RIGIDITY OF RIEMANNIAN MANIFOLDS WITH POSITIVE SCALAR CURVATURE

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Abstract. For the Bach-flat closed manifold with positive scalar curvature, we prove a rigidity result under a given inequality involving the Weyl curvature and the traceless Ricci curvature. Moreover, under an inequality involving $L^n_2$-norm of the Weyl curvature, the traceless Ricci curvature and the Yamabe invariant, we also provide a similar rigidity result. As an application, we obtain some rigidity results on 4-dimensional manifolds.

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1. Introduction

We recall that a Riemannian manifold $(M^n, g)$ is a gradient shrinking Ricci soliton if there exists a smooth function $f$ such that

$$R_{ij} + f_{ij} = \lambda g_{ij}$$

for some positive constant $\lambda$. In [5], relying on sharp algebraic curvature estimates, Catino proved some rigidity results for closed gradient shrinking Ricci soliton satisfying a $L^n$-pinching condition. From (1.1), it is easy to see that the Ricci curvature can be expressed by the Hessian of function $f$, in some sense. However, on a common Riemannian manifold $(M^n, g)$, we can not find a function which is related to the Ricci curvature. It is nature to ask whether can one obtain some rigidity results under the $L^n_2$-pinching condition, which are analogous to those of Catino in [5] on a given Riemannian manifold.

In order to study conformal relativity, R. Bach [2] in early 1920s’ introduced the following Bach tensor:

$$B_{ij} = \frac{1}{n-3} W_{ijkl,kl} + \frac{1}{n-2} W_{ijkl} R_{kl},$$

where $n \geq 4$, $W_{ijkl}$ denotes the Weyl curvature. A metric $g$ is called Bach-flat if $B_{ij} = 0$. The aim of this paper is to achieve some rigidity results under the $L^n_2$-pinching condition on a given Riemannian manifold. In order to state our results, throughout this paper, we always denote $\hat{R}_{ij}$ the traceless Ricci curvature.

Now, we can state our first result as follows:

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**Theorem 1.1.** Let \((M^4, g)\) be a closed Bach-flat manifold with positive constant scalar curvature. If
\[
|W + \frac{\sqrt{2}}{3} \Ric \otimes g| < \frac{2}{3\sqrt{3}} R, \tag{1.3}
\]
then \(M^4\) is isometric to a quotient of the round sphere \(S^4\).

**Remark 1.1.** Under the condition
\[
|W| + |\hat{R}_{ij}| < \frac{R}{4n}, \tag{1.4}
\]
Fang and Yuan, in [6], have proved (see [6, Theorem A]) the same result as Theorem 1.1. It is easy to see from (1.3)
\[
|W|^2 + |\hat{R}_{ij}|^2 \leq 2(|W|^2 + |\hat{R}_{ij}|^2) < \frac{4R^2}{27}, \tag{1.5}
\]
which gives
\[
|W| + |\hat{R}_{ij}| \leq \sqrt{2(|W|^2 + |\hat{R}_{ij}|^2)} < \frac{4R}{3\sqrt{6}} \tag{1.6}
\]
Clearly, we have
\[
\frac{4R}{3\sqrt{6}} > \frac{R}{16} \tag{1.7}
\]
and hence, for \(n = 4\), our Theorem 1.1 generalizes Theorem A of Fang and Yuan [6].

The Yamabe invariant \(Y(M, [g])\) associated to \((M^n, g)\) is defined by
\[
Y(M, [g]) = \inf_{\tilde{g} \in [g]} \frac{\int_M \hat{R} \, dv_{\tilde{g}}}{\left( \int_M dv_{g} \right)^{\frac{n-2}{n}}} \tag{1.8}
\]
where \([g]\) is the conformal class of the metric \(g\). For closed manifolds, \(Y(M, [g])\) is positive if and only if there exists a conformal metric in \([g]\) with everywhere positive scalar curvature. Therefore, for any closed manifold with positive scalar curvature, from (1.8), we obtain
\[
\frac{n-2}{4(n-1)} Y(M, [g])(\int_M |u|^\frac{2n}{n-2} \, dv_g)^\frac{n-2}{n} \leq \int_M |\nabla u|^2 \, dv_g + \frac{n-2}{4(n-1)} \int_M R u^2 \, dv_g. \tag{1.9}
\]

By the aid of the Yamabe invariant given in (1.3), we can prove the following result:

**Theorem 1.2.** Let \((M^n, g)\) be a closed Bach-flat manifold with positive constant scalar curvature. If either \(4 \leq n \leq 5\) and
\[
\left( \int_M \left| W + \frac{\sqrt{n}}{\sqrt{8(n-2)}} \Ric \otimes g \right|^\frac{n}{2} \, dv_g \right)^\frac{2}{n} < \frac{1}{4} \sqrt{\frac{n-2}{2(n-1)}} Y(M, [g]); \tag{1.10}
\]
or $n = 6$ and
\[
\left( \int_M |W + \frac{\sqrt{3}}{8} \text{Ric} \otimes g|^3 dv_g \right)^{\frac{1}{3}} < \frac{1}{25} \sqrt{21} 10 Y(M, [g]),
\]
then $M^n$ is isometric to a quotient of the round sphere $S^n$.

For manifolds with harmonic curvature, we also obtain the following similar rigidity results:

**Theorem 1.3.** Let $M^4$ be a closed manifold with harmonic curvature. If the scalar curvature is positive and
\[
\int_M \left( |W| + \frac{1}{\sqrt{2}} \text{Ric} \otimes g \right)^2 dv_g < \frac{25}{486} Y^2(M, [g]),
\]
then $M^4$ is isometric to a quotient of the round sphere $S^4$.

When $n = 4$, we can get the following results:

**Corollary 1.4.** Let $(M^4, g)$ be a closed manifold. If it is Bach-flat and satisfies
\[
\int_M \left( |W|^2 + \frac{5}{4} |\tilde{R}|^2 \right) dv_g \leq \frac{1}{48} \int_M R^2 dv_g,
\]
then it is isometric to a quotient of the round sphere $S^4$. If the curvature is harmonic and
\[
\int_M \left( |W|^2 + \frac{374}{81} |\tilde{R}|^2 \right) dv_g \leq \frac{25}{486} \int_M R^2 dv_g,
\]
then it is also isometric to a quotient of the round sphere $S^4$.

**Remark 1.2.** The pinching conditions (1.13) and (1.14) are equivalent to
\[
\frac{13}{8} \int_M |W|^2 dv_g + \frac{1}{12} \int_M R^2 dv_g \leq 20 \pi^2 \chi(M) \quad (1.15)
\]
and
\[
\frac{268}{81} \int_M |W|^2 dv_g + \frac{1}{3} \int_M R^2 dv_g \leq \frac{5984}{81} \pi^2 \chi(M), \quad (1.16)
\]
respectively. Here $\chi(M)$ is the Euler-Poincaré characteristic of $M^4$.

### 2. Some lemmas

Recall that the Weyl curvature $W_{ijkl}$ is defined by
\[
W_{ijkl} = R_{ijkl} - \frac{1}{n-2} (R_{ik} g_{jl} - R_{il} g_{jk} + R_{jl} g_{ik} - R_{jk} g_{il})
\]
\[
+ \frac{R}{(n-1)(n-2)} (g_{ik} g_{jl} - g_{il} g_{jk})
\]
\[
= R_{ijkl} - \frac{1}{n-2} (\tilde{R}_{ik} g_{jl} - \tilde{R}_{il} g_{jk} + \tilde{R}_{jl} g_{ik} - \tilde{R}_{jk} g_{il})
\]
\[
- \frac{R}{n(n-1)} (g_{ik} g_{jl} - g_{il} g_{jk}),
\]

(2.1)
where $R$ is the scalar curvature. Since the divergence of the Weyl curvature tensor is related to the Cotton tensor by

$$-\frac{n-3}{n-2}C_{ijk} = W_{ijkl},$$

(2.2)

where the Cotton tensor is given by

$$C_{ijk} = R_{kj,i} - R_{ki,j} - \frac{1}{2(n-1)}(R_{,i}g_{jk} - R_{,j}g_{ik})$$

$$= \hat{R}_{kj,i} - \hat{R}_{ki,j} + \frac{n-2}{2n(n-1)}(R_{,i}g_{jk} - R_{,j}g_{ik}),$$

(2.3)

the formula (1.2) can be written as

$$B_{ij} = \frac{1}{n-2}(C_{kij,k} + W_{ikjl}R_{kl}).$$

(2.4)

As in [6], for any $\theta \in \mathbb{R}$ and a symmetric 2-tensor $\varphi$, we introduce the following $\theta$-tensor:

$$C^\theta_{ijk} := \varphi_{kj,i} - \theta\varphi_{ki,j}.$$  

(2.5)

We call a $\theta$-tensor $C^\theta_{ijk}$ the $\theta$-Codazzi tensor if $C^\theta_{ijk} = 0$. In particular, a Codazzi tensor is a special case of $\theta$-Codazzi tensor with $\theta = 1$ and the $\theta$-Codazzi tensor with $\theta = -1$ is called the anti-Codazzi tensor.

**Lemma 2.1.** Let $M^n$ be a closed manifold. Then for the $\theta$-tensor defined by (2.5), we have

$$\int_M [(\theta^2 + 1)|\nabla \varphi|^2 - |C^\theta_{ij}|^2] dv_g = 2\theta \int_M (|\text{div}\varphi|^2 + W_{ijkl}\dot{\varphi}_{jl}\dot{\varphi}_{ik}$$

$$- \frac{n}{n-2} \hat{R}_{ij}\dot{\varphi}_{jk}\dot{\varphi}_{ki} - \frac{R}{n-1} |\dot{\varphi}_{ij}|^2) dv_g,$$

(2.6)

where $\dot{\varphi}_{ij} = \varphi_{ij} - \frac{\text{tr}\varphi}{n} g_{ij}$. In particular, taking $\varphi_{ij} = R_{ij}$ in (2.6), we obtain

$$\int_M |\nabla \dot{R}_{ij}|^2 dv_g \geq \frac{2\theta}{\theta^2 + 1} \int_M \left( W_{ijkl}\dot{R}_{jl}\dot{R}_{ik} - \frac{n}{n-2} \hat{R}_{ij} \dot{R}_{jk} \dot{R}_{ki}$$

$$- \frac{1}{n-1} R |\dot{R}_{ij}|^2 + \frac{(n-2)^2}{4n^2} |\nabla R|^2 \right) dv_g.$$

(2.7)

**Proof.** For any $\theta \in \mathbb{R}$, we have

$$\int_M |C^\theta_{ij}|^2 dv_g = \int_M |\varphi_{kj,i} - \theta\varphi_{ki,j}|^2 dv_g$$

$$= \int_M \left[ (1 + \theta^2)|\nabla \varphi|^2 - 2\theta\varphi_{kj,i}\varphi_{ki,j} \right] dv_g.$$

(2.8)
By the integration by parts, we have

$$
\int_M \varphi_{k,j,i} \varphi_{k,i,j} \, dv_g = - \int_M \varphi_{k,j,i} \varphi_{k,i,j} \, dv_g
$$

$$
= - \int_M (\varphi_{k,j,i} + \varphi_{i,j} R_{kij} + \varphi_{k,t} R_{ijj}) \varphi_{k,i,j} \, dv_g
$$

$$
= \int_M (-\varphi_{k,j,i} + \varphi_{i,j} R_{kiji} - \varphi_{k,t} R_{kij}) \varphi_{k,i,j} \, dv_g
$$

$$
= \int_M \left[ |\nabla \varphi|^2 + R_{ijkl} \varphi_{jli} \varphi_{k} - \hat{R}d \varphi_{k} \varphi_{k} - \frac{2}{n} (\text{tr} \varphi) \hat{R}_{ij} \hat{\varphi}_{ij}
$$

$$
- \frac{R}{n} \left( |\hat{\varphi}_{ij}|^2 + \frac{(\text{tr} \varphi)^2}{n} \right) \right] \, dv_g.
$$

(2.9)

Applying

$$
R_{ijkl} \varphi_{jli} \varphi_{k} = W_{ijkl} \varphi_{jli} \varphi_{k} + \frac{2}{n-2} \left[ (\text{tr} \varphi) R_{ij} - R_{ij} \varphi_{j} \varphi_{k} \right]
$$

$$
+ \frac{R}{n(n-1)} [(\text{tr} \varphi)^2 - |\varphi_{ij}|^2] \right)
$$

(2.10)

in (2.9), we obtain

$$
\int_M \varphi_{k,j,i} \varphi_{k,i,j} \, dv_g = \int_M \left[ |\nabla \varphi|^2 + W_{ijkl} \varphi_{jli} \varphi_{k} + \frac{2}{n} (\text{tr} \varphi) \hat{R}_{ij} \hat{\varphi}_{ij}
$$

$$
- \frac{R}{n-2} \hat{R}_{ij} \hat{\varphi}_{kj} \varphi_{k} - \frac{R}{n-1} |\hat{\varphi}_{ij}|^2 \right] \, dv_g \quad (2.11)
$$

and the desired (2.6) follows from by putting (2.11) into (2.8).

Using the second Bianchi identity, we have $R_{ijkl} \varphi_{jli} \varphi_{k} \varphi_{k} = R_{ijkl} W_{ijkl} \varphi_{jli} \varphi_{k} + \frac{n-2}{n} (\text{tr} \varphi) R_{ij} \varphi_{j} \varphi_{k}$.

Hence, (2.7) follows by letting $\varphi_{ij} = R_{ij}$ in (2.6). Thus, we conclude the proof of Lemma 2.1. □

**Lemma 2.2.** Let $M^n$ be a closed Bach-flat Riemannian manifold. Then we have

$$
\int_M |\nabla \hat{R}_{ij}|^2 \, dv_g = \int_M \left( 2W_{ijkl} \hat{R}_{ij} \hat{R}_{jk} - \frac{n}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki}
$$

$$
- \frac{1}{n-1} \hat{R} \hat{R}_{ij} \hat{R}_{ji} + \frac{(n-2)^2}{4n(n-1)} |\nabla \hat{R}|^2 \right) \, dv_g. \quad (2.12)
$$

**Proof.** Using the formula (2.1), we can derive

$$
\hat{R}_{kl} R_{ikjl} = \hat{R}_{kl} W_{kj} + \frac{1}{n-2} (|\hat{R}_{ij}|^2 g_{ij} - 2 \hat{R}_{ij} \hat{R}_{jk}) - \frac{1}{n(n-1)} \hat{R} \hat{R}_{ij}. \quad (2.13)
$$
which shows

\[
\tilde{R}_{kj,ik} = \tilde{R}_{kj,ki} + R_{ij} R_{ik} - R_{kl} R_{ijk} \\
= \frac{n-2}{2n} R_{ij} + \frac{1}{n} R \tilde{R}_{ij} - \left[ R_{kl} W_{ikjl} + \frac{1}{n(n-1)} R \tilde{R}_{ij} \right] \\
+ \frac{1}{n-2} \left( |\tilde{R}_{ij}|^2 g_{ij} - 2 \tilde{R}_{ik} \tilde{R}_{jk} \right) - \frac{1}{n-2} R \tilde{R}_{ij},
\]

Thus, from (2.3) and (2.14), we have

\[
C_{kij,k} = \Delta \tilde{R}_{ij} - \tilde{R}_{kj,ik} + \frac{n-2}{2n(n-1)} (g_{ij} \Delta R - R_{ij}) \\
= \Delta \tilde{R}_{ij} - \left( \frac{n-2}{2n} R_{ij} + \frac{1}{n-2} R \tilde{R}_{ij} - \frac{1}{n-1} R \tilde{R}_{ij} \right) \\
- \frac{1}{n-2} |\tilde{R}_{ij}|^2 g_{ij},
\]

and

\[
0 = (n-2) B_{ij} \tilde{R}_{ij} \\
= C_{kij,k} \tilde{R}_{ij} + W_{ikjl} \tilde{R}_{ij} \tilde{R}_{kl} \\
= \tilde{R}_{ij} \Delta \tilde{R}_{ij} - \frac{n-2}{n-2} \tilde{R}_{ij} \tilde{R}_{jk} \tilde{R}_{ki} - \frac{1}{n-1} R |\tilde{R}_{ij}|^2 \\
+ 2 W_{ikjl} \tilde{R}_{ij} \tilde{R}_{kl} - \frac{n-2}{2(n-1)} R_{ij} \tilde{R}_{ij},
\]

which gives

\[
\int_M |\nabla \tilde{R}_{ij}|^2 dv_g = - \int_M \tilde{R}_{ij} \Delta \tilde{R}_{ij} dv_g \\
= \int_M \left( 2 W_{ijkl} \tilde{R}_{ij} \tilde{R}_{ik} - \frac{n-2}{n-2} \tilde{R}_{ij} \tilde{R}_{jk} \tilde{R}_{ki} \\
- \frac{1}{n-1} R |\tilde{R}_{ij}|^2 + \frac{(n-2)^2}{4n(n-1)} |\nabla R|^2 \right) dv_g.
\]

We complete the proof of Lemma 2.2. □
Lemma 2.3. On every $n$-dimensional Riemannian manifold $(M^n, g)$, for any $\rho \in \mathbb{R}$, the following estimate holds

$$
|-W_{ijkl} \hat{R}_{jl} \hat{R}_{ik} + \frac{\rho}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki}| \\
\leq \sqrt{\frac{n-2}{2(n-1)}} \left( |W|^2 + \frac{2\rho^2}{n(n-2)} |\hat{R}_{ij}|^2 \right)^{\frac{1}{2}} |\hat{R}_{ij}|^2 \\
= \sqrt{\frac{n-2}{2(n-1)}} \left( W + \frac{\rho}{\sqrt{2n(n-2)}} \hat{\text{Ric}} \otimes g \right) |\hat{R}_{ij}|^2.
$$

(2.18)

Proof. We will prove Lemma 2.3 only by some modifications for the proof of Proposition 2.1 of Catino in [5]. Following [3, Lemma 4.7], we have

$$(\hat{\text{Ric}} \otimes g)_{ijkl} = \hat{R}_{ik} g_{jl} - \hat{R}_{il} g_{jk} + \hat{R}_{jl} g_{ik} - \hat{R}_{jk} g_{il},$$

and

$$(\hat{\text{Ric}} \otimes \hat{\text{Ric}})_{ijkl} = 2(\hat{R}_{ik} \hat{R}_{jl} - \hat{R}_{il} \hat{R}_{jk}),$$

It is easy to see

$$W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} = \frac{1}{4} W_{ijkl}(\hat{\text{Ric}} \otimes \hat{\text{Ric}})_{ijkl},$$

$$\hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} = -\frac{1}{8} (\hat{\text{Ric}} \otimes g)_{ijkl}(\hat{\text{Ric}} \otimes \hat{\text{Ric}})_{ijkl},$$

which shows that

$$-W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} + \frac{\rho}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} = -\frac{1}{4} \left( W + \frac{\rho}{2(n-2)} \hat{\text{Ric}} \otimes g \right)_{ijkl}(\hat{\text{Ric}} \otimes \hat{\text{Ric}})_{ijkl}.$$  

(2.19)

Let

$$T = \hat{\text{Ric}} \otimes \hat{\text{Ric}} \otimes \text{U} - \text{V},$$

(2.20)

where

$$U_{ijkl} = -\frac{2}{n(n-1)} |\hat{R}_{ij}|^2 (g \otimes g)_{ijkl},$$

$$V_{ijkl} = -\frac{2}{n-2} (\hat{\text{Ric}}^2 \otimes g)_{ijkl} + \frac{4}{n(n-2)} |\hat{R}_{ij}|^2 (g \otimes g)_{ijkl}$$

with $(\text{Ric}^2)_{ij} = \hat{R}_{ik} \hat{R}_{kj}$. Then $T$ is totally tracefree. Taking the square norm, we have

$$|\hat{\text{Ric}} \otimes \hat{\text{Ric}}|^2 = 8 |\hat{R}_{ij}|^4 - 8 |\text{Ric}^2|^2, $$

$$|U|^2 = \frac{8}{n(n-1)} |\hat{R}_{ij}|^4,$$

$$|V|^2 = \frac{16}{n-2} |\text{Ric}^2|^2 - \frac{16}{n(n-2)} |\hat{R}_{ij}|^4$$

and

$$|T|^2 + \frac{n}{2} |V|^2 = |\hat{\text{Ric}} \otimes \hat{\text{Ric}}|^2 + \frac{n-2}{2} |V|^2 - |U|^2 = 8 \frac{(n-2)}{n-1} |\hat{R}_{ij}|^4. $$

(2.21)
Since both $W$ and $T$ are totally tracefree, using the Cauchy-Schwarz inequality, we obtain
\[
\left| \left( W + \frac{\rho}{2(n-2)} \hat{\text{Ric}} \otimes g \right)_{ijkl} \left( \hat{\text{Ric}} \otimes \hat{\text{Ric}} \right)_{ijkl} \right|^2 \\
= \left| \left( W + \frac{\rho}{2(n-2)} \hat{\text{Ric}} \otimes g \right)_{ijkl} (T + V)_{ijkl} \right|^2 \\
= \left| \left( W + \frac{\rho}{2(n-2)} \sqrt{\frac{2}{n}} \hat{\text{Ric}} \otimes g \right)_{ijkl} (T + \sqrt{\frac{n}{2}} V)_{ijkl} \right|^2 \\
\leq \left| W + \frac{\rho}{2(n-2)} \sqrt{\frac{2}{n}} \hat{\text{Ric}} \otimes g \right|^2 \left( |T|^2 + \frac{n}{2} |V|^2 \right) \\
= \frac{8(n-2)}{n-1} \left( |W|^2 + \frac{2\rho^2}{n(n-2)} |\hat{R}_{ij}|^2 \right) |\hat{R}_{ij}|^4.
\]

Applying (2.22) in (2.19) and using that
\[
\left| \left( W + \frac{\rho}{\sqrt{2n(n-2)}} \hat{\text{Ric}} \otimes g \right)_{ijkl} \right|^2 = |W|^2 + \frac{2\rho^2}{n(n-2)} |\hat{R}_{ij}|^2,
\]
we obtain the desired estimate (2.18).

We complete the proof of Lemma 2.3. □

Following the proof in [8], for manifolds of which the metric $g$ is Einstein, we can prove the following (see [8, Equation (5)])
\[
\Delta W_{ijkl} = \frac{2}{n} R W_{ijkl} - 2(2 W_{ipkq} W_{pjql} + \frac{1}{2} W_{klpq} W_{pqij}),
\]
which gives
\[
\frac{1}{2} \Delta |W|^2 = |\nabla W|^2 + \frac{2}{n} R |W|^2 - 2(2 W_{ijkl} W_{ipkq} W_{pjql} + \frac{1}{2} W_{ijkl} W_{klpq} W_{pqij}).
\]

The following estimate comes from [9, Lemma 2.5] (or see [5, 7]):

**Lemma 2.4.** On every $n$-dimensional Riemannian manifold $(M^n, g)$, there exists a positive constant $C(n)$ such that the following estimate holds
\[
2 W_{ijkl} W_{ipkq} W_{pjql} + \frac{1}{2} W_{ijkl} W_{klpq} W_{pqij} \leq C(n) |W|^3,
\]
where $C(n)$ is defined by
\[
C(n) = \begin{cases} 
\frac{\sqrt{6}}{1}, & n = 4 \\
\frac{1}{4\sqrt{10}}, & n = 5 \\
\frac{\sqrt{10}}{2\sqrt{3}}, & n = 6 \\
\frac{3}{2}, & n \geq 7.
\end{cases}
\]

Now we recall the following result which follows from the proof of [5, Theorem 3.3]:


Lemma 2.5. Let $C(n)$ be defined by (2.26). Then for the Einstein manifold $(M^n, g)$, $n \geq 4$, with positive scalar curvature, we have

$$
\begin{align*}
&\left[\frac{n+1}{n-1} \left( Y(M, [g]) - \frac{8(n-1)}{n-2} C(n) \left( \int_M |W|^\frac{n}{2} \, dv_g \right)^\frac{2}{n} \right) \right] \int_M |\nabla |W|^2 \, dv_g \\
&+ \left[ \frac{2}{n} Y(M, [g]) - 2C(n) \left( \int_M |W|^\frac{n}{2} \, dv_g \right)^\frac{2}{n} \right] \int_M R|W|^2 \, dv_g \leq 0.
\end{align*}
$$

(2.27)

Proof. Using (2.25), we can deduce from (2.24)

$$
\frac{1}{2} \Delta |W|^2 \geq |\nabla |W|^2 + \frac{2}{n} R|W|^2 - 2C(n)|W|^3
$$

(2.28)

Putting the following refined Kato inequality of any Einstein manifold

$$
|\nabla |W|^2 \geq \frac{n+1}{n-1} |\nabla |W||^2
$$

(2.29)

into (2.28) yields

$$
\frac{1}{2} \Delta |W|^2 \geq \frac{n+1}{n-1} |\nabla |W||^2 + \frac{2}{n} R|W|^2 - 2C(n)|W|^3.
$$

(2.30)

Using the Hölder inequality and (1.9) with $u = |W|$, we get

$$
0 \geq \frac{n+1}{n-1} \int_M |\nabla |W||^2 \, dv_g + \frac{2}{n} \int_M R|W|^2 \, dv_g - 2C(n) \int_M |W|^3 \, dv_g
$$

$$
\geq \frac{n+1}{n-1} \int_M |\nabla |W||^2 \, dv_g + \frac{2}{n} \int_M R|W|^2 \, dv_g
$$

$$
- 2C(n) \left( \int_M |W|^\frac{n}{2} \, dv_g \right)^\frac{n}{2} \left( \int_M |W|^\frac{2n}{n-2} \, dv_g \right)^\frac{n-2}{n}
$$

$$
\geq \frac{n+1}{n-1} \int_M |\nabla |W||^2 \, dv_g + \frac{2}{n} \int_M R|W|^2 \, dv_g
$$

$$
- 2C(n) \left( \int_M |W|^\frac{n}{2} \, dv_g \right)^\frac{n}{2} \left( \int_M |\nabla |W||^2 \, dv_g \right)
$$

$$
- \int_M R|W|^2 \, dv_g
$$

$$
= \left[ \frac{n+1}{n-1} Y(M, [g]) - \frac{8(n-1)}{n-2} C(n) \left( \int_M |W|^\frac{n}{2} \, dv_g \right)^\frac{2}{n} \right] \int_M |\nabla |W||^2 \, dv_g
$$

$$
+ \left[ \frac{2}{n} Y(M, [g]) - 2C(n) \left( \int_M |W|^\frac{n}{2} \, dv_g \right)^\frac{2}{n} \right] \int_M R|W|^2 \, dv_g,
$$

(2.31)

and the proof of Lemma 2.5 is completed.
3. Proof of Theorems

3.1. Proof of Theorem 1.1. Now, with the help of Lemma 2.1 and Lemma 2.2, we will complete the proof of Theorem 1.1.

Combining (2.7) with (2.12), we obtain

\[
0 \geq \int_{M} \left[ 2 \left( -W_{ijkl} \tilde{R}_{ij} \tilde{R}_{kl} + \frac{n(\theta - 1)^2}{2(\theta^2 - \theta + 1)} \frac{1}{n-2} \tilde{R}_{ij} \tilde{R}_{jk} \tilde{R}_{ki} \right) + \frac{(\theta - 1)^2}{(n-1)(\theta^2 - \theta + 1)} R|\tilde{R}_{ij}|^2 \right] dv_g.
\]

(3.1)

Since for any \( \theta \), we have \( \theta^2 - \theta + 1 > 0 \), using (2.18), we can deduce from (3.1)

\[
0 \geq \int_{M} \left[ -\sqrt{\frac{2(n-2)}{n-1}} |W| + \frac{n(\theta - 1)^2}{2\sqrt{2(n-2)(\theta^2 - \theta + 1)}} \tilde{\text{Ric}} \otimes g \right.
\]

\[
+ \left. \frac{(\theta - 1)^2}{(n-1)(\theta^2 - \theta + 1)} R \right]|\tilde{R}_{ij}|^2 dv_g
\]

\[
\geq 0
\]

under the condition

\[
\sqrt{\frac{2(n-2)}{n-1}} |\theta^2 - \theta + 1| W + \frac{n}{2\sqrt{2(n-2)}} \tilde{\text{Ric}} \otimes g \left| \tilde{\text{Ric}} \otimes g \right| < \frac{1}{n-1} R
\]

(3.3)

with a given \( \theta \) (\( \theta \neq 1 \)). Minimizing the coefficient of \( W \) with respect to the function \( \theta \) by taking

\[
\theta = -1,
\]

(3.4)

we obtain that if

\[
|W + \frac{\sqrt{2n}}{3(n-2)} \tilde{\text{Ric}} \otimes g| < \frac{4}{3\sqrt{2(n-1)(n-2)}} R
\]

(3.5)

holds, then (3.2) shows \( \tilde{R}_{ij} = 0 \) and the metric \( g \) must be Einstein.

On the other hand, integrating (2.28), we have

\[
\int_{M} \left( \frac{1}{n} R - C(n)|W| \right)|W|^2 dv_g \leq 0.
\]

(3.6)

Under the assumption (1.3), we have that the metric \( g \) is Einstein and the condition (3.5) becomes

\[
|W| < \frac{4}{3\sqrt{2(n-1)(n-2)}} R.
\]

(3.7)

In this case, we have

\[
C(n)|W| < \frac{4C(n)}{3\sqrt{2(n-1)(n-2)}} R < \frac{1}{n} R,
\]

(3.8)
for $n = 4$, and (3.6) yields

$$0 \leq \int_M \left( \frac{1}{n} R - C(n)|W| \right) |W|^2 \, dv_g \leq 0. \tag{3.9}$$

This shows that $W_{ijkl} = 0$ and $M^4$ is isometric to a quotient of the round sphere $S^4$.

We complete the proof of Theorem 1.1.

3.2. **Proof of Theorem 1.2** Replacing $u$ in (1.9) with $|\dddot{R}_{ij}|$ yields

$$\frac{n - 2}{4(n - 1)} Y(M, [g])(\int_M |\dddot{R}_{ij}|^{2n-2} \, dv_g)^{\frac{n-2}{n}} \leq \int_M |\nabla \dddot{R}_{ij}|^2 \, dv_g + \frac{n - 2}{4(n - 1)} \int_M R \dddot{R}_{ij}^2 \, dv_g. \tag{3.10}$$

On the other hand, using the Kato inequality, (2.12) and (2.18), we obtain

$$\int_M |\nabla \dddot{R}_{ij}|^2 \, dv_g \leq \int_M |\dddot{R}_{ij}|^2 \, dv_g$$

$$= \int_M \left[ 2W_{ijkl} \dddot{R}_{ij} \dddot{R}_{ik} - \frac{n}{n - 2} \dddot{R}_{ij} \dddot{R}_{jk} \dddot{R}_{ki} - \frac{1}{n - 1} R |\dddot{R}_{ij}|^2 \right] \, dv_g$$

$$\leq \int_M \left[ \sqrt{\frac{2(n - 2)}{n - 1}} |W + \sqrt{\frac{n}{8(n - 2)}} \text{Ric} \otimes g| - \frac{1}{n - 1} R |\dddot{R}_{ij}|^2 \right] \, dv_g. \tag{3.11}$$

Inserting (3.11) into (3.10) gives

$$\frac{n - 2}{4(n - 1)} Y(M, [g])(\int_M |\dddot{R}_{ij}|^{2n-2} \, dv_g)^{\frac{n-2}{n}} \leq \int_M \left[ \sqrt{\frac{2(n - 2)}{n - 1}} \left| W + \sqrt{\frac{n}{8(n - 2)}} \text{Ric} \otimes g \right| - \frac{1}{n - 1} R |\dddot{R}_{ij}|^2 \right] \, dv_g. \tag{3.12}$$

Hence, for $4 \leq n \leq 6$, using Hölder inequality for (3.12) yields

$$\left[ \frac{n - 2}{4(n - 1)} Y(M, [g]) - \sqrt{\frac{2(n - 2)}{n - 1}} \left( \int_M \left| W + \sqrt{\frac{n}{8(n - 2)}} \text{Ric} \otimes g \right|^n \, dv_g \right)^{\frac{2}{n}} \right]$$

$$\times \left( \int_M |\dddot{R}_{ij}|^{2n-2} \, dv_g \right)^{\frac{n-2}{n}} \leq 0, \tag{3.13}$$

which, under the condition

$$\left( \int_M \left| W + \sqrt{\frac{n}{8(n - 2)}} \text{Ric} \otimes g \right|^n \, dv_g \right)^{\frac{2}{n}} < \frac{1}{4} \sqrt{\frac{n - 2}{2(n - 1)} Y(M, [g])} \tag{3.14}$$
(3.13) shows that the metric $g$ must be Einstein. Since the metric $g$ is Einstein, then (3.14) becomes

$$
\left( \int_M |W|^\frac{n}{2} \, dv_g \right)^\frac{2}{n} < \frac{1}{4} \sqrt{\frac{n-2}{2(n-1)}} Y(M, [g]).
$$

(3.15)

Noticing for $4 \leq n \leq 5$,

$$
\frac{1}{4} \sqrt{\frac{n-2}{2(n-1)}} = \min \left\{ \frac{1}{4} \sqrt{\frac{n-2}{2(n-1)}}, \frac{(n+1)(n-2)}{8(n-1)^2 C(n)}, \frac{1}{n C(n)} \right\},
$$

(3.16)

and for $n = 6$,

$$
\frac{(n+1)(n-2)}{8(n-1)^2 C(n)} = \min \left\{ \frac{1}{4} \sqrt{\frac{n-2}{2(n-1)}}, \frac{(n+1)(n-2)}{8(n-1)^2 C(n)}, \frac{1}{n C(n)} \right\},
$$

(3.17)

hence under the assumption (1.10) or (1.11), we have

$$
0 \leq \left[ \frac{n+1}{n-1} Y(M, [g]) - \frac{8(n-1)}{n-2} C(n) \left( \int_M |W|^\frac{n}{2} \, dv_g \right)^\frac{2}{n} \right] \int_M |\nabla |W||^2 \, dv_g
$$

+ \left[ \frac{2}{n} Y(M, [g]) - 2C(n) \left( \int_M |W|^\frac{n}{2} \, dv_g \right)^\frac{2}{n} \right] \int_M R|W|^2 \, dv_g \leq 0,
$$

(3.18)

which shows that $W_{ijkl} = 0$ and $M^n$ is isometric to a quotient of the round sphere $S^n$.

3.3. Proof of Theorem 1.3. If the curvature of $(M^n, g)$ is harmonic, then we have

$$
R_{ki,j} - R_{k,j,i} = R_{ijkl,l} = 0,
$$

(3.19)

This shows that the Ricci curvature is Codazzi and

$$
R_{i} = R_{kk,i} = R_{ik,k} = \frac{1}{2} R_{i},
$$

(3.20)

which shows the scalar curvature is constant. Therefore, (2.7) becomes

$$
\int_M |\nabla \hat{R}_{ij}|^2 \, dv_g = \int_M \left[ W_{ijkl} \hat{R}_{jl} \hat{R}_{ik} - \frac{n}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} - \frac{1}{n-1} R|\hat{R}_{ij}|^2 \right] \, dv_g.
$$

(3.21)

As first observed by Bourguignon [4], the Codazzi tensor $\hat{R}_{ij}$ satisfies the following sharp inequality (for a proof, see for instance [8]):

$$
|\nabla \hat{R}_{ij}|^2 \geq \frac{n+2}{n} |\nabla |\hat{R}_{ij}||^2.
$$

(3.22)
It follows from (3.21) that
\[
0 \geq \frac{n+2}{n} \int_M |\nabla R_{ij}|^2 \, dv_g + \frac{1}{n-1} R |\tilde{R}_{ij}|^2 \, dv_g,
\]
which combining with (3.10) gives
\[
0 \geq \frac{n+2}{n} \left[ \frac{n-2}{4(n-1)} Y(M, [g]) \left( \int_M |\tilde{R}_{ij}|^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}} - \frac{n-2}{4(n-1)} \int_M R |\tilde{R}_{ij}|^2 \, dv_g \right]

\[
- \int_M \left[ \frac{n-2}{2(n-1)} |W + \frac{\sqrt{n}}{\sqrt{2(n-2)}} \text{Ric} \otimes g|| \tilde{R}_{ij}||^2 \, dv_g + \frac{1}{n-1} \int_M R |\tilde{R}_{ij}|^2 \, dv_g \right]

\[
\geq \left[ \frac{n^2-4}{4n(n-1)} Y(M, [g]) - \frac{n-2}{2(n-1)} \left( \int_M |W + \frac{\sqrt{n}}{\sqrt{2(n-2)}} \text{Ric} \otimes g||^{\frac{2}{n}} \right)^{\frac{n}{2}} \right]

\[
\times \left( \int_M |\tilde{R}_{ij}|^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}} - \frac{n^2-4n-4}{4n(n-1)} \int_M R |\tilde{R}_{ij}|^2 \, dv_g
\]
\[
\geq 0.
\]
under (1.12) for \( n = 4 \). Hence, we obtain that the metric \( g \) must be Einstein and hence (1.12) becomes
\[
\left( \int_M |W|^{\frac{n}{2}} \, dv_g \right)^{\frac{n}{2}} \leq \frac{n+2}{2n} \sqrt{\frac{n-2}{2(n-1)}} Y(M, [g]).
\]
Noticing for \( n = 4 \),
\[
\frac{(n+1)(n-2)}{8(n-1)^2 C(n)} = \min \left\{ \frac{n+2}{2n} \sqrt{\frac{n-2}{2(n-1)}}, \frac{(n+1)(n-2)}{8(n-1)^2 C(n)}, \frac{1}{n C(n)} \right\},
\]
hence, under the condition (1.12), we have that (3.24) holds and
\[
0 \leq \left[ \frac{n+1}{n-1} Y(M, [g]) - \frac{8(n-1)}{2(n-2)} C(n) \left( \int_M |W|^{\frac{n}{2}} \, dv_g \right)^{\frac{n}{2}} \right] \int_M |\nabla W|^2 \, dv_g
\]
\[
+ \left[ \frac{2}{n} Y(M, [g]) - 2C(n) \left( \int_M |W|^{\frac{n}{2}} \, dv_g \right)^{\frac{n}{2}} \right] \int_M R |W|^2 \, dv_g \leq 0
\]
which shows \( W_{ijkl} = 0 \) and \( M^4 \) is isometric to a quotient of the round sphere \( S^4 \).

We complete the proof of Theorem 1.3.

4. SOME PROOF FOR FOUR DIMENSIONAL MANIFOLDS

When \( n = 4 \), it has been prove by Catino (see [5, Lemma 4.1]) that
Lemma 4.1. Let \((M^4, g)\) be a closed manifold. Then
\[
Y^2(M, [g]) \geq \int_M (R^2 - 12|\tilde{R}_{ij}|^2) \, dv_g, \tag{4.1}
\]
with the inequality is strict unless \((M^4, g)\) is conformally Einstein.

Since (1.11) can be written as
\[
\int_M (|W|^2 + |\tilde{R}_{ij}|^2) \, dv_g < \frac{1}{48} Y^2(M, [g]). \tag{4.2}
\]
Using (1.1), it is easy to see
\[
\int_M (|W|^2 + |\tilde{R}_{ij}|^2) \, dv_g - \frac{1}{48} Y^2(M, [g]) \leq \int_M \left( |W|^2 + \frac{5}{4}|\tilde{R}_{ij}|^2 - \frac{1}{48} R^2 \right) \, dv_g. \tag{4.3}
\]
Moreover, (1.12) can be written as
\[
\int_M (|W|^2 + 4|\tilde{R}_{ij}|^2) \, dv_g < \frac{25}{486} Y^2(M, [g]), \tag{4.4}
\]
which combining with (1.12) gives
\[
\int_M (|W|^2 + 4|\tilde{R}_{ij}|^2) \, dv_g - \frac{25}{486} Y^2(M, [g]) \leq \int_M \left( |W|^2 + \frac{374}{81}|\tilde{R}_{ij}|^2 - \frac{25}{486} R^2 \right) \, dv_g. \tag{4.5}
\]
Hence, Corollary 1.4 follows from (4.3) immediately.

When \(n = 4\), the following Chern-Gauss-Bonnet formula (see [1, Equation 6.31])
\[
\int_M \left( |W|^2 - 2|\tilde{R}_{ij}|^2 + \frac{1}{6} R^2 \right) \, dv_g = 32\pi^2 \chi(M) \tag{4.6}
\]
is well-known. This shows
\[
\int_M |\tilde{R}_{ij}|^2 \, dv_g = \int_M \left( \frac{1}{2} |W|^2 + \frac{1}{12} R^2 \right) \, dv_g - 16\pi^2 \chi(M). \tag{4.7}
\]
Therefore, (1.15) and (1.16) follow from inserting (4.7) into (1.13) and (1.14), respectively. This proves Remark 1.2.

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