DIMENSION ESTIMATE FOR THE TWO-SIDED POINTS OF PLANAR SOBOLEV EXTENSION DOMAINS

JYRKI TAKANEN

Abstract. In this paper we give an estimate for the Hausdorff dimension of the set of two-sided points of the boundary of bounded simply connected Sobolev $W^{1,p}$-extension domain for $1 < p < 2$. Sharpness of the estimate is shown by examples. We also prove the equivalence of different definitions of two-sided points.

1. Introduction

This paper is part of the study of the geometry of the boundary of Sobolev extension domains in Euclidean spaces. Recall that a domain $\Omega$ is a $W^{1,p}$-extension domain if there exists a bounded operator $E: W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$ with the property that $E|_{\Omega} = u$ for each $u \in W^{1,p}(\Omega)$. Here, for $1 \leq p \leq \infty$, we denote by $W^{1,p}(\Omega)$ the set of all functions in $L^p(\Omega)$ whose first distributional derivatives are in $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is normed by

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

Additionally, in the case $p > 1$ the operator may be chosen to be linear (see [7]). When $p = 1$ linearity of the operator is known only for bounded simply connected domains ([11]).

Several classes of domains are known to be $W^{1,p}$-extension domains. For instance, Lipschitz domains (Calderón [1] $1 < p < \infty$, and Stein [18] $p = 1, \infty$). Jones [8] introduced a wider class of $(\epsilon, \delta)$-domains and proved that every $(\epsilon, \delta)$-domain is a $W^{1,p}$-extension domain. Notice that the Hausdorff dimension of the boundary of a Lipschitz domain is $n - 1$ and the boundary is rectifiable. For an $(\epsilon, \delta)$-domain the Hausdorff dimension of the boundary may be strictly greater than $n - 1$ and it may not be locally rectifiable (for example the Koch snowflake). However, an easy argument shows that the boundary of an $(\epsilon, \delta)$-domain can not self-intersect.

In the case where $\Omega \subset \mathbb{R}^2$ is bounded and simply connected, the $W^{1,p}$-extendability has been characterized. From the results in [4], [5], [6], [8], we know that a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ is a $W^{1,2}$-extension domain if and only if $\Omega$ is a quasidisk, or equivalently an uniform domain.

In [17] Shvartsman proved the following characterization for $W^{1,p}$-extension domains. For $2 < p < \infty$ and $\Omega$ a finitely connected bounded planar domain, then $\Omega$ is a Sobolev $W^{1,p}$-extension domain if and only if for some $C > 1$ the following condition is satisfied: for every $x, y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ joining $x$ to $y$ such that

$$\int_\gamma \text{dist}(z, \partial \Omega)^{1-p} \, ds(z) \leq C\|x - y\|^{p-2}.$$

Date: June 24, 2021.

2000 Mathematics Subject Classification. Primary 46E35, 28A75.

The author acknowledges the support from the Academy of Finland, grant no. 314789.
In particular, when $2 \leq p < \infty$, a finitely connected bounded $W^{1,p}$-extension domain $\Omega$ is quasiconvex, meaning that there exists a constant $C \geq 1$ such that any pair of points in $z_1, z_2 \in \Omega$ can be connected with a rectifiable curve $\gamma \subset \Omega$ whose length satisfies $\ell(\gamma) \leq C|z_1 - z_2|$.

In paper [10] the case $1 < p < 2$ was characterized: a bounded simply connected $\Omega \subset \mathbb{R}^2$ is a Sobolev $W^{1,p}$-extension domain if and only if there exists a constant $C > 1$ such that for every $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting $z_1$ and $z_2$ and satisfying

$$
\int_\gamma \mathrm{dist}(z, \partial \Omega)^{1-p} \, ds(z) \leq C\|z_1 - z_2\|^{2-p}.
$$

(1.1)

The above geometric characterizations give bounds for the size of the boundary of Sobolev extension domains. The following estimate for the Hausdorff dimension of the boundary for simply connected $W^{1,p}$-extension domain $\Omega$ in the case $p \in (1, 2)$ was given in [12]:

$$\dim_{\mathcal{H}}(\partial \Omega) \leq 2 - \frac{M}{C},$$

where $C$ is the constant in (1.1) and $M > 0$ is an universal constant. Recall that for $s > 0$, the $s$-dimensional Hausdorff measure of a subset $A \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

where $\mathcal{H}^s_\delta(A) = \inf \{ \sum_i \mathrm{diam}(E_i)^s : A \subset \bigcup_i E_i, \mathrm{diam}(E_i) \leq \delta \}$. The Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is then given by

$$\dim_{\mathcal{H}}(A) = \inf \{ t : \mathcal{H}^t(A) < \infty \}.$$

In this paper, we are interested in the case $1 < p < 2$, when the boundary of $\Omega$ may self-intersect, (for examples see [11] Example 2.5], [2], and Section 4). More accurately, we study the size of the set of two-sided points. In the case of $2 \leq p \leq \infty$, there are no such points which can be seen from the quasiconvexity. The case $p = 1$ has been characterized in [11] as a variant of quasiconvexity of the complement. In this case the dimension of the set of two-sided points does not depend on the constant in quasiconvexity. Let us define what we mean by a two-sided point. Here we give a definition which generalizes to $\mathbb{R}^n$, but the proof of our main theorem will use an equivalent formulation based on conformal maps, see Section 2.

**Definition 1.1** (Two-sided points of the boundary of a domain). Let $\Omega \subset \mathbb{R}^n$ be a domain. A point $x \in \partial \Omega$ is called two-sided, if there exists $R > 0$ such that for all $r \in (0, R)$ there exist connected components $\Omega^1_r$ and $\Omega^2_r$ of $\Omega \cap B(x, r)$ that are nested: $\Omega^1_r \subset \Omega^2_r$, for $0 < r < s < R$ and $i \in \{1, 2\}$.

We denote by $\mathcal{T}$ the two-sided points of $\Omega$. Note, that the nestedness condition in Definition 1.1 for the connected components $\Omega^i_r$ implies that $x \in \partial \Omega^i_r$. We establish the following dimension estimate for $\mathcal{T}$ for simply connected planar $W^{1,p}$-extension domains.

**Theorem 1.2.** Let $1 < p < 2$ and $\Omega \subset \mathbb{R}^2$ a simply connected, bounded Sobolev $W^{1,p}$-extension domain. Let $\mathcal{T}$ be the set of two-sided points of $\Omega$. Then

$$\dim_{\mathcal{H}}(\mathcal{T}) \leq 2 - p + \log_2 \left( 1 - \frac{2^{p-1}}{2^{p-1} - 1} \right),$$

(1.2)

where $C$ is the constant in (1.1).
Recalling that necessarily $C \geq 1$, the estimate (1.2) implies the existence of a constant $M_1(p) > 0$ depending only on $p$, such that

$$\dim_{\mathcal{H}}(\mathcal{J}) \leq 2 - p - \frac{M_1(p)}{C}.$$  

In Section 1 we show the existence of another constant $M_2 > 0$ and examples $\Omega_{p,C}$ of Sobolev $W^{1,p}$-extension domains for every $1 < p < 2$ and every $C > C(p)$ for which

$$\dim_{\mathcal{H}}(\mathcal{J}\Omega_{p,C}) \geq 2 - p - \frac{M_2}{C}.$$  

This shows the sharpness of Theorem 1.2.

2. Equivalent definitions for two-sided points

In this section we give equivalent conditions for the set of two-sided points in the case that the domain is John. We note that a bounded simply connected planar domain satisfying the condition (1.1) is John ([5, Chapter 6 Thm 3.5]). Recall, that $\Omega$ is a $J$-John domain, if there exists a constant $J > 0$ and a point $x_0 \in \Omega$ so that for every $x \in \Omega$ there exists a unit speed curve $\gamma: [0, \ell(\gamma)] \to \Omega$ such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$, and

$$\text{dist} (\gamma(t), \partial\Omega) \geq Jt \quad \text{for all } t \in [0, \ell(\gamma)].$$  

(2.1)

We denote the open unit disk of the plane by $\mathbb{D}$. For a bounded simply connected John domain $\Omega \subset \mathbb{R}^2$, a conformal map $f: \mathbb{D} \to \Omega$ can always be extended continuously to a map $f: \overline{\mathbb{D}} \to \overline{\Omega}$. This is because a John domain is finitely connected along its boundary [14] and a conformal map from the unit disk to $\Omega$ can be extended continuously onto the closure $\overline{\Omega}$ if and only if the domain is finitely connected along its boundary [15].

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected John domain (especially, if $\Omega$ is a bounded simply connected $W^{1,p}$-extension domain for $1 < p < 2$). Let $f: \mathbb{D} \to \Omega$ be a conformal map extended continuously to a function $\overline{\mathbb{D}} \to \overline{\Omega}$ still denoted by $f$. Define

$$E = \{x \in \partial\Omega : f^{-1}(\{x\}) \text{ disconnects } \partial\mathbb{D}\}$$

and

$$\tilde{E} = \{x \in \partial\Omega : \text{card } (f^{-1}(\{x\})) > 1\}.$$  

Then

$$\mathcal{J} = E = \tilde{E},$$

where $\mathcal{J}$ is the set of two-sided points according to Definition [14].

In the proof of Theorem 2.1 we need the following lemma.

**Lemma 2.2.** Let $\Omega \subset \mathbb{R}^2$ be a simply connected John domain, let $x \in \partial\Omega$, and $r \in (0, \text{diam}(\Omega))$. If there exist two disjoint open sets $U_1, U_2 \subset \Omega \cap B(x, r)$ such that $x \in \partial U_1 \cap \partial U_2$ and both of the sets $U_1$ and $U_2$ are unions of connected components of $\Omega \cap B(x, r)$. Then there exist connected components $U'_1$ and $U'_2$ of $U_1$ and $U_2$ respectively, such that $x \in \partial U'_1 \cap \partial U'_2$.

**Proof.** Let us first show that there exists $N \in \mathbb{N}$ such that

$$\text{card } \{\hat{\Omega} : \hat{\Omega} \text{ connected component of } \Omega \cap B(x, r) \text{ such that } \hat{\Omega} \cap B(x, r/2) \neq \emptyset \} \leq N.$$  

(2.2)

Take $M \in \mathbb{N}$ components $\hat{\Omega}_i$ as in (2.2), and choose from each one a point $x_i \in \hat{\Omega}_i \cap B(x, r/2)$. Let $\gamma_i$ be a John curve connecting $x_i$ to a fixed John center $x_0$ of $\Omega$. For each $i$ for which $x_0 \notin \hat{\Omega}_i$, the curve $\gamma_i$ must exit $B(x, 2r/3)$. For these $i$ we consider points $y_i \in \gamma_i \cap S(x, 2r/3)$,
which then exist for all but maybe one of the indexes $i$. By the John condition there exists balls $B_i = B(y_i, Jr/6) \subset \Omega_i$. As the balls $B_i$ are disjointed and $B_i$ covers an arc of $S(x, 2r/3)$ of length at least $Jr/3$, we have $(M - 1)Jr/3 \leq \frac{4}{3} \pi r$, hence $M - 1 \leq \left( \frac{6}{4 \pi} \right)^{-1}$.

Next we show that (2.2) implies the claim of the lemma. Define

$$\{A_j\}_{j=1}^{k} := \{\tilde{\Omega} \subset U_1 : \tilde{\Omega} \text{ connected component of } \Omega \cap B(x, r) \text{ such that } \tilde{\Omega} \cap B(x, r/2) \neq \emptyset\}.$$

By (2.2) we have $k \leq N$. Since $U_1$ consists of connected components of $\Omega \cap B(x, r)$, we have

$$U_1 \cap B(x, r/2) \subset \bigcup_{j=1}^{k} A_j.$$

Now, because $x \in \bigcup_{j=1}^{k} A_j = \bigcup_{j=1}^{k} \overline{A}_j$ there exists $j$ such that $x \in \overline{A}_j$. We call this $A_j$ the set $U'_1$. Similarly we find $U'_2$ for $U_2$. □

Notice that Lemma (2.2) does not hold for general simply connected domain $\Omega$, for example consider the topologist’s comb.

**Proof of Theorem 2.1.** We divide the proof into several claims. Showing that

$$\tilde{E} \subset E \subset \mathcal{J} \subset \tilde{E}.$$

**Claim 1:** $\tilde{E} \subset E$.

Let $z \in \partial \Omega$ such that $\text{card} \left( f^{-1}(z) \right) > 1$, and $A = \partial \mathbb{D} \setminus f^{-1}(z)$. Let $x_1, x_2 \in f^{-1}(z)$. By [16] Theorem 10.9, the set $f^{-1}(z)$ has Hausdorff dimension zero. Therefore, we find points of $A$ from both components of $\partial \mathbb{D} \setminus \{x_1, x_2\}$. Hence $A$ is disconnected in $\partial \mathbb{D}$, and thus $z \in E$.

**Claim 2:** $\mathcal{J} \subset \tilde{E}$.

Let $z \in \mathcal{J}$. By assumption there exists $R > 0$ such that for each $0 < r < R$ there exists disjoint connected components $\Omega^i_r, \Omega^2_r \subset \Omega \cap B(z, r)$, with the property that $\Omega^i_r \subset \Omega^2_r$ when $0 < r < s$. Towards a contradiction, assume that $f^{-1}(z)$ is a singleton ($w = f^{-1}(z)$).

By continuity of $f$ (up to the boundary) there exists $\varepsilon > 0$ such that $f(B(w, \varepsilon) \cap \mathbb{D}) \subset B(z, r)$. As a continuous image of a connected set $f(B(w, \varepsilon) \cap \mathbb{D})$ is connected. We show that $f^{-1}(\Omega^i_r) \cap B(w, \varepsilon) \neq \emptyset$ for $j = 1, 2$, which gives a contradiction with $\Omega^i_r$ being the disjoint connected components of $B(z, r) \cap \Omega$. Let $(z^j_i)_{i=1}^{\infty} \subset \Omega^j_r$ be a sequence such that $z^j_i \rightarrow z$. By going to a subsequence, we may assume that $(f^{-1}(z^j_i))_{i=1}^{\infty}$ converges to a point $w^j \in f^{-1}(\Omega^j_r)$. Since $f$ is continuous, $f(w^j) = z$. But then $w^j = w$ by the uniqueness of the preimage of $z$. Hence, $f^{-1}(z^j_i) \rightarrow w$ meaning that for some $i$ we have $f^{-1}(z^j_i) \in B(w, \varepsilon)$ showing $f^{-1}(\Omega^j_r) \cap B(w, \varepsilon) \neq \emptyset$. Therefore, $\Omega^j_r \cap f(B(w, \varepsilon) \cap \mathbb{D}) \neq \emptyset$, connecting sets $\Omega^j_r$. This completes the proof.

**Claim 3:** $E \subset \mathcal{J}$.

Let $z \in E$. We will show that $z \in \mathcal{J}$. We do this by first showing by induction that there exists $i_0 \in \mathbb{N}$ so that for all $i \geq i_0$ there exist connected components $\Omega^{j\rightarrow i}_r$ of $\Omega \cap B(z, 2^{-i})$, $j \in \{1, 2\}$, that are nested for fixed $j \in \{1, 2\}$. At each step of the induction we will have to make sure that $z \in \partial \Omega^{1\rightarrow i}_r \cap \partial \Omega^{2\rightarrow i}_r$.

**Initial step:** Let us show that there exists $r > 0$ such that $B(z, r) \cap \Omega$ may be written as an union of two disjointed open sets such that $z$ is contained in the boundary of both sets. First, since $f^{-1}(z) = \cap_{\gamma > 0} f^{-1}(B(z, r) \cap \partial \Omega)$, there exists $R > 0$ such that $H = f^{-1}(B(z, R) \cap \partial \Omega)$ is disconnected in $\partial \mathbb{D}$. By the continuity of $f$, $K = f^{-1}(\overline{B(z, R/2)})$ is a closed set in the
closed disk $\overline{D}$. Let $y_1, y_2 \in \partial D \setminus H$ such that $y_1$ and $y_2$ are in different connected components of $\partial D \setminus H$. Define $e = \min(\text{dist}(y_1, K), \text{dist}(y_2, K))/2$. Now $K \setminus B(0, 1 - e)$ is disconnected in $\overline{D}$. Next we notice that $\text{dist}(f(B(0, 1 - e)), \partial \Omega) = R' > 0$. Thus the original claim holds with the radius $r = \min(R, R')/2$. Let us now define $i_0 \in \mathbb{N}$ to be the smallest integer for which $2^{-i_0} \leq r$. Call $U_1$ and $U_2$ the two disjoint open sets for which $z \in \partial U_1 \cap \partial U_2$ and $\Omega \cap B(z, 2^{-i_0}) = U_1 \cup U_2$. By Lemma 2.2 we have connected components $\Omega_{2-i_0}^1 \subset U_1$ and $\Omega_{2-i_0}^2 \subset U_2$ of $\Omega \cap B(z, 2^{-i_0})$ such that $z \in \partial \Omega_{2-i_0}^1 \cap \partial \Omega_{2-i_0}^2$.

**Induction step:** Assume that for some $i \in \mathbb{N}$ there exist disjoint connected components $\Omega_{2-i}^1$ and $\Omega_{2-i}^2$ of $\Omega \cap B(z, 2^{-i})$ such that $z \in \partial \Omega_{2-i}^1 \cap \partial \Omega_{2-i}^2$. Let $U_1 = \Omega_{2-i}^1 \cap B(z, 2^{-i-1})$. Let $V$ be a connected component of $U_2$. It suffices to show that $V$ is a connected component of $\Omega \cap B(z, 2^{-i-1})$. Take a connected component $V' \supset V$ of $\Omega \cap B(z, 2^{-i-1})$. There exists connected component $W'$ of $\Omega \cap B(z, 2^{-i})$ such that $W' \supset V'$. Since $\emptyset \neq V \subset W' \cap \Omega_{2-i}^1$, we have $W' = \Omega_{2-i}^1$. Furthermore $V' \subset \Omega_{2-i}^1 \cap B(z, 2^{-i-1}) = U_1$. As $V'$ is connected we have $V' = V$.

Similarly for $U_2$, Now, by Lemma 2.2 we may choose connected components $U_1' \subset U_1$ and $U_2' \subset U_2$ (of $\Omega \cap B(z, 2^{-i-1})$) such that $z \in \partial U_1' \cap \partial U_2'$.

**General $r \in (0, 2^{-i_0})$:** Let $2^{-i-1} < r < 2^{-i}$. Let $\Omega_{2-i}^1$ be the connected component of $\Omega \cap B(z, r)$ containing $\Omega_{2-i}^1$. Since $\Omega_{2-i}^1$ is connected component of $\Omega \cap B(z, 2^{-i})$ containing $\Omega_{2-i}^1$, we have $\Omega_{2-i}^1 \subset \Omega_{2-i}^1$. Let us show that $\Omega_{2-i}^1 \subset \Omega_{2-i}^1$ for all $0 < r < s$. Let $0 < r < s$. We consider two cases: (1) If $2^{-i-1} < r < s < 2^{-i}$ the sets $\Omega_{2-i}^1$ and $\Omega_{2-i}^1$ are connected components of $\Omega \cap B(z, r)$ and $\Omega \cap B(z, s)$, respectively, both containing $\Omega_{2-i}^1$. Since $\Omega_{2-i}^1 \subset \Omega \cap B(z, s)$ and $\Omega_{2-i}^1$ is connected we have $\Omega_{2-i}^1 \subset \Omega_{2-i}^1$.

(2) If $2^{-i-1} \leq r \leq 2^{-i} \leq 2^{-j-1} \leq s \leq 2^{-j}$ sets $\Omega_{2-j}^1$ and $\Omega_{2-j}^1$ are connected components of $\Omega \cap B(z, r)$ and $\Omega \cap B(z, s)$ which contain $\Omega_{2-j-1}^1$ and $\Omega_{2-j-1}^1$, respectively. Similarly as in (1) we have $\Omega_{2-j}^1 \subset \Omega_{2-j}^1 \subset \cdots \subset \Omega_{2-j}^1 \subset \Omega_{2-j}^1$.

3. **Upper bound for the dimension of the set of two-sided points**

In this section we prove Theorem 1.2 that establishes an upper bound for the Hausdorff dimension of the set of two-sided points $\mathcal{F}$ for a planar simply connected $W^{1,p}$-extension domain when $1 < p < 2$. We do this by using one of the equivalent definitions of two-sided points given in Theorem 2.1. Namely, we consider

$$E = \{x \in \partial \Omega : f^{-1}\{x\} \text{ disconnects } \partial D\},$$

where $f: \overline{D} \to \Omega$ is a conformal map extended continuously to a function $\overline{D} \to \overline{\Omega}$ still denoted by $f$ (see the beginning of Section 2).

The idea of the proof is to reduce the dimension estimate of $E$ to a dimension estimate of $E$ along a single curve $\gamma$ satisfying (1.1). This is possible by Lemma 3.1. Then, on each $\gamma$ the dimension estimate is obtained via Lemma 3.2 by estimating the number of balls needed to cover the set $E \cap \gamma$ at different scales.

Following the ideas of [10, Lemma 4.6] we first show that the set of two-sided points can be covered by a countable union of curves fulfilling condition (1.1).

**Lemma 3.1.** Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be bounded simply connected $W^{1,p}$ Sobolev extension domain. Then there exists a countable collection $\Gamma$ of curves satisfying (1.1) so
that for the set $E$ of two-sided points we have

$$E \subset \bigcup_{\gamma \in \Gamma} \gamma^o \cap \partial \Omega,$$

where $\gamma^o$ denotes the curve $\gamma$ without its endpoints.

Proof. Let $f : \mathbb{D} \to \overline{\Omega}$ be continuous and conformal in $\mathbb{D}$. Let $\{x_j\} \subset \partial \mathbb{D}$ be dense and for each pair $(x_j, x_i)$, $i \neq j$ select a curve $\gamma_{i,j}$ satisfying [1] between the points $f(x_i)$ and $f(x_j)$. Define $\Gamma = \{\gamma_{i,j} : i \neq j\}$.

Now, let $z \in E$. By the definition of $E$ there exist $x_a, x_b \in f^{-1}(z)$, $x_a \neq x_b$ which divide $\partial \mathbb{D}$ into two components $I_a$ and $I_b$, so that $f(I_a) \neq \{z\} \neq f(I_b)$. By the continuity of $f$, there exist $i, j$, $i \neq j$ such that $x_i \in I_a$ and $x_j \in I_b$ and $f(x_i) \neq z \neq f(x_j) \neq f(x_i)$. Let $\gamma_{i,j} \in \Gamma$ be the curve connecting $f(x_i) =: z_i$ and $f(x_j) =: z_j$. Let $\tilde{\gamma} := f([x_i,0] \cup [0, x_j])$. The curve $[x_i,0] \cup [0, x_j]$ divides $\mathbb{D}$ into two components $A$ and $B$. By interchanging $A$ and $B$ if necessary, we have $x_a \in \overline{A}$ and $x_b \in \overline{B}$, and by continuity $z \in \overline{f(A)} \cap \overline{f(B)}$.

Since, the curve $\gamma_{i,j}$ may be assumed to be injective ([3, Lemma 3.1]), and since $z_i \neq z_j$, the curve $\tilde{\gamma} \cup \gamma_{i,j}$ is Jordan. Let $\tilde{A}$ and $\tilde{B}$ be the corresponding Jordan components. Since $f(A) \subset \overline{\tilde{A}}$, $f(B) \subset \overline{\tilde{B}}$ we have $z \in \overline{\tilde{A}} \cap \overline{\tilde{B}} = \gamma_{i,j} \cup \tilde{\gamma}$. Furthermore, since $\tilde{\gamma} \subset f(\mathbb{D}) \cup \{z_i, z_j\} = \Omega \cup \{z_i, z_j\}$, we have $z \in \gamma_{i,j}$. □

To prove Theorem 1.2 we need the following sufficient condition for an upper bound of the Hausdorff dimension.

**Lemma 3.2.** Let $\gamma : J \to \mathbb{R}^2$ be a rectifiable curve from a compact interval $J \subset \mathbb{R}$ and $E \subset \gamma(J)$. Let $0 < \lambda < 1$ and $i_0 \in \mathbb{N}$. Define for each $i \geq i_0$ a maximal $\lambda^i$-separated net

$$\{x_k^i\}_{k \in I_i} \subset E \cap \gamma(J).$$

Assume that the following holds: For each $i \geq i_0$ and $k \in I_i$ there exists $j > i$, such that

$$N_j < \lambda^{-(j-i)s},$$

where $N_j = \text{card} \{\{l \in I_j : B(x_l^j, \lambda^j) \cap B(x_k^i, \lambda^i) \neq \emptyset\}\}$. Then $\dim_{H}(E) \leq s$.

Proof. Define $\mathcal{B}_{i_0} = \{B(x_k^{i_0}, \lambda^{i_0}) : k \in I_{i_0}\}$ and inductively for $n > i_0$ by

$$\mathcal{B}_n = \bigcup_{B(x_k^i, \lambda^i) \in \mathcal{B}_{n-1}} \{B(x_m^j, \lambda^j) : B(x_m^j, \lambda^j) \cap B(x_k^i, \lambda^i) = \emptyset\},$$

where $j = j(i, k) > i$ is given by the assumption. Clearly $\mathcal{B}_n$ is a cover of $E$ for each $n \geq i_0$, and for all $B \in \mathcal{B}_n$

$$\text{diam} \ B \leq 2\lambda^n.$$

By assumption, for each $B = B(x_k^i, \lambda^i) \in \mathcal{B}_{n-1}$ and with $j = j(i, k)$ again given by the assumption

$$\sum_{B \in \mathcal{B}_n} \text{diam} \ (B(x_m^j, \lambda^j))^s = N_j(2\lambda^j)^s < (2\lambda^i)^s = \text{diam} \ (B)^s,$$

and therefore

$$\sum_{B \in \mathcal{B}_n} \text{diam} \ (B)^s \leq \sum_{B \in \mathcal{B}_{n-1}} \text{diam} \ (B)^s.$$
Let \( \delta > 0 \) and choose \( n \in \mathbb{N} \) such that \( 2\lambda^n < \delta \). Now
\[
\mathcal{H}_0^s(E) \leq \sum_{B \in \mathcal{B}_n} \text{diam}(B)^s \leq \sum_{B \in \mathcal{B}_{n-1}} \text{diam}(B)^s \leq \ldots
\]
\[
\leq \sum_{B \in \mathcal{B}_{i_0}} \text{diam}(B)^s \leq \text{card}(I_{i_0})(2\lambda^{i_0})^s < \infty.
\]
By letting \( \delta \to 0 \), we get \( \mathcal{H}^s(E) \leq \text{card}(I_{i_0})(2\lambda^{i_0})^s < \infty \), and consequently \( \dim_{\mathcal{H}}(E) \leq s \). \( \square \)

**Proof of Theorem 1.2.** By Theorem 2.1 we have \( E = \mathcal{J} \). Let \( \Gamma \) be the set of curves given in Lemma 3.1. Let \( \gamma \in \Gamma \) and define the set
\[
\{x^i_k\}_{k \in I_i} \subset E \cap \gamma
\]
to be a maximal \( 2^{-i} \) separated net for all \( i \in \mathbb{N} \). Take \( s < \dim_{\mathcal{H}}(E \cap \gamma) \). Then, by Lemma 3.2 there exists \( i \in \mathbb{N} \) and \( k \in I_i \) such that \( N_j \geq 2^{(j-i)s} \) for all \( j > i \), where
\[
N_j = \text{card}(\{l \in I_j : B(x^i_k, 2^{-i}) \cap B(x^i_k, 2^{-i}) \neq \emptyset\}).
\]
Note that, trivially also \( N_i \geq 1 \). Denote \( B = B(x^i_k, 2^{-i+1}) \). For all \( j > i + 1 \) the ball \( B \) contains at least \( N_{j-1} \) pairwise disjoint balls \( B(x^{i-1}_{j-1}, 2^{-j}) \) centered at \( E \cap \gamma \), and so we have
\[
\mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < 2^{-j}\}) \geq N_{j-1}2^{-j}. \tag{3.1}
\]
Using (1.1), Cavalieri’s principle, (3.1), and Lemma 3.2 we estimate
\[
C2^{-(i-2)(2-p)} \geq \int_{\gamma \cap B} \text{dist}(z, \partial\Omega)^{1-p} \, dz
\]
\[
= \int_0^\infty \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega)^{1-p} > t\}) \, dt
\]
\[
= \int_0^\infty \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < t^{1/p}\}) \, dt
\]
\[
= \sum_{j \in \mathbb{Z}} \int_{2^{-(j-1)(1-p)}}^{2^{-j(1-p)}} \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < t^{1/p}\}) \, dt
\]
\[
\geq \sum_{j=i+1}^{\infty} \int_{2^{-(j-1)(1-p)}}^{2^{-j(1-p)}} \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < 2^{-j}\}) \, dt
\]
\[
\geq \sum_{j=i+1}^{\infty} 2^{-j(1-p)}(1-2^{1-p})N_{j-1}2^{-j}
\]
\[
\geq \sum_{j=i+1}^{\infty} (2^{p-1}-1)2^{-j(1-p)}2^{j-1-i}s2^{-j},
\]
which implies
\[
C \geq (2^{p-1}-1)2^{2p-5} \sum_{j=i+1}^{\infty} 2^{j-1-i}(s+p-2)
\]
\[
= (2^{p-1}-1)2^{2p-5} \frac{1}{1 - 2^{-(2-(p+s))}}. \tag{3.2}
\]
A reordering of (3.2) gives
\[ s \leq 2 - p + \log_2 \left( 1 - \frac{2^{2p-5}(2p-1)}{C} \right). \]
Since \( s < \dim_{\mathcal{H}}(E \cap \gamma) \) was arbitrary, we have
\[ \dim_{\mathcal{H}}(E \cap \gamma) \leq 2 - p + \log_2 \left( 1 - \frac{2^{2p-5}(2p-1)}{C} \right). \]
Recalling that \( \Gamma \) is countable, and that by Lemma 3.1
\[ E \subset \bigcup_{\gamma \in \Gamma} E \cap \gamma, \]
the claim follows. \( \square \)

4. Sharpness of the dimension estimate

In this section we show the sharpness of the estimate given in Theorem 1.2. We do this by constructing a domain whose set of two-sided points contains a Cantor type set.

Let \( 0 < \lambda < 1/2 \). Let \( C_\lambda \) be the standard Cantor set obtained as the attractor of the iterated function system \( \{ f_1 = \lambda x, f_2 = \lambda x + 1 - \lambda \} \). For later use we fix some notation. Let \( I_0^1 = [0,1] \), and \( \tilde{I}_1^1 := (\lambda,1 - \lambda) \) be the first removed interval. We denote by \( I_j^i \) the \( 2^j \) closed intervals left after \( j \) iterations, and similarly the \( 2^j - 1 \) removed open intervals by \( \tilde{I}_j^i \).

The lengths of the intervals are
\[ |I_j^i| = \lambda^j, \quad i = 1, \ldots, 2^j, j = 0, 1, 2, \ldots \]
and
\[ |\tilde{I}_j^i| = (1 - 2\lambda)\lambda^{j-1}, \quad i = 1, \ldots, 2^{j-1}, j = 1, 2, 3, \ldots. \]
Recall that, \( C_\lambda \) is of zero \( \mathcal{H}^1 \)-measure, and \( \dim_{\mathcal{H}}(C_\lambda) = \frac{\log 2}{\log \lambda} \) (see e.g. [13] p.60–62]).

Define
\[ \Omega_\lambda = (-1,1)^2 \setminus \{ (x,y) : x \geq 0, |y| \leq d(x,C_\lambda) \}. \]
Set \( \Omega_\lambda \) is clearly a domain and the set of two-sided points is \( C_\lambda \setminus \{ (0,0) \}. \)

Lemma 4.1. The domain \( \Omega_\lambda \) above satisfies the curve condition \( \text{(1.1)} \) for \( 1 < p < 2 + \frac{\log 2}{\log \lambda} \).
That is, for each \( x, y \in \Omega_\lambda^c \) there exists rectifiable curve \( \gamma : [0, l(\gamma)] \to \Omega_\lambda^c \) connecting \( x, y \) such that
\[ \int_{\gamma} \text{dist} (z, \partial \Omega_\lambda)^{1-p} \, ds(z) \leq C(p,\lambda)|x-y|^{2-p}. \tag{4.1} \]
Moreover, we have the estimate
\[ C(p,\lambda) \leq \frac{c}{(2 - p)\lambda^{2-p}(1 - 2\lambda^{2-p})}, \]
where \( c \) is an absolute constant.

Proof. We consider three cases: (i) Assume first that \( x, y \in \mathbb{R}^2 \setminus (-1,1)^2 \). Define \( \gamma \) as a path of minimal length made of at most four line segments \( L_k \) with slope \( \pm \frac{\pi}{4} \) such that \( \gamma \subset \Omega^c \). Now for each segment \( L_k \)
\[ \int_{L_k} \text{dist} \left( z, \partial \Omega \right)^{1-p} \, ds(z) \leq 2 \int_0^{\left| L_k \right|} \left( \frac{t}{\sqrt{2}} \right)^{1-p} \, dt = \frac{2^{1-p}}{2-p} \left| L_k \right|^{2-p} \leq \frac{c}{2-p} |x - y|^{2-p}, \]

where \( |L_k| \) is the length of the segment \( L_k \).

(ii) Assume \( x, y \in \Omega^c \cap (-1,1)^2 \). Denote \( x = (x_1, x_2) \), \( y = (y_1, y_2) \), \( \tilde{x} = (x_1, 0) \) and \( \tilde{y} = (y_1, 0) \) and define \( \gamma = [x, \tilde{x}] * [\tilde{x}, \tilde{y}] * [\tilde{y}, y] \). For \( [x, \tilde{x}] \) (and similarly for \( [\tilde{y}, y] \)) by the geometry of the set \( \Omega \)

\[ \int_{[x, \tilde{x}]} \text{dist} \left( z, \partial \Omega \right)^{1-p} \, ds(z) \leq \int_0^{\left| \tilde{x} - x \right|} \left( \frac{t}{\sqrt{2}} \right)^{1-p} \, dt \]

\[ = \frac{2^{\frac{1}{2}(p-1)}}{2-p} \left| x - \tilde{x} \right|^{2-p} \]

\[ \leq \frac{2^{\frac{1}{2}(p-1)}}{2-p} \left| x - y \right|^{2-p}. \] (4.2)

For the segment \( [\tilde{x}, \tilde{y}] \) let \( j \in \mathbb{N} \) be such that

\[ \lambda^j < |\tilde{x} - \tilde{y}| \leq \lambda^{j-1}. \]

Now, \( [\tilde{x}, \tilde{y}] \) intersects at most two of the intervals \( I_j^i \) and one interval \( \tilde{I}_j \), where \( I_j^i \) and \( \tilde{I}_j \) are the closed and open intervals, respectively, related to the \( j \)-th step of the construction of the Cantor set. For every \( j \) and \( i \) we have

\[ \int_{I_j^i} \text{dist} \left( z, \partial \Omega \right)^{1-p} \, ds(z) = 2 \int_0^{\frac{1}{2}(1-2\lambda)\lambda^{j-1}} \left( \frac{t}{\sqrt{2}} \right)^{1-p} \, dt \]

\[ = \frac{2^{\frac{3}{2}(p-1)}}{2-p} (1 - 2\lambda)^{2-p} \lambda^{(j-1)(2-p)} \]

\[ = \frac{2^{\frac{3}{2}(p-1)}}{2-p} |I_j^i|^{2-p}. \] (4.3)

Since \( C_\lambda \cap I_j^i \) has a zero \( \mathcal{H}^1 \)-measure, by using (4.3) for all \( k > j, i = 1, \ldots, 2^k \), we get

\[ \int_{I_j^i} \text{dist} \left( z, \partial \Omega \right)^{1-p} \, ds(z) = \sum_{k=j}^{\infty} \sum_{I_{k+1} \subset I_j^i} \int_{I_k^i} \text{dist} \left( z, \partial \Omega \right)^{1-p} \, ds(z) \]

\[ = \sum_{k=j}^{\infty} 2^{k-j} \cdot \frac{1}{2} \int_0^{\frac{1}{2}(1-2\lambda)\lambda^{k}} \left( \frac{t}{\sqrt{2}} \right)^{1-p} \, dt \]

\[ = \frac{2^{\frac{3}{2}(p-1)}}{2-p} (1 - 2\lambda)^{2-p} \sum_{k=j}^{\infty} 2^{k-j} (\lambda^{p-2})^{-k} \]

\[ = \frac{2^{\frac{3}{2}(p-1)}}{2-p} \frac{(1 - 2\lambda)^{2-p} \sum_{k=j}^{\infty} \lambda^{(2-p)j}}{1 - 2\lambda^{2-p}} \]

\[ = \frac{2^{\frac{3}{2}(p-1)}}{2-p} \frac{|\tilde{I}_{j+1}|^{2-p}}{1 - 2|I_1|^{2-p}}, \]
where the last sum converges by the assumption $p < 2 - \log_3 2$.

Therefore,
\[
\int_{[x, y]} \operatorname{dist}(z, \partial \Omega)^{1-p} \, ds(z) \leq \frac{2^{2(p-1)} - 2}{p} \left( |\bar{I}|^{2-p} + 2 \cdot \frac{|\bar{I}|^{2-p}}{1 - 2 |I|^{2-p}} \right)
\leq \frac{2^{2(p-1)} - 2}{p} \frac{(1 - 2\lambda)^{2-p}}{\lambda^2 - p(1 - 2\lambda^{2-p})} |\tilde{x} - \tilde{y}|^{2-p} \tag{4.4}
\]

Combining $(4.2)$ and $(4.4)$ we have
\[
\int_{\gamma} \operatorname{dist}(z, \partial \Omega)^{1-p} \, ds(z) \leq 2^{2(p-1)} \frac{(1 - 2\lambda)^{2-p}}{p} \frac{\lambda^2 - p(1 - 2\lambda^{2-p})}{x - y}^{2-p}.
\]

(iii): Finally, assume that $x \in \Omega^c \cap (-1, 1)^2$ and $y \in \mathbb{R}^2 \setminus (-1, 1)^2$. Connect $x$ and $w = (1, 0)$ with $\gamma_1$ as in (ii) and $w$ and $y$ with $\gamma_2$ as in (i). Let us check that the curve $\gamma$ defined as $\gamma = \gamma_1 \ast \gamma_2$ fulfills $(4.1)$. This follows from the fact that $x$ and $y$ can not be close to each other without being close to $z$ i.e. $|x - w| < c_1 |x - y|$ and $|y - w| < c_2 |x - y|$ with absolute constants $c_1, c_2 > 0$. Thus
\[
\int_{\gamma} \operatorname{dist}(z, \partial \Omega)^{1-p} \, ds(z) = \int_{\gamma_1} \operatorname{dist}(z, \partial \Omega)^{1-p} \, ds(z) + \int_{\gamma_2} \operatorname{dist}(z, \partial \Omega)^{1-p} \, ds(z)
\leq \int_{\gamma_2} \operatorname{dist}(z, \partial \Omega)^{1-p} \, ds(z) \leq 2 \frac{c}{p} |x - y|^{2-p}.
\]

Using Lemma $(4.1)$ we can now show the existence of constants $M_2 > 0$ and $C(p) > 0$ so that $(4.3)$ holds for $C \geq C(p)$.

Fix $p \in (1, 2)$, and let $M_2 = \frac{8c}{\log 2}$ where $c$ is the absolute constant from Lemma $(4.1)$. In order to make estimates, we use the construction for $\lambda \in \left[ \frac{1}{2} \sqrt{p - 1}, 2^{\frac{1}{p - 2}} \right]$. By Lemma $(4.1)$ we know that the domain $\Omega_\lambda$ satisfies the curve condition with the constant
\[
\frac{c}{(2 - p)\lambda^2 - p(1 - 2\lambda^{2-p})}.
\tag{4.5}
\]

Let us define
\[
C(p) = \frac{c}{(2 - p)2^{p-3}(1 - 2^{p-2})}.
\]

Note that, $C(p)$ equals $(4.5)$ with $\lambda = \frac{1}{2} \sqrt{p - 1}$. Now, for $C \geq C(p)$, by the continuity of the constant in $(4.3)$ as a function of $\lambda$ and the fact that it tends to infinity as $\lambda \nearrow 2^{\frac{1}{p-1}}$, there exists $\lambda_C \in \left[ \frac{1}{2} \sqrt{p - 1}, 2^{\frac{1}{p-2}} \right]$ such that
\[
C = \frac{c}{(2 - p)\lambda_C^2 - p(1 - 2\lambda_C^{2-p})}.
\]
We show that

$$\dim \mathcal{C}_C = -\frac{\log 2}{\log \lambda_C} \geq 2 - \frac{M_2}{C}. \quad (4.6)$$

By the assumption \(\lambda_C \geq \frac{1}{2}^{\frac{1}{2p-2}}\), we have

$$C \leq \frac{c^{2^3-p}}{(2 - p)(1 - 2\lambda^{2-p})} \leq \frac{4c}{(2 - p)(1 - 2\lambda^{2-p})}. $$

In order to see that (4.6) holds, we show that

$$f_p(\lambda) = 2 - p - \frac{M_2}{4c} (2 - p) (1 - 2\lambda^{2-p}) + \frac{\log 2}{\log \lambda}$$

is non-positive on the interval \([\frac{1}{2}^{\frac{1}{2p-2}}, 2^{\frac{1}{2p-2}}]\). This follows from

$$\min_{\lambda \in [\frac{1}{2}^{\frac{1}{2p-2}}, 2^{\frac{1}{2p-2}}]} f'_p(\lambda) \geq 2 \frac{M_2}{4c} (2 - p)^2 \left(\frac{1}{2}^{\frac{1}{2p-2}}\right)^{1-p} - \frac{\log 2}{2^{1-2p}\log 2 (2^{\frac{1}{2p-2}})}$$

$$= \frac{M_2}{2c} (2 - p)^2 \left(\frac{3-p}{2p-2}\right)^{1-p} - \frac{(p - 2)^2}{2^{p-2}\log 2}$$

$$= \frac{(2 - p)^2}{2^{\frac{p-2}{2}}} \left(\frac{M_2}{2c} \left(\frac{3-p}{2p-2}\right)^{2-p} - \frac{1}{\log 2}\right)$$

$$\geq \frac{(2 - p)^2}{2^{\frac{p-2}{2}}} \left(\frac{M_2}{8c} - \frac{1}{\log 2}\right) \geq 0,$$

and

$$f_p(\lambda) \leq f_p(2^{\frac{1}{2p-2}}) = 0.$$

Hence, (4.6) holds.

Acknowledgements

The author thanks his advisor Tapio Rajala for helpful comments and suggestions. The author thanks Miguel García-Bravo for his comments, suggestions, and corrections, which improved this paper.

References

[1] A. P. Calderón, Lebesgue spaces of differentiable functions and distributions, Proc. Sympos. Pure Math. IV (1961) 33–49.
[2] T. deheuvels, Sobolev extension property for tree-shaped domains with self-contacting fractal boundary. Annali della Scuola Normale Superiore di Pisa, Classe di Scienze. 2016;15(special):209-47.
[3] K. J. Falconer, The geometry of fractal sets. Cambridge Tracts in Mathematics, vol. 85. Cambridge University Press, Cambridge, 1986.
[4] V. M. Gol’dshtein, T. G. Latfullin, and S. K. Vodop’yanov, A criterion for the extension of functions of the class \(L^2\) from unbounded plain domains (Russian), Sibirsk. Mat. Zh. 20 (1979), 416–419.
[5] V. M. Gol’dshtein and Yu G. Reshetnyak., Quasiconformal mappings and Sobolev spaces. Mathematics and its Applications (Soviet Series), 54 (1990), Kluwer Academic Publishers Group, Dordrecht.
[6] V. M. Gol’dshtein and S. K. Vodop’yanov Prolongement de fonctions différentiables hors de domaines planes (French), C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), 581–584.
[7] P. Hajłasz, P. Koskela, and H. Tuominen. Sobolev embeddings, extensions and measure density condition. J. of Funct. Anal. 254, no. 5 (2008): 1217–1234.
[8] P. W. Jones, *Quasiconformal mappings and extendability of Sobolev functions*, Acta Math. 47 (1981), 71–88.
[9] P. Koskela, *Capacity extension domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No. 73 (1990), 42 pp.
[10] P. Koskela, T. Rajala, and Y.-Y. Zhang, *A geometric characterization of planar sobolev extension domains*, preprint.
[11] P. Koskela, T. Rajala, and Y.-Y. Zhang, *Planar W^{1,1}-Extension Domains*, preprint.
[12] D. Lučić, T. Rajala, and J. Takanen, *Dimension estimates for the boundary of planar Sobolev extension domains*, preprint.
[13] P. Mattila, *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. No. 44. Cambridge university press, 1999.
[14] R. Nääkö and J. Väisälä, *John disks*, Exposition. Math. 9 (1991), 3–43.
[15] B. Palka, *An Introduction to Complex Function Theory*, Undergraduate Texts in Mathematics, New York: Springer-Verlag, 1991.
[16] C. Pommerenke, *Boundary Behaviour of Conformal Maps*. Springer Berlin Heidelberg, 1992.
[17] P. Shvartsman, *On Sobolev extension domains in $\mathbb{R}^n$*, J. Funct. Anal. 258 (2010), no. 7, 2205–2245.
[18] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970.

University of Jyväskylä, Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland

Email address: jyrki.j.takanen@jyu.fi