Growth of matter density perturbations in 4D Einstien-Gauss-Bonnet gravity

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Recently, a novel Einstein-Gauss-Bonnet gravity in four dimension has been introduced [Phys. Rev. Lett. 124 (2020) 081301]. We will investigate cosmological consequences of this model in details. We will consider linear matter density perturbations and also estimate relevant modified gravity parameters. Specially we will concentrate on the growth of baryonic matter perturbations on top of FRW geometry and show that for a narrow range of the Gauss-Bonnet coupling constant, observational data will be satisfied. In this paper, it is shown that for enough large values of the coupling constant, the suggested $\sigma_8$ will be greater than $\Lambda$CDM value.

I. INTRODUCTION

It is well-known that the Einstein-Hilbert term plus cosmological constant is not the unique healthy second order gravitational Lagrangian in higher than four dimensional space-times. The Lovelock theorem [1] states that in $D \geq 4$ space-time dimensions, for a generic metric field the unique healthy action with second order field equation can be given by Lovelock-Lanczos action. In four dimensions however, one has left with only the Ricci scalar plus a constant due to the fact that other Lovelock invariants become total derivative. Generally, in $d+1$ dimensions where $d$ denoted spatial dimensions, one has $(d-1)/2$ (even dimensions) or $d/2$ (odd dimensions) Lovelock invariant terms. From the viewpoint of the field equations, one can prove that the Lovelock tensor in $D$ dimensions has coefficients $(d-n)$, where $n = 3, \ldots, d$ making the Lovelock tensor to vanish in lower dimensions [2].

Many attempts has been done in the literature to make higher order Lovelock invariants to contribute in four dimensions, including the introduction of extra degrees of freedom, non-minimally coupled to the Lovelock invariants [3], or making non-linear function out of the Lovelock invariants [4].

Recently, a novel way to make the Lovelock tensors to appear in 4 dimensional equations of motion is introduced, which is based on the introduction of non-standard coupling constants. In this approach, we consider the theory in an arbitrary $d+1$ dimensional space-time and modify the coupling constant of the Lovelock invariants to have an extra $(d-3)$ factor. As we have discussed above the extra prefactor cancels the same factor which is obtained from the variation of the Lovelock invariant, so that when one take a limit $d \rightarrow 3$, the Lovelock tensor arises in metric field equation [5]. The simplest possibility is to consider the effects Gauss-Bonnet invariant in four dimensions. One can see that the Gauss-Bonnet tensor make a contribution proportional to $H^4$ in the Friedmann equation. This term would modify the effective cosmological constant of the theory and also shift the Planck mass by a constant while dynamical degrees of freedom of the theory remains 2 [5]. Tensor perturbations around FRW geometry is also considered in this context and it is shown that the sound speed and the Hubble friction will be modified by a factor proportional to $H^2/M_P^2$. As a result the theory can satisfy the recent observations on the gravitational waves produced by merging two Black holes/White dwarfs [6]. Dynamical system analysis of the theory has also been considered in [7] and the authors have suggested that the universe can evolve from non-flat to flat geometries in this theory. Many aspects of the theory has been investigated in the literature, including black holes [8], and cosmology [9]. Theoretical aspects and generalizations of the idea has also been considered extensively [10].

In this paper, we are going to consider the effects of this new 4D Gauss-Bonnet term on the growth rate of the baryonic matter density perturbations. We have assumed that the matter content of the universe can be described by a perfect fluid with barotropic equation of state $p = \omega \rho$. We will show that the Gauss-Bonnet term modify the behavior of Hubble parameter and the acceleration rate of the universe at early times. Matter density perturbations around the FRW geometry shows that the Gauss-Bonnet term will affect the growth rate at early times and also modify the $f\sigma_8$ value. Also, we will show that the growth rate of matter density perturbations is faster than $\Lambda$CDM model.

The paper is organized as follows: In the next section, there is a brief review of the theory. In section II, the background cosmological evolution of the theory in the presence of dust and radiations is considered. In section IV we study the pressureless matter density perturbations of the theory in subhorizon limit and investigate the effect of the 4D Gauss-Bonnet term in the growth rate of the matter perturbations. Section V will be devoted to conclusions and final remarks.
II. THE MODEL

The 4D Einstein-Gauss-Bonnet (EGB) action in \((d+1)\) dimensional space time is

\[
S = \int d^{d+1}x \sqrt{-g} \left( \kappa^2 (R - 2\Lambda) + \mathcal{L}_m + \frac{\alpha}{d-3} \mathcal{G} \right),
\]

where \(\Lambda\) is the cosmological constant, \(\mathcal{L}_m\) is the Lagrangian of the matter field, \(d\) is the dimension of the constant-time hypersurface and \(\mathcal{G}\) is the Gauss-Bonnet Lagrangian defined as

\[
\mathcal{G} = R^\mu\nu\alpha\beta R_{\mu\nu\alpha\beta} - 4R^\mu\nu R_{\mu\nu} + R^2.
\]

As was discussed in the Introduction, the coupling constant of the Gauss-Bonnet term is written in such a way that it cancels the \(d-3\) factor in the Gauss-Bonnet tensor. One can obtain the field equations of the action \((1)\) as

\[
2\kappa^2 \left( G_{\mu\nu} + \Lambda g_{\mu\nu} \right) + \frac{\alpha}{d-3} \left( R R_{\mu\nu} - 8 R^\rho_{\mu\nu} R^\alpha_{\rho} - 8 R_{\mu\alpha\nu\beta} R^\beta_{\nu\sigma} g^\alpha_{\mu} \mathcal{G} \right) = T_{\mu\nu},
\]

where the energy-momentum tensor \(T_{\mu\nu}\) of the matter field is defined as

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \left( \sqrt{-g} \mathcal{L}_m \right)}{\delta g^{\mu\nu}}.
\]

The expression in the second parenthesis of equation \((3)\) is the Gauss-Bonnet tensor with the property that it contains at most second order derivatives. As a result the above theory is free from Ostrogradski instability.

One easily verify that the energy-momentum tensor of the matter field is conserved due to the Bianchi identity applied to the Gauss-Bonnet tensor. This is however trivial since we did not introduce non-minimal matter geometry couplings in the action.

III. COSMOLOGY

In this section, we will investigate the cosmological implications of the 4D EGB theory by adopting the homogeneous flat FRW metric for the geometry of the Universe, given by

\[
ds^2 = a(t)^2 \eta_{\mu\nu} dx^\mu dx^\nu,
\]

where \(a(t)\) is the scale factor, the parameter \(t\) stands for the conformal time and \(\eta_{\mu\nu}\) is the Minkowski metric. We assume that the Universe is filled with perfect fluid which is characterized by energy density \(\rho\) and thermodynamic pressure \(p\), with the energy-momentum tensor given by

\[
T^{\mu\nu} = (\rho + p) u^\mu u^\nu + pg^{\mu\nu}.
\]

Assume that the equation of state the matter source is the barotropic equation of state with \(p = \omega \rho\), where \(\omega\) is a constant.

With these assumptions, the Friedmann and Raychaudhuri equations for 4D EGB theory reduce as

\[
2\kappa^2 \left( 3\mathcal{H}^2 - a^2 \Lambda \right) + \frac{6\alpha \mathcal{H}^4}{a^2} = a^2 \rho,
\]

and

\[
2\kappa^2 \left( 2\mathcal{H} - a^2 \Lambda + \mathcal{H}^2 \right) - \frac{2\alpha}{a^2} \mathcal{H}^2 \left( \mathcal{H}^2 - 4\mathcal{H} \right) = -a^2 p,
\]

where \(\mathcal{H} = \frac{\dot{a}}{a}\) is the Hubble parameter and dot denotes derivative with respect to the conformal time \(t\). The conservation equation of the matter field \((5)\) can be written as

\[
\dot{\rho} + 3\mathcal{H}(p + \rho) = 0.
\]

Now, suppose that the Universe is filled with non relativistic matter with \(\omega_m = 0\) and the relativistic matter with \(\omega_r = 1/3\) such that

\[
\rho = \rho_m + \rho_r, \quad p = p_r = \frac{1}{3} \rho_r.
\]

Let us define the following set of dimensionless parameters

\[
t = H_0 t, \quad \mathcal{H} = H_0 \mathcal{H}, \quad \beta = \frac{H_0^2}{\kappa^2} \alpha,
\]

\[
\Omega_\Lambda = \frac{1}{3} H_0^2 \Lambda, \quad \Omega_i = \frac{1}{6\kappa^2 H_0^2} \rho_i, \quad i = r, m.
\]

where \(H_0\) is the current Hubble parameter. We suppose that the radiation and non-relativistic matter are conserved separately, so from equation \((5)\) we can obtain the behavior of \(\Omega_i\)’s in terms of the scale factor as

\[
\Omega_r = \frac{\Omega_{r0}}{a^4}, \quad \Omega_m = \frac{\Omega_{m0}}{a^3},
\]

where the constants \(\Omega_{r0}\) and \(\Omega_{m0}\) are the present time density parameters of radiation and dust, respectively. The vales of these parameters from the Plank data \([11]\) are \(\Omega_{m0} = 0.305\) and \(\Omega_{r0} = 0.531 \times 10^{-4}\).

To compare the cosmological behavior of the model with cosmological observations we use the redshift parameter \(z\) instead of the conformal time defined as

\[
1 + z = \frac{1}{a},
\]

where we have used the normalized scale factor by taking \(a(0) = 1\). To investigate the cosmological evolution of the Universe, considering the evolution of the Hubble parameter \(\mathcal{H}\) and of the deceleration parameter \(q\) are necessary. The deceleration parameter determines that the
expansion of the Universe is whether accelerating or decelerating. In terms of the redshift $z$ this parameter can be obtained as
\[ q = (1 + z) \frac{d \ln h}{dz}. \] (15)

To investigate the evolution of the cosmological parameters $h$ and $q$ in the 4D EGB theory, we consider the numerical solutions of the field equations (8) and (9) for three different values of the parameter $\beta$, $(\beta = -0.001, 0.001 \text{ and } 0.005)$. By the use of Friedmann equation at the present time $z = 0$, and adopting the $h(0) = 1$, for each value of the parameter $\beta$ one can easily obtain the corresponding value of the parameter $\Omega_\Lambda$ in this model. The variation of the Hubble Parameter $h$ are depicted with respect to the redshift $z$ in FIG. 1. In this figure, the solid curve shows the evolution of the Hubble parameter in the standard $\Lambda$CDM model and the error bars are the experimental data [12]. One can see that 4D EGB model predicts that the Universe is expanding. The late time behavior of the Hubble parameter is independent on the value of the model parameter $\beta$ and in this era the evolution of this parameter is well matched with the standard $\Lambda$CDM model and observational data.

FIG. 1: Variation of the Hubble parameter $h$ as a function of redshift $z$ for different values of the parameter $\beta$, $\beta = -0.001$ (dashed curve), $\beta = 0.001$ (dotted curve), $\beta = 0.005$ (dot-dashed curve) and $\Lambda$CDM (solid-shadowed curve). The observational data are shown by error bars.

FIG. 2: Variation of deceleration parameter $q$ as a function of redshift $z$ for different values of the parameter $\beta$, $\beta = -0.001$ (dashed curve), $\beta = 0.001$ (dotted curve), $\beta = 0.005$ (dot-dashed curve) and $\Lambda$CDM (solid-shadowed curve).

The investigation of the cosmological evolution of the Hubble parameter and deceleration parameter shows that the 4D EGB model can satisfy the observational data for some special values of the parameter $\beta$. This is not sufficient for a model to be viable since there are observational data on perturbative limit in which the model should satisfy. As a result, in the next section, we will consider the matter density perturbations of the model around FRW model and compare them with the observational data.

### IV. MATTER DENSITY PERTURBATIONS

In this section, we will consider the scalar perturbations of the field equations (8), (9) and (10) in the Newtonian gauge to obtain the matter density perturbation of the 4D EGB model. The perturbed conformal FRW metric in Newtonian gauge can be written as
\[ ds^2 = a^2(t) \left[ -(1 + 2\varphi)dt^2 + (1 - 2\psi)d\vec{x}^2 \right], \] (16)
where $\varphi$ and $\psi$ are the metric perturbations. The perturbed energy momentum tensor of ordinary matter is defined as
\[ \delta T^0_0 = -\delta \rho + \rho \delta, \quad \delta T^0_i = (1 + \delta)\rho \partial_i \varphi, \]
\[ \delta T^i_j = \delta \rho \delta_v \delta, \] (17)
where, $\rho$ is the background density, $\delta$ is the matter density contrast defined as $\delta = \delta \rho / \rho$ and $v$ is the scalar mode of the velocity perturbation. We assume that the perturbations of the pressure is given by $\delta p / \delta \rho = c_s^2$, where $c_s$ is the adiabatic sound speed of the fluid. The equation of state at the background level is $\omega = p / \rho$.

In the following, we will restrict ourselves to the matter density perturbations of the pressureless matter source.
As a result we expect that equation of state and the sound speed are zero. So we set \( \omega = 0 = c_s \). With this assumption we use the notation \( \delta_m \equiv \delta \) for matter density perturbation in the matter dominated epoch. Also for the ease of calculations, in the following we will Fourier transform the perturbed equation.

Because the energy-momentum tensor in this model is conserved, the perturbation of the conservation equation (5) is identical to the one in the Einstein general relativity. The perturbed temporal and spatial components of equation (5) can be written as

\[
\theta = 3 \dot{\psi} - \dot{\delta}_m, \tag{18}
\]

and

\[
\dot{\theta} + \mathcal{H} \theta - k^2 \varphi = 0, \tag{19}
\]

where \( \theta = \nabla_i \nabla^i \psi \) is the velocity divergence. The (00) component of the metric field equation is

\[
a^4 \rho \delta_m + 4 \mathcal{A} (k^2 \psi + 3 \mathcal{H}^2 \varphi + 3 \mathcal{H} \dot{\psi}) = 0, \tag{20}
\]

where we have denoted

\[
\mathcal{A} = 2 \alpha H^2 + a^2 \kappa^2, \quad \mathcal{B} = 2 \alpha H^2 - a^2 \kappa^2. \tag{21}
\]

The spatial off-diagonal component of equation (3) leads to the relation

\[
\mathcal{A} \varphi + (\mathcal{B} - 4 \alpha \dot{\mathcal{H}}) \psi = 0. \tag{22}
\]

Using above equation, one can obtain the anisotropic stress as

\[
\eta = \frac{\varphi}{\psi} = \frac{1}{\mathcal{A}} \left( 4 \alpha \mathcal{H} - \mathcal{B} \right). \tag{23}
\]

It should be noted that the anisotropic stress becomes unity in the case of vanishing \( \alpha \). The evolution of anisotropic stress as a function of redshift is depicted in FIG. 3 for three different value of \( \beta \). It can be seen from the figure that at early times, the deviations of the anisotropic stress from unity becomes large. Since the observational data for this parameter are still weak [13], it can not be used to restrict modifications of the gravitational field.

The (ii) component of the metric field equation is

\[
2 \mathcal{H} A \varphi + \left( 4 \mathcal{A} \left( 2 A - a^2 \kappa^2 \right) - k^2 A - 2 \mathcal{H} \mathcal{B} \right) \varphi + 2 \mathcal{A} \dot{\psi} + 4 \mathcal{H} \left( a^2 \kappa^2 + 2 \alpha \mathcal{H} \right) \dot{\psi} + k^2 \left( 4 \alpha \mathcal{H} - \mathcal{B} \right) \psi = 0, \tag{24}
\]

We are interested in the evolution of the matter density perturbation \( \delta_m \). In the following we will restrict our considerations to the sub-horizon scales in which the Hubble radius is much greater than the physical wavelength.

![FIG. 3: Variation of the anisotropic stress \( \eta \) as a function of redshift \( z \) for different values of the parameter \( \beta \), \( \beta = -0.001 \) (dashed curve), \( \beta = 0.001 \) (dotted curve), \( \beta = 0.005 \) (dot-dashed curve).](image)

**A. Subhorizon limit**

In the subhorizon limit, where \( \mathcal{H} \ll \frac{1}{2 \pi a} \), equation (20) takes the form

\[
a^4 \rho \delta_m + 4 A k^2 \psi = 0. \tag{25}
\]

Using equation (22) one can obtain the generalized Poisson equation as

\[
\rho \delta_m + 4 \frac{k^2}{a^2} \frac{\mathcal{A}^2}{A^2} \phi = 0. \tag{26}
\]

This relation can be written in Fourier space as

\[
\phi = -4 \pi G_{\text{eff}} \frac{a^2}{k^2 \rho} \delta_m, \tag{27}
\]

where we have defined the effective gravitational constant as

\[
\frac{G_{\text{eff}}}{G} = \frac{\kappa^2 a^2 (4 \alpha \mathcal{H} - \mathcal{B})}{\mathcal{A}^2}. \tag{28}
\]

The deviation of \( G_{\text{eff}} \) from the Newtonian gravitational constant \( G \) in terms of redshift is shown in FIG. 4. One can see that at \( z > 0.5 \), the effective gravitational constant differs significantly with the Newtonian value. It should be noted that for both quantities \( \eta \) and \( G_{\text{eff}} \), one can see that at late times their values become approximately equal to the standard ΛCDM value. As a result, one expects that the qualitative behavior of our model will be identical to the ΛCDM model at these times.

By using the dimensionless parameters, equation (20) in the subhorizon limit in terms of redshift yields

\[
2 \frac{k^2}{3 \mathcal{H}_0^2} \left( 2 (z + 1)^2 + 1 \right)^2 \varphi = (z + 1) \Omega_{m0} \times \left( 2 (z + 1)^2 + 4 \beta h (z + 1)^3 h' - 1 \right) \delta_m, \tag{29}
\]
where prime denotes derivative with respect to the redshift $z$. By substituting (22) into (18) one can easily obtain $\theta$ in terms of the matter density contrast $\delta_m$ and the Newtonian potential $\varphi$. As a result, one can obtain a differential equation for matter density contrast as

$$
\delta_m'' + \frac{h'}{h} \delta_m' + \frac{3\Omega_{m0}}{2(z+1)} \left( 2\beta h^2 (z+1)^2 + 4\beta h (z+1)^3 h' - 1 \right) \delta_m = 0.
$$

(30)

To solve the above equation we use the same initial conditions as one in $\Lambda$CDM model in deep matter dominated phase

$$
d\delta_m|_{z_0} = \delta_m|_{z_\star}, \quad \delta_m|_{z_\star} = a_*,
$$

(31)

where $a_*$ is the value of the scale factor at the redshift $z = z_\star$ and we choose $z_\star = 7.1$. In FIG. 5 we have plotted the evolution of the matter density contrast for different values of $\beta$ in terms of redshift $z$. The red solid curve shows the variation of matter density perturbation for $\Lambda$CDM model. In $z < 1$ the behavior of $\delta_m$ in 4D EGB model mimics the $\Lambda$CDM model. However for larger values of the redshift the deviation from $\Lambda$CDM becomes obvious.

The growth rate of matter perturbation is defined as

$$
f = \frac{d\ln \delta_m}{d\ln a} = -(1 + z) \frac{\delta_m'}{\delta_m}.
$$

(32)

Equation (30) can be written in terms of $f$ as

$$
f' + \left( \frac{h'}{h} - \frac{1 + f}{1 + z} \right) f
- \frac{3\Omega_{m0}}{2} \left( 2\beta h^2 (z+1)^2 + 4\beta h (z+1)^3 h' - 1 \right) = 0
$$

(33)

To consider the compatibility of the model with observations we use the available data for $f\sigma_8$ [14]. This parameter is defined as

$$
f\sigma_8 = \sigma_8(z) \frac{\delta_m'}{\delta_m},
$$

(34)

where $\sigma_8(z) = \sigma_8^0 \frac{\delta_m(z)}{\delta_m(0)}$. The constant $\sigma_8^0$ is a model-dependent parameter. So to determine this parameter from observations we should at first specify the underlying model. The observational data from weak lensing...
TABLE I: The values of $\sigma_8^0$ for different values of $\beta$  

| $\beta$  | $\sigma_8^0$     |
|---------|----------------|
| 0.001   | 0.731191       |
| 0.001   | 0.778723       |
| 0.005   | 0.856066       |

In FIG. 6, we have plotted $f\sigma_8$ for 4D EGB model. In this figure the red solid line corresponds to the $\Lambda$CDM model and the error bars show the observational data for $f\sigma_8$ [14]. One can see that as the value of the coupling constant becomes greater, the quantity $f\sigma_8$ decay faster at early times. Choosing larger than $\beta = 0.005$ will no satisfy the observational data. As a result, one has an upper bound for the value of the parameter $\beta$ from observational data on $f\sigma_8$.

V. CONCLUSIONS AND FINAL REMARKS

In this paper, we have considered the cosmological implications of a novel four dimensional Einstein-Gauss-Bonnet gravity. The model is constructed in such a way that the prefactor $d-3$ of the Gauss-Bonnet tensor, where $d$ is the spatial dimensions, is compensated by the Gauss-Bonnet coupling constant. This makes the Gauss-Bonnet tensor to contribute in four dimensions. In this paper, we have consider the cosmology of such a theory in the presence of baryonic matter sources. The late time behavior of the model is equivalent to the standard $\Lambda$CDM model and as a result the 4D EGB gravity can satisfy background level observational data. We have also considered the perturbations of the matter density fluctuations around FRW geometry and obtained growth rate of the matter density perturbations in this model. The 4D EGB theory makes the anisotropic stress to differs from unity. This is in fact a general behavior in higher order modifications of the gravitational action. However, we have shown that the deviations from unity of the anisotropic stress increases as one goes back to early times. This also happens for the effective gravitational constant, larger values of the EGB coupling constant results in a smaller gravitational constant. However for negative values of $\beta$, the effective gravitational constant becomes greater that the standard Newtonian value.

The growth of matter density perturbations can be tested by experimental data through the behavior of $f\sigma_8$. Since the growth of gravitational seeds starts in the sub-horizon scale, we have considered sub-horizon limit of the matter dominated epoch in this paper. We have obtained the $\sigma_8^0$ value for the present model. It is shown that for larger values of $\beta$ the value of $\sigma_8^0$ can exceed the corresponding in $\Lambda$CDM model. For negative values of the coupling constant $\beta$ the value of $\sigma_8^0$ is always smaller than its $\Lambda$CDM value. Also we have shown that the $f\sigma_8$ can satisfy observational data. However, growth rate of the matter density perturbations in the 4D EGB theory is faster than $\Lambda$CDM predictions.
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