Gauge invariant variables and the Yang-Mills-Chern-Simons theory

DIMITRA KARABALI
Department of Physics and Astronomy
Lehman College of the City University of New York
New York, NY 10468
and
Physics Department, Rockefeller University
New York, New York 10021
karabali@theory.rockefeller.edu

CHANJU KIM
Korea Institute for Advanced Study
130-012 Seoul, Korea
cjkim@kias.re.kr

V.P. NAIR
Physics Department
City College of the City University of New York
New York, New York 10031
vpn@ajanta.sci.ccny.cuny.edu

Abstract
A Hamiltonian analysis of Yang-Mills (YM) theory in (2+1) dimensions with a level $k$ Chern-Simons term is carried out using a gauge invariant matrix parametrization of the potentials. The gauge boson states are constructed and the contribution of the dynamical mass gap to the gauge boson mass is obtained. Long distance properties of vacuum expectation values are related to a Euclidean two-dimensional YM theory coupled to $k$ flavors of Dirac fermions in the fundamental representation. We also discuss the expectation value of the Wilson loop operator and give a comparison with previous results.
1. Introduction

In recent papers, we have carried out a nonperturbative analysis of Yang-Mills (YM) gauge theories in two spatial dimensions in a Hamiltonian formulation [1-3]. The use of a matrix parametrization for the gauge potentials enabled us to derive a formulation in terms of gauge invariant variables. This framework and the use of some techniques from conformal field theory helped to simplify the analysis of the spectrum of the theory, leading to results on the mass gap and the vacuum wavefunction. A recent analytic calculation of the string tension gave values within 3% of the values from Monte Carlo simulations.

In this paper, we extend our gauge invariant Hamiltonian analysis to YM theory with a Chern-Simons (CS) mass term added, i.e., the YMCS theory. In particular we shall focus on how the pure YM results on the mass gap, the vacuum wavefunction and the expectation value of Wilson loop are affected by the addition of the CS term.

It is well known, that the presence of the CS term generates a perturbative mass for the gauge bosons [4]. The question of interest here is whether and how the perturbative mass gets augmented or modified by the dynamical generation of mass, which is known to occur for the pure YM case. Our approach to this problem is the construction of gauge invariant states corresponding to the dynamical gauge bosons and the analysis of the corresponding Schrödinger equation. The situation is essentially analogous to our analysis in the case of pure YM case.

There have, of course, been many papers analyzing different aspects of the YMCS theory. A particular way of constructing the lower excited states of the YMCS theory has been given by Grignani et al [5]. Later we shall comment on the extent to which our results differ from this work.

In section 2, the reduction to gauge invariant variables is carried out. Some of the eigenstates of the kinetic energy operator are obtained in section 3. Section 4 deals with the inclusion of the potential energy, vacuum wavefunction, screening, etc. In section 5, a comparison to other recent work is made. The paper concludes with a short discussion.

2. Gauge invariant variables for YMCS

We shall discuss an \( SU(N) \)-gauge theory. The gauge potentials can be written as \( A_\mu = -it^a A^a_\mu \), \( \mu = 0, 1, 2 \), where \( t^a \) are hermitian \( (N \times N) \)-matrices which form a basis of the Lie algebra of \( SU(N) \) with \( [t^a, t^b] = if^{abc}t^c \), \( \text{Tr}(t^a t^b) = \frac{N}{2} \delta^{ab} \). The action for the
YMCS theory is given by $S = S_{YM} + S_{CS}$, with

$$
S_{YM} = \frac{1}{4e^2} \int d^3x \ F_{\mu\nu}^a F^{a\mu\nu}
$$

$$
S_{CS} = -\frac{k}{4\pi} \int d^3x \ Tr \left( A_{\mu} \partial_{\nu} A_\alpha + \frac{2}{3} A_{\mu} A_{\nu} A_\alpha \right) \epsilon^{\mu\nu\alpha}
$$

$$
F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + f^{abc} A_{\mu}^b A_{\nu}^c
$$

where $e$ is the coupling constant; $e^2$ has the dimension of mass. The parameter $k$ is an integer, the level number of the CS term. From now on we shall work in the $A_0 = 0$ gauge, which is convenient for a Hamiltonian formulation.

The canonical momenta are easily identified and the electric field operators are given by

$$
E^a = \frac{\Pi^a}{2} + \frac{ik}{8\pi} A^a = -\frac{i}{2} \frac{\delta}{\delta A^a} + \frac{ik}{8\pi} A^a
$$

$$
\bar{E}^a = \frac{\bar{\Pi}^a}{2} - \frac{ik}{8\pi} \bar{A}^a = -\frac{i}{2} \frac{\delta}{\delta \bar{A}^a} - \frac{ik}{8\pi} \bar{A}^a
$$

where we use complex components, $E = \frac{1}{2}(E_1 + iE_2)$, $A = \frac{1}{2}(A_1 + iA_2)$, $E_i = \frac{1}{e^2} F_{0i}$. The commutation rule for $E^a, \bar{E}^a$ is given by

$$
[\bar{E}^a(\vec{x}), E^b(\vec{y})] = \frac{k}{8\pi} \delta^{ab} \delta(\vec{x} - \vec{y})
$$

The Hamiltonian can be written as

$$
H = T + V
$$

$$
T = 2e^2 \int d^2x \ E^a \bar{E}^a, \quad V = \frac{1}{2e^2} \int d^2x \ B^a B^a
$$

$$
B^a = \frac{1}{2} \epsilon_{ij} F_{ij}^a
$$

We have normally ordered the kinetic energy operator $T$ in accordance with (3).

The Gauss law operator $I^a$ is given by

$$
I^a = (D\bar{\Pi} + \bar{D}\Pi)^a + \frac{ik}{4\pi} (\partial \bar{A}^a - \partial A^a)
$$

The physical states must obey the condition $\int \theta^a(\vec{x}) I^a(\vec{x}) |\psi\rangle = 0$, for functions $\theta^a(\vec{x})$ which vanish at spatial infinity. This requirement, along with equations (2-5), will define the theory.
In carrying out a similar analysis for the pure Yang-Mills case [1-3] we used a particular matrix parametrization of the gauge fields, which eventually led to a gauge invariant formulation. We shall use the same parametrization here, namely
\[ A = -\partial M M^{-1} \quad \text{and} \quad \bar{A} = M^{\dagger -1}\bar{\partial}M^{\dagger} \] (6)
where \( M \) is a complex \( SL(N,\mathbb{C}) \)-matrix. The wavefunctions in the \( A \)-diagonal representation are thus functions of \( M, M^{\dagger} \). From (6) it is clear that the gauge transformation \( A_i \to A_i^h = hA_i h^{-1} - \partial_i h h^{-1} \), is expressed in terms of \( M, M^{\dagger} \) by \( M \to hM, M^{\dagger} \to M^{\dagger}h^{-1} \) for \( h(x) \in SU(N) \). Since the Gauss law operator in (5) generates gauge transformations, we see that the Gauss law condition is equivalent to
\[ \Psi(hM, M^{\dagger}h^{-1}) = \left[ 1 + \frac{k}{2\pi} \int \Tr(M^{\dagger -1}\bar{\partial}M^{\dagger}\partial\theta + \bar{\partial}\theta\partial M M^{-1}) \right] \Psi(M, M^{\dagger}) \] (7)
where \( h(x) \approx 1 + \theta(x) \), \( \theta = -it^a\theta^a \). The general form of the wavefunction obeying (7) can be written as
\[ \Psi(M, M^{\dagger}) = \exp\left[ \frac{k}{2} (S(M^{\dagger}) - S(M)) \right] \chi(H) \] (8)
where \( \chi \) is gauge invariant and depends on \( M, M^{\dagger} \) only via the gauge invariant combination \( H = M^{\dagger}M \). \( S(M) \) is the Wess-Zumino-Witten (WZW) action for \( M \) given by \[ S(M) = \frac{1}{2\pi} \int d^2x \Tr(\partial\bar{\partial}M^{-1}) + \frac{i}{12\pi} \int \Tr(M^{-1} dM)^3 \] (9)
By virtue of the Polyakov-Wiegmann (PW) identity \[ S(hM) = S(h) + S(M) - \frac{1}{\pi} \int d^2x \Tr(h^{-1}\bar{\partial}h\partial M M^{-1}) \] (10)
it is easily checked that (8) is the general solution to the Gauss law condition (7).

We have previously calculated the volume measure for gauge invariant configurations \[1,8] as
\[ d\mu(C) = d\mu(H)e^{2c_A S(H)} \] (11)
where \( C \) is the space of gauge potentials modulo gauge transformations, \( d\mu(H) \) is the product of the Haar measure for \( H \) over all space points and \( c_A \) is the quadratic Casimir for the adjoint representation of \( SU(N) \). In calculating the inner product of two states,
notice that since the phase $\omega$ in (8) is real, $\Psi_1^* \Psi_2 = \chi_1^* \chi_2$ is gauge invariant. Hence the inner product for two states with wavefunctions of the form (8) is given by

$$\langle 1|2 \rangle = \int d\mu(H) e^{2e_A S(H)} \chi_1^* \chi_2$$

(12)

In computing matrix elements of operators involving $E$, $\bar{E}$, the phase $e^{i\omega}$ does contribute since $E$, $\bar{E}$ do not commute with it. We can however use $\chi(H)$ as the wavefunction of the state (with the inner product (12)) provided every operator $O$ acting on $\Psi$ is redefined as $e^{-i\omega} O e^{i\omega}$ in terms of its action on $\chi$'s.

In terms of the $\chi$’s, the corresponding Schrödinger equation reads

$$\mathcal{H}' \chi(H) = E \chi(H)$$

(13)

where $\mathcal{H}' = e^{-i\omega} \mathcal{H} e^{i\omega} = e^{-i\omega} T e^{i\omega} + V = T' + V$. $\mathcal{H}'$ can be expressed in terms of the gauge invariant variable $H = M^\dagger M$.

$$T' = \frac{e^2}{2} \int H_{ab}(\vec{x}) \left( \int_y \tilde{G}(\vec{x}, \vec{y}) \tilde{p}(\vec{y}) - \frac{i k}{4\pi} (\partial HH^{-1})(\vec{x}) \right)_a \left( \int_u G(\vec{x}, \vec{u}) p(\vec{u}) + \frac{i k}{4\pi} (H^{-1} \bar{\partial} H)(\vec{x}) \right)_b$$

$$V = \frac{2}{e^2} \int \bar{\partial} (\partial HH^{-1}) \bar{\partial} (\partial HH^{-1})$$

(14)

In arriving at (14) we have used the fact that with the parametrization (6) of the potentials, we have

$$\Pi^a(\vec{x}) = -i \frac{\delta}{\delta A^a(\vec{x})} = i M_{ba}^\dagger(\vec{x}) \int_y \tilde{G}(\vec{x}, \vec{y}) \tilde{p}_b(\vec{y})$$

$$\bar{\Pi}^a(\vec{x}) = -i \frac{\delta}{\delta A^a(\vec{x})} = -i M_{ab}(\vec{x}) \int_y G(\vec{x}, \vec{y}) p_b(\vec{y})$$

(15)

where $M_{ab} = 2Tr(t^a M^b M^{-1})$ is the adjoint representation of $M$. (Similarly $H_{ab} = 2Tr(t^a H^b H^{-1})$ is the adjoint representation of $H$.) $G(\vec{x}, \vec{y})$, $\tilde{G}(\vec{x}, \vec{y})$ are the Green’s functions for the operators $\partial$, $\bar{\partial}$ respectively.

$$\partial_x G(\vec{x}, \vec{y}) = \bar{\partial}_x \tilde{G}(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y})$$

$$G(\vec{x}, \vec{y}) = \frac{1}{\pi (\vec{x} - \vec{y})} \quad \tilde{G}(\vec{x}, \vec{y}) = \frac{1}{\pi (\vec{x} - \vec{y})}$$

(16)

where $x$, $y$ and $\vec{x}$, $\vec{y}$ are holomorphic and antiholomorphic variables respectively. $p$ acts as the right translation operator on $M$ or $H$ and $\bar{p}$ as the left translation operator on $M^\dagger$ or $H$, namely

$$[p_a(\vec{y}), M(\vec{x})] = M(\vec{x})(-it_a) \delta(\vec{x} - \vec{y})$$

$$[\bar{p}_a(\vec{y}), M^\dagger(\vec{x})] = (-it_a) M^\dagger(\vec{y}) \delta(\vec{x} - \vec{y})$$

(17)
$p_a$, $\bar{p}_a$ can be written as functional differential operators once a parametrization of $H$ is chosen.

$\mathcal{G}$, $\tilde{\mathcal{G}}$ in (14) are the regularized versions of the Green’s functions $G(\vec{x}, \vec{y})$ and $\tilde{G}(\vec{x}, \vec{y})$. They have been defined in ref. [2] as

$$\tilde{G}(\vec{x}, \vec{y}) = \tilde{G}(\vec{x}, \vec{y})[1 - e^{-|\vec{x} - \vec{y}|^2/\epsilon}H(x, \bar{y}) H^{-1}(y, \bar{y})]$$

$$\mathcal{G}(\vec{x}, \vec{y}) = G(\vec{x}, \vec{y})[1 - e^{-|\vec{x} - \vec{y}|^2/\epsilon}H^{-1}(y, \bar{x}) H(\bar{y}, y)]$$

As $\epsilon \to 0$, for finite $|\vec{x} - \vec{y}|$, $(\mathcal{G}, \tilde{\mathcal{G}}) \to (G, \tilde{G})$. Since regularization is needed for calculations involving $T$ or $T'$ we have included it at this stage.

In deriving (14) the following relations expressing the action of the operators $p_a$, $\bar{p}_a$ on the Wess-Zumino actions $S(M)$, $S(M^\dagger)$ were used

$$\bar{p}_a S(M^\dagger) = -\frac{i}{2\pi} \overline{\partial}(\partial M^\dagger M^\dagger^{-1})_a$$

$$p_a S(M) = -\frac{i}{2\pi} \partial(M^{-1} \overline{\partial} M)_a$$

Comparing (14) with the corresponding expression for the Hamiltonian of the pure Yang-Mills case, we see that we can write $\mathcal{H}'$ as (up to an overall constant)

$$\mathcal{H}' = \mathcal{H}_{YM} + \frac{ie^2 k}{8\pi} [((\overline{\partial} H H^{-1})_a \tilde{G} \bar{p}_a - (H^{-1} \partial H)_a \mathcal{G} p_a]$$

$$+ \frac{e^2 k^2}{16\pi^2} \text{Tr}(\partial H H^{-1} \overline{\partial} H H^{-1})$$

Under a parity transformation, $H \to H^{-1}$, $p \to -\bar{p}$, $\bar{p} \to -p$ and we see that the term in (14) or (20) which involves just one factor of $p, \bar{p}$ violate parity conservation, as expected. $\mathcal{H}_{YM}$ is of course parity-invariant.

The expression (14) or (20) for $T'$ is suitable for analyzing symmetries and for comparison with perturbative analysis, but there is an alternative form which is more convenient for our purposes. From the definition of the electric field operators in (2), we see that

$$\overline{E}^a \Psi = e^{-\frac{k^2}{4\pi} \int A^a \bar{A}^a} \left( -\frac{i}{2\delta A^a} \right) \left( e^{\frac{k^2}{4\pi} \int A^a \bar{A}^a} \Psi \right)$$

Because of the extra factor of $\exp\left(\frac{k^2}{4\pi} \int A^a \bar{A}^a\right)$ and the PW identity

$$S(H) = S(M) + S(M^\dagger) - \frac{1}{2\pi} \int A^a \bar{A}^a$$

(22)
we can simplify various formulae by defining \( \chi(H) = \exp\left(\frac{1}{2}kS(H)\right) \Phi(H) \). Thus

\[
\Psi = \exp \left[ \frac{k}{2} (S(M^\dagger) - S(M) + S(H)) \right] \Phi(H)
\]

\[
= \exp \left[ kS(M^\dagger) - \frac{k}{4\pi} \int A^a \bar{A}^a \right] \Phi(H)
\]

In terms of the \( \Phi \)'s, the corresponding Schrödinger equation reads

\[
\hat{H}\Phi(H) = E\Phi(H)
\]

where \( \hat{H} = e^{-\frac{k}{2}S(H)}\mathcal{H}'e^{\frac{k}{2}S(H)} = \hat{T} + V \). Using (19) we can write \( \hat{T} \) as

\[
\hat{T} = \frac{e^2}{2} \int H_{ab}(\vec{x}) \left( \int_y \bar{G}(\vec{x}, y)\bar{p}_a(y) - \frac{ik}{2\pi} (\partial H H^{-1})_a \right) \int_u G(\vec{x}, u)p_b(u)
\]

\[
= \frac{e^2}{2} \int H_{ab} e^{-kS(H)} \bar{G}\bar{p}_a e^{kS(H)} Gp_b
\]

\[
= T_{YM} - \frac{ie^2k}{4\pi} \int H_{ab} (\partial H H^{-1})_a \bar{G}p_b
\]

In terms of \( \Phi \)'s, the inner product is given by

\[
\langle 1|2 \rangle = \int d\mu(H)e^{(k+2c_a)S(H)} \Phi_1^* \Phi_2
\]

This inner product for the YMCS theory agrees with what is obtained for the pure CS theory as well [8]. Compared to the pure YM case, the essential difference in the measure is to replace \( 2c_A \) by \( k + 2c_A \) as the coefficient of the WZW action \( S(H) \).

While expression (25) is self-adjoint with respect to the measure (26), it is not manifestly so. An alternative expression for the kinetic energy operator \( \hat{T} \) can be obtained, which is manifestly self-adjoint. We can write a general matrix element of \( \hat{T} \) as

\[
\langle 1|\hat{T}|2 \rangle = \frac{e^2}{2} \int d\mu(H)e^{(k+2c_a)S(H)} \Phi_1^* H_{ab} e^{-kS(H)} \bar{G}\bar{p}_a e^{kS(H)} Gp_b \Phi_2
\]

From previous analysis [2] we found that

\[
\left[ \bar{G}\bar{p}_a(\vec{x}) , H_{ab}(\vec{x}) e^{2c_A S(H)} \right] = 0
\]

Using this we can rewrite (27) as

\[
\langle 1|\hat{T}|2 \rangle = \frac{e^2}{2} \int d\mu(H)e^{(k+2c_a)S(H)} \Phi_1^* e^{-(k+2c_a)S(H)} (\bar{G}\bar{p})_a H_{ab} e^{(k+2c_a)S(H)} (Gp)_b \Phi_2
\]
which leads to a self-adjoint expression for \( \tilde{T} \) as

\[
\tilde{T} = \frac{e^2}{2} \int e^{-(k+2c_A)}S(H)(Gp)_aH_{ab}e^{(k+2c_A)}S(H)(Gp)_b
\]

(28)

The expression for \( \tilde{T} \) has the same form as in the pure YM theory [2] except for the \( 2c_A \rightarrow k + 2c_A \) shift in the coefficient of \( S(H) \).

In the case of the pure Yang-Mills case we defined the currents

\[
J_a = \frac{c_A}{\pi} \partial H H^{-1}, \quad \bar{J}_a = \frac{c_A}{\pi} H^{-1} \partial H
\]

(29)

It turned out that these current operators generated the whole spectrum of the theory. The situation is not exactly the same in the case of YMCS theory as we shall comment later; however for both YM and YMCS theories, \( J_a \) plays the role of the nonperturbative gluon.

The kinetic energy operator can be simplified in the case of \( \Phi \)'s being purely functions of the current \( J_a \), rather than \( H \) in general; such a simplification will be useful later in the evaluation of the vacuum wavefunction. We find

\[
\tilde{T} = T_{YM} + \frac{e^2 k}{4\pi} \int J^a \frac{\delta}{\delta J^a}
\]

(30a)

\[
T_{YM} = \frac{e^2 c_A}{2\pi} \left[ \int \frac{\delta}{\delta J^a(\bar{u})} \frac{\delta}{\delta J^a(\bar{u})} + \int \Omega_{ab}(\bar{u}, \bar{v}) \frac{\delta}{\delta J^a(\bar{u})} \frac{\delta}{\delta J^b(\bar{v})} \right]
\]

(30b)

\[
\Omega_{ab}(\bar{u}, \bar{v}) = \frac{c_A}{\pi^2 (u-v)^2} - i \frac{f_{abc}J^c(\bar{v})}{\pi (u-v)}
\]

(30c)

We see that the coefficient of the \( \int J\delta/\delta J \)-term is \( (k + 2c_A)e^2/4\pi \), giving a mass of this value to every factor of \( J^a \), which is consistent with the shift \( 2c_A \rightarrow k + 2c_A \).

3. Eigenstates of \( T \)

In analyzing the Schrödinger equation and the eigenstates of the Hamiltonian, we shall follow the strategy we used for the pure Yang-Mills case, namely, we shall consider eigenstates of \( T \) first, neglecting the potential energy term \( V \). This is essentially a strong coupling limit, \( e^2 \gg p \), where \( p \) is the typical momentum scale. In the next section we shall see how the effects of the potential energy term can be included.

The commutation rule (3) shows that \( \bar{E}^a \) is like an annihilation operator while \( E^a \) is like a creation operator. Further, since \( T = 2e^2 \int E^a \bar{E}^a \), we see that the ground state or
vacuum state for $T$ is given by $T\Psi_0 = 0$ with $\bar{E}^a\Psi_0 = 0$. This is equivalent to $\Phi = \Phi_0 = \text{constant}$. In other words, upto a normalization constant,

$$\Psi_0 = \exp \left[ k\mathcal{S}(M^\dagger) - \frac{k}{4\pi} \int A^a \bar{A}^a \right]$$  (31)

This is normalizable with the inner product (12).

Since $E^a$ behaves like a creation operator, we should expect that the excited states can be obtained by successive applications of $E^a$ on $\Psi_0$. In particular, the first excited state may be expected to be of the form $E^a\Psi_0$. However, this is not gauge invariant. Gauge invariant combinations are given by $E^b(x)M^\dagger_{ab}(\bar{x})\Psi_0$ and $M^\dagger_{ab}(\bar{x})E^b(x)\Psi_0$. These can be evaluated as follows.

$$M^\dagger_{ab}(\bar{x})E^b(x)\Psi_0 = \frac{k}{4cA} J_a(\bar{x})\Psi_0$$  (32a)

$$E^b(x)M^\dagger_{ab}(\bar{x})\Psi_0 = M^\dagger_{ab}(\bar{x})E^b(x)\Psi_0 - \frac{i}{2} \left[ \frac{\delta M^\dagger_{ab}(\bar{x})}{\delta A^b(y)} \right]_{\bar{y} \rightarrow \bar{x}} \Psi_0$$

$$= \left( \frac{k}{4cA} + \frac{1}{2} \right) J_a(\bar{x})\Psi_0 = \frac{k + 2cA}{4cA} J_a(\bar{x})\Psi_0$$  (32b)

where we have used the fact that, with proper regularization,

$$-\frac{i}{2} \left[ \frac{\delta M^\dagger_{ab}(\bar{x})}{\delta A^b(y)} \right]_{\bar{y} \rightarrow \bar{x}} = \frac{1}{2} J_a(\bar{x})$$

Thus, apart from constant factors, both $E^bM^\dagger_{ab}$ and $M^\dagger_{ab}E^b$ involve the current $J_a$. From (30), we thus have

$$T( M^\dagger_{ab}(\bar{x})E^b(x)\Psi_0) = \frac{e^2}{4\pi} (k + 2cA) ( M^\dagger_{ab}(\bar{x})E^b(x)\Psi_0)$$  (33)

It is also useful to work through this directly without first relating it to $J$’s. We have

$$T( M^\dagger_{ab}(\bar{x})E^b(x)\Psi_0) = 2e^2 \int y E^k(y)[\bar{E}^k(y), M^\dagger_{ab}(\bar{x})E^b(x)]\Psi_0$$

$$= \frac{e^2k}{4\pi} E^b(x)M^\dagger_{ab}(\bar{x})\Psi_0$$  (34)

We rewrite this in terms of $M^\dagger E$ using (32). The above equation then reduces to

$$T( J_a\Psi_0) = \frac{e^2}{4\pi} (k + 2cA)(J_a\Psi_0)$$  (35)
in agreement with (33). If we start with $E^b(\vec{x})M^\dagger_{ab}(\vec{x})\Psi_0$, the same result is obtained with a careful treatment of potential singularities.

In the large $k$, or semiclassical limit, the lowest excited state $J_a\Psi_0$ has a mass $e^2k/4\pi$ which is the perturbative mass acquired by the gauge boson due to the CS term. $J_a\Psi_0$ may thus be identified as the state corresponding to a single gauge boson which now has a mass $(k + 2c_A)e^2/4\pi$. It is clear that states with many gauge bosons may be obtained by a suitable product of many $J$’s.

4. Potential energy, screening, etc.

We now turn to the inclusion of the potential energy term $V = \frac{1}{2e^2} \int B^2$ and how this modifies the vacuum wavefunction. In the case of the pure YM theory, this could be done in a power series in $1/e^2$ and the resulting series summed up to obtain the vacuum wavefunction in the form $\Phi_0 = e^P$ with

$$P = -\frac{2\pi^2}{e^2c_A^2} \int \delta J_a \left( \frac{1}{m + \sqrt{m^2 - \nabla^2}} \right) \delta J_a + 3J - \text{terms}$$

where $m = e^2c_A/2\pi$. The $3J$-terms involve at least 3 powers of the current $J_a$. Such terms were shown to be subdominant, compared to the $2J$-term displayed, for modes of $J$ in the large and small momentum regimes. For low momentum modes, $P$ simplifies and gives

$$\Phi_0 \approx \exp \left[ -\frac{1}{2me^2} \int \text{Tr} B^2 \right]$$

Vacuum expectation values are thus reduced to correlators in a Euclidean two-dimensional YM theory of coupling constant $g^2 = me^2 = e^4c_A/2\pi$. The expectation value of the Wilson loop operator then follows an area law. The string tension $\sigma$ was obtained as $\sqrt{\sigma} = e^2\sqrt{(N^2 - 1)/8\pi}$, which is in very good agreement with Monte Carlo estimates [9].

In the present case of YMCS theory, the kinetic energy operator has a structure similar to the pure YM case, viz.,

$$\tilde{T} = \tilde{m} \int \delta J_a(\vec{u}) \delta J_a(\vec{u}) + m \int \Omega_{ab}(\vec{u}, \vec{v}) \frac{\delta}{\delta J_a(\vec{u})} \frac{\delta}{\delta J_b(\vec{v})}$$

$$= \tilde{m} \left[ \int \xi^a(\vec{u}) \frac{\delta}{\delta \xi^a(\vec{u})} + \int \Omega_{ab}(\vec{u}, \vec{v}) \frac{\delta}{\delta \xi^a(\vec{u})} \frac{\delta}{\delta \xi^b(\vec{v})} \right]$$

where $\tilde{m} = (k + 2c_A)e^2/4\pi$ and $\xi = \sqrt{\tilde{m}/m} \cdot J$. The inclusion of the potential energy can thus be done in a way similar to the pure YM case. For the term with two currents in $\Phi_0$
we get $\Phi_0 = e^P$, where $P$ is given by (36) with $m \to \bar{m}$ and $J \to \xi$. In other words,

$$\Phi_0 = \exp\left[-\frac{\pi}{m c_A} \int \bar{\partial} \xi \left(\frac{1}{\bar{m} + \sqrt{\bar{m}^2 - \nabla^2}}\right) \bar{\partial} \xi\right] = \exp\left[-\frac{\pi}{m c_A} \int \bar{\partial} J \left(\frac{1}{\bar{m} + \sqrt{\bar{m}^2 - \nabla^2}}\right) \bar{\partial} J\right]$$  \hspace{1cm} (39)

For low momentum modes, we also have the approximation (37), again with $m \to \bar{m}$. We thus find $|\Phi_0|^2 = e^{-S^{(2)}}$, where $S^{(2)}$ corresponds to the action of a Euclidean two-dimensional YM theory of $g^2 = \bar{m} e^2 = e^4 (k + 2 c_A)/4\pi$.

However, we have more than a 2d-YM theory in the evaluation of correlators. The volume measure for gauge invariant configurations is $d\mu(H) \exp(2 c_A S(H))$. For the YMCS theory, the inner product as given by (26) involves $d\mu(H) \exp[(k + 2 c_A) S(H)]$. Thus the vacuum expectation value of an operator $O$ in the $(2 + 1)$-dimensional YMCS theory can be written, for long wavelength modes, as

$$\langle O \rangle = \int d\mu(H) e^{(k + 2 c_A) S(H)} e^{-\frac{1}{4 g^2} \int F^2} O$$

$$= \int d\mu(C) \ e^{k S(H)} e^{-\frac{1}{4 g^2} \int F^2} O$$

(40)

The extra factor $e^{k S}$ can also be expressed in terms of integration over two-dimensional fermions as

$$e^{S(H)} = \det(D\bar{D}) = \int [dQ] e^{-\int (\bar{Q}_L D Q_L + \bar{Q}_R D Q_R)}$$

$$= \int [dQ] e^{\int \bar{Q}_i \gamma^i D Q}$$

(41)

where the $Q$’s are fermions in the fundamental representation of $SU(N)$. Thus with $k$ flavors of such fermions we get the factor $e^{k S(H)}$. An alternative way of writing (40) is therefore

$$\langle O \rangle = \int [dQ] d\mu(H) e^{2 c_A S(H)} e^{-F} O$$

$$= \int [dQ] d\mu(C) \ e^{-F} O$$

(42a)

$$F = \int d^2 x \left[\frac{1}{4 g^2} F_{\mu\nu}^a F^{a\mu\nu} + \sum_{i=1}^k \bar{Q}_i^i \gamma^i \cdot D Q_i^i\right]$$

(42b)

where $g^2 = e^4 (k + 2 c_A)/4\pi$. In other words, the problem is equivalent to the computation of functional averages in a two-dimensional YM theory coupled to $k$ flavors of massless Dirac fermions in the fundamental representation.
Two-dimensional YM theory coupled to fermions in the fundamental representation has been analyzed by a number of authors [10]. Some of their results can be taken over to the present context. For example, the presence of massless dynamical fermions leads to screening of Wilson loop operators. We may thus conclude that the average of the Wilson loop operator for the YMCS theory will not obey an area law; rather we should have $\langle W_C \rangle = e^{-w}$, with $w/A_C \to 0$ as the area $A_C \to \infty$.

The appearance of fermions in the fundamental representation in (42) makes the screening of charges plausible from an intuitive point of view. Nevertheless, (42) ultimately provides only a mathematically useful representation; there are, of course, no fermions in the YMCS theory we are considering. In fact, for the expectation value of $W_C$, one can directly do the integration using (40); the term $e^{kS}$ leads to short range propagators for the gauge potentials and hence to screening.

Since screening effects generally require the presence of charged particles in the physical spectrum, one might ask the question of whether these can be understood in terms of the gauge fields themselves. The inner product (26) shows that matrix elements in the YMCS theory are obtained in terms of a hermitian WZW model of level $(k + 2c_A)$. The correlators in this model are the analytic continuation of the correlators of the level $k$ $SU(N)$ WZW-model with $\kappa = k + c_A$ replaced by $-\kappa = -(k + c_A)$. The states of finite norm in YMCS theory can thus be constructed in terms of the integrable primary operators of the $SU(N)$-theory and not just the currents; such primary operators other than the identity do exist for $k \neq 0$ [1,2,8]. An example of such a state would be

$$\alpha = U(\infty, \vec{x})M(\vec{x})$$

$$U(\infty, \vec{x}) = P \exp \left[ \int_{\vec{x}}^{\infty} A \right]$$

(43)

$\alpha$ is gauge invariant under transformations which go to the identity at spatial infinity. Under gauge transformations which go to a constant $h \neq 1$ at spatial infinity, $\alpha \to h\alpha$, which is what is expected of a charged state.

5. Comparison with previous results

As mentioned before, there have been many papers analyzing different aspects of the YMCS theory. A Hamiltonian analysis, which is closest in spirit to ours, is in a recent paper by Grignani et al where a gauge invariant construction of eigenstates of $T$ is presented [5].
The construction of the states is as follows. Define
\[ S(g, A, B) = S(g) + \frac{1}{\pi} \int \text{Tr}[g^{-1} \partial gB - \bar{A} \partial gg^{-1} + \bar{A}gBg^{-1} - \frac{1}{2}(\bar{A}A + \bar{B}B)] \] (44)
where \( g(x) \) is an \( SU(N) \)-valued field, \( S(g) \) is the WZW action and \( B \) is an auxiliary field variable. Equation (44) is like a “gauged” WZW action with \((\bar{A}, B)\) acting as the gauge field. Using (2), we can check immediately that \( \bar{E}^a e^{-kS(g,A,B)} = 0 \). The strategy used in ref. [5] is to use this fact and define the vacuum wavefunction for \( T \) as
\[ \Psi_0[A, B] = \int [dg] e^{-kS(g,A,B)} \] (45)
where the group integration is done with the Haar measure. The resulting \( \Psi_0[A, B] \) still depends on the auxiliary variable \( B \) but by defining the inner product as
\[ \langle 1|2 \rangle = \int \frac{[dAdB]}{\text{vol}\mathcal{G}_s} \Psi_1^*(A, B)\Psi_2(A, B) \] (46)
one has a systematic way to compute matrix elements. (\( \mathcal{G}_s \) is the space of all gauge transformations which approach a constant at spatial infinity.) Under a gauge transformation, \( M \to hM, \ M^\dagger \to M^\dagger h^{-1} \), we have \( g \to hg \). Since \( E^b(\vec{x}) \) behaves as a “creation” operator by virtue of (3), one can use the gauge transformation property of \( g \) and define a gauge invariant excited state
\[ \Psi_a[\vec{x}; A, B] = E_b(\vec{x}) \int [dg] g_{ba}(\vec{x}) e^{-kS(g,A,B)} \] (47)
Higher excited states were also defined in ref. [5]; for comparison with our results, the lowest excited state (47) is adequate to illustrate the main points. Since \( \bar{E} \) has the commutation rule (3) with \( E \) and does not affect \( g_{ba} \), it was argued in ref. [5], that \( \Psi_a[\vec{x}; A, B] \) is an eigenstate of \( T \) with eigenvalue \( e^2k/4\pi \). It was indicated that the proper regularization of \( E\bar{E} \) in \( T \) might modify the result, although no calculations to this effect were given. In comparing with our results in section 3 the following questions arise: How are the states \( \Psi_a[\vec{x}; A, B] \) related to our excited states created by the action of the current \( J_a \) on the \( \Psi_0 \), eq. (32)? If they are related what is the correct energy eigenvalue? These questions were already posed in ref. [5]. In what follows we present an analysis which essentially provides an answer to these questions and the source of the discrepancy between our results and the ones in ref. [5].

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We begin with the observation that $\Psi_{0}[A, B]$ in (45) is proportional to our $\Psi_{0}$ in (31). We can write $B = -\partial NN^{-1}$ for some $SL(N, C)$-matrix $N$ and then (44) can be written as

$$S(g, A, B) = S(M^\dagger gN) - S(M^\dagger) - S(N) - \frac{1}{2\pi} \int \text{Tr}(A\bar{A} + B\bar{B})$$  

Equation (45) now becomes

$$\Psi_{0}[A, B] = e^{kS(M^\dagger)} - \frac{k}{4\pi} \int A^a\bar{A}^a e^{kS(N)} - \frac{k}{4\pi} \int B^a\bar{B}^a \int dg e^{-kS(M^\dagger gN)}$$

where, by invariance of the Haar measure, $dg = d(M^\dagger gN)$ and hence $C$ is independent of $A, \bar{A}$. $\Psi_{0}[A, B]$ in (45) agrees upto a normalization constant with our $\Psi_{0}$ in (31).

Consider now the excited state (47). Carrying out the action of $E_b$ on $S(g, A, B)$ we find

$$\Psi_{a}[\vec{x}; A, B] = \frac{ik}{4\pi} \int [dg] g_{ba}(\vec{x})[A^b - (gBg^{-1} - \partial gg^{-1})^b](\vec{x})e^{-kS(g, A, B)}$$

There are singularities involved in this expression. $\Psi_{a}[\vec{x}; A, B]$ involves averages with the level $k$ $SU(N)$ WZW model and, in particular, it involves the combination $g_{ba}(\vec{x})\partial gg^{-1}(\vec{x})$. Since $\partial gg^{-1}$ is the current of the WZW model, there are singularities in the WZW correlator $\langle \partial gg^{-1}(\vec{y})g_{ba}(\vec{x}) \rangle$ as $\vec{y} \to \vec{x}$, as is evident from the standard operator product expansions. This is also seen from the use of (48) to simplify (50). We find

$$\Psi_{a}[\vec{x}; A, B] = \frac{ik}{4\pi} \int [dg] [(M^\dagger^{-1}PN^{-1})_{ba}(\vec{x})(-\partial MM^{-1} - M^\dagger^{-1}\partial M^\dagger)_b(\vec{x})] + g_{ba}(\vec{x})(M^\dagger^{-1}\partial PP^{-1}M^\dagger)_b(\vec{x})] e^{-kS(g, A, B)}$$

where $P = M^\dagger gN$.

The operator product expansion gives

$$\text{Tr}(t_a \partial PP^{-1})(\vec{y}) = \frac{-i}{k} \frac{M^\dagger_{ab}(\vec{x})f_{lbm}g_{ma}(\vec{x})}{(y - x)}$$

We see that (47), as written, is not well defined. To avoid the singularity in the second term of (51), we must introduce some sort of point-splitting. We can define a proper regularized version of (47) by

$$\Psi_{a}[\vec{x}; A, B] = R_{cb}(\vec{x}, \vec{y})E_{c}(\vec{x}) \int [dg] g_{ba}(\vec{y})e^{-kS(g, A, B)}|_{\vec{y} \to \vec{x}}$$
where
\[ R_{cb}(\vec{x}, \vec{y}) = \left[ e^{-A(x-y) - \bar{A}(\vec{x}-\vec{y})} \right]_{cb} \] (53b)

is the Wilson line from \( y \) to \( x \) (written above for small separation). With this regularization we can now evaluate \( \Psi_a[x; A, B] \) to get
\[ \Psi_a[x; A, B] = \frac{k + 2c_A}{2c_A} \langle (PN^{-1})_{ba} \rangle J_b(x) \Psi_0 \] (54a)

where
\[ \langle (PN^{-1})_{ba} \rangle = \int [dP] P_{bs} e^{-kS(P)} N_{sa}^{-1} \] (54b)

We see that \( \Psi_a[x; A, B] \) is indeed proportional to our excited state (32). However two remarks concerning (54) are in order. First of all, regularization is important, not just for the evaluation of \( T \), but for defining the \( \Psi_a \) itself via (47). Secondly, the average of \( P \) in (54b) is actually zero. Thus the state would be zero unless this trivial (independent of \( A, \bar{A} \)) factor can be removed by some suitable procedure. It may be possible to do this by defining “renormalized” \( g_{ab} \)’s with factors depending on infrared and ultraviolet cutoffs (similar to what is done in minimal conformal models \cite{11}).

Since \( \Psi_a[x; A, B] \) is proportional to \( J_a(x) \Psi_0 \) we would expect that its energy eigenvalue at the strong coupling limit (its \( T \) eigenvalue) should be \( \frac{e^2}{4\pi} (k + 2c_A) \), according to our analysis. Thus regularization should indeed shift the eigenvalue from the value \( e^2 k / 4\pi \) found in ref. [5]. We shall now go over the salient features of such a regularized calculation.

First of all, instead of (53b) we use the more systematic expression
\[ R_{cb}(\vec{x}, \vec{y}) = \left[ M^{-1}(x, \vec{x})H(x, \vec{y})H^{-1}(y, \vec{y})M^{-1}(y, \vec{y}) \right]_{cb} \sigma(\vec{x}, \vec{y}, \epsilon) \]
\[ \sigma(\vec{x}, \vec{y}, \epsilon) = \frac{1}{\pi \epsilon} \exp \left( -|\vec{x} - \vec{y}|^2 / \epsilon \right) \] (55)

As \( \epsilon \to 0 \), \( \sigma(\vec{x}, \vec{y}, \epsilon) \to \delta(\vec{x}, \vec{y}) \) and \( R_{cb}(\vec{x}, \vec{y}) \to \delta_{cb} \delta(\vec{x}, \vec{y}) \) as desired. Expansion of (55) for small separations coincides with (53b). The regularized state is given by
\[ \Psi_a[\vec{x}; A, B] = \int_y \int [dg] R_{cb}(\vec{x}, \vec{y}) E_{c}(\vec{x}) g_{ba}(\vec{y}) e^{-kS(g, A, B)} \] (56)

The action of \( E \) on \( S(g, A, B) \) produces the following expression for (56)
\[ \Psi_a[\vec{x}; A, B] = \frac{k}{2c_A} \int_y \sigma(\vec{x}, \vec{y}, \epsilon) \int [dg] e^{-kS(g)} (g N^{-1})_{ba}(y) [H(x, \vec{y}) H^{-1}(y, \vec{y})]_{cb} \left[ J - \frac{c_A}{\pi} (\partial gg^{-1}) \right]_e (\vec{x}) \Psi_0 \] (57)

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In applying $T$ on $\Psi_a[x; A, B]$ we encounter the following terms.

$$
T\Psi_a[\vec{x}; A, B] = I + II + III + IV + V
$$

$$
I = \frac{ke^2}{c_A} \int \sigma e^{-kS(gN^{-1})_{ba}} \left[ E^k, [\bar{E}^k, [HH^{-1}]_{cb}] \right] \left( J - \frac{c_A}{\pi} (\partial gg^{-1}) \right)_c \Psi_0
$$

$$
II = \frac{ke^2}{c_A} \int \sigma e^{-kS(gN^{-1})_{ba}} \left[ \bar{E}^k, [HH^{-1}]_{cb} \right] [E^k, J_c] \Psi_0
$$

$$
III = \frac{ke^2}{c_A} \int \sigma e^{-kS(gN^{-1})_{ba}} \left[ \bar{E}^k, [HH^{-1}]_{cb} \right] \left( J - \frac{c_A}{\pi} (\partial gg^{-1}) \right)_c E^k \Psi_0
$$

$$
IV = \frac{ke^2}{c_A} \int \sigma e^{-kS(gN^{-1})_{ba}} \left[ E^k, [HH^{-1}]_{cb} \right] \bar{E}^k J_c \Psi_0
$$

$$
V = \frac{ke^2}{c_A} \int \sigma e^{-kS(gN^{-1})_{ba}} \left[ HH^{-1} \right]_{cb} E^k \bar{E}^k J_c \Psi_0
$$

(58)

Term $V$ has been essentially calculated in (35). We find

$$
V = \frac{k}{2c_A} \frac{e^{2(k+2c_A)}}{4\pi} J_b \Psi_0 \int e^{-kS(g)(gN^{-1})_{ba}}
$$

(59)

In evaluating the other terms we make the following observations. The commutator of $H(x, \bar{y})H^{-1}(y, \bar{y})$ with $E$ or $\bar{E}$ is of order $(x - y)$ and it vanishes as $x \rightarrow y$. However there are singularities of the order $\frac{1}{x-y}$ coming from the operator product expansion $g_{be}(\bar{y})(\partial gg^{-1})_c(\vec{x})$ in terms $I$ and $III$. As a result one might expect a finite contribution from $I$ and $III$. There are also possible new singularities due to the coincidence limit taken when $\bar{E}(\vec{w})$ and $E(\vec{w})$ act on the operators at the same position, which needs more careful treatment. With the regularized expressions for $E, \bar{E}$ from (2), (15) and (18) we have

$$
[\bar{E}_k(\vec{w}), H(x, \bar{y})H^{-1}(y, \bar{y})] = -\frac{1}{2} M_{kl}(\vec{w}) \left[ G_{lm}(\vec{w}, x, \bar{y}) - G_{lm}(\vec{w}, \bar{y}) \right] H(x, \bar{y}) T^m H^{-1}(y, \bar{y})
$$

(60)

where $T^m$ are the Lie algebra generators in the adjoint representation. After tedious but straightforward calculations, the commutator $\int_w [E, [\bar{E}, HH^{-1}]]$ can be shown to be

$$
\int_w [E(\vec{w}), [\bar{E}(\vec{w}), H(x, \bar{y})H^{-1}(y, \bar{y})]] = \frac{1}{4} (x - y) J(\vec{x}) + O((x - y)^2)
$$

(61)

Combining this with (52) we get

$$
I = \frac{c_A e^2}{2\pi} J_b \Psi_0 \int e^{-kS(g)(gN^{-1})_{ba}}
$$

(62)
Similarly, we get a nonzero finite result from term $III$, 

$$III = \frac{ke^2}{4\pi} J_b \Psi_0 \int e^{-kS(g)}(gN^{-1})_{ba}$$

(63)

The remaining terms $II$ and $IV$ need careful evaluation; the calculations are once again quite long since most of the integrals involved do not admit approximations for the parameter-regimes of interest. Eventually the terms $II$ and $IV$ are seen to vanish. Using the results (58-63), we then find

$$T\Psi_a[\vec{x}; A, B] = \frac{e^2(k + 2c_A)}{4\pi} \frac{k + 2c_A}{2c_A} J_b \Psi_0 \int e^{-kS(g)}(gN^{-1})_{ba}$$

$$= \frac{e^2}{4\pi}(k + 2c_A)\Psi_a[\vec{x}; A, B]$$

(64)

The eigenvalue of $T$ is indeed $(e^2/4\pi)(k + 2c_A)$ in agreement with our results. Thus the analysis of ref. [5], if carried further, taking account of regularizations, will give results identical to ours.

Similar results hold for the higher excited states considered in [5]. As an example we consider the properly regularized excited state with two $E$ 's, namely 

$$\Psi_{a_1a_2}[\vec{x}_1, \vec{x}_2; A, B] = \int [dg] \prod_{i=1}^{2} R_{c_i b_i} (\vec{x}_i, \vec{y}_i) E_{c_i} (\vec{x}_i) g_{b_i a_i} (\vec{y}_i) e^{-kS(g,A,B)} =$$

$$\left(\frac{k}{2c_A}\right)^2 \int \prod_{i=1}^{2} \sigma(\vec{x}_i, \vec{y}_i, \epsilon)[H(x_i, y_i)H^{-1}(y_i, y_i)]_{c_i b_i} \left[J - \frac{c_A}{\pi}(\partial gg^{-1})\right]_{c_1} (\vec{x}_i) g_{b_i a_i} (\vec{y}_i) \Psi_0 [A, B]$$

(65)

This expression can be simplified by carrying out the operator product expansion as before. In this case, we need to reduce three- and four-point functions of the type

$$\int e^{-kS(g)} g_{b_1 a_1} (\vec{y}_1) g_{b_2 a_2} (\vec{y}_2) (\partial gg^{-1})_{c_1} (\vec{x}_1)$$

(66a)

$$\int e^{-kS(g)} g_{b_1 a_1} (\vec{y}_1) g_{b_2 a_2} (\vec{y}_2) (\partial gg^{-1})_{c_1} (\vec{x}_1) (\partial gg^{-1})_{c_2} (\vec{x}_2)$$

(66b)

into two-point functions of two $g$'s. After taking singularities from operator product expansions into account, we find

$$\Psi_{a_1a_2}[\vec{x}_1, \vec{x}_2; A, B] = \left\{ \begin{array}{l}
(1 + \frac{k}{2c_A}) J_{b_1} (\vec{x}_1) J_{b_2} (\vec{x}_2) + \left(1 + \frac{k}{2c_A}\right) G(\vec{x}_1, \vec{x}_2) [J(\vec{x}_1) + J(\vec{x}_2)] \delta_{b_1 b_2}

+ \frac{k}{2} [G(\vec{x}_1, \vec{x}_2)]^2 \delta_{b_1 b_2} \int e^{-kS(g)}(gN^{-1})_{b_1 a_1} (\vec{y}_1) (gN^{-1})_{b_2 a_2} (\vec{y}_2) \Psi_0
\end{array} \right\}$$

(67)
where the factor 1 in the first two terms has come from regulators as in the lowest excited $J$ state. Using the expression (38) of $T$ in terms of $J, T$ on $\Psi_{a_1a_2}[\vec{x}_1, \vec{x}_2; A, B]$ becomes

$$T\Psi_{a_1a_2} = 2\tilde{m}\Psi_{a_1a_2} + 2\tilde{m}c_A[G(\vec{x}_1, \vec{x}_2)]^2\delta_{b_1b_2} \int e^{-kS(g)}(gN^{-1})_{b_1a_1}(\vec{x}_1)(gN^{-1})_{b_2a_2}(\vec{x}_2)\Psi_0$$

(68)

Therefore the state $\Psi_{a_1a_2}$ is essentially an eigenstate of $T$ (by redefining the constant part in $\Psi_{a_1a_2}$ to absorb the last term in (68)). The corresponding eigenvalue is $2\frac{e^2}{4\pi}(k + 2c_A)$. In obtaining this result, it is crucial that the ratio of the coefficients of the two-$J$ and one-$J$ terms in $\Psi_{a_1a_2}$ is as in (67); otherwise one does not get an eigenstate of $T$ of the form (38). Although we did not explicitly calculate eigenvalues of $T$ for even higher excited states, we expect the eigenvalue of the state involving $n$ $E$’s or $n$ $J$’s to be $n\frac{e^2}{4\pi}(k + 2c_A)$. Thus we see that the contribution of the regulator used in defining states is important.

An interesting point of the above calculation is that the eigenvalues add up without any correction, irrespective of the positions $\vec{x}_1$ and $\vec{x}_2$. Therefore, in the strong coupling limit where the potential term is neglected, there is no interaction between $J$’s which modifies the eigenvalue. In other words, the $J$’s can be far separated from each other. This is consistent with the fact that there is no confinement in YMCS theory. One might ask the question of whether such states can also occur in pure YM theory. We expect these states to be nonnormalizable in this case, in agreement with the expected confinement in the pure Yang-Mills case.

Our calculation which gives the shift $2c_A \rightarrow k + 2c_A$ takes account of the nonperturbative corrections to the mass. This is different from the shift $k \rightarrow k + c_A$ expected from perturbative calculations [12]. Such perturbative corrections to our calculation may also exist; they can in principle be calculated using perturbation theory in the Hamiltonian formalism. The explicitly self-adjoint expression (28) for $\tilde{T}$ is most suited for this analysis; the mixing of parity-violating and conserving terms, as in the calculation of Pisarski and Rao, should produce the perturbative corrections. Cornwall has recently pointed out that the inclusion of both perturbative and nonperturbative effects for the mass can perhaps lead to some sort of critical behaviour at a certain value of $k$ [13]. Our calculations do not go far enough to make more precise statements regarding this question.

### 6. Conclusion

We have performed a Hamiltonian analysis of Yang-Mills theory with a level $k$ Chern-Simons term in terms of gauge invariant variables. The low lying spectrum of this theory
at the strong coupling limit has been obtained. The gauge invariant version of the massive
gauge boson states is given in terms of the current $J^a$. The mass of the gauge boson
is $\frac{e^2}{4\pi}(k + 2c_A)$. Long distance properties of vacuum expectation values are related to
a Euclidean two-dimensional YM theory coupled to $k$ flavors of Dirac fermions in the
fundamental representation. The expectation value of the Wilson loop operator should
exhibit screening rather than an area law. Related comments and comparison with previous
results are also given.

Acknowledgements

This work was supported in part by the NSF grant PHY-9605216. CK thanks Lehman
College of CUNY and Rockefeller University for hospitality facilitating the completion of
this work.

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