GAUSSIAN INTEGRAL MEANS OF ENTIRE FUNCTIONS

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ABSTRACT. For an entire function \( f : \mathbb{C} \to \mathbb{C} \) and a triple \((p, \alpha, r) \in (0, \infty) \times (-\infty, \infty) \times (0, \infty)\), the Gaussian integral means of \( f \) (with respect to the area measure \( dA \)) is defined by
\[
M_{p, \alpha}(f, r) = \left( \int_{|z| < r} e^{-\alpha|z|^2} dA(z) \right)^{-1} \int_{|z| < r} |f(z)|^p e^{-\alpha|z|^2} dA(z).
\]

Via deriving a maximum principle for \( M_{p, \alpha}(f, r) \), we establish not only Fock-Sobolev trace inequalities associated with \( M_{p, p/2}(z^m f(z), \infty) \) (as \( m = 0, 1, 2, \ldots \)), but also convexities of \( r \mapsto \ln M_{p, \alpha}(z^m, r) \) and \( r \mapsto M_{2, \alpha < 0}(f, r) \) in \( \ln r \) with \( 0 < r < \infty \).

1. INTRODUCTION

Let \( dA \) be the Euclidean area measure on the finite complex plane \( \mathbb{C} \). Suppose \( \alpha \) is real and \( 0 < p < \infty \). For any entire function \( f : \mathbb{C} \to \mathbb{C} \), we consider its Gaussian integral means
\[
M_{p, \alpha}(f, r) = \frac{\int_{|z| < r} |f(z)|^p e^{-\alpha|z|^2} dA(z)}{\int_{|z| < r} e^{-\alpha|z|^2} dA(z)} \quad \forall \ r \in (0, \infty).
\]

Upon writing
\[
\begin{align*}
M(r) &= \int_0^{2\pi} |f(re^{i\theta})|^p d\theta; \\
v(r) &= r e^{-\alpha r^2}; \\
i &= \sqrt{-1} \quad \text{the imaginary unit},
\end{align*}
\]
we get
\[
\frac{d}{dr} M_{p, \alpha}(f, r) = \frac{v(r) \int_0^r (M(r) - M(s)) v(s) ds}{2\pi \left( \int_0^r v(s) ds \right)^2} \geq 0,
\]
and hence the function \( r \mapsto M_{p, \alpha}(f, r) \) is strictly increasing on \((0, \infty)\) unless \( f \) is constant. Consequently, letting \( r \to 0 \) and \( r \to \infty \) in \( M_{p, \alpha}(f, r) \)

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respectively, we find the following maximum principle for $r \in (0, \infty)$:

$$|f(0)|^p = M_{p,\alpha}(f,0) \leq M_{p,\alpha}(f,r) \leq M_{p,\alpha}(f,\infty) = \int_{\mathbb{C}} |f(z)|^p e^{-\alpha |z|^2} dA(z)$$

with equality if and only if $f$ is a constant.

Besides the above maximum principle we are here motived mainly by [15, 6, 7, 13, 12, 14, 2] to take a further look at the Gaussian integral means $M_{p,\alpha}(f, r)$ from two perspectives. The first is to treat the last inequality as a space embedding: if $d\mu_r(z) = 1_{|z| < r} dA(z)$ (with $1_E$ being the characteristic function of $E \subset \mathbb{C}$) then

$$\int_{\mathbb{C}} |f(z)|^p e^{-\frac{|z|^2}{2}} d\mu_r(z) \leq \left( \int_{|z| < r} e^{-\frac{|z|^2}{4}} dA(z) \right) M_{p,p/2}(f, \infty).$$

Such an interpretation leads to characterizing a given nonnegative Borel measure $\mu$ on $\mathbb{C}$ such that the following Fock-Sobolev trace inequality

$$\|f\|_{L^q(\mathbb{C}, \mu)} \equiv \left( \int_{\mathbb{C}} |f(z)|^q d\mu(z) \right)^{\frac{1}{q}} \lesssim \left( M_{p,p/2}(z^m f(z), \infty) \right)^{\frac{1}{p}} \approx \left( \int_{\mathbb{C}} |z^m f(z)|^p e^{-\frac{|z|^2}{2}} dA(z) \right)^{\frac{1}{p}} \equiv \|f\|_{\mathcal{F}^{p,m}}$$

holds for all holomorphic functions $f : \mathbb{C} \mapsto \mathbb{C}$ in $\mathcal{F}^{p,m}$. In the above and below:

- $0 < p, q < \infty$;
- $X \lesssim Y$ (i.e. $Y \gtrsim X$) means that there is a constant $c > 0$ such that $X \leq c Y$ - moreover - $X \approx Y$ is equivalent to $X \lesssim Y \lesssim X$;
- $m$ is nonnegative integer;
- $\mathcal{F}^p = \mathcal{F}^{p,0}$ and $\mathcal{F}^{p,m}$ stand for the so-called Fock space and Fock-Sobolev space of order $m \geq 1$ respectively. Interestingly, for an entire function $f : \mathbb{C} \mapsto \mathbb{C}$ one has (cf. [2]):

$$f \in \mathcal{F}^{p,m} \iff |f(0)| + \cdots + |f^{(m-1)}(0)| + \|f^{(m)}\|_{\mathcal{F}^{p,0}} < \infty.$$

- $B(a,r) = \{z \in \mathbb{C} : |z - a| < r\}$ is the Euclidean disk centered at $a \in \mathbb{C}$ with radius $r > 0$. 

As stated in Theorem 3 of Section 2, the above-required measure is fully determined by
\[
\sup_{a \in \mathbb{C}} \mu(B(a,r)) \frac{(1+|a|)^q}{(1+|a|)^m} < \infty \quad \text{as} \quad 0 < p \leq q < \infty;
\int_{\mathbb{C}} \left( \frac{\mu(B(a,r))}{(1+|a|)^m} \right)^{\frac{p}{p-q}} dA(a) < \infty \quad \text{as} \quad 0 < q < p < \infty.
\]

As a particularly interesting and natural by-product of this characterization, we can also use the Taylor expansion of an entire function at the origin to get the optimal Gaussian Poincaré inequality (see [8, (1.6)] as well as [5, p. 115] and [16, Theorem 1] for the endpoint case corresponding to \( f \in \mathcal{F}_{1,1} \) with \( f(0) = 0 \)
\[
\int_{\mathbb{C}} |f(z)e^{-\frac{|z|^2}{2}}|^2 dA(z) - \pi |f(0)|^2 \leq \int_{\mathbb{C}} |f'(z)e^{-\frac{|z|^2}{2}}|^2 dA(z) \quad \forall \ f \in \mathcal{F}^{2,1}
\]
which, plus the foregoing maximum-principle-based estimate (cf. [2, (1)])
\[
|f'(z)|e^{-\frac{|z|^2}{2}} \leq (2\pi)^{-1} \int_{\mathbb{C}} |f'(z)e^{-\frac{|z|^2}{2}}| dA(z) \quad \forall \ f \in \mathcal{F}^{1,1},
\]
derives the following Gaussian isoperimetric-Sobolev inequality \( f \in \mathcal{F}^{1,1} \):
\[
\int_{\mathbb{C}} |f(z)e^{-\frac{|z|^2}{2}}|^2 dA(z) - \pi |f(0)|^2 \leq (2\pi)^{-1} \left( \int_{\mathbb{C}} |f'(z)e^{-\frac{|z|^2}{2}}| dA(z) \right)^2
\]
whose sharp form is
\[
\int_{\mathbb{C}} |f(z)e^{-\frac{|z|^2}{2}}|^2 dA(z) - \pi |f(0)|^2 \leq (4\pi)^{-1} \left( \int_{\mathbb{C}} |f'(z)e^{-\frac{|z|^2}{2}}| dA(z) \right)^2
\]
since this inequality can be proved valid for the entire functions \( f(z) = z^k \) with \( k = 1, 2, 3, \ldots \) through a direct computation with the polar coordinate system, the mathematical induction and the inequality for the gamma function \( \Gamma(\cdot) \) below:
\[
\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \leq \sqrt{\frac{k+1}{2}}.
\]

The second is to decide: when \( \ln r \mapsto \ln M_{p,\alpha}(z^k, r) \) is convex for \( r \in (0, \infty) \), namely, when the Gaussian Hadamard Three Circle Theorem below
\[
\left( \ln \frac{r_2}{r_1} \right) \ln M_{p,\alpha}(z^k, r) \leq \left( \ln \frac{r_2}{r} \right) \ln M_{p,\alpha}(z^k, r_1) + \left( \ln \frac{r}{r_1} \right) \ln M_{p,\alpha}(z^k, r_2)
\]
holds for \( 0 < r_1 \leq r \leq r_2 < \infty \). The expected result is presented in Theorem 7 of Section 3, saying that for a nonnegative integer \( k \) and a positive
number \( p \),
\[
\begin{align*}
\ln r \mapsto \ln M_{p,\alpha}(z^k, r) & \text{ is concave as } r \in (0, \infty) \text{ under } 0 < \alpha < \infty; \\
\exists c \in (0, \infty) & \exists \ln r \mapsto \ln M_{p,\alpha}(z^k, r) \text{ is convex and concave} \\
\text{as } r \in (0, c] \text{ and } r \in [c, \infty) \text{ respectively under } - \infty < \alpha \leq 0.
\end{align*}
\]

As a consequence, we have that if \( -\infty < \alpha, -p < 0 \) then the function \( \ln r \mapsto \ln M_{p,\alpha}(z^k, r) \) is convex as \( r \in (0, \sqrt{(2 + pk)/(2-\alpha)}) \) and hence the function \( \ln r \mapsto \ln M_{2,\alpha}(f, r) \) is convex as \( r \in (0, \sqrt{1/(1-\alpha)}) \) for any entire function \( f : \mathbb{C} \mapsto \mathbb{C} \). In other words,
\[
\left( \ln \frac{r_2}{r_1} \right) \ln M_{2,\alpha}(f, r) \leq \left( \ln \frac{r_2}{r_1} \right) \ln M_{2,\alpha}(f, r_1) + \left( \ln \frac{r}{r_1} \right) \ln M_{2,\alpha}(f, r_2)
\]
when \( 0 < r_1 \leq r \leq r_2 < \sqrt{1/(1-\alpha)} \). However, as proved in Remark 9 via considering the entire function \( 1 + z \), the last convexity cannot be extended to \((0, \infty)\).

2. Trace inequalities for Fock-Sobolev spaces

We need two lemmas. The first lemma comes from [2] and [18, 17, 4, 11].

**Lemma 1.** Let \( p, \sigma, a, t, \lambda \in (0, \infty) \).

(i) If \( m \) is a nonnegative integer, \( p_m(z) \) is the Taylor polynomial of \( e^z \) of order \( m - 1 \) (with the convention that \( p_0 = 0 \)), and \( b > - (mp + 2) \), then
\[
\int_\mathbb{C} |e^{z\overline{w}} - p_m(z\overline{w})|^p |e^{-a|w|^2}|w|^p dA(w) \lesssim |z|^p e^{\frac{p}{2}|z|^2} \quad \forall \quad |z| \geq \sigma.
\]
Furthermore, this last inequality holds also for all \( z \in \mathbb{C} \) when \( b \leq pm \).

(ii) If \( f : \mathbb{C} \mapsto \mathbb{C} \) is an entire function, then
\[
\left| f(z) e^{-\frac{\sigma}{2}|z|^2} \right|^p \lesssim \int_{B(z, \sigma)} \left| f(w) e^{-\frac{\sigma}{2}|w|^2} \right|^p dA(w) \quad \forall \quad z \in \mathbb{C}.
\]

(iii) There exists a positive constant \( r_0 \) such that for any \( 0 < r \leq r_0 \), the Fock space \( F^p \) exactly consists of all functions \( f = \sum_{w \in \mathbb{Z}^2} c_w k_w, \) where
\[
\begin{align*}
\{ k_w(z) = \exp(z\overline{w} - |w|^2/2); \\
c_w : w \in r\mathbb{Z}^2 \} & \in L^p; \\
\| \{ c_w \} \|_p & = \left( \sum_{w \in \mathbb{Z}^2} |c_w|^p \right)^{\frac{1}{p}}; \\
\mathbb{Z}^2 & = \{ n + im : n, m = 0, \pm1, \pm2, \ldots \}; \\
r\mathbb{Z}^2 & = \{ r(n + im) : n, m = 0, \pm1, \pm2, \ldots \}.
\end{align*}
\]

Moreover
\[
\| f \|_{F^p} \approx \inf \| \{ c_w \} \|_p \quad \forall \quad f \in F^p,
\]
where the infimum is taken over all sequences \( \{ c_w \} \) giving rise to the above decomposition.
The second lemma is the so-called Khinchine’s inequality, which can be found, for example, in [7].

**Lemma 2.** Suppose $p \in (0, \infty)$ and $c_j \in \mathbb{C}$. For the integer part $[t]$ of $t \in (0, \infty)$ let

$$r_0(t) = \begin{cases} 1, & 0 \leq t - [t] < 1/2 \\ -1, & 1/2 \leq t - [t] < 1 \end{cases}$$

and

$$r_j(t) = r_0(2^jt) \quad \forall \quad j = 1, 2, \cdots .$$

Then

$$\left( \sum_{j=1}^{m} |c_j|^2 \right)^{\frac{1}{2}} \approx \left( \int_0^1 \left| \sum_{j=1}^{m} c_j r_j(t) \right|^p dt \right)^{\frac{1}{p}}.$$

As the main result of this section, the forthcoming family of analytic-geometric trace inequalities for the Fock-Sobolev spaces is a natural generalization of the so-called diagonal Carleson measures for the Fock-Sobolev spaces in [2].

**Theorem 3.** Let $m$ be a nonnegative integer, $r \in (0, \infty)$, and $\mu$ be a non-negative Borel measure on $\mathbb{C}$.

(i) If $0 < p \leq q < \infty$, then

$$\|f\|_{L^q(\mathbb{C}, \mu)} \lesssim \|f\|_{\mathcal{F}^{p,m}} \quad \forall \quad f \in \mathcal{F}^{p,m}$$

when and only when

$$\sup_{a \in \mathbb{C}} \mu(B(a, r)) \frac{\mu(B(a, r))}{(1 + |a|)^{m}q} < \infty.$$  

Equivalently, $a \mapsto \mu(B(a, r))(1 + |a|)^{-m}$ is of class $L^\infty(\mathbb{C})$.

(ii) If $0 < q < p < \infty$, then

$$\|f\|_{L^q(\mathbb{C}, \mu)} \lesssim \|f\|_{p,m} \quad \forall \quad f \in \mathcal{F}^{p,m}$$

when and only when

$$\sum_{a \in \mathbb{Z}^2} \left( \frac{\mu(B(a, r))}{(1 + |a|)^{mq}} \right)^{\frac{p}{m}} < \infty \quad \text{where} \quad s \in (0, \infty).$$

Equivalently, $a \mapsto \mu(B(a, r))(1 + |a|)^{-mq}$ is of class $L^{p/(p-q)}(\mathbb{C})$.

**Proof.** (i) Suppose $0 < p \leq q < \infty$. The following argument is similar to that of Theorem 10 in [2].

Assume firstly that $\|f\|_{L^q(\mathbb{C}, \mu)} \lesssim \|f\|_{\mathcal{F}^{p,m}}$ holds for all $f \in \mathcal{F}^{p,m}$. Taking $f = 1$ shows that $\mu(K) \lesssim 1$ for any compact set $K \subset \mathbb{C}$.

Fix any $a \in \mathbb{C}$ and let

$$f(z) = (e^{az} - p_m(z\overline{a}))/z^m$$
in the last assumption. Then Lemma 1 (i) implies
\[ \int_C \left| e^{\frac{\pi z \bar{\alpha}}{2m}} e^{-\frac{1}{2}|z|^2} \right|^q d\mu(z) \lesssim (e^{\frac{q}{2}|\alpha|^2})^\frac{q}{p} = e^{\frac{q}{2}|\alpha|^2}. \]

In particular,
\[ \int_{B(a,r)} \left| e^{\frac{\pi z \bar{\alpha}}{2m}} e^{-\frac{1}{2}|z|^2} \right|^q d\mu(z) \lesssim e^{\frac{q}{2}|\alpha|^2}. \]

If \(|a| > 2r\), then \(|z|^m\) is comparable to \((1 + |a|)^m\) for \(B(a, r)\). So
\[ \int_{B(a,r)} |e^{\frac{\pi |z||z-a|}{2m}} e^{-\frac{1}{2}|z-a|^2}| d\mu(z) \lesssim (1 + |a|)^m e^{\frac{q}{2}|\alpha|^2} \]
holds for all \(|a| > 2r\). Note that
\[ \lim_{|a| \to \infty} \inf_{z \in B(a,r)} |1 - e^{-\frac{\pi |z|^2}{2m}}| = 1. \]

Thus
\[ \int_{B(a,r)} e^{\frac{\pi |z|^2}{2m}} e^{-\frac{1}{2}|z|^2} d\mu(z) \lesssim (1 + |a|)^m e^{\frac{q}{2}|\alpha|^2} \]
holds for the sufficiently large \(|a|\). But this last inequality is clearly true for smaller \(|a|\) as well. So we have
\[ \int_{B(a,r)} |e^{\frac{\pi |z|^2}{2m}} e^{-\frac{1}{2}|z|^2}| d\mu(z) \lesssim (1 + |a|)^m e^{\frac{q}{2}|\alpha|^2} \quad \forall \quad a \in \mathbb{C}. \]

Completing a square in the exponent, we can rewrite the inequality above as
\[ \int_{B(a,r)} e^{-\frac{1}{2}|z-a|^2} d\mu(z) \lesssim (1 + |a|)^mq \]
thereby deducing
\[ \mu(B(a, r)) \lesssim (1 + |a|)^mq e^{\frac{q}{2}r^2} \quad \forall \quad a \in \mathbb{C}. \]

Conversely, assume that
\[ \mu(B(a, r)) \lesssim (1 + |a|)^mq \quad \forall \quad a \in \mathbb{C}. \]

We proceed to estimate the integral
\[ \|f\|_{L^q(C, \mu)}^q = \int_C |f(z)e^{-\frac{1}{2}|z|^2}|^q d\mu(z) \]
of any given function \(f \in \mathcal{F}^{p,m}\). For any positive number \(s\) let \(Q_s\) denote the following square in \(\mathbb{C}\) with vertices \(0, s, si, \) and \(s + si:\)
\[ Q_s = \{ z = x + iy : 0 < x \leq s \quad \& \quad 0 < y \leq s \}. \]

It is clear that
\[ \mathbb{C} = \bigcup_{a \in \mathbb{Z}^2} (Q_s + a) \]
is a decomposition of \( \mathbb{C} \) into disjoint squares of side length \( s \). Thus
\[
\|f\|_{L^q(\mathbb{C}, \mu)}^q = \sum \int_{Q_s + a} |f(z)e^{-\frac{1}{2}|z|^2}|q d\mu(z).
\]

Fix positive numbers \( s \) and \( t \) such that \( t + \sqrt{s} = r \). By Lemma 1(ii)
\[
|f(z)e^{-\frac{1}{2}|z|^2}|p \lesssim \int_{B(z,t)} |f(w)e^{-\frac{1}{2}|w|^2}|p dA(w)
\]
\[
\lesssim \frac{1}{(1 + |z|)^{mp}} \int_{B(z,t)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|p dA(w)
\]
holds for all \( z \in \mathbb{C} \). Now if \( z \in Q_s + a \), where \( a \in \mathbb{Z}^2 \) implies \( B(z,t) \subset B(a, r) \) by the triangle inequality, and hence \( 1 + |z| \approx 1 + |a| \). Consequently,
\[
|f(z)e^{-\frac{1}{2}|z|^2}|p \lesssim \frac{1}{(1 + |a|)^{mp}} \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|p dA(w).
\]
This amounts to
\[
|f(z)e^{-\frac{1}{2}|z|^2}|q \lesssim \frac{1}{(1 + |a|)^{mq}} \left( \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|p dA(w) \right)^{\frac{q}{p}}.
\]
Therefore,
\[
\|f\|_{L^q(\mathbb{C}, \mu)}^q \lesssim \sum_{a \in \mathbb{Z}^2} \frac{\mu(B(a,r))}{(1 + |a|)^{mq}} \left( \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|p dA(w) \right)^{\frac{q}{p}}.
\]
Combining this last estimate with the previous assumption on \( \mu \) and \( p \leq q \), we obtain
\[
\|f\|_{L^q(\mathbb{C}, \mu)}^q \lesssim \sum_{a \in \mathbb{Z}^2} \left( \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|p dA(w) \right)^{\frac{q}{p}}
\]
\[
\lesssim \left( \sum_{a \in \mathbb{Z}^2} \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|p dA(w) \right)^{\frac{q}{p}}.
\]
Note that there exists a positive integer \( N \) such that each point in \( \mathbb{C} \) belongs to at most \( N \) of the disks \( B(a, r) \), where \( a \in \mathbb{Z}^2 \). So, one gets
\[
\|f\|_{L^q(\mathbb{C}, \mu)}^q \lesssim \left( \int_{\mathbb{C}} |w^m f(w) e^{-\frac{1}{2}|w|^2}|p dA(w) \right)^{\frac{q}{p}} \approx \|f\|_{L^{2p,q}}^{2p},
\]
as desired.

(ii) Suppose \( 0 < q < p < \infty \). The following proof is inspired by [13].
First assume that \( \|f\|_{L^q(C, \mu)} \lesssim \|f\|_{F^{p,m}} \) holds for all \( f \in F^{p,m} \). For any \( \{c_j\} \in l^p \), we may choose \( \{r_j(t)\} \) as in Lemma 2, thereby getting
\[
\{c_j r_j(t)\} \in l^p \quad \& \quad \|\{c_j r_j(t)\}\|_{l^p} = \|\{c_j\}\|_{l^p}.
\]
Then by Lemma 1 (iii) we know that
\[
\sum_{j=1}^{\infty} c_j r_j(t) k_{a_j}(z) \equiv z^m f(z)
\]
is in \( F^p \) with norm \( \|f\|_{F^{p,m}} \approx \inf \|\{c_j\}\|_{l^p} \). Here \( \{a_j\} \) is the sequence of all complex numbers of \( s \mathbb{Z}^2 \) and \( k_a(z) = e^{s \pi \frac{|a|^2}{2}} \). In particular,
\[
f(z) = \sum_{j=1}^{\infty} c_j r_j(t) k_{a_j}(z) z^{-m}.
\]
According to the assumption we have
\[
\int_C \left| \sum_{j=1}^{\infty} c_j r_j(t) \frac{k_{a_j}(z)}{z^m e^{s|z|^2/2}} \right|^q d\mu(z) = \|f\|_{L^q(C, \mu)}^q \lesssim \|f\|_{F^{p,m}}^q,
\]
whence getting by Lemma 2,
\[
\int_C \left( \sum_{j=1}^{\infty} |c_j k_{a_j}(z) e^{-\frac{s}{2} |z|^2} |z|^{-2m} \right)^{\frac{q}{2}} d\mu(z) \lesssim \|f\|_{F^{p,m}}^q.
\]
Also, note that if \( |a| > 2r \) then \( |z|^m \) is comparable to \( (1 + |a|)^m \) for \( z \in B(a, r) \). So
\[
\int_C \left( \sum_{j=1}^{\infty} |c_j k_{a_j}(z) e^{-\frac{s}{2} |z|^2} |z|^{-2m} \right)^{\frac{q}{2}} d\mu(z)
\]
\[
= \sum_{l=1}^{\infty} \int_{Q_{s+a_l}} \left( \sum_{j=1}^{\infty} |c_j e^{-\frac{s}{2} |z-a_j|^2} |z|^{-2m} \right)^{\frac{q}{2}} d\mu(z)
\]
\[
\geq \sum_{l=1}^{\infty} \int_{Q_{s+a_l}} |c_l|^q |z|^{-mq} e^{-\frac{s}{2} |z-a_l|^2} d\mu(z)
\]
\[
\geq \sum_{j=1}^{\infty} \int_{B(a_j, r)} |c_j|^q |z|^{-mq} e^{-\frac{s}{2} |z-a_j|^2} d\mu(z)
\]
\[
\geq \sum_{j=1}^{\infty} |c_j|^q \frac{\mu(B(a_j, r))}{(1 + |a_j|)^{mq}}.
\]
So, a combination of the previously-established inequalities gives

\[
\sum_{j=1}^{\infty} |c_j|^q \frac{\mu(B(a_j, r))}{(1 + |a_j|)^{mq}} \lesssim \|\{c_j\}\|_{l^p}^q = \|\{c_j^q\}\|_{l^{p/q}}.
\]

Since \(p/(p - q)\) is the conjugate number of \(p/q\), an application of the Riesz representation theorem yields

\[
\left\{ \frac{\mu(B(a_j, r))}{(1 + |a_j|)^{mq}} \right\} \in l^{\frac{p}{p - q}}.
\]

Conversely, assume that the last statement holds. Note that the first part of the argument for the above (i) tells that

\[
\|f\|_{L^q(C, \mu)}^q \lesssim \sum_{a \in s \mathbb{Z}^2} \frac{\mu(B(a, r))}{(1 + |a|)^{mq}} \left( \int_{B(a, r)} |w|^m f(w) e^{-\frac{1}{2}|w|^2} |w|^p dA(w) \right)^{\frac{p}{q}}
\]

holds for all \(f \in \mathcal{F}^{p,m}\). Applying H"older’s inequality to the last summation we obtain

\[
\|f\|_{L^q(C, \mu)}^q \lesssim \left( \sum_{a \in s \mathbb{Z}^2} \left( \frac{\mu(B(a, r))}{(1 + |a|)^{mq}} \right)^{\frac{p}{p - q}} \right)^{\frac{p - q}{p}} \times \left( \sum_{a \in s \mathbb{Z}^2} \int_{B(a, r)} |w|^m f(w) e^{-\frac{1}{2}|w|^2} |w|^p dA(w) \right)^{\frac{q}{p}}.
\]

Once again, notice that there exists a positive integer \(N\) such that each point in \(\mathbb{C}\) belongs to at most \(N\) of the disks \(B(a, r)\), where \(a \in s \mathbb{Z}^2\). So,

\[
\|f\|_{L^q(C, \mu)}^q \lesssim \left( \sum_{a \in s \mathbb{Z}^2} \left( \frac{\mu(B(a, r))}{(1 + |a|)^{mq}} \right)^{\frac{p}{p - q}} \right)^{\frac{p - q}{p}} \|f\|_{\mathcal{F}^{p,m}}^q.
\]

This completes the argument. \(\square\)

The following extends [1, Theorem 5] (cf. [10, Theorem 1]), and [3, Theorem 1], respectively.

**Corollary 4.** Let \(\phi : \mathbb{C} \mapsto \mathbb{C}\) be an entire function. For \(p \in (0, \infty)\) and a nonnegative integer \(m\) define two linear operators acting on an entire function \(f : \mathbb{C} \mapsto \mathbb{C}\):

\[
\begin{align*}
C_\phi f(z) &= f \circ \phi(z) \quad \forall \quad z \in \mathbb{C}; \\
T_\phi f(z) &= \int_0^z f(w) \phi'(w) \, dw \quad \forall \quad z \in \mathbb{C}.
\end{align*}
\]
(i) The composition operator $C_{\phi} : \mathcal{F}_p^{m} \mapsto \mathcal{F}_q$ exists as a bounded operator if and only if

$$\begin{align*}
\sup_{a \in \mathbb{C}} \int_{\mathbb{R}^2} e^{-q|z|^2/2} dA(z) &< \infty \text{ when } 0 < p \leq q < \infty; \\
\int_{\mathbb{C}} \left( \frac{\int_{B(a,r)} e^{-q|z|^2/2} dA(z)}{(1+|z|)^m_q} \right)^{p/(p-q)} dA(a) &< \infty \text{ when } 0 < q < p < \infty.
\end{align*}$$

(ii) The Riemann-Stieltjes integral operator $T_{\phi} : \mathcal{F}_p^{m} \mapsto \mathcal{F}_q$ exists as a bounded operator if and only if

$$\begin{align*}
\sup_{a \in \mathbb{C}} \int_{B(a,r)} \left( \frac{\phi'(z)}{(1+|z|)^m_q} \right)^q dA(z) &< \infty \text{ when } 0 < p \leq q < \infty; \\
\int_{\mathbb{C}} \left( \frac{\phi'(z)}{(1+|z|)^m_q} \right)^q dA(a) &< \infty \text{ when } 0 < q < p < \infty.
\end{align*}$$

Proof. (i) For any Borel set $E \subset \mathbb{C}$ let $\phi^{-1}(E)$ be the pre-image of $E$ under $\phi$ and

$$\mu(E) = \int_{\phi^{-1}(E)} \exp \left( -\frac{q|z|^2}{2} \right) dA(z).$$

Then

$$\|C_{\phi} f\|_{L^q(\mathbb{C}, \mu)}^q = \int_{\mathbb{C}} |f(z) e^{-\frac{|z|^2}{2}}|^q d\mu(z) \quad \forall \ f \in \mathcal{F}_p^{m}.$$ 

An application of Theorem [8] with the above formula gives the desired result.

(ii) According to [3, Proposition 1], an entire function $f : \mathbb{C} \mapsto \mathbb{C}$ belongs to $\mathcal{F}_q$ if and only if

$$\int_{\mathbb{C}} \left( \frac{|f'(z)| e^{-\frac{|z|^2}{2}}}{1+|z|} \right)^q dA(z) < \infty.$$ 

So, $T_{\phi} f \in \mathcal{F}_q$ is equivalent to

$$\int_{\mathbb{C}} \left( \frac{|f(z) \phi'(z)| e^{-\frac{|z|^2}{2}}}{1+|z|} \right)^q dA(z) < \infty.$$ 

Now, choosing

$$d\mu(z) = \left( \frac{\phi'(z)}{1+|z|} \right)^q dA(z)$$

in Theorem [8] we get the boundedness result for $T_{\phi}$. \qed
3. CONVEXITIES OR CONCAVITIES IN LOGARITHM

We also need two lemmas. The first one comes directly from [12, Lemmas 2, 1, 6] with \((0, 1)\) being replaced by \((0, \infty)\).

**Lemma 5.**

(i) Suppose \(f\) is positive and twice differentiable on \((0, \infty)\). Let

\[
D(f(x)) \equiv \frac{f'(x)}{f(x)} + x \frac{f''(x)}{f(x)} - x \left( \frac{f'(x)}{f(x)} \right)^2.
\]

Then the function \(\ln f(x)\) is concave in \(\ln x\) if and only if \(D(f(x)) \leq 0\) on \((0, \infty)\) and \(\ln f(x)\) is convex in \(\ln x\) if and only if \(D(f(x)) \geq 0\) on \((0, \infty)\).

(ii) Suppose \(f\) is twice differentiable on \((0, \infty)\). Then \(f(x)\) is convex in \(\ln x\) if and only if \(f(x^2)\) is convex in \(\ln x\) and \(\ln f(x)\) is concave in \(\ln x\) if and only if \(f(x^2)\) is concave in \(\ln x\).

(iii) Suppose \(\{h_k(x)\}\) is a sequence of positive and twice differentiable functions on \((0, \infty)\) such that the function

\[
H(x) = \sum_{k=0}^{\infty} h_k(x)
\]

is also twice differentiable on \((0, \infty)\). If for each natural number \(k\) the function \(\ln h_k(x)\) is convex in \(\ln x\), then \(\ln H(x)\) is also convex in \(\ln x\).

The second lemma as below is elementary.

**Lemma 6.** Suppose \(f\) is continuous differentiable on \([0, \infty)\). If \(f'(\infty) \equiv \lim_{x \to \infty} f'(x) = -\infty\), then \(f(\infty) \equiv \lim_{x \to \infty} f(x) = -\infty\).

The main result of this section is the following log-convexity theorem.

**Theorem 7.** Suppose \(k\) is a nonnegative integer and \(0 < p < \infty\).

(i) If \(0 < \alpha < \infty\), then the function \(r \mapsto \ln M_{p,\alpha}(z^k, r)\) is concave in \(\ln r\).

(ii) If \(-\infty < \alpha \leq 0\), then there exists some \(c\) (depending on \(k\) and \(\alpha\)) on \((0, \infty)\) such that the function \(r \mapsto \ln M_{p,\alpha}(z^k, r)\) is convex in \(\ln r\) on \((0, c]\) and concave in \(\ln r\) on \([c, \infty)\).

**Proof.** The case \(\alpha = 0\) is a straightforward by-product of the classical Hardy convexity theorem (cf. [9]). So, for the rest of the proof we may assume \(\alpha \neq 0\).

By the polar coordinates and an obvious change of variables, we have

\[
M_{p,\alpha}(z^k, r) = \frac{\int_0^{r^2} t^{pk/2}e^{-\alpha t} \, dt}{\int_0^{r^2} e^{-\alpha t} \, dt}.
\]
For any nonnegative parameter $\lambda$ we define
\[
f_\lambda(x) = \int_0^x t^\lambda e^{-\alpha t} \, dt \quad \forall x \in (0, \infty).
\]

To prove Theorem 7, by Lemma 5 (i)-(ii), we need only to consider the function
\[
\Delta(\lambda, x) = \frac{f'_\lambda}{f_\lambda} + x \frac{f''_\lambda}{f_\lambda} - x \left( \frac{f'_0}{f_0} \right)^2 - \left( \frac{f'_0}{f_0} + x \frac{f''_0}{f_0} - x \left( \frac{f'_0}{f_0} \right)^2 \right).
\]

Here and henceforth, the derivatives $f'_\lambda(x)$ and $f''_\lambda(x)$ are taken with respect to $x$ not $\lambda$.

To simplify notation, we write $h = f_\lambda(x)$ and denote by $h', h'', h'''$ to the various derivatives of $f_\lambda(x)$ with respect to $x$. Meanwhile, $\partial / \partial \lambda$ stands for the derivative with respect to $\lambda$.

Thanks to $h = \int_0^x t^\lambda e^{-\alpha t} \, dt$, we get
\[
\begin{align*}
    h' &= x^\lambda e^{-\alpha x}; \\
    h'' &= (\lambda - \alpha x)x^{\lambda-1}e^{-\alpha x}; \\
    h''' &= x^{\lambda-2}e^{-\alpha x}(\lambda^2 - \lambda - 2\alpha x + \alpha^2 x^2).
\end{align*}
\]

At the same time, we have
\[
\begin{align*}
    \frac{\partial h}{\partial \lambda} &= \int_0^x t^\lambda e^{-\alpha t} \ln t \, dt; \\
    \frac{\partial h'}{\partial \lambda} &= \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial \lambda} \right) = h' \ln x; \\
    \frac{\partial h''}{\partial \lambda} &= \frac{h'}{x} + h'' \ln x.
\end{align*}
\]

Note that the function inside the brackets in $\Delta(\lambda, x)$ is independent of $\lambda$. So,
\[
\begin{align*}
    \frac{\partial \Delta}{\partial \lambda} &= \frac{1}{h^2} \left( h^2 \frac{\partial h'}{\partial \lambda} + xh \frac{\partial h''}{\partial \lambda} - 2xh' \frac{\partial h'}{\partial \lambda} \right) \\
    &\quad - \frac{1}{h^3} \frac{\partial h}{\partial \lambda} (hh' + xhh'' - 2x(h')^2) \\
    &= \frac{1}{h^2} \left( hh' \ln x + hh' + xhh'' \ln x - 2x(h')^2 \ln x \right) \\
    &\quad - \frac{1}{h^3} \frac{\partial h}{\partial \lambda} (hh' + xhh'' - 2x(h')^2) \\
    &= \frac{h'}{h} + \frac{1}{h^3} \left( h \ln x - \frac{\partial h}{\partial \lambda} \right) (hh' + xhh'' - 2x(h')^2).
\end{align*}
\]
From now on, we use the notation $X \sim Y$ to represent that $X$ and $Y$ have the same sign. Let us consider the following two functions (with $\lambda$ fixed):

$$
\begin{align*}
d_1(x) &= h \ln x - \frac{\partial \lambda}{\partial x}; \\
d_2(x) &= \frac{h h' + x h h'' - 2x(h')^2}{h'} = (\lambda + 1 - \alpha x)h - 2x^{\lambda + 1}e^{-\alpha x}.
\end{align*}
$$

Since $d_1'(x) = h/x > 0$, one has $d_1(x) \geq d_1(0) = 0$. Now we want to prove that $d_2(x) < 0$ for all $x > 0$. By direct computations, we obtain

$$
\begin{align*}
d_2'(x) &= -\alpha h - (\lambda + 1 - \alpha x)x^\lambda e^{-\alpha x}; \\
d_2''(x) &= (\lambda + 1 - \alpha x)(-\lambda + \alpha x)x^{\lambda - 1}e^{-\alpha x}.
\end{align*}
$$

(i) If $\alpha > 0$, then under $0 < x \leq \frac{\lambda + 1}{\alpha}$ we have $d_2'(x) \leq -\alpha h < 0$. When $x > \frac{\lambda + 1}{\alpha}$, it is easy to obtain $d_2''(x) < 0$, and then

$$
d_2'(x) \leq d_2'' \left( \frac{\lambda + 1}{\alpha} \right) = -\alpha h \left( \frac{\lambda + 1}{\alpha} \right) < 0.
$$

Hence $d_2(x) < d_2(0) = 0$ for all $x > 0$.

(ii) If $\alpha < 0$, then it is easy to see $d_2''(x) < 0$, and hence $d_2'(x) \leq d_2'(0) = 0$. This in turn implies $d_2(x) < d_2(0) = 0$ for all $x > 0$.

With the help of the above analysis, we deduce

$$
\frac{\partial \Delta}{\partial \lambda} \sim -\frac{h^2 h'}{h h' + x h h'' - 2x(h')^2} - h \ln x + \frac{\partial h}{\partial \lambda} =: \delta(x).
$$

Further computations derive

$$
\delta'(x) = -\frac{2h(h')^2 + h^2 h''}{h h' + x h h'' - 2x(h')^2} + \frac{h^2 h'(2h h'' + x h h''' - 3x h' h'' - (h')^2)}{(h h' + x h h'' - 2x(h')^2)^2} - \frac{h}{x}
= \left( \frac{h^2}{x(h h' + x h h'' - 2x(h')^2)} \right) \times \left( -((h')^2 + x h' h'' + 2x^2(h'')^2 - x^2 h'' h'' h') h + x(h')^2(h' + x h'') \right)
= \left( \frac{h h'}{h h' + x h h'' - 2x(h')^2} \right)^2 \times \left( \frac{((\lambda + 1)^2 - (2\lambda + 1)\alpha x + \alpha^2 x^2) \delta_1(x)}{x} \right).
$$

Here

$$
\delta_1(x) = -h + \frac{x^{\lambda + 1}e^{-\alpha x} (\lambda + 1 - \alpha x)}{(\lambda + 1)^2 - (2\lambda + 1)\alpha x + \alpha^2 x^2}.
$$
And, a computation implies
\[
\delta_1'(x) = \frac{-\alpha x^{\lambda+1} e^{-\alpha x (\lambda + 1 + \alpha x)}}{((\lambda + 1)^2 - (2\lambda + 1)\alpha x + \alpha^2 x^2)^2}
\]
and then
\[
\delta_1'(0) = \delta_1(0) = 0.
\]
With details deferred to after the proof, we also have \(\delta_1'(0) = 0\) and when \(\alpha < 0\) we have \(\delta_1'(\infty) = -\infty\). Without loss of generality, we may just handle the case \(\lambda > 0\) in what follows.

(i) If \(\alpha > 0\), then \(\delta_1'(x) < 0\) for all \(x \in (0, \infty)\), and hence \(\delta_1(x) < \delta_1(0) = 0\) on \((0, \infty)\). This implies
\[
\frac{\partial \Delta(\lambda, x)}{\partial \lambda} < 0 \quad \forall \ x \in (0, \infty).
\]
Therefore, \(\Delta(\lambda, x) \leq \Delta(0, x) = 0\), and the desired result follows.

(ii) If \(\alpha < 0\), then \(\delta_1'(x)\) has only one zero \(-\frac{\lambda + 1}{\alpha}\) on \((0, \infty)\) and \(\delta_1(x)\) is increasing on \((0, -\frac{\lambda + 1}{\alpha})\) and decreasing on \((-\frac{\lambda + 1}{\alpha}, \infty)\). Noticing \(\delta_1'(\infty) = -\infty\), we use Lemma 6 to get \(\delta_1(\infty) = -\infty\). Hence \(\delta_1(x)\) has only one zero \(x^*\) on \((0, \infty)\) (Note that \(x^* > \frac{\lambda + 1}{\alpha}\)) and \(\delta_1(x)\) is positive on \((0, x^*)\) and negative on \((x^*, \infty)\). For \(\delta(x)\) and \(\frac{\partial \Delta}{\partial \alpha}\) we have similar results. Hence \(\frac{\partial \Delta}{\partial \alpha}\) has exactly one zero \(x_0\) (depending on \(k\) and \(\alpha\)) on \((0, \infty)\) (Note that \(x_0 > x^* > \frac{\lambda + 1}{\alpha}\)) and \(\frac{\partial \Delta}{\partial \alpha}\) is positive on \((0, x_0)\) and negative on \((x_0, \infty)\). This implies \(\Delta(\lambda, x) \leq \Delta(0, x) = 0\) on \((0, x_0)\) and \(\Delta(\lambda, x) \geq \Delta(0, x) = 0\) on \((x_0, \infty)\). Now, setting \(c = \sqrt{x_0}\) yields the desired result.

Finally, let us verify the above-claimed formulas:
\[
\left\{
\begin{array}{l}
\delta'(0) = 0; \\
\delta'(\infty) = -\infty.
\end{array}
\right.
\]
As a matter of fact, L’Hospital’s rule gives
\[
\lim_{x \to 0} \frac{h}{x} = 0, \quad \lim_{x \to 0} \frac{xh'}{h} = \lim_{x \to 0} \frac{h' + xh''}{h'} = \lambda + 1.
\]
Consequently,
\[
\lim_{x \to 0} \frac{hh'}{hh' + xhh'' - 2x(h')^2} = \lim_{x \to 0} \frac{1}{\frac{h'^2 - 2xh'}{h}} = -\frac{1}{(\lambda + 1)^{-1}}.
\]
It follows from the definition of \(\delta_1(x)\) that \(\lim_{x \to 0} \frac{\delta_1(x)}{x} = 0\). So, by the definition of \(\delta_1(x)\) we have \(\delta'(0) = 0\).

In a similar manner, another application of L’Hospital’s rule derives
\[
\lim_{x \to \infty} \frac{h'}{h} = \lim_{x \to \infty} \frac{h''}{h'} = -\alpha
\]
and consequently,
\[
\lim_{x \to \infty} \frac{x h h'}{h h' + x h h'' - 2 x (h')^2} = \lim_{x \to \infty} \frac{1}{x + \frac{h''}{h'} - \frac{2 h'}{x^2}} = \frac{1}{\alpha}.
\]
The definition of \( \delta_1(x) \) and L’Hopital’s rule imply
\[
\lim_{x \to \infty} \frac{\delta_1(x)}{x} = \lim_{x \to \infty} \delta_1'(x) = -\infty,
\]
and then \( \delta'(\infty) = -\infty \).

**Corollary 8.** Suppose \( \alpha < 0 \) and \( 0 < p < \infty \). Then the function \( r \mapsto \ln M_{p,\alpha}(z^k, r) \) is convex in \( \ln r \) on \((0, c]\), where \( c = \sqrt{(pk + 2)(-2\alpha)^{-1}} \).

Moreover, for \( p = 2 \), the function \( r \mapsto \ln M_{2,\alpha}(f, r) \) is convex in \( \ln r \) on \((0, \sqrt{(-\alpha)^{-1}}]\) for any entire function \( f : \mathbb{C} \mapsto \mathbb{C} \).

**Proof.** The first part of Corollary 8 follows from the proof of Theorem 7 with \( \lambda = pk/2 \). As for the second part, it is easy to see \( c \geq \sqrt{1 - \alpha} \). Suppose
\[
f(z) = \sum_{k=0}^{\infty} a_k z^k.
\]

It follows from an integration in polar coordinates that
\[
M_{2,\alpha}(f, r) = \sum_{k=0}^{\infty} |a_k|^2 M_{2,\alpha}(z^k, r).
\]

Now, applying Lemma 5(iii) we obtain the desired result.

**Remark 9.** Theorem 7 tells us that the integral means of all monomials are logarithmically concave when \( \alpha > 0 \). However, this is not true for all entire functions, even for linear mappings.

**Proof.** For instance, just choose \( p = 2, \alpha = 1 \) and \( f(z) = a + z \). Using polar coordinates and changing variables we have
\[
M_{p,\alpha}(f, r) = \frac{\int_0^r (c + t)e^{-t} dt}{\int_0^r e^{-t} dt} \quad \text{where} \quad c = |a|^2.
\]

By Lemma 5(i), we just need to consider the function
\[
F(x) = \frac{\int_0^x (c + t)e^{-t} dt}{\int_0^x e^{-t} dt} = \frac{c + 1 - (c + 1 + x)e^{-x}}{1 - e^{-x}} \equiv \frac{g(x)}{h(x)}.
\]

Employing the \( D \)-notation in Lemma 5(i), we have
\[
\begin{align*}
D(g(x)) &= \frac{1}{g'} ((c + 2x - cx - x^2)e^{-x}g(x) - x(c + x)^2e^{-2x}) \\
D(h(x)) &= \frac{e^{-x}}{h^2}(1 - x - e^{-x}),
\end{align*}
\]
whence getting
\[ \begin{align*}
D(F(x)) &= D(g(x)) - D(h(x)) \\
&= e^{-x}((c + 1)(1 + 3x - x^2) + (3 + 3c - 6x - 6cx - x^2)e^{-x} \\
&\quad + (-3 - 3c + 3x + 3cx + 2x^2 + cx^2 + x^3)e^{-2x} + (c + 1)e^{-3x}) \\
&\sim (c + 1)(-1 + 3x - x^2)e^{3x} + (3 + 3c - 6x - 6cx - x^2)e^{2x} \\
&\quad + (-3 - 3c + 3x + 3cx + 2x^2 + cx^2 + x^3)e^x + (c + 1) \\
&\equiv G(x).
\end{align*} \]

A direct computation gives
\[
\begin{cases}
G'(x) \\
~ (c + 1)(7 - 3x) - \frac{14+12c+2x}{e^x} + \frac{7+5c+5x+cx+x^2}{e^{2x}} \equiv H(x); \\
H'(x) = -3(c + 1) + \frac{12+12c+2x}{e^x} - \frac{9+9c+8x+2cx+2x^2}{e^{2x}}; \\
H''(x) \sim -10 - 12c - 2x + \frac{10+16c+12x+4cx+4x^2}{e^x} \equiv J(x); \\
J'(x) = -2(1 - e^{-x}) - (12c + 4x + 4cx + 4x^2)e^{-x} \leq 0.
\end{cases}
\]

Noticing that \( J(0) = 4c > 0 \) and \( J(\infty) = -\infty \), we know that there exists a number \( x_3 \in (0, \infty) \) such that \( J(x) \) is positive, and hence \( H''(x) \) is positive on \((0, x_3)\) and negative on \((x_3, \infty)\). Note that
\[
\begin{align*}
H'(0) &= H(0) = G(0) = 0; \\
H'(\infty) &= -3(c + 1) < 0; \\
H(\infty) &= G(\infty) = -\infty.
\end{align*}
\]

So, the functions \( H'' \), \( H' \), \( H \), \( G \) have similar properties. In particular, there exists a \( x_0 \in (0, \infty) \) such that \( G(x) \), and hence, \( D(F(x)) \) is positive on \((0, x_0)\) and negative on \((x_0, \infty)\). This implies that \( \ln M_{p, \alpha}(f, r) \) is convex in \( \ln r \) on \((0, \sqrt{x_0})\) and concave in \( \ln r \) on \((\sqrt{x_0}, \infty)\). Especially, when \( c = 0 \), the function \( \ln M_{p, \alpha}(f, r) \) is concave in \( \ln r \) on \((0, \infty)\).

Certainly, it is interesting to determine the maximal interval \((\lambda, \infty)\) on which \( G \) is negative, that is, the area integral means \( M_{p, \alpha}(a + z, r) \) is logarithmically concave on \((\sqrt{\lambda}, \infty)\) for any \( a \in \mathbb{C} \).

It follows from the definition of \( G \) that
\[
G(x) \sim G_0(x) + \frac{x^2e^x}{c + 1}(1 + x - e^x),
\]
where
\[
G_0(x) = (-1 + 3x - x^2)e^{3x} + (3 - 6x)e^{2x} + (-3 + 3x + x^2)e^x + 1.
\]

Note that
\[
G(x) < 0 \quad \forall \quad c = |a|^2 \iff G_0(x) < 0.
\]
Also, it is not hard to prove that $G_0(x)$ has exactly one real zero $\lambda$ in $(0, \infty)$, and $G_0(x)$ is positive on $(0, \lambda)$ and negative on $(\lambda, \infty)$. A numerical computation shows that $\lambda = 1.86047095 \cdots$. This implies that $M_{2,1}(a + z, r)$ is logarithmically concave on $(\sqrt{\lambda}, \infty)$ for any $a \in \mathbb{C}$, and the interval $(\sqrt{\lambda}, \infty)$ is maximal.

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