Testing Dispersion Relations of Quantum $\kappa$–Poincaré Algebra on Cosmological Ground

J. Kowalski–Glikman*
Institute for Theoretical Physics
University of Wroclaw
Pl. Maxa Borna 9
Pl–50-204 Wroclaw, Poland

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Abstract

Following the procedure proposed recently by Martin and Brandenberger we investigate the spectrum of the cosmological perturbations in the case when the “trans–Plackian” dispersion relations are motivated by the quantum $\kappa$-Poincaré algebra. We find that depending on the choice of initial conditions of the perturbations, the spectrum either differs from the flat one for for instantaneous Minkowski vacuum or, in the case of initial conditions minimizing energy density, leads to the observed scale-invariant Harrison–Zel’dovich spectrum in the Friedmann epoch.

*e-mail address jurekk@ift.uni.wroc.pl
In the recent years a growing mass of evidence appeared, indicating that on short (trans-Planckian) scale the usual space-time symmetries are drastically modified, e.g., by existence of a fundamental length scale (see e.g. [1, 2]). Some time ago Lukierski, Nowicki, Ruegg, and Tolstoy [3] using formalism of quantum algebras derived a quantum deformation of the Poincaré algebra, which is a high energy extension of the standard low-energy Poincaré algebra. This algebra, called $\kappa$-Poincaré algebra includes the parameter $\kappa$ of dimension of (inverse) length, usually identified with the Planck length. The algebra (in the so-called bicrossproduct basis) [4, 5]) takes the form:

$$
[M_{\mu\nu}, M_{\rho\tau}] = i (\eta_{\mu\tau} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\tau} + \eta_{\nu\rho} M_{\mu\tau} - \eta_{\mu\tau} M_{\nu\rho}),
$$

$$
[M_i, P_j] = i\epsilon_{ijk} P_k, \quad [M_i, P_0] = iP_i,
$$

$$
[N_i, P_j] = -i\delta_{ij} \left( \frac{\kappa}{2} \left( 1 - e^{2P_0/\kappa} \right) + \frac{1}{2\kappa} \vec{P}^2 \right) + \frac{i}{\kappa} P_i P_j,
$$

$$
[N_i, P_0] = iP_i,
$$

$$
[P_{\mu}, P_{\nu}] = 0,
$$

where $P_\mu = (P_i, P_0)$ are space and time components of four-momentum, and $M_{\mu\nu}$ are modified Lorentz generators with rotations $M_k = \frac{1}{2}\epsilon_{ijk} M_{ij}$, and boosts $N_i = M_{0i}$. It can be readily found that the first Casimir of this algebra, defining the mass-shell is of the form

$$
C_{bcp}^1 = (\vec{P}^2 e^{-P_0/\kappa} - \left( 2\kappa \sinh \left( \frac{P_0}{2\kappa} \right) \right)^2.
$$

Equations (1, 2), with additional structures of quantum algebra (coproducts and antipodes) form the algebraic framework for field theoretical constructions. In the first step in this construction one should identify the deformed $\kappa$-Minkowski space. This can be done by taking the set of generators dual to the $\kappa$-Poincaré algebra, $(\tilde{x}^\mu, \Lambda^{\rho\sigma})$, which form the $\kappa$-Poincaré

\footnote{There exists another realization of this quantum algebra, with slightly different coproduct, for whose the first Casimir is of the form $C_{bcp}'^1 = (\vec{P})^2 e^{+P_0/\kappa} - \left( 2\kappa \sinh \left( \frac{P_0}{2\kappa} \right) \right)^2$ and the dispersion relation $(k)^2 e^{+\omega/\kappa} - \left( 2\kappa \sinh \left( \frac{\omega}{2\kappa} \right) \right)^2 = 0$. This case differs from the cases considered in this papers by the fact that there is a cut-off for three-momentum: $k \rightarrow \kappa$ when $\omega \rightarrow \infty$. For this reason we will consider this case in a separate paper.}
group and dividing it by its Lorentz subgroup (observe that in algebra (1) the Lorentz generators are not deformed.) This has been done in the papers [4, 6]. As a result one obtains that the deformed $\kappa$-Minkowski space must be necessarily a non-commuting space:

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa} \left( \delta^\mu_0 \hat{x}^\nu - \delta^\nu_0 \hat{x}^\mu \right) \tag{3}$$

Now the question arises as to how one can define a quantum theory on such a space. Such a construction has been presented recently in [8]. This construction consists of three major steps. First one $\kappa$-deforms the classical local field theory on standard Minkowski space obtaining as a result a local $\kappa$-deformed field theory on $\kappa$-deformed, non-commutative Minkowski space. Then one observes that momenta in (1) are commutative and uses an appropriately defined $\kappa$-deformed Fourier transform to obtain a $\kappa$ deformed field theory on commutative momentum space. In the last step one makes use of the standard inverse Fourier transform to get a non-local, $\kappa$-deformed field theory on standard Minkowski space. This non-locality exhibits itself in non-polynomial structure of the Casimir operator (2), but the fact that the Minkowski space is now standard makes it possible to use the standard realization of momenta operators as derivatives over commuting Minkowski space positions, $P_\mu = i \partial/\partial x^\mu$.

One can now turn to the free massless scalar field (see [8]). The invariant wave operator on $\kappa$-deformed Minkowski space $\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu}$ can be expressed, in the momentum space as $C_1^{kcp} \left( 1 - \frac{C_1^{kcp}}{4\kappa^2} \right)$. We see that the spectrum of the modified massless wave operator contains the deformed massless mode $C_1^{kcp} \phi = 0$ and the tachyon $(C_1^{kcp} - 4\kappa^2) \phi = 0$. In this paper we will consider only the first branch, leaving the possible physical meaning of the second one to future investigations, and thus we are left with a field theory whose dynamics is governed by the Casimir operator, (2).

This Casimir operator leads to the following dispersion relation

$$(k)^2 e^{-\omega/\kappa} - \left( 2\kappa \sinh \left( \frac{\omega}{2\kappa} \right) \right)^2 = 0. \tag{4}$$

It should be noted that the wave equation leading to the above dispersion relation is non-local in time. Following the discussion in [7] we solve this equation for $\omega^2$ so that the resulting dispersion relation corresponds to an
operator of second order in time derivatives, and non-local in space:

\[ \omega^2 = \left( \kappa \log \left( 1 + \frac{k}{\kappa} \right) \right)^2 . \]  

(5)

Below we will use this expression to define a theory of modified dynamics of fluctuations in inflationary universe. It should be noted that except of the second Newton’s law there is no clear principle saying which relation is physically more fundamental. Therefore, strictly speaking one cannot say that the theory described by (5) and investigated below is derived from \( \kappa \)-Poincaré algebra, though it is not excluded that one could derive this theory from the first principles using some procedure different from that presented in [8].

It is worth pausing for a moment to show that the dispersion relation (5) indeed leads to modified wave equation covariant under \( \kappa \)-Poincaré symmetry. For the scalar field \( \phi \) this equation reads

\[ P_0^2 \phi = \left[ \kappa \log \left( 1 + \frac{|\vec{P}|}{\kappa} \right) \right]^2 \phi . \]  

(6)

It is clear that this equation is invariant under three-dimensional rotations generated by \( M_i \). Let us consider generalized boosts parameterized by infinitesimal parameter \( \varepsilon \) resulting from eq. (1).

\[ \delta_\varepsilon P_0 = \left[ \varepsilon^i N_i, P_0 \right] = i\varepsilon^i P_i \]  

(7)

\[ \delta_\varepsilon P_j = \left[ \varepsilon^i N_i, P_j \right] = -i\varepsilon_j \left( \frac{\kappa}{2} \left( 1 - e^{2P_0/\kappa} \right) + \frac{1}{2\kappa} |\vec{P}|^2 \right) + \frac{i}{\kappa} \varepsilon^i P_i P_j \]  

(8)

One can check by direct computation that eq. (6) is covariant under transformations (7), (8).\footnote{In the course of this calculation one uses eq. (6) twice: first to express \( P_0 \delta P_0 \) as \( \kappa \log \left( 1 + \frac{|\vec{P}|}{\kappa} \right) \delta P_0 \) and second to rewrite the \( e^{2P_0/\kappa} \) factor in terms of \( P_i \).}

Since the parameter \( \kappa \) is assumed to be of order of Planck length, for a long time the \( \kappa \)-Poincaré algebra remained a nice mathematical construction without any direct, testable physical applications and consequences (see however [9]). Recently however some areas have been identified, where the
behavior of matter at very high, “trans–Planckian” frequencies has a direct, low energy consequences. One of these areas is the black hole physics. The question whether modified high frequency behavior has an effect on Hawking radiation and Hawking–Beckenstein entropy formula has been considered (in the context of sonic black holes) by Unruh [10], Corley and Jacobson, and others [11] (for recent review see [12]). These authors proposed a rather ad hoc dispersion relations, to wit

\[ \omega = \kappa \tanh^{1/p} \left( \frac{k}{\kappa} \right)^{p}, \quad \text{(Unruh)} \]  

and

\[ \omega^2 = k^2 - \frac{k^4}{\kappa^2}, \quad \text{(Corley–Jacobson)}, \]  

where we again denoted by \( \kappa \) the characteristic length.

In the recent papers Niemeyer [13] and Martin and Branderberger [14] considered another setting in which there is a direct relation between trans–Planckian physics and low energy phenomenology. Namely, they considered quantum fluctuations during inflationary stage, giving rise to large scale formation in the Friedmann universe. These fluctuations are enormously red shifted in the course of inflation, and thus the wavelengths of the modes that correspond to the present large scale structure were, at the initial time, well in the realm of the trans–Planckian physics. It follows that it is justified to ask, to what extend the modified dispersion relations have their imprints on the present day large scale structure of the universe. Since we know that in order to agree with observations, the spectrum of fluctuations should be (almost) flat, this setting is perfect to test validity of modified dispersion relations, derived or motivated by other means. One should stress at this point that the relation (4) we are to investigate in this paper are motivated by considerations, having root in some fundamental physics (in the case at hands, from quantum groups).

To set the stage let us recall some fundamental facts concerning early cosmology and large scale structure formation. We consider spatially flat Friedmann universe with the conformally flat line element

\[ ds^2 = a^2(\eta) \left( -d\eta^2 + \sum_{i=1}^{3} dx_i^2 \right). \]
The conformal time $\eta$ is related to the cosmic time $t$ by relation $dt = a(\eta)d\eta$; in particular the De–Sitter universe corresponds to $\eta = H^{-1}e^{-Ht}$, $a(\eta) = l_H/\eta$.

To compute power spectra of observable quantities in the standard case (i.e., with unmodified dispersion relations) one considers equation of evolution of modes of Fourier components of massless scalar field $\mu_n$

$$\mu_n'' + \left(n^2 - \frac{a''}{a}\right)\mu_n = 0,$$

(12)

where prime denotes derivative with respect to conformal time $\eta$. When the wavelength of fluctuations is much longer than the characteristic cosmological scale, the Hubble radius, $\lambda(\eta) \equiv a(\eta)2\pi/n \gg l_H$, so that the first term in the parenthesis is small compared to $a''/a$, the solution of this equation is $\mu_n(\eta) = C_n a(\eta)$. One can then calculate the resulting spectra, to obtain

$$n^3 P = n^3 |C_n|^2$$

(13)

The observed, flat, scale invariant spectrum corresponds to the case when $C_n \sim n^{-3/2}$.

This behaviour can be readily seen from the explicit general solution of eq. (12). One has in this case

$$\mu_n(\eta) = \alpha_n \left(1 + \frac{i}{n\eta}\right)e^{i\eta} + \beta_n \left(1 - \frac{i}{n\eta}\right)e^{-i\eta}.$$  

(14)

For large $\eta$ (corresponding to early times) the second term in parentheses is small and the fluctuation behaves as a wave with slowly changing amplitude; starting from $\eta_H = 2\pi/n$, corresponding to the moment when fluctuation crosses the horizon, there are no oscillations and the wave gets frozen. Thus to find the spectrum we have to compute

$$|C_n| = \frac{1}{a(\eta_H)} |\mu_n(\eta_H)| = \frac{2\pi}{l_H n} |\mu_n(\eta_H)|.$$  

(15)

We see therefore that in order to lead to flat spectrum, the coefficients $\alpha_n$, $\beta_n$ should behave as $\sim 1/\sqrt{n}$.

Equation (12) corresponds to the standard dispersion relation

$$\omega^2 = k^2 = \frac{n^2}{a^2}.$$  


In the case of modified dispersion relation $\omega^2 = \Upsilon^2(k)$ one should replace $n^2$ with

$$n^2_{\text{mod}} = a^2(\eta)\Upsilon^2(k) = a^2(\eta)\Upsilon^2 \left( \frac{n}{a(\eta)} \right).$$

(16)

Therefore the equation we are to consider is

$$\mu'' + \left( a^2(\eta)\Upsilon^2 \left( \frac{n}{a(\eta)} \right) - \frac{a''}{a} \right) \mu_n = 0.$$

(17)

This equation should be appended by initial conditions at for $\mu_n$ and $\mu'_n$. In this paper we will consider two natural sets of initial conditions: the “minimal energy condition” of [14],

$$\mu_n(\eta) = \sqrt{\frac{a(\eta)}{2}} \Upsilon^{-1/2} \left( \frac{n}{2\pi a(\eta)} \right),$$

(18)

$$\mu'_n(\eta) = \pm i \sqrt{\frac{1}{2a(\eta)}} \Upsilon^{1/2} \left( \frac{n}{2\pi a(\eta)} \right);$$

(19)

and the instantaneous Minkowski vacuum conditions, as in the case of standard dispersion relation

$$\mu_n(\eta) = \frac{1}{\sqrt{2n}}, \quad \mu'_n(\eta) = \pm i \sqrt{\frac{n}{2}}.$$  

(20)

Observe that in the case of standard dispersion relation, $\Upsilon^2 = n^2$ these initial condition are identical.

Our goal would be therefore to solve eq. (17) with initial conditions (18, 19) or (20), and by making use of expression (13) to find the power spectrum of cosmological perturbation. We will consider two regimes: (I) when modifications of dispersion relations are relevant and $\Upsilon$ term dominates in eq. (17); (II) when the perturbation is large as compared to the scale defined by $\kappa$, in which case we have to do with standard dispersion relation and eq. (14) holds. With this equation we can compute modulus of the coefficient $C_n$.

Let us turn to the dispersion relation (5) which is valid in regime (I)

$$\Upsilon(k) = \kappa \log \left( 1 + \frac{k}{\kappa} \right).$$

(21)
In this regime, assuming $a(\eta) = l_H/\eta$ eq. (17) takes the form

$$\frac{d^2 \mu_n^{(I)}}{d\eta^2} + \left[\frac{4\pi^2 \epsilon^2}{\eta^2} \log^2 \left(1 + \frac{n\eta}{2\pi \epsilon}\right) - \frac{2}{\eta^2}\right] \mu_n^{(I)} = 0,$$

(22)

where we introduced the parameter $\epsilon = (\kappa l_H)$ which is a ratio of two relevant length scales $l_H$, the size of the cosmological horizon and $1/\kappa$. If we assume that $1/\kappa$ is of order of Planck length, then $\epsilon$ is equal to the size of the horizon expressed in Planck unit. The numerical value of this parameter depends therefore on details of the dynamics of inflation. Following [14] we assume that $\epsilon \sim 10^6$.

To solve equation (22) we make use of the fact that for equation $\mu'' + W^2(\eta)\mu$ the solution is of the approximate form

$$\mu = \frac{1}{\sqrt{2W}} \exp \left(\pm i \int W(\eta) d\eta\right).$$

This solution is valid in the adiabatic regime, where

$$\frac{1}{2} \left(\frac{W''}{W^3} - \frac{3 W'^2}{2W^4}\right) \ll 1.$$

(23)

In our case we get the condition

$$-3 n^2 \eta^2 + 4 n \pi \epsilon \eta \log(1 + \frac{n\eta}{2\pi \epsilon}) + (2 \pi \epsilon + n \eta)^2 \log(1 + \frac{n\eta}{2\pi \epsilon})^2$$

$$\leq \frac{16 \pi^2 \epsilon^2 (2 \pi \epsilon + n \eta)^2 \log(1 + \frac{n\eta}{2\pi \epsilon})^4}{16 \pi^2 \epsilon^2 (2 \pi \epsilon + n \eta)^2 \log(1 + \frac{n\eta}{2\pi \epsilon})^4} \ll 1.$$

In regime (I) the fluctuations are described therefore by

$$\mu_n^{(I)} = A_n^{(I)} \sqrt{\frac{\eta}{4\pi \epsilon \log \left(1 + \frac{n\eta}{2\pi \epsilon}\right)}} \exp \left[-2\pi i \epsilon Li_2 \left(-\frac{n\eta}{2\pi \epsilon}\right)\right]$$

$$+ B_n^{(I)} \sqrt{\frac{\eta}{4\pi \epsilon \log \left(1 + \frac{n\eta}{2\pi \epsilon}\right)}} \exp \left[+2\pi i \epsilon Li_2 \left(-\frac{n\eta}{2\pi \epsilon}\right)\right],$$

(24)

where $Li_2(x) \equiv \int dx \log(1-x)/x$ is the polylogarithm function.

Let us now turn to initial conditions (18, 19). At the time $\eta = \eta_i$ we find

$$\mu_n^{(I)}(\eta_i) = A_n^{(I)} \sqrt{\frac{\eta_i}{4\pi \epsilon \log \left(1 + \frac{n\eta_i}{2\pi \epsilon}\right)}} \exp \left[-2\pi i \epsilon Li_2 \left(-\frac{n\eta_i}{2\pi \epsilon}\right)\right]$$

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\[ +B_n^{(I)} \sqrt{\frac{\eta}{4\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}} \exp \left[ +2\pi i \epsilon Li_2 \left( -\frac{m\eta}{2\pi\epsilon} \right) \right] = \sqrt{\frac{\eta}{4\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}} \]

and

\[ \frac{1}{i} \mu_n^{(I)}(\eta) = A_n^{(I)} \sqrt{\frac{2\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}{2\eta}} \exp \left[ -2\pi i \epsilon Li_2 \left( -\frac{m\eta}{2\pi\epsilon} \right) \right] \]

\[ -B_n^{(I)} \sqrt{\frac{2\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}{2\eta}} \exp \left[ +2\pi i \epsilon Li_2 \left( -\frac{m\eta}{2\pi\epsilon} \right) \right] = \sqrt{\frac{2\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}{2\eta}}, \]

Which leads to

\[ A_n^{(I)} = \exp \left[ 2\pi i \epsilon Li_2 \left( -\frac{m\eta}{2\pi\epsilon} \right) \right], \quad B_n^{(I)} = 0, \quad (27) \]

and

\[ \mu_n^{(I)} = \sqrt{\frac{\eta}{4\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}} \exp \left[ 2\pi i \epsilon Li_2 \left( -\frac{m\eta}{2\pi\epsilon} \right) - 2\pi i \epsilon Li_2 \left( -\frac{m\eta}{2\pi\epsilon} \right) \right]. \quad (28) \]

In the case of linear standard vacuum initial conditions, (20) we similarly obtain

\[ A_n^{(I)st} = \left( \frac{1}{\sqrt{8n}} \right) \sqrt{\frac{2\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}{2\eta}} + \sqrt{\frac{n}{8}} \sqrt{\frac{\eta}{4\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}} e^{2\pi i \epsilon Li_2 \left( -\frac{m\eta}{2\pi\epsilon} \right)} \]

\[ B_n^{(I)st} = \left( \frac{1}{\sqrt{8n}} \right) \sqrt{\frac{2\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}{2\eta}} - \sqrt{\frac{n}{8}} \sqrt{\frac{\eta}{4\pi\epsilon \log \left(1 + \frac{m}{2\pi\epsilon}\right)}} e^{-2\pi i \epsilon Li_2 \left( -\frac{m\eta}{2\pi\epsilon} \right)} \]

Now we turn to regime (II), and we have to match above solution with \( \mu_n^{(II)} \) given by (14) at \( \eta = \eta_1 \). Before doing that, let us pause for a moment to find the characteristic value of the conformal time \( \eta_1 \). It corresponds to the moment when the wavelength of the perturbation equals the characteristic length \( \kappa^{-1} \). Thus

\[ \eta_1 = \frac{2\pi\epsilon}{n}. \quad (31) \]
It must be checked if at this value of conformal time we are still in the region of validity of adiabatic approximation, i.e., if condition (23) is satisfied. One finds the condition
\[
-3 + 4 \log(2)^2 + \log(4) \leq \frac{32 \pi^2 \epsilon^2 \log(2)^4}{\pi^2} \ll 1
\]
which indeed holds for any \(\epsilon \sim 10^{-1}\) and larger. On the other hand we must also satisfy the condition
\[
4\pi^2 \epsilon^2 \log(2) \gg 2
\]
so that we can safely assume that fluctuations do not feel the cosmic expansion and neglecting the factor \((-2/\eta^2)\) in eq. (22) was indeed justified. This condition is well satisfied for \(\epsilon\) slightly larger than 1. We see therefore that our approximate solution holds at the matching point \(\eta = \eta_1\). Physically this means that for such values of the parameter \(\epsilon\), the fluctuation relaxes sufficiently slowly from the regime ruled by the trans-Planckian physics and at the same time reaches the regime described by standard physics well before the fluctuation length becomes comparable with the Hubble size. This analysis of the adiabaticity conditions is in perfect agreement with the results of papers [13], [14].

Let us thus match the solution (28) with the solution (14). In view of eq. (15) we will be mainly interested in the \(n\) dependence of the coefficients. Computing \(\mu_{n}^{(f)}(\eta_1)\) and its derivative and comparing with appropriate expressions obtained from (15) we easily find
\[
\alpha_n \sim \frac{1}{\sqrt{n}} e^{2\pi i \epsilon L_{iz} \left( \frac{-n \eta_1}{2\pi \epsilon} \right)}, \quad \beta_n \sim \frac{1}{\sqrt{n}} e^{2\pi i \epsilon L_{iz} \left( \frac{-n \eta_1}{2\pi \epsilon} \right)},
\]
both multiplied by complicated numerical \((n\)-independent\) coefficients. This means that
\[
|C_n| \sim n^{-3/2}
\]
and we recover the flat spectrum of the fluctuations.

For instantaneous Minkowski vacuum initial conditions (20), the final result changes. One finds
\[
\alpha_n, \beta_n \sim \text{const} + \frac{1}{n},
\]
and thus one cannot obtain the scale invariant spectrum. This corresponds to the result of Martin and Brandenberger [14] who also concluded that these standard initial conditions do not lead to the correct spectrum of fluctuations.
This result is not very surprising. The initial conditions (20) are natural in the case of standard dispersion relation. However, it would be hardly understandable if they remain valid in the case of modified nonlinear dispersion. The choice of the minimal energy conditions of Martin and Brandenberger (18, 19), on the other hand, is justified by powerful physical principle that the fluctuations leading to large scale structure originate from vacuum fluctuations with minimal possible energy.

This result is the first test passed by $\kappa$ physics. This is important because till now $\kappa$-Poincaré symmetry was nothing but a nice algebraic construction. I hope that the result reported in this paper would encourages one to look for new grounds where predictions of modified dispersion relations might be possibly checked (for a very recent attempt along this line, see [15].)

Some open problems remain, of course. Probably the most important one is what is the source of “trans-Plackian frequencies reservoir”. In other words one should understand the initial conditions (18, 19) from the point of view of fundamental $\kappa$-quantum field theory. It is also interesting to see imprints of $\kappa$-physics on early stages of inflation, especially in the pictures where the inflation starts immediately after Planck era.

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