GLOBAL EXISTENCE OF SOLUTIONS TO REACTION DIFFUSION SYSTEMS WITH MASS TRANSPORT TYPE BOUNDARY CONDITIONS ON AN EVOLVING DOMAIN

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(Communicated by Michael Winkler)

Abstract. We consider reaction diffusion systems where components diffuse inside the domain and react on the surface through mass transport type boundary conditions on an evolving domain. Using a Lyapunov functional and duality arguments, we establish the existence of component wise non-negative global solutions.

1. Introduction. The reaction–diffusion mechanism is one of the simplest and most elegant pattern formation models. Turing (1952) [35] first proposed the mechanism in the context of biological morphogenesis, showing that reactions between two diffusible chemicals (morphogens) could give rise to spatially heterogeneous concentrations through an instability driven by diffusion. Recently there has been ample of studies on models that involve coupled bulk surface dynamics [9], [10], [12], [23], [24]. Hahn et al [11] modeled the surfactant concentration by use of coupled bulk-surface model and Rätz and Röger [28], [29] studied the symmetry breaking in a bulk surface reaction diffusion model for signalling networks. In the former work, a reaction–convection diffusion is proposed that couples the concentration of the surfactants in the bulk and on the free surfaces while in the latter work, a single diffusion partial differential equation is formulated inside the bulk of a cell, while on the cell surface a system of two membrane reaction diffusion equations is formulated.

Sharma and Morgan, [32] worked on coupled reaction diffusion system with \( m \) components in the bulk coupled with \( n \) components on the boundary and under certain conditions established the local and global wellposedness of the model. They further established the uniform boundedness of the solution. Recent advances in mathematical modelling and developmental biology identify the important role of

2020 Mathematics Subject Classification. Primary: 35K57, 35B45.

Key words and phrases. Reaction-diffusion equations, mass transport, conservation of mass, global wellposedness, linear estimates, evolving domain.

The first author’s research was supported by SEED grant of IIT Jodhpur.

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evolution of domains during the reaction process as central in the formation of patterns, both empirically (Kondo and Asai [13]) and computationally (Comanici and Golubitsky [5] ; Crampin et al. [3]). Experimental observations on the skin pigmentation of certain species of fish have shown that patterns evolve in a dynamic manner during the growth of the developing animal. Kondo and Asai [13] describe observations on the marine angelfish Pomacanthus semicirculatus, where juveniles display a regular array of vertical stripes which increase in number during growth, with new stripes appearing in the gaps between existing ones as the animal doubles in length. Further in [3], Crampin et al investigated the sequence of patterns generated by a reaction–diffusion system on growing domain. They derived a general evolution equation to incorporate domain growth in reaction–diffusion models and considered the case of slow and isotropic domain growth in one spatial dimension. The results that Crampin et al [3] present, suggest at least in one-dimension, that growth may in fact stabilize the frequency-doubling sequence and subsequently that it may be a mechanism for robust pattern formation. Also, in this respect, many numerical studies, such as in Barrass et al. [2]; and Madzvamuse and Maini [22] ; Madzvamuse [19], of RDS’s on evolving domains are available. We also observed that Kulesa et al. [15] have incorporated exponential domain growth into a model for the spatio-temporal sequence of initiation of tooth primordia in the Alligator Mississipiensis. In the model, domain growth plays a central role in establishing the order in which tooth precursors appear.

A specific feature of reaction-diffusion patterns on growing domains is the tendency for stripe patterns to double in the number of stripes each time the domain doubles in length, called mode-doubling. Since their seminal introduction by Turing [35], reaction-diffusion systems (RDS’s) have constituted a standard framework for the mathematical modelling of pattern formation in chemistry and biology. Numerous studies on the stability of solutions of RDS’s on fixed domains are available, for example, Hollis et al. [12] ; Rothe [30]; Sharma and Morgan [32], but very little literature regarding the global wellposedness of solutions of RDS’s on evolving domains. In direction of stability, Madzvamuse et al.[21] provides a linear stability analysis of RDS’s on continuously evolving domains, and Labadie [16] examines the stability of solutions of RDS’s on monotonically growing surfaces. Chandrashekar et al [36] showed that RDS fulfils a restricted version of certain stability conditions, introduced by Morgan [23] for fixed domain, then the RDS fulfills the same stability conditions on any bounded spatially linear isotropic evolution of the domain. They proved that, under certain conditions, the existence and uniqueness for a RDS on a fixed domain implies the existence and uniqueness for the corresponding RDS on an evolving domain. This is, to our best knowledge, the first result that holds independently of the growth rate and is thus valid on growing or contracting domains as well as domains that exhibit periods of growth and periods of contraction. Again these models arise in the area of tissue engineering and regenerative medicine, electrospun membrane which are useful in applications such as filtration systems and sensors for chemical detection.

In [4], [25], [14], [7] and the references therein, the authors derived the equation for the reaction diffusion equation on a growing manifold with or without boundary. They imposed special growth conditions such as isotropic (including exponential) or anisotropic and studied the behaviour of solutions. More precisely, they studied pattern formation on a manifold beginning with an initial static pattern and compared it with the final pattern after the manifold stops growing. The main focus
of their work has been stability analysis and numerical simulations to study the
development of patterns with growth, on curved surfaces.

In this paper, we prove the global existence for solutions of reaction diffusion
system on a domain in $\mathbb{R}^n$ evolving with time. In [12], the authors had proved
global existence and uniform boundedness for a class of two component reaction
diffusion system where one of the components is given to be apriori bounded as
long as the solutions exists. This was extended in [31] to more general system
involving two components and with Neumann boundary conditions using Lyapunov
type functional for deriving the apriori estmates. Keeping in mind the possible
applications to systems such as Brusselator (see Section 7), here we use techniques
of [31] to obtain the global existence for a two component reaction diffusion system
on an evolving domain in the case when one of the component remains apriori
bounded. Extension of the estimates of Fabes-Riviere [8] to a more general operator
and construction of a suitable Lyapunov functional are crucial ingredients in our
proof to obtain Hölder and $L_p$ estimates. These results as well as the local existence
is proved here for $m$ component system of reaction diffusion equation. We also define
a Lyapunov functional different from the one used in [32] and [1] which can be used
to obtain $L_p$ estimates for the $m$ component system, as in these references. Once
this is done, the global existence for the general case of system of $m$ components
on evolving domain will follow from arguments similar to [32].

As in many of the existing works, we consider here dilational anisotropic as well as
isotropic growth, though the arguments extend to a more general growth. Consider
compact domains $\Omega_t \subset \mathbb{R}^n$, $t \geq 0$ with boundary $\partial \Omega_t = \Gamma_t$ evolving according to
the given law (flow) $y(x,t)$ so that we can represent $\Omega_t = y_t(\Omega_0) = y_t(\Omega)$, $t \geq 0$
where $\Omega_0 = \Omega$ is the initial domain. We assume that $y$ is a diffeomorphism and
as in [14], it is separable in $t$ and $x$ variable. In practice, one expects that for an
arbitrary domain, at a future time $t$, the boundary $\Gamma_t$ may begin self intersecting, or
the domain $\Omega_t$ may split. Here we are interested in modeling situations where the
domain does not break up and the boundary evolves in such a way that $\Gamma_t$ continues
to remain smooth. So, without loss of generality we assume that the domain and
hence the boundary remain asymptotically close to a fixed domain, which we denote
by $\Omega_\infty$ with boundary $\Gamma_\infty$, and that for each $t \geq 0$, $\Gamma_t$ is $C^{2+\mu}$. Letting $c_i$ denote the concentration of the $i$-th component, we consider the system of equations

$$
\begin{align*}
\frac{\partial c_i}{\partial t} &= d_i \Delta c_i + f_i(c_1, \ldots, c_m) \quad y \in \Omega_t, t \in (0, T), 1 \leq i \leq m, \\
d_i \frac{\partial c_i}{\partial \eta_t} &= g_i(c_1, \ldots, c_m) \quad y \in \Gamma_t, t \in (0, T), 1 \leq i \leq m, \\
c_i &= c_i^0 \quad y \in \Omega_0 = \Omega, t = 0, 1 \leq i \leq m
\end{align*}
$$

(1.1)

where $\eta_t$ denotes the outward unit normal vector to the boundary $\Gamma_t$ and $\Omega_0 = \Omega$
is the initial domain.

Note that our results can be generalised to a manifold $(\Omega, g)$ with boundary
where Laplacian $\Delta$ is replaced by the Laplace-Beltrami operator $\Delta_g$ corresponding
to the Riemannian metric $g$. This will be done in our future work [34], where we
prove the global existence of solutions for volume-surface reaction diffusion systems
on manifolds.

The first step is to transform the system of equations (1.1) on $\Omega_t$ to an equivalent
system on the initial domain $\Omega$, as in [4], [25], [14]. The information on how the
domain is evolving is captured in the diffusion term of the transformed equation,
and generally, the evolution of domain is described by a flow which is separable in time and spatial variables. We note that in [14], a suitable transformation was used so that the diffusion term in the resulting equation does not have time dependent term. Whereas we do analysis of the transformed equation with diffusion term depending on the time variable.

The plan of paper is as follows. We begin by the derivation of the equations on the evolving domain and reduction to a pull back system on the initial domain Ω in Section 2. Section 3 contains primary assumptions on the vector fields \( f \) and \( g \), and statements of our local and global existence results. In addition to quasi-positivity assumptions that guarantee the componentwise nonnegativity of solutions, we also assume polynomial bounds, and that the reaction vector fields satisfy a condition that is similar to the condition given in [1] and [31]. In Section 4 we discuss the Hölder estimates which will be useful in establishing the global wellposedness of the model on growing domain. Local existence is established in Section 5 and in Section 6 we develop a boot strapping process based upon duality estimates, and provide a proof of our global existence result. Section 7 contains a few examples.

2. Equation for evolving domains. Here we show how to reduce the system (1.1) on \( \Omega_t \) to a system on the fixed domain \( \Omega \). For simplicity of notations and keeping in mind practical applications, we show the derivation for domains \( \Omega_t \subset \mathbb{R}^3 \).

Let \( y_t : \Omega \to \mathbb{R}^3 \) be a one parameter family of diffeomorphisms such that \( y_0 = \text{Id} \), the identity map and \( \Omega_t = y_t(\Omega) \) denote the domain evolving with time \( t \geq 0 \) such that \( \Omega_0 = \Omega \). We obtain a parametrization for \( \Omega_t \) by writing \( y_t(x,t) = (y_1(x,t),y_2(x,t),y_3(x,t)) = y_t(x) \) for \( t \geq 0 \), so that \( y(x,0) = x \in \Omega_0 = \Omega \).

If \( c \) denotes the chemical concentration in the domain \( \Omega_t \), then the diffusion process for \( c \) is driven by the equation

\[
\frac{d}{dt} \int_{\Omega_t} (c(y) \, d\Omega_t) = D \int_{\partial \Omega_t} \nabla c(y) \cdot \nu_t \, d\sigma_t,
\]

where \( d\Omega_t = \, dy \) is the volume element in \( \Omega_t \), \( \sigma_t \) is a parametrization for \( \partial \Omega_t \) and \( d\sigma_t \) is the surface area element for \( \partial \Omega_t \). Since \( y \) is a diffeomorphism, we have \( \, dy = \sqrt{\det(Dy(x,t))} \, dx \) and hence

\[
\frac{d}{dt} \int_{\Omega_t} (c(y) \, dy) = \frac{d}{dt} \int_{\Omega} c(y(x,t),t) \sqrt{\det(Dy(x,t))} \, dx
\]

\[
= \int_{\Omega} \left\{ \frac{dy}{dt}(x,t) \cdot \nabla c(y(x,t),t) + \frac{dc}{dt}(y(x,t),t) \right\} \sqrt{\det(Dy(x,t))} \, dx
\]

\[
+ \int_{\Omega} c(y(x,t),t) \frac{d}{dt} \left( \sqrt{\det(Dy(x,t))} \right) \, dx,
\]

while using the Stokes theorem and change of variables, we see that

\[
D \int_{\partial \Omega_t} \nabla c(y) \cdot \nu_t \, d\sigma_t(y) = D \int_{\Omega_t} \Delta c(y) \, dy
\]

\[
= D \int_{\Omega} \Delta_t c((y(x),t),t) \sqrt{\det(Dy(x,t))} \, dx.
\]
Note, with \( \frac{\partial}{\partial y_i} \) so that we can express parametrization for \( \partial \) and \( \sigma \)

\[
\frac{\partial^2}{\partial y_i^2} c(y, t) = \sum_{j=1}^{3} \frac{\partial x_j}{\partial y_i} \frac{\partial^2 c}{\partial y_j} (y(x, t), t) + \sum_{j=1}^{3} \frac{\partial^2 x_j}{\partial y_i^2} \frac{\partial c}{\partial x_j} (y(x, t), t).
\]

Thus,

\[
\Delta_t = \sum_{i=1}^{3} \sum_{j,k=1}^{3} \frac{\partial x_j}{\partial y_i} \frac{\partial x_k}{\partial y_i} \frac{\partial^2}{\partial y_j \partial y_k} + \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2 x_j}{\partial y_i^2} \frac{\partial}{\partial x_j}.
\]

(2.4)

Combining equations (2.1)-(2.2) we get that the concentration in the domain \( \Omega_t \) satisfies the equation

\[
\frac{dc}{dt} (y(x, t), t) + \frac{du}{dt} (x, t) \cdot \nabla c(y(x, t), t)
\]

\[
= D \Delta_t c(y(x, t), t) - \frac{1}{\sqrt{\det(Dy(x, t))}} \frac{d}{dt} \left( \sqrt{\det(Dy(x, t))} \right) c(y(x, t), t)
\]

(2.5)

for \( x \in \Omega \) and \( t \geq 0 \). Define the function \( u : \Omega \times [0, T) \) as

\[
u(x, t) := c(y(x, t), t),
\]

then \( \frac{d}{dt} u(x, t) = \frac{dc}{dt} (y(x, t), t) = \frac{dc}{dt} (y(x, t), t) + \frac{du}{dt} (x, t) \cdot \nabla c(y(x, t), t) \). The equation (2.5) can now be written in terms of \( u \) as

\[
\frac{du}{dt} (x, t) = D \Delta_t u(x, t) - \frac{1}{\sqrt{\det(Dy(x, t))}} \frac{d}{dt} \left( \sqrt{\det(Dy(x, t))} \right) u(x, t), \text{ for } (x, t) \in \Omega \times [0, T).
\]

(2.6)

In particular, for the flow \( y(x, t) := A(t)x, t \geq 0 \) where \( A(t) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is a family of diffeomorphism such that \( A(0) = I_d \), the identity map so that \( \Omega_t = A(t)\Omega \). As a special case we let

\[
A(t) = \begin{pmatrix}
\lambda_1(t) & 0 & 0 \\
0 & \lambda_2(t) & 0 \\
0 & 0 & \lambda_3(t)
\end{pmatrix}
\]

so that there is dilational growth- which is isotropic if \( \lambda_1(t) = \lambda_2(t) \neq \lambda_3(t) \) and anisotropic otherwise. The equation (2.6) in this case is

\[
\frac{du}{dt} (x, t) = D \Delta_t u(x, t) - \frac{(\sqrt{\lambda_1(t)\lambda_2(t)\lambda_3(t)})'}{\sqrt{\lambda_1(t)\lambda_2(t)\lambda_3(t)}} u(x, t), \text{ for } (x, t) \in \Omega \times [0, T) (2.7)
\]

with

\[
\Delta_t = \sum_{i=1}^{3} \frac{1}{\lambda_i(t)^2} \frac{\partial^2}{\partial x_i^2}.
\]

Let \( \sigma : [\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \rightarrow \mathbb{R}^3 \) be a parametrization of \( \partial \Omega = \Gamma \) with

\[
\sigma(\alpha, \beta) = (x(\alpha, \beta), y(\alpha, \beta), z(\alpha, \beta))
\]

so that we can express parametrization for \( \partial \Omega_t = \Gamma_t \) as \( \sigma^t : [\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \rightarrow \mathbb{R}^3 \) with

\[
\sigma^t(\alpha, \beta) = \sigma(\alpha, \beta, t) = (x(\alpha, \beta, t), y(\alpha, \beta, t), z(\alpha, \beta, t))
\]

where for \( t = 0 \),

\[
\sigma^0(\alpha, \beta) = \sigma(\alpha, \beta, 0) = \sigma(\alpha, \beta).
\]
For a point \( p_t = \sigma^t(\alpha_0, \beta_0) \in \Gamma_t \), the tangent plane is
\[
T_{p_t, \Gamma_t} = \text{span}\{\sigma^t_\beta(\alpha_0, \beta_0), \sigma^t_\alpha(\alpha_0, \beta_0)\}
\]
and the normal at this point is
\[
\nu_t(\alpha_0, \beta_0) = \sigma^t_\beta(\alpha_0, \beta_0) \times \sigma^t_\alpha(\alpha_0, \beta_0).
\]
(2.8)
The compatibility condition is given by
\[
D\nabla c \cdot \nu_t = G(u, v, t) \quad \text{on} \quad \Gamma_t,
\]
which can be transformed to fixed boundary \( \Gamma \) as
\[
D\nabla_t(\nu_t(\alpha, \beta, t)) \cdot \nu = G(u, v, t) \quad \text{on} \quad \Gamma
\]
where \( \nu_t \) is defined as in (2.8). With our parametrization, \( \Gamma_t = y_t(\Gamma) \) can be simply expressed as
\[
\sigma_t(\alpha, \beta) = y_t(x(\alpha, \beta), y(\alpha, \beta), z(\alpha, \beta))
\]
so that the area element for \( \Gamma_t \) is \( \sqrt{\det y_t} \, d\sigma \) and hence the pull back of this surface measure on \( \Gamma \) will be \( \sqrt{\det A(t)} \, d\sigma \). We may also consider \( A(t) = (a_{ij}(t)) \), in which case we obtain a more complicated expression for \( \Delta_t \) and rest of the arguments will follow similarly.

3. Notations and main results. Throughout this paper, \( n \geq 2 \) and \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \Gamma (\partial \Omega) \) belonging to the class \( C^{2+\mu} \) with \( \mu > 0 \) such that \( \Omega \) lies locally on one side of its boundary, \( \eta \) is the unit outward normal (from \( \Omega \)) to \( \partial \Omega \), and \( \Delta \) is the Laplace operator. In addition, \( m, k, n, i \) and \( j \) are positive integers.

3.1. Basic function spaces. Let \( \mathcal{U} \) be a bounded domain on \( \mathbb{R}^m \) with smooth boundary such that \( \mathcal{U} \) lies locally on one side of \( \partial \mathcal{U} \). We define all function spaces on \( \mathcal{U} \) and \( \mathcal{U}_T = \mathcal{U} \times (0, T) \). \( \mathcal{L}_q(\mathcal{U}) \) is the Banach space consisting of all measurable functions on \( \mathcal{U} \) that are \( q^{th} (q \geq 1) \) power summable on \( \mathcal{U} \). The norm is defined as
\[
\|u\|_{q, \mathcal{U}} = \left( \int_{\mathcal{U}} |u(x)|^q \, dx \right)^{\frac{1}{q}}.
\]
Also,
\[
\|u\|_{\infty, \mathcal{U}} = \text{ess sup}\{\|u(x)\| : x \in \mathcal{U}\}.
\]
Measurability and summability are to be understood everywhere in the sense of Lebesgue.

If \( p \geq 1 \), then \( \mathcal{W}_p^2(\mathcal{U}) \) is the Sobolev space of functions \( u : \mathcal{U} \rightarrow \mathbb{R} \) with generalized derivatives, \( \partial^s_x u \) (in the sense of distributions) \( |s| \leq 2 \) belonging to \( L_p(\mathcal{U}) \). Here \( s = (s_1, s_2, \ldots, s_n), |s| = s_1 + s_2 + \ldots + s_n, \) \( |s| \leq 2 \), and \( \partial^s_x = \partial^s_1 \partial^s_2 \ldots \partial^s_n \) where \( \partial_i = \frac{\partial}{\partial x_i} \). The norm in this space is
\[
\|u\|_{p, \mathcal{U}}^{(2)} = \sum_{|s| = 0}^2 \|\partial^s_x u\|_{p, \mathcal{U}}.
\]
Similarly, \( \mathcal{W}^{2,1}_p(\mathcal{U}_T) \) is the Sobolev space of functions \( u : \mathcal{U}_T \rightarrow \mathbb{R} \) with generalized derivatives, \( \partial^r_y \partial^s_x u \) (in the sense of distributions) where \( 2r + |s| \leq 2 \) and each derivative belonging to \( L_p(\mathcal{U}_T) \). The norm in this space is
\[
\|u\|_{p, \mathcal{U}_T}^{(2)} = \sum_{2r + |s| = 0}^2 \|\partial^r_y \partial^s_x u\|_{p, \mathcal{U}_T}.
In addition to $W^{2,1}_p(\mathcal{U}_T)$, we will encounter other spaces with two different ratios of upper indices, $W^{1,0}_2(\mathcal{U}_T)$, $W^{1,1}_2(\mathcal{U}_T)$, $V_2(\mathcal{U}_T)$, $V^{1,0}_2(\mathcal{U}_T)$, and $V^{1,1}_2(\mathcal{U}_T)$ as defined in [18].

We also introduce $W^l_p(\mathcal{U})$, where $l > 0$ is not an integer, because initial data will be taken from these spaces. The space $W^l_p(\mathcal{U})$ with nonintegral $l$, is a Banach space consisting of elements of $W^{[l]}_p$ ([l] is the largest integer less than $l$) with the finite norm

$$
\|u\|_{p, l}^{([l])} = \langle u \rangle_{p, l}^{([l])} + \|u\|_{p, l}^{([l])}
$$

where

$$
\|u\|_{p, l}^{([l])} = \sum_{s=0}^{[l]} \|\partial_s^u u\|_{p, l}
$$

and

$$
\langle u \rangle_{p, l}^{([l])} = \sum_{s=0}^{[l]} \left( \int_{\mathcal{U}} dx \int_{\mathcal{U}} |\partial_s^u u(x) - \partial_s^y u(y)|^p \frac{dy}{|x - y|^{n+|[l]-[s]|}} \right)^{\frac{1}{p}}.
$$

The $W^l_p(\partial \mathcal{U}_T)$ spaces with non integral $l$ also play an important role in the study of boundary value problems with nonhomogeneous boundary conditions, especially in the proof of exact estimates of their solutions. It is a Banach space when $p \geq 1$, which is defined by means of parametrization of the surface $\partial \mathcal{U}$. For a rigorous treatment of these spaces, we refer the reader to page 81 of Chapter 2 of [18].

The use of the spaces $W^l_p(\partial \mathcal{U}_T)$ is connected to the fact that the differential properties of the boundary values of functions from $W^{2,1}_p(\mathcal{U}_T)$ and of certain of its derivatives, $\partial_s^u \partial_t^u$, can be exactly described in terms of the spaces $W^l_p(\partial \mathcal{U}_T)$, where $l = 2 - 2r - s - \frac{1}{p}$.

For $0 < \alpha, \beta < 1$, $C^{\alpha, \beta}(\overline{\mathcal{U}_T})$ is the Banach space of Hölder continuous functions $u$ with the finite norm

$$
|u|^{(\alpha)}_{\overline{\mathcal{U}_T}} = \sup_{(x,t) \in \mathcal{U}_T} |u(x,t)| + [u]^{(\alpha)}_{x, \mathcal{U}_T} + [u]^{(\beta)}_{t, \mathcal{U}_T}
$$

where

$$
[u]^{(\alpha)}_{x, \mathcal{U}_T} = \sup_{(x,t), (x',t') \in \mathcal{U}_T, x \neq x'} \frac{|u(x,t) - u(x',t')|}{|x - x'|^\alpha}
$$

and

$$
[u]^{(\beta)}_{t, \mathcal{U}_T} = \sup_{(x,t), (x',t') \in \mathcal{U}_T, t \neq t'} \frac{|u(x,t) - u(x,t')|}{|t - t'|^\beta}.
$$

We shall denote the space $C^{\frac{\alpha}{2}, \frac{\beta}{2}}(\overline{\mathcal{U}_T})$ by $C^{\frac{\alpha}{2}}(\overline{\mathcal{U}_T})$. $C(\mathcal{U}_T, \mathbb{R}^n)$ is the set of all continuous functions $u : \mathcal{U}_T \to \mathbb{R}^n$, and $C^{1,0}(\mathcal{U}_T, \mathbb{R}^n)$ is the set of all continuous functions $u : \mathcal{U}_T \to \mathbb{R}^n$ for which $u_{s_i}$ is continuous for all $1 \leq i \leq n$. $C^{2,1}(\mathcal{U}_T, \mathbb{R}^n)$ is the set of all continuous functions $u : \mathcal{U}_T \to \mathbb{R}^n$ having continuous derivatives $u_{x_i}, u_{x_ix_i}$, and $u_t$ in $\mathcal{U}_T$. Note that similar definitions can be given on $\overline{\mathcal{U}_T}$. 
Assumptions on the system: Let $\Omega \subset \mathbb{R}^n$ with $C^{2+\mu}$ boundary $\Gamma$ for some $\mu > 0$. We consider the system

\[
\begin{align*}
\frac{du}{dt}(x,t) &= \mathcal{L}u(x,t) + f_1(u,v), \text{ in } \Omega \times [0,T) \\
\frac{dv}{dt}(x,t) &= \tilde{\mathcal{L}}v(x,t) + f_2(u,v), \text{ in } \Omega \times [0,T) \\
\nabla_t u \cdot \eta &= g_1(u,v), \\
\nabla_t v \cdot \eta &= g_2(u,v), \text{ on } \partial \Omega \times [0,T) \\
u = u_0, \\
v = v_0, \text{ in } \Omega \times \{0\}
\end{align*}
\]

with the operator

\[
\mathcal{L} = D \sum_{i=1}^{n} \frac{1}{\lambda_i(t)} \frac{\partial^2}{\partial x_i^2} - \frac{\left( \prod_{i=1}^{n} \lambda_i(t) \right)'}{\prod_{i=1}^{n} \lambda_i(t)} = D\Delta_t - a(t)
\]

and

\[
\tilde{\mathcal{L}} = \tilde{D}\Delta_t - a(t)
\]

where

\[
a(t) = \left( \frac{\prod_{i=1}^{n} \lambda_i(t)}{\prod_{i=1}^{n} \lambda_i(t)} \right)'.
\]

We assume that there exists constants $\Lambda_1, \Lambda_2 > 0$ and $k_1, k_2 > 0$ such that

\[
\Lambda_1 \leq \frac{1}{\Lambda_2(t)} \leq \Lambda_2, \quad i = 1, \ldots, n
\]

Here

\[
\nabla_t = A(t)^{-1} \nabla = \left( \frac{1}{\lambda_1(t)} \partial_{x_1}, \frac{1}{\lambda_2(t)} \partial_{x_2}, \ldots, \frac{1}{\lambda_n(t)} \partial_{x_n} \right)
\]

and $\eta$ is the unit outward normal vector on $\Gamma$.

For sake of completeness, we also mention here the extension to $m$ components for evolving domains. That is, let $\mathbf{u} = (u_1, \ldots, u_m)$ be solution of the system

\[
\begin{align*}
\frac{\partial u_i}{\partial t}(x,t) &= \mathcal{L}u_i(x,t) + f_i(u), \text{ in } \Omega \times [0,T) \\
\nabla_t u_i \cdot \eta &= g_i(u) \text{ on } \partial \Omega \times [0,T) \\
u_i &= u_{i0}, \text{ in } \Omega \times \{0\}
\end{align*}
\]

with

\[
\mathcal{L} = D \sum_{i=1}^{n} \frac{1}{\lambda_k(t)} \frac{\partial^2}{\partial x_k^2} - \frac{\left( \prod_{i=1}^{n} \lambda_i(t) \right)'}{\prod_{i=1}^{n} \lambda_i(t)} = D\Delta_t - a(t)
\]

$a(t)$ and $\nabla_t$ as before. We remark that throughout, $\mathbb{R}_+^m$ is the nonnegative orthant in $\mathbb{R}^m$, $m \geq 2$. Here we list the assumptions required to prove our results for a general $m$ component system, with the understanding that for $m = 2$, we denote $u_1 = u$, $u_2 = v$ and $d_1 = D$, $d_2 = \tilde{D}$. 

(V\textsubscript{N}) $u_0 = (u_{0i}) \in C^2(\bar{\Omega})$ and is componentwise nonnegative on $\bar{\Omega}$. Moreover, $u_0$ satisfies the compatibility condition

$$d_i \frac{\partial u_0}{\partial n} = g_i(u_0) \quad \text{on } \Gamma, \quad \text{for all } i = 1, \ldots, m.$$ 

(V\textsubscript{F}) $f_i, g_i : \mathbb{R}^m \to \mathbb{R}$, for $i = 1, \ldots, m$ are locally Lipschitz.

(V\textsubscript{QP}) $f$ and $g$ are quasi positive. That is, for each $i = 1, \ldots, m$, if $u \in \mathbb{R}_+^m$ with $u_i = 0$ then $f_i(u), g_i(u) \geq 0$.

(V\textsubscript{L1}) There exists constants $b_j > 0$ and $L_1 > 0$ such that

$$\sum_{j=1}^m b_j f_j(z), \sum_{j=1}^m b_j g_j(z) \leq L_1 \left( \sum_{j=1}^m z_j + 1 \right) \quad \text{for all } z \in \mathbb{R}_+^m.$$ 

(V\textsubscript{Poly}) For $i = 1, \ldots, m$, $f_i$ and $g_i$ are polynomially bounded. That is, there exists $K_{fg} > 0$ and a natural number $l$ such that

$$f_i(u,v), g_i(u,v) \leq K_{fg} (u + v + 1)^l \quad \text{for all } (u,v) \in \mathbb{R}_+^m.$$ 

Under the assumption that $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_m)$ are locally Lipschitz, we are able to prove the following local existence result.

**Theorem 3.1. (Local Existence)** Suppose (V\textsubscript{N}), (V\textsubscript{F}) and (V\textsubscript{QP}) hold. Then there exists $T_{\text{max}} > 0$ such that (3.1) has a unique, maximal, component-wise nonnegative solution $(u,v)$ with $T = T_{\text{max}}$. Moreover, if $T_{\text{max}} < \infty$ then

$$\limsup_{t \to T_{\text{max}}} \|u(\cdot,t)\|_{\infty,\Omega} + \limsup_{t \to T_{\text{max}}} \|v(\cdot,t)\|_{\infty,\Omega} = \infty.$$ 

We remark that this local existence result is true for $m$ components with $m \geq 2$ though we have indicated the proof here only for $m = 2$. The following result gives global existence of solutions of (3.1) in case we know that one of the components is bounded by a suitable function.

**Theorem 3.2.** Suppose (V\textsubscript{N}), (V\textsubscript{F}), (V\textsubscript{QP}), (V\textsubscript{L1}) and (V\textsubscript{Poly}) hold, and let $T_{\text{max}} > 0$ be given in Theorem 3.1. If there exists a nondecreasing function $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $\|v(\cdot,t)\|_{\infty,\Omega} \leq h(t)$ for all $0 \leq t < T_{\text{max}}$, and there exists $K > 0$ so that whenever $a \geq K$ there exists $L_a \geq 0$ so that

$$af_1(u,v) + f_2(u,v), ag_1(u,v) + g_2(u,v) \leq L_a(u + v + 1), \quad \text{for all } (u,v) \in \mathbb{R}_+^2,$$

(3.9)

then (3.1) has a unique component-wise nonegative global solution.

In order to prove global wellposedness, we need Hölder estimates of the solution of the associated linearized problem. The Hölder estimates in Theorem 3.6 of [32] are extended to a more general operator $\mathcal{L}$ described in (3.2).

Thus, consider the equation

$$d \nabla \varphi \cdot \eta = \mathcal{L} \varphi + \theta \quad \text{for } (x,t) \in \Omega \times (0,T)$$

$$\varphi(x,0) = \varphi_0(x) \quad \text{for } x \in \Omega$$

(3.10)

where $\eta$ denotes the outward unit normal vector on $\Gamma$. Then,
Theorem 3.3. Let $p > n + 1$, $T > 0$, $\theta \in L_p(\Omega \times (0,T))$, $\varphi_1 \in L_p(\Gamma \times (0,T))$ and $\varphi_0 \in W^2_p(\Omega)$ such that

$$d\frac{\partial \varphi_0}{\partial \eta} = \varphi_1(x,0) \text{ on } \Gamma.$$  

Then, there exists a unique weak solution $\varphi(x)$ such that

$$\varphi \in V^{1.5}_2(\Omega_T) \text{ of } (3.10) \text{ and a constant } C_{p,T,\|\det A(t)\|_\infty} > 0 \text{ independent of } \theta, \varphi_1, \varphi_0 \text{ such that for } 0 < \beta < 1 - \frac{n+1}{p}$$

$$|\varphi|^{\beta}_{\Omega_T} \leq C_{p,T,\|\det A(t)\|_\infty} \left(\|\theta\|_{p,\Omega_T} + \|\varphi_1\|_{p,\Gamma_T} + \|\varphi_0\|_{p,\Omega}^2\right).$$  

Theorem 3.4. (Global Existence) Suppose $(V_N)$, $(V_F)$, $(V_{QP})$, $(V_{Poly})$ and the condition $(V_L)$ described below hold.

$$(V_L) \text{ There exists a constant } K > 0, \text{ such that for any } a = (a_1, ..., a_{m-1}, a_m) \in \mathbb{R}^m \text{ with } a_1, ..., a_{m-1} \geq K, \text{ and } a_m = 1, \text{ there exists a constant } L_a \geq 0 \text{ such that}$$

$$\sum_{j=1}^{m} a_j f_j(z), \sum_{j=1}^{m} a_j g_j(z) \leq L_a \left(\sum_{j=1}^{m} z_j + 1\right) \text{ for all } z \in \mathbb{R}^m.$$  

Then, $(3.7)$ has a unique component-wise nonegative global solution.

Note that defining $c(y,t) = u(A(t)^{-1}y,t)$ for $y \in \Omega_t$, above results can be translated to the solutions $c = (c_1, ..., c_m)$ of $(1.1)$ as follows:

Theorem 3.5. (Local existence for evolving domain) Suppose $(V_N)$, $(V_F)$ and $(V_{QP})$ hold. Then there exists $T_{\text{max}} > 0$ such that $(1.1)$ has a unique, maximal, component-wise nonegative solution $c$ with $T = T_{\text{max}}$. Moreover, if $T_{\text{max}} < \infty$ then for all $i = 1, ..., m$,

$$\lim_{t \to T_{\text{max}}} \sup \|c_i(\cdot,t)\|_{\infty,\Omega_t} = \infty.$$  

Theorem 3.6. For $m = 2$, suppose $(V_N)$, $(V_F)$, $(V_{QP})$, $(V_{L1})$ and $(V_{Poly})$ hold, and let $T_{\text{max}} > 0$ be given in Theorem 3.1. If there exists a nondecreasing function $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $\|c_2(\cdot,t)\|_{\infty,\Omega_t} \leq h(t)$ for all $0 \leq t < T_{\text{max}}$, and there exists $K > 0$ so that whenever $a \geq K$ there exists $L_a \geq 0$ so that

$$a f_1(u,v) + f_2(u,v), a g_1(u,v) + g_2(u,v) \leq L_a(u + v + 1), \text{ for all } (u,v) \in \mathbb{R}^2_+,$$  

then $(1.1)$ has a unique component-wise nonegative global solution $c = (c_1, c_2)$.

Theorem 3.7. (Global existence for evolving domain) Suppose $(V_N)$, $(V_F)$, $(V_{QP})$, $(V_{Poly})$ and the condition $(V_L)$ described below hold.

$$(V_L) \text{ There exists a constant } K > 0, \text{ such that for any } a = (a_1, ..., a_{m-1}, a_m) \in \mathbb{R}^m \text{ with } a_1, ..., a_{m-1} \geq K, \text{ and } a_m = 1, \text{ there exists a constant } L_a \geq 0 \text{ such that}$$

$$\sum_{j=1}^{m} a_j f_j(z), \sum_{j=1}^{m} a_j g_j(z) \leq L_a \left(\sum_{j=1}^{m} z_j + 1\right) \text{ for all } z \in \mathbb{R}^m.$$  

Then, $(1.1)$ has a unique component-wise nonegative global solution.

In the next section we obtain estimates for the linearized problem associated to $(3.7)$. 
4. Hölder estimates: Proof of Theorem 3.3. We begin by listing some of the results from [18] which will be used in this as well as upcoming sections. Using the notations therein, we let

\[ \mathcal{L}(x,t,\partial_x,\partial_t) = \partial_t - \sum_{i,j=1}^{n} a_{i,j}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t) \frac{\partial}{\partial x_i} + a(x,t) \]

denote a uniformly parabolic operator. Consider the Dirichlet problem

\[ \mathcal{L}u = f(x,t) \]

\[ u|_{\partial\Omega_T} = \Phi(x,t), \quad u|_{t=0} = \phi(x) \tag{4.1} \]

then Theorem 9.1 from [18] states

Lemma 4.1. Let \( q > 1 \). Suppose that the coefficients \( a_{ij} \) of the operator \( \mathcal{L} \) are bounded continuous function in \( \mathcal{U}_T \), while \( a_i \) and \( a \) have finite norms \( \|a_i\|_{r,\mathcal{U}_T} \) and \( \|a\|_{s,\mathcal{U}_T} \) respectively, where

\[ r = \begin{cases} \max(q,n+2) & \text{for } q \neq n+2 \\ n+2+\epsilon & \text{for } q = n+2 \end{cases} \]

and

\[ s = \begin{cases} \max(q,\frac{n+2}{2}) & \text{for } q \neq \frac{n+2}{2} \\ \frac{n+2}{2} + \epsilon & \text{for } q = \frac{n+2}{2} \end{cases} \]

Let \( \partial \Omega \subset C^{2+\mu} \) and \( \epsilon > 0 \) is very small. Suppose the quantities \( \|a_i\|_{r,\mathcal{U}_T,t+\tau} \) and \( \|a\|_{s,\mathcal{U}_T,t+\tau} \) tend to zero for \( \tau \to 0 \). Then for any \( f \in L_q(\mathcal{U}_T) \), \( \phi \in W^{2,-\frac{2}{2}}_q(\Omega) \) and \( \Phi \in W^{2,-\frac{1}{2},1-\frac{1}{2}}_q(\partial\mathcal{U}_T) \) with \( q \neq \frac{3}{2} \), satisfying the case \( q > \frac{3}{2} \) the compatibility condition of zero order

\[ \phi|_{\partial\Omega} = \Phi|_{t=0} \]

system has a unique solution \( u \in W^{2,1}_q(\mathcal{U}_T) \). Moreover it satisfies the estimate

\[ \|u\|_{q,\mathcal{U}_T}^{(2)} \leq c \left( \|f\|_{q,\mathcal{U}_T} + \|\phi\|_{q,\Omega}^{(2-\frac{q}{2})} + \|\Phi\|_{q,\partial\mathcal{U}_T}^{(2-\frac{1}{2})} \right) \].

Now for the Neumann problem

\[ \mathcal{L}u = f(x,t) \]

\[ u|_{t=0} = \phi(x) \tag{4.2} \]

\[ \sum_{i=1}^{n} b_i(x,t) \partial_x u + b(x,t)u \mid_{\partial\mathcal{U}_T} = \Phi(x,t) \]

where we assume \( \sum_{i=1}^{n} b_i(x,t)\eta_i(x) \geq \delta > 0 \) everywhere on \( \partial\mathcal{U}_T \), \( \eta \) denoting the unit outward normal vector to \( \partial\mathcal{U}_T \). Then Neumann counterpart of above Lemma can be written as follows.

Lemma 4.2. Let \( p > 1 \) and suppose that \( \theta \in L_p(\Omega \times (0,T)) \), \( \varphi_0 \in W^{2,-\frac{2}{2}}_p(\Omega) \), and \( \gamma \in W^{1-\frac{1}{2},-\frac{1}{2}}_p(\Gamma \times (0,T)) \) with \( p \neq 3 \). In addition, when \( p > 3 \) assume

\[ d \frac{\partial \varphi_0}{\partial \eta} = \gamma(x,0) \quad \text{on } \Gamma. \]
Then (4.2) has a unique solution \( \varphi \in W^{2,1}_p(U_T) \) and there exists \( C \) independent of \( \theta, \varphi_0 \) and \( \gamma \) such that
\[
||\varphi||^{(2)}_{p,U_T} \leq C \left( ||\theta||_{p,U_T} + ||\varphi||^{(2-\frac{q}{2})}_{p,\Omega} + ||\gamma||^{(1-\frac{q}{2})}_{p,\partial\Omega} \right).
\]

We will also need the following Corollary from [18].

**Corollary 1.** If the conditions of Lemma 4.1 are fulfilled for \( q > \frac{n+2}{2} \) then the solution of problem (4.1) satisfies a Hölder condition in \( x \) and \( t \). Moreover, when \( q > n + 2 \) then the derivatives of the associated Neumann boundary value problem will also satisfy Hölder condition in \( x \) and \( t \).

Next, we will prove the Hölder estimates for the solution of the linearized Neumann problem associated to (3.7) corresponding to the operator \( \Delta_t \). The ideas for these estimates were developed in Section 5 of [32] and here we adapt those techniques for our operator \( \Delta_t \). For this section, we will make further reduction by writing \( \tilde{u} = \left( \sqrt{\prod_{i=1}^{n} \lambda_i(t)} \right) u \) so that \( \tilde{u} \) solves the equation
\[
\frac{d\tilde{u}}{dt}(x,t) = D\Delta_t \tilde{u}(x,t) \quad \text{for} \quad (x,t) \in \Omega \times [0,T)
\]
where
\[
\Delta_t = \sum_{i=1}^{n} \frac{1}{\lambda_i(t)^2} \frac{\partial^2}{\partial x_i^2}.
\]

With this reduction, instead of working with equation (3.10), it suffices to obtain estimates of the equation
\[
\begin{align*}
\frac{\partial \varphi}{\partial t} &= \Delta_t \varphi + \theta \quad \text{for} \quad (x,t) \in \Omega \times (0,T) \\
d\nabla \varphi \cdot \eta &= \varphi_1 \quad \text{for} \quad (x,t) \in \Gamma \times (0,T) \\
\varphi(x,0) &= \varphi_0(x) \quad \text{for} \quad x \in \Omega.
\end{align*}
\]

Infact, the results of this section hold for a general \( \Delta_t = \sum_{i,j=1}^{n} a_{i,j}(t) \frac{\partial^2}{\partial x_i \partial x_j} \) with a positive definite \( A(t) = (a_{i,j}(t)) \) where the coefficient matrix is function of \( t \) alone. Extension of these results to more general operator will appear in a forthcoming work.

The proof of Theorem 3.3 will follow arguing as in the proof of Theorem 3.6 of [32] for (4.5). Here we point out the necessary changes when we replace the usual Laplacian with \( \Delta_t \) in the results used. Firstly, the following Lemma from Pg 351, [18] gives \( W^{2,1}_p(U_T) \) on the solutions of (3.10).

**Lemma 4.3.** Let \( p > 1 \). Suppose \( \theta \in L_p(U_T) \), \( \varphi_0 \in W^{(2-\frac{q}{2})}(\Omega) \) and let \( \theta \in W^{(1-\frac{q}{2})}(\Gamma \times (0,T)) \) with \( p \neq 3 \). In case \( p > 3 \) we further assume that
\[
\frac{d}{d\eta} \varphi_0 = \varphi_1 \quad \text{on} \quad \Gamma \times \{0\}.
\]

Then equation (3.10) has a unique solution \( \varphi \in W^{2,1}_p(U_T) \) and there exists a constant \( C(\Omega,p,T) \), independent of \( \theta, \varphi_0 \) and \( \varphi_1 \), such that
\[
||\varphi||^{(2)}_{p,U_T} \leq C(\Omega,T,p,T) \left( ||\theta||_{p,U_T} + ||\varphi_0||^{(2-\frac{q}{2})}_{p,\Omega} + ||\varphi_1||^{(1-\frac{q}{2})}_{p,\partial\Omega} \right).
\]
where $\tilde{A}(s) = A(s)^{-1}$.

For $0 < \varepsilon < t$, we can define the operator $J_\varepsilon(f)$ corresponding to our equation as

$$J_\varepsilon(f)(Q,t) = \int_0^{t-\varepsilon} \int_{\Gamma} \frac{<\tilde{A}(s)(y-Q),\eta_Q>}{\sqrt{\det A(s)(t-s)^{2+1}}} \exp\left(-\frac{\langle\tilde{A}(s)(Q-y),(Q-y)\rangle}{4\pi(t-s)}\right) f(s,y) d\sigma ds. \tag{4.7}$$

Noticing that the change of variables gives

$$J_\varepsilon(f) = \int_0^{t-\varepsilon} \int_{\Gamma} \frac{<\tilde{A}(s)(y-Q),\eta_Q>}{(t-s)^{2+1}} \exp\left(-\frac{\langle(Q-y),(Q-y)\rangle}{4\pi(t-s)}\right) F(s,y) d\sigma ds$$

where $F(s,y) = f(s,A(s)^{-1}y)$, the estimates and properties of the operator $J_\varepsilon$ can be summarized as follows.

**Proposition 1. (Fabes-Riviere)** Assume $\Omega$ is a $C^1$ domain, $Q \in \Gamma$ and $\eta_Q$ denote the unit outward normal to $\Gamma$ at $Q$. For $0 < \varepsilon < t$, let the functional $J_\varepsilon(f)$ be defined as in (4.7). Then

1. for $1 < p < \infty$ there exists $C(p,||A||_\infty) > 0$ such that

$$J(f)(Q,t) = \sup_{0 < \varepsilon < t} |J_\varepsilon(f)(Q,t)|$$

satisfies

$$||J(f)||_{L_p(\Gamma \times (0,T))} \leq C(p,||A||_\infty)||f||_{L_p(\Gamma \times (0,T))} \text{ for all } f \in L_p(\Gamma \times (0,T));$$

2. $\lim_{\varepsilon \to 0^+} J_\varepsilon(f) = J(f)$ exists in $L_p(\Gamma \times (0,T))$ and pointwise for almost every $(Q,t) \in \Gamma \times (0,T)$ provided $f \in L_p(\Gamma \times (0,T), 1 < p < \infty$;

3. $cI + J$ is invertible on $L_p(\Gamma \times (0,T))$ for each $1 < p < \infty$ where $I$ is the identity operator and $c \not= 0$ in $\mathbb{R}$.

Note that now the constants will also depend on the matrix $A$. For $Q \in \Gamma$, $(x,t) \in U_T$ and $t > s$, define

$$W(t-s,x,Q) = \frac{1}{\sqrt{\det A(s)(t-s)^{2+1}}} \exp\left(-\frac{\langle\tilde{A}(s)(Q-y),(Q-y)\rangle}{4\pi(t-s)}\right),$$

and $g(Q,t) = -2(-c_n I + J)^{-1} \gamma(Q,t)$ \tag{4.8}

where $c_n = \frac{\omega_n H(0)}{2}$, $\omega_n = \text{surface area of unit sphere in } \mathbb{R}^n$ and

$$H(0) = \int_0^\infty \frac{1}{s^{n/2+1}} \exp\left(-\frac{1}{4s}\right) ds.$$

Referring to the Theorem 2.4 in [8] we have the following definition.
**Definition 4.4.** A function $\varphi$ is a classical solution of the system \((3.10)\) with $d = 1$ and $\gamma \in L_p(\Gamma \times (0, T))$ for $p > 1$ if and only if

$$
\varphi(x, t) = \int_0^t \int_{\Gamma} W(t-s, x, Q)g(Q, s) \, d\sigma \, ds \quad \text{for all } (x, t) \in U_T. \tag{4.9}
$$

We claim that the classical solution $\varphi$ of \((4.5)\) defined as in \((4.9)\) is Hölder continuous. For $(x, T), (y, \tau) \in U_T$, $0 < \tau < T$, consider the difference

$$
\varphi(x, T) - \varphi(y, \tau) = \int_0^T \int_{\Gamma} [W(T-s, x, Q) - W(\tau-s, y, Q)] \, g(Q, s) \, d\sigma \, ds.
$$

The following three Lemmas provide the required estimates.

**Lemma 4.5.** Let $p > n + 1$. Suppose $(x, T), (y, \tau) \in U$ with $0 < \tau < T$ and

$$
\mathcal{R}^c = \{(Q, s) \in \Gamma \times (0, \tau) : |x - Q| + |T - s|^{\frac{1}{2}} < 2|x - y| + |T - \tau|^{\frac{1}{2}}\}.
$$

Then for $0 < a < 1 - \frac{n+1}{p}$, there exists $C(p, n, \Omega, T, \|A\|_{\infty}) > 0$ independent of $g \in L_p(\Gamma \times (0, T))$ such that

$$
\int_{\mathcal{R}^c} \left| (W(T-s, x, Q) - W(\tau-s, y, Q)) \, g(Q, s) \right| \, d\sigma \, ds \leq C \left( |x - y| + |T - \tau|^{\frac{1}{2}} \right)^a \|g\|_{p, \Gamma \times [0, \tau]}. \tag{4.10}
$$

**Lemma 4.6.** Let $p > n + 1$. Suppose $(x, T), (y, \tau) \in U$ with $0 < \tau < T$ and

$$
\mathcal{R} = \{(Q, s) \in \Gamma \times (0, \tau) : 2(|x - y| + |T - \tau|^{\frac{1}{2}}) < |x - Q| + |T - s|^{\frac{1}{2}}\}.
$$

Then for $0 < a < 1 - \frac{n+1}{p}$, there exists $C(p, n, \Omega, T, \|A\|_{\infty}) > 0$ independent of $g \in L_p(\Gamma \times (0, T))$ such that

$$
\int_{\mathcal{R}} \left| (W(T-s, x, Q) - W(\tau-s, y, Q)) \, g(Q, s) \right| \, d\sigma \, ds \leq C \left( |x - y| + |T - \tau|^{\frac{1}{2}} \right)^a \|g\|_{p, \Gamma \times [0, \tau]}. \tag{4.11}
$$

**Lemma 4.7.** Let $p > n + 1$ and suppose $(x, T), (y, \tau) \in U$ with $0 < \tau < T$. Then for $0 < a < 1 - \frac{n+1}{p}$, there exists $C(p, n, \Omega, T, \|A\|_{\infty}) > 0$ independent of $g \in L_p(\Gamma \times (0, T))$ such that

$$
\int_\tau^T \int_{\Gamma} |W(T-s, x, Q)g(Q, s)| \, d\sigma \, ds \leq C(|T - \tau|^{\frac{a}{2}} \|g\|_{p, \Gamma \times [\tau, T]}). \tag{4.12}
$$

We refer to the proofs of Lemmas 5.5, 5.6 and 5.7 respectively in [32] which can be repeated verbatim for the above three Lemmas. Similar to Proposition 5.8 in [32], we have the following Hölder estimates for the solution of \((3.10)\).

**Proposition 2.** Let $\gamma \in L_p(\Gamma \times (0, T))$ for $p > n + 1$. Then a solution of \((3.10)\) is Hölder continuous on $\Omega \times (0, T)$ with Hölder exponent $0 < a < 1 - \frac{n+1}{p}$ and there is a constant $K_p > 0$, depending on $p, \Omega, T$ and $\|A\|_{\infty}$, independent of $\varphi_1$ such that

$$
|\varphi(x, T) - \varphi(y, \tau)| \leq K_p \left( |T - \tau|^{\frac{1}{2}} + |x - y| \right)^a \|\varphi_1\|_{p, \Gamma \times (0, T)}. \tag{4.13}
$$
for all \((x, T), (y, \tau) \in \mathcal{U}\).

The proof of Theorem 3.3 can now be completed.

5. Local existence of the solution. Here we illustrate the proof of local existence of solutions of (3.7) for the case \(m = 2\), which can be easily extended to \(m\) component case. In order to prove local existence of the solution we need the following result.

**Theorem 5.1.** If \(f = (f_1, f_2), g = (g_1, g_2)\) are Lipschitz functions then the equation (3.1) has a unique global solution.

**Proof.** Here we sketch first few steps of the proof to indicate that the linear term can be controlled. Let \(T > 0\) and \(u_0, v_0 \in W^2_p(\Omega) \times W^2_p(\Omega)\) such that they satisfy the compatibility condition

\[
\frac{\partial u_0}{\partial \eta} = g_1(u_0, v_0) \quad \text{and} \quad \frac{\partial v_0}{\partial \eta} = g_2(u_0, v_0).
\]

Set 

\[
X = \{(u, v) \in C([0, T]) \times C([0, T]) : u(x, 0) = 0 \text{ and } v(x, 0) = 0 \text{ for all } x \in \Omega\}.
\]

Note that \((X, \| \cdot \|_{\infty})\) is a Banach space. Let \((u, v) \in X\) and consider the problem

\[
\begin{align*}
\frac{dU}{dt}(x, t) &= LU(x, t) + f_1(u + u_0, v + v_0), \quad \text{in } \Omega \times [0, T) \\
\frac{dV}{dt}(x, t) &= LV(x, t) + f_2(u + u_0, v + v_0), \quad \text{in } \Omega \times [0, T) \\
\nabla_t U \cdot \eta &= g_1(u + u_0, v + v_0), \quad \nabla_t V \cdot \eta = g_2(u + u_0, v + v_0) \quad \text{on } \partial \Omega \times [0, T) \quad (5.1) \\
U &= u_0, \quad V &= v_0 \quad \text{in } \Omega \times \{0\}.
\end{align*}
\]

From Lemma 4.1, (5.1) possess a unique solution \((U, V) \in W^{2,1}_q(\Omega_T) \times W^{2,1}_q(\Omega_T)\). Furthermore, from embedding \((U, V) \in C(\Omega_T \times [0, T]) \times C(\Omega_T \times [0, T])\). Define 

\[
S : X \to X \quad \text{as}
\]

\[
S(u, v) = (U - u_0, V - v_0)
\]

where \((U, V)\) solves (5.1). We will see \(S\) is continuous and compact. Using linearity, \((U - \tilde{U}, V - \tilde{V})\) solves \([0, T)\) the system

\[
\begin{align*}
\frac{d}{dt}(U - \tilde{U})(x, t) &= \mathcal{L}(U - \tilde{U})(x, t) + f_1(u + u_0, v + v_0) - f_1(\tilde{u} + u_0, \tilde{v} + v_0), \\
\frac{d}{dt}(V - \tilde{V})(x, t) &= \mathcal{L}(V - \tilde{V})(x, t) + f_2(u + u_0, v + v_0) - f_2(\tilde{u} + u_0, \tilde{v} + v_0), \\
\nabla_t (U - \tilde{U}) \cdot \eta &= g_1(u + u_0, v + v_0) - g_1(\tilde{u} + u_0, \tilde{v} + v_0) \quad \text{on } \partial \Omega \times [0, T) \\
\nabla_t (V - \tilde{V}) \cdot \eta &= g_2(u + u_0, v + v_0) - g_2(\tilde{u} + u_0, \tilde{v} + v_0) \quad \text{on } \partial \Omega \times [0, T) \\
U - \tilde{U} &= 0, \quad V - \tilde{V} = 0 \quad \text{in } \Omega \times \{0\}. \quad (5.2)
\end{align*}
\]

From Corollary 1, if \(q > n + 2\) then solution of (5.2) is Hölder continuous. Therefore there exists \(C\) independent of \(f\) and \(g\), \(i = 1, 2\), such that

\[
\|U - \tilde{U}\|_{\infty, \Omega_T} + \|V - \tilde{V}\|_{\infty, \Omega_T} \leq C \left\{ \|f_1(u + u_0, v + v_0) - f_1(\tilde{u} + u_0, \tilde{v} + v_0)\|_{q, \Omega_T} \\
+ \|f_2(u + u_0, v + v_0) - f_2(\tilde{u} + u_0, \tilde{v} + v_0)\|_{q, \Omega_T} \\
+ \|g_1(u + u_0, v + v_0) - g_1(\tilde{u} + u_0, \tilde{v} + v_0)\|_{q, \partial \Omega_T} \\
+ \|g_2(u + u_0, v + v_0) - g_2(\tilde{u} + u_0, \tilde{v} + v_0)\|_{q, \partial \Omega_T} \right\}.
\]
Using boundedness of $\Omega$, there exists $\tilde{C} > 0$ such that
\[
\|U - \tilde{U}\|_{\infty,T} + \|V - \tilde{V}\|_{\infty,T} \\
\leq \tilde{C} \left( \|f_1(u + u_0, v + v_0) - f_1(\tilde{u} + u_0, \tilde{v} + v_0)\|_{\infty,T} \\
+ \|f_2(u + u_0, v + v_0) - f_2(\tilde{u} + u_0, \tilde{v} + v_0)\|_{\infty,T} \\
+ \|g_1(u + u_0, v + v_0) - g_1(\tilde{u} + u_0, \tilde{v} + v_0)\|_{\infty,\partial\Omega_T} \\
+ \|g_2(u + u_0, v + v_0) - g_2(\tilde{u} + u_0, \tilde{v} + v_0)\|_{\infty,\partial\Omega_T} \right).
\]

Since $f_i, g_i, i = 1, 2$ are Lipschitz, $S$ is continuous with respect to the sup norm. Now it remain to show that this $S$ is compact. Moreover, $p > n + 2$ from Corollary 1 imples that solution is in fact Hölder continuous therefore $S$ maps bounded sets in $X$ to precompact sets, hence $S$ is compact with respect to sup norm. The uniqueness of the solution follows by deriving the Gronwall’s inequality on $\Omega_T$ by arguments similar to as in the proof of Theorem 6.1 of [32]. Since $T > 0$ was arbitrary, we further conclude the existence of unique global solution.

**Proof of Theorem 3.1:** The proof of the theorem involves truncating the given functions $f, g$ so that the truncated functions are Lipschitz. Precisely, for each $r > k$, we define cut off functions $\phi_r \in C^\infty_0(\mathbb{R}^2, [0, 1])$ and $\psi_r \in C^\infty_0(\mathbb{R}^2, [0, 1])$ such that $\phi_r(z, w) = 1$ when $|z| \leq r$ and $|w| \leq r$, and $\phi_r(z, w) = 0$ for all $|z| > 2r$ or $|w| \geq 2r$. Define $f_r = f\phi_r$ and $g_r = g\psi_r$. We also have $u_0 \in W^2_p(\Omega)$ and $v_0 \in W^2_p(\partial\Omega)$ with $p > n$ and $u_0, v_0$ satisfy the compatibility condition for $p > 3$. Hence, from the Sobolev imbedding theorem, $u_0$ and $v_0$ are bounded functions, i.e., there exists $k > 0$ such that $\|u_0(\cdot)\|_{\infty,\Omega} \leq k$ and $\|v_0(\cdot)\|_{\infty,\partial\Omega} \leq k$. Applying Theorem 5.1, we obtain global solution $(u_r, v_r)$ for each $r$. Then letting $r \to \infty$ we obtain the solution $(u, v)$ with required properties. We refer to [32] for details.

**6. Existence of global solution.** In this section we will prove global existence of solutions of the system (3.1) under given conditions. We begin by proving apriori estimates, in particular, $L_1$ estimate for the solutions of (3.1).

**Lemma 6.1. (L₁-estimates)** Let $(u, v)$ be the unique maximal nonnegative solution to (3.1) and suppose that $T_{\text{max}} < \infty$. If $V_N, V_F$ and $V_{L,1}$ hold, then there exists $C_1(D, \tilde{D}, L_1, k_2)$ such that
\[
\|u(\cdot, t)\|_{1, \Omega} \leq C_1(t) \quad \forall \ 0 \leq t < T_{\text{max}}.
\]

Proof. Adding the two equations in (3.1) and integrating the equation over $\Omega$, we get
\[
\frac{d}{dt} \int_\Omega (u + v) = \int_\Omega \tilde{D}\Delta u + \int_\Omega \tilde{D}\Delta v - \int_\Gamma a(t)(u + v) + \int_\Omega (f_1(u, v) + f_2(u, v)) \\
\leq \int_\Omega (f_1(u, v) + f_2(u, v)) + \int_\Gamma (g_1(u, v) + g_2(u, v)) - \int_\Gamma a(t)(u + v) \\
\leq \int_\Omega (L + k_2)(u + v + 1) + \int_\Gamma (L + k_2)(u + v + 1),
\]
where recall $a(t) = \frac{\lambda_1(t)\lambda_2(t)\lambda_3(t)}{(\lambda_1(t)\lambda_2(t)\lambda_3(t))^r}$ and $a(t) \leq k_2$ for all $t$ by assumption. Fix $0 < T < T_{\text{max}}$, $d > 0$ a constant (to be chosen later), $L_1 > 0$ and consider the
system

\[
\begin{align*}
\varphi_t &= -d\Delta_t \varphi - (L_1 + k_2)\varphi \quad (x, t) \in \Omega \times (0, T) \\
d\nabla_t \varphi \cdot \eta &= (L_1 + k_2)\varphi + 1 \quad (x, t) \in \Gamma \times (0, T) \\
\varphi &= \varphi_T \quad x \in \Omega, \ t = T.
\end{align*}
\] (6.3)

Here, \(\varphi_T \in C^{2+\gamma}(\overline{\Omega})\) for some \(\gamma > 0\) is strictly positive and satisfies the compatibility condition

\[
d\nabla_T \varphi_T \cdot \eta = (L_1 + k_2)\varphi_T + 1 \text{ on } \Gamma \times \{T\}.
\]

From Theorem 5.3 in chapter 4 of [18], \(\varphi \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\Omega} \times [0, T])\), and therefore \(\varphi \in C^{2+\gamma,1+\frac{\gamma}{2}}(\Gamma \times [0, T])\). Moreover, arguing as in the previous section, we conclude \(\varphi \geq 0\). Now, consider

\[
0 = \int_0^T \int_\Omega u(-\varphi_t - d\Delta_t \varphi - (L_1 + k_2)\varphi) \, dt \, dx
\]

\[
= \int_0^T \int_\Omega \varphi(u_t - D\Delta_t u) - (L_1 + k_2)\int_0^T \int_\Omega u\varphi - \int_0^T \int_\Omega uD\nabla_t \varphi \cdot \eta
\]

\[
+ (D - d) \int_0^T \int_\Omega u\Delta_t \varphi + \int_0^T \int_\Omega \varphi D\nabla_t u \cdot \eta + \int_\Omega u(x, 0)\varphi(x, 0) - \int_\Omega u(x, T)\varphi(\cdot, T)
\]

\[
\leq \int_0^T \int_\Omega \varphi(f_1(u, v) - L_1 u) - \frac{D}{d} \int_0^T \int_\Omega (u(L_1 + k_2)\varphi + u) + (D - d) \int_0^T \int_\Omega u\Delta_t \varphi
\]

\[
+ D \int_0^T \int_\Gamma \varphi g_1(u, v) + \int_\Omega u(x, 0)\varphi(x, 0) - \int_\Omega u(x, T)\varphi(\cdot, T). \quad (6.4)
\]

For \(v\) we have the similar equation with \(f_1\) replaced by \(f_2\) and \(g_1\) replaced by \(g_2\), i.e.,

\[
0 \leq \int_0^T \int_\Omega \varphi(f_2(u, v) - L_1 v) - \frac{\dot{D}}{d} \int_0^T \int_\Gamma (v(L_1 + k_2)\varphi + v) + (\dot{D} - d) \int_0^T \int_\Omega v\Delta_t \varphi
\]

\[
+ \dot{D} \int_0^T \int_\Gamma \varphi g_2(u, v) + \int_\Omega v(x, 0)\varphi(x, 0) - \int_\Omega v(x, T)\varphi(\cdot, T). \quad (6.5)
\]
Summing these equations, and making use of \((V_{L1})\) and choosing \(d = \min\{D, \tilde{D}\}\), gives
\[
\int_0^T \int_{\Gamma} (u + v) \leq \int_0^T \int_{\Gamma} (u + v)(1 + (L_1 + k_2)\varphi) \leq \\
\int_0^T \int_{\Omega} L_1 \varphi + \int_0^T \int_{\Gamma} L_1 \varphi + (D - d) \int_0^T \int_{\Omega} u \Delta_t \varphi + \int_0^T \int_{\Gamma} v \Delta_t \varphi \\
+ \int_{\Omega} u_0(x) \varphi(x, 0) - \int_{\Gamma} u(x, T) \varphi_T(x) + \int_{\Omega} v_0(x) \varphi(x, 0) - \int_{\Gamma} v(x, T) \varphi_T(x). \tag{6.6}
\]
Since \(\varphi_T\) is strictly positive, we can choose a \(\delta > 0\) such that \(\delta \leq \varphi(x)\) for all \(x \in \Omega\).
Then (6.6) implies
\[
\delta \int_{\Gamma} (u(x, T) + v(x, T)) + \int_0^T \int_{\Omega} (u + v) \leq \int_0^T \int_{\Omega} L_1 \varphi + \int_0^T \int_{\Gamma} L_1 \varphi \\
+ (D - d) \int_0^T \int_{\Omega} u \Delta_t \varphi + (\tilde{D} - d) \int_0^T \int_{\Gamma} v \Delta_t \varphi + \int_{\Omega} (u_0 + v_0) \varphi(x, 0). \tag{6.7}
\]
Thus, there exist constants \(C_1, C_2 > 0\), depending on \(L_1, d, \varphi_T, u_0, v_0, D, \tilde{D}\), and at most exponentially on \(T\), such that
\[
\delta \int_{\Gamma} (u(x, T) + v(x, T)) + \int_0^T \int_{\Omega} (u + v) \leq C_1 + C_2 \int_0^T \int_{\Omega} (u + v). \tag{6.8}
\]
Now, return to (6.2), and integrate both sides with respect to \(t\) to obtain
\[
\int_0^T \int_{\Omega} (u + v) dx \leq L_1 \left( \int_0^t \int_{\Omega} (u + v) + \int_0^t \int_{\Gamma} (u + v) + t |\Gamma| + t |\Omega| \right) + \int_{\Omega} (u_0 + v_0). \tag{6.9}
\]
The second term on the right hand side of (6.9) can be bounded above by \(L_1\) times the right hand side of (6.8). Using this estimate, and Gronwall’s inequality, we can obtain a bound for \(\int_0^T \int_{\Omega} (u + v)\) that depends on \(T\). Placing this on the right hand side of (6.8) gives a bound for \(\int_{\Gamma} (u(x, T) + v(x, T))\) that depends on \(T\). Applying this to the second integral on the right hand side of (6.2), and using Gronwall’s inequality, gives the result. \(\square\)

**Remark 1.** The above proof can be imitated for \(m\) components to obtain \(L_1\) estimates for solutions of (3.7).

For sake of completeness of our arguments, we state below the Lemma 3.3 proved in [31].

**Lemma 6.2.** Given \(\gamma \geq 1\) and \(\epsilon > 0\), there exists \(C_{\epsilon, \gamma} > 0\) such that
\[
\|v\|_{2, \partial\Omega}^2 \leq \epsilon \|\nabla v\|_{2, \Omega}^2 + C_{\epsilon, \gamma} \|v^\gamma\|_{1, \Omega}^\gamma \tag{6.10}
\]
and
\[ \|v\|_{L^2(\Omega)}^2 \leq \epsilon \|
abla v\|_{L^2(\Omega)}^2 + C_{\epsilon,\gamma} \|v\|_{L^1(\Omega)}^{\frac{2}{\gamma}} \]
for all \( v \in H^1(\Omega) \).

**Proof of Theorem 3.2:** If \( T_{\text{max}} = \infty \), then there is nothing to prove. So, assume \( T = T_{\text{max}} < \infty \). We first claim that under the given assumptions,
\[ \|u\|_{L^p(\Omega_T)} \leq C(p, h(T), L_1, D, \tilde{D}, \Omega, \|\det A(t)\|_{\infty}). \]

(6.12)

We can assume without loss of generality that \( b_1 = b_2 = 1 \) in \((V_{L^1})\). Let \( 1 < p < \infty \), set \( p' = \frac{p}{p-1} \) and choose \( \xi \in L^{p'}(\Omega_T) \) such that
\[ \xi \geq 0 \text{ and } \|\xi\|_{L^{p'}(\Omega_T)} = 1. \]

(6.13)

Furthermore, let \( L_2 \geq \max\{\frac{D_{L_1}}{\tilde{D}}, L_1\} \) and suppose \( \varphi \) solves
\[ \varphi_t + D\Delta_t \varphi = -L_1 \varphi - \xi \quad \text{in } \Omega_T, \]
\[ D\nabla \varphi \cdot \eta = L_2 \varphi \quad \text{on } \Gamma_T, \]
\[ \varphi = 0 \quad \text{in } \Omega \times \{T\}. \]

(6.14)

Though (6.14) may appear to be a backwards heat equation, the substitution \( \tau = T - t \) immediately reveals that it is actually a forward heat equation. Arguing as in the proof of the Theorem 3.1, we conclude that \( \varphi \geq 0 \). In addition, from Lemma 4.1, there is a constant \( C(p, D, \tilde{D}, \Omega, L_1) > 0 \), and independent of \( \xi \) such that
\[ \|\varphi\|_{L^{p'}(\Omega_T)} \leq C. \]

(6.15)

Multiply (6.14) with \((u + v)\) and integrating by parts we have
\[
\int_0^T \int_\Omega (u + v)\xi dx dt
\]
\[
= \int_0^T \int_\Omega (u + v)(-\varphi_t - D\Delta_t \varphi - L_1 \varphi) dx dt
\]
\[
\leq \int_\Omega (u_0 + v_0)\varphi(x, 0) dx - \int_0^T \int_\Omega (u + v)D\Delta_t \varphi dx dt - \int_0^T \int_\Omega L_1(u + v) \varphi dx dt. \]

(6.16)

Multiplying equation (3.1), integrating by parts and using \((V_{L^1})\) we get
\[
\int_0^T \int_\Omega \varphi(u_t + v_t) dx dt
\]
\[
= \int_0^T \int_\Omega \varphi(f_1 + f_2) dx dt + \int_0^T \int_\Omega \varphi(D\Delta_t u + \tilde{D}\Delta_t v) dx dt - \int_0^T \int_\Omega a(t)(u + v) dx dt
\]
\[
\int_0^T \int_\Omega \phi L_1(u + v + 1)dxdt + \int_0^T \int_\Omega \phi (D\Delta_t u + \tilde{D}\Delta_t v)dxdt - \int_0^T \int_\Omega a(t)(u + v)dxdt
\]

\[
\leq \int_0^T \int_\Omega \phi L_1(u + v + 1)dxdt + \int_0^T \int_\Omega \phi (u\Delta_t \phi + \tilde{D}v\Delta_t \phi)dxdt
\]

\[
- \int_0^T \int_\Gamma \int_\Gamma L_2 \phi u - \int_0^T \int_\Gamma \frac{\tilde{D}}{D} L_2 \phi v.
\] (6.17)

Combining (6.16) and (6.17), we have

\[
\int_0^T \int_\Omega (u + v)\xi dxdt \leq \int_\Omega (u_0 + v_0)\phi(x, 0)dx + \int_\Omega (\tilde{D} - D)v\Delta \phi dxdt
\]

\[
+ \int_0^T \int_\Omega L_1 \phi dxdt + \int_0^T \int_\Gamma L_1 \phi d\sigma - \int_0^T \int_\Omega a(t)(u + v)dxdt
\]

\[
- \int_0^T \int_\Gamma L_2 \phi u - \int_0^T \int_\Gamma \frac{\tilde{D}}{D} L_2 \phi v
\]

\[
\leq \int_\Omega (u_0 + v_0)\phi(x, 0)dx + \int_\Omega (\tilde{D} - D)v\Delta \phi dxdt
\]

\[
+ \int_0^T \int_\Omega L_1 \phi dxdt + \int_0^T \int_\Gamma L_1 \phi d\sigma - \int_0^T \int_\Omega a(t)(u + v)dxdt.
\] (6.18)

By assumption, \(\|v(\cdot, t)\|_{\infty, \Omega} \leq h(t)\) and (6.15) implies \(\|\phi\|_{(2,1)}^{(P, \Omega_T)} \leq C_0\). Also, integrating (6.14) reveals that

\[
\int_\Omega \phi(\cdot, 0) = -\int_0^T \int_{\partial\Omega} L_2 \phi - \int_0^T \int_\Omega L_1 \phi + \int_0^T \int_\Omega \xi.
\] (6.19)

Therefore, \(\|\phi(\cdot, t)\|_{1, \Omega}\) can be bounded independent of \(\xi\) by using the norm bound on \(\phi\) and the fact that \(\|\xi\|_{P', \Omega_T} = 1\). In addition, the trace embedding theorem implies \(\|\phi\|_{1, \Omega_T}\) can be bounded in terms of \(\|\phi\|_{(2,1)}^{(P, \Omega_T)}\), which can be bounded independent of \(\xi\), for the same reason as above.

Therefore, by applying duality to (6.18), we see that

\[
\|u\|_{p, \Omega_T} \leq C(p, h(T), L_1, D, \tilde{D}, \Omega).
\] (6.20)

Also, since \(1 < p < \infty\) is arbitrary, we have this estimate for every \(1 < p < \infty\). Moreover, the sup norm bound on \(v\), the \(L_p(\Omega_T)\) bounds on \(u\) for all \(1 < p < \infty\), and \((V_{Polg})\) imply we have \(L_q(\Omega_T)\) bounds on \(f(u, v)\) and \(g(u, v)\) for all \(1 < q < \infty\).

Now, we use the bounds above and assumption (3.9) to show \(\|u\|_{p, \Omega_T}\) is bounded for all \(1 < p < \infty\). To this end, we employ a modification of an argument given
in [1] for the case \( m = 2 \). Suppose \( p \in \mathbb{N} \) such that \( p \geq 2 \), and choose a constant 
\( \Theta > \max \left\{ K, \frac{D + \tilde{D}}{2\sqrt{DD}} \right\} \). For \( a, b \geq 0 \) we denote \( w^{(a,b)} := u^a v^b \) and define the polynomial 
\[
P(u, v, p, \Theta)^{\beta^2} = \sum_{\beta = 0}^{p} \frac{p!}{\beta!(p - \beta)!} \Theta^\beta w^{(\beta, p-\beta)}.
\] (6.21)

In general, to fix notation we let 
\[
P(u, v, p, \Theta)^{c(\beta)} = \sum_{\beta = 0}^{p} \frac{p!}{\beta!(p - \beta)!} \Theta^{c(\beta)} w^{(\beta, p-\beta)},
\] (6.22)

where \( c(p) \) is a prescribed function of \( p \). Note that 
\[
\frac{\partial P}{\partial t} = \sum_{\beta = 0}^{p} \frac{p!}{\beta!(p - \beta)!} \Theta^\beta \left( \beta w^{(\beta-1, p-\beta)} u_t + (p - \beta) w^{(\beta, p-\beta-1)} v_t \right)
\] 
\[
= \left( p\Theta^p v_t + p\Theta^p v_t \right) dx + X_1 + X_2,
\] (6.23)

where
\[
X_1 = \sum_{\beta = 1}^{p-1} \frac{p!}{(\beta - 1)!(p - \beta)!} \Theta^\beta w^{(\beta-1, p-\beta)} u_t
\] 
\[
= p\Theta^p v_t u_t + \sum_{\beta = 2}^{p-1} \frac{p!}{(\beta - 1)!(p - \beta)!} \Theta^\beta w^{(\beta-1, p-\beta)} u_t
\] 
\[
= p\Theta^p v_t u_t + \sum_{\beta = 1}^{p-2} \frac{p!}{\beta!(p - \beta - 1)!} \Theta^{(\beta+1)} w^{(\beta, p-\beta-1)} (u_t)_t
\] 
\[
= p\Theta^p v_t u_t + \sum_{\beta = 1}^{p-2} \frac{p!}{\beta!(p - \beta - 1)!} \Theta^\beta w^{(\beta, p-\beta-1)} \Theta^{2\beta+1} u_t
\] (6.24)

and
\[
X_2 = \sum_{\beta = 1}^{p-1} \frac{p!}{\beta!(p - \beta - 1)!} \Theta^\beta w^{(\beta, p-\beta-1)} v_t
\] 
\[
= p\Theta^{(p-1)} v_t u_t + \sum_{\beta = 1}^{p-2} \frac{p!}{\beta!(p - \beta - 1)!} \Theta^\beta w^{(\beta, p-\beta-1)} v_t.
\] (6.25)

Combining (6.23)-(6.25) we get 
\[
\frac{\partial P}{\partial t} = \sum_{\beta = 0}^{p-1} \frac{p!}{\beta!(p - \beta)!} \Theta^\beta w^{(\beta, p-\beta-1)} \left( \Theta^{2\beta+1} u_t + v_t \right).
\] (6.26)

Clearly, above steps hold even if we differentiate with respect to any variable \( x_i, \) 
\( i = 1, 2, 3, \) i.e.,
\[
\frac{\partial P}{\partial x_i} = \sum_{\beta = 0}^{p-1} \frac{p!}{\beta!(p - \beta)!} \Theta^\beta w^{(\beta, p-\beta-1)} \left( \Theta^{2\beta+1} \frac{\partial u}{\partial x_i} + \frac{\partial v}{\partial x_i} \right).
\] (6.27)
Using the fact that \((u,v)\) satisfies the equation (3.1).

\[
\frac{\partial P}{\partial t} = \sum_{\beta=0}^{p-1} \frac{p!}{\beta!(p-1-\beta)!} \Theta^{2\beta} w^{(\beta,p-1-\beta)} \left( \Theta^{2\beta+1} (Lu + f_1(u,v)) + \tilde{L}v + f_2(u,v) \right). 
\]

Integrating over \(\Omega\), we have

\[
\int_{\Omega} \frac{\partial P}{\partial t} \, dx = I + II + III \quad (6.29)
\]

where

\[
I = \int_{\Omega} \sum_{\beta=0}^{p-1} \frac{p!}{\beta!(p-1-\beta)!} \Theta^{2\beta} w^{(\beta,p-1-\beta)} \left( \Theta^{2\beta+1} f_1(u,v) + f_2(u,v) \right) \, dx \quad (6.30)
\]

and

\[
II = \int_{\Omega} \sum_{\beta=0}^{p-1} \frac{p!}{\beta!(p-1-\beta)!} \Theta^{2\beta} w^{(\beta,p-1-\beta)} \left( \Theta^{2\beta+1} D\Delta_t u + \tilde{D}\Delta_t v \right) \, dx. \quad (6.31)
\]

\[
III = - \int_{\Omega} \sum_{\beta=0}^{p-1} \frac{p!}{\beta!(p-1-\beta)!} \Theta^{2\beta} w^{(\beta,p-1-\beta)} a(t) \left( \Theta^{2\beta+1} u + v \right) \, dx \leq -k_1 C(p,\Theta) \int_{\Omega} (u + v)^p \, dx. \quad (6.32)
\]

Choosing \(\Theta \geq K\) and applying (3.9), we have

\[
I \leq \int_{\Omega} \sum_{\beta=0}^{p-1} \frac{p!}{\beta!(p-1-\beta)!} \Theta^{2\beta} w^{(\beta,p-1-\beta)} L_\Theta (u + v + 1) \, dx \leq L_\Theta \int_{\Omega} [(u + v)^p + (u + v)^{p-1}] \, dx \leq C(p,h(T),L_1,D,\tilde{D},\Omega,||det A(t)||_\infty, L_\Theta) \quad \text{from } (6.20). \quad (6.33)
\]

While

\[
II = \int_{\Omega} \sum_{\beta=0}^{p-1} \frac{p!}{\beta!(p-1-\beta)!} \Theta^{2\beta} w^{(\beta,p-1-\beta)} \left( \Theta^{2\beta+1} D\Delta_t u + \tilde{D}\Delta_t v \right) \, dx
\]

\[
= \int_{\Omega} \sum_{\beta=0}^{p-1} \frac{p!}{\beta!(p-1-\beta)!} \Theta^{2\beta} w^{(\beta,p-1-\beta)} \left( \Theta^{2\beta+1} g_1(u,v) + g_2(u,v) \right) \, dx
\]

\[
- \int_{\Omega} \sum_{\beta=0}^{p-1} \frac{p!}{\beta!(p-1-\beta)!} \Theta^{2\beta} \left< \Theta^{2\beta+1} D\nabla_t w^{(\beta,p-1-\beta)}, \nabla_t u \right> \, dx
\]

\[
- \int_{\Omega} \sum_{\beta=0}^{p-1} \frac{p!}{\beta!(p-1-\beta)!} \Theta^{2\beta} \left< \tilde{D}\nabla_t w^{(\beta,p-1-\beta)}, \nabla_t v \right> \, dx. \quad (6.34)
\]
We have \( \nabla_t w^{(β,p-1-β)} = βu^{β-1}v^{(p-1-β)}\nabla_t u + u^β(p-1-β)v^{(p-2-β)}\nabla_t v \), hence we can write
\[
\int_Ω \sum_{β=0}^{p-2} \frac{p_1}{β!(p-1-β)!} Θ^{β^2} \left( <Θ^{2β+1}Dνt w^{(β,p-1-β)}, νt u > + <Dνt w^{(β,p-1-β)}, νt v > \right) dx
\geq \int_Ω \sum_{β=0}^{p-2} \frac{p_1}{β!(p-2-β)!} Θ^{β^2} w^{(β,p-2-β)} \sum_{k=1}^{3} \frac{1}{λ_k(l)^2} \left( B(Θ, D, ˜D) \left( \frac{∂x_k u}{∂x_k v} \right), (∂x_k u, ∂x_k v) \right)
\]
where
\[
B(Θ, D, ˜D) = \left( \frac{DΘ^{4β+4}}{(D+ ˜D)^{2β+1}} \frac{(D+ ˜D)}{2} \right).
\]
Again choosing \( Θ \) sufficiently large so that the matrix \( B(Θ, D, ˜D) \) is positive definite and recalling that \( 0 < Λ_1 ≤ \frac{1}{λ_k(l)} ≤ Λ_2 \), there exists \( α_{Θ,p} > 0 \) such that
\[
\int_Ω \sum_{β=0}^{p-2} \frac{p_1}{β!(p-1-β)!} Θ^{β^2} \left( <Θ^{2β+1}Dνt w^{(β,p-1-β)}, νt u > + <Dνt w^{(β,p-1-β)}, νt v > \right) dx
\geq Λ_1^2 α_{Θ,p} \int_Ω \sum_{β=0}^{p-2} \frac{p_1}{β!(p-2-β)!} Θ^{β^2} w^{(β,p-2-β)} \sum_{k=1}^{3} (∂x_k u^2 + ∂x_k v^2)
\geq Λ_1^2 α_{Θ,p} \int_Ω \left( |∇(u)^{p/2}|^2 + |∇(v)^{p/2}|^2 \right) dx.
\]
Substituting (6.32), (6.33), (6.34) and (6.35) in (6.29) we get
\[
\frac{∂P}{∂t} + Λ_1^2 α_{Θ,p} \int_Ω \left( |∇(u)^{p/2}|^2 + |∇(v)^{p/2}|^2 \right) dx + k_1 C(p, Θ) \int_Ω (u + v)^p dx
\leq C(p, h(T), L_1, D, ˜D, Ω, L_Θ) + N_{p, Θ, Γ} \left[ \int_Γ (u^p + v^p) dσ + 1 \right], \quad (6.35)
\]
for some constant \( N_{p, Θ, Γ} \). Applying Lemma 6.2 to the functions \( u^{p/2} \) and \( v^{p/2} \) with \( γ = p \) and using (6.10), there exists \( ˜N_{p, Θ, Γ} > 0 \) such that
\[
2N_{p, Θ, Γ} \int_Γ (u^p + v^p) dσ ≤ α_{Θ,p} Λ_1 \int_Ω \left( |∇(u)^{p/2}|^2 + |∇(v)^{p/2}|^2 \right) dx
\]
\[
+ ˜N_{p, Θ, Γ} \left( \int_Ω (u + v) dx \right)^p. \quad (6.36)
\]
Adding (6.35) with (6.36), we get
\[
\frac{∂P}{∂t} + N_{p, Θ, Γ} \int_Γ (u^p + v^p) dσ ≤ C(p, h(T), L_1, D, ˜D, M, L_Θ)
\]
\[
+ ˜N_{p, Θ, Γ} \left( \int_Ω (u + v) dx \right)^p + N_{p, Θ, Γ}. \quad (6.37)
\]
Finally, if we integrate over time, we find that \( \|u\|_{p,Γ,T} \) is bounded in terms of \( p, Γ, Ω, Θ, h(T), w_1, w_2 \) and \( \|v\|_{p,Ω,T} \). Since this holds for every natural number \( p ≥ 2 \), we can use the assumption (V_{P,Ω}) and the bounds above, along with Proposition
2 to conclude that \( \|(u,v)\|_{\infty, \Omega_T} < \infty \). From Theorem 3.1, this contradicts our assumption that \( T_{\text{max}} < \infty \). Therefore, \( T_{\text{max}} = \infty \), and Theorem 3.2 is proved.

For \( m \geq 2 \) components, we first obtain the following \( L_p \) estimates.

**Lemma 6.3.** Suppose that \( (V_N), (V_F), (V_{QP}) \) and \( (V_L) \) are satisfied, and \( u \) is the unique, componentwise nonnegative, maximal solution to (3.1). If \( 1 < p < \infty \) and \( T = T_{\text{max}} < \infty \), then \( \|u\|_{p, \Omega_T} \) and \( \|u\|_{p, \Gamma_T} \) are bounded.

The proof of Lemma 6.3 is using the Lyapunov function which is an extension of the polynomial (6.21) to \( m \) components, i.e.,

\[
P = \sum_{|\beta|=0}^{p} \frac{p!}{\beta!(p-|\beta|)!} \beta_1^{\beta_1} \ldots \beta_{m-1}^{\beta_{m-1}} u_1^{\beta_1} \ldots u_{m-1}^{\beta_{m-1}} u_m^{p-|\beta|}
\]

where \( |\beta| = |\beta_1| + \ldots + |\beta_{m-1}| \) and \( \beta! = \beta_1! \ldots \beta_{m-1}! \). The estimates are obtained following steps of Lemma 5.3 of [31] using this polynomial, which is relatively simpler than \( H \) defined in [1] and used in [31].

**Proof of Theorem 3.4:** From Theorem 3.1, we already have a componentwise nonnegative, unique, maximal solution of (3.1). If \( T_{\text{max}} = \infty \), then we are done. So, by way of contradiction assume \( T_{\text{max}} < \infty \). From Lemma 6.3, we have \( L_p \) estimates for our solution for all \( p \geq 1 \) on \( \Omega \times (0, T_{\text{max}}) \) and \( M \times (0, T_{\text{max}}) \). We know from \( (V_{Poly}) \) that the \( F_i \) and \( G_i \) are polynomially bounded above for each \( i \). Then proceeding as in the proof of Theorem 3.3 in [32] with the bounds from Lemma 6.3 we have \( T_{\text{max}} = \infty \). □

7. Examples.

**Example 1.** Here, we give an example related to the well known Brusselator. Consider the system

\[
\begin{align*}
  u_1 &= d_1 \Delta u_1 & y \in \Omega_1, t > 0 \\
  u_2 &= d_2 \Delta u_2 & y \in \Omega_1, t > 0 \\
  d_1 \frac{\partial u_1}{\partial \eta} &= \alpha u_2 - u_2^2 u_1 & y \in \partial \Omega_1, t > 0 \\
  d_2 \frac{\partial u_2}{\partial \eta} &= \beta - (\alpha + 1)u_2 + u_2^2 u_1 & y \in \partial \Omega_1, t > 0 \\
  u_i(y, 0) &= w_i(y) & y \in \Omega_0
\end{align*}
\]

where \( d_1, d_2, \alpha, \beta > 0 \) and \( w \) is sufficiently smooth and componentwise nonnegative. If we define

\[
  f(u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad g(u) = \begin{pmatrix} \alpha u_2 - u_2^2 u_1 \\ \beta - (\alpha + 1)u_2 + u_2^2 u_1 \end{pmatrix}
\]

for all \( u \in \mathbb{R}^2_+ \), then \( (V_N), (V_F), (V_{QP}) \) and \( (V_{Poly}) \) are satisfied with \( a_1 \geq 1 \) and \( L_a = \max\{\beta, \alpha \cdot a_1\} \). Therefore, Theorem 3.4 implies (7.1) has a unique, componentwise nonnegative, global solution.
Example 2. We next consider a general reaction mechanism of the form

\[ R_1 + R_2 \rightarrow P_1 \]

where \( R_i \) and \( P_i \) represent reactant and product species, respectively. If we set \( u_i = [R_i] \) for \( i = 1, 2 \), and \( u_3 = [P_1] \), and let \( k_f, k_r \) be the (nonnegative) forward and reverse reaction rates, respectively, then we can model the process by the application of the law of conservation of mass and the second law of Fick (flow) with the following reaction–diffusion system:

\[
\begin{align*}
  u_i &= d_i \Delta u_i \quad y \in \Omega, t > 0, i = 1, 2, 3 \\
  d_1 \frac{\partial u_1}{\partial \eta} &= -k_f u_1 u_2 + k_r u_3 \quad y \in \partial \Omega, t > 0 \\
  d_2 \frac{\partial u_2}{\partial \eta} &= -k_f u_1 u_2 + k_r u_3 \quad y \in \partial \Omega, t > 0 \quad (7.2) \\
  d_3 \frac{\partial u_3}{\partial \eta} &= k_f u_1 u_2 - k_r u_3 \quad y \in \partial \Omega, t > 0 \\
  u_i(y, 0) &= w_i(y) \quad y \in \Omega_0, i = 1, 2, 3,
\end{align*}
\]

where \( d_i > 0 \) and the initial data \( w \) is sufficiently smooth and componentwise nonnegative. If we define

\[
\begin{align*}
  f(u) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
  g(u) &= \begin{pmatrix} -k_f u_1 u_2 + k_r v_3 \\ -k_f u_1 u_2 + k_r v_3 \\ k_f u_1 u_2 - k_r v_3 \end{pmatrix}
\end{align*}
\]

for all \( u \in \mathbb{R}_+^3 \), then \((V_N), (V_F), (V_{QP})\) and \((V_{Poly})\) are satisfied. In addition, \((V_{L_1})\) is satisfied with \( L_1 = 0 \) since

\[
\frac{1}{2} f_1(z) + \frac{1}{2} f_2(z) + f_3(z) = 0 \quad \text{and} \quad \frac{1}{2} g_1(z) + \frac{1}{2} g_2(z) + g_3(z) = 0
\]

for all \( z \in \mathbb{R}_+^3 \). Therefore, the hypothesis of Theorems 3.4 is satisfied. As a result \((7.2)\) has a unique, componentwise nonnegative, global solution.

Example 3. Finally, we consider a system that satisfies the hypothesis of the Theorem 3.4, where the boundary reaction vector field does not satisfy a linear intermediate sums condition. Let

\[
\begin{align*}
  u_1 &= d_1 \Delta u \quad y \in \Omega, t > 0 \\
  u_2 &= d_2 \Delta u \quad y \in \Omega, t > 0 \\
  d_1 \frac{\partial u_1}{\partial \eta} &= \alpha u_1 u_2^3 - u_1 u_2^2 \quad y \in \partial \Omega, t > 0 \\
  d_2 \frac{\partial u_2}{\partial \eta} &= u_1 u_2^2 - \beta u_1 u_2 \quad y \in \partial \Omega, t > 0 \quad (7.3) \\
  u(y, 0) &= w(y) \quad y \in \Omega_0
\end{align*}
\]

where \( d_1, d_2, \alpha, \beta > 0 \) and \( w \) is sufficiently smooth and componentwise nonnegative. In this setting

\[
\begin{align*}
  f(u) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
  g(u) &= \begin{pmatrix} \alpha u_1 u_2^3 - u_1 u_2^2 \\ u_1 u_2^2 - \beta u_1 u_2 \end{pmatrix}
\end{align*}
\]
for all \( u \in \mathbb{R}^2_+ \). It is simple matter to see that \((V_N), (V'_P), (V_{QP})\) and \((V_{Poly})\) are satisfied. Also, if \( a \geq 1 \) then

\[
af_1(u) + f_2(u) = 0 \quad \text{and} \quad ag_1(u) + g_2(u) \leq (a\alpha - \beta)u_1(u_2^3 - u_6^3) \leq \frac{a\alpha}{4}u_1
\]

for all \( u \in \mathbb{R}^2_+ \). Consequently, \((V_L)\) is satisfied. Therefore, Theorem 3.4 implies (7.3) has a unique, componentwise nonnegative, global solution.

**Acknowledgments.** The authors thank Professor Kumaresan for introducing them academically.

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Received December 2020; revised February 2021; early access October 2021.

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