Local Discontinuous Galerkin Method for the Time-Fractional KdV Equation with the Caputo-Fabrizio Fractional Derivative

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Abstract

This paper studies the time-fractional Korteweg-de Vries (KdV) equations with Caputo-Fabrizio fractional derivatives. The scheme is presented by using a finite difference method in temporal variable and a local discontinuous Galerkin method (LDG) in space. Stability and convergence are demonstrated by a specific choice of numerical fluxes. Finally, the efficiency and accuracy of the scheme are verified by numerical experiments.

Keywords

Caputo-Fabrizio Fractional Derivative, Local Discontinuous Galerkin Method, Stability, Error Analysis

1. Introduction

Fractional differential equations have become increasingly important due to their deep scientific and engineering background to correctly model challenging phenomena such as long-range time memory effects, mechanical systems, control systems, etc. [1] [2]. In recent years, variable-order fractional calculus has been found in some physical processes such as algebraic structure and noise reduction. Variable-order fractional calculus is a natural choice to provide an effective mathematical framework for describing complex problems and has many advantages in describing the memory properties of systems [3]-[9].

Fractional partial differential equations can describe abnormal physical phenomena more accurately than integer partial differential equations, which have attracted more and more attention. However, it is difficult to obtain analytical solutions to fractional partial differential equations when the fractional derivatives are known. Therefore, we need to consider efficient numerical methods...
such as the finite element method [10] [11] [12] [13], discontinuous Galerkin method [14] [15] [16] [17] [18], spectral method [19] [20], and finite difference method [21] [22] [23], finite volume method [24] [25]. Wei [26] studied the exact numerical scheme of a class of variable-order fractional diffusion equations, using the fractional derivatives of Caputo-Fabrizio and the theoretical analysis by the local discontinuous Galerkin method. Du [27] proposed different difference schemes for multi-dimensional variable-order time fractional subdiffusion equations and found a special point approximation for the variable-order time Caputo derivative. It is proved that the resulting difference scheme is uniquely solvable. Li et al. [28] carried out a numerical study on three typical Caputo-type partial differential equations using the finite difference method/local discontinuous Galerkin finite element method.

The KdV equation was first proposed by Boussinesq in 1877, and it is a typical dispersion nonlinear partial differential equation. The nonlinear KdV equation was derived by Korteweg and de Vries in 1895 [29], and it describes the propagation of waves in various nonlinear dispersive media. Since then, the KdV equation has been widely used in various physical phenomena and engineering modeling, such as nonlinear wave interactions [30], interfacial electrohydrodynamics [31], plasma physics, geology, etc. Numerous numerical methods have been proposed to solve this equation, such as finite difference schemes [32] [33], pseudospectral methods [34], thermal equilibrium integration methods [35], and discontinuous Galerkin methods [36] [37]. For sufficiently smooth solutions, the following literature does some numerical work on the fractional time KdV equation. Wei et al. [38] proposed the LDG finite element method of the KdV-Burgers-Kuramoto equation, using variable-order Riemann-Liouville fractional derivatives, and proved the unconditional stability and convergence of the scheme. Zhang [39] constructed an efficient numerical scheme for solving linearized fractional KdV equations on unbounded spaces. The non-local fractional derivatives are obtained by exponentiating the convolution kernel and approximately evaluating the initial boundary value problem.

In this paper, the Korteweg-de Vries equation (KdV) with Caputo-Fabrizio fractional derivatives is constructed

\[
\begin{align*}
D^{\alpha(x)}_{t} u(x,t) + \delta u_{xxx}(x,t) + \lambda g(u)_{x} &= f(x,t), \quad (x,t) \in (a,b) \times (0,T], \\
u(x,0) &= v_{0}(x), \quad x \in [a,b],
\end{align*}
\]

(1.1)

where the fractional derivative orders $\alpha (t) \in (0,1)$, $f$, $g(u)$ and $u_0$ are smooth functions. $\delta$ and $\lambda$ are positive constants. In addition, the solutions in this paper are periodic or compactly supported.

The Caputo-Fabrizio fractional derivative in (1.1) is defined as

\[
D^{\alpha(x)}_{t} u(x,t) = \frac{1}{\alpha(t)} \int_{0}^{t} \frac{\partial u(x,s)}{\partial s} \exp \left[\frac{\alpha(t)-1}{\alpha(t)} (t-s)\right] \, ds, \quad s \in (0,t].
\]

(1.2)

There are many definitions of fractional derivatives, of which the most widely used are Riemann-Liouville fractional derivatives and Caputo fractional deriva-
The Caputo-Fabrizio fractional derivative used in this paper was proposed by Caputo and Fabrizio [40] in 2015. Compared with the Caputo fractional derivative model, the Caputo-Fabrizio fractional derivative model can describe different scales and configurations of matter. The Caputo-Fabrizio fractional-order derivatives have been widely used by researchers such as Ann Al Sawoor et al. [41] who studied the asymptotic stability of linear and interval linear fractional-order neutral delay differential systems described by the Caputo-Fabrizio fractional derivatives.

The key to the KdV equation LDG method is to rewrite the equation into a first-order equation system by introducing two auxiliary variables. The LDG method was first introduced by Cockburn and Shu to solve the convection-diffusion equation [42]. One of its advantages is that its solution and spatial derivatives have optimal \((k + 1)\) order convergence on the \(L^2\) norm. Yan and Shu [43] developed a numerical method for LDG for general KdV-type equations involving third-order derivatives. Wei and He [44] used the LDG finite element method to solve the time-fractional KdV equation problem, discretized using finite differences in time and local discontinuous Galerkin methods in space. In [45], the authors established the \(L^2\) conservative LDG numerical scheme and compared it with the dissipative LDG scheme of the KdV type equation to show the dissipative induced phase error. In [46], Baccouch investigated the nonlinear KdV partial differential equation LDG numerical scheme. The results show that the LDG solution is superconvergent to a special Gauss-Radau projection of the exact solution.

The structure of this paper is as follows. In Section 2, some basic notation and mathematical foundations are introduced. Section 3 mainly introduces discrete methods and constructs the LDG scheme. Section 4 presents the stability and convergence results of the scheme. In Section 5, we give numerical experiments to illustrate the accuracy of our proposed format. Finally, we summarize and discuss our results in Section 6.

### 2. Preliminaries

#### 2.1. Notations and Projection

Divide the interval \(\Omega = [a, b]\) as \(J : a = x_1 < x_2 < \cdots < x_{N+1} = b\). For \(j = 1, \cdots, N\), define \(I_j = \left[ x_{j-1/2}, x_{j+1/2} \right]\), and \(\Delta x_j = x_{j+1/2} - x_{j-1/2}, h = \max_{j=1}^{N} \Delta x_j\).

We divide the interval \([0, T]\) evenly into time steps \(\Delta t = T/M = t_n - t_{n-1}\), \(t_n = n\Delta t, n = 0, 1, \cdots, M\) are mesh points.

The left and right limits of \(u\) at \(x_{j+1/2}\) are denoted by \(u_{j+1/2}^-\) and \(u_{j+1/2}^+\), respectively. Where \(u_{j+1/2}^+\) is in the right cell \(I_{j+1}\), and \(u_{j+1/2}^-\) is in the left cell \(I_j\). Define

\[
\left[u_h\right]_{j+1/2} = u_{j+1/2}^+ - u_{j+1/2}^-.
\]
The associated discontinuous Galerkin space $V_h^k$ is defined as follows
\[ V_h^k = \{ v : v \in P^k(I_j), x \in I_j, j = 1, 2, \ldots, N \}. \]

In proving the error estimate, we will use two projections on the one-dimensional interval $[a, b]$.

Denoted as $P^*$,
\[ \int_{I_j} (P^* \omega(x) - \omega(x))v(x) = 0, \forall v \in P^k(I_j), \]
and $P^*$,
\[ \int_{I_j} (P^- \omega(x) - \omega(x))v(x) = 0, \forall v \in P^{k-1}(I_j), P^+ \omega \left( \frac{x^+}{j+\frac{1}{2}} \right) = \omega \left( \frac{x^-}{j+\frac{1}{2}} \right). \]

For the above projection $P, P^*$, it can be obtained from the standard approximation theory [47] [48] [49] [50],
\[ \| \rho \|_h + h^2 \| \rho \|_{\tau_h} \leq Ch^{k+1}, \]
where $\rho = P^\omega - \omega$ or $\rho = P^+ \omega - \omega$. We want to denote all element boundary points in one-dimensional space by $\tau_h$. Furthermore, we have the following definition [51]
\[ \| \rho \|_s = \left( \frac{1}{2} \sum_{I \in \mathcal{N}_h} \left( \left( \rho_{I^+} \right)^2 + \left( \rho_{I^-} \right)^2 \right) \right)^{\frac{1}{2}}. \]

In this paper, $C$ is a positive constant, which may take different values in different positions. $(\cdot, \cdot)_D$ represents the scalar inner product over $L^2(D)$, $\| \cdot \|_D$ represents the correlation norm. When $D = \Omega$, we drop it.

2.2. Numerical Flux

In this paper, we will use the flux $\hat{g}(\psi^+, \psi^-)$, which is related to the discontinuous Galerkin spatial discretization. $\hat{g}(\psi^+, \psi^-)$ is a monotonic numerical flux that depends on the left and right limits of the function $\psi$ at point $x_{j+\frac{1}{2}}$, satisfying the following conditions:

1) It is local Lipschitz continuous, so $\hat{g}(\psi^+, \psi^-)$ is bounded when the function $\psi$ is in a bounded region;

2) It is consistent with the flux $g(\psi)$, i.e., $\hat{g}(\psi^+, \psi^-) = g(\psi)$;

3) It is a function with monotonic properties, the first parameter is a non-decreasing function, and the second parameter is a non-increasing function.

3. The LDG Schemes

This section introduces the LDG method for the time-fractional KdV Equation (1.1).

First, we discretize the fractional derivative in the time direction
\begin{align}
D_{\phi(t)}^{\alpha}u(x,t_n) &= \frac{1}{\alpha(t_n)} \int_0^{t_n} \frac{\partial u(x,s)}{\partial s} \exp \left[ \frac{\alpha(t_n) - 1}{\alpha(t_n)} (t_n - s) \right] ds \\
&= \frac{1}{(1 - \alpha(t_n)) \Delta t} \sum_{k=1}^{n} \left( u(x,t_k) - u(x,t_{k-1}) \right) \left[ \exp \left( \frac{\alpha(t_n) - 1}{\alpha(t_n)} (n - k) \right) - \exp \left( \frac{\alpha(t_n) - 1}{\alpha(t_n)} (n - k + 1) \right) \right] \\
&\quad + \frac{1}{\alpha(t_n)} \sum_{k=1}^{n} \left( s - t_{k-1} + \frac{1}{2} \right) \frac{\partial^2 u(x,c_k)}{\partial s^2} \exp \left[ \frac{\alpha(t_n) - 1}{\alpha(t_n)} (t_n - s) \right] ds \\
&= \frac{1}{(1 - \alpha(t_n)) \Delta t} \sum_{k=1}^{n} (u(x,t_k) - u(x,t_{k-1})) W_k^n + R^n(x),
\end{align}

where $R^n(x)$ is the truncation error in the time direction,

\begin{align}
R^n(x) &= \frac{1}{\alpha(t_n)} \sum_{k=1}^{n} \left( s - t_{k-1} + \frac{1}{2} \right) \frac{\partial^2 u(x,c_k)}{\partial s^2} \exp \left[ \frac{\alpha(t_n) - 1}{\alpha(t_n)} (t_n - s) \right] ds, \\
W_k^n &= \exp \left[ \frac{\alpha(t_n) - 1}{\alpha(t_n)} (n - k) \right] - \exp \left[ \frac{\alpha(t_n) - 1}{\alpha(t_n)} (n - k + 1) \right],
\end{align}

and $c_k \in (t_{k-1}, t_k)$.

By further calculation we can get

\begin{align}
D_{\phi(t)}^{\alpha}u(x,t_n) &= \frac{1}{(1 - \alpha(t_n)) \Delta t} \left( W_n^n u(x,t_n) - W_1^n u(x,t_0) \right) \\
&\quad + \sum_{k=1}^{n-1} \left( W_k^n - W_{k+1}^n \right) u(x,t_k) + R^n(x).
\end{align}

**Lemma 3.1.** [52] [53] When $0 < \alpha(t) < 1$, the truncation error $R^n(x)$ satisfies the following estimation

\begin{align}
\|R^n(x)\| \leq C (\Delta t)^\gamma.
\end{align}

$W_k^n$ has the following properties

\begin{align}
0 < W_1^n < W_2^n < \cdots < W_n^n, \\
\|W_{k+1}^n - W_k^n\| < C, \quad \forall k \leq n - 1,
\end{align}

and

\begin{align}
\sum_{s=2}^{n} W_s^n &= \sum_{s=2}^{n} \left[ \exp \left( \frac{\alpha(t_n) - 1}{\alpha(t_n)} (n - 1) \right) - \exp \left( \frac{\alpha(t_n) - 1}{\alpha(t_n)} (n) \right) \right] \\
&= \exp \left( \frac{\alpha(t_n) - 1}{\alpha(t_n)} \right) - \exp \left( \frac{\alpha(t_n) - 1}{\alpha(t_n)} \right) J < C.
\end{align}

Rewrite the Equation (1.1) as a first-order system of equations,

\begin{align}
\begin{cases}
D_t^{\alpha}u(x,t) + \delta q_t(x,t) + \lambda g(u)_t = f(x,t), \\
p = u_t, \\
q = p_t.
\end{cases}
\end{align}
\( u^n_h, p^n_h, q^n_h \in V_h^k \) represent approximate solutions of \( u(\cdot, t_n), p(\cdot, t_n), q(\cdot, t_n) \), respectively. \( f^n = f(\cdot, t_n) \). Find \( u^n_h, p^n_h, q^n_h \in V_h^k \) such that for the test function \( v, \phi, \varphi \in V_h^k \), we have

\[
W_n^h \int_{\Omega} u^n_h v dx - \beta \left( \int_{\Omega} q^n_h v dx - \sum_{j=1}^{N} \left( q^n_j v \right)_{\frac{1}{2}} - \left( q^n_j v \right)_{\frac{1}{2}} + \right) \\
= \sum_{k=1}^{\infty} (W_{k+1}^n - W_k^n) \int_{\Omega} u^n_h v dx + W_k^n \int_{\Omega} u^n_h v dx + \beta \int_{\Omega} f^n v dx,
\]

(3.7)

\[
\int_{\Omega} q^n_h \phi dx + \int_{\Omega} p^n_h \phi dx - \sum_{j=1}^{N} \left( p^n_j \phi \right)_{\frac{1}{2}} - \left( p^n_j \phi \right)_{\frac{1}{2}} = 0,
\]

\[
\int_{\Omega} p^n_h \varphi dx + \int_{\Omega} u^n_h \varphi dx - \sum_{j=1}^{N} \left( u^n_j \varphi \right)_{\frac{1}{2}} - \left( u^n_j \varphi \right)_{\frac{1}{2}} = 0,
\]

where \( \beta = (1 - \alpha(t_n)) \Delta t \).

The hat function in the element boundary term resulting from the integral by parts in (3.7) is the numerical flux. To ensure stability, we can take the following alternating numerical fluxes

\[
\hat{u}_h^n = \left( u_h^n \right)^-, \quad \hat{p}_h^n = \left( p_h^n \right)^+, \quad \hat{q}_h^n = \left( q_h^n \right)^+.
\]

(3.8)

The choice of flux (3.8) is not unique, only \( \hat{u}_h^n \) and \( \hat{p}_h^n, \hat{q}_h^n \) can take the opposite sides [54].

The fluxes \( \hat{g}\left( u_h^n, u_h^n \right)^* \) are monotonic fluxes as described in Section 2.2. Examples of monotonic fluxes suitable for local discontinuous Galerkin methods can be found [55] [56]. For example, we can use the Lax-Friedrich flux, which consists of

\[
\hat{g}^{LF} (\psi^- , \psi^+) = \frac{1}{2} \left( g(\psi^-) + g(\psi^+) - \lambda_0 (\psi^+ - \psi^-) \right), \quad \lambda_0 = \max_{\psi} |g'(\psi)|.
\]

In the next section, we discuss the stability and convergence of the numerical Scheme (3.7).

4. Stability and Convergence

To simplify the notation, we consider the case of \( f = 0 \) in the numerical analysis.

**Theorem 4.1.** Under periodic or tightly supported boundary conditions, the fully discrete LDG scheme (3.7) is unconditionally stable, and the numerical solution \( u_h^n \) satisfies

\[
W_n^h \int_{\Omega} u^n_h v dx - \beta \left( \int_{\Omega} q^n_h v dx - \sum_{j=1}^{N} \left( q^n_j v \right)_{\frac{1}{2}} - \left( q^n_j v \right)_{\frac{1}{2}} + \right) \\
= \sum_{k=1}^{\infty} (W_{k+1}^n - W_k^n) \int_{\Omega} u^n_h v dx + W_k^n \int_{\Omega} u^n_h v dx + \beta \int_{\Omega} f^n v dx,
\]

(3.7)
Proof. Add the three equations in the scheme (3.7),

\[
W^n_x \int u^n_x v \, dx + \int \Omega q^n_x \phi \, dx + \int \Omega p^n_x \varphi \, dx - \beta \delta \left( \int \Omega q^n_x v \, dx - \sum_{j=1}^N \left( \frac{\left( q^n_x \right)^j}{\frac{1}{2}} \right) \right)
\]

\[-\beta \delta \left( \int \Omega g(u^n_x) v \, dx - \sum_{j=1}^N \left( \frac{\left( g(u^n_x) \right)^j}{\frac{1}{2}} \right) \right)\]

\[+ \int \Omega p^n_x \phi \, dx - \sum_{j=1}^N \left( \frac{\left( p^n_x \phi \right)^j}{\frac{1}{2}} \right) \]

\[+ \int \Omega u^n_x \varphi \, dx - \sum_{j=1}^N \left( \frac{\left( u^n_x \varphi \right)^j}{\frac{1}{2}} \right) \]

\[= \sum_{k=1}^N \left( W^n_k - W_k^n \right) \int u^n_x v \, dx + W^n_1 \int u^n_0 \, dx.
\]

Substitute the test function \( v = u^n_x, \phi = \beta \delta p^n_x, \varphi = -\beta \delta q^n_x \) into the scheme (4.2), using flux (3.8) and the Cauchy-Schwarz inequality, we get

\[
W^n_x \left\| u^n_x \right\| + \beta \delta \left\| p^n_x \right\| - \beta \delta \left\| q^n_x \right\|
\]

\[-\beta \delta \left( \int \Omega q^n_x (u^n_x)_s \, dx - \sum_{j=1}^N \left( \left( q^n_x \right)^j \left( u^n_x \right)^j \right) \right) \]

\[-\beta \delta \left( \int \Omega g(u^n_x) (u^n_x)_s \, dx - \sum_{j=1}^N \left( \left( g(u^n_x) \right)^j \left( u^n_x \right)^j \right) \right) \]

\[+ \beta \delta \left( \int \Omega p^n_x (p^n_x)_s \, dx - \sum_{j=1}^N \left( \left( p^n_x \right)^j \left( p^n_x \right)^j \right) \right) \]

\[+ \beta \delta \left( \int \Omega q^n_x (q^n_x)_s \, dx - \sum_{j=1}^N \left( \left( q^n_x \right)^j \left( q^n_x \right)^j \right) \right) \]

\[\leq \sum_{k=1}^N \left( W^n_k - W_k^n \right) \left\| u^n_x \right\| + W^n_1 \left\| u^n_0 \right\|,
\]

which is

\[
W^n_x \left\| u^n_x \right\| + \beta \lambda \tilde{G}(u^n_x)
\]

\[+ \sum_{j=1}^N \beta \delta \left( \Psi \left( u^n_s, p^n_s, q^n_s \right)_{\frac{1}{2}} - \Psi \left( u^n_s, p^n_s, q^n_s \right)_{\frac{1}{2}} \right) \]

\[\leq \sum_{k=1}^N \left( W^n_k - W_k^n \right) \left\| u^n_s \right\| + W^n_1 \left\| u^n_0 \right\|.
\]

here

\[
\tilde{G}(u^n_x) = - \left( \int \Omega g(u^n_x) (u^n_x)_s \, dx - \sum_{j=1}^N \left( \left( g(u^n_x) \right)^j \left( u^n_x \right)^j \right) \right).
\]
\[ \Psi(u^n, p^n, q^n) = -(u^n)^{+}(q^n)^{-} + (q^n)^{+}(u^n)^{-} + \frac{1}{2}(p^n)^{+}^2 \]
\[ - (p^n)^{+}(p^n)^{-} + (u^n)^{-}(q^n)^{-}, \]
\[ \Theta(u^n, p^n, q^n) = -(u^n)^{+}(q^n)^{-} + (q^n)^{+}(u^n)^{-} + \frac{1}{2}(p^n)^{+}^2 - (p^n)^{+}(p^n)^{-} \]
\[ + (u^n)^{-}(q^n)^{-} + (u^n)^{+}(q^n)^{+} - (q^n)^{+}(u^n)^{-} - \frac{1}{2}(p^n)^{+}^2 \]
\[ + (p^n)^{+}(p^n)^{-} - (u^n)^{-}(q^n)^{-}. \]

The above scheme can be calculated by
\[ \sum_{j=1}^{N} \beta \delta \left( \Psi(u^n, p^n, q^n) + \frac{1}{2}(p^n)^{+}^2 \right) \]
\[ - \Psi(u^n, p^n, q^n) \right)_{j=1}^{2} = 0, \quad (4.4) \]
\[ \Theta(u^n, p^n, q^n) \right)_{j=1}^{2} = \frac{1}{2}(p^n)^{+}^2 \]

For nonlinear terms, let \( G(u) = \int_{0}^{u} g(u) \, du \). Using the mean value theorem and the monotonicity of liquidity yields \( \tilde{G}(u^n) = (G' (\xi) - \xi) [u^n] \geq 0 \), where \( \xi \) is a value between \( u^n^- \) and \( u^n^+ \).

Substituting (4.4) into (4.3), we get
\[ W^n_a \|u^n\| + \sum_{j=1}^{N} \beta \delta \left( |p^n|^{+} \right) ^2 \leq \sum_{j=1}^{N} (W_{n+1}^a - W_{n}^a) \|u^n\|^{+} \|p^n\|^{+} + W_{n+1}^a \|u^n\|^{+} \|\tilde{u}^n\| \]  
(4.5)

Prove Theorem 4.1 by mathematical induction. Let \( n = 1 \) in the scheme (4.5), we have
\[ W_{1}^a \|u_{1}^n\| + \sum_{j=1}^{N} \beta \delta \left( |p_{1}^n|^{+} \right) ^2 \leq W_{1}^a \|u_{1}^n\|^{+} \|\tilde{u}_{1}^n\| \]

since
\[ \int_{\Omega} u_{1}^n u_{1}^n \, dx \leq \frac{1}{2} \|u_{1}^{n+1}\|^{2} + \frac{1}{2} \|u_{1}^{n+1}\|^{2} \]
we can get the following result
\[ W_{1}^a \|u_{1}^n\| \leq W_{1}^a \left( \frac{1}{2} \|u_{1}^{n+1}\|^{2} + \frac{1}{2} \|u_{1}^{n+1}\|^{2} \right), \]
which means
\[ \|u_{1}^n\| \leq \|u_{1}^{n+1}\| \]

Suppose the following inequalities hold
\[ \|u_{m}^n\| \leq \|u_{m}^n\|, \quad m = 1, 2, 3, \cdots, n-1. \]

Next prove \( \|u_{n}^n\| \leq \|u_{n}^n\| \).

From (4.5) we get
\[ W_{n}^a \|u_{n}^n\| \leq \sum_{k=1}^{n-1} (W_{k+1}^a - W_{k}^a) \|u_{k}^n\| + W_{n+1}^a \|u_{n}^n\| \]
\[ \leq \sum_{k=1}^{n} (W_{k+1}^a - W_{k}^a) + W_{n+1}^a \|u_{n}^n\| \]
\[ = W_{n}^a \|u_{n}^n\|. \]

DOI: 10.4236/jamp.2022.106132
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Journal of Applied Mathematics and Physics
Therefore, we have

$$\|u_h\| \leq \|u_h^0\|.$$  

In summary, Theorem 4.1 is proved.

Next, we will state the error estimates of the equation $g(u) = u$ in the linear case, and use (3.8) as the flux choice. We have the following theorem.

**Theorem 4.2.** Let $u(x, t_n)$ be the exact solution of the problem (1.1), $u(t) \in H^{m+1}(D)$ is $0 \leq m \leq k + 1$ is smooth enough. Let $u_h^0$ be the numerical solution of the fully discrete LDG scheme (3.7), then there are the following error estimates

$$\|u(x, t_n) - u_h^0\| \leq C \left( \frac{\Delta}{\Delta} + \frac{h^{k+1}}{\Delta} \right),$$  

(4.7)

$C$ is a positive constant that depends on $u, T$.

**Proof.** Denote

$$e^u = u(x, t_n) - u_h^0 = \xi^u_0 - \eta^u_0, \quad \xi^u_0 = \mathcal{P} e^u, \quad \eta^u_0 = \mathcal{P} u(x, t_n) - u(x, t_n),$$

$$e^p = p(x, t_n) - p_h^0 = \xi^p_0 - \eta^p_0, \quad \xi^p_0 = \mathcal{P} e^p, \quad \eta^p_0 = \mathcal{P} p(x, t_n) - p(x, t_n),$$

$$e^q = q(x, t_n) - q_h^0 = \xi^q_0 - \eta^q_0, \quad \xi^q_0 = \mathcal{P} e^q, \quad \eta^q_0 = \mathcal{P} q(x, t_n) - q(x, t_n).$$

The above $\eta^u_0, \eta^p_0, \eta^q_0$ can be estimated by the inequality (2.4). Next, we mainly discuss $\xi^u_0, \xi^p_0, \xi^q_0$.

We can easily verify that the exact solution of the partial differential Equation (1.1) satisfies the following

$$W_n \int_{\Omega} u(x, t_n) v dx - \beta \delta \left( \int_{\Omega} q(x, t_n) v dx - \sum_{j=1}^{N} \left( (q(x, t_n) v^j)_{\frac{1}{2}} - (q(x, t_n) v^j)_{\frac{1}{2}} \right) \right)$$

$$- \beta \lambda \left( \int_{\Omega} (u(x, t_n) v dx - \sum_{j=1}^{N} \left( (u(x, t_n) v^j)_{\frac{1}{2}} - (u(x, t_n) v^j)_{\frac{1}{2}} \right) \right)$$

$$= \sum_{k=1}^{n} \left( W^u_{k} - W^u_{k+1} \right) \int_{\Omega} u(x, t_k) v dx + W_n \int_{\Omega} u(x, t_0) v dx + \beta \int_{\Omega} f(x, t_n) v dx - \beta \int_{\Omega} R^u(x) v dx,$$  

(4.9)

$$\int_{\Omega} q(x, t_n) \phi dx + \int_{\Omega} p(x, t_n) \phi dx - \sum_{j=1}^{N} \left( (p(x, t_n) \phi^j)_{\frac{1}{2}} - (p(x, t_n) \phi^j)_{\frac{1}{2}} \right) = 0,$$

$$\int_{\Omega} p(x, t_n) \phi^j dx + \int_{\Omega} u(x, t_n) \phi^j dx - \sum_{j=1}^{N} \left( (u(x, t_n) \phi^j)_{\frac{1}{2}} - (u(x, t_n) \phi^j)_{\frac{1}{2}} \right) = 0.$$

Select the flux (3.8), and subtract the Equations (3.7) and (4.9) to get the error equation

$$W_n \int_{\Omega} e^u v dx - \beta \delta \left( \int_{\Omega} e^u v dx - \sum_{j=1}^{N} \left( (e^u v^j)_{\frac{1}{2}} - (e^u v^j)_{\frac{1}{2}} \right) \right)$$

$$- \beta \lambda \left( \int_{\Omega} e^u v dx - \sum_{j=1}^{N} \left( (e^u v^j)_{\frac{1}{2}} - (e^u v^j)_{\frac{1}{2}} \right) \right)$$

$$- \sum_{k=1}^{n-1} \left( W^u_{k} - W^u_{k+1} \right) \int_{\Omega} e^u v dx - W_n \int_{\Omega} e^u v dx + \beta \int_{\Omega} R^u(x) v dx$$

$$= \sum_{j=1}^{N} \left( W^u_{k} - W^u_{k+1} \right) \int_{\Omega} e^u v dx - \beta \delta \left( \int_{\Omega} e^u v dx - \sum_{j=1}^{N} \left( (e^u v^j)_{\frac{1}{2}} - (e^u v^j)_{\frac{1}{2}} \right) \right)$$

$$- \beta \lambda \left( \int_{\Omega} e^u v dx - \sum_{j=1}^{N} \left( (e^u v^j)_{\frac{1}{2}} - (e^u v^j)_{\frac{1}{2}} \right) \right)$$

$$- \sum_{k=1}^{n-1} \left( W^u_{k} - W^u_{k+1} \right) \int_{\Omega} e^u v dx - W_n \int_{\Omega} e^u v dx + \beta \int_{\Omega} R^u(x) v dx.$$
\[ W^n \int_\Omega \zeta^n \varphi \, dx - \beta \delta \int_\Omega \zeta^n \varphi \, dx - \sum_{j=1}^{N} \left[ \left( \zeta^n_j \right)^- \varphi \right]_{\frac{\sqrt{3}}{2}}^1 - \left( \left( \zeta^n_j \right)^- \varphi \right)_{\frac{\sqrt{3}}{2}}^1 = 0. \]

Substitute (4.2) into (4.10) to get

\[ W^n \int_\Omega \zeta^n \varphi \, dx - \beta \delta \int_\Omega \zeta^n \varphi \, dx - \sum_{j=1}^{N} \left[ \left( \zeta^n_j \right)^- \varphi \right]_{\frac{\sqrt{3}}{2}}^1 - \left( \left( \zeta^n_j \right)^- \varphi \right)_{\frac{\sqrt{3}}{2}}^1 = 0. \]

Using the projection property (2.1) - (2.3) and the test functions \( v = \zeta^n \), \( \phi = \beta \delta \zeta^n \), and \( \varphi = -\beta \delta \zeta^n \) in (4.11), the following equality holds

\[ W^n \int_\Omega \zeta^n \varphi \, dx - \beta \delta \int_\Omega \zeta^n \varphi \, dx - \sum_{j=1}^{N} \left[ \left( \zeta^n_j \right)^\phi \right]_{\frac{\sqrt{3}}{2}}^1 - \left( \left( \zeta^n_j \right)^\phi \right)_{\frac{\sqrt{3}}{2}}^1 = 0. \]
Note that $\xi_n^0 = 0$, the following equation can be obtained
\[
W_n^a \| \xi_n^a \| + \frac{\beta \delta}{2} \sum_{j=1}^{N} [\xi_n^a]^2 \cdot \frac{1}{2} + \frac{\beta^2}{2} \sum_{j=1}^{N} [\xi_n^a]^2 \cdot \frac{1}{2} \\
\leq \sum_{j=1}^{N} (W_n^a - W_n^a) \int_{\Omega} \xi_n^a \xi_n^a dx - \beta \int_{\Omega} R^a (x) \xi_n^a dx + \ell \\
+ \beta \delta \sum_{j=1}^{N} \left( \left( (\eta_n^a)^{+} [\xi_n^a] \right)_{j \cdot \frac{1}{2}} - \left( (\eta_n^a)^{+} [\xi_n^a] \right)_{j \cdot \frac{1}{2}} \right) \\
- \beta \delta \sum_{j=1}^{N} \left( \left( (\eta_n^a)^{+} [\xi_n^a] \right)_{j \cdot \frac{1}{2}} - \left( (\eta_n^a)^{+} [\xi_n^a] \right)_{j \cdot \frac{1}{2}} \right)_{j \cdot \frac{1}{2}},
\]
where
\[
\ell = W_n^a \int_{\Omega} \eta_n^a \xi_n^a dx \leq \sum_{j=1}^{N} (W_n^a - W_n^a) \int_{\Omega} \eta_n^a \xi_n^a dx - W_n^a \int_{\Omega} \eta_n^a \xi_n^a dx \leq Ch^{k+1} \| \xi_n^a \|
\]
Using the Cauchy-Schwarz inequality, we have
\[
W_n^a \| \xi_n^a \| + \frac{\beta \delta}{2} \sum_{j=1}^{N} [\xi_n^a]^2 \cdot \frac{1}{2} + \frac{\beta^2}{2} \sum_{j=1}^{N} [\xi_n^a]^2 \cdot \frac{1}{2} \\
\leq \sum_{j=1}^{N} (W_n^a - W_n^a) \| \xi_n^a \| + \beta \| R^a \| \| \xi_n^a \| + Ch^{k+1} \| \xi_n^a \| \quad (4.13)
\]
Use $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, we have
\[
W_n^a \| \xi_n^a \| + \frac{\beta \delta}{2} \sum_{j=1}^{N} [\xi_n^a]^2 \cdot \frac{1}{2} + \frac{\beta^2}{2} \sum_{j=1}^{N} [\xi_n^a]^2 \cdot \frac{1}{2} \\
\leq \frac{W_n^a}{2} \| \xi_n^a \| + \frac{1}{2W_n^a} \left( \sum_{j=1}^{N} (W_n^a - W_n^a) \| \xi_n^a \| + \beta \| R^a \| + Ch^{k+1} \right) \quad (4.14)
\]
which is
\[
W_n^a \| \xi_n^a \| \leq \frac{W_n^a}{2} \| \xi_n^a \| + \frac{1}{2W_n^a} \left( \sum_{j=1}^{N} (W_n^a - W_n^a) \| \xi_n^a \| + \beta \| R^a \| + Ch^{k+1} \right) \quad (4.15)
\]
Multiply both sides of the formula by $2W_n^a$,
\[
\left( W_n^a \| \xi_n^a \| \right)^2 \leq \left( \sum_{j=1}^{N} (W_n^a - W_n^a) \| \xi_n^a \| + \beta \| R^a \| + Ch^{k+1} \right) \quad (4.16)
\]
According to $a^2 + b^2 \leq (a + b)^2$, we can get
\[ W_a^n \| \zeta_a^n \| \leq \sum_{k=1}^{N-1} (W_{a_{k+1}}^{n} - W_a^n) \| \zeta_a^k \| + \beta \| R_a^n \| + Ch^{k+1} \]
\[ + \sqrt{\beta \delta \sum_{j=1}^{N} \left( (\eta_{a_{j}}^n)^2 + (\eta_{a_{j+1}}^n)^2 \right) \frac{1}{j}}. \] (4.16)

From Lemma 3.1, it can be known that \( \| R_a^n \| \leq C(\Delta t)^2 \), and \( \beta = O(\Delta t) = C(\Delta t) \) are defined for simplicity.

Assume that the following estimates hold
\[ \| \zeta_a^n \| \leq Cn \left( (\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}} \right), \] (4.17)

We also prove it by mathematical induction. When \( n = 1 \), the following inequality holds,
\[ W_1 \| \zeta_1 \| \leq C(\Delta t)^2 + Ch^{k+1} + \sqrt{C(\Delta t)} \left( Ch^{k+\frac{1}{2}} \right), \]
therefore
\[ \| \zeta_1 \| \leq C \left( (\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}} \right). \] (4.18)

Then assume that
\[ \| \zeta_j \| \leq Cj \left( (\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}} \right), \quad j = 1, 2, \ldots, n-1. \] (4.19)

According to (4.16) and (4.18), we can get
\[ W_a^n \| \zeta_a^n \| \leq W_a^n (n-1) C \left( (\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}} \right) \]
\[ + C(\Delta t)^2 + Ch^{k+1} + C(\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}}, \]
so
\[ \| \zeta_a^n \| \leq Cn \left( (\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}} \right). \] (4.20)

Since \( j \Delta t \leq n \Delta t = T \),
\[ \| \zeta_a^n \| \leq Cn \Delta t \left( (\Delta t)^{-1} h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}} \right) \]
\[ = CT \left( (\Delta t)^{-1} h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}} \right) \] (4.21)
\[ \leq C \left( (\Delta t)^{-1} h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}} \right). \]

Combining the triangle inequality and the projection property (2.4), it can be seen that the Theorem 4.2 holds.

5. Numerical Experiment

In this section, discussing the effectiveness of the above scheme for solving KdV
equations, we consider the following numerical example with initial values and periodic boundary conditions

\[
\begin{aligned}
&\left[D_t^{\alpha(t)}u(x,t) + \delta u_{xxx}(x,t) + \lambda g(u)_x = f(x,t), \quad (x,t) \in (0,1) \times (0,1),
\right. \\
&\left. u(x,0) = \sin(2\pi x), \quad x \in [0,1],
\end{aligned}
\]  

(5.1)

where \( \delta = 2, \lambda = 9, g(u) = \frac{1}{3}u^2, \)

\[
f(x,t) = \exp[\Gamma]\left[1 - \exp\left[-\frac{t}{\alpha(t)}\right]\right]\sin(2\pi x) - 16\pi^2 \exp[t] \cos(2\pi x) \\
+ 6\pi \exp[2t] \sin(4\pi x).
\]

Now we can check that the exact solution is

\[u(x,t) = e^t \sin(2\pi x).\]

In the following numerical calculations, we will provide the results of the above examples under different \( \alpha(t) \) conditions using piecewise \( P^k \) polynomials to validate our method. The detailed results for the time and space directions are listed below, with \( h = 1/N, \Delta t = 1/M \) for time step and space step, respectively.

In order to reflect the spatial accuracy of the scheme, Figure 1 and Figure 2 adopt a fixed small time step \( \Delta t = \frac{1}{1000} \) and the variable space step \( N = 5,10,20,40 \). Selecting different \( \alpha(t) \), the accuracy of \( L^2 \) norm and \( L^\infty \) norm of piecewise \( P^k \) polynomial can reach the optimal order. Table 1 examines the convergence rate in the time direction of the LDG method, we choose a sufficiently small space step \( h = \frac{1}{1000} \) and a variable time step \( \Delta t = 5,10,20,40 \). It can be seen from Table 1 that it has first-order convergence in time, which is also consistent with the theoretical results.

Table 1. For different order \( \alpha(t) \) when \( N = 1000, T = 1 \), use the piecewise \( P^k \) polynomial to test the time accuracy.

| \( a(t) \) | \( M \) | \( L^2 \)-error | order | \( L^\infty \)-error | order |
|---|---|---|---|---|---|
| \( a(t) = 2\cos(t) \) | 5 | 2.342556643646625e−02 | - | 3.84443535521065e−02 | - |
| | 10 | 1.287283762777858e−02 | 0.86 | 1.784166238464656e−02 | 1.01 |
| | 20 | 6.828570270036006e−03 | 0.91 | 9.105420610539252e−03 | 0.97 |
| | 40 | 3.497426279254679e−03 | 0.97 | 4.694853168636008e−03 | 0.96 |
| \( a(t) = \frac{2}{7} + \frac{t}{7} \) | 5 | 4.662548602534104e−02 | - | 6.552106547740656e−02 | - |
| | 10 | 2.479318097735330e−02 | 0.91 | 3.451685864768518e−02 | 0.92 |
| | 20 | 1.270402131562550e−02 | 0.96 | 1.726411528972149e−02 | 0.99 |
| | 40 | 6.405110600169740e−03 | 0.99 | 8.833848912488842e−03 | 0.97 |
Figure 1. $L^2$ errors and $L^\infty$ errors VS $h$, order for $\alpha(t) = \frac{2t + \sin(t)}{7}$, $M = 1000$, piecewise $P^1$ and $P^2$ polynomial.

Figure 2. $L^2$ errors and $L^\infty$ errors VS $h$, order for $\alpha(t) = \frac{e^t}{4\pi}$, $M = 1000$, piecewise $P^1$ and $P^2$ polynomial.

6. Conclusion

This paper discusses the solution of a class of time-fractional KdV equations by the LDG method under the Caputo-Fabrizio fractional derivative. We derive the stability and error estimates of the proposed scheme. Numerical results demonstrate the effectiveness and good numerical performance of the method. In the future, we will consider generalizing this scheme to two-dimensional or high-dimensional cases.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.
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