General rogue wave solutions to the Sasa-Satsuma equation

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Abstract

General rogue wave solutions to the Sasa-Satsuma equation are constructed by the Kadomtsev-Petviashvili (KP) hierarchy reduction method. These solutions are presented in three different forms. The first form is expressed in terms of recursively defined differential operators while the second form shares a similar solution structure except that the differential operators are no longer recursively defined. Instead of using differential operators, the third form is expressed by Schur polynomials.

1 Introduction

The Sasa-Satsuma (SS) equation is one of the nontrivial integrable extensions of the nonlinear Schrödinger (NLS) equation. It was discovered by Sasa and Satsuma \cite{Sas93} when they were searching integrable cases of a higher-order NLS equation proposed by Kodama and Hasegawa \cite{Kod86, Has92}. This equation can be written in the form \cite{Sas93}

\begin{equation}
\begin{aligned}
&i \frac{\partial q}{\partial T} + \frac{1}{2} \frac{\partial^2 q}{\partial X^2} + |q|^2 q + i \varepsilon \left\{ \frac{\partial^3 q}{\partial X^3} + 6|q|^2 \frac{\partial q}{\partial X} + 3q \frac{\partial |q|^2}{\partial X} \right\} = 0,
\end{aligned}
\end{equation}

where \(q\) corresponds to the complex envelope of the wave field and the real constant \(\varepsilon\) scales the integrable perturbations of the NLS equation. For the case of \(\varepsilon = 0\), the SS equation reduces to the NLS equation. Like the NLS equation, the SS equation contains essential terms that are commonly involved in nonlinear optics \cite{Fen99, Ves01}, such as the third-order dispersion, the self-steepening and the self-frequency shift. As a result of its integrability and physical implications,
the SS equation has been studied comprehensively. Soliton solutions of the SS equation have been obtained in a number of works \([6, 7, 8]\) while the long-time asymptotic behaviour of the SS equation with decaying initial data was analyzed in \([9]\) by formulating a Riemann-Hilbert problem. Very recently, breather solutions of the SS equation were derived based on the Kadomtsev-Petviashvili (KP) hierarchy reduction method.

Rogue waves have been part of folklore for centuries in the maritime community. They refer to unusually large, unpredictable and suddenly appearing surface waves. These features indicate that they may result in tremendous danger to ships, oil platforms and coastal structures. Since the first record in 1995, rogue waves have attracted much attention from the scientific community \([10, 11, 12]\). They have been experimentally reported in diverse nonlinear systems, such as optical fibers \([10]\), Bose-Einstein condensates \([13]\) and plasmas \([14]\). After that, there have been rapidly growing research interests in rogue waves. From the mathematical viewpoint, the first model for rogue waves is the Peregrine soliton \([15]\), a particular rational solution of the NLS equation. Since the discovery of higher-order rogue waves by Akhmediev et al. \([16]\), there has been an explosion of research activities on the study of higher-order rogue waves. Explicit expressions of rogue wave solutions have been derived in a variety of integrable equations as well as their multi-component generalizations, such as the Davey-Stewartson equations \([17, 18]\), the Yajima-Oikawa equation \([19, 20]\), a long-wave-short-wave model of Newell type \([21]\), the derivative NLS equation \([22]\), the three-wave equation \([23]\), the Manakov system \([24, 25]\) and the coupled Hirota system \([26]\). While rogue waves on the periodic background \([27, 28, 29, 30]\) have been extensively studied, rogue waves of infinite order have been revealed \([31]\) with the help of Riemann-Hilbert approach \([32]\). In addition, higher-order rogue waves may exhibit universal patterns. Preliminary results on rogue wave patterns of the NLS equation were obtained in \([33, 34, 35]\). Very recently, Yang and Yang \([36]\) proved their results analytically by connecting rogue waves solutions of the NLS equation with the Yablonskii-Vorob’ev polynomial hierarchy. Moreover, they \([37]\) showed that similar rogue wave patterns also appear in many other integrable equations.

Rogue wave solutions of the NLS equation have been investigated comprehensively and numerous satisfactory results have been achieved. Therefore, it is natural to extend these results to the SS equation that serves as an integrable extension of the NLS equation. Fundamental rogue wave solutions of the SS equation were first derived by Bandelow and Akhmediev \([38, 39]\) by taking the long wave limit of breather solutions. With the help of Darboux dressing method, Chen \([40]\) revealed a distinctive characteristic of the fundamental rogue wave of the SS equation, which is the so-called twisted-rogue wave pair. Explicit second order rogue waves were obtained by Mu et al. \([41]\) and their structure is much more complicated than the case of the NLS equation. Higher-order rogue waves were considered in \([42, 43]\). Nevertheless, explicit forms of higher-order rogue waves of the SS equation are still lack of exploring.
The main objective of this paper is to construct general rogue wave solutions to the Sasa-Satsuma equation \[8, 44\] 
\[u_t = u_{xxx} - 6c|u|^2u_x - 3cu(|u|^2)_x,\]  
(2)

where \(c\) is a real constant. This form is more convenient for our consideration and for the case of \(c = -1\), it can be obtained from (1) via Gauge, Galilean and scale transformations. We will adopt the Kadomtsev-Petviashvili (KP) hierarchy reduction method. This is a very powerful method to construct explicit solutions of integrable equations and has found extensive applications in the derivation of rogue wave or soliton solutions of various integrable equations \[15, 17, 18, 20, 22, 23\]. However, several difficulties are encountered when applying this method to derive rogue wave solutions to the Sasa-Satsuma equation due to its complexity. Unlike the NLS equation which belongs to the AKP hierarchy, the SS equation belongs to the CKP hierarchy which is a symmetry reduction and the sub-hierarchy of the AKP hierarchy \[46\]. This forces us to start with a kernel of 2 \(\times\) 2 matrix which is one of the novelties of the present paper. On the other hand, the bilinear equations of the Sasa-Satsuma equation correspond to eleven bilinear equations in the KP hierarchy. Therefore, as explained in subsequent sections, the reduction procedure and the rogue wave solutions are much more complicated.

The structure of this paper is as follows. In Section 2, we present the general rogue wave solutions to the Sasa-Satsuma equation. These solutions are given in three different forms. The first two forms are expressed in terms of differential operators. The difference between them is that the differential operators are recursively defined for one form whereas they are no longer recursively defined for the other form. Instead of using differential operators, the third form is represented by Schur polynomials. Then the solution dynamics are briefly examined in Section 3 and the proofs of our main results are provided in Sections 4-6. Finally, we conclude this paper in Section 7.

2 General rogue wave solutions to the Sasa-Satsuma equation

In this section, we present the general rogue wave solutions to the SSE \[2\]. These solutions are presented in three different forms. The first two forms are expressed in terms of differential operators while the third form is expressed by Schur polynomials. The difference between the first two forms is that the differential operators are recursively defined in one of them whereas the other one does not involve recursive relations among the differential operators.

**Theorem 2.1.** Let \(a = i\kappa\), where \(\kappa\) is real, \(F(p)\) be a function defined by 
\[F(p) = \frac{1}{p - a} + \frac{1}{p + a} - \frac{p}{c}\]  
(3)

and \(\xi\) be a root of 
\[F'(p) = -\frac{1}{(p - a)^2} - \frac{1}{(p + a)^2} - \frac{1}{c} = 0.\]  
(4)
Then the Sasa-Satsuma equation (2) admits the general rogue wave solutions

$$u = \frac{g}{f} e^{i(\kappa(x - 6ct) - \kappa^3 t)} \quad (5)$$

where

$$f(x, t) = \tau_0(x - 6ct, t), \quad g(x, t) = \tau_1(x - 6ct, t)$$

and \( \tau_k (k = 0, 1) \) is defined as

$$\tau_k = \det \left[ M^k_{ij} \right]_{1 \leq i,j \leq N}$$

where

$$M^k_{ij} = \left[ \begin{array}{ll} A^{(N-i)}_{2i-1} (p_1) B^{(N-j)}_{2j-1} (q_1) m_{11}^k & A^{(N-i)}_{2i-1} (p_1) B^{(N-j)}_{2j-1} (q_2) m_{12}^k \\ A^{(N-i)}_{2i-1} (p_2) B^{(N-j)}_{2j-1} (q_1) m_{21}^k & A^{(N-i)}_{2i-1} (p_2) B^{(N-j)}_{2j-1} (q_2) m_{22}^k \end{array} \right] \bigg| \begin{array}{ll} q_1 = \xi, \\ p_1 = q_2 = \xi^* \end{array}$$

with \((\alpha, \beta = 1, 2, \mu, \nu = 0, 1, \ldots, N - 1)\)

$$m_{\alpha \beta}^k = \frac{1}{p_\alpha + q_\beta} \left( \frac{-p_\alpha - a}{q_\beta + a} \right)^k e^{\xi_\alpha + \eta_\beta}$$

$$\xi_\alpha = p_\alpha x + p_\beta^3 t + \xi_{\alpha 0}, \quad \eta_\beta = q_\beta x + q_\beta^3 t + \eta_{\beta 0}$$

$$A^{(\mu)}_{2i-1} (p) = \sum_{n=0}^{2i-1} \frac{a^{(\mu)}_{2i-1-n}(p)}{n!} (p \partial_p)^n, \quad B^{(\nu)}_{2j-1} (q) = \sum_{n=0}^{2j-1} \frac{b^{(\nu)}_{2j-1-n}(q)}{n!} (q \partial_q)^n$$

$$a^{(\mu+1)}_k (p) = \sum_{n=0}^{k} \frac{\left( p \partial_p \right)^{n+2} F(p)}{(n+2)!} a^{(\mu)}_k (p), \quad b^{(\nu+1)}_k (q) = \sum_{n=0}^{k} \frac{(q \partial_q)^{n+2} F(q)}{(n+2)!} b^{(\nu)}_k (q),$$

Here, the parameters \(p_j, q_j, \xi_{j0}, \eta_{j0}, a_n^{(0)}, b_n^{(0)} (j = 1, 2, n = 0, 1, \ldots, 2N - 1)\) satisfy the constraints

$$p_j = q_j, \quad p_j = p_{j^*}, \quad \xi_{j0} = \eta_{j0}, \quad \xi_j = \xi_{-j0}, \quad a_n^{(0)}(p_j) = b_n^{(0)}(q_j), \quad a_n^{(0)}(p_j) = [a_n^{(0)}(p_{3-j})]^*.$$  \(6\)

Remark 1. We note that the algebraic equation (4) can be rewritten as

$$p^4 + 2 \left( c + \kappa^2 \right) p^2 + \kappa^2 (\kappa^2 - 2c) = 0.$$  \(7\)

It is also easy to see that the root \(\xi\) of the above equation can be neither real nor purely imaginary otherwise non-trivial rogue wave solutions (5) do not exist.

Further, the equation (7) has at least one pair of complex conjugate roots \((\xi, \xi^*)\) with \(\xi^2 \notin \mathbb{R}\) only when \(c (c + 4\kappa^2) < 0\). Hence, the rogue wave solution (5) exists if and only if \(c\) and \(\kappa\) satisfy the conditions

$$c < 0, \quad c + 4\kappa^2 > 0.$$  \(8\)

On the other hand, as (7) is a quartic equation in \(\xi\), it has four roots (counting multiplicity). Due to the fact that all coefficients of (7) are real, these roots demonstrate a symmetric structure
when $c$ and $\kappa$ satisfy (8), that is, if $\xi$ is a root of (7), then the other three roots are $-\xi$ and $\pm \xi^*$. As a result, they can be explicitly expressed as

$$\pm \frac{1}{\sqrt{2}} \left[ \left( |\kappa| (\kappa^2 - 2c)^{1/2} - c - \kappa^2 \right)^{1/2} \pm i \left( |\kappa| (\kappa^2 - 2c)^{1/2} + c + \kappa^2 \right)^{1/2} \right].$$

Apart from this, the symmetric structure indicates that all the roots of $F'(p) = 0$ are simple.

In Theorem 2.1, we use recursive relations to define the differential operators that appear in the matrix elements of the $\tau$ function. To avoid these recursive relations, Yang and Yang have proposed the so-called ‘$W$-treatment’ method which has been applied to derive rogue wave solutions of the Boussinesq equation \[47\] and the three-wave equations \[36\]. The key point of this method is to introduce more general differential operators in the $\tau$ function. It turns out that this method applies to the Sasa-Satsuma equation (2) as well and the result is give as follows.

**Theorem 2.2.** The Sasa-Satsuma equation (2) admits the general rogue wave solutions

$$u = g f e^{i(\kappa(x-6ct) - \kappa^3 t)}$$

where $\kappa$ is real,

$$f(x,t) = \tau_0(x-6ct,t), \quad g(x,t) = \tau_1(x-6ct,t)$$

and $\tau_k (k = 0, 1)$ is defined as

$$\tau_k = \det_{1 \leq i,j \leq N} \left( M_{2i-1,2j-1}^k \right)$$

where

$$M_{ij}^k = \begin{pmatrix} A_{1i} B_{j1} m_{11}^k & A_{1i} B_{j2} m_{12}^k \\ A_{2i} B_{j1} m_{21}^k & A_{2i} B_{j2} m_{22}^k \end{pmatrix}$$

with $(\alpha, \beta = 1, 2)$

$$A_{ij} = \frac{1}{\xi} \left[ f_{1\alpha}(p_{1\alpha}) \partial_{p_{1\alpha}} \right]^j, \quad B_{ij} = \frac{1}{\xi} \left[ f_{2\beta}(q_{1\beta}) \partial_{q_{1\beta}} \right]^j.$$  \hspace{1cm} (10)

Here

$$f_{1i}(z) = \pm \frac{F^2(z) - F^2(\xi)}{F'(z)}, \quad f_{2i}(z) = \pm \frac{F^2(z) - F^2(\xi^*)}{F'(z)}, \quad i = 1, 2.$$ \hspace{1cm} (11)

All other functions and parameters are the same as in Theorem 2.1 except that

$$\xi_{10} = \sum_{n=1}^{\infty} a_n \ln^n U_1(p_1), \quad \eta_{10} = \sum_{n=1}^{\infty} a_n \ln^n \nu_1(q_1),$$

where

$$U_1(p_1) = \frac{F(p_1) \pm \sqrt{F^2(p_1) - F^2(\xi)}}{F(\xi)}, \quad \nu_1(q_1) = \frac{F(q_1) \pm \sqrt{F^2(q_1) - F^2(\xi)}}{F(\xi)}$$ \hspace{1cm} (12)

and $a_r$ are free complex constants.
Finally, we show that the solutions obtained in Theorem 2.2 can be converted into a more explicit manner by means of Schur polynomials. These Schur polynomials are defined via the generating function
\[ \sum_{n=0}^{\infty} S_n(x) \lambda^n = \exp \left( \sum_{k=1}^{\infty} x_k \lambda^k \right), \]
where \( x = (x_1, x_2, \cdots). \) To be more explicit, we have
\[ S_0(x) = 1, \quad S_1(x) = x_1, \quad S_2(x) = \frac{1}{2} x_1^2 + x_2, \ldots, \quad S_j(x) = \sum_{l_1+2l_2+\cdots+ml_m = j} \left( \prod_{i=1}^{m} \frac{x_i^{l_i}}{l_i!} \right). \] (13)

**Theorem 2.3.** Let \( a = i\kappa, \) where \( \kappa \) is real, \( p(\kappa) \) be the function defined by
\[ F[p(\kappa)] = F(\xi) \cosh(\kappa), \] (14)
and \( \hat{p}_1 = p'(0), \) where \( F(p) \) and \( \xi \) are the same as in Theorem 2.1 and \( p(0) = \xi. \) Further, let \( x(k) = (x_1, x_2, \cdots), h^\pm = (h^+_1, h^+_2, \cdots), s = (s_1, s_2, \cdots), \hat{s} = (\hat{s}_1, \hat{s}_2, \cdots), b = (b_1, b_2, \cdots) \) and \( \hat{b} = (\hat{b}_1, \hat{b}_2, \cdots) \) be the vectors defined by
\[
\begin{align*}
x_n(k) &= c_n x + d_n t + a_n, \quad (15) \\
\sum_{n=1}^{\infty} b_n \kappa^n &= \ln \left[ \frac{p(\kappa) + \xi}{2\xi} \right], \quad \sum_{n=1}^{\infty} \hat{b}_n \kappa^n = \ln \left[ \frac{p(\kappa) + \xi^*}{\xi + \xi^*} \right], \quad (16) \\
\sum_{n=1}^{\infty} s_n \kappa^n &= \ln \left[ \frac{p(\kappa) - \xi}{\hat{p}_1 \kappa} \right], \quad \sum_{n=1}^{\infty} \hat{s}_n \kappa^n = \ln \left[ \frac{p(\kappa) - \xi}{\hat{p}_1 \kappa \frac{p(\kappa) + \xi^*}{p(\kappa) + \xi}} \right], \quad (17) \\
\sum_{n=1}^{\infty} h_+ \kappa^n &= \ln \left[ \frac{p(\kappa) - a}{\xi - a} \right], \quad \sum_{n=1}^{\infty} h_- \kappa^n = \ln \left[ \frac{p(\kappa) + a}{\xi + a} \right]. \quad (18)
\end{align*}
\]
where \( a_n \) is arbitrary, and \( c_n, d_n \) are defined by the expansions
\[ \sum_{n=1}^{\infty} c_n \kappa^n = p(\kappa) - \xi, \quad \sum_{n=1}^{\infty} d_n \kappa^n = p^3(\kappa) - \xi^3. \] (19)

Then the Sasa-Satsuma equation (2) admits the general rogue wave solutions
\[ u = \frac{g}{f} e^{i(\kappa(x-6ct)-\kappa^3 t)} \] (20)
where
\[ f(x,t) = \sigma_0(x-6ct,t), \quad g(x,t) = \sigma_1(x-6ct,t) \]
and \( \sigma_k (k = 0, 1) \) is defined as
\[
\sigma_k = \det_{1 \leq i,j \leq N} (M_{2i-1,2j-1}^k) \\
M_{ij}^k = \left( \begin{array}{cc} m_{2i-1,2j-1}^k & m_{2i-1,2j}^k \\ m_{2i,2j-1}^k & m_{2i,2j}^k \end{array} \right) \]
with
and

\[
\begin{align*}
\m_{2i-1,2j-1}^k &= \sum_{\nu=0}^{\min(i,j)} \frac{1}{2\xi} \left( \frac{b_1^2}{4\xi^2} \right)^\nu S_\nu \left( x + \nu s - \hat{b} + k h^+ \right) S_{N-\nu} \left( x + \nu s - \hat{b} - k h^- \right) \\
\m_{2i-1,2j}^k &= \sum_{\nu=0}^{\min(i,j)} \frac{1}{\xi + \xi^*} \left( \frac{|\hat{b}_1|^2}{(\xi + \xi^*)^2} \right)^\nu S_\nu \left( x + \nu \hat{s} - \hat{b} + k h^+ \right) S_{N-\nu} \left( x + \nu \hat{s} - \hat{b} - k h^- \right) \\
\m_{2i-2j-1}^k &= \sum_{\nu=0}^{\min(i,j)} \frac{1}{\xi^* + \xi} \left( \frac{|\hat{b}_1|^2}{(\xi^* + \xi)^2} \right)^\nu S_\nu \left( x^* + \nu \hat{s}^* - \hat{b}^* + k(h^-)^* \right) S_{N-\nu} \left( x^* + \nu \hat{s}^* - \hat{b}^* - k(h^-)^* \right) \\
\m_{2i-2j}^k &= \sum_{\nu=0}^{\min(i,j)} \frac{1}{2\xi^*} \left( \frac{|\hat{b}_1|^2}{4(\xi^*)^2} \right)^\nu S_\nu \left( x^* + \nu s^* - \hat{b}^* + k(h^-)^* \right) S_{N-\nu} \left( x^* + \nu s^* - \hat{b}^* - k(h^-)^* \right).
\end{align*}
\]

**Remark 2.** The function \( \sigma_k \) \((k = 0, 1)\) in Theorem 2.3 is represented by a \( N \times N \) block matrix where each block is a \( 2 \times 2 \) matrix. In fact, by applying row and column operations, we are able to rewrite \( \sigma_k \) as a \( 2 \times 2 \) block matrix where each block is a \( N \times N \) matrix, that is

\[
\sigma_k = \det \begin{pmatrix}
\sigma_k^{[1,1]} & \sigma_k^{[1,2]} \\
\sigma_k^{[2,1]} & \sigma_k^{[2,2]}
\end{pmatrix},
\]

where

\[
\sigma_k^{[1,1]} = \left( \m_{2i-1,2j-1}^k \right), \quad \sigma_k^{[1,2]} = \left( \m_{2i-1,2j}^k \right), \quad \sigma_k^{[2,1]} = \left( \m_{2i-2j-1}^k \right), \quad \sigma_k^{[2,2]} = \left( \m_{2i-2j}^k \right),
\]

and \( m_{I,J} \) \((1 \leq I, J \leq 2N)\) are defined in Theorem 2.3. In a similar way, we may express the \( \tau \)-function in Theorems 2.1 and 2.2 as a \( 2 \times 2 \) block matrix with each block being a \( N \times N \) matrix.

**Remark 3.** The function \( \sigma_k \) \((k = 0, 1)\) in Theorem 2.3 is a polynomial in \( x \) and \( t \). It can be computed that for the \( N \)-th order rogue wave, the degree of \( \sigma_k(k = 0, 1) \) is \( 2N(N + 1) \) for both variables \( x \) and \( t \). Since the computations are very similar to those developed by Yang and Yang (see Appendix A in [15]), we omit the details.

**Remark 4.** The rogue wave solutions of order \( N \) presented in Theorem 2.3 contain \( 2N - 1 \) free parameters \( a_1, a_2, \ldots, a_{2N-1} \). By a shift of \( x \) and \( t \), we may normalize \( a_1 = 0 \). Using similar arguments as in [15] [22], it can be shown that these rogue wave solutions are independent of the parameters \( a_{2n} \), where \( n \) is a positive integer. As a consequence, we will set \( a_2, a_4, \ldots, a_{2n}, \ldots \) to be 0 in the subsequent discussions.

### 3 Dynamics of rogue wave solutions

In this section, we analyze the dynamics of rogue wave solutions of the Sasa-Satsuma equation derived in Theorem 2.3.
To obtain the fundamental rogue wave solutions, we set \( N = 1 \) in Theorem 2.3. In this case, we have
\[
\sigma_0 = \begin{bmatrix}
m_{11}^{(0)} & m_{12}^{(0)} \\
m_{21}^{(0)} & m_{22}^{(0)}
\end{bmatrix}, \quad \sigma_1 = \begin{bmatrix}
m_{11}^{(1)} & m_{12}^{(1)} \\
m_{21}^{(1)} & m_{22}^{(1)}
\end{bmatrix},
\]
where \((k = 0, 1)\)
\[
m_{11}^{(k)} = \frac{1}{2\xi} \left[ (c_1 x + d_1 t + a_1 - b_1 + kh_1^+) (c_1 x + d_1 t + a_1 - b_1 - kh_1^-) + \left( \frac{\hat{p}_1}{2\xi} \right)^2 \right],
\]
\[
m_{12}^{(k)} = \frac{1}{\xi + \xi^*} \left[ (c_1 x + d_1 t + a_1 - \hat{b}_1 + kh_1^+) (c_1 x + d_1 t + a_1 - \hat{b}_1 - k(h_1^+)^*) + \frac{|\hat{p}_1|^2}{(\xi + \xi^*)^2} \right],
\]
\[
m_{21}^{(k)} = \frac{1}{\xi + \xi^*} \left[ (c_1 x + d_1 t + a_1 - \hat{b}_1 + k(h_1)^*) (c_1 x + d_1 t + a_1 - \hat{b}_1 - kh_1^-) + \frac{|\hat{p}_1|^2}{(\xi + \xi^*)^2} \right],
\]
\[
m_{22}^{(k)} = \frac{1}{2\xi} \left[ (c_1^* x + d_1^* t + a_1^* - b_1^* + h_1^+) (c_1^* x + d_1^* t + a_1^* - b_1^* - k(h_1^+)^*) + \left( \frac{\hat{p}_1}{2\xi^*} \right)^2 \right],
\]
and
\[
\hat{p}_1 = \pm \sqrt{\frac{F(\xi) (\xi^2 - a^2)^3}{4(3a^2 \xi + \xi^3)}}, \quad b_1 = \frac{\hat{p}_1}{2\xi}, \quad \hat{b}_1 = \frac{\hat{p}_1}{\xi + \xi^*}, \quad c_1 = \hat{p}_1, \quad d_1 = 3\xi^2 \hat{p}_1, \quad h_1^+ = \frac{\hat{p}_1}{\xi - a}, \quad h_1^- = \frac{\hat{p}_1}{\xi + a}.
\]
Without loss of generality, we may set \( a_1 = 0 \) by a shift of \( x \) and \( t \). Then the fundamental rogue wave solutions of the Sasa-Satsuma equation (2) are given by
\[
u = \frac{g}{f} e^{i(\kappa x - 6\alpha t - \kappa^3 t)}, \quad (21)
\]
where
\[
f(x, t) = \left[ \frac{1}{2\xi} \left( x + 3(\xi^2 - 2c)t - \frac{1}{2\xi^2} \frac{1}{4\xi^2} \right)^2 - \frac{1}{(\xi + \xi^*)^2} \left[ x + 3(\xi^2 - 2c)t - \frac{1}{\xi + \xi^*} \frac{1}{\xi^* + i\kappa} \right] + \frac{1}{(\xi + \xi^*)^2} \right]^2,
\]
\[
g(x, t) = \left[ \frac{1}{4\xi^2} \left( x + 3(\xi^2 - 2c)t - \frac{1}{2\xi^2} + \frac{1}{\xi^* - i\kappa} \right) \left( x + 3(\xi^2 - 2c)t - \frac{1}{2\xi^2} - \frac{1}{\xi^* + i\kappa} \right) + \frac{1}{4\xi^2} \right]^2.
\]
We note that the above solutions are the same as the fundamental rogue wave solutions derived in Theorem 2.1 up to multiplication of a constant with unit modulus. It is clear that the solution set of (2) is invariant under multiplication of \( e^{i\theta} \), where \( \theta \) is real, so this is reasonable. Since the dynamics of the fundamental rogue wave solutions corresponding to Theorem 2.1 have been thoroughly examined in [19], we will omit the details. Instead, we only recall some distinctive
features. On the one hand, the $\tau$-function of the fundamental rogue wave solutions $(21)$ is a polynomial of degree 4 in $x$ and $t$ while it is degree 2 for other integrable systems. Therefore, the patterns of the fundamental rogue wave in the Sasa-Satsuma equation are far more diverse than most other integrable systems. As shown in $(49)$, these fundamental rogue waves can be classified into at least four distinctive types which are determined by the values of the free parameter $a = i\kappa$. On the other hand, among these four types, the so-called twisted rogue wave (TRW) pair (see Fig. $1$), which was first reported in $(40)$, is a distinctive characteristic of the Sasa-Satsuma equation in contrast with many other integrable equations. As illustrated in Fig. $1$ this TRW pair possesses four zero-amplitude points $P_1, P_2, Q_1, Q_2$ and two maximum amplitude points $M_1$ and $M_2$. The appearance demonstrates that it comprises two extended rogue-wave components bending towards each other and displaying an identical but antisymmetric structure, which is the reason to name it as TRW pair. As depicted in $(44)$, the fundamental rogue waves have three other intrinsic structures (see Fig. $2$) which can be obtained by altering the parameter values that appear in the solutions.

To get higher-order rogue waves, we may take $N \geq 2$ in Theorem $2.3$. In particular, when $N = 2$, the $\tau$-function is the determinant of a $4 \times 4$ matrix. Similar to the fundamental rogue waves, after some computations, we can show that the second-order rogue waves presented in Theorem $2.3$ are equivalent to those given in Theorem $2.1$. As the solution dynamics of the second- and third-order rogue waves have been thoroughly investigated in $(49)$, we only provide the graphs of second-order rogue waves in Figs $3-4$.

Recently Yang and Yang $(36, 37)$ have performed a comprehensive study on the rogue wave patterns of several integrable equations including the nonlinear Schrödinger equation, the Bossinesq equation and the Manakov system. They have shown that when one of the internal parameters in the rogue wave solutions is large enough, the rogue waves may display universal patterns. In particular, they have shown that the fourth-order rogue wave may consist of ten separating fundamental rogue waves which are far away from the origin otherwise it is comprised of a second-order order super rogue wave near the origin and seven separating fundamental rogue waves that are far away from the origin. Similarly, the fifth-order rogue wave is either comprised of fifteen separating fundamental rogue waves that are far away from the origin or made up of a third-order order super rogue wave near the origin and nine separating fundamental rogue waves that are far away from the origin. It turns out that this also occurs in the fourth- and fifth-order rogue wave solutions of the SS equation. As there are four types of fundamental rogue waves, we have totally four types of fourth-order rogue waves for each pattern. For simplicity, we merely show one type in Figs. $5$ and $6$. In addition, the rogue wave pattern up to fifth-order of the SS equation is summarized in Fig. $7$. 
Figure 1: (Color online) The twisted rouge wave pair with parameter values $a = 0.45i, c = -0.5$.

4 Proof of Theorem 2.1

In this section, we derive the rogue wave solutions presented in Theorem 2.1. First, we transform the Sasa-Satsuma equation (2) into bilinear forms. Then rogue wave solutions will be derived by connecting these bilinear equations with those coming from the KP hierarchy.

Proposition 4.1 ([44]). The Sasa-Satsuma equation

$$u_t = u_{xxx} - 6c|u|^2u_x - 3cu(|u|^2)_x$$

can be transformed into the system of bilinear equations

$$\begin{align*}
(D^2_x - 4c) f \cdot f &= -4cgg^* \\
(D^3_x - D_t + 3i\kappa D^2_x - 3(\kappa^2 + 4c) D_x - 6i\kappa c) g \cdot f + 6i\kappa cgg &= 0 \\
(D_x + 2i\kappa) g \cdot g^* &= 2i\kappa f
\end{align*}$$

(22)
via the variable transformation
\[ u = \frac{g}{f} e^{i(\kappa(x-6ct)-\kappa^3 t)}, \]
where \( \kappa \) is real, \( f \) is a real-valued function, \( g \) is a complex-valued function, \( q \) is an auxiliary function and \( D \) is the Hirota’s bilinear operator \([50]\) defined by
\[ D_x^m D_t^n f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n \left[ f(x,t)g(x',t') \right]_{x'=x,t'=t}. \]

Suppose \( \tilde{m}_{ij}^{kl}, \varphi_{i}^{kl}, \psi_{j}^{kl} \), which are functions of \( x, y, t, r \) and \( s \), satisfy the differential and difference relations

\[ \partial_x \tilde{m}_{ij}^{kl} = \varphi_{i}^{kl} \psi_{j}^{kl}, \]
\[ \partial_x \varphi_{i}^{kl} = \varphi_{i}^{k+1,l} + a \varphi_{i}^{k,l} = \varphi_{i}^{k+1,l} + b \varphi_{i}^{k,l}, \]
\[ \partial_x \psi_{j}^{kl} = -\psi_{j}^{k-1,l} - a \psi_{j}^{k,l} = -\psi_{j}^{k-1,l} - b \psi_{j}^{k,l}, \]
\[ \partial_y \varphi_{i}^{kl} = \partial_x^2 \varphi_{i}^{kl}, \quad \partial_y \psi_{j}^{kl} = -\partial_x^2 \psi_{j}^{kl}, \]
\[ \partial_t \varphi_{i}^{kl} = \partial_x \varphi_{i}^{kl}, \quad \partial_t \psi_{j}^{kl} = \partial_x \psi_{j}^{kl}, \]
\[ \partial_r \varphi_{i}^{kl} = \varphi_{i}^{k-1,l}, \quad \partial_r \psi_{j}^{kl} = -\psi_{j}^{k+1,l}, \]
\[ \partial_s \varphi_{i}^{kl} = \varphi_{i}^{k,l-1}, \quad \partial_s \psi_{j}^{kl} = -\psi_{j}^{k,l+1}, \]

Figure 2: (Color online) First-order rogue waves under parameter values \( c = -0.5, a_1 = 0 \) (a) \( a = 0.5i \), (c) \( a = 1.04i \) and (e) \( a = 1.84i \). (d), (e) and (f) are the corresponding density plots of (a), (b) and (c) respectively.
Figure 3: (Color online) Second-order rogue wave with parameter values $a = 1.4i, c = -0.7, a_1 = a_2 = 0, a_3 = 100 \exp(7\pi i/4)$.

Figure 4: (Color online) Second-order rogue wave with parameter values $a = 1.4i, c = -0.7, a_1 = a_2 = 3 = 0$.

which imply the relations

\[
\partial_y \tilde{m}_{ij}^{kl} = \varphi_i^{k+1,l} \psi_j^l + \varphi_i^{k,l} \psi_j^{k-1,l} + 2a \varphi_i^{k,l} \psi_j^l = \varphi_i^{k+1,l} \psi_j^l + \varphi_i^{k,l} \psi_j^{k-1,l} + 2b \varphi_i^{k,l} \psi_j^l, \tag{31}
\]

\[
\partial_t \tilde{m}_{ij}^{kl} = \varphi_i^{k+2,l} \psi_j^l + 3a \varphi_i^{k+1,l} \psi_j^{k+1,l} + \varphi_i^{k-1,l} \psi_j^{k-1,l} + 3a^2 \varphi_i^{k,l} \psi_j^l + 3a \varphi_i^{k,l} \psi_j^{k-1,l} + \varphi_i^{k-2,l} \psi_j^{k-2,l}, \tag{32}
\]

\[
\partial_r \tilde{m}_{ij}^{kl} = -\varphi_i^{k-1,l} \psi_j^{k+1,l}, \quad \partial_s \tilde{m}_{ij}^{kl} = -\varphi_i^{k,l-1} \psi_j^{k+1,l}, \tag{34}
\]

\[
\tilde{m}_{ij}^{k+1,l} = \tilde{m}_{ij}^{k,l} + \varphi_i^{k,l} \psi_j^{k+1,l}, \quad \tilde{m}_{ij}^{k,l+1} = \tilde{m}_{ij}^{k,l} + \varphi_i^{k,l} \psi_j^{k,l+1} \tag{35}
\]

where $a, b$ are constants, then it can be calculated that \[45, 48\] the determinant

\[
\tilde{\tau}_{kl} = \det_{1 \leq i, j \leq 2N} \left( \tilde{m}_{ij}^{kl} \right)
\]
would satisfy the bilinear equations

\[
(D_x D_x - 2) \tau_{kl} \cdot \tau_{kl} = -2 \tau_{k+1,l} \tau_{k-1,l} 
\]  
\[
(D_x D_x - 2) \tau_{kl} \cdot \tau_{kl} = -2 \tau_{k,l+1} \tau_{k,l-1} 
\]  
\[
(D_x^2 - D_y + 2a D_x) \tau_{k+1,l} \cdot \tau_{kl} = 0 
\]  
\[
(D_x^2 - D_y + 2b D_x) \tau_{k,l+1} \cdot \tau_{kl} = 0 
\]  
\[
(D_x^2 + 3D_x D_y - 4D_t + 3a (D_x^2 + D_y) + 6a^2 D_x) \tau_{k+1,l} \cdot \tau_{kl} = 0 
\]  
\[
(D_x^3 + 3D_x D_y - 4D_t + 3b (D_x^2 + D_y) + 6b^2 D_x) \tau_{k,l+1} \cdot \tau_{kl} = 0 
\]  
\[
(D_x (D_x^2 - D_y + 2a D_x) - 4D_x) \tau_{k+1,l} \cdot \tau_{kl} = 0 
\]  
\[
(D_x (D_x^2 - D_y + 2b D_x) - 4D_x) \tau_{k,l+1} \cdot \tau_{kl} = 0 
\]  
\[
(D_x (D_x^2 - D_y + 2a D_x) - 4 (D_x + a - b)) \tau_{k+1,l} \cdot \tau_{kl} + 4(a - b) \tau_{k+1,l+1} \tau_{k,l-1} = 0 
\]  
\[
(D_x (D_x^2 - D_y + 2b D_x) - 4 (D_x + b - a)) \tau_{k,l+1} \cdot \tau_{kl} + 4(b - a) \tau_{k+1,l+1} \tau_{k-1,l} = 0 
\]  
\[
(D_x + a - b) \tau_{k+1,l} \cdot \tau_{k,l+1} = (a-b) \tau_{k+1,l+1} \tau_{k,l-1}. 
\]  

Denote by

\[
m_{\alpha \beta}^{kl} = \frac{1}{p_{\alpha} + q_{\beta}} \left( -\frac{p_{\alpha} - a}{q_{\beta} + a} \right)^k \left( -\frac{p_{\alpha} - b}{q_{\beta} + b} \right)^l e^{e^{\xi_{\alpha}} + \eta_{\beta}} 
\]

\[
\varphi_{\alpha}^{kl} = (p_{\alpha} - a)^k (p_{\alpha} - b)^l e^{\xi_{\alpha}} 
\]

\[
\psi_{\beta}^{kl} = \left[ -(q_{\beta} + a) \right]^{-k} \left[ -(q_{\beta} + b) \right]^{-l} e^{\eta_{\beta}} 
\]

\[
\xi_{\alpha} = p_{\alpha} x + p_{\alpha}^2 y + p_{\alpha}^3 t + \frac{1}{p_{\alpha} - a} r + \frac{1}{p_{\alpha} - b} s + \xi_{\alpha 0} 
\]

\[
\eta_{\beta} = q_{\beta} x - q_{\beta}^2 y + q_{\beta}^3 t + \frac{1}{q_{\beta} + a} r + \frac{1}{q_{\beta} + b} s + \eta_{\beta 0} 
\]

where \(\alpha, \beta = 1, 2,\) and \(p_{\alpha}, q_{\beta}, \xi_{\alpha 0}, \eta_{\beta 0}\) are constants, then direct computations indicate that \(m_{\alpha \beta}^{kl}, \varphi_{\alpha}^{kl}, \psi_{\beta}^{kl}\) satisfy the differential and difference relations \([24]: [30].\)

Next, we define

\[
M_{ij}^{(k,l,\mu_i,\nu_j)} = \begin{pmatrix} \sim m_{2l-1,2j-1}^{(k,l,\mu_i,\nu_j)} & \sim m_{2l-1,2j}^{(k,l,\mu_i,\nu_j)} \\ \sim m_{2l,2j-1}^{(k,l,\mu_i,\nu_j)} & \sim m_{2l,2j}^{(k,l,\mu_i,\nu_j)} \end{pmatrix} 
\]

\[
= \begin{pmatrix} A_{i}^{(\mu_i)} (p_1) B_{j}^{(\nu_j)} (q_1) m_{11}^{kl} & A_{i}^{(\mu_i)} (p_1) B_{j}^{(\nu_j)} (q_2) m_{12}^{kl} \\ A_{i}^{(\mu_i)} (p_2) B_{j}^{(\nu_j)} (q_1) m_{21}^{kl} & A_{i}^{(\mu_i)} (p_2) B_{j}^{(\nu_j)} (q_2) m_{22}^{kl} \end{pmatrix} 
\]

and

\[
\begin{pmatrix} \varphi_{2l-1}^{(k,l,\mu_i)} \\ \varphi_{2l}^{(k,l,\mu_i)} \end{pmatrix} = \begin{pmatrix} A_{i}^{(\mu_i)} (p_1) \varphi_{1}^{kl} \\ A_{i}^{(\mu_i)} (p_2) \varphi_{2}^{kl} \end{pmatrix}, 
\]

\[
\begin{pmatrix} \psi_{2j-1}^{(k,l,\nu_j)} \\ \psi_{2j}^{(k,l,\nu_j)} \end{pmatrix} = \begin{pmatrix} B_{j}^{(\nu_j)} (q_1) \psi_{1}^{kl} \\ B_{j}^{(\nu_j)} (q_2) \psi_{2}^{kl} \end{pmatrix} 
\]
Figure 5: (Color online) Fourth-order rouge waves under parameter values $a = 1.4i, c = -0.7, a_1 = a_2 = a_4 = a_6 = 0$ (a) $a_3 = 200 \exp(0.75\pi i), a_5 = 0, a_7 = 0$; (b) $a_3 = 0, a_5 = 3000 \exp(0.85\pi i), a_7 = 0$ and (c) $a_3 = 0, a_5 = 0, a_7 = -25000$.

where $i, j = 1, 2 \cdots, N$ and

$$A_i^{(\mu_i)}(p) = \sum_{n=0}^{\mu_i} \frac{a_i^{(\mu_i)}(p)}{n!} (p\partial_p)^n, \quad B_j^{(\nu_j)}(q) = \sum_{n=0}^{\nu_j} \frac{b_j^{(\nu_j)}(q)}{n!} (q\partial_q)^n$$

then the functions $\tilde{m}_{ij}^{(k,l,\mu_i)}, \tilde{\varphi}_i^{(k,l,\mu_i)}, \psi_j^{(k,l,\nu_j)}$ would satisfy the differential and difference relations (24)-(30) as the operators $A_i^{(\mu_i)}$ and $B_j^{(\nu_j)}$ commute with the partial differentials with respect to $x, y, t, r, s$. Hence, the determinant

$$\tau_{kl} = \det_{1 \leq \kappa, \lambda \leq N} \left( M_{\kappa, \lambda}(k,l,\mu_i,\nu_j) \right)$$

satisfies the bilinear equations (36)-(46) for any sequence of indices $(i_1, i_2, \ldots, i_N; j_1, j_2, \ldots, j_N)$. In particular, the determinant

$$\tau_{kl} = \det_{1 \leq i, j \leq N} \left( M_{2i-1,2j-1}(k,l,\mu_i,\nu_j) \right)$$
solves the bilinear equations (36)-(46).

\[ \text{Figure 6: (Color online) Fifth-order rogue waves under parameter values} \]

\[ \begin{align*}
\text{a)} & a_3 = 100 \exp(0.75\pi i), a_5 = 0, a_7 = 0, a_9 = 0, \]
\[ \text{b)} & a_3 = 0, a_5 = 1500 \exp(0.85\pi i), a_7 = 0, a_9 = 0, \]
\[ \text{c)} & a_3 = 0, a_5 = 0, a_7 = -20000, a_9 = 0, \]
\[ \text{d)} & a_3 = 0, a_5 = 0, a_7 = 0, a_9 = -200000. \]

From now on, for some reason that will be clear later, we set \( \mu_i = N - i, \nu_j = N - j \) for fixed \( N \) and

\[ \begin{align*}
a_k^{(\mu+1)}(p) &= \sum_{n=0}^{k} \frac{(p\partial_p)^{n+2} F(p)}{(n+2)!} a_{k-n}^{(\nu)}(p), \\
b_k^{(\nu+1)}(p) &= \sum_{n=0}^{k} \frac{(q\partial_q)^{n+2} F(q)}{(n+2)!} b_{k-n}^{(\nu)}(q), \quad \mu, \nu = 0, 1, \ldots, N - 1.
\end{align*} \]

For brevity, we denote \( M_{ij}^{(k,l,\mu_i,\nu_j)} \) and \( \tilde{m}_{ij}^{(k,l,\mu_i,\nu_j)} \) by \( M_{ij}^{kl} \) and \( \tilde{m}_{ij}^{kl} \) respectively.

In what follows, we will establish the reductions from the bilinear equations (36)-(46) in the KP hierarchy to the bilinear equations (22). The reduction procedure consists of several steps.
Figure 7: Predicted rogue patterns $u_N(x, t)$ for the orders $2 \leq N \leq 5$ and the large parameter $a_{2m+1}$ from $m = 1$ to $N - 1$. For each panel, the large parameter $a_{2m+1}$ in the rogue wave is displayed in the above table, with the other internal parameters set as zero.
We start with the reduction from AKP to CKP \cite{[51]}. To achieve this, we impose the parameter relations

\[ q_j = p_j, \quad b = -a, \quad \xi_{j0} = \eta_{j0}, \quad a_n^{(0)}(p_j) = b_n^{(0)}(q_j), \]

where \( j = 1, 2 \) and \( n = 0, 1, \ldots, 2N - 1 \), then we obtain

\[ \xi_j(x, y, t, r, s) = \eta_j(x, -y, t, s, r) \]

and

\[ a_k^{(\mu+1)}(p_j) = b_k^{(\mu+1)}(q_j). \]

Therefore, we have

\[ m_{\beta\alpha}^{-l-k}(x, -y, t, s, r) = m_{\alpha\beta}^{kl}(x, y, t, r, s), \quad 1 \leq \alpha, \beta \leq 2 \]

and

\[
\begin{align*}
M_{ji}^{-l-k}(x, -y, t, s, r) &= \left( A_{2j-1}^{(N-j)}(p_1)B_{2i-1}^{(N-i)}(q_1)m_{11}^{-l-k} A_{2j-1}^{(N-j)}(p_1)B_{2i-1}^{(N-i)}(q_2)m_{12}^{-l-k} \
&\quad A_{2j-1}^{(N-j)}(p_2)B_{2i-1}^{(N-i)}(q_1)m_{12}^{-l-k} A_{2j-1}^{(N-j)}(p_2)B_{2i-1}^{(N-i)}(q_2)m_{22}^{-l-k} \right)(x, -y, t, s, r) \\
&= \left( B_{2j-1}^{(N-j)}(q_1)A_{2i-1}^{(N-i)}(p_1)m_{11}^{kl} B_{2j-1}^{(N-j)}(q_1)A_{2i-1}^{(N-i)}(p_2)m_{21}^{kl} \
&\quad B_{2j-1}^{(N-j)}(q_2)A_{2i-1}^{(N-i)}(p_1)m_{12}^{kl} B_{2j-1}^{(N-j)}(q_2)A_{2i-1}^{(N-i)}(p_2)m_{22}^{kl} \right)(x, y, t, r, s) \\
&= (M_{ij}^{kl})^T(x, y, t, r, s),
\end{align*}
\]

where \( M^T \) denotes the transpose of the matrix \( M \). Then it follows that

\[
\tau_{-l,-k}(x, -y, t, s, r) = |M_{ij}^{-l,-k}(x, -y, t, s, r)|
= |(M_{ij}^{kl})^T(x, y, t, r, s)|
= |(\tilde{M}_{ij}^{kl})^T|
= \tau_{kl}(x, y, t, r, s).
\tag{52}
\]

Now we consider the dimension reduction that is a crucial step in the derivation of rogue wave solutions. We first introduce the operator defined by

\[
\tilde{D}_{r,s,x} = \left( \partial_r + \partial_s - \frac{1}{c} \partial_x \right).
\tag{53}
\]

Let

\[
\begin{align*}
F(p) &= \frac{1}{p - a} + \frac{1}{p + a} - \frac{p}{c}, \\
G(p) &= p \partial_p F(p) = -p \left[ \frac{1}{(p-a)^2} + \frac{1}{(p+a)^2} + \frac{1}{c} \right],
\end{align*}
\]
then we have

\[ B_1^{(N-1)} \tilde{D}_{r,s,m} = B_1^{(N-1)}[F(p) + F(q)]m \]

\[ = [F(p) + F(q)]B_1^{(N-1)}m + B_0^{(N-1)}G(q)m \]

and for \( 2 \leq j \leq N \),

\[ B_{2j-1}^{(\nu)}(q)\tilde{D}_{r,s,m} = B_{2j-1}^{(\nu)}(q)[F(p) + F(q)]m \]

\[ = B_{2j-1}^{(\nu)}(q) \sum_{n=0}^{2j-1} b_{2j-1-n}^{(\nu)}(q)\frac{(q\partial_q)^n[F(p) + F(q)]}{n!}m \]

\[ = B_{2j-1}^{(\nu)}(q) \sum_{n=0}^{2j-1} b_{2j-1-n}^{(\nu)}(q) \frac{1}{n!} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \{(q\partial_q)^{n-k}[F(p) + F(q)]\}(q\partial_q)^{n-k}m \]

\[ \sum_{n=0}^{2j-1} b_{2j-1-n}^{(\nu)}(q) \frac{1}{n!} [F(p) + F(q)](q\partial_q)^{n-1}m \]

\[ + \sum_{n=2}^{2j-1} b_{2j-1-n}^{(\nu)}(q) \frac{1}{n!} \sum_{k=0}^{n-2} \left( \frac{n}{k} \right) \{(q\partial_q)^{n-k}[F(p) + F(q)]\}(q\partial_q)^{n-k}m \]

\[ = [F(p) + F(q)]B_{2j-1}^{(\nu)}(q)m + [(q\partial_q)F(q)] \sum_{n=1}^{2j-1} b_{2j-1-n}^{(\nu)}(q)\frac{(q\partial_q)^{n-1}m}{(n-1)!} \]

\[ + \sum_{k=0}^{2j-3} \frac{1}{k!} \left[ \sum_{n=k+2}^{2j-1} \frac{b_{2j-1-n}^{(\nu)}(q)}{(n-k)!} (q\partial_q)^{n-k}F(q) \right] (q\partial_q)^{k}m \]

\[ = [F(p) + F(q)]B_{2j-1}^{(\nu)}(q)m + [(q\partial_q)F(q)] \sum_{n=1}^{2j-1} b_{2j-1-n}^{(\nu)}(q)\frac{(q\partial_q)^{n-1}m}{(n-1)!} \]

\[ + \sum_{k=0}^{2j-3} \frac{1}{k!} \left[ \sum_{m=0}^{2j-3-k} \frac{(q\partial_q)^{m+2}F(q)}{(m+2)!} b_{2j-3-k-m}^{(\nu)}(q) \right] (q\partial_q)^{k}m \]

\[ = [F(p) + F(q)]B_{2j-1}^{(\nu)}(q)m + [(q\partial_q)F(q)] \sum_{n=1}^{2j-1} b_{2j-1-n}^{(\nu)}(q)\frac{(q\partial_q)^{n-1}m}{(n-1)!} \]

\[ + \sum_{k=0}^{2j-3} \frac{1}{k!} b_{2j-3-k}^{(\nu+1)}(q)(q\partial_q)^{k}m \]

\[ = [F(p) + F(q)]B_{2j-1}^{(\nu)}(q)m + G(q) \sum_{n=1}^{2j-1} b_{2j-1-n}^{(\nu)}(q)\frac{(q\partial_q)^{n-1}m}{(n-1)!} \]

\[ + B_{2j-3}^{(\nu+1)}(q)m. \]

\( \text{(54)} \)
Similarly, we can get

\[
A_{1}^{(N-1)} \tilde{D}_{r,s,x} m = [F(p) + F(q)] A_{1}^{(N-1)} m + a_{0}^{(N-1)} G(p)m
\]

\[
A_{2i-1}^{(\mu)} (p) \tilde{D}_{r,s,x} m = [F(p) + F(q)] A_{2i-1}^{(\mu)} (p)m + G(p) \sum_{n=1}^{2i-1-n} a_{2i-1-n}^{(\mu)} (p) (p \partial_{p})^{n-1} m
\]

\[
+ A_{2i-3}^{(\mu+1)} (p)m,
\]

where \(2 \leq i \leq N\).

Since \(\xi\) and \(\xi^*\) are a pair of complex conjugate roots of \(F'(p) = 0\), we have

\[
G(\xi) = G(\xi^*) = 0.
\]

Then combing the equations (54)-(58) yields that

\[
\tilde{D}_{r,s,x} A_{1}^{(\mu)} (p)B_{1}^{(\nu)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*} = [F(p) + F(q)] A_{1}^{(\mu)} (p)B_{1}^{(\nu)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*}
\]

\[
\tilde{D}_{r,s,x} A_{1}^{(\mu)} (p)B_{2j-1}^{(\nu)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*} = [F(p) + F(q)] A_{1}^{(\mu)} (p)B_{2j-1}^{(\nu)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*}
\]

\[
+ A_{1}^{(\mu)} (p)B_{2j-3}^{(\nu+1)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*}, \quad 2 \leq j \leq N
\]

\[
\tilde{D}_{r,s,x} A_{2i-1}^{(\mu)} (p)B_{1}^{(\nu)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*} = [F(p) + F(q)] A_{2i-1}^{(\mu)} (p)B_{1}^{(\nu)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*}
\]

\[
+ A_{2i-3}^{(\mu+1)} (p)B_{1}^{(\nu)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*}, \quad 2 \leq i \leq N
\]

\[
\tilde{D}_{r,s,x} A_{2i-1}^{(\mu)} (p)B_{2j-1}^{(\nu)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*} = [F(p) + F(q)] A_{2i-1}^{(\mu)} (p)B_{2j-1}^{(\nu)} (q)m + A_{2i-3}^{(\mu+1)} (p)B_{2j-1}^{(\nu)} (q)m
\]

\[
+ A_{2i-3}^{(\mu)} (p)B_{2j-3}^{(\nu+1)} (q)m \bigg|_{p_{1}=q_{1}=\xi, \quad p_{2}=q_{2}=\xi^*}, \quad 2 \leq i, j \leq N.
\]

As a consequence, if we define

\[
\tilde{m}_{k+1,i,j}^{kl} = \tilde{m}_{k,i-1}^{kl} = 0, \quad i, j = 1, 2, \ldots, 2N,
\]
then it follows that

\[\tilde{D}_{r,s,x} \tau_{kl} \bigg|_{p_1=q_1=\xi, \quad p_2=q_2=\xi^*} = \sum_{i,j=1}^{2N} \Delta_{ij} \left( \tilde{D}_{r,s,x} \tilde{m}_{ij}^{kl} \right) \bigg|_{p_1=q_1=\xi, \quad p_2=q_2=\xi^*} \]

\[\sum_{i,j=1}^{N} \Delta_{2i-1,2j-1} \left( [F(p_1) + F(q_1)] \tilde{m}_{2i-1,2j-1}^{kl} + \tilde{m}_{2i-3,2j-1}^{kl} + \tilde{m}_{2i-1,2j-3}^{kl} \right) \]

\[+ \sum_{i,j=1}^{N} \Delta_{2i-1,2j} \left( [F(p_1) + F(q_2)] \tilde{m}_{2i-1,2j}^{kl} + \tilde{m}_{2i-3,2j}^{kl} + \tilde{m}_{2i-1,2j-2}^{kl} \right) \]

\[+ \sum_{i,j=1}^{N} \Delta_{2i,2j-1} \left( [F(p_2) + F(q_1)] \tilde{m}_{2i,2j-1}^{kl} + \tilde{m}_{2i,2j-1}^{kl} + \tilde{m}_{2i,2j-3}^{kl} \right) \]

\[+ \sum_{i,j=1}^{N} \Delta_{2i,2j} \left( [F(p_2) + F(q_2)] \tilde{m}_{2i,2j}^{kl} + \tilde{m}_{2i-2,2j-2}^{kl} + \tilde{m}_{2i-2,2j-2}^{kl} \right) \bigg|_{p_1=q_1=\xi, \quad p_2=q_2=\xi^*} \]

\[= F(p_1) \sum_{i=1}^{2N} \sum_{j=1}^{2N} \Delta_{2i-1,j} \tilde{m}_{2i-1,j}^{kl} + F(p_2) \sum_{i=1}^{2N} \sum_{j=1}^{2N} \Delta_{2i,j} \tilde{m}_{2i,j}^{kl} \]

\[+ F(q_1) \sum_{i=1}^{2N} \sum_{j=1}^{N} \Delta_{i,2j-1} \tilde{m}_{i,2j-1}^{kl} + F(q_2) \sum_{i=1}^{2N} \sum_{j=1}^{N} \Delta_{i,2j} \tilde{m}_{i,2j}^{kl} \]

\[+ \sum_{i,j=1}^{N} \Delta_{2i-1,2j-1} \left( \tilde{m}_{2i-3,2j-1}^{kl} + \tilde{m}_{2i-1,2j-3}^{kl} \right) \]

\[+ \sum_{i,j=1}^{N} \Delta_{2i-1,2j} \left( \tilde{m}_{2i-3,2j}^{kl} + \tilde{m}_{2i-1,2j-2}^{kl} \right) \]

\[+ \sum_{i,j=1}^{N} \Delta_{2i,2j-1} \left( \tilde{m}_{2i,2j-1}^{kl} + \tilde{m}_{2i,2j-3}^{kl} \right) \]

\[+ \sum_{i,j=1}^{N} \Delta_{2i,2j} \left( \tilde{m}_{2i-2,2j-2}^{kl} + \tilde{m}_{2i-2,2j-2}^{kl} \right) \bigg|_{p_1=q_1=\xi, \quad p_2=q_2=\xi^*} \]

\[= [F(p_1) + F(q_1) + F(p_2) + F(q_2)] N \tau_{kl} \bigg|_{p_1=q_1=\xi, \quad p_2=q_2=\xi^*} \]

Thus, with the above result, we can replace the derivatives in \( r \) and \( s \) by derivatives in \( x \) in the bilinear equations (36)-(46) and obtain

\[(D_x^2 - 4c) \tau_{kl} \cdot \tau_{kl} = -2c (\tau_{k+1,l} \tau_{k-1,l} + \tau_{k,l+1} \tau_{k,l-1}) \]

\[(D_x^3 - D_t + 3aD_x^2 + 3(a^2 - 2c) D_x - 6ac) \tau_{k+1,l} \cdot \tau_{kl} + 6ac \tau_{k+1,l+1} \tau_{k,l-1} = 0 \]

\[(D_x^3 - D_t - 3aD_x^2 + 3(a^2 - 2c) D_x + 6ac) \tau_{k,l+1} \cdot \tau_{kl} - 6ac \tau_{k+1,l+1} \tau_{k-1,l} = 0 \]

\[(D_x + 2a) \tau_{k+1,l} \cdot \tau_{k,l+1} = 2a \tau_{k+1,l+1} \tau_{kl}. \]
Since the bilinear equations (72)-(75) do not involve derivatives with respect to \( y, r \) and \( s \), we may take \( y = r = s = 0 \). Then according to (52), we have

\[
\tau_{kl}(x,t) = \tau_{-l,-k}(x,t).
\]

(76)

Finally, we consider the complex conjugate reduction. Note that the parameter relations (6) imply that

\[
a_k^{(\mu)}(p_1) = [a_k^{(\mu)}(p_2)]^*, \quad b_k^{(\nu)}(q_1) = [b_k^{(\nu)}(p_2)]^*.
\]

Then using the parameter relations (6), (77) and the assumption that \( a = i\kappa \) is purely imaginary, we may deduce that

\[
(M_{ij}^{0k})^* = \begin{pmatrix}
A_{2i-1}(N-i)(p_1)B_{2j-1}(N-j)(q_1)m_{11}^0 & A_{2i-1}(N-i)(p_1)B_{2j-1}(N-j)(q_2)m_{12}^0 \\
A_{2i-1}(N-i)(p_2)B_{2j-1}(N-j)(q_1)m_{21}^0 & A_{2i-1}(N-i)(p_2)B_{2j-1}(N-j)(q_2)m_{22}^0
\end{pmatrix}^*.
\]

(77)
As a consequence, we have

\[ \tilde{r}_{0k}^* = \left| \tilde{m}_{ij}^{0k} \right| \]

\[
\begin{array}{cccccccc}
\tilde{m}_{21}^{k0} & \tilde{m}_{21}^{k0} & \cdots & \tilde{m}_{21,2j}^{k0} & \tilde{m}_{21,2j+1}^{k0} & \cdots & \tilde{m}_{21,2N}^{k0} & \tilde{m}_{21,2N-1}^{k0} \\
\tilde{m}_{12}^{k0} & \tilde{m}_{12}^{k0} & \cdots & \tilde{m}_{12,j}^{k0} & \tilde{m}_{12,j+1}^{k0} & \cdots & \tilde{m}_{12,12}^{k0} & \tilde{m}_{12,12-1}^{k0} \\
& & \ddots & & \vdots & \cdots & \vdots & \vdots \\
\tilde{m}_{21,2}^{k0} & \tilde{m}_{21,2}^{k0} & \cdots & \tilde{m}_{21,2j}^{k0} & \tilde{m}_{21,2j+1}^{k0} & \cdots & \tilde{m}_{21,2N}^{k0} & \tilde{m}_{21,2N-1}^{k0} \\
\tilde{m}_{21,1}^{k0} & \tilde{m}_{21,1}^{k0} & \cdots & \tilde{m}_{21,1,j}^{k0} & \tilde{m}_{21,1,j+1}^{k0} & \cdots & \tilde{m}_{21,1,12}^{k0} & \tilde{m}_{21,1,11}^{k0} \\
& & \ddots & & \vdots & \cdots & \vdots & \vdots \\
\tilde{m}_{2N,2}^{k0} & \tilde{m}_{2N,2}^{k0} & \cdots & \tilde{m}_{2N,2,2j}^{k0} & \tilde{m}_{2N,2,2j+1}^{k0} & \cdots & \tilde{m}_{2N,2,2N}^{k0} & \tilde{m}_{2N,2,2N-1}^{k0} \\
\tilde{m}_{2N,1}^{k0} & \tilde{m}_{2N,1}^{k0} & \cdots & \tilde{m}_{2N,2,1}^{k0} & \tilde{m}_{2N,2,1}^{k0} & \cdots & \tilde{m}_{2N,2,2N}^{k0} & \tilde{m}_{2N,2,2N-1}^{k0} \\
\end{array}
= (-1)^N 
\begin{array}{cccccccc}
\tilde{m}_{12}^{k0} & \tilde{m}_{12}^{k0} & \cdots & \tilde{m}_{12,2j}^{k0} & \tilde{m}_{12,2j+1}^{k0} & \cdots & \tilde{m}_{12,12}^{k0} & \tilde{m}_{12,12-1}^{k0} \\
\tilde{m}_{21}^{k0} & \tilde{m}_{21}^{k0} & \cdots & \tilde{m}_{21,2j}^{k0} & \tilde{m}_{21,2j+1}^{k0} & \cdots & \tilde{m}_{21,12}^{k0} & \tilde{m}_{21,11}^{k0} \\
& & \ddots & & \vdots & \cdots & \vdots & \vdots \\
\tilde{m}_{21,2}^{k0} & \tilde{m}_{21,2}^{k0} & \cdots & \tilde{m}_{21,2j}^{k0} & \tilde{m}_{21,2j+1}^{k0} & \cdots & \tilde{m}_{21,12}^{k0} & \tilde{m}_{21,11}^{k0} \\
\tilde{m}_{21,1}^{k0} & \tilde{m}_{21,1}^{k0} & \cdots & \tilde{m}_{21,1,j}^{k0} & \tilde{m}_{21,1,j+1}^{k0} & \cdots & \tilde{m}_{21,1,12}^{k0} & \tilde{m}_{21,1,11}^{k0} \\
& & \ddots & & \vdots & \cdots & \vdots & \vdots \\
\tilde{m}_{2N,2}^{k0} & \tilde{m}_{2N,2}^{k0} & \cdots & \tilde{m}_{2N,2,2j}^{k0} & \tilde{m}_{2N,2,2j+1}^{k0} & \cdots & \tilde{m}_{2N,2,12}^{k0} & \tilde{m}_{2N,2,11}^{k0} \\
\tilde{m}_{2N,1}^{k0} & \tilde{m}_{2N,1}^{k0} & \cdots & \tilde{m}_{2N,2,1}^{k0} & \tilde{m}_{2N,2,1}^{k0} & \cdots & \tilde{m}_{2N,2,12}^{k0} & \tilde{m}_{2N,2,11}^{k0} \\
\end{array}
= \tau_{k0}.

Similar method as above demonstrates that

\[ \tau^*_k = \tau_{kk}, \]

which implies that \( \tau_{kk} \) is real. Define

\[ \tilde{f} = \tau_{00}, \quad \tilde{g} = \tau_{10}, \quad \tilde{h} = \tau_{01}, \quad \tilde{q} = \tau_{11}, \]

then it is clear that \( \tilde{f} \) and \( \tilde{g} \) are real-valued functions and \( \tilde{g}^* = \tilde{h} \). From (76), we find that

\[ \tau_{-1,0} = \tilde{g}^*, \quad \tau_{0,-1} = \tilde{g}. \]
Therefore, the bilinear equations (72)-(75) become
\[
\begin{align*}
(D_x^2 - 4c) \tilde{f} \cdot \tilde{f} &= -4c \tilde{g} \tilde{g}^*, \\
(D_x^3 - D_t + 3i\kappa D_x^2 - 3(\kappa^2 + 2c) D_x - 6i\kappa c) \tilde{g} \cdot \tilde{f} + 6i\kappa c \tilde{g}^* &= 0, \\
(D_x + 2i\kappa) \tilde{g} \cdot \tilde{g}^* &= 2i\kappa \tilde{f}.
\end{align*}
\]

Let
\[
f(x, t) = \tilde{f}(x - 6ct, t), \quad g(x, t) = \tilde{g}(x - 6ct, t), \quad q(x, t) = \tilde{q}(x - 6ct, t),
\]
then the system of bilinear equations (77) reduces to (22), and thus we can obtain the following solution to the Sasa-Satsuma equation (2)
\[
u = \frac{g}{f} e^{i(\kappa(x - 6ct) - \kappa^2 t)},
\]
where
\[
f(x, t) = \tau_0(x - 6ct, t), \quad g(x, t) = \tau_1(x - 6ct, t)
\]
and
\[
\tau_k = \tau_{k0}, \quad k = 0, 1.
\]
This completes the proof.

5 Proof of Theorem 2.2

In Theorem 2.1, the rogue wave solutions contain differential operators that are recursively defined. In fact, we are able to avoid these recursive relations by modifying the differential operators. To achieve this, we introduce the notations
\[
\mathcal{M}_{ij}^{kl} = \begin{pmatrix}
  m_{2i-1,2j-1}^{kl} & m_{2i,2j}^{kl} \\
  m_{2i,2j-1}^{kl} & m_{2i+1,2j}^{kl}
\end{pmatrix}, \quad \mathcal{N}_{ij}^{kl} = \begin{pmatrix}
  A_{i1} B_{j1} m_{11}^{kl} & A_{i1} B_{j2} m_{12}^{kl} \\
  A_{i2} B_{j1} m_{21}^{kl} & A_{i2} B_{j2} m_{22}^{kl}
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
  \varphi_{2i-1}^{kl} \\
  \varphi_{2i}^{kl}
\end{pmatrix} = \begin{pmatrix}
  A_{i1} \varphi_1^{kl} \\
  A_{i2} \varphi_2^{kl}
\end{pmatrix}, \quad \begin{pmatrix}
  \psi_{2i-1}^{kl} \\
  \psi_{2i}^{kl}
\end{pmatrix} = \begin{pmatrix}
  B_{j1} \psi_1^{kl} \\
  B_{j2} \psi_2^{kl}
\end{pmatrix},
\]
where \(i, j = 1, 2, \ldots, N, \quad \eta_{\alpha\beta}^{kl}, \varphi_{\alpha}^{kl}, \psi_{\beta}^{kl} (\alpha, \beta = 1, 2)\) are given in (47)-(49) and the differential operators \(A_{i\alpha}, B_{j\beta}\) are defined by
\[
A_{i\alpha} = \frac{1}{i!} [f_{1\alpha}(p_\alpha) \partial_{p_\alpha}]^i, \quad B_{j\beta} = \frac{1}{j!} [f_{2\beta}(q_\beta) \partial_{q_\beta}]^j
\]
with $f_{1\alpha}(p_{\alpha}), f_{2\beta}(q_{\beta})$ being arbitrary functions. Since the operators $A_{i\alpha}$ and $B_{j\beta}$ commute with the partial differentials with respect to $x, y, t, r, s$, the functions $m_{ij}^{kl}, \varphi_{i}^{kl}, \psi_{j}^{kl}$ would satisfy the differential and difference relations (24)-(30). Hence, for any sequence of indices $(i_1, i_2, \ldots, i_N; j_1, j_2, \ldots, j_N)$, the determinant

$$
\tau_{kl} = \det_{1 \leq \mu, \nu \leq N} \left( M_{\mu \nu}^{ik,jl} \right) \tag{79}
$$

satisfies the bilinear equations (36)-(46).

Next, by imposing the parameter relations

$$
f_{1j} \equiv f_{2j}, \quad q_j = p_j, \quad b = -a, \quad \xi_j = \eta_j, \quad j = 1, 2, \tag{80}
$$

which yield the reduction from AKP to CKP, and using similar arguments as in the proof of Theorem 2.1 we obtain

$$
\tau_{-l,-k}(x, -y, t, s, r) = \tau_{kl}(x, y, t, r, s). \tag{81}
$$

In what follows, we will deal with a crucial step in applying the KP hierarchy reduction method, that is, dimension reduction. In this process, several conditions are needed to accomplish this step. As a result of these conditions, the indices in the determinant (79) will be selected. In addition, we will determine the functions $f_{1\alpha}, f_{2\beta}$ as well as the values of $p_{\alpha}, q_{\beta}$ that are involved in the matrix entries of the $\tau$ function. We also remark that the method we will utilize was named ‘W-p treatment’ which was developed by Yang and Yang in the derivations of rogue wave solutions of the Boussinesq equation [47] and the three-wave equations [36].

As shown in the proof of Theorem 2.1 by imposing the condition

$$
\tilde{D}_{r,s,x} \tau_{kl} = C \tau_{kl}, \tag{82}
$$

where $C$ is certain constant and $\tilde{D}_{r,s,x}$ is the differential operator defined in (53), we can reduce the bilinear equations (36)-(40) to (72)-(75). Therefore, it suffices to find conditions under which the equation (82) holds.

It is clear that

$$
\tilde{D}_{r,s,x} M_{ij}^{kl} = 
\begin{pmatrix}
A_{i1}B_{j1}[F(p_1)+F(q_1)]m_{11}^{kl} & A_{i1}B_{j2}[F(p_1)+F(q_2)]m_{12}^{kl} \\
A_{i2}B_{j1}[F(p_2)+F(q_1)]m_{21}^{kl} & A_{i2}B_{j2}[F(p_2)+F(q_2)]m_{22}^{kl}
\end{pmatrix}
$$

where

$$
F(p) = \frac{1}{p-a} + \frac{1}{p+a} - \frac{p}{c},
$$

Inspired by the work of Yang and Yang [47, 36], we rewrite the functions $f_{1\alpha}(p_{\alpha})$ and $f_{2\beta}(q_{\beta})$ in the form

$$
f_1(p_{\alpha}) = \frac{U_{\alpha}(p_{\alpha})}{U_{\alpha}^*(p_{\alpha})}, \quad f_2(q_{\beta}) = \frac{V_{\beta}(q_{\beta})}{V_{\beta}^*(q_{\beta})}. \tag{83}
$$
where the new variables $\mathcal{U}_\alpha$ and $\mathcal{V}_\beta$ are introduced such that

$$f_{\alpha}\partial_{p_\alpha} = \partial_{u\alpha}\mathcal{U}_\alpha, \quad f_{\beta}\partial_{q_\beta} = \partial_{u\beta}\mathcal{V}_\beta.$$

The reason for this representation of $f_{1\alpha}(p_\alpha)$ and $f_{2\beta}(q_\beta)$ is as follows. With the above relations, we may get (see [36])

$$\mathcal{A}_\alpha F(p_\alpha) = \sum_{k=0}^{i} \frac{1}{k!} \left[(f_{1\alpha}\partial_{p_\alpha})^k F(p_\alpha)\right] \mathcal{A}_{i-k}, \quad \alpha = 1, 2, \quad (84)$$

and

$$\mathcal{B}_{j\beta} F(q_\beta) = \sum_{l=0}^{i} \frac{1}{l!} \left[(f_{2\beta}\partial_{q_\beta})^l F(q_\beta)\right] \mathcal{B}_{l-j}, \quad \beta = 1, 2. \quad (85)$$

Consequently, it follows that

$$\bar{D}_{r,s,x} \mathcal{M}_{ij}^{kl} = \left( \begin{array}{cc} a_{2i-1,2j-1}^{kl} & a_{2i-1,2j}^{kl} \\ a_{2i,2j-1}^{kl} & a_{2i,2j}^{kl} \end{array} \right)$$

where

$$a_{2i-1,2j-1}^{kl} = \sum_{m=0}^{i} \frac{1}{m!} \left[(f_{11}\partial_{p_1})^m F(p_1)\right] \mathcal{M}_{2i-1,2j-1}^{kl} + \sum_{n=0}^{j} \frac{1}{n!} \left[(f_{21}\partial_{q_1})^n F(q_1)\right] \mathcal{M}_{2i-1,2j-1}^{kl}, \quad (86)$$

$$a_{2i-1,2j}^{kl} = \sum_{m=0}^{i} \frac{1}{m!} \left[(f_{11}\partial_{p_1})^m F(p_1)\right] \mathcal{M}_{2i-1,2j}^{kl} + \sum_{n=0}^{j} \frac{1}{n!} \left[(f_{22}\partial_{q_2})^n F(q_2)\right] \mathcal{M}_{2i-1,2j}^{kl}, \quad (87)$$

$$a_{2i,2j-1}^{kl} = \sum_{m=0}^{i} \frac{1}{m!} \left[(f_{12}\partial_{p_2})^m F(p_2)\right] \mathcal{M}_{2i,2j-1}^{kl} + \sum_{n=0}^{j} \frac{1}{n!} \left[(f_{21}\partial_{q_1})^n F(q_1)\right] \mathcal{M}_{2i,2j-1}^{kl}, \quad (88)$$

$$a_{2i,2j}^{kl} = \sum_{m=0}^{i} \frac{1}{m!} \left[(f_{12}\partial_{p_2})^m F(p_2)\right] \mathcal{M}_{2i,2j}^{kl} + \sum_{n=0}^{j} \frac{1}{n!} \left[(f_{22}\partial_{q_2})^n F(q_2)\right] \mathcal{M}_{2i,2j}^{kl}, \quad (89)$$

According to the work of Ohta and Yang [45], in order to realize the dimensional reduction condition (82), we can choose proper functions $f_{1\alpha}, f_{2\beta}$ and the values of $p_\alpha, q_\beta$ such that the terms with odd $m$ or $n$ on the right-hand side of equations (86)-(89) vanish. It turns out that this can be guaranteed if we choose $\xi, \tilde{\xi}, \eta$ and $\tilde{\eta}$ such that

$$F'(\xi) = 0, \quad F'(\tilde{\xi}) = 0, \quad F'(\eta) = 0, \quad F'(\tilde{\eta}) = 0 \quad (90)$$

and impose the conditions

$$(f_{1\alpha}\partial_{p_\alpha})^2 F(p_\alpha) = F(p_\alpha), \quad (f_{2\beta}\partial_{q_\beta})^2 F(q_\beta) = F(q_\beta), \quad \alpha, \beta = 1, 2. \quad (91)$$
With these conditions, we find that

\[
\begin{align*}
a_{2i-1,2j-1}^{kl} &= F(p_1) \sum_{m=0}^{l-1} \frac{1}{m!} m_{2(i-m)-1,2j-1}^{kl} |_{p_1=\xi, q_1=\eta} + F(q_1) \sum_{n=0}^{l} \frac{1}{n!} m_{2i-1,2(j-n)-1}^{kl} |_{p_1=\xi, q_1=\eta} \\
a_{2i,2j}^{kl} &= F(p_2) \sum_{m=0}^{l-1} \frac{1}{m!} m_{2(i-m),2j-1}^{kl} |_{p_1=\xi, q_1=\eta} + F(q_2) \sum_{n=0}^{l} \frac{1}{n!} m_{2i,2(j-n)-1}^{kl} |_{p_1=\xi, q_1=\eta} \\
a_{2i,2j}^{kl} &= F(p_2) \sum_{m=0}^{l-1} \frac{1}{m!} m_{2(i-m),2j}^{kl} |_{p_1=\xi, q_1=\eta} + F(q_2) \sum_{n=0}^{l} \frac{1}{n!} m_{2i,2(j-n)}^{kl} |_{p_1=\xi, q_1=\eta} \\
F(p_1) &= \sum_{m=0}^{l} \frac{1}{m!} m_{2i-1,2(j-n)-1}^{kl} |_{p_1=\xi, q_1=\eta} \\
F(q_1) &= \sum_{n=0}^{l} \frac{1}{n!} m_{2i-1,2(j-n)}^{kl} |\eta_1 \eta_2 = \eta_2 \eta_1 \end{align*}
\]

Then, by restricting the indices of the general determinant \( [79] \) to the determinant

\[
\tau_{kl} = \det_{1 \leq i, j \leq N} \left( M_{2i-1,2j-1}^{kl} |_{p_1=\xi, q_1=\eta} \right)
\]

and using the relations \([92] - [95] \), a similar argument as in the proof of Theorem \(2.1 \) yields

\[
\tilde{D}_{r,s,x} \tau_{kl} = \left[ F(\xi) + F(\eta) + F(\tilde{\xi}) + F(\tilde{\eta}) \right] N \tau_{kl}.
\]

Therefore, the function \( \tau_{kl} \) given in \([96] \) satisfies the dimensional reduction condition \([82] \) as long as \([90] \) and \([91] \) hold.

Now we move to determine the values of \( \xi, \tilde{\xi}, \eta, \tilde{\eta} \) and the expressions of \( f_{1\alpha}, f_{2\beta} \) such that they satisfy \([90] \) and \([91] \). From \([80] \), we can get \( \xi = \eta \) and \( \tilde{\xi} = \tilde{\eta} \). Similar to the proof of Theorem \(2.1 \), we should require \( \tilde{\xi} = \xi^* \) and \( \tilde{\eta} = \eta^* \) to make sure that the complex conjugate condition is fulfilled. On the other hand, the terms \( 1/(p_\alpha + q_\beta) \) in \([17] \) indicate that \( \xi + \tilde{\eta} \neq 0 \) and \( \tilde{\xi} + \eta \neq 0 \) while the expressions of \( \tau_{kl} \) in \([96] \) show that \( \xi \neq \tilde{\xi} \). Therefore \( \xi^2 \) cannot be real. Since the equations \([7] \) and \([90] \) are equivalent, from Remark \([1] \) we find that this occurs if and only if

\[
c < 0, \quad c + 4\kappa^2 > 0.
\]
In this case, the roots of the equation (7) demonstrate a symmetric structure (see Remark [1]). Hence, no matter which root we choose, we get the same solution. Apart from this, the symmetric structure indicates that all the roots of \( F'(p) = 0 \) are simple.

In order to find \( f_{11} \), it suffices to solve the differential equation

\[
(f_{11} \partial_{p_1})^2 F(p_1) = F(p_1).
\]

With (83), it becomes

\[
\partial_{\mathcal{U}_4}^2 F(p_1) = F(p_1).
\]

Normalize \( \mathcal{U}_4(\xi) = 1 \), then the unique solution to this second-order differential equation with the initial condition \( F(\xi) = F(\xi), F'(\xi) = 0 \) is

\[
F(p_1) = \frac{1}{2} F(\xi) \left( \mathcal{U}_1(p_1) + \frac{1}{\mathcal{U}_4(p_1)} \right).
\]

Hence, with (83), it follows that

\[
\mathcal{U}_1(p_1) = \frac{F(p_1) \pm \sqrt{F^2(p_1) - F^2(\xi)}}{F(\xi)}
\]

which gives

\[
f_{11}(p_1) = \pm \frac{\sqrt{F^2(p_1) - F^2(\xi)}}{F'(p_1)}.
\]

By changing \((f_{11}, p_1, \xi)\) to \((f_{12}, p_2, \xi^*)\), \((f_{21}, q_1, \xi)\) or \((f_{22}, q_2, \xi^*)\), we may solve for \( f_{12}, f_{21} \) and \( f_{22} \) as well, which share the same form as (102). This completes the dimension reduction procedure. As a consequence, the bilinear equations (36)-(46) are reduced to (72)-(75) and hence, we can take \( y = r = s = 0 \) because the bilinear equations (72)-(75) do not involve derivatives with respect to \( y, r \) and \( s \). Then according to (81), we have

\[
\tau_{kl}(x,t) = \tau_{l-k}(x,t).
\]

Now we consider the complex conjugate reduction. To this end, we set \( a \) to be purely imaginary and impose the parameter constraint

\[
\xi_{10} = \xi_{20}^*.
\]

In view of \([F(p)]^* = F(p^*)\), we find \([F^{(n)}(p)]^* = F^{(n)}(p^*)\) holds for any positive integer \( n \). This implies that

\[
[f_{11}(p)]^* = f_{12}(p^*), \quad [f_{21}(p)]^* = f_{22}(p^*), \quad [f_{11}^{(n)}(p)]^* = f_{12}^{(n)}(p^*), \quad [f_{21}^{(n)}(p)]^* = f_{22}^{(n)}(p^*),
\]

where \( n \) is a positive integer. It then follows that

\[
\left( \mathcal{M}_{ij}^{kl} \right)^*_{p_1=q_1=\xi, p_2=q_2=\xi^*} = \left( \begin{array}{cc}
(m_{2j-1,2j-1}^{kl})^* & (m_{2i-1,2j}^{kl})^* \\
(m_{2i-1,2j}^{kl})^* & (m_{2i-1,2j})^*
\end{array} \right)_{p_1=q_1=\xi, p_2=q_2=\xi^*}.
\]

\[
= \left( \begin{array}{cc}
m_{2i-1,2j}^{lk} & m_{2i-1,2j}^{lk} \\
m_{2i-1,2j}^{lk} & m_{2i-1,2j}^{lk}
\end{array} \right)_{p_1=q_1=\xi, p_2=q_2=\xi^*}.
\]
A similar argument as in the proof of Theorem 2.1 gives
\[ \tau_{kl}^* = \tau_{lk}. \] (106)

Define
\[ \tilde{f} = \tau_{00}, \quad \tilde{g} = \tau_{10}, \quad \tilde{h} = \tau_{01}, \quad \tilde{q} = \tau_{11}, \]
then we find that \( \tilde{f} \) and \( \tilde{q} \) are real-valued functions and \( \tilde{g}^* = \tilde{h} \). With (103), we get
\[ \tau_{-1,0} = \tilde{g}^*, \quad \tau_{0,-1} = \tilde{g}. \]

Therefore, the bilinear equations (72)-(75) are reduced to (77). Let
\[ f(x,t) = \tilde{f}(x - 6ct, t), \quad g(x,t) = \tilde{g}(x - 6ct, t), \quad q(x,t) = \tilde{q}(x - 6ct, t), \]
then \( f, g \) and \( g \) satisfy the bilinear equations (22). Hence, with \( \tau_k = \tau_{0k}, k = 0, 1 \), rational solutions to the Sasa-Satsuma equation (2) are obtained via the transformation (23).

Finally, we apply the method proposed in [22] to introduce free parameters into these solutions. This is accomplished by choosing \( \xi_{10} \) as
\[ \xi_{10} = \sum_{r=1}^{\infty} a_n \ln^n U_1(p_1), \] (107)
where \( U_1(p_1) \) is defined in (101) and \( a_n \) are free complex constants. Thus the proof of Theorem 2.2 is completed.

6 Proof of Theorem 2.3

In order to represent the rational solutions of the Sasa-Satsuma equation derived in Theorem 2.2 in terms of Schur polynomials, we first consider the generator of the differential operators
\[ \left[ f_{1\alpha}(p_\alpha) \partial_{p_\alpha} \right]^i \left[ f_{2\beta}(q_\beta) \partial_{q_\beta} \right]^j \]
given by
\[ G = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\kappa^i \lambda^j}{i! j!} (f_{1\alpha} \partial_{p_\alpha})^i (f_{2\beta} \partial_{q_\beta})^j. \] (108)

Using the relations (83), the above generator can be rewritten as
\[ G = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\kappa^i \lambda^j}{i! j!} (\partial_{\ln U_\alpha})^i (\partial_{\ln V_\beta})^j = \exp \left( \kappa \partial_{\ln U_\alpha} + \lambda \partial_{\ln V_\beta} \right). \]

This implies that, for any function \( H(X,Y) \), we have
\[ GH(X,Y) = H(e^\kappa X, e^\lambda Y). \] (109)

From (100), we see that \( p_\alpha \) and \( q_\beta \) are related to \( U_\alpha \) and \( V_\beta \) respectively and hence
\[ p_\alpha = p_\alpha(U_\alpha), \quad q_\beta = q_\beta(V_\beta). \]
In addition, from the proof of Theorem 2.2, we may deduce that
\[ \mathcal{U}_1(\xi) = \mathcal{U}_2(\xi^*) = \mathcal{V}_1(\xi) = \mathcal{V}_2(\xi^*) = 1. \]  
(110)

Therefore, by denoting
\[ p_\alpha(\kappa) = p_\alpha(e^\kappa), \quad q_\beta(\lambda) = q_\beta(e^\lambda) \]  
and acting \( \mathcal{G} \) on \( m_{\alpha,\beta}^k \) defined in Theorem 2.2, we can get
\[
\mathcal{G} m_{\alpha,\beta}^k \bigg|_{p_1=q_1=\xi} = \frac{1}{p_\alpha(\kappa) + q_\beta(\lambda)} \frac{\left( p_\alpha(\kappa) - a \right) k}{q_\beta(\lambda) + a} \exp \left[ \sum_{n=1}^{\infty} \left( \hat{a}_{n\alpha} \kappa^n + \hat{a}_{n\beta} \lambda^n \right) \right] 
\exp \left[ (p_\alpha(\kappa) + q_\beta(\lambda)) x + (p_\alpha^3(\kappa) + q_\beta^3(\lambda)) t \right],
\]  
(112)
where
\[ \hat{a}_{n1} = a_n, \quad \hat{a}_{n2} = a_n^*. \]

Denote by \( p(\kappa) = p_1(\kappa), q(\lambda) = q_1(\lambda) \), then it follows from (6) that
\[ p_2(\kappa) = p^*(\kappa), \quad q(\lambda) = p(\lambda), \quad q_2(\lambda) = p^*(\lambda), \]
where \( \kappa, \lambda \) are real variables. With (112), we have
\[
\frac{1}{m_{\alpha,\beta}} \mathcal{G} m_{\alpha,\beta}^k \bigg|_{p_1=q_1=\xi} = \frac{p_\alpha + q_\beta}{p_\alpha(\kappa) + q_\beta(\lambda)} \frac{\left( p_\alpha(\kappa) - a \right)^k}{q_\beta(\lambda) + a} \exp \left[ \sum_{n=1}^{\infty} \left( \hat{a}_{n\alpha} \kappa^n + \hat{a}_{n\beta} \lambda^n \right) \right] 
\exp \left[ (p_\alpha(\kappa) - p_\alpha + q_\beta(\lambda) - q_\beta) x + (p_\alpha^3(\kappa) - p_\alpha^3 + q_\beta^3(\lambda) - q_\beta^3) t \right] \bigg|_{p_1=q_1=\xi^*}.
\]  
(113)

In order to connect the right hand side of the above equation with Schur polynomials, we have to expand it as a series in \( \kappa \) and \( \lambda \).

Without loss of generality, we may assume \( \alpha = \beta = 1 \) and denote by
\[ \hat{p}_1 = p^1(0), \quad \hat{q}_1 = q^1(0). \]  
(114)

Note that when \( (\kappa, \lambda) \) is around \( (0, 0) \), we have
\[
\frac{1}{g(\kappa) + h(\lambda)} = \frac{g(0) + h(0)}{[g(\kappa) + h(0)](h(\lambda) + g(0))} \frac{1}{1 - \frac{g(\kappa) - g(0) h(\lambda) - h(0)}{g(\kappa) + h(0) h(\lambda) + g(0)}}
\]  
\[
= \frac{g(0) + h(0)}{[g(\kappa) + h(0)](h(\lambda) + g(0))} \sum_{\nu=0}^{\infty} \left[ \frac{g(\kappa) - g(0) h(\lambda) - h(0)}{g(\kappa) + h(0) h(\lambda) + g(0)} \right]^\nu.
\]  
(115)
Comparing the coefficients of $\kappa^A$ then substituting (117) and (118) into (113) yields

$$\frac{2\xi}{p + q} = \frac{4\xi^2}{(p + \xi)(q + \xi)} \sum_{\nu = 0}^{\infty} \left( \frac{q - \xi}{p + \xi} \right)^\nu$$

while the remaining terms on the right hand side of (113) can be rewritten as

$$\sum_{\nu = 0}^{\infty} \left( \frac{\hat{p}_1 \hat{q}_1}{4\xi^2} \kappa \lambda \right)^\nu \exp \left( \sum_{n=1}^{\infty} (\nu s_n - b_n) \kappa^n + (\nu s_n - b_n) \lambda^n \right)$$  (117)

using the above identity, the first term on the right hand side of (113) can be rewritten as

$$\exp \left\{ \sum_{n=1}^{\infty} \kappa^n (c_n x + d_n t + k h^+_n) + \sum_{n=1}^{\infty} \lambda^n (c_n x + d_n t - k h^-_n) + \sum_{n=1}^{\infty} (a_n \kappa^n + a_n \lambda^n) \right\}.$$  (118)

Then substituting (117) and (118) into (113) yields

$$\frac{1}{m_{11}^k} G m_{11}^k \Bigg|_{p_1 = q_1 = \xi, p_2 = q_2 = \xi^*} = \sum_{\nu = 0}^{\infty} \left( \frac{\hat{p}_1 \hat{q}_1}{4\xi^2} \kappa \lambda \right)^\nu \exp \left( \sum_{n=1}^{\infty} (x_n + \nu s_n - b_n + k h^+_n) \kappa^n + (x_n + \nu s_n - b_n - k h^-_n) \lambda^n \right),$$

Comparing the coefficients of $\kappa^i \lambda^j$ on both sides of the above equation, it follows that

$$\frac{A_{i1} B_{j1} m_{11}^{(k,l)} \Bigg|_{p_1 = q_1 = \xi, p_2 = q_2 = \xi^*}}{(p_1 + q_1) m_{11}^{(k,l)}} = \sum_{\nu = 0}^{\min(i,j)} \left( \frac{\hat{p}_1^2}{4\xi^2} \right)^\nu S_{i-\nu} (x + \nu s - b + k h^+ \hat{b}) S_{j-\nu} (x + \nu s - b - k h^- \hat{b}),$$

where $A_{i1}, B_{j1}$ are defined in (10). Using similar arguments, it can be shown that

$$\frac{A_{i2} B_{j2} m_{12}^{(k,l)} \Bigg|_{p_1 = q_1 = \xi, p_2 = q_2 = \xi^*}}{(p_1 + q_1) m_{12}^{(k,l)}} = \sum_{\nu = 0}^{\min(i,j)} \left( \frac{\hat{p}_1^2}{4\xi^2} \right)^\nu S_{i-\nu} (x + \nu s - b + k h^+ \hat{b}) S_{j-\nu} (x + \nu s - b - k h^- \hat{b})$$

$$\frac{A_{i2} B_{j2} m_{22}^{(k,l)} \Bigg|_{p_1 = q_1 = \xi, p_2 = q_2 = \xi^*}}{(p_2 + q_1) m_{22}^{(k,l)}} = \sum_{\nu = 0}^{\min(i,j)} \left( \frac{\hat{p}_1^2}{4\xi^2} \right)^\nu S_{i-\nu} (x + \nu s - b + k h^+ \hat{b}) S_{j-\nu} (x + \nu s - b - k h^- \hat{b})$$

Let

$$\sigma_k = \frac{\tau_k}{\prod_{\alpha, \beta = 1}^{2} (p_\alpha + q_\beta) m_{\alpha, \beta}^{(k,l)} \Bigg|_{p_1 = q_1 = \xi, p_2 = q_2 = \xi^*} \bigg)^N}.  \quad (119)$$
Considering the fact that the solution set of (2) is invariant under multiplication of \( e^{i\theta} \), where \( \theta \) is real, the function (20) still satisfies the Sasa-Satsuma equation (2). In other words, the rational solutions presented in Theorem 2.2 can be expressed in terms of Schur polynomials. Thus the proof of Theorem 2.3 is completed.

7 Conclusions

In summary, we have derived general rogue wave solutions to the Sasa-Satsuma equation by applying the KP hierarchy reduction method. These solutions are expressed in terms of Gram-type determinants in three different forms. The first two forms are expressed in terms of differential operators. The difference between them is that the differential operators are recursively defined for one form whereas they are no longer recursively defined for the other form. Instead of using differential operators, the third form is represented by Schur polynomials based on the solutions involving differential operators without recursive relations. Owing to the complexity of the Sasa-Satsuma equation, it is shown that its solution structure is more complicated compared with many other integrable equations including the NLS equation which can be reduced from the Sasa-Satsuma equation. As such, we propose a kernel of \( 2 \times 2 \) matrix in the Gram-type determinant solutions of the bilinear equations in the KP hierarchy. To the best of knowledge, this solution form has not been reported before. In addition, the intermediate computations are much more involved compared with most of the integrable equations that can be solved by the same method due to the multiple corresponding bilinear equations in the KP hierarchy.

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