Constructive expansion for quartic vector fields
theories
I. Low dimensions

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Abstract

This paper is the first of a series aiming at proving rigorously the analyticity and the Borel summability of generic quartic bosonic and fermionic vector models (generalizing the \(O(N)\) vector model) in diverse dimensions. Both non-relativistic (Schrödinger) and relativistic (Klein–Gordon and Dirac) kinetic terms are considered. The 4-tensor defining the interactions is constant but otherwise arbitrary, up to the symmetries imposed by the statistics of the field. In this paper, we focus on models of low dimensions: bosons and fermions for \(d = 0, 1\), and relativistic bosons for \(d = 2\). Moreover, we investigate the large \(N\) and massless limits along with quenching for fermions in \(d = 1\). These results are established using the loop vertex expansion (LVE) and have applications in different fields, including data sciences, condensed matter and string field theory. In particular, this establishes the Borel summability of the SYK model both at finite and large \(N\).
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1 Introduction

A vector model corresponds to a theory of multiple bosonic (commuting) or fermionic (anticommuting) fields interacting with each other. Among those models, quartic vector models have a distinguished role since they are universal: quartic terms correspond to the first stable correction to the quadratic (Gaussian) term (a cubic term would make the path integral ill-defined). Such an interaction represents the first non-trivial term coupling together the different fields (or more precisely the eigenstates of the kinetic term). Moreover, any potential (with a parity symmetry) can be approximated by a quartic model by performing a Taylor expansion. As a consequence, quartic vector models are ubiquitous in physics and beyond. Of particular interest to us are applications to data science, condensed matter (and more precisely the SYK model) and to string field theory.\(^1\) These are reviewed in Section 1.1.

This paper is a first of a series whose goal is to prove rigorously the analyticity and the Borel summability of a large class of such quartic vector models in \(d\) dimensions using the loop vertex expansion (LVE) [1, 2] and its generalizations. We consider a constant but otherwise arbitrary (up to a stability condition and the symmetry imposed by the field statistics) quartic tensor of coupling constants. Both the Schrödinger (non-relativistic) and Klein–Gordon / Dirac (relativistic) types of propagators are considered. As such, our analysis is the first to exhaust all the possible cases of renormalizable models with a purely quartic interaction.

We outline the main steps of the method in Section 1.2. In this paper, we study the analyticity and Borel summability of super-renormalizable bosonic and fermionic models in \(d = 0, 1\), and of relativistic bosonic models in \(d = 2\). This allows to introduce, for the simplest examples, the different types of constructive expansions along with all the tools needed. Since constructive methods have the reputation of being esoteric and because this paper targets different public, we will provide many details which the expert reader can skip. The following models are studied:

- \(d = 0\) bosonic models (Section 3.1) with identically distributed values (Propositions 2 and 3) and with a non-trivial kernel (Remark 2)
- \(d = 0\) fermionic models (Section 3.2 and proposition 4)
- \(d = 1\) relativistic bosonic models (Section 4.1) at finite temperature (Proposition 5) and zero temperature (Remark 6)

\(^1\)Since this work is located at the intersection of different fields, we develop in some details notions that may be well-known to some of the readers, but less to others. We also are also explicit in the derivations and we use the different dimensions to introduce step by step the needed tools.
• \( d = 1 \) massless fermionic models at finite temperature: Majorana fermions (Section 4.2.1 and proposition 6)

• \( d = 1 \) SYK model (massless fermionic model with quenching) at finite and large \( N \) (Section 4.2.2 and proposition 7)

• \( d = 1 \) massive fermionic models at finite temperature (Remark 8)

• \( d = 1 \) non-relativistic bosonic models at finite temperature (Remark 9)

• \( d = 2 \) relativistic bosonic models at finite temperature (Section 5 and proposition 9)

The cases of real and complex for bosons, and Majorana and Dirac for fermions, differ marginally. For simplicity, we will mostly focus on Majorana fields, while we will alternate between real and complex bosonic fields to show how the arguments are modified.

The remaining super-renormalizable and just-renormalizable models will be considered in a future paper.

1.1 Quartic vector models

We review briefly some of the theories which motivated the current analysis.

**Statistical physics and data science** Given a large number of interacting particles, it is generically not possible to describe exactly the dynamics of each individual particle. Statistical physics replaces this exact knowledge of each variable \( \phi_i \) (which can be positions, velocities...) by a probability distribution

\[
p(\phi_i) = \frac{1}{Z} e^{-S(\phi_i)},
\]

where \( S \) is the Hamiltonian (Euclidean action) and \( Z \) is a normalisation factor (called the partition function). Since the variables \( \phi_i \) form a vector in the internal (or flavor) space, this explains the terminology of vector model.

Due to the central limit theorem, the probability distribution is well approximated by a Gaussian distribution for a large number of independent variables. The corresponding Hamiltonian is quadratic\(^2\)

\[
S = \phi_i C^{-1}_{ij} \phi_j,
\]

and deviations to this law correspond to higher-order polynomial terms

\[
S = \phi_i C^{-1}_{ij} \phi_j + W_{ijk} \phi_i \phi_j \phi_k + W_{ijklm} \phi_i \phi_j \phi_k \phi_l + \cdots
\]

\(^2\)We adopt Einstein convention: sums over repeated indices are implicit.
If the probability distribution is symmetric, then odd powers are absent (in particular, $W_{ijk} = 0$ in the previous equation).

When correlations are not negligible, for example close to a phase transition, the central limit theorem breaks down and the corrections to the Gaussian approximation are not necessarily small. However, additional considerations (such as taking into account the temperature) leads to consistent truncations of the Hamiltonian (1.3). In most cases, it is sufficient to consider only a polynomial of order 4, leading to a 0-dimensional (because there is no spacetime) quartic vector model.

In machine learning, one aim is to reconstruct the underlying distribution of a dataset (typically through unsupervised learning). This corresponds formally to the same problem as the one described in statistical physics [3]. A specific application of quartic vector models to PCA can be found in [4].

**Condensed matter and SYK model**  Quartic vector models are common in condensed matter. The standard textbook example is the effective theory of electrons in a superconductor [5, sec. 21.6]. The action of this model reads

$$S = -\int \! d^4x \psi_1(t, \vec{x}) \left( -i \frac{\partial}{\partial t} - A_0(t, \vec{x}) + E \left( -i \nabla + \vec{A}(t, \vec{x}) \right) \right) \psi_1(t, \vec{x})$$

$$+ \int \! dt \int \! d^3\vec{x} \psi_1(t, \vec{x}) \left( -i \frac{\partial}{\partial t} - A(t, \vec{x}) + E \left( -i \nabla + \vec{A}(t, \vec{x}) \right) \right) \psi_1(t, \vec{x})$$

$$+ \int \! dt \int \! d^3\vec{x}_1 d^3\vec{x}_2 d^3\vec{x}_3 d^3\vec{x}_4 \mathcal{J}_{i_1i_2i_3i_4} (\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$$

$$\times \psi_{i_1}(t, \vec{x}_1) \psi_{i_2}(t, \vec{x}_2) \psi_{i_3}(t, \vec{x}_3) \psi_{i_4}(t, \vec{x}_4).$$

where $\psi_i$ is the electron field, $A_0$ and $\vec{A}$ are the scalar and vector potentials of the external electromagnetic field, and $\mathcal{J}_{i_1i_2i_3i_4}$ is the position-dependent coupling tensor. This model falls beyond the scope of this paper due to the presence of the electromagnetic field and because the interaction is non-local (each field is at a different spatial point and the interaction tensor is not constant). Moreover, this model is non-renormalizable (see Section 2.2) and thus cannot be treated with the method of this paper.\(^3\)

On the other hand, the SYK model [6–9] (see [10, 11] for recent reviews) falls in the class of models studied in this paper. It received recently a lot of attention due to its distinguished features: it is solvable in the strong coupling regime, it displays a near-conformal invariance and it saturates the chaos bound proposed in [12]. All together these characteristics make the SYK model a good candidate to describe black holes (with near-horizon geometry $\text{AdS}_2$) through the $\text{AdS}/\text{CFT}$ correspondence [13]. The SYK model, as originally defined in [6, 7], corresponds to $d = 1$ quantum mechanical system (time only)

\(^3\)However, if the spatial dimensions are compact, the action in the momentum space reduces to the $d = 1$ case. Indeed, after Fourier transformation, the spatial momentum are discrete and can be collected together with the field index $i$.\)
of $N$ Majorana $\psi_i$ with a quartic interaction

$$S = \int dt \left( \frac{1}{2} \dot{\psi}_i \frac{d}{dt} \psi_i + \frac{1}{4!} J_{ijkl} \psi_i \psi_j \psi_k \psi_l \right).$$  \hspace{1cm} (1.5)$$

where the index $i$ runs from 1 to $N$, with random couplings (quenching), which is achieved by integrating over $J_{ijkl}$ with a Gaussian measure.

**String field theory**  String theory provides an attempt towards a theory of quantum gravity together with a unification of the matter and interactions. The standard formulation of string theory uses first-quantization (worldsheet formalism) and as such displays various limitations \[14\], in particular related to renormalization and multi-particle effects. The aim of string field theory is to provide a second-quantized version (i.e. a field theory) of string theory, allowing to use all the standard tools of QFT. Such a formulation is believed to be necessary for defining properly string theory and addressing its most fundamental issues (such as the question of the landscape and the definition of M-theory).

The complete definition of string field theory is highly technical and the interested reader is referred to the literature \[14, 15\]: we only provide the main ideas in order to connect to the models studied in this paper. The classical action of the bosonic closed string field theory reads

$$S = \sum_{n \geq 2} g_s^{n-2} \frac{n!}{n!} V_n(\Psi, ..., \Psi)$$ \hspace{1cm} (1.6)$$

where $\Psi$ is the string field and $g_s$ is the string coupling constant. The role of the $n$-string vertex $V_n$ is to combine $n$ strings and to output a complex number — it describes how $n$ strings interact with each other.

A string is an extended object and as such it can vibrates in many different ways, in addition to moving in spacetime. This can be studied by developing the string field into (discrete and continuous) Fourier modes

$$\Psi = \sum_i \int d^dk \phi_i(k) \Psi_i(k)$$ \hspace{1cm} (1.7)$$

where $\{\Psi_i(k)\}$ is a basis for the string field space, $i$ denotes a set of discrete labels and $k$ is the spacetime momentum. The coefficients $\phi_i(k)$ of the expansion will be interpreted as particle-like spacetime fields (of all integer spins in the bosonic string, of integer and half-integer spins for the superstring). Inserting
this expansion in the action up to order $O(g^2_s)$ yields

\begin{align}
S & = \int d^d k \phi_i(k) C^{-1} \phi_j(k) \\
& + \int d^d k_1 d^d k_2 d^d k_3 \mathcal{W}_{i\bar{i}j\bar{j}k\bar{k}}(k_1, k_2, k_3) \phi_{i\bar{i}}(k_1) \phi_{j\bar{j}}(k_2) \phi_{i\bar{k}}(k_3) \\
& + \int d^d k_1 d^d k_2 d^d k_3 d^d k_4 \mathcal{W}_{i\bar{i}j\bar{j}k\bar{k}l\bar{l}}(k_1, k_2, k_3, k_4) \\
& \quad \times \phi_{i\bar{i}}(k_1) \phi_{j\bar{j}}(k_2) \phi_{i\bar{k}}(k_3) \phi_{i\bar{l}}(k_4)
\end{align}

(1.8)

(it is always possible to choose the basis such that the quadratic term has only one integral). The cubic and quartic interaction tensors are defined by

\begin{align}
\mathcal{W}_{i\bar{i}j\bar{j}k\bar{k}}(k_1, k_2, k_3) & := \mathcal{V}_3\left(\Psi_{i\bar{i}}(k_1), \Psi_{j\bar{j}}(k_2), \Psi_{i\bar{k}}(k_3)\right), \\
\mathcal{W}_{i\bar{i}j\bar{j}k\bar{k}l\bar{l}}(k_1, k_2, k_3, k_4) & := \mathcal{V}_4\left(\Psi_{i\bar{i}}(k_1), \Psi_{j\bar{j}}(k_2), \Psi_{i\bar{k}}(k_3), \Psi_{i\bar{l}}(k_4)\right).
\end{align}

(1.9a, 1.9b)

Thus, the quartic theories studied in this paper provide a (crude) toy model$^5$ for string field theory (one has to keep only the scalar fields, to forget the cubic interaction and to truncate the momentum dependence of the interaction).

Recently, many progresses have been made in proving consistency properties (such as unitarity, crossing symmetry, etc.) of string theory by writing a general QFT which includes string field theory as a subcase and generalizing standard QFT methods [14, 16–20]. Following this fruitful approach, it would be interesting to generalize the models – and techniques – considered in this paper to make them as close as possible to string field theory, and, ultimately, to try to extend the constructive techniques to string field theory itself.

**Particle physics and Higgs sector**

As an additional motivation, renormalizability implies that interactions can be at most quartic in $d = 4$ dimensions. This is relevant for the Higgs sector: while the Standard Model contains a single Higgs field, considering several Higgs fields is common in Beyond Standard Model phenomenology (such as in supersymmetric extension [21, 22] or in multi-Higgs doublet models [23]). Such models are of the form studied this paper, but the case $d = 4$ is just renormalizable and will be postponed to future work.

### 1.2 Constructive QFT and LVE

Because constructive field theory is not part of most theoretical physics curriculum, we begin by a brief summary of the aim – dealing with the sum-

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$^4$These quantities are fixed once the basis $\{\Psi_i(k)\}$ is chosen. The latter determines the background on which the string theory is defined (number of non-compact dimensions, geometry of the compactified dimensions, etc.).

$^5$After performing a Fourier transformation from the momentum space to the position space.
mability of perturbative expansions – and the general strategy of constructive techniques, of which the loop vertex expansion (LVE) is one example.

In a QFT (that is, a theory with thermal or quantum fluctuations), the partition function $Z$ (or, equivalently, the free energy $F$) – from which all relevant informations can be extracted – is defined by a path integral, that is, a functional integration over the fields describing the degrees of freedom of the system together with a weight (usually corresponding to the classical action).

Usually this partition function cannot be computed exactly and one resorts to a perturbative expansion in the coupling constants. Then, exchanging the integration and sum leads to a sum of Gaussian integrals which have a natural interpretation in terms of amplitudes $A_G$ indexed by Feynman diagrams $G$:

$$F = \sum_{G \text{ connected}} A_G .$$  \hfill (1.10)

However, this is an asymptotic series which diverges (said another way, its radius of convergence in the coupling constants vanishes) even if each individual amplitude is finite (or renormalized if UV divergences occur) [24]

$$\sum_{G} |A_G| = \infty .$$ \hfill (1.11)

In terms of Feynman diagrams, this can be understood from the fact that general graphs proliferate rapidly and lead to a combinatorial divergence. In turn, this divergence can be tracked back to the illegal manipulation of exchanging sums and integrations in the perturbative expansion.

The free energy must be defined in another way, and making sense of this procedure is the goal of constructive field theory [25–27].

Since the divergence arises due to the number of graphs that exist at each order, this hints towards finding another expansion in terms of other objects which number grows slower: trees. The standard strategy is to consider, for any pair made of a connected Feynman diagram $G$ and of a spanning tree $T \subset G$ in it, a universal non-trivial weight\footnote{This weight is the percentage of Hepp’s sectors of $G$ in which $T$ is leading in the sense of Kruskal’s greedy algorithm.} $\varpi(G, T)$. These weights, being percentage by definition, are normalized such that:

$$\sum_{T \subset G} \varpi(G, T) = 1 .$$ \hfill (1.12)

They allow to rewrite the Feynman expansion as a sum indexed by spanning trees rather than Feynman diagrams:

$$F = \sum_{G} A_G = \sum_{G} \sum_{T \subset G} \varpi(G, T) A_G = \sum_{T} A_T ,$$ \hfill (1.13)
where:

\[ A_T := \sum_{G \supset T} \varpi(G, T) A_G . \]  

(1.14)

Now, since the trees do not proliferate as fast as Feynman graphs, in good cases it can be shown that:

\[ \sum_T |A_T| < \infty , \]  

(1.15)

at least in a certain summability domain. The loop vertex expansion (LVE), that we will use in this paper, provides an explicit realization of this general strategy.

For this, one introduces first an auxiliary field (called the intermediate or Hubbard–Stratonovich field) in order to decompose the quartic interactions into cubic interactions [28, 29]. Since the resulting action is quadratic in the original field, the latter can be integrated out to obtain an effective action for the intermediate field. The idea of the loop vertex expansion (LVE) [1, 2, 27, 30, 31] is to expand this action in a very specific manner – in terms of the effective vertex –, such that the resulting terms of the series are interpreted as trees. Then, standard tools from constructive field theory allows to show that the LVE series is analytic in the coupling constant in some non-empty domain and has thus a finite radius of convergence [32, 33]. In turn, this implies that the original path integral is uniquely defined by this expansion in its domain of convergence. This also shows the existence of the path integral itself (which is not guaranteed since functional integrals are defined formally as the question of convergence in infinite-dimensional space is subtle). Finally, Nevanlinna’s theorem is used to show the Borel summability of the perturbative expansion. As a consequence, the Borel transform of the perturbative series has a finite radius of convergence and is equivalent to the LVE, showing that it makes sense to use it for practical purposes.

When there is at least one spatial dimension, new divergences appear due to quantum effects. Accordingly, one has to refine the analysis by introducing a decomposition of the graphs into scales – leading to the MLVE (multi-scale LVE) – such that divergences can be renormalized [27, 34] (see also [35, 36]).

Earlier studies of relativistic bosons include: [37–39] for \( d = 0 \), [40] for \( d = 1 \), [41–44] for \( d = 2 \), [45] for \( d = 3 \), and [2, 46] for \( d = 4 \). A general but less rigorous argument for all super-renormalizable relativistic scalar fields with generic polynomial interactions has been given in [47, 48]. The Borel summability in the large \( N \) limit is discussed in [49, 50], and the non-relativistic case in [51]. For a perspective on higher-order interactions, see [52–54]. Constructive methods – and, especially, the LVE – have also been generalized to quartic matrix [1, 55–58] and tensor [59–65] models.

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\(^7\)UV divergences in \( d = 1 \) are not physical as they can be removed under proposer regularization. For this reason, the MLVE is not necessary in this case.
1.3 Outline

In Section 2, we introduce the bosonic and fermionic quartic vector models studied in this paper, along with some terminology. Moreover, we describe in details the power-counting in Section 2.2. Section 3 performs the LVE analysis for the $d = 0$ bosonic (Section 3.1) and fermionic (Section 3.2) models. Next, Section 4 extends the analysis to the $d = 1$ models at finite temperature, considering the zero-temperature as a limiting case. Quenching leads to the SYK model, which is studied at both at finite and large $N$ in Section 4.2.2. Finally, the $d = 2$ relativistic bosonic models are studied in Section 5, which introduces the MLVE.

2 Quartic vector models

In this section, we summarise the properties of the $d$-dimensional bosonic and fermionic vector models, including in particular a description of the propagators and of the power-counting.

2.1 Bosonic and fermionic models

This paper is devoted to a family of statistical models for random $N$-component vector fields $\{\phi_i\}$ in $d$ dimensions, $\phi_i : \mathbb{R}^d \rightarrow \mathbb{R}, \mathbb{C}$, $i = 1, \cdots, N$, collectively denoted as $\Phi := \{\phi_i\}$. The value at the point $x \in \mathbb{R}^d$ is denoted as $\phi_i(x)$. The field components $\phi_i$ can be either ordinary commutative numbers ($\phi_i \phi_j = \phi_j \phi_i$) or non-commutative (Grassmann valued) numbers ($\phi_i \phi_j = -\phi_j \phi_i$), depending if we describe bosonic or fermionic random degrees of freedom. Note that we focus on statistical models, that is to say, quantum field theory in imaginary or Euclidean time.

To put in a nutshell, without lost of generality, the probability law $p(\Phi)$ for a configuration $\Phi$ is fixed by the choice of a classical action $S(\Phi)$:

$$p(\Phi) = \frac{1}{Z} e^{-S(\Phi)},$$

(2.1)

where $Z$, the partition function is a normalization factor for the probability distribution, ensuring that $\int p(\Phi) d\Phi = 1$:

$$Z := \int \prod_{i=1}^{N} d\phi_i e^{-S(\Phi)},$$

(2.2)

where $d\phi_i$ denotes the generalized Lebesgue measure defining path integration. The choice of the classical action $S(\Phi)$ characterizes the probability distribution, and the dominant configurations for the set $\{\phi_i\}$. These configurations

---

The index $i$ is sometimes called "flavour".
are fixed from the saddle-point equation:

\[
\frac{\partial S}{\partial \phi_i}(\Phi) = 0.
\]  

(2.3)

Usually, the choice of the classical action comes from general principles as well as specific conditions due to the nature of the random variables. In this respect, we have to distinguish between bosonic and fermionic fields. Among the common features, the asymptotic positivity is a minimal requirement, ensuring integrability of the probability density. Then, we assume that \( S(\Phi) \) can be expanded in power of the fields; and a weakly necessary condition for the realization of the first condition is to keep only the even terms in the expansion. For our considerations, we will limit the expansion to the first term beyond the Gaussian regime, the quartic term. For the rest of this paper and to clarify the notations, we will denote as \( \phi_i \) the bosonic variables and as \( \psi_i \) the fermionic ones. \( \Phi \) then denote the collective bosonic vector, and \( \Psi := \{\psi_i\} \) the analogous for fermions fields. Finally, without specification about the nature of the random variable, we will denote them as \( \Xi := \{\chi_i\} \).

For real fields \( \chi_i : \mathbb{R}^d \to \mathbb{R} \) we choose a classical action of the form:

\[
S(\Xi) = \int dx \left[ \frac{1}{2} \sum_{i,j} \chi_i(x) C_{ij}^{-1} \chi_j(x) + \frac{u}{4!} \sum_{ijkl} \mathcal{W}_{ijkl} \chi_i(x) \chi_j(x) \chi_k(x) \chi_l(x) \right].
\]

(2.4)

In this expression \( C_{ij} \) designates the covariance matrix, \( u \) the coupling constant and \( \mathcal{W}_{ijkl} \) the coupling tensor. Note that the interaction is local is the usual meaning in field theory. Moreover, in these notations, \( C_{ij} \) is understood as a differential operator with respect to the space variable \( x \):

\[
C_{ij} := M_{ij} + a_{ij,\alpha}^{(1)} \frac{\partial}{\partial x_\alpha} + a_{ij,\alpha\beta}^{(2)} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} + \cdots,
\]

(2.5)

where the Greek indices \( \alpha, \beta \) run from 1 to \( d \) and the Einstein convention for repeated indices is assumed. Note that the mass matrix \( M_{ij} \) as well as the weight matrices \( a_{ij}^{(n)} \) do not depend on the positions \( x \). For bosonic fields, \( \chi_i \chi_j \) and \( \chi_i \chi_j \chi_k \chi_l \) are completely symmetric tensors, whereas they are completely antisymmetric tensors for fermionic fields. As a result:

- **bosonic fields**: \( C_{ij} = C_{ji} \) and \( \mathcal{W}_{ijkl} \) is completely symmetric.
- **fermionic fields**: \( C_{ij} = -C_{ji} \) and \( \mathcal{W}_{ijkl} \) is completely anti-symmetric.

For complex fields \( \chi_i : \mathbb{R}^d \to \mathbb{C} \), we choose the classical action of the form:

\[
S(\Xi) = \int dx \left[ \sum_{i,j} \bar{\chi}_i(x) C_{ij}^{-1} \chi_j(x) + \frac{u}{4!} \sum_{ijkl} \mathcal{W}_{ijkl} \bar{\chi}_i(x) \chi_j(x) \chi_k(x) \bar{\chi}_l(x) \right].
\]

(2.6)
Once again, the permutation symmetries of the covariance and coupling tensors depend on the nature of the fields. Requiring the classical action to be a real functional of the fields, we get the conditions:

- **bosonic fields:** \( C_{ij} = C_{ij}^\dagger \) and:
  \[
  W_{ijkl} = W_{klij} = W_{ilkj} = W_{kjil} = W_{jilk}^*.
  \]  
  \[(2.7)\]

- **fermionic fields:** \( C_{ij} = -C_{ij}^\dagger \) and:
  \[
  W_{ijkl} = W_{klij} = -W_{ilkj} = -W_{kjil} = W_{jilk}^*.
  \]  
  \[(2.8)\]

In this paper, we focus on UV finite and super-renormalizable field theories in low dimensions. Obviously, these characteristics depend on the form of the classical action: especially, on the degree of the interactions and on the choice of the bare propagator. For quartic models, the degree of the interaction is fixed and therefore only the bare propagator can be relevant for characterizing the renormalizability of the theory. Usually, physics imposes some strong constraint on the choice of this propagator, which is generically of the form \( (2.5) \). Among the constraints fixing the bare propagator, the natural symmetry group of the spacetime is generically relevant. In particular, the equations of movement are expected to be independent of the specific orientation of the coordinate system, and therefore invariants with respect to the rotation group \( \text{SO}(d) \). But it may happens that all the coordinates do not play the same role. For condensed matter models, one coordinate is distinguished as an Euclidean time, related to the temperature, and the rotational invariance is expected only with respect to the subgroup \( \text{SO}(d-1) \). In this paper, we will therefore distinguish between the non-relativistic or Schrödinger-type propagators, invariants with respect to \( \text{SO}(d-1) \), and the relativistic propagators, invariant with respect to \( \text{SO}(d) \). As a result, \( C^{-1} \) has to be build in terms of \( \text{SO}(D) \) invariants, \( D \) being equals to \( d \) or \( d - 1 \).

A realization of this condition is to impose that the matrix coefficients \( a^{(n)}_{\alpha_1 \cdots \alpha_n} \) or a sub-block of them transform, with respect to the Greek indices, like a tensor representations of the symmetry group \( \text{SO}(D) \). Other constraints comes from the standard axioms in field theory. One of them, the so-called Osterwalder-Schrader (OS) positivity, imposes a strong restriction on the degree of the derivative operators involved in \( C^{-1} \), which can not exceed two. As a result, the propagators that we will consider for the rest of this paper will all have the form of a direct product:

\[
C^{-1}_{ij} := K_{ij} C^{-1} \quad \text{with} \quad C^{-1} := \mu + a^{\alpha} \partial_{\alpha} + b^{\alpha\beta} \partial_{\alpha} \partial_{\beta},
\]  
\[(2.9)\]

where \( \mu \in \mathbb{R} \) is a pure number, \( a^{\alpha} \) and \( b^{\alpha\beta} \) are the components of tensors in \( d \) dimensions, and \( K_{ij} \) is a real invertible \( N \times N \) matrix. Note that, in addition to the external indices \( i \) and \( j \), the matrix \( K \) may depend on some internal
indices like spin, as it will be the case for Majorana fermions (see Section 3.2) or for Dirac fermions discussed at the end of this section. The choice of this matrix depends on the specificity of the model that we consider, however, we can expect that it has to be bounded:

\[ |K_{ij}| \leq \kappa, \forall i, j, \quad \kappa \in \mathbb{R}. \tag{2.10} \]

For large \( N \), additional conditions could be imposed to make the sums over loops not so big. Indeed, we will see in Section 3.1, that for \( d = 0 \) the radius of convergence of the LVE constructive expansion goes to zero as \( 1/N \) for identically distributed vector fields \( K_{ij} = \delta_{ij} \). For this reason, it may be reasonable to impose a condition such that \( K_{ij} \) goes to zero for \( i, j \gg 1 \). Convergence of the sums is expected for exponential decay, but it has to be discussed for slower decays, especially for power decays as:

\[ |K_{ij}| \lesssim \min_{i,j>1} \left( \frac{1}{i^\epsilon}, \frac{1}{j^\epsilon} \right) \tag{2.11} \]

where \( \epsilon \) is an arbitrary real number, possibly small. In most of this paper (the exception being Section 4.2), we will assume \( N \) finite and not so big, and the field components identically distributed \( K_{ij} = \delta_{ij} \). Except for the choice of the matrix \( K_{ij} \), the different types of models that we will consider will be distinguished from the choice of the spatial covariance \( \tilde{C} \). This choice depends on general considerations about the considered model. However, physical models, that is to say, models closely related to a concrete physical system, are generally minor variations around a given fundamental choice. For instance, the propagator for Dirac fermions is not so far from the propagator for Majorana spinors. In particular, the details are not relevant for the power-counting theorem proved in the next section, and we can consider a restricted set of representative choices, keeping in mind that minor differences may occur for cousin models. From these considerations, we will focus on three representative families of models, reflecting the choice of the symmetry group and statistics:

- Schrödinger propagator: non-relativistic bosons and fermions in \((d-1)\)-dimensional Euclidean space:

\[ \tilde{C}^{-1} = \frac{\partial}{\partial t} - \kappa \Delta + m \tag{2.12} \]

where \( \Delta \) is the usual Laplacian over \( \mathbb{R}^{d-1} \): \( \Delta := \sum_{i=1}^{d-1} \frac{\partial^2}{\partial x_i^2} \), and \( \kappa \) is a positive constant. For physical particles, it may be expressed in term of the Planck constant \( \hbar \) and the effective mass as \( m^* \)

\[ \kappa = \frac{\hbar^2}{2m^*}. \tag{2.13} \]
• *Klein–Gordon* propagator: relativistic bosons in $d$–dimensional Euclidean space-time:

$$\tilde{C}^{-1} = -\frac{\partial^2}{\partial t^2} - \Delta + m^2 .$$

(2.14)

• *Dirac* propagator: relativistic fermions in $d$–dimensional Euclidean space-time:

$$\tilde{C}^{-1} = \gamma^\alpha \partial_\alpha + m .$$

(2.15)

where $\gamma^\alpha$, $\alpha = 1, \ldots, d$ denote the Dirac matrices, satisfying the Dirac algebra $\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}$ and $\text{tr}(\gamma^\alpha) = 0$ for $d > 0$. Dirac matrices are $2^n \times 2^n$ matrices$^9$, meaning that relativistic fermions $\psi_i$ have to be understood as $2^n$- anti-commutative vectors:

$$\psi_i = \begin{pmatrix} \psi_{i,1} \\ \vdots \\ \psi_{i,2^n} \end{pmatrix} .$$

(2.16)

### 2.2 Power-counting and classification

In order to investigate the divergences of the theory, we will introduce a slicing in the interior of the Feynman amplitudes, by decomposing each propagator edge into slices, following the standard *multi-scale decomposition*. First of all, we rewrite each propagator using Schwinger trick, and we introduce a slice index $i \geq 1$. The propagator in Fourier space in the slice $i \geq 1$ is written as $\tilde{C}^{(i)}$. For each model, they read:

- non-relativistic bosons and fermions:

  $$\tilde{C}^{(i)}(\omega, \vec{p}^2) := (-i\omega + E_c) \int_{M-2i}^{M-2(i-1)} d\alpha e^{-\alpha(\omega^2+E_c^2)} ,$$

  (2.17)
  where we defined $E_c$, the kinetic energy, as $E_c := \kappa \vec{p}^2 + m$.

- relativistic bosons:

  $$\tilde{C}^{(i)}(\omega^2, \vec{p}^2) := \int_{M-2i}^{M-2(i-1)} d\alpha e^{-\alpha(\omega^2+\vec{p}^2+m^2)} .$$

  (2.18)

- relativistic fermions:

  $$\tilde{C}^{(i)}(\omega, \vec{p}) := (-\gamma^\mu \partial_\mu + m) \int_{M-2i}^{M-2(i-1)} d\alpha e^{-\alpha(\omega^2+\vec{p}^2+m^2)} .$$

  (2.19)

  where for the last equality we used the relation $(-\gamma^\mu \partial_\mu + m)(\gamma^\mu \partial_\mu + m) = -\Delta + m^2$.

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$^9n$ being such that $d = 2n$ for even $d$ and $d = 2n + 1$ for odd $d$. 

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The complete propagator is then given by the sum
\[ \tilde{C} = \sum_{i=1}^{i_{\text{max}}} \tilde{C}^{(i)} + \tilde{C}^{(0)} , \] (2.20)
where \( i_{\text{max}} \) is some fixed integer. With these notations, \( M^{i_{\text{max}}} \) has to be interpreted as the UV regulator, ensuring finiteness of the loop integrals for large momenta. The remainder \( \tilde{C}^{(0)} \) includes integration over Schwinger parameter \( \alpha \) from 1 to \( \infty \), and is completely regular in the UV. Similarly, the decomposition could be used to introduce a cut-off in the region \( \alpha \gg 1 \) in order to prevent IR divergences. However, for our considerations, such a prescription remains unnecessary, the mass parameter ensuring that IR divergences are discarded.

To enunciate the main statement of this section, we have to specify the range of the momenta \( \omega \) and \( \vec{p} \), which do not play necessarily the same role. For non-relativistic models at finite temperature \( \beta \), the time coordinate is periodic \( t \sim t + \beta \) and a fundamental interval is \( t \in [-\beta/2, \beta/2] \). Then, the conjugate momentum \( \omega \) takes discrete values:
\[ \omega = \frac{2\pi n}{\beta} \quad \text{(bosons)}, \quad \omega = \frac{(2n + 1)\pi}{\beta} \quad \text{(fermions)} , \] (2.21)
where \( n \in \mathbb{N} \). These two constraints for bosons and fermions come respectively from the usual periodicity and anti-periodicity imposed on Fourier basis functions. The zero-temperature limit \( \beta \to \infty \) corresponds to non-compact Euclidean time \( t \in \mathbb{R} \) (without periodic conditions). For spatial momenta \( \vec{p} \), however, there is no restriction, except if we impose to the system to leave into a box of finite size. For our purpose, we assume \( \vec{p} \in \mathbb{R}^{d-1} \) to be a continuous variable covering the entire space, without restriction.

These considerations covered, we will establish the following power-counting theorem from this scale decomposition:

**Theorem 1** Let \( \mathcal{A}_G \) be the amplitude of the connected Feynman graph \( G \) with \( V \) vertices and \( L \) internal propagator edges. Introducing a slicing for each propagator edge, we denote by \( \mu = \{i_1, \ldots, i_L\} \) a scale assignment for edges labelled from 1 to \( L \). The amplitude can then decomposed as a sum over scale assignments so that:
\[ \mathcal{A}_G = \sum_{\mu} \mathcal{A}_{G,\mu} . \] (2.22)

For fixed \( \mu \), let \( G_i = \bigcup_{k=1}^{k(i)} G^k_i \subset G \) be the subgraph of \( G \) made of edges with scale assignment higher or equal to \( i \), and \( G^k_i \) its \( k \)th connected component, and where \( k(i) \) is the total number of connected components. We have the following statement in the UV sector, that is, for scale assignments higher or equals to 1:
\[ |\mathcal{A}_{G,\mu}| \leq K^{L(G)} \prod_{i,k} M^{\Omega(G^k_i)} \] (2.23)
the bound being uniform for some constant $K$, and where:

- non-relativistic bosons and fermions:
  \[ \Omega(G_i^k) = -L(G_i^k) + \frac{1}{2} (d - 1) (L(G_i^k) - V(G_i^k) + 1), \]  \[(2.24)\]

- relativistic bosons:
  \[ \Omega(G_i^k) = -2L(G_i^k) + d(L(G_i^k) - V(G_i^k) + 1), \]  \[(2.25)\]

- relativistic fermions:
  \[ \Omega(G_i^k) = -L(G_i^k) + d(L(G_i^k) - V(G_i^k) + 1), \]  \[(2.26)\]

In these expressions, $L(G)$ and $V(G)$ denotes respectively the numbers of internal edges and vertices of the diagram $G$.

**Proof.** We will consider in full details the bound for non-relativistic particles. First, we have to bound the propagator into the slice $i$. In the UV sector, we have the trivial uniform bound:

\[ |\tilde{C}^{(i)}(\omega, \mathbf{p}^2)| \leq K M^{-i} e^{-M^{-i}(|\omega| + E_c)}. \]  \[(2.27)\]

From the Feynman rules, we then deduce the first bound

\[ |A_{G,\mu}| \leq K^L \prod_{e \in G} M^{-i_e} \times \int_{\ell=1}^{\pi(G)} \prod_{e \in \partial \ell} e^{-M^{-i_e}(|\omega_e| + E_{c,e})}, \]  \[(2.28)\]

where $e$ are the edges of the graph, $\pi(G)$ denotes the set of loops in $G$, and $\partial \ell$ the boundary of the loop $\ell$, that is, the edges building the loop. $\omega_e$ and $\mathbf{p}_\ell$ denote the energy and momentum along the loop $\ell$. For $e \in \partial \ell$, $\omega_e = \omega_\ell + \delta_e$, where the translation $\delta_e$ comes from the energy conservation at each vertex. The same thing holds for momenta. Let $i_\ell$ be the smallest scale assignment in the boundary of $\ell$. We then have:

\[ \int_{\ell=1}^{\pi(G)} \prod_{e \in \partial \ell} e^{-M^{-i_e}(|\omega_e| + E_{c,e})} \leq K M^{\frac{1}{2} i_{\ell} (d+1)}, \]  \[(2.29)\]

so that the bound for $A_{G,\mu}$ becomes:

\[ |A_{G,\mu}| \leq K^L K^{i_{\pi(G)}} \prod_{e \in G} M^{-i_e} \times \prod_{\ell=1}^{\pi(G)} M^{\frac{1}{2} i_{\ell} (d+1)}. \]  \[(2.30)\]

Then, because of the obvious relations:

\[ \prod_{e \in G} M^{-i_e} = \prod_{i=1}^{i_e} M^{-1} = \prod_{i} \prod_{k} \prod_{e \in G_i^k} M^{-1} = \prod_{i,k} M^{-L(G_i^k)}, \]  \[(2.31)\]
and
\[ \prod_{\ell=1}^{\pi(G)} M^{\frac{1}{2}L \mu(d+1)} = \prod_{i,k} M^{\frac{1}{2}L\pi(G^i_k)(d+1)} , \] (2.32)
we get:
\[ |A_{G,\mu}| \leq K L K' |\pi(G)| \prod_{i,k} M^{-L(G^i_k)+\frac{1}{2}L\pi(G^i_k)}(d+1) , \] (2.33)

Where $|\pi(G)|$ is the number of loops in $G$. Finally, because $|\pi(G)| = L(G) - V(G) + 1$, and that for a quartic model $4V \geq 2L$ we then deduce the power-counting (2.24). The two bounds (2.25) and (2.26) come from the same strategy, the main changing coming from the bound of the propagator into the slice $i$.

For relativistic bosons, the propagator (2.18) admits the uniform bound
\[ |\tilde{C}^{(i)}(\omega^2, \vec{p}^2)| \leq KM^{-2i}e^{-M^{-2i}(\omega^2+\vec{p}^2)} . \] (2.34)
Firstly, with respect to the non-relativistic bound (2.28), the factor $M^{-2i}$ generates the contribution $\prod_e M^{-2i_e}$, replacing the contribution $\prod_e M^{-i_e}$ in (2.28).
Secondly, optimizing the integration along each loop, we generate a factor $\prod_{\ell} M^{-d\ell}$. Then, from the same trick as in equations (2.31) and (2.32), we obtain the power-counting (2.25).

For the relativistic fermions, the propagator into slice $i$ (2.19) has the bound
\[ |\tilde{C}^{(i)}(\omega^2, \vec{p}^2)| \leq KM^{-i}e^{-M^{-2i}(\omega^2+\vec{p}^2)} . \] (2.35)
we recover both the $M^{-i}$ of the non-relativistic theory in front of the exponential, and the Gaussian integration of the relativistic bosons. The power-counting (2.26) follows.

\[ \square \]

From this theorem, we can establish a criterion allowing to classify theories following their renormalizability, from super-renormalizable theories to non-renormalizable theories. Indeed, because for a graph $G$ with $V$ 4-valent vertices, $L$ internal edges and $N$ external edges:
\[ 4V = 2L + N , \] (2.36)
the powers counting (2.24), (2.25) and (2.26) may be rewritten as:
- non-relativistic bosons and fermions
\[ \Omega = \frac{1}{2}(d - 3)V - \frac{d - 1}{4}N + \frac{1}{2}(d + 1) \] (2.37)
• relativistic bosons

\[ \Omega = (d - 4)V - N \left( \frac{d}{2} - 1 \right) + d \]  

(2.38)

• relativistic fermions

\[ \Omega = (d - 2)V - \frac{d-1}{2}N + d \]  

(2.39)

Note that these power-counting are well known in field theory, we recall them for the self-consistency of this paper. In particular, the power-counting and the normalizability criteria that we will deduce are independent of the specific structure of the interaction tensors.

We deduce the following renormalizability criteria for each case studied in this paper:

• Non-relativistic bosons and fermions are super-renormalizable for \( d < 3 \), just-renormalizable for \( d = 3 \) and non-renormalizable otherwise. The case \( d = 1 \) is UV finite: the fermionic case is studied in Section 4.2 and the bosonic case in Remark 9. The power-counting becomes

\[ \Omega = -\frac{V}{2} + \frac{6 - N}{4}, \]  

(2.40)

and there are only a finite set of divergent diagrams. For \( N = 0 \), \( \Omega = (3 - V)/2 \), and the divergent vacuum diagrams are for \( V = 1, 2 \) and 3. For \( N = 2 \), \( \omega = (2 - V)/2 \) and the divergent diagrams are for \( V = 1 \) and \( V = 2 \). Finally, for \( N = 4 \), there is in principle one divergent diagram for \( V = 1 \), but such a cannot have a loop and thus is not a divergent diagram.

• Relativistic bosons are super-renormalizable for \( d < 4 \), just renormalizable for \( d = 4 \) and non-renormalizable otherwise. The \( d = 1 \) case is finite since \( \Omega < 0 \). In the case \( d = 2 \), the power-counting becomes \( \Omega = 2(1 - V) \). There are two divergent diagrams, for \( V = 1 \) and \( N = 0, 2 \).

• Relativistic fermions are super-renormalizable for \( d < 2 \), just renormalizable for \( d = 2 \), and non-renormalizable otherwise. For \( d = 1 \), there is only one divergent vacuum diagram made of a single vertex.

In this paper, we will focus on low dimensions because the proof complexity increases with the number of dimensions (due to the UV divergences). Moreover, the LVE technique has to be considerably improved for just-renormalizable models. We will focus mostly on \( d = 0, 1 \), adding the proof for relativistic bosons in \( d = 2 \) in order to also discuss the MLVE. Other models are postponed to a future paper.
Moreover, we will not cover all the possible cases which are well-defined (real/complex, finite/zero temperatures, massive/massless) since the differences are often inessential for the proof and taking into account all possibilities would weight the calculations unnecessarily. Instead, we will detail the proofs for a subset of representative models, and comment how the analyticity bounds and Borel summability for the remaining models can be obtained. We focus on Majorana fields because these models are related to the SYK models.

3 Constructive expansion for \( d = 0 \)

We first investigate the bosonic case with real fields, before studying the fermionic case (Majorana fermions). We will construct the LVE constructive expansion explicitly, and show the existence of a finite domain of analyticity, allowing to prove Borel summability of the free energy and connected correlation functions. On the way towards these results, we will fix some conventions and notations useful for the rest of this paper. Moreover, we will introduce some useful tools, such as the BKAR forest formula, the intermediate field decomposition as well as some key theorems for the proofs.

3.1 Bosonic models

3.1.1 Intermediate field formalism and BKAR forest formula

Let \( Z(u) \) the partition function of the statistical model defined as

\[
Z(u) = \int d\mu_C(\phi) e^{-\frac{u}{4!} \sum_{ijkl} W_{ijkl} \phi_i \phi_j \phi_k \phi_l},
\]

where \( d\mu_C(\phi) \) is a shorthand notation for the Gaussian measure:

\[
d\mu_C(\phi) := \prod_{i=1}^{N} d\phi_i e^{-\frac{1}{2} \phi_i C_{ij}^{-1} \phi_j}.
\]

For \( d = 0 \) \( C_{ij} \equiv K_{ij} \) and we use of \( C_{ij} \) for this section. The interaction tensor is completely symmetric. It can be viewed as a matrix \( W_{IJ} \), where the big indices \( I,J \equiv (ij),(kl) \) run from 1 to \( N(N+1)/2 \). As a symmetric matrix with real coefficients, \( W_{IJ} \) can be diagonalized with eigenvalues \( \{\Lambda_L\} \):

\[
W_{IJ} = \sum_{L} \Lambda_L O_{IL} O_{LJ}^T,
\]

where \( O \) is an orthogonal matrix. Defining the new field \( \Psi_L \) as:

\[
\Psi_L := \sum_{I=(ij)} O_{IL} \phi_i \phi_j,
\]

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the partition function becomes:

$$Z(u) = \int d\mu_C(\phi) \prod_L e^{-\frac{u}{4} \Lambda_L |\Psi_L|^2}.$$  \hspace{1cm} (3.5)

If we suppose $u$ positive, a simple way to ensure stability is to assume that the eigenvalues $\Lambda_L$ are all positive. In fact, we will remove formally this condition venturing into the complex plane $u = \rho e^{i\theta}$, and show that summability makes sense when $u$ becomes complex.

Now, we introduce the matrix–like intermediate fields $\sigma^{(L)}$, indexed with a pair of indices $L$, with the normalized Gaussian measure:

$$d\nu(\sigma) := \prod_L e^{-\frac{1}{2} (\sigma^{(L)})^2} d\sigma^{(L)},$$  \hspace{1cm} (3.6)

in order to break the quartic interaction $|\Psi_L|^2$ as a three-body interaction:

$$Z(u) = \int d\mu_C(\phi) d\nu(\sigma) \prod_L e^{\sqrt{-\frac{u}{4\Lambda_L}} \lambda_L \Psi_L \sigma^{(L)}},$$  \hspace{1cm} (3.7)

where $\lambda^2_L := \Lambda_L$. Note that $\lambda_L \Psi_L := A_L$ is a symmetric matrix in the original little indices $L \equiv (kl)$. Viewed as a matrix, $\sigma^{(kl)}$ can be considered symmetric without loss of generality. Indeed, decomposing it in symmetric and anti-symmetric parts as

$$\sigma = \sigma_S + \sigma_{AS},$$  \hspace{1cm} (3.8)

the contraction $\sum_{kl} A_{kl} \sigma^{(kl)}$ only selects the symmetric part: $\sum_{kl} A_{kl} \sigma^{(kl)} = \sum_{kl} A_{kl} \sigma_S^{(kl)}$. Moreover, the kinetic action for the matrices $\sigma$ reads:

$$\frac{1}{2} \sum_{ij} (\sigma^{(ij)})^2 = \frac{1}{2} \text{Tr}(\sigma_S)^2 - \frac{1}{2} \text{Tr}(\sigma_{AS})^2.$$  \hspace{1cm} (3.9)

Then, the anti-symmetric part in completely decoupled from the rest of the action, and can be integrated out.\(^1\) The integration over the original field variable is now a simple Gaussian integration for the effective covariance:

$$C_I^{-1} \rightarrow C_I^{-1} - \sqrt{-\frac{u}{3}} \sum_L O_{IL} \lambda_L \sigma^{(L)} =: C_I^{-1} - \sqrt{-\frac{u}{3}} O_I(\sigma),$$  \hspace{1cm} (3.10)

which can be trivially performed, leading to the factor:

$$\det \left( 1 - \sqrt{-\frac{u}{3}} C O(\sigma) \right)^{-1/2} = e^{-\frac{1}{2} \text{Tr} \ln \left( 1 - \sqrt{-\frac{u}{3}} C O(\sigma) \right)}.$$  \hspace{1cm} (3.11)

At this stage, there are two strategies:

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\(^1\)Even if the sign in front of the kinetic action for anti-symmetric part in negative, it is well integrable, because all the eigenvalues of a real anti-symmetric matrix are imaginary.
• **Perturbative expansion:** We expand the exponential (3.11) in power of $\sqrt{u}$. This strategy corresponds to the standard perturbative expansion, with the difficulty pointed out. The Feynman graphs proliferate rapidly, and the radius of convergence of the perturbative series vanishes.

• **Loop vertex expansion:** We expand the exponential (3.11) in powers of 
\[ \mathcal{V}(\sigma) := \frac{1}{2} \text{Tr} \ln \left( 1 - \frac{u}{3} C \mathcal{O} (\sigma) \right). \] (3.12)

It has the advantage to sum automatically a significant part of the perturbative expansion as an effective vertex $\mathcal{V}$, an important step toward the goal to sum the perturbative expansion. We then retain the expansion:
\[ Z(u) = \int d\nu(\sigma) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} [\mathcal{V}(\sigma)]^n. \] (3.13)

Note that for the moment we do not do any illegal manipulation.

At this stage, we introduce an important tool for constructive goal: the BKAR forest interpolation formula. The BKAR (Brydges–Kennedy–Abdesselam–Rivasseau) formula, nicknamed the “constructive swiss knife”, is the heart of the LVE. A forest formula expands a quantity defined on $n$ points in terms of forests built on these points. There are in fact many forest formulas, but the BKAR formula seems the only one which is both *symmetric* under permutation of the $n$ points and *positive*.

Let $[1, \ldots, n]$ be the finite set of points considered above. An edge $l$ between two elements $i, j \in [1, \ldots, n]$ is a couple $(i, j)$ for $1 \leq i < j \leq n$, and the set of such edges can be identified with the set of lines of $K_n$, the complete graph with $n$ vertices. Consider the vector space $S_n$ of $n \times n$ symmetric matrices, whose dimension is $n(n+1)/2$ and the *compact and convex* subset $PS_n$ of *positive* symmetric matrices whose diagonal coefficients are all equal to 1, and off-diagonal elements are between 0 and 1. Any $X \in PS_n$ can be parametrized by $n(n-1)/2$ elements $X_l$, where $l$ run over the edges of the complete graph $K_n$. Let us consider a smooth function $f$ defined in the interior of $PS_n$ with continuous extensions to $PS_n$ itself. The BKAR forest formula states that:

**Theorem 2 (The BKAR forest formula)**

\[ f(1) = \sum_{\mathcal{F}} \int d\nu f X^\mathcal{F}(w_{\mathcal{F}}) \] (3.14)

where $1$ is the matrix with all entries equal to 1, and:

• The sum is over the forests $\mathcal{F}$ over $n$ labelled vertices, including the empty forest.
- The integration over $dw_F$ means integration from 0 to 1 over one parameter for each edge of the forest. Note that there are no integration for the empty forest since by convention an empty product is 1.

- $\partial_F := \prod_{l \in F} \partial_l$ means a product of partial derivatives with respect to the variables $X_l$ associated to the edge $l$ of $F$.

- The matrix $X^F(w_F) \in PS_n$ is such that $X^F_{ii}(w_F) = 1 \forall i$, and for $i \neq j$ $X^F_{ij}(w_F)$ is the infimum of the $w_l$ variables for $l$ in the unique path from $i$ to $j$ in $F$. If no such path exists, by definition $X^F_{ij}(w_F) = 0$.

We return on the equation (3.13). First, we use the replica trick, replacing the intermediate fields $\sigma^{(L)}(L)$ by a $n$-vector field $\sigma^{(L)}_p$, $p$ running from 1 to $n$, with covariance inverse matrix $1$; the matrix with all entries are equals to 1.

The partition function can be rewritten as:

$$Z(u) = \sum_n \int d\nu_1(\sigma) \frac{(-1)^n}{n!} \prod_{p=1}^n \mathcal{V}(\sigma_p),$$

where the Gaussian measure $d\nu_1(\sigma)$ is defined as:

$$d\nu_1(\sigma) := \frac{\prod_L e^{-\frac{1}{2} \sum_{pq} \sigma^{(L)}_p x_{pq} \sigma^{(L)}_q} \prod_{p=1}^n d\sigma^{(L)}_p}{\int \prod_L e^{-\frac{1}{2} \sum_{pq} \sigma^{(L)}_p x_{pq} \sigma^{(L)}_q} \prod_{p=1}^n d\sigma^{(L)}_p}_{x_{pq}=1, \forall p, q}.\quad (3.16)$$

Note that we have exchanged the sum and integration here. This is an important step, which can be understood as a definition of the partition function. Indeed, without permutation, we know that the function is bounded, with permutation, we will show the existence of a small domain of convergence. The last ingredient that we need is the following proposition:

**Proposition 1 (Derivative representation)** Let $F(\Phi)$ be an analytic function of $\Phi$. We have:

$$\int d\mu_C(\phi) F(\Phi) = e^{\frac{1}{2} \sum_{ij} C_{ij} \frac{\partial}{\partial \phi_i} F(\Phi)} \bigg|_{\Phi=0}.\quad (3.17)$$

**Proof.** We write the left hand side as:

$$\int d\nu_C(\phi) F(\Phi) = \int d\nu_C(\phi) F \left( \left\{ \frac{\partial}{\partial j_i} \right\} \right) e^{\sum_j j_i \phi_i} \bigg|_{j_i=0} \quad (3.18)$$

$$= F \left( \left\{ \frac{\partial}{\partial j_i} \right\} \right) \int d\nu_C(\phi) e^{\sum_j j_i \phi_i} \bigg|_{j_i=0}.\quad (3.19)$$

Now, let us consider the remaining Gaussian integration. Instead of computing it directly, we expand the exponential in power of $j$:

$$\int d\nu_C(\phi) e^{\sum_j j_i \phi_i} = \sum_n \frac{1}{n!} \int d\nu_C(\phi) \left( \sum_i j_i \phi_i \right)^n.\quad (3.20)$$
The Wick’s theorem then teaches us two things. The first one is that the terms odd in \( \phi \) vanish. The second one, that all possible contractions between pairs of \( \phi \) have to be taken into account. A moment of reflection shows that this is exactly what we do by applying \( n \) times the operator \( \frac{\partial}{\partial \phi_i} C_{ij} \frac{\partial}{\partial \phi_j} \). More precisely:

\[
\left( \frac{\partial}{\partial \phi_i} C_{ij} \frac{\partial}{\partial \phi_j} \right)^n \left( \sum_i j_i \phi_i \right)^n = 2^n n! \int d\nu_C(\phi) \left( \sum_i j_i \phi_i \right)^n, \tag{3.21}
\]

where the factor \( 2^n n! \) in front of the right hand side is the number of permutations for the derivative contributing to the same Wick contraction (we can exchange the two derivatives in the block \( \frac{\partial}{\partial \phi_i} C_{ij} \frac{\partial}{\partial \phi_j} \) and any of the \( n \) blocks with another). The left hand side can then be rewritten as:

\[
\sum_p \frac{1}{p!} \left( \frac{1}{2} \frac{\partial}{\partial \phi_i} C_{ij} \frac{\partial}{\partial \phi_j} \right)^p \left( \sum_i j_i \phi_i \right)^n \bigg|_{\phi=0} = \int d\nu_C(\phi) \left( \sum_i j_i \phi_i \right)^n, \tag{3.22}
\]

implying:

\[
\sum_n \frac{1}{n!} \int d\nu_C(\psi) \left( \sum_i j_i \phi_i \right)^n = e^{\frac{1}{2} \sum \frac{\partial}{\partial \phi_i} C_{ij} \frac{\partial}{\partial \phi_j} \sum_i j_i \phi_i} \bigg|_{\phi=0} \tag{3.23}
\]

and then:

\[
F \left( \frac{\partial}{\partial j} \right) \int d\nu_C(\phi) e^{\sum_i j_i \phi_i} \bigg|_{j_i=0} = e^{\frac{1}{2} \sum \frac{\partial}{\partial \phi_i} C_{ij} \frac{\partial}{\partial \phi_j} F(\Phi)} \bigg|_{\Phi=0}. \tag{3.24}
\]

Then, formula (3.15) can be rewritten as:

\[
Z(u) = \prod_L e^{\frac{1}{2} \sum_{pq} \frac{\partial}{\partial \sigma(p)} x_{pq} \frac{\partial}{\partial \sigma(q)}} \sum_{n=1}^\infty \frac{(-1)^n}{n!} \prod_{p=1}^n \mathcal{V}(\sigma_p) \bigg|_{\sigma_p(x)=0, x_{pq}=1}. \tag{3.25}
\]

Applying the BKAR forest formula, we get the decomposition:

\[
Z(u) = \sum_n \frac{1}{n!} \sum_{\mathcal{F}} \prod_{\ell \in \mathcal{F}} \left( \int_0^1 dx_\ell \right) \prod_L e^{\frac{1}{2} \sum_{pq} \frac{\partial}{\partial \sigma(p)} X_{pq}(x_\ell) \frac{\partial}{\partial \sigma(q)}} \prod_{\ell \in \mathcal{F}} \frac{1}{2} \sum_{L} \left[ \frac{\partial}{\partial \sigma_i(L)} \frac{\partial}{\partial \sigma_j(L)} \mathcal{V}(\sigma_p) \right]_{\sigma_p(x)=0}, \tag{3.26}
\]

where \( i(\ell) \) and \( j(\ell) \) are respectively the source and target for the edge \( \ell \in \mathcal{F} \), the sum over \( \mathcal{F} \) being over the forests with \( n \) vertices. \( X_{pq}(x_\ell) \) is the infimum over the parameters \( x_\ell \) in the unique path in the forest \( \mathcal{F} \), connecting \( p \) and \( q \), and the infimum is set to 1 if \( p = q \) and to zero if \( p \) and \( q \) are not connected by
the forest. Taking the logarithm, we obtain the free energy $F(u)$. Moreover, because the free energy is expanded in connected graphs, we expect that the sum over the forests reduces to a sum over trees with $n$ vertices (denoted as $T_n$):

$$F(u) = \sum_n \frac{1}{n!} \sum_{\ell \in T_n} \left( \int_0^1 dx_\ell \right) \prod_{L} e^{\frac{1}{2} \sum_{pq} \frac{\partial}{\partial \sigma_p} X_{pq}(x_\ell) \frac{\partial}{\partial \sigma_q}} \prod_{\ell \in T_n} \frac{1}{2} \sum_{L} \frac{\partial}{\partial \sigma_i^{(L)}} \frac{\partial}{\partial \sigma_j^{(L)}} \prod_{p=1}^n \mathcal{V}(\sigma_p) \bigg|_{\sigma_p^{(t)} = 0}. \quad (3.27)$$

### 3.1.2 Analyticity for i.i.d vectors

We restrict our attention on the case $C_{ij} = \delta_{ij}$, that is, for i.i.d vector fields. The effective vertex then becomes simply:

$$\mathcal{V}(\sigma) := \frac{1}{2} \text{Tr} \ln \left( 1 - \sqrt{-\frac{u}{3}} \mathcal{O}(\sigma) \right). \quad (3.28)$$

Defining $\chi^{(L)}_p := \lambda_L \sigma^{(L)}_p$, the partition function for $\sigma$ fields becomes:

$$Z(u) = \prod_{L} e^{\frac{1}{2} \sum_{pq} \frac{\partial}{\partial \sigma_p} \chi^{(L)}_I X_{pq}(x_\ell) \frac{\partial}{\partial \sigma_q} \chi^{(L)}_J} \sum_{n=1}^\infty \frac{(-1)^n}{n!} \prod_{p=1}^n \mathcal{V}(\chi_p) \bigg|_{\chi_p^{(L)} = 0}. \quad (3.29)$$

Finally, $\mathcal{O}(\chi)$ is nothing but a global rotation. Then, redefining $\chi$ as $\chi' := O\chi$, we get (omitting the $'$ and summing over repeated indices):

$$Z(u) = e^{\frac{1}{2} \sum_{pq} \frac{\partial}{\partial \sigma_p} \chi^{(L)}_I O_{IL} X_{pq}(x_\ell) \frac{\partial}{\partial \sigma_q} \chi^{(L)}_J} \sum_{n=1}^\infty \frac{(-1)^n}{n!} \prod_{p=1}^n \mathcal{V}(\chi_p) \bigg|_{\chi_p^{(L)} = 0}, \quad (3.30)$$

where we defined

$$\varpi_{I}^{L} = \sum_{L} O_{IL} \frac{1}{\lambda_L^2} O_{LJ}^{T} \rightarrow \varpi_{IJ} = \sum_{L} O_{IL} \chi_{I}^{L} O_{LJ}^{T}. \quad (3.31)$$

Applying forest formula, we then get for the free energy:

$$F(u) = \sum_n \frac{1}{n!} \sum_{\ell \in T_n} \left( \int_0^1 dx_\ell \right) e^{\frac{1}{2} \sum_{pq} \frac{\partial}{\partial \chi_p} \varpi_{IJ} X_{pq}(x_\ell) \frac{\partial}{\partial \chi_q}} \prod_{\ell \in T_n} \frac{1}{2} \sum_{L} \frac{\partial}{\partial \chi_i^{(L)}} \frac{\partial}{\partial \chi_j^{(L)}} \prod_{p=1}^n \mathcal{V}(\chi_p) \bigg|_{\chi_p^{(L)} = 0}, \quad (3.32)$$

where

$$\mathcal{V}(\chi) := \frac{1}{2} \text{Tr} \ln \left( 1 - \sqrt{-\frac{u}{3}} \chi \right). \quad (3.33)$$
The term with \( n = 0 \) will be treated separately, and we define \( \tilde{F}(u) = \sum_{n=1}^{\infty} \frac{1}{n!} F_n(u) \), with:

\[
F_n(u) := \sum_{T_n} \prod_{\ell \in T_n} \left( \int_0^1 dx_\ell \right) e^{\frac{1}{2} \sum_{pq} \frac{\partial}{\partial x_p} \varpi_{IJ} X_{pq}(x_\ell) \frac{\partial}{\partial x_I^{(J)}}} \prod_{\ell \in T_n} \left( \frac{1}{2} \frac{\partial}{\partial \chi^{(I)}_{i(\ell)}} \frac{\partial}{\partial \chi^{(J)}_{j(\ell)}} \prod_{p=1}^n \mathcal{V}(\chi_p) \right)_{\chi_p^{(L)} = 0}.
\]

We have to understand more about the structure of the trees. For each edge in the tree, we have a couple of derivatives \( \frac{\partial}{\partial \chi^{(I)}_{i(\ell)}} \frac{\partial}{\partial \chi^{(J)}_{j(\ell)}} \), moreover:

\[
\frac{\partial^k}{\partial (\chi^{(L)}_p)^k} \mathcal{V}(\chi_p) = -\frac{1}{2} (k-1)! \left( \frac{u}{3} \right)^k \left( 1 - \frac{1}{1 - \sqrt{\frac{u}{3}} \chi} \right)^k.
\]

As a result, computing each derivative, we get:

\[
F_n(u) := \left( \frac{-1}{2} \right)^n \sum_{T_n} \prod_{\ell \in T_n} \int_0^1 dx_\ell \prod_{v \in T_n} c(v) - 1! \left( \frac{u}{3} \right)^{c(v)} \prod_{\ell \in T_n} \varpi_{I_v(\ell)J_v(\ell)} \prod_{v \in T_n} \mathcal{V}_{I_{v,1},\ldots,I_{v,c(v)}}(\chi_p) \right)_{\chi_p^{(L)} = 0},
\]

where the \( s(\ell) \) (resp. \( t(\ell) \)) are sources (resp. targets) of the edge \( \ell \), which can be written as couples \( s(\ell) = (v(\ell), i) \) \( 1 \leq i \leq c(v) \) (and similarly \( t(\ell) = (v(\ell), i) \)). The couple \( (v, i) \) with \( 1 \leq i \leq c(v) \) denotes the boundaries of each of the \( c(v) \) edges hooked to \( v \). The vertices \( \mathcal{V}_{I_{v,1},\ldots,I_{v,c(v)}} \) are defined as:

\[
\mathcal{V}_{I_{v,1},\ldots,I_{v,c(v)}} := \left[ \prod_{\ell \in T_v} \frac{\partial}{\partial \chi^{(j)}_{j(\ell)}} \right] \mathcal{V}(\chi_v).
\]

Explicitly, if we denote as \( (i_{v,n}, j_{v,n}) \equiv I_{v,n} \) the pair of indices labelled with \( I_{v,n} \), we have:

\[
\mathcal{V}_{I_{v,1},\ldots,I_{v,c(v)}} = \frac{1}{2} \left[ \prod_{n=1}^{c(v)} \delta_{j_{v,n}, n+1} \right] [R^{c(v)-1}(\chi_v)]_{i_{v,1},j_{v,c(v)}},
\]

where we introduced the \textit{resolvent}:

\[
R(\chi) := \frac{1}{1 - \sqrt{\frac{u}{3}} \chi}.
\]
Lemma 1 Let $u = \rho e^{i\varphi}$, $\varphi \in ]-\pi, \pi]$, and $D$ be a real symmetric $N \times N$ operator with eigenvalues $u_i$. Defining $R$ as

$$R := (1 - iDe^{i\varphi/2})^{-1},$$

we have the following uniform bound for the standard operator norm:

$$\|R(\chi)\| \leq \cos^{-1}(\varphi/2).$$

Proof. From definition $R$ as eigenvalues $R_i = (1 - iu_ie^{i\varphi/2})^{-1}$. Let $|u_i\rangle \in \mathbb{R}^N$ be a basis of orthogonal eigenvectors. From definition:

$$\|R(\chi)\| = \sup_v \frac{\sqrt{\langle Rv|Rv \rangle}}{\sqrt{\langle v|v \rangle}}.$$ (3.42)

Expanding $|v\rangle = \sum_i c_i(v)|u_i\rangle$ along the eigenbasis, we get:

$$\|R(\chi)\| = \sup_v \sum_i |R_i| \frac{c_i^2(v)}{\sum_j c_j^2(v)} \leq \max |R_i|.$$ (3.43)

To bound the absolute value of the eigenvalues $R_i$, we begin by expanding them as follow:

$$\left| \frac{1}{1 - iu_i e^{i\varphi/2}} \right|^2 = \frac{1}{1 + 2u_i \sin(\varphi/2) + u_i^2}.$$ (3.44)

Let us denote as $f(x) := x^2 - 2x \sin(\varphi/2) + 1$. This function vanishes for: $x_{\pm} = \sin(\varphi/2) \pm \sqrt{\sin^2(\varphi/2) - 1}$, which is a purely imaginary number, except for $\varphi = \pi$, where $x_c = 1$. The denominator then does not vanish everywhere except for this value. Let $x_0$ the minimum of the function $f(x)$, reached for $x_0 = \sin(\varphi/2) \leq x_c$. Making explicit the condition $f(x) \geq f(x_0)$, we get:

$$f(x) \geq (\sin(\varphi/2))^2 - 2(\sin(\varphi/2))^2 + 1 = \cos^2(\varphi/2),$$

as a result:

$$|R_i| \leq \cos^{-1}(\varphi/2),$$

the cosinus function being positive into the interval ]$-\pi/2, \pi/2$].

□

Remark 1 The lemma generalizes when $D$ becomes a $N \times N$ hermitian matrix.
For convenience, we adopt the following graphical representation for the vertex $V_{I_v,1,...,I_v,c(v)}$:

\[
V_{I_v,1,...,I_v,c(v)} = \quad ,
\]

where the black circles represent a resolvent insertion $R(\chi_v)$ and the dotted edges in the interior of the vertex indicate the index contraction. Figure 1 below provides an example of a tree with five vertices.

\[
\text{Figure 1: An example of a tree with five vertices. The edges between the vertices correspond to contractions with } \omega_{IJ}.\]

The aim is to find an optimal bound for each term of the series $F_n = \frac{(-1)^n}{2^n - 1} \sum_{\mathcal{T}_n} \mathcal{A}_{\mathcal{T}_n}$. We define:

\[
\mathcal{A}_{\mathcal{T}_n} := \prod_{\ell \in \mathcal{T}_n} \left( \int_0^1 dx_\ell \right) \int d\mu_{\omega \otimes x}(\chi_{p}^{(L)}) \prod_{v \in \mathcal{T}_n} (c(v) - 1)! \left( \sqrt{-\frac{\mu}{3}} \right)^{c(v)} \times \mathcal{B}_{\mathcal{T}_n} \bigg|_{\chi_p^{(L)} = 0, x_{pq} = 1} ,
\]

where:

\[
\mathcal{B}_{\mathcal{T}_n} := \prod_{\ell \in \mathcal{T}_n} \omega_{I(\ell)I(\ell)} \prod_{v \in \mathcal{T}_n} V_{I_v,1,...,I_v,c(v)}(\chi_p).
\]
connected together with the bridge $\ell$, so that the amplitude $A_T$:

$$
\mathcal{B}_T = \sum_{I,J} V_I^{(v)} \omega_{IJ} \tilde{V}_J^{(w)} = \sum_L \mathcal{O}(V)_L^{(v)} (\lambda_L)^2 \mathcal{O}(\bar{V})_L^{(w)},
$$

(3.49)

where:

$$
\mathcal{O}(V)_L^{(v)} := \sum_I V_I^{(v)} O_{IL}.
$$

(3.50)

Let us denote as $\lambda_0 \leq \sum_I \lambda_{IL}^2$ the highest eigenvalue $\lambda_L$, we get

$$
\mathcal{B}_T \leq (\lambda_0)^2 \sum_L \mathcal{O}(V)_L^{(v)} \mathcal{O}(\bar{V})_L^{(w)} = (\lambda_0)^2 \sum_L V_L^{(v)} \bar{V}_L^{(w)}.
$$

(3.51)

Finally, using standard Cauchy–Schwarz inequality:

$$
\left| \sum_L V_L^{(v)} \bar{V}_L^{(w)} \right| \leq \sqrt{\left( \sum_I (V_I^{(v)})^2 \right) \left( \sum_J (V_J^{(w)})^2 \right)} \leq \left( \sum_I |V_I^{(v)}| \right) \left( \sum_J |V_J^{(w)}| \right).
$$

(3.52)

Figure 2: The structure of the tree $T$.

Recursively for all edges of the tree, we get the first bound:

$$
\mathcal{B}_{T_n} \leq (\lambda_0)^{2(n-1)} \prod_{v \in T_n} \sum \left| \mathcal{V}_{I_v,1,...,I_v,c(v)}(\chi_p) \right|.
$$

(3.53)

Each vertex $\mathcal{V}_{I_v,1,...,I_v,c(v)}(\chi_p)$ involves a resolvent $R_c(v)(\chi_p)$, and from the structure of the vertices given in (3.46), the sum is nothing but:

$$
\sum_{\{I_v,1 \leq i \leq c(v)\}} |\mathcal{V}_{I_v,1,...,I_v,c(v)}(\chi_p)| = \frac{1}{2} \text{Tr} |R_c(v)(\chi_p)|,
$$

(3.54)

and from Lemma 1:

$$
\text{Tr} |R_c(v)(\chi_p)| \leq \left| \frac{1}{\cos \frac{\sigma}{2}} \right|^{c(v)} \text{Tr I},
$$

(3.55)
where $I$ is the $N \times N$ identity matrix, and we get the second bound:

$$B_{\mathcal{T}_n} \leq \frac{(\lambda_0)^{2(n-1)}}{2^n} \left| \frac{1}{\cos^2 \frac{\theta}{2}} \right|^{n-1} (\text{Tr} \ I)^n = \frac{(\lambda_0)^{2(n-1)}}{2^n} \left| \frac{1}{\cos^2 \frac{\theta}{2}} \right|^{n-1} N^n \quad (3.56)$$

where we used the relation $\sum_v c(v) = 2(n-1)$. It comes from the fact that each edge of the tree is hooked to two different vertices, meaning that the sum of the coordination number is twice the number of edges. The number of edges being $n-1$ for a tree with $n$ vertices, the sum follows. For the amplitude $A_{\mathcal{T}_n}$, equation (3.1.2), we get:

$$|A_{\mathcal{T}_n}| \leq N^n \frac{(\lambda_0)^{2(n-1)}}{2^n} \left| \frac{1}{\cos^2 \frac{\theta}{2}} \right|^{n-1} \left( \frac{u}{3} \right)^{n-1} \prod_{v \in \mathcal{T}_n} (c(v) - 1)! \quad (3.57)$$

where we use the fact that the Gaussian integrations are normalized to 1. From these bounds, we get for $F_n$:

$$\frac{1}{n!} |F_n| \leq N^n \frac{1}{2^{n-1}} \frac{(\lambda_0)^{2(n-1)}}{2^n} \left| \frac{1}{\cos^2 \frac{\theta}{2}} \right|^{n-1} \left( \frac{u}{3} \right)^{n-1} \sum_{\mathcal{T}_n} \prod_{v \in \mathcal{T}_n} (c(v) - 1)! \quad \frac{1}{n!} \cdot (3.58)$$

The last sum can be easily bounded using Cayley formula. The number of labelled trees with $n$ vertices and a fixed configuration for coordination numbers $\{c(v)\}$ is:

$$\Omega(n, \{c(v)\}) = \frac{n!}{\prod_v (c(v) - 1)!} \cdot (3.59)$$

We then have:

$$\sum_{\mathcal{T}_n} \prod_{v=1}^{n} \frac{(c(v) - 1)!}{n!} = \sum_{\{c(v)\} \in \Omega(n, \{c(v)\})} \prod_{v=1}^{n} \frac{(c(v) - 1)!}{n!} \times \Omega(n, \{c(v)\}) = \sum_{\{c(v)\} \in \Omega(n, \{c(v)\})} 1 \cdot (3.60)$$

Finally, the last sum is trivially bounded with the area of the $(n-1)$-dimensional sphere of radius $\sqrt{2(n-1)}$:

$$\sum_{\{c(v)\} \in \Omega(n, \{c(v)\})} 1 \leq \frac{2\pi^{n/2}}{(n/2 - 1)!} (2n - 2)^{n/2} \leq 2\sqrt{2e} \left( 2\sqrt{\frac{\pi}{e}} \right)^{n-1} \cdot (3.61)$$

where we used the Stirling formula for the last step. As a result:

$$\frac{1}{n!} |F_n| \leq \sqrt{\frac{2}{\pi}} \frac{24e^{2 \frac{\cos(\theta/2)}{2}^2}}{u \lambda_0^2} \left| \frac{N \lambda_0^2 u \pi^{n/2}}{\cos^2 \frac{\theta}{2} 6 \sqrt{\pi}} \right| \cdot (3.62)$$
We have then obtained a bound for all the terms of the expansion of the free-energy, except for the vacuum term $F_0$ with zero vertex:

$$F_0 := \int d\mu \otimes x (\chi^{(I)}) \text{Tr} \ln \left( 1 - \sqrt{-\frac{u}{3}} \chi \right).$$

But it can be easily rewritten as a convergent expression. First of all, we have:

$$\left| \int d\mu \otimes x (\chi^{(I)}) \text{Tr} \ln \left( 1 - \sqrt{-\frac{u}{3}} \chi \right) \right|$$

$$= \left| \int d\mu \otimes x (\chi^{(I)}) \int_0^1 dt \text{Tr} \left[ \frac{\sqrt{-\frac{u}{3}} \chi}{1 - \sqrt{-\frac{u}{3}} \chi t} \right] \right|$$

$$\leq (\lambda_0)^2 \left| \int d\mu \otimes x (\chi^{(I)}) \int_0^1 dt \frac{\rho}{(1 - \sqrt{-\frac{u}{3}} \chi t)^2} \right|,$$

where we used the fact that, for a single vertex, we have no edge and then $x = 1$. To conclude this section, we have proved the following proposition ensuring the existence of a non-vanishing analyticity domain:

**Proposition 2** The free energy $F(u)$ for the zero-dimensional bosonic model is analytic in the complex variable $u = \rho e^{i\theta}$, at least in the interior of the cardioid domain:

$$\rho \leq \frac{6}{N^2 \lambda_0^2} \sqrt{\frac{e}{\pi}} \cos^2 \left( \frac{\theta}{2} \right).$$

With respect to the 'standard' zero-dimensional $\phi^4$ theory

$$S[\phi] = \frac{1}{2} \phi^2 + \frac{u}{4!} \phi^4,$$

we can point out two differences:

- The presence of the square of the highest eigenvalue of the interaction matrix $W_{ij}$. This should not be a surprise, since the relevant coupling is intuitively understood as $u||W||$.

- The factor $1/N$, implying that the size of the convergence domain decreases with the number of degrees of freedom. Note that this is not a limitation of the theory, but a limitation of the method. When the number of interacting degrees of freedom becomes infinite, some divergences may occur in the expansion, requiring more a sophisticated analysis (see Section 4.2).
Remark 2 For the derivation of the bound, we assumed the i.i.d Gaussian measure $K_{ij} = \delta_{ij}$. Note that this condition can be easily relaxed. First of all, the Lemma 1 holds and the resolvent $R(\chi)$ remains bounded as $\cos^{-1}(\theta/2)$. The expression (3.28) remains unchanged as well. Each effective vertex involves one $K^{-1}$ insertion per derivative with respect to the intermediate field. If we assume the kernel $K$ bounded as $\|K\| \leq \kappa$, the bound (3.55) is then replaced by: $\kappa - c_v \cos^{-1}(\theta/2) Tr' I$. The notation $Tr'$ refers to the fact that the original trace is reduced into the support $D(K)$ of $K_{ij}$, whose cardinality equals the number of non-zero eigenvalues. The radius of convergence then remains the one given by Proposition 2, up to the replacement $N \rightarrow D(K)$ and $\cos(\theta/2) \rightarrow \kappa \cos(\theta/2)$, that is to say:

$$\rho \leq \frac{6}{D(K)} \lambda_0^2 \sqrt{\frac{e}{\pi}} \kappa^2 \cos^2 \left(\frac{\theta}{2}\right).$$  

(3.67)

3.1.3 Borel Summability

In this section, we discuss the summability of the free energy, by checking all the requirement of the standard Nevanlinna’s theorem that we recall here:

**Theorem 3 (Nevanlinna)** A series $\sum_{n=1}^{\infty} a_n \lambda^n$ is Borel summable to a function $f(\lambda)$ if the following conditions are met:

- $f(\lambda)$ is analytic in a disk $\text{Re}(\lambda^{-1}) > R^{-1}$ with $R \in \mathbb{R}^+$.
- $f(\lambda)$ admits a Taylor expansion at the origin:

$$f(\lambda) = \sum_{k=0}^{r-1} a_k \lambda^k + R_r f(\lambda), \quad |R_r f(\lambda)| \leq K\sigma^r r! |\lambda|^r,$$

(3.68)

for some constants $K$ and $\sigma$ independent of $r$.

If $f(\lambda)$ is Borel summable in $\lambda$, then:

$$B = \sum_{n=0}^{\infty} \frac{1}{n!} a_n t^n$$

(3.69)

is an analytic function for $|t| < \sigma^{-1}$ which admits an analytic continuation in the strip $\{ z \mid |\text{Im}(z)| < \sigma^{-1} \}$ such that $|B| \leq Be^{tR}$ for some constant $B$ and $f(\lambda)$ is represented by the absolutely convergent integral:

$$f(\lambda) = \frac{1}{\lambda} \int_0^{+\infty} dt \, Be^{-t/\lambda}.$$

(3.70)

In the previous section, we checked the first requirement of the Nevanlinna’s theorem: we found a non-empty domain of analyticity. We then have to check
the second point: to find a bound for the remainder of the Taylor expansion. To this end, we have to complicate our trees by adding loops.

The remainder of the expansion of the free energy reads \((g = u/4!):\)

\[
R_r F'(g) := g^{r+1} \int_0^1 \frac{(1-t)^r}{r!} (F^n(tg)) dt, \quad (3.71)
\]

with \(F' := \sum_{n=1}^{\infty} F_n = F - F_0\). The remainder can be expanded as a sum over labelled trees thanks to the BKAR formula, and we get:

\[
R_r F'(g) := -\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{T_n} \prod_{\ell \in T_n} \left( \int_0^1 dx_{\ell} \right) \int d\mu_{\infty}(\chi_p^{(\ell)}) R_r [\mathcal{Y}_T] ; \quad (3.72)
\]

where

\[
\mathcal{Y}_T := g^{n-1} \prod_{v \in T_n} (c(v) - 1) \prod_{\ell \in T_n} \varpi_{I(\ell)I(\ell)} \prod_{v \in T_n} \vartheta_{I_{v1}, \cdots, I_{vc(v)}} (\chi_p)^n \bigg|_{x_{pq} = 1}. \quad (3.73)
\]

First, note that when \(n - 2 \geq r\), \(R_r [\mathcal{Y}_T] = \mathcal{Y}_T\). Each of these terms admits a bound of the form \(K^n|\lambda|^n\), and the sum converges at least in the interior of the cardioid. When \(n - 2 < r\) however, the remainder has to be computed following a Taylor expansion of the resolvents in \(\mathcal{Y}_T\). This expansion dresses the trees with loops. Concretely, we extract the global factor \(g^{n-1}\) in front of \(\mathcal{Y}_T\), and we write:

\[
\mathcal{Y}_T = g^{n-1} \tilde{\mathcal{Y}}_T , \quad (3.74)
\]

such that:

\[
R_r [\mathcal{Y}_T] = g^{n-1} R_r [\mathcal{Y}_T] . \quad (3.75)
\]

Defining \(z := \sqrt{-u/3}\), we have:

\[
\frac{d}{dz} \text{Tr} R(\chi) = n! \text{Tr} [\chi^n R^{n+1}(\chi)] . \quad (3.76)
\]

As announced, the tree \(T_n\) becomes a spanning tree for a graph with loops, generated from the new Gaussian integrations of the fields \(\chi_v\) coming from the Taylor expansion of the resolvent (see Figure 3 below).

Each power of the resolvent can be bounded using Lemma 1 so that the Taylor expansion of \(\mathcal{Y}_T\) admits the following bounds, involving only the new Gaussian integrations coming from the Taylor expansion:

\[
\tilde{\mathcal{Y}}_T \leq \frac{1}{2^n} \left[ \frac{\lambda^2_0}{\cos^2 \frac{\theta}{2}} \right]^{2(n-1)} \sum_{k=0}^{\infty} \frac{k!}{\prod_{l=1}^{k} k_l!} \frac{z^k}{k!} \prod_{v \in T_n} (c(v) + k_v - 1)! \times \text{Tr} \left[ \prod_{v \in T_n} \chi_v^{k_v} \right] \bigg|_{x_{pq} = 1} . \quad (3.77)
\]
Next, from formula (3.71), we get:

\[
R_{r-n+1}[\bar{Y}_n] \leq \frac{z^{2(r-n)}}{2^n} \int_0^1 dt \left( \frac{1-t}{2r-2n+2} \right)^{2(r-2n+2)} \sum_{\{k_i\}|\sum_{k_i}=2r-2n+3} \frac{(2r-2n+3)!}{\prod_{i=1}^{n} k_i!} \\
\times \prod_{v=1}^{n} (c(v) + k_v - 1)! \times \text{Tr} \left[ I \prod_{v \in T_n} \frac{\chi_v^{k_v}}{\chi_v^{(t)_{=0}}} \right]. \quad (3.78)
\]

We can now report this expression in equation (3.72). First, note that because \( C_n^p \leq 2^n \), the factor \( c(m)(c(m) + k_m)!/[k_m!c(m)!] \) is bounded by

\[
e^{\ln(c(m))}2e^{c(m) + k_m} \leq (2e)^{c(m)2k_m},
\]

and the product over \( m \) gives a factor \((2e)^{2n-22r-2n+3}\). Secondly, we can perform the Gaussian integration. Because there are \( 2r-2n \) fields, the number of Wick contractions is \((2r-n)!! = 2^n r! \). Lastly, the remaining integration over \( t \) gives

\[
\int_0^1 (1-t)^{2r-2n+2} dt = \frac{1}{(2r-2n+3)},
\]

which together with the denominator \((2r-2n+2)!\) exactly compensates the combinatorial factor \((2r-2n+3)!\). As in the previous section, using Cayley’s theorem for the number of trees with \( n \) vertices and Stirling’s formula, we find a bound of the form: \( AB_1^n B_2^n r! \) for some constants \( A, B_1 \) and \( B_2 \). Because \( n-2 < r \), summing over \( n \), we find the final bound: \( A' |g|^r B^r r! \) for the contributions in (3.72) for which \( n-2 < r \). As explained before, the contributions for \( n-2 \geq r \) are all bounded by bounds of the form: \( |g|^n K^n \), and the sum behaves as: \( A'' |g|^r K^r \). Ultimately, because, for positive constants \( k_1 \) and \( k_2 \), \( k_1 r! + k_2 \leq (k_1 + k_2)r! \), we find that \( |R_n F(g)| \leq A''(B')^r |g|^r r! \), which corresponds to the second condition of Theorem 3. This completes the proof of the following statement:
**Proposition 3** The free energy $F(u)$ for zero-dimensional bosonic model is Borel summable in $u$.

### 3.1.4 Correlation functions

Connected correlation functions are obtained from the free energy with external sources $F(u, J)$ by taking functional derivatives with respect to the sources $J$. Let $F(u, J) = \ln Z(u, J)$, with:

$$Z(u, J) := \int d\mu_C(\phi) e^{-\frac{u}{2} \sum_{ijkl} W_{ijkl} \phi_i \phi_j \phi_k \phi_l + \sum_i j_i \phi_i} \quad (3.79)$$

being the sourced partition function, $J := \{j_i\}$ and $j_i : \mathbb{R}^d \to \mathbb{R}$. The connected correlation functions with $n$ external edges $G_{i_1, \ldots, i_n}$ are then given by:

$$G_{i_1, \ldots, i_n} := \left[ \prod_{k=1}^n \frac{\partial}{\partial j_{i_k}} \right] F(u, J) \bigg|_{J=0} . \quad (3.80)$$

The sourced partition function can be decomposed as three-body interactions using the intermediate field trick:

$$Z(u, J) = \int d\mu_C(\phi) d\nu(\sigma) \prod_{L} e^{\sqrt{-u} \lambda_L \Psi_L^{(\sigma)}(\sigma) + \sum_i j_i \phi_i} . \quad (3.81)$$

Up to the rescalings $\chi = \lambda_L \sigma^{(L)}$ and global rotations $\chi \to O\chi$ already discussed at the beginning of Section 3.1.2, the Gaussian integration over the fields $\phi_i$ can be performed; leading to

$$Z(u, J) = \int d\nu(\chi) e^{-V(\chi) + \frac{1}{2} J^T R(\chi) J} \quad (3.82)$$

where $V(\chi)$ and $R(\chi)$ are given by equations (3.33) and (3.39). Computing the derivatives with respect to the source, we get a non-zero correlation functions only for $n = 2p$. Moreover, we get a sum over index permutations of a product of $p$ resolvents. Finally, expanding this expression with the same replica trick than for the partition function $Z(u)$, using BKAR forest formula, and keeping only the connected contributions labelled by indexed trees, we get a sum involving exactly the same terms like in the expansion of the free energy. Once more, using Lemma 1 each resolvent or derivative of resolvent can be bounded by a constant in the cardioid domain, in the same way as for the free energy, and the absolute convergence of the expansion of any connected correlation function follows.

**Corollary 1** The connected correlation functions for the bosonic model in zero dimension are analytic at least in the interior of the cardioid domain, and Borel summable.
3.2 Fermionic models

In this section, we investigate the Borel summability of quartic fermionic models in zero dimension. As recalled in the introduction, the path integral description of fluctuating fermionic fields requires Grassmann-valued fields, such that:

\[ \psi_i \psi_j = -\psi_j \psi_i. \] (3.83)

As for bosons, the indices \( i, j \) run from 1 to \( N \), and we will assume that \( N \) could be large but finite (the limit \( N \to \infty \) requires more care, see Section 4.2). In this way, we assume that all the sums over \( N \) are bounded, and \( N \) to be a relevant parameter of the theory. For bosons we started with i.i.d real model; but there is no equivalent fermionic model, because for real Grassmann numbers the commutation relation (3.83) implies \( \psi_i \delta_{ij} \psi_j = 0 \). There are many way to circumvent this difficulty without getting too far from the simplest i.i.d distribution. The simplest way to solve this difficulty is to consider complex Grassmann fields, for which a kinetic action of the form \( \bar{\psi} \delta_{ij} \psi \) makes sense.

We will briefly extend our results to this case at the end of this section. However, we make the choice to focus on the real field case, essentially because of its close relation with physically relevant models like SYK models discussed in Section 3.

3.2.1 Majorana field theory

In order to get a well i.i.d real field model, we have to use a physical feature of fermionic fields. Indeed, physically relevant fermionic fields share some internal discrete degrees of freedom, like spin. Assuming that such a spin exists for our fields, we can introduce a new index \( \sigma = \pm \) such that \( \psi_i \) becomes a bi-vector:

\[ \psi_i = \begin{pmatrix} \psi_{i,+} \\ \psi_{i,-} \end{pmatrix} \] (3.84)

and we will denote the components as \( \psi_{i,\sigma} \). The total number of degrees of freedom then goes from \( N \) to \( 2N \) in this simplest model; and we can choose the following kinetic action:

\[ S_{\text{kin}}(\Psi) := \sum_{i,\sigma,j,\sigma'} \psi_{i,\sigma} \left( K_{ij}^{-1} \tau_{\sigma,\sigma'} \right) \psi_{j,\sigma'}, \] (3.85)

where the matrix \( \tau = \tau^{-1} \) couples the up and down spins:

\[ \tau := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \] (3.86)

In zero dimension, real fields correspond to Majorana spinors, and \( \tau \) may be freely interpreted as a charge conjugation matrix. With this model, i.i.d the
distribution $K_{ij} = \delta_{ij}$ makes sense. Note that we do not need the correspondence to be perfect with a realistic physical model; we only require our model be sufficiently close to realistic ones so that our constructive analysis will be relevant for those models. To complete the definition of the real (or Majorana) model, we have to precise the interaction part of the action $S_{\text{int}}$. We set:

$$S_{\text{int}}(\Psi) := \frac{u}{4!} \sum_{\sigma} \sum_{ijkl} J_{ijkl} \psi_{i,\sigma} \psi_{j,\sigma} \psi_{k,\sigma} \psi_{l,\sigma}, \quad (3.87)$$

using the notation $J_{ijkl}$ for the (totally anti-symmetric) fermionic tensor coupling:

$$J_{ijkl} = J_{klij} = -J_{jikl} = -J_{ijlk} = -J_{ikjl}. \quad (3.88)$$

### 3.2.2 LVE for Majorana i.i.d fermions

We can then use of the same strategy as for the bosonic case. We can diagonalize the matrix $J_{IJ}$ for the blocks indices $I = (ij)$ and $J = (kl)$:

$$J_{IJ} = \sum_{L} \xi_{L}^{2} O_{IL} O_{LJ}^{T}, \quad (3.89)$$

where $O$ is an orthogonal transformation (we keep the same notation as in the previous section). The eigenvalues $\xi_{L}$ are matrices in the original little indices.

For $d = 0$ and $K_{ij}^{-1} = \delta_{ij}$, the strategy used for the bosonic bound holds. We get for the partition function:

$$Z(u) = \int d\mu C(\psi) d\nu(\sigma) \prod_{L} e^{\sqrt{-u} \xi_{L} \Xi_{L}^{\sigma(L)}} \psi_{i,\sigma} \psi_{j,\sigma}, \quad (3.90)$$

where

$$\Xi_{L} := \sum_{i=j} O_{IL} \psi_{i} \psi_{j}. \quad (3.91)$$

and $\int d\mu C(\psi)$ is the normalized Gaussian Grassmann measure:

$$\int d\mu C(\psi) = 1, \quad (3.92)$$

where $C_{ij} := \delta_{ij} \tau$ is a matrix valued operator into the two-dimensional spin space. Performing the integration over the original fields $\psi_{i}$ using standard results about Berezin integral,

$$\int \prod_{i}^{2n} d\psi_{i} \exp \left( \Psi^{T} A \Psi \right) = 2^{n} \sqrt{\det A}, \quad (3.93)$$

we get the following effective bosonic theory:

$$Z(u, J) = \int d\nu(\chi) e^{V(\chi)} \quad (3.94)$$
where $\mathcal{V}(\chi)$ is given by equation (3.33). Except the difference of sign in front of the effective interaction $\mathcal{V}$, it is exactly what we obtained for the bosonic fields. In particular, our conclusions about the bound of terms involved in the tree expansion of the free energy hold, and our proofs are insensitive to the relative signs for each term (we focus on the absolute values). The only difference comes from the presence of the matrix $\tau$. From Lemma 1, it follows that the norm of the resolvent $R(\chi)$ remains bounded as $\cos^{-1}(\theta/2)$. However, the trace over external indices, $\text{Tr}$, involved in the definition of the vertex function $\mathcal{V}(\chi)$ (c.f. equation (3.33)) becomes over the whole index structure, including spin indices. Each derivative involved in the definition of the effective vertex introduces a $\tau$ insertion. Let $v$ and $w$ two effective vertices and $\ell$ a link between them. If we denote as $\text{tr}$ the trace over spin indices, for the edge $\ell$ correspond the trace $\text{tr}[R(v)\tau R(w)\tau]$. The external index structure with respect to the indices $i, j$ remaining unchanged, the bound given by equation (3.55) holds. Fixing the external indices, and because the operator norm of $\tau$ is $\|\tau\| = 1$, the traces over spin indices can be bounded as:

$$|\text{tr}[R(v)\tau R(w)\tau]| \leq \text{tr}|R(v)R(w)| \leq \text{tr}|R(v)| \times \text{tr}|R(w)|. \quad (3.95)$$

As a consequence, the trace $\text{Tr}|R^{(v)}|$ involved on the right hand side of the bound (3.55) becomes $\text{Tr}(\text{tr}|R|)\epsilon^{(v)} \leq \|R\|\text{tr}I_2 \times \text{Tr}I = \|R\| \times 2N$; where we introduced a subscript 2 for the $2 \times 2$ identity matrix $I_2$. Taking into account this new factor 2 coming from traces over internal indices, we state that:

**Proposition 4** The free energy $F(u)$ and the connected correlation functions of the fermionic model in zero dimension is analytic in the complex variable $u$ at least in the interior of the cardioid domain:

$$|u| \leq \frac{3}{N\xi_0^2}\sqrt{\frac{e}{\pi}}\cos^2\left(\frac{\theta}{2}\right), \quad (3.96)$$

where $\xi^2_0$ denotes the highest eigenvalue $\xi^2_L$, and Borel summable.

**Remark 3** Note that internal space of dimensions higher than 2 may be considered, and the bound follows trivially from our proof.

**Remark 4** For Dirac fermions, the $\psi_i$ become complex Grassmann numbers, and we may consider i.i.d propagators of the form $\bar{\psi}_i\delta_{ij}\psi_j$. The interacting tensor $\mathcal{J}$ can be diagonalized in a similar way (see the next section for complex bosonic field) as for real fields, and the same bound holds, up to the factor 1/2 coming from the sums over internal spin index.
4 Regularized constructive expansion for $d = 1$

In this section, we address the questions of analyticity and summability of one-dimensional models. We will consider two specific cases in details: the relativistic complex boson (Section 4.1) and the massless Majorana fermion – which is itself related to the SYK model – at finite and large $N$ (sec:exp-1d:fermion), both at finite temperature. The other models are not considered in details because they are either direct extensions or limits of the two other models; some are discussed in Remarks 6, 8 and 9. Moreover, the Schrödinger and Dirac operators are equal in $d = 1$ such that the non-relativistic bosons and fermions, and relativistic fermions are basically equivalent, the few signs due to the different statistics not making a big difference.\footnote{However, real bosons are trivial in $d = 1$ since the kinetic term is a total derivative and vanishes due to the boundary conditions (periodic at finite temperature, fall-off at infinity at zero-temperature). Hence, when speaking of a non-relativistic boson we implicitly consider complex fields.}

Note that for $d = 1$ our power-counting theorem (equations (2.37), (2.38) and (2.39)), the divergent degree $\Omega$ of the relativistic boson is absolutely negative:

$$\Omega_{RB} = -3V + \frac{N}{2} + 1. \quad (4.1)$$

In contrast, for non-relativistic bosons and (relativistic or non-relativistic) fermions, the power-counting for $d = 1$ depends only on the number of vertices:

$$\Omega_{NRB,F} = 1 - V. \quad (4.2)$$

Hence, superficially logarithmic divergences occur for $V = 1$. They correspond to sums of the type:

$$\sum_{\omega} \frac{1}{i\omega + m}, \quad (4.3)$$

which appear for instance in the computation of the one-loop self energy $\Sigma_{ij}(\omega, \omega')$:

$$\Sigma_{ij}(\omega, \omega') = -\frac{u}{2} i \tilde{\omega} j \quad (4.4)$$

leading to (the repeated indices $k$ are summed):

$$i \tilde{\omega} j = \frac{1}{\beta} \delta(\omega - \omega') \mathcal{W}_{ijk} \sum_{\omega''} \frac{1}{i\omega'' + m}. \quad (4.5)$$

The sum (4.3) is not absolutely convergent, but however, cannot be considered on the same footing than usual UV divergences. In fact, we have to distinguish divergences coming from “physics” and divergences which are only a problem of regularization. For the sum (4.3), the bad behavior is a consequence of using a continuous time in the definition of the path integral. In the discretized
version of the path integral, the fields $\phi$ and $\bar{\phi}$ are not evaluated exactly at the same time, and the difference generates a factor $e^{i\omega\epsilon}$ in front of $m$, modifying the sum as:

$$\lim_{\epsilon \to 0} \sum_{\omega} \frac{e^{-i\omega\epsilon}}{i\omega e^{-i\omega\epsilon} + m},$$

(4.6)

which is well-defined. In condensed matter physics, the general strategy to compute sums of the form $\sum_{\omega} g(i\omega)$ is to replace them with a contour integration into the complex plane, weighting the analytic function $g(z)$ with another analytic function $f(z)$ generating the poles on the imaginary axis from the original discrete sum. The cotangent function is a standard choice, but for $\epsilon > 0$, $f(z) := \beta/(1 \pm e^{-\beta z})$ seems to be more appropriate to ensure the vanishing of the integral for large $|z|$, after deformation of the integration contour.\(^\text{12}\) The net result is ($y \in \mathbb{R}^+$):

$$S_1 := \sum_{n \in \mathbb{Z}} \frac{1}{in + y} \equiv \mp 2\pi n_\pm(y),$$

(4.7)

where $n_\pm(y)$ is the statistical weight, the sign selecting the Bose–Einstein or Fermi–Dirac statistics:

$$n_\pm(y) := \frac{1}{e^{-2\pi y} \mp 1}.$$  

(4.8)

With this additional regularization, all the loop sums are convergent, and the theory becomes divergence-free. The tadpole contribution for the self energy reads:

$$\begin{array}{c}
\begin{array}{c}
\mathbb{O}
\end{array}
\end{array}
= \mp \delta(\omega - \omega') W_{ijkk} n_\pm(m \beta/2\pi).$$

(4.9)

The sum (4.7) will be useful for Majorana fermions. For relativistic bosons, the corresponding tadpole involves the obviously convergent sum

$$S_2 := \sum_{n \in \mathbb{Z}} \frac{1}{n^2 + y^2} = \frac{\pi}{y} \coth(\pi y),$$

(4.10)

for strictly positive $y$.

### 4.1 Relativistic bosonic models

We consider a complex\(^\text{13}\) scalar field $\phi \in \mathbb{C}$ at finite temperature $\beta$. The zero-temperature limit $\beta \to \infty$ ($t \in \mathbb{R}$, non-periodic) is discussed in Remark 6.

We will work in Fourier space, the field $\phi$ having the Fourier series:

$$\phi_i = \frac{1}{\sqrt{\beta}} \sum_{n \in \mathbb{Z}} \varphi_i(n) e^{i\frac{2\pi n}{\beta}}.$$  

(4.11)

\(^\text{12}\)The sign $+$ being for bosons and the sign $-$ for fermions.

\(^\text{13}\)We consider the case of a complex field to introduce some variation.
Rewritten in Fourier components, the classical action (2.4) becomes:

\[
S(\Phi) = \sum_{i,n \in \mathbb{Z}} \bar{\phi}_i(n) \left( \left( \frac{2\pi n}{\beta} \right)^2 + m^2 \right) \varphi_i(n)
\]

\[
+ \frac{u}{4\beta} \sum_{\{n_{\ell}\} \in \mathbb{Z}^4} \delta \left( \sum_{\ell} \epsilon_{\ell} n_{\ell} \right) \sum_{ijkl} W_{ijkl} \bar{\phi}_i(n_1) \varphi_j(n_2) \bar{\varphi}_k(n_3) \varphi_l(n_4),
\]

where \( \epsilon_{\ell} = \pm 1 \) following if the index \( \ell \) refers to \( \varphi \) or \( \bar{\varphi} \) fields, and the conservation delta \( \delta \left( \sum_{\ell} \epsilon_{\ell} n_{\ell} \right) \) is a discrete Kronecker delta:

\[
\delta \left( \sum_{\ell} \epsilon_{\ell} n_{\ell} \right) := \delta_0 \sum_{\ell} \epsilon_{\ell} n_{\ell}.
\]

The propagator of the theory is then \( C(\omega, \omega') = C(\omega)\delta(\omega - \omega') \), with:

\[
C(\omega) = \frac{1}{\omega^2 + m^2}.
\]

The decomposition (3.5) remains formally unchanged. Focusing on real tensor couplings \( W_{ijkl} \), the outlines of the proofs of Section 3 are essentially the same. Grouping indices two by two, the matrix \( W_{IJ} \) may be diagonalized into the super-index space with orthogonal real matrices \( O_{IJ} \); and as in Section 3, we introduce \( \Psi_L \) given by:

\[
\Psi_L := \sum_{I=ij} O_{IL} \phi_i \bar{\phi}_j
\]

such that

\[
Z(u) = \int d\mu_C(\phi) \prod_L e^{-\frac{u}{2} \lambda_L \int d[\Psi_S]^2}.
\]

In order to break the square, we introduce a time-dependent intermediate field \( \sigma^{(L)}_i \), with normalized Gaussian measure:

\[
d\nu(\sigma) := \prod_L e^{-\frac{1}{2} \int d[\sigma^{(L)}] \, d[\sigma^{(L)}]} \int d[\Psi_S] e^{-\frac{1}{2} \int d[\Psi_S]^2}.
\]

so that the quartic interaction is broken into a three-body interaction as:

\[
Z(u) = \int d\mu_C(\phi, \bar{\phi}) d\nu(\sigma) \prod_L e^{\sqrt{-u} \lambda_L \int d[\Psi_S] \sigma^{(L)}}.
\]

Written in Fourier representation, the corresponding classical action becomes:

\[
S(\varphi, \sigma) = \frac{1}{2} \sum_{i,j,\omega} \sigma_{ij}(\omega) \sigma_{ij}(-\omega) + \sum_{i,\omega} \bar{\phi}_i(\omega)(\omega^2 + m^2) \phi_i(\omega)
\]

\[
- \sqrt{\frac{u}{2\beta}} \sum_{ijkl} W_{ijkl} \bar{\phi}_i(\omega) \bar{\varphi}_j(\omega') \sigma_{kl}(\omega' - \omega),
\]

40
where
\[ \mathcal{W}^j_{ijkl} := O_{ij,mn} \lambda_{mn} O^T_{mn,kl}. \] (4.20)

The effective kinetic term for the fields \( \varphi_i(\omega) \) reads:
\[ \sum_{ij} \sum_{\omega} \overline{\varphi}_i(\omega) \left( (\omega^2 + m^2) \delta_{ij} \delta_{\omega \omega'} - \sqrt{-\frac{u}{2\beta}} \sum_{kl} \mathcal{W}^j_{ijkl} \sigma_{kl}(\omega - \omega') \right) \varphi_j(\omega'). \]

Introducing the kinetic kernel \( \mathcal{K} \) as follows:
\[ \mathcal{K}_{ij}(\omega, \omega') := \left( \delta_{ij} \delta_{\omega \omega'} - \sqrt{-\frac{u}{2\beta}} \frac{1}{\omega^2 + m^2} \sum_{kl} \mathcal{W}^j_{ijkl} \sigma_{kl}(\omega - \omega') \right), \] (4.21)
and taking into account the normalization of the measure \( d\mu_C \), the integration over the original field variable leads to the determinant:
\[ \det \mathcal{K} = e^{-\text{Tr} \ln \mathcal{K}}, \] (4.22)
providing the effective matrix-field theory:
\[ Z(u) = \int d\nu(\sigma) e^{-\text{Tr} \ln \mathcal{K}[\sigma]}, \] (4.23)
where \( \text{Tr} \) means trace over the complete set of indices. That is to say, for a matrix \( A_{ij,\omega\omega'} \):
\[ \text{Tr} A := \sum_{i,\omega} A_{ii,\omega\omega}. \] (4.24)
Equation (4.23) is formally identical to the zero-dimensional case, except for an additional integration over \( \omega \). Because all the integrals are convergent, the tree expansion of the free energy has exactly the same structure as the one for \( d = 0 \). The only changes come from the dependence on \( \omega \) of the intermediate fields, as well as the additional bound coming from \( \omega \)-integrations. Introducing the fields \( \chi(\omega) \) like in (3.33) and (3.39), the new resolvent is:
\[ R[\chi](\omega, \omega') := \mathcal{K}^{-1}[\chi](\omega, \omega'), \] (4.25)
Assuming \( u = \rho e^{it}, \theta \in \left[ -\pi, \pi \right] \), we get the following bound for the operator norm, extending the Lemma 1:

**Lemma 2** Let \( \mathcal{E} \equiv \mathbb{C}^N \otimes L_2(S^1, \mathbb{R}) \), where \( L_2(S^1, \mathbb{R}) \) denote the space of square summable functions spanned by \( \{ e^{int} \} \), with \( n \in \mathbb{Z} \) and \( t \in [0, 1] \). Let \( \{|j,n\} \) be an orthogonal basis on \( \mathcal{E} \), for \( j \) running from 1 to \( N \). Let \( \mathcal{H} \) an hermitian operator on \( \mathcal{E} \). By definition \( \mathcal{H} \) act on the basis states \( |i, n\rangle \) as:
\[ \mathcal{H}|i, n\rangle = \sum_{j,m} \mathcal{H}_{ji, mn}|j, m\rangle. \] (4.26)
Let \( u = |u|e^{i\varphi}, \varphi \in [-\pi, \pi] \) and \( R \) the resolvent, defined as
\[
R^{-1} = \text{id} + i\sqrt{u}H,
\] (4.27)
where \( \text{id} \) denote the identity operator on \( \mathcal{E} \). We have the following uniform bound for the standard induced operator norm on \( \mathcal{E} \):
\[
\|R[\chi]\| \leq \cos^{-1}(\varphi/2).
\] (4.28)

The proof of this lemma follows the same strategy as for Lemma 1, of which it is a special extension. Our aim is to use this lemma to bound the resolvent (4.25). The corresponding operator \( H \) has matrix elements:
\[
H_{ij,\omega\omega'} := -\frac{1}{\sqrt{2\beta}}(C\Sigma)_{ij,\omega\omega'}
\] (4.29)
where \( \Sigma \) has matrix elements \( \Sigma_{ij,\omega\omega'} := \sum_{kl} W_{ijkl}^2 \sigma_{kl}(\omega - \omega') \) and \( C_{\omega\omega'} := C(\omega)\delta_{\omega\omega'}. \) The product \( C\Sigma \) is not hermitian, as it is easy to check. However, \( H \) appears in the trace:
\[
\text{Tr} \ln \mathcal{K} = \text{Tr} \ln \left( \text{id} + i\sqrt{\frac{u}{2\beta}} C\Sigma \right),
\] (4.30)
which can be rewritten as:
\[
\text{Tr} \ln \mathcal{K} = \text{Tr} \ln \left( \text{id} + i\sqrt{\frac{u}{2\beta}} C^{1/2}\Sigma C^{1/2} \right),
\] (4.31)
and the association \( C^{1/2}\Sigma C^{1/2} \) is obviously hermitian, because \( \sigma_{ij} \) is a real field, \( \sigma_{ij}^*(\omega) = \sigma_{ij}(-\omega) \):
\[
\left( C^{1/2}\Sigma C^{1/2} \right)^\dagger = C^{1/2}\Sigma C^{1/2}.
\] (4.32)

In order to understand the modifications coming from the new propagator, let us consider the simpler tree, made of two vertices:

\[
\begin{array}{c}
\circ \quad \circ \\
\end{array}
\]

with explicit expression:
\[
A_1 = \frac{1}{2} \sum_\omega \left( \int_0^1 dx \right) \int \mathcal{D}\chi \left[ \chi \right] \frac{\partial}{\partial \chi_1^{(1)}(\omega)} \overline{\varphi}_{IJ} \frac{\partial}{\partial \chi_2^{(2)}(-\omega)} \prod_{p=1}^2 \psi_\omega(\chi_p) \bigg|_{\chi_p^{(L)} = 0},
\] (4.33)
where the Gaussian measure is for the propagator $X_{ij}^{IJ}(\omega, \omega') = \varpi_{IJ}x_{ij}\delta(\omega - \omega')$. Computing the derivatives with respect to $\chi_2^{(j)}$ and $\chi_1^{(I)}$, we get, for each one:

$$\frac{\partial}{\partial \chi_1^{(I)}(\omega)} V(\chi_1) = -\sum_{\omega, \omega''} \sqrt{-\frac{u}{2\beta}} \frac{1}{\omega''^2 + m^2} \delta(\omega - \omega'' + \omega') V_I[\chi_1](\omega', \omega'').$$

Formally, the last term has the structure of a trace over $\omega$, $\sqrt{-\frac{u}{2\beta}} \text{Tr} B V_I[\chi_1]$, involving the strictly positive matrix

$$B_{\omega''}(\omega) := \frac{1}{\omega''^2 + m^2} \delta(\omega - \omega'' + \omega')$$

(4.34)

For $A$ bounded and $B$ strictly positive $|\text{Tr}(AB)| \leq \|A\| \text{Tr} B$. However, $B$ is not diagonalizable. In fact, only the $k$th diagonal of the matrix is non-vanishing (counting from the principal diagonal): $B_{ij} = b_i \delta_{i,j-k}$ with $b_i > 0$ and $k \neq 0$. For our purpose, $A \equiv R$, and the resolvent is unitary diagonalizable as a function of the hermitian operator $H$. Let us denote as $U$ the unitary operator diagonalizing $A$ and as $a_i$ its eigenvalues, we get:

$$|\text{Tr}(AB)| \leq \left| \sum_{i,l} a_i |U_{i-k,l}U_{i,l}'b_i| \right| \leq \|A\| \sum_{i,l} |U_{i-k,l}U_{i,l}'b_i|.$$  

(4.35)

Using Cauchy–Schwarz inequality, the last product can be bounded as:

$$\sum_{i,l} |U_{i-k,l}U_{i,l}'b_i| = \sum_i b_i \sum_l |U_{i-k,l}| |U_{i,l}| \leq \sum_i b_i \sqrt{\sum_l |U_{i-k,l}|^2 \sum_l |U_{i,l}|^2},$$

(4.36)

that is to say, because $UU^\dagger = \text{id}$:

$$|\text{Tr}(AB)| \leq \|A\| \left( \sum_i b_i \right).$$

(4.37)

As a result:

$$\sum_i \left| \frac{\partial}{\partial \chi_1^{(I)}(\omega)} V(\chi_1) \right| \leq \frac{1}{\cos(\theta/2)} \left| \frac{u}{2\beta} \right|^{1/2} (\text{Tr} I) \frac{\beta}{2m} \coth \left( \frac{\beta m}{2} \right),$$

(4.38)

where we used Lemma 1. From the from Cauchy–Schwarz inequality, and like for the $d = 0$ case, we deduce that

$$|A_1| \leq dx d\nu_X[\chi] \left| \frac{u \lambda_0^2}{4\beta} \right| \left( \frac{\beta}{2m} \coth \left( \frac{\beta m}{2} \right) \right)^2 \frac{1}{\cos^2(\theta/2)} (\text{Tr} I)^2 \sum_\omega 1,$$

(4.39)

and taking into account the normalization of the integrals, we get the final bound for $A_1$:

$$|A_1| \leq \left| \frac{u \lambda_0^2}{16} \right| \left( \text{Tr} I \right)^2 \frac{\beta}{m^2} \coth^2 \left( \frac{\beta m}{2} \right) \sum_\omega 1.$$  

(4.40)
As we will see, the last sum over all the degrees of freedom appears in front of all the amplitude bounds, and we expect it is a consequence of the translation invariance of the theory for a very pessimistic bound. However, the fact that it is only a constant constant factor, without dependence over the order of the expansion allows to discard it from the discussion about the existence of a finite analyticity domain. We only assume implicitly some regularization procedure to make the sum finite.

Now, let us consider a tree with three vertices, like this one:

![Tree Diagram]

Up to Gaussian and $x$ integrations, we get the explicit expression:

$$
\frac{1}{2^2} \left( \frac{u}{2\beta} \right)^2 \sum_{\omega_1,\omega_2} U_{i;\omega_1} \omega_{IJ} U_{JK;\omega_1,\omega_2} \omega_{KL} U_{L;\omega_2}
$$

where we omitted the explicit dependence on $\chi_i$ for the vertex $i$, and introduced the notation:

$$
U_{i_1,\ldots,i_n;\omega_1,\ldots,\omega_n} := \left( \sqrt{\frac{3\beta}{-u}} \right)^n \left[ \prod_{i=1}^{n} \frac{\partial}{\partial \chi_{i}(\epsilon_i \omega_i)} \right] V(\chi_i),
$$

where $\epsilon_i = \pm$. Explicitly, the structure of such a vertex is the following:

$$
U_{i_1,\ldots,i_n;\omega_1,\ldots,\omega_n} = R_{j_1 j_2;\omega_1 \omega'_1} (C \chi) \omega'_2 \omega_1 R_{j_2 j_3;\omega_2 \omega'_2} \ldots R_{j_n i_1;\omega_n \omega'_n} (C \chi) \omega'_1 \omega_1 i_1 \omega_n
$$

where we integrated over repeated $\omega$–indices, and where:

$$
(C \chi) \omega'_i \omega_{i+1;I;\omega_i} = \frac{1}{\omega_i^2 + m^2} \frac{\partial}{\partial \chi_{i}(\epsilon_i \omega_i)} (C \chi_{i}) \omega'_i \omega_2.
$$

Explicitly:

$$
(C \chi) \omega'_i \omega_{i+1;I;\omega_i} = \frac{1}{\omega_i^2 + m^2} \delta(\epsilon_i \omega_i - \omega'_i + \omega_{i+1}).
$$

We then deduce the bounds

$$
\sum_{I,J} |U_{ij;\omega_1,\omega_2}| \leq \frac{\text{Tr} I}{\cos^2(\theta/2)} \delta(\epsilon_1 \omega_1 + \epsilon_2 \omega_2) \left( \frac{\beta}{2m} \coth \left( \frac{\beta \omega_0}{2} \right) \right)^2,
$$

and

$$
\sum_{I} |U_{i;\omega_1}| \leq \frac{\delta(\omega_1)}{\cos(\theta/2)} \text{Tr} I \left( \frac{\beta}{2m} \coth \left( \frac{\beta \omega_0}{2} \right) \right).
$$
As a result, the amplitude $A_2$ for the two-vertex contribution is bounded as:

$$|A_2| \leq \left| \frac{u \lambda_0^2}{4 \beta} \right|^2 \frac{(\text{Tr } I)^3}{\cos^4(\theta/2)} \left( \frac{\beta}{2m} \right)^4 \coth^4 \left( \frac{\beta m}{2} \right) \sum_\omega 1. \quad (4.48)$$

To generalize for an arbitrary number of vertices, we have:

$$\sum_{I_1, \ldots, I_k} |U_{I_1, \ldots, I_k; \omega_1, \ldots, \omega_k}| \leq \frac{\text{Tr } I}{\cos^k(\theta/2)} \delta \left( \sum_{i=1}^k \epsilon_i \omega_i \right) \left( \frac{\beta}{2m} \right)^k \coth^k \left( \frac{\beta m}{2} \right),$$

from which we deduce the bound for the amplitude $A_{n-1}$ having $n$-vertices:

$$|A_{n-1}| \leq \left| \frac{u \lambda_0^2}{4 \beta} \right|^{n-1} \frac{(\text{Tr } I)^n}{\cos^{2n-2}(\theta/2)} \left( \frac{\beta}{2m} \right)^{2n-2} \coth^{2n-2} \left( \frac{\beta m}{2} \right) \sum_\omega 1. \quad (4.49)$$

The last factor $\sum_\omega$ is easy to check. We have one sum per link and $c(v) - 1$ delta per vertex $v$. Because $\sum_{v}(c(v) - 1) = L - 1$, where $L$ denotes the number of links, the result follows. Counting the number of trees for each configuration like for the $d = 0$ case, we deduce the following statement:

**Proposition 5** For the bosonic one-dimensional quartic models at finite temperature, the free energy $F(u)$ and the correlation functions are analytic with respect to the coupling $u = \rho e^{i \theta}$, at least in the interior of the cardioid domain:

$$\rho \leq \frac{16 \sqrt{e/\pi} m^2}{N \lambda_0^2 \beta \coth^2 \left( \frac{\beta m}{2} \right)} \cos^2 \left( \frac{\theta}{2} \right), \quad (4.51)$$

and Borel summable.

**Remark 5** In the $\beta \to 0$ limit (high-temperature limit), to obtain a non-trivial theory, we have to assume the following rescaling: $\lambda_0^2 \to \lambda_0^2 / \beta; m^2 \to m^2 / \beta$. As a result,

$$\beta \coth^2 \left( \sqrt{\beta m/2} \right) \to 4/m^2,$$

and we recover the $d = 0$ domain given by Proposition 2, setting $m = 1$ (up to numerical factors coming from the different normalization between complex and scalar fields).

**Remark 6** The opposite limit $\beta \to \infty$ (zero-temperature limit) is more difficult to track. Fixing $m$, and because $\coth(x) \to 1$ when $x \to \infty$, we expect that the size of the cardioid goes to zero as $1/\beta$. It is easy to see that this dependence is a calculation artefact. Indeed, in the low temperature limit, $t \in ] - \infty, \infty[$, and the sums over $\omega$ become integration like:

$$\sum_\omega f(\omega) \to \frac{\beta}{2 \pi} \int d\omega f(\omega). \quad (4.52)$$
In that limit, \( \beta \) completely disappears, and the Fourier series becomes continuous Fourier transform:

\[
\phi_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \varphi_i(\omega) e^{i\omega t}.
\]

Because:

\[
\int_{-\infty}^{+\infty} \frac{d\omega}{\omega^2 + m^2} = \frac{\pi}{m},
\]

the rule to pass from finite to infinite \( \beta \) is to replace \( \beta \) by \( 2\pi \) everywhere, and set \( \coth(\beta m/2) = 1 \). In the zero-temperature limit, we then get the following cardioid domain:

\[
\rho \leq \frac{8\sqrt{e/\pi^3} m^2}{N\lambda_0^2} \cos^2 \left( \frac{\theta}{2} \right), \quad \beta = +\infty.
\]

**Remark 7** Note that the massless limit of (4.51) is ill-defined as the RHS goes to zero.

### 4.2 Massless fermionic and SYK models

We now turn to the fermionic case, focusing on Majorana fields. As mentioned several times, this model is closely related to the famous SYK model [6–9], and we are going to review the differences.

For the standard SYK model, the tensor coupling \( J \) is itself a random variable, such that the partition function behaves as follow (which is referred to as quenching):

\[
Z(u) = \int dJ \ e^{-S(J)} \int d\psi e^{-S(J,\psi)} =: \int dJ \ e^{-S(J)} Z(J),
\]

where we call \( Z(J) \) the *partial partition function*, integrating over the fermionic degrees of freedom only. Once again, note that \( J \) is assumed to be real, with strictly positive eigenvalues \( \xi_i^2 \). The measure for \( J \), \( dJ e^{-S(J)} \) is Gaussian:

\[
S(J) := \frac{\alpha(N)}{2\bar{g}^2} \sum_{ijkl} J_{ijkl} J_{ijkl},
\]

where \( \alpha(N) \) is a certain power of \( N \). Note that the generalized SYK model involves an interaction of degree \( q, \ J_{i_1 \cdots i_q} \psi_{i_1} \cdots \psi_{i_q}; \) our conclusion about the Borel summability holds only for the original \( q = 4 \) SYK model. The function \( \alpha(N) \) is fixed to \( N^{q-1} \) usually, and we will set \( \alpha(N) = N^3 \) for our considerations.

The classical action for the fermions is chosen to be:

\[
S(J, \psi) := \int_{-\beta/2}^{\beta/2} L(\Psi, J) dt
\]
with the Lagrangian density:

\[ \mathcal{L}(\Psi, J) := \frac{1}{2} \sum_i \psi_i \frac{d}{dt} \psi_i - \frac{u}{4!} \sum_{ijkl} J_{ijkl} \psi_i \psi_j \psi_k \psi_l. \]  

(4.59)

Note the minus sign in front of the interaction term, which arises from \(-1 = i\pi/2\) for \(q = 4\). It will be crucial for the convergence of the resolvent, and hence to obtain a finite analyticity domain, by allowing \(H\) in (4.64) to be Hermitian.

Our proof will proceed in two main steps. As a first step, we prove the Borel summability of the partial free energy \(\ln Z(J)\), using the same strategy as for the bosonic case. As a second step, we prove Borel summability of the full free energy \(\ln Z(u)\), with respect to \(u\).

### 4.2.1 Majorana fermions with fixed coupling tensor

The Fourier representation of the propagator of the theory, \(C(\omega)\) is then fixed to be:

\[ C(\omega) = \frac{1}{i\omega}. \]  

(4.60)

Note that the model does not require a mass term to be UV regularized because of the boundary condition \(\psi_t = -\psi(t + \beta)\), which excludes the zero-frequency mode \(\omega = 0\). The spectrum for fermions is given by formula (2.21):

\[ \sum_\omega \frac{1}{i\omega} = \frac{\beta}{2i\pi} \sum_n \frac{1}{n + 1/2} = \frac{\beta}{2}. \]  

(4.61)

As in the previous section, bounding the amplitude requires to bound sums like:

\[ f(\omega) := \sum_{\omega', \omega''} \frac{1}{\omega'} \delta(\omega - \omega'' + \omega') \mathcal{V}_i[\chi_{1}](\omega', \omega'') := \text{Tr} B(\omega) R[\chi_{1}], \]  

(4.62)

with

\[ B_{\omega', \omega''}(\omega) := \frac{1}{\omega''} \delta(\omega - \omega'' + \omega'). \]  

(4.63)

There is an additional subtlety coming from the fact that the propagator is not real. Indeed, the resolvent becomes:

\[ R_{\omega, \omega'}^{-1} = \delta_{\omega, \omega'} - \frac{u}{3\beta} C^{1/2}(\omega) \Sigma_{\omega'} C^{1/2}(\omega') = \delta_{\omega, \omega'} + i \frac{u}{3\beta} \frac{1}{\sqrt{\omega}} \frac{1}{\sqrt{\omega'}}. \]

The operator

\[ \mathcal{H} := \frac{1}{\sqrt{\omega'}} \Sigma_{\omega'} \frac{1}{\sqrt{\omega}}, \]  

(4.64)
is hermitian, once again due to the reality condition $\Sigma_{\omega\omega'}^* = \Sigma_{-\omega,-\omega'}$. As a result, $R$ admits the operator bound:

$$\|R\| \leq \cos^{-1}\left(\frac{\theta}{2}\right).$$

(4.65)

with $\theta = \arg(u)$. Note that, due to the sign in front of $u$ in the definition (4.59), the cardioid domain is rotated by $\pi$. Because $B(\omega)$ is not positive defined, we do not use the same trick as in the previous section to bound $f(\omega)$. However, we can use the properties of the operator bound. In particular, let $U$ be the unitary transformation acting on the Hilbert space $\mathbb{C}^N \otimes L^2(S_1, \mathbb{R})$ which diagonalizes $V_I$, and let $r_{i\omega}$ be its eigenvalues. $f(\omega)$ may be rewritten as:

$$f(\omega) = \sum_{i\omega'} r_{i\omega'} \tilde{B}_{i\omega'}(\omega)$$

(4.66)

where $\tilde{B}_{i\omega'}(\omega) := (U^\dagger B(\omega)U)_{i,i\omega'}$. Obviously, $\tilde{B}_{i\omega'} \in \mathbb{C}^N \otimes L^2(S_1, \mathbb{R})$. Indeed:

$$\|\tilde{B}(\omega)\|_2 = \sum_{i\omega'} \tilde{B}_{i\omega'}^\dagger \tilde{B}_{i\omega'} \leq \left(\sum_{i\omega'} |\tilde{B}_{i\omega'}|^2\right)^{1/2} \leq \left(\frac{\beta}{2}\right)^2.$$  

(4.67)

As a result, because $R$ is bounded, and the the definition of the operator norm:

$$|f(\omega)| \leq \|R\| \times \|\tilde{B}(\omega)\|_2 = \frac{\beta}{2} \cos^{-1}\left(\frac{\theta}{2}\right).$$

(4.68)

Then, the bound deduced in the previous section follows, up to the replacements $\beta \coth(\beta m/2)/2m \to \beta/2$ and $\sqrt{u/2\beta} \to \sqrt{u/3\beta}$; and taking into account the $1/2$ per vertex coming from the substitution $e^{-\operatorname{Tr} \ln} \to e^{-\frac{1}{3}\operatorname{Tr} \ln}$. Therefore:

**Proposition 6** The perturbative free energy $F(u)$ and the correlation functions of the one-dimensional Majorana model are analytic, at least in the interior of the cardioid domain:

$$\rho \leq \frac{48\sqrt{e/\pi}}{N\xi_0^2\beta} \cos^2\left(\frac{\theta}{2}\right),$$

(4.69)

and corresponds to their Borel sums.

**Remark 8** The case of the $d = 1$ massive fermion and non-relativistic boson with a finite mass follows directly from the previous computations, replacing the factor $\beta n_-(0) = \beta/2$ by $\beta n_-(m\beta/2\pi) = \beta/(1 + e^{-m\beta})$ according to (4.7)
and (4.61). As a consequence, the free energy $F(u)$ and correlation functions are analytic, at least in the interior of the cardioid domain:

$$
\rho \leq \frac{24\sqrt{e/\pi}}{N\xi_{0}^2\beta} \left( 1 + e^{-m\beta} \right) \cos^2\left( \frac{\theta}{2} \right),
$$

(4.70)

and Borel summable.

**Remark 9** The case of the $d = 1$ non-relativistic (complex) boson is a simple extension of Remark 8, replacing $n_-$ by $n_+$ and up to some factor $O(1)$. We find that the free energy and correlation functions are analytic at least in the domain:

$$
\rho \leq O(1)\frac{\sqrt{e/\pi}}{N\lambda_{0}^2\beta} \left( 1 - e^{-m\beta} \right) \cos^2\left( \frac{\theta}{2} \right),
$$

(4.71)

and Borel summable.

### 4.2.2 SYK model and large $N$ limit

We proved the existence of a finite analyticity domain for the fermionic part of the SYK model. We will prove that the Gaussian integration over the coupling $J_{ijkl}$ preserves the existence of a non-vanishing analyticity domain, which is also finite in the large $N$ limit. In the other sections, we contented ourselves of a pessimistic bound for the amplitude, because we were only interested by the existence of a convergence domain of finite size. In this section, we have to be more scrupulous especially on the counting of the powers of $N$, to ensure finiteness of the large $N$ limit.

As a first step, let us examine a naive bound, integrating directly the bound of the fermionic part. To simplify the integration, we can replace $\xi_0^2$ by $\sum I \xi_{I}^2 \geq \xi_0^2$. The amplitude bound for the fermionic part takes the form:

$$
|A_n| \leq K(u, N) \left( \sum I \xi_{I}^2 \right)^n \rho^n(u, N),
$$

(4.72)

As a result, into the analytic domain $\rho(u, N) \sum I \xi_{I}^2 \leq 1$,

$$
|F(u, J) - F_0(u, J)| \leq K(u, N) \frac{\rho(u, N) \sum I \xi_{I}^2}{1 - \rho(u, N) \sum I \xi_{I}^2}.
$$

(4.73)

The integration over $J$ does not take into account the proper symmetries assumed for the coupling $J$. These symmetries come from the form of the coupling with the Majorana field, and the components of $J$ which do not satisfy these symmetries factorizes in front of the fermionic integral. More precisely, denoting as $J_0$ the part of the tensor satisfying the symmetries required by
the quartic interaction and $\mathcal{J}_\perp$ the complementary tensor: $\mathcal{J}_\perp := \mathcal{J} - \mathcal{J}_0$, we get:

$$Z(u) = \int d\mu_\perp(\mathcal{J}) \times \int d\mu_0(\mathcal{J}) \int d\psi e^{-S(\mathcal{J}_0,\psi)} =: \left( \int d\mu_\perp(\mathcal{J}) \right) Z_0(u). \quad (4.74)$$

The first factor is a purely Gaussian integration, and we can discard it from our investigations. Because $|e^x| \leq e^{|x|}$ we obtain:

$$\left| \int d\mu_0(\mathcal{J}) \int d\psi e^{-S(\mathcal{J}_0,\psi)} \right| \leq \int d\mu_0(\mathcal{J}) e^{\sum L \xi^4_L}. \quad (4.75)$$

Because $|F(u, \mathcal{J})| \leq |F_0(u, \mathcal{J})| + |F(u, \mathcal{J}) - F_0(u, \mathcal{J})|$, and using the positivity of the Gaussian measure, we get, from the Cauchy–Schwarz inequality:

$$Z^2_0(u) \leq \left( \int d\mu_0(\mathcal{J}) e^{2|F_0(u, \mathcal{J})|} \right) \left( \int d\mu_0(\mathcal{J}) e^{2|F(u, \mathcal{J}) - F_0(u, \mathcal{J})|} \right). \quad (4.76)$$

The Gaussian measure reads explicitly:

$$d\mu_0(\mathcal{J}) = e^{-\frac{N^3}{2\pi} \sum L \xi^4_L}. \quad (4.77)$$

We recall that $F_0(u, \mathcal{J})$ is the first term of the tree expansion, and we proved it is finite for the $d = 0$ case. Moreover, the bound is linear with respect to $\sum L \xi^2_L$, and the Gaussian integration takes the form:

$$\int \prod L d\xi^2_L e^{-\frac{N^3}{2\pi} \sum L \xi^4_L} e^{K(u, N) \sum L \xi^2_L} \propto e^{(K(u, N))^2 \frac{N^2}{2\pi}}. \quad (4.78)$$

Explicitly, the second factor of (4.76) reads:

$$e^{-K(u, N)} \int \prod L d\xi^2_L e^{-\frac{N^3}{2\pi} \sum L \xi^4_L} e^{\frac{K(u, N)}{1 - \rho(u, N) \sum I \xi^2_I}}. \quad (4.79)$$

At this stage, one can think to build a constructive expansion in power of $\rho(u, N)$. Unfortunately, $\mathcal{Q} := (1 - \rho(u, N) \sum I \xi^2_I)^{-1}$, which plays the same role as the resolvent in the previous constructive expansion, is not bounded.

The problem essentially arises because we bounded the fermionic amplitudes after integration over the couplings. Let us return on the building of the first bound, see equation (3.49). Let $\mathcal{T}$ the tree pictured on the Figure 4, with $n$ vertices, and let $\mathcal{O}(V^{(1)})_I$, $\mathcal{O}(V^{(2)})_{IJ}$ and $\mathcal{O}(V^{(3)})_J$ three connected components such that the amplitude $\mathcal{A}_T$ reads (we are only interested by the internal index structure of the amplitude, and we forget the sums over frequencies $\omega$):

$$\mathcal{A}_T \propto \sum_{I,J} \mathcal{O}(V^{(1)})_I \xi^2_I \mathcal{O}(V^{(2)})_{IJ} \xi^2_J \mathcal{O}(V^{(3)})_J. \quad (4.80)$$

Due to Wick theorem, the computation of the Gaussian integral over $\mathcal{J}$ generates "pairings" between $\xi^2_I$ variables, and then add loops on the tree. There are two of sources for $\xi^2_I$ variables: the resolvent and the links, generating three types of pairing:
Figure 4: A typical tree contributing to the constructive expansion of the fermionic integral.

1. Wick contractions between two edge variables, represented as the dashed–dotted line on Figure 5 (a).

2. Wick contractions between two resolvents, as on Figure 5 (b).

3. Wick contractions between resolvent and edge variables, as on Figure 5 (c).

In contrast to the two other contractions, the contractions of type (1) are exact Wick contractions, whereas the other ones are effective, in the sense that the resolvents have to be expanded in power of $\xi_2^2$. To distinguish between exact and effective contractions, we denote the first ones with dashed–dotted lines, and the second ones with dashed lines on Figure 5.

Ultimately, we are interested to deal with the limit $N \to \infty$, and some of the allowed contractions will be discarded in this limit. For instance, the contractions of type (1) between edge variables introduce additional Kronecker delta, which reduces the strength of the sums over internal indices, and ultimately the final dependence on $N$ of the amplitude $A_T$. The same thing holds for contractions between edge variable and resolvent which are not hooked on the boundary of the corresponding edge, as well as between different resolvents.

Figure 5: Allowed Wick-contractions on the tree $T$. Contraction between two edge variables (a), between two corners (b) and between edge and corner (c).
The contractions of the edge variables which optimize the $N$ dependence of $A_T$ are then necessarily between one of the two resolvents hooked on the boundary of the corresponding edge; and the remaining contractions have to be "self loops" between resolvents. As a result, for each edge, there are two allowed contractions:

\[ \bullet \sim \sim \sim \sim \sim \bullet + \bullet \sim \sim \sim \sim \bullet, \quad (4.81) \]

and the leading order contribution in $N$ for $\langle A_T \rangle_J$ – the averaged amplitude, reads:

\[ \langle A_T \rangle_J = \left( \begin{array}{c}
\text{+ perm}
\end{array} \right), \quad (4.82) \]

where the notation $\langle X \rangle_J$ means Gaussian averaging with respect to the tensor coupling $J$, normalized to 1: $\langle 1 \rangle_J = 1$.

The permutations count all the allowed contractions per edges, as explained on equation (4.81). There are two type of contractions per edges, and $n - 1$ edges. Therefore, we get the bound for $\langle A_T \rangle_J$:

\[ |\langle A_T \rangle_J| = 2^{n-1} \quad (4.83) \]

The final step to bound the amplitude is then to find a bound for this typical tree with short loops. Let us consider the simpler tree $T_2$ with two vertices, on the right of equation (4.81). Denoting $\tilde{R}_{ij} = R_{ij} \delta_{ij}$, the amplitude becomes proportional to:

\[ |\langle A_{T_2} \rangle_J| \propto \left| \sum_I \langle |\tilde{R}_I| \rangle_J \langle \xi^2_I \tilde{R}_I \rangle_J \right| \leq \sqrt{\sum_I \langle |\tilde{R}_I| \rangle_J^2 \times \sum_I \langle |\xi^2_I \tilde{R}_I| \rangle_J^2}. \quad (4.84) \]

Due to the normalization of the Gaussian integration, the first factor is uniformly bounded as

\[ \sum_I \langle |\tilde{R}_I| \rangle_J^2 \leq \cos^{-2} \left( \frac{\theta}{2} \right) \text{Tr } I. \quad (4.85) \]

For the second factor, we use Cauchy–Schwarz inequality to get:

\[ \sum_I \langle |\xi^2_I \tilde{R}_I| \rangle_J^2 \leq \sum_I \langle |\xi_I^2| \rangle_J^2 \sum_I \langle |\tilde{R}_I| \rangle_J^2 \propto \frac{|g|^2}{\cos^2 \left( \frac{\theta}{2} \right)} N^{-3} N^2 \times N. \quad (4.86) \]

As a result $|\langle A_{T_2} \rangle_J| \propto \sqrt{N}$. The same argument can be extended for "big traces" occurring for vertices having coordination numbers $c(v)$ higher than 1, like we considered in equation (3.46), and we can prove the following lemma:
Lemma 3 The bound of the averaged amplitude \( \langle \mathcal{A}_{T_n} \rangle_{\mathcal{J}} \) for any tree with \( n > 1 \) vertices scales as:
\[
|\langle \mathcal{A}_{T_n} \rangle_{\mathcal{J}}| \sim \sqrt{N}.
\] (4.87)

Proof. To prove this, we proceed recursively on \( n \). Any tree can be built from the elementary tree having 1 vertex adding leaves one by one. As a result, any tree \( T_{n+1} \) may be obtained from a tree \( T_n \) with \( n \) vertices, adding a single leaf. Assuming that the property hold for any tree of size \( n \), we have to prove that it survives adding a leaf. The amplitude \( \mathcal{A}_{T_{n+1}} \) reads:
\[
\mathcal{A}_{T_{n+1}} = \sum_I V_I^{(n+1)} \xi_I^2 \tilde{R}_I,
\] (4.88)
where \( V_I^{(n)} \) denotes the rest of the tree with \( n \) vertices. Denoting as \( v \) the vertex to which the leaf is added and as \( c(v) + 1 \) its coordination number, \( V_I^{(n)} \) has the following structure:
\[
V_I^{(n)} = \sum_{\{I_k\}} V_{I_{c(v)},\ldots,I_{c(v)}} \prod_{k=1}^{c(v)} \xi_I^2 V^{(m(k))}_k \xi_I^2 \tilde{R}_I.
\] (4.89)
Along each of the \( c(v) \) edges, we can bound using our favorite Cauchy–Schwarz inequality. For \( k = 1 \), we define
\[
V_{I_1}^{(n)} = \sum_{\{I_k\}, k \neq 1} V_{I_{c(v)},\ldots,I_{c(v)}} \prod_{k=2}^{c(v)} \xi_I^2 V^{(m(k))}_k \xi_I^2 \tilde{R}_I.
\] (4.90)
Then, assuming we averaged on \( \mathcal{J} \) only on the connected component \( V^{(m(1))}_{I_1} \), we have two configurations, corresponding to the two allowed Wick contractions along the edge 1. However, even in the case when the contraction is on the component \( V^{(n)}_{I_1} \), we can replace the contracted resolvent with the resolvent of the component \( V^{(m(1))}_{I_1} \), so that in any cases we have the bound:
\[
\sum_{I_1} \langle V_{I_1}^{(n)} \rangle_{\mathcal{J}} \langle \xi_{I_1}^2 V^{(m(1))}_{I_1} \rangle_{\mathcal{J}} \leq \sqrt{\sum_{I_1} \langle |V_{I_1}^{(n)}| \rangle_{\mathcal{J}}^2 \sum_{I_1} \langle |\xi_{I_1}^2 V^{(m(1))}_{I_1}| \rangle_{\mathcal{J}}^2} \leq \sum_{I_1} \langle |V_{I_1}^{(n)}| \rangle_{\mathcal{J}} \sqrt{\sum_{I_1} \langle |\xi_{I_1}^2 V^{(m(1))}_{I_1}| \rangle_{\mathcal{J}}^2}.
\] (4.91)
Recursively, we get:
\[
\sum_{I_1} \langle V_{I_1}^{(n)} \rangle_{\mathcal{J}} \langle \xi_{I_1}^2 V^{(m(1))}_{I_1} \rangle_{\mathcal{J}} \leq \sum_{\{I_k\}} \langle |V_{I_{c(v)},\ldots,I_{c(v)}}| \rangle_{\mathcal{J}} \prod_{k=1}^{c(v)} c_k,
\] (4.92)
with: \( C_k := \sqrt{\sum_{I}(\|\xi^2 I_{I_{k}} V^{(m(k))}\|^2)} \). Finally, returning on the sum (4.88), we get:

\[
\sum I \langle | V^{(n)} I_{I_{k}} \xi^2 \bar{R} | \rangle \leq \prod_{k=1}^{c(v)} C_k \times \sqrt{\sum_{I,I_{k}} \langle | V^{(n)} I_{I_{k}} \xi^2 \bar{R} | \rangle} \sum I \langle | \xi^2 I_{I_{k}} \rangle \rangle^2. \tag{4.93}
\]

From equation (4.86), the second term into the square-root is of order 1. Moreover, the first term is nothing but a big trace, like for (3.54), as a result:

\[
\sum I \langle | V^{(n)} I_{I_{k}} \xi^2 \bar{R} | \rangle \leq \left( \frac{1}{2} \cos^{-c(v)-1}\left( \frac{\theta}{2} \right) \right) \times N. \tag{4.94}
\]

Finally, because of our recursion hypothesis, all the \( C_k \) have to be of order 1 in \( N \), implying \( |\langle A_{T+1}\rangle| \sim \sqrt{N} \).

\( \square \)

To conclude, the only changes from the bound for the fermionic part coming from the Gaussian integration are:

1. The replacement \( \xi^2 \rightarrow \bar{g} \),
2. the combinatorial factor \( 2^{n-1} \) coming from the Wick contractions.

As a result, the averaged amplitude \( \langle A_{T_n}\rangle \) admits the bound:

\[
|\langle A_{T_n}\rangle| \leq \bar{K}\sqrt{N} \times |u|^n(\bar{\rho})^{-2n}(\frac{\theta}{2}), \tag{4.95}
\]

for some constant \( \bar{K} \) and with

\[
(\bar{\rho})^{-1} = \frac{24\sqrt{e/\pi}}{\beta g}. \tag{4.96}
\]

Therefore:

**Proposition 7** In the large \( N \) limit, the quartic SYK model is analytic in \( u = \rho e^{i\theta} \) at least in the interior of the cardioid domain

\[
\rho \leq \bar{\rho} \cos^2(\theta/2) = \frac{\beta g}{24\sqrt{e/\pi}} \cos^2(\theta/2). \tag{4.97}
\]
5 Renormalized constructive expansion in $d = 2$ relativistic bosons

In this section, we extend our results for a super-renormalizable model having physical divergences and requiring non-trivial subtractions to become well-defined. To subtract these divergences, we require renormalization prescription, and we have to establish a list of the divergent diagrams. We focus our attention on the two-dimensional relativistic complex bosonic model. We show that, in order to get a constructive expansion allowing to bound the amplitudes, we need to improve the standard LVE used in the first part of this paper. This is achieved by considering the multi-scale loop vertex expansion (MLVE) [27, 34], which allows to subtract scale by scale the divergent subgraphs, from an expansion requiring “forests into forests”, and known as jungle expansion.

5.1 Divergent graphs and renormalized model

From the power-counting theorem, equation (2.38), the degree of divergence for two-dimensional relativistic bosons becomes independent of the number of external edges $N$, and reads:

$$\Omega_{RB} = 2(1 - V). \quad (5.1)$$

As a result, there are two logarithmic divergent graphs, respectively for $(V, N) = (1, 2)$ and $(V, N) = (1, 0)$. From an elementary perturbative calculation using Feynman rules for the interaction:

$$S_{\text{int}} := \frac{1}{2} \sum_{ijkl} \int \frac{dt}{-\beta/2} \int_{-\infty}^{+\infty} dx \ W_{ijkl} \phi_i(x, t) \bar{\phi}_j(x, t) \phi_k(x, t) \bar{\phi}_l(x, t), \quad (5.2)$$

we get the explicit expressions of the two divergent diagrams:

1. The vacuum diagram $(V, N) = (1, 0)$:

$$\infty = -\frac{u}{2\pi\beta} \left( \sum_{\omega} \int_{-\infty}^{+\infty} \frac{dp}{\omega^2 + p^2 + m^2} \right)^2 \sum_{ij} W_{iijj}, \quad (5.3)$$

2. The 2-point diagram $(V, N) = (1, 2)$:

$$i \quad \circ \quad j = -\frac{u}{\pi\beta} \left( \sum_{\omega} \int_{-\infty}^{+\infty} \frac{dp}{\omega^2 + p^2 + m^2} \right) \sum_{kk} W_{kkkk}. \quad (5.4)$$

Both of these diagrams scale as $\ln(\Lambda)$, with $\Lambda$ designating some arbitrary UV cut-off in the sums and integrals. To obtain the renormalized action, we proceed
following two steps, beginning with non-vacuum diagrams. From the previous calculation, the 2-point divergent diagrams are renormalized with the mass counter-term:

$$
\delta V_1(\bar{\Phi}, \Phi) := \sum_\omega \int_{-\infty}^{+\infty} dp \bar{\phi}_i(\omega, p) \delta m_{ij}^2 \phi_j(\omega, p),
$$

with $\delta m_{ij}^2 := \Delta^2 \sum_k W_{jikk}$, and:

$$
\Delta^2 := -\frac{u}{\pi \beta} \sum_\omega \int_{-\infty}^{+\infty} \frac{dp}{\omega^2 + p^2 + m^2},
$$

so that the partially renormalized classical action $S_{PR}(\bar{\Phi}, \Phi)$ reads:

$$
S_{PR}(\bar{\Phi}, \Phi) = S(\bar{\Phi}, \Phi) + \delta V_1(\bar{\Phi}, \Phi).
$$

With this counter-term, all the divergent perturbative non-vacuum diagrams generated by the functional

$$
Z_{PR}(J, \bar{J}) = \int d\Phi d\bar{\Phi} e^{-S_{PR}(\bar{\Phi}, \Phi) + \sum_i \bar{J}_i \phi_i + \sum_i \bar{\phi}_i J_i}
$$

are cancelled. In order to obtain a theory without divergences, we have to subtract the vacuum divergences. As discussed before, this subtraction requires the counter-term $\delta V_2$:

$$
\delta V_2 := -\frac{u}{2\pi \beta} \left( \sum_\omega \int_{-\infty}^{+\infty} \frac{dp}{\omega^2 + p^2 + m^2} \right)^2 \sum_{ij} W_{ijjj}.
$$

However, an additional contribution arises from the mass counter-term, which introduce itself vacuum divergences requiring another counter-term $\delta V_3$, and it is easy to check that $\delta V_3 = -2\delta V_2$, so that the remaining vacuum counter-term is $-\delta V_2$ and the completely renormalized model reads:

$$
Z_R(J, \bar{J}) = e^{\delta V_2} \int d\Phi d\bar{\Phi} e^{-S_{PR}(\bar{\Phi}, \Phi) + \sum_i \bar{J}_i \phi_i + \sum_i \bar{\phi}_i J_i}
$$

As for the previous cases, the first step is to decompose the quartic interaction into three-body interaction using the intermediate field representation. The interaction part of the action, including mass counter-term reads (we simplify the notations for the sums and integrals, and discard the arguments of the functions)

$$
S_{\text{int},PR}(\bar{\Phi}, \Phi) = \frac{1}{2} \frac{u}{2\pi \beta} \int W_{ijkl} \phi_i \bar{\phi}_j \phi_k \bar{\phi}_l + \Delta^2 \sum \int \left( \sum_k W_{ijkk} \right) \phi_i \bar{\phi}_j.
$$

Completing the square, we get:

$$
S_{\text{int}}(\bar{\Phi}, \Phi) = \frac{1}{2} \frac{u}{2\pi \beta} \sum \left( \mathcal{M}_l + \frac{2\pi \beta \Delta^2}{u} \mathcal{X}_l \right)^2 - \frac{\pi \beta}{2u} (\Delta^2)^2 \sum \mathcal{X}_l^2,
$$

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where we defined $M_{kl} := \sum_{i,j} W_{ijkl}^{1/2} \phi_i \phi_j$ and $X_{ij} := \sum_k W_{ijjk}^{1/2}$, and the last term writes explicitly as:

$$\frac{\pi \beta}{2u} (\Delta^2)^2 \sum_I X_I^2 = \frac{u}{2\pi \beta} \left( \sum_{\omega} \int_{-\infty}^{+\infty} \frac{dp}{\omega^2 + p^2 + m^2} \right)^2 \sum_{ij} W_{ij} = -\delta V_2, \quad (5.12)$$

so that the renormalized partition function (5.10) may be rewritten as:

$$Z_R(J, \bar{J}) = \int d\Phi d \bar{\Phi} e^{-S_{\text{kin}}(\bar{\Phi}, \Phi)} e^{-\frac{1}{2} \frac{u}{2\pi \beta} \sum_I (M_I + 2\pi \beta \Delta^2 u X_I)^2}. \quad (5.13)$$

Breaking the square using the Gaussian trick, we get:

$$Z(u) = \int d\nu(\sigma) d\mu_C(\bar{\Phi}, \Phi) e^{-S_{\text{int}}(\bar{\Phi}, \Phi, \sigma)} e^{-\frac{1}{2} \frac{u}{2\pi \beta} \sum_I (M_I + 2\pi \beta \Delta^2 u X_I) \sigma_I}. \quad (5.14)$$

Integrating over the fields $\Phi$ and $\bar{\Phi}$, and rewriting the determinant as a $\text{Tr-log}$ effective interaction, we finally get the effective model for the field $\sigma$:

$$Z(u) = \int d\nu(\sigma) e^{-\text{Tr} \ln K[\sigma]} \sqrt{\frac{2 \pi \beta}{u}} \Delta^2 \int \sum_I X_I \sigma_I, \quad (5.15)$$

where $K$ is the matrix with elements:

$$K_{ij}(p, p') := \delta_{ij} \delta(p, p') - \sqrt{\frac{-u}{2\pi \beta}} \frac{1}{p^2 + m^2} W_{ijkl}^{1/2} \sigma_k(p - p') \quad (5.16)$$

with $\delta(p, p') := \delta_{\omega,\omega'} \delta(p - p')$ and $\text{Tr}$ in formula (5.16) means

$$\text{Tr} K := \sum_i \int dp K_{ii}(p, p). \quad (5.17)$$

with the short notation $\int dp := \sum_{\omega} \int dp$. Note that the linear terms in $\sigma$ into the equation (5.16) exactly compensate the first term in the Taylor expansion of the logarithm; in such a way that the effective partition function may be rewritten as:

$$Z(u) = \int d\nu(\sigma) e^{-\text{Tr} \ln K[\sigma]} \quad (5.18)$$

The notation $\ln 2$ in this expression makes sense because $K$ has the form $K := \text{Id}/H$, the explicit expression for the operator $H$ being given on equation (5.17).
Moreover, we recall the standard definition $\ln_2(1-x) = \ln(1-x) + x$. Defining the resolvent $R = K^{-1}$, we have once again the operator bound:

$$\|R\| \leq \cos^{-1} \left( \frac{\theta}{2} \right)$$

(5.20)

where $\theta = \arg(u)$, $|u| \leq \pi$. This bound follows from the Lemma 2, using the same symmetrized form like in equation (4.31) to define the (anti-hermitian) operator

$$\mathcal{H} := -\sqrt{-\frac{u}{2\pi \beta}} C^{1/2} \Sigma C^{1/2},$$

(5.21)

to transform $\mathcal{H}$ in an hermitian operator on $\mathbb{C}^N \otimes L_2(S^1, \mathbb{R}) \otimes L_2(\mathbb{R})$.

### 5.2 Multi-scale loop vertex expansion

In this section, we have to fix our conventions about the UV cut-off. A natural choice, taking into account the $\text{SO}(2)$ invariance of the Laplacian is to choose a disk: $0 \leq p^2 \leq \Lambda^2$, with the short notations $p := (\omega, p)$ and $p^2 := \omega^2 + p^2$. We refer as "momentum space" the set of $p$ into the disk of radius $\Lambda$. In order to use the multi-scale expansion, we need to introduce a slicing into the momentum space. To this end, we consider a pair of real and integer $(M, j_{\text{max}})$ and assume that $\Lambda = M^{j_{\text{max}}}$. Intermediate scales are then defined as $M^j$ for $j \leq j_{\text{max}}$, and we introduce the slice function $\chi_j(p)$ such that:

$$\chi_j(p) := \chi_{\leq j}(p) - \chi_{\leq j-1}(p) \quad j \geq 2,$$

(5.22)

where $\chi_{\leq j}(p)$ denotes the step function $\chi_{\leq j}(p) := \theta(M^{2j} - p^2)$. Defining

$$V_{\leq j} := \text{Tr} \ln_2 (\text{id} + \chi_{\leq j}\mathcal{H}\chi_{\leq j}),$$

(5.23)

and

$$V_j := V_{\leq j} - V_{\leq j-1},$$

(5.24)

such that $\sum_j V_j = V \equiv \text{Tr} \ln_2 (\text{id} + \mathcal{H})$. In terms of this slice decomposition, the partition function (5.19) reads:

$$Z(u) = \int d\nu(\sigma) \prod_{j=0}^{j_{\text{max}}} e^{-V_j[\sigma]}.$$

(5.25)

The first MLVE trick is then to view the product as a determinant, and to rewrite it as a Grassmann integral. More precisely, defining $W_j := e^{V_j} - 1$, we may rewrite (5.25) as:

$$Z(u) = \int d\nu(\sigma) \int \prod_{j=0}^{j_{\text{max}}} d\mu_1(\bar{\chi}_j, \chi_j) e^{-\sum_{j=0}^{j_{\text{max}}} \bar{\chi}_j W_j \chi_j},$$

(5.26)

\footnote{We use the same symbol $\mathcal{H}$ is both cases, the context avoiding any ambiguity.}
with the identity Gaussian Grassmann measure
\[ d\mu_1(\bar{\chi}_j, \chi_j) := d\bar{\chi}_j d\chi_j \exp (-\bar{\chi}_j \chi_j). \]

(5.27)

We are now in position to perform the MLVE of the model. We will only reproduce the main step of the general method, referring on the standard reference [34] for technical details. Let us define \( S = [0, j_{\text{max}}] \) the set of scales and \( I_S \) the \( S \times S \) identity matrix, allowing to rewrite the partition function \( Z(u) \) in the compact form
\[ Z(u) = \int d\nu_S e^{-W} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu (-W)^n, \]

(5.28)

where \( d\nu := d\nu(\sigma) d\mu_1(\{\bar{\chi}_j, \chi_j\}) \). The quotes around the equality symbol refers to the illegal permutation of sum and Gaussian integrations. In fact, the aim of our procedure is to give a sense for this equality. As for the \( d = 0 \) and \( d = 1 \) cases, we introduce replicas for the vertices in the set \( V := \{1, \cdots, n\} \):

\[ Z(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_{S,V} \prod_{a=1}^{n} (-W_a), \]

(5.29)

in such a way that the vertex \( W_a \) express in terms of bosonic intermediate fields \( \sigma^a \), and the Gaussian measure \( d\nu_{S,V} \) becomes \( d\nu_{S,V} = d\nu_1(\sigma) d\mu_2(\{\bar{\chi}_j, \chi_j\}) \), the covariance becoming the \( V \times V \) matrix \( \mathbf{1}_V \) with all entries equals to 1. The MLVE trick to factorize the integral is to perform two successive forest formula. The first one over the bosonic intermediate field follows the same way as for the previous cases. We introduce the covariance \( x_{ab} = x_{ba} \), with \( x_{aa} = 1 \) between the bosonic replicas, and use the forest formula (2) to write, in the derivative representation (holding as well for Grassmann fields):

\[ Z(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{\ell \in \mathcal{F}} dx_{\ell} \left[ e^{\frac{1}{2} \sum_{a,b} X_{ab}(x_{\ell}) \frac{\partial}{\partial \sigma^a} \frac{\partial}{\partial \sigma^b} \sum_{j=0}^{j_{\text{max}}}} \sum_{j=0}^{j_{\text{max}}} \bar{\chi}_j W_j(\sigma^a) \chi_j \right] \prod_{\ell \in \mathcal{F}} \left( \frac{\partial}{\partial \sigma^a(\ell)} \frac{\partial}{\partial \sigma^b(\ell)} \right) \prod_{a=1}^{n} \left( -\sum_{j=0}^{j_{\text{max}}} \bar{\chi}_j W_j(\sigma^a) \chi_j \right) \bigg|_{\sigma = \chi = \bar{\chi}} = 0. \]

(5.30)

where \( a(\ell) \) and \( b(\ell) \) denote the end points of the edge \( \ell \). The forest \( \mathcal{F} \) defines a natural partition of the set \( V \) into blocks building of its connected spanning trees. We call such a block as \( \mathcal{B} \), and \( V/\mathcal{F} \) the reduced set building of bosonic spanning trees \( \mathcal{B} \). Introducing the same replica trick for fermionic fields between these blocks, such that \( \chi_j \rightarrow \chi^\mathcal{B}_j \), as well as covariance \( y_{BS'} = y_{SB} \), setting all equals to 1. Following standard notations, we call \( L_F \) a generic fermionic edge between two blocks \( \mathcal{B} \) and \( \mathcal{B}' \). Using the Forest formula a
second time, we obtain an expansion indexed by two-level jungles $\mathcal{J}$ rather than forests:

$$Z(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{J}} \sum_{j_1} \cdots \sum_{j_n} \int_{0}^{1} dx_{\mathcal{J}} \int dv_{\mathcal{J}} \partial_{\mathcal{J}} \left[ \prod_{B} \prod_{a \in B} \left( \lambda_{j_a}^{B} W_{j_a}(\sigma^{a}) \chi_{j_a}^{B} \right) \right],$$

(5.31)

where:

- The two level jungles $\mathcal{J} = (\mathcal{F}_{B}, \mathcal{F}_{F})$ are ordered pairs of bosonic and fermionic disjoint forests of the set $V$, denoted as $\mathcal{F}_{B}$ and $\mathcal{F}_{F}$. We denote as $\ell_{B}$ and $L_{F}$ the bosonic and fermionic edges of the two components of the jungle $\mathcal{J}$. The notation $\ell$ being kept to denote the generic edges of $\mathcal{J}$.

- $dx_{\mathcal{J}}$ means integration from 0 to 1 over parameters $x_{\ell}$, for each $\ell \in \mathcal{J}$.

- $\partial_{\mathcal{J}}$ is a compact notation for:

$$\partial_{\mathcal{J}} := \prod_{\ell_{B} = (a, b) \in \mathcal{F}_{B}} \left( \frac{\partial}{\partial \sigma^{a}} \frac{\partial}{\partial \sigma^{b}} \right)$$

$$\times \prod_{L_{F} = (p, q) \in \mathcal{F}_{F}} \delta_{j_{p} j_{q}} \left( \frac{\partial}{\partial \chi_{j_{a}}^{B}(a)} \frac{\partial}{\partial \chi_{j_{b}}^{B}(b)} + \frac{\partial}{\partial \chi_{j_{b}}^{B}(b)} \frac{\partial}{\partial \chi_{j_{a}}^{B}(a)} \right).$$

(5.32)

- The Gaussian measure $dv_{\mathcal{J}}$ has covariance $X(x_{\ell}) \otimes 1_{S}$ for bosons and $Y(x_{\ell}) \otimes 1_{S}$ for fermions, that is to say, in derivative representation:

$$dv_{\mathcal{J}} \equiv \left[ \frac{1}{e} \sum_{a,b} X_{ab}(x_{\ell}) \frac{\partial}{\partial \sigma^{a}} \frac{\partial}{\partial \sigma^{b}} + \sum_{B B'} Y_{B B'}(x_{\ell}) \sum_{a \in B, b \in B'} \delta_{j_{a} j_{b}} \frac{\partial}{\partial \chi_{j_{a}}^{B}} \frac{\partial}{\partial \chi_{j_{b}}^{B'}} \right]_{\sigma = \chi = \bar{\chi} = 0}.$$

(5.33)

- $X_{ab}(x_{\ell})$ is the infimum of the parameters $x_{\ell}$ for the bosonic component of the Jungle, in the unique path from $a$ to $b$. It is set to be equal to zero if the path does not exist, and to 1 if $a = b$.

- $Y_{B B'}(x_{\ell})$ is the infimum of the fermionic parameters along the fermionic edges $L_{F}$, between the blocks $B$ and $B'$.

As for the cases discussed in the previous section, our aim is now to bound the amplitudes indexed with jungles. More precisely, we will prove the existence of a finite analyticity domain for the free energy $F(u) = \ln Z(u)$, expanding in terms of connected jungles:

$$F(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\text{trees}} \sum_{j_1} \cdots \sum_{j_n} \int_{0}^{1} dx_{\mathcal{J}} \int dv_{\mathcal{J}} \partial_{\mathcal{J}} \left[ \prod_{B} \prod_{a \in B} \left( \lambda_{j_a}^{B} W_{j_a}(\sigma^{a}) \chi_{j_a}^{B} \right) \right].$$

(5.34)
In the next section, we will prove the existence of a finite domain of analyticity, bounding separately the bosonic and fermionic contributions. The final result allows to extend our conclusions of the previous section about Borel summability for a model admitting non-trivial divergences.

## 5.3 Bounds and convergence

In this section, we will bound the bosonic and fermionic contributions of the decomposition (5.34). The technical parts of the bounds are essentially the same as for the general treatment given in [34], and we refer to this paper for the technical subtleties, only indicating the main steps of the proof and focusing on the specificity of the model.

### 5.3.1 Fermionic integrals

The fermionic part of the expansion (5.34) is exactly the same as in [34]. Due to the standard properties of Grassmann integration, the Gaussian integration over these variables can be written as:

\[
\prod_B \prod_{a \in B} \left( \frac{\partial}{\partial \bar{\chi}^B_{ja}} \frac{\partial}{\partial \chi^B_{ja}} \right) e^{\sum_{B,B'} Y_{BB'}(x_{L_F}) \sum_{a \in B, b \in B'} \delta_{ja_{j_a}} \bar{\chi}^B_{ja} \chi^B_{jb}} 
\times \prod_{L_F \in F_F} \delta_{j_a(L_F)j_b(L_F)} \left( \bar{\chi}^B_{ja(L_F)} \chi^B_{ja(L_F)} + \bar{\chi}^B_{jb(L_F)} \chi^B_{jb(L_F)} \right) \bigg|_{\bar{\chi}, \chi = 0}.
\]

Denoting \( Y_{ab} := Y_{B(a)B(b)\delta_{ja,jb}} \), and taking into account that this matrix is symmetric, the previous Gaussian integral writes into the more familiar form:

\[
\int \prod_B \prod_{a \in B} d\bar{\chi}^B_{ja} d\chi^B_{ja} e^{-\sum_{a, b} \bar{\chi}^B_{ja} Y_{ab} \chi^B_{ja}} \prod_{L_F \in F_F} \delta_{j_a(L_F)j_b(L_F)} \left( \bar{\chi}^B_{ja(L_F)} \chi^B_{ja(L_F)} + \bar{\chi}^B_{jb(L_F)} \chi^B_{jb(L_F)} \right) \bigg|_{\bar{\chi}, \chi = 0}.
\]

Defining:

\[
Y^{p_1, \ldots, p_k}_{m_1, \ldots, m_k} := \int \prod_B \prod_{a \in B} d\bar{\chi}^B_{ja} d\chi^B_{ja} e^{-\sum_{a, b} \bar{\chi}^B_{ja} Y_{ab} \chi^B_{ja}} \prod_{r=1}^k \bar{\chi}^{B(r)}_{j_{r}} \chi^{B(r)}_{j_{r}},
\]

as the minor of the matrix \( Y \), having the lines \( p_1 \cdots p_k \) and the columns \( m_1 \cdots m_k \) deleted; and taking into account the hard core constraint inside each block, meaning that the integral (5.36) vanishes as soon as two vertices
are belonging to the same bosonic block $B$ with the same scale attribution, one can rewrite the equation (5.36) as:
\[
\left( \prod_{B} \prod_{a,b \in B \atop a \neq b} (1 - \delta_{j_a,j_b}) \right) \left( \prod_{L \in F_F} \delta_{j_a(L_F)j_b(L_F)} \right) \left( Y^{p_1,\ldots,p_k}_{a_1,\ldots,a_k} + Y^{m_1,\ldots,m_k}_{p_1,\ldots,p_k} + \ldots + Y^{m_1,\ldots,m_k}_{a_1,\ldots,a_k} \right)
\]
\[
(5.38)
\]
where the sum runs over the $2^k$ ways to exchange the upper and lower indices, and $k := |F_F|$ is the cardinal of the fermionic forest, and the first product over factors $(1 - \delta_{j_a,j_b})$ implements the hard core constraint. To bound the fermionic contribution, we have the important lemma:

**Lemma 4** Due to the positivity of the covariance $Y$, for any $\{m_i\}$ and $\{p_i\}$ the minor $Y^{p_1,\ldots,p_k}_{m_1,\ldots,m_k}$ defined in (5.37) satisfies:
\[
|Y^{p_1,\ldots,p_k}_{m_1,\ldots,m_k}| \leq 1.
\]
\[
(5.39)
\]
This lemma can be easily checked, and the proof may be found in [34].

### 5.3.2 Bosonic integrals

We now move on to the problem of the bosonic integrals. From the two level trees decomposition of the free energy, it is clear that bosonic integrations factorizes over each block $B$. As a result, we can only consider a single of such blocks. To this end, let us consider such a block $B$. It involves the Gaussian integration:
\[
\int d\nu_B F_B(\sigma) = e^{\frac{1}{2} \sum_{a,b \in B} X_{ab}(w) \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} F_B(\sigma)} \bigg|_{\sigma = 0}
\]
\[
(5.40)
\]
with $F_B(\sigma)$ defined as:
\[
F_B(\sigma) = \int \prod_{\ell \in B} dp_{\ell} \left( \frac{\partial^2}{\partial \sigma_a(\ell) \partial \sigma_b(\ell)} \left( -p_{\ell_a} p_{\ell_b} \right) \right) \prod_{a \in B} W_{j_a}(\sigma_a),
\]
\[
(5.41)
\]
where we introduce the explicit momentum dependence and the integrations over momenta along each bosonic edge. The derivatives $\partial / \partial \sigma$ can be evaluated using the famous *Faà di Bruno* formula, which extends the standard derivation rule for composed functions:
\[
\frac{\partial^q}{\partial x^q} f(g(x)) = \sum_\pi f^{[\pi]}(g(x)) \prod_{B \in \pi} g^{[B]}(x).
\]
\[
(5.42)
\]
In this formula, $\pi$ runs over the partitions of the set $\{1,\ldots,q\}$ and $B$ runs through the blocks of the partition $\pi$. Computing the first derivative of the potential $V_j$, we get:
\[
\partial_{\sigma} (-V_j) = \text{Tr} ( (\partial_{\sigma} \mathcal{H}) \chi_{\leq j} (R_{\leq j} - 1) - (\partial_{\sigma} \mathcal{H}) \chi_{\leq j-1} (R_{\leq j-1} - 1) ) = \text{Tr} ( \mathcal{H} \chi_{\leq j} (\partial_{\sigma} \mathcal{H}) \chi_{\leq j} R_{\leq j} - \mathcal{H} \chi_{\leq j-1} (\partial_{\sigma} \mathcal{H}) \chi_{\leq j-1} R_{\leq j-1} )
\]
\[
(5.43)
\]
where:
\[
R_{\leq j} := (1 + \chi_{\leq j} \mathcal{H} \chi_{\leq j})^{-1}.
\] (5.44)

The formula can be easily extended for a derivative of degree \( k > 0 \) as:
\[
\prod_{l=1}^{k} \partial_{\sigma_a(l)} (-V_j) = \sum_{\pi} \text{Tr} \left( \frac{(\partial_{\sigma_a(\pi(1))} \mathcal{H}) \chi_{\leq j} R_{\leq j} \cdots (\partial_{\sigma_a(\pi(k))} \mathcal{H}) \chi_{\leq j} R_{\leq j}}{k \text{ times}} \right),
\]
\[
- \frac{(\partial_{\sigma_a(\pi(1))} \mathcal{H}) \chi_{\leq j - 1} R_{\leq j - 1} \cdots (\partial_{\sigma_a(\pi(k))} \mathcal{H}) \chi_{\leq j - 1} R_{\leq j - 1}}{k \text{ times}},
\] (5.45)

where the sum over \( \pi \) runs over permutation of \( k \) elements, up to cyclic permutations. As a result \( \sum_{\pi} = (k - 1)! \). Then, the \( k \)-th derivative of \( W_j \) can be deduced from the Faà di Bruno formula (5.42). For \( k > 0 \) we get:
\[
\prod_{l=1}^{k} \partial_{\sigma_a(l)} (-W_j) = e^{-V_j} \sum_{\{m_l\}} \frac{k!}{\prod_{i \geq 1} m_i!(l!m_i)} \prod_{l \geq 1} [\partial_{\sigma_a(l)} (-V_j)]^{m_l},
\] (5.46)

where we used the compact notation \( \prod_{l \geq 1} [\partial_{\sigma_a(l)} (-V_j)]^{m_l} \) to represent a block of derivative of size \( l \). In (5.41), we can rewrite the product as a product over the arcs of the vertices:
\[
F_{B}(\tilde{\sigma}) = \prod_{v \in B} \left[ \frac{c(v)}{k \prod_{i=1}^{k} \partial_{\sigma_a(v)}(\epsilon_k p_k)} \right] W_{j_v}(\sigma_{a(v)}).
\] (5.47)

where once again \( c(v) \) denotes the coordination number of the vertex \( v \), equal to the number of half lines of the intermediate-fields hooked to this vertex, and \( \epsilon_k = \pm 1 \) is a sign depending of the orientation of the momentum. As a result, the bosonic integral (5.40) becomes:
\[
\int d\nu_{B} \left[ \prod_{m \in B} e^{-V_j} (\lambda)^{c(m)} \sum_{\{x_i^{(m)}\}} \frac{c(m)!}{\prod_{l \geq 1} x_i^{(m)!} \prod_{l \geq 1} [\partial_{\sigma_a(l)} (-V_j)]^{m_l}} \right],
\] (5.48)

where we used again the compact notations for the derivatives, and the integrations over momenta are implicit. Note that the bar over the product means that we extracted the factor \((\lambda)^{1 \times x_i^{(m)}}\), with \( \lambda^2 := -u/2\pi\beta \). Using the constraint: \( \sum_m c(m) = 2(|B| - 1) \), with \( |B| \) the number of vertices of \( B \), and because of the bound \( \|R\| \leq \cos^{-1}(\theta/2) \), the equation (5.48) satisfies the
inequality:

$$\left| \int d\nu_B F_B(\tau) \right| \leq \left( \frac{\lambda^2}{\cos^2(\phi/2)} \right)^{|B|-1} \int d\nu_B \left[ \prod_{m \in B} e^{-V_{jm}(\tau_m)} \right]$$

$$\times \sum_{\{x^{(m)}_i\}_{i \geq 1}} \frac{c(m)!}{\prod_{i \geq 1} x^{(m)}_i \prod_{l \geq 1} |\partial^l (V_j)| x^{(m)}_l} |R = \text{id}|,$$

where in the right hand side we set $R = \text{id}$. Defining:

$$G_B := \prod_{m \in B} \sum_{\{x^{(m)}_i\}_{i \geq 1}} \frac{c(m)!}{\prod_{i \geq 1} x^{(m)}_i \prod_{l \geq 1} |\partial^l (V_j)| x^{(m)}_l} |R = \text{id}|,$$

and since the Gaussian measure $d\nu_B$ is positive, we can use the Cauchy–Schwarz inequality to get:

$$\int d\nu_B \prod_{m \in B} e^{-V_{jm}(\tau_m)} G_B \leq \left( \int d\nu_B \prod_{m \in B} |e^{-2V_{jm}(\tau_m)}| \right)^{1/2} \left( \int d\nu_B |G_B|^2 \right)^{1/2}.$$  \hspace{1cm} (5.51)

We called the first term non-perturbative factor, and the second term perturbative factor, and shall to treat separately each of them.

### 5.3.3 Final bounds

**Non-perturbative bound** We begin with the first term of the bosonic integral, the non-perturbative contribution (following the notations of [63]):

$$B_1 := \int d\nu_B \prod_{a \in B} |e^{-2V_{ja}(\tau_a)}|,$$

$d\nu_B$ meaning restriction of the measure $d\nu_J$ to the block $B$. First of all, note that: $|e^{-2V_{ja}(\tau_a)}|$ is uniformly bounded by $\leq e^{2|V_{ja}(\tau_a)|}$. Secondly, because of the identity:

$$\ln_2(1 - x) = \int_0^1 dt \frac{tx^2}{1 - tx},$$  \hspace{1cm} (5.53)

and using the bound (5.20), we get the inequality:

$$|V_j| \leq \frac{1}{\cos(\theta/2)} \left| \text{Tr}(\mathcal{H}_{\leq j})^2 - \text{Tr}(\mathcal{H}_{\leq j-1})^2 \right|,$$  \hspace{1cm} (5.54)

with the definition $\mathcal{H}_{\leq j} := \chi_{\leq j} \mathcal{H} \chi_{\leq j}$. Because of the cyclicity of the trace:

$$\text{Tr}(\mathcal{H}_{\leq j})^2 - \text{Tr}(\mathcal{H}_{\leq j-1})^2 = \text{Tr} \mathcal{H} (\chi_{\leq j} \mathcal{H} \chi_{\leq j} - \chi_{\leq j-1} \mathcal{H} \chi_{\leq j-1}).$$  \hspace{1cm} (5.55)
Because of the definition \( \chi_j := \chi_{\leq j} - \chi_{\leq j-1} \), the difference may be rewritten as:
\[
\chi_{\leq j} \mathcal{H} \chi_{\leq j} - \chi_{\leq j-1} \mathcal{H} \chi_{\leq j-1} = \chi_j \mathcal{H} \chi_{\leq j-1} + \chi_{\leq j-1} \mathcal{H} \chi_j
\]  \hspace{1cm} (5.56)
and we get, because of the cyclicity of the trace:
\[
\text{Tr}(\mathcal{H} \chi_j)^2 - \text{Tr}(\mathcal{H} \chi_{\leq j-1})^2 = 2 \text{Tr}(\mathcal{H} \chi_j \mathcal{H} \chi_{\leq j-1}).
\]  \hspace{1cm} (5.57)
Because \( \chi_{\leq j}^2 = \chi_{\leq j} \) and \( \chi_j^2 = \chi_j \); defining \( C_j := \chi_j \mathcal{H} \chi_{\leq j} \) and \( C_{\leq j} := \chi_{\leq j} \mathcal{H} \chi_{\leq j} \), the trace may be explicitly computed as:
\[
\text{Tr}(\mathcal{H} \chi_j \mathcal{H} \chi_{\leq j-1}) = \frac{-uN}{2\pi\beta} \int dp' \sigma(-p') \left( \int dp C_{\leq j}(p) C_{j}(p - p') \right) \sigma(p'),
\]  \hspace{1cm} (5.58)
the factor \( N \) arising from the trace over internal indices \( \text{Tr} \mathbf{I} = N \). The covariance for bosonic variables is then modified as \( X_B^{-1} \rightarrow X_B^{-1} - 8 \cos^{-1}(\theta/2) \mathcal{I}_B \otimes \hat{L}, \hat{L} \) being the bounded operator of \( L_2(S^1, \mathbb{R}) \):
\[
\hat{L}(p') := \mathcal{O}(1) \left( \frac{|u|N}{2\pi\beta} \int dp C_{\leq j}(p) C_{j}(p - p') \right), \quad \| \hat{L} \| \leq \mathcal{O}(1) \left( \frac{|u|N}{2\pi\beta} M^{-2j} \right),
\]  \hspace{1cm} (5.59)
\( \mathcal{O}(1) \) denoting some numerical constant of order 1. Computing the Gaussian integration provides the determinant:
\[
\text{det} \left[ \mathbf{id} - \frac{8}{\cos(\theta/2)} \frac{|u|N}{2\pi\beta} X_B \otimes \hat{L} \right]^{-1/2} = e^{-\frac{1}{2} \text{Tr} \ln \left( \mathbf{id} - \frac{8}{\cos(\theta/2)} \frac{|u|N}{2\pi\beta} X_B \otimes \hat{L} \right)}.
\]  \hspace{1cm} (5.60)
For some positive bounded operator \( \hat{A} \), we have the following bound:
\[
-\text{Tr} \ln(1 - \hat{A}) = \sum_{n=1}^{\infty} \frac{\text{Tr}(\hat{A})^n}{n} \leq \text{Tr}(\hat{A}) \times \sum_{n=2}^{\infty} \frac{\| \hat{A} \|^n}{n} \leq \text{Tr}(\hat{A}) \times \sum_{n=2}^{\infty} \| \hat{A} \|^n.
\]  \hspace{1cm} (5.61)
Setting \( \hat{A} := \frac{8|u|N}{2\pi\beta} \cos^{-1}(\theta/2) \mathcal{I}_B \otimes \hat{L} \), it is easy to check that:
\[
\text{Tr}(\hat{A}) \leq \mathcal{O}(1) \left( \frac{8|u|N}{2\pi\beta} \cos^{-1}(\theta/2) |\mathcal{B}| \right), \quad \| \hat{A} \| \leq \mathcal{O}(1) \left( \frac{8|u|N}{2\pi\beta} \cos^{-1}(\theta/2) \right),
\]  \hspace{1cm} (5.62)
leading to:
\[
|\text{Tr} \ln(1 - \hat{A})| \leq |\mathcal{B}| \sum_{n=1}^{\infty} \left( \mathcal{O}(1) \left( \frac{8|u|N}{2\pi\beta} \cos^{-1}(\theta/2) \right) \right)^n.
\]  \hspace{1cm} (5.63)
As a result, we deduced the following lemma:

**Lemma 5** For \( |u| \) small enough, \( B_1 \) defined in (5.52) satisfy the following uniform bound:
\[
B_1 \leq e^{\mathcal{O}(1) \left( \frac{|u|N}{2\pi\beta} \cos^{-1}(\theta/2) \right)}.
\]  \hspace{1cm} (5.64)
Perturbative bound  We now move on to the perturbative bound:

\[ B_2 := \left( \int d\nu_B |G_B|^2 \right)^{1/2}. \]  \hspace{1cm} (5.65)

For \( l > 1 \) we have:

\[ \int dp C_j^l(p) \leq \int dp C_j^2(p) \leq \frac{1}{M^{4(j-1)}} \int dp \chi_j(p). \]  \hspace{1cm} (5.66)

The last integral is noting but the volume between the disks of radius \( p^2 = M^{2(j-1)} \) and \( p^2 = M^{2j} \), as a result:

\[ \int dp C_j^l(p) \leq \mathcal{O}(1) \frac{\pi}{M^{2j-4}}. \]  \hspace{1cm} (5.67)

The result may be generalized for integrals of the type:

\[ \mathcal{I}_{p_1, \ldots, p_l} := \int dp \left( C_{<j}(p) C_{<j}(p-p_1) \cdots C_{<j}(p-p_l) \right) - (j \to j-1). \]  \hspace{1cm} (5.68)

The domain of the integral is the intersection of disks centered in \( 0, p_1, \ldots, p_l \). In the worst case, \( p_1 = p_2 = \cdots = 0 \), and we get:

\[ \mathcal{I}_{p_1, \ldots, p_l} \leq M^{-2l+1(j-1)} \int dp \chi_j(p) \leq \mathcal{O}(1) \frac{\pi}{M^{2j+2}} \leq \mathcal{O}(1) \frac{\pi}{M^{2j-4}}. \]  \hspace{1cm} (5.69)

To compute the Gaussian integral, we have to retain only the leaves of the trees, each of them involving a single intermediate field. To say more, because of the inequality (5.69), the Gaussian block has to be uniformly bounded by a Gaussian integral of the form:

\[ H_B := \int d\tilde{\nu}_B \prod_{a \in B} \left( \mathcal{O}(1) \frac{\pi}{M^{2j_a-4}} \right)^{k_a} \times (\sigma_a)^{k_a}. \]  \hspace{1cm} (5.70)

Because we bounded the integrals over each loop with the worst bound, the remaining Gaussian integrations are momentum-independent. Then, the variables \( \sigma_a \) do not depend on the momentum, and the Gaussian measure \( \int d\tilde{\nu}_B \) has covariance \( X_B \). Because \( ||X_B|| \leq 1 \), it follows that the Gaussian integration (5.70) is bounded by:

\[ |H_B| \leq \prod_{a \in B} \left( \mathcal{O}(1) \frac{\pi}{M^{2j_a-4}} \right)^{k_a} \times \left( \sum_{a \in B} k_a \right)!! \]  \hspace{1cm} (5.71)

From equation (5.65), the double factorial admits the bound:

\[ \left( 2 \sum_{m \in B} x_1^{(m)} \right)!! \leq (4|B| - 4)!! . \]  \hspace{1cm} (5.72)
As a result, assuming $j_m > 2$, we get the following bound:\(^{15}\)

$$|B_2| \leq \sqrt{(4|B| - 4)!} \times \prod_{m \in B} \sum_{\{x_i^{(m)}\}} \frac{c(m)!}{\Pi_{i \geq 1} x_i^{(m)}! \Pi_{i \geq 1} x_i^{(m)}!} \left| \frac{x_i^{(m)}}{\lambda} \right| \times (\mathcal{O}(1)\pi)^{c(m)} M^{-(2j_m - 4)}. \quad (5.73)$$

Assuming $|\lambda| \leq 1$, and taking into account that

$$\sum_{\{x_i^{(m)}\}} \frac{1}{\Pi_{i \geq 1} x_i^{(m)}! \Pi_{i \geq 1} x_i^{(m)}!}$$

is nothing but the coefficient of $x^{c(m)}$ in the Taylor expansion of $\prod_k e^{x^k/k} = 1/(1 - x)$, and recalling that, due to the momentum conservation a global factor $\int d\mathbf{p}$ appears in front of each amplitude bound, we get the pessimistic bound for the perturbative contribution:

**Lemma 6** For $|\lambda|$ small enough, the perturbative contribution $B_2$ satisfies the following bound:

$$|B_2| \leq \sqrt{(4|B| - 4)!} \times \prod_{m \in B} (\mathcal{O}(1)\pi)^{c(m)} c(m)! M^{-2j_m} \int d\mathbf{p}. \quad (5.75)$$

As a result, taking into account the Lemmas 5 and 6, we deduce the final bound:

$$\left| \int dv_B \Pi_{m \in B} e^{-V_{jm}(\tau_m)G_B} \right| \leq e^{\mathcal{O}(1)} \frac{\lambda}{\cos(\phi/2)} |B| \sqrt{(4|B| - 4)!}$$

$$\times \prod_{m \in B} c(m)! \mathcal{O}(1)\pi^{c(m)} M^{-2j_m} \int d\mathbf{p}, \quad (5.76)$$

and the following proposition for the bosonic integration:

**Proposition 8** For $|\lambda|$ small enough, the bosonic integration admits the bound:

$$\left| \int dv_B \Pi_{m \in B} \frac{\lambda}{\cos(\phi/2)} |B| \sqrt{(4|B| - 4)!} \times \prod_{m \in B} c(m)! (\mathcal{O}(1)\pi)^{c(m)} M^{-2j_m} \int d\mathbf{p}. \quad (5.77)$$

\(^{15}\)Note that this restriction is unnecessary, the first slices can be bounded with a simple loop vertex expansion.
As for the $d = 1$ case, the last integral $\int dp$ appears as a global factor, and may be discarded from the analysis, up to a regularization procedure to make it finite. To simplify the expressions, we forget this constant factor. Finally, collecting the results for bosonic and Grassmann bounds, we get the uniform bound for the free energy $\ln(Z)$ (to simplify the expression we forget the exponential factor $e^{O(1)|\lambda \cos(\phi/2)|B|}$):

$$|\ln Z[J, \bar{J}, \lambda]| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_\mathcal{J} \prod_{k=1}^{n} \prod_{j_k=0}^{j_{\max}} 2^{L(\mathcal{F}_F)} \left( \prod_{\ell_f \in \mathcal{F}_F} \delta_{j_{\ell_f}(j_{\ell_f})} \right)^{B-1}$$

$$\times \prod_{\mathcal{B}} \prod_{m, m' \in \mathcal{B}, m \neq m'} (1 - \delta_{j_m j_{m'}}) N^{(|\mathcal{B}| - 1)}(\frac{|\lambda|}{\cos^2(\phi/2)})^{B-1} \times \sqrt{(4|\mathcal{B}| - 4)!!} \prod_{m \in \mathcal{B}} c(m)! \left(\mathcal{O}(1)\pi\right)c(m) M^{-2j_m},$$

(5.78)

the factor $2^{L(\mathcal{F}_F)}$ involving the number of fermionic edges $L(\mathcal{F}_F)$. Because of Cayley’s Theorem, the number of trees with $n$ labelled vertices and coordination numbers $c_i$ for each vertex $i = 1, \ldots, n$ is given by:

$$\frac{(n-2)!}{\prod_i (c_i - 1)!}.$$  

(5.79)

This result shows that the sum involved in (5.78) obeys

$$\sum_m c(m) \prod_{m \in \mathcal{B}} c(m) = \frac{(3|\mathcal{B}| - 3)!}{(|\mathcal{B}| - 2)!(2|\mathcal{B}| - 1)!}.$$  

(5.80)

Collecting all the factorials leads to:

$$\sqrt{(4|\mathcal{B}| - 4)!!} \frac{(3|\mathcal{B}| - 3)!}{(2|\mathcal{B}| - 1)!},$$  

(5.81)

and using Stirling’s formula we get:

$$2\sqrt{(4|\mathcal{B}| - 4)!!} \frac{(3|\mathcal{B}| - 3)!}{(2|\mathcal{B}| - 1)!} \leq (|\mathcal{B}| - 1)! 3^{|\mathcal{B}|} e^{-|\mathcal{B}| |\mathcal{B}|^{\mathcal{B}}}. $$  

(5.82)

We now move on to sum over scale attributions, taking into account the hard core constraint. As explained in full details in [34], the hard core constraint imposes that the scale assignments of vertices in a same block are all different, implying that:

$$\sum_{m \in \mathcal{B}} j_m \geq j_{\min} + (j_{\min} + 1) + \cdots + (j_{\min} + |\mathcal{B}| - 1) = j_{\min}|\mathcal{B}| + \frac{|\mathcal{B}|(|\mathcal{B}| - 1)}{4}.$$  

(5.83)
and therefore:
\[ \sum_{m \in B} (j_m - 2) \geq \frac{1}{2} \sum_{m \in B} j_m + \frac{j_{\text{min}} - 2}{2} |B| + \frac{|B|(|B| - 5)}{4}, \tag{5.84} \]
where we have introduced explicitly the minimal scale \( j_{\text{min}} > 2 \). This result implies that,
\[ \sum_{\{j_m\}} \prod_{m \neq m'} \left(1 - \delta_{j_m j_{m'}}\right) \prod_{m \in B} M^{-j_m} \leq \left( \sum_{j=j_{\text{min}}}^{j_{\text{max}}} M^{-j/2} \right)^{|B|} \frac{1}{M^{j_{\text{min}} - 2 |B| + \frac{|B|(|B| - 5)}{4}}}, \tag{5.85} \]
which, for \( j_{\text{min}} > 2 \) and \( M > 4 \), is uniformly bounded by \( M^{-|B|^2/4} \). Note that the upper bound \( j_{\text{max}} \) can now be sent to infinity without any divergence, ensuring the theory is well-defined in the ultraviolet limit.

The final step is to sum over the fermionic forest. Such a forest can be partitioned into components of cardinality \( b_k \), associated to connected blocks of size \( k \), having \( k \) sub-vertices. The number of fermionic edges is then \( \sum_k b_k - 1 \), and for each component with \( k \) sub-vertices, there are \( n^{b_k} \) ways to hook a fermionic edge. Moreover, Cayley’s theorem (without constraint on the coordinate number) states that the number of trees with \( v \) labelled vertices is \( v^{v-2} \), leading to a contribution \( n^{\sum_k b_k - 2} \). Finally, because of the constraint \( \sum_k k b_k = n \), when the number of (sub) vertices is fixed to \( n \), and from Stirling formula: \( n^{(\sum_k b_k) - 2} \leq (\sum_k b_k)!e^n \), we find:
\[ \left| \ln \mathcal{Z}[J, \bar{J}, \lambda] \right| \leq N \sum_n \frac{1}{n!} \sum_{\{b_k\}} \frac{n!}{\prod_k b_k!(k!)^{b_k}} 2^{\sum_k b_k - 1} k^{(\sum_k b_k) - 2} \prod_k n^{b_k} \sum_{k^{bh_k=n}} \left[ \mathcal{O}(1) \frac{\pi^2 |\lambda| N}{\cos^2(\phi/2)} \right]^{k-1} \sqrt{(4k - 4)!!(3k - 3)!(2k - 1)!} M^{-k^2/4} \tag{5.86} \]

Taking into account the bound (5.82) and performing the sum over the \( \{b_k\} \), we get finally:
\[ \left| \ln \mathcal{Z}[J, \bar{J}, \lambda] \right| \leq \sum_{b \geq 0} \left[ \sum_{n \geq 1} \left( \frac{\pi^2 |\lambda| N}{\cos^2(\phi/2)} \right)^{n-1} 3^3 n^n M^{-n^2/4} \right]^b \tag{5.87} \]
The power of \( M \) ensures that, for \( M \) sufficiently large, this factor compensates the bad divergence associated to \( n^n \). To conclude:
**Proposition 9** There exists $\rho$ small enough so that the tree expansion is absolutely convergent in the interior of the cardioid domain

$$|\lambda| \leq \frac{\rho}{N} \cos(\theta/2),$$

(5.88)

with $\theta := \arg(\lambda)$ and we remind the definition $\lambda^2 = -u/2\pi\beta$. Moreover, it corresponds to the Borel sum of the perturbative expansion in $\lambda$.

To prove Borel summability, we have to use the same strategy as for $d = 0$, and bound the remainder of the Taylor expansion. The bound may be straightforwardly checked by mixing the results of sections Sections 3 and 5.

### 6 Conclusion

This paper was devoted to quartic vector models for which we prove the existence of a constructive expansion for super-renormalizable models. The originality of the approach is to consider arbitrary quartic coupling, and then to provide universal conclusions about Borel summability. We adopted a progressive approach, increasing the difficulty with the number of (Euclidean) space-time dimensions. A large part of the paper was devoted to the bosonic case without ultraviolet divergences. However, the fermionic case was investigated as well, for $d = 0$ and $d = 1$, due to the relation of the latter with the SYK model. The paper concludes on the investigation of a bosonic super-renormalizable model having a finite set of divergent subgraphs. As we explained, the subtraction of the divergent subgraphs requires a non-trivial improvement of the standard LVE, the MLVE, introducing a slicing over the momenta integration and two level trees replacing ordinary trees. This paper is the first one of a series of two investigating Borel summability of quartic vector models using LVE technique. The next step is to study the other super-renormalisable models and, more interestingly, the just-renormalizable case. In the latter case, the infinite set of divergences can be removed by a finite set of counter-terms, but the LVE and MLVE are insufficient. One needs a new improvement which will be detailed in a forthcoming paper.

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