**p-RATIONAL FIELDS AND THE STRUCTURE OF SOME MODULES**

ABDELAZIZ EL HABIBI (1), M’HAMMED ZIANE (2)

Abstract. Assume that the field $K$ is $p$-rational. We study the freeness of the $\Lambda(G_{\infty,S})$-module $X = \mathcal{H}^{ab} = \text{Gal}(K_{S\cup S_p}/K_{\infty,S})^{ab}$. For numerical evidence to our result we consider the case of fields of the form $\mathbb{Q}(\sqrt{p}, \sqrt{-d})$.

1. Introduction

The cohomology theory gives subtle and deep lying arithmetic laws if we study Galois groups with restricted ramification. In this paper we consider infinite fields with restricted ramification and try to study closely the associated Galois groups. Let $K$ be a field which is of finite or infinite degree over the rationales. If $\Sigma$ is a set of places of $K$, then we denote by $K_{\Sigma}$ the maximal pro-$p$-extension of $K$ which is unramified outside $\Sigma$ and let $G_{\Sigma}(K)$ be its Galois group over $K$. If $K$ is a number field and $p$ an odd prime number such that $\Sigma$ contains the set of places of $K$ dividing $p$, then class field theory shows that the group $G_{\Sigma}(K)^{ab}$ is isomorphic to $\mathbb{Z}_p^\rho \times T_{\Sigma}$, where $\rho$ is the $\mathbb{Z}_p$-rank and $T_{\Sigma}$ is the $\mathbb{Z}_p$-torsion. Later we will study and consider examples for which $\rho = 1 + r_2$ and $T_{S_p} = 0$, such fields are called $p$-rational [MN]. Moreover, it is known that in this case $G_{\Sigma}(K)$ is of cohomological dimension at most two. We refer the reader to [NSW] for a complete study of Galois groups with restricted ramification.

In the following we consider extensions with restricted ramification of infinite fields such as the cyclotomic $\mathbb{Z}_p$ of a number field. More precisely, let $K$ be a number field and $p$ an odd prime number. Assume that $S$ is a finite set of non-archimedean places of $K$ and consider the extension $K_{\infty,S}$ where $K_{\infty}$ is the cyclotomic $\mathbb{Z}_p$-extension of $K$. For $S$ satisfying $S \cap S_p = \emptyset$ and $N(v) \equiv 1 \pmod{p}$ for every places $v \in S$, the group $G_S(K_{\infty})$ has been considered by several authors ([Sal], [MO], [It],...), its structure is used to study the Galois group $G_{\infty,S} := \text{Gal}(K_{\infty,S}/K)$. In the present paper, the Galois groups we study are considered as modules over the complete Iwasawa algebra $\Lambda(G_{\infty,S}) = \mathbb{Z}_p[[G_{\infty,S}]]$ define by:

$$\Lambda(G_{\infty,S}) = \mathbb{Z}_p[[G_{\infty,S}]] := \lim_{\leftarrow U} \mathbb{Z}_p[G_{\infty,S}/U],$$

the projective limite with respect to the open normal sub-groups.

Let $K_{S\cup S_p}$ be the maximal pro-$p$-extension of $K$ which is unramified outside $S \cup S_p$ with Galois group $G_{S\cup S_p}(K) = \text{Gal}(K_{S\cup S_p}/K)$. We consider the normal subgroup $\mathcal{H} = \text{Gal}(K_{S\cup S_p}/K_{\infty,S})$ of $G_{S\cup S_p}(K)$, its abelianizer $\mathcal{X}$ may be endowed with a structure of $\Lambda(G_{\infty,S})$-module. Using Nakayama’s lemma, we obtain that $\mathcal{X}$ is finitely generated over $\Lambda(G_{\infty,S})$ (see the proof of Theorem 3.1). In the present paper we study the freeness of the $\Lambda(G_{\infty,S})$-module $\mathcal{X}$. Roughly speaking, we prove (Theorem 3.1) that the $\Lambda(G_{\infty,S})$-module $\mathcal{X}$ is free of rank $r_2$ by using the following theorem [NSW, Proposition 5.6.13]:

**Theorem 1.1.** Let $\mathcal{G}$ be a pro-$p$-group with finite presentation and cohomological dimension at most 2. Let $\mathcal{H}$ be a normal closed subgroup of $\mathcal{G}$ with $H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Let $G = \mathcal{G}/\mathcal{H}$.

2010 Mathematics Subject Classification. 11R23, 11R37, 11S25.

Key words and phrases. Iwasawa theory, Class field theory, Cohomology of number fields, $p$-adic analytic structures.

1
and \( X = H^{ab} \). Suppose that \( X \) is finitely generated under \( \Lambda(G) \).

Suppose that \( \text{cd}(G) \leq 2 \) and \( H^2(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \). If the natural morphism \( \text{Tor}_{\mathbb{Z}_p} \mathcal{G}^{ab} \longrightarrow \mathcal{G}^{ab} \) is injective, then \( \Lambda(G) \)-module \( X \) is free.

We are considering fields \( K \) which are \( p \)-rational in the sense of [MN], [GJ], [Gra] and [Gre]. In particular, a number field \( K \) is said to be \( p \)-rational if the Galois group \( G_{S_p}(K) \) is a free \( p \)-pro-\( p \)-group on \( r_2 + 1 \) generators. The assumption that \( K \) is \( p \)-rational with the fact that the number \( s \) of places \( v \) of \( K_\infty \) above places in \( S \) is one, gives that \( G_{\infty,S} \) is a \( p \)-adic analytic group without \( p \)-torsion.

For the statement of the results, we need some standard notations. Let \( U_p \) be the \( p \)-adic compactification of the group of units of the ring of integers of \( K_v \) and denote \( U_p = \prod_{v \in S_p} U_v \) there product for the primes above \( p \). If \( T \) is a finite set of places of \( K \), let

\[
\iota_T : \mathcal{E} \longrightarrow \prod_{v \in T} U_v,
\]

the local embedding with respect to element of \( T \) of \( \mathcal{E} = \mathbb{Z}_p \otimes E_K \) the \( p \)-adic completion of the group of units of \( K \). Let \( \mathcal{E}_T \) be the kernel of \( \iota_T \), we denote \( \iota_T \) by \( \iota_p \) when \( T = S_p \).

Using the above notations we can state the following result:

**Theorem 1.2.** (Theorem 3.1) Let \( K \) be a \( p \)-rational field not containing the \( p \)-th roots of unity and satisfying the following conditions:

1. The group \( i_p(\mathcal{E}_S) \) is a direct summand of \( U_p \).
2. \( s = 1 \).

Then the \( \Lambda(G_{\infty,S}) \)-module \( X \) is free of rank \( r_2 \).

Assume that \( \mathcal{G} = G_{S_1 \cup S_p}(K) \) and \( G = G_{\infty,S} \) which are \( p \)-pro-\( p \)-groups. It is known that the cohomological dimension of \( G_{S_1 \cup S_p} \) is at most two. Since \( H \) is a normal subgroup of \( \mathcal{G} \) such that \( H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \) (see [Ng3, theorem 2.2]), and \( G_{\infty,S} \) is of cohomological dimension at most two (see the proof of theorem 3.1), we can apply theorem 1.1 to these groups where condition (1) of theorem 3.1 gives the injection

\[
\text{Tor}_{\mathbb{Z}_p} G_{S_1 \cup S_p}^{ab}(K) \longrightarrow G_{\infty,S}^{ab}.
\]

We use the reciprocity map in the pro-\( p \)-version of global class field theory to control the kernel of the above map (see [J, Theorem 2.3] and [Sa2, Proposition 1.2]).

One remark that Proposition 3.2 gives the equivalence:

\[
\mathcal{X} \text{ is free } \Leftrightarrow \text{Tor}_{\mathbb{Z}_p} \left( \prod_{v \in S_p} U_v \cap i_p(\mathcal{E}_S) \right) = \{1\}.
\]

As an application of our result we consider the fields of the form \( \mathbb{Q}(\sqrt{pq}, \sqrt{-d}) \), where we suppose to be \( p \)-rational, where \( q \) is an odd prime number and \( p > 3 \). Using pari-GP and the technique described in example 3.5 we numerical examples for such fields.

**Acknowledgement.** The first author would like to thank Christian Maire for many helpful discussions and remarks on a first draft of this paper. I am also grateful to Aurel Page for improving my program on PariGP and Bill Allombert for inviting me many times to the Atelier PariGP.

**Notations**

Fix a prime number \( p > 2 \) and a number field \( K \). We use the following notation:

- \( \mu_p \) denote the set of all \( p \)-th roots of unity.
- \( S_p = \{ \mathfrak{p} \in Pl_K, \mathfrak{p}/p \} \).
- \( S \) : finite set of primes of \( K \) which is disjoint from \( S_p \).
- \( E_K \) denote the unit group of \( K \).
• $K_\infty$ : the cyclotomic $\mathbb{Z}_p$-extension of $K$ and put $\Gamma = \text{Gal}(K_\infty/K)$. We denote by $K_n$ the $n$th layer of $K_\infty/K$.
• $K_{\infty,S}$ : the maximal pro-$p$ extension of $K_\infty$ unramified outside $S$.
• If $\Sigma$ is any finite set of places of $K$, denote by $K_\Sigma$ the maximal pro-$p$ extension of $K$ unramified outside $\Sigma$ and put $G_\Sigma(K) = \text{Gal}(K_\Sigma/K)$. Let denote $\mathcal{X}_\Sigma(K) := \text{Gal}(K_\Sigma/K)^{ab}$.
• If $A$ is a $\mathbb{Z}_p$-module, denote by $d_pA := \dim_{\mathbb{F}_p} A/p^r$ the $p$-rank of $A$.
• If $G$ is a profinite group, $H$ a closed subgroup of $G$ and $M$ a compact $\mathbb{Z}_p[[H]]$-module, then $\text{Ind}^H_G M := M \hat{\otimes}_{\mathbb{Z}_p[[H]]} \mathbb{Z}_p[[G]]$ denotes the compact induction of $M$ from $H$ to $G$.

2. Preliminaries

The results of this paragraph figure in \[B\], \[NSW\], \[Sel\], \[Ng2\] ... Let $p$ be a prime number. Let $G$ be a pro-$p$-group. Consider the complete Iwasawa algebra of $G$ over $\mathbb{Z}_p$

$$\Lambda(G) = \mathbb{Z}_p[[G]] := \lim_{\leftarrow U} \mathbb{Z}_p[G/U],$$

the projective limite with respect to the open normal sub-groups. The compact $\mathbb{Z}_p$-algebra is a local ring with maximal ideal $m_G$ generated by the augmentation ideal $I_G$ and $p\Lambda(G)$.

Let $\mathcal{C}$ be the abelian category of compact $\Lambda(G)$-modules. In the following all considered modules $\mathcal{X}$ lies in the category $\mathcal{C}$. Let recall the crucial Nakayama lemma

**Lemma 2.1.** A $\Lambda(G)$-module $\mathcal{X} \in \mathcal{C}$ is generated by $r$ elements if and only if the $\mathbb{F}_p$-vector space $\mathcal{X}/m_G \mathcal{X}$ has dimension $r$.

**Definition 2.2.** If $\mathcal{X}$ is a free and finitely generated module over $\Lambda(G)$, then its $\Lambda(G)$-rank is the unique integer $\rho_\mathcal{X}$ satisfying

$$\mathcal{X} \simeq \Lambda(G)^{\rho_\mathcal{X}}.$$ 

2.1. **Structure and cohomological dimension of $\mathcal{X}$**. Let $G$ be a pro-$p$-group. We say that $G$ has cohomological dimension $cd(G) = n$ if the group $H^{n+1}(G, \mathbb{F}_p)$ vanishes while $H^n(G, \mathbb{F}_p)$ is not.

We have the following useful equivalence for free pro-$p$-groups:

**Proposition 2.3.** A non-trivial pro-$p$-group $G$ is said to be a free pro-$p$-group if and only if $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ and $G^{ab}$ has no torsion.

**Proof.** Consider the exact sequence $1 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Z}_p/p\mathbb{Z}_p \rightarrow 1$, used to obtain the homology exact sequence

$$\ldots H_2(G, \mathbb{Z}_p) \overset{p}{\rightarrow} H_2(G, \mathbb{Z}_p) \rightarrow H_2(G, \mathbb{F}_p) \rightarrow H_1(G, \mathbb{Z}_p) \overset{p}{\rightarrow} H_1(G, \mathbb{Z}_p) \ldots$$

and hence

$$H_2(G, \mathbb{Z}_p)/p \hookrightarrow H_2(G, \mathbb{F}_p) \rightarrow G^{ab}[p] \rightarrow 1.$$ 

It follows that $d_pH_2(G, \mathbb{F}_p) = d_pH_2(G, \mathbb{Z}_p) + d_pG^{ab}[p]$.

Also we have the following lemma,

**Lemma 2.4.** Let $G$ be a pro-$p$-group such that for any $i \geq 0$ the groups $H_i(G, \mathbb{Z}_p)$ are finitely generated as $\mathbb{Z}_p$-modules. Then $H_2(G, \mathbb{Z}_p) = 0$ if and only if $rg_{\mathbb{Z}_p} G^{ab} = 1 - \chi_2(G)$.

Where $\chi_2(G)$ is the Euler-Poincaré characteristic of $G$ with coefficients in $\mathbb{F}_p$.

In particular if the group $G$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, the group $H_2(G, \mathbb{Z}_p)$ is trivial if and only if the product $\mathbb{Z}_p \times \mathbb{Z}_p$ is not direct.
Proposition 2.5. Let $\mathcal{X}$ be a finitely generated $\Lambda(G)$-module. Then $\mathcal{X}$ is free if and only if $H_1(G, \mathcal{X})$ is trivial and $\mathcal{X}_G := \mathcal{X}/I_G$ is $\mathbb{Z}_p$-free.

Proof. Let $0 \to N \to \Lambda(G)^r \to \mathcal{X} \to 1$ be a minimal presentation of $\mathcal{X}$, which gives the following sequence

$$
\begin{array}{ccccccc}
& & H_1(G, \mathcal{X}) & \to & N_G & \to & \mathbb{Z}_p^r & \to & \mathcal{X}_G & \to & 1 .
\end{array}
$$

The fact that $\mathcal{X}$ is a projective $\Lambda(G)$-module gives the conclusion. □

Lemma 2.6. [NSW, Chapter II, §4, Exercise 4] Let $G$ be a pro-$p$-group and $\mathcal{H}$ be a normal closed subgroup of $G$ with $H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Let $G := G/\mathcal{H}$. Then there is an exact sequence in homology

$$
H_3(G, \mathbb{Z}_p) \to H_1(G, \mathcal{H}^{ab}) \to H_2(G, \mathbb{Z}_p) \to H_2(G, \mathcal{Z}_p) \to \mathcal{H}_G^{ab} \to \mathcal{G}^{ab} \to C^{ab}
$$

Proof. We use the Hochschild-Serre spectral sequence

$$
H^i(G, H^j(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)) \Rightarrow H^{i+j}(G, \mathbb{Q}_p/\mathbb{Z}_p)
$$

which gives the seven terms exact sequence

$$
0 \to E_2^{1,0} \to E_1 \to E_2^{0,1} \overset{d_2^{0,1}}{\to} E_2^{2,0} \to E_2^{2,1} \overset{d_2^{2,1}}{\to} E_2^{3,0},
$$

where $E_3^{1,1} = \ker d_2^{1,1}$. Then:

$$
\begin{array}{ccccccc}
0 & \to & H^1(G, \mathbb{Q}_p/\mathbb{Z}_p) & \to & H^1(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p) & \to & H^1(\mathcal{H}, \mathcal{H}^{ab}) & \to & H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) \\
& & & & & & & \downarrow \\
& & H^3(G, \mathbb{Q}_p/\mathbb{Z}_p) & \leftarrow & H^1(G, H^1(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)) & \leftarrow & E_3^{2}.
\end{array}
$$

By definition $E_2^{0,2}$ is a subgroup of $E_2^{0,2} = H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)$, which is trivial by hypotheses, moreover $E_2^{2,1} = E_2^{0,2}$ is also trivial. We obtain that $E_2^1 = E_2^2 = H^2(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p)$ and it suffices to apply the Pontryagin duality

$$
\begin{array}{ccccccc}
H_3(G, \mathbb{Z}_p) & \to & H_1(G, H_1(\mathcal{H}, \mathbb{Z}_p)) & \to & H_2(G, \mathbb{Z}_p) & \to & H_2(G, \mathbb{Z}_p) \\
& & & & & \downarrow \\
0 & \to & H_1(G, \mathbb{Z}_p) & \to & H_1(\mathcal{G}, \mathbb{Z}_p) & \to & H_1(\mathcal{H}, \mathbb{Z}_p).
\end{array}
$$

Proof of Theorem 1.1. Since $cd(G) \leq 2$, we have an exact sequence

$$
0 \to H_2(G, \mathbb{Z}_p) \overset{p}{\to} H_2(G, \mathbb{Z}_p) \to H_2(G, \mathbb{F}_p),
$$

and from our assumption, Thus $H_2(G, \mathbb{Z}_p)$ is free of finite rank as a $\mathbb{Z}_p$-module, $H_1(G, \mathcal{X})$ is trivial. The spectral sequence $H^i(G, H^j(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)) \Rightarrow H^{i+j}(G, \mathbb{Q}_p/\mathbb{Z}_p)$ gives the exact sequence

$$
0 \to H_2(G, \mathbb{Z}_p) \to \mathcal{X}_G \to \mathcal{G}^{ab} \to G^{ab} \to 0
$$

□
(using $H_2(G, \mathbb{Z}_p)$) if moreover the natural morphism $\text{Tor}_{\mathbb{Z}_p}G^{ab} \rightarrow G^{ab}$ is injective, then, $\mathcal{X}_G$ is $\mathbb{Z}_p$-free.

This proposition is an improvement Theorem [1.4] by giving the other implication:

**Proposition 2.7.** Under the hypotheses of the theorem [1.4] suppose that $\text{cd}(G) \leq 2$ and that $H_2(\mathcal{G}, \mathbb{Z}_p) = H_2(G, \mathbb{Z}_p) = 0$. Then the $\Lambda(G)$-module $\mathcal{X}$ is free if and only if the morphism $\text{Tor}_{\mathbb{Z}_p}G^{ab} \rightarrow G^{ab}$ is injective.

**Proof.** We have that $H_2(\mathcal{G}, \mathbb{Z}_p) = 0$, then $H_1(\mathcal{G}, \mathcal{X})$ is trivial, and the spectral sequence $H^i(\mathcal{G}, H^j(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)) \Rightarrow H^{i+j}(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p))$ gives the exact sequence

$$0 \rightarrow \mathcal{X}_G \rightarrow G^{ab} \rightarrow G^{ab} \rightarrow 1$$

hence $\mathcal{X}_G$ is $\mathbb{Z}_p$-free if and only if the morphism $\text{Tor}_{\mathbb{Z}_p}G^{ab} \rightarrow G^{ab}$ is injective. □

2.2. **On analytic pro-$p$-groups.** The main references for this part are: Lazard [L], Dixon, Du Sautoy, Segal, Mann [DDMS], ...

**Definition 2.8.** A topological group $G$ is $p$-adic analytic if $G$ has a structure of $p$-adic analytic manifold for which the morphism $(G, G) \rightarrow G: (x, y) \rightarrow xy^{-1}$ is analytic.

Let $G$ be a pro-$p$-group. If $G$ is $p$-adic analytic, then $\Lambda(G)$ is noetherian (see [L V.2.2.4]). If more $G$ is without $p$-torsion, then $\Lambda(G)$ is without zero divisor (see [Ne]). If $\Lambda(G)$ is Noetherian and without zero-divisors we can form a skew field $Q(G)$ of fractions of $\Lambda(G)$ (see [GW] chapter 9). This allows us to define the rank of a $\Lambda(G)$-module:

**Definition 2.9.** Suppose that $G$ is $p$-adic analytic group without $p$-torsion, the $\Lambda(G)$-rank of a finitely generated $\Lambda(G)$-module $\mathcal{X}$, is defined by

$$\text{rank}_{\Lambda(G)}(\mathcal{X}) := \text{dim}_{Q(G)}(Q(G) \otimes_{\Lambda(G)} \mathcal{X}).$$

**Theorem 2.10.** ([Ne3, proposition 1.1 and theorem 1.4]) Let $\mathcal{G}$ be a pro-$p$-group with finite presentation and cohomological dimension at most 2. Let $\mathcal{H}$ be a normal closed subgroup of $\mathcal{G}$ and $G := \mathcal{G}/\mathcal{H}$ is a $p$-adic analytic group without $p$-torsion. Put $\mathcal{X} = \mathcal{H}^{ab}$. Then $\mathcal{X}$ and $H_2(\mathcal{H}, \mathbb{Z}_p)$ are finitely generated under $\Lambda(G)$ and

$$\text{rank}_{\Lambda(G)}(\mathcal{X}) = -\chi(\mathcal{G}) + \delta_{G,1} + \text{rank}_{\Lambda(G)}(H_2(\mathcal{H}, \mathbb{Z}_p)),$$

where $\chi(\mathcal{G}) = \sum_{i \geq 0} \text{dim}_{\mathbb{Q}} H_i(\mathcal{G}, \mathbb{F}_p)$ is the Euler-Poincaré characteristic of $G$ and $\delta_{G,1} = 1$ if $G$ is trivial and zero otherwise.

**Remark 2.11.** A more general result can be found in the paper [1] of Howson.

2.3. **On the torsion of some Iwasawa modules.** Let $\Sigma$ be a finite set of places of $K$ containing $S_p$. Let $K_\Sigma$ be the maximal $\Sigma$-ramified pro-$p$-extension of $K$. Let denote $\mathfrak{X}_\Sigma(K) := \text{Gal}(K_\Sigma/K)^{ab}$.

We denote $\Gamma$ the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of $K$.

By definition we have the following exact sequence:

$$0 \rightarrow \text{Tor}_{\Lambda(\Gamma)} \mathfrak{X}_\Sigma(K_\infty) \rightarrow \mathfrak{X}_\Sigma(K_\infty) \rightarrow fr_{\Lambda(\Gamma)} \mathfrak{X}_\Sigma(K_\infty) \rightarrow 0$$

**Theorem 2.12.** ([Iw, section 2]) Suppose that $K_\infty$ satisfy the weak Leopoldt conjecture then we have:

1. $\text{rang}_{\Lambda(\Gamma)} \mathfrak{X}_\Sigma(K_\infty) = r_2$
2. $\mathfrak{X}_\Sigma(K_\infty)$ has no non trivial finite sub-module.
We are going to study the relation between $\text{Tor}_{\Lambda(\Gamma)}(\mathfrak{X}_\Sigma(K_\infty))$ which is the projective limit of $\text{Tor}_{\mathbb{Z}_p}(\mathfrak{X}_\Sigma(K_n))$.

The next theorem is necessary to prove some results in the following.

**Theorem 2.13.** ([Ng1, Proposition 3.1] or [Ng2, Proposition 3.1]) Let $K_\infty/K$ be the cyclotomic $\mathbb{Z}_p$-extension of $K$. Suppose that the Leopoldt conjecture holds for the fields $K_n$ and the prime $p$ for any $n \geq 0$. Then

$$\text{Tor}_{\Lambda(\Gamma)}(\mathfrak{X}_\Sigma(K_\infty)) \cong \lim_{\leftarrow} \text{Tor}_{\mathbb{Z}_p}\mathfrak{X}_\Sigma(K_n).$$

**Proof.** Using the structure of $\Lambda(\Gamma)$-modules ([NSW, theorem 5.1.10]), we have the following exact sequences:

$$0 \longrightarrow \text{Tor}_{\Lambda(\Gamma)}\mathfrak{X}_\Sigma(K_\infty) \longrightarrow \mathfrak{X}_\Sigma(K_\infty) \longrightarrow \text{fr}_{\Lambda(\Gamma)}\mathfrak{X}_\Sigma(K_\infty) \longrightarrow 0$$

and

$$0 \longrightarrow \text{fr}_{\Lambda(\Gamma)}(\mathfrak{X}_\Sigma(K_\infty)) \longrightarrow \Lambda(\Gamma)^{r_2} \longrightarrow F \longrightarrow 0$$

where $F$ is a finite $\Lambda(\Gamma)$-module.

By Snake lemma we obtain the following:

$$0 \longrightarrow (\text{Tor}_{\Lambda(\Gamma)}\mathfrak{X}_\Sigma(K_\infty))_{\Gamma_n} \longrightarrow (\mathfrak{X}_\Sigma(K_\infty))_{\Gamma_n} \longrightarrow (\text{fr}_{\Lambda(\Gamma)}\mathfrak{X}_\Sigma(K_\infty))_{\Gamma_n} \longrightarrow 0,$$

and

$$0 \longrightarrow F_{\Gamma_n} \longrightarrow (\text{fr}_{\Lambda(\Gamma)}\mathfrak{X}_\Sigma(K_\infty))_{\Gamma_n} \longrightarrow (\Lambda(\Gamma)^{r_2})_{\Gamma_n} \longrightarrow F_{\Gamma_n} \longrightarrow 0,$$

it is now clear that Leopoldt’s conjecture holds for all $K_n$, $n \geq 0$ if and only if divisor of $\mathfrak{X}_\Sigma(K_\infty)$ is disjoint from all $\omega_n = \gamma^{p^n} - 1$, $n \geq 0$, where $\gamma$ is topological generator of $\Gamma$ ([Iw, §10]).

A simple calculation of $\mathbb{Z}_p$-rank (we recall that the fields $K_n$ satisfy the Leopoltd conjecture) and we have that

$$\Gamma_n \cong \mathfrak{X}_\Sigma(K_n)/(\mathfrak{X}_\Sigma(K_\infty))_{\Gamma_n},$$

which gives the exact sequence

$$0 \longrightarrow (\text{Tor}_{\Lambda(\Gamma)}\mathfrak{X}_\Sigma(K_\infty))_{\Gamma_n} \longrightarrow \text{Tor}_{\mathbb{Z}_p}((\mathfrak{X}_\Sigma(K_\infty))_{\Gamma_n}) \longrightarrow F_{\Gamma_n} \longrightarrow 0$$

furthermore since $\Gamma_n \cong \mathfrak{X}_\Sigma(K_n)/(\mathfrak{X}_\Sigma(K_\infty))_{\Gamma_n}$ we have that $\text{Tor}_{\mathbb{Z}_p}((\mathfrak{X}_\Sigma(K_\infty))_{\Gamma_n}) \cong \text{Tor}_{\mathbb{Z}_p}(\mathfrak{X}_\Sigma(K_n)).$

Passing to the projective limit gives the required result. □

The next theorem is necessary to prove some results in the following.

**Theorem 2.14.** ([NSW, Theorem 11.3.5]) Assume that the weak leopoldt conjecture holds for the $\mathbb{Z}_p$-extension $K_\infty/K$ and let $\Sigma \supset S_p$ be finite. Then there exists a canonical exact sequence of $\Lambda(\Gamma)$-modules.

$$0 \longrightarrow \bigoplus_{v \in \Sigma \setminus S_p} \text{Ind}_{\Gamma_v}^{\Gamma}(T(K_v(p)/K_v)_{G(K_\infty)_v}) \longrightarrow \mathfrak{X}_\Sigma(K_\infty) \longrightarrow \mathfrak{X}_{S_p}(K_\infty) \longrightarrow 0.$$

In particular, there is an exact sequence of $\Lambda(\Gamma)$-torsion modules
\[0 \rightarrow \bigoplus_{v \in \Sigma} S_p \text{ Ind}_{\Omega}^{\Gamma_v} \left( T(K_v(p)/K_v)G(k_\infty)_v \right) \rightarrow \text{Tor}_{\Lambda(\Gamma)}(\mathcal{X}_\Sigma(K_\infty)) \rightarrow \text{Tor}_{\Lambda(\Gamma)}(\mathcal{X}_{S_p}(K_\infty)) \rightarrow 0 ,\]

where \( K_v(p) \) is the maximal pro-\( p \)-extension of \( K_v \), \( T(K_v(p)/K_v) \) is the inertia subgroup, \( \Gamma_v \) is the decomposition subgroup of \( \Gamma \) at \( v \) and \( G(k_\infty)_v \) is the absolute Galois group of \( (K_\infty)_v \).

### 2.4. \( p \)-rational fields.

Number fields \( K \) such that \( H^2(G_{S_p}(K), \mathbb{Z}/p\mathbb{Z}) = 0 \) are called \( p \)-rational fields. In particular, a number field \( K \) is \( p \)-rational precisely when the Galois group \( G_{S_p}(F) \) of the maximal pro-\( p \)-extension of \( F \) which is unramified outside \( p \) is pro-\( p \)-free (with rank \( 1 + r_2 \), \( r_2 \) being the number of complex primes of \( K \)). They are first introduced in \([MN]\) to construct non-abelian extensions of \( \mathbb{Q} \) satisfying the Leopoldt conjecture. Recently, R.Greenberg \([Gr\alpha]\) used \( p \)-rational number fields for the construction (in a non geometric manner) of \( p \)-adic representations with open image in \( \text{GL}_n(\mathbb{Z}_p) \), \( n \geq 3 \), of the absolute Galois group \( G_{\mathbb{Q}} \).

**Proposition 2.15.** \([MN]\) The number field \( K \) is said to be \( p \)-rational if the following equivalent conditions are satisfied:

1. \( K \) satisfies Leopoldt’s conjecture and \( G_{S_p}^{ab}(K) \) is torsion-free as a \( \mathbb{Z}_p \)-module.
2. \( \left\{ \alpha \in K^\times \mid \alpha \mathcal{O}_K = a^p \text{ for some fractional ideal } a \right\} = (K^\times)^p \)
   \( \text{and } \delta(K) = \sum_{v \in S_p} \delta(K_v) \).
   Where for a field \( F \), \( \delta(F) \) is 1 if \( \mu_p \subset F \) and 0 otherwise.

In the case where \( K \) is totally real, it is \( p \)-rational if and only if \( G_{S_p}^{ab}(K) \cong \mathbb{Z}_p \cong G_{S_p}(K) \).

**Examples 2.16.** If \( p = 3 \) we additionally assume that it is unramified in the first two statements:

1. If \( K \) is an imaginary quadratic field such that \( p \nmid h_K \), then \( K \) is \( p \)-rational.
2. If \( K \) is a real quadratic field, then it is \( p \)-rational if and only if \( p \nmid h_K \) and the fundamental unit is not a \( p \)-power in \( K_v \) for all \( v|p \).
3. If the prime \( p \geq 3 \) is regular then the field \( \mathbb{Q}(\mu_p) \) is \( p \)-rational.

Let \( \Delta \) be an abelian group and let \( \hat{\Delta} \) denote the set of the \( p \)-adic irreducible characters of \( \Delta \). Let \( \mathcal{O} \) be the ring of integers of a finite extension of \( \mathbb{Q}_p \), which contains the values of all \( \chi \in \hat{\Delta} \). For any \( \mathbb{Z}_p[\Delta] \)-module \( M \) we define the \( \chi \)-quotient \( M_\chi \) of \( M \) by

\[ M_\chi = M \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}(\chi) \cong (M \otimes_{\mathbb{Z}_p} \mathcal{O}(\chi^{-1}))_\Delta \]

where \( \mathcal{O}(\chi) \) denote the ring \( \mathcal{O} \) on which \( \Delta \) acts via \( \chi \).

**Proposition 2.17.** Let \( M \) be a finite \( \mathbb{Z}_p[\Delta] \)-module. If for all \( p \)-adic irreducible characters \( \chi \), \( M_\chi = 0 \) then \( M = 0 \).

We have the following interesting proposition which is easy to prove in the semi-simple case.

**Proposition 2.18.** Let \( p \) be an odd prime number and \( K \) a finite abelian extension of \( \mathbb{Q} \) such that \( p \nmid [K : \mathbb{Q}] \), then the field \( K \) is \( p \)-rational if and only if every cyclic extension of \( \mathbb{Q} \) contained in \( K \) is \( p \)-rational.

**Proof.** It suffices to show the second implication, i.e., if every cyclic sub-extension is \( p \)-rational then \( K \) is \( p \)-rational.

Let \( \Delta \) be the Galois group of \( K \) over \( \mathbb{Q} \) and denote \( \hat{\Delta} \) the corresponding group of irreducible character. Let \( \chi \) be such an irreducible character and let \( K_\chi \) be its fixed field, if \( \Delta_\chi = \)
Gal(K/K_\chi) and using the definition of the \chi-quotient, we obtain
\[ H^2(G_{S_p}(K)\frac{\mathbb{Z}}{p\mathbb{Z}}) = (H^2(G_{S_p}(K),\mathbb{Z}/p\mathbb{Z}) \otimes O(\chi^{-1}))_\Delta = ((H^2(G_{S_p}(K),\mathbb{Z}/p\mathbb{Z}) \otimes O(\chi^{-1}))_\Delta)_{\Delta/\Delta} \]
Since K_\chi is p-rational, \[ H^2(G_{S_p}(K_\chi)\frac{\mathbb{Z}}{p\mathbb{Z}}) = 0. \] It follows that \[ H^2(G_{S_p}(K),\mathbb{Z}/p\mathbb{Z}) = 0. \]

Proposition 2.17 shows that
\[ H^2(G_{S_p}(K),\mathbb{Z}/p\mathbb{Z}) = 0. \]
This gives the desired implication since the set of cyclic extensions is one-to-one with the set of irreducible characters.

For example if p is an odd prime, then \( \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \) is p-rational if and only if the fields \( \mathbb{Q}(\sqrt{d_1}), \mathbb{Q}(\sqrt{d_2}) \) and \( \mathbb{Q}(\sqrt{d_1d_2}) \) are p-rationals.

3. On the structure of the \( \Lambda(G_{\infty,S}) \)-module \( \mathcal{X} \)

Let K be a number field and p an odd prime number. Let S be a finite set of primes of K which is disjoint from \( S_p \) and let \( \Sigma = S \cup S_p \). Recall that \( \mathcal{X} \) is the abelianizer of the group \( \mathcal{H} \), where \( \mathcal{H} = \text{Gal}(K_{\Sigma}/K_{\infty,S}). \)

3.1. The structure of the \( \Lambda(G_{\infty,S}) \)-module \( \mathcal{X} \). We are going to use local p-adic class field theory in the following, for this we introduce some notations.
Let \( v \) be a place of K and let \( \mathcal{U}_v \) be the completion of the p-adic units of \( O_v \) of the field \( K_v \) defined as the projective limit \( \lim_n O_v^\times/O_v^{p^n} \). The locally cyclotomic units \( \tilde{U}_v \) of \( K_v \) are the units of \( K_v \) which are norms in the cyclotomic \( \mathbb{Z}_p \)-extension \( K_{v,\infty} \) of \( K_v \).
In particular if \( v \nmid p \), we have \( \tilde{U}_v = U_v \). By local class field theory the group \( \tilde{U}_v \) correspond to the compositum of the cyclotomic \( \mathbb{Z}_p \)-extension of \( K_v \) and the unramified \( \mathbb{Z}_p \)-extension \( K_{v,un} \) of \( K_v \).
If \( T \) is a finite set of places of \( K \), set \( \mathcal{U}_T = \bigcup_{\nu \in T} \mathcal{U}_\nu \), \( \tilde{U}_T = \bigcup_{\nu \in T} \tilde{U}_\nu \) and \( U_p = \prod_{v \in S_p} \mathcal{U}_v \). We are interested in the freeness of the \( \Lambda(G_{\infty,S}) \)-module \( \mathcal{X} \).

Theorem 3.1. Let K be a p-rational field not containing the \( p^{th} \)-roots of unity and satisfying the following conditions:

1. The group \( \mathcal{E}_S \) is a direct summand of \( \mathcal{U}_p \).
2. \( s = 1 \).

Then the \( \Lambda(G_{\infty,S}) \)-module \( \mathcal{X} \) is free of rank \( r_2 \).

Proof. By class field theory and thanks to the weak Leopoldt conjecture (which is true for the cyclotomic \( \mathbb{Z}_p \)-extension) one has the following exact sequence (theorem 2.14)

\[ 0 \rightarrow \bigoplus_{v \in \Sigma \setminus S_p} \text{Ind}_{\Gamma_v}^\mathcal{H}(T(K_v(p)/K_v)G_{(K_{\infty,v}))} \rightarrow \text{Tor}_{\Lambda(\Gamma)}(\mathcal{X}_{\Sigma}(K_{\infty})) \rightarrow \text{Tor}_{\Lambda(\Gamma)}(\mathcal{X}_{S_p}(K_{\infty})) \rightarrow 0, \]

since \( \mu_p \subset K_v \) for \( v \in S \) and \( v \) is finitely decomposed in \( K_{\infty}/K \), \( T(K_v(p)/K_v)G_{(K_{\infty,v})} \) is isomorphic to \( \mathbb{Z}_p(1) \). Hence we obtain the exact sequence,

\[ 0 \rightarrow \mathcal{W} \rightarrow \text{Tor}_{\Lambda(\Gamma)}(\mathcal{X}_{\Sigma}(K_{\infty})) \rightarrow \text{Tor}_{\Lambda(\Gamma)}(\mathcal{X}_{S_p}(K_{\infty})) \rightarrow 0 \]
where $\mathcal{W}$ is the direct sum of $s$ copy of $\mathbb{Z}_p(1)$.

Since $K$ is $p$-rational, hence $K_n$ is $p$-rational, by theorem 2.4.3 we see that $\text{Tor}_{\Lambda(\Gamma)}(\mathcal{X}_{S_p}(K_\infty))$ is trivial, we have that $\text{Tor}_{\Lambda(\Gamma)}(\mathcal{X}_{\Sigma}(K_\infty)) \simeq \mathcal{W} \simeq \mathcal{X}_{S}(K_\infty)$, and hence $\mathcal{X}_{S}(K_\infty) \simeq \mathbb{Z}_p$, we can see that $G_{\infty,S} \simeq \mathbb{Z}_p \times \mathbb{Z}_p$; hence the cohomology group $H_2(G_{\infty,S}, \mathbb{Z}_p)$ is trivial. The $\mathbb{Z}_p$-freeness of $\mathcal{X}_{S}(K_\infty)$ gives that the group $G_S(K_\infty)$ is also $\mathbb{Z}_p$-free of rank 1, and hence

$$cd(G_{\infty,S}) \leq cd(G_S(K_\infty)) + cd(\Gamma) = 2,$$

the cohomological dimension of $G_{\infty,S}$ is at most 2. It is known that $G_\Sigma(K)$ is of cohomological dimension at most 2.

Let prove that $\mathcal{X}$ is finitely generated as $\Lambda(G_{\infty,S})$-module.

Since $G_{\infty,S} \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $\Lambda(G_{\infty,S})$ is Noetherian, then $G_{\infty,S}$ is of finite presentation. The Hochschild-Serre spectral sequence applied to the short exact sequence

$$1 \to \mathcal{H} \to G_\Sigma(K) \to G_{\infty,S} \to 1$$

shows that

$$\cdots H^1(G_\Sigma(K), \mathbb{F}_p) \to H^1(\mathcal{H}, \mathbb{F}_p)^{G_{\infty,S}} \to H^2(G_{\infty,S}, \mathbb{F}_p) \to \cdots$$

Since $H^1(G_\Sigma(K), \mathbb{F}_p)$ and $H^2(G_{\infty,S}, \mathbb{F}_p)$ are finite, $\mathcal{X}_{G_{\infty,S}}/p = (H^1(\mathcal{H}, \mathbb{F}_p)^{G_{\infty,S}})^*$ is also finite, where $*$ is the Pontrjagin dual. Then by Nakayama’s lemma, one has $\mathcal{X}$ is finitely generated as $\Lambda(G_{\infty,S})$-module.

The field $K$ satisfy the Leopoldt conjecture, then the groups $H^2(G_\Sigma(K), \mathbb{Q}_p/\mathbb{Z}_p)$ and $H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)$ are trivial (see [Ng3]). By the theorem 1.1, the $\Lambda(G_{\infty,S})$-module $\mathcal{X}$ is free if the morphism of restriction $\text{Tor}_{\mathbb{Z}_p}G_\Sigma(K)^{ab} \to G_{\infty,S}^{ab}$ is injective.

We have the following exact sequence

$$1 \to \mathcal{K}^\times \mathcal{U}_S^{\times} \to \mathcal{J}_S^{\times} \to \mathcal{J}_S \to 1,$$

where $\mathcal{K}^\times = \lim_{\leftarrow n} K^\times/(K^\times)^{p^{n}}$ is the $p$-adic compactified of $K^\times$ and $\mathcal{J}_S$ is the $p$-adic compactified of the group of ideles of $K$.

By the class field $p$-adic correspondence we have the following exact sequence

$$1 \to \text{Gal}(K_{\Sigma}/K_{\infty,S})^{ab} \to \text{Gal}(K_{\Sigma}/K)^{ab} \to \text{Gal}(K_{\infty,S}/K)^{ab} \to 1,$$

and note that

$$\frac{\mathcal{K}^\times \mathcal{U}_S^{\times}}{\mathcal{K}^\times \mathcal{U}_\Sigma} \simeq \frac{\prod_{v \in S_p} \mathcal{U}_v}{\mathcal{I}_p(\mathcal{E}_S) \cap \prod_{v \in S_p} \mathcal{U}_v}.$$

By hypotheses $\mathcal{I}_p(\mathcal{E}_S)$ is a direct summand of $\mathcal{U}_p$, and the group $\mathcal{U}_p$ is a free $\mathbb{Z}_p$-module of rank $[K : \mathbb{Q}]$, then we have also

$$\text{Tor}_{\mathbb{Z}_p} \left( \frac{\prod_{v \in S_p} \mathcal{U}_v}{\mathcal{I}_p(\mathcal{E}_S) \cap \prod_{v \in S_p} \mathcal{U}_v} \right) = \{1\}.$$

\[\square\]

**Proposition 3.2.** Let $K$ be a $p$-rational field. Suppose that $s = 1$. Then the $\Lambda(G_{\infty,S})$-module $\mathcal{X}$ is free of rank $r_2$ if and only if

$$\text{Tor}_{\mathbb{Z}_p} \left( \frac{\prod_{v \in S_p} \mathcal{U}_v}{\mathcal{I}_p(\mathcal{E}_S) \cap \prod_{v \in S_p} \mathcal{U}_v} \right) = \{1\}.$$

**Proof.** Using proposition 2.7 and the proof of theorem 3.1 one obtains the desired result. \[\square\]
3.2. Examples. We consider the following field $K = \mathbb{Q}(\sqrt{pq}, \sqrt{-d})$ where $p$ and $q$ are two distinct odd primes such that $p > 3$ and $q \equiv -1 \pmod{p}$, $d$ be a positive squarefree integer such that $p \nmid d$ and $q \nmid d$. Suppose that $-d$ is not a quadratic residue modulo $p$ and $q$. Since $p$ and $q$ are ramified in the field $K^+ = \mathbb{Q}(\sqrt{pq})$ there is one prime $\mathfrak{p}$ above $p$ and one prime $\mathfrak{q}$ above $q$. Let $\epsilon$ be the fundamental unit of $K^+$. Let $S_p = \{\mathfrak{p}\}$, $S = \{\mathfrak{q}\}$ and denote $\Sigma = S \cup S_p$, $L_1 = \mathbb{Q}(\sqrt{-dpq})$ and $L_2 = \mathbb{Q}(\sqrt{-d})$.

Proposition 3.3. The field $K$ is $p$-rational if and only if the following conditions are satisfied:

- the class numbers of $K^+$, $L_1$ and $L_2$ are prime to $p$,
- the fundamental unit $\epsilon$ is not a $p$-power in the completion $K_p^+$.

We have the following theorem on the freeness of $\mathcal{X}$.

Theorem 3.4. Suppose that $K$ is a $p$-rational field. Suppose that $\nu_p(\mathcal{E}_S)$ is a direct summand of $\mathcal{U}_p$ and let $s = 1$. Then the $\Lambda(G_{\infty,S})$-module $\mathcal{X}$ is free of rank 2:

\[ \mathcal{X} \simeq \mathbb{Z}_p[[\mathbb{Z}_p \times \mathbb{Z}_p]] \oplus \mathbb{Z}_p[[\mathbb{Z}_p \times \mathbb{Z}_p]]. \]

Example 3.5. Using Pari-GP ([Par]) we verified the following example where $K = \mathbb{Q}(\sqrt{91}, \sqrt{-2})$, $p = 7$, $q = 13$ and $d = 2$. We denote $K^+$ be the maximal real subfield of $K$, i.e., $K^+ = \mathbb{Q}(\sqrt{91})$. We have $S_p = \{\mathfrak{p}\}$, $S = \{\mathfrak{q}\}$ and $\Sigma = \{\mathfrak{p}, \mathfrak{q}\}$. Let $\mathfrak{p} | p$ and $\mathfrak{q} | q$.

Let $A_n$ (resp. $A_n^+$) be the $\ell$-part of the class group of $K_n$ (resp. $K_n^+$) and $A_{n,S}$ (resp. $A_{n,S}^+$) the $\ell$-part of the class group of $K_n$ (resp. $K_n^+$) of ray $S$. For a large enough integer $n$, we have the following (see [Gra1])

\[ \#A_{n,S} \leq \#A_n \prod_{v \in S(K_n)} p_{\text{ram}} + o(1), \]

where $\alpha_S = \sum_{v \in S(K^+)} \alpha_v$ and $\alpha_v$ is 0 or 1 such that $\alpha_v = 1$ if the following is satisfied

1. $|U_{K^+}| = 1$,
2. $|U_{K_v}| \neq 1$.

In this case we have that $\alpha_S = 1$.

The fundamental unit of the field $K^+$ is not a $p$-power in $K_p$ and $p$ is prime to the class number of $K^+$ and the class number of $L$. Then by proposition 3.3 the field $K$ is $p$-rational. Since $A_0 = \{1\}$ and $\alpha_S = 1$, we have that the invariant $\lambda_S$ of the module $\mathcal{X}_S(K_{\infty})$ equals 1, hence we obtain that $d_{\ell}(A_{0,S}) = 1$, and using Nakayama lemma we deduce the following isomorphism

\[ \mathcal{X}_S(K_{\infty}) \simeq \mathbb{Z}_p \simeq \mathcal{W}, \]

where $\mathcal{W}$ is a direct sum of $s$ copy of $\mathbb{Z}_p(1)$ and $s$ is the number of places of $K_{\infty}$ above places in $S$ such that $K_{\infty,v}$ contain $\mu_p$, in this case $s = 1$.

We have that $K_{\mathfrak{q}}$ contains the $7$-th roots of unity. It remains to prove that $\tau_7(\mathcal{E}_S)$ is a direct summand of $\mathcal{U}_p$. We have $K = \mathbb{Q}(\theta)$, where $\theta$ satisfy $\theta^4 - 178\theta^2 + 8649 = 0$. In this case $E_K = \langle \pm 1, \varepsilon \rangle$, with $\varepsilon = 55/629^3-14905/629+1574$. We verify that $v_p(\varepsilon^2-1) = v_q(\varepsilon^2-1) = 1$, we have $\tau_7(\mathcal{E}_S) = \tau_7(\langle \varepsilon^2 \rangle)$ is a direct summand of $\mathcal{U}_p$.

Then the $\Lambda(G_{\infty,S})$-module $\mathcal{X}$ is free of rank 2:

\[ \mathcal{X} \simeq \mathbb{Z}_7[[\mathbb{Z}_7 \times \mathbb{Z}_7]] \oplus \mathbb{Z}_7[[\mathbb{Z}_7 \times \mathbb{Z}_7]]. \]

Using corollary 3.2 of [Gra2] we give an algorithm for testing the $p$-rationality of a family of number fields of the form $K = \mathbb{Q}(\sqrt{pq}, \sqrt{-2})$, with $p$ and $q$ given as in the example 3.2.

The algorithm gives also the primes which satisfy the conditions of example 3.3. In the following table we give numerical evidence for the existence of fields $K$ which satisfy the conditions of theorem 3.3.
| $p$ | \{79,109,239,359,389,439,599,719,829,1039,1319,1429,1439,1879,2239,2269,2309,2399,4591,5279,5309,5879,6079,6199,6359,6599,6679,6829,6959,7109,7559,7759,7829,8389,8429,8629,8719,8999,9199,9319,9479,9679,9719,9839,9949\} |
| --- | --- |
| 7 | \{13,167,181,223,461,503,727,797,853,1021,1063,1231,1399,1511,1567,1637,1693,1847,1973,2029,2141,2351,2477,2687,3037,3527,3541,3709,3821,3863,3877,3919,4157,4423,4493,4549,4591,5039,5303,5501,5557,5879,6047,6173,6229,6271,6397,6719,6733,7013,7237,7349,7559,7573,7727,7853,7951,8287,8861,9239,9421,9463,9533,9743\} |
| 13 | \{103,181,311,389,701,727,1039,1117,1637,1663,1871,1949,2053,2287,3119,3821,4133,4159,4679,4783,5303,5407,5693,5927,6343,6551,6863,6967,7487,7591,7669,8111,8293,8423,8839,9151,9463\} |
| 23 | \{367,919,1103,1471,2069,2207,2437,2621,3541,3863,4093,4231,4783,4967,5197,5381,5519,5749,6301,6991,7589,7727,8647,8693,9831,9199,9613\} |
| 29 | \{173,463,2029,2087,2551,4639,6263,6959,9221,9511,9743\} |
| 31 | \{61,557,743,991,1301,1487,1549,2293,3037,3533,3719,3967,4463,5021,6199,7253,7687,8431,8741,9733\} |
| 37 | \{887,2663,3847,4957,5623,6733,7103,7621,8287,9397,9767\} |
| 47 | \{751,1597,1879,1973,3853,4229,5639,7237,8647,8741\} |
| 53 | \{2543,2861,3391,4133,4663,5087,6359,7207\} |
| 61 | \{487,853,1951,2927,4391,6709,8783\} |
| 71 | \{709,1277,3407,5821,6247,6389,7951,8093\} |
| 79 | \{157,631,2053,4423,7109,7583,7741,9479\} |
| 101 | \{2423,7069,8887\} |
| 103 | \{823,2677,4943,7621,9887\} |
| 109 | \{653,5231,8501,8719\} |
| 127 | \{3301,5333,9397\} |
| 149 | \{2383,7151\} |
| 151 | \{3623,4831,7247,7549\} |
| 157 | \{941,3767,5023\} |
| 167 | \{1669,2671,4007,6679,7013\} |
| 173 | \{2767\} |
| 181 | \{1447,5791\} |
| 191 | \{4583,7639\} |
| 197 | \{1181,7879\} |
| 199 | \{397,3581,6367,9551,9949\} |
| 223 | \{1783,4013,5351\} |
| 229 | \{1373,1831\} |
| 239 | \{2389,3823\} |
| 263 | \{4733,6311,8941\} |
| 271 | \{541,4877\} |
| 277 | \{3877,8863\} |
| 311 | \{3109\} |
| 317 | \{1901,7607\} |
| 349 | \{2791\} |
| 359 | \{5743\} |
| 367 | \{733,8807\} |
| 373 | \{2237,8951\} |
| 431 | \{7757\} |
References

[B] A. Brumer. Pseudocompact algebras, profinite groups and class formations. Journal of Algebra, 4(3), 442-470 (1966).

[DDMS] J.D. Dixon, M.P.F. Du Sautoy, A. Mann, D. Segal, Analytic pro-p-groups, Cambridge studies in advanced mathematics 61, Cambridge University Press, 1999.

[Gra1] G. Gras, Class Field Theory, SMM, Springer, (2003).

[Gra2] G. Gras, A program to test the p-rationality of any number field. arXiv preprint arxiv: 1709.06388v1 (2017).

[Gre] R. Greenberg, Galois representations with open image, Ann. Math. Québec, 40.1: 83-119 (2016).

[GJ] G. Gras and J. F. Jaulent. Sur les corps de nombres réguliers. Mathematische Zeitschrift, 202(3), 343-365(1989).

[GW] K.R. Goodearl and R.B. Warfield, An Introduction to Noncommutative Noetherian Rings, LMS Student texts, vol. 16, Cambridge University Press, 1989.

[H] S. Howson, Euler Characteristics as invariants of Iwasawa modules, Proc. London Math. Society 85 (2002) no. 3, 634-658.

[It] T. Ihara, On tamely ramified Iwasawa modules for the cyclotomic $\mathbb{Z}_p$-extension of abelian fields. Osaka Journal of Mathematics, 51, 513-536 (2014).

[Iw] K. Iwasawa, On $\mathbb{Z}_p$-extensions of algebraic number fields, Annals of Mathematics, Second Series, Vol 98, No.2(Sep 1973), pp.326-246.

[J] J.-F. Jaulent, Théorie $l$-adique globale du corps de classes, J. Théor. Nombres Bordeaux 10 (1998), no. 2, 355-397.

[L] M. Lazard, Groupes analytiques $p$-adiques, IHES Publ. Math. 26 (1965), 389-603.

[M] C. Maire, Cohomology of Number Fields and Analytic pro-p-groups. Moscow mathematical journal, vol. 10, no 2, p. 399-414 (2010).

[MO] Y. Mizusawa, M. Ozaki, On tame pro-$p$ Galois groups over basic $\mathbb{Z}_p$-extensions, Math. Z. 273, no. 3-4, 1161-1173 (2013).

[Ne] A. Neumann, Completed group Algebras without zero divisors, Arch. Math. 51 (1988), 496-499.

[Ng1] T. Nguyen Quang Do, T. Nguyen Quang Do, Sur la cohomologie de certains modules galoisiens $p$-ramifiés, dans «Théorie des Nombres», Laval, ed. J-M. De Koninck et C. Levesque, W. de Gruyter, 740-753 (1989).

[Ng2] T. Nguyen Quang Do, Sur la $\mathbb{Z}_p$-torsion de certains modules Galoisiens, Ann. Ins. Fourier 36, 27-46 (1986).

[Ng3] T. Nguyen Quang Do, Formations de classes et modules d’Iwasawa, Number theory, Noordwijkerhout 1983, 167-185, Lecture Notes in Math., 1068, Springer, Berlin, 1984.

[NSW] J. Neukirch, A. Schmidt and K. Wingberg, Cohomology of Number Fields, Second edition, Grundlehren der Math. Wissenschaften 323, Springer-Verlag, Berlin, (2008).

[Pari] The PARI Group-Bordeaux, Pari/gp, version 2.10.0, 2017, available from [http://pari.math.u-bordeaux.fr/]

[Sa1] L. Salle, On maximal tamely ramified pro-2-extensions over the cyclotomic $\mathbb{Z}_2$-extension of an imaginary quadratic field, Osaka J. Math. 47, 921-942 (2010).

[Sa2] L. Salle, Sur les pro-$p$-extensions à ramification restreinte au-dessus de la $\mathbb{Z}_p$-extension cyclotomique d’un corps d’ nombres, J. Théor. Nombres Bordeaux 20, no. 2, 485-523(2008).

[Se] J. P. Serre, Cohomology Galoisienne. In: Lectures Notes in Mathematics, vol. 5. Berlin (1994).

(1) ACSA Laboratory, Department of Mathematics, Faculty of Sciences, Mohammed First University, Oujda, Morocco.
E-mail address: a.elhabibi@ump.ac.ma

(2) ACSA Laboratory, Department of Mathematics, Faculty of Sciences, Mohammed First University, Oujda, Morocco.
E-mail address: ziane12001@yahoo.fr