A New Trigonometric Spline Approach to Numerical Solution of Generalized Nonlinear Klien-Gordon Equation

Shazalina Mat Zin1,2, Muhammad Abbas1,3, Ahmad Abd Majid1, Ahmad Izani Md Ismail1

1 School of Mathematical Sciences, Universiti Sains Malaysia, Pulau Pinang, Malaysia, 2 Institute for Engineering Mathematics, Universiti Malaysia Perlis, Perlis, Malaysia, 3 Department of Mathematics, University of Sargodha, Sargodha, Pakistan

Abstract
The generalized nonlinear Klien-Gordon equation plays an important role in quantum mechanics. In this paper, a new three-time level implicit approach based on cubic trigonometric B-spline is presented for the approximate solution of this equation with Dirichlet boundary conditions. The usual finite difference approach is used to discretize the time derivative while cubic trigonometric B-spline is applied as an interpolating function in the space dimension. Several examples are discussed to exhibit the feasibility and capability of the approach. The absolute errors and $L_{\infty}$ error norms are also computed at different times to assess the performance of the proposed approach and the results were found to be in good agreement with known solutions and with existing schemes in literature.

Introduction
The generalized nonlinear Klien-Gordon (KG) equation arises in various problems in science and engineering. This paper focuses on the analysis and numerical solution of the generalized nonlinear KG equation, which is given in the following form [14]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \nabla^2 u(x,t) + \beta u(x,t) + G(u(x,t)) = f(x,t),$$

$$a \leq x \leq b, \quad 0 \leq t \leq T$$

subject to initial conditions

$$\left\{ \begin{array}{l}
 u(x,0) = \omega_1(x), \\
 \frac{\partial u(x,0)}{\partial t} = \omega_2(x),
\end{array} \right. \quad a \leq x \leq b$$

and with Dirichlet boundary conditions

$$\left\{ \begin{array}{l}
 u(a,t) = \phi_1(t), \\
 u(b,t) = \phi_2(t),
\end{array} \right. \quad 0 \leq t \leq T$$

where $u = u(x,t)$ denotes the wave displacement at position and time $(x,t)$, $a$ and $\beta$ are real constants, $G(u)$ is a nonlinear function of $u$ and $f(x,t)$, $\omega_1(x)$, $\omega_2(x)$, $\phi_1(t)$ and $\phi_2(t)$ are known functions.

In particular, the KG equation is important in mathematical physics especially in quantum mechanics and it is well known as a soliton equation. A study on the interaction of soliton in collisionless plasma, the recurrence of initial states and examination of the nonlinear wave equations was in [1].

Several methods, in addition to several finite difference schemes, have been developed to solve the nonlinear KG equation. Jimenez and Vazquez [2] introduced four numerical schemes for solving the nonlinear KG equation. Ming and Guo utilized a Fourier collocation method for solving the nonlinear KG equation [3]. The KG equation was approximated using decomposition scheme by deeba and Khuri [4] and using the Legendre spectral method by Guo et al. [5]. Song et al. solved an initial value problem involving the nonlinear KG equation using fully implicit and discrete energy conserving finite difference scheme [6]. Wazwaz introduced the tanh and sine-cosine method to obtain compact and noncompact solutions for the nonlinear KG equation [7]. Sirendaoreji solved the nonlinear KG equation using the auxiliary equation method to construct new exact traveling wave solutions with quadratic and cubic nonlinearity [8]. Yucel solved the nonlinear KG equation using homotopy analysis method [9] and Chowdhury and Hashim solved the equation using homotopy-perturbation method [10].

B-spline functions can be used to solve numerically linear and non-linear differential equations. Caglar et. al. [11] has introduced a cubic B-spline interpolation method to solve the two-point boundary value problem. Hamid et al. [12] has introduced an alternative cubic trigonometric B-spline interpolation method to solve the same problem. Dehghan and Shokri [13] have obtained a numerical solution of the nonlinear KG equation using Thin Plate Splines radial basis functions. Khuri and Sayfy [14] have solved the generalized nonlinear KG equation using a finite
element collocation approach based on third degree B-spline polynomials.

In this work, a new three-time level implicit approach which combines a finite difference approach and cubic trigonometric B-spline collocation method (CTBCM) is proposed to solve generalized nonlinear KG equation. The finite difference approach is proposed to discretize time derivative and cubic trigonometric collocation method is applied to interpolate the solutions at time. Two numerical experiments are carried out to calculate the numerical solutions, absolute errors, $L_\infty$ error norms and order of convergence for each problem in order to show the accuracy of method.

Temporal Discretization

Consider a uniform mesh $\Omega$ with grid points $(x_j, t_k)$ to discretize the grid region $\Delta = [a,b] \times [0,T]$ with $x_j = a + jh$, $j = 0,1,2,\ldots,n$ and $t_k = k\Delta t$, $k = 0,1,2,3,\ldots,N$, $T = N\Delta t$. $h$ and $\Delta t$ denote mesh space size and time step size respectively. The time derivative is approximated using the central finite difference formula

$$\frac{\partial^2 u}{\partial t^2} = \frac{u^{j+1} - 2u^j + u^{j-1}}{(\Delta t)^2}$$

(4)

Using the approximation of equation (4), equation (1) becomes

$$\frac{u^{j+1} - 2u^j + u^{j-1}}{(\Delta t)^2} + x \frac{\partial^2 u}{\partial x^2} + f(x_j, t_k) = G(u(x_j, t_k))$$

(5)
Using the \( \theta \)-weighted technique, the space derivatives of equation (5) becomes

\[
\frac{u^k_j + 1 - 2u^k_j + u^k_j - 1}{(\Delta t)^2} + (1 - \theta)g^k_j + \theta g^{k+1}_j = f^k_j - G(\alpha^k_j) \tag{6}
\]

where \( g^k_j = x \frac{\partial^2 u^k_j}{\partial x^2} + \beta(u^j_k) \), \( g^{k+1}_j \) \( = x \frac{\partial^2 u^{k+1}_j}{\partial x^2} + \beta(u^{k+1}_j) \), \( 0 \leq \theta \leq 1 \) and the subscripts \( k \) and \( k+1 \) are successive time levels.

After simplification, equation (6) leads to

\[
u_{j}^{k+1} + \theta(\Delta t)^2 g_{j}^{k+1} = 2u_{j}^{k} + (\Delta t)^2 \left[ f_{j}^{k} - G(u_{j}^{k}) \right]
\]

\[
-(1 - \theta)(\Delta t)^2 g_{j}^{k} - u_{j}^{k-1}
\]

\[
(7)
\]

### Trigonometric B-Spline Collocation method

In this section, CTBCM is used to solve Klein-Gordon equation. Cubic trigonometric B-spline are used to approximate the space derivatives. To construct the numerical solution, nodal points \( (x_i,t_k) \) defined in the region \([a,b] \times [0,T]\) where \( a = x_0 < x_1 < \cdots < x_n = b \) and \( x_j + 1 - x_j = h \).

The approximate solution \( \tilde{u}(x,t) \) to the exact solution \( u(x,t) \) is defined as \([18]\):

\[
\tilde{u}(x,t) = \sum_{j=-n}^{n} C_j(t) T_{4j}(x)
\]

where \( C_j(t) \) are time dependent unknowns to be determined and \( T_{4j}(x) \) are cubic trigonometric B-spline basis function given as:

\[
\begin{align*}
T_{4j}(x) &=
\begin{cases}
p(x_j) = \sin\left(\frac{x-x_j}{2}\right), \\
q(x_j) = \sin\left(\frac{x-x_j}{2}\right), \\
\kappa = \sin\left(\frac{h}{2}\right) \sin\left(\frac{3h}{2}\right).
\end{cases}
\end{align*}
\]

Due to local support properties of B-spline basis function, there are only three non-zero basis functions \( T_{4j-3}(x_j), T_{4j-2}(x_j) \) and \( T_{4j-1}(x_j) \) are included over subinterval \([x_{j-1},x_j]\). Thus, the approximation \( u_j^k \) and its derivatives with respect to \( x \) can be simplified as:

\[
\begin{align*}
\left[u_j^k\right] &= \eta_1 C_{j-3}(t) + \eta_2 C_{j-2}(t) + \eta_3 C_{j-1}(t) \\
\left[u_x_j^k\right] &= -\eta_3 C_{j-3}(t) + \eta_3 C_{j-1}(t) \\
\left[u_{xx}_j^k\right] &= -\eta_3 C_{j-3}(t) + \eta_3 C_{j-1}(t) + \eta_4 C_{j-2}(t) + \eta_4 C_{j-1}(t)
\end{align*}
\]

where

\[
\eta_1 = \csc(h) \csc\left(\frac{3h}{2}\right) \sin^3\left(\frac{h}{2}\right),
\]

### Table 1. Numerical solution of Problem 1 at \( t = 5 \)

| \( x/n \) | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---------|---|-----|-----|-----|-----|-----|
| 10      | 0 | 0.0567 | 0.1134 | 0.1702 | 0.2269 | 0.2837 |
| 20      | 0 | 0.0567 | 0.1135 | 0.1702 | 0.2269 | 0.2837 |
| 40      | 0 | 0.0567 | 0.1135 | 0.1702 | 0.2269 | 0.2837 |
| 80      | 0 | 0.0567 | 0.1135 | 0.1702 | 0.2269 | 0.2837 |

### Table 2. Absolute error of Problem 1 at \( t = 5 \)

| \( x/n \) | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---------|---|-----|-----|-----|-----|-----|
| 10      | 0 | 2.473 x 10^{-5} | 3.810 x 10^{-5} | 3.559 x 10^{-5} | 2.152 x 10^{-5} | 0 |
| 20      | 0 | 6.486 x 10^{-6} | 1.496 x 10^{-6} | 8.992 x 10^{-6} | 5.485 x 10^{-6} | 0 |
| 40      | 0 | 1.709 x 10^{-6} | 2.490 x 10^{-6} | 2.337 x 10^{-6} | 1.442 x 10^{-6} | 0 |
| 80      | 0 | 5.119 x 10^{-7} | 7.460 x 10^{-7} | 7.003 x 10^{-7} | 4.320 x 10^{-7} | 0 |
where, $C$ computing condition is substituted into the last term of equation (7) for Table 4 into equation (7). Initially, time dependent unknowns $C_t$ are subsequently the time dependent unknowns, $C$ are calculated and shown in section 3.1. Next, the following initial condition is substituted into the last term of equation (7) for computing $C$:

$$u_j^{-1} = u_j - 2 \Delta t \phi_j(x) \quad (10a)$$

Subsequently, the time dependent unknowns, $C_k$ for $k \geq 1$ should be generated. After simplification, equation (7) leads to

$$\rho_3 u_j^{k+1} + \rho_2 (u_{x_j})_j^{k+1} = \rho_4 u_j^k - \rho_4 (u_{x_j})_j^k - u_j^{k-1} + D_j^k \quad (11)$$

where,

$$\rho_3 = 1 + \theta_1, \quad \rho_2 = \theta_2, \quad \rho_4 = 1 - (1 - \theta) \gamma_1,$$

The system obtained on simplifying (11) consists of $n + 3$ unknowns $C = (C_{k-1}, C_{k}, C_{k+1}, \ldots, C_{n+1})$ in $n + 1$ linear equations at the time level $t = t_{k+1}$. In order to obtain a unique solution, the equation (8) is applied to the boundary conditions given in equation (3) for two additional linear equations.

$$u(a, t_{k+1}) = \eta_1 C_{j-1}(t_k) + \eta_2 C_{j-2}(t_k) + \eta_3 C_{j-1}(t_k) = \phi_1(t_{k+1}) \quad (12)$$

$$u(b, t_{k+1}) = \eta_1 C_{j-1}(t_k) + \eta_2 C_{j-2}(t_k) + \eta_3 C_{j-1}(t_k) = \phi_2(t_{k+1}) \quad (13)$$

From equations (11), (12) and (13), the system can be written as

$$MC^{k+1} = NC^k - PC^{k-1} + E \quad (14)$$

where,

### Table 3. Comparison of $L_{\infty}$ errors norms with Khuri & Sayfy [14] using $h = 0.1$, $\Delta t = 0.005$ and Dehghan and Shokri [13] using $h = 0.02$, $\Delta t = 0.0001$.

|        | 1   | 2   | 3   | 4   | 5   |
|--------|-----|-----|-----|-----|-----|
| $t = 1$ | 1.254 $\times 10^{-5}$ | ------ | 1.555 $\times 10^{-5}$ | ------ | 3.379 $\times 10^{-5}$ |
| $t = 5$ | 2.838 $\times 10^{-4}$ | 3.299 $\times 10^{-4}$ | 7.005 $\times 10^{-5}$ | 3.018 $\times 10^{-4}$ | 3.249 $\times 10^{-4}$ |
| Present method | 4.552 $\times 10^{-5}$ | 4.358 $\times 10^{-5}$ | 1.359 $\times 10^{-5}$ | 5.353 $\times 10^{-5}$ | 3.868 $\times 10^{-5}$ |

doi:10.1371/journal.pone.0095774.t003

### Table 4. The maximum $L_{\infty}$ errors norms and order of convergence, $p$ for Problem 1.

|        | $t = 2$ | $t = 5$ |
|--------|---------|---------|
| $L_{\infty}$ | $p$ | $L_{\infty}$ | $p$ |
| 10  | 4.358 $\times 10^{-5}$ | ------ | 3.868 $\times 10^{-5}$ | ------ |
| 20  | 1.094 $\times 10^{-5}$ | 1.9942 | 9.643 $\times 10^{-6}$ | 2.0039 |
| 40  | 2.886 $\times 10^{-6}$ | 1.9927 | 2.532 $\times 10^{-6}$ | 1.9290 |
| 80  | 6.648 $\times 10^{-7}$ | 1.7384 | 7.592 $\times 10^{-7}$ | 1.7379 |

doi:10.1371/journal.pone.0095774.t004
A New CTBCM to Numerical Solution of Generalized Nonlinear KG Equation

Table 5. Numerical solution of Problem 1 at $t = 5$

| x/n | 0   | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|-----|-----|-----|-----|-----|-----|-----|
| 25  | 0   | 0.0567 | 0.1134 | 0.1702 | 0.2269 | 0.2837 |
| 50  | 0   | 0.0567 | 0.1135 | 0.1702 | 0.2269 | 0.2837 |
| 100 | 0   | 0.0567 | 0.1135 | 0.1702 | 0.2269 | 0.2837 |
| 200 | 0   | 0.0567 | 0.1135 | 0.1702 | 0.2269 | 0.2837 |

Half explicit and half implicit scheme is produced by choosing $\theta$ to be 0.5. This scheme is known as Crank-Nicolson scheme. System (14) becomes a tri-diagonal matrix system of dimension $(n+3) \times (n+3)$ that can be solved using the Thomas Algorithm [17].

Table 6. Absolute errors of Problem 1 at $t = 5$

| x/n | 0   | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|-----|-----|-----|-----|-----|-----|-----|
| 25  | 0   | $4.404 \times 10^{-6}$ | $6.172 \times 10^{-6}$ | $5.745 \times 10^{-6}$ | $3.524 \times 10^{-6}$ | 0   |
| 50  | 0   | $1.101 \times 10^{-6}$ | $1.550 \times 10^{-6}$ | $1.439 \times 10^{-6}$ | $8.827 \times 10^{-7}$ | 0   |
| 100 | 0   | $2.783 \times 10^{-7}$ | $3.926 \times 10^{-7}$ | $3.644 \times 10^{-7}$ | $2.236 \times 10^{-7}$ | 0   |
| 200 | 0   | $7.320 \times 10^{-8}$ | $1.033 \times 10^{-7}$ | $9.587 \times 10^{-8}$ | $5.882 \times 10^{-8}$ | 0   |

doi:10.1371/journal.pone.0095774.t005
doi:10.1371/journal.pone.0095774.t006
Table 7. Comparison of $L_{\infty}$ errors norms with Khuri & Sayfy [14] using $h=0.04$, $\Delta t=0.0001$ and Dehghan and Shokri [13] using $h=0.02$, $\Delta t=0.0001$.

|        | 1          | 2          | 3          | 4          | 5          |
|--------|------------|------------|------------|------------|------------|
|        | $1.254 \times 10^{-5}$ | $1.555 \times 10^{-5}$ | $1.634 \times 10^{-5}$ | $3.379 \times 10^{-5}$ | $3.700 \times 10^{-5}$ |
|        | $4.599 \times 10^{-5}$ | $8.053 \times 10^{-5}$ | $1.276 \times 10^{-5}$ | $7.292 \times 10^{-5}$ | $5.128 \times 10^{-5}$ |
| Present method | $7.316 \times 10^{-6}$ | $6.986 \times 10^{-6}$ | $2.089 \times 10^{-6}$ | $8.596 \times 10^{-6}$ | $6.245 \times 10^{-6}$ |

Initial vector $C^0$

The initial vectors $C^0$ can be obtained from the initial condition as well as boundary values of the derivatives of the initial condition [11,15]:

$$\begin{align*}
(u^0)_j &= -\eta_1 C^0_{j-1} + \eta_2 C^0_j = w'_j(x_1), \ j = 0 \\
P_{\eta_1} C^0_{j-1} + \eta_2 C^0_j &= w_1(x_1), \ j = 0, 1, \ldots, n \\
(u^0)_j &= -\eta_1 C^0_{j-1} + \eta_2 C^0_j = w'_j(x_1), \ j = n
\end{align*}$$

This yields a $(n+3) \times (n+3)$ matrix system:

$$AC^0 = B$$

Where $A$ is the $(n+3) \times (n+3)$ matrix and $B$ is the $(n+3) \times 1$ vector:

$$A = \begin{pmatrix}
-\eta_1 & 0 & \ldots & 0 & 0 & 0 \\
0 & \eta_1 & \eta_2 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \eta_1 & \eta_2 & \eta_3 & 0 \\
0 & 0 & 0 & \ldots & 0 & -\eta_1 & 0 & \eta_3 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
w_1(x_0) \\
w_2(x_0) \\
\vdots \\
w_n(x_0) \\
\end{pmatrix}$$

The solution can be obtained by using the Thomas Algorithm [17].

Stability analysis using von Neumann method

The von Neumann analysis of stability considers the growth of error in a single Fourier mode [19–20]

$$C^j_k = \delta^k \exp(i\eta j h)$$

where $i = \sqrt{-1}$ and $\eta$ is the mode number. It is known that this method can be used to analyze the stability of linear scheme. All the nonlinear term in (11) are assumed to be zero [19]. Thus, equation (11) becomes

$$[1 + \gamma_1 |u|^{k+1} + \gamma_2 (u_{xx})^{k+1}] = [2 - (1 - \theta)\gamma_1]u^k - (1 - \theta)\gamma_2 (u_{xx})^k - u^{k-1}$$

where $\gamma_1 = (\Delta t)^2 \beta$ and $\gamma_2 = (\Delta t)^2 \tau$. Substituting the approximate solution (8) in equation (10) leads to

$$p_1 C_j^{k+1} + p_2 C_j^{k+2} + p_3 C_{j-1}^{k+1} = p_3 C_j^{k+1} + p_4 C_{j-2}^{k+1} + p_5 C_{j-1}^{k+1}$$

where

$$p_1 = \eta_1 + \theta(\rho_1),$$
$$p_2 = \eta_2 + \theta(\rho_2),$$
$$p_3 = 2\eta_1 + (1 - \theta)(\rho_1),$$
$$p_4 = 2\eta_1 + (1 - \theta)(\rho_2),$$
$$p_5 = \eta_4 \gamma_1 + \eta_1 \gamma_2,$$
$$p_5 = \eta_4 \gamma_1 + \eta_2 \gamma_2.$$
In order to obtain the amplification factor $|\delta|$, the trial solution (17) is substituted in (19) and after some simplification, we obtain,

$$\delta^2 A + \delta B + C = 0$$

(20)

where $A = p_1(2 \cos \eta h) + p_2$, $B = p_3(2 \cos \eta h) + p_4$ and $C = \eta_1(2 \cos \eta h) + \eta_2$. Since $A, B, C \geq 0$ and $0 \leq \eta \leq 1$, the amplification factor of this scheme is

$$|\delta| = \sqrt{\frac{M}{M + 0N}} \leq 1$$

(21)

where $M = \eta_1(2 \cos \eta h) + \eta_2$ and $N = \eta_1(2 \cos \eta h) + \eta_2$. Hence, this scheme is unconditionally stable.

Numerical Results and Discussions

In this section, the CTBCM is applied on two numerical problems. In order to measure the accuracy of the method, absolute errors and $L_\infty$ error norms are calculated using the following formulas [16]

Absolute error $= |\tilde{u}_i - u_i|$  \hspace{1cm} (22)

$$L_\infty = \max_i |\tilde{u}_i - u_i|$$  \hspace{1cm} (23)

where $\tilde{u}_i$ and $u_i$ are analytical solution and approximate solution of proposed problem (1), respectively. The numerical order of convergence, $p$ is obtained by using following formula [14]
The tabulated in Tables 1–4 and Tables 5–8. The absolute errors and order of convergence of each case are compared to Dehghan and Shokri [13] and Khuri and Sayfy [14].

The final time is taken to be $T = 5.0$. Fig. 1 (b) and Fig. 2 (a) show the approximate solution and Fig. 2 (b) shows the error of this problem with $n = 40$ and $\Delta t = 0.1$. Two cases of this problem are discussed where Case 1 and Case 2 consider $\Delta t = 0.001$ and $\Delta t = 0.005$, respectively. Numerical solutions, absolute errors, and order of convergence of each case are tabulated in Tables 1–4 and Tables 5–8. The $L_\infty$ error norms are compared to Dehghan and Shokri [13] and Khuri and Sayfy [14] in Table 3 and Table 7. The comparison indicates that the present method is more accurate. The order of convergence of the present method is calculated by the use of the formula given in (24) and is tabulated in Table 4 and Table 6. An examination of these tables indicates the method has a nearly second order of convergence.

**Case 1.** Numerical solutions, absolute errors, $L_\infty$ error norms and order of convergence using time step size $\Delta t = 0.005$

**Case 2.** Numerical solutions, absolute errors, $L_\infty$ error norms and order of convergence using time step size $\Delta t = 0.001$

### Problem 2

The following nonlinear Klein-Gordon equation which is also known as the Sine-Gordon equation is considered [14]

$$u_t - u_{xx} + \sin u = 0$$

It is subject to initial conditions and boundary conditions as

$$u(x,0) = 0, \quad u_t(x,0) = 4 \tanh x; \quad 0 \leq x \leq \ln 2$$

$$u(0,t) = 4 \tan^{-1} t, \quad u(\ln 2,t) = 4 \tan^{-1} \left( \frac{4t}{5} \right); \quad 0 \leq t \leq 1$$

The analytical solution of this problem is $u(x,t) = 4 \tan^{-1} (t \tanh x)$. Fig. 3 (a) depicts a graph of this analytical solution. The final time is taken as $T = 5.0$. The approximate solutions are calculated at time step size $\Delta t = 0.01$ with different mesh space size, $h$. Numerical solutions are recorded in Table 9 and graphical solutions are plotted in Fig. 3 (b) and Fig. 4 (a). Absolute errors are calculated and shown in Table 10 while the 3D error plot is depicted in Fig. 4 (b). Table 11 shows the comparison of $L_\infty$ error norms between the present method with Khuri and Sayfy [14] method. This comparison shows that the present method gives better results.

### Table 9. Numerical solution of Problem 2 at $t = 1$.

| $x/n$ | 0 | $\ln 2/5$ | $2 \ln 2/5$ | $3 \ln 2/5$ | $4 \ln 2/5$ | $\ln 2$ |
|---|---|---|---|---|---|---|
| 5 | 3.1416 | 3.1222 | 3.0654 | 2.9733 | 2.8495 | 2.6990 |
| 10 | 3.1416 | 3.1222 | 3.0655 | 2.9734 | 2.8496 | 2.6990 |
| 20 | 3.1416 | 3.1223 | 3.0655 | 2.9734 | 2.8496 | 2.6990 |
| 40 | 3.1416 | 3.1224 | 3.0657 | 2.9736 | 2.8497 | 2.6990 |

### Table 10. Absolute error of Problem 2 at $t = 1$.

| $x/n$ | 0 | $\ln 2/5$ | $2 \ln 2/5$ | $3 \ln 2/5$ | $4 \ln 2/5$ | $\ln 2$ |
|---|---|---|---|---|---|---|
| 5 | $2.686 \times 10^{-4}$ | $3.534 \times 10^{-4}$ | $3.072 \times 10^{-4}$ | $1.942 \times 10^{-4}$ | $0$ |
| 10 | $1.744 \times 10^{-4}$ | $2.525 \times 10^{-4}$ | $1.994 \times 10^{-4}$ | $1.080 \times 10^{-4}$ | $0$ |
| 20 | $1.492 \times 10^{-4}$ | $2.266 \times 10^{-4}$ | $1.737 \times 10^{-4}$ | $8.401 \times 10^{-5}$ | $0$ |
| 40 | $1.435 \times 10^{-4}$ | $2.193 \times 10^{-4}$ | $1.678 \times 10^{-4}$ | $7.886 \times 10^{-5}$ | $0$ |

doi:10.1371/journal.pone.0095774.t010
Conclusions

In this work, Klein-Gordon equation has been successfully solved using CTBCM incorporating a finite difference scheme. Specifically, the central difference approach is used to discretize the time derivatives and cubic trigonometric B-spline is used to interpolate the solutions at displacement. Well-known two test problems were solved using the proposed method and the solution obtained were in good agreement with the known solution. Accurate solutions at intermediate points can be easily obtained.

Acknowledgments

The authors are indebted to the anonymous reviewers for their helpful, valuable comments and suggestions in the improvement of this manuscript.

References

1. Dodd RK, Eilbeck JC, Gibbon JD (1982) Soliton and Nonlinear Wave Equation. Academic, London.
2. Jiminez S, Vazquez L (1990) Analysis of four numerical schemes for nonlinear Klein-Gordon equation, Appl. Math. Comput 35: 61–94.
3. Ming W, Guo BY (1996) A Legendre spectral method for solving the nonlinear Klein-Gordon equation, Journal of Computational Physics 124: 442–448.
4. Guo BY, Li X, Vazquez L (1996) A Legendre spectral method for solving the nonlinear Klein-Gordon equation, Journal of Computational Physics 124: 442–448.
5. Wong YS, Chang Q, Gong L (1997) An initial-boundary value problem of a generalized nonlinear Klein-Gordon equation, Math. Appl. Comput. 13 (1): 19–36.
6. Wang YS, Chang Q, Gong L (1997) An initial-boundary value problem of a generalized nonlinear Klein-Gordon equation, Applied Mathematics and Computation 84: 77–93.
7. Wazwaz M (2000) The tanh sine-cosine method for compact and noncompact solutions of the nonlinear Klein-Gordon equation, Applied Mathematics and Computation 1167: 1179–1195.
8. Sirendrooji (2000) Auxiliary equation method and new solutions of Klein-Gordon equation, Chaos, Soliton and Fractals 11 (4): 943–950.
9. Yucai U (2008) Homotopy analysis method for the Duffing equation with initial conditions, Applied Mathematics and Computation 203 (1): 307–395.
10. Chowdhury MSH, Hashim I (2009) Application of homotopy-perturbation method to Klein-Gordon and Sine-Gordon equations, Chaos, Soliton and Fractals 39 (4): 1928–1935.
11. Caglar H, Caglar N, Elzainturi K (2006) B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point-boundary value problems, Applied Mathematics and Computation 175 (1): 72–79.
12. Hamid MN, Majid AA, Ismail AIM (2010) Cubic trigonometric B-spline applied to linear two-point boundary value problems of order two, World Academy of Science, Engineering and Technology 47: 478–803.
13. Deliugan M, Shokri A (2009) Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions, Journal of Computational and Applied Mathematics 230 (2): 400–410.
14. Khuri SA, Sayfy A (2010) A spline collocation approach for the numerical solution of a generalized nonlinear Klein-Gordon equation, Applied Mathematics and Computation 216 (4): 1047–1055.
15. Dag I, Irk D, Saka B (2005) A numerical solution of the Burgers’ equation using cubic B-splines, Applied Mathematics and Computation 163 (1): 199–211.
16. Sajadian M (2013) Numerical solutions of Korteweg de Vries and Korteweg-de Vries-Burger’s equations using computer programming the Sinc function 15 (1): 69–79.
17. Rosenberg DUV (1979) Methods for solution of partial differential equations, vol. 113. New York: American Elsevier Publishing Inc.
18. DeBoor C (1978) A Practical Guide to Splines, vol. 27 of Applied Mathematical Sciences, Springer, New York, NY, USA.
19. Siddiqi SS, Arshed S (2013) Quintic B-spline for the numerical solution of the good Boussinesq equation, Journal of the Egyptian Mathematical Society (in press).
20. Abbas M, Majid AA, Ismail AIM, Rashid A (2014) Numerical Method Using Cubic B-Spline for a Strongly Coupled Reaction-Diffusion System. PloS ONE 9(1): e83265. doi:10.1371/journal.pone.0083265.

Table 11. Comparison of $L_\infty$ errors norms with Khuri and Sayfy [14] using $h = (\ln 2)/5$, $\Delta t = 0.01$.

| $t$       | 0.01            | 0.02            | 0.10            | 0.50            | 1.00            |
|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Khuri & Sayfy [14] | $1.3 \times 10^{-6}$ | $6.6 \times 10^{-6}$ | $3.8 \times 10^{-4}$ | $4.8 \times 10^{-3}$ | $2.2 \times 10^{-3}$ |
| Present method            | $5.566 \times 10^{-7}$ | $1.618 \times 10^{-6}$ | $9.851 \times 10^{-5}$ | $1.323 \times 10^{-3}$ | $3.534 \times 10^{-4}$ |