New exact solutions for a coupled KdV–like model

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Abstract. An extension of a recent method is applied in order to construct new explicit exact solutions for a system of coupled KdV-like equations.

1. Introduction

Recently, see [1]–[3], we have studied the system of Korteweg–de Vries (KdV) equations

\[ u_t + u_{xxx} + 2uu_x + 2e_1vv_x + e_2(u_xv + uv_x) + e_3v_{xxx} = 0, \]
\[ c_1v_t + v_{xxx} + 2vv_x + c_2v_x + c_3(e_1(u_xv + uv_x) + 2e_2uu_x + e_3u_{xxx}) = 0, \]  

(1)

coupled through both dispersive and nonlinear terms.

The system was originally developed by Gear and Grimshaw [4] to model the strong interaction of two-dimensional, long, internal gravity waves propagating on neighboring pycnoclines in a stratified fluid.

In (1), \( u(t, x) \) and \( v(t, x) \) are the dependent variables and represent the displacement from the horizontal of the fluid interfaces; subscripts denote partial derivatives with respect to the indicated independent variables, whereas \( e_i \) (\( i = 1, 2, 3 \)) and \( c_j \) (\( j = 1, 2, 3 \)) are constant parameters. It should be noted that \( c_1 \) is positive and finite (regardless of its magnitude).

In real physics, the linear dispersion relations are generally different for the two layers of the fluids, especially for different fluids; therefore, we are interested to find solutions for different choices of the parameters in order to capture the different linear dispersion relations of the fluids.

Some mathematical questions as existence and uniqueness of the solution for the model equations (1) have been studied in [5], where it is shown that the Cauchy problem is globally well-posed in suitably strong function spaces. Moreover, Bona and Saut have also stated that the system (1) is susceptible of experiencing the dispersive blow-up phenomenon. In [6], a study of the integrability of models for coupled KdV systems with different linear dispersion relations has been performed. A study by means of quasi self-adjointness can be found in [2], while, following the guidelines of [7], an approximate symmetry reduction of the model (1) is performed in [8].

In this paper, we devise an extension of an algebraic method to uniformly construct a series of explicit exact solutions for the model.
2. Methodology
The soliton theory have been studied extensively and a lot of interesting results have been found for many coupled systems of partial differential equations. It is known that systems of coupled KdV-like equations admit soliton solutions - see for example [9] -[12] and references therein for a review - moreover, in [13] the author discusses the problem how the coupled two-component KdV systems are connected with the generalized two component Harry Dym systems, while in [14] periodic wave solutions to a coupled KdV equations with variable coefficients are obtained. The literature is also very rich in interesting results obtained on generalized systems of coupled KdV-like equations by Fordy and his co-workers [15],[16]. In [17], an extended algebraic method is used for constructing exact traveling wave solutions for some coupled nonlinear evolution equations.

These facts are confirmed by the symmetry classification performed in [1], by which it is clear that starting from the Principal Lie Algebra of (1) spanned by the operators

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \]

the model admits traveling wave solutions

\[ u(x, t) = U(z), \quad v(x, t) = V(z), \quad z = x - ct, \quad (c \text{ constant}), \]

where the functions \( U \) and \( V \) satisfy the following system of ODEs:

\[ U''' + e_3 V''' + U'(2e_2 V + 2U + c) + V'(2e_1 V + e_2 U) = 0 \]
\[ c_3 e_2 U''' + V''' + c_3(2e_2 U' + e_1 V)U + V'(c_2 + c c_1) + (c_3 e_1 U' + 2V')V = 0, \]

(3)

(4)

where the prime ′ denotes the derivative with respect to \( z \).

Due to the strong nonlinearity of the system of generalized KdV equations (1), in order to characterize a complete hierarchy of traveling wave solutions admitted by the model at hand for some values of the parameters \( e_i \) and \( c_i \) \( (i = 1, 2, 3) \), we use in [3] an algebraic method proposed recently [18].

According to the procedure outlined in [18], we introduce a function \( \phi = \phi(z) \) which is a solution of the first-order ODE

\[ \phi' = \epsilon \sqrt{\sum_{j}^{r} \lambda_j \phi^j} \]

(5)

where \( \epsilon = \pm 1 \). Then the derivatives with respect to \( z \) become the derivatives with respect to the variable \( \phi \) as follows

\[ \frac{d}{dz} \rightarrow \epsilon \sqrt{\sum_{j}^{r} \lambda_j \phi^j} \frac{d}{d\phi}, \quad \frac{d^2}{dz^2} \rightarrow \epsilon^2 \frac{1}{2} \left[ \sum_{j}^{r} j \lambda_j \phi^{j-1} \frac{d}{d\phi} + \sum_{j}^{r} \lambda_j \phi^j \frac{d^2}{d\phi^2} \right], \ldots \]

and write the solution of (4),

\[ u(x, t) = U(z) = \sum_{i}^{n_1} \alpha_i \phi^i(z), \quad v(x, t) = V(z) = \sum_{i}^{n_2} \beta_i \phi^i(z), \]

(6)

as a sum of powers of \( \phi(z) \), \( \alpha_i \) and \( \beta_i \) being arbitrary constants. Balancing the highest derivative terms with the nonlinear terms we give a relation between positive integers \( n_1 \), \( n_2 \) and \( r \): \( n_2 = n_1 = n \) and \( r = n + 2 \).
Substituting the expansion of \( U(z) \) and \( V(z) \) into (4), and setting the coefficients of all powers of \( \phi^j \) and \( \phi^j \sqrt{\sum_j \lambda_j \phi^j} \) to zero, we get a system of algebraic equations, from which the constants \( \lambda_j, \alpha_i \) and \( \beta_i \) \( (j = 0 \ldots n + 2 \text{ and } i = 0 \ldots n) \) can be found explicitly.

In the following, we ask if the model (1) admits not only the classical traveling wave solutions but also a general solution \( \phi(z) \) of (5), which depends of a variable \( z \) of the form

\[
z = x - f(t),
\]

where \( f(t) \) is a suitable function. Of course, when \( f(t) \) is a linear function of \( t \), we recover the results of [1, 3] in which the complete group classification of the model and some exact solutions in closed form have been found. In fact, in the case of PDEs, Lie symmetries may lead to the determination of invariant solutions of a variety of models and to transformations into new equivalent and more convenient forms (some examples in different fields can be found in [19]-[24]).

Now we ask if, generalizing the procedure introduced by Fan, we can obtain a more general class of solutions admitted by the model which includes the soliton solution as a particular case. We suppose that the solution of (1) is of the form

\[
u(x, t) = u(z) + g_1(t), \quad v(x, t) = V(z) + g_2(t),
\]

where \( z = x - f(t) \). Substituting (8) into the system (1), in order to obtain a system of ODEs, we have that

- \( g_1(t) \) and \( g_2(t) \) must be linear function of \( t \) linked by constraints

\[
\begin{align*}
    c_1(2g_1(t) + e_2g_2(t)) - (c_3e_1g_1(t) + 2g_2(t)) &= 0, \quad (9) \\
    e_2g_1(t) + 2c_1g_2(t) &= 0, \quad 2c_3(e_2g_1(t) + e_1g_2(t)) &= 0, \quad (10)
\end{align*}
\]

satisfied for suitable subclasses of the model \((c_3 = 0 \text{ or } e_1 = 0 \text{ or } e_2 = 0, \text{ or } e_1 = e_2 = 0)\).

- \( f(t) \) quadratic and linked to \( g_1(t) \), \( g_2(t) \) by the relation

\[
c_1f'(t) = c_3e_1g_1(t) + 2g_2(t). \quad (11)
\]

When we introduce the function \( \phi(z) \), with \( z = x - f(t) \), as solution of (5), we assume

\[
U(z) = \sum_{i} \alpha_i \phi^i(z) + f_1(z), \quad V(z) = \sum_{i} \beta_i \phi^i(z) + f_2(z)
\]

(12)

\( f_1(z) \) and \( f_2(z) \) being functions of their argument.

Setting the coefficients the different powers of \( \phi^j \sqrt{\sum_j \lambda_j \phi^j} \) to zero we get a set of constraints for \( f_1(z) \) and \( f_2(z) \) that must be compatible with the algebraic system obtained by setting the coefficients of the powers of \( \phi^j \) to zero:

\[
\begin{align*}
    (2\alpha_2 + e_2\beta_2)\phi_1 + (e_2\alpha_2 + 2e_1\beta_2)\phi_2 &= A, \quad (13) \\
    (2\alpha_1 + e_2\beta_1)\phi_1 + (e_2\alpha_1 + 2e_1\beta_1)\phi_2 &= B, \quad (14) \\
    (c_3(e_2\alpha_2 + e_1\beta_2))\phi_1 + (e_3(e_1\alpha_2 + 2\beta_2))\phi_2 &= C, \quad (15) \\
    (c_3(e_2\alpha_1 + e_1\beta_1))\phi_1 + (c_3(e_1\alpha_1 + 2\beta_1))\phi_2 &= D, \quad (16) \\
    (2\alpha_0 + e_2\beta_0)\phi_1 + (e_2\alpha_0 + 2e_1\beta_0)\phi_2 &= E, \quad (17) \\
    +[(f_1(z) + e_2f_1(z)f_2(z) + e_1f_2(z)^2)' + k_1 + f_1''(z) + e_3f_2'''(z) = 0, \quad (18) \\
    c_3(e_2\alpha_0 + e_1\beta_0)\phi_1 + (c_3(e_1\alpha_0 + 2\beta_0 + c_2))\phi_2 &= 0, \quad (19) \\
    +c_3(e_2f_1(z)^2 + e_1f_2(z)f_1(z)) + f_2(z)^2f_1' + c_1k_3 + c_3e_3f_1''(z) + f_2'''(z) &= 0 \quad (20)
\end{align*}
\]
where $g_1(t) = k_1t + k_2$ and $g_2(t) = k_3t + k_4$, and $A$, $B$, $C$ and $D$ are constants. Through the substitution of (8) into (1) and balancing the highest derivative terms with the nonlinear terms in (4) we yield the following set of algebraic equations

\begin{align}
\alpha_2^2 + (e_2\alpha_2 + e_1\beta_2)\beta_2 + 6(\alpha_2 + e_3\beta_2)\lambda_4 &= 0, \\
(2\alpha_1 + e_2\beta_1)\alpha_2 + (e_2\alpha_1 + 2e_1\beta_1)\beta_2 + 5(\alpha_2 + e_3\beta_2)\lambda_3 + 2(\alpha_1 + e_3\beta_1)\lambda_4 &= 0, \\
2((2\alpha_0 + e_2\beta_0)\alpha_2 + (e_2\alpha_0 + 2e_1\beta_0)\beta_2) + (2\alpha_1^2 + e_2\alpha_1\beta_1 + e_1\beta_1^2) + 2A \\
+ 8(\alpha_2 + e_3\beta_2)\lambda_2 + 3(\alpha_1 + e_3\beta_1)\lambda_3 &= 0, \\
(2\alpha_0 + e_2\beta_0)\alpha_1 + (e_2\alpha_0 + 2e_1\beta_0)\beta_1 + 3(\alpha_2 + e_3\beta_2)\lambda_1 + (\alpha_1 + e_3\beta_1)\lambda_2 + B &= 0, \\
\beta_2^2 + c_3(e_2\alpha_2 + e_1\beta_2)\alpha_2 + 6(c_3e_3\alpha_2 + \beta_2)\lambda_4 &= 0, \\
c_3(2e_2\alpha_1 + e_1\beta_1)\alpha_2 + (c_3e_1\alpha_1 + 2\beta_1)\beta_2 + 5(\beta_2 + c_3e_3\alpha_1)\lambda_3 + 2(\beta_1 + c_3e_3\alpha_1)\lambda_4 &= 0, \\
c_3(2e_2\alpha_0 + e_1\beta_0)\alpha_2 + (c_3e_1\alpha_0 + 2\beta_0 + c_2)\beta_2 + (c_3e_1\alpha_1\beta_1 + 2\beta_1^2 + c_3e_2\alpha_1^2) + 2c_3C \\
+ 4(\beta_2 + c_3e_3\alpha_2)\lambda_2 + 3(\beta_1 + c_3e_3\alpha_1)\lambda_3 &= 0, \\
(2\beta_0 + c_3e_1\alpha_0 + c_2)\beta_1 + c_3(2e_2\alpha_0 + e_1\beta_0)\alpha_1 + 3(\beta_2 + c_3e_3\alpha_2)\lambda_1 \\
+ (\beta_1 + c_3e_3\alpha_1)\lambda_2 + D &= 0.
\end{align}

The only possible choices which can be compatible with the method proposed by Fan are $e_1 = 0$ or $e_2 = 0$; the cases $c_3 = 0$ or the particular subcase $e_1 = e_2 = 0$ do not allow for having a compatible generalized solution.

If $e_2 = 0$, $e_1 = 2\frac{K_4}{\alpha_1}$, $g_1(t) = k_1t + k_2$ and $g_2(t) = 0$ and $f_1(z) = \delta f_2(z)$ with $\delta \neq 0, 1$ (for $\delta = 0$ or $\delta = 1$ an incompatibility with algebraic relations since $f_1$ and $f_2(z)$ turn out to be constant), the algebraic system provides the explicit solution for $f_2(z)$:

\begin{equation}
f_2(z) = \frac{(K_1 - K_2) \cosh(2z\sqrt{\lambda_2}) + (K_1 + K_2) \sinh(2z\sqrt{\lambda_2})}{2\sqrt{\lambda_2}} + k_1 \frac{z}{4\delta \lambda_2(1 - \delta)} + K_3,
\end{equation}

$K_1, K_2, K_3$ being constants with $\lambda_2 \neq 0$; moreover, for $\lambda_3 = \lambda_4 = 0$, the following expressions of parameters arise:

\begin{align}
\alpha_2 &= 2\delta^2\beta_2, \\
\alpha_1 &= 2\delta^2\beta_1, \\
\alpha_0 &= \beta_0\delta + 2\lambda_2(\delta - 1), \\
\beta_1 &= \frac{\lambda_3}{\lambda_2}\beta_2, \\
c_3 &= \frac{1}{\delta^3}, \\
c_1 &= -\frac{1}{2\delta}, \\
e_3 &= -\delta^2, \\
e_2 &= 2\lambda_2 \frac{\delta - 1}{\delta}.
\end{align}

Now, the solution of (5) reads as:

\begin{equation}
\phi(z) = -2\lambda_1 \mp (1 + \lambda_1^2 - 4\lambda_0\lambda_2) \cosh(z\sqrt{\lambda_2}) \mp (1 - \lambda_1^2 + 4\lambda_0\lambda_2) \sinh(z\sqrt{\lambda_2}).
\end{equation}

This class of solutions admitted by the model includes a series of “generalized traveling wave solutions” such as: soliton, rational, triangular generalized solutions etc.. We use the term “generalized” because the independent variable has the form $z = x - k_3t - 2k_1t^2$.

In particular, if we assume $\lambda_2 < 0$ (this is the case when $\lambda_2 = -\frac{1}{2}$, $\lambda_1 = -\frac{1}{4}$, $\lambda_0 = 0$), we obtain:

\begin{equation}
\phi(z) = -\frac{1}{2} \sin^2 \left( \frac{z}{2\sqrt{2}} \right).
\end{equation}

Therefore, we are able to write:

\begin{align}
U(z) &= k_1 \frac{z}{2(\delta - 1)} + \beta_2 \delta \frac{\cos(\sqrt{\lambda_2}z)}{3\delta^2} + \delta \frac{K_1}{\sqrt{2}} \sin(\sqrt{\lambda_2}z) + \delta \beta_0 - \delta + 1, \\
V(z) &= k_1 \frac{z}{2(\delta - 1)} + \beta_2 \frac{\cos(\sqrt{\lambda_2}z)}{3\delta^2} + \delta \frac{K_1}{\sqrt{2}} \sin(\sqrt{\lambda_2}z) + \beta_0,
\end{align}
for $K_2 = K_1$ and $K_3 = 0$.

The choices $e_1 = 0, e_2 = 2$ and $g_2(t) = k_3 t + k_4$ with $c_1 f_2(z) = \delta f_1(z)$, lead us to a family of solutions of the model (1) whose profiles are specular to the previous ones.

For suitable values of $\beta_0, \beta_2, \delta$ and $k_1$ we obtain the classical traveling wave solution ($k_1 = 0$) and the “generalized traveling waves” that are shown in the Figures 1 and 2.

Figure 1. 2D view of the solution (38) for fixed value of the independent variable with $u(x, t)$ versus $t \in [0, 20]$; when $\beta_0 = 0$, $\beta_2 = 32$, $k_2 = 1/5$, $k_1 = 1/20$, $K_1 = 10$, $\delta = 2$. We have in orange the classical solitary wave and in blue the “generalized traveling wave”.

Figure 2. 3D view of the solution (38) when $x \in [-10, 10]$ and $t \in [0, 30]$; with $\beta_0 = 0$, $\beta_2 = 32$, $k_2 = 1/5$, $k_1 = 1/20$, $K_1 = 10$, $\delta = 2$. In the picture we have in orange the classical solitary wave and in blue the “generalized traveling wave”.
[1] Ruggieri M. and Speciale M. P., 2013, *Similarity Reduction and Closed Form Solutions for a Model Derived from Two Layer Fluids*, in press on Advances in Difference Equations, Springer.

[2] Ruggieri M. and Speciale M. P., 2013 *Quasi Self-adjoint Coupled KdV-like Equations*, In Numerical Analysis and Applied Mathematics, AIP Conference Proceedings, **1558**, doi: 10.1063/1.4825730, 1220-1223.

[3] Ruggieri M. and Speciale M. P., 2013, *On a hierarchy of traveling wave solutions in a shallow stratified fluid*, In Numerical Analysis and Applied Mathematics, AIP Conference Proceedings **1558**, doi: 10.1063/1.4825873, 1793-1796.

[4] Gear J. A. and Grimshaw R., 1984, *Weak and strong interactions between internal solitary waves*, Stud. Appl. Math **(70)**, 3, 235–258.

[5] Bona J. L. *et al.*, 1992, *A Model System for Strong Interaction Between Internal Solitary Waves*, Commun. Math. Phys., **143**, 287–313.

[6] Bin T. *et al.*, 2006, *A New Coupled KdV Equation: Painlevé Test*, Commun. Theor. Phys., **45**, 965–968.

[7] Ruggieri M. and Valenti A., 2013, *Approximate symmetries in nonlinear viscoelastic media*, Boundary Value Problems, **143**, doi: 10.1186/1687-2770-2013-143.

[8] Ruggieri M. and Speciale M. P., 2013, *Approximate Analysis of a generalized KdV-like Equations*, to appear.

[9] Hu H.C. and Lou S.Y., 2004, *Exact solutions of Bogoyavlenskii coupled KdV equations*, Commun. Theor. Phys. **42**, 485–487.

[10] Fan E. and Hon Y., 2003, *A series of traveling wave solutions for two variant Boussinesq equations in shallow water waves*, Chaos Solitons Fractals **15**, 559–566.

[11] Hirota R. and Satsuma J., 1981, *Soliton solutions of a coupled Kortewegde Vries equation*, Phys. Lett. A **85**, 407–408.

[12] Hirota R. and Satsuma J., 1982, *A coupled KdV equation is one case of the four-reduction of the KP hierarchy*, J. Phys. Soc. Japan **51**, 3390–3397.

[13] Popowicz Z., 2012, *Two-Component Coupled KdV Equations and its Connection with the Generalized Harry Dym Equations*, arXiv:1210.5822v1.

[14] Zhou Y. *et al.*, 2003, *Periodic wave solutions to a coupled KdV equations with variable coefficients*, Physics Letters A, **308**, 3136.

[15] Dodd R. and Fordy A., 1982, *On the integrability of a system of coupled KdV equations*, Physics Letters A, **89**, 168-170.

[16] Antonowicz M. and Fordy A. P., 1987, *Coupled KdV equations with multi-Hamiltonian structures*, Physica D: Nonlinear Phenomena **28**, 3, 345-357.

[17] Seadawy A.R. and Rashidy El, 2013, *Traveling wave solutions for some coupled nonlinear evolution*, Mathematical and Computer Modelling equations, **57**, 1371-1379.

[18] Fan, E.G., 2003, *An algebraic method for finding a series of exact solutions to integrable and nonintegrable nonlinear equations*, Journal of Physics A, **36**, 7009–7026.

[19] Ruggieri M. and Valenti A., 2011, *Exact solutions for a nonlinear model of dissipative media*, Journal of Mathematical Physics, **52**, 043520.

[20] Ruggieri M. and Valenti A., 2009, *Symmetries and reduction techniques for dissipative models*, Journal of Mathematical Physics, **50**, 063506 - 063506-9.

[21] M. Ruggieri, 2012, *Kink solutions for a class of generalized dissipative equations*, Abstract and Applied Analysis, **2012**, 237135-7.

[22] Oliveri F. and Speciale M. P., 2012, *Equivalence Transformations of Quasilinear First Order Systems and Reduction to Autonomous and Homogeneous Form*, Acta Applicandae Mathematicae, **122**, 447–460.

[23] Margheriti L. and Speciale M. P., 2011, *Unsteady solutions of Euler equations generated by steady solutions*, Acta applicandae mathematicae, **113**, 3, 289-303.

[24] Oliveri F. and Speciale M. P., 2013,*Reduction of balance laws to conservation laws by means of equivalence transformations*, Journal of Mathematical Physics, **54**, 041506.