Transition probability estimates for subordinate random walks

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Abstract
Let $S_n$ be a symmetric simple random walk on the integer lattice $\mathbb{Z}^d$. For a Bernstein function $\phi$ we consider a random walk $S_n^\phi$ which is subordinated to $S_n$. Under a certain assumption on the behaviour of $\phi$ at zero we establish global estimates for the transition probabilities of the random walk $S_n^\phi$. The main tools that we apply are a parabolic Harnack inequality and appropriate bounds for the transition kernel of the corresponding continuous time random walk.

KEYWORDS
Bernstein function, discrete subordination, parabolic Harnack inequality, random walk, scaling condition, transition probability estimate

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1 | INTRODUCTION

The main aim of this article is to obtain global estimates for the transition probabilities for a class of random walks on the integer lattice $\mathbb{Z}^d$ that are subordinated to the simple random walk. Random walks from this class are obtained via discrete subordination which was defined in [8]. They have neither second moment nor finite support and thus studying their long time behaviour becomes very demanding. The procedure of discrete subordination can be regarded as a discrete counterpart of Bochner’s subordination for semigroups of operators which was widely applied in probability theory for continuous time Markov processes.

To be more precise, let $P$ be the one-step transition operator of the simple (symmetric) random walk $S_n$ on the space $\mathbb{Z}^d$, that is $Pf(x) = \frac{1}{2d} \sum_{j=1}^{d} f(x \pm e_j)$, where $e_j$ is the unit vector in $\mathbb{Z}^d$ with $j^{th}$ component 1. For any Bernstein function $\phi$ such that $\phi(0) = 0$ and $\phi(1) = 1$ we define a new transition operator $P^\phi$ via the following functional equation

$$I - P^\phi = \phi(I - P).$$
The operator $P^\phi - I$ generates a random walk $S^\phi_n$ which is the subordinate random walk related to the function $\phi$, see Section 2 for the probabilistic definition.

In this article we are concerned with the transition probabilities of the random walk $S^\phi_n$ which are defined as $p^\phi(n, x, y) = \mathbb{P}(S^\phi_n = y)$. In the course of study we assume that $\phi$ is a complete Bernstein function and satisfies the following scaling condition: there are some constants $c_*, c^* > 0$ and $0 < \alpha_* \leq \alpha^* < 1$ such that

$$c_* \left( \frac{R}{r} \right)^{\alpha_*} \leq \frac{\phi(R)}{\phi(r)} \leq c^* \left( \frac{R}{r} \right)^{\alpha^*}, \quad 0 < r \leq R \leq 1. \quad (1.1)$$

Under these two assumptions we establish global estimates for the function $p^\phi(n, x, y)$, that is we prove that for all $x, y \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ it holds

$$p^\phi(n, x, y) \asymp \min\left\{ \left( \frac{\phi^{-1}(n^{-1})}{n^\frac{d}{2}} \right)^{\frac{d}{2}}, \frac{n \phi(|x-y|^2)}{|x-y|^d} \right\}, \quad (1.2)$$

see Theorem 3.1, Theorem 5.1 and Theorem 6.17. In the above relation, the symbol $\asymp$ means that the ratio of the two expressions is bounded from below and from above by some positive constants.

Similar questions have already been addressed in the literature. In [4] global estimates for the transition probabilities of random walks with unbounded range on $\mathbb{Z}^d$ were established under the assumption that the one-step transition probability from $x$ to $y$ is a stable-like function, i.e. it is comparable to the regularly varying function $|x-y|^{-(d+\alpha)}$ for $\alpha \in (0, 2)$. Let us compare this result to our estimates (1.2). In [17] it was proved that under the same assumptions as in the present paper, the one-step transition probability of the subordinate random walk $S^\phi_n$ satisfies

$$p^\phi(1, x, y) \asymp |x-y|^{-d} \phi(|x-y|^2), \quad \text{for } x \neq y. \quad (1.3)$$

In particular, if $\alpha_* = \alpha^*$ in (1.1) then we are in the scope of [4] but it may well happen that $0 < \alpha_* < \alpha^* < 1$. Moreover, condition (1.1) means that the function $\phi$ is a $O$-regularly varying function at 0 with Matuszewski indices contained in (0,1), see [9, Sec. 2]. Complete Bernstein functions with such behaviour at zero can be found in the closing table of [20] and include functions: $\phi(\lambda) = \lambda^\alpha + \lambda^\beta$, $\alpha, \beta \in (0, 1)$; $\phi(\lambda) = \lambda^{\alpha}(\log(1+\lambda))^\beta$, $\alpha \in (0, 1)$, $\beta \in (0, 1-\alpha)$; $\phi(\lambda) = (\log(\cosh(\sqrt{\lambda})))^{\alpha}$, for $\alpha \in (0, 1)$, etc. It is also possible to construct examples of complete Bernstein functions which fulfil (1.1) and are not comparable to any regularly varying function, see e.g. [13]. This shows that our estimates apply to a class of random walks whose one-step transition probabilities may not be comparable to a regularly varying function which goes beyond the assumptions of [4].

In [18] global estimates for transition probabilities for a class of Markov chains on a uniformly discrete metric measure space were proved under the assumption that the one-step transition kernel is comparable to a regularly varying function times a homogenous volume growth function. We mention here further related papers and monographs which focus on estimates of transition probabilities of random walks [1–3, 12, 14, 15, 21, 22, 24].

A class of subordinate random walks was introduced in paper [8] in the context of random walks on groups. In [7] authors established asymptotics of the transition probabilities of subordinate random walks on $\mathbb{Z}^d$ under the assumption that $\phi$ is regularly varying at zero. This result is valid in a specific region which depends on the time and the space variables. It implies that the corresponding estimates hold only in that region whereas (1.2) is true for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. There are more papers where subordinate random walks were studied from potential-theoretic point of view, see [5, 6, 11, 16, 17]. Note that discrete subordination allows us to efficiently construct examples of random walks with the controlled tail behaviour. In particular, the regular variation at zero of the function $\phi$ is a necessary and sufficient condition for $S^\phi_n$ to belong to the domain of attraction of a stable law, see [7, 16].

Let us comment on the structure and methods of the article. In Section 2 we give the precise definition of the subordinate random walk and we prove some auxiliary results which include an estimate for the time to leave a ball for the random walk $S^\phi_n$. Our proof is an application of the concentration inequality from [19]. Section 3 is devoted to the proof of the on-diagonal bound for the kernel $p^\phi(n, x, y)$. For this we use the Fourier analytic approach which was previously applied in [7] to find asymptotics of $p^\phi(n, x, y)$ under the assumption that $\phi$ is a regularly varying function at zero. In Section 4 we prove a parabolic Harnack inequality which is in itself a valuable contribution and this is the main tool that we use.
to obtain off-diagonal bounds for $p^\phi(n, x, y)$. To show this inequality we follow the elegant approach of [4], which was also applied in [18]. In Section 5 we obtain the global lower bound by the application of the parabolic Harnack inequality combined with the on-diagonal estimate. Section 6 is a twofold paragraph. In the first part we study the continuous time random walk which is constructed from $S^\phi_n$ with the aid of an independent Poisson process. For such a process we find the upper heat kernel estimate. To get this result we apply the marvellous approach of [10] where the authors study stability of heat kernel estimates for jump processes on metric measure spaces. In the second part we apply estimates for the continuous time random walk to prove hitting time estimates and, finally, upper bounds for $p^\phi(n, x, y)$.

**Notation.** Throughout the paper $C, c, c_1, c_2, \ldots$ denote absolute constants. Their labelling starts anew in each statement and their dependence on the function $\phi$ and on the dimension $d$ is not mentioned explicitly. The cardinality of a set $A \subset \mathbb{Z}^d$ is denoted by $|A|$. The Euclidean distance between $x$ and $y$ is denoted by $|x - y|$. For $x \in \mathbb{R}^d$ and $r > 0$, we write $B(x, r) = \{ y \in \mathbb{Z}^d : |y - x| < r \}$ and $B_r = B(0, r)$. We use the notation $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For any two positive functions $f$ and $g$, we write $f \asymp g$ if there exist constants $c_1, c_2 > 0$ such that $c_1 \leq g/f \leq c_2$.

## 2. Preliminaries

Let $S_n = X_1 + \cdots + X_n$ be the simple (symmetric) random walk in $\mathbb{Z}^d$ which starts from the origin. This means $(X_k)_{k \geq 1}$ is a sequence of independent identically distributed random variables defined on a given probability space $(\Omega, \mathcal{A}, P)$ with distribution $P(X_k = e_i) = P(X_k = -e_i) = 1/2d$, for each $i = 1, 2, \ldots, d$. Here $e_i$ is the $i$th unit vector in $\mathbb{Z}^d$.

Let $\phi$ be a Bernstein function such that $\phi(0) = 0, \phi(1) = 1$. Such a function admits the following integral representation

$$\phi(\lambda) = \ell \lambda + \int_{(0, \infty)} (1 - e^{-\lambda t}) \mu(\text{d}t),$$

for $\ell \geq 0$ and a measure $\mu$ on $(0, \infty)$ satisfying $\int_{(0, \infty)} (1 \wedge t) \mu(\text{d}t) < \infty$, see [20, Sec. 3].

We consider a sequence of positive numbers $a^\phi_m$ which is related to the function $\phi$ and is defined as

$$a^\phi_m = \ell \delta_1(m) + \frac{1}{m!} \int_{(0, \infty)} t^m e^{-t} \mu(\text{d}t), \quad m \geq 1,$$

where $\delta_x$ is the Dirac measure at $x$. One easily verifies that

$$\sum_{m=1}^{\infty} a^\phi_m = \ell + \int_{(0, \infty)} (e^t - 1) \mu(\text{d}t) = \ell + \int_{(0, \infty)} (1 - e^{-t}) \mu(\text{d}t) = \phi(1) = 1.$$

Let $\tau_n = R_1 + \cdots + R_n$ be a random walk on $\mathbb{Z}_+$ with increments $R_i$ that are independent of the random walk $S_n$ and have the distribution given by $P(R_1 = m) = a^\phi_m$. A subordinate random walk is defined as $S^\phi_n := S_{\tau_n}$, for all $n \geq 0$. Notice that the one-step transition probability $p^\phi(1, x, y)$ of the random walk $S^\phi_n$ is of the form

$$p^\phi(1, x, y) = P^\phi(S^\phi_1 = y) = \sum_{m=1}^{\infty} P^\phi(S_{R_1} = y \mid R_1 = m) a^\phi_m = \sum_{m=1}^{\infty} p(m, x, y) a^\phi_m,$$

where $p(n, x, y) = P^\phi(S_n = y)$ stands for the $n$-step transition probability of the simple random walk $S_n$. We use the notation $p^\phi(n, x, x) = p^\phi(n, 0)$ and $p^\phi(1, x, y) = p^\phi(x, y) = p^\phi(x - y)$.

In the course of study we always assume that $\phi$ is a complete Bernstein function. Recall that this means that the measure $\mu$ from (2.1) has a completely monotone density with respect to Lebesgue measure, see [20, Def. 6.1.]. We additionally require that $\phi$ has no drift term, that is $\ell = 0$ in (2.1). Next assumption on the function $\phi$ is that it satisfies scaling condition (1.1). These assumptions are not explicitly stated in the results.
2.1 Auxiliary results

We repeatedly use the fact that
\[ c'r^d \leq |B(x, r)| \leq c''r^d, \quad x \in \mathbb{Z}^d, \] (2.4)
for constants \( c', c'' > 0 \) which depend only on the dimension \( d \).

We recall that for any Bernstein function \( \phi \) it holds \( \phi(\lambda t) \leq \lambda \phi(t) \), for all \( \lambda \geq 1, t > 0 \), which implies
\[ \frac{\phi(v)}{\phi(u)} \leq \frac{v}{u}, \quad 0 < u \leq v. \] (2.5)

We formulate bounds for the inverse function \( \phi^{-1} \) which easily follow from (1.1) and take the form
\[ \left( \frac{1}{c^*} \right)^{1/\alpha^*} \left( \frac{R}{r} \right)^{1/\alpha^*} \phi^{-1}(R) \leq \phi^{-1}(r) \leq \left( \frac{1}{c_\phi} \right)^{1/\alpha_*} \left( \frac{R}{r} \right)^{1/\alpha_*}, \quad 0 < r \leq R \leq 1. \] (2.6)

Throughout the paper we use the following decreasing function
\[ j(r) = r^{-d} \phi(r^{-2}), \quad r > 0. \] (2.7)
Notice that with this notation (1.3) becomes \( p^\phi(1, x, y) \simeq j(|x - y|), x \neq y. \)

**Lemma 2.1.** There exists a constant \( c_0 > 0 \) such that
\[ \sum_{y \in B(x, r)^c} j(|x - y|) \leq c_0 \phi(r^{-2}) \]
for every \( x \in \mathbb{Z}^d \) and \( r > 0. \)

**Proof.** Assume that \( r \geq 1. \) By (1.1), we have
\[
\sum_{y \in B(x, r)^c} j(|x - y|) \leq \sum_{l=0}^{\infty} \sum_{2^l r < |x - y| < 2^{l+1} r} j(2^l r)
\leq c'' 2^d \phi(r^{-2}) \sum_{l=0}^{\infty} \frac{\phi\left((2^l r)^{-2}\right)}{\phi(r^{-2})} \leq c_0 \phi(r^{-2}).
\] If \( r \in (0, 1) \) then \( B(x, r)^c = B(x, 1)^c. \) Therefore
\[
\sum_{y \in B(x, r)^c} j(|x - y|) = \sum_{y \in B(x, 1)^c} j(|x - y|) \leq c_0 \phi(1^{-2}) \leq c_0 \phi(r^{-2})
\]
what finishes the proof. \( \square \)

Next we prove a pair of useful estimates for the subordinate random walk.

**Lemma 2.2.** There exists a constant \( C_1 > 0 \) such that
\[ p^\phi(x, x) \geq C_1, \quad x \in \mathbb{Z}^d. \]
Proof. By [14, Thm. 1.2.1],

\[ P(S_{2m} = 0) \asymp m^{-d/2}, \ m \in \mathbb{N}. \]

This and the fact that \( P(S_{2m-1} = 0) = 0 \) combined with (2.3), [17, Lemma 3.1.] and (1.1) yield for all \( x \in \mathbb{Z}^d \)

\[
p^\phi(x, x) \geq c_1 \sum_{m=1}^{\infty} \phi\left(\frac{(2m-1)}{2m}\right) m^{-d/2} \geq c_1 \sum_{m=1}^{\infty} m^{-\alpha -d/2} > 0,
\]

as desired. \( \square \)

2.2 Estimates for probability of leaving a ball

In this paragraph we establish the following result.

Theorem 2.3. There exists a constant \( \gamma \in (0, 1) \) such that for all \( r > 0 \)

\[
P^x\left(\max_{k \leq \lfloor \gamma/\phi(r^{-2}) \rfloor} ||S_k^\phi - x|| \geq r/2\right) \leq 1/4. \tag{2.8}
\]

Our approach is based on the application of the concentration inequality from [19, Lemma on page 949]. Since we use it at numerous occasions in the paper, we formulate it as a separate result.

Lemma 2.4. Let \((X_k)_{k \geq 1}\) be a sequence of independent random variables taking values in \( \mathbb{R}^d \) and with the common distribution function denoted by \( F \). Let \( S_n = \sum_{i=1}^{n} X_i \) be the corresponding random walk and set \( M_n = \max_{k \leq n} |S_k| \). Define for \( x > 0 \)

\[
G(x) = P(|X_1| > x), \quad K(x) = x^{-2} \int_{|y| < x} |y|^2 \ dF(y),
\]

\[
M(x) = x^{-1} \left| \int_{|y| < x} y \ dF(y) \right|, \quad h(x) = G(x) + K(x) + M(x).
\]

Then there is a constant \( C > 0 \), which depends only on the dimension \( d \), such that for all \( n \in \mathbb{N} \) and all \( a > 0 \)

\[
P(M_n \geq a) \leq Cnh(a) \quad \text{and} \quad P(M_n \leq a) \leq \frac{C}{nh(a)}.
\]

Since the random walk \( S_n^\phi \) is symmetric, the associated function \( h \) is of the form

\[
h(x) = P( |S_1^\phi| > x ) + x^{-2} \int_{|y| < x} |y|^2 \ dF(y), \tag{2.9}
\]

where \( F \) is the distribution function of the random variable \( S_1^\phi \). Before we prove Theorem 2.3 we show that under the scaling condition (1.1) the function \( h \) is dominated by the function \( \phi \).

Lemma 2.5. In the above notation, there is a constant \( C \geq 1 \) such that

\[
h(x) \leq C\phi(x^{-2}), \quad x > 0.
\]
Proof. First observe that if \( x \in (0, 1) \) then \( h(x) = P(S_1^\phi \neq 0) \) and whence the result follows. Assume next that \( x \geq 1 \). Using (1.3) and (1.1) we get

\[
P\left( |S_1^\phi| > x \right) \leq c_1 \sum_{|y| > x} |y|^{-d} \phi(|y|^{-2}) \leq \frac{c_1}{c_2} \phi(x^{-2}) \sum_{|y| > x} |y|^{-d} (x/|y|)^{2\alpha_x}
\]

\[
\leq c_2x^{2\alpha_x} \phi(x^{-2}) \int_x^\infty r^{-d-2\alpha_x} r^{d-1} dr = c_3 \phi(x^{-2}).
\]

We can similarly show that

\[
x^{-2} \int_{|y| \leq x} |y|^2 \ dF(y) \leq c_4 \phi(x^{-2})
\]

for some constant \( c_4 > 0 \) and the proof is finished.

Proof of Theorem 2.3. We first consider the case \( r < 1 \). Since \( \phi \) is increasing and \( \phi(1) = 1 \), we have \( y/\phi(r^{-2}) < 1 \), for any \( \gamma \in (0, 1) \). Therefore

\[
\max_{k \leq \lfloor \gamma/\phi(r^{-2}) \rfloor} |S_k^\phi - x| = |S_0^\phi - x|
\]

and thus for any \( r < 1 \) it holds

\[
P^x\left( \max_{k \leq \lfloor \gamma/\phi(r^{-2}) \rfloor} |S_k^\phi - x| \geq r/2 \right) = 0.
\]

Assume that \( r \geq 1 \). Applying Lemma 2.4 we get

\[
P^x\left( \max_{k \leq \lfloor \gamma/\phi(r^{-2}) \rfloor} |S_k^\phi - x| \geq r/2 \right) \leq c_1 \lfloor \gamma/\phi(r^{-2}) \rfloor h(r/2),
\]

where \( c_1 \) depends only on the dimension \( d \). By Lemma 2.5 and (2.5),

\[
P^x\left( \max_{k \leq \lfloor \gamma/\phi(r^{-2}) \rfloor} |S_k^\phi - x| \geq r/2 \right) \leq 4c_1 C \lfloor \gamma/\phi(r^{-2}) \rfloor \phi(r^{-2}) \leq 4c_1 C \gamma.
\]

Choosing \( \gamma = \frac{1}{2} \wedge \frac{1}{16c_1 C} \) we obtain (2.8) for all \( r > 0 \).}

3 \quad ON-DIAGONAL BOUNDS

In this section we establish the on-diagonal bounds. We apply a Fourier analytic method which is extracted from [7].

Theorem 3.1. For all \( n \in \mathbb{N} \) it holds

\[
p^\phi(n, 0) \asymp \left( \phi^{-1}(n^{-1}) \right)^{d/2}.
\]

Proof. Let \( \Psi \) be the characteristic function of the simple random walk \( S \). Then the characteristic function of \( S_0^\phi \) is \( \Psi^\phi(\theta) = 1 - \phi(1 - \Psi(\theta)) \), see [8]. Thus, by the Fourier inversion formula,
\[ p^d(n, 0) = \frac{1}{(2\pi)^d} \int_{D_d} (1 - \phi(1 - \Psi(\theta)))^n d\theta, \]  

(3.2)

where \( D_d = [-\pi, \pi]^d \). We fix \( \epsilon > 0 \) and first we estimate the integral in (3.2) over the set

\[ D^\epsilon_d := \{ \theta \in D_d : |\theta| \geq \epsilon \}. \]

Since \( |1 - \phi(1 - \Psi(\theta))| = 1 \) if and only if \( \theta \in 2\pi \mathbb{Z}^d \), see [7, Claim 2], it holds that \( |1 - \phi(1 - \Psi(\theta))| < 1 - \eta \) for all \( \theta \in D^\epsilon_d \) and for some \( \eta \in (0, 1) \). Hence

\[ \frac{1}{(2\pi)^d} \int_{D^\epsilon_d} |1 - \phi(1 - \Psi(\theta))| d\theta \leq (1 - \eta)^n. \]

Next, we consider the remaining part of the integral in (3.2), that is over the ball \( B_\epsilon \). For \( n \in \mathbb{N} \) we set \( a_n = (\phi^{-1}(n^{-1}))^{1/2} \) and by the change of variable we get

\[ a_n^{-d} \int_{|\theta| < \epsilon} (1 - \phi(1 - \Psi(\theta)))^n d\theta = \int_{|\xi| < \epsilon/a_n} (1 - \phi(1 - \Psi(a_n \xi)))^n d\xi. \]

To finish the proof we need to show that for some \( c_1, c_2 > 0 \)

\[ c_1 \leq \int_{|\xi| < \epsilon/a_n} (1 - \phi(1 - \Psi(a_n \xi)))^n d\xi \leq c_2. \]  

(3.3)

Notice that it suffices to prove (3.3) only for \( n \) large enough, as the integrand in (3.3) is strictly positive if \( \epsilon \) is small enough, and thus in the end of the proof we can change constants appropriately to estimate the expression in (3.2) for all \( n \).

We observe that

\[ \lim_{n \to \infty} \frac{1 - \Psi(a_n \xi)}{|a_n \xi|^2/d} = \frac{1}{2}. \]  

(3.4)

Indeed, this follows easily from the fact that

\[ \Psi(\theta) = \frac{1}{d} \sum_{m=1}^d \cos(\theta_m), \quad \theta = (\theta_1, \theta_2, ..., \theta_d), \]

and, for some \( c_3 > 0 \) and for all \( x \in \mathbb{R} \),

\[ |1 - \cos(x) - x^2/2| \leq c_3 x^4. \]

We next prove that for some \( c_4, c_5 > 0 \) and for all \( n \in \mathbb{N} \)

\[ c_4 \left( |\xi|^{2\alpha_\star} \wedge |\xi|^{2\alpha_\star^*} \right) \leq n \phi(1 - \Psi(a_n \xi)) \leq c_5 \left( |\xi|^{2\alpha_\star} \vee |\xi|^{2\alpha_\star^*} \right). \]  

(3.5)

For that we establish the following simple result.

Claim 1. Let \( (a_n)_n \) and \( (b_n)_n \) be two sequences of positive numbers both tending to zero and such that \( \lim_{n \to \infty} (a_n/b_n) = 1 \). Then there exists a constant \( c_6 > 0 \) such that
\[ c_6^{-1} \leq \frac{\phi(a_n)}{\phi(b_n)} \leq c_6, \quad n \in \mathbb{N}. \] (3.6)

**Proof of Claim 1.** Scaling condition (1.1) implies that, for some \( c_7 > 0 \),

\[ c_7^{-1}((x/y)^{a^+} \land (x/y)^{x^+}) \leq \frac{\phi(x)}{\phi(y)} \leq c_7((x/y)^{a^+} \lor (x/y)^{x^+}), \quad x, y \in (0, 1). \]

With this inequality it is straightforward to obtain (3.6).

By Claim 1 and (3.4),

\[ c_8^{-1} \leq \frac{\phi(1 - \Psi(a_n \xi))}{\phi\left(\frac{|a_n \xi|^2}{2d}\right)} \leq c_8 \]

and whence

\[ n \phi(1 - \Psi(a_n \xi)) = \frac{\phi(1 - \Psi(a_n \xi)) \phi\left(\frac{|a_n \xi|^2}{2d}\right)}{\phi\left(\frac{|a_n \xi|^2}{2d}\right)} \leq \frac{\phi\left(\frac{a_n^2 |\xi|^2}{2d}\right)}{\phi(a_n^2)}. \] (3.7)

Applying scaling condition (1.1) in (3.7) we get (3.5).

Next, we notice that

\[ \lim_{n \to \infty} \frac{n \log(1 - \phi(1 - \Psi(a_n \xi)))}{-n \phi(1 - \Psi(a_n \xi))} = 1. \]

Thus, by (3.5), for \( n \) large enough,

\[ \int_{|\xi| < a_n} e^{-c_9\left(|\xi|^{2a^+} \lor |\xi|^{2x^+}\right)} d\xi \leq \int_{|\xi| < c/a_n} (1 - \phi(1 - \Psi(a_n \xi))) \ d\xi \leq \int_{|\xi| < c/a_n} e^{-c_{10}\left(|\xi|^{2a^+} \lor |\xi|^{2x^+}\right)} d\xi. \]

Since both of the side integrals converge to positive constants as \( n \) goes to infinity, we conclude that (3.3) is valid for \( n \) large enough and the proof is finished.

**Corollary 3.2.** There is a constant \( c > 0 \) such that

\[ p^\phi(n, x, y) \leq c(\phi^{-1}(n^{-1}))^{d/2}, \quad \text{for } n \in \mathbb{N} \text{ and } x, y \in \mathbb{Z}^d. \]

**Proof.** This follows by Theorem 3.1 combined with the Cauchy–Schwarz inequality.

### 4 PARABOLIC HARNACK INEQUALITY

In this section we prove the parabolic Harnack inequality which is the main tool that we use to obtain off-diagonal bounds in Sections 5 and 6. We follow closely the elegant approach of [4] but we emphasize that for the case that we undertake in the paper it requires numerous adjustments and alterations.

Let \( P = \mathbb{N}_0 \times \mathbb{Z}^d \) and consider the \( P \)-valued Markov chain \( \left(V_k, S_k^\phi\right)_{k \geq 0} \), where \( V_k = V_0 + k \). We write \( P^{(j, x)} \) for the law of \( \left(V_k, S_k^\phi\right) \) when it starts from \((j, x)\) and we set \( F_j = \sigma\left\{ V_k, S_k^\phi : k \leq j \right\} \). A bounded function \( q \) defined on \( P \) is
called parabolic on a subset $D \subseteq P$ if $q\left(V_{k+1}, S_{k+1}^{\phi}\right)$ is a martingale, where $\tau_D$ denotes the exit time of the Markov chain \( \left(V_k, S_k^{\phi}\right) \) from the set $D$. We now prove the following important observation.

**Lemma 4.1.** For each $n_0 \in \mathbb{N}$ and $x_0 \in \mathbb{Z}^d$ the function $q(k, x) = p^\phi(n_0 - k, x, x_0)$ is parabolic on the set $\{0, 1, 2, \ldots, n_0\} \times \mathbb{Z}^d$.

**Proof.** By the Markov property,

\[
\mathbb{E}\left[ q\left(V_{k+1}, S_{k+1}^{\phi}\right) \mid F_k \right] = \mathbb{E}\left[ \left(p^\phi\left(n_0 - V_1, S_1^{\phi}, x_0\right)\right) \right] = \sum_{x \in \mathbb{Z}^d} p^\phi\left(1, S_1^{\phi}, x\right) p^\phi\left(n_0 - V_1 - 1, x, x_0\right) = q \left(V_k, S_k^{\phi}\right),
\]

where the last equality follows by the semigroup relation. \(\square\)

We introduce the notation

\[ Q(k, x, r) = \{k, k+1, \ldots, k + \lfloor \gamma / \phi(r^{-2}) \rfloor\} \times B(x, r), \]

where $\gamma$ is the constant from Theorem 2.3. We fix the following two constants

\[ B = 3 \vee (2/c_\star)^{1/2\alpha_*}, \quad b = 3 \vee (\lfloor (3/c_\star)^{1/\alpha_*} \rfloor + 1). \quad (4.1) \]

The main result of this section is the following theorem.

**Theorem 4.2.** There exists a constant $C_{PH} > 0$ such that for every nonnegative, bounded function $q$ on $P$ which is parabolic on the set \( \left\{0, 1, 2, \ldots, \left\lfloor \gamma / \phi(R^{-2}) \right\rfloor \right\} \times \mathbb{Z}^d \), the following inequality holds

\[ \max_{(k, y) \in Q\left(\gamma / \phi(R^{-2})\right) \times \mathbb{Z}^d} q(k, y) \leq C_{PH} \min_{w \in B(z, R/B)} q(0, w) \quad (4.2) \]

for all $z \in \mathbb{Z}^d$ and for $R$ large enough.

Before we prove this theorem we need to establish a series of lemmas. Let

\[ \tau(k, x, r) := \min\{l \geq 0 : (V_l, S_l^{\phi}) \not\in Q(k, x, r)\} \]

and put $\tau(x, r) = \tau(0, x, r)$. We observe that $\tau(k, x, r) \leq \lfloor \gamma / \phi(r^{-2}) \rfloor + 1$. For a non-empty set $A \subseteq Q(0, x, r)$, we define

\[ A(k) = \{y \in \mathbb{Z}^d : (k, y) \in A\} \subset \mathbb{Z}^d. \]

We now fix a non-empty $A \subseteq Q(0, x, r)$ such that $A(0) = \emptyset$ and we set

\[ N(k, x) = P^{(k,x)}(S_1^{\phi} \in A(k + 1)) \mathbbm{1}_{A}(k, x). \]

For any $A \subseteq P$ we also define

\[ T_A = \min\{n \geq 0 : (V_n, S_n^{\phi}) \in A\}, \text{ and } T_\emptyset = \infty. \]
Lemma 4.3. In the above notation, let

\[ J_n = \mathbb{1}_A(V_n, S_n^\phi) - \mathbb{1}_A(V_0, S_0^\phi) - \sum_{k=0}^{n-1} N(V_k, S_k^\phi). \]

The process \( J_{n\wedge T_A} \) is a \( \mathcal{F} \)-martingale.

Proof. We have

\[ \mathbb{E}[J_{(k+1)\wedge T_A} - J_{k\wedge T_A} \mid F_k] = \mathbb{E}\left[ \mathbb{1}_A(V_{(k+1)\wedge T_A}, S_{(k+1)\wedge T_A}^\phi) - \mathbb{1}_A(V_{k\wedge T_A}, S_{k\wedge T_A}^\phi) - N(V_{k\wedge T_A}, S_{k\wedge T_A}^\phi) \mid F_k \right]. \]

If \( T_A \leq k \) then the right-hand side of the identity above is zero. If \( T_A > k \) then

\[ \mathbb{E}[J_{(k+1)\wedge T_A} - J_{k\wedge T_A} \mid F_k] = \mathbb{E}\left[ \mathbb{1}_A(V_{k+1}, S_{k+1}^\phi) \mid F_k \right] - N(V_k, S_k^\phi) = \mathbb{P}(V_k, S_k^\phi)(S_{k+1}^\phi \in A(V_k + 1)) - N(V_k, S_k^\phi) = 0, \]

as desired. \( \square \)

Proposition 4.4. There exists a constant \( \theta_1 \in (0, 1) \) such that

\[ \mathbb{P}^{(0,x)}(T_A < \tau(x, r)) \geq \theta_1 |A| j(r). \quad (4.3) \]

Proof. We claim that \( |y/\phi(r^{-2})| + 1 \leq 2y/\phi(r^{-2}) \). Indeed, we have \( A(0) = \emptyset \) and \( A \neq \emptyset \) so it follows that \( A(k) \neq \emptyset \), for some \( k \geq 1 \). Thus \( y/\phi(r^{-2}) \geq 1 \), which clearly yields the claim.

We first assume that \( \mathbb{P}^{(0,x)}(T_A \leq \tau(x, r)) \geq 1/4 \). By (2.4) we get

\[ |A| j(r) \leq c''(\lfloor |y/\phi(r^{-2})| + 1 \rfloor \phi(r^{-2}) \leq 2c''r. \]

Hence

\[ \mathbb{P}^{(0,x)}(T_A \leq \tau(x, r)) \geq \frac{1}{4} = \frac{1}{8c''r} - 2c''r \geq \frac{1}{8c''r} |A| j(r). \]

Assume that \( \mathbb{P}^{(0,x)}(T_A \leq \tau(x, r)) < 1/4 \). Let \( M := T_A \wedge \tau(x, r) \). By Lemma 4.3 and the Optional Stopping Theorem, \( \mathbb{E}[J_M] = \mathbb{E}[J_0] = 0 \). This and the fact that \( (0, X_0) \notin A \) imply

\[ \mathbb{E}^{(0,x)}\left[ \mathbb{1}_A(M, S_M^\phi) \right] = \mathbb{E}^{(0,x)}\left[ \sum_{k=0}^{M-1} N(k, S_k^\phi) \right]. \]

By (1.3), Lemma 2.2 and using monotonicity of the function \( j \), we get that for \( (k, w) \in Q(0, x, r) \cap A^c \)

\[ N(k, w) = \sum_{y \in A(k+1) \setminus \{w\}} p^\phi(w, y) + p^\phi(w, w) \mathbb{1}_{A(k+1)}(w) \geq c_1 j(2r) |A(k + 1) \setminus \{w\}| + C_1 \mathbb{1}_{A(k+1)}(w) \geq c_2 j(r) |A(k + 1)|. \]
Observe that if $M \geq [\gamma / \phi(r^{-2})]$ then $\sum_{k=0}^{M-1} |A(k+1)| = |A|$. Hence, on the set \{ $M \geq [\gamma / \phi(r^{-2})]$ \} we have

$$\sum_{k=0}^{M-1} N(k, S^\phi_k) \geq \sum_{k=0}^{M-1} c_2 |A(k+1)|j(r) = c_2 |A|j(r).$$

Since $P^{(0,x)}(T_A \leq \tau(x,r)) = E^{(0,x)}\left[1_A(M, S^\phi_M)\right]$, we get

$$P^{(0,x)}(T_A \leq \tau(x,r)) \geq c_2 |A|j(r)P^{(0,x)}(M \geq [\gamma / \phi(r^{-2})])$$

$$= c_2 |A|j(r) \left(1 - P^{(0,x)}(T_A < \tau(x,r), T_A < [\gamma / \phi(r^{-2})]) - P^{(0,x)}(\tau(x,r) < T_A, \tau(x,r) < [\gamma / \phi(r^{-2})])\right)$$

$$\geq c_2 |A|j(r) \left(1 - P^{(0,x)}(T_A < \tau(x,r)) - P^{(0,x)}(\tau(x,r) \leq [\gamma / \phi(r^{-2})])\right).$$

We notice that $\max_{k \leq [\gamma / \phi(r^{-2})]} |S^\phi_k - x| \geq r/2$ if $\tau(x,r) \leq [\gamma / \phi(r^{-2})]$. Thus (2.8) implies

$$P^{(0,x)}(\tau(x,r) \leq [\gamma / \phi(r^{-2})]) \leq P^{(0,x)}\left(\max_{k \leq [\gamma / \phi(r^{-2})]} |S^\phi_k - x| \geq r/2\right) \leq 1/4.$$

We conclude the desired result with $\theta_1 = \frac{1}{2} \wedge \frac{1}{8c} \gamma \wedge \frac{c_2}{2}$.

**Lemma 4.5.** There exists a constant $\theta_2 > 0$ such that for $(k, x) \in Q(0, z, R/2)$ and for $r > 0$ such that $k \geq [\gamma / \phi(r^{-2})] + 1$ we have

$$P^{(0,x)}(T_{U(k,x,r)} \leq \tau(z,R)) \geq \theta_2 j(R) \frac{j(r)}{j(r)}$$

where $U(k, x, r) = \{k\} \times B(x, r)$.

**Proof.** Let $Q' = \{k, k-1, \ldots, k- [\gamma / \phi(r^{-2})]\} \times B(x, r/2)$. One easily verifies that $Q'(0) = \emptyset$ and $Q' \subseteq Q(0, z, R)$. By Proposition 4.4, we get

$$P^{(0,x)}(T_{Q'} \leq \tau(z,R)) \geq \theta_1 |Q'|j(R) \geq \theta_1 c'([\gamma / \phi(r^{-2})] + 1)(r/2)^d j(R)$$

$$\geq \frac{\theta_1 c'}{2^d} \frac{\gamma}{\phi(r^{-2})} r^d j(R) = \theta_1 c' \frac{j(R)}{j(r)}.$$  

The strong Markov property yields

$$P^{(0,x)}(T_{U(k,x,r)} \leq \tau(z,R)) \geq P^{(0,x)}(T_{U(k,x,r)} < \tau(z,R), T_{Q'} < \tau(z,R))$$

$$= P\left(T_{Q'}, S^\phi_{T_{Q'}} \leq \tau(z,R)\right) P^{(0,x)}(T_{Q'} < \tau(z,R)).$$

We are left to bound from below the first term in (4.4). Observe that if the process $\left(V_k, S^\phi_k\right)$ starts from the point $\left(T_{Q'}, S^\phi_{T_{Q'}}\right)$ and $S^\phi$-coordinate stays in $B(x, r)$ for at least $[\gamma / \phi(r^{-2})]$ steps, then $\left(V_k, S^\phi_k\right)$ hits $U(k, x, r)$ before exiting $Q(0, z, R)$. We also notice that $S^\phi$-coordinate stays in $B(x, r)$ for at least $[\gamma / \phi(r^{-2})]$ steps if for all $T_{Q'} \leq k \leq T_{Q'} + [\gamma / \phi(r^{-2})]$ it holds $|S^\phi_k - S^\phi_{T_{Q'}}| < r/2$. Thus, using Theorem 2.3, we get

$$\left|S^\phi_k - S^\phi_{T_{Q'}}\right| < \frac{r}{2}.$$
\[
\mathbb{P}\left(T_{Q'} - S_{\tau(x,y)}^\phi\right) \left(T_{U(k,x,r)} < \tau(z,R)\right) \geq 3/4
\]

and we conclude that

\[
\mathbb{P}^{(0,x)}\left(T_{U(k,x,r)} < \tau(z,R)\right) \geq \frac{j(R)}{j(r)},
\]

where \(\vartheta_2 = \frac{3c_1}{4}\).

\[\square\]

**Lemma 4.6.** Let \(H(k, w) \geq 0\) be a function on \(P\) such that \(H(k, w) \mathbb{1}_{B(x,2r)(w)} = 0\). There exists a constant \(\vartheta_3 > 0\) which does not depend on \(x, r, H\) and such that

\[
\mathbb{E}^{(0,y)}\left[H\left(V_{\tau(x,r)}, S_{\tau(x,r)}^\phi\right)\right] \leq \vartheta_3 \mathbb{E}^{(0,y)}\left[H\left(V_{\tau(x,r)}, S_{\tau(x,r)}^\phi\right)\right],
\]

for all \(y \in B(x,r/2)\).

**Proof.** It suffices to check validity of (4.5) for \(H = 1_{(k,w)}\) if \(y \in B(x,r/2)\), \(w \notin B(x,2r)\) and \(1 \leq k \leq \lfloor \gamma/\phi(r^{-2}) \rfloor + 1\). With such a choice we have

\[
\mathbb{E}^{(0,y)}\left[1_{(k,w)} \left(V_{\tau(x,r)}, S_{\tau(x,r)}^\phi\right)\right] = \mathbb{E}^{(0,y)}\left[\mathbb{E}^{(0,y)}\left[1_{(k,w)} \left(V_{\tau(x,r)}, S_{\tau(x,r)}^\phi\right) | F_{k-1}\right]\right]
\]

\[
= \mathbb{E}^{(0,y)}\left[1_{\{\tau(x,r) > k-1\}} P^\phi \left(S_{k-1}^\phi, w\right)\right],
\]

(4.6)

Since \(S_{k-1}^\phi \in B(x,r)\), we have \(P^\phi \left(S_{k-1}^\phi, w\right) \geq \inf_{z \in B(x,r)} P^\phi (z,w)\). For \(z \in B(x,r)\) and \(w \notin B(x,2r), z \neq w\) and whence (1.3) implies

\[
\mathbb{E}^{(0,y)}\left[1_{(k,w)} \left(V_{\tau(x,r)}, S_{\tau(x,r)}^\phi\right)\right] \geq c_1 \mathbb{P}^{(0,y)} \left(\tau(x,r) = \lfloor \gamma/\phi(r^{-2}) \rfloor + 1\right) \inf_{z \in B(x,r)} j(|z - w|).
\]

If \(\left(V_k, S_k^\phi\right)\) starts from \((0,y)\) and \(S_k^\phi\)-coordinate stays in \(B(y, r/2)\) for \(\lfloor \gamma/\phi(r^{-2}) \rfloor\) steps then at the same time it also stays in \(B(x,r)\). Hence

\[
\frac{3}{4} \leq \mathbb{P}^{(0,y)} \left(\max_{k \leq \lfloor \gamma/\phi(r^{-2}) \rfloor} |S_k^\phi - y| < \frac{r}{2}\right) \leq \mathbb{P}^{(0,y)} \left(\tau(x,r) = \lfloor \gamma/\phi(r^{-2}) \rfloor + 1\right).
\]

For every \(z \in B(x,r)\) we have \(|z - w| \leq 2|x - w|\). By monotonicity of \(j\) and [17, Lemma 2.4], we get

\[
\inf_{z \in B(x,r)} j(|z - w|) \geq j(2|x - w|) \geq 2^{d-2} j(|x - w|),
\]

We obtain

\[
\mathbb{E}^{(0,y)}\left[1_{(k,w)} \left(V_{\tau(x,r)}, S_{\tau(x,r)}^\phi\right)\right] \geq c_2 j(|x - w|).
\]

Notice that (4.6) remains valid if the process starts from \((0,x)\) instead of \((0,y)\). Similarly we prove that

\[
\mathbb{E}^{(0,x)}\left[1_{(k,w)} \left(V_{\tau(x,r)}, S_{\tau(x,r)}^\phi\right)\right] \leq c_3 j(|x - w|).
\]

The result follows with \(\vartheta_3 = c_3/c_2\).

\[\square\]

We can now prove the parabolic Harnack inequality.
Proof of Theorem 4.2. By multiplying the function $q$ by a constant, we can assume that
\[
\min_{w \in B(z, R/B)} q(0, w) = q(0, v) = 1.
\] (4.7)

Notice that if $q(0, x) = 0$ for some $x \in B(z, R/B)$ then (4.2) is trivially satisfied, as the parabolicity of $q$ implies that
\[
\max_{(k, y) \in Q([y/\phi(R^2)], x, R/B)} q(k, y) = 0.
\]

Let $B$ be the constant defined at (4.1). By Lemma A.2 of the Appendix, there exists a constant $R_0 \geq B$ such that
\[
[\gamma/\phi(r^{-2})] \geq [\gamma/\phi((r/B)^{-2})] + 1, \quad r \geq R_0.
\] (4.8)

Let us fix $r \geq R_0$, $(k, x) \in \mathcal{P}$ and a set $G \subseteq Q(k + 1, x, r/B)$ for which it holds
\[
\frac{|G|}{|Q(k + 1, x, r/B)|} \geq \frac{1}{3}.
\]

We claim that for such a set $G$ there is a constant $c_1 \in (0, 1)$ such that
\[
\mathbb{P}^{(k, x)}(T_G < \tau(k, x, r)) \geq c_1.
\] (4.9)

Indeed, by our choice $G \subseteq Q(k, x, r)$ and $G(k) = \emptyset$. Therefore, Proposition 4.4 and relation (2.5) yield
\[
\mathbb{P}^{(k, x)}(T_G < \tau(k, x, r)) \geq \frac{\vartheta_1}{3} \frac{\gamma}{\phi((r/B)^{-2})} \left( \frac{r}{B} \right)^d r^{-d} \phi(r^{-2}) \geq \frac{\vartheta_1 \gamma c'}{3B^{d+2}} = c_1,
\]
where we can achieve that $c_1 < 1$ by decreasing $c'$ in (2.4) if necessary.

Let $\vartheta_1, \vartheta_2$ and $\vartheta_3$ be the constants from Proposition 4.4, Lemma 4.5 and Lemma 4.6 respectively. We set
\[
\eta = \frac{c_1}{3}, \quad \zeta = \frac{c_1}{3} \wedge \frac{\eta}{\vartheta_2}, \quad a = 2 \vee \left( \frac{2}{c_*} \right)^{1/\alpha_*},
\] (4.10)

where $c_1$ is the constant from relation (4.9) and $c_*, \alpha_* \in (0, 1)$ are the constants from the scaling condition (1.1).

Claim 2. There exists a constant $c_2 > 0$ such that for all $r, R, K > 0$ which satisfy
\[
\frac{r}{R} < 1 \quad \text{and} \quad \frac{r}{R} K^{1/(d+2)} \geq c_2,
\] (4.11)

the following two inequalities hold:
\[
\frac{j(2\sqrt{aR})}{j(r/R_0)} > \frac{1}{\vartheta_2 \zeta K},
\] (4.12)
\[
|Q(0, x, r/B)| j\left( \sqrt{bR} \right) > \frac{3}{\vartheta_1 \zeta K}.
\] (4.13)

We prove this claim in the end of the proof of the theorem and the value of the constant $c_2$ is specified there, see (4.25).

We construct a sequence of points $(k_i, x_i)$ such that $K_i = q(k_i, x_i)$ is large enough and under this condition the sequence $K_i = q(k_i, x_i)$ is increasing and tends to infinity, cf. (4.18). This will finally contradict the fact that $q$ is bounded.
and whence the result will follow. Let us choose \((k_1, x_1) \in Q([\gamma / \phi(R^{-2})], z, R)\) such that it holds

\[
K_1 = q(k_1, x_1) = \max_{(k, y) \in Q([\gamma / \phi(R^{-2})], z, R/B)} q(k, y).
\]

Evidently it suffices to study the case \(c_2 K_1^{-1/(d+2)} < 1 / B\). Suppose that we have already defined the points

\[
(k_1, x_1), (k_2, x_2), \ldots, (k_i, x_i) \in Q([\gamma / \phi(R^{-2})], z, R).
\]

We describe the procedure how to obtain \((k_{i+1}, x_{i+1}) \in Q([\gamma / \phi(R^{-2})], z, R)\). We first define \(r_i\) by

\[
r_i = c_2 K_i^{-1/(d+2)}.
\]

With our choice of constants and using (4.8) one can easily verify that for \(v\) defined in (4.7) it holds

\[
(k_i, x_i) \in Q(0, v, \sqrt{aR}) \quad \text{and} \quad k_i \geq 1 + [\gamma / \phi((r_i / R_0)^{-2})].
\]

Now, suppose that \(q \geq \zeta K_i\) on the set \(U_i := \{k_i\} \times B(x_i, r_i / R_0)\). Since the function \(q\) is parabolic on

\[
D = \left\{0, 1, 2, \ldots, \left[\frac{\gamma}{\phi}\left(\sqrt{aR}^{-2}\right)\right]\right\} \times \mathbb{Z}^d,
\]

we know that \(q\left(V_{k_1 \wedge \tau_D}, S_{k_1 \wedge \tau_D}^\phi\right)\) is a martingale. Thus (4.12) and Lemma 4.5 imply

\[
1 = q(0, v) = E^{(0, v)}\left[q\left(V_{T_{U_i} \wedge \tau_D}, S_{T_{U_i} \wedge \tau_D}^\phi\right)\right]
\geq E^{(0, v)}\left[q\left(V_{T_{U_i} \wedge \tau_D}, S_{T_{U_i} \wedge \tau_D}^\phi\right) \mathbb{1}\{T_{U_i} < \tau(v, 2\sqrt{aR})\}\right]
= E^{(0, v)}\left[q\left(V_{T_{U_i}}, S_{T_{U_i}}^\phi\right) \mathbb{1}\{T_{U_i} < \tau(v, 2\sqrt{aR})\}\right]
\geq \zeta K_i D^{(0, v)}\left(T_{U_i} < \tau(v, 2\sqrt{aR})\right)
\geq \zeta K_i \phi_2 \frac{j(2\sqrt{aR})}{j(r_i / R_0)}
> \zeta K_i \phi_2 \frac{1}{\zeta K_i \phi_2}
= 1,
\]

and we mention that we could apply Lemma 4.5 because of (4.15). Thus we get a contradiction, so there must exist \(y_i \in B(x_i, r_i / R_0)\) such that \(q(k_i, y_i) < \zeta K_i\). Observe that

\[
q(k_i, y_i) < \zeta K_i \leq (c_1 / 3) K_i < K_i / 3
\]

and whence \(x_i \neq y_i\). This in turn implies

\[
r_i \geq R_0.
\]

Suppose next that
\( \mathbb{E}^{(k_l, x_i)} \left[ q \left(V_{\tau(k_l, x_l, r_l)}, S^\phi_{\tau(k_l, x_l, r_l)} \right) 1 \{S^\phi_{\tau(k_l, x_l, r_l)} \notin B(x_l, 2r_l) \} \right] \geq \eta K_l. \)

By Lemma 4.6 we have

\[ \zeta K_l > q(k_l, y_l) = \mathbb{E}^{(k_l, y_l)} \left[ q \left(V_{\tau(k_l, x_l, r_l)}, S^\phi_{\tau(k_l, x_l, r_l)} \right) \right] \]
\[ \geq \mathbb{E}^{(k_l, y_l)} \left[ q \left(V_{\tau(k_l, x_l, r_l)}, S^\phi_{\tau(k_l, x_l, r_l)} \right) 1 \{S^\phi_{\tau(k_l, x_l, r_l)} \notin B(x_l, 2r_l) \} \right] \]
\[ \geq \mathbb{E}^{(k_l, y_l)} \left[ q \left(V_{\tau(k_l, x_l, r_l)}, S^\phi_{\tau(k_l, x_l, r_l)} \right) 1 \{S^\phi_{\tau(k_l, x_l, r_l)} \notin B(x_l, 2r_l) \} \right] \]
\[ \geq \frac{\eta}{\delta_3} K_l \]
\[ \geq \zeta K_l, \]

which again gives a contradiction. Therefore

\[ \mathbb{E}^{(k_l, x_i)} \left[ q \left(V_{\tau(k_l, x_l, r_l)}, S^\phi_{\tau(k_l, x_l, r_l)} \right) 1 \{S^\phi_{\tau(k_l, x_l, r_l)} \notin B(x_l, 2r_l) \} \right] < \eta K_l. \] (4.17)

Define the set

\[ A_l = \{ (j, y) \in Q(k_l + 1, x_l, r_l / B) : q(j, y) \geq \zeta K_l \}. \]

We want to apply Proposition 4.4 for \( A_l \) and \( Q \left( 0, v, \sqrt{b R} \right) \). Clearly \( A_l \subseteq Q \left( k_l + 1, x_l, r_l / B \right) \) and \( A_l(0) = \emptyset \). Moreover, with the aid of (4.8), (4.14) and (1.1) one can verify that \( Q(k_l + 1, x_l, r_l / B) \subseteq Q \left( 0, v, \sqrt{b R} \right) \). Therefore

\[ 1 = q(0, v) = \mathbb{E}^{(0, v)} \left[ q \left(V_{T_{A_l} \wedge T \left( v, \sqrt{b R} \right)}, X_{T_{A_l} \wedge T \left( v, \sqrt{b R} \right)} \right) \right] \]
\[ \geq \mathbb{E}^{(0, v)} \left[ q \left(V_{T_{A_l} \wedge T \left( v, \sqrt{b R} \right)}, X_{T_{A_l} \wedge T \left( v, \sqrt{b R} \right)} \right) 1 \{T_{A_l} < T \left( v, \sqrt{b R} \right)\} \right] \]
\[ = \mathbb{E}^{(0, v)} \left[ q \left(V_{T_{A_l}}, X_{T_{A_l}} \right) 1 \{T_{A_l} < T \left( v, \sqrt{b R} \right)\} \right] \]
\[ \geq \zeta K_l \mathbb{E}^{(0, v)} \left[ T_{A_l} < T \left( v, \sqrt{b R} \right) \right] \]
\[ \geq \zeta K_l \beta_1 |A_l| j \left( \sqrt{b R} \right) \]
\[ \geq \zeta K_l \beta_1 |A_l| j \left( \sqrt{b R} \right) \]
\[ \geq \zeta K_l \beta_1 \frac{|A_l|}{|Q(k_l + 1, x_l, r_l / B)|} \frac{3}{\zeta K_l \beta_1}, \]

where we used (4.13) in the last line. We conclude that

\[ \frac{|A_l|}{|Q(k_l + 1, x_l, r_l / B)|} \leq \frac{1}{3}. \]

Define next

\[ D_l = Q(k_l + 1, x_l, r_l / B) \setminus A_l \quad \text{and} \quad M_l = \max_{Q(k_l + 1, x_l, 2r_l)} q. \]
By \((4.17)\) combined with \((4.9)\), we obtain
\[
K_i = \mathbb{E}(k_i, x_i) \left[ q \left( \mathbb{V}_{T_D_i}, \mathbb{X}_{T_D_i} \right) 1 \{ T_{D_i} < \tau(k_i, x_i, r_i) \} \right] 
+ \mathbb{E}(k_i, x_i) \left[ q \left( \mathbb{V}_{\tau(k_i, x_i, r_i)}, \mathbb{X}_{\tau(k_i, x_i, r_i)} \right) \mathbb{1} \{ \tau(k_i, x_i, r_i) \mathbb{E}(k_i, x_i) \}
\]
\[
\leq \zeta K_i + \eta K_i + M_i \left( 1 - \mathbb{P}(k_i, x_i) \{ T_{D_i} < \tau(k_i, x_i, r_i) \} \right) 
\leq \frac{c_1}{3} K_i + \frac{c_1}{3} K_i + M_i \left( 1 - c_1 \right) 
= \frac{2c_1}{3} K_i + M_i \left( 1 - c_1 \right). 
\]

Hence \(M_i/K_i \geq 1 + \rho\), where \(\rho = c_1/\left(3 \left( 1 - c_1 \right) \right) > 0\). Finally, the point \((k_{i+1}, x_{i+1}) \in Q(k_i + 1, x_i, 2r_i)\) is chosen such that
\[
K_{i+1} = q(k_{i+1}, x_{i+1}) = M_i. 
\]
This implies
\[
K_{i+1} \geq (1 + \rho)K_i. 
\]
which together with \((4.14)\) gives
\[
r_{i+1} \leq r_i \left( 1 + \rho \right)^{-1/(d+2)}. 
\]
We want finally to show that if \(K_1\) is chosen to be sufficiently large then the new point \((k_{i+1}, x_{i+1})\) lies in \(Q([\gamma/\Phi(R^{-2})], z, R)\). Indeed, by iterating \((4.19)\), we get
\[
r_{i+1} \leq r_i \left( 1 + \rho \right)^{-1/(d+2)} \leq r_{i-1} \left( 1 + \rho \right)^{-2/(d+2)} \leq \ldots \leq r_1 \left( 1 + \rho \right)^{-i/(d+2)}. 
\]
Using \((4.20)\) and scaling condition \((1.1)\) one easily shows that
\[
k_{i+1} \leq \left[ \gamma/\Phi(R^{-2}) \right] + \left[ \gamma/\Phi((R/B)^{-2}) \right] + \frac{5c_{e}^{-1}}{1 - \kappa^{2a}} \frac{1}{\phi(r_1^{-2})}, 
\]
with \(\kappa = (1 + \rho)^{-1/(d+2)}\). In a similar fashion we get
\[
|x_{i+1} - z| \leq \frac{R}{B} + 2r_1 \sum_{j=0}^{\infty} \left( 1 + \rho \right)^{-1/(d+2)}j = \frac{R}{B} + \frac{2r_1}{1 - \kappa}. 
\]
We next need the following easy technical result which we prove later.

\(\textbf{Claim 3.}\) There is a constant \(c_3 > 0\) such that the following two relations
\[
\left[ \gamma/\Phi((R/B)^{-2}) \right] + \frac{5c_{e}^{-1}}{1 - \kappa^{2a}} \frac{1}{\phi(c_3 R^{-2})} \leq \left[ \gamma/\Phi(R^{-2}) \right] 
\]
\[
(4.23) 
\]
\[
\frac{R}{B} + \frac{2c_2R}{1 - \kappa} < R \tag{4.24}
\]

hold for all \( R \) sufficiently large.

At last, let \( c_3 \) be a constant as in Claim 3 and suppose that \( K_1 \geq (c_2/c_3)^{d+2} \). This would mean that \( r_1 \leq c_3R \). By (4.21), (4.22) and Claim 3, \( (k_{i+1}, x_{i+1}) \in Q([\gamma/\phi(R^{-2})], z, R) \). However, by (4.16) \( r_i \geq 3 \) for all \( i \). On the other hand, if we let \( i \) tend to infinity in (4.20), we would obtain that \( r_i \) approaches zero. This is a contradiction and whence \( K_1 \leq (c_2/c_3)^{d+2} \), which means that (4.2) holds with \( C_{PH} = (c_2/c_3)^{d+2} \) and for all \( R \) large enough. To finish the proof we are left to establish Claims 2 and 3.

**Proof of Claim 2.** We set

\[
c_2 = 2R_0 \sqrt{a} \left( \frac{1}{\theta_2 \xi} \right)^{1/(d+2)} \vee B \sqrt{b} \left( \frac{3}{\theta_1 \xi' c'} \right)^{1/(d+2)}, \tag{4.25}
\]

where \( \gamma \) is the constant from Theorem 2.3, \( c' \) is the constant from (2.4) and \( b \) is defined in (4.1). We show that the claim is true with such a constant. We start by showing (4.12). Combining (2.5) and (4.11) we get

\[
\frac{j \left( 2 \sqrt{aR} \right)}{j \left( r/R_0 \right)} = \left( 2R_0 \sqrt{a} \right)^{-d} \left( \frac{R}{r} \right)^{-d} \frac{\phi \left( \left( 2 \sqrt{aR} \right)^{-2} \right)}{\phi \left( \left( r/R_0 \right)^{-2} \right)} \geq \frac{1}{\left( 2R_0 \sqrt{a} \right)^{d+2}} \left( \frac{r}{R} \right)^{d+2}
\]

\[
> \frac{1}{\left( 2R_0 \sqrt{a} \right)^{d+2}} \left( 2R_0 \sqrt{R} \right)^{d+2} \frac{\theta_2 \xi}{K^{-1}}
\]

\[
= \frac{1}{\theta_2 \xi K}. \tag{4.26}
\]

Similarly, to prove (4.13) we apply (2.4) and (2.5) and obtain

\[
\left| Q(0, x, r/B) \right| j \left( \sqrt{bR} \right) \geq \frac{\gamma c' b^{-d/2}}{B^d} \left( \frac{r}{R} \right)^{d} \frac{\phi \left( \left( \sqrt{bR} \right)^{-2} \right)}{\phi \left( \left( r/B \right)^{-2} \right)} \geq \frac{\gamma c'}{B \sqrt{b}} \left( \frac{3}{\theta_1 \xi' c'} \right)^{d+2} K^{-1}
\]

\[
> \frac{\gamma c'}{B \sqrt{b}} \frac{3 \left( B \sqrt{b} \right)^{d+2}}{\theta_1 \xi \gamma c'} K^{-1}
\]

\[
= \frac{3}{\theta_1 \xi K}. \tag{4.27}
\]
Proof of Claim 3. Notice that (4.23) is equivalent to
\[
\frac{5c_c^{-1}}{1 - \kappa^{2\alpha_*}} \phi\left(\frac{1}{(c_3 R)^{-2}}\right) \leq [\gamma/\phi(R^{-2})] - [\gamma/\phi((R/B)^{-2})].
\]
Using (A.2) and (A.3) we get
\[
[\gamma/\phi(R^{-2})] - [\gamma/\phi((R/B)^{-2})] \geq \frac{\gamma}{2\phi(B^2R^{-2})}.
\]
Hence, it is enough to define \(c_3\) for which
\[
\frac{\phi(B^2R^{-2})}{\phi(c_3^{-2}R^{-2})} \leq \frac{\gamma c_c(1 - \kappa^{2\alpha_*})}{10}.
\]
This can be achieved by setting
\[
c_3 := B^{-1} \left(1 \land \left(\gamma c_c^2(1 - \kappa^{2\alpha_*})/10\right)^{1/2\alpha_*} \land (B - 1)(1 - \kappa)/3\right).
\]
Indeed, with such a choice, for \(R\) sufficiently large we apply the scaling condition and get
\[
\frac{\phi(B^2R^{-2})}{\phi(c_3^{-2}R^{-2})} \leq \frac{1}{c_c}(c_3B)^{2\alpha_*}.
\]
Clearly (4.26) follows. With such \(c_3\) the validity of (4.24) is obvious. \(\square\)

5 | LOWER BOUND

The aim of this section is to prove the global lower estimate. We use a probabilistic method based on the parabolic Harnack inequality.

Theorem 5.1. Under our assumptions, for some constant \(C > 0\)
\[
p^\phi(n, x, y) \geq C \left(\frac{n}{\phi^{-1}(n^{-1})}\right)^{d/2} \land \frac{n}{|x - y|} \phi\left(|x - y|^{-2}\right),
\]
for all \(x, y \in \mathbb{Z}^d\), for all \(n \in \mathbb{N}\).

Proof. Let us set
\[
r_n = \frac{1}{\sqrt{\phi^{-1}(n^{-1})}}, \ n \geq 1.
\]
Near-diagonal bound: We start by proving that there exists a constant \(C > 0\) such that
\[
p^\phi(n, x, y) \geq C \left(\frac{n}{\phi^{-1}(n^{-1})}\right)^{d/2},
\]
for \(n \in \mathbb{N}\) and \(|x - y| \leq d_1r_n\), where \(d_1 > 0\) is a constant to be specified. We take \(n \in \mathbb{N}\) and choose \(R\) to satisfy \(n = \gamma/\phi(R^{-2})\), where \(\gamma\) is the constant from Theorem 2.3. Let \(q(k, \omega) = p^\phi(bn - k, x, \omega)\), where \(b\) is the constant from (4.1).
By Lemma 4.1, $q$ is parabolic on $\{0, 1, 2, \ldots, bn\} \times \mathbb{Z}^d$. Since by our choice $bn \geq \gamma/\phi \left( \left( \sqrt{bR} \right)^{-2} \right)$, $q$ is also parabolic on

$$\left\{ 0, 1, 2, \ldots, \left[ \gamma/\phi \left( \left( \sqrt{bR} \right)^{-2} \right) \right] \right\} \times \mathbb{Z}^d.$$  

We now choose $d_1 = 1/B$ which implies that $B(0, d_1 r_n) \subseteq B(y, R/B)$ and whence $(n, x) \in Q(\lfloor \gamma/\phi(R^{-2}) \rfloor, y, R/B)$. By choosing $n$ big enough we can make $R$ large enough and this allows us to apply Theorem 4.2. Thus, there is $n_0 \geq 1$ such that for all $n \geq n_0$,

$$\min_{z \in B(y, d_1 r_n)} p^\phi (bn, x, z) \geq \min_{z \in B(y, R/B)} p^\phi (bn, x, z) = \min_{z \in B(y, R/B)} q(0, z) \geq C^{-1}_{PH} \max_{(k, z) \in Q(\lfloor \gamma/\phi(R^{-2}) \rfloor, y, R/B)} q(k, z) \geq C^{-1}_{PH} q(n, x).$$

Hence, by Theorem 3.1,

$$\min_{z \in B(y, d_1 r_n)} p^\phi (bn, x, z) \geq C^{-1}_{PH} q(n, x) = C^{-1}_{PH} p^\phi ((b-1)n, x, x) \geq C^{-1}_{PH} c_1 (\phi^{-1} ((b-1)n^{-1}))^{d/2} \geq C^{-1}_{PH} c_1 (\phi^{-1} (bn^{-1}))^{d/2},$$

for all $x \in \mathbb{Z}^d$ and $n \geq n_0$. Hence, we have proved (5.2) for all integers of the form $bn$ with $n \geq n_0$. For the remaining values of $n$ between $bn_0$ and $b(n_0 + 1)$ (and so forth) we use Lemma 2.2 to get

$$p^\phi (bn + 1, x, y) = \sum_{z \in \mathbb{Z}^d} p^\phi (bn, x, z)p^\phi (z, y) \geq p^\phi (bn, x, y)p^\phi (y, y) \geq C_1 p^\phi (bn, x, y) \geq C_1 c_2 (\phi^{-1} (bn^{-1}))^{d/2} \geq C_1 c_2 (\phi^{-1} ((bn + 1)^{-1}))^{d/2}.$$

For $n < bn_0$ we apply the above procedure together with (1.3), and this gives (5.2) for all $n$.

**Estimate away from the diagonal:** Let $j(r)$ be the function defined at (2.7). We now show that there is $C > 0$ such that

$$p^\phi (n, x, y) \geq C n j(|x - y|),$$

for all $n \in \mathbb{N}$ and $|x - y| \geq d_2 r_n$, where a constant $d_2 > 0$ will be specified. We first claim that there is a constant $c_3 > 0$ such that for all $x \in \mathbb{Z}^d$ and for all $k, n \in \mathbb{N}$

$$P^x \left( \max_{j \leq k} |S^\phi_j - x| \geq c_3 r_n \right) \leq \frac{1}{2} \frac{k}{n}. \tag{5.4}$$

By Lemma 2.4 and Lemma 2.5 we get

$$P^x \left( \max_{j \leq k} |S^\phi_j - x| \geq c_3 r_n \right) \leq c_4 k \phi (c_3^{-2} r_n^{-2}).$$

This is true for all constants $c_3 > 0$. We define specific constant $c_3$ as

$$c_3 = 1 \lor \left( 2 c_4 / c_+ \right)^{1/2x_+}.$$

Since $c_3 \geq 1$ we can use lower scaling to obtain (5.4).

We now set $d_2 = 3c_3$ and we notice that $d_1 < d_2$, as $d_1 = 1/B \leq 1/3$. Let

$$\tau(x, r) = \inf \left\{ k : S^\phi_k \not\in B(x, r) \right\}$$
and consider a family of sets
\[ A_k = \left\{ \tau(x, c_3 r_n) = k, \ S_{k+1}^\phi, \ldots, S_{n-1}^\phi \in B(y, c_3 r_n), \ S_n^\phi = y \right\}, \] (5.5)
for \( k = 1, 2, \ldots, n \). Observe that
\[ p^{\phi}(n, x, y) = P_z(S_n^\phi = y) \geq \sum_{k=1}^{n} P_z(A_k) \]
and our task is to estimate the last sum from below. By the time reversal of the random walk we get
\[ P_z(A_k) = \sum_{x_{k-1} \in B(x, c_3 r_n)} \left( P_z(\tau(x, c_3 r_n) > k - 1, \ S_{k-1}^\phi = x_{k-1}) p^{\phi}(x_{k-1}, x_k) \right. \]
\[ \left. \times P_y(\tau(y, c_3 r_n) > n - k, \ S_{n-k}^\phi = x_k) \right). \] (5.6)
For \( x_{k-1} \in B(x, c_3 r_n), x_k \in B(y, c_3 r_n) \) and \( |x - y| \geq d_2 r_n = 3 c_3 r_n \), we have
\[ |x_{k-1} - x_k| \leq 3 c_3 r_n + |x - y| \leq 2 |x - y|, \]
and whence, for \( |x - y| \geq d_2 r_n \), by using (1.3)
\[ p^{\phi}(x_{k-1}, x_k) \geq c_5 j(|x - y|). \] (5.7)
Thus
\[ P_z(A_k) \geq c_5 j(|x - y|) P_z(\tau(x, c_3 r_n) > k - 1) P_y(\tau(y, c_3 r_n) > n - k). \] (5.8)
Using (5.4) we get
\[ P_z(A_k) \geq c_5 \left( 1 - \frac{k - 1}{2}, \frac{1}{n} \right) \left( 1 - \frac{n - k}{2}, \frac{1}{n} \right) j(|x - y|) \geq \frac{c_5}{4} j(|x - y|) \]
and (5.3) follows for all \( n \in \mathbb{N} \) and \( |x - y| \geq d_2 r_n \).

**Intermediate estimate:** We finally show that
\[ p^{\phi}(n, x, y) \geq C \left( \phi^{-1}(n^{-1}) \right)^{d/2}, \] (5.9)
for all \( n \in \mathbb{N} \) and for \( d_1 r_n < |x - y| < d_2 r_n \). For any \( 1 \leq K \leq n \) we can write
\[ p^{\phi}(n, x, y) \geq \sum_{z \in B(y, d_1 r_n/2)} p^{\phi}(\lfloor n/K \rfloor, x, z) p^{\phi}(n - \lfloor n/K \rfloor, z, y). \]

We now state the claim which we prove later.

**Claim 4.** Let us set
\[ K = 2 \vee c^+ \left( \frac{2 d_2}{d_1} \right)^{2 \alpha^+} \vee \left( 1 - \frac{4^{-\alpha^+}}{c_2} \right)^{-1}. \] (5.10)
Then for all \( n \geq K \) the following inequalities
\[ \frac{d_1 r_n}{2} \geq d_2 r_{\lfloor n/K \rfloor}, \quad r_{n-\lfloor n/K \rfloor} \geq \frac{r_n}{2} \]

hold.

Thus, if \(|x - y| > d_1 r_n\) and \(z \in B(y, d_1 r_n/2)\) then

\[ |x - z| \geq d_2 r_{\lfloor n/K \rfloor} \quad \text{and} \quad |y - z| \leq d_1 r_{n-\lfloor n/K \rfloor}. \]

Combining this with (5.2) and (5.3) we get

\[ p^\phi(n, x, y) \geq c_6 \sum_{z \in B(y, d_1 r_n/2)} [n/K] j(|x - z|)(\phi^{-1}((n - |n/K|)^{-1}))^{d/2}. \]

Since \(|x - y| < d_2 r_n\), for every \(z \in B(y, d_1 r_n/2)\) we get \(|x - z| \leq c_7 r_n\), where \(c_7 = d_1/2 + d_2 \geq 1\). By monotonicity of \(j\) and (2.5) we get

\[ j(|x - z|) \geq c_7^{-d-2} (\phi^{-1}(n^{-1}))^{d/2} r_n^{-1} \]

and whence

\[ p^\phi(n, x, y) \geq c_8 n/K n^{-1} (\phi^{-1}(n^{-1}))^{d/2} (\phi^{-1}((n - |n/K|)^{-1}))^{d/2} |B(y, d_1 r_n/2)| \]

\[ \geq c_8 |n/K| n^{-1} \left(\frac{\phi^{-1}((n - |n/K|)^{-1})}{\phi^{-1}(n^{-1})}\right)^{d/2} (\phi^{-1}(n^{-1}))^{d/2}. \]

(5.11)

Clearly \(|n/K| n^{-1} \geq \frac{1}{2K}\) and, by (2.6),

\[ \frac{\phi^{-1}((n - |n/K|)^{-1})}{\phi^{-1}(n^{-1})} \geq \left( \frac{1}{c^*} \right)^{1/\alpha^*} \left( \frac{n - |n/K|}{n} \right)^{-1/\alpha^*} \]

\[ \geq \left( \frac{1}{c^* - c^*/(2K)} \right)^{1/\alpha^*}. \]

Combining these two bounds with (5.11) we obtain (5.9) for all \(n \geq K\) and for \(d_1 r_n < |x - y| < d_2 r_n\). For \(n < K\) we proceed as in the end of the proof of near-diagonal bound.

**Proof of Claim 4.** Since \(r_{n/K} \geq r_{\lfloor n/K \rfloor}\), it is enough to find \(K\) such that

\[ \frac{d_1 r_n}{2} \geq d_2 r_{n/K} \iff \frac{\phi^{-1}((n/K)^{-1})}{\phi^{-1}(n^{-1})} \geq \left( \frac{2d_2}{d_1} \right)^{2}. \]

By (2.6), for \(n \geq K\),

\[ \frac{\phi^{-1}((n/K)^{-1})}{\phi^{-1}(n^{-1})} \geq \left( \frac{1}{c^*} \right)^{1/\alpha^*} \left( \frac{n/K}{n^{-1}} \right)^{1/\alpha^*} = \left( \frac{K}{c^*} \right)^{1/\alpha^*}, \]

and whence \(K\) has to satisfy \(K \geq c^* \left( \frac{2d_2}{d_1} \right)^{2\alpha^*}\). Similarly, as \(r_{n-\lfloor n/K \rfloor} \geq r_{n-n/K}\), it is enough to have \(K\) such that

\[ r_{n-n/K} \geq \frac{1}{2} r_n \iff \frac{\phi^{-1}((n - n/K)^{-1})}{\phi^{-1}(n^{-1})} \leq 4. \]
We assume that $K \geq 2$ and thus (2.6) implies
\[
\frac{\phi^{-1}((n - n/K)^{-1})}{\phi^{-1}(n^{-1})} \leq \left(\frac{1}{c_*}\right)^{1/\alpha_*} \left(\frac{(n - n/K)^{-1}}{n^{-1}}\right)^{1/\alpha_*} = c_*^{-1/\alpha_*}(1 - 1/K)^{-1/\alpha_*}.
\]

We conclude that $K$ has to be such that $K \geq \left(1 - \frac{4^{-\alpha_*}}{c_*}\right)^{-1}$.

Finally, combining inequalities (5.2), (5.3) and (5.9) we obtain (5.1) and the proof is finished. □

6 | UPPER BOUND

In this section we aim to prove the global upper estimates for the transition probabilities of the random walk $S_n^\phi$. Our strategy is to study the corresponding continuous time random walk and to estimate its transition kernel and hitting time of a ball, and then to use these results to get similar identities in the discrete time. The main reason why we switch to the continuous time random walk is to prove Proposition 6.16 which is a key result to establish the off-diagonal upper estimates which are our goal. Another possible approach would be to obtain the estimate for the hitting time of a ball from Proposition 6.16 directly in the discrete setting. This, however, seems to be a hard task and we do not address this problem in the present paper.

6.1 | Estimates for the continuous time random walk

We study the continuous time version of the random walk $S_n^\phi$ which is constructed in the standard way, that is we take $(U_i)_{i \in \mathbb{N}}$ to be a sequence of independent, identically distributed exponential random variables with parameter 1 which are independent of $S^\phi$. Let $T_0 = 0$ and $T_k = \sum_{i=1}^k U_i$. Then we define $Y_t = S_n^\phi$ if $T_n \leq t < T_{n+1}$. Equivalently, we can take $(N_t)_{t \geq 0}$ to be a homogeneous Poisson process with intensity 1 independent of the random walk $S^\phi$ and then $Y_t = S_{N_t}^\phi$.

The transition probability of the process $Y$ is denoted by $q(t,x,y) = \mathbb{P}(Y_t = y)$.

We want to find the upper bound for $q(t,x,y)$.

Proposition 6.1. There is a constant $c > 0$ such that
\[
q(t,x,y) \leq c \left(\phi^{-1}(t^{-1})\right)^{d/2} \wedge \frac{t}{|x-y|^d} \phi(|x-y|^{-2}),
\]
for all $x, y \in \mathbb{Z}^d$ and for all $t \geq 1$.

We first handle the on-diagonal part.

Lemma 6.2. There exists a constant $C_2 > 0$ such that for $t > 0$ and $x, y \in \mathbb{Z}^d$
\[
q(t,x,y) \leq C_2 (\phi^{-1}(t^{-1}))^{d/2}.
\]

Proof. By independence and Theorem 3.1 we get
\[
q(t,x,x) \leq e^{-t} + c_1 e^{-t} \sum_{k=1}^\infty \frac{t^k}{k!} (\phi^{-1}(k^{-1}))^{d/2} = e^{-t} + c_1 e^{-t} (\phi^{-1}(t^{-1}))^{d/2} \left(\sum_{k>t} + \sum_{1 \leq k \leq t}\right) \frac{t^k}{k!} (\phi^{-1}(k^{-1}))^{d/2} = e^{-t} + c_1 e^{-t} (\phi^{-1}(t^{-1}))^{d/2} (\sum_1 + \sum_2).
\]
By monotonicity, $\Sigma_1 \leq e^t$. We next find a bound for $\Sigma_2$ and after that, we will show that $e^{-t} \leq c_4(\phi^{-1}(t^{-1}))^{d/2}$ for all $t > 0$ and for some constant $c_4 > 0$. Observe that $\Sigma_2 = 0$ for $t < 1$. By (2.6) we get

$$\Sigma_2 \leq c_2 t^{d/2\alpha_*} \sum_{k \in \mathbb{Z}} k^t \frac{1}{k!} k^{d/2\alpha_*} \leq c_3 e^t,$$

where in the last inequality we applied [23, Cor. 3]. It suffices to show that

$$e^{-t} \leq c_4(\phi^{-1}(t^{-1}))^{d/2}, \quad t > 0.$$

For $t \geq 1$ this follows easily from (2.6) whereas for $t \in (0,1)$ we observe that $e^{-t} < 1$ and $\phi^{-1}(t^{-1}) > 1$. Finally, by the Cauchy–Schwarz inequality we obtain

$$q(t, x, y) = \sum_{z \in \mathbb{Z}^d} q(t/2, x, z)q(t/2, y, z)$$

$$\leq \left( \sum_{z \in \mathbb{Z}^d} q(t/2, x, z)^2 \right)^{1/2} \left( \sum_{z \in \mathbb{Z}^d} q(t/2, y, z)^2 \right)^{1/2}$$

$$\leq C_2(\phi^{-1}(t^{-1}))^{d/2}$$

and the proof of (6.2) is finished. \hfill \Box

Before we prove the off-diagonal estimate in (6.1) we establish a series of auxiliary results. We follow here the elaborate approach of [10]. We use the following notation

$$\tau^Y(x, r) = \inf \{ t \geq 0 : Y_t \notin B(x, r) \}.$$

**Lemma 6.3.** For all $r \geq 1$ it holds

$$\mathbb{E}^x[\tau^Y(x, r)] \asymp \frac{1}{\phi(r^{-2})}.$$

**Proof.** Let

$$\tau^{S^x}(x, r) = \inf \{ k \geq 0 : S^x_k \notin B(x, r) \}.$$

By [17, Prop. 5.4 and Lem. 5.5],

$$\mathbb{E}^x[\tau^{S^x}(x, n)] \asymp \frac{1}{\phi(n^{-2})}, \quad n \in \mathbb{N}.$$

Then, by Wald’s identity,

$$\mathbb{E}^x[\tau^Y(x, n)] = \mathbb{E}^x(U_1 + \cdots + U_{\tau^{S^x}(x, n)}) = \mathbb{E}^x[\tau^{S^x}(x, n)].$$

Hence, for every $n \in \mathbb{N}$ we have

$$\frac{c_1}{\phi(n^{-2})} \leq \mathbb{E}^x[\tau^Y(x, n)] \leq \frac{c_2}{\phi(n^{-2})}.$$
Finally, by monotonicity of $\phi$ and by (2.5) we easily conclude the desired estimate. 

**Lemma 6.4.** There exist constants $C_3, C_4 > 0$ such that

$$P^x(\tau_Y(x, r) \leq t) \leq 1 - \frac{C_3\phi((2r)^{-2})}{\phi(r^{-2})} + C_4t\phi((2r)^{-2}),$$

(6.3)

for all $x \in \mathbb{Z}^d$ and for all $r, t > 0$

**Proof.** We first consider the case $r \in (0, 1)$. Then $Y$ exits the ball $B(x, r)$ as soon as it jumps to some point other than $x$. Observe that

$$\{\tau_Y(x, r) \leq t\} = \bigcup_{n=1}^{\infty} \{T_n \leq t, S_1^\phi = S_2^\phi = \ldots = S_{n-1}^\phi = x, S_n^\phi \neq x\}.$$

Hence

$$P^x(\tau_Y(x, r) \leq t) = \sum_{n=1}^{\infty} P(T_n \leq t)(P(S_1^\phi = 0))^{n-1}P(S_1^\phi \neq 0) \leq t,$$

where we used Lemma A.3. Choosing $C'_3 = 1/2$ we have

$$1 - \frac{C'_3\phi((2r)^{-2})}{\phi(r^{-2})} \geq \frac{1}{2}.$$

If we set $C'_4 = 1/\phi(1/4)$ we have $t \leq C'_4t\phi((2r)^{-2})$, Hence, for $r < 1$ we have

$$P^x(\tau_Y(x, r) \leq t) \leq 1 - \frac{C'_3\phi((2r)^{-2})}{\phi(r^{-2})} + C'_4t\phi((2r)^{-2}),$$

and this is precisely (6.3) with $C'_3$ and $C'_4$.

Next, assume that $r \geq 1$. Since for any $t > 0$

$$\tau_Y(x, r) \leq t + (\tau_Y(x, r) - t)1_{[\tau_Y(x,r) > t]},$$

by Markov property and Lemma 6.3 we get

$$E^x[\tau_Y(x, r)] \leq t + E^x[1_{[\tau_Y(x,r) > t]}E^{\tau_Y}(x, r) - t]] \leq t + \sup_{z \in B(x, r)} E^z[\tau_Y(x, r)]P^x(\tau_Y(x, r) > t) \leq t + \sup_{z \in B(x, r)} E^z[\tau_Y(z, 2r)]P^x(\tau_Y(x, r) > t) \leq t + \frac{c_2}{\phi((2r)^{-2})}P^x(\tau_Y(x, r) > t).$$

Using again Lemma 6.3 we have

$$\frac{c_1}{\phi(r^{-2})} \leq E^x[\tau_Y(x, r)] \leq t + \frac{c_2}{\phi((2r)^{-2})}P^x(\tau_Y(x, r) > t)$$
and whence
\[
1 - \operatorname{P}^x (\tau^Y(x,r) \leq t) \geq \frac{c_1 \phi((2r)^{-2})}{c_2 \phi(r^{-2})} - \frac{t \phi((2r)^{-2})}{c_2}.
\]
If we set \( C_3 = \min \{ C'_3, c_1/c_2 \} \leq 1/2 \) and \( C_4 = \max \{ C'_4, 1/c_2 \} \) we obtain (6.3) and the proof is finished. \( \square \)

We now study the truncated process which is built upon the process \( Y \). For any \( \rho > 0 \) we denote by \( Y^{(\rho)} \) the process obtained by removing from \( Y \) the jumps of size larger than \( \rho \). More precisely, the process \( Y^{(\rho)} \) is associated with the following Dirichlet form
\[
\mathcal{E}^{(\rho)}(u,v) = \sum_{|x-y| \leq \rho} (u(x) - u(y))(v(x) - v(y)) \phi(x,y),
\]
which is defined for functions \( u,v \) from the domain of the Dirichlet form of the random walk \( S^\phi \), cf. [2, Sec. 5]. We write \( q^{(\rho)}(t,x,y) \) for the transition probability of \( Y^{(\rho)} \) and \( Q_{t}^{(\rho)} \) for its semigroup. We work also with the killed processes. For any non-empty \( D \subseteq \mathbb{Z}^d \) we denote by \( (Q_t^D)_{t \geq 0} \) the semigroup of the killed process \( Y^D \). Similarly we write \( (Q_t^{(\rho),D})_{t \geq 0} \) for the semigroups of \( Y^{(\rho),D} \). Let
\[
\tau^{(\rho)}(x,r) = \inf \left\{ t \geq 0 : Y_t^{(\rho)} \notin B(x,r) \right\}.
\]

Lemma 6.5. There exist constants \( C_5 \in (0,1) \) and \( C_6 > 0 \) such that for any \( r, t, \rho > 0 \)
\[
\operatorname{P}^x (\tau^{(\rho)}(x,r) \leq t) \leq 1 - C_5 + C_6 t \phi((2r)^{-2}) \lor \phi(\rho^{-2})).
\]

Proof. By Lemma 6.4 and (2.5) we get that for all \( x \in \mathbb{Z}^d \) and \( r, t > 0 \)
\[
\operatorname{P}^x (\tau^Y(x,r) \leq t) \leq 1 - C_3/4 + C_4 t \phi((2r)^{-2}).
\]
According to [10, Lemma 7.8], for all \( t > 0 \)
\[
Q_{t}^{B(x,r)\mathbb{1}_{B(x,r)}}(x) \leq Q_{t}^{(\rho),B(x,r)\mathbb{1}_{B(x,r)}}(x) + c_1 t \phi(\rho^{-2}). \tag{6.4}
\]

Remark. In [10, Lemma 7.8] the authors assume more restrictive assumption on the function \( \phi \) then our condition (1.1), namely they require the global scaling. The key tool to prove (6.4) is, however, [10, Lemma 2.1] which in our case is covered by Lemma 2.1. \( \square \)

We notice that
\[
Q_{t}^{B(x,r)\mathbb{1}_{B(x,r)}}(x) = \mathbb{E}^x \left[ 1_{B(x,r)}(Y_t) \mathbb{1}_{\{\tau^Y(x,r) > t\}} \right] = \operatorname{P}^x (\tau^Y(x,r) > t),
\]
\[
Q_{t}^{(\rho),B(x,r)\mathbb{1}_{B(x,r)}}(x) = \mathbb{E}^x \left[ 1_{B(x,r)}(Y_t^{(\rho)}) \mathbb{1}_{\{\tau^{(\rho)}(x,r) > t\}} \right] = \operatorname{P}^x (\tau^{(\rho)}(x,r) > t)
\]
and whence
\[
\operatorname{P}^x (\tau^Y(x,r) > t) \leq \operatorname{P}^x (\tau^{(\rho)}(x,r) > t) + c_1 t \phi(\rho^{-2}).
\]
This and Lemma 6.4 imply
\[
\operatorname{P}^x (\tau^{(\rho)}(x,r) \leq t) \leq 1 - \frac{C_3}{4} + C_4 t \phi((2r)^{-2}) + c_1 t \phi(\rho^{-2})
\]
and the result follows if we choose $C_5 = C_3 / 4 < 1$ and $C_6 = C_4 + c_1$.

**Lemma 6.6.** There exist constants $\varepsilon \in (0, 1)$ and $C_7 > 0$ such that for $x \in \mathbb{Z}^d$ and all $r, \lambda, \rho > 0$ with $\lambda \geq C_7 \phi((r \wedge \rho)^{-2})$ it holds

$$
E^x \left[ e^{-\lambda \tau_{\phi}(x,r)} \right] \leq 1 - \varepsilon. 
$$

**Proof.** By Lemma 6.5, for any $t > 0$ and $x \in \mathbb{Z}^d$,

$$
E^x \left[ e^{-\lambda \tau_{\phi}(x,r)} \right] = E^x \left[ e^{-\lambda \tau_{\phi}(x,r)} 1_{\{\tau_{\phi}(x,r) \leq t\}} \right] + E^x \left[ e^{-\lambda \tau_{\phi}(x,r)} 1_{\{\tau_{\phi}(x,r) > t\}} \right]
\leq P^x \left( \tau_{\phi}(x,r) \leq t \right) + e^{-\lambda t}
\leq 1 - C_5 + C_6 t \left( \phi((2r)^{-2}) \vee \phi((\rho^{-2})) \right) + e^{-\lambda t}.
$$

We now choose $\varepsilon = C_5 / 4 \in (0, 1)$. We next take $t = c_1 / \phi((r \wedge \rho)^{-2})$, for some $c_1 > 0$, in such a way that $C_6 t \phi((2r)^{-2}) + C_6 t \phi((\rho^{-2})) \leq 2 \varepsilon$. Hence, we need to choose $c_1 > 0$ such that

$$
\frac{C_6 c_1 \phi((2r)^{-2})}{\phi((r \wedge \rho)^{-2})} + \frac{C_6 c_1 \phi((\rho^{-2})}{\phi((r \wedge \rho)^{-2})} \leq 2 \varepsilon.
$$

Since $\phi$ is increasing,

$$
\frac{\phi((2r)^{-2})}{\phi((r \wedge \rho)^{-2})} \leq 1 \quad \text{and} \quad \frac{\phi((\rho^{-2})}{\phi((r \wedge \rho)^{-2})} \leq 1
$$

and thus it suffices to choose $c_1 \leq \varepsilon / C_6$. At last, we claim that there is $C_7 > 0$ such that for $\lambda \geq C_7 \phi((r \wedge \rho)^{-2})$ we will have $e^{-\lambda t} \leq \varepsilon$. Indeed, with such a choice we get that $\lambda t \geq C_7 c_1$ and thus we can choose $C_7$ so big that $e^{-\lambda t} \leq C_5 / 4 = \varepsilon$. We finally obtain

$$
E^x \left[ e^{-\lambda \tau_{\phi}(x,r)} \right] \leq 1 - C_5 + C_6 t \left( \phi((2r)^{-2}) + \phi((\rho^{-2})) \right) + e^{-\lambda t}
\leq 1 - \varepsilon,
$$

as desired. \(\square\)

**Lemma 6.7.** There exist constants $C_8, C_9 > 0$ such that for $x \in \mathbb{Z}^d$ and $R, \rho > 0$

$$
E^x \left[ e^{-C_7 \phi((\rho^{-2})\tau_{\phi}(x,R)} \right] \leq C_8 e^{-C_9 R / \rho},
$$

where $C_7 > 0$ is the constant from Lemma 6.6.

**Proof.** We first observe that if $\rho \geq R / 2$ then we can choose $C_8$ and $C_9$ such that $C_8 \exp(-2C_9) \geq 1$ and result follows. Thus we study the case $\rho \in (0, R / 2)$. Let $z \in \mathbb{Z}^d, R > 0$ be fixed. We write for simplicity $\tau = \tau_{\phi}(z,R)$. For any fixed $0 < r < R / 2$ we set $n = \lfloor R / 2r \rfloor$. Let

$$
u(x) = E^x \left[ e^{-\lambda t} \right] \quad \text{and} \quad m_k = \|u\|_{L^\infty(B(z,kr))}, \ k \in \{1, 2, ..., n\}.
$$

We fix $\varepsilon$ from Lemma 6.6 and for any $0 < \varepsilon' < \varepsilon$ we choose $x_k \in B(z, kr)$ such that

$$(1 - \varepsilon') m_k < u(x_k) = m_k.$$
Since $x_k \in B(z, kr)$ and $n = \lfloor R/2r \rfloor$ it is easy to see that for any $k \leq n - 1$

$$B(x_k, r) \subseteq B(z, (k + 1)r) \subseteq B(z, R).$$

Next we consider the following function

$$v_k(x) = \mathbb{E}^x \left[ e^{-\lambda \tau_k} \right], \quad x \in B(x_k, r),$$

where we write $\tau_k = \tau(x_k, r)$. By the strong Markov property, for any $x \in B(x_k, r),

$$u(x) = \mathbb{E}^x \left[ e^{-\lambda \tau_k} e^{-\lambda (\tau - \tau_k)} \right] = \mathbb{E}^x \left[ e^{-\lambda \tau_k} \mathbb{E}^{Y_{\tau_k}} \left( e^{-\lambda \tau} \right) \right] = \mathbb{E}^x \left[ e^{-\lambda \tau_k} u(Y_{\tau_k}) \right].$$

Since $Y_{\tau_k} \in B(x_k, r + \rho)$, we get that for every $x \in B(x_k, r)$

$$u(x) \leq v_k(x) \| u \|_{L^\infty(B(x_k, r + \rho))}.$$

It follows that for any $0 < \rho \leq r$

$$u(x_k) \leq v_k(x_k) \| u \|_{L^\infty(B(x_k, r + \rho))} \leq v_k(x_k) m_{k+2}.$$

Since $u(x_k) \geq (1 - \varepsilon') m_k$, we have

$$(1 - \varepsilon') m_k \leq v_k(x_k) m_{k+2}.$$ 

In view of Lemma 6.6, if $\lambda \geq C_7 \phi (\rho^{-2})$ and $0 < \rho \leq r$ then $v_k(x_k) \leq 1 - \varepsilon$. Hence

$$m_k \leq \left( \frac{1 - \varepsilon}{1 - \varepsilon'} \right) m_{k+2}$$

and iterating yields

$$u(z) \leq m_1 \leq \left( \frac{1 - \varepsilon}{1 - \varepsilon'} \right) m_3 \leq \left( \frac{1 - \varepsilon}{1 - \varepsilon'} \right)^2 m_5 \leq \ldots \leq \left( \frac{1 - \varepsilon}{1 - \varepsilon'} \right)^{n-1} m_{2n-1}.$$

Since $u(x) \leq 1$, we have $m_{2n-1} \leq 1$. Thus

$$u(z) \leq \left( \frac{1 - \varepsilon}{1 - \varepsilon'} \right)^{n-1}.$$ 

Setting $2C_9 = \log \left( (1 - \varepsilon')/(1 - \varepsilon) \right)$ we get

$$\left( \frac{1 - \varepsilon}{1 - \varepsilon'} \right)^{n-1} \leq \left( \frac{1 - \varepsilon}{1 - \varepsilon'} \right)^{R/2r-2}$$

which gives

$$u(z) \leq C_8 \exp \left( -C_9 \frac{R}{r} \right).$$

with $C_8 = e^{4C_9}$. If we set $\lambda = C_7 \phi (\rho^{-2})$ and $\rho = r$ we conclude the result. □
Corollary 6.8. For any \( R, \rho, t > 0 \) and all \( x \in \mathbb{Z}^d \)

\[
P^x (\tau^{(\rho)}(x, R) \leq t) \leq C_8 e^{-C_7 \rho + C_7 t \phi(\rho^{-2})},
\]

where \( C_7 > 0 \) is the constant from Lemma 6.6 and \( C_8, C_9 > 0 \) from Lemma 6.7.

Proof. By Lemma 6.7,

\[
P^x (\tau^{(\rho)}(x, R) \leq t) = P^x (e^{-C_7 \phi(\rho^{-2}) \tau^{(\rho)}(x, R)} \geq e^{-C_7 \phi(\rho^{-2}) t})
\]

\[
\leq e^{C_7 \phi(\rho^{-2}) t} E^x \left[ e^{-C_7 \phi(\rho^{-2}) \tau^{(\rho)}(x, R)} \right]
\]

\[
\leq C_8 e^{-C_7 \rho + C_7 t \phi(\rho^{-2})},
\]

as desired. \( \square \)

For any \( \rho > 0 \) and \( x, y \in \mathbb{Z}^d \), we define

\[
J_\rho(x, y) = p \phi(x, y) 1_{|x-y| > \rho}.
\]

By Meyer’s decomposition and [10, Lemma 7.2(1)], the following estimate holds

\[
q(t, x, y) \leq q^{\rho}(t, x, y) + E^x \left[ \int_0^t \sum_{z \in \mathbb{Z}^d} J_\rho \left( Y^{(\rho)}_s, z \right) q(t-s, z, y) \, ds \right], \quad x, y \in \mathbb{Z}^d.
\]

(6.6)

Proposition 6.9. There exists \( C_{10} > 0 \) such that for all \( t, \rho > 0 \) and \( x \in \mathbb{Z}^d \)

\[
E^x \left[ \int_0^t \sum_{z \in \mathbb{Z}^d} J_\rho \left( Y^{(\rho)}_s, z \right) q(t-s, z, y) \, ds \right] \leq C_{10} t \rho^{-d} \phi(\rho^{-2}).
\]

Proof. By monotonicity and (1.3) we get \( J_\rho(x, y) \leq C_{10} \rho^{-d} \phi(\rho^{-2}) \), for some \( C_{10} > 0 \). This and symmetry imply the result. \( \square \)

In the next lemma we prove the upper bound for the transition kernel of the truncated process.

Lemma 6.10. For all \( t \geq 1 \) and \( x, y \in \mathbb{Z}^d \)

\[
q^{\rho}(t, x, y) \leq C_{11} \left( \frac{1}{t-1} \right)^{d/2} \exp \left( C_{12} t \phi(\rho^{-2}) - C_{13} \frac{|x-y|}{\rho} \right),
\]

(6.7)

where \( C_{11}, C_{12}, C_{13} > 0 \) are constants independent of \( \rho \).

Proof. A direct application of [10, Lemma 7.2(2)] combined with Lemma 2.1 and Lemma 6.2, imply that for all \( t > 0 \) and \( x, y \in \mathbb{Z}^d \) we have

\[
q^{\rho}(t, x, y) \leq q(t, x, y) e^{c_0 \phi(\rho^{-2})} \leq C_2 \left( \frac{1}{t-1} \right)^{d/2} \exp(c_0 t \phi(\rho^{-2})).
\]

(6.8)
We first observe that for $|x - y| < 2\rho$ relation (6.7) is trivial. Indeed, since
$$\exp\left(\frac{-C_{13}|x - y|}{\rho}\right) > \exp(-2C_{13}),$$
for any $C_{13} > 0$, we get
$$q^{(\rho)}(t, x, y) \leq C_2 (\phi^{-1}(t^{-1}))^{d/2} \exp(c_0 t \phi(\rho^{-2})) \frac{\exp(-2C_{13})}{\exp(-2C_{13})}$$
$$\leq C_{11} (\phi^{-1}(t^{-1}))^{d/2} \exp\left(C_{12} t \phi(\rho^{-2}) - C_{13} \frac{|x - y|}{\rho}\right). \quad (6.9)$$
for any $C_{11} \geq C_2 / \exp(-2C_{13})$, $C_{12} \geq c_0$.

Assume that $|x - y| \geq 2\rho$. By Corollary 6.8,
$$Q^{(\rho)}_t 1_{B(x, r)}(x) \leq P^x (r^{(\rho)}(x, r) \leq t) \leq C_8 \exp\left(-C_9 \frac{r}{\rho} + C_7 t \phi(\rho^{-2})\right). \quad (6.10)$$

We set $r = |x - y|/2$ and write
$$q^{(\rho)}(2t, x, y) = \sum_{z \in \mathbb{Z}^d} q^{(\rho)}(t, x, z) q^{(\rho)}(t, z, y)$$
$$\leq \sum_{z \in B(x, r)^c} q^{(\rho)}(t, x, z) q^{(\rho)}(t, z, y) + \sum_{z \in B(y, r)^c} q^{(\rho)}(t, x, z) q^{(\rho)}(t, z, y).$$

By (6.8) and (6.10) we get
$$\sum_{z \in B(x, r)^c} q^{(\rho)}(t, x, z) q^{(\rho)}(t, z, y) \leq C_2 (\phi^{-1}(t^{-1}))^{d/2} e^{c_0 t \phi(\rho^{-2})} \sum_{z \in B(x, r)^c} q^{(\rho)}(t, x, z)$$
$$\leq C_2 C_8 (\phi^{-1}(t^{-1}))^{d/2} e^{c_0 t \phi(\rho^{-2})} e^{-C_9 \frac{r}{\rho} + C_7 t \phi(\rho^{-2})}$$
$$= C_2 C_8 (\phi^{-1}(t^{-1}))^{d/2} e^{(c_0 + C_7)t \phi(\rho^{-2}) - C_9 \frac{|x - y|}{\rho}}.$$

We can show a similar bound for $z \in B(y, r)^c$ and thus, for every $t > 0$ and $|x - y| \geq 2\rho$ we have
$$q^{(\rho)}(2t, x, y) \leq 2C_2 C_8 (\phi^{-1}(t^{-1}))^{d/2} e^{(c_0 + C_7)t \phi(\rho^{-2}) - C_9 \frac{|x - y|}{\rho}}.$$

Replacing $t$ with $t/2$ yields (6.7). It only remains to show that
$$\frac{\phi^{-1}(t/2^{-1})}{\phi^{-1}(t^{-1})} \leq c_1,$$ \quad (6.11)

for some constant $c_1 > 0$. To prove (6.11) we have to apply scaling condition (2.6) and this is the reason why estimate (6.7) works only for $t \geq 1$. Indeed, for $t \geq 2$, by (2.6) we get
$$\frac{\phi^{-1}(t/2^{-1})}{\phi^{-1}(t^{-1})} \leq \left(\frac{2}{c_0}\right)^{1/\alpha_*}.$$

For $1 \leq t \leq 2$ we simply use monotonicity and (6.11) follows. \qed
In the rest of this section we use the notation

\[ r_t = \frac{1}{\sqrt{\phi^{-1}(t^{-1})}}, \quad t \geq 1. \]

**Lemma 6.11.** There are \( N \in \mathbb{N} \) with \( N > (2\alpha_* + d)/(2\alpha_*) \) and \( c_1 \geq 1 \) such that for all \( r > 0, t \geq 1 \) and \( x \in \mathbb{Z}^d \)

\[
\sum_{y \in B(x, r)} q(t, x, y) \leq c_1 r^{-\theta} \left( \phi^{-1} \left( \frac{t-1}{t} \right) \right)^{-\theta/2},
\]

(6.12)

where \( 0 < \theta = 2\alpha_* - (2\alpha_* + d)/N \) and \( \alpha_* \) is the constant from (1.1).

**Proof.** We first observe that for \( r \leq r_t \) relation (6.12) is trivially satisfied, as in this case \( r_t/r \geq 1 \).

We assume that \( r > r_t \). We set

\[ N = \lfloor 2 + d/(2\alpha_*) \rfloor \] (6.13)

and with this \( N \) we define a sequence

\[ \rho_n = 2^n \alpha r_t^{1/N} r_t^{-1/N}, \quad n \in \mathbb{N}, \]

where

\[
\left( \frac{d}{d+2\alpha_*} + \frac{1}{2} \right) < \alpha < 1.
\]

(6.14)

We now show that under this choice we have

\[
\frac{2^n r}{\rho_n} \leq \frac{\rho_n}{r_t},
\]

(6.15)

and

\[
t \phi \left( \rho_n^{-2} \right) \leq 1.
\]

(6.16)

Indeed, (6.15) follows from (6.13) and from the fact that \( \alpha \geq 1/2 \), and

\[
\frac{2^n r}{\rho_n} = 2^n (1-\alpha) \left( \frac{r}{r_t} \right)^{1/N}, \quad \text{and} \quad \frac{\rho_n}{r_t} = 2^n \alpha \left( \frac{r}{r_t} \right)^{1-1/N}.
\]

Similarly, (6.16) follows, since under our choice we see that \( \rho_n \geq r_t \).

Recall that by (6.6) and Proposition 6.9 we have

\[
q(t, x, y) \leq q(\rho)(t, x, y) + C_{10} f(\rho),
\]

(6.17)

for all \( \rho, t > 0 \) and \( x, y \in \mathbb{Z}^d \). Next, by Lemma 6.10, for all \( t \geq 1, x, y \in \mathbb{Z}^d \) and \( n \in \mathbb{N} \), we have

\[
q(\rho_n)(t, x, y) \leq C_{11} \left( \phi^{-1}(t^{-1}) \right)^{d/2} \exp \left( C_{12} t \phi \left( \rho_n^{-2} \right) - C_{13} \frac{|x-y|}{\rho_n} \right),
\]
where $C_{11}, C_{12}, C_{13} > 0$ are constants independent of $\rho_n$. Hence, for all $2^n r \leq |x - y| < 2^{n+1} r$ and all $t \geq 1$ we have

$$q^{(\rho_n)}(t, x, y) \leq C_{11} \left( \phi^{-1}(t^{-1}) \right)^{d/2} \exp \left( C_{12} t \phi \left( \rho_n^{-2} \right) - C_{13} \frac{2^n r}{\rho_n} \right).$$

By (6.16) we get

$$q^{(\rho_n)}(t, x, y) \leq c_2 \left( \phi^{-1}(t^{-1}) \right)^{d/2} \exp \left( -C_{13} \frac{2^n r}{\rho_n} \right). \quad (6.18)$$

Thus, by (6.17) and (6.18) we get, for $t \geq 1$ and $x \in \mathbb{Z}^d$

$$\sum_{y \in B(x, r)^c} q(t, x, y) \leq \sum_{n=0}^{\infty} \sum_{2^n r \leq |x - y| < 2^{n+1} r} \left( q^{(\rho_n)}(t, x, y) + C_{10} t \phi \left( \rho_n \right) \right) \leq c_3 \sum_{n=0}^{\infty} (2^n r)^d \left( \phi^{-1}(t^{-1}) \right)^{d/2} e^{-C_{13} \frac{2^n r}{\rho_n}} + c_4 \sum_{n=0}^{\infty} (2^n r)^d t \phi \left( \rho_n \right) = I_1 + I_2.$$

We first estimate $I_2$. Since $\rho_n^{-2} \leq \phi^{-1}(t^{-1}) \leq 1$, we can use (1.1) to get

$$t \phi \left( \rho_n^{-2} \right) \leq \frac{1}{c_*} \left( \frac{r_t}{\rho_n} \right)^{2\alpha_*}.$$

This implies

$$I_2 \leq c_3 \sum_{n=0}^{\infty} \left( \frac{2^n r}{\rho_n} \right)^d \left( \frac{r_t}{\rho_n} \right)^{2\alpha_*} = c_4 \left( \frac{r_t}{r} \right)^{2\alpha_* - (2\alpha_* + d)/N} \sum_{n=0}^{\infty} 2^{n(d - \alpha(d + 2\alpha_*))}.$$

By (6.14), $d - \alpha(d + 2\alpha_*) < 0$ and whence

$$I_2 \leq c_5 \left( \frac{r_t}{r} \right)^{2\alpha_* - (2\alpha_* + d)/N}. \quad (6.19)$$

We proceed to estimate $I_1$. There exists a constant $c_K > 0$ such that for $x \geq C_{13} e^{-x} \leq c_K x^{-K}$. Applying this, we get

$$\exp \left( -C_{13} \frac{2^n r}{\rho_n} \right) \leq c_K \left( \frac{C_{13} 2^n r}{\rho_n} \right)^{-K}, \quad K > 0.$$

We set

$$K = 1 + N(d + 2\alpha_*) \vee \frac{d}{1 - \alpha}.$$ 

For such $K$ we have $K/N > d + 2\alpha_*$ and $(1 - \alpha)K > d$ and this yields

$$I_1 \leq c_3 \sum_{n=0}^{\infty} c_K C_{13}^{-K} \left( \frac{2^n r}{r_t} \right)^d \left( \frac{2^n r}{r_t} \right)^{1/N} \left( \frac{1}{r_t} \right)^{1/N} \leq c_6 \left( \frac{r_t}{r} \right)^{2\alpha_* - (2\alpha_* + d)/N}. \quad (6.20)$$

Using the definition of $\theta$, (6.19), (6.20) and setting $c_1 = c_5 + c_6$ we conclude (6.12). \qed
Lemma 6.12. Assume that condition (6.12) holds with some \( \theta > 0 \). Then there exists a constant \( c_2 > 0 \) such that for any ball \( B(x_0, r) \) and for any \( t \geq 1 \)

\[
P^x(\tau^Y(x_0, r) \leq t) \leq c_2 \epsilon^{-\theta} (\phi^{-1}(1-r))^{-\theta/2}, \quad x \in B(x_0, r/4).
\]

Proof. For \( x \in B(x_0, r/4) \), we have \( B(x, 3r/4) \subseteq B(x_0, r) \). Using (6.12) we get

\[
P^x(\tau^Y(x_0, r) \leq t) \leq P^x(\tau^Y(x, 3r/4) \leq t) + \sup_{s \leq t} P^x(\tau^Y(x, s) \leq t)
\]

\[
\leq \sum_{y \in B(x, 3r/4)} q(2t, x, y) + \sup_{s \leq t} \sum_{y \in B(x, r/4)} q(2t - s, x, y)
\]

\[
\leq c_1 \left( \frac{r}{r/2} \right)^\theta + c_1 \sup_{s \leq t} r^{\theta/2}.
\]

Since \( t \geq 1 \), we can use (2.6) to obtain

\[
\frac{r}{2t} \leq \left( \frac{2}{c_+} \right)^{1/2} r_t.
\]

Since \( s \leq t \), we have

\[
\sup_{s \leq t} \frac{r}{2t - s} \leq \left( \frac{2}{c_+} \right)^{1/2} r_t.
\]

With these estimates used in (6.21) we get

\[
P^x(\tau^Y(x_0, r) \leq t) \leq c_1 \epsilon^{\theta/2} \left( \frac{r_t}{r} \right)^\theta + c_1 \epsilon^{\theta/2} \left( \frac{r_t}{r} \right)^\theta = c_2 \left( \frac{r_t}{r} \right)^\theta,
\]

for all \( x \in B(x_0, r/4) \). □

Lemma 6.13. Assume that condition (6.12) holds with \( 0 < \theta = 2\alpha_+ - (2\alpha_+ + d)/N \). Then for all \( t \geq 1, k \geq 1 \) and \( |x_0 - y_0| > 4k\rho \) it holds

\[
q^{(\theta)}(t, x_0, y_0) \leq c(k)(\phi^{-1}(t^{-1}))^{d/2} \exp(c_0 t \phi(\rho^{-2}))(1 + \frac{\rho}{r_t})^{-\theta(k-1)}.
\]

Proof. As observed in the proof of Lemma 6.5, for all \( t > 0 \),

\[
Q^B_t \mathbb{1}_B(x) \leq Q^B_t \mathbb{1}_B(x) + c_t \epsilon \phi(\rho^{-2})
\]

and

\[
P^x(\tau^Y(x_0, r) \leq t) = 1 - Q^B_t \mathbb{1}_B(x).
\]
This and Lemma 6.12 imply
\[
1 - Q_t^{(\rho), B} 1_B(x) - c_1 t \phi(\rho^{-2}) \leq 1 - Q_t^B 1_B(x) \leq c_2 \left( \frac{r}{r_t} \right)^{-\theta}.
\]

Hence
\[
1 - Q_t^{(\rho), B} 1_B(x) \leq c_3 \left[ \left( \frac{r}{r_t} \right)^{-\theta} + t \phi(\rho^{-2}) \right], \quad x \in B(x_0, r/4). \tag{6.23}
\]

We now proceed to prove (6.22). If \( \rho < r_t \) then clearly
\[
\left( 1 + \frac{\rho}{r_t} \right)^{(k-1)\theta} < 2^{(k-1)\theta}.
\]
and, by (6.8),
\[
q^{(\rho)}(t, x_0, y_0) \leq C_2 2^{(k-1)\theta} \left( \phi^{-1}(t^{-1}) \right)^{d/2} \exp(c_0 t \phi(\rho^{-2})) \left( 1 + \frac{\rho}{r_t} \right)^{-\theta}.
\]
as claimed.

Let us now consider the case \( \rho \geq r_t \). Fix \( k \geq 1 \), \( t \geq 1 \) and \( x_0, y_0 \in \mathbb{Z}^d \) such that \( |x_0 - y_0| > 4k\rho \). Set \( r = |x_0 - y_0| / 2 > 2k\rho \) and
\[
\psi(r, t) = c_3 \left[ \left( \frac{r}{r_t} \right)^{-\theta} + t \phi(\rho^{-2}) \right]. \tag{6.24}
\]

We take \( R = r/k > 2\rho \) and apply [10, Lemma 7.11] to get
\[
Q_t^{(\rho)} 1_{B(x_0, r)}(x) \leq \left\{ c_4 \left[ \left( \frac{r/k - \rho}{r_t} \right)^{-\theta} + t \phi(\rho^{-2}) \right] \right\}^{k-1}, \quad x \in B(x_0, R). \tag{6.25}
\]

Remark. In our case the assumption of [10, Lemma 7.11] is valid only for \( t \geq 1 \). Since the lemma is proved by induction, we could repeat the argument and get the same result. \( \square \)

Notice that
\[
\left( \frac{r}{k} - \rho \right)^{-\theta} < \rho^{-\theta}.
\]
Using this and the fact that \( R > \rho \), we obtain
\[
Q_t^{(\rho)} 1_{B(x_0, r)}(x) \leq c_1(k) \left\{ \left( \frac{\rho}{r_t} \right)^{-\theta} + t \phi(\rho^{-2}) \right\}^{k-1}, \quad x \in B(x_0, \rho). \tag{6.25}
\]

We notice that
\[
t \phi(\rho^{-2}) \leq \frac{1}{c_2} \left( \frac{\rho}{r_t} \right)^{-\theta}, \quad \rho \geq r_t.
\]
This follows easily by (1.1). Combining this with (6.25) we get

\[
Q_t^\varrho 1_{B(x_0, r)^c}(x) \leq c_2(k) \left( \frac{\varrho}{r_t} \right)^{-(k-1)\varrho}, \quad x \in B(x_0, \rho).
\] (6.26)

Moreover, since \( \rho \geq r_t \), we have

\[
\left( \frac{\rho}{r_t} \right)^{-(k-1)\varrho} \leq 2^{(k-1)\varrho} \left( 1 + \frac{\rho}{r_t} \right)^{-(k-1)\varrho}.
\]

Hence, by (6.26),

\[
Q_t^\varrho 1_{B(x_0, r)^c}(x_0) \leq c_3(k) \left( 1 + \frac{\rho}{r_t} \right)^{-(k-1)\varrho}.
\] (6.27)

Further, observe that

\[
Q_t^\varrho 1_{B(x_0, r)^c}(x_0) = \mathbb{P}_{x_0}(Y_t^\varrho \in B(x_0, r)^c) = \sum_{z \in B(x_0, r)^c} q^\varrho(t, x_0, z)
\]

and, by the semigroup property,

\[
q^\varrho(2t, x_0, y_0) = \sum_{z \in \mathbb{Z}^d} q^\varrho(t, x_0, z) q^\varrho(t, z, y_0)
\]

\[
\leq \sum_{z \in B(x_0, r)^c} q^\varrho(t, x_0, z) q^\varrho(t, z, y_0) + \sum_{z \in B(y_0, r)^c} q^\varrho(t, x_0, z) q^\varrho(t, z, y_0).
\]

Using (6.8) and (6.27) we obtain

\[
\sum_{z \in B(x_0, r)^c} q^\varrho(t, x_0, z) q^\varrho(t, z, y_0) \leq C_2 \left( \phi^{-1}(t^{-1}) \right)^{d/2} \exp(c_0 t \phi(\varrho^{-2})) Q_t^\varrho 1_{B(x_0, r)^c}(x_0)
\]

\[
\leq c_4(k) \left( \phi^{-1}(t^{-1}) \right)^{d/2} \exp(c_0 t \phi(\varrho^{-2})) \left( 1 + \frac{\rho}{r_t} \right)^{-(k-1)\varrho}.
\]

Similarly, we show that

\[
\sum_{z \in B(y_0, r)^c} q^\varrho(t, x_0, z) q^\varrho(t, z, y_0) \leq c_4(k) \left( \phi^{-1}(t^{-1}) \right)^{d/2} \exp(c_0 t \phi(\varrho^{-2})) \left( 1 + \frac{\rho}{r_t} \right)^{-(k-1)\varrho}.
\]

This yields

\[
q^\varrho(2t, x_0, y_0) \leq c_5(k) \left( \phi^{-1}(t^{-1}) \right)^{d/2} \exp(c_0 t \phi(\varrho^{-2})) \left( 1 + \frac{\rho}{r_t} \right)^{-(k-1)\varrho}.
\]

As in the proof of Lemma 6.10, we can replace \( 2t \) with \( t \) and the proof is finished.

We now finally prove the upper bound for the heat kernel of the process \( Y_t \).

**Proof of Proposition 6.1.** Our aim is to prove that for all \( t \geq 1 \)

\[
q(t, x, y) \leq c_1 t |x - y|^{-d} \phi(|x - y|^{-2}), \quad x \neq y.
\] (6.28)
We take arbitrary \( x_0, y_0 \in \mathbb{Z}^d \) such that \( x_0 \neq y_0 \) and we set \( r := |x_0 - y_0|/2 \). Assume that \( r < r_t \). We show that in this case the on-diagonal bound from Lemma 6.2 is smaller than the bound in (6.28), that is

\[
(\phi^{-1}(t^{-1}))^{d/2} \leq c_2 t r^{-d} \phi(r^{-2}).
\]  

(6.29)

Indeed, since \( 1/2 \leq r < r_t \), we can use Lemma A.1 (with \( L = 4 \)) to obtain

\[
\frac{(\phi^{-1}(t^{-1}))^{d/2}}{tr^{-d} \phi(r^{-2})} \leq 4\alpha^* \frac{r_t}{r} \left( \frac{r}{r_t} \right)^{-d} \leq 4\alpha^*.
\]

Combining (6.29) with Lemma 6.2 and using (2.5) we get

\[
q(t, x_0, y_0) \leq C_2 c_2^2 d t |x_0 - y_0|^{-d} \phi(4|x_0 - y_0|^{-2}) \leq c_3 t |x_0 - y_0|^{-d} \phi(|x_0 - y_0|^{-2}).
\]  

(6.30)

We next consider the case \( r \geq r_t \). We set \( k = 1 + (d + 2\alpha^*)/\beta \) and \( \rho = r/(8k) \). By (6.6), Proposition 6.9 and (6.22),

\[
q(t, x_0, y_0) \leq c(k)(\phi^{-1}(t^{-1}))^{d/2} \exp(c_0 t \phi(\rho^{-2}))(1 + \frac{\rho}{r_t})^{-k-1} + C_10 t \rho^{-d} \phi(\rho^{-2}).
\]

We observe that \( t \phi(\rho^{-2}) \) is bounded. This follows as \( r \geq r_t \) implies \( t \phi(\rho^{-2}) \leq 1 \), and we use \( \rho = r/(8k) \) with (2.5) to get

\[
t \phi(\rho^{-2}) = t \phi(64k^2 \rho^{-2}) \leq 64k^2 t \phi(\rho^{-2}) \leq 64k^2.
\]

Hence

\[
q(t, x_0, y_0) \leq c_6(k)(\phi^{-1}(t^{-1}))^{d/2} \exp(c_0 64k^2(1 + \frac{\rho}{r_t})^{-k-1}) + C_10 t \rho^{-d} \phi(\rho^{-2})
\]

\[
\leq c_6(k)(\phi^{-1}(t^{-1}))^{d/2} \left( 1 + \frac{\rho}{r_t} \right)^{-k-1} + C_10 t \rho^{-d} \phi(\rho^{-2}).
\]  

(6.31)

Since \( \rho = r/(8k) \) and \( r_t/r > 0 \), we get

\[
\left( 1 + \frac{\rho}{r_t} \right)^{-k-1} \leq c_7(k) \left( \frac{r}{r_t} \right)^{-k-1},
\]

and, by (2.5),

\[
\rho^{-d} \phi(\rho^{-2}) = (r/(8k))^{-d} \phi(r/(8k))^{-2} \leq (8k)^{d+2} r^{-d} \phi(r^{-2}).
\]

These inequalities together with (6.31) yield

\[
q(t, x_0, y_0) \leq c_8(k)(\phi^{-1}(t^{-1}))^{d/2} \left( \frac{r}{r_t} \right)^{-k-1} + c_8(k) t r^{-d} \phi(r^{-2})
\]

\[
= c_8(k) t r^{-d} \phi(r^{-2}) \left[ \frac{t^{-1}}{\phi(r^{-2})} \left( \frac{r}{r_t} \right)^{-2\alpha^*} + 1 \right].
\]  

(6.32)
By \( r^{-2} \leq r_t^{-2} \leq 1 \) and (1.1), we get
\[
\frac{t^{-1}}{\phi(r^{-2})} \left( \frac{r}{r_t} \right)^{-2\alpha^*} \leq c^*.
\]
Thus, (6.32) implies
\[
q(t, x_0, y_0) \leq c_0(k) 2^{d+2} t |x_0 - y_0|^{-d} \phi \left( |x_0 - y_0|^{-2} \right) .
\] (6.33)
Finally, (6.30) and (6.33) yield relation (6.28) for all \( t \geq 1 \) and \( x \neq y \). Keeping in mind Lemma 6.2 we conclude the result.

6.2 | Full upper estimate

In this paragraph we establish the upper bound for the transition probability of the random walk \( S_n^\phi \). We follow approach of [4], cf. also [18], which is based on the application of the hitting time estimates. We start with results for the process \( Y \) and then we exploit them to obtain bounds for \( S_n^\phi \). Recall that
\[
\tau_Y(x, r) = \inf \{ t \geq 0 : Y_t \not\in B(x, r) \} .
\]

**Proposition 6.14.** There exists a constant \( C_{14} > 0 \) such that
\[
P^x(\tau_Y(x, r) \leq t) \leq C_{14} t \phi(r^{-2}) ,
\]
for all \( x \in \mathbb{Z}^d, r > 0 \) and \( t \geq 1 \).

**Proof.** By Proposition 6.1 and Lemma 2.1, we get
\[
P^x(|Y_t - x| \geq r) \leq c_1 t \sum_{y \in B(x, r)^c} |x - y|^{-d} \phi \left( |x - y|^{-2} \right) \leq c_2 t \phi(r^{-2}) ,
\]
for all \( x \in \mathbb{Z}^d, r > 0 \) and \( t \geq 1 \). For simplicity we write \( \tau = \tau_Y(x, r) \). Thus, by (2.5),
\[
P^x(\tau \leq t) = P^x(\tau \leq t, |Y_{2t} - x| \leq r/2) + P^x(\tau \leq t, |Y_{2t} - x| > r/2) \\
\leq P^x(\tau \leq t, |Y_{2t} - Y_t| \geq r/2) + P^x(|Y_{2t} - x| > r/2) \\
\leq E^x \left[ 1_{\{|\tau\leq t\}} P^Y \left( |Y_{2t-\tau} - Y_0| \geq r/2 \right) \right] + c_2 t \phi((r/2)^{-2}) \\
\leq E^x \left[ 1_{\{|\tau\leq t\}} \sup_{y \in B(x, r)^c} \sup_{s \leq t} P^Y \left( |Y_{2t-s} - y| \geq r/2 \right) \right] + 2c_2 t \phi(4r^{-2}) \\
\leq 2c_2 t \phi(4r^{-2}) E^x \left[ 1_{\{|\tau\leq t\}} \right] + 2c_2 t \phi(4r^{-2}) \\
\leq C_{14} t \phi(r^{-2}) ,
\]
as desired.\[\square\]
We use the notation
\[ T^Y(x,r) = \inf \{ t \geq 0 : Y_t \in B(x,r) \} \quad \text{and} \quad T^{S^d}(x,r) = \inf \left\{ k \in \mathbb{N}_0 : S^d_k \in B(x,r) \right\} \]
and we recall that \( r_t = \left( \phi^{-1} (t^{-1}) \right)^{-1/2} \), for \( t \geq 1 \).

**Lemma 6.15.** There exists a constant \( C_{15} > 0 \) such that
\[ \mathbb{P}^x (T^Y (y, r_t) \leq t) \leq C_{15} t r^d j (|x - y|), \tag{6.34} \]
for all \( x, y \in \mathbb{Z}^d \) and \( t \geq 1 \).

**Proof.** We first show that there is \( c_1 > 0 \) such that
\[ \mathbb{P}^z (\tau^Y (z, c_1 r_t) > t) \geq 1/2. \tag{6.35} \]
Indeed, we set
\[ c_1 = 1 \vee \left( \frac{2C_{14}}{c_*} \right)^{1/2a_*}, \]
where \( C_{14} \) comes from Proposition 6.14. Using Proposition 6.14 and (1.1) we get
\[ \mathbb{P}^z (\tau^Y (z, c_1 r_t) \leq t) \leq C_{14} t \phi \left( (c_1 r_t)^{-2} \right) \leq \frac{C_{14}}{c_* c_1^{2a_*}} \leq \frac{1}{2}. \]

We now consider the case \( |x - y| \leq 2 (1 + c_1) r_t \). By monotonicity of \( j(r) \) and relation (2.5), we get
\[ tr^d j (|x - y|) \geq tr^d j (2 (1 + c_1) r_t) \geq (2 (1 + c_1))^{-(d+2)} \geq (2 (1 + c_1))^{-(d+2)} \mathbb{P}^x (T^Y (y, r_t) \leq t). \]

Therefore
\[ \mathbb{P}^x (T^Y (y, r_t) \leq t) \leq C'_{15} t r^d j (|x - y|), \tag{6.36} \]
with \( C'_{15} = (2 (1 + c_1))^{d+2} \).

Next, we consider the case \( |x - y| > 2 (1 + c_1) r_t \). We write \( T = T^Y (y, r_t) \). Using the strong Markov property and (6.35) we get
\[ \mathbb{P}^x (\tau^Y (y, r_t) \leq t) \leq C'_{15} t r^d j (|x - y|), \tag{6.36} \]
with \( C'_{15} = (2 (1 + c_1))^{d+2} \).

If \( T \leq t \) and \( \sup_{T \leq s \leq T + t} |Y_s - Y_T| \leq c_1 r_t \) then \( |Y_t - Y_T| \leq c_1 r_t \). As \( T \) is the first moment when the process \( Y_t \) hits the ball \( B(y, r_t) \), it follows that
\[ |Y_t - y| \leq |Y_t - Y_T| + |Y_T - y| \leq c_1 r_t + r_t = (1 + c_1) r_t. \]
Combining these two inequalities with (6.37), we get
\[
P^x(T \leq t) \leq 2P^x\left( |Y_t - y| \leq (1 + c_1)r_t \right) \leq 2 \sum_{z \in B(y, (1+c_1)r_t)} q(t, x, z).
\] (6.38)

Since \( x \not\in B(y, 2(1 + c_1)r_t) \) and \( z \in B(y, (1 + c_1)r_t) \), we have \( x \neq z \) and thus we can use (6.28). Notice also that \( |x - z| \geq |x - y|/2 \). This, monotonicity of \( j \), [17, Lemma 2.4] and (6.38) imply
\[
P^x(T \leq t) \leq c_2 t \sum_{z \in B(y, (1+c_1)r_t)} j(|x - z|) \leq C_1^0 t r_t^d j(|x - y|).
\] (6.39)

Relations (6.36) and (6.39) yield the result. \( \square \)

**Proposition 6.16.** There exists a constant \( C_{16} > 0 \) such that
\[
P^x\left( T^{S^\phi}(y, r_n) \leq n \right) \leq C_{16} nr_n^d j(|x - y|),
\]
for all \( x, y \in \mathbb{Z}^d \) and \( n \in \mathbb{N} \).

**Proof.** As before \( (T_k)_{k \in \mathbb{N}_0} \) stand for the arrival times of the Poisson process \( (N_t)_{t \geq 0} \) that was used to define the process \( Y \). More precisely, \( N_k = k \) for all \( T_k \leq t < T_{k+1} \). Using the Markov inequality, we easily get that \( P(T_n \leq 2n) \geq \frac{1}{2} \). By independence, Lemma 6.15 and (2.6), we obtain
\[
\frac{1}{2} P^x\left( T^{S^\phi}(y, r_n) \leq n \right) \leq P^x\left( T^{S^\phi}(y, r_n) \leq n, T_n \leq 2n \right) \leq P^x\left( T^Y(y, r_n) \leq 2n \right) \\
\leq P^x\left( T^Y(y, r_{2n}) \leq 2n \right) \leq 2C_{15} nr_n^d j(|x - y|) = C_{16} nr_n^d j(|x - y|),
\]
as claimed. \( \square \)

In the following theorem we finally prove the upper bound for the transition probability of the random walk \( S^\phi \). In the proof we again apply the parabolic Harnack inequality.

**Theorem 6.17.** There exists a constant \( C > 0 \) such that
\[
p^\phi(n, x, y) \leq C \left( (\phi^{-1}(n^{-1}))^{d/2} \wedge \frac{n}{|x - y|^d} \phi\left( |x - y|^{-2} \right) \right),
\]
for all \( x, y \in \mathbb{Z}^d \) and \( n \in \mathbb{N} \).

**Proof.** By Proposition 6.16 we have for all \( k \in \mathbb{N} \)
\[
\sum_{z \in B(y, r_k)} p^\phi(k, x, z) \leq P^x\left( T^{S^\phi}(y, r_k) \leq k \right) \leq C_{16} kr_k^d j(|x - y|).
\]
On the other hand
\[
\sum_{z \in B(y, r_k)} p^\phi(k, x, z) \geq c' r_k^d \min_{z \in B(y, r_k)} P^\phi(k, x, z).
\]
Hence
\[
\min_{z \in B(y, r_k)} P^\phi(k, x, z) \leq c_1 k j(|x - y|).
\] (6.40)
Next we apply the parabolic Harnack inequality. We choose $R > 0$ to satisfy $\gamma/\phi(R^{-2}) = n$, where $\gamma$ is the constant from Theorem 2.3. Remember that we can choose $\gamma$ to be even smaller than specified in the theorem. Thus we take $\gamma \leq B^{-2}$ where $B$ is the constant defined in (4.1). By (2.5) we easily get that $r_n \leq R/B$. By Lemma 4.1, the function $q(k, w) = p^\phi(bn - k, x, w)$ is parabolic on $\{0, 1, 2, ..., bn\} \times \mathbb{Z}^d$, where $b$ is defined at (4.1). With our choice $bn \geq \left\lceil \gamma/\phi\left(\left(\sqrt{bR}\right)^{-2}\right) \right\rceil$ and thus the function $q$ is parabolic on $\left\{0, 1, 2, ..., \left\lfloor \gamma/\phi\left(\sqrt{bR}\right) - 2 \right\rfloor \right\} \times \mathbb{Z}^d$. By (6.40), we get

$$\min_{z \in B(y, R/B)} q(0, z) = \min_{z \in B(y, R/B)} p^\phi(bn, x, z) \leq \min_{z \in B(y, r_n)} p^\phi(bn, x, z) \leq c_1 bn j(|x - y|). \tag{6.41}$$

Choosing $n$ big enough we can enlarge $R$ so that we can apply Theorem 4.2. Hence

$$\max_{(k, z) \in Q([\gamma/\phi(R^{-2})], y, R/B)} q(k, z) \leq C_{PH} \min_{z \in B(y, R/B)} q(0, z). \tag{6.42}$$

Since $n = \gamma/\phi(R^{-2})$, it is clear that $(n, y) \in Q([\gamma/\phi(R^{-2})], y, R/B)$. Combining this with (6.41), we obtain

$$p^\phi((b - 1)n, x, y) = q(n, y) \leq \max_{(k, z) \in Q([\gamma/\phi(R^{-2})], y, R/B)} q(k, z) \leq C_{PH} \min_{z \in B(y, R/B)} q(0, z) \leq C_{PH} c_1 bn j(|x - y|) \tag{6.43}$$

$$= c_2 (b - 1) n j(|x - y|). \tag{6.44}$$

Similarly as in the proof of Theorem 5.1, we can show that this is enough to get the desired upper bound for all $n \in \mathbb{N}$. Finally, we have

$$p^\phi(n, x, y) \leq c_3 n j(|x - y|),$$

for all $x, y \in \mathbb{Z}^d$, $x \neq y$ and $n \in \mathbb{N}$. This combined with Corollary 3.2 yields the result.

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**APPENDIX**

**Lemma A.1.** Let $L \geq 1$. Then for all $0 < r \leq 1 \wedge R \leq R \leq L$ we have

$$\frac{c_+}{L^{\alpha^*}} \left( \frac{R}{r} \right)^{\alpha^*} \leq \frac{\phi(R)}{\phi(r)} \leq \phi(L) \left( \frac{R}{r} \right)^{\alpha^*}. \tag{A.1}$$

**Proof.** Since $L \geq 1$, relation (A.1) follows directly from (1.1) in the case $R \leq 1$. For $0 < r \leq 1 < R \leq L$ (using (1.1) and the fact that $\phi$ is increasing) we have

$$\frac{\phi(R)}{\phi(r)} \leq \frac{\phi(L)}{\phi(r)} \leq \phi(L) \left( \frac{1}{r} \right)^{\alpha^*} \leq \phi(L) \left( \frac{R}{r} \right)^{\alpha^*},$$

and similarly

$$\frac{\phi(R)}{\phi(r)} \geq \frac{\phi(1)}{\phi(r)} \geq c_+ \left( \frac{1}{r} \right)^{\alpha^*} \geq \frac{c_+}{L^{\alpha^*}} \left( \frac{R}{r} \right)^{\alpha^*},$$

as desired. \[\square\]

**Lemma A.2.** There exists a constant $R_0 \geq B$ such that

$$\left\lfloor \frac{\gamma}{\phi(R^{-2})} \right\rfloor \geq \left\lfloor \frac{\gamma}{\phi((R/B)^{-2})} \right\rfloor + 1, \quad R \geq R_0,$$

where $B$ is defined at (4.1).

**Proof.** For every $x \in \mathbb{R}$ we write $|x| = x - m(x), m(x) \in [0, 1)$. Thus, we look for $R_0$ such that

$$\frac{\gamma}{\phi(R^{-2})} - \frac{\gamma}{\phi(B^{-2}R^{-2})} \geq 1 + m(\gamma/\phi(R^{-2})) - m(\gamma/\phi((R/B)^{-2})), \quad R \geq R_0.$$
Observe that $1 + m(\gamma / \phi(R^{-2})) - m(\gamma / \phi((R/B)^{-2})) \leq 2$. Hence, it is enough to find $R_0$ large enough and such that

$$\frac{\gamma}{\phi(R^{-2})} - \frac{\gamma}{\phi(B^2R^{-2})} \geq 2, \quad R \geq R_0.$$ 

By (1.1), we get

$$\frac{\gamma}{\phi(R^{-2})} - \frac{\gamma}{\phi(B^2R^{-2})} \geq \frac{\gamma}{\phi(B^2R^{-2})} (c_B^{2\alpha} - 1) \geq \frac{\gamma}{\phi(B^2R^{-2})} \xrightarrow{R \to \infty} \infty. \quad (A.2)$$

Therefore, there exists $R_0 \geq B$ such that

$$\frac{\gamma}{\phi(B^2R^{-2})} \geq 2, \quad R \geq R_0, \quad (A.3)$$

and the proof is finished. \hfill \Box

**Lemma A.3.** Let $(U_i)_{i \in \mathbb{N}}$ be a sequence of independent, identically distributed exponential random variables with parameter 1 and let $T_n = \sum_{i=1}^{n} U_i$. Then for all $n \in \mathbb{N}$ and $t > 0$

$$\mathbb{P}(T_n \leq t) \leq t.$$ 

**Proof.** Denote by $F_{T_n}(t) = \mathbb{P}(T_n \leq t)$ the distribution function and by $f_{T_n}$ the density of $T_n$. It is enough to prove that $f_{T_n}(t) \leq 1$, for $t > 0$. For $n = 1$ the result is obvious. For $n \geq 2$ it is easy to check that the function $f_{T_n}$ obtains maximum for $t = n - 1$ and that

$$\max f_{T_n} = \frac{(n - 1)^{n-1} e^{-(n-1)}}{(n-1)!}.$$ 

The result follows from the inequality $n! \geq \sqrt{2\pi}nn^ne^{-n}$. \hfill \Box