Modified Lévy Laplacian on manifold

Boris O. Volkov

May 31, 2022

1 Steklov Mathematical Institute, ul. Gubkina, 8, Moscow, 119991, Russia
E-mail: borisvolkov1986@gmail.com

Dedicated to Prof. I. V. Volovich on his 75th anniversary.

Abstract

An infinite dimensional Laplacian defined as the Cesàro mean of the second order
directional derivatives on manifold is considered. This Laplacian is parameterized
by the choice of a curve in the group of orthogonal rotations. It is shown that,
under certain conditions on the curve, this operator is related to instantons on a
4-dimensional manifold.

keywords: Lévy Laplacian; Yang–Mills equations; instantons.

Introduction

The Lévy Laplacian is an infinite dimensional Laplacian defined as the Cesaro mean of the
second order directional derivatives (see [1]). Let $E \subset H \subset E^*$ be a Gelfand triplet. Here $H$
is a real separable Hilbert space and $E$ is a locally convex space continuously
embedded into $H$. Let $\{e_n\}$ be an orthonormal basis in $H$ consisting of elements from $E$. Let $f$
be two times Frechet differentiable function on $E$. Then the value of the Lévy
Laplacian $\Delta^{(e_n)}_L$ on the function $f$ is

$$\Delta^{(e_n)}_L f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \langle f''(x)e_k, e_k \rangle. \quad (1)$$

Thus, the definition of the Lévy Laplacian depends on the choice of a Hilbert space $H$
and an orthonormal basis $\{e_n\}$.

It turns out that this operator is related to the Yang–Mills gauge fields. Accardi,
Gibilisco and Volovich proved the following assertion (see [2]). A connection $A$
in a vector bundle over $\mathbb{R}^d$ satisfies the Yang–Mills equations

$$D^*_A F = 0$$

if and only if a parallel transport $U^A$ associated with the connection $A$ satisfies the Laplace equation for the Lévy Laplacian $\Delta_L$:

$$\Delta_L U^A = 0.$$
Here $F = dA + A \wedge A$ is the curvature and $D_A^*\!$ is adjoint operator to the covariant exterior derivative. Note the Lévy Laplacian in [2] was defined in a different way from (1), namely as a special integral functional defined by a special form of the second derivative. But it was proved in [3, 4] that this Lévy Laplacian can be defined by (1) if we choose the Sobolev space $H^1([0,1],\mathbb{R}^d)$ as $E$, the Hilbert space $L_2([0,1],\mathbb{R}^d)$ as $H$ and some natural basis $\{e_n\}$ in $L_2([0,1],\mathbb{R}^d)$ (see also [5]). The theorem on equivalence of the Laplace equation for the Lévy Laplacian and the Yang–Mills equations was generalized for manifolds (see [6, 5]).

It is a natural question to ask if the Lévy Laplacian related to instantons and anti-instantons on 4-manifolds (see [7]), i.e. solutions of the self-duality Yang–Mills equations

$$F = - * F$$

or the anti self-duality Yang–Mills equations

$$F = * F.$$ 

It turns out that the Lévy Laplacian introduced in [2] is not invariant under the action of infinite dimensional rotations $W \in C^1([0,1],SO(4))$. So any rotation $W \in C^1([0,1],SO(4))$ generates the so-called modified Lévy Laplacian $\Delta^W_L$. The Lie group $SO(4)$ is not simple. There are two normal Lie subgroups $S_3^L$ and $S_3^R$ of $SO(4)$. The Lie algebra so(4) can be decomposed as a direct sum $so(4) = Lie(S_3^L) \oplus Lie(S_3^R)$, where $Lie(S_3^L)$ and $Lie(S_3^R)$ are the Lie algebras of the groups $S_3^L$ and $S_3^R$ respectively. This decomposition corresponds to instantons and anti-instantons. The following theorem was proved for instantons over $\mathbb{R}^4$ by the author in [8].

**Theorem 1** Let $A$ be a smooth connection in the vector bundle over $\mathbb{R}^4$ with fiber $\mathbb{C}^N$. Let the value of the Yang–Mills action functional is finite on a connection $A$:

$$-\int_{\mathbb{R}^4} \text{tr}(F_{\mu\nu}(x)F^{\mu\nu}(x))dx < \infty.$$ 

Let $U^A$ be the parallel transport generated by the connection $A$. Let $W \in C^1([0,1],S_3^L)$ ($W \in C^1([0,1],S_3^R)$) and

$$\dim \text{span}\{W^{-1}(t)\tilde{W}(t)\}_{t \in [0,1]} \geq 2.$$ 

The following two assertions are equivalent:

1. the connection $A$ on $\mathbb{R}^4$ is an anti-instanton (instanton);
2. the parallel transport $U^A$ is a solution of the Laplace equation for the modified Lévy Laplacian $\Delta^W_L$:

$$\Delta^W_L U^A = 0.$$ 

In the present paper, we generalize this theorem for instantons and anti-instantons on a 4-dimensional Riemannian manifold. The effect of the holonomy group on a manifold on the connection between Lévy Laplacians and instantons was studied in [9].

The concept of the modified Lévy Laplacian can be generalized to the case of a manifold. It is well-known fact that if $f$ is a smooth function on a $d$-dimensional Riemannian
manifold $M$ than the value of the Laplace–Beltrami operator $\Delta_M$ on $f$ at the point $x$ can be found as

$$\Delta_M f(x) = \sum_{i=1}^{d} \frac{d^2}{dt^2} \bigg|_{t=0} f(\exp_x(tZ_i)),$$

where $\exp_x$ is an exponential mapping on the manifold $M$ at the point $x \in M$ and $\{Z_1, \ldots, Z_d\}$ is an orthonormal basis in the tangent space $T_x M$. In [10] by Accardi and Smolyanov, the Lévy Laplacian was introduced on the infinite dimensional manifold by analogy with formulas (1) and (2). In the present paper, we generalize this definition for modified Lévy Laplacians.

Different approaches to the Yang–Mills equations based on the parallel transport but not based on the Lévy Laplacian were used in [11, 12, 13, 14, 15, 16]. Theory of infinite dimensional Laplacians on infinite dimensional manifolds was also considered in [17, 18]. For a recent development in the study of infinite dimensional Laplacians see [21, 22, 23].

The paper is organized as follows. In Sec. 1 we give preliminary information about infinite dimensional geometry. In Sec. 2, we give the definition of the Lévy Laplacian. In Sec. 3, we give preliminary information on the Yang–Mills equations and instantons. We formulate the main theorem on relationship between instantons on a 4-dimensional manifold and instantons in Sec. 4 and prove it in Sec. 5.

## 1 Hilbert manifold of $H^1$-curves

In this section, we give some background information about the Hilbert manifold of $H^1$-curves and two canonical Hilbert vector bundles $\mathcal{H}^1 \subset \mathcal{H}^0$ over this manifold. This is an analog of the embedding $H^1_0([0, 1], \mathbb{R}^d) \subset L_2([0, 1], \mathbb{R}^d)$ from the definition of the Lévy Laplacian (1).

Let $M$ be a smooth connected (not necessary compact) Riemannian $d$-dimensional manifold. Let $g$ denote the Riemannian metric on $M$. Let $\Gamma^r_{\lambda\mu}$ be the Christoffel symbols of the Levi-Civita connection associated with this metric on $M$. We will raise and lower indices using the metric $g$ and we will sum over repeated indices.

For any sub-interval $I \subset [0, 1]$ the symbols $H^0(I, \mathbb{R}^d)$ and $H^1(I, \mathbb{R}^d)$ denote the spaces of $L_2$-functions and $H^1$-functions on $I$ with values in $\mathbb{R}^d$ respectively. These spaces are Hilbert spaces with scalar products

$$(h_1, h_2)_0 = \int_I (h_1(t), h_2(t))_{\mathbb{R}^d} dt$$

and

$$(h_1, h_2)_1 = \int_I (h_1(t), h_2(t))_{\mathbb{R}^d} dt + \int_I (\dot{h}_1(t), \dot{h}_2(t))_{\mathbb{R}^d} dt$$

respectively. Let $H^1_0 = \{h \in H^1([0, 1], \mathbb{R}^d): h(0) = 0\}$ and $H^1_{0, 0} = \{h \in H^1_0: h(1) = 0\}$.

The curve $\gamma: [0, 1] \to M$ on the manifold $M$ is called $H^1$-curve, if $\phi_a \circ \gamma \in H^1(I, \mathbb{R}^4)$ for any interval $I \subset [0, 1]$ and for any coordinate chart $(\phi_a, W_a)$ of the manifold $M$ such that $\gamma(I) \subset W_a$. Let $\Omega_m$ be the set of all $H^1$-curves in $M$ with the origin at the point $m \in M$. The set $\Omega_m$ can be endowed with the structure of an infinite dimensional Hilbert manifold modeled over the Hilbert space $H^1_0$ (see [12, 24, 25, 26]).
For any $\gamma \in \Omega_m$ the mapping $X(\gamma; \cdot) : [0, 1] \to TM$ such that $X(\gamma; t) \in T_{\gamma(t)}M$ for any $t \in [0, 1]$ is a vector field along $\gamma$. We will also use the notation $X(\gamma)$ for $X(\gamma; \cdot)$.

The symbol $H^0_\gamma(TM)$ denotes the Hilbert space of all $H^0$-fields along $\gamma$. The scalar product on this space is defined by the formula

$$G_0(X(\gamma), Y(\gamma)) = \int_0^1 g(X(\gamma; t), Y(\gamma; t))dt.$$  

For any absolutely continuous field $X(\gamma)$ along $\gamma \in \Omega$ its covariant derivative $\nabla X(\gamma)$ is defined by

$$\nabla X(\gamma; t) = \frac{d}{dt}X(\gamma; t) + \Gamma(\gamma(t))(X(\gamma; t), \dot{\gamma}(t)),$$

where $(\Gamma(x)(X, Y))^\mu = \Gamma^\mu_{\nu\rho}(x)X^\nu Y^\rho$ in local coordinates. Let $Q(\gamma; \cdot)$ denote the parallel transport generated by the Levi-Civita connection along the curve $\gamma$. It is easy to show that

$$\nabla X(\gamma; t) = Q(\gamma; t)\frac{d}{dt}(Q(\gamma; t)^{-1}X(\gamma; t)).$$

The symbol $H^1_\gamma(TM)$ denotes the Hilbert space of all $H^1$-fields along $\gamma$. The scalar product on this space is defined by the formula

$$G_1(X(\gamma), Y(\gamma)) = \int_0^1 g(X(\gamma; t), Y(\gamma; t))dt + \int_0^1 g(\nabla X(\gamma; t), \nabla Y(\gamma; t))dt.$$  

We consider two canonical Riemannian vector bundles $H^0$ and $H^1$ over the Hilbert manifold $\Omega_m$ (see [24, 25]). The fiber of $H^0$ over $\gamma \in \Omega_m$ is the space $H^0_\gamma(TM)$. The vector bundle $H^1$ is the tangent bundle over the manifold $\Omega_m$. Its fiber over $\gamma \in \Omega_m$ is the space $H^1_\gamma(TM)$. The exponential mapping at $\gamma \in \Omega_m$ is defined by the formula

$$\exp_\gamma(X)(t) = \exp_{\gamma(t)}(X(t)), \text{ where } X \in H^1_\gamma(TM).$$

We can identify $\mathbb{R}^d$ and the Hilbert spaces $H^1([0, 1], \mathbb{R}^d)$ and with $T_mM$ and $H^1([0, 1], T_mM)$ respectively. Let fix an orthonormal basis $\{Z_1, \ldots, Z_d\}$ in $T_mM$. Let $Z_i(\gamma, t) = Q_{t,0}(\gamma)Z_i$ for $i \in \{1, \ldots, d\}$. Due to (3), for any $\gamma \in \Omega_m$ the Levi-Civita connection generates the canonical isometrical isomorphism between $H^1_0$ and $H^1_\gamma(TM)$, which action on $h \in H^1_0$ we will denote by $\tilde{h}$. This isomorphism acts by the formula

$$\tilde{h}(\gamma; t) = Q_{t,0}(\gamma)h(t) = Z_\mu(\gamma, t)h^\mu(t),$$

where $h = h^\mu Z_\mu$. Let $H^1_0$ be the vector bundle over $\Omega_m$ which fiber over $\gamma \in \Omega_m$ is $H^1_\gamma(TM)$.

## 2 Definition of modified Lévy Laplacian on manifold

In this section, we give a definition of the Lévy Laplacian on the space of functions on the Hilbert manifold of $H^1$-curves as the Cesaro mean of the second order directional derivatives.

Everywhere below we assume that $\{e_n\}$ is an orthonormal basis in $L_2([0, 1], \mathbb{R})$ such that $e_n \in H^1$ and $e_n(0) = e_n(1) = 0$ for all $n \in \mathbb{N}$. Let $e_{\mu,n} = Z_\mu e_n$. Then $\{e_{\mu,n}\}$ form the
orthonormal basis in $H^0$ and all its elements belong to $H^1_{0,0}$. Hence, $\{\tilde{e}_{\mu,n}(\gamma)\}$ form the orthonormal basis in the fiber of $\mathcal{H}^0$ over $\gamma \in \Omega_m$ which elements belong to the fiber of $\mathcal{H}^1$.

Any smooth curve $W \in C^1([0,1], SO(d))$ defines an orthogonal operator in $L_2([0,1], \mathbb{R}^d)$ by pointwise left multiplication:

$$(Wu)(t) = W(t)u(t).$$

The subspace $H^1_{0,0}([0,1], \mathbb{R}^d) \subset L_2([0,1], \mathbb{R}^d)$ is invariant under the action of $W$.

Let $\mathcal{F}(\Omega_m, \mathbb{R})$ denote the space of all functions on $\Omega_m$.

**Definition 1** The (modified) Lévy Laplacian, generalized by the orthonormal basis $\{e_{\mu,n}\}$ and the curve $W \in C^1([0,1], SO(4))$, is a linear mapping

$$\Delta^W_{L}\{e_{\mu,n}\} : \text{dom}\Delta^W_{L}\{e_{\mu,n}\} \rightarrow \mathcal{F}(\Omega_m, \mathbb{R})$$

defined by

$$\Delta^W_{L}\{e_{\mu,n}\} f(\gamma) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \frac{d^2}{ds^2} \Big|_{s=0} f(\exp(sW(\tilde{e}_{\mu,k}(\gamma)))),$$

where $\text{dom}\Delta^W_{L}\{e_{\mu,n}\}$ is the space of all sections $f \in \mathcal{F}(\Omega_m, \mathbb{R})$ such that the right side of (6) exists for all $\gamma \in \Omega_m$.

**Remark 1** This definition is a generalization of the definition of the Lévy Laplacian on a manifold introduced by Accardi and Smolyanov in [10] and the definition of the Lévy Laplacian generalized by the curve $W \in C^1([0,1], SO(4))$, introduced by the author for $M = \mathbb{R}^4$ in [8].

Recall the following definition from [1] (see also [27]).

**Definition 2** An orthonormal basis $\{e_n\}$ in $L_2([0,1], \mathbb{R})$ is called weakly uniformly dense (or equally dense) if

$$\lim_{n \rightarrow \infty} \int_0^1 h(t) \left( \frac{1}{n} \sum_{k=1}^{n} e_k^2(t) - 1 \right) dt = 0$$

for any $h \in L_\infty([0,1], \mathbb{R})$.

**Example 1** The basis $e_n(t) = \sqrt{2} \sin(n\pi t)$ is a weakly uniformly dense basis. [1, 27]

The following example belongs to Polishchuk. [27]

**Example 2** Let $r \in C([0,1], \mathbb{R})$. Let $u_n$ be the normalized $n$-th eigenfunction of the following second order differential equation:

$$u''(t) - r(t)u(t) + \lambda u(t) = 0$$

with boundary condition: $u(0) = u(1) = 0$. Then $\{u_n\}_{n=1}^{\infty}$ is an equally dense orthonormal basis in $L_2([0,1], \mathbb{R})$. 


One can show that if \( \{e_n\} \) is weakly uniformly dense basis than Definition 1 for constant \( W \) coincides with definition of the Lévy Laplacian from [6] and in the general case coincides with definition of the modified Lévy Laplacian from Ref. [4].

**Example 3** Let \( f \in C^2(M, \mathbb{R}) \). Let \( \mathcal{L}_f: \Omega_m \rightarrow \mathbb{R} \) be defined by:

\[
\mathcal{L}_f(\gamma) = \int_0^1 f(\gamma(t))dt.
\]

Then for an arbitrary \( W \in C^1([0, 1], SO(d)) \) the value of the Lévy Laplacian on the functional \( \mathcal{L}_f \) is

\[
\Delta^W L_f(\gamma) = \int_0^1 \Delta_{(M,g)} f(\gamma(t))dt,
\]

where \( \Delta_{(M,g)} \) is the Laplace-Beltrami operator on the manifold \( M \).

**Remark 2** If \( W \) from the definition of the Lévy Laplacian \( \Delta^W L \) is constant then we obtain the Lévy Laplacian which is related to the Yang–Mills equations (see [2, 6, 3, 5]).

### 3 Yang–Mills equations and instantons

In this section, \( M \) is a smooth orientable (not necessary compact) Riemannian 4-dimensional manifold (\( d = 4 \)). Let \( G \subseteq SU(N) \) be a closed Lie group and \( \text{Lie}(G) \subseteq su(N) \) be its Lie algebra. Let \( E = E(\mathbb{C}^N, \pi, M, G) \) be a vector bundle over \( M \) with the projection \( \pi: E \rightarrow M \), the fiber \( \mathbb{C}^N \) and the structure group \( G \). We will denote the fiber \( \pi^{-1}(x) \cong \mathbb{C}^N \) over \( x \in M \) by the symbol \( E_x \). Let \( P \) be the principle bundle over \( M \) associated with \( E \). Let \( \text{ad}(P) = \text{Lie}(G) \times_G M \) be the adjoint bundle of \( P \) (the fiber of \( \text{ad}P \) is isomorphic to \( \text{Lie}(G) \)).

A connection \( A(x) = A_\mu(x)dx^\mu \) in the vector bundle \( E \) is a smooth section in \( \Lambda^1(T^*M) \otimes \text{ad}P \). The curvature \( F(x) = \sum_{\mu<\nu} F_{\mu\nu}(x)dx^\mu \wedge dx^\nu \) of the connection \( A \) is a smooth section in \( \Lambda^2(T^*M) \otimes \text{ad}P \) defined by the formula \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \).

The Yang–Mills action functional has the form

\[
S_{YM}(A) = -\frac{1}{2} \int_M \text{tr}(F_{\mu\nu}(x)F^{\mu\nu}(x))Vol(dx),
\]

where \( Vol \) is the Riemannian volume measure on the manifold \( M \). The Yang–Mills equations on a connection \( A \) have the form

\[
D_A^* F = 0,
\]

where \( D_A \) is the operator of the exterior covariant derivative and \( D_A^* \) is its formally adjoint operator. In local coordinates, we have

\[
D_A^* F_\nu = -\nabla^\mu F_{\nu\mu}
\]

and

\[
\nabla_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}] - F_{\mu\kappa} \Gamma_{\lambda\nu}^\kappa - F_{\nu\kappa} \Gamma_{\lambda\mu}^\kappa,
\]

\[
\]
where $\Gamma^\nu_{\lambda\mu}$ are the Christoffel symbols of the Levi-Civita connection on $M$. The Yang–Mills equations are the Euler-Lagrange equations for the Yang–Mills action functional $S$.

If $M$ is a 4-manifold, it is possible to consider the Yang–Mills self-duality equations

$$F = *F$$

(10)

or Yang–Mills anti self-duality equations

$$F = - * F$$

(11)

on a connection $A$, where $*$ is the Hodge star. A connection is called an instanton or an anti-instanton if it is a solution of equations (11) or (10) respectively. Due to the Bianchi identities $D_AF = 0$ and equality $D^*_A = - * D_A*$, instantons and anti-instantons are solutions of the Yang–Mills equations (9).

4 Laplace equation and instantons

Let $E_m$ be a Hilbert vector bundle over the Hilbert manifold $\Omega_m$ such that its fiber over $\gamma \in \Omega_m$ is the space $\text{Hom}(E_m, E_{\gamma(1)})$. The definition of the modified Lévy Laplacian $\Delta^W_L$ can been transferred without changes to the space of sections in the bundle $E_m$.

For any $H^1$-curve $\gamma$ an operator $U^A_{t,s}(\gamma) \in \text{Hom}(E_{\gamma(s)}, E_{\gamma(t)})$, where $0 \leq s \leq t \leq 1$, is a solution of the system

$$
\begin{cases}
\frac{d}{dt} U^A_{t,s}(\gamma) = -A_\mu(\gamma(t)) \dot{\gamma}^\mu(t) U^A_{t,s}(\gamma) \\
\frac{d}{ds} U^A_{t,s}(\gamma) = U^A_{t,s}(\gamma) A_\mu(\gamma(s)) \dot{\gamma}^\mu(s) \\
U^A_{t,s}(\gamma)|_{t=s} = I_N.
\end{cases}
$$

(12)

Then $U^A_{1,0}$ is a parallel transport along the curve $\gamma$ generated by the connection $A$.

The parallel transport has the following properties (see [4]):

1. The mapping $\Omega_m \ni \gamma \mapsto U^A_{1,0}(\gamma)$ is a $C^\infty$-smooth section in the vector bundle $E_m$ (for the proof of smoothness see [11, 12]).

2. The parallel transport does not depend on the choice of parametrization of the curve. Let $\sigma : [0, 1] \to [0, 1]$ be a non-decreasing piecewise $C^1$-smooth function such that $\sigma(0) = 0$ and $\sigma(1) = 1$. Then

$$U^A_{\sigma(t), \sigma(s)}(\gamma) = U^A_{t,s}(\gamma \circ \sigma)$$

(13)

for any $\gamma \in \Omega_m$.

3. For any $\gamma \in \Omega_m$ the parallel transport satisfies the multiplicative property:

$$U^A_{t,s}(\gamma) U^A_{s,r}(\gamma) = U^A_{t,r}(\gamma)$$

(14)

for $r \leq s \leq t$.

4. If the restriction of $\gamma \in \Omega_m$ on $[s, t]$ is constant, then

$$U^A_{t,s}(\gamma) \equiv I_N.$$
The dimension \( d = 4 \) is special, because where two normal subgroups \( S^3_L \) and \( S^3_R \) of \( SO(4) \). The groups \( S^3_L \) and \( S^3_R \) consist of real matrices of the form

\[
\begin{pmatrix}
a & -b & -c & -d \\
b & a & -d & c \\
c & d & a & -b \\
d & -c & b & a
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
a & -b & -c & -d \\
b & a & d & -c \\
c & -d & a & b \\
d & c & -b & a
\end{pmatrix},
\]

where \( a^2 + b^2 + c^2 + d^2 = 1 \), respectively. The Lie algebras \( \text{Lie}(S^3_L) \) and \( \text{Lie}(S^3_R) \) of the Lie groups \( S^3_L \) and \( S^3_R \) consist of real matrices of the form

\[
\begin{pmatrix}
0 & -b & -c & -d \\
b & 0 & -d & c \\
c & d & 0 & -b \\
d & -c & b & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & -b & -c & -d \\
b & 0 & d & -c \\
c & -d & 0 & b \\
d & c & -b & 0
\end{pmatrix}
\]

respectively.

If \( W \in C^1([0, 1], SO(4)) \) let \( L_W(t) = W^{-1}(t)\bar{W}(t) \). Then \( L_W \) is continuous curve in \( \text{so}(4) \). Let \( L_W^+(t) \) and \( L_W^-(t) \) denote the orthogonal projection \( L_W(t) \) on the Lie algebras \( \text{Lie}(S^3_L) \) and \( \text{Lie}(S^3_R) \) respectively. Let us show that the operator \( \Delta_W \) is related to instantons and anti-instantons, if \( W \in C^1([0, 1], \text{Lie}(S^3_L)) \) or \( W \in C^1([0, 1], \text{Lie}(S^3_R)) \) respectively (see also [8, 9, 10, 11]). Let us prove the following theorem.

**Theorem 2** Let \( W \in C^1([0, 1], S^3_L) \) (\( W \in C^1([0, 1], S^3_R) \)) and

\[
\dim \text{span}\{L_W(t)\}_{t \in [0, 1]} \geq 2.
\]

Let \( \{e_n\} \) be a weakly uniformly dense orthonormal basis in \( L_2([0, 1], \mathbb{R}) \) such that \( e_n \in H^1 \) and \( e_n(0) = e_n(1) = 0 \) for all \( n \in \mathbb{N} \). Let there exists the sequence \( \{x_n\} \) of points in \( M \) such that

\[
\lim_{n \to \infty} F_-(x_n) = 0 \quad \lim_{n \to \infty} F_+(x_n) = 0.
\]

(16)

The following two assertions are equivalent:

1. a connection \( A \) is a solution of self-duality equations (anti self-duality equations): \( F = *F, (F = - *F) \);

2. the parallel transport \( U^{A}_{1,0} \) is a solution of the equation:

\[
\Delta_W^{\{e_n\}} U^{A}_{1,0} = 0;
\]

**Remark 3** If the base manifold \( M \) is compact than condition \( (17) \) means that there exists a point \( x \in M \) such that \( F_-(x) = 0 \) \( (F_+(x) = 0) \).

## 5 Proof of the main theorem

In this section, we will prove several lemmas, which together provide a proof of the main theorem.
We will use the following notations for all \( \gamma \in \Omega_m \) and \( t \in [0, 1] \):

\[
L(\gamma, t) = U^A_{t,0}(\gamma)^{-1}F(\gamma(t))U^A_{t,0}(\gamma). \tag{17}
\]

Its anti self-dual and self-dual parts have form

\[
L_{-}(\gamma, t) = U^A_{t,0}(\gamma)^{-1}F_{-}(\gamma(t))U^A_{t,0}(\gamma), \tag{18}
\]

\[
L_{+}(\gamma, t) = U^A_{t,0}(\gamma)^{-1}F_{+}(\gamma(t))U^A_{t,0}(\gamma). \tag{19}
\]

If \( \gamma \in \Omega_m \) and \( r, t \in [0, 1] \) let

\[
L^W(\gamma, t, r) := \sum_{\mu=1}^{4} L(\gamma, t) < \dot{W}(r)e_\mu(\gamma, t), W(r)e_\mu(\gamma, t) > .
\]

It can been checked by direct computations that

\[
L^W(\gamma, t, r) = \text{tr}_{\text{so}(4)}(\dot{W}(r)W^{-1}(r)L(\gamma, t)) = \text{tr}_{\text{so}(4)}(L^W_+(r)L_{+}(\gamma, t)) + \text{tr}_{\text{so}(4)}(L^W_-(r)L_{-}(\gamma, t)). \tag{20}
\]

The parallel transport belongs to the domain of the modified Lévy Laplacian. The following theorem can been proved by direct computations.

**Lemma 1** Let \( \{e_n\} \) be a weakly uniformly dense orthonormal basis in \( L^2([0, 1], \mathbb{R}) \) such that \( e_n \in H^1 \) and \( e_n(0) = e_n(1) = 0 \) for all \( n \in \mathbb{N} \). The value of the modified Lévy Laplacian \( \Delta^W_{L}(e_{\mu,n}) \) on the parallel transport is

\[
\Delta^W_{L}(e_{\mu,n})U^A_{1,0}(\gamma) =
\int_0^1 U^A_{1,t}(\gamma)D_A^*F(\gamma(t))\dot{\gamma}(t)U^A_{t,0}(\gamma)dt - U^A_{1,0}(\gamma) \int_0^1 L^W(\gamma, t, t)dt. \tag{20}
\]

The first term in the right side of (20) is invariant under reparameterization of the curve \( \gamma \) but the second term is not. Therefore, the following lemma is true (for the proof see [4]).

**Lemma 2** Let \( W \in C^1([0, 1], \text{SO}(4)) \). Let the parallel transport \( U^A_{1,0} \) be s a solution of the Laplace equation for the Lévy Laplacian \( \Delta^W_{L}(e_{\mu,n}) \):

\[
\Delta^W_{L}(e_{\mu,n})U^A_{1,0} = 0.
\]

Then the connection \( A \) satisfies the Yang–Mills equations and for any \( \gamma \in \Omega_m \) the following holds

\[
\int_0^1 L^W(\gamma, t, t)dt = 0.
\]

The following lemma is an analog for manifolds of Proposition 3 from Ref. [8].
**Lemma 3** Let for any $\gamma \in \Omega_m$ the following relation holds
\[
\int_0^1 L^W(\gamma, t, t)dt = 0.
\]
Then for any $\gamma \in \Omega_m$ and for any $t, r \in [0, 1]$ the following holds
\[
L^W(\gamma, t, r) = \text{tr}_\text{so}(L_W(r)F(m)).
\]

**Proof** For any $\gamma \in \Omega_m$ and for any $t \in [0, 1]$ and $r \in [0, 1]$ let the curve $\gamma \in \Omega_m$ be introduced by the following:
\[
\gamma_{r, \varepsilon}(\tau) = \begin{cases} 
  m, & \text{if } 0 \leq \tau \leq r - \varepsilon, \\
  \gamma\left(\frac{\tau - r + \varepsilon}{\varepsilon}\right), & \text{if } r - \varepsilon < \tau \leq r, \\
  \gamma(t), & \text{if } r < \tau \leq 1.
\end{cases}
\]
We can use the properties of the parallel transport. If $0 \leq \tau \leq r - \varepsilon$, then
\[
L^W(\gamma_{r, \varepsilon}, \tau, r) = \text{tr}_\text{so}(L_W(r)U_{t, 0}^A(\gamma)^{-1}F(\gamma(t))U_{t, 0}^A(\gamma)) = L^W(\gamma, t, \tau).
\]
If $r < \tau \leq 1$, then
\[
L^W(\gamma_{r, \varepsilon}, \tau, r) = \text{tr}_\text{so}(L_W(r)U_{t, 0}^A(\gamma)^{-1}F(\gamma(t))U_{t, 0}^A(\gamma)) = \text{tr}_\text{so}(L_W(\tau)F(m)).
\]
Hence,
\[
\int_0^1 L^W(\gamma_{r, \varepsilon}, \tau, \tau)d\tau =
\]
\[
= \int_r^1 L^W(\gamma, t, \tau)d\tau + \int_r^{r - \varepsilon} L^W(\gamma_{r, \varepsilon}, \tau, \tau)d\tau + \int_{r - \varepsilon}^0 \text{tr}_\text{so}(L_W(\tau)F(m))d\tau = 0.
\]
There exists $C > 0$ such that the following estimates hold
\[
\| \int_{r - \varepsilon}^r L^W(\gamma_{r, \varepsilon}, \tau, \tau)d\tau \|_{\text{so}(N)} \leq C\varepsilon,
\]
\[
\| \int_{r - \varepsilon}^r \text{tr}_\text{so}(L_W(\tau)F(m))d\tau \|_{\text{so}(N)} \leq C\varepsilon.
\]
Then for any $\varepsilon > 0$
\[
\| \int_r^1 L^W(\gamma, t, \tau)d\tau + \int_0^r \text{tr}_\text{so}(L_W(\tau)F(m))d\tau \|_{\text{su}(N)} =
\]
\[
= \| \int_r^1 L^W(\gamma, t, \tau)d\tau + \int_0^r \text{tr}_\text{so}(L_W(\tau)F(m))d\tau \|_{\text{su}(N)} \leq
\]
\[
\leq \| \int_{r - \varepsilon}^r \text{tr}_\text{so}(L_W(\tau)F(m))d\tau \|_{\text{su}(N)} \leq 2C\varepsilon.
\]
Then
\[
- \int_r^1 L^W(\gamma, t, \tau)d\tau = \int_0^r \text{tr}_\text{so}(L_W(\tau)F(m))d\tau.
\]
Differentiating the last equality we get
\[
L^W(\gamma, t, r) = \text{tr}_\text{so}(L_W(r)F(m)).
\]
Lemma 4 Let $W \in C^1([0,1], S_3^L)$ ($W \in C^1([0,1], S_3^R)$). Let for any $\gamma \in \Omega_m$ the following holds
\[ \int_0^1 L^W(\gamma, t, t)dt = 0. \]

Let there exists the sequence $\{x_n\}$ in $M$ such that
\[ \lim_{n \to \infty} F_-(x_n) = 0 \quad \left( \lim_{n \to \infty} F_+(x_n) = 0. \right) \] (21)

Then for any $\gamma \in \Omega_m$ and for any $t, r \in [0,1]$ the following holds
\[ L^W(\gamma, t, r) = 0. \]

Proof If $W \in C^1([0,1], S_3^L)$, then $L^W_-(r) = 0$ and
\[ L^W(\gamma, t, r) = \text{tr}_{so(4)}(\dot{W}(r) W^{-1}(r) L(\gamma, t)) = \text{tr}_{so(4)}(L^+_{W}(r) L_+(\gamma, t)). \]

Let there exists the sequence $\{x_n\}$ in $M$ such that
\[ \lim_{n \to \infty} F_-(x_n) = 0, \]
then for any $r \in [0,1]$ there exist the sequence of curves $\{\gamma_n\}$ and points $\{t_n\}$ such that $x_n = \gamma_n(t_n)$. Due to Lemma 3 for any $\gamma \in \Omega_m$ and for any $t, r \in [0,1]$ we have
\[ L^W(\gamma, t, r) = \text{tr}_{so(4)}(L_W(r) F(m)) = \lim_{n \to \infty} L^W(\gamma_n, t_n, r) = 0. \] (23)

Lemma 5 Let a connection $A$ satisfy the Yang–Mills equations
\[ D^*_A F = 0. \] (22)

Let $W \in C^1([0,1], S_3^L)$ ($W \in C^1([0,1], S_3^R)$) and
\[ \dim \text{span}\{L_W(t)\}_{t \in [0,1]} \geq 2. \] (23)

Let for any $\gamma \in \Omega_m$ and for any $t, r \in [0,1]$ the following holds
\[ L^W(\gamma, t, r) = 0. \]

Then the connection $A$ is a solution of self–duality equations (anti self–duality equations): $F = \ast F$, ($F = - \ast F$).

Proof Lemma 4 implies that for any $\gamma \in \Omega_m$ and for any $t, r \in [0,1]$ the following holds $L^W(\gamma, t, r) = 0$. This equality and inequality (23) together imply that we can choose the orthonormal basis $\{Z_1, Z_2, Z_3, Z_4\}$ in $T_m M$ such that for any $t \in [0,1]$
\[ \begin{cases} 
L_+(\gamma, t) \langle Z_1(\gamma, t) \wedge Z_2(\gamma, t) \rangle = 0 \\
L_+(\gamma, t) \langle Z_1(\gamma, t) \wedge Z_3(\gamma, t) \rangle = 0.
\end{cases} \] (24)

Differentiating (24) and multiplying left and right by $U^A_{l,0}(\gamma)$ and $(U^A_{l,0}(\gamma))^{-1}$, respectively we get
\[ \nabla F_+ (\gamma(t)) \langle \dot{\gamma}(t), Z_1(\gamma, t), Z_2(\gamma, t) \rangle = 0 \] (25)
and
\[ \nabla F_+(\gamma(t))\langle \gamma(t), Z_1(\gamma, t), Z_3(\gamma, t) \rangle = 0. \] (26)

The Yang–Mills equations imply that
\[ 0 = \nabla F(\gamma(t))\langle Z_1(\gamma, t), Z_1(\gamma, t), Z_4(\gamma, t) \rangle + \]
\[ + \nabla F(\gamma(t))\langle Z_2(\gamma, t), Z_2(\gamma, t), Z_4(\gamma, t) \rangle + \]
\[ + \nabla F(\gamma(t))\langle Z_3(\gamma, t), Z_3(\gamma, t), Z_4(\gamma, t) \rangle. \] (27)

The Bianchi identities imply that
\[ 0 = \nabla F(\gamma(t))\langle Z_1(\gamma, t), Z_3(\gamma, t), Z_2(\gamma, t) \rangle + \]
\[ + \nabla F(\gamma(t))\langle Z_2(\gamma, t), Z_1(\gamma, t), Z_3(\gamma, t) \rangle + \]
\[ + \nabla F(\gamma(t))\langle Z_3(\gamma, t), Z_2(\gamma, t), Z_1(\gamma, t) \rangle. \] (28)

Equalities (27) and (28) together imply
\[ \nabla F(\gamma(t))\langle Z_1(\gamma, t), Z_1(\gamma, t), Z_2(\gamma, t) \rangle - \nabla F(\gamma(t))\langle Z_1(\gamma, t), Z_3(\gamma, t), Z_2(\gamma, t) \rangle = \]
\[ = \nabla F(\gamma(t))\langle Z_2(\gamma, t), Z_1(\gamma, t), Z_3(\gamma, t) \rangle - \]
\[ - \nabla F(\gamma(t))\langle Z_2(\gamma, t), Z_2(\gamma, t), Z_4(\gamma, t) \rangle + \]
\[ + \nabla F(\gamma(t))\langle Z_3(\gamma, t), Z_2(\gamma, t), Z_1(\gamma, t) \rangle - \]
\[ - \nabla F(\gamma(t))\langle Z_3(\gamma, t), Z_3(\gamma, t), Z_4(\gamma, t) \rangle. \] (29)

Equalities (29) can be rewritten in the form
\[ \nabla F_+(\gamma(t))\langle Z_1(\gamma, t), Z_1(\gamma, t), Z_2(\gamma, t) \rangle = \]
\[ = \nabla F_+(\gamma(t))\langle Z_1(\gamma, t), Z_3(\gamma, t), Z_4(\gamma, t) \rangle = \]
\[ = \nabla F_+(\gamma(t))\langle Z_2(\gamma, t), Z_1(\gamma, t), Z_3(\gamma, t) \rangle + \]
\[ + \nabla F_+(\gamma(t))\langle Z_3(\gamma, t), Z_2(\gamma, t), Z_1(\gamma, t) \rangle. \] (30)

Due to (25) and to (26), the right side of (30) is zero. Hence,
\[ \nabla F_+(\gamma(t))\langle Z_1(\gamma, t), Z_1(\gamma, t), Z_2(\gamma, t) \rangle = \]
\[ = \nabla F_+(\gamma(t))\langle Z_1(\gamma, t), Z_3(\gamma, t), Z_4(\gamma, t) \rangle = 0. \]

By analogous reasoning, one can prove that
\[ \nabla F_+(\gamma(t))\langle Z_\mu(\gamma, t), Z_\nu(\gamma, t), Z_\lambda(\gamma, t) \rangle = 0 \]
for all \( \mu, \nu, \lambda \in \{1, 2, 3, 4\} \). Hence the tensor \( F_+ \) is parallel and \( \| F_+ \| \) is constant on the manifold \( M \). There is a sequence \( \{x_n\} \) of points in \( M \), such that \( \lim_{n \to \infty} \| F_+(x_n) \| = 0 \). Then \( F_+ \equiv 0 \) and \( A \) is an antiinstanton.

These lemmas together imply the assertion of the main theorem.

Acknowledgments

This work is supported by the Russian Science Foundation under grant 19-11-00320.
References

[1] P. Lévy. *Problèmes concrets d’analyse fonctionnelle*. Gautier-Villars, Paris, 1951.

[2] L. Accardi, P. Gibilisco, and I.V. Volovich. Yang–Mills gauge fields as harmonic functions for the Lévy-Laplacians. *Russ. J. Math. Phys.*, 2:235–250, 1994.

[3] B. O. Volkov. Lévy–Laplacian and the gauge field. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 15:1250027, 2012.

[4] B. O. Volkov. Lévy Laplacians and instantons on manifolds. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 23:2050008, 2020.

[5] B. O. Volkov. Applications of Levy differential operators in the theory of gauge fields. *J Math Sci*, 252:20–35, 2021.

[6] R. Léandre and I.V. Volovich. The stochastic Lévy Laplacian and Yang–Mills equation on manifolds. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 4:161–172, 2001.

[7] L. Accardi. Yang–Mills equations and Lévy Laplacians. In *Dirichlet Forms and Stochastic Processes, Beijing, 1993.*, pages 593–660, Berlin; New York, 1995. de Gruyter.

[8] B. O. Volkov. Lévy Laplacians and instantons. *Proc. Steklov Inst. Math.*, 290:210–222, 2015.

[9] B. O. Volkov. Levy Laplacians, holonomy group and instantons on 4-manifolds. *Potential Analysis*, page 10013, 2022.

[10] L. Accardi and O. G. Smolyanov. Feynman formulas for evolution equations with Lévy Laplacians on infinite-dimensional manifolds. *Doklady Mathematics*, 73:252–257, 2006.

[11] L. Gross. A Poincaré lemma for connection forms. *J. Funct. Anal.*, 63:1–46, 1985.

[12] B. Driver. Classifications of bundle connection pairs by parallel translation and lassos. *J. Funct. Anal.*, 83:1–46, 1989.

[13] R. O. Bauer. Characterizing Yang–Mills fields by stochastic parallel transport. *J. Funct. Anal.*, 155:536–549, 1998.

[14] R. O. Bauer. Random holonomy for Yang–Mills fields: long–time asymptotics. *Potential Anal.*, 18:43–57, 2003.

[15] M Arnaudon, R. O. Bauer, and A. Thalmaier. A probabilistic approach to the Yang–Mills heat equation. *J. Math. Pures Appl.*, 81:43–57, 2003.

[16] M Arnaudon and A. Thalmaier. Yang–Mills fields and random holonomy along Brownian bridges. *Ann. Probab.*, 31:769–790, 2003.
[17] Yu. V. Bogdanskii and A. Yu Potapenko. Laplacian with respect to a measure on the Riemannian manifold and the dirichlet problem. i. *Ukr. Math. J.*, 68:769–790, 2016.

[18] Yu. V. Bogdanskii and A. Yu Potapenko. Laplacian with respect to a measure on the Riemannian manifold and the Dirichlet problem. ii. *Ukr. Math. J.*, 68:1665–1672, 2017.

[19] B. O. Volkov. Lévy Laplacians in Hida Calculus and Malliavin Calculus. *Proc. Steklov Inst. Math.*, 301:11–24, 2018.

[20] L. Accardi, A. Hasegawa, U. C. Ji, and K. Saitô. White noise delta functions and infinite dimensional Laplacians. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 23:2050028, 2020.

[21] V. Zh. Sakbaev. Random walks and measures on Hilbert space that are invariant with respect to shifts and rotations. *Journal of Mathematical Sciences*, 241:469, 2019.

[22] V. M. Busovikov and V. Zh. Sakbaev. Sobolev spaces of functions on a hilbert space endowed with a translation-invariant measure and approximations of semigroups. *Izv. Math.*, 84:694–721, 2020.

[23] V.Z. Sakbaev and D.V. Zavadsky. Analogs of the Lebesgue measure and diffusion in a Hilbert space. *Int J Theor Phys*, 60:617–629, 2021.

[24] W. Klingenberg. *Riemannian geometry*. de Gruyter Studies in Mathematics Vol. 1, Berlin, 1982.

[25] W. Klingenberg. *Lectures on closed geodesics*. Springer-Verlag, Berlin-New York, 1978.

[26] B. O. Volkov. Lévy Laplacian on manifold and Yang–Mills heat flow. *Lobachevskii Journal of Mathematics*, 40:1615–1626, 2019.

[27] M. N. Feller. Infinite-dimensional elliptic equations and operators of Lévy type. *Russian Math. Surveys*, 41:119–170, 1986.