A Note on the Gauge Group of the Electroweak Interactions

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We propose a three-fold covering of the group U(2) as a gauge group for the electroweak interactions for the purpose of describing fields with integer and fractional electric charges with respect to the residual electromagnetic gauge group after a spontaneous breaking of the gauge symmetry. In a more general scheme we construct a three-fold covering of U(n) and consider for the case $n=2$ several representations which are used in the construction of a model of the electroweak interactions in a subsequent paper.

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1. INTRODUCTION

In the Weinberg-Salam (WS) model\textsuperscript{[1,2]} the transformation laws under the group $U(1)$ of the weak hypercharge $Y$ are different for the quark and lepton fields. Among the irreducible representations of $U(1)$, namely $e^{i\alpha} \rightarrow e^{i\alpha}$, $0 \leq \alpha \leq 2\pi$ and $n \in \mathbb{Z}$ any integer, for the left and right lepton fields one takes the transformations (for each generation)

$$\psi_L \rightarrow e^{-i\alpha}\psi_L, \quad \psi_R \rightarrow e^{-2i\alpha}\psi_R$$

(or $Y = -1$ for $\psi_L$, and $Y = -2$ for $\psi_R$), in order to obtain the correct electric charges of the leptons (in units of the elementary charge $e$). With the same purpose one sets for the quark fields

$$u_L \rightarrow e^{\frac{i\alpha}{3}}u_L, \quad d_L \rightarrow e^{\frac{i\alpha}{3}}d_L, \quad u_R \rightarrow e^{\frac{4i\alpha}{3}}u_R, \quad d_R \rightarrow e^{-\frac{2i\alpha}{3}}d_R, \quad \text{etc} \ldots ,$$

i.e. $Y = 1/3$ for $u_L$, $d_L$ and $Y = 4/3$ for $u_R$, $Y = -2/3$ for $d_R$. Obviously, these formulae do not fit with the irreducible representations of the group $U(1)$ defined as

$$U(1) = \{ e^{i\alpha} \mid 0 \leq \alpha \leq 2\pi \} .$$

In this paper we propose to use as a gauge group for the WS model a three-fold covering of $U(2) = (U(1) \times SU(2))/\mathbb{Z}_2$ in order to deal with descent representations on the fields. We apply the term metaunitary group for this three-fold covering and denote it by $MU(2)$. In a recent publication Roepstorff and Vehns\textsuperscript{[3]} propose a subgroup $G$ of $SU(5)$ as a gauge group for the standard model such that $G$ appears also as a covering of $U(2)$. Our construction for the group $MU(2)$ is motivated by an argument of Guillemin and Sternberg\textsuperscript{[4]} for the definition of a two-fold covering $L$ of the general linear group, aimed to define a representation of the type $g \rightarrow \det^{1/2}g$ of $L$. Noting that the Lie algebras of the groups $U(1)$ and $R$ coincide, we may expect that for the description of fields with electric charge proportional to $e/3$, a suitable group may be a factor group of $R \times SU(2)$. In order to give a more general framework, we present in Section II the construction of a three-fold covering $MU(n)$ of the group $U(n)$. In Section III we specialize to the case $n=2$ and consider several representations of $MU(2)$ and its Lie algebra which will turn useful for the description of leptons and quarks in a subsequent paper.
II. THE METAUNITARY GROUP $\text{MU}(N)$

As mentioned above, we are looking for a gauge group of the electroweak interactions as a suitable factor group of $\mathbb{R} \times \text{SU}(2)$. Following an argument from [4] and in order to provide a more general framework, we begin with $\mathbb{R} \times \text{SU}(n)$, $n \geq 2$, where the group composition law reads

$$(u, A) \cdot (v, B) = (u + v, AB), \quad u, v \in \mathbb{R}, \quad A, B \in \text{SU}(n)$$

and consider the subgroup of $\mathbb{R} \times \text{SU}(n)$ with elements

$$\left\{ \left( k \frac{2\pi}{n}, e^{-k \frac{2\pi i}{n}} I \right) \mid k \in \mathbb{Z} \right\},$$

which is isomorphic to $\mathbb{Z}$. This subgroup is a normal one and through the map $T : \mathbb{R} \times \text{SU}(n) \to \text{U}(n)$, defined as

$$T(u, A) = e^{iu} A,$$

we obtain an isomorphism $(\mathbb{R} \times \text{SU}(n))/\mathbb{Z} = \text{U}(n)$. Indeed

$$T\left( u + k \frac{2\pi}{n}, e^{-k \frac{2\pi i}{n}} A \right) = T(u, A).$$

The factor group $\text{MU}(n) = (\mathbb{R} \times \text{SU}(n))/3\mathbb{Z}$, consisting of the equivalence classes

$$[u, A] = \left\{ \left( u + 3k \frac{2\pi}{n}, e^{-3k \frac{2\pi i}{n}} A \right) \mid k \in \mathbb{Z} \right\},$$

we call metaunitary group. Clearly the map $T : \text{MU}(n) \to \text{U}(n)$ defines a homomorphism onto $\text{U}(n)$. Moreover, $T$ defines a three-fold covering of $\text{U}(n)$. Indeed, the kernel of $T$ as a map acting on $\text{MU}(n)$ consists of the elements

$$[0, I], \quad \left[ \frac{2\pi}{n}, e^{-\frac{2i\pi}{n}} I \right], \quad \text{and} \quad \left[ \frac{4\pi}{n}, e^{-\frac{4i\pi}{n}} I \right].$$

Certainly, the group $\text{MU}(n)$ is locally isomorphic to $\text{U}(n)$ and $\text{SU}(n) \times \text{U}(1)$. The same technique is applicable for constructing arbitrary $l$-fold covering of $\text{U}(n)$.

III. PARTICULAR REPRESENTATIONS OF $\text{MU}(2)$ AND ITS LIE ALGEBRA

A. Representations of $\text{MU}(2)$

We here specialize to the case $n = 2$. Consider the map $\text{Det}^{\frac{1}{2}} : \mathbb{R} \times \text{SU}(2) \to \text{U}(1)$ defined by

$$\text{Det}^{\frac{1}{2}}(u, A) = e^{\frac{2iu}{3}}, \quad (u, A) \in \mathbb{R} \times \text{SU}(2).$$

Due to the property

$$\text{Det}^{\frac{1}{2}}\left( u + 3k\pi, e^{-3k\pi i} A \right) = \text{Det}^{\frac{1}{2}}(u, A)$$

the map $\text{Det}^{\frac{1}{2}}$ is well defined on $\text{MU}(2)$ and gives a homomorphism of $\text{MU}(2)$ onto $\text{U}(1)$. For every integer $k$ the mapping $\text{Det}^{\frac{k}{2}} : \text{MU}(2) \to \text{U}(1)$ given by

$$\text{Det}^{\frac{k}{2}}[u, A] = \left( \text{Det}^{\frac{1}{2}}[u, A] \right)^k,$$

is also a homomorphism of $\text{MU}(2)$ on $\text{U}(1)$. For $k = 3$

$$\text{Det}[u, A] = \text{det} \circ T [u, A]$$

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where “det” stands for the usual determinant. Using the maps $T$ and $\text{Det}^k$ we define the homomorphisms $T^k : \text{MU}(2) \rightarrow \text{U}(2)$ by

$$T^k [u, A] = \text{Det}^k [u, A] T [u, A] = e^{iu(1 + \frac{k}{3})} A .$$  

(14)

Some particular cases are

$$T^0 [u, a] = T [u, A] = e^{iu} A , \quad T^{-2} [u, A] = e^{-\frac{4iu}{3}} A ,$$  

(15)

$$\text{Det}^k [u, A] = e^{\frac{2iu}{3}} , \quad \text{Det}^{-\frac{2}{3}} [u, A] = e^{-\frac{4iu}{3}} , \quad \text{Det} [u, A] = e^{2iu} .$$  

(16)

The one-parameter subgroup of $\text{MU}(2)$,

$$\text{MU}_{\text{em}}(1) = \left\{ \left[ -\frac{\alpha}{2}, \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{-i\alpha}{2}} \end{pmatrix} \right] \in \text{MU}(2) \mid \alpha \in \mathbb{R} \right\} ,$$  

(17)

has the meaning of the group generated by the electric charge generator

$$Q = \frac{1}{2} Y + I_3 = \frac{1}{2} I + \frac{1}{2} \sigma_3 ,$$  

(18)

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The image of $Q$ in each representation of $\text{MU}_{\text{em}}(1)$ in a space $V$ is identified with the electric charge in this representation, the eigenvalues $q_i$ of $Q$ being identified with the charges of the corresponding eigenvectors from $V$.

Let $e_1, e_2$ be a basis in $\mathbb{C}^2$. The second exterior degree $\Lambda^2 \mathbb{C}^2$ is a one-dimensional complex space generated by $e_1 \wedge e_2$ which carries the representations $\text{Det}^k$ for different $k$. Let $a = u e_1 + v e_2 \in \mathbb{C}^2$ and $w e_1 \wedge e_2 \in \Lambda^2 \mathbb{C}^2$. Using the notation $\mathcal{M}(\alpha)$, $0 \leq \alpha \leq 2\pi$, for $\text{MU}_{\text{em}}(1)$, one finds in the representations (15,16)

1) $T \left[ \mathcal{M}(\alpha) \right] = \left[ \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \right] , \quad Q_T = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} , \quad q_u = 0 , \quad q_v = -1 . \quad (19)$

2) $T^{-2} \left[ \mathcal{M}(\alpha) \right] = \left[ \begin{pmatrix} e^{\frac{2i\alpha}{3}} & 0 \\ 0 & e^{-\frac{2i\alpha}{3}} \end{pmatrix} \right] , \quad Q_{T^{-2}} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} , \quad q_u = \frac{2}{3} , \quad q_v = -\frac{1}{3} . \quad (20)$

3) $\text{Det}^\frac{1}{2} \left[ \mathcal{M}(\alpha) \right] = e^{-\frac{i\alpha}{3}} , \quad Q_{\text{Det}^\frac{1}{2}} = -\frac{1}{3} , \quad q_w = -\frac{1}{3} . \quad (21)$

4) $\text{Det}^{-\frac{2}{3}} \left[ \mathcal{M}(\alpha) \right] = e^{\frac{2i\alpha}{3}} , \quad Q_{\text{Det}^{-\frac{2}{3}}} = \frac{2}{3} , \quad q_w = \frac{2}{3} . \quad (22)$

5) $\text{Det} \left[ \mathcal{M}(\alpha) \right] = e^{-i\alpha} , \quad Q_{\text{Det}} = -1 , \quad q_w = -1 . \quad (23)$

The choice of these representations is justified by the reduction of $\text{MU}(2)$ to $\text{MU}_{\text{em}}(1)$ in analogy with the reduction of $\text{SU}(2) \times \text{U}(1)$ to $\text{U}_{\text{em}}(1)$ in the WS model. Then a direct sum of one-dimensional representations of $\text{MU}_{\text{em}}(1)$ appears, each of them with a fixed electric charge.

**B. Representations of the Lie Algebra of MU(2)**

The groups $\text{MU}(2)$ and $\mathbb{R} \times \text{SU}(2)$ are locally isomorphic and one has for their Lie algebras

$$\text{Lie} \text{MU}(2) = \mathbb{R} \oplus \text{Lie} \text{SU}(2) .$$  

(24)

Accordingly, a set of four generators for $\text{MU}(2)$ is given by
where $\sigma^a$ are the Pauli matrices and

$$[X^a, X^b] = i \epsilon^{abc} X^c, \quad [X^a, X^a] = 0.$$  

The subgroups of $\text{MU}(2)$, generated by $X^a$ and $X$, are

$$G_{X^a}(t) = \left\{ e^{i a^a t} \left| t \in \mathbb{R} \right. \right\}, \quad a = 1, 2, 3, \quad (26)$$

$$G_X(t) = \left\{ e^{i \frac{1}{2} t} I \left| t \in \mathbb{R} \right. \right\}. \quad (27)$$

Each representation $T$ of the group $\text{MU}(2)$ generates a representation $T_*$ of its Lie algebra. For the particular representations $(15)-(23)$ defined in the previous subsection one finds for the generators $X^a$ and $X$

1) $T \left[ 0, e^{i a^a t} \right] = e^{i a^a t}, \quad T \left[ -i a^a t, I \right] = e^{-i t} I. \quad (28)$

$$T_*(X^a) = -i \frac{d}{dt} e^{i a^a t} \mid _{t=0} = \frac{\sigma^a}{2}. \quad (29)$$

$$T_*(X) = -i \frac{d}{dt} e^{-i t} I \mid _{t=0} = -\frac{I}{2}. \quad (30)$$

2) $T^{-2} \left[ 0, e^{i a^a t} \right] = e^{i a^a t}, \quad T^{-2} \left[ -i a^a t, I \right] = e^{2i t} I. \quad (31)$

$$T^{-2}_*(X^a) = -i \frac{d}{dt} e^{i a^a t} \mid _{t=0} = \frac{\sigma^a}{2}. \quad (32)$$

$$T^{-2}_*(X) = -i \frac{d}{dt} e^{-2i t} I \mid _{t=0} = \frac{I}{3}. \quad (33)$$

3) $\det^* \left[ 0, e^{i a^a t} \right] = 1, \quad \det^* \left[ -i a^a t, I \right] = e^{-i t}. \quad (34)$

$$\det^*_*(X^a) = -i \frac{d}{dt} 1 \mid _{t=0} = 0. \quad (35)$$

$$\det^*_*(X) = -i \frac{d}{dt} e^{-i t} \mid _{t=0} = \frac{1}{3}. \quad (36)$$

4) $\det^{-\frac{1}{2}} \left[ 0, e^{i a^a t} \right] = 1, \quad \det^{-\frac{1}{2}} \left[ -i a^a t, I \right] = e^{\frac{1}{2}i t}. \quad (37)$

$$\det^{-\frac{1}{2}}_*(X^a) = -i \frac{d}{dt} 1 \mid _{t=0} = 0. \quad (38)$$

$$\det^{-\frac{1}{2}}_*(X) = -i \frac{d}{dt} e^{-\frac{1}{2}i t} \mid _{t=0} = \frac{2}{3}. \quad (39)$$

5) $\det \left[ 0, e^{i a^a t} \right] = 1, \quad \det \left[ -i a^a t, I \right] = e^{-i t}. \quad (40)$

$$\det_*(X^a) = -i \frac{d}{dt} 1 \mid _{t=0} = 0. \quad (41)$$

$$\det_*(X) = -i \frac{d}{dt} e^{-i t} \mid _{t=0} = -1. \quad (42)$$

These representations of Lie $\text{MU}(2)$ will be used in a subsequent paper for the explicit form of the covariant derivatives of the fields in a model based on the gauge group $\text{MU}(2)$. 

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