On the Hasse Principle for the Brauer group of a purely transcendental extension field in one variable over an arbitrary field

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Abstract

In this paper we show the Hasse principle for the Brauer group of a purely transcendental extension field in one variable over an arbitrary field.

1 Introduction

For a field $k$, let $k_s$ be the separable closure of $k$ and $\bar{k}$ the algebraic closure of $k$. Let $K$ be a global field (i.e., an algebraic number field or an algebraic function field of transcendental degree one over a finite field), $S$ the set of all primes of $K$ and $\widehat{K}_p$ the completion of $K$ at $p \in S$. For a ring $A$, let $\text{Br}(A)$ be the Brauer group of $A$ (see [6, p.141, IV, §2]). Then, the local-global map

$$\text{Br}(K) \to \prod_{p \in S} \text{Br}(\widehat{K}_p)$$

is injective (see [5, Theorem 8.42 (2)]). We call a statement of this form the Hasse principle. It is also known that the Hasse principle holds if $K$ is a purely transcendental extension field in one variable over a perfect field $k$ (see [8]). We show that it also holds without any assumption on $k$. The following is our main theorem.

Theorem 3.5. Let $k$ be an arbitrary field, $k(t)$ the purely transcendental extension field in one variable $t$ over $k$ and $\widehat{k(t)}_p$ the quotient field of the completion of $\mathcal{O}_{\widehat{k(t)}, p}$. Then, the local-global map

$$\text{Br}(k(t)) \to \prod_{p \in \mathbb{P}_k^1 \atop \text{ht}(p)=1} \text{Br}(\widehat{k(t)}_p)$$

is injective.

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Moreover, if \( k \) is a separably closed field, the Hasse principle for the Brauer group of any algebraic function fields in one variable over \( k \) is shown by using [2, Corollaire (5.8)] as in the case of Theorem 3.5.

For the difference between the case of perfect fields and Theorem 3.5, see Remark 3.7.

## 2 Notations

For a field \( k \) and a Galois extension field \( k' \) of \( k \), \( G(k'/k) \) denotes the Galois group of \( k'/k \) and \( k_s \) denotes the separable closure of \( k \). We denote \( G(k_s/k) \) by \( G_k \) and the category of (discrete) \( G_k \)-modules (cf. [7, p.10, I]) by \( G_k \)-mod. For a discrete \( G(k'/k) \)-module \( A \) (but the action is continuous) and a positive integer \( q \), \( H^q(k'/k, A) \) denotes the \( q \)-th cohomology group of \( G(k'/k) \) with coefficients in \( A \) (see [7, p.10, I, \( \S \)]). We put \( H^q(k, A) = H^q(k_s/k, A) \). \( \text{Res} : H^q(k, A) \to H^q(k', A) \) denotes the restriction homomorphism. For a group \( G \), we put \( G_q = \{ g \in G \mid g^q = 1 \} \) and \( X(G) \) the group of characters of \( G \).

For a scheme \( X \), \( X^{(i)} \) is the set of points of codimension \( i \) and \( X_{(i)} \) is the set of points of dimension \( i \). We denote the étale site (resp. finite étale site) on \( X \) by \( X_{et} \) (resp. \( X_{fet} \)) and the category of sheaves over \( X_{et} \) (resp. \( X_{fet} \)) by \( \mathscr{S}_{Xet} \) (resp. \( \mathscr{S}_{Xfet} \)). For \( \mathcal{F} \in \mathscr{S}_{Xet} \) (resp. \( \mathscr{S}_{Xfet} \)), we denote the \( q \)-th cohomology group of \( X_{et} \) (\( X_{fet} \)) with values in \( \mathcal{F} \) by \( H^q_{et}(X, \mathcal{F}) \) (resp. \( H^q_{fet}(X, \mathcal{F}) \)). If \( Y \subset X \) is a closed subscheme, we denote the \( q \)-th local (étale) cohomology with support in \( Y \) by \( H^q_Y(X, \mathcal{F}) \). For an integral scheme \( X \) and \( p \in X^{(1)} \), let \( R(X) \) be the function field of \( X \), \( \mathcal{O}_{X, p} \) the local ring at \( p \) of \( X \), \( \tilde{O}_{X, p} \) the completion of \( \mathcal{O}_{X, p} \), \( \tilde{R}(X)_p \) its quotient field, \( \tilde{O}_{X, \tilde{p}} \) the Henselization of \( \mathcal{O}_{X, p} \), \( \tilde{R}(X)_{\tilde{p}} \) its quotient field, \( \mathcal{O}_{X, \tilde{p}} \) the strictly Henselization of \( \mathcal{O}_{X, p} \) and \( R(X)_{\tilde{p}} \) its quotient field.

## 3 Main theorem

### Theorem 3.1

Let \( X \) be a 1-dimensional connected regular scheme, \( K \) its quotient field. Then

\[
0 \longrightarrow \text{Br}(X) \longrightarrow \text{Br}(K) \longrightarrow \prod_{p \in X^{(1)}} \text{Br}(\tilde{R}(X)_p)/\text{Br}(\tilde{O}_{X, p}) \tag{1}
\]

is exact.

**Proof.** Suppose that \( B \) is a discrete valuation ring, \( L \) is its quotient field, \( Y = \text{Spec} \ B \) and \( Z = Y \setminus \text{Spec} \ L = \{ \mathfrak{p} \} \). Then we have the exact sequence

\[
H^p(Y, \mathcal{G}_m) \to H^p(\text{Spec} \ L, \mathcal{G}_m) \to H^{p+1}_Z(Y, \mathcal{G}_m) \tag{2}
\]

by [6, p.92, III, Proposition 1.25] and \( H^2(Y, \mathcal{G}_m) \to H^2(\text{Spec} \ L, \mathcal{G}_m) \) is injective by [6, p.145, IV, \( \S \)]. Moreover we have

\[
H^p_Z(Y, \mathcal{G}_m) \simeq H^p_{\mathfrak{p}}(\text{Spec} \ (\tilde{O}_{Y, \mathfrak{p}}), \mathcal{G}_m) \tag{3}
\]
by [6] p.93, III, Corollary 1.28]. Moreover, the diagram

\[
\begin{array}{ccc}
\text{Br}(K)/\text{Br}(\mathcal{O}_{X,p}) & \xrightarrow{\sim} & \text{Br}(\widetilde{R}(X)_p)/\text{Br}(\widetilde{O}_p) \\
\downarrow & & \downarrow \\
H^3_{(p)}(\text{Spec}(\mathcal{O}_{X,p}), \mathbb{G}_m) & \xrightarrow{\text{ef}(3)} & H^3_{(p)}(\text{Spec}(\widetilde{O}_{X,p}), \mathbb{G}_m)
\end{array}
\]

is commutative. Therefore

\[
\text{Br}(K)/\text{Br}(\mathcal{O}_{X,p}) \to \text{Br}(\widetilde{R}(X)_p)/\text{Br}(\widetilde{O}_{X,p})
\]

is injective. So the statement follows from [2] p.77, II, Proposition 2.3. \hfill \Box

**Lemma 3.2.** Let \( A \) be a Henselian discrete valuation ring, \( K \) its quotient field, \( k \) its residue field and \( K_{nr} \) its maximal unramified extension. Then

\[
H^p(\text{Spec}(A), g_*(\mathbb{G}_m)) = H^p(K_{nr}/K, (K_{nr})^*)
\]

for any \( p > 0 \) and the sequence

\[
0 \to H^p(\text{Spec}(A), \mathbb{G}_m) \to H^p(K_{nr}/K, (K_{nr})^*) \to H^p(k, \mathbb{Z}) \to 0
\]

is exact.

**Proof.** Let \( i: \text{Spec}(k) \to \text{Spec}(A) \) be the natural map. Then, \( i_* \) is exact. Let (set) be the class of all separated etale morphisms and \( f: X_{et} \to X_{set} \) the continuous morphism which is induced by identity map on \( X \). Then \( f_* \) is exact by [6] p.112, (b) of Examples 3.4]. Let (\( fet \)) be the class of all finite etale morphisms and \( f': X_{set} \to X_{fet} \) the continuous morphism which is induced by identity map on \( X \).

Let \( Y \to X \) be a separated etale morphism with \( Y \) connected, \( R(Y) \) the ring of rational functions of \( Y \), \( A \to B \) the normalization of \( A \) in \( R(Y) \) and \( X' = \text{Spec}(B) \). Then \( R(Y)/K \) is a finite separable extension and \( Y \) is an open subscheme of \( X' \) by [6] p.29, I, Theorem 3.20]. Moreover \( X' \to X \) is finite by [6] p.4, I, Proposition 1.1]. Then, since \( A \) is a Henselian discrete valuation ring, \( B \) is a Henselian discrete valuation ring by [6] p.33, I, (b) of Theorem 4.2] and [6] p.34, I, Corollary 4.3]. Also \( R(X')/R(X) \) is an unramified extension. Therefore \( f'_* \) is exact by [6] p.111, III, Proposition 3.3]. So \( f'_* \circ f_* \) is exact and

\[
H^p_{fet}(X, (f' \circ f)_*(\mathcal{F})) \simeq H^p_{et}(X, \mathcal{F})
\]

for any \( \mathcal{F} \in \mathbb{S}_{X_{et}} \).

We have the isomorphism \( G_{K-\text{mod}} \simeq \mathbb{S}_{\text{Spec}(K)_{et}} \) by [6] p.53, II.§1,Theorem1.9]. Let the functor \( N \) be defined as

\[
(G_{K-\text{mod}}) \ni M \mapsto M^\text{Gal}(K_{s}/K_{nr}) \in (G_{k-\text{mod}})
\]
and $N' : \mathcal{S}_{\text{Spec}(K)_{et}} \to \mathcal{S}_{\text{Spec}(k)_{et}}$ the functor which corresponds to $N$. Let $Y'' \in X_{fet}$ be connected. Moreover, let $K'' = R(Y'')$ and $k''$ the finite extension field of $k$ which corresponds to the closed point of $Y''$. Then

$$N'(F)(\text{Spec}(k'')) = F(\text{Spec}(K''))$$

for $F \in \mathcal{S}_{\text{Spec}(K)_{et}}$ because

$$G(K_{nr}/K'') \simeq G_{k''}, \quad G(K_{nr}/K'') \simeq G_{K'/G_{K_{nr}}}.$$

Therefore the diagram

$$
\begin{array}{ccc}
G_{K-\text{mod}} & \cong & \mathcal{S}_{\text{Spec}(K)_{et}} \\
& N & \downarrow \\
G_k-\text{mod} & \cong & \mathcal{S}_{\text{Spec}(k)_{et}}
\end{array}
$$

is commutative. So

$$H^p_{et}(X, g_*(\mathbb{G}_m)) = H^p_{fet}(X, f' \circ f \circ g_*(\mathbb{G}_m))$$

$$= H^p_{fet}(X, f' \circ f \circ i_*(N'(\mathbb{G}_m)))$$

$$= H^p_{et}(X, i_*(N'(\mathbb{G}_m)))$$

$$= H^p_{et}(\text{Spec}(k), N'(\mathbb{G}_m))$$

$$= H^p(k, (K_{nr})^*) = H^p(K_{nr}/K, (K_{nr})^*).$$

If we want to show where we consider the sheaf $\mathbb{G}_m$, we use the notation such as $\mathbb{G}_{m,A}$. Then the exact sequence (4) follows from the exact sequence of sheaves

$$0 \to \mathbb{G}_{m,A} \to g_*(\mathbb{G}_{m,K}) \to i_*(\mathbb{Z}) \to 0$$

(cf, [6] p.106, III, Example 2.22). So the proof is complete.

\[ \square \]

**Corollary 3.3.** Consider the situation of Theorem 3.1 and

$$\text{Br}_{un}(X) = \text{Ker} \left( \text{Br}(K) \xrightarrow{\text{Res}} \prod_{p \in X_{(0)}} \text{Br}(\widetilde{R(X)}_{\bar{p}}) \right).$$

Then the sequence

$$0 \to \text{Br}(X) \to \text{Br}_{un}(X) \to \prod_{p \in X^{(1)}} X(G_{\kappa(p)})$$

is exact.
Proof. It follow from \[2, \text{p.76, II, Corollaire 2.2}\] and \[6, \text{p.147, IV, Proposition 2.11 (b)}\] that $\text{Br}(\mathcal{O}_{X,p}) \subset \text{Br}_{un}(\text{Spec}(\mathcal{O}_{X,p}))$. So the sequence

$$0 \rightarrow \text{Br}(\mathcal{O}_{X,p}) \rightarrow \text{Br}_{un}(\text{Spec}(\mathcal{O}_{X,p})) \rightarrow \text{Br}(\widetilde{R}(X)_p)/\text{Br}(\widetilde{O}_{X,p})$$

is exact by Theorem 3.1. Moreover, $\text{Br}(\widetilde{R}(X)_p)/\text{Br}(\widetilde{O}_{X,p}) \simeq X(G_{\kappa(p)})$ by Lemma 3.2. Therefore the sequence

$$0 \rightarrow \text{Br}(\mathcal{O}_{X,p}) \rightarrow \text{Br}_{un}(\text{Spec}(\mathcal{O}_{X,p})) \rightarrow X(G_{\kappa(p)})$$

(6)

is exact. So the statement follows from (6) and \[2, \text{p.77, II, Proposition 2.3}\].

Remark 3.4. 1. Suppose that $X$ is a regular algebraic curve over a field $k$. If $k$ is perfect, $\text{Br}_{un}(X) = \text{Br}(K)$ by \[7, \text{p.80, II, 3.3}\]. If $(n, \text{ch}(k)) = 1$, $\text{Br}_{un}(X)_n = \text{Br}(K)_n$ by \[7, \text{p.111, Appendix, §2, (2.2)}\].

2. Corollary 3.3 is true even if $\dim X \neq 1$ because

$$\text{H}^2(X, g_*(\mathbb{G}_m, K)) = \text{Ker} \left( \text{Br}(K)^{\text{Res}} \xrightarrow{\prod_{x \in X(0)}} \text{Br}(K_x) \right)$$

where $g : \text{Spec } K \rightarrow X$ is the generic point of $X$.

Theorem 3.5. Let $k$ be an arbitrary field $k$ and $k(x)$ the purely transcendental extension field in one variable $x$ over $k$. Then, the local-global map

$$\text{Br}(k(x)) \rightarrow \prod_{p \in \mathbb{P}_k^1(1)} \text{Br}(\widetilde{k(x)}_p)$$

is injective.

Proof. By using the facts \[11, \text{proof of Theorem 1}\] and \[23, \text{p.674, §3.4, Lemma 16}\], we see that $\text{Br}(\widetilde{k(x)}_p) \simeq \text{Br}(k(x)_p)$. So it is sufficient for the proof of the statement to prove that

$$\text{Br}(k(x)) \rightarrow \prod_{p \in \mathbb{P}_k^1(1)} \text{Br}(\widetilde{k(x)}_p)$$

is injective. We denote the point which corresponds to $(\frac{1}{x}) \in \text{Spec}(k[\frac{1}{x}]) \subset \mathbb{P}_k^1$ by $\infty$. Then, by Theorem 3.1

$$\text{Ker} \left( \text{Br}(k(x)) \rightarrow \prod_{p \in \mathbb{P}_k^1(1)} \text{Br}(\widetilde{k(x)}_p) \right)$$

$$\subset \text{Ker} \left( \text{Br}(k(x)) \rightarrow \prod_{p \in (\mathbb{P}_k^1(1) \setminus \infty)} \text{Br}(\widetilde{\mathbb{P}_k^1(1)})/\text{Br}(\widetilde{\mathcal{O}_{\mathbb{P}_k^1(1)}}) \right)$$

$$= \text{Br}(k[x]).$$
Moreover
\[
\ker \left( \text{Br}(k(x)) \to \prod_{p \in \mathbb{P}^1_k} \text{Br}(\mathbb{K}(x)_{\wp}) \right) \subset \ker \left( \text{Br}(k[x]) \to \text{Br}(k(x)) \to \text{Br}(\mathbb{P}^1_k) \right)
\]
and \(\ker \left( \text{Br}(k[x]) \to \text{Br}(k(x)) \to \text{Br}(\mathbb{P}^1_k) \right) = 0\) by \cite{6} p.153, IV, Exercise 2.20 (d) or \cite{9}. Therefore
\[
\ker \left( \text{Br}(k(x)) \to \prod_{p \in \mathbb{P}^1_k} \text{Br}(\mathbb{K}(x)_{\wp}) \right) = 0.
\]
So the statement follows. \(\square\)

**Corollary 3.6.** Let \(X\) be an algebraic curve over a separably closed field such that regular and proper. Then, the local-global map
\[
\text{Br}(R(X)) \to \prod_{p \in X^{(1)}} \text{Br}(R(X)_p)
\]
is injective.

**Proof.** The statement follows from Theorem 3.1 and \cite{2} III, Corollary 5.8. \(\square\)

**Remark 3.7.** If \(k\) is perfect, Theorem 3.5 is proved by using the exact sequence
\[
0 \to \text{Br}(\mathbb{P}^1_k) \to \text{Br}(k(x)) \to \bigoplus_{p \in \mathbb{P}^1_k} X(G_{\kappa(p)}) \quad (7)
\]
in \cite{8}. But it is unknown fact whether (7) is exact or not in the case where \(k\) is not perfect and Theorem 3.5 has not been proved. The sequence (5) is exact in Corollary 3.3 but the sequence (7) is not exact in the case where \(k\) is not perfect as follows.

It is known that \(k\) is perfect if and only if \(\text{Br}(k) = \text{Br}(k[x])\) (cf, \cite{11} p.389, Theorem 7.5). So \(\text{Br}(k[x]) \neq 0\) in the case where \(k\) is the separable closure of an imperfect field and \(\text{Br}(k(x)) \neq 0\) because \(\text{Br}(k[x]) \subset \text{Br}(k(x))\). On the other hand, \(X(G_{\kappa(p)}) = \{1\}\) and \(\text{Br}(\mathbb{P}^1_k) = \text{Br}(k) = \{0\}\). So the sequence (7) is not exact.

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