Harmonic states for the free particle

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Abstract

Different families of states, which are solutions of the time-dependent free Schrödinger equation, are imported from the harmonic oscillator using the quantum Arnold transformation introduced in Aldaya et al (2011 J. Phys. A: Math. Theor. 44 065302). Among them, infinite series of states are given that are normalizable, expand the whole space of solutions, are spatially multi-localized and are eigenstates of a suitably defined number operator. Associated with these states new sets of coherent and squeezed states for the free particle are defined representing traveling, squeezed, multi-localized wave packets. These states are also constructed in higher dimensions, leading to the quantum mechanical version of the Hermite–Gauss and Laguerre–Gauss states of paraxial wave optics. Some applications of these new families of states and procedures to experimentally realize and manipulate them are outlined.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In the context of the quantum free particle, the eigenstates of the Hamiltonian, which are also eigenstates of the momentum operator, are not normalizable. These states, the plane waves, are fully delocalized. However, it is customary to expand any normalizable solution of the free, time-dependent, Schrödinger equation in terms of plane waves, using the Fourier transform, building in this way wave packets which represent localized solutions and that are no longer eigenstates of the Hamiltonian. The simplest examples are the Gaussian wave packets, which have the property of minimizing the uncertainty relations between the position and the momentum operator.

In [1], the quantum Arnold transformation (QAT) was introduced, mapping states and operators from certain quantum systems to the free particle. This transformation is the quantum version of the Arnold transform [2], which is a change of variables in position and time that maps solutions $\chi(t')$ of a certain type of classical equation of motion, a nonhomogeneous...
linear second-order ordinary differential equation (LSODE), to solutions $x(t)$ of the classical equation for the free particle.

In this paper, we construct different families of states in the space of solutions of the quantum free particle which are the image under the QAT of the corresponding families in the quantum harmonic oscillator (HO). Among them, we provide discrete bases of free particle states mapped from the eigenstates of the HO. They are not eigenstates of the free particle Hamiltonian, i.e. they are not stationary states, but, rather, eigenfunctions of a certain quadratic operator $\hat{N}$ with discrete eigenvalues. The first states of these bases are Gaussian wave packets, and are localized with arbitrary initial size, related to the oscillator frequencies and chosen at will, and have initial minimal uncertainties. The following ones are ‘multi-localized’ in the sense that, for instance, in one dimension the $n$th-order state presents $n$ zeros and $n + 1$ humps which spread out with time. This mimics the situation for the HO to such an extent that, in one dimension, it is possible to build creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$. The number operator proves to be $\hat{N} \sim \hat{a}^\dagger \hat{a}$, although in the case of the free particle it is not the Hamiltonian. Going even further, we give a set of ‘coherent’ and ‘squeezed’ states which are interpreted as traveling, squeezed wave packets. This construction can easily be generalized to higher dimensions in different coordinate systems.

These families of states and their construction through the QAT can be of physical relevance in the description and preparation of initial states, and even time-evolution processes, in atom optics, ion trap physics and Bose–Einstein condensates. For instance, the process of turning off and on a harmonic potential trapping a set of atoms, as well as the production of coherent and squeezed states in this context, is easily achieved by means of the QAT. It might also be useful in describing scattering processes in a discrete basis, instead of the usual approach employing plane waves.

The content of the paper is as follows. Section 2 is devoted to the construction of the discrete bases of states in one dimension by means of the QAT, mapping the eigenstates of the HO to the free particle Hilbert space. Section 3 deals with the construction of coherent and squeezed states and the interpretation of them as traveling, squeezed wave packets. Section 4 generalizes this construction to two and three dimensions in Cartesian, polar and spherical coordinates, giving rise to Hermite–Gauss, Laguerre–Gauss and spherical-Gauss wave packets, respectively. Section 5 presents some possible physical applications and proposes experimental settings for producing and manipulating these states. Section 6 gives some possible generalizations of these constructions and outlines possible lines of research. Finally, an appendix collects the main formulas for the computation of coherent and squeezed states for the free particle.

2. Discrete bases of wave packets in one dimension

Let $\mathcal{H}$ be the Hilbert space of solutions of the free particle, time-dependent Schrödinger equation, and $\mathcal{H}_{\text{HO}}$ the one corresponding to the HO of a given frequency $\omega$. We shall denote by $\psi(x,t) \in \mathcal{H}$ the free particle solutions and by $\psi'(x',t') \in \mathcal{H}_{\text{HO}}$ the HO ones. Then, the QAT [1], relating solutions of time-dependent Schrödinger equations for the free particle and the HO, or rather, its inverse, is given by

$$
\psi'(x',t') = \hat{A}^{-1} \psi(x,t) = \frac{1}{\sqrt{u_2(t')}} e^{\frac{i}{\hbar} \int_{x}^{x'} \frac{W(x',t')}{2u_1(t')}} \psi\left(\frac{x'}{u_2(t')} , \frac{u_1(t')}{u_2(t')} \right).
$$

(1)

The classical Arnold transformation (CAT) [2] is explicitly

$$
A : \mathbb{R} \times T' \longrightarrow \mathbb{R} \times T \quad (x',t') \longmapsto (x,t) = A((x',t')) = \left(\frac{x'}{u_2(t')} , \frac{u_1(t')}{u_2(t')} \right).
$$

(2)
Figure 1. Pictorial representation of the CAT for the HO. Here, A maps the solid part of the helix (half a period of a HO trajectory) into the whole line (a free particle trajectory). Velocities have been included in the graphic for clarity. The horizontal plane represents the space of all possible initial conditions at \( t = 0 \). Note that both trajectories coincide and are tangent at this plane, due to conditions (4).

\( T \) and \( T' \) are open intervals of the real line containing \( t = 0 \) and \( t' = 0 \), respectively, and are two independent solutions of the LSODE (here dots mean derivation with respect to \( t' \)):

\[
\ddot{x}' + f(t') \dot{x}' + \omega(t')^2 x' = 0,
\]

and \( W(t') \) is the Wronskian \( W(t') = \dot{u}_1(t')u_2(t') - u_1(t')\dot{u}_2(t') \) of the two solutions. Applying the change of variables \( A \) to a solution \( x'(t') \) on a given interval of time \( T' \), a solution \( x(t) \) of the classical equation of motion \( \ddot{x} = 0 \) on an interval \( T \) is obtained, where now dots mean derivation with respect to \( t \).

We impose on \( u_1 \) and \( u_2 \) the condition that they preserve the identity of \( t \) and \( x \), i.e. that \( (x,t) \) coincide with \( (x',t') \) at an initial point \( t_0' \), arbitrarily taken to be \( t_0' = 0 \):

\[
u_1(0) = 0, \quad u_2(0) = 1, \quad \dot{u}_1(0) = 1, \quad \dot{u}_2(0) = 0.
\]

This fixes a unique form of \( A \) for a given ‘target’ LSODE-type physical system. However, the QAT is still valid if solutions \( u_1 \) and \( u_2 \) do not satisfy (4) (see [1] for details).

For the case of the HO \( f = 0 \) and \( \omega(t') = \omega \), and the two independent solutions can be chosen (see [1] for details) as \( u_1(t') = \frac{1}{\omega} \sin(\omega t') \) and \( u_2(t') = \cos(\omega t') \), with \( W(t') = 1 \). It can be checked that the change of variables results in

\[
\begin{align*}
  t' &= \frac{1}{\omega} \tan^{-1}(\omega t) \\
  x' &= \cos(\tan^{-1}(\omega t)) x = \frac{x}{\sqrt{1 + \omega^2 t'^2}}.
\end{align*}
\]

With this change, the classical equation of the free particle \( \ddot{x}(t) = 0 \) transforms into the classical equation of the HO \( \ddot{x}'(t') + \omega^2 x'(t') = 0 \) up to a time-dependent global factor.

A pictorial representation of the CAT for the HO can be seen in figure 1.

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3 The quantum transformation in the particular case of the HO was already given independently by Jackiw [3] (even with an extra \( x^{-2} \) potential term) and by Takagi [4].
Coming back to the quantum level, the QAT can be diagrammatically represented as

\[ \mathcal{H} \xrightarrow{\hat{A}} \mathcal{H}_0 \]

\[ \mathcal{H}_0 \xrightarrow{\hat{U}} \mathcal{H}_0 \]

\[ \mathcal{H}_0 \xrightarrow{\hat{U}_{\text{HO}}} \]

where \( \hat{U} \) and \( \hat{U}_{\text{HO}} \) stand for the evolution operators of the free particle and the HO, respectively, while \( \mathcal{H}_0 \) is the Hilbert space, either for the free particle or the HO, of solutions of their respective Schrödinger equations at \( t = 0 \). The fact that \( \mathcal{H}_0 \) is common to both systems is a consequence of imposing conditions (4) (see [1] for the required modifications when these conditions are not satisfied).

Thanks to the commutativity of this diagram and the unitarity of the operators appearing in it, we can map objects (wavefunctions, operators, expectation values, uncertainties) from one system to the other. In [1], we benefited from this fact for transporting the simplicity of the free particle to more involved systems, finding, for instance, an analytic expression for the evolution operator of the complicated systems (even with time-dependent Hamiltonians) in terms of the free particle evolution operator. In this paper we shall proceed the other way round, transporting objects and properties from the HO to the free particle.

Applying now the QAT to the time-dependent eigenstates of the HO Hamiltonian \( \hat{H}_{\text{HO}} \), with energy \( E_n = \hbar \omega (n + \frac{1}{2}) \),

\[
\psi_n(\chi', t') = \mathcal{N}_n e^{-i\omega(n+\frac{1}{2})t'} e^{-\frac{m\omega}{\hbar} t'} H_n \left( \sqrt{\frac{m\omega}{\hbar}} \chi' \right),
\]

(6)

where \( \mathcal{N}_n = \left( \frac{m\omega}{\hbar} \right)^{\frac{1}{4}} \sqrt{n!} \) and \( H_n \) are the Hermite polynomials, we obtain the following set of states, solutions of the Schrödinger equation for the free particle:

\[
\psi_n(x, t) = \mathcal{N}_n \frac{1}{\sqrt{|\delta|}} e^{-\frac{x^2}{2\tau^2}} \left( \frac{\delta^*}{|\delta|} \right)^{n+\frac{1}{2}} H_n \left( \frac{x}{\sqrt{2L|\delta|}} \right),
\]

(7)

where in order to obtain a more compact notation, we have introduced the quantities \( L = \sqrt{\frac{\hbar}{m\omega}} \), with dimensions of length, and \( \tau = \frac{2mL^2}{\hbar} = \omega^{-1} \), with dimensions of time. We also denote by \( \delta \) the complex, time-dependent, dimensionless expression \( \delta = 1 + i\omega t = 1 + i\frac{\hbar}{m\omega} t = 1 + i\frac{t}{\tau} \).

We have also used the fact that \( e^{-i\omega t} = e^{-i\omega t^{-1}(\omega t)} = \delta^* \). Note that with these definitions the normalization factor \( \mathcal{N}_n \) can be written as \( \mathcal{N}_n = \left( \frac{2\pi}{\sqrt{2\tau^2}} \right)^{\frac{1}{4}} \).

This set of states constitutes a basis for the space of solutions of the free Schrödinger equation, since it is mapped from a basis for the HO through \( \hat{A} \), which is unitary.

The family of wavefunctions (7) has been known in the literature as Hermite–Gauss wave packets [5], and they have been widely used, in their two-dimensional version (see section 4),

\[ \left( \frac{\delta}{|\delta|} \right)^{n+\frac{1}{2}} \]

is known as Guoy phase in paraxial wave optics [6], and as Lewis phase in the context of time-dependent quadratic Hamiltonians in quantum mechanics [7].
in paraxial wave optics [6]. However, these kinds of states and the ones constructed in sections 3 and 4 are better understood in the framework of the QAT. Note that making use of the classical solutions only, and through the QAT, we have been able to import the time evolution from the stationary states of the HO, $\psi'_n(x', t')$, to the non-stationary ones, $\psi_n(x, t)$, without solving the time-dependent Schrödinger equation.

The first state of this basis, the one mapped from the HO vacuum state, is given by

$$\psi_0(x, t) = \left( \frac{2\pi}{L} \right)^{\frac{1}{4}} e^{- \frac{x^2}{4L} \delta^2} e^{\frac{i}{\hbar \mu} \omega x^2 \frac{\partial}{\partial x}},$$

which is nothing other than a Gaussian wave packet with center at the origin and width $L$. The parameter $\tau$ is the dispersion time of the Gaussian wave packet (see, for instance, [11]).

We see that the number of ‘parts’, or humps, of the wavefunctions, determined by the number of zeros, is ‘quantized’, in the sense that there is only one hump between two consecutive zeros. They are therefore multi-localized, as the probability of finding the particle is ‘peaked’ in separate regions of space, with nearly zero probability of finding it in between.

The QAT also allows us to map invariant operators from one Hilbert space ($\mathcal{H}_{\text{HO}}$) to the other ($\mathcal{H}$). These invariant operators are also known as constant or integral of motion operators, in the sense that their matrix elements are constant with respect to their corresponding time evolution, and preserve their respective Hilbert spaces. Particularizing the general expression given in [1] or by direct computation, the conserved position and momentum operator for the HO can be written as

$${\hat{X}}' = \frac{\hat{u}_1(t')}{W(t')} x' + \frac{\hbar}{m} \frac{\hat{u}_1(t')}{W(t')} \frac{\partial}{\partial x'} = \cos \omega t' x' + \frac{\hbar}{m \omega} \sin \omega t' \frac{\partial}{\partial x'},$$

$${\hat{P}}' = -i \hbar \frac{\hat{u}_2(t')}{W(t')} \frac{\partial}{\partial x'} - m \frac{\hat{u}_2(t')}{W(t')} x' = -i \hbar \cos \omega t' \frac{\partial}{\partial x'} + m \omega \sin \omega t' x',$$

and therefore, the conserved creation and annihilation operators are

$$\hat{a}' = \left( \frac{\hbar}{2L} x' + L \frac{\partial}{\partial x'} \right),$$

$$\hat{a}'^\dagger = \left( \frac{\hbar}{2L} x' - L \frac{\partial}{\partial x'} \right).$$

Note that these are the operators acting on solutions of the time-dependent Schrödinger equation for the HO $\psi'_n(x', t')$ as ladder operators.

5 In [8] a basis of discrete states and their corresponding coherent states were constructed for the damped HO, which, in a certain limiting process, would reproduce ours. Also, in [9] the HO wavefunctions are mapped to solutions of generalized, time-dependent HO in the framework of [10], which resembles the approach in [1], although with somewhat less physical insight.
Position and momentum operators $\hat{X}'$ and $\hat{P}'$ are mapped into operators representing conserved position and momentum operators for the free particle through the QAT:

\[
\hat{X} = x + \frac{i\hbar}{m} \frac{\partial}{\partial x} \\
\hat{P} = -i\hbar \frac{\partial}{\partial x}.
\]

As a consequence, ladder operators for the HO can be mapped into ladder operators for the free particle that act as creation and annihilation operators for the (time-dependent) Hermite–Gauss states:

\[
\hat{a} = L \delta \frac{\partial}{\partial x} + \frac{x}{2L} \\
\hat{a}^\dagger = -L \delta^* \frac{\partial}{\partial x} + \frac{x}{2L}.
\]

The action of $\hat{a}$ and $\hat{a}^\dagger$ on the Hermite–Gauss wavefunctions is the usual one:

\[
\hat{a} \psi_n(x, t) = \sqrt{n} \psi_{n-1}(x, t), \quad \hat{a}^\dagger \psi_n(x, t) = \sqrt{n+1} \psi_{n+1}(x, t).
\]

It is possible to introduce this discrete basis without resorting to the QAT in a very intuitive manner. The key point is that the operator $\hat{a}$ annihilates the Gaussian wave packet, and this fact characterizes it. The whole family of states (7) can be generated by acting with the adjoint operator $\hat{a}^\dagger$ of $\hat{a}$. The rest of the construction, i.e. uncertainties, coherent and squeezed states, etc would proceed without the need of resorting to the QAT. However, as we shall see below, the QAT is very useful when performing involved computations in a very easy way.

In fact, a very useful property of the QAT is its unitarity, implying that it preserves scalar products and therefore expectation values. This will allow us to compute expectation values of operators in the free particle in terms of the corresponding ones in the HO.

Denoting by $\langle \cdot, \cdot \rangle$ the scalar product in the Hilbert space of solutions of the free particle Schrödinger equation, and by $\langle \cdot, \cdot \rangle'$ the corresponding one in the HO Hilbert space, we have that

\[
\langle \psi, \phi \rangle = \langle \psi', \phi' \rangle' \quad \langle \hat{O} \rangle_{\psi} = \langle \hat{O}' \rangle'_{\psi'},
\]

where $\psi, \phi$ and $\hat{O}$ are related to $\psi', \phi'$ and $\hat{O}'$ through the QAT, respectively. It is important to note [1] that (11) is only valid when $\hat{O}$ and $\hat{O}'$ are invariant operators. These operators are essentially $\hat{X}$ and $\hat{P}$ above, closing a Heisenberg–Weyl algebra (or $\hat{X}'$ and $\hat{P}'$ in the HO), and their powers. Among them, the most important ones are the quadratic operators, which together with the Heisenberg–Weyl algebra expand the Schrödinger algebra (see (18)).

For instance, we can apply this property to compute the uncertainties associated with each wavefunction. As a function of time, for each state $\psi_n(x, t)$, they read

\[
\Delta \hat{x}_n \Delta \hat{p}_n = \left( n + \frac{1}{2} \right) \hbar |\delta| = \frac{E_n}{\omega} |\delta|.
\]

To compute the uncertainties $\Delta \hat{x}_n = \sqrt{(\langle \hat{x}^2 \rangle_\psi - \langle \hat{x} \rangle_\psi)^2}$ and $\Delta \hat{p}_n = \sqrt{(\langle \hat{p}^2 \rangle_\psi - \langle \hat{p} \rangle_\psi)^2}$ we have used the transformation of non-conserved operators when they are written in terms of conserved ones, that is,

\[
\hat{x} = \hat{X} + \frac{t}{m} \hat{P} \overset{\hat{a} \dagger}{\rightarrow} \hat{X}' + \frac{u_1(t')}{m u_2(t')} \hat{P}' = \frac{\hat{x}'}{u_2(t')} = |\delta| \hat{x}'
\]

\[
\hat{p} = \hat{P} \overset{\hat{a} \dagger}{\rightarrow} \hat{P}' = \frac{1}{|\delta|} (\hat{p}' + m \omega^2 \hat{x}'),
\]

\[
\langle \hat{X}' \rangle = \frac{\langle \hat{x} \rangle}{|\delta|}, \quad \langle \hat{P}' \rangle = \frac{\langle \hat{p} \rangle}{|\delta|},
\]

\[
(13)
\]

\[
(10)
\]
as well as
\[
\langle (\hat{x})^2 \rangle_{\psi_n} = \frac{1}{m^2 (\omega^2)^2} \langle (\hat{x})^2 \rangle_{\psi_1} = |\delta|^2 \frac{\hbar}{m \omega} \left( n + \frac{1}{2} \right),
\]
where in the last equality of the equations the quantum Virial theorem for the HO potential \[11\], which is homogeneous of degree 2, has been used: 1 \( \frac{1}{2} m \omega^2 \langle (\hat{x})^2 \rangle_{\psi_1} = \frac{1}{2m} \langle (\hat{P})^2 \rangle_{\psi_1} = \frac{1}{2} (\hat{H}_\text{HO}) \langle \psi_1 \rangle.

For \( n = 0 \), the time evolution of the uncertainty relation (12) is the one which results from the usual Gaussian wave packet \[11\], and, among all, the minimal one.

The number operator associated with the creation and annihilation operators above will provide the position of the state in this grid of uncertainties. We can derive it (or map it from the number operator for the HO) in the usual way:
\[
\hat{N} = \frac{1}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) = \left[ -|\delta|^2 L^2 \frac{\partial^2}{\partial x^2} + i \frac{\hbar}{\tau} \left( x \frac{\partial}{\partial x} + \frac{1}{2} \right) + \frac{x^2}{4L^2} \right].
\]
By making use of the Schrödinger equation, we can turn this operator into a first-order one:
\[
\hat{\Delta} = \left[ i|\delta|^2 \tau \frac{\partial}{\partial t} + i \frac{\tau}{\tau} \left( x \frac{\partial}{\partial x} + \frac{1}{2} \right) + \frac{x^2}{4L^2} \right],
\]
the validity of which is restricted to solutions of the Schrödinger equation. The action of this operator is such that
\[
\hat{\Delta} \psi_n(x, t) = \left( n + \frac{1}{2} \right) \psi_n(x, t) = \frac{E_n}{\hbar \omega} \psi_n(x, t),
\]
thus reproducing the uncertainties (up to a factor \( \hbar \)) given in equation (12) at time \( t = 0 \).

It is quite interesting to note that this operator belongs to the algebra of the ‘maximal kinematical’ symmetry of the free particle, i.e. the Schrödinger group \[12\], built out of constants of motion operators up to quadratic order. This symmetry is the standard Galilean symmetry (with generators \( \hat{P}^2 = 2m \hat{H}, \hat{P}, \hat{X} \) and the identity \( \hat{I} \)) together with spatial dilations (generated by \( \hat{X} \hat{P} = \frac{1}{2}(\hat{X} \hat{P} + \hat{P} \hat{X}) \)) and non-relativistic ‘conformal’ transformations (with generator \( \hat{X}^2 \)). These generators can be written, in their first-order version, as
\[
\begin{align*}
\hat{P} &= -i \hbar \frac{\partial}{\partial x} \\
\hat{X} &= x + i \frac{\hbar}{m} \frac{\partial}{\partial t} \\
\hat{X}^2 &= 2i \frac{\hbar}{m} \frac{\partial}{\partial t} + 2i \frac{\hbar}{m} \frac{\partial}{\partial x} + \frac{x^2}{4} + \frac{i \hbar}{2} 
\end{align*}
\]
providing the non-trivial commutation relations
\[
\begin{align*}
[\hat{X}, \hat{P}] &= i \hbar \\
[\hat{X}, \hat{P}^2] &= 2i \hbar \hat{P} \\
[\hat{X}, \hat{X}^2] &= 0 \\
[\hat{X}, \hat{X} \hat{P}] &= i \hbar \hat{X}
\end{align*}
\]
\[
\begin{align*}
[\hat{P}, \hat{P}^2] &= 0 \\
[\hat{P}, \hat{X}^2] &= -2i \hbar \hat{X} \\
[\hat{P}, \hat{X} \hat{P}] &= -i \hbar \hat{P}
\end{align*}
\]
\[
\begin{align*}
[\hat{X}^2, \hat{P}^2] &= 4i \hbar \hat{X} \hat{P} \\
[\hat{X}^2, \hat{X} \hat{P}] &= 2i \hbar \hat{X} \\
[\hat{P}^2, \hat{X} \hat{P}] &= -2i \hbar \hat{P}^2.
\end{align*}
\]

It is easily checked that \( \hat{\Delta} \) belongs to this Lie algebra, it’s relation with the basis above being
\[
\hat{\Delta} = \frac{1}{\hbar \omega} \left( \frac{1}{2m} \hat{P}^2 + \frac{m \omega}{2} \hat{X}^2 \right) = \frac{1}{\hbar \omega} \hat{H}_\text{HO},
\]
where \( \hat{H}_\text{HO} \) is the operator mapped through the QAT from the HO Hamiltonian \( \hat{H}_\text{HO} \). See \[1\] for the relevance of quadratic operators like \( \hat{\Delta} \), but distinct from the Hamiltonian, for building bases of the Hilbert space (see also \[13, 8, 10, 14\]).
3. Coherent and squeezed states

Once we have successfully imported the eigenstates of the HO, we can go further with this scheme and map different families of coherent states from the HO to the free particle (as we shall see, they might be defined equivalently, without resorting to the QAT, as eigenstates of the annihilation operator \( \hat{a} \) in the free particle). Even more, squeezed states may be introduced in a similar fashion. These states have the property that, having minimal uncertainty relations, the uncertainty in position (or momentum) is squeezed, at the expense of increasing the uncertainty in momentum (or position). These states are used in every day experiments in quantum optics since they have low noise and allow us to measure quantities with higher precision [15].

The most general squeezed and displaced number states \(|n, \xi, a\rangle'\) for the HO are built using the squeezing operator \( \hat{S}(\xi) = e^{\frac{i}{4}(\xi^2 \hat{a}^\dagger - \xi (\hat{a})^2)} \) and the displacement operator \( \hat{D}(a) = e^{a \hat{a}^\dagger - a^\dagger \hat{a}} \):

\[
|n, \xi, a\rangle' = \hat{D}(a)\hat{S}(\xi)|n\rangle,
\]

where \( a \) is, as usual, the complex number

\[
a = \sqrt{\frac{m_0}{2\hbar}} x_0 + i \frac{1}{\sqrt{2m_0\hbar}} p_0 = \frac{x_0}{2L} + i \frac{p_0 L}{\hbar} = \frac{1}{2L}(x_0 + i t_0 \tau),
\]

with \( t_0 = \frac{p_0}{m_0} \), and \( \xi \) is the squeezing parameter that will be taken real, \( \xi = r \), for simplicity.6

The complete expression for the configuration space wavefunctions of these states for the HO, and its free particle version through the QAT, is given in the appendix. Here, we only give their general form for the free particle:

\[
\psi^a_{(\omega, r)}(x, t) = N_{\omega} e^{\frac{i\omega}{2} \frac{1}{\sqrt{|\delta_r|}}} \left( \delta_r^* \right) \frac{\delta_r}{|\delta_r|} e^{i\theta(x, t)} e^{-q^2/2} H_n(q),
\]

where \( q = \frac{e^{-\delta_r} \sqrt{L}}{\sqrt{|\delta_r|}} \) and \( \delta_r = 1 + i e^{\delta_r} \), and the phase \( \theta(x, t) \) is given in equation (A.6) in the appendix.

Now we focus ourselves on some particular cases, giving their physical interpretation in the free particle system. By putting \( n = r = 0 \) in (26), we obtain coherent states for the free particle:

\[
\phi_\omega(x, t) = \left( \frac{2\pi}{L} \right)^{-\frac{1}{4}} \left( \frac{\delta_r}{|\delta_r|} \right)^{\frac{i}{4}} e^{-i(q_n x + q_n pt) + q_n^2/2 + q_n^4/4},
\]

which could also have been obtained as eigenstates of the annihilation operator since they verify

\[
\hat{a} \phi_\omega(x, t) = a \phi_\omega(x, t).
\]

These states can also be obtained by the action of the imported displacement operator \( \hat{D}(a) = e^{a \hat{a}^\dagger - a^\dagger \hat{a}} = e^{\frac{i}{2}(p_0 x - x_0 p)} \), which is, up to a phase, a Galilean boost with parameter \( p_0 \) together with a translation of parameter \( x_0 \) on the Gaussian wave packet. They constitute an over-complete set of the Hilbert space of the free particle and represent traveling Gaussian wave packets, with mean momentum and initial position \( p_0 \) and \( x_0 \), respectively. These states are not eigenstates of the number operator \( \hat{N} \). The expectation values of this operator on these coherent states can be computed importing the result from the HO using (11) and (23):

\[
\langle \phi_\omega | \hat{N} | \phi_\omega \rangle = \hbar \left( |a|^2 + \frac{1}{2} \right).
\]

6 The difference between the real and imaginary squeezing parameter is easily seen by checking the action of the squeezing operator on the Wigner quasi-probability distribution, with domain the phase space \((x, p)\). Real \( \xi \) implies squeezing in the \( x \) or \( p \) direction, while imaginary \( \xi \) squeezes in the diagonal \( x + p \) or \( x - p \) directions. In other words, when \( \xi = \rho e^{i\phi} \), the squeezing takes place in the direction \( 2\theta \) in the plane \((x, p)\).
Setting now \( n \neq 0, r = 0 \) in (26), we obtain a family of states of the free particle that could be obtained by acting with Galilean boosts and translations on a fixed state of the basis, \( \psi_n(x, t) \), and constitutes a new over-complete set of states for the free particle Hilbert space:

\[
\phi_n^a(x, t) = N_n \frac{1}{\sqrt{8|n|}} \left( \frac{\delta_n^*}{|n|} \right)^{x_0 + \frac{1}{2}} H_n \left( \frac{x - x_0 - v_0 t}{\sqrt{2|n|}} \right) e^{-i(\omega x - \omega t)} e^{\frac{x^2}{2L^2}}.
\]

representing traveling multi-localized wave packets bearing \( n + 1 \) humps, with mean momentum and initial position \( p_0 \) and \( x_0 \), respectively, where \( a \) is given by (25). As in the case \( n = 0 \), these states are not eigenstates of the number operator, but the expectation values, computed as before, are

\[
\langle \phi_n^a | \hat{N} | \phi_n^a \rangle = h(|a|^2 + n + \frac{1}{2}).
\]

Being a set of coherent states, the uncertainty relations of \( \psi_n^a \), \( \forall a \in \mathbb{C}, \) are the same as those of \( \psi_n \) given in equation (12) (see [16]).

Finally, let us discuss the simplest case of the squeezed state that with \( r \neq 0, n = 0 \) and \( a = 0 \) corresponds to the squeezed vacuum in the HO or a ‘squeezed’ Gaussian wave packet in the free particle:

\[
\psi_r(x, t) = \frac{(2\pi)^{-1/4}}{\sqrt{L|b_r|}} e^{\frac{r_0^2}{4}} \left( \frac{\delta_0^*}{|b_r|} \right)^{x_0 + \frac{1}{2}} e^{-\frac{x^2}{4L^2}} e^{\frac{-x^2}{4L^2}}.
\]

This wavefunction corresponds to a Gaussian wave packet where the position and time variables have been rescaled, in addition to the wavefunction, as

\[
\psi_r(x, t) = e^{i\frac{r}{2}} \psi_0(e^{\frac{r}{2}} x, e^{\frac{r}{2}} t).
\]

Again, this can be interpreted as the action of the imported squeezing operator \( \hat{S}(r) = e^{\frac{i}{2}(\hat{a}^2 - \hat{a}^2)} = e^{\frac{r}{2} \hat{X} \hat{P}} \) which turns out to be a dilation.

Comparing the expressions of (8) and (32), we realize that (32) corresponds to a Gaussian wave packet with initial width \( L_r = e^{-r} L \). This can be easily understood since in the free particle there is no natural frequency, and therefore there is no natural width \( L \), so that squeezing can be absorbed in the width \( L \) of the wave packet.

### 4. Discrete bases of wave packets in higher dimensions

The generalization of the previous construction to higher dimensions is immediate. In more dimensions, due to the symmetries of the HO, we can choose different separation of variables to solve the Schrödinger equation for the HO, which amounts to finding simultaneous eigenstates for the HO Hamiltonian and other operators commuting with it and among themselves. For instance, separation of variables in Cartesian coordinates in \( N \) dimensions is equivalent to diagonalizing simultaneously the HO Hamiltonian and the 1D HO Hamiltonians in each coordinate. This can always be done even if the frequencies for each direction are different. The common basis of eigenstates is the product of 1D HO eigenstates in each variable. If we impose spherical symmetry, then all frequencies must coincide and we can search for common eigenstates of the Hamiltonian and the angular momentum operators commuting with it and among themselves (for instance, in three dimensions they would be \( \hat{L}^2 \) and \( \hat{L}_z \), and in two dimensions it would be only \( \hat{L} \)).

For the case of Cartesian coordinates, the generalization of the previous sections is immediate by a direct generalization of the QAT to higher dimensions, the discrete basis of states for the \( N \)-dimension free particle being products of \( N \) copies of (7) in each of the
Figure 3. Probability distribution for the Hermite–Gauss states $\psi_{1,0}$ and $\psi_{1,1}$ in the plane $x$–$y$ at time $t$. Whiter areas indicate higher probability; the positions of the maxima are shown.

If $\psi_\vec{n}(x_i, t)$ represents a wave packet of the form given in (7) but in the variable $x_i$ and with width $L_i$ (and with the corresponding frequency $\omega_i$), then the $N$-dimensional wave packets are

$$
\psi_{\vec{n}}(\vec{x}, t) = \prod_{i=1}^{N} \psi_\vec{n}_i(x_i, t),
$$

(34)

where $\vec{n} = (n_1, n_2, \ldots, n_N)$. If all the widths $L_i = L$ are identical, it can be written as

$$
\psi_{\vec{n}}(\vec{x}, t) = \left(\frac{2\pi}{L|\delta|}\right)^{N/2} e^{\frac{-\sum_{i=1}^{N} n_i^2}{4L^2\delta^2}} \frac{\Gamma(n + |l| + 1)}{\sqrt{\pi}^{N/2} \Gamma(n + 1)} \prod_{i=1}^{N} L_{n_i}^{(l)} \left(\frac{x_i}{\sqrt{2L|\delta|}}\right).
$$

(35)

Coherent states, or traveling wave packets, are defined similarly as products of $N$ copies of (30), with a vector $\vec{a} \in \mathbb{C}^N$:

$$
\psi_{\vec{a}}(\vec{x}, t) = \prod_{i=1}^{N} \psi_{\vec{a}_i}(x_i, t).
$$

(36)

For $N = 2$, these states have been widely used in the paraxial approximation to the Helmholtz equation of wave optics, known as the Hermite–Gauss states [6]. Due to the analogy of the paraxial approximation to the Helmholtz equation (with the $z$ coordinate acting as time) and Schrödinger equation in two dimensions, they have also been exploited in atom optics and matter waves [17].

In figure 3, a density plot of the probability distribution for the Hermite–Gauss states $\psi_{1,0}$ and $\psi_{1,1}$ is shown.

In the case of cylindrical symmetry in paraxial wave optics, the polar version of these states have been used, known as Laguerre–Gauss states [6].

The corresponding states for the free Schrödinger equation in two dimensions, derived through the QAT from the 2D HO in polar coordinates, are

$$
\psi_{n,l}(r, \phi, t) = \sqrt{\frac{n!}{2\pi \Gamma(n + |l| + 1)L^2|\delta|}} \frac{\delta^{|l|+1}}{|\delta|} \sum_{k=0}^{2n+|l|+1} \frac{1}{\Gamma(k)} \frac{r^{2k+|l|}}{2L^2|\delta|^2} e^{i\phi} e^{-\frac{r^2}{4L^2|\delta|}} H_n \left(\frac{x_i}{\sqrt{2L|\delta|}}\right),
$$

(37)

where $n, l = 0, \pm 1, \pm 2, \ldots$ and $L_n^{(l)}(x)$ are Laguerre polynomials. The state with $n = 0, l = 0$ is the Gaussian wave packet in two dimensions.
Figure 4. Probability distribution for the Laguerre–Gauss states $\psi_{0,1}^\pm$ and $\psi_{1,1}^\pm$ in the plane $x$–$y$ at time $t$. Whiter areas indicate higher probability; the positions of the maxima are shown.

These states are eigenstates of the angular momentum operator $\hat{L}$ in two dimensions, with values $\hat{L}\psi_{n,l}(r, \phi, t) = l\psi_{n,l}(r, \phi, t)$. In Figure 4, density plots of the probability distribution for the Laguerre–Gauss states $\psi_{0,1}$ and $\psi_{1,1}$ are shown. The generalization to three dimensions in spherical coordinates is straightforward, the states having the form of (37) but in terms of spherical harmonics and confluent hypergeometric functions $M(\cdot, \cdot; \cdot)$ (which are also polynomials in this case):

$$\psi_{n,l,m}(r, \theta, \phi, t) = \sqrt{\frac{\Gamma(l + 3/2 + n - 1)}{\sqrt{2}(n - 1)!\Gamma(l + 3/2)2L^3|\delta|}} \left(\frac{\delta^*}{|\delta|}\right)^{2(n-1)+l+3/2} e^{-\frac{r^2}{2L^2}} \times Y_m^l(\theta, \phi) \left(\frac{r}{\sqrt{2L}|\delta|}\right)^l M\left(-n + 1, l + 3/2; \frac{r^2}{2L^2|\delta|^2}\right),$$

(38)

None of these states are eigenstates of the free particle Hamiltonian, but can be seen to have expectation values of the energy equal to half the energy of the corresponding, through the QAT, HO eigenstates:

$$\langle \hat{H}\rangle_{\psi_{n,l,m}} = \frac{1}{2} E_{\tilde{n}} = \hbar \omega\left(\sum_{i=1}^N n_i + \frac{N}{2}\right)$$

$$\langle \hat{H}\rangle_{\psi_{n,l}} = \frac{1}{2} E_{nl} = \hbar \omega(2n + |l| + 1)$$

$$\langle \hat{H}\rangle_{\psi_{n,l,m}} = \frac{1}{2} E_{nlm} = \hbar \omega\left(2(n - 1) + l + \frac{3}{2}\right).$$

(39)

These states, as in the one-dimensional case, have many humps. For Cartesian coordinates, they have $\prod_{i=1}^N (n_i + 1)$ humps, while in polar coordinates they have $(n + 1)$ humps (seen in the radial coordinate) with the annular form.

Uncertainty relations can also be computed for these states. However, only the Cartesian version of the operators $\hat{x}$ and $\hat{p}$ can be used since the canonical momentum associated with the radial coordinate, $p_r$, is not self-adjoint. The expressions, computed as in the one-dimensional case, are

7 This is a consequence, again, of (11) and the quantum Virial theorem for the quadratic, homogeneous HO potential.
\begin{align}
(\Delta x)\psi_i(\Delta p)\psi_{\delta} &= \frac{1}{\omega}|\delta|E_{m}, \quad i = 1, \ldots, N \\
(\Delta x)\psi_{\delta}(\Delta p)\psi_{\delta} &= (\Delta y)\psi_{\delta}(\Delta p)\psi_{\delta} = \frac{1}{2\omega}|\delta|E_{\delta} \\
(\Delta x)\psi_{\delta}(\Delta p)\psi_{\delta} &= (\Delta y)\psi_{\delta}(\Delta p)\psi_{\delta} = (\Delta z)\psi_{\delta}(\Delta p)\psi_{\delta} \\
&= \frac{1}{3\omega}|\delta|E_{\delta \delta m}.
\end{align}

As in the one-dimensional case, squeezed states, interpreted as rescaled wave packets, can also be introduced in higher dimensions.

5. Physical applications

The theoretical relevance of these multi-localized, squeezed, traveling wave packets for the free particle, as well as their construction by means of the QAT, is of no doubt. From a more practical point of view, the preparation of this kind of discretized free states might be achieved by the use of a HO the potential of which is switched off at a given time. The vacuum state of this HO, when switched off, will provide the ‘vacuum’ Gaussian wave packet with width \( L = \sqrt{\frac{\hbar}{m\omega}} \), where \( m \) is the mass of the particle and \( \omega \) is the frequency of the oscillator. Note that the dispersion time \( \tau \) coincides with the inverse of the frequency of the oscillator. If the HO is in the \( m \)th excited state, the \((n + 1)\)-hump state is obtained. To obtain traveling states, the initial state should be a coherent state \( \phi(x, t) \) of equation (27) for a one-hump traveling state of \( \phi_{0}(x, t) \) for a \((n + 1)\)-hump traveling state. These coherent states can be obtained by acting with time-dependent classical forces on the HO according to Glauber [30] (see also [16, 19]). In fact, if the classical force is given by the potential \( V(x) = -f(t)x \), and the initial state is the vacuum \( |0\rangle \), then, if the force \( f(t) \) has finite duration or it is fast decaying in time, after a suitable time a standard coherent state \( |a\rangle \) is obtained with \( a = \frac{\sqrt{\delta_1}}{\omega}\hat{f}(\omega) \), where \( \hat{f}(\omega) \) is the Fourier component of \( f(t) \) in the frequency \( \omega \) of the oscillator. For an arbitrary force \( f(t) \) a time-dependent coherent state \( |a(t)\rangle \) is obtained, see [16, 19].

It is possible to ‘capture’ one of these traveling states at a later time \( t_1 \) switching on a HO potential with an appropriate frequency \( \omega_1 \) that would ‘freeze’ and keep it in an coherent state (without dispersion), until the potential is switched off again at a time \( t_2 \) and the wave packet is released, traveling again as a free wave packet that disperses in time. This way, information can be stored temporally in these ‘oscillator traps’, which can also be used for further manipulating them or even measuring the resulting state by means of adequate lasers. The frequency \( \omega_1 \) required to capture the dispersed wave packet should be fine tuned in such a way that the wave packet, at the time \( t_1 \), matches an appropriate solution (with the same \( n \) and the same \( \omega \)) of the HO with frequency \( \omega_1 \). Using the relation between \( \omega \) and \( L \) (valid at \( t = 0 \)), \( \omega = \hbar/\sqrt{2mL^2} \), and the fact that the width of the wave packet scales as \( |\delta| \), see equation (14), then \( \omega_1 = \frac{\hbar}{2mL^2}\delta_1 \), where \( \delta_1 = 1 + io\delta_1 \). This expression can be written in terms of distances using the mean velocity \( v_0 \) of the wave packet encoded in the complex number \( a \).

This process can be elegantly described by a sequence of QAT and evolution operators in the following way. Denote by \( \hat{A}_{\omega_1}(t) \) the QAT from a HO of frequency \( \omega_1 \) performed at time \( t \) and by \( \hat{U}_{\omega_1} \) the unitary time evolution operator for that HO. Then the state after the process described above is (only the time dependence is explicit)

\[ \hat{A}_{\omega_1}(t) \hat{U}_{\omega_1} \phi_{\delta} \]

8 This idea was proposed in [29] and was named ‘quantum sling’.

9 With this notation, for \( t = t_0, \hat{A}_{\omega_1}(t_0) \) is the identity (classical solutions \( \mu_1(t') \) and \( \mu_2(t') \) have been appropriately chosen, see [1]), and \( t = \frac{\omega_1(t')}{2\omega_1} \).
\[ \psi(t) = \hat{U}(t, t_2)\hat{U}_{01}'(t_2, t_1)\hat{U}(t_1, t_0)\psi(t_0) \]
\[ = \hat{A}_{01, t_2}(t)\hat{U}_{01}'(t', t_2)\hat{U}_{01}'(t_2, t_1)\hat{A}_{01, t_1}(t_1)\hat{U}_{01}'(t_1, t_0)\psi(t_0) \]
\[ = \hat{A}_{01, t_2}(t)\hat{A}_{01, t_1}(t_1)\hat{U}_{01}'(t', t_1)\hat{U}_{01}'(t_1, t_0)\psi(t_0). \]

where \( \psi(t_0) \) is the starting HO state. This way, in the last expression only evolution operators for HOs appear. See [31] for a more detailed description.

Note that if the frequency of the HO is not modified, and the same frequency \( \omega \) is used to capture the state in the HO trap, the resulting state will be a squeezed state with squeezing parameter \( r \) given by \( r = -\log(|\delta_1|) \), which is negative. This can be seen as a feasible way of producing squeezing in trapped states, simply switching off–on the trap for a period of time \( t \), resulting in a squeezing parameter \( r = -\frac{1}{2} \log(1 + \omega^2 t^2) \). In fact, a similar way of producing squeezed states in Bose–Einstein condensates was reported in [32].

The states constructed here are rather robust in the sense that the number of humps is conserved even in the presence of small perturbations. Numerical calculations in one dimension have been performed, simulating perturbations by square potentials (well or barriers), leading to the conclusion that this holds as long as the mean energy of the state is large compared with the scale of the perturbing potential and the wave packet is sharp enough in momentum space in such a way that the transmission coefficient can be considered a constant. Under these circumstances (see for instance [28]), the wave packet behaves as a plane wave and the effect of the barrier in the transmitted packet is an overall attenuation, preserving its shape, and a time delay which takes its maximum values for energies near the resonant ones (and where the transmission coefficient is one). As shown in [28], this result is valid for any bounded potential of compact support, provided that the width of the potential is small in the sense that the time to pass through the barrier is smaller than the dispersion time of the wave packet \( \tau \). Therefore, the conclusions obtained with the square potential can be generalized to any finite-range bounded potential.

These wave packets are also robust under the influence of time-dependent, homogeneous external fields, or even linear damping. In these cases, the centroid of the wave packets follows the classical trajectories, but their shapes are unaltered, apart from the unavoidable dispersion [5]. The case of linear damping is interesting due to the fact that the presence of damping prevents the dispersion of the wave packets, which asymptotically have finite width [8, 5]. The case of homogeneous external fields is particularly interesting, since it includes the case of free fall, which is a common experimental situation in atomic physics or Bose–Einstein condensates (see for instance [33]).

Under the conditions commented above, these wave packets evolve without distortion even in the presence of perturbations. However, one could be interested, acting with appropriate potentials, in obtaining transitions between wave packets with different number of humps, in such a way that, for instance, a one-hump packet splits into a two-hump packet or a two-hump packet coalesces into a one-hump packet. This would open the door to performing quantum gates acting on q-bits realized with the one-hump and the two-hump states. A way of implementing this is to benefit from the fact that wave packet dynamics is similar to wave optics in the sense that an analog to the ABCD law for optics is satisfied for wave packets [18]. Even the transmutation of Hermite–Gauss wave packets into Laguerre–Gauss wave packets can be achieved using ABCD matrices, implementing a mode converter [34].

Among further theoretical applications, we could think of expanding plane waves in terms of the discrete basis \( \{|\phi_{\tilde{a}}^{(2)}\rangle\}_{\tilde{a} \in \mathbb{C}^N} \), with fixed \( \tilde{a} \in \mathbb{C}^N \), and describing scattering process in a discrete

\(^{10}\)Regarding the dispersion effect, which should be taken into account for practical application in atomic physics or BEC, it takes place in a time scale given by the dispersion time \( \tau \). For instance, for a condensate of Na atoms with size \( \approx 1 \text{ mm} \), this time scale is \( \approx 500 \text{ s} \).
basis, or expanding arbitrary wave packets in a continuum over-complete set \( \{ \phi_\vec{n} \}_{\vec{n} \in \mathbb{N}_0^N} \), with fixed \( \vec{n} \in \mathbb{N}_0^N \), which could be discretized in a lattice \( \mathbb{Z}^N \times \mathbb{Z}^N \) of points to perform numerical computations, while keeping the over-complete character. Note that they are over-complete for \( t = 0 \) if the volume of the unit cell is smaller or equal to \( \hbar^N \), and again by the unitary time evolution they continue to be over-complete for any \( t \), see [16]. Similar constructions, like the HO method [22] or the transformed harmonic oscillator (THO) method [23], have been proposed, mainly in nuclear physics, to describe the bound states and the continuum spectrum in a discrete basis. But there the construction just appears as a mathematical tool for approximating the solutions, with no physical meaning. Our states, however, are physically meaningful (as traveling wave packets) and experimentally feasible.

6. Comments and outlook

In this paper, we have constructed different families of free particle states mapping harmonic oscillator states through the QAT. It would also be possible to generalize the construction to other important families of harmonic oscillator states, Gaussian states, which are mixed states with the property that their Wigner distribution functions are Gaussian (see, for instance, [19]). They include coherent states, squeezed states and thermal equilibrium states. Their main property is that Gaussian states keep this property under evolution even in the presence of dissipation and decoherence. That is the reason why they have been profusely used in quantum optics and quantum information in the continuous variables formalism [20, 21].

We should stress that, by using the QAT, the density matrix \( \hat{\rho}' \) of a mixed harmonic oscillator state can be mapped into the density matrix \( \hat{\rho} \) of a mixed free particle state:

\[
\hat{\rho} = \hat{A} \hat{\rho}' \hat{A}^\dagger.
\] (42)

The unitarity of \( \hat{A} \) guarantees that \( \hat{\rho} \) is a proper density matrix, provided that \( \hat{\rho}_{HO} \) is. Even more, it is easily checked using the evolution operator constructed in [1] that

\[
\frac{d\hat{\rho}'}{dt} = -\frac{i}{\hbar}[\hat{H}_{HO}, \hat{\rho}'] \Rightarrow \frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}]
\] (43)

i.e. if \( \hat{\rho}' \) satisfies the quantum Liouville equation for the harmonic oscillator Hamiltonian, then \( \hat{\rho} \) satisfies the free particle counterpart.

All the properties of the density matrix \( \hat{\rho}' \) are transferred to \( \hat{\rho} \), such as characteristic functions, quasi probability distributions, etc. In particular, if \( \hat{\rho}' \) describes a Gaussian state, also \( \hat{\rho} \) corresponds to a Gaussian state.

A deeper study of the QAT applied to mixed states and how it can be used to describe dissipation and decoherence analyzing the transformation properties of master equations like the Lindblad one under the QAT is the subject of a work in progress and will be presented elsewhere.

Another interesting point to study is whether the free particle states with many humps are physically observable and measurable, since their harmonic oscillator counterpart, the Hermite–Gauss and Laguerre–Gauss states are ‘visible’ in quantum optics using lasers [6]. Let us consider, for instance, a two-hump wave packet \( \phi_\vec{a}^{(1,0,\ldots,0)}(\vec{x}, t) \) in two or three dimensions with the humps in the transversal direction to that of the mean velocity \( \vec{v}_0 \). The separation of the two maxima of \( |\phi_\vec{a}^{(1,0,\ldots,0)}|^2 \) (see figure 2) is greater, in a factor 1.6, than the uncertainty in position \( \Delta x_1 \). Therefore the two humps should be measurable, and in fact, if this wave packet propagates in a bubble or wire chamber, two parallel, divergent tracks would be observed (if times \( t \ll \tau \) are considered). For a three-hump wave packet, the separation among consecutive maxima (see figure 2) is smaller than the uncertainty in position, although the distance between the more separated maxima is greater than the uncertainty in position. This, together with the
fact that the central maximum is smaller than the external ones, suggests that only two overlapping thick tracks would be observed in a bubble or wire chamber. A similar behavior for a larger number of humps is expected. We think that these points deserve further study and clarification.

The ideas developed here could also be applied to relativistic systems, particularly to the free particle in de Sitter spacetime, where the ordinary formulation of quantum theory does not find a natural physical vacuum [24]. In this sense, the generalization of our approach to the relativistic case would provide a hierarchy of states where the first state, the relativistic counterpart of the Gaussian wave packet [25, 26], plays the role of a vacuum [27].

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Appendix. Derivation of coherent, squeezed, number states for the free particle

In this appendix, we write down the configuration space wavefunctions for coherent, squeezed, number states for the free particle using the QAT. Let us first introduce some notation in order to simplify the expressions. Let \( \delta = 1 + i \tan(\omega t') \) and \( \delta' = 1 + i e^{i\pi} \tan(\omega t') \). These expressions, under the classical Arnold transform (5), are mapped to their corresponding unprimed versions, \( \delta = 1 + i\omega t \) and \( \delta' = 1 + i e^{i\pi} \omega t \).

Let us also define the dimensionless quantity

\[
q' = \frac{\sqrt{\hbar m}}{\omega} \left( x - x_0 \cos(\omega t') - \frac{p_0}{m} \sin(\omega t') \right) \left( e^{i\omega t'} \sin^2(\omega t') + e^{-i\omega t'} \cos^2(\omega t') \right)^{1/2} = \sqrt{\frac{m \omega}{\hbar |\delta'|}} e^{i(\omega t' - x_0 \cos(\omega t') - \frac{p_0}{m \omega} \sin(\omega t'))}.
\]

(A.1)

The configuration space wavefunctions are then written as

\[
\psi_n^{(a,r)}(x', t') = N_n e^{i\gamma / 2} \left( \frac{|\delta'|}{\delta} \right)^{n+1} e^{i\theta(\omega t')} e^{-q'^2 / 2} H_n(q'),
\]

(A.2)

where the phase \( \theta(x', t') \) is given by

\[
\theta(x', t') = \frac{1}{2} \left( \frac{1}{\hbar \omega} \frac{p_0}{m} - \frac{m \omega}{\hbar} \right) \tan(\omega t') + \frac{1}{\hbar} p_0 x_0 + e^{-i} |\delta'| \left( \frac{p_0}{\sqrt{\hbar \omega}} - \sqrt{\frac{m \omega}{\hbar} \tan(\omega t')} \right) q' + \sinh(2r) \tan(\omega t') q'^2.
\]

(A.3)

To compute the QAT of these states, \( \psi_n^{(a,r)}(x, t) \), it is useful to write down a ‘dictionary’, completing (5):

\[
\begin{align*}
\omega t' & \rightarrow \tan^{-1}(\omega t) & x' & \rightarrow \frac{x}{|\delta|} \\
\cos(\omega t') & \rightarrow \frac{1}{|\delta|} & \sin(\omega t') & \rightarrow \frac{\omega t}{|\delta|} \\
\delta' & \rightarrow \delta & \delta'_n & \rightarrow \delta_n \\
\psi' & \rightarrow \psi = \frac{1}{\sqrt{|\delta|}} e^{i\omega t' x^2 / \sqrt{\hbar \omega}} \psi'.
\end{align*}
\]

(A.4)

Using this dictionary, the adimensional quantity \( q' \) changes to

\[
q' \rightarrow q = \frac{x - x_0 + \frac{p_0}{m} t}{\sqrt{2L} e^{-i} |\delta'|}.
\]

(A.5)
and the phase $\theta(x', t')$ changes to

$$\theta'(x', t') \rightarrow \theta(x, t) = \frac{1}{|\delta_i|^2} \left( \frac{1}{m\hbar} \frac{\partial^2}{\partial x^2} - \frac{m\omega}{\hbar^2} \right) \frac{1}{\omega t} + \frac{1}{\hbar} p_0 x_0$$

$$+ e^{-r\delta_i} \left( \frac{p_0}{\sqrt{m\hbar}} - \frac{m\omega}{\hbar} x_0 \omega t \right) q + \sinh(2r \omega t q^2) \right]. \quad (A.6)$$

Therefore, the corresponding coherent, squeezed, number state for the free particle reads

$$\psi^{\text{coh}}_n(x, t) = N_n e^{\sqrt{2} \sqrt{|\delta_i|}} \left( \frac{\delta_i^+}{|\delta_i|^{n+1/2}} \right) e^{\sqrt{\omega} q^2 / 2 \omega t} e^{i(qx \omega t)} e^{-q^2/2} H_n(q). \quad (A.7)$$

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