ON THE COARSE CLASSIFICATION OF TIGHT CONTACT STRUCTURES

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Abstract. We present a sketch of the proof of the following theorems: (1) Every 3-manifold has only finitely many homotopy classes of 2-plane fields which carry tight contact structures. (2) Every closed atoroidal 3-manifold carries finitely many isotopy classes of tight contact structures.

In this article we explain how to normalize tight contact structures with respect to a fixed triangulation. Using this technique, we obtain the following results:

Theorem 0.1. Let $M$ be a closed, oriented 3-manifold. There are finitely many homotopy classes of 2-plane fields which carry tight contact structures.

Theorem 0.2. Every closed, oriented, atoroidal 3-manifold carries a finite number of tight contact structures up to isotopy.

P. Kronheimer and T. Mrowka [KM] had previously shown Theorem 0.1 for (weakly) symplectically (semi-)fillable contact structures. Our theorem is a genuine improvement of the Kronheimer-Mrowka theorem because there exist tight structures which are not fillable [EH].

Now, since every Reebless foliation is a limit of tight contact structures [Co4 ET], we obtain a new proof of a recent result of D. Gabai [Ga].

Corollary 0.3 (Gabai). There are finitely many homotopy classes of plane fields which carry Reebless foliations.

Next, shifting our attention to isotopy classes of contact structures, we see that Theorem 0.2 complements the following theorem [Co2 Co3 HKM]:

Theorem 0.4 (Colin, Honda-Kazez-Matić). Every closed, oriented, irreducible, toroidal 3-manifold carries infinitely many tight contact structures up to isomorphism.

Summarizing, we have:

Theorem 0.5. A closed, oriented, irreducible 3-manifold carries infinitely many tight contact structures (up to isotopy or up to isomorphism) if and only if it is toroidal.

A more complete account of the proofs will appear in [CGH].

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1. Contact geometry in dimension 3

In dimension 3, there are exactly two types of locally homogeneous 2-plane fields \( \xi = \ker \alpha \). The plane field \( \xi \) is a contact structure when \( \alpha \wedge d\alpha \) is nowhere vanishing, and is a foliation when \( \alpha \wedge d\alpha \equiv 0 \). Almost by definition, the classification of contact structures and foliations must reflect global properties of the ambient manifold \( M \). Contact structures, unlike foliations, are very stable objects. For example, two \( C^0 \)-close contact structures are isotopic [Co1]. Thus classifying contact structures up to isomorphism or isotopy appears to be a reasonable project and has been the subject of numerous studies in the past twenty years.

In what follows, we only consider oriented manifolds and oriented contact structures which are positive, i.e., satisfy \( \alpha \wedge d\alpha > 0 \). Also, denote the metric completion of each component of \( A \setminus B \) by \( \overline{A \setminus B} \).

1.1. Tight vs. overtwisted. The study of contact 3-manifolds reveals a dichotomy in the world of contact structures: tight vs. overtwisted [Be, El1, El2]. A contact structure \( \xi \) is overtwisted if there exists an embedded 2-disk \( D \) which is tangent to \( \xi \) at all its boundary points; otherwise \( \xi \) is tight. The boundary of \( D \) plays a role analogous to vanishing cycles in foliation theory, and there are many analogies between tight contact structures and Reebless (or taut) foliations [ET].

The classification of overtwisted contact structures up to isotopy coincides with the classification of plane fields up to homotopy by the work of Y. Eliashberg [El1]; contact structures therefore need tightness to become geometrically significant. A similar observation holds in foliation theory: Reeb components should be avoided.

1.2. Convex surfaces. The main tool for analyzing contact manifolds is the theory of convex surfaces, first introduced in [Gi1].

Let \( S \) be an oriented, properly embedded surface in a contact manifold \( (M, \xi) \). We assume that \( S \) is either closed or compact with Legendrian boundary (i.e., tangent to \( \xi \) at every point). Then the characteristic foliation \( \xi S \) on \( S \) is the singular foliation obtained by integrating the singular line field \( TS \cap \xi \). It is directed by the orientations of \( S \) and \( \xi \), and is singular precisely when \( \xi = TS \). The singularities are generically isolated and of two types: elliptic of index 1 or hyperbolic of index \(-1\). They are positive (or negative) if the orientation of \( \xi \) and \( TS \) coincide (or not). The germ of a contact structure near a surface \( S \) is completely determined by the characteristic foliation \( \xi S \).

A surface \( S \) is said to be convex if it is transversal to a vector field which preserves \( \xi \), i.e., a contact vector field. Surprisingly, this notion of convexity is generic [Gi1]: every closed surface admits a \( C^\infty \)-small perturbation into a convex surface. Moreover, the convexity can be read off from the characteristic foliation \( \xi S \); for example, if \( \xi S \) is singular Morse-Smale, then \( S \) is convex [Gi1].
Let $\gamma$ be a closed embedded Legendrian curve on an embedded surface $S$. We define the *relative Thurston-Bennequin invariant* $tb(\gamma, S)$ (also called the *twisting number* relative to $S$) to be half the number of algebraic intersections of $\xi$ and $TS$ along $\gamma$.

If $S$ is a surface with Legendrian boundary and if each component of $\partial S$ has nonpositive relative Thurston-Bennequin invariant, then $S$ can be made convex by an isotopy which fixes $\partial S$, is $C^0$-small in a neighborhood of $\partial S$, and is $C^\infty$-small outside.

Let $S$ be a surface transverse to a contact vector field $X$. Define the *dividing set* of $S$ to be

$$\Gamma_S = \{x \in S | X(x) \in \xi(x)\}.$$  

It is a smooth embedded multicurve transverse to $\xi S$ and does not depend on $X$ up to isotopy through multicurves transverse to $\xi S$. $\Gamma_S$ inherits a natural orientation from that of $\xi S$. We say that the characteristic foliation $\xi S$ is *adapted* to $\Gamma_S$.

The multicurve $\Gamma_S$ captures all of the essential information on $\xi$ in a neighborhood of $S$, according to the following *Flexibility Theorem* [Gi1]:

**Theorem 1.1** (Giroux). Let $S$ be a closed surface, $f_0 : S \hookrightarrow (M, \xi)$, $g : S \hookrightarrow (M', \xi')$ be two embeddings with $f_0(S)$ and $g(S)$ convex, and $X$ a contact vector field transverse to $f_0(S)$. If the oriented multicurves $f_0^{-1}(\Gamma_{f_0(S)})$ and $g^{-1}(\Gamma_{g(S)})$ coincide, then there exists an isotopy $(f_t)_{t \in [0, 1]}$ transverse to $X$ such that $f_1(\xi f_1(S)) = g^*(\xi g(S))$. The same holds if $\partial S \neq \emptyset$ and the characteristic foliations coincide near $\partial S$; in this case we get an isotopy rel $\partial S$.

If $S$ is a surface with Legendrian boundary, an arc $\delta$ of $\Gamma_S$ is said *boundary-parallel* if one component of $S \setminus \delta$ is a half disk which has no other intersections with $\Gamma_S$.

### 1.3. Bypasses

We now introduce the notion of a *bypass*, as in [Ho1] and [Et], which allows us to investigate the contact structure outside the neighborhood of a convex surface.

Let $D$ be a half-disk embedded in a contact manifold and $\alpha \cup \beta$ be the decomposition of $\partial D$ into two smooth arcs which meet only at their endpoints. The disk $D$ is called a *bypass* if it satisfies the following:

- $\partial D$ is Legendrian.
- $D$ is convex, without singularities in its interior.
- Along $\partial D$, we have two positive elliptic singularities at $\partial \alpha = \partial \beta$, a positive hyperbolic singularity in the interior of $\beta$ and a negative elliptic singularity in the interior of $\alpha$.

The Thurston-Bennequin invariant of $\partial D$ is $-1$, with a contribution of $0$ from $\beta$ and $-1$ from $\alpha$. In other words, $\beta$ is a more “efficient” Legendrian arc than $\alpha$.

**Lemma 1.2.** Let $S$ be a convex surface with Legendrian boundary. Assume there exists a boundary-parallel component $\delta$ of $\Gamma_S$ which cuts off a half-disk $D_0 \subset S$. (If $S$ is a disk, we assume in addition that $\Gamma_S$ is not connected.) Let $D_1$ be a tubular neighborhood of $D_0$ inside $S$, with $D_1 \cap \Gamma_S = \emptyset$. Denote $\alpha_1 = D_1 \cap \partial S$. If $X$ is a contact vector field transverse to $S$, then there exists an isotopy of $S$ rel $\partial S$ through surfaces transversal to $X$, which leads to a surface $S'$ such that the image $D' \subset S'$ of $D_1$ is a bypass.

The $\alpha_1$ in the lemma plays the role of $\alpha$ in the definition of a bypass. If $S$ has been nicely normalized along $\alpha_1$, the isotopy can be chosen to be $C^\infty$-small.
Proof. One can draw a foliation on $S$ which is adapted to $\Gamma_S$, and where $D_1$ is a bypass. The desired isotopy is then given by the Flexibility Theorem. 

Here is a fundamental result of Y. Eliashberg [El2] which classifies tight contact structures on the 3-ball $B^3$.

**Theorem 1.3.** Let $\Gamma$ be a connected dividing set on $\partial B^3$, and $\mathcal{F}$ a foliation adapted to $\Gamma$. Then there exists a unique tight contact structure on $B^3$, up to isotopy rel $\partial B^3$, whose characteristic foliation on $\partial B^3$ is $\mathcal{F}$.

1.4. The generalized Lutz modification. Let $T \subset (M, \xi)$ be an embedded 2-torus which meets $\xi$ transversely. Then there exists a closed tubular neighborhood $U = (\mathbb{R}^2/\mathbb{Z}^2) \times [0, 2\pi]$ of $T = (\mathbb{R}^2/\mathbb{Z}^2) \times \{\pi\}$ fibered by Legendrian intervals $\{pt\} \times [0, 2\pi]$. Then, with respect to the coordinate system $(x, y, t)$ for $(\mathbb{R}^2/\mathbb{Z}^2) \times [0, 2\pi]$, $\xi$ is given by:

$$\cos f(x, y, t)dx - \sin f(x, y, t)dy = 0,$$

where $f$ is a circle-valued function $(\mathbb{R}^2/\mathbb{Z}^2) \times [0, 2\pi] \to \mathbb{R}/2\pi \mathbb{Z}$. For every $n \in \mathbb{N}$, choose a smooth increasing function $g_n : [0, 2\pi] \to \mathbb{R}$ with $g_n(0) = 0$ and $g_n(2\pi) = 2n\pi$. The plane field defined on $U$ as the kernel of $\alpha_n = \cos(f + g_n)dx - \sin(f + g_n)dy$ is a contact structure. It coincides with $\xi$ on $\partial U$, and thus can be extended by $\xi$ to give a contact structure $\xi_n$ on $M$. It is called a generalized Lutz modification of $\xi$ of index $n$ along $T$. (While the contact structure $\xi_n$ depends on the choice of tubular neighborhood $U$, its isotopy class only depends on $T$.)

Consider the function $\chi : U \to [0, 1]$ which is 0 on $\partial U$ and is strictly positive on the interior of $U$. The path of 1-forms $(1-t)\alpha_0 + t\alpha_n + t(1-t)\chi dt$ defines a path of plane fields rel $\partial U$ between $\xi|_U$ and $\ker \alpha_n$. Thus $\xi$ and $\xi_n$ are homotopic through plane fields.

Note that if $(M, \xi)$ is overtwisted, then many generalized Lutz modifications do not alter the isotopy class of $\xi$. For example, if $T$ bounds a solid torus and the characteristic foliation $\xi T$ is a suspension, then the new contact structure $\xi_n$ is also overtwisted, and is therefore isotopic to $\xi$ by Eliashberg’s overtwisted classification theorem. In general, it is not clear whether a generalized Lutz modification on an overtwisted contact manifold remains overtwisted.

**Exercise:** Let $(M, \xi)$ be an overtwisted contact manifold and $T \subset M$ a pre-Lagrangian incompressible torus. If $\xi_n$ is obtained from $\xi$ by a generalized Lutz modification along $T$, then $(M, \xi_n)$ has a finite cover which is overtwisted.

**Question:** Is $(M, \xi_n)$ itself overtwisted?

1.5. Branched surfaces. A branched surface $\mathcal{B}$ is a topological space such that every $p \in \mathcal{B}$ has a neighborhood which is given by one of the three possibilities in Figure [Diagram]. A branched surface with boundary $\mathcal{B}$ is a topological space locally modeled on the three diagrams in Figure [Diagram] plus “halves” of the diagrams to the left and in the middle, so that $\partial \mathcal{B}$ is a train track.

The branch locus $L$ of a branched surface $\mathcal{B}$ is the set of points $p \in \mathcal{B}$ for which no neighborhood is modeled on a plane or a half-plane. It is a collection of embedded curves which intersect transversally at double points. If $\mathcal{B}$ is embedded inside a 3-manifold, the
neighborhood $N(B)$ of $B$ admits a fibration by intervals and a projection $\pi : N(B) \to B$ (see Figure 2). The boundary of $N(B)$ is divided into two pieces, the horizontal boundary $\partial_h N(B)$ which is transversal to the fibration and the vertical boundary $\partial_v N(B)$ which is tangent to the fibration.

We can now strengthen Theorem 0.1:

**Theorem 1.4.** Given a closed 3-manifold $M$, there exists a finite collection of contact structures $\{\xi_1, \ldots, \xi_k\}$ and, for each $1 \leq i \leq k$, a finite set of tori $T_i = \{T_i^1, \ldots, T_i^k\}$ transverse to $\xi_i$, whose union is a branched surface, such that every tight contact structure $\xi$ on $M$ is obtained from one of the $\xi_i$ by performing a generalized Lutz modification on a subset of $T_i$.

Theorem 1.4 follows immediately from Theorem 0.1 since we have already shown that a generalized Lutz modification does not alter the homotopy class of the contact structure.

In [Co2, Co3, Gi2, HKM], generalized Lutz modifications were performed along pre-Lagrangian incompressible tori to produce infinitely many nonisomorphic tight contact structures. More precisely, if $\xi_n$ is the tight contact structure given by the equation $\cos(nt)dx - \sin(nt)dy = 0$ on the thickened torus $(\mathbb{R}/\mathbb{Z})^2 \times [0, 2\pi]$ with coordinates $(x, y, t)$, then one can define the torsion of a contact manifold $(M, \xi)$ as the supremum over $n \in \mathbb{N}$ for which there exists a contact embedding $\phi_n : (T^2 \times [0, 2\pi], \xi_n) \hookrightarrow (M, \xi)$. This invariant of $(M, \xi)$,
introduced in \[\text{Gi}^2\], is intended to measure “how large” the contact manifold is. Notice that if the torsion is finite, then it can be increased by a Lutz modification.

In \[\text{Co}^2, \text{Co}^3, \text{HKM}\], we essentially prove that every closed, oriented, irreducible, and toroidal 3-manifold carries tight contact structures with arbitrarily large and finite torsion.

2. Normalizing a triangulation in a tight contact manifold

We now describe the main tool (Proposition 2.4) for proving Theorems 1.4 and 0.2, which echoes the theory of normal surfaces of Haken and Kneser \[\text{Ha, Kn}\]. This technique is also very similar to Gabai’s in \[\text{Ga}\].

Let \(\tau\) be a fixed triangulation of a 3-manifold \(M\). Denote the \(i\)-skeleton of \(\tau\) by \(\tau^i\).

2.1. Maximal triangulations. Given a tight contact structure \(\xi\) on \(M\), we can isotop \(\tau\) so that the 1-skeleton is a Legendrian graph and each face is convex. Such a triangulation will be called a contact triangulation. This is easy to accomplish because:

- every embedded graph can be \(C^0\)-approximated by a Legendrian graph;
- the Thurston-Bennequin number \(tb(\gamma)\) along each edge \(\gamma\), computed with the trivialization given by any adjacent face, can be made strictly negative by a \(C^0\)-small isotopy of \(\tau^1\) rel \(\tau^0\);
- by genericity, after a \(C^0\)-small isotopy in a neighborhood of \(\tau^1\) rel \(\tau^1\), each face can then be made convex by a \(C^\infty\)-small isotopy rel \(\tau^1\).

Note that the triangulation is now singular: at each vertex \(x\), the tangent lines to the adjacent edges are all contained in \(\xi(x)\).

Denote \(TB(\xi, \tau) = \sum_{F \in \tau^2} tb(\partial F)\). If \([\tau]\) is the set of all contact triangulations isotopic to \(\tau\), then define

\[
TB(\xi, [\tau]) = \max_{\tau' \in [\tau]} TB(\xi, \tau').
\]

We now assume, after a possible change of notation, that \(\tau\) realizes the maximum value \(TB(\xi, [\tau])\). Such a triangulation is said to be a maximal triangulation for \(\xi\) in the class \([\tau]\). In that case, we have the following:

Lemma 2.1.

(a) For every face \(F\) of \(\tau\), each component of \(\Gamma_F\) is an arc whose endpoints are on different edges, except possibly six arcs which are boundary-parallel components and whose endpoints are close to a vertex (see Figure 4).

(b) the “holonomy” of the dividing curves around the boundary of a 3-simplex \(B\) is as in Figure 4.

Proof. We only sketch the proof of (a). Closed components of \(\Gamma_F\) are not allowed because the structure is tight. The existence of a boundary-parallel component “far” from the vertices would imply the existence of a bypass which does not meet the vertices. Let \(\alpha \cup \beta\) be the boundary of the bypass, where \(\alpha \subset \partial F\), and isotop the edge \(l\) containing \(\alpha\) to \((l \setminus \alpha) \cup \beta\). It strictly increases \(TB(\xi, \tau)\). See Figure 4.

The proof of (b) also relies on the existence of bypasses inside \(B\) in case the holonomy is not as in Figure 4 (see \[\text{Ho}^3\]).
Figure 3. The dotted dividing curves can be isotoped across, leaving the solid dividing curves.

Figure 4.

2.2. Fibered prisms. A fibered prism is a polyhedron $P$ diffeomorphic to $R \times [-1, 1]$, where $R$ is a triangle or a rectangle, and $\{pt\} \times [-1, 1]$ are the fibers. The set $R \times \{-1, 1\}$ is called the horizontal boundary, and the set $(\partial R) \times [-1, 1]$ the vertical boundary.

Let $\xi$ be a tight contact structure on $M$, which has been isotoped so that $\tau$ is a maximal triangulation for $\xi$. Then we partition families of parallel arcs of $\Gamma_F$ ($F$ is a face of $\tau$) inside fibered prisms:

Lemma 2.2. For each 3-simplex $B$ of $\tau$, there are at most 5 embeddings $\phi_i : P_i \hookrightarrow B$, $i = 1, \ldots, 5$, of fibered prisms $P_i$, such that the following are satisfied.

- The vertical edges (resp. faces) of $P_i$ are sent into edges (resp. faces) of $B$.
- $\phi_i(P_i \setminus (\partial P \times [-1, 1])) \subset \text{int}(B)$.
- $\phi_i(P_i \times \{-1, 1\}) \cap \Gamma_F = \emptyset$, for every face $F$ of $B$.
- The fibered prisms coming from two different 3-simplices intersect along rectangles.
- For every face $F$ of $B$, at most $C$ components of $\Gamma_F$ are not contained in the image of the $\phi_i$’s. Here $C$ is a universal constant which does not depend on $\xi$ or on $\tau$. 
There exists a finite number of pairs \((B_i, \zeta_i)\), \(i = 1, \ldots, n\), where \(B_i\) is a branched surface and \(\zeta_i\) a contact structure on \(M \setminus N(B_i)\), such that every tight contact structure \(\xi\) on \(M\), up to isotopy, is generated by one of the \((B_i, \zeta_i)\), i.e.,

1. \(\xi|_{M \setminus N(B_i)} = \zeta_i\);

Moreover, by a combination of the Flexibility Theorem and Theorem 1.3, we can normalize \(\xi|_{\phi_1(P_i)}\) as in the following lemma, after isotoping \(\xi\). (For notational convenience, we will not distinguish \(P_i\) from its image from now on.)

**Lemma 2.3.** Fix a nonsingular foliation \(\mathcal{F}\) on the horizontal boundary of each prism (the same for each face) which coincides with the characteristic foliation near its 1-skeleton. Then \(\xi\) can be isotoped rel \(\tau^1\) so that the characteristic foliation of \(\xi\) equals \(\mathcal{F}\) on the horizontal boundary and each vertical fiber \(\{pt\} \times [-1, 1]\) is Legendrian.

**2.3. The branched surfaces.**

**Proposition 2.4.** There exists a finite number of pairs \((B_1, \zeta_1), \ldots, (B_n, \zeta_n)\), where \(B_i\) is a branched surface and \(\zeta_i\) a contact structure on \(M \setminus N(B_i)\), such that every tight contact structure \(\xi\) on \(M\), up to isotopy, is generated by one of the \((B_i, \zeta_i)\), i.e.,

1. \(\xi|_{M \setminus N(B_i)} = \zeta_i\);
(2) \( \xi|_{N(B)} \) is tangent to the fibers of \( N(B) \).

Moreover, one may assume that the branched surfaces have empty boundary.

**Proof.** Let \( \xi \) be a tight contact structure on \( M \), isotoped so that \( \tau \) is a maximal triangulation for \( \xi \). Apply Lemmas 2.2 and 2.3 to obtain the collection \( \{ P_1, \ldots, P_k \} \) of fibered prisms \( P_i = R_i \times [-1, 1] \) ranging over all the 3-simplices \( B \) of \( \tau \). For every edge \( a \in \tau^1 \), denote \( a' \) the arc obtained from \( a \) by deleting a small neighborhood of its endpoints, and \( a' \times D^2 \) a small tubular neighborhood of \( a' \) such that the fibration \( a' \times \{ pt \} \) is Legendrian and coincides with the one given on the \( P_i \)'s.

**Lemma 2.5.** There exists a branched surface \( B \) with boundary, together with a fibered neighborhood \( N(B) \) fibered by Legendrian intervals, which is obtained by smoothing

\[
K = \left( \bigcup_{1 \leq i \leq k} P_i \right) \cup \left( \bigcup_{a \in \tau^1} (a' \times D^2) \right).
\]

Moreover, every Legendrian fiber of \( N(B) \) lies inside a Legendrian fiber of \( K \).

**Proof of Lemma 2.5.** Up to reparametrization, one can always assume that \( Q_i \cap Q_j = \emptyset \) for \( i \neq j \), where we denote \( Q_i = R_i \times \{ 0 \} \). Now let \( Q'_i = Q_i \setminus \bigcup_{a \in \tau^1} (a' \times D^2) \), i.e., we remove from \( Q_i \) a neighborhood of its vertices. If \( P_i \) intersects \( P_j \), we join the corresponding edges of \( Q'_i \) and \( Q'_j \) by a small band transverse to the fibration. Denote by \( B' \) the branched surface with boundary consisting of the union of the \( Q'_i \)'s and the bands. Note that \( B' \) does not have any triple branch points.

Take a disk \( \{ pt \} \times D^2 \subset a' \times D^2 \) for each edge \( a \) of \( \tau^1 \), and squash each point of \( B' \cap (a' \times D^2) \) to the point of \( \{ pt \} \times D^2 \) situated on the same fiber. We obtain a “singular surface” that can easily be perturbed into a branched surface \( B \); this is where we need to introduce triple branch points. Once the edges of \( K \) are smoothed, we naturally obtain a fibered neighborhood \( N(B) \) of \( B \) with the desired properties. \( \square \)

The construction of Lemma 2.5 gives rise to a finite number of branched surfaces, since each simplex contains at most 5 prisms, and for each simplex there are 7 total prism positions to choose from (see Figure 6). Here we are considering two prisms \( P_i, P_j \) in a 3-simplex \( B \) to be equivalent if there is an isotopy from \( P_i \) to \( P_j \) which restricts to each edge \( a \) to be an isotopy of \( P_i \cap a \) to \( P_j \cap a \) and on each face \( F \) to be an isotopy of \( P_i \cap F \) to \( P_j \cap F \).

We may assume that all the tight contact structures \( \xi \) with the same set of fibered prisms (and hence the same branched surface \( B \)) agree on \( \partial N(B) \), because we can impose the same nonsingular characteristic foliation on the horizontal boundaries of the prisms by Lemma 2.3. On the complementary regions of \( N(B) \), we have a natural decomposition into polyhedra which is inherited from \( \tau \). On the faces of the polyhedra, the number of components of the dividing set is universally bounded by Lemma 2.2. This leads to a finite number of possible dividing curve configurations, and hence to finitely many characteristic foliations by the Flexibility Theorem. But now, according to Theorem 1.3, a tight contact structure on each polyhedron is determined by its restriction to the boundary.

Now the branched surface \( B \) may have nonempty boundary. We then use an amputation principle to excise the boundary: if a vertical face of a prism is entirely contained in \( \partial_v N(B) \),
then all tight contact structures which coincide outside of $N(B)$ with $\zeta$ and are tangent to the fibers of $N(B)$ coincide on this face. In particular, they all have the same number of components of dividing curves on this face, and the number of dividing curves on the other vertical faces of the prism is bounded. Thus, all the tight contact structures which coincide with $\zeta$ on $M \setminus N(B)$ coincide, up to isotopy, with a finite set of models on this prism, by taking into account the finite repartition of the parallel dividing curves inside adjacent prisms. Therefore, we can amputate this prism from $N(B)$, at the expense of possibly increasing the number of pairs $(B_i, \xi_i)$. This is a finite process which must eventually yield a branched surface without boundary (or an empty branched surface).

3. Tight Contact Structures Carried by the Fibered Neighborhood of a Branched Surface

We now analyze contact structures which are carried by the fibered neighborhood of a branched surface $N(B)$, i.e., are tangent to the fibers and are fixed on the boundary.

Let $S$ be the set of tight contact structures tangent to the fibers of $N(B)$ (with a given germ along $\partial N(B)$), and let $B_1, \ldots, B_d$ be the components of $B \setminus L$ (recall $L$ is the singular locus of $B$). If we fix $\zeta_0 \in S$, then for any given $\xi \in S$ one can define a weight system $w_\xi$ on $B$: $w_\xi(B_i)$, $i = 1, \ldots, d$, is the difference of rotation between $\xi$ and $\zeta_0$ along a fiber $\pi^{-1}(p)$, where $p \in B_i$. It is a set of integers which does not depend on the choice of $p \in B_i$. Note that, unlike the relative Thurston-Bennequin invariant, a left twist contributes positively to the weight. It is easy to verify that $\xi \in S$ is determined up to isotopy by its weight $w_\xi = (w_\xi(B_1), \ldots, w_\xi(B_d)) \in \ZZ^d$. Also, each $w_\xi(B_i)$ is bounded below by the positive contact structure condition. By amputating the $B_i$ for which $w_\xi(B_i)$ is negative, we may assume that $w_\xi(B_i)$ is nonnegative for each $\xi \in S$.

To each smooth edge $A$ of $L$, we associate a linear equation

$$\phi_A(x_1, \ldots, x_d) = x_i - (x_j + x_k) = 0,$$

with $(x_1, \ldots, x_d) \in \ZZ^d$, where $B_i, B_j$ and $B_k$ are the components of $B \setminus L$ which meet along $A$, and $B_j$ and $B_k$ branch out of $B_i$. Let $V$ be the subspace of $\RR^d$ of solutions to the system $\{\phi_A = 0 \mid \forall$ smooth edges $A$ of $L\}$. Each weight $w$ is an element of $V \cap \ZZ_{\geq 0}^d$, where $\ZZ_{\geq 0}^d = \{(x_1, \ldots, x_d) \in \ZZ^d \mid x_i \geq 0 \ \forall i\}$.

Define a partial order $\leq$ on $\ZZ^d$ by:

$$(x_1, \ldots, x_d) \leq (x'_1, \ldots, x'_d) \text{ if } x_i \leq x'_i \text{ for } i = 1, \ldots, d.$$

There is only a finite number of minimal elements of $V \cap \ZZ_{\geq 0}^d$ with respect to $\leq$, and we denote them $u_1, \ldots, u_k$. They generate the monoid $V \cap \ZZ_{\geq 0}^d$. On the other hand, we have the following standard fact:

**Lemma 3.1.** The isotopy classes of embedded surfaces carried by $N(B)$ (i.e., transverse to the fibers) are in bijection with elements of $V \cap \ZZ_{\geq 0}^d$.

**Remark.** Every surface carried by $N(B)$ is transverse to a contact structure and is thus either a torus or a Klein bottle.
Let \( \{T_1, \ldots, T_k\} \) be the collection of surfaces carried by \( N(B) \) and determined by the weights \( u_1, \ldots, u_k \). The following lemma, together with Proposition 2.4 and the remark below, yields Theorem 1.4.

Assume all the \( T_i \)'s are tori. Then we have:

**Lemma 3.2.** Every contact structure \( \xi \in S \) is obtained from \( \xi_0 \) by a collection of generalized Lutz modifications along \( T_1, \ldots, T_k \).

**Proof.** Given \( \xi \in S \), we can write \( w_\xi = \sum_{1 \leq i \leq k} n_i u_i, \ n_i \in \mathbb{Z}_{\geq 0} \). The contact structure \( \xi' \) obtained from \( \xi_0 \) by generalized Lutz modifications of index \( n_i \) along \( T_i \) is tangent to the fibers of \( N(B) \) and has the same weight as \( \xi \). Therefore \( \xi \) and \( \xi' \) are isotopic rel \( \partial N(B) \). □

**Remark.** If \( T_i \) is a Klein bottle, then we can replace it by the torus \( T_i' = \partial N(T_i) \), where \( N(T_i) \) is the tubular neighborhood of \( T_i \). Denote \( T_i' = T_i \) if \( T_i \) is a torus. Then the set \( S \) is still generated by Lutz modifications along the \( T_i' \), but starting from a finite set of structures (not only \( \xi_0 \)), obtained from \( \xi_0 \) by performing \( \pi \)-twists along the Klein bottles.

### 4. ISOTOPY AND BEYOND

The proof of Theorem 0.2 follows from a more in-depth analysis of tight contact structures carried by the fibered neighborhood of these branched surfaces. We use the classification of tight contact structures on the thickened torus (see [Gi4, Ho1]) to build a process which allows us to decrease the weights in case the manifold is atoroidal. In the general case, one should be able to prove that there exists a finite number of tight contact structures of bounded torsion up to isomorphism. Moreover, these techniques apply to manifolds \( M \) with boundary, provided \( \partial M \) is not a torus and we prescribe a fixed characteristic foliation along \( \partial M \). We then obtain relative versions of Theorems 0.1 and 0.2.

Finally, we have the following:

**Theorem 4.1.** Given an oriented topological knot type \( K \) in the standard tight contact 3-sphere \( (S^3, \xi_{std}) \) and an integer \( n \), there exists, up to contact isotopy, a finite number of Legendrian knots in \( K \) with Thurston-Bennequin invariant equal to \( n \).

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12 VINCENT COLIN, EMMANUEL GIROUX, AND KO HONDA

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