Igal Talmi’s scientific career

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Abstract. Igal Talmi’s scientific career started in the early years of the shell model. He was responsible for much of the standard machinery and basic approaches of the shell model and his ideas provided insights relevant to the microscopic foundations of the interacting-boson model. His papers were supplemented by two comprehensive books which enlightened practitioners of both models.

Igal Talmi received the degree of Dr.Sc.Nat. at E.T.H., Zurich, in 1952. His first published paper appeared in Phys.Rev. 82 (1951) 101. So the beginning of his scientific career coincided with the early years of the nuclear shell model, which he made the centre of his research.

The shell model is based on a set of single-particle orbitals \( \{ \phi_n \} \) plus a residual two-body interaction \( V(r_{12}) \). The basic input to any calculation involved the two-body matrix element

\[
\int \int \phi^*_n(\vec{r}_1)\phi^*_b(\vec{r}_2)V(|\vec{r}_1 - \vec{r}_2|)\phi_c(\vec{r}_1)\phi_d(\vec{r}_2)d^3 \vec{r}_1 d^3 \vec{r}_2
\]

and it was necessary to simplify the double integral. The accepted approach in the parallel atomic shell model, as espoused by Slater, was to expand the interaction in terms of products of functions of \( \vec{r}_1 \) and \( \vec{r}_2 \). Igal proposed instead to transform the double integral to one over relative and centre of mass coordinates, \( \vec{r} = \vec{r}_1 - \vec{r}_2 \) and \( \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2) \). The centre-of-mass integration would then be trivial and only a single three-dimensional integral would remain.

Only plane waves admit the double factorisation

\[
e^{i\vec{k}_1 \cdot \vec{r}_1}e^{i\vec{k}_2 \cdot \vec{r}_2} = e^{i\vec{k} \cdot \vec{r}}e^{i\vec{R} \cdot \vec{R}}
\]

with \( \vec{k} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2) \) and \( \vec{K} = \vec{k}_1 + \vec{k}_2 \). But harmonic oscillator wavefunctions admit an almost equally simple factorisation, because the two-particle oscillator Hamiltonian satisfies

\[
H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 r_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 r_2^2 = \frac{p^2}{2(m/2)} + \frac{1}{2}(m/2)\omega^2 \vec{r}^2 + \frac{\vec{P}^2}{2(2m)} + \frac{1}{2}(2m)\omega^2 \vec{R}^2,
\]

implying

\[
E = (2n_1 + \ell_1 + 2n_2 + \ell_2 + 3)\hbar\omega = (2n + \ell + 2N + L + 3)\hbar\omega
\]

and hence

\[
|n_1\ell_1n_2\ell_2; \lambda\nu\rangle = \sum_{n\ell NL} \langle n\ell NL; \lambda|n_1\ell_1n_2\ell_2; \lambda\rangle |n\ell NL; \lambda\nu\rangle,
\]

in an obvious notation, where the product of wavefunctions is coupled to total angular momentum \( \lambda \), projection \( \mu \). This is the Talmi transformation, though the transformation...
coefficients are universally known as Brody-Moshinsky brackets, in honour of their heroic
compilation. For given \( n_1, \ell_1, n_2, \ell_2, \lambda \), the sum over \( n, \ell, N, L \) is limited by the
energy condition, by the triangle condition \(|\ell - L| \leq \lambda \leq \ell + L\) and by the parity condition
\((-1)_{\ell_1+\ell_2} = (-1)^{\ell + L}\), so there are generally only a few terms in the expansion.

With the Talmi transformation in hand, the two-body matrix elements involve a simple centre-of-mass integral, an angular integral which can generally be performed by means of Racah algebra and a radial integral. The last can be simplified further, using the properties of the well-known harmonic oscillator wavefunctions, and expressed in terms of a finite set of Talmi integrals, which involve only the interaction \( V(r) \) and the \( n = 1 \) radial wavefunctions. This set of tools allowed the explicit evaluation of two-body matrix elements for a given form of the two-body interaction.

An important early application of this shell-model technology was to the very slow \( \beta \)-decay of \( ^{14}C \) to \( ^{14}N \), which could not be explained in simple terms. The ability to compute all the relevant matrix elements allowed Igal to deduce that strong tensor force effects could lead to near cancellation of the Gamow-Teller matrix element for the decay, furnishing a plausible explanation for the long lifetime.

The emergence of the two-body matrix element as a viable tool brought out the significance of the connection between many-body matrix elements and two-body matrix elements of the residual interaction. The fact that the many-body matrix elements could be expressed directly in terms of two-body matrix elements implied specific relations between different spectra, independent of the detailed nature of the interaction, provided only that it was two-body in nature. These relations could be checked directly from the data. For example, \( j^n \) and \( j^{-n} = j^{2j+1-n} \) spectra should be the same; all \( j^n \) configurations for even \( n \) should have the same \( v=2 \) spectra if seniority is conserved; \( j^n \) spectra could be predicted from observed \( j^2 \) spectra with the help of coefficients of fractional parentage. Such tests were remarkably successful.

A classic example of the comparison of measured spectra within a shell-model framework was the case of \( ^{38}Cl \) and \( ^{40}K \), both of which have a single \( f_{7/2} \) neutron, while \( ^{38}Cl \) has a single \( d_{3/2} \) proton and \( ^{40}K \) has three \( d_{3/2} \) protons, or a single \( d_{3/2} \) proton hole. Shell-model methods allow the spectrum of \( ^{40}K \) to be predicted from that of \( ^{38}Cl \). The comparison with data is shown in fig. 1. The agreement is very convincing. This is a specific example of the relationship between particle-particle and particle-hole matrix elements, generally referred to nowadays as a Pandya transformation.

The governing paradigm of the modern shell model — that the effective two-body interaction is fully characterised by its two-body matrix elements, which can be treated as the underlying parameters of the model — stems directly from the early work of Talmi and his collaborators and students. The extraction of empirical effective interaction matrix elements from fits to experimental data serves as a benchmark for microscopic calculations.

The ability to compute two-body interaction matrix elements triggered Igal’s long-term interest in nuclear Coulomb energies. His results for the \( Z \) dependence of Coulomb energies in a shell and between shells were compared with empirical systematics derived from mirror nucleus binding energy differences. Over the years, he emphasized the importance of exchange corrections to the Coulomb energy and the strong correlation between Coulomb energies and root mean square radii of nuclei. He took an abiding interest in efforts to explain the Nolen-Schiffer anomaly in computed closed-shell-plus-or-minus-one mirror nucleus Coulomb energies.

Using the fact that a short-range attractive interaction favours the states of lowest possible seniority, i.e. with the least number of identical particles not in zero-coupled pairs, so that ground states of \( j^n \) configurations have seniority \( v = 0 \) for even \( n \) and \( v = 1 \) for odd \( n \), Igal produced the Talmi mass formula

\[
B.E.(j^n) = B.E.(j^0) + nC + \frac{n(n-1)}{2}a + \left[ \frac{n}{2} \right]b,
\]
Figure 1. Spectrum of $^{40}K$, predicted from that of $^{38}Cl$, compared with measured spectrum. The largest discrepancy is 25.6 keV.

where B.E. is the binding energy of $n$ particles outside closed shells, $C$ is a single-particle energy, $a$ is a saturation term (which is found from empirical fits to be repulsive, as it must be for saturation) and $b$ is a pairing term (found to be attractive). [Recall that $[x]$ is the largest integer not greater than $x$.] The concept of simple $n$ dependence of binding energies underlies the approach to nuclear masses embodied in the Garvey-Kelson mass relations. The Talmi mass formula generalizes, for neutrons and protons in $j^n$ configurations of good isospin $T$, to

$$B(n) = B(0) + nC + \frac{n(n-1)}{2}a + \{T(T+1) - \frac{3n}{4}\}b + \left[\frac{n}{2}\right]c + \text{Coulomb energy.}$$

The $T(T+1)$ dependence contributes a Wigner term to the binding energy.

The classic application of the Talmi mass formula was to the $f_{7/2}$ shell, for $f_{7/2}^n$ configurations of protons ($Ca$ isotopes) and neutrons ($N = 28$ isotones). The fit of the formula, with its characteristic saw-toothed odd–even behaviour, to the data is excellent, the largest deviation being of order 100 keV on a scale of tens of MeV (see fig. 2).

The Talmi mass formula implies that two-nucleon separation energies of $j^n$ configurations lie on a straight line as a function of $n$. Conservation of seniority implies that the $2^+$ excitation
Figure 2. Fit of Talmi mass formula to Ca isotopes and $N = 28$ isotones. The data are represented by symbols, the fit by lines joining calculated values.

Figure 3. Dependence of $N = 82$ $2^+$ excitation energies on proton number $Z$.

energy in such configurations is independent of $n$. Igal observed that, in semi-magic nuclei, first excited $2^+$ energies are roughly independent of the number of non-magic nucleons and two-nucleon separation energies lie on a straight line as a function of this number, though the configurations involved are far from simple $j^n$ with a single $j$. The example of $N = 82$ isotones is illustrated in figs. 3 and 4.
The observed systematics suggested the notion of generalised seniority, in which the ground states of even-even semi-magic nuclei are produced by the action of a composite pair creation operator $S^\dagger = \sum_j \alpha_j (a_j^\dagger a_j^\dagger)^{(0)}$ on the doubly-magic core state $|0\rangle$, via $(S^\dagger)^n|0\rangle$. The structure constants $\alpha_j$ of the composite pair remain unchanged through the major shell (the well-known quasispin limit corresponding to equal $\alpha_j$'s). Linear two-nucleon separation energies and constant $2^+$ excitation energies follow from the generalised seniority conditions $H|0\rangle = 0$, which sets the doubly-magic ground state $|0\rangle$ at zero energy; 

$$HS^\dagger|0\rangle = V_0 S^\dagger|0\rangle,$$

which makes the single-pair state an eigenstate;

$$[\left[H, S^\dagger\right], S^\dagger] = W(S^\dagger)^2;$$

$$HD^\dagger_M|0\rangle = V_2 D^\dagger_M|0\rangle,$$

which makes the single $J = 2$ pair state an eigenstate; and

$$[\left[H, S^\dagger\right], D^\dagger_M] = W S^\dagger D^\dagger_M;$$

where $H$ is a Hamiltonian with at most two-body interactions and $D^\dagger_M = \sum_{jj'} \beta_{jj'} (a_j^\dagger a_{j'}^\dagger)^{(2)}_M$ creates a $J = 2$ pair. [Note the suggestion of an algebraic structure.] Igal spelled out the conditions to be satisfied by the two-body matrix elements of $H$ in order for generalised seniority to hold. (He also looked into the conditions to be satisfied by these matrix elements in order for a pure rotational spectrum to emerge from a shell-model calculation and concluded that at least three active single-particle orbitals would be required.) Later, he defined states of generalised seniority with good total isospin.
Based on his understanding that one type of active nucleon (i.e. semi-magic nuclei) would produce constant $2^+$ energies with nucleon number and good generalised seniority, while both types of nucleons active (away from closed shells) gave rise to a rapidly falling $2^+$ energy with nucleon number and increasing quadrupole collectivity, Igal realised that a microscopic foundation for the interacting-boson model (IBM) in terms of correlated fermion pairs would necessarily entail two types of pairs, making the neutron–proton IBM-2 unavoidable. This insight also gave rise to the “standard” IBM-2 Hamiltonian with neutron and proton $d$-boson single-particle energies reflecting isovector like-particle fermion pairing (generalised seniority) and a quadrupole–quadrupole interaction between unlike bosons reflecting the generalised-seniority-breaking isoscalar neutron–proton interaction. Igal also participated in the study of explicit fermion–boson mappings required to connect the interacting-boson model with its shell-model roots and in the introduction of the boson $F$-spin analog to nucleon isospin.

In addition to his influential papers and conference talks, Igal also produced two books that served as guides and companions to generations of nuclear structure theorists. The first [1], written with the late Amos de Shalit, was a veritable bible of shell theory and the second [2], some thirty years later, continued the tradition of being an exhaustive compendium of relevant results and derivations.

**Acknowledgments**

It is an honour and a privilege to be asked to summarize the scientific accomplishments of Igal Talmi on his eightieth birthday. Of course, it is impossible to do justice to his prodigious achievements in a restricted time and space framework, so only a skeleton outline could be managed. I am sure Igal’s collaborators will forgive me for not mentioning them by name in this context, devoted to Igal, and for not supplying precise references.

Although Igal and I have never published a paper together, we have worked in close proximity to one another for many years and I regard him as my mentor in nuclear structure theory. It is a great pleasure for me to be able to wish him a happy eightieth birthday.

**Appendix**

Igal asked to include the following historical tidbit, which was not referred to in the oral presentation. I am happy to comply.

“The attractive $T=0$ average interaction, acting between protons and neutrons, is the dominant factor in creating the potential well of the shell model. It determines its depth as well as its form. This implies that energies of single proton orbits depend on the occupation numbers of neutrons and vice versa. A dramatic demonstration of this feature is offered by the level order in the $^{11}\text{Be}$ nucleus. In the ground state, according to the simple shell model, a valence neutron is occupying the $1p_{1/2}$ orbit outside closed $1s_{1/2}$ and $1p_{3/2}$ orbits. In an experimental paper, a spin $1/2$, negative parity assignment for it was given in parentheses and it was stated that ‘it seems at least possible’. Looking at the data, it seemed to me that a $1^+$ assignment, due to the valence neutron occupying the $2s_{1/2}$ orbit, was more likely. In $^{13}\text{C}$, the $\frac{1}{2}^+$ state lies about 3 MeV above the $\frac{1}{2}^-$ ground state. If two $1p_{3/2}$ protons are removed from it to reach $^{11}\text{Be}$, the absence of their attraction, stronger to a $1p_{1/2}$ neutron than to a $2s_{1/2}$ neutron, could reverse the positions of these orbits. Using the simplest shell model configurations, matrix elements of the relevant interactions could be extracted from level positions in $^{12}\text{B}$. Using them, a simple yet exact shell model calculation could determine the relative positions when one $1p_{3/2}$ proton is removed from $^{12}\text{B}$ to form $^{11}\text{Be}$. A graphic solution of the exact shell model calculation (not just an extrapolation!) predicted the $\frac{1}{2}^+$ state to be the ground state of $^{11}\text{Be}$ and the $\frac{1}{2}^-$ state, in which the valence neutron is in a lower major shell, to lie 0.21 MeV above it. That state was later found at 0.32 MeV above the $\frac{1}{2}^+$ ground state. We also calculated the binding energy of
$^{11}$Be in fair agreement with the measured value. We did not realize, however, that the radial wave function of a loosely bound $s$-neutron should have a large tail. Thus, we missed predicting that $^{11}$Be is a halo nucleus as discovered many years later.”

References
[1] De-Shalit A and Talmi I 1963 Nuclear Shell Theory (New York and London: Academic Press)
[2] Talmi I 1993 Simple Models of Complex Nuclei (Switzerland: Harwood Academic Publishers)