Abstract

A twisted group algebra $F \star A$ of a finitely generated free abelian group $A$ over a field $F$ is in general noncommutative, but $A$ may contain nontrivial subgroups $C$ so that the subalgebra $F \star C$ is commutative. In this paper, we show that an $F \star A$-module $M$ which is finitely generated as an $F \star C$-module for any commutative subalgebra $F \star C$ must be artinian. We also show that $M$ must be torsion-free as $F \star C$-module and the Gelfand–Kirillov dimension of $M$ must equal the rank of $C$. We then apply these results to modules over finitely generated nilpotent groups of class 2.

1 Introduction

A quantum Laurent polynomial algebra $P(q, F)$ over a field $F$ is defined as the associative $F$-algebra generated over $F$ by the variables $u_1, \cdots, u_n$ and their inverses, satisfying the relations,

$$u_i u_j = q_{ij} u_j u_i,$$

where $q_{ij} \in F$ are nonzero scalars and $q$ is the $n \times n$ matrix $(q_{ij})$. The $q_{ij}$ satisfy

$$q_{ii} = 1 = q_{ij} q_{ji}, \quad i, j = 1, \cdots, n.$$

These algebras have been called by various names. They have been called the multiplicative analogues of the Weyl algebra, the McConnell–Pettit algebras, the quantum Laurent polynomial algebras and the quantum tori. They arise as localizations of group algebras (see [Br]) and also play an important role in noncommutative geometry (see [M]).

The quantum Laurent polynomial algebras are precisely the twisted group algebras (see [PT] Chapter 1) $F \star A$ of a finitely generated free abelian group $A$ over a field $F$. In this paper we shall mainly use the notation $F \star A$ for a quantum Laurent polynomial algebra. Sometimes the following more explicit notation shall be needed:

$$F_q[u_1^{\pm 1}, \cdots, u_n^{\pm 1}],$$

where $q = (q_{ij})$.

Subsequently the situation with an arbitrary number of generators $u_1^{\pm 1}, \cdots, u_n^{\pm 1}$ was studied in [MP]. In [MP], it was noted that the Krull and the global dimensions of $F \star A$ must coincide. An interesting and important criterion was given for these dimensions in [MP, Corollary 3.8] in terms of certain localizations (see [MP]).
In [AG]. Modules over quantum Laurent polynomials have been considered in [A1], [?], [?], [BG] and [MP]. In this note we employ the methods of [AG] and [MP] and the work in [BG] in order to prove the following theorem for modules over $F^*A$:

**Theorem A.** Let $M$ be a nonzero finitely generated $F^*A$-module, where $F^*A$ has center $F$. Let $C < A$ be a subgroup which contains a subgroup $C_0$ of finite index such that $F^*C_0$ is commutative. If $M$ is finitely generated as an $F^*C$-module then,

(i) $\mathcal{GK}(M) = \text{rk}(C)$,

(ii) $M$ is $F^*C$-torsion-free,

(iii) $M$ is artinian,

(iv) $M$ is cyclic.

In this theorem $\mathcal{GK}(M)$ denotes the Gelfand–Kirillov dimension of $M$. It follows from this result that each finitely generated $F^*A$-module with GK dimension one must be artinian (see Corollary 2.4).

Artinian modules arise in other situations also. For example it was shown in [Ar] that if the multiparameters $q_{ij}$, where $1 \leq i < j \leq n$, are independent in $F^*$, then each finitely generated periodic $F^*A$-module is artinian and cyclic.

In Proposition 3.7 we give an application of this theorem to finitely generated modules over finitely generated and torsion-free nilpotent groups $H$ of class 2. If $k$ is field we denote by $kH$ the group algebra of $H$ over $k$. A $kH$-module $M$ is said to be reduced if the annihilator of $M$ in $k\zeta H$ is a prime ideal $P$ and $M$ is $k\zeta H/P$-torsion-free. Every finitely generated $kH$-module is filtered by a finite series each section of which is reduced with some prime ideal of $k\zeta H$. The following theorem which is a special case of is deduced

**Theorem B.** Let $H$ be a finitely generated torsion-free nilpotent group of class 2. A finitely generated $kH$-module $Q$ which is reduced with annihilator $P$, and is finitely generated over $kL$ for some abelian subgroup $L \leq H$ is torsion-free as $kL/P,kL$-module. Moreover if $\zeta H$ is cyclic then $Q$ is also artinian.

2 **Background on $F^*A$**

The twisted group algebra $F^*A$ of a finitely generated free abelian group $A$ over a field $F$ has as an $F$-basis a copy $\overline{A} := \{\overline{a}, a \in A\}$ of $A$. The multiplication of basis elements $\overline{a}_1, \overline{a}_2$ is defined by

$$\overline{a}_1\overline{a}_2 = \lambda(a_1, a_2)\overline{a}_1\overline{a}_2,$$

where $\lambda : A \times A \to F^*$ is a function. The associativity condition requires that the map $\lambda$ be a 2-cocycle (see [P1] Chapter 1). The scalars $\kappa \in F$ commute with the basis elements. Thus $\kappa \overline{a} = \overline{a}\kappa$ for $\kappa \in F$ and $a \in A$. Thus any $\alpha \in F^*A$ can be uniquely expressed in the form $\alpha = \sum_{a \in A} \lambda_a \overline{a}$, where $\lambda_a = 0$ for all but finitely many $a \in A$. The subset of elements $a \in A$ such that $\lambda_a \neq 0$ is known as the support of $\alpha$ in $A$. For a subgroup $B$ of $A$, the subset of elements $\beta \in F^*A$ such that the support of $\beta$ lies in $B$, has the structure of a twisted group algebra of $B$ over $F$. This is denoted as $F^*B$. As already mentioned, $F^*A$ has as an $F$-basis a copy $\{\overline{a}, a \in A\}$ of $A$. By
a diagonal change in basis we may also take the subset \( \{ \kappa_a \bar{a}, a \in A, \kappa_a \in F^* \} \) as an \( F \)-basis for \( F * A \). As explained in [P1 Chapter 1], with the help of a diagonal change in basis it can be shown that \( F * Z \) is commutative whenever \( Z \) is an infinite cyclic subgroup of \( A \). By [Br Theorem A], the maximal rank of a subgroup \( B \) of \( A \) such that \( F * B \) is commutative is equal to the Krull and the global dimension of \( F * A \). Examples of \( F * A \) for which the commutativity of \( F * B \) implies that \( B \) is an infinite cyclic subgroup of \( A \) can be found in [MP].

It is known (e.g. [P2 Lemma 37.8]) that for each subgroup \( B \leq A \), the monoid of nonzero elements of \( F * B \) is an Ore subset of \( F * A \). Thus we may localize \( F * A \) at \( \{x_{k+1}, \ldots, x_n\} \), where the localised generators are enclosed in roundparentheses. As shown in [MP], the unit group \( U \) of \( F * A \) is the set

\[
U = \{ \kappa \bar{a}, \kappa \in F^*, a \in A \}.
\]

The derived subgroup \( U' \) of \( U \) is a subgroup of \( F^* \). Thus \( U \) is nilpotent of class at most 2. For subsets \( X \) and \( Y \) of \( A \), we denote by \( \langle X \rangle \) the subset \( \{ \bar{x} \bar{a}, x \in X \} \) of \( U \), and by \( \langle X, Y \rangle \) the subgroup of \( U' \) generated by the commutators \( [\bar{x}, \bar{y}] \), where \( x \in X \) and \( y \in Y \). Since \( U' \) is central in \( U \), hence by [? Chapter 5],

\[
[x_1 x_2, y] = [x_1, y][x_2, y] \quad [x, y_1 y_2] = [x, y_1][x, y_2]
\]

A dimension for studying modules over a crossed product \( D * A \) of a free finitely generated abelian group \( A \) over a division ring \( D \), denoted as \( \dim(M) \), was introduced in [BG] and was shown to coincide with the Gelfand–Kirillov dimension (\[KL\]).

**Definition 2.1** (Definition 2.1 of [BG]). Let \( M \) be a \( D * A \)-module. The dimension \( \dim(M) \) of \( M \) is defined to be the greatest integer \( r \), \( 0 \leq r \leq \text{rk}(A) \), so that for some subgroup \( B \) in \( A \) with rank \( r \), \( M \) is not \( D * B \)-torsion.

**Proposition 2.2** (Lemma 2.2(1) of [BG]). Let

\[
0 \to M_1 \to M \to M_2 \to 0,
\]

be an exact sequence of \( D * A \)-modules. Then,

\[
\dim(M) = \sup(\dim(M_1), \dim(M_2)).
\]

### 3 The proofs of Theorems A and B

The following lemma shall be key tool in the proof of Theorem A.
Lemma 3.1. Suppose that $F * A$ has a finitely generated module $M$ and $A$ has a subgroup $C$ with $A/C$ torsion free, $\text{rk}(C) = \mathcal{GK}(M)$, and $F * C$ commutative. Suppose moreover that $M$ is not $F * C$-torsion. Then $C$ has a virtual complement $E$ in $A$ such that $F * E$ is commutative. In fact given $\mathbb{Z}$-bases $\{x_1, \ldots, x_r\}$ and $\{x_1, \ldots, x_r, x_{r+1}, \ldots, x_n\}$ for $C$ and $A$ respectively, there exist monomials $\mu_j, j = r + 1, \ldots, n$, in $F * C$, and an integer $s > 0$ such that the monomials $\mu_j \bar{x}_j^s$ commute in $F * A$.

Proof. Let $\bar{x}_i \bar{x}_j = q_{ij} \bar{x}_j \bar{x}_i$, where, $i, j = 1, \ldots, n$ and $q_{ij} \in F^*$. We set $S = F * C \setminus \{0\}$ and denote the quotient field $(F * C)^{-1}$ by $\Delta$. Then $(F * A)^{-1}$ is a crossed product

$$R = F(x_1, \ldots, x_r)[x_{r+1}, \ldots, x_n].$$

The corresponding module of fractions $MS^{-1}$ is nonzero as $M$ is not $S$-torsion by the hypothesis. Furthermore, by the hypothesis, $\mathcal{GK}(M) = \text{rk}(C)$ and so in view of [BG, Lemma 2.3], $MS^{-1}$ is finite dimensional as $\Delta$-space. It is shown in [AG, Section 3] that if $R$ has a module that is one dimensional over $\Delta$ then there exist monomials $\mu_i \in F * C$ such that the monomials $\mu_i \bar{x}_i$, where $r + 1 \leq i \leq n$, commute mutually. Since the scalars in $F$ are central in $F * A$, we may assume that $\mu_i \in \bar{C}$. If $c_i$ is the element of $C$ corresponding to $\mu_i$, we may in this case take

$$E = (c_{r+1}x_{r+1}, \ldots, c_nx_n).$$

the $s$-fold exterior power $M' := \wedge^s(MS_{F_S}^{-1})$, where, $s = \text{dim}_{F_S}(MS^{-1})$ is a one dimensional module over the crossed product $R'$ obtained from $R$ by raising the cocycle of $R$ to its $s$-th power. More explicit $R'$ is the crossed product

$$R' = R = F(x_1, \ldots, x_r)[x_{r+1}, \ldots, x_n]$$

but with $[\bar{x}_i, \bar{x}_j] = q_{ij}',$ where

$$q_{ij}' = q_{ij}, \quad 1 \leq i \leq r; \quad q_{ij}' = q_{ij}^s, \quad r < i, j \leq n. \quad (1)$$

Then there are monomials $\mu_j \in F * C$ such that the monomials $\mu_j \bar{x}_j$, where $j = r + 1, \ldots, n$ commute with respect to the cocycle defined in equation $[\mathcal{G}]$. It easily follows from this that the monomials $\{\mu_j \bar{x}_j\}_{j=r+1}^{n}$ commute in $F * A$.

Remark 3.2. The above lemma remains valid if we drop the assumption that $A/C$ is torsion-free. In this case $A$ has a subgroup $A'$ of finite index such that $A'/C$ is torsion-free. Moreover in view of [BG, Lemma 2.7], $M$ is finitely generated and has GK dimension equal to $\text{rk}(C)$ as $F * A'$-module.

We are now ready to prove,

Theorem A. Let $M$ be a nonzero finitely generated $F * A$-module where $F * A$ has center $F$. Let $C < A$ be a subgroup having a subgroup $C_0$ of finite index such that $F * C_0$ is commutative. If $M$ is finitely generated as an $F * C$-module then,

(i) $\mathcal{GK}(M) = \text{rk}(C),$

(ii) $M$ is $F * C$-torsion-free,

(iii) $M$ is artinian,

(iv) $M$ is cyclic.
Proof. (i) We shall denote the module $M$ regarded as $F \ast C$-module as $M_C$. By hypothesis $M_C$ is finitely generated and so $M_{C_0}$ is also finitely generated where $M_{C_0}$ denotes $M_C$ viewed as $F \ast C_0$-module. By [BG, Lemma 2.7],

$$\mathcal{GK}(M) = \mathcal{GK}(M_C) = \mathcal{GK}(M_{C_0})$$

If $\mathcal{GK}(M) < \text{rk}(C)$ we may pick a subgroup $E_0 < C_0$ with $\text{rk}(E_0) < \text{rk}(C_0)$ such that $M_{C_0}$ is not $F \ast E_0$-torsion and $\mathcal{GK}(M) = \text{rk}(E_0)$ (section ??). Moreover $F \ast E_0$ is commutative since $F \ast C_0$ is commutative by the hypothesis. By Lemma 3.1 and Remark 3.2 $E_0$ has a (virtual) complement $E_1$ in $A$ such that $F \ast E_1$ is commutative. Since $\text{rk}(E_1) + \text{rk}(C_0)$ exceeds $\text{rk}(A)$, Hence $E_1 \cap C_0 > \langle 1 \rangle$. Moreover as $E_1 E_0$ has finite index in $A$, hence $E_1 C_0$ has finite index in $A$. But $E_1 \cap C_0$ is central in $F \ast E_1 C_0$ and hence $F \ast A$ has center larger than $F$. This is contrary to the hypothesis in the theorem.

(ii) Suppose that the $F \ast C$-torsion submodule $T$ of $M$ is nonzero. We recall that $T$ is an $F \ast A$-submodule of $M$. Applying part (i) of the theorem just established to $T$ we obtain $\mathcal{GK}(T) = \text{rk}(C)$.

We shall denote the $F \ast C$-module structure on $T$ by $T_C$. We note that $T_C$ is finitely generated. Moreover

$$\mathcal{GK}(T_C) = \text{rk}(T) = \text{rk}(C)$$

by [BG Lemma 2.7]. It then follows from [BG Proposition 2.6], that there exists $t \in T$ so that $\text{ann}_{F \ast C}(t) = 0$. But by the definition of $T$ this is $\text{ann}_{F \ast C}(t) \neq 0$. Hence $T = 0$ and $M$ is $F \ast C$-torsion-free.

(iii) We first note that each nonzero subfactor of $M$ has the same GK dimension as $M$. Let $N$ and $L$ be submodules of $M$ such that $Q := N/L$ is nonzero. As $Q$ finitely generated over $F \ast C$ it is $F \ast C$-torsion free by part (ii) of the theorem shown above. It follows from Definition 2.1 and Proposition 2.2 that

$$\mathcal{GK}(Q) = \text{rk}(C) = \mathcal{GK}(M).$$

Now let

$$M = M_0 > M_1 > M_2 > \cdots ,$$

be a strictly descending sequence of submodules of $M$. By the above $\mathcal{GK}(M_i/M_{i+1}) = \mathcal{GK}(M)$ for all $i \geq 0$. By [MP Lemma 5.6] and [MP 5.9] the sequence (2) must become constant after a finite number of steps. Hence $M$ is artinian.

(iv) Since $F \ast A$ has center $F$, it is simple by [MP Proposition 1.3]. It follows from [Ba Corollary 1.5] that $M$ is cyclic. $\square$

Corollary 3.3. Each finitely generated module with Gelfand–Kirillov dimension less than or equal to one over an $F \ast A$ with center exactly $F$ is artinian and cyclic.

Proof. We recall that an $F \ast A$-module with GK dimension zero is finite dimensional (e.g. [KL]) over the base field $F$ and so is artinian. By [W1 Lemma 2.9], an $F \ast A$-module with GK dimension one is finitely generated over some $F \ast C$ with $C$ infinite cyclic. Since $F \ast C$ must be commutative, the result follows from Theorem A. $\square$

Proposition 3.4. A finitely generated $F \ast A$-module $M$ satisfying

$$\mathcal{GK}(N) + \dim(F \ast A) = \text{rk}(A)$$

has finite length.
Proof. By [?], Theorem A, \( \dim(F \ast A) \) is equal to the maximal rank of a subgroup \( B \) of \( A \) such that \( F \ast B \) is commutative. Hence in view of [?B], the minimum possible GK dimension of a nonzero finitely generated \( F \ast A \)-module is \( \text{rk}(A) - \dim(F \ast A) = \mathcal{G}K(M) \). Thus in any (strictly) descending sequence \( M = M_0 > M_1 > \cdots > \) of submodules of \( M, \mathcal{G}K(M_i/M_{i+1}) = \mathcal{G}K(M) \) for each \( i \geq 0 \). It then p follows from [MP, Lemma 5.6] and [MP, 5.9] that this sequence must halt. Hence \( M \) has finite length. 

We now investigate the situation in which a submodule of a finitely generated \( F \ast A \)-module is finitely generated over a commutative subalgebra \( F \ast C \) for \( C < A \).

**Definition 3.5.** A nonzero \( F \ast A \)-module \( N \) is called critical when \( N/L \) has strictly smaller GK dimension than \( N \) for each nonzero proper submodule \( L \) of \( N \).

### 3.1 Applications to nilpotent groups

We shall now consider some applications to finitely generated torsion-free nilpotent groups of class 2. Throughout this section \( H \) stands for such a group and \( k \) denotes a field. We denote the group algebra of \( H \) over \( k \) by \( kH \) and by \( \zeta H \) the center of \( H \).

In [G], the following definition is given.

**Definition 3.6.** A finitely generated \( kH \)-module \( M \) is said to be reduced if there is a prime ideal \( P \) of \( k\zeta H \) such that \( M \) is annihilated by \( P \) and is \( k\zeta H/P \)-torsion-free.

It is remarked in [G] that every finitely generated \( kH \)-module is filtered by a finite series in which each section is reduced. We are now ready to prove

**Proposition 3.7.** Let \( H \) be a finitely generated torsion-free nilpotent group of class 2. Let \( M \) be a finitely generated \( kH \)-module reduced with annihilator \( P \) in \( k\zeta H \). If \( M \) is finitely generated as a \( kL \)-module for some abelian subgroup \( L < H \) then,

- (i) \( M \) is torsion-free as \( kL/P.kL \)-module,
- (ii) for any descending sequence \( M_1 \supseteq M_2 \supseteq \cdots \supseteq \) of submodules of \( M \), there is an integer \( s \) such that \( M_k/M_{k+1} \) is \( k\zeta H/P \)-torsion for all \( k \geq s \).

**Proof.** We note that there is no harm in assuming that \( L \supset \zeta H \). As in [G], let \( K \) denote the field of fractions of \( k\zeta H/P \) and set

\[
\hat{M} = M \otimes_{k\zeta H/P} K.
\]

Then \( \hat{M} \) is a finitely generated module for \( kH/P.kH \otimes_{k\zeta H/P} K \) and the latter is a twisted group algebra \( K \ast A \), where \( A = H/\zeta H \). Moreover the center of \( K \ast A \) is exactly \( K \). By the hypothesis in the theorem, \( M \) is finitely generated over \( kL \), and so over \( kL/P.kL \). Thus \( \hat{M} \) is finitely generated over \( kL/P.kL \otimes_{k\zeta H/P} K \) which is the subalgebra \( K \ast C \) for \( C = L/\zeta H \). Moreover since \( L \) is abelian, therefore, \( K \ast C \) is commutative. By Theorem A, \( \hat{M} \) is \( K \ast C \)-torsion-free. In other words \( \hat{M} \) is torsion-free as \( kL/P.kL \otimes_{k\zeta H/P} K \)-module. As \( M \) is reduced it is by definition \( k\zeta H/P \)-torsion free. Hence the natural \( kH/P.kH \)-linear map \( \mu : M \to \hat{M} \) via \( y \mapsto y \otimes 1 \) is a monomorphism. Therefore \( M \) must be \( kL/P.kL \)-torsion free. This shows part(i). \( \square \)
As a consequence we obtain

**Theorem B.** Let $H$ be a finitely generated torsion-free nilpotent group of class 2. For a finitely generated $kH$-module $N$ which is reduced with annihilator $P$ and is finitely generated over $kL$ for some abelian subgroup $L \leq H$, $N$ is torsion-free as $kL/PkL$-module. Moreover if $\zeta H$ is cyclic then $N$ has finite length.

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