Estimates for parameters and characteristics of the confining SU(3)-gluonic field in the ground state of toponium: Relativistic and nonrelativistic approaches

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Abstract

The confinement mechanism earlier proposed by author is applied to describe the (possible) ground state of toponium ηt. For this aim the nonperturbative consistent approach is elaborated in both relativistic and nonrelativistic cases. The study entails estimates for parameters of the confining SU(3)-gluonic field in the above quarkonium, those estimates being also consistent with possible width of decay ηt → 2γ. The corresponding estimates of the gluon concentrations, electric and magnetic colour field strengths are also adduced for the mentioned field at the scales of toponium.

Key words: Quantum chromodynamics, Confinement, Heavy quarkonia
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1 Introduction

In Refs. [1–3] for the Dirac-Yang-Mills system derived from QCD-Lagrangian there was found and explored an unique family of compatible nonperturbative solutions which could pretend to describing confinement of two quarks. Applications of the family to description of both the heavy quarkonia spectra [4] and a number of properties of pions, kaons and η-meson [5–7] showed that the confinement mechanism is qualitatively the same for both light mesons and heavy quarkonia. At the moment it can be decribed in the following form.

Two main physical reasons for linear confinement in the mechanism under discussion are the following ones. The first one is that gluon exchange between quarks is realized with the propagator different from the photon-like one and
existence and form of such a propagator is-direct consequence of the unique
confining nonperturbative solutions of the Yang-Mills equations [2,3]. The sec-
ond reason is that, owing to the structure of mentioned propagator, quarks
mainly emit and interchange with the soft gluons so the gluon condensate (a
classical gluon field) between quarks basically consists of soft gluons (for more
details see Refs. [2,3]) but, because of that any gluon also emits gluons (still
softer), the corresponding gluon concentrations rapidly become huge and form
the linear confining magnetic colour field of enormous strengths which leads
to confinement of quarks. This is by virtue of the fact that just magnetic part
of the mentioned propagator is responsible for larger portion of gluon con-
centrations at large distances since the magnetic part has stronger infrared
singularities than the electric one. Under the circumstances physically nonlin-
erarity of the Yang-Mills equations effectively vanishes so the latter possess the
unique nonperturbative confining solutions of the Abelian-like form (with the
values in Cartan subalgebra of SU(3)-Lie algebra) [2,3] that describe the gluon
condensate under consideration. Moreover, since the overwhelming majority
of gluons is soft they cannot leave hadron (meson) until some gluon obtains
additional energy (due to an external reason) to rush out. So we deal with
confinement of gluons as well.

The approach under discussion equips us with the explicit wave functions for
every two quarks (meson or quarkonium). The wave functions are parametrized
by a set of real constants $a_j, b_j, B_j$ describing the mentioned nonperturbative
confining gluon field (the gluon condensate) and they are nonperturbative mod-
ulo square integrable solutions of the Dirac equation in the above confining
SU(3)-field and also depend on $\mu_0$, the reduced mass of the current masses of
quarks forming meson. It is clear that under the given approach just constants
$a_j, b_j, B_j, \mu_0$ determine all properties of any meson (quarkonium), i.e., the
approach directly appeals to quark and gluonic degrees of freedom as should be
according to the first principles of QCD. Also it is clear that the mentioned
constants should be extracted from experimental data.

Such a program has been to a certain extent advanced in Refs. [4–7]. The
aim of the present paper is to return to heavy quarkonia physics to obtain
estimates for $a_j, b_j, B_j$ in the possible ground state of the heaviest quarkonium –
toponium (in what follows we denote it as $\eta_t$), where so far little is still known
about experimental spectroscopy of the system.

Under the situation a certain motivation of studying toponium is adduced
in Section 2. Section 3 contains main relations underlying both relativistic
and nonrelativistic descriptions of any quarkonia in our approach. Section
4 is devoted to computing electric form factor, the root-mean-square radius
$< r >$ and magnetic moment of the quarkonium under consideration in an
explicit analytic form. Section 5 gives an independent estimate for $< r >$ which
is used in Section 6 for obtaining estimates for parameters of the confining
SU(3)-gluonic field for the toponium ground state \( \eta_t \) in both relativistic and nonrelativistic cases. Further in Section 7 we show that estimates of Section 6 can also be consistent with possible width of two-photon decay \( \eta_t \to 2\gamma \). Section 8 employs the obtained parameters of SU(3)-gluonic field to get the corresponding estimates for such characteristics of the mentioned field as gluon concentrations, electric and magnetic colour field strengths at the scales of \( \eta_t \) while Section 9 is devoted to discussion and concluding remarks.

At last Appendices A and B contain the detailed description of main building blocks for meson wave functions in the approach under discussion, respectively: eigenspinors of the Euclidean Dirac operator on two-sphere \( S^2 \) and radial parts for the modulo square integrable solutions of Dirac equation in the confining SU(3)-Yang-Mills field while Appendix C supplements Appendices A and B with proof of the fact that the so-called nonrelativistic confining potentials do not obey the Maxwell or SU(3)-Yang-Mills equations.

Further we shall deal with the metric of the flat Minkowski spacetime \( M \) that we write down (using the ordinary set of local spherical coordinates \( r, \vartheta, \varphi \) for the spatial part) in the form

\[
d s^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu \equiv dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \tag{1}
\]

Besides, we have \(|\delta| = |\det(g_{\mu\nu})| = (r^2 \sin \vartheta)^2\) and \(0 \leq r < \infty, 0 \leq \vartheta < \pi, 0 \leq \varphi < 2\pi\).

Throughout the paper we employ the Heaviside-Lorentz system of units with \( \hbar = c = 1 \), unless explicitly stated otherwise, so the gauge coupling constant \( g \) and the strong coupling constant \( \alpha_s \) are connected by relation \( g^2/(4\pi) = \alpha_s \).

In what follows we shall denote \( L_2(F) \) the set of the modulo square integrable complex functions on any manifold \( F \) furnished with an integration measure, then \( L_n^2(F) \) will be the \( n \)-fold direct product of \( L_2(F) \) endowed with the obvious scalar product while \( \dagger \) and \( * \) stand, respectively, for Hermitian and complex conjugation. Our choice of Dirac \( \gamma \)-matrices conforms to the so-called standard representation and is the same as in Ref. [5]. At last \( \otimes \) means tensorial product of matrices and \( I_n \) is the unit \( n \times n \) matrix so that, e.g., we have

\[
I_3 \otimes \gamma^\mu = \begin{pmatrix}
\gamma^\mu & 0 & 0 \\
0 & \gamma^\mu & 0 \\
0 & 0 & \gamma^\mu
\end{pmatrix}
\]

for any Dirac \( \gamma \)-matrix \( \gamma^\mu \) and so forth.

When calculating we apply the relations \( 1 \text{ GeV}^{-1} \approx 0.1973269679 \text{ fm}, 1 \text{ s}^{-1} \approx 0.658211915 \times 10^{-24} \text{ GeV}, 1 \text{ V/m} \approx 0.230956375 \times 10^{-23} \text{ GeV}^2, 1 \text{ T} = 4\pi \times 10^{-7} \text{ H/m} \times 1 \text{ A/m} \approx 0.6925075988 \times 10^{-15} \text{ GeV}^2 \).

Finally, for the necessary estimates we shall employ the \( T_{00} \)-component (volu-
metric energy density) of the energy-momentum tensor for a SU(3)-Yang-Mills field which should be written in the chosen system of units in the form
\[ T_{\mu\nu} = -F_{\mu\alpha}^a F_{\nu\beta}^a g^{\alpha\beta} + \frac{1}{4} F_{\beta\gamma}^a F_{\alpha\delta}^a g^{\gamma\delta} g_{\mu\nu}. \] (2)

2 Motivation

Though theoretical toponium physics has been developing already during a long time (see early reviews of Refs. [8] and references therein) so far little is still known about experimental spectroscopy of the system, as we have mentioned in Section 1. According to standard model (SM) with three generations (see, e.g., Ref. [15]) main properties of top-quark are determined by summand in SM-Lagrangian of the form
\[ -g/(2\sqrt{2})(\bar{t}\gamma_{\mu}(1 + \gamma_5)V_{tb} b)W^\mu, \]
where the gauge coupling constant \( g \) is connected with the Fermi coupling constant \( G_F \) as
\[ g^2\sqrt{2} = 8G_F m_W^2 \]
\( (m_W \approx 80.403 \text{ Gev is mass of } W\text{-boson}) \) while \( V_{tb} \) is the corresponding element of the Cabibbo–Kobayashi–Maskawa mixing matrix.

In second order in \( g \) this yields the decay width \( \Gamma(t \rightarrow Wb) \sim G_F |V_{tb}|^2 m_t^3 \) so that \( \Gamma \) proves to be between \( (1.0 - 1.6) \text{ GeV} \) depending on top-quark mass \( m_t \) [15]. With its respectively short lifetime of order \( 0.5 \times 10^{-24} \text{ s} \), the top quark is expected to decay before top-flavored hadrons or \( \bar{t}t \) quarkonium bound states can form although as far back as in Ref. [9] possibilities for formation of toponium at \( m_t \sim 170 \text{ GeV} \) were discussed (see also Refs. [10] and references therein).

The above estimates, however, suppose \( |V_{tb}| \) to be of order 1. But it is known that mixings and the number of fermion generations are not fixed by SM. Under the circumstances possible existence of extra SM families may sufficiently decrease \( |V_{tb}| \) so that toponium can be formed [11]. Under this situation, one of the main signals for detecting toponium should be decay \( \eta_t \rightarrow 2\gamma \) and enough number of those decays might be observable even at LHC and its further upgrades and, of course, at possible future collider VLHC (for more details see Ref. [11]). For the sake of justice we should, however, note that present measurements still indicate that \( |V_{tb}| > 0.78 \) [15].

Also within the SM based on the two Higgs doublets formation and observation of heavy quarkonia (including toponium) might be lightened [12]. Thus, on the whole, the question of existence and observability for toponium remains open and, probably, possible discovery of toponium at future colliders might provide a connection to new or unexpected physics.

Specifically, therefore, there is an certain interest of trying to explore the toponium already now from the point of view of the above confinement mechanism which directly appeals to quark and gluonic degrees of freedom as should be
according to the first principles of QCD. Under the circumstances we shall use some previous theoretical estimates for toponium obtained from other considerations (see, e.g., Refs. [13,14] and references therein).

3 Main relations

As was mentioned above, our considerations shall be based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system (derived from QCD-Lagrangian) studied in details in Refs. [1–3]. Referring for more details to those references, let us briefly describe and specify only the relations necessary to us in the present paper.

One part of the mentioned family is presented by the unique nonperturbative confining solution of the Yang-Mills equations for \( A = A_\mu dx^\mu = A^a_\mu \lambda_a dx^\mu \) (\( \lambda_a \) are the known Gell-Mann matrices, \( \mu = t, r, \vartheta, \phi \), \( a = 1, \ldots, 8 \)) and looks as follows

\[
A_{1t} \equiv A^3_t + \frac{1}{\sqrt{3}} A^8_t = -\frac{a_1}{r} + A_1, \quad A_{2t} \equiv -A^3_t + \frac{1}{\sqrt{3}} A^8_t = -\frac{a_2}{r} + A_2,
\]

\[
A_{3t} \equiv -\frac{2}{\sqrt{3}} A^8_t = \frac{a_1 + a_2}{r} - (A_1 + A_2),
\]

\[
A_{1\varphi} \equiv A^3_\varphi + \frac{1}{\sqrt{3}} A^8_\varphi = b_1 r + B_1, \quad A_{2\varphi} \equiv -A^3_\varphi + \frac{1}{\sqrt{3}} A^8_\varphi = b_2 r + B_2,
\]

\[
A_{3\varphi} \equiv -\frac{2}{\sqrt{3}} A^8_\varphi = -(b_1 + b_2)r - (B_1 + B_2)
\]

with the real constants \( a_j, A_j, b_j, B_j \) parametrizing the family. As has been repeatedly explained in Refs. [2–5], parameters \( A_{1,2} \) of solution (3) are inessential for physics in question and we can consider \( A_1 = A_2 = 0 \). Obviously we have \( \sum_{j=1}^3 A_{jt} = \sum_{j=1}^3 A_{j\varphi} = 0 \) which reflects the fact that for any matrix \( T \) from SU(3)-Lie algebra we have \( \text{Tr} \ T = 0 \). Also, as has been repeatedly discussed by us earlier (see, e.g., Refs. [2,3]), from the above form it is clear that the solution (3) is a configuration describing the electric Coulomb-like colour field (components \( A^3_t, A^8_t \)) and the magnetic colour field linear in \( r \) (components \( A^3_\varphi, A^8_\varphi \)) and we wrote down the solution (3) in the combinations that are just needed further to insert into the Dirac equation (4).

Another part of the family is given by the unique nonperturbative modulo square integrable solutions of the Dirac equation in the confining SU(3)-field of (3) \( \Psi = (\Psi_1, \Psi_2, \Psi_3) \) with the four-dimensional Dirac spinors \( \Psi_j \) representing the \( j \)th colour component of the meson, so \( \Psi \) may describe relative motion (relativistic bound states) of two quarks in mesons and the mentioned Dirac
that yields (at $g \neq 0$)

$$\omega_j = \omega_j(n_j, t_j, \lambda_j) = \\
\frac{\Lambda_j g^2 a_j b_j \pm (n_j + \alpha_j) \sqrt{(n_j^2 + 2n_j \alpha_j + \Lambda_j^2) \mu_0^2 + g^2 b_j^2 (n_j^2 + 2n_j \alpha_j)} }{n_j^2 + 2n_j \alpha_j + \Lambda_j^2}, j = 1, 2, 3,$$

(8)

Then the unique nonperturbative modulo square integrable solutions of (4) are (with Pauli matrix $\sigma_i$)

$$\Psi_j = e^{-i\omega_j t} \psi_j \equiv e^{-i\omega_j t} \begin{pmatrix} F_{j1}(r) \Phi_j(\theta, \phi) \\ F_{j2}(r) \sigma_1 \Phi_j(\theta, \phi) \end{pmatrix}, j = 1, 2, 3 \quad (6)$$

with the 2D eigenspinor $\Phi_j = \begin{pmatrix} \Phi_{j1} \\ \Phi_{j2} \end{pmatrix}$ of the Euclidean Dirac operator $D_0$ on the unit sphere $S^2$, while the coordinate $r$ stands for the distance between quarks. The explicit form of $\Phi_j$ is discussed in Appendix A. We can call the quantity $\omega_j$ relative energy of $j$th colour component of meson (while $\psi_j$ is wave function of a stationary state for $j$th colour component) but we can see that if we want to interpret (4) as equation for eigenvalues of the relative motion energy, i.e., to rewrite it in the form $H \psi = \omega \psi$ with $\psi = (\psi_1, \psi_2, \psi_3)$ then we should put $\omega = \omega_j$ for any $j$ so that $H_j \psi_j = \omega_j \psi_j = \omega \psi_j$. Under this situation, if a meson is composed of quarks $q_{i,2}$ with different flavours then the energy spectrum of the meson will be given by $\epsilon = m_{q_i} + m_{q_i} + \omega$ with the current quark masses $m_{q_k}$ (rest energies) of the corresponding quarks. On the other hand for determination of $\omega_j$ the following quadratic equation can be obtained [1–3]

$$[g^2 a_j^2 + (n_j + \alpha_j)^2] \omega_j^2 - 2(\lambda_j - gB_j) g^2 a_j b_j \omega_j + [(\lambda_j - gB_j)^2 - (n_j + \alpha_j)^2] g^2 b_j^2 - \mu_0^2 (n_j + \alpha_j)^2 = 0,$$

(7)

that yields (at $g \neq 0$)

$$\omega_j = \omega_j(n_j, l_j, \lambda_j) = \\
\frac{\Lambda_j g^2 a_j b_j \pm (n_j + \alpha_j) \sqrt{(n_j^2 + 2n_j \alpha_j + \Lambda_j^2) \mu_0^2 + g^2 b_j^2 (n_j^2 + 2n_j \alpha_j)} }{n_j^2 + 2n_j \alpha_j + \Lambda_j^2}, j = 1, 2, 3,$$
where \( a_3 = -(a_1 + a_2), \ b_3 = -(b_1 + b_2), \ B_3 = -(B_1 + B_2), \ \Lambda_j = \lambda_j - gB_j, \alpha_j = \sqrt{\Lambda_j^2 - g^2a_j^2}, n_j = 0, 1, 2, \ldots\), while \( \lambda_j = \pm(l_j + 1) \) are the eigenvalues of Euclidean Dirac operator \( D_0 \) on unit sphere with \( l_j = 0, 1, 2, \ldots\). It should be noted that in the papers \([1–5]\) we used the ansatz (6) with the factor \( e^{i\omega_jt} \) instead of \( e^{-i\omega_jt} \) but then the Dirac equation (4) would look as \(-i\partial_t\Psi = H\Psi\) and in equation (7) the second summand would have plus sign while the first summand in numerator of (8) would have minus sign. In the papers \([6,7]\) we returned to the conventional form of writing Dirac equation and this slightly modified the equations (7)–(8). In the given paper we conform to the same prescription as in Refs. \([6,7]\).

In line with the above we should have \( \omega = \omega_1 = \omega_2 = \omega_3 \) in energy spectrum \( \epsilon = m_{q_1} + m_{q_2} + \omega \) for any meson (quarkonium) and this at once imposes two conditions on parameters \( a_j, b_j, B_j \) when choosing some experimental value for \( \epsilon \) at the given current quark masses \( m_{q_1}, m_{q_2} \).

The general form of the radial parts of (6) is considered in Appendix B. Within the given paper we need only of the radial parts of (6) at \( n_j = 0 \) (the ground state) that are \([\text{see } (B.5)]\)

\[
F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_jr} \left( 1 - \frac{gb_j}{\beta_j} \right), \quad P_j = gb_j + \beta_j;
\]

\[
F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_jr} \left( 1 + \frac{gb_j}{\beta_j} \right), \quad Q_j = \mu_0 - \omega_j
\]

with \( \beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2b_j^2} \) while \( C_j \) is determined from the normalization condition \( \int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2)dr = \frac{1}{3} \). Consequently, we shall gain that \( \Psi \in L_2^4(\mathbb{R}^3) \) at any \( t \in \mathbb{R} \) and, as a result, the solutions of (6) may describe relativistic bound states (mesons) with the energy (mass) spectrum \( \epsilon \).

### 3.1 Nonrelativistic limit

It is useful to specify the nonrelativistic limit (when \( c \to \infty \)) for spectrum (8). For that one should replace \( g \to g/\sqrt{\hbar c}, \ a_j \to a_j/\sqrt{\hbar c}, \ b_j \to b_j\sqrt{\hbar c}, \ B_j \to B_j/\sqrt{\hbar c} \) and, expanding (8) in \( z = 1/c \), we shall get

\[
\omega_j(n_j, l_j, \lambda_j) = \pm \mu_0 z^2 \left[ 1 + \frac{g^2a_j^2}{2h^2(n_j + |\lambda_j|)^2z^2} + \frac{\lambda_j g^2a_j b_j}{h(n_j + |\lambda_j|)^2} + \frac{g^3 B_j a_j^2 f(n_j, \lambda_j)}{h^3(n_j + |\lambda_j|)^2z^2} \right] z + O(z^2),
\]

where \( f(n_j, \lambda_j) = 4\lambda_j n_j(n_j^2 + \lambda_j^2) + \frac{|\lambda_j|}{\lambda_j} (n_j^4 + 6n_j^2\lambda_j^2 + \lambda_j^4) \).
As is seen from (10), at \( c \to \infty \) the contribution of linear magnetic colour field (parameters \( b_j, B_j \)) to spectrum really vanishes and spectrum in essence becomes purely nonrelativistic Coulomb one (modulo the rest energy). Also it is clear that when \( n_j \to \infty, \omega_j \to \pm \sqrt{\mu_0^2 + g^2 b_j^2}. \)

We may seemingly use (8) with various combinations of signes (\( \pm \)) before second summand in numerators of (8) but, due to (10), it is reasonable to take all signs equal to plus which is our choice within the paper. Besides, as is not complicated to see, radial parts in nonrelativistic limit have the behaviour of form \( F_{j1}, F_{j2} \sim r^{l_j+1} \), which allows one to call quantum number \( l_j \) angular momentum for \( j \)th colour component though angular momentum is not conserved in the field (3) [1,3]. So for meson (quarkonium) under consideration we should put all \( l_j = 0 \).

3.2 Eigenspinors with \( \lambda = \pm 1 \)

Finally it should be noted that spectrum (8) is degenerated owing to degeneracy of eigenvalues for the Euclidean Dirac operator \( D_0 \) on the unit sphere \( S^2 \). Namely, each eigenvalue of \( D_0 \) \( \lambda = \pm (l + 1), l = 0, 1, 2, ... \), has multiplicity \( 2(l + 1) \) so we has \( 2(l + 1) \) eigenspinors orthogonal to each other. Ad referendum we need eigenspinors corresponding to \( \lambda = \pm 1 \) (\( l = 0 \)) so here is their explicit form [see (A.16)]

\[
\lambda = -1 : \Phi = \frac{C}{2} \begin{pmatrix} e^{i\frac{\vartheta}{2}} \\ e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{i\varphi/2}, \text{ or } \Phi = \frac{C}{2} \begin{pmatrix} e^{i\frac{\vartheta}{2}} \\ -e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{-i\varphi/2},
\]

\[
\lambda = 1 : \Phi = \frac{C}{2} \begin{pmatrix} e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{i\varphi/2}, \text{ or } \Phi = \frac{C}{2} \begin{pmatrix} -e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{-i\varphi/2}
\]

(11)

with the coefficient \( C = 1/\sqrt{2\pi} \) (for more details see Appendix A).

4 Electric form factor, the root-mean-square radius and magnetic moment

As has been mentioned in Section 1, at present little is known about experimental spectroscopy of the toponium so we should choose a few quantities that are the most important from the physical point of view to characterize the toponium and then we should evaluate the given quantities within the framework of our approach. Under the circumstances let us settle on the ground state energy of toponium, the root-mean-square radius of it and magnetic moment. All three magnitudes are essentially nonperturbative ones and can be calculated only by nonperturbative techniques.
Within the present paper we shall use relations (8) at \( n_j = 0 = l_j \) so the ground state energy of toponium is given by \( \epsilon = 2m_t + \omega \) with \( \omega = \omega_j(0,0,\lambda_j) \) for any \( j = 1, 2, 3 \) whereas

\[
\omega = \frac{g^2a_1b_1}{|\Lambda_1|} + \frac{\alpha_1\mu_0}{|\Lambda_1|} = \frac{g^2a_2b_2}{|\Lambda_2|} + \frac{\alpha_2\mu_0}{|\Lambda_2|} = \frac{g^2a_3b_3}{|\Lambda_3|} + \frac{\alpha_3\mu_0}{|\Lambda_3|} = \epsilon - 2m_t \tag{12}
\]

and, as a consequence, the corresponding toponium wave functions of (6) are represented by (9) and (11). As the concrete value of \( \epsilon \) we shall take the one of Ref. [14] equal to 347.4 GeV.

4.1 Choice of t-quark mass and gauge coupling constant

It is evident for employing the above relations we have to assign some values to t-quark mass and gauge coupling constant \( g \). In accordance with Ref. [15] we take \( m_t = 173.25 \) GeV at present. Under the circumstances, the reduced mass \( \mu_0 \) of (5) will take value \( m_t/2 \). As to gauge coupling constant \( g = \sqrt{4\pi\alpha_s} \), it should be noted that recently some attempts have been made to generalize the standard formula for \( \alpha_s = \alpha_s(Q^2) = \frac{12\pi}{(33-2n_f) \ln (Q^2/\Lambda^2)} \) (\( n_f \) is number of quark flavours) holding true at the momentum transfer \( \sqrt{Q^2} \to \infty \) to the whole interval \( 0 \leq \sqrt{Q^2} \leq \infty \). We shall employ one such a generalization used in Refs. [16]. It looks as follows (\( x = \sqrt{Q^2} \) in GeV)

\[
\alpha(x) = \frac{12\pi}{(33-2n_f) \ln \frac{x^2f_1(x)}{\Lambda^2}} \tag{13}
\]

with

\[
f_1(x) = 1 + \left( \frac{(1 + x)(33 - 2n_f)}{12} \ln \frac{m^2}{\Lambda^2} - 1 \right)^{-1} + 0.6x^{1.3}, \quad f_2(x) = m^2(1 + 2.8x^2)^{-2},
\]

wherefrom one can conclude that \( \alpha_s \to \pi = 3.1415... \) when \( x \to 0 \), i.e.,

\( g \to 2\pi = 6.2831... \). We used (13) at \( m = 1 \) GeV, \( \Lambda = 0.234 \) GeV, \( n_f = 6 \),

\( x = 2m_t = 346.50 \) GeV to obtain \( g = 1.243528161 \) necessary for our further computations at the mass scale of toponium.

4.2 Electric form factor

For each meson (quarkonium) with the wave function \( \Psi = (\Psi_j) \) of (6) we can define electromagnetic current \( J^\mu = \overline{\Psi}(I_3 \otimes \gamma^\mu)\Psi = (\Psi^\dagger \Psi, \Psi^\dagger(I_3 \otimes \alpha)\Psi) = \)
with the momentum transfer $K\neq 0$, the expression (15) to depend on 3-vector $K$ is a function of $K^2$, as should be, and we can determine the root-mean-square radius of meson (quarkonium) in the form

$$<r> = \sqrt{\sum_{j=1}^{3} \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2}}$$  \hspace{1cm} (16)$$

When calculating (15) also the fact was used that by virtue of the normalization condition for wave functions we have $C_j^2[P_j^2(1 - gb_j/\beta_j)^2 + Q_j^2(1 + gb_j/\beta_j)^2] = (2\beta_j)^{2\alpha_j + 1}/[3\Gamma(2\alpha_j + 1)]$.

It is clear, we can directly calculate $<r>$ in accordance with the standard quantum mechanics rules as $<r> = \sqrt{\int r^2\Psi^\dagger\Psi d^3x} = \sqrt{\sum_{j=1}^{3} \int r^2\Psi_j^\dagger\Psi_j d^3x}$ and the result will be the same as in (16). So we should not call $<r>$ of (16) the charge radius of meson (quarkonium) – it is just the radius of meson (quarkonium) determined by the wave functions of (6) (at $n_j = 0 = l_j$) with respect to strong interaction, i.e., radius of confinement. Now we should notice the expression (15) to depend on 3-vector $K$. To rewrite it in the form holding true for any 4-vector $Q$, let us remind that according to general considerations (see, e.g., Ref. [18]) the relation (15) corresponds to the so-called Breit frame where $Q^2 = -K^2$ [when fixing metric by (1)] so it is not complicated to rewrite
for arbitrary $Q$ in the form

$$f(Q^2) = \sum_{j=1}^{3} f_j(Q^2) = \sum_{j=1}^{3} \left(\frac{(2\beta_j^{2})^{2\alpha_j+1}}{6\alpha_j}\right) \cdot \frac{\sin[2\alpha_j \arctan(\sqrt{|Q^2|}/(2\beta_j))]}{\sqrt{|Q^2|}(4\beta_j^2 - Q^2)^{\alpha_j}}$$

(17)

which passes on to (15) in the Breit frame.

4.3 Magnetic moment

We can define the volumetric magnetic moment density by $m = q(r \times J)/2 = q[(yJ_z - zJ_y)i + (zJ_x - xJ_z)j + (xJ_y - yJ_x)k]/2$ with the meson charge $q$ and $J = \Psi^\dagger(I_3 \otimes \alpha)\Psi$. Using (6) we have in the explicit form

$$J_x = \sum_{j=1}^{3} (F_{j1}^* F_{j2} + F_{j2}^* F_{j1}) \frac{\Phi_j^\dagger \Phi_j}{r^2}, \quad J_y = \sum_{j=1}^{3} (F_{j1}^* F_{j2} - F_{j2}^* F_{j1}) \frac{\Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j}{r^2},$$

$$J_z = \sum_{j=1}^{3} (F_{j1}^* F_{j2} - F_{j2}^* F_{j1}) \frac{\Phi_j^\dagger \sigma_3 \sigma_1 \Phi_j}{r^2}$$

(18)

with Pauli matrices $\sigma_{1,2,3}$. Magnetic moment of meson (quarkonium) is $M = \int_V m d^3x$, where $V$ is volume of meson (quarkonium) (the ball of radius $< r >$). Then at $n_j = l_j = 0$, as is seen from (9), (11), $F_{j1}^* = F_{j1}, F_{j2}^* = -F_{j2}, \Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j = 0$ for any spinor of (11) which entails $J_x = J_y = 0$, i.e., $m_z = 0$ while $\int_V m x y d^3x = 0$ because of turning the integral over $\varphi$ to zero, which is easy to check. As a result, magnetic moments of mesons (quarkonia) with the wave functions of (6) (at $l_j = 0$) are equal to zero, as should be according to experimental data [15].

Though we can also evaluate magnetic form factor $F(Q^2)$ of meson (quarkonium) as a function of $Q^2$ (see Refs. [6,7]) but the latter will not be used in the given paper so we shall not dwell upon it.

5 An estimate of $< r >$ from leptonic width

The question now is how to estimate $< r >$ independently to then calculate it within framework of our approach. For this aim we shall employ the possible width of leptonic decay $V \to e^+e^-$ which is approximately equal to $\Gamma_1 \approx 13$ keV according to Ref. [13] and $V$ stands for the toponium state analogous to $J/\psi$ state in charmonium. Under this situation one can use a variant of formulas originating from Ref. [19]. Such formulas are often employed in the heavy quarkonia physics (see, e. g., Ref. [20]). In their turn they are actually
based on the standard expression from the elementary kinetic theory of gases (see, e. g., Ref. [21]) for the number \( \nu \) of collisions of a molecule per unit time

\[
\nu = \sqrt{2\sigma} < v > n ,
\]

(19)

where \( \sigma \) is an effective cross section for molecules, \( < v > \) is a mean molecular velocity, \( n \) is the concentration of molecules. If replacing \( \nu \rightarrow \Gamma_1 \) we may fit (19) to estimate the leptonic width \( \Gamma_1 \) when interpreting \( \sigma \) as the cross section of creation of \( e^+e^- \) from the pair \( \bar{t}t \) due to electromagnetic interaction, \( < v > \) and \( n \) as, respectively, a mean quark velocity and concentration of quarks (antiquarks) in toponium. To obtain \( \sigma \) in the explicit form one may take the corresponding formula for the cross section of creation of \( e^+e^- \) from the muon pair \( \mu^+\mu^- \) (see, e. g., Ref. [18]) and, after replacing \( \alpha_{em} \rightarrow Q\alpha_{em} \), \( m_\mu \rightarrow m_t \) with electromagnetic coupling constant \( \alpha_{em} = 1/137.0359895 \) and muon mass \( m_\mu \), obtain

\[
\sigma = \frac{4\pi N Q^2\alpha^2_{em}}{3s} \left( 1 + \frac{2m^2_t}{s} \right) \sqrt{1 - \frac{4m^2_t}{s}} ,
\]

(20)

where electron mass \( m_e = 0.510998918 \) MeV, the Mandelstam invariant \( s = 2m_t(m_t + \epsilon/2) \) with \( \epsilon \) from (12), \( N \) is the number of colours and \( Q = 2/3 \) for toponium. To get \( < v > \) one may use the standard relativistic relation \( v = \sqrt{T(T + 2E_0)/(T + E_0)} \) with kinetic \( T \) and rest energies \( E_0 \) for velocity \( v \) of a point-like particle. Putting \( T = \epsilon/2 - m_t, E_0 = m_t \) we shall gain

\[
< v > = \sqrt{1 - \frac{4m^2_t}{\epsilon^2}} .
\]

(21)

At last, obviously, \( n = 1/V \), where the volume of quarkonium \( V = 4\pi < r >^3 /3 \) with the sought \( < r > \), the latter being yet not related to formula (16). The relations (19)–(21) entail the sought independent estimate for \( < r > \)

\[
< r > = \left( \frac{3\sigma \sqrt{2} \sqrt{1 - \frac{4m^2_t}{\epsilon^2}}}{4\pi\Gamma_1} \right)^{1/3}
\]

(22)

with \( \sigma \) of (20). When inserting \( N = 3, \epsilon = 347.4 \) GeV, \( m_t = 173.25 \) GeV, \( m_e = 0.510998918 \) MeV, \( \Gamma_1 = 13 \) keV into (22) we shall have \( < r > \approx 0.2162653913 \times 10^{-2} \) fm. In further considerations we can use this independent estimate of \( < r > \) while calculating \( < r > \) according to (16) which will impose certain restrictions on parameters of the confining SU(3)-gluonic field in toponium.

It should be noted that in the heavy quarkonia physics (see, e. g., Ref. [20]) in (19) one often puts \( n = |\psi(0)|^2 \), where \( \psi(r) \) is a wave function of the heavy quarkonium stationary state which may be obtained, for example, within the framework of potential approach as a solution of the Schrödinger type
Following this prescription in our approach with wave functions of (6) we should put \( n = \sum_{j=1}^{3} |\psi_j(0)|^2 \) with \( \psi_j \) of (6) which would entail
\[
<r> = \left[ \frac{3}{(4\pi \sum_{j=1}^{3} |\psi_j(0)|^2)^{1/3}} \right]^{1/3}
\]
instead of (16). But it is clear that (16) gives physically more correct expression for \( <r> \) since it employs all values of meson wave function rather than the only one at \( r = 0 \) (inasmuch as here
\[
<r> = \sqrt{\int r^2 |\psi|^2 d^3x} = \sqrt{\sum_{j=1}^{3} \int r^2 |\psi_j|^2 d^3x}. 
\]
So we shall use just (16) in what follows.

6 Estimates for parameters of SU(3)-gluonic field in the ground state of toponium \( \eta_t \)

Now we are able to estimate parameters \( a_j, b_j, B_j \) of the confining SU(3)-field (3) for the toponium ground state \( \eta_t \) within framework of two approaches – relativistic and nonrelativistic ones.

6.1 Relativistic approach

Under this approach we should consider (12) and (16) the system of equations which should be solved compatibly if taking \( \epsilon = 347.4 \text{ GeV}, m_t = 173.25 \text{ GeV} \) and \( <r> \approx 0.2162653913 \times 10^{-2} \text{ fm} \) in accordance with the independent estimate of Section 5. While computing for distinctness we take all eigenvalues \( \lambda_j \) of the Euclidean Dirac operator \( D_0 \) on the unit two-sphere \( S^2 \) equal to (-1). The results of numerical compatible solving of equations (12), (16) are adduced in Tables 1–2.

| Table 1 | Gauge coupling constant, reduced mass \( \mu_0 \) and parameters of the confining SU(3)- gluonic field in the toponium ground state \( \eta_t \): relativistic approach |
|----------|------------------|-----------|-----------|-----------|-----------|-----------|
| \( g \)   | \( \mu_0 \) (GeV) | \( a_1 \)  | \( a_2 \)  | \( b_1 \) (GeV) | \( b_2 \) (GeV) | \( B_1 \)  | \( B_2 \)  |
| 1.24353  | 86.2650          | 0.361253  | 0.339442  | 48.9402   | 76.7974   | -0.360    | -0.295    |

| Table 2 | Theoretical ground state energy of toponium and its radius: relativistic approach |
|----------|------------------|-----------|
| \( \epsilon = 2m_t + \omega_j(0, 0, -1) = 347.400 \) | Theoret. \( \epsilon \) (GeV) | Theoret. \( <r> \) (fm) |
| 0.213915 \times 10^{-2} |                   | 0.213915 \times 10^{-2} |
6.2 Nonrelativistic approach

If estimating the quark velocity in toponium by the relation (21) then at \( \epsilon = 347.4 \) GeV, \( m_t = 173.25 \) GeV we obtain \( < v > \approx 0.071935 \) that points out a nonrelativistic approach to be applicable. It should be noted, however, this nonrelativistic approach should be consistent with the above relativistic one. We cannot, therefore, follow the standard strategy of the heavy quarkonia physics which exploits the so-called potential approach (see, e. g., Refs. [8,20]). The essence of the latter is that the interaction between quarks is modelled on a nonrelativistic confining potential in the form \( V(R) = a/R + kR + c_0 \) with some real constants \( a, k, c_0 \) and the distance between quarks \( R \). However, parameters of such a potential, i.e. quantities \( a, k, c_0 \) are not related with QCD-Lagrangian in any way and we cannot speak about \( V(R) \) as describing some gluon configuration between quarks. It would be possible if the mentioned potential were a solution of Yang-Mills equations directly derived from QCD-Lagrangian since, from the QCD-point of view, any gluonic field should be a solution of Yang-Mills equations (as well as any electromagnetic field is by definition always a solution of Maxwell equations).

In reality, as was shown in Refs. [2,3] (see also Appendix C), potential of form \( a/R + kR + c_0 \) cannot be a solution of the Yang-Mills equations if simultaneously \( a \neq 0, k \neq 0 \). Therefore, it is impossible to obtain compatible solutions of the Yang-Mills-Dirac (Pauli, Schrödinger) system when inserting potential of the mentioned form into Dirac (Pauli, Schrödinger) equation. So, we draw the conclusion (mentioned as far back as in Refs. [4]) that the potential approach seems to be inconsistent: it is not based on compatible nonperturbative solutions for the Dirac-Yang-Mills system derived from QCD-Lagrangian in contrast to our confinement mechanism. Actually potential approach for heavy quarkonia has been historically modeled on positronium theory. In the latter case, however, one uses the unique modulo square integrable solutions of Dirac (Schrödinger) equation in the Coulomb field [condensate of huge number of (virtual) photons], i. e., one employs the unique compatible nonperturbative solutions of the Maxwell-Dirac (Schrödinger) system directly derived from QED-Lagrangian to describe positronium (or hydrogen atom) spectrum.

On the other hand, as was mentioned in Section 1, our confinement mechanism is based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system derived from QCD-Lagrangian and just magnetic colour field of solution (3) is responsible for linear confinement. But, as we have seen in Section 2 [see relation (10)], if directly taking the nonrelativistic limit \( c \to \infty \) then the contribution of linear magnetic colour field (parameters \( b_j, B_j \)) to spectrum really vanishes and spectrum in essence becomes purely nonrelativistic Coulomb one (modulo the rest energy). Consequently, as we emphasized as far back as in Refs. [4], the confinement mechanism un-
der discussion is essentially connected with relativistic effects conditioned by availability of the mentioned magnetic colour field between any two quarks. Under the circumstances the only reasonable way of constructing a nonrelativistic approach within our confinement scheme is the power series expansion of the physical magnitudes of interest in $z = 1/c$ with retaining necessary number of terms. It is clear, we then shall obtain the consistent transition from relativistic regime to nonrelativistic one. In their turn, the mentioned magnitudes may be computed within the relativistic framework with the help of wave functions (6) and then the necessary expansions should be fulfilled.

Following the just formulated receipt for description of toponium in a nonrelativistic manner we should use (10) (at $\hbar = c = 1$) with subtracting the rest energy $\mu_0 c^2$ to replace the relations (12) (at $n_j = 0, \lambda_j = -1$) by

$$\omega = -\mu_0 g^2 a_1^2/2 - g^2 a_1 b_1 + \mu_0 g^3 B_1 a_1^2 = -\mu_0 g^2 a_2^2/2 - g^2 a_2 b_2 + \mu_0 g^3 B_2 a_2^2 =$$

$$-\mu_0 g^2 a_3^2/2 - g^2 a_3 b_3 + \mu_0 g^3 B_3 a_3^2 = \epsilon - 2m_t$$

with $a_3 = -(a_1 + a_2), b_3 = -(b_1 + b_2), B_3 = -(B_1 + B_2).$

At the same time we can use (16) to compute $<r>$ with replacing the quantities $\alpha_j = \sqrt{\lambda_j - gB_j^2 - g^2a_j^2}, \beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2b_j^2} by their nonrelativistic expressions

$$\alpha_j = |\lambda_j| - |\lambda_j|/\lambda_j \hbar^2 gB_2 z + O(z^2),$$

$$\beta_j = \mu_0 c^2 + 1/2 \mu_0 \left(2g^2 b_j^2 - 1/4 \hbar^2 \mu_0^2 g^4 a_j^4\right) z^2 + O(z^3).$$

Before expanding $\beta_j$ we made replacement $\omega_j \rightarrow \omega_j - \mu_0 c^2$ in the formula $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2b_j^2},$ i.e. we subtracted the rest energy from $\omega_j$ as is required in nonrelativistic limit (see, e.g., Ref. [18]). After it we should compatibly solve equations (23) and (16) with $\alpha_j, \beta_j$ of (24)–(25) (with $\hbar = c = 1$) at $\lambda_j = -1, \epsilon = 347.4$ GeV, $m_t = 173.25$ GeV and $<r> \approx 0.2162653913 \times 10^{-2}$ fm in accordance with the independent estimate of Section 5. When solving we should impose the conditions $\alpha_j > -1/2, \beta_j > 0$ to have $\Psi_j \in L^2(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ for $\Psi_j$ of (6).

The results of numerical computation are adduced in Tables 3–4.

**Table 3**

| $g$ | $\mu_0$ (GeV) | $a_1$ | $a_2$ | $b_1$ (GeV) | $b_2$ (GeV) | $B_1$ | $B_2$ |
|-----|---------------|-------|-------|-------------|-------------|-------|-------|
| 1.24353 | 86.6250 | 0.200 | 1.32600 | -291.248 | 143.349 | -0.780 | 1.40873 |
| Theoret. $\epsilon$ (GeV) | Theoret. $<r>$ (fm) |
|--------------------------|--------------------|
| $\epsilon = 2m_t + \omega_j(0,0,-1) = 347.400$ | $0.216264 \times 10^{-2}$ |

7 Consistency with possible width of two-photon decay $\eta_t \to 2\gamma$

Let us consider whether the estimates of previous section are consistent with possible width $\Gamma_2$ of two-photon decay $\eta_t \to 2\gamma$ which might be one of the main signals for detecting toponium under certain conditions (see Section 2). To estimate $\Gamma_2$ we can use an analogy of toponium with charmonium where width $\Gamma(\eta_c \to 2\gamma) \approx 7.2$ keV for the charmonium ground state $\eta_c$ whereas leptonic width $\Gamma(J/\psi \to e^+e^-) \approx 5.55$ keV for state $J/\psi$ [15]. Accordingly, ratio $\Gamma(\eta_c \to 2\gamma)/\Gamma(J/\psi \to e^+e^-) \approx 1.297$ and if taking the same ratio for $\Gamma_2/\Gamma_1$ with $\Gamma_1 \approx 13$ keV of Section 5 for toponium state $V$ analogous to $J/\psi$ then we shall obtain $\Gamma_2 \approx 16.865$ keV.

On the other hand, actually kinematic analysis based on Lorentz- and gauge invariances gives rise to the following expression for width $\Gamma$ of the electromagnetic decay $P \to 2\gamma$ (where $P$ stands for any meson from $\pi^0$, $\eta$, $\eta'$, see, e.g., Refs. [22])

$$\Gamma = \frac{1}{4}\pi\alpha_{em}^2 g_{P\gamma\gamma}^2 \mu^3$$

(26)

with electromagnetic coupling constant $\alpha_{em} = 1/137.0359895$ and $P$-meson mass $\mu$ while the information about strong interaction of quarks in $P$-meson is encoded in a decay constant $g_{P\gamma\gamma}$. Making replacement $g_{P\gamma\gamma} = f_P/\mu$ we can reduce (26) to the form

$$\Gamma = \frac{\pi\alpha_{em}^2 f_P^2}{4\mu}.$$  

(27)

Now it should be noted that the only invariant which $f_P$ might depend on is $Q^2 = \mu^2$, i.e. we should find such a function $\mathcal{F}(Q^2)$ for that $\mathcal{F}(Q^2 = \mu^2) = f_P$. It is obvious from physical point of view that $\mathcal{F}$ should be connected with electromagnetic properties of $P$-meson. As we have seen above in Section 4, there are at least two suitable functions for this aim – electric and magnetic form factors. But there exist no experimental consequences related to magnetic form factor at present whereas electric one to some extent determines, e.g., an effective size of meson (quarkonium) in the form $<r>$ of (16). It is reasonable, therefore, to take $\mathcal{F}(Q^2 = \mu^2) = Af(Q^2 = \mu^2)$ with some constant $A$ and electric form factor $f$ of (17) for the sought relation. Under the situation we obtain additional equation imposed on parameters of the confining SU(3)-gluonic field in $P$-meson which has been used in Refs. [6,7] for to estimate the mentioned parameters in $\pi^0$- and $\eta$-mesons. Inasmuch as relation (27) is in essence nonperturbative since decay constant $f_P$ cannot be computed by perturbative techniques, we may extend (27) over heavy quarkonia states.
similar to $\pi^0, \eta, \eta'$, in particular, over $\eta_t$. As a result, using (17) we come from (27) to relation

$$\Gamma = \Gamma_2 = \frac{\pi \alpha_{em}^2 \mu}{4} \left( A \sum_{j=1}^{3} \frac{1}{6\alpha_j x_j} \sin (2\alpha_j \arctan x_j) \right)^2 \approx 16.865 \text{keV}$$

with $x_j = \mu/(2\beta_j)$ and $\mu = \epsilon = 347.4$ GeV. Under the circumstances we can employ the results of Tables 1 and 3 and compute the left-hand side of (28) in relativistic and nonrelativistic regimes respectively which entails the corresponding values $A \approx 0.0340$ and $A \approx 0.156$. Consequently, we draw the conclusion that parameters of the confining SU(3)-gluonic field in toponium from Tables 1 and 3 might be consistent with $\Gamma_2$ in both regimes.

8 Estimates of gluon concentrations, electric and magnetic colour field strengths

Now let us remind that, according to Refs. [3,5], one can confront the field (3) with $T_{00}$-component (volumetric energy density of the SU(3)-gluonic field) of the energy-momentum tensor (2) so that

$$T_{00} \equiv T_{tt} = \frac{E^2 + H^2}{2} = \frac{1}{2} \left( \frac{a_1^2 + a_1 a_2 + a_2^2}{r^4} + \frac{b_1^2 + b_1 b_2 + b_2^2}{r^2 \sin^2 \vartheta} \right) \equiv \frac{A}{r^4} + \frac{B}{r^2 \sin^2 \vartheta}$$

with electric $E$ and magnetic $H$ colour field strengths and with real $A > 0, B > 0$. One can also introduce magnetic colour induction $B = (4\pi \times 10^{-7} \text{H/m}) H$, where $H$ in A/m.

To estimate the gluon concentrations we can employ (29) and, taking the quantity $\omega = \Gamma_1 = \Gamma_1 (V \rightarrow e^+ e^-) \approx 13$ keV of Section 5 for the characteristic frequency of gluons, we obtain the sought characteristic concentration $n$ in the form

$$n = \frac{T_{00}}{\Gamma_1}$$

so we can rewrite (29) in the form $T_{00} = T_{00}^{\text{coul}} + T_{00}^{\text{lin}}$ conforming to the contributions from the Coulomb and linear parts of the solution (3). This entails the corresponding split of $n$ from (30) as $n = n_{\text{coul}} + n_{\text{lin}}$.

The parameters of Tables 1 and 3 were employed when computing and for simplicity we put $\sin \vartheta = 1$ in (29), whereas the Bohr radius $a_0 = 0.529177249 \cdot 10^5$ fm [15].

Table 5 contains the numerical results for $n_{\text{coul}}, n_{\text{lin}}, n, E, H, B$ for the quarkonium under discussion in relativistic approach while Table 6 is obtained in nonrelativistic one.
Table 5
Glue concentrations, electric and magnetic colour field strengths in toponium: relativistic approach

| $\eta_t$: $r_0 = < r > = 0.213915 \times 10^{-2}$ fm | $n_{\text{coul}}$ (m$^{-3}$) | $n_{\text{lin}}$ (m$^{-3}$) | $n$ (m$^{-3}$) | $E$ (V/m) | $H$ (A/m) | $B$ (T) |
|---|---|---|---|---|---|---|
| $r_0$ | 0.880566 $\times 10^{66}$ | 0.131833 $\times 10^{64}$ | 0.881884 $\times 10^{66}$ | 0.223572 $\times 10^{30}$ | 0.116367 $\times 10^{27}$ | 0.146231 $\times 10^{21}$ |
| $r_0$ | 0.880566 $\times 10^{62}$ | 0.131833 $\times 10^{62}$ | 0.101240 $\times 10^{63}$ | 0.223572 $\times 10^{28}$ | 0.116367 $\times 10^{26}$ | 0.146231 $\times 10^{20}$ |
| $10r_0$ | 0.880566 $\times 10^{58}$ | 0.131833 $\times 10^{60}$ | 0.140639 $\times 10^{60}$ | 0.223572 $\times 10^{26}$ | 0.116367 $\times 10^{25}$ | 0.146231 $\times 10^{19}$ |
| 1.0 | 0.184386 $\times 10^{52}$ | 0.603282 $\times 10^{56}$ | 0.603282 $\times 10^{56}$ | 0.102305 $\times 10^{23}$ | 0.248927 $\times 10^{23}$ | 0.312811 $\times 10^{17}$ |
| $a_0$ | 0.235138 $\times 10^{33}$ | 0.215429 $\times 10^{47}$ | 0.215429 $\times 10^{47}$ | 0.365339 $\times 10^{13}$ | 0.470494 $\times 10^{18}$ | 0.591127 $\times 10^{12}$ |

Table 6
Glue concentrations, electric and magnetic colour field strengths in toponium: nonrelativistic approach

| $\eta_t$: $r_0 = < r > = 0.216264 \times 10^{-2}$ fm | $n_{\text{coul}}$ (m$^{-3}$) | $n_{\text{lin}}$ (m$^{-3}$) | $n$ (m$^{-3}$) | $E$ (V/m) | $H$ (A/m) | $B$ (T) |
|---|---|---|---|---|---|---|
| $r_0$ | 0.402970 $\times 10^{67}$ | 0.680959 $\times 10^{64}$ | 0.403651 $\times 10^{67}$ | 0.478269 $\times 10^{30}$ | 0.264472 $\times 10^{27}$ | 0.332345 $\times 10^{21}$ |
| $r_0$ | 0.402970 $\times 10^{63}$ | 0.680960 $\times 10^{62}$ | 0.471066 $\times 10^{63}$ | 0.478269 $\times 10^{28}$ | 0.264472 $\times 10^{26}$ | 0.332345 $\times 10^{20}$ |
| $10r_0$ | 0.402970 $\times 10^{59}$ | 0.680960 $\times 10^{60}$ | 0.721257 $\times 10^{60}$ | 0.478269 $\times 10^{26}$ | 0.264472 $\times 10^{25}$ | 0.332345 $\times 10^{19}$ |
| 1.0 | 0.881475 $\times 10^{52}$ | 0.318486 $\times 10^{57}$ | 0.318486 $\times 10^{57}$ | 0.223572 $\times 10^{23}$ | 0.571957 $\times 10^{23}$ | 0.718742 $\times 10^{17}$ |
| $a_0$ | 0.112410 $\times 10^{34}$ | 0.113733 $\times 10^{48}$ | 0.113733 $\times 10^{48}$ | 0.798800 $\times 10^{13}$ | 0.108084 $\times 10^{19}$ | 0.135823 $\times 10^{13}$ |

9 Discussion and concluding remarks

9.1 Discussion

We can see that main reasonable characteristics of $\eta_t$ such as energy of ground state and the root-mean-square radius (in essence, radius of confinement) and also width of possible decay $\eta_t \rightarrow 2\gamma$ may be consistent with appropriate parameters of the confining SU(3)-gluonic field between quarks in toponium in both relativistic and nonrelativistic approaches within the framework of our confinement mechanism. In other words, we can obtain a description of toponium directly appealing to quark and gluonic degrees of freedom as should be from the first principles of QCD. As is seen from Tables 5 and 6, at both relativistic and nonrelativistic description the glue concentrations are huge at the characteristic scales of the toponium ground state $\eta_t$ and the corresponding fields (electric and magnetic colour ones) can be considered to be the classical ones with enormous strengths. The part $n_{\text{coul}}$ of glue concent-
tration $n$ connected with the Coulomb electric colour field is decreasing faster than $n_{\text{lin}}$, the part of $n$ related to the linear magnetic colour field, and at large distances $n_{\text{lin}}$ becomes dominant. It should be emphasized that in fact the gluon concentrations are much greater than the estimates given in Tables 5 and 6 because the latter are the estimates for sufficiently big possible gluon frequencies, i.e. for sufficiently big possible gluon impulses (under the concrete situation of toponium ground state $\eta_t$). As was mentioned in Section 1, the overwhelming majority of gluons between quarks should be soft, i.e., with frequencies $<\sim 13$ keV so the corresponding concentrations $\gg$ the ones in Tables 5 and 6. The given picture is in concordance with the one obtained in Refs. [4–7]. As a result, the confinement mechanism developed in Refs. [1–3] is also confirmed by the considerations of the present paper.

By the way, the estimate of $<r> \approx 0.216 \times 10^{-2}$ fm obtained in present paper allows one to make a suggestion about why observation of toponium finds difficulty at colliders. Let us use an analogy with classical electrodynamics where, as is well known (see e.g. Ref. [23]), the notion of classical electromagnetic field (a photon condensate) generated by a charged particle is applicable only at distances $\gg$ the Compton wavelength $\lambda_c = 1/m$ for the given point-like particle with mass $m$. Passing on to QCD, gluons and quarkonia and replacing electromagnetic field by colour one in the case of $t$-quark with mass $m_t = 173.25$ GeV we obtain $\lambda_c \approx 0.11 \times 10^{-2}$ fm. If comparing the above $<r>$ to characteristic radius of weak interaction $r_{\text{weak}} \sim 1/m_W \approx 0.2450 \times 10^{-2}$ fm ($m_W \approx 80.403$ Gev is mass of $W$-boson) and to the just obtained $\lambda_c$ then we have inequality $\lambda_c < r_{\text{weak}} \sim <r>$ so one may draw the conclusion that in pair $\bar{t}t$ when creating at colliders the most probable distance between quarks is $\leq r_{\text{weak}}$ so that quarks are more inclined to weak interaction rather than to strong one. In other words, they have not time in order to form a classical confining SU(3)-gluonic field due to strong interaction and to constitute a bound state in virtue of it. But we cannot completely exclude events where distance between $t$-quarks when creating at colliders would be much greater than $\lambda_c$ which might entail formation of $\eta_t$, for example, and, consequently, a signal for detecting toponium, e.g., in the form of decay $\eta_t \rightarrow 2\gamma$, as mentioned in Section 2.

It should be noted, however, that our results are of a preliminary character not only because of that the experimental spectroscopy of toponium is in its infancy but also in virtue of that the current quark masses (as well as the gauge coupling constant $g$) used in computation are known only within the certain limits and we can expect similar limits for the magnitudes discussed in the paper so it is necessary further specification of the parameters for the confining SU(3)-gluonic field in toponium which can be obtained when experimental situation for toponium becomes more satisfactory. We hope to then continue analysing the toponium physics.
9.2 Concluding remarks

Finally we should note the following. As has been shown in the paper, our approach allows one to conduct both relativistic and nonrelativistic description and both the cases are consistent with each other. Only experiments can, however, determine what physical picture (relativistic or nonrelativistic one) for quarkonia is really realized and enough for their complete description. Our approach works in either case since it is based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system derived from QCD-Lagrangian and, as a result, the approach is itself nonperturbative, relativistic from the outset, admits self-consistent nonrelativistic limit and may be employed for any meson (quarkonium).

Appendix A

We here represent some results about eigenspinors of the Euclidean Dirac operator on two-sphere $S^2$ employed in the main part of the paper.

When separating variables in the Dirac equation (4) there naturally arises the Euclidean Dirac operator $D_0$ on the unit two-dimensional sphere $S^2$ and we should know its eigenvalues with the corresponding eigenspinors. Such a problem also arises in the black hole theory while describing the so-called twisted spinors on Schwarzschild and Reissner-Nordströöm black holes and it was analysed in Refs. [3,24], so we can use the results obtained therein for our aims. Let us adduce the necessary relations.

The eigenvalue equation for corresponding spinors $\Phi$ may look as follows

$$D_0 \Phi = \lambda \Phi.$$  \hspace{1cm} (A.1)

As was discussed in Refs. [24], the natural form of $D_0$ in local coordinates $\vartheta, \varphi$ on the unit sphere $S^2$ looks as

$$D_0 = -i\sigma_1 \left[ i\sigma_2 \partial_{\vartheta} + i\sigma_3 \frac{1}{\sin \vartheta} \left( \partial_{\varphi} - \frac{1}{2} \sigma_2 \sigma_3 \cos \vartheta \right) \right] =$$

$$\sigma_1 \sigma_2 \partial_{\vartheta} + \frac{1}{\sin \vartheta} \sigma_1 \sigma_3 \partial_{\varphi} - \frac{\cot \vartheta}{2} \sigma_1 \sigma_2$$  \hspace{1cm} (A.2)

with the ordinary Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
so that $\sigma_1 D_0 = -D_0 \sigma_1$.

The equation (A.1) was explored in Refs. [24]. Spectrum of $D_0$ consists of the numbers $\lambda = \pm (l+1)$ with multiplicity $2(l+1)$ of each one, where $l = 0, 1, 2, \ldots$. Let us introduce the number $m$ such that $-l \leq m \leq l+1$ and the corresponding number $m' = m - 1/2$ so $|m'| \leq l + 1/2$. Then the conforming eigenspinors of operator $D_0$ are

$$\Phi = \left( \Phi_1 \Phi_2 \right) = \Phi_{\pm \lambda} = C \left( \frac{P_{m'\pm 1/2}^k \pm P_{m'1/2}^k}{P_{m'\pm 1/2}^k \pm P_{m'1/2}^k} \right) e^{-im'\varphi} \quad (A.3)$$

with the coefficient $C = \sqrt{\frac{l+1}{2\pi}}$ and $k = l + 1/2$. These spinors form an orthonormal basis in $L^2(S^2)$ and are subject to the normalization condition

$$\int_{S^2} \Phi^\dagger \Phi d\Omega = \int_0^{2\pi} \int_0^\pi (|\Phi_1|^2 + |\Phi_2|^2) \sin \vartheta d\vartheta d\varphi = 1. \quad (A.4)$$

Further, owing to the relation $\sigma_1 D_0 = -D_0 \sigma_1$ we, obviously, have

$$\sigma_1 \Phi_{\pm \lambda} = \Phi_{\pm \lambda}. \quad (A.5)$$

As to functions $P_{m'n'}^k(\cos \vartheta) \equiv P_{m',n'}^k(\cos \vartheta)$ then they can be chosen by miscellaneous ways, for instance, as follows (see, e.g., Ref. [25])

$$P_{m'n'}^k(\cos \vartheta) = i^{-m'-n'} \left( \frac{(k - m')!(k - n')!}{(k + m')!(k + n')!} \right)^{\frac{m'+n'}{2}} \times$$

$$\times \sum_{j=\max(m',n')}^k \frac{(k + j)!(j - m')!(j - n')!}{(k - j)!(j - m')!(j - n')!} \left( \frac{1 - \cos \vartheta}{2} \right)^j \quad (A.6)$$

with the orthogonality relation at $m', n'$ fixed

$$\int_0^\pi P_{m'n'}^{*k}(\cos \vartheta) P_{m'n'}^{k'}(\cos \vartheta) \sin \vartheta d\vartheta = \frac{2}{2k + 1} \delta_{kk'}. \quad (A.7)$$

It should be noted that square of $D_0$ is

$$D_0^2 = -\Delta g^2 I_2 + \sigma_2 \sigma_3 \frac{\cos \vartheta}{\sin^2 \vartheta} \partial_\varphi + \frac{1}{4 \sin^2 \vartheta} + \frac{1}{4}, \quad (A.8)$$

while Laplacian on the unit sphere is

$$\Delta g^2 = \frac{1}{\sin \vartheta} \partial_\vartheta \sin \vartheta \partial_\vartheta + \frac{1}{\sin^2 \vartheta} \partial_\varphi^2 = \partial_\vartheta^2 + \cot \vartheta \partial_\vartheta + \frac{1}{\sin^2 \vartheta} \partial_\varphi^2, \quad (A.9)$$

so the relation (A.8) is a particular case of the so-called Weitzenböck-Lichnerowicz formulas (see Refs. [26]). Then from (A.1) it follows $D_0^2 \Phi = \lambda^2 \Phi$ and, when
using the ansatz $\Phi = P(\vartheta)e^{-im'\varphi} = \left(\begin{array}{c} P_1 \\ P_2 \end{array}\right) e^{-im'\varphi}$, $P_{1,2} = P_{1,2}(\vartheta)$, the equation $\mathcal{D}_0^2\Phi = \lambda^2\Phi$ turns into

$$
\left(-\partial_0^2 - \cot \vartheta \partial_{\vartheta} + \frac{m'^2 + \frac{1}{4}}{\sin^2 \vartheta} + \frac{m' \cos \vartheta}{\sin^2 \vartheta} \sigma_1\right) P = 
$$

$$
\left(\lambda^2 - \frac{1}{4}\right) P,
$$

wherefrom all the above results concerning spectrum of $\mathcal{D}_0$ can be derived [24].

When calculating the functions $P^k_{m,n'}(\cos \vartheta)$ directly, to our mind, it is the most convenient to use the integral expression [25]

$$
P^k_{m,n'}(\cos \vartheta) = \frac{1}{2\pi} \sqrt{(k - m')!(k + m')!} \int_0^{2\pi} \left(ie^{i\varphi/2} \cos \frac{\vartheta}{2} + i e^{-i\varphi/2} \sin \frac{\vartheta}{2}\right)^{k-n'}
$$

$$
\left(i e^{i\varphi/2} \sin \frac{\vartheta}{2} + e^{-i\varphi/2} \cos \frac{\vartheta}{2}\right)^{k+n'} e^{im'\varphi} d\varphi
$$

and the symmetry relations ($z = \cos \vartheta$)

$$
P^k_{m,n'}(z) = P^k_{n',m}(z), P^k_{m',-n'}(z) = P^k_{-m',n'}(z), P^k_{m,n'}(z) = P^k_{-m',-n'}(z),
$$

$$
P^k_{m,n'}(-z) = i^{2k-2m'-2n'} P^k_{m',-n'}(z).
$$

In particular

$$
P^k_{kk}(z) = \cos^{2k} (\vartheta/2), P^k_{k,-k}(z) = i^{2k} \sin^{2k} (\vartheta/2), P^k_{k0}(z) = \frac{i^k \sqrt{(2k)!}}{2^k k!} \sin^{k} \vartheta,
$$

$$
P^k_{kn'}(z) = i^{k-n'} \sqrt{\frac{(2k)!}{(k-n)!(k+n)!}} \sin^{k-n'} (\vartheta/2) \cos^{k+n'} (\vartheta/2).
$$

If $\lambda = \pm (l + 1) = \pm 1$ then $l = 0$ and from (A.3) it follows that $k = l + 1/2 = 1/2, |m'| < 1/2$ and we need the functions $P^{1/2}_{\pm 1/2,\pm 1/2}$ that are easily evaluated with the help of (A.11)–(A.13) so the eigenspinors for $\lambda = -1$ are

$$
\Phi = \frac{C}{2} \left(\cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2}\right) e^{i\varphi/2}, \Phi = \frac{C}{2} \left(-\cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2}\right) e^{-i\varphi/2},
$$

while for $\lambda = 1$ the conforming spinors are

$$
\Phi = \frac{C}{2} \left(\cos \frac{\vartheta}{2} - i \sin \frac{\vartheta}{2}\right) e^{i\varphi/2}, \Phi = \frac{C}{2} \left(-\cos \frac{\vartheta}{2} - i \sin \frac{\vartheta}{2}\right) e^{-i\varphi/2}
$$
with the coefficient \( C = \sqrt{1/(2\pi)} \).

It is clear that (A.14)–(A.15) can be rewritten in the form

\[
\begin{align*}
\lambda = -1 : \Phi &= \frac{C}{2} \left( e^{i\frac{\varphi}{2}} \right) e^{i\varphi/2}, \quad \text{or} \quad \Phi = \frac{C}{2} \left( e^{-i\frac{\varphi}{2}} \right) e^{-i\varphi/2}, \\
\lambda = 1 : \Phi &= \frac{C}{2} \left( e^{-i\frac{\varphi}{2}} \right) e^{i\varphi/2}, \quad \text{or} \quad \Phi = \frac{C}{2} \left( e^{i\frac{\varphi}{2}} \right) e^{-i\varphi/2},
\end{align*}
\]

so the relation (A.5) is easily verified at \( \lambda = \pm 1 \).

**Appendix B**

We here adduce the explicit form for the radial parts of meson wave functions from (6). At \( n_j = 0 \) they are given by

\[
F_{j1} = C_j r^\alpha e^{-\beta_j r} \left( 1 - \frac{Y_j}{Z_j} \right), \quad F_{j2} = iC_j r^\alpha e^{-\beta_j r} \left( 1 + \frac{Y_j}{Z_j} \right),
\]

while at \( n_j > 0 \) by

\[
\begin{align*}
F_{j1} &= C_j P_j r^\alpha e^{-\beta_j r} \left[ \left( 1 - \frac{Y_j}{Z_j} \right) L_{n_j}^{2\alpha_j} + \frac{P_j Q_j}{Z_j} r_j L_{n_j+1}^{2\alpha_j+1} \right], \\
F_{j2} &= iC_j Q_j r^\alpha e^{-\beta_j r} \left[ \left( 1 + \frac{Y_j}{Z_j} \right) L_{n_j}^{2\alpha_j} - \frac{P_j Q_j}{Z_j} r_j L_{n_j-1}^{2\alpha_j+1} \right]
\end{align*}
\]

with the Laguerre polynomials \( L_n^\rho(r_j), \ r_j = 2\beta_j r, \beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 B_j^2} \) at \( j = 1, 2, 3 \) with \( b_3 = -(b_1 + b_2), P_j = gb_j + \beta_j, Q_j = \mu_0 - \omega_j, Y_j = P_j Q_j \alpha_j + (P_j^2 - Q_j^2)ga_j/2, Z_j = P_j Q_j \Lambda_j + (P_j^2 + Q_j^2)ga_j/2 \) with \( a_3 = -(a_1 + a_2), \Lambda_j = \lambda_j - gB_j \) with \( B_3 = -(B_1 + B_2), \alpha_j = \sqrt{\lambda_j^2 - g^2 a_j^2} \), while \( \lambda_j = \pm(l_j + 1) \) are the eigenvalues of Euclidean Dirac operator \( D_0 \) on unit two-sphere with \( l_j = 0, 1, 2, ... \) (see Appendix A) and quantum numbers \( n_j = 0, 1, 2, ... \) are defined by the relations

\[
n_j = \frac{gb_j Z_j - \beta_j Y_j}{\beta_j P_j Q_j},
\]

which entails the quadratic equation (7) and spectrum (8). Further, \( C_j \) of (B.1)–(B.2) should be determined from the normalization condition

\[
\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2)dr = \frac{1}{3}.
\]
As a consequence, we shall gain that in (6) $\Psi_j \in L^4_2(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, accordingly, $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ may describe relativistic bound states in the field (3) with the energy spectrum (8). As is clear from (B.3) at $n_j = 0$ we have $g_{b_j}/\beta_j = Y_j/Z_j$ so the radial parts of (B.1) can be rewritten as

$$F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left( 1 - \frac{g_{b_j}}{\beta_j} \right), \quad F_{j2} = i C_j Q_j r^{\alpha_j} e^{-\beta_j r} \left( 1 + \frac{g_{b_j}}{\beta_j} \right). \quad (B.5)$$

More details can be found in Refs. [1,3].

Appendix C

The facts adduced here have been obtained in Refs. [1–3] and we concisely give them here only for completeness of discussion in Section 6.

The Dirac-Yang-Mills system derived from QCD-Lagrangian according to the standard prescription of Lagrange approach is

$$\mathcal{D} \Psi = \mu_0 \Psi, \quad (C.1)$$

$$d * F = g(*F \wedge A - A \wedge *F) + gJ \quad (C.2)$$

with a gauge coupling constant $g$, Dirac operator $\mathcal{D}$, $F = dA + gA \wedge A$ and the Cartan’s wedge (external) product $\wedge$, whereas $*$ means the Hodge star operator conforming to a Minkowski metric, for instance, in the form of (1), while the source $J$ (a non-Abelian SU(3)-current) is

$$J = j^a_{\mu} \lambda_a * (dx^\mu) = *j = *(j^a_{\mu} \lambda_a dx^\mu) = *(j^a \lambda_a), \quad (C.3)$$

where currents

$$j^a = j^a_{\mu} dx^\mu = \bar{\Psi} (I_3 \otimes \gamma_{\mu}) \lambda^a \Psi dx^\mu,$$

so summing over $a = 1, \ldots, 8$ is implied in (C.3). Besides we have $\text{div}(j^a) = \text{div}(j) = 0$ if $\Psi$ obeys Dirac equation $C.1$ [1,3], where the divergence of the Lie algebra valued 1-form $A = A_{\mu} dx^{\mu} = A^a_{\mu} \lambda_a dx^{\mu}$ is defined by the relation (see, e.g. Refs. [26])

$$\text{div}(A) = \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} A_{\nu}).$$

Definitions of the operators $*$ and $d$ (external differentiation) can be found in Refs. [26] while explicit form of Dirac operator $\mathcal{D}$ of (C.1) depends on choice of local coordinates on Minkowski spacetime and for case of coordinates $t, r, \vartheta, \varphi$ Dirac equation (C.1) can be rewritten in form (4)–(5) of Section 3 and if we
require its modulo square integrable solutions to consist from the components of form $\Psi_j \sim r^{\alpha_j} e^{-\beta_j r}$ with some $\alpha_j > 0$, $\beta_j > 0$ then it will entail all the components of the current $J$ to be modulo $<< 1$ at each point of Minkowski space (perhaps except for a small neighbourhood of point $r = 0$). The latter allow us to put $J \approx 0$ and we come to the problem of finding the confining solutions for the Yang-Mills equations of (C.2) with $J = 0$ whose unique nontrivial form is given by (3) of Section 3 while the unique corresponding modulo square integrable solutions of C.1 are given by (6) of Section 3 (for more details see Refs. [1–3] and Appendix B).

As has been mentioned in Subsection 6.2, in meson spectroscopy one often uses the nonrelativistic confining potentials. Those confining potentials between quarks are usually modelled in the form $a/r + b r$ with some constants $a$ and $b$. It is clear, however, that from the QCD point of view the interaction between quarks should be described by the whole SU(3)-field $A_\mu = A_\mu^a \lambda_a$, genuinely relativistic object, the nonrelativistic potential being only some component of $A_\mu^a$ surviving in the nonrelativistic limit when the light velocity $c \to \infty$.

Let us explore whether such potentials may be the solutions of the Maxwell or SU(3)-Yang-Mills equations. Though this can be easily derived from the above result about uniqueness of solution (3) of Section 3, let us consider the given situation directly in view of its physical importance. We shall use the Hodge star operator action on the basis differential 1- and 2-forms on Minkowski spacetime with local coordinates $t, r, \vartheta, \varphi$ in the form

\[
* \frac{dt}{r^2} = \sin \vartheta dr \wedge d\vartheta \wedge d\varphi, ~ *dr = r^2 \sin \vartheta dt \wedge d\vartheta \wedge d\varphi, \\
* \frac{d\vartheta}{r} = -r \sin \vartheta dt \wedge dr \wedge d\varphi, ~ *d\varphi = r dt \wedge dr \wedge d\vartheta, \\
* (dt \wedge dr) = -r^2 \sin \vartheta d\vartheta \wedge d\varphi, * (dt \wedge d\vartheta) = \sin \vartheta dr \wedge d\varphi, \\
* (dt \wedge d\varphi) = -\frac{1}{\sin \vartheta} dr \wedge d\vartheta, * (dr \wedge d\vartheta) = \sin \vartheta dt \wedge d\varphi, \\
* (dr \wedge d\varphi) = -\frac{1}{\sin \vartheta} dt \wedge d\vartheta. \quad (C.4)
\]

**Maxwell Equations**

In the case of Maxwell equations (i.e., Yang-Mills equations for the case of U(1)-group looking as $d \star F = 0$ at $J = 0$) the ansatz $A = A_t dt = (a/r + br)dt$ yields $F = dA = (a/r^2 - b)dt \wedge dr$. Then with the help of (C.4) we have $*F = \sin \vartheta (b r^2 - a) d\vartheta \wedge d\varphi$ and the relation $d \star F = 2br \sin \vartheta dr \wedge d\vartheta \wedge d\varphi = 0$ entails $b \equiv 0$. 

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SU(3)-Yang-Mills Equations

We use the ansatz

$$A = A_a^i \lambda_a dt = (A'/r + B'r)dt$$  \hspace{1cm} (C.5)

with some constant matrices $A' = \alpha^a \lambda_a, B' = \beta^a \lambda_a$. Then $A \wedge A = 0, F = dA + gA \wedge A = dA = (A'/r^2 - B')dt \wedge dr$. Again with the help of (C.4) we have $*F = \sin \vartheta (B'r^2 - A')d\vartheta \wedge d\varphi, d * F = 2B'r \sin \vartheta dr \wedge d\vartheta \wedge d\varphi, *F \wedge A - A \wedge *F = -2[A', B']r dt \wedge d\vartheta \wedge d\varphi$. Under the circumstances the Yang-Mills equations (C.2) (at $J = 0$) are tantamount to the conditions

$$d * F = 0, *F \wedge A - A \wedge *F = 0.$$  \hspace{1cm} (C.6)

but $\text{div}(J) \neq 0$ and this is not consistent with the only source (C.3) derived from the QCD-Lagrangian. We can avoid this difficulty putting matrices $A', B'$ are not equal to zero simultaneously and both matrices belong to Cartan subalgebra of SU(3)-Lie algebra, i.e. commutator $[A', B'] = 0$. Then $B' = \beta_3 \lambda_3 + \beta_8 \lambda_8$ and for consistency with the only admissible source of (C.3) we should require source of (C.3) to be equal to one of (C.6) which entails

$$g \nabla(I_3 \otimes \gamma_\mu) \lambda^a \Psi \lambda_a dx^\mu = \frac{2(\beta^3 \lambda_3 + \beta^8 \lambda_8)}{r} dt,$$

wherefrom one can conclude that

$$g \nabla(I_3 \otimes \gamma_\mu) \lambda^a \Psi = 0, a \neq 3, 8, g \nabla(I_3 \otimes \gamma_\mu) \lambda^3 \Psi = \frac{2\beta^3}{r},$$

$$g \nabla(I_3 \otimes \gamma_\mu) \lambda^8 \Psi = \frac{2\beta^8}{r}, g \nabla(I_3 \otimes \gamma_\mu) \lambda^a \Psi = 0, a = 1, \ldots, 8, \mu \neq t,$$  \hspace{1cm} (C.7)

which can obviously be satisfied only at $\beta^3 \sim \beta^8 \sim \Psi \rightarrow 0$ at each point of Minkowski spacetime, i.e., really matrix $B' = 0$ again. All the above can easily be generalized to any group SU($N$) with $N > 1$ [2,3].

As a result, the potentials employed in nonrelativistic approaches do not obey the Maxwell or Yang-Mills equations. The latter ones are essentially relativistic and, as we can see from (3) of Section 3, the components linear in $r$ of the unique exact solution $A_\mu$ are different from $A_t$ and related with magnetic (colour) field vanishing in the nonrelativistic limit.
Remark about search for nonrelativistic confining potentials

The above results make us cast a new glance at search of many years for non-relativistic potentials modelling the confinement. Many efforts were devoted to the latter topic, for example, within the framework of lattice gauge theories or potential approach (see, e.g., Ref. [20] and references therein). It should be noted, however, that almost in all literature on this direction one does not bring up the question: whether such potentials could (or should) satisfy the Yang-Mills equations? As is clear from the above the answer is negative. That is why the mentioned approaches seem to be inconsistent: such potentials cannot describe any gluonic configuration between quarks since any gluonic field should be a solution of Yang-Mills equations (as well as any electromagnetic field is by definition always a solution of Maxwell equations).

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