1. Introduction

An invariant Poisson structure on a finite-dimensional principal bundle $P \to B$ descends to a Poisson structure on the base. This is immediate from the identification of functions on $B$ with invariant functions on $P$, or alternatively, because the invariant Poisson bivector field $\pi_P$ pushes down to a Poisson bivector field on $B$.

One is tempted to apply these facts to the following infinite-dimensional setting. Let $G$ be a connected Lie group. Its loop group $LG = \mathrm{Map}(S^1, G)$ acts by gauge transformations on the space $A = \Omega^1(S^1, g)$ of connections on the trivial $G$-bundle over the circle. The based loop group $L_0G \subset LG$ acts freely, and the holonomy of a connection identifies $A/L_0G$ with $G$. We will refer to the resulting principal $L_0G$-bundle

$$\text{Hol}: A \to G$$

as the \textit{holonomy fibration}. Suppose the Lie algebra carries an invariant metric, used to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. It defines a central extension $\widehat{L_0g}$ of $L_0g$ by $\mathbb{R}$, and one may regard $A$ as the affine subspace of $\widehat{L_0g}^*$ at level 1. Formally, it carries a Poisson structure called the Lie-Poisson structure, with symplectic leaves the level 1 coadjoint orbits of $LG$.

The naive attempt to push this down to a Poisson structure on $G$ runs into problems, related to the precise meaning of a Poisson structure in infinite dimensions. Indeed,
the Lie-Poisson structure on $\mathcal{A}$, viewed as a bilinear bracket $\{\cdot,\cdot\}$ on functions, cannot be defined on all functions; its domain does not even contain all pullbacks $\text{Hol}^* f$ with $f \in C^\infty(G)$. Similarly, the Lie-Poisson structure on $\mathcal{A}$ cannot be a genuine bivector field, since sections of $\wedge^2 T\mathcal{A}$, by definition, have only finite rank.

In this paper, we shall take a third viewpoint, regarding the Lie-Poisson structure on $\mathcal{A}$ as a Dirac structure. Recall that a Dirac structure on a finite-dimensional manifold $Q$ is a Lagrangian sub-bundle $E \subseteq TQ \oplus T^* Q$ satisfying a certain integrability condition. Poisson structures are Dirac structures for which $E$ is the graph of a skew-adjoint bundle map $T^* Q \to TQ$. In finite dimensions, this is equivalent to the property $E \cap TQ = 0$.

The definition of Dirac structures carries over to infinite-dimensional Hilbert manifolds, but here the conditions of $E$ being a graph or having trivial intersection with the tangent bundle are no longer equivalent. We will call a Dirac structure $E$ with the latter property a weak Poisson structure. Equivalently, the weak Poisson structures are described as a family of skew-adjoint operators $D_q: \text{dom}(D_q) \to T_q Q$, with dense domain in $T^*_q Q$. The leaves of a weak Poisson structure carry closed 2-forms that are weakly symplectic.

Taking $\mathcal{A}$ to consist of connections of a fixed Sobolev class (e.g. $L^2$ or higher), we observe that the Lie-Poisson structure is well-defined as a weak Poisson structure in the above sense. The corresponding skew-adjoint operators are the covariant derivatives $\partial_A$.

Using the reduction procedure for Dirac structures [9], this weak Poisson structure may be pushed down under the map $\text{Hol}$. We will show that the result is the well-known Cartan-Dirac structure on $G$. The Cartan-Dirac structure had been discovered independently by Alekseev, Ševera and Strobl in the late 1990s, and plays an important role in the theory of $D$-branes [13, 16, 20] as well as for quasi-Hamiltonian $G$-spaces [2, 3]. Our reduction procedure extends to Hamiltonian spaces, and clarifies the correspondence [3] between Hamiltonian loop group spaces [26] and q-Hamiltonian $G$-spaces. We also describe multiplicative properties of the Cartan-Dirac structure [2, 22] from the point of view of reduction from suitable spaces of connections.

In this article, we will mostly work with a closely related holonomy fibration $\text{Hol}: \mathcal{A}_I \to G$, given by connections on the interval $I = [0, 1]$, with an action of the gauge group $G_I = \text{Map}(I, G)$. The ‘Lie-Poisson’ structure on $\mathcal{A}_I$ is a Dirac structure described by connections $\partial_A$ as before, but whose domain involves periodic boundary conditions. The reduction by the group $G_{I,\partial I}$ of gauge transformations trivial at the boundary, results in the Cartan-Dirac structure. From this point of view, we may consider alternative boundary conditions for the family of operators $\partial_A$, given by Lagrangian Lie subalgebras $\mathfrak{s} \subseteq \mathfrak{g} \oplus \mathfrak{g}$. The corresponding weak Poisson structures on $\mathcal{A}_I$ reduce to generalizations of the Cartan-Dirac structure.

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2. Dirac structures in infinite dimensions

In this section, we review the theory of Courant algebroids, Dirac structures, and their reduction in an infinite-dimensional context. For a treatment of differential geometry on Banach manifolds and Hilbert manifolds, see e.g. [1].

Much of the material is a direct extension of the finite-dimensional theory. Special care needs to be taken due to the fact that the sum of closed subspaces of a Banach space need not be closed. These problems are already apparent in the linear version of the theory, described below.

2.1. Linear Dirac geometry in infinite dimensions. Throughout this paper, the terms Banach space and Hilbert space designate a real topological vector space whose topology is defined by a Banach norm and Hilbert inner product, respectively. The norm or inner product itself is not considered part of the structure. By [24], a Banach space is a Hilbert space if and only if every closed subspace admits a closed complement. For this reason, we will mainly work with Hilbert spaces and Hilbert manifolds.

A continuous symmetric bilinear form \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \) on a Hilbert space \( V \) is called non-degenerate if the associated map \( V \to V^*, v \mapsto \langle v, \cdot \rangle \) is an isomorphism. We will refer to \( \langle \cdot, \cdot \rangle \) as a pseudo-Riemannian metric, or simply as a metric, and call \( V \) a metrized Hilbert space. We stress that \( \langle \cdot, \cdot \rangle \) is not necessarily a Hilbert space inner product.

If \( F \) is a subspace of a metrized Hilbert space \( V \), denote by \( F^\perp \) its orthogonal relative to the metric. Accordingly, \( F \) is called isotropic if \( F \subseteq F^\perp \), co-isotropic if \( F^\perp \subseteq F \), and Lagrangian if \( F = F^\perp \). A Lagrangian splitting of \( V \) is a direct sum decomposition \( V = F_1 \oplus F_2 \) into Lagrangian subspaces. In finite dimensions, this is equivalent to \( F_1 \cap F_2 = 0 \), but in infinite dimensions this is stronger:

**Example 2.1.** Suppose \( \mathcal{H} \) is a Hilbert space, and equip \( V = \mathcal{H} \oplus \mathcal{H}^* \) with the metric
\[
\langle v_1 + \mu_1, v_2 + \mu_2 \rangle = \mu_2(v_1) + \mu_1(v_2)
\]
for \( v_1, v_2 \in \mathcal{H} \), \( \mu_1, \mu_2 \in \mathcal{H}^* \). Then \( \mathcal{H}, \mathcal{H}^* \) are Lagrangian subspaces. Suppose \( A : \text{dom}(A) \to \mathcal{H}^* \) is an unbounded linear operator with dense domain \( \text{dom}(A) \subseteq \mathcal{H} \). By definition, \( A \) is a closed operator if and only if its graph \( \text{gr}(A) \) is closed, and is an unbounded skew-adjoint operator if and only if \( \text{gr}(A) \) is Lagrangian. Now suppose that \( A \) is an unbounded skew-adjoint operator with \( \text{dom}(A) \neq \mathcal{H} \). Then \( \text{gr}(A) \), \( \mathcal{H}^* \) are Lagrangian subspaces with trivial intersection, but \( \text{gr}(A) + \mathcal{H}^* \neq \mathcal{H} \oplus \mathcal{H}^* \).

If \( C \subseteq V \) is a closed co-isotropic subspace of a metrized Hilbert space, we define a reduced space \( V_C = C/C^\perp \). It inherits a metric from the metric on \( V \). Given a subspace \( F \subseteq V \), define \( F_C = (F \cap C)/(F \cap C^\perp) \). In finite dimensions, the reduction \( L_C \) of a Lagrangian subspace \( L \) is again Lagrangian, but this need not be the case in infinite dimensions:

**Example 2.2.** In the setting of Example 2.1, pick \( v \in \mathcal{H} - \text{dom}(A) \), and let \( C = \text{span}(v) \oplus \mathcal{H}^* \). Then \( C \) is coisotropic, with \( C^\perp = \text{ann}(v) \subseteq \mathcal{H}^* \). Hence \( C/C^\perp = \text{span}(v) \oplus \text{span}(v)^* \) is 2-dimensional. The Lagrangian subspace \( L = \text{gr}(A) \) satisfies \( L \cap C = 0 \), hence \( L_C = 0 \) is not Lagrangian.
To ensure that the reduction of a Lagrangian subspace is Lagrangian, we need an additional condition:

**Proposition 2.3.** Let $V$ be a metrized Hilbert space, and $C$ a closed co-isotropic subspace of $V$. Let $L \subseteq V$ be a Lagrangian subspace with the property that $L + C$ is closed. Then $L_C = (L \cap C)/(L \cap C^\perp)$ is Lagrangian in $V_C$.

A proof is given in the Appendix, see Proposition A.1.

**Remark 2.4.** Given a metrized Hilbert space $V$, the sum $F_1 + F_2$ of subspaces is closed if and only if $F_1^\perp + F_2^\perp$ is closed. Hence, the condition in Proposition 2.3 is equivalent to the condition that $L + C^\perp$ be closed.

**Remark 2.5.** In subsequent sections, we use vector bundle versions of the results described above. We refer to a Hilbert vector bundle $V \to M$ over a Hilbert manifold, with a (pseudo-Riemannian) fiber metric $\langle \cdot, \cdot \rangle$, as a *metrized vector bundle*. Given a closed coisotropic subbundle $C \subseteq V$, the quotient $V_C = C/C^\perp$ inherits a metric. For a Lagrangian sub-bundle $L \subseteq V$ the reduction $L_C = (L \cap C)/(L \cap C^\perp)$ is a Lagrangian subbundle provided $L + C$ is a closed subbundle. In particular, this is the case if the intersection is transverse, i.e. $L + C = V$, or if $L \subseteq C$.

For any metrized Hilbert space $V$, let $\overline{V}$ denote the same Hilbert space with the opposite metric. A *Lagrangian relation* $R: V_1 \dashrightarrow V_2$ between two metrized Hilbert spaces is a linear relation whose graph $\text{gr}(R) \subseteq V_2 \times \overline{V}_1$ is Lagrangian. We will write $v_1 \sim_R v_2$ if and only if $(v_2, v_1) \in \text{gr}(R)$, and define the kernel and range of $R$ as

$$\text{ker}(R) = \{v_1 \in V_1 \mid v_1 \sim_R 0\}, \quad \text{ran}(R) = \{v_2 \in V_2 \mid \exists v_1 \in V_1: v_1 \sim_R v_2\}.$$

The space $\text{ker}(R)$ is closed, but $\text{ran}(R)$ not necessarily so. Similarly, we define $\text{ker}^*(R) = \text{ker}(R^\top)$ and $\text{ran}^*(R) = \text{ran}(R^\top)$, where $R^\top: V_2 \to V_1$ is the transpose relation. We have

$$\text{ker}(R) = \text{ran}^*(R^\perp), \quad \text{ker}^*(R) = \text{ran}(R^\perp).$$

Given another Lagrangian relation $R': V_2 \dashrightarrow V_3$, one defines $R' \circ R$ as a composition of relations. If the $V_i$ are finite-dimensional, then $R' \circ R$ is again a Lagrangian relation, but in infinite dimensions additional assumptions are needed. We say that $R', R$ have *transverse composition* if

(1) \[ \text{ran}(R) + \text{ran}^*(R') = V_2. \]

**Proposition 2.6.** If $R', R$ have transverse composition, then $R' \circ R$ is a Lagrangian relation.

**Proof.** Let $V = (V_3 \times \overline{V}_2) \times (V_2 \times \overline{V}_1)$ and $C = V_3 \times (V_2)_\Delta \times \overline{V}_1$, where $(V_2)_\Delta \subseteq V_2 \times \overline{V}_2$ is the diagonal subspace. Then

$$\text{gr}(R' \circ R) = (\text{gr}(R') \times \text{gr}(R))_C \subseteq V_C = V_3 \times \overline{V}_1.$$

Since $\text{ran}(R) + \text{ran}^*(R')$ is the image of $\text{gr}(R') \times \text{gr}(R)$ under the projection $V \to V/C \cong V_2$, this is equivalent to $(\text{gr}(R') \times \text{gr}(R)) + C = V$. By Proposition 2.3 this guarantees that $R' \circ R$ is a Lagrangian relation. \qed
Taking orthogonals, we see that the transversality (1) implies
\[(2) \quad \ker(R') \cap \ker^*(R) = 0,\]
which says that whenever \(v_1 \sim_{R^* \circ R} v_3\), then the element \(v_2\) with \(v_1 \sim_R v_2\) and \(v_2 \sim_{R'} v_3\) is uniquely determined. We will call the composition \(R' \circ R\) weakly transverse if the condition (2) holds, or equivalently \(\text{ran}(R) + \text{ran}^*(R')\) is dense in \(V_2\).

**Definition 2.7.** A pair \((V, E)\) consisting of a metrized Hilbert space and a Lagrangian subspace is called a linear Dirac structure. A linear Dirac morphism \(R: (V_1, E_1) \rightarrow (V_2, E_2)\) is a Lagrangian relation \(R: V_1 \rightarrow V_2\) such that \(E_2 = R \circ E_1\), where the composition is weakly transverse (i.e. \(E_1 \cap \ker(R) = 0\)). If the composition is transverse (i.e. \(E_1 + \text{ran}^*(R) = V_1\)), we will call \(R\) a strong linear Dirac morphism.

Here the Lagrangian subspaces \(E_i \subseteq V_i\) are regarded as linear relations \(E_i: 0 \rightarrow V_i\). In the following result, we consider \(F_i \subseteq V_i\) as Lagrangian relations \(F_i: V_i \rightarrow 0\).

**Proposition 2.8.** Suppose \(R: (V_1, E_1) \rightarrow (V_2, E_2)\) is a linear Dirac morphism, and let \(F_2\) be a Lagrangian complement to \(E_2\). Then \(F_1 = F_2 \circ R\) is a Lagrangian subspace with \(E_1 \cap F_1 = 0\). If \(R\) is a strong linear Dirac morphism, then \(V_1 = E_1 \oplus F_1\).

**Proof.** Since \(E_2 \subseteq \text{ran}(R)\), we have that \(\text{ran}(R) + F_2 = V_2\). Hence the composition is transverse, and \(F_1 = F_2 \circ R\) is a Lagrangian subspace. Suppose \(x_1 \in E_1 \cap F_1\). Since \(x_1 \in F_1\), there exists \(x_2 \in F_2\) with \(x_1 \sim_R x_2\). Since \(x_1 \in E_1\), this relation implies that \(x_2 \in E_2\). Hence \(x_2 = 0\). But \(x_1 \in E_1\), \(x_1 \sim_R 0\) means \(x_1 = 0\), by weak transversality of the composition \(R \circ E_1\).

Suppose now that the composition is transverse, so that \(V_1 = E_1 + \text{ran}^*(R)\). Let \(x_1 \in \text{ran}^*(R)\), so that \(x_1 \sim_R x_2\) for some \(x_2 \in V_2\). Write \(x_2 = x'_2 + x''_2\) with \(x'_2 \in E_2\) and \(x''_2 \in F_2\). Let \(x'_1 \in E_1\) be an element with \(x'_1 \sim_R x'_2\), and put \(x''_1 = x_1 - x'_1\). Then \(x''_1 \sim_R x''_2\), hence \(x''_1 \in F_1\). This shows \(V_1 = E_1 \oplus F_1\). \(\square\)

**Proposition 2.9.** Suppose \(R: (V_1, E_1) \rightarrow (V_2, E_2)\) and \(R': (V_2, E_2) \rightarrow (V_3, E_3)\) are strong linear Dirac morphisms. Then the composition \(R' \circ R\) is transverse, and defines a strong linear Dirac morphism \(R' \circ R: (V_1, E_1) \rightarrow (V_3, E_3)\).

**Proof.** Choose a Lagrangian complement \(F_3\) to \(E_3 \subseteq V_3\). Then \(F_2 = F_3 \circ R' \subseteq V_2\) is a Lagrangian complement to \(E_2\), and \(F_1 = F_2 \circ R\) is a Lagrangian complement to \(E_1\). We have \(E_2 \subseteq \text{ran}(R)\) and \(F_2 \subseteq \text{ran}^*(R')\), hence \(\text{ran}(R) + \text{ran}^*(R') = V_2\), proving transversality of the composition \(R' \circ R\). Similarly, \(F_1 = F_3 \circ (R' \circ R)\) shows that \(F_1 \subseteq \text{ran}^*(R' \circ R)\). Hence \(E_1 + \text{ran}^*(R' \circ R) = V_1\). \(\square\)

**2.2. Courant algebroids.** The usual definition of a Courant algebroid [25, 32] works equally well for infinite dimensional manifolds. In the remainder of this section we shall use the terms “manifold”, “vector bundle”, “Lie group”, etc. to refer to Hilbert manifold, Hilbert vector bundle, Hilbert Lie group, and so on. A metrized vector bundle is a Hilbert vector bundle with a fiberwise (pseudo-Riemannian) metric.

A Courant algebroid is a metrized vector bundle \((\mathcal{A}, \langle \cdot, \cdot \rangle)\) over a manifold \(Q\), equipped with a smooth bundle map \(a: \mathcal{A} \rightarrow TQ\) called the anchor, and a bilinear Courant bracket
$[\cdot, \cdot]: \Gamma(A) \times \Gamma(A) \to \Gamma(A)$, such that the following axioms are satisfied, for all smooth sections $\sigma_1, \sigma_2, \sigma_3$ of $A$:

$$[\sigma_1, [\sigma_2, \sigma_3]] = [[\sigma_1, \sigma_2], \sigma_3] + [\sigma_2, [\sigma_1, \sigma_3]],$$

$$a(\sigma_1)[\sigma_2, \sigma_3] = \langle [\sigma_1, \sigma_2], \sigma_3 \rangle + \langle \sigma_2, [\sigma_1, \sigma_3] \rangle,$$

$$a^*d(\sigma_1, \sigma_2) = [\sigma_1, \sigma_2] + [\sigma_2, \sigma_1].$$

Here $a^*: T^*Q \to A$ is the dual anchor composed with the isomorphism $A^* \cong A$ given by the metric.

These axioms imply the following properties [37], for all $f \in C^\infty(Q)$:

$$[\sigma_1, f\sigma_2] = f[\sigma_1, \sigma_2] + (a(\sigma_1)f)\sigma_2,$$

$$a([[\sigma_1, \sigma_2]]) = [a(\sigma_1), a(\sigma_2)].$$

A Dirac structure $(A, E)$ on $Q$ is a Courant algebroid together with a Lagrangian sub-bundle $E \subseteq A$ whose space of sections is closed under the bracket. If the Courant algebroid $A$ is fixed, we refer to $E$ itself as the Dirac structure. For any Dirac structure, the Courant bracket restricts to a Lie bracket on $\Gamma(Q)$, and $E$ is a Lie algebroid. A connected submanifold $\mathcal{O} \subseteq Q$ is called a leaf of $E$ if $a(E|_{\mathcal{O}}) = T\mathcal{O}$, and is maximal with this property. If $\dim Q < \infty$, the Stefan-Sussmann theorem [6, 35, 36] asserts that $Q$ acquires a singular foliation by leaves. In infinite dimensions, there are similar results due to Chillingworth-Stefan [14] and Pelletier [30] (the latter reference discusses foliations defined by Banach-Lie algebroids). In our main applications the foliation will be explicitly given as the orbits of a Lie group action.

**Example 2.10.**

(a) Suppose $Q$ is a manifold with a closed 3-form $\eta \in \Omega^3(Q)$. Then the direct sum $TQ \oplus T^*Q$ carries the structure of a Courant algebroid, with metric $\langle v_1 + \mu_1, v_2 + \mu_2 \rangle = \langle \mu_1, v_2 \rangle + \langle \mu_2, v_1 \rangle$, with anchor the projection to the first summand, and with the Courant bracket

$$[[v_1 + \mu_1, v_2 + \mu_2]] = [v_1, v_2] + \mathcal{L}_{v_1}\mu_2 - \iota_{v_2}\mathcal{L}_{\mu_1} + \iota_{v_1}\iota_{v_2}\eta.$$ 

We will denote this Courant algebroid by $TQ_\eta$. If $\eta = 0$, it is called the standard Courant algebroid and is denoted $TQ$. Suppose $E \subseteq TQ_\eta$ is a Dirac structure. If $i_\mathcal{O}: \mathcal{O} \to Q$ is the inclusion of a leaf of $E$, then there is a 2-form $\omega_\mathcal{O} \in \Omega^2(\mathcal{O})$, uniquely defined by the property $\omega_\mathcal{O}(v_1, v_2) = \langle \alpha_1, v_2 \rangle$ for all $v_i \in T_{v_1}\mathcal{O}$, where $\alpha_1 \in T^*Q$ is chosen so that $v_1 + \alpha_1 \in E_m$. It follows that the 2-form satisfies $d\omega_\mathcal{O} = -i_\mathcal{O}^*\eta$.

(b) Suppose $\mathfrak{g}$ is a Lie algebra with an invariant metric. Given a $\mathfrak{g}$-action on $Q$ such that the stabilizer algebras $\mathfrak{g}_m = \{ \xi \in \mathfrak{g} | \xi_Q(m) = 0 \}$ are coisotropic, the product $A = Q \times \mathfrak{g}$ becomes a Courant algebroid, with anchor the action map $Q \times \mathfrak{g} \to TQ$, and with Courant bracket extending the Lie bracket on constant sections (see [21]). This is called an action Courant algebroid. For any Lagrangian Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$, the subbundle $E = Q \times \mathfrak{s}$ is a Dirac structure in $A$. 

2.3. **Weak Poisson structures.** A Lagrangian subbundle \( E \subseteq TQ \) with the property \( E \oplus TQ = TQ \) amounts to a continuous skew-symmetric bilinear form \( \pi \) on \( T^*Q \), such that \( E = \text{gr}(\pi^*) \) is the graph of the associated map. If \( E \) is a Dirac structure with this property, we will call \( \pi \) (or \( E \) itself) a *Poisson structure* on \( Q \). In particular, \( \pi \) determines a bracket on \( \mathcal{C}^\infty(Q) \) in the usual way. For general Banach (as opposed to Hilbert) manifolds, the definition is more involved, see Odzijewicz-Ratiu [29]. Given a leaf \( \mathcal{O} \) of a Poisson structure, the 2-form \( \omega \) on that leaf is symplectic, in the strong sense that the bundle map \( \Omega^\flat : TQ \to T^*Q \) is invertible.

A Dirac structure \( E \subseteq TQ \) satisfying the weaker condition \( E \cap TQ = 0 \) will be called a *weak Poisson structure*; this may be regarded as a family of skew-adjoint unbounded operators. The resulting 2-forms \( \omega \) on leaves are only weakly symplectic, in the sense that \( \Omega^\flat \) is injective. In the finite-dimensional setting, the notions coincide. See Posthuma [31, Chapter 4.1] for another definition of weak Poisson structure. Given a weak Poisson structure \( E \), let \( \mathcal{C}^\infty_E(Q) \) be the space of smooth functions \( f \) for which there exists a vector field \( \nu_f \) with \( \nu_f + df \in \Gamma(E) \). Since \( E \cap TQ = 0 \), the vector field \( \nu_f \) is uniquely determined. The elements of \( \mathcal{C}^\infty_E(Q) \) are called *admissible* [15] or *Hamiltonian* [1] functions, and \( \nu_f \) the corresponding *Hamiltonian vector field*. The space of Hamiltonian functions is a Poisson algebra for the bracket

\[
\{f_1, f_2\} = \nu_{f_1}(f_2).
\]

2.4. **Morphisms.** Morphisms of Courant algebroids and Dirac structures are defined as Lagrangian correspondences.

For any Courant algebroid \( A \), denote by \( \overline{A} \) the Courant algebroid which is obtained from \( A \) by reversing the sign of the metric. A *morphism of Courant algebroids* \( R : A_1 \longrightarrow A_2 \)

\[
\text{is a smooth map } \Phi : Q_1 \to Q_2 \text{ of the base manifolds, together with a Lagrangian subbundle } \text{gr}(R) \subseteq A_2 \times \overline{A}_1 \text{ along the graph } \text{gr}(\Phi) \subseteq Q_2 \times Q_1, \text{ satisfying the following integrability condition: If two sections of } A_2 \times \overline{A}_1 \text{ restrict to sections of } \text{gr}(R), \text{ then so does their Courant bracket. We will depict Courant morphisms as follows}
\]

\[
\begin{array}{ccc}
A_1 & \xrightarrow{R} & A_2 \\
\downarrow & & \downarrow \\
Q_1 & \xrightarrow{\Phi} & Q_2 \\
\end{array}
\]

Composition of Courant morphisms is defined as a composition of Lagrangian relations, assuming that the composition is transverse. As shown in [22], the integrability condition is preserved under composition.

For \( x_i \in A_i \), we will write \( x_1 \sim_R x_2 \) if \( (x_2, x_1) \in \text{gr}(R) \). Similarly, if \( \sigma_i \in \Gamma(A_i) \) are sections we write \( \sigma_1 \sim_R \sigma_2 \) if \( (\sigma_2, \sigma_1) \) restricts to a section of \( \text{gr}(R) \). Consider the dual
of the tangent map $T\Phi : TQ_1 \to TQ_2$ as a relation

\begin{equation}
\begin{array}{ccc}
T^*Q_1 & \xrightarrow{T^*\Phi} & T^*Q_2 \\
\downarrow & & \downarrow \\
Q_1 & \xrightarrow{\Phi} & Q_2
\end{array}
\end{equation}

That is, $\mu_1 \sim_{T^*\Phi} \mu_2$ for $\mu_i \in T^*_m Q_i$ means $m_2 = \Phi(m_1)$ and $\mu_1 = (T_m \Phi)^* \mu_2$.

**Lemma 2.11.** Let $R: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be a Courant morphism with base map $\Phi: Q_1 \to Q_2$. Then

$$a_2^* \circ T^* \Phi = R \circ a_1^*.$$ 

That is, the dual of $a = (a_2, a_1)$ restricts to a bundle map $a^*: \text{gr}(T^* \Phi) \to \text{gr}(R)$.

**Proof.** The assertion follows by dualizing the property $T\Phi \circ a_2 = a_1 \circ R$, using that $R^* = R$ under the identification $\mathcal{A}_i^* = \mathcal{A}_i$. In detail, let $\mu_i \in T^*_m Q_i$ with $\mu_1 \sim_{T^*\Phi} \mu_2$. For all $x_i \in \mathcal{A}_m$ with $x_1 \sim_R x_2$, we have that

$$\langle a_1^*(\mu_1), x_1 \rangle = \langle \mu_2, T\Phi(a_1(x_1)) \rangle = \langle \mu_2, a_2(x_2) \rangle = \langle a_2^*(\mu_2), x_2 \rangle,$$

that is, $(a_2^*(\mu_2), a_1^*(\mu_1)) = 0$. This shows $(a_2^*(\mu_2), a_1^*(\mu_1)) \in \text{gr}(R)^\perp = \text{gr}(R)$, as desired. \(\square\)

Let $(A_i, E_i)$, $i = 1, 2$ be Dirac structures on $Q_i$. We say that (4) defines a *Dirac morphism* (or *morphism of Manin pairs*) [11]

$$R: (A_1, E_1) \rightarrow (A_2, E_2)$$

if for all $m \in Q$, every $x_2 \in (E_2)_{\Phi(m)}$ is $R$-related to a unique element $x_1 \in (E_1)_m$. Equivalently,

$$\Phi^* E_2 = R \circ E_1$$

where the composition is weakly transverse (when the composition is transverse, the Dirac morphism is called *strong*). The resulting bundle map $\Phi^* E_2 \rightarrow E_1$ defines a *comorphism* of Lie algebroids $R: E_1 \rightarrow E_2$: It is compatible with the anchor, and the map on sections $\Phi^* : \Gamma(E_2) \rightarrow \Gamma(E_1)$ preserves Lie brackets.

**Definition 2.12 ([11]).** A *Hamiltonian space* for a Dirac structure $(\mathcal{A}, E)$ on $Q$ is a manifold $M$ with a Dirac morphism

$$R: (TM, TM) \rightarrow (\mathcal{A}, E).$$

The base map $\Phi : M \to Q$ is called the *moment map*.

Given a Hamiltonian space, the resulting Lie algebroid comorphism $TM \rightarrow E$ defines an *action* of the Lie algebroid $E$ on the manifold $M$ [11]. In particular, if $E$ is the action Lie algebroid for a $\mathfrak{g}$-action on $Q$, then one obtains a $\mathfrak{g}$-action on $M$.

**Example 2.13.** Let $Q$ be a manifold with a weak Poisson structure $(\mathbb{T}Q, E)$, thus $E \cap TQ = 0$, and let $M$ be a Hamiltonian space, defined by a Dirac morphism $R: (\mathbb{T}M, TM) \rightarrow (\mathbb{T}Q, E)$. By Proposition 2.8, the backward image $F = TQ \circ R$ is a Lagrangian subbundle with $TM \cap F = 0$. We conclude that $F$ is again a weak Poisson structure. The map $\Phi$ is anti-Poisson for these Poisson structures.
2.5. Exact Courant algebroids. A Courant algebroid \( \mathbb{A} \) with base \( Q \) is called exact \cite{Severa03} if the following sequence is exact:

\[
0 \longrightarrow T^*Q \xymatrix{ \ar[r]^-{a^*} & } \mathbb{A} \xymatrix{ \ar[r]^-{a} & } TQ \xymatrix{ \ar[r] & } 0.
\]

Equivalently, \( a^* \) embeds \( T^*Q \) as a Lagrangian subbundle, defining a Dirac structure \((\mathbb{A}, \text{ran}(a^*))\). Using again that \( a(x) = \text{gr}(a) \) proves exactness at \( \text{gr}(a) \), the statement that \( a \) is equivalent to (c). The final statement follows from Lemma 2.11.

Proof. The implication (a) \( \Rightarrow \) (b) is trivial. Suppose condition (b) holds. Then the map \( a^{|_{T^*\Phi}} : \text{gr}(T^*\Phi) \rightarrow \text{gr}(R) \) is injective, with image a closed subbundle of \( \text{ker}(a|_{\text{gr}(R)}) \subseteq \text{gr}(R) \). Since the \( \mathbb{A}_i \) are exact Courant algebroids, any \( x \in \text{ker}(a|_{\text{gr}(R)}) \) is of the form \( x = a^*\mu \) for some \( \mu \in T^*(Q_2 \times Q_1) \). For all \( y \in \text{gr}(R) \), we have \( 0 = \langle a^*\mu, y \rangle = \langle \mu, a(y) \rangle \).

Using again that \( a|_{\text{gr}(R)} : \text{gr}(R) \rightarrow \text{gr}(T\Phi) \) is surjective, it follows that \( \mu \in \text{gr}(T^*\Phi) \). This proves exactness at \( \text{gr}(R) \), and hence (a). On the other hand, condition (b) amounts to the statement that \( a_1(\text{ran}^*(R)) = TQ_1 \). Since \( a_1 \) is the projection along \( \text{ran}(a_1^*) \), this is equivalent to (c). The final statement follows from Lemma 2.11.

As a consequence of the fact that (8) is strongly Dirac, exact Courant morphisms can always be composed (see Proposition 2.9). Another consequence is that one can ‘pull back’ isotropic splittings:

\[
\text{Proposition 2.14. The following conditions are equivalent:}
\]

(a) \( R \) is exact.

(b) \( R \) is full \cite{SjMon09}, that is, \( a|_{\text{gr}(R)} : \text{gr}(R) \rightarrow \text{gr}(T\Phi) \) is surjective.

(c) \( \text{ran}^*(R) + \text{ran}(a_1^*) = \mathbb{A}_1 \).

Furthermore, in this case \( R \) defines a strong Dirac morphism

\[
R : (\mathbb{A}_1, \text{ran}(a_1^*)) \longrightarrow (\mathbb{A}_2, \text{ran}(a_2^*)).
\]

Proof. The implication (a) \( \Rightarrow \) (b) is trivial. Suppose condition (b) holds. Then the map \( a^{|_{T^*\Phi}} : \text{gr}(T^*\Phi) \rightarrow \text{gr}(R) \) is injective, with image a closed subbundle of \( \text{ker}(a|_{\text{gr}(R)}) \subseteq \text{gr}(R) \). Since the \( \mathbb{A}_i \) are exact Courant algebroids, any \( x \in \text{ker}(a|_{\text{gr}(R)}) \) is of the form \( x = a^*\mu \) for some \( \mu \in T^*(Q_2 \times Q_1) \). For all \( y \in \text{gr}(R) \), we have \( 0 = \langle a^*\mu, y \rangle = \langle \mu, a(y) \rangle \).

Using again that \( a|_{\text{gr}(R)} : \text{gr}(R) \rightarrow \text{gr}(T\Phi) \) is surjective, it follows that \( \mu \in \text{gr}(T^*\Phi) \). This proves exactness at \( \text{gr}(R) \), and hence (a). On the other hand, condition (b) amounts to the statement that \( a_1(\text{ran}^*(R)) = TQ_1 \). Since \( a_1 \) is the projection along \( \text{ran}(a_1^*) \), this is equivalent to (c). The final statement follows from Lemma 2.11.
Proposition 2.15. Let \( R : \mathfrak{a}_1 \rightarrow \mathfrak{a}_2 \) be an exact Courant morphism, and \( j_2 : TQ_2 \rightarrow \mathfrak{a}_2 \) an isotropic splitting, with corresponding 3-form \( \eta_2 \). Then there is a unique isotropic splitting \( j_1 : TQ_1 \rightarrow \mathfrak{a}_1 \) such that
\[
R \circ j_1 = j_2 \circ T\Phi.
\]
The corresponding 3-form is \( \eta_1 = \Phi^*\eta_2 \).

Proof. The subbundle \( F_2 = j_2(TQ_2) \) is a Lagrangian complement to \( \text{ran}(a_2^*) \). Since (8) is strongly Dirac, Proposition 2.8 shows that its backward image \( F_1 = \Phi^*F_2 \circ R \) is a Lagrangian complement to \( \text{ran}(a_1^*) \). Hence it is of the form \( F_1 = j_1(TQ_1) \) for an isotropic splitting \( j_1 \). By construction, this splitting satisfies \( R \circ j_1 = j_2 \circ T\Phi \). Uniqueness of the isotropic splitting \( j_1 \) with this property follows from \( \ker(a_1) \cap \ker(R) = 0 \). Let \( \eta_1 \) be the corresponding 3-form. Then \( \eta = \text{pr}_1^2 \eta_2 - \text{pr}_2^1 \eta_1 \in \Omega^3(Q_2 \times Q_1) \) is the 3-form for the splitting \( j = j_2 \times j_1 \) of \( \mathfrak{a}_2 \times \mathfrak{a}_1 \). If \( v, v', v'' \) are vector fields on \( Q = Q_2 \times Q_1 \) that are tangent to \( \text{gr}(\Phi) \), then \( j(v), j(v'), j(v'') \) restrict to sections of \( \text{gr}(R) \), and so does \( [j(v'), j(v'')] \). It follows that \( \langle \eta(v, v', v'') \rangle = \langle j(v), [j(v'), j(v'')] \rangle \) vanishes along \( \text{gr}(\Phi) \), which is to say \( \eta_1 = \Phi^*\eta_2 \). \( \square \)

Proposition 2.16. Let \( \mathfrak{a}_1, \mathfrak{a}_2 \) be exact Courant algebroids over \( Q_1, Q_2 \), with isotropic splittings \( j_1, j_2 \) identifying \( \mathfrak{a}_i = TQ_i, \eta_i \). Then an exact Courant morphism \( R : \mathfrak{a}_1 \rightarrow \mathfrak{a}_2 \) with base map \( \Phi : Q_1 \rightarrow Q_2 \) is equivalently described by a 2-form \( \omega \in \Omega^2(Q_1) \) satisfying
\[
d\omega = \eta_1 - \Phi^*\eta_2.
\]
This is standard in the finite-dimensional case (see e.g. [19]), and the proof carries over to infinite dimensions. In one direction, the 2-form \( \omega \) relates the splitting \( j_1 \) to the pullback of the splitting \( j_2 \) (see Proposition 2.15). In the other direction, \( \Phi \) and \( \omega \) determine an exact Courant morphism
\[
T\Phi : TQ_{1, \eta_1} \rightarrow TQ_{2, \eta_2}
\]
by the condition
\[
v_1 + \mu_1 \sim_{T\Phi^\omega} v_2 + \mu_2 \iff v_2 = \Phi_* v_1, \quad \mu_1 = \Phi^* \mu_2 + \iota(v_1)\omega.
\]
Under composition, \( T\Phi \circ T\Phi' = T(\Phi \circ \Phi')_{\omega + \Phi^*\omega'} \).

Given an exact Dirac structure, i.e., a Dirac structure \( (\mathfrak{a}, E) \) with \( \mathfrak{a} \) exact, we define an exact Hamiltonian space for \( (\mathfrak{a}, E) \) to be a manifold \( M \) together with an exact Dirac morphism \( R : (TM, TM) \rightarrow (\mathfrak{a}, E) \).

Proposition 2.17. Let \( (\mathfrak{a}, E) \) be an exact Dirac structure over \( Q \), with a given isotropic splitting \( j : TQ \rightarrow \mathfrak{a} \) identifying \( \mathfrak{a} = TQ, \eta \). Then the exact Hamiltonian spaces for \( (\mathfrak{a}, E) \) are described by a map \( \Phi : M \rightarrow Q \), a 2-form \( \omega \in \Omega^2(M) \), and a Lie algebroid action of \( E \) along \( \Phi \), satisfying
\[
\begin{align*}
(a) \quad &d\omega = -\Phi^*\eta, \\
(b) \quad &\ker(\omega) \cap \ker(T\Phi) = 0, \\
(c) \quad &\iota(\sigma_M)\omega = -\Phi^*(j^*\sigma).
\end{align*}
\]
Here \( \Phi^* : \Gamma(E) \rightarrow \Gamma(TM) \), \( \sigma \mapsto \sigma_M \), is the Lie algebroid action, and \( j^* : \mathfrak{a} \rightarrow T^*Q \) is the bundle map dual to \( j \).
Proof. The exact Courant morphisms $R: TM \rightarrow TQ_\eta$ are of the form $R = T\Phi_\omega$, where $\omega$ satisfies (a). Since $\ker(R) = \{v - \iota_v\omega \mid v \in \ker T\Phi\}$, we see that the weak transversality condition $\ker(R) \cap TM$ for the composition $R \circ TM$ is equivalent to (b). The last property (c) is equivalent to $R \circ TM = \Phi^*E$.

Example 2.18. (See [11].) Any leaf $i_\mathcal{O}: \mathcal{O} \hookrightarrow Q$ of an exact Dirac structure $(\mathbb{A}, E)$ is naturally an exact Hamiltonian space. Here

$$R: (T\mathcal{O}, T\mathcal{O}) \rightarrow (\mathbb{A}, E)$$

is uniquely defined by its properties that $v \sim_R x$ for $v \in T\mathcal{O}$ and $x \in E$ with $(T_i\mathcal{O})(v) = a(x)$, together with $\mu \sim_R a^\mathcal{O}(\nu)$ for $\mu \in T^*\mathcal{O}$, $\nu \in T^*Q$ such that $\mu \sim_{T^*\mathcal{O}} \nu$. Given an isotropic splitting, identifying $\mathbb{A} = TQ_\eta$, we obtain a 2-form $\omega \in \Omega^2(\mathcal{O})$ with $d\omega = -i^*_\mathcal{O}\eta$.

Example 2.19. Let $Q$ be a manifold with a weak Poisson structure $(TQ, E)$, thus $E \cap TQ = 0$. Let $M$ be an exact Hamiltonian space, defined by an exact Dirac morphism $R = T\Phi_\omega: (TM, TM) \rightarrow (TQ, E)$. According to the proposition, $\ker(\omega) \cap \ker(T\Phi) = 0$. In fact, it is automatic that $\ker(\omega) = 0$. To see this, note that

$$\Phi^*E = R \circ TM, \quad \text{gr}(\omega) = TQ \circ R.$$ 

Since $TQ \cap E = 0$, Proposition 2.8 shows that $\text{gr}(\omega) \cap TM = 0$. Equivalently, $\ker(\omega) = 0$.  

2.6. The Cartan-Dirac structure. Of special interest in this paper is the Cartan-Dirac structure on a Lie group $G$. We describe here its definition as an action Courant algebroid; later we will show that the same Dirac structure arises by reduction from the Lie-Poisson structure on the space of connections.

2.6.1. Definition of the Cartan-Dirac structure. Let $G$ be a Lie group. For $X \in \mathfrak{g}$ we denote by $X^L, X^R$ the corresponding left, right-invariant vector fields. The Maurer-Cartan forms on $G$ will be denoted $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})^G$; thus $\iota(X^L)\theta^L = X = \iota(X^R)\theta^R$.

Suppose $G$ carries a bi-invariant pseudo-Riemannian metric, with corresponding Ad-invariant metric $(X_0, X_1) \mapsto X_0 \cdot X_1$ on $\mathfrak{g}$. We denote by $\overline{G}$ the Lie group $G$ with the opposite pseudo-Riemannian metric, and likewise by $\overline{\mathfrak{g}}$ the Lie algebra $\mathfrak{g}$ with the opposite metric. Let $D := \overline{G} \times G$ act on $G$ by

$$(g_0, g_1).a = g_0 a g_1^{-1}.$$

The infinitesimal action $\mathfrak{d} = \overline{\mathfrak{g}} \oplus \mathfrak{g} \rightarrow \Gamma(TG)$ reads as $(X_0, X_1) \mapsto X_0^L - X_0^R$. It has co-isotropic stabilizers, hence it defines an action Courant algebroid

$$(11) \quad \mathbb{A} = G \times \mathfrak{d}.$$

We refer to $\mathbb{A}$ as the Cartan-Courant algebroid. If $\mathfrak{s} \subseteq \mathfrak{d}$ is any subspace, the subbundle $E^{(s)} = G \times \mathfrak{s}$ is Lagrangian if and only if $\mathfrak{s}$ is Lagrangian, and is involutive if and only if $\mathfrak{s}$ is a Lie subalgebra. Thus, any Lagrangian Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{d}$ determines a Dirac structure. The Dirac structure $E = E_{\mathfrak{g}_\Delta} \subseteq \mathbb{A}$ defined by the diagonal $\mathfrak{g}_\Delta \subseteq \mathfrak{d}$ is called the Cartan-Dirac structure.
Example 2.20. If $\kappa: \mathfrak{g} \to \mathfrak{g}$ is an orthogonal Lie algebra automorphism, then the graph $\text{gr}(\kappa) = \{(\kappa(X), X) | X \in \mathfrak{g}\}$ is a Lagrangian Lie subalgebra. Hence it determines a Dirac structure $E(\kappa) = E_{\text{gr}(\kappa)}$. If the metric on $\mathfrak{g}$ is positive definite, then any Lagrangian Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{d}$ arises in this way. Indeed, any Lagrangian subspace is then given as the graph of an orthogonal transformation, and the condition that $\mathfrak{s}$ is a Lie subalgebra means that this transformation preserves Lie brackets.

2.6.2. Splitting. The Cartan-Courant algebroid (11) is exact, with an isotropic splitting $j: TG \to A$ given at the group unit by the map $\mathfrak{g} \to \mathfrak{d}$, $X \mapsto \frac{1}{2}(X, X)$. Equivalently, the map on sections $j: \Gamma(TG) \to \Gamma(A)$ is

$$j(v) = \left(-\frac{1}{2}v^0 \theta^R, \frac{1}{2}v^1 \theta^L\right),$$

for $v \in \Gamma(TG)$. By direct calculation, one find that the resulting 3-form is the Cartan 3-form

$$\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L],$$

and that $\alpha = j^* \circ \varphi: \mathfrak{d} \to \Omega^1(Q)$ is given by

$$\alpha(X_0, X_1) = \frac{1}{2}(\theta^L \cdot X_1 + \theta^R \cdot X_0), \quad (X_0, X_1) \in \mathfrak{d}. $$

Let $\varphi: A = G \times (\mathfrak{g} \oplus \mathfrak{g}) \cong TG_\eta$ be the resulting isomorphism. On the level of sections,

$$\varphi(X_0, X_1) = X_0^L - X_0^R + \alpha(X_0, X_1)$$

for $(X_0, X_1) \in \mathfrak{g} \oplus \mathfrak{g}$. Taking $X_0 = X_1 = X$, we see that the Cartan Dirac structure is spanned by the sections $X_G + \frac{1}{2}(\theta^L + \theta^R) \cdot X$ for $X \in \mathfrak{g}$, where $X_G$ is the generating vector field for the conjugation action.

2.6.3. Hamiltonian spaces. Suppose $\mathfrak{s} \subseteq \mathfrak{d}$ is a Lagrangian Lie subalgebra, defining a Dirac structure $(A, E(\mathfrak{s}))$. The data of Hamiltonian space $R: (TM, TM) \to (A, E(\mathfrak{s}))$ for this Dirac structure gives, in particular, a Lie algebra action of $\mathfrak{s}$ on $M$, such that $Y_M \sim R \varphi(Y)$ for all $Y \in \mathfrak{s}$. If $R$ is exact, one can use splittings to formulate these conditions in terms of differential forms. Indeed, Proposition 2.17 specializes to the following statement.

Proposition 2.21. An exact Hamiltonian space for the Dirac structure $(A, E(\mathfrak{s}))$ is equivalent to a triple $(M, \omega, \Phi)$, consisting of a manifold $M$ with an $\mathfrak{s}$-action, a 2-form $\omega \in \Omega^2(M)$ and an $\mathfrak{s}$-equivariant map $\Phi: M \to G$ satisfying

(a) $d\omega = -\Phi^* \eta,$
(b) $\ker(\omega) \cap \ker(T\Phi) = 0,$
(c) $\iota(Y_M)\omega = -\frac{1}{2}(Y_1 \cdot \theta^L + Y_0 \cdot \theta^R)$ for all $Y = (Y_0, Y_1) \in \mathfrak{s}$.

For the special case that $\mathfrak{s}$ is the diagonal, we recover the axioms of a q-Hamiltonian $\mathfrak{g}$-space as in [3]. If the action of $\mathfrak{s}$ integrates to an action of a Lie group $S$, and if $R$ is $S$-equivariant, we get a Hamiltonian $S$-space for $(A, E(\mathfrak{s}))$: That is, $\omega$ is $S$-invariant and $\Phi$ is $S$-equivariant. For instance, the $S$-orbits in $G$ are Hamiltonian $S$-spaces for $(A, E(\mathfrak{s}))$. Other examples are obtained by ‘fusion’, as in [3].
2.6.4. Multiplicative properties. Give \( \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \) the pair groupoid structure \( \mathfrak{d} \rightrightarrows \mathfrak{g} \), with multiplication \( (Y_0, Y_1) \circ (Y'_0, Y'_1) = (Y_0, Y'_1) \) for \( Y_1 = Y'_0 \). Taking the direct product of this groupoid with the group \( G \), we obtain a groupoid \( A \rightrightarrows \mathfrak{g} \). Since the groupoid multiplication covers the group multiplication of \( G \), this is pictured as

\[
\begin{array}{c}
A \\
\downarrow \\
G \\
\downarrow \\
pt
\end{array}
\]

Let \( \text{Mult}_A \) be the groupoid multiplication, defined on the subset of composable elements. Its graph \( \text{gr} (\text{Mult}_A) \subset A \times A \times A \) is a Dirac structure along the graph \( \text{gr} (\text{Mult}_G) \) of the group multiplication, defining a Courant morphism \([2]\),

\[
\text{gr} (\text{Mult}_A) : A \times A \longrightarrow A.
\]

Similarly, the groupoid inversion, \( \text{Inv}_A = \text{Inv}_G \times \text{Inv}_\mathfrak{d} \) defines a Courant morphism

\[
\text{gr} (\text{Inv}_A) : A \longrightarrow \overline{A}.
\]

The Cartan Dirac structure \((A, E)\) makes \( G \) into a Dirac Lie group, in the sense that the groupoid multiplication defines a morphism of Manin pairs,

\[
(15) \quad \text{gr} (\text{Mult}_A) : (A, E) \times (A, E) \longrightarrow (A, E),
\]

with underlying map the group multiplication \([2, 22]\). More generally, suppose \( \mathfrak{s}_1, \mathfrak{s}_2 \subseteq \mathfrak{d} \) are Lagrangian Lie subalgebras, and that the groupoid multiplication \( \mathfrak{s}_1 \circ \mathfrak{s}_2 \) is a transverse composition of linear relations. Then \( \mathfrak{s}_1 \circ \mathfrak{s}_2 \) is a Lagrangian Lie subalgebra, and \( \text{gr} (\text{Mult}_A) \) defines a morphism of Manin pairs

\[
(16) \quad \text{gr} (\text{Mult}_A) : (A, E_{\mathfrak{s}_1}) \times (A, E_{\mathfrak{s}_1}) \longrightarrow (A, E_{\mathfrak{s}_1 \circ \mathfrak{s}_2}).
\]

In terms of the identification \( A \cong TG_\eta \) defined by the splitting of \( A \), the multiplication morphism is given by the pair \((\text{Mult}_G, \varsigma)\), where \( \varsigma \) is a 2-form on \( G \times G \) satisfying

\[
d\varsigma = \text{Mult}_G^* \eta - \text{pr}_1^* \eta - \text{pr}_2^* \eta,
\]

where \( \text{pr}_1, \text{pr}_2 : G \times G \rightarrow G \) are the two projections. As shown in \([2]\), this 2-form is

\[
(17) \quad \varsigma = -\frac{1}{2} \text{pr}_1^* \theta^L \cdot \text{pr}_2^* \theta^R.
\]

3. Reduction of Dirac structures

In this section we continue to use the terms “manifold”, “vector bundle”, “Lie group”, etc. to refer to Hilbert manifolds, Hilbert vector bundles, Hilbert Lie groups, and so on, unless otherwise specified.
3.1. Actions on Courant algebroids. Let $\mathcal{A}$ be a Courant algebroid with base $Q$. A Courant derivation of $\mathcal{A}$ is a linear operator $\tilde{v}$ on $\Gamma(\mathcal{A})$, together with a vector field $v$ on $Q$, satisfying
\[
v(\sigma_1, \sigma_2) = \langle \tilde{v}\sigma_1, \sigma_2 \rangle + \langle \sigma_1, \tilde{v}\sigma_2 \rangle,
\]
\[
\tilde{v}[\sigma_1, \sigma_2] = [\tilde{v}\sigma_1, \sigma_2] + [\sigma_1, \tilde{v}\sigma_2],
\]
\[
a(\tilde{v}(\sigma)) = [v, a(\sigma)].
\]
for all $\sigma_1, \sigma_2 \in \Gamma(\mathcal{A})$. These properties imply the property, $\tilde{v}(f \sigma) = v(f)\sigma + f \tilde{v}(\sigma)$ for all $f \in \mathcal{C}^\infty(Q)$ and $\sigma \in \Gamma(\mathcal{A})$. Let $\text{Der}(\mathcal{A})$ be the Lie algebra of Courant derivations of $\mathcal{A}$. A Courant derivation is called inner if it is of the form $\tilde{v} = [\sigma, \cdot]$ for some $\sigma \in \Gamma(\mathcal{A})$; we refer to $\sigma$ as a generator of this Courant derivation. Note that the map $\Gamma(\mathcal{A}) \to \text{Der}(\mathcal{A})$, $\sigma \mapsto [\sigma, \cdot]$ is bracket-preserving.

A Courant automorphism of $\mathcal{A}$ is a vector bundle automorphism preserving the metric, the bracket, and compatible with the anchor. One can informally regard $\text{Der}(\mathcal{A})$ as the Lie algebra of the group $\text{Aut}(\mathcal{A})$ of Courant automorphisms.

In particular, any 1-parameter group $g_t \in \text{Aut}(\mathcal{A})$, $t \in \mathbb{R}$ of Courant automorphisms determines a Courant derivation $\tilde{v}(\sigma) = \left. \frac{d}{dt} \right|_{t=0} (g_t)^* \sigma$.

**Remark 3.1.** For exact Courant algebroids, the group $\text{Aut}(\mathcal{A})$ and the Lie algebra $\text{Der}(\mathcal{A})$ are described in [19, Section 2.1]. Choose a splitting to identify $\mathcal{A} = TQ_\eta$ for a closed 3-form $\eta$. Then $\text{Der}(TQ_\eta)$ is the Lie subalgebra of the semidirect product $\Gamma(TQ) \ltimes \Omega^2(Q)$, consisting of pairs $(v, \varepsilon)$ with $\mathcal{L}_v \eta = d\varepsilon$. The corresponding derivation $\tilde{v}$ reads as
\[
\tilde{v}(w + \mu) = [v, w] + \mathcal{L}_v \mu - \iota_w \varepsilon,
\]
for vector fields $w$ and 1-forms $\mu$. The inner derivation $[\sigma, \cdot]$ defined by $\sigma = w + \mu$ corresponds to the pair $(v, \varepsilon)$ with $v = w$ and $\varepsilon = -(d\mu + \iota_w \eta)$. (We see that that $[\sigma, \cdot] = 0$ if and only if $\sigma = a^\ast \mu$ with $d\mu = 0$; this description does not depend on the choice of splitting.) Similarly $\text{Aut}(\mathcal{A})$ is isomorphic to the subgroup of the semidirect product $\text{Diff}(Q) \ltimes \Omega^2(Q)$ consisting of pairs $(\Phi, \varepsilon)$ with $\Phi^* \eta + d\varepsilon = 0$; the action of such a pair is given by the Courant morphism $T\Phi_\varepsilon$. (Since $\Phi$ is a diffeomorphism, this morphism is given by an actual vector bundle automorphism of $\mathcal{A}$.)

**Definition 3.2.**
(a) Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-action on a Courant algebroid $\mathcal{A}$ is a Lie algebra homomorphism $\mathfrak{g} \to \text{Der}(\mathcal{A})$, $\xi \mapsto \xi_\mathcal{A}$. A bracket preserving map $\varrho: \mathfrak{g} \to \Gamma(\mathcal{A})$ is said to define generators for the $\mathfrak{g}$-action if $\xi_\mathcal{A} = [\varrho(\xi), \cdot]$ for all $\xi \in \mathfrak{g}$.

(b) Let $G$ be a Lie group, acting on $\mathcal{A}$ by Courant automorphisms. A $G$-equivariant map $\varrho: \mathfrak{g} \to \Gamma(\mathcal{A})$ is said to define generators for the $G$-action if
\[
\xi_\mathcal{A} := \left. \frac{d}{dt} \right|_{t=0} \exp(-t \xi)^* = [\varrho(\xi), \cdot]
\]
for all $\xi \in \mathfrak{g}$.

Observe that $a(\varrho(\xi)) = \xi_Q$ are the generating vector fields for an action on $Q$. We will use the same letter $\varrho$ to denote the associated bundle map
\[
\varrho: Q \times \mathfrak{g} \to \mathcal{A}, \ (m, \xi) \mapsto \varrho(\xi)_m.
\]
The dual map $\mathcal{A} \to \mathfrak{g}^*$ is sometimes referred to as a moment map for the Courant $G$ action. The set of generators for a Courant $G$-action is either empty, or is an affine space modeled on the space of $G$-equivariant maps from $\mathfrak{g}$ into the kernel of the map $\sigma \mapsto [\sigma, \cdot]$. For an exact Courant algebroid, this kernel is identified with the space of closed 1-forms.

**Example 3.3.** For any action Courant algebroid $\mathcal{A} = Q \times \mathfrak{d}$, the Lie algebra $\mathfrak{d}$ acts on $\mathcal{A}$ by derivations, with the constant sections $\varrho: \mathfrak{d} \to \Gamma(\mathcal{A})$ as generators. In particular, the Cartan-Courant algebroid $G \times (\mathfrak{g} \oplus \mathfrak{g}) \cong TG_\eta$ is $D = \mathcal{G} \times G$-equivariant, with the map (14) as generators.

**Example 3.4 (Lie-Poisson structure).** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, let $\xi_{\mathfrak{g}}^* \in \Gamma(T\mathfrak{g}^*)$ be the generating vector fields for the coadjoint action, and denote by $d\mu \in \Omega^1(\mathfrak{g}^*, \mathfrak{g}^*)$ the tautological 1-form. Then the map $\varrho: \mathfrak{g} \to \Gamma(T\mathfrak{g}^*)$,

\begin{equation}
\varrho(\xi) = \xi_{\mathfrak{g}}^* + \langle d\mu, \xi \rangle
\end{equation}

defines isotropic generators for the $G$-action on $T\mathfrak{g}^*$. The Dirac structure $E \subseteq T\mathfrak{g}^*$ spanned by the sections $\varrho(\xi), \xi \in \mathfrak{g}$ is a Poisson structure, in the strong sense that $T\mathfrak{g}^* = E \oplus T\mathfrak{g}^*$. It is known as the Lie-Poisson structure on $\mathfrak{g}^*$.

**3.2. Reduction of Dirac structures.** Let $\mathcal{A} \to Q$ be a $G$-equivariant Courant algebroid with generators $\varrho: Q \times \mathfrak{g} \to \mathcal{A}$. We assume that the action is principal, i.e. that $Q$ is a principal bundle with base manifold $Q/G$.

**Lemma 3.5.** The generators span a closed subbundle $\text{ran}(\varrho) = \varrho(Q \times \mathfrak{g}) \subseteq \mathcal{A}$.

**Proof.** The map $\varrho$ is a continuous bundle map. Since the composition $a \circ \varrho: Q \times \mathfrak{g} \to TQ$ is injective, with closed image, $\varrho$ must also have a closed image. $\square$

We will describe the reduction procedure for the case that the generators $\varrho(\xi)$ are isotropic; equivalently, $\text{ran}(\varrho) = \varrho(Q \times \mathfrak{g})$ is isotropic.

**Theorem 3.6 ([9]).** Suppose the generators are isotropic; thus $C = \varrho(Q \times \mathfrak{g})^\perp$ is coisotropic. Then $\mathcal{A}_C = C/C^\perp$ is a $G$-equivariant bundle, and the quotient bundle

\[ \mathcal{A}_{\text{red}} = \mathcal{A}_C/G \]

with the induced fiber metric, bracket and anchor map is a Courant algebroid. If $E \subseteq \mathcal{A}$ is a $G$-invariant Dirac structure and $E + C$ is a closed subbundle, then the reduction $E_C = (E \cap C)/(E \cap C^\perp)$ is a $G$-equivariant bundle, and the reduced bundle

\[ E_{\text{red}} = E_C/G \subseteq \mathcal{A}_{\text{red}} \]

defines a Dirac structure $(\mathcal{A}_{\text{red}}, E_{\text{red}})$ over $Q/G$.

This result was proved in [9] in the finite-dimensional setting, and for the case of exact Courant algebroids. However, the proof immediately carries over to the general case. A key observation is that the space $\Gamma(C)^G$ is closed under Courant bracket, containing $\Gamma(C^\perp)^G$ as a Courant ideal. Hence the Courant bracket descends to $\Gamma(\mathcal{A}_{\text{red}}) = \Gamma(C)^G/\Gamma(C^\perp)^G$. Further, since $a(C^\perp) \subseteq TQ$ lies in the $G$-orbit directions, we obtain a reduced anchor map $a_{\text{red}}: \mathcal{A}_{\text{red}} \to T(Q/G)$.

**Remarks 3.7.**
(a) The condition that $E + C$ be closed is trivially satisfied if $E \subseteq C$, i.e. $\text{ran}(g) \subseteq E$. One then has $E_{\text{red}} = (E/\text{ran}(g))/G$, and $a(E_{\text{red}}) \subseteq T(Q/G)$ is the image of $a(E) \subseteq TQ$ under the quotient.

(b) Suppose that the action of $G$ on $\mathbb{A}$ extends to an action of a Lie group $U \supseteq G$, with $U$-equivariant generators $g: u \rightarrow \Gamma(\mathbb{A})$ extending those of $g$. We assume that $G$ is a normal subgroup of $U$, and that $(g(\xi), g(\zeta)) = 0$ for all $\xi \in \mathfrak{g}$, $\zeta \in \mathfrak{u}$. Then the $G$-reduced Courant algebroid $\mathbb{A}_{\text{red}}$ inherits an action of the quotient group $U/G$, with generators $g_{\text{red}}: u/\mathfrak{g} \rightarrow \Gamma(\mathbb{A}_{\text{red}})$ induced from those on $\mathbb{A}$.

(c) There is a natural Courant morphism $q: \mathbb{A} \longrightarrow \mathbb{A}_{\text{red}}$, where $x \sim_q y$ if and only if $x \in C$, with $y$ its image in $\mathbb{A}_{\text{red}}$. If the $G$-invariant Dirac structure $E$ has the property $E + C = \mathbb{A}$, then $q$ defines a strong Dirac morphism $q: (\mathbb{A}, E) \longrightarrow (\mathbb{A}_{\text{red}}, E_{\text{red}})$. In the opposite extreme, if $\text{ran}(g) \subseteq E$, this is a Dirac comorphism.

(d) The closed subbundle $a^{-1}(\text{ran}(g_Q)) \subseteq \mathbb{A}$ decomposes as a direct sum $\ker(a) \oplus C^\perp$; hence $\ker(a) + C$ is again closed. If $\mathbb{A}$ is exact, so that $\ker(a) = \text{ran}(a^*)$ is Lagrangian, this shows that $\text{ran}(a^*) + C = \mathbb{A}$, or equivalently $a(C) = TQ$. Using these facts, we see that $\mathbb{A}_{\text{red}}$ is exact, and the Courant morphism $q$ is exact as well. Furthermore, $q: (\mathbb{A}, \text{ran}(a^*)) \longrightarrow (\mathbb{A}_{\text{red}}, \text{ran}(a^*_{\text{red}}))$ is a strong Dirac morphism.

### 3.3. Reduction of Dirac Morphisms

Let $\mathbb{A}_i \rightarrow Q_i$, $i = 1, 2$, be $G_i$-equivariant Courant algebroids over $Q_i$, with generators $g_i: \mathfrak{g}_i \rightarrow \Gamma(\mathbb{A}_i)$. A Courant morphism $R: \mathbb{A}_1 \longrightarrow \mathbb{A}_2$, with base map $\Phi: Q_1 \rightarrow Q_2$, is called equivariant with respect to a group homomorphism $f: G_1 \rightarrow G_2$ if $x_1 \sim_R x_2 \Rightarrow g \cdot x_1 \sim_R f(g) \cdot x_2$ for all $g \in G_1$ and $x_i \in \mathbb{A}_i$. It intertwines the generators if

$$g_1(\xi) \sim_R g_2(f(\xi))$$

for all $\xi \in \mathfrak{g}_1$. Let $\text{ran}(g_i) = g_i(Q_i \times \mathfrak{g}_i) \subseteq \mathbb{A}_i$ denote the closed subbundles spanned by the generators.

### Theorem 3.8

Suppose in addition that the generators for the $G_i$-actions are isotropic and that the $G_i$-actions on $Q_i$ are principal. Then:

(a) The Courant morphism $R$ descends to a Courant morphism $$(\mathbb{A}_1)_{\text{red}} \longrightarrow (\mathbb{A}_2)_{\text{red}}^R$$ $$(Q_1/G_1) \longrightarrow Q_2/G_2$$ $\Phi_{\text{red}}$$

(b) The morphism $R_{\text{red}}$ has the property $q_2 \circ R = R_{\text{red}} \circ q_1$, where $q_i: \mathbb{A}_i \longrightarrow (\mathbb{A}_{\text{red}})_i$ are the reduction morphisms.
(c) Suppose the Courant algebroids $A_i$ are exact. Then, the reduction procedure gives a one-to-one correspondence between Courant morphisms $(A_1)_{\text{red}} \rightarrow (A_2)_{\text{red}}$, and equivariant Courant morphisms $A_1 \rightarrow A_2$ intertwining the generators.

**Proof.** Consider the $G \times G_1$-equivariant Courant algebroid $A = A_2 \times \overline{A_1}$ over $Q = Q_2 \times Q_1$, with generators $\varrho = q_2 \times q_1$. Let $\pi_i: Q_i \rightarrow Q_i/G_i$ be the quotient maps, write $\pi = \pi_2 \times \pi_1$ and $G = G_2 \times G_1$, and define $C_i = g_i(Q_i \times g_i)^{\perp}$. The subbundle $C = C_1 \times C_2$ defines the reduction to $A_{\text{red}} = \overline{A}/(G_2 \times G_1) = (A_2)_{\text{red}} \times (A_1)_{\text{red}}$. The graph $\text{gr}(\Phi)$ is invariant under the action of the diagonal subgroup $(G_1)_\Delta = \{(f(g), g) | g \in G_1\} \subseteq G$. Let

$$\tilde{\text{gr}}(\Phi) = G \cdot \text{gr}(\Phi) = G \times_{(G_1)_\Delta} \text{gr}(\Phi)$$

be the flow-out under the $G$-action. Then

$$\tilde{\text{gr}}(\Phi)/G = \text{gr}(\Phi)/(G_1)_\Delta = \text{gr}(\Phi_{\text{red}}).$$

(a) By definition of a Dirac morphism, $\text{gr}(R)$ is a Dirac structure along $\text{gr}(\Phi)$: whenever two sections of $A$ restrict over $\text{gr}(\Phi)$ to sections of $\text{gr}(R)$, then so does their Courant bracket. Also, $\text{gr}(R)$ is invariant under the action of $(G_1)_\Delta$ and its flow-out

$$\tilde{\text{gr}}(R) = G \cdot \text{gr}(R) \cong G \times_{(G_1)_\Delta} \text{gr}(R) \rightarrow \tilde{\text{gr}}(\Phi)$$

is a closed Lagrangian subbundle. Since its space of sections is generated by $\Gamma(\tilde{\text{gr}}(R))^{G} \cong \Gamma(\text{gr}(R))^{|G_1)_\Delta}$, it is also involutive. Hence it is a Dirac structure along $\text{gr}(\Phi)$.

Along $\text{gr}(\Phi)$, the sum

$$\text{gr}(R) + \text{ran} (\varrho) |_{\text{gr}(\Phi)}$$

is a closed subbundle, for the following reason: since $g_1(\xi) \sim_R g_2(f(\xi))$ for all $\xi \in g_1$, it coincides with the direct sum of closed subbundles $R \oplus \text{ran}(\varrho_2)$, and this is mapped by the anchor to the closed subbundle $T \tilde{\text{gr}}(\Phi) = T \text{gr}(\Phi) \oplus \text{ran}(\varrho_2)$ in a way which preserves the direct sum decomposition and is injective on the second factor. As a result, its flow-out $\tilde{\text{gr}}(R) + \text{ran}(\varrho)|_{\tilde{\text{gr}}(\Phi)}$ under the action of $G$ is also closed.

It follows that $\tilde{\text{gr}}(R)_C = (\tilde{\text{gr}}(R) \cap C)/(\tilde{\text{gr}}(R) \cap C^\perp)$ is a Lagrangian subbundle of $A_C$ along $\text{gr}(\Phi)$ and hence that

$$\text{gr}(R)_{\text{red}} = \tilde{\text{gr}}(R)_C/G,$$

is a Lagrangian subbundle of $A_{\text{red}} = A_C/G$ along the graph of $\Phi_{\text{red}}$. To check integrability of $\text{gr}(R)_{\text{red}}$, it is enough to argue locally. Let $\sigma, \sigma'$ be sections of $A_{\text{red}}$ defined near $(\pi_2(\Phi(m)), \pi_1(m))$, and restricting to sections of $\text{gr}(R)_{\text{red}}$. Using local triviality of the principal bundle, these lift to $G$-invariant sections $\tilde{\sigma}, \tilde{\sigma}'$ of $C \subseteq A$, defined near $(\Phi(m), m)$, and restricting to sections of $\text{gr}(R)$. The Courant bracket $[\tilde{\sigma}, \tilde{\sigma}']$ has the same property, by integrability of both $C$ and $\tilde{\text{gr}}(R)$. Hence $[\sigma, \sigma']$ descends to a section of $\text{gr}(R)_{\text{red}}$.

(b) Let $x_1 \in A_1$ and $y_2 \in (A_2)_{\text{red}}$; we must show that $x_1 \sim_R x_2 \sim_{\text{red}} y_2$ for some $x_2 \in A_2$ if and only if $x_1 \sim_{q_1} y_1 \sim_{\text{red}} y_2$ for some $y_1 \in (A_1)_{\text{red}}$. Given the latter property, since $y_1 \sim_{\text{red}} y_2$ the definition of $R_{\text{red}}$ gives $\tilde{x}_1 \sim_R \tilde{x}_2$ for some $\tilde{x}_1 \in \tilde{A}_1$ with $\tilde{x}_i \sim_{q_i} x_i$. The difference $\tilde{x}_1 - x_1$ is $q_1$-related to 0, hence it is of the form
\(g_1(\xi_1)|_m\) for some \(\xi_1 \in g_1\). Put \(x_2 = \bar{x}_2 - g_2(f(\xi_1))\). Then \(x_1 \sim_R x_2 \sim_R y_2\) as desired. Conversely, given \(x_2\) with this property, so that \(x_2 \in C_2\), the condition \(x_1 \sim_R x_2\) implies that for all \(\xi_1 \in g_1\), \(\langle x_1, g_1(\xi_1) \rangle = \langle x_2, g_2(f(\xi_1)) \rangle = 0\). Hence \(x_1 \in C_1\), which determines an element \(y_1\) with \(x_1 \sim_{q_1} y_1\). By definition of \(R_{\text{red}}\), the property \(x_1 \sim_R x_2\) descends to \(y_1 \sim_{R_{\text{red}}} y_2\).

(c) Given \(R\), we show how to express \(R\) in terms of \(R_{\text{red}}\). Let \(p_i: C_i \to A_i\) be the quotient maps, and \(p = p_2 \times p_1\). The pre-image \(p^{-1}(\text{gr}(R_{\text{red}}))\) is a Lagrangian subbundle along \(\pi^{-1}(\text{gr}(\Phi_{\text{red}})) = \tilde{\text{gr}}(\Phi)\). Its intersection with \(a^{-1}(\text{gr}(T\Phi))\) is contained in \(\text{gr}(R)\). Since \(A\) is an exact Courant algebroid, \(D = a^{-1}(\text{gr}(T\Phi))\) is a closed coisotropic subbundle; the orthogonal bundle is \(D^\perp = a^*(\text{gr}(T^*\Phi))\). Note that \(p^{-1}(\text{gr}(R_{\text{red}}))|_{\text{gr}(\Phi)} + D = \text{ran}(\rho)|_{\text{gr}(\Phi)} + D\) is closed, by the same reasoning as in part (a). Reducing \(p^{-1}(\text{gr}(R_{\text{red}}))|_{\text{gr}(\Phi)}\) with respect to \(D\), and then taking the inverse image under the quotient map \(D \to D/D^\perp\), we obtain a Lagrangian subbundle

\[(p^{-1}(\text{gr}(R_{\text{red}}))|_{\text{gr}(\Phi)} \cap D) + D^\perp\]

along \(\text{gr}(\Phi)\). Since both summands lie in \(\text{gr}(R)\), the sum (20) is in fact equal to \(\text{gr}(R)\). Conversely, if the Courant morphism \(R_{\text{red}}\) is given, we can take (20) as the definition of \(R\). This \(R\) is \((G_1)_{\Delta}\)-equivariant and intertwines the generators, and an argument similar to (a) shows that it is integrable. \(\square\)

For the rest of this Section we shall focus on the case in which there a single group \(G\) acting on both \(A_i\) and equivariance holds with respect to the identity map.

**Proposition 3.9.** Consider the setting of Theorem 3.8 with \(G_1 = G_2 = G\) and \(f = \text{id}_G\). Suppose \(E_i \subseteq A_i\) are \(G\)-invariant Dirac structures such \(E_i + \text{ran}(\varrho_i)\) are closed subbundles, and that \(R\) defines a Dirac morphism \(R: (A_1, E_1) \to (A_2, E_2)\). Suppose also that for all \(m \in Q_1, \xi \in g\),

\[g_2(\xi) \Phi(m) \in E_2 \Rightarrow g_1(\xi)|_m \in E_1\]

Then \(R_{\text{red}}\) defines a Dirac morphism

\[R_{\text{red}}: ((A_1)_{\text{red}}, (E_1)_{\text{red}}) \to ((A_2)_{\text{red}}, (E_2)_{\text{red}})\].

If the composition \(R \circ E_1\) is transverse, then so is the composition \(R_{\text{red}} \circ (E_1)_{\text{red}}\).

**Proof.** We have to show that every \(y_2 \in \Phi^*_{\text{red}}(E_2)_{\text{red}}\) is \(R_{\text{red}}\)-related to a unique element \(y_1 \in (E_1)_{\text{red}}\). Let \(x_2 \in E_2 \cap C_2\) be a lift of \(y_2\). Since \(R\) is a Dirac morphism, there exists \(x_1 \in E_1\) with \(x_1 \sim_R x_2\). This element satisfies \(\langle x_1, g_1(\xi) \rangle = \langle x_2, g_2(\xi) \rangle = 0\) for all \(\xi \in g\), hence \(x_1 \in C_1\). Letting \(y_1 \in (E_1)_{\text{red}}\) be the image, we get \(y_1 \sim_{R_{\text{red}}} y_2\). For uniqueness, suppose \(y_1 \in (E_1)_{\text{red}}\) satisfies \(y_1 \sim_{R_{\text{red}}} 0\). Choose elements \(x_i \in C_i \cap E_i\) with \(x_1 \sim_R x_2, x_1 \sim_{q_1} y_1\) and \(x_2 \sim_{q_2} 0\). The last condition gives \(x_2 = g_2(\xi) \Phi(m)\) for some \(\xi \in g\). Then \(x_1 \sim_R x_2\) but also \(g_1(\xi) \sim_R x_2\). By assumption (21), and since \(R\) is a Dirac morphism, this implies \(x_1 = g_1(\xi)\). Hence \(y_1 = 0\).

Suppose that the composition of \(R\) with \(E_1\) is transverse, that is, \(\text{ran}^*(R) + E_1 = A_1\). We want to prove \(\text{ran}^*(R_{\text{red}}) + (E_1)_{\text{red}} = (A_1)_{\text{red}}\). Given \(v_1 \in (A_1)_{\text{red}}\), let \(u_1 \in C_1\) be a preimage. Write \(u_1 = x_1 + a_1\) with \(x_1 \in E_1\) and \(a_1 \in \text{ran}^*(R)\). Then \(a_1 \sim_R a_2\) for some
a_2 \in \mathbb{A}_2$. By assumption, for all \( w_2 \in E_2 \cap \text{ran}(\varrho_2) \) there exists \( w_1 \in E_1 \cap \text{ran}(\varrho_1) \) with \( w_1 \sim_R w_2 \). Therefore,
\[
\langle a_2, w_2 \rangle = \langle a_1, w_1 \rangle = 0
\]
for all \( w_2 \in E_2 \cap \text{ran}(\varrho_2) \), which proves \( a_2 \in E_2 + C_2 \). Modifying the element \( x_1 \), we may arrange that the \( E_2 \)-component of \( a_2 \) is zero. Hence \( a_2 \in C_2 \) descends to an element \( b_2 \in (\mathbb{A}_2)_{\text{red}} \). Using part (b) from Theorem 3.8, the property \( a_1 \sim_R a_2 \sim_{q_2} b_2 \) shows the existence of an element \( b_1 \) with \( a_1 \sim_{q_1} b_1 \sim_{R_{\text{red}}} b_2 \). In particular, \( b_1 \in \text{ran}^\ast(R_{\text{red}}) \). It also follows that \( a_1 \in C_1 \), and hence \( x_1 = u_1 - a_1 \in E_1 \cap C_1 \). Letting \( y_1 \in (E_1)_{\text{red}} \) be its image, we obtain \( v_1 = y_1 + b_1 \).

**Remark 3.10.** Condition (21) is automatic in the following two cases

(a) \( \text{ran}(\varrho_2) \cap E_2 = 0 \),
(b) \( \text{ran}(\varrho_1) \subseteq E_1 \).

As a special case, suppose \( R: (TM, TM) \rightarrow (\mathbb{A}, E) \) is a \( G \)-equivariant Hamiltonian space, with base map \( \Phi: M \rightarrow Q \), and where the \( G \)-action on \( TM \) is the standard lift of a \( G \)-action on \( M \). If the \( G \)-action on \( Q \) is a principal action, then by equivariance the action on \( M \) is again a principal action. Hence we obtain a Hamiltonian space \( R_{\text{red}}: (T(M/G), T(M/G)) \rightarrow (\mathbb{A}_{\text{red}}, E_{\text{red}}) \).

### 3.4. Reduction of exact Courant algebroids.

Suppose \( \mathbb{A} \rightarrow Q \) is a \( G \)-equivariant exact Courant algebroid, and let \( j: TQ \rightarrow \mathbb{A} \) be a \( G \)-equivariant isotropic splitting, identifying \( \mathbb{A} \cong TQ_q \), for a \( G \)-invariant closed 3-form \( \eta \in \Omega^3(Q) \). The following result describes isotropic generators for the action in terms of the splitting. Recall that the Cartan complex of equivariant differential forms on \( Q \) is the space of \( G \)-equivariant polynomial maps \( \beta: \mathfrak{g} \rightarrow \Omega(Q) \), with the equivariant differential
\[
(d_G \beta)(\xi) = d\beta(\xi) - \iota(\xi \eta)\beta(\xi).
\]

**Proposition 3.11.** [9] For \( \mathbb{A} \cong TQ_q \) as above, there is a 1-1 correspondence between \( G \)-equivariant isotropic generators \( \varrho: \mathfrak{g} \rightarrow \Gamma(\mathbb{A}) \) and closed equivariant extensions
\[
\eta_G(\xi) = \eta + \alpha(\xi)
\]
of the 3-form \( \eta \). That is, \( \alpha: \mathfrak{g} \rightarrow \Omega^1(Q) \) is a \( G \)-equivariant map such that \( d_G \eta_G = 0 \).

Under this correspondence,
\[
\varrho(\xi) = j(\xi \eta) + a^\ast(\alpha(\xi)).
\]
Changing the splitting by an invariant 2-form \( \varpi \in \Omega^2(Q)^G \) modifies \( \eta_G \) to \( \eta_G' = \eta_G + d_G \varpi \).

**Remark 3.12.** If the generators are not necessarily isotropic, one finds instead that
\[
d_G \eta_G(\xi) = \frac{1}{2} \langle \varrho(\xi), \varrho(\xi) \rangle.
\]

We now make the additional assumption that the \( G \)-action on \( Q \) is principal, as in Theorem 3.6, with quotient map \( \pi: Q \rightarrow Q/G \). Suppose isotropic generators \( \varrho: \mathfrak{g} \rightarrow \Gamma(\mathbb{A}) \) are given.

**Definition 3.13.** An isotropic splitting \( j: TQ \rightarrow \mathbb{A} \) is called \( \mathfrak{g} \)-horizontal if \( \varrho(Q \times \mathfrak{g}) \subseteq j(TQ) \), or equivalently \( \varrho(\xi) = j(\xi \eta) \) for all \( \xi \in \mathfrak{g} \). It is called \( G \)-basic if it is both \( G \)-invariant and \( \mathfrak{g} \)-horizontal.
Thus, an invariant isotropic splitting is $G$-basic if and only if $\alpha = 0$. There is a 1-1 correspondence between $G$-basic splittings $j$ of $\mathbb{A}$ and isotropic splittings $j_{\text{red}}$ of $\mathbb{A}_{\text{red}}$. Under this correspondence, $j(TQ)$ is the pre-image of $j_{\text{red}}(T(Q/G))$ under the quotient map. The three-form of a $G$-basic splitting $j$ coincides with its equivariant extension, and equals the pullback of the three-form of the reduced splitting $j_{\text{red}}$:

$$\eta = \pi^* \eta_{\text{red}}.$$ 

**Proposition 3.14.** Let $Q \to Q/G$ be a principal $G$-bundle with connection $\theta \in \Omega^1(Q, \mathfrak{g})^G$. Let $\mathbb{A} \to Q$ be a $G$-equivariant Courant algebroid with isotropic generators, and let $j: TQ \to \mathbb{A}$ be a $G$-invariant isotropic splitting. Put

$$\varpi = - \alpha(\theta) + \frac{1}{2} c(\theta, \theta) \in \Omega^2(Q)^G,$$

where $\alpha$ is given by (23), and $c(\xi, \xi') = \iota(\xi_Q)\alpha(\xi') \in C^\infty(Q)$. Twisting the splitting $j$ by $\varpi$, we obtain a $G$-basic splitting. The resulting 3-form on $Q/G$ satisfies

$$\pi^* \eta_{\text{red}} = \eta + d\varpi.$$ 

**Proof.** The $\varpi$-twisted splitting $j'$ is given by (7), and the corresponding 1-forms $\alpha'(\xi)$ are $\alpha'(\xi) = \alpha(\xi) - \iota(\xi_Q)\varpi$. But

$$\iota(\xi_Q)\varpi = - (\iota(\xi_Q)\alpha)(\theta) + \alpha(\xi) + c(\xi, \theta) = \alpha(\xi).$$

Thus $\alpha'(\xi) = 0$. \qed

**Remark 3.15.** Note that if the splitting $j$ was $G$-basic to begin with, then $\varpi = 0$, and hence $j' = j$, for any choice of connection $\theta$.

**Remark 3.16.** The 2-form $\varpi$ also appears in the context of lifting the structure group of the principal bundle to a central extension by $U(1)$. See Appendix B.

The reduction of an exact Courant morphism is again exact:

**Proposition 3.17.** In the setting of Theorem 3.8 with $G_1 = G_2 = G$ and $f = \text{id}_G$, suppose that Courant algebroids $\mathbb{A}_i$ and the Courant morphism $R$ are exact. Then so are $(\mathbb{A}_i)_{\text{red}}$ and the Courant morphism $R_{\text{red}}$.

**Proof.** In the exact case, $R$ defines a strong Dirac morphism $(\mathbb{A}_1, \text{ran}(\mathbb{a}_1^*)) \to (\mathbb{A}_2, \text{ran}(\mathbb{a}_2^*))$. We have $\text{ran}(\mathbb{a}_1^*)_{\text{red}} = \text{ran}(\mathbb{a}_2^*)_{\text{red}}$, hence by Proposition 3.9 applied to $E_i = \text{ran}(\mathbb{a}_i^*)$ we obtain a strong Dirac morphism $R_{\text{red}}: ((\mathbb{A}_1)_{\text{red}}, \text{ran}(\mathbb{a}_1^*))_{\text{red}} \to ((\mathbb{A}_2)_{\text{red}}, \text{ran}(\mathbb{a}_2^*))_{\text{red}}$. In turn, this means that $R_{\text{red}}$ is exact. \qed

We now describe the reduction of exact Courant morphisms in terms of isotropic splittings. Suppose $A_i \to Q_i$ for $i = 1, 2$ are $G$-equivariant exact Courant algebroids, with isotropic generators $g_i: \mathfrak{g} \to \Gamma(A_i)$. Let $j_i: TQ_i \to A_i$ be $G$-equivariant isotropic splittings, identifying $A_i = TQ_{i, \eta_i}$ for closed 3-forms $\eta_i$.

**Proposition 3.18.** A $G$-equivariant exact Courant morphism

$$T\Phi_{\omega}: TQ_{1, \eta_1} \to TQ_{2, \eta_2}$$

intertwines the generators (cf. Equation (19) with $f = \text{id}_G$) if and only if

$$d_G \omega = \eta_{1, G} - \Phi^* \eta_{2, G}.$$ 

(26)
If the $G$-actions on $Q_i$ are principal actions, and $\theta_i \in \Omega^1(Q_i, g)$ are connection 1-forms, defining 2-forms $\omega_i \in \Omega^2(Q_i)$ as in (24) and 3-forms $\eta_i, \eta_i, \eta_i$ as in (25), then the reduced Courant morphism is
\[
(\mathcal{T}\Phi_{\text{red}})_{\text{red}} : (\mathcal{T}Q_1, \text{red})_{\eta_1, \text{red}} \to (\mathcal{T}Q_2, \text{red})_{\eta_2, \text{red}}
\]
where $\Phi_{\text{red}} : Q_1/G \to Q_2/G$ is the map induced by $\Phi$, and $\omega_{\text{red}}$ is given by
\[
\pi_1^*\omega_{\text{red}} = \omega + \omega_1 - \Phi^*\omega_2.
\]

Proof. By definition, the 2-form $\omega$ relates the splitting $j_1$ with the ‘pullback’ of the splitting $j_2$. Hence, (26) follows from Proposition 3.11. Suppose now that the $G$-actions are principal. Given connection 1-forms $\theta_i$ and the corresponding 2-forms $\omega_i$, let $j_i'$ be the $G$-basic splittings obtained by twisting $j_i$ by the 2-forms $\omega_i$. The 3-forms change to $\eta_i' = \eta_i + d\omega_i$, which are $G$-basic and in particular coincide with their equivariant extensions: $\eta_{i,G} = \eta_i'$. The 2-form describing $R = \mathcal{T}\Phi_{\omega}$ relative to the new splitting is $\omega' = \omega + \omega_1 - \Phi^*\omega_2$. Equation (26) gets replaced with $d_{\text{red}} \omega' = \eta_1' - \Phi^*\eta_2'$, which shows in particular that $\omega'$ is $G$-basic. The resulting 2-form $\omega_{\text{red}}$ with $\pi_1^*\omega_{\text{red}} = \omega'$ describes the exact morphism $R_{\text{red}}$. □

4. The Hilbert principal bundle of connections

Let $G$ be a connected finite-dimensional Lie group. The holonomy fibration is defined to be the space of connections $\mathcal{A}_I$ on the trivial $G$-bundle over the interval $I = [0, 1]$. By imposing appropriate regularity conditions, the space $\mathcal{A}_I$ is a principal bundle for the Hilbert Lie group of gauge transformations which are trivial at the boundary $\partial I$. The principal bundle projection is the map to $G$ given by the holonomy along the interval. A slight modification of this holonomy fibration, which makes contact with the usual theory of loop groups, is studied in Section 6; there we consider connections on the trivial $G$-bundle over the circle instead of the interval. Also, in our study of the geometry of these fibrations it will be useful to choose principal connections, which may be done via the Caloron correspondence, reviewed in Appendix C.

4.1. Sobolev space notation. We use the following basic properties of Sobolev spaces (see e.g. [7, Section 11] or [5, Section 14]). Let $r \geq 0$. For a finite-dimensional compact manifold $M$, possibly with boundary, let $H^s(M)$ denote the order $r$ Sobolev space of functions. In particular, we have $H^0(M) = L^2(M)$. The spaces $H^s(M)$ are Hilbert spaces, with compact inclusions $H^s(M) \subseteq H^r(M)$ for $s > r$. By the Sobolev embedding theorem, $H^s(M) \subseteq C^l(M)$ for $r - \frac{1}{2} \dim(M) > l$. The space $H^s(M)$ is a Banach algebra for $r - \frac{1}{2} \dim(M) > 0$, and for $0 \leq j \leq r$ the space $H^j(M)$ is a module over this Banach algebra. If $Z \subseteq M$ is a submanifold, and $r - \frac{1}{2} \dim(Z) > 0$, then the restriction of continuous functions from $M$ to $Z$ extends to a continuous linear map $H^s(M) \to H^s_{r - \frac{1}{2} \dim(Z)}(Z)$, with a continuous right inverse.

If $N$ is another finite-dimensional manifold and $r - \frac{1}{2} \dim(M) > 0$, one defines spaces $\text{Map}_{H^s}(M, N) \subseteq C^0(M, N)$ of maps $M \to N$ of Sobolev class $r$, by choosing local charts for $N$. In particular, if $G$ is a finite-dimensional Lie group, and $r - \frac{1}{2} \dim(M) > 0$, then $\text{Map}_{H^s}(M, G)$ is a Hilbert Lie group under pointwise multiplication.

We denote by $\Omega^k_{H^s}(M)$ the sections of $\wedge^k T^* M$ of Sobolev class $r$. 
4.2. The holonomy map as a principal bundle projection. We make use of several elementary results in gauge theory (see e.g. [17, Appendix A]), specialized to the case of 1-dimensional manifolds. Fix a real number $r \geq 0$. The space of connections on the trivial $G$-bundle over the interval $I = [0, 1]$ with Sobolev class $r$ is a Hilbert manifold
\[ A_1 = \Omega^1_{H_r}(I, g). \]
Since $r \geq 0$, the space of maps
\[ G_1 = \text{Map}_{H_{r+1}}(I, G) \]
defines a Hilbert Lie group, with Lie algebra $g_1 = \Omega^0_{H_{r+1}}(I, g)$. This group acts smoothly by gauge transformations
\[ g \cdot A = \text{Ad}_g(A) - g^* \theta^R, \]
for $g \in G_1$ and $A \in A_1$. Here $\theta^R \in \Omega^1(G, g)$ is the right-invariant Maurer-Cartan form on $G$. Note that $g$ is taken to have Sobolev class $r+1$ because the involvement of derivatives implies that $g^* \theta^R$ has class $r$. Given $\xi \in g_1$, the corresponding generating vector field $\xi_{A_1}$ is given by
\[ \xi_{A_1} |_A = \partial_A \xi, \]
where
\[ \partial_A = \partial + \text{ad}(A) : \Omega^0_{H_{r+1}}(I, g) \to \Omega^1_{H_r}(I, g) \]
is the exterior covariant derivative associated to $A \in A_1$. The action (27) is transitive: given $A \in A_1$, the equation
\[ A = g^{-1} \cdot 0 = g^* \theta^L \]
is a first order ordinary differential equation for $g \in G_1$, and so has a unique solution once an initial condition $g(0)$ is chosen. Furthermore, this solution lies in $H_{r+1}$ by standard elliptic theory, as required. We define the holonomy map $\text{Hol} : A_1 \to G$ in terms of the commutative diagram
\[ \begin{array}{ccc}
G_1 & \xrightarrow{\text{g}^{-1} \cdot 0} & A_1 \\
\downarrow & & \downarrow \text{Hol} \\
G \times G & \xrightarrow{\text{Hol}} & G
\end{array} \]
where the left vertical map is given by $g \mapsto (g(0), g(1))$ and the lower horizontal map is $(a_0, a_1) \mapsto a_0^{-1} a_1$. Both horizontal maps may be seen as quotient maps for a principal $G$-action, given by multiplication from the left. All maps in the diagram are $G_1$-equivariant, where $G_1$ acts on itself by
\[ g \mapsto k . g, \quad (k.g)(t) = g(t)k(t)^{-1}, \]
on $G \times G$ by $(a_0, a_1) \mapsto (a_0 k(0)^{-1}, a_1 k(1)^{-1})$, and on $G$ by $a \mapsto k(0)ak(1)^{-1}$. In particular,
\[ \text{Hol}(k \cdot A) = k(0) \text{Hol}(A) k(1)^{-1}, \]
for $k \in G_I$. The map Hol may be regarded as the quotient map for the principal action of the subgroup
\[
G_{1,01} = \{ g \in G_I : g(0) = g(1) = e \}.
\]
By taking the differential of (31), we see that the differential $T_{\text{Hol}} : T \mathcal{A}_I \to TG$ satisfies
\[
(T_{A_{\text{Hol}}})(\xi_{\mathcal{A}_I}|_A) = (\xi(1)^L - \xi(0)^R)|_{\text{Hol}(A)}
\]
for $\xi \in \mathfrak{g}_I$.

4.3. **Principal connections for the holonomy fibration.** Any function $\chi \in C^\infty(I)$ with $\chi(0) = 0$ and $\chi(1) = 1$ defines a connection $\theta$ on the principal bundle $\mathcal{A}_I \to G$. The connection can be described in terms of the corresponding horizontal lift. Let $g \in G_I$ be any path such that $A = g^{-1} \cdot 0$. Using left-trivialization $TG = G \times \mathfrak{g}$, the horizontal lift for $\theta$ is given as $T_{\text{Hol}(A)}G \to T \mathcal{A}_I$, $X \mapsto \partial_A \xi$ where $\xi \in \mathfrak{g}_I$ is the path
\[
\xi(t) = \chi(t) \text{Ad}_{g(t)^{-1}g(1)} X.
\]
Note that this does not depend on the choice of $g$ with $g^{-1} \cdot 0 = A$. In Appendix C, we review the ‘conceptual construction’ of $\theta$, provided by the caloron correspondence. The horizontal bundle defined by $\theta$ is invariant under the full action of $G_I$ (not only of the structure group $G_{1,01}$ of the principal bundle).

In particular, one can take $\chi(t) = t$. The resulting connection $\theta$ is uniquely characterized by $G_I$-invariance together with the value at the zero connection $A = 0$, given by
\[
\iota(a)\theta(t) = \int_0^t a - t \int_0^1 a, \quad a \in T_0 \mathcal{A}_I.
\]
That is, the horizontal space at $A = 0$ is $\mathfrak{g} \subset \Omega^1(I, \mathfrak{g})$, embedded as ‘constant 1-forms’.

**Remark** 4.1. Suppose $r = 0$, so that $\mathcal{A}_I$ consists of $L^2$-connections, and suppose that $\mathfrak{g}$ comes equipped with an Ad-invariant metric (as in Section 5 below). Define a $G_I$-invariant pseudo-Riemannian metric on $\mathcal{A}_I$ via
\[
(a_1, a_2) = \int_{[0,1]} a_1 \cdot * a_2.
\]
Then the connection $\theta$ defined by $\chi(t) = t$ is the unique connection for which the horizontal spaces are orthogonal to the $G_{1,01}$-orbits for the metric (35). (This is easily verified at $A = 0$; the claim follows by invariance.)

5. **Dirac reduction for the holonomy fibration**

Let $G$ be a connected finite-dimensional Lie group with a bi-invariant pseudo-Riemannian metric, so that its Lie algebra $\mathfrak{g}$ is a *metrized Lie algebra*, that is, it comes with a non-degenerate Ad-invariant symmetric bilinear form, denoted by $(X_0, X_1) \mapsto X_0 \cdot X_1$. We use the metric to define a $G_I$-invariant “Lie-Poisson” structure on $\mathcal{A}_I$, a weak Poisson structure given by an invariant Dirac structure in the standard Courant algebroid $T \mathcal{A}_I$. We then explain how to carry out a reduction along the holonomy map.
Hol: $\mathcal{A}_1 \to G$, obtaining the Cartan-Dirac structure of Section 2.6. We also consider more general weak Poisson structures on $\mathcal{A}_1$, which reduce to other Dirac structures on $G$. Finally, we study the reduction of Hamiltonian spaces for these weak Poisson structures.

5.1. $G_1$-action on $T\mathcal{A}_1$. Let $dA$ denote the tautological $\Omega^1_{I}(I, g)$-valued 1-form on the affine space $\mathcal{A}_1$, defined by

$t(a) \, dA = a$

for all $a \in T_{\mathcal{A}1}$. Let $\mathcal{A}_1 = T\mathcal{A}_1 \oplus T^*\mathcal{A}_1$ be the standard Courant algebroid over $\mathcal{A}_1$, and define sections

$(36) \quad \phi(\xi) = \xi_{\mathcal{A}_1} + (dA, \xi) \in \Gamma(T\mathcal{A}_1), \quad \xi \in \mathfrak{g}_1,$

where $\xi_{\mathcal{A}_1}$ are the generating vector fields for the $G_1$-action on $\mathcal{A}_1$, and the 1-form component is such that

$i_a(\xi_{\mathcal{A}_1}) = \int_0^1 a \cdot \xi, \quad a \in T_{\mathcal{A}_1}.$

Note that this is similar to the formula for the sections spanning the Lie-Poisson structure on $\mathfrak{g}^*$ (cf. Equation (18)). For any subspace $s \subseteq \mathfrak{g} \oplus \mathfrak{g}$, let $\mathfrak{g}^{(s)}_1 = \{ \xi \in \mathfrak{g}_1 \mid (\xi(0), \xi(1)) \in s \}$

be the subspace of paths with end points in $s$. Let $E^{(s)} \subseteq T\mathcal{A}_1$ denote the subbundle spanned by all $\phi(\xi), \xi \in \mathfrak{g}_1^{(s)}$. For the trivial subspace $s = \{0\}$, the space $\mathfrak{g}_1^{(s)}$ coincides with $\mathfrak{g}_{I, \partial I}$, the Lie algebra of the structure group (32) of the holonomy fibration.

Proposition 5.1. The sections (36) are generators for the standard lift of the $G_1$-action to the Courant algebroid $T\mathcal{A}_1$. They satisfy

$(37) \quad \langle \phi(\xi), \phi(\zeta) \rangle = \xi(1) \cdot \zeta(1) - \xi(0) \cdot \zeta(0)$

for all $\xi, \zeta \in \mathfrak{g}_1$. Furthermore, for any subspace $s \subseteq \mathfrak{g} \oplus \mathfrak{g}$, one has

$(38) \quad (E^{(s)})^\perp = E^{(s^\perp)}.$

Proof. The map $\mathfrak{g}_1 \to \Omega^1(\mathcal{A}_1), \xi \mapsto (dA, \xi)$ is $G_1$-equivariant and takes values in closed 1-forms. Since the $\xi_{\mathcal{A}_1}$ (viewed as sections of $T\mathcal{A}_1$) are generators for the action, so are $\xi_{\mathcal{A}_1} + (dA, \xi)$. In particular,

$(39) \quad \phi(Ad_g \xi) = g \phi(\xi)$

for all $g \in G_1, \xi \in \mathfrak{g}_1$. Furthermore,

$\langle \phi(\xi), \phi(\zeta) \rangle = 2t(\xi_{\mathcal{A}_1}) \langle dA, \xi \rangle = 2 \int_0^1 \partial_A \xi \cdot \zeta = \int_0^1 \partial(\zeta \cdot \xi)$

$= \xi(1) \cdot \zeta(1) - \xi(0) \cdot \zeta(0),$

which proves (37) by polarization.

This also gives the reverse inclusion in (38). For the forward inclusion, we first show that

$(40) \quad \phi(\mathcal{A}_1 \times \mathfrak{g}_{I, \partial I})^\perp \subseteq \phi(\mathcal{A}_1 \times \mathfrak{g}_1).$
We may use the $G_1$-invariance to assume $A = 0$. Suppose $b + \beta \in T_0A_1$ is orthogonal to $g(A_1 \times g_{l,0})$. That is, for all $\xi \in g_{l,0}$,

$$0 = (b + \beta, \xi_{A_1} + \langle dA, \xi \rangle) = \beta(\partial \xi) + \int_0^1 b \cdot \xi.$$ 

Let $\zeta$ be a solution of $\partial \zeta = b$, with the unique initial condition $\zeta(0)$ such that for all $X \in g$,

$$X \cdot \int_0^1 \zeta(t) \, dt = \beta(X \, dt).$$

By elliptic regularity, $\zeta$ has Sobolev class $r + 1$, so that $\zeta \in g_1$. We will show that $\beta = \langle dA, \zeta \rangle$, which then proves that $b + \beta = g(\zeta)|_0$.

Consider the decomposition of $T_0A_1$ into horizontal and vertical directions, relative to the standard connection $\theta$ given by (34). The vertical space is spanned by elements $\xi_{A_1} = \partial \xi$ with $\xi \in g_{l,0}$, and we have

$$\iota(\xi_{A_1}) \langle dA, \zeta \rangle = \langle \partial \xi, \zeta \rangle = -\langle \partial \zeta, \xi \rangle = -\int_0^1 b \cdot \xi = \beta(\partial \xi).$$

The horizontal space is spanned by elements of the form $X \, dt$ with $X \in g$, and on such elements we have

$$\iota(X \, dt) \langle dA, \zeta \rangle = X \cdot \int_0^1 \zeta(t) \, dt = \beta(X \, dt),$$

by definition of $\zeta$, establishing (40). For the final equality (38), note that $g(A_1 \times g_{l,0}) \subseteq \mathcal{E}^{(s)}$, and so, using (40), $(\mathcal{E}^{(s)})^{\perp} \subseteq g(A_1 \times g_1)$. Then (38) follows from (37).

5.2. The Lie-Poisson structure on $A_1$. Proposition 5.1 shows that $\mathcal{E}^{(s)}$ is a Lagrangian subbundle if and only if the subspace $s$ is Lagrangian. Since the map $g: g_1 \to \Gamma(TA_1)$ is bracket preserving, it follows that $\mathcal{E}^{(s)}$ is a Dirac structure if and only if $s$ is a Lagrangian Lie subalgebra.

Remark 5.2. Since $\xi_{A_1} | A = \partial_A \xi$, the fiber $\mathcal{E}^{(s)} | A$ may be interpreted as the graph of the unbounded operator $\partial_A$ with dense domain $g_{l}^{(s)} \subseteq \Omega^0_{r+1}(I, g)$ consisting of paths of Sobolev class $r + 1$ with end points in $s$. Taking $r = 0$, the operator $\partial_A$ is skew-adjoint if and only if $s$ is Lagrangian.

Proposition 5.3. For any Lagrangian Lie subalgebra $s$, the Dirac structure $\mathcal{E}^{(s)}$ is a weak Poisson structure on $A_1$. Its leaves $\mathcal{O}$ are the orbits of the $g_{l}^{(s)}$-action on $A_1$, with weakly symplectic 2-forms $\omega_\mathcal{O}$ given on generating vector fields by

$$\omega_\mathcal{O}(\xi_1, \xi_2|_A) = \int_I \xi_1 \cdot \partial_A \xi_2,$$

for $A \in \mathcal{O}$ and $\xi_1, \xi_2 \in g_{l}^{(s)}$.

Proof. From the formula $g(\xi) = \xi_{A_1} + \langle dA, \xi \rangle$, it is immediate that $\mathcal{E}^{(s)} \cap TA_1 = 0$, and that the orbits of $\mathcal{E}^{(s)}$ are the orbits of the $g_{l}^{(s)}$-action. By definition, the 2-forms on these orbits satisfy

$$\iota(\xi \xi_\mathcal{O}) \omega_\mathcal{O}|_A = -\iota_\mathcal{O}^*(dA, \xi), \quad \xi \in g_{l}^{(s)}, \ A \in \mathcal{O}.$$
where $i_{\mathcal{O}}$ is the inclusion of $\mathcal{O}$. Hence
\[
\omega_{\mathcal{O}}(\xi_1, \xi_2)\big|_A = -\iota(\xi_2, \omega)(dA, \xi_1) = \langle \partial_A \xi_2, \xi_1 \rangle = \int_I \xi_1 \cdot \partial_A \xi_2,
\]
for all $\xi_1, \xi_2 \in g_1^{(s)}$. \hfill $\Box$

The case of the diagonal $s = g^\Delta$ (periodic boundary conditions) is particularly important. The Dirac structure $\mathcal{E} = \mathcal{E}^{(s)\Delta}$ is called the Lie-Poisson structure on $A_1$.

**Remark 5.4.** By definition, the algebra of admissible functions (cf. Section 2.3) for the weak Poisson structure $\mathcal{E}^{(s)}$ contains all affine-linear functions of the form $f(A) = t + \langle A, \xi \rangle$ with $\xi \in g_1^{(s)}$ and $t \in \mathbb{R}$; the corresponding Hamiltonian vector field is $v_f = \xi_{A_1}$. These affine-linear functions form a Lie algebra under the Poisson bracket:

\[
\{t_1 + \langle A, \xi_1 \rangle, t_2 + \langle A, \xi_2 \rangle\} = \mathcal{L}(\xi_1, A)\langle A, \xi_2 \rangle = \int_I \partial_A \xi_1 \cdot \xi_2 = \int_I \partial \xi_1 \cdot \xi_2 + \langle A, \xi_1 \xi_2 \rangle.
\]

Therefore, they define a central extension of the Lie algebra $g_1^{(s)}$, with cocycle $\int_I \partial \xi_1 \cdot \xi_2$. For $s = g^\Delta$, this is the standard central extension of the loop algebra.

Let $S \subseteq D = \mathcal{G} \times \mathcal{G}$ be a Lie subgroup whose Lie algebra $\mathfrak{s} \subseteq \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ is Lagrangian. Consider the subgroup
\[
S \equiv G_1^{(s)} \subseteq G_1
\]
consisting of paths $g \in G_1$ with endpoints $(g(0), g(1)) \in S$. Generalizing [3, Theorem 8.3], we have:

**Proposition 5.5.** An exact Hamiltonian $S$-space for the weak Poisson structure $(\mathbb{T} \mathcal{A}_1, \mathcal{E}^{(s)})$ is equivalent to a manifold $M$ with an action of $S$, together with an invariant weakly symplectic 2-form $\sigma \in \Omega^2(M)$ and an equivariant moment map $\Psi : M \to \mathcal{A}_1$ satisfying
\[
\iota(\xi_M)\sigma = -\langle d\Psi, \xi \rangle, \quad \xi \in g_1^{(s)}.
\]

Here $\langle d\Psi, \xi \rangle \in \Omega^1(M)$ denotes the pullback by $\Psi$ of the 1-form $\langle dA, \xi \rangle \in \Omega^1(A_1)$.

**Proof.** This is a special case of Proposition 2.17 together with Example 2.19. \hfill $\Box$

**5.3. Reduction of the Lie-Poisson structure on the space of connections.** In this section, we exhibit the Cartan-Dirac structure from Section 2.6 as a reduction of the Lie-Poisson structure on the space of connections over the unit interval. In Section 6, we give a similar construction for connections over the circle $S^1$. Since the standard lift of the principal $G_{1,\partial}$-action on $A_1$ to $\mathbb{T} \mathcal{A}_1$ has isotropic generators, we use the machinery of Section 3.2 to define a reduced Courant algebroid $(\mathbb{T} \mathcal{A}_1)_{\text{red}}$ over $G = A_1/G_{1,\partial}$.

**Theorem 5.6.** The reduced Courant algebroid $(\mathbb{T} \mathcal{A}_1)_{\text{red}}$ is canonically isomorphic to the Cartan-Courant algebroid $\mathfrak{a}$ over $G$. This isomorphism intertwines the $G \times G \cong G_1/G_{1,\partial}$-actions together with their generators, and restricts to an isomorphism of Dirac structures
\[
((\mathbb{T} \mathcal{A}_1)_{\text{red}}, \mathcal{E}^{(s)}_{\text{red}}) \cong (\mathfrak{a}, E^{(s)})
\]
for each Lagrangian Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{d}$. In particular, the reduction of the Lie-Poisson structure on $A_1$ is the Cartan-Dirac structure on $G$. Also, the $G_1$-basic splitting of $\mathbb{T} \mathcal{A}_1$
defined by a principal connection $\theta$ as in Section 4.3 descends to the splitting (12) of the Cartan-Courant algebroid.

**Proof.** The map $G_1 \to G \times G$, $k \mapsto (k(0), k(1))$ descends to an identification

$$G_1/G_1\partial = G \times G, \quad g_1/g_1\partial = \mathfrak{g} \oplus \mathfrak{g}.$$  

Let $C = \varrho(A_1 \times \mathfrak{g})$, thus $C^\perp = \varrho(A_1 \times g_1\partial)$ by (38). By definition, $(T \mathcal{A}_1)_{\text{red}} = (C/C^\perp)/G_{1\partial}$. Since the action of $G_{1\partial}$ on $g_1/g_1\partial$ is trivial, it follows that $(T \mathcal{A}_1)_{\text{red}}$ is an action Courant algebroid

$$(T \mathcal{A}_1)_{\text{red}} = G \times g_1/g_1\partial = G \times (\mathfrak{g} \oplus \mathfrak{g})$$

with the constant sections as the reduced generators $\varrho_{\text{red}}: g_1/g_1\partial \to \Gamma(T \mathcal{A}_1)_{\text{red}}$. The action of $[k] \in G_1/G_1\partial$ on $G$ is induced from the action of $k \in G_1$ on $A_1$, and is given by $[k].a = k(0)ak(1)^{-1}$, by the equivariance property (31) of the holonomy map. This shows that $(T \mathcal{A}_1)_{\text{red}}$ is the Cartan-Courant algebroid $\mathcal{A}$, where the isomorphism intertwines the actions and the generators. Since $g_1^{(s)}/g_1\partial = s$, it is immediate that $\mathcal{E}^{(s)} = \varrho(A_1 \times g_1^{(s)})$ has reduction $E^{(s)} = G \times s$.

We now verify the reduction of splittings. As in Section 3.4, let $\varpi \in \Omega^2(A_1)$ be the 2-form determined by the principal connection $\theta$. In the notation from that section,

$$\alpha(\xi) = \langle dA, \xi \rangle, \quad c(\xi, \xi') = \iota_{\xi_A} \alpha(\xi') = \langle \partial_A \xi, \xi' \rangle,$$

for $\xi, \xi' \in g_1\partial$, hence

$$\varpi = -\langle dA, \theta \rangle + \frac{1}{2} \langle \partial_A \theta, \theta \rangle,$$

defining the $G_1\partial$-basic splitting $j: T\mathcal{A}_1 \to T\mathcal{A}_1$ via $j(a) = a + \iota(a)\varpi$. Let $j_{\text{red}}: TG \to (T\mathcal{A}_1)_{\text{red}}$ the reduced splitting. To compute it, let $\beta: \mathfrak{g} \to \Omega^1(G)$ be the map given as

$$\varrho_{\text{red}}(0, X) - j_{\text{red}}(X^L) = a^*_{\text{red}}(\beta(X))$$

for all $X \in \mathfrak{g}$, with $a^*: (T\mathcal{A}_1)_{\text{red}} = \mathcal{A} \to TG$ the reduced anchor. Then

$$\varrho(\xi) - j(\xi_A) = \text{Hol}^* \beta(\xi(1))$$

for all $\xi \in g_1$ with $\xi(0) = 0$. We use (44) to compute the map $\beta$, which then determines $j_{\text{red}}$ via (43). Let $\theta$ be obtained from the function $\chi \in C^\infty(I)$ with $\chi(0) = 0$, $\chi(1) = 1$, as in Section 4.3. Given $X, Z \in \mathfrak{g}$, let

$$\xi(t) = \chi(t) \text{Ad}_{\varrho(t)^{-1}\varrho(0)} X, \quad \zeta(t) = \chi(t) \text{Ad}_{\varrho(t)^{-1}\varrho(0)} Z.$$  

Then $\xi, \zeta$ are the unique paths from $0$ to $X, Z$ such that $\xi_{A\{A}, \zeta_{A\{A}$ are horizontal with respect to $\theta_{\{A}$. With this choice of $\xi$, we obtain

$$\iota(\xi_{A\{A})\varpi = -\langle \xi_{A\{A}, \theta \rangle = -\langle \partial_A \xi, \theta \rangle,$$

hence $j(\xi) = \xi_{A\{A} - \langle \partial_A \xi, \theta \rangle$. It follows that $\text{Hol}^* \beta(X) = \langle dA, \xi \rangle - \langle \partial_A \xi, \theta \rangle$, thus

$$\iota(\zeta_A) \text{Hol}^* \beta(X) = \langle \partial_A \zeta, \xi \rangle = X \int_0^1 \frac{\partial \chi}{\partial t} \chi(t) dt = \frac{1}{2} Z \cdot X.$$

Since $\zeta_A \sim_{\text{Hol}} Z^L$, the left hand side can also be written $\text{Hol}^* \iota(Z^L) \beta(X)$. We conclude $\beta(X) = \frac{1}{2} X \cdot \theta^L$, and hence $j_{\text{red}}(X^L) = \varrho_{\text{red}}(0, X) - \frac{1}{2} a^*_{\text{red}} \theta^L \cdot X$. This is consistent
with the formulas (12) for the Cartan-Courant algebroid, proving that the two splittings coincide.

\[\square\]

Remark 5.7.

(a) The above theorem holds for all regularities \( r \geq 0 \) imposed on the connections \( A_I \).

It thus shows that the reduction \((T A_I)_{\text{red}}\) is insensitive to the chosen regularity \( r \geq 0 \).

(b) As shown in [4], the 2-form \( \varpi \in \Omega^2(\mathcal{A}_I)^{G_1} \) determined by the standard connection \( \theta \) on the holonomy fibration is given by the formula

\[
\varpi = \frac{1}{2} \int_{[0,1]} \text{Hol}_s^* \theta R \cdot \frac{\partial}{\partial s} (\text{Hol}_s^* \theta R) ds \in \Omega^2(\mathcal{A}_I)^{G_1},
\]

where \( \theta R \in \Omega^1(G, \mathfrak{g}) \) is the right invariant Maurer-Cartan 1-form on \( G \), and \( \text{Hol}_s : A_I \to G \) is given by \( \text{Hol}_s(A) = g(s) \), where \( g \in G_1 \) is the parallel transport for \( A \), i.e. \( g(0) = e \) and \( A = g^* \theta^E \). This 2-form \( \varpi \) also appears in [3, Section 8.1].

(c) The \((G \times G)\)-equivariant splittings of the Cartan-Courant algebroid form an affine space for the vector space of bi-invariant 2-forms on the base \( G \). If \( G \) is compact or semi-simple, then the space \( \Omega^2(G)^{G \times G} = (\wedge^2 \mathfrak{g}^*)^G \) is zero. Hence, in this case any \( G_1 \)-invariant connection 1-form \( \theta \) on \( A_I \) will lead to the same 2-form \( \varpi \), and to the same reduced splitting of \((T A_I)_{\text{red}} = \mathfrak{a}\).

5.4. Reduction of Hamiltonian spaces. Let \( S \subseteq D = G \times G \) be a Lie subgroup whose Lie algebra \( \mathfrak{s} \subseteq \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \) is Lagrangian. Consider the subgroup \( S \equiv G_1^{(S)} \subseteq G_1 \)

consisting of paths \( g \in G_1 \) with endpoints \((g(0), g(1)) \in S \). The group \( S \) contains \( G_{1,\partial} \) as a normal subgroup, with quotient \( S/G_{1,\partial} = S \). As a special case of the general result concerning reduction of Dirac structures (Proposition 3.9), we obtain:

**Proposition 5.8.** Reduction by the action of \( G_{1,\partial} \) defines a 1-1 correspondence between Hamiltonian \( S \)-spaces \( M \) for \((T A_I, E^{(s)})\) and Hamiltonian \( S \)-spaces \( M \) for \((\mathfrak{a}, E^{(s)})\). The spaces and moment maps are related by the commutative diagram

\[
\begin{array}{ccc}
M & \overset{\Psi}{\longrightarrow} & A_I \\
\downarrow{\pi} & & \downarrow{\text{Hol}} \\
M & \overset{\Phi}{\longrightarrow} & G
\end{array}
\]

Here \( \pi : M \to M \) is the quotient by the action of \( G_{1,\partial} \). The correspondence preserves exactness.

**Proof.** For any \( S \)-equivariant map \( \Psi : M \to A_I \), since the action of \( G_{1,\partial} \subseteq S \) on \( A_I \) is a principal action, the action on \( M \) is a principal action. Taking quotients by \( G_{1,\partial} \), one obtains a manifold \( M \) with an \( S = S/G_{1,\partial} \)-equivariant map \( \Phi \) to \( G = A_I/G_{1,\partial} \).

Conversely, given \( M \) with an \( S \)-equivariant map \( \Phi : M \to G \), define \( M \subseteq M \times A_I \) as the pullback of the principal bundle \( \text{Hol} : A_I \to G \) under the map \( \Phi \). The diagonal \( S \)-action on \( M \times A_I \) (where the action on \( M \) is via the quotient map to \( S \)) restricts to an
action on $\mathcal{M}$, and the projection to the second factor restricts to an $\mathcal{S}$-equivariant map $\Psi: \mathcal{M} \to \mathcal{A}_I$. Suppose now that
\[
\mathcal{R}: (\mathcal{T}M, TM) \rightarrow (\mathcal{T}\mathcal{A}_I, \mathcal{E}^{(s)})
\]
is a Dirac morphism, with base map $\Psi$. According to Proposition 3.9, the reduction by $G_{1,01}$ gives an $\mathcal{S}$-equivariant Dirac morphism
\[
R = R_{\text{red}}: (\mathcal{T}M, TM) \rightarrow ((\mathcal{T}\mathcal{A}_I)_{\text{red}}, (\mathcal{E}^{(s)})_{\text{red}}) \cong (\mathcal{A}, E^{(s)})
\]
with base map $\Phi = \Psi_{\text{red}}: M \to G$. By Proposition 3.17, the morphism $\mathcal{R}$ is exact if and only if $R$ is exact. Conversely, given the Dirac morphism $R: (\mathcal{T}M, TM) \rightarrow (\mathcal{A}, E^{(s)})$, part (c) of Theorem 3.8 shows how to recover $\mathcal{R}$. \hfill \Box

Note that if the moment map $\Psi$ is proper, then so is $\Phi$. In this case, the finite-dimensionality of $G$ implies finite-dimensionality of $M$.

Recall that the exact Hamiltonian spaces for $(\mathcal{T}\mathcal{A}_I, \mathcal{E}^{(s)})$ are described by triples $(\mathcal{M}, \sigma, \Psi)$ (see Proposition 5.5), while those for $(\mathcal{A}, E^{(s)})$ are described by triples $(\mathcal{M}, \omega, \Phi)$ (see Proposition 2.21). Under the correspondence from Proposition 5.8, these are related as follows. Let $\varpi \in \Omega^2(\mathcal{A}_I)$ be the $G_1$-invariant 2-form defined by the standard connection $\theta$ on the holonomy fibration.

**Proposition 5.9.** Let $(\mathcal{M}, \sigma, \Psi)$ be an exact Hamiltonian $\mathcal{S}$-space for $(\mathcal{T}\mathcal{A}_I, \mathcal{E}^{(s)})$, and $(\mathcal{M}, \omega, \Phi)$ the corresponding exact Hamiltonian $\mathcal{S}$-space for $(\mathcal{A}, E^{(s)})$. Then
\[
\sigma = \pi^*\omega + \Psi^*\varpi.
\]

**Proof.** In terms of the splittings, we have $\mathcal{R} = \mathcal{T}\Psi_\sigma$ and $R = \mathcal{T}\Phi_\omega$, for an $\mathcal{S}$-invariant 2-form $\sigma \in \Omega^2(\mathcal{M})$ and an $\mathcal{S}$-invariant 2-form $\omega \in \Omega^2(M)$. Since the $\varpi$-twist of the standard splitting of $\mathcal{T}\mathcal{A}_I$ descends to the splitting (12) of $\mathcal{A}$, these 2-forms are related by (46). \hfill \Box

**5.5. Multiplicative structures.** In this subsection, we obtain the multiplicative structures $\text{Mult}_\mathcal{A}$ and $\text{Inv}_\mathcal{A}$ on the Cartan-Courant algebroid $\mathcal{A}$ described in Section 2.6.4 as a reduction from appropriate spaces of connections.

We begin describing how to get group multiplication $\text{Mult}_G: G \times G \rightarrow G$ in terms of spaces of connections. Let $\mathcal{M}$ denote the space of flat $G$-connections of class $H_k$ on the trivial principal $G$-bundle over a triangle $\mathcal{T} \subseteq \mathbb{R}^2$ (i.e. a 2-simplex), with $k > 1$. Following [3, Section 9.1], $\mathcal{M}$ is a smooth infinite dimensional Hilbert manifold on which the Hilbert Lie group $G_\mathcal{T} = \text{Map}_{\mathcal{H}_{k+1}}(\mathcal{T}, G)$ acts by gauge transformations.

Let $z_0, z_1, z_2 \in \partial \mathcal{T}$ be the cyclically oriented vertices of the 2-simplex. ($\partial \mathcal{T}$ is taken positively oriented w.r.t. $\mathcal{T}$.) We thus define a map
\[
\Phi: \mathcal{M} \rightarrow \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1, A \mapsto (\gamma_2^* A, \gamma_0^* A, \gamma_1^* A)
\]
where $\gamma_i : [0,1] \rightarrow \partial \mathcal{T}$ is an orientation preserving parameterization of the edge $[z_i, z_{i+1}] \subseteq \partial \mathcal{T}$, for $i = 0, 1, 2$ ($z_3 = z_0$), and we denoted $\tilde{\gamma}(t) = \gamma(1 - t)$. Here, we take $\mathcal{A}_I$ with regularity $r = k - 1/2$ so that $\Phi$ is smooth because $k > 1$. If we consider the subgroup $G_{\mathcal{T}, z} = \{ g \in G_\mathcal{T} : g(z_i) = e \}$ acting on $\mathcal{M}$ and $G_{1,01} \times G_{1,01} \times G_{1,01}$ acting on $(\mathcal{A}_I)^3$,
the map $\Phi$ is equivariant relative to the group homomorphism $f : g \mapsto (\tilde{g}_2 g, \gamma_0^* g, \gamma_1^* g)$. The induced map

$$\Phi_{\text{red}} : M := \mathcal{M}/G_{T,Z} \to (A_1)^3/(G_{1,0})^3 \simeq G \times G \times G$$

is an embedding of $M \simeq G^2$ inside $G^3$ satisfying

$$(47) \quad \Phi_{\text{red}}(M) = \text{gr}(\text{Mult}_G) = \{(k, g, h) \in G^3 : ghk^{-1} = e\},$$

since the holonomy around $\partial \mathcal{T}$ of a flat connection on $\mathcal{T}$ is trivial.

At the level of Courant algebroids, the map $\Phi$ can be supplemented with the Atiyah-Bott presymplectic 2-form $\sigma \in \Omega^2(\mathcal{M})$ ([5]). It has the following property (see e.g. [3, Section 9.1]), for $\xi \in \mathfrak{g}_\mathcal{T} = \Omega^0_{h+1}((\mathcal{T}, \mathfrak{g})$ inducing the infinitesimal gauge transformation $\xi_M|_A \in T_A\mathcal{M}$,

$$(48) \quad i_{\xi_M|_A} \sigma = -\int_{\partial \mathcal{T}} A : \xi = \int_I \tilde{\gamma}_2^* (A : \xi) - \int_I \gamma_0^* (A : \xi) - \int_I \gamma_1^* (A : \xi).$$

The induced exact Courant morphism

$$\mathbb{T}\Phi_\sigma : \mathbb{T}\mathcal{M} \longrightarrow \mathcal{T}A_1 \times \mathbb{T}A_1 \times \mathbb{T}A_1$$

is thus equivariant relative to $f$ when considering the natural lifted $G_{T,Z}$-action on $\mathbb{T}\mathcal{M}$ and the $(G_{1,0})^3$-action defined by $\varrho \times \tilde{\varrho} \times \tilde{\varrho}$ on $(\mathcal{T}A_1)^3$. (Here $\tilde{\varrho}(\eta) = \eta_A - \langle dA, \eta \rangle$, for $\eta \in \mathfrak{g}_I$, corresponds to the generators associated to opposite metric on $\mathfrak{g}$.) We can thus apply Thm 3.8 and reduce the (exact) Courant morphism $\mathbb{T}\Phi_\sigma$ to a (exact) Courant morphism

$$(\mathbb{T}\Phi_\sigma)_{\text{red}} : \mathbb{T}\mathcal{M} \longrightarrow \mathbb{A} \times \tilde{\mathbb{A}} \times \tilde{\mathbb{A}}.$$

The Courant analogue of eq. (47) is the following:

**Proposition 5.10.** With the notations above,

$$\text{gr}(\mathbb{T}\Phi_\sigma)_{\text{red}} \circ T\mathcal{M} = \text{gr}(\text{Mult}_\mathbb{A}),$$

where $\text{Mult}_\mathbb{A} : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ was defined in Section 2.6.4.

**Proof.** We shall denote $q_M : \mathbb{T}\mathcal{M} \longrightarrow \mathbb{T}\mathcal{M}$ and $q_\mathbb{A} : \mathbb{T}A_1 \longrightarrow \mathbb{A}$ the quotient relations and $R = (\mathbb{T}\Phi_\sigma)_{\text{red}}$. (Recall that $\varrho(\xi_M|_A) \sim q_\mathbb{A} (\text{Hol}(A), \xi(0) \oplus \xi(1))$.) It is clear that

$$R \circ T\mathcal{M} = R \circ q_M \circ T\mathcal{M} = (q_\mathbb{A} \times q_\mathbb{A} \times q_\mathbb{A}) \circ \mathbb{T}\Phi_\sigma \circ T\mathcal{M}$$

at the set-theoretic level. Since the r.h.s. in the Proposition is Lagrangian, we only need to show that $R \circ T\mathcal{M}$ is included in this set. This, in turn, follows from the fact that, given $g, h \in G$ and $X_i \in \mathfrak{g}, i = 0, 1, 2$, one can find $A \in \mathcal{M}$ and $\xi \in \mathfrak{g}_\mathcal{T}$ so that $[A] \simeq (gh, g, h)$ and $\xi(z_i) = X_i$. For, then, using eq. (48), $\xi_M|_A \in T\mathcal{M}$ is related by $(q_\mathbb{A} \times q_\mathbb{A} \times q_\mathbb{A}) \circ \mathbb{T}\Phi_\sigma$ to

$$(gh, X_0 \oplus X_2) \times (g, X_0 \oplus X_1) \times (h, X_1 \oplus X_2)$$

as wanted. \hfill \Box

**Remark 5.11.** The basic splitting of $\mathbb{T}A_1$ given in Thm. 5.6 can be used to induce a splitting of $(\mathbb{T}A_1)^3$ which is basic for $\varrho \times \tilde{\varrho} \times \tilde{\varrho}$. Following Prop. 3.18, the reduced splitting takes the reduced exact Courant morphism $(\mathbb{T}\Phi_\sigma)_{\text{red}}$ to the form $\mathbb{T}\Phi_{\text{red}, \sigma_{\text{red}}}$ for an induced 2-form $\sigma_{\text{red}} \in \Omega^2(\mathcal{M}_{\text{red}})$. Using the identification $\mathcal{M}_{\text{red}} \simeq G^2$, $[A] \mapsto (\text{Hol}(\gamma_0^* A), \text{Hol}(\gamma_1^* A)), a straightforward computation shows that $\sigma_{\text{red}} = \varsigma \in \Omega^2(G \times G)$, the 2-form introduced in eq. (17).
Finally, we describe the inversion morphism \( \text{Inv}_A \) as a reduction. The diffeomorphism \( \text{Inv}_1 : \mathcal{A}_1 \to \mathcal{A}_1, \ a(s)ds \mapsto -a(1-s)ds \) is \( G_{1,G} \)-equivariant with respect to the group homomorphism \( g(s) \mapsto g(1-s) \) and covers the group inversion \( \text{Inv}_G : G \to G \) along the holonomy fibration \( \text{Hol} : \mathcal{A}_1 \to G \). Moreover, the natural lift \( \text{Inv}_A : T\mathcal{A}_1 \to T\mathcal{A}_1 \) of \( \text{Inv}_1 \) is equivariant for the actions \( g \) and \( \tilde{g} \), respectively. Recalling the definition of the quotient relation \( q_A : T\mathcal{A}_1 \to A \) as in the Proof above, the corresponding reduced morphism \( (\text{Inv}_A)_{\text{red}} : \tilde{A} \to \tilde{A} \) relates

\[
(g, X_0 \oplus X_1) \sim (g^{-1}, X_1 \oplus X_0).
\]

Then \( (\text{Inv}_A)_{\text{red}} = \text{Inv}_A \) coincides with inversion in the groupoid \( G \times \mathfrak{d} \) as described in Section 2.6.4.

6. Connections over \( S^1 \)

In the previous section, we obtained the Cartan-Courant algebroid on \( G \), together with its Cartan-Dirac structure, by reduction along the principal \( G_{1,G} \)-bundle \( \text{Hol} : \mathcal{A}_1 \to G \) for connections on a unit interval. For applications to moduli spaces of flat connections over surfaces with boundary, one is interested in a modification of this construction using the space of connections over a circle, denoted by \( \mathcal{A}_{S^1} \). In this case, the group acting on \( \mathcal{A}_{S^1} \) is the loop group \( LG \) and, unlike the \( G_{1,G} \)-action on \( \mathcal{A}_1 \), this action is not transitive.

In section 6.1, we describe an \( L_0G \)-bundle \( \text{Hol} : \mathcal{A}_{S^1} \to G \) corresponding to the quotient by the based loop group \( L_0G \) and introduce a transitive Lie algebroid \( R \) over \( \mathcal{A}_{S^1} \). Here, \( L_0G \) plays the role of \( G_{1,G} \) and \( R \) that of the transitive \( g_{1,G} \)-action on \( \mathcal{A}_1 \). In section 6.2, we introduce an \( LG \)-action on the standard Courant algebroid \( T\mathcal{A}_{S^1} \) and a weak Poisson structure \( \mathcal{E} \) analogous to the Lie-Poisson structure on \( \mathcal{A}_1 \). Finally, we show that reduction of \( (T\mathcal{A}_{S^1}, \mathcal{E}) \) under the \( L_0G \)-action also yields the Cartan-Dirac structure \( (\tilde{A}, \tilde{E}) \).

6.1. The holonomy fibration for the circle. Let \( \mathcal{A}_{S^1} = \Omega^1_{\text{Hor}}(S^1, \mathfrak{g}) \) be the space of connections on the trivial \( G \)-bundle over the circle \( S^1 = \mathbb{R}/\mathbb{Z} \). Let

\[
(49) \quad LG = G_{S^1} := \text{Map}_{\text{Hor}+1}(S^1, G)
\]

be the loop group; the subgroup \( L_0G \) of loops with \( \gamma(0) = e \) is the based loop group. We then define the path space (see Appendix C for its relation to the caloron correspondence, which also makes it clear that it is a Hilbert manifold)

\[
(50) \quad \mathcal{P}G = \{ g \in \text{Map}_{\text{Hor}+1}(\mathbb{R}, G) \mid g(t+1)g(t)^{-1} = g(1)g(0)^{-1} \text{ for all } t \}.
\]

The loop group \( LG \) acts on \( \mathcal{P}G \) by \( (k \cdot g)(t) = g(t)k(t)^{-1} \). This action is a principal action, with quotient map \( g \mapsto g(1)g(0)^{-1} \). The principal action commutes with the \( G \)-action on \( \mathcal{P}G \) by pointwise multiplication from the left; this action makes \( \mathcal{P}G \) into an \( LG \)-equivariant principal \( G \)-bundle over \( \mathcal{A}_{S^1} \), with quotient map \( g \mapsto A = g^{-1} \cdot 0 \). The holonomy \( \text{Hol} : \mathcal{A}_{S^1} \to G \) of a connection may be defined in terms of the commutative
where the left vertical map is given by $q: g \mapsto (g(0), g(1))$ and the lower horizontal map is $(a_0, a_1) \mapsto a_0^{-1}a_1$. The holonomy map has the equivariance property $\text{Hol}(k \cdot A) = \text{Ad}_{k(0)} \text{Hol}(A)$ for $k \in LG$ and $A \in A_{S^1}$. The generating vector fields for the action of $Lg = \Omega^0_{H+1}(S^1, g)$ are again given by the covariant derivatives,

\begin{equation}
(52)\quad \xi_{A_{S^1}}|A = \partial_A \xi;
\end{equation}

the differential of Hol maps these to the generators for the conjugation action. We denote by

\begin{equation}
(53)\quad T_g \mathcal{P}G \cong \left\{ \xi \in \Omega^0_{H+1}(\mathbb{R}, g) \right| \text{Ad}_{g(t)}(\xi(t+1) - \xi(t)) = \text{const} \right\}.
\end{equation}

The action $T_g \mathcal{P}G \to T_{gh^{-1}} \mathcal{P}G$ of elements $h \in LG$ is given by $\xi \mapsto \text{Ad}_h \xi$, while the action $T_g \mathcal{P}G \to T_{ga} \mathcal{P}G$ of elements $a \in G$ is $\xi \mapsto \xi$. In terms of this identification (53), and using left trivialization $TG = G \times g$, the tangent map to the left vertical map in (51) is given by

\[ T_q: T_g \mathcal{P}G \to g \oplus g, \quad \xi \mapsto q(\xi) := (\xi(0), \xi(1)). \]

**Proof.** The tangent bundle of $\mathcal{P}G$ can itself be regarded as the total space of the path fibration for the tangent group $TG$:

\[ T(\mathcal{P}G) = \mathcal{P}(TG). \]

Using left trivialization to identify $TG = G \times g$, the group structure reads as $(a_1, X_1)(a_2, X_2) = (a_1a_2, \text{Ad}_{a_1} X_1 + X_2)$, and $(a, X)^{-1} = (a^{-1}, -\text{Ad}_a X)$. Hence, the condition for a path $t \mapsto (g(t), \xi(t))$ to define an element of $\mathcal{P}(TG)$ is that

\[ (g(t+1), \xi(t+1))(g(t), \xi(t))^{-1} = (g(t+1)g(t)^{-1}, \text{Ad}_{g(t)}(\xi(t+1) - \xi(t))) \]

be constant as a function of $t$. The last claim follows since the tangent map to $q: \mathcal{P}G \to G \times G$ is the corresponding map for $\mathcal{P}(TG) \to TG \times TG$ for the group $TG$. \qed

Regard $\mathcal{P}G$ as an $LG$-equivariant principal $G$-bundle over $A_{S^1}$, and let

\[ R = T(\mathcal{P}G)/G \to A_{S^1} \]

be the corresponding $LG$-equivariant Lie algebroid.
Proposition 6.2. The fibers of the Lie algebroid $R$ have the following description,

$$R_A = \{ \xi \in \Omega^1_{R^1+1}(\mathbb{R}, \mathfrak{g}) \mid \partial_A \xi \text{ is periodic} \},$$

with anchor map $\xi \mapsto \xi_{A_{S^1}}(A) = \partial_A \xi$. The Lie bracket on sections of $R$ is given by

$$[\xi_1, \xi_2]_R = [\xi_1, \xi_2] + L(\xi_{1,A})\xi_2 - L(\xi_{2,A})\xi_1;$$

here $L(a)\xi$ denotes the Lie derivative of the function $\xi$ with respect to the vector field $a$, and $[\xi_1, \xi_2]$ is the pointwise Lie bracket.

Proof. The subspace on the right hand side of (53) depends only on $A = g^{-1} \cdot 0$; equation (54) gives a direct description in terms of $A$. (Recall that $\partial_A = \text{Ad}_{g^{-1}} \circ \partial \circ \text{Ad}_g$.) The expression for the Lie bracket follows from a similar formula for the bracket on sections of $T(PG)$.

6.2. Reduction by the $L_0 G$-action. The lift of the $LG$-action on $A_{S^1}$ to the standard Courant algebroid $\mathbb{T}A_{S^1}$ has isotropic generators $g: L\mathfrak{g} \to \Gamma(\mathbb{T}A_{S^1})$ given by the same formulas as for $A_1$:

$$g(\xi) = \xi_{A_{S^1}} + \langle dA, \xi \rangle, \quad \xi \in L\mathfrak{g}.$$  

By (52), the fiber of $E = g(A_{S^1} \times L\mathfrak{g})$ at $A \in A_{S^1}$ may be regarded as the graph of the skew-adjoint operator $\partial_A: \Omega^0(S^1, \mathfrak{g}) \to \Omega^1(S^1, \mathfrak{g})$. In particular, $E$ is a Lagrangian subbundle, and since it is involutive it is a Dirac structure $E \subseteq \mathbb{T}A_{S^1}$. Indeed, $E$ is a weak Poisson structure, which we will again refer to as a Lie-Poisson structure on $A_{S^1}$.

To describe its reduction with respect to the based loop group $L_0 G$, we extend (56) to sections of the Lie algebroid $R$:

$$g(\xi) = \xi_{A_{S^1}} + \langle dA, \xi \rangle_1, \quad \xi \in \Gamma(R);$$

here we denote by $\xi_1 \in \Omega^1_{L_0 G}((S^1, \mathfrak{g})$ the restriction to $I \subseteq \mathbb{R}$, regarded as a piecewise continuous function on $S^1$ (with a jump singularity at 0) and by $\langle dA, \xi \rangle_1$ the corresponding element of $T^*_A A_{S^1}$.

Lemma 6.3. For $\xi_1, \xi_2 \in \Gamma(R)$, the pairing of the corresponding sections is given by

$$\langle g(\xi_1), g(\xi_2) \rangle = \xi_1(1) \cdot \xi_2(1) - \xi_1(0) \cdot \xi_2(0),$$

while the Courant bracket is

$$\llbracket g(\xi_1), g(\xi_2) \rrbracket = g([\xi_1, \xi_2]_R) + \xi_2(1) \cdot d\xi_1(1) - \xi_1(0) \cdot d\xi_2(0).$$

Proof. We will write $\xi^\sharp = \xi_A$ for the vector field defined by $\xi \in \Gamma(R)$. The formula for the pairing follows from

$$\iota_{\xi_1}^* \langle dA, \xi_2 \rangle + \iota_{\xi_2}^* \langle dA, \xi_1 \rangle = \langle \partial_A \xi_1, \xi_2 \rangle + \langle \xi_1, \partial_A \xi_2 \rangle = \int_1 \partial(\xi_1 \cdot \xi_2),$$

The vector field component of the Courant bracket $\llbracket g(\xi_1), g(\xi_2) \rrbracket$ is $[\xi_1^\sharp, \xi_2^\sharp] = [\xi_1, \xi_2]^\sharp$. For the 1-form component, we have to calculate

$$\mathcal{L}_{\xi_1}^\sharp(\langle dA, \xi_2 \rangle) - \iota_{\xi_2}^* d(\langle dA, \xi_1 \rangle) = \langle dA, (\mathcal{L}_{\xi_2}^\sharp \xi_1 - \mathcal{L}_{\xi_1}^\sharp \xi_2) \rangle + \langle d\mathcal{L}_{\xi_1}^\sharp A, \xi_2 \rangle + \langle \iota_{\xi_2}^\sharp dA, d\xi_1 \rangle$$

$$= \langle dA, (\mathcal{L}_{\xi_2}^\sharp \xi_1 - \mathcal{L}_{\xi_1}^\sharp \xi_2) \rangle + \langle d\partial_A \xi_1, \xi_2 \rangle + \langle \partial_A \xi_2, d\xi_1 \rangle.$$
But \( \langle d\partial A \xi_1, \xi_2 \rangle = \langle \partial A d\xi_1, \xi_2 \rangle + \langle [dA, \xi_1], \xi_2 \rangle \). The first term combines with \( \langle \partial A \xi_2, d\xi_1 \rangle \) to give \( \int_I \partial \langle \xi_2, d\xi_1 \rangle \), while the second term combines with \( \langle dA, (L_{\xi_1} \xi_2 - L_{\xi_2} \xi_1) \rangle \) to give \( \langle dA, [\xi_1, \xi_2]_R \rangle \).

We are now in position to compute the reduction of the Lie-Poisson structure \( \mathcal{E} \subseteq \mathbb{T}A_{S^1} \) by the action of \( L_0G \). By definition, the reduced Courant algebroid is \( (C/C^\perp)/L_0G \), where \( C \) is the coisotropic subbundle with fibers \( C_A = (\varphi(L_0g)_A)^\perp \).

**Theorem 6.4 (Reduction of the weak Poisson structure on \( A_{S^1} \)).** The reduction of the Dirac structure \( (\mathbb{T}A_{S^1}, \mathcal{E}) \) under the action of the based loop group \( L_0G \) is canonically isomorphic to the Cartan-Dirac structure \( (\Lambda, E) \). In more detail, \( C \) is spanned by sections \( \varphi(\xi) \) with \( \xi \in \Gamma(R) \), and the map

\[
C \to G \times (\overline{g} \oplus g), \quad \varphi(\xi) \mapsto (\text{Hol}(A), \xi(0), \xi(1))
\]

descends to an isomorphism of Courant algebroids \( (\mathbb{T}A_{S^1})_{\text{red}} \to \Lambda \).

**Proof.** An element \( a + \langle dA, u \rangle \) with \( a \in TA_{S^1} = \Omega^1_{H_r}(S^1, g) \) and \( u \in \Omega^0_{H_{r+1}}(S^1, g) \), lies in \( C_A = \varphi(L_0g)^\perp_A \) if and only if for all \( \tau \in L_0g \),

\[
0 = \langle a + \langle dA, u \rangle, \partial_A \tau + \langle dA, \tau \rangle \rangle = \langle a - \partial_A u, \tau \rangle
\]

Equivalently, \( a - \partial_A u \) is a multiple of the \( \delta \)-distribution supported at 0. In particular, \( u \) is given by a continuous function on \( I \) (regarded as a piecewise continuous function on \( S^1 \) with a jump discontinuity at 0). Given \( a \in TA_{S^1} = \Omega^1_{H_r}(S^1, g) \), we can determine the corresponding \( u \) by integration. Furthermore, by lifting the differential equation to \( \mathbb{R} \), we see that \( u \) is the restriction to \( I \) of a function \( \xi \in \Omega^0_{H_{r+1}}(\mathbb{R}, g) \) satisfying \( \partial_A \xi = a \) (where \( A, a \) are regarded as periodic forms on \( \mathbb{R} \)). In particular, \( \partial_A \xi \) is periodic, that is, \( \xi \in R_A \).

This gives the desired identification of \( R_A \to C_A, \xi \mapsto \varphi(\xi)_A \).

Since the kernel of the map \( R_A \to \overline{g} \oplus g, \xi \mapsto (\text{Hol}(A), \xi(0), \xi(1)) \) is exactly \( L_0g \), it follows that \( (\mathbb{T}A_{S^1})_{\text{red}} = G \times (\overline{g} \oplus g) \) as a vector bundle. The metric and Courant bracket on \( (\mathbb{T}A_{S^1})_{\text{red}} \) are induced from the metric and Courant bracket on \( L_0G \)-invariant sections of \( C \); using the Lemma we obtain the metric and Courant bracket of the Cartan-Courant algebroid. Finally, since the \( L_0G \)-invariant sections \( \varphi(\xi) \) of \( \mathcal{E} \subseteq R \) are those with \( \xi(0) = \xi(1) \), we see that \( \mathcal{E}_{\text{red}} \) is the Cartan-Dirac structure.

Similarly to \( A_I \), the fibration \( A_{S^1} \to G \) has a standard connection, defined by any choice of a function \( \chi \in C^\infty(I) \) such that \( \chi \) extends to a smooth function on \( \mathbb{R} \), equal to 0 for \( t \leq 0 \) and equal to 1 for \( t \geq 1 \). The connection is best described in terms of the caloron correspondence, Appendix C. Arguing as in the case of \( A_I \), we obtain:

**Theorem 6.5.** The reduction of the Dirac structure \( (\mathbb{T}A_{S^1}, \mathcal{E}) \) with respect to the based loop group \( L_0G \) is \( G = \text{LG}/L_0G \)-equivariantly isomorphic to the Cartan-Dirac structure \( (\Lambda, E) \) over \( G \). Furthermore, the reduction of the \( L_0G \)-basic splitting of \( \mathbb{T}A_{S^1} \), defined by the standard connection \( \theta \) on the holonomy fibration, is the usual splitting of the Cartan-Courant algebroid, identifying \( \Lambda \cong \mathbb{T}G_\theta \). The reduction procedure gives a one-to-one correspondence between \( \text{LG} \)-equivariant (exact) Hamiltonian spaces for \( (\mathbb{T}A_{S^1}, \mathcal{E}) \) and \( G \)-equivariant (exact) Hamiltonian spaces for \( (\Lambda, E) \).
Appendix A. Reduction in Infinite Dimensions

Let \( V \) be a Banach space. The closure of a subspace \( F \subseteq V \) will be denoted \( \text{cl}(F) \), and the annihilator \( \text{ann}(F) \subseteq V^* \), where \( V^* \) is the topological dual space of \( V \). For Banach spaces \( V, V' \), denote by \( \mathbb{B}(V,V') \) the Banach space of continuous linear maps \( V \to V' \). More generally, given Banach spaces \( V_1, \ldots, V_l \) there is a Banach space \( \mathbb{B}(V_1, \ldots, V_l; V') \) of continuous multilinear maps \( V_1 \times \cdots \times V_l \to V' \).

Suppose \( V \) is a Hilbert space with a pseudo-Riemannian metric \( B \). Let \( B^\flat: V \to V^* \) be the associated map. For any subspace \( F \subseteq V \), we have \( B^\flat(F^\perp) = \text{ann}(F) \), and \( (F^\perp)^\perp = \text{cl}(F) \).

For the following Proposition, we observe that if \( F_1, F_2 \) are closed subspaces of a real Hilbert space \( V \), then \( F_1 + F_2 \) is closed in \( V \) if and only if \( \text{ann}(F_1) + \text{ann}(F_2) \) is closed in \( V^* \). (Proof: let \( F_1', F_2' \) be closed complements to \( F_1 \cap F_2 \) in \( F_1, F_2 \) respectively. If \( F_1 + F_2 \) is closed, let \( N \) be a closed complement to \( F_1 + F_2 \) in \( V \). Then \( V = F_1 \cap F_2 \oplus F_1' \oplus F_2' \oplus N \) is a direct sum decomposition of \( V \) into closed subspaces. By considering the dual decomposition of \( V \), it follows that the inclusion \( \text{ann}(F_1) + \text{ann}(F_2) \to \text{ann}(F_1 \cap F_2) \) is an equality.)

Thus, if \( V \) carries a metric \( B \), then \( F_1 + F_2 \) is closed if and only if \( F_1^\perp + F_2^\perp \) is closed. Criteria for \( F_1 + F_2 \) to be closed may be found in [33]; in particular, it is known that the sum of disjoint closed subspaces is closed if and only if a suitably defined ‘angle’ between these subspaces is non-zero.

**Proposition A.1.** Let \( V \) be a real Hilbert space with a metric \( B \), and \( C \) a closed co-isotropic subspace of \( V \). Then

(a) \( C \) admits a closed isotropic complement. (In particular, every Lagrangian subspace admits a Lagrangian complement.)

(b) The quotient \( V_C = C/C^\perp \) inherits a metric \( B_C \).

(c) Suppose \( L \subseteq V \) is Lagrangian. Then \( L + C \) is closed if and only if \( L + C^\perp \) is closed, and in this case \( L_C = (L \cap C)/(L \cap C^\perp) \) is Lagrangian in \( V_C \).

**Proof.** (a) Choose a closed complement \( F \) to \( C \). Then \( F^\perp \) is a closed complement to \( C^\perp \). The projection to \( C^\perp \) along \( F^\perp \) restricts to a continuous linear map \( A: F \to C^\perp \), and

\[
F' = \left\{ v - \frac{1}{2} A(v) \mid v \in F \right\}
\]

is the desired isotropic complement to \( C \). (\( F' \) is closed since it is the graph of a continuous linear map \( -\frac{1}{2} A: F \to C^\perp \subseteq C \).)

(b) The bilinear form \( B \) descends to a continuous symmetric bilinear form \( B_C: V_C \times V_C \to R \). We have to verify that \( B_C \) is non-degenerate. Let \( F \) be a closed isotropic subspace with \( V = C \oplus F \), hence \( V = C^\perp \oplus F^\perp \). Intersecting with \( C \), it follows that \( C = C^\perp \oplus (C \cap F^\perp) \), thus \( V = C^\perp \oplus F \oplus (C \cap F^\perp) \). The quotient map \( C \to V_C \) induces a topological isomorphism \( C \cap F^\perp \to V_C \), identifying \( B_C \) with the restriction of \( B \) to \( C \cap F^\perp = (C^\perp \oplus F)^\perp \). The latter is non-degenerate, hence so is \( B_C \).

(c) The inverse image of \( L_C^\perp \) in \( C \) is

\[
(L \cap C)^\perp = \text{cl}(L + C^\perp) \cap C \supseteq (L + C^\perp) \cap C = (L \cap C) + C^\perp.
\]
Applying the projection $C \to V_C$, it follows that $L_C^\perp \supseteq L_C$. If $L + C^\perp$ is closed, the inclusion becomes an equality, and we obtain $L_C^\perp = L_C$.

□

Appendix B. Lifting problems

Let $Q \to B$ be a principal $G$-bundle, and

$$1 \to U(1) \to \hat{G} \to G \to 1$$

a central extension. Consider the exact sequence of vector bundles over $B$,

$$(58) \quad 0 \to B \times \mathbb{R} \to Q \times_G \hat{g} \to Q \times_G g \to 0.$$ 

A splitting of this sequence may be regarded as a $G$-equivariant map $\nu: g \to \Omega^0(Q, \hat{g})$ whose composition with the projection $\hat{g} \to g$ is the identity. The differential of this map is scalar-valued, defining a linear map

$$\alpha: g \to \Omega^1_{cl}(Q), \quad \xi \mapsto d\nu(\xi)$$

with values in closed 1-forms. The map

$$\varrho: g \to \Gamma(TQ), \quad \xi \mapsto \xi_Q + \alpha(\xi)$$

gives isotropic generators for the natural $G$-action on $TQ$. The standard splitting of $TQ$ is not basic for this $G$-action. However, by Proposition 3.14 any principal connection $\theta$ on $Q$ defines a new $G$-basic splitting of $TQ$, giving an identification $(TQ)_{\text{red}} = TB_\eta$ for a closed 3-form $\eta \in \Omega^3(B)$. The construction also gives a 2-form $\varpi$ on $Q$ with $d\varpi = -\pi^*\eta$. These are exactly the 2-form and 3-form appearing in Brylinski’s discussion of the problem of lifting the structure group to $\hat{G}$ [8]. In particular, the cohomology class of $\eta$ is the image in de Rham cohomology of the obstruction class in $H^3(B, \mathbb{Z})$ for the existence of a lift.

Appendix C. Caloron correspondence

The caloron correspondence, due to Garland-Murray [18], Murray-Stevenson [27], and Murray-Vozzo [28], relates principal bundles over a base $B$, with structure group the (based) loop group, with (framed) principal bundles over a base $B \times S^1$, with structure group $G$. Among other things, this correspondence leads to a simple construction of principal connections on the loop group bundle.

C.1. Caloron correspondence for $A_1$. In this section we will use a version of the caloron correspondence where we work with path spaces rather than loop spaces. A framing of a principal $G$-bundle $Q \to B$ along a submanifold $Z \subseteq B$ is a trivialization along $Z$, i.e., a section $\sigma: Z \to Q|_Z$. A principal connection $\nu \in \Omega^1(Q, g)$ is a framed connection if $\sigma^*\nu = 0$. Given a manifold $M$ with two submanifolds $M_0, M_1$, we say that $\gamma: I \to M$ is a based path if $\gamma(0) \in M_0$ and $\gamma(1) \in M_1$. Let $M_I$ be the space of paths $I \to M$ of Sobolev class $r + 1$, and $M_{I,\text{bt}} \subseteq M_I$ the based paths. Given a principal bundle
as above, with framings $\sigma_i: B_i \to Q$ along $B_i \subseteq B$, and taking $Q_i = \sigma_i(B_i)$, we obtain a diagram of principal bundles

\[
\begin{array}{ccc}
Q_{1,0} \ar[d]/G_{1,0} & \ar[r] & Q_1 \ar[d]/G_1 \\
B_{1,0} \ar[r] & B_1
\end{array}
\]

Any principal connection $\nu \in \Omega^1(Q, g)$ determines a principal connection $\nu_1$ on the bundle $Q_1$. If $\nu$ is a framed connection, then $\nu_1$ restricts to a principal connection on $Q_{1,0}$.

As a special case, take $Q$ to be the trivial principal $G$-bundle $Q = B \times G$ over $B = G \times I$, with the framings along $B_0 = G \times \{0\}, B_1 = G \times \{1\}$ given by

\[
\sigma_0(a, 0) = (a, 0, e), \quad \sigma_1(a, 1) = (a, 1, a),
\]

and with the principal $G$-action $k.(a, s, g) = (a, s, gk^{-1})$. Consider the inclusion $G \to B_{1,0}$, taking $a \in G$ to the path $\gamma(t) = (a, t)$. The restriction of $Q_{1,0}$ to this submanifold $G \subseteq B_{1,0}$ is identified with $G_{1,0} = \{g \in G | g(0) = e\}$, by the map

\[
G_{1,0} \to Q_{1,0}, \quad g \mapsto (t \mapsto (g(1), t, g(t))).
\]

On the other hand, the map $G_1 \to A_1, \quad g \mapsto g^{-1}$ restricts to a diffeomorphism $G_{1,0} \cong A_1$.

In summary, we have a commutative diagram,

\[
\begin{array}{ccc}
A_1 \ar[d]/G_{1,0} & \ar[r] & (G \times I \times G)^{1,0} \ar[d]/G_{1,0} \ar[r]/G_1 & (G \times I \times G) \ar[d]/G_1 \\
G \ar[r] & (G \times I)^{1,0} \ar[r]/G_1 & (G \times I)
\end{array}
\]

To incorporate the $G_1$-action on $A_1$ in this picture, note that the principal action of $G$ on $Q$ extends to an action of $G \times G \times G$:

\[
(u, v, k).((a, s, g) = (uav^{-1}, s, ukg^{-1}).
\]

It defines a $G_1 \times G_1 \times G_1$-action on $Q_1$, given by the same formula (but with $u$, $k$, etc. as paths). The subbundle $Q_{1,0}$ is preserved by the subgroup of paths $(u, v, k)$ such that $u(0) = k(0)$ and $v(1) = k(1)$, and the subbundle $A_1$ by the subgroup $G_1 \subseteq G_1 \times G_1 \times G_1$ of paths of the form $(u, v, k)(t) = (k(0), k(1), k(t))$.

As explained above, a framed principal connection $\nu$ on $Q$ defines a principal connection $\nu_1$ on $Q_1$, which then pulls back to a connection on $Q_{1,0}$. Let $\theta$ denote its restriction to $A_1$. If $\nu$ is furthermore invariant under the action of $(u, v) \in G \times G$ by automorphisms, then $\nu_1$ will be invariant under the $G_1 \times G_1$-action. That is, the horizontal subbundle $\ker(\nu_1) \subseteq TQ_1$ is invariant not just under the gauge action, but under the full $G_1 \times G_1 \times G_1$-action. It then follows that the connection $\theta$ is $G_1$-equivariant, in the sense that the horizontal distribution $\ker(\theta)$ is $G_1$-invariant.

To get concrete formulas, we express the principal connection $\nu$ on $Q = B \times G$ in terms of its connection 1-forms $\kappa \in \Omega^1(B, g)$:

\[
\nu = \text{Ad}_{g^{-1}} \kappa + g^\ast \theta^L.
\]
Here the variable \( g \) is regarded as the projection \( g : B \times G \to G \). The connection \( \nu \) is a framed connection if and only if
\[
(59) \quad i_g^* \kappa = 0, \quad i_\nu^* \kappa = -a^* \theta^R,
\]
where \( i_g : G \to B,\ a \mapsto (a, s) \). It is furthermore invariant under the \( G \times G \)-action by automorphisms if and only if
\[
(60) \quad (u, v)^* \kappa = Ad_u \kappa.
\]

**Proposition C.1.** Let \( \nu \) be a framed connection on \( Q = B \times G \), defined by a connection 1-form \( \kappa \in \Omega^1(G \times I, g) \). For \( t \in I \), let \( \kappa_t = i_t^* \kappa \). Let \( A \in A_I \), defining a parallel transport \( g \in G_{1,0} \). Then the horizontal lift for the resulting connection 1-form \( \theta \) is given at \( A \in A_I \) by
\[
T_{\text{Hol}(A)} G \to T_A A_I, \quad X \mapsto \partial_A \xi
\]
where \( \xi \in \mathfrak{g}_I \) is the path \( \xi(t) = -Ad_{g(t)^{-1}} \kappa_t(X) \).

**Proof.** Note that \( \xi(0) = 0 \), while
\[
\xi(1) = -Ad_{\text{Hol}(A)^{-1}} \kappa_1(X) = \iota(X) \theta^{L_{\text{Hol}(A)}}.
\]
The proposition asserts that the horizontal lift of \( X \) is given by \( \partial_A \xi \in T_A A_I \), the infinitesimal action of \( \xi \in \mathfrak{g}_I \) on \( A_I \). The image of \( \partial_A \xi \) under the differential of the map \( A_I \to (G \times I \times G) \) is the infinitesimal action of \( (\xi(0), \xi(1), \xi) \in \mathfrak{g}_I \times \mathfrak{g}_I \times \mathfrak{g}_I \) at \( (\text{Hol}(A), 0, g) \), that is,
\[
(61) \quad (\xi(1)^{L_{\text{Hol}(A)}}, 0, \xi_{G_1 | g}) = (X, 0, \xi_{G_1 | g}).
\]
On the other hand, the image of \( X \in T_{\text{Hol}(A)} G \) under the differential of \( G \to (G \times I) \) is the constant vector field \( (X, 0) \in TB_I \), and by the formula for \( \nu_1 \) in terms of the connection 1-form, (61) is precisely the horizontal lift of \( (X, 0) \). \( \square \)

A convenient choice for \( \kappa \) satisfying (59) as well as the invariance (60) is given by
\[
(62) \quad \kappa = -\chi(s) \ a^* \theta^R
\]
for any function \( \chi \in C^\infty(I) \) such that \( \chi(0) = 0,\ \chi(1) = 1 \).

**C.2. Caloron correspondence for \( A_{S^1} \).** The caloron correspondence for \( A_{S^1} \) runs as follows (see [28, Example 3.4]). Consider the trivial principal \( G \)-bundle \( Q = G \times \mathbb{R} \times G \), with the principal action of \( x \in G \) given as
\[
x \cdot (a, s, y) = (a, s, yx^{-1}),
\]
for \( a, y \in G \) and \( s \in \mathbb{R} \). The group of integers \( \mathbb{Z} \) acts by principal bundle automorphisms, \( n \cdot (a, s, y) = (a, s + n, a^ny) \); the quotient is a principal bundle
\[
Q = (G \times \mathbb{R} \times G)/\mathbb{Z} \to G \times S^1, \quad [(a, s, y)] \mapsto (a, [s]),
\]
with a canonical framing along \( G \times \{0\} \), given by \( (a, [0]) \mapsto [(a, 0, e)] \), and with a \( G \times \mathbb{R} \)-action by bundle automorphisms
\[
(a', s'), [(a, s, y)] = [(Ad_{a'} a, s + s', a'a)].
\]
Taking loops of Sobolev class $r + 1$, we obtain a $G$-equivariant principal $LG$-bundle $LQ \to L(G \times S^1)$, containing the bundle of quasi-periodic paths $\mathcal{P}G$ as a $G$-equivariant subbundle:

\[
\begin{array}{ccc}
\mathcal{P}G & \xrightarrow{\pi} & LQ \\
\downarrow & & \downarrow \\
G & \xrightarrow{\pi} & L(G \times S^1)
\end{array}
\]

Here the lower horizontal map takes $a \in G$ to the loop $s \mapsto (a, s)$, while the upper horizontal map takes $g \in \mathcal{P}G$ to the loop, $s \mapsto [(\pi(g), s, g(s))]$. Similarly, working with framed loops we obtain a diagram

\[
\begin{array}{ccc}
A_{S^1} & \xrightarrow{\text{Hol}} & L_0Q \\
\downarrow & & \downarrow \\
G & \xrightarrow{\pi} & L_0(G \times S^1)
\end{array}
\]

Given a principal connection $\nu \in \Omega^1(Q, g)$ on the bundle $Q \to G \times S^1$, the loop functor determines a connection on $LQ \to L(G \times S^1)$, which then pulls back to a connection $\theta$ on the principal $LG$-bundle $\mathcal{P}G \to G$. Furthermore, if $\nu$ is a framed connection, then the resulting connection on $LQ \to LG$ restricts to a connection on $L_0Q$, and hence $\theta$ reduces to a connection on $A_{S^1} \cong \mathcal{P}_0G \to G$.

To describe framed connections on $Q$, we use the canonical trivialization of its pullback under the map $G \times I \to G \times S^1$, $(a, s) \mapsto (a, [s])$. A sufficient condition for $\chi \in C^\infty(I)$ with $\chi(0) = 0$, $\chi(1) = 1$ to define a connection on $Q$, is that $\chi$ extends to a smooth function on $\mathbb{R}$, equal to 0 for $s \leq 0$ and equal to 1 for $s \geq 1$. The resulting connection $\theta$ on the loop group bundle $\mathcal{P}G \to G$ is again referred to as a standard connection. Connections of this type were used by Carey-Mickelsson [12]. While $\theta$ depends on the choice of $\chi$, the resulting 2-form $\varpi \in \Omega^2(A_{S^1})$ is independent of that choice [4]; it is the pullback of the corresponding 2-form on $A = A_I$.

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