that $\text{cat}_{LS} X \leq n$ if and only if $X$ admits a categorical sequence of length $n+1$.

Let $U = \{U_\alpha\}_{\alpha \in A}$ be a family of sets in a topological space $X$. Formally, it is a function $U : A \to 2^X \setminus \{\emptyset\}$ from the index set to the set of nonempty subsets of $X$. Thus, it is allowed to have $U_\alpha = U_\beta$ for $\alpha \neq \beta$. The sets $U_\alpha$ in the family $U$ will be called elements of $U$. The multiplicity of $U$ (or the order) at a point $x \in X$, denoted $\text{Ord}_x U$, is the number of elements of $U$ that contain $x$. The multiplicity of $U$ is defined as $\text{Ord}_x U = \sup_{x \in X} \text{Ord}_x U$. A family $U$ is a cover of $X$ if $\text{Ord}_x U \neq 0$ for all $x$. A cover $U$ is a refinement of another cover $C$ ($U$ refines $C$) if for every $U \in U$ there exists $C \in C$ such that $U \subset C$. We recall that the covering dimension of a topological space $X$ does not exceed $n$, $\dim X \leq n$, if for every open cover $C$ of $X$ there is an open refinement $U$ with $\text{Ord}_x U \leq n+1$.

We recall that a family $F$ of subsets of a topological space $X$ is called locally finite if for every $x \in X$ there is a neighborhood $U$ of $x$ which has a nonempty intersection at most with finitely many sets from $F$. The following proposition makes the LS-category analogous to the covering dimension.

**Proposition 2.1.** For a paracompact topological space $X$, $\text{cat}_{LS} X \leq n$ if and only if $X$ admits an $X$-contractible locally finite open cover $\mathcal{V}$ with $\text{Ord}_x \mathcal{V} \leq n+1$.

**Proof.** If $\text{cat}_{LS} X \leq n$, then by the definition, $X$ admits an open contractible cover that consists of $n+1$ sets and therefore its multiplicity is at most $n+1$.

Let $\mathcal{V}$ be a contractible cover of $X$ of multiplicity $\leq n+1$. We construct a categorical sequence $O_0 \subset O_1 \subset \cdots \subset O_{n+1}$ of length $n+1$. We define $O_1 = \{x \in X \mid \text{Ord}_x \mathcal{V} = n+1\}$. Note that

$$O_1 = \bigcup_{\{V_0, \ldots, V_n\} \subset \mathcal{V}} V_0 \cap \cdots \cap V_n.$$

Note that this is a disjoint union and every nonempty summand is $X$-contractible. Thus $O_1$ is $X$-contractible. Next, we define $O_2 = \{x \in X \mid \text{Ord}_x \mathcal{V} \geq n\}$. Then

$$O_2 \setminus O_1 = \bigcup_{\{V_0, \ldots, V_{n-1}\} \subset \mathcal{V}} (V_0 \cap \cdots \cap V_{n-1} \setminus O_1)$$

is a disjoint union of closed in $O_2$ subsets. Since $\mathcal{V}$ is locally finite, the family of nonempty summands

$$\{V_0 \cap \cdots \cap V_{n-1} \setminus O_1 \mid V_0, \ldots, V_{n-1} \in \mathcal{V}, V_0 \cap \cdots \cap V_{n-1} \setminus O_1 \neq \emptyset\}$$

is a disjoint union of closed in $O_2$ subsets. Since $\mathcal{V}$ is locally finite, the family of nonempty summands
is locally finite. We recall that every disjoint locally finite family of closed subsets is discrete. Hence there are open (in $O_2$ and hence in $X$) disjoint neighborhoods $W_{V_0,\ldots,V_{n-1}}$ of these summands $V_0\cap\cdots\cap V_{n-1}\setminus O_1$. By taking $W_{V_0,\ldots,V_{n-1}}\cap V_0$ we may assume that the neighborhood of the summand $V_0\cap\cdots\cap V_{n-1}\setminus O_1$ is contained in $V_0$. Thus, we may assume that all neighborhoods $W_{V_0,\ldots,V_{n-1}}$ are $X$-contractible. Define $O_3=\{x\in X\mid \text{Ord}_x\mathcal{V}\geq n-1\}$ as the union of $(n-1)$-fold intersections and so on. In general, $O_k=\{x\in X\mid \text{Ord}_x\mathcal{V}\geq n-k+2\}$. Similarly, 

$$O_{k+1}\setminus O_k = \bigcup_{\{V_0,\ldots,V_{n-k}\}\subset\mathcal{V}} (V_0\cap\cdots\cap V_{n-k}\setminus O_k)$$

is a disjoint union of closed in $O_{k+1}$ subsets. Since the family of nonempty summands in this union is locally finite, there are open in $O_{k+1}$, and hence in $X$, disjoint neighborhoods of these summands $V_0\cap\cdots\cap V_{n-k}\setminus O_k$ such that each neighborhood lies in some $X$-contractible set $V\in\mathcal{V}$.

Then $O_{n+1}$ is the union of elements of $\mathcal{V}$ (1-fold intersections) and hence $O_{n+1}=X$. The categorical sequence conditions are satisfied. 

A family $\mathcal{U}$ of subsets of $X$ is called a $k$-cover, $k\in\mathbb{N}$ if every subfamily of $k$ elements forms a cover of $X$.

**Example.** Let

$$U = \bigcup_{i\in\mathbb{Z}} (m_i, m(i+1)-1)$$

be the union of disjoint intervals in $\mathbb{R}$ of length $m-1$ with the distance 1 between any two consecutive intervals. Let $U = \{T_rU \mid r=0,\ldots,m-1\}$ be the family of translates $T_rU = \{x+r\mid x\in U\}$ of $U$. Clearly, $\mathcal{U}$ is a 3-cover of $\mathbb{R}$ that consists of $m$ subsets.

If we take the intervals of length $m-k$ and the distance $k$,

$$U = \bigcup_{i\in\mathbb{Z}} (m_i, m(i+1)-k),$$

then $\mathcal{U} = \{T_rU \mid r=0,\ldots,m-1\}$ is a $(k+2)$-cover that consists of $m$ subsets. The proof can be derived from the following:

**Proposition 2.2.** A family $\mathcal{U}$ that consists of $m$ subsets of $X$ is an $(n+1)$-cover of $X$ if and only if $\text{Ord}_x\mathcal{U} \geq m-n$ for all $x\in X$.

**Proof.** If $\text{Ord}_x\mathcal{U} < m-n$ for some $x\in X$, then $n+1 = m - (m-n) + 1$ elements of $\mathcal{U}$ do not cover $x$.

If $n+1$ elements of $\mathcal{U}$ do not cover some $x$, then $\text{Ord}_x\mathcal{U} \leq m-(n+1) < m-n$. □

Inspired by the work of Kolmogorov on Hilbert’s 13th problem, Ostrand gave the following characterization of the covering dimension [Os].

**Theorem 2.3 (Ostrand).** A metric space $X$ is of dimension $\leq n$ if and only if for each open cover $\mathcal{C}$ of $X$ and each integer $m \geq n+1$, there exist $m$ disjoint families of open sets $\mathcal{U}_1,\ldots,\mathcal{U}_m$ such that their union $\bigcup\mathcal{U}_i$ is an $(n+1)$-cover of $X$ and it refines $\mathcal{C}$.

Let $\mathcal{U}$ be a family of subsets in $X$ and let $A \subset X$. We denote by $\mathcal{U}|_A = \{U \cap A \mid U \in \mathcal{U}\}$ the restriction of $\mathcal{U}$ to $A$. 


Definition 2.4. Let \( f : X \to Y \) be a map. An open cover \( \mathcal{U} = \{U_0, U_1, \ldots, U_n\} \) of \( X \) is called uniformly \( f \)-contractible if for every \( y \in Y \) there is a neighborhood \( V \) such that the restriction \( \mathcal{U}|_{f^{-1}(V)} \) of \( \mathcal{U} \) to the preimage \( f^{-1}(V) \) consists of \( X \)-contractible sets.

We will use uniformly \( f \)-contractible covers to give in the next section an alternative extension of the Lusternik-Schnirelmann category to mappings. The standard extension \( \text{cat}_{\text{LS}}(f) \) satisfies the equalities \( \text{cat}_{\text{LS}}(1_X) = \text{cat}_{\text{LS}}X \) and \( \text{cat}_{\text{LS}}(c) = 0 \), where \( 1_X \) and \( c : X \to * \) are the identity map and the constant map respectively. Our extension \( \text{cat}_{\text{LS}}^* \) satisfies the opposite: \( \text{cat}_{\text{LS}}^*(c) = \text{cat}_{\text{LS}}X \). Also it satisfies \( \text{cat}_{\text{LS}}^*(1_X) = 0 \) for locally contractible spaces (see §3).

Theorem 2.5. Let \( \mathcal{U} = \{U_0, \ldots, U_n\} \) be an open cover of a normal topological space \( X \). Then for any \( m = n, n+1, \ldots, \infty \) there is an open \((n+1)\)-cover of \( X \), \( U_m = \{U_0, \ldots, U_m\} \) such that for \( k > n \), \( U_k = \bigcup_{i=0}^n V_i \) is a disjoint union with \( V_i \subset U_i \).

In particular, if \( \mathcal{U} \) is \( X \)-contractible, the cover \( U_m \) is \( X \)-contractible. If \( \mathcal{U} \) is uniformly \( f \)-contractible for some \( f : X \to Z \), the cover \( U_m \) is uniformly \( f \)-contractible.

Proof. We construct the family \( U_m \) by induction on \( m \). For \( m = n \) we take \( U_m = \mathcal{U} \).

Let \( U_{m-1} = \{U_0, \ldots, U_{m-1}\} \) be the corresponding family for \( m > n \). By Proposition 2.2, \( \text{Ord}\,\mathcal{U} \geq m-n \). Consider \( Y = \{x \in X \mid \text{Ord}\,\mathcal{U} = m-n\} \). Clearly, it is a closed subset of \( X \). If \( Y = \emptyset \), then by Proposition 2.2, \( \mathcal{U} \) is an \( n \)-cover and we can add \( U_m = U_0 \) to obtain a desired \((n+1)\)-cover. Assume that \( Y \neq \emptyset \). We show that for every \( i \leq n \), the set \( Y \cap U_i \) is closed in \( X \). Let \( x \) be a limit point of \( Y \cap U_i \) that does not belong to \( U_i \). Let \( U_{i_1}, \ldots, U_{i_{m-n}} \) be the elements of the cover \( \mathcal{U} \) that contain \( x \in Y \). The limit point condition implies that \((U_{i_1} \cap \cdots \cap U_{i_{m-n}}) \cap (Y \cap U_i) \neq \emptyset \). Then \( \text{Ord}\,\mathcal{U} = m-n+1 \) for all \( y \in Y \cap U_i \cap U_{i_0} \cap \cdots \cap U_{i_{m-n}} \), a contradiction.

We define recursively \( F_0 = Y \cap U_0 \) and \( F_{i+1} = Y \cap U_{i+1} \setminus (\bigcup_{k=0}^i U_k) \). Note that \( \{F_i\}_{i=0}^n \) is a disjoint finite family of closed subsets with \( \bigcup_{i=0}^n F_i = Y \). Since \( X \) is normal, we can fix disjoint open neighborhoods \( V_i \) of \( F_i \) with \( V_i \subset U_i \). We define \( U_m = \bigcup_{i=0}^n V_i \). In view of Proposition 2.2, \( U_0, \ldots, U_{m-1}, U_m \) is an \((n+1)\)-cover.

Clearly, if all \( U_i \) are \( X \)-contractible, \( i \leq n \), then \( U_m \) is \( X \)-contractible. If all \( U_i \) are uniformly \( f \)-contractible, for some \( f : X \to Z \), then \( U_m \) is uniformly \( f \)-contractible.

Corollary 2.6. For a normal topological space \( X \), \( \text{cat}_{\text{LS}}X \leq n \) if and only if for any \( m > n \), \( X \) admits an open \((n+1)\)-cover by \( m \) \( X \)-contractible sets.

This corollary is a \( \text{cat}_{\text{LS}} \)-analog of Ostrand’s theorem. It also can be found in with further reference to [CLOT] in [CLOT].

3. Fibration theorems for \( \text{cat}_{\text{LS}} \)

Definition 3.1. The \( * \)-category \( \text{cat}_{\text{LS}}^*(f) \) of a map \( f : X \to Y \) is the minimal \( n \) if it exists, such that there is a uniformly \( f \)-contractible open cover \( \mathcal{U} = \{U_0, U_1, \ldots, U_n\} \) of \( X \).

Note that \( \text{cat}_{\text{LS}}^*(c) = \text{cat}_{\text{LS}}X \) for a constant map \( c : X \to pt \). More generally, \( \text{cat}_{\text{LS}}^*(\pi) = \text{cat}_{\text{LS}}X \) for the projection \( \pi : X \times Y \to Y \).

Theorem 3.2. The inequality \( \text{cat}_{\text{LS}}X \leq \dim Y + \text{cat}_{\text{LS}}^* f \) holds true for any continuous map of a normal space.
fibrations

Proof. The requirements to the spaces in the theorem are that the Ostrand theorem holds true for \( Y \); i.e. they are fairly general (say, \( Y \) is normal).

Let \( \dim Y = n \) and \( \cat_0 S^* f = m \). Let \( \mathcal{U} = \{ U_0, \ldots, U_m \} \) be a uniformly \( f \)-contractible cover of \( X \). For \( y \in Y \) denote by \( V_y \) a neighborhood of \( y \) from the definition of the uniform \( f \)-contractibility. In view of Theorem 2.2 there is a refinement \( \mathcal{V} = V_0 \cup \cdots \cup V_{n+m} \) of the cover \( \{ V_y \mid y \in Y \} \) of \( Y \) such that each family \( V_i \) is disjoint and \( \mathcal{V} \) is an \((n+1)\)-cover. Let \( V_i = \bigcup V_i \).

We apply Theorem 2.5 to extend the family \( \mathcal{U} \) to a uniformly \( f \)-contractible \((m+1)\)-cover \( \{ U_0, \ldots, U_{n+m} \} \). Consider the family \( \mathcal{W} = \{ f^{-1}(V_i) \cap U_i \}_{0 \leq i \leq n+m} \). Note that it is \( X \)-contractible. Thus, in order to get the inequality \( \cat_0 LS X \leq n+m \) it suffices to show that \( \mathcal{W} \) is a cover of \( X \). Since \( \mathcal{V} \) is an \((n+1)\)-cover, by Proposition 2.2 every \( y \in Y \) is covered by \( m+1 \) elements of \( \mathcal{V}, V_{y_0}, \ldots, V_{y_m} \). Since \( \{ U_0, \ldots, U_{n+m} \} \) is an \((m+1)\)-cover, the family \( U_{y_0}, \ldots, U_{y_m} \) covers \( X \). Therefore the family \( f^{-1}(V_{y_0}) \cap U_{y_0}, \ldots, f^{-1}(V_{y_m}) \cap U_{y_m} \) covers the fiber \( f^{-1}(y) \). Since \( y \in Y \) is arbitrary, \( \mathcal{W} \) covers all \( X \).

\( \square \)

Corollary 3.3 (Corollary 9.35 [CLOT], [OW]). Let \( p : X \to Y \) be a closed map of ANE. If each fiber \( p^{-1}(y) \) is contractible in \( X \), then \( \cat_0 LS X \leq \dim Y \).

Proof. In this case \( \cat_0 LS^* p = 0 \). Indeed, since \( X \) is an ANE, a contraction of \( p^{-1}(y) \) to a point can be extended to a neighborhood \( U \). Since the map \( p \) is closed there is a neighborhood \( V \) of \( y \) such that \( p^{-1}(V) \subset U \).

We recall that the homotopical dimension of a space \( X \), \( \text{hd}(X) \), is the minimal dimension of a CW-complex homotopy equivalent to \( X \) [CLOT].

Proposition 3.4. Let \( p : E \to X \) be a fibration with \((n-1)\)-connected fiber where \( n = \text{hd}(X) \). Then \( p \) admits a section.

Proof. Let \( h : Y \to X \) be a homotopy equivalence with the homotopy inverse \( g : X \to Y \), where \( Y \) is a \( C_0 \)-complex of dimension \( n \). Since the fiber of \( p \) is \((n-1)\)-connected, the map \( h \) admits a lift \( h' : Y \to E \). Let \( H \) be a homotopy connecting \( h \circ g \) with \( 1_X \). By the homotopy lifting property there is a lift \( H' : X \times I \to E \) of \( H \) with \( H|_{X \times \{0\}} = h' \circ g \). Then the restriction \( H'|_{X \times \{1\}} \) is a section. \( \square \)

We introduce a fiberwise version of Ganea’s fibration. First we recall that the \( k \)-th Ganea’s fibration \( p_k : E_k(Z, z_0) \to Z \) over a path connected space \( Z \) with a fixed base point \( z_0 \) is the fiberwise join product of \( k+1 \) copies of Serre’s path fibrations \( p_0 : PZ \to Z \). We recall that \( PZ \) consists of paths \( \phi \) in \( Z \) with the initial point \( z_0 \) and \( p_0 \) takes \( \phi \) to \( \phi(1) \). Note that \( p_0 \) is a Hurewicz fibration and since the fiberwise join of Hurewicz fibrations is a Hurewicz fibration, so are all \( p_k \) [SV]. Also we note that the fiber of \( p_0 \) is the loop space \( \Omega Z \) and therefore, the fiber of \( p_k \) is the join product \( *^{k+1} \Omega Z \) of \( k+1 \) copies of \( \Omega Z \) (see [CLOT] for more details).

Theorem 3.5 (Ganea, Svarc). For a path connected normal space \( X \) with a non-degenerate base point, \( \cat_0 LS(X) \leq k \) if and only if the Ganea fibration \( p_k : E_k(Z, z_0) \to Z \) has a section.

The proof can be found in [CLOT], [SV].

The Ganea construction can be done simultaneously for all possible choices of the base points \( z_0 \). Namely, for the path fibration we consider the map \( \tilde{p}_0 : C(I, Z) \to Z \times Z \) defined on all paths in \( Z \) as \( \tilde{p}_0(\phi) = (\phi(1), \phi(0)) \). It is easy to check that \( \tilde{p}_0 \) is a Hurewicz fibration. Therefore the (iterated) fiberwise join of \( \tilde{p}_0 \) with itself is a
Hurewicz fibration. Let $\tilde{p}_k : \tilde{E}_k \to Z \times Z$ denote the fiberwise join of $k + 1$ copies of $p_0$. We call $\tilde{p}_k$ the extended Ganea fibration. Note that for every $z_0 \in Z$, the preimage $\tilde{p}_k^{-1}(Z \times \{z_0\})$ is homeomorphic to $E_k(Z, z_0)$ and the restriction of $\tilde{p}_k$ to $\tilde{p}_k^{-1}(Z \times \{z_0\})$ is the Ganea fibration $p_k$ with the base point $z_0$.

Now let $f : X \to Y$ be a locally trivial bundle with a path connected fiber $Z$ and let $f$ admit a section $s : Y \to X$. We define a space

$$E_0 = \{ \phi \in C(I, X) \mid s(f(I)) = \{\phi(0)\} \}$$

to be the space of all paths $\phi$ in $X$ with the initial point $s(y)$ for some $y \in Y$ such that the image of $\phi$ is contained in the fiber $f^{-1}(y)$. The topology in $E_0$ is inherited from $C(I, X)$. We define a map $\xi_0 : E_0 \to X$ by the formula $\xi_0(\phi) = \phi(1)$. Then $\xi_k : E_k \to X$ is defined as the fiberwise join of $k + 1$ copies of $\xi_0$. Formally, we define $E_k$ inductively as a subspace of the join $E_0 \ast E_{k-1}$:

$$E_k = \bigcup \{ \phi \ast \psi \in E_0 \ast E_{k-1} \mid \xi_0(\phi) = \xi_{k-1}(\psi) \},$$

which is the union of all intervals $[\phi, \psi] = \phi \ast \psi$ with the endpoints $\phi \in E_0$ and $\psi \in E_{k-1}$ such that $\xi_0(\phi) = \xi_{k-1}(\psi)$. There is a natural projection $\xi_k : E_k \to X$ that takes all points of each interval $[\phi, \psi]$ to $\phi(0)$.

Note that when $f : X = Z \times Y$ is a trivial bundle and a section $s : Y \to X$ is defined by a point $z_0 \in Z$, then $E_k = E_k(Z, z_0) \times Y$ and $\xi_k = p_k \times 1_Y$ where $p_k : E_k \to Z$ is the Ganea fibration.

**Lemma 3.6.** Let $f : X \to Y$ be a locally trivial bundle between paracompact spaces with a path connected fiber $Z$ and with a section $s : Y \to X$. Then

i. For each $k$ the map $\xi_k : E_k \to X$ is a Hurewicz fibration.

ii. The fiber of $\xi_k$ is precisely the join of $k + 1$ copies of the space of paths from $s(f(x))$ to $x$ which is homeomorphic to $*^{k+1}\Omega Z$.

iii. $\xi_k$ has a section if and only if $X$ has an open cover $U = \{U_0, \ldots, U_k\}$ by sets, each of which admits a fiberwise deformation into $s(Y)$.

**Proof.**

i. In view of Dold’s theorem [Do] it suffices to show that $\xi_k$ is a Hurewicz fibration over $f^{-1}(U)$ for all $U \in U$ for some locally finite cover of $X$. We consider a cover $U$ such that $f$ admits a trivialization over $U$ for all $U \in U$, i.e., fiberwise homeomorphisms $h_U : f^{-1}(U) \to U \times Z$. Then the section $s$ defines a map

$$\sigma_U = \pi_2 \circ h_U \circ s : U \to Z$$

where $\pi_2 : U \times Z \to Z$ is the projection to the second factor. If the map $\sigma_U$ were constant, the fibration $\xi_k$ over $f^{-1}(U)$ would be a Hurewicz fibration being homeomorphic to the product $1_U \times p_k$. In the general case the fibration $\xi_k$ over $f^{-1}(U)$ is obtained as the pull-back of the extended Ganea fibration $\tilde{p}_k : \tilde{E}_k \to Z \times Z$ under the map $(\sigma_U \times 1_Z) \circ h_U : f^{-1}(U) \to Z \times Z$. Hence it is a Hurewicz fibration.

ii. We note that the map $\xi_k$ over the fiber $(f^{-1}(x), s(x))$ coincides with the Ganea fibration $p_k$ for $Z$. Therefore, the fiber of $\xi_k$ coincides with the fiber of $p_k$; i.e., it is $*^{k+1}\Omega Z$.

iii. Note that when $Y = pt$, iii turns into the Ganea–Švarc theorem. Thus, iii can be viewed as a fiberwise version of the Ganea–Švarc theorem.

Suppose $\xi_k$ has a section $\sigma : X \to E_k$. For each $x \in X$ the element $\sigma(x)$ of $*^{k+1}\Omega F$ can be presented as the $(k + 1)$-tuple

$$\sigma(x) = (\phi_0, t_0, \ldots, (\phi_k, t_k)) \mid \sum t_i = 1, t_i \geq 0).$$

We use the notation $\sigma(x)_i = t_i$. Clearly, $\sigma(x)_i$ is a continuous function.
A section $\sigma : X \to E_k$ defines a cover $U = \{U_0, \ldots, U_k\}$ of $X$ as follows:

$$U_i = \{x \in X \mid \sigma(x)_i > 0\}.$$ 

By the construction of $U_i$ for $i \leq n$ for every $x \in U_i$ there is a canonical path connecting $x$ with $sf(x)$. We use these paths to contract a fiberwise deformation of $U_i$ into $s(Y)$.

Moreover, Borel’s construction

Proof. Let $\text{cat} \leq 0(a(\cup \text{U})\ast \text{U})\ast \text{U}$ holds true:

$$\text{X} \leftarrow \text{X} \times \text{E}_\pi \rightarrow \text{E}_\pi$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\text{X} \leftarrow g\ast \text{X} \times_\pi \text{E}_\pi \rightarrow \text{B}_\pi.$$ 

We recall that $[x]$ denotes the smallest integer $n$ such that $x \leq n$.

**Theorem 3.7.** Suppose that a locally trivial fibration $f : X \to Y$ with an $r$-connected fiber $F$ admits a section. Then

$$\text{cat} \leq 0(\ast f) \leq \left\lfloor \frac{\text{hd}(X) - r}{r + 1} \right\rfloor.$$ 

Moreover,

$$\text{cat} \leq 0(X) \leq \text{cat} \leq 0(Y) + \left\lfloor \frac{\text{hd}(X) - r}{r + 1} \right\rfloor.$$ 

Proof. Let $\text{cat} \leq 0(Y) = m$ and $\text{hd}(X) = n$.

Let $s : Y \to X$ be a section. By Lemma 3.6 i–ii $\xi_k$ is a Hurewicz fibration with the fiber the join product $\ast^{k+1}\text{Omega}F$ of $k + 1$ copies of the loop space $\Omega F$. Thus, it is $(k + (k + 1)r - 1)$-connected. By Proposition 3.4 there is a section $\sigma : X \to E_k$ whenever $k(r + 1) + r \geq n$. The smallest such $k$ is equal to $\left\lceil \frac{n}{r+1} \right\rceil$.

By Lemma 3.6 ii a section $\sigma : X \to E_k$ defines a cover $U = \{U_0, \ldots, U_k\}$ by the sets fiberwise contractible to $s(Y)$. Let $U_{m+k} = \{U_0, \ldots, U_{m+k}\}$ be an extension of $U$ to a $(k+1)$-cover of $X$ from Theorem 2.4.

Let $V = \{V_0, \ldots, V_{m+k}\}$ be an open $Y$-contractible $(m+1)$-cover of $Y$. We show that the sets $W_i = f^{-1}(V_i) \cap U_i$ are contractible in $X$ for all $i$. By Theorem 2.5 $U_i$ is fiberwise contractible into $s(Y)$ for $i \leq m + k$. Hence we can contract $f^{-1}(V_i) \cap U_i$ to $s(V_i)$ in $X$. Then we apply a contraction of $s(V_i)$ to a point in $s(Y)$.

Similarly as in the proof of Theorem 2.2 we show that $\{W_i\}_{i=0}^{m+k}$ is a cover of $X$. Since $V$ is an $(m+1)$-cover, by Proposition 2.2 every $y \in Y$ is covered by at least $k + 1$ elements $V_{i_0}, \ldots, V_{i_k}$ of $V$. By the construction $U_{i_0}, \ldots, U_{i_k}$ is a cover of $X$. Hence $W_{i_0}, \ldots, W_{i_k}$ covers $f^{-1}(y)$.

**4. THE LUSTERNIK-SCHNIRELMANN CATEGORY OF COMPLEXES WITH LOW DIMENSIONAL FUNDAMENTAL GROUPS**

**Theorem 4.1.** For every complex $X$ with $\text{cd}(\pi_1(X)) \leq 2$ the following inequality holds true:

$$\text{cat} \leq 0(X) \leq \text{cd}(\pi_1(X)) + \left\lfloor \frac{\text{hd}(X) - 1}{2} \right\rfloor.$$ 

Proof. Let $\pi = \pi_1(X)$ and let $\hat{X}$ denote the universal cover of $X$. We consider Borel’s construction

$$\hat{X} \leftarrow \hat{X} \times E_\pi \rightarrow E_\pi$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$X \leftarrow g\ast \hat{X} \times_\pi \text{E}_\pi \rightarrow \text{B}_\pi.$$
We claim that there is a section $s : B\pi \to \tilde{X} \times_{\pi} E\pi$ of $f$. By the condition $cd\pi \leq 2$ we may assume that $B\pi$ is a complex of dimension $\leq 3$. Note that $f$ is a locally trivial bundle with the fiber $\tilde{X}$. Since the fiber of $f$ is simply connected, there is a lift of the 2-skeleton. The condition $cd\pi \leq 2$ implies $H^3(B\pi, E) = 0$ for every $\pi$-module. Thus, we have no obstruction for the lift of the 3-skeleton (see, for example, [Po], [Th] for the basics of obstruction theory with twisted coefficients).

We apply Theorem 3.7 to obtain the inequality

$$\text{cat}_{LS} X \leq \text{cat}_{LS}(B\pi) + \left\lceil \frac{\text{hd}(\tilde{X} \times_{\pi} E\pi) - 1}{2} \right\rceil.$$ 

Since $g$ is a fibration with the homotopy trivial fiber, the space $\tilde{X} \times_{\pi} E\pi$ is homotopy equivalent to $X$. Thus, $\text{hd}(\tilde{X} \times_{\pi} E\pi) = \text{hd}(X)$. Note that the results of Eilenberg and Ganea [EG] in view of the Stallings-Swan theorem [Sta], [Swan] imply that $\text{cat}_{LS} B\pi = cd\pi$ for all groups $\pi$. □

**Corollary 4.2.** For every complex $X$ with free fundamental group,

$$\text{cat}_{LS} X \leq 1 + \left\lceil \frac{\dim X - 1}{2} \right\rceil.$$ 

Note that this estimate is sharp on $X = S^1 \times \mathbb{C}P^n$.

**Corollary 4.3.** For every 3-dimensional complex $X$ with free fundamental group, $\text{cat}_{LS} X \leq 2$.

This corollary can also be derived from the fact that in the case of a free fundamental group every 2-complex is homotopy equivalent to the wedge of circles and 2-spheres [KR].

It is unclear whether the estimate $\text{cat}_{LS} X \leq 2 + \left\lceil \frac{\dim X - 1}{2} \right\rceil$ is sharp for complexes with $cd(\pi_1(X)) = 2$. It is sharp if the answer to the following question is affirmative.

**Question 4.4.** Does there exists a 4-complex $K$ with free fundamental group and with $\text{cat}_{LS}(K \times S^1) = 4$?

Indeed, for $X = K \times S^1$ we would have the equality $4 = 2 + \left\lceil \frac{5 - 1}{2} \right\rceil$. Note that $cd(\pi_1(X)) = 2$.

**Acknowledgement**

The author is thankful to the referee for valuable remarks.

**References**

CLOT. Cornea, O.; Lupton, G.; Oprea, J.; Tanré, D.: Lusternik-Schnirelmann category. Mathematical Surveys and Monographs, 103. American Mathematical Society, Providence, RI, 2003. MR1990857 (2004e:55001)

Cu. Cuvilliez, M.: LS-catégorie et $k$-monomorphisme. Thèse, Université Catholique de Louvain, 1998.

Do. Dold, A.: Partitions of unity in the theory of fibrations. Ann. of Math. (2) 78 (1963), 223-255. MR0155330 (27:5264)

DKR. Dranishnikov, A.; Katz, M.; Rudyak, Y.: Small values of the Lusternik-Schnirelmann category for manifolds. Geometry and Topology 12 (2008), issue 3, 1711-1727. MR2421138

EG. Eilenberg, S.; Ganea, T.: On the Lusternik-Schnirelmann category of abstract groups. Ann. of Math. (2) 65 (1957), 517-518. MR0085510 (19:552d)

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
KR. Katz, M.; Rudyak, Y.: Lusternik-Schnirelmann category and systolic category of low-dimensional manifolds. *Communications on Pure and Applied Mathematics* **59** (2006), 1433-1456. MR2248895 (2007h:53055)

OW. Oprea, J.; Walsh, J.: Quotient maps, group actions and Lusternik-Schnirelmann category, *Topology Appl.* **117** (2002), 285-305. MR1874091 (2002i:55001)

Os. Ostrand, Ph.: *Dimension of metric spaces and Hilbert’s problem 13*, *Bull. Amer. Math. Soc.* **71** (1965), 619-622. MR0177391 (31:1654)

Po. Postnikov, M.: Classification of the continuous mappings of an arbitrary $n$-dimensional polyhedron into a connected topological space which is aspherical in dimensions greater than unity and less than $n$. (Russian) *Doklady Akad. Nauk SSSR (N.S.)* **67** (1949), 427-430. MR0033522 (11:451c)

Sta. Stallings, J.: Groups of dimension 1 are locally free. *Bull. Amer. Math. Soc.* **74** (1968), 361-364. MR0223439 (36:6487)

St. Strom, J.: Lusternik-Schnirelmann category of spaces with free fundamental group. *Algebr. Geom. Topol.* **7** (2007), 1805-1808. MR2366179 (2008k:55007)

Sv. Švarc, A.: The genus of a fibered space. *Trudy Moskov. Mat. Obšč.* **10** (1961), 217-272; The genus of a fibre space. *Trudy Moskov. Mat. Obšč.* **11** (1962), 99-126; in Amer. Math. Soc. Transl. Series 2, vol. **55**, 1966. MR0154284 (27:4233), MR0151982 (27:1963)

Swan, R.: Groups of cohomological dimension one. *J. Algebra* **12** (1969), 585-610. MR0240177 (39:1531)

Th. Thomas, E.: Seminar on fiber spaces. *Lecture Notes in Mathematics*, **13**, Springer-Verlag, Berlin, New York, 1966. MR0203733 (34:3582)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, 358 LITTLE HALL, GAINESVILLE, FLORIDA 32601-8105

*E-mail address: dranish@math.ufl.edu*