Analysis of A New Variable Time-stepping Time Filter Algorithm for The Unsteady Stokes/Darcy Model *

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Abstract

In this report, we first present a first order $\theta$-scheme with the variable time steps which is one parameter family of Linear Multi-step methods for the unsteady Stokes/Darcy model. On the basis of this scheme, we use a time filter algorithm to increase the convergence order to the second order with almost no increasing the amount of computation, from which we get a new efficient algorithm. Then we analyze stabilities and the second-order accuracy of variable time-stepping algorithms of coupled and decoupled Linear Multi-step methods plus time filters, respectively. Finally, the theoretical results including effectiveness, convergence and efficiency are verified by several numerical experiments.

Keywords: Stokes/Darcy model, variable time-stepping, Linear Multi-step method, time filter

AMS Subject Classification: 76D05, 76S05, 76D03, 35D05

1 Introduction

In recent years, the coupling problem between free flow and porous media flow has attracted more and more attention. This coupling flow appears in many fields, such as environmental problems of groundwater pollution, transport problems between surface water and groundwater, protection of karst aquifers, exploitation of fracture-vuggy reservoirs, and some technologies related to fluid filtration in industry. These problems are also quite closely related to our life. The Navier-Stokes/Darcy and Stokes/Darcy models are very important for these problems in the research field, and many researchers have carried out in-depth studies on the two models. In this paper, we focus on the Stokes/Darcy model.

A lot of work has been done on the Stokes/Darcy model. Numerical methods for steady Stokes/Darcy models include finite element methods, discontinuous Galerkin methods, interface relaxation methods, Lagrange multiplier methods, two-grid or multi-grid methods, domain decomposition methods [1–17] and so on. However, for the unsteady Stokes-Darcy model, The discretization of time is still a problem that needs to be studied. Many scholars now use first-order algorithms that are more computationally efficient and easy to implement [18–22], and some scholars use higher-order algorithms with higher precision [23–25]. All the methods mentioned in papers are related to theoretical analysis and numerical experiments. At present, the research on constant time step are relatively mature, many scholars have begun to notice that the variable time-stepping algorithm which has many advantages in both time accuracy and computational efficiency. The constant time-stepping algorithm fixes the time step in the process of calculation, and cannot adjust the size of the time step according to the actual situation. Compared with it, the variable time-stepping algorithm can give different time step according to the needs of different models, and shorten the step as much as possible to ensure the computational efficiency.

In this paper, we first give a first order $\theta$-scheme with the variable time steps, which is a parameter family of Linear Multi-step methods for the unsteady Stokes/Darcy model. In particular, when $\theta = 0$, it’s the Backward Euler method, when $\theta = 1/2$, it’s the Crank-Nicolson method, and when $\theta = 1$, it’s the Forward Euler method.

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Here, let’s consider the more general case, which is 0 < \theta < 1/2. Since time filters are easy to modify and implement programatically and can improve the accuracy of algorithms, time filters are widely used \cite{24-28}. So based on the first order Linear Multi-step method, we think about the effect of adding the simple time filters for the unsteady Stokes/Darcy model. The method is modular and need to add only two additional line of code, which increases the accuracy of the Linear Multi-step Method from first to second order. Here we propose variable time-stepping algorithms for coupled and decoupled Linear Multi-step methods plus time filters, and fully discretize it in space and time. We analyze the stabilities and the second-order accuracy of the two algorithms and the results do not change as the step size increases or decreases. Finally, we do two numerical experiments. In the first test, we verify the stabilities of the variable time-stepping algorithms by three sets of different variations in time steps. In the second test, we show that the convergence order of the coupled and decoupled algorithms are increased from the first order to the second order, and by comparing the two algorithms, we get that the decoupled algorithm is more computationally efficient.

About the other parts of this article are as follows: Section 2, we review coupled Stokes-Darcy model and the weak formulation. Section 3 is divided into two small parts, one is to introduce variable time-stepping algorithms of the coupled and decoupled Linear Multi-step methods plus time filters. The other is the stability analysis. Section 4, we give the error estimates of the two variable time-stepping algorithms respectively. We verified the effectiveness, convergence and efficiency of numerical algorithms through two numerical experiments in Section 5.

2 The Stokes-Darcy model and weak formulation

This section, the coupled Stokes-Darcy model is considered in a bounded smooth domain \( \Omega \subset \mathbb{R}^d \), \( d = 2 \) or 3, which consists of a free fluid flow region \( \Omega_f \) and a porous media flow region \( \Omega_p \) with the unit outward normal vectors \( \vec{n}_f \) and \( \vec{n}_p \) on \( \partial \Omega_f \) and \( \partial \Omega_p \), where \( \Omega_f \) and \( \Omega_p \) are two disjoint, connected and bounded domains. The interface \( \Gamma = \Omega_f \cap \Omega_p \) separated the two regions \( \Omega_f \) and \( \Omega_p \), we need to pay attention to \( \vec{n}_f = -\vec{n}_p \) on \( \Gamma \) and define \( \Gamma_i = \partial \Omega_i \cap \partial \Omega \) for \( i = f, p \). We can refer to the sketch (Figure 1).

Figure 1: A bounded smooth domain \( \Omega \) consisting of a fluid flow region \( \Omega_f \) and a porous media flow region \( \Omega_p \) separated by an interface \( \Gamma \).

For the finite time interval \([0, T]\). The flow in the free fluid flow region \( \Omega_f \) we describe it using the Stokes equation, which is stated as follows: for fluid velocity \( \vec{u}_f(x,t) \) and kinematic pressure \( p_f(x,t) \)

\[
\begin{align*}
\frac{\partial \vec{u}_f}{\partial t} - \nu \nabla \vec{u}_f + \nabla p_f &= \vec{g}_f, & \text{in } \Omega_f \times (0, T], \\
\nabla \cdot \vec{u}_f &= 0, & \text{in } \Omega_f \times (0, T], \\
\vec{u}_f(x, 0) &= \vec{u}_{0f}(x), & \text{in } \Omega_f.
\end{align*}
\]

where \( \nu > 0 \) is the kinetic viscosity and \( \vec{g}_f(x,t) \) is the external force.

The flow in the porous media region \( \Omega_p \) we describe it using the following equation:

\[
\begin{align*}
S_0 \frac{\partial \phi_p}{\partial t} + \nabla \cdot \vec{u}_p &= g_p, & \text{in } \Omega_p \times (0, T], \\
\vec{u}_p &= -K \nabla \phi_p, & \text{in } \Omega_p \times (0, T], \\
\phi_p(x, 0) &= \phi_{0p}(x), & \text{in } \Omega_p.
\end{align*}
\]

Combining the equation (2.4) and (2.5), we have the Darcy equations: for the piezometric (hydraulic) head
where \( \vec{u}_p \) is the flow velocity in the porous media region which is proportional to the gradient of \( \phi_p \), namely, the Darcy’s law. \( S_0 \) is the specific mass storativity coefficient and \( \phi_p = z + \frac{p_g}{\rho_g} \), where \( z \) and \( p_g \) denote the relative depth from a fixed reference level and the dynamic pressure, \( \rho \) and \( g \) represent the density and the gravitational constant, respectively. \( g_p(\vec{x}, t) \) is a source term and \( K = [K_{ij}]_{d \times d} \) denotes a symmetric and positive definite matrix with the smallest eigenvalue \( K_{min} > 0 \), which is allowed to vary in space.

We usually assume that the fluid velocity \( \vec{u}_f(\vec{x}, t) \) and the piezometric (hydraulic) head \( \phi_p(\vec{x}, t) \) satisfy the homogeneous Dirichlet boundary conditions:

\[
\vec{u}_f = 0 \quad \text{and} \quad \phi_p = 0, \quad \text{on} \quad \Gamma_p \times (0, T].
\]

Interface conditions are important in the Stokes-Darcy model, which include the conservation of mass, the balance of normal forces, and the Beaver-Joseph Saffmann conditions on \( \Gamma \):

\[
\vec{u}_f \cdot \vec{n}_f + \vec{u}_p \cdot \vec{n}_p = 0,
\]

\[
p_f - \nu \vec{u}_f \cdot \nabla \vec{u}_f = g \phi_p,
\]

\[
-\nu \nabla \cdot \vec{u}_f = \frac{\alpha \nu \sqrt{\beta}}{\sqrt{\text{trace}(\Pi)}} \vec{u}_f \cdot \vec{n}_f, \quad i = 1, 2, \ldots, d - 1,
\]

where \( \vec{n}_f, i = 1, 2, \ldots, d - 1, \) are the orthonormal tangential unit vectors on the interface \( \Gamma \), \( \beta \) is the space dimension, \( \alpha \) is an positive parameter and the permeability \( \Pi \) has the relation \( \Pi = \frac{K}{g} \).

Now we give the Hilbert space that needs to be used in the next analysis process:

\[
X_f = \{ \vec{u}_f \in (H^1(\Omega_f))^d : \vec{u}_f = 0, \text{ on } \Gamma_f \},
\]

\[
X_p = \{ \psi_\phi \in (H^1(\Omega_p))^d : \psi_\phi = 0, \text{ on } \Gamma_p \},
\]

\[
Q_f = L^2(\Omega_f),
\]

\[
X = X_f \times X_p.
\]

In addition, we define \( X' \), \( X'_f \) and \( X'_p \) to represent the dual spaces of \( X \), \( X_f \) and \( X_p \), respectively. For the domain \( D \), we define the scalar inner product in \( D = \Omega_f \) or \( \Omega_p \) by \( (\cdot, \cdot)_D \). For the Hilbert space \( X, X'_f \) and \( X'_p \), we denote the corresponding norms:

\[
\| \vec{u} \|_X = \sqrt{\nu \| \nabla \vec{u} \|_{L^2(\Omega_f)}} + g S_0 \| \phi_p \|_{\Omega_f}, \quad \forall \vec{u} = (\vec{u}_f, \phi_p) \in X,
\]

\[
\| \vec{u} \|_X = \sqrt{\nu \| \nabla \vec{u} \|_{L^2(\Omega_p)}} + g K \| \phi_p \|_{\Omega_p}, \quad \forall \vec{u} = (\vec{u}_f, \phi_p) \in X,
\]

\[
\| \vec{u}_f \|_{X_f} = \| \vec{u}_f \|_{L^2(\Omega_f)} + \| \nabla \vec{u}_f \|_{L^2(\Omega_f)}, \quad \forall \vec{u}_f \in X_f,
\]

\[
\| \phi_p \|_{X_p} = \| \phi_p \|_{L^2(\Omega_p)}, \quad \forall \phi_p \in X_p,
\]

where the norms \( \| \cdot \|_{X_f/X_p} \) and \( \| \cdot \|_{f/p} \) denotes \( H^1(\Omega_f/p) \) and \( L^2(\Omega_f/p) \). Then for the function \( v(x, t) \), we define the norms:

\[
\| v \|_{L^2(0,T;L^2)} := (\int_0^T \| v(\cdot, t) \|_{L^2}^2 dt)^{1/2}, \quad \| v \|_{L^\infty(0,T;L^\infty)} := \sup_{t \in [0,T]} \| v(\cdot, t) \|_{L^\infty}.
\]

Based on the above related concepts, we have weak formulations of the unsteady coupled Stokes-Darcy model [2.1]–[2.11], which is expressed as: \( \vec{g}_f \in L^2(0, T; L^2(\Omega_f)) \) and \( g_p \in L^2(0, T; L^2(\Omega_p)) \), find \( \vec{u} = (\vec{u}_f, \phi_p) \in \big( L^2(0, T; X_f) \cap L^\infty(0, T; L^2(\Omega_f)) \times L^2(0, T; X_p) \cap L^\infty(0, T; L^2(\Omega_p)) \big) \) and \( p_f \in L^2(0, T; Q_f) \) such that \( \forall (\vec{u}, q_f) \in X \times Q_f \),

\[
\frac{\partial \vec{u}}{\partial t} = \vec{g}_f + a(\vec{u}, \vec{u}) + b(\vec{u}, p_f) = -\vec{f}, \quad \text{in } X',
\]

\[
b(\vec{u}, q_f) = 0,
\]

\[
\vec{u}(\vec{x}, 0) = \vec{u}_0,
\]

\[
(2.12)
\]
where

\[
\frac{\partial \vec{u}}{\partial t} = \left( \frac{\partial \vec{u}}{\partial t} \right) - \vec{v}_f + g(S_0 \phi_P, \psi_p), \\
a(\vec{u}, \vec{v}) = a_\Omega(\vec{u}, \vec{v}) + a_{\Gamma} (\vec{u}, \vec{v}) , \\
a_{\Omega}(\vec{u}, \vec{v}) = a_{\Omega_f}(\vec{u}_f, \vec{v}_f) + a_{\Omega_p}(\phi_P, \psi_P) , \\
a_{\Omega_f}(\vec{u}_f, \vec{v}_f) = \nu(\nabla(\vec{u}_f), \nabla(\vec{v}_f))_{\Omega_f} + \sum_{i=1}^{d-1} \int_{\Gamma} \frac{\alpha_i}{\sqrt{1 + \text{trace}(\Pi)}} (\vec{u}_f \cdot \vec{v}_f)(\vec{u}_f \cdot \vec{v}_f), \\
a_{\Omega_p}(\phi_P, \psi_P) = g(K \nabla \phi_P, \nabla \psi_P)_{\Omega_p} , \\
b_{\Gamma}(\vec{u}_f, \vec{v}_f) = c_{\Gamma}(\vec{v}_f, \phi_P) - c_{\Gamma}(\vec{u}_f, \psi_P) = g(\phi_P, \vec{v}_f \cdot \vec{n}_f) - g(\psi_P, \vec{u}_f \cdot \vec{n}_f) , \\
\langle \vec{F}, \vec{v} \rangle_{\Gamma} = (\vec{g}, \vec{v})_{\Omega_f} + g(g_P, \psi_P)_{\Omega_p}.
\]

The coupled Stokes-Darcy model is well-posedness, which we can find in the other papers, we mainly analyze its numerical solution in this paper. For bilinear form \(a(\cdot, \cdot)\), it is continuous and coercive:

\[
a(\vec{u}, \vec{v}) \leq C_{\text{con}} \| \vec{u} \|_{X} \| \vec{v} \|_{X}, \quad \forall \; \vec{u}, \vec{v} \in X , \\
a(\vec{u}, \vec{u}) \geq C_{\text{coer}} \| \vec{u} \|_{X}^2 , \quad \forall \; \vec{u} \in X , \\
a_{\Omega}(\vec{u}_f, \vec{v}_f) \geq \tilde{C}_{\text{coer}} \| \vec{u}_f \|_{X_f}^2 , \quad \forall \; \vec{u}_f \in X_f, \\
a_{\Omega_p}(\phi_P, \phi_P) \geq \tilde{C}_{\text{coer}} \| \phi_P \|_{X_p}^2 , \quad \forall \; \phi_P \in X_p.
\]

At the same time, for the interface term \(a_{\Gamma}(\cdot, \cdot)\), it satisfies the anti-symmetric properties:

\[
a_{\Gamma}(\vec{u}, \vec{v}) = -a_{\Gamma}(\vec{v}, \vec{u}) \quad \text{and} \quad a_{\Gamma}(\vec{u}, \vec{u}) = 0 \quad \forall \; \vec{u}, \vec{v} \in X, \\
a_{\Gamma}(\vec{u}_f, \vec{v}_f) = -a_{\Gamma}(\vec{v}_f, \vec{u}_f) \quad \text{and} \quad a_{\Gamma}(\vec{u}_f, \vec{u}_f) = 0 \quad \forall \; \vec{u}_f, \vec{v}_f \in X_f, \\
a_{\Gamma}(\phi_P, \phi_P) = -a_{\Gamma}(\phi_P, \phi_P) \quad \text{and} \quad a_{\Gamma}(\phi_P, \phi_P) = 0 \quad \forall \; \phi_P \in X_p.
\]

Then, we use the finite element methods (FEMs) to discretize the Stokes-Darcy model in space. Assuming \(h\) is an any given small positive parameter, the regular triangulations \(T_h, T_{fh}\) and \(T_{ph}\) are regular partitions of triangular or quadrilateral elements of \(\Omega\) and \(\Omega_p\). In order to facilitate our later analysis, we assume the domain \(\Omega = \Omega_f \times \Omega_p\) is smooth enough. And we choose the MINI elements (P1b-P1) \(X_{fh} \subset X_f, Q_{fh} \subset Q_f\) and the linear Lagrange elements (P1) \(X_{ph} \subset X_p\) which are finite element spaces, we denote \(X_h = X_{fh} \times X_{ph}\) and assume that the fluid velocity space \(X_{fh}\) and the pressure space \(Q_{fh}\) satisfy the discrete LBB condition:

\[
\inf_{q_f^h \in Q_{fh}} \sup_{\vec{v}_f^h \in X_{fh}} \frac{(q_f^h, \nabla \cdot \vec{v}_f^h)_{\Omega_f}}{|q_f^h|_{Q_f} \| \vec{v}_f^h \|_{X_f}} \geq \beta.
\]

We define the linear projection operator (see [19]): \(\forall \; \vec{v}_f^h \in X_{fh}, q_f^h \in Q_{fh}\) and \(t \in (0, T), \; P_h : (\vec{u}(t), p_f(t)) \in X \times Q_f \rightarrow (P_h^\vec{u}(t), P_h^{p_f}(t)) \in X_h \times Q_{fh}\) satisfies

\[
a(P_h^\vec{u}(t), \vec{v}_f^h) + b(\vec{v}_f^h, P_h^{p_f}(t)) = a(\vec{u}(t), \vec{v}_f^h) + b(\vec{v}_f^h, p_f(t)), \\
b(P_h^\vec{u}(t), q_f^h(t)) = 0.
\]

Then we assume that \((\vec{u}(t), p_f(t))\) is smooth enough and the projection operator \((P_h^\vec{u}(t), P_h^{p_f}(t))\) of \((\vec{u}(t), p_f(t))\) satisfies the approximation properties:

\[
\| P_h^\vec{u}(t) - \vec{u}(t) \|_0 \leq C h^2 \| \vec{u}(t) \|_H^2, \\
\| P_h^\vec{u}(t) - \vec{u}(t) \|_X \leq C h \| \vec{u}(t) \|_{H^2}, \\
\| P_h^{p_f}(t) - p_f(t) \|_{L^2} \leq C(h^2 \| p_f(t) \|_{H^1}).
\]

In addition, we give several inequalities, including the Poincaré, trace, Sobolev and inverse inequalities: there exist constants \(C_i\) and \(\tilde{C}_i\) \((i = p, t, s, I)\) such that for \(\forall \; \vec{v}_f \in X_f\) and \(\psi_p \in X_p\),

\[
\| \nabla \vec{v}_f \|_f \leq C_t \| \vec{v}_f \|_f, \\
\| \psi_P \|_P \leq \tilde{C}_t \| \nabla \psi_P \|_P, \\
\| \nabla \vec{v}_f \| \leq C_t \| \vec{v}_f \|_f, \\
\| \psi_P \|_{H^\frac{1}{2}(\partial \Omega_f)} \leq \tilde{C}_s \| \nabla \psi_P \|_P, \\
\| \vec{v}_f \|_{X_f} \leq C_t h^{-1} \| \vec{v}_f \|_f, \\
\| \psi_P \|_{X_p} \leq \tilde{C}_t h^{-1} \| \psi_P \|_P.
\]
Note that $C_j$ $(j = p, t, s, I)$ depend on the fluid flow domain $\Omega_f$ and $\tilde{C}_i$ $(i = p, t, s, I)$ depend on the porous media domain $\Omega_p$.

3 Numerical algorithms and stabilities

We divide this section into two parts, the first part will give the variable time-stepping algorithms of coupled and decoupled Linear Multi-step methods plus time filters. And the second part will analyze stabilities of the two algorithms separately.

Before analyzing, we need to recall several lemmas that they will use multiple times during the analysis.

**Lemma 3.1.** Let $\delta = \beta_2 - \frac{\alpha_2}{2} > 0$. Then the coefficients $\alpha_1$ and $\beta_1$ satisfy the following relation:

$$2 \left( \sum_{i=0}^{2} \alpha_i \frac{\partial t}{\partial t} \right) \left( \sum_{i=0}^{2} \beta_i \frac{\partial t}{\partial t} \right) \geq \left( \alpha_2^2 + \delta \right) \frac{\partial t}{\partial t}^2 - (2\alpha_2 - 1) \frac{\partial t}{\partial t}^2 - ((\alpha_2 - 1)^2 + \delta) \frac{\partial t}{\partial t}^2$$

$$- 2(\alpha_2(\alpha_2 - 1) + \delta) \left( \partial t \partial t_1 - \partial t \partial t_0 \right), \quad \forall \partial t_0, \partial t_1, \partial t_2 \in R.$$

**Lemma 3.2.** For $\forall \vec{v}_f \in X_f$, $\phi_p \in X_p$, there exists $C_k > 0$ such that $\forall \varepsilon > 0$,

$$|c_T(\vec{v}_f, \phi_p)| \leq \frac{1}{4\varepsilon} \|\vec{v}_f\|_{H^1}^2 + C_k \varepsilon \|\phi_p\|_{H^1}^2. \quad (3.1)$$

In addition, for $\forall \vec{v}_f \in X_f$, $\phi_p \in X_p$, there exists $C_k > 0$ such that $\forall \varepsilon > 0$,

$$|c_T(\vec{v}_f, \phi_p)| \leq \frac{1}{4\varepsilon} \|\vec{v}_f\|_{H^1}^2 + C_k \varepsilon \|\phi_p\|_{H^1}^2. \quad (3.2)$$

**Lemma 3.3.** (Discrete Gronwall Inequality) Let $\Delta t$, $C$, $a_i$, $b_i$, $c_i$, $d_i$, (for integers $n \geq 0$) be non-negative numbers such that

$$a_n + \Delta t \sum_{i=0}^{n} b_i \leq + \Delta t \sum_{i=0}^{n-1} c_i + C, \quad \forall \ n \geq 1,$$

then

$$a_n + \Delta t \sum_{i=0}^{n} b_i \leq \exp \left( \Delta t \sum_{i=0}^{n-1} d_i \right) \left( \Delta t \sum_{i=0}^{n-1} c_i + C \right), \quad \forall \ n \geq 1. \quad (3.3)$$

For the rest of the paper, $P = \{t_m\}_{m=0}^{N}$ is the partition on time interval $t_0 = 0$, $t_N = T$, $k_m = t_{m+1} - t_m$ is the time step size, and $\tau_m = \frac{k_m}{k_m}$ is a ratio for the time step and satisfies $\tau_{min} \leq \tau_m \leq \tau_{max}$. Here

$$(\hat{\mathbf{u}}_{h_{m+1}}, \hat{p}_{f_{m+1}}) = (\hat{\mathbf{u}}_{f_{m+1}}, \hat{p}_{h_{m+1}}, \hat{p}_{h_{m+1}}) = (\hat{\mathbf{u}}_{h_{m+1}}(t_{m+1}), \hat{p}_{f_{m+1}}(t_{m+1}), \hat{p}_{h_{m+1}}(t_{m+1})).$$

### 3.1 Numerical algorithms

First, we introduce variable time-stepping coupled algorithm.

**The Linear Multistep method (First Order):**

1. **Give** $(\hat{\mathbf{u}}_{f_{0}}, \hat{\mathbf{p}}_{f_{0}})$ and $(\hat{\mathbf{u}}_{h_{1}}, \hat{\mathbf{p}}_{h_{1}})$, find

$$\hat{\mathbf{u}}_{h_{m+1}} = (\hat{\mathbf{u}}_{f_{m+1}}, \hat{\mathbf{p}}_{h_{m+1}}) \in X_h$$

and $\hat{p}_{f_{m+1}} \in Q_{fh}$ with $m = 1, 2, ..., N - 1$, $\forall \hat{\mathbf{u}}_{h} \in X_h$ and $\hat{p}_{f} \in Q_{fh}$,

$$\left( \frac{\hat{\mathbf{u}}_{h_{m+1}} - \hat{\mathbf{u}}_{h_{m}}}{k_m}, \hat{\mathbf{u}}_{h_{m}} \right) + a((1 - \theta) \hat{\mathbf{u}}_{h_{m+1}} + \theta \hat{\mathbf{u}}_{h_{m}}, \hat{\mathbf{u}}_{h_{m}}) + b((1 - \theta) p_{f_{m+1}} + \theta p_{f_{m}})$$

$$= \langle (1 - \theta) \hat{\mathbf{F}}_{m+1} + \theta \hat{\mathbf{F}}_{m}, \hat{\mathbf{u}}_{h_{m}} \rangle,$$

$$b(1 - \theta) \hat{\mathbf{u}}_{h_{m+1}} + \theta \hat{\mathbf{u}}_{h_{m}}, \hat{p}_{f} \rangle = 0. \quad (3.5)$$

**The Time Filters (Second Order):**
Update the previous solutions \((\hat{\tilde{u}}^{h,m+1}, \hat{p}_f^{h,m+1})\) by time filters,
\[
\begin{align*}
\hat{\tilde{u}}^{h,m+1} &= \hat{\tilde{u}}^{h,m} - \frac{(1 - 2\theta)(1 + \tau_{m-1})\tau_{m-1}}{2(1 - \theta)\tau_{m-1} + 1} \left( \frac{1}{1 + \tau_{m-1}} \hat{\tilde{u}}^{h,m} + \frac{\tau_{m-1} - 1}{1 + \tau_{m-1}} \hat{\tilde{u}}^{h,m-1} \right), \\
\hat{p}_f^{h,m+1} &= \hat{p}_f^{h,m} - \frac{(1 - 2\theta)(1 + \tau_{m-1})\tau_{m-1}}{2(1 - \theta)\tau_{m-1} + 1} \left( \frac{1}{1 + \tau_{m-1}} \hat{p}_f^{h,m} + \frac{\tau_{m-1} - 1}{1 + \tau_{m-1}} \hat{p}_f^{h,m-1} \right).
\end{align*}
\] (3.6)

Then, we introduce variable time-stepping decoupled algorithm.

**The Linear Multistep method (First Order):**

Given \((\hat{\tilde{u}}_f^{h,0}, \hat{p}_f^{h,0})\) and \((\hat{\tilde{u}}_f^{h,1}, \hat{p}_f^{h,1})\), find \((\hat{\tilde{u}}_f^{h,m+1}, \hat{p}_f^{h,m+1}) \in (X_{fh}, Q_{fh})\) with \(m = 1, 2, \ldots, N - 1\), such that for \(\forall \hat{\tilde{u}}_f^h \in X_{fh}\) and \(q^h \in Q_{fh}\),
\[
\begin{align*}
\frac{\hat{\tilde{u}}_f^{h,m+1} - \hat{\tilde{u}}_f^{h,m}}{k_m} + a_{\Omega_f} (1 - \theta)\hat{\tilde{u}}_f^{h,m+1} + \theta\hat{\tilde{u}}_f^{h,m} + \hat{\tilde{u}}_f^{h,m-1} + (1 + (1 - \theta)\tau_{m-1})\hat{\tilde{u}}_f^{h,m-1} &+ (1 - \theta)\tau_{m-1} \hat{\tilde{u}}_f^{h,m-1} - \hat{\tilde{u}}_f^{h,m-1} + \phi_p^{m,h-1}, \psi^h) \\
&= b((1 - \theta)\hat{\tilde{u}}_f^{h,m+1} + \theta\hat{\tilde{u}}_f^{h,m}, \hat{\tilde{u}}_f^{h,m-1}) = 0.
\end{align*}
\] (3.7)

Give \(\hat{\tilde{u}}_f^{h,0}\) and \(\hat{\tilde{u}}_f^{h,1}\), find \(\hat{\tilde{u}}_f^{h,m+1} \in X_{ph}\) with \(m = 1, 2, \ldots, N - 1\), such that for \(\forall \psi^h \in X_{ph}\),
\[
\begin{align*}
g \left( \frac{\hat{\tilde{u}}_f^{h,m+1} - \phi_p^{m,h}}{k_m}, \psi^h \right) + a_{\Omega_p} (1 - \theta)\hat{\tilde{u}}_f^{h,m+1} + \theta\hat{\tilde{u}}_f^{h,m} + \phi_p^{m,h-1}, \phi_p^{m,h} \psi^h &+ (1 + (1 - \theta)\tau_{m-1})\hat{\tilde{u}}_f^{h,m-1} - (1 - \theta)\tau_{m-1} \hat{\tilde{u}}_f^{h,m-1} - \psi^h \right) \\
&= g((1 - \theta)\hat{\tilde{u}}_f^{h,m+1} + \theta\hat{\tilde{u}}_f^{h,m}, \hat{\tilde{u}}_f^{h,m-1}) = 0.
\end{align*}
\] (3.8)

**The Time Filters (Second Order):**

Update the previous solutions \((\hat{\tilde{u}}_f^{h,m+1}, \hat{p}_f^{h,m+1}, \phi_p^{h,m+1})\) by time filters,
\[
\begin{align*}
\hat{\tilde{u}}_f^{h,m+1} &= \hat{\tilde{u}}_f^{h,m} - \frac{(1 - 2\theta)(1 + \tau_{m-1})\tau_{m-1}}{2(1 - \theta)\tau_{m-1} + 1} \left( \frac{1}{1 + \tau_{m-1}} \hat{\tilde{u}}_f^{h,m} + \frac{\tau_{m-1} - 1}{1 + \tau_{m-1}} \hat{\tilde{u}}_f^{h,m-1} \right), \\
\hat{p}_f^{h,m+1} &= \hat{p}_f^{h,m} - \frac{(1 - 2\theta)(1 + \tau_{m-1})\tau_{m-1}}{2(1 - \theta)\tau_{m-1} + 1} \left( \frac{1}{1 + \tau_{m-1}} \hat{p}_f^{h,m} + \frac{\tau_{m-1} - 1}{1 + \tau_{m-1}} \hat{p}_f^{h,m-1} \right), \\
\phi_p^{h,m+1} &= \phi_p^{h,m} - \frac{(1 - 2\theta)(1 + \tau_{m-1})\tau_{m-1}}{2(1 - \theta)\tau_{m-1} + 1} \left( \frac{1}{1 + \tau_{m-1}} \phi_p^{h,m} + \frac{\tau_{m-1} - 1}{1 + \tau_{m-1}} \phi_p^{h,m-1} \right).
\end{align*}
\] (3.9)

**Remark 1:** Here, we use the second-order extrapolation method to approximate \(\hat{\tilde{u}}_f^{h,m+1}\) with \((1 + \tau_{m-1})\hat{\tilde{u}}_f^{h,m} - \tau_{m-1} \hat{\tilde{u}}_f^{h,m-1}\) and \(\phi_p^{h,m+1}\) with \((1 + \tau_{m-1})\phi_p^{h,m} - \tau_{m-1} \phi_p^{h,m-1}\) in the interface coupled term. At the same time, whether the pressure \(\hat{p}_f^{h,m+1}\) is filtered or not has little influence on the result.

We define some notations for the following analysis:
\[
\begin{align*}
A(\hat{\tilde{u}}^{h,m+1}) &= \frac{2(1 - \theta)(1 + \tau_{m-1})\tau_{m-1}}{\tau_{m-1} + 1}, \\
B(\hat{\tilde{u}}^{h,m+1}) &= \frac{2(1 - \theta)^2\tau_{m-1} + 1 - \theta}{\tau_{m-1} + 1}, \\
S(\hat{\tilde{u}}^{h,m+1}) &= \frac{(1 - \theta)(1 - 2\theta)\tau_{m-1}}{\tau_{m-1} + 1}, \\
W(\eta_p^{h,m+1}) &= \frac{2(1 - \theta)^2\tau_{m-1} + 1 - \theta}{\tau_{m-1} + 1}, \\
D(\theta, \tau_{m-1}) &= \frac{2(1 - \theta)(5 - 6\theta)\tau_{m-1} + 1 - \theta}{\tau_{m-1} + 1}, \\
H(\theta, \tau_{m-1}) &= \frac{2(3 - 4\theta)\tau_{m-1} + 1 - \theta}{\tau_{m-1} + 1}, \\
E(\theta, \tau_{m-1}) &= \frac{2(1 - \theta)(3 - 2\theta)\tau_{m-1} + 1 - \theta}{\tau_{m-1} + 1}.
\end{align*}
\]
\[ F(\theta, \tau_{m-1}) = \frac{12(1-\theta)(1-2\theta)\tau_{m-1}^2 + 2(1-2\theta)(5-2\theta)\tau_{m-1} + 2(1-2\theta)}{2(\tau_{m-1}+1)^2} > 0, \]
\[ G(\theta, \tau_{m-1}) = \frac{6(1-\theta)(1-2\theta)\tau_{m-1}^2 + (1-2\theta)(5-2\theta)\tau_{m-1} + 1 - 2\theta}{2(\tau_{m-1}+1)^2} \cdot \frac{6(1-\theta)(5-2\theta)\tau_{m-1} + 1}{2(2-\theta)\tau_{m-1} + 3 - 2\theta} > 0, \]
\[ I(\theta, \tau_{m-1}) = D(\theta, \tau_{m-1}) - G(\theta, \tau_{m-1}) = \frac{(\tau_{m-1}+1)(2\tau_{m-1}\theta - 2\tau_{m-1} - 1)}{6\tau_{m-1}^2 \theta - 4\tau_{m-1}^2 + 2\tau_{m-1}\theta - 3\tau_{m-1} - 1} > 0. \]

### 3.2 Stabilities analysis

First, we give the following stability theorem of variable time-stepping coupled algorithm.

**Theorem 3.1.** (stability for variable time-stepping coupled algorithm) Let \( \overline{u}^{h,m+1} \) be the solution of the Linear Multi-step methods plus filters with \( 0 < \theta < \frac{1}{2} \). For \( N \geq 2 \), we have

\[ I(\theta, \tau)\|\overline{u}^{h,N}\|_0^2 + C_{\text{coe}} \sum_{m=1}^{N-1} \left[ k_m \|B(\overline{u}^{h,m+1})\|_X^2 \right] \leq C(\|\overline{u}^f\|_{L^2(0,T;L^2(\Omega_h)))}^2 + \|g_0\|_{L^2(0,T;L^2(\Omega_h))}^2 + \|\overline{u}^{h,1}\|_0^2 + \|\overline{u}^{h,0}\|_0^2 + \|\overline{u}^{h,1}\|_0\|\overline{u}^{h,0}\|_0), \]

where \( C \) is a positive constant, which is independent of \( h, k_m \) or other parameters and \( \tau_{\text{min}} \leq \tau \leq \tau_{\text{max}} \).

**Proof.** From (3.6), we get

\[
\overline{v}^{h,m+1} = \frac{2(1-\theta)\tau_{m-1} + 1}{\tau_{m-1} + 1} \overline{v}^{h,m} - \frac{(1-2\theta)\tau_{m-1}}{\tau_{m-1} + 1} \overline{v}^{h,m-1},
\]

\[
\overline{p}^{h,m+1} = \frac{2(1-\theta)\tau_{m-1} + 1}{\tau_{m-1} + 1} \overline{p}^{h,m} - \frac{(1-2\theta)\tau_{m-1}}{\tau_{m-1} + 1} \overline{p}^{h,m-1},
\]

and then take them into (3.5), for \( \forall \overline{v}^h \in X_h, \overline{q}^h \in Q_h \),

\[
\frac{1}{k_m} (A(\overline{v}^{h,m+1}, \overline{v}^h) + a(B(\overline{v}^{h,m+1}, \overline{v}^h) + b(\overline{v}^h, B(\overline{p}^{h,m+1}) = \langle (1-\theta)\overline{v}^{m+1} + \theta\overline{v}^m, \overline{v}^h \rangle, \tag{3.10}
\]

\[
b(\overline{v}^{h,m+1}, \overline{q}^h) = 0.
\]

Setting \( \overline{v}^h = 2k_mB(\overline{u}^{h,m+1}, \overline{v}^h), \overline{q}^h = 2k_mB(\overline{p}^{h,m+1}) \) and analysing each term of the first equation in (3.10), first since \( \frac{2(1-\theta)^2\tau_{m-1} + 1}{\tau_{m-1}^4} - \frac{2(1-\theta)^2\tau_{m-1} + 1}{\tau_{m-1}^4} > 0 \), we can use Lemma 3.1 to handle the first term on the left-hand side

\[
2(A(\overline{u}^{h,m+1}, B(\overline{u}^{h,m+1})), \overline{v}^h) + a(B(\overline{u}^{h,m+1}, \overline{v}^h)) = D(\theta, \tau_{m-1})|\overline{u}^{h,m+1}|_0^2 - H(\theta, \tau_{m-1})|\overline{u}^{h,m}|_0^2 - E(\theta, \tau_{m-1})|\overline{u}^{h,m-1}|_0^2 - F(\theta, \tau_{m-1})|\overline{u}^{h,m+1}|_0|\overline{u}^{h,m}|_0 - |\overline{u}^{h,m}|_0|\overline{u}^{h,m-1}|_0 \geq C_{\text{min}} \left( D(\theta, \tau_{m-1})|\overline{u}^{h,m+1}|_0^2 - H(\theta, \tau)|\overline{u}^{h,m}|_0^2 - E(\theta, \tau)|\overline{u}^{h,m-1}|_0^2 - F(\theta, \tau)|\overline{u}^{h,m+1}|_0|\overline{u}^{h,m}|_0 - |\overline{u}^{h,m}|_0|\overline{u}^{h,m-1}|_0 \right). \tag{3.11}
\]

where \( C_{\text{min}} = \min\{C_D, C_H, C_E, C_F\}, (C_i = \frac{i(\theta, \tau_{m-1})}{(\theta, \tau)}, i = D, H, E, F) \) and \( \tau_{min} \leq \tau \leq \tau_{max} \).

Then using the coercivity of the bilinear form \( a(\cdot, \cdot) \), the second term on the left can be handled as

\[
2k_m a(\overline{u}^{h,m+1}, B(\overline{u}^{h,m+1})) \geq 2C_{\text{coe}}k_m \|B(\overline{u}^{h,m+1})\|_X^2. \tag{3.12}
\]
Finally, we use the Cauchy Schwarz inequality and Young’s inequality, the external force term on the right can be written as

\[
2k_m \langle (1 - \theta) \tilde{F}^{m+1}_f + \theta \tilde{F}^m, B(\tilde{u}^{h,m+1}) \rangle
\leq \frac{\theta^2 k_m}{C_{\text{coe}}} \| \tilde{F}^m \|^2_{X'} + \frac{(1 - \theta)^2 k_m}{C_{\text{coe}}} \| \tilde{F}^{m+1} \|^2_{X'} + C_{\text{coe}} k_m \| B(\tilde{u}^{h,m+1}) \|^2_{X'}.
\]

(3.13)

Combining the (3.11) and (3.13) and sum them over \( m = 1, 2, \ldots, N - 1 \), and let \( k = \max_{1 \leq m \leq N - 1} \{ k_m \} \), we have

\[
C_{\min} \left( D(\theta, \tau) \| \tilde{u}^{h,N}_\theta \|_0^2 + E(\theta, \tau) \| \tilde{u}^{h,N-1}_\theta \|_0^2 - F(\theta, \tau) \| \tilde{u}^{h,N}_\theta \|_0 \| \tilde{u}^{h,N-1}_\theta \|_0 \right) + C_{\text{coe}} \sum_{m=1}^{N-1} k_m \| B(\tilde{u}^{h,m+1}) \|^2_{X'}
\leq k \theta^2 \sum_{m=1}^{N-1} \| \tilde{F}^m \|^2_{X'} + \frac{k(1 - \theta)^2}{C_{\text{coe}}} \sum_{m=1}^{N-1} \| \tilde{F}^{m+1} \|^2_{X'}
\leq C_{\min} \left( D(\theta, \tau) \| \tilde{u}^{h,1}_\theta \|_0^2 + E(\theta, \tau) \| \tilde{u}^{h,0}_\theta \|_0^2 - F(\theta, \tau) \| \tilde{u}^{h,1}_\theta \|_0 \| \tilde{u}^{h,0}_\theta \|_0 \right).
\]

(Note that

\[-F(\theta, \tau) \| \tilde{u}^{h,N}_\theta \|_0 \| \tilde{u}^{h,N-1}_\theta \|_0 \geq -G(\theta, \tau) \| \tilde{u}^{h,N}_\theta \|_0^2 - E(\theta, \tau) \| \tilde{u}^{h,N-1}_\theta \|_0^2,
\]

so we have

\[
C_{\min} \left( D(\theta, \tau) \| \tilde{u}^{h,1}_\theta \|_0^2 + E(\theta, \tau) \| \tilde{u}^{h,0}_\theta \|_0^2 - F(\theta, \tau) \| \tilde{u}^{h,1}_\theta \|_0 \| \tilde{u}^{h,0}_\theta \|_0 \right).
\]

Thus, we end the proof.

Then, we derive the following stability theorem of variable time-stepping decoupled algorithm.

**Theorem 3.2.** (stability for variable time-stepping decoupled algorithm) Let \( \tilde{u}^{h,m+1}_f, \phi^{h,m+1}_p \) be the solution of the Linear Multi-step method plus time filters with \( 0 < \theta < \frac{1}{2} \). For \( N \geq 2 \), we have

\[
I(\theta, \tau) \| \tilde{u}^{h,N}_\theta \|_f^2 + g I(\theta, \tau) \| \phi^{h,N}_p \|_p^2 + \tilde{C}_{\text{coe}} \sum_{m=1}^{N-1} k_m \| B(\tilde{u}^{h,m+1}) \|^2_{X_f} + g \tilde{C}_{\text{coe}} \sum_{m=1}^{N-1} k_m \| B(\phi^{h,m+1}_p) \|^2_{X_p}
\leq C(T) \left( \| g_f \|^2_{L^2(\Omega_f)} + \| g_p \|^2_{L^2(\Omega_p)} \right)
+ C_{\min} \left( D(\theta, \tau) \| \tilde{u}^{h,1}_\theta \|_f^2 + E(\theta, \tau) \| \tilde{u}^{h,0}_\theta \|_f^2 - F(\theta, \tau) \| \tilde{u}^{h,1}_\theta \|_f \| \tilde{u}^{h,0}_\theta \|_f \right)
+ g C_{\min} \left( D(\theta, \tau) \| \phi^{h,1}_p \|_p^2 + E(\theta, \tau) \| \phi^{h,0}_p \|_p^2 - F(\theta, \tau) \| \phi^{h,1}_p \|_p \| \phi^{h,0}_p \|_p \right),
\]

where \( C(T) = \exp \left( \frac{2k}{C_{\text{coe}} C_{\min} I(\theta, \tau)} \right) \) and \( \tau_{\min} \leq \tau \leq \tau_{\max} \).

**Proof.** From (3.9), we have

\[
\begin{align*}
\tilde{u}^{h,m+1}_f &= \frac{2(1 - \theta) \tau_{m+1}}{\tau_{m-1} + 1} \tilde{u}^{h,m+1}_{f,m} - (1 - 2\theta) \tau_{m-1} \tilde{u}^{h,m}_{f,m} + \frac{(1 - 2\theta) \tau_{m-1}^2}{\tau_{m-1} + 1} \tilde{u}^{h,m-1}_{f,m} + \frac{(1 - 2\theta) \tau_{m-1}^2}{\tau_{m-1} + 1} \tilde{u}^{h,m-1}_{f,m-1}, \\
\phi^{h,m+1}_p &= \frac{2(1 - \theta) \tau_{m+1}}{\tau_{m-1} + 1} \phi^{h,m+1}_{p,m} - (1 - 2\theta) \tau_{m-1} \phi^{h,m}_{p,m} + \frac{(1 - 2\theta) \tau_{m-1}^2}{\tau_{m-1} + 1} \phi^{h,m-1}_{p,m} + \frac{(1 - 2\theta) \tau_{m-1}^2}{\tau_{m-1} + 1} \phi^{h,m-1}_{p,m-1}, \\
\tilde{\phi}^{h,m+1}_p &= \frac{2(1 - \theta) \tau_{m+1}}{\tau_{m-1} + 1} \tilde{\phi}^{h,m+1}_{p,m} - (1 - 2\theta) \tau_{m-1} \tilde{\phi}^{h,m}_{p,m} + \frac{(1 - 2\theta) \tau_{m-1}^2}{\tau_{m-1} + 1} \tilde{\phi}^{h,m-1}_{p,m} + \frac{(1 - 2\theta) \tau_{m-1}^2}{\tau_{m-1} + 1} \tilde{\phi}^{h,m-1}_{p,m-1},
\end{align*}
\]
then take them into (3.7) and (3.8), add them together, for \( \forall \, \vec{v}^h_f \in X_{fh}, \psi^h_p \in X_{ph} \) and \( q^h_p \in Q_{fh} \),
\[
\frac{1}{k_m} (A(\vec{u}^{h,m+1}_f, \vec{v}^h_f)_{\Omega_f} + \frac{g}{k_m} (A(\phi^h,m+1, \psi^h_p)_{\Omega_p} + a_{\Omega_f} (B(\vec{u}^{h,m+1}_f), \vec{v}^h_f) + a_{\Omega_p} (B(\phi^h,m+1), \psi^h_p) + b(\vec{v}^h_f, B(p^{h,m+1}_f)) \\
= ((1 - \theta) \vec{g}^{m+1}_f + \theta \vec{g}^m_f, \vec{v}^h_f)_{\Omega_f} + g((1 - \theta) \vec{g}^{m+1}_f + \theta \vec{g}^m_f, \psi^h_p)_{\Omega_p} \\
- c_{\Gamma} ((1 + (1 - \theta) \tau_m-1) \phi^h_p - (1 - \theta) \tau_m-1 \phi^h,m-1) \\
+ c_{\Gamma} ((1 + (1 - \theta) \tau_m-1) \vec{u}^{h,m}_f - (1 - \theta) \tau_m-1 \vec{u}^{h,m-1}_f, \vec{v}^h_f), \\
b(B(\vec{u}^{h,m+1}_f), q^h_p) = 0.
\]

Setting \( \vec{v}^h_p = 2k_m B(\vec{u}^{h,m+1}_f), \psi^h_p = 2k_m B(\phi^h,m+1) \) and \( q^h_p = 2k_m B(\phi^h,m+1) \). Looking back at the proof process of Theorem 3.1, we have the following equations to hold
\[
2(A(\vec{u}^{h,m+1}_f), B(\vec{u}^{h,m+1}_f))_{\Omega_f} \\
\geq C_{min} \left(D(\theta, \tau) \| \vec{u}^{h,m+1}_f \|_f^2 - H(\theta, \tau) \| \vec{u}^{h,m}_f \|_f^2 - E(\theta, \tau) \| \vec{u}^{h,m-1}_f \|_f^2 \\
- F(\theta, \tau) (\| \vec{u}^{h,m}_f \|_f \| \vec{u}^{h,m-1}_f \|_f - \| \vec{u}^{h,m}_f \|_f \| \vec{u}^{h,m-1}_f \|_f) \right),
\]
and
\[
2g(A(\phi^h,m+1), B(\phi^h,m+1))_{\Omega_p} \\
\geq gC_{min} \left(D(\theta, \tau) \| \phi^h,m+1 \|_p^2 - H(\theta, \tau) \| \phi^h,m \|_p^2 - E(\theta, \tau) \| \phi^h,m-1 \|_p^2 \\
- F(\theta, \tau) (\| \phi^h,m \|_p \| \phi^h,m-1 \|_p - \| \phi^h,m \|_p \| \phi^h,m-1 \|_p) \right),
\]
and
\[
2k_m a_{\Omega_f} (B(\vec{u}^{h,m+1}_f), B(\vec{u}^{h,m+1}_f)) \geq 2C_{\text{coke}} k_m \| B(\vec{u}^{h,m+1}_f) \|_{\mathcal{X}_f},
\]
and
\[
2k_m a_{\Omega_p} (B(\phi^h,m+1), B(\phi^h,m+1)) \geq 2gC_{\text{coke}} k_m \| B(\phi^h,m+1) \|_{\mathcal{X}_p},
\]
and
\[
2k_m ((1 - \theta) \vec{g}^{m+1}_f + \theta \vec{g}^m_f, B(\vec{u}^{h,m+1}_f)) \\
\leq \frac{2g^2 k_m}{\tilde{C}_{\text{coe}}} \| \vec{g}_f \|_{\mathcal{X}_f} + \frac{2(1 - \theta)^2 k_m}{\tilde{C}_{\text{coe}}} \| \vec{g}_f \|_{\mathcal{X}_f}^2 + \frac{\tilde{C}_{\text{coe}} k_m}{2} \| B(\phi^h,m+1) \|_{\mathcal{X}_f}^2,
\]
and
\[
2g k_m ((1 - \theta) \vec{g}^{m+1}_f + \theta \vec{g}^m_f, B(\phi^h,m+1)) \\
\leq \frac{2g^2 k_m}{\tilde{C}_{\text{coe}}} \| \phi^h \|_{\mathcal{X}_p}^2 + \frac{2g(1 - \theta)^2 k_m}{\tilde{C}_{\text{coe}}} \| \phi^h \|_{\mathcal{X}_p}^2 + \frac{g\tilde{C}_{\text{coe}} k_m}{2} \| B(\phi^h,m+1) \|_{\mathcal{X}_p}^2.
\]

Here, we need to analyze the interface item on the right-hand side. Using the Lemma 3.2 presented earlier,
and bringing in the appropriate parameters \( \varepsilon_1 = \frac{1}{C_{\text{coe}}} \) and \( \varepsilon_2 = \frac{\theta}{C_{\text{coe}}} \), we have

\[
\begin{align*}
- c_r (B(u_f^{h,m+1}_\tau, 1 + (1 - \theta)\tau_{m-1}) \phi_p^{h,m} - (1 - \theta)\tau_{m-1} \phi_p^{h,m-1}) \\
+ c_r ((1 + (1 - \theta)\tau_{m-1}) \tilde{u}_{f,m} - (1 - \theta)\tau_{m-1} \tilde{u}_{f,m-1}, B(\phi_p^{h,m+1}))
\end{align*}
\]

(3.21)

\[
\leq \frac{\hat{C}_{\text{coe}} k_m}{2} \|B(\tilde{u}_f^{h,m+1})\|_{X_p}^2 + 2C_1 h^{-1} k_m \|((1 + (1 - \theta)\tau_{m-1}) \phi_p^{h,m} - (1 - \theta)\tau_{m-1} \phi_p^{h,m-1})\|_p^2
\]

\[
+ g \frac{\hat{C}_{\text{coe}} k_m}{2} \|B(\phi_p^{h,m+1})\|_{X_p}^2 + 2gC_2 h^{-1} k_m \|((1 + (1 - \theta)\tau_{m-1}) \tilde{u}_f^{h,m} - (1 - \theta)\tau_{m-1} \tilde{u}_f^{h,m-1})\|_f^2
\]

\[
\leq \frac{\hat{C}_{\text{coe}} k_m}{2} \|B(\tilde{u}_f^{h,m+1})\|_{X_p}^2 + \frac{2g k m}{C_{\text{coe}}} \|((1 + (1 - \theta)\tau_{m-1}) \tilde{u}_f^{h,m} - (1 - \theta)\tau_{m-1} \tilde{u}_f^{h,m-1})\|_f^2
\]

\[
+ g \frac{\hat{C}_{\text{coe}} k_m}{2} \|B(\phi_p^{h,m+1})\|_{X_p}^2 + \frac{2k m}{C_{\text{coe}}} \|((1 + (1 - \theta)\tau_{m-1}) \tilde{u}_f^{h,m} - (1 - \theta)\tau_{m-1} \tilde{u}_f^{h,m-1})\|_f^2
\]

where the last inequality follows from properly chosen constant \( C_1 \) and \( C_2 \).

Combining the [3.15] and [3.21], and summing [3.14] over \( m = 1, 2, \ldots, N-1 \), we use the same method as the Theorem 3.1 and let \( k = \max_{1 \leq m \leq N-1} k_m \), we have

\[
C_{\min} \left( D(\theta, \tau) \|u_f^{\theta,N} \|_{f}^2 + E(\theta, \tau) \|u_f^{\theta,N-1} \|_{f}^2 - F(\theta, \tau) \|u_f^{\theta,N} \|_{f} \|u_f^{\theta,N-1} \|_{f} \right)
\]

\[
+ g C_{\min} \left( D(\theta, \tau) \|\phi_p^{\theta,N} \|_{p}^2 + E(\theta, \tau) \|\phi_p^{\theta,N-1} \|_{p}^2 - F(\theta, \tau) \|\phi_p^{\theta,N} \|_{p} \|\phi_p^{\theta,N-1} \|_{p} \right)
\]

\[
+ \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \left[ k_m \|B(u_f^{\theta,m+1})\|_{X_f} + g \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \left[ k_m \|B(\phi_p^{\theta,m+1})\|_{X_f} \right] \right]
\]

\[
\leq 2k^2 \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \|u_f^{\theta,m+1}\|_{X_f}^2 + 2k(1 - \theta)^2 \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \|u_f^{\theta,m+1}\|_{X_f}^2 + 2g k^2 \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \|\phi_p^{\theta,m+1}\|_{X_f}^2 + 2g k \|\phi_p^{\theta,m+1}\|_{p}^2
\]

\[
+ 2k \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \|((1 + (1 - \theta)\tau_{m-1}) \phi_p^{h,m} - (1 - \theta)\tau_{m-1} \phi_p^{h,m-1})\|_p^2
\]

\[
+ 2k \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \|((1 + (1 - \theta)\tau_{m-1}) \tilde{u}_f^{h,m} - (1 - \theta)\tau_{m-1} \tilde{u}_f^{h,m-1})\|_f^2
\]

\[
+ C_{\min} \left( D(\theta, \tau) \|\phi_p^{\theta,0} \|_{p}^2 + E(\theta, \tau) \|\phi_p^{\theta,0} \|_{p}^2 - F(\theta, \tau) \|\phi_p^{\theta,0} \|_{p} \|\phi_p^{\theta,0} \|_{p} \right)
\]

Then note that

\[-F(\theta, \tau) \|\phi_p^{\theta,N} \|_{p} \|\phi_p^{\theta,N-1} \|_{p} \geq -G(\theta, \tau) \|\phi_p^{\theta,N} \|_{p}^2 - E(\theta, \tau) \|\phi_p^{\theta,N-1} \|_{p}^2 \]

and

\[-F(\theta, \tau) \|\phi_p^{\theta,N} \|_{p} \|\phi_p^{\theta,N-1} \|_{p} \geq -G(\theta, \tau) \|\phi_p^{\theta,N} \|_{p}^2 - E(\theta, \tau) \|\phi_p^{\theta,N-1} \|_{p}^2 , \]

so we can get

\[
C_{\min} I(\theta, \tau) \|u_f^{\theta,N} \|_{f}^2 + g C_{\min} I(\theta, \tau) \|\phi_p^{\theta,N} \|_{p}^2 + \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \left[ k_m \|B(u_f^{\theta,m+1})\|_{X_f} \right] + g \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \left[ k_m \|B(\phi_p^{\theta,m+1})\|_{X_f} \right]
\]

\[
\leq 2k^2 \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \|u_f^{\theta,m+1}\|_{X_f}^2 + 2k(1 - \theta)^2 \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \|u_f^{\theta,m+1}\|_{X_f}^2 + 2g k^2 \hat{C}_{\text{coe}} \sum_{m=1}^{N-1} \|\phi_p^{\theta,m+1}\|_{X_f}^2 + 2g k \|\phi_p^{\theta,m+1}\|_{p}^2
\]

\[
+ \hat{C}_{\text{coe}} C_{\min} I(\theta, \tau) \|\phi_p^{\theta,m+1}\|_{p}^2 + 2k \hat{C}_{\text{coe}} C_{\min} I(\theta, \tau) \sum_{m=1}^{N-1} \left[ C_{\min} I(\theta, \tau) \|\tilde{u}_f^{\theta,m+1}\|_{f} \right]
\]

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Applying the discrete Gronwall inequality, we end the proof. □

4 Error estimates

In this section, we analyze the errors of variable time-stepping coupled and decoupled algorithms. For the sake of the later analysis, we define error functions:

\[ e^{h,m}_u = \tilde{u}^{h,m} - \tilde{u}^m = \tilde{u}^{h,m} - P^\mathbb{H}_h \tilde{u}^m + P^\mathbb{H}_h \tilde{u}^m - \tilde{u}^m = \eta^{h,m}_u, \]

\[ e^{h,m}_p = \tilde{p}^{h,m} - p^m = \tilde{p}^{h,m} - P^p h \tilde{p}^m + P^p h \tilde{p}^m - p^m = \eta^{h,m}_p, \]

\[ e^{h,m}_w = \tilde{w}^{h,m} - w^m = \tilde{w}^{h,m} - P^w h \tilde{w}^m + P^w h \tilde{w}^m - w^m = \eta^{h,m}_w, \]

\[ e^{h,m}_\phi = \phi^m - \phi^m = \phi^m - P^\phi \phi^m + P^\phi \phi^m - \phi^m = \eta^{h,m}_\phi. \]

Obviously, we have

\[ \|e^{h,m}_u\|_0 \leq Ch^2 \|\tilde{u}^h(t)\|_{H^2}, \quad \|e^{h,m}_w\|_X \leq Ch \|\tilde{u}^h(t)\|_{H^2}, \quad \|e^{h,m}_p\|_{L^2} \leq Ch \|\tilde{p}^h(t)\|_{H^1}, \]

\[ \|e^{h,m}_w\|_{L^2} \leq Ch \|\tilde{u}^h(t)\|_{H^2}, \quad \|e^{h,m}_p\|_{L^2} \leq Ch \|\phi^p(t)\|_{H^1}, \quad \|e^{h,m}_\phi\|_{L^2} \leq Ch \|\phi^p(t)\|_{H^2}. \] (4.1)

Note that \( \eta^{h,0}_u = 0, \eta^{h,0}_p = 0, \eta^{h,0}_w = 0 \) and \( \eta^{h,0}_\phi = 0 \).

Assume the solution satisfies the following regularity conditions:

\[ \tilde{u}_f \in L^\infty(0,T; X^2_f), \tilde{u}_{f,t} \in L^2(0,T; X^1_f) \cap L^\infty(0,T; L^2), \tilde{u}_{f,tt} \in L^2(0,T; L^2), \tilde{u}_{f,ttt} \in L^2(0,T; X'_f), \]

\[ \phi_p \in L^\infty(0,T; X^2_p), \phi_{p,t} \in L^2(0,T; X^1_p) \cap L^\infty(0,T; L^2), \phi_{p,tt} \in L^2(0,T; L^2), \phi_{p,ttt} \in L^2(0,T; X'_p). \] (4.2)

And the external force \( \tilde{g}_f \) and \( g_p \) also need to be satisfied

\[ \tilde{g}_{f,t} \in L^2(0,T; L^2), \tilde{g}_{f,tt} \in L^2(0,T; X'_f), g_{p,t} \in L^2(0,T; L^2), g_{p,tt} \in L^2(0,T; X'_p). \] (4.3)

First, we derive the following error estimate of variable time-stepping coupled algorithm.

**Theorem 4.1.** (second-order convergence for variable time-stepping coupled algorithm) Under the assumption of (4.2) and (4.3), for \( N \geq 2 \) we have the estimate

\[ I(\theta, \tau)\|e^N_2\|_0^2 + C_{soc} \sum_{m=1}^{N-1} [k_m \|B(e^{h,m+1}_2)\|_{X^3}^2] \leq C(\hat{k}^4 + h^4), \]

where \( 0 < \theta < \frac{1}{2} \), \( \tau_{\text{min}} \leq \tau \leq \tau_{\text{max}} \), \( \hat{k} = \max_{1 \leq m \leq N-1} \{ k_m + k_{m-1} \} \) and \( C \) is a positive constant.

**Proof.** First, let us multiply (2.12) by \( (1-\theta)^{m+1} - \theta^{m+1} \) at \( t_{m+1} \), \( t_m \) and \( t_{m-1} \), respectively. Then, subtracting the summation of three equations from (3.10), we use the error functions and the properties of the project operator (2.11), for \( \tilde{u}^h \in X_h \) and \( q^h \in Q_h \),

\[ \frac{1}{k_m} (A(\eta^{h,m+1}_u, \tilde{u}^h) + a(B(\eta^{h,m+1}_u, \tilde{u}^h) + b(\tilde{u}^h, B(\eta^{h,m+1}_p))) = - \left( \frac{1}{k_m} A(\tilde{u}^{m+1}_u, \tilde{u}^h) - B(\tilde{u}^{m+1}_u, \tilde{u}^h) + \frac{1}{k_m} A(\eta^{h,m+1}_u, \tilde{u}^h) + (\tilde{F}^{m+1}_u, \tilde{u}^h), \right. \]

\[ b(B(\eta^{h,m+1}_p), q^h) = 0. \] (4.4)
Setting \( \overline{w}^h = 2k_m B h,m+1 \), \( \varphi_f^h = 2k_m B h,m+1 \) and analysing each term of the first equation in (4.4). Similar to Theorem 3.1 we use Lemma 3.1 to handle the first term on the left-hand side, we have

\[
2 (A h,m+1, B h,m+1) \geq C_{\min} \left( D(\theta, \tau) \| h,m+1 \|_0^2 - H(\theta, \tau) \| h,m \|_0^2 - E(\theta, \tau) \| h,m-1 \|_0^2 \right) - F(\theta, \tau) (\| h,m+1 \|_0 \| h,m \|_0 - \| h,m \|_0 \| h,m-1 \|_0) \right).
\]

Then using the coercivity of the bilinear form \( a(\cdot, \cdot) \), the second term on the left can be handled as

\[
2k_m a (B h,m+1, B h,m+1) \geq 2C_{\text{coer}} k_m \| B h,m+1 \| X^2.
\]

Next, we consider the right side of (4.4). For the first term on the right-hand side, we use the Taylor expansion with the integral remainder,

\[
\overline{u}^m = \overline{u}^{m+1} - k_m \overline{u}^m + \frac{k_m^2}{2} \overline{u}^{m+1} + \frac{1}{2} \int_{t^{m+1}}^{t^m} (t^m - t)^2 \overline{u}_{tt} dt,
\]

\[
\overline{u}^{m+1} = \overline{u}^{m+1} - (k_m + k_m^2) \overline{u}_{tt}^m + \frac{(k_m + k_m^2)^2}{2} \overline{u}^{m+1} + \frac{1}{2} \int_{t^{m+1}}^{t^{m-1}} (t^m - t)^2 \overline{u}_{tt} dt,
\]

\[
\overline{u}^m = \overline{u}^{m+1} - k_m \overline{u}^m - \int_{t^{m+1}}^{t^m} (t^m - t) \overline{u}_{tt} dt,
\]

\[
\overline{u}^{m+1} = \overline{u}_{tt}^m + (k_m + k_m^2) \overline{u}_{tt}^m - \int_{t^{m+1}}^{t^{m-1}} (t^m - t) \overline{u}_{tt} dt.
\]

So we get

\[
\frac{1}{k_m} A (\overline{u}^{m+1}) - B (\overline{u}^{m+1}) = \frac{(1 - 2\theta) \tau_{m-1} + 1}{2k_m} \int_{t^m}^{t^{m+1}} (t^m - t)^2 \overline{u}_{tt} dt - \frac{(1 - 2\theta) \tau_{m-1}}{2k_m \tau_{m-1} + 1} \int_{t^{m+1}}^{t^{m-1}} (t^m - t)^2 \overline{u}_{tt} dt
\]

\[
- ((1 - \theta)(1 - 2\theta) \tau_{m-1} - \theta) \int_{t^m}^{t^{m+1}} (t^m - t) \overline{u}_{tt} dt - \frac{(1 - \theta)(1 - 2\theta) \tau_{m-1}^2}{\tau_{m-1} + 1} \int_{t^{m+1}}^{t^{m-1}} (t^m - t) \overline{u}_{tt} dt.
\]

By using Cauchy-Schwarz inequality,

\[
\left( \int_{t^m}^{t^{m+1}} (t^m - t)^2 \overline{u}_{tt} dt \right)^2 \leq \frac{k_m^5}{5} \int_{t^m}^{t^{m+1}} \overline{u}_{tt}^2 dt,
\]

\[
\left( \int_{t^{m+1}}^{t^{m-1}} (t^m - t)^2 \overline{u}_{tt} dt \right)^2 \leq \frac{(k_m + k_m^2)^5}{5} \int_{t^{m+1}}^{t^{m-1}} \overline{u}_{tt}^2 dt,
\]

\[
\left( \int_{t^m}^{t^{m+1}} (t^m - t) \overline{u}_{tt} dt \right)^2 \leq \frac{k_m^3}{3} \int_{t^m}^{t^{m+1}} \overline{u}_{tt}^2 dt,
\]

\[
\left( \int_{t^{m+1}}^{t^{m-1}} (t^m - t) \overline{u}_{tt} dt \right)^2 \leq \frac{(k_m + k_m^2)^3}{3} \int_{t^{m+1}}^{t^{m-1}} \overline{u}_{tt}^2 dt,
\]

the first term on the right can be handled as

\[
2k_m \left( \frac{1}{k_m} A (\overline{u}^{m+1}) - B (\overline{u}^{m+1}), B (h,m+1) \right) \leq \frac{3k_m}{C_{\text{coer}}} \| A (\overline{u}^{m+1}) - B (\overline{u}^{m+1}) \| X^2 + \frac{C_{\text{coer}} k_m}{3} \| B (h,m+1) \| X^2
\]

\[
\leq \frac{3C_{\text{coer}} k_m}{3} \| B (h,m+1) \| X^2 + \frac{(k_m + k_m^2)^3}{C_{\text{coer}}} \left( (1 - \theta)^2 \tau_{m-1} + ((1 - 2\theta) \tau_{m-1} + 1)^2 \right) \int_{t^{m+1}}^{t^{m-1}} \| \overline{u}_{tt} \| X^2 dt.
\]
In the same way, for the second term on the right side, we use the Taylor expansion with the integral remainder,
\[ \tilde{\mathbf{u}}^m = \mathbf{u}^{m+1} + \int_{t_{m+1}}^{t_m} \tilde{\mathbf{u}}_t dt, \]
\[ \tilde{\mathbf{u}}^{m-1} = \mathbf{u}^{m+1} + \int_{t_{m+1}}^{t_m} \tilde{\mathbf{u}}_t dt. \]

Then
\[ \frac{1}{k_m} A(\zeta_{2}^{h,m+1}) = \frac{1}{k_m} [(P_{h}^2 - I) A(\tilde{\mathbf{u}}^{m+1})] \]
\[ = \frac{1}{k_m} \left[ (1 - 2\theta) \tau_{m-1} + 1 \int_{t_{m+1}}^{t_m} (P_{h}^2 - I) \tilde{\mathbf{u}}_t dt - \frac{(1 - 2\theta) \tau_{m-1}^2}{\tau_{m-1} + 1} \int_{t_{m+1}}^{t_m} (P_{h}^2 - I) \tilde{\mathbf{u}}_t dt \right]. \]

Thus, we have
\[ 2k_m \left( \frac{1}{k_m} A(\zeta_{2}^{h,m+1}), B(\eta_{2}^{h,m+1}) \right) \]
\[ \leq C_{\text{coce}} k_m^{\frac{1}{3}} \| B(\eta_{2}^{h,m+1}) \|_{X}^{2} X + \frac{3k_m}{C_{\text{coce}}} \| \frac{1}{k_m} A(\zeta_{2}^{h,m+1}) \|_{X}. \]
(4.8)

Similarly, for the third term on the right,
\[ \tilde{F}^{m+1} = \tilde{F}^m + k_m \tilde{F}_t^m - \int_{t_{m+1}}^{t_m} (t^{m+1} - t) \tilde{F}_t dt, \]
\[ \tilde{F}^{m-1} = \tilde{F}^m - k_{m-1} \tilde{F}_t^m - \int_{t_{m+1}}^{t_m} (t^{m+1} - t) \tilde{F}_t dt, \]
then
\[ S(\tilde{F}^{m+1}) = (1 - \theta)(1 - 2\theta) \tau_{m-1} + 1 \int_{t_{m+1}}^{t_m} (m+1 - t) \tilde{F}_t dt - \frac{(1 - \theta)(1 - 2\theta) \tau_{m-1}^2}{\tau_{m-1} + 1} \int_{t_{m+1}}^{t_m} (t^{m+1} - t) \tilde{F}_t dt. \]

Thus, we have
\[ 2k_m \left( S(\tilde{F}^{m+1}), B(\eta_{2}^{h,m+1}) \right) \]
\[ \leq C_{\text{coce}} k_m^{\frac{1}{3}} \| B(\eta_{2}^{h,m+1}) \|_{X}^{2} X + \frac{k_m}{C_{\text{coce}}} \| S(\tilde{F}^{m+1}) \|_{X}. \]
(4.9)

Combining the (4.5)-(4.9) and sum the (4.4) over \( m = 1, 2, ..., N - 1 \), and we use the same method as Theorem 3.1. Let \( k = \max_{1 \leq m \leq N - 1} \{ k_{m} + k_{m-1} \} \), we have
\[ C_{\text{coce}} \left[ \frac{1}{\tau_{m-1}} \int_{t_{m+1}}^{t_m} \| \tilde{\mathbf{u}}_t \|_{X}^2 dt + \frac{1}{\tau_{m-1}} \int_{t_{m+1}}^{t_m} \| \tilde{\mathbf{F}}_t \|_{X}^2 dt \right] \]
\[ \leq \sum_{m=1}^{N-1} \left[ \frac{\tilde{k}^4}{C_{\text{coce}}} \left( (1 - \theta)^2 \tau_{m-1}^2 + ((1 - 2\theta) \tau_{m-1} + 1)^2 \right) \int_{t_{m+1}}^{t_m} \| \tilde{\mathbf{u}}_t \|_{X}^2 dt \right. \]
\[ + \left. \frac{\tilde{k}^4 (1 - \theta)^2 (1 - 2\theta)^2 \tau_{m-1}^2 \tau_{m-1}^2}{C_{\text{coce}} (\tau_{m-1} + 1)^2} \int_{t_{m+1}}^{t_m} \| \tilde{\mathbf{F}}_t \|_{X}^2 dt \right] \]
\[ + \frac{3((1 - 2\theta) \tau_{m-1} + 1)^2 + 3(1 - 2\theta)^2 \tau_{m-1}^4}{C_{\text{coce}}} \int_{t_{m+1}}^{t_m} \| (P_{h}^2 - I) \tilde{\mathbf{u}}_t \|_{X}^2 dt \right]. \]

Finally, using the triangle inequality, we end the proof.
Then, we derive the following error estimate of variable time-stepping decoupled algorithm.

**Theorem 4.2.** (second-order convergence for the variable time-stepping decoupled algorithm) Under the assumption of (4.2) and (4.3), for $N \geq 2$ we have the estimate

$$I(\theta, \tau)|h_{\hat{u}_f}^N|_f^2 + gI(\theta, \tau)|h_{\phi_p}^N|_p^2 + C_{\text{coe}} \sum_{m=1}^{N-1} [k_m\|B(c_{\hat{u}_f}^{m+1})\|_X^2 + gC_{\text{coe}} \sum_{m=1}^{N-1} [k_m\|B(e_{\phi_p}^{m+1})\|_X^2] \leq C(k^4 + h^4),$$

where $0 < \theta < \frac{1}{2}$, $\tau_{\min} \leq \tau \leq \tau_{\max}$, $\hat{k} = \max_{1 \leq m \leq N-1} \{k_m + k_{m-1}\}$ and $C$ is a positive constant.

**Proof.** First, let us multiply (2.12) by $\frac{(1-\theta)^2 \tau_{m-1} + (1-\theta)}{\tau_{m-1} + 4}$ at $t_{m+1}$, $t_m$ and $t_{m-1}$, respectively. Then, subtracting the summation of three equations from (3.14), we use the error functions and the properties of the project operator (2.16), for $\forall p\tilde{f}_j \in X_{fh}, \phi_p^h \in X_{ph}$ and $\bar{q}^h \in Q_{fh}$,

\[
\begin{align*}
\frac{1}{k_m}(A(\tilde{u}_f^{m+1}, \tilde{u}_f^{m+1})_f)_{\Omega_f} + \frac{1}{k_m}(A(\eta_{\phi_p}^{m+1}, \eta_{\phi_p}^{m+1})_p)_{\Omega_p} \\
+ a_{\Omega_f}(B(\eta_{\phi_p}^{m+1}, \phi_p^h)) + a_{\Omega_p}(B(\eta_{\phi_p}^{m+1}, \phi_p^h)) + b(\phi_{p,t}^{m+1}, \phi_p^h)
\end{align*}
\]

\[
= -\left( \frac{1}{k_m} A(\tilde{u}_f^{m+1} - B(\bar{u}_f^{m+1}), \tilde{v}_f^{m+1})_f + g(\frac{1}{k_m} A(\phi_p^{m+1} - B(\bar{u}_f^{m+1}), \phi_p^{h})_p + b(\phi_{p,t}^{m+1}, \phi_p^h))_p \right. \\
+ \left( \frac{1}{k_m} A(\zeta_{\phi_p}^{m+1}, \tilde{v}_f^{m+1})_f + g(\frac{1}{k_m} A(\phi_p^{m+1}, \phi_p^{h})_p + b(\phi_{p,t}^{m+1}, \phi_p^h))_p \right.
\]

\[
+ S(\bar{u}_f^{m+1}, \tilde{v}_f^{m+1})_f + g(S(\phi_p^{m+1}, \phi_p^{h})_p + b(\phi_{p,t}^{m+1}, \phi_p^h))_p
\]

\[
- c_{\Omega_f}(\tilde{v}_f^{m+1} - (1 + (1-\theta)\tau_{m-1})\phi_{p,m-1}^h - (1-\theta)\tau_{m-1}\phi_{p,m-1}^h - B(\phi_p^{m+1}))_f \\
+ c_{\Omega_p}(\tilde{v}_f^{m+1} - (1 + (1-\theta)\tau_{m-1})\phi_{p,m-1}^h - (1-\theta)\tau_{m-1}\phi_{p,m-1}^h - B(\phi_p^{m+1}))_p
\]

Setting $\tilde{v}_f^h = 2k_mB(\eta_{\phi_p}^{h,m+1}), \psi_p^h = 2k_mB(\eta_{\phi_p}^{h,m+1})$ and $\bar{q}_f^h = 2k_mB(\eta_{\phi_p}^{h,m+1})$. Review the proof process of Theorem 4.1 the following equations hold

\[
2(A(\eta_{\phi_p}^{h,m+1}, B(\eta_{\phi_p}^{h,m+1}))_f \\
\geq C_{\min} \left( D(\theta, \tau)|h_{\hat{u}_f}^{h,m+1}|_f^2 - H(\theta, \tau)||h_{\hat{u}_f}^{h,m}|_f^2 - E(\theta, \tau)|h_{\hat{u}_f}^{h,m-1}|_f^2 \\
- F(\theta, \tau)(||h_{\hat{u}_f}^{h,m}|_f||h_{\hat{u}_f}^{h,m-1}|_f - ||h_{\hat{u}_f}^{h,m-1}|_f) \right),
\]

\[
2g(A(\eta_{\phi_p}^{h,m+1}, B(\eta_{\phi_p}^{h,m+1}))_p \\
\geq gC_{\min} \left( D(\theta, \tau)|h_{\phi_p}^{h,m+1}|_p^2 - H(\theta, \tau)||h_{\phi_p}^{h,m}|_p^2 - E(\theta, \tau)|h_{\phi_p}^{h,m-1}|_p^2 \\
- F(\theta, \tau)(||h_{\phi_p}^{h,m}|_p||h_{\phi_p}^{h,m-1}|_p - ||h_{\phi_p}^{h,m-1}|_p) \right),
\]

\[
2k_m a_{\Omega_f}(B(\eta_{\phi_p}^{h,m+1}, B(\eta_{\phi_p}^{h,m+1})) \geq 2C_{\text{coe}}k_m\|B(\eta_{\phi_p}^{h,m+1})\|_X^2,
\]

and

\[
2k_m a_{\Omega_p}(B(\eta_{\phi_p}^{h,m+1}, B(\eta_{\phi_p}^{h,m+1})) \geq 2C_{\text{coe}}k_m\|B(\eta_{\phi_p}^{h,m+1})\|_X^2,
\]
and
\[
2k_m \left( \frac{1}{k_m} A(\tilde{u}_f^{m+1}) - B(\tilde{u}_f^{m+1}, B(\eta_n^{h,m+1})) \right)_{\Omega_f} \\
\leq \frac{C_{\text{coec}}k_m}{6} \|B(\eta_n^{h,m+1})\|^2_{X_f} + \frac{2(k_m + k_{m-1})^4}{C_{\text{coec}}} \left[ \frac{(1 - \theta)^2\tau_m^{m-1} + ((1 - 2\theta)\tau_{m-1} + 1)^2}{4} \right]^{\frac{m+1}{\tau_{m-1}}} \|\tilde{u}_{tt}\|^2_{X_f} dt,
\]

and
\[
2g_m \left( \frac{1}{k_m} A(\phi_p^{m+1}) - B(\phi_p^{m+1}, B(\phi_p^{m+1})) \right)_{\Omega_p} \\
\leq g \frac{C_{\text{coec}}k_m}{6} \|B(\phi_p^{m+1})\|^2_{X_p} + \frac{2(g_k + k_{m-1})^4}{C_{\text{coec}}} \left[ \frac{(1 - \theta)^2\tau_m^{m-1} + ((1 - 2\theta)\tau_{m-1} + 1)^2}{4} \right]^{\frac{m+1}{\tau_{m-1}}} \|\phi_{p,tt}\|^2_{X_p} dt,
\]

and
\[
2k_m \left( \frac{1}{k_m} A(\zeta_{\phi_p}^{h,m+1}), B(\eta_n^{h,m+1}) \right)_{\Omega_f} \\
\leq \frac{C_{\text{coec}}k_m}{6} \|B(\eta_n^{h,m+1})\|^2_{X_f} + 6((1 - \theta)^2\theta + 1)^2 + 6(1 - 2\theta)\theta^4 \tau_m^{m-1} (\tau_{m-1} + 1)^2 \|\phi_{p,tt}\|^2_{X_p} dt,
\]

and
\[
2g_m \left( \frac{1}{k_m} A(\phi^{h,m+1}), B(\phi^{h,m+1}) \right)_{\Omega_p} \\
\leq g \frac{C_{\text{coec}}k_m}{6} \|B(\phi^{h,m+1})\|^2_{X_p} + 6g((1 - \theta)^2\theta + 1)^2 + 6g(1 - 2\theta)\theta^4 \|\phi_{p,tt}\|^2_{X_p} dt,
\]

and
\[
2k_m \left( \frac{1}{k_m} A(\tilde{g}_f^{m+1}), B(\eta_n^{h,m+1}) \right)_{\Omega_f} \\
\leq \frac{C_{\text{coec}}k_m}{6} \|B(\eta_n^{h,m+1})\|^2_{X_f} + 2(k_m + k_{m-1})^4 (1 - \theta)^2 (1 - 2\theta)\theta^2 (\tau_m^{m-1} + 1)^2 \|\tilde{g}_{tt}\|^2_{X_f} dt,
\]

and
\[
2g_m \left( \frac{1}{k_m} A(\phi^{m+1}), B(\eta_n^{h,m+1}) \right)_{\Omega_p} \\
\leq g \frac{C_{\text{coec}}k_m}{6} \|B(\eta_n^{h,m+1})\|^2_{X_p} + 2g(k_m + k_{m-1})^4 (1 - \theta)^2 (1 - 2\theta)\theta^2 (\tau_m^{m-1} + 1)^2 \|\phi_{p,tt}\|^2_{X_p} dt.
\]

Next, we mainly analyze the interface terms at the right-hand side, and through some simply making up terms, the interface terms can be written as
\[
- 2k_m c_T \left( B(\eta_n^{h,m+1}), (1 + (1 - \theta)\tau_{m-1})\phi_p^{h,m} - (1 - \theta)\tau_{m-1}\phi_p^{h,m-1} - B(\phi_p^{m+1}) \right) \\
+ 2k_m c_T ((1 + (1 - \theta)\tau_{m-1})\tilde{u}_f^{h,m} - (1 - \theta)\tau_{m-1}\tilde{u}_f^{h,m-1} - B(\tilde{u}_f^{m+1}), B(\eta_n^{h,m+1})) \\
= - 2k_m c_T \left( B(\eta_n^{h,m+1}), B(\eta_n^{h,m+1}) \right) + 2k_m c_T \left( B(\eta_n^{h,m+1}), B(\eta_n^{h,m+1}) \right) \\
- 2k_m c_T \left( B(\eta_n^{h,m+1}), W(\phi_p^{h,m+1}) \right) + 2k_m c_T \left( W(\eta_n^{h,m+1}), B(\eta_n^{h,m+1}) \right) \\
- 2k_m c_T \left( B(\eta_n^{h,m+1}), \zeta(\phi_p^{h,m+1}) \right) + 2k_m c_T \left( \zeta(\eta_n^{h,m+1}), B(\phi_p^{h,m+1}) \right) \\
- 2k_m c_T \left( B(\eta_n^{h,m+1}), \phi_p^{m+1} \right) + 2k_m c_T \left( W(\phi_p^{m+1}), B(\phi_p^{h,m+1}) \right).
\]
By using Lemma 3.2, and taking the appropriate $\varepsilon_3 = \varepsilon_5 = \frac{3}{C_{\text{coe}}}$ and $\varepsilon_4 = \varepsilon_6 = \frac{3g}{C_{\text{coe}}}$ into them, we have

$$-2k_m\varepsilon (B(\eta_{h,n+1}^{h,m}), B(\phi_{p}^{h,m+1})) + 2k_m\varepsilon (B(\phi_{p}^{h,m+1}), B(\eta_{p}^{h,m+1}))$$

$$-2k_m\varepsilon (B(\eta_{h,n+1}^{h,m}), W(\phi_{p}^{h,m+1})) + 2k_m\varepsilon (W(\phi_{p}^{h,m+1}), B(\eta_{p}^{h,m+1}))$$

\[
\leq \frac{C_{\text{coe}}km}{6} \|B(\eta_{h,n+1}^{h,m})\|^2_{X_p} + \frac{6C_{\text{coe}}km}{6} \|B(\phi_{p}^{h,m+1})\|^2_{p} + \frac{gC_{\text{coe}}km}{6} \|B(\eta_{p}^{h,m+1})\|^2_{X_p} + \frac{6gC_{\text{coe}}km}{6} \|B(\phi_{p}^{h,m+1})\|^2_{p} + \frac{6gC_{\text{coe}}km}{6} \|W(\eta_{h,n+1}^{h,m})\|^2_{f} + \frac{6k_m}{C_{\text{coe}}} \|W(\phi_{p}^{h,m+1})\|^2_{f},
\]

where the last inequality follows from properly chosen constant $C_3, C_4, C_5$ and $C_6$.

For the next four terms, we use the Taylor expansion with the integral remainder

$$\phi_{p}^{m+1} = \phi_{p}^{m} + k_m\phi_{p,t}^{m} - \int_{t_m}^{t_{m+1}} (t_{m+1} - t)\phi_{p,tt}dt,$$

$$\phi_{n}^{m-1} = \phi_{n}^{m} - k_m\phi_{n,t}^{m} - \int_{t_m}^{t_{m-1}} (t_{m-1} - t)\phi_{n,tt}dt,$$

then

$$W(\phi_{p}^{m+1}) = -2(1 - \theta)^2 \tau_{m-1} + (1 - \theta)\tau_{m-1} - \int_{t_m}^{t_{m+1}} (t_{m+1} - t)\phi_{p,tt}dt$$

$$+ 2(1 - \theta)^2 \tau_{m-1} + (1 - \theta)\tau_{m-1} - \int_{t_m}^{t_{m+1}} (t_{m-1} - t)\phi_{n,tt}dt,$$

similarly, we have

$$W(\tilde{\phi}_{f}^{m+1}) = -2(1 - \theta)^2 \tau_{m-1} + (1 - \theta)\tau_{m-1} - \int_{t_m}^{t_{m+1}} (t_{m+1} - t)\tilde{\phi}_{f,tt}dt$$

$$+ 2(1 - \theta)^2 \tau_{m-1} + (1 - \theta)\tau_{m-1} - \int_{t_m}^{t_{m+1}} (t_{m-1} - t)\tilde{\phi}_{f,tt}dt.$$

By using Lemma 3.2 and taking $\varepsilon_7 = \frac{3}{C_{\text{coe}}}$ and $\varepsilon_8 = \frac{3g}{C_{\text{coe}}}$ into them, we get

$$-2k_m\varepsilon (B(\eta_{h,n+1}^{h,m}), W(\phi_{p}^{h,m+1})) + 2k_m\varepsilon (W(\phi_{p}^{h,m+1}), B(\eta_{p}^{h,m+1}))$$

\[
\leq \frac{C_{\text{coe}}km}{6} \|B(\eta_{h,n+1}^{h,m})\|^2_{X_p} + \frac{6C_{\text{coe}}km}{6} \|W(\phi_{p}^{h,m+1})\|^2_{p} + \frac{gC_{\text{coe}}km}{6} \|B(\eta_{p}^{h,m+1})\|^2_{X_p} + \frac{6gC_{\text{coe}}km}{6} \|W(\phi_{p}^{h,m+1})\|^2_{p} + \frac{6k_m}{C_{\text{coe}}} \|W(\eta_{h,n+1}^{h,m})\|^2_{f} + \frac{6gC_{\text{coe}}km}{6} \|W(\phi_{p}^{h,m+1})\|^2_{f},
\]

where the inequality follows from properly chosen constant $C_7$ and $C_8$. 

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Combining the above analysis, the interface terms can be handled as

\[-2k_m \mathcal{C}_{m,n} I(\theta, \tau) |\zeta_{u_{\alpha}}^{\text{h}, n+1}|^2 + 2k_m \mathcal{C}_{m,n} I(\theta, \tau) |\zeta_{\phi_p}^{\text{h}, n+1}|^2 + \mathcal{C}_{\text{coe}} \sum_{m=1}^{N-1} [k_m |B(\zeta_{u_{\alpha}}^{\text{h}, n+1})|^2]_{X_f} + g\mathcal{C}_{\text{coe}} \sum_{m=1}^{N-1} [k_m |b(\zeta_{\phi_p}^{\text{h}, n+1})|^2]_{X_p} \]

\[\leq \sum_{m=1}^{N-1} \frac{2k^4}{\mathcal{C}_{\text{coe}}} \left( (1 - \theta)^2 \tau_{m-1}^2 + (1 - 2\theta)\tau_{m-1} + 1 \right)^2 \int_{t_{m-1}}^{t_{m+1}} \| \bar{u}_{f,tt} \|_{X_f}^2 dt + \frac{2gk^4}{\mathcal{C}_{\text{coe}}} \left( (1 - \theta)^2 \tau_{m-1}^2 + (1 - 2\theta)\tau_{m-1} + 1 \right)^2 \int_{t_{m-1}}^{t_{m+1}} \| \phi_{p,tt} \|_{X_p}^2 dt + \frac{6g(1 - 2\theta)\tau_{m-1} + 1)^2 + 6g(1 - 2\theta)\tau_{m-1}^2 \int_{t_{m-1}}^{t_{m+1}} \| (P_{h}^{\bar{u}_{f}} - I) \bar{u}_{f,t} \|_{X_f}^2 dt + \frac{2k^4(1 - \theta)^2(1 - 2\theta)^2(\tau_{m-1}^2 + 1)^2 \tau_{m-1} \int_{t_{m-1}}^{t_{m+1}} \| \bar{g}_{f,tt} \|_{X_f}^2 dt + \frac{2gk^4(1 - \theta)^2(1 - 2\theta)^2(\tau_{m-1}^2 + 1)^2 \tau_{m-1} \int_{t_{m-1}}^{t_{m+1}} \| \phi_{p,t} \|_{X_p}^2 dt + \frac{2k^4(2(1 - \theta)\tau_{m-1} + 1)^2(\tau_{m-1}^2 + 1) \int_{t_{m-1}}^{t_{m+1}} \| \bar{u}_{f,tt} \|_{X_f}^2 dt + \frac{2gk^4(2(1 - \theta)\tau_{m-1} + 1)^2(\tau_{m-1}^2 + 1) \int_{t_{m-1}}^{t_{m+1}} \| \phi_{p,t} \|_{X_p}^2 dt + \frac{6gk}{\mathcal{C}_{\text{coe}}} \| B(\zeta_{\phi_p}^{\text{h}, n+1}) \|_p + \frac{6k}{\mathcal{C}_{\text{coe}}} \| B(\zeta_{u_{\alpha}}^{\text{h}, n+1}) \|_f + \frac{6gk}{\mathcal{C}_{\text{coe}}} \| W(\zeta_{\phi_p}^{\text{h}, n+1}) \|_p + \frac{6k}{\mathcal{C}_{\text{coe}}} \| W(\zeta_{u_{\alpha}}^{\text{h}, n+1}) \|_f \right].

Finally, using the triangle inequality, we end the proof.

5 Numerical experiments

In this section, we do two numerical experiments. In the first test, we verify the effectiveness of the variable time-stepping coupled and decoupled algorithms by three different sets of variation rules for time steps $k_1^m$, $k_2^m$, $k_3^m$. In order to show the convergence orders of the variable time-stepping coupled and decoupled algorithms,
we use the convergence orders results of constant time-stepping algorithm to demonstrate. Since in the paper we consider $0 < \theta < 1/2$, here we choose $\theta = 1/3$ to verify convergence and efficiency. The convergence orders are increased from the first order to the second order, and the decoupled algorithm is more efficient than the coupled algorithm in the second test. The following numerical experiments are implemented using the Software package FreeFEM++, and we set all the physical parameters $n$, $\rho$, $g$, $\nu$, $K$, $S_0$ and $\alpha$ are equal to 1, and the initial conditions, boundary conditions and the source terms follow from the exact solution.

![Figure 2: Speed contours and velocity streamlines for Linear Multistep method plus time filter with $k_{1m}^1$.](image1)

- (a) Coupled algorithm with $\theta = 0.2$
- (b) Coupled algorithm with $\theta = 0.3$
- (c) Coupled algorithm with $\theta = 0.4$
- (d) Decoupled scheme with $\theta = 0.2$
- (e) Decoupled algorithm with $\theta = 0.3$
- (f) Decoupled algorithm with $\theta = 0.4$

![Figure 3: Speed contours and velocity streamlines for Linear Multistep method plus time filter with $k_{1m}^2$.](image2)

- (a) Coupled algorithm with $\theta = 0.2$
- (b) Coupled algorithm with $\theta = 0.3$
- (c) Coupled scheme with $\theta = 0.4$
- (d) Decoupled algorithm with $\theta = 0.2$
- (e) Decoupled algorithm with $\theta = 0.3$
- (f) Decoupled algorithm with $\theta = 0.4$
5.1 Test of the effectiveness for the variable time-stepping algorithms

Here we use the numerical example from [35], let the computational domain \( \Omega \) be composed of \( \Omega_f = (0, \pi) \times (0,1) \) and \( \Omega_p = (0, \pi) \times (-1, 0) \) with the interface \( \Gamma = (0, \pi) \times 0 \). The Taylor-Hood element (P2-P1) and the piecewise quadratic polynomials (P2) are used for the free fluid equation and the porous media flow equation. The exact solution is given by

\[
\begin{align*}
\vec{u}_f &= \left[ \frac{1}{\pi} \sin(2\pi y)\cos(x)e^t, (-2 + \frac{1}{\pi^2} \sin^2(\pi y))\sin(x)e^t \right], \\
p_f &= 0, \\
\phi_p &= (e^y - e^{-y})\sin(x)e^t.
\end{align*}
\]

For this test, we change the time step size to see the effect on the experimental results and set the diameters \( h = 1/100 \) for space triangulation. We use coupled and decoupled algorithms to this test problem for 40 time steps and refer to the time step size \( k_{1m}^1, k_{2m}^2, k_{3m}^3 \) similar to that in [36]:

\[
k_{1m}^1 = 0.01 + 0.05t_m, \quad m \geq 0,
\]

and

\[
k_{2m}^2 = \begin{cases} 
0.01, & 0 \leq m \leq 10, \\
0.01 + 0.05\sin(10t_m), & m > 10,
\end{cases}
\]

and

\[
k_{3m}^3 = 0.1 - 0.05t_m, \quad m \geq 0.
\]

Figures 2-4 show speed contours and velocity streamlines of coupled and decoupled Linear Multi-step methods plus time filters for \( \theta = 0.2, 0.3, 0.4 \) with different time step size \( k_{1m}^1, k_{2m}^2, k_{3m}^3 \), respectively. From these figures, we can see that these variable time-stepping algorithms can effectively simulate fluid motion regardless of whether the time step increases or decreases.

5.2 Test of the convergence and efficiency for the variable time-stepping schemes

Here we use the example from [19], considering the model problem on \( \Omega_f = (0.1) \times (1,2) \) and \( \Omega_p = (0.1) \times (0,1) \) with the interface \( \Gamma = (0,1) \times 1 \). We use the well-known MINI elements(P1b-P1) for the fluid
Table 1: The convergence order of coupled Linear Multi-step method at time $T = 1$, with varying time step $\Delta t$ but fixed mesh size $h = \frac{1}{8}$.

| $\Delta t$ | $\|\tilde{u}_f^{h,\Delta t} - \tilde{u}_f^{h,\Delta t/2}\|_{L^2}$ | $\rho_{u_f}$ | $\|p_f^{h,\Delta t} - p_f^{h,\Delta t/2}\|_{L^2}$ | $\rho_{p_f}$ | $\|\phi_p^{h,\Delta t} - \phi_p^{h,\Delta t/2}\|_{L^2}$ | $\rho_{\phi_p}$ | CPU (s) |
|-----------|-------------------------------------------------|----------|----------------------------------|---------|----------------------------------|----------|--------|
| $\frac{1}{8}$ | 2.00123 | 0.00348072 | 2.00231 | 0.000207141 | 2.3728 | 0.668 |
| $\frac{1}{16}$ | 1.99806 | 0.00173835 | 1.99941 | 0.000420741 | 2.0028 | 0.762 |
| $\frac{1}{32}$ | 1.99853 | 0.000868654 | 2.00066 | 0.000537458 | 1.9936 | 0.742 |
| $\frac{1}{64}$ | 1.99916 | 0.00044183 | 2.00035 | 0.000537458 | 2.0049 | 0.732 |

Table 2: The convergence order of coupled Linear Multi-step method plus time filters at time $T = 1$, with varying time step $\Delta t$ but fixed mesh size $h = \frac{1}{8}$.

| $\Delta t$ | $\|\tilde{u}_f^{h,\Delta t} - \tilde{u}_f^{h,\Delta t/2}\|_{L^2}$ | $\rho_{u_f}$ | $\|p_f^{h,\Delta t} - p_f^{h,\Delta t/2}\|_{L^2}$ | $\rho_{p_f}$ | $\|\phi_p^{h,\Delta t} - \phi_p^{h,\Delta t/2}\|_{L^2}$ | $\rho_{\phi_p}$ | CPU (s) |
|-----------|-------------------------------------------------|----------|----------------------------------|---------|----------------------------------|----------|--------|
| $\frac{1}{8}$ | 4.17159 | 0.00074627 | 3.86373 | 0.000530548 | 4.1645 | 0.975 |
| $\frac{1}{16}$ | 4.08765 | 0.0001932 | 3.99793 | 8.41744 | 0.00383 | 1.973 |
| $\frac{1}{32}$ | 4.04433 | 4.8325e-05 | 3.99856 | 2.06116 | 0.00423 | 4.045 |
| $\frac{1}{64}$ | 4.022 | 1.20856e-05 | 3.99917 | 5.09989 | 0.00421 | 8.383 |

Table 3: The convergence order of decoupled Linear Multi-step method at time $T = 1$, with varying time step $\Delta t$ but fixed mesh size $h = \frac{1}{8}$.

| $\Delta t$ | $\|\tilde{u}_f^{h,\Delta t} - \tilde{u}_f^{h,\Delta t/2}\|_{L^2}$ | $\rho_{u_f}$ | $\|p_f^{h,\Delta t} - p_f^{h,\Delta t/2}\|_{L^2}$ | $\rho_{p_f}$ | $\|\phi_p^{h,\Delta t} - \phi_p^{h,\Delta t/2}\|_{L^2}$ | $\rho_{\phi_p}$ | CPU (s) |
|-----------|-------------------------------------------------|----------|----------------------------------|---------|----------------------------------|----------|--------|
| $\frac{1}{8}$ | 2.00123 | 0.00348072 | 2.00231 | 0.000207141 | 2.3728 | 0.668 |
| $\frac{1}{16}$ | 1.99806 | 0.00173835 | 1.99941 | 0.000420741 | 2.0028 | 0.762 |
| $\frac{1}{32}$ | 1.99853 | 0.000868654 | 2.00066 | 0.000537458 | 1.9936 | 0.742 |
| $\frac{1}{64}$ | 1.99916 | 0.00044183 | 2.00035 | 0.000537458 | 2.0049 | 0.732 |

Table 4: The convergence order of decoupled Linear Multi-step method plus time filters at time $T = 1$, with varying time step $\Delta t$ but fixed mesh size $h = \frac{1}{8}$.

| $\Delta t$ | $\|\tilde{u}_f^{h,\Delta t} - \tilde{u}_f^{h,\Delta t/2}\|_{L^2}$ | $\rho_{u_f}$ | $\|p_f^{h,\Delta t} - p_f^{h,\Delta t/2}\|_{L^2}$ | $\rho_{p_f}$ | $\|\phi_p^{h,\Delta t} - \phi_p^{h,\Delta t/2}\|_{L^2}$ | $\rho_{\phi_p}$ | CPU (s) |
|-----------|-------------------------------------------------|----------|----------------------------------|---------|----------------------------------|----------|--------|
| $\frac{1}{8}$ | 4.17159 | 0.000100451 | 3.89091 | 0.000311536 | 4.1683 | 0.637 |
| $\frac{1}{16}$ | 4.08764 | 0.000258169 | 4.02511 | 7.47389 | 0.00589 | 1.296 |
| $\frac{1}{32}$ | 4.04432 | 6.41395 | 0.001233 | 1.82915 | 0.00435 | 2.681 |
| $\frac{1}{64}$ | 4.02299 | 1.59856 | 0.0061 | 4.52367 | 0.00288 | 5.379 |

Table 5: The convergence order of coupled Linear Multi-step method at time $T = 1$, with varying mesh size $h$, but fixed time step $\Delta t = 0.01$.

| $h$ | $\|\tilde{u}_f - \tilde{u}_p\|_{L^2}$ | $\rho_{u_f,p}$ | $\|p_f - p_p\|_{L^2}$ | $\rho_{p_f,p}$ | $\|\phi_p - \phi_p\|_{L^2}$ | $\rho_{\phi_p}$ | CPU (s) |
|-----|---------------------------------|-------------|-----------------|-------------|-----------------|-------------|--------|
| $\frac{1}{8}$ | 0.0097303 | 1.98461 | 0.351569 | 1.70551 | 0.0065461 | 1.84055 | 1.129 |
| $\frac{1}{16}$ | 0.0176195 | 1.99466 | 0.107795 | 1.59905 | 0.0185807 | 1.9432 | 4.142 |
| $\frac{1}{32}$ | 0.00442212 | 1.98848 | 0.0355826 | 1.53809 | 0.00482795 | 1.94907 | 18.753 |
| $\frac{1}{64}$ | 0.00111417 | 1.95201 | 0.0122252 | 1.45774 | 0.00125035 | 1.84164 | 75.291 |

$\tilde{u}_f$ is the exact solution of the porous media flow equation. The exact solution is:

$$\tilde{u}_f = ((x^2(y - 1)^2 + y)\cos(t) - \frac{2}{3}x(y - 1)^3\cos(t) + (2 - \pi \sin(\pi x))\cos(t)).$$

$p_f$ is the pressure solution of the porous media flow equation.

$\phi_p$ is the pressure solution of the porous media flow equation.

The equations and the linear Lagrangian elements (P1) for the porous media flow equation. The exact solution is:

$$\tilde{u}_f = ((x^2(y - 1)^2 + y)\cos(t) - \frac{2}{3}x(y - 1)^3\cos(t) + (2 - \pi \sin(\pi x))\cos(t)).$$

$p_f$ is the pressure solution of the porous media flow equation.

$\phi_p$ is the pressure solution of the porous media flow equation.

$\phi_p = (2 - \pi \sin(\pi x))(1 - y - \cos(\pi y))\cos(t)$.
where same method as in [19]. So we define $h$ varying mesh size $h$, and Table 1 with Table 2 and Table 3 with Table 4. It can be found that the convergence orders both are $O(\Delta t^2)$. In particular, $\rho \approx 4$ for $\gamma = 2$ and $\rho \approx 8$ for $\gamma = 3$, when the corresponding order of convergence in time is of $O(\Delta t^2)$ and $O(\Delta t^3)$, respectively. So we change the time step $\Delta t$ from $\frac{1}{20}$ to $\frac{1}{30}$, but fix the mesh size $h = \frac{1}{8}$. We get a set of values for $\rho_v$ in the Table 1, Table 2 and Table 3. We can easily state that the coupled and decoupled Linear Multi-step methods plus time filters are convergent in time size $\Delta t$ and the orders of convergence both are $O(\Delta t^2)$.

At the same time, we compare Table 1 with Table 2 and Table 3 with Table 4. It can be found that the convergence orders of coupled and decoupled Linear Multi-step methods are first order, while the convergence orders of coupled and decoupled Linear Multi-step methods plus time filters are second order, that is, the convergence orders of Linear Multi-step method can be improved from first order to second order by adding time filter algorithm. And by comparing the CPU time in the tables, we can find that the computational efficiency of the Linear Multi-step method plus time filters is slightly lower than the Linear Multi-step method, but the difference between the two algorithms is not large, so we can know the Linear Multi-step method plus time filters is more efficient because it can achieve a higher order of convergence with almost the similar time. Then we calculate the convergence orders of the coupled and decoupled Linear Multi-step methods and the Linear Multi-step methods plus time filters by varying the mesh size $h$ with a fixed time step $\Delta t$. Thus, the approximation error is mainly determined by the mesh size $h$, so here we estimate the corresponding convergence

| $h$ | $\|u_f - \hat{u}_f^h\|_{L^2}$ | $\rho_{h,u_f}$ | $\|p_f - p_f^h\|_{L^2}$ | $\rho_{h,p_f}$ | $\|\phi - \phi^h\|_{L^2}$ | $\rho_{h,\phi}$ | CPU (s) |
|-----|------------------|-------------|----------------|-------------|----------------|-------------|---------|
| 1/1 | 0.0697 | 0.98548 | 0.351834 | 1.70476 | 0.0665162 | 1.84342 | 1.157 |
| 1/2 | 0.0176 | 2.00082 | 0.107936 | 1.59858 | 0.0185349 | 1.95591 | 54.21 |
| 1/4 | 0.0044 | 2.01089 | 0.0356405 | 1.54385 | 0.00477752 | 1.9703 | 17.342 |
| 1/8 | 0.0011 | 2.0247 | 0.0122236 | 1.51929 | 0.001109811 | 2.02597 | 71.904 |

Table 6: The convergence order of coupled Linear Multi-step method plus time filters at time $T = 1$, with varying mesh size $h$, but fixed time step $\Delta t = 0.01$.

| $h$ | $\|u_f - \hat{u}_f^h\|_{L^2}$ | $\rho_{h,u_f}$ | $\|p_f - p_f^h\|_{L^2}$ | $\rho_{h,p_f}$ | $\|\phi - \phi^h\|_{L^2}$ | $\rho_{h,\phi}$ | CPU (s) |
|-----|------------------|-------------|----------------|-------------|----------------|-------------|---------|
| 1/1 | 0.0697 | 0.98548 | 0.351834 | 1.70476 | 0.0665162 | 1.84342 | 1.157 |
| 1/2 | 0.0176 | 2.00082 | 0.107936 | 1.59858 | 0.0185349 | 1.95591 | 54.21 |
| 1/4 | 0.0044 | 2.01089 | 0.0356405 | 1.54385 | 0.00477752 | 1.9703 | 17.342 |
| 1/8 | 0.0011 | 2.0247 | 0.0122236 | 1.51929 | 0.001109811 | 2.02597 | 71.904 |

Table 7: The convergence order of decoupled Linear Multi-step method at time $T = 1$, with varying mesh size $h$, but fixed time step $\Delta t = 0.01$.

Table 8: The convergence order of decoupled Linear Multi-step method plus time filters at time $T = 1$, with varying mesh size $h$, but fixed time step $\Delta t = 0.01$. We get a set of values for $\rho_v$ in the Table 1, Table 2 and Table 3. We can clearly state that the coupled and decoupled Linear Multi-step methods plus time filters are convergent in time size $\Delta t$ and the orders of convergence both are $O(\Delta t^2)$. At the same time, we compare Table 1 with Table 2 and Table 3 with Table 4. It can be found that the convergence orders of coupled and decoupled Linear Multi-step methods are first order, while the convergence orders of coupled and decoupled Linear Multi-step methods plus time filters are second order, that is, the convergence orders of Linear Multi-step method can be improved from first order to second order by adding time filter algorithm. And by comparing the CPU time in the tables, we can find that the computational efficiency of the Linear Multi-step method plus time filters is slightly lower than the Linear Multi-step method, but the difference between the two algorithms is not large, so we can know the Linear Multi-step method plus time filters is more efficient because it can achieve a higher order of convergence with almost the similar time. Then we calculate the convergence orders of the coupled and decoupled Linear Multi-step methods and the Linear Multi-step methods plus time filters by varying the mesh size $h$ with a fixed time step $\Delta t$. Thus, the approximation error is mainly determined by the mesh size $h$, so here we estimate the corresponding convergence
order by

\[ \rho_{h,v} = \frac{\log(e_{v}(h_1))}{\log(h_2/h_1)} \]

where \( e_v(h) \) is the error computed by the algorithm with time step \( \Delta t \). So we change the mesh size \( h \) from \( \frac{1}{4} \) to \( \frac{1}{64} \), but fix the time step \( \Delta t = 0.01 \). Similarly, we get a set of values for \( \rho_{h,v} \) in the Table 5-8 from them we can clearly state that the coupled and decoupled Linear Multi-step methods plus time filters are convergent in mesh size \( h \) and the orders of convergence both are \( O(h^2) \). Finally, compared to the coupled Linear Multi-step method plus time filters, the decoupled Linear Multi-step method plus time filters takes less computation time, that is, it is more computationally efficient.

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