Maximizing absorption and scattering by dipole particles

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Abstract

This is a review and tutorial paper which discusses the fundamental limitations on the maximal power which can be received, absorbed, and scattered by an electrically small electrically polarizable particle and infinite periodical arrays of such particles.

1 Introduction

Absorption and scattering of electromagnetic waves by electrically (optically) small particles are basic, but very important phenomena, which we encounter in many branches of physics and engineering (light propagation in atmosphere, small antennas and sensors, plasmonics, etc.) Especially, resonant particles are interesting and important for applications due to strong interactions with electromagnetic fields. Most commonly, these are plasmonic nanoparticles (optical frequencies) and various small antennas (from radio to terahertz frequencies). On one hand, understanding of electromagnetic response of small particles appears to be very simple: After all, we deal with radiation from the simplest source, a point electric dipole. And indeed, basic properties of dipole radiators and antennas are well known for very long time, since the beginning of the 20ies century, and documented in numerous text books, e.g. [1, 2]. In particular, it is a classical result that the effective area (absorption cross section) of a small resonant wire antenna is of the order of \( \lambda^2 \), where \( \lambda \) is the wavelength of the incident radiation (e.g., [1]). If this effective area would be as small as the geometrical cross section of the thin metal wire forming the antenna, no radios would work at frequencies below microwaves. But on the other hand, it appears that in the optical literature the fact that a small resonant particle can collect and absorb much more power than is incident on its geometrical cross section was recognized only in 1983 [3, 4]. Interestingly, the same property has been recognized in 1950ies mainly in the works of S.M. Rytov in the context of studies of thermal radiation from finite-size bodies (thermal radiation from a resonant particle exceeds the black-body limit in the vicinity of resonances). Recently, with the developments of plasmonics and metamaterials,
basic phenomena of scattering and absorption in small resonant particles are again in the focus of research interests. For example, in paper [5] it was shown that a focused laser beam (the focal area of the order of $\lambda^2$) can be nearly fully reflected by a small resonant dipole particle, which follows from the fact (see above) that its scattering cross section is of the same order.

Likewise, the theory of periodical arrays of small electric dipoles at normal incidence is a very simple special case of the well-developed theory of antenna arrays (e.g., [1]). In particular, if a single small but resonant particle can absorb or reflect power “collected” from the surface of the order $\lambda^2$, it appears clear that a regular array of such particles can fully reflect plane waves if the period is smaller than $\lambda$ (the theory is given e.g., in [6]). This is the enabling phenomenon in many applications of frequency selective surfaces (e.g., [7]). The Babinet principle tells that an ideally conducting sheet with periodically arranged electrically small but resonant holes can be fully transparent at the resonant frequency (in recent literature, this phenomenon is called *extraordinary transmission*). In the optical literature, research on such simple resonant arrays is active even now: For example, in [8], this effect of nearly full reflection from an array of resonant low-loss particles is demonstrated for an array of dielectric spheres. However, general considerations on the maximum reflected and received power in such arrays appeared in the literature only recently [9, 10]. Paper [11] considers limiting receiving and scattering performance also of large and directive antennas.

In most recent literature, there are numerous publications on exotic and special properties of resonant particles and their arrays, especially in plasmonics, in connection with the developments of metasurfaces and metamaterials and new devices (cloaked sensors, light-trapping structures, etc.) This calls for good understanding of electromagnetic response of small but resonant structures and especially on fundamental limitations on their absorptive and scattering performance. This paper is a tutorial overview with the focus on fundamental limitations on absorption, extinction, and scattering cross sections and on most general results which hold for all small scatterers and planar arrays of such particles. Most (probably, even all) results and conclusions of this review are known, but this knowledge is scattered in many publications in journals and books in various disciplines. There is no attempt to make a historical overview of the subject, and the reference list is limited to papers which are necessary to understand this review. Limitations on electromagnetic response of small particles are considered also in recent papers [12–14]. Paper [15] studies extreme absorption properties of small bi-anisotropic particles.

## 2 Extracted, scattered, and absorbed powers

Let us consider a small electrically polarizable particle in free space, excited by external electric field, which can be assumed to be approximately uniform over the particle volume. The complex amplitude of the incident electric field at the position of the particle we denote as $E_{\text{inc}}$. The linear particle response we model by its polarizability $\alpha$ defined through

$$p = \alpha E_{\text{inc}}$$  \(1\)
where $p$ is the complex amplitude of the induced electric dipole moment. The particle can be anisotropic, so that the induced dipole moment is not necessarily parallel to the exciting field, but we will need only the ratio between the amplitudes of these vectors, thus, knowledge of the scalar value of the polarizability is sufficient. The power extracted by a dipole inclusion (dipole moment $p$) from a given incident field $\mathbf{E}_{inc}$ is given by the classical formula

$$P_{ext} = \frac{1}{2} \text{Re} \int_{V} \mathbf{J}^* \cdot \mathbf{E}_{inc} dV = \frac{1}{2} \text{Re}(\omega \mathbf{p}^* \mathbf{E}_{inc}) = -\frac{\omega}{2} \text{Im}(\alpha)|\mathbf{E}_{inc}|^2 = -\eta \omega \text{Im}(\alpha)P_{inc}$$

(2)

(the harmonic time dependence is in the form $\exp(j\omega t)$). Here $\mathbf{J}$ is the volumetric electric current density inside the particle. This general formula is valid for any dispersive and lossy particle, assuming only that it is a small dipole particle and $\mathbf{E}_{inc}$ is uniform over its volume. Here and below, the result in terms of the incident power density $P_{inc}$ holds if the particle is excited by a propagating plane wave. The corresponding value of the incident time-averaged power flow density is

$$P_{inc} = \frac{1}{2\eta}|\mathbf{E}_{inc}|^2$$

(3)

where $\eta = \sqrt{\mu_0/\varepsilon_0}$ is the free-space wave impedance. The power which is scattered (re-radiated) by the particle is given as the power radiated by the electric dipole $p$:

$$P_{sc} = \frac{\mu_0 \omega^4 |\alpha|^2}{12\pi c} |\mathbf{E}_{inc}|^2 = \frac{k^4}{6\pi \varepsilon_0} |\alpha|^2 P_{inc}$$

(4)

We can find the general expression for the scattering loss factor in the dipole polarizability equating the extracted and scattered powers for the case of a lossless particle (no absorption). This allows us to find an expression for the imaginary part of the inverse polarizability (e.g. [6, eq. (4.82)])

$$\frac{1}{\alpha} \frac{\text{Re}(\alpha) - j\text{Im}(\alpha)}{|\alpha|^2}$$

(5)

as

$$\text{Im} \left( \frac{1}{\alpha} \right) = -\frac{\text{Im}(\alpha)}{|\alpha|^2} = \frac{\mu_0 \omega^3}{6\pi c} = \frac{k^3}{6\pi \varepsilon_0}$$

(6)

($k$ is the free-space wave number). This is a classical result which dates back to the work of M. Plank (1902), see references in [16].

For the most general linear particle we can write for the inverse polarizability

$$\frac{1}{\alpha} = \xi' + j \xi'' + j \frac{k^3}{6\pi \varepsilon_0}$$

(7)

Here the last term is due to the scattering loss, and the complex value of $\xi = \xi' + j \xi''$ depends on the particle size, shape, material, and the frequency. For passive particles the absorption coefficient $\xi'' \geq 0$ (this ensures that the absorbed power is non-negative, see eq. (10) below), and for lossless particles $\xi'' = 0$. The real part of the inverse polarizability $\xi'$ can take any real value. $\xi' = 0$ corresponds to the resonant frequencies of the particle, and $|\xi'| \to \infty$ corresponds
to zero polarizability (particle is not excited at all, or there is no particle). The scattering-loss term remains the same also for absorbing particles, because this value depends only on the frequency and not on the particle parameters.

The absorbed power is the difference between the power extracted by the particle from the incident field (2) and the power scattered by the same particle into the surrounding space (4):

$$P_{\text{abs}} = -\frac{\omega}{2} \text{Im}(\alpha) |E_{\text{inc}}|^2 - \frac{\mu_0 \omega^4 |\alpha|^2}{12 \pi c} |E_{\text{inc}}|^2 = \frac{\omega \xi''}{\xi^2 + \left( \xi'' + \frac{k^3}{6 \varepsilon_0} \right)^2} |E_{\text{inc}}|^2 = \frac{k_0 \xi''}{\xi^2 + \left( \xi'' + \frac{k^3}{6 \varepsilon_0} \right)^2} P_{\text{inc}}$$

(8)

Here we have substituted the general expression for the polarizability $\alpha$ from (7).

Let us list the corresponding effective cross sections of the particle, defined as the ratios of these powers to the incident power density. The extinction cross section:

$$\sigma_{\text{ext}} = \frac{P_{\text{ext}}}{P_{\text{inc}}} = \frac{k}{\varepsilon_0} \frac{\xi''}{\xi^2 + \left( \xi'' + \frac{k^3}{6 \varepsilon_0} \right)^2}$$

(9)

The absorption cross section:

$$\sigma_{\text{abs}} = \frac{P_{\text{abs}}}{P_{\text{inc}}} = \frac{k}{\varepsilon_0} \frac{\xi''}{\xi^2 + \left( \xi'' + \frac{k^3}{6 \varepsilon_0} \right)^2}$$

(10)

and the total scattering cross section:

$$\sigma_{\text{sc}} = \frac{P_{\text{sc}}}{P_{\text{inc}}} = \frac{k^4}{6 \pi \varepsilon_0^2} \frac{1}{\xi^2 + \left( \xi'' + \frac{k^3}{6 \varepsilon_0} \right)^2}$$

(11)

Finally, the radar (backscattering) cross section, defined as

$$\sigma = \lim_{D \to \infty} \frac{4 \pi D^2}{4 \pi} \frac{|E_{\text{sc}}|^2}{|E_{\text{inc}}|^2}$$

(12)

(here $D$ is the distance from the particle to the observation point along the direction towards the source) equals

$$\sigma = \frac{k^4 |\alpha|^2}{4 \pi \varepsilon_0^2} = \frac{k^4}{4 \pi \varepsilon_0^2} \frac{1}{\xi^2 + \left( \xi'' + \frac{k^3}{6 \varepsilon_0} \right)^2}$$

(13)

Note that it differs from the total scattering cross section (9) only by a numerical factor.

In many papers and books, scattering from small particles is considered based on the quasi-static approximation of the particle polarizability $\alpha$. Under this approximation, when the loss parameter decreases, the amplitude of the induced dipole moment can reach arbitrary large values. This is, naturally, because scattering from the particle is neglected in the quasi-static approximation, and there is no fundamental lower limit on the degree of losses. In this
approximation there is no difference between extinction and absorption cross sections, see (9) and (10) in the assumption that \( k \to 0 \). It may be confusing that in some books (e.g. [2, Sec. 5.2], [25, Eq. (8.7)]), the extinction cross section of small dipolar particles expressed in terms of the imaginary part of the polarizability as in (2) is called the absorption cross section (although they are indeed not distinguishable in the quasi-static model).

While for determination of the near-field distribution and for calculations of the real part of the polarizability the accuracy of the quasi-static approximation is improving with decreasing the particle size, the above results for the effective cross section areas show that neglecting the scattering term proportional to \( k^3 \) leads to dramatically different results for low-loss resonant particles. Actually, the quasi-static model of the polarizability leads to qualitatively wrong (non-physical) results for the absorption cross section of low-loss particles. In the quasi-static approximation \( \sigma_{\text{abs}} \) (10) at resonance (at \( \xi' = 0 \)) diverges as \( 1/\xi'' \) when the loss parameter \( \xi'' \to 0 \), while the full-wave result shows that absorption tends to zero as \( \sigma_{\text{abs}} \sim \xi'' \) for small \( \xi'' \). Actually, the quasi-static approximation can be used for small resonant dipole scatterers only when the absorption loss dominates over the scattering loss. For plasmonic nanoparticles this means the case of small radius of the particle, see formula (26) below (for silver spheres the estimation based on the Drude model gives \( r \ll 10 \text{ nm} \) [17]). For electrically small low-loss dielectric spheres the quasi-static approximation is accurate when \( |\xi'| \ll k^3/(6\pi\epsilon_0) \), which is equivalent to

\[
\left(\frac{r}{\lambda}\right)^3 \ll \frac{3}{16\pi^3} \left|\frac{\epsilon + 2}{\epsilon - 1}\right|
\]  

where \( \epsilon \) is the relative permittivity of the sphere material.

3 Extreme values

The absorbed power (8) and the absorption cross section (10) reach their maxima at

\[
\xi' = 0, \quad \xi'' = \frac{k^3}{6\pi\epsilon_0}
\]  

The maximal possible value of absorbed power reads, upon substitution of these values,

\[
P_{\text{abs max}} = \frac{3\pi}{2k^2}P_{\text{inc}} = \frac{3}{8\pi}\lambda^2P_{\text{inc}}
\]  

and the ultimate value of the absorption cross section equals to \( \frac{3}{8\pi}\lambda^2 \). When the particle is tuned to absorb the maximum possible power, the scattered power equals to the absorbed power. Indeed, substitution of \( |\alpha|^2 \) with \( \xi' = 0 \) and \( \xi'' = \frac{k^3}{6\pi\epsilon_0} \) into the expression for the total scattered power (4) gives the same result as in (16).

Physically, this extremum of absorbed power corresponds to a particle at resonance (\( \xi' = 0 \)) whose loss parameter is such that the amount of absorbed power is equal to the amount of scattered power. This extremum has the same meaning as the extremum of power delivered to a load from a generator when the load impedance is matched to the impedance of the source.
Similarly, in that case half of the power is delivered to the load and half is dissipated in the internal resistance of the generator.

We can find the extreme value of the scattering cross section of any dipole particle from (4) or (11). Obviously, scattering is maximized when $\xi' = \xi'' = 0$. As expected, this corresponds to a lossless particle at its resonance: In this case the induced dipole moment is maximized, which obviously corresponds to the maximum scattered power. The corresponding value of the maximum scattered power reads

$$P_{\text{sc max}} = \frac{3}{2\pi} \lambda^2 P_{\text{inc}}$$

and the maximum possible total scattering cross section equals $\frac{3}{2\pi} \lambda^2$, four times as large as the maximum absorption cross section.

The extreme value of the extinction cross section we find maximizing the extracted power (2) or the extinction cross section (9). Also in this case the maximum is reached at $\xi' = \xi'' = 0$, the same as for the maximal scattering cross section. Thus, the maximum extinction cross section equals to the maximum scattering cross section ($P_{\text{sc max}} = P_{\text{ext max}}$), and it is reached when the particle resonates but does not absorb power.

### 3.1 Special cases

#### 3.1.1 Small dielectric sphere

For a special case of a small dielectric sphere (radius $r$) with the relative permittivity $\epsilon = \epsilon' - j\epsilon''$ the inverse polarizability reads, with the accuracy up to the third-order terms $(kr)^3$ (e.g. [25, Eq. (8.10)])

$$\frac{1}{\alpha} = \frac{1}{4\pi \varepsilon_0 r^3} \frac{\epsilon' + 2 - j\epsilon'' - \frac{3}{5}(\epsilon' - 2 - j\epsilon'')(kr)^2}{\epsilon' - 1 - j\epsilon''} + j\frac{k^3}{6\pi \varepsilon_0}$$

(18)

Thus, parameters $\xi'$ and $\xi''$ in (7) read

$$\xi' = \frac{\left[\epsilon' + 2 - \frac{3}{5}(kr)^2(\epsilon' - 2)\right] (\epsilon' - 1) + \epsilon'' \left[1 - \frac{3}{5}(kr)^2\right]}{4\pi \varepsilon_0 r^3[(\epsilon' - 1)^2 + \epsilon''^2]}$$

(19)

$$\xi'' = \frac{3\epsilon''}{4\pi \varepsilon_0 r^3[(\epsilon' - 1)^2 + \epsilon''^2]} \left(1 + \frac{1}{5}(kr)^2\right)$$

(20)

To write the conditions for the maximum absorbed power (15) in terms of the particle permittivity, we first equate $\xi''$ and $k^3/(6\pi \varepsilon_0)$. In the assumption of small losses, this gives

$$\epsilon'' \approx \frac{2}{9}(kr)^3(\epsilon' - 1)^2$$

(21)

Substituting into the second condition $\xi' = 0$ and keeping the terms up to the second order, we get

$$\epsilon' \approx -2 - \frac{3}{5}(kr)^2$$

(22)
This finally determines the required loss factor

\[ e'' \approx 2(kr)^3 \]  

(23)

The conditions for the maximum scattered power \( \xi' = \xi'' = 0 \) read, in the same approximation,

\[ e' \approx -2 - \frac{3}{5}(kr)^2, \quad e'' = 0 \]  

(24)

(lossless plasmonic particle at resonance).

3.1.2 Small metal sphere (Drude model)

For a special case of a metal sphere modelled by the Drude relative permittivity

\[ \epsilon = 1 - \frac{\omega_p^2}{\omega(\omega - j\Gamma)} \]  

(25)

we have, neglecting the second-order term,

\[ \xi' = \frac{\omega_p^2 - 3\omega^2}{4\pi\epsilon_0 r^3\omega_p^2}, \quad \xi'' = \frac{3\omega\Gamma}{4\pi\epsilon_0 r^3\omega_p^2} \]  

(26)

Thus, the maximum absorption corresponds to the frequency equal to the resonant frequency \( \omega = \omega_p/\sqrt{3} \) and to the loss factor equal to

\[ \Gamma = \frac{2\omega}{3}(kr)^3 \]  

(27)

3.1.3 Electrically small dipole antenna

Let us look at the maximal power received by a short electric dipole antenna (the arm length \( l \)). Instead of considering the antenna as a scatterer in terms of its polarizability, we can find the maximal received power directly. The electromotive force induced by the incident field is \( E_{inc.}l \).

To maximize the received power, we bring the antenna to resonance (by an inductive load) and assume that the antenna material is perfectly conducting (we look at the maximum possible received power, so we should maximize the efficiency to 100%). Then the input impedance is equal to the radiation resistance

\[ R_{rad} = \frac{\eta}{6\pi} (kl)^2 \]  

(28)

(e.g., [1]).

To maximize the received power, we load the antenna by a matched load whose resistance is equal to this radiation resistance. Note that if we want to maximize the extracted power, we do not need to minimize losses in the antenna itself, as the respective condition is the equality
of the total antenna resistance and the radiation resistance. The current through the load is 
\[ I = \frac{E_{\text{inc}}}{2R_{\text{rad}}} \]. The power delivered to the load reads
\[ P_{\text{abs max}} = \frac{1}{2} |I|^2 R_{\text{rad}} = \frac{6\pi}{4\eta k^2} |E_{\text{inc}}|^2 = \frac{3}{8\pi} \lambda^2 P_{\text{inc}} \] 
(29)
\[ \sigma_{\text{abs max}} = \frac{3}{8\pi} \lambda^2 \] 
(30)
is the maximal value of the effective area of this antenna, the same result as derived above for arbitrary small particles. For small electric dipoles this result is known from the antenna theory, e.g. [1, eq. (4-32)].

The inverse polarizability of a short dipole antenna loaded by a load impedance \( Z_{\text{load}} = jX + R \) and tuned close to the resonance can be written as (e.g., [18])
\[ \frac{1}{\alpha} = \frac{j\omega}{l^2} (Z_{\text{inp}} + Z_{\text{load}}) \] 
(31)
where \( Z_{\text{inp}} \) is the input impedance of the antenna. This allows us to find
\[ \xi' = -\frac{\omega}{l^2} [\text{Im}(Z_{\text{inp}}) + X] \] 
(32)
\[ \xi'' = \frac{\omega}{l^2} [\text{Re}(Z_{\text{inp}} - R_{\text{rad}}) + R] \] 
(33)
For a small antenna, the imaginary part of \( Z_{\text{inp}} \) is negative (capacitive reactance) and the real part is the sum of the radiation resistance (28) and the loss resistance in the antenna wires (for perfectly conducting wires the loss resistance is zero).

### 3.2 Generalization using the reciprocity principle

Continuing the example of the dipole antenna, we can write its maximum effective area (absorption cross section) in terms of the antenna gain. Because in this case the antenna efficiency is 100%, its gain is equal to its directivity, and the directivity of the small dipole antenna equals to \( 3/2 \) [1]. Thus, we can re-write relation (30) as
\[ \sigma_{\text{abs max}} = \frac{3}{8\pi} \lambda^2 = \frac{3}{2} \left( \frac{\lambda^2}{4\pi} \right) = G \frac{\lambda^2}{4\pi} \] 
(34)
In the antenna theory it is well known that for any reciprocal antenna located in a reciprocal environment and matched to its load the ratio of the antenna gain to its effective area \( \frac{G}{\sigma_{\text{abs}}} \) is a universal constant, which is independent from the antenna type. Thus, the relation
\[ \sigma_{\text{abs max}} = G \frac{\lambda^2}{4\pi} \] 
(35)
holds in fact for any reciprocal antenna matched to its load. This clearly tells the (obvious) fact that received power can be increased by making the antenna directive. In terms of the optical
language, this tells that the extinction cross section can be increased over the value given by (30), if higher-order modes are excited in the particle.

In the theory of superdirective antennas it is proven that antenna directivity can be made arbitrary large if arbitrary antenna current distributions (with arbitrary fast variations over the antenna volume) are allowed. This result is usually attributed to C.W. Oseen (1922), see discussion and references e.g. in [19, Sec. 2.2]. A superdirective antenna is equivalent to an electrically small particle where many higher-order modes (index \(l \neq 1\) in the Mie theory of scattering by a sphere) are strongly (resonantly) excited. This requirement obviously presents challenges in practical realizations of classical superdirective antennas [19] and recently introduced “superscatters” and “superabsorbers” [20–24]. Note strong sub-wavelength oscillations of fields clearly visible in examples presented in [21].

4 Balance between absorption and scattering

Let us consider the ratio of the absorbed power \(P_{\text{abs}}\) to the total extracted power \(P_{\text{ext}}\). Dividing (10) by (9), we find

\[
\frac{P_{\text{abs}}}{P_{\text{ext}}} = \frac{\xi''}{\xi'' + \frac{k^3}{6\pi\epsilon_0}}
\]

(36)

This ratio can vary from 0 to 1, reaching 1 when \(\xi'' \to \infty\), but of course at this limit there is no absorption nor extinction. Next, let us find the ratio

\[
\frac{P_{\text{abs}}}{P_{\text{sc}}} = \frac{\xi'' 6\pi\epsilon_0}{k^3}
\]

(37)

This ratio can take any value from zero to infinity, but, again, in the limit of infinity the particle is simply not excited at all, and in the limit of zero there is no absorption.

For a specific (desired) value of the ratio \(P_{\text{abs}}/P_{\text{sc}}\) at a given frequency there is a corresponding value of the loss factor \(\xi''\) defined by (37). For this value of the power ratio and the loss factor the absorbed power (8) is defined by the value of the real part of the inverse polarizability \(\xi'\). Parameter \(\xi'^2\) can take any real value from zero (particle at resonance) to infinity (the polarizability equals zero). The first case corresponds to the maximum allowed received power for a given value of the power ratio (37), and the received power is zero in the last case. The maximal possible received power for a specific value of the ratio \(P_{\text{abs}}/P_{\text{sc}}\) can be written in the form

\[
P_{\text{abs}}|_{\xi' = 0} = \frac{P_{\text{abs}}}{P_{\text{sc}}} \frac{3}{2\pi \lambda^2} P_{\text{inc}}
\]

(38)

Here we have used (37) to express \(\xi''\) in terms of the power ratio and the wave number, and substituted that into (8). This dependence is illustrated by Fig. 1.

As was discussed above, the point of the maximal received power corresponds to \(P_{\text{abs}}/P_{\text{sc}} = 1\). To reach the maximum of the absorbed power for any desired ratio \(P_{\text{abs}}/P_{\text{sc}}\) one should
Figure 1: Dependence of the absorbed power on the ratio of absorbed and scattered powers. The maximum of absorbed power (16) is reached at $P_{\text{abs}} = P_{\text{sc}}$.

bring the particle to resonance and tune the loss factor using equation (37). For example, a short dipole antenna can be tuned to receive maximum possible power for any desired value of $P_{\text{abs}}/P_{\text{sc}}$ by loading it with complex impedance $Z_{\text{load}} = j\omega L + R$, where the inductance $L$ is chosen so that the inductive load compensates the input capacitance of the antenna ($\xi'$ in (32) is zero), and the resistor $R$ is either larger than the radiation resistance (28) or smaller than that. In the first case we can reach any point on the curve to the right from $P_{\text{abs}}/P_{\text{sc}} = 1$, which corresponds to the regime of a “cloaked sensor” [14,26], where the ratio $P_{\text{abs}}/P_{\text{sc}}$ is maximized. Using (33), we can find an elegant design formula for cloaked sensors realized as simple loaded dipole antennas. Substituting $\xi''$ of a loaded dipole antenna into (37), we find the required load resistance to realize a sensor which receives the maximum possible power at a given level of $P_{\text{abs}}/P_{\text{sc}}$:

$$R = R_{\text{rad}} \frac{P_{\text{abs}}}{P_{\text{sc}}}$$ \hspace{1cm} (39)

(we have assumed 100% efficiency, which means that Re($Z_{\text{inp}} - R_{\text{rad}}$) = 0 in (33)). The curve in Fig. 1 also shows how much the absorbed power will drop at any level of achieved scattering reduction. For example, with increasing the ratio $P_{\text{abs}}/P_{\text{sc}}$ 40 times, the absorbed power will drop at least about 10 times. In the case when $R < R_{\text{rad}}$, we reach the regime of the maximized scattered power for any given level of loss in the antenna. Similarly, for a plasmonic sphere we can substitute $\xi''$ from (26) into (37) and find that in order to ensure the desired value of the ratio between absorbed and scattered powers at frequency $\omega$, the sphere parameters should satisfy

$$\frac{\omega \Gamma}{\omega_p^2} = \frac{2}{9} (kr)^3 \frac{P_{\text{abs}}}{P_{\text{sc}}}$$ \hspace{1cm} (40)

Using (4) we can find the maximum of the scattered power reachable for a given level of
the ratio (37):

\[
P_{\text{sc}}|_{\xi'=0} = \frac{1}{1 + \frac{P_{\text{abs}}}{P_{\text{sc}}} \frac{3}{2\pi} \lambda^2 P_{\text{inc}}} \tag{41}
\]

This simply tells that scattering is always reduced with increasing absorption in the particle. In a similar way we can find the maximal possible absorbed power for a specific (desired) value of the ratio (36):

\[
P_{\text{abs}}|_{\xi'=0} = \frac{P_{\text{abs}}}{P_{\text{sc}}} \left(1 - \frac{P_{\text{abs}}}{P_{\text{sc}}} \right) \frac{3}{2\pi} \lambda^2 P_{\text{inc}} \tag{42}
\]

This dependence is shown in Fig. 2. As discussed above, the ultimate maximum of the received power (16) corresponds to \( P_{\text{abs}} = P_{\text{sc}} \) in (38) and (42).

More careful and detailed analysis of these limits and their implications for the design of small weakly-scattering antennas (cloaked sensors) can be found in paper [14].

Considering the special case of a plasmonic nanoparticle, one can substitute \( \xi'' \) for a plasmonic nanosphere from (26) and study how these ratios depend on the sphere radius and metal parameters. Basically, \( \xi'' \) is inversely proportional to \( r^3 \), thus, for small spheres it becomes large, both ratios grow, and it looks like the spheres absorb more. But of course the absorbed power in fact becomes small for large \( \xi'' \), as we know from (8).

5 Maximal power absorbed and scattered by small particles in planar regular arrays

Let us consider an infinite two-dimensional periodical array (square unit cells) of arbitrary electrically small particles and study the same limits for particles in regular arrays. This issue was considered in [10] in terms of the antenna array theory. We assume that the array is
excited by a normally incident plane wave and the array period \( a \) is smaller than \( \lambda \), so that no diffraction lobes appear. Similarly to the previous sections, we assume that the particles are small dipole particles excited by electric fields at the position of the particles. The periodical grid can be conveniently modelled by its grid impedance \( Z_g \) [6, Ch. 4], which relates the total (surface-averaged) electric field \( E_{\text{tot}} \) in the array plane and the averaged surface current density \( J = \frac{j\omega p}{a^2} \):

\[
E_{\text{tot}} = Z_g J
\]

In terms of the polarizability of the individual particles in free space \( \alpha \), defined in (7), the grid impedance reads [6, Eq. (4.95)]

\[
Z_g = Z'_g + j Z''_g = -j \frac{a^2}{\omega} \left( \hat{\xi}' + j \xi'' \right)
\]

where \( a \) is the array period and the hat indicates that the real part of the inverse polarizability is modified due to reactive-field interactions between the particles in the infinite array. For electrically dense arrays of small particles this interaction can be estimated analytically as

\[
\hat{\xi}' = \xi' - \text{Re}(\beta) \approx \xi' - \frac{0.36}{\epsilon_0 a^2}
\]

(for our consideration here the value of the real part of the interaction constant \( \beta \) will be not important).

Next we write the reflection and transmission coefficients (e.g., [6, Eq. (4.43)])

\[
R = -\frac{\eta}{\eta^2 + Z_g}
\]

\[
T = \frac{Z_g}{\eta^2 + Z_g}
\]

where, as before, \( \eta = \sqrt{\frac{\mu_0}{\epsilon_0}} \). The absorption coefficient reads

\[
L = 1 - |R|^2 - |T|^2 = \frac{\eta Z'_g}{Z''_g + \left( \frac{\eta}{2} + Z'_g \right)^2}
\]

and the power absorbed by one unit area of the array is

\[
P_{\text{abs}} = LP_{\text{inc}} = \frac{\eta Z'_g}{Z''_g + \left( \frac{\eta}{2} + Z'_g \right)^2} P_{\text{inc}} = \frac{k}{2 \epsilon_0 a^2} \frac{\xi''}{\xi'^2 + \left( \frac{k}{2 \epsilon_0 a^2} + \xi'' \right)^2} P_{\text{inc}}
\]

Obviously, the highest value of the absorbed power is \( P_{\text{abs max}} = \frac{1}{2} P_{\text{inc}} \), and it is reached if

\[
Z''_g = 0, \quad Z'_g = \frac{\eta}{2}
\]
or, in terms of the particle polarizability,

\[ \hat{\xi}' = 0, \quad \hat{\xi}'' = \frac{k}{2\epsilon_0 a^2} \]  

The first condition physically means that the particles are at resonance (including the reactive coupling with all the other particles in the array). The second condition means that the absorption in the array is optimized so that the absorbed power is equal to the re-radiated power (this value of \( \hat{\xi}'' \) is equal to the term in the imaginary part of the interaction constant which measures the plane-wave power radiated by the unit area of the infinite array, see \([6, \text{Eq. (4.88)}]\)).

Let us discuss the balance between the absorbed and re-radiated powers in more detail. We can find the power scattered by a unit area of the infinite array in the same way as for a single particle: as the difference of the power extracted from the incident field and the absorbed power. The extracted power is, similarly to (2),

\[ P_{\text{ext}} = \frac{1}{2} \text{Re} (J^* E_{\text{inc}}) \]  

where \( J \) is the surface current density. Substituting the total tangential surface-averaged electric field in the array plane in terms of the incident field and the plane-wave field created by the induced surface currents as \( E_{\text{tot}} = E_{\text{inc}} - \eta J \) into (43), we can express the induced current density in terms of the incident field:

\[ J = \frac{1}{Z_g + \frac{\eta}{2}} E_{\text{inc}} \]  

Substitution into (52) gives the extracted power

\[ P_{\text{ext}} = \eta \frac{Z_g' + \frac{\eta}{2}}{Z_g'' + \left( \frac{\eta}{2} + Z_g' \right)^2} P_{\text{inc}} = \frac{k}{\epsilon_0 a^2} \frac{\hat{\xi}'' + \frac{k}{2\epsilon_0 a^2}}{\hat{\xi}'' + \left( \hat{\xi}'' + \frac{k}{2\epsilon_0 a^2} \right)^2} P_{\text{inc}} \]  

and the scattered (re-radiated) power

\[ P_{\text{sc}} = P_{\text{ext}} - P_{\text{abs}} = \frac{\eta^2}{Z_g'' + \left( \frac{\eta}{2} + Z_g' \right)^2} P_{\text{inc}} = \frac{1}{2} \left( \frac{k}{\epsilon_0 a^2} \right)^2 \frac{1}{\hat{\xi}'' + \left( \hat{\xi}'' + \frac{k}{2\epsilon_0 a^2} \right)^2} P_{\text{inc}} \]  

(we have used formula (49) for the absorbed power). Note that this definition implies that the array re-radiates symmetrically in the backward and forward directions (the same result (55) for the scattered power \( P_{\text{sc}} \) can be obtained as \( P_{\text{sc}} = 2|R|^2 P_{\text{inc}}, \text{see (46)} \)). The power which is actually propagating in the space behind the array is the result of interference of this re-radiated plane wave and the incident plane wave, and it is given by \( |T|^2 P_{\text{inc}} \). For this reason, the re-radiated power can take values between zero and \( 2P_{\text{inc}} \). The last value corresponds to the case of total reflection, where the array generates reflected power which is equal to the incident power and creates secondary field behind the array which cancels the incident field.
there. This is the same definition as the definition of the total scattering cross section in the theory of diffraction (the total scattering cross section of a large conducting body is equal to the double geometrical cross section).

Now we can study the ratios

\[
\frac{P_{\text{abs}}}{P_{\text{abs}} + P_{\text{sc}}} = \frac{P_{\text{abs}}}{P_{\text{ext}}} = \frac{Z'_g}{Z'_g + \frac{\eta}{2}} = \xi'' + \frac{k}{2\varepsilon_0 a^2}
\]

and

\[
\frac{P_{\text{abs}}}{P_{\text{sc}}} = \frac{2Z'_g}{\eta} = \xi'' \frac{2\varepsilon_0 a^2}{k}
\]

(we have used the relation between the surface resistance of the homogenized sheet and the loss factor of a single particle $Z'_g = \xi'' a^2/\omega$ (44), [6, eq. (4.95)]).

These results are analogous to those for single particles in free space (36) and (37): The single-particle radiation damping factor $\frac{k}{\varepsilon_0 a^2}$ which replaces the term $\frac{k}{2\varepsilon_0 a^2}$ which measures the plane-wave power radiated from a unit area of the infinite array. These are the two terms of the imaginary part of the interaction constant for infinite arrays of dipole particles, see [6, Eq. (4.89)]. We can make a similar conclusion about the ratio between the absorbed and scattered powers: Because the surface resistance $Z'_g$ and take any value between zero (lossless particles) and infinity (no particles at all), the ratio (57) can take any value between zero and infinity. On the other hand, the value of the absorbed power (49) cannot be larger than one half of the incident power, and it tends to zero when ratio (57) tend to infinity.

In the same way as for a single particle we can find the maximum received power per unit area achievable at a given value of the ratio (57). Using (49) together with (57), we arrive to

\[
P_{\text{abs}}|_{Z''_g=0} = \frac{P_{\text{abs}}}{P_{\text{sc}}} \left(1 + \frac{P_{\text{abs}}}{P_{\text{sc}}}\right)^2 2P_{\text{inc}}
\]

Tuning the particle reactance, the absorbed power changes between zero and the extremal value given by the above relation. We see again that the ultimate maximum is $0.5 P_{\text{inc}}$, reachable for $P_{\text{abs}} = P_{\text{sc}}$. One can also study how the scattering efficiency is limited by losses in the particles, calculating the maximal scattered power for a fixed level of the ratio (57):

\[
P_{\text{sc}}|_{Z''_g=0} = \frac{1}{\left(1 + \frac{P_{\text{abs}}}{P_{\text{sc}}}\right)^2} 2P_{\text{inc}}
\]

Let us look at how much power is absorbed by each particle in the array and compare with the absorption by the same particle in free space (8). The power absorbed by one unit area ($1 \text{ m}^2$) of the array we get multiplying $L$ in (48) by the incident power density $P_{\text{inc}} = \frac{1}{2} |E_{\text{inc}}|^2$. Next, the power per particle we get multiplying by the unit-cell area $a^2$. The result is

\[
P_{\text{abs}}|_{\text{per particle}} = \frac{a^2 Z'_g}{Z''_g + \left(\frac{\eta}{2} + \frac{\varepsilon''}{2}\right)} |E_{\text{inc}}|^2 = \frac{\frac{\omega}{2} \xi''}{\xi'^2 + \left(\xi'' + \frac{\eta}{2a^2}\right)^2} |E_{\text{inc}}|^2 = \frac{\frac{\omega}{2} \xi''}{\xi'^2 + \left(\xi'' + \frac{k}{2a^2}\varepsilon_0\right)^2} |E_{\text{inc}}|^2
\]

(60)
(we have substituted the value of $Z_g$ in terms of the particle polarizability using (44)). Substitution of the optimal value of the particle polarizability (51) gives the maximum power received by one particle

$$P_{\text{abs max}} = \frac{a^2}{4\eta} |E_{\text{inc}}|^2 = \frac{a^2}{2} P_{\text{inc}} \quad (61)$$

Thus, the extreme value of the effective area of each particle acting as a receiving antenna in an infinite array is

$$\sigma_{\text{abs max}} = \frac{a^2}{2} \quad (62)$$

This agrees with the fact that an infinite array of electric dipole antennas can receive at maximum one half of the incident power (the area of the unit cell is $a^2$, but the effective receiving area of each dipole is $a^2/2$). It is interesting to observe that particles in the grid absorb as much as the same particles in free space when

$$ka = \sqrt{3\pi} \quad (63)$$

For denser grids each particle dissipates smaller power than the same particle in free space illuminated by the same incident field.

The power scattered by each particle in an infinite grid we can find multiplying (55) by the unit-cell area $a^2$:

$$P_{\text{sc}}|_{\text{per particle}} = \frac{\frac{a^2 \eta^2}{2}}{Z_g''^2 + \left( \frac{\eta}{2} + Z_g' \right)^2} P_{\text{inc}} \quad (64)$$

The maximal scattered power is achieved when $Z_g = 0$, and it is equal to $2a^2P_{\text{inc}}$: The effective scattering area of each unit cell is twice as large as the physical area. This means that the array fully reflects the incident plane wave and creates a complete shadow behind the array (the reflection coefficient $R = -1$ and the transmission coefficient $T = 0$, see (46) and (47)). In terms of the particle polarizability this regime corresponds to lossless ($\xi'' = 0$) and resonant (in the array) particles ($\xi' = 0$). Continuing the comparison of properties of individual particles in free space and the same particles in regular arrays, we see that also the scattered powers are the same if (63) is satisfied. Note that the ratio between the maximum scattered and received powers for regular grids is the same as for individual particles in free space (four).

The largest distance between small particles in a regular array such that no grating lobes still appear equals $\lambda$ (for the normal incidence). At this period, the extreme values of absorption and scattering cross sections per particle in the array become $\sigma_{\text{abs max}} = \frac{1}{2} \lambda^2$, $\sigma_{\text{abs max}} = 2 \lambda^2$, which are larger values than for the same particles in free space. Thus, coherent interactions in the grid can enhance interactions of the particles with the fields. On the other hand, for dense arrays ($a \ll \lambda$), each particle in the array absorbs and scatters less than it would do individually in free space.

Finally we note that adding an array of similarly optimized magnetic dipoles to the array we can receive 100% of the incident power, which is also a known result [9, 10].
6 Conclusion

The limitations on the extinction, absorption, and scattering properties of arbitrary small scatterers or antennas come from the fact that the maximum amplitude of the induced dipole moment is limited by the inevitable scattering loss. Since the scattering loss factor is the same for all dipole scatterers (it depends only on the frequency), the limitations are very general. The only assumptions (besides linearity) are that the particle is small (reacts as an electric dipole) and that there is no bi-anisotropy and no magnetic polarization. Absorption and scattering by a magnetic dipole particle obeys the same rules, as follows from duality (the corresponding formulas can be obtained replacing $\epsilon_0$ by $\mu_0$ and vice versa. Combining a magnetic dipole with an electric dipole, we can double the received power. More general bi-anisotropic particles are considered in recent paper [15].

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