Long-Distance $1/N_c$ Corrections to
Density-Density Operators in $K \to \pi\pi$ Decays

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Abstract

In this talk we discuss the general method to calculate loop corrections to $\Delta S = 1$ density-density operators in the $1/N_c$ approach. As a result we present the long-distance evolution of the operators $Q_6$ and $Q_8$ to $O(p^0/N_c)$ in the chiral and the $1/N_c$ expansions.

1 Introduction

Within the standard model the calculation of the $K \to \pi\pi$ decay amplitudes is based on the effective low-energy hamiltonian for $\Delta S = 1$ transitions $H_{\Delta S=1}$:

$$H_{\Delta S=1}^{\text{eff}} = \frac{G_F}{\sqrt{2}} \xi_u \sum_{i=1}^{8} c_i(\mu)Q_i(\mu) \quad (\mu < m_c) ,$$

where the Wilson coefficient functions $c_i(\mu)$ of the local four-fermion operators $Q_i(\mu)$ are obtained by means of the renormalization group equation. They were computed in an extensive next-to-leading logarithm analysis by two groups $[2, 3]$. Long-distance contributions to the isospin amplitudes $A_I$ are contained in the hadronic matrix elements of the bosonized operators. Among the various four-fermions operators the gluon and the electroweak penguin

$$Q_6 = -2 \sum_{q=u,d,s} \bar{s}(1+\gamma_5)q \bar{q}(1-\gamma_5)d , \quad Q_8 = -3 \sum_{q=u,d,s} e_q \bar{s}(1+\gamma_5)q \bar{q}(1-\gamma_5)d ,$$

where $e_q = (2/3, -1/3, -1/3)$] are particularly interesting for two reasons. First, the two operators dominate the direct $CP$ violation in $K \to \pi\pi$ decays ($\varepsilon'/\varepsilon$). Secondly, they have a density-density structure different from the structure of current-current four-fermions operators widely investigated previously.

$\ast$Talk presented by T. Hambye at the XVI Autumn School and Workshop on Fermion Masses, Mixing and CP Violation, Lisboa, Portugal, 6-15 October 1997.
In this talk we focus on the method to calculate the loop (i.e., the $1/N_c$) corrections to the hadronic matrix elements (with $N_c$ the number of colors). It is of special importance to examine whether they significantly affect the large cancellation between the gluon and the electroweak penguin contributions in the ratio $\varepsilon'/\varepsilon$ obtained at the tree level in Ref. [4]. The approach we will follow is the $1/N_c$ expansion as it has been introduced in Ref. [5] to investigate the $\Delta I = 1/2$ selection rule.

To compute the hadronic matrix elements we will start from the low-energy chiral effective lagrangian for pseudoscalar mesons. Calculating the loops we have to choose in particular a regularization scheme. One possibility is to use dimensional regularization in which case strictly one applies the effective lagrangian beyond its low-energy domain of validity. This problem can be avoided by using an energy cut-off. The price to pay is the loss of translational invariance (which particularly implies a dependence of the loop integrals on the precise definition of the momentum integration variable inside the loops). In the following analysis we will use a cut-off regularization for the divergent contributions because we believe that for these contributions this procedure is more appropriate (see e.g. Ref. [8]). In particular we will argue that the problem of translational non-invariance can be treated in a satisfactory way separating the factorizable and non-factorizable contributions explicitly: a priori the non-factorizable diagrams are momentum prescription dependent, but only one prescription yields a consistent matching with the short-distance QCD contribution. The factorizable diagrams on the other hand refer to the purely strong sector of the theory. Consequently, as we will show explicitly, their sum does not contain any divergent term. Therefore they can and will be calculated within dimensional regularization (in difference to the non-factorizable diagrams) which yields an unambiguous result.

In Section 2 we specify the low-energy effective lagrangian. In Sections 3 and 4 we analyze the factorizable and non-factorizable diagrams, respectively. Finally, in Section 5 we discuss our results and summarize. We will focus here on the general method to calculate the loop corrections to $Q_6$ and $Q_8$ in a systematic way, and we will present only the divergent terms explicitly. These will be calculated at the operator level giving the evolution of the operators $Q_6$ and $Q_8$ (from our results the $K \rightarrow \pi\pi$ matrix elements can be obtained in a straightforward way). Numerical results including the non-negligible finite terms are presented by G. Köhler in these proceedings, and some additional details can be found in Ref. [7].

## 2 Low-energy Effective Lagrangian

Within our study we will use the low-energy effective chiral lagrangian for pseudoscalar mesons which involves an expansion in momenta where terms up to $O(p^4)$ are included [8],

$$L_{\text{eff}} = \frac{f^2}{4} \left( \langle \partial_\mu U^\dagger \partial^\mu U \rangle + \frac{\alpha}{4N_c} \langle \ln U^\dagger - \ln U \rangle^2 + r \langle M U^\dagger + U M^\dagger \rangle \right) + r^2 H_2 \langle M^\dagger M \rangle$$
\[ + r L_5 (\partial_\mu U^\dagger \partial^\mu U (M^\dagger U + U^\dagger M)) + r L_8 (M^\dagger U M^\dagger U + MU^\dagger MU^\dagger) \]  

(4)

with \( \langle A \rangle \) denoting the trace of \( A \) and \( M = \text{diag}(m_u, m_d, m_s) \). \( f \) and \( r \) are free parameters related to the pion decay constant \( F_\pi \) and to the quark condensate, respectively, with \( r = -2 \langle \bar{q} q \rangle / f^2 \). In obtaining Eq. (4) we used the general form of the lagrangian \[ \text{and omitted terms of } \mathcal{O}(p^4) \] which do not contribute to the \( K \to \pi \pi \) matrix elements of \( Q_6 \) and \( Q_8 \) or are subleading in the \( 1/N_c \) expansion. The fields of the complex matrix \( U \) are identified with the pseudoscalar meson nonet defined in a non-linear representation:

\[ U = \exp \frac{i}{f} \Pi, \quad \Pi = \pi^a \lambda_a, \quad \langle \lambda_a \lambda_b \rangle = 2 \delta_{ab}, \]  

(5)

where, in terms of the physical states,  

\[ \Pi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{2}} a \eta + \sqrt{\frac{2}{3}} b \eta' & \sqrt{2} \pi^+ & \sqrt{2} K^+ \\ \sqrt{2} \pi^- & -\pi^0 + \frac{1}{\sqrt{2}} a \eta + \sqrt{\frac{2}{3}} b \eta' & \sqrt{2} K^0 \\ \sqrt{2} K^- & \sqrt{2} K^0 & -\frac{2}{\sqrt{3}} b \eta' + \frac{1}{\sqrt{3}} a \eta' \end{pmatrix}, \]  

(6)

and  

\[ a = \cos \theta - \sqrt{2} \sin \theta, \quad \sqrt{2} b = \sin \theta + \sqrt{2} \cos \theta. \]  

(7)

Note that we treat the singlet as a dynamical degree of freedom and include in Eq. (4) a term for the strong anomaly proportional to the instanton parameter \( \alpha \). This term gives a non-vanishing mass of the \( \eta_0 \) in the chiral limit \( (m_q = 0) \) reflecting the explicit breaking of the axial \( U(1) \) symmetry. \( \theta \) is the \( \eta - \eta' \) mixing angle for which we take the value \( \theta = -19^\circ \). \[ ^1 \]

The bosonic representation of the quark densities is defined in terms of (functional) derivatives:

\[ (D_L)_{ij} = \frac{1}{2} \bar{q} \left( 1 - \gamma_5 \right) q \]  

\[ = -\frac{\delta \mathcal{L}_{\text{eff}}}{\delta M_{ij}} = -r \left( \frac{f^2}{4} U^\dagger + L_5 \partial_\mu U^\dagger \partial^\mu U^\dagger + 2r L_8 U^\dagger M U^\dagger + r H_2 M^\dagger \right)_{ji}, \]  

(8)

and the right-handed density \((D_R)_{ij}\) is obtained by hermitian conjugation. Eq. (8) allows us to express the operators \( Q_6 \) and \( Q_8 \) in terms of the meson fields:

\[ Q_6 = -2 f^2 r^2 \sum_q \left[ \frac{1}{4} f^2 (U^\dagger)_{dq}(U)_{qs} + (U^\dagger)_{dq}(L_5 U \partial_\mu U^\dagger \partial^\mu U + 2r L_8 U^\dagger M U^\dagger + r H_2 M^\dagger)_{dq}(U)_{qs} \right] + \mathcal{O}(p^4), \]  

(9)

\[ Q_8 = -3 f^2 r^2 \sum_q e_q \left[ \frac{1}{4} f^2 (U^\dagger)_{dq}(U)_{qs} + (U^\dagger)_{dq}(L_5 U \partial_\mu U^\dagger \partial^\mu U + 2r L_8 U^\dagger M U^\dagger + r H_2 M^\dagger)_{dq}(U)_{qs} \right] + \mathcal{O}(p^4). \]  

(10)

\[^1\text{In addition, one might note that the contribution of the contact term } \propto \langle M^\dagger M \rangle \text{ vanishes in the isospin limit } (m_u = m_d).\]
For the operator $Q_6$ the $(U^*)_{dq}(U)_{qs}$ term which is of $\mathcal{O}(p^0)$ vanishes at the tree level. This property follows from the unitarity of $U$. However, as we will see, when investigating off-shell corrections it must be included.

In the following we will consider the one-loop effects over the $\mathcal{O}(p^2)$ lagrangian, that is to say, the $\mathcal{O}(p^0/N_c)$ corrections to $Q_6$ and $Q_8$. Through the renormalization procedure, this requires to take also into account the tree level $\mathcal{O}(p^4)$ lagrangian [i.e., the $\mathcal{O}(p^2)$ terms for $Q_6$ and $Q_8$] proportional to $L_5$, $L_8$ and $H_2$ in Eq. (4).

3 Factorizable $1/N_c$ Corrections

Since factorizable and non-factorizable corrections refer to disconnected sectors of the theory (strong and weak sectors), we introduce two different scales: $\lambda_c$ is the cut-off for the factorizable diagrams and $\Lambda_c$ for the non-factorizable. We will refer to them as the factorizable and the non-factorizable scales, respectively. A similar distinction of the scales was also performed in Ref. [10] in the calculation of the $B_K$ parameter.

As the factorizable loop corrections refer to the purely strong sector of the theory for these corrections there is no matching between the long- and short-distance contributions except for the scale dependence of the overall factor $r^2 \sim 1/m_s^2$ in $Q_6$ and $Q_8$ [see Eq. (17) below]. This property follows from the fact that the evolution of $m_s$ which already appears at leading $N_c$ is the inverse of the evolution of a quark density. Therefore, except for the scale of $1/m_s^2$ which exactly cancels the factorizable evolution of the density-density operators at short distances, the only scale remaining in the matrix elements is the non-factorizable scale $\Lambda_c$. It represents the non-trivial part of the factorization scale in the operator product expansion. The only matching between long- and short-distance contributions is obtained by identifying the cut-off scale $\Lambda_c$ of the non-factorizable diagrams with the QCD renormalization scale $\mu$.

In this section we shall show explicitly, at the level of a single density operator, that the quadratic and logarithmic dependence on $\lambda_c$ which arises from the factorizable loop diagrams is absorbed in the renormalization of the low-energy lagrangian. Consequently, in the factorizable sector the chiral loop corrections do not induce ultraviolet divergent terms in addition to the $1/m_s^2$ factor.

The proof of the absorption of the factorizable scale $\lambda_c$ will be carried out in the isospin limit. This explicit demonstration is instructive for several reasons. First, we verify the validity of the general concept in the case of bosonized densities which, contrary to the currents, do not obey conservation laws (i.e., only PCAC can be used for the densities). Second, we check, within the cut-off formalism, whether there is a dependence on a given momentum shift ($q \rightarrow q \pm p$). Thirdly, including the $\eta_0$ as a dynamical degree of freedom we examine the corresponding modifications in the renormalization procedure. Finally, there remain finite terms from the factorizable $1/N_c$ corrections which explicitly enter the numerical analysis of the matrix elements. This point will be discussed at the end of this Section.

To calculate the evolution of the operators we apply the background field method
as used in Refs. [11] and [12] for current-current operators. This approach is powerful as it keeps track of the chiral structure in the loop corrections. It is particularly useful to study the ultraviolet behaviour of the theory.

In order to calculate the evolution of the density operator we decompose the matrix $U$ in the classical field $\bar{U}$ and the quantum fluctuation $\xi$,

$$U = \exp(i\xi/f) \bar{U}, \quad \xi = \xi^a \lambda_a,$$

with $\bar{U}$ satisfying the equation of motion

$$\bar{U} \partial^2 \bar{U}^\dagger - \partial^2 \bar{U} \bar{U}^\dagger + r \bar{U} \mathcal{M}^\dagger - r \mathcal{M} \bar{U} \bar{U}^\dagger \alpha N_c (\ln \bar{U} - \ln \bar{U}^\dagger) \cdot 1, \quad \bar{U} = \exp(i\pi^a \lambda_a/f).$$

The lagrangian of Eq. (4) thus reads

$$L = \bar{L} + \frac{1}{2} (\partial_\mu \xi^a \partial^\mu \xi_a) + \frac{1}{4} ([\partial_\mu \xi^a, \xi^b] \partial^\mu \bar{U} \bar{U}^\dagger) - \frac{r}{8} (\xi^2 \bar{U} \mathcal{M}^\dagger + \bar{U} \mathcal{M}^\dagger \xi^2 - \frac{1}{2} \alpha \xi^0 \xi^0 + O(\xi^3)).$$

The corresponding expansion of the meson density around the classical field yields

$$(D_L)_{ij} = (\bar{D}_L)_{ij} + if \frac{r}{4} (\bar{U} \xi)_{ji} + \frac{r}{8} (\bar{U} \xi^2)_{ji} + O(\xi^3).$$

Fig. 1. Evolution of the density operator; the black circle, square and triangle denote the kinetic, mass and $U_A(1)$ breaking terms in Eq. (13), the crossed circle the density of Eq. (14). The lines represent the $\xi$ propagators.

The evolution of $(D_L)_{ij}$ is determined by the diagrams of Fig. 1. Integrating out the fluctuation $\xi$ we obtain

$$(D_L)_{ij}(\lambda_c) = -\frac{f^2}{4} r (\bar{U}^\dagger)_{ji}(0) + \frac{3}{4} r (\bar{U}^\dagger)_{ji}(0) \lambda_c^2 (4\pi)^2 - \frac{r}{12} (\bar{U}^\dagger)_{ji}(0) \alpha \frac{\log \lambda_c^2}{(4\pi)^2}
- r^2 (\mathcal{M}^\dagger)_{ji}(0) \left[ H_2 + \frac{3 \log \lambda_c^2}{16 (4\pi)^2} \right] - 2 r^2 (\bar{U} \mathcal{M} \bar{U})^\dagger_{ji}(0) \left[ L_8 + \frac{3 \log \lambda_c^2}{32 (4\pi)^2} \right]
- r (\partial_\mu \bar{U}^\dagger \partial^\mu \bar{U} \mathcal{M})^\dagger_{ji}(0) \left[ L_5 + \frac{3 \log \lambda_c^2}{16 (4\pi)^2} \right] + \ldots,$$

where the ellipses denote finite terms (non-divergent in $\lambda_c$) coming from the loop corrections. The quadratic and logarithmic terms for the wave function and mass renormalizations can be calculated from the diagrams of Figs. 2 and 3, i.e., from the off-shell
corrections to the kinetic and the mass operator, respectively, second and third term of Eq. \((13)\). We get

\[
m_\pi^2 = r \hat{m} \left[ 1 - \frac{8m_\pi^2}{f^2} (L_5 - 2L_8) + \frac{1}{3} \frac{\alpha \log \lambda_c^2}{(4\pi)^2 f^2} \right] + \ldots ,
\]

\[
m_K^2 = r \frac{\hat{m} + m_s}{2} \left[ 1 - \frac{8m_K^2}{f^2} (L_5 - 2L_8) + \frac{1}{3} \frac{\alpha \log \lambda_c^2}{(4\pi)^2 f^2} \right] + \ldots ,
\]

\[
Z_\pi = 1 + \frac{8L_5}{f^2} m_\pi^2 - 3 \frac{\lambda_c^2}{(4\pi)^2 f^2} + \frac{3}{2} \frac{m_\pi^2 \log \lambda_c^2}{(4\pi)^2 f^2} + \ldots ,
\]

\[
Z_K = 1 + \frac{8L_5}{f^2} m_K^2 - 3 \frac{\lambda_c^2}{(4\pi)^2 f^2} + \frac{3}{2} \frac{m_K^2 \log \lambda_c^2}{(4\pi)^2 f^2} + \ldots ,
\]

with \(\hat{m} = (m_u + m_d)/2\).

**Fig. 2.** Evolution of the kinetic operator (wave function renormalization).

**Fig. 3.** Evolution of the mass operator (mass renormalization).

**Fig. 4.** Evolution of the current operator. The crossed circle here denotes the bosonized current.

Along the same lines \(F_\pi\) and \(F_K\) can be calculated, to one-loop order, from the diagrams of Fig. 4, and we obtain\(^2\)

\[
F_\pi = f \left[ 1 + \frac{4L_5}{f^2} m_\pi^2 - \frac{3}{2} \frac{\lambda_c^2}{(4\pi)^2 f^2} + \frac{3}{4} \frac{m_\pi^2 \log \lambda_c^2}{(4\pi)^2 f^2} + \ldots \right],
\]

\(^2\)The representation of the bosonized current in terms of the background field can be found in Ref. [12].
Both the quadratic and the logarithmic terms of Eqs. (15)-(21) prove to be independent of the way we define the integration variable inside the loops. This is due to the fact that the quadratically divergent integrals resulting from the diagrams of Figs. 1-4 [i.e., those of the form \(d^4q/(q \pm p)^2\)] do not induce subleading logarithms, that is to say, all quadratic and logarithmic dependence on the scale \(\lambda_c\) originates from the leading divergence of a given integral.

Now looking at Eqs. (18)-(21) we observe that the ratio \(\Pi/f\) and, consequently, the matrix field \(U\) are not renormalized (i.e., \(\pi_0/f = \pi_r/F_\pi\) and \(K_0/f = K_r/F_K\)). Defining the renormalized (scale independent) couplings \(\hat{L}_i\) through the relations

\[
\frac{F_K}{F_\pi} = 1 + \frac{4L_5}{f^2}(m^2_K - m^2_\pi) \left[ L_5 + \frac{3 \log \lambda_c^2}{16 (4\pi)^2} \right] + \ldots , \tag{22}
\]

\[
\equiv 1 + \frac{\hat{L}_5}{F_\pi^2}(m^2_K - m^2_\pi) , \tag{23}
\]

\[
\frac{m^2_K}{m^2_\pi} = \frac{\hat{m} + m_s}{2\hat{m}} \left[ 1 - \frac{8(m^2_K - m^2_\pi)}{f^2}(L_5 - 2L_8) \right] + \ldots , \tag{24}
\]

\[
\equiv \frac{\hat{m} + m_s}{2\hat{m}} \left[ 1 - \frac{8(m^2_K - m^2_\pi)}{F_\pi^2}(\hat{L}_5 - 2\hat{L}_8) \right] , \tag{25}
\]

from Eqs. (22) and (23) we find, to one-loop order,

\[
L_5 = \hat{L}_5 - \frac{3 \log \lambda_c^2}{16 (4\pi)^2} + \ldots , \tag{26}
\]

in accordance with the result from chiral perturbation theory \cite{8}. Note that Eq. (24) exhibits no explicit dependence on the scale \(\lambda_c\); i.e., the chiral loop corrections of Eqs. (16) and (17) do not contribute to the \(SU(3)\) breaking in the masses and, consequently, can be absorbed in \(r\). This implies

\[
L_5 - 2L_8 = \hat{L}_5 - 2\hat{L}_8 + \ldots . \tag{27}
\]

Then, from Eqs. (26) and (27) we get

\[
L_8 = \hat{L}_8 - \frac{3 \log \lambda_c^2}{32 (4\pi)^2} + \ldots . \tag{28}
\]

One might note that the coefficient in front of the logarithm in Eq. (28) differs from the one given in Ref. \cite{8}. This property follows from the presence of the singlet \(\eta_0\) in the calculation. Eqs. (24) and (25) define the renormalization conditions because the term \(\hat{L}_5 - 2\hat{L}_8\) plus the constant terms which appear in the ratio of the masses in
Eq. (24) determine the bare constant $L_5 - 2L_8$. Similarly Eqs. (22) and (23) with the associated finite terms determine the coupling constant $L_5$.

Then, by means of Eqs. (17) and (20), we can rewrite the density of Eq. (15) as

$$
(D_L)_{ij}(\lambda_c) = -\frac{2m_K^2}{(m + m_s)} \left[ \frac{F_\pi^2}{4} \left( 1 + \frac{8\hat{L}_5}{F_\pi^2} \left( m_K^2 - m_\pi^2 \right) - \frac{16\hat{L}_8}{F_\pi^2} m_K^2 \right) (U^\dagger)_{ji} 
+ (\partial_\mu \bar{U}^\dagger \partial^\mu \bar{U}^\dagger)_{ji} \hat{L}_5^r + 2(\bar{U}^\dagger \chi U^\dagger)_{ji} \hat{L}_8^r + (\chi^\dagger)_{ji} \hat{H}_2^r \right],
$$

(29)

with $\chi = \text{diag}(m_\pi^2, m_\pi^2, 2m_K^2 - m_\pi^2)$. In obtaining Eq. (29) we used the renormalized couplings of Eqs. (26) and (28). In addition, we introduced

$$
\hat{H}_2^r = H_2 + \frac{3}{16} \log \frac{\lambda_c^2}{(4\pi)^2} + \ldots.
$$

(30)

Note that the renormalized density exhibits no dependence on the scale $\lambda_c$, except for the scale of $1/(m + m_s)$. Note also that in Eqs. (15) and (29) we did not specify logarithmic terms induced at the one-loop order which correspond to the $L_4$, $L_6$ and $L_7$ operators in the chiral effective lagrangian of Ref. [8]. An explicit calculation of these terms shows that they give no contribution to the $K \to \pi\pi$ matrix elements of $Q_6$ and $Q_8$.

The factorizable contributions to the $Q_6$ and $Q_8$ operators can be obtained in a straightforward way from Eq. (29). As the tree level expansion of $Q_6$, due to the unitarity of the matrix field $U$, starts at the $O(p^2)$, no terms arise from the renormalization of the wave functions and masses, as well as, the bare decay constant $f$. These corrections will be of higher order. Only the renormalization of the $O(p^2)$ parameters enters the calculation. This statement does not hold for the electroweak operator $Q_8$ which, for $K^0 \to \pi^+\pi^-$, induces a non-vanishing tree matrix element at the $O(p^0)$.

In conclusion, using a cut-off regularization the evolution of the density operator up to the orders $p^2$ and $p^0/N_c$ is given, modulo finite loop corrections, by Eq. (29). Our result exhibits no explicit scale dependence. Moreover, it does not depend on the momentum prescription inside the loops. The finite terms, on the other hand, will not be absorbed completely in the renormalization of the various parameters. This can be seen, e.g., from the fact that the diagrams of Fig. 1 contain rescattering processes which induce a non-vanishing imaginary part. As the renormalized parameters are defined to be real, the latter will remain.

In addition, the real part of the finite corrections carries a dependence on the momentum prescription used to define the cut-off. However, we proved that the chiral loop diagrams do not induce ultraviolet divergent terms. Therefore we are allowed to calculate the remaining finite corrections in dimensional regularization, which is momentum translation invariant (i.e., we are allowed to take the limit $\lambda_c \to \infty$). This procedure implies an extrapolation of the low-energy effective theory for terms of $O(m_\pi^2, \lambda_c^2; m_\pi^2, \lambda_c^2; \ldots)$ up to scales where these terms are negligible. This is the usual assumption made in chiral perturbation theory for three flavors.
The non-factorizable $1/N_c$ corrections to the hadronic matrix elements constitute the part to be matched to the short-distance Wilson coefficient functions; i.e., the corresponding scale $\Lambda_c$ has to be identified with the renormalization scale $\mu$ of QCD. As the non-factorizable terms are ultraviolet divergent we calculate their contribution with a Euclidian cut-off following the discussion of the introduction. The integrals will generally depend on the momentum prescription used inside the loop.

In the existing studies of the hadronic matrix elements the color singlet boson connecting the two densities (or currents) was integrated out from the beginning [4, 3, 13, 14]. Thus the integration variable was taken to be the momentum of the meson in the loop, and the cut-off was the upper limit of its momentum. As there is no corresponding quantity in the short-distance part, in this treatment of the integrals there is no clear matching with QCD. This ambiguity is removed, for non-factorizable diagrams, by considering the two densities to be connected to each other through the exchange of the color singlet boson, as was already discussed in Refs. [6, 12, 15, 16, 17]. A consistent matching is then obtained by assigning the same momentum to the color singlet boson at long and short distances and by identifying this momentum with the loop integration variable. Consequently, the matching fixes the frame and no other translated frame is appropriate.

\[
Q_{NF}^6(\Lambda_c^2) = F_\pi^2 \left( \frac{2m_K^2}{m + m_s} \right)^2 \log \Lambda_c^2 \left[ \frac{3}{4} \left( \partial_\mu \bar{U}^\dagger \partial^\mu \bar{U} \right)_{ds} + \frac{1}{2} \left( \partial_\mu \bar{U}^\dagger \bar{U} \right)_{ds} \delta_{qq} + \frac{3}{4} \left( \bar{U}^\dagger \chi + \chi^\dagger \bar{U} \right)_{ds} + \frac{1}{2} \left( \partial_\mu \bar{U}^\dagger \partial^\mu \bar{U} \right)_{ds} \delta_{qq} + \frac{1}{3} \alpha \left( \bar{U}^\dagger \right)_{dq} (\bar{U})_{qs} \right], \tag{31}
\]

\[
Q_{NF}^8(\Lambda_c^2) = \frac{3}{2} F_\pi^2 \left( \frac{2m_K^2}{m + m_s} \right)^2 \log \Lambda_c^2 \left[ \sum_q e_q \left[ \frac{1}{4} \left( \partial_\mu \bar{U}^\dagger \partial^\mu \bar{U} \right)_{ds} \delta_{qq} + \frac{1}{2} \left( \partial_\mu \bar{U}^\dagger \bar{U} \right)_{ds} \delta_{qq} + \frac{1}{2} \left( \bar{U}^\dagger \chi + \chi^\dagger \bar{U} \right)_{ds} \delta_{qq} + \frac{1}{3} \alpha \left( \bar{U}^\dagger \right)_{dq} (\bar{U})_{qs} \right], \tag{32}
\]

Fig. 5. Non-factorizable loop diagrams for the evolution of a density-density operator.
Only the diagonal evolution of $Q_6$, i.e., the first term on the right-hand side of Eq. (31), gives a non-zero contribution to the $K \rightarrow \pi\pi$ matrix elements. In particular, the mass term which is of the $L_8$ and $H_2$ form vanishes for $K \rightarrow \pi\pi$ decays, as do the $L_8$ and $H_2$ contributions at the tree level (due to a cancellation between the tadpole and non-tadpole diagrams). In Eq. (32) for completeness we kept the terms proportional to $\delta_{qq}$ which, however, cancel through the summation over the flavor index.

Note that Eqs. (31) and (32) are given in terms of operators and, consequently, can be applied to $K \rightarrow 3\pi$ decays, too. Note also that our results, Eqs. (31) and (32), exhibit no quadratic dependence on the scale $\Lambda_c$; i.e., up to the first order corrections in the twofold expansion in $p^2$ and $1/N_c$ the matching involves only logarithmic terms from both the short- and the long-distance evolution of the four-quark operators. This is due to the fact that there is no quadratically divergent diagram in Fig. 5 apart from the first one which vanishes for the $Q_6$ and $Q_8$ operators. Moreover, for a general density-density operator there are no logarithms which are the subleading logs of quadratically divergent terms. Therefore, all the logarithms appearing in Eqs. (31) and (32) are leading divergences, which are independent of the momentum prescription. The finite terms calculated along with these logarithms depend on the momentum prescription. They are, however, uniquely determined through the matching condition with QCD which fixes the momenta in the loop as explained above.

One might note that the statements we made above do not hold for current-current operators: the $1/N_c$ corrections to these operators, performed in the first non-vanishing order of their chiral expansion, exhibit terms which are quadratic in $\Lambda_c$. Furthermore, already these terms were shown to depend on the momentum prescription [12].

We close this section by giving the long-distance evolution, at the $O(p^0)$, of a general density-density operator $Q_{D}^{abcd} \equiv -8(D_R)_{ab}(D_L)_{cd}$. As we showed in Section 3, the factorizable $1/N_c$ corrections do not affect its ultraviolet behaviour. Then, from the non-factorizable diagrams of Fig. 5 we find:

$$Q_{D}^{abcd}(\Lambda_c^2) = Q_{D}^{abcd}(0) \left[ 1 - \frac{2\alpha}{3} \log \frac{\Lambda_c^2}{\Lambda} \right] - \frac{F_\pi^2}{3} \left( \frac{2m_K^2}{m + m_s} \right)^2 \frac{\Lambda_c^2}{(4\pi)^2} \delta^{da}\delta^{bc}$$

$$\quad + \frac{F_\pi^2}{4} \left( \frac{2m_K^2}{m + m_s} \right)^2 \log \frac{\Lambda_c^2}{(4\pi)^2} \left[ (\bar{U}^\dagger \chi + \chi^\dagger \bar{U})^{da}\delta^{bc} + \delta^{da}(\chi \bar{U}^\dagger + \bar{U} \chi)^{bc} \right]$$

$$\quad + (\partial_{\mu} \bar{U}^\dagger (\partial^\mu \bar{U})^{da}\delta^{bc} + \delta^{da}(\partial_{\mu} \bar{U} (\partial^\mu \bar{U})^{bc} + 2(\partial_{\mu} \bar{U}^\dagger)^{da}(\bar{U} (\partial^\mu \bar{U})^{bc}) \right] \right). \quad (33)$$

The corresponding expressions for the non-factorizable loop corrections to the operators $Q_6$ and $Q_8$, Eqs. (31) and (32), can be obtained directly from Eq. (33).

5 Discussion

In summary, since the non-factorizable contributions contain (logarithmically) divergent terms we consider that these contributions have to be calculated within a cut-off
regularization. Therefore, at the level of the finite terms [but, as we have shown, to $O(p^0/N_c)$ not at the level of the divergent terms] the translation non-invariance could render a priori the calculation of the loops arbitrary. However, for the non-factorizable diagrams a consistent matching (in which we can identify the same quantity in the short- and long-distance pictures) fixes the momentum prescription and renders the result unambiguous. On the other hand, there is no way to establish a unique momentum prescription for the factorizable diagrams. Nevertheless, as the complete sum of the factorizable diagrams is finite, for this sum we are allowed to take the limit $\lambda_c \to \infty$ and to use dimensional regularization which yields an unambiguous result, too.

Consequently, in the factorizable sector at the level of the finite terms only the sum of all (factorizable) diagrams is meaningful. To be explicit, we have no access to the renormalization of the couplings separately as their divergences induce an arbitrariness at the level of the finite terms. The case of the operator $Q_6$ is particularly illustrative. At the tree level this operator vanishes to $O(p^0)$ due to the unitarity of the matrix $U$. Nevertheless, the one-loop corrections to the $O(p^0)$ $(U^\dagger)_d q(U)_{q_\sigma}$ term must be computed. Indeed, as long as we keep track of the density-density structure of the operator $Q_6$ (to separate the factorizable and the non-factorizable diagrams) these corrections are non-vanishing. In particular, we have shown that the non-factorizable diagrams over the $(U^\dagger)_d q(U)_{q_\sigma}$ operator yield a non-trivial dependence on the scale $\Lambda_c$ which has to be matched to the short-distance contribution. In addition, the logarithms of Eq. (15) are needed in order to cancel the scale dependence of the various bare parameters in the tree level expressions as shown in Section 3. We note that in the twofold expansion in $p^2$ and $1/N_c$ the contribution of the loops over the $O(p^0)$ matrix element must be treated at the same level as the leading non-vanishing $O(p^2)$ tree level contribution proportional to $L_5$. This statement does not hold for $Q_8$ whose $O(p^0/N_c)$ corrections are subleading with respect to the leading $O(p^0)$ tree level.

We close with a note on the comparison of the evolution of the operators $Q_6$ and $Q_8$ at long and short distances. As argued above, to $O(p^0/N_c)$ the long-distance evolution of $Q_6$ and $Q_8$ is only logarithmic as in the short-distance (QCD) picture. Except for the case where the coefficients of the logs are strictly equal in both domains, this property prevents us from determining any value of $\Lambda_c$ for which the matching is completely flat. It turns out that, even if the coefficients of the logarithms are found to be relatively moderate at long distance, they are still larger than the corresponding short-distance ones. This is to be expected as the short-distance coefficients are close to zero, and as we have calculated only the lowest order (long-distance) evolution in a theory which is truncated to the pseudoscalar mesons.

The corrections we have calculated are the first order corrections over the well established $O(p^2)$ lagrangian, and the slope obtained for the scale dependence of the matrix elements is unambiguous. The fact that the long- and short-distance coefficients are different does not necessarily mean that the effects of higher order corrections and higher resonances are large for the absolute values of the matrix elements. However, it is desirable to investigate these effects explicitly.
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