Applications of the Capelli identities in physics and representation theory

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Abstract

Capelli identities are shown to facilitate the construction of representations of various Heisenberg algebras that arise in many-particle quantum mechanics and the construction of holomorphic representations of many Lie algebras by vector coherent state methods. We consider the original Capelli identity and its generalizations by Turnbull and by Howe and Umeda.

Keywords: Capelli identities, group contraction, vector coherent state theory, induced representations, holomorphic representations, Pfaffians

1. Introduction

The Capelli identities [1–5] play central roles in invariant theory [2, 6], but have received little attention in physics. However, as this paper shows, they are powerful tools for constructing asymptotic macroscopic limits of Lie algebra representations and boson approximations, of importance in physics, and they provide the essential information needed to complete the vector coherent state (VCS) construction [7–12] of induced holomorphic representations of Lie algebras. Note that, by constructing a representation of a Lie algebra we mean more than simply defining a module for the representation. A construction includes specifying a basis for the representation, which for a unitary representation should be orthonormal, and showing how it transforms under the action of Lie algebra elements. For the purposes of this paper, basis states are needed which separate into subsets that transform irreducibly under the actions of subalgebras and other Lie algebras.

This paper will show that a Lie algebra with holomorphic representations can be contracted to a semi-direct sum of a compact subalgebra and one of the Heisenberg algebras underlying the Capelli identities. For each irreducible representation (irrep) of the compact subalgebra, a corresponding unitary holomorphic irrep is constructed for the contracted Lie algebra. A motivation for this construction is that many of the spectrum generating algebras for nuclear models belong to the class of Lie algebras considered and their contractions are...
realized in macroscopic limits. A more compelling motivation is that the holomorphic representations of these semi-direct sum Lie algebras provide the necessary ingredients for the VCS construction of the holomorphic representations of the original uncontracted Lie algebra which are needed, for example, in applications to finite nuclei. This will be shown in a following publication.

The three standard Capelli identities are expressed in terms of complex variables \( z_{ij}, i, j = 1, \ldots, n \) and derivative operators \( \partial_{ij} := \partial/\partial z_{ij} \). In the original type I identity [1, 2], the \( z_{ij} \) variables are independent and, together with the derivative operators, satisfy the commutation relations

\[
[\partial_{ij}, z_{kl}] = \delta_{ik} \delta_{jl}
\]

of a Heisenberg algebra. In the type II generalization by Turnbull [3], the variables are symmetric (\( z_{ij} = z_{ji} \)) and, with the symmetrized derivative operators \( \partial_{ij} := (1 + \delta_{ij})\partial/\partial z_{ij} \), satisfy the commutation relations

\[
[\partial_{ij}, z_{kl}] = \delta_{i,k} \delta_{j,l} + \delta_{k,i} \delta_{j,l},
\]

which are likewise those of a Heisenberg algebra. In the type III generalization by Howe and Umeda [4] (see also Kostant and Sahi [5]), the variables are anti-symmetric (\( z_{ij} = -z_{ji} \)) and, with the derivative operators \( \partial_{ij} = \partial/\partial z_{ij} \), satisfy the commutation relations

\[
[\partial_{ij}, z_{kl}] = \delta_{i,k} \delta_{j,l} - \delta_{j,k} \delta_{i,l},
\]

which are again those of a Heisenberg algebra.

Recall that the Heisenberg algebra of complex variables \( z_i, i = 1, \ldots, n \) and derivative operators \( \{\partial_i\} \), with commutation relations

\[
[\partial_i, z_j] = \delta_{ij},
\]

is realized in a Bargmann representation [13] in which the raising and lowering operators \( \{c_i^\dagger, c_i\} \) of an \( n \)-dimensional oscillator are represented

\[
c_i^\dagger \rightarrow z_i, \quad c_i \rightarrow \partial_i = \partial/\partial z_i, \quad i = 1, \ldots, n,
\]

on a Hilbert space of entire analytic functions of the \( z_i \) variables. Such a Bargmann Hilbert space is spanned by monomials \( \psi_q \equiv \prod_{i=1}^n z_i^{p_i} \) with inner products

\[
\langle \psi_q | \psi_p \rangle = \prod_{i=1}^n \partial_i^{q_i} z_i^{p_i} \big|_{z_i = 0} = \delta_{q,p} \prod_{i=1}^n p_i !.
\]

Thus, the Bargmann representation of a Heisenberg algebra with \( n \) linearly-independent \( z_i \) elements is well understood in a monomial basis, and its Hilbert space naturally decomposes into subspaces that span irreps of \( g(n) \) and its \( u(n) \) real form. However, the Heisenberg algebras with \( z_{ij} \) variables having double indices have extra structure and their Bargmann spaces are Hilbert spaces for representations of different Lie algebras. Thus, the construction of the unitary irrep of a Heisenberg algebra in such variables, in bases that belong to irreps of the different Lie algebras, is a non-trivial task which, as we show, is made possible by use of the Capelli identities.

The representation of a Heisenberg algebra with symmetric variables \( z_{ij} = z_{ji}, i, j = 1, 2, 3 \) relative to an orthonormal \( U(3) \)-coupled basis was derived by Quesne [14], for use in interacting-boson models [15–18], and by Rosensteel and Rowe [19] for use in a macroscopic approximation [20, 21] to the microscopic symplectic model of nuclear collective states [22, 23]. Following the introduction of VCS theory [7, 8], it became apparent that the unitary representations of the symmetric Heisenberg matrix algebras in \( U(3) \)
coupled bases are needed for the VCS construction of the holomorphic representations of the non-compact Lie algebra $\mathfrak{sp}(3, \mathbb{R})$ in related $U(3)$-coupled bases; these expressions of the holomorphic representations of $\mathfrak{sp}(3, \mathbb{R})$ are central to the many-nucleon theory of nuclear collective dynamics [24]. In fact, the representations of the Heisenberg matrix algebras associated with all the Capelli identities enable the construction of orthonormal basis states for the holomorphic representations of many Lie groups and their Lie algebras by algebraic methods which avoid the difficult inner products of Harish-Chandra theory [25, 26] and the restriction of holomorphic representations to those of discrete series. This subject is reviewed and developed in a following paper [27] (see also section 4). In consideration of these possibilities, it was recognized [28] that the needed representations of the Heisenberg algebras follow from the Capelli identities.

This paper extends and generalizes the results of [28], which did not consider the Heisenberg algebra associated with the type I Capelli identity and was written before the type III identity was discovered. It shows that the Capelli identities lead to algorithms for the construction of irreducible unitary representations of the corresponding Heisenberg algebras, albeit each in different but equally useful bases.

2. Constructions of a Lie algebra

A procedure for deriving contractions of a Lie algebra has been given by İnönü and Wigner [29] in which some elements of the Lie algebra contract to an Abelian algebra. Other contractions have been defined, for example, by Saletan [30], Weimer-Woods [31], and others. The following definition is appropriate for the construction of holomorphic representations, in which subsets of elements contract to Heisenberg algebras. Such contractions lead to so-called boson approximations and underlie models in physics that become accurate in limiting (e.g., large particle number) situations [20, 45]. They also provide leading-order terms in the precise boson expansions of VCS theory and essential steps in the construction of holomorphic representations of the Lie algebras to which they apply.

Let $g_0$ be a real semi-simple or reductive Lie algebra which, as a vector space, has a direct sum decomposition

$$g_0 = h_0 + p_0,$$

where $h_0$ is a subalgebra and $p_0$ is invariant under the adjoint action of $h_0$ i.e., $[h_0, p_0] \subseteq p_0$. The complex extension $g_0$ of $g$ has the parallel decomposition

$$g = h + p.$$  

A map $g \rightarrow \tilde{g}: X \mapsto \tilde{X}$, from $g$ to a semi-direct sum Lie algebra $\tilde{g}$, for which $\tilde{p}$ is an ideal, is said to be a contraction of $g$ if it maps $h \rightarrow \tilde{h}$ and $p \rightarrow \tilde{p}$ such that

(i) $\tilde{h} \rightarrow h$ is an isomorphism,  

(ii) $[\tilde{X}, \tilde{Y}] = \tilde{Z}$ if $[X, Y] = Z$,  

$$\forall X \in h \text{ and } \forall Y, Z \in p,$$

(iii) $[\tilde{Y}, \tilde{Z}] \in \mathbb{C}$,  

$$\forall Y, Z \in p,$$

where $\mathbb{C}$ denotes the complex numbers regarded as a one-dimensional subalgebra of $\tilde{p}$. Condition (ii) ensures that the elements of $\tilde{p}$ transform under the adjoint action of $\tilde{h}$ in the same way as the elements of $p$ transform under the adjoint action of $h$. Condition (iii) is defined such that the elements of $p$ contract to elements of either an Abelian or (with the inclusion of an identity element in $\tilde{p}$) a Heisenberg algebra. Such contractions of $g$ define
corresponding contractions $g_0 \rightarrow \tilde{g}_0$ of its real forms. They are particularly useful when $h_0$ is compact and $[p, p] \subset h$. If $G_0$ and $H_0$ are the maximal connected subgroups with Lie algebras given, respectively, by $g_0$ and $h_0$, then the factor space $G_0/H_0$ is a symmetric space and VCS theory gives a prescription for inducing holomorphic irreps of $g_0$ and $G_0$ from irreps of $h_0$ and $H_0$ [11, 12]. For example, $su(3)$ has a direct sum vector-space decomposition

$$su(3) = so(3) + p_0,$$

in which $p_0$ is a five-dimensional subset of elements of $su(3)$ that transform as the components of an angular-momentum $L = 2$ irrep under the adjoint action of $so(3)$, i.e., $[so(3), p_0] \subset p_0$, and $[p_0, p_0] \subset so(3)$. The $su(3)$ elements of this decomposition satisfy commutation relations

$$[L_i, L_j] \in so(3), \quad \forall L_i, L_j \in so(3),$$

$$[L_i, Q_\nu] \in p_0, \quad \forall L_i \in so(3) \quad \text{and} \quad \forall Q_\nu \in p_0,$$

$$[Q_\rho, Q_\nu] \in so(3), \quad \forall Q_\rho, Q_\nu \in p_0.$$ There is then a contraction of $su(3)$ to a semi-direct sum Lie algebra $Rs \subset sl(3)$ in which the $so(3)$ subalgebra is unchanged and the contracted elements of $p_0$ continue to transform as components of an $L = 2$ tensor under the adjoint action of $so(3)$ but span a real five-dimensional Abelian Lie algebra isomorphic to $R^5$. This contraction is of interest because it exposes the macroscopic limit of the angular-momentum states of a large-dimensional $SU(3)$ irrep to be those of a rigid-rotor [32] in which the intrinsic states of the rotor are described by a representation of the $R^5$ subalgebra and its rotational states carry representations of $so(3)$. The following examples, in which $p$ includes an identity and is a Heisenberg algebra, are of relevance to this paper

### 2.1. Contraction of $u(p + q)$ and $u(p, q)$ Lie algebras

As a first example, consider the Lie algebras $u(p + q)$ and $u(p, q)$ with a common compact subalgebra $h_0 = u(p) \oplus u(q)$ and common complex extension $g = gl(p + q)$. This extension is a matrix group with elements

$$\begin{pmatrix} X & A \\ B & Y \end{pmatrix} \quad \text{with} \quad X \in gl(p), \quad A \in p_+,$$

$$B \in p_-, \quad Y \in gl(q),$$

where $p_\pm$ are, respectively, Abelian subalgebras of raising and lowering operators and $p$ is their vector-space direct sum $p = p_+ + p_-$ with $[p_+, p_+] \subset h$. An equivalent realization is given in differential form by

$$X = \sum_{i=1}^q \sum_{j=1}^q X_{ij} x_i \frac{\partial}{\partial x_j}, \quad A = \sum_{i=1}^p \sum_{a=1}^q A_{ia} x_i \frac{\partial}{\partial y_a},$$

$$B = \sum_{a=1}^p \sum_{i=1}^q B_{ai} y_a \frac{\partial}{\partial x_i}, \quad Y = \sum_{a=1}^p \sum_{\beta=1}^q Y_{\alpha\beta} y_\alpha \frac{\partial}{\partial y_\beta}.$$
It is now seen that there are contractions in which
\[
X \rightarrow \sum_{ij} X_{ij} z_{ia} \partial_{ja}, \quad A \rightarrow k \sum_{ia} A_{ia} z_{ia}, \quad (19)
\]
\[
B \rightarrow k^2 \sum_{ia} B_{aa} \partial_{ia}, \quad Y \rightarrow \sum_{ij} Y_{ij} z_{ja} \partial_{ia}, \quad (20)
\]
where \( \partial_{ia} = \partial / \partial z_{ia} \), and \( \chi \) and \( \gamma \) are representations of the respective elements \( X \in u(p) \) and \( Y \in u(q) \) that commute with the \( \{ z_{ia} \} \) variables and their derivatives. In accordance with the above definition, this contraction conserves all the commutation relations of the \( gl + pq \) Lie algebra except for those of \( A \in p_+ \) and \( B \in p_- \) for which
\[
[B, A] \rightarrow [k]^2 \sum_{ia} B_{aa} A_{ia} \in \mathbb{C}. \quad (21)
\]
The value of \( k \) in this contraction is, in principle, arbitrary. But specific and possibly \( ia \) -dependent values will be appropriate for different irreps as they approach asymptotic limits. These contractions define a semi-direct sum of \( u(p) \oplus u(q) \) and a Heisenberg Lie algebra.

It will be shown in the following that, with a knowledge of elementary \( u(p) \) and \( u(q) \) vector-coupling coefficients (also called Clebsch–Gordan and Wigner coefficients), the unitary irreps of the Heisenberg algebra that emerges on the space of polynomials in the above \( \{ z_{ia} \} \) variables, and corresponding irreps of the \( u(p + q) \) and \( u(p, q) \) contractions, can be constructed in a \( u(p) \oplus u(q) \) basis by use of the first Capelli identity.

This \( g \rightarrow \bar{g} \) contraction can be used to derive an important approximation in physics, known as the random phase approximation (RPA) [33], which has many applications in the quantum theory of many-particle systems; see [34] for a review and an algebraic expression of the RPA in a nuclear physics context. The RPA is a quantal version of the classical theory of small-amplitude normal-mode vibrations. Let \( g_0 = u(p + q) \) be the unitary transformations of a \( (p + q) \)-dimensional Hilbert space \( H \) of single-fermion states. A Hilbert space \( H \) for \( q \) fermions is then defined as the space spanned by all exterior products of \( q \) states in \( g_0 \). A Hamiltonian \( H \) for this space, with interactions between pairs of fermions, is a Hermitian combination of linear and quadratic elements of \( g_0 = gl + pq \). A step towards determining low-energy states of \( H \) is to separate the Hilbert space \( H \) into subspaces
\[
H = H^{(q)} \oplus H^{(p)}, \quad (22)
\]
such that the energy \( \langle \phi | H | \phi \rangle \) of the \( q \)-fermion state \( | \phi \rangle \), given by the exterior product of an orthonormal basis for \( H^{(q)} \), minimizes the energy among all such states for different choices of \( H^{(p)} \). Such a minimum-energy state \( | \phi \rangle \) defines a zero-order Hartree–Fock approximation for the ground state of the Hamiltonian \( H \). It also defines a decomposition of the Lie algebra \( g = gl(p + q) \) into two parts, \( \mathfrak{h} + \mathfrak{p} \) as defined above, with \( \mathfrak{h} = gl(q) \oplus gl(p) \). It is then shown by Hartree–Fock variational methods (see [35], for example) that, because of the separation of \( H \) and \( \mathfrak{h} \) into complementary subspaces by minimization of \( \langle \phi | H | \phi \rangle \), the Hamiltonian \( H \) is expressible as a sum of linear terms in \( \mathfrak{h} \) and interaction terms that are bilinear in the elements of \( \mathfrak{p} \). It is also determined that, for many-particles (i.e., large values of \( q \) and \( p \)), there is a neighbourhood of the Hilbert space \( H \) about the state \( | \phi \rangle \) on which the action of the Lie algebra \( g \) is accurately replaced by its contraction limit. Moreover, the Hamiltonian on this neighbourhood becomes a quadratic function of the elements of the Heisenberg algebra of the coupled harmonic oscillator form.
\[ H \rightarrow H_{\text{RPA}} = E_0 + \sum_{\mu ij} V_{\mu ij} z_{ij} \frac{\partial}{\partial z_{ij}} + \sum_{\mu ij} W_{\mu ij} z_{ij} \frac{\partial^2}{\partial z_{ij} \partial z_{ij}}, \]  

(23)

and easily diagonalized to produce an improved approximation for a so-called correlated ground state and its low-energy excitations. The RPA proves to be a good approximation in many systems for which the ground state is already reasonably-well approximated by \( \phi \).

One can now anticipate that with the RPA set on a clear algebraic foundation, higher-order approximations to it will emerge naturally from the holomorphic representations of \( u(p + q) \) in a \( u(q) \oplus g(p) \) basis.

### 2.2. Contraction of \( \mathfrak{sp}(N) \) and \( \mathfrak{sp}(N, \mathbb{R}) \) Lie algebras

The compact \( \mathfrak{sp}(N) \) and non-compact \( \mathfrak{sp}(N, \mathbb{R}) \) Lie algebras have a common complex extension \( \mathfrak{sp}(N, \mathbb{C}) \) and a common compact subalgebra \( u(N) \) for which \( g(n) \) is the complex extension. The algebra \( \mathfrak{sp}(N, \mathbb{C}) \) has a simple realization with basis given in terms of variables \( x_i, i = 1, ..., N \) and corresponding derivative operators by

\[ C_{ij} = \frac{1}{2} \left( x_i \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_i} x_j \right), \quad A_{ij} = x_i x_j, \quad B_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}. \]  

(24)

It also has a contraction in which

\[ C_{ij} \rightarrow C_{ij} + \sum_s z_{is} \delta_{js}, \quad A_{ij} \rightarrow k z_{ij}, \quad B_{ij} \rightarrow k^* \delta_{ij}, \]  

(25)

for some \( k \in \mathbb{C} \), where the elements \( \{C_{ij}^{(0)}\} \) are representations of elements of \( g(n) \) that commute with the \( \{z_{ij}\} \) variables and their derivatives, and

\[ z_{ij} = z_{ji}, \quad \delta_{ij} = \left( 1 + \delta_{ij} \right) \partial / \partial z_{ij}. \]  

(26)

Thus, \( \{z_{ij}\} \) and \( \{\delta_{ij}\} \) are raising and lowering operators for a Heisenberg algebra of the type II Capelli identity and the above contractions are representations of a semi-direct sum of \( u(N) \) and such a Heisenberg Lie algebra.

The non-compact \( \mathfrak{sp}(N, \mathbb{R}) \) algebras and the Lie groups they generate have important applications in physics, e.g., in beam optics for particles and light [36, 37] and in the microscopic theory of nuclear collective dynamics [22, 24, 38]. The compact \( \mathfrak{sp}(N) \) symplectic algebras and associated Lie groups are also used extensively in the coupling schemes of atomic [39] and nuclear [40, 41] shell models. The contraction limits of the symplectic algebras characterize the quantum dynamics of these systems as they approach their macroscopic limits. Particularly important is the fact that uncontracted representations of both the compact and non-compact symplectic Lie algebras and groups can be induced, by VCS methods, in a computationally tractable manner from the representations of their contractions.

### 2.3. Contraction of \( \mathfrak{so}(2N) \) and \( \mathfrak{so}(2N)^* \) Lie algebras

The \( \mathfrak{so}(2N) \) and \( \mathfrak{so}(2N)^* \) Lie algebras have a common complex extension \( \mathfrak{so}(2N, \mathbb{C}) \) and a common compact subalgebra given again by \( u(n) \) with complex extension \( g(n) \). The Lie algebra \( \mathfrak{so}(2N, \mathbb{C}) \) has a simple realization with basis elements

\[ C_{ij} = \frac{1}{2} \left( a_i^+ a_j - a_j a_i^+ \right), \quad A_{ij} = a_i^+ a_j^+, \quad B_{ij} = a_j a_i, \]  

(27)

expressed in terms of fermion creation and annihilation operators \( \{a_i^+, i = 1, ..., N\} \) and \( \{a_i, i = 1, ..., N\} \) that obey anti-commutation relations
where the anti-commutator of two operators \( X \) and \( Y \) is defined by \( \{ X, Y \} = XY + YX \).

For a complex parameter \( k \), the \( \mathfrak{so}(2N, \mathbb{C}) \) algebra has a contraction

\[
C_{ij} \rightarrow C_{ij} + \sum_s z_{is} \partial_{js}, \quad A_{ij} \rightarrow k z_{ij}, \quad B_{ij} \rightarrow k^n \partial_{ij},
\]

where \( C_{ij} \) represents an element of \( \mathfrak{gl}(n) \) that commutes with the \( \{ z_{ij} \} \) variables and their derivatives, and

\[
z_{ij} = -z_{ji}, \quad \partial_{ij} = \partial/\partial z_{ij}.
\]

The \( \{ z_{ij} \} \) variables and the derivative operators \( \{ \partial_{ij} \} \) are now the raising and lowering operators of a Heisenberg algebra of the type III Capelli identity. These contractions are again seen to be representations of a semi-direct sum of \( \mathfrak{u}(N) \) and a Heisenberg Lie algebra.

The compact \( \text{SO}(2N) \) Lie group is of significance in the physics of fermion superconductivity in which it features as the group of Bogolyubov–Valatin transformations that leave the fermion anti-commutation relations invariant [35, 42]. Thus, this group \( \text{SO}(2N) \) as well as it extension to \( \text{SO}(2N + 1) \) feature in theories of fermion pair coupling (see, for example, [43] and references therein). The contractions of the \( \mathfrak{so}(2N) \) Lie algebra, known as the fermion pair algebra, are relevant to many-fermion models in which pairs of fermions are treated as bosons [44, 45].

3. Representations of the three classes of Heisenberg algebra

The contraction of a Lie algebra \( \mathfrak{g} = \mathfrak{h} + \mathfrak{p} \) is defined above in terms of two algebras: \( \mathfrak{h} \) and a Heisenberg algebra. A Heisenberg algebra has a single unitary irrep to within unitary equivalence [46, 47]. However, to construct a unitary irrep of the contracted Lie algebra, the irrep of the Heisenberg algebra is needed in a basis that separates its Hilbert space into irreducible subspaces of \( \mathfrak{h} \). Such a basis is said to reduce the representation of \( \mathfrak{h} \) on the given Hilbert space.

3.1. Heisenberg algebras related to type I Capelli algebras

The Heisenberg algebra that emerges in the contraction of the general linear algebra \( \mathfrak{gl}(p + q) \) contains a set of variables \( \{ z_{\alpha \lambda}; i = 1, \ldots, p, \alpha = 1, \ldots, q \} \) and their derivatives \( \{ \partial_{\alpha \lambda} := \partial/\partial z_{\alpha \lambda} \} \) that satisfy the commutation relations

\[
\left[ \partial_{\alpha \lambda}, z_{\beta \mu} \right] = \delta_{i,\mu} \delta_{\alpha,\beta}.
\]

General linear groups \( \text{GL}(p) \) and \( \text{GL}(q) \) have fundamental representations as left and right linear transformations, respectively, of the \( \{ z_{\alpha \lambda} \} \) variables. Elements of their \( \mathfrak{gl}(p) \) and \( \mathfrak{gl}(q) \) Lie algebras then have corresponding realizations given by

\[
L_{ij}^{(p)} := \sum_{\alpha=1}^p z_{\alpha \lambda} \partial_{\lambda \mu}, \quad R_{ij}^{(q)} := -\sum_{\lambda=1}^q z_{\lambda \mu} \partial_{\lambda \alpha}.
\]

Thus, in addition to carrying an irrep of a Heisenberg algebra, the Bargmann Hilbert space \( \mathbb{B}^{(pq)} \) spanned by polynomials in the \( \{ z_{\alpha \lambda} \} \) variables is a module for a representation of the direct product \( \text{GL}(p) \times \text{GL}(q) \). This representation is a sum of outer tensor product irreps \( \bigotimes \{ \nu \} \otimes \{ \nu \} \), with highest weights
where $N = \min(p, q)$. This decomposition of the representation of $GL(p) \times GL(q)$ on the Bargmann space $B^{(pq)}$ follows directly from the Schur–Weyl duality theorem [2] and is a prototype of Howe duality [48] (see also [42] for a review of duality relationships in a physics context). With an inner product for $B^{(pq)}$ defined such that $\partial_{\alpha i} = \text{Hermitian adjoint of } \alpha_{z i}$, which we denote by $\partial = \alpha_{z i}^\dagger$, the representation of $GL(p) \times GL(q)$ on $B^{(pq)}$ restricts to a unitary representation of its $U(p) \times U(q)$ real form.

### 3.1.1. Extremal states for irreps of $U(p) \times U(q)$ on subspaces of $B^{(pq)}$

Irreps of $U(p) \times U(q)$ on subspaces of $B^{(pq)}$ are conveniently characterized by extremal states, defined as states of highest weight relative to $U(p)$ and of lowest weight relative to $U(q)$. For example, the elementary polynomial $z_{11}$ is of highest $U(p)$ weight $\nu = \{1\}$ and lowest $U(q)$ weight $\nu^{-} = \{-1\}$. A general (non-normalized) extremal state for a $U(p) \times U(q)$ irrep on a subspace of $B^{(pq)}$ with highest $U(p)$ weight $\nu$ and lowest $U(q)$ weight $\nu^{-}$ is given by

$$\psi^{(N)}(z) := x_{1}^\nu x_{2}^{\nu^{-}} \ldots x_{N}^{\nu^{-}}, \quad \text{with } \nu = \sum_{j=1}^{N} \pi_{j} \text{ and } N \leq \min(p, q).$$

Thus, every such $U(p) \times U(q)$ extremal weight is uniquely defined by its $U(p)$ highest weight component.

### 3.1.2. Normalization of $U(p) \times U(q)$ extremal states.

The normalizations of the above-defined extremal states are determined by use of the type I Capelli identity. This identity states that the product of the determinants

$$x_{n} := \det \begin{pmatrix} z_{11} & z_{12} & \ldots & z_{1n} \\ z_{21} & z_{22} & \ldots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \ldots & z_{nn} \end{pmatrix}, \quad V_{n} := \det \begin{pmatrix} \partial_{11} & \partial_{12} & \ldots & \partial_{1n} \\ \partial_{21} & \partial_{22} & \ldots & \partial_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{n1} & \partial_{n2} & \ldots & \partial_{nn} \end{pmatrix},$$

for $n = 1, \ldots, \min(p, q)$. Then, $x_{n}$ is a polynomial of highest $U(p)$ weight $\{\nu\}$ and lowest $U(q)$ weight $\{-\nu^{-}\}$. A general (non-normalized) extremal state for a $U(p) \times U(q)$ irrep on a subspace of $B^{(pq)}$ with highest $U(p)$ weight $\nu$ and lowest $U(q)$ weight $-\nu$ is given by

$$x_{n} V_{n},$$

where

$$E_{ij}^{(n)} = \sum_{s=1}^{n} z_{is} \partial_{sj}, \quad i, j = 1, \ldots, n \leq N,$$

is an element of a $u(n)$ Lie algebra and it is understood that the determinant on the right side of (36) is defined with products of elements from different columns that preserve their left to right order. An equivalent expression is given by

$$V_{n} x_{n} = \det \left[ E_{ij}^{(n)} + (n + 1 - i) \delta_{ij} \right].$$

The squared norm of the wave function (35)

$$\left\langle \psi_{\nu}^{(N)} \right| \psi_{\nu}^{(N)} \right\rangle = \left\langle V_{1}^{\pi_{1}} V_{2}^{\pi_{2}} \ldots V_{N-1}^{\pi_{N-1}} V_{N}^{\pi_{N}} x_{1}^{\nu_{1}} x_{2}^{\nu_{2}} \ldots x_{N}^{\nu_{N}} \right\rangle$$

(39)
is determined from the observation that, because an extremal state \( \psi \) satisfies the equation
\[ E_{ij}^{(n)} \psi = 0 \text{ for } i < j, \]
it is an eigenstate of the operator \( V_n x_n \) of equation (38) with eigenvalue given by
\[ V_n x_n \psi = \left( E_{11}^{(n)} + n \right) \left( E_{22}^{(n)} + n - 1 \right) \cdots \left( E_{nn}^{(n)} + 1 \right) \psi. \] (40)

Also, because \( x_n \) is itself of extremal-weight \( \{ \lambda \} \), it follows that
\[ V_n x_n \psi = V_n \left( E_{11}^{(n)} + n \right) \left( E_{22}^{(n)} + n - 1 \right) \cdots \left( E_{nn}^{(n)} + 1 \right) \psi \]
\[ = V_n \left( E_{11}^{(n)} + n + 1 \right) \left( E_{11}^{(n)} + n + 1 \right) \cdots \left( E_{nn}^{(n)} + 2 \right) \psi \]
\[ = \left( E_{11}^{(n)} + n + 1 \right)! \left( E_{22}^{(n)} + n \right)! \cdots \left( E_{nn}^{(n)} + 2 \right)! \psi. \] (41)

Proceeding in this way, it is determined that
\[ V_n^{p} \psi \psi = \left( E_{11}^{(n)} + n + \pi - 1 \right)! \left( E_{22}^{(n)} + n + \pi - 2 \right)! \cdots \left( E_{nn}^{(n)} + \pi \right)! \]
\[ \left( E_{11}^{(n)} + n - 1 \right)! \left( E_{22}^{(n)} + n - 2 \right)! \cdots \left( E_{nn}^{(n)} \right)! \psi. \] (42)

The definition (37) of \( E_{ij}^{(n)} \) implies that
\[ E_{ii}^{(n)} \psi^{(m)} = E_{ii}^{(m)} \psi^{(m)}, \text{ for } n \geq m. \] (45)

It follows that
\[ \left\langle V_i^{p} x_i^{p} \right\rangle = \frac{\left( E_{11}^{(1)} + \pi_1 \right)!}{E_{11}^{(1)}!} = \pi_1!. \] (46)
\[ \left\langle V_i^{p} V_j^{p} x_j^{p} \right\rangle = \frac{\left( E_{11}^{(2)} + \pi_2 + 1 \right)! \left( E_{22}^{(2)} + \pi_2 \right)!}{E_{11}^{(2)}! E_{22}^{(2)}!} \frac{\pi_2}{\pi_1 + 1}, \] (47)
and that
\[ \left\langle V_i^{p} V_j^{p} x_j^{p} x_k^{p} \right\rangle = \frac{\left( \pi_1 + \pi_2 + \pi_3 + 2 \right)! \left( \pi_2 + \pi_3 + 3 \right)!}{\pi_1 + 1 \pi_1 + \pi_2 + 2} \frac{\pi_1}{\pi_2 + 1}. \] (48)

Thus, in terms of weight components \( \nu_i = \sum_{j=1}^{n} \pi_j \),
\[ \left\langle \psi^{(1)} \right| \psi^{(1)} \right\rangle = \nu_1!, \] (49)
\[ \left\langle \psi^{(2)} \right| \psi^{(2)} \right\rangle = \frac{\left( \nu_1 + 1 \right)! \nu_2!}{\nu_1! \nu_2 + 1!}; \] (50)
\[ \left\langle \psi^{(3)} \right| \psi^{(3)} \right\rangle = \frac{\left( \nu_1 + 2 \right)! \left( \nu_2 + 1 \right)! \nu_3!}{\left( \nu_1 + \nu_2 + 1 \right)! \left( \nu_1 + \nu_3 + 2 \right)! \left( \nu_2 + \nu_3 + 1 \right)!}, \] (51)
and the pattern becomes recognizable.
Claim I. The squared norm of the wave function $\psi^{(N)}_\nu$ defined by equation (35) is

$$\mathcal{N}_\nu^{(N)} = \left\langle \psi^{(N)}_\nu \right| \psi^{(N)}_\nu \right\rangle = \prod_{i=1}^{N} \frac{\left(\nu_i + N - i\right)!}{\prod_{j=i+1}^{N} \left(\nu_i - \nu_j + j - i\right)!}.$$  (52)

Proof. The claim has been established for $N \leq 3$. For arbitrary $N$, the wave function $\psi^{(N)}_\nu$ is a product

$$\psi^{(N)}_\nu = x^{\pi_\mu}_\nu \psi^{(N-1)}_\mu,$$  (53)

where $\psi^{(N-1)}_\mu$ is a highest-weight polynomial of weight $\mu$ and

$$\nu = \left(\mu_1 + \pi_N, \mu_2 + \pi_N, \ldots, \mu_{N-1} + \pi_N, \pi_N\right).$$  (54)

Then

$$\mathcal{N}_\nu^{(N)} = \left\langle \psi^{(N-1)}_\mu \right| \left[ \mathcal{V}^{\pi_N}_N x^{\pi_\mu}_\nu \psi^{(N-1)}_\mu \right]$$  (55)

and, by use of equation (44),

$$\mathcal{N}_\nu^{(N)} = \left(\nu_1 + N - 1\right)! \left(\nu_2 + N - 2\right)! \cdots \nu_N! \left(\nu_1 - \nu_N + N - 1\right)! \left(\nu_2 - \nu_N + N - 2\right)! \cdots \left(\nu_{N-1} - \nu_N\right)! \left\langle \psi^{(N-1)}_\mu \left| \psi^{(N-1)}_\mu \right\rangle \right\rangle.$$  (56)

consistent with the claim. Thus, because the claim is valid for $N \leq 3$, it is valid for all $N$. □

3.1.3. Matrix elements of a type I Heisenberg algebra between $U(p) \times U(q)$ extremal states.

Given some elementary vector-coupling coefficients for the $u(p)$ and $u(q)$ irreps, one can obtain all matrix elements of a type I Heisenberg algebra in a $U(p) \times U(q)$-coupled basis from the subset

$$\left\langle \nu + \Delta_k \left| z_{\ell k} \right| \nu \right\rangle = \left\langle \nu \right| \left[ \partial_{\ell k} \left( \nu + \Delta_k \right) \right]^\ast,$$  (57)

(* denotes complex conjugation) for normalized extremal states

$$\left\| \nu + \Delta_k \right\| = \frac{\left\| \psi^{(N)}_\nu \right\|}{\left\langle \psi^{(N)}_\nu \right| \left[ \psi^{(N)}_\nu \right] \rangle^{1/2}}, \left\| \nu \right\| = \frac{\left\| \psi^{(N)}_\nu \right\|}{\left\langle \psi^{(N)}_\nu \right| \left[ \psi^{(N)}_\nu \right] \rangle^{1/2}},$$  (58)

where $\psi^{(N)}_\nu$ is defined by equation (35) and $\Delta_k = \left(0, \ldots, 0, 1, 0, \ldots, 0\right)$ is the $u(N)$ weight of $z_{\ell k}$ for which all but its $k$ component is equal to zero. It follows from the expression (35) of the wave function $\psi^{(N)}_\nu$ that the corresponding wave function $\psi^{(N)}_{\nu + 0}$, with a shifted weight is given by

$$\psi^{(N)}_{\nu + 0} = x^{\pi_1}_1 \cdots x^{\pi_{k-2}}_{k-2} x^{\pi_{k-1}}_{k-1} x^{\pi_{k+1}}_k x^{\pi_{k+2}}_{k+1} \cdots x^{\pi_N}_N.$$  (59)

Consider the ratio

$$R := \frac{\left\langle \psi^{(N)}_{\nu + \Delta_k} \left| z_{\ell k} \psi^{(N)}_\nu \right\rangle}{\left\langle \psi^{(N)}_{\nu + \Delta_k} \right| \left[ \psi^{(N)}_{\nu + \Delta_k} \right] \rangle} = \frac{\left\langle \psi^{(N)}_\nu \right| \left[ \partial_{\ell k} \psi^{(N)}_{\nu + \Delta_k} \right]}{\left\langle \psi^{(N)}_\nu \right| \left[ \psi^{(N)}_\nu \right] \rangle}.$$  (60)
with
\[ W = V_x^{\pi e_1} \cdots V_x^{\pi e_N} x^{\pi e_N} \cdots x^{\pi e_1}. \] (62)

This ratio has the simplifying property that the function \( x^{\pi e_1} x^{\pi e_{k-1}} x^{\pi e_k} \cdots x^{\pi e_1} \to \) the right of the operator \( W \) is the same in both its numerator and denominator. Moreover, this function is an eigenfunction of \( W \). Thus, its eigenvalue can be factored out to give
\[ R = \frac{\langle V_1^{\pi e_1} \cdots V_k^{\pi e_k} x^{\pi e_k} x^{\pi e_{k-1}} x^{\pi e_k} \cdots x^{\pi e_1} \rangle}{\langle V_1^{\pi e_1} \cdots V_{k-1}^{\pi e_{k-1}} V_k^{\pi e_k} x^{\pi e_k} x^{\pi e_{k-1}} x^{\pi e_k} \cdots x^{\pi e_1} \rangle}, \] (63)

where use is made of the observation that \( \{\partial_{x_k}, x_k\} = 0 \) for all \( k' < k \). Now, with the identity
\[ \partial_{x_k}, x_k^{\pi e} = (\pi_k + 1)x_{k-1}x_k^{\pi e}, \] (64)

it is determined that
\[ R = (\pi_k + 1) \frac{\langle V_1^{\pi e_1} \cdots V_k^{\pi e_k} x^{\pi e_k} x^{\pi e_{k-1}} x^{\pi e_k} \cdots x^{\pi e_1} \rangle}{\langle V_1^{\pi e_1} \cdots V_{k-1}^{\pi e_{k-1}} V_k^{\pi e_k} x^{\pi e_k} x^{\pi e_{k-1}} x^{\pi e_k} \cdots x^{\pi e_1} \rangle}, \] (65)

where
\[ \psi^{(k)}_\mu := x_1^{\pi e_1} x_2^{\pi e_2} \cdots x_k^{\pi e_k}, \quad \mu_j = \sum_{j=i}^k \pi_j = \nu_i - \nu_{k+1}, \quad i = 1, \ldots, k. \] (66)

It follows that the desired matrix elements
\[ \langle \nu + \Delta_k | z_{kk} | \nu \rangle = \frac{\langle \nu^{(N)}_e | z_{kk} | \nu^{(N)}_e \rangle}{\langle \nu^{(N)}_e | \psi^{(N)}_e | \nu^{(N)}_e \rangle}, \] (67)

are given by
\[ \langle \nu + \Delta_k | z_{kk} | \nu \rangle = (\nu_k - \nu_{k+1} + 1) \frac{\langle \psi^{(k)}_\mu | \psi^{(k)}_\mu \rangle}{\langle \psi^{(k)}_\mu | \psi^{(k)}_\mu \rangle}, \] (68)

and are easily evaluated with the expression for the norms given by claim I.
3.1.4. Unitary representation of the type I Heisenberg algebra in a \( U(p) \times U(q) \)-coupled basis.\] The essential tool that makes it possible to determine all matrix elements of the above-defined Heisenberg algebra in terms of elementary \( U(p) \) and \( U(q) \) vector-coupling coefficients is the Wigner–Eckart theorem\(^1\).

The Wigner–Eckart theorem states, for example, that if \( \lambda \) labels a \( U(N) \) irrep and \( \mu \) indexes an orthonormal basis for this irrep then, if \( T_{\lambda_2} \) is a tensor operator whose components \( \{ T_{\lambda_2, \nu} \} \) transform as basis states for a \( U(N) \) irrep \( \lambda_2 \), the matrix elements of this tensor between basis states of any two \( U(N) \) irreps are expressible as a sum of products
\[
\langle \lambda_3 \mu_3 | T_{\lambda_2, \mu_2} | \lambda_1 \mu_1 \rangle = \sum_\rho \langle \lambda_1 \mu_1, \lambda_2 \mu_2 | \rho \lambda_3 \mu_3 \rangle \langle \lambda_3 \parallel T_{\lambda_2} \parallel \lambda_2 \rangle_\rho,
\]
where \( \rho \) indexes the \( n_\lambda \) occurrences of the \( U(N) \) irrep \( \lambda_3 \) in the tensor product \( \lambda_2 \otimes \lambda_1 = \oplus_{\nu} n_\nu \lambda_\nu \). \( \langle \lambda_1 \mu_1, \lambda_2 \mu_2 | \rho \lambda_3 \mu_3 \rangle \) is a vector-coupling coefficient, and the so-called reduced matrix element \( \langle \lambda_3 \parallel T_{\lambda_2} \parallel \lambda_2 \rangle_\rho \) is independent of the choice of bases. The theorem is valuable because the vector-coupling coefficients are determined by the properties of the \( U(N) \) irreps, independently of how they are realized in any particular situation. Thus, if the vector-coupling coefficients are known, one has only to calculate \( n_\lambda \) distinct non-zero matrix element \( \langle \lambda_3 \mu_3 | T_{\lambda_2, \mu_2} | \lambda_2 \mu_2 \rangle_\rho \) to determine the reduced matrix elements \( \langle \lambda_3 | T_{\lambda_2} | \lambda_2 \rangle_\rho \). From the unitarity of a \( U(N) \) representation it also follows that the matrix elements of the Hermitian adjoints of a tensor operator are given by
\[
\langle \lambda_3 \mu_3 | \left( T_{\lambda_2, \mu_2} \right)^\dagger | \lambda_1 \mu_1 \rangle = \langle \lambda_1 \mu_1 | T_{\lambda_2, \mu_2} | \lambda_3 \mu_3 \rangle^*,
\]
where * denotes complex conjugation.

Application of the Wigner–Eckart theorem to the calculation of matrix elements of the Heisenberg algebras that we consider is simplified considerably by the fact that the elements of these algebras are components of tensors \( T_{\lambda_2} \) for which the tensor product \( \lambda_2 \otimes \lambda_1 \) is always multiplicity free. Thus, for present purposes, we have no need of the multiplicity index and all matrix elements of the Heisenberg algebra under consideration are defined by single matrix elements between extremal states. A minor complication arises for the type I Heisenberg algebras because their elements are components of a direct product \( U(p) \times U(q) \) tensor. Nevertheless, as we now show, the construction remains simple.

An element \( z_{\alpha} \) of the type I Heisenberg algebra is a component of weight \( \Delta_i \) of a \( U(p) \) tensor of highest weight \( \Delta_i \) and a component of weight \( -\Delta_q \) of a \( U(q) \) tensor that is also of highest weight \( \Delta_q \). By definition \( |\nu\rangle \) is a state of highest weight \( \nu \) of a \( U(p) \) irrep and of lowest weight \( -\nu \) of a \( U(q) \) irrep. Equation (68) gives a non-vanishing matrix element \( \langle \nu | z_{\alpha} | \nu \rangle \) with \( k \) such that \( \Delta_k = \nu' = -\nu \). Thus, according to the Wigner–Eckart theorem, the matrix element of equation (68) is equal to the product
\[
\langle \nu | z_{\alpha} | \nu \rangle = \langle \nu, \Delta_i \Delta_q | \nu' \rangle \langle \nu, -\nu, \Delta_i, -\Delta_q | \nu', -\nu' \rangle \langle \nu' | z \parallel \nu \rangle,
\]
in which \( \langle \nu, \Delta_i \Delta_q | \nu' \rangle \) and \( \langle \nu, -\nu, \Delta_i, -\Delta_q | \nu', -\nu' \rangle \) are, respectively, \( U(p) \) and \( U(q) \) vector-coupling coefficients, and \( \langle \nu' | z \parallel \nu \rangle \) is a \( U(p) \times U(q) \)-reduced matrix element for their combined irreps. Thus, the matrix elements of any \( z_{\alpha} \) are obtained by the further use of

\(^1\) In its original application to \( SU(2) \), the Wigner–Eckart theorem \([59]\) states that matrix elements of a spherical tensor operator connecting the angular-momentum eigenstates of two \( SU(2) \) irreps can be expressed as products of two factors, one of which is independent of angular momentum orientation, and the other is a vector-coupling (Clebsch–Gordan) coefficient. It is now known that, subject to a few conditions, the theorem also applies to the matrix elements of other groups and Lie algebras (see for example \([60]\)). In particular, it applies to the unitary groups.
equation (69) and because $\partial_{\alpha_i}$ is the Hermitian adjoint of $z_{\alpha_i}$ in a unitary representation, its matrix elements are also obtained from the Hermiticity relationship (70).

The vector-coupling coefficients are well-known for $U(2)$ and a computer code is freely available for those of $U(3)$ [49]. Restricted sets of coefficients are also available for other groups, such as $U(4)$ [50] and $SO(5)$ [51] and algorithms have been developed [52–55] for computing the coefficients of any unitary group.

3.2. Heisenberg algebras related to type II Capelli algebras

The Heisenberg algebra that emerges in the contraction of the symplectic algebra $\mathfrak{sp}(N, \mathbb{C})$ contains a set of symmetric variables and derivative operators

$$z_{\alpha j} = z_{ji}, \quad \partial_{\alpha j} = \partial_{ji} = \left(1 + \delta_{\alpha j}\right)\partial_j z_{\alpha j}, \quad i, j = 1, \ldots, N,$$

that satisfy the commutation relations

$$\left[\partial_{\alpha j}, z_{\beta k}\right] = \delta_{\alpha k} \delta_{j,\beta} + \delta_{\alpha j} \delta_{\beta, k}.$$  

The Bargmann space $\mathbb{B}^{(N+1)}$ spanned by polynomials in the $\frac{1}{2}N(N+1)$ independent variables $\{z_{\alpha j}, i < j\}$ carries the unitary irrep of this Heisenberg algebra. It is also a module for a reducible representation of the $gl(N)$ Lie algebra with elements

$$E^{(N)}_{ij} = \sum_{i=1}^{N} z_{ij} \partial_{ji},$$

This representation is a sum of GL $(N)$ irreps $\mathfrak{gl}_\nu$, with highest weights

$$\nu = \{\nu_1, \nu_2, \ldots, \nu_N\}, \quad \nu_1 \geq \nu_2 \geq \ldots \geq \nu_N \geq 0,$$

in which, because $[E^{(N)}_{\alpha i}, z_{\beta k}] = (\delta_{\alpha k} + \delta_{\beta, j})z_{\alpha j}$ implies that $z_{\alpha j}$ is of weight $\Delta_k + \Delta_k$, each component $\nu_i$ is an even integer. With an inner product for $\mathbb{B}^{(N+1)}$ defined such that $\delta_{\alpha j}$ is the Hermitian adjoint of $z_{\alpha j}$, the representation of GL $(N)$ on $\mathbb{B}^{(N+1)}$ restricts to a unitary representation of its $U(N)$ real form.

3.2.1. Highest-weight states for irreps of $U(N)$ on subspaces of $\mathbb{B}^{(N+1)}$. The irreps of $U(N)$ on subspaces of $\mathbb{B}^{(N+1)}$ are characterized by highest-weight states, which are polynomials in the $\{z_{\alpha j}\}$ variables that are annihilated by the $\{E^{(N)}_{\alpha i}, i < j\}$ raising operators. For example, the elementary polynomial $z_{11}$ is of highest $U(N)$ weight $\{2\} \equiv \{2, 0, \ldots\}$ and the polynomial $(z_{11}z_{22} - z_{12}z_{21})$ is of highest $U(N)$ weight $\{2^2\} \equiv \{2, 2, 0, \ldots\}$. In parallel with equation (34), let $x_n$ and $V_n$ for $n = 1, \ldots, N$ denote the determinants

$$x_n := \det(z_{ij}), \quad V_n := \det(\partial_{ij}), \quad \text{with} \quad i, j = 1, \ldots, n.$$  

Then, $x_n$ is a polynomial of highest $U(N)$ weight $\{2^N\}$. A general highest-weight state of weight $\nu$ for a $U(N)$ irrep on a subspace of $\mathbb{B}^{(N+1)}$ is given by

$$\psi^{(N)}_\nu(z) := x_1^{\nu_1}x_2^{\nu_2} \ldots x_N^{\nu_N} \text{ with } \nu_i = \sum_{j=i}^{N} 2p_j.$$  

3.2.2. Normalization factors for the $U(N)$ highest-weight states. The normalizations of the above-defined highest-weight states are determined by use of the type II Capelli identity
\( x_n V_n = \det \left[ E_{ij}^{(n)} + (n - i)\delta_{ij} \right], \quad (78) \)

in which

\[ E_{ij}^{(n)} = \sum_{s=1}^{n} z_{is} \partial_{j\mu}, \quad i, j = 1, \ldots, n \leq N, \quad (79) \]

is an element of a \( \mathfrak{u}(n) \) Lie algebra and it is understood that the determinant on the right side of (78) is defined with products of elements from different columns that preserve their left to right order. An equivalent expression is given by

\[ V_n x_n = \det \left[ E_{ij}^{(n)} + (n + 2 - i)\delta_{ij} \right]. \quad (80) \]

The squared norm

\[ \langle \psi^{(N)}_\nu | \psi^{(N)}_\nu \rangle = \langle V_1^{p_1} V_2^{p_2} \ldots V_n^{p_n} x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n} \rangle, \quad (81) \]

of the wave function \( \psi^{(N)}_\nu \) of equation (77) is determined starting from the observation that, when acting on a highest-weight wave function \( \psi \), the operator \( V_n x_n \) of equation (80) simplifies to

\[ V_n x_n \psi = \left( E_{11}^{(n)} + n + 1 \right) \left( E_{22}^{(n)} + n \right) \left( E_{nm}^{(n)} + 2 \right) \psi. \quad (82) \]

Proceeding as in section 3.1, it is determined that

\[ V^{p}_1 x^{p}_1 \psi = \frac{\left( E_{11}^{(n)} + n + 2p - 1 \right) !! \left( E_{22}^{(n)} + n + 2p - 2 \right) !! \ldots \left( E_{nm}^{(n)} + 2p \right) !!}{\left( E_{11}^{(n)} + n - 1 \right) !! \left( E_{22}^{(n)} + n - 2 \right) !! \ldots \left( E_{nm}^{(n)} \right) !!} \psi. \quad (83) \]

Similarly, with the recognition that

\[ E_{ii}^{(n)} \psi^{(m)}_\mu = E_{ii}^{(m)} \psi^{(m)}_\mu, \quad \text{if } n \geq m, \quad (84) \]

and evaluation of \( \langle V_1^{p_1} V_2^{p_2} x_1^{p_1} \rangle \) and \( \langle V_1^{p_1} V_2^{p_2} V_3^{p_3} x_1^{p_1} x_2^{p_2} \rangle \), the pattern becomes recognizable.

**Claim II.** The squared norm of the wave function \( \psi^{(N)}_\nu \) of equation (77) is

\[ \mathcal{N}^{(N)}_\nu = \left\{ \psi^{(N)}_\nu \right\} \left[ \prod_{i=1}^{N} \left( \nu + N - i \right) !! \prod_{j=i+1}^{N} \frac{\left( \nu_i - \nu_j + j - i - 1 \right) !!}{\left( \nu - \nu_j + j - i \right) !!} \right], \quad (85) \]

**Proof.** The wave function \( \psi^{(N)}_\nu \) for arbitrary \( N \) is a product

\[ \psi^{(N)}_\nu = x^{N}_\nu \psi^{(N-1)}_\mu, \quad (86) \]

where \( \psi^{(N-1)}_\mu \) is a highest-weight polynomial of weight \( \mu \) and

\[ \nu = (\mu_1 + 2p_N, \ldots, \mu_{N-1} + 2p_N, 2p_N). \quad (87) \]
By use of equation (83) and assuming the claim is correct for $\mathcal{A}^{(N-1)}$, it follows that

$$
\left\{ \psi^{(N-1)}_\mu \right\} V_N^{p_N} X_N^{p_N} \psi^{(N-1)}_\mu = \frac{(\nu_1 + N - 1)! (\nu_2 + N - 2)! \cdots (\nu_N)!}{(\nu_1 - \nu_N + N - 1)! (\nu_2 - \nu_N + N - 2)! \cdots (\nu_{N-1} - \nu_N)!} \times \left\{ \psi^{(N-1)}_\mu \right\} \psi^{(N-1)}_\mu = \mathcal{A}^{(N)}_\nu.
$$

(88)

Thus, given that the claim is valid for $N \leq 3$, it is valid for all $N$.

\[ \square \]

### 3.2.3. Matrix elements of a type II Heisenberg algebra between $U(N)$ highest-weight states.

Given a knowledge of the irreps of $u(N)$ and its coupling coefficients, one can obtain all matrix elements of the contracted $\mathfrak{sp}(N, C)$ Lie algebra from the matrix elements

$$
\langle \nu + 2\Delta_k | z_{kk} | \nu \rangle = \langle \nu | \partial_{kk} | \nu + 2\Delta_k \rangle^\nu, \quad k = 1, \ldots, N
$$

(89)

between normalized highest-weight states

$$
| \nu + 2\Delta_k \rangle := \frac{\left| \psi^{(N)}_{\nu + 2\Delta_k} \right\rangle}{\left\{ \left| \psi^{(N)}_{\nu + 2\Delta_k} \right\rangle \left| \psi^{(N)}_{\nu + 2\Delta_k} \right\rangle \right\}^{1/2}}, \quad | \nu \rangle := \frac{\left| \psi^{(N)}_{\nu} \right\rangle}{\left\{ \left| \psi^{(N)}_{\nu} \right\rangle \left| \psi^{(N)}_{\nu} \right\rangle \right\}^{1/2}},
$$

(90)

where $2\Delta_k = (0, \ldots, 0, 2, 0, \ldots, 0)$ is the weight of $z_{kk}$ as a $u(N)$ raising operator with all but its $k$ component equal to zero. Thus, with $\psi^{(N)}_{\nu}$ expressed as in equation (77), the shifted wave functions take the form

$$
\psi^{(N)}_{\nu + 2\Delta_k} = X_1^{P_1} \cdots X_{k-2}^{P_{k-2}} X_{k-1}^{P_{k-1}+1} \cdots X_k^{P_k+1} X_{k+1} \cdots X_N^{P_N}.
$$

(91)

Consider first the ratio

$$
R := \frac{\left\langle \psi^{(N)}_{\nu + 2\Delta_k} | z_{kk} | \psi^{(N)}_{\nu} \right\rangle}{\left\langle \psi^{(N)}_{\nu + 2\Delta_k} | \psi^{(N)}_{\nu} \right\rangle} = \frac{\left\langle \psi^{(N)}_{\nu} | \partial_{kk} | \psi^{(N)}_{\nu + 2\Delta_k} \right\rangle}{\left\langle \psi^{(N)}_{\nu + 2\Delta_k} | \psi^{(N)}_{\nu} \right\rangle}
$$

(92)

$$
= \frac{\left\langle V_1^{P_1} \cdots V_k^{P_k} \partial_{kk} W_k^{p_k+1} x_{k-1}^{P_{k-1}+1} x_{k-2} \cdots x_1^{P_1} \right\rangle}{\left\langle V_1^{P_1} \cdots V_k^{P_k} W_k^{p_k+1} x_{k-1}^{P_{k-1}+1} x_{k-2} \cdots x_1^{P_1} \right\rangle},
$$

(93)

with

$$
W = V_1^{P_1+1} \cdots V_N^{P_N} X_N^{P_N} \cdots X_{k+1}^{P_{k+1}}.
$$

(94)

The monomial to the right of the operator $W$ is the same in both the numerator and denominator of $R$. It is an eigenfunction of $W$ and its eigenvalue can be factored out to give

$$
R = \frac{\left\langle V_1^{P_1} \cdots V_k^{P_k} \partial_{kk} | x_{k-1}^{P_{k-1}+1} x_{k-2} \cdots x_1^{P_1} \right\rangle}{\left\langle V_1^{P_1} \cdots V_k^{P_k} | x_{k-1}^{P_{k-1}+1} x_{k-2} \cdots x_1^{P_1} \right\rangle},
$$

(95)

where use is made of the observation that $[\partial_{kk}, x_k] = 0$ for all $k' < k$. The identity

$$
\left[ \partial_{kk}, x_k^{P_k+1} \right] = 2(p_k + 1)x_{k-1} x_k^{P_k},
$$

(96)
then leads to the expression

\[ R = 2\left(p_k + 1\right) \frac{\left\langle \psi^{(k)}_\mu | \psi^{(k)}_\mu \right\rangle}{\left\langle \psi^{(k)}_{\mu + 2\Delta} | \psi^{(k)}_{\mu + 2\Delta} \right\rangle}, \quad (97) \]

where

\[ \psi^{(k)}_\mu = x^p_1 \ldots x^p_k, \quad \psi^{(k)}_{\mu + 2\Delta} = x^p_1 \ldots x^p_{k-1}^{-1} x^p_k + 1; \quad (98) \]

which corresponds to setting \( \mu_i = v_i - v_{k+1} \), for \( i = 1, \ldots, k \), and \( 2p_k = v_k - v_{k+1} \).

Finally, the desired matrix elements between highest-weight states are given by

\[ \left\langle \nu + 2\Delta_k \right| z_{kk} \left| \nu \right\rangle = \left( \nu_k - \nu_{k+1} + 2 \right) \frac{\left\langle \psi^{(k)}_\mu | \psi^{(k)}_\mu \right\rangle}{\left\langle \psi^{(k)}_{\mu + 2\Delta} | \psi^{(k)}_{\mu + 2\Delta} \right\rangle} \left( \frac{\left\langle \psi^{(N)}_\nu | \psi^{(N)}_\nu \right\rangle}{\left\langle \psi^{(N)}_{\nu + 2\Delta} | \psi^{(N)}_{\nu + 2\Delta} \right\rangle} \right)^2 \quad (99) \]

and are easily evaluated with the expression for the norms given by claim II.

### 3.2.4. Unitary representation of the type II Heisenberg algebra in a U(N)-coupled basis

An element \( z_{ij} \) of the type II Heisenberg algebra is a component of weight \( \Delta_k + \Delta_i \) of a U(N) tensor of highest weight \( 2\Delta_k = (2, 0, \ldots, 0) \). Equation (99) gives a non-vanishing matrix element \( \left\langle \nu' \right| z_{kk} \left| \nu \right\rangle \) with \( k \) such that \( 2\Delta_k = \nu' - \nu \). Thus, according to the Wigner–Eckart theorem, the matrix element of equation (68) is equal to the product

\[ \left\langle \nu' \right| z_{kk} \left| \nu \right\rangle = \left( \nu, \quad 2\Delta_k, \quad 2\Delta_k \right| \left( \nu' \nu \right) \left\langle \nu' \right| z \left| \nu \right\rangle, \quad (100) \]

in which \( (\nu, \quad 2\Delta_k, \quad 2\Delta_k \left| \nu' \nu \right) \) is a U(N) vector-coupling coefficients, and \( \left\langle \nu' \right| z \left| \nu \right\rangle \) is U(N) -reduced matrix element. Thus, the matrix elements of any \( z_{ij} \) are obtained by the further use of equation (69) and because \( \partial_{ij} \) is the Hermitian adjoint of \( z_{ij} \) in a unitary representation, its matrix elements are also obtained from the Hermiticity relationship (70).

### 3.3. Heisenberg algebras related to type III Capelli algebra

The Heisenberg algebra that emerges in the contraction of an \( \mathfrak{so}(2N, \mathbb{C}) \) Lie algebra contains a set of anti-symmetric variables and derivative operators

\[ z_{ij} = -z_{ji}, \quad \partial_{ij} = -\partial_{ji} = \partial / \partial z_{ij}, \quad i, j = 1, ..., N \quad (101) \]

do that satisfy the commutation relations

\[ \left[ \partial_{ij}, \ z_{kj} \right] = \delta_{ik} \delta_{j,l} - \delta_{ij} \delta_{k,l}. \quad (102) \]

The Bargmann space \( \mathcal{B}^{(2N(N-1)/2)} \) spanned by polynomials in the \( 1/2 \) \( N(N-1) \) independent variables \( \{ z_{ij}, \ i < j \} \) carries the unique unitary irrep of this Heisenberg algebra. It is also the module for a highly reducible representation of the \( \mathfrak{gl}(N) \subset \mathfrak{so}(2N, \mathbb{C}) \) subalgebra with elements

\[ E^{(N)}_{ij} = \sum_{s=1}^{N} z_{is} \partial_{js}. \quad (103) \]

This representation is a sum of irreps \( \Theta_\nu \{ \nu \} \) of highest weight \( \nu \equiv \{ \nu_1, \nu_2, ..., \nu_N \} \) which, because
\[
\left[ E^{(N)}_j, z_{kl} \right] = (\delta_{ik} + \delta_{jl}) z_{kl}
\] (104)

implies that \( z_{kl} \) is of weight \( \Delta_k + \Delta_l \).

It will now be shown that the Bargmann space \( B^{(N(N-1))} \) of polynomials in the anti-symmetric \( \{ z_{kl} \} \) variables is a sum of irreducible subspaces of highest weights \( v = \{ v_1, v_2, \ldots, v_N \} \) with

\[
v_1 = v_2 = v_3 = v_4 = \ldots = v_{N-1} = v_N \geq 0, \quad \text{if } N \text{ is even},
\]

\[
v_1 = v_2 = v_3 = v_4 = \ldots = v_{N-2} = v_{N-1} = v_N = 0, \quad \text{if } N \text{ is odd}.
\] (105) (106)

3.3.1. Highest-weight states for irreps of \( U(N) \) on subspaces of \( B^{(N(N-1))} \). As in sections 3.1 and 3.2, we define a sequence of determinants

\[
x_n := \det(z_{ij}), \quad V_n := \det(\partial_{ij}), \quad i, j = 1, \ldots, n,
\] (107)

for each positive integer \( n \leq N \). However, \( x_1 = z_{11} \) is now identically zero and the polynomial of highest \( u(N) \) weight in the space of linear functions of the \( \{ z_{ij} \} \) variables is \( \phi_1(z) = z_{12} \).

Moreover, \( \phi_1(z) \) is of weight \( \{ 1^1 \} \equiv \{ 1, 1, 0, \ldots, 0 \} \) and is a square root of the determinant \( \phi_2(z) = z_{12}^2 \). In fact, it is known \([56, 57]\) that every determinant of an \( n \times n \) anti-symmetric matrix is identically zero when \( n \) is odd. However, when \( n = 2m \) is even,

\[
x_{2m} = \phi_m^2
\] (108)

is the square of the Pfaffian \([2]\)

\[
\phi_m(z) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) z_{\sigma_1 \sigma_2} z_{\sigma_3 \sigma_4} \ldots z_{\sigma_{2m-1} \sigma_{2m}}.
\] (109)

It is also determined from equation (104) that \( \phi_m \) is a polynomial of highest \( u(N) \) weight \( \{ 1^m, 0, \ldots, 0 \} \). Thus, the Pfaffians \( \{ \phi_m \} \) are a complete set of generators of \( \mathfrak{gl}(N) \) and \( u(N) \) highest-weight polynomials given by the products

\[
\phi^{(2m)} = \phi_1^{k_{12}^{12}} \phi_2^{k_{12}^{12}} \ldots \phi_{m-1}^{k_{2m-1}^{2m-1} k_{2m}^{2m}} \phi_m^{k_{2m}^{2m}}, \quad m = 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor.
\] (110)

The Hermitian adjoints of the Pfaffians are the derivative operators

\[
\square_m = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \partial_{\sigma_1 \sigma_2} \partial_{\sigma_3 \sigma_4} \ldots \partial_{\sigma_{2m-1} \sigma_{2m}}.
\] (111)

3.3.2. Normalization factors for the \( U(N) \) highest-weight states. The type III Capelli identity \([4]\) states that the product of the determinants \( x_n \) and \( V_n \) satisfies the expression

\[
x_n V_n = \det \left[ E^{(n)}_{ij} + (n - 1 - i) \delta_{ij} \right]
\] (112)

where

\[
E^{(n)}_{ij} = \sum_{s=1}^n z_{si} \partial_{js}, \quad i, j = 1, \ldots, n,
\] (113)

is an element of a \( u(n) \) Lie algebra and it is understood that the determinant on the right side of (112) is defined with products of elements from different columns that preserve their left to right order. An equivalent expression is given by
\[ V_{n,x_n} = \det \left[ E_{ij}^{(n)} + (n + 1 - i)\delta_{ij} \right]. \]  

(114)

However, because the highest-weight polynomials are now expressed in terms of Pfaffians rather than determinants, this identity does not provide a direct determination of the squared norms

\[ \mathcal{N}_{\nu}^{(2m)} = \left\langle \Phi_{\nu}^{(2m)} \right| \Phi_{\nu}^{(2m)} \right\rangle. \]  

(115)

There is no known expansion of the product \( \Box_m \Phi_{\nu} \) similar to that given by the Capelli identity (114) for \( V_{n,x_n} \). However, it is known [28] that the highest-weight polynomials \( \{ \Phi_{\nu}^{(2m)} \} \) are eigenfunctions of \( \Box_m \Phi_{\nu} \),

\[ \Box_m \Phi_{\nu}^{(2m)} = X_{\nu} \Phi_{\nu}^{(2m)} \]  

(116)

with eigenvalues (determined by complicated algebraic manipulations)

\[ X_{\nu} = (\nu_1 + 2m - 1)(\nu_1 + 2m - 3) \cdots (\nu_{2m-1} + 1) \]
\[ = \prod_{i=1}^{m} \frac{(\nu_i + 2m + 1 - 2i)!}{(\nu_i + 2m - 2i)!}. \]  

(117)

As we now show, the eigenvalues \( \{ X_{\nu} \} \) given by (117) are easily derived from the Capelli identity (114) for which

\[ V_{2m,x_{2m}} \Phi_{\nu}^{(2m)} = \Box^2_m \Phi_{\nu}^{(2m)} = \prod_{i=1}^{2m} (\nu_i + 2m + 1 - i) \Phi_{\nu}^{(2m)}. \]  

(118)

Because \( \Phi_{\nu} \) has \( U(N) \) weight \( \{ 1^{2m} \} \) for \( 2m \leq N \), equation (116) implies that

\[ \Box_m \Phi_{\nu}^{(2m)} = X_{\mu} \Phi_{\nu}^{(2m)}, \quad \text{for } m \geq n, \]  

(119)

with

\[ \mu_i = \nu_i + 1, \quad i = 1, \ldots, 2m. \]  

(120)

It then follows that

\[ \Box^2_m \Phi_{\nu}^{(2m)} = X_{\mu} \Box_m \Phi_{\nu}^{(2m)} = X_{\mu} X_{\nu} \Phi_{\nu}^{(2m)}, \quad \text{for } m \geq n, \]  

(121)

and, because \( \nu_{2i} = \nu_{2i-1} \), the required identity

\[ X_{\mu} X_{\nu} = \prod_{i=1}^{2m} (\nu_i + 2m + 1 - i) \]  

(122)

is obtained with \( X_{\nu} \) given by equation (117).

**Claim III.** The squared norm of the highest-weight polynomial \( \Phi_{\nu}^{(2m)} \) for a \( u(N) \) irrep on the Bargmann space \( B^{(\nu)}_{\nu} \) is given by

\[ \mathcal{N}_{\nu}^{(2m)} = \left\langle \Phi_{\nu}^{(2m)} \right| \Phi_{\nu}^{(2m)} \right\rangle \]
\[ = \prod_{i=1}^{m} \frac{\prod_{j=i+1}^{m} (\nu_{2i} + 2m - 2j)!}{\prod_{j=i+1}^{m} (\nu_{2i} - \nu_{2j} + 2j - 2i)(\nu_{2i} - \nu_{2j} + 2j - 2i - 1)}. \]  

(123)
Proof. Observe that the above extension of the expression for $\Box_m \Phi^{(n)}$ to $\Box_m \Phi^{(n)}$ for $m \geq n$ further extends to

$$\Box_m \Phi_m^{(2n)} = \prod_{i=1}^{m} \frac{(\nu_{2i} + 2m + p - 2i)!}{(\nu_{2i} + 2m - 2i)!} \Phi^{(2n)},$$

for $m \geq n$. (124)

It follows that

$$\langle \Box_m \Phi \rangle = p_1 !.$$

Because equation (124) implies that

$$\Box_2 \Phi_2 \Phi_1 \Phi_0 = \frac{(p_1 + p_2 + 2)!p_2 !}{(p_1 + 2)!} \Phi_1 \Phi_0,$$

it also follows that

$$\langle \Box_2 \Box_2 \Phi_2 \Phi_1 \Phi_0 \rangle = \frac{(p_1 + p_2 + 2)!p_2 !}{(p_1 + 2)(p_1 + 1)},$$

and that, for $\Phi_3 = \Phi_2 \Phi_1 \Phi_0$ with $\nu_1 = \nu_2 = p_1 + p_2$ and $\nu_3 = \nu_4 = p_2$,

$$\langle \Phi_3 | \Phi_4 \rangle = \frac{(\nu_2 + 2)!\nu_4 !}{(\nu_2 - \nu_4 + 2)(\nu_2 - \nu_4 + 1)}$$

as predicted by the claim. Now, if the claim is valid for $\mathcal{N}_{\nu}^{(2m-2)}$ for some value of $m$, then

$$\mathcal{N}_{\nu}^{(2m)} = \langle \Phi_3 | \Phi_4 \rangle^{(2m-2)} \mathcal{N}_{\nu}^{(2m-2)} = \langle \Phi_4 | \Phi_2 \Phi_1 \Phi_0 \rangle^{(2m-2)}$$

with

$$\nu_{2j} = \mu_{2j} + \nu_{2m}.$$ (130)

Equations (124) and (129) then give

$$\mathcal{N}_{\nu}^{(2m)} = \prod_{j=1}^{m} \frac{(\nu_{2j} + 2m - 2i)!}{(\nu_{2j} - \nu_{2m} + 2m - 2i)!}$$

$$\times \frac{\prod_{i=1}^{m-1} (\nu_{2i} - \nu_{2m} + 2m - 2i)!}{\prod_{j=1}^{m-2} \prod_{j+1}^{m-1} \left( \nu_{2j} - \nu_{2j} + 2j - 2i \right) \left( \nu_{2j} - \nu_{2j} + 2j - 2i - 1 \right)}$$

$$= \prod_{j=1}^{m-1} \frac{(\nu_{2j} + 2m - 2i)!}{(\nu_{2j} - 2j - 2i)! \left( \nu_{2j} - \nu_{2j} + 2j - 2i - 1 \right)}.$$ (133)

as claimed. Thus, given that the claim is valid for $m \leq 2$, it is value for all $m$. □

3.3.3. Matrix elements of a type III Heisenberg algebra between U(N) highest-weight states.

Having determined the normalized highest-weight states
\[ |\nu\rangle := \frac{\langle \Phi (2m) | \Phi (2m) \rangle}{\langle \Phi (2m) | \Phi (2m) \rangle}, \quad m = 1, \ldots, [N/2] \quad (134) \]

for \(u(N)\) irreps on \(B^\otimes N/(N-1)\), all matrix elements of the type III Heisenberg algebra unitary irrep are determined, in terms of \(u(N)\) vector-coupling coefficients, from the non-zero matrix elements

\[ \langle \nu' | z_{2k-1, 2k} | \nu \rangle = \langle \nu | \Phi (2k-1, 2k) | \nu' \rangle^*, \quad (135) \]

where \(z_{2k-1, 2k}\) is the element of the Heisenberg algebra with weight

\[ \Delta_{2k-1} + \Delta_{2k} = \nu' - \nu. \quad (136) \]

By use of equation (110), the highest-weight wave functions are conveniently expressed in the form

\[ \Phi (2m) = \phi_k \phi_{k+1} \ldots \phi_m \quad (137) \]

with \(\nu_{2i-1} = \nu_{2i} = \sum_m p_j\). Then, with \(\nu_{2i+1} = \nu_{2i} + \delta_{i,k}\), we have

\[ \Phi (2m) = \phi_k \phi_{k+1} \ldots \phi_{2k-1} \phi_k \phi_{k+1} \ldots \phi_m. \quad (138) \]

The ratio \(R = \langle \Phi (2m) | \partial_{2k-1, 2k} \Phi (2m) \rangle / \langle \Phi (2m) | \Phi (2m) \rangle\) has expansion

\[ R = \frac{\langle \Phi (2m) | \partial_{2k-1, 2k} W \phi_k \phi_{k+1} \phi_{k-1} \phi_{k+1} \ldots \phi_m \rangle}{\langle \Phi (2m) | \partial_{2k-1, 2k} W \phi_k \phi_{k+1} \phi_{k-1} \phi_{k+1} \ldots \phi_m \rangle}, \quad (139) \]

where

\[ W = \phi_{k+1} \ldots \phi_m \phi_k \phi_{k+1} \ldots \phi_m. \quad (140) \]

Now, because the monomial to the right of the operator \(W\) is an eigenfunction of \(W\) and is the same in both the numerator and the denominator of this expression, \(W\) can be factored out of this ratio to give

\[ R = \frac{\langle \Phi (2m) | \partial_{2k-1, 2k} \phi_k \phi_{k+1} \phi_{k-1} \phi_{k+1} \ldots \phi_m \rangle}{\langle \Phi (2m) | \partial_{2k-1, 2k} \phi_k \phi_{k+1} \phi_{k-1} \phi_{k+1} \ldots \phi_m \rangle} \]

\[ = (p_k + 1) \frac{\langle \Phi (2m) | \phi_k \phi_{k+1} \phi_{k+1} \ldots \phi_m \rangle}{\langle \Phi (2m) | \phi_k \phi_{k+1} \phi_{k+1} \phi_{k-1} \phi_{k+1} \ldots \phi_m \rangle} \]

\[ = (p_k + 1) \frac{\langle \Phi (k) | \phi_k \rangle}{\langle \Phi (k) | \phi_k \rangle}, \quad (142) \]

where

\[ \Phi (k) = \phi_k \phi_{k+1} \phi_{k+1} \ldots \phi_{k-1} \phi_k, \quad \Phi (k) = \phi_k \phi_{k+1} \phi_{k+1} \ldots \phi_{k-1} \phi_k \]

with \(\mu_{2i} = \sum_j p_j\) and \(\mu'_{2i-1} = \mu_{2i} = \mu_{2i} + \delta_{i,k}\).
Finally, the desired matrix elements
\[
\langle \nu' \mid z_{2k-1,2k} \mid \nu \rangle = \frac{\langle \Phi_\nu^{(m)} \mid z_{2k-1,2k} \Phi_\nu^{(n)} \rangle}{\left[ \langle \Phi_\nu^{(m)} \mid \Phi_\nu^{(m)} \rangle \langle \Phi_\nu^{(n)} \mid \Phi_\nu^{(n)} \rangle \right]^{1/2}},
\]
relative to normalized states, are given by
\[
\langle \nu' \mid z_{2k-1,2k} \mid \nu \rangle = (\nu_{2k} - \nu_{2k+2} + 1) \frac{\langle \Phi_\nu^{(k)} \mid \Phi_\nu^{(k)} \rangle \langle \Phi_\nu^{(k)} \mid \Phi_\nu^{(m)} \rangle \langle \Phi_\nu^{(m)} \mid \Phi_\nu^{(m)} \rangle}{\left[ \langle \Phi_\nu^{(m)} \mid \Phi_\nu^{(m)} \rangle \langle \Phi_\nu^{(n)} \mid \Phi_\nu^{(n)} \rangle \right]^{1/2}}
\]
and easily evaluated with the expression for the norms given by claim III.

3.3.4. Unitary representation of the type III Heisenberg algebra in a U(N)-coupled basis. An element \( z_{k\ell} \), with \( k \neq \ell \), of the type III Heisenberg algebra is a component of weight \( \Delta_k + \Delta_\ell \) of a U(N) tensor of highest weight \( (1, 1, 0, \ldots) \). Equation (146) gives a non-vanishing matrix element \( \langle \nu' \mid z_{2k-1,2k} \mid \nu \rangle \) with \( k \) such that \( \Delta_{2k-1} + \Delta_{2k} = \nu' - \nu \). Thus, according to the Wigner–Eckart theorem,
\[
\langle \nu' \mid z_{2k-1,2k} \mid \nu \rangle = \nu \Delta_k, \Delta_\ell + \Delta_2, \nu' - \nu \parallel \nu' \rangle \langle \nu' \parallel z \parallel \nu \rangle,
\]
where \( (\nu \nu', \Delta_1 + \Delta_2, \nu' - \nu) \) is a U(N) vector-coupling coefficients, and \( \langle \nu' \parallel z \parallel \nu \rangle \) is a U(N)-reduced matrix element. Thus, the matrix elements of any \( z_{k\ell} \) are obtained by the further use of equation (69) and because \( \partial \partial \) is the Hermitian adjoint of \( z_{k\ell} \) in a unitary representation, its matrix elements are also obtained from the Hermiticity relationship (70).

4. Summary and application of the results

The Capelli identities have been used to construct the representations of three classes of Heisenberg algebras:

(A) Heisenberg algebras with elements \( \{ z_{\alpha i}, \partial_{\alpha i}; i = 1, \ldots, p, \alpha = 1, \ldots, q \} \) and commutation relations
\[
[z_{\alpha i}, z_{\beta j}] = [\partial_{\alpha i}, \partial_{\beta j}] = 0, \quad [\partial_{\alpha i}, z_{\beta j}] = \delta_{ij} \delta_{\alpha \beta}
\]
in bases that reduce the representations of the Lie algebras \( u(p) \) and \( u(q) \) with elements given, respectively, by
\[
E^{(p)}_{ij} = \sum_{a=1}^{q} z_{\alpha a} \partial_{\alpha j}, \quad E^{(q)}_{ij} = -\sum_{a=1}^{p} z_{\alpha i} \partial_{\alpha a};
\]

(B) symmetric Heisenberg algebras with elements \( \{ z_{ij} = z_{ji}, \partial_{ij} = \partial_{ji}; i, j = 1, \ldots, N \} \) and commutation relations
\[
[z_{ij}, z_{kl}] = [\partial_{ij}, \partial_{kl}] = 0, \quad [\partial_{ij}, z_{kl}] = \delta_{ij} \delta_{kl} + \delta_{kl} \delta_{ij}
\]
in bases that reduce the representation of \( u(N) \) with elements
(C) anti-symmetric Heisenberg algebras with elements 
\[ z_{ij} = -z_{ji}, \quad \partial_{ij} = -\partial_{ji}; \quad i, j = 1, \ldots, N \]
and commutation relations
\[ [z_{ij}, z_{kl}] = \left[ \partial_{ij}, \partial_{kl} \right] = 0, \quad \left[ \partial_{ij}, z_{kl} \right] = \delta_{i,k} \delta_{j,l} - \delta_{i,l} \delta_{j,k}, \]  
(152)
in bases that likewise reduce the representation of \( u(N) \) with elements
\[ E_{ij}^{(N)} = \sum_{s=1}^{N} z_{is} \partial_{sj}, \]  
(153)

Note that by ‘constructing a representation’ we mean much more than simply defining a module for the representation as explained in the introduction. Constructions are also given for elementary irreps on a Bargmann space \( \mathcal{B} \) of the semi-direct sums of these Heisenberg algebras and the associated unitary Lie algebras spanned, for example, by elements \( \{ E_{ij}^{(N)}, z_{ij}, \partial_{ij}, I, i, j = 1, \ldots, N \} \), where \( I \) is the identity element of the Heisenberg algebra. Moreover, they define asymptotic limits of the holomorphic representations of the Lie algebras from which the semi-direct sums are obtained by contraction. For example, for suitably chosen values of \( \{ k^{(x)} \} \), the contraction of \( \mathfrak{sp}(N, \mathcal{R}) \) given by equation (25) leads to an asymptotic limit
\[ C_{ij} \rightarrow C_{ij}^{(x)} + \sum_{s} z_{is} \partial_{sj}, \quad A_{ij} \rightarrow k^{(x)} z_{ij}, \quad B_{ij} \rightarrow k^{(x)^*} \partial_{ij} \]  
(154)
of an \( \mathfrak{sp}(N, \mathcal{R}) \) irrep for which the lowest-weight state belongs to an irrep \( \{ \chi \} \) of the \( u(N) \subset \mathfrak{sp}(N, \mathcal{R}) \) subalgebra. This asymptotic limit is defined on a tensor product space \( F^{(x)} := \mathcal{B}^{(\mathfrak{sp}(N) \cup \mathfrak{h}^{(N)})} \otimes \mathcal{H}^{(x)} \) of holomorphic vector-valued functions, where \( \mathcal{H}^{(x)} \) is the Hilbert space for the irrep \( \{ \chi \} \) of \( u(N) \), on which the operators \( C_{ij}^{(x)} \) act.

Such asymptotic limits correspond to macroscopic limits of microscopic models in physics. They also lead to a construction of the holomorphic representations of the Lie algebras from which the contracted Lie algebras derive. For example, VCS theory \([8, 11, 12]\), shows that a holomorphic vector-valued representation of \( \mathfrak{sp}(N, \mathcal{R}) \)
\[ \Gamma^{(x)}(C_{ij}) = C_{ij}^{(x)} + \sum_{k=1}^{d} z_{ik} \partial_{jk}, \quad \Gamma^{(x)}(A_{ij}) = \Lambda, \quad \partial_{ij}, \]  
(155)
is obtained on the same Hilbert space \( F^{(x)} \) as the contracted irrep where \( \Lambda \) is a \( U(N) \)-invariant Hermitian operator that has known eigenvalues on the \( U(N) \)-coupled basis for \( F^{x} \).

The holomorphic representations obtained from contractions in this way are precise, as opposed to approximations given by asymptotic limits that are often appropriate in applications to large many-particle systems. They are not unitary with respect to an orthonormal basis for a unitary representation of the contracted algebras. Neither, in general, are they irreducible. However, VCS theory includes a K-matrix procedure \([8, 12, 58]\) for orthogonalizing and renormalizing the bases to give explicit unitary representations on an irreducible subspace of \( F^{x} \). This procedure is simplified by a recent observation of an equivalent VCS representation, \( \Theta^{(x)} \), of the \( \mathfrak{sp}(N, \mathcal{R}) \) algebra of the form
\[ \Theta^{(k)}(C_{ij}) = C_{ij}^{(k)} + \sum_{k=1}^{q} z_{ik} \partial z_{jk}, \quad \Theta^{(k)}(A_{ij}) = z_{ij}, \quad \Theta^{(k)}(B_{ij}) = \left[ \partial_{ij}, A \right]. \tag{156} \]

which, like \( \Gamma^{(k)} \), is defined on the Hilbert space \( F^{\kappa} \) for a unitary representation of the contracted Lie algebra. In fact, as will be shown explicitly in [27], \( \Theta^{(k)} \) and \( \Gamma^{(k)} \) are bi-orthogonal duals of one another relative to the inner product appropriate for the contracted \( \text{sp}(N, \mathbb{R}) \) algebra. Thus, from the paired representations \( \Theta^{(k)} \) and \( \Gamma^{(k)} \), it is a straightforward procedure to construct orthonormal bases and irreducible unitary holomorphic representations of all the standard Lie algebras with holomorphic representations.

A practical advantage of the K-matrix procedure of VCS theory is that it makes use of algebraic methods to determine an orthonormal basis. It thereby avoids use of the standard integral form of the inner product, based on a resolution of the identity, which converges only for discrete series representations [27]. In contrast, the VCS construction of holomorphic representations is not restricted to discrete series representations.

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