Computing Constrained Approximate Equilibria in Polymatrix Games

Argyrios Deligkas\(^1\), John Fearnley\(^2\), and Rahul Savani\(^2\)

\(^1\) Technion, Israel
\(^2\) University of Liverpool, UK

Abstract. This paper is about computing constrained approximate Nash equilibria in polymatrix games, which are succinctly represented many-player games defined by an interaction graph between the players. In a recent breakthrough, Rubinstein showed that there exists a small constant \(\epsilon\), such that it is PPAD-complete to find an (unconstrained) \(\epsilon\)-Nash equilibrium of a polymatrix game. In the first part of the paper, we show that it is \(\text{NP}\)-hard to decide if a polymatrix game has a constrained approximate equilibrium for 9 natural constraints and any non-trivial approximation guarantee. These results hold even for planar bipartite polymatrix games with degree 3 and at most 7 strategies per player, and all non-trivial approximation guarantees. These results stand in contrast to similar results for bimatrix games, which obviously need a non-constant number of actions, and which rely on stronger complexity-theoretic conjectures such as the exponential time hypothesis. In the second part, we provide a deterministic QPTAS for interaction graphs with bounded treewidth and with logarithmically many actions per player that can compute constrained approximate equilibria for a wide family of constraints that cover many of the constraints dealt with in the first part.

1 Introduction

In this paper we study polymatrix games, which provide a succinct representation of a many-player game. In these games, each player is a vertex in a graph, and each edge of the graph is a bimatrix game. Every player chooses a single strategy and plays it in all of the bimatrix games that he is involved in, and his payoff is the sum of the payoffs that he obtains from each individual edge game.

A fundamental problem in algorithmic game theory is to design efficient algorithms for computing Nash equilibria. Unfortunately, even in bimatrix games, this is PPAD-complete [10,15], which probably rules out efficient algorithms. Thus, attention has shifted to approximate equilibria. There are two natural notions of an approximate equilibrium. An \(\epsilon\)-Nash equilibrium (\(\epsilon\)-NE) requires that each player has an expected payoff that is within \(\epsilon\) of their best response payoff. An \(\epsilon\)-well-supported Nash equilibrium (\(\epsilon\)-WSNE) requires that all players only play pure strategies whose payoff is within \(\epsilon\) of the best response payoff.

Constrained approximate equilibria. Sometimes, it is not enough to find an approximate NE, but instead we want to find one that satisfies certain constraints, such as having high social welfare. For bimatrix games, the algorithm
of Lipton, Markakis, and Mehta (henceforth LMM) can be adapted to provide a quasi-polynomial time approximation scheme (QPTAS) for this task [27]: we can find in $m^{O\left(\frac{\ln m}{\epsilon^2}\right)}$ time an $\epsilon$-NE whose social welfare is at least as good as any $\epsilon'$-NE where $\epsilon' < \epsilon$.

A sequence of papers [1, 9, 19, 24] has shown that polynomial time algorithms for finding $\epsilon$-NEs with good social welfare are unlikely to exist, subject to various hardness assumptions such as ETH. These hardness results carry over to a range of other properties, and apply for all $\epsilon < \frac{1}{8}$ [19].

Our contribution. We show that deciding whether there is an $\epsilon$-NE with good social welfare in a polymatrix game is $\mathsf{NP}$-complete for all $\epsilon \in [0, 1]$. We then study a variety of further constraints (Table 1). For each one, we show that deciding whether there is an $\epsilon$-WSNE that satisfies the constraint is $\mathsf{NP}$-complete for all $\epsilon \in (0, 1)$. Our results hold even when the game is a planar bipartite graph with degree at most 3, where each player has at most 7 actions.

To put these results into context, let us contrast them with the known lower bounds for bimatrix games, which also apply directly to polymatrix games. Those results [1, 9, 19, 24] imply that one cannot hope to find an algorithm that is better than a QPTAS for polymatrix games when $\epsilon < \frac{1}{8}$. In comparison, our results show a stronger $\mathsf{NP}$-hardness result, apply to all $\epsilon$ in the range $(0, 1)$, and hold even when the players have constantly many actions.

We then study the problem of computing constrained approximate equilibria in polymatrix games with restricted graphs. Although our hardness results apply to a broad class of graphs, bounded treewidth graphs do not fall within their scope. A recent result of Ortiz and Irfan [29] has provided a QPTAS for finding $\epsilon$-NEs in polymatrix games with bounded treewidth where every player has at most logarithmically many actions. We devise a dynamic programming algorithm for finding approximate equilibria in polymatrix games with bounded treewidth. Much like the algorithm in [29], we discretize both the strategy and payoff spaces, and obtain a complexity result that matches theirs. However, our algorithm works directly on the game, avoiding the reduction to a CSP used in their result.

The main benefit is that this algorithm can be adapted to provide a QPTAS for constrained approximate Nash equilibria. We introduce one variable decomposable (OVD) constraints, which are a broad class of optimization constraints, including many of the problems listed in Table 1. We show that our algorithm can be adapted to find good approximate equilibria relative to an OVD constraint. Initially, we do this for the restricted class of $k$-uniform strategies: we can find a $k$-uniform 1.5$\epsilon$-NE whose value is better than any $k$-uniform $\epsilon/4$-NE. Note that this is similar to the guarantee given by the LMM technique in bimatrix games. We extend this beyond the class of $k$-uniform strategies for constraints that are defined by a linear combination of the payoffs, such as social welfare. In this case, we find a 1.5$\epsilon$-NE whose value is within $O(\epsilon)$ of any $\epsilon/8$-NE.

Related work. Barman, Ligett and Piliouras [4] have provided a randomised QPTAS for polymatrix games played on trees. Their algorithm is also a dynamic programming algorithm that discretizes the strategy space using the notion of a $k$-uniform strategy. Their algorithm is a QPTAS for general polymatrix games.
on trees and when the number of pure strategies for every player is bounded by a constant they get an expected polynomial-time algorithm (EPTAS).

The work of Ortiz and Irfan [29] applies to a much wider class of games that they call graphical multi-hypermatrix games. They provide a QPTAS for the case where the interaction hypergraph has bounded hypertreewidth. This class includes polymatrix games that have bounded treewidth and logarithmically many actions per player. For the special cases of tree polymatrix games and tree graphical games they go further and provide explicit dynamic programming algorithms that work directly on the game, and avoid the need to solve a CSP.

Gilboa and Zemel [23] showed that it is \( \text{NP} \)-complete to decide whether there exist Nash equilibria in bimatrix games with some properties, such as high social welfare. Conitzer and Sandholm [11] extended the list of \( \text{NP} \)-complete problems of [23]. Recently, Garg et al. [22] and Bilo and Mavronicolas [5,6] extended these results to many-player games and provided analogous \( \text{ETR} \)-completeness results.

Computing approximate equilibria in bimatrix games has been well studied [8,12,16,17,21,26,31], but there has been less work for polymatrix games [3,13,20]. Rubinstein [30] has shown that there is a small constant \( \epsilon \) such that finding an \( \epsilon \)-NE of a polymatrix game is \( \text{PPAD} \)-complete. For constrained \( \epsilon \)-NE, the only positive results were for bimatrix games and gave algorithms for finding \( \epsilon \)-NE with constraints on payoffs [13,14].

2 Preliminaries

We start by fixing some notation. We use \([k]\) to denote the set of integers \( \{1, 2, \ldots, k\} \), and when a universe \([k]\) is clear, we will use \( \bar{S} = \{ i \in [k], i \notin S \} \) to denote the complement of \( S \subseteq [k] \). For a \( k \)-dimensional vector \( x \), we use \( x_{\bar{S}} \) to denote the elements of \( x \) with indices \( \bar{S} \), and in the case where \( \bar{S} = \{ i \} \) has only one element, we simply write \( x_i \) for \( x_{\bar{S}} \).

An \( n \)-player polymatrix game is defined by an undirected graph \( G = (V, E) \) with \( n \) vertices, where each vertex is a player. The edges of the graph specify which players interact with each other. For each \( i \in [n] \), we use \( N(i) = \{ j : (i, j) \in E \} \) to denote the neighbors of player \( i \). Each edge \( (i, j) \in E \) specifies a bimatrix game to be played between players \( i \) and \( j \). Each player \( i \in [n] \) has a fixed number of pure strategies \( m \), so the bimatrix game on edge \( (i, j) \in E \) is specified by an \( m \times m \) matrix \( A_{ij} \), which gives the payoffs for player \( i \), and an \( m \times m \) matrix \( A_{ji} \), which gives the payoffs for player \( j \). We allow the individual payoffs in each matrix to be an arbitrary rational number. We make the standard normalisation assumption that the maximum payoff each player can obtain under any strategy profile is 1 and the minimum is zero, unless specified otherwise. This can be achieved for example by using the procedure described in [20]. A subgame of a polymatrix game is obtained by restricting ignoring edges that are not contained within a given subgraph of the game’s interaction graph \( G \).

A mixed strategy for player \( i \) is a probability distribution over player \( i \)’s pure strategies. A strategy profile specifies a mixed strategy for every player. Given a strategy profile \( s = (s_1, \ldots, s_n) \), the pure strategy payoffs, or the payoff vector, of player \( i \) under \( s \), where only \( s_{-i} \) is relevant, is the sum of the pure strategy payoffs that he obtains in each of the bimatrix games that he plays. Formally, we define: \( p_i(s) := \sum_{j \in N(i)} A_{ij} s_j \). The expected payoff of player \( i \) under the strategy profile
| Problem description | Problem definition |
|---------------------|--------------------|
| Problem 1 | Large total payoff $u \in (0, n)$ Is there an $\epsilon$-NE $s$ such that $\sum_{i \in [n]} p_i(s) \geq u$? |
| Problem 2 | Small total payoff $u \in (0, n)$ Is there an $\epsilon$-WSNE $s$ such that $\sum_{i \in [n]} p_i(s) \leq u$? |
| Problem 3 | Small payoff $u \in [0, 1)$ Is there an $\epsilon$-WSNE $s$ such that $\min_i p_i(s) \leq u$? |
| Problem 4 | Restricted support $S \subset [n]$ Is there an $\epsilon$-WSNE $s$ with $\text{supp}(s_1) \subseteq S$? |
| Problem 5 | Two $\epsilon$-WSNE $d \in (0, 1]$ apart in Total Variation (TV) distance Are there two $\epsilon$-WSNE with TV distance $\geq d$? |
| Problem 6 | Small largest probability $p \in (0, 1)$ Is there an $\epsilon$-WSNE $s$ with $\max s_j(j) \leq p$? |
| Problem 7 | Large total support size $k \in [n \cdot m]$ Is there an $\epsilon$-WSNE $s$ such that $\sum_{i \in [n]} |\text{supp}(s_i)| \geq k$? |
| Problem 8 | Large smallest support size $k \in [n]$ Is there an $\epsilon$-WSNE $s$ such that $\min_i |\text{supp}(s_i)| \geq k$? |
| Problem 9 | Large support size $k \in [n]$ Is there an $\epsilon$-WSNE $s$ such that $|\text{supp}(s_1)| \geq k$? |

Table 1: The decision problems that we consider. All problems take as input an $n$-player polymatrix game with $m$ actions for each player and an $\epsilon \in [0, 1]$. 

$s$ is defined as $s_i \cdot p_i(s)$. The regret of player $i$ under $s$ is the difference between $i$’s best response payoff against $s_{-i}$ and between $i$’s payoff under $s$. If a strategy has regret $\epsilon$, we say that the strategy is an $\epsilon$-best response. A strategy profile $s$ is an $\epsilon$-Nash equilibrium, or $\epsilon$-NE, if no player can increase his utility more than $\epsilon$ by unilaterally from $s$, i.e., if the regret of every player is at most $\epsilon$. Formally, $s$ is an $\epsilon$-NE if for every player $i \in [n]$ it holds that $s_i \cdot p_i(s) \geq \max p_i(s) - \epsilon$. A strategy profile $s$ is an $\epsilon$-well-supported Nash equilibrium, or $\epsilon$-WSNE, if the regret of every pure strategy played with positive probability is at most $\epsilon$. Formally, $s$ is an $\epsilon$-WSNE if for every player $i \in [n]$ it holds that for all $j \in \text{supp}(s_i) = \{k \in [m] \mid s_{ik} > 0\}$ we have $(p_i(s))_j \geq \max p_i(s) - \epsilon$.

### 3 Decision problems for approximate equilibria

In this section, we show $\mathsf{NP}$-completeness for nine decision problems related to constrained approximate Nash equilibria in polymatrix games. Table 1 contains the list of the problems we study. For Problem 1, we show hardness for all $\epsilon \in [0, 1]$. For the remaining problems, we show hardness for all $\epsilon \in (0, 1)$, i.e., for all approximate equilibria except exact equilibria ($\epsilon = 0$), and trivial approximations ($\epsilon = 1$). All of these problems are contained in $\mathsf{NP}$ because a “Yes” instance can be witnessed by a suitable approximate equilibrium (or two in the case of Problem 5). The starting point for all of our hardness results is the $\mathsf{NP}$-complete problem Monotone 1-in-3 SAT.

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3 Given probability distributions $x$ and $x'$, the TV distance between them is $\max(|x_i - x'_i|)$. The TV distance between strategy profiles $s = (s_1, \ldots, s_n)$ and $s' = (s_1', \ldots, s_n')$ is the maximum TV distance of $s_i$ and $s'_i$ over all $i$. 
Definition 1 (Monotone 1-in-3 SAT). Given a monotone boolean CNF formula $\phi$ with exactly 3 distinct variables per clause, decide if there exists a satisfying assignment in which exactly one variable in each clause is true. We call such an assignment a 1-in-3 satisfying assignment.

Every formula $\phi$, with variables $V = \{x_1, \ldots, x_n\}$ and clauses $C = \{y_1, \ldots, y_m\}$, can be represented as a bipartite graph between $V$ and $C$, with an edge between $x_i$ and $y_j$ if and only if $x_i$ appears in clause $y_j$. We assume, without loss of generality, that this graph is connected. We say that $\phi$ is planar if the corresponding graph is planar. Recall that a graph is called cubic if the degree of every vertex is exactly three. We use the following result of Moore and Robson [28].

Theorem 2 (Section 3.1 [28]). Monotone 1-in-3 SAT is NP-complete even when the formula corresponds to a planar cubic graph.

From now on, we assume that $\phi$ is a monotone planar cubic formula. We say that $\phi$ is a "Yes" instance if $\phi$ admits a 1-in-3 satisfying assignment.

Problem 1 asks whether there exists an $\epsilon$-NE with high social welfare. We show that this is NP-complete for every $\epsilon \in [0,1]$, even when the interaction graph for the polymatrix game is planar, bipartite, cubic, and each player has at most three pure strategies.

Construction. Given a formula $\phi$, we construct a polymatrix game $G$ with $m + n$ players as follows. For each variable $x_i$ we create a player $v_i$ and for each clause $y_j$ we create a player $c_j$. We now use $V$ to denote the set of variable players and $C$ to denote the clause players. The interaction graph is the bipartite graph between $V$ and $C$ described above. Each edge game has the same structure. Every player in $V$ has two pure strategies called True and False, while every player in $C$ has three pure strategies that depend on the three variables in the clause. If clause $y_j$ contains variables $x_i, x_k, x_l$, then player $c_j$ has pure strategies $i, k$ and $l$. The game played between $v_i$ and $c_j$ is shown on the left in Figure 1. The bimatrix games for $v_k$ and $v_l$ are defined analogously.

Correctness. Observe that the constructed game is not normalised. We prove our result for every possible $\epsilon$, and thus in the normalised game the result will hold for every $\epsilon \in [0,1]$. Our goal is to show that, for every possible $\epsilon$, there is an $\epsilon$-NE with social welfare $m$ if and only if $\phi$ is a "Yes" instance. We begin by showing that if there is a solution for $\phi$, then there is an exact NE for $G$ with social welfare $m$, and therefore there is also an $\epsilon$-NE for every possible $\epsilon$ with social welfare $m$. We start with a simple observation about the maximum and minimum payoffs that players can obtain in $G$.

Lemma 3. In $G$, the total payoff for every variable player is at most 0, and the total payoff for every clause player $c_j$ is at most 1. Moreover, if $c_j$ gets payoff 1, then $c_j$ and the variable players connected to $c_j$ play pure strategies.

Lemma 4. If $\phi$ is a "Yes" instance, there is an NE for $G$ with social welfare $m$.

We now prove that if there is a strategy profile of $G$ with social welfare $m$ then $\phi$ is a "Yes" instance. Clearly, if this statement holds for any strategy profile, it also holds for all $\epsilon$-NE for any $\epsilon$. 
Lemma 5. If there is a strategy profile for $G$ with social welfare $m$, then $\phi$ is a "Yes" instance.

Together, Lemma 4 and Lemma 5 show that for all possible values of $\epsilon$, it is \textbf{NP}-complete to decide whether there exists an $\epsilon$-NE for $G$ with social welfare $m$. When we normalise the payoffs in $[0, 1]$, this holds for all $\epsilon \in [0, 1]$.

Theorem 6. Problem 1 is \textbf{NP}-complete for all $\epsilon \in [0, 1]$, even for degree-3 bipartite planar polymatrix games in which each player has at most 3 pure strategies.

Hardness of Problems 2–9. To show the hardness Problems 2–9, we modify the game constructed in the previous section. We use $G'$ to denote the new polymatrix game. The interaction graph for $G'$ is exactly the same as for the game $G$. The bimatrix games are extended by an extra pure strategy for each player, the strategy Out, and the payoffs are adapted. If variable $x_i$ is in clause $y_j$, then the bimatrix game between clause player $c_j$ and $v_i$ is shown on the right in Figure 1. To fix the constants, given $\epsilon \in (0, 1)$, we choose $c$ to be in the range $(\max(1 - \frac{3}{4} \epsilon, 0), 1)$, and we set $\kappa = \frac{1}{1 + 2c}$. Observe that $0 < c < 1$, and that $\kappa + 2c \cdot \kappa = 1 - \epsilon$. Furthermore, since $c > 1 - \frac{3}{4} \epsilon$ we have $0 < \kappa < \frac{1}{4}$.

In the next lemma we prove that if $\phi$ is a “Yes” instance, then the strategy profile in which all players play according to the assignment is an $\epsilon$-WSNE. No player uses the strategy Out in this strategy profile.

Lemma 7. If $\phi$ is a “Yes” instance, then $G'$ possesses an $\epsilon$-WSNE such that no player uses strategy Out.

On the other hand, we can prove that if $\phi$ is a “No” instance, then in every $\epsilon$-WSNE of $G'$ all players play the pure strategy Out.

Lemma 8. If $\phi$ is a “No” instance, then $G'$ possesses a unique $\epsilon$-WSNE where every player plays Out.
The combination of these two properties allows us to show that Problems 2–5 are \(\text{NP}\)-complete. For example, for Problem 4, we can ask whether there is an \(\epsilon\)-WSNE of the game in which player one does not play Out.

**Theorem 9.** Problems 2–5 are \(\text{NP}\)-complete for all \(\epsilon \in (0, 1)\), even on degree-3 planar bipartite polymatrix games where each player has at most 4 pure strategies.

**Duplicating strategies.** To show hardness for Problems 6–9, we slightly modify the game \(G'\) by duplicating every pure strategy except Out for all of the players. Since each player \(c_j \in C\) has the pure strategies \(i, k, l\) and Out, we give player \(c_j\) pure strategies \(i', k'\) and \(l'\), which each have identical payoffs as the original strategies. Similarly for each player \(v_i \in V\) we add the pure strategies True' and False'. Let us denote the game with the duplicated strategies by \(\tilde{G}\). Then, if \(\phi\) is a “Yes” instance, we can construct an \(\epsilon\)-WSNE in which no player plays Out, where each player places at most 0.5 probability on each pure strategy, and where each player uses a support of size 2. These properties are sufficient to show that Problems 6–9 are \(\text{NP}\)-complete.

**Theorem 10.** Problems 6–9 are \(\text{NP}\)-complete for all \(\epsilon \in (0, 1)\), even on degree-3 planar bipartite polymatrix games where each player has at most 7 pure strategies.

4 Constrained equilibria in bounded treewidth games

In this section we show that some constrained equilibrium problems can be solved in quasi-polynomial time if the input game has bounded treewidth and at most logarithmically many actions per player. We first present a dynamic programming algorithm for finding approximate Nash equilibria in these games, and then show that it can be modified to find constrained equilibria.

**Tree Decompositions.** A tree decomposition of a graph \(G = (V, E)\) is a pair \((\mathcal{X}, T)\), where \(T = (I, F)\) is a tree and \(\mathcal{X} = \{X_i \mid i \in I\}\) is a family of subsets of \(V\) such that (1) \(\bigcup_{i \in I} X_i = V\), (2) for every edge \((u, v) \in E\) there exists an \(i \in I\) such that \(\{u, v\} \subseteq X_i\), and (3) for all \(i, j, k \in I\) if \(j\) is on the path from \(i\) to \(k\) in \(T\), then \(X_i \cap X_k \subseteq X_j\). The width of a tree decomposition \((\mathcal{X}, T)\) is \(\max_i |X_i| - 1\). The treewidth of a graph is the minimum width over all possible tree decompositions of the graph. In general, computing the treewidth of a graph is \(\text{NP}\)-hard, but there are fixed parameter tractable algorithms for the problem. In particular Bodlaender [7] has given an algorithm that runs in \(O(f(w) \cdot n)\) time, where \(w\) is the treewidth of the graph, and \(n\) is the number of nodes.

4.1 An algorithm to find approximate Nash equilibria

Let \(G\) be a polymatrix game and let \((\mathcal{X}, T)\) be a tree decomposition of \(G\)'s interaction graph. We assume that an arbitrary node of \(T\) has been chosen as the root. Then, given some node \(v\) in \(T\), we define \(G(X_v)\) to be the subgame that is obtained when we only consider the players in the subtree of \(v\). More formally, this means that we only include players \(i\) that are contained in some set \(X_u\) where \(u\) is in the subtree of \(v\) in the tree decomposition. Furthermore, we will use \(\tilde{G}(X_v)\) to denote the players in \(G(X_v) \setminus X_v\). For every player \(i \in X_v\), we will use \(N_i(X_v)\) to denote the neighbours of \(i\) in \(G(X_v)\).
**k-uniform strategies.** A strategy \( s \) is said to be \( k \)-uniform if there exists a multi-set \( S \) of \( k \) pure strategies such that \( s \) plays a uniformly over the pure strategies in \( S \). These strategies naturally arise when we sample, with replacement, \( k \) pure strategies from a distribution, and play the sampled strategies uniformly. The following is a theorem of [2].

**Theorem 11.** Every \( n \)-player \( m \)-action game has a \( k \)-uniform \( \epsilon \)-NE whenever \( k \geq 8 \cdot \ln m + \ln n - \ln \epsilon + \ln 8 \).

**Candidates and witnesses.** For each node \( v \) in the tree decomposition, we compute a set of witnesses, where each witness corresponds to an \( \epsilon \)-NE in \( G(X_v) \). Our witnesses have two components: \( s \) provides a \( k \)-uniform strategy profile for the players in \( X_v \), while \( p \) contains information about the payoff that the players in \( X_v \) obtain from the players in \( G(X_v) \). By summarising the information about the players in \( G(X_v) \), we are able to keep the number of witnesses small.

There is one extra complication, however, which is that the number of possible payoff vectors that can be stored in \( p \) depends on the number of different strategies for the players in \( G(X_v) \), which is exponential, and will cause our dynamic programming table to be too large. To resolve this, we round the entries of \( p \) to a suitably small set of rounded payoffs.

Formally, we first define \( P = \{ x \in [0, 1] : x = \frac{\epsilon}{2^n} \cdot k \text{ for some } k \in \mathbb{N} \} \), to be the set of rounded payoffs. Then, given a node \( v \) in the tree decomposition, we say that a tuple \( (s, p) \) is a \( k \)-candidate if:

- \( s \) is a set of strategies of size \( |X_v| \), which gives one strategy for each player in \( X_v \).
- Every strategy in \( s \) is \( k \)-uniform.
- \( p \) is a set of payoff vectors of size \( |X_v| \). Each element \( p_i \in p \) is of the form \( P^n \), and assigns a rounded payoff to each pure strategy of player \( i \).

The set of candidates gives the set of possible entries that can appear in our dynamic programming table. Every witness is a candidate, but not every candidate is a witness. The total number of \( k \)-candidates for each tree decomposition node \( v \) can be derived as follows. Each player has \( m^k \) possible \( k \)-uniform strategies, and so there are \( m^k w \) possibilities for \( s \). We have that \( |P| = \frac{\epsilon}{2^n} \cdot k \), and that \( p \) contains \( m \cdot w \) elements of \( P \), so the total number of possibilities for \( p \) is \((2 \cdot \frac{\epsilon}{2^n})^m w \).

Hence, the total number of candidates for \( v \) is \( m^k w \cdot (2 \cdot \frac{\epsilon}{2^n})^m w \).

Next, we define what it means for a candidate to be a witness. We say that a \( k \)-candidate is an \( \epsilon, k, r \)-witness if there exists a profile \( s' \) for \( G(X_v) \) where:

- \( s' \) agrees with \( s \) for the players in \( X_v \).
- Every player in \( G(X_v) \) is \( \epsilon \)-happy, which means that no player in \( G(X_v) \) can increase their payoff by more than \( \epsilon \) by unilaterally deviating from \( s' \). Note that this does not apply to the players in \( X_v \).
- Each payoff vector \( p \in p \) is within \( r \) of the payoff that player \( i \) obtains from the players in \( G(X_v) \). More accurately, for every pure strategy \( l \) of player \( i \) we have that: \( \| p - \sum_{j \in G(X_v)} (A_{ij} \cdot s'_j) l \| \leq r \). Note that \( p \) does not capture the payoff obtained from players in \( X_v \), only those in the subtree of \( v \).
The algorithm. Our algorithm computes a set of witnesses for each tree decomposition node by dynamic programming. At every leaf, the algorithm checks every possible candidate to check whether it is a witness. At internal nodes in the tree decomposition, if a vertex is forgotten, that is, if it appears in a child of a node, but not in the node itself, then we use the set of witnesses computed for the child to check whether the forgotten node is \( \epsilon \)-happy. If this is the case, then we create a corresponding witness for the parent node. The complication here is that, since we use rounded payoff vectors, this check may declare that a player is \( \epsilon \)-happy erroneously due to rounding errors. So, during the analysis we must be careful to track the total amount of rounding error that can be introduced.

Once a set of witnesses has been computed for every tree decomposition node, a second phase is then used to find an \( \epsilon \)-NE of the game. This phase picks an arbitrary witness in the root node, and then unrolls it by walking down the tree decomposition and finding the witnesses that were used to generate it. These witnesses collectively assign a \( k \)-uniform strategy profile to each player, and this strategy profile will be the \( \epsilon \)-NE that we are looking for.

Due to space constraints, we do not give a full description of the algorithm here, but we give a complete specification in Appendix H. The following lemma summarises the key properties of the algorithm.

**Lemma 12.** There is a dynamic programming algorithm that runs in time \( O(n \cdot m^{2kw} \cdot (\frac{1}{\epsilon})^{2mw}) \) that, for each tree decomposition node \( v \), computes a set of candidates \( C(v) \) such that:

- Every candidate \((s, p) \in C(v)\) is an \( \epsilon_v \)-witness for \( v \) for some \( \epsilon_v \leq 1.5\epsilon \) and \( r_v \leq \frac{\epsilon}{4} \).
- If \( s \) is a \( k \)-uniform \( \epsilon/4 \)-NE then \( C(v) \) will contain a witness \((s', p)\) such that \( s' \) agrees with \( s \) for all players in \( X_v \).

The running time bound arises from the total number of possible candidates for each tree decomposition node. The first property ensures that the algorithm always produces a \( 1.5\epsilon \)-NE of the game, provided that the root node contains a witness. The second property ensures that the root node will contain a witness provided that game has a \( k \)-uniform \( \epsilon/4 \)-NE. Theorem 11 tells us how large \( k \) needs to be for this to be the case. These facts yield the following theorem.

**Theorem 13.** Let \( \epsilon > 0 \), \( G \) be a polymatrix game with treewidth \( w \), and \( k = 128 \cdot \frac{\ln m + \ln n - \ln \epsilon + \ln s}{\epsilon^2} \). There is an algorithm that finds a \( 1.5\epsilon \)-NE of \( G \) in time \( O(n \cdot m^{2kw} + (\frac{1}{\epsilon})^{2mw}) \). Note that if \( m \leq \ln n \) (and in particular if \( m \) is constant), this is a QPTAS.

**Corollary 14.** Let \( \epsilon > 0 \), and \( G \) be a polymatrix game with treewidth \( w \), and \( m \leq \ln n \). There is an algorithm that finds a \( 1.5\epsilon \)-NE of \( G \) in \( (\frac{1}{\epsilon})^{O(\frac{m}{\sqrt{\epsilon}})} \) time.

4.2 Constrained approximate Nash equilibria

**One variable decomposable constraints.** We now adapt the algorithm to find a certain class of constrained approximate Nash equilibria. As a motivating
example, consider Problem 1 which asks us to find an approximate NE with high social welfare. Formally, this constraint assigns a single rational number (the social welfare) to each strategy profile, and asks us to maximize this number. This constraint also satisfies a decomposability property: if a game $G$ consists of two subgames $G_1$ and $G_2$, and if there are no edges between these two subgames, then we can maximize social welfare in $G$ by maximizing social welfare in $G_1$ and $G_2$ independently. We formalise this by defining a constraint to be one variable decomposable (OVD) if the following conditions hold.

- There is a polynomial-time computable function $g$ such that maps every strategy profile in $G$ to a rational number.
- Let $s$ be a strategy for game $G$, and suppose that we want to add vertex $v$ to $G$. Let $s'$ be an extension of $s$ that assigns $s$ to $v$. There is a polynomial-time computable function add such that $g(s') = \text{add}(G, v, s, g(s))$.
- Let $G_1$ and $G_2$ be two subgames that partition $G$, and suppose that there are no edges between $G_1$ and $G_2$. Let $s_1$ be a strategy profile in $G_1$ and $s_2$ be a strategy profile in $G_2$. If $s$ is the strategy profile for $G$ that corresponds to merging $s_1$ and $s_2$, then there is a polynomial-time computable function merge such that $g(s) = \text{merge}(G_1, G_2, g(s_1), g(s_2))$.

Intuitively, the second condition allows us to add a new vertex to a subgame, and the third condition allows us to merge two disconnected subgames. Moreover, observe that the functions add and merge depend only on the value that $g$ assigns to strategies, and not the strategies themselves. This allows our algorithm to only store the value assigned by $g$, and forget the strategies themselves.

Examples of OVD constraints. Many of the problems in Table 1 are OVD constraints. Problems 1 and 2 refer to the total payoff of the strategy profile, and so $g$ is defined to be the total payoff of all players, while the functions add and merge simply add the total payoff of the two strategy profiles. Problems 5 and 6 both deal with minimizing a quantity associated with a strategy profile, so for these problems the functions add and merge use the min function to minimize the relevant quantities. Likewise, Problems 7, 8, and 9 seek to maximize a quantity, and so the functions add and merge use the max function. In all cases, proving the required properties for the functions is straightforward.

Finding OVD $k$-uniform constrained equilibria. We now show that, for every OVD constraint, the algorithm presented in Section 4.1 can be modified to find a $k$-uniform 1.5-$\epsilon$-NE that also has a high value with respect to the constraint. More formally, we show that the value assigned by $g$ to the 1.5-$\epsilon$-NE is greater than the value assigned to $g$ to all $k$-uniform $\epsilon/4$-NE in the game.

Given an OVD constraint defined by $g$, add, and merge, we add an extra element to each candidate to track the variable from the constraint: each candidate has the form $(s, p, x)$, where $s$ and $p$ are as before, and $x$ is a rational number. The definition of an $\epsilon, k, r, g$-witness is extended by adding the condition:

- Recall that $s'$ is a strategy profile for $G(X_v)$ whose existence is asserted by the witness. Let $s''$ be the restriction of $s'$ to $\tilde{G}(X_v)$. We have $x = g(s'')$. 
We then modify the algorithm to account for this new element in the witness. At each stage we track the value correct value for $x$. At the leaves, we use $g$ to compute the correct value. At internal nodes, we use add and merge to compute the correct value using the values stored in the witnesses of the children.

If at any point two witnesses are created that agree on the $s$ and $p$ components, but disagree on the $x$ component, then we only keep the witness whose $x$ value is higher. This ensures that we only keep witnesses corresponding to strategy profiles that maximize the constraint. When we reach the root, we choose the strategy profile with maximal value for $x$ to be unrolled in phase 2. The fact that we only keep one witness for each pair $s$ and $p$ means that the running time of the algorithm is unchanged. Again, we defer the technical description of the algorithm to Appendix I, but the following theorem summarises the results.

**Theorem 15.** For every $\epsilon > 0$ let $k = 128 \cdot \ln m + \ln n - \ln \epsilon + \ln \frac{8}{\epsilon}$. If $G$ is a polymatrix game with treewidth $w$, then there is an algorithm that runs in $O(n \cdot m^{2kw} + (\frac{m}{\epsilon})^{2m\epsilon})$ time and finds a k-uniform 1.5$\epsilon$-NE $s$ such that $g(s) \geq g(s')$ for every strategy profile $s'$ that is an $\epsilon/4$-NE.

**Results for non-k-uniform strategies.** The guarantee given by Theorem 15 is given relative to the best value achievable by a $k$-uniform $\epsilon/4$-NE. It is also interesting to ask whether we can drop the $k$-uniform constraint. In the following theorem, we show that if $g$ is defined to be a linear function of the payoffs in the game, then a guarantee can be given relative to every $\epsilon/8$-NE of the game. Note that this covers Problems 1, 2, and 3.

**Theorem 16.** Suppose that, for a given a OVD constraint, the function $g$ is a linear function of the payoffs. Let $s$ be the 1.5$\epsilon$-NE found by our algorithm when $G$ is a polymatrix game with treewidth $w$, then there is an algorithm that runs in $O(n \cdot m^{2kw} + (\frac{m}{\epsilon})^{2m\epsilon})$ time and finds a k-uniform 1.5$\epsilon$-NE $s$ such that $g(s) \geq g(s')$ for every strategy profile $s'$ that is an $\epsilon/4$-NE.

## 5 Further work

There are two clear directions for further work: Can we extend $\mathsf{NP}$-hardness of Problems 2–9 to $\epsilon$-NE? We believe hardness will not hold for all $\epsilon \in (0, 1)$ as for our results, but will hold for all $\epsilon$ less than a constant. Secondly, can we extend the family of constraints for which we can efficiently find constrained approximate equilibria that compare well with all other (non-$k$-uniform) approximate equilibria, beyond the constraints we can already deal with?

## References

1. P. Austrin, M. Braverman, and E. Chlamtac. Inapproximability of $\mathsf{NP}$-complete variants of Nash equilibrium. *Theory of Computing*, 9:117–142, 2013.
2. Y. Babichenko, S. Barman, and R. Peretz. Simple approximate equilibria in large games. In *EC*, pages 753–770, 2014.
3. S. Barman and K. Ligett. Finding any nontrivial coarse correlated equilibrium is hard. In *Proc. of EC*, pages 815–816, 2015.
4. S. Barman, K. Ligett, and G. Piliouras. Approximating Nash equilibria in tree polymatrix games. In *Proc. of SAGT*, pages 285–296, 2015.
5. V. Bilò and M. Mavronicolas. A catalog of $\exists\mathbb{R}$-complete decision problems about Nash equilibria in multi-player games. In *Proc. of STACS*, pages 17:1–17:13, 2016.
6. V. Bilò and M. Mavronicolas. 3-R-complete decision problems about symmetric Nash equilibria in symmetric multi-player games. In Proc. of STACS, pages 13:1–13:14, 2017.
7. H. L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. SIAM Journal on Computing, 25(6):1305–1317, 1996.
8. H. Bosse, J. Byrka, and E. Markakis. New algorithms for approximate Nash equilibria in bimatrix games. Theoretical Computer Science, 411(1):164–173, 2010.
9. M. Braverman, Y. Kun-Ko, and O. Weinstein. Approximating the best Nash equilibrium in n^[O(log n)]-time breaks the exponential time hypothesis. In Proc. of SODA, pages 970–982, 2015.
10. X. Chen, X. Deng, and S.-H. Teng. Settling the complexity of computing two-player Nash equilibria. Journal of the ACM, 56(3):14:1–14:57, 2009.
11. V. Conitzer and T. Sandholm. New complexity results about Nash equilibria. Games and Economic Behavior, 63(2):621 – 641, 2008.
12. A. Czumaj, A. Deligkas, M. Fasoulakis, J. Fearnley, M. Jurdziński, and R. Savani. Distributed methods for computing approximate equilibria. In Proc. of WINE, pages 15–28, 2016.
13. A. Czumaj, M. Fasoulakis, and M. Jurdziński. Approximate Nash equilibria with near optimal social welfare. In Proc. of IJCAI, pages 504–510, 2015.
14. A. Czumaj, M. Fasoulakis, and M. Jurdziński. Approximate plutocratic and egalitarian Nash equilibria. In Proc. of AAMAS, pages 1409–1410, 2016.
15. C. Daskalakis, P. W. Goldberg, and C. H. Papadimitriou. The complexity of computing a Nash equilibrium. SIAM Journal on Computing, 39(1):195–259, 2009.
16. C. Daskalakis, A. Mehta, and C. H. Papadimitriou. Progress in approximate Nash equilibria. In Proc. of EC, pages 355–358, 2007.
17. C. Daskalakis, A. Mehta, and C. H. Papadimitriou. A note on approximate Nash equilibria. Theoretical Computer Science, 410(17):1581–1588, 2009.
18. A. Deligkas, J. Fearnley, T. P. Igwe, and R. Savani. An empirical study on computing equilibria in polymatrix games. In Proc. of AAMAS, pages 186–195, 2016.
19. A. Deligkas, J. Fearnley, and R. Savani. Inapproximability results for approximate Nash equilibria. In Proc. of WINE, pages 29–43, 2016.
20. A. Deligkas, J. Fearnley, R. Savani, and P. G. Spirakis. Computing approximate Nash equilibria in polymatrix games. Algorithmica, 77(2):487–514, 2017.
21. J. Fearnley, P. W. Goldberg, R. Savani, and T. B. Sørensen. Approximate well-supported Nash equilibria below two-thirds. In SAGT, pages 108–119, 2012.
22. J. Garg, R. Mehta, V. V. Vazirani, and S. Yazdanbod. Etr-completeness for decision versions of multi-player (symmetric) Nash equilibria. In Proc. of ICALP, pages 554–566, 2015.
23. I. Gilboa and E. Zemel. Nash and correlated equilibria: Some complexity considerations. Games and Economic Behavior, 1(1):80 – 93, 1989.
24. E. Hazan and R. Krauthgamer. How hard is it to approximate the best Nash equilibrium? SIAM J. Comput., 40(1):79–91, 2011.
25. T. Kloks. Treewidth, Computations and Approximations, volume 842 of Lecture Notes in Computer Science. Springer, 1994.
26. S. C. Kontogiannis and P. G. Spirakis. Well supported approximate equilibria in bimatrix games. Algorithmica, 57(4):653–667, 2010.
27. R. J. Lipton, E. Markakis, and A. Mehta. Playing large games using simple strategies. In Proc. of EC, pages 36–41, 2003.
28. C. Moore and J. M. Robson. Hard tiling problems with simple tiles. Discrete & Computational Geometry, 26(4):573–590, 2001.
29. L. E. Ortiz and M. T. Irfan. FPTAS for mixed-strategy Nash equilibria in tree graphical games and their generalizations. *CoRR*, abs/1602.05237, 2016.

30. A. Rubinstein. Inapproximability of Nash equilibrium. In *STOC*, pages 409–418, 2015.

31. H. Tsaknakis and P. G. Spirakis. An optimization approach for approximate Nash equilibria. *Internet Mathematics*, 5(4):365–382, 2008.
A Proof of Lemma 3

Proof. That every variable player gets at most 0 in total follows from the fact that the largest payoff entry for the variable player in the bimatrix games is 0.

Now consider a clause player $c_j$ that interacts with variable players $i, k$ and $l$. Denote by $p_i, p_k$ and $p_l$ the probabilities that player $i, k$ and $l$ play the pure strategy True respectively. Pure strategy $i$ yields payoff $p_i - p_l - p_k$ for player $c_j$, $l$ yields payoff $-p_i + p_l - p_k$, and $k$ yields payoff $-p_i - p_l + p_k$. Since $p_i, p_k, p_l \in [0, 1]$, the maximum value for each expression is 1. Moreover, the maximum value of 1 is achieved only when exactly one of the variable players plays his pure strategy True with probability 1 and the other two players play their pure strategy False, and thus at most one of these expressions if 1, so the clause player must play a pure strategy to get 1 in total.

B Proof of Lemma 4

Proof. A 1-in-3 satisfying assignment for $\phi$ directly corresponds to a pure strategy profile for $G$: variable player $v_i \in V$ plays the truth value of $x_i$; clause player $c_j \in C$ plays the pure strategy that corresponds to the single true member variable of $y_j$. We call such a strategy profile $s$. We prove that $s$ is an NE for $G$ and that its social welfare is $m$.

First, we show that $s$ is an NE. Observe that each player in $C$ gets payoff 1 under $s$. Since, by Lemma 3, this is best payoff that a clause player can obtain, no player from $C$ has a reason to deviate. Now consider a variable player $v_i \in V$. If $v_i$ plays the pure strategy True, then he gets a total payoff of 0 irrespective of what the clause players do. On the other hand, if player $v_i$ plays the pure strategy False in $s$, then he gets payoff 0 from every bimatrix game he participates in because clause players only play strategy $i$ if $v_i$ plays True. Thus, in either case, player $v_i$ cannot gain by deviating, since, by Lemma 3, 0 is the largest payoff that $v_i$ can obtain.

Finally, observe that the social welfare of the strategy profile is $m$, because as we have argued, every clause player gets 1 and every variable player gets 0.

C Proof of Lemma 5

Proof. Suppose that there is a strategy profile $s$ with social welfare $m$. By Lemma 3 in order to achieve social welfare $m$, every clause player must get payoff 1, and this is only possible under pure strategy profiles. So, $s$ is a pure strategy profile, and naturally defines an assignment to the variables in $\phi$ according to the pure strategies played by the variable players. We argue that this is a satisfying 1-in-3 assignment of $\phi$. The reason is that, when clause player $c_j$ plays strategy $i$, variable player $v_i$ must pick True, and the other two players must pick False, because otherwise $c_j$ would not obtain payoff 1. Since this holds for all clauses, and every clause gets payoff 1, every clause must have exactly one true literal.

D Proof of Lemma 7

Proof. In exactly the same way as in the proof of Lemma 4 we interpret an 1-in-3 satisfying assignment for $\phi$ as a pure strategy profile $s$ for $G$. We must argue
that such an $s$ is an $\epsilon$-WSNE. Under $s$, every clause player gets payoff $\kappa + 2c \cdot \kappa$; he gets $\kappa$ from the game he plays with the player that plays the pure strategy True and $c \cdot \kappa$ from each of the two games he plays with the players that play the pure strategy False. The expected payoff from Out is 1 (because every clause player has degree exactly 3). Thus, every clause player plays a pure strategy that is a $(1 - \kappa - 2c \cdot \kappa)$-best response, which is an $\epsilon$-best response by the choice of $\kappa$ and $c$. Under $s$, every variable player $v_i$ gets payoff $1 - \epsilon$ since $v_i$ plays three bimatrix games with clause players that either all play $i$ if $v_i$ plays True or none of them play $i$ if $v_i$ plays False. The maximum payoff that $v_i$ could get is 1 from pure strategy Out. Thus, every variable player plays a pure strategy that is a $\epsilon$-best response. Hence, the constructed strategy profile is a $\epsilon$-WSNE. 

**E  Proof of Lemma 8**

**Proof.** Assume that $\phi$ is a “No” instance. We show that in this case in every $\epsilon$-WSNE there is at least one player who plays Out as a pure strategy. Towards a contradiction, assume that no clause player plays Out, and consider a clause player $c_j$, who is connected to $i$, $k$, and $l$. The maximum payoff that $c_j$ can get is $\kappa + 2c \cdot \kappa = 1 - \epsilon$, if and only if exactly one of $v_i$, $v_k$, and $v_l$ plays True and the other two play False. If every clause player could achieve payoff $1 - \epsilon$, we would then have an 1-in-3 satisfying assignment for $\phi$, which would be a contradiction. Thus at least one clause player, say $c_j$, gets payoff less than $1 - \epsilon$. However, if $c_j$ has payoff strictly less than $1 - \epsilon$ and does not play Out, then we do not have an $\epsilon$-WSNE, since Out always gives $c_j$ payoff 1.

Having established that in any $\epsilon$-WSNE there exists at least one player $c_j \in C$ that plays Out as a pure strategy, we now prove that all other players must also play Out as a pure strategy. There are two cases to consider.

- Let $v_i$ be a variable player, $c_j$ be a clause player who has an edge to $v_i$, and suppose that $c_j$ plays Out as a pure strategy. Observe that, if $v_i$ plays either True or False, his payoff can be at most $\frac{2}{3} - \frac{2 \epsilon}{3}$, whereas he can always obtain payoff 1 from playing Out, since at least one of his neighbours plays Out by assumption. So $v_i$ must play Out as a pure strategy in order to be in a $\epsilon$-WSNE.

- Let $v_i$ be a variable player, $c_j$ be a clause player who has an edge to $v_i$, and suppose that $v_i$ plays Out as a pure strategy. Observe that if this is the case, then $c_j$ cannot obtain payoff $\kappa + 2c \cdot \kappa = 1 - \epsilon$ from the strategies $i$, $j$, and $k$. On the other hand, $c_j$ obtains payoff 1 from Out under all strategies profiles, and so $c_j$ must play Out as a pure strategy in order to be in a $\epsilon$-WSNE.

By iteratively applying these two arguments we can prove that, since there is at least one player playing Out, all other players must also play Out. Hence, the strategy profile in which every player plays Out is the unique $\epsilon$-WSNE of the game. 

**F  Proof of Theorem 9**

**Proof.** From Lemmas 7 and 8 we can get the following two facts about $G'$. If $\phi$ is a “Yes” instance, then $G'$ possesses an $\epsilon$-WSNE such that:
(a) no player plays the pure strategy Out,
(b) every player gets payoff $1 - \epsilon$.

If $\phi$ is a “No” instance, then for every $\epsilon$-WSNE of $G'$ we have that:

(i) every player places all probability on the strategy Out,
(ii) every player gets payoff $1$.

We will use these properties to show that each of the problems is \( \text{NP} \)-complete.

Problem 2 is \( \text{NP} \)-complete because of (b) with (ii) when we set $u = (m+n) \cdot (1 - \epsilon)$, where $n$ and $m$ are the number of variables and clauses in $\phi$ respectively. This is because, if $\phi$ is a “Yes” instance, from (b) we get that there exists an $\epsilon$-WSNE where the sum of players payoffs sums up to $u$, while if $\phi$ is a “No” instance, then in the unique $\epsilon$-WSNE of the game the players’ payoffs sum up to $n + m$.

Problem 3 with $u = 1 - \epsilon$ is also \( \text{NP} \)-complete because of (b) and (ii).

Problem 4 is \( \text{NP} \)-complete because of (a) and (i). We focus on a variable player and set $S = \{\text{True}, \text{False}\}$. If $\phi$ is a “Yes” instance, then from (a) there is an $\epsilon$-WSNE where the variable player plays strategies only from $S$. If $\phi$ is a “No” instance, then from (i) in the unique $\epsilon$-WSNE the variable player must play Out.

Problem 5 is \( \text{NP} \)-complete because of (a) and (i) when we set $d = 1$. Observe that the strategy profile where every player plays his pure strategy Out is an exact NE, so it is an $\epsilon$-WSNE irrespective of whether $\phi$ is a “Yes” or “No” instance. So, from (a) we know that when $\phi$ is a “Yes” instance, there is another $\epsilon$-WSNE where no player plays his pure strategy Out, and there are two WSNEs with TV distance 1. On the other hand, if $\phi$ is a “No” instance, the “all Out” strategy profile is the unique $\epsilon$-WNSE.

\[\square\]

G Proof of Theorem 10

Proof. Lemma 7 holds also for the game $\tilde{G}$, since it differs from $G'$ only in the duplication of pure strategies other than Out. Extending the claim of Lemma 7 we have that if $\phi$ is a “Yes” instance, then there is an $\epsilon$-WSNE in $\tilde{G}$ in which no player places probability on Out, and in which the amount of probability on each pair of duplicate strategies is split evenly, i.e., probability one half on each, and $|\text{supp}(s_j)| = 2$ for every player $j$ of the game. If $\phi$ is a “No” instance, then for every $\epsilon$-WSNE of $\tilde{G}$ every player places all probability on the strategy Out, so $|\text{supp}(s_j)| = 1$ for every player $j$ of the game.

These properties then imply that our problems are \( \text{NP} \)-complete.

Problem 6 is \( \text{NP} \)-complete when we set $p = 1/2$.
Problem 7 is \( \text{NP} \)-complete when we set $k = 2(n + m)$, where $n$ and $m$ are the number of variables and clauses in $\phi$ respectively. If $\phi$ is a “Yes” instance, then there is an $\epsilon$-WSNE where the support sizes of the played strategies sum up to $k$. If on the other hand the instance does not have a solution, then in the unique $\epsilon$-WSNE of the game the support sizes sum up to $m + n < k$.

Problem 8 is \( \text{NP} \)-complete when we set $k = 2$.
Problem 9 is \( \text{NP} \)-complete when we set $k = 2$.

\[\square\]
Proof of Lemma 12

In this section, we fully describe the dynamic programming algorithm that we referred to in Section 4.1. The algorithm will proceed in two phases. Phase 1 will compute a set of candidates for each node in the tree decomposition, by starting at the leaves and working upwards. Phase 2 will then use these sets to compute an approximate Nash equilibrium of the game.

Nice tree decompositions. It has been shown that we can restrict ourselves to only considering nice tree decompositions [25]. A tree decomposition \((X, T)\) is nice if the following conditions are satisfied:

1. every node of \(T\) has at most two children,
2. if a node \(i\) has two children \(j\) and \(k\), then \(X_i = X_j = X_k\),
3. if a node \(i\) has one child \(j\), then
   - either \(|X_i| = |X_j| + 1\) and \(X_j \subseteq X_i\)
   - or \(|X_i| = |X_j| - 1\) and \(X_i \subseteq X_j\).

In a nice tree decomposition, \(T\) is a rooted binary tree, and each node \(i\) has one of the following types.

- **Start.** The node \(i\) is a leaf in \(T\).
- **Join.** The node \(i\) has exactly two children \(j\) and \(k\) in \(T\), and \(X_i = X_j = X_k\).
- **Introduce.** The node \(i\) has exactly one child \(j\) in \(T\) and \(|X_i| = |X_j| + 1\). That is, exactly one new vertex is added as we move from \(j\) to \(i\).
- **Forget.** The node \(i\) has exactly one child \(j\) in \(T\) and \(|X_i| = |X_j| - 1\). That is, exactly one vertex is removed as we move from \(j\) to \(i\).

It is a well known fact (see, e.g., [25]) that if we have a tree decomposition of width \(w\) for a given graph, then we can construct a nice tree decomposition of width \(w\) in polynomial time. Furthermore, if the original tree decomposition had \(n\) nodes, then the nice tree decomposition will have at most \(4n\) nodes. When we are working with nice tree decompositions, we can assume that if \(r\) is the root of the tree, then \(X_r = \emptyset\), since we can keep adding forget nodes to make this the case. We will assume from now on that our tree decomposition is nice.

**Phase 1.** The first phase of the algorithm will compute a set of \(k\)-candidates \(C(v)\) for each tree decomposition node \(v\). We will later prove that \(C(v)\) is in fact a set of witnesses. Since we work with nice tree decompositions, we only need to consider four types of nodes:

- **Start.** If \(v\) is a start node, then we produce \(C(v)\) by listing all possible \(k\)-uniform strategy profiles for the players in \(X_v\), and for each combination producing a candidate \((s, p)\) where \(s\) plays the strategies, and \(p\) assigns payoff 0 to every pure strategy. This can be done in \(O(m^{kw})\) time.

- **Introduce.** If \(v\) is an introduce node, then let \(u\) be the child of \(v\) in \(T\). We consider every \(k\)-uniform strategy \(s_i\) of player \(i\), and every \((s, p) \in C(u)\), and for each pair we produce \((s', p')\) where:
  - \(s'_i = s_i\), and \(s'_j = s_j\) for all \(j \neq i\),
  - \(p'_i\) is the 0 vector and \(p_j = p_j\) for \(j \neq i\),
and we add \((s', p')\) to \(C(v)\). This operation takes \(O(m^k \cdot |C(u)|)\) time.
Forget. If \( v \) is a forget node, then let \( u \) be the child of \( v \) in \( T \). Let \( i \) be the player who is removed by \( v \). For each \((s, p) \in C(u)\) we perform the following operation. First we check that player \( i \) is \( \varepsilon \)-happy, which is true if the following inequality holds:

\[
s_i \cdot \left( p_i + \sum_{j \in X_v \cap N(i)} A_{ij} \cdot s_j \right) \geq \max \left( p_i + \sum_{j \in X_v \cap N(i)} A_{ij} \cdot s_j \right) - \varepsilon.
\] (1)

This inequality simply checks whether player \( i \) is \( \varepsilon \)-happy, using the data about player \( i \)'s payoff stored in \( p \), and the strategies chosen by the other players in \( X_v \), using the data stored in \( s \).

If the test is passed, then we create \((s', p')\) as follows:

- For all \( j \neq i \) we set \( s'(j) = s(j) \).
- For all \( j \neq i \) we obtain \( p'(j) \) by computing \( p(j) + A_{ji} \cdot s_i \), and then rounding it to the closest element of \( P \).

This operation discards the strategy and payoff vector for player \( i \), and updates the payoff vectors for the players in \( X_v \) using the strategy played by player \( i \) in \( s \). We then add \((s', p')\) to \( C(v) \). This operation takes \( O(|C(u)|) \) time.

Join. Finally, if \( v \) is a join node, then let \( u_1 \) and \( u_2 \) be the two children of \( v \) in \( T \). We consider every pair of candidates \((s_{u_1}, p_{u_1}) \in C(u_1)\) and \((s_{u_2}, p_{u_2}) \in C(u_2)\). For each pair, we first check that \( s_{u_1} \) and \( s_{u_2} \) are the same, and if so we create \((s', p')\) as follows:

- \( s'(i) = s_{u_1}(i) = s_{u_2}(i) \) for all \( i \in X_v \).
- \( p'(i) = p_{u_1}(i) + p_{u_2}(i) \) for all \( i \in X_v \).

This operation copies the strategy profile, and adds the two payoff vectors. Note that no rounding is required, since \( p_{u_1}(i) \) and \( p_{u_2}(i) \) are both already rounded. This operation takes \( O(|C(u_1)| \cdot |C(u_2)|) \) time.

Running time. As we have argued, the total number of candidates for each tree decomposition node can be at most \( m^{kw} \cdot (\frac{4}{\varepsilon})^{m^{kw}} \). Since there are \( O(n) \) nodes in our tree decomposition, the overall running time of the algorithm is \( O(n \cdot m^{kw} \cdot (\frac{4}{\varepsilon})^{2m^{kw}}) \).

Correctness. We first prove that every candidate computed by the algorithm is a witness. For each node \( v \) in the tree decomposition, let \( f(v) \) be the total number of forget nodes over all paths from \( v \) to a leaf (including \( v \) itself if it is a forget node).

**Lemma 17.** For every node \( v \) in the tree decomposition, let \( r_v = \frac{\varepsilon f(v)}{4n} \) and \( \varepsilon_v = \varepsilon + 2 \cdot r_v \). Every candidate \((s, p) \in C(v)\) is an \( \varepsilon_v \)-\( k \)-\( r_v \)-witness if and only if \( s \) contains only \( k \)-uniform strategies, and \( p \) contains only k-uniform strategies. **Proof.** Our proof will be by induction over the nodes in the tree decomposition. For the base case, we consider the case where \( v \) is a start node. In this case, observe that the game \( G(X_v) \) is empty, so a \( k \)-candidate \((s, p) \) is an \( \varepsilon_v \)-\( k \)-\( r_v \)-witness if and only if \( s \) contains only \( k \)-uniform strategies, and \( p \) contains only k-uniform strategies.
0 vectors. Since $C(v)$ only contains candidates that satisfy these criteria, every member of $C(v)$ is an $\epsilon_v, k, r_v$-witness.

For the inductive step, there are three possibilities, depending on the type of $v$. In each of these cases, we assume, as inductive hypothesis, that we have proved that every member of $C(u)$ is an $\epsilon_u, k, r_u$-witness for all children $u$ of $v$.

**Introduce nodes.** Let $v$ be an introduce node, and let $(s, p) \in C(u)$ be a witness for $u$. We will show that the candidate $(s', p')$ created by our algorithm is an $\epsilon_u, k, r_u$-witness for $v$. Observe that, since $X_u$ is a separator in $G$, player $i$ has no edges to any player in $G(X_u)$. This means that the strategy played by player $i$ cannot affect whether any player in $G(X_u)$ is happy. Therefore, we can use the inductive hypothesis to show that every player in $G(X_u)$ is $\epsilon_u$-happy, and then observe that $\epsilon_u = \epsilon_v$. Moreover, player $i$ cannot obtain any payoff from the players in $G(X_u)$, and so setting $p_i$ to be the all-zero vector is correct. Thus, $(s', p')$ is an $\epsilon_v, k, r_v$-witness for $v$.

**Forget nodes.** We now consider the case where $v$ is a forget node. Let $(s, p)$ be a member of $C(u)$ that passes the test in Inequality 1. Let $s_f$ be the strategy profile for $G(X_u)$ in which every player is $\epsilon_u$-happy, whose existence is witnessed by $(s, p)$. We argue that if we add $s_i \in s$ to $s_f$, then we obtain a strategy profile for $G(X_v)$ in which every player is $\epsilon_v$ happy:

- Every player other than $i$ continues to be $\epsilon_u$-happy, because by definition $s_f$ agrees with $s_i$. Since $\epsilon_u < \epsilon_v$, these players are also $\epsilon_v$-happy.
- On the other hand, we must explicitly prove that player $i$ is $\epsilon_v$-happy. By the inductive hypothesis, the payoffs stored in $p$ are within $r_v$ of the true payoff to player $i$ under $s_f$. Thus, in the worst case, Inequality 1 ensures that:

$$s_i \cdot \left( \sum_{j \in G(X_v) \cap N(i)} A_{ij} \cdot s_j \right) + r_v \geq \max \left(p_i + \sum_{j \in G(X_v) \cap N(i)} A_{ij} \cdot s_j \right) - r_v - \epsilon.$$

This implies that player $i$ is $(\epsilon + 2r_v)$-happy, as required.

Next we argue that $p'$, the new payoff vector constructed by our algorithm, is correct. Consider a player $j \in X_v$. By the inductive hypothesis, $p_j$ gives the payoff to $j$ from the players in $G(X_u)$ with an additive error of $r_u$. We add $A_{ij} \cdot s_j$ to this vector, obtaining the payoff to $j$ from the players in $G(X_v)$ with an additive error of $r_u$. We then round to the closest element of $P$, which adds an additional error of at most $\epsilon/4n$. Since $r_v = r_u + \epsilon/4n$, this operation is correct. Thus, we have shown that $(s', p')$ is an $\epsilon_v, k, r_v$-witness for $v$.

**Join nodes.** Finally, we consider the case where $v$ is a join node. Let $(s_{u_1}, p_{u_1}) \in C(u_1)$ and $(s_{u_2}, p_{u_2}) \in C(u_2)$ be a pair of candidates. Observe that, since $X_v$ separates $G(X_{u_1})$ and $G(X_{u_2})$, no player in $G(X_{u_1})$ can influence a player in $G(X_{u_2})$, and vice versa. Therefore, when we merge these two witnesses in $(s', p')$, we do not affect whether any player in $G(X_v)$ is $\epsilon$-happy. Furthermore, when we add $p_{u_1}(i)$, which has an additive error of $r_{u_1}$, to $p_{u_2}(i)$, which has an additive
error of \( r_u \), then we obtain a vector that has an additive error of \( r_u + r_v \).

Since \( r_u \) depends on the total number of forget nodes in both subtrees of \( v \), the resulting payoff vector has an additive error of \( r_v \). So, we have shown that \((s', p')\) is an \( \epsilon_v, k, r_v \)-witness for \( v \). \(\square\)

In the other direction we must also show that the algorithm does not throw away too many witnesses. We do this in the following lemma.

**Lemma 18.** If \( s \) is a \( k \)-uniform \( \epsilon/4 \)-NE then \( C(v) \) will contain a witness \((s', p')\) such that \( s' \) agrees with \( s \) for the vertex in \( X_v \).

**Proof.** We will show, by induction, the stronger property that every tree node \( v \) will have a witness \((s, p)\) where:

- For every player \( i \in X_v \) we have \( s_i \in s \) is the same as the strategy assigned to player \( i \) by \( s_{\text{NE}} \).
- The payoffs in \( p \) are within \( r_v = \frac{\epsilon f(v)}{4n} \) of the true values given by \( t^v_{\text{NE}} = \sum_{j \in G(X_v)} A_{ij} \cdot s_j \).

For the base case, where \( v \) is a start node, observe that by assumption \( s \) only uses \( k \)-uniform strategies, and so \( C(v) \) will contain a candidate satisfying the required properties. For the inductive step, we again have three possibilities, depending on the type of \( v \).

If \( v \) is a introduce node, then let \((s, p)\) be the candidate for \( u \) whose existence is implied by the inductive hypothesis. Let \((s', p')\) be the witness that the algorithm creates using \((s, p)\) and \((s_{\text{NE}})\). Since \( X_u \) separates \( i \) from every player in \( \tilde{G}(X_v) \), we have that \( t^v_{\text{NE}} = t^v_{i} \) for all players \( i \neq j \), and \( t^v_{i} \) assigns the 0 vector to player \( i \). Since, by the inductive hypothesis, we have that \( p \) close to \( t^v_{i} \) with an additive error of at most \( r_u = r_v \), the witness \((s', p')\) created by our algorithm does indeed satisfy the required properties.

If \( v \) is a forget node, then let \((s, p)\) be the candidate for \( u \) whose existence is implied by the inductive hypothesis. We begin by arguing that \((s, p)\) is not discarded by the check from Inequality (1). Since \( s_{\text{NE}} \) is a \( \frac{\epsilon}{4} \)-NE we have:

\[
s_i \cdot \left( t^v_{i} + \sum_{j \in X_v \cap N(i)} A_{ij} \cdot s_j \right) \geq \max \left( t^v_{i} + \sum_{j \in X_v \cap N(i) \setminus s_j} A_{ij} \cdot s_j \right) = \frac{\epsilon}{4}.
\]

By the inductive hypothesis, we have that \( p \) is within \( t^v_{u} \) with an additive error of \( r_u \). Observe that, since there are at most \( n \) forget nodes in the tree decomposition, we have \( r_u \leq n \cdot \frac{\epsilon f(v)}{4n} \leq \frac{\epsilon}{4} \). Hence, we have:

\[
s_i \cdot \left( p_i + \sum_{j \in X_v \cap N(i)} A_{ij} \cdot s_j \right) \geq \max \left( p_i + \sum_{j \in X_v \cap N(i) \setminus s_j} A_{ij} \cdot s_j \right) = \frac{\epsilon}{4} - 2 \cdot \frac{\epsilon}{4}.
\]

Thus, Inequality (1) is satisfied and so \((s, p)\) will not be discarded.

We must still show that the vector \( p' \) constructed by the algorithm satisfies the requirements. Again, by the inductive hypothesis we have that \( p \) is within
$t^\text{NE}_u$ with an additive error of $r_u$. We also have that $(t^\text{NE}_u)_j = (t^\text{NE}_u)_j + A_{ji} \cdot s_i$ for all players $j$. Our algorithm does the same operation, but we must account for the extra rounding step, which can add an additional error of at most $\epsilon/4n$. Since $r_v = r_u + \epsilon/4n$, the resulting candidate satisfies the required properties.

If $v$ is a join node, then let $(s_{u1}, p_{u1})$ and $(s_{u2}, p_{u2})$ be the two candidates whose existence is implied by the inductive hypothesis. We argue that the candidate $(s', p')$ produced by the algorithm from these two candidates will satisfy the requirements. The first property is obviously true. For the property regarding $p'$, observe that we add a payoff vector with additive error $r_{u1}$ to a payoff vector with additive error $r_{u2}$, and so the resulting error will be $r_{u1} + r_{u2}$. Since $f(v) = f(u_1) + f(u_2)$, the required property is satisfied. □

Let $r$ be the root node of $T$. As we explained, we can assume that $X_r$ is the empty set, and so contains no players. Thus, $C(r)$ is either empty, or it contains a single witness (a pair of empty sets). The previous lemma implies that if the game contains a $k$-uniform $\epsilon/4$-NE, then $C(r)$ will not be empty, which will allow us to proceed to phase 2.

**Phase 2.** In phase 2, we assume that $C(r)$ contains a candidate, and we will use this candidate to find an approximate NE for the game. By Lemma 17 we know that if the $C(r)$ contains a candidate, then that candidate witnesses the existence of an $(\epsilon + 2r)$-NE, where $r_r = \frac{\epsilon \cdot |X_r|}{4n}$. Since $f(r) \leq n$ we have that there exists an $1.5\epsilon$-NE in the game, and the goal of phase 2 is to find one.

The algorithm is straightforward. It walks down all branches of the tree starting at the root. In each step, it has a candidate for the current vertex, and needs to find corresponding candidates for each child of that node. More precisely, if $v$ is a node and $(s, p)$ is the candidate that has been chosen for that node (where the sole element of $C(r)$ is chosen for the root initially), then the algorithm does the following:

- If $v$ is a join node with two children $u_1$ and $u_2$, then the algorithm finds the candidates $(s_{u1}, p_{u1}) \in C(u_1)$ and candidates $(s_{u2}, p_{u2}) \in C(u_2)$ that were used to generate $(s, p)$ in phase 1, and then continues recursively on both $u_1$ and $u_2$.

- $v$ is a forget node or an introduce node, and $u$ is the child of $v$ then the algorithm finds the candidate $(s', p')$ that was used to generate $(s, p)$ and continues recursively on $u$. If $v$ is a forget node, and player $i$ is forgotten at $v$, then once we have found $(s', p')$, we assign the strategy $s'_i$ to player $i$.

- If $v$ is a start node, then algorithm stops.

Each of these steps can be carried out by re-running the phase 1 algorithm at each node to determine how $(s, p)$ was created, so the running time of phase 2 is no more than the running time of phase 1.

After the algorithm terminates each player has been assigned a strategy, and we claim that the resulting strategy profile $s$ is an $1.5\epsilon$-NE. This follows from the fact that when we assign strategy $s_i$ to player $i$ in the forget node $v$, the test from Inequality 11 ensures that player $i$ is $\epsilon$-happy with respect to the payoff vector $p$, and the algorithm subsequently constructs a strategy profile whose
payoffs are within $r_v < \frac{\epsilon}{2}$ of $p$, so player $i$ is 1.5-\epsilon-happy in the resulting strategy profile.

I Proof of Theorem 15

With reference to the algorithm defined in Appendix H, we make the following modifications.

- At every Start node, the value of $x$ is initialized to $g(s_b)$, where $s_b$ denotes the strategy profile of an empty game.
- When we create a witness for an Introduce node $v$ with child $u$, we copy the value for $x$ given in the witness for $u$.
- When we create a witness for a Forget node $v$ with child $u$, we set the value of $x$ by computing $\text{add}(\tilde{G}(X_u), i, s_i, x)$, where $i$ is the player that is being forgotten, $s_i$ is the strategy for $i$, and $x$ is the value from the witness for $u$.
- When we create a witness for a Join node $v$ with children $u_1$ and $u_2$, we set the value of $x$ by computing $\text{merge}(\tilde{G}(X_{u_1}), \tilde{G}(X_{u_2}), x_1, x_2)$, where $x_1$ and $x_2$ are the values from the witnesses for $u_1$ and $u_2$, respectively.

Observe that the definition of the functions add and merge ensure that $x$ indeed capture the value that $g$ would assign to the witnessed strategy profile on $\tilde{G}(X_v)$.

If at any point two witnesses are created that agree on the $s$ and $p$ components, but disagree on the $x$ component, then we keep the witness whose $x$ value is higher, and discard the other one. This ensures that the number of possible witnesses computed for each tree decomposition node does not increase relative to the original algorithm, so the running time of the algorithm remains unchanged (ignoring the extra polynomial factors needed to compute $g$, add, and merge during the algorithm.)

Phase two of the algorithm is unchanged. We simply select the (unique) witness $(s, p, x) \in C(r)$ and unroll it to obtain a strategy profile $s'$ for $G$. The properties of add and merge ensure that $g(s') = x$.

We can now proceed to prove Theorem 15.

Proof. Since the modifications to the algorithm only deal with the extra parameter $x$, every witness $(s, p, x)$ computed by the algorithm corresponds to a witness $(s, p)$ that would have been computed by the original algorithm. Hence, the fact that the algorithm finds an 1.5-\epsilon-NE follows immediately from Theorem 13.

For the second property, we will use Lemma 18. This lemma proves that for every $\epsilon/4$-NE $s'$ of the game, every tree decomposition node will have a witness that corresponds to $s'$. The value of $x$ for this witness will correspond to the value that $g$ would assign to the restriction of $s'$ to $\tilde{G}(X_v)$. Since the algorithm always discards witnesses with smaller values of $x$, this ensures that the approximate equilibrium $s$ must have $g(s) \geq g(s')$.

J Proof of Theorem 16

Proof. The proof of this theorem requires a deeper understanding of the techniques used to prove Theorem 11. They show that, when $k$ is chosen according to the required bound, one can take any strategy profile $s'$ and randomly sample
a $k$-uniform strategy profile $s$. The crucial property is that, with high probability, the payoff of each pure strategy under $s'$ are close to the payoff of the corresponding pure strategy under $s$. In particular, the value of $k$ used in our algorithm ensures that the payoffs can move by at most $\epsilon/16$. This, in turn, means that any linear function over the payoffs can change by at most $c\epsilon/16$, for some constant $c$, which gives the required inequality.