Constructions of Dicke states in high spin multi-particle systems

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We study the constructions of Dicke states of identical particles of spin-1, 3/2 and 2 in the number representation with given particle number $N$ and magnetic quantum number $M$. The complete bases and corresponding coefficients in the Dicke states are given, in terms of which the Dicke states are explicitly expressed in the number representation. As a byproduct, a rule of how to construct all the anti-symmetric states in these high spin systems is given. Finally, by employing the negativity as the entanglement measure, we explore the entanglement properties for spin-1 cases including certain pure states of two particles and many-particle Dicke states.

I. INTRODUCTION

The Dicke state, put forward by Dicke in 1954, is a multi-particle state of spin-1/2 with the maximal total angular momentum [1]. During the past decades, it is under extensive researches and some new features have been found. Especially, it has become a basic state as the development of quantum information science. Based on the Dicke states, one can construct several new quantum states such as GHZ states, W states, squeezed spin states and spin coherence states, which are very important in quantum information theory [2–6]. The original Dicke state focuses on the spin-1/2 case. However, the situations of spin $s$ or angular momentum $j$ more than 1/2 have emerged their importance and attracted much attention as the development of low temperature physics. The system of many $^{23}$Na atoms trapped in an optical lattice is spin-1 [10], and the system of many $^{132}$Cs or $^{135}$Ba atoms is spin-3/2 [11, 12]. For these high-spin systems, there may exist some hidden symmetries, strong quantum fluctuations and novel phases [13]. For instance, Haldane predicted that the one dimensional Heisenberg chain has a spin gap for integer value of spin [15, 16]. Wang et al. studied the entanglement properties in a spin-1 Heisenberg chain [17, 18]. The eigenstates and magnetic response in spin-1 and 2 Bose-Einstein condensates were discussed by Koashi [19]. These systems with high spin have more spin orientations and more quantum eigenstates such that richer physical phenomena can emerge. Therefore, it is desirable to construct the basic quantum states based on Dicke states for the high-spin cases.

It is well known that the single particle is a qubit with only two magnetic components for a spin-1/2 many-body system. Thus, the configuration of the Dicke state $|J, M\rangle$ for spin-1/2 is simplest. For given total spin $J$ and total magnetic component $M$, one can find the explicit form for the its Dicke state by means of a binary linear equation group accompanied with normalization and symmetry constraints. However, for high-spin many-body systems, the construction of the Dicke state is not a easy task due to much more spin components. The conventional approach of $3j$ symbol in quantum mechanics is appropriate only for the case of small particle number $N$ and one should seek a different route for the case of large particle number $N$. In a word, the investigation of Dicke states for high-spin cases is a nontrivial but troublesome task which may be considered as a supplement to the modern quantum mechanics. Motivated by the construction of spin-1/2 Dicke state, in this work, we explore the Dicke states of identical spin-$s$ particles with $s = 1$, 3/2 and 2.

This paper is organized as follows. In Sec. V, we find those complete bases and corresponding coefficients in the Dicke state $|J, M\rangle$. In Sec. III, we study the anti-symmetric states in high spin systems. In Sec. IV, in terms of the negativity, the entanglement of two spin-1 particle is discussed. We conclude in Sec. V.

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II. CONSTRUCTION OF DICKE STATES IN HIGH SPIN MULTI-PARTICLE SYSTEM

A. Dicke states in spin-1/2 multi-particle system

First, let us recall the derivation process of Dicke states in spin-1/2 multi-particle system. For a multi-particle system consisting of identical spin-1/2 particles, the states $|J, M\rangle$, which possess the maximal total spin angular momentum, are termed as the Dicke states. Obviously, the states $|J, J\rangle$ and $|J, -J\rangle$ possess the simple form

$$|J, J\rangle = |\frac{1}{2}, \cdots \otimes \frac{1}{2}\rangle,$$

$$|J, -J\rangle = |\frac{1}{2}, \cdots \otimes -\frac{1}{2}\rangle.$$  \hspace{1cm} (1)

By introducing the collective raising and lowering operators $J_{\pm} = \sum s_{i\pm}$ and using the relation

$$J_{\pm}|J, M\rangle = \sqrt{(J \pm M)(J \pm M + 1)}|J, M \pm 1\rangle,$$

$$s_{i\pm}|s, m_s\rangle = \sqrt{(s \pm m_s)(s \pm m_s + 1)}|s, m_s \pm 1\rangle,$$  \hspace{1cm} (2)

one can obtain all the other Dicke states $|J, M\rangle$ from the states $|J, \pm J\rangle$ although the repeating process may be tedious. Here, $s$ is the spin of single particle, $s_{i\pm}$ are raising and lowering operators for the $i$th particle, and $m_s$ is the eigenvalue of $s_z$. However, this method does not have any advantage for multi-qubit system, especially, for the high-spin systems. Thereby, it is desirable to find other ways to express Dicke states of multi-particle systems. In what follows, we will demonstrate an alternative approach via the number representation.

Supposing the eigenstates of the operator $s_z$ of spin-1/2 angular momentum are $|\frac{1}{2}\rangle$ and $| -\frac{1}{2}\rangle$ and the numbers of particles occupying the two states are $n_1$ and $n_2$, respectively, we employ the number representation \{\{n_1, n_2\}\} with

$$|n_1, n_2\rangle = \sqrt{n_1!n_2!N!} \sum_{n_1} P\left(\begin{array}{c} 1/2 \cdot \cdots \cdot 1/2 \\ n_1 \end{array} \right) \cdot \sum_{n_2} P\left(\begin{array}{c} 1/2 \cdot \cdots \cdot 1/2 \\ n_2 \end{array} \right),$$  \hspace{1cm} (3)

where $N = n_1 + n_2$ is the total particle number and $P$ denotes permutation operations between two particles with different states. Then, the Dicke state $|J, M\rangle$ can be expressed as

$$|J, M\rangle = |n_1, n_2\rangle,$$  \hspace{1cm} (4)

where $J = \frac{N}{2}$ is the maximal azimuthal quantum number of $S = \sum s_i$, $M$ is the spin magnetic quantum number of $S_z$ whose value can be $M = J, J - 1, \cdots, 1 - J, -J$. It is straightforward to obtain

$$|J, \frac{N}{2}\rangle = |N, 0\rangle, \hspace{0.5cm} |J, -\frac{N}{2}\rangle = |0, N\rangle.$$  \hspace{1cm} (5)

In terms of Eq. (3) and the conservation of quantum numbers in different single particle states, one can easily obtain the constraint equation set

$$n_1 + n_2 = N, \hspace{0.5cm} \frac{n_1}{2} - \frac{n_2}{2} = M.$$  \hspace{1cm} (6)

Therefore, once $N$ and $M$ are given, the explicit form of the Dicke state can be easily derived according to Eq. (4) and Eq. (3). Finally, the Dicke state has a form

$$|J, M\rangle = |n_1, n_2\rangle = \sqrt{\frac{(\frac{N}{2} + M)!((\frac{N}{2} - M))!}{N!}} \sum_{n_1} P\left(\begin{array}{c} 1/2 \cdot \cdots \cdot 1/2 \\ \frac{N}{2} + M \end{array} \right) \cdot \sum_{n_2} P\left(\begin{array}{c} 1/2 \cdot \cdots \cdot 1/2 \\ \frac{N}{2} - M \end{array} \right).$$  \hspace{1cm} (7)
B. Dicke states of identical spin-1 particles

For simplicity, the states with maximal total angular momentum for \( N \) identical particles of high spin (\( s_i > 1/2 \)) are called generalized Dicke states \( |J, M\rangle \) here. Following the approach for the above spin-1/2 case, we can express \( |J, M\rangle \) in the number representation as

\[
|J, M\rangle = \sum_{k=0}^{\text{max}} C_{k, n_1, n_0, n-1} |n_1, n_0, n-1\rangle,
\]

where \( k \) is a parameter directly related with \( n_0 \),

\[
\text{max} = \frac{1}{2}(J - |M| - \text{min}),
\]

\[
\text{min} = \frac{1}{2}((-1)^{J-|M|+1} + 1),
\]

and \( C_{k, n_1, n_0, n-1} \) are the superposition coefficients with \( n_1, n_0 \) and \( n-1 \) denoting the numbers of particles in states \( |\uparrow\rangle, |0\rangle \) and \( |\downarrow\rangle \), respectively, and

\[
|n_1, n_0, n-1\rangle = \sqrt{\frac{n_1! n_0! n-1!}{N!}} \sum_{p} P(|1\ldots10\ldots01\ldots1\rangle).
\]

It should be noted that the form of Eq. (8) is different from that of Eq. (4) such that the values of \( n_1, n_0 \) and \( n-1 \) are not unique for specific \( N \) and \( M \), and they shall be determined with the help of Eq. (9). This is the very difference for Dicke states of high-spin systems from those of the spin-1/2 system. For the special cases \( M = \pm J \) and \( M = \pm (J - 1) \), it is easy to check that \( \text{max} = \text{min} = 0 \) and \( \text{max} = 0, \text{min} = 1 \), respectively, in which case \( n_1, n_0 \) and \( n-1 \) are uniquely determined. For other cases, we should find all the values of \( n_1, n_0 \) and \( n-1 \) as well as \( C_{k, n_1, n_0, n-1} \).

Generally, three equations are needed to determine \( n_1, n_0 \) and \( n-1 \), and it is not difficult to find the first and the second equations as follows

\[
n_1 + n_0 + n-1 = N,
\]
\[
n_1 - n-1 = M.
\]

(11)

Here, we find the third equation given by

\[
n_0 = \text{min} + 2k,
\]

(12)

with \( k = 0, 1, \ldots \text{max} \). According to Eq. (9) and Eq. (12), the number of elementary states contained in the basis \( \{|n_1, n_0, n-1\}\) for state \( |J, M\rangle \) is max +1. Using the normalization condition, we also obtain the coefficients

\[
C_{k, n_1, n_0, n-1} = \frac{(J - |M|)!}{2^{-n_0} n_1! n_0! n-1!} \prod_{l=1}^{J-|M|} \frac{1}{\sqrt{2N - l + 1}}.
\]

(13)

It is worth nothing that an arbitrary combination of max+1 elementary states does not change the value of \( M \). As a result, the arbitrary combination also applies to the construction of the Dicke state \( |J, M\rangle \). However, since \( |J, M - 1\rangle \) and \( |J, M\rangle \) must satisfy the relation

\[
|J, M - 1\rangle = \frac{j_g |J, M\rangle}{\sqrt{(J + M)(J - M + 1)}},
\]

(14)

the basis of \( |J, M\rangle \) and that of \( |J, M - 1\rangle \) also satisfy certain relations. Thereby, once the basis of the Dicke state \( |J, M\rangle \) is determined, that of \( |J, M - 1\rangle \) is specified as well.

For an illustration, we consider the case of particle number \( N = 10 \) and obtain all \( \{|n_1, n_0, n-1\}\) and the corresponding \( C_{k, n_1, n_0, n-1} \) respect to different values of \( M \), which is listed in the following Table I and Table II. For convenience, we have omitted the subscripts \( k, n_1, n_0, n-1 \) in the coefficients \( C_{k, n_1, n_0, n-1} \).

For the case of negative \( M \), we need only to exchange the values between \( n_1 \) and \( n-1 \). For instance, when \( J = 10, M = -1 \), in terms of the above table, we can conveniently construct the Dicke state \(|10, -1\rangle\) as

\[
|10, -1\rangle = 0.1225|4, 1, 5\rangle + 0.4473|3, 3, 4\rangle + 0.6929|2, 5, 3\rangle + 0.5238|1, 7, 2\rangle + 0.1746|0, 9, 1\rangle.
\]

(15)
TABLE I: The superposition coefficient and the combination \((n_1, n_0, n_{-1})\) for every \(M \in [5, 9]\) in the spin-1 system. We set \(N = 10\).

TABLE II: The superposition coefficient and the combination \((n_1, n_0, n_{-1})\) for every \(M \in [0, 4]\) in the spin-1 system. We set \(N = 10\).

C. Dicke states of identical spin-3/2 particles

For the case of spin-3/2, the states have the form similar to the case of spin-1, which reads

\[
|J, M\rangle = \sum_{k=k_0}^{\max} C_{k,n_1,n_2,n_3,n_4} |n_1, n_2, n_3, n_4\rangle,
\]

where \(C_{k,n_1,n_2,n_3,n_4}\) are the coefficients with \(n_1, n_2, n_3\) and \(n_4\) denoting the occupation numbers of particles in states \(|\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |\frac{-1}{2}\rangle\) and \(|\frac{-3}{2}\rangle\) respectively, and \(k\) related to \(n_2\) and \(n_3\). The basis \(|n_1, n_2, n_3, n_4\rangle\) has the form as

\[
|n_1, n_2, n_3, n_4\rangle = \sqrt{\frac{n_1!n_2!n_3!n_4!}{N!}} \sum_P P\left(\frac{1}{2} \cdots \frac{1}{2} \mid \frac{1}{2} \cdots \frac{1}{2} \mid \frac{-1}{2} \cdots \frac{-1}{2} \mid \frac{-3}{2} \cdots \frac{-3}{2}\right).
\]

With some calculations, we also derive

\[
k_0 = \frac{1}{2}\left\{\frac{1}{2}(\alpha_1 + |\alpha_1|) + \frac{1}{2}[1 - (-1)^\frac{3}{2}(\alpha_1 + |\alpha_1|)]\right\},
\]

with \(\alpha_1 = \frac{N}{2} - |M|\), and

\[
\max = \frac{1}{2}(J - |M| - \min),
\]

\[
\min = \frac{1}{2}((-1)^{J-|M|+1} + 1).
\]

First, the two basic constraint equations is as follows

\[
n_1 + n_2 + n_3 + n_4 = N,
\]

\[
3n_1 + n_2 - n_3 - 3n_4 = 2M.
\]

The third constraint equation is found to be

\[
n_2 - n_3 = (-1)^{\frac{|M|-M}{2|M|}} \gamma_1,
\]
with \( \gamma_i = J - |M| - 3k \). For specific value of \( k \), we also find the fourth constraint equation as

\[
n_2 + n_3 = |\gamma_i| - 2(k_1 - 1),
\]

with

\[
m_k = \frac{1}{2} \left( \frac{1}{2} (\beta_1 - |\beta_1|) + k + 1 \right) + \frac{1}{2} (\beta_1 - |\beta_1|) + k + 1) + \frac{1}{2} (\gamma_1 + |\gamma_1|)
\]

and \( \beta_1 = k - \frac{1}{2}(\alpha_1 + |\alpha_1|) \). Then, we obtain a set of constraint equations

\[
\sum_{k=k_0}^{\text{max}} \sum_{k_1=1}^{m_k} \begin{cases} 
  n_1 + n_2 + n_3 + n_4 = N \\
  3n_1 + n_2 - n_3 - 3n_4 = 2M \\
  n_2 - n_3 = (-1)^{\frac{|M|}{2}} M \gamma_i \\
  n_2 + n_3 = |\gamma_i| - 2(k_1 - 1)
\end{cases}
\]

with \( k_0 \leq k \leq \text{max} \) and \( 1 \leq k \leq m_k \). The number of states to form a complete basis is \( \sum_{k=k_0}^{\text{max}} m_k \) and the normalized coefficients are

\[
C_{k,n_1,n_2,n_3,n_4} = \frac{(J - |M|)!}{3^{-(n_2+n_3)/2}} \sqrt{\frac{N!}{n_1!n_2!n_3!n_4!}} \prod_{l=1}^{J-|M|} \frac{1}{\sqrt{(3N - l + 1)!}}.
\]

Again, for an illustration, we consider the case of particle number \( N = 6 \), and list all these parameters in the Dicke states for different values of \( M \) in Table III and IV:

| \( M = 9 \) | \( M = 8 \) | \( M = 7 \) | \( M = 6 \) | \( M = 5 \) |
|---|---|---|---|---|
| \( C \) | \( n_1, n_2, n_3, n_4 \) | \( C \) | \( n_1, n_2, n_3, n_4 \) | \( C \) | \( n_1, n_2, n_3, n_4 \) | \( C \) | \( n_1, n_2, n_3, n_4 \) | \( C \) | \( n_1, n_2, n_3, n_4 \) |
| 1 6, 0, 0, 0 | 1 5, 1, 0, 0 | 0.9393 4, 2, 0, 0 | 0.8135 3, 3, 0, 0 | 0.6301 2, 4, 0, 0 |
| 0.3430 5, 0, 1, 0 | 0.0857 5, 0, 0, 1 | 0.1715 4, 1, 0, 1 |
| 0.6752 4, 1, 1, 0 | 0.2100 4, 0, 2, 0 |
| & | & | & | & | 0.7276 3, 2, 1, 0 |

**TABLE III:** The superposition coefficient and the combination \( (n_1, n_2, n_3, n_4) \) for every \( M \in [5, 9] \) in the spin-3/2 system. We set \( N = 6 \).

| \( M = 4 \) | | \( M = 3 \) | | \( M = 2 \) | | \( M = 1 \) | | \( M = 0 \) |
|---|---|---|---|---|---|---|
| \( C \) | \( n_1, n_2, n_3, n_4 \) | \( C \) | \( n_1, n_2, n_3, n_4 \) | \( C \) | \( n_1, n_2, n_3, n_4 \) | \( C \) | \( n_1, n_2, n_3, n_4 \) | \( C \) | \( n_1, n_2, n_3, n_4 \) |
| 0.4125 1, 5, 0, 0 | 0.1982 0, 6, 0, 0 | 0.2763 1, 4, 0, 1 | 0.1825 0, 5, 0, 1 | 0.1825 1, 3, 0, 2 |
| 0.2516 3, 2, 0, 1 | 0.3594 2, 3, 0, 1 | 0.0752 3, 1, 0, 2 | 0.1361 2, 2, 0, 2 | 0.0203 3, 0, 0, 3 |
| 0.1025 4, 0, 1, 1 | 0.0284 4, 0, 0, 2 | 0.2803 3, 0, 2, 1 | 0.0641 3, 0, 1, 2 | 0.3872 1, 4, 0, 1 |
| 0.5752 2, 3, 0, 1 | 0.1706 3, 0, 2, 0 | 0.3707 0, 5, 1, 0 | 0.1666 2, 0, 4, 0 | 0.1825 2, 3, 0, 1 |
| 0.4348 3, 1, 1, 0 | 0.6267 1, 4, 1, 0 | 0.3908 2, 2, 1, 1 | 0.4713 1, 3, 1, 1 | 0.1825 1, 2, 1, 2 |
| & | & | & & 0.2142 3, 3, 1, 1 | 0.3008 2, 1, 3, 0 | 0.3333 2, 1, 2, 1 |
| & | & | & & 0.6267 2, 2, 2, 0 | 0.6769 1, 3, 2, 0 | 0.4999 0, 4, 2, 0 |
| & | & | & & & | 0.5476 1, 2, 2, 1 |
| & | & | & & & & | 0.5772 1, 2, 3, 0 |
| & | & | | & | & & | 0.5476 0, 3, 3, 0 |

**TABLE IV:** The superposition coefficient and the combination \( (n_1, n_2, n_3, n_4) \) for every \( M \in [0, 4] \) in the spin-3/2 system. We set \( N = 6 \).

For negative values of \( M \), one may perform the exchanges \( n_1 \leftrightarrow n_4 \), \( n_2 \leftrightarrow n_3 \) in the case of positive \( M \). For instance, when \( J = 6, M = -1 \), using the results of above table, we obtain

\[
|6, -1\rangle = 0.1825 |1, 0, 5, 0\rangle + 0.1361 |2, 0, 2, 2\rangle + 0.0641 |2, 1, 0, 3\rangle + 0.1666 |0, 4, 0, 2\rangle + 0.4713 |1, 1, 3, 1\rangle + 0.3333 |1, 2, 1, 2\rangle + 0.4999 |0, 2, 4, 0\rangle + 0.5772 |0, 3, 2, 1\rangle.
\]
D. Dicke states of identical spin-2 particles

For the case of spin-2, the Dicke states in the number representation are given by

$$|J, M\rangle = \sum_{k=k_0}^{\text{max}} C_{k,n_1,n_2,n_3,n_4,n_5} |n_1, n_2, n_3, n_4, n_5\rangle,$$

(27)

with $J = 2N$ and the number states

$$|n_1, n_2, n_3, n_4, n_5\rangle = \sqrt{\frac{n_1!n_2!n_3!n_4!n_5!}{N!}} \sum_{\text{sym}} P(|2\ldots2|1\ldots1|0\ldots0|1\ldots1|2\ldots2),$$

(28)

where $n_1, n_2, n_3, n_4$ and $n_5$ denote the numbers of particles in states $|2\rangle, |1\rangle, |0\rangle, |1\rangle$ and $|2\rangle$ respectively, $k$ is related to $n_2$ and $n_4$, and $k_0$ and max are given by

$$k_0 = \begin{cases} \frac{1}{2}(\alpha_2 + |\alpha_2|), & \text{if } \frac{1}{2}(\alpha_2 + |\alpha_2|) = 3n', \\
\frac{1}{2}(\alpha_2 + |\alpha_2| + 1), & \text{if } \frac{1}{2}(\alpha_2 + |\alpha_2|) = 3n' + 1, \\
\frac{1}{2}(\alpha_2 + |\alpha_2| + 2), & \text{if } \frac{1}{2}(\alpha_2 + |\alpha_2|) = 3n' + 2, \end{cases}$$

(29)

$$\text{max} = \begin{cases} \frac{2}{3}(2N - |M|), & \text{if } 2N - |M| = 3n'', \\
\frac{4}{3}(2N - |M| - \frac{3}{2}), & \text{if } 2N - |M| = 3n'' - 1, \\
\frac{4}{3}(2N - |M| - 1), & \text{if } 2N - |M| = 3n'' - 2. \end{cases}$$

(30)

Here, $n'$ and $n''$ are integers and $\alpha_2 = N - |M|$. It is easy to verify that

$$|J, J\rangle = |N, 0, 0, 0, 0\rangle, |J, -J\rangle = |0, 0, 0, 0, N\rangle,$$

To specify $n_1, n_2, n_3, n_4$ and $n_5$ as well as $C_{k,n_1,n_2,n_3,n_4,n_5}$, five equations are needed. First, the two basic equations are

$$n_1 + n_2 + n_3 + n_4 + n_5 = N,$$

$$2n_1 + n_2 - n_4 - 2n_5 = 2M.$$

(31)

We derive the other three equations as

$$n_2 - n_4 = (-1)^{\frac{M+M}{2}} \gamma_2, \quad k_0 \leq k \leq \text{max}, \quad \gamma_2 = J - |M| - 2k,$$

$$n_2 + n_4 = \gamma_2 + 2(k_1 - 1), \quad 1 \leq k_1 \leq m_k,$$

$$n_3 = 2(k_2 + 1) + \frac{1}{2}[1 - (-1)^k] - 2, \quad 0 \leq k_2 \leq m_k - k_1,$$

(32)

where

$$m_k = \frac{1}{2}\left(\frac{1}{2}(\beta_2 + |\beta_2|) + \frac{2k + 3 + (-1)^k}{4} + \frac{1}{2}(\beta_2 + |\beta_2|) + \frac{2k + 3 + (-1)^k}{4}\right) + \frac{1}{2}(\gamma_2 - |\gamma_2|),$$

(33)

with $\beta_2 = k - \frac{1}{2}(\alpha_2 + |\alpha_2|)$. Note that $k, k_1$ and $k_2$ are all nonnegative integers. Besides, we find that the number of states to form a complete basis is $\sum_{k=k_0}^{\text{max}} \frac{1}{2}m_k (m_k + 1)$ and the normalized coefficients read as

$$C_{k,n_1,n_2,n_3,n_4,n_5} = \frac{3}{2} n_3/2 \cdot \frac{(J - |M|)!}{3^{(n_2 + n_3 + n_4)/2} \sqrt{n_1!n_2!n_3!n_4!n_5} \prod_{l=1}^{J-|M|} 1/(4N-l+1)}.$$

(34)

Here, we focus on the case $N = 5$ for example and derive all the $(n_1, n_2, n_3, n_4, n_5)$ and $C_{k,n_1,n_2,n_3,n_4,n_5}$ for different values of $M$ as follows. In terms of the table above, all the five-particle Dicke states can be obtained.
By solving Eq. (6) to obtain

and substituting Eq. (36) into Eq. (35), the coefficients are simplified as

which agrees with the setup in Eq. (4).

### III. ANTI-SYMMETRIC STATES IN HIGH SPIN SYSTEMS

As a natural byproduct, we proceed to discuss the anti-symmetric states in these high spin systems. It is well known that any two particles should not be in the same state in an anti-symmetric state. Therefore, we can conclude that anti-symmetric states exist only for particle number less than 2. Similarly, one can check that the
upper limits for particle numbers of spin $1$, $3/2$ and $2$ are $3$, $4$ and $5$, respectively. Based on this fact, the number of anti-symmetric states for high spin-$s$ systems is

$$C(2s + 1, 2s + 1) + C(2s + 1, 2s) + \cdots + C(2s + 1, 2) = 2^{2s+1} - (2s + 2),$$  \hspace{1cm} (38)

where $C(n,k)$ means the $k$-combinations of $n$.

For convenience, we mark the states of single particle as $|\alpha\rangle, |\beta\rangle \cdots \in \{|-s\rangle, \cdots , |s\rangle\}$. Then, for the case of two particles, the $C_{2s+1}^2$ elementary anti-symmetric states can be write as

$$|\psi(i, j)\rangle_{AS} = \sqrt{\frac{1}{2}} \sum_{\hat{P}} \delta_{\hat{P}} P(|\alpha\rangle|\beta\rangle), \hspace{1cm} |\alpha\rangle \neq |\beta\rangle,$$  \hspace{1cm} (39)

where the symbol $P$ denotes all possible permutations and $\delta_{\hat{P}}$ (with the initial value $+1$) changes its sign between $+1$ and $-1$ after every permutation. Due to the fact that the state $|J = 2s - 1, 2s - 1\rangle$ is anti-symmetric, so all states belonging to the subspace with total spin $J = 2s - 1$ are anti-symmetric. It is obvious that in the two-qubit system there is only a elementary anti-symmetric state

$$|J = 0, M = 0\rangle = \sqrt{\frac{1}{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle),$$  \hspace{1cm} (40)

and in two-qudit system with $J = 1$ there is three elementary anti-symmetric states as

$$|J = 1, 1\rangle = \sqrt{\frac{1}{2}}(|\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle),$$

$$|J = 1, 0\rangle = \sqrt{\frac{1}{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle),$$

$$|J = 1, -1\rangle = \sqrt{\frac{1}{2}}(|0\rangle|\downarrow\rangle - |\downarrow\rangle|0\rangle),$$  \hspace{1cm} (41)

For the case with $s > 1$, in the subspace with total spin $J = 2s - 1$, the four states

$$|J = 2s - 1, \pm(2s - 1)\rangle = \sqrt{\frac{1}{2}}(|\pm s\rangle|\pm s + 1\rangle - |\pm s + 1\rangle|\pm s\rangle),$$

$$|J = 2s - 1, \pm(2s - 2)\rangle = \sqrt{\frac{1}{2}}(|\pm s\rangle|\pm s + 2\rangle - |\pm s + 2\rangle|\pm s\rangle),$$  \hspace{1cm} (42)

are elementary anti-symmetric states given by Eq. (39), and the rest $4s - 5$ states are linear superposition of those elementary anti-symmetric states. For the case including three particles, there is $C_{2s+2}^3$ elementary anti-symmetric states

$$|\psi(i, j, k)\rangle_{AS} = \sqrt{\frac{1}{3!}} \sum_{\hat{P}} \delta_{\hat{P}} P(|\alpha\rangle|\beta\rangle|\gamma\rangle), \hspace{1cm} |\alpha\rangle \neq |\beta\rangle \neq |\gamma\rangle.$$  \hspace{1cm} (43)

To conclude, for the $2s + 1$ particles, there is only one elementary anti-symmetric state as

$$|\psi[1, 2 \cdots (2s + 1)\rangle\rangle_{AS} = \sqrt{\frac{1}{(2s + 1)!}} \sum_{\hat{P}} \delta_{\hat{P}} P(|-s\rangle \otimes |s + 1\rangle \otimes \cdots \otimes |s - 1\rangle \otimes |s\rangle),$$  \hspace{1cm} (44)

which is just the state $|J = 0, M = 0\rangle$. In particular, for the case of spin-$1$, the anti-symmetric state can be written as

$$|\psi(1, 2, 3)\rangle_{AS} = \sqrt{\frac{1}{3!}}(|\uparrow\rangle|0\rangle|\downarrow\rangle + |\downarrow\rangle|0\rangle|\uparrow\rangle - |\downarrow\rangle|\uparrow\rangle|0\rangle + |\uparrow\rangle|\downarrow\rangle|0\rangle - |\downarrow\rangle|0\rangle|\downarrow\rangle + |0\rangle|\downarrow\rangle|\uparrow\rangle).$$  \hspace{1cm} (45)
For instance, the particle number may be 2, 3, 4 and 5 for $s = 2$, and one can construct all the anti-symmetric states as

$$|\psi(i, j)\rangle_{AS} = \sqrt{\frac{1}{2}} \sum_{p} \delta_{p} P(|\alpha\rangle|\beta\rangle), \ |\alpha\rangle \neq |\beta\rangle,$$

$$|\psi(i, j, k)\rangle_{AS} = \sqrt{\frac{1}{3!}} \sum_{p} \delta_{p} P(|\alpha\rangle|\beta\rangle|\gamma\rangle), \ |\alpha\rangle \neq |\beta\rangle \neq |\gamma\rangle,$$

$$|\psi(i, j, k, l)\rangle_{AS} = \sqrt{\frac{1}{4!}} \sum_{p} \delta_{p} P(|\alpha\rangle|\beta\rangle|\gamma\rangle|\eta\rangle), \ |\alpha\rangle \neq |\beta\rangle \neq |\gamma\rangle \neq |\eta\rangle,$$

$$|\psi(i, j, k, l, m)\rangle_{AS} = \sqrt{\frac{1}{5!}} \sum_{p} \delta_{p} P(|\alpha\rangle|\beta\rangle|\gamma\rangle|\eta\rangle|-1| -2),$$

where $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, |\eta\rangle \in \{2, 1, 0, -1, -2\}$, and $|2\rangle, |1\rangle, |0\rangle, |-1\rangle$ and $|-2\rangle$ are the eigenstates of single spin magnetic quantum number $m_s$ with eigenvalues 2, 1, 0, -1 and -2, respectively. According Eq. (38), there totally exist 26 different anti-symmetric states for spin-2 systems. Generally speaking, due to the Pauli exclusion principle, the number of anti-symmetric states compared to that of all the Dicke states is very limited.

### IV. ENTANGLEMENT OF TWO QU DITS IN SYSTEM WITH MANY PARTICLES

In this section, we study the entanglement for the case of two qudits. The entanglement criteria proposed by Peres-Horodecki [20, 21] is adopted. For states with certain symmetries in the high spin systems, this criterion is good enough to measure the entanglement. However, the states discussed by us are beyond this requirement. In order to quantify the entanglement, Vidal and Werner proposed a entanglement measure termed as negativity [26]. The first thing need to do is that one obtains the density matrix $\rho_{ij}$ of two qudits in the basis $\{|\uparrow\downarrow\rangle, |00\rangle, |\downarrow\rangle|\uparrow\rangle, |\uparrow\rangle|\downarrow\rangle, |\uparrow\rangle|0\rangle, |0\rangle|\uparrow\rangle, |\uparrow\rangle|\downarrow\rangle, |\downarrow\rangle|0\rangle, |\downarrow\rangle|\downarrow\rangle\}$. Next, one can perform partial transpose (PT) to $\rho_{ij}$, and obtain the matrix $\rho_{ij}^{T}$ in the basis spanned by $\{|\uparrow\rangle|\uparrow\rangle, |0\rangle|0\rangle, |\downarrow\rangle|\downarrow\rangle, |\uparrow\rangle|\downarrow\rangle, |\downarrow\rangle|\uparrow\rangle, |\uparrow\rangle|\downarrow\rangle, |\downarrow\rangle|\downarrow\rangle, |\downarrow\rangle|\downarrow\rangle\}$. The negativity is then defined as

$$N(\rho_{ij}) = \sum_{i} |\lambda_{i}|,$$

where $\lambda_{i}$ are the negative eigenvalues of $\rho_{ij}^{T}$. If $N(\rho_{ij}) > 0$, then the two particles stay in the entangled state. However, in the 9 basis, the density of two particles generally has 81 elements. Different from the case of many spin-1/2 particles, these elements cannot be represented by the expectation value of the collective operators of system. This leads to certain difficulties in the calculation of the entanglement. However, in the following, we will show that the number of effective elements will be greatly reduced for some special states.

#### A. Entanglement of specific states in the system with two spin-1 particles

Let us begin with the system with two spin-1 particles. We discuss the entanglement of two particle with spin-1. The first state considered is the generalized symmetric Bell state of two qudits, which is an important state for the qudit teleportation scheme [20, 21]. Its form is given by

$$|B_{G}\rangle = \sqrt{\frac{1}{3}} (|\uparrow\rangle|\uparrow\rangle + |0\rangle|0\rangle + |\downarrow\rangle|\downarrow\rangle),$$

where $s_{\uparrow}|\uparrow\rangle = |\uparrow\rangle$, $s_{\downarrow}|0\rangle = 0$, and $s_{\downarrow}|\downarrow\rangle = |\downarrow\rangle$. It is easy to check that this state is a maximally entangled state and the negativity equal 1 in this state. In order to compare with the state presented above, we study the negativity of another state with a generalized form

$$|\psi_{1}\rangle = \sqrt{\frac{1}{3}} (|\uparrow\rangle|\uparrow\rangle + c_{1} \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)) + c_{2}|0\rangle|0\rangle + |\downarrow\rangle|\downarrow\rangle),$$

where coefficients $c_{1}$ and $c_{2}$ satisfy the relation $|c_{1}|^2 + |c_{2}|^2 = 1$. After the PT, we can give the matrix density of two particles in a block diagonal form as

$$\rho_{ij}^{T} = \text{diag}(C_{5 \times 5}, D_{4 \times 4}),$$
with \( C \) and \( D \) given by

\[
C = \begin{pmatrix}
\frac{1}{3} & 0 & \frac{1}{2} \\
0 & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{2}
\end{pmatrix}, \quad D = \begin{pmatrix}
0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}.
\]  

(51)

Specially, for combination \((c_1 = \sqrt{1/3} \text{ and } c_2 = \sqrt{2/3})\), the state can be reduced to even spin coherent state of two qudits

\[
|\psi_e\rangle = \sqrt{\frac{1}{3}}(|2, 2\rangle + |2, 0\rangle + |2, -2\rangle).
\]  

(52)

The negativity in this state is 0.8221, where \(|2, 0\rangle\) is the Dicke state

\[
|2, 0\rangle = \sqrt{\frac{1}{6}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) + \sqrt{\frac{2}{3}}|0\rangle|0\rangle,
\]  

(53)

in which state the negativity is 0.833. It is worth mentioning that this even spin coherent state can be generate by one-axis twisting model or the two-axis counter model with the initial state \(|2, -2\rangle\). For another state

\[
|\psi_2\rangle = \frac{1}{2}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) + \frac{1}{\sqrt{2}}|0\rangle|0\rangle,
\]  

(54)

with \( M = 0 \), we obtain the negativity with value 0.9571.

There are two interested generalized singlet Bell states which are the elementary states of dimmer states [26, 22, 23]

\[
|B_{z\pm}\rangle = \sqrt{\frac{1}{3}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \pm |0\rangle|0\rangle.
\]  

(55)

Also, we obtain that the negativity is 1.

B. Entanglement of two qudits in the Dicke states of many spin-1 particles

In the nine bases, the reduced density matrix of the two spins in the Dicke states can be written as

\[
\rho_{ij} = \text{diag}(T_1, T_2, T_3, a_8, a_9)
\]  

(56)

with \( T_1, T_2 \) and \( T_3 \)

\[
T_1 = \begin{pmatrix}
a_1 & c_1 & b_3 \\
c_1^* & a_2 & c_2 \\
b_3 & c_2^* & a_3
\end{pmatrix}, \quad T_2 = \begin{pmatrix}
a_4 & b_1 \\
b_1^* & a_5 \\
a_6 & b_2 & a_7
\end{pmatrix}, \quad T_3 = \begin{pmatrix}
a_4 & c_1 \\
c_1^* & a_6 \\
a_5 & c_2 & a_7
\end{pmatrix}.
\]  

(57)

After the PT, the density matrix \( \rho_{ij} \) transfers to

\[
\rho_{ij}^T = \text{diag}(T_1', T_2', T_3', a_1, a_3).
\]  

(58)

with

\[
T_1' = \begin{pmatrix}
a_8 & b_1 & b_3 \\
b_1^* & a_2 & b_2 \\
b_3 & b_2^* & a_9
\end{pmatrix}, \quad T_2' = \begin{pmatrix}
a_4 & c_1 \\
c_1^* & a_6 \\
a_5 & c_2 & a_7
\end{pmatrix}, \quad T_3' = \begin{pmatrix}
a_5 & c_2
\end{pmatrix}.
\]  

(59)
Here, we have considered that the elements $a_1$ and $a_3$ are always positive since they characterize the probability of finding two particles in the states $|\uparrow\rangle|\downarrow\rangle$ and $|\downarrow\rangle|0\rangle$, which have no relationship with entanglement. In order to obtain the entanglement, we calculated the 17 useful elements as follows

$$a_2 = 1 - 2 \sum_k |C_{k,n_1,n_0,n-1}|^2 \frac{N - n_0}{N} + \frac{2}{N(N - 1)} \sum_k |C_{k,n_1,n_0,n-1}|^2 \{n_1 n_1 + \frac{1}{2}[n_1(n_1 - 1) + n-1(n-1 - 1)]\},$$

$$a_3 = \frac{1}{N(N - 1)} \sum_k |C_{k,n_1,n_0,n-1}|^2 n_1 (n_1 - 1),$$

$$a_4 = a_5 = \frac{1}{2} \sum_k |C_{k,n_1,n_0,n-1}|^2 \frac{N - n_0}{N} + \frac{M}{2N} - \frac{1}{N(N - 1)} \sum_k |C_{k,n_1,n_0,n-1}|^2 [n_1 n_1 + n_1(n_1 - 1)],$$

$$a_6 = a_7 = \frac{1}{2} \sum_k |C_{k,n_1,n_0,n-1}|^2 \frac{N - n_0}{N} - \frac{M}{2N} - \frac{1}{N(N - 1)} \sum_k |C_{k,n_1,n_0,n-1}|^2 [n_1 n_1 + n_1(n_1 - 1)],$$

$$a_8 = \frac{1}{N(N - 1)} \sum_k |C_{k,n_1,n_0,n-1}|^2 n_1 (n_1 - 1),$$

$$a_9 = \frac{1}{N(N - 1)} \sum_k |C_{k,n_1,n_0,n-1}|^2 n_1 (n_1 - 1),$$

$$c_1 = c_2 = \frac{1}{N(N - 1)} \sum_k |C^*_{k,n_1,n_0,n-1} C_{k,n_1+1,n_0-2,n-1+1}| \sqrt{n_0(n_0 - 1)(n_1 + 1)(n_1 - 1)},$$

$$b_1 = \frac{1}{N(N - 1)} \sum_k |C_{k,n_1,n_0,n-1}|^2 n_0 n_1,$$

$$b_2 = \frac{1}{N(N - 1)} \sum_k |C_{k,n_1,n_0,n-1}|^2 n_0 n_1,$$

$$b_3 = \frac{1}{N(N - 1)} \sum_k |C_{k,n_1,n_0,n-1}|^2 n_1 n_1. \quad (60)$$

![Negativity of the Dicke states](image.png)

**FIG. 1:** The negativity of the Dicke states $|J, M\rangle$ with positive $M$. The number of particles from top to bottom is 20, 30, ..., 80.

We can obtain the negative eigenvalues of three matrices by solving the eigenvalue equation. Substituting those eigenvalues into the formula of the negativity, the entanglement can be calculated. Specially, for the case $M = 0$, the negativity can be reduced to simpler form given by (7). The properties of entanglement in the Dicke states with $20–80$ particles are shown in Fig. 1. We observe that as $|M|$ decreases, the negativity is a monotone increasing function, and comparing with others states, the state $|J, 0\rangle$ possesses the maximal entanglement. This property is different from the concurrence of the Dicke states of multi-particle for the spin-1/2 case. In addition, as $|M|$ increases, the maximal value of the negativity decreases. Considering that those Dicke states $|J, M\rangle$ with $|M| < N - 1$ are linear combination of different states $\{|n_1, n_0, n-1\}\rangle$, the states $|J, M\rangle$ with $|M| < N - 1$ actually forms a subspace. We can construct the states which are equal probability combination of different states $|n_1, n_0, n-1\rangle$ in the subspace mentioned above. The negativities in these states are compared with those of the Dicke states. The consequences show that, with the equal
probability combination, there are some advantages in the generation of negativity, specially for the cases $M = 0$. Here, we consider $N = 30, 80$, and present the negativities of two cases, as shown Fig. 2.

FIG. 2: Comparison of negativity between the Dicke states and the states with equal probability combination of all states $|n_1, n_2, n_3\rangle$ in the subspace $M$. We take $N = 30$ (on the left side) and $N = 80$ (on the right side). The solid line and dashed line correspond to the Dicke states.

V. CONCLUSIONS

In summary, we have investigated the construction of Dicke states for high-spin particles based on that of Dicke states for the spin-1/2 case. For three high-spin cases (spin-1, 3/2 and 2) with given particle numbers and spin magnetic quantum numbers, the sets of constraint equations are found to determine all the basis states in the number representation as well as the corresponding normalized superposition coefficients, in terms of which the Dicke states are explicitly expressed in the number representation. As a byproduct, we give a rule to construct all the anti-symmetric states in these high-spin systems and show that the number of anti-symmetric states is rather limited. Finally, in terms of the negativity, the entanglement properties for spin-1 cases including specific pure states of two particles and the Dicke states of many particles are discussed. Our results may contribute to the applications of high-spin systems in quantum information science due to the crucial importance of Dicke states.

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