We derive and analyze new diffusion approximations of stationary distributions of Markov chains that are based on second- and higher-order terms in the expansion of the Markov chain generator. Our approximations achieve a higher degree of accuracy compared to diffusion approximations widely used for the past fifty years, while retaining a similar computational complexity. To support our approximations, we present a combination of theoretical and numerical results across three different models. Our approximations are derived recursively through Stein/Poisson equations, and the theoretical results are proved using Stein’s method.

**Key words**: Stein’s method; diffusion approximation; steady-state; convergence rate; moderate deviations

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1. Introduction

We propose a new class of approximations for stationary distributions of Markov chains. The new approximations will be numerically demonstrated to be accurate in three models: the $M/M/n$ queue known as the Erlang-C model, the hospital model proposed in Dai and Shi (2017), and the autoregressive (AR(1)) model studied in Blanchet and Glynn (2018). In addition to numerical results, for the Erlang-C model we provide theoretical guarantees that our approximation achieves higher-order accuracy.

Consider a one-dimensional, positive-recurrent, discrete-time Markov chain (DTMC) $X = \{X(n), n \geq 0\}$ taking values on a subset of $\mathbb{R}$. We introduce our approach in the DTMC setting, but continuous-time Markov chains (CTMC) can be treated analogously; see Section 3 where we treat the Erlang-C model. Call $\mathbb{E}(X(1) - X(0)|X(0) = x)$ the drift of the DTMC. We center and scale our DTMC by defining $\tilde{X} = \{\tilde{X}(n), n \geq 0\}$, where $\tilde{X}(n) = \delta(X(n) - R)$ for some constants $\delta > 0$ and $R \in \mathbb{R}$. We typically take $R$ to be the point where the drift of $X$ equals zero, which also happens to be the equilibrium of the corresponding fluid model; c.f., Stolyar (2015) or Ying (2016). The scaling parameter $\delta$ is related to stochastic fluctuations around $R$.

Let $\tilde{X}(0)$ have the stationary distribution of $\tilde{X}$, let $W = \tilde{X}(0)$, and let $W' = \tilde{X}(1)$. Stationarity implies that

$$\mathbb{E}f(W') - \mathbb{E}f(W) = 0$$

for all $f : \mathbb{R} \to \mathbb{R}$ such that the expectations exist. Setting $\Delta = W' - W$, for sufficiently smooth $f(x)$ we can expand the left-hand side to get

$$0 = \mathbb{E}f(W') - \mathbb{E}f(W) = \mathbb{E}\left[ \sum_{i=1}^{n} \frac{1}{i!} \Delta^i f^{(i)}(W) + \frac{1}{(n+1)!} \Delta^{n+1} f^{(n+1)}(\xi) \right], \quad n \geq 0,$$
where $\xi = \xi^{(n)}$ lies between $W$ and $W'$. Note that $\Delta$ equals our scaling term $\delta$ multiplied by the displacement of the unscaled DTMC. Informally, the DTMCs we consider are those where the moments of the (unscaled) displacement are bounded by a constant independent of $\delta$, while $\delta$ itself is close to zero. In this setting, the right-hand side of (2) is governed by its lower-order terms when $\delta$ is small. This motivates our approximations of $W$.

Letting $\mathcal{W}$ be the state space of $\tilde{X}$, for each $x \in \mathcal{W}$ let $b(x) = \mathbb{E}(\Delta | W = x)$ be the drift of the DTMC at state $x$. Let $(\underline{w}, \overline{w})$ be the smallest interval containing $\mathcal{W}$, and assume $b(x)$ is extended to be defined on all of $(\underline{w}, \overline{w})$. The precise form of the extension is unimportant for the time being and we will see in our examples that this extension often has a natural form. We approximate $W$ by a continuous random variable $Y \in (\underline{w}, \overline{w})$ with density

$$
\frac{\kappa}{v(x)} \exp \left( \int_{0}^{x} \frac{b(y)}{v(y)} dy \right), \quad x \in (\underline{w}, \overline{w}),
$$

where $\kappa$ is the normalizing constant and $v(x) : (\underline{w}, \overline{w}) \to \mathbb{R}_+$ is some function to be specified. We note that the distribution of $Y$ is determined for a given fixed set of system parameters of the Markov chain. In particular, $Y$ is well defined even when no limit is studied, so the stationary distribution of the unscaled DTMC $X$ would then be approximated by $Y/\delta + R$.

To discuss how to choose $v(x)$, suppose $(\underline{w}, \overline{w}) = \mathbb{R}$ and consider the diffusion process $\{Y(t), t \geq 0\}$ given by

$$
Y(t) = Y(0) + \int_{0}^{t} b(Y(s)) ds + \int_{0}^{t} \sqrt{2v(Y(s))} dB(s),
$$

where $\{B(t), t \geq 0\}$ is the standard Brownian motion. Under mild regularity conditions on $b(x)$ and $v(x)$, the above diffusion process is well defined and has a unique stationary
distribution whose density is given by (3); for a proof, see Chapter 15.5 of Karlin and Taylor (1981). Furthermore, the stationary density in (3) is characterized by

$$E_b(Y)f'(Y) + E[v(Y)f''(Y)] = 0$$

for all suitable $f : \mathbb{R} \to \mathbb{R}$. (5)

When one or both of the endpoints of $(w, \overline{w})$ are finite, we would account for this by adding suitable boundary reflection terms.

In this paper we think of $Y$ as being a *diffusion approximation* of $W$. The characterization equation (5) is well known for Markov processes; c.f., Ethier and Kurtz (1986). A related version is called the *basic adjoint relationship* in the context of multidimensional reflecting Brownian motions by Harrison and Williams (1987). Equation (5) is known in the Stein research community as the *Stein equation*; see, for example, Chen et al. (2011).

Ensuring that $Y$ is a good approximation of $W$ requires a careful choice of $v(x)$. If we consider (2) with $n = 2$ and ignore the third-order error term, then a natural choice is to use $v(x) = v_1(x)$, where $v_1(x)$ is an extension of $\frac{1}{2}E(\Delta^2|W = x)$ to all of $(w, \overline{w})$. Choosing a diffusion approximation in such a way was done in Mandelbaum et al. (1998) and Ward and Glynn (2003), as well as more recently in Dai and Shi (2017).

Despite this natural choice, most of the literature in the last fifty years did not use $v_1$ to develop diffusion approximations. Instead, the typical choice is $v(x) = v_0$, where

$$v_0 = v_1(0) = \frac{1}{2}E(\Delta^2|W = 0);$$

(6)

i.e., $v_0$ is $v_1(x)$ evaluated at the fluid equilibrium $x = 0$. For examples, see Halfin and Whitt (1981), Harrison and Nguyen (1993), Gurvich (2014a), Ward (2012). It is usually the case that $v_0$ and $v_1(W)$ are asymptotically close, so using $v_0$ is enough to prove a limit theorem,
which is the focus of most of the diffusion approximation literature. We however, show that using $v_0$ instead of $v_1$ can lead to significant excess error.

One such case is the Erlang-C model. It was shown in Braverman et al. (2016) that for a large class of performance measures, the $v_0$ approximation error is at most $C/\sqrt{R}$, where $R$ is a parameter known as the offered load and $C > 0$ is a constant. In Section 3 we prove this upper bound is tight. On the other hand, the $v_1$ error vanishes at a faster rate of $1/R$. Moreover, the $v_1$ error is much smaller than the $v_0$ error, even in cases when $R$ is small.

Given the performance of the $v_1$ approximation in the Erlang-C model, it is natural to wonder whether the $v_1$ error vanishes at a faster rate (compared to the $v_0$ error) for other models as well. The answer is mixed; e.g., it is not true for the model in Section 4.

In this paper we provide other options for $v(x)$ beyond $v_0$ and $v_1(x)$. For $n \geq 1$, we define a $v_n$ approximation to be one that uses information from the first $n+1$ terms of the Taylor expansion in (2); $v_n$ approximations are not unique. We adopt the convention that $v_n$ can refer to either the function $v_n(x)$, or the $v_n$ approximation itself. As a preview, we can use third-order information from the Taylor expansion is by setting

$$v(x) = v_2(x) = v_2^{(0)}(x) = \max \left\{ \frac{a(x)}{2} - \frac{b(x)c(x)}{3a(x)} - \frac{a(x)}{6} \left( \frac{c(x)}{a(x)} \right)', \eta \right\} \quad \text{for } x \in (w, \bar{w}),$$

where $a(x)$ and $c(x)$ are extensions of $E(\Delta^2|W = x)$ and $E(\Delta^3|W = x)$ to $(w, \bar{w})$, respectively, and $\eta > 0$ is a tuneable parameter selected to keep $v_2(x)$ positive.

We formally motivate and derive (7) in Section 2, where we also elaborate on the need for $\eta$ and how to choose it. Going beyond $v_2$, we derive $v_3$ for the hospital model of Section 4 and the AR(1) model of Section 5. In both cases, numerical work suggests that finding an approximation that achieves either a faster convergence rate of the error to zero, or
a significantly lower approximation error than $v_0$, requires us to go as far as $v_3$. For a discussion on how to determine which $v_n$ to use, see Section 4.1.

This paper is limited to the setting where the Markov chain is one-dimensional because the derivation of $v_n$ for $n \geq 2$ exploits the one-dimensional nature of the Poisson equation; for an example, see Section 2. At present we do not know how to generalize this to the multidimensional setting.

The theoretical framework underpinning our work is Stein’s method, which was pioneered by Stein (1972). Specifically, we use the generator comparison framework of Stein’s method, which dates back to Barbour (1988) and was popularized recently in queueing theory by Gurvich (2014a). We remark that deriving the $v_n$ approximations requires only algebra, which is handy from a practical standpoint as one can derive and implement the approximations numerically without worrying about justifying them theoretically.

In addition to deriving the $v_n$ approximations, we also use Stein’s method to provide theoretical guarantees. For the Erlang-C model, Theorem 3 establishes Cramér-type moderate-deviations error bounds. If $Y$ is an approximation of $W$, then moderate-deviations bounds refer to bounds on the relative error

$$\left| \frac{\mathbb{P}(Y \geq z)}{\mathbb{P}(W \geq z)} - 1 \right| \quad \text{and} \quad \left| \frac{\mathbb{P}(Y \leq z)}{\mathbb{P}(W \leq z)} - 1 \right|.$$ 

Compared to the Kolmogorov distance $\sup_{z \in \mathbb{R}} |\mathbb{P}(W \geq z) - \mathbb{P}(Y \geq z)|$, the relative error is a much more informative measure when the value being approximated is small, as is the case in the approximation of small tail probabilities. For many stochastic systems modeling service operations such as customer call centers and hospital operations, these small probabilities represent important performance metrics; e.g., at most 1% of customers waiting more than 10 minutes before getting into service.
To summarize, our main contribution is to present a new family of $v_n$ approximations for Markov chains. Using a combination of theoretical and numerical results, we show that the $v_n$ approximations perform significantly better than the traditional $v_0$ approximation across three separate models. Our results suggest that $v_1, v_2, v_3, \ldots$ can, and should, be used whenever possible to achieve much greater approximation accuracy. Before moving on to the main body of the paper, we first provide a brief review of related literature.

1.1. Literature Review

*Steady-state diffusion approximations.* In the last fifty years, diffusion approximations have been a major research theme in the applied probability community for approximate steady-state analysis of many stochastic systems; c.f., Kingman (1961), Halfin and Whitt (1981), Harrison and Nguyen (1993), Mandelbaum and Zeltyn (2009). Some of these approximations were initially motivated by *process-level limit theorems* that establish functional central limits in certain asymptotic parameter regions; e.g., Reiman (1984), Bramson (1998), Williams (1998). The pioneering paper of Gamarnik and Zeevi (2006) initiated a wave of research providing *steady-state limit theorems*, justifying steady-state approximations on top of process-level convergence. For some examples of these, see Tezcan (2008), Zhang and Zwart (2008), Budhiraja and Lee (2009), Katsuda (2010), Gamarnik and Stolyar (2012), Dai et al. (2014), Gurvich (2014b), Ye and Yao (2016).

Steady-state limit theorems do not provide a rate of convergence or an error bound. Recently, building on earlier work by Gurvich et al. (2014), Gurvich (2014a) developed a general approach to proving the rate of convergence for steady-state performance measures of many stochastic systems. In the setting of the $M/\text{Ph}/n + M$ queue with phase-type
service time distributions, Braverman and Dai (2017) refined the approach in Gurvich (2014a), casting it into the Stein framework that has been extensively studied in the last fifty years. The Stein framework allows one to obtain an error bound, not just a limit theorem, for approximate steady-state analysis of a stochastic system with a fixed set of system parameters. Readers are referred to Braverman et al. (2016) for a tutorial introduction to using Stein’s method for steady-state diffusion approximations of Erlang-A and Erlang-C models, where error bounds were established under a variety of metrics, including the Wasserstein distance, Kolmogorov distance, and moment difference.

Stein’s method and moderate deviations. Stein’s method was first introduced by Stein (1972). We refer the reader to the book by Chen et al. (2011) for an introduction to Stein’s method. Moderate deviations date back to Cramér (1938), who obtained expansions for tail probabilities for sums of independent random variables about the normal distribution. Stein’s method for moderate deviations for general dependent random variables was first studied in Chen et al. (2013b). See Chen et al. (2013a), Shao et al. (2018), Zhang (2019), Fang et al. (2019) for further developments.

Refined mean-field approximations. First-order approximations, such as mean-field, or fluid model approximations capture the deterministic flow of the Markov chain while ignoring the stochastic effects. A recent series of papers, Gast and Van Houdt (2017), Gast et al. (2018), Gast et al. (2019), explored refined mean-field approximations for computing moments of the Markov chain stationary distribution. In those papers, the authors were able to explicitly compute correction terms to the mean-field approximation, which significantly improves the accuracy of the approximation and speeds up the rate at which the approximation error converges to zero. However, the computation of these correction
terms rests on assuming that the mean-field model is globally exponentially stable and that the drift of the Markov chain is differentiable. These assumptions fail to hold even for some basic queueing models; e.g., the Erlang-C model.

1.2. Notation and Organization of the Paper

For \( a, b \in \mathbb{R} \), we use \( a^+, a^-, a \land b \), and \( a \lor b \) to denote \( \max(a, 0) \), \( \max(-a, 0) \), \( \min(a, b) \), and \( \max(a, b) \), respectively. We adopt the convention that \( \sum_{i=k_2}^{k_1} = 0 \) if \( k_2 < k_1 \). In Section 2, we derive several versions of \( v_2 \) and discuss how to analyze the approximation error using Stein’s method. In Sections 3–5, we study the performance of various \( v_n \) approximations for three different Markov chains. To keep the main paper a reasonable length, some details of the proofs are left to the Appendix.

2. Deriving the Diffusion Approximations

In the previous section, we said that for \( n \geq 1 \), a \( v_n \) approximation is one that uses information from the first \( n + 1 \) terms of the Taylor expansion in (2). In this section, we justify \( v_2(x) \) proposed in (7) by tapping into the third-order terms in (2). For examples of accessing fourth-order terms, we refer the reader to the derivations of \( v_3 \) for the models in Sections 4 and 5. What follows can be repeated for continuous-time Markov chains (CTMC), with the identity \( \mathbb{E}Gf(W) = 0 \) replacing \( \mathbb{E}f(W') - \mathbb{E}f(W) = 0 \), where \( G \) is the generator of the CTMC. As our starting point, we recall from (2) that

\[
0 = \mathbb{E}f(W') - \mathbb{E}f(W) = \mathbb{E} \left[ \sum_{i=1}^{n} \frac{1}{i!} \Delta^i f^{(i)}(W) + \frac{1}{(n+1)!} \Delta^{n+1} f^{(n+1)}(\xi) \right],
\]
where $\Delta = W' - W$, and that $b(x), a(x)$, and $c(x)$ are extensions of $E(\Delta|W = x), E(\Delta^2|W = x)$, and $E(\Delta^3|W = x)$ to $(\underline{w}, \overline{w})$, respectively. Let $d(x)$ be an extension of $E(\Delta^4|W = x)$ to $(\underline{w}, \overline{w})$. Setting $n = 3$ in the expansion above yields

$$\mathbb{E} b(W)f'(W) + \frac{1}{2}\mathbb{E}a(W)f''(W) + \frac{1}{6}\mathbb{E}c(W)f'''(W) = -\frac{1}{24}\mathbb{E}d(W)f^{(4)}(\xi_1)$$

where $\xi_1$ lies between $W$ and $W'$. We implicitly assume $f(x)$ is sufficiently differentiable and the expectations above exist. Since $\Delta$ is small, we treat the right-hand side as error and use the left-hand side to derive a diffusion approximation. The challenge to overcome is that the stationary density of the diffusion is characterized by (5), which considers only the first two derivatives of a function $f(x)$, whereas the left-hand side of (8) contains three derivatives. We therefore convert $f'''(W)$ into an expression involving $f''(W)$ plus some error. Consider (2) again, but with $n = 2$:

$$\mathbb{E} b(W)f'(W) + \frac{1}{2}\mathbb{E}a(W)f''(W) = -\frac{1}{6}\mathbb{E}c(W)f'''(\xi_2),$$

for some $\xi_2$ between $W$ and $W'$. Fix $f(x)$ and let $g(x) = \int_0^x \frac{c(y)}{a(y)} f''(y) dy$. Note that

$$g''(x) = \left( \frac{c(x)}{a(x)} f''(x) \right)' = \left( \frac{c(x)}{a(x)} \right)' f''(x) + \frac{c(x)}{a(x)} f'''(x).$$

Evaluating (9) with $g(x)$ in place of $f(x)$ there yields

$$\mathbb{E} \frac{b(W)c(W)}{a(W)} f''(W) + \mathbb{E} \frac{a(W)}{2} \left( \frac{c(W)}{a(W)} \right)' f''(W) + \frac{1}{2}\mathbb{E}c(W)f'''(W) = -\frac{1}{6}\mathbb{E}c(W)g''(\xi_2).$$

Rearranging terms, we have

$$\frac{1}{6}\mathbb{E}c(W)f'''(W) = -\mathbb{E} \left( \frac{b(W)c(W)}{3a(W)} + \frac{a(W)}{6} \left( \frac{c(W)}{a(W)} \right)' \right) f''(W) - \frac{1}{18}\mathbb{E}c(W)g''(\xi_2).$$
Substituting (10) into (8), we obtain
\[
E b(W) f'(W) + E \left( \frac{a(W)}{2} - \frac{b(W)c(W)}{3a(W)} - \frac{a(W)}{6} \left( \frac{c(W)}{a(W)} \right)' \right) f''(W)
= \frac{1}{18} E c(W) g'''(\xi_2) - \frac{1}{24} E d(W) f^{(4)}(\xi_1).
\]
(11)

The left-hand side resembles the generator of a diffusion process. Define
\[
v_2(x) = \frac{a(x)}{2} - \frac{b(x)c(x)}{3a(x)} - \frac{a(x)}{6} \left( \frac{c(x)}{a(x)} \right)', \quad x \in (\underline{w}, \overline{w}),
\]
(12)
and let \( v_2(x) = (v_2(x) \vee \eta) \) for some \( \eta > 0 \) to recover the \( v_2(x) \) in (7). The value of \( \eta \) should be chosen close to zero, and if \( \inf_{x \in (\underline{w}, \overline{w})} v_2(x) > 0 \), then we can pick \( v_2(x) = v_2(x) \).

We enforce \( v(x) > 0 \) because there may be issues with the integrability of the density in (3) if \( v(x) \) is allowed to be negative. For instance, in all three examples considered in this paper, \( b(x) > 0 \) when \( x \) is to the left of the fluid equilibrium of \( W \), and \( b(x) < 0 \) when \( x \) is to the right of the fluid equilibrium; i.e., the DTMC drifts back toward its equilibrium. This drift toward the equilibrium is intimately tied to the positive recurrence of the DTMC and can therefore be thought of as a reasonable assumption even if we go beyond this paper’s three examples. Now, if \( v(x) \) is allowed to be negative, it may be that \( \kappa = \infty \) in (3); e.g., if \( v(x) < 0 \) for \( x > K \) for some threshold \( K \). Conversely, \( \inf_{x \in (\underline{w}, \overline{w})} v(x) > 0 \) is sufficient to ensure that \( \kappa < \infty \) in all three of our examples. Another, more intuitive, reason that \( v(x) > 0 \) is that a diffusion coefficient cannot be negative.

2.1. The \( v_2 \) Approximation Error

Let us discuss the error of our \( v_2 \) approximation. For simplicity, let us assume that \( (\underline{w}, \overline{w}) = \mathbb{R} \) and that \( \inf_{x \in \mathbb{R}} v_2(x) > 0 \), i.e., \( v_2(x) \) equals the untruncated version \( v_2(x) \). We discuss
in Section 5.1 what happens when the latter assumption does not hold. Suppose \( Y \) is a random variable with density as in (3) and with \( v(x) \) there equal to \( v_2(x) \), i.e.,

\[
\frac{\kappa}{v_2(x)} \exp \left( \int_0^x \frac{b(y)}{v_2(y)} \, dy \right), \quad x \in (\underline{w}, \overline{w}),
\]

and assume for simplicity that \((\underline{w}, \overline{w}) = \mathbb{R}\). Fix a test function \( h : \mathbb{R} \to \mathbb{R} \) with \( E |h(Y)| < \infty \), and let \( f_h(x) \) be the solution to the Poisson equation

\[
b(x)f_h'(x) + v_2(x)f_h''(x) = Eh(Y) - h(x), \quad x \in \mathbb{R}. \tag{13}
\]

Assume that \( E |f_h(W)| < \infty \), which is typically true in practice, and take expected values with respect to \( W \) to get

\[
Eh(Y) - Eh(W) = Eb(W)f_h'(W) + Ev_2(W)f_h''(W) = \frac{1}{18} Ec(W)g_h''(\xi_2) - \frac{1}{24} Ed(W)f_h^{(4)}(\xi_1).
\]

The last equality follows from (11), and \( g_h(x) = \int_0^x \frac{c(y)}{a(y)} f_h''(y) \, dy \). We have again made an implicit assumption that \( f_h(x) \) is sufficiently regular. The regularity of \( f_h(x) \) is entirely determined by the regularity of \( b(x) \), \( v(x) \), and \( h(x) \). The right-hand side equals

\[
\frac{1}{18} \mathbb{E} \left[ c(W) \left( \frac{c(x)}{a(x)} \right)'' \bigg|_{x=\xi_2} f_h''(\xi_2) \right] + \frac{2}{18} \mathbb{E} \left[ c(W) \left( \frac{c(x)}{a(x)} \right)' \bigg|_{x=\xi_2} f_h'''(\xi_2) \right]
\]

\[
+ \frac{1}{18} \mathbb{E} \left[ c(W) \frac{c(\xi_2)}{a(\xi_2)} f_h^{(4)}(\xi_2) \right] - \frac{1}{24} \mathbb{E} d(W)f_h^{(4)}(\xi_1) \tag{14}
\]

because

\[
g_h''''(x) = \left( \frac{c(x)}{a(x)} f_h''(x) \right)'' = \left( \frac{c(x)}{a(x)} \right)'' f_h''(x) + 2 \left( \frac{c(x)}{a(x)} \right)' f_h'''(x) + \frac{c(x)}{a(x)} f_h^{(4)}(x).
\]

Note that (14) contains a term involving \( f_h''(x) \) that is not captured by \( v_2(x) \). To capture that term, we can consider

\[
v_2(x) = \frac{a(x)}{2} - \frac{b(x)c(x)}{3a(x)} - \frac{a(x)}{6} \left( \frac{c(x)}{a(x)} \right)' - \frac{1}{18} c(x) \left( \frac{c(x)}{a(x)} \right)'' \tag{14}, \quad x \in \mathbb{R}.
\]
Truncating $v_2(x)$ produces yet another $v_2$ approximation with error

$$
\frac{1}{18} \mathbb{E}[c(W)\left(\left(\frac{c(x)}{a(x)}\right)''\right)_{x=\xi_2} f''_h(\xi_2) - \left(\frac{c(W)}{a(W)}\right)'' f''_h(W)] \\
+ \frac{2}{18} \mathbb{E}[c(W)\left(\frac{c(x)}{a(x)}\right)'_{x=\xi_2} f''_h(\xi_2)] + \frac{1}{18} \mathbb{E}\left[c(W)\frac{c(\xi_2)}{a(\xi_2)} f''_h(\xi_2)\right] - \frac{1}{24} \mathbb{E}d(W)f''(\xi_1)
$$

in place of (14). In order to decide between $v_2(x)$ and $\pi_2(x)$, let us compare the two error terms in (14) and (15). We stress that the following is an informal discussion meant to develop intuition. Theoretical guarantees for $v_2$ must be established on a case-by-case basis and fall outside the scope of this paper.

Consider first the error term (14). Recall that $a(x), c(x),$ and $d(x)$ equal $\mathbb{E}(\Delta_k|W = x)$ for $k = 2, 3, 4$, respectively. Now $\Delta = W' - W$ equals $\delta$ times the one-step displacement of the Markov chain. Let us assume that the displacement is bounded by a constant independent of $\delta$, in which case $\mathbb{E}(\Delta^k|W = x)$ shrinks at the rate of at least $\delta^k$ as $\delta \to 0$. In particular, $d(x)$ shrinks at least as fast as $\delta^4$. Since $a(x)$ is the extension of the strictly positive function $\mathbb{E}(\Delta^2|W = x)$, we assume that this extension is also strictly positive. Furthermore, we assume that $a(x)$ is of order $\delta^2$, as opposed to merely shrinking at a rate of at least $\delta^2$. Formally, we assume that $\inf\{\delta^{-2} | a(x)| : \delta \in (0, 1), \ x \in (w, w)\} > 0$, which implies that, provided the derivatives exist, $c(x)\left(\frac{c(x)}{a(x)}\right)'$, $c(x)\left(\frac{c(x)}{a(x)}\right)''$, and $c(x)\left(\frac{c(x)}{a(x)}\right)'''$ all shrink at a rate of at least $\delta^4$ as $\delta \to 0$, making them comparable to $d(x)$.

Now consider the error term (15), focusing on the first line there. Provided $a(x), c(x),$ and $f_h(x)$ are sufficiently differentiable, the mean value theorem implies

$$
\mathbb{E}[c(W)\left(\left(\frac{c(x)}{a(x)}\right)''\right)_{x=\xi_2} f''_h(\xi_2) - \left(\frac{c(W)}{a(W)}\right)'' f''_h(W)] \\
= \mathbb{E}[c(W)(\xi_2 - W)\left(\left(\frac{c(x)}{a(x)}\right)'' f''_h(x)\right)_{x=\xi_2}].
$$
Under the two assumptions from before, the terms in front of the derivatives of $f_h(x)$ above shrink at a rate of at least $\delta^5$. If the rest of the terms in (15) shrink at the rate of $\delta^4$, then using $\bar{v}_2(x)$ instead of $\bar{v}_2(x)$ as the $v_2$ approximation would not make the error converge to zero faster. For this reason and also because $v_2(x)$ is simpler than $\bar{v}_2(x)$, we work with $v_2(x)$ in the models we consider.

3. Erlang-C Model

In this section we consider the Erlang-C model. We prove that the $v_1$ error converges to zero at a faster rate than the $v_0$ error. We also conduct numerical experiments where we observe that the $v_1$ error is much smaller, often by a factor of 10, than the $v_0$ error. After defining the model, we introduce the approximations in Section 3.1 and then present theoretical and numerical results in Sections 3.2 and 3.3, respectively.

The Erlang-C, or $M/M/n$, system has a single buffer served by $n$ homogeneous servers working in a first-come-first-served manner. Customers arrive according to a Poisson process with rate $\lambda$, and service times are i.i.d., exponentially distributed with mean $1/\mu$. We let $R = \lambda/\mu$ and $\rho = \frac{\lambda}{n\mu} = R/n$ be the offered load and utilization, respectively.

Let $X(t)$ be the number of customers in the system at time $t$. We assume that $\rho < 1$, implying that $X = \{X(t), t \geq 0\}$ a positive recurrent CTMC. Set $\delta = 1/\sqrt{R}$, $\tilde{X} = \{\tilde{X}(t) = \delta(X(t) - R), \ t \geq 0\}$, and let $W$ be the random variable having the stationary distribution of $\tilde{X}$. The support of $W$ is $W = \{\delta(k - R) : k \in \mathbb{Z}_+\}$, so we let

$$ (\underline{w}, \overline{w}) = (-\delta R, \infty) = (-\sqrt{R}, \infty). $$

The generator of $\tilde{X}$ satisfies (cf. Eq. (3.6) of Braverman et al. (2016))

$$ G_{\tilde{X}} f(x) = \lambda (f(x + \delta) - f(x)) + \mu \left[ \frac{x}{\delta + R} \wedge n \right] (f(x - \delta) - f(x)), $$

(17)
where \( x = \delta(k - R) \) for some integer \( k \geq 0 \). Proposition 1.1 in Henderson (1997) states that

\[
\mathbb{E} G_x f(W) = \mathbb{E} \left[ \lambda (f(W + \delta) - f(W)) + \mu [(W/\delta + R) \wedge n] (f(W - \delta) - f(W)) \right] = 0 \tag{18}
\]

for all \( f(x) \) such that \( \mathbb{E} |f(W)| < \infty \).

### 3.1. The \( v_0 \) and \( v_1 \) Approximations

Let us perform Taylor expansion on the left-hand side of (18):

\[
\mathbb{E} b(W)f'(W) + \mathbb{E} \frac{a(W)}{2} f''(W) = -\frac{1}{6} \delta^3 \lambda f'''(\xi_1) - \delta^3 \mu [(W/\delta + R) \wedge n] f'''(\xi_2), \tag{19}
\]

where \( \xi_1 \in (W, W + \delta) \), \( \xi_2 \in (W - \delta, W) \),

\[
b(x) = \delta (\lambda - \mu [(x/\delta + R) \wedge n]), \quad \text{and} \quad a(x) = \delta^2 (\lambda + \mu [(x/\delta + R) \wedge n]) = 2\mu - \delta b(x), \quad x \in \mathcal{W}. \tag{20}
\]

The second equality in (21) holds because \( \delta^2 = 1/R = \mu/\lambda \). Let

\[
\beta = \delta(n - R) > 0, \quad \text{or} \quad n = R + \beta \sqrt{R}. \tag{22}
\]

When \( \beta \) is fixed and \( R, n \to \infty \), the asymptotic regime is known as the Halfin-Whitt regime; see Halfin and Whitt (1981). It is also known as the quality and efficiency–driven regime because in this parameter region, the system simultaneously achieves short average waiting time (quality) and high server utilization (efficiency); Gans et al. (2003). Some of our results assume that \( \beta \) is fixed, while others do not.

By considering the cases when \( x \leq \beta \) and \( x > \beta \) in (20), we see that \( b(x) = -(\mu x \wedge \mu \beta) \) for \( x \in \mathcal{W} \), and we extend \( b(x) \) to the entire real line via

\[
b(x) = -(\mu x \wedge \mu \beta), \quad x \in \mathbb{R}. \tag{23}
\]
We also want a strictly positive extension of \( a(x) \) to \( \mathbb{R} \). Since \( W \subset [-\sqrt{R}, \infty) \), we define
\[
a(x) = 2\mu - \delta b(-\sqrt{R} \vee x), \quad x \in \mathbb{R},
\]
and since \( b(x) \) is nonincreasing and \( b(-\sqrt{R}) = \mu \sqrt{R} = \mu \delta \), we have \( a(x) \geq a(-\sqrt{R}) = \mu \).

Recall from (3) that our diffusion approximations all have density of the form
\[
\frac{\kappa}{v(x)} \exp \left( \int_0^x \frac{b(y)}{v(y)} \, dy \right), \quad x \in \mathbb{R},
\]
for some normalizing constant \( \kappa > 0 \). The \( v_0 \) and \( v_1 \) approximations are obtained by setting
\[
v(x) = v_0 = \frac{1}{2} a(0) = \mu \quad \text{and} \quad v(x) = v_1(x) = \frac{1}{2} a(x), \quad x \in \mathbb{R}.
\]

Let \( Y_0 \) and \( Y_1 \) be the random variables corresponding to \( v_0 \) and \( v_1 \), respectively.

**Remark 1.** To better approximate \( W \), we can use a diffusion process defined on \([ -\sqrt{R}, \infty) \) with a reflecting condition at the left boundary of \( x = -\sqrt{R} \). However, our theorems in Section 3.2 are intended for the asymptotic regimes when \( R \to \infty \). Since the probability of an empty system shrinks rapidly as \( R \) grows, the choice between a reflected diffusion on \([ -\sqrt{R}, \infty) \) and a diffusion defined on \( \mathbb{R} \) is inconsequential.

### 3.2. Theoretical Guarantees for the Approximations

We now present several theoretical results showing that the \( v_1 \) error vanishes faster than the \( v_0 \) error. Define the class of all Lipschitz-1 functions by
\[
\text{Lip}(1) = \left\{ h: \mathbb{R} \to \mathbb{R} \mid |h(x) - h(y)| \leq |x - y| \text{ for all } x, y \in \mathbb{R} \right\}.
\]

It was shown in Braverman et al. (2016) that
\[
\sup_{h \in \text{Lip}(1)} \left| \mathbb{E} h(W) - \mathbb{E} h(Y_0) \right| \leq \frac{205}{\sqrt{R}}, \quad \text{if } R < n.
\]

(25)
The quantity on the left-hand side above is known as the Wasserstein distance and, as was shown in Gibbs and Su (2002), convergence in the Wasserstein distance implies convergence in distribution. To add to the result of Braverman et al. (2016), we prove the following lower bound in Section EC.1.1 of the electronic companion.

**Proposition 1.** Assume \( n = R + \beta \sqrt{R} \) for some fixed \( \beta > 0 \). There exists a constant \( C(\beta) > 0 \) depending only on \( \beta \) such that

\[
|\mathbb{E}W - \mathbb{E}Y_0| \geq \frac{C(\beta)}{\sqrt{R}}.
\]

An immediate implication of Proposition 1 is that the Wasserstein distance between \( W \) and \( Y_0 \) is at least \( C(\beta)/\sqrt{R} \). The assumption that \( \beta \) is fixed can likely be removed (with additional effort), but that is not the focus of our paper. We turn to the \( \nu_1 \) approximation. Define \( W_2 = \{ h : \mathbb{R} \to \mathbb{R} \mid h(x), h'(x) \in \text{Lip}(1) \} \) and for two random variables \( U, V \), define the \( W_2 \) distance as

\[
d_{W_2}(U, V) = \sup_{h \in W_2} |\mathbb{E}h(U) - \mathbb{E}h(V)|.
\]

Although \( W_2 \subset \text{Lip}(1) \), it still rich enough to imply convergence in distribution. In particular, Lemma 3.5 of Braverman (2017) shows that by approximating the indicator function of a half line by Lipschitz functions with bounded second derivatives, convergence in the \( d_{W_2} \) distance implies convergence in distribution. The following result first appeared as Theorem 3.1 in Braverman (2017).

**Theorem 1.** There exists a constant \( C > 0 \) (independent of \( \lambda, n, \) and \( \mu \)) such that for all \( n \geq 1, \lambda > 0, \) and \( \mu > 0 \) satisfying \( 1 \leq R < n \),

\[
\sup_{h \in W_2} |\mathbb{E}h(W) - \mathbb{E}h(Y_1)| \leq \frac{C}{R}.
\]
Note that $h(x) = x$ belongs to $W_2$, so Theorem 1 and Proposition 1 tell us that the the $v_1$ approximation error of $E(W)$ is guaranteed to vanish faster than the $v_0$ error as $R \to \infty$.

Error bounds of the flavor of Theorem 1 were established in Gurvich et al. (2014), Gurvich (2014a), Braverman and Dai (2017), Braverman et al. (2016), all of which studied convergence rates for steady-state diffusion approximations of various models. The rate of $1/R$ is an order of magnitude better than the rates in any of the previously mentioned papers, where the authors obtained rates that would be equivalent to $1/\sqrt{R}$ in our model.

Going beyond error bounds for smooth test functions, we now present moderate-deviations bounds for our two approximations. Namely, we are interested in the relative error of approximating the cumulative distribution function (CDF) and complementary CDF (CCDF). We define the relative error of the right tail to be

$$\left| \frac{P(Y_i \geq z)}{P(W \geq z)} - 1 \right|, \quad i = 0, 1.$$ 

The relative error for the left tail is defined similarly. The first result is for $v_0$.

**Theorem 2.** Assume that $n = R + \beta \sqrt{R}$ for some fixed $\beta > 0$. There exist positive constants $c_0$ and $C$ depending only on $\beta$ such that

$$\left| \frac{P(Y_0 \geq z)}{P(W \geq z)} - 1 \right| \leq \frac{C}{\sqrt{R}} (1 + z) \quad \text{for } 0 < z \leq c_0 R^{1/2} \text{ and }$$

$$\left| \frac{P(Y_0 \leq -z)}{P(W \leq -z)} - 1 \right| \leq \frac{C}{\sqrt{R}} (1 + z^3), \quad \text{for } 0 < z \leq \min\{c_0 R^{1/6}, R^{1/2}\}. \quad (26)$$

The second result presents analogous bounds for the $v_1$ approximation.

**Theorem 3.** Assume $n = R + \beta \sqrt{R}$ for some fixed $\beta > 0$. There exist positive constants $c_1$ and $C$ depending only on $\beta$ such that

$$\left| \frac{P(Y_1 \geq z)}{P(W \geq z)} - 1 \right| \leq \frac{C}{\sqrt{R}} \left(1 + \frac{z}{\sqrt{R}}\right) \quad \text{for } 0 < z \leq c_1 R \text{ and }$$

$$\left| \frac{P(Y_1 \leq -z)}{P(W \leq -z)} - 1 \right| \leq \frac{C}{\sqrt{R}} \left(1 + z^3\right), \quad \text{for } 0 < z \leq \min\{c_0 R^{1/6}, R^{1/2}\}. \quad (27)$$
\[
\left| \frac{\mathbb{P}(Y_1 \leq -z)}{\mathbb{P}(W \leq -z)} - 1 \right| \leq \frac{C}{\sqrt{R}} \left( 1 + z + \frac{z^4}{\sqrt{R}} \right), \text{ for } 0 < z \leq \min\{c_1 R^{1/4}, R^{1/2}\}. \tag{29}
\]

Inequality (28) follows from Theorem 4.1 of Braverman (2017). We prove (29) in Section EC.1.2; Theorem 2 follows from a similar and simpler proof in Section EC.1.3.

These are called moderate deviations bounds because they cover the case when \( z \) is “moderately” far from the origin, with “moderately” being quantified by intervals of the form \( z \in [0, c_0 R^{1/2}], \, z \in [0, c_1 R] \), etc. In contrast, large-deviations results focus on understanding the behavior of \( \mathbb{P}(W \geq z) \) as \( z \to \infty \). To compare the two theorems, suppose \( z = c_0 \sqrt{R} \) and consider the upper bounds in (26) and (28). The \( v_0 \) error is guaranteed to be bounded as \( R \) grows, while the \( v_1 \) error shrinks at a rate of at least \( 1/\sqrt{R} \).

### 3.3. Numerical Results

Although the Erlang-C system depends on three parameters \( \lambda, \mu, \) and \( n \), the stationary distribution depends on only \( \rho \) and \( n \); see Appendix C in Allen (1990). Figure 1 displays the relative errors of the \( v_0 \) and \( v_1 \) approximations of \( \mathbb{E}(W) \) when \( 5 \leq n \leq 100 \) and \( 0.5 \leq \rho \leq 0.99 \). Note that the \( v_1 \) error is about ten times smaller. We also compare how well \( v_0 \) and \( v_1 \) approximate the CCDF of \( W \). Figure 2 plots the relative error of approximating \( \mathbb{P}(W/\delta + R \geq z) \) for various values of \( z \) when \( n = 10 \) and \( \rho \in [0.5, 0.99] \), and shows that the \( v_1 \) error is again much smaller. In results not reported in the paper, we observed that the \( v_1 \) error remains much smaller even as we vary \( n \).

### 4. Hospital Model

In this section we consider the discrete-time model for hospital inpatient flow proposed by Dai and Shi (2017). Numerical experiments presented later suggest that both the \( v_1 \)
Figure 1  The errors increase towards the bottom-right corner of each plot. This is due to the fact that \( \mathbb{E}(W) \) is very close to zero in that region and not because approximations perform poorly.

Figure 2  The \( v_1 \) approximation is much more accurate.

and \( v_2 \) errors vanish at the same rate as the \( v_0 \) approximation error. To observe a faster convergence rate, we have to resort to the \( v_3 \) approximation.

Consider a discrete-time queueing model with \( N \) identical servers. Let \( X(n) \) be the number of customers in the system at the end of time unit \( n \). Given \( X(0) \), we define

\[
X(n) = X(n-1) + A(n) - D(n), \quad n \geq 1,
\]
where $A(n) \sim \text{Poisson}(\Lambda)$ represents the number of new arrivals in the time period $[n-1,n)$.

At the end of each time period, every customer in service flips a coin and, with probability $\mu \in (0,1)$, departs the system at the start of the next time period. Thus, conditioned on $X(n-1) = k$, we have $D(n) \sim \text{Binomial}(k \land N, \mu)$. Assuming $\Lambda < N\mu$, Dai and Shi (2017) showed that $X = \{X(n) : n = 1, 2, \ldots\}$ is a positive-recurrent DTMC.

This DTMC is similar to the Erlang-C model, but unlike the Erlang-C model where the customer count only changes by one at a time, the jump size $X(1) - X(0)$ is not bounded because $A(n)$ is unbounded. As a result, computing the stationary distribution takes a long time when $\Lambda$ is large and the utilization $\rho = \Lambda/(N\mu)$ is near one because the state-space truncation has to be large to account for potential arrivals.

We are interested in the scaled DTMC $\tilde{X} = \{\tilde{X}(n) = \delta(X(n) - N)\}$. To stay consistent with Dai and Shi (2017), we center $X(n)$ around $N$. We consider the parameter ranges studied in Dai and Shi (2017), which, given some constant $\beta > 0$, are

$$\Lambda = \sqrt{N} - \beta, \quad \mu = \delta = 1/\sqrt{N}$$

(30)

4.1. Motivating the Need for a $v_3$ Approximation

This section contains an informal discussion aimed at explaining why the $v_0$, $v_1$, and $v_2$ errors vanish at the same rate of $\delta = 1/\sqrt{N}$, and why we need the $v_3$ approximation to observe a convergence rate of $\delta^2 = 1/N$.

Initialize $\tilde{X}(0)$ according to the stationary distribution of $\tilde{X}$, let $W = \tilde{X}(0)$, $W' = \tilde{X}(1)$, and set $\Delta = W' - W$. The support of $W$ is $W = \{\delta(k - N) : k \in \mathbb{N}\} \subset [-\delta N, \infty)$. As we are accustomed to doing by this point, we let $b(x) = \mathbb{E}(\Delta|W = x)$. We know from (37) and (38) of Dai and Shi (2017) that for $x \in W$,

$$b(x) = \mathbb{E}(\Delta|W = x) = \delta(x^- - \beta), \quad \text{and}$$

(31)
\[ \mathbb{E}(\Delta^2|W = x) = 2\delta + (b^2(x) - \delta b(x) - \delta^2 - 2\delta^2 \beta) + \delta^3 x. \]  

(32)

For the higher moments of \( \Delta \), let us use \( \epsilon(x) \) to represent a generic function that may change from line to line but always satisfies the property 

\[ |\epsilon(x)| \leq C(1 + |x|)^5 \]  

(33)

for some constant \( C > 0 \) that depends only on \( \beta \). We show in Section EC.2.1 that 

\[ \mathbb{E}(\Delta^n|W = x) = \delta^n \epsilon(x), \quad x \in W. \quad (34) \]

All of our \( v_n \) approximations share the same drift \( b(x) \), but the diffusion coefficients vary. As always, \( v_1(x) = \frac{1}{2} \mathbb{E}(\Delta^2|x) \). Since \( b(x) = 0 \) at \( x = -\beta \), (32) implies that 

\[ v_0 = v_1(-\beta) = \frac{1}{2} (2\delta - \delta^2 (1 + 2\beta) + \delta^3 \beta). \]  

(35)

The following informal discussion assumes that all functions are sufficiently differentiable and that all expectations exist. Our starting point, as always, is the Poisson equation 

\[ b(x)f_h'(x) + v(x)f_h''(x) = \mathbb{E}h(Y) - h(x), \quad x \in \mathbb{R}, \]  

(36)

where \( v(x) \) is a temporary placeholder and \( Y \) has density given by (3). The Taylor expansion in (2) tells us that 

\[ \mathbb{E}b(W)f_h'(W) + \mathbb{E} \left[ \sum_{i=2}^{n} \frac{1}{i!} \Delta^i f_h^{(i)}(W) + \frac{1}{(n+1)!} \Delta^{n+1} f_h^{(n+1)}(\xi) \right] = 0, \]

where \( \xi = \xi^{(n)} \) lies between \( W \) and \( W' \). Subtracting this equation from (36) and taking expected values there with respect to \( W \) we see that for \( n \geq 1 \), 

\[ \mathbb{E}h(W) - \mathbb{E}h(Y) = - \mathbb{E}v(W)f_h''(W) + \mathbb{E} \left[ \sum_{i=2}^{n} \frac{1}{i!} \Delta^i f_h^{(i)}(W) + \frac{1}{(n+1)!} \Delta^{n+1} f_h^{(n+1)}(\xi) \right]. \]  

(37)
Consider the $v_0$ error. When $v(x) = v_0$, equation (37) with $n = 2$ there becomes
\[
E h(W) - E h(Y) = E \left\{ \frac{1}{2} \Delta^2 - v_0 \right\} f_h''(W) + \frac{1}{6} E \Delta^3 f_h'''(\xi). \tag{38}
\]
Note that $f_h(x)$ depends on $v(x)$ because it solves (36). In Lemma 3 of Dai and Shi (2017), the authors proved that $|f_h''(x)| \leq C/\delta$ and $|f_h'''(x)| \leq C/\delta$ for some constant $C > 0$ dependent only on $\beta$. Assuming that $f_h''(x)$ and $f_h'''(x)$ indeed grow at the rate of $C/\delta$ as $\delta \to 0$, the forms of $E(\Delta^2|W = x)$ and $v_0$ in (32) and (35) yield
\[
\frac{1}{2} E(\Delta^2|W = x) - v_0 = \frac{1}{2} \left( b^2(x) - \delta b(x) + \delta^3(x^- - \beta) \right) = \frac{1}{2} \left( b^2(x) - \delta b(x) + \delta^2 b(x) \right).
\]
Since $b(x) = \delta(x^- - \beta)$, this quantity is of order $\delta^2$. Now (34) says that $E(\Delta^3|W = x)$ is also of order $\delta^2$. Therefore, we expect both $E(\Delta^2/2 - v_0)f_h''(W)$ and $E\Delta^3 f_h'''(\xi)$ to be of order $\delta$, so even if $E(\Delta^2/2 - v_0)f_h''(W)$ were not present in (38), the approximation error would still be of order $\delta$ due to $E\Delta^3 f_h'''(\xi)$. We believe this is why the $v_0$ and $v_1$ errors appear to vanish at the same rate despite $v_1(x)$ capturing the entire second order term of
\[
E \left[ \sum_{i=2}^{n} \frac{1}{i!} \Delta^i f_h^{(i)}(W) + \frac{1}{(n+1)!} \Delta^{n+1} f_h^{(n+1)}(\xi) \right]. \tag{39}
\]
Going beyond $v_0$ and $v_1$, we see that for the error to be of order $\delta^2$, the diffusion approximation must capture all the terms in (39) that are of order $\delta$. If we assume for the moment that the derivatives of $f_h(x)$ are all of order $1/\delta$, we see that our approximation has to capture all terms of order $\delta^2$ or larger in the functions $\{E(\Delta^i|x)\}_{i=1}^{\infty}$.

From (31), (32), and (34) we see that $E(\Delta^1|x)$ through $E(\Delta^4|x)$ are all of order $\delta$ or $\delta^2$, while $E(\Delta^5|x)$ is of order $\delta^3$. Thus, if we want the error to be of order $\delta^2$, our approximation must capture the terms of order $\delta$ and $\delta^2$ in $E(\Delta^1|x)$ through $E(\Delta^4|x)$ and can ignore terms.
of order $\delta^3$ like $\mathbb{E}(\Delta^5|x)$. We remark that $v_2(x)$ in (7) depends only on $\mathbb{E}(\Delta^1|x)$ through $\mathbb{E}(\Delta^3|x)$ and not on $\mathbb{E}(\Delta^4|x)$. We suspect this is why we observe the $v_2$ error to be of order $\delta$. In Section EC.2 we derive a $v_3$ approximation of the form

$$v_3(x) = \max \left\{ \delta + \frac{1}{2} \left( \delta^2 1(x < 0) - \delta b(x) - \delta^2 - 2\delta^2 \beta \right), \delta/2 \right\}$$

(40)

and in the following section we present numerical results that suggest that the error of this approximation converges to zero at a rate of $\delta^2$.

4.2. Numerical Results

In Figure 3 we compare the $v_n$ approximations for $\mathbb{E}(W)$ when $\beta = 1$ and $N \in \{4, 16, 64, 256\}$. The values of $\mathbb{E}(W)$ are estimated using a simulation; the width of the 95% confidence intervals (CIs) is on the order of $10^{-4}$. Though we do not report them, the $v_0$, $v_1$, and $v_2$ approximation errors appear to decay at the rate of $1/\sqrt{N}$, but the $v_3$ error in the table appears to decay linearly in $N$. When it comes to approximating the CCDF, $v_3$ also outperforms the other approximations; see Figure 4 for an example when $N = 64$ and $\beta = 1$. Our findings were consistent for other values of $\beta$ and $N$.

![Figure 3](image-url)  

**Figure 3** $\beta = 1$. $Y_n$ corresponds to the $v_n$ approximation.
5. AR(1) Model

In this section we consider the first-order autoregressive model with random coefficient and general error distribution, which we refer to as the AR(1) model. Blanchet and Glynn (2018) studied this model and used an Edgeworth expansion to approximate its stationary distribution. Following the notation of Blanchet and Glynn (2018), we compare the performance their expansion to our diffusion approximations.

We encounter the issue that for large values of $x$, the untruncated $v_3(x)$ becomes sufficiently negative to make our usual solution of truncation from below perform poorly when approximating the tail of the distribution. We resolve this via a hybrid approximation that uses both $v_3(x)$ and $v_2(x)$ to construct a diffusion coefficient $\hat{v}_3(x)$ that combines the extra accuracy of $v_3(x)$ with the positivity of $v_2(x)$.

To introduce the model, let $\{X_n, n \geq 1\}, \{Z_n, n \geq 1\}$ be two independent sequences of i.i.d. random variables. We assume that both $X_1$ and $Z_1$ are exponentially distributed with unit mean so that we can compare our approximations to those of Blanchet and Glynn (2018). Given $D_0 \in \mathbb{R}$ and $\alpha > 0$, consider the DTMC $D = \{D_n, n \geq 0\}$ defined as

$$D_{n+1} = e^{-\alpha Z_{n+1}} D_n + X_{n+1}$$

(41)

Figure 4 $N = 64$ and $\beta = 1$. $P(W \geq x) \approx 10^{-6}$ for $x$ at the right endpoint of the $x$-axis.
and let $D_\infty > 0$ denote the random variable having the stationary distribution of $D$. Using (41) we can see that $D_\infty$ is equal in distribution to $\sum_{k=0}^{\infty} X_k e^{-\alpha \sum_{j=0}^{k-1} Z_j}$, which is the random variable studied in Section 4 of Blanchet and Glynn (2018).

We consider $\hat{D} = \{\hat{D}_n = \delta(D_n - R), \ n \geq 0\}$, where $\delta = \sqrt{\alpha}$ and, to be consistent with Blanchet and Glynn (2018), we choose $R = 1/\alpha$. The asymptotic regime we consider is $\alpha \to 0$, so going forward we assume that $\alpha \in (0, 1)$. It follows from (41) that

$$\hat{D}_{n+1} = e^{-\alpha Z_{n+1}} \hat{D}_n + \delta \left( X_{n+1} + R(e^{-\alpha Z_{n+1}} - 1) \right).$$

Let $W = \delta(D_\infty - R)$ and $W' = e^{-\alpha Z} W + \delta \left( X + R(e^{-\alpha Z} - 1) \right)$, where $(X, Z)$ is an independent copy of $(X_1, Z_1)$, which is also independent of $W$. Since $D_\infty > 0$, the support of $W$ is $\mathcal{W} = (-1/\sqrt{\alpha}, \infty)$, which grows as $\alpha \to 0$. Stationarity implies that $\mathbb{E}f(W') - \mathbb{E}f(W) = 0$ provided $\mathbb{E}|f(W)| < \infty$. Note that the one-step jump size

$$\Delta = W' - W = W(e^{-\alpha Z} - 1) + \delta \left( X + R(e^{-\alpha Z} - 1) \right) = \delta(D_\infty(e^{-\alpha Z} - 1) + X)$$

does not depend on the choice of $R$. To present our diffusion approximations, we need expressions for $\mathbb{E}(\Delta^k|W = x)$. The following lemma is proved in Section EC.3.

**Lemma 1.** Recall that $\delta = \sqrt{\alpha}$. For any $k \geq 1$,

$$\mathbb{E}(\Delta^k|D_\infty = d) = \delta^k k! \left( 1 + \sum_{i=1}^{k} (-1)^i d^i \prod_{j=1}^{i} \frac{\alpha}{1 + j\alpha} \right), \ d > 0.$$

The relationship between $D_\infty$ and $W$ implies that

$$\mathbb{E}(\Delta^k|W = x) = \mathbb{E}(\Delta^k|D_\infty = x/\delta + R) = \delta^k k! \left( 1 + \sum_{i=1}^{k} (-1)^i \left( x\sqrt{\alpha} + 1 \right)^i \prod_{j=1}^{i} \frac{1}{1 + j\alpha} \right),$$

for $x \in \mathcal{W}$, where we used the facts that $\delta = \sqrt{\alpha}$ and $R = 1/\alpha$ in the second equality. Extending $\mathbb{E}(\Delta^k|W = x)$ to all $x \in \mathbb{R}$ in the obvious way, we now state the $v_0$, $v_1$, and
\( v_2 \) approximations, whose forms are all standard. Namely, \( v_2(x) \) follows from (7), \( v_1(x) = \mathbb{E}(\Delta^2 | W = x) / 2 \), and \( v_0 = \mathbb{E}(\Delta^2 | W = x^*) \), where \( x^* = 1 + 1/\alpha \) solves \( \mathbb{E}(\Delta | W = x) = 0 \). To present \( v_3(x) \), let us note that \( \mathbb{E}(\Delta^k | W = x) \) takes the form \( \delta^k p_k(x) \) for some degree-\( k \) polynomial \( p_k(x) \); we omit the dependence on \( \alpha \) to ease notation. Given a truncation level \( \eta > 0 \), we let \( v_3(x) = (v_3(x) \vee \eta) \), where

\[
\begin{align*}
v_3(x) &= \delta^2 \left( \frac{p_2(x)}{2} - \frac{p_1(x)\bar{p}_3(x)}{p_2(x)} - \delta p_3(x) \left( \frac{\bar{p}_3(x)}{p_2(x)} \right)' \right), \\
\bar{p}_3(x) &= \frac{1}{6} \left( p_3(x) - \frac{p_1(x)p_4(x)}{2p_2(x)} - \frac{1}{4} \delta p_2(x) \left( \frac{p_4(x)}{p_2(x)} \right)' \right), \\
p_2(x) &= \left( \frac{p_2(x)}{2} - \frac{p_1(x)p_3(x)}{3p_2(x)} - \frac{p_2(x)}{6} \left( \frac{p_3(x)}{p_2(x)} \right)' \right).
\end{align*}
\]

We derive \( v_3(x) \) by successively applying the “trick” we used in Section 2 to derive the \( v_2(x) \) approximation in order to gain access to higher order terms in the Taylor expansion. The details are left to Section EC.3 of the electronic companion. Before presenting our numerical results, we discuss how to modify the \( v_3 \) approximation to overcome the issue that \( v_3(x) \) becomes negative for large values of \( x \).

### 5.1. Hybrid Approximation

Figure 5 displays \( v_3(x) \) for several values of \( \alpha \). When \( \alpha = 0.001 \) or 0.01, the plots of \( v_3(x) \) are very close to zero but remain nonnegative. However, \( v_3(x) \) is negative when \( \alpha = 0.1, 0.5, \) or 0.9. The behavior of \( v_3(x) \) in the left tail is not as important because the left boundary of the support of \( W = \sqrt{\alpha}(D - 1/\alpha) \) is \(-1/\sqrt{\alpha}\); we therefore ignore the negativity in the left part of \( v_3(x) \) for \( x < 0 \). The farther \( v_3(x) \) drops below zero, the worse we expect the truncated \( v_3(x) \) to perform. For example, the plot with \( \alpha = 0.9 \) in Figure 6 of Section 5.2
shows that while \( v_3 \) performs well in regions where \( v_3(x) > 0 \), it does not perform as well when estimating \( P(W > x) \) for large \( x \).

To improve upon the \( v_3 \) approximation, we propose the hybrid approximation \( \hat{v}_3(x) = v_3(x)1(x \leq K) + v_2(x)1(x > K) \). The threshold \( K \) is numerically chosen to equal the right-most point of intersection of \( v_2(x) \) and \( v_3(x) \). The idea is for \( \hat{v}_3(x) \) to enjoy the increased accuracy of \( v_3(x) \) in the center with the performance of \( v_2(x) \) far in the tail. We expect \( \hat{v}_3(x) \) to outperform a truncated \( v_3(x) \) when \( v_3(x) \) drops far below zero; e.g., when \( \alpha = 0.9 \). If \( v_3(x) \) is nonnegative, we expect little benefit from \( \hat{v}_3(x) \); e.g., when \( \alpha = 0.001 \). Our expectations are consistent with our numerical findings in Section 5.2. Lastly, we remark on what can be done in the case when both \( v_2(x) \) and \( v_3(x) \) are negative in the same region: instead of falling back on \( v_2(x) \), we can combine \( v_4(x) \) with \( v_1(x) \), which is always positive because \( v_1(x) = \frac{1}{2} \mathbb{E}(\Delta^2|W = x) > 0 \) for all \( x \in \mathcal{W} \).

5.2. Numerical Results

It is well known that Edgeworth expansions, obtained for the probability distribution at a particular point, can suffer from two issues: (1) they may not be a proper probability dis-
| $\alpha = 0.64$ | $\alpha = 0.32$ | $\alpha = 0.16$ | $\alpha = 0.08$ | $\alpha = 0.04$ |
|----------------|----------------|----------------|----------------|----------------|
| $|E_f(Y_0) - E_f(W)|$ | 0.095 | 0.039 | 0.011 | 0.002 | $3.6 \times 10^{-4}$ |
| $|E_f(Y_1) - E_f(W)|$ | 0.104 | 0.037 | 0.008 | $9.4 \times 10^{-4}$ | $1.0 \times 10^{-4}$ |
| $|E_f(Y_2) - E_f(W)|$ | 0.034 | 0.011 | 0.003 | $4.7 \times 10^{-4}$ | $7.4 \times 10^{-5}$ |
| $|E_f(Y_3) - E_f(W)|$ | 0.020 | 0.005 | $7.9 \times 10^{-4}$ | $8.9 \times 10^{-6}$ | $7.8 \times 10^{-6}$ |
| $|E_f(Y_3) - E_f(W)|$ | 0.0194 | 0.005 | $7.8 \times 10^{-4}$ | $8.9 \times 10^{-5}$ | $7.8 \times 10^{-6}$ |
| $|E_f(Y_3) - E_f(W)|$ | 0.148 | 0.053 | 0.009 | $7.3 \times 10^{-4}$ | $6.3 \times 10^{-4}$ |
| $E_f(W)$ | 0.510 | 0.721 | 0.994 | 1.302 | 1.629 |

Table 1: $f(W) = \log(W + \delta R)$. The random variable $Y_n$ corresponds to the $v_n$-approximation, $\hat{v}_3$ corresponds to the $\hat{v}_3$-approximation, and $Y_e$ corresponds to the Edgeworth expansion estimate.

6. Conclusion

We have outlined a general procedure to derive $v_n$ approximations for one-dimensional Markov chains. Although the expressions for $v_n(x)$ get more complicated as $n$ increases, the diffusion approximations remain computationally tractable. A natural question is how to extend this work to the multi-dimensional setting.

Another direction worth exploring relates to establishing theoretical guarantees for the approximations. The only results we have are for the $v_1$ error in Section 3, a key ingredient
Figure 6  The plots exclude \( v_0 \) and \( v_1 \), the worst performing approximations.

of which are bounds on the derivatives to the solution of the Poisson equation, also known as Stein factor bounds; c.f., Lemma EC.3 of the electronic companion. Since the Poisson equation depends on the diffusion coefficient, we have to reestablish Stein factor bounds for each new \( v_n(x) \); the difficulty of this grows with the complexity of the expression for \( v_n(x) \). The prelimit generator approach, recently proposed by Braverman (2022), may offer a simpler avenue for theoretical guarantees because it uses Stein factor for the Markov chain instead of the diffusion.
Accompanying Proofs

This e-companion contains the proofs of certain theoretical results in the paper. It is divided into three main sections. The first section is about the Erlang-C model, and contains the proofs of Proposition 1, Theorem 3, and Theorem 2. The second and third sections derive the $v_3$ approximation for the hospital model and AR(1) model, respectively.

**EC.1. Companion for the Erlang-C Model**

To prepare for the arguments to come, let us recall the notation related to the Erlang-C model. The Erlang-C model is defined by the customer arrival rate $\lambda > 0$, the service rate $\mu > 0$, and the number of servers $n > 0$. Additional important quantities include

$$R = \frac{\lambda}{\mu} < n, \quad \beta = \frac{n - R}{\sqrt{R}} > 0, \quad \text{and} \quad \delta = \frac{1}{\sqrt{R}}.$$  

We study $W$, which has the stationary distribution of the CTMC $\{ \bar{X}(t) = \delta (X(t) - R) \}$, where $X(t)$ is the number of customers in the system at time $t \geq 0$. Equation (18) states that

$$\mathbb{E} G_{\bar{X}} f(W) = \mathbb{E} \left[ \lambda (f(W + \delta) - f(W)) + \mu \left( \left( W/\delta + R \right) \land n \right) (f(W - \delta) - f(W)) \right] = 0 \quad (\text{EC.1})$$

for all $f(x)$ satisfying $\mathbb{E} |f(W)| < \infty$. We also note that the support of $W$ is

$$W = \{-\sqrt{R}, -\sqrt{R} + \delta, -\sqrt{R} + 2\delta, \ldots\}. \quad (\text{EC.2})$$

To define $v_0(x)$ and $v_1(x)$, we recall from (20) and (21) that for $x \in W$,

$$b(x) = \delta (\lambda - \mu [(x/\delta + R) \land n]) \quad \text{and} \quad a(x) = \delta^2 (\lambda + \mu [(x/\delta + R) \land n]), \quad (\text{EC.3})$$
and from (23) and (24) that the extensions of these to $\mathbb{R}$ are

$$b(x) = -(\mu x \wedge \mu \beta) \quad \text{and} \quad a(x) = 2\mu - \delta b(-\sqrt{R} \vee x), \quad x \in \mathbb{R}.$$  \hspace{1cm} (EC.4)

We define $v_1(x) = \frac{1}{2}a(x)$ and $v_0(x) = v_0 = v_1(0) = \mu$, and for $n \in \{0,1\}$ we define the $v_n$ approximation to be the random variable $Y_n$ with density

$$\frac{\kappa}{v_n(x)} \exp \left( \int_0^x \frac{b(y)}{v_n(y)} dy \right), \quad x \in \mathbb{R},$$  \hspace{1cm} (EC.5)

where $\kappa > 0$ is a normalization constant that depends on $n$. Lastly, assuming that $-z$ belongs to $W$ and setting $f(x) = 1(x \geq -z)$ in (EC.1), we get

$$\lambda \mathbb{P}(W = -z - \delta) = \mu[(-z/\delta + R) \wedge n] \mathbb{P}(W = -z),$$  \hspace{1cm} (EC.6)

which are the flow-balance equations for the CTMC.

**EC.1.1. Proving Proposition 1**

We repeat the statement of Proposition 1 for convenience.

**Proposition EC.1.** Assume that $n = R + \beta \sqrt{R}$ for some fixed $\beta > 0$. There exists a constant $C(\beta) > 0$ depending only on $\beta$ such that

$$|\mathbb{E}W - \mathbb{E}Y_0| \geq \frac{C(\beta)}{\sqrt{R}}.$$

We prove the proposition with the help of four auxiliary lemmas. The lemmas are proved at the end of this section after we prove Proposition EC.1. We use $C = C(\beta) > 0$ to denote a constant that may change from line to line, but does not depend on anything other than $\beta$. 
Lemma EC.1. For any $\beta > 0$,
\[
\beta \mathbb{P}(W \geq \beta) = -\mathbb{E}(W 1(W < \beta)) \quad \text{and} \quad \beta \mathbb{P}(Y_0 \geq \beta) = -\mathbb{E}(Y_0 1(Y_0 < \beta)). \tag{EC.7}
\]
Consequently,
\[
\mathbb{E}W = \mathbb{E}(W - \beta)^+ \quad \text{and} \quad \mathbb{E}Y_0 = \mathbb{E}(Y_0 - \beta)^+.
\]
Lemma EC.1 implies that
\[
|\mathbb{E}W - \mathbb{E}Y_0| = |\mathbb{E}(W - \beta)^+ - \mathbb{E}(Y_0 - \beta)^+|.
\]
The next lemma rewrites the right-hand side above using the Poisson equation so that we can bound it from below.

Lemma EC.2. Fix $h \in \text{Lip}(1)$ and let $f_h(x)$ be the solution the the Poisson equation
\[
b(x)f'_h(x) + v_0f''_h(x) = \mathbb{E}h(Y_0) - h(x), \quad x \in \mathbb{R}. \tag{EC.8}
\]
Then $f''_h(x-) = \lim_{y \uparrow x} f''_h(y)$ is defined for all $x \in \mathbb{R}$ and
\[
\mathbb{E}h(W) - \mathbb{E}h(Y_0) = -\frac{1}{2}\delta \mathbb{E}b(W)f''_h(W) + \mathbb{E}\varepsilon(W),
\]
where
\[
\varepsilon(W) = \frac{1}{6}\delta^2 b(W)f''_h(W-) + \lambda(\varepsilon_+(W) + \varepsilon_-(W)) - \frac{1}{\delta}b(W)\varepsilon_-(W),
\]
\[
\varepsilon_+(W) = \frac{1}{2}\int_W^{W+\delta}(W + \delta - y)^2(f''_h(y) - f''_h(W-))dy,
\]
\[
\varepsilon_-(W) = -\frac{1}{2}\int_{W-\delta}^W(y - (W - \delta))^2(f''_h(y) - f''_h(W-))dy.
\]
Our plan is to show that for any \( h \in \text{Lip}(1) \), the term \( \mathbb{E}\varepsilon(W) \) vanishes at a rate of at least \( \delta^2 \). We then fix \( h(x) = (x - \beta)^+ \) and show that \( |\mathbb{E}b(W)f''_h(W)| \) can be bounded away from zero by a constant independent of \( \delta \), which implies Proposition EC.1. The following two lemmas are needed for this. The first one is for the upper bound on \( \mathbb{E}\varepsilon(W) \), and the second is for the lower bound on \( |\mathbb{E}b(W)f''_h(W)| \).

**Lemma EC.3.** Assume that \( n = R + \beta \sqrt{R} \) for some fixed \( \beta > 0 \). There exists \( C = C(\beta) > 0 \) depending only on \( \beta \) such that for any \( h \in \text{Lip}(1) \),

\[
\left| f''_h(x-) \right| \leq \frac{C}{\mu} \quad \text{and} \quad \mathbb{E}|b(W)| \leq \mu \mathbb{E}|W| \leq \mu C, \quad x \in \mathbb{R}.
\]

Additionally, if \( h(x) = (x - \beta)^+ \), then for all \( x \neq \beta \), \( f^{(4)}_h(x) \) exists and \( \left| f^{(4)}_h(x) \right| \leq \frac{C}{\mu} (1 + |x|) \).

**Lemma EC.4.** If \( h(x) = (x - \beta)^+ \), then \( f''_h(x) = \frac{1}{\mu \beta} \) for \( x \geq \beta \) and

\[
f''_h(x) = \frac{1 + xe^{x^2/2} \int_{-\infty}^{x} e^{-y^2/2} dy}{1 + \beta e^{\beta^2/2} \int_{-\infty}^{0} e^{-y^2/2} dy}, \quad x \leq \beta.
\]

Before proving Proposition EC.1, let us remark that we restrict ourselves to \( h(x) = (x - \beta)^+ \) to keep the proof simple. Our arguments can likely be extended to work for other \( h(x) \) at the expense of added complexity.

**Proof of Proposition EC.1** Lemma EC.2 implies that

\[
\mathbb{E}h(W) - \mathbb{E}h(Y_0) = -\frac{1}{2} \delta \mathbb{E}b(W)f''_h(W) + \mathbb{E}\varepsilon(W).
\]

In the first part of the proof, we show that \( \mathbb{E}|\varepsilon(W)| \leq C \delta^2 \). For convenience, we recall that

\[
\varepsilon(W) = \frac{1}{6} \delta^2 b(W)f'''_h(W-) + \lambda(\varepsilon_+(W) + \varepsilon_-(W)) - \frac{1}{\delta} b(W)\varepsilon_-(W),
\]

\[
\varepsilon_+(W) = \frac{1}{2} \int_{W}^{W+\delta} (W + \delta - y)^2 (f'''_h(y) - f'''_h(W-)) dy, \quad y > W.
\]

\[
\varepsilon_-(W) = -\frac{1}{2} \int_{W-\delta}^{W} (y - (W - \delta))^2 (f'''_h(y) - f'''_h(W-)) dy, \quad y < W.
\]
Lemma EC.3 immediately implies that \( \mathbb{E} |b(W)f_h''(W-)| \leq C \). Next, we show that \( \mathbb{E} |\varepsilon_+(W)| \leq \frac{C}{\mu} \delta^4 \). By considering the cases when \( W = \beta \) and \( W \neq \beta \) and using Lemma EC.3, we see that

\[
\mathbb{E} |\varepsilon_+(W)| \leq \frac{1}{2} \mathbb{P}(W = \beta) \int_{\beta}^{\beta + \delta} (\beta + \delta - y)^2 (|f_h''(y)| + |f_h''(\beta - y)|) dy
\]

\[
+ \frac{1}{2} \mathbb{E}
\begin{cases}
1(W \neq \beta) \int_{W}^{W + \delta} (W + \delta - y)^2 \int_{y}^{\infty} f_h''(u) dudy
\end{cases}
\]

\[
\leq \frac{C}{\mu} \mathbb{P}(W = \beta) \int_{\beta}^{\beta + \delta} (\beta + \delta - y)^2 dy
\]

\[
+ \frac{C}{\mu} \mathbb{E}
\begin{cases}
1(W \neq \beta) \int_{W}^{W + \delta} \delta (1 + |W| + \delta)(W + \delta - y)^2 dy
\end{cases}
\]

\[
\leq \frac{C}{\mu} \delta^3 \mathbb{P}(W = \beta) + \frac{C}{\mu} \delta^4.
\]

This argument can be repeated to show \( \mathbb{E} |\varepsilon_-(W)| \leq \frac{C}{\mu} \delta^4 \). It was shown in (3.29) of Braverman (2017) that \( \mathbb{P}(W = \beta) \leq C \delta \), but for completeness we repeat the argument at the end of the proof. Combining these results, we arrive at

\[
\mathbb{E} |\varepsilon(W)| \leq \mathbb{E} |\varepsilon+(W)| + \mathbb{E} |\varepsilon-(W)| \leq C \delta^2 + \frac{\lambda}{\mu} \delta^4 + \mathbb{E} |b(W)| \frac{C}{\mu} \delta^3 \leq C \delta^2,
\]

where in the last inequality we use the bound on \( \mathbb{E} |b(W)| \) from Lemma EC.3 and the fact that \( \delta^2 = 1/R = \mu/\lambda \). We now show that \( |\mathbb{E} b(W)f_h''(W)| \geq C \). Combining the fact that \( b(x) = -(\mu x \wedge \mu \beta) \) with the form of \( f_h''(x) \) from Lemma EC.4, we have

\[
\mathbb{E} b(W)f_h''(W) = \frac{-\mu \beta}{\mu \beta} \mathbb{P}(W \geq \beta) - \frac{\mu}{\mu \beta} \mathbb{E}
\begin{cases}
W \left(1 + W e^{W^2/2} \int_{-\infty}^{W} e^{-y^2/2} dy\right) 1(W < \beta)
\end{cases}
\]

\[
\leq - \mathbb{P}(W \geq \beta) - \frac{\mathbb{E}(W 1(W < \beta))}{\beta 1 + \beta e^{\beta^2/2} \int_{-\infty}^{\beta} e^{-y^2/2} dy}.
\]
From Lemma EC.1 we know that $\mathbb{E}(W1(W < \beta)) = -\beta \mathbb{P}(W \geq \beta)$, so

$$\mathbb{E}b(W)f''_h(W) = -\mathbb{P}(W \geq \beta) + \frac{\beta \mathbb{P}(W \geq \beta)}{\beta} \mathbb{E} \left[ \left( W^2 e^{W^2/2} \int_{-\infty}^{W} e^{-y^2/2} dy \right) 1(W < \beta) \right]$$

$$\leq \mathbb{P}(W \geq \beta) \left( -1 + \frac{1}{1 + \beta e^{\beta^2/2} \int_{-\infty}^{\beta} e^{-y^2/2} dy} \right)$$

$$\leq -C \mathbb{P}(W \geq \beta).$$

Proposition 1 of Halfin and Whitt (1981) tells us that $\mathbb{P}(W \geq \beta)$ converges to a positive constant (depending on $\beta$) as $R \to \infty$. This implies the lower bound on $|\mathbb{E}b(W)f''_h(W)|$.

Lastly, we prove that $\mathbb{P}(W = \beta) \leq C\delta$. Let $\phi_0(x)$ be the density of $Y_0$. Since $W$ is grid valued, for any $z \in (0, \delta)$ we have

$$\mathbb{P}(W = \beta) = \mathbb{P}(\beta - z \leq W \leq \beta + z)$$

$$\leq \mathbb{P}(\beta - z \leq Y_0 \leq \beta + z) + |\mathbb{P}(\beta - z \leq W \leq \beta + z) - \mathbb{P}(\beta - z \leq Y_0 \leq \beta + z)|$$

$$\leq 2z \sup_{x \in \mathbb{R}} \phi_0(x) + 2 \sup_{x \in \mathbb{R}} |\mathbb{P}(W \leq x) - \mathbb{P}(Y_0 \leq x)|.$$

To reach the desired conclusion, we use Lemma 7 and Theorem 3 of Braverman et al. (2016). The former says that $\phi_0(x) \leq \sqrt{2/\pi}$, while the latter result says that

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(W \leq x) - \mathbb{P}(Y_0 \leq x)| \leq C\delta.$$

**EC.1.1.1. Proof of Lemma EC.1.**

Proof of Lemma EC.1 If $f(x) = x$, then $\mathbb{E}G_{\beta}f(W) = 0$ because $\mathbb{E}|W| \leq \infty$; see Lemma 2 of Braverman et al. (2016) for a proof of the latter fact. It follows from (EC.1) that

$$\lambda - \mathbb{E}\left( \mu \left[ (W/\delta + R) \wedge \beta \right] \right) = \mathbb{E}b(W) = -\mu \mathbb{E}(W \wedge \beta) = 0,$$
implying that

\[-\beta \mathbb{P}(W \geq \beta) = \mathbb{E}(W1(W < \beta)). \tag{EC.9}\]

Adding \(\mathbb{E}(W1(W \geq \beta))\) to both sides proves that \(\mathbb{E}W = \mathbb{E}(W - \beta)^+\). The claim about \(Y_0\) follows similarly. The density of \(Y_0\) is given by (EC.5), so

\[\mathbb{E}b(Y_0) = \kappa \int_{-\infty}^{\infty} \frac{b(x)}{\mu} \exp \left( \int_0^x \frac{b(y)}{\mu} \, dy \right) \, dx = 0.\]

The last equality follows from integration by parts and the fact that \(\lim_{x \to \pm \infty} \exp \left( \int_0^x \frac{b(y)}{\mu} \, dy \right) = 0\). Therefore

\[\frac{1}{\mu} \mathbb{E}b(Y_0) = -\beta \mathbb{P}(Y_0 \geq \beta) - \mathbb{E}(Y_01(Y_0 < \beta)) = 0\]

and \(\mathbb{E}Y_0 = \mathbb{E}(Y_0 - \beta)^+\). \(\square\)

**EC.1.1.2. Proof of Lemma EC.2.**

Proof of Lemma EC.2  First, note that \(f_h'''(x) = -\frac{b(x)}{v_0} f_h'(x) + \frac{1}{v_0} (\mathbb{E}h(Y_0) - h(x))\) is differentiable almost everywhere because \(f_h'(x)\) is continuously differentiable and both \(b(x)\) and \(h(x)\) are Lipschitz functions. The former statement follows, for example, from (B.1) of Braverman et al. (2016). Therefore, \(f_h'''(x)\) exists almost everywhere. Now (EC.1) implies that

\[\mathbb{E}G_h f_h(W) = \mathbb{E} \left[ \lambda (f_h(W + \delta) - f_h(W)) + \mu [ (W/\delta + R) \wedge n ] (f_h(W - \delta) - f_h(W)) \right] = 0,\]

provided \(\mathbb{E}|f_h(W)| < \infty\). The integrability of \(f_h(W)\) has already been established in Braverman et al. (2016); see Lemma 1 and Remark 2 there. Since \(f_h'''(x)\) does not exist everywhere
(for instance at \(x = \beta\)), to perform Taylor expansion we need to use the integral form of the remainder term. We claim that

\[
f_h(x + \delta) - f_h(x) = \delta f'_h(x) + \frac{1}{2} \delta^2 f''_h(x) + \frac{1}{6} \delta^3 f'''_h(x-) + \varepsilon_+(x),
\]

\[
f_h(x - \delta) - f_h(x) = -\delta f'_h(x) + \frac{1}{2} \delta^2 f''_h(x) - \frac{1}{6} \delta^3 f'''_h(x-) + \varepsilon_-(x).
\]

(EC.10)

To verify the claim, note that

\[
f_h(x + \delta) - f_h(x) = \int_x^{x+\delta} f'_h(y)dy = \delta f'_h(x) + \int_x^{x+\delta} (f'_h(y) - f'_h(x))dy
\]

\[
= \delta f'_h(x) + \int_x^{x+\delta} \int_y^x f''_h(u)du dy
\]

\[
= \delta f'_h(x) + \int_x^{x+\delta} (x+\delta-u)f''_h(u)du.
\]

A similar treatment of \(f_h(x+\delta) - f_h(x)\) yields (EC.10). Letting \(s(W) = \mu[(W/\delta + R) \land n]\), we therefore have

\[
\mathbb{E}G_X f_h(W) = \mathbb{E}\left[\delta(\lambda - s(W))f'_h(W) + \frac{1}{2} \delta^2(\lambda + s(W))f''_h(W)
\right.\]

\[
+ \frac{1}{6} \delta^3(\lambda - s(W))f'''_h(W-) + \lambda \varepsilon_+(W) + s(W)\varepsilon_-(W)\right].
\]

We know from (EC.3) that \(\delta(\lambda - s(W)) = b(W)\) and \(\delta^2(\lambda + s(W)) = a(W)\), and consequently \(s(W) = \lambda - b(W)/\delta\). Therefore,

\[
\mathbb{E}G_X f_h(W)
\]

\[
= \mathbb{E}\left[b(W)f'_h(W) + \frac{1}{2} a(W)f''_h(W) + \frac{1}{6} \delta b(W)f'''_h(W-) + \lambda(\varepsilon_+(W) + \varepsilon_-(W)) - \frac{1}{\delta} b(W)\varepsilon_-(W)\right] = 0.
\]

Taking expected values in the Poisson equation (EC.8), we get

\[
\mathbb{E}h(Y_0) - \mathbb{E}h(W) = \mathbb{E}\left[b(W)f'_h(W) + v_0f''_h(W)\right] - \mathbb{E}G_X f_h(W)
\]

\[
= -\frac{1}{2} \mathbb{E}(a(W) - a(0))f''_h(W) - \mathbb{E}\varepsilon(W).
\]

We conclude by noting that \(a(W) - a(0) = -\delta b(W)\). \(\square\)
EC.1.1.3. Proof of Lemma EC.3.

Proof of Lemma EC.3 To bound $|f^{(4)}(x)|$, first note that the Poisson equation (EC.8) implies that

$$f'''_h(x) = -\frac{b(x)}{v_0} f''_h(x) - \frac{b'(x)}{v_0} f'_h(x) - \frac{1}{v_0} h'(x).$$

Since $b(x)$ and $h(x)$ are piece-wise linear with a kink at $x = \beta$, the derivative above exists for all $x \neq \beta$. Differentiating again, we get

$$f^{(4)}_h(x) = -\frac{b(x)}{v_0} f'''_h(x) - \frac{b'(x)}{v_0} f''_h(x) - \frac{b'(x)}{v_0} f'_h(x), \quad x \neq \beta.$$ 

We conclude that $|f^{(4)}_h(x)| \leq (C/\mu)(1 + |x|)$ for $x \neq \beta$ because $v_0 = \mu$, $|b(x)| \leq \mu |x|$, $|b'(x)| \leq \mu$, $|f''_h(x)| \leq C/\mu$, and $|f'_h(x)| \leq C/\mu$, where the last two inequalities follow from Lemma 3 of Braverman et al. (2016). To conclude the proof, we note that $\mathbb{E}|b(W)| \leq \mu \mathbb{E}|W| \leq \mu C$, where the last inequality follows from Lemma 2 of Braverman et al. (2016), which tell us that $\mathbb{E}|W| \leq C$. \hfill \Box

EC.1.1.4. Proof of Lemma EC.4.

Proof of Lemma EC.4 First assume that $x \geq \beta$. In (B.8) of Braverman et al. (2016) it is shown that

$$f''_h(x) = e^{-\int_0^x \frac{b(u)}{v_0} du} \int_x^\infty \frac{1}{\mu} (h'(y) + f'_h(y)b'(y)) e^{\int_y^\infty \frac{b(u)}{v_0} du} dy = e^{-\int_0^\beta \frac{b(u)}{v_0} du} \int_x^\infty \frac{1}{\mu} e^{\beta b(u)/v_0} dy,$$

where in the second equality we use $b'(x) = 0$ and $h'(x) = 1$ for $x \geq \beta$. Since $b(x)/v_0 = -\beta$ for $x \geq \beta$,

$$f''_h(x) = e^{\beta(x-\beta)} \int_x^\infty \frac{1}{\mu} e^{-\beta(y-\beta)} dy = \frac{1}{\mu \beta}.$$
Now suppose that $x \leq \beta$. The Poisson equation (EC.8) and the fact that $h(x) = 0$ imply that

$$f''_h(x) = -\frac{b(x)}{v_0} f'_h(x) + \frac{1}{v_0} \mathbb{E}h(Y_0) = xf'_h(x) + \frac{\mathbb{E}h(Y_0)}{\mu}.$$ 

One can verify by differentiating that

$$f'_h(x) = e^{-\int_0^x \frac{b(u)}{v_0} du} \int_{-\infty}^x \frac{1}{v_0} (\mathbb{E}h(Y_0) - h(y)) e^{\int_y^\beta \frac{b(u)}{v_0} du} dy = \frac{\mathbb{E}h(Y_0)}{\mu} e^{\frac{1}{2}x^2} \int_{-\infty}^x e^{-\frac{1}{2}y^2} dy.$$ 

The first equality appears as equation (B.1) in Braverman et al. (2016). The second equality follows from the form of $b(x)$ in (EC.4) and the fact that $h(x) = 0$ for $x \leq \beta$. Lastly, since the density of $Y_0$ is given by (EC.5), we have

$$\mathbb{E}h(Y_0) = \int_{-\infty}^\infty h(y) \frac{e^{\int_y^\beta \frac{b(u)}{v_0} du}}{\sqrt{2\pi}} dy = \frac{\int_{-\infty}^\beta (y - \beta)^+ e^{\int_y^\beta \frac{b(u)}{v_0} du} dy}{\int_{-\infty}^\beta e^{\int_y^\beta \frac{b(u)}{v_0} du} dy} + \frac{\int_{-\infty}^\infty e^{-\frac{1}{2}(y^2 - \beta^2)} dy}{\int_{-\infty}^\beta e^{-\frac{1}{2}(y^2 - \beta^2)} dy + \int_{\beta}^\infty e^{-\beta(y - \beta)} dy} = \frac{1}{\beta^2} \frac{1}{\beta^2} \int_{-\infty}^\beta e^{-\frac{1}{2}y^2} dy + 1/\beta.$$ 

This verifies the form of $f''_h(x)$ when $x \leq \beta$. □

**EC.1.2. Proving Theorem 3**

We first recall Theorem 3.

**Theorem EC.1.** Assume $n = R + \beta \sqrt{R}$ for some fixed $\beta > 0$. There exist positive constants $c_1$ and $C$ depending only on $\beta$ such that

$$\left| \frac{\mathbb{P}(Y \geq z)}{\mathbb{P}(W \geq z)} - 1 \right| \leq C \frac{1 + \frac{z}{\sqrt{R}}}{\sqrt{R}} \quad \text{for } 0 < z \leq c_1 R \quad \text{(EC.11)}$$

$$\left| \frac{\mathbb{P}(Y \leq -z)}{\mathbb{P}(W \leq -z)} - 1 \right| \leq C \frac{1 + \frac{z^4}{\sqrt{R}}}{\sqrt{R}} , \quad \text{for } 0 < z \leq \min\{c_1 R^{1/4}, R^{1/2}\}. \quad \text{(EC.12)}$$
We use $c_1$, $C$, $C_1$, $K$ to denote positive constants which may differ from line to line, but will only depend on $\beta$. We first prove (EC.12), and then (EC.11).

Note that if $R \leq K$ for some $K > 0$, then $n$ is also bounded because $\beta$ is fixed and $n = R + \beta \sqrt{R}$. We argue that (EC.12) holds trivially in such a case. Observe that for any $K > 0$, because $\{W = -\sqrt{R}\}$ corresponds to an empty system,

$$\inf_{0 < x \leq 0} \mathbb{P}(W \leq -z) \geq \inf_{0 < x \leq K} \mathbb{P}(W = -\sqrt{R}) \geq L(K, \beta) > 0,$$

where $L(K, \beta)$ is a positive constant depending only on $K$ and $\beta$. The second-last inequality is true because $\inf_{0 < x \leq K} \mathbb{P}(W = -\sqrt{R})$ is at least as large as $\mathbb{P}(W = -\sqrt{R})$ when $R = K$, because for each $n$, the probability of an empty system decreases in $R$. We can choose a sufficiently large $C$ (that depends on $K$) to ensure (EC.12) trivially holds. Therefore, in the following, we assume $R \geq C_1$ for a sufficiently large $C_1$. Since (EC.12) requires $0 < z \leq c_1 R^{1/4}$, we can also assume that

$$\delta = \frac{1}{\sqrt{R}} < \min\{1/2, \beta\}, \quad 0 < z < \sqrt{R} - 2, \quad \delta(z + 1) < 1/2. \quad \text{(EC.13)}$$

If not, we simply increase the value of $C_1$ and decrease the value of $c_1$ until (EC.13) holds. Without loss of generality let us therefore assume (EC.13) going forward. Given $z \in \mathbb{R}$, we let $f_z(x)$ be the solution (cf. (EC.34)) to the Poisson equation

$$\frac{1}{2} a(x) f''_z(x) + b(x) f'_z(x) = \mathbb{P}(Y_1 \leq -z) - 1(x \leq -z), \quad x \in \mathbb{R}. \quad \text{(EC.14)}$$

The following object will be of use. Define, for $W$ in its support (EC.2),

$$K_W(y) = \begin{cases} 
(\lambda - b(W)/\delta)(y + \delta) \geq 0, & y \in [-\delta, 0], \\
\lambda(\delta - y) \geq 0, & y \in [0, \delta].
\end{cases} \quad \text{(EC.15)}$$
It can be checked that
\[
\int_{-\delta}^{\delta} K_W(y)dy = \frac{1}{2} \delta^2 \lambda - \frac{1}{2} \delta b(W), \quad \int_{0}^{\delta} K_W(y)dy = \frac{1}{2} \delta^2 \lambda,
\]
\[
\int_{-\delta}^{\delta} K_W(y)dy = \frac{1}{2} a(W) = \mu - \frac{\delta}{2} b(W), \quad \text{and} \quad \int_{-\delta}^{\delta} yK_W(y)dy = \frac{\delta^2 b(W)}{6}. \tag{EC.16}
\]

Our first result is an expression for \( P(Y_1 \leq -z) - P(W \leq -z) \), and is proved in Section EC.1.2.1.

**Lemma EC.5.** For any \( z \in \mathbb{R} \),
\[
P(Y_1 \leq -z) - P(W \leq -z) = \mathbb{E} \left[ \int_{-\delta}^{\delta} \left( \frac{2b(W+y)}{a(W+y)} f_z'(W+y) - \frac{2b(W)}{a(W)} f_z'(W) \right) K_W(y)dy \right] \\
- \mathbb{E} \left[ \int_{-\delta}^{\delta} \left( \frac{2}{a(W)} 1(W \leq -z) - \frac{2}{a(W+y)} 1(W+y \leq -z) \right) K_W(y)dy \right] \\
- P(Y_1 \leq -z) \mathbb{E} \left[ \int_{-\delta}^{\delta} \left( \frac{2}{a(W+y)} - \frac{2}{a(W)} \right) K_W(y)dy \right]. \tag{EC.17}
\]

The bulk of the effort to prove (EC.12) comes from the first term. The following lemma is proved in Section EC.1.2.2.

**Lemma EC.6.** For \( x \in \mathbb{R} \), define \( r(x) = 2b(x)/a(x) \). There exists constants \( c_1, C, C_1 > 0 \) depending only on \( \beta \) such that for any \( R \geq C_1 \) and \( 0 < z \leq c_1 R^{1/4} \) satisfying (EC.13),
\[
\left| \mathbb{E} \left[ \int_{-\delta}^{\delta} (r(W+y)f_z'(W+y) - r(W)f_z'(W)) K_W(y)dy \right] \right| \leq C \delta^2 (z \vee 1)^{4} P(Y_1 \leq -z). \tag{EC.18}
\]

**Proof of (EC.12)** We first bound the third term in (EC.17). Using the form of \( a(x) \) in (EC.4) and the assumption that \( \delta < 1/2 \) in (EC.13), it is not hard to check that
\[
\mu \leq a(x) \leq 2\mu + \delta \mu \beta \leq C \mu, \quad \text{and} \quad |a'(x)| \leq \delta \mu, \tag{EC.19}
\]
from which it follows that
\[
\frac{1}{a(x)} \leq \frac{1}{\mu} \quad \text{and} \quad \left| \frac{1}{a(x)} - \frac{1}{a(y)} \right| = \frac{\left| a(y) - a(x) \right|}{a(y)a(x)} \leq \frac{\delta |y-x|}{\mu}.
\] (EC.20)

Therefore, the third term in (EC.17) satisfies
\[
\mathbb{P}(Y_1 \leq -z) \left| \mathbb{E} \left[ \int_{-\delta}^{\delta} \left( \frac{2}{a(W+y)} - \frac{2}{a(W)} \right) K_W(y) dy \right] \right| \leq \mathbb{P}(Y_1 \leq -z) \left| \mathbb{E} \left[ \frac{2\delta^2}{\mu} \int_{-\delta}^{\delta} K_W(y) dy \right] \right|
\]
\[
= \mathbb{P}(Y_1 \leq -z) \left| \mathbb{E} \left[ \frac{2\delta^2}{\mu} \frac{1}{2a(W)} \right] \right|
\]
\[
\leq C\delta^2 \mathbb{P}(Y_1 \leq -z). \] (EC.21)

The equality is due to (EC.16). The second term in (EC.17) is bounded similarly. Namely,
\[
\left| \mathbb{E} \left[ \int_{-\delta}^{\delta} \left( \frac{2}{a(W)} 1(W \leq -z) - \frac{2}{a(W+y)} 1(W+y \leq -z) \right) K_W(y) dy \right] \right|
\]
\[
\leq \left| \mathbb{E} \left[ \int_{-\delta}^{\delta} \left( \frac{2}{a(W)} - \frac{2}{a(W+y)} \right) 1(W \leq -z) K_W(y) dy \right] \right|
\]
\[
+ \left| \mathbb{E} \left[ \int_{-\delta}^{\delta} \frac{2}{a(W+y)} \left( 1(W \leq -z) - 1(W+y \leq -z) \right) K_W(y) dy \right] \right|
\]
\[
\leq C\delta^2 \mathbb{P}(W \leq -z) + \frac{2}{\mu} \mathbb{E} \left[ \int_{-\delta}^{\delta} \left| 1(W \leq -z) - 1(W+y \leq -z) \right| K_W(y) dy \right]. \] (EC.22)

Letting \(-\tilde{z}\) denote the smallest value in the support of \(\{W : W > -z\}\) (cf. (EC.2)), we have
\[
\frac{2}{\mu} \mathbb{E} \left[ \int_{-\delta}^{\delta} \left| 1(W \leq -z) - 1(W+y \leq -z) \right| K_W(y) dy \right] \leq C\mathbb{P}(W = -\tilde{z}) + C\mathbb{P}(W = -\tilde{z} - \delta).
\] (EC.23)

At the end of this proof we argue that
\[
\mathbb{P}(W = -\tilde{z}) + \mathbb{P}(W = -\tilde{z} - \delta) \leq C\delta(z \vee 1)\mathbb{P}(W \leq -z). \] (EC.24)
Combining the bounds in (EC.21), (EC.22), (EC.23) and (EC.24) with Lemma EC.6 yields

\[ |P(Y_1 \leq -z) - P(W \leq -z)| \leq C\delta^2(z \vee 1)^4P(Y_1 \leq -z) + C\delta(z \vee 1)P(W \leq -z), \]

Dividing both sides by \(P(W \leq -z)\), which is allowed because our assumption that \(z < \sqrt{R} - 2\) in (EC.13) implies that \(P(W \leq -z) \geq P(W = -\sqrt{R}) > 0\), we arrive at

\[ \frac{|P(Y_1 \leq -z)|}{P(W \leq -z)} - 1 \leq C\left(\delta^2(z \vee 1)^4P(Y_1 \leq -z) + \delta(z \vee 1)\right). \quad \text{(EC.25)} \]

Since we assumed \(z \leq c_1R^{1/4}\) and \(R \geq C_1\), it follows that \(\delta^2(z \vee 1)^4 \leq c_1^4\vee (1/C_1)\), which can be made arbitrarily small (without affecting \(C\) above) by decreasing \(c_1\) and increasing \(C_1\). Therefore, without loss of generality we can assume \(C\delta^2(z \vee 1)^4 < 1/2\), so

\[ \frac{1}{2} \frac{P(Y_1 \leq -z)}{P(W \leq -z)} \leq (1 - C\delta^2(z \vee 1)^4)\frac{P(Y_1 \leq -z)}{P(W \leq -z)} \leq 1 + C\delta(z \vee 1) \leq C, \]

where the second inequality is due to (EC.25) and the last inequality follows from (EC.13). Combining the upper bound above with (EC.25) proves (EC.12). It remains to verify (EC.24). Equation (EC.6) implies that for \(-y\) in the support of \(W\),

\[ \lambda P(W = -y - \delta) = \mu[(-y/\delta + R) \wedge n]P(W = -y). \]

Since \(\beta = \delta(n - R)\), the set \(\{-y/\delta + R \leq n\}\) equals \(\{-y \leq \beta\}\), so dividing both sides above by \(\lambda\) and using the fact that \(\mu/\lambda = 1/R\), it follows that

\[ P(W = -y - \delta) = (1 - \delta y)P(W = -y) \quad \text{for } -y \leq \beta. \quad \text{(EC.26)} \]

Recall that \(\tilde{z}\) denotes the smallest value in the support of \(\{W : W > -z\}\), so \(-\tilde{z} - \delta \leq -z < -\tilde{z}\) and the set \(\{W \leq -z\}\) equals \(\{W \leq -\tilde{z} - \delta\}\). Also recall that we assumed in (EC.13) that \(0 < z < \sqrt{R} - 2\) and \(\delta < \beta\). Therefore,

\[ P(W \leq -z) = P(W = -\tilde{z} - \delta) + P(W = -\tilde{z} - 2\delta) + \cdots + P(W = -\sqrt{R}). \]
\[ P(W = -\tilde{z}) \left( (1 - \delta \tilde{z}) + ((1 - \delta \tilde{z})(1 - \delta (\tilde{z} + \delta))) \right.
\[ \geq P(W = -\tilde{z}) \left( (1 - \delta \tilde{z}) + ((1 - \delta \tilde{z})(1 - \delta (\tilde{z} + \delta))) \right.
\[ \geq P(W = -\tilde{z}) \left( (1 - \delta \tilde{z}) \cdots (1 - \delta (\tilde{z} + \left\lfloor \frac{1}{\delta} \right\rfloor \delta)) \right). \]

The second equality follows from (EC.26) with \( \tilde{z} \) in place of \( y \). This requires that \(-\tilde{z} \leq \beta\), which follows from the fact that \(-\tilde{z} - \delta \leq -z < 0\) and therefore \(-\tilde{z} < \beta\) by our assumption that \( \delta < \beta \) in (EC.13). In the last inequality, \( \left\lfloor \cdot \right\rfloor \) denotes the integer part and the inequality itself follows from the fact that \(- (\tilde{z} + \left\lfloor \frac{1}{\delta} \right\rfloor \delta) \geq -(\tilde{z} + 1) > - (z + 1) > -\sqrt{R}\). Now for any \( 0 \leq k \leq \left\lfloor \frac{1}{\delta} \right\rfloor \) we have \( 1 - \delta (\tilde{z} + \delta k) \geq 1 - \delta (\tilde{z} + 1) > 1 - \delta (z + 1) > 1/2\), where the last inequality follows from our assumption in (EC.13). Therefore, the right-hand side above is bounded from below by

\[ P(W = -\tilde{z}) \left( (1 - \delta (\tilde{z} + 1)) + (1 - \delta (\tilde{z} + 1))^2 + \cdots + (1 - \delta (\tilde{z} + 1))^{|\frac{1}{\delta}| + 1} \right) \]

\[ \geq P(W = -\tilde{z})(1 - \delta (\tilde{z} + 1)) \frac{1 - (1 - \delta (\tilde{z} + 1))^{|\frac{1}{\delta}| + 1}}{\delta (\tilde{z} + 1)} \]

\[ \geq P(W = -\tilde{z})(1 - \delta (\tilde{z} + 1)) \frac{1 - (e^{-\delta (\tilde{z} + 1)})^{|\frac{1}{\delta}| + 1}}{\delta (\tilde{z} + 1)} \]

\[ \geq P(W = -\tilde{z})(1 - \delta (\tilde{z} + 1)) \frac{1 - e^{-1/2}}{\delta (\tilde{z} + 1)} \]

\[ \geq P(W = -\tilde{z}) \frac{1 - e^{-1/2}}{2\delta (z + 1)}. \]

The first inequality is true because \( 1 - x \leq e^{-x} \) and \( \delta (\tilde{z} + 1) \geq \delta (z - \delta + 1) > 0 \). The second inequality is true because \( 0 < \delta (\tilde{z} + 1) < \delta (z + 1) \) and \( \delta (\tilde{z} + 1)(|1/\delta| + 1) > (\delta/2)(|1/\delta| + 1) > 1/2 \) (because \( |1/\delta| + 1 > 1/\delta, \delta < 1/2 \) by (EC.13), and \(-\tilde{z} - \delta \leq -z < 0\) from it follows that \( \tilde{z} > -\delta > -1/2\)). The last inequality follows from \( \delta (z + 1) < 1/2 \) in (EC.13). This
implies that \( \mathbb{P}(W = -\tilde{z}) \leq C \delta (z \lor 1) \mathbb{P}(W \leq -z) \). Using \(-\tilde{z} < \delta < 1/2\) and (EC.26) again we get

\[
\mathbb{P}(W = -\tilde{z} - \delta) = (1 - \delta \tilde{z}) \mathbb{P}(W = -\tilde{z}) \leq (1 + \delta^2) \mathbb{P}(W = -\tilde{z}) \leq \frac{5}{4} \mathbb{P}(W = -\tilde{z}),
\]

This proves (EC.24).  \( \square \)

**EC.1.2.1. Proof of Lemma EC.5.**

*Proof of Lemma EC.5* Lemma EC.8, which we state in Section EC.1.2.2, implies that \( f'_z(x) \) is bounded and absolutely continuous with bounded \( f''_z(x) \). Lemma EC.3 implies \( \mathbb{E}|W| < \infty \), which when combined with the fact that \( f'_z(x) \) is bounded implies \( \mathbb{E}|f_z(W)| < \infty \), and in turn \( \mathbb{E}G_X f_z(W) = 0 \) due to (EC.1). Letting

\[
G_Y f(x) = \frac{1}{2} a(x)f''(x) + b(x)f'(x),
\]

taking expected values with respect to \( W \) in the Poisson equation (EC.14) and subtracting \( \mathbb{E}G_X f_z(W) = 0 \) from it, we get

\[
\mathbb{E}G_Y f_z(W) - \mathbb{E}G_X f_z(W) = \mathbb{P}(Y_1 \leq -z) - \mathbb{P}(W \leq -z).
\]  \( \text{(EC.27)} \)

To prove the lemma we work on the left-hand side. Similar to (EC.10), for any function \( f: \mathbb{R} \to \mathbb{R} \) with an absolutely continuous derivative, and any \( x, \delta \in \mathbb{R} \),

\[
f(x + \delta) - f(x) = \delta f'(x) + \int_x^{x+\delta} (x + \delta - y) f''(y) dy = \delta f'(x) + \int_0^\delta (\delta - y) f''(x + y) dy.
\]

Applying this expansion to \( G_X \) in (EC.1), we get

\[
G_X f_z(W) = \delta \left( \lambda - \mu \left[ (W/\delta + R) \land n \right] \right) f'_z(W)
+ \lambda \int_0^\delta (\delta - y) f''_z(W + y) dy + \mu \left[ (W/\delta + R) \land n \right] \int_{-\delta}^0 (y + \delta) f''_z(W + y) dy.
\]
Recall from (EC.3) that $b(W) = \delta(\lambda - \mu[(W/\delta + R) \land n])$, and consequently $\mu[(W/\delta + R) \land n] = \lambda - b(W)/\delta$. We recall $K_W(y)$ from (EC.15), so

$$G_X f_z(W) = b(W) f'_z(W) + \int_{-\delta}^{\delta} f''_z(W + y) K_W(y) dy$$

$$= b(W) f'_z(W) + \frac{1}{2} a(W) f''_z(W) + \int_{-\delta}^{\delta} (f''_z(W + y) - f''_z(W)) K_W(y) dy. \quad (\text{EC.28})$$

The last equality follows from $\int_{-\delta}^{\delta} K_W(y) dy = \frac{1}{2} a(W)$ in (EC.16). Thus,

$$P(Y_1 \leq -z) - P(W \leq -z) = \mathbb{E}G_Y f_z(W) - \mathbb{E}G_X f_z(W)$$

$$= \mathbb{E} \left[ \int_{-\delta}^{\delta} (f''_z(W) - f''_z(W + y)) K_W(y) dy \right]. \quad (\text{EC.29})$$

In the last equality above we used (EC.28). The lemma follows from (EC.14); i.e., $f''_z(x) = -\frac{2b(x)}{a(x)} f'_z(x) + \frac{2}{a(x)} (P(Y_1 \leq -z) - 1(x \leq -z)). \quad \square$

**EC.1.2.2. Proof of Lemma EC.6.** We first present a series of intermediary lemmas that represent the main steps in the proof, and then use them to prove Lemma EC.6. We remind the reader that (EC.4) implies that

$$r(x) = \begin{cases} -2x, & x \leq -1/\delta, \\ \frac{-2x}{2 + \delta x}, & x \in [-1/\delta, \beta], \\ \frac{-2\delta}{2 + \delta \beta}, & x \geq \beta, \end{cases}$$

$$r'(x) = \begin{cases} -2, & x \leq -1/\delta, \\ \frac{-4}{(2 + \delta x)^2}, & x \in (-1/\delta, \beta], \\ 0, & x > \beta, \end{cases} \quad (\text{EC.30})$$

where $r'(x)$ is interpreted as the derivative from the left at the points $x = -1/\delta, \beta$. In particular, note that $|r'(x)| \leq 4$. The first lemma decomposes

$$\mathbb{E} \left[ \int_{-\delta}^{\delta} (r(W + y) f'_z(W + y) - r(W) f'_z(W)) K_W(y) dy \right]$$

into a more convenient form. It is proved in Section EC.1.2.3.
**Lemma EC.7.** Let $f_z(x)$ solve (EC.14), then

\[
\int_{-\delta}^{\delta} (r(W+y)f_z'(W+y) - r(W)f_z'(W)) K_W(y)dy \\
= \int_{-\delta}^{\delta} K_W(y)r(W+y)g f_z''(W)dy \\
- \int_{-\delta}^{\delta} K_W(y)r(W+y) \int_{0}^{y} \int_{0}^{y} r(W+u)f_z''(W+u)dudsdy \\
- \int_{-\delta}^{\delta} K_W(y)r(W+y) \int_{0}^{y} \int_{0}^{y} r'(W+u)f_z'(W+u)dudsdy \\
- \int_{-\delta}^{\delta} K_W(y)r(W+y) \int_{0}^{y} \left[ \frac{1(W+s \leq -z)}{a(W+s)/2} - \frac{1(W \leq -z)}{a(W)/2} \right]dsdy \\
+ \mathbb{P}(Y_1 \leq -z) \int_{-\delta}^{\delta} K_W(y)r(W+y) \int_{0}^{y} \left[ \frac{2}{a(W+s)} - \frac{2}{a(W)} \right]dsdy \\
+ 1(W = -\frac{1}{\delta}) f_z'(W) \int_{0}^{\delta} K_W(y) \int_{0}^{y} r'(W+s)dsdy \\
+ 1(W = \beta) f_z'(W) \int_{-\delta}^{0} K_W(y) \int_{0}^{y} r'(W+s)dsdy \\
+ 1(W \in [-\frac{1}{\delta}, \frac{\beta - \delta}] f_z'(W) \int_{-\delta}^{\delta} K_W(y) \int_{0}^{y} \int_{0}^{y} r''(W+u)dudsdy \\
+ 1(W \in [-\frac{1}{\delta}, \frac{\beta - \delta}] f_z'(W)r'(W) \delta^2 b(W) \frac{6}{6}.
\]

Next we need a bound on $f_z'(x)$ and $f_z''(x)$ as provided by the following lemma.

**Lemma EC.8.** Let $f_z(x)$ solve (EC.14). There exists a constant $C$ that depends only on $\beta$ such that for any $z > 0$,

\[
|f_z'(x)| \leq C \left( \frac{1(x \leq -z)}{\mu} \left( \frac{\mu}{|b(x)|} \wedge 1 \right) + \mathbb{P}(Y_1 \leq -z) \left( 1(-z < x < 0)e^{-\int_{0}^{x} r(u)du} + 1(x \geq 0) \left( \frac{\mu}{|b(x)|} \wedge 1 \right) \right) \),
\]

\[
|f_z''(x)| \leq C \left( 1(x \leq -z) + \mathbb{P}(Y_1 \leq -z) \left( 1(-z < x < 0)(1+|x|)e^{-\int_{0}^{x} r(u)du} + 1(x \geq 0) \right) \right).
\]
Proof of Lemma EC.8  We recall from (EC.5) that the density of $Y_1$ is

$$\kappa \frac{2}{a(x)} \exp \left( \int_0^x \frac{2b(y)}{a(y)} dy \right), \quad x \in \mathbb{R},$$

where $\kappa$ is the normalizing constant. One may verify that the solution to the Poisson equation $\frac{1}{2}a(x)f''_z(x) + b(x)f'_z(x) = \mathbb{P}(Y_1 \leq -z) - 1(x \leq -z)$ satisfies

$$f'_z(x) = \begin{cases} -\mathbb{P}(Y_1 \geq -z)e^{-\int_0^z r(u)du} \frac{1}{\kappa} \mathbb{P}(Y_1 \leq x), & x \leq -z, \\ -\mathbb{P}(Y_1 \leq -z)e^{-\int_0^z r(u)du} \frac{1}{\kappa} \mathbb{P}(Y_1 \geq x), & x \geq -z. \end{cases} \quad (EC.34)$$

Rearranging the Poisson equation and using (EC.34) yields

$$f''_z(x) = -r(x)f'_z(x) + 2\left( \mathbb{P}(Y_1 \leq -z) - 1(x \leq -z) \right) \frac{a(x)}{a(x)} = \begin{cases} -\mathbb{P}(Y_1 \geq -z) \left[ -\frac{2}{a(x)} - \frac{r(x)}{\kappa} e^{-\int_0^z r(u)du} \mathbb{P}(Y_1 \leq x) \right], & x \leq -z, \\ -\mathbb{P}(Y_1 \leq -z) \left[ -\frac{2}{a(x)} - \frac{r(x)}{\kappa} e^{-\int_0^z r(u)du} \mathbb{P}(Y_1 \geq x) \right], & x > -z. \end{cases} \quad (EC.35)$$

We will start by proving the bound on $f''_z(x)$. The form of the density of $Y_1$ implies

$$\frac{1}{\kappa} \mathbb{P}(Y_1 \leq x) = \int_{-\infty}^x \frac{2}{a(y)} e^{\int_0^y r(u)du} dy.$$

Since $b(x)$ is nonincreasing and $b(x) > 0$ when $x < 0$, then for $x < 0$,

$$\frac{1}{\kappa} \mathbb{P}(Y_1 \leq x) = \int_{-\infty}^x \frac{2}{a(y)} e^{\int_0^y r(u)du} dy \leq \frac{1}{b(x)} \int_{-\infty}^x \frac{2b(y)}{a(y)} e^{\int_0^y r(u)du} dy = \frac{1}{b(x)} e^{\int_0^x r(u)du}, \quad (EC.36)$$

and since $b(x) < 0$ for $x > 0$, then for $x > 0$,

$$\frac{1}{\kappa} \mathbb{P}(Y_1 \geq x) \leq \frac{1}{b(x)} \int_x^{\infty} \frac{2b(y)}{a(y)} e^{\int_0^y r(u)du} dy = \frac{-1}{b(x)} e^{\int_0^x r(u)du} = \frac{1}{|b(x)|} e^{\int_0^x r(u)du}. \quad (EC.37)$$
Applying (EC.36) and (EC.37) to the form of \( f''_z(x) \) above, we get the desired upper bound when \( x \leq -z \) or \( x \geq 0 \). When \(-z < x < 0\), we use the fact that \( |r(x)| \leq C|x| \) to get

\[
|f''_z(x)| \leq \frac{2}{a(x)} \mathbb{P}(Y_1 \leq -z) + C \mathbb{P}(Y_1 \leq -z) |x| e^{-\int_0^x f'_z r(u) du} dy \int_x^\infty \frac{2}{a(y)} e^{f'_z r(u) du} dy \leq \frac{C}{\mu} \mathbb{P}(Y_1 \leq -z) (1 + |x|) e^{-\int_0^x f'_z r(u) du}.
\]

The last inequality follows from the facts that \( a(x) \geq \mu \), and that \( \int_x^\infty e^{f'_z r(u) du} dy \) can be bounded by a constant that depends only on \( \beta \), which is evident from the form of \( r(x) \).

This establishes the bound on \( f''_z(x) \). Repeating the same procedure with \( f'_z(x) \) gives us the bound

\[
|f'_z(x)| \leq C \left( 1(x \leq -z) \frac{1}{b(x)} + \mathbb{P}(Y_1 \leq -z) \left( 1(-z < x < 0) \frac{1}{\mu} e^{-\int_0^x f'_z r(u) du} + 1(x \geq 0) \frac{1}{|b(x)|} \right) \right).
\]

To conclude the proof, we require the following two inequalities from Lemma B.8 of Braverman (2017):

\[
e^{-\int_0^x f'_z r(u) du} \int_{-\infty}^x \frac{2}{a(y)} e^{f'_z r(u) du} dy \leq \begin{cases} \frac{3}{\mu}, & x \leq 0, \\ \frac{1}{\mu} e^{\beta^2 (3 + \beta)}, & x \in [0, \beta], \\ \frac{1}{\mu} e^{\beta^2 (3 + \beta)}, & x \in [0, \beta], \\ \frac{1}{\mu}, & x \geq \beta. \end{cases} \tag{EC.38}
\]

\[
e^{-\int_0^x f'_z r(u) du} \int_x^\infty \frac{2}{a(y)} e^{f'_z r(u) du} dy \leq \begin{cases} \frac{3}{\mu}, & x \leq 0, \\ \frac{1}{\mu} e^{\beta^2 (3 + \beta)}, & x \in [0, \beta], \\ \frac{1}{\mu}, & x \geq \beta. \end{cases} \tag{EC.39}
\]

By repeating the bounding procedure discussed above, but using (EC.38) and (EC.39) in place of (EC.36) and (EC.37), we arrive at

\[
|f'_z(x)| \leq \frac{C}{\mu} \left( 1(x \leq -z) + \mathbb{P}(Y_1 \leq -z) \left( 1(-z < x < 0) e^{-\int_0^x f'_z r(u) du} + 1(x \geq 0) \right) \right),
\]

which establishes the desired bound on \( f'_z(x) \). \( \square \)
Lastly, we will need the following lemmas, which are proved in Section EC.1.2.4.

**Lemma EC.9.** There exist constant $c_1, C_1 > 0$ such that for any integer $k \geq 0$, some $C(k) > 0$, and any $R \geq C_1$ and $0 < z \leq c_1 R^{1/4}$,

$$E|1(W \leq -z)W^k| \leq C(k)(z \vee 1)^{k+1} \mathbb{P}(Y_1 \leq -z), \quad (EC.40)$$

The constants $c_1, C_1, C(k)$ depend on $\beta$.

**Lemma EC.10.** There exist constant $c_1, C_1 > 0$ such that for any integer $k \geq 0$, some $C(k) > 0$, and any $R \geq C_1$ and $0 < z \leq c_1 R^{1/4}$,

$$E|1(-z \leq W \leq 0)W^k e^{-\int_{-\delta}^{W} r(W + y) f''_z(W) dy}| \leq C(k)(z \vee 1)^{k+1}, \quad (EC.41)$$

The constants $c_1, C_1, C(k)$ depend on $\beta$.

We are now ready to prove Lemma EC.6.

**Proof of Lemma EC.6.** Again, in the following, we C to denote a constant whose value may change from line to line but only depends on $\beta$. We take expected values on both sides of (EC.31) and bound the terms on the right-hand side one line at a time. For the first line, we need to bound

$$\left| E \left[ \int_{-\delta}^{\delta} K_W(y)r(W + y)y f''_z(W)dy \right] \right| \leq \left| E \left[ r(W)f''_z(W) \int_{-\delta}^{\delta} yK_W(y)dy \right] \right| + \left| E \left[ f''_z(W) \int_{-\delta}^{\delta} yK_W(y)(r(W + y) - r(W))dy \right] \right|.$$

Using (EC.16), it follows that

$$\left| E \left[ r(W)f''_z(W) \int_{-\delta}^{\delta} yK_W(y)dy \right] \right| = \left| E \left[ r(W)f''_z(W) \frac{\delta^2 b(W)}{6} \right] \right| \leq C_\mu \delta^2 E\left| W^2 f''_z(W) \right|,$$
where we used $|r'(x)| \leq C |x|$ and $|b(x)| \leq \mu |x|$ in the inequality. Furthermore, since $|a(x)| \leq C\mu$ and recalling from (EC.30) that $|r'(x)| \leq 4$, we have

$$
\left| \mathbb{E} \left[ f''_z(W) \int_{-\delta}^{\delta} y K_W(y)(r(W+y) - r(W)) dy \right] \right| \leq \mathbb{E} \left[ C |f''_z(W)| \delta^2 \int_{-\delta}^{\delta} K_W(y) dy \right] \\
= \mathbb{E} \left[ C |f''_z(W)| \delta^2 \frac{\mu(W)}{2} \right] \\
\leq C \mu \delta^2 |f''_z(W)|.
$$

Therefore,

$$
\left| \mathbb{E} \left[ \int_{-\delta}^{\delta} K_W(y)r(W+y) y f''_z(W) dy \right] \right| \leq C \mu \delta^2 (1 + W^2) |f''_z(W)|.
$$

Applying the bound on $|f''_z(x)|$ from Lemma EC.8, we see that the right-hand side above is bounded by

$$
C \delta^2 \mathbb{E} \left[ (1 + W^2) 1(W \leq -z) \right. \\
+ \mathbb{P}(Y_1 \leq -z)(1 + W^2) \left( (1 - z < W < 0)(1 + |W|) e^{-\int_0^W r(u) du} + 1(W \geq 0) \right) \right] \\
\leq C \mathbb{P}(Y_1 \leq -z) \delta^2 (z \vee 1)^4,
$$

where the inequality is due to Lemmas EC.9 and EC.10 and the fact that $\mathbb{E} W^2 \leq C$, which was proved in Lemma A.1 of Braverman (2017). Following the same argument, the second line

$$
\left| \int_{-\delta}^{\delta} K_W(y)r(W+y) \int_0^y \int_0^s r(W+u)f''_z(W+u) dudsdy \right| \\
\leq C \mu \delta^2 (1 + W^2) \sup_{-\delta \leq u \leq \delta} |f''_z(W+u)| \\
\leq C \delta^2 \mathbb{E} \left[ (1 + W^2) 1(W \leq -z + \delta) \right. \\
+ \mathbb{P}(Y_1 \leq -z)(1 + W^2) \left( (1 - z - \delta < W < \delta)(1 + |W|) e^{sup_{-\delta \leq u \leq \delta} (-\int_0^{W+u} r(v) dv)} + 1(W \geq -\delta) \right) \right],
$$
which is also bounded by $C\mathbb{P}(Y_1 \leq -z)\delta^2(z \vee 1)^4$ from simple modifications of Lemmas EC.9 and EC.10. For the third line, we use the representation $f_z'(W + u) - f_z'(W) = \int_0^u f_z''(W + v)dv$ and the triangle inequality to get

\[
\left| \mathbb{E} \left[ \int_{-\delta}^{\delta} K_W(y)(W + y) \int_0^y \int_0^s r'(W + u) f_z'(W + u) dudsdy \right] \right|
\leq \left| \mathbb{E} \left[ \int_{-\delta}^{\delta} K_W(y) r(W + y) \int_0^y \int_0^s r'(W + u) f_z''(W + v) dv dudsdy \right] \right|
+ \left| \mathbb{E} \left[ \int_{-\delta}^{\delta} K_W(y) f_z'(W) (r(W + y) - r(W)) \int_0^y \int_0^s r'(W + u) dudsdy \right] \right|
+ \left| \mathbb{E} \left[ \int_{-\delta}^{\delta} K_W(y) f_z'(W) r(W) \int_0^y \int_0^s r'(W + u) dudsdy \right] \right|
\leq C\mathbb{E} \left[ (|W| + \delta) \int_{-\delta}^{\delta} K_W(y) \int_0^y \int_0^s |f_z''(W + v)| dv dudsdy \right]
+ C\delta^3 \mathbb{E} \left[ |f_z'(W)| \int_{-\delta}^{\delta} K_W(y) dy \right]
+ C\delta^3 \mathbb{E} \left[ |r(W) f_z'(W)| \int_{-\delta}^{\delta} K_W(y) dy \right]
\leq C\mathbb{E} \left[ (|W| + \delta) \int_{-\delta}^{\delta} K_W(y) \int_0^y \int_0^s |f_z''(W + v)| dv dudsdy \right]
+ C\mu \delta^3 \mathbb{E} \left[ |f_z'(W)| \right]
+ C\delta^2 \mathbb{E} \left[ b(W) f_z'(W) \right].
\]

In the second inequality we used $|r'(x)| \leq 4$ and the last inequality is due to (EC.16), that $r(x) = 2b(x)/a(x)$ and the fact that $|a(x)| \leq C\mu$. The right-hand side can be bounded by $C\mathbb{P}(Y_1 \leq -z)\delta^2(z \vee 1)^4$ as follows. The first term on the right-hand side above can be bounded by repeating the procedure used to bound lines one and two of (EC.31). The second and third terms can be bounded by combining the bound on $f_z'(x)$ from Lemma EC.8 with Lemmas EC.9 and EC.10. For the fourth line, repeating the arguments...
from (EC.22), (EC.23) and (EC.24), we have

\[
\left| \mathbb{E} \left[ \int_{-\delta}^{\delta} K_W(y) r(W + y) \int_0^y \left[ 1(W + s \leq -z) - \frac{1(W \leq -z)}{a(W)/2} \right] ds \right] \right| \\
\leq \left| \mathbb{E} \left[ \int_{-\delta}^{\delta} K_W(y) r(W + y) \int_0^y \left[ \frac{2}{a(W + s)} - \frac{2}{a(W)} \right] 1(W \leq -z) ds \right] \right| \\
+ \left| \mathbb{E} \left[ \int_{-\delta}^{\delta} K_W(y) r(W + y) \int_0^y \left[ \frac{2(1(W + s \leq -z) - 1(W \leq -z))}{a(W + s)} \right] ds \right] \right| \\
\leq C\delta^3 \mathbb{E}(1 + |W|) 1(W \leq -z) + C\delta(z \vee 1) \mathbb{P}(W = -\bar{z}) + \mathbb{P}(W = -\bar{z} - \delta) \\
\leq C\mathbb{P}(Y_1 \leq -z)\delta^3(z \vee 1)^2 + C\delta^2(z \vee 1)^2 \mathbb{P}(W \leq -z) \\
\leq C\mathbb{P}(Y_1 \leq -z)\delta^2(z \vee 1)^3,
\]

where in the last two inequalities we used Lemma EC.9. The fifth line is bounded as follows. Using (EC.16), (EC.20) and the fact that \(|r(x)| \leq C|x|\), we get

\[
\left| \mathbb{E} \left[ \int_{-\delta}^{\delta} K_W(y) r(W + y) \mathbb{P}(Y_1 \leq -z) \int_0^y \left[ \frac{2}{a(W + s)} - \frac{2}{a(W)} \right] ds \right] \right| \\
\leq C\mathbb{P}(Y_1 \leq -z)\delta^3 \mathbb{E}(1 + |W|) \leq C\mathbb{P}(Y_1 \leq -z)\delta^3.
\]

The sixth line is bounded as follows. Using \(|r'(x)| \leq 4\), (EC.16), and \(|1(W = -1/\delta)f'_z(W)| \leq 1(W = -1/\delta)C/\mu\), which follows from Lemma EC.8 and the fact that \(z < \sqrt{R} - 2 < \sqrt{R} = 1/\delta\) from (EC.13), we have

\[
\left| \mathbb{E} \left[ 1(W = -1/\delta)f'_z(W) \int_0^\delta K_W(y) \int_0^y r'(W + s) ds \right] \right| \leq C\delta \mathbb{P}(W = -1/\delta),
\]

and now we argue that

\[
C\delta \mathbb{P}(W = -1/\delta) \leq C\delta^2(z \vee 1) \mathbb{P}(Y_1 \leq -z). \tag{EC.42}
\]

For any integer \(0 < k < R\), from (EC.6)

\[
\mathbb{P}(W = -1/\delta) = \mathbb{P}(W = -\sqrt{R}) \leq \frac{1}{k} \sum_{i=0}^{k} \mathbb{P}(W = \delta(i - R)) = \frac{1}{k} \mathbb{P}(W \leq \delta(k - R)).
\]
Choose \( k \) such that \( \delta(k - R) \) is the largest element in the support of \( W \) that is less than or equal to \(-z\), and use the fact that \( z \leq c_1 R^{1/4} \) to conclude that

\[
P(W = -1/\delta) \leq C\delta P(W \leq -z).
\]

Using Lemma EC.9 implies (EC.42). The seventh line is bounded as follows. Using \( |r'(x)| \leq 4 \), (EC.16), and \(|1(W = \beta) f'_x(W)| \leq 1(W = \beta) C/\mu \), which follows from Lemma EC.8, we have

\[
\left| \mathbb{E} \left[ 1(W = \beta) f'_x(W) \int_{-\delta}^{0} K_y(y) \int_{0}^{y} r'(W + s) ds dy \right] \right| 
\leq C\delta P(W = \beta) \mathbb{P}(Y_1 \leq -z) \leq C\delta^2 \mathbb{P}(Y_1 \leq -z),
\]

where we used \( P(W = \beta) \leq C\delta \), which was argued at the end of the proof of Proposition EC.1. The eighth line is bounded as follows. Using \( |r''(x)| \leq C\delta \) and Lemma EC.8,

\[
\left| \mathbb{E} \left[ 1(W \in [-1/\delta + \delta, \beta - \delta]) f'_x(W) \int_{-\delta}^{0} K_y(y) \int_{0}^{y} \int_{0}^{u} r''(W + u) du ds dy \right] \right| 
\leq C\delta^3 P(W \leq -z) + C\delta^3 \mathbb{P}(Y_1 \leq -z) \mathbb{E} |1(-z \leq W \leq 0) e^{-\int_{0}^{W} r(u) du}| + C\delta^3 \mathbb{P}(Y_1 \leq -z)
\leq C\delta^3 (z \lor 1) \mathbb{P}(Y_1 \leq -z),
\]

where in the last inequality we used Lemmas EC.9 and EC.10. The ninth line is bounded similarly. Using \( |r'(x)| \leq 4 \) and the bound on \( |f'_x(x)b(x)| \) from Lemma EC.8, we have

\[
\left| \mathbb{E} \left\{ 1(W \in [-1/\delta + \delta, \beta - \delta]) f'_x(W) r'(W) \frac{\delta^2 b(W)}{6} \right\} \right| 
\leq C\delta^2 \mathbb{E} |W1(W \leq -z)| + C\delta^2 \mathbb{P}(Y_1 \leq -z) \mathbb{E} |1(-z \leq W \leq 0) W e^{-\int_{0}^{W} r(u) du}| + C\delta^2 \mathbb{P}(Y_1 \leq -z)
\leq C\delta^2 (z \lor 1)^2 \mathbb{P}(W \leq -z) + C\delta^2 \mathbb{P}(Y_1 \leq -z) \leq C\delta^2 (z \lor 1)^3 \mathbb{P}(Y_1 \leq -z).
\]

The second inequality is due to Lemmas EC.9 and EC.10. The last inequality is due to Lemma EC.9 with \( k = 0 \) there. Combining the bounds proves Lemma EC.6. \( \square \)
EC.1.2.3. Proof of Lemma EC.7.

Proof of Lemma EC.7. Assume for now that for all \( y \in (-\delta, \delta) \),

\[
r(W + y)f'_z(W + y) - r(W)f'_z(W)
\]

\[
= yr(W + y)f''_z(W) - r(W + y) \int_0^y \int_0^s \left( r(W + u)f''_z(W + u) + r'(W + u)f'_z(W + u) \right) duds
\]

\[
- r(W + y) \int_0^y \left( \frac{2}{a(W + s)} 1(W + s \leq -z) - \frac{2}{a(W)} 1(W \leq -z) \right) ds
\]

\[
+ \mathbb{P}(Y_1 \leq -z) r(W + y) \int_0^y \left( \frac{2}{a(W + s)} - \frac{2}{a(W)} \right) ds + f'_z(W) \int_0^y r'(W + s) ds. \quad (EC.43)
\]

We postpone verifying (EC.43) to the end of this proof, but (EC.43) implies

\[
\int_{-\delta}^\delta \left( r(W + y)f'_z(W + y) - r(W)f'_z(W) \right) K_W(y) dy
\]

\[
= \int_{-\delta}^\delta K_W(y)yr(W + y)f''_z(W)dy
\]

\[
- \int_{-\delta}^\delta K_W(y)r(W + y) \int_0^y \int_0^s r(W + u)f''_z(W + u)dudsdy
\]

\[
- \int_{-\delta}^\delta K_W(y)r(W + y) \int_0^y \int_0^s r'(W + u)f'_z(W + u)dudsdy
\]

\[
- \int_{-\delta}^\delta K_W(y)r(W + y) \int_0^y \left( \frac{2}{a(W + s)} 1(W + s \leq -z) - \frac{2}{a(W)} 1(W \leq -z) \right) dsdy
\]

\[
+ \mathbb{P}(Y_1 \leq -z) \int_{-\delta}^\delta K_W(y)r(W + y) \int_0^y \left( \frac{2}{a(W + s)} - \frac{2}{a(W)} \right) dsdy
\]

\[
+ f'_z(W) \int_{-\delta}^\delta K_W(y) \int_0^y r'(W + s)dsdy.
\]

We are almost done, but the last term on the right-hand side above requires some additional manipulations. Since \( r'(x) = 0 \) for \( x > \beta \) and \( K_W(y) = 0 \) for \( W = -1/\delta \) and \( y \in [-\delta, 0] \),

\[
\int_{-\delta}^\delta K_W(y) \int_0^y r'(W + s)dsdy
\]

\[
= 1(W = -1/\delta) \int_{-\delta}^\delta K_W(y) \int_0^y r'(W + s)dsdy
\]
\[ + 1(W = \beta) \int_{-\delta}^{0} K_W(y) \int_{0}^{y} r'(W + s) ds dy + 1(W \in [-1/\delta + \delta, \beta - \delta]) \int_{-\delta}^{\delta} K_W(y) \int_{0}^{y} r'(W + s) ds dy, \]

and for \( W \in [-1/\delta + \delta, \beta - \delta], \)
\[
\int_{-\delta}^{\delta} K_W(y) \int_{0}^{y} r'(W + s) ds dy = \int_{-\delta}^{\delta} K_W(y) \int_{0}^{y} (r'(W + s) - r'(W)) ds dy + r'(W) \int_{-\delta}^{\delta} y K_W(y) dy = \int_{-\delta}^{\delta} K_W(y) \int_{0}^{y} r''(W + u) du ds dy + r'(W) \frac{1}{6} \delta^2 b(W). \]

To conclude the proof, we verify (EC.43):
\[
r(W + y) f'_z(W + y) - r(W) f'_z(W) = r(W + y) f'_z(W) + r(W + y) \int_{0}^{y} f''_z(W + s) ds - r(W) f'_z(W) = r(W + y) \int_{0}^{y} f''_z(W + s) ds + f'_z(W) \int_{0}^{y} r'(W + s) ds.
\]

Now
\[
\int_{0}^{y} f''_z(W + s) ds = y f''_z(W) + \int_{0}^{y} (f''_z(W + s) - f''_z(W)) ds = y f''_z(W) - \int_{0}^{y} \left( r(W + s) f'_z(W + s) - r(W) f'_z(W) \right) ds - \int_{0}^{y} \left( \frac{2}{a(W + s)} 1(W + s \leq -z) - \frac{2}{a(W)} 1(W \leq -z) \right) ds + \mathbb{P}(Y_1 \leq -z) \int_{0}^{y} \left( \frac{2}{a(W + s)} - \frac{2}{a(W)} \right) ds,
\]

and the fundamental theorem of calculus tells us that
\[
r(W + s) f'_z(W + s) - r(W) f'_z(W) = \int_{0}^{s} \left( r(W + u) f''_z(W + u) + r'(W + u) f'_z(W + u) \right) du,
\]

which proves (EC.43). \qed
**EC.1.2.4. Proving Lemmas EC.9 and EC.10.** We first prove an upper bound on $Ee^{-tW}$ for $0 \leq t \leq c_1 R^{1/4}$. Note that $Ee^{-tW} < \infty$ for $t \geq 0$ because $W \geq -\sqrt{R}$ (cf. (EC.2)).

**Lemma EC.11.** There exist $c_1, C_1 > 0$ such that if $R \geq C_1$ and $0 \leq t \leq c_1 R^{1/4}$, then

$$Ee^{-tW} \leq Ce^t \left(2 - \frac{\delta t^3}{6}\right).$$

**(EC.44)**

**Proof of Lemma EC.11** For $0 \leq s \leq t$, define $h(s) = Ee^{-sW}$, so $h'(s) = -E(We^{-sW})$.

We will shortly prove that

$$\left(1 + \frac{\delta s}{2} + \frac{\delta^2 s^2}{6}\right)h'(s) \leq (s + C\delta^2 s^3)h(s).$$

**(EC.45)**

from which we have

$$h'(s) \leq \left(\frac{s}{1 + \frac{\delta s}{2} + \frac{\delta^2 s^2}{6}} + \frac{C\delta^2 s^3}{1 + \frac{\delta s}{2} + \frac{\delta^2 s^2}{6}}\right)h(s) \leq \left(\frac{s}{1 + \frac{\delta s}{2} + \frac{\delta^2 s^2}{6}} + C\delta^2 t^3\right)h(s) \leq \left(s(1 - \delta s/2 + (\delta s/2)^2) + C\delta^2 t^3\right)h(s) \leq \left(s - \delta s^2/2 + C\delta^2 t^3\right)h(s).$$

Above we have used $0 \leq s \leq t$. The third inequality follows from the inequality $1/(1+x) \leq (1 - x + x^2)$ for $x \geq 0$. Since $h(0) = 1$,

$$\log(h(t)) = \int_0^t \frac{h'(s)}{h(s)} ds \leq \int_0^t (s - \frac{\delta s^2}{2} + C\delta^2 t^3) ds \leq \frac{t^2}{2} - \frac{\delta t^3}{6} + C\delta^2 t^4 \leq \frac{t^2}{2} - \frac{\delta t^3}{6} + Cc_1,$$

which implies (EC.44). We are left to prove (EC.45). Recall from (EC.1) that $EG_X f(W) = 0$ provided $E|f(W)| < \infty$. Choose $f(x) = -e^{-sx}/s$, so that $f'(x) = e^{-sx}$ and $f''(x) = -se^{-sx}$. The form of $G_X f(x)$ in (EC.28) implies

$$E\left[\frac{a(W)}{2} f''(W) + b(W) f'(W) + \int_{-\delta}^{\delta} (f''(W+y) - f''(W)) K_W(y) dy\right] = 0$$

**(EC.46)**
Using the form of $b(x)$ from (EC.4), we have
\[ \mathbb{E}[b(W)f'(W)] = \mathbb{E}[1(W \leq \beta)(-\mu W)e^{-sW} + 1(W > \beta)(-\mu\beta)e^{-sW}] = \mu h'(s) + \mathbb{E}[1(W > \beta)\mu(W - \beta)e^{-sW}] \] (EC.47)
\[ \geq \mu h'(s). \]

Similarly, using the form of $a(x)$ from (EC.4) and the fact that $W \geq -1/\delta$, we have
\[ -\mathbb{E}\left[ \frac{a(W)}{2} f''(W) \right] = \mu \mathbb{E}\left[ 1\left(-\frac{1}{\delta} \leq W \leq \beta\right)(1 + \frac{\delta W}{2})s e^{-sW} + 1\left(W > \beta\right)(1 + \frac{\delta W}{2})se^{-sW} \right] = \mu \mathbb{E}\left[ (1 + \frac{\delta W}{2})se^{-sW} \right] - \mu \mathbb{E}\left[ 1\left(W > \beta\right)\frac{\delta(W - \beta)}{2}se^{-sW} \right] \leq \mu \mathbb{E}\left[ (1 + \frac{\delta W}{2})se^{-sW} \right] = \mu sh(s) - \frac{\delta s}{2} h'(s). \] (EC.48)

Lastly,
\[ -\mathbb{E}\left[ \int_{-\delta}^{\delta} (f''(W+y) - f''(W))K_W(y)dy \right] = \mathbb{E}\left[ \int_{-\delta}^{\delta} s(e^{-s(W+y)} - e^{-sW})K_W(y)dy \right] \leq \mathbb{E}\left[ se^{-sW} \int_{-\delta}^{\delta} (-sy + s^2 y^2 e^{\left|sy\right|})K_W(y)dy \right] \leq \mathbb{E}\left[ se^{-sW} \int_{-\delta}^{\delta} (-sy + C s^2 y^2)K_W(y)dy \right]. \]

The first inequality is due to $e^{-x} - 1 \leq -x + x^2 |x|$, and the second inequality is due to $|sy| \leq c_1 R^{1/4} \frac{1}{\sqrt{R}} \leq C$. Recall from (EC.16) that $\int_{-\delta}^{\delta} K_W(y)dy = \frac{1}{2} a(W)$ and that $\int_{-\delta}^{\delta} yK_W(y)dy = \frac{\delta^2 b(W)}{6}$, and from (EC.19) that $|a(x)| \leq C \mu$. Also note from (EC.4) that $b(x) = -\mu x - 1(x > \beta)(\mu\beta - \mu x)$. Therefore, the right-hand side above is bounded by
\[ \mathbb{E}\left[ s^2 e^{-sW} \frac{-\delta^2 b(W)}{6} \right] + C \mu \delta^2 s^3 \mathbb{E}e^{-sW} \]
\[ = \frac{\delta^2 s^2}{6} \mathbb{E}[\mu W e^{-sW}] + \frac{\delta^2 s^2}{6} \mathbb{E}[1(W > \beta)(\mu\beta - \mu W)e^{-sW}] + C \mu \delta^2 s^3 \mathbb{E}e^{-sW} \]
\[ = -\mu \frac{\delta^2 s^2}{6} h'(s) + \frac{\delta^2 s^2}{6} \mathbb{E}[1(W > \beta)(\mu\beta - \mu W)e^{-sW}] + C \mu \delta^2 s^3 h(s) \leq -\mu \frac{\delta^2 s^2}{6} h'(s) + C \mu \delta^2 s^3 h(s). \] (EC.49)
Combining (EC.47)–(EC.49) with (EC.46) concludes the proof. □

Now we are ready to prove Lemmas EC.9 and EC.10.

Proof of Lemma EC.9 Suppose $k \geq 1$. We prove the lemma by showing that

$$
\mathbb{E}[1(W \leq -z)W^k] \leq C(k) (z \vee 1) e^{-\frac{z^2}{2} - \frac{\delta z^3}{6}},
$$

and

$$
\frac{1}{z} e^{-\frac{z^2}{2} - \frac{\delta z^3}{6}} \leq \mathbb{P}(Y_1 \leq -z).
$$

Let us start with the first inequality. For $0 < z < 1$, the first inequality holds because of Lemma EC.11. Therefore, we only need to consider the case $z \geq 1$. Integration by parts yields

$$
\mathbb{E}[1(W \leq -z)W^k] = z^k \mathbb{P}(W \leq -z) + \int_{-\infty}^{-z} k(-y)^{k-1} \mathbb{P}(W \leq y) dy.
$$

For the first term, note that

$$
z^k \mathbb{P}(W \leq -z) \leq z^k \mathbb{E}\left(e^{(z-z-W)} 1(W \leq -z)\right) \leq z^k e^{-z^2} \mathbb{E}e^{-zW} \leq C z^k e^{-\frac{z^2}{2} - \frac{\delta z^3}{6}},
$$

where the last inequality is due to Lemma EC.11. For the second term, we have

$$
\int_{-\infty}^{-z} k(-y)^{k-1} \mathbb{P}(W \leq y) dy \leq \int_{-\infty}^{-z} k(-y)^{k-1} \mathbb{E}\left(e^{z(y-W)} 1(W \leq y)\right) dy
$$

$$
\leq \int_{-\infty}^{-z} k(-y)^{k-1} e^{zy} \mathbb{E}e^{-zW} dy
$$

$$
\leq Ce^{\frac{z^2}{2} - \frac{\delta z^3}{6}} \int_{-\infty}^{-z} k(-y)^{k-1} e^{zy} dy
$$

$$
\leq C(k) z^{k-2} e^{-\frac{z^2}{2} - \frac{\delta z^3}{6}}.
$$

The second-last inequality is due to Lemma EC.11 and in the last inequality we used

$$
\int_{-\infty}^{-z} (-y)^{k-1} e^{zy} dy \leq C(k) z^{k-2} e^{-z^2}
$$

for $k \geq 1$ and $z \geq 1$. This proves the first inequality in (EC.50), and we now argue the second one. Recall that $r(x) = 2b(x)/a(x)$ and that the density of $Y_1$ in (EC.5) implies

$$
\mathbb{P}(Y_1 \leq -z) = \int_{-\infty}^{-z} \frac{2\kappa}{a(y)} e^{\int_{0}^{y} r(u) du} dy \geq \int_{-\infty}^{-z} \frac{2\kappa}{a(y)} e^{\int_{0}^{y} r(u) du} dy = \int_{-\infty}^{-z} \frac{2\kappa}{a(y)} e^{\int_{0}^{y} \frac{2u}{\pi + a(u)} du} dy.
$$
The last equality follows from the assumption $z \leq \sqrt{R} - 2$ in (EC.13) so that $-z - 1 \geq -\sqrt{R}$, meaning $a(x) = 2\mu - \delta b(x)$ for $x \in [-z - 1, 0]$. We now argue that $2\kappa/a(x) \geq C$ for some constant $C > 0$ that depends only on $\beta$. The density of $Y_1$ is given by (EC.5), so we know that the normalizing constant $\kappa$ satisfies

$$
\frac{1}{\kappa} = \int_{-\infty}^{\infty} \frac{2}{a(x)} e^{\int_{-\infty}^{x} 2b(y)/a(y)dy} dx
= \int_{-\infty}^{0} \frac{2}{a(x)} e^{\int_{0}^{x} 2|b(y)|/a(y)dy} dx + \int_{0}^{\infty} \frac{2}{a(x)} e^{\int_{0}^{x} -2|b(y)|/a(y)dy} dx
\leq \frac{C}{\mu} \int_{-\infty}^{0} e^{\int_{0}^{x} -2|b(y)|/\mu dy} dx + \frac{C}{\mu} \int_{0}^{\infty} e^{\int_{0}^{x} -2|b(y)|/\mu dy} dx.
$$

The second equality is true because $b(x)$ in (EC.4) is nonincreasing and $b(0) = 0$, meaning $b(x) = -|b(x)|1(x \geq 0) + |b(x)|1(x < 0)$. The inequality is true because $a(x) \geq \mu$ and is nondecreasing with $a(x) \leq a(\beta) = \mu(2 + \delta \beta)$. Since $b(x)/\mu$ is a function that depends only on $\beta$, the right-hand side is a quantity that increases in $\delta$. In (EC.13) we assumed $\delta < 1/2$, so $1/\kappa \leq \sup_{\delta \in (0, 1/2)} 1/\kappa \leq C/\mu$. Combining this with the fact that $a(x) \leq C\mu$ from (EC.19) we get $2\kappa/a(x) \geq C$. Therefore,

$$
P(Y_1 \leq -z) \geq C \int_{-z-1}^{-z} e^{\int_{0}^{y} -2u/\mu dy} dy = C \int_{-z-1}^{-z} e^{-y^2/2 - 2y} - 4\log(2)/\delta^2 dy.
$$

Taylor expansion tells us that

$$
\frac{4\log(\delta y + 2) - 2\delta y}{\delta^2} - \frac{4\log(2)}{\delta^2} = \frac{-y^2}{2} + \frac{\delta y^3}{6} - \frac{\delta^2 \xi^4}{16},
$$

for some $\xi$ between 0 and $y$. Recall that $\delta = 1/\sqrt{R}$ and $z \leq c_1 R^{1/4}$, meaning $\delta^2 |\xi|^4 \leq C$ when $y \in [-z - 1, -z]$, so

$$
P(Y_1 \leq -z) \geq C \int_{-z-1}^{-z} e^{-y^2/2 + \frac{\delta y^3}{6}} dy = C \int_{-1}^{0} e^{-\frac{(-z+y)^2}{2}} - \frac{\delta(-z+y)^3}{6} dy.
$$
Now for \( y \in [-1, 0] \), and \( z \leq c_1 R^{1/4} \) we use the fact that \( \delta z^2 \leq C \) to get

\[
-\frac{(z+y)^2}{2} + \frac{\delta (z+y)^3}{6} = -\frac{z^2}{2} + zy - \frac{y^2}{2} + \frac{\delta z^2 y}{2} - \frac{\delta y^3}{6} - \frac{\delta z^3}{6}
\]

\[
\geq -\frac{z^2}{2} + zy - \frac{y^2}{2} (y - \delta z^2) - \frac{\delta z^3}{6} - C
\]

\[
\geq -\frac{z^2}{2} + zy - \frac{\delta z^3}{6} - C,
\]

so

\[
P(Y_1 \leq -z) \geq C \int_{-1}^{0} e^{-\frac{z^2}{2} + zy - \frac{\delta z^3}{6}} dy \geq C e^{-\frac{z^2}{2} - \frac{\delta z^3}{6}}. \tag{EC.53}
\]

This proves (EC.50) for \( k \geq 1 \). For \( k = 0 \), the lemma follows from (EC.51) and (EC.53). □

**Proof of Lemma EC.10** The bound (EC.41) trivially holds if \( 0 < z < 1 \). Therefore, we assume \( z \geq 1 \) in the following. Using integration by parts, we have for \( k \geq 1 \),

\[
E[1(-z \leq W \leq 0)W^ke^{-\int_{0}^{W} r(u)du}] = -z^k e^{-\int_{-z}^{0} r(u)du}P(W \leq -z) + \int_{-z}^{0} (k(-y)^{k-1} + (-y)^k r(y)) e^{-\int_{0}^{y} r(u)du} P(W \leq y) dy
\]

\[
\leq C(k) \int_{-z}^{0} ((-y)^{k-1} + (-y)^{k+1}) e^{-\int_{0}^{y} r(u)du} P(W \leq y) dy. \tag{EC.54}
\]

The last inequality is due to \( r(y) = 2b(y)/a(y) = 2\mu(-y)/a(y) \leq C(-y) \) when \( y \leq 0 \), because \( a(y) \geq \mu \). The same argument tells us that for \( k = 0 \),

\[
E[1(-z \leq W \leq 0)e^{-\int_{0}^{W} r(u)du}] \leq 1 + C \int_{-z}^{0} (-y) e^{-\int_{0}^{y} r(u)du} P(W \leq y) dy. \tag{EC.55}
\]

In what follows we prove that for \( k \geq 0 \),

\[
\int_{-z}^{0} (-y)^k e^{-\int_{0}^{y} r(u)du} P(W \leq y) dy \leq C z^{k+1}. \tag{EC.56}
\]
Lemma EC.10 follows from (EC.54), (EC.55) and (EC.56). Without loss of generality assume that \( z \) is an integer. If not, increase it to the nearest integer. The taylor expansion in (EC.52) implies that for \( 0 \leq -y \leq z \leq c_1 R^{1/4} \),

\[
- \int_0^y r(u) du = - \int_0^y \frac{2b(u)}{a(u)} du = - \frac{4 \log(\delta y + 2) - 2 \delta y}{\delta^2} + \frac{4 \log(2)}{\delta^2} \leq \frac{y^2}{2} - \frac{\delta y^3}{6} + C.
\]

Thus,

\[
\begin{align*}
\int_{-z}^0 (-y)^k e^{-\int_0^y r(u) du} P(W \leq y) dy & \leq C \sum_{j=-z}^{-1} |j|^k \int_j^{j+1} e^{\frac{y^2}{2} - \frac{\delta y^3}{6} e^{-|j|y}} e^{-|j|y} P(W \leq y) dy \\
& \leq C \sum_{j=-z}^{-1} |j|^k e^{-\frac{y^2}{2}} \sup_{j \leq y \leq j+1} \left[ e^{\frac{\delta y^3}{6}} \sup_{j \leq y \leq j+1} e^{-|j|y} \right] \int_j^{j+1} e^{-|j|y} P(W \leq y) dy \\
& \leq C \sum_{j=-z}^{-1} |j|^k e^{-\frac{y^2}{2}} e^{-\frac{\delta y^3}{6}} e^{-|j|y} P(W \leq y) dy \\
& = C \sum_{j=-z}^{-1} |j|^k e^{-\frac{y^2}{2}} e^{-\frac{\delta y^3}{6}} e^{-\frac{|j|^3}{6}} e^{-|j|y} P(W \leq y).
\end{align*}
\]

We used \( \sup_{j \leq y \leq j+1} \left[ e^{\frac{\delta y^3}{6}} \right] \) in the last inequality and integration by parts in the last equality. Invoking Lemma EC.11, we have

\[
\int_{-z}^0 (-y)^k e^{-\int_0^y r(u) du} P(W \leq y) dy \leq C \sum_{j=-z}^{-1} |j|^{k-1} e^{-\frac{y^2}{2}} e^{-\frac{\delta y^3}{6}} e^{-|j|^3} \leq C \sum_{j=-z}^{-1} |j|^{k-1} \leq C z^{k/1},
\]

where we used

\[
\sup_{j \leq y \leq j+1} \left[ -\frac{\delta y^3}{6} - \frac{\delta |j|^3}{6} \right] \leq C \delta j^2 \leq C, \text{ for } 1 \leq -j \leq z \leq c_1 R^{1/4}.
\]

This proves (EC.56). \( \square \)
EC.1.2.5. Proof of (EC.11).

Proof of (EC.11). Inequality (EC.11) contains an upper bound on \( \left| \frac{P(Y_1 \geq z)}{P(W \geq z)} - 1 \right| \). A similar upper bound on \( \left| \frac{P(W \geq z)}{P(Y_1 \geq z)} - 1 \right| \) is a consequence of Theorem 4.1 of Braverman (2017). The following simple modification of the argument in Braverman (2017) implies (EC.11).

It follows from (4.8), (4.9), (4.11), and (4.12) of Braverman (2017) that there exist some \( c_1, C_1 > 0 \) such that for \( R \geq C_1 \) and \( 0 < z \leq c_1 R \),

\[
|P(Y_1 \geq z) - P(W \geq z)| \\
\leq C\delta^2 P(Y_1 \geq z) + C\delta^2 P(W \geq z) + C\delta^2 \min\{(z \vee 1), \frac{1}{\delta^2}\} P(Y_1 \geq z) + C\delta P(W \geq z).
\]

Dividing both sides by \( P(W \geq z) \) we have

\[
\left| \frac{P(Y_1 \geq z)}{P(W \geq z)} - 1 \right| \leq C\delta^2 (z \vee 1) \frac{P(Y_1 \geq z)}{P(W \geq z)} + C\delta.
\]

Note that \( C\delta^2 (z \vee 1) \leq C(\delta^2 z + \delta^2) = C(z/R + 1/R) \leq C(c_1 + 1/C_1) \), and choose \( c_1 \) small enough and \( C_1 \) large enough so that \( C(c_1 + 1/C_1) < 1/2 \). Then, repeating the argument used below (EC.25) implies (EC.11) for \( R \geq C_1 \), and the argument above (EC.13) implies (EC.11) for \( R < C_1 \). □

EC.1.3. Proof of Theorem 2

We first recall Theorem 2.

Theorem EC.2. Assume \( n = R + \beta \sqrt{R} \) for some fixed \( \beta > 0 \). There exist positive constants \( c_0 \) and \( C \) depending only on \( \beta \) such that

\[
\left| \frac{P(Y_0 \geq z)}{P(W \geq z)} - 1 \right| \leq \frac{C}{\sqrt{R}} (1 + z) \quad \text{for} \quad 0 < z \leq c_0 R^{1/2} \quad \text{and} \quad (EC.57)
\]

\[
\left| \frac{P(Y_0 \leq -z)}{P(W \leq -z)} - 1 \right| \leq \frac{C}{\sqrt{R}} (1 + z^3), \quad \text{for} \quad 0 < z \leq \min\{c_0 R^{1/6}, R^{1/2}\}. \quad (EC.58)
\]
Theorem EC.2 follows from a similar and simpler proof than that of Theorem EC.1. In particular, the bound (EC.58) can be proved by a simple adaption of the arguments in Chen et al. (2013b) and therefore its proof is omitted. The proof of (EC.57) below may be useful for other exponential approximation problems. We will use \( c_0, C, C_0 \) to denote positive constants that may differ from line to line, but will only depend on \( \beta \). We require the following two lemmas. The first one is proved in Section EC.1.3.1, and the second in Section EC.1.3.2.

**Lemma EC.12.** There exist \( C, C_0 > 0 \) that depend only on \( \beta \), such that for any \( R \geq C_0 \) and any \( z > \beta \),

\[
P(W \geq z + \delta) \leq Ce^{-(\beta - 3\beta \delta)z}.
\] (EC.59)

To state the second lemma we let \( f_\delta(x) \) solve the Poisson equation

\[
b(x)f'_\delta(x) + \mu f''_\delta(x) = 1(x \geq z) - \mathbb{P}(Y_0 \geq z).
\] (EC.60)

**Lemma EC.13.** There exist \( C_0, C > 0 \) depending on \( \beta \) such that for all \( R \geq C_0 \) and any \( z > \beta \),

\[
\mathbb{E} \left( \int_{-\delta}^{\delta} (b(W + y)f'_\delta(W + y) - b(W)f'_\delta(W)) \frac{K_W(y)}{\mu} dy \right) \\
\leq C\delta \mathbb{P}(Y_0 \geq z) \mathbb{E} \left( \sup_{|s| \leq \delta} \left( 1 + e^{\beta(W + s)}1(\beta \leq W + s \leq z) \right) \right),
\] (EC.61)

where \( K_W(y) \) is defined in (EC.15). Furthermore,

\[
-\frac{\delta}{2\mu} \mathbb{E}(b^2(W)f'_\delta(W)) \leq C\delta \left( \mathbb{P}(W \geq z) + \mathbb{P}(Y_0 \geq z) \left( 1 + \mathbb{E}e^{\beta W}1(\beta \leq W \leq z) + |W| \right) \right),
\] (EC.62)

\[
-\frac{\delta}{2\mu} \mathbb{E}(b(W)(\mathbb{P}(Y_0 \geq z) - 1(W \geq \delta + z))) \leq C\delta \mathbb{P}(W \geq z + \delta).
\] (EC.63)
Proof of (EC.57)  Note that if $R$ is bounded, then, for fixed $\beta$, $n$ is also bounded and $P(W \geq z)$ is bounded away from 0 for $z$ in the bounded range $0 < z \leq cR^{1/2}$. By choosing a sufficiently large $C$, (EC.57) trivially holds. Therefore, in the following, we assume $R \geq C_0$ for a sufficiently large $C_0$. Additionally, the result for finite range $z \in (0, \beta + \delta]$ follows from the Berry-Esseen bound in Theorem 3 of Braverman et al. (2016), so we assume $z > \beta + \delta$.

Recall that the density of $Y_0$ is given in (EC.5), and that $b(x) = - (\mu x \wedge \mu \beta)$. Just like in (EC.34), one may verify that the solution to the Poisson equation (EC.60) satisfies

$$f_z'(x) = \left\{ \begin{array}{ll} -\mathbb{P}(Y_0 \geq z) e^{-\int_0^z \frac{b(u)}{\mu} du} \int_{-\infty}^{x} \frac{1}{\mu} e^{\int_{y}^{\infty} \frac{b(u)}{\mu} du} dy, & x < z, \\ -\mathbb{P}(Y_0 \leq z) e^{-\int_0^z \frac{b(u)}{\mu} du} \int_{x}^{\infty} \frac{1}{\mu} e^{\int_{y}^{\infty} \frac{b(u)}{\mu} du} dy, & x \geq z, \end{array} \right. \tag{EC.64}$$

so (EC.1) implies $\mathbb{E}G_\delta f_z(W) = 0$. Taking expected values in (EC.60) then gives us

$$\mathbb{P}(W \geq z) - \mathbb{P}(Y_0 \geq z)$$

$$= \mathbb{E}
\left( b(W) f_z'(W) + \mu f_z''(W) \right) - \mathbb{E}G_\delta f_z(W)$$

$$= \mathbb{E}
\left( \mu f_z''(W) - \int_{-\delta}^{\delta} f_z''(W + y) K_W(y) dy \right)$$

$$= \mathbb{E}(1(W \geq z) - \mathbb{P}(Y_0 \geq z) - b(W) f_z'(W))$$

$$- \mathbb{E}
\left( \int_{-\delta}^{\delta} (1(W + y \geq z) - \mathbb{P}(Y_0 \geq z) - b(W + y) f_z'(W + y)) \frac{1}{\mu} K_W(y) dy \right),$$

where $K_W(y)$ is defined in (EC.15), the second equality is due to (EC.28), and the final equality follows from $\mu f_z''(x) = -b(x) f_z'(x) + 1(x \geq z) - \mathbb{P}(Y_0 \geq z)$. Using $K_W(y) \geq 0$ and recalling from (EC.16) that $\int_{-\delta}^{\delta} K_W(y) dy = \mu - \frac{\delta}{2} b(W)$, we have

$$b(W) f_z'(W) = \int_{-\delta}^{\delta} b(W) f_z'(W) \frac{K_W(y)}{\mu} dy + \frac{\delta}{2\mu} b^2(W) f_z'(W),$$

$$- \mathbb{E}
\left( \int_{-\delta}^{\delta} 1(W + y \geq z) \frac{1}{\mu} K_W(y) dy \right) \leq - \mathbb{E}
\left( 1(W \geq z + \delta) (1 - \frac{\delta}{2\mu} b(W)) \right),$$

$$\mathbb{E}
\left( \int_{-\delta}^{\delta} \mathbb{P}(Y_0 \geq z) \frac{1}{\mu} K_W(y) dy \right) = \mathbb{P}(Y_0 \geq z) \mathbb{E}
\left( 1 - \frac{\delta}{2\mu} b(W) \right),$$
and therefore,
\[ P(W \geq z) - P(Y_0 \geq z) \]
\[ \leq P(z \leq W < z + \delta) + \mathbb{E}\left( \int_{-\delta}^{\delta} (b(W + y)f_z'(W + y) - b(W)f_z'(W)) \frac{K_W(y)}{\mu} dy \right) \]
\[ - \frac{\delta}{2\mu} \mathbb{E}\left( b^2(W)f_z'(W) \right) - \frac{\delta}{2\mu} \mathbb{E}\left( (P(Y_0 \geq z) - 1(W \geq z + \delta)) \right). \] (EC.65)

Subtracting \( P(z \leq W < z + \delta) \) from both sides and using Lemma EC.13 to bound the remaining three terms, we get
\[ P(W \geq z + \delta) - P(Y_0 \geq z) \]
\[ \leq C\delta \sup_{|s| \leq \delta} \left( P(W \geq z) + P(Y_0 \geq z)(1 + \mathbb{E}|W| + \mathbb{E}e^{\beta(W+s)}1(\beta \leq W + s \leq z)) \right). \]

We now argue that the right-hand side above can be bounded by \( C\delta P(Y_0 \geq z)(z \vee 1) \). Note that \( \mathbb{E}|W| \leq C \) due to Lemma EC.3. Next, since \( b(x)/\mu = -(x \wedge \beta) \) only depends on \( \beta \) and \( z \geq \beta \), then \( P(Y_0 \geq z) = \frac{\int_{\infty}^{z} e^{-\beta y} dy}{\int_{\infty}^{\infty} e^{b_0(u)} m(u)/\mu du} = Ce^{-\beta z} \) for some \( C > 0 \) depending only on \( \beta \). Thus, Lemma EC.12 tells us that for \( R \geq C_0 \) and \( \beta + \delta < z \leq c_0 R^{1/2} \),
\[ P(W \geq z) = \frac{P(W \geq z)}{P(Y_0 \geq z)} P(Y_0 \geq z) \leq CP(Y_0 \geq z). \]

Moreover, for \( |s| \leq \delta \),
\[ \mathbb{E}(e^{\beta(W+s)}1(\beta \leq W + s \leq z)) \]
\[ = e^{\beta^2} P(\beta \leq W + s \leq z) + \int_{\beta}^{\infty} e^{\beta y} P(y < W + s \leq z) dy \]
\[ \leq C + \beta \int_{\beta}^{\infty} e^{\beta y} P(W + s \geq u) du \]
\[ \leq C + \int_{\beta}^{\infty} C du \]
\[ \leq C(z \vee 1). \]
The first equality follows from integration by parts, and the second-last inequality is due to Lemma EC.12. Thus we have shown that

$$\mathbb{P}(W \geq z + \delta) - \mathbb{P}(Y_0 \geq z) \leq C \delta \mathbb{P}(Y_0 \geq z)(z \lor 1).$$

Note that $|\mathbb{P}(Y_0 \geq z)/\mathbb{P}(Y_0 \geq z + \delta) - 1| = |e^{\beta \delta} - 1| \leq \delta C$. Therefore,

$$\mathbb{P}(W \geq z + \delta) - \mathbb{P}(Y_0 \geq z + \delta)(1 + \delta C) \leq C \delta \mathbb{P}(Y_0 \geq z + \delta)(1 + \delta C)(z \lor 1).$$

Dividing both sides of the above inequality by $\mathbb{P}(W \geq z + \delta)$, we obtain

$$1 - \frac{\mathbb{P}(Y_0 \geq z + \delta)}{\mathbb{P}(W \geq z + \delta)}(1 + \delta C) \leq C \delta \frac{\mathbb{P}(Y_0 \geq z + \delta)}{\mathbb{P}(W \geq z + \delta)}(1 + \delta C)(z \lor 1).$$

Choose $C_0$ large enough and $c_0$ small enough and since $R \geq C_0$ and $0 < z \leq c_0 R^{1/2}$, the constant in front of $\mathbb{P}(Y_0 \geq z + \delta)/\mathbb{P}(W \geq z + \delta)$ on the right-hand side can be made less than 1/2, implying

$$1 - \frac{\mathbb{P}(Y_0 \geq z + \delta)}{\mathbb{P}(W \geq z + \delta)} \leq C \delta (z \lor 1).$$

A similar argument can be repeated to show $\frac{\mathbb{P}(Y_0 \geq z + \delta)}{\mathbb{P}(W \geq z + \delta)} - 1 \leq C \delta (z \lor 1)$, implying (EC.57).

\[\square\]

**EC.1.3.1. Proof of Lemma EC.12.** We require the following auxiliary result, whose proof is provided after the proof of Lemma EC.12.

**Lemma EC.14.** There exist constants $C, C_0 > 0$ depending only on $\beta$ such that for $R \geq C_0$,

$$\mathbb{E} e^{(\beta - 3\beta^2 \delta)W} \leq C/\delta.$$  \hfill (EC.66)
Proof of Lemma EC.12 Since \( \delta = 1/\sqrt{R} \), we can choose \( C_0 \) large enough so that \((\beta - 3\beta^2\delta) > \beta/2\). For notational convenience, we set \( \nu = 3\beta^2\delta \). Define

\[
f''(x) = \begin{cases} 
(\beta - \nu)e^{(\beta - \nu)x}, & x \leq z \\
\text{linear interpolation}, & z < x \leq z + \Delta \\
0, & x > z + \Delta,
\end{cases}
\]

with \( 0 < \Delta \leq \delta \) to be chosen, \( f'(x) = 1 + \int_0^x f''(y)dy \) and \( f(x) = \int_0^x f'(y)dy \). Note that \( f(x) \)
grows linearly in \( x \) when \( x \geq z + \Delta \) because \( f'(x) = f'(z + \Delta) \) for \( x \geq z + \Delta \), so \( \mathbb{E}|f(W)| < \infty \) because \( W \) is bounded from below and \( \mathbb{E}|W| < \infty \). Therefore, \( \mathbb{E}G_{\tilde{X}}f(W) = 0 \) due to (EC.1), implying \( \mathbb{E}(-b(W)f'(W)) = \mathbb{E}\left(\int_{-\delta}^\delta f''(W+y)K_W(y)dy\right) \) if we use the form of \( G_{\tilde{X}}f(x) \) in (EC.28). Note also that \( f'(0) = 1, f'(x) \geq 0, \) and \( f'(x) \geq e^{(\beta - \nu)x} \) for \( x \geq z + \Delta \).

Using \( b(x) = -((\mu x \wedge \mu \beta) \) and the assumption that \( z > \beta \) we therefore have

\[
\mathbb{E}(-b(W)f'(W)) \geq \mathbb{E}(\mu We^{(\beta - \nu)W}1(W < \beta)) \\
+ \mathbb{E}(\mu\beta e^{(\beta - \nu)W}1(\beta \leq W \leq z)) + \mathbb{E}(\mu\beta e^{(\beta - \nu)z}1(W > z + \Delta)) \\
\geq -\mu C + \mathbb{E}(\mu\beta e^{(\beta - \nu)W}1(\beta \leq W \leq z)) + \mathbb{E}(\mu\beta e^{(\beta - \nu)z}1(W > z + \Delta)),
\]
where the second inequality is due to $-|x|e^{-(\beta - \nu)|x|}1(x < \beta) \geq -C$. Recalling from (EC.16) that $\int_{-\delta}^{\delta} K_W(y)dy = \mu - \delta b(W)/2$, we have

$$\mathbb{E}\left(\int_{-\delta}^{\delta} f''(W + y)K_W(y)dy\right) \leq \mathbb{E}\left(\sup_{|s| \leq \delta} f''(W + s) \int_{-\delta}^{\delta} K_W(y)dy\right)$$

$$= \mathbb{E}\left(\mu \sup_{|s| \leq \delta} f''(W + s) \left(1 - \frac{\delta}{2\mu} b(W)\right) 1(W \leq \Delta + \delta)\right)$$

$$\leq \mathbb{E}\left(\mu(\beta - \nu)e^{(\beta - \nu)(W + \delta)} \left(1 - \frac{\delta}{2\mu} b(W)\right) 1(W \leq \Delta + \delta)\right)$$

$$= \mathbb{E}\left(\mu(\beta - \nu)e^{(\beta - \nu)(W + \delta)} \left(1 + \frac{\delta}{2} \beta\right) 1(\beta \leq W \leq \Delta + \delta)\right)$$

$$+ \mathbb{E}\left(\mu(\beta + C\delta)e^{(\beta - \nu)W} 1(\beta \leq W \leq z)\right)$$

Combining the inequalities above, we have

$$\mathbb{E}\left(\beta e^{(\beta - \nu)z}1(W > z + \Delta + \delta)\right)$$

$$\leq C + C\delta \mathbb{E}\left(e^{(\beta - \nu)W} 1(\beta \leq W \leq z + \Delta + \delta)\right) + \beta e^{(\beta - \nu)z} \mathbb{P}(z \leq W \leq z + \Delta).$$

Without loss of generality we assume $z$ does not belong to the support of $W$ and let $\Delta \to 0$, and observe that $\mathbb{P}(z \leq W \leq z + \Delta) \to 0$. Therefore, we have

$$\beta e^{(\beta - \nu)z} \mathbb{P}(W > z + \delta) \leq C + C\delta \mathbb{E}\left(e^{(\beta - \nu)W}\right) \leq C$$

where we have used Lemma EC.14. □

**Proof of Lemma EC.14**  Since $\delta = 1/\sqrt{R}$, we can choose $C_0$ large enough so that $(\beta - 3\beta^2 \delta) > \beta/2$. For notational convenience, we set $\nu = 3\beta^2 \delta$. Fix $M > \beta$ and let $f(x) = \int_{0}^{x} e^{(\beta - \nu)(u^{\wedge}M)}dy$. We recall from (EC.28) that $G_x f(x) = b(W)f'(W) + \int_{-\delta}^{\delta} f''(W + y)K_W(y)dy$ where $K_W(y)$ is defined in (EC.15). Since $\beta - \nu > \beta/2$ by assumption, the
function \( f(x) \) grows linearly for \( x \geq M \), so \( \mathbb{E}|f(W)| < \infty \) because \( \mathbb{E}|W| < \infty \). Therefore, \( \mathbb{E}G_x f(W) = 0 \), or \( \mathbb{E}(-b(W)f'(W)) = \mathbb{E}\left(\int_{-\delta}^{\delta} f''(W+y)K_W(y)\,dy\right) \), due to (EC.1). Now
\[
\mathbb{E}\left(-b(W)f'(W)\right) = \mathbb{E}\left(\mu \beta e^{(\beta-\nu)\left(W^\wedge M\right)}1(W \geq \beta)\right) + \mathbb{E}\left(\mu We^{(\beta-\nu)W}1(W < \beta)\right)
\]
\[
\geq \mathbb{E}\left(\mu \beta e^{(\beta-\nu)\left(W^\wedge M\right)}\right) - \mu C
\]
where in the last inequality we used the fact that \(|(x-\beta)e^{(\beta-\nu)x}|1(x < \beta) \leq C\) if \((\beta - \nu) > \beta/2\). Furthermore, since \( f''(x) = (\beta - \nu)e^{(\beta-\nu)(x\wedge M)}1(x < M) \) and \( \int_{-\delta}^{\delta} K_W(y)\,dy = \mu - \delta b(W)/2 \), we have
\[
\mathbb{E}\left(\int_{-\delta}^{\delta} f''(W+y)K_W(y)\,dy\right) \leq \mathbb{E}\left(\int_{-\delta}^{\delta} (\beta - \nu)e^{(\beta-\nu)\left((W+y)^\wedge M\right)}K_W(y)\,dy\right)
\]
\[
\leq (\beta - \nu)e^{(\beta-\nu)\delta}\mathbb{E}\left(e^{(\beta-\nu)\left(W^\wedge M\right)}(\mu - \frac{\delta}{2} b(W))\right)
\]
\[
= \mu(\beta - \nu)e^{(\beta-\nu)\delta}\mathbb{E}\left(e^{(\beta-\nu)\left(W^\wedge M\right)}(1 + \frac{\delta}{2} W)1(W \leq \beta)\right)
\]
\[
+ \mu(\beta - \nu)e^{(\beta-\nu)\delta}\mathbb{E}\left(e^{(\beta-\nu)\left(W^\wedge M\right)}(1 + \frac{\delta}{2} \beta)1(W \geq \beta)\right)
\]
\[
\leq \mu C + \mu(\beta - \nu)e^{(\beta-\nu)\delta}(1 + \frac{\delta}{2} \beta)\mathbb{E}(e^{(\beta-\nu)\left(W^\wedge M\right)}).
\]

(\text{EC.68})

Divide both sides of (\text{EC.67}) and (\text{EC.68}) by \( \mu \delta \) and combine these two inequalities, and also substitute \( 3\beta^2 \delta \) for \( \nu \), to get
\[
\frac{\left(\beta - (\beta - 3\beta^2 \delta)e^{(\beta-3\beta^2 \delta)\delta}(1 + \frac{\delta}{2} \beta)\right)}{\delta} \mathbb{E}(e^{(\beta-3\beta^2 \delta)\left(W^\wedge M\right)}) \leq C/\delta.
\]

Since the coefficient in front of the expected value on the left-hand side converges to a positive constant as \( \delta \to 0 \), for sufficiently small \( \delta \) (or sufficiently large \( C_0 \)), we have
\[
\mathbb{E}(e^{(\beta-3\beta^2 \delta)\left(W^\wedge M\right)}) \leq C/\delta.
\]

We conclude by letting \( M \to \infty \). \( \square \)
EC.1.3.2. Proof of Lemma EC.13. We begin by proving (EC.61).

\[
\mathbb{E}\left( \int_{-\delta}^{\delta} \left( b(W + y) f'_z(W + y) - b(W) f'_z(W) \right) \frac{K_W(y)}{\mu} \, dy \right)
= \mathbb{E}\left( \int_{-\delta}^{\delta} \frac{K_W(y)}{\mu} \left( b(x) f'_z(x) \right)'_{|x=W+s} \, ds \, dy \right)
\leq \delta \mathbb{E}\left( \sup_{|s| \leq \delta} \left| \left( b(x) f'_z(x) \right)'_{|x=W+s} \right| (1 - \frac{\delta}{2\mu} b(W)) \right)
\leq C \delta \mathbb{E}\left( \sup_{|s| \leq \delta} \left| \left( b(x) f'_z(x) \right)'_{|x=W+s} \right| \right).
\]

The first inequality is due to $K_W(y) \geq 0$ and $\int_{-\delta}^{\delta} K_W(y)/\mu \, dy = 1 - \delta b(W)/(2\mu)$ from (EC.16), and the last inequality is true because $|\frac{\delta}{2\mu} b(W)| = |\frac{\delta}{2}(W \land \beta)| \leq C$ since $W \geq -1/\delta$. To bound the right-hand side we note that (cf. (EC.64))

\[
-b(x) f'_z(x) = \begin{cases} 
\mathbb{P}(Y_0 \geq z) b(x) e^{-\int_{-\infty}^{x} \frac{b(u)}{\mu} \, du} \int_{-\infty}^{x} \frac{1}{\mu} e^{\int_{0}^{y} \frac{b(u)}{\mu} \, du} \, dy, & x < z, \\
\mathbb{P}(Y_0 \leq z) b(x) e^{-\int_{-\infty}^{x} \frac{b(u)}{\mu} \, du} \int_{x}^{\infty} \frac{1}{\mu} e^{\int_{0}^{y} \frac{b(u)}{\mu} \, du} \, dy, & x \geq z,
\end{cases}
\]

so for $x > z > \beta$,

\[
(-b(x) f'_z(x))' = -\beta \mathbb{P}(Y_0 \leq z) \left( -1 + \beta e^{-\int_{-\infty}^{x} \frac{b(u)}{\mu} \, du} \int_{x}^{\infty} e^{\int_{0}^{y} \frac{b(u)}{\mu} \, du} \, dy \right)
= -\beta \mathbb{P}(Y_0 \leq z) \left( -1 + \beta e^{-\frac{d^2}{2} + \beta x} \int_{x}^{\infty} e^{\frac{d^2}{2} - \beta y} \, dy \right)
= 0.
\]

For $\beta < x < z$,

\[
|(-b(x) f'_z(x))'| = \beta \mathbb{P}(Y_0 \geq z) \left( 1 + \beta e^{-\frac{d^2}{2} + \beta x} \int_{-\infty}^{x} e^{\int_{0}^{y} \frac{b(u)}{\mu} \, du} \, dy \right) \leq \beta \mathbb{P}(Y_0 \geq z)(1 + Ce^{\beta x}).
\]

In the inequality above we used the fact that $\int_{-\infty}^{x} e^{\int_{0}^{y} \frac{b(u)}{\mu} \, du} \, dy \leq C$ because $b(x)/\mu = -(x \land \beta)$ depends only on $\beta$. Lastly, for $x < \beta$,

\[
|(-b(x) f'_z(x))'| = \mathbb{P}(Y_0 \geq z) \left| x + (e^{x^2/2} + x^2 e^{x^2/2}) \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \, dy \right|.
\]
When $-1 \leq x < \beta$, the right-hand side is bounded by $CP(Y_0 \geq z)$ and when $x < -1$, we use the bound $\frac{1}{x-1}e^{-\frac{x^2}{2}} \leq \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \leq \frac{1}{x}e^{-\frac{x^2}{2}}$ to conclude that

$$P(Y_0 \geq z) \left| x + (e^{\frac{x^2}{2}} + x^2 e^{\frac{x^2}{2}}) \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \leq P(Y_0 \geq z) \left| x + C \right| \leq CP(Y_0 \geq z).$$

Combining the three cases yields (EC.61). We now prove the bound on $-\frac{\delta}{2\mu}E(b^2(W)f_z'(W))$ in (EC.62). From the form of $b(x)f_z'(x)$ above, we have for $x > z > \beta$,

$$-\frac{1}{\mu}b^2(x)f_z'(x) = \beta P(Y_0 \leq z),$$

for $\beta \leq x \leq z$,

$$-\frac{1}{\mu}b^2(x)f_z'(x) = P(Y_0 \geq z) \beta e^{-\frac{y^2}{2} + \beta x} \int_{-\infty}^{x} e^{\int_{y}^{\mu} b(u) du} dy \leq Ce^\beta P(Y_0 \geq z),$$

and for $x < \beta$,

$$-\frac{1}{\mu}b^2(x)f(x) = P(Y_0 \geq z) x^2 e^{\frac{x^2}{2}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \leq P(Y_0 \geq z)(1 + |x|).$$

Combining the three cases implies (EC.62). Lastly we prove (EC.63). In the proof of Lemma EC.1 we showed that $Eb(W) = 0$. Furthermore, $-\mu \beta \leq b(x) \leq 0$ for $x \geq 0$, so

$$-\frac{\delta}{2\mu}E(b(W)(P(Y_0 \geq z) - 1(W \geq z + \delta))) = \frac{\delta}{2\mu}E(b(W)1(W \geq z + \delta)) \leq C\delta P(W \geq z + \delta).$$

□

EC.2. Companion for the Hospital Model

In this portion of the electronic companion, we motivate the $v_3$ approximation for the hospital model presented in Section 4.1, where we suggested using

$$v_3(x) = \max \left\{ \delta + \frac{1}{2}(\delta^2 1(x < 0) - \delta^2 (x^- - \beta) - \delta^2 - 2\delta^2 \beta), \delta/2 \right\}. \quad (EC.69)$$
We recall that $\tilde{X} = \{\tilde{X}(n) = \delta(X(n) - N)\}$, where $X(n)$ is the customer count at the end of time unit $n$, that $W$ and $W'$ have the distributions of $\tilde{X}(0)$ and $\tilde{X}(1)$ when $\tilde{X}(0)$ is initialized according to the stationary distribution of $\tilde{X}$, and that $\Delta = W' - W$. We use $\epsilon(x)$ and $\epsilon_i(x)$ to denote generic functions satisfying

$$|\epsilon(x)| \leq C(1 + |x|)^5.$$  \hfill (EC.70)

The following lemma gives us the conditional moments of $\Delta$. It is proved in Section EC.2.1.

**Lemma EC.15.** For the hospital model with $N$ servers, $\Lambda = \sqrt{N} - \beta$ and $\mu = \delta = 1/\sqrt{N}$,

$$b(x) = \mathbb{E}(\Delta|W = x) = \delta(x^- - \beta),$$  \hfill (EC.71)

$$\mathbb{E}(\Delta^2|W = x) = 2\delta + \left(b^2(x) - \delta b(x) - \delta^2 - 2\delta^2 \beta\right) + \delta^3 x^-$$  \hfill (EC.72)

$$\mathbb{E}(\Delta^3|W = x) = 6\delta b(x) + \delta^3 \epsilon(x),$$  \hfill (EC.73)

$$\mathbb{E}(\Delta^4|W = x) = 12\delta^2 + \delta^3 \epsilon(x),$$  \hfill (EC.74)

$$\mathbb{E}(\Delta^5|W = x) = \delta^3 \epsilon(x).$$  \hfill (EC.75)

To derive $v_3(x)$, we begin with the Taylor expansion in (2) with $n = 4$, which says that for sufficiently smooth $f(x)$,

$$-\mathbb{E}\Delta f'(W) = \mathbb{E}\left[\sum_{i=2}^{4} \frac{1}{i!} \mathbb{E}\Delta^i f(i)(W) + \frac{1}{5!} \mathbb{E}\Delta^5 f(5)(\xi_1)\right],$$  \hfill (EC.76)

where $\xi_i$ denote numbers lying between $W$ and $W'$. By combining (EC.76) with Lemma EC.15, we will show that

$$-\mathbb{E}b(W)f'(W) - \frac{1}{2} \mathbb{E}\left(2\delta + \delta^2 1(W < 0) - \delta^2(W^- - \beta) - \delta^2 - 2\delta^2 \beta\right)f''(W)$$

$$= \delta^3 \left(\frac{1}{2} \mathbb{E}\epsilon_0(W)f''(W) + \frac{1}{6} \mathbb{E}\epsilon_3(W)f'''(W) + \frac{1}{24} \mathbb{E}\epsilon_4(W)f^{(4)}(W) + \frac{1}{120} \mathbb{E}\epsilon_5(W)f^{(5)}(\xi_1)\right)$$

$$+ \frac{1}{2} \delta^3 \mathbb{E}\left(\epsilon_1(W)f^{(4)}(W) + \epsilon_2(W)f^{(5)}(\xi)\right)$$

$$+ \frac{1}{2} \delta^3 \mathbb{E}\left(\epsilon_6(W)f''(W) + \epsilon_7(W)f'''(W) + \epsilon_2(W)\left(\frac{d^2}{dx^2}((x^- - \beta)f''(x))\right|_{x=\xi_3}\right).$$  \hfill (EC.77)
Truncating the term in front of $f''(W)$ on the left-hand side from below by $\delta/2$ gives us $v_3(x)$ in (EC.69). The truncation level $\delta/2$ is chosen because the support of $W$ is in $[-\delta N, \infty)$ and the term in front of $f''(W)$ on the left-hand side of (EC.77) equals $\delta(\frac{1}{2} - \frac{1}{2}\delta\beta) \approx \frac{\delta}{2}$ when evaluated at the point $W = -\delta N$. We could have chosen $\delta(\frac{1}{2} - \frac{1}{2}\delta\beta)$ instead (when this quantity is positive), but in practice this does not make a big difference.

Let us now prove (EC.77). Combining (EC.76) with Lemma EC.15 yields

$$-Eb(W)f'(W) = \frac{1}{2}E\left(2\delta + b^2(W) - \delta b(W) - \delta^2 - 2\delta^2\beta + \delta^3W^-ight)f''(W)$$
$$+ \frac{1}{6}E(6\delta b(W) + \delta^3\epsilon_3(W))f'''(W) + \frac{1}{24}E(12\delta^2 + \delta^3\epsilon_4(W))f^{(4)}(W)$$
$$+ \frac{1}{120}\delta^3\epsilon_5(W)f^{(5)}(\xi_1).$$

(EC.78)

Let us write $W^- f''(W)$ as $\epsilon_0(W)f''(W)$ and rearrange the right-hand side above into the more convenient form:

$$-Eb(W)f'(W) - \frac{1}{2}E\left(2\delta + b^2(W) - \delta b(W) - \delta^2 - 2\delta^2\beta\right)f''(W)$$
$$= \frac{1}{2}\delta Eb(W)f'''(W) + \frac{1}{2}\delta(Eb(W)f''(W) + \delta Ef^{(4)}(W))$$
$$+ \delta^3\left(\frac{1}{2}E\epsilon_0(W)f''(W) + \frac{1}{6}E\epsilon_3(W)f'''(W) + \frac{1}{24}E\epsilon_4(W)f^{(4)}(W) + \frac{1}{120}E\epsilon_5(W)f^{(5)}(\xi_1)\right).$$

The last row is considered as error because of the $\delta^3$ there. We wish to transform the first row on the right-hand side into an expression involving $f''(x)$ plus error. To this end we require the following lemma, which is proved at the end of this section.

**Lemma EC.16.** Suppose that $g \in C^3(\mathbb{R})$ is such that $Eg(W') - Eg(W) = 0$. Then

$$Eb(W)g'(W) + \delta Eg''(W) = \delta^2E\left(\epsilon_1(W)g''(W) + \epsilon_2(W)g'''(\xi)\right),$$

where $\epsilon_i(x)$ are generic functions satisfying (EC.70) and $\xi$ lies between $W$ and $W'$. 
We apply Lemma EC.16 with \( g(x) = \frac{1}{2} \delta f''(x) \) to get
\[
\frac{1}{2} \delta \left( \mathbb{E} b(W) f'''(W) + \delta \mathbb{E} f^{(4)}(W) \right) = \frac{1}{2} \delta^3 \mathbb{E} \left( \epsilon_1(W) f^{(4)}(W) + \epsilon_2(W) f^{(5)}(\xi) \right). \tag{EC.79}
\]

The left-hand side above coincides with one of the terms in the second row of (EC.78). Next we choose \( g(x) = \int_0^x b(y) f''(y) \, dy \) and note that \( g''(x) = b'(x) f''(x) + b(x) f'''(x) \). Applying Lemma EC.16 with our new choice of \( g(x) \), we get
\[
\mathbb{E} b^2(W) f''(W) + \delta \mathbb{E} \left( b(W) f''(W) + b(W) f'''(W) \right)
= \delta^2 \mathbb{E} \left( \epsilon_1(W) \left( b'(W) f''(W) + b(W) f'''(W) \right) + \epsilon_2(W) \left( \frac{d^2}{dx^2} (b(x) f''(x)) \right)_{x=\xi_3} \right)
= \delta^3 \mathbb{E} \left( \epsilon_6(W) f''(W) + \epsilon_7(W) f'''(W) + \epsilon_2(W) \left( \frac{d^2}{dx^2} ((x-\beta) f''(x)) \right)_{x=\xi_3} \right). \tag{EC.80}
\]

The last equality follows from the fact that \( b(x) = \delta (x-\beta) \). Multiplying both sides by \( 1/2 \) and rearranging terms, we get
\[
\frac{1}{2} \delta \mathbb{E} b(W) f'''(W)
= -\frac{1}{2} \mathbb{E} \left( b^2(W) + \delta b'(W) \right) f''(W)
+ \frac{1}{2} \delta^3 \mathbb{E} \left( \epsilon_6(W) f''(W) + \epsilon_7(W) f'''(W) + \epsilon_2(W) \left( \frac{d^2}{dx^2} ((x-\beta) f''(x)) \right)_{x=\xi_3} \right). \tag{EC.80}
\]

Plugging (EC.79) and (EC.80) into (EC.78), we conclude that
\[
- \mathbb{E} b(W) f'(W) - \frac{1}{2} \mathbb{E} \left( 2\delta + b^2(W) - \delta b(W) - \delta^2 - 2\delta^2 \beta \right) f''(W)
= -\frac{1}{2} \mathbb{E} \left( b^2(W) + \delta b'(W) \right) f''(W)
+ \frac{\delta^3}{2} \mathbb{E} \left( \frac{1}{2} \epsilon_6(W) f'''(W) + \frac{1}{6} \mathbb{E} \epsilon_3(W) f'''(W) + \frac{1}{24} \mathbb{E} \epsilon_4(W) f^{(4)}(W) + \frac{1}{120} \mathbb{E} \epsilon_5(W) f^{(5)}(\xi_1) \right)
+ \frac{\delta^3}{2} \mathbb{E} \left( \epsilon_1(W) f^{(4)}(W) + \epsilon_2(W) f^{(5)}(\xi) \right)
+ \frac{1}{2} \delta^3 \mathbb{E} \left( \epsilon_6(W) f''(W) + \epsilon_7(W) f'''(W) + \epsilon_2(W) \left( \frac{d^2}{dx^2} ((x-\beta) f''(x)) \right)_{x=\xi_3} \right).
To conclude (EC.77), we move $-\frac{1}{2}\mathbb{E}(b^2(W) + \delta b'(W))f''(W)$ to the left-hand side and note that

$$2\delta + b^2(W) - \delta b(W) - \delta^2 - 2\delta^2\beta - b^2(W) - \delta b'(W)$$

$$= 2\delta + \delta^2(W^- - \beta)^2 - \delta^2(W^- - \beta) - \delta^2 - 2\delta^2\beta - \delta^2(W^- - \beta)^2 + \delta^21(W < 0)$$

$$= 2\delta - \delta^2(W^- - \beta) - \delta^2 - 2\delta^2\beta + \delta^21(W < 0),$$

which coincides with the term in front of $f''(W)$ on the left-hand side of (EC.77).

Proof of Lemma EC.16 Since $\mathbb{E}g'(W') - \mathbb{E}g(W) = 0$, performing a third-order Taylor expansion gives us

$$0 = \mathbb{E}\Delta g'(W) + \frac{1}{2}\mathbb{E}\Delta^2 g''(W) + \frac{1}{6}\mathbb{E}\Delta^3 g'''(\xi)$$

$$= \mathbb{E}b(W)g'(W) + \frac{1}{2}\mathbb{E}\left(2\delta + b^2(W) - \delta b(W) - \delta^2 - 2\delta^2\beta + \delta^3W^-\right)g''(W)$$

$$+ \frac{1}{6}\mathbb{E}(6\delta b(W) + \delta^3\epsilon_0(W))g'''(\xi).$$

The second equality is due to Lemma EC.15. We set

$$\epsilon_1(x) = -\frac{b^2(x) - \delta b(x) - \delta^2 - 2\delta^2\beta + \delta^3x^-}{2\delta^2}$$

and

$$\epsilon_2(x) = -\frac{6\delta b(x) + \delta^3\epsilon_0(x)}{6\delta^2}$$

and note that both $\epsilon_1(x)$ and $\epsilon_2(x)$ satisfy (EC.70) because $b(x) = \delta(x^- - \beta)$. □

EC.2.1. Proof of Lemma EC.15

By the definition of the hospital model the change in customers $\Delta = W' - W$ satisfies

$$\Delta = \delta(A - D),$$

where $A$ is a mean $\Lambda$ Poisson random variable, and conditioned on $W = x = \delta(k - N)$, $D \sim \text{Binomial}(k \wedge N, \mu)$; see Dai and Shi (2017). To prove the lemma, we utilize the following
Stein identities for a mean $\Lambda$ Poisson random variable $X$ and a Binomial$(k, \mu)$ random variable $Y$:

$$
\mathbb{E} X f(X) = \Lambda \mathbb{E} f(X + 1) \quad \text{for each } f : \mathbb{Z}_+ \to \mathbb{R} \text{ with } \mathbb{E}|X f(X)| < \infty, \quad \text{(EC.81)}
$$

$$
\mathbb{E} Y f(Y) = \mu k \mathbb{E} f(Y + 1) - \mu \mathbb{E}[Y (f(Y + 1) - f(Y))]
\quad \text{for each } f : \mathbb{Z}_+ \to \mathbb{R}. \quad \text{(EC.82)}
$$

See, for example, Lectures VII and VIII of Stein (1986).

**Proof of Lemma EC.15** We prove (EC.71)–(EC.75) in sequence. Using the facts that

$$
\Lambda = \sqrt{N} - \beta \quad \text{and} \quad \mu = \delta = 1/\sqrt{N}, \quad \text{for } x = \delta(k - N) \text{ we have}
$$

$$
\mathbb{E}(A - D|W = x) = \Lambda - (k \land N) \mu = \mu(\Lambda/\mu - N - (k - N \land 0)) = -\beta - (x \land 0) = x^- - \beta.
\quad \text{(EC.83)}
$$

For the remainder of the proof we adopt the convention that all expectations are conditioned on $W = x$. We now prove (EC.72):

$$
\mathbb{E}\Delta^2 = \delta^2 \left( \mathbb{E}[A(A - D)] - \mathbb{E}[D(A - D)] \right)
= \delta^2 \left( \Lambda \mathbb{E}(A - D + 1) - \mathbb{E}[D(A - D)] \right)
= \delta^2 \left( \Lambda \mathbb{E}(A - D + 1) - \mu(n \land N)\mathbb{E}(A - D - 1) - \mu\mathbb{E}D \right)
= \delta^2 \left( 2\Lambda + (\Lambda - \mu(n \land N))\mathbb{E}(A - D - 1) + \mu(x^- - \sqrt{N}) \right)
= \delta^2 \left( 2\Lambda + (x^- - \beta)^2 - (x^- - \beta) + \mu(x^- - \sqrt{N}) \right). \quad \text{(EC.84)}
$$

We used (EC.81) in the second equality and (EC.82) in the third equality. The fourth and fifth equalities are due to

$$
\mathbb{E}D = \mathbb{E}(D - A) + \mathbb{E}A = -(x^- - \beta) + \Lambda = -x^- + \sqrt{N}
$$

and $\Lambda - \mu(n \land N) = \mathbb{E}(A - D) = x^- - \beta$, respectively.
Lastly, bringing the $\delta^2$ term in (EC.84) inside the parenthesis and recalling that $\delta^2 \Lambda = \delta - \delta^2 \beta$, $\delta(x^- - \beta) = b(x)$, and $\mu = 1/\sqrt{N}$ yields (EC.72). We now prove (EC.73). Note that $\mathbb{E}\Delta^3 = \delta^3 \mathbb{E}(A - D)^3$ equals

$$
\delta^3 \left( \mathbb{E}[A(A - D)^2] - \mathbb{E}[D(A - D)^2] \right) = \delta^3 \left( 4\Lambda \mathbb{E}(A - D) + (A - (n \land N) \mu) \mathbb{E}(A - D - 1)^2 + \mu \mathbb{E}D \left[ -2(A - D) + 1 \right] \right).
$$

The first equality is due to (EC.81) and (EC.82) and to get the second equality we use $\mathbb{E}(A - D + 1)^2 = \mathbb{E}(A - D - 1 + 2)^2 = \mathbb{E}[(A - D - 1)^2 + 4(A - D)]$. Let us analyze the terms above one by one. First, we have

$$
\delta^3 4 \Lambda \mathbb{E}(A - D) = 4(x^- - \beta) \delta^3 \Lambda = 4(x^- - \beta)(\delta^2 - \delta^3 \beta) = 4 \delta^2 (x^- - \beta) + \delta^3 e(x).
$$

Second, since $\delta(A - D) = \Delta$, we have

$$
\delta^3 (A - (n \land N) \mu) \mathbb{E}(A - D - 1)^2 = \delta(x^- - \beta) \mathbb{E}\left( \Delta^2 - 2 \delta \mathbb{E}\Delta + \delta^2 \right) = \delta(x^- - \beta) \mathbb{E}\Delta^2 + \delta^3 \epsilon(x)
$$

The last equality is due to (EC.72), which says that $\mathbb{E}\Delta^2 = 2 \delta + \delta^2 \epsilon(x)$. Lastly,

$$
\delta^3 \mu \mathbb{E}D \left[ -2(A - D) + 1 \right] = \delta^4 \left( 2 \mathbb{E}(A - D)^2 - 2 \mathbb{E}A(A - D) + \mathbb{E}D \right) = \delta^2 \left( 2 \mathbb{E}\Delta^2 - 2 \delta^2 \Lambda \mathbb{E}(A - D + 1) + \delta^2 (x^- + \sqrt{N}) \right) = \delta^3 \epsilon(x).
$$
The second equality is due to (EC.81) and the last equality follows from \( \mathbb{E} \Delta^2 = \delta \epsilon(x) \)
and \( \delta^2 \Lambda \mathbb{E}(A - D + 1) = \delta(1 - \delta \beta)(x^- - \beta + 1) = \delta \epsilon(x) \). Putting the pieces together yields
\[ \mathbb{E} \Delta^3 = 6 \delta^2 (x^- - \beta) + \delta^3 \epsilon(x), \]
which proves (EC.73). We now prove (EC.74):
\[
\mathbb{E} \Delta^4 = \delta^4 \left( \mathbb{E} \left[ A(A - D)^3 \right] - \mathbb{E} \left[ D(A - D)^3 \right] \right)
= \delta^4 \left( \Lambda \mathbb{E} \left[ (A - D + 1)^3 \right] - \mu(n \wedge N) \mathbb{E} (A - D - 1)^3 \right)
+ \mu \mathbb{E} \left[ D(A - D - 1)^3 - D(A - D)^3 \right]
= \delta^4 \left( \Lambda \mathbb{E} \left[ (A - D + 1)^3 - (A - D - 1)^3 \right] + (A - (n \wedge N) \mu) \mathbb{E} (A - D - 1)^3 \right)
+ \mu \mathbb{E} \left[ (A - D - 1)^3 - (A - D)^3 \right]
= \delta^4 \left( \Lambda \mathbb{E} \left[ 6(A - D)^2 + 2 \right] + (x^- - \beta) \mathbb{E} (A - D - 1)^3 \right)
+ \mu \mathbb{E} \left[ -3(A - D)^2 + 3(A - D) - 1 \right].
\]
Let us analyze the terms above one by one. First,
\[
\delta^4 \Lambda \mathbb{E} \left[ 6(A - D)^2 + 2 \right] = \delta^2 \Lambda \mathbb{E} \left[ 6 \Delta^2 + 2 \delta^2 \right] = 12 \delta^2 + \delta^3 \epsilon(x).
\]
Second,
\[
\delta^4 (x^- - \beta) \mathbb{E} (A - D - 1)^3 = \delta (x^- - \beta) \left( \mathbb{E} \Delta^3 - 3 \delta \mathbb{E} \Delta^2 + 3 \delta^2 \mathbb{E} \Delta - 1 \right) = \delta^3 \epsilon(x),
\]
and third,
\[
\delta^4 \mu \mathbb{E} \left[ -3(A - D)^2 + 3(A - D) - 1 \right]
= \delta^5 \mathbb{E} \left[ 3(A - D)^3 - 3(A - D)^2 + D \right] + \delta^5 \mathbb{E} \left[ -3(A - D)^2 + 3(A - D) \right]
= \delta^2 \mathbb{E} \left[ 3 \Delta^3 - 3 \delta \Delta^2 + \delta^3 D \right] + \delta^5 \Lambda \mathbb{E} \left[ -3(A - D + 1)^2 + 3(A - D + 1) \right]
= \delta^3 \epsilon(x).
\]
Putting the pieces together yields \( \mathbb{E} \Delta^4 = 12 \delta^2 + \delta^3 \epsilon(x) \). The proof of (EC.75) is analogous
to the proof of (EC.74) and is omitted. \( \square \)
EC.3. Companion for the AR(1) Model

In this section we derive the $v_3$ approximation for the AR(1) model and then prove Lemma 1 in Section EC.3.1. We recall that $W = \delta(D_\infty - R), \ W' = e^{-\alpha Z} W + \delta(X + R(e^{-\alpha Z} - 1)), \text{ and } \Delta = W' - W$, where $\delta = \sqrt{\alpha}, \ R = 1/\alpha, \ X$ and $Z$ are independent unit-mean exponentially distributed random variables that are also independent of $W$, and $D_\infty > 0$ has the limiting distribution of the AR(1) model defined by (41). The asymptotic regime we consider is $\alpha \to 0$, so we assume that $\alpha \in (0, 1)$. We recall Lemma 1:

**Lemma EC.17.** Recall that $\delta = \sqrt{\alpha}$. For any $k \geq 1$,

$$
\mathbb{E}(\Delta^k | D_\infty = d) = \delta^k k! \left( 1 + \sum_{i=1}^{k} (-1)^i d^i \prod_{j=1}^{i} \frac{\alpha}{1 + j\alpha} \right), \quad d > 0.
$$

We also recall that for $x \geq -1/\sqrt{\alpha}$,

$$
\mathbb{E}(\Delta^k | W = x) = \mathbb{E}(\Delta^k | D_\infty = x/\delta + R) = \delta^k k! \left( 1 + \sum_{i=1}^{k} (-1)^i \left( x\sqrt{\alpha} + 1 \right)^i \prod_{j=1}^{i} \frac{1}{1 + j\alpha} \right),
$$

and observe that $\mathbb{E}(\Delta^k | W = x) = \delta^k p_k(x)$ for some degree-$k$ polynomial $p_k(x)$. To derive $v_3(x)$, we start with the Taylor expansion in (2) with $n = 4$; i.e., for any function $f : \mathbb{R} \to \mathbb{R}$ satisfying $\mathbb{E} f(W') - \mathbb{E} f(W) = 0$,

$$
\delta \mathbb{E} p_1(W) f'(W) + \frac{1}{2} \delta^2 \mathbb{E} p_2(W) f''(W) + \frac{1}{6} \delta^3 \mathbb{E} p_3(W) f'''(W) + \frac{1}{24} \delta^4 \mathbb{E} p_4(W) f^{(4)}(W) = -\frac{1}{120} \delta^5 \mathbb{E} p_5(W) f^{(5)}(\xi). \quad (EC.85)
$$

Since $\sup_{\alpha \in (0,1)} |p_k(x)| < \infty$ for each $x \in \mathbb{R}$, the right-hand side is of order $\delta^5$. When deriving $v_3$, we want it to account for all terms of order $\delta, \ldots, \delta^4$ and treat terms of order $\delta^5$ as error. The following lemma is the basis for the $v_3$ approximation. It converts the third and fourth derivative terms in (EC.85) into expressions involving $f''(x)$ plus error. Its proof is similar to the $v_2$ derivation in Section 2, so we postpone it until the end of this section.
Lemma EC.18. Define

\[ \tilde{p}_3(x) = \frac{1}{6} \left( p_3(x) - \frac{p_1(x)p_4(x)}{2p_2(x)} - \frac{1}{4} \delta p_2(x) \left( \frac{p_4(x)}{p_2(x)} \right)' \right), \]
\[ p_2(x) = \left( \frac{p_2(x)}{2} - \frac{p_1(x)p_3(x)}{3p_2(x)} - \frac{p_2(x)}{6} \left( \frac{p_3(x)}{p_2(x)} \right)' \right). \]

Let \( W \) and \( W' \) be as in Section 5. If \( f \in C^5(\mathbb{R}) \) is such that \( \mathbb{E} f(W') - \mathbb{E} f(W) = 0 \), then

\[ \delta \mathbb{E} p_1(W) f''(W) + \delta^2 \mathbb{E} \left( \frac{p_2(W)}{2} - \frac{p_1(W)p_3(W)}{p_2(W)} - \delta \frac{p_2(W)}{p_3(W)} \left( \frac{p_3(W)}{p_2(W)} \right)' \right) f''(W) \]
\[ = - \frac{1}{120} \delta^5 \mathbb{E} p_5(W) f^{(5)}(\xi_1) + \frac{1}{72} \delta^5 \mathbb{E} p_3(W) \left( \frac{p_4(x)}{p_2(x)} f'''(x) \right)' \bigg|_{x=\xi_2} \]
\[ + \frac{1}{24} \delta^5 \mathbb{E} p_4(W) \left( \frac{p_3(x)}{p_2(x)} f''(x) \right)''' \bigg|_{x=\xi_3} - \frac{1}{18} \delta^5 \mathbb{E} p_3(W) \left( \frac{p_4(x)}{p_2(x)} \frac{p_3(x)}{p_2(x)} f''(x) \right)'' \bigg|_{x=\xi_4}. \]

(EC.86)

We choose

\[ v_3(x) = \delta^2 \left( \frac{p_2(x)}{2} - \frac{p_1(x)p_3(x)}{p_2(x)} - \delta \frac{p_2(x)}{p_3(x)} \left( \frac{p_3(x)}{p_2(x)} \right)' \right) \]

based on the term in front of \( f''(W) \) on the left-hand side of (EC.86), which is the basis for the \( v_3 \) approximation in Section 5. Our choice is based on the presumption that all the terms on the right-hand side of (EC.86) are of order \( \delta^5 \) when \( \delta \) is close to zero. We do not prove this claim rigorously in this paper. Nevertheless, our \( v_3 \) approximation performs quite well numerically.

Proof of Lemma EC.18 In Section 2, we used \( a(x), \ldots, d(x) \) to represent \( \mathbb{E}(\Delta^k|W = x) \); i.e.,

\[ b(x) = \mathbb{E}(\Delta|W = x), \quad a(x) = \mathbb{E}(\Delta^2|W = x), \quad c(x) = \mathbb{E}(\Delta^3|W = x), \]
\[ d(x) = \mathbb{E}(\Delta^4|W = x), \quad e(x) = \mathbb{E}(\Delta^5|W = x). \]

(EC.87)
Since this proof relies heavily on Section 2, we use this notation and then convert to use \( \delta^k p_k(x) = E(\Delta^k | W = x) \) at the end. Our starting point is equation (EC.85), which we recall for convenience:

\[
\mathbb{E}b(W)f'(W) + \frac{1}{2} \mathbb{E}a(W)f''(W) + \frac{1}{6} \mathbb{E}c(W)f'''(W) + \frac{1}{24} \mathbb{E}d(W)f^{(4)}(W)
\]

\[
= - \frac{1}{120} \mathbb{E}e(W)f^{(5)}(\xi). \tag{EC.88}
\]

Our proof relies on several key equations from Section 2, which we recall as we go. Let \( g_1(x) = \int_0^x \frac{d(y)}{a(y)} f'''(y) dy \) and use (9), or

\[
\mathbb{E}b(W)f'(W) + \frac{1}{2} \mathbb{E}a(W)f''(W) = - \frac{1}{6} \mathbb{E}c(W)f'''(\xi_2), \tag{EC.89}
\]

with \( g_1(x) \) in place of \( f(x) \) there to get

\[
\mathbb{E} \frac{b(W)d(W)}{a(W)} f'''(W) + \mathbb{E} \frac{a(W)}{2} \left( \frac{d(W)}{a(W)} \right)' f'''(W) + \frac{1}{2} \mathbb{E}d(W)f^{(4)}(W) = - \frac{1}{6} \mathbb{E}c(W)g_1'''(\xi_2).
\]

We multiply both sides by 1/12 and subtract the result from (EC.88) to get

\[
\mathbb{E}b(W)f'(W) + \frac{1}{2} \mathbb{E}a(W)f''(W) + \mathbb{E}c(W)f'''(W)
\]

\[
= \frac{1}{72} \mathbb{E}c(W)g_1'''(\xi_2) - \frac{1}{120} \mathbb{E}e(W)f^{(5)}(\xi_1), \tag{EC.90}
\]

where \( \bar{c}(x) = \frac{1}{6} c(x) - \frac{b(x)d(x)}{12a(x)} - \frac{a(x)}{24} \left( \frac{d(x)}{a(x)} \right)' \). Note that \( \bar{c}(x) = \delta^3 p_3(x) \). Next, let \( g_2(x) = \int_0^x \frac{\bar{c}(y)}{\bar{a}(y)} f''(y) dy \), where

\[
\bar{v}_2(x) = \frac{a(x)}{2} - \frac{b(x)c(x)}{3a(x)} - \frac{a(x)}{6} \left( \frac{c(x)}{a(x)} \right)', \quad x \in \mathbb{R}
\]

is identical to \( v_2(x) \) defined in (12), and note that \( v_2(x) = \delta^2 \bar{p}_2(x) \). We use (11), or

\[
\mathbb{E}b(W)f'(W) + \mathbb{E} \left( \frac{a(W)}{2} - \frac{b(W)c(W)}{3a(W)} - \frac{a(W)}{6} \left( \frac{c(W)}{a(W)} \right) \right) f''(W)
\]

\[
= \frac{1}{18} \mathbb{E}c(W)g'''(\xi_2) - \frac{1}{24} \mathbb{E}d(W)f^{(4)}(\xi_1), \tag{EC.91}
\]
with $g_2(x)$ in place of $f(x)$ there to get
\[
\mathbb{E}b(W) \frac{\tilde{c}(W)}{\tilde{v}_2(W)} \frac{f''(W)}{2} + \mathbb{E}v_2(W) \left( \frac{\frac{\tilde{c}(W)}{\tilde{v}_2(W)}}{2} \right) f''(W) + \mathbb{E}v_2(W) \frac{\frac{\tilde{c}(W)}{\tilde{v}_2(W)}}{2} f''(W)
\]
\[
= \frac{1}{18} \mathbb{E}c(W) \left( \frac{c(x)}{a(x)} g_2''(x) \right)'' \bigg|_{x=\xi_4} - \frac{1}{24} \mathbb{E}d(W) g_2^{(4)}(\xi_3).
\]
Subtracting the equation above from (EC.90), we conclude that
\[
\mathbb{E}b(W) f'(W) + \mathbb{E} \left( \frac{a(W)}{2} - b(W) \frac{c(W)}{v_2(W)} - v_2(W) \left( \frac{\frac{c(W)}{v_2(W)}}{W} \right)' \right) f''(W)
\]
\[
= - \frac{1}{18} \mathbb{E}c(W) \left( \frac{c(x)}{a(x)} g_2''(x) \right)'' \bigg|_{x=\xi_4} + \frac{1}{24} \mathbb{E}d(W) g_2^{(4)}(\xi_3)
\]
\[
+ \frac{1}{72} \mathbb{E}c(W) g_1'''(\xi_2) - \frac{1}{120} \mathbb{E}c(W) f^{(5)}(\xi_1).
\]
To conclude, we note that $g_2^{(4)}(\xi_3) = \left( \frac{c(x)}{a(x)} f''(x) \right)''' \bigg|_{x=\xi_3}$ and $g_1'''(\xi_2) = \left( \frac{d(x)}{a(x)} f'''(x) \right)''' \bigg|_{x=\xi_2}$

and then substitute $\delta p_1(x)$ for $b(x)$, $\delta^2 p_2(x)$ for $a(x)$, etc., where they appear above.

**EC.3.1. Proof of Lemma 1**

Recall that $\Delta = W' - W = \delta \left( D_{\infty}(e^{-\alpha Z} - 1) + X \right)$, so
\[
\mathbb{E}(\Delta^k | D_{\infty} = d) = \delta^k \mathbb{E}(d(e^{-\alpha Z} - 1) + X)^k = \delta^k \sum_{i=0}^k \binom{k}{i} \mathbb{E}(X^i(e^{-\alpha Z} - 1)^{k-i}) d^{k-i}.
\]
Since $X$ and $Z$ are independent and exponentially distributed with mean 1, we have
\[
\mathbb{E}X^i = i! \text{ and }
\]
\[
\binom{k}{i} \mathbb{E}(X^i(e^{-\alpha Z} - 1)^{k-i}) = \frac{k!}{(k-i)!} \mathbb{E}(e^{-\alpha Z} - 1)^{k-i} = \frac{k!}{(k-i)!} \int_0^\infty (e^{-\alpha z} - 1)^{k-i} e^{-z} dz.
\]
Using integration by parts,
\[
\int_0^\infty (e^{-\alpha z} - 1)^{k-i} e^{-z} dz
\]
\[ (k-i)(-\alpha) \int_0^\infty (e^{-\alpha z} - 1)^{k-i-1} e^{-(1+\alpha)z} \, dz \]
\[ = (k-i)(k-i-1)(-\alpha)^2 \frac{1}{1+\alpha} \int_0^\infty (e^{-\alpha z} - 1)^{k-i-2} e^{-(1+2\alpha)z} \, dz \]
\[ \ldots \]
\[ = (k-i)!(-\alpha)^{k-i} \frac{1}{1+\alpha} \frac{1}{1+2\alpha} \ldots \frac{1}{1+(k-i)\alpha} \int_0^\infty e^{-(1+(k-i)\alpha)z} \, dz \]
\[ = (k-i)!(-\alpha)^{k-i} \frac{1}{1+\alpha} \frac{1}{1+2\alpha} \ldots \frac{1}{1+(k-i)\alpha}. \]

\[ \square \]

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