On a Grauert-Riemenschneider vanishing theorem for Frobenius split varieties in characteristic $p$.

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1 Introduction

It is known that the Grauert-Riemenschneider vanishing theorem is not valid in characteristic $p$ (\cite{1}). Here we show that it may be restored in the presence of a suitable Frobenius splitting. The proof uses interchanging two projective limits, one involving iterated Frobenius maps, cf. \cite{3} and \cite{4}, the other coming from Grothendieck’s theorem on formal functions. That leads to the following general vanishing theorem which we then apply in the situation of the Grauert-Riemenschneider theorem.

**Theorem 1.1** Let $\pi : X \to Y$ be a proper morphism of schemes of finite type over a perfect field of characteristic $p > 0$. Let $D$ be a closed subscheme of $X$ with ideal sheaf $\mathcal{I}$, let $E$ be a closed subscheme of $Y$ and let $i \geq 0$ such that
1. $D$ contains the geometric points of $\pi^{-1}E$.
2. $R^i\pi_*(\mathcal{I})$ vanishes off $E$.
3. $X$ is Frobenius split, compatibly with $D$.
Then $R^i\pi_*(\mathcal{I})$ vanishes on all of $Y$.

**Theorem 1.2** (Grauert-Riemenschneider with Frobenius splitting.) Let $\pi : X \to Y$ be a proper birational morphism of varieties in characteristic $p > 0$ such that:
1. $X$ is non-singular and there is $\sigma \in H^0(X, K_X^{-1}) = H^0(X, c_1(X))$ such that $\sigma^{p-1}$ splits $X$. (cf. [5].)

2. $D = \text{div}(\sigma)$ contains the exceptional locus of $\pi$ set theoretically. Then $R^i\pi_*K_X = 0$ for $i > 0$.

**Remark 1.3** It will be clear from the proof that many variations on our Grauert-Riemenschneider theorem are possible. For instance, one may replace $D$ by some subdivisor which still contains the exceptional locus, and thus replace $K_X$ in the conclusion by the new $O_X(-D)$. Similarly, the birationality assumption may be weakened, as it is used only to conclude that condition 2 of [1.1] is satisfied.

## 2 Proofs

### 2.1 Proof of [1.2]

We assume theorem [1.1]. For $E$ we take the image of the exceptional locus. Dualizing $\sigma$ we get a short exact sequence

$$0 \rightarrow K_X \rightarrow O_X \rightarrow O_D \rightarrow 0,$$

so $K_X$ may be identified with the ideal sheaf $\mathcal{I}$ of $D$. That $D$ is compatibly split is clear from local computations, cf. Remark on page 36 of [3].

**Lemma 2.2** Let $\cdots \rightarrow M_2 \rightarrow M_1 \rightarrow M_0$ be a projective system of artinian modules over some ring $R$, with transition maps $f_i^j : M_j \rightarrow M_i$. If $f_i^0$ is nonzero for all $i$, then the projective limit is nonzero.

**Proof.** Put $M_i^{\text{stab}} = \bigcap_{j \geq i} f_{i}^{j}(M_j)$. Then $M_i^{\text{stab}} = f_{i}^{k}(M_k)$ for $k \gg 0$. So

$$f_{i}^{i+1}(M_{i+1}^{\text{stab}}) = f_{i}^{i+1}f_{i+1}^{k}(M_k) = f_{i}^{k}(M_k) = M_{i}^{\text{stab}}$$

for $k \gg 0$. Therefore we have a subsystem $(M_i^{\text{stab}})$ with nonzero surjective maps, whence the result.

### 2.3 Proof of [1.1]

We argue by contradiction. We may assume $Y$ is affine, so that $R^i\pi_*(\mathcal{I})$ equals $H^i(X, \mathcal{I})$. Choose an irreducible component, with generic point $y$ say, of the support on $Y$ of $H^i(X, \mathcal{I})$, which we suppose
to be nonzero. Observe that \(y \in E\). The Frobenius map \(F\) as well as its splitting act on the exact sequence of sheaves

\[
0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_D \to 0.
\]

Therefore the Frobenius and its iterates act by split injective endomorphisms, \(p\)-linear over \(A = \Gamma(Y, \mathcal{O}_Y)\), on \(H^1(X, \mathcal{I})\), and the same remains true after localisation and completion at \(y\). Let \(R\) be a regular ring of the form \(L[[X_1, \ldots, X_m]]\) mapping onto \(A_y^\wedge\), where \(L\) is a field of representatives in the completed local ring \(A_y^\wedge\). In the projective system of artinian modules

\[
\cdots \to R \otimes^{p^r} H^i(X, \mathcal{I})_y^\wedge \to R \otimes^{p^{r-1}} H^i(X, \mathcal{I})_y^\wedge \to \cdots
\]

all maps towards \(R \otimes^{p^0} H^i(X, \mathcal{I})_y^\wedge = H^i(X, \mathcal{I})_y^\wedge\) are nonzero. Here \(R \otimes^{p^r}\) refers to base change along the \(r\) times iterated Frobenius endomorphism of the regular ring \(R\), and the projective system is thus the one defining the “leveling” \(G(H^i(X, \mathcal{I})_y^\wedge)\), in the sense of [4], of \(H^i(X, \mathcal{I})_y^\wedge\) as an \(R\) module. The projective limit is nonzero by the Lemma. On the other hand, as \(R\) is a finite free module over \(R\) via \(F^r\), one may also compute

\[
G(H^i(X, \mathcal{I})_y^\wedge) = \lim_{\leftarrow r} R \otimes^{p^r} H^i(X, \mathcal{I})_y^\wedge
\]

as follows

\[
\lim_{\leftarrow r} R \otimes^{p^r} H^i(X, \mathcal{I})_y^\wedge = \lim_{\leftarrow r} R \otimes^{p^r} \lim_{\leftarrow s} H^i(X_s, \mathcal{I}_s) = \lim_{\leftarrow r} \lim_{\leftarrow s} R \otimes^{p^r} H^i(X_s, \mathcal{I}_s)
\]

where \(X_s\) and \(\mathcal{I}_s\) are the usual thickenings from Grothendieck’s theorem on formal functions. But by the Artin-Rees lemma the Frobenius map acts nilpotently on \(\mathcal{I}_s\), (note that some power of \(\mathcal{I}\) is contained in the pull back of the ideal sheaf of \(E\)), so \(\lim_{\leftarrow r} R \otimes^{p^r} H^i(X_s, \mathcal{I}_s)\) vanishes. But then \(G(H^i(X, \mathcal{I})_y^\wedge)\) is both nonzero and zero. \(\square\)

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