List-decoding homomorphism codes with arbitrary codomains

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Abstract

The codewords of the homomorphism code $a\text{Hom}(G,H)$ are the affine homomorphisms between two finite groups, $G$ and $H$, generalizing Hadamard codes. Following the work of Goldreich–Levin (1989), Grigorescu et al. (2006), Dinur et al. (2008), and Guo and Sudan (2014), we further expand the range of groups for which local list-decoding is possible up to $\text{mindist}$, the minimum distance of the code. In particular, for the first time, we do not require either $G$ or $H$ to be solvable. Specifically, we demonstrate a $\text{poly}(1/\varepsilon)$ bound on the list size, i.e., on the number of codewords within distance $(\text{mindist} - \varepsilon)$ from any received word, when $G$ is either abelian or an alternating group, and $H$ is an arbitrary (finite or infinite) group. We conjecture that a similar bound holds for all finite simple groups as domains; the alternating groups serve as the first test case.

The abelian vs. arbitrary result then permits us to adapt previous techniques to obtain efficient local list-decoding for this case. We also obtain efficient local list-decoding for the permutation representations of alternating groups (i.e., when the codomain is a symmetric group $S_m$) under the restriction that the domain $G = A_n$ is paired with codomain $H = S_m$ satisfying $m < 2^{n-1}/\sqrt{n}$.

The limitations on the codomain in the latter case arise from severe technical difficulties stemming from the need to solve the homomorphism extension (HOMEXT) problem in certain cases; these are addressed in a separate paper (Wuu 2018).

However, we also introduce an intermediate “semi-algorithmic” model we call Certificate List-Decoding that bypasses the HOMEXT bottleneck and works in the alternating vs. arbitrary setting.

Our new combinatorial tools allow us to play on the relatively well-understood top layers of the subgroup lattice of the domain, avoiding the dependence on the codomain, a bottleneck in previous work.

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1 Introduction

1.1 Brief history

Let $G$ and $H$ be finite groups, to be referred to as the domain and the codomain, respectively. A map $\psi : G \rightarrow H$ is an affine homomorphism if it is a translate of a homomorphism, i.e., if there exists a homomorphism $\varphi : G \rightarrow H$ and an element $h \in H$ such that $(\forall g \in G)(\psi(g) = \varphi(g) \cdot h)$. We write $\text{Hom}(G,H)$ and $\text{aHom}(G,H)$ to denote the set of homomorphisms and affine homomorphisms, respectively. Let $H^G$ denote the set of all functions $f : G \rightarrow H$. We view $\text{aHom}(G,H)$ as a (nonlinear) code within the code space $H^G$ (the space of possible “received words”) and refer to this class of codes as homomorphism codes.

Homomorphism codes are candidates for efficient local list-decoding up to minimum distance (mindist) and in many cases it is known that their minimum distance is (asymptotically) equal to the list-decoding bound.

This line of work goes back to the celebrated paper by Goldreich and Levin (1989) \cite{GL89} who found local list-decoders for Hadamard codes, i.e., for homomorphism codes with domain $G = \mathbb{Z}_2^n$ and codomain $H = \mathbb{Z}_2$. This result was extended to homomorphism codes of abelian groups (both the domain and the codomain abelian) by Grigorescu, Kopparty, and Sudan (2006) \cite{GKS06} and Dinur, Grigorescu, Kopparty, and Sudan (2008) \cite{DGKS08} and to the case of supersolvable domain and nilpotent codomain by Guo and Sudan (2014) \cite{GS14}, cf. \cite{BGSW18}.

While homomorphism codes have low (logarithmic) rates, they tend to have remarkable list-decoding properties.

In particular, in all cases studied so far (including the present paper), for an arbitrary received word $f \in H^G$, and any $\varepsilon > 0$, the number of codewords within radius (mindist $- \varepsilon$) is bounded
by $\text{poly}(1/\varepsilon)$ (as opposed to some faster-growing function of $\varepsilon$, as permitted in the theory of list-decoding). This is an essential feature for the complexity-theoretic application (hard-core predicates) by Goldreich and Levin.

We call the $\text{poly}(1/\varepsilon)$ bound economical, and a homomorphism code permitting such a bound combinatorially economically list-decodable (CombEcon).

By efficient decoding we mean $\text{poly}(\log |G|, 1/\varepsilon)$ queries to the received word and $\text{poly}(\log |G|, \log |H|, 1/\varepsilon)$ additional work. We call a CombEcon code AlgEcon (algorithmically economically list-decodable) if it permits efficient decoding in this sense. So the cited results show that homomorphism codes with abelian domain and codomain, and more generally with supersolvable domain and nilpotent codomain, are CombEcon and AlgEcon.

In all work on the subject, this efficiency depends on the computational representation of the groups used (presentation in terms of generators and relators, black-box access, permutation groups, matrix groups). We shall make the representation required explicit in all algorithmic results.

1.2 Our contribution – combinatorial bounds

In this paper we further expand the range of groups for which efficient local list-decoding is possible up to the minimum distance. In particular, for the first time, we do not require either $G$ or $H$ to be solvable. In fact, in our combinatorial and semi-algorithmic results (see below), the codomain is an arbitrary (finite or infinite) group. We say that a class $\mathcal{G}$ of finite groups is universally CombEcon if for all $G \in \mathcal{G}$ and arbitrary (finite or infinite) $H$, the code $\text{aHom}(G, H)$ is CombEcon. This paper is the first to demonstrate the existence of significant universally CombEcon classes.

Convention 1.1. When speaking of a homomorphism code $\text{aHom}(G, H)$, the domain $G$ will always be a finite group, but the codomain $H$ will, in general, not be restricted to be finite.

Theorem 1.2 (Main combinatorial result). Finite abelian and alternating groups are universally CombEcon.

We explain this result in detail. By “distance” in a code we mean normalized Hamming distance.

(Restatement of Theorem 1.2.) Let the domain $G$ be a finite abelian or alternating group and $H$ an arbitrary (finite or infinite) group. Let $\text{mindist}$ denote the minimum distance of the homomorphism code $\text{aHom}(G, H)$ and let $\varepsilon > 0$. Let $f \in H^G$ be an arbitrary received word. Then the number of codewords within $\text{mindist} - \varepsilon$ of $f$ is at most $\text{poly}(1/\varepsilon)$.

The degree of the polynomial in the $\text{poly}(1/\varepsilon)$ expression for abelian domains $G$ is $C + 4$ where $C$ is the degree in the corresponding $\{\text{abelian} \to \text{abelian}\}$ result (currently $C \approx 105$ [GS14]). For alternating domains $G$, we prove a bound of 9 on the degree of the polynomial; with additional work, this can be improved to 7.

Our choice of the alternating groups as the domain is our test case of what we believe is a general phenomenon valid for all finite simple groups.

Conjecture 1.3. The class of finite simple groups is universally CombEcon.

The following problem is also open.

Problem 1.4. Is the class of finite groups universally CombEcon?

We suspect the answer is “no.”

Theorem 1.2 also holds for a hierarchy of wider classes of finite groups we call shallow random generation groups or “SRG groups” (see Sec. 4.4). This class includes the alternating groups. The
defining feature of these groups is that a bounded number of random elements generate, with extremely high probability, a “shallow” subgroup, i.e., a subgroup at bounded distance from the top of the subgroup lattice.

Our new combinatorial tools allow us to play on the relatively well-understood top layers of the subgroup lattice of the domain, avoiding the dependence on the codomain, a bottleneck in previous work.

Remark 1.5. Our results list-decode certain classes of codes up to distance \(\text{mindist} - \epsilon\) for positive \(\epsilon\). In many cases, \(\text{mindist}\) is the list-decoding boundary; examples show that the length of the list may blow up when \(\epsilon\) is set to zero. Classes of such examples with abelian domain and codomain were found by Guo and Sudan [GS14]. We add classes of examples with alternating domains (Section 9.4).

1.3 Our contribution – algorithms

On the algorithmic front, the combinatorial bound in the \{abelian→arbitrary\} case permits us to adapt the algorithm of [GKS06] to obtain efficient local list-decoding. We say that a class \(\mathcal{G}\) of finite groups is universally AlgEcon if for all \(G \in \mathcal{G}\) and arbitrary finite \(H\), the code \(\text{aHom}(G,H)\) is AlgEcon. The validity of such a statement depends not only on the class \(\mathcal{G}\) but also on the representation of the domain and codomain.

Corollary 1.6. Let \(G\) be a finite abelian group and \(H\) an arbitrary finite group. Under suitable assumptions on the representation of \(G\) and \(H\), the homomorphism code \(\text{aHom}(G,H)\) is AlgEcon.

In other words, abelian groups are universally AlgEcon.

In fact, the algorithm is so efficient that in the unit-cost black-box-access model for \(H\) (elements of \(H\) can be named and operations on them performed at unit cost) the work required is only \(\text{poly}(\log |G|, 1/\epsilon)\). (The cost does not depend on \(|H|\); indeed, in this case, infinite \(H\) is also allowed).

We need to clarify the “suitable representation.” It suffices to assume that \(G\) is a finite abelian group given in any presentation by generators and relators, assuming in addition that a superset of the prime divisors of the order of \(G\) is available. Without the prime divisors, we need a factoring oracle. We need black-box access to \(H\).

A permutation representation of degree \(m\) of a group \(G\) is a homomorphism \(G \to S_m\), where the codomain is the symmetric group of degree \(m\). We also obtain efficient local list-decoding for the permutation representations of alternating groups under a rather generous restriction on the size of the permutation domain.

The limitations on the codomain arise from severe technical difficulties encountered.

In contrast to all previous work, in the alternating case the minimum distance does not necessarily correspond to a subgroup of smallest index (modulo the “irrelevant kernel,” see Sec. 4.2). This necessitates the introduction of the homomorphism extension (HomExt) problem, a problem of interest in its own right, which remains the principal bottleneck for algorithmic progress. The problem was solved by Wu [Wuu18] in the special case stated above.

To bypass the HomExt bottleneck, we introduce a new model we call Certificate List-Decoding. In this model the output is a short (\(\text{poly}(1/\epsilon)\)) list of partial maps from \(G\) to \(H\) that includes, for each affine homomorphism \(\varphi\) within \(\text{mindist} - \epsilon\) of the received word, a certificate of \(\varphi\), i.e., a partial affine homomorphism that uniquely extends to \(\varphi\).
We say that a homomorphism code is **economically certificate-list-decodable** (CertEcon) if such a list can be efficiently generated.

Note that, by definition, AlgEcon $\implies$ CertEcon $\implies$ CombEcon.

We say that a class $\mathcal{G}$ of finite groups is **universally CertEcon** if for all $G \in \mathcal{G}$ and arbitrary (finite or infinite) $H$, the code $a\text{Hom}(G, H)$ is CertEcon.

**Theorem 1.8** (Main semi-algorithmic result). *Alternating groups are universally CertEcon.*

In fact we show that SRG groups are universally CertEcon.

Finally, we show that certificate list-decoding, combined with a HomExt oracle for the top layers of the subgroup lattice of $G$, suffices for list-decoding $a\text{Hom}(G, H)$.

This is the route we take to proving Theorem 1.7.

We give more formal statements of these results in Section 4.

### 1.4 The structure of the paper

Much of our conceptual framework can be interpreted for codes in general, not just for homomorphism codes. In Section 2 we develop the general terminology. This includes the notions of economy in local list-decoding as well as the new concepts of certificate-list decoding (Sec. 2.4), our semi-algorithmic intermediate concept, and mean-list decoding, our main tool for domain relaxation (Sec. 2.5), motivated by Guo and Sudan’s use of repeated codes [GS14]. We also introduce subword extenders, which constitute the bridge between certificate-list decoding and algorithmic list-decoding (Sec. 2.6).

In Section 3 we present notation and terminology from group theory and computational group theory, including our access models, i.e., computational representations of groups (black-box, generator-relator presentations, etc.).

Section 4 gives formal statements of our results and occasional minor proofs that contribute to the conceptual development. The section includes a discussion of shallow-random-generation (SRG) groups (Section 4.4). Section 4.7 explains the role of the Homomorphism Extension problem in bridging the gap between certificate-list decoding and algorithmic list-decoding.

Section 5 describes a simple combinatorial lemma (“Bipartite covering lemma”) and applies it in two separate contexts: connecting mean-list-size to list-size, from which we infer our domain relaxation principle, and the equivalence (both combinatorial and algorithmic) of $\text{Hom}$ and $a\text{Hom}$.

Section 6 outlines our basic strategy for the combinatorial bounds. It indicates the differences between the approach to abelian domains and to alternating (and SRG) domains. We indicate that the same strategy also produces certificate-list-decoders.

Section 7 describes the tools for the combinatorial bounds. We compare one of our tools, a sphere packing argument via a strong negative correlation inequality, to the Johnson bound.

Section 8 gives the full technical development of our results for abelian domains.

The rest of the paper, Sections 9 to 11, provides the proofs for alternating domains and their generalizations, the SRG groups.

We give two proofs that alternating groups are CombEcon.

The first proof, in Section 9, is based on a sphere packing argument and is non-constructive, but the method applies under quite general circumstances. The second, in Section 11, depends on structure specific to the alternating groups (or more generally, to SRG groups), that proof directly translates to a semi-algorithmic result (CertEcon), and under restrictions of the codomain, also provides an algorithmic result (AlgEcon).
2 Terminology for general codes

2.1 List-decoding

We introduce some terminology that applies to codes in general and not just homomorphism codes.

Let $\Sigma$ be an alphabet and $\Omega$ a set we think of as the set of positions. We view $\Sigma^\Omega$, the set of $\Omega \to \Sigma$ functions, as our code space; we call its elements words. We write $\text{dist}(u,w)$ for the normalized Hamming distance between two words $u,w \in \Sigma^\Omega$ (so $0 \leq \text{dist}(u,w) \leq 1$) and refer to it simply as distance. Let $C \subseteq \Sigma^\Omega$ be a code; we call its elements codewords.

Words we wish to decode are referred to in the literature as received words. We refer to the set of codewords within a specified distance $\rho$ of a received word $f \in \Sigma^\Omega$ as “the list” and denote it by $\mathcal{L} = \mathcal{L}(C,f,\rho)$. We write $\ell(C,\rho) := \max_f |\mathcal{L}(C,f,\rho)|$.

2.2 Combinatorial list-decoding

The list-decoding problem splits into a combinatorial and an algorithmic part.

The combinatorial problem, to which we refer as combinatorial list-decoding, asks to bound the size of the list. Typically, we take $\rho = (\text{mindist} - \varepsilon)$ and we wish to obtain a bound $\ell(C,\rho) \leq c(\varepsilon)$, that depends only on $\varepsilon$ and the class $\mathcal{C}$ of codes under discussion ($C \in \mathcal{C}$).

We say that a class $\mathcal{C}$ of codes is CombEcon (“combinatorially economically list-decodable”) if $c(\varepsilon) = \text{poly}(1/\varepsilon)$ for $C \in \mathcal{F}$. (With some abuse of terminology, we shall refer to a code $C$ as a CombEcon code if the class $\mathcal{C}$ of codes is understood from the context.)

2.3 Algorithmic list-decoding

We shall describe algorithms with certain performance guarantees typically guaranteeing properties of the output with specified probability.

A list-decoder is an algorithm that, given the received word $f \in \Sigma^\Omega$ and the distance $\rho$, lists a superset $\mathcal{L}$ of the list $\mathcal{L} = \mathcal{L}(C,f,\rho)$. Typically, we take $\rho = (\text{mindist} - \varepsilon)$ and we wish to produce a list of size $|\mathcal{L}| \leq \tilde{c}(\varepsilon)$ for some $\tilde{c}(\varepsilon)$ that depends only on $\varepsilon$ and the class $\mathcal{C}$ of codes under discussion ($C \in \mathcal{C}$).

Adapting the terminology of [GKS06] and [DGKS08], we say that a local algorithm is a probabilistic algorithm that has oracle access to the received word $f$. We say that $\mathcal{C}$ is an AlgEcon (“algorithmically economically list-decodable”) class of codes if there exists a local list-decoder with the following features.

Input: mindist, $\varepsilon > 0$, oracle access to $f \in \Sigma^\Omega$.

Notation: $\mathcal{L} = \mathcal{L}(C,f,\text{mindist} - \varepsilon)$.

Output: A list $\mathcal{L}$ of codewords in $C$ of length $|\mathcal{L}| = \text{poly}(1/\varepsilon)$.

Guarantee: With probability $\geq 3/4$, we have $\mathcal{L} \supseteq \mathcal{L}$.

Cost:

(i) $\text{poly}(|\Omega|, 1/\varepsilon)$ queries to the received word $f$.

(ii) $\text{poly}(|\Omega|, \log|\Sigma|, 1/\varepsilon)$ amount of work.

Access: The meaning of this definition depends also on the access model to $\Sigma$ and $\Omega$. We shall clarify this in each application.
Strong AlgEcon

In the unit cost model for $\Sigma$, we charge unit cost to name an element of $\Sigma$.

We say that $C$ is a strong AlgEcon code if there exists a list-decoder satisfying the conditions of AlgEcon, except with (ii) replaced by the following.

$$(ii') \quad \text{poly}(\log{|\Omega|}, 1/\varepsilon) \text{ amount of work in the unit cost model for } \Sigma.$$ 

Typically, elements of $\Sigma$ are encoded by strings of length $\log{|\Sigma|}$ and therefore $(ii')$ implies $(ii)$ with linear dependence on $\log{|\Sigma|}$. The AlgEcon results proved in prior work [DGKS08, GS14, BGSW18] are actually strong AlgEcon results for those classes of pairs of groups. Our AlgEcon result for abelian domain is also strong AlgEcon (see Section 4.3). On the other hand, our AlgEcon result for alternating domain does not meet the “strong” requirement.

Remark 2.1. The unit cost model can also be used in the case of infinite $\Sigma$. In fact, our AlgEcon result for abelian domains holds even for infinite codomains in the unit cost model, i.e., it satisfies $(ii')$.

2.4 Certificate list-decoding

In the light of technical difficulties arising from algorithmic list-decoding, we introduce a new type of list-decoding that is intermediate between the combinatorial and algorithmic. We call it “certificate list-decoding.” We shall refer to results of this type as “semi-algorithmic.”

A partial map $\gamma$ from $\Omega$ to $\Sigma$, denoted $\gamma : \Omega \rightarrow \Sigma$, is a map of a subset of $\Omega$ to $\Sigma$. In particular, $\text{dom}(\gamma) \subseteq \Omega$.

Definition 2.2 (Certificate). We say that a partial map $\gamma : \Omega \rightarrow \Sigma$ is a certificate for the codeword $\varphi \in C$ if $\gamma = \varphi|_{\text{dom}(\gamma)}$ and $\varphi$ is the unique codeword that extends $\gamma$. A certificate for the code $C$ is a certificate for some codeword in $C$. 

Definition 2.3 (Certificate-list). We say that a list $\Gamma$ of $\Omega \rightarrow \Sigma$ partial maps is a certificate-list for the set $K \subseteq C$ of codewords if $\Gamma$ contains a certificate for each codeword in $K$. A certificate-list for $C$ up to distance $\rho$ of the received word $f : \Omega \rightarrow \Sigma$ is a certificate-list for the list $L = L(C, f, \rho)$.

Remark 2.4. Note that we permit the certificate-list $\Gamma$ to contain redundancies (more than one certificate for the same codeword) and irrelevant items (partial functions that are not certificates of any codeword in $K$, or not even certificates of any codeword at all).

Definition 2.5. A certificate-list-decoder is an algorithm that, given the received word $f \in \Sigma^\Omega$ and the distance $\rho$,

constructs a certificate-list of $C$ up to distance $\rho$ of $f$.

Definition 2.6. We say that $C$ is a CertEcon (“certificate-economically list-decodable”) code if there exists a local certificate-list-decoder with the following features.

Input: $\varepsilon > 0$, oracle access to $f \in \Sigma^\Omega$.

Notation: Again, let $L = L(C, f, \text{mindist} - \varepsilon)$.

Output: A list $\Gamma$ of $\Omega \rightarrow \Sigma$ partial maps of length $|\Gamma| = \text{poly}(1/\varepsilon)$.

Guarantee: With probability $\geq 3/4$, we have that $\Gamma$ is a certificate-list for $L$.

Cost:

(i) $\text{poly}(\log{|\Omega|}, 1/\varepsilon)$ queries to the received word $f$.

(ii) $\text{poly}(\log{|\Omega|}, \log{|\Sigma|}, 1/\varepsilon)$ amount of work.
**Access:** The meaning of this definition depends also on the access model to $\Sigma$ and $\Omega$. We shall clarify this in each application.

**Remark 2.7.** Note that $\text{mindist}$ is not part of the input. In our results, we are likely to find a certificate of $C$ up to distance $(\text{mindist} - \varepsilon)$ of the received word $f$, regardless of the actual value of $\text{mindist}$. We note that, depending on the access model, we may not be able to find $\text{mindist}$.

**Remark 2.8.** CertEcon is intermediate between AlgEcon and CombEcon. Indeed, CertEcon implies CombEcon, by the length bound of the Output and the Guarantee. Moreover, AlgEcon implies CertEcon, as the AlgEcon Output $\mathcal{L}$ satisfies the definition of a certificate, under the same Guarantee and Cost bound.

**Strong CertEcon**

**Definition 2.9.** We say that $C$ is a **strong CertEcon** code if there exists a certificate-list-decoder satisfying the conditions of CertEcon, except with (ii) replaced by the following.

$$(\text{ii}') \quad \text{poly}(\log|\Omega|, 1/\varepsilon) \text{ amount of work in the unit cost model for } \Sigma.$$  

All CertEcon results in this paper are actually strong CertEcon results.

**Remark 2.10.** Strong CertEcon does not follow from AlgEcon, though it does follow from strong AlgEcon.

**Remark 2.11.** As in the AlgEcon context, the unit cost model can also be used in the case of infinite $\Sigma$. In fact, all our CertEcon results hold for infinite codomain in the unit cost model, i.e., they satisfy (ii').

2.5 Mean-list-decoding

Let $\mathcal{F} = \{f_i : i \in I\}$ be a family of received words $f_i \in \Sigma^\Omega$. By the size $|\mathcal{F}|$ we mean the size $|I|$ of the index set $I$. The **average distance** $\text{dist}(w, \mathcal{F})$ of a word $w \in \Sigma^\Omega$ to $\mathcal{F}$ is the average distance of $w$ to elements of $\mathcal{F}$, given by $\text{dist}(w, \mathcal{F}) = \mathbb{E}_{i \in I}[\text{dist}(w, f_i)]$. (The expectation $\mathbb{E}$ is taken with respect to the uniform distribution over $I$.)

**Definition 2.12 (Mean-lists).** We define the **mean list** $\mathcal{L}$ as the set of codewords within a specified average distance $\rho$ of the received words $\mathcal{F}$, i.e.,

$$\mathcal{L} = \mathcal{L}(C, \mathcal{F}, \rho) := \{w \in C : \text{dist}(w, \mathcal{F}) \leq \rho\}. \quad (1)$$

We write $\text{m\ell}(C, \rho) := \max_{\mathcal{F}} \mathcal{L}(C, \mathcal{F}, \rho)$ for the maximum mean-list size for a given distance $\rho$.

This concept was inspired by the use of repeated codes by Guo and Sudan [GS14], see Remark 5.4. As we shall see, the mean list-decoding concept helps expand the scope of our results, without making them more difficult to prove. We adapt above terminology to the context of mean-list-decoding.

**Combinatorial.** We wish to bound mean-list size by $|\mathcal{L}(C, \mathcal{F}, \rho)| \leq c'(\varepsilon)$ for some $c'(\varepsilon)$ that depends only on $\varepsilon$ and the class $\mathcal{C}$ of codes under discussion ($C \in \mathcal{C}$). We say that the class $\mathcal{C}$ of codes is **CombEconM** ("combinatorially economically mean-list-decodable") if $c'(\varepsilon) = \text{poly}(1/\varepsilon)$ for $C \in \mathcal{C}$. 
Algorithmic. We say that the class $\mathcal{C}$ of codes is $\text{AlgEconM}$ ("algorithmically economically mean-list-decodable") if it satisfies the definition of AlgEcon classes of codes, with the following modifications.

For each $\mathcal{C} \in \mathcal{C}$, the received word $f$ is replaced by a family $\mathcal{F}$ of received words and the list $\mathcal{L}$ becomes $\mathcal{L} = \mathcal{L}(\mathcal{C}, \mathcal{F}, \rho)$. Oracle access to $\mathcal{F}$ means that, given $i \in I$ and $\omega \in \Omega$, the oracle returns $f_i(\omega)$. Condition (ii) is replaced by the following.

(ii-M) $\text{poly}(\log|\Omega|, \log|\Sigma|, \log|\mathcal{F}|, 1/\varepsilon)$ amount of work.

Note that the number of queries to the family $\mathcal{F}$ remains $\text{poly}(\log|\Omega|, 1/\varepsilon)$.

Certificate. We say that a class $\mathcal{C}$ of codes is $\text{CertEconM}$ ("certificate economically mean-list-decodable") if it satisfies the definition of CertEcon codes, with the same modifications as AlgEconM.

Theorem 2.13. For a class $\mathcal{C}$ of codes, we have the following.

(i) $\mathcal{C}$ is $\text{CombEconM}$ if and only if it is $\text{CombEcon}$.

(ii) $\mathcal{C}$ is $\text{AlgEconM}$ if and only if it is $\text{AlgEcon}$.

(iii) $\mathcal{C}$ is $\text{CertEconM}$ if and only if it is $\text{CertEcon}$.

For more detailed statements and proofs see Section 5.2.

Remark 2.14 (Significance of mean-list-decoding). Dinur et al. show the CombEcon and AlgEcon list-decodability of $\{\text{abelian} \to \text{abelian}\}$ homomorphism codes [DGKS08]. We shall see that Theorem 2.13 quickly leads to the conclusion of CombEcon list-decodability of $\{\text{arbitrary} \to \text{abelian}\}$ homomorphism codes. The same inference can be made about AlgEcon list-decodability, assuming natural conditions about representation of the domain group. See Section 5.2 for details.

Strong mean-list-decoding

We say that $\mathcal{C}$ is a strong $\text{AlgEconM}$ code if it satisfies the definition of AlgEconM, except with (ii-M) replaced by (ii'-M) below. Similarly, we say that $\mathcal{C}$ is a strong $\text{CertEconM}$ code if it satisfies the definition of CertEconM, except with (ii-M) replaced by (ii'-M) below.

(ii'-M) $\text{poly}(\log|\Omega|, 1/\varepsilon)$ amount of work in the unit cost model for $\Sigma$ and unit sampling cost model for $\mathcal{F}$.

In the unit sampling cost model for $\mathcal{F} = \{f_i : i \in I\}$, we charge unit cost for naming any $i \in I$ and for generating a uniform random $i \in I$.

2.6 Subword extension

In this section we introduce terminology to formalize our strategy to advance from certificate-list-decoding to algorithmic list-decoding (Observation 2.24 below).

Definition 2.15 (Subword extension problem). Let $\mathcal{C}$ be a code. The subword extension problem asks, given a partial map $\gamma : \Omega \to \Sigma$, whether $\gamma$ extends to a codeword in $\mathcal{C}$.

A subword extender is an algorithm that answers this question and returns a codeword in $\mathcal{C}$ extending $\gamma$, if one exists.
Observation 2.16. A certificate-list-decoder and a subword extender combine to a list-decoder.

Remark 2.17. This observation describes our two-phase plan to prove algorithmic list-decodability results for homomorphism codes with alternating domains. In the case of homomorphism codes, the subword extension problem corresponds to the homomorphism extension problem (see Section 4.9). The algorithmic difficulty of the homomorphism extension problem is a major bottleneck to further progress.

In fact, the plan suggested by this observation is too ambitious. We have no hope to solve the subword extension problem in cases of interest for all subwords.

Therefore, we relax the subword extender concept; correspondingly, we strengthen the notion of certificates required.

Let $W$ be a set of $\Omega \to \Sigma$ partial maps.

Definition 2.18 ($W$-subword extender). The $W$-subword extension problem asks to solve the subword extension problem on inputs from $W$. A $W$-subword extender is an algorithm $\mathcal{A}$ that takes as input any partial map $\gamma : \Omega \to \Sigma$ and returns a yes/no answer; and in the case of a “yes” answer, it also returns a codeword $\mathcal{A}(\gamma) \in C$, such that

- if $\gamma \in W$ then the answer is “yes” if and only if $\gamma$ extends to a codeword, and in this case, $\mathcal{A}(\gamma)$ is a codeword that extends $\gamma$.

Remark 2.19. Note that $\mathcal{A}$ is not required to decide whether $\gamma \in W$. $\mathcal{A}$ must correctly decide extendability of $\gamma$ for all $\gamma \in W$; in case $\gamma \notin W$, the algorithm may return an arbitrary answer.

Definition 2.20 ($W$-certificate). A $W$-certificate is a certificate that belongs to $W$.

Definition 2.21 ($W$-certificate-list). We say that a list $\Gamma$ of $\Omega \to \Sigma$ partial maps is a $W$-certificate-list for the set $\mathcal{K} \subseteq C$ of codewords if $\Gamma$ contains a $W$-certificate for each codeword in $\mathcal{K}$. A $W$-certificate-list for $\mathcal{C}$ up to distance $\rho$ of the received word $f : \Omega \to \Sigma$ is a $W$-certificate-list for the list $\mathcal{L} = \mathcal{L}(\mathcal{C}, f, \rho)$.

Remark 2.22. Note that, as mentioned in Remark 2.4, we permit the $W$-certificate-list $\Gamma$ to contain redundancies and irrelevant items, including partial functions $\gamma$ that do not belong to $W$.

Definition 2.23. A $W$-certificate-list-decoder is an algorithm that, given the received word $f \in \Sigma^\Omega$ and the distance $\rho$, constructs a $W$-certificate-list of $\mathcal{C}$ up to distance $\rho$ of $f$.

Our overall strategy for the case when $G$ is “far from abelian” is summarized in the following observation.

Observation 2.24. For any set $W$ of $\Omega \to \Sigma$ partial maps, a $W$-certificate-list-decoder and a $W$-subword extender combine to a list-decoder.

Definition 2.25. We say that $\mathcal{C}$ is a $W$-CertEcon (“$W$-certificate-economically list-decodable”) code if there exists a local $W$-certificate-list-decoder with the features listed in Definition 2.6.

Definition 2.26. We say that $\mathcal{C}$ is a strong $W$-CertEcon code if there exists a strong $W$-certificate-list-decoder, i.e., a $W$-certificate-list-decoder that is a strong certificate-list-decoder (see Definition 2.9).
2.7 Minimum distance versus maximum agreement

Recall that our code space is $\Sigma^\Omega$, the set of $\Omega \to \Sigma$ functions. In the theory of error-correcting codes, the usual measure of distance between two functions (strings) is the (normalized) Hamming distance, the fraction of symbols on which they differ. Following [GKS06], we find it convenient to consider the measure complementary to normalized Hamming distance, the (normalized) agreement,

$$\text{agr}(f, g) := \frac{1}{|\Omega|}|\{\omega \in \Omega \mid f(\omega) = g(\omega)\}|,$$

the fraction of positions on which the two functions $f, g : \Omega \to \Sigma$ agree.

**Definition 2.27.** The maximum agreement of the code $\mathcal{C}$ is given by

$$\Lambda_{\mathcal{C}} := \max_{\varphi, \psi \in \mathcal{C}} \text{agr}(\varphi, \psi).$$

**Fact 2.28.** The minimum distance is the complement of the maximum agreement, i.e.,

$$\text{mindist} = 1 - \Lambda_{\mathcal{C}}.$$

So, the codewords within distance ($\text{mindist} - \varepsilon$) of a received word $f$ are the same as the codewords with agreement at least $\Lambda_{\mathcal{C}} + \varepsilon$ with $f$.

Classes of examples for the infeasibility of list-decoding outside this range were provided by Guo and Sudan [GS14] for abelian domain and codomain, and we provide such classes for alternating domain (see Section 9.4), so the list-decoding radius is $\text{mindist}$ for those classes.

3 Preliminaries

Let $G$ be a set. For any subset $S \subseteq G$, define the density of $S$ in $G$ by $\mu_G(S) = \frac{|S|}{|G|}$. We call $G$ the “ambient set” and write $\mu(S) = \mu_G(S)$ when $G$ is understood. The ambient set will generally be a group $G$.

3.1 Groups

In this paper we will denote the class of all groups (finite or infinite) by $\text{Groups}$. We write $\text{Abel}$ to denote the class of finite abelian groups and $\text{Alt}$ for the class of (finite) alternating groups.

Our group theory reference is [Rob95]. We review some definitions and facts.

Let $G$ be a group. We write $H \leq G$ to express that $H$ is a subgroup; we write $H \trianglelefteq G$ if $H$ is a normal subgroup. We refer to cosets of subgroups of $G$ as subcosets. For the subcoset $aH$ of $G$ (where $H \leq G$), let $|G : aH| := |G : H|$ denote the index of $H$ in $G$. For a subset $S$ of a group $G$, the subgroup $\langle S \rangle$ generated by $S$ is the smallest subgroup of $G$ containing $S$. If $\langle S \rangle = G$, then $S$ generates $G$. A subset $K \subseteq G$ is affine-closed if $(\forall a, b, c \in K)(ab^{-1}c \in K)$. An affine-closed subset is either empty or it is a subcoset. The intersection of affine-closed subsets is affine-closed. The affine closure $\langle S \rangle_{\text{aff}}$, affinely generated by $S$, is the the smallest affine-closed subset containing $S$. Note that the affine closure of the empty set is empty. The affine closure of a nonempty set is a subcoset; indeed, for any $q \in S$, we have that $\langle S \rangle_{\text{aff}} = q \cdot \langle q^{-1}r \mid r \in S \rangle$. 

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3.2 Homomorphism codes

3.2.1 Affine homomorphisms as codewords

Let $G$ be a finite group and $H$ a group. Denote the set of homomorphisms from $G$ to $H$ by $\text{Hom}(G,H)$.

**Definition 3.1.** Let $G_1$ and $H_1$ be affine-closed subsets of $G$ and $H$, resp. A function $\varphi: G_1 \to H_1$ is an affine homomorphism if

$$(\forall a, b, c \in G_1)(\varphi(a)\varphi(b)^{-1}\varphi(c) = \varphi(ab^{-1}c)).$$

We write $\text{aHom}(G_1, H_1)$ to denote the set of affine homomorphisms from $G_1$ to $H_1$.

**Fact 3.2.** Let $G_1 \leq G$ and $H_1 \leq H$. Let $a \in G$ and $b \in H$. A function $\varphi: aG_1 \to bH_1$ is an affine homomorphism if and only if there exists $h \in H$ and $\varphi_0 \in \text{Hom}(G_1, H_1)$ such that

$$\varphi(g) = h \cdot \varphi_0(g)$$

for every $g \in G_1$. The element $h$ and the homomorphism $\varphi_0$ are unique.

The analogous statement also holds with $h$ on the right of $\varphi_0$.

**Definition 3.3.** For sets $G, H$ and functions $f, g: G \to H$, the equalizer $\text{Eq}(f, g)$ is the subset of $G$ on which $f$ and $g$ agree, i.e.,

$$\text{Eq}(f, g) := \{x \in G \mid f(x) = g(x)\}.$$  

More generally, if $\Phi$ is a collection of functions from $G$ to $H$, then the equalizer $\text{Eq}(\Phi)$ is the set

$$\text{Eq}(\Phi) := \{x \in G \mid (\forall f, g \in \Phi)(f(x) = g(x))\}.$$  

**Fact 3.4.** (a) If $\varphi, \psi \in \text{Hom}(G, H)$ then $\text{Eq}(\varphi, \psi) \leq G$.

(b) If $\varphi, \psi \in \text{aHom}(G, H)$ then $\text{Eq}(\varphi, \psi)$ is affine-closed. Moreover, if $\varphi_0, \psi_0 \in \text{Hom}(G, H)$ are the corresponding homomorphisms (see (3)) then either $\text{Eq}(\varphi, \psi)$ is empty or $\text{Eq}(\varphi, \psi) = g \cdot \text{Eq}(\varphi_0, \psi_0)$ for any $g \in \text{Eq}(\varphi, \psi)$.

Recall that the (normalized) agreement $\text{agr}(f, g)$ between two functions $f, g: G \to H$ is given by

$$\text{agr}(f, g) := \frac{|\text{Eq}(f, g)|}{|G|}.$$  

Specializing Def. 2.27 to homomorphism codes, we write

$$\Lambda_{G,H} := \Lambda_{\text{aHom}(G, H)}$$

for the maximum agreement of $\text{aHom}(G, H)$. In other words,

$$\Lambda_{G,H} := \max_{\varphi, \psi \in \text{aHom}(G, H)} \text{agr}(\varphi, \psi)$$

If the groups $G$ and $H$ are understood, we often write $\Lambda$ in place of $\Lambda_{G,H}$. Using this terminology, the min distance of the homomorphism code $\text{aHom}(G, H)$ is $(1 - \Lambda_{G,H})$.

The following statement appears in [Guo15, Prop. 3.5]. We include the proof for completeness.
Proposition 3.5 (Guo). Let $G, H$ be groups. The maximum agreement $\Lambda_{G,H}$ can equivalently be defined with $\text{aHom}$ replaced by $\text{Hom}$, i.e.,

$$\Lambda_{\text{Hom}(G,H)} = \Lambda_{\text{aHom}(G,H)}.$$ 

Here we use the convention that the maximum of the empty set (of nonnegative numbers) is zero. Otherwise we would need to make the additional assumption $|\text{Hom}(G,H)| > 1$.

Proof. Let $\Lambda'_{G,H} = \Lambda_{\text{Hom}(G,H)}$.

Obviously $\Lambda_{G,H} \geq \Lambda'_{G,H}$. Now let $\varphi, \psi \in \text{aHom}(G,H)$ be such that for all $g \in G$ we have $\varphi(g) = h_1 \varphi_0(g)$ and $\psi(g) = h_2 \psi_0(g)$.

It follows that if $g \in \text{Eq}(\varphi, \psi)$ then $\text{Eq}(\varphi, \psi) = g \text{Eq}(\varphi_0, \psi_0)$. Hence $\text{agr}(\varphi, \psi)$ is either zero or equal to $\text{agr}(\varphi_0, \psi_0)$, proving that $\Lambda_{G,H} \leq \Lambda'_{G,H}$. \[\square\]

Corollary 3.6. Let $G$ be a finite group and $H$ a group. Then, $\Lambda \leq \max\{\mu(K) \mid K \leq G\}$, the largest density of a proper subgroup of $G$.

Fact 3.7. Let $G$ and $H$ be groups and $S \subseteq G$ a subset. If $\varphi, \psi \in \text{aHom}(G,H)$ and $\varphi(x) = \psi(x)$ for all $x \in S$, then $\langle S \rangle_{\text{aff}} \subseteq \text{Eq}(\varphi, \psi)$. \[\square\]

Corollary 3.8. Let $G$ be a finite group, $H$ a group, and $S \subseteq G$, such that $\mu(\langle S \rangle_{\text{aff}}) > \Lambda_{G,H}$. If $\varphi, \psi \in \text{aHom}(G,H)$ are such that $\varphi(x) = \psi(x)$ for all $x \in S$, then $\varphi = \psi$.

Remark 3.9 (Why affine?). The reader may ask, why we (and all prior work) consider affine homomorphisms rather than homomorphisms. The reason is that affine homomorphisms are simply the more natural objects in this context. To begin with, this object is more homogeneous. For instance, for finite $H$, under random affine homomorphisms, the images of any element $g \in G$ are uniformly distributed over $H$.

This uniformity also serves as an inductive tool: when extending the domain from a subgroup $G_0$ to a group $G$, the action of any homomorphism $\varphi \in \text{Hom}(G,H)$ can be split into actions on the cosets of $G_0$ in $G$. Those actions are affine homomorphisms. On the other hand we also note that list-decoding of $\text{Hom}(G,H)$ and $\text{aHom}(G,H)$ are essentially equivalent tasks; see Section 5.5.

3.3 Computational representations of groups and homomorphisms

In this section we discuss the models of access to groups required by our algorithms. The choice of the model significantly impacts the running time and even the feasibility of an algorithm.

The models include oracle models (black-box access, black-box groups), generator-relator presentations, and various explicit models such as permutation groups, matrix groups, direct products of cyclic groups of known orders.

Recall that our domain groups are always finite but the codomain may be infinite (Convention 1.1).

Recall also that homomorphisms will be represented by the list of their values on a set of generators.

3.3.1 Black-box models

If the codomain is infinite, and even if it is finite but very large, the black-box-group model with its fixed-length encoding [BS84a] is not appropriate (see “encoded groups” below). We start with an extension of that model.
Definition 3.10 (Black-box access). An unencoded black-box representation of a (finite or infinite) group $K$ is an ordered 5-tuple $(U, r, \text{mult}, \text{inv}, \text{id})$ where

- $U$ is a (possibly infinite) set;
- $r: U \rightarrow K \cup \{\ast\}$ with $r(U) \supseteq K$;
- $\text{mult}: r^{-1}(K) \times r^{-1}(K) \rightarrow r^{-1}(K)$ with $r(\text{mult}(x, y)) = r(x)r(y)$ for all $x, y \in r^{-1}(K)$;
- $\text{inv}: r^{-1}(K) \rightarrow r^{-1}(K)$ with $r(\text{inv}(x)) = r(x)^{-1}$ for $x \in r^{-1}(K)$; and
- $\text{id}: r^{-1}(K) \rightarrow \{\text{yes, no}\}$ with $\text{id}(x) = \text{yes}$ if and only if $r(x)$ is the identity in $K$.

We say that an algorithm has black-box access to the group $K$ if the algorithm can store elements of $U$ and query the functions (oracles) $\text{mult}, \text{inv}, \text{id}$. We say that $K$ is given as an (unencoded) black-box group if in addition a list of generators of $K$ is given.

Remark 3.11. We emphasize that the difference between black-box access to a group $G$ and the group $G$ being given as a black-box group is that in the latter model, a list of generators of $G$ is given, whereas no elements of $G$ may be a priori known in the former.

If $U = \{0, 1\}^n$ then we talk about an encoded group, of encoding length $n$. This of course implies that $K$ is finite, namely, $|K| \leq 2^n$. (This is the model introduced in [BS84b].)

In an abuse of notation, when black-box access to a group $K$ is given, we may refer to elements of $r^{-1}(K)$ by their images under $r$, we may write $gh$ in place of $\text{mult}(g, h)$, we may write $g^{-1}$ in place of $\text{inv}(g)$, and we may write $g = 1$ in place of $\text{id}(g) = \text{yes}$.

Access to domain and codomain. In general we shall not need generators of the codomain, $H$, just black-box access. On the other hand, we do need generators of the domain, $G$; homomorphisms will be defined by their values on a set of generators. So our access to the domain will be assumed to be at least as strong as an (encoded) black-box group.

The black-box unit cost model. The (unencoded) black-box access model is particularly well suited to the unit-cost model where we assume that we can copy and store an element of $U$ and query an oracle at unit cost. We shall analyze our algorithms in the unit-cost model for the codomain $H$. This essentially counts the operations performed in $H$, so its bit-cost will incur an additional factor of $O(\log |H|)$ (if $H$ is finite and nearly optimally encoded).

Random generation. In encoded black-box groups, independent nearly uniform random elements can be generated in time, polynomial in the encoding length [Bab91].

Remark 3.12. Black-box groups have been studied in a substantial body of literature, both in the theory of computing and in computational group theory (see the references in [BBS09]). It is common to make additional access assumptions to a black-box group (assume additional oracles) such as an oracle for the order of the elements.

Given a black-box group $H$, we cannot determine the order $|H|$ or the order of a given element $h \in H$. In fact, even with an oracle for the order of elements, $\mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_p$ cannot be distinguished in fewer than $\Omega(\sqrt{p})$ randomized black-box identity queries.
To avoid such obstacles, it is common to assume additional information beyond black-box access. In finding $\Lambda_{G,H}$ for abelian domain $G$, one needs to decide if a given prime divides $|H|$. To accomplish this, we assume additional information about the group $H$ such as the order $|H|$ or the list of primes dividing $|H|$.

### 3.3.2 Generator-relator presentation, homomorphism checking

By “presentation” of a group we mean generator-relator presentation.

For a group given by a presentation, basic questions, such as whether the group has order 1, are undecidable. However, special types of presentations, such as polycyclic presentations of finite solvable groups, are often helpful. Note, however, that it is not known how to efficiently perform group operations in a finite solvable group given by a polycyclic presentation, so such presentations cannot answer basic black-box queries.

Any presentation, however, can be used for homomorphism checking, a critical operation in decoding homomorphism codes.

**Proposition 3.13** (Homomorphism checking). Let $S \subseteq G$ be a list of generators of $G$. Assume a presentation of $G$ is given in terms of $S$. Let $\varphi : S \to H$ be a function. Then $\varphi$ extends to a homomorphism $\bar{\varphi} : G \to H$ if and only if the list $(\varphi(s) \mid s \in S)$ satisfies the relations.

Note that this gives an efficient way to check whether $\varphi$ extends to a homomorphism if the relators are short or are given as short straight-line programs, assuming black-box access to the codomain.

**Definition 3.14.** Let $G$ be a group and $S = \{s_1, \ldots, s_k\}$ a list of elements of $G$. A straight-line program in $G$ from $S$ to $g \in G$ is a sequence $P = (x_1, \ldots, x_m)$ of elements of $G$ such that each $x_i$ is either a member of $S$ or a product of the form $x_j x_k$ for some $j, k < i$ or $x_i^{-1}$ for some $i < k$. We say that the element $x_m$ is given in terms of $S$ by the straight-line program $P$.

The following is well known.

**Proposition 3.15.** Let $G \leq S_n$ be a permutation group and $S$ a set of generators of $G$. Given $S$, a presentation of $G$ in terms of $S$ can be computed in polynomial time, where the relators returned are represented as straight-line programs.

### 3.3.3 Abelian groups

The invariant factor decomposition of a finite abelian group $G$ is a decomposition as a direct product of cyclic groups, $G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, where for each $i$, the integer $n_i$ divides $n_{i+1}$. Such a decomposition can be further split into a direct product of cyclic groups of prime power order; the result is a primary decomposition.

Any abelian presentation of a finitely generated abelian group can be converted into an invariant factor decomposition in polynomial time, using the Smith normal form. However, moving to a primary decomposition requires factoring the order of the group; to this end, knowing a superset of the prime divisors of the order suffices. All prior algorithmic results as well as those of the present paper on homomorphism codes with abelian domain require the primary decomposition.
4 Formal statements

4.1 List-decoding homomorphism codes

Let $\mathcal{D}$ be a class of pairs $(G, H)$ of groups. We say that $\mathcal{D}$ is CombEcon if the class $\{\text{aHom}(G, H) \mid (G, H) \in \mathcal{D}\}$ of codes is CombEcon. We define CertEcon and AlgEcon classes of pairs of groups analogously.

Denote by Groups the class of all groups, finite or infinite. Recall that we say that a class $\mathcal{G}$ of finite groups is universally CombEcon if $\mathcal{G} \times \text{Groups}$ is CombEcon. We define universally CertEcon and universally AlgEcon analogously, under access models to be specified.

A common feature of the prior work reviewed in Section 1.1 is that each class of pairs of groups considered was CombEcon and AlgEcon.

The present work continues to maintain this feature.

All previously existing results put structural restrictions both on the domain and the codomain. In particular, they were restricted to subclasses of the solvable groups. In this paper we extend the economical list-decodability (both combinatorial and algorithmic) in the following three directions.

1. We give a general principle for removing certain types of constraints on the domain (see Section 4.2). It will follow that the previously known results extend to arbitrary domains.

2. We find universally economically list-decodable classes of groups

   Specifically, abelian and alternating groups are universally CombEcon. Moreover, abelian groups are universally AlgEcon, and alternating groups are universally CertEcon, under modest access assumptions.

3. We exhibit the first (nontrivial) classes of examples where the domain is not solvable.

   We note that no CombEcon bounds appear to be known for the much-studied classical linear codes (Reed–Solomon, Reed–Muller, BCH) (cf., e.g., [BL15]).

   The poly$(1/\varepsilon)$ CombEcon bound for Hadamard codes is quadratic [GL89]. For abelian and nilpotent groups, it currently has degree 105 [DGKS08, GS14].

4.2 Extending the domain: the irrelevant kernel

In the prior work reviewed, both the domain and the codomain was abelian or close to abelian (nilpotent or supersolvable). It is natural to ask how to further relax the structural constraints on the groups involved.

We point out that structural constraints such as nilpotence or solvability (or any other hereditary property) play a very different role if imposed on the domain as on the codomain. For instance, a combinatorial list-decoding bound on $\{\text{abelian} \to \text{abelian}\}$ homomorphism codes implies the same bound for $\{\text{arbitrary} \to \text{abelian}\}$ homomorphism codes. This is shown by reducing list-decoding $\text{aHom}(G, H)$ for arbitrary $G$ and abelian $H$ to mean-list-decoding $\text{aHom}(G/G', H)$, where $G'$ is the commutator subgroup of $G$, so $G/G'$ is the largest abelian quotient of $G$. A similar argument extends the bounds for $\{\text{nilpotent} \to \text{nilpotent}\}$ homomorphism codes to $\{\text{arbitrary} \to \text{nilpotent}\}$, working through the largest nilpotent quotient of $G$.

   Similar results hold for certificate and algorithmic list-decoding.

   In general, we can replace $G$ by its relevant quotient $G/N$, where $N$ is the irrelevant kernel (intersection of the kernels of all $G \to H$ homomorphisms), see Sec. 5.4.
While this observation extends the reach of the results of Dinur et al. [DGKS08] and Guo and Sudan [GS14], it also shows that, in a sense, the gains by extending the class of groups serving as the domains, without relaxing the structural constraints on the codomains, are \textit{virtual}, and the main impediment to extending these results to wider classes of pairs of groups is the structural constraints on the codomain.

Our main contribution is the \textbf{elimination of all constraints on the codomain.} This also opens up the question of meaningfully (as opposed to “virtually”) \textbf{removing structural constraints on the domain side.} Of particular interest becomes the case where the domain is a \textit{finite simple group} and the codomain is arbitrary. We initiate this direction by studying the class of \textbf{alternating groups as domains.}

\textbf{Definition 4.1 (Irrelevant kernel).} Let $G$ and $H$ be groups. The $(G, H)$-\textbf{irrelevant kernel} (or “irrelevant kernel” if $G$ and $H$ are clear) is the intersection of the kernels of all $G \rightarrow H$ homomorphisms, i.e.,

$$\bigcap_{\varphi \in \text{Hom}(G, H)} \ker(\varphi).$$

(4)

We call elements and subgroups of the irrelevant kernel \textbf{irrelevant.}

For instance, if $H$ is abelian, then the commutator subgroup $G'$ is irrelevant.

\textbf{Theorem 4.2.} Let $N$ be an irrelevant normal subgroup of $G$. Then, $\Lambda_{G/N,H} = \Lambda_{G,H}$. Moreover,

(i) if $\text{aHom}(G/N, H)$ is CombEcon then $\text{aHom}(G, H)$ is CombEcon;

(ii) if $\text{aHom}(G/N, H)$ is CertEcon then $\text{aHom}(G, H)$ is CertEcon;

(iii) if $\text{aHom}(G/N, H)$ is AlgEcon then $\text{aHom}(G, H)$ is AlgEcon.

For items (ii) and (iii) we need to make suitable assumptions on access to the domain.

For the proofs and discussion, see Section 5.2. The proofs rely on mean-list-decoding (Theorem 2.13).

\textbf{Corollary 4.3.} The code $\text{aHom}(G, H)$ is AlgEcon for any finite group $G$ and any finite nilpotent group $H$.

\textbf{Proof.} Combine Theorem 4.2 and the main result of [GS14]. For abelian $H$, use [DGKS08] instead. \hfill $\square$

\subsection*{4.3 List-decoding: abelian $\rightarrow$ arbitrary}

We state our main result about abelian domains.

\textbf{Theorem 4.4.} If $G$ is a finite abelian group, then $G$ is universally CombEcon and strong AlgEcon list-decodable.

The degree of the $\text{poly}(1/\varepsilon)$ list-size bound is $C+4$ where $C$ is the bound for \{abelian $\rightarrow$ abelian\} homomorphism codes (currently $C \approx 105$ [GS14]).

The proof of the CombEcon bound is based on the following structural result that says the range of all relevant homomorphisms is covered by a small number of finite abelian subgroups of $H$.

\textbf{Theorem 4.5 (Structure of range).} Let $G$ be a finite abelian group, $H$ an arbitrary group (finite or infinite), $f \in H^G$ a function, and $\varepsilon > 0$. Then there exists a set $\mathcal{A}$ of finite abelian subgroups of $H$ with $|\mathcal{A}| < \frac{1}{4(\Lambda + \varepsilon)\varepsilon^2} + \frac{1}{\varepsilon}$ such that for all $\varphi \in \mathcal{L}($Hom$(G, H), f, \Lambda + \varepsilon)$, there is $M \in \mathcal{A}$ such that $\varphi(G) \leq M$. 

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Access model. We need to clarify how the algorithm accesses the domain and codomain. Following [DGKS08, GS14, BGSW18], we assume the domain is given explicitly as a direct product of cyclic groups of prime power order. We remark that representing the domain in terms of a presentation by generators and abelian relations would suffice, if we are also given a superset of the prime divisors of the order of the domain. Without that additional information, factoring would be required (see Section 3.3.3). — We only require black-box access to the codomain (see Definition 3.10).

Pointer. We prove Theorems 4.4 and 4.5 in Section 8. The essential new result is the CombEcon bound, proved in Section 8.2. The algorithm is an adaptation of the algorithm of [DGKS08, GS14], based on our CombEcon bound. This adaptation will be discussed in Section 8.3.

4.4 Shallow random generation and list-decodability

We shall consider groups with the property that a bounded number of random elements tend to generate a subgroup of bounded depth (see Definitions 4.9 and 4.10 below). This class includes the alternating groups. We show that groups in this class are CombEcon, and under minimal assumptions on access they are also CertEcon.

It will be useful to consider an $H$-independent lower bound on the quantity $\Lambda_{G,H}$. Definition 4.6. We define $\Lambda^*_G = \min\{\Lambda_{G,H} : \Lambda_{G,H} \neq 0, H \in \text{Groups}\}$.

Observation 4.7. For simple groups the following three quantities are equal: (a) $\Lambda^*_G$, (b) $\Lambda_{G,G}$, and (c) the largest fraction of elements of $G$ fixed by an automorphism.

Observation 4.8. For $G = A_n$, $n \geq 5$, we have $\Lambda^*_G = 1/\binom{n}{2}$.

The depth of a subgroup $M$ in a group $G$ is the length $d$ of the longest subgroup chain $M = M_0 < M_1 < \cdots < M_d = G$. We say that a subgroup is “shallow” if its depth is bounded. It follows from a result of citeBabai1989 that already a pair of elements in $A_n$ generates a subgroup of depth at most 6. This is the property that we generalize.

Definition 4.9 (Shallow random generation). Let $k, d \in \mathbb{N}$. We say that a finite group $G$ is $(k,d)$-shallow generating if

$$\Pr_{g_1, \ldots, g_k \in G}\left[\text{depth}(\langle g_1, \ldots, g_k \rangle) > d\right] < (\Lambda^*_G)^k.$$  \hfill (5)

Definition 4.10 (SRG groups). We say that a class $\mathcal{G}$ of finite groups has shallow random generation ($\mathcal{G}$ is SRG) if there exist $k, d \in \mathbb{N}$ such that all $G \in \mathcal{G}$ are $(k, d)$-shallow generating.

Lemma 4.11. The alternating groups are SRG groups. In particular, for sufficiently large $n$, the alternating group $A_n$ is $(2,6)$-shallow generating.

We prove this lemma in Section 10.1. We note that certain classes of Lie type simple groups are also SRG. We shall elaborate on this in a separate paper.

Now we can state one of the main results of this paper.

Theorem 4.12. If $G$ is an SRG group, then $G$ is universally CombEcon list-decodable.

For the case of alternating groups, we show that the degree of the poly$(1/\epsilon)$ list-size bound is at most 9; with further work this can be reduced to 7.

Theorem 4.13. If $G$ is an SRG group, then $G$ is universally strong CertEcon list-decodable.
In fact, SRG groups are universally strong $W_{\Lambda_{G,H}}$-CertEcon list-decodable (see Section 2.6 for the definition of $W$-certificates and section 4.5 for the definition of $W_{\Lambda_{G,H}}$). This restriction on the type of certificates we obtain is necessary for extensions to AlgEcon results (cf. comment before Definition 2.18). Section 4.5 discusses $W$-certificates in the context of homomorphism codes. A formal statement of the $W_{\Lambda_{G,H}}$-CertEcon result is given in Section 4.6.

Access model. For the CertEcon results, we assume access to (nearly) uniform random elements of the domain. We do not multiply elements of the domain, so we do not need black-box access to the domain. However, representing the domain as an encoded black-box group suffices for random generation (see Sec. 3.3.1).

We need no access to the codomain.

Pointers. We prove the CombEcon result in Section 11.1 and the CertEcon result in Section 11.2. For alternating groups we also give another, non-algorithmic, proof of the CombEcon result in Section 9. That proof relies on a generic sphere packing argument to split the sphere into more tractable bins (see Lemma 7.3 and Section 9.2).

4.5 Certificate list-decoding for homomorphism codes

First we translate the concepts associated with certificate list-decoding (Section 2.4) to the context of homomorphism codes. A certificate $\gamma$ is a $G \rightarrow H$ partial map that extends uniquely to an affine homomorphism $\varphi \in a\text{Hom}(G,H)$.

A subword extender is an algorithm that extends a $G \rightarrow H$ partial map to a full homomorphism if possible.

Recall that for a subset $S \subseteq G$, we denote by $\mu_G(S) := |S|/|G|$ the density of $S$ in $G$. For notational simplicity, we write $\Lambda$ for $\Lambda_{G,H}$.

Notation 4.14. Let $W_\Lambda$ (resp. $W_\Lambda^a$) be the set of $G \rightarrow H$ partial maps $\gamma$ such that $\mu(\langle \text{dom}(\gamma) \rangle) > \Lambda$ (resp. $\mu(\langle \text{dom}(\gamma) \rangle_{\text{aff}}) > \Lambda$).

Recall that we have introduced certificate list-decoding as an intermediate step towards algorithmic list-decoding, to address technical difficulties that arise in algorithmic list-decoding in the alternating case. Our plan is to apply Observation 2.24 on subword extension with $W = W_\Lambda^a$.

Observation 4.15. If a partial map $\gamma: G \rightarrow H$ belongs to $W_\Lambda^a$, then $\gamma$ extends to at most one affine homomorphism in $a\text{Hom}(G,H)$.

We will find $W_\Lambda^a$-certificate-list-decoders for a large class of homomorphism codes, and we wish to find corresponding $W_\Lambda^a$-subword-extenders.

Let $\gamma$ be a $G \rightarrow H$ partial map. We present three conditions on $\gamma$, then discuss their relationships to each other as well as to list-decoding.

1. If $\gamma$ extends to an affine homomorphism in $a\text{Hom}(G,H)$, then the extension is unique, i.e., $\gamma$ is a certificate for some affine homomorphism.

2. $\mu(\langle \text{dom}(\gamma) \rangle_{\text{aff}}) > \Lambda$.

3. The affine closure of $\text{dom}(\gamma)$ is $G$. 

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Clearly, Condition (3) implies Condition (2), which implies Condition (1). Implications in the other direction do not hold in general. In particular, neither reverse implication holds for the alternating groups.

Algorithmic list-decoding requires a list of full affine homomorphisms. (Recall that affine homomorphisms are represented as partial maps satisfying Condition (3).)

Certificate list-decoding requires the list of partial maps to satisfy Condition (1). Our CertEcon algorithms actually return certificates satisfying Condition (2), i.e., they are $W_a^\Lambda$-certificate-list-decoders.

In the case of abelian $G$, Condition (3) is equivalent to Condition (1) if the irrelevant kernel is trivial (see Definition 4.1). So, in this case certificate list-decoding and algorithmic list-decoding are equivalent. We introduced the mean-list-decoding machinery to address the case of nontrivial irrelevant kernel (see Theorems 2.13 and 5.18).

4.6 Certificate list-decoding: SRG $\rightarrow$ arbitrary

Recall that, in the context of list-decoding $a\Hom(G,H)$, $W_a^\Lambda$ denotes the set of $G \rightarrow H$ partial maps $\gamma$ such that $\mu((\text{dom}(\gamma))_{\text{aff}}) > \Lambda$, where $\Lambda = \Lambda_{G,H}$. We state the promised strengthening of Theorem 4.13.

**Theorem 4.16** (SRG certificate, abridged). If $G$ is an SRG group, then $G$ is universally strong $W_a^\Lambda$-CertEcon list-decodable.

**Access model.** We assume access to (nearly) uniform random elements of the domain. We do not multiply elements of the domain. We remark that representing the domain as a black-box group would suffice for random generation citeBab91BBpolygen.

We need no access to the codomain. We get ahold of elements of the codomain by querying the received word. We shall not perform any group operations in the codomain.

Actually our conclusion is much stronger than what would be implied by Theorem 4.16.

**Theorem 4.17** (SRG certificate, unabridged). Let $G$ be a $(k,d)$-shallow generating group and $H$ an arbitrary group. We have a local algorithm with the following features.

**Input:**
Values $\varepsilon, \eta > 0$.

**Output:** A set $\Pi \subseteq G^{k+d+1}$ of $(k + d + 1)$-tuples in $G$, where

$$|\Pi| = \left\lceil \frac{1}{\varepsilon^{k+d+1}} \ln \left( \frac{1}{\eta \varepsilon^{k+d+1}} \right) \right\rceil.$$

**Cost:** $\text{poly}(1/\varepsilon, \ln(1/\eta))$ amount of work.

**Performance guarantee:** For every received word $f \in H^G$, with probability at least $(1 - \eta)$, the set $\Gamma := \{f|_R : R \in \Pi\}$ is $W_a^\Lambda$-certificate-list for $a\Hom(G,H)$ up to distance $(\text{mindist} - \varepsilon)$ of $f$.

**Access model.** Same as in Theorem 4.16.

**Pointer.** The proof of Theorems 4.16 and 4.17 can be found in Section 11.2.
Remark 4.18. Given that $A_n$ is $(2,6)$-shallow generating (Lemma 4.11), Theorem 4.17 applies to $A_n$ with $k + d + 1 = 9$. We think of $A_n$ being given in its natural permutation representation. We note that a representation of $A_n$ as a black-box group would suffice, because the natural permutation representation of an alternating group can be efficiently extracted from a black-box group representation citeBBtoAlt.

4.7 Algorithmic list-decoding, assuming certificate list-decoding and homomorphism extension

In the light of Observation 2.24
(a certificate-list-decoder and a subword extender combine to a list-decoder) and the CertEcon results stated above, we need subword extenders for homomorphism codes.

The homomorphism extension problem is the same as the subword extension problem for $\text{Hom}(G, H)$. We shall see below that it can also be used to solve the subword extension problem for $\text{aHom}(G, H)$.

The Homomorphism Extension Problem asks whether a $G \rightarrow H$ partial map extends to a homomorphism on the whole group. As before, let $\Lambda = \Lambda_{G,H}$.

Definition 4.19. (Homomorphism Extension, $\text{HOMExt}(G, H)$)

Instance: A partial map $\gamma : G \rightarrow H$.

Solution: A homomorphism $\varphi \in \text{Hom}(G, H)$ that extends $\gamma$, i. e., $\varphi|_{\text{dom}\gamma} = \gamma$.

The Homomorphism Extension Decision Problem asks whether a solution exists. The Homomorphism Extension Search Problem asks to determine whether a solution exists and, if so, to find one.

Let $M$ denote the subgroup of $G$ generated by the domain of $\gamma$. The Homomorphism Extension problem splits into the following two questions.

(a) Does $\gamma$ extend to an $M \rightarrow H$ homomorphism? (If such an extension exists, it is clearly unique.)

(b) Given an $M \rightarrow H$ homomorphism, does it extend to a $G \rightarrow H$ homomorphism?

Question (a) can be solved efficiently if a presentation of $M$ is available in terms of the set $\text{dom}(\gamma)$ of generators and we have black-box access to $H$ (see Prop. 3.13). Such presentation can always be found efficiently if $G$ is given as a permutation group. (Prop. 3.15).

The difficult problem is to extend a homomorphism from $M$ to $G$. For $G = A_n$, we are only able to do this when $M$ has polynomial index in $G$. Therefore we consider the threshold version of the problem.

Definition 4.20. (Homomorphism Extension with Threshold, $\text{HOMExt}_{\lambda}(G, H)$)

Instance: A number $\lambda > 0$ and a partial map $\gamma : G \rightarrow H$ satisfying $\mu((\text{dom}\gamma)) > \lambda$.

Solution: A homomorphism $\varphi \in \text{Hom}(G, H)$ that extends $\gamma$, i. e., $\varphi|_{\text{dom}\gamma} = \gamma$.

Note that, if $\lambda_1 \leq \lambda_2$, then an oracle for $\text{HOMExt}_{\lambda_1}(G, H)$ can answer the $\text{HOMExt}_{\lambda_2}(G, H)$ queries.

Next we reduce the extension problem for affine homomorphisms to the $\text{HOMExt}$ problem, i. e., the extension problem for homomorphisms.

Proposition 4.21. Let $G$ and $H$ be groups to which we are given black-box access. Then, a subword extender for $\text{aHom}(G, H)$ can be implemented in poly($\text{enc}(G)$)-time in the unit-cost model for $H$, assuming we are given an oracle for the $\text{HOMExt}(G, H)$ search problem — that is, we have a subword extender for $\text{Hom}(G, H)$.
Proof. Let \( \gamma : G \to H \) be a partial map. If \( \text{dom}(\gamma) = \emptyset \) then the map \( G \to \{1_H\} \) to the identity element of \( H \) extends \( \gamma \). Otherwise, fix \( a \in \text{dom}(\gamma) \). Let \( \gamma_0 : a^{-1} \cdot \text{dom}(\gamma) \to H \) by \( \gamma_0(g) = \gamma(a)^{-1} \gamma(ag) \).

Then, \( \gamma_0 \) extends to a homomorphism \( \varphi_0 \) if and only if \( \gamma \) extends to an affine homomorphism \( \varphi \), with \( \varphi(g) = \gamma(a) \varphi_0(a^{-1}g) \) for all \( g \in G \).

Since Theorem 4.13 guarantees \( W^A_\Lambda \)-certificate-lists, we need only provide a \( W^A_\Lambda \)-subword extender (see Observation 2.24). In this case, the HOMEXT oracle may be relaxed to account for this restriction on certificates. Further elaborating on the comment after Remark 2.17, we note that this relaxation is critical to our application to the alternating group. While we are able to solve \( \text{HOMEXT}(A_n, S_m) \) for partial maps whose domain generates subgroups of polynomial index, we see little hope to solving it for all partial maps on \( A_n \).

The next result is the \( W^A_\Lambda \)-relaxation of Proposition 4.21.

**Proposition 4.22.** Let \( G \) and \( H \) be groups to which we are given black-box access. Suppose we are given an oracle for the \( \text{HOMEXT}_\Lambda(G, H) \) search problem. Then, a \( W^A_\Lambda \)-subword extender for \( \text{aHom}(G, H) \) can be implemented in \( \text{poly}(\text{enc}(G)) \) time in the unit-cost model for \( H \).

The proof is the same as that of Proposition 4.21.

**Remark 4.23.** In practice we may not be able to determine the value of \( \Lambda \), while we may be able to determine a rather large lower bound \( \lambda \leq \Lambda \). So, we instead ask for an oracle for \( \text{HOMEXT}_\lambda(G, H) \). This is the procedure we follow in this paper for the alternating group.

**Corollary 4.24.** Let \( G \) be an SRG group and \( H \) be an arbitrary group. Under the assumptions of Proposition 4.22, \( \text{aHom}(G, H) \) is \( \text{AlgEcon} \).

**Proof.** Combine Proposition 4.22 and Theorem 4.16.

**4.8 Homomorphism extension from alternating groups**

The following theorem addresses the HOMEXT Search Problem for the permutation representations of the alternating groups. This is the main result of [Wuu18].

**Theorem 4.25** (Wuu). Let \( G = A_n \), \( H = S_m \) and \( \lambda = 1/\text{poly}(n) \). If \( m < 2^{n-1}/\sqrt{n} \), then the HOMEXT\(_\Lambda(G, H) \) search problem can be solved in \( \text{poly}(n, m) \) time.

**Remark 4.26.** In fact, under the assumptions of Theorem 4.25, the number of extensions can be counted in \( \text{poly}(n, m) \) time.

This result is proved by looking at the orbits in \( [m] \) of the group \( M \) generated by the domain of the partial function, then deciding how they may combine to form orbits of \( G \). We reformulate HOMEXT with symmetric codomain as an exponentially large instance of a generalized Subset Sum Problem to which we have oracle access. The technical assumption \( m < 2^{n-1}/\sqrt{n} \) guarantees that the arising instance of generalized Subset Sum is tractable. Answering oracle queries amounts to solving certain problems of computational group theory such as the conjugacy problem for permutation groups.

**4.9 Algorithmic list-decoding: alternating \( \to \) symmetric, restricted cases**

We need one more ingredient before we can prove our main algorithmic result.

**Lemma 4.27.** Let \( n \geq 10 \). Let \( G = A_n \) and let \( H \) be a group. If \( \Lambda_{G,H} \neq 0 \), then either \( \Lambda_{G,H} = 1/(\binom{n}{2}) \) or \( \Lambda_{G,H} = 1/n \).
Proof. We note that $\Lambda_{G,G} \geq 1/(\binom{n}{2})$, since the identity automorphism of $G$ and the automorphism that sends $g$ to its conjugation by the transposition $(12)$ agree on $G_{\{1,2\}}$ which has index $\binom{n}{2}$ (In fact, $\Lambda_{G,G} = 1/(\binom{n}{2})$).

Suppose $\Lambda_{G,H} \neq 0$, so $\text{Hom}(G,H)$ is nontrivial. Since $A_n$ is simple, $H$ contains an isomorphic copy of $A_n$ (The image of a nontrivial homomorphism is isomorphic to $A_n$). So, $\Lambda_{G,H} \geq \Lambda_{G,G} \geq 1/(\binom{n}{2})$. By Fact 3.4 and the Jordan-Liebeck Theorem (see Section 9.1), $\Lambda_{G,H} = 1/(\binom{n}{2})$ or $1/n$.

We remark that Guo [Guo15, Proposition 6.1] proved that $1/(\binom{n}{2}) \leq \Lambda_{A_n,A_n} \leq 1/n$ for $n \geq 5$.

We have now stated all the ingredients needed for the AlgEcon result for alternating domains.

**Theorem 4.28 (AlgEcon for alternating domains).** If $G = A_n$ is an alternating group and $H = S_m$ is a symmetric group, then $\text{aHom}(G,H)$ is AlgEcon, assuming $m < 2^{n-1}/\sqrt{n}$.

**Access model.** We assume both $A_n$ and $S_m$ are given in their natural permutation representations.

**Proof.** The proof follows the “CertEcon with HomExt implies AlgEcon” approach discussed in Section 4.7.

Lemma 4.11 shows that alternating groups are SRG groups, which are universally $\mathcal{W}_\Lambda$-CertEcon by Theorem 4.16. Now $\Lambda_{G,H} \geq 1/(\binom{n}{2})$ by Lemma 4.27. Theorem 4.25 shows that $\text{HomExt}_{1/(\binom{n}{2})}(G,H)$ can be solved in $\text{poly}(n,m)$ time, under the stated restrictions on the codomain $H$.

This $\text{poly}(n,m)$-time subword extender combines with the $\mathcal{W}_\Lambda$-CertEcon claim to justify the AlgEcon claim via Observation 2.24.

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5 Bipartite covering arguments

5.1 Bipartite covering lemma

In this section we describe a simple combinatorial lemma that will be used in two separate contexts throughout this section (mean-list decoding with application to the domain-relaxation principle and the equivalence of efficiency of list-decoding Hom and aHom. The applications are both combinatorial and semi-algorithmic.

We write $X = (V,W;E)$ to denote a bipartite graph with given vertex partition $(V,W)$ (all edges go between $V$ and $W$). We denote the set of neighbors of vertex $u$ by $N(u)$.

**Lemma 5.1.** Let $\delta, \eta, L > 0$. Let $X = (V,W;E)$ be a bipartite graph. Suppose $\deg(v) \leq L$ for all $v \in V$ and $\deg(w) \geq \delta|V|$ for all $w \in W$. Then the following hold.

(a) (double counting lemma) $|W| \leq L/\delta$.

(b) (bipartite covering lemma) Set

$$s = \left[ \frac{4}{3\delta} \ln(L/(\eta \delta)) \right].$$

Choose a sequence $(u_1, \ldots, u_s) \in V^s$ uniformly at random. Create a set $U \subseteq V$ by independently including each $u_i$ with probability $3/4$. Then with probability $\geq (1-\eta)$, we have $W = \bigcup_{u \in U} N(u)$.
Proof. (a) Count edges two ways.

\[ L \cdot |V| \geq \sum_{v \in V} \deg(v) = \sum_{w \in W} \deg(w) \geq \delta |W||V|. \]

So, \(|W| \leq L/\delta\).

(b)

Given \(u_1, \ldots, u_s\), choose \(\hat{u}_1, \ldots, \hat{u}_s \in V \cup \{\star\}\) independently as follows. For each \(i = 1, \ldots, s\), let \(\hat{u}_i\) be \(u_i\) with probability \(3/4\) and \(\star\) otherwise. Define the neighbor set of \(\star\) by \(N(\star) = \emptyset\).

Fix \(w \in W\). We have \(\Pr_{v \in V}(w \in N(v)) \geq \delta\) by assumption. So, for each \(i\), \(\Pr_{\hat{u}_i}(w \in N(\hat{u}_i)) = \frac{3}{4} \cdot \Pr_{u_i}(w \in N(u_i)) \geq \frac{3}{4} \delta\). Since the \(\hat{u}_i\) were chosen independently,

\[ \Pr \left( w \not\in \bigcup_{i=1}^{s} N(\hat{u}_i) \right) \leq \left(1 - \frac{3}{4} \delta\right)^s. \]

Taking the union bound over \(w \in W\), we find that

\[ \Pr \left( W \not\subseteq \bigcup_{i=1}^{s} N(\hat{u}_i) \right) \leq |W| \left(1 - \frac{3}{4} \delta\right)^s \leq \frac{L}{\delta} \left(1 - \frac{3}{4} \delta\right)^s \leq \eta. \]

\(\square\)

### 5.2 Mean-list-decoding

This is achieved using the concept of mean-list-decoding, introduced in Section 2.5.

### 5.3 List size versus mean-list size

In this section we discuss results that apply to all codes, not just to homomorphism codes.

The main result of this section is Lemma 5.5, which shows that mean-lists are contained in a small number of random lists, with a slight degradation of the parameters. That mean-list size is bounded via list-size is shown by item (i) of Lemma 5.5. It follows immediately that the concepts of CombEconM and CombEcon are equivalent (Corollary 5.8). Lemma 5.5 item (ii) shows the equivalence of AlgEconM with AlgEcon and CertEconM with CertEcon, completing the proof of Theorem 2.13. Further consequences of Lemma 5.5 will follow in Section 5.4, leading to the constraint-relaxation principle on the domain.

Recall that \(\ell(C, \lambda)\) denotes the maximum list size for \(C\) with agreement \(\lambda\). Now we define the analogous quantities for mean-lists, slightly refining Def. 2.12. Let \(C\) be a code, \(r\) and \(s\) natural numbers, and \(\lambda, \delta > 0\).

**Definition 5.2 (Mean-list-size).** The **maximum \(r\)-mean-list size** for \(C\) with agreement \(\lambda\), denoted \(m_r \ell(C, \lambda)\), is the maximum size of the mean-lists \(L(C, \mathcal{F}, \lambda)\) over all families \(\mathcal{F}\) of \(r\) received words, i.e.,

\[ m_r \ell(C, \lambda) = \max\{|L(C, \mathcal{F}, \lambda)| : |\mathcal{F}| = r\}. \]

The **maximum mean-list size** for \(C\) with agreement \(\lambda\) is the maximum over the \(r\)-mean-list sizes for \(C\) with agreement \(\lambda\), i.e.,

\[ m \ell(C, \lambda) = \max_r m_r \ell(C, \lambda). \]

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Note that $m_1 \ell(C, \lambda) = \ell(C, \lambda)$.

From the definitions it follows that aHom$(G, H)$ is CombEconM if and only if

$$m \ell(\text{aHom}(G, H), \Lambda_{G, H} + \varepsilon) = \text{poly}(1/\varepsilon).$$

**Notation 5.3.** For a word $w$, we denote by $w \ast r = (w \ldots w)$ the word found by concatenating $r$ copies of $w$. For a set $S$ of words, we write $S \ast r := \{w \ast r : w \in S\}$.

**Remark 5.4** (Mean-list-decoding versus repeated codes). Let $\mathcal{F} = \{f_i : i \in [r]\}$ be a family of $r$ received words. Notice that $\mathcal{L}(\mathcal{C} \ast r, (f_1, \ldots, f_r), \lambda)$ is the $r$-fold repetition of $\mathcal{L}(\mathcal{C}, \mathcal{F}, \lambda)$, i.e.,

$$\mathcal{L}(\mathcal{C}, \mathcal{F}, \lambda) \ast r = \mathcal{L}(\mathcal{C} \ast r, (f_1, \ldots, f_r), \lambda).$$

It follows that $m \ell(\mathcal{C}, \lambda) = \ell(\mathcal{C} \ast r, \lambda)$. In this way, mean-list-decoding can be viewed as list-decoding repeated codes.

Next we state the central result of this section: every mean-list is covered by a small number of lists.

**Lemma 5.5** (Concentration of mean-lists). Let $\mathcal{C}$ be a code and $\lambda, \delta, \eta > 0$. Let $\mathcal{F} = \{f_i : i \in I\}$ be a family of received words. Let $\mathcal{L} = \mathcal{L}(\mathcal{C}, \mathcal{F}, \lambda + \delta)$. Then the following hold.

(i) $|\mathcal{L}| \leq \ell(\mathcal{C}, \lambda)/\delta$.

(ii) Set $s = \lceil \frac{1}{3\delta} \left( \ln \ell(\mathcal{C}, \lambda) + \ln(1/\eta\delta) \right) \rceil$. Choose a sequence $(j_1, \ldots, j_s) \in I^s$ uniformly at random. For each $i$ ($1 \leq i \leq s$) independently apply the list-decoder to the received word $f_{j_i}$ with agreement threshold $\lambda$. Let $\mathcal{L}_i$ denote the output list. Then, with probability $\geq 1 - \eta$, we have $\mathcal{L} \subseteq \bigcup_{i=1}^s \mathcal{L}_i$.

Not only does this lemma allow us to give combinatorial bounds for mean-lists in terms of lists, it will also be used to construct a (certificate-)mean-list-decoder from a (certificate-)list-decoder.

The proof will follow from the Bipartite covering lemma (Lemma 5.1) together with the following observation.

**Lemma 5.6** (Markov degradation). Fix a codeword $\varphi$. Let $\mathcal{F} = \{f_i : i \in I\}$ be a family of received words in the codespace. Assume $E_i(\text{agr}(\varphi, f_i)) \geq \lambda + \delta$. Then $\text{Pr}_i(\text{agr}(\varphi, f_i) > \lambda) > \delta$.

**Proof.** Let $x_i = \text{dist}(\varphi, f_i) = 1 - \text{agr}(\varphi, f_i)$. Then $E_i(x_i) \leq 1 - \lambda - \delta$. Therefore, by Markov’s inequality, $\text{Pr}(\text{agr}(\varphi, f_i) \leq \lambda) = \text{Pr}(\text{dist}(\varphi, f_i) \geq 1 - \lambda) \leq \frac{1}{1 - \lambda} = 1 - \frac{\delta}{1 - \lambda} < 1 - \delta$. \hfill $\square$

**Proof of Lemma 5.5.** We apply Lemma 5.1 to the bipartite graph $X = (I, \mathcal{L}; E)$ where the edge set $E$ consist of the pairs $(i, \varphi) \in I \times \mathcal{L}$ satisfying $\text{agr}(f_i, \varphi) > \lambda$. Then, $\text{deg}(f) \leq \ell(\mathcal{C}, \lambda)$ by the definition of max list size $\ell$ and $\text{deg}(\varphi) \geq \delta|I|$ by Lemma 5.6.

The decoder succeeds with probability at least $3/4$, so $\mathcal{L}_i \supseteq \mathcal{L}(\mathcal{C}, f_{j_i}, \lambda)$ happens with probability $\geq 3/4$ independently over $i = 1, \ldots, s$. The lemma follows from Lemma 5.1. \hfill $\square$

**Corollary 5.7.** For $\mathcal{C}$ a code, $r$ a natural number, and $\lambda, \delta > 0$, we have

$$m \ell(\mathcal{C}, \lambda + \delta) \leq \frac{1}{\delta} m_r \ell(\mathcal{C}, \lambda).$$
Proof. Let $s$ be a natural number. By the definition of $m\ell$, it suffices to show that $m_s\ell(C, \lambda + \delta) \leq \frac{1}{\delta}m_r\ell(C, \lambda)$.

But, we find that $m_s\ell(C, \lambda + \delta) \leq m_{sr}\ell(C, \lambda + \delta)$ by [GS14, Lemma 3.3] (though their lemma is stated in terms of repeated codes). By Lemma 5.5, we find that $m_{sr}\ell(C, \lambda + \delta) \leq \frac{1}{\delta}m_r\ell(C, \lambda)$. □

The following is now immediate.

**Corollary 5.8.** For $C$ a code and $\varepsilon > 0$, we have

$$m\ell(C, 1 - \text{mindist} + \varepsilon) \leq \frac{2}{\varepsilon}\ell(C, 1 - \text{mindist} + \varepsilon/2).$$

Consequently, if a class of codes is CombEcon with degree $c$, then it is CombEcon with degree $c + 1$.

Next, we derive the algorithmic versions of this result. We shall make the following assumption on access to our family $\mathcal{F} = \{f_i : i \in I\}$ of received words.

**Access 5.9.** An oracle provides uniform random elements of the index set $I$ of $\mathcal{F}$.

**Theorem 5.10.** Under Access 5.9, if a class $\mathcal{C}$ of codes is AlgEcon then it is AlgEconM. Under the same assumptions, if $\mathcal{C}$ is CertEcon then it is CertEconM.

**Remark 5.11.** The bounds on cost in the result above deteriorate as follows.

- A $2/\varepsilon$ multiplicative factor in list size.
- An $O(\frac{1}{\varepsilon}\ln(1/\varepsilon))$ multiplicative factor in queries to the received word $f$.
- An $O(\frac{1}{\varepsilon}\ln(1/\varepsilon))$ multiplicative factor in amount of work.

We show that, to (certificate-)mean-list-decode a family $\mathcal{F}$ of functions, (certificate-)list-decoding a small random subset of the functions in $\mathcal{F}$ suffices. Lemma 5.1 already contains the machinery to guarantee the necessary probability of success.

**Proof of Theorem 5.10.** We first prove the claim for AlgEcon. Let DECODE be a list-decoder for the class $\mathcal{C}$ satisfying AlgEcon assumptions. We denote by $\text{DECODE}(C, f, 1 - \text{mindist} + \varepsilon)$ the output of DECODE on the input $f$ and $\varepsilon > 0$, where $f$ is a received word in the code space of a code $C \in \mathcal{C}$.

We describe here a mean-list-decoder that satisfies AlgEconM. It takes as input $\mathcal{F}$ and $\varepsilon > 0$, where $\mathcal{F}$ is the family of received words in the code space of the code $C \in \mathcal{C}$.

Denote by $I$ the index set of $\mathcal{F}$. Via the provided oracle, generate a subset $S \subseteq I$ by picking $s$ elements of $I$ independently and uniformly. The value of $s$ will be determined later. Return the list given by

$$\bigcup_{i \in S} \text{DECODE}(C, f_i, 1 - \text{mindist} + \varepsilon/2).$$

(7)

We show that this is a list-decoder satisfying the conditions of AlgEconM through direct application of Lemma 5.5.

The output $\text{DECODE}(C, f, 1 - \text{mindist} + \varepsilon/2)$ contains $\mathcal{L}(C, f, 1 - \text{mindist} + \varepsilon/2)$ with probability $3/4$ by the definition of list-decoder. If $s$ is set as in Lemma 5.5 (ii) with $\eta = 1/4$ and $\delta = \varepsilon/2$, we find that the the desired mean-list is returned with probability at least $3/4$.

Notice that the list-decoder DECODE is called $s = \left\lceil \frac{8}{\varepsilon^2} (\ln(\ell(C, 1 - \text{mindist} + \varepsilon/2) + \ln(8/\varepsilon)) \right\rceil$ times as a subroutine. Very little processing is done outside of these calls. Moreover, since $\mathcal{C}$ is AlgEcon, it is CombEcon, so $\ell(C, 1 - \text{mindist} + \varepsilon/2) = \text{poly}(\varepsilon/2)$ and DECODE is called $O(\frac{1}{\varepsilon}\ln(1/\varepsilon))$
The proof for CertEcon is similar, found mainly by replacing the occurrences of “list-decoder” with “certificate-list-decoder.” The mean-certificate-list-decoder returns the union of \( s \) output lists by \( \text{DECODE} \), which would denote the assumed certificate-list-decoder.

\[ \square \]

The same conclusions follow for the strong versions of these concepts.

**Remark 5.12.** Knowledge of \( \text{mindist} \) is not needed in the conversion from CertEcon to CertEconM. Even in the AlgEcon case, \( \text{mindist} \) is only needed if required by the list-decoder \( \text{DECODE} \). The crucial knowledge for this conversion is \( \epsilon \), so that the deterioration factor (denoted \( \delta \) above) can be controlled. This deterioration factor is set to \( \delta = \epsilon/2 \) in our proofs.

### 5.4 Irrelevant normal subgroups and the domain relaxation principle

In this section we present the principle of lifting constraints on the domain. An example is the automatic extension of \{abelian\( \to \)abelian\} results to the \{arbitrary\( \to \)abelian\} context (Theorem 5.22. (See the discussion in Section 4.2 and Remark 2.14.)

The key concept is the *irrelevant kernel* \( N \) for a pair \((G, H)\) of groups, defined as the intersection of the kernels of all \( G \to H \) homomorphisms. We shall find that extending \( G/N \) to \( G \) retains economical list-decodability.

We first identify the code \( a\text{Hom}(G, H) \) with a repeated code found from \( a\text{Hom}(G/N, H) \). This hinges on \( N \) being an irrelevant normal subgroup. Recall that \( N \) is \((G, H)\)-irrelevant if \( N \) is contained in the kernel of every \( G \to H \) homomorphism (see Definition 4.1).

For groups \( K \) and \( H \), an enumeration \( K = \{ k_1, \ldots, k_{|K|} \} \) induces a bijection between the set of functions \( H^K \) and the set of words \( H^{[K]} \) by \( f \mapsto (f(k_1), \ldots, f(k_{|K|})) \).

**Observation 5.13** (Identification of \( a\text{Hom}(G, H) \) lists with \( a\text{Hom}(G/N, H) \) mean-lists). Let \( G, H \) and \( N \) be groups such that \( N \) is a \((G, H)\)-irrelevant normal subgroup. Let \( f : G \to H \). There are enumerations of \( G \) and \( G/N \), and a family \( \mathcal{F} \) of functions \( G/N \to H \) such that

\[ \mathcal{L}(a\text{Hom}(G, H), f, \lambda) = \mathcal{L}(a\text{Hom}(G/N, H), \mathcal{F}, \lambda) \ast |N|. \]

**Proof.** Let \( S = \{ s_1, \ldots, s_{|G:N|} \} \) be a set of coset representatives of \( N \) in \( G \). We write \( N = \{ g_1, \ldots, g_{|N|} \} \). We enumerate the elements of \( G/N \) as \( \{s_1N, \ldots, s_{|G:N|}N\} \), and we enumerate the elements of \( G \) by concatenating \( (s_1g_i, \ldots, s_{|G:N|}g_i) \) for \( i = 1, \ldots, |N| \). We thereby think of any function \( G/N \to H \) as a word in \( H^{[G:N]} \), and any function \( G \to H \) as a word in \( H^{[G]} \).

For \( i = 1, \ldots, |N| \), let \( f_i : G/N \to H \) by \( f_i(sN) = f(sg_i) \) for all \( s \in S \). Let \( \mathcal{F} = \{ f_i : i = 1, \ldots, |N| \} \).

Let \( \pi : G \to G/N \) be the projection onto cosets. We note that \( a\text{Hom}(G, H) = \{ \varphi \circ \pi : \varphi \in a\text{Hom}(G/N, H) \} \), since \( N \) is a \((G, H)\)-irrelevant subgroup. Then, interpreting functions as codewords as above, \( a\text{Hom}(G, H) = a\text{Hom}(G/N, H) \ast |N| \). In other words, the codeword \( \varphi \circ \pi \) is a concatenation \( (\varphi, \ldots, \varphi) \) of \(|N|\) copies of the codeword \( \varphi \).
Furthermore, for any $\varphi \in \text{aHom}(G/N, H)$, we have that

\[
\text{agr}(f, \varphi \circ \pi) = \frac{1}{|G|} |\{ g : f(g) = (\varphi \circ \pi)(g) \}|
\]

\[
= \frac{1}{|N|} \sum_{i=1}^{|N|} \frac{1}{|G/N|} |\{ s \in S : f(sg_i) = (\varphi \circ \pi)(sg_i) \}|
\]

\[
= \frac{1}{|N|} \sum_{i=1}^{|N|} \text{agr}(f_i, \varphi)
\]

\[
= \mathbb{E}_i \text{agr}(f_i, \varphi).
\]

This shows that $\varphi \circ \pi \in \mathcal{L}(\text{aHom}(G, H), f, \lambda)$ exactly if $\varphi \in \mathcal{L}(\text{aHom}(G/N, H), F, \lambda)$. So,

\[
\mathcal{L}(\text{aHom}(G, H), f, \lambda) = \{ \varphi \circ \pi : \varphi \in \mathcal{L}(\text{aHom}(G/N, H), F, \lambda) \}
\]

\[
= \mathcal{L}(\text{aHom}(G/N, H), F, \lambda) \ast |N|.
\]

\[\square\]

Remark 5.14. The proof of Observation 5.13 shows more: There is a bijection between functions $f \in H^G$ and families $\mathcal{F}$ of $|N|$ functions so that the equation holds.

Corollary 5.15. If $G, H$ and $N$ are groups such that $N$ is a $(G, H)$-irrelevant normal subgroup of $G$, then

\[
\ell(\text{aHom}(G, H), \lambda) = m_{|N|} \ell(\text{aHom}(G/N, H), \lambda).
\]

We first illustrate the relaxation principle combinatorially, through bounds on list-size. The goal would be to conclude that, if $\mathcal{G} \times \mathcal{G}$ is CombEconM for a class $\mathcal{G}$ of groups, then $\mathcal{Groups} \times \mathcal{G}$ is CombEcon. This principle will generalize to CertEcon and AlgEcon as well.

Remark 5.16. If $N$ is a $(G, H)$-irrelevant normal subgroup, then $\Lambda_{G/N, H} = \Lambda_{G, H}$.

Lemma 5.17 (Irrelevant normal subgroup lemma). Let $G, H$ and $N$ be groups such that $N$ is a $(G, H)$-irrelevant normal subgroup. Then,

\[
\ell(\text{aHom}(G, H), \Lambda_G + \varepsilon) \leq \frac{2}{\varepsilon} \cdot \ell(\text{aHom}(G/N, H), \Lambda_G + \varepsilon/2).\]

Proof. Calculate

\[
\ell(\text{aHom}(G, H), \Lambda + \varepsilon) = m_{|N|} \ell(\text{aHom}(G/N, H), \Lambda + \varepsilon)\text{ Corollary 5.15}
\]

\[
\leq m \ell(\text{aHom}(G/N, H), \Lambda)\text{ definition of } m \ell
\]

\[
\leq \frac{2}{\varepsilon} \cdot m \ell(\text{aHom}(G/N, H))\text{ Corollary 5.7 with } r = 1
\]

\[
= \frac{2}{\varepsilon} \cdot \ell(\text{aHom}(G/N, H), \Lambda + \varepsilon/2).
\]

\[\square\]

This implies that, if $\text{aHom}(G/N, H)$ is CombEconM, then $\text{aHom}(G, H)$ is CombEcon. This principle holds for CertEcon and AlgEcon as well.

Theorem 5.18. Let $G, H$ and $N$ be groups such that $N$ is a $(G, H)$-irrelevant normal subgroup of $G$.  

29
(i) If $a\text{Hom}(G/N, H)$ is CombEcon, then $a\text{Hom}(G, H)$ is CombEcon.

(ii) Under suitable access assumptions (Access 5.19 (ii)), if $a\text{Hom}(G/N, H)$ is CertEcon, then $a\text{Hom}(G, H)$ is CertEcon.

(iii) Under suitable access assumptions (Access 5.19 (iii)), if $a\text{Hom}(G/N, H)$ is AlgEcon, then $a\text{Hom}(G, H)$ is AlgEcon.

The deterioration in cost is as described in Remark 5.11.

Access 5.19. (ii) (a) Elements of $N$ can be generated uniformly. (b) A transversal, i.e., an injection $G/N \to G$ that assigns a representative element to each coset, is given. (c) $G/N$ is known well enough to satisfy the CertEcon access assumptions on $a\text{Hom}(G/N, H)$.

(iii) (a') Elements of $N$ can be generated uniformly and generators for $N$ are given. (b') Same as (b). (c') Same as (c), for AlgEcon.

Remark 5.20. If the access assumptions on $N$ are at least as strong as having $N$ as a black-box group, then generating (nearly) uniform elements and being given a set of generators are equivalent. If $N$ is a black-box group, generators are given by definition. Nearly uniform random elements in black-box groups can be generated in polynomial time (polynomial in the encoding length of groups elements).

Remark 5.21. For the proof of this theorem, we will actually need the –EconM versions of the assumptions, which we may assume as a consequence of Theorem 2.13.

Proof of Theorem 5.18. Set $\Lambda = \Lambda_{G,H} = \Lambda_{G/N,H}$. Let $\pi : G \to G/N$ be the projection onto cosets.

(i) $a\text{Hom}(G/N, H)$ is CombEconM, so $a\text{Hom}(G, H)$ is CombEcon by Corollary 5.15.

(ii) A certificate-list-decoder satisfying the conditions of CertEconM exists for $a\text{Hom}(G/N, H)$. Its output list $\Gamma$ is a certificate list for $L(a\text{Hom}(G/N, H), F, \Lambda + \varepsilon)$, where $\varepsilon > 0$ and $F = \{f_1, \ldots, f_{|N|}\}$ is constructed from $f$ as in Observation 5.13. We construct a certificate list $\tilde{\Gamma}$ for $L(a\text{Hom}(G, H), f, \Lambda + \varepsilon)$, by replacing each $G/N \mapsto H$ partial map $\gamma \in \Gamma$ with a $G \mapsto H$ partial map $\tilde{\gamma}$ defined as follows.

Denote by $\tau : G/N \to G$ the injection guaranteed by the assumption. Let $\tilde{\gamma}$ have domain $\text{dom}(\tilde{\gamma}) = \tau(\text{dom}(\gamma))$, and define $\tilde{\gamma}(g) = \gamma(\pi(g))$ for each $g \in \text{dom}(\gamma)$. If $\gamma$ is a certificate for $\varphi \in a\text{Hom}(G/N, H)$, then $\tilde{\gamma}$ is a certificate for $\varphi \circ \pi \in a\text{Hom}(G, H)$.

(iii) A list-decoder satisfying the conditions of AlgEconM exists for $a\text{Hom}(G/N, H)$. By Observation 5.13, it suffices to, given the list $\mathcal{L} = L(a\text{Hom}(G/N, H), \mathcal{F}, \Lambda + \varepsilon)$, return the list $\tilde{\mathcal{L}} = \{\varphi \circ \pi : \varphi \in \mathcal{L}\}$. We address algorithmic issues of defining $\varphi \circ \pi$ from $\varphi$.

Denote by $X$ the given set of generators of $N$ and by $\tau : G/N \to G$ the given injective map. Each homomorphism $\varphi$ is represented by its values on a set $Y$ of generators of $G/N$. The set $X \cup \tau(Y)$ is a set of generators for $G$. Define $\tilde{\varphi}$ on this set by the following.

$$
\tilde{\varphi}(g) = \begin{cases} 
1 & g \in X \\
(\varphi \circ \pi)(g) & g \in \tau(Y).
\end{cases}
$$

A class $\mathcal{G}$ of finite groups is a quasivariety if it is closed under subgroups and direct products. The classes of abelian, nilpotent, and solvable groups are examples of quasivarieties.

Theorem 5.22. Let $\mathcal{G}$ be a quasivariety of finite groups.
(i) Suppose $a\text{Hom}(G, H)$ is CombEcon for every $G, H \in \mathcal{G}$. Then, $a\text{Hom}(G, H)$ is CombEcon for $H \in \mathcal{G}$ and arbitrary $G$.

(ii) Under suitable access assumptions (Access 5.24 (ii)), if $a\text{Hom}(G, H)$ is CertEcon for every $G, H \in \mathcal{G}$, then $a\text{Hom}(G, H)$ is CertEcon for $H \in \mathcal{G}$ and arbitrary $G$.

(iii) Under suitable access assumptions (Access 5.24 (ii)), if $a\text{Hom}(G, H)$ is AlgEcon for every $G, H \in \mathcal{G}$, then $a\text{Hom}(G, H)$ is AlgEcon for $H \in \mathcal{G}$ and arbitrary $G$.

The deterioration in cost is as described in Remark 5.11.

Remark 5.23. In fact, the class $\mathcal{G}$ need only be closed under subdirect products.

The access assumptions mirror those of Access 5.19 of Theorem 5.18.

Access 5.24. For every $G \in \text{Groups}$ and $H \in \mathcal{G}$, denote by $N$ the $(G, H)$-irrelevant kernel and assume we have access to $N$ and $G/N$ as follows.

(ii) (a) Random elements of $N$ can be generated uniformly. (b) A transversal of $G/N$ in $G$ can be found. (c) $G/N$ can be found well enough to satisfy the access model of the assumed CertEcon list-decodability of pairs in $\mathcal{G} \times \mathcal{G}$.

(iii) (a') Random elements of $N$ can be generated uniformly and a set of generators for $N$ can be found. (b') Same as (b). (c') Same as (c) but for AlgEcon list-decodability.

Proof. Fix $H \in \mathcal{G}$ and $G \in \text{Groups}$. Let $N$ be the $(G, H)$-irrelevant kernel. By Theorem 5.18, all desired conclusions will follow if we show that $G/N \in \mathcal{G}$.

Let

$$\tilde{H} = \prod_{\varphi \in a\text{Hom}(G, H)} H.$$ 

Define the map $\tau : G \to \tilde{H}$ given by $\tau(g) = (\varphi(g))_{\varphi}$. Notice that $\tau(G)$ is subgroup of $\tilde{H}$ and is thus a subdirect product of copies of the group $H \in \mathcal{G}$. Since $\mathcal{G}$ is closed under subdirect products, it follows that $\tau(G) \in \mathcal{G}$.

Since $\ker(\tau) = N$, we have $\tau(G) \cong G/N$, so $G/N \in \mathcal{G}$. 

The AlgEcon list-decodability of $\{\text{abelian} \to \text{abelian}\}$ and $\{\text{nilpotent} \to \text{nilpotent}\}$ homomorphism codes is shown in [DGKS08] and [GS14], respectively. As the class of abelian groups and the class of nilpotent groups both form quasivarieties, we conclude the following using the mentioned results and Theorem 5.22.

Corollary 5.25. If $G$ is a group and $H$ an abelian group (or, more generally, nilpotent), then we have AlgEcon (and therefore CombEcon) list-decoding of $a\text{Hom}(G, H)$.

5.5 Hom versus $a\text{Hom}$

We show that the code $\text{Hom}(G, H)$ is CombEcon if and only if $a\text{Hom}(G, H)$ is CombEcon, and similarly for CertEcon, and AlgEcon under modest assumptions of the representation of the groups. Therefore we can use these two types of codes interchangeably. This reflects a phenomenon similar to our results on mean-list-decoding.

We fix terminology for this section. For an affine homomorphism $\varphi \in \text{Hom}(G, H)$, we denote its base homomorphism by $\varphi^0 \in \text{Hom}(G, H)$ (the unique homomorphism satisfying $\varphi = h\varphi^0$ for
For an element \( a \in G \) and function \( f : G \to H \), we denote by \( f^a : G \to H \) the function
\[
f^a(g) = f(a)^{-1}f(ag).
\]

We state the central result of this section, that every aHom list is contained within a small number of translated Hom lists. It is similar in spirit to Lemma 5.5, and it is similarly proved by the Bipartite Covering Lemma (Lemma 5.1). The deterioration in list size and cost will be addressed in Remark 5.29 below.

**Lemma 5.26 (Concentration of aHom lists).** Let \( G \) and \( H \) be groups, \( f : G \to H \) a received word, and \( 0 < \lambda \leq 1 \). Let \( L = L(\text{aHom}(G, H), f, \lambda) \). We conclude the following.

(i) \(|L| \leq \frac{1}{\lambda} \ell(\text{Hom}(G, H), \lambda)\).

(ii) Let \( S \) be a subset of \( G \) formed by choosing \( \lceil \frac{4}{3}\lambda \ln|L| + \ln(1/\eta \lambda) \rceil \) elements in \( G \) independently and uniformly. Suppose that, independently for each \( a \in G \), the subset \( D_a \) of \( L(\text{Hom}(G, H), f^a, \lambda) \) with probability \( \geq \frac{3}{4} \). We denote \( \widetilde{D}_a = \{ f(a)\psi(a^{-1})\psi \in \text{aHom}(G, H) : \psi \in D_a \} \). Then, with probability \( \geq 1 - \eta \), we have
\[
L \subseteq \bigcup_{a \in S} \widetilde{D}_a.
\]

We remark that \( \widetilde{D}_a \subseteq \text{aHom}(G, H) \) is found by translating elements of \( D_a \subseteq \text{Hom}(G, H) \), but not all by the same element.

We defer the proof to state the main result of this section, which is an immediate consequence.

**Access 5.27.** An oracle provides uniform random elements of \( G \).

**Corollary 5.28 (Hom versus aHom).** Let \( G \) and \( H \) be groups. If \( \text{Hom}(G, H) \) is CombEcon, then \( \text{aHom}(G, H) \) is CombEcon. Under Access 5.27, if \( \text{Hom}(G, H) \) is CertEcon, then \( \text{aHom}(G, H) \) is CertEcon. Under Access 5.27, if \( \text{Hom}(G, H) \) is AlgEcon, then \( \text{aHom}(G, H) \) is AlgEcon.

**Remark 5.29 (Deterioration).** The bounds on cost in the result above deteriorate as follows.

- A \( \frac{1}{\lambda} \) multiplicative factor in list size.
- An \( O(\frac{1}{\lambda} \ln(1/\lambda)) \) multiplicative factor in queries to the received word \( f \).
- An \( O(\frac{1}{\lambda} \ln(1/\lambda)) \) multiplicative factor in amount of work.

Towards proving Lemma 5.26, we state a few facts relating affine homomorphisms to their base homomorphisms.

**Observation 5.30.** Let \( G \) and \( H \) be groups and \( \varphi \in \text{aHom}(G, H) \). Then,
\[
\varphi(a)^{-1}\varphi(ag) = \varphi^0(g) \quad \forall a, g \in G.
\]

**Corollary 5.31.** Let \( G \) and \( H \) be groups, \( f : G \to H \), and \( \varphi \in \text{aHom}(G, H) \). If \( f(a) = \varphi(a) \), then
\[
f(ag) = \varphi(ag) \iff f(a)^{-1}f(ag) = \varphi^0(g) \iff f^a(g) = \varphi^0(g).
\]

It follows that \( \text{agr}(f, \varphi) = \text{agr}(f^a, \varphi^0) \).

We can now prove our concentration lemma.
Proof of Lemma 5.26. Fix the function \( f : G \rightarrow H \). Consider the bipartite graph with vertices \( V = G \) and \( W = \mathcal{L}(\text{aHom}(G, H), f, \lambda) \), where the edge set contains \((a, \varphi) \in G \times \text{aHom}(G, H)\) if \( f(a) = \varphi(a) \). We wish to apply Lemma 5.1, so we first check the conditions are satisfied.

For every \( \varphi \in W \), we have \( \text{agr}(\varphi, f) > \lambda \) by definition, so \( \text{deg}(\varphi) > \lambda \).

We show that \( \text{deg}(a) \leq \ell(\text{Hom}(G, H), \lambda) \), by showing that the map \( \varphi \mapsto \varphi^0 \) is an injection from \( N(a) = \{ \varphi \in W : f(a) = \varphi(a) \} \) to \( \mathcal{L}(\text{Hom}(G, H), f^0, \lambda) \). This map is well-defined since \( f(a) = \varphi(a) \) implies \( \text{agr}(f, \varphi) = \text{agr}(f^0, \varphi^0) \), by Corollary 5.31. We show this map is injective. If \( \varphi_1, \varphi_2 \in \text{aHom}(G, H) \) satisfy \( \varphi_1(a) = f(a) = \varphi_2(a) \) and \( \varphi_1^0 = \varphi_2^0 \), then \( \varphi_1(g) = \varphi_1(a)\varphi_1^0(a^{-1}g) = \varphi_2(a)\varphi_2^0(a^{-1}g) = \varphi_2(g) \) for all \( g \in G \), by Observation 5.30.

Apply the two parts of Lemma 5.1 to find the following.

(i) We conclude that \( |\mathcal{L}(\text{aHom}(G, H), f, \lambda)| \leq \frac{1}{\ell}(\text{Hom}(G, H), \lambda) \).

(ii) The subset \( U \) is the chosen subset \( S \) of \( G \). The subset \( \hat{U} \) contains the element \( a \in U \subseteq G \) if list-decoding for \( \mathcal{L}(\text{Hom}(G, H), f^0, \lambda) \) succeeds, which happens with probability \( \geq 3/4 \) independently over \( a \).

It remains to show that if \( D_a = \mathcal{L}(\text{Hom}(G, H), f^0, \lambda) \), then \( \tilde{D}_a \supseteq N(a) \). But, if \( \varphi \in N(a) \), we have already established that \( \varphi^0 \in \mathcal{L}(\text{Hom}(G, H), f^0, \lambda) \). Moreover, since \( f(a) = \varphi(a) \), we have \( \varphi(g) = f(a)\varphi^0(a^{-1}g) \) for all \( g \in G \), by Observation 5.30. So, \( \tilde{D}_a \supseteq N(a) \) by the definition of \( D_a \).

\( \square \)

6 Strategy

In this section, we outline our strategy for proving CombEcon results.

Let \( f \in \text{aHom}(G, H) \) be a received word and let \( \mathcal{L} \) be the set of codewords within distance \( (\text{mindist} - \varepsilon) \) of \( f \). The combinatorial problem is to bound \( |\mathcal{L}| \leq \text{poly}(1/\varepsilon) \). First, we partition \( \mathcal{L} \) into more manageable subsets (which we call buckets). We bound the number of buckets using a sphere-packing argument, and we bound the maximum size of the buckets.

6.1 The sphere packing argument: strong negative correlation

We recall that the agreement \( \text{agr}(f, g) \) of two functions \( f, g \) in the code space \( H^G \) is the proportion of inputs on which \( f \) and \( g \) agree; i.e., \( \text{agr}(f, g) = (1/|G|)|\{x \in G \mid f(x) = g(x)\}| \). So, the distance between \( f \) and \( g \) is \( 1 - \text{agr}(f, g) \). And, \( \Lambda = \Lambda_{G, H} \) is the maximum agreement between elements of \( \text{aHom}(G, H) \); so the minimum distance of the code \( \text{aHom}(G, H) \) is \( 1 - \Lambda \).

Let \( \Psi \) be a maximal set of elements of \( \text{aHom}(G, H) \) such that their pairwise distance is at least \( 1 - \Lambda^2 \). For each \( \psi \in \Psi \), we let the bucket \( \mathcal{L}_\psi \) consist of all homomorphisms in \( \mathcal{L} \) that are within a ball of radius \( 1 - \Lambda^2 \) around \( \psi \). That is, every pair of homomorphisms in \( \Psi \) has agreement at most \( \Lambda^2 \), and every homomorphism in \( \mathcal{L}_\psi \) has agreement greater than \( \Lambda^2 \) with \( \psi \).

We note that every homomorphism in \( \mathcal{L} \) is in at least one bucket.

To bound the number of buckets, \( |\Psi| \), we use a sphere-packing argument based on strong negative correlation (the sets \( \text{Eq}(f, \psi) \) for \( \psi \in \Psi \) are strongly negatively correlated; see Section 7.1). We find that \( |\Psi| \) is at most \( O(1/\varepsilon^2) \).

We discuss the division into buckets in detail in Sections 7.4 and 7.5.

Our strategy to bound the sizes of the buckets is different depending on the type of the domain \( G \). In all cases, we further divide each bucket into smaller units (which we call sub-buckets), but
the method of division will differ. We discuss abelian $G$ in Section 6.2 (in detail in Section 8), and alternating and SRG groups $G$ in Section 6.3 (in detail in Sections 9, 10, and 11).

6.2 Bounding the list size for abelian groups

To prove that abelian groups are universally CombEcon, we prove the following structure theorem. The theorem asserts that the codomain has a small number of abelian subgroups so that each homomorphism in the list $\mathcal{L}$ maps domain $G$ into one of those abelian subgroups.

**Theorem 6.1.** There exists a set $\mathcal{A}$ of finite abelian subgroups of $H$ with $|\mathcal{A}| \leq \frac{1}{4(\Lambda + \varepsilon)^2} + \frac{1}{\varepsilon}$ such that for all $\varphi \in \mathcal{L}$,

there is $M \in \mathcal{A}$ such that $\varphi(G) \leq M$.

This is restated in Section 8 as Theorem 8.7, and proved there.

This result reduces the problem to showing CombEcon of $\{\text{abelian} \rightarrow \text{abelian}\}$, which was done by Dinur, Grigorescu, Kopparty, and Sudan [DGKS08].

For find the set $\mathcal{A}$ of subgroups, we introduce the concept of an abelian enlargement. The abelian enlargement of a subset $T \subseteq H$ by a group $B \leq H$ is the group generated by $T$ and the elements of $B$ that are in the centralizer of $T$; that is,

$$\text{enl}_B(T) = \langle T, C_H(T) \cap B \rangle.$$ 

Let $\mathcal{A}_\psi$ be the set of all groups $M$ that occur as the abelian enlargement of $f(g)$ by $\psi(G)$ for at least an $\varepsilon$ proportion of $g \in G$.

We shall show

that every homomorphism $\varphi$ in the bucket $\mathcal{L}_\psi$ has its image contained in one of these subgroups. The idea is that since $\varphi$ and $\psi$ have large agreement, most of $\varphi(G)$ is contained in $\psi(G)$. So even if we take a single random element of $\varphi(G)$, it is likely that its enlargement by $\psi(G)$ already contains all of $\varphi(G)$. Specifically,

**Proposition 6.2.** Let $\varphi, \psi \in \text{Hom}(G, H)$ and $g \in G$ such that $\langle g, \text{Eq}(\psi, \varphi) \rangle = G$.

Then $\varphi(G) \leq \text{enl}_{\psi(G)}(\varphi(G)) = \text{enl}_{\psi\langle G \rangle}(\varphi(g))$.

This is restated in Section 8 as Proposition 8.4, and proved there.

We can then use the CombEcon bound to adapt the Dinur et. al. algorithm to this more general class of codes.

6.3 Bucket estimation for alternating and SRG

We describe our strategy to bound the size of a bucket in the case that $G$ is alternating. We carry out this strategy in Section 9.3.

All homomorphisms in one bucket $\mathcal{L}_\psi$ have high agreement with one representative homomorphism $\psi$. However, we have no control over where in $G$ this agreement occurs. We split each bucket $\mathcal{L}_\psi$ further into sub-buckets $\mathcal{L}_{\psi,K}$, this time so that the homomorphisms in each sub-bucket agree on a specified large

(low-depth) subgroup $K$ of $G$.

**Bound on the number of sub-buckets per bucket.** Fortunately, low-index subgroups of alternating groups are well understood, and there are few of them. There are at most $\text{poly}(1/\Lambda)$ large subgroups of $G$.

So, each bucket is divided into at most $\text{poly}(1/\Lambda)$ sub-buckets.
It is a consequence of the strong negative correlation principle that \( \text{poly}(1/\epsilon) = \text{poly}(1/\Lambda, 1/\epsilon) \). We will find \( \Lambda = \text{poly}(n) \) for alternating groups.

**Bound on the size of each sub-bucket.**

We will bound the size of a sub-bucket \( \mathcal{L}_{\psi,K} \). We describe a process for choosing a random homomorphism in the sub-bucket.

For a positive integer \( d \), we choose \( d \) random elements of \( G \). If there is a unique homomorphism \( \varphi \) that agrees with \( f \) on the \( d \) random inputs, and agrees with \( \psi \) on \( K \), we choose this homomorphism.

If \( d \) is at least the depth of the subgroup \( K \) in \( G \), then each homomorphism in the sub-bucket gets chosen with probability at least \( \epsilon^d \). So, the size of the sub-bucket is at most \( 1/\epsilon^d \). Conveniently, the depth of large subgroups of the alternating group \( A_n \) is bounded (see Section 9).

**Generalization.** This strategy for list decoding \{alternating \( \rightarrow \) anything\} works more broadly; it still works when the group in the domain comes from a much larger class of groups, which we call \( \text{SRG groups} \) (Section 10). A group \( G \) is SRG if, roughly, a small number of random elements chosen from \( G \) are likely to generate a low depth subgroup. See Section 4.4 for a precise definition.

This view allows us not only to combinatorially list-decode \{SRG \( \rightarrow \) anything\}, but also to certificate list-decode: Certificates are given by \( f \) restricted to a small number of random elements of \( G \).

### 7 Tools for CombEcon

In this section, we introduce the tools we need for proving CombEcon results. In Section 7.1, we introduce the strong negative correlation bound for bounding the number of sets that have small pairwise intersection. In Section 7.2, we compare this bound to the classical Johnson Bound. In Section 7.3, we use this bound to show that we may assume that \( \epsilon \) is small in our CombEcon proofs. In Sections 7.4 and 7.5, we start carrying out the strategy outlined in Section 6 for proving CombEcon results by the method of bucket splitting.

#### 7.1 Strong negative correlation and sphere packing

We define strongly negatively correlated families of subsets and give a simple proof for a bound on their sizes. We apply this bound via a sphere-packing argument to divide lists into a small number of “buckets.” Another consequence is that we may without loss of generality assume that \( \epsilon < \sqrt{2\Lambda} \), where we again let \( \Lambda = \Lambda_{G,H} \).

**Definition 7.1** (Strong negative correlation). Let \( 0 < \rho \leq 1 \) and \( \tau > 0 \). Let \( X \) be a finite set and let \( S_1, \ldots, S_r \subseteq X \). We say that \( S_1, \ldots, S_r \) are \((\rho, \tau)\)-strongly negatively correlated if

1. \( \mu(S_i) \geq \rho \) for all \( i \), and
2. \( \mu(S_i \cap S_j) \leq \rho^2 - \tau \) for all \( i \neq j \).

**Lemma 7.2** (Strong negative correlation bound). Let \( 0 < \rho < 1 \) and \( \tau > 0 \). Let \( X \) be a finite set and let \( S_1, \ldots, S_r \subseteq X \) be \((\rho, \tau)\)-strongly negatively correlated subsets. Then, \( r \leq \frac{1}{4\rho^2} + 1 \).

**Proof.** Choose \( x \) uniformly from \( X \). For \( 1 \leq i \leq r \), let \( Z_i(x) = \chi[x \in S_i] \) be the indicator random variable for the event that \( x \in S_i \). Notice that \( \text{Var}(Z_i) = \mu(S_i)(1 - \mu(S_i)) \leq \frac{1}{4} \).

For \( i \neq j \),

\[
\text{Cov}(Z_i, Z_j) = \mathbb{E}[Z_i Z_j] - \mathbb{E}[Z_i] \mathbb{E}[Z_j] \leq (\rho^2 - \tau) - \rho^2 = -\tau.
\]
So,

\[
0 \leq \text{Var} \left( \sum_i Z_i \right) = \sum_i \text{Var}(Z_i) + \sum_{i \neq j} \text{Cov}(Z_i, Z_j) \leq \frac{r}{4} + r(r - 1)(-\tau).
\]

Solving for \( r \) gives the bound as claimed. \( \square \)

When applied to equalizers, Lemma 7.2 will bound the size of a set of homomorphisms, each with high agreement with the received word but also with low pairwise agreement. This will serve as our base tool to split \( L(\text{aHom}(G, H), f, \Lambda + \varepsilon) \) into buckets.

**Lemma 7.3 (Sphere packing bound).** Let \( G \) be a finite group, \( H \) a group, and \( \varepsilon > 0 \). Let \( f: G \rightarrow H \) be a received word. Let \( \Psi \subseteq L(\text{aHom}(G, H), f, \Lambda + \varepsilon) \) be a subset of the list. Suppose that \( \Psi \) is maximal under the condition that its members have low pairwise agreement, specifically, \( \text{agr}(\psi_1, \psi_2) \leq \Lambda^2 \) for all distinct \( \psi_1, \psi_2 \in \Psi \). Then, the size of \( \Psi \) is bounded by

\[
|\Psi| \leq \frac{1}{4(\Lambda + \varepsilon)\varepsilon} + 1. \tag{8}
\]

Notice that the result also holds with \( \text{Hom} \) in place of \( \text{aHom} \), as \( \text{Hom} \subseteq \text{aHom} \).

**Proof.** The sets \( \text{Eq}(f, \psi) \) for \( \psi \in \Psi \) are \((\Lambda + \varepsilon, (\Lambda + \varepsilon)\varepsilon)\)-strongly negatively correlated, so the result follows by Lemma 7.2. \( \square \)

**Remark 7.4.** While Lemma 7.3 applies to all groups, it is an existential result. We cannot algorithmically find the homomorphisms chosen in \( \Psi \). So, although we use this lemma in proving combinatorial results, we cannot directly translate those proofs into algorithms.

The way we overcome this varies by setting. The list-decoder for \{abelian \( \rightarrow \) arbitrary\} is indifferent to how CombEcon is proved; the AlgEcon proof relies on already having proved that abelian groups are universally CombEcon, but the algorithm is is unconnected to the method of proof. In other cases, including \{alternating \( \rightarrow \) arbitrary\}, we overcome this difficulty by using “shallow random generation” (see Section 10).

### 7.2 Comparison between strong negative correlation and the Johnson Bound

The sphere packing bound (Lemma 7.3) can be rephrased as a statement about codes, and is essentially equivalent to a version of the classical Johnson bound.

Consider a code \( C \subseteq \Sigma^n \) of length \( n \) over an alphabet of size \( |\Sigma| = q \). Suppose the maximum agreement at most \( \Lambda \), so that the distance of the code is \( 1 - \Lambda \); let \( d = (1 - \Lambda)n \). Suppose that \( 0 \in \Sigma \), and that in every codeword in \( C \), at exactly \( \rho n \) of the \( n \) symbols are 0; one says that a codeword has weight \( w = (1 - \rho)n \). Let \( A_q(n, d, w) \) be the maximum size of such a code \( C \).

The Restricted Johnson Bound (see [HP03, Section 2.3.1]) says that

\[
A_q(n, d, w) \leq \frac{nd(q - 1)}{qw^2 - 2(q - 1)nw + nd(q - 1)},
\]

provided that the denominator is greater than 0.

Suppose that instead of requiring each codeword to have exactly \( \rho n \) zeros, we instead require it to have at least \( \rho n \) zeros (that is, weight at most \( (1 - \rho)n \)). Let \( A'_q(n, d, w) \) be the maximum size of such a code \( C \). The Restricted Johnson Bound still holds with \( A' \) in place of \( A \).
Equivalently,

\[ A'_q(n, (1 - \Lambda)n, (1 - \rho)n) \leq \frac{1 - \Lambda}{\rho^2 - \Lambda + \frac{(\rho-1)^2}{q-1}}, \]

again provided that the denominator is greater than 0.

So, for all \( q \geq 2 \), for all \( 0 < \rho \leq 1 \), and for all \( 0 \leq \Lambda < \rho^2 \),

\[ A'_q(n, (1 - \Lambda)n, (1 - \rho)n) \leq \frac{1 - \Lambda}{\rho^2 - \Lambda}. \]

The condition that each codeword has at least \( \rho n \) zeros can be rephrased as a condition that each codeword has at agreement at least \( \rho \) with the all-zero word. The all-zero word can be replaced with any word, and the bound still holds. Thus, this bound can be rephrased as saying that for any code \( C \) with maximum agreement \( \Lambda \), and any \( 0 < \rho \leq 1 \) such that \( \Lambda < \rho^2 \), we have that

\[ \ell(C, \rho) \leq \frac{1 - \Lambda}{\rho^2 - \Lambda}. \]

We compare this to the following generalization of the sphere packing bound, which is proved the same way, using the strong negative correlation bound (Lemma 7.2).

**Lemma 7.5.** Let \( C \) be a code with maximum agreement \( \Lambda \). Let \( \sqrt{\Lambda} < \rho \leq 1 \). Then, \( \ell(C, \rho) \leq \frac{1}{4(\rho^2 - \Lambda)} + 1 \).

**Proof.** Consider any word \( f \). To bound the size of \( \mathcal{L} = \mathcal{L}(C, f, \rho) \), we note that the sets \( \text{Eq}(f, \varphi) \) for \( \varphi \in \mathcal{L} \) are \((\rho, \rho^2 - \Lambda)\)-strongly negatively correlated, and we apply the strong negative correlation bound (Lemma 7.2).

When \( \rho > \frac{1}{2} \), this can be further improved to \( \ell(C, \rho) \leq \frac{\rho^2}{\rho^2 - \Lambda} + 1 \), by being more careful in the proof of the strong negative correlation bound.

### 7.3 Large \( \varepsilon \)

As a first consequence of strong negative correlation (Lemma 7.2), we will see that we may assume \( \varepsilon \) is “small” — specifically, \( \varepsilon < \sqrt{2\Lambda} \) — in our CombEcon proofs. So, to show CombEcon it suffices to show a list-size bound of \( \text{poly}(1/\varepsilon, 1/\Lambda) \) rather than \( \text{poly}(1/\varepsilon) \).

**Lemma 7.6** (Large \( \varepsilon \) lemma). Let \( G \) be a finite group and \( H \) a group. Suppose that \( \Lambda \leq \frac{1}{2} \varepsilon^2 \). Then, \( \ell(\text{aHom}(G, H), \Lambda + \varepsilon) \leq \frac{1}{2\varepsilon^2} + 1 \). In particular, \( \text{aHom}(G, H) \) is combinatorially \((\Lambda + \varepsilon, \text{poly}(1/\varepsilon))\)-list-decodable.

**Proof.** The sets \( \text{Eq}(f, \varphi) \) for \( \varphi \in \mathcal{L}(\text{aHom}(G, H), \Lambda + \varepsilon) \) are \((\Lambda + \varepsilon, \varepsilon^2/2)\)-strongly negatively correlated, so the result follows from Lemma 7.2.

**Corollary 7.7.** Let \( G \) be a finite group and \( H \) a group. If \( \ell(\text{aHom}(G, H), \Lambda + \varepsilon) \leq \text{poly}(1/\varepsilon, 1/\Lambda) \), then \( \text{aHom}(G, H) \) is CombEcon.

The result also holds with \( \text{Hom} \) in place of \( \text{aHom} \).
7.4 Bucket splitting

In Section 6, we outlined a strategy for showing that certain classes of homomorphism codes are CombEcon. In this section, we begin carrying out this strategy, and introduce our tools.

Let $G$ be a finite group, $H$ a group (finite or infinite), $\Lambda = \Lambda_{G,H}$ the maximum agreement, $f : G \to H$ a received word, $\varepsilon > 0$ a real, and $L = L(a\text{Hom}(G,H), f, \Lambda + \varepsilon)$ the list.

Our goal is to bound the size $|L|$ of the list. To do this, we split $L(a\text{Hom}(G,H), f, \Lambda + \varepsilon)$ into sets called buckets, which we label with elements of the set $\Psi \subseteq L$ introduced in Lemma 7.3. Each bucket, denoted $L_\psi$, will contain the sphere centered at the homomorphism $\psi \in \Psi$ with radius $(1 - \Lambda^2)$.

Definition 7.8 (Bucket $L_\psi$). Let $G$ be a finite group, $H$ a group, $\psi \in a\text{Hom}(G,H)$, $f : G \to H$, and $\varepsilon > 0$.

The bucket $L_\psi$ is

$$L_\psi := \{ \varphi \in L \mid \text{agr}(\varphi, \psi) > \Lambda^2 \}.$$ 

Lemma 7.9 (Bucket-splitting lemma). Let $G$ be a finite group, $H$ a group, $f : G \to H$, $\psi \in a\text{Hom}(G,H)$, and $\varepsilon > 0$. Then, there exists a subset $\Psi \subseteq L(a\text{Hom}(G,H), f, \Lambda + \varepsilon)$, with size $|\Psi| \leq \frac{1}{4(\Lambda + \varepsilon)^2} + 1$, such that

$$L(a\text{Hom}(G,H), f, \Lambda + \varepsilon) \subseteq \bigcup_{\psi \in \Psi} L_\psi.$$ 

Proof. Let $\Psi$ be as in Lemma 7.3, that is, a subset of $L(a\text{Hom}(G,H), f, \Lambda + \varepsilon)$ that is maximal under the conditions that distinct $\psi_1, \psi_2 \in \Psi$ have small agreement $\text{agr}(\psi_1, \psi_2) \leq \Lambda^2$. By the maximality of $\Psi$, every $\varphi \in L(a\text{Hom}(G,H), f, \Lambda + \varepsilon)$ has high agreement $\text{agr}(\varphi, \psi) > \Lambda^2$ with some homomorphism $\psi \in \Psi$.

We bound the size of each bucket $L_\psi$ by further subdividing it into smaller sets, which we refer to as sub-buckets. We then bound both the number of sub-buckets per bucket, and the size of each sub-bucket. The method of subdivision will differ depending on the type of group.

For abelian groups, we label each sub-bucket with an abelian subgroup $M$ of the codomain, $H$; the sub-bucket $L_\psi^M$ consists of all homomorphisms $\varphi \in L_\psi$ whose image is contained in $M$.

Definition 7.10 (Sub-bucket $L_\psi^M$ for abelian groups). Suppose $G$ is an abelian group. Let $\psi \in a\text{Hom}(G,H)$ and $M \leq H$. The sub-bucket $L_\psi^M$ is

$$L_\psi^M = \{ \varphi \in L_\psi \mid \varphi(G) \leq M \}.$$ 

For alternating groups, we label each sub-bucket with a subgroup $K$ of the domain, $G$; the sub-bucket $L_\psi^K$ consists of all homomorphisms $\varphi \in L_\psi$ whose equalizer with $\psi$ contains $K$.

Definition 7.11 (Sub-bucket $L_\psi^K$ for alternating groups). Suppose $G$ is an alternating group. Let $\psi \in L$ and $K \leq H$. The sub-bucket $L_\psi^K$ is

$$L_\psi^K = \{ \varphi \in L \mid K \leq \text{Eq}(\varphi, \psi) \}.$$
To bound the size of the buckets, we in both cases use the fact that elements of $L_\psi$ agree with $\psi$ on a subgroup of large density. Moreover, in abelian groups and alternating groups, subgroups with large density have small depth. We leverage this in the next lemma. The lemma will help us bound the number of sub-buckets $L_\psi$ per bucket in the abelian case (but we will need to bound the size of each sub-bucket another way). And, it will help us bound the size of each sub-bucket $L_\psi,K$ in the alternating case (but we will need to bound the number of sub-buckets per bucket another way).

The set $S$ in the lemma should be thought of as $\text{Eq}(f,\varphi)$ for some homomorphism $\varphi$ of interest.

**Lemma 7.12.** Let $0 \leq \lambda < 1$. Let $G$ be a finite group, $K \leq G$ a subgroup, and $S \subseteq G$ a subset. Suppose that $\mu_G(S) > \lambda$. Let $\varepsilon = \mu(S) - \lambda$ and $d = \text{depth}_G(K)$. Then,

$$\Pr_{s_1,\ldots,s_d \in S}[\mu(\langle K,s_1,\ldots,s_d \rangle) > \lambda] \geq \left(\frac{\varepsilon}{\lambda + \varepsilon}\right)^d.$$

It follows that

$$\Pr_{g_1,\ldots,g_d \in G}[g_1,\ldots,g_d \in S \text{ and } \mu(\langle K,g_1,\ldots,g_d \rangle) > \lambda] \geq \varepsilon^d.$$

This is proved by repeated application of Bayes’ rule.

**Proof.** Pick $s_1,s_2,s_3,\ldots$ independently and uniformly from $S$.

We proceed by induction on $|G : K|$.

Suppose $\mu(K) > \lambda$. Then, $\Pr[\mu(\langle K,s_1,\ldots,s_d \rangle) > \lambda] = 1$.

Suppose $\mu(K) \leq \lambda$. Then, with probability $\frac{\mu(S \setminus K)}{\mu(S)} \geq \frac{\varepsilon}{\lambda + \varepsilon}$, we have that $s_1 \notin K$, so $\langle K,s_1 \rangle > K$, and $\text{depth}_G(K,s_1) \leq d - 1$. Then, by the induction hypothesis,

$$\Pr[\mu(\langle K,s_1,\ldots,s_d \rangle)] \geq \Pr[\mu(\langle K,s_1,\ldots,s_d \rangle) | s_1 \notin K] \cdot \Pr[s_1 \notin K] \geq \left(\frac{\varepsilon}{\lambda + \varepsilon}\right)^{d-1} \cdot \left(\frac{\varepsilon}{\lambda + \varepsilon}\right).$$

This completes the inductive step.

**Remark 7.13.** With a little more care in the proof (separating out the case $\lambda/2 < \mu(K) \leq \lambda$), one can prove the stronger conclusion

$$\Pr[\mu(\langle K,s_1,\ldots,s_d \rangle) > \lambda] \geq \left(\frac{\frac{1}{2}\lambda + \varepsilon}{\lambda + \varepsilon}\right)^{d-1} \cdot \frac{\varepsilon}{\lambda + \varepsilon}. \quad (11)$$

### 7.5 Bipartite generation-graphs

We retain the notation $G,H,f,\varepsilon,L$ from the previous subsection. Let $\Psi \subseteq \text{aHom}(G,H)$ be as defined in Lemma 7.3.

In the case that $G$ is abelian or alternating, we divided the list $L$ into buckets $L_{\psi}$. To bound the size $|L|$ of the list, we just need to bound the size $|L_{\psi}|$ of each bucket. If we fix $\psi \in \Psi$ and consider some homomorphism $\varphi \in L_{\psi}$ that is hidden to us, we can almost recover $\varphi$ with some decent probability if we have access to $\psi$ and a few random elements of $\text{Eq}(f,\varphi)$. By “almost recover,” we mean there is a small list of homomorphisms which contains $\varphi$.

To make this more precise, we define the following bipartite graph.
Definition 7.14. Let $d$ be a positive integer. We define a bipartite graph $X_{\psi,d}$. The left vertex set is $V = G^d$ and the right vertex set is $W = L_\psi$. The vertices $(g_1, \ldots, g_d) \in V$ and $\varphi \in W$ are adjacent if $g_1, \ldots, g_d \in \text{Eq}(f, \varphi)$ and $\mu(\langle \text{Eq}(\psi, \varphi), g_1, \ldots, g_d \rangle) > \Lambda$.

We will bound $L_\psi$ by applying part (a) of Lemma 5.1 (double counting) to $X_{\psi,d}$. So, we would like to bound the degree of a left vertex from above, and the degree of a right vertex from below.

The next lemma bounds below the degree of a right vertex in certain cases, and will be useful when $G$ is abelian or alternating.

Lemma 7.15. Let $d$ be a positive integer, and $\varphi \in L_\psi$ be a right vertex of $X_{\psi,d}$ such that $\text{depth}_G \text{Eq}(\psi, \varphi) \leq d$. Then $\varphi$ is adjacent to at least an $\epsilon^d$ fraction of the left vertices.

Proof. By Lemma 7.12 with $\lambda = \Lambda$, and $S = \text{Eq}(f, \varphi)$, and $K = \text{Eq}(\psi, \varphi)$, we have that

$$\Pr_{g_1, \ldots, g_d \in G} \left[ g_1, \ldots, g_d \in \text{Eq}(f, \psi) \text{ and } \mu(\langle \text{Eq}(\psi, \varphi), g_1, \ldots, g_d \rangle) > \Lambda \right] \geq \epsilon^d.$$ 

So, $\varphi$ is adjacent to at least $\epsilon^d$ fraction of the tuples $(g_1, \ldots, g_d)$. □

We would also like to bound the degree of a left vertex from above. For abelian groups, we will do this in Section 8.2, in Corollary 8.11. For alternating groups, we do this by splitting the graph $X_{\psi,d}$ into subgraphs based on the sub-buckets $L_{\psi,K}$ defined in the previous subsection. We will use the following lemma.

Lemma 7.16. Let $K \leq G$ be a subgroup, $\psi \in \text{Hom}(G, H)$ a homomorphism, $d$ a nonnegative integer, and $g_1, \ldots, g_d \in G$. If $\mu(\langle K, g_1, \ldots, g_d \rangle) > \Lambda$, then there is at most one homomorphism $\varphi \in \text{Hom}(G, H)$ such that $K \leq \text{Eq}(\psi, \varphi)$ and $g_1, \ldots, g_d \in \text{Eq}(f, \varphi)$.

Proof. If there were two such homomorphisms $\varphi_1$ and $\varphi_2$, we would have that $K \leq \text{Eq}(\psi, \varphi_1) \cap \text{Eq}(\psi, \varphi_2) \leq \text{Eq}(\varphi_1, \varphi_2)$, and $g_1, \ldots, g_d \in \text{Eq}(f, \varphi_1) \cap \text{Eq}(f, \varphi_2) \leq \text{Eq}(\varphi_1, \varphi_2)$ so $\text{agr}(\varphi_1, \varphi_2) \geq \mu(\langle K, g_1, \ldots, g_d \rangle) > \Lambda$, so $\varphi_1 = \varphi_2$. □

As an application of Lemma 7.16, we can bound the size of the sub-buckets $L_{\psi,K}$ for $K$ of low depth, which will be useful in proving that alternating groups are universally CombEcon. We do this in the next corollary. This lemma will also be used when $G$ is a shallow random generation group (See Section 10).

Corollary 7.17 (Sub-bucket bound for low-depth label subgroups). Let $G$ be a finite group, $H$ a group, $K \leq G$ a subgroup, $f : G \to H$, and $\epsilon > 0$.

Then,

$$|L_{\psi,K}| \leq 1/\epsilon^{\text{depth}_G(K)}.$$ 

Proof. Let $d = \text{depth}_G(K)$. Consider the induced subgraph of $X_{\psi,d}$ with the same left vertex set, and with right vertex set $L_{\psi,K}$. By Lemma 7.16, each left vertex has degree at most 1. By Lemma 7.15, each right vertex is adjacent to at least an $\epsilon^d$ fraction of the left vertices. Apply part (a) of Lemma 5.1 (double counting). □

We will use this corollary in Section 9.2.
8 Homomorphism Codes with finite abelian domain and arbitrary codomain

In this section we show that finite abelian groups are universally combinatorially and algorithmically economically list-decodable. The key technical result is Theorem 8.7, which says that there are a small number of abelian subgroups of the codomain such that every homomorphism in the list maps into one of these subgroups.

In Section 8.1 we introduce a tool called an abelian enlargement. Using this tool, in Section 8.2 we prove Theorem 8.7 (the key result mentioned in the previous paragraph) and infer that abelian groups are universally CombEcon. In Section 8.3 we adapt the algorithm of [DGKS08, GS14], to give an algorithm to locally list-decode these codes.

In Section 8.4 we describe $\Lambda_{G,H}$ for these and a few other codes, slightly generalizing a result of Guo [Guo15]. Our proof of CombEcon only uses that taking a subset of the codomain does not increase $\Lambda$.

We remark that these codes usually cannot be list-decoded beyond radius $1 - (\Lambda_{G,H} + \varepsilon)$ (see Remark 1.5).

8.1 Abelian enlargements

Throughout this section, let $G$ be a finite abelian group, and $H$ a group (finite or infinite).

To prove that abelian groups are universally CombEcon, we will follow the outline given in Section 7.4. That is, we divide the list $L = L(\text{aHom}(G,H), f, \Lambda + \varepsilon)$ into buckets $L_{\psi}$, and then subdivide each bucket into sub-buckets $L_{\psi}^M$ for a small number of abelian subgroups $M \leq H$. (Actually, we will use Hom in place of aHom, but it does not matter by Lemma 5.26.)

To select the subgroups $M$, we introduce an operation that we call an abelian enlargement. In Section 8.2, we will use this operation in our proof that abelian groups are universally CombEcon. For a subset $T$ and a finite abelian subgroup $B$ of a group $H$, the abelian $B$-enlargement of $T$ in $H$ is the group generated by $T$ along with every element of $B$ that commutes with $T$.

If 

$\varphi, \psi \in \text{Hom}(G,H)$ are homomorphisms, then the abelian $\psi(G)$-enlargement of $\varphi(G)$ will certainly still include $\varphi(G)$. But also, if $\varphi$ and $\psi$ have large agreement, then most of $\varphi(G)$ is contained in $\psi(G)$, so even if we take a single element of $\varphi(G)$, then when we enlarge it by $\psi(G)$ it is likely that the result will already contain all of $\varphi(G)$.

This will help us bound the number of subgroups $M \leq H$ we need, and thus the number of sub-buckets.

For any $\varphi$ in the bucket $L_{\psi}$,

its image is likely to be contained

in the enlargement by $\psi(G)$ of a single random element of $f(G)$.

This is the method by which we choose our subgroups $M$.

Definition 8.1. For $H$ a group, $B \leq H$ a subgroup, and $T \subseteq H$ a subset, define the abelian $B$-enlargement of $T$ in $H$ to be

$$\text{enl}_B(T) = \langle T, C_H(T) \cap B \rangle,$$

where $C_H(T)$ denotes the centralizer of $T$ in $H$.

For $h \in H$, we may write $\text{enl}_B(h)$ in place of $\text{enl}_B(\{h\})$.

We note that if $\langle T_1 \rangle = \langle T_2 \rangle$, then $\text{enl}_B(T_1) = \text{enl}_B(T_2)$.
Lemma 8.2. For $B \leq H$ a finite abelian subgroup and $T \subseteq H$ a set such that $\langle T \rangle$ is a finite abelian group, $\text{enl}_B(T)$ is a finite abelian group.

In fact, we will only be concerned with the case that $B \leq H$ is a finite abelian subgroup, and $\langle T \rangle$ is abelian.

Proof. Every element of $T$ commutes with every element of $C_H(T)$ by the definition of the centralizer. So, $\text{enl}_B(T)$ is the direct product of $\langle T \rangle$ and $C_H(T) \cap B$. The group $\langle T \rangle$ is finite abelian by assumption, and $C_H(T) \cap B$ is finite abelian because it is a subgroup of $B$. So, $\text{enl}_B(T)$ is a finite abelian group. □

Lemma 8.3. For $B$ and $T$ as above, and $U \subseteq \text{enl}_B(T)$, we have that $\text{enl}_B(T) = \text{enl}_B(T \cup U)$.

Proof. First, we show that $\text{enl}_B(T) \leq \text{enl}_B(T \cup U)$. Since $\text{enl}_B(T)$ is abelian, we have that $C_H(T) \cap B \leq \text{enl}_B(T) \leq C_H(\text{enl}_B(T)) \leq C_H(U)$.

So,

$$C_H(T) \cap B \leq C_H(T) \cap C_H(U) \cap B = C_H(T \cup U) \cap B \leq \text{enl}_B(T \cup U).$$

Since also $T \subseteq \text{enl}_B(T \cup U)$, we have that $\text{enl}_B(T) \leq \text{enl}_B(T \cup U)$.

Next, we show that $\text{enl}_B(T \cup U) \leq \text{enl}_B(T)$. We have that $T \subseteq \text{enl}_B(T)$, that $U \subseteq \text{enl}_B(T)$, and $C_H(T \cup U) \cap B \leq C_H(T) \cap B \leq \text{enl}_B(T)$. So, $\text{enl}_B(T \cup U) = \langle T \cup U, C_H(T \cup U) \cap B \rangle \leq \text{enl}_B(T)$. □

Proposition 8.4. Let $\varphi, \psi \in \text{Hom}(G, H)$ and $A \subseteq G$ such that $\langle A, \text{Eq}(\psi, \varphi), \ker \varphi \rangle = G$.

Then $\text{enl}_{\psi(G)}(\varphi(A)) = \text{enl}_{\psi(G)}(\varphi(G))$.

Proof. Since $G$ is finite abelian, so are $\varphi(G)$ and $\psi(G)$. Let $B = \psi(G)$. Let $T = \varphi(A)$. Let $U = \varphi(\text{Eq}(\psi, \varphi))$. Since $T, U \subseteq \varphi(G)$, and $\varphi(G)$ is abelian, $U \leq C_H(T)$. And, since $U = \psi(\text{Eq}(\psi, \varphi))$, we have that $U \leq \psi(T) = B$. Thus, $U \leq C_H(T) \cap B \leq \text{enl}_B(T)$.

Also, $\langle T \cup U \rangle = \langle T, U, 1 \rangle = \langle \varphi(A), \varphi(\text{Eq}(\psi, \varphi)), \varphi(\ker \varphi) \rangle = \varphi(\langle A, \text{Eq}(\psi, \varphi), \ker \varphi \rangle) = \varphi(G)$.

Therefore, by Lemma 8.3,

$$\text{enl}_{\psi(G)}(\varphi(A)) = \text{enl}_B(T) = \text{enl}_B(T \cup U) = \text{enl}_B(\langle T \cup U \rangle) = \text{enl}_{\psi(G)}(\varphi(G)).$$

(12) □

Corollary 8.5. Let $\varphi, \psi, A$ be as above. Then $\varphi(G) \leq \text{enl}_{\psi(G)}(\varphi(A))$.

8.2 Combinatorial list-decodability, finite abelian to anything

In this section, we establish that finite abelian groups are universally CombEcon.

Throughout this section, let $G$ be a finite abelian group, and $H$ an arbitrary group (finite or infinite). Let $f: G \rightarrow H$ be a received word. Let $\varepsilon > 0$. Let $\mathcal{L} = \mathcal{L}(\text{Hom}(G, H), f, \lambda + \varepsilon)$ be the list (note that in this section we deal with the code of homomorphisms, rather than affine homomorphisms; however, we can convert between the two; see Section 5.5). The list $\mathcal{L}$ is divided into buckets $\mathcal{L}_\psi$ for $\psi \in \Psi$, where $\Psi$ is as in Lemma 7.3.

We will see that there is a small set of abelian subgroups $M \leq H$ such that every $\varphi \in \mathcal{L}$ has its image in some $M$.

Dinur, Grigorescu, Kopparty, and Sudan [DGKS08] proved that $\text{aHom}(G, H)$ is CombEcon (and in fact, AlgEcon) for all finite abelian groups $G$ and $H$. 42
Theorem 8.6 (DGKS 2008). The class $\text{Abel} \times \text{Abel}$ of pairs of abelian groups is CombEcon.

The following theorem, combined with the DGKS result, lets us conclude that $\text{Hom}(G, H)$ (and thus $\text{aHom}(G, H)$) is CombEcon.

**Theorem 8.7.** There exists a set $\mathcal{A}$ of finite abelian subgroups of $H$ with $|\mathcal{A}| \leq \frac{1}{4(\Lambda + \varepsilon)^2} + \frac{1}{\varepsilon}$ such that for all $\varphi \in \mathcal{L}$, there is $M \in \mathcal{A}$ such that $\varphi(G) \leq M$.

**Corollary 8.8.** Finite abelian groups are universally CombEcon. Specifically, let $C$ be a constant such that $\ell(\text{aHom}(G, H), \Lambda + \varepsilon) \leq \left(\frac{1}{\varepsilon}C\right)^3$ for $G, H$ finite abelian groups. Then $\ell(\text{aHom}(G, H), \Lambda + \varepsilon) \leq O\left(\left(\frac{1}{\varepsilon}C\right)^4\right)$ for $G$ a finite abelian group and $H$ arbitrary group.

By [GS14, BGSW18], the constant $C$ is approximately 105.

**Proof of Corollary 8.8.** Let $\mathcal{A}$ be the collection of subgroups of $H$ guaranteed by Theorem 8.7. Then, $\mathcal{L} \subseteq \bigcup_{M \in \mathcal{A}} \mathcal{L}(\text{Hom}(G, M), f, \Lambda + \varepsilon)$ (on the right hand side we let $f$ be redefined arbitrarily at points in its domain that do not map to $M$). So, $|\mathcal{L}| \leq \sum_{M \in \mathcal{A}} \ell(\text{Hom}(G, M), \Lambda + \varepsilon) \leq \left(\frac{1}{4(\Lambda + \varepsilon)^2} + \frac{1}{\varepsilon}\right)\left(\frac{1}{\varepsilon}C\right)^3$.

We then apply Lemma 5.26.

In the remainder of this subsection, we prove Theorem 8.7.

Recall our strategy from Sections 7.4 and 7.5 of dividing the list $\mathcal{L}$ into buckets $\mathcal{L}_\psi$, and bounding the size of each bucket by considering the graph $X_{\psi,d}$. We will now carry out this strategy. We will use $d = 1$. We next bound the degree of each right vertex.

**Lemma 8.9.** Let $\psi \in \Psi$. Each right vertex of $X_{\psi,1}$ is adjacent to at least an $\varepsilon$ fraction of the left vertices.

**Proof.** We know that $1/\Lambda$ is the least integer that divides $|G : N|$, where $N$ is the $(G, H)$-irrelevant kernel.

For all $\varphi \in \mathcal{L}_\psi$, we have that $\text{Eq}(\psi, \varphi)$ contains the irrelevant kernel and has density greater than $\Lambda^2$, so has depth at most 1. We apply Lemma 7.15.

The next lemma tells us about the connected components of $X_{\psi,1}$. It also helps us bound the degree of a left vertex via the DGKS result.

**Lemma 8.10.** Let $\psi \in \Psi$. Let $g_1 \in G$ be a left vertex of $X_{\psi,1}$, and $\varphi \in \mathcal{L}_\psi$ a right vertex. If $g_1$ is adjacent to $\varphi$ then $\text{enl}_{\psi(G)}(f(g_1)) = \text{enl}_{\psi(G)}(\varphi(G))$.

**Proof.** We have $g_1 \in \text{Eq}(f, \varphi)$ and $\langle \text{Eq}(\psi, \varphi), g_1 \rangle$ has density greater than $\Lambda$, so is equal to $G$. So, by Proposition 8.4, $\text{enl}_{\psi(G)}(f(g_1)) = \text{enl}_{\psi(G)}(\varphi(g_1)) = \text{enl}_{\psi(G)}(\varphi(G))$.

Lemma 8.10 allows us to associate an abelian subgroup of $H$ to each connected component of $X_{\psi,1}$.

**Corollary 8.11.** Each left vertex of of $X_{\psi,1}$ has degree at most $\left(\frac{1}{\varepsilon}\right)^C$.
Proof. All the neighbors of $g_1$ are elements of $\mathcal{L}(\text{Hom}(G, M), f, \Lambda + \varepsilon)$ for $M = \text{enl}_{\psi(g_1)}(f(g_1))$. We apply the DGKS result.

\[\square\]

**Remark 8.12.** We can now prove Corollary 8.8, bypassing Theorem 8.7, by applying part (a) of Lemma 5.1 (double counting). We can use Lemma 8.9 to bound the degree of a right vertex, and Corollary 8.11 to bound the degree of a left vertex.

We need one more lemma before we prove Theorem 8.7.

**Lemma 8.13.** Let $\psi \in \Psi$. There is a set $\mathcal{A}_\psi$ of finite abelian subgroups of $H$ with $|\mathcal{A}_\psi| \leq \frac{1}{\varepsilon}$ such that for all $\varphi \in \mathcal{L}_\psi$, there is $M \in \mathcal{A}_\psi$ for which $\varphi(G) \leq M$.

**Proof.** Let $\mathcal{A}_\psi = \{\text{enl}_{\psi(G)}(\varphi(G)) \mid \varphi \in \mathcal{L}_\psi\}$, which is a set of finite abelian subgroups of $H$ by Lemma 8.2. By Lemma 8.10, we can associate an abelian subgroup to each connected component of $X_{\psi,1}$. Each element of $\mathcal{A}_\psi$ is associated to at least one connected component that contains a right vertex. By Lemma 8.9, there are at most $1/\varepsilon$ such components, so $|\mathcal{A}_\psi| \leq 1/\varepsilon$.

We are ready to prove the main theorem of this section.

**Proof of Theorem 8.7.** Let $\mathcal{A}_\psi$ be as in Lemma 8.13. Let $\mathcal{A} = \bigcup_{\psi \in \Psi} \mathcal{A}_\psi$. By Lemma 7.3, $|\Psi| \leq \frac{1}{4(\Lambda + \varepsilon)^2} + 1$. By Lemma 8.13, $|\mathcal{A}_\psi| \leq \frac{1}{\varepsilon}$. So, $|\mathcal{A}| \leq \frac{1}{4(\Lambda + \varepsilon)^2} + \frac{1}{\varepsilon}$.

For $\varphi \in \mathcal{L}$, we have that $\varphi \in \mathcal{L}_\psi$ for some $\psi \in \Psi$, and so the image of $\varphi$ is in $\text{enl}_{\psi(G)}(\varphi(G)) \in \mathcal{A}_\psi \subseteq \mathcal{A}$.

\[\square\]

### 8.3 Algorithm

For $G$ a finite abelian group given explicitly by a primary decomposition, and $H$ a group with black-box access, we can locally list-decode aHom$(G, H)$ using essentially the same algorithm as the one by Dinur Grigorescu, Kopparty, and Sudan in Section 5 of [DGKS08]. We make only slight modifications. Thus, such codes are AlgEcon.

**Theorem 8.14.** Let $\mathcal{D}$ be the class of pairs $(G, H)$ where $G$ is an abelian group given explicitly by an primary decomposition, and $H$ is a group with black-box access. Then there is an algorithm to locally list-decode $\mathcal{D}$ in time $\text{poly}(\log |G| \cdot \frac{1}{\varepsilon})$.

We assume black-box access to $H$. We do not assume black-box access to $G$; if only black-box access were assumed, then for $p$ a prime, it would take $p + 1$ queries to a group to determine whether the group were isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}_p^2$. Like [DGKS08], we assume that $G$ is given explicitly by an primary decomposition.

We assume that we have an algorithm determining $\Lambda_{G,H}$, although this assumption can be removed.

Next, [DGKS08] reduces to the case where $H = \mathbb{Z}_p$. We don’t make this reduction. We let $p$ be the prime such that $\Lambda = \frac{1}{p}$. Every mention of $\mathbb{Z}_p$ should be replaced by $H$. As in their algorithm, we take $G = G_1, \ldots, G_k$, with each $G_i = \mathbb{Z}_{p_i}$. We order the $G_i$ such that $p_1 = p$. For them, the only important coordinates are the ones where $p_i = p$, but for our purposes, instances of $\mathbb{Z}_p^{k_i}$ should be replaced with $\mathbb{Z}_{p_i}^{k_i}$.

In the algorithm Extend of [DGKS08], the statement “If $c_1 - c_2$ is not divisible by $p_i$” should be replaced with “If $c_1 - c_2$ is not divisible by $p_i$, and if $f(y_1, c_1, s)$ and $f(y_2, c_2, s)$ commute with each
other and with \( \varphi(e_1), \ldots, \varphi(e_{i-1})' \). Here \( e_j \) denotes a generator of \( G_j \). The system of equations that follows should be solved under the assumption that the order of \( a \) divides \( p_i^{e_i} \).

We note that when solving the system of equations in EXTEND, we are working in an abelian subgroup of \( H \). Actually, even this does not matter; we can solve the system of equations without assuming elements of \( H \) commute.

### 8.4 \( \Lambda_{G,H} \) when \( G \) or \( H \) is solvable

We give a combinatorial description of \( \Lambda_{G,H} \) when \( G \) is a finite abelian group and \( H \) is an arbitrary group.

**Proposition 8.15.** Let \( G \) be a finite abelian group and \( H \) a group. Then \( \Lambda_{G,H} = 1/p \), where \( p \) is the smallest prime number such that \( p \) divides \( |G| \) and \( H \) has an element of order \( p \). If no such \( p \) exists, then \( |\text{Hom}(G,H)| = 1 \) and \( \Lambda_{G,H} = 0 \).

This proposition is a special case of the following theorem, which describes \( \Lambda_{G,H} \) when \( G \) or \( H \) is a solvable group. This is a slight generalization of a result of Guo [Guo15, Theorem 1.1].

**Theorem 8.16.** Let \( G \) be a finite group and \( H \) a group, such that at least one of \( G \) or \( H \) is solvable. Then \( \Lambda_{G,H} = 1/p \), where \( p \) is the smallest prime number such that \( G \) has a normal subgroup of index \( p \) and \( H \) has an element of order \( p \). If no such \( p \) exists, then \( |\text{Hom}(G,H)| = 1 \) and \( \Lambda_{G,H} = 0 \).

We will prove Theorem 8.16 in this subsection. Guo proved Theorem 8.16 in the case where \( H \) is finite, and either \( G \) is solvable or \( H \) is nilpotent.

Our proof relies on the following lemma about \( \prod_{\varphi \in \text{Hom}(G,H)} \ker \varphi \).

**Lemma 8.17.** Let \( G \) be a finite group and \( H \) a group. Let \( K = \prod_{\varphi \in \text{Hom}(G,H)} \ker \varphi \). Then every prime factor of \( |G : K| \) is the order of an element of \( H \).

**Proof.** Consider any prime factor \( p \) of \( |G : K| \). Then there is \( g \in G \) such that \( gK \) has order \( p \) in \( G/K \). Since \( g \notin K \), there is \( \varphi \in \text{Hom}(G,H) \) such that \( g \notin \ker \varphi \). We have \( g^p \in K \), so \( \varphi(g)^p \in \ker \varphi(K) = 1 \), so \( |\varphi(g)| \) divides \( p \). Since \( \varphi(g) \neq 1 \), we have \( |\varphi(g)| = p \).

We also use the following well-known fact and a theorem by Berkovich (see [Isa08]).

**Fact 8.18.** In a solvable group, a normal subgroup is a maximal normal subgroup if and only if it has prime index.

**Theorem 8.19 (Berkovich).** Let \( G \) be a finite solvable group and \( K \) a proper subgroup of smallest index. Then \( K \trianglelefteq G \).

Next, we prove Theorem 8.16 in the case when \( G \) is solvable. Guo [Guo15, Theorem 5.5] proved this in the case that also \( H \) is finite.

Guo’s proof can be modified slightly to also accommodate infinite groups. We include a compact proof for completeness.

**Proof of Theorem 8.16 in the case where \( G \) is solvable.** Let \( K = \prod_{\varphi \in \text{Hom}(G,H)} \ker \varphi \). If \( \Lambda > 0 \), then there is a nontrivial homomorphism \( \varphi \in \text{Hom}(G,H) \). Then \( \ker \varphi \) is a proper normal subgroup of \( G \), so is contained in a maximal normal subgroup \( M \). By Fact 8.18, \( |G : M| \) is prime, and since \( M \geq \ker \varphi \geq K \), we have that \( |G : M| \) divides \( |G : K| \). So, by Lemma 8.17, we have that \( H \) has an element of order \( |G : M| \). So, if \( \Lambda > 0 \), then \( p \) exists.

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Henceforth, assume $p$ exists. Let $N$ be a normal subgroup of $G$ of index $p$, and $h$ an element of $H$ of order $p$. We show that $\Lambda \geq 1/p$ by exhibiting a pair of homomorphisms that achieve this agreement. Let $\varphi_1: G \to H$ be the trivial homomorphism. Since $|G/N| = |\langle h \rangle| = p$ prime, there is a group isomorphism $G/N \to \langle h \rangle$. This lifts to a group homomorphism $\varphi_2: G \to \langle h \rangle$. Then $\text{Eq}(\varphi_1, \varphi_2) = N$, so $\text{agr}(\varphi_1, \varphi_2) = 1/p$. So, $\Lambda_{G,H} \geq 1/p$.

We next show that $\Lambda \leq 1/p$. Since $N = \ker \varphi_2 \geq K$, we have that $N/K \leq G/K$ and $|G/K : N/K| = p$. Furthermore, we claim that $N/K$ is a proper normal subgroup of smallest prime index in $G/K$—if there were a proper normal $\hat{N}/K$ of prime index $q < p$ (with $K \leq \hat{N} \leq G$), then $\hat{N}$ would be a normal subgroup of $G$ of index $q$, and $H$ would have an element of order $q$ by Lemma 8.17, which would contradict the definition of $p$. By Fact 8.18, $N/K$ is in fact a proper normal subgroup of smallest index in $G/K$ (removing “prime”). By Theorem 8.19, we further have that $N/K$ is a proper subgroup of smallest index in $G/K$ (removing “normal”). Thus, $N$ is the has the smallest index of any subgroup of $G$ that contains $K$. Any equalizer of two homomorphisms in $\text{Hom}(G,H)$ contains $K$, so no equalizer can have smaller index than $N$. So, $\Lambda \leq \mu(N) = 1/p$.

To prove Theorem 8.16 in the case where $H$ is solvable, we use the following fact.

**Lemma 8.20.** Let $G$ be a group and $H$ a solvable group. Let $K = \bigcap_{\varphi \in \text{Hom}(G,H)} \ker \varphi$. Then $G/K$ is solvable.

**Proof.** For each $\varphi \in \text{Hom}(G,H)$, we have that $\varphi(G)$ is solvable with derived length at most the derived length of $H$. Also, $G/\ker \varphi \cong \varphi(G)$. So, $\prod_{\varphi \in \text{Hom}(G,H)} G/\ker \varphi$ is a direct product of solvable subgroups with bounded derived length, so is solvable. Let $\psi: G \to \prod_{\varphi \in \text{Hom}(G,H)} G/\ker \varphi$ be the projection onto each coordinate. Then $\ker \psi = K$. And, $\psi(G)$ is a subgroup of a solvable group, so is solvable. Thus, $G/K = G/\ker \psi \cong \psi(G)$ is solvable.

We can now prove Theorem 8.16 in the case where $H$ is solvable.

**Proof of Theorem 8.16 in the case where $H$ is solvable.** Let $K = \bigcap_{\varphi \in \text{Hom}(G,H)} \ker \varphi$. Let $p$ be the smallest prime divisor of $|G|$ such that $G$ has a normal subgroup of index $p$ and $H$ has an element of order $p$. If $N$ is a normal subgroup of prime index and $H$ contains an element $h$ of order $|G : N|$, then $K \leq N$, since the isomorphism $G/N \to \langle h \rangle$ lifts to a homomorphism $G \to \langle h \rangle$ with kernel $N$. So, $p$ is the smallest prime index of a normal subgroup of $G$ that contains $K$. So, $p$ is the smallest prime index of a normal subgroup of $G/K$.

We have that $G/K$ is solvable by Lemma 8.20. So, the case of Theorem 8.16 in which the domain is solvable, we have that $\Lambda_{G/K,H} = 1/p$. Then, by Lemma 5.17, we have $\Lambda_{G,H} = \Lambda_{G/K,H} = 1/p$. □

### 9 Alternating domain, combinatorial list-decoding

In this section, we will find that homomorphism codes with alternating domain are CombEcon. The exact constant is stated in Theorem 9.7. We remark that the constant in the poly(1/ε)-bound on list size can be improved using the SRG methods of Section 10. The proof here utilizes the sphere packing bound, the sub-bucket bound, and a previous result on length of subgroup chains in symmetric groups [Bab86], which do most of the heavy lifting.

First, in Section 9.1, we present some background on the structure of alternating groups. In Section 9.2, we present a corollary to the sphere packing bound, Lemma 7.3. In Section 9.3 we show that $\Lambda_{\text{Alt},H}$ can only take the values $1/n$ and $1/(n^2)$ and prove the claim that $\text{Alt} \times \text{Groups}$ is CombEcon. Section 9.4 addresses the list-decoding radius of homomorphism codes with alternating domain, by exhibiting a “blowup” in list size when agreement is exactly $\Lambda$, or when radius is $(1-\Lambda)$.
9.1 Background on structure of alternating groups

For a set \( \Omega \), let \( \text{Alt}(\Omega) \) denote the alternating group on \( \Omega \). Similarly, let \( \text{Sym}(\Omega) \) denote the symmetric group on \( \Omega \). We denote \( A_n = \text{Alt}([n]) \) and \( S_n = \text{Sym}([n]) \).

Let \( G \leq \text{Sym}(\Omega) \). For \( \pi \in G \) and \( x \in \Omega \), we denote by \( x^\pi \) the action of \( \pi \) on \( x \). For \( x \in \Omega \), denote by \( G_x = \{ \pi \in G \mid x^\pi = x \} \) the point stabilizer of \( x \). Let \( \Delta \subseteq \Omega \). Denote by \( G(\Delta) = \{ \pi \in G \mid (\forall x \in \Delta)(x^\pi = x) \} \) the pointwise stabilizer of \( \Delta \). Denote by \( G(\Delta) = \{ \pi \in G \mid \Delta^\pi = \Delta \} \) the setwise stabilizer of \( \Delta \), where \( \Delta^\pi := \{ x^\pi : x \in \Delta \} \).

We present a few useful structural results for alternating and symmetric groups. The following theorem, due to Liebeck (see [DM96, Theorem 5.2A]), describes the large subgroups of \( A_n \).

**Theorem 9.1** (Jordan-Liebeck). Let \( n \geq 10 \) and let \( r \) be an integer with \( 1 \leq r < n/2 \). Suppose that \( K \leq A_n \) has index \( [A_n : K] < \binom{n}{r} \). Then, for some \( \Delta \subseteq [n] \) with \( |\Delta| < r \), we have \( (A_n)(\Delta) \leq K \leq (A_n)(\Delta) \).

We will need the following result from [Bab86], which describes the length of subgroup chains.

**Theorem 9.2** (Babai). The length of any subgroup chain in \( \text{Sym}(\Omega) \) is at most \( 2n - 3 \).

**Corollary 9.3.** The length of every subgroup chain between \( A_{n-k} \) and \( S_n \) is at most \( 2k \).

9.2 Sphere packing by low-depth subgroups

We present a consequence (Lemma 9.6) of the sphere packing bound (Lemma 7.3), which we will use to prove that alternating groups are universally CombEcon. This lemma bounds the list size in terms of a “starting set” of subgroups and the subgroup depth of its members. This approach depends very little on the codomain \( H \).

Throughout this section, we let \( G \) be a finite group, \( H \) a group (finite or infinite), \( \Lambda = \Lambda_{G,H} \) the maximum agreement, \( f : G \to H \) a received word, \( \varepsilon > 0 \) a real, \( \mathcal{L} = \mathcal{L}(a\text{Hom}(G,H), f, \Lambda + \varepsilon) \) the list, and \( \Psi \subseteq a\text{Hom}(G,H) \) as defined in Lemma 7.3.

Recall our strategy for proving CombEcon using the Sphere Packing Lemma — We divided the list \( \mathcal{L} \) into buckets \( \mathcal{L}_\psi \) for \( \psi \in \Psi \), where

\[
\mathcal{L}_\psi = \{ \varphi \in a\text{Hom}(G,H), f, \Lambda + \varepsilon \mid \text{agr}(\psi, \varphi) > \Lambda^2 \}.
\]

We then split the bucket \( \mathcal{L}_\psi \) into further sub-buckets according to the location of agreement with \( \psi \). Each sub-bucket of \( \mathcal{L}_\psi \) is labeled by a subgroup \( K \) of \( G \) which we call the label subgroup. We defined \( \mathcal{L}_{\psi,K} \subseteq \mathcal{L}_\psi \) to be the subset of homomorphisms whose equalizer with \( \psi \) contain \( K \); that is,

\[
\mathcal{L}_{\psi,K} = \{ \varphi \in \mathcal{L}_\psi \mid K \leq \text{Eq}(\varphi, \psi) \}.
\]

We concern ourselves now with the the set of label subgroups; we call such a set a starting set. Intuitively, a set \( \mathcal{S} \) of subgroups is a starting set if the upper range of the subgroup lattice of \( G \) contains only supergroups of elements in \( \mathcal{S} \).

With an appropriate notion of “upper range,” these starting sets form a sufficient set of label subgroups so that the sub-buckets \( \mathcal{L}_{\psi,K} \) cover the bucket \( \mathcal{L}_\psi \) (see Remark 9.5).

**Definition 9.4** \(((G, \lambda))-starting-set\). Let \( \mathcal{S} \) be a set of subgroups of \( G \). Let \( \lambda \in (0,1) \). We say that \( \mathcal{S} \) is a \((G, \lambda)\)-starting-set if

\[
(\forall K \leq G)(\mu_G(K) > \lambda ) \Rightarrow (\exists S \in \mathcal{S})(S \leq K)).
\]

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Remark 9.5. Suppose that \( S \) is a \((G, \Lambda^2)\)-starting set. Then, for any \( f : G \to H \) and \( \psi \in \text{aHom}(G, H) \),

\[
\mathcal{L}_\psi = \bigcup_{K \in S} \mathcal{L}_{\psi, K}. \tag{13}
\]

Combining this with the bucket-splitting lemma (Lemma 7.9),

\[
\mathcal{L} = \bigcup_{\psi \in \Psi} \bigcup_{K \in S} \mathcal{L}_{\psi, K}. \tag{14}
\]

This allows us to use our bound on \( \mathcal{L}_{\psi, K} \) from Corollary 7.17 to bound the size of the list.

Lemma 9.6 (Sphere packing via low-depth subgroups). Let \( G \) be a finite group, \( H \) a group, and \( \varepsilon > 0 \). Let \( S \) be a \((G, \Lambda^2)\)-starting-set. Then,

\[
\ell(\text{Hom}(G, H), \Lambda + \varepsilon) \leq \left( \frac{1}{4(\Lambda + \varepsilon)^6} + 1 \right) \cdot \sum_{K \in S} 1/\varepsilon^{\text{depth}(K)}. \tag{15}
\]

Proof. By Remark 9.5, we have that \( |\mathcal{L}| \leq \sum_{\varphi \in \Phi} \sum_{K \in S} |\mathcal{L}_{\varphi, K}| \). By Lemma 7.3, \( |\Psi| \leq \frac{1}{4(\Lambda + \varepsilon)^6} \), and by Corollary 7.17, \( |\mathcal{L}_{\varphi, K}| \leq 1/\varepsilon^{\text{depth}(K)} \).

We will use Lemma 9.6 in the proof that alternating groups are universally CombEcon.

9.3 Proof \( A_n \) is universally CombEcon

We prove that \( A_n \) is CombEcon by proving Theorem 9.7 below, which states a constant for the CombEcon claim. (This constant is improved via the methods of SRG groups in Section 10.)

Theorem 9.7. For every group \( H \), integer \( n \geq 38 \) and \( \varepsilon > 0 \), we find that

\[
\ell(\text{Hom}(A_n, H), \Lambda_{A_n, H} + \varepsilon) \leq 1/\varepsilon^{16}.
\]

Proof of Theorem 9.7. By Lemma 4.27, we find that \( \Lambda^2 \geq 1/(\binom{n}{2})^2 \geq 1/(n^5) \). We use Lemma 9.6.

We first define a starting set

\[
S = \{(A_n)(\Delta) : \Delta \subseteq [n], |\Delta| = 5\}.
\]

That \( S \) is an \((A_n, \Lambda^2)\)-starting-set follows by Jordan-Liebeck, Theorem 9.1. By Corollary 9.3, we find that \( \text{depth}_{A_n}(K) \leq 2 \cdot 5 - 2 = 8 \) for all \( K \in S \). We assume \( \varepsilon^2 < 2\Lambda \) by Lemma 7.6. Since \( |S| = \binom{n}{5} < (\binom{n}{2})/2 = (\Lambda/2)^3 \leq 1/\varepsilon^6 \), Theorem follows from Lemma 9.6.

9.4 Upper bound on list-decoding radius

We showed in Section 9.3 that \( \text{Alt} \times \text{Groups} \), and all of its subclasses, have list-decoding radius greater than \( 1 - (\Lambda + \varepsilon) \) for all \( \varepsilon > 0 \).

In contrast, \( \text{Alt} \times \text{Groups} \) and many of its subclasses have list-decoding radius at most \( 1 - \Lambda \). In this section, we demonstrate such a subclass. The number of homomorphisms within a closed ball of radius \( 1 - \Lambda \) of a received word will be exponential in \( \log|G| \) and \( \log|H| \). We note that \( |H| \geq |G| \) unless \( \Lambda = 0 \).
Proposition 9.8. For any $n$, and $\lambda \in \{1/n, 1/(\binom{n}{2})\}$, there exists a finite group $H_n$ such that $\Lambda_{A_n,H_n} = \lambda$ and
\[
\ell(\text{Hom}(A_n, H_n), \Lambda) = 2^{\Omega(n)} \geq 2^{\Omega\left(\frac{1}{\sqrt{\log |H|}}\right)}.
\]

Moreover, for any fixed $n \geq 10$, and any integer $M$, there is a finite group $H$ such that
\[
\ell(\text{Hom}(A_n, H), \Lambda) \geq M.
\]

Proof. We use the same construction for both parts. To prove the first claim, let $k = n$. To prove the second claim, let $k \geq \log_2 M$.

Suppose $\lambda = 1/n$. Let $H_n = A_{n+1}^k$, the direct product of $k$ copies of $A_{n+1}$. Then $\Lambda_{A_n,H_n} = 1/n$. Let $f: A_n \rightarrow H_n$ by $f(g) = (g, \ldots, g)$, the diagonal identity map, where $A_n$ is embedded in $A_{n+1}$. For nonempty $S \subseteq [n]$ and $j \in [n]$, let $h = h(S,j) = (h_1, \ldots, h_k) \in H_n$, where $h_i$ is the transposition $(j, n+1)$ if $i \in S$ and 1 otherwise. For each such $h$, let $\varphi_h \in \text{Hom}(A_n, H_n)$ be given by $\varphi_h(g) = h^{-1}f(g)h$. Each $\varphi_h$ has agreement $\text{agrf}(\varphi_h, f) = 1/n = \Lambda$ with $f$. There are $n(2^k - 1)$ such $h$, so $\ell(\text{Hom}(A_n, H_n), \Lambda) \geq n(2^k - 1)$.

Suppose $\lambda = 1/(\binom{n}{2})$. Let $H_n = A_k^n$. Then, $\Lambda_{A_n,H_n} = 1/(\binom{n}{2})$. Let $f: A_n \rightarrow H_n$ by $f(g) = (g, \ldots, g)$, the diagonal identity map. For nonempty $S \subseteq [n]$ and $\tau \in S_n$ is a transposition, let $h = h_{S,\tau} = (h_1, \ldots, h_k) \in A_k^n$, where $h_i = \tau$ if $i \in S$ and 1 otherwise. For each such $h$, let $\varphi_h \in \text{Hom}(A_n, H_n)$ be given by $\varphi_h(g) = h^{-1}f(g)h$. Each such $\varphi_h$ has agreement $\text{agrf}(\varphi_h, f) = 1/(\binom{n}{2})$. There are $\binom{n}{2}(2^k - 1)$ such $h$, so $\ell(\text{Hom}(A_n, H_n), \Lambda) \geq \binom{n}{2}(2^k - 1)$.

We remark that $\ell(\text{Hom}(A_n, H), \Lambda_{A_n,H})$ is not bounded as a function of $n$ for a wide variety of classes of $H$.

10 Shallow random generation

In this section, we prove results about SRG groups, defined in Section 4.4), for which few random elements tend to generate a shallow (low depth) subgroup. In Section 10.1, we will show that alternating groups are SRG. In Section 10.3 we will prove that SRG groups are also “KLC” groups (another generation property, defined in Section 10.2). The consequences of SRG will be proved using the KLC assumption in Section 11.

Recall that Section 5.5 showed

the code $\text{Hom}(G, H)$ is CombEcon if and only if $a\text{Hom}(G, H)$ is CombEcon, and similarly for CertEcon, and AlgEcon under modest assumptions of the representation of the groups. All our results about SRG groups (universal CombEcon and CertEcon, see Section 11) will take advantage of this equivalence, as our proofs will argue about $\text{Hom}(G, H)$ instead of $a\text{Hom}(G, H)$. To reflect this, concepts in this section are defined in terms of subgroup generation using $\langle \cdot \rangle$ instead of affine generation using $\langle \cdot \rangle_{\text{aff}}$.

Further recall Proposition 3.5 which states that, if $\Lambda_{G,H} \neq 0$, then ‘aHom’ can be replaced by ‘Hom’ in the definition of $\Lambda_{G,H}$, i.e.,
\[
\Lambda_{G,H} = \max_{\varphi, \psi \in \text{Hom}(G, H)} \text{agrf}(\varphi, \psi).
\]

If we used instead the affine version of our tools and arguments to reason directly about $a\text{Hom}(G, H)$ (instead of reasoning about $\text{Hom}(G, H)$ then using the material of Section 5.5 to get bounds for $a\text{Hom}(G, H)$), the degrees of the polynomials in the $\text{poly}(1/\varepsilon)$ expression for the SRG results would be identical.
10.1 Alternating groups are SRG

In this subsection, we prove that $\text{Alt}$ is SRG.

**Theorem 10.1.** The class of alternating groups is SRG. In fact, for all $k \geq 2$ there is an integer $n_k$ such that for all $n \geq n_k$, the alternating group $A_n$ is $(k, 4k - 2)$-shallow generating.

The quantity $n_k$ may be large as a function of $k$; however, there is an integer $n_0$ such that for all $k \geq 2$ and all $n \geq \max(n_0, (3k)^3)$, the alternating group $A_n$ is $(k, 6k - 2)$-shallow generating.

**Consequences.** Before proving Theorem 10.1, we first discuss its consequences. In Section 11 we prove facts that hold for all SRG groups. Here are the implications of those facts for alternating groups.

From Theorem 11.1, we find that $\text{aHom}(A_n, H)$ is CombEcon with degree 8, i.e., $\ell(\text{aHom}(A_n, H), \Lambda + \varepsilon) < 1/\varepsilon^8$ for all $H \in \text{Groups}$. We remark that the constant 8 can be improved to 6. This is by improving the $(8, \Lambda, 7)$-generated claim to $(6, \Lambda, 5)$-generated by going through the proof with a “depth$_\Lambda$” notion instead of depth$_G$ as written. This depth$_\Lambda(K)$ refers to maximal length of a subgroup chain from $K$ to a subgroup of density greater than $\Lambda$.

By Theorem 11.2, $\text{Alt}$ is CertEcon. More specifically, $\text{Alt}$ is universally strong certificate-list-decodable using $O(\ln(1/\varepsilon)/\varepsilon^9)$ queries and computation time. Certificates are generated by querying the received word on sets of uniform size. This is formalized in Section 11.2.

If $H = S_m$ and $m < 2^n/\sqrt{1.6n}$, then calls to the subword extender $\text{HomExt}$ can be executed in poly$(n, m)$-time, by Theorem 4.25. We combine $\text{HomExt}$ with the certificate-list-decoder to find a list-decoder. One call to $\text{HomExt}$ is made per certificate in the returned certificate-list, so the list-decoder runs in time poly$(1/\varepsilon, n, m)$ while using poly$(1/\varepsilon)$-queries.

**Presentation of proof.** To prove Theorem 10.1 we first present a useful result [Bab89, Theorem 1.5] that two random elements of the symmetric group generate a ‘large’ subgroup with high probability. This event is denoted $E(n, k)$ in the statement.

**Theorem 10.2** (Babai). Let $\pi, \sigma$ be a pair of independent uniform random elements from $S_n$. For $0 \leq k \leq n/3$, let $E(n, k)$ denote the following event: The subgroup $K = \langle \pi, \sigma \rangle$ acts as $S_r$ or $A_r$ on $r$ elements of the permutation domain for some $r \geq n - k$. Then,

$$\Pr(E(n, k)) = 1 - \left( \frac{n}{k + 1} \right)^{-1} + O\left( \left( \frac{n}{k + 2} \right)^{-1} \right).$$  \hspace{1cm} (18)

The constant implied by the big-O notation is absolute.

**Remark 10.3.** Suppose that we choose $\pi$ and $\sigma$ from $A_n$ (instead of $S_n$) in Theorem 10.2. The same conclusion is still true. However, using only Theorem 10.2 as justification, the conclusion is slightly weaker – there will be a coefficient of 4 in front of $(\binom{n}{k+1})^{-1}$. In our application, this coefficient makes no difference to our argument.

We now bound the depth of the subgroup generated by two elements, given that $E(n, 2k)$ occurred.

**Claim 10.4.** There exists $n_0$ such that the following holds: Let $E, k, \pi, \sigma$ be defined as in Theorem 10.2. If $n \geq n_0$ and $E(n, k)$ occurs, then $\text{depth}_{A_n}(\langle \pi, \sigma \rangle) \leq 2k - 2$. 

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Proof. Let \( K = \langle \pi, \sigma \rangle \). If \( E(n, k) \) occurs, then \( K \) acts as \( S_r \) or \( A_r \) on some subset of \( r \) elements of \([n]\), for some \( r \geq n - k > n/2 \). So, \( A_r \leq K \).

By Corollary 9.3, we find that \( \text{depth}_{A_n}(K) \leq \text{depth}_{A_n}(A_r) \leq 2k - 2 \).

\( \square \)

We can now prove Theorem 10.1.

Proof of Theorem 10.1. We prove the first part.

By Theorem 10.2 and Claim 10.4, it holds for large \( n \) that

\[
\text{Pr}_{\pi,\sigma \in A_n} \left[ \text{depth}_{A_n}(\langle \pi, \sigma \rangle) > 4k - 2 \right] \leq \text{Pr}_{\pi,\sigma \in A_n} \left[ -E(n, 2k) \right] \leq \frac{4}{(n/2) + 1} \leq \frac{1}{(n/2)^k} = (A_n^{*})^k.
\]

It follows that \( A_n \) is \((k, 4k - 2)\)-shallow generating.

The proof of the second part is similar. \( \square \)

10.2 Subset-generation

In this section we define a useful technical “KLC” condition on groups, which SRG groups satisfy (shown in the next section). The connection between KLC and universal CombEcon or CertEcon is more direct, and our “SRG implies universally CombEcon and CertEcon” results are proved through the KLC property (Section 11).

Definition 10.5 \((k, \lambda, c)\)-subset-generated). Let \( G \) be a finite group, \( k \) a nonnegative integer, \( 0 \leq \lambda < 1 \), and \( c \geq 0 \). We say that \( G \) is \((k, \lambda, c)\)-subset-generated if, for all subsets \( S \subseteq G \) with \( \mu(S) > \lambda \), we have that

\[
\text{Pr}_{s_1, \ldots, s_k \in S} \left[ \mu(\langle s_1, \ldots, s_k \rangle) > \lambda \right] \geq \left( 1 - \frac{\lambda}{\mu(S)} \right)^c,
\]

where \( s_1, \ldots, s_k \) are chosen independently and uniformly from \( S \).

Note that, if we define \( \varepsilon = \mu(S) - \lambda \), then \( 1 - \frac{\lambda}{\mu(S)} = \frac{\varepsilon}{\lambda + \varepsilon} \), so Equation (19) mirrors the expression of Lemma 7.12.

We say that \( G \) is \((k, \lambda, c)\)-affine-generated if it satisfies Definition 10.5 but with \( \langle s_1, \ldots, s_k \rangle \) replaced by \( \langle s_1, \ldots, s_k \rangle_{\text{aff}} \).

We will make a few remarks on these definitions below, but first we define KLC classes of groups.

Definition 10.6 (KLC). Let \( \mathcal{G} \) be a class of finite groups. We say that \( \mathcal{G} \) is KLC if there exists a positive integer \( k \) and a constant \( c > 0 \) such that, for all \( G \in \mathcal{G} \) and for all groups \( H \), we have that \( G \) is \((k, \Lambda_G, H, c)\)-subset-generated.

The notion of “KLC-affine” can be defined analogously. But, according to Remark 10.7 (c) below, the two conditions are equivalent on a class of groups.

We make a few remarks on the definitions of \((k, \lambda, c)\)-subset-generated groups.

Remark 10.7. (a) For every \( k \geq 1 \) and \( c \geq 0 \), the class of all finite groups is \((k, 0, c)\)-subset-generated.

(b) Classes of \((k, \lambda, c)\)-subset-generated groups are monotone in both \( k \) and \( c \). More specifically, for \( k' > k \) and \( c' > c \), if \( G \) is \((k, \lambda, c)\)-subset-generated, then \( G \) is also \((k', \lambda, c)\)-subset-generated and \((k, \lambda, c')\)-subset-generated.

(c) If \( G \) is \((k, \lambda, c)\)-affine-generated, then it is \((k, \lambda, c)\)-subset-generated. If \( G \) is \((k, \lambda, c)\)-affine-generated, then it is \((k + 1, \lambda, c)\)-generated.
10.3 SRG implies subset-generation

We prove that SRG implies KLC, using a straightforward application of Bayes’ rule.

**Theorem 10.8** (SRG implies KLC). If a class \( \mathcal{G} \) of groups is SRG, then \( \mathcal{G} \) is KLC.

In particular, let \( G \) be a finite group, \( k, d \in \mathbb{N} \), and \( \lambda > 0 \). If \( G \) is \((k, d)\)-shallow generating, then \( G \) is \((k + d, \lambda, 1 + d)\)-subset-generated for all \( \lambda \geq \Lambda^*_G \).

**Proof.** All groups are trivially \((1, 0, 1)\)-subset-generated, which covers the case where \( \lambda = 0 \).

Let \( \lambda > 0 \). By assumption, we know that

\[
\Pr_{g_1, \ldots, g_k \in G} \left[ \text{depth}(\langle g_1, \ldots, g_k \rangle) > d \right] < \lambda^k.
\]  

(20)

We check the definition of \((k + d, \lambda, 1 + d)\)-subset-generated.

Let \( S \subseteq G \) be such that \( \mu(S) > \lambda \) and let \( \varepsilon = \mu(S) - \lambda \). We will pick a \( k \)-tuple \( g = (g_1, \ldots, g_k) \) and a \( d \)-tuple \( s = (s_1, \ldots, s_d) \) from \( S \). We write \( \langle g \rangle \) to mean \( \langle g_1, \ldots, g_k \rangle \). We write \( \langle s \rangle \) and \( \langle s, g \rangle \) similarly.

Observe that

\[
\Pr_{g \in S^k, s \in S^d} \left[ \mu(\langle g, s \rangle) > \lambda \right] \geq \Pr_{s \in S^d} \left[ \mu(\langle g, s \rangle) > \lambda \mid \text{depth}(\langle g \rangle) \geq d \right] \cdot \Pr_{g \in S^k} \left[ \text{depth}(\langle g \rangle) \geq d \right].
\]

We bound the two components of the right hand side separately. When we drop the subscript on \( \Pr \), that means the elements are chosen at random from \( G \). First, we consider the second component.

\[
\Pr_{g \in S^k} \left[ \text{depth}(\langle g \rangle) \geq d \right] = \frac{\Pr_{g \in S^k} \left[ \text{depth}(\langle g \rangle) \geq d \mid g \in S^k \right]}{\Pr_{g \in S^k} \left[ g \in S^k \right]}
\]

\[
\geq \frac{\Pr_{g \in S^k} \left[ g \in S^k \right] + \Pr_{\text{depth}(\langle g \rangle) \geq d} - 1}{\Pr_{g \in S^k} \left[ g \in S^k \right]}
\]

\[
> \frac{\mu(S)^k - (\lambda)^k}{\mu(S)^k} = 1 - \left( \frac{\lambda}{\mu(S)} \right)^k
\]

\[
\geq 1 - \frac{\lambda}{\lambda + \varepsilon} = \frac{\varepsilon}{\lambda + \varepsilon}.
\]

Now, it suffices to show that the first component has probability bounded by \( \left( \frac{\varepsilon}{\lambda + \varepsilon} \right)^d \). But, if \( \text{depth}(\langle g \rangle) \geq d \), then it follows from Lemma 7.12, with \( K = \langle g \rangle \) and \( \lambda = \lambda \), that

\[
\Pr_{s \in S^d} \left[ \mu(\langle g, s \rangle) > \lambda \mid \text{depth}(\langle g \rangle) \geq d \right] > \left( \frac{\varepsilon}{\lambda + \varepsilon} \right)^d.
\]

We conclude that \( G \) is \((k + d, \lambda, 1 + d)\)-subset-generated, or,

\[
\Pr_{s_1, \ldots, s_{k+d} \in S} \left[ \mu(\langle s_1, \ldots, s_{k+d} \rangle) > \lambda \right] = \Pr_{g \in S^k, s \in S^d} \left[ \mu(\langle g, s \rangle) > \lambda \right] > \left( \frac{\varepsilon}{\lambda + \varepsilon} \right)^{d+1}.
\]

\( \square \)
11 Consequences of shallow random generation

We proved the claimed consequences of shallow random generation, using the KLC property. These consequences include universal CombEcon (Section 11.1) and universal CertEcon (Section 11.2).

Universal AlgEcon follows if a subword extender (HomExt(G, H) oracle) is provided, as discussed in Section 4.7. Stronger subword extenders provide better guarantees on the output list and lower bounds on \( \Lambda \) (Section 11.3).

11.1 SRG implies CombEcon

KLC implies universally CombEcon.

**Theorem 11.1.** If a class \( \mathcal{G} \) is SRG, then \( \mathcal{G} \) is universally CombEcon. More precisely, let \( k \in \mathbb{N} \) and \( c > 0 \). If \( G \) is a \((k, \Lambda_{G,H}, c)\)-subset-generated group, then \( \ell(\text{Hom}(G,H), \Lambda_{G,H} + \varepsilon) \leq 1/\varepsilon \max\{c,k\} \) for all \( \varepsilon \in (0,1-\Lambda) \) and groups \( H \).

**Proof.** By Theorem 10.8, if \( \mathcal{G} \) is SRG, then \( \mathcal{G} \) is KLC. We define a bipartite graph \( X \). The left vertex set of \( X \) is \( G^k \). The right vertex set is \( \mathcal{L} = \text{L}(\text{Hom}(G,H), f, \Lambda + \varepsilon) \). There is an edge between \((g_1, \ldots, g_k) \in G^k \) and \( \varphi \in \mathcal{L} \) if \( g_1, \ldots, g_k \in \text{Eq}(f, \varphi) \), and \( \mu(\langle g_1, \ldots, g_k \rangle) > \Lambda \).

By the definition of \((k, \Lambda_{G,H}, c)\)-subset-generated group, each right vertex has degree at least \( \varepsilon (\Lambda + \varepsilon)c |S|^k \), which is at least \( \varepsilon (\Lambda + \varepsilon)c |G|^k \geq \varepsilon \max\{c,k\} |G|^k \).

By Lemma 7.16 with \( K = 1 \), each left vertex of \( X \) has degree at most one. So, by part (a) of Lemma 5.1 (double counting), \( |\mathcal{L}| \leq 1/\varepsilon \max\{c,k\} \). \( \square \)

11.2 SRG implies CertEcon

A KLC class of groups is also universally CertEcon. The argument is conceptually similar to that of CombEcon, but we need to formalize algorithmic issues.

Again, let \( W_\Lambda \) denote the set of \( G \rightarrow H \) partial maps \( \gamma \) satisfying \( \mu(\langle \text{dom}\, \gamma \rangle) > \Lambda_{G,H} \).

**Theorem 11.2.** If \( \mathcal{G} \) be an SRG class of groups, then \( \mathcal{G} \) is universally strong \( W_\Lambda \)-CertEcon. We assume that all groups in \( \mathcal{G} \) are encoded groups, that (nearly) uniform elements of \( G \) are provided, and that we have oracle access to the entries of the received word.

If a partial map \( \gamma \) can be extended to some homomorphism and satisfies \( \mu(\langle \text{dom}\, \gamma \rangle) > \Lambda_{G,H} \), then \( \gamma \) is a \( W \)-certificate. However, \( \text{dom}(\gamma) \) may fail to generate the entire group, i.e., \( \langle \text{dom}\, \gamma \rangle \not\subseteq G \).

Following the strategy of Section 2.6, we may combine a certificate list-decoder and a HOMOMORPHISM EXTENSION oracle \( \text{HomExt}_\Lambda \) (a \( W_\Lambda \)-subword extender). The next section discusses how to take this strategy a step further using a stronger HOMOMORPHISM EXTENSION oracle to lower bound \( \Lambda \).

**Domain certificates versus certificates.**

We develop some terminology for Theorem 11.2, based on the natural idea of generating certificates by querying the received word \( f \). “Domain certificates” (dependent on \( f \)) are subsets of the domain that define a certificate when restricting \( f \) to that set.

Let \( G \) and \( H \) be groups and \( f \in H^G \) be a received word in the codespace of \( \text{Hom}(G,H) \). Let \( S \subseteq G \) be a subset. Denote by \( f_S \) the restriction of \( f \) to \( S \), i.e., the \( H \rightarrow G \) partial map with domain \( S \) defined by \( f_S(g) = f(g) \) for \( g \in S \).
Definition 11.3 (Domain certificate). When the code \( \text{Hom}(G, H) \) and the received word \( f \) are understood, we say that a subset \( S \subseteq G \) is a domain certificate if the \( G \rightarrow H \) partial map \( f_S \) is a certificate for \( \text{aHom}(G, H) \).

For a set \( \mathcal{W} \) of \( G \rightarrow H \) partial maps, we say that \( S \) is a domain \( \mathcal{W} \)-certificate if \( f_S \) is a \( \mathcal{W} \)-certificate for \( \text{Hom}(G, H) \).

Note that a domain certificate \( S \subseteq G \) is a domain \( \mathcal{W}_\Lambda \)-certificate if and only if \( \mu(\langle S \rangle) > \Lambda_{G,H} \).

Definition 11.4 (Domain-certificate-list). We say that a list \( \Upsilon \) of subsets of \( G \) is a domain-certificate-list for a subset \( L \subseteq \text{aHom}(G, H) \) of affine homomorphisms if \( \Upsilon \) contains a domain certificate for each codeword in \( L \). We define domain-\( \mathcal{W} \)-certificate-lists similarly.

Domain certificate result.

Now, we can restate the unabridged SRG result (Theorem 4.17) in terms of domain certificates.

Theorem 11.5 (SRG implies CertEcon, via domain certificates). Let \( k \in \mathbb{N} \) and \( c > 0 \). Let \( G \) be a \((k, \Lambda_{G,H}, c)\)-subset-generated group and \( H \) a group. Let \( f : G \rightarrow H, \varepsilon > 0 \) and \( \eta > 0 \). Let \( \Upsilon \) be a list of \[ \left\lfloor \frac{1}{c} \ln \left( \frac{1}{\varepsilon \eta c} \right) \right\rfloor \] independently chosen subsets of \( G \), each of size \( \max\{c, k\} \). Then, with probability at least \((1 - \eta)\), \( \Upsilon \) is a domain-\( \mathcal{W}_\Lambda \)-certificate-list of \( \mathcal{L}(\text{aHom}(G, H), f, \Lambda + \varepsilon) \).

The proof is delayed to first discuss its implications and access model.

Remark 11.6 (Access model). To generate the domain-\( \mathcal{W} \)-certificate-list, we need access only the domain, only in the ability to generate random elements. No knowledge of \( H \) is required. The dependence on \( H \) appears only in the \( \Lambda_{G,H} \) of the assumption that \( G \) is \((k, \Lambda_{G,H}, c)\)-subset generated, but the KLC assumption means that \( G \) is \((k, \Lambda_{G,H}, c)\)-subset generated for every \( H \). Knowledge of \( \Lambda_{G,H} \) is also not required.

Theorem 11.5 produces domain \( \mathcal{W} \)-certificates. No work is involved other than generating these \( \text{poly}(1/\varepsilon) \) uniform random elements of \( G \). To then produce actual \( \mathcal{W} \)-certificates, simply query \( f \) on the domain \( \mathcal{W} \)-certificates, an additional \( \text{poly}(1/\varepsilon) \) queries to \( f \). Theorem 11.2 follows.

Remark 11.7 (Amount of work). Theorem 11.2 implies a stronger result than strong CertEcon, as only a \( \text{poly}(1/\varepsilon) \) amount of work is required in the unit cost model (no dependency on \(|G|\)).

Analysis.

To prove Theorem 11.5 we check the definition of domain \( \mathcal{W}_\Lambda \)-certificate-list. In other words, with probability \((1 - \eta)\), for every \( \varphi \in \mathcal{L}(\text{Hom}(G, H), f, \Lambda + \varepsilon) \) the list \( \Upsilon \) contains a domain \( \mathcal{W}_\Lambda \)-certificate \( S_\varphi \subseteq G \) for \( \varphi \). A sufficient condition to be a domain \( \mathcal{W}_\Lambda \) certificate is given below.

Observation 11.8. If the conditions \( \mu(\langle S \rangle) > \Lambda \) and \( S \subseteq \text{Eq}(\varphi, f) \) are satisfied, then \( S \) is a domain \( \mathcal{W}_\Lambda \)-certificate for \( \varphi \).

Proof of Theorem 11.5. Let \( \mathcal{L} = \mathcal{L}(\text{Hom}(G, H), f, \Lambda + \varepsilon) \). Recall that \( G \) is \((k, \Lambda_{G,H}, c)\)-subset generated. Let \( b = \max\{c, k\} \). Let \( \Upsilon = \{S_1, \ldots, S_t\} \) be the list of uniformly chosen subsets of \( G \) as assumed. Then, \( t = \left\lfloor \frac{1}{c} \ln \left( \frac{1}{\varepsilon \eta c} \right) \right\rfloor \) and each \( S_i \) consists of \( b \) uniformly and independently chosen elements of \( G \).

\footnote{Two incomparable sufficient conditions for the access model to \( G \) are black-box access and polycyclic presentations. In a black-box group, \( \varepsilon \)-uniform elements can be generated in polynomial time [Bab91]. Given a polycyclic presentation, exactly uniform elements can be generated.}
Fix $\varphi \in \mathcal{L}$. Fix $S \in \Upsilon$. We calculate the following.

$$
\Pr[S \text{ is a domain } W_\Lambda\text{-certificate for } \varphi] \geq \Pr[S \subseteq \text{Eq}(\psi, f) \cap \mu(\langle S \rangle) > \Lambda] \\
= \Pr[\mu(\langle S \rangle) > \Lambda | S \subseteq \text{Eq}(\psi, f)] \cdot \Pr[S \subseteq \text{Eq}(\psi, f)] \\
> \left(\frac{\varepsilon}{\Lambda + \varepsilon}\right)^{c} (\Lambda + \varepsilon)^{k} \\
> \varepsilon^{b}.
$$

The first inequality follows from Observation 11.8, and the second inequality follows from the definition of $(k, \Lambda, c)$-subset generated.

The probability that $\Upsilon$ is not a domain $W$-certificate-list for $L$, i.e., there is $\varphi \in \mathcal{L}$ such that $\Upsilon$ contains no domain-$W$-certificate for $\varphi$, is bounded by

$$
|\mathcal{L}| \cdot \left(1 - \varepsilon^{b}\right)^{t} \leq \frac{1}{\varepsilon^{b}} \exp\left(-\varepsilon^{b} \cdot t\right) < \eta,
$$

where we have used that $|\mathcal{L}| \leq 1/\varepsilon^{b}$ by the CombEcon result Theorem 11.1.

11.3 Improvements on $\Lambda$

In this section we first discuss the role of $\Lambda$ in the relationship between CertEcon, HomExt, and AlgEcon. Then, we will give an algorithm that improves our lower bounds on $\Lambda$ and discuss its benefits.

Role of $\Lambda$ lower bounds in subword extenders.

As discussed in generality (Section 2.6), for two sets $W_1$ and $W_2$ of partial maps $\Omega \rightarrow \Sigma$, if they satisfy $W_1 \subseteq W_2$ then a $W_1$-certificate-list-decoder and a $W_2$-subword-extender combine to a list-decoder. In our context, we consider the sets $W_\lambda$ consisting of $G \rightarrow H$ partial maps whose domain generate a subgroup of $\lambda$ density, i.e.,

$$
W_\lambda := \{\gamma : G \rightarrow H | \mu(\langle \text{dom } \gamma \rangle) > \lambda\}.
$$

Since we have $W_{\Lambda,G,H}$-CertEcon results for SRG groups (Theorem 11.2), it suffices to find $W_\lambda$-subword extenders, i.e., solve $\text{HomExt}_\lambda(G, H)$, for $a\text{Hom}(G, H)$ with $\lambda \leq \Lambda_{G,H}$.

So, a stronger lower bound for $\Lambda_{G,H}$ allows use of a weaker $\text{HomExt}$ oracle.

Role of subword extenders in $\Lambda$ lower bounds.

Conversely, a stronger $\text{HomExt}(G, H)$ oracle can be used to update lower bounds on $\Lambda_{G,H}$, which allows better pruning of the output list. We will explain both clauses of this sentence below.

First, we discuss finding better lower bounds on $\Lambda_{G,H}$, using a stronger version of $\text{HomExt}$ we call $\text{HomExt}012$. The $\text{HomExt}012$ Problem asks to distinguish between the cases of no extension, unique extension, and multiple extensions. In the case of a unique extension, it asks for the extension.

Definition 11.9. ($\text{HomExt}012(G, H)$)

Instance: A partial map $\gamma : G \rightarrow H$.

Solutions: The set defined by

$$
\text{HExt}(\gamma) := \{\varphi \in \text{Hom}(G, H) : \varphi|_{\text{dom } \gamma} = \gamma\}.
$$
Output: \[
\begin{array}{ll}
\text{‘none’} & \text{if } |\text{HExt}(\gamma)| = 0 \\
\varphi \in \text{HExt}(\gamma) & \text{if } |\text{HExt}(\gamma)| = 1 \\
\text{‘multiple’} & \text{if } |\text{HExt}(\gamma)| \geq 2
\end{array}
\]

The HOMEXT012\(_\lambda\)(\(G, H\)) problem is defined similarly, but requiring only correct answers on the \(G\rightarrow H\) partial maps in \(W_\lambda\).

**Proposition 11.10.** Let \(G\) and \(H\) be groups to which we are given black-box access. Suppose that we are given an oracle for HOMEXT012\(_\lambda\)(\(G, H\)) and an order oracle for subgroups. Then, a subword extender for aHom\(_\lambda\)(\(G, H\)) can be implemented in poly(enc(\(G\))-time in the unit-cost model for \(H\). Moreover, for any \(G\rightarrow H\) partial map \(\gamma\) on which HOMEXT012(\(G, H\)) returns ‘multiple,’ the value of \(\mu(\langle \text{dom } \gamma \rangle)\) is a lower bound for \(\Lambda\).

**Proof.** The first conclusion of this proposition is the same as Proposition 4.21.

The second conclusion is trivial, since multiple extensions of \(\gamma\) would be distinct homomorphisms in Hom\(_\lambda\)(\(G, H\)) that agree on \(\langle \text{dom } \gamma \rangle\), so \(\mu(\langle \text{dom } \gamma \rangle)\) lower bounds \(\Lambda\). (Recall Proposition 3.5 which states that, if \(\Lambda \neq 0\), then ‘aHom’ can be replaced by ‘Hom’ in the definition of \(\Lambda\).)

The order oracle can be used to calculate the value of \(\mu(\langle \text{dom } \gamma \rangle)\).

Recall that our main algorithmic result (Theorem 1.7) hinges on a solution [Wuu18] for HOMEXT\(_\lambda\)(\(G, H\)) Search in the cases considered (\(G = A_n\) and \(H = S_m\) for exponentially bounded \(m\)), for the lower bound \(\lambda = 1/\binom{n}{2}\) of \(\Lambda\). In fact [Wuu18] provides a solution for HOMEXT012\(_\lambda\)(\(G, H\)) with the same \(\lambda\) (or the version of HOMExt that counts solutions until some threshold). This allows a stronger lower bounding of \(\Lambda\) as promised.

We discuss the “better pruning” consequences of these updated lower bounds on \(\Lambda\).

1. Better pruning of the final output list: The definition of list-decoder requires only that the output list \(\tilde{\mathcal{L}}\) be a superlist of the desired list \(\mathcal{L} = \mathcal{L}(\text{aHom}(G, H), f, \Lambda + \varepsilon)\). The output can be pruned, using any lower bound \(\lambda\) on \(\Lambda\), to contain only affine homomorphisms \(\varphi \in \text{aHom}(G, H)\) that have high agreement \(\text{agr}(\varphi, f) > \lambda + \varepsilon/2\) with \(f\). The pruning can be accomplished by sampling agreement.

2. Faster processing of certificate-lists (output of certificate-list-decoder) into output lists (output of list-decoder): While the value of \(\Lambda\) may be unknown, the certificate-list-decoder of Theorem 11.2 guarantees certificates in \(W_{\Lambda,G,H}\). A known lower bound \(\lambda\) for \(\Lambda\) allows pruning of partial maps \(\gamma\) that do not satisfy \(\mu(\langle \text{dom } \gamma \rangle) > \lambda\). The subword extender need not be called on these maps.

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