THE MÖBIUS-POMPEİU METRIC PROPERTY

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In the paper we consider an extension of Möbius-Pompeiu theorem of the elementary geometry over metric spaces. We specially take into consideration Ptolemaic metric spaces.

1 The Möbius-Pompeıu theorem and metric spaces

In this paper we consider the following statement of elementary geometry [1], [2]:

**Theorem 1.1 (Möbius, Pompeiu)** Let ABC be an equilateral triangle and M any point in its plane. Then segments MA, MB and MC are sides of a triangle.

Let us consider analogous problem for the metric space $(X, d)$ with at least four points. Let $A, B, C \in X$ be three fixed points. Then, for the point $M \in X$ we suppose that a triangle can be formed from the distances $d_1 = d(M, A)$, $d_2 = d(M, B)$ and $d_3 = d(M, C)$ iff the following conjunction of inequalities is true:

$$(1.1) \quad d_1 + d_2 - d_3 \geq 0 \quad \text{and} \quad d_2 + d_3 - d_1 \geq 0 \quad \text{and} \quad d_3 + d_1 - d_2 \geq 0.$$ 

If in conjunction (1.1) at least one equality is true, then we suppose that a degenerative triangle can be formed. If in (1.1) sharp inequalities are true:

$$(1.2) \quad d_1 + d_2 - d_3 > 0 \quad \text{and} \quad d_2 + d_3 - d_1 > 0 \quad \text{and} \quad d_3 + d_1 - d_2 > 0,$$

then we suppose that a non-degenerative triangle can be formed. In this case, for the point $M$, for which the conjunction (1.2) is true, we define that point have Möbius-Pompeıu metric property. The main subject of this paper is to determine points $M$ which do not have Möbius-Pompeıu metric property, i.e. these points which fulfill the following disjunction of the inequalities:

$$(1.3) \quad d_1 + d_2 - d_3 \leq 0 \quad \text{or} \quad d_2 + d_3 - d_1 \leq 0 \quad \text{or} \quad d_3 + d_1 - d_2 \leq 0.$$ 

Let us notice that the point $M \in \{A, B, C\}$ do not have Möbius-Pompeıu metric property. Thus in consideration which follows, we assume that the metric space $(X, d)$ has at least four points.

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2 Ptolemaic metric spaces

A metric space \((X, d)\) is called Ptolemaic metric space if Ptolemaic inequality holds:

\[
d(x_1, x_2)d(x_3, x_4) \leq d(x_2, x_4)d(x_1, x_3) + d(x_1, x_4)d(x_2, x_3)
\]

for every \(x_1, x_2, x_3, x_4 \in X\) [3]. A normed space \((X, |.|)\) is Ptolemaic normed space if metric space \((X, d)\) is Ptolemaic with the distance \(d(x, y) = |x - y|\). Let us notice that the following lemma is true [3]:

**Lemma 2.1** A normed space is Ptolemaic iff it is an inner product space.

We give two basic examples of Ptolemaic spaces [3].

**Example 2.2** 1°. The space \(\mathbb{R}^n\) with the Euclidean metric \(d(x, y) = |x - y|\) is a Ptolemaic metric space.

2°. The space \(\mathbb{R}^n\) with the chordal metric on the unit Riemann sphere

\[
d(x, y) = \frac{2|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}
\]

is a Ptolemaic metric space.

We will illustrate following considerations with the previous examples of Ptolemaic metric spaces in the case of dimension \(n = 2\).

3 The main results

Let \((X, d)\) be a metric space. Let us fix three points \(A, B, C \in X\) and form distances:

\[
a = d(B, C), \quad b = d(C, A), \quad c = d(A, B).
\]

For any point \(M \in X\) let us form distances:

\[
d_1 = d(M, A), \quad d_2 = d(M, B), \quad d_3 = d(M, C).
\]

**Inequality** \(d_2 + d_3 \leq d_1\)

Let us determine a set of \(M\) points of metric spaces \(X\) for which the following inequality is true:

\[
d_2 + d_3 \leq d_1.
\]

Let us form two functions:

\[
\alpha_1 = \alpha_1(M) = 4d_2^2d_3^2 - (d_1^2 - (d_2^2 + d_3^2))^2,
\]

\[
\beta_1 = \beta_1(M) = d_2^2 + d_3^2 - d_1^2.
\]
The Möbius-Pompeiu metric property

Lemma 3.1 For points $A$, $B$ and $C$ inequality $\alpha_1 \leq 0$ is true.

Proof. For point $A$: $d_1 = 0$ and $\alpha_1 = -(c^2 - b^2)^2 \leq 0$ are true. Similarly, the previous inequality is true for the points $B$ and $C$. $\blacksquare$

Example 3.2 Let vertices $A$, $B$, $C$ of the triangle $ABC$ in the plane $\mathbb{R}^2$ be given by coordinates $A(a_1,b_1)$, $B(a_2,b_2)$, $C(a_3,b_3)$ and let $M(x,y)$ be any point in its plane.

1°. Let us in the plane $\mathbb{R}^2$ use Euclidean metric $d$. Let us specify the form of term $\alpha_1$ and $\beta_1$ which correspond to functions (3.4) and (3.5) respectively. It is true:

$$\alpha_1 = k(x^2+y^2)^2 + \left( A_1 x + B_1 y \right) (x^2+y^2) + C_1 x^2 + D_1 xy + E_1 y^2 + F_1 x + G_1 y + H_1,$$

for some coefficients $k, A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1 \in \mathbb{R}$ ($k = 3$). Equality $\alpha_1 = 0$ determines the algebraic curve of the fourth order. By inequality $\alpha_1 < 0$ we determine the interior of the previous curve. Also, it is true:

$$\beta_1 = A_2 (x^2+y^2) + B_2 x + C_2 y + D_2,$$

for some coefficients $A_2, B_2, C_2, D_2 \in \mathbb{R}$ ($A_2 = 1$). If $B_2^2 + C_2^2 > 4D_2$ equality $\beta_1 = 0$ is possible and determines the circle. Then by inequality $\beta_1 < 0$ we determine the interior of the circle.

2°. Let us in the plane $\mathbb{R}^2$ use chordal metric $\overline{d}$. Let us specify the form of the term $\alpha_1$ and $\overline{\beta}_1$ which correspond to functions (3.4) and (3.5) respectively. It is true:

$$\alpha_1 = \frac{k(x^2+y^2)^2 + (\overline{A}_1 x + \overline{B}_1 y) (x^2+y^2) + \overline{C}_1 x^2 + \overline{D}_1 xy + \overline{E}_1 y^2 + \overline{F}_1 x + \overline{G}_1 y + \overline{H}_1}{(1 + a_1^2 + b_1^2)(1 + a_2^2 + b_2^2)(1 + a_3^2 + b_3^2)},$$

for some coefficients $\overline{k}, \overline{A}_1, \overline{B}_1, \overline{C}_1, \overline{D}_1, \overline{E}_1, \overline{F}_1, \overline{G}_1, \overline{H}_1 \in \mathbb{R}$. If $\overline{k} \neq 0$ equality $\alpha_1 = 0$ determines the algebraic curve of the fourth order. Then by inequality $\alpha_1 < 0$ we determine the interior of the previous curve. Also, it is true:

$$\overline{\beta}_1 = \frac{\overline{A}_2 (x^2+y^2) + \overline{B}_2 x + \overline{C}_2 y + \overline{D}_2}{(1 + a_1^2 + b_1^2)(1 + a_2^2 + b_2^2)(1 + a_3^2 + b_3^2)},$$

for some coefficients $\overline{A}_2, \overline{B}_2, \overline{C}_2, \overline{D}_2 \in \mathbb{R}$. If $\overline{A}_2 \neq 0$ and $\overline{B}_2^2 + \overline{C}_2^2 > 4\overline{A}_2 \overline{D}_2$ equality $\overline{\beta}_1 = 0$ is possible and determines the circle. Then by the inequality $\overline{\beta}_1 < 0$ we determine the interior of the circle.

Further, let us notice that for the function $\alpha_1$:

$$\alpha_1 = (d_2 + d_3 - d_1)(d_3 + d_1 - d_2)(d_1 + d_2 - d_3)(d_1 + d_2 + d_3).$$

According to (3.10) equality $\alpha_1 = 0$ is equivalent with union of equalities:
Lemma 3.3 1°. For the point B: \(d_2 + d_3 \leq d_1\) iff \(c \geq a\). 2°. For the point C: \(d_2 + d_3 \leq d_1\) iff \(b \geq a\).

Remark 3.4 If \(a > b, c\) then for points B and C: \(\alpha_1 \leq 0\) and \(\alpha_1^{(1)} > 0\).

Lemma 3.5 If for point M: \(d_2 + d_3 \leq d_1\), then we have inequalities:

\[
(3.14) \quad d_1 + d_2 \geq d_3, \text{ where equality is true for } M = B \text{ and } a = c
\]

and

\[
(3.15) \quad d_3 + d_1 \geq d_2, \text{ where equality is true for } M = C \text{ and } a = b.
\]

Proof. It is true

\[
(3.16) \quad (d_1) + d_2 - d_3 \geq (d_2 + d_3) + d_2 - d_3 = 2d_2 \geq 0.
\]

Hence, the inequality (3.14) follows. Thus, the equality is true only if \(M = B\) \((d_2 = 0)\) and \(a = c\). Analogously, it is true

\[
(3.17) \quad d_3 + (d_1) - d_2 \geq d_3 + (d_2 + d_3) - d_2 = 2d_3 \geq 0.
\]

Hence, the inequality (3.15) follows. Thus, the equality is true only if \(M = C\) \((d_3 = 0)\) and \(a = b\).

Lemma 3.6 1°. If the point M fulfills \(d_2 + d_3 \leq d_1\) then the following implication is true:

\[
(3.18) \quad \alpha_1 \leq 0 \implies \beta_1 \leq 0.
\]

2°. If the point M fulfills \(d_3 + d_1 \leq d_2\) or \(d_1 + d_2 \leq d_3\) then the following implication is true:

\[
(3.19) \quad \alpha_1 \leq 0 \implies \beta_1 \geq 0.
\]

Proof. The implications (3.18) and (3.19) have the same assumptions:

\[
(3.20) \quad \alpha_1 = 4d_2^2d_3^3 - (d_1^2 - d_2^2 - d_3^2)^2 = (2d_2d_3 - d_1^2 + d_2^2 + d_3^2)(2d_2d_3 + d_1^2 - d_2^2 - d_3^2) \leq 0,
\]

which follow if the following conjunction is true

\[
(3.21) \quad (2d_2d_3 - d_1^2 + d_2^2 + d_3^2) \leq 0 \text{ and } (2d_2d_3 + d_1^2 - d_2^2 - d_3^2) \geq 0
\]

or the conjunction

\[
(3.22) \quad (2d_2d_3 - d_1^2 + d_2^2 + d_3^2) \geq 0 \text{ and } (2d_2d_3 + d_1^2 - d_2^2 - d_3^2) \leq 0.
\]
Lemma 3.7 In the metric space $X$ the condition $d_2 + d_3 \leq d_1$ is equivalent to the conjunction $\alpha_1 \leq 0$ and $\beta_1 \leq 0$.

Proof. ($\Rightarrow$) Let for the point $M$ the condition $d_2 + d_3 \leq d_1$ be true. On the basis of equality (3.10) and on the basis of lemma 3.5 it follows $\alpha_1 \leq 0$. Therefore, on the basis of lemma 3.6, it follows $\beta_1 \leq 0$.

($\Leftarrow$) Let for the point $M$ conjunction $\alpha_1 \leq 0$ and $\beta_1 \leq 0$ be true. Then from the conjunction

$$\alpha_1 = (d_2 + d_3 - d_1)(d_2 + d_3 + d_1)(2d_2d_3 - \beta_1) \leq 0 \text{ and } \beta_1 \leq 0$$

follows the condition $d_2 + d_3 \leq d_1$. □

Lemma 3.8 In Ptolemaic metric space $X$ an inequality $\alpha_1^{(1)} \leq 0$ is true iff $b \geq a$ or $c \geq a$.  

1°. Let $d_2 + d_3 \leq d_1$ be true. For $M = B$ or $M = C$ implication (3.18) is directly verified. Especially for $M = B$ and $a = c$ or for $M = C$ and $a = b$ equality $\beta_1 = 0$ is true. Let us assume that $M \neq B, C$ and let us assume that $\alpha_1 \leq 0$ in (3.18) be true. On the basis of $d_2 + d_3 \leq d_1$, according to lemma 3.5 it follows that $d_1 + d_2 > d_3$ and $d_3 + d_1 > d_2$. Therefore

(3.23) \[ 2d_2d_3 - d_1^2 + d_2^2 + d_3^2 = (d_2 + d_3)^2 - d_1^2 \leq 0 \]

and (3.24) \[ 2d_2d_3 + d_1^2 - d_2^2 - d_3^2 = (d_1 - d_2 + d_3)(d_1 + d_2 - d_3) > 0. \]

From (3.23) and (3.24) we can conclude that the conjunction (3.21) is true and conjunction (3.22) is not true. From the conjunction (3.21) it follows that $d_1^2 - d_2^2 - d_3^2 \geq 2d_2d_3 > d_2^2 + d_3^2 - d_1^2$ and from there $d_1^2 > d_2^2 + d_3^2$, i.e. $\beta_1 < 0$.

2°. Let $d_3 + d_1 \leq d_2$ be true. For $M = B$ or $M = C$ implication (3.19) is directly verified. Especially for $M = B$ and $a = c$ or for $M = C$ and $a = b$ equality $\beta_1 = 0$ is true. Let us assume that $M \neq B, C$ and let us assume that $\alpha_1 \leq 0$ in (3.19) be true. On the basis of $d_3 + d_1 \leq d_2$, according to the lemma analogous to lemma 3.5, it follows $d_2 + d_3 > d_1$ and $d_1 + d_2 > d_3$. Therefore

(3.25) \[ 2d_2d_3 - d_1^2 + d_2^2 + d_3^2 = (d_2 + d_3)^2 - d_1^2 > 0 \]

and (3.26) \[ 2d_2d_3 + d_1^2 - d_2^2 - d_3^2 = (d_1 - d_2 + d_3)(d_1 + d_2 - d_3) \leq 0. \]

From (3.25) and (3.26) we can conclude that conjunction (3.22) is true and conjunction (3.21) is not true. From conjunction (3.22) follows $d_2^2 + d_3^2 - d_1^2 \geq 2d_2d_3 > d_2^2 + d_3^2 - d_1^2$ and therefore, $d_2^2 + d_3^2 > d_1^2$, i.e. $\beta_1 > 0$. The implication (3.19) is similarly verified in the case of the inequality $d_1 + d_2 \leq d_3$. □

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Proof. On the basis of lemma 3.3 if \( a \leq c \) then for the point \( B \) we have: \( \alpha_1^{(1)} = a - c \leq 0 \) or if \( a \leq b \) then for the point \( C \) we have: \( \alpha_1^{(1)} = b - a \leq 0 \). Conversely, let \( a > b, c \) be true. Let \( M \in X \setminus \{A, B, C\} \) be any point. Then on the basis of Ptolemaic inequality
\[
(3.28) \quad c \cdot d_3 + b \cdot d_2 \geq a \cdot d_1
\]
and assumption \( a > b, c \) we can conclude
\[
(3.29) \quad \alpha_1^{(1)} = d_2 + d_3 - d_1 > 0.
\]
By contraposition the statement follows.

On the basis of the previous lemmas we can conclude the following theorem is true.

**Theorem 3.9** In the metric space \( X \) a point \( M \) fulfills
\[
\alpha_1^{(1)} = d_2 + d_3 - d_1 \leq 0 \text{ iff } \alpha_1 \leq 0 \text{ and } \beta_1 \leq 0 \text{ are true. In Ptolemaic metric space } X \text{ the set of these points } M \text{ is non-empty iff:}
\]
\[
(3.30) \quad b \geq a \text{ or } c \geq a.
\]

**Inequalities**
\[
d_2 + d_3 \leq d_1, \quad d_3 + d_1 \leq d_2, \quad d_1 + d_2 \leq d_3
\]

Let us determine set of points \( M \) in (Ptolemaic) metric spaces for which some inequalities in (1.3) are true. With respect to point \( A \) we formed functions (3.4) and (3.5). Next, with respect to point \( B \) let us form functions:
\[
(3.31) \quad \alpha_2 = \alpha_2(M) = 4d_3^2d_1^2 - (d_2^2 - (d_3^2 + d_1^2))^2,
\]
\[
(3.32) \quad \beta_2 = \beta_2(M) = d_3^2 + d_1^2 - d_2^2
\]
and with respect to point \( C \) let us form functions:
\[
(3.33) \quad \alpha_3 = \alpha_3(M) = 4d_2^2d_1^2 - (d_3^2 - (d_1^2 + d_2^2))^2,
\]
\[
(3.34) \quad \beta_3 = \beta_3(M) = d_1^2 + d_2^2 - d_3^2.
\]
The following equality \( \alpha_1 = \alpha_2 = \alpha_3 \) is true. Analogously to the theorem 3.9 we can conclude the following theorems are true.

**Theorem 3.10** In the metric space \( X \) point \( M \) fulfills \( \alpha_1^{(2)} = d_3 + d_1 - d_2 \leq 0 \) iff \( \alpha_1 \leq 0 \) and \( \beta_2 \leq 0 \) are true. In Ptolemaic metric space \( X \) the set of these points \( M \) is non-empty iff:
\[
(3.35) \quad c \geq b \text{ or } a \geq b.
\]

**Theorem 3.11** In the metric space \( X \) point \( M \) fulfills \( \alpha_1^{(3)} = d_1 + d_2 - d_3 \leq 0 \) iff \( \alpha_1 \leq 0 \) and \( \beta_3 \leq 0 \) are true. In Ptolemaic metric space \( X \) the set of these points \( M \) is non-empty iff:
\[
(3.36) \quad a \geq c \text{ or } b \geq c.
\]
For (Ptolemaic) metric space $X$ the set of the points $M$ with Möbius-Pompeiu metric property fulfill a conjunction:

\[ \alpha^{(1)}_1 > 0 \quad \text{and} \quad \alpha^{(2)}_1 > 0 \quad \text{and} \quad \alpha^{(3)}_1 > 0. \]

(3.37)

Using theorems 3.9, 3.10 and 3.11 we can determine when some inequalities in (3.37) are not true.

Finally, in the following example let us illustrate a set of points in $\mathbb{R}^2$ with Möbius-Pompeiu metric property, with respect to three fixed points $A, B, C \in \mathbb{R}^2$, if we use metrics $d$ and $\overline{d}$ from the example 2.2.

**Example 3.12 1°.** Let in the plane $\mathbb{R}^2$ the Euclidean metric $d$ is used. By picture 1 we illustrate the case of the triangle $ABC$ for which $a > c > b$ is true. Then $\alpha^{(1)}_1 > 0$ is true (the curve $\alpha^{(1)}_1 = 0$, on the basis of the theorem 3.9, has empty interior and border), otherwise the curves $\alpha^{(2)}_1 = 0$, $\alpha^{(3)}_1 = 0$ have non-empty interior and border. We can form a non-degenerate triangle from the remaining points.
In the case of the equilateral triangle $ABC$ the curves $\alpha_1^{(1)} = 0$, $\alpha_1^{(2)} = 0$ and $\alpha_1^{(3)} = 0$ transform onto the (smaller) arcs $BC$, $CA$ and $AB$ of the circumcircle. Hence, we have Möbius-Pompeiu theorem in the following form: for equilateral triangle $ABC$ the set of points $M$ in the plane, such that from distances $d_1 = d(M, A)$, $d_2 = d(M, B)$ and $d_3 = d(M, C)$ one can form a degenerative triangle, is circumcircle; from the other points in the plane we can form non-degenerate triangle.

20. Let in the plane $\mathbb{R}^2$ the chordal metric $\widetilde{d}$ is used. Let $A, B, C \in \mathbb{S}\setminus\{(0, 0, 1)\}$ be points on the unit Riemann sphere $\mathbb{S}$, with uniquely determined projections:

$$A' = \mathcal{P}^{-1}(A) = a_1 + b_1i, \quad B' = \mathcal{P}^{-1}(B) = a_2 + b_2i, \quad C' = \mathcal{P}^{-1}(C) = a_3 + b_3i \in \mathbb{C}$$

with inversely stereographical projection from the north pole:

$$\mathcal{P}^{-1} = \mathcal{P}^{-1}(x, y, z) = \left(\frac{x}{1 - z}\right) + \left(\frac{y}{1 - z}\right)i : \mathbb{S}\setminus\{(0, 0, 1)\} \rightarrow \mathbb{C}.$$  

Through points $A, B, C$ on the Riemann sphere let us set great circles (picture 2). In the complex plane we uniquely determine images of great circles as corresponding circles through points $A', B', C'$ (picture 3). By picture 3 we illustrate the case of points $A', B', C'$ for which $b > c > a$ and $k \neq 0$ are true. Then $\alpha_1^{(2)} > 0$ (the curve $\alpha_1^{(2)} = 0$, on the basis of the theorem 3.10, has empty interior and border), otherwise curves $\alpha_1^{(1)} = 0, \alpha_1^{(3)} = 0$ have non-empty interior and border. From the remaining points we can form a non-degenerate triangle.

Picture 2.

Picture 3.
Let us consider the case when $A$, $B$, $C$ are chordally equidistantly arranged points on the Riemann sphere $S$. Then the set of points $M$ on the Riemann sphere, being such that from chordal distances $d_1 = \overline{d}(M, A)$, $d_2 = \overline{d}(M, B)$ and $d_3 = \overline{d}(M, C)$ one can form a degenerative triangle, is circumcircle; from other points on the Riemann sphere one can form a non-degenerative triangle. Using inverse stereographical projection $P^{-1}$ we can conclude that analogous statement in complex plane $C$ is valid if we use chordal metric $\overline{d}$.

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