Equivalence of Classical Skyrmions and Coherent States of Baryons
I. Constrained Quantization on the $SU(2)$ and $SO(3)$ manifolds

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In the Skyrme model, the Lagrangian can be quantized in several ways using the collective coordinate approach. Not all of which produce quantum states that can be interpreted as physical particles. For example the $SU(2)$ collective coordinate approach produces both integral and half-integral spin and isospin states. Only half of these are the physical baryons. Less well known is the fact that it is equally possible to quantize the system using the $SO(3)$ collective coordinates. This produces only unphysical integral spin and isospin states. To fulfill the goal of being able to express a classical skyrmion as a coherent state of baryons directly in terms of the baryon states, surprisingly a combination of both collective coordinate approaches is required. To prepare for the subsequent application to skyrmion formation through disoriented chiral condensates in heavy ion collisions, the Skyrme model is rigorously quantized using the Dirac prescription for constrained systems. This is shown for both the $SU(2)$ as well as the $SO(3)$ collective coordinate approach.

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I. INTRODUCTION

Four decades ago contrary to the popular trend at the time to build theories in terms of fermions, Skyrme succeeded in building a model with a Lagrangian density based on meson fields. Not that it was particularly difficult to construct meson field theories but rather this one now known as the Skyrme model has the remarkable feature that fermions can come out of boson fields in the form of topological solitons \[1, 2\]. Those of the Skyrme model are known as skyrmions. The topological charge or winding number $B$ of skyrmions was correctly identified by Skyrme as the baryon number and the $B = 1$ skyrmion was conjectured to be a fermion \[1, 2\]. It was only until much later through the method of Ref. \[1\] that the topological charge was worked out to be equal to the baryon number \[1, 2\]. Ref. \[1\] was also able to rewrite the latter in the form of the winding number and thus made the connection. As to the conjecture of a $B = 1$ skyrmion is a fermion, this can only be settled by considering three or more flavors of quarks when the Wess-Zumino effective action \[2\] was introduced to break an unphysical symmetry in the Skyrme model \[2\]. Only with this action can one show that under a rotation by an angle of $2\pi$, a skyrmion acquires a phase change of $\exp(iN\pi)$ \[2\]. A $B = 1$ skyrmion is therefore a fermion and a baryon. These distinctive works eliminated any doubt on the species of particle that a skyrmion belongs. Therefore the Skyrme model in the $B = 1$ sector became a non-trivial model of baryon or nucleon. Many further works have been produced using this model and many applications were found. For example for studying the properties such as masses, magnetic moments, sizes, interactions, structure, and form factors of nucleons and baryons \[1, 2, 3, 4, 5\]. Other remarkable points about the Skyrme model are that it can explain the fact that most spin in a proton is not carried by the valence quarks and that there is the existence of the $q\bar{q}$ sea inside the nucleon whereas the more basic non-relativistic quark model completely fails in this regard \[15, 16\]. There are also attempts to go beyond the two-flavor Skyrme model. This has proven to be not so simple. There are at least two ways to do this. One is to continue with the collective coordinate approach initiated in Ref. \[1\] by embedding the flavor $SU(2)$ skyrmion in the higher flavor $SU(N)$ theory \[12, 17, 18\], the other is to bind a kaon to a flavor $SU(2)$ skyrmion and treating strange quark as distinct from the light quarks \[19, 20\]. Unfortunately both approaches have their own problems in getting a reasonable estimates of the baryon mass spectrum. Some reviews of both approaches as well as on other applications of the three-flavor skyrmions can be found in Ref. \[20, 22\]. On two- or more flavor skyrmions these have been considered as deuteron or other massive nuclei, one can consult for example Ref. \[22\].

Our interest in the skyrmion stems from an apparently rather remote subfield of physics. While most of the works mentioned above were interested in the general properties, static or otherwise, of the baryons, we are more concerned with the dynamical production of skyrmions in the seemingly rather unlikely context of heavy ion collisions. The motivation behind this is the possibility of observing skyrmion production. This calls for an examination of the skyrmion in a very different light. Rather than asking the questions of what are the physical properties of the quantum states of the Skyrme model, the question we ask ourselves is what exactly is the quantum analog of the classical skyrmion solution. If one does not know what the classical solution corresponds to in terms of physical baryon states, there is little chance for their observation.

This possibility of skyrmion production in heavy ion...
collisions was first raised in Ref. [26]. Strangely this was at least several years before the same possibility was considered in jets [27, 28] and reapplied to heavy ion collisions [29, 30]. Within the context of heavy ion collisions skyrmions are considered as topological defects which might form when the system passes from the chiral symmetric phase back into the broken phase just as strings and monopoles might form during the electroweak phase transition in the early universe through the Kibble mechanism [31]. However unlike other type of topological defects skyrmions are subject to strict boundary conditions which significantly modify their formation probability [32]. This must be kept in mind. Common to both heavy ion collisions and the electroweak phase transition there is an order parameter or vacuum expectation value $\langle \phi \rangle$. Below the phase transition temperature this takes on a value and a direction in the normal vacuum. But at temperatures above the phase transition, there is sufficient energy available so that $\phi$ can point in any directions in the order parameter space. As the system cools down different spatial regions can have different $\langle \phi \rangle$’s since they need not be correlated. In heavy ion collisions from this point onward there are two scenarios which can happen and they are not necessarily mutually exclusive.

The first scenario is the system would try to restore the vacuum value of $\langle \phi \rangle$, $\langle \phi \rangle = \langle \sigma \rangle$ for the chiral phase transition, by eliminating spatial domains with anomalous values of $\langle \phi \rangle$. This is done through pion radiation. For example if $\phi$ is pointing in the $\pi^+$ direction in a domain, this is simply eliminated by emitting low energy $\pi^+$. This has led to the proposal of detecting the chiral phase transition by detecting the formation of these domains. One examines the fluctuation in the yield of the three species of pions on a collision event by collision event basis. This idea was first raised in Ref. [33, 34] and gained much momentum through the work of Ref. [35, 36]. Much of its appeal came from cosmic ray data where in a few event there was the absence of $\Sigma^0$. Such isospin breaking occurrences are known as the Centauro events. It was thought that one could recreate these Centauro events in heavy ion collisions or in high energy hadron-hadron collision experiments [37, 38, 39, 40]. Soon it was shown that in order to detect these pion radiations amongst the many copiously produced pions and hadrons via other mechanisms, large domains of fixed chiral orientation pointing away from the vacuum direction were necessary [39, 40, 41]. Due to the fact that misalignment of the chiral condensates with the vacuum condensate is central to this phenomenon, it was coined Disoriented Chiral Condensates (DCC) soon after the first proposal [38, 40]. Motivated by the Centauro events, the primary observable for DCC is not surprisingly the ratio of the yield of charged-to-neutral pions. Much work has been done on DCC and especially analysis on how to extract information from data. Based on the fact that the chiral condensates can point in any directions in the order parameter space which is $S^3$, one expect a distribution

$$\frac{1}{N} \frac{dN}{df} = \frac{1}{2\sqrt{f}}$$

(1.1)

for the fraction of neutral-to-total pion $f$ [34, 36, 38]. Unfortunately so far all searches for DCC found no trace of the existence of these chiral condensates: none in heavy ion collisions experiments conducted at the Super Proton Synchrotron (SPS) at CERN [22, 23, 44] and none at the Tevatron in Fermilab [13]. Instead of the Eq. (1.1) these experiments only found a binomial distribution for $f$ signalling the absence of large chiral domains.

The second scenario is instead of getting rid of the anomalous order parameter by radiating pions away, the different chiral orientations of the DCC at the interfaces of the domains may be sufficiently different that non-trivial topological defects can develop as the system evolves towards the normal vacuum. Such defects would be prevented from collapsing altogether and converting into pion radiations due to the nonlinearity of the underlying field theory that dictates the chiral dynamics. These defects would emerge as anomalous production of baryons and antibaryons via $B = 1$ skyrmion and $B = -1$ antiskyrmion through DCC. Although baryon production through skyrmion is not new, it has only been connected to the much discussed DCC and to hyperon data at the SPS recently in Ref. [46, 47, 48]. The data at the SPS may be showing an anomalous baryon-antibaryon production mechanism which leads to deviation from the expected rare hyperon yields [46, 49].

These works use an approach based much on intuition and rough estimates while evidently a more solid theoretical foundation is preferred. For example one does not know exactly what the classical skyrmion solutions correspond to in terms of physical baryons. Therefore one has no concrete way to determine what baryons will be produced if skyrmions are formed in heavy ion collisions. For instance which baryons exactly among the octet or decuplet baryons would be produced? If more than one is produced, what is the probability of a $\Delta$ being produced rather than a $\Sigma$? Why should a proton be produced but not a $\Xi$? Our attempt to concretize the idea of skyrmion production and better connecting it to experimental data has met with a serious challenge.

A survey of the existing literature revealed that there are various past attempts at identifying the skyrmion or chiral soliton with a superposition of baryon states. In Ref. [50, 51, 52, 53, 54] soliton solutions were found from the linear sigma model and they were identified with the hedgehog baryons. In these papers each hedgehog is a $N_f = 3$ valence quark state surrounded by a coherent pion cloud. The known baryon states are then projected out of the hedgehog. This method enabled these authors to study the static properties of the baryons. It also gave the probabilities of the distribution of baryons inside the chiral soliton. However this method is based on the linear sigma model and cannot be used in the Skyrme model because the latter is is essentially the non-linear sigma
model with additional terms. Another approach is that of Ref. [55, 56] which identifies the skyrmions as $U(4)$ group theoretic coherent states of the Perelomov type [57]. The group algebra is realized by some auxiliary boson fields and the coherent states are states created by these bosons from the vacuum. To study the baryons, one projects out the states with the appropriate quantum numbers from these bosonic coherent states. This method is flexible in that the boson number can be equated with $N_c$, the number of colors. Thus one can go to the large $N_c$ limit. In spite of some very nice features of these approaches, we will adopt another. One that is more closely connected to the original Skyrme model and the baryon wavefunctions derived from it. The collective quantization of Ref. [6] produced a set of baryon wavefunctions that exist on the compact manifold of $S^3$. As we will explain in this paper, we are of the opinion that it is better to identify a skyrmion directly as a coherent state of baryons. There will be no auxiliary boson unlike Ref. [55, 56], no explicit coherency in a pion cloud but only one directly of baryon states. To solve this problem coherent states of baryons on compact manifolds will be required. Very fortunately the long duration between the skyrmion was first discovered and our arrival at this problem has permitted much advances in the field of mathematical physics. Because of which it is our opinion that the time is ripe for tackling this problem using such an approach.

Previously the crucial idea of the classical skyrmion as a coherent state of baryons and higher resonances was presented in Ref. [10]. An outline to the solution was given. The main goal is therefore to show how to solve this problem in detail. Since this is evidently of interest in its own right, we will concentrate exclusively on obtaining the solution. This will be done in two stages. The quantization on the compact manifolds will be in this paper. The actual construction of the coherent states of baryons will be done in a second paper [11]. Applications to heavy ion collisions and other considerations will be pursued elsewhere [12].

Because of the indices in space and the abstract group space in the paper, it is clearest to fix the notations throughout. We will use $\mu, \nu$ for Lorentz indices, $b, c, d, e = 0, 1, 2, 3$ for indices in the four-dimensional space of $SU(2)$, and $i, j, k, \cdots = 1, 2, 3, \cdots$ for spatial indices. Repeated indices imply the Einstein summation convention.

We will first briefly review the Skyrme Model in Sec. II followed by quantization via the collective coordinate approach. The eigenstates will be obtained which is the first step in making connection with the physical states. In the following sections, quantization using both the $SU(2)$ and $SO(3)$ collective coordinates will be shown. While the first one has been done before, the $SO(3)$ quantization in all its details has not appeared in any literature that we know of. Showing them together here permits a direct comparison. In both cases, the classical form of the theory will be shown first, then the Dirac brackets will be constructed for a proper quantization of a constrained system. The quantum commutators then follow from the classical Dirac brackets. Differential representations to the operators will be found next and the eigenstates will be derived from the Schrödinger equations. Finally in the appendices some technical details and wavefunctions are given.

II. THE SKYRME MODEL

In 1962 Skyrme introduced a Lagrangian density in his attempt to unify mesons and baryons. The Lagrangian density is

$$\mathcal{L}_S = \frac{f_\pi^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32g^2} \text{tr}[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2 \quad (2.1)$$

where

$$U = \exp[ip_\tau \cdot \phi / f_\pi] = (\sigma + i \tau \cdot \pi) / f_\pi \quad (2.2)$$

$f_\pi$ is the pion decay constant and $g$ is now known to be the $\rho-\pi-\pi$ coupling. The first term is the usual nonlinear sigma model and the second term was introduced by Skyrme [1, 2]. It is necessary in order for soliton-like classical solutions to the Euler-Lagrange equation to exist. To find the solution to the equation of motion, Skyrme used the so-called Hedgehog ansatz to impose maximal symmetry to the solution. It has the form

$$U = U_S = \exp[i \tau \cdot \phi \cdot F(r)] \quad (2.3)$$

where $F(r)$ is a radial function. The energy or mass of the static soliton can be worked out by substituting this ansatz into the Eq. (2.1)

$$M = - \int d^3 r \mathcal{L}_S(U = U_S)$$

$$= 4\pi \int_0^\infty dr \left\{ \frac{1}{2} f_\pi^2 [r^2 (dF / dr)^2 + 2 \sin^2 F] + \frac{1}{2g^2} \sin^2 F \left[ 2(4\pi F^2)^2 + \sin^2 F \right] \right\} \quad (2.4)$$

In order for the solution to be stable, the energy associated with $U_S$ is minimized by a functional variation of $F$ with the fixed boundary conditions

$$F(r \to \infty) \to 0 \quad \text{and} \quad F(r = 0) = B\pi \quad (2.5)$$

This ensures the stability of the solution. The boundary conditions are essential for the topological charge to be given by the winding number $B$ [1, 2, 3, 4]. Because the Skyrme model belongs to a class of effective Lagrangians that approximate QCD at low energies, the solitons or skyrmions of the theory have physical meaning. In the introduction $B$ has been identified as the baryon number and a skyrmion is generally considered the “classical limit” of a baryon [1, 2, 4, 5, 6, 7, 8]. Unlike a baryon, a skyrmion is a classical object without any quantum number other than $B$. Therefore a first step towards identifying the classical skyrmion is to quantize it.
A. The Hamiltonian

Quantization of the skyrmion amounts to finding the Hamiltonian and the eigenstates. A static skyrmion has no dependence and therefore cannot be quantized. Such a dependence must be introduced. One cannot, however, have an arbitrary time evolution of $U$ without the risk of losing the skyrmion completely or introducing unwanted complications during the time evolution (though the topological charge is conserved due to the fact that the third homotopy group of $SU(N)$ is the group of integers $\pi_3(SU(N)) = \mathbb{Z}$). A solution is to let the time evolution be restricted to a change from one skyrmion solution to another. Given that $U_S$ is a solution of the Euler-Lagrange equation,

$$ U'_S = A(t) U_S A^{-1}(t) \tag{2.6} $$

is also a solution for any element $A(t)$ of $SU(N)$ because of the form of Eq. (2.3). This method of providing a time dependence to the skyrmion and yet preserving the essential form of the solution Eq. (2.3) was first shown in Ref. \cite{9} and is the well-known collective coordinate method of quantization. For simplicity, we will avoid the problems of three-flavor skyrmions \cite{17, 18, 19, 20, 21, 22} and deal exclusively with only two flavors.

To quantize the theory, the Hamiltonian is required which can be obtained from the Lagrangian. Inserting now $U'_S$ into Eq. (2.1) and integrating over spatial volume gives the Lagrangian

$$ L = \lambda \text{tr}(\dot{A} \dot{A}^\dagger) - M \tag{2.7} $$

where $\lambda = \frac{8\pi}{3gf^2} \Lambda$ and

$$ \Lambda = \int_0^\infty d\tilde{r} \tilde{r}^2 \sin^2 \theta \left\{ 1 + \left( \frac{df^2}{d\theta} \right)^2 + \frac{\sin^2 \theta}{f^2} \right\} \tag{2.8} $$

with $\tilde{r} = g f \theta r$. The skyrmion solution $U_S$ is now completely hidden inside the constants $\lambda$ and the mass $M$. With only the kinetic term depending on the collective coordinate $A(t)$, the quantization is straightforward. It is more convenient to rewrite the element of $SU(2)$ as

$$ A = a_0 + i a \cdot \tau. \tag{2.9} $$

Because of the unitary nature of $SU(2)$, the components $a_b$ are subject to the constraint

$$ c_1 = a_0^2 + a^2 = a_b a_b = 1. \tag{2.10} $$

These coordinates are therefore restricted to the surface of a four-dimensional unit sphere or $S^3$. In terms of $a_b$’s Eq. (2.7) becomes

$$ L = 2 \lambda \dot{a}_b \dot{a}_b - M. \tag{2.11} $$

The conjugate momentum to the coordinate $a_b$ is readily found to be $\pi_b = 4 \lambda \dot{a}_b$. The Hamiltonian is therefore

$$ H = \frac{1}{8\lambda} \pi_b \pi_b + M. \tag{2.12} $$

Ordinarily quantization would be straightforward, one simply imposes the commutators

$$ [\hat{a}_b, \hat{a}_c] = 0, \quad [\hat{a}_b, \hat{\pi}_b] = i \delta_{bc}, \tag{2.13} $$

then finds a differential representation for $\pi_b$ and solves the resulting Schrödinger equation for the eigenstates. This would be essentially the approach of \cite{14} which we have followed closely up to now. The presence of the constraint Eq. (2.10) changes everything. Eq. (2.13) are the correct commutation relations only if $A$ specifies coordinates in Euclidean space $\mathbb{R}^4$ which is not the case. Because the space is $S^3$, we will follow the prescription of Dirac for quantization in the presence of constraints \cite{14, 15}.

B. The Dirac Brackets

To prepare for the quantization, one first find all the secondary constraints from the given number of primary ones. Only after all the constraints have been found can the Dirac brackets be constructed from the Poisson brackets. It is the Dirac brackets and not the Poisson brackets that will be carried through into the quantum theory as commutators upon quantization \cite{14, 15}. In our problem, there is only one primary constraint which is Eq. (2.10). To find the secondary constraints, one constructs the total Hamiltonian

$$ H_T = \frac{1}{8\lambda} \pi_b \pi_b + M + \rho(a_b a_b - 1). \tag{2.14} $$

This, in general, is simply the Hamiltonian supplemented with the primary constraints via the use of one Lagrange multiplier per constraint. To derive the second class constraints, one makes use of the basic Poisson bracket

$$ \{ a_b, \pi_c \} = \delta_{bc}, \tag{2.15} $$

and the recursion formula

$$ c_{i+1} = \{ c_i, H_T \} = 0, \tag{2.16} $$

starting with $i = 1$, repeatedly until no more new constraints are produced. After dropping non-essential constant factors, we find

$$ c_2 = \frac{1}{2} (a_b \pi_b + \pi_b a_b) = a_b \pi_b = \pi_b a_b = 0 \tag{2.17} $$

and

$$ c_3 = \frac{1}{4\lambda} \pi_b \pi_b - 2\rho a_b a_b = 0. \tag{2.18} $$

This process terminates at $c_3$ because $c_2$ is reproduced at $i = 4$

$$ c_4 = c_2 = 0. \tag{2.19} $$

According to Ref. \cite{14, 15} $c_1$ and $c_2$ are the second-class constraints necessary for constructing the Dirac brackets while $c_3$ is to provide a solution to the Lagrange
multiplier $\rho$. To arrive at the Dirac brackets, the non-singular anti-symmetric matrix of Poisson brackets between second-class constraints must be constructed. This is

$$m = \begin{pmatrix} \{c_1, c_1\} & \{c_1, c_2\} \\ \{c_2, c_1\} & \{c_2, c_2\} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}. \quad (2.20)$$

Equipped with this matrix, the Dirac bracket between any two quantities, say $A$ and $B$ is

$$\{A, B\}_D = \{A, B\} - \{A, c_1\} m^{-1}_{ij} \{c_j, B\}. \quad (2.21)$$

$m^{-1}$ is the inverse matrix of $m$ and is given by

$$m^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.22)$$

Applying this result to all combinations of $a_b$ and $\pi_c$, one gets

\begin{align*}
\{a_b, a_c\}_D &= \{a_b, a_c\} = 0 \quad (2.23a) \\
\{a_b, \pi_c\}_D &= \{a_b, \pi_c\} - \{a_b, c_2\} m^{-1}_{21} \{c_1, \pi_c\} \\
&= \delta_{bc} - a_b a_c \quad (2.23b) \\
\{\pi_b, \pi_c\}_D &= -\{\pi_b, c_1\} m^{-1}_{12} \{c_2, \pi_c\} \\
&\quad -\{\pi_b, c_2\} m^{-1}_{21} \{c_1, \pi_c\} \\
&= -a_b \pi_c + a_c \pi_b. \quad (2.23c)
\end{align*}

These are the Dirac brackets between the basic positions and conjugate momenta on $S^3$.

### C. Angular Momentum

On $S^3$ rotation is no different from that in $\mathbb{R}^4$ and the group is $SO(4)$ which has six independent generators. These six angular momenta are direct generalization of the three generators of rotation in the familiar three-dimensional Euclidean space

$$L_{bc} = a_b \pi_c - a_c \pi_b. \quad (2.24)$$

Using $c_2$, one can verify that

$$\pi_d^2 = \frac{1}{2} L_{bc} L_{bc} \quad (2.25)$$

so that the Hamiltonian can also be written as

$$H = \frac{1}{16\lambda} L_{bc} L_{bc} + M. \quad (2.26)$$

Based on the geometry of the compact space one can expect that the angular momentum is more fundamental than the linear ones. This will be clearly seen later on. This is also the reason why Eq. (2.20) is preferred over Eq. (2.12).

### D. Constrained Quantization

Quantization in the presence of constraints amounts to making the change from classical Dirac brackets to commutators. That is

$$[A, B]_D \rightarrow \frac{1}{i} [\hat{A}, \hat{B}]. \quad (2.27)$$

The circumflex here over $A$ and $B$ on the right designate them to be operators. This reproduces the result for the commutators between $\hat{a}_b$ and $\hat{\pi}_c$ in Ref. [10]

\begin{align*}
[\hat{a}_b, \hat{a}_c] &= 0 \quad (2.28a) \\
[\hat{a}_b, \hat{\pi}_c] &= i(\delta_{bc} - \hat{a}_b \hat{a}_c) \quad (2.28b) \\
[\hat{\pi}_b, \hat{\pi}_c] &= i(\hat{a}_c \hat{\pi}_b - \hat{a}_b \hat{\pi}_c) = i L_{cb} \quad (2.28c)
\end{align*}

$L_{bc}$ are the operators of the angular momenta. These are the basic commutators of the $SU(2)$ collective coordinate quantization. One can easily find a differential representation that satisfies the commutators. In fact the representation

\begin{align*}
\hat{a}_b &= a_b, \quad \hat{a}_c = 0 \quad (2.29a) \\
\hat{\pi}_b &= -i(\delta_{bc} - a_b a_c) \partial_c + i \alpha a_b \quad (2.29b)
\end{align*}

with any constant $\alpha$ will satisfy them. The choice of $\alpha$ and some related issues will be discussed in the Appendix A. This is an unfamiliar representation compared to the usual derivative form for $a_b$. However in spite of this, the angular momentum operators are still given by

$$\hat{L}_{bc} = \hat{a}_b \hat{\pi}_c - \hat{a}_c \hat{\pi}_b = -i(a_b \partial_c - a_c \partial_b). \quad (2.30)$$

### E. Eigenstates

With the differential representation of $\hat{L}_{bc}$, the Hamiltonian in Eq. (2.24) becomes

$$\hat{H} = -\frac{1}{8\lambda} [\delta_{bc} - a_b a_c] \partial_b \partial_c - 3a_d \partial_d] + M \quad (2.31)$$

where the $\nabla^2$ can be thought of as the Laplacian on $S^3$. Although its form is not of a Laplacian, it does act like one on $S^3$. The fact that $\nabla^2$ respects the compact space of $S^3$ permits a problem-free (see Appendix B) differentiation procedure. As a result, differentiation can now be performed in exactly the same way as in Euclidean space. The Schrödinger equation can be solved. The wavefunctions are polynomials in powers of the combinations of products of $(a_b + i a_c)^{\ell}$. For example with integral power $l$

$$-\nabla^2 (a_b + i a_c)^l = l(l+2)(a_b + i a_c)^l \quad (2.32)$$

where $b \neq c$ or with $b \neq c \neq d \neq e$

$$-\nabla^2 (a_b + i a_c)^{l-p}(a_d + i a_e)^p \quad (2.33)$$

$$= l(l+2)(a_b + i a_c)^{l-p}(a_d + i a_e)^p(2.33)$$
so the energy only depends on the degree of the polynomial \( l \) and not on the which \( a \)’s that make up the polynomial. With the various combinations of \((a_l + ia_c)\) for a given \( l \), there is a high degree of degeneracy in the energy eigenstates. To label these states, there must be other quantum numbers beside energy.

While the momentum operators \( \hat{p} \) acquired unfamiliar form in Eq. (2.29), \( L_{bc} \) remains the same as those in Euclidean space. From their combinations spin \( J_l \) and isospin \( I_l \) operators can be constructed (see Appendix I). Using the corresponding differential operators, it can be easily checked with Eq. (2.31) that

\[- \nabla^2 = 2(\hat{J}^2 + \hat{I}^2) = 4\hat{J}^2 = 4\hat{I}^2.\] (2.34)

The Hamiltonian then becomes

\[\hat{H} = \frac{1}{4\lambda}(\hat{J}^2 + \hat{I}^2) + M = \frac{1}{2\lambda}\hat{J}^2 + M = \frac{1}{2\lambda}\hat{I}^2 + M\] (2.35)

and the energy eigenvalues are

\[E_l = \frac{1}{8\lambda}l(l + 2) + M.\] (2.36)

Since

\[\hat{J}^2, \hat{I}^2 = [\hat{J}, \hat{I}] = [\hat{J}, \hat{I}] = 0,\] (2.37)

the Hamiltonian commutes with spin and isospin operators

\[\hat{H}, \hat{J}^2 = \hat{H}, \hat{I}^2 = 0\] (2.38)

so eigenstates can be labeled by quantum numbers of energy \( l \), the third component of spin \( m \) and isospin \( n \) such as \( l', m, n' \). For reasons that will become clear below, it is better to use \( l'/2, m, n' \) instead.

Acting on a test wavefunction with \( \hat{J}_3 \) and \( \hat{I}_3 \), we have

\[\hat{J}_3(a_l + ia_c)^l = \pm \frac{l}{2}(a_l + ia_c)^l\] (2.39)
\[\hat{I}_3(a_l + ia_c)^l = \pm \frac{l}{2}(a_l + ia_c)^l\] (2.40)

or

\[\hat{J}_3(a_l + ia_c)^l = \pm \frac{l}{2}(a_l + ia_c)^l\] (2.41)
\[\hat{I}_3(a_l + ia_c)^l = \pm \frac{l}{2}(a_l + ia_c)^l\] (2.42)

The degree of the polynomial \( l \) only gives the energy quantum number \( l \), but also the highest or lowest spin and isospin \( \pm l/2 \) of a (iso)spin multiplet. This can also be seen by varying the integer \( p \) between 0 and \( l \). Also from Eqs. (2.32), (2.33) and (2.34) we see that the eigenvalues of \( J^2 \) and \( I^2 \) for the above wavefunctions are identical and equal to \( J(j + 1) = l(l/2 + 1) \) or \( j = l/2 \). Other examples of the (iso)spin operators acting on test wavefunctions can be found in the Appendix I. Even \( l \) gives integral and odd \( l \) gives half-integral spin and isospin states. Both bosons and fermions are therefore admitted in this system which was shown to be possible long ago in Ref. [67]. However not all these states are physical. Baryons are fermions and must have half-integral (iso)spin. Therefore only odd degree polynomials are physical wavefunctions. For example a spin up and spin one-half state has \( l = 1 \) or \( j = m = n = 1/2 \) and is represented by

\[\langle a|\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle = \frac{1}{\pi}(a_1 + ia_2)\] (2.43)

and a state with spin \(-1/2 \) and isospin \( 3/2 \), \( l = 3 \) or \( j = n = 3/2 \) and \( m = -1/2 \) has the wavefunction

\[\langle a|\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle = -\sqrt{\frac{6}{\pi}}(a_1 + ia_2)(a_0 - ia_3)^2\] (2.44)

where the Dirac ket notation \(|a\rangle\) is used to represent the position vector on \( S^3 \). Other spin-half wavefunctions can be written down. Clearly \((a_5 \pm ia_5)^f\) is the highest or lowest (iso)spin state of a multiplet. One can reproduce the wavefunctions of the whole multiplet for a given degree \( l \) by using the (iso)spin raising or lowering operators as usual. In Appendix I the complete set of normalized spin one-half and three-half wavefunctions are given. Quantization therefore permits the physical states to be found as the eigenstates of the Skyrmion Hamiltonian.

### III. SKYRME MODEL WITH SO(3) COLLECTIVE COORDINATES

#### A. From SU(2) to SO(3)

The general form of the skyrme solution introduced in Sec. II in Eq. (2.3) is invariant under a simultaneous \( SU(2) \) and spatial rotation. Therefore applying a \( SU(2) \) rotation alone must generate a rotation in space. This is equivalent to mapping from \( SU(2) \) to \( SO(3) \). Using the trigonometric form of Eq. (2.3)

\[U_S = \cos F(r) + i\tau \cdot \hat{r} \sin F(r)\] (3.1)

the action of a \( SU(2) \) rotation by an element \( A \) is

\[AUA^\dagger = \cos F(r) + i(A \tau \cdot \hat{r} A^\dagger) \sin F(r).\] (3.2)

\( \mathbb{I} \) denotes generally the identity matrix of a group throughout this paper. Its dimension depends on the context. Here it is the \( 2 \times 2 \) identity matrix of \( SU(2) \). The second term can be further decomposed in terms of the Pauli spin matrices as

\[A\tau_i A^\dagger = \tau_j R_{ij}(A).\] (3.3)

The coefficients of this decomposition \( R_{ij} \) which is a function of \( A \) form a \( 3 \times 3 \) matrix which can be considered as an element of the \( SO(3) \) group. Imagine now that the \( A \) are the collective coordinates for quantizing the
skyrmion. Then the map Eq. (3.3) will permit us to rewrite everything in terms of the rotation matrix $R$. The collective coordinates will now have nine components $R_{ij}$ (not all independent) instead of the four $a_k$ (also not all independent). This method of mapping from SU(2) to SO(3) is known in the literature \[34\]. In the previous section we have already seen how the SU(2) Skyrme theory allows only coherent states formed of both boson and fermion states. These states cannot be the equivalence of a classical skyrmion which must only consist of physical baryon states. Let us now construct instead the SO(3) equivalence of the classical Skyrme Lagrangian and Hamiltonian. Then quantization will give us operators and eigenstates in SO(3).

**B. The Classical Theory**

To derive the SO(3) equivalence of Eq. (2.7), we first rewrite this Lagrangian using the unitary property of $A$ as

$$L = -\lambda \text{tr}(A^\dagger \dot{A})^2 - M .$$

(3.4)

Next we use the identity \[33\]

$$A^\dagger \dot{A} = \frac{1}{2} \tau_\lambda \text{tr}(\tau_i A^\dagger \dot{A}) ,$$

(3.5)

which can be easily verified so that we have

$$L = -\frac{1}{4} \lambda \left(\text{tr}(\tau_i A^\dagger \dot{A})\right)^2 - M .$$

(3.6)

To proceed further differentiating both sides of the map Eq. (3.3) with time, followed by multiplying $A^\dagger$ from the left and $A$ from the right and using Eq. (3.3) again to get

$$A^\dagger \dot{A} + \tau_i \dot{A}^\dagger A = \tau_i R_{ij}(A^\dagger) \dot{R}_{ij}(A) .$$

(3.7)

Taking the trace after multiplying on both sides with $\tau_k$, one arrives at

$$R_{kj}(A^\dagger) \dot{R}_{ji}(A) = -i \epsilon_{kij} \text{tr}(\tau_j A^\dagger \dot{A}) .$$

(3.8)

The first term in Eq. (3.8) can therefore be replaced to give us a Lagrangian in terms of the rotation matrix $R$

$$L = -\frac{1}{7} \lambda \text{tr}(R^{-1} \dot{R}^2) - M .$$

(3.9)

We have use the fact that $R_{ij}(A^\dagger) = R_{ij}^{-1}(A)$ here. The nine elements of $R_{ij}$ become the (collective) coordinates. Similar to the $a_k$ these are not independent but are subject to the constraints of SO(3)

$$\text{det} R = 1$$

(3.10a)

$$RR^{-1} = R^{-1}R = 1$$

(3.10b)

$$R^{-1} = R^T .$$

(3.10c)

The superscript $T$ in the last line stands for transpose. Because of the last constraint one can get rid of $R^{-1}$ everywhere. More explicitly in component form these constraints are

$$\epsilon_{ijk} R_{i1} R_{j2} R_{k3} = \epsilon_{ijk} R_{i1} R_{j2} R_{k3} = 1$$

(3.11a)

$$\epsilon_{ijk} R_{i1} R_{m2} R_{n3} = \epsilon_{ijk} R_{i1} R_{m2} R_{n3} = \epsilon_{lmn}$$

(3.11b)

$$R_{ik} R_{kj} = R_{ki} R_{kj} = \delta_{ij} .$$

(3.11c)

The first and second constraints are equivalent but the second is useful for many algebraic manipulations.

From the above constraints we have the familiar

$$R_{ij}^{-1} \dot{R}_{kj} = R_{ki} \dot{R}_{kj} = -\dot{R}_{ki} R_{kj} .$$

(3.12)

Therefore the Lagrangian can be rewritten as

$$L = \frac{1}{7} \lambda \dot{R}_{ij} \dot{R}_{ij} - M .$$

(3.13)

The conjugate momenta to $R_{ij}$ are

$$\Pi_{ij} = \frac{1}{7} \lambda \dot{R}_{ij} .$$

(3.14)

The Hamiltonian is readily derived to be

$$H = \frac{1}{7} \lambda \dot{R}_{ij} \dot{R}_{ij} + M = \frac{1}{2} \Pi_{ij} \Pi_{ij} + M .$$

(3.15)

**C. The Dirac Brackets**

As a first step toward quantization, we construct the Dirac brackets. To do this, we construct the total Hamiltonian in the same way as before from Eq. (3.11) via the Lagrange multiplier method. Let us see how many constraints there are. The first one in Eq. (3.11) appears to contain $2 \times 3 \times 3 \times 3 = 54$ of them but in fact there is only one because of antisymmetry. The first constraint is therefore

$$c_1 = \epsilon_{ijk} R_{i1} R_{j2} R_{k3} = 1 .$$

(3.16)

The second constraint apparently contains eighteen but because of the symmetry and equivalence of the transposed matrix, there are really only six

$$c_n = R_{ik} R_{jk} - \delta_{ij} = 0$$

(3.17)

where $n = 2, 3, \ldots, 7$ for $(ij) = (11), (12), (22), (13), (23)$ and (33), respectively. That makes seven constraints for nine $R_{ij}$ components. But SO(3) is a three-parameter compact Lie group so there can only be six constraints of the type of $c_n$ for $n = 1–7$ (constraints that involve only $R_{ij}$ and no $\Pi_{ij}$). After dropping $c_7$ we have the total Hamiltonian

$$H_T = \frac{1}{2} \Pi_{ij} \Pi_{ij} + M + \sum_{n=1}^{6} \rho_n c_n .$$

(3.18)

Remember from the SU(2) theory previously that these are not the only constraints. The total number of constraints can only be determined by repeated application of

$$c_{i+6} = \{c_i, H_T\} = 0$$

(3.19)
for \( i \geq 1 \). This will produce many constraints and we are likely to have to calculate the inverse of a very large matrix of Poisson brackets in order to construct the Dirac brackets. This step is very arduous if not impossible. We shall try another strategy.

1. The Special Case of SO(2): The Constraints

To keep the calculation within manageable size, we calculate the SO(2) Dirac brackets and then extrapolate the results to SO(3) at the end. The primary constraints in this case are

\[
\begin{align*}
    c_1 &= \epsilon_{ij} R_{1j} R_{2j} - 1 = 0, \\
    c_n &= R_{ik} R_{jk} - \delta_{ij} = 0,
\end{align*}
\]

for \( n = 2, 3, 4 \) for \((ij) = (11), (12), (22)\) respectively. Here apparently we have four unknowns and four constraint equations so that each \( R_{ij} \) is fixed and can be calculated. However SO(2) is a one parameter Lie group. One of the constraints must be dependent of the other three and so can be dropped.

There is a choice here of what form of the constraints should be used. One could drop one of them, say \( c_4 \), but keeping the first three to ensure that there will not be any redundancy, or as is well known there are some simple relations between the matrix elements of SO(2) which are equivalent to the above constraint equations. Let us try the following constraints which are complete but simpler

\[
\begin{align*}
    c_1' &= \epsilon_{ij} R_{1j} R_{2j} - 1 = 0, \\
    c_2' &= R_{11} - R_{22} = 0, \\
    c_3' &= R_{12} + R_{21} = 0.
\end{align*}
\]

These evidently imply

\[
R_{11}^2 + R_{12}^2 = R_{22}^2 + R_{21}^2 = 1
\]

which shows that they are equivalent to the previous version of \( c_n \)'s.

The total Hamiltonian can be written as

\[
H_T = \frac{1}{\lambda} \Pi_{ij} \Pi_{ij} + M + \rho_1 (\cos \epsilon_{ij} R_{1j} R_{2j} - 1) + \rho_2 (R_{11} - R_{22}) + \rho_3 (R_{12} + R_{21}).
\]

Now using the basic Poisson brackets

\[
\{ R_{ij}, \Pi_{kl} \} = \delta_{ij} \delta_{kl},
\]

( all other brackets vanish) and after dropping the factor \( \rho_1/\lambda \), repeated application of

\[
c_1' + c_2' \{ c_1', H_T \} = 0
\]

gives

\[
\begin{align*}
    c_4' &= \frac{1}{\epsilon} \epsilon_{ij} (\Pi_{1j} R_{2j} + R_{2j} \Pi_{1i}) \\
    &= \epsilon_{ij} (\Pi_{1j} R_{2j} + R_{2j} \Pi_{1i}) = 0.
\end{align*}
\]

The simpler \( c_2' \) and \( c_3' \) give

\[
c_5' = \Pi_{11} - \Pi_{22} = 0
\]

and

\[
c_6' = \Pi_{12} + \Pi_{21} = 0.
\]

Continuing with \( i \geq 4 \), we get

\[
c_7' = \frac{4}{\lambda} \epsilon_{ij} \Pi_{1j} \Pi_{2j} - 2 \rho_1 - \rho_2 (R_{22} - R_{11}) + \rho_3 (R_{12} + R_{11}) = 0
\]

where Eq. (3.25) has been used. According to Dirac, one is permitted to make use of the constraints only after the Poisson brackets have been evaluated \( [\pi, \pi] \). Doing this now using \( c_2' \) and \( c_4' \) reduces \( c_7' \) to

\[
c_7' = \frac{2}{\lambda} \epsilon_{ij} \Pi_{1j} \Pi_{2j} - \rho_1 = 0.
\]

Next for \( c_8' \) and \( c_9' \), again we impose the constraints after the brackets have been evaluated to get

\[
c_8' = \rho_2 = 0
\]

and

\[
c_9' = \rho_3 = 0.
\]

The constraints \( c_7', c_8' \) and \( c_9' \) provide solution to the Lagrange multipliers. There are no first class constraints since none of them has vanishing Poisson bracket with every other constraints. Therefore the second class constraints are \( c_n' \), \( n = 1, \ldots, 6 \). We have a \( 6 \times 6 \) matrix of Poisson brackets of second class constraints \( \{ c_i', c_j' \} \).

2. The Special Case of SO(2): The Dirac Brackets

The Poisson brackets amongst \( c_1', c_2' \) and \( c_3' \) evidently vanish. The same is also true between \( c_5' \) and \( c_6' \). The only potentially non-vanishing elements are those Poisson brackets involving \( c_4 \) and the \( \{ c_2', c_5' \} \) and \( \{ c_3', c_6' \} \) pairs. Working them out explicitly, the matrix of Poisson brackets \( m_{ij} = \{ c_i', c_j' \} \) is

\[
m = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

This can be easily inverted to give

\[
m^{-1} = \frac{1}{2} \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Trying out a few Dirac brackets using the definition Eq. (2.21) produces vanishing brackets between the components of the coordinates

\[
\{ R_{ij}, R_{kl} \}_D = 0.
\]
Between coordinate and momentum pairs, we can have for example
\[
\{R_{11}, \Pi_{11}\}_D = 1 - \{R_{11}, c'_1\}m^{-1}_{11}\{c'_1, \Pi_{11}\}
\]
\[
- \{R_{11}, c'_2\}m^{-1}_{22}\{c'_2, \Pi_{11}\}
\]
\[
= \frac{1}{2}(1 - R_{22}R_{22}) = \frac{1}{2}(1 - R_{11}R_{11}) ,
\]
(3.39)

\[
\{R_{11}, \Pi_{12}\}_D = - \{R_{11}, c'_1\}m^{-1}_{11}\{c'_1, \Pi_{12}\}
\]
\[
- \{R_{11}, c'_2\}m^{-1}_{22}\{c'_2, \Pi_{12}\}
\]
\[
= \frac{1}{2}R_{22}R_{21} = - \frac{1}{2}R_{11}R_{12} ,
\]
(3.40)

\[
\{R_{12}, \Pi_{11}\}_D = 1 - \{R_{12}, c'_1\}m^{-1}_{11}\{c'_1, \Pi_{11}\}
\]
\[
- \{R_{12}, c'_2\}m^{-1}_{22}\{c'_2, \Pi_{11}\}
\]
\[
= \frac{1}{2}(1 - R_{21}R_{21}) = \frac{1}{2}(1 - R_{12}R_{12}) .
\]
(3.41)

and also
\[
\{R_{12}, \Pi_{11}\}_D = - \{R_{12}, c'_1\}m^{-1}_{11}\{c'_1, \Pi_{11}\}
\]
\[
= \frac{1}{2}R_{21}R_{22} = - \frac{1}{2}R_{12}R_{11} .
\]
(3.42)
The same must hold if the indices have been interchanged (1, 2) → (2, 1) everywhere. One could now attempt to deduce the general expression for the Dirac brackets between \(R_{ij}\) and \(\Pi_{kl}\)
\[
\{R_{ij}, \Pi_{kl}\}_D = \delta_{ik}\delta_{jl} - \frac{1}{2}(\delta_{ik}\delta_{jl} + R_{ij}R_{kl}) + \ldots .
\]
(3.43)
The first term is from the first Poisson bracket in the definition of the Dirac bracket Eq. (2.21) and the third term seems to hold everywhere. The second term is there so that this also agrees with Eqs. (3.39) and (3.41). Lastly there might be other terms hence the ellipses.

Other Dirac brackets between \(R_{ij}\) and \(\Pi_{kl}\) can be deduce using the constraints. From Eq. (3.39)
\[
\{R_{11}, \Pi_{22}\}_D = \{R_{22}, \Pi_{11}\}_D = \frac{1}{2}(1 - R_{11}R_{22}) ,
\]
(3.44)
the first term of which deviates from Eq. (3.43). Then from Eq. (3.40)
\[
\{R_{22}, \Pi_{12}\}_D = - \{R_{11}, \Pi_{21}\}_D = - \{R_{22}, \Pi_{21}\}_D
\]
\[
= - \frac{1}{2}R_{22}R_{12} = \frac{1}{2}R_{11}R_{21} = \frac{1}{2}R_{22}R_{21} .
\]
(3.45)
These are all consistent with Eq. (3.43). Next there is also Eq. (3.41) which can be combined with the constraints to give
\[
\{R_{12}, \Pi_{12}\}_D = \{R_{21}, \Pi_{21}\}_D = - \frac{1}{2}(1 + R_{12}R_{21})
\]
(3.46)
and
\[
\{R_{21}, \Pi_{21}\}_D = \frac{1}{2}(1 - R_{21}R_{21}) .
\]
(3.47)
Of these equations the second is consistent with our guess in Eq. (3.43) but not the first. This seems to hint at a term of \(-\delta_{il}\delta_{kl}/2\) when \(i \neq k\) or \(j \neq l\) otherwise this would contradict Eq. (3.39). Then Eq. (3.44) suggests another term of the form \(\delta_{ij}\delta_{kl}/2\) under the same condition of \(i \neq k\) or \(j \neq l\). The sum of these two terms would ensure all conditions are met. This requires extending Eq. (3.43) to
\[
\{R_{ij}, \Pi_{kl}\}_D = \delta_{ik}\delta_{jl} - \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) + R_{ij}R_{kl} .
\]
(3.48)
This is consistent with all the above Dirac brackets. Simplifying as much as possible finally gives
\[
\{R_{ij}, \Pi_{kl}\}_D = \frac{1}{2}\epsilon_{ijkl} + \epsilon_{ik}\epsilon_{jl} - R_{ij}R_{kl} .
\]
(3.49)
We have made use of the identity
\[
\epsilon_{ijkl} = \delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}
\]
(3.50)
which is only a special case of the three dimensional expression for \(\epsilon_{mn}c_{mn3}\).

There still remain the Dirac brackets between momenta to be worked out
\[
\{\Pi_{ij}, \Pi_{kl}\}_D = - \{\Pi_{ij}, c'_1\}m^{-1}_{14}\{c'_4, \Pi_{kl}\}
\]
\[
- \{\Pi_{ij}, c'_2\}m^{-1}_{41}\{c'_1, \Pi_{kl}\} .
\]
(3.51)

For example between equal-index pairs, all Dirac brackets vanish
\[
\{\Pi_{11}, \Pi_{11}\}_D = \{\Pi_{22}, \Pi_{22}\}_D = \{\Pi_{11}, \Pi_{22}\}_D = 0 .
\]
(3.52)

Between equal pairs, they also vanish
\[
\{\Pi_{12}, \Pi_{12}\}_D = - \{\Pi_{12}, \Pi_{21}\}_D = 0 .
\]
(3.53)

Next between unequal-index pairs
\[
\{\Pi_{11}, \Pi_{12}\}_D = \frac{1}{2}(R_{22}R_{21} - R_{21}R_{22})
\]
\[
= -\frac{1}{2}(R_{11}R_{12} - R_{12}R_{11}) .
\]
(3.54)

To arrive at the last expression, one uses \(c'_2\) and \(c'_3\). This last line seems to be more closely related to the LHS. From here the rest of the brackets can be deduced
\[
\{\Pi_{11}, \Pi_{12}\}_D = \{\Pi_{22}, \Pi_{21}\}_D = - \{\Pi_{11}, \Pi_{21}\}_D
\]
\[
= - \{\Pi_{22}, \Pi_{21}\}_D \neq 0 .
\]
(3.55)

Aiming now for a general expression, from these examples we can infer that the Dirac brackets between conjugate momenta are
\[
\{\Pi_{ij}, \Pi_{kl}\}_D = - \frac{1}{2}(R_{ij}R_{kl} - R_{kl}R_{ij}) .
\]
(3.56)

This is satisfied by the Dirac brackets above. This can be further verified using Eq. (3.49) and the Jacobi identity. The latter is one of the basic requirements for the Dirac brackets in the Dirac bracket between conjugate momenta are in general non-vanishing. This is different from the familiar examples when the different components of the momenta are independent.
3. Extrapolating The Dirac Brackets from SO(2) to SO(3)

Returning now to SO(3), we still have the original primary constraints of \(c_i, i = 1-6\) which ultimately dictate what the Dirac brackets will be. We start by considering the simplest ones — those involving only components of \(R\). The vanishing of the Dirac brackets between them

\[
\{ R_{ij}, R_{kl} \}_D = 0 \tag{3.57}
\]

is expected to continue to hold for general SO(\(N\)). This can be understood as follows. Since the only non-vanishing Poisson brackets between \(R_{ij}\) and the constraints are those \(c_n\) which involve \(\Pi_{ij}\), these must all be secondary constraints derived from the \(c_i, i \leq 6\) via \(\{ c_i, H_T \}\). In this way, each primary constraint will lead to a secondary constraint \(c_n\), \(6 < n < 13\) such as

\[
c_7 = \epsilon_{ijk}(\Pi_{1i}R_{2j}R_{3k} + R_{1i}\Pi_{2j}R_{3k} + R_{1i}R_{2j}\Pi_{3k}) = 0 \tag{3.58}
\]

and

\[
c_n = \Pi_{ik}R_{jk} + R_{ik}\Pi_{jk} = 0 \tag{3.59}
\]

for \(8 \leq n \leq 12\) with \((i, j) = (11), (12), (13), (22), (23)\). The next level of constraints \(c_n\) with \(n > 12\) generated from these will therefore be equations involving the Lagrange multiplier \(\rho_i, i \leq 6\) since the presence of \(\Pi\) will bring the multipliers into the equations. We expect no more new constraints after that. The matrix of Poisson brackets between second class constraints \(m_{ij} = \{ c_i, c_j \}\) will be a \(12 \times 12\) square matrix. It must have a \(6 \times 6\) block of zeros in the upper left-hand corner between row one to six and column one to six because primary constraints have mutually vanishing Poisson brackets. After taking the inverse, this block will reappear on the lower right-hand corner of \(m^{-1}\). All cofactors in this \(6 \times 6\) corner vanishes because of the block of zeros in \(m\). But the only way to have non-vanishing Dirac brackets between components of \(R\) is if the entries in this block are not all zeros because \(\{ R_{ij}, \epsilon_p \} \neq 0\) only if \(7 \geq p \geq 12\). Thus the \(R\) components must commute with each other.

Let us now extrapolate Eq. (3.49). The only part of this equation that cannot readily be extended to SO(3) is the second term with the antisymmetric tensors. In SO(3) the structure constants are the \(\epsilon_{ijk}\) instead of \(\epsilon_{ij}\). The latter is implicitly equal to \(\epsilon_{ij3}\) when \(i, j\) are restricted only to 1 and 2. The \(\epsilon_{ik}\) \(\epsilon_{jl}\) in Eq. (3.43) must be replaced by \(\epsilon_{ik3}\epsilon_{jl3}\). But this would leave us with the two 3 indices when there is no reason for 3 to be any more special than either 1 and 2. If one remembers that SO(2) is a subgroup of SO(3) which is equal to the former when the constraint \(R_3 = 1\) is imposed. With this piece of information, one can now remove the 3 indices by first identifying \(\epsilon_{ik3}\epsilon_{jl3}\) with \(\epsilon_{ik}\epsilon_{jl}\) \(R_{3j}\). Then one can replace this with \(\epsilon_{ikm}\epsilon_{jln}R_{mn}\) when the first pair of indices are no longer restricted to 1 and 2. The Dirac brackets become

\[
\{ R_{ij}, \Pi_{kl} \}_D = \frac{1}{2}(\delta_{ik}\delta_{jl} + \epsilon_{ikm}\epsilon_{jln}R_{mn} - R_{ij}R_{kl}) \tag{3.60}
\]

This must hold because one could imagine using the indices, say 2 and 3, for SO(2) instead of 1 and 2 then this would reduce to the correct form even for this artificial case when \(R_{11} = 1\) and \(R_{13} = R_{31} = 0\) for \(i = 2, 3\).

Lastly the brackets Eq. (3.56) seem to be ready for generalization to SO(3) by letting the indices running from 1 to 3. Are these the correct Dirac brackets for SO(3)? In fact for SO(2), the form of Eq. (3.56) is not unique. In this special case, one could equally rewrite it after some experimentation with Eq. (3.54) using \(c_2'\) and \(c_3'\) as

\[
\{ \Pi_{ij}, \Pi_{kl} \}_D = \frac{1}{2}\delta_{ik}(R_{ml}\Pi_{mj} - R_{mj}\Pi_{ml}) + \frac{1}{2}\delta_{jl}(R_{km}\Pi_{im} - R_{im}\Pi_{ki}) \tag{3.61}
\]

Which of these are the correct Dirac brackets? It can be easily checked by using the Jacobi identity that Eq. (3.61) are the correct ones.

D. Spin and Isospin Operators

The Lagrangian Eq. (3.9) manifests symmetry under the left and right SO(3) rotation. An infinitesimal transformation with parameter \(\eta_k\), where \(k\) index labels the direction of rotation, gives

\[
R_{ij} \to R_{ij} + \eta_k\delta^L_{ik}R_{ij} = R_{ij} + \eta_k\epsilon_{kl}R_{ij} \tag{3.62}
\]

\[
R_{ij} \to R_{ij} + \eta_k\delta^R_{ik}R_{ij} = R_{ij} + \eta_k\epsilon_{kl}R_{kl} \tag{3.63}
\]

Because \(R\) has only time dependence, these generate the conserved left and right Noether charges but no three-current

\[
Q^L_i = \frac{\partial L}{\partial R_{jk}}\delta^L_{ik}R_{jk} = \Pi_{jk}\epsilon_{ijl}R_{lk} = \epsilon_{ijk}\epsilon_{klj}R_{lk} \tag{3.64a}
\]

\[
Q^R_i = \frac{\partial L}{\partial R_{jk}}\delta^R_{ik}R_{jk} = \Pi_{jk}\epsilon_{ijl}R_{lk} = \epsilon_{ijk}\Pi_{lk}R_{lj} \tag{3.64b}
\]

Time evolution of the primary constraints in Eq. (3.11) leads to a set of secondary constraints [64, 65]. For example the second equation in Eq. (3.11) gives

\[
R_{ik}\Pi_{jk} + \Pi_{ik}R_{jk} = R_{ki}\Pi_{kj} + \Pi_{ki}R_{kj} = 0 \tag{3.65}
\]

This together with the primary constraints allow us to derive

\[
Q^L_iQ^L_i = Q^R_iQ^R_i = 2\Pi_{ij}\Pi_{ij} \tag{3.66}
\]

The Hamiltonian acquires yet other equivalent forms

\[
H = \frac{1}{4\pi} Q^L_iQ^L_i + M = \frac{1}{4\pi} Q^R_iQ^R_i + M \tag{3.67}
\]

The significant of the conserved charges will become clear in the next section.
E. The Connection between the $SU(2)$ and $SO(3)$ collective variables

To get an idea of the physics of the conserved charges, we now derive the connections between the $SU(2)$ conjugate pairs $(a_0, \pi_0)$ and those pairs $(R_{ij}, \Pi_{ij})$ of $SO(3)$. From the map Eq. (3.34), one can deduce that

$$R_{ij} = \frac{1}{2} \text{tr}(\tau_i A^r A^l) .$$

(3.68)

To make the connection more explicit, one can use the component form of $A$ in Eq. (2.9) to get the expression

$$R_{ij} = 2a_i a_j + \delta_{ij}(2a_0^2 - 1) - 2\epsilon_{ijk} a_0 a_k .$$

(3.69)

This expression together with Eq. (3.14) gives us the conjugate momentum $\Pi_{ij}$ in terms of $a_0$ and $\pi_0$

$$\Pi_{ij} = \frac{1}{4}(a_i \pi_j + a_j \pi_i + \delta_{ij}(a_0 \pi_0 + \pi_0 a_0) - \epsilon_{ijk} a_0 a_k + \pi_0 a_k) .$$

(3.70)

Substituting these functions into Eq. (3.64) after some algebra, we have

$$Q^L_{ij} = \frac{1}{2}(a_i \pi_j - a_j \pi_i - \epsilon_{ijk} a_0 a_k)$$

$$Q^R_{ij} = \frac{1}{2}(a_i \pi_j - a_j \pi_i + \epsilon_{ijk} a_0 a_k) .$$

(3.71)

(3.72)

Comparing the expressions for $J_i$ and $I_i$ in Eqs. (C4) and (C5) (temporarily replacing $\pi_0$ with $-i\partial_0$ to see this) reveals that

$$Q^L_{ij} = -J_i , \quad Q^R_{ij} = I_i .$$

(3.73)

The conserved charges are actually spin and isospin. Note that all variables here are classical and commuting so that there is no ambiguity as to the ordering. For this reason, we will refrain from carrying the connections here to the quantum level when one does not know which operator ordering is the correct one. However one still identifies the charge operators with spin and isospin operators as shown here. Therefore the Hamiltonian can be written in terms of spin and isospin as in the $SU(2)$ case

$$H = \frac{1}{2\lambda} F^2 + M = \frac{1}{2\lambda} F^2 + M .$$

(3.74)

F. Constrained Quantization

Quantization is accomplished by simply replacing the Dirac brackets [5] by $-i$ times the commutators. Therefore

$$[\hat{R}_{ij}, \hat{R}_{kl}] = 0$$

(3.75a)

$$[\hat{R}_{ij}, \hat{\Pi}_{kl}] = 2i(\delta_{ik}\delta_{jl} + \epsilon_{kml}\epsilon_{jlm} - \hat{R}_{mn}\hat{R}_{nk})$$

(3.75b)

$$[\hat{\Pi}_{ij}, \hat{\Pi}_{kl}] = \frac{1}{4} \hat{\epsilon}_{ikm}(\hat{R}_{ml}\hat{\Pi}_{m} - \hat{R}_{ml}\hat{\Pi}_{m})$$

$$+ \frac{1}{4} \hat{\epsilon}_{jl}(\hat{R}_{km}\hat{\Pi}_{m} - \hat{R}_{km}\hat{\Pi}_{m}) .$$

(3.75c)

A simple multiplication and differential representation of the operators are

$$\hat{R}_{ij} = R_{ij}$$

$$\hat{\Pi}_{ij} = -\frac{1}{2}i(\delta_{ik}\delta_{jl} + \epsilon_{kml}\epsilon_{jlm}R_{mn} - R_{ij}R_{kl})\partial_{kl} + i\alpha R_{ij}$$

(3.76)

(3.77)

where the shorthand $\partial_{ij} = \partial / \partial R_{ij}$ is used. Promoting Eq. (3.64) first to operators, then after some algebra using the identities

$$\epsilon_{mja}\epsilon_{nlb}\hat{R}_{kl}\hat{R}_{ab} = \hat{R}_{mn}\delta_{jk} - \hat{R}_{jn}\delta_{mk}$$

(3.78a)

$$\epsilon_{mka}\epsilon_{nlb}\hat{R}_{ij}\hat{R}_{ab} = \hat{R}_{mn}\delta_{jk} - \hat{R}_{mk}\delta_{jn}$$

(3.78b)

and the above representation, it can be shown that

$$\hat{J}_i = \frac{1}{2}\epsilon_{ijk}(\hat{R}_{jl}\hat{\Pi}_{kl} - \hat{R}_{kl}\hat{\Pi}_{jl})$$

$$- \frac{1}{2}\epsilon_{ijk}(\hat{R}_{ij}\partial_{kl} - \hat{R}_{kl}\partial_{ij})$$

(3.79)

$$\hat{I}_i = \frac{1}{2}\epsilon_{ijk}(\hat{R}_{ij}\hat{\Pi}_{lk} - \hat{R}_{lk}\hat{\Pi}_{ij})$$

$$= -\frac{1}{2}\epsilon_{ijk}(\hat{R}_{lj}\partial_{ik} - \hat{R}_{ik}\partial_{lj}) .$$

(3.80)

One can easily check that they satisfy all the commutation relations expected of the spin and isospin operators (these are of course the same as those for $SU(2)$ see Appendix C).

G. Eigenstates

From Eq. (3.74) and the $\hat{J}$ and $\hat{I}$ operators, the differential form of the Hamiltonian is

$$\hat{H} = -\frac{1}{2\lambda}(\partial_{ij} - 2R_{ij} - R_{im}R_{kj}\partial_{km})\partial_{ij} + M$$

$$= -\frac{1}{2\lambda} \nabla^2 + M .$$

(3.81)

To find the wavefunctions, it is more convenient to use specific combinations of the components. They are

$$R^{++} = R_{11} - R_{22} + i(R_{12} + R_{21})$$

(3.82a)

$$R^{+-} = R_{11} + R_{22} - i(R_{12} - R_{21})$$

(3.82b)

$$R^{-+} = R_{11} + R_{22} + i(R_{12} - R_{21})$$

(3.82c)

$$R^{-} = R_{11} - R_{22} - i(R_{12} + R_{21})$$

(3.82d)

$$R^{\pm 0} = R_{13} \pm iR_{23}$$

(3.82e)

$$R^{0\pm} = R_{31} \pm iR_{32}$$

(3.82f)

$$R^{00} = R_{33} .$$

(3.82g)

Using these, it can be easily verified that

$$-\nabla^2 (R^{pq})^l = l(l+1)(R^{pq})^l$$

(3.83)

where $l \in \mathbb{Z}$ with the exception of $p = q = 0$ when $l > 1$, or

$$-\nabla^2 (R^{pq})^{l-r}(R^{pq})^r = l(l+1)(R^{pq})^{l-r}(R^{pq})^r$$

(3.84)

with $r \leq l$. The energy eigenvalues are then

$$E_l = \frac{1}{2\lambda}(l(l+1) + M .$$

(3.85)
It can thus be established that the wavefunctions are polynomials in the products of the components of $R$. They are also eigenstates of the third component of spin and isospin

$$\hat{J}_3(R^{pq})^l = p_l(R^{pq})^l \quad (3.86)$$
$$\hat{I}_3(R^{pq})^l = q_l(R^{pq})^l \quad (3.87)$$
$$\hat{J}_3(R^{pq})^{-l-r}(R^{\pm 0})^r = \pm l(R^{pq})^{-l-r}(R^{\pm 0})^r \quad (3.88)$$
$$\hat{I}_3(R^{pq})^{-l-r}(R^{\pm 0})^r = q(l-r)(R^{pq})^{-l-r}(R^{\pm 0})^r \quad , \quad (3.89)$$

The polynomial degree $l$ dictates the energy, spin and isospin eigenvalues just as in the $SU(2)$ theory. This time, however, there are only integer values of $l$ (functions with half-integer $l$ are not defined). The Hilbert space of this system consists purely of bosonic states $|j, m, n\rangle$ where $j, m, n \in \mathbb{Z}$. These are not physical states expected from the Skyrme model.

**IV. SUMMARY AND OUTLOOK**

In this paper we have achieved the quantization of the Skyrme model on the $S^3$ as well as on the $SO(3)$ compact manifolds. These have been done in a rigorous manner using the Dirac prescription for a constrained system. While the results from the $SU(2)$ collective coordinate quantization are mostly known, this is not true for the same on the $SO(3)$ manifold. For example, the commutators Eq. (2.72) in this case have not appeared in any literature that we know of. This may be related to the fact the eigenstates are not physical and therefore there is little incentive to pursue this second method in any detail. This will turn out not to be completely true. In the next paper to come and indeed was mentioned in [58], we will need the $SO(3)$ quantized operators, if not the unphysical eigenstates, to construct coherent states of baryons. The method of coherent states on a compact manifold is a mixture of methods used in the context of functional analyses [58] and in the studies of the classical limit of quantum gravity [59]. All these will be shown in the work to come.

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**APPENDIX A: CHOICE OF THE SECONDARY CONSTRAINT**

Quantizing the theory produces certain ambiguities associated with the constraints. These ambiguities have to do with the ordering of the classical $c$-numbers turned quantum $q$-numbers. For example while the primary constraint

$$\hat{c}_1 = \hat{a}_b \hat{a}_b - \hat{1} = 0 \quad , \quad (A1)$$

does not pose a problem, the secondary constraint Eq. (2.17) can be expressed as any of the following:

$$\hat{c}_2 = \hat{a}_b \hat{\pi}_b = 0 \quad (A2)$$
or

$$\hat{c}_2 = \hat{\pi}_b \hat{a}_b = 0 \quad (A3)$$
or

$$\hat{c}_2 = \frac{1}{2} (\hat{a}_b \hat{\pi}_b + \hat{\pi}_b \hat{a}_b) = 0 \quad . \quad (A4)$$

They are not equivalent because of Eq. (2.28b). Let us not insist on any one of them. From Eq. (2.28b) they imply

$$\hat{\pi}_b \hat{a}_b = -i(d - 1) \quad , \quad (A5a)$$
$$\hat{a}_b \hat{\pi}_b = i(d - 1) \quad , \quad \text{or} \quad (A5b)$$
$$\hat{a}_b \hat{\pi}_b = \frac{1}{2} i(d - 1) \quad , \quad \hat{\pi}_b \hat{a}_b = -\frac{1}{2} i(d - 1) \quad , \quad (A5c)$$

respectively. We have decided to introduce the dimension $d$ during the intermediate steps of the calculation which can be set to $d = 4$ at any time. We have seen in the differential representation of $\pi_b$ in Eq. (2.29) that there is a free parameter $\alpha$. Using this representation and the above equations, one can see that $\alpha$ is in fact not really a free parameter. Its value is actually restricted by how the secondary constraint is implemented. In the three cases, we have

$$\alpha = 0 \quad , \quad (A6a)$$
$$\alpha = d - 1 \quad , \quad (A6b)$$
$$\alpha = \frac{1}{2} (d - 1) \quad , \quad (A6c)$$

or equivalently

$$\hat{\pi}_b = -i(\delta_{bc} - a_b a_c) \partial_c \quad , \quad (A7a)$$
$$\hat{\pi}_b = -i(\delta_{bc} - a_b a_c) \partial_c + i(d - 1) \hat{a}_b \quad , \quad (A7b)$$
$$\hat{\pi}_b = -i(\delta_{bc} - a_b a_c) \partial_c + i(d - 1) \hat{a}_b \quad , \quad (A7c)$$

respectively. Among the three choices, the third one seems more natural in the sense that Eq. (A7) is a direct consequence quantum mechanically of the time-evolved form of Eq. (A1).

In the main text, the readers might have noticed that we always expressed the classical Hamiltonian in terms of $J^2$ first and then upon quantization substituting everything with their corresponding operators. This is slightly
different from quantizing with the classical Hamiltonian expressed in terms of the \( \pi_a \pi_b \). We have been careful to avoid the latter because of the non-uniqueness of the differential representation of \( \pi_b \). On the other hand the differential representation of \( \hat{J}_l \) is independent on the choice of \( \alpha \). We feel that the Hamiltonian should be free from such ambiguity and in any case \( \hat{J}_l \) are more fundamental on \( S^3 \) than \( \pi_b \). In favoring \( \hat{J}_l \) over \( \pi_b \), the Hamiltonian operator is

\[
\hat{H} = \frac{1}{2\lambda} \hat{J}^2 + M = \frac{1}{8\lambda} \hat{\pi}_b \hat{\pi}_b + M + \frac{1}{8\lambda} \alpha (\alpha - d + 1) .
\]

The last equality is more cumbersome looking. The extra terms are needed so that different values of \( \alpha \) will give the same Hamiltonian because the \( \hat{\pi}_b \) will change with \( \alpha \).

**APPENDIX B: QUANTIZATION ON \( S^3 \)**

Quantization of the Hamiltonian Eq. (2.12) is problematic because the coordinates \( A \) lives on \( S^3 \) which is a curved compact space. It is tempting to follow the usual procedure of canonical quantization and use the commutators in Eq. (2.13)

\[
[\hat{a}_b, \hat{a}_c] = 0 , \quad [\hat{a}_b, \hat{\pi}_c] = i\delta_{bc} . \quad (B1)
\]

This would give the familiar differential representation for \( \hat{\pi}_b \)

\[
\hat{\pi}_b = -i \frac{\partial}{\partial \hat{a}_b} = -i \partial_b . \quad (B2)
\]

The problem of following this standard procedure is that in the subsequent use of \( \hat{\pi}_b \), it is always necessary to keep in mind that the derivative is meant to be on \( S^3 \). Thus the straightforward rules of differentiation does not necessarily apply and indeed may be modified. For example while \( \partial_b a_c = \delta_{bc} \) in \( \mathbb{R}^n \), this is not true on \( S^3 \) which can be very confusing. In fact one must use the strange result \( \partial_b a_c = \delta_{bc} - a_c \delta_{ba} \) to get the right answer. The problem is that the commutators Eq. (2.13) do not respect the constraint of this system which is \( a_b a_b = 1 \). A correct quantization procedure must yield

\[
[\hat{a}_b \hat{a}_b - \hat{1}, \hat{O}] = [\hat{a}_b \hat{a}_b, \hat{O}] = 0 , \quad (B3)
\]

where \( \hat{O} \) is any operator, whereas Eq. (2.13) gives

\[
[\hat{a}_b \hat{a}_b - \hat{1}, \hat{\pi}_c] = 0 , \quad [\hat{a}_b \hat{a}_b, \hat{\pi}_c] = 2i\hat{a}_c . \quad (B4)
\]

The first one of these is fine but not the second. In the presence of constraints, one quantizes the Dirac brackets instead of the Poisson brackets. In short the Dirac bracket is the Poisson bracket but with some subtraction so that every constraint of the system is respected by the Dirac bracket. Then upon quantization they are automatically respected by the commutators. In essence this is not dissimilar to the way of constructing a vector from a given vector so that the result is orthogonal to a third. In the main text we showed how to do this properly.

**APPENDIX C: ANGULAR MOMENTUM, SPIN, ISOISPIN OPERATORS AND THEIR EIGENFUNCTIONS**

In the text we gave the six generators of rotation in \( \mathbb{R}^4 \) or those of the group \( SO(4) \)

\[
\hat{L}_{bc} = \hat{a}_b \hat{\pi}_c - \hat{a}_c \hat{\pi}_b = -i(a_b \partial_c - a_c \partial_b) . \quad (C1)
\]

Rewriting them more conveniently in three-vector form

\[
\hat{L}_i = \frac{1}{2} \varepsilon_{ijk} \hat{L}_{jk} \quad (C2a) \\
\hat{K}_i = \hat{L}_{0i} \quad (C2b)
\]

they obey the following commutators

\[
[\hat{L}_i, \hat{L}_j] = i\varepsilon_{ijk} \hat{L}_k \quad (C3a) \\
[\hat{L}_i, \hat{K}_j] = i\varepsilon_{ijk} \hat{K}_k \quad (C3b) \\
[\hat{K}_i, \hat{K}_j] = i\varepsilon_{ijk} \hat{L}_k . \quad (C3c)
\]

From them one can construct generators of two separate copies of \( SU(2) \)

\[
\hat{J}_i = \frac{1}{2}(\hat{L}_i + \hat{K}_i) = \frac{1}{2} i \left( a_i \partial_0 - a_0 \partial_i - \varepsilon_{ijk} a_j \partial_k \right) \quad (C4) \\
\hat{I}_i = \frac{1}{2}(\hat{L}_i - \hat{K}_i) = \frac{1}{2} i \left( a_0 \partial_i - a_i \partial_0 - \varepsilon_{ijk} a_j \partial_k \right) \quad (C5)
\]

They are the spin and isospin operators with the familiar commutators

\[
[\hat{J}_i, \hat{J}_j] = i\varepsilon_{ijk} \hat{J}_k \quad (C6a) \\
[\hat{I}_i, \hat{I}_j] = i\varepsilon_{ijk} \hat{I}_k \quad (C6b) \\
[\hat{J}_i, \hat{J}_j] = 0 . \quad (C6c)
\]

One can form raising and lowering operators for spin and isospin as usual

\[
\hat{J}_\pm = \hat{J}_1 \pm i \hat{J}_2 , \quad \hat{I}_\pm = \hat{I}_1 \pm i \hat{I}_2 . \quad (C7)
\]

Their commutation relations are

\[
[\hat{J}_+, \hat{J}_-] = 2\hat{J}_3 , \quad [\hat{J}_3, \hat{J}_\pm] = \pm \hat{J}_\pm , \quad (C8a) \\
[\hat{I}_+, \hat{I}_-] = 2\hat{I}_3 , \quad [\hat{I}_3, \hat{I}_\pm] = \pm \hat{I}_\pm \quad (C8b)
\]

which follows naturally from Eq. (C6). As is well known, the squared operators can be expressed in terms of them

\[
\hat{J}^2 = \hat{J}_+ \hat{J}_- + \hat{J}_3(\hat{J}_3 + 1) = \hat{J}_+ \hat{J}_- + \hat{J}_3(\hat{J}_3 - 1) \quad (C9a) \\
\hat{I}^2 = \hat{I}_+ \hat{I}_- + \hat{I}_3(\hat{I}_3 + 1) = \hat{I}_+ \hat{I}_- + \hat{I}_3(\hat{I}_3 - 1) . \quad (C9b)
\]
APPENDIX D: WAVEFUNCTIONS ON $S^3$

The operators $\hat{J}_3$ and $\hat{I}_3$ acting on some general representative wavefunctions is enough to show the range and relation of the spin and isospin quantum numbers to the polynomial degree $l$. Acting on wavefunctions of $(a_b + ia_c)$ raised to some power give

\[
\hat{J}_3(a_1 \pm ia_2)^l = \pm \frac{i}{2}(a_1 \pm ia_2)^l
\]
(D1)
\[
\hat{J}_3(a_0 \pm ia_3)^l = \pm \frac{i}{2}(a_0 \pm ia_3)^l
\]
(D2)
\[
\hat{I}_3(a_1 \pm ia_2)^l = \pm \frac{i}{2}(a_1 \pm ia_2)^l
\]
(D3)
\[
\hat{I}_3(a_0 \pm ia_3)^l = \mp \frac{i}{2}(a_0 \pm ia_3)^l
\]
(D4)

and on a product of two different $(a_b + ia_c)'s$ give

\[
\hat{J}_3(a_1 \pm ia_2)^l(a_0 \pm ia_3)^p
= \pm \frac{i}{2}(a_1 \pm ia_2)^l(a_0 \pm ia_3)^p
\]
(D5)
\[
\hat{J}_3(a_1 \pm ia_2)^l(a_0 \mp ia_3)^p
= \pm (\frac{i}{2} - p)(a_1 \pm ia_2)^l(a_0 \mp ia_3)^p
\]
(D6)
\[
\hat{I}_3(a_1 \pm ia_2)^l(a_0 \pm ia_3)^p
= \pm \frac{i}{2}(a_1 \pm ia_2)^l(a_0 \pm ia_3)^p
\]
(D7)
\[
\hat{I}_3(a_1 \pm ia_2)^l(a_0 \mp ia_3)^p
= \pm \frac{i}{2}(a_1 \pm ia_2)^l(a_0 \mp ia_3)^p
\]
(D8)

where $l \geq p$ and $p \geq 0$. Varying $p$ between 0 and $l$ shows that $l/2$ is the largest modulus of the third component of spin and isospin of the multiplet.

In flavor SU(2) Adkins et al. [3] showed that one could obtain the normalized wavefunctions on $S^3$. We give the complete set for $j = 0, 1/2, 1$ and 3/2 wavefunctions. The relative phases of these wavefunctions in a multiplet are important. For $j = 0$ this is simply a constant

\[
\langle a | 0, 0, 0 \rangle = \frac{1}{\pi^{1/2}}.
\]
(D9)

For $j = 1/2$

\[
\langle a | +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2} \rangle = \frac{i}{\pi^2}(a_1 + ia_2)
\]
(D10)
\[
\langle a | +\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2} \rangle = -\frac{i}{\pi^2}(a_0 - ia_3)
\]
(D11)
\[
\langle a | +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2} \rangle = \frac{i}{\pi^2}(a_0 + ia_3)
\]
(D12)
\[
\langle a | +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \rangle = -\frac{i}{\pi^2}(a_1 - ia_2)
\]
(D13)

For $j = 1$

\[
\langle a | 1, +1, +1 \rangle = \frac{\sqrt{3}}{\pi^{1/2}}(a_1 + ia_2)^2
\]
(D14)
\[
\langle a | 1, +1, 0 \rangle = \frac{\sqrt{3}}{\pi^{1/2}}(a_1 + ia_2)(a_0 + ia_3)
\]
(D15)
\[
\langle a | 1, +1, -1 \rangle = -\frac{\sqrt{3}}{\pi^{1/2}}(a_0 + ia_3)^2
\]
(D16)
\[
\langle a | 1, 0, +1 \rangle = -i\frac{\sqrt{3}}{\pi}(a_1 + ia_2)(a_0 - ia_3)
\]
(D17)
\[
\langle a | 1, 0, 0 \rangle = \frac{\sqrt{3}}{\pi^{1/2}}((a_0^2 + a_3^2) - (a_1^2 + a_2^2))
\]
(D18)
\[
\langle a | 1, 0, -1 \rangle = -i\frac{\sqrt{3}}{\pi}(a_1 - ia_2)(a_0 + ia_3)
\]
(D19)
\[
\langle a | 1, -1, +1 \rangle = -\frac{\sqrt{3}}{\pi^{1/2}}(a_0 - ia_3)^2
\]
(D20)
\[
\langle a | 1, -1, 0 \rangle = i\frac{\sqrt{3}}{\pi}(a_1 - ia_2)(a_0 - ia_3)
\]
(D21)
\[
\langle a | 1, -1, -1 \rangle = \frac{\sqrt{3}}{\pi^{1/2}}(a_1 - ia_2)^2
\]
(D22)

For $j = 3/2$

\[
\langle a | +\frac{3}{2}, +\frac{3}{2}, +\frac{3}{2} \rangle = \frac{i}{\pi^3}(a_1 + ia_2)^3
\]
(D23)
\[
\langle a | +\frac{3}{2}, +\frac{3}{2}, -\frac{3}{2} \rangle = \frac{i}{\pi^{3/2}}(a_1 + ia_2)^2(a_0 + ia_3)
\]
(D24)
\[
\langle a | +\frac{3}{2}, -\frac{3}{2}, +\frac{3}{2} \rangle = -\frac{i}{\pi^{3/2}}(a_1 + ia_2)(a_0 + ia_3)^2
\]
(D25)
\[
\langle a | +\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2} \rangle = -i\frac{\sqrt{3}}{\pi}(a_0 + ia_3)^3
\]
(D26)
\[
\langle a | +\frac{3}{2}, +\frac{3}{2}, -\frac{3}{2} \rangle = -i\frac{\sqrt{6}}{\pi}(a_1 + ia_2)^2(a_0 - ia_3)
\]
(D27)
\[
\langle a | +\frac{3}{2}, -\frac{3}{2}, +\frac{3}{2} \rangle = -\frac{\sqrt{6}}{\pi}(a_1 + ia_2)(1 - 3(a_0^2 + a_3^2))
\]
(D28)
\[
\langle a | +\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2} \rangle = \frac{i\sqrt{6}}{\pi}(a_0 + ia_3)(1 - 3(a_0^2 + a_3^2))
\]
(D29)
\[
\langle a | +\frac{3}{2}, +\frac{3}{2}, -\frac{3}{2} \rangle = i\frac{\sqrt{6}}{\pi}(a_1 + ia_2)(a_0 + ia_3)^2
\]
(D30)
\[
\langle a | +\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2} \rangle = -\frac{\sqrt{6}}{\pi}(a_1 + ia_2)(a_0 - ia_3)^2
\]
(D31)
\[
\langle a | +\frac{3}{2}, +\frac{3}{2}, -\frac{3}{2} \rangle = \frac{i\sqrt{6}}{\pi}(a_1 + ia_2)(1 - 3(a_0^2 + a_3^2))
\]
(D32)
\[
\langle a | +\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2} \rangle = \frac{\sqrt{6}}{\pi}(a_1 - ia_2)^2(a_0 + ia_3)
\]
(D33)
\[
\langle a | +\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2} \rangle = \frac{\sqrt{6}}{\pi}(a_1 - ia_2)^2(a_0 + ia_3)
\]
\[ \langle a|\frac{j}{2}, \frac{j}{2}, -\frac{j}{2} \rangle = \frac{i\sqrt{2}}{\pi} (a_0 - ia_3)^3 \quad (D35) \]
\[ \langle a|\frac{j}{2}, \frac{j}{2}, -\frac{j}{2} \rangle = \frac{\sqrt{6}}{\pi} (a_0 - ia_3)^2 (a_1 - ia_2) \quad (D36) \]
\[ \langle a|\frac{j}{2}, -\frac{j}{2}, \frac{j}{2} \rangle = -i\frac{\sqrt{6}}{\pi} (a_1 - ia_2)^2 (a_0 - ia_3) \quad (D37) \]
\[ \langle a|\frac{j}{2}, -\frac{j}{2}, -\frac{j}{2} \rangle = -\frac{\sqrt{2}}{\pi} (a_1 - ia_2)^3 \quad (D38) \]

**APPENDIX E: WAVEFUNCTIONS ON THE \( SO(3) \) MANIFOLD**

The lowest state in this case has \( j = m = n = 0 \) and is simply a constant

\[ \langle R| 0, 0, 0 \rangle = 1 . \quad (E1) \]

The next higher states are those with \( j = 1 \). There are nine of these. Those with \( j = 1, m = 1 \)

\[ \langle R| 1, +1, +1 \rangle = R_{11} - R_{22} + i(R_{12} + R_{21}) = R^+ \quad (E2) \]
\[ \langle R| 1, +1, 0 \rangle = -(R_{13} + iR_{23}) = -R^0 \quad (E3) \]
\[ \langle R| 1, +1, -1 \rangle = -(R_{11} + R_{22} - i(R_{12} - R_{21})) = -R^{+-} \quad (E4) \]

Higher wavefunctions with total (iso)spin \( j \) are made up of products of \( R^{pq} \) to power \( j \). The wavefunctions here have not been normalized.

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