Augmentations, Fillings, and Clusters

Daping Weng

Michigan State University

November 2020

Joint work with Honghao Gao and Linhui Shen

arXiv:2008.10793, arXiv:2009.00499
Table of Contents

1. Legendrian Links and Exact Lagrangian Fillings
2. Chekanov-Eliashberg DGA and Augmentation Variety
3. Double Bott-Samelson Cells and Cluster Varieties
4. Admissible Fillings and Cluster Charts
Legendrian Links and Exact Lagrangian Fillings
Definition

Equip $\mathbb{R}^3_{xyz}$ with the standard contact 1-form $\alpha = dz - ydx$.

**Definition**

A *Legendrian link* is an embedded closed 1-dimensional submanifold $\Lambda \subset \mathbb{R}^3_{xyz}$ such that $\alpha|_{\Lambda} = 0$.

There are two useful projections when studying Legendrian links.

- **Front projection** $\pi_F : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xz}$; $y$ can be recovered by $y = \frac{dz}{dx}$.
- **Lagrangian projection** $\pi_L : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xy}$; $z$ can be recovered by $z = \int ydx$.

**Example**

A *Legendrian unknot*.

**Example**

For any positive braid $\beta \in Br_n^+$, its rainbow closure $\Lambda_{\beta}$ is naturally a Legendrian link.
A contact manifold can be viewed as the boundary of a symplectic manifold. The symplectic manifold $\mathbb{R}^4_{xyzt}$ with symplectic form $\omega = d(e^t \alpha)$ is a *symplectization* of the standard contact $\mathbb{R}^3_{xyz}$.

**Definition**

Let $\Lambda_\pm$ be two Legendrian links. An *exact Lagrangian cobordism* $L : \Lambda_- \to \Lambda_+$ is a Lagrangian surface $L \subset \mathbb{R}^4_{xyzt}$ such that

- $L$ is asymptotically $\Lambda_\pm$ at $\mathbb{R}^3_{xyz} \times (\pm \infty)$;
- the 1-form $e^t \alpha|_L = df$ for some function $f$ on $L$ that are asymptotically constant.

An exact Lagrangian cobordism $L : \emptyset \to \Lambda$ is also called an *exact Lagrangian filling*.
Distinguishing Exact Lagrangian Fillings

- By a result of Chantraine [Cha10], all exact Lagrangian fillings are topologically the same.
- We would like to distinguish exact Lagrangian fillings up to Hamiltonian isotopy.
- Another question: does there exist a Legendrian link with infinitely many non-Hamiltonian isotopic fillings?
- In 2020, different groups came up with examples with infinitely many fillings, answering the last question:
  - R. Casals and H. Gao [CG20] (January): any torus \((n, m)\)-link except \((2, m)\), \((3, 3)\), \((3, 4)\) and \((3, 5)\).
  - R. Casals and E. Zaslow [CZ20] (July): rainbow closures of an infinite family of 3-strand braids.
  - H. Gao, L. Shen and W. [GSW20] (September): rainbow closures of positive braids \(\beta\) other than those are associated with quivers of finite type.
  - R. Casals and L. Ng (upcoming): certain family of Legendrian links including some \((-1)\)-closures of positive braids that are not necessarily rainbow closures of positive braids.

- Among the four projects above, the first two are based on the theory of microlocal sheaves [STZ17, STWZ19], and the latter two are based on symplectic field theory [EHK16]. Surprisingly (or not surprisingly), all of the four projects have direct or indirect connection to cluster theory.
Quiver from Positive Braids

Consider $\Lambda_{(2,1,1,1,2,1,1,1,2,2)} \in \text{Br}_3^+$.

Below are what we call *standard ADE links*.

| $\text{Br}_2^+$ | $\text{Br}_3^+$ |
|-----------------|-----------------|
| $A_r$           | $D_r$           |
| $E_6$           | $E_7$           |
| $E_8$           | $s_1^{r+1}$     |
| $s_1^{r-2} s_2 s_1^2 s_2$ | $s_1^3 s_2 s_1^3 s_2$ |
| $s_1^4 s_2 s_1^3 s_2$ | $s_1^5 s_2 s_1^3 s_2$ |

Theorem (Gao-Shen-W)

*If the rainbow closure of a positive braid is not Legendrian isotopic to a split union of unknots and connect sums of standard ADE links, then it admits infinitely many non-Hamiltonian isotopic exact Lagrangian fillings.*
Chekanov-Eliashberg DGA and Augmentation Variety
Reeb Chords

Definition
Double points in a Lagrangian projection $\pi_L(\Lambda)$ are called Reeb chords. In other words, each Reeb chord corresponds to a pair of points in $\Lambda$ with the same $x$- and $y$-coordinates.

Example
Below are depictions of the unique Reeb chord of the Legendrian unknot example in its Lagrangian projection and front projection.
Definition of the Chekanov-Eliashberg dga

We follow [Che02, ENS02] for the construction of the CE dga. Let \( \Lambda \) be an \textbf{oriented} Legendrian link \textbf{each of whose components has rotation number 0}. We decorate \( \pi_L(\Lambda) \) with marked points \( t_i \) away from crossings, such that each link component of \( \Lambda \) contains at least (a lift of) one such marked point.

- As an algebra, the CE dga \( \mathcal{A}(\Lambda) \) is a \( \mathbb{Z}_2 \)-algebra freely generated by the Reeb chords in \( \pi_L(\Lambda) \) and formal variables \( t_i^{\pm 1} \) associated with the marked points.
- The degrees of the formal variables \( t_i^{\pm 1} \) are 0; the degrees of the Reeb chords are given by the Maslov potential.
- The differential is given by a counting of pseudo-holomorphic disks in \( \pi_L(\Lambda) \).

\[
\partial b = \sum_{c_1, \ldots, c_n} \sum_{u \in \mathcal{M}(b; c_1, \ldots, c_n)} w(u)
\]

\[
w(u) = c_1 t_1^{-1} c_2 c_3 t_2 c_4 c_5
\]
Example

Consider the following Legendrian trefoil $\Lambda$.

- As an algebra, $A(\Lambda)$ is generated freely over $\mathbb{Z}_2$ by $t_i^\pm$, $b_i$, and $a_i$.
- The grading on the generators are
  \[ |t_i^\pm| = |b_i| = 0, \quad |a_i| = 1. \]
- Note that $A(\Lambda)$ is concentrated in non-negative degrees, so automatically
  \[ \partial b_i = \partial t_i^\pm = 0. \]

For $\partial a_1$ and $\partial a_2$:

\[ \partial a_1 = t_1^{-1} + b_1 + b_3 + b_1 b_2 b_3, \]
\[ \partial a_2 = t_2^{-1} + b_2 + t_1 + b_2 b_3 t_1 + t_1 b_1 b_2 + b_2 b_3 t_1 b_1 b_2. \]
Augmentation Variety

**Definition**

Let $\Lambda$ be a Legendrian link and let $\mathcal{A}(\Lambda)$ be its CE-dga. Let $\mathbb{F}$ be an algebraically closed field of characteristic 2. An *augmentation* is a dga homomorphism $\epsilon : \mathcal{A}(\Lambda) \to \mathbb{F}$ where $\mathbb{F}$ is concentrated at degree 0 and equipped with the trivial differential. The *augmentation variety* $\text{Aug}(\Lambda)$ is defined to be the moduli space of augmentations of $\mathcal{A}(\Lambda)$.

- Note that an augmentation $\epsilon$ basically assigns $\mathbb{F}$-values to the degree 0 generators of $\mathcal{A}(\Lambda)$ subject to the conditions $\partial a = 0$ for degree 1 Reeb chords $a$. Therefore augmentation varieties are always affine varieties.
- If $\mathcal{A}(\Lambda)$ is concentrated in non-negative degrees, then

$$\text{Aug}(\Lambda) \cong \text{Spec } H_0 (\mathcal{A}^c(\Lambda)),$$

where $c$ stands for the commutatization of $\mathcal{A}(\Lambda)$.

**Example**

In our last example, we have $\partial a_1 = t_1^{-1} + b_1 + b_3 + b_1 b_2 b_3$ and $\partial a_2 = t_2^{-1} + b_2 + b_1 + b_2 b_3 t_1 + t_1 b_1 b_2 + b_2 b_3 t_1 b_1 b_2$. Setting these two equations to 0 cuts out a 3-dimensional affine variety in $\mathbb{F}^3_{b_1, b_2, b_3} \times (\mathbb{F}^\times)^2_{t_1, t_2}$. 
Augmentation Varieties of Positive Braid Closures

For a positive braid $\beta \in \text{Br}_n^+$, $\Lambda_\beta$ has $n + l$ number of Reeb chords in total. There are $l$ Reeb chords coming from the crossings of $\beta$, which we denote by $b_k$ ($1 \leq k \leq l$), and $n$ Reeb chords coming from the right cusps, which we denote by $a_i$. Their gradings are $|b_k| = 0$ and $|a_i| = 1$.

For the crossing $i_k$ between the $i_k$th and $i_k + 1$st strands, we associate an $n \times n$ matrix $Z_{i_k}(b_k)$ of the form

$$
Z_{i_k}(b_k) = \begin{pmatrix}
1 & & & & \\
& \ddots & & & \\
& & b_k & 1 & \\
& & 1 & 0 & \\
& & & & \ddots \\
& & & & 1
\end{pmatrix}
$$
Augmentation Varieties of Positive Braid Closures

Let $\beta = (i_1, \ldots, i_l)$ be a positive braid in $\mathcal{B}_n^+$. Define $M := Z_{i_1}(b_1) Z_{i_2}(b_2) \cdots Z_{i_l}(b_l)$.

Theorem (Gao-Shen-W)

The homology $H_0(\mathcal{A}(\Lambda_\beta))$ is generated by $b_1, \cdots, b_l, t_1^{\pm 1}, \cdots, t_n^{\pm 1}$, modulo the relations $t_k^{-1} = M_k$ for $1 \leq k \leq n$, where $M_k$ is the Gelfand-Retakh quasi-determinant [GR91] of the upper-left $k \times k$ submatrix of $M$ with respect to the $(k, k)$-entry.

Corollary (Gao-Shen-W.)

Let $M^c$ be the matrix obtained by abelianizing entries of the matrix $M$. Then the augmentation variety $\text{Aug}(\Lambda_\beta)$ is isomorphic to the non-vanishing locus of $\prod_{m=1}^n \Delta_m(M^c)$ in $\mathbb{F}_{b_1, \ldots, b_l}$, where $\Delta_m$ denotes the determinant of the $m \times m$ submatrix at the upper left corner.

We recognize that the non-vanishing locus stated above is isomorphic to a double Bott-Samelson cell, which is known to be a cluster variety.
Double Bott-Samelson Cells and Cluster Varieties
Double Bott-Samelson Cells

Double Bott-Samelson cells were introduced in a joint work with L. Shen [SW19]. Let $G$ be a Kac-Peterson group with a pair of opposite Borel subgroups $B_\pm$. Denote $x B_\pm \xrightarrow{w} y B_\pm$ if $x^{-1} y \in B_\pm w B_\pm$ and denote $xB_- \xrightarrow{W} yB_+$ if $x^{-1} y \in B_- w B_+$.

$$ P_{i_1} \times P_{i_2} \times \cdots \times P_{i_l} = \bigsqcup_{j \leq i} (B_+ s_{j_1} B_+) \times (B_+ s_{j_2} B_+) \times \cdots \times (B_+ s_{j_m} B_+) $$

\[ [x_1, \ldots, x_m] \in (B_+ s_{j_1} B_+) \times \cdots \times (B_+ s_{j_m} B_+) \] gives rise to a unique sequence

$$ B_+ \xrightarrow{s_{j_1}} x_1 B_+ \xrightarrow{s_{j_2}} x_1 x_2 B_+ \xrightarrow{s_{j_3}} \cdots \xrightarrow{s_{j_m}} x_1 \cdots x_m B_+ . $$

Denote left cosets of $B_+$ as $B^i$ and denote left cosets of $B_-$ as $B^j$. Let $U_\pm := [B_\pm, B_\pm]$. Denote left cosets of $U_+$ as $A^i$ and denote left cosets of $U_-$ as $A^j$.

**Definition (Shen-W.)**

Let $(\beta, \gamma)$ be a pair of positive braids associated to $G$ and let $i = (i_1, \ldots, i_l)$ and $j = (j_1, \ldots, j_m)$ be words of $\beta$ and $\gamma$. The decorated double Bott-Samelson cell is

$$ \text{Conf}^\gamma_\beta (C) = \left\{ A_0 \xrightarrow{s_{j_1}} B_1 \xrightarrow{s_{j_2}} \cdots \xrightarrow{s_{j_m}} B_m \right\} \bigsqcup \left\{ B^0 \xrightarrow{s_{i_1}} B^1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_l}} B^l \right\} \bigsqcup G. $$
For a point in $\text{Conf}^e_\beta(C)$, we can use the $G$-action to move the configuration such that the left vertical edge is $U_- \overset{U_-}{\longrightarrow} B_+$. There are $\mathbb{A}^1$-many flags that are $s_{i_1}$ away from $B_+$, and they can be parametrized as $z_{i_1}(q_1)B_+$, where $z_i(q) = \varphi_i \begin{pmatrix} q & -1 \\ 1 & 0 \end{pmatrix}$.

In order for $U_- \overset{U_-}{\longrightarrow} z_{i_1}(q_1) \cdots z_{i_l}(q_l)B_+$ to hold, we need all principal minors of $z_{i_1}(q_1) \cdots z_{i_l}(q_l)$ to be non-zero.

**Theorem (Gao-Shen-W.)**

For $G = \text{SL}_n$, the map $\gamma^*(q_i) = b_i$ defines a canonical isomorphism $\gamma : \text{Aug} \left( \Lambda_\beta \right) \xrightarrow{\cong} \text{Conf}^e_\beta(C)$ (over characteristic 2). Moreover, this canonical isomorphism does not depend on the choice of word $i$ for $\beta$.

We proved in [SW19] that double Bott-Samelson cells, both decorated and undecorated, are cluster varieties. By a pull-back through $\gamma$, we obtain a cluster variety structure on $\text{Aug} \left( \Lambda_\beta \right)$. 
Cluster theory began with the invention of cluster algebras by Fomin and Zelevinsky [FZ02]. Fock and Goncharov introduced cluster varieties [FG09] as geometric counterparts of cluster algebras.

There are two types of cluster varieties, one is called $K_2$ or $\mathcal{A}$, and the other one is called Poisson or $\mathcal{X}$. We will focus on the $\mathcal{A}$ type today.

A cluster variety is an affine variety together with an atlas (up to codimension 2) of open torus charts called cluster charts.

Each torus chart $\alpha$ is equipped with a set of cluster coordinates $(A_i; \alpha)$ and a quiver $Q_\alpha$. The cluster coordinates are indexed by the vertex set of the quiver $Q_\alpha$.

Charts are glued via a process called cluster mutation. For any (unfrozen) vertex $k$ of $Q_\alpha$, the cluster mutation $\mu_k$ relates $\alpha$ to another cluster chart $\alpha'$, and their cluster coordinates are related by

$$A_{i; \alpha'} = \begin{cases} \frac{1}{A_{k; \alpha}} \left( \prod_{\substack{j \to k \ \text{in} \ Q_\alpha}} A_{j; \alpha} + \prod_{\substack{k \to j \ \text{in} \ Q_\alpha}} A_{j; \alpha} \right) & \text{if } i = k, \\ A_{i; \alpha} & \text{otherwise.} \end{cases}$$

On the other hand, their quivers $Q_{\alpha'}$ and $Q_\alpha$ are related by a quiver mutation at the quiver vertex $k$. 

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \quad \bullet \\
\end{array}
\]
Admissible Fillings and Cluster Charts
Admissible Fillings

- Ekholm, Honda, and Kálmán [EHK16] constructed a contravariant functor $\Phi^*$ from the cobordism category of Legendrian links in the symplectization $\mathbb{R}^4_{xyzt}$ to the category of dga’s, mapping $\Lambda$ to $\mathcal{A}(\Lambda)$.
- They further proved that if two exact Lagrangian cobordisms $L, L' : \Lambda_- \to \Lambda_+$ are Hamiltonian isotopic, then the induced dga homomorphisms $\Phi^*_L$ and $\Phi^*_L'$ are chain-homotopic, which then implies that they induce equal homomorphisms between the 0th homologies
  \[ \Phi^*_L = \Phi^*_L' : H_0(\mathcal{A}(\Lambda_+)) \to H_0(\mathcal{A}(\Lambda_-)). \]
- Consequently, if the dga $\mathcal{A}(\Lambda_{\pm})$ are concentrated in non-negative degrees, then Hamiltonian isotopic exact Lagrangian cobordisms induce an equal augmentation variety morphisms
  \[ \Phi_L = \Phi_{L'} : \text{Aug}(\Lambda_-) \to \text{Aug}(\Lambda_+). \]
- In this project, we focus on a special family of exact Lagrangian fillings which we call *admissible fillings*. They are compositions of the following:
  - saddle cobordisms;
  - cyclic rotations;
  - braid moves;
  - filling an unknot.

We choose these as building blocks because we can compute their functorial morphisms explicitly by based on methods developed by Eckholm-Honda-Kálmán [EHK16] and Sivek [Siv11].
Theorem (Gao-Shen-W.)

If $L$ is an admissible filling of $\Lambda_\beta$, then $\Phi_L : \text{Aug} (\emptyset, P) \to \text{Aug} (\Lambda_\beta)$ is an open embedding of an algebraic torus, and the image of $\Phi_L$ coincides with a cluster chart of $\text{Aug} (\Lambda_\beta)$. Moreover, if two admissible fillings $L$ and $L'$ correspond to two distinct cluster seeds, then $L$ and $L'$ are not Hamiltonian isotopic.
Aug (∅, P) ≃ (ℝ×)³_p₁,p₂,p₃ → Aug (∧(1,1,1)) ⊂ ℝ³_b₁,b₂,b₃.
The Legendrian trefoil augmentation variety $\text{Aug}(\Lambda_{(1,1,1)})$ is cut out from $\mathbb{F}_{b_1,b_2,b_3}^3 \times \mathbb{F}_{t_1}^\times$ by the equation $t_1^{-1} + b_1 + b_3 + b_1 b_2 b_3 = 0$. It has five clusters.

We have a calculator to compute $\Phi_L$ and the corresponding cluster for any admissible filling $L$ of the rainbow closure of a positive braid.
Full Cyclic Rotation and Cluster DT Transformation

Definition
We define the \textit{full cyclic rotation} \( R : \Lambda_\beta \to \Lambda_\beta \) to be the exact Lagrangian cobordism that rotates the crossings of \( \beta \) one-by-one from right to left.

Theorem (Gao-Shen-W.)
The functorial isomorphism \( \Phi_R : \text{Aug}(\Lambda_\beta) \to \text{Aug}(\Lambda_\beta) \) equals \( DT^2 \), where \( DT \) is the cluster Donaldson-Thomas transformation on \( \text{Aug}(\Lambda_\beta) \).

The cluster DT transformation is a biregular automorphism on a cluster variety and it permutes the cluster charts on a cluster variety. It is known that if \( Q \) is an acyclic quiver of infinite type, then this permutation is of infinite order [LLMSS20].

Corollary (Gao-Shen-W.)
Suppose \( \text{Aug}(\Lambda_\beta) \) has a quiver that is acyclic and of infinite type. Then for any admissible filling \( L \) of \( \Lambda_\beta \), the set \( \{R^k \circ L \mid k \geq 0\} \) is an infinite family of non-Hamiltonian isotopic exact Lagrangian fillings.

\[
\begin{array}{c}
\downarrow \quad t \\
L \quad \quad R \quad \quad R \quad \quad R \quad \quad \cdots
\end{array}
\]
For a general positive braid $\beta$, we employ an exhaustive argument in [GSW20] to show that if $\Lambda_\beta$ is not Legendrian isotopic to a split union of unknots and connect sums of standard ADE links, then there exists a positive braid $\gamma$ with an acyclic quiver of infinite type and with an admissible cobordism $K : \Lambda_\gamma \to \Lambda_\beta$.

We also prove that the functorial morphism $\Phi_K : \text{Aug}(\Lambda_\gamma) \to \text{Aug}(\Lambda_\beta)$ is an open embedding that preserves the cluster structure.

Therefore, by composing $K$ with the infinitely many non-Hamiltonian isotopic admissible fillings of $\Lambda_\gamma$ we get infinitely many non-Hamiltonian isotopic admissible fillings of the form $K \circ R^k \circ L$ for $\Lambda_\beta$ as well.
There is a characteristic 0 version of CE dga’s and augmentation varieties. When $\Lambda_\beta$ is decorated with only one marked point per link component, the characteristic 0 augmentation variety $\text{Aug}(\Lambda_\beta)$ is a symplectic variety.

There is a Deohdar stratification on double Bott-Samelson cells [SW19], and there is a stratification on augmentation variety by normal rulings [HR14]. These two stratifications coincide under the canonical isomorphism $\gamma$.

For any Legendrian link $\Lambda$ one can also define an augmentation stack $\text{Aug}_{st}(\Lambda)$ [NRSSZ15], and the augmentation stack possesses an unfrozen cluster $\mathcal{X}$ structure [STWZ19, SW19]. For any positive braid $\beta$, the projection map $\text{Aug}(\Lambda_\beta) \to \text{Aug}_{st}(\Lambda_\beta)$ coincides with the cluster theoretical map $p : \mathcal{A} \to \mathcal{X}^{uf}$. 
Thank You!
Roger Casals and Honghao Gao.
Infinitely many Lagrangian fillings.
Preprint, 2020.
arXiv:2001.01334.

Baptiste Chantraine.
Lagrangian concordance of Legendrian knots.
*Algebr. Geom. Topol.*, 10(1):63–85, 2010.
arXiv:math/0611848, doi:10.2140/agt.2010.10.63.

Yuri Chekanov.
Differential algebra of Legendrian links.
*Invent. Math.*, 150(3):441–483, 2002.
arXiv:math/9709233, doi:10.1007/s002220200212.

Roger Casals and Eric Zaslow.
Legendrian weaves: N-graph calculus, flag moduli and applications, 2020.
arXiv:2007.04943.

Tobias Ekholm, Ko Honda, and Tamás Kálmán.
Legendrian knots and exact Lagrangian cobordisms.
*J. Eur. Math. Soc. (JEMS)*, 18(11):2627–2689, 2016.
arXiv:1212.1519, doi:10.4171/JEMS/650.

John Etnyre, Lenhard Ng, and Joshua Sabloff.
Invariants of Legendrian knots and coherent orientations.
*J. Symplectic Geom.*, 1(2):321–367, 2002.
URL: http://projecteuclid.org/euclid.jsg/1092316653, arXiv:math/0101145.

Vladimir Fock and Alexander Goncharov.
Cluster ensembles, quantization and the dilogarithm.
*Ann. Sci. Éc. Norm. Supér. (4)*, 42(6):865–930, 2009.
arXiv:math/0311245, doi:10.24033/asens.2112.
Sergey Fomin and Andrei Zelevinsky.
Cluster algebras. I. Foundations.
*J. Amer. Math. Soc.*, 15(2):497–529, 2002.
arXiv:math/0104151, doi:10.1090/S0894-0347-01-00385-X.

I. M. Gelfand and V. S. Retakh.
Determinants of matrices over noncommutative rings.
*Funktsional. Anal. i Prilozhen.*, 25(2):13–25, 96, 1991.
doi:10.1007/BF01079588.

Honghao Gao, Linhui Shen, and Daping Weng.
Positive braid links with infinitely many fillings.
Preprint, 2020.
arXiv:2009.00499.

Michael B. Henry and Dan Rutherford.
Ruling polynomials and augmentations over finite fields.
*Journal of Topology*, 8(1):1–37, 07 2014.
arXiv:1308.4662.
arXiv:https://academic.oup.com/jtopol/article-pdf/8/1/1/5096021/jtu013.pdf, doi:10.1112/jtopol/jtu013.

Kyungyong Lee, Li Li, Matthew Mills, Ralf Schiffler, and Alexandra Seceleanu.
Frieze varieties: a characterization of the finite-tame-wild trichotomy for acyclic quivers.
*Adv. Math.*, 367:107130, 33, 2020.
arXiv:1803.08459, doi:10.1016/j.aim.2020.107130.

Lenhard Ng.
Computable Legendrian invariants.
*Topology*, 42(1):55–82, 2003.
arXiv:math/0011265, doi:10.1016/S0040-9383(02)00010-1.

Lenhard Ng, Dan Rutherford, Vivek Shende, Steven Sivek, and Eric Zaslow.
Augmentations are sheaves.
Preprint, 2015.
arXiv:1502.04939.
Steven Sivek.
A bordered Chekanov-Eliashberg algebra.
*J. Topol.*, 4(1):73–104, 2011.
arXiv:1004.4929, doi:10.1112/jtopol/jtq035.

Vivek Shende, David Treumann, Harold Williams, and Eric Zaslow.
Cluster varieties from Legendrian knots.
*Duke Math. J.*, 168(15):2801–2871, 2019.
arXiv:1512.08942, doi:10.1215/00127094-2019-0027.

Vivek Shende, David Treumann, and Eric Zaslow.
Legendrian knots and constructible sheaves.
*Invent. Math.*, 207(3):1031–1133, 2017.
arXiv:1402.0490, doi:10.1007/s00222-016-0681-5.

Linhui Shen and Daping Weng.
Cluster structures on double Bott-Samelson cells.
Preprint, 2019.
arXiv:1904.07992.