Local Instability Critical Condition in Continuous Medium and its Parameters Inversion

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Abstract. In this paper, earthquake process is described as the process of local destabilization of system in mathematical and physical view and the critical conditions of local instability are given by variational principle. The unknown of basic properties, evolution characters and disturbance characters of crustal medium in deep and the non-completeness of human knowledge make great hardships in directly determining the stiffness and evolution of lithospheric medium. So an inversion strategy is proposed based on the distribution of a dense number of GPS sites and strain gauges.

1. Introduction

Earthquake description may be equivalent to the mechanical problem about internal stability in solid under boundary value conditions[2,3], and the macroscopic local deformation of lithospheric materials due to defect accumulation at different temperatures in the process of continuous loading of far-field boundary, or under the action of mantle material movement. In the analysis domain, many defects undergo irreversible growth, such as the convergence of voids and existing cracks, dislocation propagation, and new defect nucleation. At the same time, strong interaction occurs between different types of defects, prompting rapid evolution and expansion of dislocations and cracks till the final convergence of these defects, leading to the gradual deterioration of the material, thus local instability occurs. The sudden release of the strain energy accumulated in the limited area causes the focus surface ruptures rapidly, and the elastic rebound produces medium vibration which will propagate in the form of waves, and then the analysis domain enters the adjustment period and the new accumulation period.

The continuous loading of boundary stress is a long-term process, while instability occurs instantaneously. Regarding the definition of stability, Hill has boiled it down to that the equilibrium state can be considered as stable if the additional displacement created by applying any small disturbance to the equilibrium body remains infinitely small[4,5]. Conversely, if any small disturbance can cause the displacement of the object to be a finite value, the equilibrium state is unstable. Obviously, a sufficient condition for maintaining stability is that after applying a small disturbance to the equilibrium position, the internal energy of the object should be greater than the work done by the external force on the system.

That means, there is a potential function $\Pi$ in the analysis domain, and the potential function obtains an increment $\Delta \Pi$ if any small disturbance exists in certain equilibrium state, and the increment is expanded at this equilibrium position with the Tayler series.
\[ \Delta \Pi = \delta \Pi + \frac{1}{2} \delta^2 \Pi + ... \]  

(1)

Where, \( \Delta \Pi = 0 \) is a sufficient condition that the equilibrium state remains stable. When \( \Delta \Pi > 0 \) and \( \delta^2 \Pi > 0 \), the system is stable; when \( \delta^2 \Pi < 0 \), the system is unstable; and when \( \delta^2 \Pi = 0 \), the system is at a critical point, and it may be stable or unstable.

Based on the principle of energy, this paper describes the phenomenon of earthquake occurrence as the local instability of the system in mathematics and mechanics, gives critical conditions of local instability in the analysis domain, analyzes the basic properties, evolution, and uncertainty of disturbance properties of the crustal medium, the incompleteness of human knowledge, and the difficulty in directly determining the stiffness and evolution of the medium in the lithosphere, proposing a medium stiffness inversion strategy based on a certain number of GPS and strain gauge distribution.

2. Critical Conditions for Earthquake Occurrence

The analysis domain volume is \( V \), the force is \( b \), the stress boundary is \( S_\sigma \), the stress is \( \phi \), the displacement boundary is \( S_u \), the displacement is \( \bar{u} \), and the equilibrium equation is:

\[
\sigma_{ij} + b_i = \begin{cases} 
0 & \text{Earthquake} \\
\rho \ddot{u}_i & \text{Earthquake} 
\end{cases} 
\]

(2)

The solid constitutive relations are all in the form below:

\[
\sigma_{ij} = D_{ijkl} \varepsilon_{kl} \quad \text{or} \quad \tilde{\sigma}_{ij} = D_{ijkl} \tilde{\varepsilon}_{kl} 
\]

(3)

Where, \( D_{ijkl} = D_{ijkl}^e \) in the linear elastic phase; and \( D_{ijkl} = D_{ijkl}^{p} \) in nonlinear phase. \( D_{ijkl}^{p} \) is related to the factors like loading conditions, yield conditions, damage conditions and plastic stiffness of the material.

The purpose of this paper is to study the macroscopic phenomena in different plasticities and damage development stages in the analysis domain under boundary loading conditions, and establish a mechanical description of the earthquake occurrence process in the domain meeting the conditions of Formulas (2)–(3) (see Fig.1). Where, the pseudo-static process of strain energy accumulation and the earthquake-triggered dynamic process are included.

**Fig 1** Instability Failure Process in Solid

The displacement boundary is substituted by Lagrangian multiplier, and the corresponding constrained potential energy principle \(^\text{[6]}\) is

\[
\Pi = \int_V W dV - \int_V b \cdot u dV - \int_{S_u} \phi u dS - \int_{S_\sigma} n_i \sigma_{ij} (u_i - \bar{u}_i) dS
\]

(4)
Where, \[ W = \int_0^1 \sigma_y \, d\varepsilon_y \] \( \varepsilon \) is the density of strain energy

\[ \delta \Pi = 0, \] then:

\[ -\int \delta u_i (\sigma_{ij} + b_i) \, dV + \int \delta u_i (n_i \sigma_{ij} - \phi_i) \, dS + \int \delta (n_i \sigma_{ij}) (u_i - \bar{u}_i) \, dS = 0 \] \tag{5}

The left side of Formula (5) corresponds to the equilibrium equation, the stress boundary condition and the displacement boundary condition, respectively.

Known by Formula (1), the critical stability condition is

\[ \delta^2 \Pi = \int_V \delta \sigma_{ij} \delta \varepsilon_{ij} \, dV = 0 \] \tag{6}

The analysis domain is divided into a finite number of subdomains \( V_i \), and the formula above is

\[ \delta^2 \Pi = \sum_1 \int_{V_i} \delta \sigma_{ij} \delta \varepsilon_{ij} \, dV = 0 \] \tag{7}

When the sub-domain is small enough and there is no discontinuity in it, the second-order work positive definiteness of the equivalent material of Formula (7) is lost, namely:

\[ \delta \sigma_{ij} \delta \varepsilon_{ij} = 0 \] \tag{8}

Formula (3) is substituted into (8), then

\[ x_y D_{yln} x_l = 0 \] \tag{9}

When a particular vector is considered

\[ x_y = \frac{1}{2} (g_j n_j + g_l n_l) \quad \forall g_j \neq 0, \quad \forall n_i \in \{ |n_l| = 1 \} \] \tag{10}

Generally, the description of Formula (10) is weaker relative to Formula (9). If \( D \) in Formula (3) is symmetrical, and Formulas (3) and (10) are substituted into Formula (9), the critical condition of instability along the plane is obtained, i.e.

\[ (n_j D_{yln} n_l) \cdot g_i = 0 \] \tag{11}

The components of the materials in the nature are always of instability, including earthquakes, landslides, avalanches, sand liquefaction, mudslides and other seismic geological disasters. According to the definition of structural instability, the above natural phenomena must meet:

\[ (n_j D_{yln} n_l) \cdot g_i < 0 \] \tag{12}

The start time of the condition meeting Formula (12) in the analysis domain corresponds to the earthquake occurrence process of Formula (2).

3. Constitutive Relations of Rock Mass Materials

The rock mass material is uneven and is the result of the accumulation of long-term geological changes in the natural environment. We know very little about what harsh environment the materials in the lithosphere have experienced. Therefore, at present, the constitutive structure of rock mass materials is generally treated by continuous medium, and great research progress has been made, but some also use special structures and discrete bodies for simulation. This research on this aspect is generally only carried out for some specific problems.

Whether the continuum mechanics can be applied depends on the scale of problem to be solved, which is reflected in micro, meso and macro perspectives. From a macro perspective, the engineering, geological and geodesic scales are different. As for the issue studied in this paper, due to the large space considered, the crust can be taken as a continuous medium. At a certain temperature, its constitutive structure is considered as plasticity and damage, and the two are coupled.
In general, rock mass materials have two distinct mechanisms of material stiffness degradation, i.e. plasticity and damage. These two mechanisms can be described by plastic theory and damage theory. There are two damage mechanisms: one damage mechanism is non-related to plastic coupling, and the other damage mechanism is non-related to plastic deformation. Their damage dissipation functions are distributed in the elastic and plastic domains. Based on the thermodynamic framework, the coupled plastic-damage constitutive equation is obtained, but no more details of the constitutive theory is provided herein, we only discuss the main part [8].

The strain in the analytical domain strain is resolved as follows:

\[ \varepsilon_{ij} = (\varepsilon_{ij}^e + \varepsilon_{ij}^{ed}) + (\varepsilon_{ij}^p + \varepsilon_{ij}^{id}) \]  

(13)

Where, \( \varepsilon_{ij}^e \) is the elastic strain, \( \varepsilon_{ij}^{ed} \) is the elastic damage strain, \( \varepsilon_{ij}^p \) is the plastic strain, \( \varepsilon_{ij}^{id} \) is the plastic damage strain, \( \varepsilon_{ij}^r \) is the recoverable strain, and \( \varepsilon_{ij}^u \) is the unrecoverable strain, \( \varepsilon_{ij}^r = \varepsilon_{ij}^e + \varepsilon_{ij}^{ed} \).

Their constitutive relations of them are as follows:

\[ \sigma_{ij} = D_{ijkl} (\varepsilon_{ij}^e - \varepsilon_{ij}^r) - \beta_{ij}(T - T_e) \]  

(14)

It is assumed that the plastic yield surface \( f \) and the damage growth surface \( g \) are known. By applying the state equation of pair force of the inner variable, with the pair force of the inner variable described as dissipative function, the unrecoverable strain rate evolves into:

\[ \dot{\varepsilon}_{ij}^r = \dot{\lambda}^r \frac{\partial f}{\partial \sigma_{ij}} + \dot{\lambda}^e \frac{\partial g}{\partial \sigma_{ij}} \]  

(15)

The damage rate evolves to

\[ \dot{d}_{ij} = \dot{\lambda}^r \frac{\partial f}{\partial Y_{ij}} + \dot{\lambda}^e \frac{\partial g}{\partial Y_{ij}} \]  

(16)

From the consistency condition of plasticity and damage, a linear equation in two unknowns with \( \dot{\lambda}^r \) and \( \dot{\lambda}^e \) as variables can be obtained. Solve the equation and import it back into Formula (15), as we know \( \dot{\varepsilon}_{ij}^e = P_{ijkl}\dot{e}_{kl}, \dot{\varepsilon}_{ij}^{ed} = Z_{ijkl}\dot{e}_{kl} \), then \( \dot{\varepsilon}_{ij}^r = U_{ijkl}\dot{e}_{kl}, U_{ijkl} = P_{ijkl} + Z_{ijkl} \). The constitutive equation is

\[ \dot{\sigma}_{ij} = D_{ijkl}^{ed} \dot{e}_{ij}^r - \beta_{ij}\hat{T} \]  

(17)

Where, \( D_{ijkl}^{ed} = (D_{ijkl}^{ed} - D_{ijkl}^{um} U_{mn}) \), \( D_{ijkl}^{ed} \) is the elastic modulus of the medium at a certain temperature.

Strictly speaking, Formula (17) cannot be called a constitutive equation, but merely a relation between stress and strain. The rock mass materials are a combination of various components, and have various structural forms. Different structures interact with each other, conversion may occur between different components and different components may generate new structures under the action of fluid and temperature. Therefore, even if Formula (17) described by the incomplete constitutive structure, it is not easy to obtain the media parameters of the lithosphere at different stages.

4. Factors affecting D

Destructive earthquakes mostly occur within 30km, and an in-depth understanding of the characteristics of rock media within 30km is too profound compared to modern human’s knowledge. In addition, the boundary of the plate theory is also very wide, and there are numerous large and small fracture boundaries inside the plate.

Now, let’s discuss \( \dot{D} \) in Formula (17). For the one-dimensional problems, \( \dot{D} \) is namely the slope of the \( \sigma \sim \varepsilon \) relation curves. The slope is related not only to nonlinear deformation, medium damage,
temperature change $T$, but also to loading conditions and disturbance modes $(\delta \sigma, \delta \varepsilon)$. Therefore, generally we have

$$D = D^{\text{meq}}(\sigma, \varepsilon, \varepsilon', d, T, \delta \sigma, \delta \varepsilon)$$  \hspace{1cm} (18)

Where, $\sigma = \sigma(t)$, $\varepsilon = \varepsilon(t)$, $\varepsilon' = \varepsilon'(t)$, $d = d(t)$, $t \in (t_s, t_{\text{now}})$ and $t_s = 1 \sim 5 \text{ka BP}$.

In principle, there are two ways to determine $D$. For the first method, ideally, assume that the stresses and strains throughout the medium and their history can be directly determined; but this is not the case, we only know the displacement, deformation of some surface points and the stress field direction of the structure at large scales. The second method is to analyze the region and regional structure, invert the regional earthquake case, and study the evolutionary relations between the observed data of physical and mechanical parameters and $D$ by applying the numerical method under the conditions that the displacement and deformation, boundary displacement input and the stress field direction of the structure are known.

4.1 Basic properties of crustal media

The physical properties of the earth medium are generally determined by obtaining the shear wave velocity and compression wave velocity with the seismic wave inversion method, while the authenticity and accuracy of the shear wave velocity and compression wave velocity need to be tested through boreholes. However, deep boreholes are still rare today, and the number of which is insufficient to cover earthquake-hazardous areas around the world. In addition, the nucleation of large earthquakes around 10 km is also less possible [9].

The temperatures of media in different depths and regions in the lithosphere are quite different as well, especially in the low-speed body anomaly area. Theoretically, it is believed that the temperature and pressure in the crust increase with depth, and the temperature rises by 1 °C with the increase of every 100m of depth. The drilling results in recent years show that when the depth is more than 3km, the temperature rises by 2.5°C with the increase of every 100m of depth, and reaches 200°C at the depth of 11km. Due to the different geological structures and constituent materials in the subsurface, the geothermal gradients vary from place to place. Without consideration of phase transition and fluid-solid coupling problems in rock materials under high temperature environment, the elastic constant of the rock along the depth variation can be obtained as a general value.

4.2 Evolution of crustal medium

In the early stage of strain energy accumulation, the medium is in the elastic stage, but as the strain energy accumulates, the medium enters the nonlinear stage, and plasticity and damage may exist at the same time. The evolution of plasticity is discussed here, because the damage evolution problem is more prominent than damage.

The initial stage of strain energy accumulation will last for a long time, meaning that, the elastic stage is very long, and the damage gradually evolves only within a short time before earthquake; at the moment immediately before the earthquake, the damage is connected quickly, causing serious degradation of $D$, and the unstability will occur in a certain form.

The evolutionary research on sparse damage is deeply conducted [11], mainly emphasizing the mathematical description of the main features of damage, the law of dynamics of damage and the relationship between statistical effects of damage and macroscopic mechanical properties. In general continuous damage mechanics applications, the former two are usually ignored in the study. The main features of damage should include scale, orientation and coordinates. The law of dynamics of damage should include connection and interaction of damage. The damage variable $d$ can be defined as damage density, total scale effect and release deformation energy. However, the damage connection mechanism is very complicated and is now generally limited to the evolution of sparse damage.

In the region of high strain energy accumulation, the damage interaction and connection mechanics mechanism are still not clear, but the damage connection may cause major changes in the main characteristics of the damage. The work on this part is generally carried out by numerical experiments [12], and the basic thought of the catastrophe theory may also be adopted.
4.3 Disturbance

It is directly related to stress and strain. Therefore, \( D \) will change when all far-field and near-field events pose disturbance on the stress or strain in the area. In the far field, the impact of large earthquakes on the analysis domain is an important issue to research \[13\]; in the near-field, reservoir storage and drainage, oilfield water injection and mining, nuclear testing, etc. are important issues. Besides, the temperature change of the medium in the lithosphere may directly affect the change of \( D \).

The commonalities of the basic properties, evolution and disturbance characteristics of the above three parts of the crustal medium are all that the parameters of the medium are not clear. The first method, namely directly determining \( D \) using stress and strain is less possible.

5. Inversion Method of \( D \) in the Analysis Domain

In the analysis domain, according to the variational principle of taking \( u_i \), \( \varepsilon_{ij} \) bivariate, the finite-point GPS observations, strain observations and local structure stress fields are used to perform inversion solution for \( D \). Specifically, the variational principle of bivariate

\[
\Pi'_p(\varepsilon_{ij}, u_i) = \Pi + \int_V \frac{\partial W}{\partial \varepsilon_{ij}} (\varepsilon_{ij} - \frac{1}{2} u_{ij} - \frac{1}{2} u_{ji}) \, dV
\]

(19)

Wherein, \( \Pi \) is found in Formula (4), the GPS observations at the domain boundary are used as displacement boundaries.

The strain measurement at the measurement point in the domain is substituted into the functional equation (19) of the generalized variational principle as the constraint, and the penalty function is used.

\[
\Pi''_p = \Pi'_p + \alpha \sum_{k=1}^{n_{\text{min}}} (\varepsilon_{ij}^k - \bar{\varepsilon}_{ij}^k)^2
\]

(20)

The finite element of strain and displacement is discretized, and the overall finite element format in the calculation domain is obtained. The stiffness \( D \) of medium at each point in the variation region is obtained. The displacement field and strain field are obtained by solving the above equations.

The objective function is defined as the square of the difference between the measured point displacement and the measured GPS value in the domain.

\[
F = \sum_{k=1}^{n_{\text{min}}} (u_i^k - \bar{u}_i^k)^2
\]

(21)

When \( F \to F_{\text{min}} \), it can be considered that the \( D \) in the area is real. In fact, it is crucial to carry out the correlation analysis on the selection of the analysis domain, whether there is localization acceleration of the GPS field and the strain field before calculation, which involves the determination on the accumulation rate, accumulation time, accumulation range, and blocking range, the size of the asperities based on the density of the observation point.

In theory, this inversion is feasible. However, the constraint is basically in the shallow part, and there is no deep constraint; if a certain amount of constraints exist both in the shallow part and in the deep part, it will mean huge calculation workload. In addition, the inversion results are generally of multi-solution, and the identification of correct solutions also requires rich expertise. However, with the improvement of cognition, observation equipment and quantity of crustal medium, the evolution of \( D \)-based on thermodynamic framework will get more clear.

The regional seismic hazard analysis can also be performed using the above numerical method. By modeling based on the tectonic environment, which does not need accurate inversion of \( D \); using the empirical method to determine the stiffness tensor of the medium and taking GPS data as input, the strain and stress fields will be obtained. Where strain and stress are concentrated, it is often a possible area for earthquakes \[14\]
6. Conclusions and Recommendations
Starting from the energy principle, the phenomena occurring during earthquake is transformed into the description of the local instability in the system, and the critical conditions of local instability in the analysis domain are given.

The critical condition of instability depends on the constitutive relation of the crustal medium. There are two damage mechanisms in the lithosphere. Based on the thermodynamic framework, the coupled plastic-damage constitutive equation is given. However, the basic properties and evolution parameters of the crustal medium are not clear, and the direct determination of $D$ using stress and strain is less possible.

In view of the unknown constitutive parameters of lithosphere materials, based on the variational principle of bivariate, the inversion method of $D$ is proposed with the finite-point GPS observations, strain observations and local tectonic stress fields and the possibility of inversion is analyzed.

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