Inverse Problems

The conical Radon transform with vertices on triple line segments

Sunghwan Moon¹,³ and Markus Haltmeier²

¹ Department of Mathematics, Kyungpook National University, Daegu 41566, Republic of Korea
² Department of Mathematics, University of Innsbruck, Technikerstrasse 13, Innsbruck 6020, Austria
E-mail: sunghwan.moon@knu.ac.kr and markus.haltmeier@uibk.ac.at

Received 25 February 2020, revised 31 July 2020
Accepted for publication 11 August 2020
Published 23 October 2020

Abstract
We study the inversion of the conical Radon transform which integrates a function on the surface of a cone. The conical Radon transform recently got significant attention due to its relevance in various imaging applications such as Compton camera imaging and single scattering optical tomography. The unrestricted conical Radon transform is over-determined because the manifold of all cones depends on six variables: the center position, the axis orientation and the opening angle of the cone. In this work, we consider a particular restricted conical Radon transform using triple linear sensor of finite length where integrals over a three-dimensional set of cones are collected, determined by a one-dimensional vertex set, a one-dimensional set of central axes, and a one-dimensional set of opening angle. As the main result in this paper we derive an analytic inversion formula for the restricted conical Radon transform. Along that way we define a certain ray transform adapted to the triple line sensor for which we establish an analytic inversion formula.

Keywords: conical Radon transform, reconstruction formula, inversion, Compton camera

(Some figures may appear in colour only in the online journal)
1. Introduction

The conical Radon transform maps a function \( f : \mathbb{R}^3 \to \mathbb{R} \) in a three-dimensional space to its integrals over one-sided circular cones,

\[
Cf(u, \beta, \psi) = \int_{\mathbb{R}^3} \int_{0}^{\infty} f(u + r\alpha) r \delta(\alpha \cdot \beta - \cos \psi) dr \ dS(\alpha)
\]

for \((u, \beta, \psi) \in \mathbb{R}^3 \times S^2 \times [0, \pi]\). Here the cones over which the function is integrated are described by the vertex \( u \in \mathbb{R}^3 \), the central axis \( \beta \in S^2 \) and the opening angle \( \psi \in [0, \pi] \), and \( \delta \) denotes the one-dimensional delta-distribution. Inverting the unrestricted conical Radon transform is over-determined as \( Cf \) depends on six variables whereas the unknown function only depends on three. Various forms of conical Radon transforms arise by restricting to certain subsets of cones. Several inversion formulas for various types of conical Radon transforms have been derived in [4–6, 8–10, 15–19, 21, 23, 24, 26, 27, 30]. Also, as a special two-dimensional version of the conical Radon transform, the V-line transform is studied in [3, 14]. For recent reviews of conical Radon transforms see [2, 22, 28]. In this paper we restrict the cones of integration to a three-dimensional submanifold of conical surfaces associated to linear detectors where the vertices and the axes directions are restricted to one dimension.

Among others, inverting the conical Radon transform is relevant for Compton camera imaging. A Compton camera (also called electronically collimated \( \gamma \)-camera) has been proposed in [25, 29] for single photon emission computed tomography (SPECT) offering increased efficiency compared to a conventional \( \gamma \)-camera. A standard Compton camera consists of two planar detectors: a scatter detector and an absorption detector, positioned one behind the other. A photon emitted from a radioactive source toward the camera undergoes Compton scattering in the scatter detector, and is absorbed in the absorption detector positioned behind (see figure 1). In each detector plane, the positions \( u, u_a \) and the energy of the photon are measured. The energy difference determines the scattering angle \( \psi \) under which the photon path has been scattered at the scattering detector. Therefore, the measurements allow to conclude that photon has been emitted on a conical surface with a vertex \( u \), an axis direction pointing from \( u_a \) to \( u \) and an opening angle \( \psi \). In a similar manner, assuming a continuous source distribution of emitting photons, the Compton camera yields the conical Radon transform of the source distribution with vertices restricted to the scattering plane. The corresponding data depend on five variables.

Instead of planar Compton cameras in this paper we consider Compton cameras consisting of two parallel detector lines (left image in figure 2). Basically, the data acquisition with linear detectors is the same as in the standard one. The main difference is that the vertices are restricted to the one-dimensional scattering detector and the axes directions are restricted to the one-dimensional set of all directions pointing from the linear absorption detector to the linear scattering detector. Thus, the corresponding data depend on three variables and thus are no longer formally over-determined. Note however that still, some redundancy in the data is to be expected, similar to the classical Radon transform. Deriving corresponding range conditions is beyond the scope of the present paper. As the main theoretical result of this paper, we derive an analytic inversion formula for triple linear sensor of finite length. As shown in the right image in figure 2, the triple line sensor consists of three one-dimensional vertex sets \( \Xi_1, \Xi_2, \Xi_3 \) of finite length each associated with a one-dimensional set axis directions. Compton cameras consisting of two parallel detector lines (left image in figure 2) suggested in [16, 20] do not provide compact build up (because their ideal size is infinite) unlike the proposed triple linear sensors which consist of finite line segments. When finding the inversion formula in [16,
we take the Fourier transform with respect to the variable of vertex position. Thus the set of the vertex variable should contain an infinite line in order to prevent severe truncation artefacts. Note that in our setting, the set of the vertex variables does not contain infinite lines and thus we cannot apply the strategy in [16] to this setting. In this paper we therefore establish new inversion strategy that is able to deal with vertices on finite line segments.

In practice it may be easier to build linear detectors than planar detectors because the former requires less physical space and less complicated electronics. Moreover, to the high dimensionality of the data obtained from the planar detectors methods only using partial data have been derived (see e.g. [4–6, 10, 19, 23, 26]). On the other hand, due to the random emission of gamma-rays, Compton camera data are considerably noisy and utilizing full five-dimensional data is advised to obtain accurate reconstruction results [1, 7]. In pure form, a one-dimensional Compton camera suffers from a low SNR. However, the data of planar sensors can be grouped in data sets of several virtual linear detectors. Therefore, inversion methods for linear detectors can be applied to give several reconstructions for planar detectors that can be aggregated for noise reduction. Detailed investigations of noise characteristics is an interesting issue of future research.

The rest of this paper is organized as follows. The conical Radon transform with triple linear detector introduced in section 2. In section 3 we derive an analytic inversion formula. As main ingredient of the proof we reduce the conical Radon transform to a weighted ray transform and provide a novel inversion formula for this weighted ray transform. Section 4 is devoted to the inversion of the $n$-dimensional conical Radon transform. The paper concludes with a short summary and outlook presented in section 5.
2. The conical Radon transform

In this section, we formally define the conical Radon transform with vertices on triple lines of finite length. Let

\[ \Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3 \]  

be the set of vertices where

\[ \Xi_1 := \{(y_1,0,0) : y_1 \in [0,1]\} \]
\[ \Xi_2 := \{(0,y_2,0) : y_2 \in [0,1]\} \]
\[ \Xi_3 := \{(0,0,y_3) : y_3 \in [0,1]\}. \]

We denote by \( \psi \in [0,\pi] \) the opening angle of the circular cone and consider for each of the sets \( \Xi_j \) a different one-dimensional set of central axes. More precisely we parametrize each of the sets of central axes with \( \bar{\beta} = (\beta_1, \beta_2) \in S^1 \) and define

\[ \beta_u = \begin{cases} 
(\beta_1, 0, \beta_2) & \text{if } u \in \Xi_1 \\
(0, \beta_1, \beta_2) & \text{if } u \in \Xi_2 \\
(\beta_1, 0, \beta_2) & \text{if } u \in \Xi_3.
\end{cases} \]

Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be the distribution of the radioactivity sources. For the following it is convenient to work with \( s = \cos \psi \in [-1,1] \). We then define the conical Radon transform \( C_k f \) of a function \( f \in C(\mathbb{R}^3) \) with compact support as follows.

**Definition 1 (the conical Radon transform).** For given \( k \in \mathbb{N} \) we define the conical Radon transform \( C_k f : \Xi \times S^1 \times \mathbb{R} \to \mathbb{R} \) with vertices on triple lines by

\[ C_k f(u, \bar{\beta}, s) := \int_{\Xi} \int_{\mathbb{R}} f(u + r\alpha) d^2 \delta(\alpha \cdot \bar{\beta}_u - s) dr \, dS(\alpha) \]  

if \( s \in [-1,1] \), otherwise.

Here \( \Xi \) and \( \bar{\beta}_u \) are defined by (1)–(3), \( \delta \) is the one-dimensional delta-distribution and \( dS \) is the standard area measure on the unit sphere \( S^2 \),

\[ dS(\alpha) = \delta \left( 1 - \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} \right) \, d\alpha_1 d\alpha_2 d\alpha_3 \quad \text{for } \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3. \]

Assuming that the density of photons decreases geometrically and proportional to the distance from the source to detectors, then the data measured by a Compton camera are given by the transform \( C_1 f \). When the density decreases at a different power of distance, we need different values of \( k \), see [15].

3. Exact inversion formula

In this section we derive an explicit inversion formula for the conical Radon transform. Along that way we introduce a weighted ray transform, show how the conical Radon transform can be reduced to the weighted ray transform and derive an explicit inversion formula for the weighted
ray transform. A similar strategy for a conical Radon transform with an unrestricted central axis has already been used in [13, 16].

**Definition 2** (the weighted ray transform). Let $\Xi$ be defined by (1) (or (9) for an $n$-dimensional space) and set $\text{conv}(\Xi) := \{ x \in [0, 1]^n : \sum_{i=1}^n x_i \leq 1 \}$. We define the weighted ray transform $P_{k,f} : \Xi \times \mathbb{R}^n \to \mathbb{R}$ of a continuous compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$\forall (u, w) \in \Xi \times \mathbb{R}^n : \quad P_{k,f}(u, w) := \int_0^\infty f(u + rw)r^k \, dr.$$  

(5)

Actually, $P_{k,f}$ is the special instance of the momentum ray transforms dealt with in [11, 12]. As mentioned in [16], it is easy to check that $P_{k,f}$ is homogeneous of degree $-(k + 1)$ in the variable of $w$, i.e.

$$P_{k,f}(u, w) = |w|^{-(k + 1)}P_{k,f}(u, w/|w|)$$

for $w \neq 0$.

**Lemma 3** (reduction of the conical Radon transform to the weighted ray transform). For $f \in C_\infty(\mathbb{R}^3)$ with compact support in $\text{conv}(\Xi)$, we have

$$\forall (u, w) \in \Xi \times \mathbb{R}^3 : \quad P_{k,f}(u, w) = \frac{\mu_u}{4\pi|w|^k+2} \int_{\mathbb{S}^1} H_\beta(C_{k,f}) \left( u, \beta, \beta \cdot \frac{w_u}{|w|} \right) \, d\beta.$$  

Here $H_\beta(s) = \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{\cos(t-\beta s)}{t^2} \, dt$ is the Hilbert transform and we have written

$$\bar{w}_u := \begin{cases} (w_1, w_3) \in \mathbb{R}^2 & \text{if } u \in \Xi_1 \\ (w_2, w_3) \in \mathbb{R}^2 & \text{if } u \in \Xi_2 \\ (w_1, w_3) \in \mathbb{R}^2 & \text{if } u \in \Xi_3 \end{cases}$$

and $w_u$ for the component of $w$ missing in $\bar{w}_u$.

We omit the proof here since it can be similarly proved as theorem 5 in [16] with some minor modifications. Actually we do not need compact support in $\text{conv}(\Xi)$, but only compact support in the first octant as the assumption. This assumption is necessary because when we reduce the conical Radon transform to a weighted ray transform, we use the inversion formula for the vertical slice transform

$$V\phi(\theta, s) = \int_{\omega = s} \phi(\omega)dS(\omega) \quad \text{for } \phi \in C(\mathbb{S}^1)$$

and for its application we use the one-side compact support assumption. For more details, we refer the readers to [16].

Now we are ready to obtain the inversion formula for $C_{k,f}$.

**Theorem 4** (inversion formula for the conical Radon transform). Let $f \in C_\infty(\mathbb{R}^3)$ have compact support in $\text{conv}(\Xi)$. Then,

$$f(x) = -\frac{1}{2^2(2\pi)^3} \Delta_x \int_{\mathbb{S}^2} F_k(x, n)dS(n).$$  

(6)
Here for any point $\Lambda(x, n) \in \Xi \cap \{ y \in \mathbb{R}^3 : (x - y) \cdot n = 0 \}$, if $k > 1$

$$F_k(x, n) = \frac{1}{(k - 2)!} \left\{ \begin{array}{ll}
\int_{n \in \mathbb{R}^2 \setminus \partial \beta} \left( \int_{x \in \mathbb{R}^3} \alpha_2 (\alpha_1 - \omega) f \left( \frac{\partial_k^{-1} L_k f}{|\omega|} \right) \left( \Lambda(x, n), \beta, \beta \cdot (\alpha_1, \alpha_3) \right) \right. \\
\times dS(\beta) d\omega \ dS(\alpha) & \text{if } \Lambda(x, n) \in \Xi_1, \\
\int_{n \in \mathbb{R}^2 \setminus \partial \beta} \left( \int_{x \in \mathbb{R}^3} \alpha_2 (\alpha_1 - \omega) f \left( \frac{\partial_k^{-1} L_k f}{|\omega|} \right) \left( \Lambda(x, n), \beta, \beta \cdot (\alpha_1, \alpha_3) \right) \right. \\
\times dS(\beta) d\omega \ dS(\alpha) & \text{if } \Lambda(x, n) \in \Xi_2, \\
\int_{n \in \mathbb{R}^2 \setminus \partial \beta} \left( \int_{x \in \mathbb{R}^3} \alpha_2 (\alpha_1 - \omega) f \left( \frac{\partial_k^{-1} L_k f}{|\omega|} \right) \left( \Lambda(x, n), \beta, \beta \cdot (\alpha_1, \alpha_3) \right) \right. \\
\times dS(\beta) d\omega \ dS(\alpha) & \text{if } \Lambda(x, n) \in \Xi_3,
\end{array} \right\}
$$

and if $k = 1$

$$F_1(x, n) = \left\{ \begin{array}{ll}
\int_{n \in \mathbb{R}^2 \setminus \partial \beta} \left( \int_{x \in \mathbb{R}^3} \alpha_1 (\alpha_1 - \omega) f \left( \frac{\partial_k^{-1} L_k f}{|\omega|} \right) \left( \Lambda(x, n), \beta, \beta \cdot (\alpha_1, \alpha_3) \right) \right. \\
\times dS(\beta) d\omega \ dS(\alpha) & \text{if } \Lambda(x, n) \in \Xi_1, \\
\int_{n \in \mathbb{R}^2 \setminus \partial \beta} \left( \int_{x \in \mathbb{R}^3} \alpha_1 (\alpha_1 - \omega) f \left( \frac{\partial_k^{-1} L_k f}{|\omega|} \right) \left( \Lambda(x, n), \beta, \beta \cdot (\alpha_1, \alpha_3) \right) \right. \\
\times dS(\beta) d\omega \ dS(\alpha) & \text{if } \Lambda(x, n) \in \Xi_2, \\
\int_{n \in \mathbb{R}^2 \setminus \partial \beta} \left( \int_{x \in \mathbb{R}^3} \alpha_1 (\alpha_1 - \omega) f \left( \frac{\partial_k^{-1} L_k f}{|\omega|} \right) \left( \Lambda(x, n), \beta, \beta \cdot (\alpha_1, \alpha_3) \right) \right. \\
\times dS(\beta) d\omega \ dS(\alpha) & \text{if } \Lambda(x, n) \in \Xi_3.
\end{array} \right\}
$$

Proof. Notice that when $k \geq 2$, by the chain rule, we have for $u \in \Xi_1$

$$(\partial_{u_1}^{-1} P_k f)(u, w) = \int_{0}^{\infty} r \partial_{u_1}^{-1} f(u + rw)dr
$$

$$= \int_{0}^{\infty} \partial_{u_1}^{-1} f(u + rw) r \ dr = (\partial_{u_1}^{-1} P_k f)(u, w).$$

Similarly, we have $(\partial_{u_2}^{-1} P_k f)(u, w) = (\partial_{u_2}^{-1} P_k f)(u, w)$ for $u \in \Xi_2$ and $(\partial_{u_3}^{-1} P_k f)(u, w) = (\partial_{u_3}^{-1} P_k f)(u, w)$ for $u \in \Xi_3$. Together with Cauchy’s formula for repeated integration we obtain

$$P_k f(u, w) = \frac{1}{(k - 2)!} \left\{ \begin{array}{ll}
\int_{-\infty}^{w_1} (w_1 - \omega)^{k-2}(\partial_{u_1}^{-1} P_k f)(u, \omega, w_2, w_3) d\omega & \text{if } u \in \Xi_1, \\
\int_{-\infty}^{w_2} (w_2 - \omega)^{k-2}(\partial_{u_2}^{-1} P_k f)(u, w_1, \omega, w_3) d\omega & \text{if } u \in \Xi_2, \\
\int_{-\infty}^{w_3} (w_3 - \omega)^{k-2}(\partial_{u_3}^{-1} P_k f)(u, w_1, w_2, \omega) d\omega & \text{if } u \in \Xi_3.
\end{array} \right\}
$$

(7)

Recall that $Rf$ be the three-dimensional regular Radon transform, i.e.

$$Rf(n, s) = \int_{\{x : n \cdot x = s\}} f(x) dx \quad \text{for} \quad (n, s) \in \mathbb{S}^2 \times \mathbb{R}.$$
Then, using the polar coordinates, one can easily verify that
\[
Rf(n, x \cdot n) = \int_0^{\infty} \int_{n \perp \mathbb{S}^2} f(\Lambda(x, n) + r\alpha) r \, dS(\alpha) \, dr = \int_{n \perp \mathbb{S}^2} P_1 f(\Lambda(x, n), \alpha) \, dS(\alpha). \tag{8}
\]
If \( k = 1 \), (8) with lemma 3 yields
\[
Rf(n, x \cdot n) = \frac{1}{4\pi} F_1(x, n).
\]
If \( k > 1 \), (8) with (7) and lemma 3 yields
\[
Rf(n, x \cdot n) = \frac{1}{(k-2)!} \begin{cases} 
\int_{n \perp \mathbb{S}^2} \int_{-\infty}^{\alpha_1} (\alpha_1 - \omega)^{-k} (\partial_{\alpha_1}^{k-1} F_1 f)(\Lambda(x, n), \omega, \alpha_2, \alpha_3) \, d\omega \, dS(\alpha) \\
\int_{n \perp \mathbb{S}^2} \int_{-\infty}^{\alpha_2} (\alpha_2 - \omega)^{-k} (\partial_{\alpha_2}^{k-1} F_1 f)(\Lambda(x, n), \alpha_1, \omega, \alpha_3) \, d\omega \, dS(\alpha) \\
\int_{n \perp \mathbb{S}^2} \int_{-\infty}^{\alpha_3} (\alpha_3 - \omega)^{-k} (\partial_{\alpha_3}^{k-1} F_1 f)(\Lambda(x, n), \alpha_1, \alpha_2, \omega) \, d\omega \, dS(\alpha)
\end{cases}
\]

Applying the well-known inversion formula for \( Rf \), i.e. for \( x \in \mathbb{R}^3 \),
\[
f(x) = -\frac{1}{8\pi^2} \Delta_x \left( \int_{\mathbb{S}^2} Rf(n, x \cdot n) \, dS(n) \right),
\]
we have our assertion. \( \square \)

**Remark 5 (generalization to different vertex sets).** We point out that an inversion formula similar to (6) can also be derived for other arrangements of triple line detectors of finite length. In such a case, one reconstructs a function \( f \in C^\infty(\mathbb{R}^3) \) with compact support in a certain set \( K \) depending on \( \Xi \) by deriving generalizations of lemma 3 and theorem 4. For such results, the following condition has to be satisfied: for every \( x \in K \), every plane passing through \( x \) intersects the vertex set \( \Xi \).

### 4. The \( n \)-dimensional conical Radon transform with \( n \geq 3 \)

In this section, we formally define the \( n \)-dimensional conical Radon transform with vertices on \( n \)-multiple lines. For \( n \geq 3 \), let
\[
\Xi = \bigcup_{i=1}^n \Xi_i \tag{9}
\]
be the set of vertices where
\[ \Xi_i := \{(0, \ldots, 0, y_i, 0, \ldots, 0) : y_i \in [0, 1]\}. \tag{10} \]

We parametrize each of the sets of central axes with \( \beta = (\beta_1, \beta_2, \ldots, \beta_{n-1}) \in S^{n-2} \) and define
\[
\beta_u = \begin{cases} 
(\beta_1, 0, \beta_2, \beta_3, \ldots, \beta_{n-1}) & \text{if } u \in \Xi_1, \\
(0, \beta_1, \beta_2, \ldots, \beta_{n-1}) & \text{if } u \in \Xi_2, \\
\vdots & \\
(\beta_1, \ldots, \beta_{n-2}, 0, \beta_{n-1}) & \text{if } u \in \Xi_n. 
\end{cases} \tag{11} 
\]

For \( f : \mathbb{R}^n \to \mathbb{R} \), we define the conical Radon transform \( C_k f \) of a function \( f \in C(\mathbb{R}^n) \) with compact support as follows.

**Definition 6 (the \( n \)-dimensional conical Radon transform).** For a given \( k \in \mathbb{N} \) we define the conical Radon transform \( C_k f : \Xi \times S^{n-2} \times \mathbb{R} \to \mathbb{R} \) with vertices on \( n \)-multiple lines by
\[
C_k f(u, \beta, s) := \begin{cases} 
\int_0^\infty \int_0^\infty f(u + r\alpha) \delta(\alpha \cdot \beta_u - s) \, dS(\alpha) & \text{if } s \in [-1, 1], \\
0 & \text{otherwise.} 
\end{cases} \tag{12} 
\]

Here \( \Xi \) and \( \beta_u \) are defined by (9)–(11), \( \delta \) is the one-dimensional delta-distribution and \( dS \) is the standard area measure on the unit sphere \( S^{n-1} \).

We now extend Lemma 3 to the \( n \)-dimensional case.

**Lemma 7 (reduction of the \( n \)-dimensional conical Radon transform to the weighted ray transform).** For \( f \in C^\infty(\mathbb{R}^n) \) with compact support in \( \text{conv}(\Xi) \), we have for all \( (u, w) \in \Xi \times \mathbb{R}^d \),
\[
P_k f(u, w) = \frac{w_u}{2(2\pi)^{n-2}|w|^{k+2}} \begin{cases} 
(1) \int_0^{\infty} H_s \partial_s^{n-2}(C_k f) \left( u, \beta, \frac{w_u}{|w|} \right) \, dS(\beta) & \text{if odd}, \\
-1 \int_0^{\infty} \partial_s^{n-2}(C_k f) \left( u, \beta, \frac{w_u}{|w|} \right) \, dS(\beta) & \text{if even}. 
\end{cases} 
\]

Here \( H_s f(s) = \frac{1}{\pi} \int_0^{\infty} \frac{f(t)}{t^2 + s^2} \, dt \) is the Hilbert transform and we have written
\[
w_u := \begin{cases} 
(u_1, w_1, u_2, \ldots, w_n) \in \mathbb{R}^{n-1} & \text{if } u \in \Xi_1, \\
(u_2, w_2, u_3, \ldots, w_n) \in \mathbb{R}^{n-1} & \text{if } u \in \Xi_2, \\
\vdots & \\
(u_{i-1}, w_{i-1}, u_i, w_{i+1}, \ldots, w_n) \in \mathbb{R}^{n-1} & \text{if } u \in \Xi_i, \\
\vdots & \\
(u_1, w_2, \ldots, w_{n-2}, w_n) \in \mathbb{R}^{n-1} & \text{if } u \in \Xi_n. 
\end{cases} 
\]
and $w_u$ for the component of $w$ missing in $\bar{w}_u$.

With the same reason as for the three-dimensional case, we omit the proof.

**Theorem 8 (inversion formula for the $n$-dimensional conical Radon transform).**

Let $k \geq n-2$ and $f \in C^\infty(\mathbb{R}^n)$ have compact support in $\text{conv}(\Xi)$. Then we have

$$f(x) = \frac{1}{2^k(2\pi)^{n-1}} (-\Delta)^{\frac{k+1}{2}} \int_{S^{n-1}} F_k(x, n) dS(n),$$

where $F_k(x, n)$, $k > n-2$ is defined by

(n odd)

$$\left\{ \begin{array}{l}
\frac{(-1)^{\frac{m}{2}}}{(k-(n-1))!} \\
\times \int_{\mathbb{R}^n} \int_{-\infty}^{\alpha_1} \int_{\mathbb{R}^2} \int_{-\infty}^{\alpha_2} \left( \frac{\alpha_2(\alpha_1 - \omega)^{k-(n-1)}}{(\omega, \alpha_2, \alpha_3, \ldots, \alpha_n)^{k+2}} \right) \\
\times (f^H_1(\alpha)(-2) f^{\bar{H}}_1(-2) C f) \left( \Lambda(x, n), \vec{\beta}_k \beta_{\Lambda(x,n)} \right) \\
\times dS(\vec{\beta}) d\omega \ dS(\alpha) \quad \text{if } \Lambda(x, n) \in \Xi_1
\end{array} \right.$$
and \( \Lambda(x, n) \) is any point in \( \Xi \cap \{ y \in \mathbb{R}^n : (x - y) \cdot n = 0 \} \). Also, \( F_{n-2}(x, n) \) is equal to

\[
(\text{n odd}) \quad \left\{ \begin{array}{ll}
\int \int_{n^{-1} \mathbb{S}^{n-2}} & \alpha_2(H_i \partial^{n-2} \mathcal{C}_{n-2} f) \left( \Lambda(x, n), \bar{\beta}, \beta_{\Lambda(x,n)} \cdot \alpha \right) dS(\bar{\beta}) dS(\alpha) \\
& \text{if } \Lambda(x, n) \in \Xi_1 \\
\int \int_{n^{-1} \mathbb{S}^{n-2}} & \alpha_{i-1}(H_i \partial^{n-2} \mathcal{C}_{n-2} f) \left( \Lambda(x, n), \bar{\beta}, \beta_{\Lambda(x,n)} \cdot \alpha \right) dS(\bar{\beta}) dS(\alpha) \\
& \text{if } \Lambda(x, n) \in \Xi_i, \quad i = 2, 3, \ldots, n
\end{array} \right.
\]

\[
(\text{n even}) \quad \left\{ \begin{array}{ll}
\int \int_{n^{-1} \mathbb{S}^{n-2}} & \alpha_2(H_i \partial^{n-2} \mathcal{C}_{n-2} f) \left( \Lambda(x, n), \bar{\beta}, \beta_{\Lambda(x,n)} \cdot \alpha \right) dS(\bar{\beta}) dS(\alpha) \\
& \text{if } \Lambda(x, n) \in \Xi_1 \\
\int \int_{n^{-1} \mathbb{S}^{n-2}} & \alpha_{i-1}(H_i \partial^{n-2} \mathcal{C}_{n-2} f) \left( \Lambda(x, n), \bar{\beta}, \beta_{\Lambda(x,n)} \cdot \alpha \right) dS(\bar{\beta}) dS(\alpha) \\
& \text{if } \Lambda(x, n) \in \Xi_i, \quad i = 2, 3, \ldots, n
\end{array} \right.
\]

**Proof.** Notice that like the three-dimensional case by the chain rule, we have for \( u \in \Xi \) and \( k > n - 2 \),

\[
(\partial_u^{k-(n-2)} \mathcal{P}_{n-2} f)(u, w) = \int_0^\infty r^{n-2} \partial_u^{k-(n-2)} f(u + rw) dr = \int_0^\infty \partial_u^{k-(n-2)} f(u + rw) r^k dr = (\partial_u^{k-(n-2)} \mathcal{P}_k f)(u, w).
\]

Cauchy’s formula for repeated integration gives for \( u \in \Xi_i \),

\[
\mathcal{P}_{n-2} f(u, w) = \frac{1}{(k - (n-1))!} \int_0^{w_i} \cdots \int_0^{w_{n-1}} (w_1 - w)^{k-(n-1)} \\
\times (\partial_u^{k-(n-2)} \mathcal{P}_k f)(u, w_1, w_2, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_n) dw.
\]

Using the spherical coordinates like the three-dimensional case, one can easily verify that if \( \Lambda(x, n) \in \Xi_i \),

\[
R f(n, x \cdot n) = \int_{(x \cdot n) = 0} f(x) dx = \int_{n^{-1} \mathbb{S}^{n-1}} \int_0^\infty f(\Lambda(x, n) + r\alpha) r^{n-2} dr dS(\alpha) \\
= \int_{n^{-1} \mathbb{S}^{n-1}} \mathcal{P}_{n-2} f(\Lambda(x, n), \alpha) dS(\alpha).
\]
Also, we notice that \( \bar{\beta} \cdot \bar{w} \Lambda(x, n) = \beta \Lambda(x, n) \cdot w \). If \( k = n - 2 \), (15) with lemma 7 gives

\[
Rf(n, x \cdot n) = \frac{1}{2(2\pi)^{n-2}} F_{n-2}(x, n).
\]

If \( k > n - 2 \), (15) with (14) and lemma 7 gives

\[
Rf(n, x \cdot n) = \frac{1}{(k - (n - 1))!} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{1}{\omega^{k-1}} \left( \partial_{\omega}^{k-1} \int_{\mathbb{R}^{n-1}} Rf(n, x \cdot n) dS(n) \right) d\omega \ dS(\alpha)
\]

\[
= \frac{1}{2(2\pi)^{n-2}} F_{k}(x, n).
\]

Applying the well-known inversion formula for \( Rf \), i.e., for \( x \in \mathbb{R}^n \),

\[
f(x) = \frac{1}{2(2\pi)^{n-1}} \left( -\Delta_{x} \right)^{\frac{n}{2}} \left( \int_{\mathbb{R}^{n-1}} Rf(n, x \cdot n) dS(n) \right),
\]

we have our assertion. \( \square \)

**Remark 9.** If \( k < n - 2 \), we can get the inversion formula by applying the strategy of theorem 8 with

\[
(\partial_{u_{i}}^{(n-2)-k} P_{n-2} f)(u, w) = \int_{0}^{\infty} \rho^{n-2-k} \partial_{u_{i}}^{(n-2)-k} f(u + rw) dr
\]

\[
= \int_{0}^{\infty} \partial_{u_{i}}^{(n-2)-k} f(u + rw)^{\rho} dr = (\partial_{u_{i}}^{(n-2)-k} P_{k} f)(u, w)
\]

and

\[
P_{n-2} f(u, w) = \frac{1}{((n-3) - k)!} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} ((u_{i} - u)^{(n-3)-k} (\partial_{u_{i}}^{(n-2)-k} P_{k} f)
\]

\[
\times (u_{1}, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_{n}, w) du,
\]

instead of (13) and (14), respectively.

### 5. Conclusion

In this paper we derived an explicit inversion formula for inverting the conical Radon transform with vertices on triple line segments. The considered geometry does not use formally over-determined data and uses a bounded vertex set. As a main auxiliary result we derived an inversion formula for a weighted ray transform adjusted to the triple linear detector. While the used data was motivated by SPECT imaging with one-dimensional Compton cameras our results are applicable to other settings as well. An extension of our results to higher dimension has been provided. In future work we will investigate the numerical implementation of the derived inversion approach and compare with other inversion methods.
Acknowledgment

The authors are thankful to the referees for multiple suggestions that helped to improve this paper. The work of SM was supported by the National Research Foundation of Korea Grant funded by the Korea Government (MSIP) (2018R1D1A3B07041149). The work of MH has been supported by the Austrian Science Fund (FWF), project P 30747-N32.

ORCID iDs

Sunghwan Moon https://orcid.org/0000-0003-3384-0391

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