Renormalized new solutions for the massless Thirring model

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Abstract

We present a non-perturbative study of the (1+1)-dimensional massless Thirring model by using path integral methods. The regularization ambiguities -coming from the computation of the fermionic determinant- allow to find new solution types for the model. At quantum level the Ward identity for the 1PI 2-point function for the fermionic current separates such solutions in two phases or sectors, the first one has a local gauge symmetry that is implemented at quantum level and the other one without this symmetry. The symmetric phase is a new solution which is unrelated to the previous studies of the model and, in the non-symmetric phase there are solutions that for some values of the ambiguity parameter are related to well-known solutions of the model. We construct the Schwinger-Dyson equations and the Ward identities. We make a detailed analysis of their UV divergence structure and, after, we perform a non-perturbative regularization and renormalization of the model.

Keywords: Theory of quantized fields, Gauge field theories, Symmetry and conservation laws.
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1 Introduction

The massless Thirring model is a well-known exactly solvable quantum field theory in (1+1) dimensions, as it has been shown, for the first time, by Thirring in his seminal paper [1]. Thirring constructed the eigenstates of the Hamiltonian and calculated some observable quantities and found them to be finite after renormalization. From that date, many studies and extensive investigation of the model have been done. Hagen [2] introduced an external field and found the general solution of the Thirring model. Of course, the exact solution of the massless Thirring model means the possibility of an explicit evaluation of any fermionic correlation function, and it was carried out, for the first time, by Hagen [2], and afterwards by Klaiber [3]. In the Klaiber’s analysis of the Thirring model, he found the operator solutions which are expressed in terms of a free massless Dirac field. On the other hand, Nakanishi [4] expressed the solution in terms of the free massless bosonic field and asserted that all Heisenberg operators should be expressed in terms of asymptotic fields from the standpoint of the general principles of quantum field theory.

Recently a study has been made on the massless Thirring model has been made whose purpose was to investigate the equivalence between the massive Thirring model and the sine-Gordon model [5]. On the other hand an extensive and detailed analysis of the model in its chiral symmetric phase was performed in [6] and its operator bosonic representation in [7].

In previous works, the ambiguities have been observed in the fermionic current definition, as it was noted by Johnson [8], Hagen [2] and Klaiber [3], and it affects the coupling constant, therefore, in the massless Thirring model, the coupling constant is determined only when we define the current regularization. It was proved that the coupling constants are related to each other depending on the regularization. For example the coupling constant of the Schwinger definition $g_s$ is given by

$$g_s = \frac{g_J}{1 - \frac{4\pi}{g_J}},$$

where $g_J$ is the coupling constant of the Johnson definition [8]. Another study of ambiguity of the current regularization was performed in [10]. These current ambiguities also appear in the massive Thirring model which still remains to be understood [11].

The aim of the present work is to show renormalized new solutions for the model obtained by using non-perturbative techniques via the functional integral method. The key to get such solutions are the regularization ambiguities that appear in the calculation of the fermionic determinant. Once the fermionic determinant is well-defined we are capable to solve the model without having to define the fermionic current [12, 13].
Then, the regularization ambiguities allow us to show that, at quantum level, there are two ways to get the full quantization of the massless Thirring model. The first one has solutions that are related to the various studies of the model just performed over the last years in which they do not consider the possibility of implementing a local gauge symmetry at the quantum level by exploring the ambiguities that appear in the calculation of the fermionic current. And the second one is a solution, as it will be shown, which results when we impose a local gauge symmetry at the quantum level. This imposition leads to an equation which fixes the ambiguity parameter as a simple function of the coupling constant. The unexpected result comes from the results when we impose a local gauge symmetry at the quantum level. This imposition leads to an equation which determines the ambiguity parameter as a simple function of the coupling constant. The unexpected result comes from the results when we impose a local gauge symmetry at the quantum level. This imposition leads to an equation which determines the ambiguity parameter as a simple function of the coupling constant.

The paper is displayed in the following way. In section 2, we present the model and shown the consequences of the regularization ambiguities in the solutions for the model. In section 3 and 4, we give our conclusion and discuss their consequences. In section 5, we give our conclusion and discuss their consequences. In section 6, we give our conclusion and discuss their consequences.

We use the following conventions:

\[ g^{00} = 1 = g^{11}, \quad g^{01} = 1 = \epsilon_{01}, \]
\[ \{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}, \quad \{\gamma_\mu, \gamma_5\} = 0, \quad \gamma_5 = \gamma^0 \gamma^1, \quad P_\pm = \frac{1}{2} (1 \pm \gamma_5). \]
\[ \tilde{\nabla}^\mu = e^{\mu\nu} V_\nu. \]

## 2 The massless Thirring model

The massless Thirring model \[ \text{II} \] is a theory of a self–coupled Dirac field defined by the following Lagrangian density

\[ \mathcal{L}[\bar{\psi}, \psi] = \bar{\psi} i \gamma^\mu \partial_\mu \psi - \frac{g}{2} \left( \bar{\psi} \gamma^\mu \psi \right)^2, \tag{1} \]

where \( g \neq 0 \). The Lagrangian is invariant under the global chiral group \( U_\nu(1) \times U_\lambda(1) \) at classical level,

\[ \psi \rightarrow e^{i \alpha \psi}, \quad \psi \rightarrow e^{i \beta \gamma_5 \psi}. \tag{2} \]

We quantize this fermionic model following the functional integral method, thus we write its generating functional as

\[ Z[\eta, \bar{\eta}] = \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp \left( i \int dx \bar{\psi} i \partial_\mu \psi - \frac{g}{2} \left( \bar{\psi} \gamma^\mu \psi \right)^2 + \bar{\eta} \psi + \bar{\psi} \eta \right). \tag{3} \]

The first step in the quantization procedure is to linearize the quadratic term in the fermionic current introducing an auxiliary vector field \( A_\mu \), such that the generating functional is expressed as

\[ Z[\eta, \bar{\eta}] = \int \mathcal{D} A_\mu \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp \left( i \int dx \bar{\psi} (i \partial_\mu + A_\mu) \psi + \frac{1}{2g} A_\mu A^\nu + \bar{\eta} \psi + \bar{\psi} \eta \right), \tag{4} \]

where the presence of the massive term for the vector field precludes the existence (at classical level) of the following \( U(1) \) local gauge symmetry,

\[ \psi \rightarrow e^{i \alpha \psi}, \quad \bar{\psi} \rightarrow e^{-i \alpha} \bar{\psi}, \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha. \tag{5} \]

The fact that at classical level the model does not have a local gauge symmetry allows us to compute the fermionic determinant using a generalized regularization prescription \[ \text{[12] [13] [14] [15] [16]} \] to control the UV divergences. By following the point-splitting regularization implemented in \[ \text{[15]} \], we compute the fermionic determinant to be

\[ \det(i \partial + A) = \exp \left( i \int dx A_\mu \left[ \frac{\pi}{g^2} + \frac{a+1}{2} \right] g^{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} A_\nu \right), \tag{6} \]

where the parameter \( a \) characterizes the different regularization procedures used to control the UV divergences. When \( a = 1 \) the fermionic determinant is invariant under the local gauge transformation \( A_\mu \rightarrow A_\mu + \partial_\mu \alpha \).

In the following, we study the model for all the possible values of the ambiguity parameter \( a \neq 1 \).

After the fermionic integration the generating functional \[ \text{II} \] reads as

\[ Z[\eta, \bar{\eta}] = \int \mathcal{D} A_\mu \exp \left( i \int dx \frac{1}{2\pi} A_\mu \left[ \frac{\pi}{g^2} + \frac{a+1}{2} \right] g^{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} A_\nu \right) \exp \left( -i \int dx dy \bar{\eta}(x) G(x, y; A) \eta(y) \right), \tag{7} \]

where \( \mathcal{Z} \) characterizes the different regularization procedures used to control the UV divergences. When \( a = 1 \) the fermionic determinant is invariant under the local gauge transformation \( A_\mu \rightarrow A_\mu + \partial_\mu \alpha \).
where $G(x, y; A)$ is the Green’s function of the Dirac’s equation: $(i\partial + A)G(x, y; A) = \delta(x - y)$, and it can be exactly computed,

$$G(x, y; A) = \exp \left(-i \int dz \ A_{\mu}(z) j_{\mu}^{A}(z, x, y)\right) P_+ G_F(x - y) + \exp \left(i \int dz \ A_{\mu}(z) j_{\mu}^{A}(z, x, y)\right) P_- G_F(x - y), \quad (8)$$

with $G_F(x - y)$ being the Green’s function of the massless Dirac’s equation: $i\partial \psi_F(x - y) = \delta(x - y)$; and the contact current $j_{\mu}^{A}(z, x, y)$ is

$$j_{\mu}^{A}(z, x, y) = (\partial_{\mu} + \bar{\partial}_{\mu})[D_F(z - x) - D_F(z - y)], \quad (9)$$

where $D_F(x)$ is the Green’s function of the massless Klein-Gordon equation: $\Box D_F(x - y) = \delta(x - y)$.

At quantum level the Ward identity for the 1PI 2-point function of the fermionic current is given by

$$q_{\mu} \Gamma^{\mu\nu}(q) = -\left(g + \frac{2\pi}{a - 1}\right) q^\nu. \quad (10)$$

then, we will have a local $U(1)$ gauge symmetry if and only if

$$a = 1 - \frac{2\pi}{g} \quad (11)$$

such symmetry is reflected in the effective action $\mathcal{A}$ which is invariant under the local $U(1)$ gauge transformation

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\alpha. \quad (12)$$

The equation (11) is important since the ambiguity becomes a simple function of the fundamental coupling of the model. The ambiguity that appears in the regularization process of the fermionic determinant or in the definition of the fermionic current is fixed by the requirement of a new local gauge symmetry at quantum level. Therefore, the model at quantum level displays two phases: one of them does not have the local $U(1)$ gauge symmetry and the other one has. In the following we analyze both phases separately.

3 The non–gauge invariant case: $a \neq 1 - \frac{2\pi}{g}$

We begin the quantization process from the generating functional defined in (4) and we include the external source for the fermionic current, thus, we have

$$Z[\eta, \bar{\eta}, C_{\mu}] = \int \mathcal{D}A_{\mu} \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left(i \int dx \ \bar{\psi}(i\partial + \bar{\partial} + C)\psi + \frac{1}{2g} A_{\mu} A^{\mu} + \bar{\eta} \psi + \bar{\psi} \eta\right). \quad (13)$$

In this situation we do not need a gauge fixing term for the vector field because its effective action $\mathcal{A}$ is not gauge invariant under (12). Thus, considering the fermionic determinant in (4) for $a \neq 1 - \frac{2\pi}{g}$ we can compute all the correlation functions and we can also construct the Schwinger-Dyson equations (SDE) and the Ward identities.

3.1 The Green’s functions

3.1.1 The fermion propagator $G(x - y) = \langle 0 | \bar{\psi}(x) \psi(y) | 0 \rangle$

The fermion propagator reads as

$$G(x - y) = i \exp \left(-\frac{i \pi}{b(b - 1)} \int \frac{dk}{(2\pi)^2} \frac{1 - e^{-ik\cdot(x-y)}}{k^2}\right) G_F(x - y), \quad (14)$$

where

$$b = \frac{\pi}{g} + \frac{a + 1}{2} \quad (15)$$

The fermion propagator has ultraviolet divergences and it will be shown that after its regularization and renormalization procedures guarantee the full finiteness of the model.
3.1.4 The 1PI functions

The propagator is defined as

\[ n(\Delta) = -\frac{1}{2\pi} \int \frac{d^4k}{e^{ik\cdot\xi} - \delta} \] (16)

and it is free of UV divergences and, the tensors \( T^\mu_{\nu} \) and \( L^\mu_{\nu} \) are the transversal and longitudinal projectors defined as

\[ T^\mu_{\nu} = \frac{q^\mu q^\nu}{q^2} , \quad \text{and} \quad L^\mu_{\nu} = \frac{q^\nu q^\mu}{q^2} . \] (17)

3.1.3 The vertex function \( G^\mu(x,y,z) = \langle 0| T \psi(x) \bar{\psi}(y) j^\mu(z) |0 \rangle \)

It is calculated to be

\[ G^\mu(x,y,z) = i \int \frac{dk}{(2\pi)^2} \left( - \frac{\pi}{g(b-1) k^2} + \frac{\pi}{gb k^2} \gamma_5 \right) \left( e^{-ik \cdot (z-x)} - e^{-ik \cdot (z-y)} \right) G(x-y). \] (18)

In momentum space it reads as

\[ G^\mu(p,-p-q,q) = i \left( - \frac{\pi}{g(b-1) q^2} + \frac{\pi}{gb q^2} \gamma_5 \right) \left[ G(p+q) - G(p) \right] . \] (19)

where \( G(p) \) is the Fourier transforms of the fermionic propagator \( G \). Thus, we can note that the UV divergences of this Green’s function come from the fermionic propagator.

It can be shown that 2-point Green’s functions for the pure fermionic current are UV finite, and all the 2n+m-point Green’s functions involved 2n fermions and m current insertions have logarithmic UV divergences.

3.1.2 The current propagator \( J_{\mu\nu}(x-y) = \langle 0| T j^\mu_{\mu}(x) j^\nu_{\nu}(y) |0 \rangle \)

In momentum space the propagator of the fermionic current \( j^\mu = \bar{\psi} \gamma^\mu \psi \) is calculated to be

\[ i J^\mu_{\nu}(q) = \frac{T^\mu_{\nu}}{g} + \frac{L^\mu_{\nu}}{g} \] (16)

and it is free of UV divergences and, the tensors \( T^\mu_{\nu} \) and \( L^\mu_{\nu} \) are the transversal and longitudinal projectors defined as

\[ T^\mu_{\nu} = g^\mu_{\nu} - \frac{q^\mu q^\nu}{q^2} , \quad \text{and} \quad L^\mu_{\nu} = \frac{q^\nu q^\mu}{q^2} . \] (17)

3.1.1 The 1PI functions

We write the 1PI in momentum space. Thus, the 1PI 2-point fermion function \( \Gamma(p) \) which is the inverse of the fermionic propagator is defined as

\[ \Gamma(p) = iG^{-1}(p) . \] (20)

The 1PI function 2-point function of the fermion current \( \Gamma^\mu_{\nu}(q) \) is given by

\[ \Gamma^\mu_{\nu}(q) = i J_{\nu\mu}^{-1}(q) = - \left( g + \frac{2\pi}{a+1} \right) T^\mu_{\nu} - \left( g + \frac{2\pi}{a-1} \right) L^\mu_{\nu} \] (21)

The 1PI 3-point function \( \Gamma^\mu(p,k,q) \) related to the vertex function \( G^\mu(p,k,q) \) is

\[ \Gamma^\mu(p,k,q) = \left( \frac{2\pi}{a-1} \frac{q^\mu}{q^2} + \frac{2\pi}{a+1} \frac{q^\mu}{q^2} \gamma_5 \right) \left[ \Gamma(p,k) - \Gamma(p) \right] \] (22)

3.2 Schwinger–Dyson equations

We start from the generating functional \( Z[\eta, \bar{\eta}, C] \) that after integration of the vector field can reads as

\[ Z[\eta, \bar{\eta}, C] = \int D\psi D\bar{\psi} \exp \left( i \int dx \bar{\psi}(i\partial + G)\psi - \frac{g}{2}(\bar{\psi} \gamma^\mu \psi)^2 + \bar{\eta} \psi + \bar{\psi} \eta \right) , \] (23)

Making the functional variation of the generating functional with respect to the fermionic field \( \bar{\psi} \) we get the quantum equation of motion

\[ \left( \delta \frac{\partial}{\delta \bar{\eta}}(x) + g \gamma^\mu \delta \frac{\partial}{\delta \eta}(x) \delta C^\mu(x) \right) - iC^\mu(x) \gamma^\mu \delta \frac{\partial}{\delta \eta}(x) + \eta(x) \right) Z[\eta, \bar{\eta}, C] = 0 \] (24)

where we have used the functional representations for the fields

\[ \psi = -i \frac{\delta}{\delta \bar{\eta}} , \quad \bar{\psi} = i \frac{\delta}{\delta \eta} , \quad j^\mu = -i \frac{\delta}{\delta C^\mu} . \] (25)

We perform a functional derivative with respect to η(y) in the functional equation \( i\frac{\partial}{\partial x} G(x - y) = i\delta(x - y) + g\gamma^\mu G_\mu(x, y; x) \) (26) and setting the external sources to zero we obtain the SDE for the fermion propagator

\[ G(p) = i\frac{i}{p} - i\int \frac{dk}{(2\pi)^2} f(k) \frac{1}{p - k} G(p - k). \] (27)

where the function \( f(k) \) reads for

\[ f(k) = -\frac{\pi}{b(b - 1)k^2}. \] (28)

Iterating the SDE we generate an expansion in \( \hbar \) or equivalently in powers of \( f(k) \)

\[ G(p) = i\frac{i}{p} + \int \frac{dk}{(2\pi)^2} f(k) \frac{1}{p - k} \frac{1}{p - k} - i\int \frac{dk}{(2\pi)^2} \frac{dq}{(2\pi)^2} f(q) f(k) \frac{1}{p - k} \frac{1}{p - k} \frac{1}{p - k - q} + \ldots. \] (29)

which is a semi–perturbative expansion just as it was shown in [17] for the anomalous and chiral Schwinger model cases. A power counting analysis of the series clearly shows the existence of a logarithmic UV divergence in each order or loop, such property of the series guarantees that the model is a fully renormalizable field theory.

### 3.3 The Ward identities

We start from the generating functional [13] and we make the following fermionic local phase transformation

\[ \psi \to e^{i\lambda(x)} \psi, \quad \bar{\psi} \to \bar{\psi} e^{-i\lambda(x)}. \] (30)

In our framework the fermionic measure is not invariant under such transformation changing as

\[ \mathcal{D}\psi\mathcal{D}\bar{\psi} \to J[A, \lambda] \mathcal{D}\psi\mathcal{D}\bar{\psi} \] (31)

where \( J[A, \lambda] \) is the Jacobian given by

\[ J[A, \lambda] = \frac{\text{det}(i\partial + A + \gamma)}{\text{det}(i\partial + A + \gamma - i\partial\lambda)} = \exp\left(i\int dx \frac{a - 1}{2\pi} (A_\mu + C_\mu)\partial^\mu\lambda - \frac{a - 1}{4\pi} (\partial^\mu\lambda)^2\right), \] (32)

then, the generating functional in [13] reads as

\[ Z[\eta, \bar{\eta}, C_\mu] = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} J[A] \exp\left(i\int dx \bar{\psi}(i\partial + A + \gamma - i\partial\lambda)\psi + \frac{1}{2g} A_\mu A^\mu + \bar{\eta} e^{i\lambda(x)} \psi + \bar{\psi} e^{-i\lambda(x)} \eta\right). \] (33)

Next, we do the following transformation in the vector field \( A_\mu \to A_\mu + \partial_\mu\lambda \) and after integrating it we obtain

\[ Z[\eta, \bar{\eta}, C_\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(i\int dx \bar{\psi}i\partial\psi - \frac{g}{2} (\bar{\psi}\gamma^\mu\psi)^2 + \bar{\eta} e^{i\lambda(x)} \psi + \bar{\psi} e^{-i\lambda(x)} \eta + C_\mu j^\mu\right) \exp\left(i\int dx \left[\frac{a - 1}{2\pi} C_\mu - \frac{g - b - 1}{\pi} j^\mu\right] \partial^\mu\lambda + \frac{1}{2} \left[\frac{b - 1}{\pi} - \frac{g - (b - 1)^2}{\pi^2}\right] (\partial^\mu\lambda)^2\right). \] (34)

By considering \( \lambda(x) \) as an infinitesimal function and expanding to first order in \( \lambda \), we get the functional equation satisfied by the generating functional [13]

\[ \left(\frac{\delta}{\delta\eta} - \frac{a - 1}{2\pi} \partial^\mu C_\mu - i\frac{b - 1}{\pi} \partial^\mu \delta \right) Z[\eta, \bar{\eta}, C_\mu] = 0 \] (35)

where we have used the functional representation \( 25 \) for the fields.

The generating functional of the connected Green’s functions \( W[\eta, \bar{\eta}, C_\mu] \) is defined as being \( W = -i\ln Z \) and the generating functional of the 1PI functions \( \Gamma[\psi, j^\mu] \), both are related via the following Legendre transformation

\[ \Gamma[\bar{\psi}, \psi, j^\mu] = W[\eta, \bar{\eta}, C_\mu] - \int dx \left(\bar{\eta}\psi + \bar{\psi}\eta + C_\mu j^\mu\right), \] (36)
and using the functional relations

\[ \psi = \frac{\delta W}{\delta \eta}, \quad \bar{\psi} = -\frac{\delta W}{\delta \eta}, \quad j_\mu = \frac{\delta W}{\delta C^\mu}, \quad \bar{\eta} = \frac{\delta \Gamma}{\delta \psi}, \quad \eta = -\frac{\delta \Gamma}{\delta \bar{\psi}}, \quad C_\mu = \frac{\delta \Gamma}{\delta j^\mu}, \quad (37) \]

we obtain the functional equation satisfied by the generating functional \( \Gamma[\bar{\psi}, \psi, j^\mu] \) from which we get all Ward identities

\[ i \frac{\delta \Gamma}{\delta \bar{\psi}(x)} \psi(x) - i \frac{\delta \Gamma}{\delta \psi(x)} \bar{\psi}(x) + \frac{a-1}{2\pi} \partial_\mu \frac{\delta \Gamma}{\delta j^\mu(x)} + g \frac{b-1}{\pi} \partial_\mu j_\mu(x) = 0. \quad (38) \]

The first Ward identity is for the 1PI 2-point function \( \Gamma^{\mu\nu}(q) \) of the fermionic current which reads as

\[ q_\mu \Gamma^{\mu\nu}(q) = - \left( g + \frac{2\pi}{a-1} \right) q^\nu. \quad (39) \]

as it can be shown explicitly from the equation (21). We can note that the current propagator is transverse for the critical value \( a = 1 - \frac{2\pi}{g} \) and, as we mentioned in Section 2, it implies in the existence of a local \( U(1) \) gauge symmetry for this value of the ambiguity parameter.

The second Ward identity which relates the 1PI 3-point function \( \Gamma^\mu(p,-p-q; q) \) and the 1PI 2-point fermion function \( \Gamma(p) \) is given by

\[ q_\mu \Gamma^\mu(p; q) = \frac{2\pi}{a-1} \left[ \Gamma(p+q) - \Gamma(p) \right]. \quad (40) \]

it can also be shown easily from the equation (22). This identity guarantees that all the divergences come from the fermion propagator. Thus, we can see that the ambiguity parameter does not renormalize and consequently the coupling constant either.

### 3.4 Regularization and renormalization of the UV divergences

We will study the UV divergences structure of the pure fermionic \( 2n \)-point Green’s functions (correlation functions without current insertions) to identify their origin and later to perform their regularization. The \( (2n + m) \)-point Green’s functions, where \( m \) is the number of current insertions, have similar divergence structure. Thus, as we will see, a simple renormalization of fermionic wave function is sufficient for all Green’s functions becomes finite.

Then, in the generating functional (7) we decompose the vector field \( A_\mu \) in their longitudinal and transversal components

\[ A_\mu = \partial_\mu \rho - \partial_\mu \phi, \quad (41) \]

getting the following expression for the generating functional

\[ Z[\eta, \bar{\eta}] = \int \mathcal{D} \rho \mathcal{D} \phi \exp \left( i \int dx \left[ \frac{b-1}{2\pi} \partial_\mu \rho \partial^\mu \rho - \frac{b}{2\pi} \partial_\mu \phi \partial^\mu \phi \right] \right) \exp \left( -i \int dx dy \bar{\eta}(x)G(x, y; \rho, \phi)\eta(y) \right), \quad (42) \]

where \( G(x, y; \rho, \phi) \) is given by

\[ G(x, y; \rho, \phi) = \sum_{\epsilon = \pm} \exp \left( i \int dz \left[ \rho(z) - \epsilon \phi(z) \right] j(z, x, y) \right) P_G(x - y), \quad (43) \]

and the contact current \( j(z, x, y) = \delta(z - x) - \delta(z - y) \). From the generating functional (12) we can compute all pure fermionic Green’s functions, for example, the fermionic propagator

\[ G(x - y) = \int \mathcal{D} \rho \exp \left( i \int dz \left[ \frac{b-1}{2\pi} \partial_\mu \rho \partial^\mu \rho + \rho(z)j(z, x, y) \right] \right) \sum_{\epsilon = \pm} \int \mathcal{D} \phi \exp \left( -i \int dz \left[ \frac{b}{2\pi} \partial_\mu \phi \partial^\mu \phi + \epsilon \phi(z)j(z, x, y) \right] \right) P_G(x - y). \quad (44) \]

We can note that the functional integration on the fields \( \rho \) and \( \phi \) are factorized. After performing the functional integration we can see that the ultraviolet divergences appear due to the bad behavior at high energy of the propagator of the transversal and longitudinal components of the vector field. The same situation happened when we compute all others higher order \( 2n \)-point and \( (2n + m) \)-point Green’s functions.
As we have already known the origin of the ultraviolet divergences, our next step will be to construct a non-perturbative scheme of regularization inside the generating functional. Thus, we begin the regularization process from the generating functional (13) setting $C_\mu = 0$ which reads

$$Z[\eta, \bar{\eta}] = \int DA_\mu D\psi D\bar{\psi} \exp \left( i \int dx \bar{\psi} (i \partial + A) \psi + \frac{i}{2g} A_\mu A^\mu + \bar{\eta} \psi + \bar{\psi} \eta \right).$$  \hspace{1cm} (45)

We decompose the vector field as in (11) and make the following fermionic transformation

$$\psi \to e^{i \rho - i \gamma_5 \phi} \psi , \quad \bar{\psi} \to \bar{\psi} e^{-i \rho - i \gamma_5 \phi}.$$  \hspace{1cm} (46)

As it is well-known, the fermionic measure is not invariant under a chiral transformation [20]. On the other hand, we can also show that it is not invariant under a phase transformation [14, 19] due to the generalized prescription used to compute the fermionic determinant. Thus, the fermionic measure under the transformation (46) changes as

$$D\psi D\bar{\psi} \rightarrow J[\rho, \phi] D\psi D\bar{\psi},$$

where $J[\rho, \phi]$ is the Jacobian given by

$$J[\rho, \phi] = \frac{\det(i \partial + \partial \rho - i \phi)}{\det(i \partial)} = \exp \left( \int dx \frac{a - 1}{4\pi} \partial_\mu \rho \partial^\mu \rho - \frac{a + 1}{4\pi} \partial_\mu \phi \partial^\mu \phi \right).$$  \hspace{1cm} (47)

Inserting (41), (46) and (47) into the generating functional (15), we get

$$Z[\eta, \bar{\eta}] = \int D\rho D\phi D\psi D\bar{\psi} \exp \left( \int dx \bar{\psi} i \partial \psi + \frac{b - 1}{2\pi} \partial_\mu \rho \partial^\mu \rho - \frac{b}{2\pi} \partial_\mu \phi \partial^\mu \phi + \bar{\eta} e^{i \rho - i \gamma_5 \phi} \psi + \bar{\psi} e^{-i \rho - i \gamma_5 \phi} \eta \right).$$  \hspace{1cm} (48)

The fundamental observation here, is that we could make everything finite if we had a better UV behavior for the $\rho$ and $\phi$ propagators. Then, by following the scheme showed in [17], we add to the generating function two regulator fields $\alpha$ and $\beta$ with large masses $\Lambda_\alpha$ and $\Lambda_\beta$ respectively [20] ($\Lambda^2 \rightarrow +\infty$). Also, the interacting terms of $\rho$ and $\phi$ fields are changed as $\rho \rightarrow \rho + \alpha$ and $\phi \rightarrow \phi + \beta$. In this way, we get to define the regularized generating functional as being

$$Z^\lambda[\eta, \bar{\eta}] = \int D\rho D\phi D\psi D\bar{\psi} D\alpha D\beta \exp \left( \int dx \left[ \frac{b}{2\pi} \partial_\mu \rho \partial^\mu \rho + \frac{b}{2\pi} \partial_\mu \phi \partial^\mu \phi - \frac{b - 1}{2\pi} \partial_\mu \rho \partial^\mu \rho - \frac{b - 1}{2\pi} \partial_\mu \phi \partial^\mu \phi - \frac{b - 1}{2\pi} \partial_\mu \alpha \partial^\mu \alpha - \frac{b - 1}{2\pi} \partial_\mu \beta \partial^\mu \beta \right] \right) \exp \left( \int dx \bar{\psi} i \partial \psi + \bar{\eta} e^{i \rho - i \gamma_5 \phi} \psi + \bar{\psi} e^{-i \rho - i \gamma_5 \phi} \eta \right).$$  \hspace{1cm} (49)

Now, before carrying out the integration of $\alpha$ and $\beta$ fields, we make a change of variables $\rho \rightarrow \rho - \alpha$ and $\phi \rightarrow \phi - \beta$. Thus, we obtain the following expression for the regularized generating functional

$$Z^\lambda[\eta, \bar{\eta}] = \int D\rho D\phi D\psi D\bar{\psi} \exp \left( \int dx \bar{\psi} i \partial \psi + \bar{\eta} e^{i \rho - i \gamma_5 \phi} \psi + \bar{\psi} e^{-i \rho - i \gamma_5 \phi} \eta \right) \exp \left( \int dx \frac{b}{2\pi \Lambda_\alpha^2} \phi \Box (\Box + \Lambda_\alpha^2) \phi - \frac{b - 1}{2\pi \Lambda_\alpha^2} \rho \Box (\Box + \Lambda_\alpha^2) \rho \right).$$  \hspace{1cm} (50)

In this way we were able to improve the ultraviolet behavior of both propagators. Being able to regularize those propagators, we are going to express the regularized generating functional in function of the original fields as in (15). Doing the change of fermionic variables inverse to that one made in (46), and expressing fields $\rho$ and $\phi$ as functions of the field $A_\mu$ by using (41), we obtain

$$Z^\lambda[\eta, \bar{\eta}] = \int DA_\mu D\psi D\bar{\psi} \exp \left( \int dx \bar{\psi} (i \partial + A) \psi + \frac{i}{2g} A_\mu A^\mu + \frac{b}{2\pi \Lambda_\alpha^2} (\partial \cdot A)^2 - \frac{b - 1}{2\pi \Lambda_\alpha^2} (\partial \cdot A)^2 + \bar{\eta} \psi + \bar{\psi} \eta \right).$$  \hspace{1cm} (51)

Comparing the Lagrangian densities that appear in the generating functionals (15) and (11), we can see that the regularization process has given rise to two new terms which depend explicitly on the Pauli-Villars cut-offs, such terms are enough to turn completely finite the model. We can to compute all the regularized fermionic Green’s functions starting from (50) or (15).

By integrating the auxiliary field $A_\mu$, we get the regularized generating functional in terms of the fermion fields alone,

$$Z^\lambda[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} \exp \left( \int dx \bar{\psi} i \partial \psi - \frac{g}{2} (\bar{\psi} \gamma^\mu \psi)^2 + \frac{g}{2} (\bar{\psi} \gamma^\mu \psi D^\lambda_{\mu\nu} \bar{\psi} \gamma^\nu \psi + \bar{\eta} \psi + \bar{\psi} \eta) \right),$$  \hspace{1cm} (52)
where the operator $D_{\mu\nu}^\Lambda$ carries all the information about the non–local counter-terms which regularize the fermionic theory,

$$D_{\mu\nu}^\Lambda = \frac{\partial_{\mu} \partial_{\nu}}{\Box + \frac{\pi \Lambda^2}{g(b-1)}} + \frac{g_{\mu\nu} \Box - \partial_{\mu} \partial_{\nu}}{\Box + \frac{\pi \Lambda^2}{gb}},$$

as we can see, it shows explicitly the longitudinal and transversal contributions that regularize the respective components of the fermionic current–current interaction.

### 3.4.1 Regularized and renormalized fermion propagator

We start from the regularized generating functional, integrating over the fermionic fields it reads as

$$Z^\Lambda[\eta, \bar{\eta}] = \int \mathcal{D} \rho \mathcal{D} \phi \exp \left( i \int dx \frac{b}{2\pi \Lambda^2} \phi \Box \left( \Box + \Lambda^2 \right) \phi - \frac{b-1}{2\pi \Lambda^2} \rho \Box \left( \Box + \Lambda^2 \right) \rho \right) \exp \left( -i \int dx dy \bar{\eta}(x) G(x, y; \rho, \phi) \eta(y) \right),$$

where the Green’s function $G(x, y; \rho, \phi)$ is given by. Then, the regularized fermion propagator in configuration space is given by

$$G^\Lambda(x - y) = i \exp \left( i \int \frac{dk}{(2\pi)^2} \left[ \frac{\pi \Lambda^2}{b-1} \frac{1}{k^2(k^2 - \Lambda^2)} - \frac{\pi \Lambda^2}{b} \frac{1}{k^2(k^2 - \Lambda^2)} \right] [1 - e^{-ik(x-y)}] \right) G_F(x - y),$$

performing the momentum integration in the exponential and introducing the arbitrary mass scale $M$, the propagator is

$$G^\Lambda(x, y) = \frac{1}{2\pi i} \left( \frac{x - y}{(x - y)^2 - i\varepsilon} \right) \left( \frac{\Lambda^2}{M^2} \right) - \frac{1}{4(b-1)} \left( \frac{\Lambda^2}{M^2} \right) \frac{1}{4b} \left( - M^2(x - y)^2 + i\varepsilon \right)^{-1/c},$$

where the parameter $c$ is defined as being

$$c = 4b(b-1) = \left( \frac{2\pi}{g} + a \right)^2 - 1,$$

we can see that $c \in [-1, +\infty)$ and $c^{-1} \in (-\infty, -1] \cup < 0, +\infty >$.

In this way, we can renormalize exactly in configuration space. The renormalization procedure establishes the following relation between renormalized and bare (regularized) 1PI Green’s function, $\Gamma^R = Z_\psi \Gamma^\Lambda$, where $Z_\psi$ is the fermionic wave function renormalization constant that relates the bare and renormalized fermionic fields, $\psi_{bare} = Z_\psi \psi_R$. Thus, from, we choose the fermionic wave function renormalization constant to be

$$Z_\psi = \left( \frac{\Lambda^2}{M^2} \right) - \frac{1}{4(b-1)} \left( \frac{\Lambda^2}{M^2} \right) \frac{1}{4b}.$$

It can be shown that a renormalization of the fermionic wave function is sufficient to get renormalize all the Green’s functions of the model.

Then, the renormalized propagator of massless Thirring model in configuration space is read as

$$G^R(x - y) = \frac{1}{i 2\pi (x - y)^2 - i\varepsilon} \left( - M^2(x - y)^2 + i\varepsilon \right)^{-1/c},$$

in the limit $g \to 0$ we also obtain the propagator of a free massless fermion field.

Then, by computing the Fourier transform we get the renormalized fermion propagator in momentum space,

$$G^R(p) = \frac{i}{p} \left( \frac{M^2}{p^2} \right) \frac{1}{c} e^{-ip/c} \frac{\Gamma(1 - 1/c)}{\Gamma(1 + 1/c)},$$

where $M = 2M$. The dynamical or anomalous dimension of the massless Thirring fermion field is $1/c$. By choosing an adequate value for the ambiguity parameter we can reproduce results such as has been obtained in.
This value is got when we set \( c = 2\pi/g \), thus, the ambiguity parameter which reproduces the result to the anomalous dimension found in the literature is

\[
a_\epsilon := -\frac{2\pi}{g} \pm \sqrt{1 + \frac{2\pi}{g}}. \tag{61}
\]

In this value we will obtain a result which agrees well with Glaser’s analysis of the massless Thirring model \cite{22, 23} and spectral analysis of two-point correlation functions in 1+1-dimensional quantum field theories developed by Schröer \cite{24}.

The renormalized propagator \( \langle 60 \rangle \) has singularities in \( \frac{1}{g} = \pm 1, \pm 2, \ldots \) And, if we set \( c = 2\pi/g \) the propagator \( \langle 60 \rangle \) has singularities in \( g = \pm 2\pi \) such as it was shown by Klaiber analysis \cite{23}.

4 The gauge invariant case: \( a = 1 - \frac{2\pi}{g} \)

As we have observed in Section 2, the Ward identity for the 1PI 2-point function of the fermionic current reveals the existence of a local \( U(1) \) gauge symmetry for \( a = 1 - \frac{2\pi}{g} \). For this value such symmetry is explicit in the effective action for the vector field in the generating functional \( \langle 7 \rangle \), i.e., we can not to perform the functional integration. Therefore, we need to construct a well-defined generating functional for this gauge theory. Such construction is making via the Faddeev-Popov procedure \cite{25} with Lorentz gauge fixing condition \( f(A_\mu) = \frac{1}{\sqrt{\xi}} \partial \cdot A \) in the generating functional \( \langle 4 \rangle \). Thus, the resulting generating functional for this gauge invariant theory reads as

\[
Z'[\eta, \bar{\eta}, C_\mu] = \int DA_\mu D\psi D\bar{\psi} \exp \left( i \int dx \bar{\psi}(i\partial + A + \mathcal{G})\psi + \frac{1}{g} A_\mu A^\mu - \frac{1}{2\xi} (\partial \cdot A)^2 + \bar{\eta}\psi + \bar{\psi}\eta \right), \tag{62}
\]

where we introduce the external source \( C_\mu \) for the fermionic current. From this generating functional we can compute all the Green’s functions and we can also to construct the Schwinger–Dyson equations (SDE) and the Ward identities.

4.1 The Green’s functions

4.1.1 The fermion propagator \( G^c(x - y) = \langle 0| T\psi(x)\bar{\psi}(y)|0 \rangle \)

We set \( C_\mu = 0 \) in the generating functional \( \langle 62 \rangle \) and integrating the fermionic degrees of freedom we obtain

\[
Z^c[\eta, \bar{\eta}] = \int DA_\mu \exp \left( i \int dx \frac{1}{2} A_\mu \left[ \frac{1}{\pi} g^{\mu\nu} - \frac{1}{\pi} \partial^\mu \partial^\nu + \frac{1}{\xi} \partial^\mu \partial^\nu \right] A_\nu \right) \exp \left( -i \int dx dy \bar{\eta}(x) G(x, y; A)\eta(y) \right), \tag{63}
\]

where the Green’s function \( G(x, y; A) \) is given in \( \langle 7 \rangle \). From this, we calculate the fermion propagator which becomes

\[
G^c(x - y) = i \exp \left( i \int \frac{dk}{(2\pi)^2} \left( \frac{\xi}{k^2} + \frac{\pi}{k^2} \right) \left[ 1 - e^{-ik \cdot (x - y)} \right] \right) G_F(x - y), \tag{64}
\]

We can see the existence of logarithmic UV divergences coming from the term in the integral proportional to \( k^{-2} \) and, we also observe that the term proportional to the gauge parameter \( \xi \) generates serious infrared divergences which can not be renormalized. These infrared divergences proportional to \( \xi \) are also observed in the all Green’s functions computed from the generating functional \( \langle 62 \rangle \), then, we use a gauge free of infrared divergences \( \xi = 0 \).

4.1.2 The current propagator \( J^c_{\mu\nu} = \langle 0| Tj_\mu(x) J_{\nu}(y)|0 \rangle \)

In momentum space the propagator of the fermionic current is calculated to be

\[
i J^c_{\mu\nu}(q) = \frac{1}{g} \left( 1 - \frac{\pi}{g} \right) T_{\mu\nu} + \frac{1}{g} \left( \frac{\xi}{gk^2} - 1 \right) L_{\mu\nu} \tag{65}
\]

it is free of UV divergences.

4.1.3 The vertex function \( G^c_\mu(x, y; z) = \langle 0| T\psi(x)\bar{\psi}(y)J_{\mu}(z)|0 \rangle \)

The vertex function is calculated to be

\[
G^c_\mu(x, y; z) = i \int \frac{dk}{(2\pi)^2} \left( \frac{\xi}{gk^2} - 1 \right) \bar{k}_\mu \frac{k_\mu}{k^2} + \frac{\pi}{gk^2} \gamma_5 \left( e^{-ik \cdot (z - x)} - e^{-ik \cdot (z - y)} \right) G^c(x - y), \tag{66}
\]
in momentum space it reads as
\[
G^\xi_\mu(p, -p - q; q) = i \left[ \left( \frac{\xi}{g q^2} - 1 \right) \frac{q_\mu}{q^2} + \frac{\bar{q}_\mu}{g q^2} \gamma_5 \right] \left[ G^\xi(p + q) - G^\xi(p) \right].
\]

where \(G^\xi(p)\) is the Fourier transforms of the fermionic propagator \(\mathcal{Z}^\xi\). Thus, we can note that the UV divergences of this Green’s function come from the fermionic propagator.

4.1.4 The 1PI functions

In momentum space the 1PI 2-point fermion function \(\Gamma^\xi(p)\) is
\[
\Gamma^\xi(p) = i \left[ G^\xi(p) \right]^{-1}.
\]

Also, the 1PI 2-point function for the fermionic current \(\Gamma^\xi_{\mu\nu}(q)\) is
\[
\Gamma^\xi_{\mu\nu}(q) = -\frac{g}{1 - \frac{\xi}{g q^2}} T_{\mu\nu} - \frac{g}{\xi - 1} L_{\mu\nu}
\]

The 1PI 3-point function, \(\Gamma^\xi_{\mu}(p, k; q)\) related to the vertex function \(G^\xi_{\mu}(p, k; q)\) is given by
\[
\Gamma^\xi_{\mu}(p, -p - q; q) = \left( -g \frac{q_\mu}{q^2} + \frac{\bar{q}_\mu}{g q^2} \gamma_5 \right) \left[ \Gamma^\xi(p + q) - \Gamma^\xi(p) \right]
\]

4.2 The Schwinger-Dyson equations

We start from the well-defined gauge fixed generating functional \(\mathcal{Z}^\xi\) which after the integration of the vector field reads
\[
\mathcal{Z}^\xi[\eta, \bar{\eta}, C_{\mu}] = \int\!D\psi D\bar{\psi} \exp \left( i \int\!dx \bar{\psi}(i\partial + \xi\bar{\eta})\psi - \frac{g}{2} j_\mu j^\mu + \frac{g}{2} j_\mu j^\nu D_{\mu\nu}^\xi + \bar{\eta}\psi + \bar{\psi}\eta \right),
\]

where the third term appears due to the gauge fixing condition, and the operator \(D_{\mu\nu}^\xi\) is given as
\[
D_{\mu\nu}^\xi = \frac{\partial_{\mu} \partial_{\nu}}{\Box + \frac{\xi}{g}}.
\]

By using the functional representation for the fields \(\bar{\psi}, \psi\) in the equation of motion for the fermionic field, we get the quantum equation of motion for the generating functional \(\mathcal{Z}^\xi\)

\[
\left( \delta \frac{\delta}{\delta \bar{\eta}(x)} + g \gamma^\mu \frac{\delta}{\delta C^\mu(x)} - i C_{\mu}(x) \gamma^\mu \frac{\delta}{\delta \bar{\eta}(x)} + \eta(x) - g \gamma^\mu \frac{\delta}{\delta \bar{\eta}(x)} D_{\mu\nu}^\xi \frac{\delta}{\delta C_{\nu}(x)} \right) \mathcal{Z}^\xi[\eta, \bar{\eta}, C^\mu] = 0
\]

The last term makes a difference between the SDE’s \(\mathcal{Z}^\xi\) and \(\mathcal{Z}^\xi_{\text{non-gauge}}\) but it arises due to the introduction of a gauge fixing term into the generating functional \(\mathcal{Z}^\xi\) for well-defined it. Thus, in a formal treatment the gauge fixing term is not present and therefore the SDE for the gauge invariant phase is exactly the same as in \(\mathcal{Z}^\xi_{\text{non-gauge}}\) for the non-gauge invariant phase .

Thus, the SDE for the fermion propagator in the gauge invariant phase is given by
\[
i \partial_x G^\xi(x - y) = i \delta(x - y) + g \gamma^\mu G^\xi_{\mu}(x, y; z) - g \gamma^\mu \lim_{z \to x} \partial_x G^\xi_{\mu}(x, y; z)
\]

Both the fermionic propagator \(\mathcal{Z}^\xi\) and the vertex function \(\mathcal{Z}^\xi_{\text{non-gauge}}\) computed from the generating functional \(\mathcal{Z}^\xi\) are also solutions for the quantum equation of motion of the massless Thirring model.

In momentum space the SDE for the fermion propagator reads as
\[
G^\xi(p) = i \frac{1}{p} - i \frac{dk}{(2\pi)^2} f^\xi(k) \frac{1}{p} G^\xi(p - k),
\]

where the function \(f^\xi(k)\) reads
\[
f^\xi(k) = \frac{\xi}{k^4} + \frac{\pi}{k^2}.
\]

Similarly to the case non–gauge invariant \(\mathcal{Z}^\xi\) by iterating the SDE it generates an expansion in \(\hbar\) or equivalently in powers of \(f^\xi(k)\). The power counting analysis of the series shows clearly the existence of the same logarithmic UV divergences in each order or loop what indicates that the model is a fully renormalizable field theory.
4.3 The Ward identities

We start from the well-defined generating functional \( \Gamma_{\mu\nu}(x) \) and we make the following fermionic transformation

\[
\psi \to e^{i\lambda(x)}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-i\lambda(x)},
\]

one more time it is worthwhile to note that in our framework the fermionic measure is not invariant under this transformation changing as \( D\psi D\bar{\psi} \to J^x[A,\lambda] D\psi D\bar{\psi} \) where \( J^x[A,\lambda] \) is the Jacobian given by

\[
J^x[A,\lambda] = \exp \left( i \int \! dx \, - \frac{1}{2g} (A_\mu + C_\mu) \partial^\mu \lambda + \frac{1}{2g} (\partial_\mu \lambda)^2 \right).
\]

The generating functional after the fermionic transformation reads as

\[
Z^x[\eta, \bar{\eta}, C_\mu] = \int D A_\mu D\psi D\bar{\psi} J^x[A] \exp \left( i \int \! dx \, \bar{\psi} (i\partial + A + \xi - \partial\lambda) \psi + \frac{1}{2g} AA - \frac{1}{2g} (\partial A)^2 + \bar{\eta} e^{i\lambda(x)} \psi + \bar{\psi} e^{-i\lambda(x)} \eta \right).
\]

At once we make the transformation \( A_\mu \to A_\mu + \partial_\mu \lambda \) and after integrating over the vector field we obtain

\[
Z^x[\eta, \bar{\eta}, C_\mu] = \int D\psi D\bar{\psi} \exp \left( i \int \! dx \, \bar{\psi} (i\partial + A + \xi - \partial\lambda) \psi - \frac{g}{2} (\bar{\psi} \gamma^\mu \psi)^2 + \frac{g}{2} j^\mu D^x_{\mu\nu} j^{\nu} + \bar{\eta} e^{i\lambda(x)} \psi + \bar{\psi} e^{-i\lambda(x)} \eta \right)
\]

\[
\exp \left( i \int \! dx \, - \frac{1}{g} C_\mu \partial^\mu \lambda + \frac{\Box}{g} \partial_\mu j^{\mu} - \frac{1}{2g} \lambda \frac{\Box^2}{g} \right).
\]

Considering \( \lambda(x) \) as an infinitesimal function and expanding the exponential to first order in \( \lambda \) we get the functional equation satisfied by the generating functional \( \Gamma_{\mu\nu}(x) \),

\[
\left( \bar{\eta} \frac{\delta}{\delta \eta} - \eta \frac{\delta}{\delta \bar{\eta}} + \frac{1}{g} \partial_\mu C^\mu - i \frac{\Box}{g} \partial_\mu \frac{\delta}{\delta C^\mu} \right) Z^x[\eta, \bar{\eta}, C_\mu] = 0
\]

where we have used the functional representations for the fields \( \psi, \bar{\psi} \).

We define the generating functional of the connected Green’s functions \( W^x[\eta, \bar{\eta}, C_\mu] \) as being \( W^x = -i \ln Z^x \) and the generating functional of the 1PI functions \( \Gamma^x[\bar{\psi}, \psi, j^\mu] \) which are related via a Legendre transformation similar to \( \delta \ln Z^x = \bar{\eta} \frac{\delta}{\delta \eta} + \eta \frac{\delta}{\delta \bar{\eta}} \). Using the functional relations given in \( \delta \ln Z^x \), we obtain the functional equation satisfied by the generating functional \( \Gamma^x[\bar{\psi}, \psi, j^\mu] \) from which we get all Ward identities

\[
i \frac{\delta \Gamma^x}{\delta \bar{\psi}(x)} \psi(x) - i \frac{\delta \Gamma^x}{\delta \psi(x)} \bar{\psi}(x) - \frac{1}{g} \partial_\mu \frac{\delta \Gamma^x}{\partial j^{\mu}(x)} + \frac{\Box}{g} \partial_\mu j^{\mu}(x) = 0.
\]

If we observe the same functional equation in the non–gauge invariant phase \( \xi \), the equation above can be obtained by setting \( a = 1 - \frac{2\pi}{g} \). The last term appears due to the gauge fixing condition used to define the generating functional of the Green’s functions in this phase.

The first Ward identity is to the 1PI 2-point function \( \Gamma_{\mu\nu}(q) \) for the fermionic current which reads as

\[
q^\mu \Gamma_{\mu\nu}(q) = g \frac{q^2}{q^2 - \xi} q^\nu
\]

as it can be shown explicitly from the equation \( \delta \ln Z^x \). Thus, the right-handed side above is gauge dependent what guarantees that the current propagator is transverse for the critical value \( a = 1 - \frac{2\pi}{g} \), as it was noted in \( \delta \ln Z^x \).

The second Ward identity relates the 1PI 3-point function \( \Gamma_{\mu}(p; q) \) and the 1PI 2-point fermion function \( \Gamma^x(p) \) is given by

\[
q^\mu \Gamma_{\mu}(p; q) = -g \left[ \Gamma^x(p + q) - \Gamma^x(p) \right]
\]

likewise it can also be shown easily from the equation \( \delta \ln Z^x \). This identity shows that the coupling constant does not renormalize due to a renormalization of the fermionic propagator guarantees the renormalization of this 1PI 3-point function.
4.4 Regularization and renormalization of the UV divergences

We will study the divergences of the pure fermionic Green’s functions (Green’s functions without current insertions) following the procedure achieved in the subsection 3.3 for the non–gauge invariant phase. Therefore, starting from the generating functional (83) and decomposing the vectorial field $A_\mu$ in their longitudinal and transversal parts (11) we arrive to the following expression for the generating functional for the all pure fermionic Green’s functions

$$Z[\eta, \bar{\eta}] = \int D\rho D\phi \exp \left( i \int dx - \frac{1}{2\pi} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2\xi} (\Delta \phi)^2 \right) \exp \left( -i \int dy \, \bar{\eta}(x) G(x, y; \rho, \phi) \eta(y) \right),$$  \tag{85}

where the Green’s function $G(x, y; \rho, \phi)$ is given by equation (13). Thus, in order to compute the fermionic propagator we take functional derivatives with respect to the external fermionic sources and setting $\eta = \bar{\eta} = 0$ we obtain

$$G^\xi(x - y) = \int D\rho D\phi \exp \left( i \int dx - \frac{1}{2\pi} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2\xi} (\Delta \phi)^2 \right) G(x, y; \rho, \phi),$$  \tag{86}

where the integrations over the field $\rho$ and $\phi$ are factorized. It can easily be observed that the UV divergences come from the integration on the transversal field $\phi$ due to the bad behavior of its propagator at high energy, thus, we need to improve its behavior at high energy to be able to control the UV divergences.

Therefore, we start the regularization process from the generating functional (12) setting $C_\mu = 0$, then, we decompose the vector field as in (11) and make the fermionic transformation (10) which changes the fermionic measure as $D\psi D\bar{\psi} \rightarrow J^x[\rho, \phi] D\psi D\bar{\psi}$, where $J^x[\rho, \phi]$ is the Jacobian given by

$$J^x[\rho, \phi] = \exp \left( i \int dx - \frac{1}{2g} \partial_\mu \rho \partial^\mu \rho - \frac{1}{2g} \left( \frac{g}{\pi} - 1 \right) \partial_\mu \phi \partial^\mu \phi \right).$$  \tag{87}

Finally, we get the following expression for the generating functional

$$Z^\xi[\eta, \bar{\eta}] = \int D\rho D\phi D\psi D\bar{\psi} \exp \left( i \int dx \, \bar{\psi} i \partial_\mu \psi - \frac{1}{2\xi} (\Delta \phi)^2 - \frac{1}{2\pi} \partial_\mu \phi \partial^\mu \phi + \bar{\eta} e^{i\rho - i\gamma_5 \phi} \psi + \bar{\psi} e^{i\rho - i\gamma_5 \phi} \eta \right)$$  \tag{88}

Then, following the regularization procedure described between the equations (13) and (11) -for the non–gauge invariant phase- but improving only the $\phi$–propagator. Such procedure leads to the following regularized generating functional for the gauge invariant phase

$$Z^M[\eta, \bar{\eta}] = \int D\rho D\phi D\psi D\bar{\psi} \exp \left( i \int dx \, \bar{\psi} i \partial_\mu \psi - \frac{1}{2\xi} (\Delta \phi)^2 + \frac{1}{2\pi \Lambda_3^2} \partial_\mu (\Delta + \Lambda_3^2) \phi + \bar{\eta} e^{i\rho - i\gamma_5 \phi} \psi + \bar{\psi} e^{i\rho - i\gamma_5 \phi} \eta \right).$$  \tag{89}

Such regularized generating functional expressed in terms of the original fields reads as

$$Z^\Lambda[\eta, \bar{\eta}] = \int DA_\mu D\psi D\bar{\psi} \exp \left( i \int dx \, \bar{\psi} (i \partial_\mu + A_\mu) \psi + \frac{1}{2g} A_\mu A^\mu + \frac{1}{2\pi \Lambda^2_3} (\partial^\mu A_\mu)^2 - \frac{1}{2\xi} (\partial_\mu A_\mu)^2 + \bar{\eta} \psi + \bar{\psi} \eta \right)$$  \tag{90}

or, we can also express it as a pure fermionic theory integrating the auxiliary field $A_\mu$

$$Z^\Lambda[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} \exp \left( i \int dx \, \bar{\psi} i \partial_\mu \psi - \frac{g}{2} (\gamma^\mu \psi)^2 + \frac{g}{2} \bar{\psi} \gamma^\mu \psi D^\Lambda_{\mu \nu} \gamma^\nu \psi + \bar{\eta} \psi + \bar{\psi} \eta \right),$$  \tag{91}

where the operator $D^\Lambda_{\mu \nu}$ is defined as

$$D^\Lambda_{\mu \nu} = \frac{\partial_\mu \partial_\nu}{\Box + \frac{\xi}{g}} + \frac{g_{\mu \nu} \Box - \partial_\mu \partial_\nu}{\Box + \frac{\pi \Lambda^2_3}{g}}.$$  \tag{92}

The third term in the regularized generating functional (91) has two contributions: the longitudinal one which has already observed in (14) is due to the gauge fixing condition and the transversal one that arises due to the regularization procedure of the UV divergences. Thus, the gauge fixing condition acts as a natural regulator of the longitudinal part of the current–current interaction and the Pauli-Villars method was necessary to regularize the transversal component of the interaction.
4.4.1 Regularized and renormalized fermion propagator

From the regularized generating functional (89) we can compute all the regularized pure fermionic Green’s functions of the local gauge invariant model. Thus, for arbitrary gauge $\xi$ the regularized fermion propagator is given by

$$G^\Lambda (x-y) = \exp \left( i \int \frac{d^D k}{(2\pi)^D} \left[ \frac{\xi}{k^2} - \frac{\pi \Lambda_s^2}{k^2(k^2 - \Lambda_s^2)} \right] \left[ 1 - e^{-i k \cdot (x-y)} \right] \right) G_F (x-y),$$  \hspace{1cm} (93)

Now, going to the $\xi = 0$ gauge we compute the remaining integration to obtain the following expression for the regularized fermion propagator

$$G^\Lambda (x-y) = \frac{1}{2\pi i} \frac{x-y}{(x-y)^2 - i\epsilon} \left( -M^2(x-y)^2 + i\epsilon \right)^{1/4}$$  \hspace{1cm} (94)

where we have introduced an arbitrary mass scale $M$.

Immediately, we renormalize in configuration space by choosing the fermionic wave function renormalization constant as being

$$Z^\xi = \left( \frac{\tilde{\Lambda}^2}{M^2} \right)^{1/4}.$$  \hspace{1cm} (95)

Thus, in configuration space the renormalized propagator of massless Thirring model in, its gauge invariant phase, reads as

$$G^{\xi} (x-y) = \frac{1}{2\pi i} \frac{x-y}{(x-y)^2 - i\epsilon} \left( -M^2(x-y)^2 + i\epsilon \right)^{1/4}.$$  \hspace{1cm} (96)

Computing the Fourier transform we obtain renormalized fermion propagator as being

$$\tilde{G}^{\xi} (p) = \frac{i}{p} \left( \frac{\tilde{M}^2}{p^2} \right)^{1/4} e^{i\pi/4} \frac{\Gamma(5/4)}{\Gamma(3/4)}.$$  \hspace{1cm} (97)

This propagator deserves some comments. It does not have any explicit dependence in the coupling constant nor in the ambiguity parameter. Moreover, it is similar to the infrared behavior of the fermionic propagator of the Schwinger model [26, 27, 28], which has the confinement phenomenon as one of their characteristics. And, the Schwinger model presents chiral symmetry breaking due to the correlation functions of the chiral densities break the cluster property which leads to $\theta$—vacuum structure. Thus, two important quantum field theoretical phenomena could also be natural characteristics of the massless Thirring model in its gauge invariant phase.

5 Remarks and conclusions

The massless Thirring model was analyzed in a non–perturbative way by using functional integral method. The key of the present analysis is the calculus of the fermionic determinant which was made using generalized prescription to control the UV divergences that appear in its computation. This arbitrary procedure introduces an ambiguity or Jackiw–Rajaraman parameter $a$ and it allows to obtain the new solutions for the massless Thirring model.

Next, we explain the contribution of the ambiguity for the new solutions: the fermionic determinant is defined by the following functional integral

$$\det (i\partial + A) = \int D\psi D\overline{\psi} \exp \left( i \int dx \overline{\psi} (i\partial + A) \psi \right)$$  \hspace{1cm} (98)

if we make the following local phase transformation

$$\psi \rightarrow e^{i\rho(x)} \psi, \quad \overline{\psi} \rightarrow \overline{\psi} e^{-i\rho(x)}$$  \hspace{1cm} (99)

in our prescription the fermionic measure is not invariant under such transformation and there is a nontrivial Jacobian given by [12 13]

$$J_\rho = \frac{\det (i\partial + A)}{\det (i\partial + A - \partial \rho)} = \exp \left( i \int dx \frac{a-1}{2\pi} A^\mu \partial_\mu \rho - \frac{a-1}{4\pi} \partial_\mu \rho \partial^\mu \rho \right).$$  \hspace{1cm} (100)
It shows the explicit breaks of a local $U(1)$ symmetry which, in general, is associated to the conservation of the fermionic current. But such one local symmetry does not exist for the massless Thirring model at classical level, thus, we are free to compute the fermionic determinant in an arbitrary prescription characterized by the ambiguity parameter $a$.

At quantum level the Ward identity for the 1PI 2–point for the fermionic current, for $a 
eq 1$, allows to choose a value for the ambiguity parameter in which it is possible to implement a local $U(1)$ gauge symmetry for the model. This fact is reflected after performing the fermionic integration, at this point it arises a problem when we wish to perform the functional integration over the auxiliary vector field for all values of the ambiguity parameter due to there is a critical value $a = 1 - \frac{2\pi}{g}$ where it is not possible to be performed. For this value, the effective action of the vector field gains a local $U(1)$ gauge symmetry and we use the Faddeev–Popov technique to well–define the functional integration. Consequently, we named as non–gauge invariant phase to the solutions with $a 
eq 1 - \frac{2\pi}{g}$ and gauge invariant phase to the solution with $a = 1 - \frac{2\pi}{g}$. We also show that the Green’s functions, in both phases, are solutions of the Schwinger-Dyson equation for the massless Thirring model which guarantees the validity of our approach.

We identify clearly the origin of the UV logarithmic divergences present in the Green functions and make a non–perturbative regularization into their generating functional via the Pauli–Villars method. Such detailed analysis of the divergence structure is fundamental to carry out the covariant non–perturbative regularization of the UV divergences, as it was observed in [17]. The UV divergences has a non–perturbative character due to they cannot be obtained via the usual perturbative technique which uses the coupling constant as parameter expansion; this divergence type has already observed in the anomalous and chiral Schwinger model [19]. Accordingly, the regularization procedure modifies the components transversal and longitudinal of the current–current interaction yielding us the non–perturbative counter–terms which make possible the full renormalization of the model in both phases.

A straightforward calculation for the higher fermionic correlation functions from the regularized Lagrangian [12] and [9] shows that they have already regularized. This fact is notably important for the full renormalization of the model, because it implies in the non existence of additional divergences in the theory. Thus, in both phases, the coupling constant $g$ does not renormalize such as it can be shown from the Ward’s identities [10] and [8] what was also observed in the literature [6]. The Green’s functions for the composite operators, as for example, chiral densities need an additional renormalization procedure what will be reported in a following paper.

In the non–gauge invariant phase there are some values of the ambiguity parameter that reproduce the solutions found in the literature computed by different methods or techniques. On the other hand, the gauge invariant phase is a solution which is implemented, for the first time, in the present work and it can not be reproduced for no value of the ambiguity parameter or coupling constant related to the non symmetric phase.

In the gauge invariant phase, we note that for a general value of the gauge parameter $\xi$, the Green’s functions are plagued of infrared divergences but in the $\xi = 0$ gauge they are infrared finite, then, we can work in this gauge. On the other hand, the non–symmetric phase is free of infrared divergences.

The gauge invariant phase shows some peculiar features, the first feature is a simple equation (11) which relates the ambiguity parameter $a$ and the coupling constant $g$. Possibly, it could be fixed the ambiguity problem in the definition of the fermionic current in the present model or given a guide to solve it in the massive Thirring model [14]. This possibility comes at quantum level where the model gets a local gauge symmetry which allows to fix the ambiguity related to regularization freedom originated from the classical level. The other one is the fermionic propagator structure which resembles the behavior of the infrared limit for the fermion propagator of the Schwinger model [26] [27] [28]. As it is well-known the Schwinger model displays many characteristics of (1+3) dimensional more realistic gauge theories, such as confinement or $\theta$–vacuum structure, such characteristics could emerge in the present phase. The study of the Hilbert space structure of the symmetric phase would allow understanding if there is fermion confinement and would identify the physical operators establishing clearly which symmetries are present and which are violated at quantum level. These questions mentioned above are at the moment under investigation and results will be reported elsewhere.

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