THE BERRY PHASE FOR SIMPLE HARMONIC OSCILLATORS

SERGEI K. SUSLOV

ABSTRACT. We evaluate the Berry phase for a “missing” family of the square integrable wavefunctions for the linear harmonic oscillator, which cannot be derived by the separation of variables (in a natural way). Instead, it is obtained by the action of the maximal kinematical invariance group on the standard solutions. A simple closed formula for the phase (in terms of elementary functions) is found by integration with the help of a computer algebra system.

Recent reports on observations of the dynamical Casimir effect [27], [51] strengthens the interest to ‘nonclassical’ states in quantum optics and generalized harmonic oscillators [10], [11], [13], [14], [15], [17], [33], [34] and [37]. The amplification of quantum fluctuations by modulating parameters of an oscillator is closely related to the process of particle production in quantum fields [11], [24], [34] and [37]. Other dynamical amplification mechanisms include the Unruh effect [47] and Hawking radiation [20], [21].

The purpose of this paper is to evaluate the Berry phase for certain “missing” solutions of the time-dependent Schrödinger equation for the linear harmonic oscillator as an instructive example. Applications will be discussed elsewhere.

1. Hidden Solutions

The time-dependent Schrödinger equation for the simple harmonic oscillator,

\[ 2i\psi_t + \psi_{xx} - x^2\psi = 0, \quad (1.1) \]

has the following six-parameter family of (square integrable) solutions [32]:

\[ \psi_n(x,t) = \frac{e^{i(\alpha(t)x^2+\delta(t)x+\kappa(t))+(2n+1)\gamma(t))}}{\sqrt{2n!\mu(t)\sqrt{\pi}}} \cdot e^{-(\beta(t)x+\varepsilon(t))^2/2} H_n(\beta(t)x+\varepsilon(t)), \quad (1.2) \]

where \( H_n(x) \) are the Hermite polynomials [40] and

\[ \mu(t) = \mu_0 \sqrt{\beta_0^4 \sin^2 t + (2\alpha_0 \sin t + \cos t)^2}, \quad (1.3) \]

\[ \alpha(t) = \frac{\alpha_0 \cos 2t + \sin 2t}{\beta_0^4 \sin^2 t + (2\alpha_0 \sin t + \cos t)^2}, \quad (1.4) \]

\[ \beta(t) = \frac{\beta_0}{\sqrt{\beta_0^4 \sin^2 t + (2\alpha_0 \sin t + \cos t)^2}}. \quad (1.5) \]

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\[ \gamma (t) = \gamma_0 - \frac{1}{2} \arctan \frac{\beta_0^2 \sin t}{2 \alpha_0 \sin t + \cos t}, \quad (1.6) \]

\[ \delta (t) = \frac{\delta_0 (2 \alpha_0 \sin t + \cos t) + \varepsilon_0 \beta_0^3 \sin t}{\beta_0^4 \sin^2 t + (2 \alpha_0 \sin t + \cos t)^2}, \quad (1.7) \]

\[ \varepsilon (t) = \frac{\varepsilon_0 (2 \alpha_0 \sin t + \cos t) - \beta_0 \delta_0 \sin t}{\sqrt{\beta_0^4 \sin^2 t + (2 \alpha_0 \sin t + \cos t)^2}}, \quad (1.8) \]

\[ \kappa (t) = \kappa_0 + \sin^2 t \frac{\varepsilon_0 \beta_0^2 (\alpha_0 \varepsilon_0 - \beta_0 \delta_0) - \alpha_0 \delta_0^2}{\beta_0^4 \sin^2 t + (2 \alpha_0 \sin t + \cos t)^2} + \frac{1}{4} \sin 2t \frac{\varepsilon_0^2 \beta_0^2 - \delta_0^2}{\beta_0^4 \sin^2 t + (2 \alpha_0 \sin t + \cos t)^2}, \quad (1.9) \]

\((\mu_0 \neq 0, \alpha_0, \beta_0 \neq 0, \gamma_0, \delta_0, \varepsilon_0, \kappa_0\) are real initial data). These solutions have been derived analytically in the framework of a unified approach to generalized harmonic oscillators (see, for example, [8], [9], [29], [52], [53] and the references therein). They are also verified by a direct substitution with the aid of \textit{Mathematica} computer algebra system [26], [31]. (The simplest special case \(\mu_0 = \beta_0 = 1\) and \(\alpha_0 = \gamma_0 = \delta_0 = \varepsilon_0 = \kappa_0 = 0\) reproduces the textbook solution obtained by the separation of variables [42], [18], [28], [35]. The shape-preserving oscillator evolutions occur when \(\alpha_0 = 0\) and \(\beta_0 = 1\) and a special case when \(\alpha_0 = 0\) is discussed in [22]. More details on the derivation of these formulas and some \textit{Mathematica} animations, revealing a new feature – an oscillation in space of the probability density \(|\psi(x,t)|^2\) – of these solutions, can be found in Refs. [26], [30] and [31].)

The “dynamic harmonic oscillator states” (1.2)–(1.9) are eigenfunctions,

\[ E (t) \psi_n (x, t) = \left( n + \frac{1}{2} \right) \psi_n (x, t), \quad (1.10) \]

of the time-dependent quadratic invariant,

\[ E (t) = \frac{1}{2} \left[ \frac{(p - 2 \alpha x - \delta)^2}{\beta^2} + (\beta x + \varepsilon)^2 \right], \quad \frac{d}{dt} \langle E \rangle = 0, \quad (1.11) \]

where \( p = i^{-1} \partial / \partial x \) and the required operator identity,

\[ \frac{\partial E}{\partial t} + i^{-1} [E, H] = 0, \quad H = \frac{1}{2} (p^2 + x^2), \quad (1.12) \]

holds [41].

The (isomorphic) maximum kinematical invariance groups of the free particle and harmonic oscillator were introduced in [1], [2], [19], [23], [38] and [39] (see also [7], [25], [36], [48] and the references therein). We have established a connection with certain Ermakov-type system which allows us to bypass a complexity of the traditional Lie algebra approach [30], [32]. (A general procedure of obtaining new solutions by acting on any set of given ones by enveloping algebra of generators of the Heisenberg–Weyl group is described in [15]; see also [3], [4] and [14].)

2. Evaluation of the Phase

The holonomic effect in quantum mechanics known as Berry’s phase [5], [6], [43], [50] has received considerable attention over the years (see, for example, [16], [49] and the other references in [41]).
The derivative of Berry’s phase has been recently calculated for the generalized harmonic oscillators as follows [41]:

\[
\frac{d\theta_n}{dt} = -\beta^{-2} \left( \varepsilon^2 + n + \frac{1}{2} \right) \frac{d\alpha}{dt} + \varepsilon \beta^{-1} \frac{d\delta}{dt} - \frac{d\kappa}{dt},
\]

(2.1)

where we are going to use (1.4)–(1.9) and simplify. Integrating by parts, one gets

\[
\theta_n = -\left( n + \frac{1}{2} \right) \int \beta^{-2} \frac{d\alpha}{dt} dt - \left( \frac{\varepsilon}{\beta} \right)^2 \alpha + \varepsilon \delta - \kappa
\]

(2.2)

+ \int \left[ \alpha \frac{d}{dt} \left( \frac{\varepsilon}{\beta} \right)^2 - \delta \frac{d}{dt} \left( \frac{\varepsilon}{\beta} \right) \right] dt.

Here,

\[
4\beta_0^2 \left[ \left( \frac{\varepsilon}{\beta} \right)^2 \alpha - \frac{\varepsilon \delta}{\beta} + \kappa \right] = 2\beta_0 \left( 2\beta_0 \kappa_0 - \delta_0 \varepsilon_0 \right)
\]

\[+ 2\varepsilon_0 (2\alpha_0 \varepsilon_0 - \beta_0 \delta_0) \cos 2t + \left[ (2\alpha_0 \varepsilon_0 - \beta_0 \delta_0)^2 - \varepsilon_0^2 \right] \sin 2t,
\]

(2.3)

\[
4\beta_0^2 \int \left[ \alpha \frac{d}{dt} \left( \frac{\varepsilon}{\beta} \right)^2 - \delta \frac{d}{dt} \left( \frac{\varepsilon}{\beta} \right) \right] dt = 2t \left[ (2\alpha_0 \varepsilon_0 - \beta_0 \delta_0)^2 + \varepsilon_0^2 \right]
\]

\[+ 2\varepsilon_0 (2\alpha_0 \varepsilon_0 - \beta_0 \delta_0) \cos 2t + \left[ (2\alpha_0 \varepsilon_0 - \beta_0 \delta_0)^2 - \varepsilon_0^2 \right] \sin 2t,
\]

(2.4)

\[
\int \beta^{-2} \frac{d\alpha}{dt} dt = -t \frac{4\alpha_0^2 + \beta_0^4 + 1}{2\beta_0^2} + \arctan \left( \frac{2\alpha_0 + (4\alpha_0^2 + \beta_0^4) \tan t}{\beta_0^2} \right)
\]

(2.5)

with the aid of Mathematica (the notebook is available from the author’s website [45]).

Finally, we evaluate Berry’s phase in a closed form:

\[
\theta_n (t) = -\left( n + \frac{1}{2} \right) \left[ \arctan \left( \frac{2\alpha_0 + (4\alpha_0^2 + \beta_0^4) \tan t}{\beta_0^2} \right) - \arctan \left( \frac{2\alpha_0}{\beta_0^2} - t \frac{4\alpha_0^2 + \beta_0^4 + 1}{2\beta_0^2} \right) \right]
\]

\[+ \left(2\alpha_0 \varepsilon_0 - \beta_0 \delta_0\right)^2 + \varepsilon_0^2, \quad \theta_n (0) = 0.
\]

(2.6)

(This expression has been verified by differentiation with the help of Mathematica once again [45]. Examples are presented in Figure 1.) To the best of our knowledge, this formula is also missing in the available literature — in the simplest case $\beta_0 = 1$ and $\alpha_0 = \gamma_0 = \delta_0 = \varepsilon_0 = \kappa_0 = 0$, one obtains $\theta_n = 0$, which is a well-known result for the textbook solutions. Our formula implies that for the shape-preserving oscillator evolutions, when $\alpha_0 = 0$ and $\beta_0 = 1$, the phase does not depend on $n$.

On the second hand, Eq. (42) of Ref. [41] gives an alternative formula for evaluation of the phase,

\[
\theta_n = (2n + 1) \gamma + \int \langle H \rangle dt,
\]

(2.7)

where

\[
\langle H \rangle = \frac{1}{2} \left[ \langle p^2 \rangle + \langle x^2 \rangle \right] = \left( n + \frac{1}{2} \right) \frac{1 + 4\alpha_0^2 + \beta_0^4}{2\beta_0^2} + \frac{(2\alpha_0 \varepsilon_0 - \beta_0 \delta_0)^2 + \varepsilon_0^2}{2\beta_0^2}
\]

(2.8)
by (A.3)–(A.5) of Ref. [32]. As a result one gets

\[ \theta_n = -\left(n + \frac{1}{2}\right) \arctan \frac{\beta_0 \tan t}{1 + 2\alpha_0 \tan t} + \left(n + \frac{1}{2}\right) \frac{1 + 4\alpha_0^2 + \beta_0^4}{2\beta_0^2} t + \frac{(2\alpha_0 \varepsilon_0 - \beta_0 \delta_0)^2 + \varepsilon_0^2}{2\beta_0^2} t, \] (2.9)

which is equivalent to our previous expression (2.6) up to an elementary transformation.

3. A Conclusion

In addition to the oscillation in space of the probability density \( |\psi(x, t)|^2 \), which has already been computer animated in [31] and [32], the “dynamic harmonic states” (1.2)–(1.9) possess the nontrivial Berry phase. These two distinguished features of the quantum motion under consideration might be observed in a clever experiment.

Moreover, the electromagnetic field quantization presents the EM field in nonstationary media as a set of harmonic oscillators [11] and [12]. Thus the Berry phase evaluated in this paper is somehow related to the squeezed states of light which are produced in the process of parametric amplification. (See also Ref. [46] for other possible applications.)

![Figure 1. The phases \( \theta_0(t) \) and \( \theta_1(t) \) with \( \alpha_0 = \gamma_0 = \varepsilon_0 = 0, \beta_0 = 2/3 \) and \( \delta_0 = 1 \) (thick and thin lines, respectively).](image)

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REFERENCES

[1] R. L. Anderson, S. Kumei and C. E. Wulfman, Invariants of the equations of wave mechanics. I, Rev. Mex. Fís. 21 (1972), 1–33.
[2] R. L. Anderson, S. Kumei and C. E. Wulfman, Invariants of the equations of wave mechanics. II One-particle Schrödinger equations, Rev. Mex. Fís. 21 (1972), 35–57.
[3] V. G. Bagrov, V. V. Belov and I. M. Ternov, Quasiclassical trajectory-coherent states of a particle in an arbitrary electromagnetic field, J. Math. Phys. 24 (1983) #12, 2855–2859.
[4] V. V. Belov and A. G. Karavaev, Higher approximations for quasiclassical trajectory-coherent states, Izvestiya Vysshikh Uchebnykh Zavedenij Fizika, 31 (1987) #10, 14–18 [in Russian]; see also English transl.: Sov. Phys. Journal 1989, 30 #10, 819–822.
[5] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. Roy. Soc. London, A392 (1984) # 1802, 45–57.
[6] M. V. Berry, Classical adiabatic angles and quantum adiabatic phase, J. Phys. A: Math. Gen. 18 (1985) # 1, 15–27.
[7] C. P. Boyer, R. T. Sharp and P. Winternitz, Symmetry breaking interactions for the time dependent Schrödinger equation, J. Math. Phys. 17 (1976) #8, 1439–1451.
[8] R. Cordero-Soto, R. M. López, E. Suazo and S. K. Suslov, Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields, Lett. Math. Phys. 84 (2008) #2–3, 159–178.
[9] R. Cordero-Soto, E. Suazo and S. K. Suslov, Quantum integrals of motion for variable quadratic Hamiltonians, Ann. Phys. 325 (2010) #9, 1884–1912.
[10] V. V. Dodonov, ‘Nonclassical’ states in quantum optics: a ‘squeezed’ review of the first 75 years, J. Opt. B: Quantum Semiclass. Opt. 4 (2002), R1–R33.
[11] V. V. Dodonov, Current status of dynamical Casimir effect, Physica Scripta 82 (2010) #3, 038105 (10 pp).
[12] V. V. Dodonov, A. B. Klimov and D. E. Nikonov, Quantum phenomena in nonstationary media, Phys. Rev. A. 47 (1993) # 5, 4422–4429.
[13] V. V. Dodonov, I. A. Malkin and V. I. Man’ko, Integrals of motion, Green functions, and coherent states of dynamical systems, Int. J. Theor. Phys. 14 (1975) # 1, 37–54.
[14] V. V. Dodonov and V. I. Man’ko, Coherent states and the resonance of a quantum damped oscillator, Phys. Rev. A 20 (1979) # 2, 550–560.
[15] V. V. Dodonov and V. I. Man’ko, Invariants and correlated states of nonstationary quantum systems, in: Invariants and the Evolution of Nonstationary Quantum Systems, Proceedings of Lebedev Physics Institute, vol. 183, pp. 71-181, Nauka, Moscow, 1987 [in Russian]; English translation published by Nova Science, Commack, New York, 1989, pp. 103-261.
[16] V. V. Dodonov and V. I. Man’ko, ‘Nonclassical’ states in quantum physics: brief historical review, in: Theory of Nonclassical States of Light, (V. V. Dodonov and V. I. Man’ko, Eds.), Taylor & Francis, London and New York, 2003, pp. 1–94.
[17] S. Flügge, Practical Quantum Mechanics, Springer–Verlag, Berlin, 1999.
[18] C. H. Hagen, Scale and conformal transformations in Galilean-covariant field theory, Phys. Rev. D 5 (1972) #2, 377–388.
[19] S. W. Hawking, Black hole explosions?, Nature, London 248 (1974), 30–31.
[20] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. 43 (1975) #3, 199–220.
[21] K. Husimi, Miscellanea in elementary quantum mechanics: I–II, Prog. Theor. Phys. 9 (1953) #3, 238–244; Prog. Theor. Phys. 9 (1953) #4, 381–402.
[22] R. Jackiw, Dynamical symmetry of the magnetic monopole, Ann. Phys. 129 (1980), 183–200.
[23] T. A. Jacobson, Introduction to quantum fields in curved spacetime and the Hawking effect, arXiv:0308048v3 [gr-qc] 9 April 2004.
[24] E. G. Kalnins and W. Miller, Lie theory and separation of variables. 5. The equations \( iU + U_{xx} + c/x^2U = 0 \), J. Math. Phys. 15 (1974) #10, 1728–1737.
[25] C. Koutschan, http://hahn.la.asu.edu/~suslov/currev/index.htm; see Mathematica notebook: Koutschan.nb; see also http://www.risc.jku.at/people/ckoutsch/pekeris/
[27] P. Lähteenmäki, G. S. Paraoanu, J. Hassel and P. J. Hakonen, *Dynamical Casimir effect in a Josephson metamaterial*, arXiv:1111.5608v2 [cond-mat.mes-hall] 1 Dec 2011.

[28] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory*, Pergamon Press, Oxford, 1977.

[29] N. Lanfear, R. M. López and S. K. Suslov, *Exact wave functions for generalized harmonic oscillators*, Journal of Russian Laser Research 32 (2011) #4, 352–361; see also arXiv:11002.5119v2 [math-ph] 20 Jul 2011.

[30] R. M. López, S. K. Suslov and J. M. Vega-Guzmán, *On the harmonic oscillator group*, arXiv:1111.5569v2 [math-ph] 4 Dec 2011.

[31] R. M. López, S. K. Suslov and J. M. Vega-Guzmán, http://hahn.la.asu.edu/~suslov/curres/index.htm; see Mathematica notebook: HarmonicOscillatorGroup.nb

[32] R. M. López, S. K. Suslov and J. M. Vega-Guzmán, *On a hidden symmetry of quantum harmonic oscillators*, Journal of Difference Equations and Applications, 2012, http://dx.doi.org/10.1080/10236198.2012.658384; see also arXiv:1112.2586v2 [quant-ph] 2 Jan 2012.

[33] I. A. Malkin and V. I. Man’ko, *Dynamical Symmetries and Coherent States of Quantum System*, Nauka, Moscow, 1979 [in Russian].

[34] V. I. Man’ko, *The Casimir effect and quantum vacuum generator*, Journal of Soviet Laser Research 12 (1991), 383–385.

[35] E. Merzbacher, *Quantum Mechanics*, third edition, John Wiley & Sons, New York, 1998.

[36] W Miller, Jr., *Symmetry and Separation of Variables*, Encyclopedia of Mathematics and Its Applications, Vol. 4, Addison–Wesley Publishing Company, Reading etc, 1977.

[37] P. D. Nation, J. R. Johansson, M. P. Blencowe and F. Nori, *Stimulationg uncertainty: Amplifying the quantum vacuum with superconducting circuits*, Rev. Mod. Phys. 84 (2012), January–March, 1–24.

[38] U. Niederer, *The maximal kinematical invariance group of the free Schrödinger equations*, Helv. Phys. Acta 45 (1972), 802–810.

[39] U. Niederer, *The maximal kinematical invariance group of the harmonic oscillator*, Helv. Phys. Acta 46 (1973), 191–200.

[40] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer–Verlag, Berlin, New York, 1991.

[41] B. Sanborn, S. K. Suslov and L. Vinet, *Dynamic invariants and the Berry phase for generalized driven harmonic oscillators*, Journal of Russian Laser Research 32 (2011) #5, 486–494; see also arXiv:1108.5144v1 [math-ph] 25 Aug 2011.

[42] E. Schrödinger, *Der stetige Übergang von der Mikro-zur Makro Mechanik*, Die Naturwissenshaften, 14 (1926), 664–666; see also Collected Papers on Wave Mechanics, Blackie & Son Ltd, London and Glasgow, 1928, pp. 41–44, for English translation of Schrödinger’s original paper.

[43] B. Simon, *Holonomy, the quantum adiabatic theorem, and Berry’s phase*, Phys. Rev. Lett. 51 (1983) #24, 2167–2170.

[44] S. K. Suslov, *Dynamical invariants for variable quadratic Hamiltonians*, Physica Scripta 81 (2010) #5, 055006 (11 pp); see also arXiv:1002.0144v6 [math-ph] 11 Mar 2010.

[45] S. K. Suslov, http://hahn.la.asu.edu/~suslov/curres/index.htm; see Mathematica notebook: BerrySummary.nb.

[46] D. Xiao, M.-Ch. Chang and Q. Niu, *Berry phase effects on electronic properties*, Rev. Mod. Phys. 82 (2010), July–September, 1959–2007.

[47] W. G. Unruh, *Notes on black-hole evaporation*, Phys. Rev. D 14 (1976) #4, 870–892.

[48] L. Vinet and A. Zhedanov, *Representations of the Schrödinger group and matrix orthogonal polynomials*, J. Phys. A: Math. Theor. 44 (2011) #35, 355201 (28 pages).

[49] S. I. Vinitskiĭ, V. L. Derbov, V. N. Dubovik, B. L. Markovskiĭ, and Yu. P. Stepanovskii, *Topological phases in quantum mechanics and polarization optics*, Sov. Phys. Usp. 33 (1990) #6, 403–428.

[50] F. Wilczek and A. Zee, *Appearance of gauge structure in simple dynamical systems*, Phys. Rev. Lett. 52 (1984) #24, 2111–2114.

[51] C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J. R. Johansson, T. Duty, F. Nori and P. Delsing, *Observation of the dynamical Casimir effect in a superconducting circuit*, Nature 479 (2011) November 17, 376–379.

[52] K. B. Wolf, *On time-dependent quadratic Hamiltonians*, SIAM J. Appl. Math. 40 (1981) #3, 419–431.

[53] K-H. Yeon, K-K. Lee, Ch-I. Um, T. F. George and L. N. Pandey, *Exact quantum theory of a time-dependent bound Hamiltonian systems*, Phys. Rev. A 48 (1993) #4, 2716–2720.
School of Mathematical and Statistical Sciences & Mathematical, Computational and Modeling Sciences Center, Arizona State University, Tempe, AZ 85287–1804, U.S.A.

E-mail address: sks@asu.edu

URL: http://hahn.la.asu.edu/~suslov/index.html