Generalized Beta Divergence

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Abstract—This paper generalizes beta divergence beyond its classical form associated with power variance functions of Tweedie models. Generalized form is represented by a compact definite integral as a function of variance function of the exponential dispersion model. This compact integral form simplifies derivations of many properties such as scaling, translation and expectation of the beta divergence. Further, we show that beta divergence and (half of) the statistical deviance are equivalent measures.

Index Terms—Beta divergence, alpha divergence, Tweedie models, dispersion models, variance functions, deviance.

I. INTRODUCTION

Divergences and distributions are deeply related concepts studied extensively in various fields. This paper is another attempt that casts their relations specifically into that of beta divergences and dispersion models and studies accordingly. The main consequence of this study is that beta divergence and (half of) statistical deviance are represented identical equations and they are, therefore, equivalent measures. In this respect, formulation of beta divergence is generalized and thus is extended beyond its Tweedie related classical forms [1], [2], [3], [4] and is aligned with exponential dispersion models. This is achieved by defining beta divergences as a function of so-called variance functions of exponential dispersion models. One immediate consequence is that we can compute beta divergence for non-Tweedie models such as negative binomial or hyperbolic secant distribution. Another consequence is compact integral representation of beta divergence

\[ d_\beta(x, \mu) = \int_{\mu}^{x} \frac{x - t}{v(t)} dt \]

where \( v(t) \) is variance functions of the dispersion models. This form gives a simple and intuitive way for statistical interpretation of beta divergence by decomposing it into the difference of two integrals

\[ d_\beta(x, \mu) = \int_{\mu}^{x} \frac{x - t}{v(t)} dt = \int_{\mu_0}^{\mu} \frac{x - t}{v(t)} dt - \int_{\mu_0}^{\mu} \frac{x - t}{v(t)} dt \]

which is equal to the log-likelihood ratio of the full model to the parametric model

\[ \mathcal{L}_x(x) - \mathcal{L}_x(\mu) = \frac{1}{2} d_v(x, \mu) \]

and that is half of the unit deviance \( d_v \) by definition. This way, beta divergence is linked to half of statistical deviance. Interestingly, quasi-log-likelihood, a deeply related concept to deviance, is defined by an identical integral form by Wedderburn in 1974 [5] as (adapted notation)

\[ d_\nu(x, \mu) = 2d_\beta(x, \mu) = 2 \int_{\mu}^{x} \frac{x - t}{v(t)} dt \]

There is a rich literature on connection of divergences and distributions. In one study Banerjee et al. showed the bijection between regular exponential family distributions and the Bregman divergences [6]. As a special case of this bijection, connection of beta divergences and Tweedie distributions has been briefly remarked by [7] and has been specifically studied in a recent report [8].

The attractive point of studying beta divergence is its generalization capability for learning algorithms. This capability has already been exploited in matrix and tensor factorizations by the researchers such as [9], [3], [10], and very recently [11]. The key point of this generalization is minimization of beta divergence, whose underlying distribution is parametrized by a scalar index, as in the case of Tweedie models. The net effect of our work would be replacing this distribution index by so-called variance function to deal with broad class of distributions.

To sum up, intended purpose of this paper is to facilitate generalized approach for designing learning algorithms that optimize beta divergences. For this, the paper aims to gain some insight into various properties of beta divergence. The main contributions are as follows.

- The beta divergence is extended and linked to exponential dispersion models beyond Tweedie family. As a result, a statement like 'beta divergence for binomial distribution' becomes reasonable and it is equal to

\[ d_\beta(x, \mu) = x \log \frac{x}{\mu} + (1 - x) \log \frac{1 - x}{1 - \mu} \]

- Various functions including beta divergences, alpha divergences, cumulant functions, dual cumulant functions are all expressed in similar definite integral forms.

- Derivations of certain properties of beta divergences are simplified by using their integral representations. For example, connection of beta divergence \( d_\beta(x, \mu) \) and its scaled form \( d_\beta(x/c, \mu/c) \), that is

\[ d_\beta(x, \mu) = \frac{c^2}{f(c)} d_\beta(x/c, \mu/c) \]

can be simply shown by change of variables in the integral.

- The relation of beta divergence and unit deviance has already been studied in the scope of Tweedie models [8]. Here this connection is shown in a broader scope as

\[ d_\nu(x, \mu) = 2d_\beta(x, \mu) = 2 \int_{\mu}^{x} \frac{x - t}{v(t)} dt \]
• Finally, we present many examples that apply the results for the Tweedie models, which most could be considered as corollaries.

This paper is organized as follows. Section 2 introduces the notation very briefly. Section 3 gives some background information about dispersion models, cumulant functions, Legendre duality and Bregman divergences. Section 4 identifies basic elements such as canonical parameter $\theta$ and dual cumulant function in definite integral forms. Section 5 is all about beta divergence as its generic integral forms, along with its properties such as scaling and transformation. Section 6, then, links beta divergence to log-likelihood and statistical deviance. Finally, in Appendix, similar integral forms are given for the alpha divergence.

II. NOTATION

In particular, $d_{\phi}(x,\mu)$ is the Bregman divergence generated by the convex function $\phi(\cdot)$. Likewise, $d_{f}(x,\mu)$ denotes $f$-divergence generated by the convex function $f(\cdot)$. As special cases, $d_{\alpha}(x,\mu)$ and $d_{\beta}(x,\mu)$ denote alpha and beta divergences. Similarly $d_{\phi}(x,\mu)$ denotes the statistical deviance. Related with the deviance, quasi-log-likelihood of the parametric model is denoted by $L_{\mu}(\mu)$ whereas $L_{\mu}(x)$ denotes the quasi-log-likelihood of the 'full' model. Without loss of generality, we consider only scalar valued functions, and consider only univariate variables whereas the work can easily be extended to multivariate case. In other words, we assume $\mu, \theta \in \mathbb{R}$ and correspondingly when referred we consider only ordinary scalar product as $\langle \mu, \theta \rangle$ rather than the inner product $\langle \mu, \theta \rangle$.

III. BACKGROUND

A. Dispersion Models

(Reproductive) exponential dispersion models EDM($\theta, \varphi$) are two-parameter linear exponential family distributions defined as

$$p(x; \theta, \varphi) = h(x, \varphi) \exp \left\{ \varphi^{-1}(\theta x - \psi(\theta)) \right\}$$

(2)

Here $\theta \in \Theta$ is the canonical (natural) parameter with $\Theta$ being the canonical parameter domain, $\varphi$ is the dispersion parameter as $\varphi > 0$ and $\psi$ is the cumulant function or cumulant generator that is inherently related to cumulant generating function. The expectation (mean) parameter is denoted by $\mu \in \Omega$ with $\Omega$ being the mean parameter domain. In this paper we assume mean parameter domain is identical to the convex support of the variable $X$ and thus we write $x, \mu \in \Omega$. The expectation parameter $\mu$ is the first cumulant and is tied to the canonical parameter $\theta$ with the differential equation so-called mean value mapping $[12]$

$$\mu(\theta) = \frac{d\psi(\theta)}{d\theta}$$

(3)

which can be obtained after differentiating both side of the equation wrt $\theta$

$$1 = \int h(x, \varphi) \exp \left\{ \varphi^{-1}(\theta x - \psi(\theta)) \right\} dx$$

(4)

Likewise the inverse function $(\mu(\theta))^{-1}$ is so-called inverse mean value mapping $[12]$.

B. Dual Cumulant Function

Similar to one-to-one correspondence between $\theta$ and $\mu$ that span dual spaces $\Theta$ and $\Omega$, cumulant function $\psi(\cdot)$ has a conjugate dual form denoted by $\phi(\cdot)$. This is the function that casts and specializes Bregman divergence to beta divergence. Similarly, when applied to Csiszar $f$-divergence, it is specialized to Amari alpha divergence $[4]$. One other interesting property of this function is that it is connected to the entropy of the underlying distribution $[13]$, hence it is also called as (negative) entropy function. In this paper we simply refer as dual cumulant function, defined as follows

$$\phi(\mu) = \sup_{\theta \in \Theta} \{ \mu \theta - \psi(\theta) \}$$

(5)

Dual (conjugates) cumulant function has the following properties.

1) Derivative of the dual cumulant function is the differential equation

$$\frac{d\phi(\mu)}{d\mu} = \theta(\mu) = \arg \sup_{\theta} \mu \theta - \psi(\theta)$$

(6)

2) The maximizing argument is as

$$\theta^* = \theta(\mu) = (\psi'(\theta))^{-1}(\mu)$$

(7)

3) The dual cumulant function can be computed as

$$\phi(\mu) = \mu \theta(\mu) - \psi(\theta(\mu))$$

(8)

This transformation is known as Legendre transform. For an extensive mathematical treatment, Legendre transform and convex analysis are in [14]. Dual space and duality of exponential families are in [15] and in [16].

Example 1. Dual of the cumulant function $\psi(\theta) = \exp \theta$ for the Poisson can be found as below after applying (7)

$$\theta(\mu) = (\exp(\theta))^{-1}(\mu) = \log \mu$$

(9)

Then by (8), the dual cumulant function becomes

$$\phi(\mu) = \mu \log \mu - \exp(\log \mu) = \mu \log \mu - \mu$$

(10)

C. Variance Functions

The relationship between $\theta$ and $\mu$ is more direct and given as $[17]$

$$\frac{d\theta(\mu)}{d\mu} = \frac{d^2 \phi(\mu)}{d\mu^2} = v(\mu)^{-1}$$

(11)

Here $v(\mu)$ is the variance function $[17]$–$[19]$, and is related to the variance of the distribution by dispersion parameter $\varphi$ as

$$Var(x) = \varphi v(\mu)$$

(12)

As a special case of dispersion models, Tweedie distributions also called Tweedie models assume that the variance function is in the form of power function $[17], [19]$ given as

$$v(\mu) = \mu^p$$

(13)

that fully characterizes one-parameter dispersion model. Here, the exponent is $p = 0, 1, 2, 3$ for well known distributions
as Gaussian, Poisson, gamma and inverse Gaussian. For \( 1 < p < 2 \), they can be represented as the Poisson sum of gamma distributions so-called compound Poisson distribution. Indeed, they exist for all real values of \( p \) except for the range \( 0 < p < 1 \) [17].

Variance functions play important roles in characterizing one-parameter distributions similar to the role moment generating functions play. Given variance function, density for one-parameter distributions similar to the role moment generating function and characteristic equation and next inverting characteristic equation via Fourier inversion formula [12].

D. Bregman Divergences

By definition, for any real valued differentiable convex function \( \phi \) the Bregman divergence [20]

\[
d_{\phi}(x, \mu) = \phi(x) - \phi(\mu) - (x - \mu)\phi'(\mu)
\]

It is equal to tail of first-order Taylor expansion of \( \phi(x) \) at \( \mu \). In addition, it enjoys convex duality and can be equally expressed as

\[
d_{\phi}(x, \mu) = \phi(x) + \psi(\theta(\mu)) - x\theta(\mu)
\]

that can be showed by plugging \( \phi(\mu) \) using the dual relation

\[
\phi(\mu) = \mu\theta(\mu) - \psi(\theta(\mu))
\]

in (14) and identifying \( \phi'(\mu) = \theta(\mu) \) as given in (6).

The Bregman divergences are non-negative quantities as \( d_{\phi}(x, \mu) \geq 0 \) and equality holds only for \( x = \mu \). However, they provide neither symmetry nor triangular inequality in general and hence are not considered to be metrics. For a special choice of function \( \phi \), Bregman divergence turns to Euclidean distance that exhibits metric properties.

### Table I

| \( v(\mu) \) | Tweedie | \( v(\mu) \) | Non-Tweedie |
|----------|---------|----------|-------------|
| \( \mu^0 \) | Gaussian | \( \mu - \mu^2 \) | Bernoulli |
| \( \mu^1 \) | Poisson | \( \mu + \mu^2 \) | Neg. Binomial |
| \( \mu^2 \) | Gamma | \( 1 + \mu^2 \) | Hyperbolic secant |
| \( \mu^3 \) | Inv. Gaussian | | |
| \( \mu^p \) | General Tweedie | | |

Figure 1. Beta divergence illustration \( d_{\beta}(x, \mu) \) as the length of line segment \( [CD] \).

### IV. Basic Elements

A. The Canonical Parameter \( \theta(\mu) \)

We first derive the canonical parameter \( \theta \) as the definite integral on interval \([\mu_0, \mu]\) by solving the differential equation

\[
\frac{d\theta(\mu)}{d\mu} = \frac{1}{v(\mu)} \Rightarrow \theta(\mu) = \int_{\mu_0}^\mu \frac{1}{v(t)} \, dt
\]

Here the choice of the lower bound \( \mu_0 \) is arbitrary but fixed. Indeed, any fixed value \( \mu_0 \in \Omega \) can be specified as lower bound. As it will be clear, this choice has no effect on the computation of the beta divergence.

B. Dual Cumulant Function \( \phi(\mu) \)

Dual cumulant function plays central role when defining and generating beta divergences as special cases of Bregman divergences.

**Lemma 1.** Let \( v \) be variance function of a dispersion model. Generalized dual cumulant function is equal to

\[
\phi(\mu) = \int_{\mu_0}^\mu \frac{\mu - t}{v(t)} \, dt
\]

**Proof:**

By solving the differential equation, we obtain the dual cumulant function \( \phi(\mu) \) as

\[
\frac{d\phi(\mu)}{d\mu} = \theta(\mu) \Rightarrow \phi(\mu) = \int_{\mu_0}^\mu \theta(t) \, dt
\]

that is redefined after plugging \( \theta(\mu) \) in

\[
\phi(\mu) = \int_{\mu_0}^\mu \left( \int_{\mu_0}^t \frac{1}{v(z)} \, dz \right) \, dt
\]

Here bounds for the variables can be replaced as

\[
\mu_0 < z < t \quad \Rightarrow \quad \mu_0 < z < \mu
\]

\[
\mu_0 < t < \mu \quad \Rightarrow \quad z < t < \mu
\]

where we change the order of the integration and end up with the integral form for the dual cumulant function \( \phi \).

As will be needed in further sections, the first and second derivatives of the dual cumulant function wrt \( \mu \) are computed as follows

\[
\phi'(\mu) = \int_{\mu_0}^\mu \frac{1}{v(t)} \, dt \quad \phi''(\mu) = \frac{1}{v(\mu)}
\]

noting that the derivative of the dual cumulant function is inverse mean value mapping as \( \phi'(\mu) = \theta(\mu) \).

The dual cumulant function \( \phi \) corresponds to area under the curve \( \theta(t) \) parametrized by \( t \in \mathbb{R} \), in the interval \([t = \mu_0, t = \mu]\) given by the equation

\[
\phi(\mu) = \int_{\mu_0}^\mu \theta(t) \, dt
\]

as illustrated by Figure 2. The curve equation \( \theta(t) \) in the figure is obtained as given in the following example.
whereas the function $\phi(\mu)$ contains non-linear terms wrt $\mu$ as

$$\phi(\mu) = \frac{\mu^{2-p}}{(1-p)(2-p)}$$

whereas the function $\phi_0(\mu)$ contains linear terms wrt $\mu$

$$\phi_0(\mu) = -\mu^{1-p} \frac{2-p}{2}$$

The point of separating linear and non-linear terms is that linear terms are canceled smoothly and disappear when generating beta divergence by the Bregman divergence as also reported by [3], [21]. Indeed, very often we only use non-linear part of dual cumulant function $\phi_1(\mu)$ that has special values for $p = 0, 1, 2$

$$\phi_1(\mu) = \begin{cases} \frac{1}{2} \mu^2 & p = 0 \\
\mu \log \mu & p = 1 \\
-\log \mu & p = 2 \end{cases}$$

Finally, as the general formulation, after setting $\mu_0 = 1$, dual cumulant function $\phi(\mu)$ for Tweedie models becomes as

$$\phi(\mu) = \frac{\mu^{2-p}}{(1-p)(2-p)} - \frac{\mu}{1-p} + \frac{1}{2-p}$$

where the limits can be found by l’Hôpital’s. This form is reported by various authors such as [4], [22] with the index parameter $q$ usually adjusted as $q = 2 - p$.

V. Generalized Beta Divergence

This section is all about beta divergence where we first obtain its integral form and then study its scaling and translation properties.

A. Extending Beta Divergence

In the literature beta divergence is formulated as (with usually adjusted index parameter as $\beta = 2 - p$ or as $\beta = 1 - p$ [7])

$$d_\beta(x, \mu) = \frac{x^{2-p}}{(1-p)(2-p)} - \frac{x \mu^{1-p}}{1-p} + \frac{\mu^{2-p}}{2-p}$$

This definition is linked to power variance functions of Tweedie models. In this section its definition is extended beyond Tweedie models.

Definition 1. Let $x, \mu \in \Omega$. (Generalized) beta divergence $d_\beta(x, \mu)$ is specialized Bregman divergence generated by (generalized) dual cumulant function $\phi$, which, in turn is induced by variance function $v(\mu)$.

This definition implies that beta divergence is generated or induced by some variance functions. In this way, via the variance functions, beta divergence is linked to the exponential dispersion models. To differentiate this definition from well-known power function related definition it is qualified as ‘generalized’ whereas for the rest of the paper, we drop the term ‘generalized’ and simply refer as beta divergence.

Lemma 2. Beta divergence defined in Definition 1 $d_\beta(x, \mu)$ is equal to

$$d_\beta(x, \mu) = \int_\mu^x \frac{x - t}{v(t)} dt$$

Proof: By simply substituting the dual cumulant function and its first derivative

$$\phi(\mu) = \int_{\mu_0}^\mu \frac{\mu - t}{v(t)} dt \quad \phi'(\mu) = \int_{\mu_0}^\mu \frac{1}{v(t)} dt$$

in the Bregman divergence, we obtain the beta divergence. 

We note that beta divergence is specialized to the dual cumulant function as a special case when we measure divergence of $\mu$ from the base measure $\mu_0$

$$\phi(\mu) = d_\beta(\mu, \mu_0) = \int_{\mu_0}^\mu \frac{\mu - t}{v(t)} dt$$

Here, we abuse the notation and still use $d_\beta(\mu, \mu_0)$. This is indeed awkward since $\mu_0$ is no longer an independent variable but instead just a parameter and hence the notation $d_\beta(\mu | \mu_0)$ would be better choice.
Remark 1. Remark that the integral form of the beta divergence can be obtained from Taylor expansion. Bregman divergence is equal to the first order Taylor expansion [22]

$$\phi(x) = \phi(\mu) + \phi'(\mu)(x - \mu) + R_{\phi}(x, \mu)$$

(36)

where $R_{\phi}$ is the remainder term expressed as

$$R_{\phi}(x, \mu) = \int_{\mu}^{x} (x - t)\phi''(t)\,dt$$

(37)

The remainder is interpreted as the divergence from $x$ to $\mu$. As a special case for the dual cumulant function $\phi$ the Bregman divergence is specialized as beta divergence as

$$R_{\phi}(x, \mu) = d_\beta(x, \mu) = \int_{\mu}^{x} \frac{x - t}{v(t)}\,dt$$

(38)

where $\phi''(t)$ is the inverse variance function $v(\mu)^{-1}$.

Example 4. By following Definition 1 we easily find beta divergences for distributions including Tweedie and non-Tweedie families by decomposing the integral into two parts and computing each separately as

$$d_\beta(x, \mu) = \int_{\mu}^{x} \frac{x - t}{v(t)}\,dt = \int_{\mu}^{x} \frac{1}{v(t)}\,dt - \int_{\mu}^{x} \frac{t}{v(t)}\,dt$$

For example, for Tweedie models with VF $v(\mu) = \mu^p$ this results to the classical form of beta divergence as in (32). For others, for Bernoulli (Binomial with $m = 1$) with VF given as $v(\mu) = \mu - \mu^2$ [23] generates the beta divergence

$$d_\beta(x, \mu) = x \log \frac{x}{\mu} + (1 - x) \log \frac{1 - x}{1 - \mu}$$

(39)

whereas we compute beta divergences for negative binomial distribution with VF $v(\mu) = \mu + \mu^2$

$$d_\beta(x, \mu) = x \log \frac{x(1 + \mu)}{\mu(1 + x)} + \log \frac{1 + \mu}{1 + x}$$

(40)

and for hyperbolic secant distribution with $v(\mu) = 1 + \mu^2$

$$d_\beta(x, \mu) = x(\arctan x - \arctan \mu) + \frac{1}{2} \log \frac{1 + \mu^2}{1 + x^2}$$

by simply following integrals (ignoring the constant)

$$\int_{\mu}^{x} \frac{dt}{1 + t^2} = \arctan t$$

$$\int_{\mu}^{x} \frac{t}{1 + t^2} dt = \frac{1}{2} \log(1 + t^2)$$

B. Geometric Interpretation of Beta Divergence

As a special case of Bregman divergence, beta divergence can be interpreted as the length of the line segment in Figure 1 as already known. As an alternative interpretation, beta divergence $d_\beta(x, \mu)$ for $x \geq \mu$ is area of the region surrounded by at the top the curve $\theta(t)$, at the bottom by the horizontal line $y = \theta(\mu)$ and at right by the vertical line $x = \theta(x)$. To show this, we use the identities

$$\phi(\mu) = \int_{\mu}^{0} \theta(t)\,dt$$

and

$$\phi'(\mu) = \theta(\mu)$$

(41)

in the Bregman divergence as

$$d_\beta(x, \mu) = \phi(x) - \phi(\mu) - (x - \mu)\phi'(\mu)$$

$$= \int_{\mu}^{x} \theta(t)\,dt - \int_{\mu}^{x} \theta(t)\,dt - (x - \mu)\theta(\mu)$$

$$= \left(\int_{\mu}^{x} \theta(t)\,dt\right) - (x - \mu)\theta(\mu)$$

where this subtraction corresponds to the area in Figure 3 as $d_\beta(x, \mu) = $ Area of BCFD $-$ Area of BCED $=$ Area of DEF

In addition, in Figure 3 the dual cumulant functions are identified as

$$\phi(\mu) = $ Area of ABD

(42)

$$\phi(x) = $ Area of ACF

(43)

C. Scaling of Beta Divergence

In this section we find the scaling of beta divergence by $1/c$, i.e. we relate

$$\int_{\mu}^{x} \frac{x - t}{v(t)}\,dt \propto \int_{\mu/c}^{x/c} \frac{x/c - t}{v(t)}\,dt$$

(44)

One immediate use of the result of scaling analysis is that by substituting $c = \mu$, we connect beta divergence to alpha divergence as shown later in this paper.

Lemma 3. Let $x, \mu \in \Omega$, and $v(\mu)$ be the variance function. Then beta divergence $\ d_\beta(x, \mu)$ and its scaled form $d_\beta(x/c, \mu/c)$ with the scalar $c \in \mathbb{R}_+$ are related as

$$d_\beta(x, \mu) = \frac{c^2}{f(c)}d_\beta(x/c, \mu/c)$$

(45)

where the variance function $v(\cdot)$ is written as in the following form

$$f(c) = \frac{v(ct)}{v(t)}.$$  

(46)

Proof:

Here we use change of variable of the integrals. Consider the integral

$$\int_{\mu}^{x} f(g(t))g'(t)\,dt$$

(47)
where the function \( g \) is set to
\[
g(t) = \frac{t}{c} \quad \text{with} \quad g'(t) = \frac{1}{c}
\]
and we look for \( f \). The hint is that we want \( f(g(t))g'(t) \) to match to beta divergence function inside the integral
\[
f'(t) \frac{1}{c} = \frac{x - t}{v(t)} \quad \text{or} \quad f(r) = \frac{xc - cr}{v(cr)}
\]

Example 5. For Tweedie models where \( v(\mu) = \mu^p, f(c) = c^p \) since
\[
v(c\mu) = (c\mu)^p = c^p v(\mu)
\]
and hence
\[
d_\beta(x, c) = \frac{c^2}{c^p} d_\beta(x/c, \mu/c) = c^{2-p} d_\beta(x/c, \mu/c)
\]
For specifically \( c = \mu \), we have
\[
d_\beta(x, \mu) = \mu^{2-p} d_\beta(x/\mu, 1)
\]

D. Translation of Beta Divergence

Similar to scaling property here we analyse translation property of the beta divergence by the scalar \( c \in \mathbb{R} \). In other words, now, we relate
\[
\int_\mu^x \frac{x - t}{v(t)} \, dt \propto \int_{\mu + c}^{x + c} \frac{x + c - t}{v(t)} \, dt
\]

Lemma 4. beta divergence \( d_\beta(x, \mu) \) and its translated form \( d_\beta(x + c, \mu + c) \) with the scalar \( c \in \mathbb{R} \) are related as
\[
d_\beta(x, \mu) = \frac{1}{f(c)} d_\beta(x + c, \mu + c)
\]

where the function \( f \) is the ratio of translated variance function to the original variance function as
\[
f(c) = \frac{v(\mu - c)}{v(\mu)}.
\]

Example 6. For an immediate example, for the Gaussian distribution the variance function \( v(\mu) = 1 \) implies that \( f(c) = 1 \). Thus, for any displacement \( c \) we have
\[
d_\beta(x, \mu) = d_\beta(x + c, \mu + c)
\]
which means beta divergence is invariant for translation under the Gaussian distribution (or equivalently saying under the Euclidean distance).

Proof: The proof is fairly similar to that of scaling of beta divergence above. To apply change of variable of the integrals, we choose function \( g \) as
\[
g(t) = t + c \quad \text{with} \quad g'(t) = 1
\]
and function \( f \) becomes
\[
f(t + c) = \frac{x - t}{v(t)}
\]
Then by plugging \( r = t + c \) or \( t = r - c \) and getting rid of \( t \) it is simplified as
\[
f(r) = \frac{x - r + c}{v(r - c)}
\]

Finally the integral turns to beta divergence as
\[
d_\beta(x, \mu) = \int_{g(\mu)}^{\mu} f(t) \, dt
\]
\[
= \int_{\mu + c}^{x + c} \frac{(x + c) - t}{v(t - c)} \, dt
\]
To simplify this integration further we consider a special case
\[
v(t - c) = f(c)v(t)
\]
By this beta divergence can be expressed as
\[
d_\beta(x, \mu) = \frac{1}{f(c)} \int_{\mu + c}^{x + c} \frac{(x + c) - t}{v(t)} \, dt
\]
where we obtain the relation
\[
d_\beta(x, \mu) = \frac{1}{f(c)} d_\beta(x + c, \mu + c)
\]

Example 7. For exponential variance functions \( v(\mu) = \gamma^\mu \), for any \( \gamma \in \mathbb{R}_+ \), we find \( f(c) = \gamma^{-c} \) since
\[
v(\mu - c) = \gamma^{\mu-c} = \gamma^{-c} \gamma^\mu = \gamma^{-c} v(\mu)
\]
Hence translated beta divergence by \( c \) becomes
\[
d_\beta(x, \mu) = \gamma^c d_\beta(x + c, \mu + c)
\]

VI. LOG-LIKELIHOOD, DEVIANECE AND BETA DIVERGENCE

This section links beta divergence to log-likelihood and statistical deviance.
A. Unit Quasi-Log-Likelihood

Consider one-parameter dispersion model parameterized by \( \mu \):

\[
p(x; \mu) = h(x, \varphi) \exp \left\{ \varphi^{-1} (\theta(\mu)x - \psi(\theta(\mu))) \right\}
\]

that the dispersion \( \varphi \) is assumed to be an arbitrary but fixed parameter.

**Definition 2.** Let \( x, \mu \in \Omega, \theta \in \Theta \) and \( \Omega = \Theta \). Let \( \mu \) be ML estimate of \( x \), i.e. \( \mu \) and \( \theta \) are duals, then unit quasi-log-likelihood denoted by \( L_x(\mu) \) is defined as

\[
L_x(\mu) = \theta(\mu)x - \psi(\theta(\mu))
\]

(69)

The unit quasi-log-likelihood is quasi-log-likelihood given by Wedderburn [5] with the addition of 'unit' term. Compared to a log-likelihood function, the constant term \( \log h(x, \varphi) \) wrt \( \mu \) is dropped and it is scaled by the dispersion \( \varphi \) that turns the quasi-log-likelihood into 'unit' quasi-log-likelihood. On the other hand, \( L_x(\mu) \) has many properties in common with the log-likelihood as given by Wedderburn [5].

**Lemma 5.** Unit quasi-log-likelihood function \( L_x(\mu) \) is equal to

\[
L_x(\mu) = \int_{\mu_0}^{\mu} \frac{x-t}{v(t)} \, dt
\]

(70)

**Proof:**

First, getting rid of \( \psi(\theta(\mu)) \) as plugging the duality

\[
\psi(\theta(\mu)) = \mu \theta(\mu) - \phi(\mu)
\]

in the definition of \( L_x(\mu) \)

\[
L_x(\mu) = \theta(\mu)x - \psi(\theta(\mu)) = \theta(\mu)x - \mu \theta(\mu) + \phi(\mu)
\]

then, second, by substituting the terms \( \theta(\mu) \) and \( \phi(\mu) \), we obtain

\[
L_x(\mu) = (x-\mu) \int_{\mu_0}^{\mu} \frac{1}{v(t)} \, dt + \int_{\mu_0}^{\mu} \frac{\mu - t}{v(t)} \, dt
\]

(72)

\[
= \int_{\mu_0}^{\mu} \frac{x-t}{v(t)} \, dt
\]

(73)

For practical purposes in the examples, we take the lower bound \( \mu_0 = 0 \).

We show that there are following connections between dual cumulant function and unit quasi-log-likelihood.

**Corollary 1.** Let \( \phi \) be the dual cumulant function and \( \mathcal{L} \) be the unit quasi-log-likelihood. Then there are the following connections between \( \phi(\cdot) \) and \( \mathcal{L}(\cdot) \)

i) \( \phi(\mu) = \mathcal{L}_\mu(\mu) = \int_{\mu_0}^{\mu} \frac{\mu - t}{v(t)} \, dt \)

(74)

ii) \( \phi(x) = \mathcal{L}_x(x) = \int_{\mu_0}^{x} \frac{x-t}{v(t)} \, dt \)

(75)

Here \( \mathcal{L}_x(x) \) is the quasi-log-likelihood of the 'full' model where data speaks about data whereas \( \mathcal{L}_x(\mu) \) is the quasi-log-likelihood of the parametric model where data speaks about model parameters. \( \mathcal{L}_\mu(\mu) \) is the same function as \( \mathcal{L}_x(x) \) where it only makes sense when both of \( x \) and \( \mu \) are independent variables of the same function such as \( d(x, \mu) \) that can be written as differences of \( \mathcal{L}_x(x) \) and \( \mathcal{L}_\mu(\mu) \).

**Proof:**

Proof of i) and ii) immediately follow from the integral definitions of \( \phi \) and \( \mathcal{L} \).

Scaled quasi-log-likelihood \( \mathcal{L}_x(\mu) \) provides the following properties.

**Corollary 2.** The unit quasi-log-likelihood \( \mathcal{L}_x(\mu) \) and its first two derivatives have the following expectations

i) \( E [\mathcal{L}_x(\mu)] = \phi(\mu) \)

(76)

ii) \( E \left[ \frac{\partial \mathcal{L}_x(\mu)}{\partial \mu} \right] = 0 \)

(77)

iii) \( E \left[ \frac{\partial^2 \mathcal{L}_x(\mu)}{\partial \mu^2} \right] = - \frac{1}{v(\mu)} \)

(78)

**Proof:**

For i), simply we take the expectation of \( \mathcal{L}_x(\mu) \)

\[
E [\mathcal{L}_x(\mu)] = E [x \theta(\mu) - \psi(\theta(\mu))] = \phi(\mu).
\]

(79)

For ii), it turns to the expectation

\[
E \left[ \frac{x - \mu}{v(\mu)} \right] = 0
\]

(80)

For iii), it is the expectation of the second derivative as

\[
E \left[ - \frac{1}{v(\mu)} + (x - \mu)(-1) \frac{1}{v(\mu)^2} \phi'(\mu) \right] = - \frac{1}{v(\mu)}
\]

(81)

In fact, \( \mathcal{L}_x(\mu) \) is scaled form of quasi-log-likelihood, and thus they provide similar properties, and properties ii) and iii) have already been shown in [5].

B. Statistical Deviance

By definition, unit deviance is two times of the log-likelihood ratio scaled by the dispersion, i.e. twice ratio of unit log-likelihood of the 'full' model to the that of parametric model [12], [23].

**Definition 3.** Let \( x, \mu \in \Omega, \mu \) be ML estimate of \( x \) and \( \mathcal{L}_x(\mu) \) be the unit quasi-log-likelihood function. Then unit deviance denoted by \( d_v(x, \mu) \) is

\[
d_v(x, \mu) = 2 \{ \mathcal{L}_x(x) - \mathcal{L}_x(\mu) \}
\]

(82)

This definition leads to the integral representation of the unit deviance that is also given in [12], [23].

**Lemma 6.** The unit deviance denoted by \( d_v(x, \mu) \) is equal to

\[
d_v(x, \mu) = 2 \int_{\mu}^{x} \frac{x-t}{v(t)} \, dt
\]

(83)

**Proof:**

Proof is simply completed by subtracting integral forms of \( \mathcal{L}_x(x) \) and \( \mathcal{L}_x(\mu) \).

The following lemma states the equivalence of beta divergence and unit deviance concepts.
**Lemma 7.** Let \( d_\beta(x, \mu) \) be the beta divergence and \( d_\nu(x, \mu) \) be the unit (scaled) deviance. Then unit deviance is twice of the beta divergence as

\[
d_\nu(x, \mu) = 2d_\beta(x, \mu) = 2 \int_{\mu}^{x} \frac{x - t}{\nu(t)} \, dt \tag{84}
\]

**Proof:** Proof is immediately implied by the equality of the integral representations of both functions.

This lemma implies the connection between beta divergence and unit quasi-log-likelihood.

**Corollary 3.** Beta divergence \( d_\beta(x, \mu) \) is equal to the following difference

\[
d_\beta(x, \mu) = \mathcal{L}_x(x) - \mathcal{L}_x(\mu) \tag{85}
\]

**Proof:** Proof is trivial consequence of the lemma above.

We note that this result is independent of the lower bound \( \mu_0 \).

**C. Density Representation via Beta Divergence**

One immediate consequence of Lemma 7 that states equivalence of beta divergence and divergence is that the standard form of density of dispersion model \( DM(\mu, \varphi) \) [12]

\[
p(x; \mu, \varphi) = g(x, \varphi) \exp \left\{ - \frac{1}{2} \varphi^{-1} d_\nu(x, \mu) \right\} \tag{86}
\]

can be written in terms of beta divergence as

\[
p(x; \mu, \varphi) = g(x, \varphi) \exp \left\{ - \varphi^{-1} d_\beta(x, \mu) \right\} \tag{87}
\]

**Remark 2.** Note that by plugging dual form of beta divergence

\[
x\theta - \psi(\theta) = \phi(x) - d_\beta(x, \mu) \tag{88}
\]

in density of exponential dispersion models

\[
p(x; \theta, \varphi) = h(x, \varphi) \exp \{ \varphi^{-1}(x \theta - \psi(\theta)) \}
\]

we obtain density form in (87). This is special case of the generalized method in [6] that exploits the bijection between Bregman divergences and exponential family of distributions. Here the functions \( h \) and \( g \) are related as

\[
g(x, \varphi) = h(x, \varphi) \exp \{ \varphi^{-1} \phi(x) \} \tag{89}
\]

In the followings we illustrate various densities expressed via beta divergences [8].

**Example 8.** The density of the Gaussian distribution with dispersion \( \varphi = \sigma^2 \) is given as [12]

\[
p(x; \mu, \sigma^2) = \left(2\pi\sigma^2\right)^{-\frac{1}{2}} \exp \left\{ \frac{-x^2}{2\sigma^2} \right\} \frac{1}{\theta(\mu)} \exp \left\{ \frac{1}{\theta(\mu)} \left( x \mu - \mu^2 \right) \right\}
\]

that is equivalently expressed as via beta divergence

\[
p(x; \mu, \sigma^2) = \left(2\pi\sigma^2\right)^{-\frac{1}{2}} \exp \left\{ - \frac{1}{\sigma^2} d_\beta(x, \mu) \right\}
\]

The density of the gamma distribution with \( a \) and \( b \) shape and (inverse) scale parameters is

\[
p(x; a, b) = \frac{x^{a-1}}{\Gamma(a)} \exp\{-bx + a \log x\} \tag{90}
\]

Using the gamma distribution convention such that \( \mu = a/b \) and \( \text{Var}(x) = a/b^2 \) dispersion becomes \( \varphi = 1/a \) we re-write the density in terms of mean and inverse dispersion as

\[
p(x; \mu, a) = \frac{x^{a-1}}{\Gamma(a)} a^a \exp\left\{ a(- \frac{1}{\mu} x - \log \mu) \right\} \tag{91}
\]

Then, by adding and subtracting \( \log x + 1 \) in the exponent we obtain

\[
p(x; \mu, a) = \frac{x^{a-1} \exp(-a)}{\Gamma(a)} \exp\{-a d_\beta(x, \mu)\} \tag{92}
\]

For the Poisson distribution with the dispersion \( \varphi = 1 \) the density is [12]

\[
p(x; \mu) = \frac{1}{x!} \exp\left\{ x \log \mu - \mu - \frac{\mu}{\psi(\mu)} \right\} \tag{93}
\]

that by adding and subtracting \( x \log x - x \) in the exponent we obtain beta representation of the density

\[
p(x; \mu) = \frac{x^x \exp x}{x!} \exp\{-d_\beta(x, \mu)\} \tag{94}
\]

**D. Expectation of Beta Divergence**

An interesting quantity is the expected beta divergence. It opens connections to the relating beta divergence and Jensen divergence.

**Lemma 8.** Expectation of beta divergence is

\[
E[d_\beta(x, \mu)] = E[\phi_1(x)] - \phi_1(\mu) \tag{95}
\]

where \( \phi_1(\cdot) \) is the non linear part of the dual cumulant function as defined earlier.

In this context, \( \phi_1(\cdot) \) can be simply regarded as dual cumulant function.

**Proof:** By taking the expectation of Bregman divergence

\[
E[d_\beta(x, \mu)] = E[\phi(x) - \phi(\mu) - (x - \mu) \phi'(\mu)] \tag{96}
\]

\[
= E[\phi(x)] - \phi(\mu) - E[(x - \mu) \phi'(\mu)] \tag{97}
\]

\[
= E[\phi(x)] - \phi(\mu) \tag{98}
\]

Now by plugging \( \phi = \phi_1 + \phi_0 \) we end up with

\[
E[d_\beta(x, \mu)] = E[\phi_1(x)] + E[\phi_0(x)] - \phi_1(\mu) - \phi_0(\mu) \tag{99}
\]

\[
= E[\phi_1(x)] + \phi_0(\mu) - \phi_1(\mu) - \phi_0(\mu) \tag{100}
\]

\[
= E[\phi_1(x)] - \phi_1(\mu) \tag{101}
\]

Note that \( E[\phi_0(x)] = E[\phi_0(\mu)] \) due to that \( E[x] = \mu \) and that \( \phi_0(x) \) has only linear terms wrt \( x \).
Corollary 4. Expected beta divergence is equal to Jensen gap for the dual cumulant function $\phi_1$ as

$$E[d_\beta(x, \mu)] = E[\phi_1(x)] - \phi_1(E[x])$$

(99)

Proof: Jensen inequality for the convex function $\phi_1$ is

$$E[\phi_1(x)] \geq \phi_1(E[x]) = \phi_1(\mu)$$

(100)

where the gap is the expected beta divergence given in (95).

Example 9. Expectation of beta divergence for Tweedie models is computed by using Lemma 8 as follows.

For Tweedie models non-linear part of dual cumulant function $\phi_1(x)$ is

$$\phi_1(x) = \frac{x^{2-p}}{(1-p)(2-p)}$$

(101)

Hence, expected beta divergence for Tweedie models is as

$$E[d_\beta(x, \mu)] = E\left[\frac{x^{2-p}}{(1-p)(2-p)} - \frac{\mu^{2-p}}{(1-p)(2-p)}\right]$$

(102)

For the limits we can either use l’Hospital or simply apply non-linear part of dual cumulant function $\phi_1(x)$ as given in Example 3 that results to the following special case for $p = 0$ and limits for $p = 1, 2$

$$E[d_\beta(x, \mu)] = \begin{cases} \frac{1}{2} \left( E[x^2] - \mu^2 \right) & p = 0 \\
- E[\log x] + \log \mu & p = 1 \\
- E[\log x] + \log \mu & p = 2 
\end{cases}$$

where for $p = 0$ it is equal to $\frac{1}{2} \sigma^2$.

Clearly, expectation of beta divergence has connection to entropy, and hence using this expectation we can write entropy for dispersion models

$$\mathcal{H}_\alpha(\mu) = - E[\log g(x, \varphi)] + \varphi^{-1} E[d_\beta(x, \mu)]$$

(103)

Example 10. Equation (103) gives another way of computing the entropy as given in these examples. For the Gaussian, the dispersion is $\varphi = \sigma^2$ and

$$E[\log g(x, \sigma^2)] = E\left[\log(2\pi\sigma^2)^{-\frac{1}{2}}\right]$$

(104)

$$E[d_\beta(x, \mu)] = \frac{1}{2}\sigma^2$$

(105)

and plugging them in (103) we end up with the entropy for the Gaussian as

$$\mathcal{H}_\alpha(\mu) = \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2}$$

(106)

For the gamma the dispersion is $\varphi = 1/\alpha$, the expectation parameter is $\mu = a/b$ and the expectation $E[\log x] = \psi(a) - \log b$. Identifying the rest as

$$E[\log g(\cdot)] = E\left[\log \frac{x^{-a}\exp(-a)}{\Gamma(a)}\right]$$

$$= - E[\log x] + \log \frac{a\exp(-a)}{\Gamma(a)}$$

$$E[d_\beta(\cdot, \cdot)] = - E[\log x] + \log \mu = - \psi(a) + \log b + \log(a/b)$$

and plugging them all in (103) we end up with the entropy for the Gamma as

$$\mathcal{H}_\alpha(\mu) = a - \log b + \log \Gamma(a) + (1 - a)\psi(a)$$

(107)

Table II

| Integration | Range |
|-------------|-------|
| Dual cumulant of $x$ | $\phi(x) = d_\beta(x|\mu_0) = \int_{\mu_0}^{\mu} \frac{x-t}{\sqrt{2\pi} \sigma} \ dt$ | $[\mu_0, x]$ |
| Canonical parameter | $\theta(\mu) = \int_{\mu_0}^{\mu} \frac{1}{\sqrt{2\pi} \sigma} \ dt$ | $[\mu_0, \mu]$ |
| Cumulant | $\psi(\theta(\mu)) = \int_{\mu_0}^{\mu} \frac{1}{\sqrt{2\pi} \sigma} \ dt$ | $[\mu_0, \mu]$ |
| Dual cumulant of $\mu$ | $d_\beta(x|\mu_0) = \int_{\mu_0}^{\mu} \frac{x-t}{\sqrt{2\pi} \sigma} \ dt$ | $[\mu_0, \mu]$ |
| Beta divergence | $d_\alpha(x, \mu) = \int_{\mu_0}^{\mu} \psi\left(\frac{\theta(x|t)}{\mu}\right) \ dt$ | $[\mu, x]$ |
| Log likelihood | $\mathcal{L}_x(\mu) = \int_{\mu_0}^{\mu} \frac{x-1}{\sqrt{2\pi} \sigma} \ dt$ | $[\mu_0, \mu]$ |
| Full log Likelihood | $\mathcal{L}_x(x) = \int_{\mu_0}^{x} \frac{x-1}{\sqrt{2\pi} \sigma} \ dt$ | $[\mu_0, x]$ |

VIII. Appendix - Alpha Divergence

Likewise beta divergence, another specialized divergence is alpha divergence which is a type of the $f$-divergence [24]. It has strong connection to beta divergence, and hence here as we show it enjoys similar compact integral representation as beta divergence. Before that we introduce $f$-divergence briefly.

A. $f$-Divergence

The $f$-divergences are generalized KL divergences, and are introduced independently by authors Csiszár [25], Morimoto [26] and Ali & Silvey [27] during 1960s. By definition, for any real valued convex function $f$ the $f$-divergence is defined as [25]

$$d_f(x, \mu) = \mu f\left(\frac{x}{\mu}\right) \quad \text{with} \quad f(1) = 0$$

(108)

For the setting $x = 1$, the divergence $d_f(1, \mu)$ becomes only a function of $\mu$

$$f^*(\mu) = \mu f(1/\mu)$$

(109)

where $f^*$ is called as Csiszár dual of the function $f$.

Likewise the Bregman divergence $f$-divergences are non-negative quantities as $d_f(x, \mu) \geq 0$ and iff $d_f(x, x) = 0$. As a special case, Hellinger distance is a type of symmetric alpha divergence with $p = 3/2$ that exhibits metric properties.
B. Alpha Divergence

In the literature alpha divergence has many different forms [4, 22, 24, 28] where all are equivalent. The one that index variable aligns with Tweedie models index parameter is given in [8] as

\[ d_\alpha(x, \mu) = \frac{x^{2-p}\mu^p-1}{(1-p)(2-p)} - \frac{x}{1-p} + \frac{\mu}{2-p} \tag{110} \]

Here by changing \( p = 2 - \alpha \) we obtain alpha divergence form given in [4] whereas with \( p = (\alpha + 3)/2 \) we obtain Amari alpha that generates another form given in [29]. Likewise, for \( p = 2 - \delta \), it turns to \( \delta \)-divergence, that is identical to alpha divergence with \( \delta \) as the index parameter [28].

\( f \)-divergence is specialized to alpha divergence when dual cumulant function \( \phi \) is used.

**Definition 4.** Let \( x, \mu \in \Omega \). Alpha divergence of \( x \) from \( \mu \), denoted by \( d_\alpha(x, \mu) \) is the \( f \)-divergence generated by the dual cumulant function \( \phi \) induced by variance function \( v(\mu) \).

\( f \)-divergence requires that the function \( f \) provides \( f(1) = 0 \). The dual cumulant function \( \phi \) accomplishes this by choosing the base lower bound as \( \mu_0 = 1 \) so that the function becomes

\[ \phi(\mu) = \int_1^\mu \frac{\mu - t}{v(t)} \, dt \tag{111} \]

In this way, the function \( \phi \) provides that \( \phi(1) = 0 \) as well as \( \phi(1) = 0 \).

**Lemma 9.** Alpha divergence \( d_\alpha(x, \mu) \) is equal to

\[ d_\alpha(x, \mu) = \mu \int_1^{x/\mu} \frac{(x/\mu) - t}{v(t)} \, dt \tag{112} \]

**Proof:** By simply specializing the \( f \)-divergence definition \( d_f(x, \mu) = \mu f(x/\mu) \) as

\[ d_\alpha(x, \mu) = \mu \phi(x/\mu) \tag{113} \]

In the following we show well known symmetry condition for alpha divergence.

**Corollary 5.** For functions that provide \( \phi(r) = r\phi(1/r) \), alpha divergence is symmetric, i.e.

\[ d_\alpha(x, \mu) = d_\alpha(\mu, x) \implies \phi(r) = r\phi(1/r) \tag{114} \]

**Proof:**

We want to find functions \( \phi \) that generates symmetric alpha divergences such as

\[ d_\alpha(x, \mu) = d_\alpha(\mu, x) \tag{115} \]

Then we plug the definition of alpha divergence as

\[ \mu \int_1^{x/\mu} \frac{x/\mu - t}{v(t)} \, dt = x \int_1^{\mu/x} \frac{\mu/x - t}{v(t)} \, dt \tag{116} \]

and then setting \( r = \mu/x \) we identify the function \( \phi() \)

\[ x = \int_1^{x/\mu} \frac{x/\mu - t}{v(t)} \, dt \]

\[ x = \frac{\int_1^{x/\mu} x/\mu - t \, dt}{\int_1^{x/\mu} v(t) \, dt} = \frac{\phi(r)}{\phi(1/r)} \implies \phi(r) = r\phi(1/r) \]

**Example 11.** For Tweedie models with \( v(\mu) = \mu^p \) we find that

1. \( \phi_p(\mu) = \mu \phi_p(1/\mu) \implies p = 3/2 \tag{117} \)
2. \( \phi_p(\mu) = \mu \phi_q(1/\mu) \implies p + q = 3 \tag{118} \)

Alpha divergence has an interesting compact integral representation in terms of the canonical parameter and the cumulant function. In the following we first obtain integral representation of the cumulant function.

**Lemma 10.** The cumulant function is expressed in integral form as

\[ \psi(\theta(\mu)) = \int_1^\mu \frac{t}{v(t)} \, dt \tag{119} \]

**Proof:** The cumulant can be found trivially using the duality as

\[ \psi(\theta(\mu)) = \mu \theta(\mu) - \phi(\mu) \tag{120} \]

and plugging in the relevant terms we obtain the cumulant function

\[ \psi(\theta(\mu)) = \mu \int_1^\mu \frac{1}{v(t)} \, dt - \int_1^\mu \frac{\mu - t}{v(t)} \, dt = \int_1^\mu \frac{t}{v(t)} \, dt \tag{121} \]

**Lemma 11.** The alpha divergence can also be expressed in terms of the cumulant function as

\[ d_\alpha(x, \mu) = \int_\mu^x \psi(\theta(x/t)) \, dt \tag{122} \]

**Proof:** We indeed prove that alpha divergence can be written as in the following

\[ d_\alpha(x, \mu) = \mu \int_1^{x/\mu} \frac{x/\mu - t}{v(t)} \, dt = \int_\mu^x \left( \int_1^{x/t} \frac{z}{v(z)} \, dz \right) \, dt \]

by changing the bounds for the variables in the double integral

\[ 1 < z < x/t \implies 1 < z < x/\mu \]

\[ \mu < t < x \implies \mu < t < x/z \]

and then by changing the order of the integration. Then by definition the term inside the parenthesis turns to \( \psi(\theta(x/t)) \).

**Example 12.** We obtain alpha divergence for Tweedie models using directly by (122)

\[ d_\alpha(x, \mu) = \int_\mu^x \psi(\theta(x/t)) \, dt = \int_\mu^x \frac{(x/t)^{2-p} - 1}{2 - p} \, dt \tag{123} \]

where for Tweedie models \( \psi(\theta(\mu)) \) is

\[ \psi(\theta(\mu)) = \int_1^\mu \frac{t}{p+1} \, dt = \frac{\mu^{2-p} - 1}{2 - p} \]
C. Connection of Alpha and Beta Divergences

**Corollary 6.** Alpha divergence can be written in terms of beta divergence \( x/\mu \) from 1

\[
d_\alpha(x, \mu) = \mu d_\beta(x/\mu, 1)
\]

**(124)**

**Proof:** Integral form of alpha divergence trivially regarded as beta divergence scaled by \( \mu \).

The connection between beta and alpha divergences can be interpreted as Csiszar’s duality such that alpha divergence is Csiszar dual of beta divergence implied by the definition.

**Corollary 7.** Connection of alpha and beta divergences is given as

\[
d_\beta(x, \mu) = \frac{\mu}{f(\mu)} d_\alpha(x, \mu)
\]

**(125)**

**Proof:** Using the relations

\[d_\alpha(x, \mu) = \mu d_\beta(x/\mu, 1)
\]

**(126)**

\[d_\beta(x, \mu) = \frac{\mu^2}{f(\mu)} d_\beta(x/\mu, 1)
\]

**(127)**

we obtain connection of alpha and beta given in (125) where \( f \) provides

\[f(c) = \frac{v(c)}{v(1)}, \quad f(1) = 1
\]

**(128)**

**Example 13.** For Tweedie models with \( v(\mu) = \mu^p \) the connection is

\[
d_\beta(x, \mu) = \mu^{1-p} d_\alpha(x, \mu)
\]

**(129)**

since \( f(\mu) = \mu^p \).

**Corollary 8.** Any variance function \( v(\mu) = k\mu, \quad k \in \mathbb{R}_+ \) generates alpha and beta divergences that are equal in value

\[
d_\beta(x, \mu) = d_\alpha(x, \mu) \quad \text{for} \quad v(\mu) = k\mu.
\]

**(130)**

**Proof:** Alpha and beta divergences are equal when

\[
\frac{\mu}{f(\mu)} = 1 \quad \Rightarrow \quad f(\mu) = \mu.
\]

**(131)**

Then the variance functions turns to

\[f(c) = \frac{v(c\mu)}{v(\mu)} \quad \Rightarrow \quad c = \frac{v(c\mu)}{v(\mu)} \quad \Rightarrow \quad v(\mu) = k\mu.
\]

**(132)**

For \( k = 1 \), the variance function becomes \( v(\mu) = \mu \) that corresponds to the Poisson distribution.

**References**

[1] A. Basu, I. R. Harris, N. L. Hjort, and M. C. Jones, “Robust and efficient estimation by minimising a density power divergence,” *Biometrika*, vol. 85, no. 3, pp. 549–559, 1998.

[2] S. Eguchi and Y. Kano, “Robustifying maximum likelihood estimation,” Institute of Statistical Mathematics in Tokyo, Tech. Rep., 2001.

[3] A. Cichocki and S. Amari, “Families of alpha-beta-and gamma-divergences: Flexible and robust measures of similarities,” *Entropy*, vol. 12, pp. 1532–1568, 2010.

[4] A. Cichocki, S. Cruces, and S. Amari, “Generalized alpha-beta divergences and their application to robust nonnegative matrix factorization,” *Entropy*, vol. 13, no. 1, pp. 134–170, 2011.

[5] R. W. M. Wedderburn, “Quasi-likelihood functions, generalized linear models, and the gauss-newton method,” *Biometrika*, vol. 61, pp. 439–447, 1974.

[6] A. Banerjee, S. Merugu, I. S. Dhillon, and J. Ghosh, “Clustering with Bregman divergences,” *JMLR*, vol. 6, pp. 1705–1749, 2005.

[7] A. Cichocki, R. Zdunek, A. H. Phan, and S. Amari, *Nonnegative Matrix and Tensor Factorization*. Wiley, 2009.

[8] Y. K. Yilmaz and A. T. Cemgil, “Alpha/beta divergences and Tweedie models,” Tech. Rep. arXiv:1209.4280, 2012.

[9] C. Fevotte and J. Idier, “Algorithms for nonnegative matrix factorization with the beta divergence,” *Neural Comput.*, pp. 2421–2456, 2011.

[10] Y. K. Yilmaz and A. T. Cemgil, *Algorithms for Probabilistic Latent Tensor Factorization*, 2011, accessed at October 2011. [Online]. Available: http://www.science2direct.com/science/article/pii/S0165168411003537

[11] Learning the Beta-Divergence in Tweedie Compound Poisson Matrix Factorization Models, 2013.

[12] B. Jørgensen, *The Theory of Dispersion Models*. Chapman Hall/CRC Monographs on Statistics and Applied Probability, 1997.

[13] M. Wainwright and M. I. Jordan, “Graphical models, exponential families, and variational inference,” Foundations and Trends in Machine Learning, vol. 1, pp. 1–305, 2008.

[14] R. T. Rockafellar, *Convex Analysis*. Princeton University Press, 1970.

[15] O. E. Barndorff-Nielsen, *Information and Exponential Families: in Statistical Theory*. Wiley, 1978.

[16] S. Amari and H. Nagaoka, *Methods of Information Geometry*. American Mathematical Society, 2000.

[17] B. Jørgensen, “Exponential dispersion models,” *J. of the Royal Statistical Society. Series B*, vol. 49, pp. 127–162, 1987.

[18] M. C. Tweedie, “An index which distinguishes between some important exponential families,” *Statistics: applications and new directions*, Indian Statist. Inst., Calcutta, pp. 579–604, 1984.

[19] S. K. Bar-Lev and P. Enis, “Reproducibility and natural exponential families with power variance functions,” *Annals of Stat.*, vol. 14, 1986.

[20] L. M. Bregman, “The relaxation method of finding the common points of convex sets and its application to the solution of problems in convex programming,” *USSR Computational Mathematics and Mathematical Physics*, vol. 7, pp. 200–217, 1967.

[21] M. A. Reid and R. C. Williamson, “Information, divergence and risk for binary experiments,” *JMLR*, vol. 12, pp. 731–817, 2011.

[22] F. Liese and I. Vajda, “On divergences and informations in statistics and information theory,” *IEEE Transactions on Information Theory*, vol. 52, no. 10, 2006.

[23] C. E. McCullouch and J. A. Nelder, *Generalized Linear Models*, 2nd ed. Chapman and Hall, 1989.

[24] S. Amari, *Differential-Geometrical Methods in Statistics*, Editor, Ed. Springer Verlag, 1985.

[25] J. Csiszar, “Eine informations theoretische ungleichung und ihre anwendung auf den beweis der ergodizitt von markoffschen ketten,” *Publ. Math. Inst. Hungar. Acad.*, vol. 8, pp. 85–108, 1963.

[26] T. Morimoto, “Markov processes and the h-theorem,” *J. Phys. Soc. Jap.*, vol. 12, pp. 328–331, 1963.

[27] S. M. Ali and S. D. Silvey, “A general class of coefficients of divergence of one distribution from another,” *J. Roy. Statist. Soc. Ser B*, vol. 28, pp. 131–146, 1966.

[28] H. Zhu and R. Rohwer, “Information geometric measurements of generalization,” *Aston University, Tech. Rep. NCRG/4350*, 1995.

[29] A. Amari, “Information Geometry in Optimization, Machine Learning and Statistical Inference,” *Frontiers of Electrical and Electronic Engineering in China*, vol. 5, no. 3, pp. 241–260, 2010.

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