On the Explicit Height Distribution and Expected Number of Local Maxima of Isotropic Gaussian Random Fields

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March 5, 2015

Abstract

The explicit formulae for the height distribution and expected number of local maxima have been obtained for isotropic Gaussian random fields on certain low-dimensional Euclidean space or low-dimensional spheres.

Keywords: Height distribution; local maxima; Gaussian orthogonal ensemble; isotropic; Gaussian random field; sphere.

1 Introduction

It has been known that, by using the Kac-Rice formula, one can obtain implicit formulae for the height distribution and expected number of local maxima of smooth Gaussian random fields parameterized over an $N$-dimensional space. See, for example, Adler and Taylor (2007) and Cheng and Schwartzman (2015). In particular, if the Gaussian field is isotropic, then the application of certain Gaussian random matrices involving Gaussian orthogonal ensemble (GOE) makes the above implicit formulae computable [cf. Cheng and Schwartzman (2015) and Fyodorov(2004)]. However, for an arbitrary dimension $N$, the general explicit formulae are still hard to formulate due to the difficulty of evaluating certain GOE computation, see Theorems 3.1 and 4.1 below. Motivated by statistical applications of detection of peaks in Cheng and Schwartzman (2014), we investigate here the explicit formulae for the height distribution and expected number of local maxima of isotropic Gaussian random fields on $N$-dimensional Euclidean space or $N$-dimensional spheres, where $N = 1, 2, 3$.

2 Preliminary GOE Computation

Recall that an $N \times N$ random matrix $M_N$ is said to have the Gaussian Orthogonal Ensemble (GOE) distribution if it is symmetric, with centered Gaussian entries $M_{ij}$ satisfying $\text{Var}(M_{ii}) = \frac{1}{2}$.

*Research partially supported by NIH grant R01-CA157528.
1, \( \text{Var}(M_{ij}) = 1/2 \) if \( i < j \) and the random variables \( \{M_{ij}, 1 \leq i \leq j \leq N\} \) are independent. Moreover, the explicit formula for the distribution \( Q_N \) of the eigenvalues \( \lambda_i \) of \( M_N \) is given by

\[
Q_N(d\lambda) = \frac{1}{c_N} \prod_{i=1}^{N} e^{-\frac{1}{2} \lambda_i^2} d\lambda_i \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j| \mathbb{1}_{\{\lambda_1 \leq \ldots \leq \lambda_N\}}, \tag{2.1}
\]

where the normalization constant \( c_N \) can be computed from Selberg’s integral

\[
c_N = \frac{1}{N!} (2\sqrt{2})^N \prod_{i=1}^{N} \Gamma\left(1 + \frac{i}{2}\right). \tag{2.2}
\]

Denote by \( \mathcal{N}(\mu, \sigma^2) \) a Gaussian distribution with mean \( \mu \) and variance \( \sigma^2 \). Let \( \Phi(x) = \mathbb{P}(X \leq x) \) and \( \Phi_{\Sigma}(x_1, x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) \), where \( X \sim \mathcal{N}(0, 1) \) and \( (X_1, X_2) \) is a centered bivariate Gaussian vector with covariance \( \Sigma \). For \( \gamma > 0 \), let

\[
G_\gamma(x) = \int_{-\infty}^{x} e^{-\gamma t^2} dt. \tag{2.3}
\]

It can be seen that \( G_\gamma(x) = \sqrt{\pi/\gamma} \Phi(\sqrt{2\gamma}x) \).

**Lemma 2.1** For constants \( \gamma > 0, \sigma > 0, \alpha > 0 \) and \( \beta \in \mathbb{R} \),

\[
\int_{-\infty}^{\infty} e^{-\alpha(x-\beta)^2} G_\gamma(x) dx = \frac{\pi}{\sqrt{\alpha\gamma}} \Phi\left(\frac{\beta \sqrt{2\alpha\gamma}}{\sqrt{\alpha + \gamma}}\right),
\]

\[
\int_{-\infty}^{\infty} e^{-\alpha(y-\beta)^2} \int_{-\infty}^{y} e^{-\sigma x^2} G_\gamma(x) dx dy = \frac{\pi^{3/2}}{\sqrt{\alpha\sigma\gamma}} \Phi_{\Sigma}(0, \beta),
\]

where

\[
\Sigma = \left(\begin{array}{cc}
\frac{\sigma + \gamma}{2\sigma \gamma} & -1 \\
\frac{1}{2\sigma} & \frac{\sigma + \alpha}{2\sigma \alpha}
\end{array}\right). \tag{2.4}
\]

**Proof** Let \( Z_1 \sim \mathcal{N}(0, \frac{1}{2\gamma}) \), \( Z_2 \sim \mathcal{N}(0, \frac{1}{2\sigma}) \) and \( Z_3 \sim \mathcal{N}(\beta, \frac{1}{2\alpha}) \) be independent. Then \( (Z_1 - Z_3) \sim \mathcal{N}(-\beta, \frac{\alpha + \gamma}{2\alpha \gamma}) \) and hence

\[
\int_{-\infty}^{\infty} g(x) e^{-\alpha(x-\beta)^2} dx = \frac{\pi}{\sqrt{\alpha\gamma}} \mathbb{P}(Z_1 - Z_3 < 0) = \frac{\pi}{\sqrt{\alpha\gamma}} \Phi\left(\frac{\beta \sqrt{2\alpha\gamma}}{\sqrt{\alpha + \gamma}}\right).
\]

Let \( \mu = (0, -\beta) \) and let \( \Sigma \) be as defined in (2.4). Then \( (Z_1 - Z_2, Z_2 - Z_3) \sim \mathcal{N}(\mu, \Sigma) \), which implies

\[
\int_{-\infty}^{\infty} e^{-\alpha(y-\beta)^2} \int_{-\infty}^{y} e^{-\sigma x^2} g(x) dx dy = \frac{\pi^{3/2}}{\sqrt{\alpha\sigma\gamma}} \mathbb{P}(Z_1 - Z_2 < 0, Z_2 - Z_3 < 0) = \frac{\pi^{3/2}}{\sqrt{\alpha\sigma\gamma}} \Phi_{\Sigma}(0, \beta).
\]

\( \square \)
Lemma 2.2 Let $N = 1$. Then for constants $a > 0$ and $b \in \mathbb{R}$,

$$
\mathbb{E}^{N+1}_{\text{GOE}} \left\{ \exp \left[ \frac{1}{2} \lambda_{N+1}^2 - a(\lambda_{N+1} - b)^2 \right] \right\} = \frac{\sqrt{4a + 2}}{4a} e^{-\frac{ab^2}{2a+1}} + \frac{b\sqrt{\pi}}{\sqrt{2a}} \Phi \left( \frac{b\sqrt{2a}}{\sqrt{2a + 1}} \right).
$$

Proof By (2.1) and integration by parts,

$$
c_{N+1} \mathbb{E}^{N+1}_{\text{GOE}} \left\{ \exp \left[ \frac{1}{2} \lambda_{N+1}^2 - a(\lambda_{N+1} - b)^2 \right] \right\} = \int_{-\infty}^{\infty} e^{-a(\lambda_2 - b)^2} d\lambda_2 \int_{-\infty}^{\lambda_2} e^{-\frac{\lambda_2^2}{2}} (\lambda_2 - \lambda_1) d\lambda_1
$$

Applying Lemma 2.1 and noting that $c_{N+1} = 2\sqrt{\pi}$ when $N = 1$, we obtain the desired result. □

Lemma 2.3 Let $N = 2$. Then for constants $a > 0$ and $b \in \mathbb{R}$,

$$
\mathbb{E}^{N+1}_{\text{GOE}} \left\{ \exp \left[ \frac{1}{2} \lambda_{N+1}^2 - a(\lambda_{N+1} - b)^2 \right] \right\} = \frac{1}{a + 2b^2 - 1} \left[ 1 + \frac{b\sqrt{2a}}{\sqrt{a + 1}} \Phi \left( \frac{\sqrt{2ab}}{(2a + 1)(a + 1)} \right) + \frac{\sqrt{2}}{\sqrt{2a + 1}} e^{-\frac{ab^2}{2a+1}} \right].
$$

Proof By (2.1) and integration by parts,

$$
c_{N+1} \mathbb{E}^{N+1}_{\text{GOE}} \left\{ \exp \left[ \frac{1}{2} \lambda_{N+1}^2 - a(\lambda_{N+1} - b)^2 \right] \right\} = \int_{-\infty}^{\infty} e^{-a(\lambda_3 - b)^2} d\lambda_3 \int_{-\infty}^{\lambda_3} e^{-\frac{\lambda_3^2}{2}} (\lambda_3 - \lambda_2) d\lambda_2 \int_{-\infty}^{\lambda_2} e^{-\frac{\lambda_2^2}{2}} (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) d\lambda_1
$$

Applying integration by parts and Lemma 2.1 and noting that $c_{N+1} = \sqrt{2\pi}$ when $N = 2$, we obtain the desired result. □
Lemma 2.4 Let $N = 3$. Then for constants $a > 0$ and $b \in \mathbb{R}$,

\[
\mathbb{E}_{GOE}^{N+1} \left\{ \exp \left[ \frac{1}{2} \lambda_{N+1}^2 - a(\lambda_{N+1} - b)^2 \right] \right\}
\]

\[
= \left[ \frac{24a^3 + 12a^2 + 6a + 1}{2a(2a + 1)^2} b^2 + \frac{6a^2 + 3a + 2}{4a^2(2a + 1)} + \frac{3}{2} \right] \int_{-\infty}^{\lambda_4} e^{-\frac{\lambda_4^2}{4}} \left( \lambda_4 - \lambda_3 \right) d\lambda_3 \int_{-\infty}^{\lambda_4} e^{-\frac{\lambda_3^2}{4}} \left( \lambda_4 - \lambda_2 \right) d\lambda_2
\]

\[
+ \frac{1}{\sqrt{2(a + 1)}} e^{-\frac{a^2}{2(a + 1)}} \Phi \left( \frac{\sqrt{2}ab}{\sqrt{(a + 1)(2a + 3)}} \right)
\]

\[
+ \left[ \frac{a + 1}{2a} b^2 + \frac{1 - a}{2a^2} - 1 \right] \frac{1}{\sqrt{2(a + 1)}} e^{-\frac{a^2}{2(a + 1)}} \Phi \left( \frac{\sqrt{2}ab}{\sqrt{(a + 1)(2a + 3)}} \right) + \frac{b}{2\sqrt{2\pi(2a + 1)\sqrt{2a + 3}}} e^{-\frac{ab^2}{2a + 3}}
\]

\[
+ \left[ \frac{b^2 + 3(1 - a)}{2a} \right] \sqrt{\pi} b \left[ \Phi_{\Sigma_1}(0, b) + \Phi_{\Sigma_2}(0, b) \right],
\]

where

\[
\Sigma_1 = \left( \begin{array}{cc} \frac{3}{2} & -1 \\ \frac{1}{2} & \frac{1}{2a} \end{array} \right), \quad \Sigma_2 = \left( \begin{array}{cc} \frac{3}{2} & -1 \\ -1 & \frac{1}{2a} \end{array} \right).
\]

Proof By (2.1) and integration by parts,

\[
c_{N+1} \mathbb{E}_{GOE}^{N+1} \left\{ \exp \left[ \frac{1}{2} \lambda_{N+1}^2 - a(\lambda_{N+1} - b)^2 \right] \right\}
\]

\[
= \int_{-\infty}^{\infty} e^{-a(\lambda_4 - b)^2} d\lambda_4 \int_{-\infty}^{\lambda_4} e^{-\frac{\lambda_4^2}{4}} (\lambda_4 - \lambda_3) d\lambda_3 \int_{-\infty}^{\lambda_4} e^{-\frac{\lambda_3^2}{4}} (\lambda_4 - \lambda_2)(\lambda_3 - \lambda_2) d\lambda_2 \int_{-\infty}^{\lambda_2} e^{-\frac{\lambda_2^2}{4}} (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1) d\lambda_1
\]

\[
= \left[ \frac{3b}{2a} + b^3 - \frac{3}{2} \right] \int_{-\infty}^{\infty} e^{-a(\lambda_4 - b)^2} \int_{-\infty}^{\lambda_4} e^{-\frac{\lambda_4^2}{4}} G_1(\lambda_4) d\lambda_4
\]

\[
+ \left[ \frac{b^3 - \frac{3}{2} b}{2a} \right] \int_{-\infty}^{\infty} e^{-a(\lambda_4 - b)^2} \int_{-\infty}^{\lambda_4} e^{-\frac{\lambda_4^2}{4}} G_1(\lambda_4) d\lambda_4
\]

\[
+ \left[ \frac{1}{2} \int_{-\infty}^{\lambda_4} \frac{a + 1 + 2a^2 b^2}{2(a + 1)^2} + \frac{b}{2a} \frac{3b}{2a} + \frac{3b}{2a} \right] \int_{-\infty}^{\lambda_4} e^{-a(\lambda_4 - b)^2} G_{1/2}(\lambda_4) d\lambda_4
\]

Applying integration by parts and Lemma 2.1 and noting that $c_{N+1} = 2\pi$ when $N = 3$, we obtain the desired result. \qed
3 Isotropic Gaussian Random Fields on Euclidean Space

In this section, we consider smooth isotropic Gaussian fields on Euclidean space. Let \( \{ f(t) : t \in \mathbb{R}^N \} \) be a real-valued, \( C^2 \), centered, unit-variance isotropic Gaussian field. Due to isotropy, we can write the covariance function of the field as \( \mathbb{E}\{ f(t)f(s) \} = \rho(\|t - s\|^2) \) for an appropriate function \( \rho(\cdot) : [0, \infty) \to \mathbb{R} \), and denote

\[
\rho' = \rho'(0), \quad \rho'' = \rho''(0), \quad \kappa = -\rho'/\sqrt{\rho''}. \tag{3.1}
\]

Let \( f_i(t) = \frac{\partial f(t)}{\partial t_i} \) and \( f_{ij}(t) = \frac{\partial^2 f(t)}{\partial t_i \partial t_j} \). Denote by \( \nabla f(t) \) and \( \nabla^2 f(t) \) the column vector \( (f_1(t), \ldots, f_N(t))^T \) and the \( N \times N \) matrix \( (f_{ij}(t))_{i,j=1,\ldots,N} \), respectively. By isotropy again, the covariance of \( \nabla f(t), \nabla^2 f(t) \) only depends on \( \rho' \) and \( \rho'' \). In particular, \( \text{Var}(f_i(t)) = -2\rho' \) and \( \text{Var}(f_{ij}(t)) = 12\rho'' \) for any \( i \in \{1, \ldots, N\} \), which implies \( \rho' < 0 \) and \( \rho'' > 0 \) and hence \( \kappa > 0 \).

Over the unit open cube \( (0,1)^N \) in \( \mathbb{R}^N \), define respectively the expected number of local maxima of \( f \) and the expected number of local maxima of \( f \) exceeding level \( u \) as

\[
M(f) = \# \{ t \in (0,1)^N : \nabla f(t) = 0, \text{index}(\nabla^2 f(t)) = N \},
\]

\[
M_u(f) = \# \{ t \in (0,1)^N : f(t) \geq u, \nabla f(t) = 0, \text{index}(\nabla^2 f(t)) = N \},
\]

where the index of a matrix is regarded as the number of its negative eigenvalues. Define the height distribution of a local maximum of \( f \) at some point, say the origin, as

\[
F(u) := \lim_{\varepsilon \to 0} \mathbb{P}\left\{ f(0) > u \mid \exists \text{ a local maximum of } f(t) \text{ in } (-\varepsilon, \varepsilon)^N \right\}.
\]

Then we have the following result in Cheng and Schwartzman (2015).

**Theorem 3.1** Let \( \{ f(t) : t \in \mathbb{R}^N \} \) be a centered, unit-variance, isotropic Gaussian random field satisfying the conditions (C1), (C2) and (C3) in Cheng and Schwartzman (2015). Then for each \( u \in \mathbb{R} \),

\[
\mathbb{E}\{ M(f) \} = \left( \frac{2}{\pi} \right)^{(N+1)/2} \Gamma \left( \frac{N+1}{2} \right) \left( -\frac{\rho''}{\rho'} \right)^{N/2} \mathbb{E}^{N+1}_{\text{GOE}} \left\{ \exp \left[ -\frac{\lambda_{N+1}^2}{2} \right] \right\}
\]

and

\[
F(u) = \frac{\mathbb{E}\{ M_u(f) \}}{\mathbb{E}\{ M(f) \}} = \frac{(1 - \kappa^2)^{-1/2} \int_u^\infty \phi(x) \mathbb{E}^{N+1}_{\text{GOE}} \left\{ \exp \left[ \lambda_{N+1}^2/2 - (\lambda_{N+1} - \kappa x/\sqrt{2})^2/(1 - \kappa^2) \right] \right\} dx}{\mathbb{E}^{N+1}_{\text{GOE}} \left\{ \exp \left[ -\lambda_{N+1}^2/2 \right] \right\}}, \tag{3.2}
\]

where \( \phi(x) \) is the density of standard Gaussian variable and \( \rho' \), \( \rho'' \) and \( \kappa \) are defined in (3.1).
Remark 3.2 Recall the condition (C3) in Cheng and Schwartzman (2015) that \( \kappa \leq 1 \). We show below that, the factors \( (1 - \kappa^2)^{-1/2} \) and \( (1 - \kappa^2)^{-1} \) in (3.2) will be cancelled out, implying that (3.2) also holds when \( \kappa = 1 \). In fact, by (2.1),

\[
(1 - \kappa^2)^{-1/2} \mathbb{E}_{GOE}^{N+1} \left\{ \exp \left[ \lambda_{N+1}^2/2 - \left( \lambda_{N+1} - \kappa x / \sqrt{2} \right)^2 / (1 - \kappa^2) \right] \right\}
\]

\[
= \frac{1}{c_{N+1} \sqrt{1 - \kappa^2}} \int_{\mathbb{R}^{N+1}} e^{\lambda_{N+1}^2 - \kappa x \lambda_{N+1} / \sqrt{2} / (1 - \kappa^2)} \prod_{i=1}^{N} e^{-\frac{1}{2} \lambda_i^2} d\lambda_i \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j| 
\times \prod_{1 \leq k \leq N} |\lambda_k - \lambda_{N+1}| \mathbb{1}_{\{\lambda_1 \leq \ldots \leq \lambda_N \leq \lambda_{N+1}\}} d\lambda_1 \cdots d\lambda_N d\lambda_{N+1}
\]

\[
= \frac{1}{c_{N+1}} \int_{\mathbb{R}^{N+1}} e^{-\lambda_{N+1}^2} \prod_{i=1}^{N} e^{-\frac{1}{2} \lambda_i^2} d\lambda_i \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j| \prod_{1 \leq k \leq N} |\lambda_k - \sqrt{1 - \kappa^2} \lambda_{N+1} - \kappa x / \sqrt{2}| 
\times \mathbb{1}_{\{\lambda_1 \leq \ldots \leq \lambda_N \leq \sqrt{1 - \kappa^2} \lambda_{N+1} + \kappa x / \sqrt{2}\}} d\lambda_1 \cdots d\lambda_N d\lambda_{N+1},
\]

where we have made change of variables \( \tilde{\lambda}_{N+1} = (\lambda_{N+1} - \kappa x / \sqrt{2}) / \sqrt{1 - \kappa^2} \).

From now on, we denote by \( h \) the density of the height distribution of local maxima of \( f \), i.e. \( h(x) = -F'(x) \) or \( F(x) = \int_x^\infty h(t) dt \). Also notice that, due to the formula \( \mathbb{E}\{M_u(f)\} = F(u) \mathbb{E}\{M(f)\} \) in (3.2), we only need to find \( F(u) \) and \( \mathbb{E}\{M(f)\} \).

**Proposition 3.3** Let the assumptions in Theorem 3.1 hold. If \( N = 1 \), then

\[
\mathbb{E}\{M(f)\} = \frac{\sqrt{6}}{2\pi} \sqrt{-\frac{\rho''}{\rho'}}
\]

and

\[
h(x) = \frac{\sqrt{3 - \kappa^2}}{\sqrt{6\pi}} e^{-\frac{3x^2}{2(3 - \kappa^2)}} + \frac{2\kappa x \sqrt{\pi}}{\sqrt{6}} \phi(x) \Phi \left( \frac{\kappa x}{\sqrt{3 - \kappa^2}} \right).
\]

If \( N = 2 \), then

\[
\mathbb{E}\{M(f)\} = -\frac{\rho''}{\sqrt{3\pi}\rho'}
\]

and

\[
h(x) = \sqrt{3}\kappa^2(x^2 - 1) \phi(x) \Phi \left( \frac{\kappa x}{\sqrt{2 - \kappa^2}} \right) + \frac{\kappa x \sqrt{3(2 - \kappa^2)}}{2\pi} e^{-\frac{x^2}{2 - \kappa^2}}
\]

\[+ \frac{\sqrt{6}}{\sqrt{\pi(3 - \kappa^2)}} \frac{e^{-\frac{3x^2}{2(3 - \kappa^2)}} \Phi \left( \frac{\kappa x}{\sqrt{(3 - \kappa^2)(2 - \kappa^2)}} \right)}{\Phi \left( \frac{\kappa x}{\sqrt{(3 - \kappa^2)(2 - \kappa^2)}} \right)}.
\]

If \( N = 3 \), then

\[
\mathbb{E}\{M(f)\} = \frac{29\sqrt{6} - 36}{36\pi^2} \left( -\frac{\rho''}{\rho'} \right)^{3/2}
\]
and
\[
\begin{aligned}
    h(x) &= \frac{144\phi(x)}{29\sqrt{6} - 36} \left\{ \frac{\kappa^2 [(1 - \kappa^2)^3 + 6(1 - \kappa^2)^2 + 12(1 - \kappa^2) + 24]}{4(3 - \kappa^2)^2} x^2 \\
    &+ \frac{2(1 - \kappa^2)^3 + 3(1 - \kappa^2)^2 + 6(1 - \kappa^2)}{4(3 - \kappa^2)} + \frac{3}{2} \right\} e^{\frac{-\kappa^2 x^2}{2(3 - \kappa^2)}} \Phi \left( \frac{2\kappa x}{\sqrt{(3 - \kappa^2)(5 - 3\kappa^2)}} \right) \\
    &+ \left[ \frac{\kappa^2(2 - \kappa^2)}{4} x^2 - \frac{\kappa^2(1 - \kappa^2)}{2} - 1 \right] e^{\frac{-\kappa^2 x^2}{2(2 - \kappa^2)}} \Phi \left( \frac{\kappa x}{2\sqrt{(2 - \kappa^2)(5 - 3\kappa^2)}} \right) \\
    &+ \left[ 7 - \kappa^2 + \frac{(1 - \kappa^2)}{2(3 - \kappa^2)} \left[ 3(1 - \kappa^2)^2 + 12(1 - \kappa^2) + 28 \right] \right] e^{\frac{-\kappa^2 x^2}{2(3 - \kappa^2)}} \Phi \left( \frac{\kappa x}{4\sqrt{\pi}(3 - \kappa^2)\sqrt{5 - 3\kappa^2}} \right) \\
    &+ \frac{\sqrt{\pi} \kappa^3}{4} x(x^2 - 3) \left[ \Phi_{\Sigma_1}(0, \kappa x/\sqrt{2}) + \Phi_{\Sigma_2}(0, \kappa x/\sqrt{2}) \right],
\end{aligned}
\]

where
\[
\Sigma_1 = \left( \begin{array}{cc}
    \frac{3}{2} & -1 \\
    -1 & \frac{3 - \kappa^2}{2}
\end{array} \right), \quad \Sigma_2 = \left( \begin{array}{cc}
    \frac{3}{2} & -\frac{1}{2} \\
    -\frac{1}{2} & \frac{3 - \kappa^2}{2}
\end{array} \right).
\]

Proof Let \( N = 1 \). By Theorem 3.1, we see that \( \mathbb{E}\{M(f)\} \) can be obtained by applying Lemma 2.2 with \( a = 1 \) and \( b = 0 \), while \( h(x) \) can be obtained by applying Lemma 2.2 with \( a = (1 - \kappa^2)^{-1} \) and \( b = \kappa x/\sqrt{2} \). The cases for \( N = 2 \) and \( N = 3 \) can be proved similarly by applying Lemmas 2.3 and 2.4 respectively. \( \square \)

Figure 1: Density functions of the height distribution of local maxima on Euclidean space. All three cases \( N = 1 \) (black), \( N = 2 \) (blue) and \( N = 3 \) (red) are displayed. Solid lines represent \( \kappa = 1 \), dashed lines represent \( \kappa = 0.5 \) and dotted lines represent \( \kappa = 0.1 \).
Remark 3.4 The regularity condition (C2) in Cheng and Schwartzman (2015) implies that \( \kappa^2 < 3 \) when \( N = 1 \), \( \kappa^2 < 2 \) when \( N = 2 \) and \( \kappa^2 < 5/3 \) when \( N = 3 \). Although Theorem 3.1 holds for \( \kappa \leq 1 \) as shown in Cheng and Schwartzman (2015), we conjecture that the formulae obtained in Proposition 3.3 are valid for any isotropic Gaussian fields satisfying conditions (C1), (C2) and (C3) in Cheng and Schwartzman (2015).

4 Isotropic Gaussian Random Fields on Sphere

In this section, we denote by \( S^N \) the \( N \)-dimensional unit sphere and let \( f = \{ f(t) : t \in S^N \} \) be a real-valued, \( C^2 \), centered, unit-variance isotropic Gaussian random field on \( S^N \). Due to isotropy, we may write the covariance function of \( f \) as \( C((t, s)) \), \( t, s \in \mathbb{S}^N \), where \( C(\cdot) : [-1, 1] \to \mathbb{R} \) and \( \langle \cdot, \cdot \rangle \) is the inner product in \( \mathbb{R}^{N+1} \). See Cheng and Schwartzman (2015) and Cheng and Xiao (2014) for more results on the covariance function of an isotropic Gaussian field on \( S^N \). We define

\[
C' = C'(1), \quad C'' = C''(1), \quad \kappa_1 = C'/C'', \quad \kappa_2 = C''/C''.
\]

It is known from Cheng and Schwartzman (2015) that \( C' > 0 \) and \( C'' > 0 \).

Let \( B(t_0, \delta) \) be a geodesic (open) ball on \( S^N \) with radius \( \delta \) centered at \( t_0 \in \mathbb{S}^N \). Let \( \delta_0 \) be the number such that the area of \( B(t_0, \delta_0) \) equals 1. Following the notation in Cheng and Schwartzman (2015), over the unit geodesic ball \( B(t_0, \delta_0) \) in \( S^N \), define respectively the expected number of local maxima of \( f \) and the expected number of local maxima of \( f \) exceeding level \( u \) as

\[
M(f) = \# \{ t \in B(t_0, \delta_0) : \nabla f(t) = 0, \text{index}(\nabla^2 f(t)) = N \},
\]

\[
M_u(f) = \# \{ t \in B(t_0, \delta_0) : f(t) \geq u, \nabla f(t) = 0, \text{index}(\nabla^2 f(t)) = N \}.
\]

Define the height distribution of a local maximum of \( f \) at some point, say \( t_0 \), as

\[
F(u) := \lim_{\varepsilon \to 0} \mathbb{P} \{ f(t_0) > u | \exists \text{ a local maximum of } f(t) \text{ in } B(t_0, \varepsilon) \}.
\]

Then we have the following result in Cheng and Schwartzman (2015).

**Theorem 4.1** Let \( \{ f(t) : t \in \mathbb{S}^N \} \) be a centered, unit-variance, isotropic Gaussian random field satisfying the conditions (C1'), (C2') and (C3') in Cheng and Schwartzman (2015). Then for each \( u \in \mathbb{R} \),

\[
\mathbb{E}\{M(f)\} = \frac{\sqrt{2}}{\pi^{(N+1)/2} \kappa_1^{N/2}} \Gamma \left( \frac{N + 1}{2} \right) \mathbb{E}_{\text{GOE}}^{N+1} \left\{ \exp \left[ \frac{\lambda_{N+1}^2}{2} - \frac{\lambda_{N+1}^2}{1 + \kappa_1} \right] \right\}
\]

and

\[
F(u) = \frac{\mathbb{E}\{M_u(f)\}}{\mathbb{E}\{M(f)\}} = \left( \frac{1 + \kappa_1}{1 + \kappa_1 - \kappa_2} \right)^{1/2} \int_{-\infty}^{\infty} \phi(x) \mathbb{E}_{\text{GOE}}^{N+1} \left\{ \exp \left[ \frac{\lambda_{N+1}^2}{2} - \frac{(\lambda_{N+1} - \sqrt{\kappa_2} x / \sqrt{\kappa_1})^2}{1 + \kappa_1 - \kappa_2} \right] \right\} dx,
\]

(4.2)
where $\kappa_1$ and $\kappa_2$ are defined in (4.1).

**Remark 4.2** Recall the condition $(C3')$ in Cheng and Schwartzman (2015) that $\kappa_2 - \kappa_1 \leq 1$. Similarly to Remark 3.2, it can be shown that, the factors $(1 + \kappa_1 - \kappa_2)^{-1/2}$ and $(1 + \kappa_1 - \kappa_2)^{-1}$ in (4.2) will be cancelled out, implying that (4.2) also holds when $\kappa_2 - \kappa_1 = 1$.

![Figure 2: Density functions of the height distribution of local maxima on sphere. All three cases $N = 1$ (black), $N = 2$ (blue) and $N = 3$ (red) are displayed. Solid lines represent $\kappa_1 = 1$ and $\kappa_2 = 2$, dashed lines represent $\kappa_1 = 1$ and $\kappa_2 = 1$ and dotted lines represent $\kappa_1 = 0.1$ and $\kappa_2 = 0.1$.](image)

Now, we can obtain the following explicit formulae for $E\{M(f)\}$ and the height density $h$, which imply $F$ and $E\{M_u(f)\}$.

**Proposition 4.3** Let the assumptions in Theorem 4.1 hold. If $N = 1$, then

$$E\{M(f)\} = \frac{\sqrt{3 + \kappa_1}}{2\pi \sqrt{\kappa_1}}$$

and

$$h(x) = \frac{1}{\sqrt{3 + \kappa_1}} \left[ \sqrt{3 + \kappa_1 - \kappa_2} e^{-\frac{(3 + \kappa_1)x^2}{2(3 + \kappa_1 - \kappa_2)}} + \sqrt{2\pi \kappa_2} x \phi(x) \Phi \left( \frac{\sqrt{\kappa_2} x}{\sqrt{3 + \kappa_1 - \kappa_2}} \right) \right].$$

If $N = 2$, then

$$E\{M(f)\} = \frac{1}{4\pi} + \frac{1}{2\pi \kappa_1 \sqrt{3 + \kappa_1}}$$
and
\[ h(x) = \frac{2\sqrt{3}+\kappa_1}{2+2\kappa_1\sqrt{3}+\kappa_1} \left\{ [\kappa_1+\kappa_2(x^2-1)] \phi(x) \Phi \left( \frac{\sqrt{\kappa_2}x}{\sqrt{2+\kappa_1-\kappa_2}} \right) ight. \\
+ \frac{\sqrt{\kappa_2(2+\kappa_1-\kappa_2)}}{2\pi} xe^{-\frac{(2+\kappa_1)x^2}{2(2+\kappa_1-\kappa_2)}} \\
+ \frac{\sqrt{2}}{\sqrt{\pi(3+\kappa_1-\kappa_2)}} e^{-\frac{(3+\kappa_1)x^2}{2(3+\kappa_1-\kappa_2)}} \Phi \left( \frac{\sqrt{\kappa_2}x}{\sqrt{(2+\kappa_1-\kappa_2)(3+\kappa_1-\kappa_2)}} \right) \left\} , \right.
\]

If \( N = 3 \), then
\[ \mathbb{E}\{M(f)\} = \frac{1}{\pi^2\kappa_1^{3/2}} \left\{ \frac{1}{2\sqrt{3}+\kappa_1} \left[ \frac{3}{2} + \frac{(1+\kappa_1)(2\kappa_1^2+7\kappa_1+11)}{4(3+\kappa_1)} \right] + \frac{(\kappa_1-1)(\kappa_1+2)}{4\sqrt{2+\kappa_1}} \right\} \]
and
\[ h(x) = \left\{ \frac{1}{2\sqrt{2}(3+\kappa_1)} \left[ \frac{3}{2} + \frac{(1+\kappa_1)(2\kappa_1^2+7\kappa_1+11)}{4(3+\kappa_1)} \right] + \frac{(\kappa_1-1)(\kappa_1+2)}{4\sqrt{2+\kappa_1}} \right\}^{-1} \phi(x) \]
\[ \times \left\{ \frac{\kappa_2}{4} \left[ \frac{(1+\kappa_1-\kappa_2)^3+6(1+\kappa_1-\kappa_2)^2+12(1+\kappa_1-\kappa_2)+24}{4(3+\kappa_1-\kappa_2)^2} \\
+ \frac{2(1+\kappa_1-\kappa_2)^3+3(1+\kappa_1-\kappa_2)^2+6(1+\kappa_1-\kappa_2)}{4(3+\kappa_1-\kappa_2)} \right] + \frac{3}{2} \right\} \frac{1}{\sqrt{2(3+\kappa_1-\kappa_2)}} \\
\times e^{-\frac{\kappa_2x^2}{2(3+\kappa_1-\kappa_2)}} \Phi \left( \frac{2\sqrt{\kappa_2}x}{\sqrt{(3+\kappa_1-\kappa_2)(5+3\kappa_1-3\kappa_2)}} \right) \\
+ \left[ \frac{\kappa_2(2+\kappa_1-\kappa_2)}{4} x^2 + \frac{(\kappa_1-1)(\kappa_1-\kappa_2)}{2} - 1 \right] \sqrt{2(2+\kappa_1-\kappa_2)} \\
\times e^{-\frac{\kappa_2x^2}{2(2+\kappa_1-\kappa_2)}} \Phi \left( \frac{\sqrt{\kappa_2}x}{\sqrt{(2+\kappa_1-\kappa_2)(5+3\kappa_1-3\kappa_2)}} \right) \\
+ \left[ 7+\kappa_1-\kappa_2 + \frac{3(1+\kappa_1-\kappa_2)^3+12(1+\kappa_1-\kappa_2)^2+28(1+\kappa_1-\kappa_2)}{2(3+\kappa_1-\kappa_2)} \right] \\
\times \frac{\sqrt{\kappa_2}}{4\sqrt{\pi(3+\kappa_1-\kappa_2)}} xe^{-\frac{3\kappa_2x^2}{2(3+\kappa_1-\kappa_2)}} \Phi \left( 0, \sqrt{\kappa_2}x \right) + \Phi \left( 0, \sqrt{\kappa_2}x \right) \right\} , \]
where
\[ \Sigma_1 = \begin{pmatrix} \frac{3}{2} & -1 \\ 3+\kappa_1-\kappa_2 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & 2+\kappa_1-\kappa_2 \end{pmatrix}. \]

**Proof** Let \( N = 1 \). By Theorem 4.1, we see that \( \mathbb{E}\{M(f)\} \) can be obtained by applying Lemma 2.2 with \( a = (1+\kappa_1)^{-1} \) and \( b = 0 \), while \( h(x) \) can be obtained by applying Lemma 2.2 with \( a = (1+\kappa_1-\kappa_2)^{-1} \) and \( b = \sqrt{\kappa_2}x/\sqrt{2} \). The cases for \( N = 2 \) and \( N = 3 \) can be proved similarly by applying Lemmas 2.3 and 2.4 respectively. \( \square \)
Remark 4.4 The regularity condition \((C2')\) in Cheng and Schwartzman (2015) implies that \(\kappa_2 - \kappa_1 < 3\) when \(N = 1\), \(\kappa_2 - \kappa_1 < 2\) when \(N = 2\) and \(\kappa_2 - \kappa_1 < 5/3\) when \(N = 3\). Although Theorem 4.1 holds for \(\kappa_2 - \kappa_1 \leq 1\) as shown in Cheng and Schwartzman (2015), we conjecture that the formulae obtained in Proposition 4.3 are valid for any isotropic Gaussian fields satisfying conditions \((C1')\), \((C2')\) and \((C3')\) in Cheng and Schwartzman (2015).

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