Density of Zeros of the Cartwright Class Functions 
and the Helson–Szegő Type Condition

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Abstract—B. Ya. Levin has proved that the zero set of a sine type function can be represented as a union of finitely many separated sets, which is an important result in the theory of exponential Riesz bases. In the present paper, we extend Levin’s result to a more general class of entire functions $F(z)$ with zeros in a strip $\sup |\text{Im} \lambda_n| < \infty$ such that $|F(x)|^2$ satisfies the Helson–Szegő condition. Moreover, we show that instead of the last condition one can require that $\log |F(x)|$ belongs to the BMO class.

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1. INTRODUCTION

The theory of Riesz bases of complex exponentials $E = \{e^{i\lambda_n t}\}$, or, in other words, nonharmonic Fourier series in $L^2$ on an interval, began with the classical work by Paley and Wiener [1], which has motivated a great deal of work by many mathematicians (see, e.g., the references in [2]–[5]). In [6] (also see [7, Lectures 21–23]) Levin developed a set of techniques that permits one to establish a relationship between the basis property of the family $E$ and the properties of an entire function with zero set $\{\lambda_n\}$. Following this approach and using the geometry of the Hardy space $H_2^+$ in the upper half-plane, Pavlov [8] obtained a full description of the exponential Riesz bases. For various generalizations of Pavlov’s result, see [9], [3], [10], and [11].

In the present paper, we denote by $\Lambda = \{\lambda_n\}$ a sequence in $\mathbb{C}$ (multiple points are allowed) lying in a strip $S_\Lambda$, parallel to the real axis, $\sup |\text{Im} \lambda_n| < \infty$. Without loss of generality of our results (and for convenience of notation), we also assume that this strip lies in the upper half-plane; i.e., $\inf \text{Im} \lambda_n > 0$. A sequence $\Lambda$ is said to be separated, or uniformly discrete, if $\inf_{k \neq n} |\lambda_k - \lambda_n| > 0$. We say that $\Lambda$ is \textit{relatively uniformly discrete} if it can be decomposed into finitely many uniformly discrete subsequences.

The following notion plays an important role in the theory of exponential bases. A function $F$ of exponential type is called a \textit{sine type function} if its zeros $\lambda_n$ lie in a strip $S_\Lambda$ and $|F(x)| \asymp 1$, $x \in \mathbb{R}$. Levin [6] and Golovin [12] proved the following.

\textbf{Proposition 1.} The family $E$ is a Riesz basis in $L^2(0, T)$ if there exists a sine type function with indicator diagram of width $T$ and uniformly discrete zero set $\Lambda$. 

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In the case of a sine type function \( f \), the width of the indicator diagram is the sum of exponential types of \( f \) in the upper and lower half-planes.

Pavlov’s result can be stated as follows.

**Proposition 2.** The family \( \mathcal{E} \) is a Riesz basis in \( L^2(0, T) \) if and only if there exists an entire function \( F \) of exponential type with indicator diagram of width \( T \) and uniformly discrete zero set \( \Lambda \) such that \( |F(x)|^2 \) satisfies the Helson–Szegő condition: there exist functions \( u, v \in L^\infty(\mathbb{R}) \), \( \|v\|_{L^\infty(\mathbb{R})} < \pi/2 \), such that

\[
|F(x)|^2 = \exp\{u(x) + Hv(x)\}.
\]

Here the map \( v \mapsto Hv \) is the Hilbert transform for bounded functions,

\[
Hv(x) = \frac{1}{\pi} \text{v.p.} \int_{-\infty}^{\infty} v(t) \left\{ \frac{1}{x-t} + \frac{t}{t^2+1} \right\} dt.
\]

The function \( F \) in this theorem is called the generating function of the family \( \{e^{i\lambda_n t}\} \) on the interval \((0, T)\). This notion plays a central role in the modern theory of nonharmonic Fourier series [2], [3].

If the sequence \( \Lambda \) is not uniformly discrete but relatively uniformly discrete, then the family \( \mathcal{E} \) is not a Riesz basis of exponentials, but it may form a Riesz basis of finite-dimensional subspaces (basis with parentheses). A first result of this type was obtained by Levin [6]. He proved that zero set \( \Lambda \) of a sine type function \( F \) with indicator diagram of width \( T \) is relatively uniformly discrete and the corresponding family \( \mathcal{E} \) forms a Riesz basis with parentheses in \( L^2(0, T) \). Necessary and sufficient conditions for the Riesz basis property with parentheses were stated as the Helson–Szegő condition for the generating function \( F \) under the condition that its zero set \( \Lambda \) is relatively uniformly discrete [13]. In the present paper, we show that the last condition is excessive: if \( F \) satisfies the Helson–Szegő condition, then its zero set is relatively uniformly discrete. Moreover, we prove that the conclusion is true if \( \log |F(x)| \) belongs to the BMO class. This condition is weaker than the Helson–Szegő condition for \( F \) (see the details below).

From Riesz bases of subspaces we can return to bases of individual functions if instead of exponentials we consider a family of exponential divided differences. The theory of Riesz bases of exponential divided differences was developed in [13], [14]. This theory has important applications to control theory of hybrid systems and delayed equations of neutral type; see, e.g., [15], [16]. In applications of Riesz bases of the exponential divided differences, one needs to check two conditions:

(i) The generation function of the family \( \mathcal{E} \) with zero set \( \Lambda \) satisfies the Helson–Szegő condition.

(ii) The set \( \Lambda \) is relatively uniformly discrete.

The proof of (ii) is typically rather complicated, since it requires a detailed information about the asymptotic behavior of the sequence \( \lambda_n \). In the present paper, we prove that (i) implies (ii).

We need to introduce some more definitions. The upper uniform density \( D_+(\Lambda) \) of a sequence \( \Lambda \) is defined by the formula

\[
D_+(\Lambda) = \lim_{r \to +\infty} \sup_{x \in \mathbb{R}} \frac{\#(\text{Re} \Lambda \cap [x, x+r])}{r}.
\]

This notion, as well as the similar notion of the lower uniform density \( D_-(\Lambda) \), plays an important role in the theory of exponential bases \( \mathcal{E} \) on an interval; see [17], [13], [14]. If \( D_+(\Lambda) = C < \infty \), then in any strip \( I \times \mathbb{R} \) there are at most \( C|I| \) points of \( \Lambda \). That is equivalent to \( \Lambda \) being relatively uniformly discrete. Here \( I \) is an interval of the real axis with length \( |I| \).

**Definition 1.** An entire function \( F \) of exponential type belongs to the Cartwright class if

\[
\int_{\mathbb{R}} \frac{\max\{\log |F(x)|, 0\}}{1+x^2} \, dx < \infty.
\]
The Cartwright class functions are studied in detail in the classical monograph [18].

**Definition 2.** The *mean oscillation* $p_I(f)$ of a locally integrable function $f$ over an interval $I$ is defined as

$$p_I(f) = \frac{1}{|I|} \int_I |f(x) - f_I| \, dx,$$

where $f_I$ is the mean value of $f$ on $I$,

$$f_I = \frac{1}{|I|} \int_I f \, dx.$$

A locally integrable function on $\mathbb{R}$ belongs to the BMO class if the supremum $\sup_{I \in J} p_I(f)$ of its mean oscillation taken over the set of all intervals of the real axis is finite.

It is well known that a locally integrable function $f$ on $\mathbb{R}$ belongs to the BMO class if and only if it can be written as

$$f = u + \mathcal{H}v,$$

where $u, v \in L^\infty(\mathbb{R})$.

In the present paper, we show that if a function $F$ with zero set $\Lambda$ belongs to the Cartwright class and

$$\log |F(x)| \in \text{BMO},$$

then $\Lambda$ has a finite upper uniform density, $D_+(\Lambda) < \infty$.

It is convenient to prove this result in contrapositive form (as the converse of the inverse theorem), which is stated as follows.

**Theorem 1.** Let the function $F$ with zero set $\Lambda$ in the strip $0 < \inf \text{Im } z \leq \sup \text{Im } z < \infty$ belong to the Cartwright class and $D_+(\Lambda) = \infty$. Then

$$\log |F(x)| \notin \text{BMO}.$$ 

In particular, it follows that $F$ does not satisfy the Helson–Szegő condition.

**Remark 1.** The authors are grateful to the anonymous referee for examples of Cartwright class functions with zero set of infinite upper uniform density lying in a strip parallel to the real axis.

The first example demonstrates a function with zeros of unbounded multiplicity:

$$f(z) = \prod_{n=1}^{\infty} \cos^n \left( \frac{z}{3^n} \right).$$

One may get rid of the multiplicity condition. For that, in the second example we consider the function

$$f_n(z) = \cos \left[ \frac{\pi}{2} \left( 3^{-n} + 3^{-n^2} \right) z \right],$$

which is zero at the points

$$z_{kn} = 3^k - \frac{3^{k-n^2}}{3^{-n} + 3^{-n^2}}$$

for all integer $k$ and $n$ such that $k > n$. One can readily verify that the function

$$f(z) = \prod_{n=1}^{\infty} f_n(z)$$

has at least $k/2$ simple zeros on the interval $(3^k - 1, 3^k)$ for all sufficiently large $k$. 
2. PROOF OF THE MAIN RESULT (THEOREM 1)

Suppose that \( D_+ (\Lambda) = \infty \). Following [19], we introduce a continuous branch \( \varphi_z (t) \) of \( \arg b_z (t) \) for the Blaschke factor

\[
b_z (t) = \frac{1 - t/z}{1 - t/\bar{z}}.
\]

For \( z = x + iy, y > 0 \), and

\[
\psi_z (t) = \arctan \frac{yt}{|z|^2 - xt},
\]

we set

\[
\varphi_z (t) = \begin{cases} 
\psi_z (t) + \pi, & t > \frac{|z|^2}{x}, \\
\pi/2, & t = \frac{|z|^2}{x}, \\
\psi_z (t), & t < \frac{|z|^2}{x},
\end{cases}
\]

if \( x > 0 \) and

\[
\varphi_z (t) = \begin{cases} 
\psi_z (t), & t > \frac{|z|^2}{x}, \\
-\pi/2, & t = \frac{|z|^2}{x}, \\
\psi_z (t) - \pi, & t < \frac{|z|^2}{x},
\end{cases}
\]

if \( x < 0 \).

The following result was obtained in [19].

**Proposition 3.** Let a function \( F \) with zero set \( \{ z_n \} \) belong to the Cartwright class. Then the Hilbert transform \( \mathcal{H} \) of \( \log |F| \) can be represented in the form

\[
\mathcal{H} [\log |F|] (t) = \theta + \frac{T}{2} t - \sum_n \varphi_{z_n} (t),
\]

(2.1)

where \( T \) is the width of the indicator diagram of \( F \) and \( \theta \) is a constant.

Note that the series converges, because the sequence \( \{ z_n \} \) satisfies the Blaschke condition

\[
\sum \frac{\text{Im} \ z_n}{|z_n|^2} < \infty.
\]

As the first step in proving Theorem 1, we show that the sum

\[
\Phi (t) = \sum_n \varphi_{z_n} (t)
\]

rapidly increases if \( D_+ (\Lambda) = \infty \).

**Lemma 1.** If \( D_+ (\Lambda) = \infty \), then for every \( M > 0 \) there exists an \( a \) such that

\[
\Phi (a + 1) - \Phi (a) \geq M.
\]
Proof of the lemma. Straightforward calculations give

\[
\psi_{x+iy}'(t) = \frac{1}{1 + \left\{ \frac{yt}{(x^2 + y^2) - xt} \right\}^2} \cdot \left[ \frac{y}{(x^2 + y^2) - xt} + \frac{yxt}{(x^2 + y^2) - xt} \right]
\]

\[
= \frac{y(x^2 + y^2)}{(x^2 + y^2 - xt)^2 + y^2 t^2}.
\]

For \( t = x + \delta, \ |\delta| \leq 1 \), we obtain

\[
\psi_{x+iy}'(t) = \frac{y}{y^2 + \delta^2}.
\]

For \( z \) in the strip \( 0 < \alpha \leq \text{Im} \, z \leq \beta < \infty \), the minimum of the function \( \frac{y}{y^2 + \delta^2} \) is positive,

\[
\psi_{x+iy}'(t) \geq \frac{y}{y^2 + 1} \geq \min \left\{ \frac{\alpha}{\alpha^2 + 1}, \frac{\beta}{\beta^2 + 1} \right\} = c > 0 \quad \text{for} \quad t \in [x - 1, x + 1]. \tag{2.2}
\]

Now set

\[
\Lambda_a = \{ z_n \mid \text{Re} \, z_n \in [a, a + 1] \}
\]

and take \( [a, a + 1] \) such that \( |\Lambda_a| \geq M/c \); this is possible by the assumption of the lemma. Obviously,

\[
\Phi(a + 1) - \Phi(a) \geq \sum_{z_n \in \Lambda_a} \varphi_{z_n}(a + 1) - \sum_{z_n \in \Lambda_a} \varphi_{z_n}(a) = \sum_{z_n \in \Lambda_a} [\varphi_{z_n}(a + 1) - \varphi_{z_n}(a)].
\]

By inequality (2.2)

\[
\varphi_{z_n}(a + 1) - \varphi_{z_n}(a) \geq c,
\]

and therefore,

\[
\Phi(a + 1) - \Phi(a) \geq M.
\]

The proof of the lemma is complete. \( \square \)

Lemma 2. Let \( g \) be an increasing function on \( \mathbb{R} \), and assume that for any \( M > 0 \) there exists an \( a \in \mathbb{R} \) such that

\[
g(a + 1) - g(a) \geq M. \tag{2.3}
\]

Then

\[
\sup_{I = [a-1, a+2]} p_I(f) = \infty,
\]

and therefore, \( g \) does not belong to BMO.

Proof. The proof of this lemma is based on straightforward calculations. Let \( M \) and \( a \) satisfy inequality (2.3). Set \( I = I_M = [a - 1, a + 2] \). We will show that

\[
p_I(g) \geq \frac{M}{6}. \tag{2.4}
\]

Obviously, the mean value \( g_I \) belongs to the interval \( [g(a - 1), g(a + 2)] \), and

\[
g(a - 1) \leq g(a) \leq g(a + M) \leq g(a + 1) \leq g(a + 2).
\]

Consider two cases:

(i) \( g_I \in [g(a - 1), g(a + M/2)] \).

(ii) \( g_I \in [g(a + M/2), g(a + 2)] \).
Case (i). If \( x \in [a+1, a+2] \), then
\[
g(x) \geq g(a+1) \geq g(a) + M \geq g(a) + \frac{M}{2} \geq g_I,
\]
\[
|g(x) - g_I| = g(x) - g_I \geq g(a+1) - g_I \geq g(a) + M - g_I \geq \frac{M}{2}.
\]
Therefore,
\[
p_I(g) \geq \frac{1}{3} \int_{a+1}^{a+2} |g(x) - g_I| \, dx \geq \frac{M}{6}.
\]

Case (ii). If \( x \in [a-1, a] \), then
\[
g_I \geq g(a) \geq g(x),
\]
\[
|g(x) - g_I| = -g(x) + g_I \geq g(a) - \frac{M}{2} - g(x) \geq g(a) - \frac{M}{2} - g(a) \geq \frac{M}{2}.
\]
Therefore,
\[
p_I(g) \geq \frac{1}{3} \int_{a-1}^{a} |g(x) - g_I| \, dx \geq \frac{M}{6}.
\]

We arrive at (2.4) and the proof of the lemma is complete.

Now we are in a position to complete the proof of Theorem 1. The functional \( p_I \) of the term \( Tt/2 \) is constant on intervals of length 3. Then it follows from (2.1) and Lemmas 1 and 2
\[
\mathcal{H}[\log |F|] \notin \text{BMO}.
\]
The Hilbert transform maps the BMO class into itself, and \( \mathcal{H}^2 = -\text{Id} \). Therefore,
\[
\log |F| = -\mathcal{H}[\log |F|],
\]
and hence \( \log |F| \) does not belong to BMO. The proof of the theorem is complete.

Remark 2. In fact, to prove the main theorem, the explicit formulas in Proposition 3 are not necessary. In the case of infinite density, we only need to show the “rapid growth” of the continuous branch of the argument of \( F \) (in the sense of Lemma 1). This was proved by the authors in a different way, but the result in [19] substantially simplifies the proof of the main result.

Note also that the connections between the continuous branches of the argument of the Blaschke product, the BMO space, and the basis property of exponential families were discussed in [3, Sec. I.4].

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