Internal Force Analysis of Parabolic Arch Considering Shear Effect under Gradient Temperature

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Abstract. This paper theoretical analysis the internal force of the fixed parabolic arches under radiant temperature gradient field incorporating shear deformations. The effective centroid of the arch-section under linear temperature gradient is derived. Based on force method and energy method, the analytical solutions of the internal force of fixed parabolic arches at pre-buckling under linear temperature gradient field are derived. A parameter study was carried out to study the influence of linear temperature gradient on the internal force of the fixed parabolic arches with different rise-span ratio and varying slenderness ratio. It is found that the temperature gradient and the rise-span ratio has a significant influence on the internal force of the parabolic arches, the influence of shear deformation causes the bending moment increase while the axial force decreases, and the axial force of parabolic arches decreases as the rise-span ratio increases.

Key words: Parabolic arches; theoretical analysis; temperature gradient field; shear deformation; analysis of the internal force.

1. Introduction

Parabolic arch is widely used in arch structure engineering, however, due to the complex linear and temperature effects, the theoretical study of the arch structure lags behind the actual needs of the project. The internal force analysis of parabolic arch is an important part of arch design, construction and maintenance, linear gradient temperature is a special idealized load, but there are similar load effects in practical engineering, such as sunshine and fire. In addition, the rise span ratio, slenderness ratio and shear deformation may have a significant impact on the internal force of the arch structure [1]. Therefore, it is necessary to study the internal force of parabolic arch under the action of linear gradient temperature.

The influence of non-uniform temperature field on the properties of arch materials, the initial homogeneous parabola arch is transformed into heterogeneous parabola arch, and the centroid of the section of the biaxial symmetrical I-shaped arch is no longer coincident with the geometric center of the section, so that the original parabola geometric arch axis cannot continue to be the effective central axis, so it is necessary to solve the effective centroid offset to ensure the accuracy of internal force analysis. So, the problem of effective centroid of parabolic arch under the action of linear gradient temperature field also needs to be solved.
Conduct relevant research. The results show that slenderness ratio and rise span ratio have certain influence on the mechanical properties of arch structure. The research on the arch structure under the action of temperature is scarce, PI and Lu, etc. have carried out systematic theoretical research on the stability of arch structure under the action of uniform temperature and beam structure under the action of gradient temperature [2-7]. The results show that the effect of temperature on the mechanical properties of the structure is significant, the analytical solution of critical instability temperature is given. In addition, song's theoretical research on the in-plane stability of the fixed circular arc arch under the action of gradient temperature and the mechanical behavior before the instability of the elastic constrained circular arc arch [8], the research shows that the gradient temperature field and boundary conditions have obvious influence on the mechanical properties of circular arc arch. However, the analysis of internal force of parabolic arch under the action of linear gradient temperature field is still blank up to now.

The purpose of this paper is to analyze the effect of shear deformation on the internal force of I-shaped parabola arch under the action of linear gradient temperature field. The analytical solution of the effective centroid offset of the parabolic arch section under the action of the linear gradient temperature field is studied and derived. Based on the force method and energy method, the analytical solution of the internal force of the fixed parabolic arch under the action of the linear gradient temperature field considering the shear deformation effect is obtained, and the accuracy of the above theory is verified by numerical simulation using the finite element software ANSYS. The influences of linear gradient temperature, rise span ratio, slenderness ratio and shear deformation on the internal forces of the fixed parabolic arch are obtained, which provides theoretical support for the related projects of the parabolic arch.

2. Basis Analysis

2.1. Parabolic Arch Model

In this paper, I-section fixed parabolic arch is taken as the research object, as shown in Figure 1. $O^{*} x^{*} y^{*} s^{*}$ is the initial coordinate system of parabolic arch. Where, $O^{*}$ is located at the geometric center of the section of the parabolic arch crown, which is the origin of the initial coordinate of the parabolic arch. $O^{*} x^{*}$, $O^{*} y^{*}$ and $O^{*} s^{*}$ are respectively expressed as the transverse coordinate axis, the longitudinal coordinate axis and the parabolic geometric arch axis of the initial coordinate system. In addition, $f$, $L$, $l$, $T_1$ and $T_2$ are respectively expressed as the rise height, span, half span, dome top temperature and arch bottom temperature of parabolic arch.

The parabolic arch coordinate equation studied in this paper is as follows

\[ y^{*} = \frac{a}{2} \frac{l}{x_{v}^{2}} \]  

Where, $a$ is the shape coefficient of parabolic arch, and its expression is
In addition, the coordinate parameter of parabola arch is expressed as \( x_i = l x^* \), \( dx_i = l dx \) so, the arc differential \( ds \) of parabola arch can be expressed as

\[
ds = l \sqrt{1 + (a_i x_i)^2} \, dx_i
\]

(3)

2.2. Basic Assumptions

The gradient temperature and parabolic arch studied in this paper meet the following assumptions:

1. The ambient temperature studied in this paper is 20°C;
2. The deformation of parabolic arch and the temperature of cross section are independent of time;
3. Poisson's ratio \( \nu \) and temperature expansion coefficient \( \alpha \) are independent of temperature;
4. The temperature of parabolic arch section changes uniformly along the \( o \cdot \hat{x} \) and \( o \cdot \hat{s} \) axes, and along the axis of \( o \cdot \hat{y} \) coordinate, as shown in Figure 2.

Fig. 2 Positive trapezoid gradient temperature field

Therefore, the temperature at any position on the parabolic arch section is expressed as

\[
T(y) = T_a + \frac{y^* \Delta T_g}{h}
\]

(4)

Where, \( h \) is the height of the arch section, and the average temperature of the arch section \( T_a \) and the temperature difference between the top and the bottom \( \Delta T_g \) can be expressed as

\[
T_a = \frac{T_1 + T_2}{2}
\]

(5)

\[
\Delta T_g = T_2 - T_1
\]

(6)

2.3. Modulus of Elasticity

As we all know, the latest code for fire protection of building steel structures gives the relationship between the elastic modulus of steel and temperature. Elastic modulus of steel is \( E_T = \xi_T \cdot E_o \), where, \( E_o \) is the modulus of elasticity of Q235 steel at 20°C, \( \xi_T \) is the reduction coefficient of elastic modulus of steel under the action of temperature. Its expression is

\[
\xi_T = \frac{7T - 4780}{6T - 4760} \quad 0°C < T \leq 600°C
\]

(7)

2.4. Effective Centroid and Effective Stiffness

The inconsistency of the elastic modulus of the arch section under the linear gradient temperature field causes the effective centroid no longer to coincide with the geometric center of the arch section.
Therefore, in order to analyze the internal force of parabola arch, the derivation of the centroid offset of parabola arch section under the action of linear gradient temperature field must be developed first.

![Fig. 3 Micro element of I-shaped arch structure](image)

The cross section of parabola arch studied in this paper is I-section, and the effective coordinate system $oxyz$ is established. Among them, $ox, oy$ and $os$ represent the effective horizontal axis, the effective vertical axis and the effective arch axis respectively, because the temperature does not change along the $o^*x^*$ and $o^*s^*$ axis, Therefore, the coordinate offset of the effective centroid to the geometric center about the $ox$ and $oz$ axes is $x_c = 0$, $y_c = 0$. The coordinate offset of $oy$ axis can be resolved as:

$$y_c = - \int_A E(y^*) y^* dA \int_A E(y^*) dA$$  \hspace{1cm} (8)

In order to ensure the accurate analysis of internal force of parabola arch under the action of gradient temperature, the effective stiffness $\overline{EA}$ and $\overline{EI}$ of arch section are derived, which can be expressed as:

$$\overline{EA} = \int_A E(y) dA \hspace{1cm} (9)$$

$$\overline{EI} = \int_A E(y) y^2 dA \hspace{1cm} (10)$$

Where $t_f$ is the thickness of I-section flange plate and $t_w$ is the thickness of I-section Web, While $b$, $h$ and $A$ are the width, height and area of the arch section respectively. In addition, the parameters $T_i$ and $\Xi_{1,2}$ can be expressed mathematically as:

$$T_i = 2380 - 3T_g$$  \hspace{1cm} (11)
3. Analytical Solution of Internal Force

3.1. Thermoelastic Analysis

The total energy equation of parabolic arch considering shear deformation under the action of linear gradient temperature field can be expressed as

\[
U_{s0} = \frac{1}{2} \int_{A} \left[ \int_{A} \left[ E\varepsilon_{\alpha0}^2 + \mu, G\gamma_{\alpha0}^2 \right] dAdx \right] + \int_{A} E\alpha \left( \Delta T_{\alpha} - \frac{\gamma \Delta T_{\gamma}}{h} \right) \varepsilon_{\alpha0} dAdx
\]

(14)

Where, \( \Delta T_{\alpha} = T_{\alpha} - 20^\circ C \), \( T_{\alpha} \) is the effective centroid temperature of the section, \( \varepsilon_{\alpha0} \) and \( \gamma_{\alpha0} \) are the axial strain and shear strain of parabolic arch respectively. \( E \) and \( G \) are expressed as the elastic modulus and shear modulus of the material respectively, And meet \( E = 2(1 + \nu)G \), \( \nu \) is poisson's ratio. Based on the theory of structural mechanics, the linear coefficient \( V_{i} \), axial force \( N \), bending moment \( M \) and shear force \( Q_{y} \) of parabolic arch can be expressed as

\[
V_{i} = \sqrt{1 + (a_{i} x_{i})^2}
\]

(15)

\[
N = -\int_{A} E\varepsilon_{\alpha0} dAdA
\]

(16)

\[
M = -\int_{A} y\varepsilon_{\alpha0} dAdA
\]

(17)

\[
Q_{y} = -\mu \int_{A} G\gamma_{\alpha0} dAdA = -\frac{1}{\beta} \int_{A} E\gamma_{\alpha0} dAdA
\]

(18)

Where, the section shear influence coefficient \( \beta \) can be expressed as

\[
\beta = \frac{2(1 + \nu)}{\mu}
\]

(19)

Where \( \mu \) is the section shear coefficient, The total energy equation of parabolic arch expressed by internal force can be obtained by substituting formula (16) - (18) into formula (14)

\[
U_{s0} = \frac{1}{2} \int_{A} \left[ \frac{N^2}{EA} + \frac{M^2}{EI} + \frac{\beta Q_y^2}{EA} \right] dAdx
\]

\[
-\alpha \Delta T_{\gamma} \int_{A} NIV_{i} dAdA + \frac{\alpha \Delta T_{\gamma}}{h} \int_{A} MIV_{i} dAdx
\]

(20)

According to the principle of force method, parabola arch is divided into two statically determinate structures along the arch crown section, as shown in Figure 6.
According to the sketch of the fixed parabola arch force method shown in Figure 4 and based on the second theorem of Carlson, the relative axial displacement $\Delta_{Nc}$, relative radial displacement $\Delta_{Qyc}$ and relative angular displacement $\Delta_{Mc}$ of the arch crown can be expressed as

$$\Delta_{Nc} = \frac{\partial U_{\text{ax}}}{\partial N_c} = 0 \quad (21)$$

$$\Delta_{Qyc} = \frac{\partial U_{\text{ax}}}{\partial Q_{yc}} = 0 \quad (22)$$

$$\Delta_{Mc} = \frac{\partial U_{\text{ax}}}{\partial M_c} = 0 \quad (23)$$

The corresponding force method equation can be obtained by substituting formula (20) into formula (21) - (23)

$$\int_{-1}^{1} \left[ \frac{N}{EA} \frac{\partial N}{\partial N_c} + \frac{M}{EI} \frac{\partial M}{\partial N_c} + \frac{\beta Q_c}{EA} \frac{\partial Q_c}{\partial N_c} \right] V_i ds \quad (24)$$

$$+ \int_{-1}^{1} \left( \frac{\alpha \Delta T_c}{h} \frac{\partial M}{\partial N_c} - \alpha \Delta T_c \frac{\partial N}{\partial N_c} \right) V_i ds = 0$$

$$\int_{-1}^{1} \left[ \frac{N}{EA} \frac{\partial N}{\partial Q_{yc}} + \frac{M}{EI} \frac{\partial M}{\partial Q_{yc}} + \frac{\beta Q_c}{EA} \frac{\partial Q_c}{\partial Q_{yc}} \right] V_i ds \quad (25)$$

$$+ \int_{-1}^{1} \left( \frac{\alpha \Delta T_c}{h} \frac{\partial M}{\partial Q_{yc}} - \alpha \Delta T_c \frac{\partial Q_c}{\partial Q_{yc}} \right) V_i ds = 0$$

$$\int_{-1}^{1} \left[ \frac{N}{EA} \frac{\partial N}{\partial M_c} + \frac{M}{EI} \frac{\partial M}{\partial M_c} + \frac{\beta Q_c}{EA} \frac{\partial Q_c}{\partial M_c} \right] V_i ds \quad (26)$$

$$+ \int_{-1}^{1} \left( \frac{\alpha \Delta T_c}{h} \frac{\partial M}{\partial M_c} - \alpha \Delta T_c \frac{\partial Q_c}{\partial M_c} \right) V_i ds = 0$$

In addition, according to the principle of force balance of parabolic arch structure, it can be obtained.

$$N = -\frac{N_c}{V_i} \quad (27)$$

$$M = M_c - \frac{N_c a t}{2} \quad (28)$$

$$Q_c = \frac{N_c a t}{V_i} \quad (29)$$

Formula (27) - (29) is substituted into formula (24) - (26) to solve the nonlinear equations. The bending moment $M_c$, axial force $N_c$ and shear force $Q_{yc}$ of the vault can be expressed as
The internal force calculation parameters $\Gamma_{1,2}$ and $\Psi$ can be expressed as

$$\Gamma_1 = \ln\left(\sqrt{a_i^2 + 1} - a_i\right)$$

$$\Gamma_2 = \ln\left(\sqrt{a_i^2 + 1} + a_i\right)$$

$$\Psi = 6\left(\Gamma_1 - \Gamma_2 - \Gamma_3\right)\left[l^2 - 64a_i^2r_i^2(\beta - 2)\right]$$

$$- 4a_i^2(a_i^2 + 1)[l^2(4a_i^4 - 9) + 8a_i^2(24br_i^2 - l^2)]$$

$$+ 4(\Gamma_1 - \Gamma_2)a_il^2\sqrt{a_i^2 + 1}(8a_i^4 + 3)$$

$$+ 32(\Gamma_1 - \Gamma_2)a_i^4\sqrt{a_i^2 + 1}(l^2 + 24r_i^2)$$

By substituting formula (30) - (31) into formula (27) - (28), the analytical solution of internal force of fixed parabolic arch under the action of linear gradient temperature field can be obtained.

$$N = \frac{768\overline{E}la\Delta T_1a_i^3}{\Psi}(\Gamma_1 - \Gamma_2 - 2a_i\sqrt{a_i^2 + 1})$$

$$M = \frac{96\overline{E}la\Delta T_1a_i^3l}{\Psi}(\Gamma_1 - \Gamma_2) - \frac{\overline{E}la\Delta T_1}{h}$$

$$+ \frac{192\overline{E}la\Delta T_1a_i^3l}{\Psi}(2a_i^2 + 1)\sqrt{a_i^2 + 1}$$

$$- \frac{768\overline{E}la\Delta T_1a_i^3x_{i1}^3l}{2\Psi}(\Gamma_1 - \Gamma_2 - 2a_i\sqrt{a_i^2 + 1})$$

As shown in Fig. 5 and Fig. 6, the variation of the internal force of the non dimensional arch crown with the rise span ratio under different slenderness ratio and different gradient temperature conditions under the action of linear gradient temperature field. It can be seen from the figure that: the axial force of arch crown decreases with the increase of rise span ratio or slenderness ratio, the bending moment of arch crown first decreases with the increase of rise span ratio, then increases with the increase of rise span ratio, and decreases with the increase of temperature difference between the top and bottom of parabolic arch section.
3.2. Finite Element Analysis

In order to verify the accuracy of the above theoretical research, the finite element software ANSYS is used to carry out a comparative analysis of its modeling. The material properties of Q235 steel and biaxial symmetrical I-section with width of 150 mm, height of 250 mm, span length of 5000 mm, flange plate thickness of 10 mm and web thickness of 6 mm are selected for modeling, as shown in Fig 7.
In order to ensure the accurate finite element analysis, the equivalent simulation method is used to simulate the linear gradient temperature field: the arch section is divided into several temperature layers, and the material properties of each layer are defined according to formula (7). The model is established and the numerical solution is expanded, and then compared with the theoretical solution.

As shown in Fig.8 and Fig.9, the theoretical analytical solutions of the non dimensional axial force and the non dimensional bending moment of the parabolic arch under the action of linear gradient temperature field are compared with their finite element results under the condition of considering shear deformation or not. It can be seen from the figure that the obtained analytical solution of internal force theory is in good agreement with the results of finite element method, which shows that the analytical theory of internal force of parabolic arch under linear gradient temperature field calculated by force method has high accuracy. It can also be seen from the figure that the axial force of the arch crown with shear deformation is smaller than that without shear deformation, and the bending moment of the arch crown with shear deformation is larger than that without shear deformation.

4. Conclusion
In this paper, the internal force of parabola arch with I-section under the action of linear gradient temperature field is studied. The analytical solution of the effective centroid of the arch section under the linear gradient temperature is derived, and the high-precision analytical solution of the temperature
internal force of the parabolic arch is given, which enriches the theoretical research system of the arch structure. The results show that gradient temperature, slenderness ratio, rise span ratio and shear deformation have significant influence on the internal force of parabolic arch.

1. Under the action of linear gradient temperature field, the bending moment of fixed parabola arch decreases with the increase of section temperature difference.
2. Under the action of linear gradient temperature field, the axial force of the fixed parabola arch decreases with the increase of slenderness ratio.
3. The axial force of fixed parabola arch decreases with the increase of rise span ratio. In addition, the moment of the vault decreases with the increase of the rise span ratio at first, and then increases with the increase of the rise span ratio.
4. The axial force of the vault considering the shear deformation is larger than that without considering the shear deformation, but the bending moment of the vault considering the shear deformation is larger than that without considering the shear deformation.

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