Research Article

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Multiplicity of Positive Solutions for a Quasilinear Schrödinger Equation with an Almost Critical Nonlinearity

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Abstract: In this paper we prove an existence result of multiple positive solutions for the following quasilinear problem:
\[
\begin{aligned}
-\Delta u - \Delta (u^2)u &= |u|^{p-2}u & \text{in } \Omega, \\
\quad u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]
where \(\Omega\) is a smooth and bounded domain in \(\mathbb{R}^N, N \geq 3\). More specifically we prove that, for \(p\) near the critical exponent \(2^* = 4N/(N - 2)\), the number of positive solutions is estimated below by topological invariants of the domain \(\Omega\): the Lusternik–Schnirelmann category and the Poincaré polynomial. With respect to the case involving semilinear equations, many difficulties appear here and the classical procedure does not apply immediately. We obtain also en passant some new results concerning the critical case.

Keywords: Quasilinear Schrödinger Equation, Variational Methods, Lusternik–Schnirelmann Category, Morse Theory, Multiplicity of Solutions

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1 Introduction

It is well known that the general quasilinear Schrödinger equation
\[
i\partial_t \psi = -\Delta \psi + V(x)\psi - \bar{h}(|\psi|^2)\psi - \kappa \Delta [\rho(|\psi|^2)]\rho'(|\psi|^2)\psi,
\]
where \(\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}\) is the unknown, \(\kappa\) is a real constant, serves as model for several physical phenomena depending on the form of the given potential \(V = V(x)\) and the given nonlinearities \(\rho(s)\) and \(\bar{h}\).

In the case \(\rho(s) = (1 + s)^{1/2}\), the equation models the self-channeling of a high-power ultra short laser in matter, see [8, 30]. It also appears in fluid mechanics [21], in the theory of Heisenberg ferromagnets and magnons [36], in dissipative quantum mechanics and in condensed matter theory [27].

When \(\rho(s) = s\), which is the case we are interested here, the above equation reduces to
\[
i\partial_t \psi = -\Delta \psi + V(x)\psi - \kappa \Delta [|\psi|^2] \psi - \bar{h}(|\psi|^2)\psi.
\] (1.1)
It was shown that a system describing the self-trapped electron on a lattice can be reduced in the continuum limit to (1.1) and numeric results on this equation are obtained in [9]. In [20], motivated by the nanotubes and fullerene related structures, it was proposed and shown that a discrete system describing the interaction of a 2-dimensional hexagonal lattice with an excitation caused by an excess electron can be reduced to (1.1); moreover numeric results have been obtained on domains of disc, cylinder or sphere type. The superfluid film equation in plasma physics has also the structure (1.1), see [22].

The search of standing wave solutions \( \psi(t, x) = \exp(-iFt)u(x) \), \( F \in \mathbb{R} \) of, (1.1) under a power type non-linearity \( h \) reduces the equation to

\[
- \Delta u - \Delta(u^2)u + W(x)u = h(u),
\]

where \( W(x) = V(x) - F, h(u) = \tilde{h}(u^2)u \) and we have assumed that \( \kappa = 1 \).

The quasilinear equation (1.2) in the whole \( \mathbb{R}^N \) has received special attention in the past years and various devices have been used: for example the method of Lagrange multipliers, which gives a solution with an unknown multiplier \( \lambda \) in front of the nonlinear term (see [29]) and the remarkable change of variable to get a semilinear equation in appropriate Orlicz space framework (see [12, 13, 24]). We refer the reader also to the papers [15, 25, 26, 31] and references therein.

In this paper we are interested in a special case of (1.2), that is, we study the equation in a smooth and bounded domain \( \Omega \subset \mathbb{R}^N, N \geq 3 \), with constant potential \( V(x) = F \) (hence \( W(x) = 0 \)) and with homogeneous Dirichlet boundary conditions; in other words, we are interested in the search of positive solutions for the problem

\[
\begin{cases}
- \Delta u - \Delta(u^2)u = |u|^{p-2}u & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with \( p \in (4, 2^*) \). As usual \( 2^* = \frac{2N}{N-2} \) is the Sobolev critical exponent of the embedding of \( H^1_0(\Omega) \) into Lebesgue spaces, and \( 2^* \) turns out to be the critical exponent for the problem, as it is shown in [12, Remark 3.3].

The main goal of this paper is to show that for \( p \) near the critical exponent \( 2^* \), the topology of the domain influences the number of positive solutions in the sense of Theorem 1 and Theorem 2 below.

Before to state our main results we recall that if \( Y \) is a closed set of a topological space \( X \), we denote the Ljusternik-Schnirelmann category of \( Y \) in \( X \) by \( \text{cat}_X(Y) \), which is the least number of closed and contractible sets in \( X \) that cover \( Y \). Moreover, \( \text{cat}_X \) denotes \( \text{cat}_X(X) \). Then we have the first multiplicity result.

**Theorem 1.** There exists \( \overline{p} \in (4, 2^*) \) such that for any \( p \in [\overline{p}, 2^*) \), problem (1.3) has at least \( \text{cat}_\Omega \) positive weak solutions. Moreover, if \( \Omega \) is not contractible in itself, then (1.3) has at least \( \text{cat}_\Omega + 1 \) positive weak solutions.

By implementing the Morse theory, we are able to prove also the following multiplicity result. Here \( \mathcal{P}_1(\Omega) \) is the Poincaré polynomial of \( \Omega \), whose definition we recall later.

**Theorem 2.** There exists \( \overline{p} \in (4, 2^*) \) such that for any \( p \in [\overline{p}, 2^*) \), problem (1.3) has at least \( 2\mathcal{P}_1(\Omega) - 1 \) positive solutions, possibly counted with their multiplicity.

In whole this paper, a function \( u : \Omega \to \mathbb{R} \) is called a (weak) solution of (1.3) if \( u \in H^1_0(\Omega) \cap L^{2^*}_{loc}(\Omega) \) and satisfies

\[
\int_{\Omega} [(1 + 2|u|^2)\nabla u \cdot \nabla \varphi + 2|\nabla u|^2 u \varphi] = \int_{\Omega} |u|^{p-2}u \varphi \quad \text{for all } \varphi \in C_0^{\infty}(\Omega).
\]

We point out that among the solutions we find there is the ground state, called hereafter \( g_p \), that is, the solution with minimal energy \( m_p \), in the sense specified in Section 3.

It is worth to mention that problem (1.3) has been studied recently in [23] in a bounded domain and the authors prove, by using Morse theory, existence results of multiple solutions. However the number of solutions found in [23] is not in relation with the topology of the domain \( \Omega \), and nothing is said on the sign of the solutions. So our paper is the first one to relate the number of positive solutions to the topology of the domain when the exponent is near the critical one \( 2^* \).
1.1 Motivation, Main Ideas and New Auxiliary Results

1.1.1 Motivation

The inspiration to study (1.3) comes from the papers [5–7] where the authors consider the model problem
\[
\begin{aligned}
-\Delta u + \lambda u &= |u|^{p-2}u \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]
and ask how the topology of the domain \( \Omega \) affects the number of positive solutions depending on suitable “limit” values of the parameters \( \lambda, p \). They introduced new techniques in order to have a “picture” of \( \Omega \) in a suitable sublevel of the energy functional associated to the problem, and then they use the Ljusternick–Schnirelmann and Morse theory in order to deduce a multiplicity result. Actually the authors treat two cases:

(i) when \( p \) is fixed and the parameter \( \lambda \) is made sufficiently large,

(ii) when \( \lambda \) is fixed and the parameter \( p \) tends to the critical value \( 2^* \),

and find solutions for \( \lambda \) large in the first case, and for \( p \) near \( 2^* \) in the second case.

After the mentioned papers [5–7], these techniques have been successfully used to prove multiplicity of positive solutions for equations involving also different operators then the Laplacian, and even in presence of a potential.

However, the existing literature mainly concerns with case (i): with this respect, many papers appeared also where the parameter \( \lambda \) can be moved, after a rescaling, into the potential or even as a factor which expands the domain \( \Omega \). For more details and results in this direction we refer the reader to the papers [1, 11] for the \( p \)-Laplacian, [2] for the magnetic Laplacian in expanding domains, [16] for the fractional Laplacian in expanding domains, [3, 4, 10] for quasilinear operators: in all these papers multiplicity result, depending on the topology of the domain, have been proved for \( \lambda \) large. We point out that there are also papers where the parameter is in a suitable potential appearing in the equation (and not in the nonlinearity). In this case multiplicity results are obtained whenever the parameter is sufficiently small and the number of solutions is affected by the topology of the set of minima of the potential. See e.g. [17, 19, 28] and the references therein.

However, case (ii) in which the role of the parameter is played by the exponent of the nonlinearity, that we believe to be very interesting too, has been much less explored. To the best of our knowledge, there are just two other papers (besides [5]) which consider the case when the parameter \( p \) approaches the critical exponent obtaining multiplicity of solutions depending on the topology of the domain: they are [34] where the Schrödinger–Poisson system is studied and [18] where the fractional Laplacian is considered. In these papers a pure power (hence homogeneous) nonlinearity is considered.

Indeed, the poorness of results in the literature concerning case (ii) has motivated the present paper.

We point out that the ideas of Benci, Cerami and Passaseo in [5–7] are not immediately applicable to our problem due to the fact that, to deal with a well-defined functional in \( H^1_0(\Omega) \), we use the change of variable \( f \) introduced in [12] which has to be treated very carefully. Indeed, the problem is transformed into a semilinear one but a nonhomogeneous nonlinearity to the power \( p \) appears. To treat this lack of homogeneity, we need some new properties, with respect to the known in the literature, of the change of variable. Moreover, this lack of homogeneity forces us to study the related functional on the Nehari manifold and not on the \( L^p \)-constraint as in [5–7]; in fact, we can not stretch the Lagrange multiplier once it appears.

1.1.2 Main Ideas

Coming back to problem (1.3), we first use the change of variable \( u = f(v) \) introduced by [12] to transform the problem into the equivalent one
\[
\begin{aligned}
-\Delta v &= |f(v)|^{p-2}f(v)f'(v) \quad \text{in } \Omega, \\
\quad v &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
Then we find its solutions as critical points of a $C^1$ functional on the so-called Nehari manifold, which is a natural constraint. In particular, we show that the functional on the Nehari manifold is bounded below, achieves the ground state level $m_p$, for $p \in (4, 2^{*})$, and by means of the Ljusternik–Schnirelmann and Morse theories we prove the multiplicity results.

Since our approach in proving Theorem 1 and Theorem 2 is variational, it will be important to have a compactness condition that we recall here once for all for the reader’s convenience. If $H$ is an Hilbert space, $\mathcal{N} \subset H$ a submanifold and $I : H \to \mathbb{R}$ a $C^1$ functional, we say that $I$ satisfies the Palais–Smale (PS) condition on $\mathcal{N}$ at level $a \in \mathbb{R}$ if any sequence $\{u_n\} \subset \mathcal{N}$ such that
\[ I(u_n) \to a \quad \text{and} \quad (I|_{\mathcal{N}})'(u_n) \to 0 \quad \text{(1.4)} \]
possesses a subsequence converging to some $u \in \mathcal{N}$. We will also say that $I|_{\mathcal{N}}$ satisfies the (PS) condition at the level $a$.

In general, a sequence satisfying the conditions in (1.4) is named Palais–Smale sequence at level $a$. If the value $a$ is not really important or is clear from the context, we will simply speak of (PS) condition and/or (PS) sequences.

### 1.1.3 New Auxiliary Results

However, in order to achieve the main results, Theorem 1 and Theorem 2, we are naturally driven to obtain some auxiliary results, mostly related to the “limiting cases”, which have an independent interest and may be useful also in other different contexts. We list them briefly here.

First of all, additional properties of the change of variable are found, see property (iii) in Corollary 1 and Lemma 2.

Moreover, we will need to study the behavior and profile decomposition of the (PS) sequences for the functional related to the critical problem, that is,
\[
\begin{align*}
-\Delta v &= |f(v)|^{2^{*} - 2}f(v)f'(v) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\quad \text{(1.5)}
\]
(or its equivalent quasilinear version). Hence as a byproduct of our proofs we obtain the representation of the (PS) sequences (known also as Splitting Lemma, see Lemma 10) for the critical problem (1.5). To the best of our knowledge, this representation is new.

Furthermore, concerning the critical case, we show in Lemma 4 a nonexistence result for problem (1.5) in a star-shaped domain; this means that the exponent $2^{*}$ is critical also with respect to the existence of solutions and implies that the ground state level $m_*$ is not achieved in this case.

We show also that $m_*$ does not depend on the domain and that for every domain $\lim_{p \to 2^*} m_p = m_*$, see Theorem 3 and Theorem 4. In particular, in Theorem 3 an estimate on $m_*$ in term of “best constants” is given. While the estimate by below of $m_*$ is quite natural and we think it is optimal, the estimate by above we believe can be improved, and indeed we think this is an interesting open problem.

Related to the critical problem in $\mathbb{R}^N$,
\[
-\Delta v = |f(v)|^{2^{*} - 2}f(v)f'(v) \quad \text{in } \mathbb{R}^N
\]
(or its equivalent quasilinear version) we deduce an passant the existence of the ground state solution at a positive energy level $m_*$, see Theorem 5, which is also new.

We believe that this paper may serve as a motivation to study the multiplicity of solutions for problems with an “almost critical” growth whenever the nonlinearity is not a pure power.

### 1.2 Structure of the Paper

The paper is organized as follows.
In Section 2 we give the variational setting of the problem. In particular, the change of variable given in [12] is introduced in order to have a well defined and $C^1$ functional whose critical points are exactly the solutions we are looking for.

In Section 3 we introduce the Nehari manifold associated to the problems settled in the domain $\Omega$, in both cases of $p$ subcritical and critical. This section is quite technical since we need to perform projections of nontrivial functions on different Nehari manifolds, and compare in some sense the Nehari manifolds of the subcritical problem with the Nehari manifold of the critical one. A “local” (PS) condition is proved for the critical case. Finally, we give also a Splitting Lemma involving the critical problem on the whole $\mathbb{R}^N$. We say that most of all the results are fundamental in order to achieve the Proposition 6 which is a key step in order to employ the Ljusternick–Schnirelmann theory.

In Section 4, the barycenter map à la Benci–Cerami is introduced and some properties are proved.

In Section 5 the proof of Theorem 1 is given by using the Ljusternick–Schnirelmann theory.

In Section 6, after recalling some basic notions in Morse theory and show that the second derivative of the functional is “of type” isomorphism minus a compact operator, we prove Theorem 2.

1.3 Final Comments

As a matter of notations, we will use the letters $C, C', \ldots, C_1, C_2, \ldots$ to denote suitable positive constants which do not depend on the functions neither on $p$. Moreover, their values, irrelevant for our purpose, are allowed to change on every estimate.

The letter $S$ will be deserved for the embedding constant of $H^1_0(\Omega)$ in $L^2^*(\Omega)$.

The symbol $o_n(1)$ stands for a vanishing sequence.

We will use sometimes the notation $|u|^p$ for the usual $L^p$-norm of the function $u$: no confusion should arise for what concerns the underlying domain.

Other notations will be introduced whenever we need.

Finally, without no loss of generality, we assume throughout the paper that $0 \in \Omega$.

2 Variational Framework

As observed in [32, 33], there are some technical difficulties to apply directly variational methods to the functional associated to (1.3), which formally should be given by

$$ J_p(u) = \frac{1}{2} \int_\Omega (1 + 2|u|^2)|\nabla u|^2 - \frac{1}{p} \int_\Omega |u|^p. $$

The main difficulty related to $J_p$ is that it is not well defined in the whole $H^1_0(\Omega)$. For example, if $u$ diverges near 0 as $|x|^{(2-N)/4}$ and then is glued to a smooth, radial, and vanishing function, we have $u \in H^1_0(B)$, while the function $|u|^2|\nabla u|^2$ does not belong to $L^1(B)$. Here $B \subset \Omega$ is a ball containing the origin in $\mathbb{R}^N$.

To overcome this difficulty, we use the arguments developed in [12]. More precisely, we make the change of variables $v = f^{-1}(u)$, where $f$ is defined by

$$ f'(t) = \frac{1}{(1 + 2f(t)^2)^{1/2}} \quad \text{on } (0, +\infty), \quad f(t) = -f(-t) \quad \text{on } (-\infty, 0]. $$

Therefore, after the change of variables, the functional $I_p$ can be rewritten in the following way:

$$ I_p(v) := I_p(f(v)) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \frac{1}{p} \int_\Omega |f(v)|^p $$

which is well defined on the space $H^1_0(\Omega)$ endowed with the usual norm

$$ \|v\|^2 = \int_\Omega |\nabla v|^2. $$
A straightforward computation shows that the functional (2.1) is of class $C^1$ with
\[
I_\mu(v)[w] = \int_\Omega \nabla v \nabla w - \int_\Omega |f(v)|^{p-2} f(v) f'(v) w
\]
for $v, w \in H^1_0(\Omega)$. Thus, the critical points of $I_\mu$ correspond exactly to the weak solutions of the semilinear problem involving a nonlinearity which is not of power type:
\[
\begin{align*}
-\Delta v &= |f(v)|^{p-2} f(v) f'(v) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\tag{2.2}
\]
This problem has a close relation with problem (1.3). In fact, if $v \in H^1_0(\Omega) \cap L^{\infty}_{loc}(\Omega)$ is a critical point of the functional $I_\mu$, hence a weak solution of (2.2), then $u = f(v)$ is a weak solution of (1.3). By the same arguments used to prove [3, Proposition 3.6], we have that each critical point $v$ of $I_\mu$ belongs to $H^1_0(\Omega) \cap L^{\infty}_{loc}(\Omega)$.

Summing up we are reduced to find nontrivial critical points of $I_\mu$. Actually, as we will see in Section 6 where the Morse theory is used, the functional is even $C^2$.

Now we show some results about the change of variable $f: \mathbb{R} \to \mathbb{R}$ that are essential in the next sections.

**Lemma 1** (see [32, 33]). The function $f$ and its derivative enjoy the following properties:

(i) $f$ is uniquely defined, $C^2$ and invertible.

(ii) $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$.

(iii) $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$.

(iv) $\frac{1}{t} f(t) \to 1$ as $t \to 0$.

(v) $|f(t)| \leq 2^{1/4} |t|^{1/2}$ for all $t \in \mathbb{R}$.

(vi) $\frac{1}{t} f(t) < tf'(t) < f(t)$ for all $t > 0$, and the reverse inequalities hold for $t < 0$.

(vii) $\frac{1}{t} f(t) \to a > 0$ as $t \to +\infty$.

(viii) There exists a positive constant $C$ such that
\[
|f(t)| \geq \begin{cases} C/t, & |t| \leq 1, \\ C|t|^{1/2}, & |t| \geq 1. \end{cases}
\]

(ix) $|f(t)f'(t)| \leq \frac{1}{2^{1/4}}$ for all $t \in \mathbb{R}$.

Particularly useful will be the inequalities
\[
\frac{f(t)^2}{2} \leq f'(t)f(t)t \leq f(t)^2 \quad \text{for all } t \in \mathbb{R}. \tag{2.3}
\]

simply obtained by (vi) of Lemma 1.

We deduce the following result.

**Corollary 1.** The following properties involving the function $f$ hold:

(i) The function $f(t)f'(t)^{-1}$ is decreasing for $t > 0$.

(ii) The function $f(t)^3 f'(t)^{-1}$ is increasing for $t > 0$.

(iii) For any $p > 4$, the function $|f(t)|^{p-2} f(t)f'(t)^{-1}$ is increasing for $t > 0$.

Properties (i) and (ii) were indeed proved in [14] however we give the proof for completeness.

**Proof.** By using (vi) of Lemma 1, it is easy to see that $\frac{1}{t} f(t)$ is non-increasing for $t > 0$. Thus,
\[
\frac{d}{dt} \left( \frac{f(t)f'(t)}{t} \right) = \frac{d}{dt} \left( \frac{f(t)}{t} \right)f'(t) + \frac{f(t)}{t} f''(t) < 0
\]
for all $t > 0$, which shows (i).

To prove (ii), we observe that since
\[
f'(t) = \frac{1}{(1 + 2f^2(t))^{1/2}} \quad \text{and} \quad f''(t) = -\frac{2f(t)f'(t)}{(1 + 2f^2(t))^{3/2}} = -2f(t)(f'(t))^4,
\]
we have
\[
\frac{d}{dt} \left( \frac{f(t)^3 f'(t)}{t} \right) = \frac{d}{dt} \left( \frac{f(t)}{t} \right)f'(t)^3 + \frac{f(t)}{t} f''(t) > 0
\]
for all $t > 0$, which shows (ii).
we have
\[ \frac{d}{dt} \left( \frac{f(t)^3f'(t)}{t} \right) = \frac{3f(t)^2(f'(t))^2 t - 2f(t)^3(f'(t))^4 t - f(t)^3f'(t)}{t^2} \geq f'(t)f(t)^2 \frac{2f(t)^2 t - f(t)}{t^2}. \]

Therefore, by (vi) and (ix) of Lemma 1, we have for all \( t > 0 \),
\[ \frac{d}{dt} \left( \frac{f(t)^3f'(t)}{t} \right) \geq f'(t)f(t)^2 \frac{2f(t)^2 t - f(t)}{t^2} > 0, \]
which proves (ii).

Finally, by using (ii) and the equality
\[ \frac{|f(t)|^{p-2}f(t)f'(t)}{t} = |f(t)|^{p-4}f(t)^3f'(t) \quad \text{for} \ t > 0, \]
we obtain (iii).

As we already said, in contrast to the classical case studied by Benci, Cerami and Passaseo, which involves a pure power nonlinearity, our nonlinearity is nonhomogeneous. Hence the next new properties will be fundamental.

**Lemma 2.** The following hold true:

(i) \( f(t)f'(t) \) is increasing. In particular,
\[ f(\lambda t)f'(\lambda t)\lambda t \leq \lambda f(t)f'(t) t \quad \text{for all} \ \lambda \in [0, 1] \ \text{and} \ t \geq 0, \]
\[ f(\lambda t)f'(\lambda t)\lambda t \geq \lambda f(t)f'(t) t \quad \text{for} \ \lambda \geq 1 \ \text{and} \ t \geq 0. \]

Moreover, for \( \alpha > 0 \), we have:

(ii) \( f(\lambda t)^\alpha \geq \lambda^\alpha f(t)^\alpha \) for all \( \lambda \in [0, 1] \) and \( t \geq 0. \)

(iii) \( f(\lambda t)^\alpha \leq \lambda^{\alpha/2} f(t)^\alpha \) for all \( \lambda \in [0, 1] \) and \( t \geq 0. \)

(iv) \( f(\lambda t)^\alpha \leq \lambda^{\alpha} f(t)^\alpha \) for all \( \lambda \geq 1 \) and \( t \geq 0. \)

(v) \( f(\lambda t)^\alpha \geq \lambda^{\alpha/2} f(t)^\alpha \) for all \( \lambda \geq 1 \) and \( t \geq 0. \)

**Proof.** By using that \( f''(t) = -2f(t)(f'(t))^4 \) (see the proof of (ii) in Corollary 1) and (ix) of Lemma 1, we find that
\[ \frac{d}{dt} \left( f(t)f'(t) \right) = (f'(t))^2 - 2f(t)(f'(t))^4 = (f'(t))^2(1 - 2f(t)^2(f'(t))^2) \geq 0 \]
proving (i).

Of course, if \( \lambda = 0 \) or \( t = 0 \), (ii) and (iv) are satisfied. Now for \( t > 0 \) fixed we have, in virtue of (vi) of Lemma 1,
\[ \frac{d}{d\lambda} \left( \frac{f(\lambda t)}{\lambda^\alpha} \right) = \left| \frac{f(\lambda t)}{\lambda^\alpha} \right| \leq 0 \]
and hence \( f(\lambda t)^\alpha \) is a non-increasing function in \( \lambda \). This gives (ii) and (iv).

The proofs of (iii) and (v) follow in a similar way, by computing the derivative with respect to \( \lambda \) of \( f(\lambda t)^\alpha \).

**Remark 1.** The inequalities in (ii)–(v) of Lemma 2 are true for any \( t \in \mathbb{R} \) if we consider the absolute values.

Let us see briefly the case \( \alpha = 1 \) that will be used explicitly in Theorem 3 and Lemma 5.

We claim that
\[ \left| \frac{f(\lambda t)}{\lambda^{1/2}} \right| \leq \left| f(t) \right| \leq \left| \frac{f(\lambda t)}{\lambda^{1/2}} \right| \quad \text{for all} \ t \in \mathbb{R}, \ \lambda \in [0, 1]. \tag{2.4} \]

We just check the second inequality since the first one follows analogously by using (iii). Indeed, for \( t \geq 0 \) it is \( f(t) \leq \frac{f(\lambda t)}{\lambda} \) which is just (ii) of Lemma 2 and then the second inequality above follows for positive \( t \). On the other hand if \( t < 0 \), then
\[ -f(t) = f(-t) \leq \frac{f(-\lambda t)}{\lambda} \leq \left| \frac{f(-\lambda t)}{\lambda} \right| = \left| \frac{f(\lambda t)}{\lambda} \right| \]
and then the second inequality above follows also for negative \( t \).

Analogously, by (iv) and (v) we deduce
\[ \left| \frac{f(\lambda t)}{\lambda} \right| \leq |f(t)| \leq \left| \frac{f(\lambda t)}{\lambda} \right| \quad \text{for all} \ t \in \mathbb{R}, \ \lambda \geq 1. \tag{2.5} \]
3 The Nehari Manifolds and Compactness Results

In this section we study the Nehari manifolds which appear in relation to our problem. In particular, we need to consider, beside problem (1.3) also some limit cases with the associated Nehari manifolds.

Associated to the functional (2.1), that is,
\[ I_p(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \frac{1}{p} \int_{\Omega} |f(v)|^p, \]
we have the set, usually called the Nehari manifold associated to (1.3),
\[ \mathcal{N}_p = \{ v \in H^1_0(\Omega) \setminus \{0\} : G_p(v) = 0 \}, \quad \text{where } G_p(v) := I'_p(v)[v] = \|v\|^2 - \int_{\Omega} |f(v)|^{p-2} f(v) f'(v) v, \]

In particular, all the critical points of \( I_p \) lie in \( \mathcal{N}_p \). In the next lemma we show basic properties of \( \mathcal{N}_p \). For completeness we present also the proof of some of its properties since we were not able to find them in the literature.

**Proposition 1.** For all \( p \in (4, 22^*) \), we have:
(i) \( \mathcal{N}_p \) is a \( C^1 \) manifold.
(ii) There exists \( c_p > 0 \) such that \( \|v\| \geq c_p \) for every \( v \in \mathcal{N}_p \).
(iii) It holds \( \inf_{u \in \mathcal{N}_p} I_p(u) > 0 \).
(iv) For every \( v \neq 0 \), the map
\[ g : t \in [0, +\infty) \mapsto I_p(tv) \in \mathbb{R} \]
has a unique global maximum \( t_p = t_p(v) > 0 \) such that \( t_p v \in \mathcal{N}_p \). In particular, when \( p = 22^* \), we will write for brevity \( t_* := t_{22^*} \).
(v) \( \mathcal{N}_p \) is homeomorphic to the unit sphere \( S = \{ v \in H^1_0(\Omega) : \|v\| = 1 \} \).
(vi) The following equalities are true:
\[ \inf_{v \in \mathcal{N}_p} I_p(v) = \inf_{v \neq 0} \max_{t > 0} I_p(tv) = \inf_{g \in \Gamma_p} \max_{t \in [0,1]} I_p(g(t)), \]
where
\[ \Gamma_p = \{ g \in C([0,1]; H^1_0(\Omega)) : g(0) = 0, I_p(g(1)) \leq 0, g(1) \neq 0 \}. \]

**Proof.** Since
\[ G'_p(v)[v] = 2 \int_{\Omega} |\nabla v|^2 - (p - 1)\int_{\Omega} |f(v)|^{p-2} f'(v)^2 v^2 - \int_{\Omega} |f(v)|^{p-2} f(v) f''(v) v^2 - \int_{\Omega} |f(v)|^{p-2} f(v) f'(v) v, \]

and \( G_p(v) = 0 \) if \( v \in \mathcal{N}_p \), we obtain
\[ G'_p(v)[v] = -\int_{\Omega} (p - 1) |f(v)|^{p-2} f'(v)^2 v^2 - \int_{\Omega} |f(v)|^{p-2} f(v) f''(v) v^2 + \int_{\Omega} |f(v)|^{p-2} f(v) f'(v) v. \]

Using that \( f''(t) = -2f(t) f'(t)^2 \) and inequalities in (ix) and (vi) of Lemma 1, we get
\[ G'_p(v)[v] = \int_{\Omega} |f(v)|^{p-2} f(v) f'(v) v - (p - 1) \int_{\Omega} |f(v)|^{p-2} f'(v)^2 v^2 + 2 \int_{\Omega} |f(v)|^p (f'(v))^4 v^2 \]
\[ = -\int_{\Omega} |f(v)|^{p-2} f'(v) v (p - 1) f'(v) v - 2f(v)^2 f'(v)^3 v - f(v) \]
\[ \leq -\int_{\Omega} |f(v)|^{p-2} f'(v) v (p - 1) f'(v) v - f'(v) v - f(v) \]
\[ = -\int_{\Omega} |f(v)|^{p-2} f'(v) v ((p - 2) f'(v) v - f(v)) \]
\[ \leq -\int_{\Omega} |f(v)|^{p-2} f'(v) v \left( \frac{p - 2}{2} f(v) - f(v) \right) = -\frac{p - 4}{2} \int_{\Omega} |f(v)|^{p-2} f(v) f'(v) v. \]
Finally, by using (2.3), we arrive at
\[ G_p^p(v)[v] \leq -\frac{p-4}{2} \int_\Omega |f(v)|^{p-2}f(v)f'(v)v \leq -\frac{p-4}{4} \int_\Omega |f(v)|^p < 0 \]
which proves (i).

Let now \( v \in N_p \). Then, by using successively (2.3) and (v) of Lemma 1, we get
\[ |v|^2 = \int_\Omega |f(v)|^{p-2}f(v)f'(v)v \leq 2^{p/4} \int_\Omega |v|^{p/2} \leq 2^{p/4} |\Omega|^{\frac{2^2-p}{2^2}} |v|^p \leq \frac{p}{4p} \int_\Omega |f(v)|^p \geq 0. \]
and hence we infer
\[ |v| \geq \left( \frac{1}{2^{p/4} |\Omega|^{\frac{2^2-p}{2^2}}} \right)^{2/(p-4)} =: c_p > 0, \]
which shows (ii).

On \( N_p \) the functional is positive since, by using (2.3), we have
\[ I_p(v) = \frac{1}{2} \int_\Omega |f(v)|^{p-2}f(v)f'(v)v - \frac{1}{p} \int_\Omega |f(v)|^p \geq \frac{p-4}{4p} \int_\Omega |f(v)|^p \geq 0. \]
Moreover, for every \( v \in N_p \) by (2.3) and (v) of Lemma 1, we have
\[ \int_\Omega |\nabla v|^2 = \int_\Omega |f(v)|^{p-2}f(v)f'(v)v \leq \int_\Omega |f(v)|^p \leq 2^{p/4} \int_\Omega |v|^{p/2} \leq C \left( \int_\Omega |\nabla v|^2 \right)^{p/2} \]
and then
\[ \int_\Omega |\nabla v|^2 \geq C' > 0. \]  \tag{3.1}

Then, if it were \( \inf_{v \in N_p} I_p(v) = 0 \), there would exist \( \{v_n\} \subset N_p \) such that, by using again (2.3),
\[ o_n(1) = I_p(v_n) \]
\[ = I_p(v_n) - \frac{2}{p} l_p^p(v_n)[v_n] \]
\[ = \frac{p-2}{2p} \int_\Omega |\nabla v_n|^2 + \frac{2}{p} \int_\Omega |f(v_n)|^{p-2}f(v_n)f'(v_n)v_n - \frac{1}{p} \int_\Omega |f(v_n)|^p \]
\[ \geq \frac{p-4}{2p} \int_\Omega |\nabla v_n|^2, \]  \tag{3.2}
which contradicts (3.1) and concludes the proof of (iii).

Let \( v \neq 0 \) and, for \( t \geq 0 \), define the map
\[ g(t) := I_p(tv) = \frac{t^2}{2} \int_\Omega |\nabla v|^2 - \frac{1}{p} \int_\Omega |f(tv)|^p. \]
It is easy to see that \( g(0) = 0 \) and \( g(t) < 0 \) for suitably large \( t \), by (viii) of Lemma 1. Clearly \( g'(t) = I_p^p(tv)[v] = 0 \) if and only if \( tv \in N_p \). Moreover, \( g'(t) = 0 \) means
\[ \int_\Omega |\nabla v|^2 = \frac{1}{t} \int_\Omega |f(tv)|^{p-2}f(tv)f'(tv)v = \int_{\{x \in \Omega : |v(x)| = 0\}} \frac{f(tv)|^{p-2}f'(tv)|v|^2}{t|v|} \]
and the right-hand side is an increasing function in \( t \). Since by (v) of Lemma 1,
\[ \lim_{t \to 0^+} \int_\Omega t^2 f(tv)|v|^2 \leq 2^{p/4} |\Omega|^{\frac{2^2-p}{2^2}} |v|^p = 0, \]
we easily see that \( g(t) > 0 \) for suitably small \( t > 0 \). Consequently, there is a unique \( t_p \neq 0 \) such that
\[ g'(t_p) = 0 \]
and \( g(t_p) = \max_{t>0} g(t) \), i.e. \( t_p v \in N_p \), proving (iv).

The proof of (v) and (vi) follows by standard arguments. \[ \square \]
Remark 2. Actually in (ii) of Proposition 1 the constant $c_p$ can be made independent on $p$ far away from 4. Indeed it is easily seen that it is possible to take a small $\eta > 0$ such that
\[ \xi := \min_{p \in [4 + \eta, 22^*]} c_p > 0. \]
In other words, all the Nehari manifolds $\mathcal{N}_p$ are bounded away from zero, independently on $p \in [4 + \eta, 22^*]$, i.e. there exists $\xi > 0$ such that
\[ \forall p \in [4 + \eta, 22^*] : v \in \mathcal{N}_p \implies \|v\| \geq \xi. \]

In the remaining part of the paper, the symbol $\eta$ will be deserved for the small positive constant given above.

The Nehari manifold well-behaves with respect to the (PS) sequences. Again, since at this stage no compactness condition is involved, we can even state the result for $p \in (4, 22^*].$

Lemma 3. Let $p \in (4, 22^*]$ be fixed and let $\{v_n\} \subset \mathcal{N}_p$ be a (PS) sequence for $I_p|_{\mathcal{N}_p}$. Then $\{v_n\}$ is a (PS) sequence for the free functional $I_p$ on $H^1_0(\Omega)$.

Now for $p \in (4, 22^*)$ it is known that the free functional $I_p$ satisfies the (PS) condition on $H^1_0(\Omega)$ and also when restricted to $\mathcal{N}_p$, see e.g. [4, Lemma 3.2 and Proposition 3.3]. In addition to the properties listed in Proposition 1, the manifold $\mathcal{N}_p$ is a natural constraint for $I_p$ in the sense that any $u \in \mathcal{N}_p$ critical point of $I_p|_{\mathcal{N}_p}$ is also a critical point for the free functional $I_p$ (see, for instance, [4, Corollary 3.4]). Hence the (constraint) critical points we will find will be solutions of our problem since no Lagrange multipliers appear.

In particular, as a consequence of the (PS) condition, we have the following fact true in the subcritical case:
\[ m_p := \min_{v \in \mathcal{N}_p} I_p(v) = I_p(g_p) > 0 \quad \text{for all } p \in (4, 22^*), \quad (3.3) \]
i.e. $m_p$ is achieved on a function, hereafter denoted with $g_p$. Since $g_p$ minimizes the energy $I_p$, it will be called a ground state. Observe that $g_p \geq 0$ and are indeed positive by the Maximum Principle.

Remark 3. We note that if $\{w_p\}_{p \in [4 + \eta, 22^*]} \subset H^1_0(\Omega)$ is such that for all $p \in [4 + \eta, 22^*]$, $w_p \in \mathcal{N}_p$, then
\[ 0 < \xi \leq \|w_p\|_p = \int_{\Omega} |f_w|^p w_p - \int_{\Omega} f w_p \leq \int_{\Omega} |f(w_p)|^p \leq 2^{1/4} \int_{\Omega} |w_p|^{p/2} \leq C|w_p|^{p/2}, \]
where $C$ can be chosen independent on $p$. We deduce that the sequences
\[ \{\|w_p\|_{p \in [4 + \eta, 22^*]}\}, \quad \{\|f(w_p)\|_{p \in [4 + \eta, 22^*]}\} \quad \text{and} \quad \{\|w_p\|_{p \in [4 + \eta, 22^*]}\} \]
are bounded away from zero.

In particular, this is true for the family of ground states $\{g_p\}_{p \in [4 + \eta, 22^*]}$. This last fact will be used in the next sections and in particular in Proposition 3.

We address now two limit cases related to our equation. They involve the critical problems both in the domain $\Omega$ and in the whole space $\mathbb{R}^N$.

### 3.1 The Critical Problem in a Domain

We introduce the critical problem in the domain $\Omega$. This is done in order to evaluate the limit of the ground state levels $\{m_p\}_{p \in (4, 22^*)}$ when $p \to 22^*$. The main theorem in this subsection is Theorem 4, which requires first some preliminary work.

Let us introduce the $C^1$ functional associated to $p = 22^*$,
\[ I_*(v) := I_{22^*}(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \frac{1}{22^*} \int_{\Omega} |f(v)|^{22^*}, \quad v \in H^1_0(\Omega), \]

whose critical points are the solutions of
\[
\begin{aligned}
-\Delta v &= |f(v)|^{2^{*}-2}f(v)f'(v) \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
(3.4)

It is known that the lack of compactness of the embedding of $H^1_0(\Omega)$ in $L^{2^*}(\Omega)$ is due to the conformal scaling
\[ v(\cdot) \mapsto v_R(\cdot) := R^{N/2^*}v(R\cdot) \quad (R > 1), \]
which leaves invariant the $L^2$-norm of the gradient as well as the $L^{2^*}$-norm, i.e.
\[ |\nabla v_R|_2^2 = |\nabla v|_2^2 \quad \text{and} \quad |v_R|_{2^*}^2 = |v|_{2^*}^2. \]

Related to the critical problem we have the following:

**Lemma 4.** If $\Omega$ is a star-shaped domain, then there exists only the trivial solution to (3.4).

**Proof.** Let $v \in H^1_0(\Omega)$ be a solution to (3.4). Setting $h(s) = |f(s)|^{2^{*}-2}f(s)f'(s)$, we have $H(v) \in L^1(\Omega)$, where $H(s) = \frac{1}{2^{*}}|f(s)|^{2^{*}}$. Moreover, according to Brézis–Kato theorem and by elliptic regularity theory, it is easily seen that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Thus, by using the Pohozaev identity (see e.g. [37, Theorem B.1]), we obtain
\[
\frac{1}{2} \int_{\Omega} |\nabla v|^2 \sigma \cdot v \, d\sigma + \frac{N-2}{2} \int_{\Omega} |\nabla v|^2 = \frac{N}{2^{*}} \int_{\Omega} |f(v)|^{2^{*}},
\]
where $\sigma$ denotes the unit outward normal to $\partial \Omega$. Since $v$ is a solution, one also has
\[
\int_{\Omega} |\nabla v|^2 = \int_{\Omega} |f(v)|^{2^{*}-2}f(v)f'(v)v.
\]
Now, combining the last two equalities we reach
\[
\frac{1}{2} \int_{\Omega} |\nabla v|^2 \sigma \cdot v \, d\sigma = \frac{N}{2^{*}} \int_{\Omega} |f(v)|^{2^{*}} - \frac{N-2}{2} \int_{\Omega} |f(v)|^{2^{*}-2}f(v)f'(v)v.
\]
Using that $f(v)f'(v)v \geq \frac{f(v)}{2}$, it follows that $\int_{\partial \Omega} |\nabla v|^2 \sigma \cdot v \, d\sigma \leq 0$ and we must have $v = 0$ provided that $\sigma \cdot v > 0$ on $\partial \Omega$. \hfill $\Box$

Let
\[ N_v = \{ v \in H^1_0(\Omega) \setminus \{0\} : G_v(v) := I_*'(v)[v] = 0 \} \]
be the Nehari manifold associated to the critical problem (3.4). By Lemma 1 it results
\[
m_* := \inf_{v \in N_v} I_*'(v) = \inf_{v \in H^1_0 \setminus \{0\}} \max_{0 < t < \infty} I_*'(tv) > 0.
\]
(3.5)

**Remark 4.** By Lemma 4, $m_*$ is not achieved on star shaped domains.

In analogy to the classical case, given any domain $D \subset \mathbb{R}^N$, we can define also the quantities
\[
Q(D) := \inf_{v \in H^1_0(D) \setminus \{0\}} \frac{|v|^2}{\left( \int_D |f(v)|^{2^{*}} \right)^{1/2}},
\]
and
\[
\overline{Q}(D) := \inf_{v \in H^1_0(D) \setminus \{0\}} \frac{|v|^2}{\left( \int_D |f(v)|^{2^{*}} \right)^{2/2^{*}}},
\]
which in some sense takes the role of best constant. This is corroborated by the next two results.

**Lemma 5.** The values $Q$ and $\overline{Q}$ does not depend on the domain.

**Proof.** The first part of the proof follows the classical one, so we just sketch it. Let $D$ be a domain in $\mathbb{R}^N$ and denote with $Q(D)$ and $Q(\mathbb{R}^N)$ the analogous quantities defined in (3.6) with respect to $D$ and $\mathbb{R}^N$. Since any function in $H^1_0(D)$ can be “seen” as a function in $H^1(\mathbb{R}^N)$, one has
\[
Q(\mathbb{R}^N) \leq Q(D).
\]
On the other hand, given $\varepsilon > 0$ let (by density) $\overline{v} \in C^{0}_0(\mathbb{R}^N)$ such that

$$\frac{\|\overline{v}\|^2}{(\int_{\mathbb{R}^N} |f(\overline{v})|^{2^{*}})^{1/2^{*}}} \leq Q(\mathbb{R}^N) + \varepsilon.$$ 

After applying the conformal scaling and eventually a translation (assume $0 \in D$) for large $R$ we have $\overline{v}_R \in H^1_0(D)$. By (2.5) in Remark 1, we infer

$$Q(D) \leq \frac{\|\overline{v}_R\|^2}{(\int_{D} |f(\overline{v}_R)|^{2^{*}})^{1/2^{*}}} = \frac{\|\overline{v}\|^2}{(\int_{\Omega} |f(\overline{v})|^{2^{*}})^{1/2^{*}}} \leq \frac{\|\overline{v}\|^2}{(\int_{\Omega} |f(\overline{v})|^{2^{*}})^{1/2^{*}}}$$

and the conclusion follows. The proof for $\overline{Q}$ is similar. \end{proof}

To obtain some bounds for $m_\ast$, we first observe that, given $A, B > 0$, then

$$\max_{t > 0} \left\{ \frac{t^2}{2} A - \frac{t^{2^{*}}}{2^{*}} B \right\} = \frac{1}{N} \left( A \frac{N}{B^{(N-2)/2}} \right)^{N/2},$$ \hspace{1cm} (3.7)

$$\max_{t > 0} \left\{ \frac{t^2}{2} A - \frac{t^{2^{*}}}{2^{*}} B \right\} = \frac{N + 2}{4N} A^{2N/(N+2)}.$$ \hspace{1cm} (3.8)

Then we have:

**Theorem 3.** The value $m_\ast$ does not depend on the domain. Moreover,

$$\frac{1}{N} S_{N/2} \leq m_\ast \leq \max \left\{ \frac{N + 2}{4N} Q^{2N/(N+2)}, \frac{2^{(N-2)/2}}{N} \right\}.$$ 

**Proof.** Let $D$ be a domain in $\mathbb{R}^N$ and denote with $m_\ast(D)$ and $m_\ast(\mathbb{R}^N)$ the analogous quantities defined in (3.5) with respect to $D$ and $\mathbb{R}^N$. As in the proof of Lemma 5 we have that

$$m_\ast(\mathbb{R}^N) \leq m_\ast(D).$$

On the other hand, given $\varepsilon > 0$, let (by density) $\overline{v} \in C^{0}_0(\mathbb{R}^N)$ such that

$$\frac{1}{2} \|\overline{v}\|^2 - \frac{1}{2^{*}} \int_{\mathbb{R}^N} |f(\overline{v})|^{2^{*}} \leq m_\ast(\mathbb{R}^N) + \varepsilon.$$ 

Then defining the rescaled function $\overline{v}_R$ (eventually after having translated everything in 0 that we can assume in $D$), for $R$ large it is $\overline{v}_R \in H^1_0(D)$. Then by taking into account (2.5) in Remark 1, we get

$$m_\ast(D) \leq \frac{1}{2} \|\overline{v}_R\|^2 - \frac{1}{2^{*}} \int_{D} |f(\overline{v}_R)|^{2^{*}} \leq \frac{1}{2} \|\overline{v}\|^2 - \frac{1}{2^{*}} \int_{D} \frac{|f(\overline{v})|^{2^{*}}}{R^{N/2^{*}}} \leq m_\ast(\mathbb{R}^N) + \varepsilon$$

so that $m_\ast(D) = m_\ast(\mathbb{R}^N)$. Moreover, fixed $v \neq 0$, by (v) of Lemma 1 for every $t > 0$ we have

$$I_\ast(tv) = \frac{t^2}{2} \|v\|^2 - \frac{1}{2^{*}} \int_{\Omega} |f(tv)|^{2^{*}} \geq \frac{t^2}{2} \|v\|^2 - \frac{2^{1/4} t^{2^{*}}}{2^{*}} \int_{\Omega} \|v\|^2 \geq \frac{t^2}{2} \|v\|^2 - \frac{t^{2^{*}}}{2^{*}} \int_{\Omega} \|v\|^2.$$ 

The lower bound of $m_\ast$ then follows in a standard way by (3.7).

To obtain the upper bound for $m_\ast$, we fix $v \in H^1_0(\Omega) \setminus \{0\}$. We know by (iv) of Proposition 1 that $t \mapsto I_\ast(tv)$ achieves its unique maximum in $t_\ast = t_\ast(v) > 0$ so that

$$\max_{t > 0} I_\ast(tv) = \frac{t^2}{2} \|v\|^2 - \frac{1}{2^{*}} \int_{\Omega} |f(t_\ast v)|^{2^{*}}.$$ 

We have now two cases.
Case 1: \( t_* \in (0, 1) \). Then by (2.4) in Remark 1 and (3.8) we have
\[
\max_{t \geq 0} I_*(tv) \leq \frac{t_*^2}{2} \|v\|^2 - \frac{t_*^{22^*}}{22^*} \int_{\Omega} |f(v)|^{22^*}
\]
\[
\leq \max_{t \geq 0} \left[ \frac{t^2}{2} \|v\|^2 - \frac{t^{22^*}}{22^*} \int_{\Omega} |f(v)|^{22^*} \right]
\]
\[
= \frac{N + 2}{4N} \left( \frac{\|v\|^2}{\left( \int_{\Omega} |f(v)|^{22^*} \right)^{1/2^*}} \right)^{2N/(N+2)}.
\]

Case 2: \( t_* > 1 \). Then by (2.5) in Remark 1 and (3.7) we have
\[
\max_{t \geq 0} I_*(tv) \leq \frac{t^2}{2} \|v\|^2 - \frac{t^{22^*}}{22^*} \int_{\Omega} |f(v)|^{22^*}
\]
\[
\leq \max_{t \geq 0} \left[ \frac{t^2}{2} \|v\|^2 - \frac{t^{22^*}}{22^*} \int_{\Omega} |f(v)|^{22^*} \right]
\]
\[
= \frac{2^{(N-2)/2}}{N} \left( \frac{\|v\|^2}{\left( \int_{\Omega} |f(v)|^{22^*} \right)^{1/2^*}} \right)^{N/2}.
\]

We conclude that
\[
m_* \leq \max \left\{ \frac{N + 2}{4N} Q^{2N/(N+2)}, \frac{2^{(N-2)/2}}{N} Q^{N/2} \right\}
\]
and the theorem is proved. \( \Box \)

### 3.2 Behavior of the Family of Ground State Levels

The value \( m_* \) in (3.5) turns out to be an upper bound for the sequence of ground states levels \( \{m_p\}_{p \in (4, 22^*)} \) given in (3.3) as we will prove below.

First we need a lemma.

**Lemma 6.** Let \( w \in H^1_0(\Omega) \backslash \{0\} \) be fixed. For every \( p \in (4, 22^*) \) let \( t_p = t_p(w) > 0 \) given in (iv) of Lemma 1, i.e. such that \( t_p w \in N_p \). Then
\[
\lim_{p \to 22^*} t_p(w) = \bar{t} > 0 \quad \text{and} \quad \bar{t}w \in N_*.
\]

Moreover, if \( w \in N_* \), then \( \lim_{p \to 22^*} t_p(w) = 1 \).

**Proof.** By definition,
\[
t_p^2 \int_{\Omega} |\nabla w|^2 = \int_{\Omega} |f(t_p w)|^{p-2} f(t_p w) f'(t_p w) t_p w \quad \text{(3.9)}
\]
and then, by (2.3) and (v) of Lemma 1, we have
\[
t_p^2 \int_{\Omega} |\nabla w|^2 \leq \int_{\Omega} |f(t_p w)|^p \leq 2^{p/4} t_p^{p/2} \int_{\Omega} |w|^{p/2}.
\]

Then
\[
t_p^{(p-4)/2} \geq \frac{\|w\|^2}{2^{p/4} \int_{\Omega} |w|^{p/2}}
\]
from which it follows
\[
\liminf_{p \to 22^*} t_p \geq \left( \frac{\|w\|^2}{2^{2/4} |w|_{2^*}^{2/2}} \right)^{(1/(2^*-2))} > 0.
\]

Assume now that \( t_p \to +\infty \) as \( p \to 22^* \). Using again (2.3), by (3.9), we infer
\[
t_p^2 \|w\|^2 \geq \frac{1}{2} \int_{\Omega} |f(t_p w)|^p
\]
and then
\[
\|w\|^2 \geq \frac{1}{2} \int_{\{x \in \Omega : w(x) \neq 0\}} f(t_p |w|)^p \frac{t_p |w|^2}{t_p^p |w|^2} |w|^2 \\
= \frac{1}{2} \int_{\{x \in \Omega : w(x) \neq 0\}} f(t_p |w|)^p \frac{(t_p |w|)^{p/2} |w|^2}{t_p^p |w|^2} |w|^2 \\
= \frac{1}{2} \int_{\{x \in \Omega : w(x) \neq 0\}} \left( \frac{f(t_p |w|)}{\sqrt{t_p |w|}} \right)^p \frac{t_p (p-\delta)/2 |w|^{p/2}}{2^{2p-\delta} t_p^{p-\delta} |w|^{p/2}} \\
\geq \frac{1}{2} C t_p^{(p-\delta)/2} \int_{\Omega} |w|^{p/2} \to +\infty \quad \text{as} \quad p \to 2^*,
\]
where in the last inequality we have used item (v) of Lemma 2. This contradiction implies that \(\{t_p\}_{p \in (4, 2^{2^*})}\) has to be bounded. Then we can assume \(\lim_{p \to 2^{2^*}} t_p = \tilde{t} > 0\) and passing to the limit in (3.9), by the Dominated Convergence Theorem we get
\[
\tilde{t}^2 \|w\|^2 = \int_{\Omega} |f(\tilde{t} w)|^{2^{2^*}-2} f(\tilde{t} w)f'(\tilde{t} w)\tilde{t} w, \tag{3.10}
\]
i.e. \(\tilde{t} w \in \mathcal{N}_*\), proving the first part of the lemma.

In the case \(w \in \mathcal{N}_*\), by definition,
\[
\|w\|^2 = \int_{\Omega} |f(w)|^{2^{2^*}-2} f(w)f'(w)w,
\]
which joint with (3.10) gives
\[
\int_{\{x \in \Omega : w(x) \neq 0\}} \frac{|f(\tilde{t} w)|^{2^{2^*}-2} f(\tilde{t} w)f'(\tilde{t} w)}{\tilde{t} w} w^2 = \int_{\{x \in \Omega : w(x) \neq 0\}} \frac{|f(w)|^{2^{2^*}-2} f(w)f'(w)}{w} w^2.
\]
The conclusion now follows since, if \(w(x) \neq 0\), by item (iii) of Corollary 1 the map \(f(t)^{2^{2^*}-1} f(t) f'(t) t^{-1}\) is increasing for \(t > 0\).

Then we have the desired upper bound for the ground state levels \(\{m_p\}_{p \in (4, 2^{2^*})}\).

**Proposition 2.** We have
\[
\limsup_{p \to 2^{2^*}} m_p \leq m_*.
\]

**Proof.** Fix \(\epsilon > 0\). By the definition of \(m_*\), there exists \(\mathcal{V} \in \mathcal{N}_*\) such that
\[
I_* (\mathcal{V}) = \frac{1}{2} \|\mathcal{V}\|^2 - \frac{1}{2^{2^*}} \int_{\Omega} |f(\mathcal{V})|^{2^{2^*}} < m_* + \epsilon.
\]

For every \(p \in (4, 2^{2^*})\) there exists a unique \(t_p = t_p(\mathcal{V}) > 0\) such that \(t_p \mathcal{V} \in \mathcal{N}_p\) and by Lemma 6 we know that \(\lim_{p \to 2^{2^*}} t_p = 1\). Then
\[
m_p \leq I_p(t_p \mathcal{V}) = \frac{t_p^2}{2} \|\mathcal{V}\|^2 - \frac{t_p^p}{p} \int_{\Omega} |f(t_p \mathcal{V})|^p
\]
and so \(\limsup_{p \to 2^{2^*}} m_p \leq I_* (\mathcal{V}) < m_* + \epsilon\) concluding the proof. In particular, we deduce the following:

**Corollary 2.** The family of minimizers \(\{g_p\}_{p \in (4, 2^{2^*})}\) is bounded in \(H^1_0(\Omega)\).

**Proof.** Since as in (3.2)
\[
m_p = I_p(g_p) - \frac{2}{p} I'_p(g_p)[g_p] \geq \frac{p - q}{2p} \|g_p\|^2,
\]
the conclusion follows from Proposition 2.
On the other hand, using item (v) of Lemma 2, bounded, we infer by (3.12) that
\[ \{ \]
and this gives that Corollary 2 can be generalized to arbitrary functions in the Nehari manifolds \( N_p \), not necessary
the ground states, as long as the functionals converge.

More specifically, let \( p_n \to 2^* \) as \( n \to +\infty \). If \( \{ w_n \} \subset H_0^1(\Omega) \) is such that \( w_n \in N_p \), for every \( n \), and
\( I_p(w_n) \to I \in (0, +\infty) \) as \( n \to \infty \), then \( \{ w_n \} \) is bounded in \( H_0^1(\Omega) \).

Indeed, similarly to the proof of Corollary 2, this easily follows from
\[ l = I_p(w_n) - \frac{2}{p_n} I'_p(w_n)[w_n] + o_n(1) \geq \frac{p_n - 4}{2p_n} \| w_n \|^2 + o_n(1). \]

We need now a technical lemma about the “projections” of the minimizers \( g_p \) on the Nehari manifold of the
critical problem \( N_* \) (see Proposition 3). Let us first observe the following result which generalizes Lemma 6.

**Lemma 7.** If \( \{ w_p \}_{p \in (4, 2^*)} \subset H_0^1(\Omega) \) is such that
(a) for every \( p \in (4, 2^*) \), \( w_p \in N_p \),
(b) there exist \( C_1, C_2 > 0 \) such that for every \( p \in (4, 2^*) \),
\[ 0 < C_1 \leq \| w_p \| \quad \text{and} \quad \| w_p \| \leq C_2. \]

Then setting \( t_*(w_p) := t_{2^*}(w_p) > 0 \) such that \( t_*(w_p)w_p \in N_* \) (see (iv) of Lemma 1), it holds
\[ 0 < \liminf_{p \to 2^*} t_*(w_p) \leq \limsup_{p \to 2^*} t_*(w_p) < +\infty. \]  

**Proof.** Indeed, we can argue similarly as in Lemma 6 (where we had just a single function \( w \)). By definition, 
(2.3) and (v) of Lemma 1, we have
\[ t_*(w_p)^2 \| w_p \|^2 = \int_{\Omega} |f(t_*(w_p)w_p)|^{2^* - 2} f(t_*(w_p)w_p)t_*(w_p)w_p \]
\[ \leq \int_{\Omega} |f(t_*(w_p)w_p)|^{2^*} \]
\[ \leq 2^{2^*/2} t_*(w_p)^{2^*} \int_{\Omega} |w_p|^{2^*}. \]

obtaining
\[ t_*(w_p)^{2^* - 2} \geq \frac{\| w_p \|^2}{2^{2^*/2} \| w_p \|^{2^* - 2}}. \]

Since by assumption, as \( p \to 2^* \), \( \{ w_p \}_{p \in (4, 2^*)} \) does not tend to zero in \( H_0^1(\Omega) \) and \( \| w_p \|^{2^*} \) is bounded, we infer by (3.12) that \( \{ t_*(w_p) \}_{p \in (4, 2^*)} \) is bounded away from zero and then \( \liminf_{p \to 2^*} t_*(w_p) > 0 \). On the other hand, using item (v) of Lemma 2,
\[ C_2 \geq \| w_p \|^2 \geq \frac{1}{2} \int_{\{ x \in \Omega : w_p(x) \neq 0 \}} \frac{f(t_*(w_p)|w_p|)^{2^*}}{t_*(w_p)^{2^*}|w_p|^2} |w_p|^2 \]
\[ = \frac{1}{2} \int_{\{ x \in \Omega : w_p(x) \neq 0 \}} \frac{f(t_*(w_p)|w_p|)^{2^*}}{t_*(w_p)^{2^*}|w_p|^2} |w_p|^2 \frac{t_*(w_p)|w_p|^{2^*}}{|w_p|^2} \]
\[ = \frac{1}{2} \int_{\{ x \in \Omega : w_p(x) \neq 0 \}} \left( \frac{f(t_*(w_p)|w_p|)}{t_*(w_p)|w_p|} \right)^{2^*} t_*(w_p)^{2^* - 2} |w_p|^2 \]
\[ \geq \frac{1}{2} t_*(w_p)^{2^* - 2} \int_{\Omega} |w_p|^2 \]
\[ \geq C t_*(w_p)^{2^* - 2} \]
and this gives that \( \{ t_*(w_p) \}_{p \in (4, 2^*)} \) has also to be bounded above when \( p \to 2^* \), proving (3.11).
The next proposition deals with the projection of the minimizers $g_p$ on the Nehari manifold of the critical problem.

**Proposition 3.** Assume that $\{w_p\}_{p \in (4, 22^*)} \subset H^1_0(\Omega)$ is such that

(a) for every $p \in (4, 22^*)$, $w_p \in \mathcal{N}_p$,

(b) there exist $C_1, C_2 > 0$ such that for all $p \in (4, 22^*)$,

$$0 < C_1 \leq |w_p|_{2^*} \quad \text{and} \quad \|w_p\| \leq C_2,$$

(c) $w_p \geq 0$ for every $p \in (4, 22^*)$.

Let $t_*(w_p) > 0$ be the unique value such that $t_*(w_p)w_p \in \mathcal{N}_*$. Then

$$\lim_{p \to 22^*} t_*(w_p) = 1.$$

In particular,

$$\lim_{p \to 22^*} t_*(g_p) = 1.$$

**Proof.** We assume that $p_n \to 22^*$ as $n \to +\infty$ and $w_n := w_{p_n} \in \mathcal{N}_{p_n}$. In virtue of (c) it is $f(w_n), f'(w_n) \geq 0$. Moreover, by Lemma 7 we can assume that

$$\lim_{n \to +\infty} t_*(w_n) = t_0 > 0.$$

Let us begin by proving the following:

**Claim.** We have

$$A_n := \left| \int_{\Omega} (f(w_n)^{22^*-2} - f(w_n) + f'(w_n)) w_n - \int_{\Omega} (f(w_n)^p - f(w_n)) f'(w_n) w_n \right| = o_n(1).$$

Indeed, let us fix $\gamma \in (0, 1)$ and let $n_0 \in \mathbb{N}$ such that $22^* - p_{n_0} < \gamma$. We have

$$A_n \leq \int_{\Omega} |(f(w_n)^{22^*-2} - f(w_n)) f'(w_n) w_n|$$

$$= \int_{\{0 < f(w_n) < \gamma\}} |(f(w_n)^{22^*-2} - f(w_n)) f'(w_n) w_n|$$

$$+ \int_{\{\gamma < f(w_n) \leq 1\}} |(f(w_n)^{22^*-2} - f(w_n)) f'(w_n) w_n|$$

$$+ \int_{\{f(w_n) > 1\}} |(f(w_n)^{22^*-2} - f(w_n)) f'(w_n) w_n|$$

$$=: a_n + b_n + c_n.$$

Let us estimate $a_n, b_n, c_n$. Using (2.3), and being $p_n$ and $22^*$ greater than 4, we have

$$a_n \leq \int_{\{0 < f(w_n) < \gamma\}} f(w_n)^{22^*} + \int_{\{0 < f(w_n) < \gamma\}} f(w_n)^p \leq \gamma^{22^*} |\Omega| + \gamma^p |\Omega| \leq 2\gamma^p |\Omega|.$$

By the Mean Value Theorem, for some $\xi_n \in (p_n - 2, 22^* - 2)$ it is (again by (2.3))

$$b_n = \int_{\{\gamma < f(w_n) \leq 1\}} f(w_n)^{\xi_n}(p_n - 22^*) \ln(f(w_n)) f(w_n) f'(w_n) w_n$$

$$\leq (p_n - 22^*) \ln(\gamma) \int_{\{\gamma < f(w_n) \leq 1\}} f(w_n)^{\xi_n + 2}$$

$$\leq (p_n - 22^*) |\Omega| \ln(\gamma)$$

$$= o_n(1).$$
Finally,

\[ c_n = \int_{\{f(w_n) > 1\}} (f(w_n)22^*-2 - f(w_n)^{p_0-2})f(w_n)f'(w_n)w_n \]
\[ \leq \int_{\{f(w_n) > 1\}} (f(w_n)22^* - f(w_n)^{p_0}) \]
\[ \leq \int_{\{f(w_n) > 1\}} (f(w_n)22^* - f(w_n)^{p_0}), \]

where we used the fact that \( f(w_n)^{p_0} > f(w_n)^{p_0} \) for \( n > n_0 \). Using again the Mean Value Theorem, for some \( \xi_0 \in (p_{n_0}, 22^*) \), and the fact that for \( s > 1 \) it is \( \ln(s) \leq C s^{22^*-\xi_0} \), we have for every \( n > n_0 \),

\[ c_n \leq \int_{\{f(w_n) > 1\}} f(w_n)\xi_0 \ln(f(w_n))(22^* - p_{n_0}) \]
\[ \leq (22^* - p_{n_0})C \int_{\{f(w_n) > 1\}} f(w_n)^{22^*} \]
\[ \leq Cy, \]

where we used that \( \int_{\{f(w_n) > 1\}} f(w_n)^{22^*} \leq C \). Summing up, \( A_n \leq 2y^4|\Omega| + o_n(1) + Cy \) and then

\[ \limsup_{n \to +\infty} A_n \leq 2y^4|\Omega| + Cy \]

which proves the Claim.

Observe now that, by (2.3),

\[ 0 \leq \int_{\Omega} f(w_n)^{22^*-2}f(w_n)f'(w_n)w_n \leq \int_{\Omega} f(w_n)^{22^*} \leq C \int_{\Omega} w_n^{22^*} \leq C', \]

then up to subsequences we have

\[ \int_{\Omega} f(w_n)^{22^*-2}f(w_n)f'(w_n)w_n \to L \quad (3.13) \]

and then the above Claim gives

\[ \|w_n\|^2 = \int_{\Omega} f(w_n)^{p_0-2}f(w_n)f'(w_n)w_n \to L \geq 0. \quad (3.14) \]

Since \( w_n \rightharpoonup 0 \) in \( H^1_0(\Omega) \) (being the Nehari manifolds uniformly bounded away from zero, see Remark 2), it has to be \( L > 0 \).

Suppose that \( t_0 > 1 \); hence, for large \( n \), we have \( t_*(w_n) > 1 \). Since \( w_n \in N_{p_n} \) and \( t_*(w_n)w_n \in N_* \), it is, by (3.14) and (i) and (v) of Lemma 2,

\[ \|w_n\|^2 = \int_{\Omega} f(w_n)^{p_0-2}f(w_n)f'(w_n)w_n \to L, \]
\[ t_*(w_n)^2\|w_n\|^2 = \int_{\Omega} f(t_*(w_n)w_n)^{22^*-2}f(t_*(w_n)w_n)f'(t_*(w_n)w_n)t_*(w_n)w_n \]
\[ \geq t_*(w_n)^{22^*} \int_{\Omega} f(w_n)^{22^*-2}f(w_n)f'(w_n)w_n. \]

Passing to the limit above and using (3.13), we deduce

\[ t_0^2L \geq t_0^{22^*}L \]

and then \( t_0 \leq 1 \) which is a contradiction.
On the other hand, if \( t_0 < 1 \), we can assume that \( t_0 (w_n) < 1 \). Then as before,
\[
\|w_n\|^2 = \int_\Omega f(w_n)w_n \to L,
\]
\[
t_*(w_n)^2 \|w_n\|^2 = \int_\Omega f(t*(w_n)w_n)^{22^* - 2} f(t*(w_n)w_n)f'(t*(w_n)w_n)t*(w_n)w_n \leq t*(w_n)^{22^*} \int_\Omega f(w_n)^{22^* - 2} f(w_n)f'(w_n)w_n
\]
and passing to the limit, by (3.13), we deduce
\[
t_0^2 L \leq t_0^{22^*} L
\]
and then \( t_0 \geq 1 \) which is again a contradiction.

Finally, recalling that \( \|q_p\|_{\mathcal{P}} \) is bounded away from zero by the final part in Remark 3, and that \( \{q_p\}_{p \in [4,22^*)} \) are bounded in \( H_0^1(\Omega) \) by Corollary 2, we have that \( \{q_p\}_{p \in [4,22^*)} \) satisfy (b), and clearly (a) and (c). Then the conclusion follows.

Thanks to the previous result we get the lower bound.

**Proposition 4.** We have
\[
m_* \leq \lim \inf_{p \to 22^*} m_p.
\]

**Proof.** For \( p_n \to 22^* \), by Corollary 2 we have \( g_n := g_{p_n} \to v \) in \( H_0^1(\Omega) \) and consequently
\[
\int_\Omega |f(g_n)|^{p_n} \to \int_\Omega |f(v)|^{22^*}, \quad \int_\Omega |f(t*(g_n)g_n)|^{22^*} \to \int_\Omega |f(v)|^{22^*}.
\]
Furthermore, by Proposition 3 we have \( t_*(g_n) \to 1 \). Since by definition
\[
\frac{1}{2} \|g_n\|^2 = m_{p_n} + \frac{1}{p_n} \int_\Omega |f(g_n)|^{p_n},
\]
we get
\[
m_* \leq I_*(t_*(g_n)g_n)
\]
\[
= \frac{t_*(g_n)^2}{2} \|g_n\|^2 - \frac{1}{22^*} \int_\Omega |f(t*(g_n)g_n)|^{22^*}
\]
\[
= t_*(g_n)^2 m_{p_n} + \frac{t_*(g_n)^2}{p_n} \int_\Omega |f(g_n)|^{p_n} - \frac{1}{22^*} \int_\Omega |f(t*(g_n)g_n)|^{22^*}
\]
and passing to the limit, we deduce \( m_* \leq \lim \inf_{n \to +\infty} m_{p_n} \).

By Proposition 2 and Proposition 4 we deduce the desired result.

**Theorem 4.** For any bounded domain \( \Omega \), it holds
\[
\lim_{p \to 22^*} m_p = m_*.
\]

### 3.3 A Local Palais–Smale Condition for \( I \).

Recall the following two Brezis–Lieb-type splitting involving the function \( f \), available when \( w_n := v_n - v \to 0 \) in \( H_0^1(\Omega) \). These splitting will be useful in the next two results.

The first splitting we recall is
\[
\int_\Omega |f(v_n)|^{22^*} = \int_\Omega |f(v)|^{22^*} + \int_\Omega |f(w_n)|^{22^*} + o_n(1).
\]
See [3, equation (3.11)] and observe that the splitting also holds in the critical case \( p = 22^* \).
The second one is
\[
\int_{\mathbb{R}^d} \left| |f(w_n)|^{2s-2}f(w_n)f'(w_n)w_n - |f(v_n)|^{2s-2}f(v_n)f'(v_n)v_n + |f(v)|^{2s-2}f(v)f'(v)v |^a \right| = o_n(1),
\]
which holds for some \(a \in (2, 2^*)\), see [3, equation (3.14)]. However, in a bounded domain we can even allow \(a = 1\) and consequently we obtain
\[
\int_{\Omega} |f(w_n)|^{2s-2}f(w_n)f'(w_n)w_n = \int_{\Omega} |f(v_n)|^{2s-2}f(v_n)f'(v_n)v_n - |f(v)|^{2s-2}f(v)f'(v)v + o_n(1). \tag{3.16}
\]

Recall that \(S\) is the best Sobolev constant of the embedding \(H^1_0(\Omega)\) into \(L^{2^*}(\Omega)\).

**Lemma 8.** Let \(\{v_n\}\) be a (PS) sequence for the functional \(I_*\) at level \(d \in \mathbb{R}\). Then, up to subsequences,

1. \(v_n \rightharpoonup v\) in \(H^1_0(\Omega)\),
2. \(I_*'(v_n) = 0\), i.e. \(v\) is a solution of (3.4),
3. setting, \(w_n := v_n - v\), then
   \[
   I_*(v_n) = I_*(v) + I_*(w_n) + o_n(1) \quad \text{and} \quad I_*'(w_n) \to 0.
   \]

In particular, \(\{w_n\}\) is a (PS) sequence for \(I_*\) at level \(d - I_*(v)\).

**Proof.** If \(d \in \mathbb{R}\), \(I_*(v_n) \to d\) and \(I_*'(v_n) \to 0\), then
\[
I_*(v_n) - \frac{1}{2^*} I_*'(v_n)\|v_n\| \leq C(1 + \|v_n\|).
\]
On the other hand, by the computation above
\[
I_*(v_n) - \frac{1}{2^*} I_*'(v_n)\|v_n\| \geq \frac{2^* - 2}{2^*} \int_{\Omega} |\nabla v_n|^2
\]
and the boundedness of \(\{v_n\}\) follows. Then we can assume that \(v_n \to v\) in \(H^1_0(\Omega)\) with strong convergence \(L^s(\Omega), s \in [1, 2^*)\) and \(v_n \to v\) a.e. in \(\Omega\). Note now, using (ix) of Lemma 1, that
\[
\int_{\Omega} |f(v_n)|^{2s-2}f(v_n)f'(v_n)|^{2N/(N+2)} \leq C \int_{\Omega} |v_n|^{2N(2s-1)/(N+2)} = \int_{\Omega} |v_n|^{2^*} \leq C'.
\]
Then there exists some \(w \in L^{2N/(N+2)}(\Omega)\) such that, up to subsequence,
\[
|f(v_n)|^{2s-2}f(v_n)f'(v_n) \to w \quad \text{in} \quad L^{2N/(N+2)}(\Omega).
\]
But it is easy to see, due to the unicity of the weak limit, that
\[
|f(v_n)|^{2s-2}f(v_n)f'(v_n) = |f(v)|^{2s-2}f(v)f'(v) \quad \text{in} \quad L^{2N/(N+2)}(\Omega)
\]
(note that \(\frac{2N}{N+2} = (2^*)^*/2\)). This allows to conclude that, for every \(\varphi \in H^1_0(\Omega)\), \(I_*'(v_n)[\varphi] \to I_*(v)[\varphi]\) and then, since \(I_*'(v_n) \to 0\), we conclude that \(v\) is a critical point of \(I_*\).

Now, by the Brezis–Lieb splitting (3.15), we have
\[
I_*(v_n) = \frac{1}{2} \int_{\Omega} |\nabla w_n + \nabla v|^2 - \frac{1}{22^*} \int_{\Omega} |f(w_n + v)|^{22^*}
\]
\[
= \frac{1}{2} \int_{\Omega} |\nabla w_n|^2 + \int_{\Omega} \nabla w_n \nabla v + \frac{1}{2} \left( \int_{\Omega} |\nabla v|^2 - \frac{1}{22^*} \int_{\Omega} |f(v)|^{22^*} - \frac{1}{22^*} \int_{\Omega} |f(w_n)|^{22^*} + o_n(1) \right)
\]
\[
= I_*(v) + I_*(w_n) + o_n(1). \tag{3.17}
\]
Moreover, since \(w_n \to 0\) in \(H^1_0(\Omega)\) and
\[
|f(w_n)|^{2s-2}f(w_n)f'(w_n) \to 0 \quad \text{in} \quad H^{-1}(\Omega),
\]
we deduce that
\[
\|I_*'(w_n)\| = \sup_{\|\varphi\| = 1} \left| \int_{\Omega} \nabla w_n \nabla \varphi - \int_{\Omega} |f(w_n)|^{2s-2}f(w_n)f'(w_n)\varphi \right| \to 0
\]
concluding the proof.
Then we have the local (PS) condition for the functional $I_s$.

**Proposition 5.** The functional $I_s$ satisfies the (PS) condition at level $d \in \mathbb{R}$, for

$$d < \frac{1}{N} \left( \frac{N}{2} \right)^{N/2}.$$

**Proof.** Let $\{v_n\}$ be a (PS)$d$ sequence for $I_s$. We know that $v_n \rightharpoonup v$ in $H^1_0(\Omega)$, $I_s'(v) = 0$ and $I_s(v) \geq 0$.

By defining $w_n := v_n - v$, and using that $\int_{\Omega} |f(w_n)|^{2^*} \leq C \int_{\Omega} |\nabla w_n|^2$, we have

$$\int_{\Omega} |\nabla w_n|^2 \to A \geq 0, \quad \int_{\Omega} |f(w_n)|^{2^*} \to B \geq 0.$$  \hspace{1cm} (3.18)

All that we need to show is that $A = 0$.

By using the Brezis–Lieb splitting (3.16), we have

$$I_s'(w_n)[w_n] = I_s'[v_n][v_n] - I_s'[v][v] + o_n(1) = o_n(1),$$

which explicitly is

$$\int_{\Omega} |\nabla w_n|^2 - \int_{\Omega} |f(w_n)|^{2^*} f'(w_n) w_n = o_n(1).$$  \hspace{1cm} (3.19)

So, in virtue of (3.19), we deduce

$$\int_{\Omega} |f(w_n)|^{2^*} f'(w_n) w_n \to A.$$  \hspace{1cm} (3.20)

Then (3) of Lemma 8 and (3.17) imply

$$d = I_s(v) + \frac{A}{2} - \frac{B}{2^*} \geq \frac{A}{2} - \frac{B}{2^*}.$$  \hspace{1cm} (3.21)

By (vi) of Lemma 1 it holds

$$\frac{1}{2} \int_{\Omega} |f(w_n)|^{2^*} \leq \int_{\Omega} |f(w_n)|^{2^*} f'(w_n) w_n \leq \int_{\Omega} |f(w_n)|^{2^*}$$

so that, by (3.18) and (3.20),

$$\frac{1}{2} B \leq A \leq B.$$  \hspace{1cm} (3.22)

Then, coming back to (3.21), we infer

$$\frac{1}{N} A = \frac{A}{2} - \frac{B}{2^*} \leq d.$$  \hspace{1cm} (3.23)

Now, by the Sobolev inequality applied to $|f(w_n)|^2$ and (ix) of Lemma 1 we get

$$S \left( \int_{\Omega} |f(w_n)|^{2^*} \right)^{2/2^*} \leq \int_{\Omega} |\nabla (f(w_n))|^2 = \int_{\Omega} |2f(w_n)f'(w_n)\nabla w_n|^2 \leq 2 \int_{\Omega} |\nabla w_n|^2$$

and then, passing to the limit and making use of (3.22), we arrive at

$$\frac{S}{2} A^{2/2^*} \leq \frac{S}{2} b^{2/2^*} \leq A.$$

If it were $A > 0$, then we deduce

$$\left( \frac{S}{2} \right)^{N/2} \leq A.$$

But then using (3.23), we get

$$\frac{1}{N} \left( \frac{S}{2} \right)^{N/2} \leq \frac{1}{N} A \leq d < \frac{1}{N} \left( \frac{S}{2} \right)^{N/2}$$

and this contradiction implies that $A = 0$, concluding the proof. \qed
3.4 A Global Compactness Result

In order to prove our multiplicity results we need to deal with another “limit” functional, now related to the critical problem in the whole \( \mathbb{R}^N \).

Let us introduce the space \( D^{1,2}(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N) \} \), which can also be characterized as the closure of \( C_0^{\infty}(\mathbb{R}^N) \) with respect to the (squared) norm

\[
\| u \|_{D^{1,2}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\nabla u|^2.
\]

A function in \( H_0^1(\Omega) \) can be thought as an element of \( D^{1,2}(\mathbb{R}^N) \).

Let us define the functional

\[
\tilde{I}(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{1}{22^*} \int_{\mathbb{R}^N} |f(v)|^{22^*},
\]

whose critical points are the weak solutions of

\[
\begin{cases}
-\Delta u = |f(v)|^{22^*-2} f(v) f'(v), \\
v \in D^{1,2}(\mathbb{R}^N).
\end{cases}
\]

(3.24)

Setting as usual

\[
\overline{N} = \{ v \in D^{1,2}(\mathbb{R}^N) \setminus \{ 0 \} : \tilde{I}'(v)[v] = 0 \},
\]

all the solutions of (3.24) are in \( \overline{N} \); it is a differentiable manifold, is bounded away from zero, and

\[
\overline{m} := \inf_{v \in \overline{N}} \tilde{I}(v) > 0.
\]

The proof of these facts is exactly as in (i)--(iii) of Lemma 1. It worth noticing, for future references, that by Theorem 3 it is

\[
m_* = \overline{m}.
\]

As a matter of notation, in the rest of the paper given a function \( z \in D^{1,2}(\mathbb{R}^N) \), \( x \in \mathbb{R}^N \) and \( R > 0 \), we define the conformal rescaling \( z_{R,x} \) as

\[
z_{R,x}(z) := R^{N/2} z(R(x-x_0)).
\]

Of course, \( \| z \|_{D^{1,2}(\mathbb{R}^N)} = \| z_{R,x} \|_{D^{1,2}(\mathbb{R}^N)} \).

We need the following important lemma whose proof is omitted since it is like in [35, Lemma 3.2]. Note that the conclusion of item (e) simply follows by Proposition 5.

**Lemma 9.** Let \( \{ w_n \} \) be a (PS) sequence for \( I_* \) at level \( \beta \) such that \( w_n \rightharpoonup 0 \) in \( H_0^1(\Omega) \). Then there exist sequences \( \{ x_n \} \subset \Omega, \{ R_n \} \subset (0, +\infty) \) with \( R_n \to +\infty \), and a nontrivial solution \( \overline{v} \) of (3.24) such that, up to subsequences,

(a) \( \overline{w}_n := w_n - \overline{v}_{R_n,x_n} + o_n(1) \) is a (PS) sequence for \( I_* \) in \( H_0^1(\Omega) \),

(b) \( \overline{w}_n \rightharpoonup 0 \) in \( H_0^1(\Omega) \),

(c) \( I_*(\overline{w}_n) = I_*(w_n) - \tilde{I}(')(\overline{v}) + o_n(1) \),

(d) \( R_n d(x_n, \partial \Omega) \to +\infty \),

(e) if \( \beta < \beta^* := \frac{1}{N+2} \), then \( \{ w_n \} \) is relatively compact; in particular, \( w_n \to 0 \) in \( H_0^1(\Omega) \) and \( I_*(w_n) \to \beta = 0 \).

Now we can prove the following “splitting lemma”, which is useful to study the behavior of the (PS) sequences for the limit functional \( I_* \) related to the critical problem in the domain \( \Omega \).

In particular, it says that, if the (PS) sequences do not converge strongly to their weak limit, then this is due to the solutions of the problem in the whole \( \mathbb{R}^N \).

**Lemma 10 (Splitting).** Let \( \{ v_n \} \subset H_0^1(\Omega) \) be a (PS) sequence for the functional \( I_* \). Let \( v_n \rightharpoonup v_0 \) in \( H_0^1(\Omega) \), where \( v_0 \) is a weak solution of (3.4) (see Lemma 8). Then:

(I) either \( \{ v_n \} \) is strongly convergent in \( H_0^1(\Omega) \) to \( v_0 \),

(II) or there exist

(i) a number \( k \in \mathbb{N} \), \( k \geq 1 \), \( k \) sequences of points \( \{ x_j^k \} \subset \Omega \) and \( k \) sequences of radii \( \{ R_j^k \} \) with \( R_j^k \to +\infty \),

where \( j = 1, \ldots, k \),

(ii) nontrivial solutions \( \{ \psi_j \}_{j=1,\ldots,k} \subset D^{1,2}(\mathbb{R}^N) \) of problem (3.24)
such that, up to subsequences,
\[ v_n - v_0 = \sum_{j=1}^{k} w_j^i + o_n(1) \quad \text{in} \quad D^{1,2}(\mathbb{R}^N), \]
(3.25)
\[ I_* (v_n) = I_*(v_0) + \sum_{j=1}^{k} \mathcal{I}(v_j) + o_n(1). \]
(3.26)

**Proof.** As we already know (see Lemma 8), \(|v_n|\) is bounded and then (we can assume that) \(v_n \rightharpoonup v_0\) in \(H^1_0(\Omega)\), \(v_0\) is a weak solution of (3.4), \(|I_*(v_n)| \leq C\) and \(v_0\) can be seen as an element of \(D^{1,2}(\mathbb{R}^N)\). Assume that \(|v_n|\) does not converge strongly to \(v_0\).

Let \(w_1^0 := v_n - v_0 \rightharpoonup 0\). Then, by Lemma 8, \([w_1^0]\) is a (PS) sequence for \(I_*\) and
\[ I_*(v_n) = I_*(v_0) + I_*(w_1^0) + o_n(1). \]
(3.27)

By Lemma 9 applied to \([w_1^0]\), we get the existence of sequences \(\{x_n^1\} \subset \Omega, \{R_n^1\} \subset (0, +\infty)\) with \(R_n^1 \to +\infty\) and \(v^j \in D^{1,2}(\mathbb{R}^N)\) solution of (3.4) such that
\[ I_* (w^1_j) = I_*(w^1_j) - \mathcal{I}(v^j) + o_n(1), \]
(3.28)

Note that, by definitions, \(w^1_j = v_n - v_0 - v^j_{R_n^1, x_n^1} + o_n(1)\). Hence, if \([w^1_j]\) is strongly convergent to zero, the theorem is proved with \(k = 1\). Otherwise, in virtue of (1a) and (1b), we can apply Lemma 9 to the sequence \([w^1_j]\): then we get the existence of sequences \(\{x^1_n\} \subset \Omega, \{R^1_n\} \subset (0, +\infty)\) with \(R^1_n \to +\infty\) and \(v^j \in D^{1,2}(\mathbb{R}^N)\) solution of (3.4) such that
\[ I_* (w^1_j) = I_*(w^1_j) - \mathcal{I}(v^j) + o_n(1), \]
(3.29)

It is \(w^1_j = v_n - v_0 - v^j_{R_n^1, x_n^1} + v^2_{R_n^1, x_n^1} + o_n(1)\). If \([w^1_j]\) is strongly convergent to zero, the theorem is proved with \(k = 2\), otherwise we go on.

By arguing in this way, at the \((j - 1)\)st stage \((j > 1)\) we have \(w^{j-1}_n \rightharpoonup 0\) in \(H^1_0(\Omega)\) and we get the existence of sequences \(\{x^{j-1}_n\} \subset \Omega, \{R^{j-1}_n\} \subset (0, +\infty)\) with \(R^{j-1}_n \to +\infty\) and \(v^{j-1} \in D^{1,2}(\mathbb{R}^N)\) solution of (3.4) such that
\[ I_* (w^{j-1}_n) = I_*(w^{j-1}_n) - \mathcal{I}(v^{j-1}) + o_n(1), \]
(3.30)

As before it is
\[ w^j_n = v_n - v_0 - \sum_{i=1}^{j-1} v^i_{R_n^i, x_n^i}, \]
(3.30)
Recalling that \( I_*(\nu_0) \geq 0 \), the previous identity gives
\[
C \geq I_*(\nu_n) \geq I_*(w_n^1) + (j - 1)\tilde{m} + o_n(1).
\] (3.31)

On the other hand, since \( |w_n^1| \) is a bounded (PS) sequence for \( I_* \),
\[
I_*(w_n^1) = I_*(w_n^1) - \frac{1}{2^{*}_*} I'_*(w_n^1)[w_n^1] + o_n(1)
\]
\[
= \frac{2^{*}_* - 2}{2^{2*}_*} \int_{\Omega} |V w_n^1|^2 + \frac{1}{2} \int_{\Omega} |f(w_n^1)|^{2^{*}_*-2}f(w_n^1)w_n^1 - \frac{1}{2^{*}_*} \int_{\Omega} |f(w_n^1)|^{2^{*}_*} + o_n(1)
\]
\[
\geq \frac{2^{*}_* - 2}{2^{2*}_*} \int_{\Omega} |V w_n^1|^2 + o_n(1)
\]
\[
\geq o_n(1)
\]
so that, by (3.31), being \( \tilde{m} > 0 \), we deduce that the process has to finish after a finite number of steps, let us say at some index \( k \). This means, see (3.29), that
\[
w_n^{k+1} = \nu_n - \sum_{i=1}^{k} v_i^j \rightarrow 0,
\]
giving (3.25). Moreover, as in (3.30), it is
\[
I_*(\nu_n) = I_*(\nu_0) + I_*(w_n^{k+1}) + \sum_{i=1}^{k} I(v_i) + o_n(1)
\]
and we deduce (3.26), concluding the proof. \( \square \)

Let \( \{\nu_n\} \) be any (PS) sequence for \( I_* \) at level \( m_* \). If case (I) in Lemma 10 happens, then we have a solution \( \nu_0 \in H_0^1(\Omega) \) with \( I_*(\nu_0) = m_* > 0 \) so \( \nu_0 \) is not trivial and belongs to the Nehari manifold of the critical problem in \( \Omega \). Consider now a big ball \( B \) containing \( \Omega \) and extend \( \nu_0 \) by zero obtaining \( \bar{\nu}_0 \in H_0^1(B) \) which belongs to the Nehari manifold of the critical problem on \( B \) and is such that
\[
\frac{1}{2} \int_{B} |V\bar{\nu}_0|^2 - \frac{1}{2^{*}_*} \int_{B} |f(\bar{\nu}_0)|^{2^{*}_*} = \frac{1}{2} \int_{\Omega} |V\nu_0|^2 - \frac{1}{2^{*}_*} \int_{\Omega} |f(\nu_0)|^{2^{*}_*} = m_*.
\]

Then \( \bar{\nu}_0 \) is a nontrivial function attaining \( m_* \) (which does not depend on the domain) on the star shape domain \( B \), and this is absurd (see Remark 4). Hence by Lemma 10 we deduce that necessarily case (II) occurs. Moreover, \( k = 1 \) (since \( \inf_{\tilde{I}} \tilde{I} = \tilde{m} = m_* \)) and then, setting for simplicity \( V := \nu_1 \) (nontrivial solution of (3.24)), the identity
\[
m_* = I_*(\nu_0) + \tilde{I}(V)
\]
holds. Even more, the above relation implies that \( I_*(\nu_0) = 0 \) and so,
\[
\nu_0 = 0 \quad \text{and} \quad m_* = \tilde{I}(V).
\]

Therefore, we have the representation
\[
\nu_n = V_{R_n,x_n} + o_n(1) \quad \text{in} \quad D^{1,2}(\mathbb{R}^N).
\] (3.32)
As a byproduct we have the following result for the critical problem in \( \mathbb{R}^N \).

**Theorem 5.** Problem (3.24) admits a nontrivial ground state solution at level \( \tilde{m} (= m_*) \). Then the problem
\[
-\Delta u - \Delta(u^2)u = |u|^{2^{*}_*-2}u \quad \text{in} \quad \mathbb{R}^N
\]
admits a nontrivial ground state solution at level \( \tilde{m} (= m_*) \).

**Proof.** Indeed, the second statement simply follows by applying the change of variable \( f \). \( \square \)
4 The Barycenter Map

The aim of this section is to localize the barycenters of functions on $\mathcal{N}_p$ which are almost at the ground state level. Indeed, thanks to the results proved in the previous sections, we are able to show that, roughly speaking, the functions in the Nehari manifold $\mathcal{N}_p$ (at least for $p$ near the critical exponent $2^*$) which are almost at the ground state level $m_p$ (see (3.3)), have barycenter “near” $\Omega$. This is the main result of this section (see Proposition 6) and will be fundamental in the next Section in order to prove the multiplicity results for our problem.

We begin by introducing the barycenter map that will allow us to compare the topology of $\Omega$ with the topology of suitable sublevels of $I_p$, precisely sublevels with energy near the minimum level $m_p$.

For $u \in H^1(\mathbb{R}^N)$ with compact support, let us denote with the same symbol $u$ its trivial extension out of supp $u$. In particular, a function in $H^1_0(\Omega)$ can be thought also as an element of $D^{1,2}(\mathbb{R}^N)$.

The barycenter of $u$ (see [6]) is defined as

$$
\beta(u) = \frac{\int_{\mathbb{R}^N} x|\nabla u|^2}{\int_{\mathbb{R}^N} |\nabla u|^2} \in \mathbb{R}^N.
$$

From now on, we fix $r > 0$ a radius sufficiently small such that $B_r \subset \Omega$ and the sets

$$
\Omega^+_r = \{ x \in \mathbb{R}^3 : d(x, \Omega) \leq r \}, \quad \Omega^-_r = \{ x \in \Omega : d(x, \partial \Omega) \geq r \}
$$

are homotopically equivalent to $\Omega$. $B_r$ stands for the ball of radius $r > 0$ centered in 0. We denote by

$$
h : \Omega^+_r \rightarrow \Omega^-_r
$$

the homotopic equivalence map such that $h|_{\Omega^+_r}$ is the identity.

Now we have the following:

**Proposition 6.** There exists $\varepsilon > 0$ such that if $p \in (2^* - \varepsilon, 2^*)$, it follows

$$
v \in \mathcal{N}_p \text{ and } I_p(v) < m_p + \varepsilon \implies \beta(v) \in \Omega^+_r.
$$

**Proof.** We argue by contradiction. Assume that there exist sequences $\varepsilon_n \to 0$, $p_n \to 2^*$ and $w_n \in \mathcal{N}_{p_n}$ such that

$$
m_{p_n} \leq I_{p_n}(w_n) \leq m_{p_n} + \varepsilon_n \quad \text{and} \quad \beta(w_n) \notin \Omega^+_r.
$$

Then, by Theorem 4, we deduce

$$
I_{p_n}(w_n) \to m_*
$$

and then by Remark 5, $\{w_n\}$ is bounded in $H^1_0(\Omega)$. We can suppose that $w_n \rightharpoonup w$ in $H^1_0(\Omega)$. Since all the Nehari manifolds $\mathcal{N}_p$ are bounded away from zero (see Lemma 1 and Remark 2), we know that $w_n \nrightarrow 0$ in $H^1_0(\Omega)$ and then, by Remark 3, we deduce $|w_n|_2 \not\to 0$.

Since the functions $|f(t)|^{p-2} f(t)$ are even, it is $I_{p_n}^p(|v| = I_{p_n}(|v|)||v||)$; hence we can assume, without loss of generality, that $w_n \geq 0$.

Let $t_*(w_n) > 0$ such that $t_*(w_n) w_n \in \mathcal{N}_*$. By Proposition 3 we have $\lim_{n \to +\infty} t_*(w_n) = 1$.

The proof now consists of the following steps:

- **Step 1:** Prove that $\{t_*(w_n) w_n\} \subset \mathcal{N}_*$ is a minimizing sequence for $I_*$ on $\mathcal{N}_*$.
- **Step 2:** Use the Ekeland Variational Principle and write $t_*(w_n) w_n = V_{R_n, x_n} + z_n$, where $V_{R_n, x_n}$ satisfies (3.32) and $z_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$.
- **Step 3:** Compute the barycenter of $t_*(w_n) w_n$ by using the representation obtained in Step 2 and contradict (4.2), finishing the proof of the proposition.

**Step 1:** $\lim_{n \to +\infty} I_*(t_*(w_n) w_n) = m_*$. Observe that by the Hölder inequality, (ii) and (v) of Lemma 2 one has

$$
I_*(t_*(w_n) w_n) - I_{p_n}(w_n) = \frac{t_*(w_n)}{2} \|w_n\|^2 - \frac{1}{2} 2^{2^*} \int_{\Omega} f(t_*(w_n) w_n)^{2^{2^*}} - \frac{1}{2} 2^{2^*} \|w_n\|^2 + \frac{1}{p_n} \int_{\Omega} f(w_n)^{p_n}
$$

$$
\leq \frac{t_*(w_n)^2 - 1}{2} \|w_n\|^2 - \frac{\tau_n}{2^{2^*}} \int_{\Omega} f(w_n)^{2^{2^*}} + \frac{1}{p_n} |\Omega|^{2^{2^*}-2^n} \left( \int_{\Omega} f(w_n)^{2^{2^*}} \right)^{p_n/2^{2^*}},
$$
where $\tau_n := \max \{t_\ast(w_n)^{2^*}, t_\ast(w_n)^{2^{*\ast}}\}$. Then passing to the limit in $n$, by using that $\lim_{n \to +\infty} t_\ast(w_n) = 1$, that \{\text{w}_n\} is bounded and that $\int_\Omega f(w_n)^{2^{*\ast}} \to M > 0$, we infer $I_\ast(t_\ast(w_n)w_n) - I_{\beta_n}(w_n) \leq o_n(1)$. Then

$$0 < \beta_n \leq I_\ast(t_\ast(w_n)w_n) \leq I_{\beta_n}(w_n) + o_n(1)$$

and by (4.3) we conclude $I_\ast(t_\ast(w_n)w_n) \to \beta_n$ for $n \to +\infty$.

**Step 2:** Representation of the minimizing sequence $\{t_\ast(w_n)w_n\}$. Since $\{t_\ast(w_n)w_n\}$ is a minimizing sequence for $I_\ast$, the Ekeland’s Variational Principle implies that there exist $\{\nu_n\} \subset N_\ast$ and $\{\mu_n\} \subset \mathbb{R}$, a sequence of Lagrange multipliers, such that

$$\|t_\ast(w_n)w_n - \nu_n\| \to 0,$$

$$I_\ast(\nu_n) \to \beta_n,$$

$$I_\ast'(\nu_n) - \mu_n G_\ast'(\nu_n) \to 0,$$

and Lemma 3 ensures that $\{\nu_n\}$ is a (PS) sequence for the free functional $I_\ast$ on the whole space $H_0^1(\Omega)$ at level $\mu_n$. By the arguments after Lemma 10 applied to this sequence $\{\nu_n\}$, we have

$$v_n - V_{R_n,x_n} \to 0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^3),$$

where $\{x_n\} \subset \Omega$, $R_n \to +\infty$. Then we can write

$$v_n = V_{R_n,x_n} + z_n$$

with a remainder $z_n$ such that $\|z_n\|_{D^{1,2}(\mathbb{R}^3)} \to 0$. It is clear that

$$t_\ast(w_n)w_n = v_n + t_\ast(w_n)w_n - v_n = v_n + o_n(1);$$

so, renaming the remainder again $z_n$, we have

$$t_\ast(w_n)w_n = V_{R_n,x_n} + z_n.$$

**Step 3:** Computing the barycenter and finishing the proof. By using the representation obtained in Step 2, the $i$th coordinate of the barycenter of $t_\ast(w_n)w_n$ satisfies

$$\beta(w_n)^i \|t_\ast(w_n)w_n\|_{D^{1,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^N} x^i |\nabla V_{R_n,x_n}|^2 + \int_{\mathbb{R}^N} x^i |\nabla z_n|^2 + 2 \int_{\mathbb{R}^N} x^i \nabla V_{R_n,x_n} \nabla z_n, \quad (4.4)$$

where $x^i$ is the $i$th coordinate of $x \in \mathbb{R}^N$. In order to localize the barycenters we need to pass to the limit in each term in the above expression; however, at this stage, the computation of each term is completely analogous to the estimates made in [34, pp. 296–297]: it just involves changes of variables in the integrals. We just recall here the final results: it is

$$\|t_\ast(w_n)w_n\|_{D^{1,2}(\mathbb{R}^3)}^2 = \|v_n\|_{D^{1,2}(\mathbb{R}^3)}^2 + o_n(1),$$

$$\int_{\mathbb{R}^N} x^i |\nabla V_{R_n,x_n}|^2 = x^i_n \int_{\mathbb{R}^N} |\nabla v|^2,$$

$$\int_{\mathbb{R}^N} x^i |\nabla z_n|^2 = \int_{\mathbb{R}^N} x^i \nabla V_{R_n,x_n} \nabla z_n = o_n(1).$$

Then by (4.4) we find for the $i$th coordinate of the barycenter,

$$\beta(w_n)^i = \frac{x^i_n \int_{\mathbb{R}^N} |\nabla v|^2 + o_n(1)}{\|v_n\|_{D^{1,2}(\mathbb{R}^3)}^2 + o_n(1)}.$$

Since $\{x_n\} \subset \Omega$ the above equation implies that for large $n$ is $\beta(w_n) \in \overline{\Omega}$: this is in contrast with (4.2) and proves the proposition.\[\square\]
5 Proof of Theorem 1

Here we complete the proof of our theorem but first we need a slight modification to the previous notations. Let \( r > 0 \) be the one fixed at the beginning of Section 4, that is, in such a way that \( \Omega^+_r = \{ x \in \mathbb{R}^3 : d(x, \Omega) \leq r \} \) and \( \Omega^-_r = \{ x \in \Omega : d(x, \partial \Omega) \geq r \} \) are homotopically equivalent to \( \Omega \). We add a subscript \( r \) to denote the same quantities defined in the previous sections when the domain \( \Omega \) is replaced by \( B_r \); namely, integrals are taken on \( B_r \), and norms are taken for functional spaces defined on \( B_r \). Hence for example, for all \( p \in (4, 22^*) \) we set

\[
\mathcal{N}_{p,r} = \left\{ v \in H^1_0(B_r) : \| v \|^2_{H^1_0(B_r)} = \int_{B_r} |f(v)|^{p-2} f(v) f'(v) v' \right\},
\]

\[
I_{p,r}(v) = \frac{1}{2} \| v \|^2_{H^1_0(B_r)} - \frac{1}{p} \int_{B_r} |f(v)|^p,
\]

\[
m_{p,r} = \min_{v \in \mathcal{N}_{p,r}} I_{p,r}(v) = I_{p,r}(g_{p,r}).
\]

Observe that, by means of the Palais Symmetric Criticality Principle, the ground state \( g_{p,r} \) is radial and the ground state level \( m_{p,r} \) coincides with the radial ground state level \( m^\text{rad}_{p,r} \) (that is, the one taken on radial functions). Moreover, let

\[
I^m_{p,r} = \{ u \in \mathcal{N}_p : I_p(u) \leq m_{p,r} \},
\]

which is non-vacuous since \( m_p < m_{p,r} \).

Define also, for \( p \in (4, 22^*) \), the map \( \Psi_{p,r} : \Omega^-_r \to \mathcal{N}_p \) such that

\[
\Psi_{p,r}(y)(x) = \begin{cases} g_{p,r}(x - y) & \text{if } x \in B_r(y), \\ 0 & \text{if } x \in \Omega \setminus B_r(y), \end{cases}
\]

and note that we have

\[
\beta(\Psi_{p,r}(y)) = y \quad \text{and} \quad \Psi_{r,p}(y) \in I^m_{p,r}.
\]

Moreover, since \( m_p + k_p = m_{p,r} \), where \( k_p > 0 \) and tends to zero if \( p \to 22^* \) (see Theorem 3 and Theorem 4), in correspondence of \( \varepsilon > 0 \) provided by Proposition 6, there exists a \( p \in [4, 22^*) \) such that for every \( p \in [\hat{p}, 22^*) \) it results \( k_p < \varepsilon \); so if \( v \in I^m_{p,r} \), we have

\[
I_p(v) \leq m_{p,r} < m_p + \varepsilon,
\]

at least for \( p \) near \( 22^* \). Hence we can define the following maps:

\[
\Omega^-_r \xrightarrow{\Psi_{p,r}} I^m_{p,r} \xrightarrow{h \beta} \Omega^-_r
\]

with \( h \) given by (4.1). Since the composite map \( h \circ \beta \circ \Psi_{p,r} \) is homotopic to the identity of \( \Omega^-_r \), by a property of the category we have

\[
\text{cat}_{I^m_{p,r}}(I^m_{p,r}) \geq \text{cat}_{\Omega^-_r}(\Omega^-_r)
\]

and due to our choice of \( r \), it follows \( \text{cat}_{\Omega^-_r}(\Omega^-_r) = \text{cat}_{\Omega^-}(\Omega^-) \). Then we have found a sublevel of \( I_p \) on \( \mathcal{N}_p \) with category greater than \( \text{cat}_{\Omega^-}(\Omega^-) \) and since the (PS) condition is verified on \( \mathcal{N}_p \), the Lusternik–Schnirelmann theory guarantees the existence of at least \( \text{cat}_{\Omega^-}(\Omega^-) \) critical points for \( I_p \) on the manifold \( \mathcal{N}_p \) which give rise to solutions of (1.3).

The existence of another solution is obtained with the same arguments of Benci, Cerami and Passaseo [7]. We recall here the arguments for the reader’s convenience. Since by assumption \( \Omega \) is not contractible in itself, by the choice of \( r \) it results \( \text{cat}_{\Omega^-_r}(\Omega^-_r) > 1 \), namely \( \Omega^-_r \) is not contractible in \( \Omega^-_r \).

**Claim.** The set \( \Psi_{p,r}(\Omega^-_r) \) is not contractible in \( I^m_{p,r} \).
Indeed, let us assume by contradiction that \( \text{cat}_{I_p^M} (\Psi_{p,r}(\Omega^-)) = 1 \): this means that there exists a map \( \mathcal{H} \in C([0, 1] \times \Psi_{p,r}(\Omega^-); I_p^{m_{p,r}}) \) such that
\[
\mathcal{H}(0, u) = u \quad \text{for all } u \in \Psi_{p,r}(\Omega^-)
\]
and there exists \( w \in I_p^{m_{p,r}} \) such that
\[
\mathcal{H}(1, u) = w \forall u \in \Psi_{p,r}(\Omega^-).
\]
Then \( F = \Psi_{p,r}^{-1}(\Psi_{p,r}(\Omega^-)) \) is closed, contains \( \Omega^- \) and is contractible in \( \Omega^- \); since one can define the map
\[
\mathcal{G}(t, x) = \begin{cases} 
\beta(\Psi_{r,p}(x)) & \text{if } 0 \leq t \leq \frac{1}{2}, \\
\beta(\beta(2t - 1, \Psi_{p,r}(x))) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
Indeed, let us assume by contradiction that \( \text{cat}_{I_p^M} (\Omega^-) = 1 \): this means that there exists a map \( \mathcal{H} \in C([0, 1] \times \Psi_{p,r}(\Omega^-); I_p^{m_{p,r}}) \) such that
\[
\mathcal{H}(0, u) = u \quad \text{for all } u \in \Psi_{p,r}(\Omega^-)
\]
and there exists \( w \in I_p^{m_{p,r}} \) such that
\[
\mathcal{H}(1, u) = w \forall u \in \Psi_{p,r}(\Omega^-).
\]
Then also \( \Omega^- \) is contractible in \( \Omega^- \) and this gives a contradiction.

On the other hand we can choose a function \( z \in N_p \setminus \Psi_{p,r}(\Omega^-) \) so that the cone
\[
\mathcal{C} = \{ \theta z + (1 - \theta)u : u \in \Psi_{p,r}(\Omega^-), \theta \in [0, 1] \}
\]
is compact and contractible in \( H_0^1(\Omega) \) and \( 0 \notin \mathcal{C} \). For every \( v \neq 0 \) let \( t_p(v) \) be the unique positive number provided by (iv) in Lemma 1; it follows that if we set
\[
\mathcal{C} := \{ t_p(v)v : v \in \mathcal{C} \}, \quad M_p := \max_{\mathcal{C}} t_p,
\]
then \( \mathcal{C} \) is contractible in \( I_p^{M_p} \) and \( M_p > m_{p,r} \). As a consequence also \( \Psi_{p,r}(\Omega^-) \) is contractible in \( I_p^{M_p} \).

In conclusion the set \( \Psi_{p,r}(\Omega^-) \) is contractible in \( I_p^{M_p} \) and not in \( I_p^{m_{p,r}} \) and this is possible, since the (PS) condition holds, only if there is another critical point with critical level between \( m_{p,r} \) and \( M_p \).

It remains to prove that these solutions are positive. Note that we can apply all the previous machinery replacing the functional \( I_p \) with
\[
I_p^*(u) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \frac{1}{p} \int_\Omega |f(v^+)|^{p-2}f(v^+)f'(v^+)v^+,
\]
where \( v^+ = \max\{v, 0\} \). Then we obtain again at least \( \text{cat}_{\Omega}(\Omega^-) \) (or \( \text{cat}_{\Omega}(\Omega^-) + 1 \)) nontrivial solutions that now are positive by the Maximum Principle.

6 Proof of Theorem 2

Before prove the theorem, we first recall some basic facts of Morse theory and fix some notations.

For a pair of topological spaces \( (X, Y) \), \( Y \subset X \), let \( H_*(X, Y) \) be its singular homology with coefficients in some field \( \mathbb{F} \) (from now on omitted) and
\[
\mathcal{P}_*(X, Y) = \sum_k \dim H_k(X, Y)t^k
\]
the Poincaré polynomial of the pair. If \( Y = \emptyset \), it will be always omitted in the objects which involve the pair.

Recall that if \( H \) is an Hilbert space, \( I : H \to \mathbb{R} \) a \( C^2 \) functional and \( v \) an isolated critical point with \( I(v) = c \), the polynomial Morse index of \( v \) is
\[
\mathcal{J}_I(v) = \sum_k \dim \text{ker} I'(v)t^k,
\]
where \( \text{ker} I'(v) = H_k(I^* \cap U, (I^* \setminus \{v\}) \cap U) \) are the critical groups. Here \( I^* = \{ v \in H : I(v) \leq c \} \) and \( U \) is a neighborhood of the critical point \( u \). The multiplicity of \( v \) is the number \( \mathcal{J}_I(v) \).

It is known that for a non-degenerate critical point \( v \) (that is, the selfadjoint operator associated to \( I'(v) \) is an isomorphism) it is \( \mathcal{J}_I(v) = \mathcal{J}_I(v^*) \), where \( i(v) \) is the (numerical) Morse index of \( v \): the maximal dimension of the subspaces where \( I'(v)[\cdot, \cdot] \) is negative definite.
Coming back to our functional, it is straightforward to see that \( I_p \) is of class \( C^2 \) and for \( v, w, u \in H^1_0(\Omega) \),
\[
I''_p(v)[w, u] = \int_{\Omega} \nabla v \nabla u - (p - 1) \int_{\Omega} |f(v)|^{p-2}(f'(v))^2wu - \int_{\Omega} |f(v)|^{p-2}f(v)f''(v)wu.
\]
Hence \( I''_p(v) \) is represented by the operator
\[
L_p(v) := R(v) - K_p(v) : H^1_0(\Omega) \to H^{-1}(\Omega),
\]
where \( R(v) \) is the Riesz isomorphism
\[
R(v) : w \in H^1_0(\Omega) \mapsto R(v)[w] \in H^{-1}(\Omega)
\]
acting as
\[
(R(v)[w])[u] = \int_{\Omega} \nabla v \nabla u \quad \text{for all } u \in H^1_0(\Omega)
\]
and
\[
K_p(v) : w \in H^1_0(\Omega) \mapsto K_p(v)[w] \in H^{-1}(\Omega)
\]
acts as
\[
(K_p(v)[w])[u] = (p - 1) \int_{\Omega} |f(v)|^{p-2}(f'(v))^2wu - \int_{\Omega} |f(v)|^{p-2}f(v)f''(v)wu \quad \text{for all } u \in H^1_0(\Omega).
\]

**Lemma 11.** The operator \( K(v) \) is compact, that is, if \( w_n \to 0 \), then \( \|K(v)[w_n]\| \to 0 \) in \( H^{-1}(\Omega) \).

**Proof.** Indeed, for every \( u \in H^1_0(\Omega) \), by (ii) and (v) of Lemma 1,
\[
\left| \int_{\Omega} |f(v)|^{p-2}(f'(v))^2w_nu \right| \leq 2^{1/4} \int_{\Omega} |v|^{(p-2)/2}|w_nu|
\leq 2^{1/4} \|v|^{(p-2)/2}\|_{22^*/(p-2)} \|w_n\|_{22^*/p}\|u\|_{22^*/(22^*-(2p-2))}
\leq C|v|^{(p-2)/2}\|w_n\|_{22^*/p}\|u\| \to 0
\]
being
\[
\frac{p-2}{22^*} + \frac{p}{22^*} + \frac{22^* - (2p-2)}{22^*} = 1, \quad \frac{22^*}{p} \in (1, 2^*) \quad \text{and} \quad \frac{22^*}{22^* - (2p-2)} \in (1, 2^*).
\]
The second integral in \( K(v) \) can be reduced to the first one. Indeed, by using first that \( |f''(t)| = 2|f(t)||f'(t)|^4 \) (see the proof of (ii) of Corollary 1) and then (ix) of Lemma 1, we get
\[
\left| \int_{\Omega} |f(v)|^{p-2}f''(v)w_nu \right| \leq \int_{\Omega} |f(v)|^{p-2}f'(v)^2(f'(v))^4|w_nu|
\leq \frac{1}{2} \int_{\Omega} |f(v)|^{p-2}(f'(v))^2|w_nu|,
\]
concluding as before. Then we deduce that
\[
\|K_p(v)[w_n]\| = \sup_{\|u\|=1} |(K_p(v)[w_n])[u]| \to 0
\]
and the proof is completed. \( \square \)

Now for \( a \in (0, +\infty) \), let us define the sets
\[
I^a_p := \{ v \in H^1_0(\Omega) : I_p(v) \leq a \}, \quad N^a_p := N_p \cap I^a_p,
\]
\[
\mathcal{S}_p := \{ v \in H^1_0(\Omega) : I'_p(v) = 0 \}, \quad \mathcal{S}^a_p := \mathcal{S}_p \cap I^a_p, \quad (\mathcal{S}_p)_a := \{ v \in \mathcal{S}_p : I_p(v) > a \}.
\]
In the remaining part of this section we will follow [6]. Let \( \overline{p} \) as in Section 5 and let \( p \in [\overline{p}, 22^*) \) be fixed. In particular, \( I_p \) satisfies the Palais–Smale condition. We are going to prove that \( I_p \) restricted to \( N_p \) has at least \( 2\beta_1(\Omega) - 1 \) critical points.
We can assume, of course, that there exists a regular value $b_p^* > m_{p,r}$ for the functional $I_p$ and then

$$
\Psi_{p,r} : \Omega_p^r \rightarrow \mathbb{N}_{p,r}^\infty \subset \mathbb{N}_p^b.
$$

Since $\Psi_{p,r}$ is injective, it is easily seen that it induces injective homomorphisms between the homology groups. Then $\dim H_k(\Omega) = \dim H_k(\Omega^r) \leq \dim H_4(N_p^b)$ and consequently

$$
P_t(N_p^b) = P_t(\Omega) + \Omega(t), \quad \Omega \in \mathbb{P},
$$

where hereafter $\mathbb{P}$ denotes the set of polynomials with nonnegative integer coefficients.

The following result is analogous to [6, Lemma 5.2]; we omit the proof.

**Lemma 12.** Let $\bar{r} \in (0, m_{p,r})$ and $a \in (\bar{r}, \infty]$ a regular level for $I_p$. Then

$$
P_t(I_p^{b_p^*}, I_p^{b_p^*}) = tP_t(N_p^b).
$$

In particular, we have the following:

**Corollary 3.** Let $\bar{r} \in (0, m_{p,r})$. Then

$$
P_t(I_p^{b_p^*}, I_p^{b_p^*}) = t(\Psi_t(\Omega) + \Omega(t)), \quad \Omega \in \mathbb{P},
$$

$$
P_t(H_4^1(\Omega), I_p^{b_p^*}) = t.
$$

**Proof.** The first identity follows by (6.1) and (6.2) by choosing $a = b_p^*$. The second one follows by (6.2) with $a = \infty$ and noticing that the Nehari manifold $N_p$ is contractible in itself (see (v) in Lemma 1). \hfill \Box

To deal with critical points above the level $b_p^*$, we need also the following result whose proof is purely algebraic and is omitted. The interested reader may consult [6, Lemma 5.6].

**Lemma 13.** It holds

$$
P_t(H_4^1(\Omega), I_p^{b_p^*}) = t^2(\Psi_t(\Omega) + \Omega(t) - 1), \quad \Omega \in \mathbb{P}.
$$

As a consequence of these facts we have:

**Corollary 4.** Suppose that the set $S_p$ is discrete. Then

$$
\sum_{u \in S_p^{b_p^*}} J_t(u) = t^2(\Psi_t(\Omega) + \Omega(t) + (1 + t)\Omega_1(t)
$$

and

$$
\sum_{u \in (S_p^{b_p^*})_2} J_t(u) = t^2(\Psi_t(\Omega) + \Omega(t) + (1 + t)\Omega_2(t),
$$

where $\Omega, \Omega_1, \Omega_2 \in \mathbb{P}$.

**Proof.** Indeed, the Morse Theory gives

$$
\sum_{u \in S_p^{b_p^*}} J_t(u) = \Psi_t(I_p^{b_p^*}, I_p^{b_p^*}) + (1 + t)\Omega_1(t)
$$

and

$$
\sum_{u \in (S_p^{b_p^*})_2} J_t(u) = \Psi_t(H_4^1(\Omega), I_p^{b_p^*}) + (1 + t)\Omega_2(t)
$$

so that, by using Corollary 3 and Lemma 13, we easily conclude. \hfill \Box

Finally, by Corollary 4 we get

$$
\sum_{u \in S_p} J_t(u) = t\Psi_t(\Omega) + t^2(\Psi_t(\Omega) - 1) + t(1 + t)\Omega(t)
$$

for some $\Omega \in \mathbb{P}$. We easily deduce that, if the critical points of $I_p$ are non-degenerate, then they are at least $2\Psi_1(\Omega) - 1$, if counted with their multiplicity.

The proof of Theorem 2 is thereby complete.
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