A Theory of Stable Market Segmentations

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Abstract

We study monopolistic market segmentation as the outcome of a cooperative game, and introduce stable segmentations as a solution concept. For any alternative segmentation, a stable segmentation contains a consumer segment that prefers the original segmentation to the alternative one. We characterize stable segmentations as efficient and saturated, which means that shifting consumers from a segment with a higher price to a segment with a lower price leads the seller to optimally increase the lower price. We constructively show that stable segmentations exist, relate stability to existing cooperative solution concepts, and show that stable segmentations satisfy many of these concepts.

1 Introduction

Sellers in a variety of markets use consumer data to conduct market segmentation and price discrimination. Various factors affect the data sellers can access and how finely they can segment the market. These factors include consumers’ data disclosure decisions and other aspects of consumer behavior on the one hand, and whether a seller can approach consumers directly or through third parties (such as an employer or a worker’s union) on the other hand. More recently, the possibility of joining online platforms or forming data cooperatives has increased consumers’ ability to control their

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data, coordinate their disclosure decisions with other consumers, and form blocs that interact with sellers.

We are interested in understanding which market segmentations can arise when consumers can form market segments in their interaction with a seller by, for example, coordinating their data disclosure decisions or creating consumer blocs.

We consider a market with a monopolistic seller of a single product and a continuum of consumers with unit demand, each characterized by their positive value for the product, which belongs to a finite set of possible values. A segmentation of the market is a partition of the consumers into segments, where each segment is a set of consumers and a price for these consumers that maximizes the seller’s profit, that is, the monopoly price given the distribution of consumer values in the segment.

If the seller has access to detailed consumer data and can unilaterally determine the segmentation, he will segment the market based on consumers’ value for the product and implement first-degree price discrimination. If instead consumers can determine and commit to a market segmentation before they learn their value for the product, so the segmentation assigns consumers to segments based on their realized value, they will rank the possible segmentations by the expected consumer surplus they generate and choose a segmentation that generates the highest expected consumer surplus. Such segmentations were identified by [Bergemann, Brooks, and Morris (2015)]. But which segmentations will arise if consumers know their value for the product at the outset, so that different consumers may prefer different market segmentations?

To investigate this question, we study the interaction among consumers and the seller in reduced form by focusing on properties of the resulting segmentation. Our main conceptual contribution is a notion of stability that corresponds to a segmentation being immune to deviations to other segmentations. The idea is that once a segmentation forms, any segment of consumers can object to transitioning to another segmentation, which prevents the latter segmentation from forming.

More precisely, a segmentation $S$ is stable if for any segmentation $S'$ that is not equivalent to $S$, that is, in which some consumers face a price different from the price they face in $S$, there is some segment in $S$ that objects to $S'$, that is, all consumers in the segment weakly prefer $S$ to $S'$ and some consumers in the segment strictly prefer $S$ to $S'$. This notion of stability captures a kind of “coalitional individual rationality (IR):” once a segment forms, its members cannot be regrouped into a different segment or segments if they all oppose this change.
As an illustration, consider a market that consists of the faculty and students of two universities, A and B. Consider a “university-based segmentation” of the market, in which each university forms a market segment. The seller sets two potentially different prices for the product, $p_A$ and $p_B$, such that the faculty and students in university $i$ face price $p_i$. Price $p_i$ maximizes the seller’s profit given the distribution of values for the product among the individuals in university $i$. In this university-based segmentation, each individual can only buy the product for the price offered at their university, and cannot unilaterally decide to join the other segment. A group of individuals, however, might be able to deviate from the segmentation by forming a new segment.

One possible deviation from this university-based segmentation is the formation of a “faculty” segment. This segment, if it forms, will consist of the faculty from both universities, and its members will face price $p_F$ that maximizes the seller’s profit given the distribution of faculty values. The thrust of our stability notion is that if the individuals in some university $i$, which includes both faculty and students, all oppose the formation of the faculty segment, then the new segment will not form. To determine whether this is the case, we must describe how the students are rearranged if the faculty segment forms. One possibility is that two new segments form, one for the students in each university. Another possibility is that the students in both universities form a single “student” segment. Our notion of stability states that the university-based segmentation is stable if for both possibilities (as well as any other segmentation) all the individuals in one of the universities object to the new segmentation.

Formally, we model the interaction between consumers and the seller as a cooperative game with non-transferable utility (NTU). The set of players is the set of consumers. The set of feasible utility vectors associated with a measurable subset $C$ of players (a coalition of consumers) consists of the profiles of payoffs of consumers in $C$ across all segmentations of $C$ (that is, when $C$ is taken to be the set of consumers partitioned into segments). The core of this game consists of utility vectors corresponding to segmentations to which no segment objects. Thus, for any segmentation we can think of the core as “prioritizing” segments that object to the segmentation. In con-

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1That is, the seller can only distinguish between individuals based on their university, perhaps by asking them to present their university ID or by selling to each university through its representatives, and the product cannot easily be resold.

2There are, of course, other possible deviations, including a segment that consists of the faculty from only one of the universities.

3In particular, indifferent faculty in university $i$ show solidarity with their fellow faculty and students in university $i$ whose preference is strict and oppose the change.
contrast, our notion of stability prioritizes the prevailing segmentation, in the sense that it suffices that the segments that form the prevailing segmentation object to any other segmentation. As we discuss below, the core is often empty but stable segmentations always exist, and when the core is not empty it essentially coincides with the set of stable segmentations.

Our main result characterizes stable segmentations. The characterization shows that a segmentation is stable if and only if it is efficient and saturated. Efficiency means that every consumer buys the product, and saturation means that consumers in each segment are not willing to accept additional consumers from segments with higher prices because doing so will increase the price in their own segment. We also show that stable segmentations are Pareto undominated, that is, there is no other segmentation that makes all consumers better off.

One interpretation of our characterization is that stability is the “right” notion for capturing coalitional IR. Indeed, a stable segmentation \( S \) respects coalitional IR in that for every non-equivalent segmentation \( S' \) it contains an objecting segment, and this objecting segment prevents \( S' \) from forming because the consumers in the segment cannot be forced to regroup to form \( S' \). But this requirement may appear too strong, since segmentation \( S' \) may not contain any segment that objects to \( S \) (so \( S' \) is not an attractive alternative to \( S \)), and even if it does, regrouping consumers to form \( S' \) may be difficult if \( S' \) differs substantially from \( S \). Our characterization clarifies these issues. First, the characterization implies that for a segmentation \( S \) to be stable it suffices that for every segmentation \( S' \) that contains a segment that objects to \( S \), segmentation \( S \) contains a segment that objects to \( S' \). Second, if a segmentation \( S \) is not stable, then the characterization implies that the segmentation is either not efficient or not saturated. In both cases it is easy to construct a segmentation \( S' \) that is a relatively minor modification of \( S \), is intuitively easy to obtain from \( S \), that contains a segment that objects to \( S \), and such that no segment in \( S \) objects to \( S' \).

The characterization also helps address the following question. Suppose that consumers can choose a segmentation before they learn their value for the product but cannot commit to the segmentation, so that some consumers may want to deviate to another segmentation after they learn their value. Which segmentation would consumers

\[\text{efficient}\]
choose initially? We formalize this as asking which stable segmentations maximize expected consumer surplus among all stable segmentations. We answer this question by constructing a stable segmentation that maximizes average consumer surplus among all segmentations (stable or not). This stable segmentation is the *maximal equal-revenue segmentation*, identified by Bergemann, Brooks, and Morris (2015). Our result that the maximal equal-revenue segmentation is stable shows that stability does not reduce the surplus consumers can achieve even in the absence of commitment or a central planner that enforces the segmentation. The result also implies that stable segmentations always exist. We then show that multiple stable segmentations may exist and that maximizing average consumer surplus neither implies nor is implied by stability.

In contrast to the existence and possible multiplicity of stable segmentations, the core of the game may be empty. We show that the core is non-empty if and only if it is profit-maximizing for the seller to set the efficient price (equal to the lowest consumer value) in the unsegmented market. In this case, the utility vector corresponding to the unsegmented market is essentially the unique element in the core. Moreover, any stable segmentation generates the same utility vector.

Beyond the core, our notion of stability refines several solution concepts for NTU games applied to our setting. We show that a stable segmentation together with its equivalent segmentations form a Morgenstern and Von Neumann (1953) stable set, a Harsanyi (1974) farsighted stable set, a Ray and Vohra (2015) farsighted stable set, and a Ray and Vohra (2019) maximal farsighted stable set. We point out that stable sets and farsighted stable sets do not always exist in NTU games. And when they do exist they may necessarily includes multiple, non-equivalent utility vectors or coalitions of players. Our results show that in our market game, “singleton” (up to equivalent segmentations) stable and farsighted stable sets always exist. In particular, for any deviation from a stable segmentation $S$ to a non-equivalent segmentation $S'$, there exists a path of “credible” and “maximal” (in the sense of Ray and Vohra, 2019) segmentations that leads back from $S'$ to $S$. This provides another justification for our notion of stability.

Our analysis of stable segmentations may be relevant to policy discussions regarding monopolies, price discrimination, data sharing, and data intermediaries. Monopolies

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5 Bergemann, Brooks, and Morris (2015) characterized the set of average consumer-producer surplus pairs achievable across all segmentations, and did not consider stability or the question of which segmentations will arise.

6 Our notion of “coalitional IR” is closely related to the notion of “coalitional sovereignty” in Ray and Vohra (2015) applied to our setting.
lead to inefficiencies, and these inefficiencies may be reduced with regulation or increased competition. Market segmentation arising from the monopolist’s access to consumer data can also reduce inefficiency, but may harm consumers, as first-degree price discrimination demonstrates. Our results show that as long as consumers have enough control over their data to achieve “coalitional IR,” market segmentation leads to efficiency and Pareto undominated outcomes for consumers. If consumers can choose a segmentation before they know their value, they can achieve the highest possible expected surplus without the help of a social planner even if they cannot commit to maintaining the segmentation after they learn their value. This indicates that policies or information intermediaries, such as data cooperatives, that support coalitional IR while allowing the seller to use consumer data to price discriminate may offer an alternative or complimentary tool to addressing the inefficiencies associated with monopolistic markets.

The rest of the paper is organized as follows. Section 1.1 describes the relationship of our work to the existing literature. Section 2 describes the model. Section 3 introduces our notion of stability and the main results. Section 4 casts our model as a cooperative game and relates our notion of stability to cooperative solution concepts. Section 5 concludes.

1.1 Related literature

Peivandi and Vohra (2021) consider stability of centralized markets against deviations by coalitions of agents. They show that fragmentation of such markets is unavoidable, despite its efficiency costs, except in special circumstances. They study a bilateral trade setting, whereas we study a setting with a population of consumers and a seller. When centralized markets are fragmented, Peivandi and Vohra (2021) do not predict what the resulting segmentation looks like, whereas characterizing stable segmentations is a main focus of our paper. Another important difference is that in their setting a coalition chooses the trading mechanism, whereas in our setting each coalition of consumers faces a profit-maximizing price set by the seller.

A recent literature on third-degree price discrimination studies consumer and producer surplus across all possible segmentations of a given market. Bergemann, Brooks, and Morris (2015) identify the set of average producer and consumer surplus pairs that result from all segmentations of a given market. Their results also identify segmentations that maximize average consumer surplus. Cummings et al. (2020) study an
extension in which only certain segmentations may be chosen. Glode, Opp, and Zhang (2018) study optimal disclosure by an informed agent in a bilateral trade setting, and show that the optimal disclosure policy leads to socially efficient trade, even though information is revealed only partially. Haghpanah and Siegel (2022b) study when a market served by a multi-product seller can be segmented in a way that is Pareto improving. Yang (2022) studies how a profit-maximizing data broker sells market segmentations to a monopolist. Ichihashi (2020), Hidir and Vellodi (2021), Braghieri (2017), and Haghpanah and Siegel (2022a) consider maximum average consumer surplus when a multi-product seller offers different products in each market segment. These papers can be seen as identifying segmentations that are chosen \textit{ex ante} by a consumer who does not know her type, because such a consumer chooses the segmentation that maximizes her expected payoff. In contrast, in this paper we study market segmentation when consumers know their type.

The related papers that study disclosure decisions by consumers who know their type model these interactions as non-cooperative games. Ali, Lewis, and Vasserman (2023) consider voluntary disclosure of data by a single consumer, and analyze the welfare implications of various disclosure policies in both a monopolistic and a competitive environment. Sher and Vohra (2015) study a disclosure setting in which the seller can commit to the mechanism that he will use after receiving information. In our setting, the seller cannot commit and chooses a profit-maximizing price for each segment. Acemoglu et al. (2019), Bergemann, Bonatti, and Gan (2022), Baumann and Dutta (2022), and Galperti and Perego (2023) also study the consequences of consumers’ disclosure decisions on prices and other market outcomes. Kuang et al. (2022) study the formation of market segmentations as the equilibrium of a non-cooperative game in which each consumer chooses which segment to join. In their setting consumers can unilaterally move between segments, whereas in our setting, which focuses on coalitional IR, once a segmentation arises consumers cannot join an existing segment if all consumers in that segment oppose this change. These differences lead to different results: in the setting of Kuang et al. (2022), the seller’s surplus in any equilibrium segmentation is equal to the seller’s surplus in the unsegmented market and consumers are better off, but the outcome need not be efficient. In contrast, our stable segmentations are Pareto undominated, efficient, and increase the seller’s surplus.
2 Model

A monopolistic seller faces a unit mass of consumers uniformly distributed on the unit interval \([0, 1]\). Consumers have unit demand for the monopolist’s product. The value of the product for consumer \(c \in [0, 1]\) is \(v(c) \in V = \{v_1, \ldots, v_n\} \subseteq R_{>0}\), where \(v\) is a measurable function and \(v_i\) increases in \(i\).\(^7\) Let \(\mu\) be the Borel measure on the unit interval. The measure of consumers with value \(v_i\) is \(f(v_i) = \mu(\{c : c \in [0, 1], v(c) = v_i\})\). We assume without loss of generality that \(f(v_i) > 0\) for every \(i \in \{1, \ldots, n\}\), and normalize the seller’s production cost to zero.

A coalition is a measurable subset \(C \subseteq [0, 1]\) of consumers. Let \(f^C(v_i) = \mu(\{c : c \in C, v(c) = v_i\})\) denote the measure of consumers with value \(v_i\) in coalition \(C\). We say that consumers with value \(v_i\) are in \(C\) (or that \(C\) contains consumers with value \(v_i\)) if \(f^C(v_i) > 0\). A price \(p \in V\) is optimal for coalition \(C\) if it maximizes the revenue from selling the product to consumers in \(C\), that is, for any other price \(p' \in V\),

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    p \sum_{i : v_i \geq p} f^C(v_i) \geq p' \sum_{i : v_i \geq p'} f^C(v_i).
\]

We restrict attention to prices in \(V\) because for any other price there exists a price in \(V\) with a weakly higher revenue.

A segment is a pair \((C, p)\), where \(C\) is a coalition and \(p\) is an optimal price for that coalition.\(^8\) A segmentation \(S\) is a finite set of segments \(\{(C_j, p_j)\}_{j=1}^{k}\) such that \(C_1, \ldots, C_k\) partitions the set of consumers \([0, 1]\). That is, a segmentation partitions the set of consumers into coalitions, and assigns an optimal price for each coalition.

Denote by \(CS(c, p) = \max\{v(c) - p, 0\}\) the surplus of consumer \(c\) who is offered the product at price \(p\). If consumer \(c\) belongs to segment \((C, p)\), then her surplus is \(CS(c, p)\). Given a segmentation \(S\), denote by \(p_S(c)\) the price in the unique segment that includes consumer \(c\). Let \(CS(c, S) = CS(c, p_S(c))\) denote the surplus of consumer \(c\) in segmentation \(S\).

Consumers’ preferences over segmentations may differ because the prices different consumers face vary within and across segmentations. Consumers’ data disclosure and other decisions, along with their interaction with the seller, which we do not explicitly model, determine the resulting segmentation. Data cooperatives or other

\(^7\)Appendix II discusses variants of our model with a continuum of possible values.

\(^8\)That is, the seller cannot price discriminate between the different consumers in \(C\), and sets a price for these consumers that maximizes his revenue.
online platforms may facilitate coordination among consumers, and consumers may also interact with the seller through an employer or a worker’s union. We describe this process in reduced form as a cooperative game with non-transferable utility (NTU). For each coalition $C$ of consumers, the set of utility vectors feasible for $C$ comprises the payoff profiles of the consumers in $C$ across all segmentations of $C$ (when $C$, instead of $[0, 1]$, is taken to be the set of consumers).

The next section introduces our solution concept, stability. Section 4 compares stability to the core and other existing solution concepts for cooperative games. We show that the core is empty except in special cases, and that in those special cases the core and stability coincide. We also show that stability refines the other solution concepts.

3 Stable Segmentations

We develop a notion of stable segmentations and show that such segmentations always exist. To this end we first formalize what it means for a segment to object to a segmentation. For the definition, it is helpful to think of $(C, p)$ as an existing segment and segmentation $S'$ as a proposed deviation.

**Definition 1 (Objection)** A segment $(C, p)$ objects to a segmentation $S'$ if $CS(c, p) \geq CS(c, S')$ for all consumers $c$ in $C$, with a strict inequality for a positive measure of consumers $c$ in $C$

In other words, a segment $(C, p)$ objects to a segmentation $S'$ if all the consumers in $C$ are weakly worse off in $S'$ and some consumers in $C$ are strictly worse off. In particular, $(C, p)$ is not in $S'$. Accommodating a weak preference for some consumers in this definition captures indifferent consumers’ willingness to passively go along with the preferences of the other members of their coalition if all those members strictly prefer to remain in their original segment $(C, p)$. Notice that the definition would be vacuous if we required the preference to be strict for all (or almost all) consumers in $C$, because the optimality of $p$ for $C$ requires that the surplus of the consumers with the lowest value in $C$ is zero.

Next we formalize the notion of a blocking segmentation. For the definition, it is helpful to think of segmentation $S$ as the prevailing segmentation and segmentation $S'$ as a proposed deviation.
**Definition 2 (Blocking)** A segmentation $S$ blocks a segmentation $S'$ if there exists a segment $(C,p)$ in $S$ that objects to $S'$.

We are now ready to define stability. For the definition, we say that segmentations $S$ and $S'$ are *equivalent* if almost all consumers face the same price in the two segmentations, that is, for almost all consumers $c$ in $[0,1]$, $c$ is in a segment with price $p$ in segmentation $S$ if and only if $c$ is in a segment with price $p$ in segmentation $S'$.\(^9\)

**Definition 3 (Stability)** A segmentation is stable if it blocks any non-equivalent segmentation.

Stability captures a kind of “coalitional individual rationality (IR),” in that no segment can be forced to regroup into one or more different segments if all its members oppose this change. A segmentation is stable if moving to any other non-equivalent segmentation violates coalitional IR. Like other notions of stability, the definition does not specify the details of the interaction among the consumers and the seller and how a stable segmentation is reached. Instead, it can be thought of as ruling out deviations from any candidate segmentation that is the outcome of the unmodeled process.

Stability may appear to demand more than just coalitional IR, because it requires that a segmentation block *every* non-equivalent segmentation, even those that are not attractive alternatives (because they do not block the original segmentation), and even those that may be “difficult to reach” from the original segmentation. Section 3.2 uses our characterization of stable segmentations from the next subsection to address these issues and show that stability is in fact the right notion for capturing coalitional IR.

### 3.1 Characterization of Stable Segmentations

We start by introducing the notion of a *canonical* segmentation. A segmentation is canonical if no two segments in it have the same price. Each segmentation $S$ is equivalent to a unique canonical segmentation, which we call the *induced canonical*

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\(^9\) Notice that two equivalent segmentations need not be identical when viewed as two sets of segments. For example, given a segmentation, if we replace a segment $(C,p)$ with two segments $(C\setminus C',p)$ and $(C',p)$ for some subset $C'$ of $C$ (so that price $p$ is optimal for both coalitions $C'$ and $C'\setminus C'$), then we obtain a new segmentation that is not identical to the original segmentation but is equivalent to it.
segmentation of \( S \). The induced canonical segmentation is obtained by merging all segments that have the same price into a single segment with that price.

Our characterization says that a segmentation is stable if and only if its induced canonical segmentation satisfies two properties: efficiency and saturation. A segmentation is efficient if all consumers buy the product, that is, for any segment \((C, p)\) in the segmentation, the price \( p \) is equal to the lowest value \( v(C) := \min\{v : f^C(v) > 0\} \) in \( C \). A segmentation is saturated if for any segment \((C, p)\) in the segmentation, whenever we add consumers to coalition \( C \) from a segment with a price strictly higher than \( p \), price \( p \) is sub-optimally low for this larger coalition. That is, for any two segments \((C, p)\) and \((C', p')\) in the segmentation with \( p < p' \) and any positive-measure set \( C'' \subseteq C' \) of consumers, any optimal price for coalition \( C \cup C'' \) is strictly higher than \( p \).

The following lemma shows that saturation can be expressed more succinctly by looking at the set of prices that are optimal for different segments.

**Lemma 1** A segmentation is saturated if and only if for any two segments \((C, p)\) and \((C', p')\) in the segmentation with \( p < p' \), there exists a price \( \hat{p} \) that is optimal for \( C \) such that \( p < \hat{p} \leq v(C') \).

**Proof.** If such a \( \hat{p} \) exists, then by adding consumers from \( C' \) to \( C \), all of whose values are at least \( v(C') \), and therefore at least \( \hat{p} \), the revenue from price \( \hat{p} \) increases more than the revenue from price \( p \). And because both \( p \) and \( \hat{p} \) are optimal for \( C \), \( p \) is no longer optimal when we add these consumers. Conversely, if no such \( \hat{p} \) exists, then \( p \) is the highest optimal price for \( C \) that does not exceed \( v(C') \), so if we add a small measure of consumers with value \( v(C') \) from \( C' \) to \( C \), price \( p \) remains optimal for \( C \).

To provide the characterization of stable segmentations, we first relate stability to Pareto dominance. We say that segmentation \( S' \) **Pareto dominates** segmentation \( S \) if \( CS(c, S') \geq CS(c, S) \) for all consumers \( c \) in \([0, 1]\), with a strict inequality for a positive measure of consumers. A segmentation \( S \) is **Pareto undominated** if no segmentation Pareto dominates \( S \).

**Lemma 2** Stable segmentations are Pareto undominated. Pareto undominated segmentations are efficient.

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10Suppose that price \( p \) is optimal for disjoint coalitions \( C^1, \ldots, C^k \), that is, \( p \sum_{i : v_i \geq p} f^{C^j}(v_i) \geq p' \sum_{i : v_i \geq p'} f^{C^j}(v_i) \) for all \( C^j \) and \( p' \). Summing over all \( j \), and letting \( C = \bigcup_j C^j \), we have \( p \sum_{i : v_i \geq p} f^{C}(v_i) \geq p' \sum_{i : v_i \geq p'} f^{C}(v_i) \) for all \( p' \), so price \( p \) is optimal for coalition \( C \).
Proof. For the first statement, if \( S' \) Pareto dominates \( S \), then no segment in \( S \) objects to \( S' \), and \( S' \) is not equivalent to \( S \). Therefore, \( S \) is not stable.

For the second statement, suppose that \( S \) is inefficient, so there is a segment \((C, p)\) in \( S \) with \( p > v(C) \). Consider a coalition \( \bar{C} \subseteq C \) that consists of the consumers in \( C \) with values strictly lower than \( p \) and a positive measure of the highest value consumers in \( C \) that is small enough that any optimal price for \( \bar{C} \) is strictly lower than \( p \). Denote by \( p' < p \) an optimal price for \( \bar{C} \), so \((\bar{C}, p')\) is a segment. Observe that \( p \) remains optimal for \( C \setminus \bar{C} \). Indeed, removing from \( C \) consumers with values strictly lower than \( p \), who do not purchase the product, does not change the revenue from \( p \); and removing from \( C \) some consumers with the highest value in \( C \) can only lower the optimal price, but \( p \) is already the lowest value of consumers in \( C \) after removing the consumers with values lower than \( p \), so \( p \) remains optimal. Now consider segmentation \( \bar{S} \) obtained from segmentation \( S \) by replacing segment \((C, p)\) with the two segments \((C \setminus \bar{C}, p)\) and \((\bar{C}, p')\). The consumers in \( \bar{C} \) with the highest value in \( C \) have a strictly higher surplus in \( \bar{S} \) than in \( S \), and all the other consumers in \( C \) have a weakly higher surplus in \( \bar{S} \) than in \( S \). Thus, \( \bar{S} \) Pareto dominates \( S \).

We now state and prove our main result.

**Theorem 1** A segmentation is stable if and only if its induced canonical segmentation is efficient and saturated.

**Proof.** To see the necessity of efficiency, consider a segmentation \( S' \) with an induced canonical segmentation \( S \). If \( S \) is inefficient, then \( S' \) is also inefficient, so by Lemma 2 \( S' \) is Pareto dominated and is not stable.

To see the necessity of saturation, consider a segmentation \( S' \) with an induced canonical segmentation \( S \). Suppose that \( S \) is not saturated. If \( S \) is inefficient, then the argument above implies that \( S' \) is not stable. Suppose that \( S \) is efficient, which together with non-saturation implies (by Lemma 1) that there are two segments \((C, v(C))\) and \((C', v(C'))\) in \( S \) with \( v(C) < v(C') \) such that no \( \hat{p} \) with \( v(C) < \hat{p} \leq v(C') \) is optimal for \( C \). In particular, if we add a positive-measure set \( C'' \subset C' \) of consumers with value \( v(C') \) to \( C \), price \( v(C) \) remains optimal, provided that the measure of \( C'' \) is sufficiently small. Let \( C'' \) be such a set that contains a positive measure of consumers with value \( v(C') \) from every segment in \( S' \) in which the price is \( v(C') \) (recall that \( S' \) need not be canonical). Let \( \bar{C} = C \cup C'' \), and consider a segmentation \( \bar{S} \) that is obtained from segmentation \( S \) by replacing \((C, v(C))\) and \((C', v(C'))\) with \((\bar{C}, v(C))\) and \((C' \setminus \bar{C}, p)\),
where \( p \) is any optimal price for \( C'' \setminus \bar{C} \). Segmentations \( S' \) and \( \bar{S} \) are not equivalent because consumers in \( C'' \) face a strictly lower price in \( \bar{S} \) than in \( S' \).

We now argue that \( S' \) does not block \( \bar{S} \). The surplus of consumers from any segment in \( S' \) with a price different from \( \nu(C) \) or \( \nu(C') \) does not change in \( \bar{S} \), so these segments do not object to \( \bar{S} \). Segments in \( S' \) with price \( \nu(C) \) do not object to \( \bar{S} \) because the consumers in these segments are in segment \((\bar{C}, \nu(C)) \) in \( \bar{S} \) and therefore face price \( \nu(C) \) in both segmentations. Finally, by construction of \( C'' \), for any segment \((C''', \nu(C''')) \) in \( S' \), some of the value \( \nu(C') \) consumers, whose surplus is zero in \( S' \), are in segment \((\bar{C}, \nu(C)) \) and obtain a strictly positive surplus, so \((C''', \nu(C''')) \) does not object to \( \bar{S} \). We conclude that \( S' \) does not block \( \bar{S} \) so \( S' \) is not stable.

We now turn to sufficiency. Consider a segmentation \( S' \) with an induced canonical segmentation \( S \) that is efficient and saturated. Let \( \bar{S} \) be a segmentation that is not blocked by \( S' \). We will show that \( \bar{S} \) is equivalent to \( S' \). Since the canonical representation of \( \bar{S} \) is also not blocked by \( S' \) and is equivalent to \( S' \) if and only if \( \bar{S} \) is equivalent to \( S' \), we suppose without loss of generality that \( \bar{S} \) is canonical. Write the two canonical segmentations as \( S = \{(C_1, v_1), \ldots, (C_n, v_n)\} \) and \( \bar{S} = \{(\bar{C}_1, v_1), \ldots, (\bar{C}_n, v_n)\} \), where for each \( i \) either \( C_i \) is empty or \( \nu(C_i) = v_i \) (because \( S \) is efficient), and each \( \bar{C}_i \) may be empty. We will show by induction that \( C_i = \bar{C}_i \) for all \( i \), which will prove that \( \bar{S} \) is equivalent to \( S' \). (For the rest of this proof, \( C_i = \bar{C}_i \) is in the “almost all” sense, that is, the measure of consumers in \( C_i \) but not in \( \bar{C}_i \) is zero, and the measure of consumers in \( \bar{C}_i \) but not in \( C_i \) is zero).

Suppose that \( C_j = \bar{C}_j \) for all \( j < i \) (the basis of the induction is \( i = 1 \)). We show that \( C_i = \bar{C}_i \). If \( i = n \), then we are done because \( S \) and \( \bar{S} \) partition the same set \([0, 1] \) of consumers. Suppose that \( i < n \). Since \( C_j = \bar{C}_j \) for all \( j < i \), a consumer faces a price \( p \geq v_i \) in \( S \) if and only if she faces a price \( p' \geq v_i \) in \( \bar{S} \) (up to a measure zero of consumers). In particular, consumers in \( C_i \) face a price of at least \( v_i \) in \( \bar{S} \). Consumers in \( C_i \) with values higher than \( v_i \) must be in \( \bar{C}_i \), otherwise these consumers face a price strictly higher than \( v_i \) in \( \bar{S} \), so any segment in \( S' \) that contains some of these consumers objects to \( \bar{S} \) (and thus \( S' \) blocks \( \bar{S} \)): the consumers in this segment are in \( C_i \) so face price \( v_i \) in \( S' \) (because \( S \) is the canonical representation of \( S' \)), and in \( \bar{S} \) the consumers in this segment face prices no lower than \( v_i \) (by the claim earlier in the paragraph). So \( C_i \) and \( \bar{C}_i \) are identical (up to a measure zero of consumers), except that \( \bar{C}_i \) possibly does not contain some consumers of value \( v_i \) from \( C_i \) and may contain some consumers from coalitions \( C_{i+1}, \ldots, C_n \), all of whom have values strictly higher than \( v_i \) (because \( S \)
is efficient). But, as we now argue, if $C_i$ and $\bar{C}_i$ are not identical, then the fact that $S$ is saturated contradicts the fact that $v_i$ is optimal for $\bar{C}_i$ (so $(\bar{C}_i, v_i)$ is not a segment).

To see this, suppose first that $\bar{C}_i$ does not contain some consumers of value $v_i$ from $C_i$. Since $S$ is saturated and $i < n$, by Lemma 1 some price $p > v_i$ is optimal for $C_i$ and $p$ is lower than the value of all consumers in coalitions $C_{i+1}, \ldots, C_n$. Removing from $C_i$ some consumers with value $v_i$ reduces the revenue from price $v_i$ but not the revenue from price $p$, which makes price $v_i$ sub-optimal. Then, if needed, adding to $C_i$ consumers from coalitions $C_{i+1}, \ldots, C_n$, all of whose values are at least $p$, to obtain $\bar{C}_i$ makes price $v_i$ even worse (weakly) relative to price $p$, so $v_i$ is not optimal for $\bar{C}_i$. Now suppose that $\bar{C}_i$ differs from $C_i$ only because $\bar{C}_i$ contains some consumers from coalitions $C_{i+1}, \ldots, C_n$, all of whom have value strictly higher than $v_i$. Adding these consumers to $C_i$ makes price $v_i$ sub-optimal because $S$ is saturated, so $v_i$ is not optimal for $\bar{C}_i$.

The proof of Theorem 1 in fact shows that if a segmentation is stable, then it is efficient and saturated, regardless of whether it is canonical. But the other direction of the proof relies on the segmentation being canonical. Example 4 in Appendix A describes a non-canonical segmentation that is efficient and saturated but not stable.

### 3.2 Discussion of Stability and Implications of Theorem 1

As mentioned after its definition (Definition 3), stability may appear too demanding because it requires that a stable segmentation block every non-equivalent segmentation. The following corollary of Theorem 1, proved in Appendix B, shows that stability is in fact equivalent to a weaker requirement.

**Corollary 1** A segmentation $S$ is stable if and only if it blocks every segmentation $S'$ that blocks $S$.

**Corollary 1** shows that stability is equivalent to only requiring that a segmentation block “attractive” alternative segmentations, that is, ones that contain segments that object to the original segmentation.

**Theorem 1** also implies that a segmentation that is not stable has blocking segmentations that it does not block and are “straightforward” modifications of the segmentation. To see this, consider a segmentation $S$ that is not stable, and its induced canonical segmentation $S'$. By Theorem 1, $S'$ is either not efficient or not saturated (or both). If $S'$ is not efficient, then $S$ is not efficient. The proof of Lemma 2 then
describes a segmentation that blocks $S$ and differs from $S$ only in that one segment in $S$ is split into two segments. This segmentation Pareto dominates $S$ so is not blocked by $S$. If $S'$ is efficient but not saturated, then the arguments in the second paragraph of the proof of Theorem 1 show that a segmentation $S''$ exists that blocks $S$ and is not blocked by $S$ and can be obtained from $S$ in two steps. First, for some price $p$, segments in $S$ with price $p$ merge to form one segment with price $p$. Second, for some higher price $p'$, some consumers from segments in $S$ with price $p'$ leave their segment and join the segment with price $p$, without affecting the optimality of $p$. No other change is needed.

Taken together, these observations show that stability is an appropriate notion for capturing coalitional IR: stability only requires that a segmentation block attractive alternative segmentations, and for any non-stable segmentation $S$ there exists an attractive alternative segmentation that is a straightforward modification of $S$ and respects coalitional IR because no segment in $S$ objects to it (so $S$ does not block it).

Finally, stability can be thought of in a procedural way that links the different segments of a blocking segmentation. To see this, consider a prevailing segmentation $S$ and a proposed objecting segment $(C', p')$. Each consumer in $C'$ weakly prefers $(C', p')$ to $S$, and for some consumers the preference is strict. Consumers in $C'$ for whom the preference is not strict can be thought of as consulting their coalition partners in $S$, saying “I was asked to join a deviating segment $(C', p')$; if I leave our current segment in $S$ to do so, would you be able to form new segments, perhaps with consumers from other segments in $S$, so that not all of you are hurt (some weakly and some strictly)? I will only join the deviating segment if your answer is affirmative.” If these coalition partners can form new segments so that not all of the coalition partners are hurt, the process continues; if these new segments include consumers from additional segments in $S$, then some consumers in those segments may consult their coalition partners in $S$, and so on. In this way, the formation of a blocking segmentation can be thought of as a “contagion” process that starts from the initial objecting segment $(C', p')$. If all consumers are able to successfully rearrange into new segments, then a blocking segmentation forms and $S$ is not stable. But if $S$ is stable, then for any initial objecting segment, this process will get ”stuck:” at some point in the process, some indifferent consumers who are considering deviating from $S$ belong to a segment that objects to any rearrangement of the remaining consumers. We formalize this process.

---

11 This does not change the price any consumer faces, and may be simpler to administer for consumers and their representatives.
Values 1 2 3

| Consumers | 0 | 1/3 | 1/2 | 1 |
|-----------|---|-----|-----|---|
| C_1       | 0 | 4/9 | 7/9 | 1 |
| C_2       |   | 4/9 | 11/18 |   |
| C_3       |   | 11/18 | 7/9 |   |

Figure 1: Example 1

in Appendix D.

3.3 Existence of Stable Segmentations

We show that stable segmentations always exist by using the characterization from Theorem 1 to construct a stable segmentation. We start with an example that demonstrates the construction.

Example 1 (Maximal equal-revenue segmentation) There are three values, 1, 2, 3, with measures \( \frac{1}{3}, \frac{1}{6}, \frac{1}{2} \), respectively, as shown in Figure 1.

Consider the segmentation \( S = \{(C_1, 1), (C_2, 2), (C_3, 3)\} \) shown in Figure 1. Coalition \( C_1 \) is the largest “equal-revenue” coalition that includes all values. That is, the measures \( \frac{1}{3}, \frac{1}{6}, \frac{2}{3} \) of the three values in coalition \( C_1 \) are such that prices 1, 2, and 3 are all optimal, and \( C_1 \) is the largest such coalition because it contains all the consumers with value 1.\(^{12}\) Segment \( (C_1, 1) \) is efficient. Consumers not in \( C_1 \) have value either 2 or 3, so adding them to \( C_1 \) makes price 1 no longer optimal.

Having put all the consumers with value 1 in segment \( C_1 \), we define the rest of the segmentation recursively to guarantee efficiency and saturation. The values of the remaining consumers are 2 and 3, and the measures of these consumers are \( \frac{1}{18} \) and \( \frac{5}{18} \), respectively. Coalition \( C_2 \), in which values 2 and 3 have measures \( \frac{1}{18} \) and \( \frac{2}{18} \), is

\(^{12}\)Any two coalitions \( C \) and \( C' \) for which all three prices are optimal are proportional, that is, \( f^C(v) = \alpha f^{C'}(v) \) for some \( \alpha > 0 \) and every value \( v \), so the largest such coalition is well-defined.
the largest coalition for which prices 2 and 3 are both optimal. The segment \((C_2, 2)\) is efficient, and adding any of the remaining consumers, all of whom have value 3, increases the optimal price. The last segment is \((C_3, 3)\), which is efficient. Thus, segmentation \(S\) is efficient and saturated. Because it is also canonical, it is stable by Theorem 1.

We now formally define the maximal equal-revenue segmentation (MERS). Let \(\bar{F}^C(v_i)\) be the cumulative measure of consumers with values \(v_i\) or higher in coalition \(C\). If \(v_i \bar{F}^C(v_i) = v_j \bar{F}^C(v_j)\), then prices \(v_i\) and \(v_j\) generate the same revenue for coalition \(C\). Coalition \(C\) is an equal-revenue coalition if all consumer values in the coalition generate the same revenue, that is, \(v_i \bar{F}^C(v_i)\) is the same for all \(v_i\) with \(f^C(v_i) > 0\). The MERS is defined recursively. The first coalition, \(C_1\), is the largest equal-revenue coalition that includes all the values. To construct \(C_1\), let \(\lambda_1\) be the eventual revenue in coalition \(C_1\) from each of the values, that is, \(\lambda_1 = v_i \bar{F}^{C_1}(v_i)\) for all \(v_i\) in \(V\). Recalling that \(f^{C_1}(v_i)\) is the measure of consumers with value \(v_i\) in coalition \(C\), and \(f(v_i)\) is the overall measure of consumers with value \(v_i\), we have that \(f(v_i) \geq f^{C_1}(v_i) = \bar{F}^{C_1}(v_i) - \bar{F}^{C_1}(v_{i+1}) = \lambda_1 \left( \frac{1}{v_i} - \frac{1}{v_{i+1}} \right)\) for all \(v_i\), where \(\frac{1}{v_{n+1}} \equiv 0\). Therefore, the highest value that \(\lambda_1\) can take is such that \(f^{C_1}(v_i) = f(v_i)\) for some \(i\). That is, \(\lambda_1\) is the smallest value such that \(\lambda_1 \left( \frac{1}{v_i} - \frac{1}{v_{i+1}} \right) = f(v_i)\) for some \(i\). Denote the index of this value by \(i_1\), so \(\lambda_1 \left( \frac{1}{v_{i_1}} - \frac{1}{v_{i_1+1}} \right) = f(v_{i_1})\). Then, more succinctly, we define \(C_1\) by letting

\[
\lambda_1 = \min_{v_i \in V} \frac{f(v_i)}{\frac{1}{v_i} - \frac{1}{v_{i+1}}} = \frac{f(v_{i_1})}{\frac{1}{v_{i_1}} - \frac{1}{v_{i_1+1}}},
\]

and letting \(\bar{F}^{C_1}(v_i) = \lambda_1/v_i\) for all \(v_i\).

Coalition \(C_1\) contains all the consumers with value \(v_{i_1}\), and adding a positive measure of consumers with other values to \(C_1\) makes price \(v_{i_1}\) sub-optimal. Therefore coalition \(C_1\) cannot be any larger and still be an equal-revenue coalition. The first segment in the MERS is \((C_1, v_{i_1})\), and the rest of the segmentation is defined recursively, where coalition \(C_j\) in the \(j\)'th segment is the largest equal-revenue coalition that includes all the values that remain after removing the consumers in \(C_1, \ldots, C_{j-1}\), that is, \(\{v_i : f^{C_j}(v_i) > 0\} = \{v_i : f^{[0,j]\cup\cup' < C_j'}(v_i) > 0\}\), and the price in the \(j\)'th segment is \(\min\{v_i : f^{C_j}(v_i) > 0\}\). This process ends because in each step the number of remaining values decreases by at least 1.

The MERS is not necessarily canonical. For example, if the first equal-revenue coalition \(C_1\) exhausts some value other than \(v_{i_1}\), then the second coalition, \(C_2\), will also
include consumers with value \( v_1 \), so the MERS will contain two segments, \((C_1, v_1)\) and \((C_2, v_1)\), that have the same price. By Theorem 1, to establish that the MERS is stable, we need show that its induced canonical segmentation is efficient and saturated.

**Proposition 1** The maximal equal-revenue segmentation (MERS) is stable.

**Proof.** The price in each segment of the MERS is equal to the lowest consumer value in the segment, so the segmentation is efficient and the same is true for its induced canonical segmentation. It remains to show that the induced canonical segmentation is saturated.

By construction of the MERS, for any two segments \((C_i, v_i)\) and \((C_j, v_j)\) with \( i < j \), the set of consumer values in \( C_j \) is a subset of that in \( C_i \). Since coalition \( C_j \) contains consumers with value \( v_j \) \((-E^C(v_j) > 0\), so does coalition \( C_i \). Because \( C_i \) is an equal-revenue segmentation, price \( v_j \) is optimal for coalition \( C_i \). Consider the segment with price \( v_i \) in the induced canonical segmentation. By definition of the induced canonical segmentation, the coalition in this segment is the union of all the coalitions with price \( v_i \) in the MERS. Price \( v_j \) is optimal for each of these coalitions, as argued above, and is therefore optimal for the union of these coalitions (see footnote 10). Thus, the induced canonical segmentation is saturated by Lemma 1.

The MERS is not the unique stable segmentation. Here is an informal description of another construction of a stable segmentation. Put all consumers with value \( v_1 \) in the first coalition, and continually add consumers with the lowest remaining value to the first coalition until some price \( v_i \) other than \( v_1 \) also becomes optimal. This forms the first coalition, \( C_1 \). The first segment is \((C,v_1)\). Repeat this process with the remaining consumers (the last segment may have only one optimal price). The resulting segmentation is canonical, efficient, and saturated. Saturation follows because given a segment \((C,v_j)\) so constructed, a value \( v_k > v_j \) becomes optimal for coalition \( C \) only when we have already added all the available consumers with values lower than \( v_k \) to \( C \), so the value of any consumer in a segment with a higher price is at least \( v_k \), and adding such consumers to \( C \) makes price \( v_j \) sub-optimal. This segmentation is also typically different from the MERS because the first segment does not generally include all values.\(^{13}\)

\(^{13}\)This alternative stable segmentation is related to the greedy algorithm of Ali, Lewis, and Vasserman (2023), adapted to the case of a finite number of values. However, our construction starts from the lowest type, whereas theirs starts from the highest type. As a result, their segmentation may not be stable. For example, suppose that there are three possible values, 1, 2, 4, each with measure \( \frac{1}{3} \).
3.4 Consumer-Optimal Stable Segmentation

Proposition 1 also helps address the following question. Suppose that consumers choose a segmentation before they learn their value for the product (by, for example, coordinating their data disclosure decisions or interacting with the seller through a third party) but can deviate from the segmentation after they learn their value if the segmentation is not stable. Which segmentation will they choose?

Before they learn their value, all consumers rank segmentations by the average consumer surplus they generate. Thus, a segmentation that maximizes average consumer surplus across all segmentations (stable or not) is preferred by all consumers. And if such a segmentation is stable, then consumers will not deviate from it after they learn their value. Bergemann, Brooks, and Morris (2015), who first introduced the MERS, showed that the MERS maximizes average consumer surplus across all segmentations. Proposition 1 shows that the MERS is stable, so consumers can achieve the maximal average surplus by choosing the MERS (or another equivalent stable segmentation) before they learn their value, even in the absence of commitment or a central planner that enforces the segmentation.

Bergemann, Brooks, and Morris (2015) also showed that segmentations other than the MERS maximize average consumer surplus. This raises the question of the relationship between stability and maximization of average consumer surplus. The following two examples show that stability is neither necessary nor sufficient for maximization of average consumer surplus.

Example 2 (A segmentation that maximizes average consumer surplus and is not stable) Consider again the setting from Example 1, with three values, 1, 2, 3, and measures $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$, and segmentation $S = \{(C_1, 1), (C_2, 2)\}$ with coalitions $C_1 = [0, \frac{1}{3}] \cup [\frac{5}{6}, 1]$ and $C_2 = (\frac{1}{3}, \frac{5}{6})$.

Coalition $C_1$ contains all the value 1 consumers and some value 3 consumers in a proportion that makes prices 1 and 3 optimal. Coalition $C_2$ contains the remaining

The greedy algorithm from Ali, Lewis, and Vasserman (2023) puts consumers with values 2 and 4 in one segment with price 2, and those with value 1 in another segment with price 1. This segmentation is not saturated because we can add consumers from the segment with price 2 to the one with price 1 without increasing the price in the latter segment. Our construction puts consumers with values 1 and 2 in one segment with price 1, and those with value 4 in another segment with price 4. This segmentation is stable.
consumers, whose proportions are such that prices 2 and 3 are optimal. Segmentation $S$ maximizes average consumer surplus across all segmentations.$^{14}$

But the segmentation is not stable. Since it is canonical and efficient, to show that it is not stable, we show that it is not saturated. Indeed, adding a small measure $\epsilon > 0$ of value 2 consumers to $C_1$ does not make price 1 sub-optimal: price 2 is not optimal for $C_1$, so if $\epsilon$ is small enough, price 2 remains sub-optimal, and the addition increases the revenue from price 1 but does not change the revenue from price 3.

Notice that since the segmentation in this example maximizes average consumer surplus, it is Pareto undominated. Thus, the example shows that there are Pareto undominated segmentations that are not stable$^{15}$

**Example 3 (A stable segmentation that does not maximize average consumer surplus)** There are three values, 1, 2, 3, each with measure $\frac{1}{3}$, as shown in Figure 2.

Consider the segmentation $S = \{(C_1, 1), (C_2, 3)\}$ in Figure 2. Coalition $C_1$ consists of the consumers with values 1 and 2; Coalition $C_3$ consists of the consumers with value 3. This segmentation is canonical, efficient, and saturated. It is therefore stable.

14The segmentation is efficient so it maximizes total surplus. It also minimizes the seller’s revenue across all segmentations. This is because price 3, which is optimal for the set of all consumers, is also optimal for each coalition.

15Such segmentations have induced canonical segmentations that are not saturated, since Pareto undominated segmentations are efficient (by Lemma 2). In order for an efficient segmentation that is not saturated to be Pareto undominated, it has to be that when consumers are made better off by moving from a segment with a higher price to a segment with a lower price without increasing the latter segment’s price, the price in the segment with the higher price necessarily increases. This is what happens in this example.
The average consumer surplus of this segmentation is \(\frac{1}{3}\), whereas the surplus of the MERS is \(\frac{2}{3}\)\(^{16}\). For some intuition, it is illuminating to study the marginal improvement in the average consumer surplus of \(S\) obtained by swapping the same measure of value 2 and 3 consumers. For this, consider a coalition \(C'_1\) obtained from \(C_1\) by removing measure \(\epsilon\) of value 2 consumers and adding a measure \(\epsilon\) of value 3 consumers, and a coalition \(C'_2\) that contains the remaining consumers. If \(\epsilon > 0\) is small enough, price 1 is optimal for \(C'_1\) and price 3 is optimal for \(C'_3\), so \(S' = \{(C'_1, 1), (C'_3, 3)\}\) is a segmentation. To compare the average consumer surplus of \(S\) and \(S'\), it suffices to consider the swapped consumers. Each value 2 consumer loses 1 unit of surplus: their surplus is 1 in \(S\) and 0 in \(S'\). Each value 3 consumer gains 2 units of surplus: their surplus is 0 in \(S\) and 2 in \(S'\)\(^{17}\).

4 Relation to Existing Cooperative Concepts

We first examine the core of the game and relate it to our notion of stability. We show that the core is empty except when the lowest price is optimal for the entire market (so the unsegmented market is efficient). We then relate stability to several other solution concepts for NTU games, including various stable set notions and the bargaining set\(^{18}\). We show that stability is a strict refinement of all these concepts. Stability therefore inherits the justifications for these concepts that often rely on farsighted behavior of agents, but has the additional (myopic) justification based on coalitional IR.

4.1 Core

We define the core to be the set of segmentations to which there is no objection\(^{19}\).

\(^{16}\)The MERS is \(\{(C'_1, 1), (C'_2, 2)\}\), where \((f^{C'_1}(1), f^{C'_1}(2), f^{C'_1}(3)) = (\frac{1}{3}, \frac{2}{3})\) and \((f^{C'_2}(1), f^{C'_2}(2), f^{C'_2}(3)) = (0, \frac{2}{3}, \frac{1}{3})\). The surplus of value 2 and 3 consumers is 1 and 2 in the first segment, and the surplus of value 3 consumers is 1 in the second segment. The average consumer surplus is therefore \(\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 + \frac{2}{3} \cdot 1 = \frac{5}{3}\).

\(^{17}\)Notice that the change in the offered price is 2 for the value 2 consumers (from 1 to 3) and \(-2\) for the value 3 consumers (from 3 to 1). But even though this change has the same absolute value, the surplus change for the value 3 consumers is higher than for the value 2 consumers because value 2 consumers do not buy the product at a price higher than 2 (so increasing the price they face from 2 to 3 does not change their surplus.)

\(^{18}\)The cooperative solution concepts we discuss, which are typically defined for games with a finite number of players, require minor adjustments because we have a continuum of consumers.

\(^{19}\)For our purposes it is more convenient to refer to a set of segmentations instead of the payoff vectors they induce.
Definition 4 (Core) The core is the set of segmentations $S$ to which no segment objects.

We can think of the core as “prioritizing” objecting segments in proposed deviations, so any segmentation blocking $S$ prevents $S$ from being in the core. In contrast, our notion of stability prioritizes segments that are part of the prevailing segmentation, in the sense that to prevent a deviation to segmentation $S'$ it suffices that some segment in $S$ object to $S'$.

The core is a demanding solution concept. We characterize when the core is not empty and show that when the core is not empty it contains only “trivial segmentations,” in which every consumer faces price $v_1$.

Proposition 2 If the market is efficient, that is, price $v_1$ is optimal for the set $[0, 1]$ of all consumers, then the core consists of all segmentations that are equivalent to the segmentation $\{(0, 1), v_1\}$. Otherwise, the core is empty.

Proof. Suppose the market is efficient. Then $\{(0, 1), v_1\}$ is a segment and there is no objection to the segmentation $\{(0, 1), v_1\}$ because in any segment $(C, p)$ the price is at least $v_1$. For the same reason, any segmentation that is equivalent to $\{(0, 1), v_1\}$ is also in the core. Now consider a segmentation $S$ that is not equivalent to $\{(0, 1), v_1\}$, which means that the price $v$ in some segment exceeds $v_1$. Because $v$ is optimal for the coalition in the segment, the coalition contains a positive measure of consumers with value $v$. And because the price in any segment is at least $v_1$, the segment $\{(0, 1), v_1\}$ objects to $S$, so $S$ is not in the core. We conclude that if the market is efficient, then the core consists of all segmentations that are equivalent to the segmentation $\{(0, 1), v_1\}$.

Now suppose that the market is not efficient, so $v_1$ is not optimal for the set $[0, 1]$ of all consumers. Then any segmentation includes a segment with a price strictly higher than $v_1$. Otherwise, the prices in all segments of the segmentation are equal to $v_1$ (because they cannot be less than $v_1$), so price $v_1$ remains optimal after we merge all coalitions into a single coalition $[0, 1]$, which contradicts the assumption that $v_1$ is not optimal for the set of all consumers. We conclude that any segmentation $S$ has a segment $(C, p)$ with $p > v_1$.

Take a segmentation $S$ and a segment $(C, p)$ in it with $p > v_1$. Because $p$ is optimal for $C$, $C$ contains a positive measure of consumers with value $p$. Consider a coalition $C'$ that consists of a positive measure of consumers with value $p$ from $C$ and a positive measure of consumers with value $v_1$ (from any segment). If $f^{C'}(p)$ is small enough
relative to $f^{C'}(v_1)$, then price $v_1$ is optimal for $C'$, so $(C', v_1)$ is a segment. The surplus of value $p$ consumers in $C'$ is $p - v_1 > 0$, whereas their surplus in $S$ is zero. The surplus of consumers with value $v_1$ in any segment is zero. Therefore, segment $(C', v_1)$ objects to $S$, so $S$ is not in the core. Since $S$ was any segmentation, the core is empty.

Even though the core may be empty, we saw that stable segmentations always exist. However, it is not immediately obvious that stability is a less demanding notion than the core. It is in principle possible that for a segmentation $S$ in the core there is another segmentation $S'$ that is not equivalent to $S$ such that neither segmentation contains a segment that objects to the other segmentation. We show that stability in fact generalizes the the core by showing that the two solution concepts coincide when the core is non-empty.

**Proposition 3** If the core is non-empty, then it is equal to the set of all stable segmentations.

**Proof.** Recall from [Proposition 2](#) that if the core is non-empty, then it consists of the segmentation $\{(0, 1], v_1)\}$ and all its equivalent segmentations. These segmentations are clearly stable because any such segmentation $S$ only contains segments of the form $(C, v_1)$, so in any non-equivalent segmentation a positive measure of consumers are offered a price higher than $v_1$, and then there is a segment in $S$ that objects to the non-equivalent segmentation. Any segmentation that is not equivalent to the segmentation $\{(0, 1], v_1)\}$ is not stable because it does not block this segmentation (since $v_1$ is the lowest price that any consumer faces in any segmentation).

Before moving on to the next cooperative solution concept, we comment on the role that weak and strict improvements in the definition of an objecting segment play in the definition of the core. Weak improvements leads to many objecting segments, which excludes many segmentations from being in the core. But, unlike for stability, where it is natural for consumers to stay in their current segment if they do not strictly benefit from deviating, for the core it may be natural to require strict improvements for all consumers in an objecting segment, since forming this segment requires actively deviating from their current segment. In our setting with a finite number of values, it is never optimal for the seller to set a price that gives all consumers in a segment positive surplus, so no objections with strict improvements for all its members exist. As we discuss in [Appendix I](#), modifying the model to accommodate a continuum of values and requiring strict improvements does not solve the issue of the core being empty.

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uninteresting; rather, it makes the core “too large” instead of “too small” (and often empty) as in our baseline model.

4.2 Stable Set

Our notion of stability is related to the notion of a stable set from Morgenstern and Von Neumann (1953). The stable set is defined for any cooperative game; we present its application to our game. Notice that whereas the stability notion of Morgenstern and Von Neumann (1953), stated below, is a property of a set of segmentations, our notion of stability is a property of a single segmentation.

**Definition 5 (Stable set, Morgenstern and Von Neumann, 1953)** A set of segmentations $S$ is a stable set if it satisfies the following two properties:

1. **Internal Stability:** For any $S \in S$, no $S' \in S$ blocks $S$.
2. **External Stability:** For any $S \notin S$, some $S' \in S$ blocks $S$.

If a segmentation $S$ is stable, then the set of all segmentations that are equivalent to $S$ is a stable set. This is easy to see: internal stability is trivially satisfied because a segmentation does not block an equivalent segmentation, and external stability is satisfied by definition of stability. Because stable segmentations always exist, stable sets exist in our setting. This is noteworthy because stable sets do not exist for some cooperative games. Moreover, even when stable sets exist, they may necessarily contain multiple elements. In contrast, our characterization of stable sets in Appendix F shows that any stable set in our setting contains an essentially unique element in the sense that it consists of all the segmentations that are equivalent to some segmentation $S$.

Perhaps surprisingly, our characterization of stable sets shows that the set of segmentations that are equivalent to a segmentation $S$ may be a stable set even if $S$ is not stable. This is because stability requires that a single segmentation block any other non-equivalent segmentation; for the set of segmentations that are equivalent to $S$ to be a stable set, on the other hand, requires that any segmentation that is not equivalent to $S$ be blocked by some segmentation that is equivalent to $S$. It may be that $S$ does not block $S'$ but a segmentation that is equivalent to $S$ does. To see this, suppose that $S$ is canonical but not stable, and consider another segmentation $S'$ that is not blocked by $S$. Take a segment $(C, p)$ in $S$. Since $(C, p)$ does not object to $S'$, $C$ may contain some consumers who prefer $S$ to $S'$ and some consumers who prefer...
If coalition $C' \subseteq C$ is such that $(C', p)$ and $(C \setminus C', p)$ are segments and we replace $(C, p)$ with $(C', p)$ and $(C \setminus C', p)$, it could be that $(C', p)$ objects to $S'$, yielding a segmentation that is equivalent to $S$ and blocks $S'$. We provide an example of this in Appendix F.

4.3 Farsighted Stable Set

Our stability notion is also related to two other notions motivated by farsighted stability: the Harsanyi stable set and the Ray and Vohra farsighted stable set (henceforth RV stable set).

Both notions define a stable set as one that satisfies internal and external stability, just like the stable set of Morgenstern and Von Neumann (1953). But the notion of blocking used to define internal and external stability is “farsighted.” A segmentation blocks another segmentation if there is a sequence of segmentations that begins with the segmentation to be blocked and ends with the blocking segmentation such that each intermediate segmentation contains a coalition that prefers the blocking segmentation to the one that preceded the intermediate segmentation. These objecting coalitions allow the blocking segmentation to be “reached” starting from the original segmentation. The two notions differ in what is assumed about the segments along the sequence other than the objecting segments, with the RV stable set assuming a kind of “coalitional autonomy” similar to the “coalitional IR” that motivates our definition of stability.

Our notion of stability satisfies these two notions which, although differing in general, coincide in our setting. Moreover, although these are set notions, in our setting they are satisfied only by singleton sets. Importantly, however, these notions are not particularly useful in our setting because they are too permissive. More precisely, for each notion we have a weak and a strong version; any segmentation that does not eliminate all consumer surplus satisfies the weak versions, and any Pareto undominated segmentation satisfied the strong versions. We provide the details in Appendix G.

4.4 Bargaining Set

Our notion of stability is closely related to the bargaining set. Roughly speaking, a segmentation $S$ is in a bargaining set if for any objection to $S$, there is another objection to $S$ that would in a sense “cancel” the original objection. In contrast, stability requires
that $S$ contain an objection to any other non-equivalent segmentation. Appendix H contains an example with two values in which all segmentations are in the bargaining set. In contrast, Appendix E shows that with two values, stable segmentations are essentially unique.

5 Conclusions

We study market segmentation of a monopolistic market. Because different consumers rank the possible segmentations differently, it is not clear which market segmentation would arise. We model the interaction that determines the segmentation as a cooperative game and develop a notion of stability that captures a segmentation being immune to deviations to other segmentations. A stable segmentation is one that, for each segmentation considered as a possible deviation, contains a segment of consumers that object to the deviation. This captures a kind of “coalitional individuals rationality (IR).”

Our main result characterizes stable segmentations as those that are efficient and saturated, in that enlarging any segment by adding consumers who face higher prices necessarily increases the profit-maximizing price for the segment. We use this characterization to show that stable segmentations always exist by showing that a particular segmentation that maximizes average consumer surplus (the MERS), identified by Bergemann, Brooks, and Morris (2015), is stable. We also show that efficiency and maximizing consumer surplus are neither necessary nor sufficient for a segmentation to be stable. We relate our notion of stability to several existing cooperative solution concepts and show that it satisfies them. Applied to our framework, these solution concepts are not particularly useful because they are either too restrictive (and may be empty) or too permissive, that is, a large set of segmentations satisfy them.

We discuss several additional results in the appendices. Appendix C highlights the separate roles that efficiency and saturation play by showing that efficient segmentations are those that are “fragmentation proof,” in that they are immune to objections by sets of consumers that are subsets of existing coalitions. The relationship between the various notions is illustrated in Figure 3. Appendix E shows that when there are only two consumer values, stability, maximizing average consumer surplus, and Pareto undominance coincide. Moreover, there is essentially a single segmentation that satisfies these properties, in a sense slightly weaker than equivalence. Example 2 shows that
Figure 3: Summary of the relation between different notions.

with more than two types there may be multiple non-equivalent stable segmentations by constructing a stable segmentation that is not equivalent to the MERS. Appendix J describes an algorithm that identifies all the stable segmentations. Appendix I discusses a variant of our model with a continuum of values, and shows that our results extend to this setting. We also discuss the effect of requiring strict improvements for all members of an objecting coalition, and show that it does not make the core a more interesting solution concept because the core becomes too large and includes all efficient segmentations; strict improvements also make stability too demanding and less interesting by not allowing objections that include indifferent consumers who “go along” with members of their existing segment who strictly prefer the existing segment.

Our results indicate that a monopolist’s use of consumer data to segment the market could be considered as a policy tool to overcome the loss of efficiency associated with monopoly pricing. While efficiency is also achieved with first-degree price discrim-
ination, our results show that as long as “coalitional IR” is maintained, the resulting efficient segmentation is Pareto undominated and may increase consumer surplus up to the highest amount possible in the “surplus triangle” of Bergemann, Brooks, and Morris (2015). If consumers can cooperatively choose a segmentation before they learn their value, then even if they can deviate after they learn their value, they can obtain the highest consumer surplus in the surplus triangle by choosing the MERS (or another suitable stable segmentation). Thus, monopolistic price discrimination subject to “coalitional IR” can be viewed as a possible alternative or addition to standard anti-trust regulation. While a detailed analysis of how to implement “coalitional IR” is beyond the scope of this paper, consumer blocs, employee unions, online platforms, and data cooperatives, which serve as intermediaries that collect data from consumers and negotiate with companies on consumers’ behalf, may be ways to achieve this.

Appendix

A An Example from Section 3.1

The following example describes a non-canonical segmentation that is efficient and saturated but not stable. Thus, to verify the stability of a segmentation, efficiency and saturation must be checked for its induced canonical segmentation.

Example 4 There are three values, 1, 2, 3, with measures 0.5, 0.25, 0.25, respectively, as shown in Figure 4.

Consider the segmentation $S = \{(C_1, 1), (C_2, 1), (C_3, 3)\}$ with coalitions $C_1, C_2, C_3$, shown in Figure 4.

For coalition $C_1$, prices 1 and 2 are optimal. For coalition $C_2$, prices 1 and 3 are optimal. Adding consumers from segment $(C_3, 3)$ to either segment $(C_1, 1)$ or $(C_2, 1)$ necessarily increases the optimal price in the latter segments. Thus, the segmentation is saturated. The segmentation is also clearly efficient.

For coalition $C_1 \cup C_2 = [0, \frac{7}{8})$, however, price 1 is the unique optimal price, so the segmentation $\{(C_1 \cup C_2, 1), (C_3, 3)\}$ is efficient but not saturated: we can add some consumers from $C_3$, all of whom have value 3, to $C_1 \cup C_2$ without changing the optimal price. To see that $S$ is not stable, note that segmentation $S' = \{([0, \frac{7}{8} + \epsilon), 1), ([\frac{7}{8} + \epsilon, 1], 3)\}$ for some small $\epsilon > 0$ Pareto dominates $S$, so $S$ is not stable by Lemma 2.
B Proof of Corollary 1

Proof. If $S$ is stable and $S'$ blocks $S$, then $S$ and $S'$ are not equivalent, so by definition of stability, $S$ blocks $S'$. For the other direction, suppose that $S$ is not stable. Let $S''$ be the induced canonical segmentation of $S$. By Theorem 1 $S''$ is either inefficient or is not saturated. In each case, we show that there is some segmentation that blocks $S$ but is not blocked by $S$.

First suppose that $S''$ is inefficient, which means that $S$ is also inefficient. Lemma 2 shows that there exists a segmentation $S'$ that Pareto dominates $S$. Therefore, $S'$ blocks $S$ but is not blocked by $S$.

Now suppose that $S''$ is efficient but not saturated. We construct a blocking segmentation $S'$ similarly to the construction in the proof of Theorem 1. Because $S''$ is not saturated, there are two segments, $(C, v(C))$ and $(C', v(C'))$, in $S''$ with $v(C) < v(C')$ such that no $\hat{p}$ with $v(C) < \hat{p} \leq v(C')$ is optimal for $C$. In particular, if we add a set $C''$ of consumers with value $v(C')$ from $C'$ to $C$, price $v(C)$ remains optimal provided that the measure of $C''$ is small. Let $C''$ be such a set that contains a positive measure of consumers with value $v(C')$ from every segment in $S$ in which the price is $v(C')$ (recall that $S$ need not be canonical). Notice that $C'' \subseteq C'$ because $S''$ is the induced canonical segmentation of $S$. Let $\bar{C} = C \cup C''$ and consider a segmentation $S'$ that is obtained from segmentation $S''$ by replacing $(C, v(C))$ and $(C', v(C'))$ with $(\bar{C}, v(C))$ and $(C' \setminus \bar{C}, p)$, where $p$ is any optimal price for $C' \setminus \bar{C}$. Segmentation $S'$ blocks...
$S$ because it contains the segment $(\bar{C}, v(C))$ that objects to $S$.

We now argue that $S$ does not block $S'$. The surplus of consumers from any segment in $S$ with a price different from $v(C)$ or $v(C')$ does not change in $S'$, so these segments do not object to $S'$. Segments in $S$ with price $v(C)$ do not object to $S'$ because the consumers in these segments are in segment $(\bar{C}, v(C))$ in $S'$ and therefore face price $v(C)$ in both segmentations. Finally, by construction of $C''$, for any segment $(C'', v(C'))$ in $S$, some of the value $v(C')$ consumers, whose surplus is zero in $S$, are in segment $(\bar{C}, v(C))$ and obtain a strictly positive surplus, so $(C'', v(C'))$ does not object to $S'$. We conclude that $S$ does not block $S'$.

\section{Fragmentation-Proofness and Efficiency}

We have seen that stability is equivalent to saturation and efficiency of the induced canonical segmentation. To further understand the interaction between saturation and efficiency, we now describe efficient segmentations as segmentations that are immune to certain objections. To this end we define \textit{fragmentation-proofness}, which excludes objections by segments with coalitions that include consumers from more than one segment.

\textbf{Definition 6} A segmentation $S$ is fragmentation-proof if there exists no objection $(C,p)$ to $S$ such that $C \subseteq C'$ for some segment $(C',p')$ in $S$.

A fragmentation-proof segmentation does not have an objection by any segment whose coalition is a subset of consumers in an existing coalition. We show that fragmentation-proofness is equivalent to efficiency.

\textbf{Proposition 4} A segmentation is fragmentation-proof if and only if it is efficient.

\textbf{Proof.} Consider an efficient segmentation and a segment $(C,p)$ in the segmentation. Because the segmentation is efficient, $p = v(C)$ is the lowest consumer value in $C$. Thus, the optimal price for any subset of $C$ is at least $p$ and there exists no objecting segment $(C',p')$ with $C' \subseteq C$.

Now consider an inefficient segmentation and a segment $(C,p)$ in the segmentation such that $p > v(C)$. Consider a coalition $C' \subseteq C$ that contains all the value $v(C)$ consumers in $C$ and a small enough measure of the consumers with the highest value in $C$ so that the unique optimal price for $C'$ is $v(C)$. The segment $(C', v(C))$ objects to $S$, so $S$ is not fragmentation-proof.
D A Procedural Way to Describe Stability

We formalize the procedural description of stability that was discussed informally in Section 3.2. Given a prevailing segmentation $S$ and an objecting segment $(C', p')$, the following iterative procedure attempts to construct a segmentation that includes $(C', p')$ and which $S$ does not block.

Let $C^0_d = C'$ (consumers already assigned to a deviating segment) and set $k = 0$.

(*) Let $C_k^l = [0, 1] \setminus C_d^k$ (consumers not yet assigned to a deviating segment), $S_d^k = \{(C, p) \in S : C \cap C_d^k \neq \emptyset\}$ (original segments that include consumers already assigned to a deviating segment), $F_k = \bigcup_{(C, p) \in S_d^k} C$ (the consumers in those original segments), and $D_k = F_k \setminus C_d^k$ (the consumers in those original segments not yet assigned to a deviating segment).

If $D_k = \emptyset$, then stop and output the segmentation that consists of the deviating segments so far constructed, whose coalitions contain precisely the consumers in $C_d^k = F_k$, and the segments in $S \setminus S_d^k$.

If $D_k \neq \emptyset$, continue the construction by, for some set of consumers $E_k \subset C_k^l$, assigning all the consumers in $D_k \cup E_k$ to new deviating segments, in a way that no segment in $S_d^k$ objects to the deviating segments. If there is no way to do this for any $E_k \subset C_k^l$, then stop and output 'stuck.' Otherwise, set $C_d^{k+1} = C_d^k \cup D_k \cup E_k$, increase $k$ by 1, and repeat from (*).

This procedure ends after finite number of iterations because for any $k$, if $D_k^{k+1} \neq \emptyset$ then $S_d^{k+1} \setminus S_d^k$ contains at least one segment from $S$, and $S$ consists of a finite number of segments.

Claim 1 A segmentation $S$ is stable if and only if for any segment $(C', p')$ that objects to $S$ the procedure always outputs ‘stuck.’

Proof. Suppose that for some segment $(C', p')$ that objects to $S$ the procedure does not output ‘stuck,’ that is, it outputs a segmentation $S'$. By construction, $S'$ contains $(C', p')$, so $S'$ is not equivalent to $S$, and no segment in $S$ objects to $S'$. Thus, $S$ is not stable.

For the other direction, suppose that for any segment $(C', p')$ that objects to $S$ the procedure outputs ‘stuck.’ By Corollary 1, it is enough to show that $S$ blocks every segmentation $S'$ that blocks $S$. Suppose that this is not the case, that is, there exists

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20 This assignment implies in particular that all the consumers in $F_k$ have been assigned to deviating segments, so we can determine whether any segment in $S_d^k$ objects to the deviating segments.
a segmentation $S'$ that blocks $S$ but $S$ does not block $S'$. Consider a segment $(C', p')$ in $S'$ that objects to $S$ and run the procedure. We will show that for appropriate sets $E^0, E^1, \ldots$, the procedure will output a segmentation and not get stuck. Let $E^k = \bigcup \{ (C'', p'') \in S'' : C'' \cap F_k \}$, where $F_k = \bigcup_{(C, p) \in S} C$ as defined in the procedure. By running the procedure with $E^0, E^1, \ldots$ and, if the procedure has not stopped by step $k$, constructing in step $k$ the set of deviating segments $\{ (C'', p'') \in S' : C'' \cap F_k \}$, we see that at each step the procedure adds at least one more segment from $S'$ (starting with the objecting segment $(C', p')$). Since $S$ does not object to $S'$, the procedure will not get stuck. Indeed, when it stops, the procedure outputs a segmentation $S''$ that includes $(C', p')$, possibly additional segments from $S'$, and possibly segments from $S$ (if the procedure stops at step $k$ with $D_k = \emptyset$ and $C^k_d \neq [0, 1]$). This completes the proof. ■

E Environments with Two Values

We have seen that, in general, stability, maximizing average consumer surplus, and Pareto undominance are different concepts. When there are only two consumer values, however, these concepts coincide. Moreover, there is essentially a single segmentation that satisfies these properties, in a sense slightly weaker than equivalence. Formally, two segmentations are weakly surplus-equivalent if each of the two corresponding induced canonical segmentations can be obtained from the other via a measure-preserving mapping that, for each value, maps the set of consumers with that value to itself. More precisely, for any two segments $(C, p)$ and $(C', p)$ with the same price in the two canonical segmentations, $f^C(v) = f^{C'}(v)$ for all $v$. Clearly, any two equivalent segmentations are weakly surplus-equivalent because equivalence requires that $C = C'$. 

Proposition 5 Suppose that there are only two values. For any segmentation $S$, the following three statements are equivalent:

1. $S$ is stable.

2. $S$ is Pareto undominated.

3. $S$ maximizes average consumer surplus.

Segmentations that satisfy these three equivalent properties are weakly surplus-equivalent.

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Proof. Suppose first that $v_1$ is optimal for the coalition $[0, 1]$. Then the segmentation $(\{[0, 1]\}, v_1)$ gives the highest possible surplus to all consumers, so a segmentation is Pareto undominated if and only if it maximizes average consumer surplus if and only if it is equivalent (and therefore weakly surplus-equivalent) to this segmentation.

Now suppose that $v_1$ is not optimal for $[0, 1]$, that is, $v_1(f^{[0,1]}(v_1) + f^{[0,1]}(v_2)) < v_2 f^{[0,1]}(v_2)$. Consider any segmentation $S = \{(C_1, v_1), (C_2, v_2)\}$ such that $C_1$ contains all the value $v_1$ consumers a measure of value $v_2$ consumers so that $v_1(f^{C_1}(v_1) + f^{C_1}(v_2)) = v_2 f^{C_1}(v_2)$, and $C_2$ contains the remaining value $v_2$ consumers. We show that any segmentation that satisfies either of the three properties, stability, Pareto undominance, and maximizing average consumer surplus, is weakly surplus-equivalent to $S$.

Consider some segmentation $S'$ that has induced canonical segmentation $S'' = \{(C''_1, v_1), (C''_2, v_2)\}$. The surplus of value $v_2$ consumers in $C''_1$ is $v_2 - v_1$, and the surplus of all other consumers is zero. So $S'$ maximizes average consumer surplus if and only if $C''_1$ has the maximal possible measure of value $v_2$ consumers, that is, if and only if it is Pareto undominated. And $C''_1$ has the maximal possible measure of value $v_2$ consumers if and only if $f^{C''_2}(v_1) = 0$ and $v_1(f^{C''_1}(v_1) + f^{C''_1}(v_2)) = v_2 f^{C''_1}(v_2)$, that is, if and only if $S'$ is weakly surplus-equivalent to $S$.21 Also, if $f^{C''_2}(v_1) = 0$ and $v_1(f^{C''_1}(v_1) + f^{C''_1}(v_2)) = v_2 f^{C''_1}(v_2)$ mean that $S''$ is saturated and efficient so $S'$ is stable. □

F Appendix for Section 4.2

We first provide an example where a segmentation $S$ does not block $S'$ but a segmentation that is equivalent to $S$ does.

Example 5 There are three values, 1, 2, 3, with measures $\frac{6}{21}, \frac{4}{21}, \frac{11}{21}$, respectively, as shown in Figure 3 and a segmentation $S = \{(C_1, 1), (C_2, 2)\}$ with $C_1 = [0, \frac{6}{21}] \cup [\frac{18}{21}, 1]$ and $C_2 = [\frac{6}{21}, \frac{18}{21}]$.

Segmentation $S$ is not stable because it is not saturated. This is because we can add some consumers with value 2 from $C_2$ to $C_1$ without increasing the optimal price $p = 1$ in the first segment. It is also easy to see directly that $S$ is not stable. For example, segmentation $S' = \{(C'_1, 1), (C'_2, 3)\}$ with coalitions $C'_1 = [0, \frac{7}{21}] \cup [\frac{18}{21}, 1]$ and $C'_2 = [\frac{7}{21}, \frac{18}{21}]$.21 Indeed, if $f^{C''_2}(v_1) > 0$, then we can add some consumers of value $v_1$ and $v_2$ from $C''_1$ to $C''_2$; and if $v_1(f^{C''_1}(v_1) + f^{C''_1}(v_2)) > v_2 f^{C''_1}(v_2)$, then we can add some consumers with value $v_2$ from $C''_2$ to $C''_1$.
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example5.png}
\caption{Example 5}
\end{figure}

$C_2 = \left[ \frac{7}{21}, \frac{18}{21} \right)$ shown in Figure 5 is not blocked by $S$. Segment $(C_1, 1)$ in $S$ does not object to $S'$ because all consumers in $C_1$ are indifferent between the two segmentations. Segment $(C_2, 2)$ does not object to $S'$ because the value 2 consumers who join the first segment in $S'$ strictly prefer $S'$ to $S$. However, segmentation $S'$ is blocked by segmentation $S'' = \{(C''_1, 1), (C''_2, 2), (C'''_2, 2)\}$ with coalitions $C''_1 = [0, \frac{6}{21}] \cup \left[ \frac{18}{21}, 1 \right]$, $C''_2 = \left[ \frac{6}{21}, \frac{7}{21} \right] \cup \left[ \frac{16}{21}, \frac{18}{21} \right]$, and $C'''_2 = \left[ \frac{7}{21}, \frac{16}{21} \right]$, which is equivalent to $S$. In particular, segment $(C'''_2, 2)$ objects to $S'$ because the consumers in $C'''$ face price 2 in $S''$ and price 3 in $S'$.

The proposition below characterizes stable sets and shows that in this example the set of all segmentations that are equivalent to $S$ is in fact a stable set. To state the proposition, we first define two weak notions of objection and blocking.

**Definition 7 (Weak Objections)** A segment $(C, p)$ weakly objects to a segmentation $S$ if $CS(c, p) > CS(c, S)$ for a positive measure of consumers $c$ in $C$, and, for every price $p'$ that is optimal for $C$, $CS(c, p) \geq CS(c, S)$ for a positive measure of consumers $c$ in $C$ whose value is $p'$.

Any objection is also a weak objection. To see this, observe that both objections
and weak objections require that some consumers strictly prefer the segment to the segmentation. But objections also require that all consumers in the segment weakly prefer the segment. Weak objections do not require this for consumers whose value is not an optimal price for the segment. And for values that are optimal prices for the segment, only some consumers with such values are required to prefer the segment. We now define the corresponding notion of weak blocking.

**Definition 8 (Weak Blocking)** A segmentation $S$ weakly blocks a segmentation $S'$ if there exists a segment $(C, p)$ in $S$ that weakly objects to $S'$.

Segmentation $S$ in [Example 5] weakly blocks (but does not block) segmentation $S'$ because segment $(C_2, 2)$ weakly objects to $S'$: consumers with value 3 in $C_2$ strictly prefer the segment to $S'$, consumers with value 2 in $C_2 \cap C'_2$ weakly prefer the segment to $S'$, and 2 is an optimal price for $C_2$.

The following proposition characterizes the stable sets.

**Proposition 6** A set of segmentations $S$ is a stable set if and only if it comprises all the segmentations that are equivalent to some canonical segmentation $S$ that weakly blocks any segmentation $S'$ that is not equivalent to $S$.

**Proof.** To see the necessity of these conditions, consider a stable set $S$ of segmentations. We first show that any segmentation in $S$ is Pareto undominated. Suppose for contradiction that a segmentation $S$ in $S$ is Pareto dominated by another segmentation $S'$. If $S'$ is in $S$, then internal stability is violated because $S'$ blocks $S$. If $S'$ is not in $S$, then, by external stability, there is a segmentation $S''$ in $S$ that blocks $S'$. But then $S''$ also blocks $S$, which violates internal stability. Pareto undominance implies that any segmentation in $S$ is efficient.

We now show that any two segmentations in $S$ are equivalent. Suppose for contradiction that segmentations $S_1$ and $S_2$ in $S$ are not equivalent. Their induced canonical segmentations $S'_1$ and $S'_2$ are also not equivalent, so there is a price $p$ and segments $(C_1, p)$ in $S'_1$ and $(C_2, p)$ in $S'_2$ with $C_1 \neq C_2$ (in the “almost all” sense), where $C_1$ or $C_2$ may be empty. Suppose without loss of generality that $p$ is the lowest such price, so any consumer in $C_1$ is either in $C_2$ or in a segment of $S'_2$ with a higher price, and similarly any consumer in $C_2$ is either in $C_1$ or in a segment of $S'_1$ with a higher price.

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22$(C_2, 2)$ does not object to $S'$ because consumers in $C_2 \setminus C'_2$ face a price of 2 in $(C_2, 2)$ and a price of 1 in $S'$. 

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(up to a set of consumers of measure 0). Because $C_1 \neq C_2$, either $C_1 \setminus C_2$ or $C_2 \setminus C_1$ has positive measure. Suppose without loss of generality that $C_1 \setminus C_2$ has positive measure. First observe that $C_1 \setminus C_2$ cannot contain a positive measure of consumers with value $p$. Indeed, such consumers would be in segments of $S'_2$ with prices strictly higher than $p$, so $S'_2$ would not be efficient, contradicting the efficiency of $S_2$. Therefore, $C_1 \setminus C_2$ contains a positive measure of consumers with values higher than $p$.

Consider any segment $(C'', p)$ in $S_1$ that contains some such consumers, that is, $C'' \cap (C_1 \setminus C_2)$ has positive measure. Because $C'' \subseteq C_1$, the consumers in $C''$ face prices no lower than $p$ in $S_2$, and the consumers in $C'' \cap (C_1 \setminus C_2)$ face prices strictly higher than $p$ in $S_2$. So $S_1$ blocks $S_2$, which contradicts internal stability.

We have established that $S$ may only contain segmentations that are equivalent to a Pareto undominated segmentation $S$. If some $S'$ that is equivalent to $S$ is not in $S$, then no segmentation in $S$ blocks $S'$ so external stability is violated. So $S$ must contain all segmentations that are equivalent to a Pareto undominated segmentation $S$, which we can assume to be canonical without loss of generality. To complete the necessity direction, it remains to show that the canonical segmentation $S$ weakly blocks any non-equivalent segmentation.

Suppose for contradiction that $S$ does not weakly block some non-equivalent segmentation $S'$. Because $S$ is a stable set and contains all segmentations that are equivalent to $S$, there is a segmentation $S''$ that blocks $S'$ and is equivalent to $S$. Consider a segment $(C'', p)$ in $S''$ that objects to $S'$, and the unique segment $(C, p)$ in $S$ in which the price is $p$. Because $(C'', p)$ objects to $S'$ and $C'' \subseteq C$, there is a positive measure of consumers in $(C, p)$ that strictly prefer $S$ to $S'$. Because $S$ does not weakly block $S'$, there exists some optimal price $v$ for $C$ such that all consumers with value $v$ in $C$ strictly prefer $S'$ to $(C, p)$. We claim that $v$ is also optimal for any segment in $S''$ with price $p$, and therefore for $C''$. To see this, consider all the segments $(C''_1, p), \ldots, (C''_{k}, p)$ in $S''$ with price $p$, so $C$ is the union of all these coalitions, one of which is $C''$. Because $p$ is optimal for $C''_j$, $j = 1, \ldots, k$, we have $vF^C_{C''_j}(v) \leq pF^{C''_j}(p)$. If price $v$ is not optimal for some $C''_j$, then $vF^C_{C''_j}(v) < pF^{C''_j}(p)$. In this case, summing up over all $j$, we have $vF^C(v) < pF^C(p)$, which contradicts the optimality of price $p$ for coalition $C$. So $v$ must be optimal for $C''$. Therefore, $v$ is optimal for $C''$, which means $C''$ contains a positive measure of consumers with value $v$. And because all consumers with value $v$ in $C$ strictly prefer $S'$ to $S$, and $S''$ is equivalent to $S'$, all these consumers strictly prefer $S'$ to $(C'', p)$ so $(C'', p)$ cannot object to $S'$, a contradiction.
To establish sufficiency, consider any canonical segmentation \( S = \{(C_1, v_1), \ldots, (C_n, v_n)\} \) that weakly blocks any non-equivalent segmentation. The set of segmentations that are equivalent to \( S \) satisfies internal stability because no segmentation blocks an equivalent segmentation. For external stability, we show that for any segmentation \( S' \) that is not equivalent to \( S \), there is a segmentation \( S'' \) that is equivalent to \( S \) and blocks \( S' \).

Consider the segment \((C, p)\) in \( S \) that weakly objects to \( S' \). We will construct a coalition \( C'' \subseteq C \) and show that the segmentation \( S'' \) that is the same as \( S \) except that \((C, p)\) is replaced with \((C \setminus C'', p)\) and \((C'', p)\), and is therefore equivalent to \( S \), objects to \( S' \). The construction of \( C'' \) has two steps. First, let \( C_1'' \) be a small coalition that comprises consumers with all the values that are optimal prices for \( C \) in proportions that make these values optimal prices for \( C_1'' \). That is, \( \epsilon = vF^{C_1''}(v) \) for some small \( \epsilon \) and all \( v \) that are optimal prices for \( C \). For the second step, let \( v' \) be such that a positive measure of consumers in \( C \) with value \( v' \) strictly prefer \((C, p)\) to \( S' \). For some \( \delta > 0 \), add to \( C_1'' \) a measure \( \delta \) of consumers in \( C \) with value \( v' \) that strictly prefer \((C, p)\) to \( S' \), and remove from \( C_1'' \) the same measure \( \delta \) of consumers with the highest value in \( C_1'' \) that is at most \( v' \) (a positive measure of these consumers exists because some consumers in \( C_1'' \) have value \( p \) and \( p < v' \), otherwise consumers with value \( v' \) have zero surplus in \( S \) so do not strictly prefer \((C, p)\) to \( S' \)). The resulting coalition is \( C'' \), which, if \( \delta \) is small relative to \( \epsilon \), satisfies that \( \epsilon = vF^{C''}(v) \) for all prices \( v \) that are optimal for \( C \). So if \( \delta \) is small relative to \( \epsilon \), then \( C'' \) has the same set of optimal prices as \( C_1'' \), and \((C'', p)\) is a segment. Similarly, if \( \epsilon \) and \( \delta \) are small enough, then \((C \setminus C'', p)\) has the same set of optimal prices as \( C \), so \((C \setminus C'', p)\) is a segment. By construction, \((C'', p)\) objects to \( S' \), so \( S'' \), which is equivalent to \( S \), blocks \( S' \). ■

The canonical segmentation \( S \) in Example 5 weakly blocks any non-equivalent segmentation, so the set of segmentations that are equivalent to \( S \) is a stable set. To see that \( S \) weakly blocks any non-equivalent segmentation, consider some segmentation \( S' \) that is not weakly blocked by \( S \). Suppose without loss of generality that \( S' \) is canonical, so \( S' = \{(C_1', 1), (C_2', 2), (C_3', 3)\} \). Because all consumers with value 1 have zero surplus, if some consumers with value 3 in \( C_1 \) strictly preferred \( S \) to \( S' \), then \((C_1, 1)\) would weakly object to \( S' \). Thus, the consumers with value 3 in \( C_1 \) are in \( C_1' \). It is impossible for all value 2 consumers in \( C_2 \) to be in \( C_1' \), because then the revenue from price 2, \( 2 \cdot (\frac{4}{21} + \frac{11}{21}) \), would be strictly higher than the revenue from price 1, which is at most \( \frac{6}{21} + \frac{4}{21} + \frac{11}{21} \). Thus, some consumers with value 2 in \( C_2 \) weakly prefer \( S \) to \( S' \). This implies that the consumers with value 3 in \( C_2 \) must be in \( C_1' \cup C_2' \), otherwise
some such consumers, those in $C'_3$, would strictly prefer $S$ to $S'$ and then $(C_2, 2)$ would weakly object to $S'$. Therefore $C'_3$ is empty ($C'_3$ cannot only contain consumers with value other than 3 because then price 3 cannot be optimal). Let $\delta_1 \geq 0$ be the measure of value 1 consumers in $C'_2$, and let $\delta_2, \delta_3 \geq 0$ be the measures of value 2 and value 3 consumers from $C_2$ that are in $C'_1$. For price 1 to be optimal for $C'_1$, the revenue from this price, $\frac{6}{21} - \delta_1 + \delta_2 + \frac{3}{21} + \delta_3$, must be no lower than the revenue from price 3, $3 \cdot (\frac{3}{21} + \delta_3)$, which means that $-\delta_1 + \delta_2 + 3\delta_3 \geq 3\delta_3$. Similarly, for price 2 to be optimal for $C'_2$ we must have $2 \cdot (\frac{4}{21} - \delta_2 + \frac{8}{21} - \delta_3) \geq 3 \cdot (\frac{8}{21} - \delta_3)$, which means that $3\delta_3 \geq 2(\delta_2 + \delta_3)$. These two inequalities hold if and only if $\delta_1 = \delta_2 = \delta_3 = 0$. We therefore have that $C_1 = C'_1$ and $C_2 = C'_2$, so $S'$ is equivalent to $S$.

G Appendix for Section 4.3

To apply the Harsanyi and RV stable sets to our setting, we need to address two technical issues. First, these notions are defined for a finite number of players. Second, they involve a definition of objection that requires a strict improvement for all members of the objecting coalition. In our setting, consumers with the lowest value in a coalition have zero surplus, so these notions become trivial (every segmentation satisfies them) if we require a strict improvement for every consumer. We define modified versions of these notions below, allowing for a continuum of players and weak improvements. Because farsighted stability considers sequences of deviations, there are two ways to allow for weak improvements. We therefore define two versions of each solution concept.

**Definition 9** A segmentation $S$ Harsanyi blocks a segmentation $S'$ if there is a sequence $S^0 = S', S^1, \ldots, S^n = S$ of segmentations and a sequence $(C^1, p^1), \ldots, (C^n, p^n)$ of segments such that for $i = 1 \ldots n$, $(C^i, p^i) \in S^i$ and $CS(c, S^{i-1}) \leq CS(c, S)$ for all consumers $c \in C^i$, with a strict inequality for a positive measure of consumers $c \in C^i$ for some $i$. If, in addition, $CS(c, S^{i-1}) < CS(c, S)$ for a positive measure of consumers $c \in C^i$ for all $i = 1 \ldots n$, we say that $S$ strongly Harsanyi blocks $S'$.

**Definition 10** A set of segmentations $S$ is a (strong) Harsanyi stable set if it satisfies the following two properties:

1. Internal Stability: For all $S \in S$, there exists no $S' \in S$ that (strong) Harsanyi blocks $S$. 

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2. **External Stability**: For all $S \notin S$, there exists $S' \in S$ that (strong) Harsanyi blocks $S$.

**Definition 11** A segmentation $S$ RV blocks a segmentation $S'$ if there is a sequence $S^0 = S', S^1, \ldots, S^n = S$ of segmentations and a sequence $(C^1, p^1), \ldots, (C^n, p^n)$ of segments such that for $i = 1 \ldots n$, $(C^i, p^i) \in S^i$ and $(C, p) \in S^i$ whenever $(C, p) \in S^{i-1}$ and $C \cap C^i = \emptyset$, and $CS(c, S^{i-1}) \leq CS(c, S)$ for all consumers $c \in C^i$, with a strict inequality for a positive measure of consumers $c \in C^i$ for some $i$. If, in addition, $CS(c, S^{i-1}) < CS(c, S)$ for a positive measure of consumers $c \in C^i$ for all $i = 1 \ldots n$, we say that $S$ strongly RV blocks $S'$.

**Definition 12** A set of segmentations $S$ is a (strong) RV stable set if it satisfies the following two properties:

1. **Internal Stability**: For all $S \in S$, there exists no $S' \in S$ that (strong) RV blocks $S$.

2. **External Stability**: For all $S \notin S$, there exists $S' \in S$ that (strong) RV blocks $S$.

For the following characterization of Harsanyi and RV stable sets we denote by $ACS(S)$ the average consumer surplus in segmentation $S$.

**Proposition 7** The following are equivalent for any set of segmentations $S$:

- $S$ is a Harsanyi stable set
- $S$ is a RV stable set
- $S = \{S\}$ for some $S$ with $ACS(S) > 0$.

The proof of Proposition 7 uses the following lemma.

**Lemma 3** For any two segmentations $S$ and $S'$, the following are equivalent:

- $S$ Harsanyi blocks $S'$.
- $S$ RV blocks $S'$.
- $ACS(S) > 0$.
**Proof.** If \( ACS(S) = 0 \), then \( CS(c, S) = 0 \) for all consumers. Therefore, \( S \) cannot Harsanyi block or RV block any segmentation.

Suppose that \( ACS(S) > 0 \). We show that \( S \) RV blocks any segmentation \( S' \), which also implies that \( S \) Harsanyi blocks \( S' \). We do so by constructing a sequence of segmentations in several steps that gradually transform \( S' \) to an elementary segmentation in which each segment includes consumers with a single value. We then proceed from the elementary segmentation to \( S \).

In step 0 we set \( S^0 = S' \). In each following step \( i > 0 \), we take the segmentation \( S^{i-1} \) and a segment \((C, p)\) in \( S^{i-1} \) that contains consumers of at least two types. For each value \( v_j \), we let \( C^i_{v_j} \) be the set of all consumers with value \( v_j \) in \( C \). \( S^i \) is constructed from \( S^{i-1} \) by replacing \((C, p)\) with the segments \((C^i_{v_j}, v_j)\) for all \( j \) such that \( f^C(v_j) > 0 \).

Let \( C^i = C^i_p \). The first phase ends with a segmentation in which every segment contains consumers of only a single type, so the surplus of all consumers is zero. The second phase has one step per segment in \( S \). In particular, for each \((C, p)\) in \( S \), given \( S^{i-1} \), we remove the consumers in \( C \) from the segments in \( S^{i-1} \) with single values and the new segment \((C, p)\) to construct \( S^i \) with \( C^i = C \). The second phase ends with \( S^n = S \).

To see that \( S \) RV blocks \( S' \), notice that in each step \( i = 1, \ldots, n \), consumers in \( C^i \) have zero surplus in \( S^{i-1} \). Therefore, they weakly prefer \( S \) to \( S^{i-1} \). Additionally, because \( ACS(S) > 0 \), there is a segment \((C, p)\) in \( S \) in which a positive measure of consumers obtain positive surplus. As a result, a positive measure of consumers strictly prefer \( S = S^n \) to \( S^{n-1} \).

**Proof of Proposition 7.** Suppose that \( S = \{S\} \) for some \( S \) with \( ACS(S) > 0 \). Then, by Lemma 3, \( S \) RV blocks and Harsanyi blocks any \( S' \neq S \), so \( S \) is a RV stable set and a Harsanyi stable set.

Consider any Harsanyi (respectively RV) stable set \( S \). The set \( S \) must contain at least one segmentation \( S \) with \( ACS(S) > 0 \), otherwise a segmentation \( S' \notin S \) is not Harsanyi (RV) blocked by any segmentation in \( S \) by Lemma 3. If the set contains more than one segmentation, then, by Lemma 3, the segmentation \( S \) that satisfies \( ACS(S) > 0 \) Harsanyi (RV) blocks the other segmentations in the set. Therefore, \( S \) contains a single segmentation \( S \), and \( ACS(S) > 0 \).

**Proposition 8** The following are equivalent for any set of segmentations \( S \):

- \( S \) is a strong Harsanyi stable set
- \( S \) is a strong RV stable set
• \( S \) is the set of all segmentations that are equivalent to some segmentation \( S \) that is Pareto undominated.

The proof uses the following lemma. The lemma uses a weak notion of equivalence of segmentations. Namely, we say that two segmentations \( S \) and \( S' \) are surplus-equivalent if almost all consumers have the same surplus in the two segmentations, that is, for almost all \( c \in [0,1] \), \( CS(c, S) = CS(c, S') \). Any two equivalent segmentations are surplus-equivalent.

**Lemma 4** For any two segmentations \( S \) and \( S' \), the following are equivalent:

- Some surplus-equivalent segmentation to \( S \) strong Harsanyi blocks \( S' \).
- Some surplus-equivalent segmentation to \( S \) strong RV blocks \( S' \).
- There exist a positive measure of consumers \( c \) such that \( CS(c, S) > CS(c, S') \).

**Proof.** If some segmentation \( S'' \) that is surplus-equivalent to \( S \) strong Harsanyi (RV) blocks \( S' \), then, by definition, a positive measure of consumers strictly prefer \( S'' \), and therefore \( S \), to \( S' \).

Suppose that a positive measure of consumers strictly prefer \( S \) to \( S' \). We show that some segmentation \( S'' \) that is surplus-equivalent to \( S \) strong RV blocks segmentation \( S' \), which also implies that \( S'' \) strong Harsanyi blocks \( S' \).

We do so by constructing a sequence of segmentations in two phases. The first phase consists of two steps. In the first step, consider some segment \((C,p)\) in \( S \) that contains a positive measure of consumers that strictly prefer \( S \) to \( S' \). Let coalition \( C^1 \) contain a positive measure of consumers with value \( p \) from \( C \), a positive measure of (but not all the) consumers from \( C \) that strictly prefer \( S \) to \( S' \), and, for every segment in \( S' \), a positive measure of consumers with the lowest value in that segment, where the proportions of consumers in \( C^1 \) are such that \((C^1,p)\) is a segment. Consider a segmentation \( S^1 \) that consists of \((C^1,p)\) and, for each consumer value, a segment that contains only the consumers in \([0,1] \setminus C^1 \) with that value, so their surplus is zero. In the second step, replace \((C^1,p)\) with \((C,p)\) and, for each consumer value, put the consumers in \( C^1 \setminus C \) with that value in a separate segment (all other segments remain intact). Denote the resulting segmentation by \( S^0 \).

The second phase consists of (potentially) several steps. In each step \( i > 0 \), take segmentation \( S^{i-1} \) and, for some segment \((C',p')\) in \( S \) that is not already in \( S^{i-1} \) and
contains a positive measure of consumers with positive surplus, let $C^i = C'$. $S^i$ is constructed from $S^{i-1}$ by taking all segments $(C'', p'')$ that contain a positive measure of consumers from $C'$ (so $C''$ contains only consumers with value $p''$) and replacing them with $(C'' \setminus C', p'')$, and finally adding segment $(C', p')$ to $S^i$. This process ends with a final segmentation $S^n$ that may differ from $S$ but is surplus-equivalent to it because for any segment in $S$ that is not in $S^n$, all consumers in that segment obtain zero surplus in both segmentations. So, for the remainder of the proof, suppose without loss of generality that $S = S^n$.

To see that $S$ RV blocks $S'$, notice that in the first step of the first phase, coalition $C^1$ contains some consumers that strictly prefer $S$ to $S'$, and all other consumers in $C^1$ weakly prefer $S$ to $S'$ because they have surplus zero in $S'$. In the second step of the first phase, by definition, some consumers in $C$ strictly prefer $S$ to $S^1$ and all the consumers in $C \setminus C^1$ because they have zero surplus in $S^1$. Similarly, in each step $i$ of the second phase, consumers in $C^i$ have surplus zero in $S^{i-1}$, and some consumers in $C^i$ strictly prefer $S$ to $S^{i-1}$ because they have a positive surplus in $S$. ■

**Proof of Proposition 8.** We first show for any Pareto undominated $S$, the set of segmentations that are equivalent to $S$ is the same as the set of segmentations that are surplus-equivalent to $S$. For this, we show that a segmentation $S'$ is equivalent to Pareto undominated $S$ if and only if it is surplus-equivalent to $S$. If $S'$ is equivalent to it, then almost all consumers have the same surplus in the two segmentations, and therefore they are surplus-equivalent. Suppose $S$ and $S'$ are surplus equivalent. Because $S$ is Pareto undominated, so is $S'$. By Lemma 2 both segmentations must be efficient. Let $\{(C_1, v_1), \ldots, (C_n, v_n)\}$ and $\{(C'_1, v_1), \ldots, (C'_n, v_n)\}$ be the canonical representations of $S$ and $S'$, respectively. Because the two segmentations are efficient, all consumers in value $v_1$ are in both $C_1$ and $C'_1$. If some consumer with value higher than $v_1$ is in $C_1$ but not $C'_1$, the the consumer’s surplus is different across the two segmentations, violating surplus-equivalence. A similar argument applies to consumers in $C'_1$ but not $C_1$. So we must have $C_1 = C'_1$. An inductive argument implies that the two segmentations are equivalent.

Suppose that $S$ is the set of all segmentations that are equivalent, and hence surplus-equivalent, to some Pareto undominated segmentation $S$. Because $S$ is Pareto undominated, for any segmentation that is not in $S$, and hence is not surplus-equivalent to $S$, there are some consumers that strictly prefer $S$ to $S'$. By Lemma 4 $S$ strong RV blocks and strong Harsanyi blocks any such $S'$, so external stability is satisfied.
Any two segmentations in $S$ are surplus-equivalent and therefore by Lemma 4, neither strong RV blocks nor strong Harsanyi blocks the other, and therefore internal stability is satisfied. So $S$ is a strong RV stable set and a strong Harsanyi stable set.

Consider a strong Harsanyi (RV) stable set $S$. If the set contains two segmentations $S$ and $S'$ in $S$ that are not surplus-equivalent, then there is a positive measure of consumers that either prefer $S$ to $S'$ or $S'$ to $S$. Then, by Lemma 4, one of the two segmentations strong Harsanyi (RV) blocks the other one, violating internal stability. Therefore, $S$ contains only surplus-equivalent segmentations. Further, if $S$ and $S'$ are surplus-equivalent and one of them is in $S$, then other other one must be too, because otherwise again by Lemma 4 the segmentation that is not in $S$ is not Harsanyi (RV) blocked by any segmentation in $S$, violating external stability. A segmentation $S$ in $S$ cannot be Pareto dominated by any segmentation $S'$ not in $S$ because otherwise, by Lemma 4, $S$ would not strong Harsanyi (RV) block $S'$, violating external stability. So $S$ must be the set of all surplus-equivalent segmentations, and therefore equivalent segmentations, to the Pareto undominated segmentation $S$.

The stable set notions we study in this section allow for weak inequalities along a sequence of segmentations. If we require strict improvement in every step, then no segmentation can block any other segmentation because in any segmentation some consumers get zero surplus. So in that case, the unique RV stable set (and also Harsanyi and also maximal RV stable set) is the set of all segmentations.

Ray and Vohra (2019) define a notion of maximality of a stable set and show that any single-payoff RV stable set is also a maximal RV stable set. Because both Proposition 7 and Proposition 8 characterize stable sets as ones that contain a single segmentation (or surplus-equivalent ones), those stable sets are also maximal. Roughly speaking, maximality requires that in a chain of segmentations defined in Definition 11 that ends in $S$, at each step the move specified by the chain is “optimal” in the sense that no coalition $C$ has another move that would lead to another segmentation in the stable set that the coalition $C$ prefers to $S$. If the stable set is a singleton, then all chains necessarily end in the same segmentation, and therefore maximality is trivially satisfied.

\[\text{Theorem 1 together with Remark 1 in Ray and Vohra (2019) show that in their setting with a finite number of players and objections that are defined to require strict improvements, any singleton RV stable set is also a maximal RV stable set. A similar argument shows that this is also the case in our setting.}\]
H Bargaining Set

There are many ways to define a bargaining set depending on whether or not we require strict or weak preferences. We only state one possibility here.

**Definition 13** The bargaining set is a set of segmentations $S$ with the property that for any objection $(C, p)$ to $S$, there exist a segment $(C', p')$ such that, $C \not\subseteq C'$, $C' \not\subseteq C$, $CS(c, p') \geq CS(c, p)$ for almost all $c \in C \cap C'$, and $CS(c, p') \geq CS(c, S)$ for almost all $c \in C' \setminus C$.

For simplicity, suppose that there are only two values, $v_1$ and $v_2$. Suppose that $v_1$ is not optimal for the set of all consumers $[0, 1]$, that is, $v_1 < v_2 f(v_2)$. We show that all segmentations are in the bargaining set.

Consider any segmentation $S$ and an objection $(C, p)$ to $S$. We must have $p = v_1$ because otherwise the surplus of all consumers in $(C, p)$ is zero. The coalition $C$ contains a positive measure, but not all, of value $v_2$ consumers. Construct a coalition $C'$ by removing an $\epsilon$ measure of value $v_2$ consumers from $C$ and replace them with the same measure of value $v_2$ consumers that are not in $C$. Notice that $(C', v_1)$ is a segment. The consumers in $C \cap C'$ have the same surplus in $(C, v_1)$ and in $(C', v_1)$. Consumers in $C' \setminus C$ have a weakly higher surplus in $(C', v_1)$ than in $S$. Therefore, $S$ is in the bargaining set.

I Continuum of Values

One modeling assumption we make is that the number of possible consumer values is finite. We consider a variant of our setting with a continuum of values and show that our main results extend: the core is empty unless the unsegmented market is efficient, and stability is characterized by efficiency and saturation.

In this setting with a continuum of values, we also study what happens if we strengthen the definition of an objection to require that almost all consumers in a segment strictly benefit from deviating to the segment. This reduces the number of deviations relative to the setting that allows for objections with indifferences, which makes the core a less demanding solution concept. With this stronger form of objections, the core becomes too permissive in the sense that it contains all finite efficient segmentations. We also study what stability looks like with this stronger form of objections. Because we require a given segmentation to veto any deviation in a stronger
sense, we should expect stability to become more demanding. In fact, stability becomes too demanding in the sense that no stable segmentations exist. Our view is that indifferences are consistent with an intuitive notion of coalitional IR and stability: a consumer who is indifferent could reasonably agree to passively go along with the objecting votes of her current coalition members who are strictly harmed by a proposed deviation. In contrast, when defining the core, it may be reasonable to require strict improvements: to be willing to actively deviate from a segmentation, consumers in the coalition should strictly benefit.

In the variant of our setting we consider there is a unit mass of consumers whose values are distributed on an interval $[v, \bar{v}]$, $0 < v < \bar{v} < \infty$ according to a distribution $F$ with a derivative that is bounded from above and away from 0. Each consumer $c \in [v, \bar{v}]$ is identified by her unique value for the product. A coalition $C$ consists of a finite union of sub-intervals of $[v, \bar{v}]$. A segment $(C, p)$ consists of a coalition $C$ and a price $p$ that is optimal (revenue-maximizing) when the values are drawn according to $F$ conditional on being in $C$. The assumption that the derivative of $F$ is bounded from above and away from 0 means that there exists some $\delta > 0$ such that for any consumer $c$, price $p = c$ is uniquely optimal for the set of consumers $[c, c + \epsilon]$ for any $\epsilon \leq \delta$.

We focus on finite segmentations. A (finite) segmentation is a finite set of segments $\{(C_j, p_j)\}_{j=1,\ldots,k}$ such that $C_1, \ldots, C_k$ partition the set of all consumers $[v, \bar{v}]$. Let $CS(c, p)$ denote the surplus of consumer $c$ from being offered price $p$, and $CS(c, S)$ the surplus of this consumer in segmentation $S$.

We first study the core and stability according to the notion of objection used throughout the paper, requiring that all consumers in the segment weakly, and some of them strictly, prefer the segment to the segmentation. Recall that stability is defined using a notion of equivalence. Here we say that two segmentations are equivalent if, for any segment $(C, p)$ in one segmentation, there exists a segment $(C', p)$ in the other segmentations such that $C$ and $C'$ are almost identical, that is, both $C \setminus C'$ and $C' \setminus C$ have zero measure. Our characterization is unchanged: the core is empty unless

\[\text{In this formulation each consumer has a unique value. An alternative formulation, inspired by the information design literature, would be as follows. An unsegmented market is a distribution } F \text{ over values } [v, \bar{v}]. \text{ A segment is a pair } (G, p), \text{ where } p \text{ is an optimal price when values are distributed according to } G. \text{ A segmentation is a finite set of segments } \{(G_j, p_j)\}_{j=1,\ldots,k} \text{ and a distribution over the segments given by probabilities } \alpha_1, \ldots, \alpha_k \text{ satisfying Bayes-plausibility, } F = \sum_j \alpha_j G_j. \text{ In this formulation, it is possible for multiple segments to contain consumers of some value. The issue now is that it is ambiguous to talk about a segment } (G, p) \text{ objecting to a segmentation } S, \text{ because there is no way to keep track of which segments in the original segmentation the consumers in the segment are coming from. We leave an appropriate formalization of such a model for future work.}\]
the unsegmented market is efficient, and stability is characterized by efficiency and saturation.

**Proposition 9** If the unsegmented market is efficient, that is, price \( v \) is optimal for the set \([v, \bar{v}]\) of all consumers, then the core consists of the segmentation \( \{(v, \bar{v}), v\} \). Otherwise, the core is empty. A segmentation is stable if and only if it is efficient and saturated.

**Proof.** If the unsegmented market is efficient, then in the segmentation \( \{(v, \bar{v}), v\} \) all consumers are offered the lowest price \( v \) and so there is no objection to the segmentation. Further, in any segmentation in the core, all consumers must be offered price \( v \), otherwise the segment that contains all consumers together with price \( v \) is an objection, so the segmentation \( \{(v, \bar{v}), v\} \) is the unique segmentation in the core.

Suppose the unsegmented market is inefficient. Then, in any segmentation, there must exist a segment with price \( p \) strictly higher than \( v \). Take a small measure \( \delta \) of consumers with value at least \( p \) from that segment, and add them to coalition \([v, v + \epsilon] \) to form a new coalition \( C \). For small enough \( \epsilon \), price \( v \) is uniquely optimal for \([v, v + \epsilon] \), so if \( \delta \) is small enough, price \( v \) is also optimal for coalition \( C \) so \((C, v)\) is an objection.

We now turn to stability. If a segmentation \( S \) is inefficient, then it contains some segment \((C, p)\) in which the price \( p \) is higher than the lowest value. Replace \((C, p)\) with two segments: a new segment \((C', p')\) with consumers whose values are lower than \( p \), and another segment containing the remaining consumers \((C' \setminus C', p)\). We must have \( p' < p \) because all consumers in \( C' \) have value at most \( p \). Also \( C' \) must contain a positive measure of consumers with value strictly higher than \( p' \) (otherwise the revenue is zero). Call the resulting segmentation \( S' \). Notice that \((C, p)\) does not object to \( S' \) because some of the consumers, those in \((C', p')\) with values above \( p' \), strictly prefer \( S' \) to \( S \). So \( S \) is not stable.

Suppose a segmentation \( S \) is efficient but not saturated. Saturation means there are two segments \((C, v(C))\) and \((C', v(C'))\) with prices \( v(C) < v(C') \) in \( S \) such that we can add some consumers from \( C' \) to \( C \) without increasing the price in the first segment. Call the resulting segmentation \( S' \). The segment \((C, v(C))\) does not object to \( S' \) because the consumers in it are indifferent. The segment \((C', v(C'))\) does not object to \( S' \) because some of its consumers, those added to the segment with the lower price, strictly prefer \( S' \) to \( S \). So \( S \) is not stable.

\(^{25}\)In this setting with a continuum of values, any segmentation is canonical because any two segments in any segmentation have different prices. As a result, any two segmentations are non-equivalent.
Suppose that a segmentation $S = \{(C_1, p_1), \ldots, (C_k, p_k)\}$, $p_1 < \ldots < p_k$ is efficient and saturated. Suppose there is some segmentation $S'$ that is not blocked by $S$. Let $C'_1, \ldots, C'_k$ be the coalitions in $S'$ belonging to segments with prices $p_1, \ldots, p_k$, respectively (these coalitions might be empty). Because $S$ is efficient, $p_1$ is equal to the lowest possible value $v$. Therefore, if $C_1$ contains a positive measure of consumers that are not in $C'_1$, then $(C_1, p_1)$ objects to $S'$. By saturation, $C'_1$ cannot be a superset of $C_1$, therefore we must have $C'_1 = C_1$ (in the almost all sense). Now again because $S$ is efficient, $p_2$ is equal to the lowest value among consumers that are not in $C_1$. A similar argument implies that we must therefore have $C'_2 = C_2$. A recursive argument implies that $S$ and $S'$ must be equivalent.

We now consider a stronger notion of an objection in which almost all consumers are required to strictly prefer the segment.

**Definition 14 (Strong objection)** A segment $(C, p)$ strongly objects to a segmentation $S$ if $CS(c, p) > CS(c, S)$ for almost all consumers in $C$.

**Definition 15 (Strong core)** The strong core is the set of segmentations $S$ to which no segment strongly objects.

**Definition 16 (Strong stability)** A segmentation is strongly stable if it contains a strong objection to any non-equivalent segmentation.

We first demonstrate the main ideas with an example. Suppose values are uniformly distributed on $[1, 3]$. For any $\delta \leq 1$, price $v$ is optimal for a coalition $[v, v + \delta]$.

So $S = \{([1, 1 + \delta], 1), ([1 + \delta, 1 + 2\delta], 1 + \delta), \ldots, ([1 + k\delta, 3], 1 + k\delta)\}$ is a segmentation and is efficient.

We argue that $S$ is in the core for any $\delta \leq 1$. So suppose $\delta \leq 1$ and there exists some strong objection $(C, p)$ to $S$. Because consumers in segment $([1, 1 + \delta], 1)$ are offered the lowest possible price, they cannot be a part of any strong objection, so $[1, 1 + \delta) \cap C$ has measure zero. This in turn implies that $p \geq 1 + \delta$. But then consumers in segment $([1 + \delta, 1 + 2\delta], 1 + \delta)$ cannot be a part of a strong objection either. An inductive argument shows that a strong objection does not exist.

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26 This notion is in fact weaker than the corresponding notion with weak objections. We call it “strong” to clarify that it is defined based on strong objections.

27 The revenue from price $p$ is proportional to $p(v + \delta - p)$, and its derivative is $v + \delta - 2p \leq \delta - v$ because $p \leq v$, and $\delta - v \leq 0$ because $\delta \leq 1 \leq v$. 

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We show below that the strong core is the set of all efficient segmentations. Our interpretation is that the strong core is a weak solution concept because there are many efficient segmentations. Recall that because the derivative of $F$ is bounded above and away from 0, there exists $\delta > 0$ such that for any consumer $c$, price $c$ is revenue-maximizing for the set of consumers $[c, c + \epsilon]$ for any $\epsilon \leq \delta$. Now we construct a class of efficient segmentations. Starting from $c = v$, choose an arbitrary $\epsilon \leq c + \delta$ and let $C = [c, c + \epsilon]$ and add segment $(C, c)$ to the segmentation. Repeat until all consumers are in the segmentation.

**Proposition 10** A segmentation is in the strong core if and only if it is efficient.

**Proof.** Suppose first that a segmentation $S$ is inefficient. Therefore, there exists a segment $(C, p)$ in $S$ such that $p$ is strictly higher than $v(C) = \inf \{v(c)|c \in C\}$. Let $p'$ be the optimal price for the set $C \cap \{v|v \leq p\}$ (which is a positive-measure set because $p > v(C)$). Notice that $p'$ is also optimal for the set $C' := C \cap \{v|p' \leq v \leq p\}$. The segment $(C', p')$ is a strong objection to $S$ because all these consumers get zero surplus in $S$ but almost all of them get a positive surplus in $(C', p')$.

Now suppose that a segmentation $S$ is not in the core, so it has an objection $(C, p)$. Because $S$ is finite and the coalition in each segment in $S$ consists of a finite union of intervals, there is an $\epsilon > 0$ such that all consumers in $(p, p + \epsilon)$ belong to a single segment in $S$, say $(C', p')$. Because price $p$ is optimal for $C$, a positive measure of consumers in $(p, p + \epsilon)$ must be in $C$ (otherwise we can increase the price without decreasing revenue). So because $(C, p)$ is a strong objection to $S$, the consumers in $(p, p + \epsilon) \cap C$ must strictly prefer $(C, p)$ to the segmentation, and it must be that $p < p'$. But this implies that $p'$ is less than the value of a positive measure of consumers in $C'$, so $(C', p')$ is inefficient and therefore $S$ is also inefficient.\[28\]

With strong objections, stable segmentations do not exist.

**Proposition 11** There exists no strongly stable segmentation.

**Proof.** Consider any segment $(C, p)$ in any segmentation $S$. Let $C'$ be a subset of $C$ consisting of consumers with value at least $c' > p$ (with $c'$ chosen so that $C'$ has positive measure). Any optimal price $p'$ for $C'$ is strictly higher than $p$. Because we removed consumers with the highest values from $C$, any optimal price $p''$ for $C \setminus C'$ is at most $p$.

\[28\] Notice that this argument does not require the objecting coalition $C$ to consist of a finite union of intervals.
Now replace \((C,p)\) with \((C',p')\) and \((C\setminus C',p'')\), and call the resulting segmentation \(S'\).
The two segmentations are not equivalent because \(p' \neq p\). But \((C,p)\) is not a strong objection to \(S'\) because the consumers in \(C\setminus C'\) weakly prefer \(S'\) to \(S\). ■

### J An Algorithmic Characterization of Stability

We describe a recursive algorithm that characterizes all stable segmentations. The characterization relies on a key structural property that relates the optimal prices across different stable segmentations. Throughout this section we will assume without loss of generality that consumers are sorted according to their value, so \(v(c) \leq v(c')\) for any two consumers \(c < c'\).

We start by describing the algorithm, summarized in [Algorithm 1]. The algorithm starts with the set of all consumers \(C = [0,1]\) and an empty segmentation. In each iteration, given a set \(C\) of remaining consumers, we add a new efficient segment \((C'',\overline{v}(C''))\), \(\overline{v}(C'') = \overline{v}(C), C'' \subseteq C\), to the segmentation.

The construction of \(C''\) has two steps. First, we define \(C'\) to be the “largest efficient prefix of \(C\).” Formally, we let \(c = \sup\{c' \leq 1 \mid \overline{v}(C) \text{ is optimal for } C \cap [0,c']\}\) be the highest consumer \(c\) for which \(C \cap [0,c]\) is efficient, and let \(C' = C \cap [0,c]\) be this largest efficient coalition. Intuitively, we start with putting into \(C'\) all consumers from \(C\) with the lowest value \(\overline{v}(C)\), and then we continually add more consumers to \(C''\) until either \(C''\) becomes equal to \(C\) or some price strictly higher than \(\overline{v}(C)\) becomes optimal (in addition to price \(\overline{v}(C)\)). If \(C = C''\), this means that \(C\) is already efficient, so the segmentation is \(\{(C,\overline{v}(C))\}\). Otherwise, we move on to the second step. In the second step, we define \(C''\) to be any subset of \(C\) that is a “revenue-preserving \(v^*(C')\)-upward shift of \(C'\),” where \(v^*(C')\) is the smallest price higher than \(\overline{v}(C')\) that is optimal for \(C'\). Formally, \(C''\) satisfies three properties: (1) \(C''\) is a subset of \(C\); (2) \(C'\) and \(C''\) are identical below \(v^*(C')\), that is \(\{c \mid v(c) < v^*(C')\} \cap C' = \{c \mid v(c) < v^*(C')\} \cap C''\); and (3) \(C''\) moves some mass of consumers with value \(v^*(C')\) or higher upwards while preserving revenue, which formally means that the distribution \(\tilde{F}^{C''}\) majorizes \(\tilde{F}^{C'}\) for values \(v^*(C')\) and above, but there is an upper bound on how much mass we can move up because the optimal revenue needs to be preserved: \(\tilde{F}^{C''}(v) \geq \tilde{F}^{C'}(v)\) and \(v\tilde{F}^{C''}(v) \leq v^*(C')\tilde{F}^{C'}(v^*(C'))\) for all \(v \geq v^*(C')\), with equality in both cases for \(v = v^*(C')\), where \(\tilde{F}^{C}(v) = \sum_{v' \geq v} f^{C}(v')\) is the measure of types \(v\) and higher in coalition \(C\).
Algorithm 1 Returns all canonical stable segmentations

| Line | Description |
|------|-------------|
| 1:   | $C \leftarrow [0, 1]$ |
| 2:   | return STABLESEGMENTATIONS($C$) |
| 3:   | procedure STABLESEGMENTATIONS($C$) \(\triangleright \) stable segmentations of $C$ |
| 4:   | $C' \leftarrow$ largest efficient prefix of $C$ |
| 5:   | if $C' = C$ then \(\triangleright C$ is already efficient |
| 6:   | return $\{(C, v(C))\}$ |
| 7:   | else \(\triangleright C$ is not efficient |
| 8:   | $v^*(C') \leftarrow$ smallest optimal price for $C'$ larger than $v(C)$ |
| 9:   | $C'' \leftarrow$ any subset of $C$ that is a revenue-preserving $v^*(C')$-upward shift of $C'$ |
| 10:  | return $\{(C'', v(C''))\} \cup$ STABLESEGMENTATIONS($C \setminus C''$) |
| 11:  | end if |
| 12:  | end procedure |

One special case of the algorithm is when we set $C'' = C'$. This special case is the greedy algorithm we described before in Section 3.3. The other special case is where we choose $C''$ such that $\bar{F}C''$ is the largest distribution according to the majorization order among those that are revenue-preserving $v^*(C')$-upward shifts of $C''$. This means that $\bar{F}C''$ is pointwise as large as possible given the constraints. That is, for all $v \geq v^*(C')$, either $v\bar{F}C''(v) = v^*(C')\bar{F}C'(v^*(C'))$ or all consumers with value $v$ in $C$ are in $C''$. This special case produces the canonical representation of the MERS. In fact, it is possible to define Algorithm 1 by letting $C'$ be the largest equal-revenue subset of $C$ and then allowing a revenue-preserving downward shift of $C'$, but this would require some additional steps to convert the segmentation into a canonical one. So for simplicity we present Algorithm 1 where $C'$ is a prefix of $C$.

We show that Algorithm 1 finds all the canonical stable segmentations.

**Proposition 12** A segmentation is stable if and only if it is equivalent to a segmentation calculated by Algorithm 1.

The proof of the proposition relies on a key structural property that relates the optimal prices across different stable segmentations. We first state and prove this property and then return to the proof of Proposition 12.

Consider any stable segmentation $S$ and its induced canonical segmentation $S''$. Consider the segment $(C_1, v_1)$ in $S''$, where $C_1$ is nonempty by efficiency (so it must contain at least all value $v_1$ consumers). Let $p_1(S) = \min\{p \mid p > v_1, p$ is optimal for $C_1\}$ be the lowest optimal price other than $v_1$ for coalition $C_1$ (define $p_1(S) = \infty$ if no such price exists). We show that this price must be the same across all stable segmentations.
Lemma 5 If $S$ and $S'$ are stable, then $p_1(S) = p_1(S')$.

Proof. Throughout this proof it is useful to use a specific way to write the optimality condition of a price. Suppose price $v_1$ is optimal for a coalition $C$. So for any other price $p$, we have $v_1 F^C(v_1) \geq p F^C(p)$, with equality if and only if $p$ is also optimal for $C$. Subtracting $v_1 F^C(p)$ from both sides, we can rewrite this as

$$v_1(F^C(v_1) - F^C(p)) \geq (p - v_1)F^C(p),$$

with equality if and only if $p$ is also optimal for $C$.

Suppose for contradiction that $S$ and $S'$ are stable but $p_1(S) \neq p_1(S')$. Suppose without loss of generality that $S$ and $S'$ are canonical and that $p_1(S) > p_1(S')$, and for simplicity let $p_1 = p_1(S')$ and $p'_1 = p_1(S')$. Our characterization of stable segmentations in Theorem 1 says that $S$ and $S'$ are efficient and saturated. Saturation implies $C_1$ contains all the consumers with value less than $p_1$. Otherwise, if we add some small measure consumers with value less than $p_1$ to $C_1$, price $v_1$ becomes the unique optimal price for the new coalition, contradicting saturation. Similarly $C'_1$ must contain all the consumers with value less than $p'_1$.

Because $p_1$ is the lowest optimal price higher than $v_1$ for $C_1$ and $v_1 < p'_1 < p_1$, price $p'_1$ is not optimal for $C_1$. Using Inequality (2) we have

$$(p'_1 - v_1) F^{C_1}(p'_1) < v_1 (F^{C_1}(v_1) - F^{C_1}(p'_1)) = v_1 (F^{C'_1}(v_1) - F^{C'_1}(p'_1)) = (p'_1 - v_1) F^{C'_1}(p'_1),$$

where the first equality followed because $C_1$ and $C'_1$ both contain all consumers with value less than $p'_1$, which means that $F^{C_1}(v_1) - F^{C_1}(p'_1)$ and $F^{C'_1}(v_1) - F^{C'_1}(p'_1)$ are equal to each other, and the second inequality followed because $p'_1$ is optimal for coalition $C'_1$ and using Inequality (2). The above sequence of inequalities implies that $F^{C_1}(p'_1) < F^{C'_1}(p'_1)$. In addition, we must have $F^{C_1}(p'_1) - F^{C_1}(p_1) \geq F^{C'_1}(p'_1) - F^{C'_1}(p_1)$ because $C_1$ contains all consumers with value at least $p'_1$ and less than $p_1$. Combine this with $F^{C_1}(p'_1) < F^{C'_1}(p'_1)$ to write

$$F^{C_1}(p_1) = (F^{C_1}(p_1) - F^{C_1}(p'_1)) + F^{C_1}(p'_1) < (F^{C'_1}(p_1) - F^{C'_1}(p'_1)) + F^{C'_1}(p'_1) = F^{C'_1}(p_1),$$

so $F^{C_1}(p_1) < F^{C'_1}(p_1)$, that is, $C'_1$ contains a strictly larger measure of consumers with value $p_1$ and above than does $C_1$.

We show that the fact that $F^{C_1}(p_1) < F^{C'_1}(p_1)$ implies that price $p_1$ has a higher
revenue for coalition $C'_1$ than price $v_1$, leading to a contradiction to optimality of $v_1$. For this we use $\bar{F}^{C_1'}(p_1) < \bar{F}^{C_1}(p_1)$ to write

$$(p_1 - v_1)\bar{F}^{C_1'}(p_1) > (p_1 - v_1)\bar{F}^{C_1}(p_1) = v_1(\bar{F}^{C_1}(v_1) - \bar{F}^{C_1}(p_1)) \geq v_1(\bar{F}^{C_1}(v_1) - \bar{F}^{C_1}(p_1)),$$

where the equality followed from the optimality of $p_1$ and $v_1$ for coalition $C_1$ rewritten using Inequality (2), and the weak inequality followed because $C_1$ contains all consumers with value less than $p_1$. But $(p_1 - v_1)\bar{F}^{C_1'}(p_1) > v_1(\bar{F}^{C_1}(v_1) - \bar{F}^{C_1}(p_1))$ means that price $p_1$ has strictly more revenue than price $v_1$ for coalition $C'_1$, contradicting the optimality of price $v_1$. ■

We now use Lemma 5 to prove Proposition 12.

**Proof of Proposition 12.** The proof is by induction on the number of values contained in $C$.

First suppose that $C$ contains only consumers with a single value. Then, the unique stable canonical segmentation contains a single segment, which is what Algorithm 1 returns.

Now suppose $C$ contains values $v_1, \ldots, v_n$, and suppose Algorithm 1 returns all canonical stable segmentations of any subset of $C$ with less than $n$ values.

First consider any segmentation produced by the algorithm. If $C$ is already efficient, then the set of all canonical stable segmentations is equal to the segmentation $\{(C, v(C))\}$, which is what the algorithm returns.

So suppose $C$ is inefficient and let $(C'', v(C))$ be the first segment that the algorithm creates. By construction, $C''$ contains all the consumers from $C$ whose values are less than the lowest optimal price $v^*(C'') > v(C)$ for $C''$. By induction, the algorithm returns some segmentation $S$ of the remaining set of values $C \setminus C''$ that is efficient, saturated, and canonical. The algorithm adds an efficient segment $(C'', v(C))$ to $S$, so the new segmentation is efficient. Also, because all consumers in $C \setminus C''$ have value $v^*(C'')$ or higher and price $v^*(C'')$ is optimal for $C''$, the new segmentation is saturated. The first segment is also the unique segment with price $v(C)$, so the new segmentation is canonical. Therefore the algorithm creates a canonical stable segmentation of $C$.

Now consider any canonical stable segmentation $S$ of $C$. Again, if $C$ is already efficient, we are done because the algorithm returns the segmentation $\{(C, v(C))\}$ which is the set of all canonical stable segmentations. So suppose $C$ is not efficient and let $(C'', v(C))$ be the unique segment with price $v(C)$ in $S$. We show that $C''$ must be a revenue-preserving $v^*(C')$-upward shift of $C'$, where $C'$ is the largest efficient prefix of
\(C\) and \(v^*(C')\) is the smallest price larger than \(v(C)\) that is optimal for \(C'\). Consider the segmentation \(S'\) that contains a segment \((C', v(C))\) in addition to any canonical stable segmentation \(S''\) of the remaining consumers \(C \setminus C'\). This segmentation \(S'\) is efficient because \(S''\) is efficient and also the segment \((C', v(C))\) is efficient. This segmentation \(S'\) is also saturated because \(S''\) is saturated and \(C \setminus C'\) contains only consumers with value \(v^*(C')\) or higher, and price \(v^*(C')\) is optimal in the first segment \((C', v(C))\). Finally, this segmentation is canonical because \(S''\) is canonical and \((C', v(C))\) is the only segment with price \(v(C)\).

Now compare the two segments \((C', v(C))\) and \((C'', v(C))\). Because both segmentations \(S\) and \(S'\) are canonical and saturated, Lemma 5 implies that \(v = v^*(C') = v^*(C'')\) is the lowest optimal price higher than \(v(C)\) in for both coalitions \(C'\) and \(C''\). Because \(S\) and \(S'\) are stable, \(C'\) and \(C''\) contain all consumers of value at most \(v\), so \(C'\) and \(C''\) are identical below \(v\), \(\{c \mid v(c) < v\} \cap C' = \{c \mid v(c) < v\} \cap C''\). The fact that \(v(C)\) and \(v\) are optimal for \(C'\) and \(C''\) and both these coalitions contain the same measure of consumers with value below \(v\) implies that these two coalitions have the same revenue and in particular \(\bar{F}C'(v) = \bar{F}C''(v)\). To see this, notice that because both \(v\) and \(v(C)\) are optimal for \(C''\), we have \(v\bar{F}C'(v) = v(C)\bar{F}C'(v)\). Subtracting \(v(C)\bar{F}C'(v)\) from both sides, we can write \((v - v(C))\bar{F}C'(v) = v(C)(\bar{F}C'(v) - \bar{F}C'(v))\). A similar equation can be written for \(C''\). Combining these two equalities with the fact that \(C'\) and \(C''\) contain the same measure of types below \(v\), \(F_C'(v(C)) - F_C'(v) = F_C''(v(C)) - F_C''(v)\), we have

\[
(v - v(C))\bar{F}C'(v) = v(C)(\bar{F}C'(v) - \bar{F}C'(v)) = v(C)(\bar{F}C''(v(C)) - \bar{F}C''(v)) = (v - v(C))\bar{F}C''(v),
\]

which implies that \(\bar{F}C'(v) = \bar{F}C''(v)\). Because \(v\) is an optimal price for both coalitions and \(v\bar{F}C'(v) = v\bar{F}C''(v)\), the two coalitions have the same optimal revenue. And optimality of price \(v\) for coalition \(C''\) can be written as \(v'\bar{F}C''(v') \leq v\bar{F}C''(v) = v\bar{F}C'(v)\) for all \(v' \geq v\). Because \(F_C'(v) = F_C''(v)\) and \(F_C'(v(C)) - F_C'(v) = F_C''(v(C)) - \bar{F}C''(v)\), the two coalitions have the same measure, \(\bar{F}C'(v(C)) = \bar{F}C''(v(C))\). And because \(C'\) is a prefix of \(C\) and \(C''\) is a subset of \(C\), the distribution \(\bar{F}C''\) majorizes the distribution \(\bar{F}C'\). So \(C''\) is a revenue-preserving \(v^*(C')\)-upward shift of \(C'\), completing the proof.

Let us apply Algorithm 1 to a parametric generalization of Example 3 with three values, 1, 2, 3, that are distributed with measures \(\left(\frac{1}{3}, \frac{1}{3} + \lambda, \frac{1}{3} - \lambda\right)\), respectively, for some parameter \(\lambda\), \(0 \leq \lambda \leq \frac{1}{9}\). The values are uniformly distributed if \(\lambda = 0\), as in
Example 3

The largest efficient prefix of the set of all consumers is coalition $C' = [0, \frac{2}{3}]$, which contains all consumers with values 1 and 2. Both prices 1 and 2 are optimal for $C'$, so $v^*(C') = 2$, and the optimal revenue is $\frac{2}{3}$. A revenue-preserving 2-upward shift $C_1$ of $C'$ is one where we substitute some consumers with value 2 with the same measure of some value 3 consumers, as long as optimal revenue is preserved, which means that $3f^{C_1}(3) \leq \frac{2}{3}$. So the first segment is $(C_1, 1)$ with measures $(\frac{1}{3}, \frac{1}{3} - \delta, \delta)$, for $\delta \leq \frac{2}{9}$. The remaining consumers $C = [0, 1] \setminus C_1$ have values 2 or 3 with measures $(0, \lambda + \delta, \frac{1}{3} - (\lambda + \delta))$. If $\lambda + \delta \geq \frac{1}{9}$, then price 2 is optimal for coalition $C$ and so the algorithm terminates by returning $(C_2, 2)$ where $C_2 = C$. If $\lambda + \delta \leq \frac{1}{9}$, then the largest efficient prefix of $C$ contains all remaining value 2 consumers, with measure $\lambda + \delta$, and a measure $2(\lambda + \delta)$ of value 3 consumers. Because value 3 is already the highest value, any 3-upward shift will result in the same measures. So the algorithm returns a segment $(C_2, 2)$ with measures $(0, \lambda + \delta, 2(\lambda + \delta))$, and another segment $(C_3, 3)$ with measure $(0, 0, \frac{1}{3} - 3(\lambda + \delta))$. To summarize, there are two types of stable segmentations. The first type of segmentation is $\{(C_1, 1), (C_2, 2)\}$ where $f^{C_1} = (\frac{1}{3}, \frac{1}{3} - \delta, \delta)$ and $f^{C_2} = (0, \lambda + \delta, \frac{1}{3} - (\lambda + \delta))$ for $\frac{1}{9} - \lambda \leq \delta \leq \frac{2}{9}$, and the second type of segmentation is $\{(C_1, 1), (C_2, 2), (C_3, 3)\}$ where $f^{C_1} = (\frac{1}{3}, \frac{1}{3} - \delta, \delta)$, $f^{C_2} = (0, \lambda + \delta, 2(\lambda + \delta))$, $f^{C_3} = (0, 0, \frac{1}{3} - 3(\lambda + \delta))$ for $0 \leq \delta \leq \frac{1}{9} - \lambda$.

Let us discuss the welfare properties of these stable segmentations. Segmentations of the first type vary in the measure $\delta$ of value 3 consumers that are contained in the first segment. Consumers in the first segment get the highest possible surplus (1 for value 2 consumers, and 2 for value 3 consumers). So these segmentations vary the fraction of type 2 versus 3 consumers who obtain their highest possible surplus. However, all these segmentations lead to the same average consumer surplus. In fact, all these segmentations lead to the highest possible average consumer surplus and achieve the lower right vertex of the “surplus triangle” of Bergemann, Brooks, and Morris (2015), shown in Figure 6. To see this, notice that price 2 is an optimal price in each segment of such a segmentation, and is also an optimal price for the unsegmented market. So these segmentations achieve efficiency without increasing revenue, and therefore lead to the highest possible consumer surplus. The segmentation is equal to the MERS for $\delta = \frac{2}{9}$, but is otherwise different from the MERS.

Segmentations of the second type, on the other hand, do not achieve the highest possible consumer surplus. This is because they contain a segment $(C_3, 3)$ that contains only value 3 consumers, for which price 2 is not optimal. These segmentations increase
Figure 6: The set of all consumer-producer surplus pairs of stable segmentations in relation to that of all possible segmentations.

the seller’s revenue relative to the unsegmented market. The average consumer surplus of these segmentations is $\frac{1}{3} + 2\lambda + 3\delta$. For a fixed $\lambda$, the highest average consumer surplus for this type of segmentation is when $\delta = \frac{1}{9} - \lambda$, in which case the third segment is empty, and the lowest average consumer surplus is when $\delta = 0$, in which case consumer surplus is $\frac{1}{3} + 2\lambda$.\textsuperscript{29}

References

Acemoglu, Daron, Ali Makhdoumi, Azarakhsh Malekian, and Asuman Ozdaglar. 2019. “Too much data: Prices and inefficiencies in data markets.” Tech. rep., National Bureau of Economic Research.

Ali, S Nageeb, Greg Lewis, and Shoshana Vasserman. 2023. “Voluntary disclosure and personalized pricing.” The Review of Economic Studies, forthcoming.

Baumann, Leonie and Rohan Dutta. 2022. “Strategic Evidence Disclosure in Networks and Equilibrium Discrimination.” Available at SSRN 4305083.

Bergemann, Dirk, Alessandro Bonatti, and Tan Gan. 2022. “The economics of social data.” The RAND Journal of Economics.

\textsuperscript{29}The fact that the average consumer surplus of all stable segmentations is higher than that of the un-segmented market is a coincidence. It is possible to construct examples where some stable segmentations have lower average consumer surplus than the un-segmented market.
Bergemann, Dirk, Benjamin Brooks, and Stephen Morris. 2015. “The limits of price discrimination.” American Economic Review 105 (3):921–57.

Braghieri, Luca. 2017. “Targeted Advertising and Price Discrimination in Intermediated Online Markets.” Working paper .

Cummings, Rachel, Nikhil R Devanur, Zhiyi Huang, and Xiangning Wang. 2020. “Algorithmic price discrimination.” In Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms. SIAM, 2432–2451.

Galperti, Simone and Jacopo Perego. 2023. “Competitive Markets for Personal Data.” Working paper .

Glode, Vincent, Christian C Opp, and Xingtan Zhang. 2018. “Voluntary disclosure in bilateral transactions.” Journal of Economic Theory 175:652–688.

Haghpanah, Nima and Ron Siegel. 2022a. “The Limits of Multiproduct Price Discrimination.” American Economic Review: Insights 4 (4):443–58.

———. 2022b. “Pareto improving segmentation of multi-product markets.” Journal of Political Economy, forthcoming .

Harsanyi, John C. 1974. “An equilibrium-point interpretation of stable sets and a proposed alternative definition.” Management science 20 (11):1472–1495.

Hidir, Sinem and Nikhil Vellodi. 2021. “Privacy, personalization, and price discrimination.” Journal of the European Economic Association 19 (2):1342–1363.

Ichihashi, Shota. 2020. “Online Privacy and Information Disclosure by Consumers.” American Economic Review 110 (2):569–95.

Kuang, Zhonghong, Sanxi Li, Yi Liu, and Yang Yu. 2022. “Stable Market Segmentation against Price Discrimination.” Working Paper .

Morgenstern, Oskar and John Von Neumann. 1953. Theory of games and economic behavior. Princeton university press.

Peivandi, Ahmad and Rakesh V Vohra. 2021. “Instability of Centralized Markets.” Econometrica 89 (1):163–179.
Ray, Debraj and Rajiv Vohra. 2015. “The farsighted stable set.” *Econometrica* 83 (3):977–1011.

———. 2019. “Maximality in the farsighted stable set.” *Econometrica* 87 (5):1763–1779.

Sher, Itai and Rakesh Vohra. 2015. “Price discrimination through communication.” *Theoretical Economics* 10 (2):597–648.

Yang, Kai Hao. 2022. “Selling consumer data for profit: Optimal market-segmentation design and its consequences.” *American Economic Review* 112 (4):1364–1393.