RESULTS ON THE SPECTRAL SCHWARTZ DISTRIBUTION

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ABSTRACT. The resolvent of an operator in a Banach space is defined on an open subset of the complex plane and is holomorphic. It obeys the resolvent equation. A generalization of this equation to Schwartz distributions is defined and a Schwartz distribution, which satisfies that equation is called a resolvent distribution. Its restriction to the subset, where it is continuous, is the usual resolvent function. Its complex conjugate derivative is, but a factor, the spectral Schwartz distribution, which is carried by a subset of the spectral set of the operator. The spectral distribution yields a spectral decomposition. The spectral distribution of a matrix and a unitary operator are given. If the the operator is a self-adjoint operator on a Hilbert space, the spectral distribution is the derivative of the spectral family. We calculate the spectral distribution of the multiplication operator and some rank one perturbations. These operators are not necessarily self adjoint and may have discrete real or imaginary eigenvalues or a nontrivial Jordan decomposition.

Keywords: Schwartz distributions, Distribution Kernels, Resolvent, Spectral Decomposition, Spectral Distribution, Multiplication Operator.

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1. Definition and Basic Properties

1.1. Introduction. In a study of radiation transfer Garyi V. Efimov, Rainer Wehrse and the author developed a method, how to calculate the spectral decomposition of a multiplication operator in \( \mathbb{R}^n \) perturbed by a rank one operator. By Krein’s formula it is possible to calculate the resolvent \( R(z) \) and then one investigates the singularities of the function \( z \mapsto \langle f_1 | R(z) | f_2 \rangle \), where \( f_1, f_2 \) are test-functions. In [1] chapter 3 the complex derivative \( 1/\pi \partial \langle f_1 | R(z) | f_2 \rangle \) is called the bracket \( \langle f_1 | M(z) | f_2 \rangle \) of the spectral Schwartz distribution \( M(z) \), which therefor was only scalarly defined. There was developed a rudimentary theory and calculated four explicit examples [9, chapter 4], originated from the theory of quantum stochastic processes. Inspecting the examples it turned out, that the resolvent it self, not only the brackets between test functions can be extended to an operator valued Schwartz distribution and under this much stronger regularity condition a deeper theory of the spectral Schwartz distribution can be established. We calculate the spectrum of the multiplication operator, and two of its perturbations by a rank one operator. The examples are not necessarily self adjoint. In the second example we have imaginary eigenvalues and a nontrivial Jordan decomposition.

1.2. Schwartz distributions. We recall the definition of a distribution given by L. Schwartz [3]. A distribution on on an open set \( G \subset \mathbb{R}^n \) is a linear functional \( T \) on the space of test functions \( \mathcal{D}(G) \), that is the space of infinitely differentiable functions of compact support ( or \( C^\infty_c \)-functions) with support in \( G \). The functional \( T \) is continuous in the following way: if the functions \( \varphi_n \in \mathcal{D}(G) \) have their support in a common compact set and if they and all their derivatives converge to 0
uniformly, then the $T(\varphi_n) \to 0$. The space of distributions on $\mathbb{R}^n$ is denoted by $\mathcal{D}'(\mathbb{R}^n)$. We write often

$$T(\varphi) = \int T(x)\varphi(x)dx.$$  

The integral notation is the usual notation of physicists and has been adopted by Schwartz with minor modifications in his articles on vector-valued distributions [7][8]. It emphasizes the fact, that a distribution can be considered as a generalized function. This is the notation in the Russian literature [2]. Similar is de Rham’s formulation [5]: $T$ can be considered as a current of degree 0 applied to a form of degree $n$ namely $\varphi(x_1, \cdots, x_n)dx_1 \cdots dx_n$. We use variable transforms of distributions accordingly.

In order to include Dirac’s original ideas, L.Schwartz introduced the distribution kernels [7]. A distribution kernel $K(x,y)$ on $\mathbb{R}^m \times \mathbb{R}^n$ is a distribution on that space. If $T$ is a distribution on $\mathbb{R}^n$ one may associate to it a kernel $T(x-y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\int \int T(x-y)\varphi(x)\psi(y)dxdy = \int dxT(x)\varphi(x)\int dy\psi(y-x) = \int dx(T \ast \psi)(x)\varphi(x),$$

where $T \ast \psi$ denotes the convolution. We have often to deal with the so called $\delta$-function given by $\delta(\varphi) = \varphi(0)$ and the generalized function $P/x = (d/dx)\ln|x|$, where $P$ stands for principal value. We have

$$\int \int \delta(x-y)\varphi(x)\psi(y)dxdy = \int dx\varphi(x,x).$$

So $\delta(x-y) = \delta(y-x)$.

If $S(x,y)$ and $T(y,z)$ are two kernels one may form a distribution of three variables $S(x,y)T(y,z)$, if this exists. If the kernels are $S(x-y)$ and $T(y-z)$ then $S(x-y)T(y-z)$ exists and is given by

$$\int \int \int dxdydzS(x-y)T(y-z)\varphi(x,y,z) = \int \int \int duvS(u)T(v)\varphi(u+y,v+y,v-z).$$

We obtain for any distribution $T$ the relation

$$\delta(x-y)T(y-z) = \delta(x-y)T(x-z).$$

We have the formulae (for the second equation see [9 p.74])

$$\int \int \int \delta(x-y)\delta(y-z)\varphi(x,y,z)dxdydz = \int \varphi(x,x,x)dx$$

and denote by

$$\frac{P}{x-\omega} = \frac{1}{y-x} \left( \frac{P}{x-\omega} - \frac{P}{y-\omega} \right) + \pi^2 \delta(x-\omega)\delta(y-\omega)$$

1.3. Notion of the Spectral Distribution. Assume we have a Banach space $V$ and denote by $L(V)$ the space of all bounded linear operators from $V$ to $V$ provided with the usual operator norm. Assume a subspace $D \subset V$ and an operator $A : D \to V$. Then the complex plane splits into two sets, the resolvent set, where the resolvent $R(z) = (z - A)^{-1}$ exists, and the spectral set, where it does not. The resolvent set is open, the spectral set is closed. In the resolvent set the resolvent obeys one of the two equivalent resolvent equations

$$R(z_1) - R(z_2) = (z_2 - z_1)R(z_1)R(z_2)$$

Go the other way round and assume an open set $G \subset \mathbb{C}$ and a function $R(z) : G \to L(V)$ satisfying the resolvent equation. The function $R(z)$ is holomorphic in $G$. The subspace $D = R(z)V$ is a
subset independent of $z \in G$. If $R(z_0)$ is injective for one $z_0 \in G$, then $R(z)$ is injective for all $z \in G$ and there exists a mapping $A : D \to V$ such that

$$(z - A)R(z)f = f \text{ for } f \in V \quad \text{and} \quad R(z)(z - A)f = f \text{ for } f \in D$$

The operator $A$ is closed and $R(z)$ is the resolvent of $A$. There exists not always such an operator corresponding to a resolvent, e.g. the function $R(z) = 0$ for all $z$ fulfills the resolvent equation and $R(z)$ is surely not injective.

We use the second equation and formulate:

**Definition 1.** Assume a Schwartz distribution $\varphi \mapsto R(\varphi)$ on an open set $G \subset \mathbb{C}$, which satisfies the equation

$$R(\varphi_1)R(\varphi_2) = \int d^2z_1R(z_1)\varphi_1(z_1)\int d^2z_2\varphi_2(z_2)/(z_2 - z_1)$$

$$- \int d^2z_2R(z_2)\varphi_2(z_2)\int d^2z_1\varphi_1(z_1)/(z_2 - z_1)$$

then we say, that $R(\varphi)$ satisfies the distribution resolvent equation and call $R(\varphi)$ a resolvent distribution. Here $d^2z = dx dy$ is the surface element on the complex plane for $z = x + iy$.

If $R(z)$ obeys the resolvent equation in $G$ it might be, that there is Schwartz distribution on an open set $G' \supset G$ extending $R(z)$ and obeying the distribution resolvent equation. Assume a distribution $R(z)$ satisfying the distribution resolvent equation, whose restriction on an open subset of the complex plane equals a continuous function, then the restriction satisfies the usual resolvent equation and is holomorphic. We call the set, where the distribution $R(z)$ is not continuous, the spectral set. It is closed in $G'$.

If $G \subset \mathbb{C}$ is open and $f : G \to \mathbb{C}, f(z) = f(x + iy)$ has a continuous derivative, set

$$(6) \quad \partial f = \frac{df}{dz} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad \overline{\partial} f = \frac{df}{dz} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

The operator $\partial$ is the usual derivative, $\overline{\partial}$ is called the complex conjugate derivative. The function $f$ is holomorphic if and only if $\overline{\partial} f = 0$.

In an analogous way one defines these derivatives for Schwartz distributions. If $T$ is a distribution on $\mathbb{C}$, then $\overline{\partial} T(\varphi) = -T(\overline{\partial} \varphi)$. The function $z \mapsto 1/z$ is locally integrable and one obtains

$$(7) \quad \overline{\partial}(1/z) = \pi \delta(z).$$

Assume $f$ to be defined and holomorphic for the elements $x + iy \in G, y \neq 0$, and that $f(x \pm i0)$ exists and is continuous, then

$$(8) \quad \overline{\partial} f(x + iy) = (i/2)(f(x + i0) - f(x - i0))\delta(y)$$

The last equation holds for operator valued functions as well. To prove these assertions, we need the following variant of Gauss’ theorem

**Lemma 1.** Assume $G \subset \mathbb{C}$ an open subset, $f : G \to \mathbb{C}$ continuously differentiable and $G_0 \subset G$ an open subset with compact closure and smooth border $G_0'$, then

$$\int_{G_0} \overline{\partial} f(z) d^2z = -\frac{i}{2} \int_{G_0'} f(z) dz$$
Here $dz$ denotes the line element directed in the sense, that the interior of $G_0$ is on the left hand side.

**Definition 2.** Assume $R(z)$ satisfying the distribution resolvent equation, then we call

$$M(z) = (1/\pi)\overline{\partial}R(z)$$

the spectral Schwartz distribution of $R(z)$.

1.4. **Basic properties.** If the resolvent distribution $R(z)$ is in an open set the extension of the resolvent function of an operator $A$, we call $M(z)$ a spectral distribution of $A$. Remark, that there may be many spectral distributions of $A$. As $R(z)$ is holomorphic on an open subset, where it is continuous, $M(z)$ is supported by the spectral set of $R(z)$.

Using the abbreviation $r(z) = 1/z$ we may the the distribution resolvent equation write in the form

$$R(\varphi_1)R(\varphi_2) = -R((r \ast \varphi_1)\varphi_2 + \varphi_1(r \ast \varphi_2))$$

The following theorem shows, that the spectral distribution can be considered as the generalization of the family of eigenprojectors in finite dimensional case. The operator $M(z)$ corresponds to eigenprojecto of the eigenvalue $z$. We have a generalized orthogonality relation and the fact, that the product with the resolvent means inserting the corresponding eigenvalue.

**Theorem 1.** The spectral distribution is multiplicative. So assume two $C^\infty_c$ functions $\varphi_1, \varphi_2$, then

$$M(\varphi_1)M(\varphi_2) = M(\varphi_1\varphi_2) \quad \text{or} \quad M(z_1)M(z_2) = \delta(z_1 - z_2)M(z_1).$$

Conversely assume a distribution $M(z)$ in $G \subset \mathbb{C}$ obeying the equation $M(z_1)M(z_2) = \delta(z_1 - z_2)M(z_1)$. Assume that the integral

$$R(z) = \int d^2\zeta M(\zeta)/(z - \zeta), \quad \int R(z)\varphi(z)d^2z = \int M(\zeta)d^2\zeta \int \varphi(z)/(z - \zeta)d^2z$$

exists for $\varphi \in C^\infty_c$, then $R(z)$ fulfills the resolvent equation for distributions. Furthermore

$$M(z_1)R(z_2) = M(z_1)/(z_2 - z_1),$$

$$\int M(z_1)\varphi_1(z_1)d^2z_1 \int R(z_2)\varphi_2(z_2)d^2z_2 = \int M(z_1)\varphi_1(z_1) \int \varphi_2(z_2)/(z_2 - z_1)d^2z_2d^2z_1.$$

**Proof.** Use the equation $\overline{\partial}r = \pi \delta$ and hence $r \ast \overline{\partial}\varphi = r \ast \overline{\partial}\delta \ast \varphi = \overline{\partial}\delta \ast r \ast \varphi = \overline{\partial}r \ast \varphi = \pi \varphi$ and calculate

$$M(\varphi_1)M(\varphi_2) = \pi^{-2}R(\overline{\partial}\varphi_1)R(\overline{\partial}\varphi_2) = -\pi^{-2}R((r \ast \overline{\partial}\varphi_1)\overline{\partial}\varphi_2 + \overline{\partial}\varphi_1(r \ast \overline{\partial}\varphi_2))$$

$$= -\pi R(\varphi_1\overline{\partial}\varphi_2 + \overline{\partial}\varphi_1\varphi_2) = -\pi R(\overline{\partial}(\varphi_1\varphi_2)) = M(\varphi_1\varphi_2).$$

In order to prove the converse assertion, observe that it means that $r \ast M$ is a resolvent distribution. Now $M(\varphi) = -1/\pi R(\overline{\partial}\varphi)$ and $(r \ast M)(\varphi) = -M(r \ast \varphi)$. Hence

$$R(\varphi_1)R(\varphi_2) = M(r \ast \varphi_1)M(r \ast \varphi_2) = M((r \ast \varphi_1)(r \ast \varphi_2))$$

$$= -1/\pi R((r \ast \varphi_1)(r \ast \varphi_2)) = -R((r \ast \varphi_1)\varphi_2 + \varphi_1(r \ast \varphi_2))$$

$$= -R((r \ast \varphi_1)\varphi_2 + \varphi_1(r \ast \varphi_2)).$$
The last equation says $M(\varphi_1)R(\varphi_2) = -M(\varphi_1(\varphi_2)).$ In fact
\[
M(\varphi_1)R(\varphi_2) = \frac{1}{\pi}R(\overline{\varphi_1})R(\varphi_2) = \frac{1}{\pi}R((\varphi_2 \varphi_1 + \overline{\varphi_1}(\varphi_2)) = \frac{1}{\pi}R(\varphi_1(\varphi_2)) = -M(\varphi_1(\varphi_2)).
\]

The operator $M(z)$ is a generalized eigen-projector. In fact if $R(z) = 1/(z - A)$ then $M(z_1)(1/(z_2 - A) = M(z_1)/(z_2 - z_1)$. The relation $M(z_1)M(z_2) = \delta(z_1 - z_2)M(z_1)$ is a generalized orthonormality relation.

**Remark 1.** If $A \in L(V)$ is an operator and if the distribution $R(z)$ fulfills the equation $AR(z) = -1 + zR(z)$ in the sense of distributions, then it fulfills the resolvent equation $R(z_1) - R(z_2) = (z_2 - z_1)R(z_1)R(z_2)$, and it is not necessarily the resolvent distribution of $A$. If $z$ is a resolvent distribution, by differentiation one obtains in this case $AM(z) = zM(z)$, as $\partial z = 0$. So $M(z)$ is an eigenprojector for the eigenvalue $z$.

One proves by a partition of unity

**Proposition 1.** Assume a bounded set $G \subset \mathbb{C}$ and a family $G_i, i = 1, \cdots, l$ of open, pairwise disjoint subsets of $G$, furthermore an open subset $G_0 \supset (G \setminus \bigcup_i G_i)$ and assume a distribution $M$ on $G$, whose restriction to $G_0$ vanishes and whose restriction to $G_i$ is multiplicative for $i = 1, \cdots, l$. Then $M$ is multiplicative on $G$ if and only if $M(\varphi)M(\psi) = 0$, under the condition that $\varphi$ has its support in $G_i$ and $\psi$ has its support in $G_j$ with $i, j \in [1, l], i \neq j$.

**Proposition 2.** Assume an open bounded set $G \subset \mathbb{C}$ and a family $G_i, i = 1, \cdots, l$ of open, pairwise disjoint subsets of $G$. Assume in each subset $G_i$ a subset $G_{i,0}$ of Lebesgue measure 0 and a resolvent function $R(z)$ on $G \setminus \bigcup G_{i,0}$ and for each $i$ a sequence of subsets $G_i \supset G_{i,n} \subset G_{i,0}$, such that
\[
R_i(\varphi) = \lim_{n \to \infty} \int_{G_i \setminus G_{i,n}} R(z) \varphi(z) d^2 z
\]
for $\varphi \in \mathcal{D}(G_i)$ exists and defines a resolvent distribution on $G_i$. Then the distribution $R$ defined by the function $R(z)$ on $G \setminus \bigcup G_{i,0}$ and by the distributions $R_i$ on $G_i$ fulfills the resolvent distribution equation.

**Proof.** Call $M_i = 1/\pi \partial R_i$. We have to show, that $M_i(\varphi_1)M_j(\varphi_2) = 0$, if the support of $\varphi_1$ is in $G_i$ and the support of $\varphi_2$ is in $G_j$ for $i \neq j$. We have using $[\overline{\varphi}]$
\[
\int_{G_i \setminus G_{i,m}} d^2 z_1 \int_{G_j \setminus G_{j,m}} d^2 z_2 \varphi_1(z_1) \varphi_2(z_2) R(z_1) R(z_2) = I + II
\]
with
\[
I = -\int_{G_i \setminus G_{i,m}} d^2 z_1 \int_{G_j \setminus G_{j,m}} d^2 z_2 \varphi_1(z_1) \varphi_2(z_2) R(z_1) \frac{1}{z_1 - z_2} = -\int_{G_i \setminus G_{i,m}} d^2 z R(z) \psi_1(z)
\]
The sequence
\[
\varphi_1(z_1) \int_{G_j \setminus G_{j,m}} d^2 z_2 \varphi_2(z_2)/(z_2 - z_1) = \psi_1(z_1),
\]
in the sense of $\mathcal{D}(G_i)$. We have $I \to -R_i(\psi) = -R(\psi) = -R(\varphi_1(\varphi_2))$. Similarly $II \to -R(\varphi_2(\varphi_1))$. By the proof of theorem $[\overline{\varphi}]$ one sees that $M_i(\varphi_1)M_j(\varphi_2) = M(\varphi_1 \varphi_2) = 0$. \[\square\]
Proposition 3. If the resolvent $R(z)$ has a pole of order $n$ in a point $z_0$, then $R(z)$ is holomorphic for $z$ in a neighborhood of $z_0$, $z \neq z_0$ and behaves like $b_0(z-z_0)^{-1} + \cdots + b_{n-1}(z-z_0)^{-n}$ near $z_0$. We have $b_0 = p$ and $b_j = a^j$, where $p^2 = p$ and $a^n = 0$ and $ap = pa$. Define

$$R(z) = \sum_{k=0}^{n-1} b_k \frac{P}{(z-z_0)^{k+1}},$$

where the distribution [6]

$$\int d^2z \frac{P}{z^k} \varphi(z) = \lim_{\varepsilon \to 0} \int_{|z| > \varepsilon} \frac{1}{z^k} \varphi(z) d^2z = (-1)^k \int \frac{1}{z} \partial^k \varphi(z) d^2z.$$

The distribution $R(z)$ fulfills the resolvent distribution equation. The spectral distribution is

$$M(z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} pa^k \partial^k \delta(z-z_0)$$

The proof follows directly from Leibniz’s formula.

Proposition 4. Assume a compact set $K$ in the real line and an open neighborhood $G = G_0 \times I$ of $K$, where $G_0$ is an open set containing $K$ and $I$ is an open interval containing $\emptyset$, and a resolvent function $R(z)$ holomorphic in $G \setminus K$. Define the function $R_u$ on $I$ by $R_u(x) = R(x + i u)$ and assume, that for any $C^\infty$ function $\varphi$ with support in $G_0$ the limits of $R_u(\varphi) = \int dR_u(x) \varphi(x)$ for $u \to 0+$ and for $u \to 0-$ exist. If $\varphi$ is a $C^\infty_-$ function on $G$, define $\varphi_u(x) = \varphi(x + i u)$. Then $u \to R_u(\varphi_u)$ is integrable and defines the distribution $R(\varphi)$ on $G$ by

$$R(\varphi) = \int du R_u(\varphi_u) = \int du \int dx R(x + i u) \varphi(x + i u).$$

The integral defines a resolvent distribution on $G$.

Proof. Define for $m = 0, 1, \cdots$ and for any test function $\varphi$ on $G_0$ the norm

$$\|\varphi\|_m = \max \{ |\partial^k \varphi(x)| : x \in G_0, k = 1, \cdots, m \}.$$

For a distribution $T$ on $G_0$ define accordingly $N_m(T) = \sup \{ \|T(\varphi)\| : |\varphi|_m \leq 1 \}$ Denote by $R_{0+}$ the limits of $R_u$. As for distributions weak convergence implies strong convergence, there exists an $m$ such that $N_m(R_u - R_{0+}) \to 0$ and $N_m(R_u - R_{0-}) \to 0$ for $u \to 0+$ or $u \to 0-$ resp.. The function $u \to R_u(\varphi_u)$ is continuous for $u \neq 0$ and has right and left limits at $u = 0$. Hence it is integrable. Define a $C^\infty$ function $\alpha$ on the real line with $\alpha \geq 0$, $\alpha(x) = 0$ for $|x| \geq 1$, $\alpha(x) = 1$ for $|x| \leq 1/2$. Put $\alpha_\varepsilon(u) = \alpha(u/\varepsilon)$. Then for $z_j = x_j + i u_j$, $j = 1, 2$

$$R(\varphi_1)R(\varphi_2) = \int \int du_1 du_2 R_{u_1}(\varphi_{1,u_1}) R_{u_2}(\varphi_{2,u_2}) = \lim_{\varepsilon \to 0} \int \int du_1 du_2 R_{u_1}(\varphi_{1,u_1}) R_{u_2}(\varphi_{2,u_2}) (1 - \alpha_\varepsilon(u_1 - u_2))$$

On the other hand

$$\int \int du_1 du_2 R_{u_1}(\varphi_{1,u_1}) R_{u_2}(\varphi_{2,u_2})(1 - \alpha_\varepsilon(u_1 - u_2)) = \int d^2z_1 d^2z_2 R(z_1)R(z_2)\varphi_1(z_1)\varphi_2(z_2)(1 - \alpha_\varepsilon(u_1 - u_2)).$$

In the second term the quantities $R(z_1), R(z_2)$ are usual resolvent functions. We apply the resolvent equation (6) and split the integral into two terms $I_\varepsilon$ and $II_\varepsilon$.

$$I_\varepsilon = - \int d^2z_1 d^2z_2 R(z_1)1/(z_1 - z_2)\varphi_1(z_1)\varphi_2(z_2)\alpha_\varepsilon(u_1 - u_2) = - \int du R(\psi_{u,\varepsilon})$$
Examples.

1.5. Theorem 2. If with \( R \) we have to estimate where \( \Gamma \) is the circle of radius \( R \) defined on the whole complex plane, then the support of \( C, K \). The integral can be estimated with some constants

\[
\int dx_2 du_2 \varphi_2(x_2 + iu_2) \frac{1}{x + iu - x_2 - iu_2} (1 - \alpha_\varepsilon(u - u_2)).
\]

Put

\[
\psi_u(x) = \varphi_1(x + iu) \int dx_2 du_2 \varphi_2(x_2 + iu_2) \frac{1}{x + iu - x_2 - iu_2}.
\]

We have to estimate

\[
\partial_x^k (\psi_u(x) - \psi_{u, \varepsilon}(x)) = \sum_{j=1}^k \binom{k}{j} \partial_x^{k-j} \varphi_1(x + iu) \int dx_2 du_2 \partial_x^j \varphi_2(x_2 + iu_2) \frac{1}{x + iu - x_2 - iu_2} \alpha_\varepsilon(u - u_2)
\]

The integral can be estimated with some constants \( C, K \)

\[
| \int dx_2 du_2 \partial_x^j \varphi_2(x + iu - x_2 - iu_2) | x_2 + iu_2 \alpha_\varepsilon(u_2) | \leq C \int |x^\prime|^\leq K, |u^\prime| \leq \varepsilon dx' du' \frac{1}{\sqrt{x'^2 + u'^2}} = O(\varepsilon \ln \varepsilon)
\]

Hence \( \| \psi_u - \psi_{u, \varepsilon} \|_m = O(\varepsilon \ln \varepsilon) \) uniformly in \( u \) and

\[
I_\varepsilon \to I = - \int d^2z_1 d^2z_2 R(z) \frac{1}{1/(z_1 - z_2)} \varphi_1(z_1) \varphi_2(z_2)
\]

By interchanging the roles of 1 and 2, one obtains the corresponding result for \( II_\varepsilon \). □

Theorem 2. If \( R \) is the resolvent of a bounded operator and \( M \) is a resolvent distribution extending \( R \) defined on the whole complex plane, then the support of \( M \) is compact and \( M(1) \) is defined, where here 1 is the constant function 1, and

\[
M(1) = \int d^2z M(z) = 1_{L(V)}
\]

In this case we call \( M \) complete.

Proof. Assume, that the support of \( M \) is contained in the circle of radius \( r \). Assume a test function \( \varphi \) constant 1 on this circle, then

\[
M(1) = M(\varphi) = -\frac{1}{\pi} \int d^2z R(z) \overline{\varphi}(z) = -\frac{1}{\pi} \int_{|z| > r} d^2z R(z) \overline{\varphi}(z) =
\]

\[
-\frac{1}{\pi} \int_{|z| > r} d^2z \overline{\varphi}(R(z)) \varphi(z) = \frac{1}{2\pi i} \int_{\Gamma} dz R(z) = 1_{L(V)},
\]

where \( \Gamma \) is the circle of radius \( r \) run in the anti-clockwise sense. Now the residuum in infinity of \( R(z) \) equals \( 1_{L(V)} \). □

1.5. Examples.

(1) Finite dimensional matrix. By Jordan’s normal form one obtains that

\[
M(z) = \sum_i p_i \sum_k (1/k!) (-1)^k a_i^k \partial^k \delta(z - \lambda_i),
\]

where the \( \lambda_i \) are the eigenvalues, the \( p_i \) are the eigenprojectors, \( p_i p_j = \delta_{ij} \) and the \( a_i \) are nilpotent and \( a_i p_j = \delta_{ij} a_i \). Instead of \( \partial \) we could have chosen any other linear combination \( D \) of \( \partial_x \) and \( \partial_y \), such that \( Dz = 1 \). This an example, that there are many resolvent distributions extending a resolvent function.
We cite another example affirming remark 1. Assume \( A = 0 \), then \( z R(z) = 1 \), so
\[
R(z) = \frac{1}{z} - \pi C \delta(z),
\]
where \( C \) is an arbitrary matrix. Then \( M(z) = \delta(z) - C \delta(z) \) and
\[
M(\varphi) M(\psi) = M(\varphi \psi)
\]
if and only if \( C^2 = 0 \), this might not be the case.

(2) Assume that \( V \) is a Hilbert space and that \( U \) is a unitary operator.
\[
\int M(z) \varphi(z) d^2 z = \sum_{l=-\infty}^{\infty} U^l \frac{1}{2 \pi} \int e^{-i \theta} \varphi(e^{i \theta}) d \theta.
\]

(3) Assume \( V \) to be a Hilbert space and \( A \) to be a self adjoint operator. Let \( E(x), x \in \mathbb{R} \) be the spectral distribution, then
\[
M(x + iy) = E'(x) \delta(y),
\]
where the derivative is in the sense of distributions.

2. Discussion of the Multiplication Operator and some Rank One Perturbations

2.1. \( C^\infty \) Multiplication Operator. Assume an open bounded set \( G \subset \mathbb{R}^n \) and a \( C^\infty \) bounded function \( P : G \to \mathbb{R} \), which is bonded with all its derivatives and with \( \nabla P(y) \neq 0 \) for all \( y \in G \). Consider \( V = L^2(G) \) and the operator \( \Omega : V \to V \) with \( \Omega f(y) = P(y) f(y) \). The operator \( \Omega \) is bounded and its resolvent is \( R(z) = 1/(z - \Omega) \) for \( \text{Im} z \neq 0 \). For any test function on the complex plane the integral
\[
R(\varphi) = \int d^2 z \varphi(z)/(z - \Omega), \quad (R(\varphi)f)(y) = \int d^2 z \varphi(z)/(z - P(y)) f(y)
\]
effects and we define the resolvent distribution in that way. A short calculation shows, that the resolvent distribution equation is fulfilled. We find for the spectral distribution
\[
M(\varphi)f(y) = \varphi(P(y)) f(y).
\]
The open mapping theorem ensures that the set \( P(G) = \{ x \in \mathbb{R} : \exists y \in G, P(y) = x \} \) is open. The set
\[
S(x) = \{ y \in G : P(y) = x \}
\]
is a \((n - 1)\)-dimensional \( C^\infty \)-submanifold of \( \mathbb{R}^n \). With the help of the theorem of implicit functions one establishes the lemma

**Lemma 2.** Assume a point \( y_0 \in G \) and set \( x_0 = P(y_0) \). The exists an open neighborhood \( U_1 \subset \mathbb{R} \) of \( x_0 \) and an open subset \( U_2 \subset \mathbb{R}^{n-1} \) and an injective \( C^\infty \) mapping \( \Psi : U_1 \times U_2 \to G \), such that
\[
P(\Psi(x,u)) = x
\]
and \( \Psi : U_1 \times U_2 \mapsto \Psi(U_1 \times U_2) \) is a diffeomorphism. Then \( \Psi(U_1 \times U_2) \) is an open neighborhood of \( y_0 \) and for fixed \( x \) the mapping \( \Psi_x : u \in U_2 \mapsto \Psi(x,u) \) is a \( C^\infty \)-chart of \( S(x) \). If
\[
J(\Psi) = | \det(\partial \Psi/\partial x, (\partial \Psi/\partial u_i)_{i=1,\ldots,n-1}) |
\]
is the absolute value of the Jacobi’s determinant and
\[
\Gamma(\Psi_x) = (\det(\partial \Psi_x/\partial u_i, \partial \Psi_x/\partial u_j)_{j=1,\ldots,n-1})^{1/2}
\]
is the square root of Gram’s determinant and \( d\sigma_x(y) \) is the euclidean surface element on \( S(x) \) for \( y \in S(x) \), where the point denotes the scalar product. Then
\[
J(\Psi)(x,u) = \Gamma(\Psi_x)(x,u)/|\nabla P(x,u)|
\]
and \( dy = dx \sigma_x(y)/|\nabla P(y)| \).
We introduce $d\tau(y) = d\sigma(y)/|\nabla P(y)|$. So
\begin{equation}
dy = dx d\sigma(y)/|\nabla P(y)| = dx d\tau(y)
\end{equation}

As any function of compact support on $G$ can be represented as a finite sum of functions which have their support in a chart like in the preceding lemma, we obtain the proposition

**Corollary 1.** If $f$ is a continuous function of compact support on $G$, then
\begin{equation}
\int f(y)dy = \int dx \int_{y \in \mathcal{S}(x)} d\tau(y)
\end{equation}

**Lemma 3.** Assume a function $f \in \mathcal{D}(G)$, then
\begin{equation}
\omega \in P(G) \mapsto \int_{y \in \mathcal{S}(\omega)} f(y)d\tau(y)
\end{equation}
is in $\mathcal{D}(P(G))$ where $P(G)$ is the image of $G$ in $\mathbb{R}^n$. Using again this lemma we have
\begin{equation}
\int_{y \in \mathcal{S}(\omega)} f(y)d\tau(y) = \int du (\Psi(u)J)\delta_{(u)}
\end{equation}
and this is surely $C^\infty$. A partition of unity finishes the proof. \[ \Box \]

If $T$ is a distribution on the open set $P(G)$, then consider $T(P(y))$. If $f$ has its support in $U_1 \times U_2$, then as distributions transform like functions
\begin{equation}
\int T(P(y))f(y)dy = \int dx d\tau(x)T(\Psi(x))f(\Psi(x))J(\Psi(x))u = \int dx T(x)\int du (\Psi(u)J)\Gamma(\Psi(x))u
\end{equation}
Hence we define for test function $f \in \mathcal{D}(G)$
\begin{equation}
T(\Omega)(f) = \int T(P(y))f(y)dy = \int dx T(x)\int_{y \in \mathcal{S}(x)} f(y)d\tau(y).
\end{equation}
Especially
\begin{equation}
\delta(x - \Omega)(f) = \int_{y \in \mathcal{S}(x)} f(y)d\tau(y)
\end{equation}
and $M(z) = \delta(x - \Omega)\delta(y)$. This measure on $\mathbb{R}^n$ has been treated by Gelfand-Schilow under the name $\delta(x - P)$ \[ \mathbb{3} \]. We cite the definition \[ \mathbb{3} \].

**Definition 3.** Assume a vector space $V$ and a linear mapping $A : V \to V$. A linear functional $F : V \to \mathbb{C}$ is called a generalized eigenvector for the eigenvalue $x$ if $F(Af) = xF(f)$ for all $f \in V$.

Obviously the operator $\Omega$ leaves $\mathcal{D}(G)$ invariant. We define the left generalized ket-vector $\langle \delta_y \rangle$ as the kernel $\langle \delta_y \rangle(w) = \delta(y - w)$ applied to $f$ in the following way
\begin{equation}
\langle \delta_y \rangle(f) = \int \delta(y - w)f(w)dw = f(y)
\end{equation}
hence
\begin{equation}
\langle \delta_y \rangle \Omega(f) = P(y)f(y) = P(y)\langle \delta_y \rangle(f)
\end{equation}
and $\langle \delta_y \rangle$ is a generalized left eigenvector for the eigenvalue $P(y)$. Similar we define the generalized right eigenvector $|\delta_y \rangle$ by $\langle f|\delta_y \rangle = \overline{f}(y)$. For $f \in L^2(G)$ the bra-vector $\langle f \rangle$ is given by the functional
exists an open neighborhood \( G \) for \( z \) to the resolvent set. We assume, that the jumps on \( \Omega \) and define the operator

\[
\delta(x - \Omega) = \int_{y \in \mathcal{S}(x)} |\delta_y| \langle \delta_y \rangle \, d\tau_x(y).
\]

This is the orthogonality relation for the generalized eigenvectors. The relation

\[
\langle f \rangle g = \int \mathcal{T}(y) g(y) \, dy.
\]

Similar relations hold for the ket-vector \( |f\rangle \). Following this idea we define \( \langle \delta_y | \delta_{y'} \rangle = \int d\omega \delta(y - \omega) \delta(y' - \omega) \) and finally

\[
\langle \delta_y | \delta_{y'} \rangle = \delta(y - y').
\]

This is the orthogonality relation for the generalized eigenvectors. The relation

\[
\langle f | g \rangle = \int \int dy \, dy' \, \mathcal{T}(y) g(y') \langle \delta_y | \delta_{y'} \rangle.
\]

states the completeness of the eigenvectors. Finally

\[
\delta(x - \Omega) = \int_{y \in \mathcal{S}(x)} |\delta_y| \langle \delta_y \rangle \, d\tau_x(y).
\]

2.2. Perturbations of the Multiplication Operator 1. We perturb the multiplication operator \( \Omega \) and define the operator

\[
H = \Omega + |g\rangle \langle h|
\]

with \( g, h \in \mathcal{D}(G) \). The resolvent can be easily calculated

\[
R(z) = R_\Omega(z) + R_\Omega(z) |g\rangle \langle h| R_\Omega(z) / C(z), \quad C(z) = 1 - \langle h| R_\Omega(z) |g\rangle,
\]

where \( R_\Omega(z) = 1/(z - \Omega) \) is the resolvent of \( \Omega \). The ket-vector \( R_\Omega(z) |g\rangle \) is the functional \( f \in \mathcal{D}(G) \rightarrow \langle f | R_\Omega(z) |g\rangle \) and similarly defined is the bra-vector \( \langle h| R_\Omega(z) \).

Recall the formula

\[
\frac{1}{x \pm i\varepsilon} \to \frac{1}{x \pm i0} = \mathcal{P} \frac{1}{x} = i\pi \delta(x) \text{ for } \varepsilon \downarrow 0,
\]

where \( \mathcal{P} \) denotes the principal value and obtain the lemma

**Lemma 4.** Assume \( f_1, f_2 \in \mathcal{D}(G) \), then for \( z \to x \pm i0 \) uniformly in \( x \)

\[
\int dy \, \frac{\mathcal{T}(y) g(y)}{z - \Omega} = \langle f_1 | \frac{1}{z - \Omega} | f_2 \rangle \to \langle f_1 | \frac{1}{x \pm i0} - \frac{1}{\Omega} | f_2 \rangle = \langle f_1 | \mathcal{P} \frac{1}{x - \Omega} | f_2 \rangle \mp i\pi \langle f_1 | \delta(x - \Omega) | f_2 \rangle
\]

with

\[
\langle f_1 | \mathcal{P} \frac{1}{x - \Omega} | f_2 \rangle = \int d\omega \mathcal{P} \frac{1}{x - \omega} \int_{y \in \mathcal{S}(\omega)} \mathcal{T}(y) f_2(y) \, d\tau(y), \quad \langle f_1 | \delta(x - \Omega) | f_2 \rangle = \int_{y \in \mathcal{S}(x)} \mathcal{T}(y) f_2(y) \, d\tau(y)
\]

We use Dirac’s notation \( \langle f_1 | A | f_2 \rangle = \langle f_1 | A f_2 \rangle \), which is in many cases convenient. We calculate for \( z \to x \pm i0 \)

\[
C(z) \to 1 - \langle h| \mathcal{P} \frac{1}{x - \Omega} |g\rangle \pm i\pi \langle h| \delta(x - \Omega) |g\rangle = C_1(x) \pm i\pi C_2(x)
\]

So there are jumps of \( C(z) \) on the real axis contained in \( P(K) \), if the supports of \( g, h \) are contained in the compact set \( K \subset G \). If \( z \notin P(K) \) and \( C(z) \neq 0 \), then \( R(z) \) exists as function and \( z \) belongs to the resolvent set. We assume, that the jumps on \( P(K) \) are isolated, more precisely that there exists an open neighborhood \( G_0 \subset \mathcal{C} \) of \( P(K) \), such that \( C(z) \) is \( \neq 0 \) and holomorphic in \( G_0 \setminus P(K) \) and \( C(z \pm i0) \neq 0 \) for \( x \in G_0 \cap \mathbb{R} \). We may assume, that \( G_0 = G_1 \times I \), where \( G_1 \) is an open subset of the real line containing \( P(K) \) and \( I \) is an open interval containing 0.

**Lemma 5.** Assume a \( C^\infty \) test function \( \varphi \) with support in \( G_1 \) and put \( R_u(x) = R(x + iu) \) then for \( \varphi \in \mathcal{D}(G_1) \) the operator \( \int dx R(x + iu) \varphi(x) = R_u(\varphi) \) on \( L^2(G) \) converges for \( u \to 0^+ \) or \( u \to 0^- \) in operator norm to an operator called \( R_{\pm i0}(\varphi) \).
Proof. Rewrite equation \[ (13) \]

\[
R(z) = \frac{1}{z - \Omega} + \int \int dw_1 dw_2 \frac{1}{z - w_1} \frac{1}{z - w_2} \delta(w_1 - \Omega) \langle h \delta(w_2 - \Omega) \rangle / C(z)
\]

\[
= \frac{1}{z - \Omega} + \int \int dw_1 dw_2 \frac{1}{w_2 - w_1} (\frac{1}{z - w_1} - \frac{1}{z - w_2}) \delta(w_1 - \Omega) \langle h \delta(w_2 - \Omega) \rangle / C(z)
\]

Then for \( u \neq 0 \)

\[
\int dx R(x + u) \varphi(x) = R_u(\varphi) = \psi_u(\Omega) + \int \int dw_1 dw_2 \chi_u(w_1, w_2) \delta(w_1 - \Omega) \langle h \delta(w_2 - \Omega) \rangle
\]

The function \( \psi_u(w) = \int dx \varphi(x)/(x + iu - w) \) is in \( \mathcal{D}(G_1) \) and converges in this sense to \( \psi_{+0} \) or \( \psi_{-0} \).

\[
\chi_{u}(w_1, w_2) = \int dx \varphi(x) \frac{1}{w_2 - w_1} (\frac{1}{x + iu - w_1} - \frac{1}{x + iu - w_2}) / C(x + iu)
\]

\[
= - \int \frac{dx}{C(x + iu)} \int_0^1 dt \frac{d}{dx} x + iu - w_1 - t(w_2 - w_1)
\]

\[
= \int \frac{dx}{C(x + iu)} \int_0^1 dt \frac{1}{x + iu - w_1 - t(w_2 - w_1)} = \int \int dx dt \frac{\omega_{u}(x + (1 - t)w_1 + tw_2)}{x + iu}
\]

\[
\omega_u(x) = \frac{d}{dx} \left( \frac{\varphi(x)}{C(x + iu)} \right)
\]

The function \( x \mapsto \omega_u(x) \) is in \( \mathcal{D}(G_1) \) and converges for \( u \to \pm 0 \) in this sense, as the function \( x \mapsto C(x + iu) \) is in \( C^\infty \) and converges to \( x \to C(x \pm 0) \) for \( x \to +0 \) or \( x \to -0 \) uniformly, the analogue holds for all derivatives. Hence we obtain

\[
\chi_{\pm 0}(w_1, w_2) = \int \int dx dt \frac{\omega_{\pm 0}(x + (1 - t)w_1 + tw_2)}{x \pm i0}.\]

Assume \( f_1, f_2 \in L^2(G) \), then for \( \vartheta = \pm 0 \)

\[
| \int \int dw_1 dw_2 (\chi_u(w_1, w_2) - \chi_{\vartheta}(w_1, w_2)) (f_1 \delta(w_1 - \Omega) \langle h \delta(w_2 - \Omega) \rangle f_2) |
\]

\[
\leq \max \{ |\chi_u(w_1, w_2) - \chi_{\vartheta}(w_1, w_2)| : w_1, w_2 \in P(K) \} \int \int dw_1 dw_2 |(f_1 \delta(w_1 - \Omega) \langle h \delta(w_2 - \Omega) \rangle f_2)|.
\]

Observe, using corollary \[ (11) \], that e.g. \( \int dw |\langle f \delta(w - \Omega) \rangle g| \leq N(f) N(g) \), where \( N(.) \) is the \( L^2 \)-norm, and conclude from there that the last expression converges to 0 in operator norm. \( \square \)

A consequence of proposition \[ (4) \] and lemma \[ (5) \]

**Proposition 5.** Define for a \( C^\infty \) test function \( \varphi \) with support in \( G_0 \) the operator valued distribution

\[
R(\varphi) = \lim_{\varepsilon \to 0} \int_{|u| > \varepsilon} dx du R(x, u) \varphi(x, u) = \lim_{\varepsilon \to 0} \int_{|u| > \varepsilon} du R_u(\varphi_u)
\]

with \( \varphi_u(x) = \varphi(x + iu) \). Then \( R(\varphi) \) extends the function \( R(z) \) on \( G_0 \setminus P(K) \) to a distribution on \( G_0 \), which fulfills the resolvent distribution equation. We have

\[
M(x + iy) = \mu(x) \delta(y), \quad \mu(x) = \frac{1}{2\pi i} (R(x - i0) - R(x + i0)).
\]
We consider for \( u \neq 0 \) the sesquilinear form
\[
f_1, f_2 \in \mathcal{D}(G) \rightarrow \mathcal{B}(R_u(x))(f_1, f_2) = \langle f_1 | R_u(x) | f_2 \rangle.
\]
By lemma 4, the bracket converges uniformly in \( x \) for \( x \rightarrow 0 \) to a sesquilinear form \( \mathcal{B}(R_{u0})(x) \). We obtain
\[
\langle f_1 | R_{u0}(\varphi) | f_2 \rangle = \int dx \varphi(x) \mathcal{B}(R_{u0})(x)(f_1, f_2),
\]
and define
\[
\langle f_1 | \kappa(x) | f_2 \rangle = \mathcal{B}(\mu(x))(f_1, f_2) = \frac{1}{2\pi i} (\mathcal{B}(R_{-0})(x)(f_1, f_2) - \mathcal{B}(R_{+0})(x)(f_1, f_2)).
\]
Hence
\[
\langle f_1 | \mu(\varphi) | f_2 \rangle = \int dx \varphi(x) \langle f_1 | \kappa(x) | f_2 \rangle
\]
and \( \kappa(x) \) appears as the restriction of the sesquilinear form \( f_1, f_2 \in L^2(G) \rightarrow \langle f_1 | \mu(x) | f_2 \rangle \), which is a distribution, to \( f_1, f_2 \in \mathcal{D}(G) \), where \( x \mapsto \langle f_1 | \kappa(x) | f_2 \rangle \) is a continuous function.

From the formula
\[
\mathcal{B}(R_{u0})(x) = \frac{\mathcal{P}}{x - \Omega} \mp i\pi \delta(x - \Omega) + \frac{1}{C_1 + i\pi C_2} \left( \frac{\mathcal{P}}{x - \Omega} \mp i\pi \delta(x - \Omega) \right) |g| \langle h | \frac{\mathcal{P}}{x - \Omega} \mp i\pi \delta(x - \Omega) \rangle.
\]
one establishes by straightforward calculations

**Proposition 6.** Recall
\[
C_1(x) = 1 - \langle h | \frac{\mathcal{P}}{x - \Omega} | g \rangle, \quad C_2(x) = \langle h | \delta(x - \Omega) | g \rangle
\]
Assume, that \( g, h \) are real and that
\[
|C(x + i0)|^2 = |C_1(x) + i\pi C_2(x)|^2 = C_1(x)^2 + \pi^2 C_2(x)^2 \neq 0
\]
for \( x \in G_0 \cap \mathbb{R} \). We define the following bra- and ket-vectors as functionals over \( \mathcal{D}(G) \).

\[
\begin{align*}
A &= A(x) = \frac{\mathcal{P}}{x - \Omega} |g\rangle, & A' &= A'(x) = \langle h | \frac{\mathcal{P}}{x - \Omega} | g \rangle
\end{align*}
\]
\[
\begin{align*}
B &= B(x) = \delta(x - \Omega) |g\rangle, & B' &= B'(x) = \langle h | \delta(x - \Omega) | g \rangle
\end{align*}
\]
For \( x \in P(K) \) one obtains
\[
\kappa(x) = \delta(x - \Omega) + \frac{1}{C_1^2 + \pi^2 C_2^2} (AC_2 A' + AC_1 B' + BC_1 A' - \pi^2 BC_1 B')
\]
If \( C_2(x) \neq 0 \), we may write
\[
\kappa(x) = \delta(x - \Omega) - \frac{BB'}{C_2} + \frac{1}{(C_1^2 + \pi^2 C_2^2) C_2} (AC_2 A' + AC_1 C_2 B' + BC_1 C_2 A' + C_1^2 BB')
\]
\[
= \delta(x - \Omega) - \frac{BB'}{C_2} + \frac{1}{(C_1^2 + \pi^2 C_2^2) C_2} (C_1 B + AC_2)(C_1 B' + A' C_2) = p(x) + |\alpha(x)\rangle \langle \alpha'(x)|
\]
with
\[
p(x) = \delta(x - \Omega) - \frac{\delta(x - \Omega)|g\rangle \langle h | \delta(x - \Omega)|g\rangle}{\langle h | \delta(x - \Omega) | g \rangle}
\]
\[
|\alpha(x)\rangle = N(x)(C_1(x) \delta(x - \Omega) |g\rangle + C_2(x) \frac{\mathcal{P}}{x - \Omega} | g \rangle)
\]
\[
\langle \alpha'(x)| = N(x)(C_1(x) \langle h | \delta(x - \Omega) + C_2(x) \langle h | \frac{\mathcal{P}}{x - \Omega} | g \rangle)
\]
and \( N(x)^2 = 1/(C_1^2 + \pi^2 C_2^2) C_2 \).

We discuss the case \( C_2(x) = 0 \). If \( h = g \), then \( C_2(x) = \int_S(x) d\tau(y) g(y)^2 = 0 \) implies that \( g \) on \( S(x) \) vanishes and hence \( B = 0, B' = 0 \). In the general case, remark that \( B = 0 \) or \( B' = 0 \) imply that \( C_2 = 0 \). Let us assume, that \( g, h \) are in such form, that \( C_2(x) = 0 \) implies \( B(x) = 0, B'(x) = 0 \). Then

\[
\kappa(x) = \begin{cases} 
  p(x) + |\alpha(x)|\langle \alpha'(x) \rangle & \text{for } C_2(x) \neq 0 \\
  \delta(x - \Omega) & \text{for } C_2(x) = 0
\end{cases}
\]

If \( g = h \), then \( \mu(x) \) is positive definite.

**Proposition 7.** The operator \( H \) maps \( \mathcal{D}(G) \to \mathcal{D}(G) \). So we may formulate

\[
\begin{align*}
H\kappa(x) &= \kappa(x)H = x\kappa(x) \\
Hp(x) &= p(x)H = xp(x) \\
H|\alpha(x)\rangle &= x|\alpha(x)\rangle \\
\langle\alpha'(x)|H &= x\langle\alpha'(x)\rangle
\end{align*}
\]

These equations have to be understood as equations for sesquilinear forms bracketed between functions in \( \mathcal{D}(G) \) or as functionals on \( \mathcal{D}(G) \). So \( \kappa(x) \) and \( p(x) \) are generalized eigensprojectors and \( |\alpha(x)\rangle \) is a generalized right eigenvector and \( \langle\alpha'(x)| \) is a generalized left eigenvector, all for the eigenvalue \( x \).

The proof is done by straightforward computation.

**Proposition 8.** We have the orthogonality relations

\[
\begin{align*}
\kappa(x)\kappa(x') &= \delta(x - x')\kappa(x) \\
p(x)p(x') &= \delta(x - x')p(x) \\
p(x)|\alpha(x')\rangle &= 0, \quad \langle\alpha'(x)|p(x') = 0 \\
\langle\alpha'(x)|\alpha(x')\rangle &= \delta(x - x')
\end{align*}
\]

These relations have to be understood in the sense of distributions, e.g. the first one signifies

\[
(\int dx_1 \varphi_1(x_1)\langle f_1|\kappa(x_1)\rangle)(\int dx_2 \varphi_2(x_2)\kappa(x_2)|f_2\rangle) = (\int dx \varphi_1(x)\varphi_2(x)\langle f_1|\kappa(x)\rangle|f_2\rangle)
\]

for \( f_1, f_2 \in \mathcal{D}(G) \) and \( \varphi_1, \varphi_2 \in \mathcal{D}(P(G)) \) and we show, that the expressions make sense.

**Proof.** Assume \( f_1 \in \mathcal{D}(G) \), by the mapping \( f_2 \in \mathcal{D}(G) \to \langle f_1|\kappa(x)\rangle|f_2\rangle \) we define a distribution called \( \langle f_1|\kappa(x)\rangle \). For a test function \( \varphi \in \mathcal{D}(P(G)) \) the integral \( \int dx \varphi(x)\langle f_1|\kappa(x)\rangle \) exists and we have

\[
(\int dx \varphi(x)\langle f_1|\kappa(x)\rangle)(f_2) = (\int dx \varphi(x)\langle f_1|\kappa(x)\rangle|f_2\rangle = (f_1|\mu(\varphi)|f_2\rangle).
\]

Hence

\[
\int dx \varphi(x)\langle f_1|\kappa(x)\rangle = (f_1|\mu(\varphi)|) \in L^2(G).
\]

Similarly define \( \kappa(x)|f\rangle \) and obtain

\[
\int dx \varphi(x)\kappa(x)|f\rangle = \mu(\varphi)f_2 \in L^2(G).
\]

So we can form the scalar product

\[
(\int dx_1 \varphi_1(x_1)\langle f_1|\kappa(x_1)\rangle)(\int dx_2 \varphi_2(x_2)\kappa(x_2)|f_2\rangle = (f_1|\mu(\varphi_1)\mu(\varphi_2)|f_2\rangle = (f_1|\mu(\varphi_1\varphi_2)|f_2\rangle)
\]
This proves the first equation. We check the last equation
\[
\langle \alpha'(x) | \alpha(x') \rangle = N(x)N(x')(C_1(x)h|\delta(x-\Omega)+C_2(x)(h|\frac{P}{x-\Omega}))(C_1(x')\delta(x'-\Omega)|g)+C_2(x')\frac{P}{x'-\Omega}|g) \\
= N(x)N(x') \int dw_1dw_2\langle h|\delta(w_1-\Omega)\delta(w_2-\Omega)|g) \\
(C_1(x)\delta(x-w_1)+C_2(x)\frac{P}{x-w_1})(C_1(x')\delta(x'-w_2)+C_2(x')\frac{P}{x'-w_2}) \\
Use \delta(w_1-\Omega)\delta(w_2-\Omega) = \delta(w_1-w_2)\delta(w_1-\Omega) and continue \\
= N(x)N(x') \int dw_2\langle C_1(x)'(x-w)+C_2(x)(x-w)\frac{P}{x-w})(C_1(x')\delta(x'-w)+C_2(x')\frac{P}{x'-w}). \\
Use the properties of the \(\delta\)-function and equation (3) and continue \\
= N(x)N(x')(C_2(x)C_1(x)\delta(x-x')+2C_2(x)C_2(x')(C_1(x')\frac{P}{x-x'}+C_1(x)\frac{P}{x'-x})+\pi^2C_2(x)\delta(x-x')) \\
+ C_2(x)C_2(x') \int dw_2\langle 1 \frac{P}{x'-w}-(\frac{P}{x-w}) \rangle \\
Now \\
C_1(x')\frac{P}{x-x'}+C_1(x)\frac{P}{x'-x} = C_1(x')-C_1(x) = \langle h|\frac{1}{x-x'}(-\frac{P}{x-x'}+\frac{P}{x-\Omega})|g) \\
= \int dw_2\langle 1 \frac{P}{x'-w}-(\frac{P}{x-w}) \rangle \\
The verification of the two other relations are left to the reader. □

2.3. Perturbation of the Multiplication Operator 2. This example is a caricature of the eigenvalue problem arising in the theory of radiation transfer in a gray atmosphere in plan parallel geometry [1]. We consider for some \(c > 1\) the set \(G = \{ -c, -1[0,1], c \subseteq R \}, \) the multiplication operator \(\Omega f(y) = yf(y)\) and two real \(C^\infty\) functions \(f, g\) on \(R\) with \(g(x) > 0\) for \(1 < x < c\) and \(-c < x < -1\) and 0 outside these two open interval. We assume \(g(y) = g(-y)\) and \(h(y) = -g(y)\) for \(y > 1\) and \(h(y) = g(y)\) for \(y < -1\). We study \(H = \Omega + |g|\langle h\rangle\) and obtain
\[
C(z) = 1 - \int_G dy \frac{g(y)h(y)}{z-y} = 1 + \int_1^c dy \frac{2yg(y)^2}{z^2-y^2}. \\
We have for \(z = x + iu\)
\[
\text{Im}C(z) = \int_1^c dy \frac{4y^2xug(y)^2}{(x^2-u^2-y^2)^2+4x^2u^2} \\
Hence \(C(z) = 0\) implies \(xu = 0\), so either \(x\) or \(u\) or both vanish. We have
\[
C(0) = 1 - \int_1^c dy \frac{2yg(y)^2}{y}, \quad C(iu) = 1 - \int_1^c dy \frac{2yg(y)^2}{u^2+y^2} \\
So \(C(iu)\) is monotonic increasing for increasing \(u\) and goes to 1 for \(u^2 \to \infty\). If \(C(0) < 0\), there exists exactly one \(u_0 > 0\) such that \(C(iu_0) = 0\), if \(C(0) > 0\), then \(C(iu) > 0\) for all \(u\).
For \(|x| \leq 1\) we have
\[
C(x) = 1 - \int_1^c dy \frac{2yg(y)^2}{y^2-x^2}.
and is monotonic decreasing for increasing \( x \). If \( C(0) > 0 \) and \( C(1) < 0 \) there exists exactly one \( x_0 \) with \( 0 < x_0 < 1 \), such that \( C(x_0) = 0 \). If \( C(0) < 0 \) there does not exist such an \( x \). We do not discuss the case \( C(0) > 0 \) and \( C(1) \geq 0 \). In case \( C(0) = 0 \) we have a double zero. For \( |x| \geq c \) we have \( C(x) \geq 1 \). Hence \( C(x) \) does not vanish for \( |x| \geq c \).

The singularities of the resolvent are the slits \([-c, -1]\) and \([1, c]\) and the zeros of \( C(z) \). In the neighborhood of a zero of \( C(z) \) we may define a resolvent distribution with the help of proposition 4.

We discuss the behavior of the resolvent in the neighborhood of the slits. We have

\[
C(x \pm i0) = C_1(x) \pm i\pi C_2(x) = 1 - \int \frac{dy g(y) h(y)}{x-y} \pm i\pi g(x) h(x).
\]

There exists a neighborhood \( G_1 \subset \mathbb{R} \) of \([-1, -c] \cup [1, c] \) and an open interval \( I \) containing 0, such that \( R(z) \) is holomorphic in \((G_1 \times I) \setminus ([1, c] \cup [1, c]) \) and \( C(x \pm i0) \neq 0 \) for \( x \in G_1 \). So we can apply prop. 4 and define a distribution \( R \) extending the resolvent function \( R(z) \) to \( G_1 \times I \) and fulfilling the distribution resolvent equation. From the local definition in the neighborhood of the singularities we can define a resolvent distribution extending \( R(z) \) to \( \mathbb{C} \) with the help of proposition 4.

**Proposition 9.** We calculate the spectral distribution. If there are two zeros \( \neq 0 \) we obtain

\[
M(z) = M(x + iu) = r_+ \delta(z - z_0) + r_- \delta(z + z_0) + \delta(u) \mu(x)
\]

where \( r_\pm \) are the residues of \( R(z) \) at the points \( \pm z_0 \) and

\[
\mu(x) = \frac{1}{2\pi i} (R(x - i0) - R(x + i0)).
\]

We identify \( \mu(x) \) with its restriction \( \kappa(x) \) as bilinear form on \( D(\mathbb{C}) \) and obtain

\[
M(z) = |\alpha_+ \rangle \langle \alpha'_+ | \delta(z - z_0) + |\alpha_- \rangle \langle \alpha'_- | \delta(z + z_0) + \delta(u) |\alpha_x \rangle \langle \alpha'_x |
\]

with the right resp. left usual eigenvectors

\[
|\alpha_\pm \rangle = \frac{1}{\sqrt{\langle h| (\pm z_0 - \Omega)^2 |g \rangle}} \frac{1}{\langle h| \pm z_0 - \Omega \rangle} \langle g| \frac{1}{\langle h| (\pm z_0 - \Omega)^2 |g \rangle} \frac{1}{\langle h| \pm z_0 - \Omega \rangle}
\]

and for \( x \in G \) the right, resp. left generalized eigenvectors

\[
|\alpha_x \rangle = \frac{1}{\sqrt{C_1 + \pi^2 C_2}} (C_1(x) |\delta_x \rangle + h(x) A(x)) \quad \langle \alpha'_x | = \frac{1}{\sqrt{C_1 + \pi^2 C_2}} (C_1(x) |\delta_x \rangle + g(x) A'(x))
\]

with

\[
C_1(x) = 1 - \int \text{d}y g(y) h(y) \frac{P}{x-y} \quad C_2(x) = g(x) h(x)
\]

\[
A(x) = \frac{P}{x-\Omega} |g\rangle \quad A'(x) = \langle h| \frac{P}{x-\Omega}
\]

\[
B(x) = g(x) |\delta_x \rangle \quad B'(x) = h(x) |\delta_x \rangle
\]

In the case \( C(0) = 0 \) we obtain

\[
R(z) = z^{-2} \frac{\Omega^{-1} |g\rangle \langle h|}{\langle h| \Omega^{-1} |g\rangle} + z^{-1} \frac{\Omega^{-2} |g\rangle \langle h| \Omega^{-1} |g\rangle + \Omega^{-1} |g\rangle \langle h| \Omega^{-2} |g\rangle}{\langle h| \Omega^{-3} |g\rangle} + O(1) = z^{-2} a + z^{-1} p_0 + O(1)
\]

and

\[
M(z) = M(x + iu) = p_0 \delta_2(z) - a \delta_2(z) + \mu(x) \delta(u),
\]
where \( \mu(x) = |\alpha_x\rangle\langle \alpha_x'| \) is given by the formula above. Here \( p^2 = p \) and \( a^2 = 0 \) and \( ap = pa = a \).

We obtain for \( \vartheta, \vartheta' = \pm \) and \( x, x' \in \mathbb{R} \) the orthogonality relations

\[
\langle \alpha'_\vartheta | \alpha_{\vartheta'} \rangle = \delta_{\vartheta, \vartheta'} \quad \langle \alpha'_\vartheta | \alpha_\vartheta \rangle = 0 \quad \langle \alpha_{\vartheta} | \alpha_{\vartheta'} \rangle = \delta(x - x')
\]

Analogous relations hold for the case \( C(0) = 0 \). The spectral distribution is complete, i.e.

\[
M(1) = \int \mathrm{d}^2z M(z) = \sum_{\vartheta = \pm} |\alpha'_\vartheta \rangle \langle \alpha_{\vartheta}| + \int \mathrm{d}x |\alpha'_x \rangle \langle \alpha_x| = 1
\]

**Proof.** Assume at first, that \( C(\vartheta) \) has two zeros \( \pm z_0 = \pm i\vartheta_0 \) or \( \pm z_0 = \pm x_0 \). The residuum of the complex function \( R(z) \) at the points \( \pm z_0 \) is given by

\[
r_\vartheta = \left( \frac{1}{\pm z_0 - \Omega} \langle h| \frac{1}{\pm z_0 - \Omega} \right) / C'(\pm z_0) = \left( \frac{1}{\pm z_0 - \Omega} \langle h| \frac{1}{\pm z_0 - \Omega} \right) \langle h|(\pm z_0 - \Omega)^{-2}|g\rangle
\]

Using the results of proposition 3 we obtain

\[
M(z) = M(x + i\vartheta) = r_+ \delta_2(z - z_0) + r_- \delta_2(z + z_0) + \delta(z) \mu(x)
\]

where

\[
\mu(x) = \frac{1}{2\pi i} (R(x - i\vartheta) - R(x + i\vartheta))
\]

We consider the case \( C(0) = 0 \). We expand at the origin

\[
C(z) = 1 + \sum_{n=0}^{\infty} z^n \langle h| \Omega^{-(n+1)}|g\rangle = z^2 \langle h| \Omega^{-3}|g\rangle + O(z^3) = -z^2 \int_{0}^{\infty} 2g(y)/y^3 \mathrm{d}y + O(z^3)
\]

\[
R(z) = z^{-2} \langle h| \Omega^{-1}|g\rangle / \langle h| \Omega^{-3}|g\rangle + z^{-1} \frac{\langle h| \Omega^{-2}|g\rangle \langle h| \Omega^{-1} + \Omega^{-1}|g\rangle}{\langle h| \Omega^{-3}|g\rangle} + O(1) = z^{-2} a + z^{-1} p_0 + O(1)
\]

and

\[
M(z) = M(x + i\vartheta) = p_0 \delta_2(z) - a \delta_2(z) + \mu(x) \delta(x).
\]

Here \( p^2 = p \) and \( a^2 = 0 \) and \( ap = pa = a \).

In both cases we can use for \( f_1, f_2 \in \mathcal{D}(\mathbb{R}) \) the relation \( \langle f_1 | \mu(x) | f_2 \rangle = \langle f_1 | \kappa(x) | f_2 \rangle \) and \( \kappa(x) \) is given by the formula

\[
\kappa(x) = \delta(x - \Omega) + \frac{1}{C_1^2 + \pi^2 C_2^2} (AC_2 A' + AC_1 B' + BC_1 A' - \pi^2 BC_2 B')
\]

\[
= \frac{1}{C_1^2 + \pi^2 C_2^2} (Ah(x) g(x) A' + AC_1 h(x) \delta_x |g(x)| \delta_x + g(x) |\delta_x| C_1 A' + C_2^2 h(x) g(x)^2 |\delta_x| |\delta_x|) = |\alpha_x\rangle \langle \alpha_x'|
\]

as \( \delta(x - \Omega) = |\delta_x\rangle \langle \delta_x| \). The proof of the orthogonality relations can be deduced from the relation \( \kappa(x) \kappa(x') = \delta(x - x') \kappa(x) \), like in the proof of prop 3 or can be done by hand. That is left to the reader. For the completeness refer to theorem 2.

\[\square\]
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