Light bullets

S V Sazonov
National Research Centre «Kurchatov Institute», 123182 Moscow, Russia

e-mail: sazonov.sergey@gmail.com

Abstract. Light bullets in media with saturating nonlinearity and quadratic nonlinearity under generation of the second terahertz harmonics are considered with the use of the averaged Lagrangian method.

1. Introduction
The term "light bullet" for the first time appeared in work [1]. This object denotes a clot of light energy localized in all directions extending in the nonlinear environment with a constant velocity. Focusing nonlinearity, negative dispersion of group velocity (DGV) and diffraction are necessary for its formation. If to distract from some canons of mathematical character, it is possible to call a light bullet a three-dimensional soliton. There are spatial and temporary optical solitons. Spatial solitons are optical bunches, almost unlimited in the direction of propagation and limited in the cross directions. The mechanism of formation of such solitons consists in the fact, that the process of cross self-focusing of a bunch due to nonlinearity of the environment is compensated by its diffraction divergence [2]. Temporary solitons are optical pulses of limited temporary duration. These objects are unlimited in the cross directions and limited in the direction of propagation. Here the nonlinear capsizing of the pulse is compensated by its dispersive spreading. A light bullet can be considered as a symbiosis of spatial and temporary solitons. Compensation of nonlinear self-compression by dispersion (in the direction of propagation) and by diffraction (in the cross directions) leads to the formation of light bullets.

The present work is devoted to a short discussion and physical analysis of the situations at which formation of these optical objects is possible.

2. Light bullets in isotropic media
Let the light pulse propagate in an isotropic dielectric in the direction of the \( z \)-axis. Its electric field \( E \) is determined through the complex envelope \( \psi \) as follows

\[
E(z, r, t) = \psi(z, r, t)e^{i(\omega t - kz)} + \psi^*(z, r, t)e^{-i(\omega t - kz)},
\]

(1)

where \( r \) is the radius vector of transversal coordinates, \( t \) is time, \( \omega \) and \( k \) are the carrier frequency and wave number, respectively.

Using the approach of slowly varying envelopes (SVE) in Maxwell's equations and in the constitutive equations, we come to the wave equation which looks like [3]:

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI. Published under licence by IOP Publishing Ltd
\[ i \frac{\partial \psi}{\partial z} + \frac{k_2}{2} \frac{\partial^2 \psi}{\partial \tau^2} - F(|\psi|^2) \psi = \frac{c}{2n_0} \Delta_\perp \psi. \] (2)

Here \( k_2 = \frac{\partial^2 k}{\partial \omega^2} = \frac{\partial v_g^{-1}}{\partial \omega} \) is the parameter of DGV, \( \tau = t - z/v_g \) is local time, \( c \) is the speed of light in vacuum, \( n \) is the refractive index corresponding to the frequency \( \omega \), \( \Delta_\perp \) is a transversal Laplacian, \( F(|\psi|^2) \) is a positively determined function characterizing the focusing nonlinearity of polarizing response of the medium.

There are various mechanisms of suppression of self-focusing which is necessary for the formation of light bullets. One of them consists in the use of media with saturated nonlinearity [3]. In this case we have

\[ F(|\psi|^2) \sim \frac{|\psi|^2}{1 + |\psi|^2 / |\psi|^2_s}, \] (3)

where \( |\psi|^2_s \) is an amplitude which corresponds to the intensity of saturation.

Except saturated nonlinearity, mechanisms connected with gradient wave guides and also with linear dispersion of higher orders, nonlinearity dispersion, Raman nonlinearity, etc. [3] are used.

We won't discuss these mechanisms here, and we will concern only the mechanism connected with saturated nonlinearity with the use of the method of the averaged Lagrangian [4 – 6]. Its essence is as follows. In the beginning the exact one-dimensional solitonic solution of the considered wave equation is found. Then in order to consider the influence of transversal spatial dimensions, the assumption of dependence of some parameters on coordinates in these solutions is introduced. As a result, we have

the so-called trial solutions. These solutions are substituted into the Lagrangian, corresponding to the wave equation including derivatives with respect to transversal coordinates. Further, integration of the resulting expression with respect to time is carried out. As a result, we come to an averaged Lagrangian containing the dependence on variable parameters. Writing down the Euler – Lagrange equation for the varied parameters with use of this averaged Lagrangian we will obtain the corresponding differential equations for these parameters.

We will describe equation (3) as follows

\[ i \frac{\partial \psi}{\partial z} + \frac{k_2}{2} \frac{\partial^2 \psi}{\partial \tau^2} - a |\psi|^2 \psi + b |\psi|^4 \psi = \frac{c}{2n_0} \Delta_\perp \psi, \] (4)

where \( a \) and \( b \) are the constants, first of which corresponds to the Kerr nonlinearity, and the second considers its saturation.

At first we will write down the one-dimensional (\( \Delta_\perp = 0 \)) solution of equation (4) in the form of a temporary soliton

\[ \psi = \frac{1}{\tau_p} \sqrt{\frac{2|k_2|}{a}} \left[ \frac{1}{\sqrt{1 - q / \tau_p^2 \text{ch}(z / \tau_p) + 1}} \right]^{1/2} \exp \left( -i \frac{|k_2| z}{2 \tau_p^2} \right), \] (5)

where \( q = \frac{8b|k_2|}{3a^2} \), \( \tau_p \) is the temporal pulse duration.

According to (5), for taking into account transversal coordinates we will choose the trial solution of equation (4) in the following form:
\[ \psi = \sqrt{\frac{2|k|}{a}} \left[ \frac{1}{\sqrt{1-qQ^2 \text{ch}(2Q\tau)+1}} \right]^{1/2} \exp(-i\Phi). \]  

(6)

Here \( Q \) and \( \Phi \) are functions of coordinates which are subject to search. As in the one-dimensional case, \( Q=1/\tau_p = \text{const.} \), and \( \Phi \sim z \), now \( Q \) and \( \Phi \) are the "slow" and "fast" functions of coordinates, respectively.

The Lagrangian that corresponds to equation (10) has an appearance:

\[ L = \frac{i}{2} \left( \psi \cdot \frac{\partial \psi}{\partial z} - \psi \frac{\partial \psi^*}{\partial z} \right) - \frac{a}{2} |\psi|^4 + \frac{b}{3} |\psi|^6 - \frac{k_1}{2} \left| \frac{\partial \psi}{\partial \tau} \right|^2 + \frac{c}{2n_0\omega} |\nabla_{\perp} \psi|^2. \]  

(7)

Substituting (6) into (7), and integrating the resulting expression with respect to \( \tau \), we obtain an averaged Lagrangian

\[ \Lambda = a \frac{\sqrt{q}}{2|k|} \int_{-\infty}^{\infty} Ld\tau = \Lambda_R + \Lambda_D , \]  

(8)

where the refraction \( \Lambda_R \) and diffraction \( \Lambda_D \) parts of an averaged Lagrangian have respectively an appearance

\[ \Lambda_R = \rho \frac{\partial \varphi}{\partial z} + \rho \left( \nabla_{\perp} \varphi \right)^2 - \frac{c|k|}{2n_0\omega} \left( \rho - \text{th} \rho \right), \]  

(9)

\[ \Lambda_D = \left( \frac{c}{n_\omega} \right)^2 G(\rho) \left( \nabla_{\perp} \rho \right)^2, \]  

(10)

\[ \rho = \frac{1}{2} \ln \left( \frac{1+\sqrt{qQ}}{1-\sqrt{qQ}} \right), \quad \varphi = \frac{c}{n_\omega} \Phi, \]  

(11)

\[ G(\rho) = \text{cth} \rho + \frac{2}{\text{sh} \rho \text{ch} \rho} - \frac{\rho}{\text{sh}^2 \rho} + \frac{\pi^2 + 3 \rho^2}{3 \text{sh}^3 \rho \text{ch} \rho} - \rho \frac{\pi^2 + \rho^2}{3 \text{sh}^4 \rho \text{ch}^2 \rho}. \]  

(12)

The refraction part of the Lagrangian corresponds to the approach of geometrical optics for solitons [4, 6]. Writing down the Euler – Lagrange equations for \( \rho \) and \( \varphi \) with the use of (8) – (12), we will have

\[ \frac{\partial \rho}{\partial z} + \nabla_{\perp} \left( \rho \nabla_{\perp} \varphi \right) = 0, \]  

(13)

\[ \frac{\partial \varphi}{\partial z} + \frac{\left( \nabla_{\perp} \varphi \right)^2}{2} + f(\rho) = \left( \frac{c}{n_\omega} \right)^2 D_\perp, \]  

(14)

where

\[ D_\perp = 2(\Delta_{\perp} \rho)G(\rho) + (\nabla_{\perp} \rho)^2 G'(\rho), \quad f(\rho) = -\frac{c|k|}{2n_0\omega} \text{th}^2 \rho. \]  

(15)

When the right-hand part in (14) equals zero, system (13), (14) is formally similar to the system of equations of hydrodynamics for two-dimensional ideal liquid. Thus, (13) represents the continuity
equation, and (14) is the Cauchy's integral. The right-hand part of (14) describes the influence of diffraction on the propagation of a pulse [7]. There are bases to believe that diffraction is capable to compensate the self-focusing of a pulse.

Equation (13) is satisfied by the solutions corresponding to an axial-symmetric light bullet:

\[
\rho = \rho_0 \frac{R_0^2}{R^2(z)} F\left(\frac{r}{R(z)}\right), \quad \varphi = f'(z) + \frac{r^2}{2} \frac{R'(z)}{R(z)}.
\]  

(16)

Here \( r \) is a radial component of a cylindrical coordinate system, making meaning of a distance from the focal \( z \)-axis to a supervision point, \( R \) is the characteristic transversal radius (aperture) of a light bunch depending on \( z \), \( R_0 \) and \( \rho_0 \) are the radius and \( \rho \) of the bunch at the input into the medium (at \( z = 0 \)), \( F \) is the function defined after substitution (16) into (14).

Following Ref. [2], we will consider a transversal profile of a soliton as a Gaussian, having chosen \( F \) in the form:

\[
F = \exp\left(-\frac{r^2}{R^2}\right).
\]  

(17)

Besides, by the substitution (16) and (17) into (14) we will take into account following approximation [2]

\[
\mu = \frac{r^2}{R^2} \ll 1.
\]  

(18)

This approximation enables a series expansion up to the first degree of the small parameter \( \mu \). Then, equating expressions in the left-hand and right-hand parts of equation (16) respectively at the zero point and by the first degrees of \( \mu \) leads to the following differential equations

\[
f'_1(z) = \frac{c[k_2]}{2n\omega q} \frac{1}{\varphi} \left(\rho_0 \frac{R_0^2}{R^2}\right) + F_D(R),
\]  

(19)

\[
R^*(z) = -\frac{\partial U}{\partial R},
\]  

(20)

\[
U(R) = U_R(R) + U_{D}(R),
\]  

(21)

\[
U_R = -\frac{c[k_2]}{2n\omega q} \frac{1}{\varphi} \left(\rho_0 \frac{R_0^2}{R^2}\right).
\]  

(22)

The functions \( F_D(R) \) and \( U_D(R) \) have very bulky appearance. Therefore, we don't write out them here. The function \( U_D(R) \) is positive and monotonously decreases by the increase of the variable \( R \).

Equation (20) is similar to Newton’s second law for a particle of unit mass with the coordinate \( R \) where \( z \) plays the role of time. This movement corresponds the "potential energy" \( U \).

The dependence \( U(R) \) is shown in figure 1. The local minimum of the function \( U(R) \) corresponds to an equilibrium aperture \( R_m \) of light bullets. It is easy to see that under the conditions \( R'(0) = 0 \) and \( R_0 > R_m \) a light bullet is stable. As can be seen from the second of expressions (16), the value \( R'(z) \) defines the curvature of wave fronts of the pulse. Therefore, wave fronts of a light bullet at \( R = R_m \) are flat.
3. **Light bullets in uni-axial crystals at second harmonic generation**

Under the fulfillment of the conditions of phase synchronism and group synchronism, the set of equations for second harmonic generation looks like [8]

\[
\frac{i}{2} \frac{\partial \psi_1}{\partial z} + \frac{k_2^{(1)}}{2} \frac{\partial^2 \psi_1}{\partial \tau^2} - d_1 \psi_1 \psi_2 = \frac{c}{2n\omega} \Delta \psi_1, \tag{23}
\]

\[
\frac{i}{2} \frac{\partial \psi_2}{\partial z} + \frac{k_2^{(2)}}{2} \frac{\partial^2 \psi_2}{\partial \tau^2} - d_2 \psi_1^2 = \frac{c}{4n\omega} \Delta \psi_2. \tag{24}
\]

Here \(d_1\) and \(d_2\) are the coefficients of quadratic nonlinearity proportional to the nonlinear susceptibility of the second order \(\chi^{(2)}\) for both harmonics, \(k_2^{(1)}\) and \(k_2^{(2)}\) are the DGV coefficients of the first and second harmonics respectively.

Let the condition

\[k_2^{(2)} = 2k_2^{(1)}\]  

be satisfied. Then set (23), (24) possesses one-dimensional (\(\Delta_\perp = 0\)) solitonic solutions [9]:

\[
\psi_1 = \pm \frac{3k_2}{4\tau_p^2} \sqrt{\frac{2}{d_1 d_2}} \exp \left( \frac{i k_2 z}{2\tau_p^2} \right) \sech^2 \left( \frac{\tau}{2\tau_p} \right), \tag{26}
\]

\[
\psi_2 = -\frac{3k_2}{4\tau_p^2} \exp \left( \frac{i k_2 z}{\tau_p^2} \right) \sech^2 \left( \frac{\tau}{2\tau_p} \right). \tag{27}
\]

Here \(k_2 = k_2^{(1)}\).

Now we will consider the transversal dynamics determined by the right-hand parts in (23) and (24). After the scale transformation \(\psi_1 = \sqrt{d_1 / (2d_2)} \Phi_1, \psi_2 = \Phi_2\), we will write down the Lagrangian which corresponds to set (23), (24):
where

\begin{equation}
L = L_1 + L_2 + L_{\text{int}},
\end{equation}

\begin{align}
L_1 &= \frac{i}{2} \left( \Phi_1^* \frac{\partial \Phi_1}{\partial z} - \Phi_1 \frac{\partial \Phi_1^*}{\partial z} \right) - \frac{k_1}{2} \left[ \frac{\partial \Phi_1}{\partial \tau} \right]^2 + \frac{c}{2n\omega} \left| \nabla \Phi_1 \right|^2, \\
L_2 &= \frac{i}{2} \left( \Phi_2^* \frac{\partial \Phi_2}{\partial z} - \Phi_2 \frac{\partial \Phi_2^*}{\partial z} \right) - \frac{k_2}{2} \left[ \frac{\partial \Phi_2}{\partial \tau} \right]^2 + \frac{c}{4n\omega} \left| \nabla \Phi_2 \right|^2,
\end{align}

\begin{equation}
L_{\text{int}} = -\frac{d_1}{2} \left( \Phi_1^2 \Phi_2 + \Phi_1^* \Phi_2^* \right). \tag{31}
\end{equation}

According to (26) and (27), we will choose trial solutions in the form

\begin{align}
\Phi_1 &= \pm \frac{6k_2}{d_1} \rho^{2/3} \exp \left(-i \frac{n\omega}{c} \varphi \right) \text{sech}^2 \left( \rho^{1/3} \tau \right), \tag{32} \\
\Phi_2 &= -\frac{3k_2}{d_1} \rho^{2/3} \exp \left(-2i \frac{n\omega}{c} \varphi \right) \text{sech}^2 \left( \rho^{1/3} \tau \right). \tag{33}
\end{align}

Here $\rho$ and $\varphi$ are unknown functions of coordinates.

After substitution (32) and (33) into (28) – (31) and integration with respect to $\tau$ we will have

\begin{equation}
\int_{-\infty}^{+\infty} L dt = 216 \left( \frac{k_2}{\gamma} \right)^2 \Lambda,
\end{equation}

where the averaged Lagrangian is

\begin{equation}
\Lambda = \rho \frac{\partial \varphi}{\partial z} \left[ \left( \frac{\nabla \varphi}{\rho} \right)^2 \right] + \frac{2ck_2}{5n\omega} \rho^{8/3} + \left( 1 + \frac{\pi^2}{30} \right) \left( \frac{c}{n\omega} \right)^2 \left( \frac{\nabla \rho}{\rho} \right)^2.
\end{equation}

Writing down the corresponding Euler – Lagrange equations for $\rho$ and $\varphi$, again we will come to system of hydrodynamic type (13,14), where

\begin{equation}
D_s = 3 \left( 1 + \frac{\pi^2}{30} \right) \Delta \sqrt[3]{\rho}, \quad f(\rho) = \frac{2c}{n\omega} k_2 \rho^{2/3}. \tag{35}
\end{equation}

By analogy with the previous section, we will put $F = \exp \left(-3r^2 / 2R^2 \right)$. Using expressions (16) and approximation (26) we will come to equation (20) for $R$, where

\begin{equation}
U = \frac{3ck_2}{n\omega} \rho^{2/3} R_0^{4/3} \frac{R_0^{4/3}}{R^{4/3}} + \frac{9}{4} \left( 1 + \frac{\pi^2}{30} \right) \left( \frac{c}{n\omega} \right)^2 \frac{1}{R^2}.
\end{equation}

From this it follows that $U(R)$ possesses a minimum, only if $k_2 < 0$. Here the dependence of $U(R)$ qualitatively coincides with that represented in figure 1. The position of the minimum of $U$ there corresponds to the value of the "bullet" radius:
Here \( \tau_0 = 1/\rho_0 \).

Thus, by the propagation of a bullet its transversal size oscillates near the value \( R_m \). Putting in the last expression \( R_m = R_0 \), we obtain

\[
R_0 = 0.706 \sqrt{\frac{c}{n_0 |k_z|}} \tau_0 .
\]

(37)

In this case oscillations of the bullet parameters don't happen, and the transversal size of a bullet is proportional to its temporary duration.

For the results received in this section to be correct, the conditions of phase synchronism and group synchronism are important. Besides, equality (25) has to be carried out. Most likely, it is impossible to strictly meet all these conditions simultaneously. However, it can be made approximately, if frequencies \( \omega \) and \( 2\omega \) lie below characteristic frequencies of the resounding absorption. This is the case, for example, for the terahertz range. In this case the wave number \( k \approx n_0 \omega / c + s\omega^3 \), where \( s \) is a constant, and the second term is little in comparison with the first one. Then, approximately, it is possible to consider phase and group velocities of both harmonics by equal \( c / n_0 \). At the same time, \( k_z = \partial^2 k / \partial \omega^2 = 6s\omega \). Then condition (25) is satisfied automatically. However, in the presence of only temporary dispersion the parameter \( s \) is positive. Therefore, \( k_z > 0 \). At the same time, it is necessary for the formation of "bullets", that \( k_z < 0 \). In a micro-dispersive (granulated) medium spatial dispersion is important. Then under certain conditions the parameter \( s \) can become negative [10]. Thereof, we have \( k_z < 0 \) at simultaneous performance of condition (25). Then the formation of bullets on both harmonics is possible.

4. Concluding remarks

Let us note, that light bullets in quadratic-nonlinear media were predicted earlier, in particular with the use of variation approach (see [3] and the references quoted there). However, the reduction of such a research to the equations of hydrodynamic type (13), (14) and the further search for their solutions is executed here for the first time.

Unlike other approaches the method of an averaged Lagrangian allows separating the nonlinear refraction determined by the type of the function in (14) [6] from the diffraction determined by the type of the right-hand part in (14) [7].

Thus, the approach offered above, describes dynamics of spatial-temporal solitons with the possibility of light bullets formation more exhaustively.

Acknowledgments

This work is supported by the Russian Foundation for Basic Research (grant No. 16-02-00453-a).

References

[1] Silberberg Ya 1990 Optics Letters 15 1282
[2] Akhmanov S A, Sukhorukov A P and Khokhlov R V 1968 Sov. Phys.-Usp. 10 609
[3] Kivshar Yu S and Agrawal G P 2003 Optical solitons (New York: Academic Press)
[4] Zhdanov S K and Trubnikov B A 1987 Sov. Phys. JETP 66 904
[5] Anderson D, Desaix M, Lisak M and Quiroga-Teixeiro M L 1992 *J. Opt. Soc. Am. B* 9 1358
[6] Zhdanov S K and Trubnikov B A 1991 *Quasi-Gaseous Unstable Media* (in Russian, Moscow: Nauka)
[7] Sazonov S V 2004 *JETP* 98 1237
[8] Karamzin Yu N, Sukhorukov A P and Trofimov V A 1989 *Mathematical Modeling in Nonlinear Optics* (in Russian, Moscow: Lomonosov Moscow State University)
[9] Karamzin Yu N and Sukhorukov A P 1974 *Sov. JETP Lett.* 20 339
[10] Sazonov S V 1995 *Opt. Spectrosc.* 79 260