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ON THE CONNECTED COMPONENTS OF MODULI SPACES OF FINITE FLAT MODELS

By Naoki Imai

Abstract. We prove that the nonordinary component is connected in the moduli spaces of finite flat models of two-dimensional local Galois representations over finite fields. This was conjectured by Kisin. As an application to global Galois representations, we prove a theorem on the modularity comparing a deformation ring and a Hecke ring.

Introduction. Let $K$ be a $p$-adic field for $p > 2$, and let $V_{F}$ be a two-dimensional continuous representation of the absolute Galois group $G_{K}$ over a finite field $F$ of characteristic $p$. The projective scheme $\mathcal{R}_{V_{F},0}$ over $F$ is the moduli of finite flat models of $V_{F}$ with some determinant condition. From the viewpoint of the application to the modularity problem, we are interested in the connected components of $\mathcal{R}_{V_{F},0}$. The ordinary component of $\mathcal{R}_{V_{F},0}$ was determined in [Kis], and Kisin conjectured that the nonordinary component is connected. We prove this conjecture, and the main theorem is the following.

Theorem. Let $F'$ be a finite extension of $F$. Suppose $x_{1}, x_{2} \in \mathcal{R}_{V_{F},0}(F')$ correspond to objects $\mathcal{M}_{1,F'}, \mathcal{M}_{2,F'}$ of (Mod/$\mathfrak{S})_{F'}$ respectively. If $\mathcal{M}_{1,F'}$ and $\mathcal{M}_{2,F'}$ are both nonordinary, then $x_{1}$ and $x_{2}$ lie on the same connected component of $\mathcal{R}_{V_{F},0}$.

When $K$ is totally ramified over $\mathbb{Q}_{p}$, this was proved in [Kis]. If the residue field of $K$ is bigger than $F_{p}$, the situation changes greatly because $\mathfrak{S} \otimes_{\mathbb{Z}_{p}} F$ can be split into a direct product. When $K$ is a general $p$-adic field, the case of $V_{F}$ being the trivial representation was treated in [Gee].

As an application to global Galois representations, we prove a theorem on the modularity, which states that a deformation ring is isomorphic to a Hecke ring up to $p$-power torsion kernel. This completes Kisin’s theory for $GL_{2}$.

Notation. Throughout this paper, we use the following notation. Let $p > 2$ be a prime number, and $k$ be a finite extension of $\mathbb{F}_{p}$ of cardinality $q = p^{n}$. The Witt ring of $k$ is denoted by $W(k)$, and let $K_{0} = W(k)[1/p]$. Let $K$ be a totally ramified extension of $K_{0}$ of degree $e$, and $\mathcal{O}_{K}$ be the ring of integers of $K$. Let $I_{K}$ be the inertia group of the absolute Galois group $G_{K}$, and $Fr_{q}$ be the
q-th power Frobenius of the absolute Galois group $G_k$. Let $\mathbb{F}$ be a finite field of characteristic $p$. The formal power series ring of $u$ over $\mathbb{F}$ is denoted by $\mathbb{F}[[u]]$, and its quotient field is denoted by $\mathbb{F}((u))$. Let $\nu_u$ be the valuation of $\mathbb{F}((u))$ normalized by $\nu_u(u) = 1$.

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1. Preliminaries. First of all, we recall some notation from [Kis], and the interested reader should consult [Kis] for more detailed definitions.

For each $\mathbb{Q}_p$-algebra embedding $\psi: K \rightarrow \overline{K}_0$, we put $\nu_\psi = 1$ and set $\nu = (\nu_\psi)_\psi$. Let $F$ be a finite Galois extension of $\mathbb{Q}_p$ containing $K_0$, and $F$ be the residue field of $F$. Let $V_F$ be a continuous two-dimensional representation of $G_K$ over $F$. We assume that $V_F$ arises as the generic fiber of a finite flat group scheme over $O_K$.

Let $R$ be a complete local Noetherian $O_F$-algebra with residue field $F$, and fix $\xi \in D^\text{fl}_{V_F}(R)$. We make the following assumption:

The morphism $\xi \rightarrow D^\text{fl}_{V_F}$ of groupoids over $\mathcal{A}R_{O_F}$ is formally smooth.

Now we can construct $\Theta_v^{\nu}: \mathcal{B}^\text{loc}_V \rightarrow \text{Spec} R^\nu$. Let $\mathcal{A}^\text{x,y}_V$ be its fiber over the closed point of $\text{Spec} R^\nu$. We assume $\text{Spec} R^\nu \neq \emptyset$, and this assumption assures that the action of $I_K$ on $\det V_F$ is the reduction mod $p$ of the cyclotomic character.

The fundamental character of level $m$ is given by

$$\omega_m: I_K \rightarrow \overline{k}^\times; \quad g \mapsto g(p^m \sqrt[p^m]{\pi}) \mod m_{O_K}.$$

Here $\pi$ is a uniformizer of $O_K$, and $m_{O_K}$ is the maximal ideal of $O_K$. If $K'/K$ is a finite unramified extension that contains the $(p^m - 1)$-st roots of unity, then the same formula as above defines a character of $G_{K'}$, which is again denoted by $\omega_m$. Note that this extension depends on the choice of the uniformizer $\pi$. From now on, we fix a uniformizer $\pi$.

**Lemma 1.1.** If $V_F$ is absolutely irreducible and $\mathbb{F}_q^2 \subset \mathbb{F}$, then

$$V_F | I_K \sim \omega_2^s \oplus \omega_2^s$$

for a positive integer $s$ such that $(q + 1) \nmid s$.

**Proof.** Let $I_p \subset I_K$ be the wild inertia group. Then $V^I_{F^p} \neq 0$ and $V^I_{F^p}$ is $G_K$-stable, so $V^I_{F^p} = V_F$. As the action of $I_K$ on $V_F$ factors through the tame inertia group, we get $V_F | I_K \sim \omega_2^{s_1} \oplus \omega_2^{s_2}$ for some nonnegative integers $s_1, s_2$ and some positive integers $m_1, m_2$. Now we fix a lifting $\tilde{Fr}_q \in G_K$ of the
contradiction. We use the same notation for any sublattice defined over \( G_K \) of \( \omega \). So we may assume \( m_1 = n \). As \( \omega \) is defined over \( G_K \), we can consider the representation \( V_\varphi \otimes \omega_n^{s_1} \) of \( G_K \). Then this representation is absolutely irreducible and factors through \( G_k \). This is a contradiction.

So we may assume \( \omega_m^{s_1} \neq \omega_m^{s_2} \). As \( V_\varphi \) is an irreducible representation, \( \omega_m^{s_1} = \omega_m^{qs_1} \) and \( \omega_m^{s_2} = \omega_m^{qs_2} \). Hence we may assume \( \omega_m^{s_1} = \omega_m^{qs_1} \) and we may assume \( m_1 = 2n \). Thus we get \( V_\varphi |_{G_k} \sim \omega_n^{2s} \otimes \omega_n^{qs} \).

If \( (q + 1) \mid s \), then \( V_\varphi |_{G_k} \sim \omega_n^{s'} \otimes \omega_n^{s'} \) where \( s' = s/(q + 1) \). This contradicts the absolutely irreducibility of \( V_\varphi \) by considering \( V_\varphi \otimes \omega_n^{-s'} \). So we get \( (q + 1) \nmid s \). □

Let \( \mathcal{S} = W(k)[1/u] \), and \( \mathcal{O}_E \) be the \( p \)-adic completion of \( \mathcal{S}[1/u] \). We choose elements \( \pi_m \in \mathcal{S} \) such that \( \pi_0 = \pi \) and \( \pi_m^p = \pi_m \) for \( m \geq 1 \), and put \( K_n = \bigcup_{m \geq 0} K(\pi_m) \). Let \( M_\varphi \in \mathcal{O}_E \mathcal{O}_{E,F} \) be the \( \varphi \)-module that corresponds to the \( G_{K_n} \)-representation \( V_\varphi(-1) \). Here \(-1\) denotes the inverse of the Tate twist.

From now on, we assume \( q_2 \subset \mathbb{F} \) and fix an embedding \( k \hookrightarrow \mathbb{F} \). This assumption does not matter, because we may extend \( \mathbb{F} \) to prove the main theorem. We consider the isomorphism

\[
\mathcal{O}_E \otimes_{k_2} \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \cong \prod_{\sigma \in \text{Gal}(k/\mathbb{F}_p)} \mathbb{F}(u); \left( \sum a_i u^i \right) \otimes \sigma \mapsto \left( \sum \sigma(a_i) b u^i \right)_\sigma
\]

and let \( \epsilon_\sigma \in k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \) be the primitive idempotent corresponding to \( \sigma \). Take \( \sigma_1, \ldots, \sigma_n \in \text{Gal}(k/\mathbb{F}_p) \) such that \( \sigma_{i+1} = \sigma_i \circ \varphi^{-1} \). Here we regard \( \varphi \) as the \( p \)-th power Frobenius, and use the convention that \( \sigma_{n+1} = \sigma_1 \). In the following, we often use such conventions. Then we have \( \varphi(\epsilon_{\sigma_i}) = \epsilon_{\sigma_{i+1}} \), and \( \phi \colon M_\varphi \to M_\varphi \) determines \( \phi \\: \epsilon_\sigma M_\varphi \to \epsilon_{\sigma_{i+1}} M_\varphi \). For \( (A_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}(u))^n \), we write

\[
M_\varphi \sim (A_1, A_2, \ldots, A_n) = (A_i)_i
\]

if there is a basis \( \{ e_1^i, e_2^i \} \) of \( \epsilon_{\sigma_i} M_\varphi \) over \( \mathbb{F}(u) \) such that \( \phi \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} = A_i \begin{pmatrix} e_1^{i+1} \\ e_2^{i+1} \end{pmatrix} \).

We use the same notation for any sublattice \( M_\varphi \subset M_\varphi \) similarly. Here and in the following, we consider only sublattices that are \( (\mathcal{S} \otimes_{k_2} \mathbb{F}) \)-modules.

Finally, for any sublattice \( M_\varphi \subset M_\varphi \) with a chosen basis \( \{ e_1^i, e_2^i \}_{1 \leq i \leq n} \) and \( B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}(u))^n \), the module generated by the entries of \( B \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} \) with the basis given by these entries is denoted by \( B \cdot M_\varphi \). Note that \( B \cdot M_\varphi \) depends on the choice of the basis of \( M_\varphi \).
Lemma 1.2. Suppose $V_{\mathbb{F}}$ is absolutely irreducible. If $\mathbb{F}'$ is the quadratic extension of $\mathbb{F}$, then

$$M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}' \sim \begin{pmatrix} 0 & \alpha_1 & \ldots & \alpha_n \\ \alpha_1 u & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n & 0 & \ldots & 0 \end{pmatrix}$$

for some $\alpha_i \in (\mathbb{F}')^\times$ and a positive integer $s$ such that $(q + 1) \nmid s$.

Proof. Let $K'$ be the quadratic unramified extension of $K$, and $k'$ be the residue field of $K'$. Then

$$V_{\mathbb{F}}(-1)|_{G_{K'}} \sim \lambda' \omega_{2n}^{-s} \oplus \lambda' \omega_{2n}^{-qs'}$$

for an unramified character $\lambda': G_{K'} \to \mathbb{F}^\times$ and a positive integer $s$ such that $(q + 1) \nmid s$ by applying Lemma 1.1 to $V_{\mathbb{F}}(-1)$. By taking the quadratic extension $\mathbb{F}'$ of $\mathbb{F}$, we can extend $\lambda'$ to $\lambda: G_{K} \to (\mathbb{F}')^\times$. We take a lifting $\tilde{\pi}_{\mathbb{F}} \in G_{K_{\infty}}$ of $\lambda$. Now we fix a $(q^2 - 1)$-st root of $\pi$, which is denoted by $\sqrt[2]{q^2-1}/\sqrt[2]{q}$. Then we put $\tilde{\alpha} = \tilde{\pi}_{\mathbb{F}}(\sqrt[2]{q^2-1}/\sqrt[2]{q}) \in \mathcal{O}_{\tilde{K}}$, and let $\alpha$ be the reduction of $\tilde{\alpha}$ in $\tilde{K}$. We have $\alpha \in \mathbb{F}'$, because $\alpha q^2 - 1 = 1$. Considering $V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$, we may assume $\mathbb{F} = \mathbb{F}'$.

We put $K'_\infty = K' \cdot K_{\infty}$. Then $(\tilde{\pi}_{\mathbb{F}})^{2} \in G_{K'_{\infty}}$. Now we have

$$\frac{(\tilde{\pi}_{\mathbb{F}})^{2}(\sqrt[2]{q^2-1}/\sqrt[2]{q})}{\sqrt[2]{q^2-1}/\sqrt[2]{q}} = \frac{(\tilde{\pi}_{\mathbb{F}})^{2}(\sqrt[2]{q^2-1}/\sqrt[2]{q})}{\sqrt[2]{q^2-1}/\sqrt[2]{q}} \cdot \frac{\tilde{\pi}_{\mathbb{F}}(\sqrt[2]{q^2-1}/\sqrt[2]{q})}{\sqrt[2]{q^2-1}/\sqrt[2]{q}} = \frac{\tilde{\pi}_{\mathbb{F}}(\tilde{\alpha} \sqrt[2]{q^2-1}/\sqrt[2]{q})}{\sqrt[2]{q^2-1}/\sqrt[2]{q}} \cdot \frac{\tilde{\pi}_{\mathbb{F}}(\tilde{\alpha} \sqrt[2]{q^2-1}/\sqrt[2]{q})}{\sqrt[2]{q^2-1}/\sqrt[2]{q}} = \tilde{\pi}_{\mathbb{F}}(\tilde{\alpha} \sqrt[2]{q^2-1}/\sqrt[2]{q})$$

and $\omega_{2n}((\tilde{\pi}_{\mathbb{F}})^{2}) = \alpha q^{s + 1}$. Hence we can take $v_1, v_2 \in V_{\mathbb{F}}(-1)$ so that

$$\tilde{\pi}_{\mathbb{F}}(v_1) = \lambda(\tilde{\pi}_{\mathbb{F}}) \alpha^{-qs} v_1, \quad \tilde{\pi}_{\mathbb{F}}(v_2) = \lambda(\tilde{\pi}_{\mathbb{F}}) \alpha^{-qs} v_1$$

and

$$g(v_1) = \lambda(g) \omega_{2n}^{-s}(g)v_1, \quad g(v_2) = \lambda(g) \omega_{2n}^{-qs}(g)v_2$$

for all $g \in G_{K'_\infty}$. We take an element $w_\lambda$ of $(\overline{k} \otimes_{\mathbb{F}_p} \mathbb{F})^\times$ so that $g(w_\lambda) = (1 \otimes \lambda(g))w_\lambda$ for all $g \in G_{K}$. By this condition, $w_\lambda$ is determined up to $(\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F})^\times$.

By the definition of the action of $G_{K_{\infty}}$ on $O_{E^a}/O_{E^a}$, we can choose an element $u_{2n}$ of $O_{E^a}/pO_{E^a}$ so that $u_{2n}^{-1} = u$ and $\tilde{\pi}_{\mathbb{F}}(u_{2n}) = \alpha u_{2n}$. We consider the isomorphism

$$k' \otimes_{\mathbb{F}_p} \mathbb{F} \to \prod_{\sigma \in \text{Gal}(k'/\mathbb{F}_p)} \mathbb{F} \ ; \ a \otimes b \mapsto (\sigma(a)b)_{\sigma}$$

and let $e_0 \in k' \otimes_{\mathbb{F}_p} \mathbb{F}$ be the primitive idempotent corresponding to $\text{id}_{k'}$. For $0 \leq r \leq 2n - 1$, we put $e_r = \phi' e_0$. Note that $(aw' \otimes 1)e_r = (1 \otimes a) e_r$ for all $a \in k'$.
We put

\[ e_1 = w^{-1}_\lambda \{(u_{2n}^s \otimes 1)(\epsilon_0 v_1 + \epsilon_n v_2) + (u_{2n}^{p_1 s} \otimes 1)(\epsilon_1 v_1 + \epsilon_{n+1} v_2) \right. \]
\[ + \cdots + (u_{2n}^{p_{n-1} s} \otimes 1)(\epsilon_{n-1} v_1 + \epsilon_{2n-1} v_2) \left. \}, \]

\[ e_2 = w^{-1}_\lambda \{(u_{2n}^{p_1 s} \otimes 1)(\epsilon_n v_1 + \epsilon_0 v_2) + (u_{2n}^{p_{n+1} s} \otimes 1)(\epsilon_{n+1} v_1 + \epsilon_1 v_2) \right. \]
\[ + \cdots + (u_{2n}^{p_{2n-1} s} \otimes 1)(\epsilon_{2n-1} v_1 + \epsilon_{n-1} v_2) \left. \} \]

in \((\mathcal{O}_{\mathbb{C}^w / p}\mathcal{O}_{\mathbb{C}^w}) \otimes \mathbb{F}_p V_{\mathbb{F}_p}(-1)\). Then \(e_1\) and \(e_2\) are fixed by \(g \in G_{K^\infty}\) and \(\tilde{\text{Fr}}_q\). Hence \(e_1, e_2\) are fixed by \(G_{K^\infty}\), and these are a basis of \(\Phi_{M_{\mathcal{O}_{\mathbb{C}^w} / \mathbb{F}_p}} \otimes \mathbb{F}\) over \(\mathbb{O}_{\mathbb{E}^w} \otimes \mathbb{Z}_p \mathbb{F}\).

We put \(\alpha_\lambda = u_{\lambda}/\phi(u_{\lambda})\). As \(\phi(u_{\lambda})\) satisfies the condition determining \(u_{\lambda}\), the element \(\alpha_\lambda\) of \((k \otimes \mathbb{F}_p \mathbb{F})^\times\) is in \((k \otimes \mathbb{F}_p \mathbb{F})^\times\). Now we have

\[ \phi(e_1) = \alpha_\lambda \{(\epsilon_1 + \epsilon_{n+1}) + \cdots + (\epsilon_{n-1} + \epsilon_{2n-1})\}e_1 + \alpha_\lambda(\epsilon_0 + \epsilon_n)e_2, \]
\[ \phi(e_2) = \alpha_\lambda u'(\epsilon_0 + \epsilon_n)e_1 + \alpha_\lambda \{(\epsilon_1 + \epsilon_{n+1}) + \cdots + (\epsilon_{n-1} + \epsilon_{2n-1})\}e_2. \]

If we put

\[ \sigma_1 = \phi, \ \sigma_2 = \text{id}_k, \ \sigma_3 = \phi^{-1}, \ldots, \ \sigma_n = \phi^2, \]

then we have

\[ \epsilon_{\sigma_1} = \epsilon_{n-1} + \epsilon_{2n-1}, \ \epsilon_{\sigma_2} = \epsilon_0 + \epsilon_n, \ldots, \ \epsilon_{\sigma_n} = \epsilon_{n-2} + \epsilon_{2n-2} \]

and

\[ M_{\mathbb{F}'} \sim \begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 u' & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n \end{pmatrix}. \]

Here \(\alpha_i\) is the \(\sigma_{i+1}\)-th component of \(\alpha_\lambda\) in \(\prod_{\sigma \in \text{Gal}(k/\mathbb{F}_p)} \mathbb{F}^\times\). \(\square\)

2. Main theorem.

\textbf{Lemma 2.1.} If \(\mathbb{F}'\) is a finite extension of \(\mathbb{F}\), the elements of \(\mathcal{I}_{\mathcal{R}_{\mathbb{F}_p, 0}(\mathbb{F}')}\) naturally correspond to free \((k[[u]] \otimes \mathbb{F}_p \mathbb{F}'))\)-submodules \(\mathcal{M}_{\mathbb{F}'} \subset M_{\mathbb{F}'} \otimes \mathbb{F}'\) of rank 2 that satisfy the following:

1. \(\mathcal{M}_{\mathbb{F}'}\) is \(\phi\)-stable.

\(\square\)
(2) For some (so any) choice of \((k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}'))-basis for \(M_{\mathbb{F}'}\), and for each \(\sigma \in \text{Gal}(k/\mathbb{F}_p)\), the map

\[ \phi: \epsilon_\sigma M_{\mathbb{F}'} \rightarrow \epsilon_{\sigma \circ \phi^{-1}} M_{\mathbb{F}'} \]

has determinant \(\alpha\) for some \(\alpha \in \mathbb{F}'[[u]]^\times\).

**Proof.** This is [Gee, Lemma 2.2]. □

**Lemma 2.2.** Suppose \(x_1, x_2 \in \mathcal{R}_{V_{\mathbb{F}},0}(\mathbb{F})\) correspond to objects \(M_{1,\mathbb{F}}, M_{2,\mathbb{F}}\) of \((\text{Mod}/\mathcal{S})_{\mathbb{F}}\) respectively. Let \(N = (N_j)_{1 \leq j \leq n}\) be a nilpotent element of \(M_2(\mathbb{F}((u)))^n\) such that \(M_{2,\mathbb{F}} = (1 + N) \cdot M_{1,\mathbb{F}}\), and \(A = (A_i)_{1 \leq i \leq n}\) be an element of \(\text{GL}_2(\mathbb{F}((u)))^n\) such that \(M_{1,\mathbb{F}} \sim A\). If \(\phi(N)A_iN_i^{-1} M_{2,\mathbb{F}} \in M_2(\mathbb{F}[[u]])\) for all \(i\), then there is a morphism \(\mathbb{P}^1 \rightarrow \mathcal{R}_{V_{\mathbb{F}},0}\) sending \(x_1\) and \(x_1\) to \(x_2\).

**Proof.** This is [Gee, Lemma 2.4]. □

**Lemma 2.3.** Suppose \(n \geq 2\). Let \(M_{\mathbb{F}}\) be the object of \((\text{Mod}/\mathcal{S})_{\mathbb{F}}\) corresponding to a point \(x \in \mathcal{R}_{V_{\mathbb{F}},0}(\mathbb{F})\). Fix a basis of \(M_{\mathbb{F}}\) over \(k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}\). Consider \(U^{(i)} = (U_j^{(i)})_{1 \leq j \leq n} \in \text{GL}_2(\mathbb{F}((u)))^n\) such that \(U_i^{(i)} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\) and \(U_j^{(i)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) for all \(j \neq i\). If \(U^{(i)} \cdot M_{\mathbb{F}}\) is \(\phi\)-stable, it corresponds to a point \(x' \in \mathcal{R}_{V_{\mathbb{F}},0}(\mathbb{F})\), and \(x'\) lies on the same connected component of \(\mathcal{R}_{V_{\mathbb{F}},0}\) as \(x\).

**Proof.** First, \(U^{(i)} \cdot M_{\mathbb{F}}\) corresponds to a point \(x' \in \mathcal{R}_{V_{\mathbb{F}},0}(\mathbb{F})\), because it satisfies the conditions of Lemma 2.1.

Next, we consider \(N^{(i)} = (N_j^{(i)})_{1 \leq j \leq n} \in M_2(\mathbb{F}((u)))^n\) such that

\[ N_i^{(i)} = \begin{pmatrix} 1 & -u \\ u^{-1} & -1 \end{pmatrix} \text{ and } N_j^{(i)} = 0 \text{ for all } j \neq i. \]

Then \(U^{(i)} \cdot M_{\mathbb{F}} = (1 + N^{(i)}) \cdot M_{\mathbb{F}}\), because \(\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2u \end{pmatrix} \begin{pmatrix} 2 & -u \\ u^{-1} & 0 \end{pmatrix}\).

So we can apply Lemma 2.2. □

**Theorem 2.4.** Let \(\mathbb{F}'\) be a finite extension of \(\mathbb{F}\). Suppose \(x_1, x_2 \in \mathcal{R}_{V_{\mathbb{F}'},0}(\mathbb{F}')\) correspond to objects \(M_{1,\mathbb{F}'}, M_{2,\mathbb{F}'}\) of \((\text{Mod}/\mathcal{S})_{\mathbb{F}'}\) respectively. If \(M_{1,\mathbb{F}'}\) and \(M_{2,\mathbb{F}'}\) are both nonordinary, then \(x_1\) and \(x_2\) lie on the same connected component of \(\mathcal{R}_{V_{\mathbb{F}'},0}\).

**Proof.** When \(n = 1\), this was proved in [Kis]. If \(e < p - 1\), then \(\mathcal{R}_{V_{\mathbb{F}'},0}(\mathbb{F}')\) is one point by [Ray, Theorem 3.3.3]. So we may assume \(n \geq 2\) and \(e \geq p - 1\). Furthermore, replacing \(V_{\mathbb{F}}\) by \(V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'\), we may assume \(\mathbb{F} = \mathbb{F}'\).

Suppose first that \(V_{\mathbb{F}}\) is reducible. We can choose a basis so that \(M_{1,\mathbb{F}} \sim A = (A_i)_{1 \leq i \leq n} \in M_2(\mathbb{F}[[u]])^n\) where \(A_i = \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix}\) for \(a_i, b_i, c_i \in \mathbb{F}[[u]]\), because
$M_{\mathcal{F}}$ is reducible and $\mathcal{M}_{1,\mathcal{F}}$ is $\phi$-stable. By the Iwasawa decomposition and the determinant conditions, we can take $B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}((u)))^n$ such that

$$\mathcal{M}_{2,\mathcal{F}} = B \cdot \mathcal{M}_{1,\mathcal{F}}$$

and $B_i = \begin{pmatrix} u^{-s_i} & v_i \\ 0 & u^{s_i} \end{pmatrix}$ for $s_i \in \mathbb{Z}$ and $v_i \in \mathbb{F}((u))$. Then $\mathcal{M}_{2,\mathcal{F}} \sim (\phi(B_i)A_iB_{i+1}^{-1})_i$, and we have

$$\phi(B_i)A_iB_{i+1}^{-1} = \begin{pmatrix} u^{-ps_i} & \phi(v_i) \\ 0 & u^{ps_i} \end{pmatrix} \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} \begin{pmatrix} u^{s_{i+1}} & \cdot \\ 0 & u^{-s_{i+1}} \end{pmatrix} = \begin{pmatrix} a_iu^{-ps_{i+1}+s_{i+1}} & -a_iv_{i+1}u^{-ps_i} + b_iu^{-ps_i-s_{i+1}} + c_i\phi(v_i)u^{-s_{i+1}} \\ 0 & c_iu^{ps_{i+1}-s_{i+1}} \end{pmatrix}.$$}

In the last matrix, every component is integral because $\mathcal{M}_{2,\mathcal{F}}$ is $\phi$-stable.

First of all, we want to reduce the problem to the case where $s_i = 0$ for all $i$. When $e = p - 1$, we have $0 \leq v_u(c_i) \leq p - 1$ and $0 \leq v_u(c_i) + ps_i - s_{i+1} \leq p - 1$ for all $i$ by the determinant conditions. From the second set of inequalities, we obtain

$$0 \leq \sum_{j=0}^{n-1} (v_u(c_{i-1-j}) + ps_{i-1-j} - s_{i-j})p^j \leq p^n - 1,$$

and we have

$$\sum_{j=0}^{n-1} (v_u(c_{i-1-j}) + ps_{i-1-j} - s_{i-j})p^j = (p^n - 1)s_i + \sum_{j=0}^{n-1} v_u(c_{i-1-j})p^j.$$

Combining these with $0 \leq v_u(c_i) \leq p - 1$, we get $-1 \leq s_i \leq 1$. If $s_i = 1$ for some $i$, the second sign of the above inequality must be the equality sign. So we get $v_u(c_i) = 0$ for all $j$. This contradicts the nonordinarity of $\mathcal{M}_{1,\mathcal{F}}$. If $s_i = -1$ for some $i$, the first sign of the above inequality must be the equality sign. So we get $v_u(c_i) + ps_j - s_{i+1} = 0$ for all $j$. This contradicts the nonordinarity of $\mathcal{M}_{2,\mathcal{F}}$. Hence, we have $s_i = 0$ for all $i$. So we may assume $e \geq p$.

We consider $U^{(0)}$ as in Lemma 2.3. If $s_i > 0$ and $U^{(0)} \cdot \mathcal{M}_{2,\mathcal{F}}$ is $\phi$-stable, we may replace $\mathcal{M}_{2,\mathcal{F}}$ with $U^{(0)} \cdot \mathcal{M}_{2,\mathcal{F}}$ by Lemma 2.3. This replacement changes $s_i$ into $s_i - 1$ and $v_i$ into $uv_i$. If $s_i < 0$, switching $\mathcal{M}_{1,\mathcal{F}}$ with $\mathcal{M}_{2,\mathcal{F}}$ so that we have $s_i > 0$, we consider the same replacement as above. Note that these replacements decrease $|s_i|$ by 1. We prove that we can continue these replacements until we get to the case where $s_i = 0$ for all $i$. Suppose that we cannot continue the replacements and there is some nonzero $s_i$. Take an index $i_0$ such that $|s_{i_0}|$ is the greatest. By switching $\mathcal{M}_{1,\mathcal{F}}$ with $\mathcal{M}_{2,\mathcal{F}}$, we may assume $s_{i_0} > 0$. As we cannot continue the replacements, we cannot decrease $s_{i_0}$ keeping the $\phi$-stability, that is,

$$v_u(c_{i_0}) + ps_{i_0} - s_{i_0+1} \leq p - 1 \text{ or } v_u(a_{i_0-1}) - ps_{i_0-1} + s_{i_0} = 0.$$
If $v_b(c_i) + ps_i = s_{i+1} \leq p - 1$, we have $s_i = 1$, $v_b(c_i) = 0$ and $s_{i+1} = 1$, because $v_b(c_i) + (p - 1)s_i + (s_i - s_{i+1}) \leq p - 1$. Now we have $v_b(a_i) - ps_i + s_{i+1} \geq 1$, because $e \geq p$ and $v_b(c_i) + ps_i - s_{i+1} \leq p - 1$. As $s_{i+1}$ cannot be decreased, $v_b(c_{i+1}) + ps_{i+1} - s_{i+2} \leq p - 1$. The same argument shows that $v_b(c_i) = 0$ and $s_i = 1$ for all $i$. This contradicts the nonoddarity of $\mathcal{M}_{1,F}$.

If $v_b(a_{i-1}) - ps_i - s_i = 0$, then $s_{i-1} > 0$ and $v_b(c_{i-1}) + ps_{i-1} - s_i = e \geq p$. As $s_{i-1}$ cannot be decreased, $v_b(a_{i-1}) = ps_{i-1} - s_{i-1} = 0$. The same argument shows that $v_b(a_i) = ps_i + s_{i+1} = 0$ for all $i$. So we have that $\mathcal{M}_{2,F}$ is an extension of a multiplicative module by an étale module. We show that such an extension splits.

Now we have $\mathcal{M}_{2,F} \sim \left( \begin{array}{cc} a'_i & b'_i \\ 0 & u^e c'_i \end{array} \right)_i$ for $a'_i, c'_i \in \mathbb{F}[u]^\times$ and $b'_i \in \mathbb{F}[u]$. Then

$$\left( \begin{array}{cc} 1 & u'_i \\ 0 & 1 \end{array} \right)_i \cdot \mathcal{M}_{2,F} \sim \left( \begin{array}{cc} a'_i & -a'_i v_{i+1} + b'_i + u^e c'_i \phi(v'_i) \\ 0 & u^e c'_i \end{array} \right)_i$$

for $u'_i \in \mathbb{F}[u]$. It suffices to show that there is $(v'_i)_{1 \leq i \leq n} \in \mathbb{F}[u]^n$ such that $a'_i v'_{i+1} = b'_i + u^e c'_i \phi(v'_i)$ for all $i$, and we can solve the system of equations by finding $v'_i$ successively in ascending order of their degrees. Hence we have that $\mathcal{M}_{2,F}$ is ordinary, and this is a contradiction.

Thus we may assume $s_i = 0$ for all $i$. Consider $N = (N_i)_{1 \leq i \leq n} \in M_2(\mathbb{F}(u))$ such that $N_i = \left( \begin{array}{cc} 0 & v_i \\ 0 & 0 \end{array} \right)$ for $v_i \in \mathbb{F}((u))$. Then we have $\mathcal{M}_{2,F} = (1 + N) \cdot \mathcal{M}_{1,F}$ and

$$\phi(N_i) \left( \begin{array}{cc} a_i & b_i \\ 0 & c_i \end{array} \right) N_{i+1} = 0.$$

Hence $x_1$ and $x_2$ lie on the same connected component by Lemma 2.2. This completes the proof in the case where $V_F$ is reducible.

From now on, we consider the case where $V_F$ is irreducible. If $V_F$ is reducible after extending the base field $\mathbb{F}$, we can reduce this case to the reducible case. So we may assume $V_F$ is absolutely irreducible. Extending the field $\mathbb{F}$, we have

$$M_F \sim \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \ldots, \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right)$$

for some $\alpha_i \in \mathbb{F}^\times$ and a positive integer $s$ by Lemma 1.2. This basis gives a sublattice $\mathcal{M}_F$. By the Iwasawa decomposition, we can take $s'_i, t'_i \in \mathbb{Z}$ and $t'_i \in \mathbb{F}(u)$ so that $\mathcal{M}_{1,F} = \left( \begin{array}{cc} u^{s'_i} & v'_i \\ 0 & u^{t'_i} \end{array} \right)_i \cdot M_F$. Changing the basis by $\left( \begin{array}{cc} u^{s'_i} & 0 \\ 0 & u^{t'_i} \end{array} \right)_i$, we get

$$M_F \sim \left( \begin{array}{cc} 1 & 0 \\ 0 & u^{s'_i} \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \ldots, \left( \begin{array}{cc} 0 & 1 \\ 0 & u^{s'_i} \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right).$$
Here we have $0 \leq t_1, 0 \leq s_i, t_i \leq e$ for $2 \leq i \leq n$, and $s_i + t_i = e$ for all $i$ by the $\phi$-stability and the determinant conditions of $\mathcal{M}_{1,F}$.

We are going to change the basis so that we have moreover $t_1 \leq e$. Changing the basis of the $i$-th component by $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$, we get the following transformations:

- $T_i$: $t_i \sim t_i - p, \ t_{i-1} \sim t_{i-1} + 1$ for $i \neq 2$,
- $T_2$: $t_2 \sim t_2 - p, \ t_1 \sim t_1 - 1$.

If $t_1 > e$, we put $m = \max\{ 1 \leq i \leq n \mid t_i \neq e \}$, and carry out $T_1$ when $m = n$, and $T_{m+1}, T_{m+2}, \ldots, T_n, T_1$ when $m \neq n$. Then $0 \leq s_i, t_i \leq e$ for $2 \leq i \leq n$, and $t_1$ decrease by $p$ when $m \neq 1$, by $p + 1$ when $m = 1$. Repeat this until we get to the situation where $t_1 \leq e$. If $e \geq p$, we get to the situation where $0 \leq s_1, t_1 \leq p - 1$. If $e = p - 1$ and we do not get to the situation where $0 \leq s_1, t_1 \leq p - 1$, then we have

$$M_{\mathcal{F}} \sim \begin{pmatrix} \alpha_1 \begin{pmatrix} 0 & u^{-1} \\ u^{-1} & 0 \end{pmatrix}, & \alpha_2 \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}, & \ldots, & \alpha_n \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} \end{pmatrix}.$$ 

In this case, changing the basis by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get

$$M_{\mathcal{F}} \sim \begin{pmatrix} \alpha_1 \begin{pmatrix} 1 & 0 \\ u^{-1} & -u^{-1} \end{pmatrix}, & \alpha_2 \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}, & \ldots, & \alpha_n \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} \end{pmatrix}.$$ 

This contradicts that $M_{\mathcal{F}}$ is irreducible. Hence we obtain a basis such that $M_{\mathcal{F}} \sim \begin{pmatrix} \alpha_1 \begin{pmatrix} 0 & u^{s_i} \\ u^{t_i} & 0 \end{pmatrix}, & \alpha_2 \begin{pmatrix} u^{e_2} & 0 \\ 0 & u^{e_2} \end{pmatrix}, & \ldots, & \alpha_n \begin{pmatrix} u^{e_n} & 0 \\ 0 & u^{e_n} \end{pmatrix} \end{pmatrix}$

for some $s_i$ and $t_i$ satisfying $s_i + t_i = e$ and $0 \leq s_i, t_i \leq e$ for all $i$. Let $\mathcal{M}_{0,\mathcal{F}}$ be the sublattice of $M_{\mathcal{F}}$ determined by this basis. Note that $\mathcal{M}_{0,\mathcal{F}}$ satisfies the conditions of Lemma 2.1, and let $x_0$ be the point of $G_{R_{V,0}}$ corresponding to $\mathcal{M}_{0,\mathcal{F}}$.

We prove that we can change $(t_i)_{1 \leq i \leq n}$ furthermore by $T_i$’s or $T_i^{-1}$’s keeping $0 \leq t_i \leq e$ for all $i$, and get to the situation where $|s_i - t_i| \leq p + 1$ for all $i$. By Lemma 2.3, these changes do not affect which of the connected components $x_0$ lies on. If $e \leq p + 1$, this is satisfied automatically. So we may assume $e \geq p + 2$. We prove that if there is an index $j$ such that $|s_j - t_j| \geq p + 2$, then there is an
index \( j_0 \) such that \( |s_{j_0} - t_{j_0}| \geq p + 2 \) and we can change \( t_{j_0} \) by \( T_{j_0} \) or \( T_{j_0}^{-1} \) so that \( |s_{j_0} - t_{j_0}| \) decreases keeping \( 0 \leq t_i \leq e \) for all \( i \). We put \( h_i = (-1)^{(i-2)/n}(s_i - t_i) \) for \( i \in \mathbb{Z} \). By assumption, there is an integer \( j_0 \) such that \( 1 \leq j_0 \leq 2n \), \( h_{j_0} \geq p + 2 \) and \( h_{j_0-1} < e \). If \( 2 \leq j_0 \leq n + 1 \), we can change \( t_{j_0} \) by \( T_{j_0}^{-1} \), otherwise by \( T_{j_0} \), so that \( |s_{j_0} - t_{j_0}| \) decreases keeping \( 0 \leq t_i \leq e \) for all \( i \). Thus we have proved the claim. Hence if \( |s_j - t_j| \geq p + 2 \) for an index \( j \), we can carry out \( T_{j_0} \) or \( T_{j_0}^{-1} \) for an index \( j_0 \) as above and this operation decreases \( \sum_{i=1}^{n} |s_i - t_i| \) by at least 2. So after finitely many operations, we get to the situation where \( |s_i - t_i| \leq p + 1 \) for all \( i \).

Hence we may assume that \( s_i \) and \( t_i \) satisfy \( s_i + t_i = e \), \( 0 \leq s_i, t_i \leq e \) and \( |s_i - t_i| \leq p + 1 \) for all \( i \). We are going to prove that \( x_0 \) and \( x_1 \) lie on the same connected component. We can prove that \( x_0 \) and \( x_2 \) lie on the same connected component by the same argument.

By the Iwasawa decomposition and the determinant conditions, we can take \( B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}(u)) \) such that \( M_{1,\mathbb{F}} = B \cdot M_{0,\mathbb{F}} \) and \( B_i = \begin{pmatrix} u^{a_i} & v_i \\ 0 & u^{e_i} \end{pmatrix} \) for \( a_i \in \mathbb{Z} \) and \( v_i \in \mathbb{F}(u) \). Then we put \( t_i = v_i(u_i) \). Now we have

\[
\phi(B_i) \begin{pmatrix} 0 & u^{a_i} \\ u^{e_i} & 0 \end{pmatrix} B_i^{-1} = \begin{pmatrix} \phi(v_i)u^{a_i} & u^{a_1 - pa_1 - a_2} - \phi(v_i)v_i u^{a_1} \\ u^{a_1 + pa_1 - a_2} & -v_i u^{a_1 + pa_1} \end{pmatrix},
\]

\[
\phi(B_i) \begin{pmatrix} u^{a_i} & 0 \\ 0 & u^{e_i} \end{pmatrix} B_i^{-1} = \begin{pmatrix} u^{a_1 - pa_1 + a_{i+1}} - v_i u^{a_1 - pa_1} \\ 0 & u^{a_1 + pa_1 - a_{i+1}} \end{pmatrix}
\]

for \( 2 \leq i \leq n \). On the right-hand sides, every component of the matrices is integral because \( M_{1,\mathbb{F}} \) is \( \phi \)-stable.

First, we consider the case \( t_1 + pa_1 + a_2 > e \). In this case,

\[
(pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e, \quad s_1 - pa_1 - a_2 = pr_1 + r_2 + t_1 < 0
\]

by the \( \phi \)-stability and the determinant conditions of \( M_{1,\mathbb{F}} \). We have \( a_1 > r_1 \), because \( t_1 + pa_1 + a_2 > e \geq pr_1 + t_1 + a_2 \). Similarly, we have \( a_2 > r_2 \), because \( t_1 + pa_1 + a_2 > e \geq r_2 + t_1 + pa_1 \).

We consider the following operations:

\[
a_i \to a_i - 1, \quad v_i \to u v_i, \text{ if it preserves the } \phi \text{-stability of } B \cdot M_{0,\mathbb{F}}.
\]

These operations replace \( x_1 \) by a point that lies on the same connected component as \( x_1 \) by Lemma 2.3. We prove that we can continue these operations until we get to the situation where \( t_1 + pa_1 + a_2 \leq e \). In other words, we reduce the problem to the case \( t_1 + pa_1 + a_2 \leq e \). If we can continue the operations endlessly, we get to the situation where \( t_1 + pa_1 + a_2 \leq e \), because the conditions \( s_i - pa_1 + a_{i+1} \geq 0 \) for \( 2 \leq i \leq n \) exclude that both \( a_1 \) and \( a_2 \) remain bounded below. Suppose we
cannot continue the operations. This is equivalent to the following condition:

\[ s_n - pa_n + a_1 = 0 \text{ or } r_2 + t_1 + pa_1 \leq p - 1, \]
\[ pr_1 + t_1 + a_2 = 0 \text{ or } t_2 + pa_2 - a_3 \leq p - 1, \]
\[ s_{i-1} - pa_{i-1} + a_i = 0 \text{ or } t_i + pa_i - a_{i+1} \leq p - 1 \text{ for each } 3 \leq i \leq n. \]

If \( e \geq p \), there are only the following two cases, because \((pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e \) and \((s_i - pa_i + a_{i+1}) + (t_i + pa_i - a_{i+1}) = e \) for \( 2 \leq i \leq n \).

Case 1: \( pr_1 + t_1 + a_2 = 0, \ s_i - pa_i + a_{i+1} = 0 \) for \( 2 \leq i \leq n \).

Case 2: \( r_2 + t_1 + pa_1 \leq p - 1, \ t_i + pa_i - a_{i+1} \leq p - 1 \) for \( 2 \leq i \leq n \).

If \( e = p - 1 \), clearly it is in Case 2.

In the Case 1. Suppose that there is an index \( i \) such that \( 2 \leq i \leq n \) and \( pr_i + t_i - a_{i+1} \neq r_{i+1} + s_i - pa_i \). Then both sides are nonnegative, because \( u_i (\phi(t_i) u_i^{d_i - a_{i+1}} - v_{i+1} u_i^{v_i - pa_i}) \geq 0 \). Comparing \( r_{i+1} + s_i - pa_i \geq 0 \) with \( s_i - pa_i + a_{i+1} = 0 \), we get \( r_{i+1} \geq a_{i+1} \). Then \( pr_{i+1} + t_{i+1} - a_{i+2} \geq pa_{i+1} + t_{i+1} - a_{i+2} \geq 0 \), and \( r_{i+2} + s_{i+1} - pa_{i+1} \geq 0 \).

Comparing \( r_{i+2} + s_{i+1} - pa_{i+1} \geq 0 \) with \( s_{i+1} - pa_{i+1} + a_{i+2} = 0 \), we get \( r_{i+2} \geq a_{i+2} \). The same argument goes on and shows \( r_1 \geq a_1 \). This is a contradiction. Thus \( pr_i + t_i - a_{i+1} = r_{i+1} + s_i - pa_i \) for all \( 2 \leq i \leq n \).

Now we change the basis of

\[ \begin{pmatrix} 0 & u^1 \\ u^1 & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^2 & 0 \\ 0 & u^2 \end{pmatrix}, \ldots, \alpha_n \begin{pmatrix} u^n & 0 \\ 0 & u^n \end{pmatrix} \]

by \( \begin{pmatrix} u^{-a_1} & u^{r_1} \\ 0 & u^{a_1} \end{pmatrix} \). Then we have

\[ \begin{pmatrix} 1 & 0 \\ u^{t_i + pa_i + a_2} & -u^e \end{pmatrix}, \alpha_2 \frac{1}{u^e} \alpha_2 \frac{1}{u^e}, \ldots, \alpha_n \frac{1}{u^e} \alpha_n \frac{1}{u^e}, \]

and this contradicts that \( M_F \) is irreducible.

In the Case 2. Suppose that there is an index \( i \) such that \( 2 \leq i \leq n \) and \( pr_i + t_i - a_{i+1} \neq r_{i+1} + s_i - pa_i \). Then both sides are nonnegative, because \( u_i (\phi(t_i) u_i^{d_i - a_{i+1}} - v_{i+1} u_i^{v_i - pa_i}) \geq 0 \). Comparing \( pr_i + t_i - a_{i+1} \geq 0 \) with \( t_i + pa_i - a_{i+1} \leq p - 1 \), we get \( r_i \geq a_i \). Then \( r_{i+1} - pa_{i+1} \geq s_{i+1} - pa_{i+1} + a_i \geq 0 \), and \( pr_{i+1} + t_{i+1} - a_i \geq 0 \) because \( u_i (\phi(t_i) u_i^{d_i - a_i} - v_{i+1} u_i^{v_i - pa_i}) \geq 0 \). Comparing \( pr_{i+1} + t_{i+1} - a_i \geq 0 \) with \( t_{i+1} + pa_{i+1} - a_i \leq p - 1 \), we get \( r_{i+1} \geq a_{i+1} \). The same argument goes on and shows that \( r_2 \geq a_2 \). This is a contradiction.
The above argument shows that
\[ r_i < a_i, \quad pr_i + t_i - a_{i+1} = r_{i+1} + s_i - pa_i < 0 \text{ for } 2 \leq i \leq n. \]

Combining these equations with \( s_1 - pa_1 - a_2 = pr_1 + r_2 + t_1 \), we get
\[
-(p^n + 1) r_1 = (p^n + 1)a_1 + (s_n - t_n) + p(s_{n-1} - t_{n-1}) \\
+ \cdots + p^{n-3}(s_3 - t_3) + p^{n-2}(s_2 - t_2) - p^{n-1}(s_1 - t_1),
\]
\[
-(p^n + 1) r_2 = (p^n + 1)a_2 - (s_1 - t_1) - p(s_n - t_n) \\
- \cdots - p^{n-3}(s_4 - t_4) - p^{n-2}(s_3 - t_3) - p^{n-1}(s_2 - t_2),
\]
\[
-(p^n + 1) r_3 = (p^n + 1)a_3 + (s_2 - t_2) - p(s_1 - t_1) \\
- \cdots - p^{n-3}(s_5 - t_5) - p^{n-2}(s_4 - t_4) - p^{n-1}(s_3 - t_3),
\]
\[ \vdots \]
\[
-(p^n + 1) r_n = (p^n + 1)a_n + (s_{n-1} - t_{n-1}) + p(s_{n-2} - t_{n-2}) \\
+ \cdots + p^{n-3}(s_2 - t_2) - p^{n-2}(s_1 - t_1) - p^{n-1}(s_n - t_n).
\]

As \( |s_i - t_i| \leq p + 1 \) and
\[
(p + 1) + p(p + 1) + \cdots + p^{n-1}(p + 1) = \left(\frac{p^n - 1}{p - 1}\right)(p + 1) < 2(p^n + 1),
\]
we get \(-a_i - 1 \leq r_i \leq -a_i + 1\). When \( e = p - 1 \), as \( |s_i - t_i| \leq p - 1 \) and
\[
(p - 1) + p(p - 1) + \cdots + p^{n-1}(p - 1) = \left(\frac{p^n - 1}{p - 1}\right)(p - 1) < (p^n + 1),
\]
we get \( r_i = -a_i \).

As \( r_2 + t_1 + pa_1 \leq p - 1 \), we have
\[ pa_1 \leq t_1 + pa_1 \leq p - 1 - r_2 \leq p + a_2. \]

For \( 2 \leq i \leq n \), as \( t_i + pa_i - a_{i+1} \leq p - 1 \), we have
\[ pa_i \leq t_i + pa_i \leq p - 1 + a_{i+1}. \]

Take an index \( i_0 \) such that \( a_{i_0} \) is the greatest. As \( pa_{i_0} \leq a_{i_0+1} + p \leq a_{i_0} + p \), we
get $a_0 \leq \frac{p}{p-1} < 2$. Combining $-a_i - 1 \leq r_i$ and $r_i < a_i$, we get $a_i \geq 0$. Hence

$$a_i = 0, \quad r_i = -1, \quad \text{or} \quad a_i = 1, \quad -2 \leq r_i \leq 0$$

for every $i$.

In the case $a_2 = 0$, we have $r_2 = -1$. Comparing $t_1 + pa_1 + a_2 > e$ with $r_2 + t_1 + pa_1 \leq p - 1$, we get $e < p$. When $e = p - 1$, we have $r_2 = -a_2$. This is a contradiction.

In the case $a_2 = 1$. As $0 \leq t_1 + pa_i - a_{i+1} \leq p - 1$ for $2 \leq i \leq n$, we have $a_i = 1$ for all $i$ and $t_i = 0$ for $2 \leq i \leq n$. As $r_2 + pa_1 + t_1 \leq p - 1$, we have $r_2 \leq -1$.

As $pr_2 + t_2 - a_3 = r_3 + s_2 - pa_2$, we have $r_3 = pr_2 + p - 1 - e \leq -e - 1 \leq -3$. This is a contradiction.

Thus we may assume $t_1 + pa_1 + a_2 \leq e$. We put $M_{3, \varphi} = \left( \begin{pmatrix} u^{-a_i} & 0 \\ 0 & u^a_i \end{pmatrix} \right) \cdot M_{0, \varphi}$, then

$$M_{3, \varphi} \sim \left( \begin{array}{cccc}
\alpha_1 & 0 & 0 & \cdots \\
0 & u^{a_1 - pa_1 - a_2} & 0 & \cdots \\
0 & 0 & u^{a_2 - pa_2 + a_3} & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & u^{a_n - pa_n + a_1} \\
\end{array} \right)$$

and $M_{1, \varphi} = \left( \begin{pmatrix} 1 & 0 & \cdots \\ 0 & \phi^{-a_i} & v_i \end{pmatrix} \right) \cdot M_{3, \varphi}$. Note that $M_{3, \varphi}$ satisfies the conditions of Lemma 2.1, and let $x_3$ be the point of $\mathcal{B} R^v_{V_y, 0}$ corresponding to $M_{3, \varphi}$. If we put $N_i = \begin{pmatrix} 0 & 0 & \cdots & v_i^{-a_i} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, then

$$\phi(N_1) = \left( \begin{pmatrix} 0 & 0 & \cdots \\ 0 & u^{a_1 - pa_1 - a_2} & 0 & \cdots \\
0 & 0 & u^{a_2 - pa_2 + a_3} & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & u^{a_n - pa_n + a_1} \end{pmatrix} \right) N_2 = \left( \begin{pmatrix} 0 & \phi(v_1) & v_2 & u^1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

for $2 \leq i \leq n$. Here we have $v_i \left( \phi(v_1) \phi(v_2 u^1) \right) \geq 0$, because $s_1 - pa_1 - a_2 \geq 0$ and $v_i \left( u^{a_1 - pa_1 - a_2} - \phi(v_1) \phi(v_2 u^1) \right) \geq 0$. Hence $x_1$ and $x_3$ lie on the same connected component by Lemma 2.2.

We are going to compare $M_{0, \varphi}$ and $M_{3, \varphi}$. Recall the previous operations on the basis of $M_{0, \varphi}$ that changed $(t_i)_{1 \leq i \leq n}$ so that $|s_i - t_i| \leq p + 1$ keeping $0 \leq t_i \leq e$ for all $i$. Apply the same operations to the basis of $M_{3, \varphi}$. By Lemma 2.3, these operations do not affect which of the connected components $x_3$ lies on. So we
may assume that

\[ s_1 - pa_1 - a_2, s_2 - pa_2 + a_3, \ldots, s_n - pa_n + a_1 \]

are all in \([ (e - p - 1)/2, (e + p + 1)/2]\). As \((e - p - 1)/2 \leq s_i \leq (e + p + 1)/2\), we have that

\[ |pa_1 + a_2| \leq p + 1, \quad |pa_2 - a_3| \leq p + 1, \ldots, \quad |pa_n - a_1| \leq p + 1. \]

Summing up the above inequalities after multiplying some \(p\)-powers so that we can eliminate \(a_j\) for \(j \neq i\), we get \((p^n + 1) |a_i| \leq \{(p^n - 1)/(p - 1)\}(p + 1)\). So we have \(|a_i| \leq 1\) for all \(i\).

In the case \(e \geq p\). We consider the operations that decrease \(|a_i|\) by 1 for an index \(i\) keeping the condition of \(\phi\)-stability. By Lemma 2.3, these operations do not affect which of the connected components \(x_3\) lies on. We prove that we can continue the operations until we have \(a_i = 0\) for all \(i\), that is, \(x_0\) and \(x_3\) lie on the same connected component. Suppose that we cannot continue the operations and there is some nonzero \(a_i\). The condition of \(\phi\)-stability is equivalent to

\[ C_i: 0 \leq s_1 - pa_1 - a_2 \leq e, \]
\[ C_2: 0 \leq s_2 - pa_2 + a_3 \leq e, \]
\[ \ldots, \]
\[ C_n: 0 \leq s_n - pa_n + a_1 \leq e. \]

Note that if \(a_i \neq 0\) or \(a_i+1 \neq 0\), we can decrease \(|a_i|\) or \(|a_i+1|\) keeping \(C_i\).

We put

\[ c_i = \sharp\{i \leq j \leq i + 1 \mid \text{we can decrease } |a_j| \text{ keeping } C_i\}, \]

and claim that \(\sharp\{j \mid a_j \neq 0\} = \sum_{i=1}^{n} c_i\). First, if \(a_i \neq 0\), we have \(c_{i-1} \geq 1\) and \(c_i \geq 1\) from the above remark. So we have \(\sharp\{j \mid a_j \neq 0\} \leq \sum_{i=1}^{n} c_i\). Second, we count \(a_i \neq 0\) in not both of \(C_{i-1}\) and \(C_i\), because we cannot continue the operations. So we have \(\sharp\{j \mid a_j \neq 0\} \geq \sum_{i=1}^{n} c_i\). Hence we have equality. From this equality, we have \(a_i \neq 0\) and \(c_i = 1\) for all \(i\). For \(2 \leq i \leq n\), we have \(a_i = a_{i+1} \neq 0\) because \(c_i = 1\). So we have \(a_1 = a_2 \neq 0\), but this contradicts \(c_1 = 1\).

In the case \(e = p - 1\). We have \(|pa_1 + a_2| \leq p - 1\) by \(C_1\), and \(|pa_i - a_{i+1}| \leq p - 1\) by \(C_i\) for \(2 \leq i \leq n\). Summing up these inequalities after multiplying some \(p\)-powers so that we can eliminate \(a_j\) for \(j \neq i\), we get \(|(p^n + 1)a_i| \leq p^n - 1\). So we have \(a_i = 0\) for all \(i\).

Hence \(x_0\) and \(x_3\) lie on the same connected component. This completes the proof. \(\square\)
3. Application. As an application of Theorem 2.4, we can improve a theorem in [Kis] comparing a deformation ring and a Hecke ring. We recall some notation from [Kis], and the interested reader should consult [Kis] for more detailed definitions.

Let $F$ be a totally real field, and $D$ be a totally definite quaternion algebra with center $F$. Let $\Sigma$ be the set of finite primes where $D$ is ramified. We assume that $\Sigma$ does not contain any primes dividing $p$. We put $\Sigma_p = \Sigma \cup \{p\} \cap p$, and fix a maximal order $O_D$ of $D$. Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F)_{\Sigma}$ be a compact open subgroup contained in $\prod_v (O_D)_{\Sigma}^\vee$, and we assume that $U_v = (O_D)^\vee_v$ for all $v \in \Sigma_p$. Let $O$ be the ring of integers of a $p$-adic field. We fix a continuous character $\psi: (\mathbb{A}_F)^\vee / F^\times \to O^\times_p$ such that $\psi$ is trivial on $U_v \cap O_D^\times$ for any finite place $v$ of $F$. Let $S$ be a finite set of primes containing the infinite primes, $\Sigma_p$, and the finite primes $v$ of $F$ such that $U_v \subset D^\times_v$ is not maximal compact. We fix a decomposition group $G_{F_v} \subset G_{F,S}$ for each $v \in S$. Let $T_{\psi,O}(U)$ (resp. $T_{\psi,O}^{univ}(U)$) denote the image of $\nu^1_{S,O} (\Sigma)$ (resp. $\nu^{univ}_{S,O}$) in the endomorphism ring of $S_2,\psi(U,O)$. Let $m$ be a maximal ideal of $T_{\psi,O}(U)$ that induces a non-Eisenstein maximal ideal of $T_{\psi,O}^{univ}$, and put $m' = m \cap T_{\psi,O}^{univ}(U)$. Then there exists a continuous representation $\rho_{m'}: G_{F,S} \to GL_2(T_{\psi,O}(U)m')$ such that the characteristic polynomial of $\rho_{m'}(\text{Frob}_v)$ is $X^2 - T_v X + N(v) S_v$ for $v \notin S$. Here $N(v)$ denotes the order of the residue field at $v$. Let $\mathbb{F}$ be the residue field of $T_{\psi,O}(U)m'$. Let $\bar{\rho}_{m'}: G_{F,S} \to GL_2(\mathbb{F})$ denote the representation obtained by reducing $\rho_{m'}$ modulo $m'$.

Now we suppose that $\bar{\rho}_{m'}$ satisfies the following conditions:

1) $\bar{\rho}_{m'}$ is unramified outside the primes of $F$ dividing $p$.

2) The restriction of $\bar{\rho}_{m'}$ to $G_{F(p)}$ is absolutely irreducible.

3) If $p = 5$, and $\bar{\rho}_{m'}$ has projective image isomorphic to $PGL_2(\mathbb{F}_5)$, then the kernel of $\text{proj} \bar{\rho}_{m'}$ does not fix $F(\zeta_5)$.

4) For each finite prime $v \in S \setminus \Sigma_p$, we have

$$ (1 - N(v)) \left( (1 + N(v))^2 \right. \text{det} \bar{\rho}_{m'}(\text{Frob}_v) - N(v) \left( \text{tr} \bar{\rho}_{m'}(\text{Frob}_v) \right)^2 \right) \in \mathbb{F}^\times. $$

Let $R_{F,S}$ (resp. $R_{F,S}^{\square}$) be the universal deformation $O$-algebra (resp. the universal framed deformation $O$-algebra) of $\bar{\rho}_{m'}$, and put $\square = R_{F,S}^{\square} \otimes_{R_{F,S}} T_{\psi,O}(U)m'$. We take a subset $\sigma'$ of the set of primes of $F$ dividing $p$, and an unramified character $\chi_p$ of $G_{F_v}$ for each $p \in \sigma'$, such that $m$ is $\sigma'$-ordinary when we put $\sigma = (\sigma', \{\chi_p\}_{p \in \sigma'})$. Now we can define a deformation ring $\bar{R}_{F,S}^{\sigma',\psi}$ and a map $\bar{R}_{F,S}^{\sigma',\psi} \to \square$ as in (3.4) of [Kis].

**Theorem 3.1.** With the above notation and the assumptions, $\bar{R}_{F,S}^{\sigma',\psi} \to \square$ is an isomorphism up to $p$-power torsion kernel.

**Proof.** Applying the Theorem 2.4, the proof goes on as in the proof of [Kis, Theorem 3.4.11].
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