CLUSTER ALGEBRAS AND SYMMETRIZABLE MATRICES

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ABSTRACT. In the structure theory of cluster algebras, principle coefficients are parametrized by a family of integer vectors, called c-vectors. Each c-vector with respect to an acyclic initial seed is a real root of the corresponding root system and the c-vectors associated with any seed defines a symmetrizable quasi-cartan companion for the corresponding exchange matrix. We establish basic combinatorial properties of these companions. In particular, we show that c-vectors define an admissible cut of edges in the associated diagrams.

1. Introduction

In the structure theory of cluster algebras, principle coefficients are parametrized by a family of integer vectors, called c-vectors. Each c-vector with respect to an acyclic initial seed is a real root of the corresponding root system; furthermore, the c-vectors associated with any seed defines a symmetrizable quasi-cartan companion for the corresponding exchange matrix [8, Corollary 3.29]. In this paper, we study basic combinatorial properties of these companions. In particular, we show that c-vectors define an admissible cut of edges in the associated diagrams.

To state our results, we need some terminology. Let us recall that an \( n \times n \) integer matrix \( B \) is skew-symmetrizable if there is a diagonal matrix \( D \) with positive diagonal entries such that \( DB \) is skew-symmetric. We denote by \( T_n \) an \( n \)-regular tree whose edges are labeled by the numbers \( 1, \ldots, n \) such that the \( n \) edges incident to each vertex have different labels. The notation \( t \xleftarrow{k} t' \) indicates that vertices \( t, t' \in T_n \) are connected by an edge labeled by \( k \). We fix a vertex \( t_0 \) in \( T_n \) and assign the pair \((c_0, B_0)\), where \( c_0 \) is the tuple of standard basis and \( B_0 \) is a skew-symmetrizable matrix. Then, to every vertex \( t \in T_n \) we assign a pair, called a Y-seed, \((c_t, B_t)\), where \( c_t = (c_1, \ldots, c_n) \) with each \( c_i = c_{it} = (c_1, \ldots, c_n) \in \mathbb{Z}^n \) being non-zero and having either all entries nonnegative or all entries nonpositive; we write \( \text{sgn}(c_i) = +1 \) or \( \text{sgn}(c_i) = -1 \) respectively and call it a c-vector. Furthermore, for any edge \( t \xleftarrow{k} t' \), the Y-seed \((c', B') = (c_{t'}, B_{t'})\) is obtained from \((c, B) = (c_t, B_t)\) by the Y-seed mutation \( \mu_k \) defined as follows, where we denote \( [b]_+ = \max(b, 0) \):

- The entries of the matrix \( B' = (B'_{ij}) \) are given by

\[
B'_{ij} = \begin{cases} 
-B_{ij} & \text{if } i = k \text{ or } j = k; \\
B_{ij} + [B_{ik}]_+ [B_{kj}]_+ - [-B_{ik}]_+ [-B_{kj}]_+ & \text{otherwise.}
\end{cases}
\]
• The tuple \( c' = (c'_1, \ldots, c'_n) \) is given by

\[
(1.2) \quad c'_i = \begin{cases} 
-c_i & \text{if } i = k; \\
-c_i + [\text{sgn}(c_k)B_{k,i}] + c_k & \text{if } i \neq k.
\end{cases}
\]

By [4] Corollary 5.5], each \( c'_i = (c'_1, \ldots, c'_n) \) also has either all entries nonnegative or all entries nonpositive. The matrix \( B' \) is skew-symmetrizable with the same choice of \( D \); we write \( B' = \mu_k(B) \) and call the transformation \( B \rightarrow B' \) the matrix mutation. For the \( Y \)-seeds, we denote \( \mu_k(c, B) = (c', B') \); we call \((c_0, B_0)\) the initial \( Y \)-seed. It is well known that mutation is an involutive operation.

Let us also recall that the diagram of a skew-symmetrizable \( n \times n \) matrix \( B \) is the directed graph \( \Gamma(B) \) whose vertices are the indices \( 1, 2, \ldots, n \) such that there is a directed edge from \( i \) to \( j \) if and only if \( B_{ij} > 0 \), and this edge is assigned the weight \( |B_{ij}| \). The diagram \( \Gamma(B) \) is called acyclic if it has no oriented cycles. Then there is a corresponding generalized Cartan matrix \( A \) such that \( A_{i,i} = 2 \) and \( A_{i,j} = -|B_{i,j}| \) for \( i \neq j \). There is also the associated root system in the root lattice spanned by the simple roots \( \alpha_i \) [6]. For each simple root \( \alpha_i \), the corresponding reflection \( s_{\alpha_i} = s_i \) is the linear isomorphism defined on the basis of simple roots as \( s_i(\alpha_j) = \alpha_j - A_{i,j}\alpha_i \). Then the real roots are defined as the vectors obtained from the simple roots by a sequence of reflections. It is well known that the coordinates of a real root with respect to the basis of simple roots are either all nonnegative or all nonpositive, see [6] for details.

On the other hand, an \( n \times n \) matrix \( A \) is called symmetrizable if there exists a symmetrizing diagonal matrix \( D \) with positive diagonal entries such that \( DA \) is symmetric. A quasi-Cartan companion (or "companion" for short) of a skew-symmetrizable matrix \( B \) is a symmetrizable matrix \( A \) such that \( A_{i,i} = 2 \), \( |A_{i,j}| = |B_{i,j}| \) for all \( i \neq j \). A fundamental relation between \( Y \)-seeds and symmetrizable matrices has been given in [8] Corollary 3.29] as follows:

**Theorem 1.1.** [8] Corollary 3.29] Suppose that the initial seed \((c_0, B_0)\) is acyclic. Then, for any \( Y \)-seed \((c_t, B_t)\), \( t \in \mathbb{T}_n \), each \( c \)-vector \( c_i = c_{i,t} \) is the coordinate vector of a real root with respect to the basis of simple roots in the corresponding root system. Furthermore, \( A = A_t = (\langle c_j, c'_j \rangle) \), the matrix of the pairings between the roots and the coroots, is a quasi-Cartan companion of the skew-symmetrizable matrix \( B = B_t \).

(The matrices \( A_t \) are symmetrizable with the same choice of a symmetrizing matrix \( D \), which is also skew-symmetrizing for all \( B_t \).)

An important combinatorial property related to quasi-Cartan companions is admissibility [9, 10], which is a generalization of the notion of a generalized Cartan matrix. More precisely, a quasi-Cartan companion \( A \) of a skew-symmetrizable matrix \( B \) admissible if, for any oriented (resp. non-oriented) cycle \( Z \) in \( \Gamma(B) \), there is exactly an odd (resp. even) number of edges \{\( i, j \)\} such that \( A_{i,j} > 0 \). If \( \Gamma(B) \) is acyclic, then the associated generalized Cartan matrix is admissible. Our first result generalizes this property by showing that the quasi-Cartan companions defined by \( c \)-vectors are also admissible:

**Theorem 1.2.** In the set-up of Theorem [11], the quasi-Cartan companion \( A \) has the following properties:
Every directed path of the diagram $\Gamma(B)$ has at most one edge $\{i, j\}$ such that $A_{i,j} > 0$.

Every oriented cycle of the diagram $\Gamma(B)$ has exactly one edge $\{i, j\}$ such that $A_{i,j} > 0$.

Every non-oriented cycle of the diagram $\Gamma(B)$ has an even number of edges $\{i, j\}$ such that $A_{i,j} > 0$.

In particular, the quasi-Cartan companion $A$ is admissible. Furthermore, any admissible quasi-Cartan companion of $B$ can be obtained from $A$ by a sequence of simultaneous sign changes in rows and columns.

The special case of this theorem when $B$ is skew-symmetric was obtained in [10, Theorem 1.4] by the author. Let us also recall from [10] that a set $C$ of edges in $\Gamma(B)$ is called an "admissible cut" if every oriented cycle contains exactly one edge that belongs to $C$ and every non-oriented cycle contains exactly an even number of edges in $C$. Thus, in the setup of the theorem, the $c$-vectors define an admissible cut of edges: the set of edges $\{i, j\}$ in $\Gamma(B)$ such that $A_{i,j} > 0$ is an admissible cut.

For skew-symmetric matrices, this notion has been applied to the representation theory of algebras in [5, ?].

Our next result is the following explicit description of the quasi-Cartan companions defined by the $c$-vectors:

**Theorem 1.3.** In the set-up of Theorem 1.1, the quasi-Cartan companion $A$ has the following properties:

1. If $\text{sgn}(B_{j,i}) = \text{sgn}(c_j)$, then $A_{j,i} = -\text{sgn}(c_j)B_{j,i} = -|B_{j,i}|$.

2. If $\text{sgn}(B_{j,i}) = -\text{sgn}(c_j)$, then $A_{j,i} = \text{sgn}(c_i)B_{j,i} = -\text{sgn}(c_i)\text{sgn}(c_j)|B_{j,i}|$.

In particular, if $\text{sgn}(c_j) = -\text{sgn}(c_i)$, then $B_{j,i} = \text{sgn}(c_i)A_{j,i}$.

Let us note that the special case of this theorem when $B$ is skew-symmetric was obtained in [10, Theorem 1.4] by the author. We will prove this more general theorem using [8, Corollary 3.29], which has been given above as Theorem 1.1 (Note that the statement [8, Corollary 3.29] was not present in the earlier versions of [8]).

**Corollary 1.4.** In the setup of Theorem 1.3, suppose that $t \xrightarrow{k} t'$ in $T_n$. Then, for $\mu_k(c, B) = (c', B')$, we have the following: if $c'_i \neq c_i$, then $c'_i = s_{c_k}(c_i)$, where $s_{c_k}$ is the reflection with respect to the real root $c_k$ and $\mathbb{Z}^n$ is identified with the root lattice.

Let us also note that Theorem 1.3 could be useful for recognizing mutation classes of acyclic diagrams: a diagram that does not have an admissible quasi-Cartan companion can not be obtained from any acyclic diagram by a sequence of mutations. An example of such a diagram is given in Figure 1. (We refer to [9, Section 2] for properties of diagrams of skew-symmetrizable matrices). Another application of the admissibility property to the corresponding Weyl groups can be found in [11], where a fundamental class of relations have been shown to be satisfied by the reflections of the c-vectors.

### 2. Proofs of main results

Let us first recall the following well-known property of root systems: For a generalized Cartan matrix $A$ with symmetrizing matrix $D = \text{diag}(d_1, ..., d_n)$, there...
is an invariant symmetric bilinear form (,) defined on the simple roots as \( (\alpha_i, \alpha_j) = d_{ij} A_{ij} = d_{ij} A_{ji} = (\alpha_j, \alpha_i) \). Let us note that, for any real root \( \alpha \), the corresponding reflection \( s_\alpha \) is defined on the real roots as \( s_\alpha(\beta) = \beta - (\beta, \alpha^\vee) \alpha \), with \( (\beta, \alpha^\vee) = 2(\alpha, \beta) / (\alpha, \alpha) \). In particular, \( s_\alpha(\alpha_j) = \alpha_j - (\alpha_j, \alpha_i^\vee) \alpha_i = \alpha_j - A_{ij} \alpha_i \).

Let us also recall the mutation of quasi Cartan companions [10 Definition 1.6]. Suppose that \( B \) is a skew-symmetrizable matrix and let \( A \) be a quasi-Cartan companion of \( B \). Let \( k \) be an index. For each sign \( \epsilon = \pm 1 \), "the \( \epsilon \)-mutation of \( A \) at \( k \)" is the quasi-Cartan matrix \( \mu_k^\epsilon(A) = A' \) such that for any \( i, j \neq k \): \( A'_{i,k} = \epsilon \text{sgn}(B_{k,i}) A_{i,k}, \ A'_{j,k} = \epsilon \text{sgn}(B_{k,j}) A_{j,k}, \ A'_{i,j} = A_{i,j} - \text{sgn}(A_{i,k} A_{j,k}) |B_{i,k} B_{k,j}| \). In the setup of Theorem 1.1, suppose that \( t \xrightarrow{k} t' \) in \( \mathbb{T}_n \) and let \( A \) and \( A' \) be the associated quasi-Cartan companions. Then \( A' = \mu_k^\epsilon(A) \) for \( \epsilon = \text{sgn}(c_k) \).

We first prove Theorem 1.3 for convenience:

Proof of Theorem 1.3. To prove the first part, let us suppose that \( \text{sgn}(B_{j,i}) = \text{sgn}(c_j) \). Let \( \mu_j(c, B) = (c', B') \) with \( B' = \mu_j(B) \). Then \( c'_j = c_j + \text{sgn}(c_k) B_{k,j} = c_j + \text{sgn}(B_{k,j}) B_{j,k} \). We denote by (,) the invariant symmetric bilinear form defined by \( A_0 \) on the root lattice and let \( D = diag(d_1, \ldots, d_n) \) be the symmetrizing matrix for \( A_0 \). Note that, by Theorem 1.1, we have the following:

\[
2d_i = (c'_j, c'_j) = (c_j, c_j), \quad 2d_j = (c_j, c_j), \quad (c_i, c_j) = (c_i, c_j) = d_i A_{ij} = d_j A_{ji}.
\]

Then \( 2d_i = (c'_j, c'_j) = (c_i + |B_{j,i}| c_j + |B_{j,i}| c_j, c_j) = (c_i, c_j) + (c_i, |B_{j,i}| c_j) + (|B_{j,i}| c_j, c_j) = 2d_i + 2|B_{j,i}|^2 c_j c_j = 2d_i + 2|B_{j,i}|^2 c_j c_j = 2d_i + 2|B_{j,i}|^2 d_j = 2d_i + 2d_j \).

To prove the second part of the theorem, let us suppose that \( \text{sgn}(B_{j,i}) = -\text{sgn}(c_j) \). Let \( \mu_j(c, B) = (c', B') \) with \( B' = \mu_j(B) \). Note that \( \text{sgn}(B'_{j,i}) = -\text{sgn}(B_{j,i}) \) and \( |B'_{j,i}| = |B_{j,i}| \) (by the definition of mutation). Let \( A' \) be the quasi-Cartan companion associated to the Y-seed \((c', B')\) (Theorem 1.1). (Note then that \( A' = \mu_k^\epsilon(A) \) where \( \epsilon = \text{sgn}(c_i) \)).

For the proof, we first assume that \( \text{sgn}(c_j) = -\text{sgn}(c_i) \). Then we have \( \text{sgn}(c_j) = \text{sgn}(B_{j,i}) \), so \( c'_j = c_j \) and \( c'_j = -c_i \), implying \( \text{sgn}(c'_j) = \text{sgn}(c_j) = -\text{sgn}(B_{j,i}) \), i.e. for the Y-seed \((c', B')\), we have \( \text{sgn}(B'_{j,i}) = \text{sgn}(c'_j) \). Thus, by the first part of the theorem, we have \( -|B'_{j,i}| = A'_{j,i} = -A_{j,i} \). Thus \( A_{j,i} = |B_{j,i}| = |B'_{j,i}| = -\text{sgn}(c_i) \text{sgn}(c_j) |B_{j,i}| \).

Let us now assume that \( \text{sgn}(c_j) = \text{sgn}(c_i) \). Then, since we have assumed \( \text{sgn}(B_{j,i}) = -\text{sgn}(c_j) \), we have \( \text{sgn}(c_i) = -\text{sgn}(B_{j,i}) = \text{sgn}(B_{j,i}) \). Then, by
the first part of the theorem, we have $A_{i,j} = -|B_{i,j}|$. Thus, since $A$ is symmetrizable and a quasi-Cartan companion, we also have $A_{j,i} = -|B_{j,i}|$, which is equal to $-\text{sgn}(c_i)\text{sgn}(c_j)|B_{j,i}|$.

On the other hand, our assumption $\text{sgn}(B_{j,i}) = -\text{sgn}(c_j)$ implies the following:

$$-\text{sgn}(c_i)\text{sgn}(c_j)|B_{j,i}| = -\text{sgn}(c_i)\text{sgn}(c_j)\text{sgn}(B_{j,i})B_{j,i} = -\text{sgn}(c_i)\text{sgn}(c_j)(-\text{sgn}(c_j))B_{j,i} = \text{sgn}(c_i)B_{j,i}. $$

This completes the proof.

**Proof of Corollary 1.4** Let us note that for $\mu = \text{sgn}$ we have the following:

$$\text{sgn}(c_i)\text{sgn}(c_j)|B_{j,i}| = \text{sgn}(c_i)\text{sgn}(c_j)\text{sgn}(B_{j,i})B_{j,i} = -\text{sgn}(c_i)\text{sgn}(c_j)(-\text{sgn}(c_j))B_{j,i} = \text{sgn}(c_i)B_{j,i}. $$

This completes the proof.

**Proof of Theorem 1.2** As we discussed in Section 1, the special case of this theorem when $B$ is skew-symmetric was obtained in [10, Theorem 1.4] by the author. The proof in [10] uses only the general properties of the mutations of skew-symmetrizable matrices with quasi-Cartan companions and the properties given in Theorem 1.3 (which was obtained for skew-symmetric matrices in [10, Theorem 1.3]); note that in this case the companion $A$ is symmetric and $A_{i,j} = c_i^T A_0 c_j$). Since we have proved Theorem 1.3 above for skew-symmetrizable matrices, the proof of [10, Theorem 1.4] also holds for the skew-symmetrizable matrices. Thus, for the proof of Theorem 1.2 we refer the reader to the proof of [10, Theorem 1.4].

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