A CONDITION IN MEAN CURVATURE PRESCRIPTIONS FOR CONFORMAL METRICS ON THE BALL

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Abstract. This paper considers the prescribed zero scalar curvature and mean curvature problem on the $n$-dimensional Euclidean ball for $n \geq 3$. Given a rotationally symmetric function $H : \partial B^n \rightarrow \mathbb{R}$, in this work, we will prove that if $H'(r)$ changes signs where $H > 0$ and $H(r)$ also satisfies a flatness condition then there exists a metric $g$ conformal to the Euclidean metric, with zero scalar curvature in the ball and mean curvature $H$ on its boundary.

1. Introduction

Let us take the $n^{th}$ dimensional ball $(B^n, g_0)$, where $n \geq 3$ and $g_0$ is the Euclidean metric that has flat scalar curvature inside the ball and constant mean curvature $H_0 = 1$ on the boundary $\partial B^n$. A classical problem of differential geometry is the characterization of a pair of functions $R$ and $H$, with $R$ defined inside the ball and $H$ defined in the boundary, such that there is a metric $g$, conformal to $g_0$, with $R$ as the prescribed scalar curvature on the ball and $H$ the prescribed mean curvature on $\partial B^n$.

Given the functions $R$ and $H$, the existence of the metric $g$ is equivalent to the existence of a smooth positive function $u$, which satisfies the nonlinear problem in the Sobolev critical exponent:

$$\begin{align*}
\Delta_g u + \frac{(n-2)}{4(n-1)} Ru^{\frac{n+2}{n-2}} &= 0 \quad \text{on} \quad B^n, \\
\frac{\partial u}{\partial \eta} + \frac{(n-2)}{2} u &= H \frac{(n-2)}{2} u^{n/(n-2)} \quad \text{on} \quad \partial B^n.
\end{align*}$$

(1.1)

This type of question got a lot of attention in the past decades. It began to be explored in [4], [6] and [9], known as The Yamabe Problem on Manifolds with Boundary.

The problem of solution existence on compact manifolds (not conformally equivalent to the unit ball) was developed by Escobar J. in [5], [8], and [10]. Later, the problem in the ball was addressed in [6], where subcritical solutions were characterized under the consideration of prescribed mean curvature and zero scalar curvature. Both [5] and [6] showcase the fact that an obstruction of the Kazdan-Warner type condition occurs in a general case for solving the problem. We cannot solve the question affirmatively for all $H$ functions.

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Considering the regularity question in [4], Cherrier showed that solutions to the problem in the adequate space would be smooth. In [1], Chang and Yang find a perturbation result via a variational approach when \( H \) is smooth; the condition is a sufficient criterion, but questions remain on the necessary conditions.

In [11], Garcia and Escudero proved that if \( R \) is flat and the function \( H : \partial R^n_+ \rightarrow \mathbb{R} \) is radially symmetric and satisfies

\[
\begin{align*}
H(x) &> 0 \quad \text{and} \quad \frac{\partial H}{\partial r} \leq 0 \quad \text{if} \quad |x| < 1, \\
H(x) &\leq 0 \quad \text{if} \quad |x| \geq 1,
\end{align*}
\]

Then, the associated differential equation system has no solution.

If \( R \) is flat, in this work, we show that a sufficient condition to guarantee the existence of the metric \( g \) is that \( \frac{\partial H}{\partial r} \) changes signs where \( H \) is positive and has a flatness condition.

Based on the ideas of Chen and Li for the sphere from [2], [3] and Yan Li on [12] and with a parallel method to the used in [2] for prescribing scalar curvature in \( S^n \) in this paper we prove the following theorem:

**Theorem 1.1.** Let \( n \geq 3 \) and let \( H = H(r) \) be a smooth function on \( \partial B^n \) symmetric along the \( x_n \) axis. Assume that \( H \) has at least two positive local maximums and satisfies a flatness condition near every critical point \( \tau_0 \) as follows.

\[
\begin{align*}
H(r) &= H(\tau_0) + a|r-r_0|^\alpha + k(|r-r_0|) \quad \text{with} \quad a \neq 0 \quad \text{and} \quad n - 2 < \alpha < n - 1.
\end{align*}
\]

If \( H'(r) \) change signs where \( H > 0 \) then the problem

\[
\begin{align*}
\Delta_g u &= 0 \quad \text{in} \quad B^n, \\
\frac{\partial u}{\partial \eta} + \frac{(n-2)}{2} u &= H\left(\frac{n-2}{2}\right) u^p \quad \text{in} \quad \partial B^n,
\end{align*}
\]

where \( 1 < p \leq \frac{n}{n-2} \) and \( k'(s) = o(s^{\alpha-1}) \), have a smooth positive solution.

2. Preliminaries

Let us consider the functionals:

\[
J_p(u) = \int_{\partial B^n} H u^{p+1} d\sigma
\]

and

\[
E(u) = \int_{B^n} |\nabla u|^2 dv + \int_{\partial B^n} \gamma_n u^2 d\sigma,
\]

where \( \gamma_n = \frac{n-2}{2} \).

Let \( ||u|| = \sqrt{E(u)} \) be the norm in the Hilbert space \( H^2_1(B^n) \). We seek critical points of \( J_p(u) \) under the constraint

\[
S = \{ u \in H^2_1(B^n) : E(u) = E(1) = \gamma_n |S^{n-1}|, \ u \geq 0 \}
\]

where \( |S^{n-1}| \) is the volume of \( S^{n-1} \).

It is easy to prove that a scalar multiple of a critical point of \( J_p \) in \( S \) is a solution of (1.3). We take a coordinate system in \( R^n \) so that the south pole of \( S^{n-1} \) is at
the origin $O$, and the center of the ball is at the point $(0, 0, ..., 1)$. Define the center of mass of the function $u : B^n \rightarrow \mathbb{R}$ as

$$ q = q(u) = \frac{\int_{B^n} zu^2 \tau(z) dv}{\int_{B^n} u^2 \tau(z) dv} $$

Let $u_q$ be the standard solution with its center of mass at $q \in B^n$. That is $u_q$ satisfies the problem.

$$\begin{align*}
\Delta_g u_q &= 0 \quad \text{in } B^n, \\
\frac{\partial u_q}{\partial n} + \gamma_n u_q &= \gamma_n u_{q}^p \quad \text{in } \partial B^n
\end{align*}$$

Let $\tilde{q}$ be the intersection of $S^{n-1}$ with the ray passing the center and the point $q$. The solutions $u_q$ depends on two parameters, the point $\tilde{q}$ and a number $\beta$ with $1 \leq \beta < \infty$. When $\tilde{q} = 0$, it can be seen that the solutions $u_q$ are given by

$$ u_q(z) = \left( \frac{4\beta}{(\beta - 1)^2 \|z\|^2 + 4s(\beta - 1) + 4\beta} \right)^{\frac{n-2}{2}} $$

where $s = z_n$ is the last component of the point $z$. These solutions solve the problem of prescribing zero scalar curvature and mean curvature $h = 1$ in the ball.

If $z \in S^{n-1}$, we get

$$ u_q(z) = \left( \frac{\beta}{(\beta^2 - 1) \|z\|^2 + 1} \right)^{\frac{n-2}{2}}. $$

For $z \in S^{n-1}$, we can write, in the spherical polar coordinates $z = (r, \theta)$ of $S^{n-1}$ centered at the south pole, with $0 \leq r \leq \pi$ and $\theta \in S^{n-1}$, and

$$ u_q(z) = \left( \frac{\beta}{(\beta^2 - 1) \sin^2 \frac{r}{2} + 1} \right)^{\frac{n-2}{2}}, $$

with $1 \leq \beta < \infty$. If $\lambda = \frac{1}{\beta}$ then $0 < \lambda \leq 1$ and

$$ u_q(z) = \left( \frac{\lambda}{\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{\frac{n-2}{2}}. $$

Using $n$-dimensional spherical coordinates, it can be shown that the volume of $S^{n-1}$ and $S^{n-2}$ are related by the formula

$$ |S^{n-1}| = |S^{n-2}| \int_0^\pi \sin^{n-2}(t) dt. $$

From here, it can be proven that the solutions $u_q$ belong to the set $S$.

Let us consider the function

$$ \psi(z) = \frac{4(z - N)}{\|z - N\|^2} + N, $$
where \( z \in \mathbb{R}^n \setminus N \) and \( N \) is the north pole, and the function \( T : \mathbb{R}^n_{-1} \rightarrow \mathbb{R}^n_{-1} \) given by \( T(x,t) = (\beta x,\beta(t+1)-1) \), with \( \beta > 0 \). The function \( \varphi = \psi^{-1} \circ T \circ \psi \) restricted to the ball \( B^n \) is referred to as a centered dilation function from the ball to the ball.

Now consider the transformation \( T_\varphi : H^1_1(B^n) \longrightarrow H^1_1(B^n) \) given by

\[
T_\varphi(u(x)) = u(\varphi(x)), \quad [\det(d\varphi(x))]^{\frac{n-2}{2(n-1)}}
\]

This family of conformal transformations leaves the equation (2.1), the energy \( E(.) \), and the functional \( J_p \) invariant as proved in [9]. Consequently, if \( u \) is a solution of (1.3), then \( T_\varphi u \) is a solution too.

We will assume that the function \( H \) has a local maximum at the south pole and estimate near this point. The calculations in other local maxima are similar.

**Definition 2.1.** Let us define the set

\[
\Sigma = \left\{ u \in S : |q(u)| \leq \rho_0, ||v|| = \min_{t,q}||u - tu|| \leq \rho_0, t \in \mathbb{R} \right\}
\]

the set of functions in \( S \) with centers of mass near the south pole \( O \) of \( S^{n-1} \).

The following section will estimate this neighborhood’s functional \( J_p \). To do this, it is necessary to show first some preliminary results on \( B^n \) whose proofs mirror those given in [2] for \( S^n \).

**Lemma 2.2.** (About the mass center)

1. Let \( q, \lambda \) and \( \bar{q} \) as previously. Then if \( q \) small enough:

\[
|q|^2 \leq C (|\bar{q}|^2 + \lambda^4)
\]

2. Let \( \rho_0 \) and \( v \) defined in (2.1). then for \( \rho_0 \) small enough:

\[
\rho_0 \leq |q| + C||v||
\]

**Lemma 2.3.** If \( u \in \Sigma \) and \( v = u - t_0u_q \) as defined in (2.1) then \( u_q \) and \( v \) are orthogonal and hold:

\[
\int_{S^{n-1}} u_q^\tau v d\sigma = 0
\]

**Lemma 2.4.** Let \( u \in \Sigma \) and \( v = u - t_0u_q \) and \( T_\varphi \) as previously defined. Then:

\[
\int_{S^{n-1}} T_\varphi v \, dv = 0 \quad \text{and} \quad \int_{S^{n-1}} x_i T_\varphi v \, dv = 0,
\]

where the functions \( x_i \) represent the coordinate functions.

From here, it follows

**Lemma 2.5.** Let \( u \in \Sigma, v = u - t_0u_q \) and \( T_\varphi : H^2_1(B^n) \longrightarrow H^2_1(B^n) \) as previously defined. We have the following:

\[
\langle T_\varphi v, K \rangle_{H^2_1(B^n)} = 0 \quad \text{and} \quad \langle T_\varphi v, x_i \rangle_{H^2_1(B^n)} = 0,
\]

with \( x_i \) the coordinate functions and \( K \) a constant function.
3. The Sub-critical Case

In this section, we will find a solution to the problem.

\[
\begin{cases}
\Delta_g u = 0 & \text{in } B^n, \\
\frac{\partial u}{\partial n} + \gamma_n u = H \gamma_n u^p & \text{in } \partial B^n
\end{cases}
\tag{3.1}
\]

for each \(1 < p < \tau\), where \(\gamma_n = \frac{n-2}{2}, \tau = \frac{n}{n-2}\) and \(k'(s) = o(s^{\alpha-1})\), has a solution.

Let us consider the functionals:

\[
J_p(u) = \int_{\partial B^n} Hu^{p+1} d\sigma
\]

and

\[
E(u) = \int_{B^n} |\nabla u|^2 dv + \int_{\partial B^n} \gamma_n u^2 d\sigma.
\]

Let \(|u| = \sqrt{E(u)}\) be the norm in the Hilbert space \(H^2_1(B^n)\). We seek critical points of \(J_p(u)\) under the constraint

\[
S = \{ u \in H^2_1(B^n) : E(u) = E(1) = \gamma_n |S^{n-1}|, u \geq 0 \}
\]

where \(|S^{n-1}|\) is the volume of \(S^{n-1}\).

3.1. Estimates on \(J_p\)

We will show estimates on \(J_p\) near the south pole \((O, \theta)\) where we assume a positive local maximum; the estimates near another local maximum will be the same. Now by the hypothesis in (1.1) then \(H(r) = H(0) - ar^\alpha\) for some \(a > 0, n-3 < \alpha < n-1\) in an open set that has the coordinate origin \(O\).

**Proposition 3.1.** For all \(\delta_1 > 0\) there exists a positive number \(p_1 \leq \tau\) such that for all \(p_1 \leq p \leq \tau\) holds:

\[
\sup_{\Sigma} J_p(u) > H(0)|S^{n-1}| - \delta_1
\]

**Proof.** Let us prove first that \(J_\tau(u_{\lambda,O}) \longrightarrow H(0)|S^{n-1}|\) when \(\lambda \longrightarrow 0\). Let us take a fixed number \(\tau_0 \neq 0\) near enough to zero then:

\[
J_\tau(u_{\lambda,O}) = \int_{S^{n-1}} H(r) u_{\lambda,O}^{\tau+1} d\sigma = \int_{S^{n-1}} H(r) \left( \frac{\lambda}{(\lambda^2 \cos^2 \frac{\tau}{2} + \sin^2 \frac{\tau}{2})} \right)^{n-1} d\sigma
\]

\[
= |S^{n-2}| \int_0^\pi \frac{\lambda^{n-1} H(r) \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{\tau}{2} + \sin^2 \frac{\tau}{2})^{n-1}} dr,
\]

where we have used that

\[
|S^{n-1}| = |S^{n-2}| \int_0^\pi \sin^{n-2}(t) dt.
\]
Hence,
\[
J_\tau(u_{\lambda,O}) = |S^{n-2}| \left\{ \int_0^{\tau_0} \frac{\lambda^{n-1}H(r) \sin^{n-2}r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr + \int_{\tau_0}^{\pi} \frac{\lambda^{n-1}H(r) \sin^{n-2}r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right\}
\]

\[
= |S^{n-2}| \left\{ \int_0^{\tau_0} \frac{\lambda^{n-1}(H(0) - ar^\alpha) \sin^{n-2}r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr + \int_{\tau_0}^{\pi} \frac{\lambda^{n-1}H(r) \sin^{n-2}r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right\}
\]

Adding and subtracting in the right-hand
\[
\int_{\tau_0}^{\pi} \frac{\lambda^{n-1}H(0) \sin^{n-2}r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr
\]
we get
\[
J_\tau(u_{\lambda,O}) = H(0)|S^{n-1}| + |S^{n-2}| \left\{ \int_0^{\tau_0} \frac{-ar^\alpha \lambda^{n-1} \sin^{n-2}r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr + \int_{\tau_0}^{\pi} \frac{\lambda^{n-1}(H(r) - H(0)) \sin^{n-2}r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right\}
\]

Let us call
\[
I_1 = \left| \int_0^{\tau_0} \frac{r^\alpha \lambda^{n-1} \sin^{n-2}r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right|
\]

and
\[
I_2 = \left| \int_{\tau_0}^{\pi} \frac{\lambda^{n-1}(H(r) - H(0)) \sin^{n-2}r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right|
\]

We will show that \( I_1 \to 0 \) and \( I_2 \to 0 \) when \( \lambda \to 0 \). Clearly:
\[
I_1 = \left| \int_0^{\tau_0} \frac{r^\alpha \lambda^{n-1}(2 \sin \frac{r}{2} \cos \frac{r}{2})^{n-2}}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right| \leq \lambda 2^{n-2} \left| \int_0^{\tau_0} \frac{r^\alpha (\sin \frac{r}{2})^{n-2}(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{1-n}}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right|
\]

\[
\leq \lambda 2^n \left| \int_0^{\tau_0} \frac{(\sin \frac{r}{2})^{n-2}}{(\sin^2 \frac{r}{2})^{n/2}} dr \right| \leq \lambda 2^n \left| \int_0^{\tau_0} \frac{(\sin \frac{r}{2})^{n-2}}{(\sin^2 \frac{r}{2})^{n/2}} dr \right| \leq \lambda 2^n \int_0^{\tau_0} r^{\alpha-2} dr = \frac{c_1 \lambda^{\frac{\alpha-1}{\alpha}}}{\alpha-1}
\]

A straightforward calculation also shows that.
\[
I_2 = \left| \int_{\tau_0}^{\pi} \frac{\lambda^{n-1}(H(r) - H(0))(2 \sin \frac{r}{2} \cos \frac{r}{2})^{n-2}}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right| \leq C_2 \lambda (\pi - \tau_0)
\]

Hence if \( \lambda \to 0 \) \( I_2 \to 0 \) and \( J_\tau(u_{\lambda,O}) \to H(0)|S^{n-1}| \).

Then if we take \( \delta_1 > 0 \), we can choose \( \lambda_0 \) such as \( u_{\lambda,O} \in \overset{o}{\Sigma} \), the interior of \( \Sigma \), and
\[
J_\tau(u_{\lambda,O}) > H(0)|S^{n-1}| - \frac{\delta_1}{2}
\]

Since \( J_p \) is continuous respect \( p \), given a fixed \( u_{\lambda_0,O} \) there is \( P_1 \) such as for all \( P \geq P_1 \):

sup \bar{J}_p(u) > H(0)|S^{n-1}| - \delta_1 \quad \square

Now we show.

Lemma 3.2. Let

$$\bar{B}_\epsilon(0) = \{ x \in S^{n-1}(e_n), \ |x| \leq \epsilon \}.$$  

for \( \epsilon > 0 \) and \( \alpha < n - 1 \), \( p \) nearly enough to \( \tau \) and for \( \lambda \) and \( |\bar{q}| \) small enough we have:

$$J_p(u_{\lambda, \bar{q}}) \leq (H(0) - C_1|\bar{q}|^\alpha)|S^{n-1}|(1 + O_p(1)) - C_1 \lambda^{\alpha + \delta_p},$$

where \( \delta_p > \tau - p \) and \( O_p(1) \to 0 \) when \( p \to \tau \).

Now we show that \( J_p \) is bounded over the boundary of \( \Sigma \).

Proposition 3.3. There are some positive constants \( \rho_0, p_0, \delta_0 \) such as for all \( P \geq p_0 \) and \( u \in \partial \Sigma \) it holds:

$$J_p(u) \leq H(0)|S^{n-1}| - \delta_0$$

Proof. Let’s consider

$$\bar{H}(x) = \begin{cases} H(x) & \text{in } B_{2\rho_0(0)}, \\ m & \text{in } S^{n-1}\{B_{2\rho_0(0)} \} \end{cases} \quad (3.2)$$

Where \( m = H|\partial B_{2\rho_0(0)} \). Now, let us define:

$$\tilde{J}_p(u) = \int_{S^{n-1}\{B_{2\rho_0(0)} \}} \bar{H}(x)u^{p+1}d\sigma.$$  

The estimates will be divided into two steps; in step one, we use inequality.

$$|J_p(u) - \tilde{J}_p(u)| \leq |\bar{J}_p(u) - \tilde{J}_\tau(u)| + |J_p(u) - \tilde{J}_p(u)|.$$  

To show the difference between \( J_p(u) \) and \( \tilde{J}_\tau(u) \) is small. Step two will carry estimates on \( \tilde{J}_\tau(u) \).

Step 1. It can be seen that:

$$\tilde{J}_p(u) \leq \tilde{J}_\tau(u)(1 + o_p(1)) \quad (3.3)$$

Where \( o_p(1) \to 0 \) when \( p \to \tau \).

Now lets check the difference between \( J_p(u) \) and \( \tilde{J}_p(u) \).

$$|J_p(u) - \tilde{J}_p(u)| = \int_{S^{n-1}\{B_{2\rho_0(0)} \}} |H(x) - m|u^{p+1}d\sigma \quad (3.4)$$
\[ \leq C_1 \int_{S^{n-1}|B_{2\rho_0}(0)} w^{p+1} d\sigma \leq C_2 \int_{S^{n-1}|B_{2\rho_0}(0)} (t_0 u_q)^{p+1} d\sigma + C_2 \int_{S^{n-1}|B_{2\rho_0}(0)} v^{p+1} d\sigma \]

\[ \leq C_3 \lambda^{(n-1) - \frac{n-2}{2} \delta_p} + C_2 \int_{S^{n-1}|B_{2\rho_0}(0)} v^{p+1} d\sigma \leq C_3 \lambda^{(n-1) - \frac{n-2}{2} \delta_p} + C ||v||^{p+1}. \]

In the last inequality, we have used the Beckner-Escobar Sobolev inequality as follows:

\[ \left( \int_{S^{n-1}|B_{2\rho_0}(0)} v^{p+1} d\sigma \right)^{\frac{1}{p+1}} \leq C \left( \int_{S^{n-1}} |\nabla v|^2 dv + \int_{\partial B^n} \gamma_n v^2 d\sigma \right)^{1/2} \leq C ||v||. \]

Now, as \( \lambda^{(n-1) - \frac{n-2}{2} \delta_p} \) and \( ||v||^{p+1} \) are small, using (3.3) and (3.4), the difference between \( J_p(u) \) and \( \tilde{J}_p(u) \) is small.

**Step 2.**

Now we will carry out the estimates on \( J_r(u) \).

Let \( u = v + t_0 u_q \in \partial \Sigma \). From (2.3), we have \( v \) and \( u_q \) are orthogonal respect the inner product associated to \( E(\cdot) \); that is,

\[ 0 = \int_{B^n} (\nabla (u - t_0 u_q)) \cdot \nabla u_q dv + \gamma_n \int_{S^{n-1}} (u - t_0 u_q) u_q d\sigma. \]

then,

\[ t_0 E(u_q) = \int_{B^n} \nabla u \nabla u_q dv + \gamma_n \int_{S^{n-1}} uu_q d\sigma. \]

Now,

\[ ||v||^2 = E(u - t_0 u_q) = E(u) + t_0^2 E(u_q) - 2t_0 \left( \int_{B^n} \nabla u \nabla u_q dv + \gamma_n \int_{S^{n-1}} uu_q d\sigma \right) \]

\[ = E(u) + t_0^2 E(u_q) - 2t_0^2 E(u_q) = E(u) - t_0^2 E(u_q) \]

Moreover \( E(u) = E(u_q) = \gamma_n |S^{n-1}| \), therefore

\[ ||v||^2 = (1 - t_0^2) \gamma_n |S^{n-1}| \]

and

\[ t_0^2 = 1 - \frac{||v||^2}{\gamma_n |S^{n-1}|}. \]

Now,

\[ \bar{J}_r(u) = \int_{S^{n-1}} \bar{H}(x) u^{\tau+1} d\sigma \]

\[ \leq t_0^{\tau+1} \int_{S^{n-1}} \bar{H}(x) u_q^{\tau+1} d\sigma + (\tau+1) \int_{S^{n-1}} \bar{H}(x) u_q^{\tau+1} v d\sigma + \frac{\tau(\tau+1)}{2} \int_{S^{n-1}} \bar{H}(x) u_q^{\tau-1} v^2 d\sigma + O(||v||^2) \]
Taking the value of $t_0$ found on (3.2), we find that the first term on the right-hand side is bounded above by
\[
\left(1 - \frac{\tau + \frac{1}{2} \frac{||v||^2}{\gamma_n|S^{n-1}|}}{S^{n-1}}\right) H(0)|S^{n-1}| (1 - k_1|\bar{q}|^{\alpha} - k_1\lambda^{\alpha}) + o(||v||^2),
\]
for some constant $k_1$.

Using the orthogonality between $v$ and $u_q^\tau$ (see lemma (2.3), lemma (2.2)), and the fact that $T_\varphi u_q$ is bounded, we find that
\[
\int_{S^{n-1}} \bar{H}(x) u_q^\tau v d\sigma = \int_{S^{n-1}} \bar{H}(x) u_q^\tau v d\sigma - m \int_{S^{n-1}} u_q^\tau v d\sigma
\]
(3.6)

Now, we will use that in the $n$-dimensional ball $B^n$, the Neumann eigenvalues of the Laplacian operator holds the inequality $0 = \lambda_1 < \lambda_2 \leq \lambda_3, ..., \lambda_n$, where the first nonzero eigenvalue can be variationally seen as:
\[
\lambda_2 = \inf \left\{ \frac{\int_{B^n} |\nabla u|^2 dv}{\int_{S^{n-1}} u^2 d\sigma} : f \in H^1_0(B^n) \setminus \{0\} \right\}.
\]
Our first nonzero Neumann eigenvalue is $\lambda = 1$ and as $T_\varphi v$ is orthogonal to the coordinate functions and constants (see (2.5)), then for some $c > 0$:
\[
1 + c \leq \frac{\int_{B^n} |\nabla T_\varphi v|^2 dv}{\int_{S^{n-1}} (T_\varphi v)^2 d\sigma}.
\]

Adding $\gamma_n$ in both sides we have:
\[
1 + c + \gamma_n \leq \frac{E(T_\varphi v)}{\int_{S^{n-1}} (T_\varphi v)^2 d\sigma}.
\]
Since $E(T_\varphi v) = E(v)$,
\[
||v||^2 = ||T_\varphi v||^2 \geq (\gamma_n + 1 + c) \int_{S^{n-1}} (T_\varphi v)^2 d\sigma.
\]

On the other hand,
\[
\int_{S^{n-1}} u_q^\tau v^2 d\sigma = \int_{S^{n-1}} u_q^{\tau-1} (\varphi(x)) v^2 (\varphi(x)) d\sigma
\]
\[
= \int_{S^{n-1}} \frac{(T_\varphi u_q)^{\tau-1}}{\det(d\varphi(x))^{\frac{n-2}{2(n-1)}}} \frac{(T_\varphi v)^2}{\det(d\varphi(x))^{\frac{n-2}{2(n-1)^2}}} d\sigma
\]
\[
= \int_{S^{n-1}} (T_\varphi v)^2 d\sigma.
\]

Hence,
\[
\int_{S^{n-1}} \bar{H}(x) u_q^\tau v^2 d\sigma \leq
\]
\[ H(0) \int_{S^{n-1}} u^{-1}_q v^2 d\sigma = H(0) \int_{S^{n-1}} (T\varphi v)^2 d\sigma \leq \frac{H(0)}{\gamma_n + 1 + c} ||v||^2. \] 

(3.7)

Now using the inequalities (3.5), (3.6) and (3.7), we find that there exists \( \beta > 0 \) such as:

\[ -J_\tau(u) \leq H(0) |S^{n-1}| \left[ 1 - \beta(|\tilde{q}|^\alpha + \lambda^\alpha + ||v||^2) \right]. \] 

(3.8)

Then

\[ J_p(u) \leq |\tilde{J}_p(u) - J_\tau(u)| + |J_p(u) - \tilde{J}_p(u)| + J_\tau(u) \]

\[ \leq a_p(1) + C_3 \lambda^{(n-1)-(n-2)\delta_\tau} + C_3 ||v||^{p+1} + H(0) |S^{n-1}| \left[ 1 - \beta \left( ||q||^\alpha + \lambda^\alpha + ||v||^2 \right) \right]. \]

When \( p \to \tau \), we obtain the result. \( \square \)

### 3.2. Proof of the Sub-critical Case.

In the following, we will prove the existence of a solution to the problem (3.1) for each \( p < \tau \). It can be proved that the set

\[ S = \left\{ u \in H^2_1(B^n) : ||u||^2 = \int_{B^n} |\nabla u|^2 dv + \int_{S^{n-1}} \gamma_n u^2 d\sigma = \gamma_n |S^{n-1}|, u \geq 0 \right\}. \]

is closed and that the functionals \( J_p(u) = \int_{S^{n-1}} Hu^{p+1} d\sigma \) are compact and Lipschitz continuous.

By hypothesis, \( H \) has at least two positive local maxima. Let \( r_1 \) and \( r_2 \) be the two least positive local maxima of \( H \). By propositions (3.1) and (3.3), there is two disjoint open sets \( \tilde{\Sigma}_1, \tilde{\Sigma}_2 \subset S, \psi_i \in \tilde{\Sigma}_i, p_0 < \tau \) and \( \delta > 0 \) such as for all \( p \geq p_0 \):

\[ J(\psi_i) > H(r_i)|S^{n-1}| - \frac{\delta}{2}, \quad i = 1, 2; \]

and

\[ J(u) \leq H(r_i)|S^{n-1}| - \delta, \forall u \in \partial \Sigma, \quad i = 1, 2; \] 

(3.9)

Let \( \gamma \) be a path in \( S \) linking the functions \( \psi_1 \) and \( \psi_2 \). Let us define the path family:

\[ \Gamma = \{ \gamma \in C([0,1], S) : \gamma(0) = \psi_1, \gamma(1) = \psi_2 \} \]

Now define

\[ c_p = \sup_{\gamma \in \Gamma} \left\{ \min_{u \in \gamma} J_p(u) \right\} \]

By the Mountain Pass Theorem, there exists a critical function \( u_p \) of \( J_p \) in \( S \) such that:

\[ J_p(u_p) = c_p \]

As a consequence of (3.9) and the definition of \( c_p \), we have:
\[ J_p(u_p) \leq \min_i \left\{ H(r_i)^1S^{n-1} | - \delta \right\} \] (3.10)

From here, it is easy to show that there exists a positive real number \( \lambda(p) \) such that \( \lambda(p)u_p \) is a solution to the problem (3.1); moreover, for all \( p \) close to \( \tau \), the constant multiples \( \lambda(p) \) are uniformly bounded from above and bounded away from 0.

Observe that since \( u_p \in S \) and the multiples \( \lambda(p) \) are uniformly bounded, the energy of the solutions \( w_p = \lambda(p)u_p \), \( p \) close to \( \tau \), are uniformly bounded.

4. A priori Estimates

In the last section, we proved the existence of a positive solution \( u_p \) to the subcritical equation (3.1) for each \( p < \tau \). Now we prove that as \( p \to \tau \), there is a subsequence of \( \{u_p\} \), which converges to a solution \( \frac{n}{n-2} \), the functions \( \{u_p\} \) are uniformly bounded; Since the functions \( u_p \) are harmonic, it is enough to do this analysis on the boundary \( S^{n-1} \).

**Theorem 4.1.** Assume that \( H \) Satisfies the flatness condition, then there is a \( p_0 < \tau \), such that for all \( p_0 < p < \tau \) the solutions of (3.1) obtained in the variational scheme are uniformly bounded.

To prove the theorem, we estimate the solutions on three regions: \( H \) negative and away from zero, \( H \) close to zero, and \( H \) positive and away from zero.

4.1. \( H \) Negative and away from zero

In this section, following the ideas of [3], we derive a priori estimates in the region where \( H \) is negative and bounded away from zero for all positive solutions of (1.1).

**Proposition 4.2.** The solutions of (3.1) are uniformly bounded in the region where \( H(x) \leq -\delta \), for every \( \delta > 0 \). The bound depends on \( \delta \), \( \text{dist} \{x|H(x) \leq -\delta\} , S_0 \), and the lower bound of \( H \), where \( S_0 = \{x|H(x) = 0\} \).

Using the conformal extension of the stereographic projection to the ball, the problem to solve is the following:

\[
\begin{cases}
-\Delta u = 0 \quad &\text{in} \quad \mathbb{R}_+^n, \\
\frac{\partial u}{\partial n} = Hu^{\frac{n}{n-2}} \quad &\text{over} \quad \partial \mathbb{R}_+^n.
\end{cases}
\] (4.1)

With asymptotic growth of the solutions at infinity.

\[ u \sim \|x\|^{2-n}. \]

Here \( H \) is the projection of the original function \( H \) to \( \partial \mathbb{R}_+^n \). To prove proposition (4.2), we use the following lemma from [11]:
Lemma 4.3. Let \( w \in C^2(\mathbb{R}_+^n) \cap C^1(\partial \mathbb{R}_+^n) \) be a nonnegative function, \( B^+_1 \) the unit ball in \( \mathbb{R}_+^n \) and \( C(x) \) a bounded function on \( \mathbb{R}_+^n \). If \( w \geq 0 \) in \( \partial' B_1^+ \) and satisfies:

\[
\begin{cases}
\Delta g w = 0 & \text{on } B_1^+, \\
\frac{\partial w}{\partial \eta} - C(x)w \geq 0 & \text{on } \partial' B_1^+ \cap \partial \mathbb{R}_+^n,
\end{cases}
\]

(4.2)

where \( \frac{\partial w}{\partial \eta} - C(x)w \) is not equal zero on all \( \partial' B_1^+ \); then \( w > 0 \) in \( B_1^+ \).

and the following Harnack inequality:

Lemma 4.4. Let \( x_0 \) be a point where \( H \) is negative. Let \( 3\epsilon_0 = \text{dist}(x_0, S_0) \) and

\[ H(x) \leq -\delta_0 \quad \text{for all } x \in \partial B_{2\epsilon_0}^+(x_0) = B_{2\epsilon_0}^+(x_0) \cap \partial \mathbb{R}_+^n. \]

Assume that \( H(x) \geq -M \) for all \( x \in \partial \mathbb{R}_+^n \). Then there exists a constant \( C = C(\epsilon_0, \delta_0, M) \), such that for any point \( x_1 \) we have:

\[ u(x_0) \leq C (|x_1 - x_0| + \epsilon_0)^{n-2} u(x_1). \]

Proof. We will present only the proof for \( p = \frac{n}{n-2} \); the proof for \( p < \frac{n}{n-2} \) is similar. Let \( \pi \in \partial B_{\epsilon_0}^+(x_0) \). Let us translate coordinates so that the point \( \pi \) becomes the origin. Let

\[ u_0(x) = \epsilon_0^{\frac{n-2}{2}} u(\epsilon_0 x) \]

A simple calculation shows that \( u_0 \) satisfies:

\[
\begin{cases}
\Delta u_0 = 0 & \text{in } \mathbb{R}_+^n, \\
\frac{\partial u_0}{\partial \eta} = \tilde{H} u_0^{n/n-2} & \text{over } \partial \mathbb{R}_+^n,
\end{cases}
\]

(4.3)

where \( \tilde{H}(x) = H(\epsilon_0 x) \).

Let \( v(x) = \frac{1}{|x|^{n-2}} u_o(\frac{x}{|x|^2}) \) the Kelvin Transform of \( u_0 \), A direct calculation proved that \( v \) satisfies the equations:

\[
\begin{cases}
\Delta v = 0 & \text{in } \mathbb{R}_+^n, \\
\frac{\partial v}{\partial \eta} = \tilde{H} \left( \frac{x}{|x|^2} \right)^{n/n-2} v^{n/n-2} & \text{over } \partial \mathbb{R}_+^n,
\end{cases}
\]

(4.4)

For \( \alpha \) big enough, let us compare the function \( \alpha v(x) \) with \( u_0 \) in the unit ball \( B_1^+(0) \). For this, let us define the function

\[ w(x) = \alpha v(x) - u_0(x). \]
Then $w(x)$ satisfies the equations:

\[
\begin{cases}
\Delta w = 0 & \text{in } B_1^+(0), \\
\frac{\partial w}{\partial \eta} = \alpha \tilde{H} \left( \frac{x}{|x|^2} \right) v^{n/n-2} - \tilde{H}(x) u_0^{n/n-2} & \text{over } \partial' B_1^+(0) = \partial B_1^+(0) \cap \partial \mathbb{R}^n_+,
\end{cases}
\] (4.5)

A straightforward calculation shows that.

\[
\frac{\partial w}{\partial \eta} - \alpha^{-\frac{2}{n-2}} \tilde{H} \left( \frac{x}{|x|^2} \right) \left[ (\alpha v)^{n/n-2} - u_0^{n/n-2} \right] = \left( \alpha^{-\frac{2}{n-2}} \tilde{H} \left( \frac{x}{|x|^2} \right) - \tilde{H}(x) \right) u_0^{n/n-2}
\]

No, w, by the Mean value theorem, there is a continuous function $\phi$ valued between $\alpha v$ and $u$ such that:

\[
\frac{\partial w}{\partial \eta} - \alpha^{-\frac{2}{n-2}} \tilde{H} \left( \frac{x}{|x|^2} \right) \phi w = \left( \alpha^{-\frac{2}{n-2}} \tilde{H} \left( \frac{x}{|x|^2} \right) - \tilde{H}(x) \right) u_0^{n/n-2}
\]

Since $-M \leq H(x) \leq -\delta$, then:

\[
\tilde{H}(x) - \alpha^{-\frac{2}{n-2}} \tilde{H} \left( \frac{x}{|x|^2} \right) \leq -\delta_0 + M \alpha^{-\frac{2}{n-2}} < 0,
\]

for $\alpha$ sufficiently large. Therefore,

\[
\frac{\partial w}{\partial \eta} - \alpha^{-\frac{2}{n-2}} \tilde{H} \left( \frac{x}{|x|^2} \right) \phi w = \left( \alpha^{-\frac{2}{n-2}} \tilde{H} \left( \frac{x}{|x|^2} \right) - \tilde{H}(x) \right) u_0^{n/n-2} > 0.
\]

From lemma (4.3), $w \geq 0$ in $B_1^+(0)$. This implies, for $\alpha$ depending only on $\delta_0$ and $M$, and $x \in B_{e_0}(\bar{x})$ that

\[
u(x) \leq \alpha \frac{e_0^{-n-2} u}{{|x - \bar{x}|}^n} \left( e_0^2 \frac{|x - \bar{x}|}{|x - \bar{x}|^2} + \bar{x} \right)
\]

Given a point $x_1$, let us take a point $\tilde{x}$, such that the three points $x_0$, $x_1$ and $\tilde{x}$, are on the same line with $x_0$ in between, and satisfy

\[|x_0 - \tilde{x}| |x_1 - \tilde{x}| = e_0^2\]

From here, it follows that.

\[u(x_0) \leq C \left( |x_1 - x_0| + \epsilon \right)^{n-2} u(x_1)\]

where $C$ only depends on $\delta_0$, $e_0$ and $M$. $\square$

Now let us prove proposition 4.2

Proof. Since the functional of energy is bounded for the subcritical solutions $u_p$, $p$ close to $\tau$, found in the previous section, given a point $x_0$ as in lemma (4.4), the integrals

\[\int_{B_0^+(x_0)} u_p^{p+1} d\sigma\]
are uniformly bounded.

Applying lemma 4.4 for \( x \in \partial' B_{\varepsilon_0}(x_0) \) we get

\[
  u_p(x) \leq C \inf_{x \in \partial' B_{\varepsilon_0}(x_0)} u_p(x) \leq \frac{C}{|\partial' B_{\varepsilon_0}(x_0)|} \int_{\partial' B_{\varepsilon_0}(x_0)} u_p d\sigma \leq \frac{C}{n} \left( \int_{\partial' B_{\varepsilon_0}(x_0)} u_p^{p+1} d\sigma \right)^{1/p} \leq K_2
\]

Where \( K_2 \) only depends on \( \delta_0, \text{dist} \{ \{ x | H(x) \leq -\delta_0 \}, \{ x | H(x) = 0 \} \} \) and the inferior bound of \( H \). Hence, the solutions \( \{ u_p \} \) are uniformly bounded where \( H \) is negative. \( \square \)

4.2. \( H \) SMALL AND CLOSE TO ZERO

**Proposition 4.5.** Let \( \{ u_p \} \) be the solutions of the subcritical equation obtained by the variational approach, there exist \( p_0 < \tau \) and \( \delta > 0 \), such that for all \( p_0 < p \leq \tau \), \( \{ u_p \} \) are uniformly bounded in the regions where \( |H(x)| \leq \delta \).

**Proof.** First, recall that the energy of the subcritical solutions of the problem is uniformly bounded. Arguing by contradiction, assume that there exists a subsequence \( \{ u_i \} \) with \( u_i = u_{p_i}, p_i \to \frac{\varepsilon_i}{\varepsilon_0} \), and a sequence of local maxima \( \{ x_i \} \) of the functions \( u_i \), with \( x_i \to x_0 \) and \( H(x_0) = 0 \) such that \( u_i(x_i) \to \infty \).

Let \( K \) be a big number and \( r_i = 2K [u_i(x_i)]^{\frac{p_i-1}{2}} \). Taking the restriction of the functions \( u_i \) on the half ball \( B_{r_i}^+(x_i) \) and defining the function

\[
  v_i(x) = \frac{1}{u_i(x_i)} u_i\left( \frac{r_i}{2K} x + x_i \right)
\]

then \( v_i(x) \) is bounded on the half ball \( B_{r_i}^+(0) \) y \( v_i(0) = 1 \) for all \( i \). The family of functions \( v_i(x) \) is equicontinuous on \( B_{r_i}^+(0) \), and by the Arzela-Ascoli Theorem \( \{ v_i \} \) has a subsequence that converges to a harmonic function \( v_0 \) in the closure of \( B_{r_i}^+(0) \subset \mathbb{R}^n_+ \) and \( v_0(0) = 1 \). Moreover, since \( H(x_0) = 0 \), then \( \frac{\partial v_0}{\partial n} = 0 \) on \( \partial' B_{K}^+(0) \). Therefore the function \( v_0 \) satisfies the mean value equality

\[
  v_0(y) = \frac{1}{2|B_{K}^+(y)|} \int_{B_{K}^+(y)} v_0 dx.
\]

From here and the fact that

\[
  |v_0|^{p+1} \leq C(|v_i|^{p+1} + |v_0 - v_i|^{p+1}),
\]

we get for \( i \) big enough that

\[
  \int_{B_{r_i}^+(0)} |v_i|^{p+1}(x) dx \geq CK^n \tag{4.6}
\]

For some positive constant \( C \). On the other hand, since the energy \( E(u_i) \) is uniformly bounded

\[
  \int_{S^{n-1}} u_i^{p+1} dV \leq C,
\]
and any $K > 0$ we have that
\[
\int_{S^{n-1}} u_i^{p+1} dV = \int_{\mathbb{R}^n_+} u_i^{p+1}(\pi(x)) dx \geq \int_{B_K^+(0)} v_i^{p+1}(x) dx,
\]
we get the inequalities
\[
C \geq \int_{S^{n-1}} u_i^{p+1} dV \geq \int_{B_K^+(0)} v_i^{p+1}(x) dx \geq CK^n,
\]
where $\pi$ is the inversion function.

If we take $K$ big enough, we get a contradiction. Consequently, the sequence $u_i$ is uniformly bounded in the regions where $H$ is small. \[\square\]

4.3. $H$ Positive and Away From Zero

**Proposition 4.6.** Let \( \{u_p\} \) be solutions of the subcritical problem (3.1) obtained by the variational approach. Then there exists a $p_0 < \tau$. Such that for all $p_0 < p < \tau$ and for any $\delta > 0$, \( \{u_p\} \) are uniformly bounded in the regions where $H(x) \geq \delta$.

**Proof.** The argument starts in the case $|H| < \delta$. Let \( \{x_i\} \) be sequence of points such that $u_i(x_i) \to \infty$ and $x_i \to x_0$ with $H(x_0) > 0$. Let $r_i(x)$ and $v_i(x)$ be defined as for the case $|H| < \delta$ and similarly $v_i(x)$ converges to standard function $v_0(x)$ in $\mathbb{R}^n_+$ with

\[
\begin{cases}
-\Delta v_0 = 0 & \text{in } \mathbb{R}^n_+,

\frac{\partial v_0}{\partial \eta} = H(x_0)u^{n/n-2} & \text{over } \partial \mathbb{R}^n_+,
\end{cases}
\]

(4.7)

It follows that
\[
\int_{B_{r_i}(x_0)} u^{p+1} dV \geq c_0 > 0.
\]

Because the total energy of $u_i$ is bounded, we can only have finitely many such points $x_0$. Hence \( \{u_i\} \) has finite isolated blow-up points. As consequence of a result in [6] (proposition 4.11) we have:

**Lemma 4.7.** Let $u_i$ be a solution of (3.1) for $n \geq 3$. Assume that for each critical point, $x_0$ we have the flatness condition $\alpha$ for some $\alpha > n - 2$, then the sequence $u_i$ can have at most one simple blow-up point, and this point must be a local maximum of $H$. Moreover, $u_i$ behaves almost like a family of the standard functions $u_q$.

However, based on the results in the subcritical case, even one point of blow-up is not possible. Let’s take $u_i$, the sequence of critical points of the functional $J_p$, obtained in section 2. We can get that from the proof of proposition 3.3

\[
J_\tau(u_i) \leq \min_k \left\{ H(r_k)|S^{n-1}| - \delta \right\}
\]

for all the positive local maxima $r_k$ of $H$. Now if $\{u_i\}$ blow up at $x_0$, then by lemma 4.7 we have
\[ J_r(u_i) \rightarrow H(x_0)|S^{n-1}|, \]

And we get a contradiction. This proves proposition 4.6. \( \square \)

From the three previous cases, we can conclude that the sequence \( u_i \) is uniformly bounded, finishing the proof of theorem 4.1. By the Arzela-Ascoli Theorem, the sequence \( u_i \) has a subsequence converging to a solution of (1.1). Hence Theorem 1.1 has been proven.

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