Huygens’ principle, the free Schrödinger particle and the quantum anti–centrifugal force

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Huygens’ principle following from the d’Alembert wave equation is not valid in two–dimensional space. A Schrödinger particle of vanishing angular momentum moving freely in two dimensions experiences an attractive force – the quantum anti–centrifugal force – towards its center. We connect these two phenomena by comparing and contrasting the radial propagators of the d’Alembert wave equation and of a free non–relativistic quantum mechanical particle in two and three dimensions.

I. INTRODUCTION

In the year 1917 Paul Ehrenfest addressed the question: Why is the space we live in three–dimensional? He provided three physical phenomena known at his time that crucially depend on the number of space dimensions: (i) The generalized electrostatic attraction leads only in two and three dimensions to stable circular orbits, (ii) only in three dimensions do we find as many electric as magnetic field components and (iii) Huygens’ principle, based on the d’Alembert wave equation, is only valid in a world with an odd number of spatial dimensions.

In the present paper we connect this phenomenon of the breakdown of Huygens’ principle in two dimensions to the surprising appearance of an attractive force acting on a free particle of vanishing angular momentum in two dimensions. We show that both phenomena have the same physical origin: Interference of waves.

We compare and contrast the Green’s functions of the free Schrödinger equation corresponding to vanishing angular momentum in two and three dimensions. Since the rotational symmetry of the initial wave function is preserved in the time evolution, we can confine ourselves to the Green’s function of the radial motion. We derive analytical expressions for these quantities by taking the product of the Green’s functions of the individual one–dimensional motion along each Cartesian axis and integrating over the angular part.

We show that the radial Green’s function in three dimensions is the difference of two Green’s functions of two free one–dimensional motions: One Green’s function describes the free motion from the initial radius \( \rho \) to the radius \( r \). The other Green’s function corresponds to the motion of the particle starting at the mirror image \( -\rho \) of the initial radius \( \rho \) reaching \( r \). In the three–dimensional space the phase difference between these two Green’s functions is \( \pi \).

In contrast, the corresponding Green’s function of the two-dimensional radial motion is more complicated. It is again the difference of two one-dimensional Green’s functions. However, they do not correspond to free motion anymore. Moreover, the phase difference between the two contributions is now \( \pi/2 \). This feature reflects the fact that in two dimensions the radial Schrödinger equation for vanishing angular momentum enjoys an additional potential that is attractive and of quantum origin. It corresponds to the quantum anti-centrifugal force.

Whereas in one and three dimensions this potential is absent, it reappears in four and higher dimensions. However, it is repulsive. In contrast to the breakdown of Huygens’ principle, which takes place for all even dimensions, this additional potential displays the unique behaviour of being attractive only in two dimensions.

Our article is organized as follows: In section 2 we briefly review the physical origin of the breakdown of Huygens’ principle and summarize the essential features of the quantum anti–centrifugal potential. Here we confine our discussion to two–and three–dimensional situations. In section 3 we derive analytical expressions for the radial Green’s function of the free Schrödinger equation in two and three dimensions. In the appendix we make contact with the corresponding expressions for the d’Alembert wave equation.

II. TWO DIMENSIONS ARE VERY DIFFERENT FROM THREE

In the present section we review two phenomena which appear to be unrelated at first sight– the breakdown of Huygens’ principle applied to d’Alembert waves in two dimensions and the quantum anti–centrifugal force acting on a Schrödinger particle moving freely in two dimensions. Here we do not dwell on mathematical formalism but focus on the key ideas. For the sake of completeness we include detailed derivations of the propagators of the d’Alembert wave equation in the appendix.
A. Breakdown of Huygens’ principle

In [1] Ehrenfest summarized earlier work by J Hadamard on the propagation of a signal according to the d’Alembert wave equation. When we express his findings in today’s language, he investigated the Green’s function of the d’Alembert equation: A delta function perturbation originating at time $t = 0$ from the origin travels in three dimensions as a spherical wave which at time $t$ had reached $r$. In this case the perturbation always keeps its shape, since for $t > 0$ the propagator in three dimensions reads

$$G^{(3)}_{\text{ret}}(t, r) = \frac{\delta(ct - r)}{4\pi r}$$

However, in two dimensions the delta function perturbation does not propagate as a well localized circular wave, but contains a tail, a so–called wake field [3]. As shown in the appendix, the propagator takes the form

$$G^{(2)}_{\text{ret}}(t, r) = \Theta((ct)^2 - r^2) \sqrt{(ct)^2 - r^2}.$$ 

Due to the Heaviside step function there exists a sharp wave front. Moreover, together with the square root in the denominator it creates a long tail of excitation in the inside domain of the propagating circular ring [1]. This wake field can be observed with water waves after a stone has excited a wave in an initially quiet pond.

It is interesting to trace the mathematical reason for the occurrence of the square root in the propagator back to the dimensionality of space. Indeed, we show in the appendix that the square root stems from the fact that the two–dimensional volume element reads $dr r d\phi$. In contrast the three–dimensional volume element $dr r d\theta d\phi$ has in the angle variable an additional factor $\sin \theta$. It is because of this extra factor that the interference of plane waves in three dimensions gives rise to a spherical Bessel function

$$\frac{1}{2} \int_0^\pi d\theta \sin \theta e^{-ikr \cos \theta} = \frac{\sin kr}{kr} \equiv j_0(kr)$$

whereas in two dimensions we find an ordinary Bessel function

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi e^{-ikr \cos \varphi} \equiv J_0(kr)$$

The spherical Bessel function

$$j_0(kr) = \frac{1}{2ikr} (e^{ikr} - e^{-ikr})$$

originating from the interference in three dimensions is an exact superposition of an incoming and an outgoing spherical wave with a fixed phase difference of $\pi$.

In contrast, the familiar asymptotic expansion [5] of the ordinary Bessel function

$$J_0(kr) \approx \frac{2}{\sqrt{\pi kr}} \cos (kr - \pi/4) = \frac{1}{2\pi kr} e^{-i\pi/4} (e^{ikr} - e^{-ikr})$$

suggests that the interference of plane waves in two dimensions gives rise to an approximate superposition of an incoming and an outgoing cylindrical wave with an asymptotic phase difference of $\pi/2$. However, equation (4) is only the most elementary asymptotic approximation of the Bessel function. In the literature there exists a variety of techniques [6] to find more accurate expansions. Independent of their specific form, they clearly indicate that the result of the interference of plane waves in two dimensions is not a pure circular wave. This also manifests itself in the fact that the zeros of $J_0$ are not equally distributed like $j_0$ but rather bunch towards the origin [2].

1 In his novel *Flatland* E Abbott [4] describes the phenomenon that people living in a two–dimensional world have a dim vision because of fog. They see unsharp surroundings. We do not know if the author was aware of the mathematical and physical details of Green’s functions in two dimensions.
B. Quantum anti–centrifugal force

We now turn to a seemingly different topic: The Schrödinger equations of a free particle of mass $M$ and energy $E > 0$ in two and three dimensions. We recall that in two dimensions the radial wave equation for vanishing angular momentum ($m = 0$) contains an attractive potential. This potential is related to interference of plane waves in two dimensions as expressed by the Sommerfeld representation [2] of the zeroth Bessel function. In this way we make the connection to Huygens’ principle.

We start from the time independent Schrödinger equation

$$\frac{-\hbar^2}{2M} \Delta \Psi(x, y) = E \Psi(x, y)$$

of the free particle and introduce polar coordinates $r$ and $\varphi$ by the transformation $x \equiv r \cos \varphi$ and $y \equiv r \sin \varphi$. With the help of the ansatz

$$\Psi(x, y) = e^{im\varphi} \frac{u_m(r)}{\sqrt{r}}$$

we arrive at the radial Schrödinger equation

$$\left\{ \frac{d^2}{dr^2} + \frac{2M}{\hbar^2} \left[ E - V_m^{(2)}(r) \right] \right\} u_m^{(2)}(r) = 0$$

Here we have introduced the effective potential [7]

$$V_m^{(2)}(r) \equiv \frac{\hbar^2}{2M} \frac{m^2 - 1/4}{r^2}$$

The first term in $V_m^{(2)}$, proportional to $m^2$, is the potential which describes the familiar centrifugal force. The negative correction term $-1/4$ gives rise to a centripetal force which from this point on we shall call a quantum anti-centrifugal force to emphasize that its binding power arises from quantum mechanics.

The effect of this contribution stands out most clearly for a particle with zero angular momentum, that is $m = 0$. In this case the effective potential

$$V_Q(r) \equiv V_0^{(2)}(r) = -\frac{\hbar^2}{2M} \frac{1}{4r^2}$$

(5)

becomes attractive and the radial wave function reads

$$u_0(r) = \sqrt{r} J_0(kr)$$

Here and throughout the paper the subscript 0 on the wave functions indicates vanishing angular momentum.

Hence, the bunching effect of the zeros of $J_0$ takes on a new meaning. It reflects the fact that there is an attractive force towards the center $x = y = 0$. Indeed, the force corresponding to $V_Q$ causes a ring–shaped wave packet to spread in asymmetric way: It initially spreads faster towards the center than towards the outside [8].

In three dimensions no such quantum potential exists. We recall [9] that in this case the effective potential is proportional to the quantum mechanical square $l(l + 1)$ of the angular momentum quantum number $l$. Hence, for $l = 0$ the potential vanishes and no asymmetric spreading takes place.

III. PROPAGATORS

This dimensionality–dependent spreading is a consequence of the propagators being different in different dimensions. In this section we compare and contrast the radial propagators in two and three dimensions. Throughout this section we confine ourselves to radially symmetric wave functions, that is, wave functions of vanishing angular momentum. This symmetry is preserved under time evolution, therefore it suffices to consider the radial propagator.

We show that the radial propagator of a free quantum mechanical particle in three dimensions is the difference between two propagators of the free one–dimensional radial motion. However, in two dimensions the propagator is different, reflecting the attractive quantum anti–centrifugal force. The mathematical reason for this difference is exactly the same as in the d’Alembert propagators: It is the difference between a spherical and an ordinary Bessel function.
A. One dimension

In this section we lay the foundations for the following sections by briefly recalling the Green’s function $G^{(1)}_{S}$ of the free Schrödinger particle of mass $M$. The Green’s function

$$G^{(1)}_{S} (x, t | \tilde{x}, t = 0) \equiv \mathcal{N}(t) e^{i\alpha(t)(x - \tilde{x})^2} \tag{6}$$

with the normalization constant

$$\mathcal{N}(t) \equiv \sqrt{\frac{\alpha(t)}{i\pi}} \tag{7}$$

and

$$\alpha(t) \equiv \frac{M^2}{2\hbar t} \tag{8}$$

allows us to propagate the initial wave function $\Psi^{(1)}_0 (x, t = 0) \equiv \Phi^{(1)}_0 (x)$ to a later time, such that

$$\Psi^{(1)}(x, t) = \int_{-\infty}^{\infty} d\tilde{x} \ G^{(1)}_{S} (x, t | \tilde{x}, t = 0) \Phi^{(1)}_0 (\tilde{x}) \tag{9}$$

For the remainder of the article it is useful to recall that for $t \rightarrow 0$ the parameter $\alpha$ and the normalization constant $\mathcal{N}$ approach infinity.

B. Two dimensions

We now use the expression (6) for the propagator $G^{(1)}_{S}$ in one dimension to derive the Green’s function of the radial motion in two dimensions. For this purpose we take the product of the one-dimensional Green’s functions of the motions along the $x$- and the $y$-axes together with the initial wave function and integrate over the angular and radial variables. We then express the wave functions by the radial wave functions.

1. Radial propagator

We start from the generalization

$$\Psi^{(2)}_0 (x, y; t) = \int_{-\infty}^{\infty} d\tilde{x} \int_{-\infty}^{\infty} d\tilde{y} \ G^{(1)}_{S} (x, t | \tilde{x}, 0) G^{(1)}_{S} (y, t | \tilde{y}, 0) \Phi^{(2)}_0 (\sqrt{\tilde{x}^2 + \tilde{y}^2}) \tag{10}$$

of the propagation equation (9) to two dimensions. Here we have considered a rotationally symmetric initial wave function $\Psi^{(1)}_0 (\tilde{x}, \tilde{y}; t = 0) \equiv \Phi^{(1)}_0 (\rho)$, which depends on the radial variable $\rho \equiv \sqrt{\tilde{x}^2 + \tilde{y}^2}$ only.

When we substitute the expression (6) for the one-dimensional Green’s function $G^{(1)}_{S}$ into (10) and introduce polar coordinates $\rho$ and $\varphi$ we arrive at

$$\Psi^{(2)}_0 (x, y; t) = \Psi^{(2)}_0 (r; t)$$

$$= \int_0^{\infty} d\rho \mathcal{N} e^{i\alpha(r^2 + \rho^2)} \mathcal{N} \sqrt{\rho} \int_0^{2\pi} d\varphi \ e^{-i2\alpha r \rho \cos \varphi} \sqrt{\rho} \Phi^{(2)}_0 (\rho) \tag{11}$$

The wave function $\Psi^{(2)}_0 (x, y; t)$ depends on the coordinates $x$ and $y$ only through the radial coordinate $r \equiv \sqrt{x^2 + y^2}$. Hence, the axial symmetry of the initial wave function is preserved under time evolution.

We recall the Sommerfeld representation (2) of the ordinary Bessel function $J_0$, and the propagation equation reduces to

$$\Psi^{(2)}_0 (r; t) = \int_0^{\infty} d\rho \mathcal{N} e^{i\alpha(r^2 + \rho^2)} \frac{1}{\sqrt{\rho}} \sqrt{\pi \alpha \rho} J_0 [2\alpha r \rho] \sqrt{\rho} \Phi^{(2)}_0 (\rho) \tag{12}$$

where we have made use of the definition (7) of $\mathcal{N}(t)$. With the help of the radial wave functions
Another manifestation of the interference of plane waves in two dimensions giving rise to an ordinary Bessel function.

High potential well at \( \rho \) that the radial coordinate is only defined for positive values. In other words, there is an infinitely steep and infinitely vanishing angular momentum is not simply that of free motion, but is modified by \( H \).

We find

\[
\psi^{(2)}_0 (r; t) = \int_0^\infty d\rho N e^{i\alpha (r^2 + \rho^2)} \frac{1}{\sqrt{\pi}} e^{i\alpha r^2} J_0 [2 \alpha \rho^2] \psi^{(2)}_0 (\rho)
\]

This relation suggests the formula

\[
g^{(2)} (r, t | \rho) \equiv N(t) e^{i\alpha (r^2 + \rho^2)} \frac{1}{\sqrt{\pi}} e^{i\alpha r^2} 2 J_0 [2 \alpha \rho^2]
\]

for the Green’s function of the radial motion of a free particle in two dimensions. The key feature of this result is the emergence of the ordinary Bessel function \( J_0 \).

2. Alternative form

It is instructive to represent \( \psi^{(2)}_0 \) the Bessel function

\[
J_0 (\xi) = \frac{1}{2} \left[ H_0^{(1)} (\xi) + H_0^{(2)} (\xi) \right]
\]

in terms of the two Hankel functions

\[
H_0^{(1)} (\xi) = \sqrt{\frac{2}{\pi \xi}} e^{i(\xi - \pi/4)H} (\xi)
\]

and

\[
H_0^{(2)} (\xi) = \sqrt{\frac{2}{\pi \xi}} e^{-i(\xi - \pi/4)H} (\xi)
\]

where we have introduced the abbreviation

\[
H (\xi) \equiv \frac{1}{\sqrt{\pi}} \int_0^\infty d\xi \frac{e^{-\xi}}{\sqrt{1 - i\xi/2\xi}} = \frac{2}{\sqrt{\pi}} \int_0^\infty d\tau \frac{e^{-\tau^2}}{\sqrt{1 - i\tau^2/(2\xi)}}
\]

We substitute these expressions into the formula (14) for the propagator and complete the squares. We arrive at

\[
g^{(2)} = N e^{i\alpha (r^2 + \rho^2)} H (2 \alpha \rho^2) - i N e^{i\alpha (r + \rho)^2 H^* (2 \alpha \rho^2)
\]

where we have used the relation

\[
\frac{1}{\sqrt{t}} = \left( e^{-i\pi/2} \right)^{1/2} = e^{-i\pi/4}
\]

When we compare the radial Green’s function \( g^{(2)} \) in two dimensions (10) to the Green’s function \( G_S^{(1)} \) in one dimension (8) we find the representation

\[
g^{(2)} (r, t | \rho) = G_S^{(1)} (r, t | \rho) \mathcal{H} [2 \alpha (t) r \rho] - i G_S^{(1)} (r, t | -\rho) \mathcal{H}^* [2 \alpha (t) r \rho]
\]

Hence the Green’s function \( g^{(2)} \) of the radial motion is the difference of two one-dimensional Green’s functions: One corresponds to the motion starting at \( \rho \), the other at the mirror image \(-\rho\). The two functions reflect the fact that the radial coordinate is only defined for positive values. In other words, there is an infinitely steep and infinitely high potential well at \( \rho = 0 \) and the radial wave function has to vanish there. This fact guarantees that the radial momentum operator is Hermitian for \( r \geq 0 \).

Moreover, there is a phase difference of \( \pi/2 \) between the two contributions in (17). This phase difference is just another manifestation of the interference of plane waves in two dimensions giving rise to an ordinary Bessel function as discussed in section (8).

However, the most important feature of (17) is the fact that each contributing term is the product of the Green’s function of a free motion and the function \( \mathcal{H} \). Hence the propagator of a free particle in two dimensions and of vanishing angular momentum is not simply that of free motion, but is modified by \( \mathcal{H} \).
3. Short time limit

We can gain deeper insight into the correction factor $\mathcal{H}$ by considering the short time limit, that is $t \to 0$. From section II, we recall that in this case $\alpha \to \infty$. The argument $\mathcal{H}$ is the product of $\alpha$ and the radial coordinates $r$ and $\rho$. Hence we are dealing with a rather subtle limit.

For $r \neq 0$ and $\rho \neq 0$ we can consider the asymptotic limit of $\mathcal{H}$ for large argument $\xi$. When we expand the second square root in the definition (15) of $\mathcal{H}$ we find

$$\mathcal{H}(\xi) \approx \frac{2}{\sqrt{\pi}} \int_0^\infty d\tau e^{-\tau^2} \left( 1 + \frac{i\tau^2}{4\xi} \right)$$

With the help of the integral relations

$$\frac{2}{\sqrt{\pi}} \int_0^\infty d\tau e^{-\tau^2} = 1$$

and

$$\frac{2}{\sqrt{\pi}} \int_0^\infty d\tau e^{-\tau^2} \tau^2 = \frac{1}{2}$$

we arrive at

$$\mathcal{H}(\xi) \approx 1 + \frac{1}{8\xi}$$

We recall the definition (8) of $\alpha$ and the correction factor $\mathcal{H}$ to the free propagator reads

$$\mathcal{H}[2\alpha(t)rp] = \mathcal{H}\left( \frac{M}{\hbar} rp \right) = 1 + \frac{i}{\hbar} \frac{1}{2M} \frac{1}{4\pi r} t$$

The second term here is proportional to the quantum anti-centrifugal potential $V_Q$ defined in (5) and the function $\mathcal{H}$ takes the compact form

$$\mathcal{H}[2\alpha(t)rp] = 1 - \frac{i}{\hbar} V_Q(\sqrt{rp})t \simeq \exp\left[ -\frac{i}{\hbar} V_Q(\sqrt{rp})t \right]$$

(18)

This formula shows clearly that for short times the radial motion in two dimensions is determined by the quantum anti-centrifugal potential $V_Q$.

We emphasize that the expression (18) is only valid for short times and for radial coordinates $r \neq 0$ and $\rho \neq 0$. For $t \neq 0$ and $r = \rho = 0$ the radial Green’s function $g^{(2)}$ vanishes, that is

$$g^{(2)}(r = 0,t | \rho) = g^{(2)}(r, t | \rho = 0) = 0$$

as required by the boundary condition $v_0^{(2)}(r = 0) = 0$ and $v_0^{(2)}(\rho = 0) = 0$. Indeed, this property follows from the exact expression (14) for $g^{(2)}$ in terms of the Bessel function.

C. Three dimensions

We now turn to the case of a free particle in three dimensions. Again we consider a wave function $\Psi^{(3)}_0(\tilde{x}, \tilde{y}, \tilde{z} ; t = 0) = \Phi^{(3)}_0(\sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2})$ that at $t = 0$ is rotationally symmetric, corresponding to vanishing angular momentum, that is, $l = 0$. Time evolution preserves this symmetry. In order to find the Green’s function in three dimensions we take the product of the Green’s functions $G_S^{(1)}$ of the free motions along the $x$-, $y$-, and $z$-axes together with the initial wave function, and integrate over spherical coordinates $\rho$, $\theta$ and $\varphi$.

We start from the expression

$$\Psi^{(3)}_0(x, y, z ; t) = \int_0^\infty d\tilde{x} \int_0^\infty d\tilde{y} \int_0^\infty d\tilde{z}
\ G_S^{(1)}(x, t | \tilde{x}) G_S^{(1)}(y, t | \tilde{y}) G_S^{(1)}(z, t | \tilde{z}) \Phi^{(3)}_0 \left( \sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} \right)$$

(19)
for the propagation of the wave function \( \Phi_0^{(3)} \). With the help of the explicit formula (18) for the Green’s function \( G_S^{(1)} \) and spherical coordinates we find

\[
\Psi_0^{(3)}(x, y, z; t) = \Psi_0^{(3)}(r; t) = \int_0^\infty d\rho N e^{i\alpha(r^2+\rho^2)} N^2 \rho \int_0^\pi d\theta \sin \theta e^{-i2\alpha r \rho \cos \theta} \int_{-\pi}^\pi d\varphi \rho \Phi_0^{(3)}(\rho)
\]  

(20)

Since the right hand side depends on the coordinates \( x, y, \) and \( z \) only via the radial coordinate \( r = \sqrt{x^2 + y^2 + z^2} \), the wave function \( \Psi_0^{(3)} \) at time \( t \) only depends on \( r \). Hence, the radial symmetry of the initial wave function is preserved during time evolution.

The integrand is independent of \( \varphi \) and we can immediately perform this and the \( \theta \) integration. Indeed, with the help of the integral representation (14) of the spherical Bessel function \( j_0 \) we arrive at

\[
\Psi_0^{(3)}(r; t) = \int_0^\infty d\rho N e^{i\alpha(r^2+\rho^2)} 4\pi N^2 \rho j_0(2\alpha r \rho) \rho \Phi_0^{(3)}(\rho)
\]

When we introduce the radial wave functions

\[
u_0^{(3)}(\rho) \equiv \Psi_0^{(3)}(\rho, t = 0) \equiv \rho \Phi_0^{(3)}(\rho)
\]

in three dimensions corresponding to vanishing angular momentum we find the propagation equation

\[
u_0^{(3)}(r, t) = \int_0^\infty d\rho \rho \nu_0^{(3)}(\rho, t) = \frac{2}{i} 2\alpha(t) r \rho j_0(2\alpha(t) r \rho)
\]

This approach yields the Green’s function

\[
g^{(3)}(r, t \mid \rho) \equiv N(t) e^{i\alpha(t)(r^2+\rho^2)} \frac{2}{i} 2\alpha(t) r \rho j_0(2\alpha(t) r \rho)
\]  

(21)

for the radial motion in three dimensions. Here we have used the definition (12) of the normalization constant \( N \).

When we compare the expression (22) for the radial propagator in three dimensions to the corresponding Green’s function (14) in two dimensions we find, apart from some factors, two changes: (i) In the transition from two to three space dimensions we have replaced the ordinary Bessel function \( J_0 \) by a spherical Bessel function \( j_0 \), (ii) moreover, the square root of \( i \) in the denominator in equation (14) is now replaced by \( i \).

These two at first sight minor substitutions have dramatic consequences. As discussed in equation (13) the spherical Bessel function is a superposition of an incoming and an outgoing spherical wave with a phase difference of \( \pi \). When we use this representation of \( j_0 \), we find the Green’s function

\[
g^{(3)}(r, t \mid \rho) \equiv N(t) \left[ e^{i\alpha(t)(r^2-\rho^2)} - e^{i\alpha(t)(r^2+\rho^2)} \right]
\]

for the radial motion in three dimensions. We emphasize that this result is valid for all times.

Similar to the Green’s function (17) in two dimensions, the corresponding expression in three dimensions is again the difference

\[
g^{(3)}(r, t \mid \rho) = G_S^{(1)}(r, t \mid \rho) - G_S^{(1)}(r, t \mid -\rho)
\]

between two propagators of one-dimensional motion: One corresponds to the motion starting at \( \rho \) and the other one at \(-\rho\). However, in contrast to the Green’s function in two dimensions, now we deal with a purely free motion represented by the Green’s function \( G_S^{(1)} \): The modification factor \( H \) of two dimensions is absent since in three dimensions there is no quantum anti-centrifugal potential.

Moreover, there is a phase shift of \( \pi \) between the two contributions. Its origin is clearly the interference of plane waves in three dimensions giving rise to a spherical Bessel function.
IV. CONCLUSIONS

We conclude by emphasizing that the radial propagator of a free particle of vanishing angular momentum moving in three dimensions is determined by the spherical Bessel function. It originates from the interference of plane waves in three dimensions. Its explicit representation in spherical waves shows that the radial propagator is the interference of two propagators of free motion in one dimension. The phase difference between the two contributions is $\pi$.

In two dimensions the radial propagator is determined by the ordinary Bessel function. It originates from interference of plane waves in two dimensions. Since an ordinary Bessel function is not a superposition of two cylindrical waves but involves correction terms, the propagator is not a superposition of propagators corresponding to free, one-dimensional motion. This feature reflects the presence of the quantum anti-centrifugal force in two dimensions.

In mathematical terms, the dramatic difference between the radial propagators in two and in three dimensions results from the different area or volume elements. A comparison between the corresponding equations (11) and (20) brings this feature out most clearly: In two as well as in three dimensions the phases of the interfering waves, that is the phases of the exponential in the angular integration are identical. However, the integration measures are different: In two dimensions we have $d\varphi$ whereas in three dimensions we have $d\theta \sin \theta = -d(\cos \theta)$. This difference leads to an ordinary Bessel function $J_0$ in two dimensions but to the spherical Bessel function in three dimensions.

It is interesting to note that this very same reason also causes the difference between the Green’s functions of the d’Alembert wave equation in two and three dimensions, and the breakdown of Huygens’ principle in two dimensions.

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APPENDIX. GREEN’S FUNCTION FOR THE D’ALEMBERT WAVE EQUATION

In this appendix we briefly derive the Green’s function $G^{(N)}$ of the $N$-dimensional d’Alembert wave equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta^{(N)}\right) G^{(N)}(t, \mathbf{r}) = \delta(t) \delta(\mathbf{r})$$

(22)

Here $\mathbf{r}$ and $\Delta^{(N)}$ denote the $N$-dimensional position vector and Laplacian, respectively.

In order to facilitate the comparison between the Green’s function of the d’Alembert wave equation and the Schrödinger equation of a free particle in the main body of the paper we focus on the cases of two and three dimensions. We show that in three dimensions the Green’s function of the d’Alembert equation is a delta function. In contrast, in two dimensions the Green’s function carries a tail behind a sharp edge. We trace this difference back to the difference between the spherical and the ordinary Bessel functions.

Fourier solution in $N$ dimensions

In order to solve the inhomogeneous wave equation (22) we make the Fourier ansatz

$$G^{(N)}(t, \mathbf{r}) = \int dk_0 \int d^N k \tilde{G}(k_0, \mathbf{k}) e^{i(k_0 c t - \mathbf{k} \cdot \mathbf{r})}$$

where $\mathbf{k}$ denotes the $N$-dimensional wave vector.

When we substitute this ansatz into (22) and use the Fourier representation

$$\delta(t) \delta(\mathbf{r}) = \frac{1}{(2\pi)^{N+1}} \int dk_0 \int d^N k e^{i k_0 c t} e^{-i \mathbf{k} \cdot \mathbf{r}}$$

of the product of delta functions, we find the algebraic equation

$$\left[ k_0^2 - |\mathbf{k}|^2 \right] \tilde{G}(k_0, \mathbf{k}) = -\frac{1}{(2\pi)^{N+1}}$$
with the solution

\[ \hat{G}(k_0, \vec{k}) = - \frac{1}{(2\pi)^{N+1}} \frac{1}{k_0^2 - k^2} = - \frac{1}{(2\pi)^{N+1}} \frac{1}{2k} \left[ \frac{1}{k_0 - k} - \frac{1}{k_0 + k} \right] \]

where \( k \equiv |\vec{k}| \).

Hence, the Fourier transform \( \hat{G} \) enjoys simple poles at \( k_0 = \pm k \). Different paths in the complex plane \( k_0 \) define different Green’s functions. Since in the present paper we are only interested in the dependence of the Green’s function on the dimensions we choose one particular propagator, namely the retarded propagator \( G^{(N)}_{\text{ret}} \).

The retarded Green’s function is defined by a path just below the real \( k_0 \)-axis. For \( t < 0 \) we can close the path by a circle at infinity in the lower complex plane without changing the value of the integral. Since in this case we do not include any poles we find

\[ G^{(N)}_{\text{ret}}(t, \vec{r}) = 0 \]

On the other hand when \( t > 0 \) we can close the path in the upper complex plane without changing the value of the integral. The residue theorem

\[ \oint dz \, \frac{f(z)}{z - z_0} = 2\pi i f(z_0) \]

where the integration path circumvents the pole \( z_0 \) in the counter–clockwise direction allows us to perform the integration over \( k_0 \). We find the expression

\[ G^{(N)}_{\text{ret}} = \frac{(-i)}{(2\pi)^N} \int d^N k \, \frac{k}{2k} \left( e^{ikct} - e^{-ikct} \right) e^{-i\vec{k} \cdot \vec{r}} \]

We can combine the results for positive and negative values of \( t \) in the form

\[ G^{(N)}_{\text{ret}}(t, \vec{r}) = \Theta(t) \frac{1}{(2\pi)^N} \int d^N k \, \frac{\sin(kct)}{k} e^{-i\vec{k} \cdot \vec{r}} \]  (23)

It is instructive to cast this formula into a slightly different form by separating the \( N \)-dimensional volume element into the integration over the length of the \( N \)-dimensional wave vector and the \( N - 1 \) angles contained in \( d\omega \).

The Green’s function then reads

\[ G^{(N)}_{\text{ret}}(t, \vec{r}) = \Theta(t) \frac{1}{(2\pi)^N} \int_0^\infty dk \, k^{N-2} \sin(kct) \int d^{N-1} \omega \, e^{-i\vec{k} \cdot \vec{r}} \]

The retarded Green’s function of the \( N \)-dimensional d’Alembert wave equation is determined by a two–fold interference: (i) the interference of plane waves in the \( N - 1 \) dimensional space of angles and (ii) the interference of these expressions due to the integration over all wave numbers. In (i) all plane waves have the same amplitude and the same wave number. However, they differ in their propagation direction. All directions appear with equal weight but the surface element can put different weight on different directions. It is this contribution that creates the difference between the propagator in two and in three dimensions, as we show now.

**Two dimensions**

We first consider the case of two dimensions, that is \( N = 2 \) and perform the remaining integrations over the two–dimensional wave vector \( \vec{k} \) in polar coordinates. With the help of the area element

\[ d^2 k = dk \, k \, d\varphi \]

and the Sommerfeld representation \( [2] \) of the Bessel function \( J_0 \), the Green’s function
following from (23) takes the form

$$G_{\text{ret}}^{(2)} = \Theta(t) \frac{1}{2\pi} \int_0^\infty dk \sin(kct) \frac{1}{2\pi} \int_{-\pi}^\pi d\varphi \ e^{-ikr \cos \varphi}$$

The integral relation

$$\int_0^\infty d\xi \sin(\xi a) J_0(\xi b) = \Theta(a^2 - b^2) \sqrt{a^2 - b^2}$$

finally yields the expression

$$G_{\text{ret}}^{(2)}(t, \vec{r}) = \Theta(t) \frac{1}{\sqrt{(ct)^2 - r^2}}$$

for the retarded Green’s function of the d’Alembert wave equation in two dimensions.

This Green’s function enjoys a sharp wave front due to the Heavside step function. Moreover, it is followed by a decay which obeys an inverse square root dependence.

Three dimensions

We now turn to the three–dimensional case where the volume element in wave vector space reads

$$d^3k = dk \ k^2 \ d\theta \ \sin \theta \ d\varphi$$

The retarded Green’s function then takes the form

$$G_{\text{ret}}^{(3)} = \Theta(t) \frac{1}{(2\pi)^2} \int_0^\infty dk \sin(kct) k \int_0^\pi \ d\theta \ \sin \theta e^{-ikr \cos \theta} \frac{1}{2\pi} \int_{-\pi}^\pi d\varphi$$

which after simple angular integrations with the help of the integral representation (1) of the spherical Bessel function $j_0$ reduces to

$$G_{\text{ret}}^{(3)} = \Theta(t) \frac{1}{\pi r} \frac{1}{2\pi} \int_0^\infty dk \sin(kct) \ j_0(kr)$$

When we compare this expression to the corresponding formula (24) for two dimensions we recognize that the ordinary Bessel function $J_0$ has been replaced in three dimensions by the spherical Bessel functions $j_0$. This has important consequences for the remaining integration over $k$. Indeed, when we use the representation (1) of the spherical Bessel function as a sine function we find

$$G_{\text{ret}}^{(3)} = \Theta(t) \frac{1}{\pi r} \frac{1}{2\pi} \int_0^\infty dk \sin(kct) \ \sin(kr)$$

which with the help of the trigonometric relation

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

yields

$$G_{\text{ret}}^{(3)} = \Theta(t) \frac{1}{2\pi r} \frac{1}{2\pi} \int_0^\infty dk \{\cos[k(ct - r)] - \cos[k(ct + r)]\}$$

or

$$G_{\text{ret}}^{(3)} = \Theta(t) \frac{1}{4\pi r}[\delta(ct - r) - \delta(ct + r)]$$

Due to the $\Theta$ function, the retarded Green’s function is only non–vanishing for $t > 0$. Hence, only the first delta function makes a contribution and we arrive at the expression

$$G_{\text{ret}}^{(3)} = \Theta(t) \frac{\delta(ct - r)}{4\pi r}$$

for the retarded Green’s function of the d’Alembert wave equation in three dimensions. The integration of the spherical Bessel function instead of the ordinary Bessel function has created the delta function in the propagator instead of the square root and $\Theta$ function.
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