The phase topology and bifurcation tori of the Hydrogen atom subjected to external fields

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Abstract. In this paper, we performed an adapted canonical transformation, and we analysed the phase space topology and the bifurcation of Liouville tori of the Hydrogen atom subjected to three static external fields: Van der Waals potential, electric and magnetic fields. In particular, for all values of the parameters of the system under consideration, the bifurcation diagrams of the momentum mapping are constructed, bifurcations of the common level sets of the first integrals are described and the all-generic bifurcations are computed for all singular points of the bifurcation diagrams. However no author has combined these three fields and studied their behavior. Numerical investigations are performed for the integrable case by means of Poincaré surfaces of section and the phase space trajectories method, and we observed the chaos-order-chaos transition

1. Introduction

Most natural phenomena are described by linear or nonlinear differential equations. These can be transformed into a Hamiltonian form and its behavior is chaotic earlier. A central concept in Hamiltonian mechanics is that of integrable systems that have a regular behavior. The study of the integrability of these systems is done by different mathematical methods such as the criterion of Ziglin \cite{1} \cite{2}, the theorem of Liouville \cite{3}, the analysis of Painlevé \cite{4} \cite{5} \cite{6} \cite{7}, the differential theory of Galois \cite{8} and the sections of Poincaré \cite{9}. Through the use of these methods, several integrable systems are discovered, such as the Paul Trap system \cite{10} \cite{11}, the Hénon-Heiles system \cite{12} \cite{13} \cite{14}, the Fokker system. Planck \cite{15} \cite{16}. Even these systems are not fully integrable but are integrable for specific cases.

In classical mechanics, integrable systems are rare, all one-dimensional and time-independent Hamiltonian systems are integrable, but for those with two or more dimensions, integrable cases are rare and exceptional.

However, we can find Hamiltonian systems completely integrable, the best example of these systems is the hydrogen atom; for this reason, we find several experimental and theoretical works aiming to study this Hamiltonian system and to try to destroy their integrability by using external fields \cite{17} - \cite{21}. Sometimes, even if the hydrogen atom is disturbed by an external field, the system remains completely integrable as in the case of the Stark effect \cite{22}, for this effect the Hamiltonian system is globally separable.
But for the quadratic effect of Zeeman [23], the Hamiltonian system has become non-integrable and their behavior is chaotic, if the hydrogen atom is disturbed by the Van der Waals potential [24] [25], the system Hamiltonian is integrable in three exceptional cases.

In this work, we studied the hydrogen atom disturbed by three external fields: an electric field (Stark effect), a magnetic field (Zeeman effect) and a Van der Waals potential. The Hamiltonian system (in units such that $m_e = h = e = c = 1$) can write in this form:

$$H = \frac{1}{2} p^2 - \frac{1}{r} + \alpha (x^2 + y^2 + \lambda^2 z^2) + \gamma (x^2 + y^2) + \beta z \quad (1)$$

and the corresponding equations of motion are

$$\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p_x} = p_x \\
\dot{y} &= \frac{\partial H}{\partial p_y} = p_y \\
\dot{z} &= \frac{\partial H}{\partial p_z} = p_z \\
\dot{p}_x &= \frac{\partial H}{\partial x} = x \left[ \frac{1}{r^{3/2}} + 2(\alpha + \gamma) \right] \\
\dot{p}_y &= \frac{\partial H}{\partial y} = y \left[ \frac{1}{r^{3/2}} + 2(\alpha + \gamma) \right] \\
\dot{p}_z &= \frac{\partial H}{\partial z} = z \left[ \frac{1}{r^{3/2}} + 2\alpha \lambda^2 \right] + \beta 
\end{align*} \quad (2)$$

Where $(\dot{X}) = (dx/dt)$, $\dot{r} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, $\dot{P} = P_x^2 + P_y^2 + P_z^2$. And $\alpha$, $\lambda$, $\gamma$ and $\beta$ are parameters representing the magnitude of the applied external fields. $\alpha$ is a constant, $\gamma$ measured in units of magnetic field $B_0 = 2.35 \times 10^5$ T, controls the quadratic Zeeman effect, $\lambda$ is the a-dimensional number, controls the anharmonicity associated to the Van der Waals potential [24][25][26], and $\beta$ measured in units of electric field $F_0 = 5.14 \times 10^11$ V/cm, controls the Stark effect.

The potential of this system combined three external fields, and the effect of each of them on the hydrogen atom is different compared to the others, for this reason, this potential is very interesting, as well as this potential is similar to fields seen by an ion confined in a Paul trap [10] [11].

The present paper is organized as follows: In Section 2, we separated the Hamiltonian system from two canonical transformations and determined their equations of motion. Section 3 presents a detailed description of the common-level set of the first integrals (invariant level sets) in order to describe the phase space of the problem, and according to the classical Liouville theorem for non-critical values of the first integrals, the invariant level set consist of two-dimensional tori. The Fomenko theory is used to describe all generic bifurcations of these two-dimensional tori, corresponding to critical values of the first integrals. In section 4 we complete this study by numerical investigation via Poincaré surfaces of sections to bring out the order-chaos transition.

2. Separation of the problem

In the Hamiltonian (1), there is a singularity at $r = 0$. It is, therefore, necessary to introduce a suitable coordinate transformation to remove it. To do this, we can use two canonical transformations, the first is

$$\begin{align*}
x &= \rho \cos \theta \\
y &= \rho \sin \theta \\
z &= z \\
P_x &= \cos \theta P_\rho - \frac{\sin \theta}{\rho} P_\theta \\
P_y &= \sin \theta P_\rho + \frac{\cos \theta}{\rho} P_\theta \\
P_z &= P_z 
\end{align*} \quad (3)$$

The Hamiltonian function takes the form

$$H = \frac{1}{2} \left( \frac{P_\rho^2}{\rho^2} + \frac{P_\theta^2}{\rho^2} + P_z^2 \right) - \frac{1}{\sqrt{\rho^2 + z^2}} + \alpha (\rho^2 + \lambda^2 z^2) + \gamma \rho^2 + \beta z \quad (4)$$

Where $P_\rho$, $P_\theta$, and $P_z$ are the canonical momenta conjugate to the coordinates $\rho$, $\theta$ and $z$ respectively. In this equation, $\theta$ is a cyclic variable, and the corresponding canonically conjugate momenta $P_\theta$ is conserved, $P_\theta = n$ with $n$ a constant.

Then, equation (4) can be rewritten as
\[ H = \frac{1}{2} \left( P_\theta^2 + \rho^2 \right) - \frac{1}{\sqrt{\rho^2 + z^2}} + \frac{n^2}{2\rho^2} + \alpha (\rho^2 + \lambda^2 z^2) + \gamma \rho^2 + \beta z \quad (5) \]

after ignoring the cyclic integral associated to the cyclic coordinate \( \theta \), \( P_\theta = n = 0 \), the equation (5) takes the form

\[ H = \frac{1}{2} \left( P_\rho^2 + \rho^2 \right) - \frac{1}{\rho^2 + z^2} + \alpha (\rho^2 + \lambda^2 z^2) + \gamma \rho^2 + \beta z \quad (6) \]

The second canonical transformation is:

\[
\left\{
\begin{array}{l}
\rho = \frac{1}{k} \sqrt{(\xi^2 - k^2)(\lambda^2 - \lambda^2)} \\
\rho = \frac{1}{k} \sqrt{(\xi^2 - k^2)(\lambda^2 - \lambda^2)}
\end{array}
\right.
\]

\[
\begin{aligned}
P_\rho &= \frac{1}{k} (\xi^2 - k^2)(\lambda^2 - \lambda^2) [\mu(\xi^2 - k^2)P_\xi + \xi(k^2 - \lambda^2)P_\mu] \\
P_\rho &= \frac{1}{k} (\xi^2 - k^2)(\lambda^2 - \lambda^2) \left[ \xi P_\xi - \mu P_\mu \right]
\end{aligned}
\]

Where \( k = \pm 1 \)

The Hamiltonian (6) has a collision singularity at \( r^2 = \rho^2 + z^2 = 0 \), which can be removed with the following change in the independent variable

\[
\tau = \int^t dt / \left( \xi(t)^2 - \mu(t)^2 \right)
\]

After this change, the equation (6) takes the form

\[
\begin{aligned}
\bar{H} &= 0 = \frac{1}{2} \left[ (\xi^2 - k^2)P_\xi^2 + (k^2 - \lambda^2)P_\mu^2 \right] - (\xi - \mu) \\
&+ (\xi^2 - \mu^2) \left[ -h + (\alpha \lambda^2 - (\alpha + \gamma)) \left( \frac{\xi^2 \mu^2}{k^2} + k^2 \right) + (\alpha + \gamma)(\xi^2 + \mu^2) \right. \\
&+ \left. \left( 2 \alpha \lambda^2 + \frac{\beta}{k} \right) \xi \mu + k \beta \right]
\end{aligned}
\]

The equations of motion associated with \( \bar{H}(\xi, \mu) \) in the new time \( \tau \) are:

\[
\begin{aligned}
\dot{\xi} &= (\xi^2 - k^2)P_\xi \\
\dot{P}_\xi &= -\xi P_\xi^2 + 1 + 2\xi(h - \beta k) - 4(\alpha + \gamma)\xi^3 + 2\xi \left( \alpha + \gamma - \alpha \lambda^2 \right) (2\xi^2 \mu^2 - \mu^4 + k^4) \\
&+ \mu \left( 2 \alpha \lambda^2 + \frac{\beta}{k} \right) \left( 3 \xi^2 - \mu^2 \right) \\
&+ \mu \left( k^2 - \lambda^2 \right) P_\mu \\
\dot{P}_\mu &= \mu P_\mu^2 - 1 + 2\mu(\beta k - h) + 4(\alpha + \gamma)\mu^3 + 2\mu \left( \alpha \lambda^2 - \alpha - \gamma \right) (2\xi^2 \mu^2 - \xi^4 + k^4) \\
&+ \xi \left( 2 \alpha \lambda^2 + \frac{\beta}{k} \right) \left( 3 \mu^2 - \xi^2 \right)
\end{aligned}
\]

3. The phase space topology

The two-dimensional Hamiltonian (8) is separable if the relations (10) are verified

\[
\alpha + \gamma - \alpha \lambda^2 = 0 \quad \text{and} \quad 2 \alpha \lambda^2 + \frac{\beta}{k} = 0 \quad \iff \quad \lambda = \pm \sqrt{1 + \gamma / \alpha} \quad \text{and} \quad \beta = -2k(\alpha + \gamma) \quad (10)
\]

in this case, the second integral of motion reads, where \( k^2 = 1 \)

\[
F = f = \frac{1}{\mu^2 - \xi^2} \left[ \frac{\mu^2}{2}(\xi^2 - 1)P_\xi^2 - \frac{\xi^2}{2}(\mu^2 - 1)P_\mu^2 + (\alpha + \gamma)(\xi^4 \mu^2 - \xi^2 \mu^4) + \xi \mu \left( \xi^2 - \mu^2 \right) \right] \]
It is easy to verify that Equation (7) becomes

\begin{equation}
\begin{aligned}
F = f &= \frac{1}{2} (z \pm 1)^2 p_\rho^2 + \frac{1}{2} (\rho^2 + 1) p_\mu^2 \mp \rho (1 \pm z) p_\rho p_\mu - \frac{(1 \pm z)}{\sqrt{\rho^2 + z^2}} \\
&\quad + (\alpha + \gamma)(1 \pm z)^2
\end{aligned}
\end{equation}

(12)

where

\begin{equation}
\begin{aligned}
G(\xi) &= (\xi^2 - 1)^2 p_\xi^2 = 2(\xi^2 - 1)(f - (\alpha + \gamma)\xi^4 + 2(\alpha + \gamma)\xi^2 + h\xi^2 + \xi) \\
G(\mu) &= (\mu^2 - 1)^2 p_\mu^2 = 2(\mu^2 - 1)(f - (\alpha + \gamma)\mu^4 + 2(\alpha + \gamma)\mu^2 + h\mu^2 + \mu)
\end{aligned}
\end{equation}

(14)

For the following calculations, we used \( G(u) \) instead of \( G(\xi) \) and \( G(\mu) \) because having the same form.

in these conditions, the equations of motion are

\begin{equation}
\begin{aligned}
\dot{\xi} &= \sqrt{G(\xi)} \\
\dot{p}_\xi &= -\frac{G(\xi)}{(\xi^2 - 1)^2} + 1 + 2\xi(h - \beta k) - 4(\alpha + \gamma)\xi^2 \\
\dot{\mu} &= -\sqrt{G(\mu)} \\
\dot{p}_\mu &= \mu \frac{G(\mu)}{(\mu^2 - 1)^2} + 1 + 2\mu(\beta k - h) + 4(\alpha + \gamma)\mu^3
\end{aligned}
\end{equation}

(16)

\( H = h \) and \( F = f \) are the first integrals of motion, functions of \((\xi, \mu, p_\xi, p_\mu)\) which are constant along the solutions of Equation (16).

3.1. Topology of Regular Level Set

In this section, we will give the admissible regions on the bifurcation diagrams as well as a detailed description of the topology of real-level sets, that is, the topology of the real phase space;

\[ \mathcal{A}_R = \{ (\xi, \mu, p_\xi, p_\mu) \in \mathbb{R}^4; H = h, F = f \} \subset \mathbb{R}^4 \]

the set of the critical values of the energy-momentum mapping

\[ (\xi, \mu, p_\xi, p_\mu) \rightarrow (H, F) \]

**Definition.** The bifurcation diagram of an integrable system is defined to be the region of possible motion depicted on the plane of first integrals \((h, f)\) [27]

It appears, according to several works carried out in this sense (for details, see [28]-[30]), that \( B \) is exactly the discriminant locus of the polynomial \( G(u) \) whose coefficients are functions of \( h, f \) and the parameters of the system \((\alpha, \gamma)\).

\[ B = \{ (h, f, \alpha, \gamma, \beta, \lambda) \in \mathbb{R}^6; \text{discr}(G(u)) = 0 \} \]

all parameters are positive except \( \alpha \) which can be positive or negative, that’s why we consider two cases \( \alpha > 0 \) and \( \alpha < 0 \).

3.1.1. For the case \( \alpha > 0 \)
In order to plot the bifurcation diagram, the polynomial \( G(u) \) must have real roots, so it is necessary that constants \( \alpha, \beta, \gamma \) and \( \lambda \) verified the equation (10)

\[
\mathcal{N} = \left\{ (h, f, \alpha > 0, \gamma, 2(\alpha + \gamma), \sqrt{1 + \frac{\gamma}{\alpha}}) \in \mathbb{R}^4; \text{discr}(G(u)) = 0 \right\}
\]

(17)

The set \( \{ \mathcal{A}_R \setminus B \} \cap \{ \alpha > 0 \} \) consists of 9 connected components (as it is shown in Figure 1). Thus, in each connected component of the set \( \mathcal{A}_R \setminus B \) the level set \( \mathcal{A}_R \) has the same topological type, and this latter may be changed only if \((h, f)\) passes through \( B \cap \{ \alpha > 0 \} \).

Figure 1. Bifurcation diagram \( B \cap \{ \alpha = \text{const} \} \) for \( \alpha > 0 \)

**Theorem 1.** The set \( \{ \mathcal{A}_R \setminus B \} \cap \{ \alpha > 0 \} \) consists of 9 connected and nonintersecting with each other domains. The sections of these components with the plane \( \{ \alpha = \text{const} \} \) are shown on figure 1. The topological type of \( \mathcal{A}_R \) is a disjoint union of two-dimensional two-tori \( 2T^2 \), two-dimensional tori \( T^2 \) and the empty set \( \emptyset \) as it is shown in table 1.

| Domain | Roots of \( G(u) \) | Projection of the admissible ovals on \( z \)-plane | Topological Type \( \mathcal{A}_R \) |
|--------|---------------------|-----------------------------------------------|---------------------------------|
| 1      | \(-1 < 1\)          | \([-1,1]\)                                  | \(\emptyset\)                        |
| 2      | \(-1 < u_1 < u_2 < 1\) | \([-1, u_1] \cup [u_2, 1]\) | \(\emptyset\)                        |
| 3      | \(-1 < u_1 < 1 < u_2\) | \([-1, u_1]\) \cup [1, u_2] | \(T^2\)                           |
| 4      | \(u_1 < -1 < 1 < u_2\) | \(\emptyset\) \cup [u_1, -1] | \(\emptyset\)                        |
| 5      | \(-1 < u_1 < u_2 < u_3 < 1 < u_4\) | \([-1, u_1] \cup [u_2, u_3]\) \cup [1, u_4] | \(2T^2\)                           |
| 6      | \(u_1 < -1 < u_2 < u_3 < 1 < u_4\) | \([u_2, u_4]\) \cup [u_4, -1] \cup [1, u_4] | \(2T^2\)                           |
| 7      | \(u_1 < u_2 < -1 < 1 < u_3 < u_4\) | \([-1, u_3]\) \cup [u_1, u_2] \cup [1, u_4] | \(2T^2\)                           |
| 8      | \(u_1 < u_2 < -1 < 1 < u_3 < u_4\) | \([-1,1]\) \cup [u_1, u_2] \cup [u_1, u_4] | \(2T^2\)                           |
| 9      | \(-1 < 1 < u_1 < u_2\) | \([-1,1]\) \cup [u_1, u_2] | \(T^2\)                           |

**Proof.** Consider the complexified system
Consider also the hyperelliptic curves
\[ \Omega_1: \{ \omega_1^2 = G(\xi) \} \quad \text{and} \quad \Omega_2: \{ \omega_2^2 = G(\mu) \} \]
and the corresponding Riemann surfaces \( R_1 \) and \( R_2 \) of the same genus \( j_1 = j_2 = 2 \). We obtain the explicit solutions of the initial problem (16) by solving the Jacobi inversion problem [31]. Thus \( \rho, z, P, \bar{P} \) can be expressed in terms of hyperelliptic functions living in the Jacobi variety \( \Omega = \Omega_1 \otimes \Omega_2 \) (where \( \otimes \) is the symmetric product). These functions however are not single valued as can be seen from formulae (13) and (16).

Indeed, to each point on the symmetric product \( \Omega_1 \otimes \Omega_2 \) there correspond two values of \( (\rho, z, P, \bar{P}) \).

Thus, we define the natural projection
\[ \mathcal{A}_C \rightarrow \Omega_1 \otimes \Omega_2 \]
Corresponding to the involution \( i: (\rho, z, P, \bar{P}) \rightarrow (\rho, z, -P, -\bar{P}) \)
the real level sets \( \mathcal{A}_R = \mathcal{R}e(\mathcal{A}_C) \) is the set of fixed points of the complex conjugation on \( \mathcal{A}_C \):
\[ \chi: (\rho, z, P, \bar{P}) \rightarrow (\bar{\rho}, \bar{z}, \bar{P}, \bar{\bar{P}}) \]
It induces an involution on the Jacobi variety and hence on \( \mathcal{A}_C \) by the natural projection \( \sigma \). Formulæ (11) and (12) imply that this involution \( \psi \) coincides with the complex conjugation (22) on \( \mathcal{A}_C \) the upshot is that in order to describe \( \mathcal{A}_R \) it is enough to study the projection \( \sigma \) and the pair \( (R, \psi) \)
\[ \sigma: \mathcal{A}_C \rightarrow \text{Jac}(R) = \Omega_1 \otimes \Omega_2 \]

Remark. The pair \( (R, \psi) \) where \( R \) is a Riemann surface and \( \sigma \) is an involution on \( R \) is called Klein surface [32].

Definition. A connected component of the set of fixed points of \( \chi \) on the curve \( \Omega_1 \) and \( \Omega_2 \) is called an oval.

To determine the ovals of \( \Omega_1 \) and \( \Omega_2 \) it suffices to study the real roots of the polynomial \( G(u) \) for different values of \( h, f \) and \( \alpha \). These roots are shown on Table 1. Using the formulæ (7) and (13) and the condition \( (\rho, z, P, \bar{P}) \in \mathbb{R}^4 \), we find exactly two admissible ovals whose projections on the \( \xi \)-plane and the \( \mu \)-plane are given by \( \Delta_1 \) and \( \Delta_2 \) (see Table 1). The product of the admissible ovals in \( \Omega_1 \otimes \Omega_2 \) and the projection \( \sigma \) of \( \mathcal{A}_M \) such as, \( \mathcal{A}_M = \sigma^{-1}(\Omega_1 \otimes \Omega_2) = \Delta_1 \times \Delta_2 \), gives:
- \( \mathcal{A}_R \) is a two-dimensional tori 2T in domain 5, 6, 7 and 8.
- \( \mathcal{A}_R \) is a two-dimensional tori T in domain 3 and 9.
- \( \mathcal{A}_R \) is the empty set \( \phi \) in domain 1, 2 and 4.

3.1.2 For the case \( \alpha < 0 \)
In this case, the bifurcation diagram is defined by (25)
\[ B' = \left\{ (h, f, \alpha < 0, |\alpha| > \gamma, -2(\alpha + \gamma), \sqrt{1 + \frac{\gamma}{\alpha}} \mathbb{R}^4 : \text{discr}(G(u)) = 0 \right\} \]

Theorem 2. The set \( \{ \mathcal{A}_R \} \cap \{ \alpha < 0 \} \) consists of 9 connected and nonintersecting with each other domains. The sections of these components with the plane \( \{ \alpha = \text{const} \} \) are shown on Figure 2. The topological type of \( \mathcal{A}_R \) is a two-dimensional tori \( T^2 \), to a disjoint union of cylinders \( C \), or it is the empty set \( \phi \) as it is shown in Table 2.

The product of the admissible ovals in \( \Omega_1 \otimes \Omega_2 \) and the projection \( \sigma \) of \( \mathcal{A}_R \) such as, \( \mathcal{A}_R = \sigma^{-1}(\Omega_1 \otimes \Omega_2) = \Delta_1 \times \Delta_2 \), gives:
\[ \mathcal{A}_C = (\xi, \mu, P, \bar{P}) \in \mathbb{C}^4 : h = cte, f = cte \]
A is a two-cylinders $2C$ in domain 2, 3 and 4.
A is a four-cylinders $4C$ in domain 5 and 6.
A is a two-dimensional tori + two cylinders $T^2 + 2C$ in domain 7.
A is the empty set $\emptyset$ in domain 1, 8 and 9.

**Table 2.** Real roots of the polynomial $G(u)$, admissible ovals for $(h, f) \in \mathbb{R}^2 \setminus B'$ and topological type of $A_R$

| Domain | Roots of $G(u)$ | Projection of the admissible ovals on z-plane | Topological Type $A_R$ |
|--------|-----------------|---------------------------------------------|----------------------|
| 1      | $-1 < 1$        | $\emptyset$                                 | $[\infty, -1] \cup [1, +\infty[$ | $\emptyset$ |
| 2      | $-1 < u_1 < u_2 < 1$ | $[u_1, u_2]$                                | $[\infty, -1] \cup [1, +\infty[$ | $2C$ |
| 3      | $u_1 < -1 < u_2 < 1$ | $[-1, u_2]$                                | $[\infty, u_2] \cup [1, +\infty[$ | $2C$ |
| 4      | $u_1 < -1 < 1 < u_2$ | $[-1, 1]$                                   | $[\infty, u_2] \cup [u_2, +\infty[$ | $2C$ |
| 5      | $u_1 < -1 < u_2 < u_3 < u_4 < 1$ | $[-1, u_2] \cup [u_3, u_4]$ | $[\infty, u_4] \cup [1, +\infty[$ | $4C$ |
| 6      | $u_1 < -1 < u_2 < 1 < u_4$ | $[-1, u_2] \cup [u_3, 1]$ | $[\infty, u_4] \cup [u_4, +\infty[$ | $4C$ |
| 7      | $u_1 < -1 < u_2 < 1 < u_3 < u_4$ | $[-1, u_3]$                                | $[\infty, u_4] \cup [1, u_5] \cup [u_6, +\infty[$ | $T^2 + 2C$ |
| 8      | $u_1 < u_2 < -1 < 1 < u_3 < u_4$ | $\emptyset$                                | $[\infty, u_4] \cup [u_2, -1] \cup [1, u_5] \cup [u_6, +\infty[$ | $\emptyset$ |
| 9      | $u_1 < u_2 < -1 < 1$ | $\emptyset$                                | $[\infty, u_4] \cup [u_2, -1] \cup [1, +\infty[$ | $\emptyset$ |

**Figure 2.** Bifurcation diagram $B' \cap \{\alpha = const\}$ for $\alpha < 0$

### 3.2. Topology of Singular Level Sets

Suppose now that the constants $h, f$ are changed in such a way that $(h, f)$ passes through the bifurcation diagram $B$. Then the topological type of $A_R$ may change and the bifurcation of Liouville tori takes place.

In this section, we will give the description of all generic bifurcations of the topological types of $A_R$ using the Fomenko bifurcation surgery on Liouville Tori [33].

In the first case $\alpha > 0$, we can have three types of bifurcation of the level set $(A_R)$ passing from a domain $i$ to domain $j$ (see table 3). To prove that, it suffices to look at the bifurcations of the roots of the polynomial $G(u)$. The Correspondence between bifurcation of the roots of the polynomial and bifurcations of invariant Liouville tori is shown in figure 3.
Figure 3. Correspondence between bifurcation of the roots of the polynomial $G(u)$ for $(h, f) \notin B$ and bifurcations of invariant Liouville tori, where $(S \wedge S)$ is a union of two circles having exactly one common point.

Table 3. Generic bifurcation of the level set $\mathcal{A}_\mathbb{R}$ passing from a domain $i$ to domain $j$

| Domain $i$ | Domain $j$ |
|------------|------------|
| $3 \to 1$  | $5 \to 3$  |
| $3 \to 2$  | $5 \to 4$  |
| $3 \to 4$  | $6 \to 4$  |
| $9 \to 1$  | $8 \to 3$  |
| $\mathbb{T}^2 \to \phi$ | $2\mathbb{T}^2 \to \phi$ | $27\mathbb{T}^2 \to \mathbb{T}$ |

We have three types of bifurcation:

- **Bifurcation $2\mathbb{T}^2 \to S \times (S \wedge S) \to \mathbb{T}^2$**: The two-dimensional two-tori $2\mathbb{T}^2$ merge into two dimensional tori $\mathbb{T}^2$ by passing through the complex $S \times (S \wedge S)$ where $(S \wedge S)$ is a union of two circles having exactly one common point.

- **Bifurcation $2\mathbb{T}^2 \to 2S \to \phi$**: The two-dimensional two-tori $2\mathbb{T}^2$ are contracted to two circles corresponding to two periodic solutions, and then vanishes.

- **Bifurcation $\mathbb{T}^2 \to S \to \phi$**: The two-dimensional tori $\mathbb{T}^2$ shrinks to a circle corresponding to the periodic solution, and vanishes.

In the second case $\alpha < 0$, the Fomenko surgery of bifurcation on Liouville tori [33] cannot be applied as its invariant level sets contain a non-compact component (cylinder $C$). We can have the type of bifurcation of the level set $\mathcal{A}_\mathbb{R}$ passing from a domain $i$ to domain $j$ (see table 4.). To prove that, it suffices to look at the bifurcations of the roots of the polynomial $G(u)$ as shown in figure 4.
Table 4. Generic bifurcation of the level set $A_R$ passing from a domain $i$ to domain $j$

| $7 \rightarrow 8$ | $6 \rightarrow 3$ | $2 \rightarrow 1$ |
|-----------------|-----------------|-----------------|
| $7 \rightarrow 9$ | $7 \rightarrow 6$ | $7 \rightarrow 3$ | $3 \rightarrow 1$ |
|                 | $5 \rightarrow 3$ |
|                 | $5 \rightarrow 4$ |

$T^2 + 2C \rightarrow \phi$ $T^2 + 2C \rightarrow 4C$ $T^2 + 2C \rightarrow 2C$ $4C \rightarrow 2C$ $2C \rightarrow \phi$

Figure 4. Correspondence between bifurcation of the roots of the polynomial $G(u)$ for $(h, f) \in B'$ and bifurcations of invariant Liouville tori

4. Numerical illustration

By using a set of software routines, implemented in Maple, for plotting 2D and 3D projections of Poincare surfaces of section map, this map is constructed using a clever method introduced by Poincare and extended by Henon [9]. We give a numerical analysis of the topological analysis studied in section 3 and show the integrable behavior of the system. For fixed values of constants $h, f, \beta, \lambda$ and $\alpha, \gamma$ vary, the Liouville tori contained in the level set $\{H=h, F=f\}$ change their topological type. However, with the change of one of the parameters of the system (control parameter), we observe the random dispersion of the points in the sections which show that the system has changed their behavior from the regularity to quasi-regularity and to chaotical behavior.

The Poincare surfaces of section are plotted in the plane $(q_2, p_2) = (z, P_z)$ or $(q_1, p_1) = (\rho, P_\rho)$.
5. Conclusion

In this paper, we have studied the complete description of the real phase topology of the Hydrogen atom subjected to three static external fields: Van der Waals potential, electric and magnetic fields. By means of a canonical transformation the system is reduced to a separable Hamiltonian system with two
degrees of freedom. The system is characterized by a characteristic polynomial depends on the invariants of motion $H$ and $F$. We studied the topological analysis of the real invariant manifolds $\mathcal{A}_h$ ($H = h, F = f$) of the system, which is why we used the Fomenko surgery theory and the Liouville tori bifurcations. For non-critical values of $H$ and $F$, we have distinguished two different cases: $\alpha > 0$ where $\mathcal{A}_h$ is a two-dimensional tori, to a disjoint union of two-dimensional two-tori, or it is the empty set. On the other hand, for the case $\alpha < 0$ it is a two-dimensional tori, a cylinders or it is the empty set. Finally, we proved the analytical results by a numerical illustration via the Poincaré sections, we also perform the order_chaos transition if one of the parameters of the controls varies.
Figure 6. Surfaces of section map in 2D for different values of $(h, f) \in \mathbb{R}^2 \setminus B$, $\alpha > 0$. The three surfaces (from left to right) correspond to $\gamma = 1$, $\beta = 4$, $\lambda = \pm \sqrt{2}$ and $\alpha = 1; 3; 6$. (a) domain 3 ($h = -2.231$, $f = 0.728$); (b) domain 5 ($h = -1.6$, $f = 0.061$); (c) domain 6 ($h = 0.422$, $f = 0.091$); (d) domain 7 ($h = 4.812$, $f = -7.069$); (e) domain 8 ($h = 6.855$, $f = -11.769$); (f) domain 9 ($h = 3.791$, $f = -7.603$).

Figure 7. Surfaces of section map for $(h, f) \in \mathbb{R}^2 \setminus B'$, $\alpha < 0$ and fixed values $\gamma = 2$, $\beta = 4$, $\lambda = \pm \sqrt{1/2}$. (a) view 3D of domain 7 ($h = -4.166$, $f = 5.815$, $\alpha = -4$); (b) view 2D of domain 7 ($h = -4.166$, $f = 5.815$, $\alpha = -4$); (c) domain 7 ($\alpha = -6$); (d) domain 7 ($\alpha = -8$).

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