Existence and nonexistence of traveling waves for a nonlocal monostable equation

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Abstract We consider the nonlocal analogue of the Fisher-KPP equation
\[ u_t = \mu * u - u + f(u), \]
where \( \mu \) is a Borel-measure on \( \mathbb{R} \) with \( \mu(\mathbb{R}) = 1 \) and \( f \) satisfies \( f(0) = f(1) = 0 \) and \( f > 0 \) in \( (0,1) \). We do not assume that \( \mu \) is absolutely continuous with respect to the Lebesgue measure. We show that there is a constant \( c_* \) such that it has a traveling wave solution with speed \( c \) when \( c \geq c_* \) while no traveling wave solution with speed \( c \) when \( c < c_* \), provided \( \int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y) < +\infty \) for some positive constant \( \lambda \). We also show that it has no traveling wave solution, provided \( f'(0) > 0 \) and \( \int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y) = +\infty \) for all positive constants \( \lambda \).

Keywords: spreading speed, convolution model, integro-differential equation, discrete monostable equation, nonlocal evolution equation, Fisher-Kolmogorov equation.

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1 Introduction

In 1930, Fisher [8] introduced the reaction-diffusion equation \( u_t = u_{xx} + u(1-u) \) as a model for the spatial spread of an advantageous form of a single gene in a population. He [9] found that there is a constant \( c_* \) such that the equation has a traveling wave solution with speed \( c \) when \( c \geq c_* \) while it has no such solution when \( c < c_* \). Kolmogorov, Petrovsky and Piskunov [16] obtained the same conclusion for a monostable equation \( u_t = u_{xx} + f(u) \) with a more general nonlinearity \( f \), and investigated long-time behavior in the model. Since the pioneering works, there have been extensive studies on traveling waves and long-time behavior for monostable evolution systems.

In this paper, we consider the following nonlocal analogue of the Fisher-KPP equation:

\[
(1.1) \quad u_t = \mu * u - u + f(u).
\]

Here, \( \mu \) is a Borel-measure on \( \mathbb{R} \) with \( \mu(\mathbb{R}) = 1 \) and the convolution is defined by

\[(\mu * u)(x) := \int_{y \in \mathbb{R}} u(x - y) d\mu(y)\]

for a bounded and Borel-measurable function \( u \) on \( \mathbb{R} \). The nonlinearity \( f \) is a Lipschitz continuous function on \( \mathbb{R} \) with \( f(0) = f(1) = 0 \) and \( f > 0 \) in \((0, 1)\). Then, \( G(u) := \mu * u - u + f(u) \) is a map from the Banach space \( L^\infty(\mathbb{R}) \) into \( L^\infty(\mathbb{R}) \) and it is Lipschitz continuous. (We note that \( u(x - y) \) is a Borel-measurable function on \( \mathbb{R}^2 \), and \( \|u\|_{L^\infty(\mathbb{R})} = 0 \) implies \( \|\mu * u\|_{L^1(\mathbb{R})} \leq \int_{y \in \mathbb{R}} (\int_{x \in \mathbb{R}} |u(x - y)| dx) d\mu(y) = 0 \). So, because the standard theory of ordinary differential equations works, we have well-posedness of the equation (1.1) and it generates a flow in \( L^\infty(\mathbb{R}) \).

For the nonlocal monostable equation, Atkinson and Reuter [1] first studied existence and nonexistence of traveling wave solutions. Schumacher [21, 22] showed that there is the minimal speed \( c_* \) of traveling wave solutions and it has a traveling wave solution with speed \( c \) when \( c \geq c_* \), provided the extra condition \( f(u) \leq f'(0)u \) and some little ones. Here, we say that the solution \( u(t, x) \) is a traveling wave solution with profile \( \psi \) and speed \( c \), if \( u(t, x) \equiv \psi(x - x_0 + ct) \) holds for some constant \( x_0 \) with \( 0 \leq \psi \leq 1 \), \( \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \). Further, Coville, Dávila and Martínez [6] proved the following theorem:
Theorem ([6]) Suppose the nonlinearity $f \in C^1(\mathbb{R})$ satisfies $f'(1) < 0$ and the Borel-measure $\mu$ has a density function $J \in C(\mathbb{R})$ with

$$\int_{y \in \mathbb{R}} (|y| + e^{-\lambda y})J(y)dy < +\infty$$

for some positive constant $\lambda$. Then, there exists a constant $c_*$ such that the equation (1.1) has a traveling wave solution with monotone profile and speed $c$ when $c \geq c_*$ while it has no such solution when $c < c_*$. Recently, the author [28] also obtained the following:

Theorem ([28]) Suppose there exists a positive constant $\lambda$ such that

$$\int_{y \in \mathbb{R}} e^{\lambda |y|}d\mu(y) < +\infty$$

holds. Then, there exists a constant $c_*$ such that the equation (1.1) has a traveling wave solution with monotone profile and speed $c$ when $c \geq c_*$ while it has no periodic traveling wave solution with average speed $c$ when $c < c_*$. Here, a solution $\{u(t, x)\}_{t \in \mathbb{R}} \subset L^\infty(\mathbb{R})$ to (1.1) is said to be a periodic traveling wave solution with average speed $c$, if there exists a positive constant $\tau$ such that $u(t + \tau, x) = u(t, x + c\tau)$ holds for all $t$ and $x \in \mathbb{R}$ with $0 \leq u(t, x) \leq 1$, $\lim_{x \to +\infty} u(t, x) = 1$ and $\|u(t, x) - 1\|_{L^\infty(\mathbb{R})} \neq 0$.

The goal of this paper is to improve this result of [28], and the following two theorems are the main results:

**Theorem 1** Suppose there exists a positive constant $\lambda$ such that

$$\int_{y \in \mathbb{R}} e^{-\lambda y}d\mu(y) < +\infty$$

holds. Then, there exists a constant $c_*$ such that the equation (1.1) has a traveling wave solution with monotone profile and speed $c$ when $c \geq c_*$ while it has no periodic traveling wave solution with average speed $c$ when $c < c_*$. Here, a solution $\{u(t, x)\}_{t \in \mathbb{R}} \subset L^\infty(\mathbb{R})$ to (1.1) is said to be a periodic traveling wave solution with average speed $c$, if there exists a positive constant $\tau$ such that $u(t + \tau, x) = u(t, x + c\tau)$ holds for all $t$ and $x \in \mathbb{R}$ with $0 \leq u(t, x) \leq 1$, $\lim_{x \to +\infty} u(t, x) = 1$ and $\|u(t, x) - 1\|_{L^\infty(\mathbb{R})} \neq 0$. 


Theorem 2  Suppose the nonlinearity \( f \in C^1(\mathbb{R}) \) satisfies
\[
f'(0) > 0.
\]
Suppose the measure \( \mu \) satisfies
\[
\int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y) = +\infty
\]
for all positive constants \( \lambda \). Then, the equation (1.1) has no periodic traveling wave solution.

In these results, we do not assume that the measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure. For example, not only the integro-differential equation
\[
\frac{\partial u}{\partial t}(t, x) = \int_0^1 u(t, x - y)dy - u(t, x) + f(u(t, x))
\]
but also the discrete equation
\[
\frac{\partial u}{\partial t}(t, x) = u(t, x - 1) - u(t, x) + f(u(t, x))
\]
satisfies the assumption of Theorem 1 for the measure \( \mu \). In order to prove these results, we employ the recursive method for monotone dynamical systems by Weinberger [25] and Li, Weinberger and Lewis [17]. We note that the semiflow generated by the equation (1.1) does not have compactness with respect to the compact-open topology.

Schumacher [21, 22], Carr and Chmaj [3] and Coville, Dávila and Martínez [6] also studied uniqueness of traveling wave solutions. In [6], we could see an interesting example of nonuniqueness, where the equation (1.1) admits infinitely many monotone profiles for standing wave solutions but it admits no continuous one. See, e.g., [5, 7, 10, 11, 12, 13, 14, 15, 18, 19, 23, 24, 26, 27] on traveling waves and long-time behavior in various monostable evolution systems, [2, 4] nonlocal bistable equations and [20] Euler equation.

In Section 2, we recall abstract results for monotone semiflows from [28]. In Section 3, we give basic facts for nonlocal equations in \( L^\infty(\mathbb{R}) \). In Section 4, we prove Theorem 1. In Section 5, we recall a result on spreading speeds by Weinberger [25]. In Section 6, we prove Theorem 2.
2 Abstract results for monotone semiflows

In this section, we recall some abstract results for monotone semiflows from [28]. Put a set of functions on $\mathbb{R}$:

$$\mathcal{M} := \{ u \mid u \text{ is a monotone nondecreasing and left continuous function on } \mathbb{R} \text{ with } 0 \leq u \leq 1 \}.$$ 

The followings are basic conditions for discrete dynamical systems on $\mathcal{M}$:

**Hypotheses 3** Let $Q_0$ be a map from $\mathcal{M}$ into $\mathcal{M}$.

(i) $Q_0$ is continuous in the following sense: If a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$ converges to $u \in \mathcal{M}$ uniformly on every bounded interval, then the sequence $\{Q_0[u_k]\}_{k \in \mathbb{N}}$ converges to $Q_0[u]$ almost everywhere.

(ii) $Q_0$ is order preserving; i.e.,

$$u_1 \leq u_2 \implies Q_0[u_1] \leq Q_0[u_2]$$

for all $u_1$ and $u_2 \in \mathcal{M}$. Here, $u \leq v$ means that $u(x) \leq v(x)$ holds for all $x \in \mathbb{R}$.

(iii) $Q_0$ is translation invariant; i.e.,

$$T_{x_0}Q_0 = Q_0T_{x_0}$$

for all $x_0 \in \mathbb{R}$. Here, $T_{x_0}$ is the translation operator defined by $(T_{x_0}[u])(\cdot) := u(\cdot - x_0)$.

(iv) $Q_0$ is monostable; i.e.,

$$0 < \alpha < 1 \implies \alpha < Q_0[\alpha]$$

for all constant functions $\alpha$.

**Remark** If $Q_0$ satisfies Hypothesis 3 (iii), then $Q_0$ maps constant functions to constant functions.

We add the following conditions to Hypotheses 3 for continuous dynamical systems on $\mathcal{M}$:

**Hypotheses 4** Let $Q := \{Q^t\}_{t \in [0, +\infty)}$ be a family of maps from $\mathcal{M}$ to $\mathcal{M}$.

(i) $Q$ is a semigroup; i.e., $Q^t \circ Q^s = Q^{t+s}$ for all $t$ and $s \in [0, +\infty)$.

(ii) $Q$ is continuous in the following sense: Suppose a sequence $\{t_k\}_{k \in \mathbb{N}} \subset [0, +\infty)$ converges to 0, and $u \in \mathcal{M}$. Then, the sequence $\{Q^{t_k}[u]\}_{k \in \mathbb{N}}$ converges to $u$ almost everywhere.
From [28], we recall the following two results for continuous dynamical systems on $\mathcal{M}$:

**Theorem 5** Let $Q^t$ be a map from $\mathcal{M}$ to $\mathcal{M}$ for $t \in [0, +\infty)$. Suppose $Q^t$ satisfies Hypotheses 3 for all $t \in (0, +\infty)$, and $Q := \{Q^t\}_{t \in [0, +\infty)}$ Hypotheses 4. Then, the following holds:

Let $c \in \mathbb{R}$. Suppose there exist $\tau \in (0, +\infty)$ and $\phi \in \mathcal{M}$ with $(Q^\tau[\phi])(x - ct) \leq \phi(x)$, $\phi \not\equiv 0$ and $\phi \not\equiv 1$. Then, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $(Q^t[\psi])(x - ct) \equiv \psi(x)$ holds for all $t \in [0, +\infty)$.

**Theorem 6** Let $Q^t$ be a map from $\mathcal{M}$ to $\mathcal{M}$ for $t \in [0, +\infty)$. Suppose $Q^t$ satisfies Hypotheses 3 for all $t \in (0, +\infty)$, and $Q := \{Q^t\}_{t \in [0, +\infty)}$ Hypotheses 4. Then, there exists $c^* \in (-\infty, +\infty]$ such that the following holds:

Let $c \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $(Q^t[\psi])(x - ct) \equiv \psi(x)$ holds for all $t \in [0, +\infty)$ if and only if $c \geq c^*$.

3 Basic facts for nonlocal equations in $L^\infty(\mathbb{R})$

In this section, we give some basic facts for the equation

$$(3.1) \quad u_t = \hat{\mu} \ast u + g(u)$$

on the phase space $L^\infty(\mathbb{R})$. First, we have the comparison theorem for (3.1) on $L^\infty(\mathbb{R})$:

**Lemma 7** Let $\hat{\mu}$ be a Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) < +\infty$. Let $g$ be a Lipschitz continuous function on $\mathbb{R}$. Let $T \in (0, +\infty)$, and two functions $u^1$ and $u^2 \in C^1([0, T], L^\infty(\mathbb{R}))$. Suppose that for any $t \in [0, T]$, the inequality

$$(u^1_t - (\hat{\mu} \ast u^1 + g(u^1)) \leq u^2_t - (\hat{\mu} \ast u^2 + g(u^2))$$

holds almost everywhere in $x$. Then, the inequality $u^1(T, x) \leq u^2(T, x)$ holds almost everywhere in $x$ if the inequality $u^1(0, x) \leq u^2(0, x)$ holds almost everywhere in $x$.

**Proof.** Put $K \in \mathbb{R}$ by

$$(3.2) \quad K := -\inf_{h>0,u \in \mathbb{R}} \frac{g(u + h) - g(u)}{h},$$

6
and \( v \in C^1([0, T], L^\infty(\mathbb{R})) \) by
\[
(3.3) \quad v(t) := e^{Kt}(u^2 - u^1)(t).
\]
Then, we have the ordinary differential equation
\[
(3.4) \quad \frac{dv}{dt} = F(t, v)
\]
in \( L^\infty(\mathbb{R}) \) with \( v(0) = (u^2 - u^1)(0) \) as we define a map \( F: [0, T] \times L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}) \) by
\[
F(t, w) := \hat{\mu} \ast w + K w + e^{Kt} \left(g(u^1(t)) + e^{-Kt}w - g(u^1(t))\right) + e^{Kt}a(t),
\]
where
\[
a := \left( \frac{du^2}{dt} - (\hat{\mu} \ast u^2 + g(u^2)) \right) - \left( \frac{du^1}{dt} - (\hat{\mu} \ast u^1 + g(u^1)) \right).
\]
For any \( t \in [0, T] \), we see the inequality
\[
(3.5) \quad a(t, x) \geq 0
\]
almost everywhere in \( x \). Take the solution \( \tilde{v} \in C^1([0, T], L^\infty(\mathbb{R})) \) to
\[
(3.6) \quad \tilde{v}(t) = v(0) + \int_0^t \max\{F(s, \tilde{v}(s)), 0\} ds.
\]
Then, for any \( t \in [0, T] \), we have
\[
(3.7) \quad \tilde{v}(t, x) \geq v(0, x) = (u^2 - u^1)(0, x) \geq 0
\]
almost everywhere in \( x \). By using (3.2), (3.5) and (3.7), for any \( t \in [0, T] \), we also have the inequality \( F(t, \tilde{v}(t)) \geq 0 \) almost everywhere in \( x \). Hence, from (3.6), \( \tilde{v}(t) \) is the solution to the same ordinary differential equation (3.4) in \( L^\infty(\mathbb{R}) \) as \( v(t) \) with \( \tilde{v}(0) = v(0) \). So, in virtue of (3.3) and (3.7),
\[
(u^2 - u^1)(T, x) = e^{-KT}v(T, x) = e^{-KT}\tilde{v}(T, x) \geq 0
\]
holds almost everywhere in \( x \).

The following lemma gives a invariant set and some positively invariant sets of the flow on \( L^\infty(\mathbb{R}) \) generated by the equation (3.1):
Lemma 8 Let $\mu$ be a Borel-measure on $\mathbb{R}$ with $\mu(\mathbb{R}) < +\infty$. Let $g$ be a Lipschitz continuous function on $\mathbb{R}$. Then, the followings hold:

(i) For any $u_0 \in BC(\mathbb{R})$, there exists a solution $\{u(t)\}_{t \in \mathbb{R}} \subset BC(\mathbb{R})$ to (3.1) with $u(0) = u_0$. Here, $BC(\mathbb{R})$ denote the set of bounded and continuous functions on $\mathbb{R}$.

(ii) Suppose a constant $\gamma$ satisfies $\gamma \mu(\mathbb{R}) + g(\gamma) = 0$. If $u_0 \in L^\infty(\mathbb{R})$ satisfies $\gamma \leq u_0$, then there exists a solution $\{u(t)\}_{t \in [0, +\infty)} \subset L^\infty(\mathbb{R})$ to (3.1) with $u(0) = u_0$ and $\gamma \leq u(t)$. If $u_0 \in L^\infty(\mathbb{R})$ satisfies $u_0 \leq \gamma$, then there exists a solution $\{u(t)\}_{t \in [0, +\infty)} \subset L^\infty(\mathbb{R})$ to (3.1) with $u(0) = u_0$ and $u(t) \leq \gamma$.

(iii) If $u_0$ is a bounded and monotone nondecreasing function on $\mathbb{R}$, then there exists a solution $\{u(t)\}_{t \in [0, +\infty)} \subset L^\infty(\mathbb{R})$ to (3.1) with $u(0) = u_0$ such that $u(t)$ is a bounded and monotone nondecreasing function on $\mathbb{R}$ for all $t \in [0, +\infty)$. If $u_0$ is a bounded and monotone nonincreasing function on $\mathbb{R}$, then there exists a solution $\{u(t)\}_{t \in [0, +\infty)} \subset L^\infty(\mathbb{R})$ to (3.1) with $u(0) = u_0$ such that $u(t)$ is a bounded and monotone nonincreasing function on $\mathbb{R}$ for all $t \in [0, +\infty)$.

Proof. We could see (i), because $BC(\mathbb{R})$ is a closed sub-space of the Banach space $L^\infty(\mathbb{R})$ and $u \in BC(\mathbb{R})$ implies $\mu * u + g(u) \in BC(\mathbb{R})$.

We could also see (ii) by using Lemma 7, because the constant $\gamma$ is a solution to (3.1).

We show (iii). Suppose $u_0$ is a bounded and monotone nondecreasing function on $\mathbb{R}$. We take a solution $\{u(t)\}_{t \in [0, +\infty)} \subset L^\infty(\mathbb{R})$ to (3.1) with $u(0) = u_0$. Let $t \in [0, +\infty)$ and $h \in [0, +\infty)$. Then, by Lemma 7, we see $u(t, x) \leq u(t, x + h)$ almost everywhere in $x$. We take a cutoff function $\rho \in C^\infty(\mathbb{R})$ with

$$|x| \geq 1/2 \implies \rho(x) = 0,$$

$$|x| < 1/2 \implies \rho(x) > 0$$

and

$$\int_{x \in \mathbb{R}} \rho(x)dx = 1.$$

As we put

$$v_n(x) := \int_{y \in \mathbb{R}} 2^n \rho(2^n(x - y))u(t, y)dy$$

for $n \in \mathbb{N}$, we see $v_n(x) \leq v_n(x + h)$ for all $x \in \mathbb{R}$. Therefore, $v_n$ is smooth, bounded and monotone nondecreasing. By Helly’s theorem, there
exist a subsequence \(n_k\) and a bounded and monotone nondecreasing function \(\psi\) on \(\mathbb{R}\) such that \(\lim_{k \to \infty} v_{n_k}(x) = \psi(x)\) holds for all \(x \in \mathbb{R}\). Then, \(\|u(t, x) - \psi(x)\|_{L^1([-C, +C])} \leq \lim_{k \to \infty} (\|u(t, x) - v_{n_k}(x)\|_{L^1([-C, +C])} + \|v_{n_k}(x) - \psi(x)\|_{L^1([-C, +C])}) = 0\) holds for all \(C \in (0, +\infty)\). Hence, we obtain \(\|u(t, x) - \psi(x)\|_{L^\infty(\mathbb{R})} = 0\).

**Lemma 9** Let \(\hat{\mu}\) be a Borel-measure on \(\mathbb{R}\) with \(\hat{\mu}(\mathbb{R}) < +\infty\). Let \(\{u_n\}_{n=1}^\infty\) be a sequence of bounded and continuous functions on \(\mathbb{R}\) with

\[
\sup_{n \in \mathbb{N}, x \in \mathbb{R}} |u_n(x)| < +\infty.
\]

Suppose the sequence \(\{u_n\}_{n=1}^\infty\) converges to 0 uniformly on every bounded interval. Then, the sequence \(\{\hat{\mu} * u_n\}_{n=1}^\infty\) converges to 0 uniformly on every bounded interval.

**Proof.** Let \(\varepsilon \in (0, +\infty)\). We take a positive constant \(C\) such that

\[
\left( \sup_{n \in \mathbb{N}, x \in \mathbb{R}} |u_n(x)| \right) \hat{\mu}(\mathbb{R} \setminus (-C, +C)) \leq \varepsilon
\]

holds. Then, because

\[
|\hat{\mu} * u_n(x)| \leq \int_{y \in (-C, +C)} |u_n(x-y)|d\hat{\mu}(y) + \int_{y \in \mathbb{R} \setminus (-C, +C)} |u_n(x-y)|d\hat{\mu}(y)
\]

\[
\leq \left( \sup_{y \in (-C, +C)} |u_n(x-y)| \right) \hat{\mu}(\mathbb{R}) + \left( \sup_{y \in \mathbb{R}} |u_n(x-y)| \right) \hat{\mu}(\mathbb{R} \setminus (-C, +C))
\]

holds, we have

\[
\sup_{x \in [-I, +I]} |\hat{\mu} * u_n(x)| \leq \left( \sup_{y \in (-I+I), +(I+I)} |u_n(y)| \right) \hat{\mu}(\mathbb{R}) + \varepsilon
\]

for all \(I \in (0, +\infty)\). Hence, we obtain

\[
\limsup_{n \to \infty} \sup_{x \in [-I, +I]} |\hat{\mu} * u_n(x)| \leq \varepsilon
\]

for all \(I \in (0, +\infty)\). ■
Proposition 10  Let \( \hat{\mu} \) be a Borel-measure on \( \mathbb{R} \) with \( \hat{\mu}(\mathbb{R}) < +\infty \), \( g \) a Lipschitz continuous function on \( \mathbb{R} \), and \( T \) a positive constant. Let a sequence \( \{u_n\}_{n=0}^{\infty} \subset C^1([0,T], L^\infty(\mathbb{R})) \) of solutions to the equation (3.1) satisfy

\[
\sup_{n \in \mathbb{N}, x \in \mathbb{R}} |u_n(0, x) - u_0(0, x)| < +\infty.
\]

Suppose

\[
\lim_{n \to \infty} \sup_{x \in [-I, I]} |u_n(0, x) - u_0(0, x)| = 0
\]

holds for all positive constants \( I \). Then,

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \|u_n(t, x) - u_0(t, x)\|_{L^\infty([-J,J])} = 0
\]

holds for all positive constants \( J \).

Proof.  First, we take a sequence \( \{w_n\}_{n=1}^{\infty} \) of nonnegative, bounded and continuous functions on \( \mathbb{R} \) with

(3.8) \[
\sup_{n \in \mathbb{N}, x \in \mathbb{R}} |w_n(x)| < +\infty
\]

such that \( \{w_n\}_{n=1}^{\infty} \) converges to 0 uniformly on every bounded interval and

(3.9) \[
|u_n(0, x) - u_0(0, x)| \leq w_n(x)
\]

holds for all \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \). Let \( \hat{A} \) denote the bounded and linear operator from the Banach space \( BC(\mathbb{R}) \) to \( BC(\mathbb{R}) \) defined by

\[
\hat{A}w := \hat{\mu} * w.
\]

From (3.8), we see \( \sup_{n \in \mathbb{N}, x \in \mathbb{R}} |(\hat{A}^k w_n)(x)| < +\infty \) for all \( k = 0, 1, 2, \ldots \). Hence, because of \( \lim_{n \to \infty} \sup_{x \in [-I, I]} |w_n(x)| = 0 \) for all \( I \in (0, +\infty) \), by Lemma 9, we have

(3.10) \[
\lim_{n \to \infty} \sup_{x \in [-J, J]} |(\hat{A}^k w_n)(x)| = 0
\]

for all \( J \in (0, +\infty) \) and \( k = 0, 1, 2, \ldots \).

Let \( \gamma \) denote the constant defined by

\[
\gamma := \sup_{h > 0, u \in \mathbb{R}} \frac{g(u + h) - g(u)}{h}.
\]
Then, we consider the following two sequences \( \{v_n\}_{n=1}^{\infty} \) and \( \{\overline{v}_n\}_{n=1}^{\infty} \subset C^1([0,T],L^\infty(\mathbb{R})) \) defined by
\[
v_n(t,x) := u_0(t,x) - e^{\gamma t}(e^{A t}w_n)(x)
\]
and
\[
\overline{v}_n(t,x) := u_0(t,x) + e^{\gamma t}(e^{A t}w_n)(x).
\]
Because \( (e^{A t}w_n)(x) \) is nonnegative for all \( n \in \mathbb{N} \), \( t \in [0,\infty) \) and \( x \in \mathbb{R} \), the function \( v_n \) is a sub-solution to (3.1) and \( \overline{v}_n \) is a super-solution to (3.1) for all \( n \in \mathbb{N} \). So, by Lemma 7 and (3.9), for any \( n \in \mathbb{N} \) and \( t \in [0,T] \),
\[
|u_n(t,x) - u_0(t,x)| \leq e^{\gamma t}(e^{A t}w_n)(x)
\]
holds almost everywhere in \( x \).

Let \( \varepsilon \in (0,\infty) \). We take \( N \in \mathbb{N} \) such that
\[
(1 + e^{\gamma T}) \left( \sum_{k=N}^{\infty} \frac{T\|\hat{A}\|_{BC(\mathbb{R}) \to BC(\mathbb{R})}^k}{k!} \right) \left( \sup_{n \in \mathbb{N}, x \in \mathbb{R}} |w_n(x)| \right) \leq \varepsilon
\]
holds. Then, in virtue of (3.11), we see
\[
\|u_n(t,x) - u_0(t,x)\|_{L^\infty([-J,+J])} \leq \sup_{x \in [-J,+J]} |e^{\gamma t}(e^{A t}w_n)(x)|
\]
\[
= e^{\gamma t} \left( \sup_{x \in [-J,+J]} \left| \left( \sum_{k=0}^{N-1} \frac{T^k}{k!} (\hat{A}^k w_n)(x) \right) + \left( \left( \sum_{k=N}^{\infty} \frac{T^k}{k!} \hat{A}^k \right) w_n \right)(x) \right| \right)
\]
\[
\leq (1 + e^{\gamma T}) \left( \sum_{k=0}^{N-1} \frac{T^k}{k!} \left( \sup_{x \in [-J,+J]} |(\hat{A}^k w_n)(x)| \right) \right) + \varepsilon
\]
for all \( J \in (0,\infty) \), \( n \in \mathbb{N} \) and \( t \in [0,T] \). So, by (3.10), we obtain
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \|u_n(t,x) - u_0(t,x)\|_{L^\infty([-J,+J])} \leq \varepsilon
\]
for all \( J \in (0,\infty) \). ■
4  Proof of Theorem 1

In this section, we prove Theorem 1 by using the results of Sections 2 and 3. The argument in this section is almost similar as in [28]. First, we recall that $\mu$ is a Borel-measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$, $f$ is a Lipschitz continuous function on $\mathbb{R}$ with $f(0) = f(1) = 0$ and $f > 0$ in $(0, 1)$ and the set $\mathcal{M}$ has been defined at the beginning of Section 2. Then, in virtue of Lemmas 7, 8 and Proposition 10, $Q^t$ ($t \in (0, +\infty)$) satisfies Hypotheses 3 and $Q$ Hypotheses 4 for the semiflow $Q = \{Q^t\}_{t \in [0, +\infty)}$ on $\mathcal{M}$ generated by (1.1). So, Theorems 5 and 6 can work for this semiflow on $\mathcal{M}$.

If the flow on $L^\infty(\mathbb{R})$ generated by (1.1) has a periodic traveling wave solution with average speed $c$ (even if the profile is not a monotone function), then it has a traveling wave solution with monotone profile and speed $c$:

**Theorem 11** Let $c \in \mathbb{R}$. Suppose there exist a positive constant $\tau$ and a solution $\{u(t, x)\}_{t \in \mathbb{R}} \subset L^\infty(\mathbb{R})$ to (1.1) with $0 \leq u(t, x) \leq 1$, $\lim_{x \to +\infty} u(t, x) = 1$ and $\|u(t, x) - 1\|_{L^\infty(\mathbb{R})} \neq 0$ such that

$$u(t + \tau, x) = u(t, x + c\tau)$$

holds for all $t$ and $x \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $\{\psi(x + ct)\}_{t \in \mathbb{R}}$ is a solution to (1.1).

**Proof.** Put two monotone nondecreasing functions $\varphi(x) := \max\{\alpha \in \mathbb{R} \mid \alpha \leq u(0, y) \text{ holds almost everywhere in } y \in (x, +\infty)\}$ and $\phi(x) := \lim_{h \to +0} \varphi(x - h)$. Then, $\phi \in \mathcal{M}$, $\phi(-\infty) < 1$ and $\phi(+\infty) = 1$ hold. We take a cutoff function $\rho \in C^\infty(\mathbb{R})$ with

$$|x + 1/2| \geq 1/2 \implies \rho(x) = 0,$$

$$|x + 1/2| < 1/2 \implies \rho(x) > 0$$

and

$$\int_{x \in \mathbb{R}} \rho(x) dx = 1.$$ 

As we put

$$v_n(x) := \int_{y \in \mathbb{R}} 2^n \rho(2^n(x - y))u(0, y)dy$$

for $n \in \mathbb{N}$, we see $\phi \leq v_n$. Let $N \in \mathbb{N}$. Because of $\lim_{n \to +}\|v_n(x) - u(0, x)\|_{L^1([-N, +N])} = 0$, there exists a subsequence $n_k$ such that $\lim_{k \to +}\|v_{n_k}(x) - u(0, x)\|_{L^1([-N, +N])} = 0$, there exists a subsequence $n_k$ such that $\lim_{k \to +}\|v_{n_k}(x) - u(0, x)\|_{L^1([-N, +N])} = 0$, there exists a subsequence $n_k$ such that $\lim_{k \to +}\|v_{n_k}(x) - u(0, x)\|_{L^1([-N, +N])} = 0$, there exists a subsequence $n_k$ such that $\lim_{k \to +}\|v_{n_k}(x) - u(0, x)\|_{L^1([-N, +N])} = 0$. Theorem 11
= u(0, x) almost everywhere in \( x \in [-N, +N] \). Therefore, we have \( \phi(x) \leq u(0, x) \) almost everywhere in \( x \in \mathbb{R} \). So, by Lemma 7, we obtain \( Q^r[\phi](x - c\tau) \leq u(\tau, x - c\tau) = u(0, x) \) almost everywhere in \( x \). Hence, because \( Q^r[\phi](x - c\tau) \leq \varphi(x) \) holds, we get \( Q^r[\phi](x - c\tau) \leq \phi(x) \). Therefore, by Theorem 5, there exists \( \psi \in \mathcal{M} \) with \( \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \) such that \( Q^t[\psi](x - ct) \equiv \psi(x) \) holds for all \( t \in [0, +\infty) \).

The infimum \( c_* \) of the speeds of traveling wave solutions is not \(-\infty\), and there is a traveling wave solution with speed \( c \) when \( c \geq c_* \):

**Lemma 12** There exists \( c_* \in (-\infty, +\infty) \) such that the following holds:

Let \( c \in \mathbb{R} \). Then, there exists \( \psi \in \mathcal{M} \) with \( \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \) such that \( \{\psi(x + ct)\}_{t \in \mathbb{R}} \) is a solution to (1.1) if and only if \( c \geq c_* \).

**Proof.** It follows from Theorem 6.

**Proof of Theorem 1.**

Let \( c_* \) be the infimum of the speeds of traveling wave solutions with monotone profile. Then, in virtue of Theorem 11 and Lemma 12, it is sufficient if we show \( c_* \neq +\infty \).

Take \( K \in [0, +\infty) \) such that

\[
K \geq \max \left\{ \int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y), \mu(\mathbb{R}) \right\} - 1 + \sup_{h > 0} \frac{f(h)}{h}.
\]

As we put \( \phi(x) := \min\{e^{\lambda x}, 1\} \in \mathcal{M} \), we see

\[
(\mu * \phi)(x) \leq \min \left\{ \left( \int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y) \right) e^{\lambda x}, \mu(\mathbb{R}) \right\}
\]

\[
\leq \max \left\{ \int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y), \mu(\mathbb{R}) \right\} \phi(x).
\]

So, \( e^{Kt}\phi(x) \) is a super-solution to (1.1), because of

\[
e^{Kt}(\mu * \phi) - e^{Kt}\phi + f(e^{Kt}\phi) \leq Ke^{Kt}\phi.
\]

Hence, by Lemma 7, we obtain \( Q^t[\phi](x) \leq e^{Kt}\phi(x) \leq e^{\lambda(x + \frac{K}{\lambda})} \) and \( Q^t[\phi](x - \frac{K}{\lambda}t) \leq \phi(x) \). Therefore, from Theorem 5, there exists \( \psi \in \mathcal{M} \) with \( \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \) such that \( Q^t[\psi](x - \frac{K}{\lambda}t) \equiv \psi(x) \) holds for all \( t \in [0, +\infty) \).

So, \( c_* \leq \frac{K}{\lambda} \) holds.
5 A result on spreading speeds by Weinberger

In this section, we recall a result by Weinberger [25]. It is used to prove Theorem 2 in Section 6. Put a set of functions on \( \mathbb{R} \):

\[
\mathcal{B} := \{ u \mid u \text{ is a continuous function on } \mathbb{R} \text{ with } 0 \leq u \leq 1 \}.
\]

**Hypotheses 13** Let \( \tilde{Q}_0 \) be a map from \( \mathcal{B} \) into \( \mathcal{B} \).

(i) \( \tilde{Q}_0 \) is continuous in the following sense: If a sequence \( \{u_k\}_{k \in \mathbb{N}} \subset \mathcal{B} \) converges to \( u \in \mathcal{B} \) uniformly on every bounded interval, then the sequence \( \{(\tilde{Q}_0[u_k])(x)\}_{k \in \mathbb{N}} \) converges to \((\tilde{Q}_0[u])(x)\) for all \( x \in \mathbb{R} \).

(ii) \( \tilde{Q}_0 \) is order preserving; i.e.,

\[
u_1 \leq u_2 \implies \tilde{Q}_0[u_1] \leq \tilde{Q}_0[u_2]
\]

for all \( u_1, u_2 \in \mathcal{B} \). Here, \( u \leq v \) means that \( u(x) \leq v(x) \) holds for all \( x \in \mathbb{R} \).

(iii) \( \tilde{Q}_0 \) is translation invariant; i.e.,

\[
T_{x_0}\tilde{Q}_0 = \tilde{Q}_0 T_{x_0}
\]

for all \( x_0 \in \mathbb{R} \). Here, \( T_{x_0} \) is the translation operator defined by \((T_{x_0}[u])(\cdot) := u(\cdot - x_0)\).

(iv) \( \tilde{Q}_0 \) is monostable; i.e.,

\[
0 < \alpha < 1 \implies \alpha < \tilde{Q}_0[\alpha]
\]

for all constant functions \( \alpha \), and \( \tilde{Q}_0[0] = 0 \).

**Remark** If \( \tilde{Q}_0 \) satisfies Hypotheses 13 (ii) and (iii), then \( \tilde{Q}_0 \) maps monotone functions to monotone functions.

**Theorem 14** Let a map \( \tilde{Q}_0 : \mathcal{B} \to \mathcal{B} \) satisfy Hypotheses 13. Let a continuous and monotone nonincreasing function \( \varphi \) on \( \mathbb{R} \) with \( 0 < \varphi(-\infty) < 1 \) satisfy \( \varphi(x) = 0 \) for all \( x \in [0, +\infty) \). For \( c \in \mathbb{R} \), define the sequence \( \{a_{c,n}\}_{n=0}^{\infty} \) of continuous and monotone nonincreasing functions on \( \mathbb{R} \) by the recursion

\[
a_{c,n+1}(x) := \max\{(\tilde{Q}_0[a_{c,n}])(x + c), \varphi(x)\}
\]
with $a_{c,0} := \varphi$. Then,
\[ 0 \leq a_{c,n} \leq a_{c,n+1} \leq 1 \]
holds for all $c \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$. For $c \in \mathbb{R}$, define the bounded and monotone nonincreasing function $a_c$ on $\mathbb{R}$ by
\[ a_c(x) := \lim_{n \to \infty} a_{c,n}(x). \]

Let $\tilde{\nu}$ be a Borel-measure on $\mathbb{R}$ with $1 < \tilde{\nu}(\mathbb{R}) < +\infty$. Suppose there exists a positive constant $\varepsilon$ such that the inequality
\[ \tilde{\nu} \ast u \leq \tilde{Q}_0[u] \]
holds for all $u \in \mathcal{B}$ with $u \leq \varepsilon$. Then, the inequality
\[ \inf_{\lambda > 0} \frac{1}{\lambda} \log \int_{y \in \mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) \leq \sup \{ c \in \mathbb{R} | a_c(+\infty) = 1 \} \]
holds.

**Proof.** It follows from Lemma 5.4 and Theorem 6.4 in [25] with $N := 1$, $\mathcal{H} := \mathbb{R}$, $\pi_0 := 0$, $\pi_1 = \pi_+ := 1$, $S^{N-1} := \{ \pm 1 \}$ and $\xi := +1$. ■

From Theorem 14, we have the following:

**Proposition 15** Let $\hat{\mu}$ be a Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) = 1$. Let $c_0 \in \mathbb{R}$, and $\hat{\psi}$ be a monotone nonincreasing function on $\mathbb{R}$ with $\hat{\psi}(-\infty) = 1$ and $\hat{\psi}(+\infty) = 0$. Suppose $\{ \hat{\psi}(x - c_0 t) \}_{t \in \mathbb{R}} \subset L^\infty(\mathbb{R})$ is a solution to
\[ u_t = \hat{\mu} \ast u - u + f(u). \]
(5.1)

Let $\tilde{Q}_0 : \mathcal{B} \to \mathcal{B}$ be the time 1 map of the semiflow on $\mathcal{B}$ generated by the equation (5.1). Let $\tilde{\nu}$ be a Borel-measure on $\mathbb{R}$ with $1 < \tilde{\nu}(\mathbb{R}) < +\infty$. Suppose there exists a positive constant $\varepsilon$ such that the inequality
\[ \tilde{\nu} \ast u \leq \tilde{Q}_0[u] \]
holds for all $u \in \mathcal{B}$ with $u \leq \varepsilon$. Then, the inequality
\[ \inf_{\lambda > 0} \frac{1}{\lambda} \log \int_{y \in \mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) \leq c_0 \]
holds.
Proof. We take a continuous and monotone nonincreasing function \( \varphi \) on \( \mathbb{R} \) with \( 0 < \varphi(-\infty) < 1 \) and \( \varphi(x) = 0 \) for all \( x \in [0, +\infty) \). For \( c \in \mathbb{R} \), we define the sequence \( \{a_{c,n}\}_{n=0}^{\infty} \) of continuous and monotone nonincreasing functions on \( \mathbb{R} \) by the recursion
\[
a_{c,n+1}(x) := \max\{(\tilde{Q}_0[a_{c,n}])(x + c), \varphi(x)\}
\]
with \( a_{c,0} := \varphi \). We also take \( x_0 \in \mathbb{R} \) such that
\[
\varphi(x) \leq \hat{\psi}(x - x_0)
\]
holds for all \( x \in \mathbb{R} \).

Let \( c \in [c_0, +\infty) \). Then, we show \( a_{c,n}(x) \leq \hat{\psi}(x - x_0) \) for all \( n = 0, 1, 2, \cdots \). We have \( a_{c,0}(x) = \varphi(x) \leq \hat{\psi}_0(x - x_0) \). As \( a_{c,n}(x) \leq \hat{\psi}(x - x_0) \) holds almost everywhere in \( x \),
\[
a_{c,n+1}(x) \leq \max\{(\tilde{Q}_0[a_{c,n}])(x + c_0), \varphi(x)\} \\
\leq \max\{\hat{\psi}(x - x_0), \varphi(x)\} = \hat{\psi}(x - x_0)
\]
also holds almost everywhere in \( x \), because \( \hat{\psi}(x - x_0 - c_0t) \) is a solution to (5.1). So, for any \( n = 0, 1, 2, \cdots \), the inequality \( a_{c,n}(x) \leq \hat{\psi}(x - x_0) \) holds almost everywhere in \( x \). Hence, because \( a_{c,n} \) is continuous and \( \hat{\psi} \) is monotone, we have
\[
a_{c,n}(x) \leq \hat{\psi}(x - x_0)
\]
for all \( x \in \mathbb{R} \), \( c \in [c_0, +\infty) \) and \( n = 0, 1, 2, \cdots \). Therefore, by Theorem 14, (5.2) and \( \hat{\psi}(+\infty) = 0 \), the inequality
\[
\inf_{\lambda > 0} \frac{1}{\lambda} \log \int_{y \in \mathbb{R}} e^{\lambda y} d\bar{\nu}(y) \leq \sup(\mathbb{R} \setminus [c_0, +\infty)) = c_0
\]
holds. \( \blacksquare \)

6 Proof of Theorem 2

In this section, we prove Theorem 2. First, we give a basic fact for the linear equation
\[
v_t = \hat{\mu} * v
\]
on the phase space \( BC(\mathbb{R}) \):
Lemma 16  Let $\hat{\mu}$ be a Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) < +\infty$. Let $\hat{P} : BC(\mathbb{R}) \to BC(\mathbb{R})$ be the time 1 map of the flow on $BC(\mathbb{R})$ generated by the linear equation (6.1). Then, there exists a Borel-measure $\hat{\nu}$ on $\mathbb{R}$ with $\hat{\nu}(\mathbb{R}) < +\infty$ such that

$$\hat{P}[v] = \hat{\nu} * v$$

holds for all $v \in BC(\mathbb{R})$. Further, if $v$ is a nonnegative, bounded and continuous function on $\mathbb{R}$, then the inequality

$$v + \hat{\mu} * v \leq \hat{\nu} * v$$

holds.

Proof. Put a functional $\tilde{P} : BC(\mathbb{R}) \to \mathbb{R}$ as

$$\tilde{P}[v] := (\hat{P}[v])(0).$$

Then, the functional $\tilde{P}$ is linear, bounded and positive. Hence, there exists a Borel-measure $\tilde{\nu}$ on $\mathbb{R}$ with $\tilde{\nu}(\mathbb{R}) < +\infty$ such that if a continuous function $v$ on $\mathbb{R}$ satisfies $\lim_{|x| \to \infty} v(x) = 0$, then

$$(6.2) \quad \tilde{P}[v] = \int_{y \in \mathbb{R}} v(y)d\tilde{\nu}(y)$$

holds.

Let $v \in BC(\mathbb{R})$. Then, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset BC(\mathbb{R})$ with $\sup_{n \in \mathbb{N}, x \in \mathbb{R}} |v_n(x)| < +\infty$ and $\lim_{|x| \to \infty} v_n(x) = 0$ for all $n \in \mathbb{N}$ such that $v_n \to v$ as $n \to \infty$ uniformly on every bounded interval. From Proposition 10, (6.2) and $\tilde{\nu}(\mathbb{R}) < +\infty$, we have

$$\tilde{P}[v] = \lim_{n \to \infty} \tilde{P}[v_n] = \lim_{n \to \infty} \int_{y \in \mathbb{R}} v_n(y)d\tilde{\nu}(y) = \int_{y \in \mathbb{R}} v(y)d\tilde{\nu}(y).$$

We take a Borel-measure $\hat{\nu}$ on $\mathbb{R}$ with $\hat{\nu}(\mathbb{R}) < +\infty$ such that

$$\hat{\nu}((\infty, y)) = \hat{\nu}((-y, +\infty))$$

holds for all $y \in \mathbb{R}$. Then, for any $v \in BC(\mathbb{R})$, we have

$$(\hat{P}[v])(x) \equiv \tilde{P}[v(\cdot + x)] \equiv \int_{y \in \mathbb{R}} v(y + x)d\hat{\nu}(y) \equiv (\hat{\nu} * v)(x).$$
Let \( v \) be a nonnegative, bounded and continuous function on \( \mathbb{R} \). Then, in \( t \in [0, +\infty) \), the function
\[
u(t, x) := v(x) + t(\hat{\mu} * v)(x)
\]
is a sub-solution to (6.1), because of \( v(x) \leq \nu(t, x) \). Hence,
\[
v + \hat{\mu} * v \leq \tilde{P}[v]
\]
holds.

Lemma 17 Let \( \hat{\mu} \) be a Borel-measure on \( \mathbb{R} \) with \( \hat{\mu}(\mathbb{R}) < +\infty \). Suppose a constant \( \gamma \) and a Lipschitz continuous function \( g \) on \( \mathbb{R} \) with \( g(0) = 0 \) satisfy \( \gamma < g'(0) \). Let \( \tilde{P} : BC(\mathbb{R}) \to BC(\mathbb{R}) \) be the time 1 map of the flow on \( BC(\mathbb{R}) \) generated by the linear equation
\[
u_t = \hat{\mu} * \nu + \gamma \nu.
\]
(6.3)
Let \( \tilde{P}_0 : BC(\mathbb{R}) \to BC(\mathbb{R}) \) be the time 1 map of the flow on \( BC(\mathbb{R}) \) generated by the equation
\[
u_t = \hat{\mu} * \nu + g(\nu).
\]
(6.4)
Then, there exists a positive constant \( \varepsilon \) such that the inequality
\[
\tilde{P}[\nu] \leq \tilde{P}_0[\nu]
\]
holds for all \( \nu \in BC(\mathbb{R}) \) with \( 0 \leq \nu \leq \varepsilon \).

Proof. We take a positive constant \( \varepsilon \) such that
\[
h \in [0, (1 + e^{\hat{\mu}(\mathbb{R}) + \gamma})\varepsilon] \implies \gamma h \leq g(h)
\]
(6.5)
holds. Let a function \( \nu \in BC(\mathbb{R}) \) satisfy \( 0 \leq \nu \leq \varepsilon \). Then, we take the solution \( \tilde{v}(t, x) \) to (6.3) with \( \tilde{v}(0, x) = \nu(x) \). We see
\[
0 \leq \tilde{v}(t, x) \leq e^{(\hat{\mu}(\mathbb{R}) + \gamma)t}\varepsilon \leq (1 + e^{\hat{\mu}(\mathbb{R}) + \gamma})\varepsilon
\]
for all \( t \in [0, 1] \). Hence, from (6.5), in \( t \in [0, 1] \), the function \( \tilde{v}(t, x) \) is a sub-solution to (6.4). So, the inequality
\[
(\tilde{P}[\nu])(x) = \tilde{v}(1, x) \leq (\tilde{P}_0[\nu])(x)
\]
holds.

We use Proposition 15, Lemmas 16 and 17 to show the following:
Lemma 18 Let \( f'(0) > 0 \). Suppose there exist \( c \in \mathbb{R} \) and \( \psi \in \mathcal{M} \) with \( \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \) such that \( \{ \psi(x + ct) \}_{t \in \mathbb{R}} \) is a solution to (1.1). Then, there exists a positive constant \( \lambda \) such that
\[
\int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y) < +\infty
\]
holds.

**Proof.** Let \( \hat{\mu} \) be the Borel-measure on \( \mathbb{R} \) with \( \hat{\mu}(\mathbb{R}) = 1 \) such that
\[
\hat{\mu}((-\infty, y)) = \mu((-y, +\infty))
\]
holds for all \( y \in \mathbb{R} \). Let \( \hat{P} : BC(\mathbb{R}) \to BC(\mathbb{R}) \) be the time 1 map of the flow on \( BC(\mathbb{R}) \) generated by the linear equation (6.1). Then, by Lemma 16, there exists a Borel-measure \( \hat{\nu} \) on \( \mathbb{R} \) with \( \hat{\nu}(\mathbb{R}) < +\infty \) such that for any \( v \in BC(\mathbb{R}) \),
\begin{equation}
\hat{P}[v] = \hat{\nu} * v
\end{equation}
holds and for any nonnegative, bounded and continuous function \( v \) on \( \mathbb{R} \),
\begin{equation}
\hat{\mu} * v \leq \hat{\nu} * v
\end{equation}
holds. Let \( \tilde{P} : BC(\mathbb{R}) \to BC(\mathbb{R}) \) be the time 1 map of the flow on \( BC(\mathbb{R}) \) generated by the linear equation
\[
v_t = \hat{\mu} * v - v + \frac{f'(0)}{2} v.
\]
Then, from (6.6) and (6.7), as \( \tilde{\nu} \) is the Borel-measure on \( \mathbb{R} \) defined by
\[
\tilde{\nu} := e^{-1 + \frac{f'(0)}{2}} \hat{\nu},
\]
we have
\begin{equation}
\tilde{P}[v] = \tilde{\nu} * v
\end{equation}
for all \( v \in BC(\mathbb{R}) \) and
\begin{equation}
\hat{\mu} * v \leq e^{1 - \frac{f'(0)}{2}} (\tilde{\nu} * v)
\end{equation}
for all \( v \in BC(\mathbb{R}) \).
for all nonnegative, bounded and continuous functions \( v \) on \( \mathbb{R} \). Because 
\[
\tilde{\nu}(\mathbb{R}) = (\tilde{\nu} \ast 1)(0) = (\tilde{P}[1])(0) = e^{\frac{f'(0)}{2}}
\]
holds from (6.8), we also have
\[
(6.10) \quad 1 < \tilde{\nu}(\mathbb{R}) < +\infty.
\]

Let \( \tilde{Q}_0 : B \to B \) be the time 1 map of the semiflow on \( B \) generated by the equation (5.1). Then, from Lemma 17 and (6.8), there exists a positive constant \( \varepsilon \) such that the inequality
\[
\tilde{\nu} \ast u = \tilde{P}[u] \leq \tilde{Q}_0[u]
\]
holds for all \( u \in B \) with \( u \leq \varepsilon \). Further, \( \hat{\psi}(x - ct) := \psi(-(x - ct)) \) is a solution to (5.1). Therefore, by Proposition 15 and (6.10), we obtain the inequality
\[
\inf_{\lambda > 0} \frac{1}{\lambda} \log \int_{y \in \mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) \leq c.
\]
So, there exists a positive constant \( \lambda \) such that
\[
\int_{y \in \mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) \leq e^{\lambda(c+1)} < +\infty
\]
holds. Hence, from (6.9),
\[
\int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y) = \int_{y \in \mathbb{R}} e^{\lambda y} d\hat{\mu}(y) = \lim_{n \to \infty} \int_{y \in \mathbb{R}} \min\{e^{\lambda y}, n\} d\hat{\mu}(y)
\]
\[
= \lim_{n \to \infty} (\hat{\mu} \ast \min\{e^{-\lambda x}, n\})(0) \leq e^{1 - \frac{f'(0)}{2}} \lim_{n \to \infty} (\tilde{\nu} \ast \min\{e^{-\lambda x}, n\})(0)
\]
\[
= e^{1 - \frac{f'(0)}{2}} \lim_{n \to \infty} \int_{y \in \mathbb{R}} \min\{e^{\lambda y}, n\} d\tilde{\nu}(y) = e^{1 - \frac{f'(0)}{2}} \int_{y \in \mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) < +\infty
\]
holds.

\[\blacksquare\]

**Proof of Theorem 2.**

It follows from Theorem 11 and Lemma 18.

\[\blacksquare\]

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