A MOLECULAR RECONSTRUCTION THEOREM FOR $H^p_{\omega'(\cdot)}(\mathbb{R}^n)$

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Abstract

In this article we give a molecular reconstruction theorem for $H^p_{\omega'(\cdot)}(\mathbb{R}^n)$. As an application of this result and the atomic decomposition developed in [5] we show that classical singular integrals can be extended to bounded operators on $H^p_{\omega'(\cdot)}(\mathbb{R}^n)$. We also prove, for certain exponents $q(\cdot)$ and certain weights $\omega$, that Riesz potential $I_{\alpha}$, with $0 < \alpha < n$, can be extended to a bounded operator from $H^p_{\omega'(\cdot)}(\mathbb{R}^n)$ into $H^q_{\omega'(\cdot)}(\mathbb{R}^n)$, for $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$.

1 Introduction

In the celebrated paper [4], Fefferman and Stein defined the Hardy space $H^p(\mathbb{R}^n)$. Since then, the study of Hardy spaces has received the attention of a substantial number of researchers. One of the most remarkable results for the study of Hardy spaces is the atomic characterization of $H^p(\mathbb{R}^n)$ (see [1, 7]). Roughly speaking, every distribution $f \in H^p$ can be expressed of the form

$$f = \sum_j \lambda_j a_j,$$

where the $a_j$’s are $p$-atoms, $\{\lambda_j\} \in \ell^p$ and $\|f\|_{H^p}^p \approx \sum_j |\lambda_j|^p$. For $0 < p \leq 1$, an $p$-atom is a function $a(\cdot)$ supported on a cube $Q$ such that

$$\|a\|_{\infty} \leq |Q|^{-1/p} \text{ and } \int x^\alpha a(x)dx = 0 \text{ for all } |\alpha| \leq n \left(\frac{1}{p} - 1\right).$$
Such decomposition allows to study the behavior of certain operators on $H^p$ by focusing one’s attention on individual atoms. In principle, the continuity of an operator $T$ on $H^p$ can often be proved by estimating $Ta$ when $a(\cdot)$ is an atom. In [2] was observed that, in general, the atoms are not mapped into atoms. However, for many convolution operators (e.g.: singular integrals or Riesz potentials), $m = Ta$ behaves like an atom. These functions $m$ were called *molecules*, their properties as well as the molecular characterization of $H^p(\mathbb{R}^n)$ were established in [15]. Then, in essence, the continuity $H^p \to H^p$ of an operator reduces to showing that it maps atoms into molecules.

The theory of variable Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ was first developed by Nakai and Sawano in [8], where they proved the infinite atomic decomposition and molecular decomposition for $H^{p(\cdot)}$. As a corollary of these decompositions, they obtained the boundedness of singular integral operators on $H^{p(\cdot)}$. Later, Cruz-Uribe and Wang in [3] independently considered the same problem under a weaker condition for variable exponents $p(\cdot)$. They gave a finite atomic decomposition for $H^{p(\cdot)}$ and also proved the boundedness of singular integral operators on $H^{p(\cdot)}$. Both theories prove equivalent definitions in terms of maximal operators using different approaches. In [12], the author jointly with Urciuolo proved the $H^{p(\cdot)} - L^{q(\cdot)}$ boundedness of certain generalized Riesz potentials and the $H^{p(\cdot)} - H^{q(\cdot)}$ boundedness of Riesz potential via the infinite atomic and molecular decomposition developed in [8]. In [10], the author gave another proof of the results obtained in [12], but by using the finite atomic decomposition given in [3].

Recently, Kwok-Pun Ho in [5] developed the weighted theory for variable Hardy spaces on $\mathbb{R}^n$. He established the atomic decompositions for the weighted Hardy spaces with variable exponents $H^{p(\cdot)}(\mathbb{R}^n)$ and also revealed some intrinsic structures of atomic decomposition for Hardy type spaces. His results generalize the infinite atomic decomposition obtained in [8]. By means of the atomic decomposition given in [5], the author in [11] proved, for certain exponents $q(\cdot)$ and certain weights $\omega$, the $H^{p(\cdot)}_\omega - L^{q(\cdot)}_\omega$ boundedness for Riesz potential.

The purpose of this article is to generalize the molecular decomposition obtained in [8, Theorem 5.2] to weighted variable Hardy spaces by using the framework developed in [5]. As an application of our molecular decomposition, we prove two theorems concerning the $H^{p(\cdot)}_\omega - H^{p(\cdot)}_{\omega'}$ boundedness of a class of singular integral operators and the $H^{p(\cdot)}_\omega - H^{q(\cdot)}_{\omega'}$ boundedness of Riesz potential. These results are established in Theorems 4.2 and 5.2 below.

This paper is organized as follows. Section 2 gives weighted variable Hardy spaces $H^{p(\cdot)}_\omega$ and some of their preliminary results. Section 3 contains a molecular reconstruction theorem for $H^{p(\cdot)}_\omega$. The boundedness of singular integrals on $H^{p(\cdot)}_\omega$ is established in Section 4. The $H^{p(\cdot)}_\omega - H^{q(\cdot)}_{\omega'}$ boundedness of Riesz potential is proved in Section 5.
Notation. The symbol $A \lesssim B$ stands for the inequality $A \leq cB$ for some positive constant $c$. The symbol $A \approx B$ stands for $B \lesssim A \lesssim B$. We denote by $Q(z,r)$ the cube centered at $z \in \mathbb{R}^n$ with side length $r$. Given a cube $Q = Q(z,r)$, we set $kQ = Q(z,kr)$ and $\ell(Q) = r$. For a measurable subset $E \subseteq \mathbb{R}^n$ we denote by $|E|$ and $\chi_E$ the Lebesgue measure of $E$ and the characteristic function of $E$ respectively. Given a real number $s \geq 0$, we write $\lfloor s \rfloor$ for the integer part of $s$. As usual we denote with $S(\mathbb{R}^n)$ the space of smooth and rapidly decreasing functions and with $S'(\mathbb{R}^n)$ the dual space. If $\beta$ is the multiindex $\beta = (\beta_1, \ldots, \beta_n)$, then $|\beta| = \beta_1 + \ldots + \beta_n$.

2 Preliminaries

We begin with the definition of weighted variable Lebesgue spaces $L^p(\cdot)(\mathbb{R}^n)$.

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function. Given a measurable set $E$, let

$$ p_-(E) = \text{ess inf}_{x \in E} p(x), \quad \text{and} \quad p_+(E) = \text{ess sup}_{x \in E} p(x). $$

When $E = \mathbb{R}^n$, we will simply write $p_- := p_- (\mathbb{R}^n)$ and $p_+ := p_+ (\mathbb{R}^n)$.

Given a measurable function $f$ on $\mathbb{R}^n$, define the modular $\rho$ associated with $p(\cdot)$ by

$$ \rho(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx. $$

We define the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ to be the set of all measurable functions $f$ such that, for some $\lambda > 0$, $\rho(f/\lambda) < \infty$. This becomes a quasi normed space when equipped with the Luxemburg norm

$$ \|f\|_{L^{p(\cdot)}} = \inf \{\lambda > 0 : \rho(f/\lambda) \leq 1\}. $$

Given a weight $\omega$, i.e.: a locally integrable function on $\mathbb{R}^n$ such that $0 < \omega(x) < \infty$ almost everywhere, we define the weighted variable Lebesgue space $L^{p(\cdot)}_\omega(\mathbb{R}^n)$ as the set of all measurable functions $f$ such that $\|f\omega\|_{L^{p(\cdot)}} < \infty$. If $f \in L^{p(\cdot)}_\omega(\mathbb{R}^n)$, we define its "norm" by

$$ \|f\|_{L^{p(\cdot)}_\omega} := \|f\omega\|_{L^{p(\cdot)}}. $$

The following result follows from the definition of the $L^{p(\cdot)}_\omega$-norm.

**Lemma 2.1.** Given a measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ with $0 < p_- \leq p_+ < \infty$ and a weight $\omega$, then for every $s > 0$,

$$ \|f\|_{L^{p(\cdot)}_\omega}^s = \|f^s\|_{L^{(p(\cdot)/s)^s}_\omega}. $$
Next, we introduce the weights used in [5] to define weighted Hardy spaces with variable exponents.

**Definition 2.2.** (See [11, Remark 2.5]) Let \( p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty) \) be a measurable function with \( 0 < p_- \leq p_+ < \infty \). We define \( \mathcal{W}_{p(\cdot)} \) as the set of all weights \( \omega \) such that

(i) there exists \( 0 < p_* < \min\{1, p_-\} \) such that \( \|\chi_Q\|_{L^{p(\cdot)/p_*}} < \infty \), and

\[
\|\chi_Q\|_{L_{\omega^{-p_*}}^{(p(\cdot)/p_*)'}} < \infty, \text{ for all cube } Q;
\]

(ii) there exist \( \kappa > 1 \) and \( s > 1 \) such that Hardy-Littlewood maximal operator \( M \) is bounded on \( L_{\omega^{-s_*/\kappa}}^{(sp(\cdot)/s)'} \).

For a measurable function \( p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty) \) such that \( 0 < p_- \leq p_+ < \infty \) and \( \omega \in \mathcal{W}_{p(\cdot)} \), in [5] the author give a variety of distinct approaches, based on differing definitions, all lead to the same notion of weighted variable Hardy space \( H_{p(\cdot)}^\omega \).

We recall some terminologies and notations from the study of maximal functions. Given \( N \in \mathbb{N} \), define

\[
\mathcal{F}_N = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sum_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\beta \varphi(x)| \leq 1 \right\}.
\]

For any \( f \in \mathcal{S}'(\mathbb{R}^n) \), the grand maximal function of \( f \) is given by

\[
\mathcal{M}_N f(x) = \sup_{t > 0} \sup_{\varphi \in \mathcal{F}_N} |(\varphi_t * f)(x)|,
\]

where \( \varphi_t(x) = t^{-n} \varphi(t^{-1}x) \). It is common to denote the grand maximal by \( \mathcal{M} \).

**Definition 2.3.** Let \( p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty) \), \( 0 < p_- \leq p_+ < \infty \), and \( \omega \in \mathcal{W}_{p(\cdot)} \). The weighted variable Hardy space \( H_{p(\cdot)}^{\omega}(\mathbb{R}^n) \) is the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) for which \( \|\mathcal{M} f\|_{L_{p(\cdot)}^\omega} < \infty \). In this case we define \( \|f\|_{H_{p(\cdot)}^{\omega}} := \|\mathcal{M} f\|_{L_{p(\cdot)}^\omega} \).

**Definition 2.4.** Let \( p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty) \), \( 0 < p_- \leq p_+ < \infty \), \( p_0 > 1 \), and \( \omega \in \mathcal{W}_{p(\cdot)} \). Fix an integer \( N \geq 1 \). A function \( a(\cdot) \) on \( \mathbb{R}^n \) is called a \( \omega - (p(\cdot), p_0, N) \) atom if there exists a cube \( Q \geq 1 \) such that

1. \( \text{supp}(a) \subset Q \),
2. \( \|a\|_{L^{p_0}} \leq \frac{|Q|^{1/p_0}}{\|\chi_Q\|_{L_{p(\cdot)}^\omega}}, \)
3. \( \int x^\beta a(x) \, dx = 0 \) for all \( |\beta| \leq N \).
Now, we introduce two indices, which are related to the intrinsic structure of the atomic decomposition of $H_\omega^{p(\cdot)}$. Given $\omega \in \mathcal{W}_{p(\cdot)}$, we write
\[
s_{\omega, p(\cdot)} := \inf \left\{ s \geq 1 : M \text{ is bounded on } L^{(sp(\cdot))'}_{\omega^{-1/s}} \right\}
\]
and
\[
S_{\omega, p(\cdot)} := \left\{ s \geq 1 : M \text{ is bounded on } L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}} \text{ for some } \kappa > 1 \right\}.
\]
Then, for every fixed $s \in S_{\omega, p(\cdot)}$, we define
\[
\kappa^s_{\omega, p(\cdot)} := \sup \left\{ \kappa > 1 : M \text{ is bounded on } L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}} \right\}.
\]
The index $\kappa^s_{\omega, p(\cdot)}$ is used to measure the left-openness of the boundedness of $M$ on the family $\left\{ L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}} \right\}_{\kappa > 1}$. The index $s_{\omega, p(\cdot)}$ is related to the vanishing moment condition and the index $\kappa^s_{\omega, p(\cdot)}$ is related to the size condition of the atoms (see [5] Theorems 5.3 and 6.3). These indices are also related to the structure of our molecular decomposition (see Definition 3.1 and Theorem 3.4 below).

**Definition 2.5.** Let $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ be a measurable function such that $0 < p_- \leq p_+ < \infty$, and let $\omega \in \mathcal{W}_{p(\cdot)}$. Given a sequence of scalars $\{\lambda_j\}_{j=1}^\infty$, a family of cubes $\{Q_j\}_{j=1}^\infty$ and $0 < \theta < \infty$, we define
\[
A \left( \{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot), \omega, \theta \right) := \left\| \sum_{j=1}^\infty \left( \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \right)^\theta \chi_{Q_j} \right\|_{L^{p(\cdot)/\theta}_\omega}^{1/\theta}.
\]

The following theorem is a version of the atomic decomposition for $H_\omega^{p(\cdot)}$ obtained in [5].

**Theorem 2.6.** Let $1 < p_0 < \infty$, $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ be a measurable function with $0 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{W}_{p(\cdot)}$. Then, for every $f \in H_\omega^{p(\cdot)}(\mathbb{R}^n) \cap L^{p_0}(\mathbb{R}^n)$ and every integer $N \geq \lfloor n s_{\omega, p(\cdot)} - n \rfloor$ fixed, there exist a sequence of scalars $\{\lambda_j\}_{j=1}^\infty$, a sequence of cubes $\{Q_j\}_{j=1}^\infty$ and $\omega - (p(\cdot), p_0, N)$ atoms $a_j$ supported on $Q_j$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in $L^{p_0}(\mathbb{R}^n)$ and
\[
A \left( \{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot), \omega, \theta \right) \lesssim \| f \|_{H_\omega^{p(\cdot)}}, \text{ for all } 0 < \theta < \infty,
\]
where the implicit constant in (1) is independent of $\{\lambda_j\}_{j=1}^\infty$, $\{Q_j\}_{j=1}^\infty$, and $f$.

**Proof.** The existence of a such atomic decomposition is guaranteed by [5] Theorem 6.2]. Its construction is analogous to that given for classical Hardy spaces (see [14] Chapter III]). So, following the proof in [9] Theorem 3.1], we obtain the convergence of the atomic series to $f$ in $L^{p_0}(\mathbb{R}^n)$. \[\square\]
The following three results will be useful in what follows.

**Lemma 2.7.** ([11, Lemma 3.4]) Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function with $0 < p_- \leq p_+ < \infty$. If $\omega \in \mathcal{W}_{p(\cdot)}$, then, for every cube $Q \subset \mathbb{R}^n$,

$$\|\chi_{2\sqrt{n}Q}\|_{L^p(\cdot)} \approx \|\chi_Q\|_{L^p(\cdot)}.$$

We say that an exponent function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that $0 < p_- \leq p_+ < \infty$ belongs to $P_{\log}(\mathbb{R}^n)$, if there exist two positive constants $C$ and $C_{\infty}$ such that $p(\cdot)$ satisfies the local log-Hölder continuity condition, i.e.:

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}, \quad |x-y| \leq \frac{1}{2},$$

and is log-Hölder continuous at infinity, i.e.:

$$|p(x) - p_{\infty}| \leq \frac{C_{\infty}}{\log(e+|x|)}, \quad x \in \mathbb{R}^n,$$

for some $p_- \leq p_{\infty} \leq p_+$.

We define the set $S_0(\mathbb{R}^n)$ by

$$S_0(\mathbb{R}^n) = \left\{ \varphi \in S(\mathbb{R}^n) : \int x^\beta \varphi(x) dx = 0, \text{ for all } \beta \in \mathbb{N}_0^n \right\}.$$

**Proposition 2.8.** ([6, Proposition 2.1]) Let $p(\cdot) \in P_{\log}(\mathbb{R}^n)$ with $0 < p_- \leq p_+ < \infty$. If $\omega \in \mathcal{W}_{p(\cdot)}$, then $S_0(\mathbb{R}^n) \subset H_{p(\cdot)}^\omega(\mathbb{R}^n)$ densely.

We conclude this preliminaries with a supporting result, which will allow us to study the behavior of Riesz potential on $H_{p(\cdot)}^\omega(\mathbb{R}^n)$.

**Proposition 2.9.** ([11, Corollary 4.2]) Let $0 < \alpha < n$, $q(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function with $0 < q_- \leq q_+ < \infty$ and $\omega \in \mathcal{W}_{q(\cdot)}$. If $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$ and $\|\chi_Q\|_{L^q_{p(\cdot)}} \approx |Q|^{-\alpha/n}\|\chi_Q\|_{L^p_{p(\cdot)}}$ for every cube $Q$, then for any sequence of scalars $\{\lambda_j\}_{j=1}^\infty$, any family of cubes $\{Q_j\}_{j=1}^\infty$, and any $\theta \in (0, \infty)$ fixed we have

$$A \left( \{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, q(\cdot), \omega, \theta \right) \lesssim A \left( \{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot), \omega, \theta \right).$$

## 3 Molecular reconstruction for $H_{p(\cdot)}^\omega(\mathbb{R}^n)$

Our definition of $\omega$-molecule is a slight modification from that given in [8] by Nakai and Sawano for variable Hardy spaces. Namely, we replace the amount $\|\chi_Q\|_{L^p(\cdot)}$ by $\|\chi_Q\|_{L^p_{p(\cdot)}}$. 
Definition 3.1. Let \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \) be a measurable function with \( 0 < p_+ \leq p_- < \infty \) and let \( \omega \in \mathcal{W}_{p(\cdot)} \). Let \( p_0 > 1 \) and \( d_{p(\cdot)} := [ns_{\omega, p(\cdot)} - n] \). A function \( m(\cdot) \) on \( \mathbb{R}^n \) is called a \( \omega - (p(\cdot), p_0, d_{p(\cdot)}) \) molecule centered at a cube \( Q = Q(z, r) \) (with side length \( \ell(Q) = r \)) if

\[
m_1 \| m \|_{L^{p_0}(2\sqrt{n}Q)} \leq \frac{|Q|^{\frac{1}{p_0}}}{\| \chi_Q \|_{L^{p(\cdot)}(\mathbb{R}^n)}},
\]

\[
m_2 |m(x)| \leq \| \chi_Q \|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} \left( 1 + \frac{|x - z|}{\ell(Q)} \right)^{-2n - 2d_{p(\cdot)} - 3} \quad \text{for all} \ x \in \mathbb{R}^n \setminus Q(z, 2\sqrt{nr}),
\]

\[
m_3 \int x^\beta m(x) \, dx = 0 \quad \text{for all multi-index} \ \beta \ \text{with} \ |\beta| \leq d_{p(\cdot)}.
\]

Remark 3.2. The conditions \( m_1 \) and \( m_2 \) imply that \( \| m \|_{L^{p_0}(\mathbb{R}^n)} \leq C \frac{|Q|^{\frac{1}{p_0}}}{\| \chi_Q \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \), where \( C \) is a positive constant independent of the molecule \( m \).

Next, we give a molecular reconstruction theorem for \( H^{p(\cdot)}_{\omega}(\mathbb{R}^n) \). For them, we need to introduce the following discrete maximal: given \( \phi \in S(\mathbb{R}^n) \) and \( f \in S'(\mathbb{R}^n) \), we define

\[M^{\text{dis}} f(x) = \sup_{j \in \mathbb{Z}} |(\phi^j \ast f)(x)|,\]

where \( \phi^j(x) = 2^n \phi(2^j x) \). From [8, Lemma 3.2 and Proof of Theorem 3.3], it follows that for all \( f \in S'(\mathbb{R}^n) \) and all \( 0 < \theta < 1 \)

\[
M^\theta_{\omega} \mathcal{M}_N f(x) \leq C \left[ M \left( \left( M^{\text{dis}} f \right)^{\theta} \right)(x) \right]^{\frac{1}{\theta}}, \quad \text{for all} \ x \in \mathbb{R}^n,
\]

if \( N \) is sufficiently large. This inequality gives the following result.

Lemma 3.3. Let \( \phi \in S(\mathbb{R}^n) \) and \( f \in S'(\mathbb{R}^n) \). If \( \omega \in \mathcal{W}_{p(\cdot)} \), then \( \| f \|_{H^{p(\cdot)}_{\omega}(\mathbb{R}^n)} \leq C \| M^{\text{dis}} f \|_{L^{p(\cdot)}_{\omega}(\mathbb{R}^n)} \), where \( C \) is a positive constant which does not depend on \( f \).

Proof. From [2] above and Lemma 2.1, we have

\[
\| f \|_{H^{p(\cdot)}_{\omega}(\mathbb{R}^n)} = \| \mathcal{M}_N f \|_{L^{p(\cdot)}_{\omega}(\mathbb{R}^n)} \lesssim \left\| M \left( \left( M^{\text{dis}} f \right)^{\theta} \right)(\cdot) \right\|_{L^{p(\cdot)}_{\omega}(\mathbb{R}^n)}}^{\frac{1}{\theta}} \|
M \left( \left( M^{\text{dis}} f \right)^{\theta} \right) \|_{L^{p(\cdot)}_{\omega}(\mathbb{R}^n)}}^{1/\theta}.
\]

By taking \( \frac{1}{\theta} > s_{\omega, p(\cdot)} \), by [3] Theorem 3.1 and Lemma 2.1, we get

\[
\| f \|_{H^{p(\cdot)}_{\omega}(\mathbb{R}^n)} \lesssim \left\| M \left( \left( M^{\text{dis}} f \right)^{\theta} \right) \right\|_{L^{p(\cdot)}_{\omega}(\mathbb{R}^n)}}^{1/\theta} \lesssim \left\| \left( M^{\text{dis}} f \right)^{\theta} \right\|_{L^{p(\cdot)}_{\omega}(\mathbb{R}^n)}}^{1/\theta} = \| M^{\text{dis}} f \|_{L^{p(\cdot)}_{\omega}(\mathbb{R}^n)}}^{1/\theta}.
\]

Consequently, we obtain the desired result. \( \square \)
Theorem 3.4. Let \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \) be a measurable function with \( 0 < p_- \leq p_+ < \infty \), and \( \omega \in \mathcal{W}_{p(\cdot)} \). Suppose that \( 0 < \theta < 1 \) satisfies \( \frac{1}{\theta} \in \mathcal{S}_{\omega, p(\cdot)} \). Let \( \{\lambda_j\}_{j=1}^\infty \) be a sequence of scalars and let \( \{Q_j\}_{j=1}^\infty \) be a family of cubes such that

\[
\mathcal{A}\left\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot), \omega, \theta\right\} := \left\| \sum_{j=1}^\infty \left( \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L_p(\omega)}} \right)^\theta \chi_{Q_j} \right\|_{L_{p(\omega)/(\theta)}}^{1/\theta} < \infty.
\]

If \( \{m_j\}_{j=1}^\infty \) is a sequence of \( \omega - (p(\cdot), p_0, \lfloor n s_\omega, p(\cdot) - n \rfloor) \) molecules, with \( p_0 > \theta \left( \kappa_{\omega, p(\cdot)} \right)' \), such that \( m_j \) is centered at \( Q_j \) for every \( j \in \mathbb{N} \) and the series

\[
g := \sum_{j=1}^\infty \lambda_j m_j
\]

converges in \( S'(\mathbb{R}^n) \), then \( g \in H^p_{\omega}(\mathbb{R}^n) \) with

\[
\|g\|_{H^p_{\omega}} \lesssim \mathcal{A}\left\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot), \omega, \theta\right\},
\]

where the implicit constant in (3) does not depend on \( g, \{\lambda_j\}_{j=1}^\infty \), and \( \{Q_j\}_{j=1}^\infty \).

Proof. Let \( \phi \in C_c^\infty(\mathbb{R}^n) \) such that \( \chi_{B(0,1)} \leq \phi \leq \chi_{B(0,2)} \), we set \( \phi_{2^k}(x) = 2^{kn} \phi(2^k x) \) where \( k \in \mathbb{Z} \). Since \( g = \sum_j \lambda_j m_j \) in \( S'(\mathbb{R}^n) \), it follows that

\[
|\left(\phi_{2^k} \ast g\right)(x)| \leq \sum_j |\lambda_j| |\left(\phi_{2^k} \ast m_j\right)(x)|, \quad \text{for all } x \in \mathbb{R}^n, \text{ and all } k \in \mathbb{Z}.
\]

We observe that the argument used in [8] Proof of Theorem 5.2] to obtain the pointwise inequality (5.2) works in this setting, but considering now the conditions \( m_1 \), \( m_2 \) and \( m_3 \) in Definition 3.1. Therefore we get

\[
M_{\phi}^\text{dis} g(x) \lesssim \sum_j |\lambda_j| \chi_{2^\sqrt{n} Q_j}(x) M(m_j)(x) + \sum_j |\lambda_j| \left[ \frac{M(\chi_{Q_j})(x)}{\|\chi_{Q_j}\|_{L_p(\omega)}} \right]^n, \quad (x \in \mathbb{R}^n),
\]

where \( M \) is the Hardy-Littlewood maximal operator and \( d_{p(\cdot)} = \lfloor n s_{\omega, p(\cdot)} - n \rfloor \).

Then, from (4) above and Lemma 3.3 we have

\[
\|g\|_{H^p_{\omega}} \lesssim \left\| M_{\phi}^\text{dis} g \right\|_{L_p(\omega)}
\]

\[
\lesssim \left\| \sum_j |\lambda_j| \chi_{2^\sqrt{n} Q_j} \cdot M(m_j) \right\|_{L^p(\omega)} + \left\| \sum_j |\lambda_j| \left[ \frac{M(\chi_{Q_j})(\cdot)}{\|\chi_{Q_j}\|_{L_p(\omega)}} \right]^n \right\|_{L_p(\omega)} = J_1 + J_2.
\]
Now, we consider $J_1$. The boundedness of the Hardy-Littlewood maximal operator $M$ on $L^{p_0}$, Remark 3.2 and Lemma 2.7 yield
\begin{equation}
\| [M(m_j)]^\theta \|_{L^{p_0}/(2\sqrt{n}Q_j)} = \| M(m_j) \|_{L^{p_0}(2\sqrt{n}Q_j)} \lesssim \| m_j \|_{L^{p_0}(\mathbb{R}^n)}^{\theta}
\end{equation}
where $0 < \theta < 1$ and \( \frac{1}{\theta} \in \mathbb{S}_{\omega,p(\cdot)} \), and $p_0 > \theta \left( \kappa^1_{\omega,p(\cdot)} \right)'$. By applying the $\theta$-inequality and \cite[Lemma 5.4]{5} with $b_j = (\chi_{2\sqrt{n}Q_j} \cdot M(m_j))^{\theta}$, \cite[3]{5} and $A_j = \| \chi_{2\sqrt{n}Q_j} \|_{L^{p_0}(\mathbb{R}^n)}^{-1}$, we get
\begin{equation}
J_1 \lesssim \left\| \sum_j \left( \frac{|\lambda_j|}{\| \chi_Q \|_{L^{p(\cdot)}}} \chi_{2\sqrt{n}Q_j} \right)^{\frac{1}{\theta}} \right\|_{L^{p(\cdot)/\theta}} \lesssim \left\| \sum_j \left( \frac{|\lambda_j|}{\| \chi_{2\sqrt{n}Q_j} \|_{L^{p(\cdot)}}} \chi_{2\sqrt{n}Q_j} \right)^{\frac{1}{\theta}} \right\|_{L^{p(\cdot)/\theta}}.
\end{equation}
Being $\chi_{2\sqrt{n}Q_j} \leq M(\chi_{Q_j})^2$, by Lemma 2.7 and \cite[Theorem 3.1]{5}, we have
\begin{equation}
J_1 \lesssim A \left( \left\{ \lambda_j \right\}_{j=1}^\infty, \left\{ Q_j \right\}_{j=1}^\infty, p(\cdot), \omega, \theta \right) < \infty.
\end{equation}
To estimate $J_2$, we write $r = \frac{n + d_{p(\cdot)} + 1}{n}$. Thus
\begin{equation}
J_2 \lesssim \left\| \left\{ \sum_j \frac{|\lambda_j|}{\| \chi_Q \|_{L^{p(\cdot)}}} M(\chi_{Q_j}(\cdot))^{1/r} \right\}^r \right\|_{L^{r^{p(\cdot)/\omega^1/r}}(\mathbb{R}^n)}.
\end{equation}
Since
\[ r = \frac{n + d_{p(\cdot)} + 1}{n} = \frac{n + \left[ ns_{\omega,p(\cdot)} - n \right] + 1}{n} \geq s_{\omega,p(\cdot)}, \]
to apply again \cite[Theorem 3.1]{5} followed by the $\theta$-inequality we obtain
\begin{equation}
J_2 \lesssim \left\| \sum_j \left( \frac{|\lambda_j|}{\| \chi_{Q_j} \|_{L^{p(\cdot)}}} \chi_{Q_j} \right)^{1/r} \right\|_{L^{r^{p(\cdot)/\omega^1/r}}(\mathbb{R}^n)} \lesssim \left\| \sum_j \left( \frac{|\lambda_j|}{\| \chi_{Q_j} \|_{L^{p(\cdot)}}} \chi_{Q_j} \right)^{1/r} \right\|_{L^{r^{p(\cdot)/\omega^1/r}}(\mathbb{R}^n)}.
\end{equation}
Hence,
\begin{equation}
J_2 \lesssim A \left( \left\{ \lambda_j \right\}_{j=1}^\infty, \left\{ Q_j \right\}_{j=1}^\infty, p(\cdot), \omega, \theta \right) < \infty.
\end{equation}
Finally, \cite[6]{5} and \cite[7]{5} give \cite[3]{5} and, with that, $g \in H^{p(\cdot)}_{\omega}(\mathbb{R}^n)$. \hfill \Box
4 Weighted variable estimates for singular integrals

Let $\Omega \in C^\infty(S^{n-1})$ with $\int_{S^{n-1}} \Omega(u) d\sigma(u) = 0$. We define the operator $T$ by

$$
Tf(x) = \lim_{\epsilon \to 0^+} \int_{|y| > \epsilon \atop |y| < 2\epsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x - y) \, dy, \quad x \in \mathbb{R}^n.
$$

It is well known that the operator $T$ is bounded on $L^{p_0}(\mathbb{R}^n)$ for all $1 < p_0 < +\infty$ and of weak-type $(1, 1)$ (see e.g. [13]).

We start studying the behavior of the operator $T$ on atoms. Then, we prove the boundedness of $T$ on $H^p_a$.

**Proposition 4.1.** Let $T$ be the operator given by (8) and let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function with $0 < p_- \leq p_+ < \infty$, and $\omega \in W_{p(\cdot)}$. Then, for some universal constant $C > 0$, $C(Ta)(\cdot)$ is a $\omega - (p(\cdot), p_0, [n s_{p(\cdot)}])$ molecule for each $\omega - (p(\cdot), p_0, n + 2[n s_{p(\cdot)}] + 2)$ atom $a(\cdot)$.

**Proof.** It is well known that $\widehat{Tf}(\xi) = m(\xi) \hat{f}(\xi)$, where the multiplier $m$ is homogeneous of degree $0$ and is indefinitely differentiable on $\mathbb{R}^n \setminus \{0\}$ (see e.g. [13]). Moreover, for $K(y) = \frac{\Omega(y/|y|)}{|y|^n}$ we have

$$
|\partial_y^\alpha K(y)| \leq C|y|^{-n-|\alpha|}, \quad \text{for all } y \neq 0 \text{ and all multi-index } \alpha.
$$

Let $p_0 > 1$ and $d_{p(\cdot)} = [n s_{p(\cdot)}] - n$. Given a $w - (p(\cdot), p_0, n + 2d_{p(\cdot)} + 2)$ atom $a(\cdot)$ with support in the cube $Q = Q(z, r)$ (with side length $\ell(Q) = r$), we have that

$$
\|Ta\|_{L^{p_0}(2\sqrt{n}Q)} \leq C\|a\|_{L^{p_0}} \leq C \frac{|Q|^{1/p_0}}{\|\chi_Q\|_{L^{p(\cdot)}_x}},
$$

since $T$ is bounded on $L^{p_0}(\mathbb{R}^n)$. In view of the moment condition of $a(\cdot)$ we obtain

$$
Ta(x) = \int_Q K(x - y)a(y)dy = \int_Q [K(x - y) - q_{n+2d_{p(\cdot)}+2}(x, y)]a(y)dy, \quad x \notin 2\sqrt{n}Q
$$

where $y \to q_{n+2d_{p(\cdot)}+2}(x, y)$ is the degree $n + 2d_{p(\cdot)} + 2$ Taylor polynomial of the function $y \to K(x - y)$ expanded around $z$. From the estimate in (9), and the standard estimate of the remainder term of the Taylor expansion, there exists $\xi$ between $y$ and $z$ such that

$$
|Ta(x)| \leq C\|a\|_1 \frac{\ell(Q)^{n+2d_{p(\cdot)}+3}}{|x - \xi|^{2n+2d_{p(\cdot)}+3}} \leq C \frac{\ell(Q)^{2n+2d_{p(\cdot)}+3}}{\|\chi_Q\|_{L^{p(\cdot)}_x}} |x - z|^{-2n-2d_{p(\cdot)}-3}, \quad x \notin 2\sqrt{n}Q,
$$

this inequality and a simple computation allow us to obtain

$$
|Ta(x)| \leq C\|\chi_Q\|_{L^{p(\cdot)}_x}^{-1} \left(1 + \frac{|x - z|}{\ell(Q)}\right)^{-2n-2d_{p(\cdot)}-3}, \quad \text{for all } x \notin 2\sqrt{n}Q.
$$
From the estimate in (11) we obtain that the function \( x \to x^\alpha Ta(x) \) belongs to \( L^1(\mathbb{R}^n) \) for each \( |\alpha| \leq d_{p(\cdot)} \), so

\[
\left| \left((-2\pi ix)^\alpha Ta\right)\hat{\cdot}(\xi) \right| = \left| \partial_{\xi}^\alpha (m(\xi)\hat{a}(\xi)) \right| = \left| \sum_{\beta \leq \alpha} c_{\alpha,\beta} (\partial_{\xi}^{\alpha-\beta} m)(\xi) (\partial_{\xi}^\beta \hat{a})(\xi) \right| = \left| \sum_{\beta \leq \alpha} c_{\alpha,\beta} (\partial_{\xi}^{\alpha-\beta} m)(\xi) \left((-2\pi ix)^\beta \hat{a}(\xi)\right) \right|,
\]

from the homogeneity of the function \( \partial_{\xi}^{\alpha-\beta} m \) we obtain that

\[
\left| \left((-2\pi ix)^\alpha Ta\right)\hat{\cdot}(\xi) \right| \leq C \left| \sum_{\beta \leq \alpha} |c_{\alpha,\beta}| \left| (\partial_{\xi}^{\alpha-\beta} m)(\xi) \right| \right|, \quad \xi \neq 0.
\]

Since \( \lim_{\xi \to 0} \frac{\left| (-2\pi ix)^\beta \hat{a}(\xi) \right|}{|\xi|^{|\alpha|-|\beta|}} = 0 \) for each \( \beta \leq \alpha \) (see 5.4, pp. 128, in [14]), taking the limit as \( \xi \to 0 \) at (13), we get

\[
\int_{\mathbb{R}^n} (-2\pi ix)^\alpha Ta(x) \, dx = ((-2\pi ix)^\alpha Ta)^{\hat{\cdot}}(0) = 0, \quad \text{for all } |\alpha| \leq d_{p(\cdot)}.
\]

From (10), (12) and (14) it follows that there exists an universal constant \( C > 0 \) such that \( C(Ta)(\cdot) \) is a \( w - (p(\cdot), p_0, d_{p(\cdot)}) \) molecule if \( a(\cdot) \) is a \( w - (p(\cdot), p_0, n + 2d_{p(\cdot)} + 2) \) atom.

**Theorem 4.2.** Let \( p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \), \( \omega \in \mathcal{W}_{p(\cdot)} \) and let \( T \) be the singular integral operator given by (8). Then \( T \) can be extended to a bounded operator on \( H^{p(\cdot)}_\omega(\mathbb{R}^n) \).

**Proof.** Given \( \omega \in \mathcal{W}_{p(\cdot)} \), by Definition 2.2 there exists \( 0 < \theta < 1 \) such that \( \frac{1}{\theta} \in \mathcal{S}_{\omega,p(\cdot)} \). Let \( p_0 > \max \left\{ \frac{1}{\theta} \left( \kappa^{1/\theta}_{\omega,p(\cdot)} \right)', 1 \right\} \). Given \( f \in \mathcal{S}_0(\mathbb{R}^n) \), by Theorem 2.6 there exist a sequence of real numbers \( \{\lambda_j\}_{j=1}^\infty \), a sequence of cubes \( \{Q_j\}_{j=1}^\infty \), and \( \omega - (p(\cdot), p_0, n + 2d_{p(\cdot)} + 2) \) atoms \( a_j \) supported on \( Q_j \), such that \( f = \sum_{j=1}^\infty \lambda_j a_j \) converges in \( L^{p_0}(\mathbb{R}^n) \) and

\[
A(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot), \omega, \theta) \lesssim \|f\|_{H^{p(\cdot)}_\omega} < \infty.
\]

Since \( f = \sum_{j=1}^\infty \lambda_j a_j \) converges in \( L^{p_0}(\mathbb{R}^n) \) and \( T \) is bounded on \( L^{p_0} \), we have that

\[
Tf = \sum_{j=1}^\infty \lambda_j Ta_j \in S'(\mathbb{R}^n).
\]

Now, by (16) and (15) above and Proposition 4.1 we can apply Theorem 3.4 to obtain

\[
\|Tf\|_{H^{p(\cdot)}_\omega} \lesssim A(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot), \omega, \theta) \lesssim \|f\|_{H^{p(\cdot)}_\omega},
\]

for all \( f \in \mathcal{S}_0(\mathbb{R}^n) \). Finally, the theorem follows from the density of \( \mathcal{S}_0(\mathbb{R}^n) \) in \( H^{p(\cdot)}_\omega(\mathbb{R}^n) \) (see Proposition 2.8).

In particular, if \( p(\cdot) \in \mathcal{P}^{\log} \) and \( \omega \in \mathcal{W}_{p(\cdot)} \), then Hilbert transform and Riesz transforms admit a continuous extension on \( H^{p(\cdot)}_\omega(\mathbb{R}) \) and on \( H^{p(\cdot)}_\omega(\mathbb{R}^n) \) respectively.
5 Weighted variable estimates for Riesz potential

Let $0 < \alpha < n$. The **Riesz potential** of order $\alpha$ is the operator $I_\alpha$ defined, say on $S(\mathbb{R}^n)$, by

\begin{equation}
I_\alpha f(x) = \int_{\mathbb{R}^n} f(y) |x - y|^{\alpha - n} dy, \quad x \in \mathbb{R}^n.
\end{equation}

In [11], the author proved that the operator $I_\alpha$ extends to a bounded operator $H^{p,\alpha}_\omega(\mathbb{R}^n) \to L^{q,\alpha}_\omega(\mathbb{R}^n)$, for $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$, under the following assumptions:

1) $q(\cdot) \in P_{\text{log}}(\mathbb{R}^n)$ and $\omega \in W_{q(\cdot)}$;
2) for every cube $Q \subset \mathbb{R}^n$, $\|\chi_Q\|_{L^{p,\alpha}_\omega} \approx |Q|^{-\alpha/n} \|\omega\|_{L^{p,\alpha}_\omega}$.

We observe that if $q(\cdot) \in P_{\text{log}}(\mathbb{R}^n)$ and $\omega \equiv 1$, then the condition 2) holds. This was proved in [12]. In [11], the author gave non-trivial examples of power weights satisfying 2). So, the condition 2) is an admissible hypothesis.

Next, assuming the conditions 1) and 2) above, we will prove that Riesz potential $I_\alpha$ extends to a bounded operator $H^{p,\alpha}_\omega(\mathbb{R}^n) \to H^{p,\alpha}_\omega(\mathbb{R}^n)$. For them, we will first show that $I_\alpha$ maps atoms into molecules.

**Proposition 5.1.** Let $0 < \alpha < n$, $q(\cdot): \mathbb{R}^n \to (0, \infty)$ be a measurable function with $0 < q_\omega \leq q_\alpha < \infty$, and $\omega \in W_{q(\cdot)}$. If $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$ and $\|\chi_Q\|_{L^{p,\alpha}_\omega} \approx |Q|^{-\alpha/n} \|\omega\|_{L^{p,\alpha}_\omega}$ for every cube $Q$, then, for some universal constant $C > 0$, $C(I_\alpha a(\cdot))$ is a $\omega - (q(\cdot), q_0, \lfloor ns_{\omega,q(\cdot)} - n \rfloor)$ molecule for each $\omega - (p(\cdot), p_0, 2\lfloor ns_{\omega,q(\cdot)} - n \rfloor + \lfloor \alpha \rfloor + 3 + n)$ atom $a(\cdot)$, where $q_0 > \frac{\alpha}{n - \alpha}$ and $\frac{1}{p_0} := \frac{1}{q_0} + \frac{\alpha}{n}$.

**Proof.** By [11] Proposition 3.1, we have that $W_{q(\cdot)} \subset W_{p(\cdot)}$ and $s_{\omega,p(\cdot)} \leq s_{\omega,q(\cdot)} + \frac{\alpha}{n}$. Let $d_{q(\cdot)} := \lfloor ns_{\omega,q(\cdot)} - n \rfloor$ and $N = 2d_{q(\cdot)} + \lfloor \alpha \rfloor + 3 + n$, we observe that $N \geq d_{p(\cdot)}$. Fix $q_0 > \frac{\alpha}{n - \alpha}$ and we put $\frac{1}{p_0} := \frac{1}{q_0} + \frac{\alpha}{n}$. Given a $\omega - (p(\cdot), p_0, N)$ atom $a(\cdot)$ supported on a cube $Q = (z, r)$, we will show that there exists an universal constant $C > 0$ such that $CI_\alpha a(\cdot)$ is a $\omega - (q(\cdot), q_0, d_{q(\cdot)})$ molecule centered at $Q$. Indeed,

$m_1$) by Sobolev’s theorem, the condition $a_1)$ of the atom $a(\cdot)$, and since $\|\chi_Q\|_{L^{p,\alpha}_\omega} \approx |Q|^{-\alpha/n} \|\omega\|_{L^{p,\alpha}_\omega}$ for every cube $Q$, we have

$$
\|I_\alpha a\|_{L^{0,2\sqrt{\pi}Q}} \lesssim \|a\|_{L^{p_0}} \leq \frac{|Q|^{\frac{1}{q_0}}}{\|\chi_Q\|_{L^{p,\alpha}_\omega}} \lesssim \frac{|Q|^{\frac{1}{p_0}}}{\|\chi_Q\|_{L^{p,\alpha}_\omega}};
$$

$m_2$) by doing use of the conditions $a_3)$ and $a_2)$ of the atom $a(\cdot)$, as in the estimate
(24) obtained in [11, Theorem 5.1], we get, for \( x \) outside \( 2\sqrt{n}Q \), that
\[
| (I_\alpha a)(x) | \lesssim \frac{\ell(Q)^{N+1}}{|x-z|^{n-\alpha+N+1}} \| a \|_{L^1}
\]
\[
\lesssim \frac{1}{\| \chi_Q \|_{L^q(Q)}^q} \left( \frac{\ell(Q)}{|x-z|} \right)^{n-\alpha+N+1}
\]
\[
\lesssim \frac{1}{\| \chi_Q \|_{L^q(Q)}^q} \left( \frac{\ell(Q) + |x-z|}{\ell(Q)} \right)^{2n+2d_{q(\cdot)}+3}
\]
\[
= \frac{1}{\| \chi_Q \|_{L^q(Q)}^q} \left( 1 + \frac{|x-z|}{\ell(Q)} \right)^{-2n-2d_{q(\cdot)}-3},
\]
where the third inequality follows since \( x \) outside \( 2\sqrt{n}Q \) implies \( \ell(Q) + |x-z| < 2|x-z| \).

\( m_3 \) the moment condition was proved by Taibleson and Weiss in [15].

Thus \( I_\alpha a \) satisfies Definition 3.1 with an universal implicit constant. This completes the proof.

**Theorem 5.2.** Let \( 0 < \alpha < n, q(\cdot) \in P^{\log}(\mathbb{R}^n) \) with \( 0 < q_- \leq q_+ < \infty \) and \( \omega \in \mathcal{W}_{q(\cdot)} \). If \( \frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n} \) and \( \| \chi_Q \|_{L^q(Q)} = |Q|^{-\alpha/n} \| \chi_Q \|_{L^q(Q)} \) for every cube \( Q \), then the Riesz potential \( I_\alpha \) given by (14) can be extended to a bounded operator \( H^0_{q(\cdot)}(\mathbb{R}^n) \to H^0_{q(\cdot)}(\mathbb{R}^n) \).

**Proof.** Let \( \omega \in \mathcal{W}_{q(\cdot)} \), by Definition 2.2, there exists \( 0 < \theta < 1 \) such that \( \frac{1}{\theta} \in S_{\omega, q(\cdot)} \).

Now, we take \( q_0 > \max \left\{ \theta \left( \frac{1/n}{\omega(\cdot) q(\cdot)} \right)’, \frac{n}{n-\alpha} \right\} \), and define \( \frac{1}{p_0} := \frac{1}{q_0} + \frac{\alpha}{n} \). By [11] Proposition 3.1, we have that \( \mathcal{W}_{q(\cdot)} \subset \mathcal{W}_{p(\cdot)} \) and \( s_{\omega, p(\cdot)} \leq s_{\omega, q(\cdot)} + \frac{\alpha}{n} \). So, given \( f \in S_0(\mathbb{R}^n) \), by Theorem 2.6 there exist a sequence of real numbers \( \{ \lambda_j \}_{j=1}^\infty \), a sequence of cubes \( \{ Q_j \}_{j=1}^\infty \), and \( \omega - (p(\cdot), p_0, 2[ns_{\omega, q(\cdot)} - n] + [\alpha] + 3 + n) \) atoms \( a_j \) supported on \( Q_j \), satisfying
\[
A(\{ \lambda_j \}_{j=1}^\infty, \{ Q_j \}_{j=1}^\infty, p(\cdot), \omega, \theta) \lesssim \| f \|_{H^0_{q(\cdot)}} < \infty,
\]
and \( f = \sum_{j=1}^\infty \lambda_j a_j \) converges in \( L^{p_0}(\mathbb{R}^n) \). Then, by Sobolev’s Theorem, \( I_\alpha f = \sum_{j=1}^\infty \lambda_j I_\alpha(a_j) \) in \( L^{\frac{n p_0}{n-\alpha p_0}}(\mathbb{R}^n) \) thus
\[
I_\alpha f = \sum_{j=1}^\infty \lambda_j I_\alpha(a_j) \text{ in } \mathcal{S}'(\mathbb{R}^n).
\]

Now, by (19), (18) and Proposition 2.9 and Proposition 3.1 we can apply Theorem 3.4 to obtain
\[
\| I_\alpha f \|_{H^0_{q(\cdot)}} \lesssim A(\{ \lambda_j \}_{j=1}^\infty, \{ Q_j \}_{j=1}^\infty, q(\cdot), \omega, \theta) \lesssim A(\{ \lambda_j \}_{j=1}^\infty, \{ Q_j \}_{j=1}^\infty, p(\cdot), \omega, \theta) \lesssim \| f \|_{H^0_{q(\cdot)}}
\]
for all \( f \in \mathcal{S}_0(\mathbb{R}^n) \). Finally, since \( p(\cdot) \in P^{\log}(\mathbb{R}^n) \), \( 0 < p_- \leq p_+ < \infty \) and \( \omega \in \mathcal{W}_{p(\cdot)} \), the theorem follows from the density of \( \mathcal{S}_0(\mathbb{R}^n) \) in \( H^0_{q(\cdot)}(\mathbb{R}^n) \) (see Proposition 2.8). 

\( \Box \)
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