QUADRIC SURFACES OF COORDINATE FINITE TYPE GAUSS MAP

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ABSTRACT. We study quadric surfaces in the 3-dimensional Euclidean space which are of coordinate finite type Gauss map with respect to the second fundamental form $II$, i.e., their Gauss map vector $n$ satisfies the relation $\Delta^I n = \Lambda n$, where $\Delta^I$ denotes the Laplace operator of the second fundamental form $II$ of the surface and $\Lambda$ is a square matrix of order 3. We show that helicoids and spheres are the only class of surfaces mentioned above satisfying $\Delta^I n = \Lambda n$.

1. INTRODUCTION

In 1983 B. Y. Chen introduces the notion of Euclidean immersions of finite type, [18], [21] and from that time on the research into surfaces of finite type has grown up as one can see in the literature in this subject. Denote by $\Delta^I$ the second Beltrami-Laplace operator corresponding to the first fundamental form $I$ of a (connected) submanifold $M^m$ in the $n$-dimensional Euclidean space $E^n$. Let $x, H$ be the position vector field and the mean curvature field of $M^m$ respectively. Then, it is well known that [18]

$$\Delta^I x = -nH.$$

From this formula one can see that $M^m$ is a minimal submanifold if and only if all coordinate functions, restricted to $M^m$, are eigenfunctions of $\Delta^I$ with eigenvalue $\lambda = 0$. Moreover in [29] T. Takahashi showed that a submanifold $M^m$ for which $\Delta^I x = \lambda x$, i.e. for which all coordinate functions are eigenfunctions of $\Delta^I$ with the same eigenvalue $\lambda$, is either minimal in $E^n$ with eigenvalue $\lambda = 0$ or minimal in a hypersphere of $E^n$ with eigenvalue $\lambda > 0$.

The class of finite type submanifolds in an arbitrary dimensional Euclidean spaces is very large, on the other hand very little is known about surfaces of finite type in the Euclidean 3-space $E^3$. Actually, so far, the only known surfaces of finite type corresponding to the first fundamental form in the Euclidean 3-space are the minimal surfaces, the circular cylinders and the spheres. So in [19] B.-Y. Chen laid out the problem to determine finite type surfaces in the Euclidean 3-space $E^3$ other than the ordinary spheres and minimal ones.

In [24] O. Garay generalized T. Takahashi’s condition studied surfaces in $E^3$ for which all coordinate functions $(x_1, x_2, x_3)$ of $x$ satisfy $\Delta^I x_i = \lambda x_i$, $i = 1, 2, 3$, not necessarily with the same eigenvalue. Another generalization was studied in [22] for which surfaces in $E^3$ satisfy the condition $\Delta^I x = Ax + B$ ($\S$) where $A \in R^{3\times 3}$, $B \in R^{3\times 1}$. It was shown that a surface $S$ in $E^3$ satisfies ($\S$) if and only if it is an open
part of a minimal surface, a sphere, or a circular cylinder. Surfaces satisfying (§) are said to be of coordinate finite type.

In this context, one can also study surfaces in \( \mathbb{E}^3 \) whose Gauss map \( n \) satisfies a relation of the form
\[
\Delta^I n = A n
\]
where \( A \in \mathbb{R}^{3 \times 3} \).

Concerning this, in [14] it was shown that planes and circular cylinders are the only ruled surfaces whose Gauss map satisfies (1.1). It is also known that [25] planes, spheres and circular cylinders are the only surfaces of revolution whose Gauss map satisfies (1.1). In [16] Ch. Baikoussis and L. Verstraelen proved that planes, spheres and circular cylinders are the only helicoidal surfaces in \( \mathbb{E}^3 \) whose Gauss map satisfies (1.1). Next in [17], this problem was solved for the class of translation surfaces in \( \mathbb{E}^3 \), it was shown that the only translation surfaces whose Gauss map satisfies (1.1) are the planes and the circular cylinders. Finally, in [14] this problem was studied for the compact and noncompact cyclides of Dupin, it was proved that neither for the compact nor for the noncompact cyclides of Dupin, there exist a matrix \( A \in \mathbb{R}^{3 \times 3} \) such that the relation (1.1) is satisfied.

In this area, S. Stamatakis and H. Al-Zoubi in 2003 introduced the notion of surfaces of finite type with respect to the second or third fundamental forms [27] and it has been a topic of active research since then. As an extension of the studies mentioned above, we raise the following two questions which seem to be very interesting:

**Problem 1.** Classifying all surfaces of coordinate finite type in the Euclidean 3-space with respect to the second or third fundamental form.

**Problem 2.** Classifying all surfaces of coordinate finite type Gauss map in the Euclidean 3-space with respect to the second or third fundamental form.

Many results concerning these two problems one can find in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 26, 28].

In this paper we will pay attention to surfaces of finite II-type. First, we will establish a formula for \( \Delta^{II} n \) by using tensors calculations. Further, we continue our study by studying the quadric surfaces in \( \mathbb{E}^3 \) which are connected, complete and of which their Gauss map satisfy the following relation
\[
\Delta^{II} n = A n,
\]

2. Preliminaries

In the Euclidean 3-space \( \mathbb{E}^3 \) we consider a \( C^r \)-surface \( S \), \( r \geq 3 \), defined by an injective \( C^r \)-immersion \( x = x(u^1, u^2) \) on a region \( U := I \times \mathbb{R} \) \( (I \subset \mathbb{R} \) open interval) of \( \mathbb{R}^2 \) whose Gaussian curvature never vanishes. We denote by
\[
I = g_{ij} du^i du^j, \quad II = b_{ij} du^i du^j, \quad III = e_{ij} du^i du^j, \quad i, j = 1, 2,
\]
the first, second and third fundamental forms of \( S \) respectively. Let \( f(u^1, u^2) \) and \( h(u^1, u^2) \) be two sufficiently differentiable functions on \( S \). Then the first differential parameter of Beltrami corresponding to the fundamental form \( J = I, II, III \) is defined by [25]
\[
\nabla^J(f, h) := a^{ij} f_{i,j} h_{j,i}
\]
where \( f_{/i} := \frac{\partial f}{\partial u^i} \), and \((a_{ij})\) denotes the components of the inverse tensor of \((g_{ij}), (b_{ij})\) and \((e_{ij})\) for \( J = I, II \) and \( III \) respectively. The second differential parameter of Beltrami corresponding to the fundamental form \( J = I, II, III \) of \( S \) is defined by \[ \triangle^J f := -a^{ij} \nabla_i^J f_j, \] (2.2)

where \( \nabla_i^J \) is the covariant derivative in the \( u^i \) direction corresponding to the fundamental form \( J \).

We compute now \( \triangle^H n \). Firstly, we mention the following relation for later use \[ \nabla^H_i (h, n) + \text{grad}^I h = 0. \] (2.3)

Applying (2.2) for the position vector \( n \) of \( S \) we have

\[ \triangle^H n = -b^{ij} (\nabla_i^H b_{ij}) n_j, \] (2.4)

Taking into consideration the equations (2.5), p.128

\[ \nabla_i^H b_{ij} = -\frac{1}{2} b^{kr} (\nabla^I r b_{ij}) n_j - e_{ij} n, \]

so that (2) takes the form

\[ \triangle^H n = \frac{1}{2} b^{kr} b_{ij} (\nabla^I r b_{ij}) n_j + b^{ij} e_{ij} n. \]

We consider the Christoffel symbols of the second kind corresponding to the first, second and third fundamental form, respectively

\[ \Gamma^k_{ij} := \frac{1}{2} g^{kr} (-g_{ij/r} + g_{ir/j} + g_{jr/i}), \]
\[ \Pi^k_{ij} := \frac{1}{2} b^{kr} (-b_{ij/r} + b_{ir/j} + b_{jr/i}), \]
\[ \Lambda^k_{ij} := \frac{1}{2} e^{kr} (-e_{ij/r} + e_{ir/j} + e_{jr/i}), \]

and we put

\[ T^k_{ij} := \Gamma^k_{ij} - \Pi^k_{ij}, \]
\[ \tilde{T}^k_{ij} := \Lambda^k_{ij} - \Pi^k_{ij}. \]

(2.5) (2.6)

It is known that (see [25], p.22)

\[ T^k_{ij} := -\frac{1}{2} b^{kr} \nabla_i^J f_{/j}, \]
\[ \tilde{T}^k_{ij} := -\frac{1}{2} b^{kr} \nabla_i^H b_{ij}, \]

(2.7) (2.8)

and

\[ \tilde{T}^k_{ij} + T^k_{ij} = 0. \]

(2.9)

On account of

\[ 2H = b_{/ij} g^{ij} = e_{ij} b^{ij}, \]

and (2.8) we obtain

\[ \triangle^H n = -b^{ij} \tilde{T}^k_{ij} n_{/k} + 2H n. \]

From the Mainardi-Codazzi equations (see [25], p.128)

\[ \nabla^J_i b_{ij} - \nabla^J_i b_{jk} = 0, \]

(2.10)
and using (2.7) and (2.9) we have
\[ \nabla^{II} n = b^{kr} T_{rj}^{j} n_{k} + 2H n. \] (2.11)

For the Christoffel symbols $\Gamma^{j}_{kj}$ and $\Pi^{j}_{kj}$ we have (see [25], p.125)
\[ \Gamma^{j}_{ij} := \frac{g_{ji}}{2g}, \quad \Pi^{j}_{ij} := \frac{b_{ji}}{2b}, \] (2.12)
where $g := \det(g_{ij})$ and $b := \det(b_{ij})$. On the other hand, the Gauss curvature $K$ of $S$ is given by
\[ K = \frac{b}{g}. \]

Hence, we have
\[ \frac{K_{jk}}{K} = \frac{b_{jk}}{b} - \frac{g_{jk}}{g}. \] (2.13)

On the other hand using (2.5), (2.7), (2.12) and (2.13) we have
\[ b^{kr} T_{rj}^{j} n_{k} = -\frac{1}{2K} b^{kr} K_{j} n_{k} = -\frac{1}{2K} \nabla^{II}(K, n). \]

Inserting this in (2.11) we find in view of (2.3)
\[ \nabla^{II} n = \frac{1}{2K} \text{grad}^{I}(K) + 2H n. \] (2.14)

Now we mention our main result which is the following

**Proposition 1.** The only quadric surfaces in the 3-dimensional Euclidean space that satisfy (1.2), are the helicoids and the spheres.

Our discussion is local, which means that we show in fact that any open part of a quadric satisfies (1.2), if it is an open part of a helicoid or a sphere.

### 3. Proof of Proposition 1

Let now $S$ be a quadric surface in the Euclidean 3-space $E^{3}$. Then $S$ is either ruled, or of one of the following two kinds

\[ z^2 - ax^2 - by^2 = c, \quad abc \neq 0 \] (3.1)
or

\[ z = \frac{a}{2} x^2 + \frac{b}{2} y^2, \quad a > 0, \quad b > 0. \] (3.2)

We first show that helicoids are the only ruled surface in the three-dimensional Euclidean space such that its Gauss map satisfies condition (1.2). Next we show that the Gauss map of a quadric of the kind (3.1) satisfies (1.2) if and only if $a = -1$ and $b = -1$, which means that $S$ is a sphere. Finally, we show that the Gauss map of a quadric of the kind (3.2) is never satisfying (1.2).
3.1. **Ruled surfaces.** Let $S$ be a ruled surface in $E^3$. We suppose that $S$ is a non-cylindrical ruled surface. This surface can be expressed in terms of a directrix curve $\sigma(s)$ and a unit vector field $\tau(s)$ pointing along the rulings as

$$S : x(s, t) = \sigma(s) + t\tau(s), \quad s \in J, \quad -\infty < t < \infty.$$ 

Moreover, we can take the parameter $s$ to be the arc length along the spherical curve $\tau(s)$. Then we have

$$\langle \sigma', \tau \rangle = 0, \quad \langle \tau, \tau \rangle = 1, \quad \langle \tau', \tau' \rangle = 1,$$

(3.3)

where the prime denotes the derivative in $s$. The first fundamental form of $S$ is

$$I = q ds^2 + dt^2,$$

while the second fundamental form is

$$II = \frac{p}{\sqrt{q}} ds^2 + \frac{2A}{\sqrt{q}} ds dt,$$

where

$$q = t^2 + 2lt + k, \quad p = mt^2 + nt + r.$$ 

For convenience, we put

$$k = \langle \sigma', \sigma' \rangle, \quad l := \langle \sigma', \tau \rangle, \quad m := \langle \tau', \tau, \tau'' \rangle, \quad n := \langle \sigma', \tau, \tau'' \rangle + \langle \tau', \tau, \sigma'' \rangle, \quad r := \langle \sigma', \tau, \sigma'' \rangle,$$

and thus we have

$$q = t^2 + 2lt + k, \quad p = mt^2 + nt + r.$$ 

Furthermore, the Gaussian curvature $K$ of $S$ is given by

$$K = -\frac{A^2}{q}.$$

(3.4)

Since the Gaussian curvature $K$ never vanishes. Thus we have

$$A \neq 0.$$ 

(3.5)

A parametric representation of the Gauss map $n$ of $S$ is given by

$$n(s, t) = \frac{1}{\sqrt{q}}(\sigma' \times \tau + t\tau' \times \tau).$$

(3.6)

If, for simplicity, we put

$$P := P(s) = \sigma' \times \tau, \quad Q := Q(s) = \tau' \times \tau,$$

then the Gauss map $n$ of $S$ becomes

$$n(s, t) = \frac{1}{\sqrt{q}}(P + tQ).$$

(3.7)
The Beltrami operator with respect to the second fundamental form, after a lengthy computation, can be expressed as follows

\[
\Delta^H = \frac{2\sqrt{q}}{A} \frac{\partial^2}{\partial s \partial t} - \frac{p\sqrt{q}}{A^2} \frac{\partial^2}{\partial t^2} - \frac{\sqrt{q} p_t \partial}{A^2 \partial t}, \quad (3.8)
\]

where

\[ p_t := \frac{\partial p}{\partial t} \]

Applying (3.8) for the position vector \( \mathbf{n} \) one finds

\[
\Delta^H \mathbf{n} = \frac{1}{q^2} \left[ \left( \frac{tpqq_{st}}{2A^2} - \frac{3t^2pq^2}{4A^2} + \frac{tqq_{st}}{A^2} - \frac{q^2 p_t}{A^2} + \frac{qpq_{tt}}{A^2} \right) \mathbf{Q} + \left( \frac{2q^2}{A} - \frac{tpq_{tt}}{A} \right) Q' - \frac{qq_{ss} P'}{A} \right]. \quad (3.9)
\]

Here again we have

\[ p_s := \frac{\partial p}{\partial s}, \quad q_t := \frac{\partial q}{\partial t}, \quad q_s := \frac{\partial q}{\partial s} \]

and the prime stands, as we mention before, the derivative with respect to \( s \), that is

\[ P' = \frac{dP}{ds}, \quad Q' = \frac{dQ}{ds}. \]

Equation (3.9) can be expressed as follows

\[
\Delta^H \mathbf{n} = \frac{1}{q^2} \left[ \frac{1}{A^2} f_1(t) Q(s) + \frac{1}{A} f_2(t) Q'(s) \right.
\]

\[
+ \left. \frac{1}{A^2} f_3(t) P(s) + \frac{1}{A} f_4(t) P'(s) \right] \quad (3.10)
\]

where \( f_i(t), i = 1, 2, 3, 4 \) are polynomials in \( t \) with functions in \( s \) as coefficients, and \( \deg(f_i) \leq 4 \). More precisely, we have

\[
f_1(t) = 2mt^3 + (n + 4lm)t^4 + (3kl - 3l^2m + 2ln + 2l'A)t^3 + (3kn - 3l^2n - 4klm + 2k'A - 2l'A)t^2 + (3kr - 3l^2r - 2kl - 2k^2m + k'IA - 4kl'A)t - k^2n - kk'A, \quad (3.11)
\]

\[
f_2(t) = 2lt^3 + (4l^2n + 2k)t^2 + 6kl + 2k^2, \quad (3.12)
\]

\[
f_3(t) = (2ln - n)t^3 + (4l + l^2m - ln - 2r + 3km)t^2 + (2kn - l^2n + 2kl + 3k'A + 2l' - 4l')t + kr + kl - 2k' - 3l^2r + 3k'l, \quad (3.13)
\]

\[
f_4(t) = 2t^3 + 6lt^2 + (4l^2 + 2k)t + 2kl. \quad (3.14)
\]
Let now \( n = (n_1, n_2, n_3), \ P = (P_1, P_2, P_3) \) and \( Q = (Q_1, Q_2, Q_3) \) be the coordinate functions of \( n, P \) and \( Q \) respectively. By virtue of (3.10) we obtain

\[
\Delta^{II} n_i = \frac{1}{q^2} \left[ \frac{1}{A^2} f_1(t) Q_i(s) + \frac{1}{A} f_2(t) Q'_i(s) \right. \\
+ \frac{1}{A^2} f_3(t) P_i(s) + \frac{1}{A} f_4(t) P'_i(s) \left. \right],
\]

(3.15)

\[
i = 1, 2, 3.
\]

We denote by \( \lambda_{ij}, i, j = 1, 2, 3 \) the entries of the matrix \( A \). Using (3.15) and condition (1.2) we have

\[
\frac{1}{A^2} f_1(t) Q_i(s) + \frac{1}{A} f_2(t) Q'_i(s) + \frac{1}{A^2} f_3(t) P_i(s) + \frac{1}{A} f_4(t) P'_i(s) = \lambda_{i1} q^2 (P_i + tQ_i) + \lambda_{i2} q^2 (P_i + tQ_i) + \lambda_{i3} q^2 (P_i + tQ_i),
\]

(3.16)

\[
i = 1, 2, 3.
\]

Consequently

\[
-3m^2 \tau_i t^5 + [(m'A - mA' - 4mn - 7\lambda m^2) \tau_i + 3mA \sigma_i]'t^4 \\
+ [mA \sigma_i' - A^2 \sigma_i'' + (2nA + 7\lambda mA) \sigma_i' + (n'A - nA') \\
+ 2\lambda m'A - 2\lambda mA' - \lambda' mA - A^2 - 10\lambda mn - 2\lambda n - n^2 \\
- 4km^2) \tau_i]t^3 + [(km'A - kmA') - \frac{1}{2} \lambda' mA + 2\lambda' A \\
- 2\lambda A' - \lambda' nA - rA' + r'A - 3\lambda A^2 - 3\lambda n^2 - 6\lambda mn \\
- 6kmn) \tau_i + 3\lambda m A A \sigma_i' - 2\lambda A^2 \sigma_i'' - A^2 \sigma_i'' + (\lambda' A + 5\lambda n \\
+ 4km + r) A \sigma_i]'t^2 + [(\frac{1}{2} \lambda' A + 3kn + 3\lambda r) A \sigma_i'] \\
+ (\lambda'A - knA' - \frac{1}{2} \lambda' nA + 2\lambda r'A - 2\lambda rA' - \lambda' rA \\
- kA^2 - 2\lambda A^2 - 2kn^2 + r^2 - 2\lambda nr - 4kmn) \tau_i \\
- 2\lambda A^2 \sigma_i'' - ka^2 A \sigma_i' + (\lambda n - r + 2km + \lambda'A) A \sigma_i]'t \\
- A^4 (\lambda_{i1} \tau_1 + \lambda_{i2} \tau_2 + \lambda_{i3} \tau_3) t + (kr'A - krA' \\
- \frac{1}{2} \lambda' rA + \lambda r^2 - k\lambda A^2 - 2\lambda nr) \tau_i - kA^2 \sigma_i'' \\
+ 2kr A \sigma_i' + (\frac{1}{2} \lambda' A - \lambda r + kn) A \sigma_i' \\
- A^4 (\lambda_{i1} \sigma_1 + \lambda_{i2} \sigma_2 + \lambda_{i3} \sigma_3) = 0.
\]

(3.17)

For \( i = 1, 2, 3 \), (3.17) is a polynomial in \( t \) with functions in \( s \) as coefficients. This implies that the coefficients of the powers of \( t \) in (3.17) must be zeros, so we obtain, for \( i = 1, 2, 3 \), the following equations

\[
3m^2 \tau_i = 0,
\]

(3.18)

\[
(m'A - mA' - 4mn - 7\lambda m^2) \tau_i + 3mA \sigma_i' = 0,
\]
\[ m\sigma'_i - A^2\tau''_i + (2nA + 7\lambda mA)\tau'_i + (n'A - nA' + 2\lambda m' - 2\lambda m'A - \lambda'mA - A^2 - 10\lambda mn - 2mr - n^2 - 4km^2)\tau_i = 0, \tag{3.19} \]

\[ (km'A - kmA' - \frac{1}{2}k'mA + 2\lambda n'A - 2\lambda m'A - \lambda' nA - rA' + r'A - 3\lambda A^2 - 3\lambda n^2 - 6\lambda mnr - 6kmn)\tau_i + 3\lambda m\sigma'_i - 2\lambda A^2\tau''_i - A^2\sigma''_i + (\lambda'A + 5\lambda n + 4km + r)A\tau_i' = 0, \tag{3.20} \]

\[ (kn'A - knA' - \frac{1}{2}k'nA + 2\lambda r'A - 2\lambda rA - \lambda' rA - \lambda A^2 - 2kn^2 + r^2 - 2\lambda knr - 4kmn)\tau_i + 2\lambda A^2\sigma''_i - kA^2\tau''_i + (\frac{1}{2}k'A + 2kn + 4\lambda r)A\tau_i' + (\lambda n - r + 2km + \lambda'A)A\sigma'_i = A^4(\lambda_1\tau_1 + \lambda_2\tau_2 + \lambda_3\tau_3), \tag{3.21} \]

\[ (kr'A - krA' - \frac{1}{2}k'rA + \lambda r^2 - k\lambda A^2 - 2kn)\tau_i + 2krA\tau_i' + (\frac{1}{2}k'A - \lambda r + kn)A\sigma'_i - kA^2\sigma''_i = A^4(\lambda_1\sigma_1 + \lambda_2\sigma_2 + \lambda_3\sigma_3). \tag{3.22} \]

From (3.18) one finds

\[ m = \langle \tau', \tau, \tau'' \rangle = 0, \tag{3.23} \]

which implies that the vectors \( \tau', \tau, \tau'' \) are linearly dependent, and hence there exist two functions \( \sigma_1 = \sigma_1(s) \) and \( \sigma_2 = \sigma_2(s) \) such that

\[ \tau'' = \sigma_1 \tau + \sigma_2 \tau'. \tag{3.24} \]

On differentiating \( \langle \tau', \tau \rangle = 1 \), we obtain \( \langle \tau', \tau'' \rangle = 0 \). So from (3.21) we have

\[ \tau'' = \sigma_1 \tau. \tag{3.25} \]

By taking the derivative of \( \langle \tau, \tau \rangle = 1 \) twice, we find that

\[ \langle \tau', \tau' \rangle + \langle \tau, \tau'' \rangle = 0. \]

But \( \langle \tau', \tau' \rangle = 1 \), and taking into account (3.24) we find that \( \sigma_1(s) = -1 \). Thus (3.26) becomes \( \tau'' = -\tau \) which implies that

\[ \tau''_i = -\tau_i, \quad i = 1, 2, 3. \tag{3.26} \]

Using (3.20) and (3.26) equation (3.19) reduces to

\[ 2nA\tau'_i + (n'A - nA' - n^2)\tau_i = 0, \quad i = 1, 2, 3 \]

or, in vector notation

\[ 2nA\tau' + (n'A - nA' - n^2)\tau = 0. \tag{3.27} \]

By taking the derivative of \( \langle \tau, \tau \rangle = 1 \), we find that the vectors \( \tau, \tau' \) are linearly independent, and so from (3.27) we obtain that \( nA = 0 \). We note that \( A \neq 0 \), since
from (3.4) the Gauss curvature vanishes, so we are left with \( n = 0 \). Then equation (3.20) becomes

\[-A^2 \sigma'' + (\lambda' A + r) A \tau' + (r' A - r A' - \lambda A^2) \tau = 0, \quad i = 1, 2, 3\]

or, in vector notation

\[-A^2 \sigma'' + (\lambda' A + r) A \tau' + (r' A - r A' - \lambda A^2) \tau = 0. \quad (3.28)\]

Taking the inner product of both sides of the above equation with \( \tau' \) we find in view of (3.3) that

\[-A^2 \langle \sigma'', \tau' \rangle + r A + \lambda' A^2 = 0. \quad (3.29)\]

On differentiating \( \lambda = \langle \sigma', \tau' \rangle \) with respect to \( s \), by virtue of (3.26) and (3.3), we get

\[\lambda' = \langle \sigma'', \tau' \rangle + \langle \sigma', \tau'' \rangle = \langle \sigma'', \tau' \rangle - \langle \sigma', \tau' \rangle = \langle \sigma'', \tau' \rangle. \quad (3.30)\]

Hence, (3.29) reduces to

\[r A = 0, \quad (3.31)\]

which implies that \( r = 0 \).

Thus the vectors \( \sigma', \tau, \sigma'' \) are linearly dependent, and so there exist two functions \( \sigma_3 = \sigma_3(s) \) and \( \sigma_4 = \sigma_4(s) \) such that

\[\sigma'' = \sigma_3 \tau + \sigma_4 \sigma'. \quad (3.35)\]

We distinguish two cases

**Case 1:** \( \lambda = 0 \). Because of \( r = 0 \) equation (3.28) would yield \( A = 0 \), which is clearly impossible for the surfaces under consideration.

**Case 2:** \( \lambda \neq 0 \). From (3.28), (3.31) and \( r = 0 \) we find that

\[-\lambda' A^2 \sigma' + \lambda' A^2 \tau' = 0\]

which implies that \( \lambda' (\sigma' - \lambda \tau') = 0 \).

If \( \lambda' \neq 0 \), then \( \sigma' = \lambda \tau' \). Hence \( \sigma', \tau' \) are linearly dependent, and so \( A = 0 \) which contradicts our previous assumption. Thus \( \lambda' = 0 \). From (3.35) we have

\[\sigma'' = -\lambda \tau. \quad (3.35)\]
On the other hand, by taking the derivative of $k$ and using the last equation we obtain that $k$ is constant. Hence equations (3.21) and (3.22) reduce to

$$\lambda_{i1}\tau_1 + \lambda_{i2}\tau_2 + \lambda_{i3}\tau_3 = 0,$$
$$\lambda_{i1}\sigma_1 + \lambda_{i2}\sigma_2 + \lambda_{i3}\sigma_3 = 0, \quad i = 1, 2, 3$$

and so $\lambda_{ij} = 0, i, j = 1, 2, 3$.

Since the parameter $s$ is the arc length of the spherical curve $\tau(s)$, and because of (3.23) we suppose, without loss of generality, that the parametrization of $\tau(s)$ is $\tau(s) = (\cos s, \sin s, 0)$.

Integrating (3.35) twice we get

$$\sigma(s) = (c_1 s + c_2 + \lambda \cos s, c_3 s + c_4 + \lambda \sin s, c_5 s + c_6),$$

where $c_i, i = 1, 2, \ldots, 6$ are constants.

Since $k = (\sigma', \sigma')$ is constant, it’s easy to show that $c_1 = c_3 = 0$. Hence $\sigma(s)$ reduces to

$$\sigma(s) = (c_2 + \lambda \cos s, c_4 + l \sin s, c_5 s + c_6).$$

Thus we have

$$S : x(s) = (c_2 + (l + t) \cos s, c_4 + (l + t) \sin s, c_5 s + c_6)$$

which is a helicoid.

### 3.2. Quadrics of the first kind.

This kind of quadric surfaces can be parametrized as follows

$$x(u, v) = \left( u, v, \sqrt{c + au^2 + bv^2} \right).$$

Let’s denote the function $c + au^2 + bv^2$ by $\omega$. Then, using the natural frame $\{x_u, x_v\}$ of $S$ defined by

$$x_u = \left( 1, 0, \frac{au}{\sqrt{\omega}} \right),$$
$$x_v = \left( 0, 1, \frac{bv}{\sqrt{\omega}} \right),$$

the components $g_{ij}$ of the first fundamental form in (local) coordinates are the following

$$g_{11} = \langle x_u, x_u \rangle = 1 + \frac{(au)^2}{\omega},$$
$$g_{12} = \langle x_u, x_v \rangle = \frac{abuv}{\omega},$$
$$g_{22} = \langle x_v, x_v \rangle = 1 + \frac{(bv)^2}{\omega}.$$
where $\Phi = c + a(a + 1)u^2 + b(b + 1)v^2$. The components $b_{ij}$ of the second fundamental form in (local) coordinates are the following
\begin{align*}
b_{11} &= \langle x_{uu}, n \rangle = \frac{a(bv^2 + c)}{\omega \sqrt{\Phi}}, \\
b_{12} &= \langle x_{uv}, n \rangle = -\frac{abuv}{\omega \sqrt{\Phi}}, \\
b_{22} &= \langle x_{vv}, n \rangle = \frac{b(au^2 + c)}{\omega \sqrt{\Phi}},
\end{align*}

By using the natural frame $\{n_u, n_v\}$ of $S$ defined by
\begin{align*}
n_u &= \left( -\frac{a(b(b + 1)v^2 + c)}{\Phi \frac{\sqrt{\Phi}}{2}}, \frac{ab(a + 1)uv}{\Phi \frac{\sqrt{\Phi}}{2}}, \frac{au[b(b + 1)v^2 - ac]}{\sqrt{\omega \Phi \sqrt{\Phi}}}, \right), \\
n_v &= \left( \frac{ab(b + 1)uv}{\Phi \frac{\sqrt{\Phi}}{2}}, \frac{b[a(a + 1)u^2 + c]}{\Phi \frac{\sqrt{\Phi}}{2}}, \frac{bv[a(a + 1)u^2 - bc]}{\sqrt{\omega \Phi \sqrt{\Phi}}}, \right).
\end{align*}

The second Beltrami differential operator with respect to the second fundamental form is defined by
\[\Delta^{II} f = -\frac{1}{\sqrt{|b|}} \frac{\partial}{\partial w^j} \left( \sqrt{|b|} b^{ij} \frac{\partial f}{\partial w^i} \right),\]
where $b := \det(b_{ij})$. After a long computation the Beltrami operator $\Delta^{II}$ of $S$ can be expressed as follows:
\[\Delta^{II} = -\frac{\sqrt{\Phi}}{c} \left[ \frac{au^2 + c}{a} \frac{\partial^2}{\partial u^2} - 2uv \frac{\partial^2}{\partial u \partial v} + \frac{bv^2 + c}{b} \frac{\partial^2}{\partial v^2} + 2u \frac{\partial}{\partial u} + 2v \frac{\partial}{\partial v} \right], \tag{3.36}\]

Applying (3.36) for the functions $n_1 = -\frac{au}{\sqrt{\Phi}}$ and $n_2 = -\frac{bv}{\sqrt{\Phi}}$, one finds
\begin{align*}
\Delta^{II} n_1 &= \Delta^{II} \left( -\frac{au}{\sqrt{\Phi}} \right) = \frac{au}{\Phi \sqrt{\Phi}} \left[ 3b(b + 1)(b - a)v^2 - 3ac - (b + 2)\Phi \right], \tag{3.37} \\
\Delta^{II} n_2 &= \Delta^{II} \left( -\frac{bv}{\sqrt{\Phi}} \right) = \frac{bv}{\Phi \sqrt{\Phi}} \left[ 3a(a + 1)(a - b)u^2 - 3bc - (a + 2)\Phi \right]. \tag{3.38}
\end{align*}

We denote by $\lambda_{ij}, i, j = 1, 2, 3$ the entries of the matrix $\Lambda$. On account of (3.36) we get
\begin{align*}
\Delta^{II} n_1 &= \Delta^{II} \left( -\frac{au}{\sqrt{\Phi}} \right) = \lambda_{11} \left( -\frac{au}{\sqrt{\Phi}} \right) + \lambda_{12} \left( -\frac{bv}{\sqrt{\Phi}} \right) + \lambda_{13} \left( \sqrt{\omega} \right), \tag{3.39} \\
\Delta^{II} n_2 &= \Delta^{II} \left( -\frac{bv}{\sqrt{\Phi}} \right) = \lambda_{21} \left( -\frac{au}{\sqrt{\Phi}} \right) + \lambda_{22} \left( -\frac{bv}{\sqrt{\Phi}} \right) + \lambda_{23} \left( \sqrt{\omega} \right), \tag{3.40}
\end{align*}

We denote by $l_{ij}, i, j = 1, 2, 3$ the entries of the matrix $\Lambda$. On account of (1.2) we get
\[\Delta^{III} x_1 = \Delta^{III} u = l_{11}u + l_{12}v + l_{13}\sqrt{\omega}, \tag{3.41}\]
\[ \Delta^{III} x_2 = \Delta^{III} v = l_{21}u + l_{22}v + l_{23}\sqrt{\omega}, \quad (3.42) \]
\[ \Delta^{III} x_3 = \Delta^{III} \sqrt{\omega} = l_{31}u + l_{32}v + l_{33}\sqrt{\omega}. \]

Applying \((?)\) on the coordinate functions \(x_i, i = 1, 2\) of the position vector \(x\) and by virtue of \((3.41)\) and \((3.42)\), we find respectively

\[ \Delta^{III} u = -\frac{u^T}{c^2} \left[ 3(a + 1)u^2 + 3(b + 1)v^2 + \frac{c(3b + a + 2ab)}{ab} \right] \]
\[ = l_{11}u + l_{12}v + l_{13}\sqrt{\omega}, \quad (3.43) \]
\[ \Delta^{III} v = -\frac{v^T}{c^2} \left[ 3(a + 1)u^2 + 3(b + 1)v^2 + \frac{c(b + 3a + 2ab)}{ab} \right] \]
\[ = l_{21}u + l_{22}v + l_{23}\sqrt{\omega}, \quad (3.44) \]

Putting \(v = 0\) in \((3.43)\), we obtain that

\[ -\frac{3a(a + 1)^2}{c^2} u^5 - \frac{(a + 1)(6b + a + 2ab)}{bc} u^3 - \frac{(3b + a + 2ab)}{ab} u \]
\[ = l_{11}u + l_{13}\sqrt{c + au^2}. \]

Since \(a \neq 0\) and \(c \neq 0\) this implies that \(a = -1\).

Similarly, if we put \(u = 0\) in \((3.44)\) we obtain that

\[ -\frac{3b(b + 1)^2}{c^2} v^5 - \frac{(b + 1)(b + 6a + 2ab)}{ac} v^3 - \frac{(b + 3a + 2ab)}{ab} v \]
\[ = l_{22}v + l_{23}\sqrt{c + bv^2}. \]

This implies that \(b = -1\). Hence \(S\) must be a sphere.

3.3. Quadrics of the second kind. For this kind of surfaces we can consider a parametrization

\[ x(u, v) = \left( u, v, \frac{a}{2}u^2 + \frac{b}{2}v^2 \right). \]

Then the components \(g_{ij}, b_{ij}\) and \(e_{ij}\) of the first, second and third fundamental tensors are the following

\[ g_{11} = 1 + (au)^2, \quad g_{12} = abuv, \quad g_{22} = 1 + (bv)^2, \]
\[ b_{11} = \frac{a}{\sqrt{g}}, \quad b_{12} = 0, \quad b_{22} = \frac{b}{\sqrt{g}}, \]
\[ e_{11} = \frac{a^2}{g^2} (1 + b^2v^2), \quad e_{12} = -\frac{a^2b^2}{g^2} uv, \quad e_{22} = \frac{b^2}{g^2} (1 + a^2u^2), \]

where \(g := \det(g_{ij}) = 1 + (au)^2 + (bv)^2\).

A straightforward computation shows that the Beltrami operator \(\Delta^{III}\) of \(S\) takes the following form
\[ \Delta^{III} = -\frac{g(1 + a^2 u^2)}{a^2} \frac{\partial^2}{\partial u^2} - \frac{g(1 + b^2 v^2)}{b^2} \frac{\partial^2}{\partial v^2} - 2uvg \frac{\partial^2}{\partial u \partial v} - 2ug \frac{\partial}{\partial u} - 2vg \frac{\partial}{\partial v}. \] (3.45)

On account of (1.2) we get

\[ \Delta^{III} x_1 = \Delta^{III} u = l_{11} u + l_{12} v + l_{13} \left( \frac{a}{2} u^2 + \frac{b}{2} v^2 \right), \] (3.46)

\[ \Delta^{III} x_2 = \Delta^{III} v = l_{21} u + l_{22} v + l_{23} \left( \frac{a}{2} u^2 + \frac{b}{2} v^2 \right), \] (3.47)

\[ \Delta^{III} x_3 = \Delta^{III} \sqrt{\omega} = l_{31} u + l_{32} v + l_{33} \left( \frac{a}{2} u^2 + \frac{b}{2} v^2 \right). \]

Applying (3.45) on the coordinate functions \( x_i, i = 1, 2 \) of the position vector \( x \) and by virtue of (3.46) and (3.47) we find respectively

\[ \Delta^{III} u = -2ug = l_{11} u + l_{12} v + l_{13} \left( \frac{a}{2} u^2 + \frac{b}{2} v^2 \right), \] (3.48)

\[ \Delta^{III} v = -2vg = l_{21} u + l_{22} v + l_{23} \left( \frac{a}{2} u^2 + \frac{b}{2} v^2 \right). \] (3.49)

Putting \( v = 0 \) in (3.48), we obtain that

\[ -2a^2 u^3 - 2u = l_{11} u + l_{13} \frac{a}{2} u^2. \]

This implies that \( a \) must be zero. Putting \( u = 0 \) in (3.49), we obtain that

\[ -2b^2 v^3 - 2v = l_{22} v + l_{23} \frac{b}{2} v^2. \]

This implies that \( b \) must be zero, which is clearly impossible, since \( a > 0 \) and \( b > 0 \).

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