Second Main Theorem and Unicity of Meromorphic Mappings for Hypersurfaces in Projective Varieties

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Abstract Let $V$ be a projective subvariety of $\mathbb{P}^n(\mathbb{C})$. A family of hypersurfaces $\{Q_i\}_{i=1}^q$ in $\mathbb{P}^n(\mathbb{C})$ is said to be in $N$-subgeneral position with respect to $V$ if for any $1 \leq i_1 < \cdots < i_{N+1} \leq q$, $V \cap (\bigcap_{j=1}^{N+1} Q_{i_j}) = \emptyset$. In this paper, we will prove a second main theorem for meromorphic mappings of $\mathbb{C}^m$ into $V$ intersecting hypersurfaces in subgeneral position with truncated counting functions. As an application of the above theorem, we give a uniqueness theorem for meromorphic mappings of $\mathbb{C}^m$ into $V$ sharing a few hypersurfaces without counting multiplicity. In particular, we extend the uniqueness theorem for linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ sharing $2n + 3$ hyperplanes in general position to the case where the mappings may be linearly degenerated.

Keywords Holomorphic curves · Algebraic degeneracy · Defect relation · Nochka weight

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1 Introduction and Main Results

This article is a continuation of our studies in [2]. To formulate the main result in [2], we recall the following.
Let $N \geq n$ and $q \geq N + 1$. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. The hypersurfaces $D_1, \ldots, D_q$ are said to be in $N$-subgeneral position in $\mathbb{P}^n(\mathbb{C})$ if $D_{j_0} \cap \cdots \cap D_{j_N} = \emptyset$ for every $1 \leq j_0 < \cdots < j_N \leq q$.

Throughout this paper, sometimes we will identify a hypersurface in $\mathbb{P}^n(\mathbb{C})$ with one of its defining homogeneous polynomials if there is no confusion. In [2], the authors proved the following result.

**Theorem 1** Let $f$ be an algebraically nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $\{Q_i\}_{i=1}^q$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with $\deg Q_i = d_i$ $(1 \leq i \leq q)$. Let $d = \lcm(d_1, \ldots, d_q)$ and $M = \left(\frac{n+d}{n+1}\right) - 1$. Assume that $q > \left(\frac{(M+1)(2N-n+1)}{n+1}\right)$. Then, we have

$$\| \left( q - \frac{(M + 1)(2N - n + 1)}{n + 1} \right) T_f(r) \| \leq \sum_{i=1}^q \frac{1}{d_i} N_{Q_i(f)}(r) + o(T_f(r)).$$

Here, by the notation “$\| P \|$” we mean that the assertion $P$ holds for all $r \in [0, \infty)$ excluding a Borel subset $E$ of the interval $[0, \infty)$ with $\int_E dr < \infty$.

The first aim of this article is to generalize the above second main theorem to meromorphic mappings into projective varieties sharing hypersurfaces in subgeneral position.

We now give the following.

**Definition 1** Let $V$ be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k$ $(k \leq n)$. Let $Q_1, \ldots, Q_q$ $(q \geq k + 1)$ be $q$ hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. The family of hypersurfaces $\{Q_i\}_{i=1}^q$ is said to be in $N$-subgeneral position with respect to $V$ if for any $1 \leq i_1 < \cdots < i_{N+1} \leq q$,

$$V \cap \left( \bigcap_{j=1}^{N+1} Q_{i_j} \right) = \emptyset.$$

If $\{D_i\}_{i=1}^q$ is in $N$-subgeneral position then we say that it is in general position with respect to $V$.

Now, let $V$ be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k$ $(k \leq n)$. Let $d$ be a positive integer. We denote by $\mathcal{I}(V)$ the ideal of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ defining $V$ and by $H_d$ the $\mathbb{C}$-vector space of all homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ of degree $d$. Define

$$\mathcal{I}_d(V) := \frac{H_d}{\mathcal{I}(V) \cap H_d} \quad \text{and} \quad H_V(d) := \dim \mathcal{I}_d(V).$$

Then $H_V(d)$ is called the Hilbert function of $V$. Each element of $\mathcal{I}_d(V)$ which is an equivalent class of an element $Q \in H_d$ will be denoted by $[Q]$.

**Definition 2** Let $f : \mathbb{C}^m \rightarrow V$ be a meromorphic mapping. We say that $f$ is degenerate over $\mathcal{I}_d(V)$ if there is $[Q] \in \mathcal{I}_d(V) \setminus \{0\}$ such that $Q(f) \equiv 0$. Otherwise, we say that $f$ is nondegenerate over $\mathcal{I}_d(V)$. It is clear that if $f$ is algebraically nondegenerate, then $f$ is nondegenerate over $\mathcal{I}_d(V)$ for every $d \geq 1$.

Our main theorem is stated as follows.
Theorem 2 Let $V$ be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k$ ($k \leq n$). Let $\{Q_i\}_{i=1}^q$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with respect to $V$ with $\deg Q_i = d_i$ ($1 \leq i \leq q$). Let $d$ be the least common multiple of $d_i$, $d = \operatorname{lcm}(d_1, \ldots, d_q)$. Let $f$ be a meromorphic mapping of $\mathbb{C}^n$ into $V$ such that $f$ is nondegenerate over $I_d(V)$. Assume that $q > \frac{(2N - k + 1)H_V(d)}{k+1}$. Then, we have

$$\left\| \left( q - \frac{(2N - k + 1)H_V(d)}{k+1} \right) T_f(r) \right\| \leq \sum_{i=1}^q \frac{1}{d_i} N^{[H_V(d)-1]}(r) + o(T_f(r)).$$

We note that the second main theorem for algebraically nondegenerate meromorphic mappings into projective subvarieties was firstly given by Min Ru [9] in 2004. In his result, the family of hypersurfaces is assumed to be in general position and there is no truncation level for the counting functions, but the total defect is $n + 1$, which is the sharp number.

Remark (i) In the case where $V$ is a linear space of dimension $k$ and each $H_i$ is a hyperplane, i.e., $d_i = 1$ ($1 \leq i \leq q$), then $H_V(d) = k + 1$ and Theorem 2 gives us the classical second main theorem of Cartan-Nochka (see [7] and [8]).

(ii) It is easy to see that $H_V(d) - 1 \leq \left( \frac{n + d}{n} \right) - 1$. Furthermore, the truncated level $(H_V(d) - 1)$ of the counting function in Theorem 2 is much smaller than the previous results of all other authors (cf. [1, 4]).

(iii) By a direct computation from Theorem 2, it is easy to see that the total defect is $\frac{(2N - k + 1)H_V(d)}{k+1}$. Unfortunately, this defect is $\geq n + 1$.

(iv) Also, the above notion of $N$-subgeneral position is a natural generalization of similar notion in the case of hyperplanes. Therefore, in order to prove Theorem 2, we give a generalization of Nochka weights for hypersurfaces in complex projective varieties.

(v) From Cartan-Nochka’s theorem, we may obtain a second main theorem by using Veronese embedding which embeds $\mathbb{P}^n(\mathbb{C})$ into $\mathbb{P}^{(n+d)}(\mathbb{C})$. But in that case, we need the condition that the family of hyperplanes corresponding to the initial family of hypersurfaces is still in subgeneral position in $\mathbb{P}^{(n+d)}(\mathbb{C})$, which is not satisfied if $N < \left( \frac{n + d}{n} \right)$.

As an application of Theorem 2, the second aim of this article is to give a uniqueness theorem for meromorphic mappings of $\mathbb{C}^m$ into $V$ sharing a few hypersurfaces without counting multiplicity.

Theorem 3 Let $V$ be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k$ ($k \leq n$). Let $\{Q_i\}_{i=1}^q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with respect to $V$ and $\deg Q_i = d_i$ ($1 \leq i \leq q$). Let $d$ be the least common multiple of $d_i$, $d = \operatorname{lcm}(d_1, \ldots, d_q)$. Let $f$ and $g$ be meromorphic mappings of $\mathbb{C}^m$ into $V$ which are nondegenerate over $I_d(V)$. Assume that

(i) $\dim(\operatorname{Zero} Q_i(f) \cap \operatorname{Zero} Q_j(f)) \leq m - 2$ for every $1 \leq i < j \leq q$,

(ii) $f = g$ on $\bigcup_{i=1}^q (\operatorname{Zero} Q_i(f) \cup \operatorname{Zero} Q_i(g))$.

Then the following assertions hold:

a) If $q > \frac{2H_V(d) - 1}{d} + \frac{(2N - k + 1)H_V(d)}{k+1}$, then $f = g$.

b) If $q > \frac{2(2N - k + 1)H_V(d)}{k+1}$, then there exist $N + 1$ hypersurfaces $Q_{i_0}, \ldots, Q_{i_N}$, $1 \leq i_0 < \cdots < i_N \leq q$, such that

$$\frac{Q_{i_0}(f)}{Q_{i_0}(g)} = \cdots = \frac{Q_{i_N}(f)}{Q_{i_N}(g)}.$$
N.B. (i) Since the truncated level of the counting function in Theorem 2 is better, the number of hypersurfaces in Theorem 3 is much smaller than the previous results on unicity of meromorphic mappings sharing hypersurfaces (cf. [4, 5]).

(ii) In the case where \( d = 1 \), Theorem 3b) immediately gives us the following uniqueness theorem for meromorphic mappings into \( \mathbb{P}^n(\mathbb{C}) \), which may be linearly degenerated, sharing few hyperplanes in general position.

**Corollary 1** Let \( \{H_i\}_{i=1}^q \) be hyperplanes in \( \mathbb{P}^n(\mathbb{C}) \) in general position. Let \( f \) and \( g \) be meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \). Assume that

(i) \( \dim(\operatorname{Zero} H_i(f) \cap \operatorname{Zero} H_j(f)) \leq m - 2 \) for every \( 1 \leq i < j \leq q \).

(ii) \( f = g \) on \( \bigcup_{i=1}^q (\operatorname{Zero} H_i(f) \cup \operatorname{Zero} H_i(g)) \).

Let \( k \) be the dimension of the smallest linear subspace containing \( f(\mathbb{C}^m) \). If \( q > 2(2n - k + 1) \), then \( f = g \).

We see that if \( f \) is linear nondegenerate, i.e., \( k = n \), then the condition of the above corollary is satisfied with \( q = 2n + 3 \). Therefore, Corollary 1 is a natural extension of the uniqueness for linear nondegenerate meromorphic mappings sharing \( 2n + 3 \) hyperplanes in \( \mathbb{P}^n(\mathbb{C}) \) in general position given by Yan-Chen [3].

**Proof** Let \( f = (f_0 : \cdots : f_n) \) and \( g = (g_0 : \cdots : g_n) \) be two reduced representations of \( f \) and \( g \), respectively. Let \( V(f) \) and \( V(g) \) be the smallest linear subspaces of \( \mathbb{P}^n(\mathbb{C}) \) containing \( f(\mathbb{C}^m) \) and \( g(\mathbb{C}^m) \), respectively. It is easy to see that \( V(f) \) (resp. \( V(g) \)) is the intersection of all hyperplanes which contain \( f(\mathbb{C}^m) \) (resp. \( g(\mathbb{C}^m) \)). We may consider \( f \) (resp. \( g \)) as a meromorphic mapping into \( V(f) \) (resp. \( V(g) \)) which is nondegenerate over \( I_1(V(f)) \) (resp. \( I_1(V(g)) \)). Of course, \( H_1, \ldots, H_q \) are in \( N \)-subgeneral position with respect to both \( V(f) \) and \( V(g) \).

Now let \( H \) be a hyperplane in \( \mathbb{P}^n(\mathbb{C}) \) such that \( f(\mathbb{C}^m) \subset H \). We denote again by \( H \) the homogeneous linear form defining the hyperplane \( H \). Suppose that \( g(\mathbb{C}^m) \not\subset H \), i.e., \( H(g) \neq 0 \). Then we have \( H(g) = H(f) = 0 \) on \( \bigcup_{i=1}^q \operatorname{Zero} H_i(g) \), and hence,

\[
T_g(r) \geq N_{H(g)}(r) \geq \sum_{i=1}^q N_{H_i(g)}^{[1]}(r) + o(T_g(r))
\]

\[
\geq \frac{1}{H_{V(g)}(1) - 1} \sum_{i=1}^q N_{H_i(g)}^{[H_{V(g)}(1) - 1]}(r) + o(T_g(r))
\]

\[
\geq \frac{1}{H_{V(g)}(1) - 1} (q - 2n + (H_{V(g)} - 1) - 1) T_g(r) + o(T_g(r))
\]

\[
\geq \frac{H_{V(g)} + 1}{H_{V(g)} - 1} T_g(r) + o(T_g(r))
\]

(here, note that \( H_{V(g)}(1) - 1 = \dim V(g) \) and \( q \geq 2n + 3 \). This is a contradiction. Therefore, \( g(\mathbb{C}^m) \subset H \). This implies that \( g(\mathbb{C}^m) \subset V(f) \), and hence, \( V(g) \subset V(f) \). Similarly, we have \( V(f) \subset V(g) \). Then \( V(f) = V(g) = V \).

We see that \( q > \frac{2(2n-k+1)}{H_1(1)} \), since \( H_1(1) = k + 1 \). Therefore, from Theorem 3b), there exist \( n + 1 \) hyperplanes \( H_{i_0}, \ldots, H_{i_n}, 1 \leq i_0 < \cdots < i_n \leq q \) such that

\[
\frac{H_{i_0}(f)}{H_{i_0}(g)} = \cdots = \frac{H_{i_n}(f)}{H_{i_n}(g)}.
\]

This implies that \( f = g \).
2 Basic Notions and Auxiliary Results from Nevanlinna Theory

2.1 Counting Function of Divisor

We set \( |||z|| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2} \) for \( z = (z_1, \ldots, z_m) \in \mathbb{C}^m \) and define
\[
B(r) := \{ z \in \mathbb{C}^m : |||z|| < r \}, \quad S(r) := \{ z \in \mathbb{C}^m : |||z|| = r \} \quad (0 < r < \infty).
\]
Define
\[
v_{m-1}(z) := (dd^c|||z||^2)^{m-1} \quad \text{and} \quad \sigma_m(z) := d^c \log |||z||^2 \wedge (dd^c \log |||z||^2)^{m-1} \text{ on } \mathbb{C}^m \setminus \{0\}.
\]
For a divisor \( \nu \) on \( \mathbb{C}^m \) and for a positive integer \( M \) or \( M = \infty \), define the counting function of \( \nu \) by
\[
\nu^{[M]}(z) = \min\{M, \nu(z)\},
\]
\[
n(t) = \begin{cases} \int_{|z| \leq t} \nu(z)v_{m-1} & \text{if } m \geq 2, \\ \sum_{|z| \leq t} \nu(z) & \text{if } m = 1. \end{cases}
\]
Similarly, we define \( n^{[M]}(t) \).

Define
\[
N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}} dt \quad (1 < r < \infty).
\]
Similarly, define \( N(r, \nu^{[M]}) \) and denote it by \( N^{[M]}(r, \nu) \).

Let \( \varphi : \mathbb{C}^m \to \mathbb{C} \) be a meromorphic function. Denote by \( \nu_{\varphi} \) the zero divisor of \( \varphi \).

Define
\[
N_{\varphi}(r) = N(r, \nu_{\varphi}), \quad N_{\varphi}^{[M]}(r) = N^{[M]}(r, \nu_{\varphi}).
\]

For brevity, we will omit the character \([M]\) if \( M = \infty \).

2.2 Characteristic Function of Meromorphic Mapping

Let \( f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates \( (w_0 : \cdots : w_n) \) on \( \mathbb{P}^n(\mathbb{C}) \), we take a reduced representation \( f = (f_0 : \cdots : f_n) \), which means that each \( f_i \) is a holomorphic function on \( \mathbb{C}^m \) and \( f(z) = (f_0(z) : \cdots : f_n(z)) \) outside the analytic subset \{ \( f_0 = \cdots = f_n = 0 \) \} of codimension \( \geq 2 \). Set \( ||f|| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2} \).

The characteristic function of \( f \) is defined by
\[
T_f(r) = \int_{S(r)} \log ||f|| \sigma_m - \int_{S(1)} \log ||f|| \sigma_m.
\]

2.3 Proximity Function of Meromorphic Function

Let \( \varphi \) be a nonzero meromorphic function on \( \mathbb{C}^m \), which is occasionally regarded as a meromorphic map into \( \mathbb{P}^1(\mathbb{C}) \). The proximity function of \( \varphi \) is defined by
\[
m(r, \varphi) = \int_{S(r)} \log \max(|\varphi|, 1) \sigma_m.
\]
The Nevanlinna’s characteristic function of \( \varphi \) is defined as follows:

\[
T(r, \varphi) = N_{\varphi}(r) + m(r, \varphi).
\]

Then

\[
T_{\varphi}(r) = T(r, \varphi) + O(1).
\]

The function \( \varphi \) is said to be small (with respect to \( f \)) if \( ||T_{\varphi}(r)|| = o(T_f(r)) \). Here, by the notation “\( ||P|| \)” we mean the assertion \( P \) holds for all \( r \in [0, \infty) \) excluding a Borel subset \( E \) of the interval \( [0, \infty) \) with \( \int_E dr < \infty \).

### 2.4 Lemma on Logarithmic Derivative (see [10, Lemma 3.11])

Let \( f \) be a nonzero meromorphic function on \( \mathbb{C}^m \). Then

\[
\left| m\left( r, \frac{D^{\alpha}(f)}{f} \right) \right| = O(\log^+ T(r, f)) \quad (\alpha \in \mathbb{Z}^m_+).
\]

Repeating the argument in [6, Proposition 4.5], we have the following.

### 2.5 Proposition

Let \( \Phi_0, \ldots, \Phi_k \) be meromorphic functions on \( \mathbb{C}^m \) such that \( \{\Phi_0, \ldots, \Phi_k\} \) are linearly independent over \( \mathbb{C} \). Then there exists an admissible set

\[
\{\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im})\}_{i=0}^k \subset \mathbb{Z}^m_+
\]

with \( |\alpha_i| = \sum_{j=1}^m |\alpha_{ij}| \leq k \) \((0 \leq i \leq k)\) such that the following are satisfied:

(i) \( \{D^{\alpha_i}\Phi_0, \ldots, D^{\alpha_i}\Phi_k\}_{i=0}^k \) is linearly independent over \( \mathcal{M} \), i.e.,

\[
\det(D^{\alpha_i}\Phi_j) \neq 0.
\]

(ii) \( \det(D^{\alpha_i}(h\Phi_j)) = h^{k+1} \cdot \det(D^{\alpha_i}\Phi_j) \) for any nonzero meromorphic function \( h \) on \( \mathbb{C}^m \).

### 3 Generalization of Nochka Weights

Let \( V \) be a complex projective subvariety of \( \mathbb{P}^n(C) \) of dimension \( k \) \((k \leq n)\). Let \( \{Q_i\}_{i=1}^q \) be \( q \) hypersurfaces in \( \mathbb{P}^n(C) \) of the common degree \( d \), which are regarded as homogeneous polynomials in variables \( (x_0, \ldots, x_n) \). We regard \( I_d(V) = \frac{H_d}{T(V) \cap H_d} \) as a complex vector space. It is easy to see that

\[
\text{rank}\{Q_i\}_{i \in R} \geq \dim V - \dim \left( \bigcap_{i \in R} Q_i \cap V \right).
\]

Set \( \dim(\emptyset) = -1 \). Then, if \( \{Q_i\}_{i=1}^q \) is in \( N \)-subgeneral position, we have

\[
\text{rank}\{Q_i\}_{i \in R} \geq \dim V - \dim \left( \bigcap_{i \in R} Q_i \cap V \right) = k + 1
\]

for any subset \( R \subset \{1, \ldots, q\} \) with \( |R| = N + 1 \).
Taking a $\mathbb{C}$-basis of $I_d(V)$, we may consider $I_d(V)$ as a $\mathbb{C}$-vector space $\mathbb{C}^M$ with $M = H_Y(d)$.

Let $\{H_i\}_{i=1}^q$ be $q$ hyperplanes in $\mathbb{C}^M$ passing through the coordinate origin. Assume that each $H_i$ is defined by the linear equation

$$a_{ij}z_1 + \cdots + a_{iM}z_M = 0,$$

where $a_{ij} \in \mathbb{C}$ ($j = 1, \ldots, M$), not all zeros. We define the vector associated with $H_i$ by

$$v_i = (a_{i1}, \ldots, a_{iM}) \in \mathbb{C}^M.$$

For each subset $R \subset \{1, \ldots, q\}$, the rank of $\{H_i\}_{i \in R}$ is defined by

$$\text{rank}(\{H_i\}_{i \in R}) = \text{rank}(\{v_i\}_{i \in R}).$$

Recall that the family $\{H_i\}_{i=1}^q$ is said to be in $N$-subgeneral position if for any subset $R \subset \{1, \ldots, q\}$ with $|R| = N + 1$, $\bigcap_{i \in R} H_i = \{0\}$, i.e., $\text{rank}(\{H_i\}_{i \in R}) = M$.

By Lemmas 3.3 and 3.4 in [8], we have the following.

**Lemma 1** Let $\{H_i\}_{i=1}^q$ be $q$ hyperplanes in $\mathbb{C}^{k+1}$ in $N$-subgeneral position and assume that $q > 2N - k + 1$. Then there are positive rational constants $\omega_i$ ($1 \leq i \leq q$) satisfying the following:

i) $0 < \omega_j \leq 1$ $\forall i \in \{1, \ldots, q\}$,

ii) Setting $\tilde{\omega} = \max_{j \in Q} \omega_j$, one gets

$$\sum_{j=1}^q \omega_j = \tilde{\omega}(q - 2N + k - 1) + k + 1.$$

iii) $\frac{k + 1}{2N - k + 1} \leq \frac{1}{\tilde{\omega}} \leq \frac{k}{N}$.

iv) For $R \subset Q$ with $0 < |R| \leq N + 1$, then $\sum_{i \in R} \omega_i \leq \text{rank}(\{H_i\}_{i \in R})$.

v) Let $E_i \geq 1$ ($1 \leq i \leq q$) be arbitrarily given numbers. For $R \subset Q$ with $0 < |R| \leq N + 1$, there is a subset $R^o \subset R$ such that $\sum_{i \in R^o} \omega_i \leq \text{rank}(\{H_i\}_{i \in R}) = \text{rank}(\{H_i\}_{i \in R})$ and

$$\prod_{i \in R} E_i^{\omega_i} \leq \prod_{i \in R^o} E_i.$$

The above $\omega_j$ are called Nochka weights and $\tilde{\omega}$ is called Nochka constant.

**Lemma 2** (cf. [2, Lemma 3.2]) Let $H_1, \ldots, H_q$ be $q$ hyperplanes in $\mathbb{C}^M$ ($M \geq 2$), passing through the coordinate origin. Let $k$ be a positive integer such that $k \leq M$. Then there exists a linear subspace $L \subset \mathbb{C}^M$ of dimension $k$ such that $L \nsubseteq H_i$ ($1 \leq i \leq q$) and

$$\text{rank}(H_{i_1} \cap L, \ldots, H_{i_l} \cap L) = \text{rank}(\{H_{i_1}, \ldots, H_{i_l}\})$$

for every $1 \leq l \leq k$, $1 \leq i_1 < \cdots < i_l \leq q$. 
Lemma 3 Let $V$ be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k$ ($k \leq n$). Let $Q_1, \ldots, Q_q$ be $q$ ($q > 2N - k + 1$) hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with respect to $V$ of the common degree $d$. Then there are positive rational constants $\omega_i$ ($1 \leq i \leq q$) satisfying the following:

i) $0 < \omega_i \leq 1$, $\forall i \in \{1, \ldots, q\}$,

ii) Setting $\tilde{\omega} = \max_{j \in Q} \omega_j$, one gets

$$\sum_{j=1}^{q} \omega_j = \tilde{\omega}(q - 2N + k - 1) + k + 1.$$ 

iii) $\frac{k + 1}{2N - k + 1} \leq \tilde{\omega} \leq \frac{k}{N}$.

iv) For $R \subset \{1, \ldots, q\}$ with $\sharp R = N + 1$, then $\sum_{i \in R} \omega_i \leq k + 1$.

v) Let $E_i \geq 1$ ($1 \leq i \leq q$) be arbitrarily given numbers. For $R \subset \{1, \ldots, q\}$ with $\sharp R = N + 1$, there is a subset $R^o \subset R$ such that $\sharp R^o = \text{rank}[Q_i]_{i \in R^o} = k + 1$ and

$$\prod_{i \in R^o} E_i^{\omega_i} \leq \prod_{i \in R} E_i.$$ 

Proof We assume that each $Q_i$ is given by

$$\sum_{I \in \mathcal{I}_d} a_{i1} x^I = 0,$$

where $\mathcal{I}_d = \{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}_0 : i_0 + \cdots + i_n = d\}$, $I = (i_0, \ldots, i_n) \in \mathcal{I}_d$, $x^I = x_0^{i_0} \cdots x_n^{i_n}$ and $a_{i1} \in \mathbb{C}$ ($1 \leq i \leq q$, $I \in \mathcal{I}_d$). Set $Q_i^+(x) = \sum_{I \in \mathcal{I}_d} a_{i1} x^I$. Then $Q_i^+ \in H_d$.

Taking a $\mathbb{C}$-basis of $I_d(V)$, we may identify $I_d(V)$ with the $\mathbb{C}$-vector space $\mathbb{C}^M$, where $M = H_V(d)$. For each $Q_i$, denote by $v_i$ the vector in $\mathbb{C}^M$ which corresponds to $[Q_i^+]$ by this identification. Denote by $H_i$ the hyperplane in $\mathbb{C}^M$ associated with the vector $v_i$.

Then for each arbitrary subset $R \subset \{1, \ldots, q\}$ with $\sharp R = N + 1$, we have

$$\dim \left( \bigcap_{i \in R} Q_i \cap V \right) \geq \dim V - \text{rank}[Q_i]_{i \in R} = k - \text{rank}[H_i]_{i \in R}.$$ 

Hence,

$$\text{rank}[H_i]_{i \in R} \geq k - \dim \left( \bigcap_{i \in R} Q_i \cap V \right) \geq k - (-1) = k + 1.$$

By Lemma 2, there exists a linear subspace $L \subset \mathbb{C}^M$ of dimension $k + 1$ such that $L \not\subset H_i$ ($1 \leq i \leq q$) and

$$\text{rank}[H_i \cap L, \ldots, H_i \cap L] = \text{rank}[H_i, \ldots, H_i]$$

for every $1 \leq l \leq k + 1$, $1 \leq i_1 < \cdots < i_l \leq q$. Since $\text{rank}[H_i]_{i \in R} \geq k + 1$, it implies that for any subset $R \subset \{1, \ldots, q\}$ with $\sharp R = N + 1$, there exists a subset $R' \subset R$ with $\sharp R' = k + 1$ and $\text{rank}[H_i]_{i \in R'} = k + 1$. Hence, we get

$$\text{rank}[H_i \cap L]_{i \in R} \geq \text{rank}[H_i \cap L]_{i \in R'} = \text{rank}[H_i]_{i \in R'} = k + 1.$$ 

This yields that $\text{rank}[H_i \cap L]_{i \in R} = k + 1$, since $\dim L = k + 1$. Therefore, $\{H_i \cap L\}_{i=1}^q$ is a family of $q$ hyperplanes in $L$ in $N$-subgeneral position.
By Lemma 1, there exist Nochka weights \( \{ \omega_i \}_{i=1}^q \) for the family \( \{ H_i \cap L \}_{i=1}^q \) in \( L \). It is clear that assertions (i)–(iv) are automatically satisfied. Now for \( R \subset \{ 1, \ldots, q \} \) with \#R = N + 1, by Lemma 1(v) we have
\[
\sum_{i \in R} \omega_i \leq \text{rank}\{ H_i \cap L \}_{i \in R} = k + 1
\]
and there is a subset \( R^0 \subset R \) such that:
\[
\#R^0 = \text{rank}\{ H_i \cap L \}_{i \in R^0} = k + 1,
\]
\[
\prod_{i \in R^0} E_i^{\omega_i} \leq \prod_{i \in R^0} E_i \forall E_i \geq 1 (1 \leq i \leq q),
\]
\[
\text{rank}\{ Q_i \}_{i \in R^0} = \text{rank}\{ H_i \cap L \}_{i \in R^0} = k + 1.
\]
Hence, the assertion (v) is also satisfied. The lemma is proved. \( \square \)

4 Second Main Theorems for Hypersurfaces

Let \( \{ Q_i \}_{i \in R} \) be a set of hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) of the common degree \( d \). Assume that each \( Q_i \) is defined by
\[
\sum_{I \in \mathcal{I}_d} a_{I} x^I = 0,
\]
where \( \mathcal{I}_d = \{ (i_0, \ldots, i_n) \in \mathbb{N}_0^{n+1} : i_0 + \cdots + i_n = d \}, I = (i_0, \ldots, i_n) \in \mathcal{I}_d, x^I = x_0^{i_0} \cdots x_n^{i_n} \) and \((x_0 : \cdots : x_n)\) is homogeneous coordinates of \( \mathbb{P}^n(\mathbb{C}) \).

Let \( f : \mathbb{C}^m \longrightarrow V \subset \mathbb{P}^n(\mathbb{C}) \) be an algebraically nondegenerate meromorphic mapping into \( V \) with a reduced representation \( f = (f_0 : \cdots : f_n) \). We define
\[
Q_i(f) = \sum_{I \in \mathcal{I}_d} a_{I} f^I,
\]
where \( f^I = f_0^{i_0} \cdots f_n^{i_n} \) for \( I = (i_0, \ldots, i_n) \). Then we see that \( f^* Q_i = v_{Q_i(f)} \) as divisors.

**Lemma 4** Let \( \{ Q_i \}_{i \in R} \) be a set of hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) of the common degree \( d \) and let \( f \) be a meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \). Assume that \( \bigcap_{i \in R} Q_i \cap V = \emptyset \). Then there exist positive constants \( \alpha \) and \( \beta \) such that
\[
\alpha ||f||^d \leq \max_{i \in R} |Q_i(f)| \leq \beta ||f||^d.
\]

**Proof** Let \((x_0 : \cdots : x_n)\) be homogeneous coordinates of \( \mathbb{P}^n(\mathbb{C}) \). Assume that each \( Q_i \) is defined by
\[
\sum_{I \in \mathcal{I}_d} a_{I} x^I = 0.
\]
Set \( Q_i(x) = \sum_{I \in \mathcal{I}_d} a_{I} x^I \) and consider the following function
\[
h(x) = \frac{\max_{i \in R} |Q_i(x)|}{||x||^d},
\]
where \( ||x|| = (\sum_{i=0}^{n} |x_i|^2)^{\frac{1}{2}} \).

Since the function \( h \) is positive continuous on \( V \), by the compactness of \( V \), there exist positive constants \( \alpha \) and \( \beta \) such that \( \alpha = \min_{x \in \mathbb{P}^n(\mathbb{C})} h(x) \) and \( \beta = \max_{x \in \mathbb{P}^n(\mathbb{C})} h(x) \). Thus,
\[
\alpha ||f||^d \leq \max_{i \in R} |Q_i(f)| \leq \beta ||f||^d.
\]
The lemma is proved. □

The following lemma is similar to Lemma 4.2 in [2] with a slight modification.

Lemma 5 (cf. [2, Lemma 4.2]) Let \( \{Q_i\}_{i=1}^{q} \) be a set of \( q \) hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) of the common degree \( d \). Then there exist \((H_V(d) - k - 1)\) hypersurfaces \( \{T_i\}_{i=1}^{H_V(d)-k-1} \) in \( \mathbb{P}^n(\mathbb{C}) \) such that for any subset \( R \in \{1, \ldots, q\} \) with \( \sharp R = \text{rank}(\{Q_i\}_{i \in R} = k + 1 \), we get \( \text{rank}(\{Q_i\}_{i \in R} \cup \{T_i\}_{i=1}^{M-k}) = H_V(d) \).

Proof For each \( R \subset \{1, \ldots, q\} \) with \( \sharp R = \text{rank}(\{Q_i\}_{i \in R} = k + 1 \), denote by \( V_R \) the set of all vectors \( v = (v_1, \ldots, v_{H_V(d)-k-1}) \in (I_d(V))^{H_V(d)-k-1} \) such that \( \{(Q_i)_{|i \in R}, v_1, \ldots, v_{H_V(d)-k-1}\} \) is linearly dependent over \( \mathbb{C} \). Then \( V_R \) is an algebraic subset of \((I_d(V))^{H_V(d)-k-1}\). Since \( \dim I_d(V) = H_V(d) \) and \( \text{rank}(\{Q_i\}_{i \in R} = k + 1 \), there exists an element

\[
v = (v_1, \ldots, v_{H_V(d)-k-1}) \in (I_d(V))^{H_V(d)-k-1}
\]

such that the family of vectors \( \{(Q_i)_{|i \in R}, v_1, \ldots, v_{H_V(d)-k-1}\} \) is linearly independent over \( \mathbb{C} \), i.e., \( v \notin V_R \). Therefore, \( V_R \) is a proper algebraic subset of \((I_d(V))^{H_V(d)-k-1}\) for each \( R \). This implies that

\[
(I_d(V))^{H_V(d)-k-1} \setminus \bigcup_{R} V_R \neq \emptyset.
\]

Hence, there is \( (T_1^+, \ldots, T_{H_V(d)-k-1}^+) \in (I_d(V))^{H_V(d)-k-1} \setminus \bigcup_{R} V_R \).

For each \( T_i^+ \), take a representation \( T_i \in H_d \) of \( T_i^+ \). Then

\[
\text{rank}(\{Q_i\}_{i \in R} \cup \{T_i\}_{i=1}^{H_V(d)-k-1}) = \text{rank}(\{Q_i\}_{i \in R} \cup \{T_i\}_{i=1}^{H_V(d)-k-1}) = H_V(d)
\]

for every subset \( R \in \{1, \ldots, q\} \) with \( \sharp R = \text{rank}(\{Q_i\}_{i \in R} = k + 1 \).

The lemma is proved. □

Proof of Theorem 2 We first prove the theorem in the case where all \( Q_i \) \( (i = 1, \ldots, q) \) have the same degree \( d \). It is easy to see that there is a positive constant \( \beta \) such that \( \beta \|f\| \geq |Q_i(f)| \) for every \( 1 \leq i \leq q \). Set \( Q := \{1, \ldots, q\} \). Let \( \{\omega_i\}_{i=1}^{q} \) be as in Lemma 3 for the family \( \{Q_i\}_{i=1}^{q} \). Let \( \{T_i\}_{i=1}^{M-k} \) be \((M - k)\) hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \), which satisfy Lemma 5.

Take a \( \mathbb{C} \)-basis \( \{(A_i)_{H_V(d)}\}_{i=1}^{M} \) of \( I_d(V) \), where \( A_i \in H_d \). Since \( f \) is nondegenerate over \( I_d(V) \), it implies that \( \{A_i(f) ; 1 \leq i \leq H_V(d)\} \) is linearly independent over \( \mathbb{C} \). Then there is an admissible set \( \{\alpha_1, \ldots, \alpha_{H_V(d)}\} \subset \mathbb{Z}^n_+ \) such that

\[
W \equiv \det((D^\alpha_j A_i(f)(1 \leq i \leq H_V(d)))_{1 \leq j \leq H_V(d)}) \not\equiv 0
\]

and \( |\alpha_j| \leq H_V(d) - 1 \) for all \( 1 \leq j \leq H_V(d) \).

For each \( R^o = \{r_1^0, \ldots, r_{k+1}^0\} \subset \{1, \ldots, q\} \) with \( \text{rank}(\{Q_i\}_{i \in R^o} = \sharp R^o = k + 1 \), set

\[
W_{R^o} \equiv \det((D^\alpha_j Q_{r_i^0}(f)(1 \leq u \leq k + 1), D^\alpha_j T_i(f)(1 \leq l \leq H_V(d) - k - 1))_{1 \leq j \leq H_V(d)}.
\]

Since \( \text{rank}(\{Q_{r_i^0} \leq u \leq k + 1 \), \( T_i(1 \leq l \leq H_V(d) - k - 1)\) = \( H_V(d) \), there exists a nonzero constant \( C_{R^o} \) such that \( W_{R^o} = C_{R^o} \cdot W \).

We denote by \( R^o \) the family of all subsets \( R^o \) of \( \{1, \ldots, q\} \) satisfying

\[
\text{rank}(\{Q_i\}_{i \in R^o} = \sharp R^o = k + 1.
\]

Let \( z \) be a fixed point. For each \( R \subset Q \) with \( \sharp R = N + 1 \), we choose \( R^o \subset R \) such that \( R^o \in R^o \) and \( R^o \) satisfy Lemma 3(v) with respect to numbers \( \{\frac{\beta \|f(z)\|}{|Q_i(f)(z)|} \}_{i=1}^{q} \). On the
other hand, there exists $\bar{R} \subset Q$ with $\sharp\bar{R} = N + 1$ such that $|Q_i(f)(z)| \leq |Q_j(f)(z)| \forall i \in \bar{R}, j \notin \bar{R}$. Since $\bigcap_{i \in \bar{R}} Q_i = \emptyset$, by Lemma 4, there exists a positive constant $\alpha_{\bar{R}}$ such that

$$\alpha_{\bar{R}} ||f||^{d} (z) \leq \max_{i \in R} |Q_i(f)(z)|.$$

Then, we get

$$\frac{||f(z)||^{d(\sum_{i=1}^{q} \omega_i)} |W(z)|}{|Q_1^{\omega_1}(f)(z) \cdots Q_q^{\omega_q}(f)(z)|} \leq \frac{\alpha_{\bar{R}}^{-N-1} \beta^{N+1} \prod_{i \in R} (\beta ||f||^{d})^{\omega_i}}{A_R |W(z)| \cdot ||f||^{d(k+1)}(z)} \leq A_R |W(z)| \cdot ||f||^{dH_V(d)}(z) \leq B_R \prod_{i \in R_0} |Q_i(f)(z)| \prod_{i=1}^{\#R-\#\bar{R}-N+1} |T_i(f)(z)|,$$

where $A_R, B_R$ are positive constants.

Put $S_R = B_R \prod_{i \in R_0} |Q_i(f)| \prod_{i=1}^{\#R-\#\bar{R}-N+1} |T_i(f)|$. By the lemma on logarithmic derivative, it is easy to see that

$$\int_{S(R)} \log^+ S_R(z) \sigma_m = o(T_f(r)).$$

Therefore, for each $z \in \mathbb{C}^m$, we have

$$\log \left( \frac{||f(z)||^{d(\sum_{i=1}^{q} \omega_i)} |W(z)|}{|Q_1^{\omega_1}(f)(z) \cdots Q_q^{\omega_q}(f)(z)|} \right) \leq \log \left( ||f||^{dH_V(d)}(z) \right) + \sum_{R \subset Q, \sharp R = N+1} \log^+ S_R.$$

Since $\sum_{i=1}^{q} \omega_i = \sum_{i=1}^{q} \omega_i (q - 2N + k - 1) + k + 1$ and by integrating both sides of the above inequality over $S(r)$, we have

$$||d\left(q - 2N + k - 1 - \frac{H_V(d) - k - 1}{\omega} \right)T_f(r) \leq \sum_{i=1}^{q} \frac{\omega_i}{\omega} N_{Q_i(f)}(r) - \frac{1}{\omega} N_W(r) + o(T_f(r)).$$

Claim $\sum_{i=1}^{q} \omega_i N_{Q_i(f)}(r) - N_W(r) \leq \sum_{i=1}^{q} \omega_i N_{Q_i(f)}^{[H_V(d)-1]}(r)$.

Indeed, let $z$ be a zero of some $Q_i(f)(z)$ and $z \notin I(f) = \{f_0 = \cdots = f_n = 0\}$. Since $\{Q_i\}_{i=1}^{q}$ is in $N$-subgeneral position, $z$ is not zero of more than $N$ functions $Q_i(f)$. Without loss of generality, we may assume that $z$ is zero of $Q_i(f)$ for each $1 \leq i \leq k \leq N$ and $z$ is not zero of $Q_i(f)$ for each $i > N$. Put $R = \{1, \ldots, N + 1\}$. Choose $R^1 \subset R$ such that $\sharp R^1 = \text{rank} \{Q_i\}_{i \in R^1} = k + 1$ and $R^1$ satisfies Lemma 3(v) with respect to numbers $\{e^{\max_{r \in R} (d - H_V(d) + 1.0)} \}_{i=1}^{q}$. Then we have

$$\sum_{i \in R^1} \omega_i \max_{r \in R^1} \{v_{Q_i(f)}(z) - H_V(d) + 1, 0 \} \leq \sum_{i \in R^1} \max_{r \in R^1} \{v_{Q_i(f)}(z) - H_V(d) + 1, 0 \}.$$

This yields that

$$v_W(z) = v_{W^l}(z) \geq \sum_{i \in R^1} \max_{r \in R^1} \{v_{Q_i(f)}(z) - H_V(d) + 1, 0 \} \geq \sum_{i \in R} \omega_i \max_{r \in R} \{v_{Q_i(f)}(z) - H_V(d) + 1, 0 \}.$$
Hence,
\[
\sum_{i=1}^{q} \omega_i v_{Q_i(f)}(z) - v_W(z) = \sum_{i \in R} \omega_i v_{Q_i(f)}(z) - v_W(z)
\]
\[
= \sum_{i \in R} \omega_i \min\{v_{Q_i(f)}(z), H_V(d) - 1\}
\]
\[
+ \sum_{i \in R} \omega_i \max\{v_{Q_i(f)}(z) - H_V(d) + 1, 0\} - v_W(z)
\]
\[
\leq \sum_{i \in R} \omega_i \min\{v_{Q_i(f)}(z), H_V(d) + 1\}
\]
\[
= \sum_{i=1}^{q} \omega_i \min\{v_{Q_i(f)}(z), M\}.
\]

Integrating both sides of this inequality, we get
\[
\sum_{i=1}^{q} \omega_i N_{Q_i(f)}(r) - N_W(r) \leq \sum_{i=1}^{q} \omega_i N^{[H_V(d) - 1]}_{Q_i(f)}(r).
\]

This proves the claim.

Combining the claim and (1), we obtain
\[
\| d(q - 2N + k - 1 - \frac{H_V(d) - k - 1}{\tilde{\omega}})T_f(r) \|
\]
\[
\leq \sum_{i=1}^{q} \frac{\omega_i}{\tilde{\omega}} N^{[H_V(d) - 1]}_{Q_i(f)}(r) + o(T_f(r))
\]
\[
\leq \sum_{i=1}^{q} N^{[H_V(d) - 1]}_{Q_i(f)}(r) + o(T_f(r)).
\]

Since \( \tilde{\omega} \geq \frac{k+1}{2N-k+1} \), the above inequality implies that
\[
\left\| d \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) T_f(r) \right\| \leq \sum_{i=1}^{q} N^{[H_V(d) - 1]}_{Q_i(f)}(r) + o(T_f(r)).
\]

Hence, the theorem is proved in the case where all \( Q_i \) have the same degree.

We now prove the theorem in the general case where \( \deg Q_i = d_i \). Applying the above case for \( f \) and the hypersurfaces \( Q^d_{i,i} \) \( (i = 1, \ldots, q) \) of the common degree \( d \), we have
\[
\left\| \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) T_f(r) \right\| \leq \frac{1}{d} \sum_{i=1}^{q} N^{[H_V(d) - 1]}_{Q^d_{i,i}}(r) + o(T_f(r))
\]
\[
\leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[H_V(d) - 1]}_{Q_i(f)}(r) + o(T_f(r))
\]
\[
= \sum_{i=1}^{q} \frac{1}{d_i} N^{[H_V(d) - 1]}_{Q_i(f)}(r) + o(T_f(r)).
\]

The theorem is proved.
5 Unicity of Meromorphic Mappings Sharing Hypersurfaces

Lemma 6 Let \( f \) and \( g \) be nonconstant meromorphic mappings of \( \mathbb{C}^m \) into a complex projective subvariety \( V \) of \( \mathbb{P}^n(\mathbb{C}) \), \( \dim V = k \) (\( k \leq n \)). Let \( Q_i \) (\( i = 1, \ldots, q \)) be moving hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) in \( N \)-subgeneral position with respect to \( V \), \( \deg Q_i = d_i, N \geq n \).

Put \( d = \text{lcm}(d_1, \ldots, d_q) \) and \( M = \binom{n+d}{n} - 1 \). Assume that both \( f \) and \( g \) are nondegenerate over \( I_d(V) \). Then \( \| T_f(r) = O(T_g(r)) \) and \( \| T_g(r) = O(T_f(r)) \) if \( q > \frac{(2N-k+1)H_V(d)}{k+1} \).

Proof Using Theorem 2 for \( f \), we have

\[
\left\| f - \frac{(2N-k+1)H_V(d)}{k+1} \right\| T_f(r) \\
\leq \sum_{i=1}^{q} \frac{H_V(d)}{d_i} N^{[1]}_{Q_i(f)}(r) + o(T_f(r)) \\
\leq \sum_{i=1}^{q} \frac{H_V(d) - 1}{d_i} N^{[1]}_{Q_i(f)}(r) + o(T_f(r)) \\
= \sum_{i=1}^{q} \frac{H_V(d) - 1}{d_i} N^{[1]}_{Q_i(g)}(r) + o(T_f(r)) \\
\leq q(H_V(d) - 1) T_g(r) + o(T_f(r)).
\]

Hence, \( \| T_f(r) = O(T_g(r)) \). Similarly, we get \( \| T_g(r) = O(T_f(r)) \). \( \square \)

Proof of Theorem 3 Assume that \( f = (f_0 : \cdots : f_n) \) and \( g = (g_0 : \cdots : g_n) \) are reduced representations of \( f \) and \( g \), respectively. Replacing \( Q_i \) by \( Q_i^{d_i} \) if necessary, without loss of generality, we may assume that \( d_i = d \) for all \( 1 \leq i \leq q \).

(a) By Lemma 6, we have \( \| T_f(r) = O(T_g(r)) \) and \( \| T_g(r) = O(T_f(r)) \). Suppose that \( f \neq g \). Then there exist two indices \( s, t \) with \( 0 \leq s < t \leq n \) such that \( H := f_s g_t - f_t g_s \neq 0 \). By the assumption (ii) of the theorem, we have \( H = 0 \) on \( \bigcup_{i=1}^{d_i} (\text{Zero } Q_i(f) \cup \text{Zero } Q_i(g)) \). Therefore, we have

\[
\nu^0_H \geq \sum_{i=1}^{q} \min\{1, \nu^0_{Q_i(f)}\}
\]

outside an analytic subset of codimension at least two. It follows that

\[
N_H(r) \geq \sum_{i=1}^{q} N_{Q_i(f)}^{[1]}(r). \tag{2}
\]

On the other hand, by the definition of the characteristic function and by the Jensen formula, we have

\[
N_H(r) = \int_{S(r)} \log |f_s g_t - f_t g_s| \sigma_m \\
\leq \int_{S(r)} \log \| f \| \sigma_m + \int_{S(r)} \log \| g \| \sigma_m \\
= T_f(r) + T_g(r).
\]
Combining this and (2), we obtain
\[ T_f(r) + T_g(r) \geq \sum_{i=1}^{q} N_{Q_i(f)}^{[1]}(r). \]

Similarly, we have
\[ T_f(r) + T_g(r) \geq \sum_{i=1}^{q} N_{Q_i(g)}^{[1]}(r). \]

Summing up both sides of the above two inequalities, we have
\[ 2(T_f(r) + T_g(r)) \geq \sum_{i=1}^{q} N_{Q_i(f)}^{[1]}(r) + \sum_{i=1}^{q} N_{Q_i(g)}^{[1]}(r). \] (3)

From (3) and applying Theorem 2 for \( f \) and \( g \), we have
\[ 2(T_f(r) + T_g(r)) \geq d \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) (T_f(r) + T_g(r)) + o(T_f(r) + T_g(r)). \]

Letting \( r \to +\infty \), we get
\[ 2 \geq d \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right), \]
i.e.,
\[ q \leq \frac{2(H_V(d) - 1) + (2N - k + 1)H_V(d)}{d}. \]

This is a contradiction. Hence, \( f = g \). The assertion (a) is proved.

(b) Again, by Lemma 6, we have \( ||T_f(r) = O(T_g(r)) \) and \( ||T_g(r) = O(T_f(r)) \).

Suppose that the assertion (b) of the theorem does not hold.

By changing indices if necessary, we may assume that
\[ \begin{align*}
\frac{Q_1(f)}{Q_1(g)} &= \cdots = \frac{Q_{k_1}(f)}{Q_{k_1}(g)} & \neq & \frac{Q_{k_1+1}(f)}{Q_{k_1+1}(g)} &= \cdots = \frac{Q_{k_2}(f)}{Q_{k_2}(g)} \\
\neq & \frac{Q_{k_2+1}(f)}{Q_{k_2+1}(g)} &= \cdots = \frac{Q_{k_3}(f)}{Q_{k_3}(g)} & \neq & \frac{Q_{k_3+1}(f)}{Q_{k_3+1}(g)} &= \cdots = \frac{Q_{k_4}(f)}{Q_{k_4}(g)},
\end{align*} \]

where \( k_s = q \).

Since the assertion (b) of the theorem does not hold, the number of elements of each group is at most \( N \). For each \( 1 \leq i \leq q \), we set
\[ \sigma(i) = \begin{cases} 
i + N & \text{if } i + N \leq q, \\
i + N - q & \text{if } i + N > q
\end{cases} \]
and
\[ P_i = Q_i(f)Q_{\sigma(i)}(g) - Q_i(g)Q_{\sigma(i)}(f). \]
Then $Q_i(f)$ and $Q_{\sigma(i)}(f)$ belong to two distinct groups, and hence, $P_i \neq 0$ for every $1 \leq i \leq q$. It is easy to see that

$$v_P(z) \geq \min\{v_{Q_i(f)}(z), v_{Q_i(g)}(z)\} + \min\{v_{Q_{\sigma(i)}(f)}(z), v_{Q_{\sigma(i)}(g)}(z)\}$$

$$+ \sum_{j=1}^{q} \min\{v_{Q_j(f)}(z), 1\}$$

$$\geq \sum_{j=1, \sigma(i)} \left( \min\{v_{Q_j(f)}(z), H_V(d) - 1\} + \min\{v_{Q_j(g)}(z), H_V(d) - 1\} \right)$$

$$- (H_V(d) - 1) \min\{v_{Q_j(f)}(z), 1\} + \sum_{j=1}^{q} \min\{v_{Q_j(f)}(z), 1\}$$

for all $z$ in $\mathbb{C}^m$.

Integrating both sides of this inequality, we get

$$\| N_{P_i}(r) \| \geq \sum_{j=1, \sigma(i)} \left( N_{Q_j(f)}^{[H_V(d)-1]}(r) + N_{Q_j(g)}^{[H_V(d)-1]}(r) - (H_V(d) - 1)N_{Q_j(f)}^{[1]}(r) \right)$$

$$+ \sum_{j=1}^{q} N_{Q_j(f)}^{[1]}(r). \quad (4)$$

Repeating the same argument as in the proof of Theorem 3, by Jensen’s formula and by the definition of the characteristic function, we have

$$\| N_{P_i}(r) \| \leq d(T_f(r) + T_g(r)) \quad (5)$$

From (4) and (5), we get

$$\| d(T_f(r) + T_g(r)) \| \geq \sum_{j=1, \sigma(i)} \left( N_{Q_j(f)}^{[H_V(d)-1]}(r) + N_{Q_j(g)}^{[H_V(d)-1]}(r) - (H_V(d) - 1)N_{Q_j(f)}^{[1]}(r) \right)$$

$$\quad + \sum_{j=1}^{q} N_{Q_j(f)}^{[1]}(r).$$

Summing-up both sides of this inequality over all $1 \leq i \leq q$, we obtain

$$\| dq(T_f(r) + T_g(r)) \| \geq 2 \sum_{j=1}^{q} \left( N_{Q_j(f)}^{[H_V(d)-1]}(r) + N_{Q_j(g)}^{[H_V(d)-1]}(r) \right) + (q - 2H_V(d)) \sum_{j=1}^{q} N_{Q_j(f)}^{[1]}(r)$$

$$\quad \geq 2d \left( q - \frac{2N - k + 1}{k + 1} H_V(d) \right) \left( T_f(r) + T_g(r) \right) + o(T_f(r)).$$

Letting $r \rightarrow +\infty$, we get

$$dq \geq 2d \left( q - \frac{2N - k + 1}{k + 1} H_V(d) \right),$$

i.e.,

$$q \leq \frac{2(2N - k + 1) H_V(d)}{k + 1}.$$

This is a contradiction.

Hence, the assertion (b) holds. The theorem is proved. \qed
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