Statistical mechanics of two-dimensional point vortices: relaxation equations and strong mixing limit

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1 Introduction

Point vortices in two-dimensional (2D) hydrodynamics provide an interesting example of systems with long-range interactions with application to geophysical and astrophysical flows [1]. Furthermore, they display many analogies (despite, of course, differences) with other systems with long-range interactions such as plasmas [2], self-gravitating systems [3], and the Hamiltonian mean field (HMF) model [4]. These systems are actively studied from the viewpoint of statistical mechanics and kinetic theory.

An isolated system made of a large number \( N \gg 1 \) of point vortices is expected to achieve a statistical equilibrium state for \( t \to +\infty \). As first recognized by Onsager [5], this statistical equilibrium state has very peculiar properties. Point vortices may be found at positive or negative temperatures and behave very differently depending on the sign of the temperature\(^1\). At negative temperatures, point vortices of the same sign tend to “attract” each other and form coherent structures similar to the large-scale vortices observed in the atmosphere of giant planets. This self-organization of point vortices into coherent structures is similar to the self-organization of stars into galaxies or globular clusters [1]. At positive temperatures, point vortices of the same sign tend to “repel” each other. In that case, the situation is similar to that of a plasma or an electrolyte studied by Debye and Hückel [6,7]. The kinetic theory of point vortices is also interesting and non-trivial [8]. The relaxation towards the statistical equilibrium state for \( t \to +\infty \) is a two-stage process.

In a first stage, the “collisions” (correlations) between point vortices are negligible and the evolution of the smooth vorticity field is described by the 2D Euler equation. This equation is exact in a proper thermodynamic limit \( N \to +\infty \) in which the domain area is fixed and the circulation of the point vortices scales as \( 1/N \) [9]. Starting from an unstable or unsteady initial condition, the 2D Euler equation develops a complicated mixing process and undergoes a violent relaxation towards a quasi stationary state (QSS) on the coarse-grained scale. This process takes place on a few dynamical times \( t_D \). The QSS is a steady state of the 2D Euler equation that depends on the initial conditions in a non-trivial manner. Miller [10] and Robert and Sommeria [11] have proposed to describe this QSS as a

\(^1\) We emphasize that a Hamiltonian system of point vortices evolves at fixed energy, so that the energy (not the temperature) is the relevant control parameter. Negative temperature states correspond to high energies and positive temperature states correspond to low energies.
statistical equilibrium state of the 2D Euler equation. The MRS theory is similar to the statistical theory of violent relaxation developed by Lynden-Bell [12] in astrophysics for the Vlasov equation describing collisionless stellar systems (see [1,13] for a development of the analogy between 2D vortices and stellar systems).

In a second stage, the “collisions” (correlations, graininess, finite \( N \)) effects between the point vortices must be taken into account and the evolution of the smooth vorticity field departs from the pure 2D Euler equation. It is expected to be governed by a kinetic equation that relaxes for \( t \rightarrow +\infty \) towards the mean field Boltzmann distribution derived by Joyce and Montgomery [14]. Different kinetic equations have been proposed. However, the relaxation towards the Boltzmann distribution, and the scaling of the relaxation time with the number \( N \) of point vortices, is still a subject of investigation (see [8] and references therein). In any case, the collisional relaxation is very slow since the relaxation time scales as \( N t_D \) or may be even longer.

We note that the limits \( N \rightarrow +\infty \) and \( t \rightarrow +\infty \) do not commute. If we take the \( N \rightarrow +\infty \) limit before the \( t \rightarrow +\infty \) limit, the evolution of the system is described by the 2D Euler equation that relaxes towards a non-Boltzmannian QSS described by the MRS theory. If we take the \( t \rightarrow +\infty \) limit before the \( N \rightarrow +\infty \) limit, the system is expected to relax towards the mean field Boltzmann distribution described by the Joyce-Montgomery theory. For large but finite \( N \), the system quickly reaches a non-Boltzmannian QSS that is a steady state of the 2D Euler equation, then slowly relaxes towards the Boltzmann distribution of statistical equilibrium.

These two successive regimes of “violent collisionless relaxation” towards a non-Boltzmannian QSS and “slow collisional relaxation” towards the Boltzmann distribution were predicted in references [1,15] from the kinetic theory of point vortices. They have been confirmed and illustrated numerically in references [16,17]. Self-gravitating systems and the HMF model display a similar two-stages evolution [1,4,18].

In this paper we consider exclusively the final statistical equilibrium state of 2D point vortices resulting from the “collisional” relaxation. The statistical mechanics of 2D point vortices has been discussed in several papers, both in the physics [14,19–22] and mathematical [23–28] literature, and a good understanding is now achieved.

In the present paper, we precise some results and provide complements that have not been given previously. In Section 2, we derive the Boltzmann distribution of the multi-species point vortex gas by using the maximum entropy principle\(^3\) (this variational problem is justified from the theory of large deviations in Appendix A). We discuss the proper thermodynamic limit of the point vortex gas, the notion of ensemble inequivalence, and the Virial theorem. We also stress some limitations of the celebrated sink-Poisson equation. In Section 3, we derive a set of relaxation equations towards the statistical equilibrium state by using a maximum entropy production principle (MEPP)\(^4\). We consider the multi-species case and take all the constraints into account (circulation of each species, energy, angular momentum, and linear impulse). We simplify these equations for the globally neutral two-species system and for the single species system. We mention the connections (as well as the differences) between these relaxation equations and the Fokker-Planck equations derived by Debye and Hückel [7] in their theory of electrolytes. We also discuss the analogy with the Smoluchowski-Poisson system describing self-gravitating Brownian particles [30] and with the Keller-Segel model describing the chemotaxis of bacterial populations [31]. In Section 4 we consider a limit of strong mixing (or low energy) and simplify the equations of the statistical theory. To leading order, the \( \omega - \psi \) relationship between the smooth vorticity and the stream function becomes linear and the maximization of the entropy becomes equivalent to the minimization of the enstrophy. This expansion is similar to the Debye-Hückel approximation for electrolytes, except that the temperature is negative instead of positive so that the effective interaction between like-sign vortices is attractive instead of repulsive. This leads to an organization at large scales presenting geometry-induced phase transitions, instead of Debye shielding. We mention the connection with the phenomenological minimum enstrophy principle [32] and with the phenomenon of “condensation” put forward by Kraichnan [33] in his statistical theory of 2D turbulence in spectral space. We also compare the results obtained for point vortices with those obtained by Chavanis and Sommeria [34] in the context of the statistical mechanics of continuous vorticity fields described by the MRS theory. At linear order, we get the same results but differences appear at the next order. In particular, the MRS theory predicts a transition between sinh and tanh-like \( \omega - \psi \) relationships depending on the sign of \( \text{Ku} - 3 \) (where Ku is the Kurtosis) while there is no such transition for point vortices which always show a

\[^2\] It is sometimes argued that the MRS theory (valid for continuous vorticity fields described by the 2D Euler equation) is a refinement of the Joyce-Montgomery theory (valid for point vortices described by Hamiltonian equations). This is not the case at all. In the context of point vortices, the MRS theory and the Joyce-Montgomery theory are fundamentally different because they apply to different regimes (collisionless relaxation vs collisional relaxation) with very different timescales (see Appendix F). Therefore, they have their own domain of validity. The same distinction applies between the Lynden-Bell distribution (describing collisionless stellar systems) and the mean field Boltzmann distribution (describing collisional stellar systems) in astrophysics.

\[^3\] We generalize previous works [14,19–22] that consider, for simplicity, only one type of point vortices with circulation \( \gamma \) or two types of point vortices with circulation \( +\gamma \) and \( -\gamma \). This generalization, although straightforward, is important in order to compare, in Section 4, the limit of strong mixing for point vortices and continuous vorticity fields.

\[^4\] The MEPP was introduced in the context of the MRS theory [29] for continuous vorticity fields. We apply it here to the point vortex gas in order to see the similarities and the differences.
2 Statistical equilibrium state of a multi-species point vortex gas

2.1 The Hamiltonian equations

The dynamical evolution of a multi-species system of point vortices in two dimensions is described by the Hamiltonian equations [35]:

\[ \frac{d\gamma_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial \gamma_i}, \]

\[ H = -\frac{1}{2\pi} \sum_{i<j} \gamma_i \gamma_j \ln |r_i - r_j|, \]

where \( \gamma_i \) is the circulation of point vortex \( i \). For simplicity, we have written the Hamiltonian in an infinite domain. In a bounded domain, it has to be modified so as to take into account the contribution of vortex images.

A particularity of this Hamiltonian system, first noticed by Kirchhoff [36], is that the coordinates \((x, y)\) of the point vortices are canonically conjugate. Another particularity of the point vortex model is that the Hamiltonian (2) does not possess the usual kinetic term \( \sum_i m_i v_i^2 / 2 \) present for material particles. This is because point vortices have no inertia. Therefore, a point vortex produces a velocity, not an acceleration (or a force), contrary to other systems of particles in interaction like electric charges in a plasma [2] or stars in a galaxy [3]. In a sense, this corresponds to the conception of motion according to Descartes in contrast to Newton.

The point vortex system conserves the energy \( E = H \) and the number \( N_\alpha \) of vortices of each species \( \alpha \) (this is equivalent to the total circulation of each species \( \Gamma_\alpha = N_\alpha \gamma_\alpha \)). There are additional conserved quantities depending on the geometry of the domain. The angular momentum \( L = \sum_i \gamma_i r_i^2 \) is conserved in an infinite domain and in a disk due to the invariance by rotation. The linear impulse \( P = \sum_i \gamma_i r_i \) is conserved in an infinite domain and in a channel due to the invariance by translation.

2.2 The Onsager theory and the different approach of Ruelle

Assuming ergodicity, the point vortex gas is expected to achieve a statistical equilibrium state for \( t \to +\infty \). The statistical mechanics of point vortices is very peculiar and was first discussed by Onsager [5] in a bounded domain.

At sufficiently large energies, point vortices of the same sign tend to “attract” each other and form large-scale vortices observed in the atmosphere of giant planets (e.g. Jupiter’s great red spot). When all the vortices have the same sign, this leads to monopoles (cyclones or anticyclones) and when vortices have positive and negative signs, this leads to dipoles (a pair of cyclone/anticyclone) or even tripoles. These organized states are characterized by negative temperatures. The existence of negative temperatures for point vortices should not cause surprise. For material particles, the temperature is a measure of the velocity dispersion and it must be positive. Indeed, it appears in the Boltzmann factor \( e^{-\beta v^2/2} \) which can be normalized only for \( \beta > 0 \). However, since point vortices have no inertia, there is no such term in the equilibrium distribution of point vortices (see, e.g., Eq. (13) below) and the temperature can be negative.

At sufficiently large negative energies, like-sign vortices tend to “repel” each other. This corresponds to positive temperature states. When all the vortices of the system have the same sign, they accumulate on the boundary of the domain. Alternatively, when the system is neutral, opposite-sign vortices tend to “attract” each other resulting in a spatially homogeneous distribution with strong correlations between the vortices. In that case, the point vortex gas is similar to a Coulombian plasma. This is the situation considered by Fröhlich and Ruelle [37]. As discussed by Ruelle [38], we may expect a phase transition (related to the Kosterlitz-Thouless transition) as a function of the temperature. At large positive temperatures, the system is in a “conducting phase” (in the plasma analogy) with free vortices that can screen other vortices. In that case, there is Debye shielding like for a Coulombian plasma. At low positive temperatures, opposite-sign vortices tend to form pairs (+,–) and the gas is in a “dielectric phase” where all the charges are bound, forming dipolar pairs. These “dipoles” are similar to “atoms” (+e,–e) in plasma physics. In that case, there is no screening.

In summary, at negative temperatures, point vortices of the same sign tend to “attract” each other and form large-scale structures. This is similar to the case of self-gravitating systems, interacting by Newtonian gravity, in astrophysics. By contrast, at positive temperatures, point vortices of the same sign tend to “repel” each other, and
opposite-sign vortices tend to “attract” each other. This is similar to the case of electric charges, interacting by the Coulombian force, in plasma physics and in electrolytes.

2.3 The thermodynamic limit and the mean field approximation

To obtain more quantitative results, Joyce and Montgomery [14] and Lundgren and Pointin [22] have considered a mean field approximation. For a single species system of point vortices, this is valid in a proper thermodynamic limit $N \to +\infty$ in such a way that the normalized energy $\epsilon = E/N^2\gamma^2$, the normalized entropy $s = S/N$, and the normalized temperature $\eta = \beta N\gamma^2$ are of order unity. On physical grounds, it is reasonable to consider that the area $\mathcal{A}$ of positive and negative point vortices with circulation $\pm\gamma$ is similar to the case of electric charges, interacting by the smooth vorticity field $\omega = \nabla \times \mathbf{u}$.

In this paper, we exclusively consider the case of spatially inhomogeneous systems described by the Boltzmann-Poisson equation [14,22]. We call $\omega_a(r,t) = \gamma_a n_a(r,t)$ the vorticity of species $a$ and $\omega(r,t) = \sum_a \omega_a(r,t)$ the total vorticity ($n_a$ is the density of point vortices of species $a$ and $n = \sum_a n_a$ is the total density). The vorticity is related to the stream function by the Poisson equation

$$-\Delta \psi = \omega = \sum_a \omega_a.$$ 

(3)

The velocity field is given by $u = -\mathbf{z} \times \nabla \psi$. In the mean field approximation, the conservation laws can be written as:

$$E = \frac{1}{2} \int \omega \psi \, dr, \quad \Gamma_a = \int \omega_a \, dr,$n

(4)

$$L = \int \omega r^2 \, dr, \quad P = -\mathbf{z} \times \int \omega r \, dr.$$ 

(5)

2.4 The Boltzmann entropy

In order to determine the statistical equilibrium state of a system of point vortices with different circulations, we generalize the maximum entropy approach of Joyce and Montgomery [14]. Following the Boltzmann procedure, we divide the domain into a very large number of microcells with size $h$ (ultimately $h \to 0$). A microcell can be occupied by an arbitrary number of point vortices. We now group these microcells into macrocells each of which contains many microcells but remains nevertheless small compared to the extension of the whole system. We call $\nu$ the number of microcells in a macrocell. A macrostate is specified by precise position $\{r_1, \ldots, r_N\}$ of the point vortices while a macrostate is specified by the number $\{n_{ia}\}$ of point vortices of each species in each macrocell (irrespective of their position in the cell). Using a combinatorial analysis, the number of microstates corresponding to the macrostate $\{n_{ia}\}$ is:

$$W(\{n_{ia}\}) = \prod_a N_a! \prod \frac{\nu^{n_{ia}}}{n_{ia}!}.$$ 

(6)

We introduce $n_a(r)$ the smooth density of point vortices of species $a$ at $r$. The smooth vorticity of species $a$ is then $\omega_a(r) = \gamma_a n_a(r)$. If we define the Boltzmann entropy by $S = \ln W$, use the Stirling formula for $n_{ia} \gg 1$, and take the continuum limit, we get:

$$S[\omega_a] = -\sum_a \int \frac{\omega_a}{\gamma_a} \ln \frac{\omega_a}{N_a \gamma_a} \, dr.$$ 

(7)

The Boltzmann entropy (7) measures the number of microstates $W[\omega_a]$ corresponding to the macrostate $\{\omega_a(r)\}$. At statistical equilibrium, the system is expected to be in the most probable macrostate, i.e. the one that is the most represented at the microscopic level.
2.5 The microcanonical ensemble

Since the energy is conserved, the proper statistical ensemble is the microcanonical ensemble. In the microcanonical ensemble, all the accessible microstates (those satisfying all the constraints) are equiprobable at statistical equilibrium. This is the fundamental postulate of statistical mechanics, assuming ergodicity. Therefore, the probability of an accessible macrostate is proportional to \( W[\omega_a] = e^{S[\omega_a]} \). As a result, the most probable macrostate is obtained by maximizing the Boltzmann entropy (7) while conserving the mean field energy (4a), the circulation of each species (4b), the angular momentum (5a), and the impulse (5b).

We thus have to solve the maximization problem (see Appendix A):

\[
S(E, \Gamma_a, L, P) = \max_{\omega_a} \{ S[\omega_a] \mid E, \Gamma_a, L, P \text{ fixed} \}. \tag{8}
\]

The critical points of the Boltzmann entropy at fixed \( E, \Gamma_a, L \) and \( P \) are determined by the condition

\[
\delta S - \beta \delta E - \sum_a \alpha_a \delta \Gamma_a - \beta \frac{\Omega}{2} \delta L + \beta U \cdot \delta P = 0, \tag{9}
\]

where \( \beta \) (inverse temperature), \( \alpha_a \) (chemical potentials), \( \Omega \) (angular velocity), and \( U \) (linear velocity) are appropriate Lagrange multipliers. Using

\[
\delta S = -\sum_a \int \frac{\delta \omega_a}{\gamma_a} \left( \ln \frac{\omega_a}{N_a \gamma_a} + 1 \right) \, d\mathbf{r}, \tag{10}
\]

\[
\delta E = \int \psi \delta \omega \, d\mathbf{r}, \quad \delta \Gamma_a = \int \delta \omega_a \, d\mathbf{r}, \tag{11}
\]

\[
\delta L = \int \delta \omega r^2 \, d\mathbf{r}, \quad \delta P = -\mathbf{z} \times \int \delta \omega r \, d\mathbf{r}, \tag{12}
\]

we obtain the mean field Boltzmann distribution for each species

\[
\omega_a = \gamma_a n_a = A_a \gamma_a e^{-\beta \gamma_a \psi_{\text{eff}}}, \tag{13}
\]

where \( A_a = N_a e^{-1-\alpha_a \gamma_a} > 0 \) is determined by the normalization condition (conservation of the number of point vortices of each species) yielding

\[
A_a = \frac{N_a}{\int e^{-\gamma_a \psi_{\text{eff}}} \, d\mathbf{r}}. \tag{14}
\]

The total vorticity is then given by:

\[
\omega = \sum_a A_a \gamma_a e^{-\beta \gamma_a \psi_{\text{eff}}} = -\frac{1}{\beta} \frac{dn}{d \psi_{\text{eff}}} = f_{\alpha, \beta}(\psi_{\text{eff}}). \tag{15}
\]

In these expressions \( \psi_{\text{eff}} \) is the relative stream function

\[
\psi_{\text{eff}} = \psi + \frac{\Omega}{2} \mathbf{r} \cdot \mathbf{r}, \tag{16}
\]

where \( \mathbf{U}_\perp = \mathbf{z} \times \mathbf{U} \). This corresponds to a relative velocity field \( \mathbf{u}_{\text{eff}} = \mathbf{u} - \Omega \times \mathbf{r} - \mathbf{U} \).

Differentiating equation (15) with respect to \( \psi_{\text{eff}} \), we obtain the identity

\[
\omega' (\psi_{\text{eff}}) = -\beta \omega_2(\psi_{\text{eff}}), \tag{17}
\]

where

\[
\omega_2 = \sum_a \gamma_a \omega_a = \sum_a \gamma_a^2 n_a \geq 0 \tag{18}
\]

is the local microscopic enstrophy. Since \( \omega = \omega(\psi_{\text{eff}}) \), the statistical theory predicts that the vorticity \( \omega(\mathbf{r}) \) is a stationary solution of the 2D Euler equation\(^7\) in a frame rotating with angular velocity \( \Omega \) and/or translating with a constant velocity \( \mathbf{U} \) (see Appendix B). Explicit examples of such structures are given in reference [43]. On the other hand, according to equations (17) and (18), the \( \omega - \psi_{\text{eff}} \) relationship is a monotonic function that is increasing at negative temperatures \( \beta < 0 \) and decreasing at positive temperatures \( \beta > 0 \).

Substituting equation (15) in the Poisson equation (3), the equilibrium stream function is obtained by solving the multi-species Boltzmann-Poisson equation

\[
-\Delta \psi = \sum_a A_a \gamma_a e^{-\beta \gamma_a \psi_{\text{eff}}} = f_{\alpha, \beta}(\psi_{\text{eff}}) \tag{19}
\]

with adequate boundary conditions, and relating the Lagrange multipliers \( \beta, \alpha_a, \Omega, \) and \( U \) to the constraints \( E, \Gamma_a, L, \) and \( P \). We also have to make sure that the distribution (13) is a (local) maximum of entropy, not a minimum or a saddle point. One can show [44] that the Boltzmann distribution (13) is a (local) maximum of constrained entropy if, and only if,

\[
\delta^2 J = \frac{1}{2} \sum_a \int \frac{(\delta \omega_a)^2}{\gamma_a \omega_a} \, d\mathbf{r} - \frac{\beta}{2} \int \delta \omega \delta \psi \, d\mathbf{r} < 0
\]

\[
\forall \delta \omega_a \mid \delta \Gamma_a = \delta L = \delta P = \delta E = 0, \tag{20}
\]

i.e. for all perturbations that conserve the circulations of each species, the angular momentum, the linear impulse, and the energy at first order. This is the criterion of thermodynamical stability in the microcanonical ensemble.

We note that the vorticity profiles of different species of point vortices are related to each other by:

\[
\frac{\omega_a}{A_a \gamma_a} \left( \frac{\omega_b}{A_b \gamma_b} \right)^{1/\gamma_b} = \left( \frac{\omega_b}{A_b \gamma_b} \right)^{1/\gamma_b}, \tag{21}
\]

and hence

\[
\omega_a(\mathbf{r}) = C_{ab} |\omega_b(\mathbf{r})|^{\gamma_b/\gamma_a}, \tag{22}
\]

\(^6\) In an unbounded domain, we must consider vortices of the same circulation otherwise they would form pairs (dipoles) and ballistically escape to infinity, implying an absence of statistical equilibrium. Furthermore, due to the \( e^{-\beta \gamma_a \mathbf{r}^2/2} \) factor, the distribution (13) can possibly be normalizable only if \( \text{sgn}(\gamma) \beta \gamma \geq 0 \).

\(^7\) The 2D Euler equation describes the dynamical evolution of the smooth vorticity field \( \omega(\mathbf{r}, t) \) of the point vortex gas in the limit \( N \rightarrow +\infty \) [8,9]. It is analogous to the Vlasov equation in plasma physics [2] and stellar dynamics [3].
where \( C_{ab} \) is independent on the position. Assuming that \( \gamma_a > 0, \gamma_b > 0 \) and that \( \omega_b(r) \) decreases with the distance (which corresponds to equilibrium states with \( \beta < 0 \)), this relation indicates that intense vortices \( (\gamma_a > \gamma_b) \) are more concentrated at the center, on average, than weaker vortices. On the other hand, equation (22) shows that opposite-sign vortices tend to separate (at negative temperatures). More generally, equation (22) characterizes the segregation between vortices with different circulations.

**Remark.** The mean field Boltzmann distribution (13) was first derived by Joyce and Montgomery [14] from a maximum entropy principle and by Lundgren and Pointin [22] from the Yvon-Born-Green (YBG) hierarchy (see also the discussion in Ref. [41]). A rigorous derivation of the mean field equations is given in references [23–28]. We stress that the statistical equilibrium state relies on an hypothesis of ergodicity (or at least efficient mixing) assuming that all the accessible microstates are equiprobable. There is no guarantee that the evolution of the point vortex gas is ergodic in all cases (on the contrary, there are strong arguments that it is not ergodic) and, therefore, the relaxation towards the Boltzmann distribution remains an open problem. To determine whether the point vortex gas truly relaxes towards the Boltzmann distribution, and to establish the scaling of the relaxation time with \( N \), we must develop a kinetic theory of point vortices [8].

### 2.6 The canonical ensemble

In the canonical ensemble, where \( \beta, \Omega \) and \( U \) are assumed to be fixed, the probability of a microstate with energy \( H \) is proportional to the Boltzmann factor \( e^{-\beta H} \), where \( H_{\text{eff}} = H + \frac{\Omega}{2} L = U \cdot \mathbf{P} \) is the effective Hamiltonian in the moving frame where the flow is steady. Therefore, the probability of an accessible macrostate is proportional to \( W[\omega][e^{-\beta E_{\text{eff}}[\omega]} = e^{S[\omega] - \beta E_{\text{eff}}[\omega]} = e^{J[\omega]}] \). As a result, the most probable macrostate is obtained by maximizing the Boltzmann free energy (more properly the Massieu function):

\[
J[\omega] = S[\omega] - \beta E[\omega] - \frac{\beta}{2} L[\omega] + \beta \mathbf{U} \cdot \mathbf{P}[\omega] \tag{23}
\]

while conserving the circulation of each species (4b).

We thus have to solve the maximization problem (see Appendix A):

\[
J(\beta, \Gamma, \Omega, U) = \max_{\omega} \{ J[\omega] \mid \Gamma_a \text{ fixed} \}. \tag{24}
\]

The critical points of the Boltzmann free energy at fixed \( \Gamma_a \) are determined by the condition

\[
\delta J - \sum_a \alpha_a \delta \Gamma_a = 0, \tag{25}
\]

where \( \alpha_a \) (chemical potentials) are appropriate Lagrange multipliers. This leads to the mean field Boltzmann distribution (13) as in the microcanonical ensemble.

The Boltzmann distribution (13) is a (local) maximum of constrained free energy \( J \) if, and only if,

\[
\delta^2 J = -\frac{1}{2} \sum_a \int \frac{\omega_a}{\gamma_a} \delta \omega_a \, dr - \beta \frac{1}{2} \int \delta \omega \delta \psi \, dr < 0 \quad \forall \delta \omega_a \mid \delta \Gamma_a = 0, \tag{26}
\]

i.e. for all perturbations that conserve the circulations of each species: \( \delta \Gamma_a = 0 \). This is the criterion of thermodynamical stability in the canonical ensemble.

### 2.7 Generic ensemble inequivalence for systems with long-range interactions

The variational problems (8) and (24) have the same critical points. However, these variational problems may not be equivalent. The set of solutions of (8) may not coincide with the set of solutions of (24). It can be shown that a solution of a variational problem is always the solution of a more constrained dual variational problem [45]. Therefore, a solution of (24) with given \( \beta, \Omega, \) and \( U \) is always a solution of (8) with the corresponding \( E, L, \) and \( \mathbf{P} \). Thus, canonical stability implies microcanonical stability:

\[
(24) \Rightarrow (8). \tag{26}
\]

However, the converse is wrong: a solution of (8) is not necessarily a solution of (24). When this happens, we speak of ensemble inequivalence. Ensemble inequivalence is generic for systems with long-range interactions but it is not compulsory.

For isolated systems that evolve at fixed energy, like the point vortex model, the proper statistical ensemble is the microcanonical ensemble. Since the energy is non-additive, the canonical ensemble is not physically justified to describe a subpart of the system (contrary to systems with short-range interactions) [4]. However, since the canonical variational problem (24) provides a sufficient condition of microcanonical stability it can be useful in

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6. This can be checked at the level of the second order variations. Indeed, if inequality (26) is satisfied for all perturbations that conserve the circulations of each species (canonical stability criterion), it is a fortiori satisfied for all perturbations that conserve in addition the angular momentum, the linear impulse, and the energy at first order (microcanonical stability criterion).

7. For self-gravitating systems, there exists a region of ensemble inequivalence in a range of energies in which the configurations have negative specific heats (these states are accessible in the microcanonical ensemble but not in the canonical ensemble) [46]. For the Hamiltonian Mean Field (HMF) model, the ensembles are equivalent for all energies even though the particles have long-range interactions [4]. For 2D vortices, the equivalence or inequivalence of the statistical ensembles depends on the shape of the domain and on the value of the control parameters \( \Gamma_a, E, L \) and \( \mathbf{P} = 0 \) in a non-trivial manner (one has to solve the equations of the statistical theory and study the variational problems (8) and (24) specifically). Note that the stability of the solutions in each ensemble (and consequently the possible occurrence of ensemble inequivalence) may be decided by plotting the series of equilibria and using the Poincaré theorem (see, e.g., [44,46] for more details).
that respect. Indeed, if we can show that the system is canonically stable, then it is granted to be microcanonically stable. Therefore, we can start by studying the canonical stability problem, which is simpler, and consider the microcanonical stability problem only if the canonical ensemble does not cover the whole range of parameters \((E, \Gamma_a, L, P)\).

Remark. The canonical ensemble is mathematically justified for a system of “Brownian point vortices” [40] described by \(N\) coupled stochastic Langevin equations (instead of Hamiltonian equations). However, this system is essentially academic and it is not clear whether it can have physical applications.

### 2.8 The Virial theorem

We can derive a form of Virial theorem for point vortices at statistical equilibrium [41]. Taking the logarithmic derivative of equation (13), we obtain

\[
\nabla \omega_a = -\beta \gamma_a \omega_a \nabla \psi_{\text{eff}}
\]

(27)

with

\[
\nabla \psi_{\text{eff}} = \nabla \psi + \Omega r - U_{\perp}.
\]

(28)

Dividing equation (27) by \(\gamma_a\), summing over the species, and introducing the local “pressure” \(p(r) = n(r)T = \sum_a (\omega_a(r)/\gamma_a)T\), we get

\[
\nabla p + \omega \nabla \psi_{\text{eff}} = 0.
\]

(29)

This equation is similar to the condition of hydrostatic equilibrium for self-gravitating systems in a rotating frame [3]. The pressure is positive at temperature and negative at negative temperatures. Taking the scalar product of equation (29) with \(r\), integrating over the entire domain, and integrating by parts the pressure term, we obtain the Virial theorem

\[
2 \int p \, dr - \mathcal{V} = \Omega L + U \cdot P = 2PV,
\]

(30)

where

\[
\mathcal{V} = \int \omega r \cdot \nabla \psi \, dr
\]

(31)

is the Virial of the point vortex gas, \(V\) is the area of the domain, and \(P = \frac{1}{2\pi} \int p \, r \cdot dS\) is the average pressure on the boundary of the domain (if the pressure is uniform on the boundary, with value \(p_b\), then \(P = p_b\)). We note that the angular momentum \(L = \int \omega r^2 \, dr\) of point vortices is similar to the moment of inertia of material particles. Using the isothermal equation of state \(p(r) = n(r)T\), we can rewrite the Virial theorem as:

\[
2NT - \mathcal{V} - \Omega L + U \cdot P = 2PV.
\]

(32)

In an unbounded domain, and for axisymmetric flows in a disk of radius \(R\), the Virial \(\mathcal{V}\) is given by [41]:

\[
\mathcal{V} = -\frac{1}{4\pi}.
\]

(34)

In that case, the Virial theorem can be rewritten as

\[
PV = N(T - T_c) - \frac{1}{2} \Omega L
\]

(35)

with \(P = p(R) = n(R)T\) and \(V = \pi R^2\). We have introduced the critical temperature

\[
T_c = \frac{\Gamma^2}{8\pi N^2}.
\]

(36)

For a single species system of point vortices, we have

\[
T_c = \frac{N \gamma^2}{8\pi}.
\]

(37)

The Virial theorem (35) relates the angular velocity \(\Omega\) to the angular momentum \(L\), the temperature \(T\), and the pressure \(P\) on the boundary. When \(\Omega = 0\), it reduces to:

\[
PV = N(T - T_c).
\]

(38)

On the other hand, in an unbounded domain, using \(PV \to 0\) at large distances, we obtain the identity

\[
\frac{1}{2} \Omega L = N(T - T_c).
\]

(39)

Additional results and discussion are given in reference [41].

### 2.9 The sinh-Poisson equation

If we consider a globally neutral system of point vortices with \(N/2\) point vortices of circulation \(\gamma\) and \(N/2\) point vortices of circulation \(-\gamma\), we find that the Boltzmann-Poisson equation (19) takes the form

\[
\omega = -\Delta \psi = \gamma \left(A_+ e^{\beta \gamma \psi_{\text{eff}}} - A_- e^{\beta \gamma \psi_{\text{eff}}}ight)
\]

(40)

with

\[
A_\pm = \frac{N}{2(e^{\mp \beta \gamma \psi_{\text{eff}}})},
\]

(41)

where the brackets denote a domain average. In a bounded domain, this equation has a non-trivial solution \(\psi \neq 0\) only if \(\beta < \beta_c < 0\) [47]. If we assume that \(A_+ = A_- = A\) (which corresponds to the equality of the chemical potential of the two species), we obtain the celebrated sinh-Poisson equation [14]:

\[
\omega = -\Delta \psi = 2\gamma A \sinh(-\beta \gamma \psi_{\text{eff}}).
\]

(42)

\(^{10}\) In an unbounded domain, we can impose \(P = 0\) by taking the origin at the center of vorticity. In a disk the linear impulse is not conserved so that \(U = 0\).
However, there is no fundamental reason why \( A_+ = A_- \) so this assumption is restrictive. For example, equation (42) holds if \( \psi_{2n+1} = 0 \) for all \( n \in \mathbb{N} \) but this condition is not always satisfied. In particular, some important solutions may be forgotten by using the sinh-Poisson equation (42) instead of the Boltzmann-Poisson equation (40), as discussed in Section 4.

3 Relaxation equations based on a maximum entropy production principle

In this section, using a MEPP, we derive a set of equations for a multi-species point vortex gas that relax towards statistical equilibrium.

3.1 Relaxation equations for a multi-species point vortex gas

The conservation of the number of point vortices of each species can be expressed in the local form by an equation of the form

\[
\frac{\partial \omega_a}{\partial t} + u \cdot \nabla \omega_a = -\nabla \cdot J_a, \quad (43)
\]

where \( J_a(r,t) \) is the current of species \( a \) to be determined. The evolution of the total vorticity \( \omega = \sum_a \omega_a \) is given by:

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = -\nabla \cdot J, \quad (44)
\]

where \( J = \sum_a J_a \) is the total current of vorticity.

We can express the time variation of energy, angular momentum, and linear impulse in terms of \( J \) using equations (4a), (5), and (44). This leads to the constraints

\[
\dot{E} = \int \mathbf{J} \cdot \nabla \psi \, d\mathbf{r} = 0, \quad (45)
\]

\[
\dot{L} = 2 \int \mathbf{J} \cdot \mathbf{r} \, d\mathbf{r} = 0, \quad (46)
\]

\[
\dot{\mathbf{P}} = -2\varepsilon \times \int \mathbf{J} \, d\mathbf{r} = 0. \quad (47)
\]

Using equations (7) and (43), we similarly express the rate of entropy production as:

\[
\dot{S} = -\sum_a \int \frac{\nabla \omega_a}{\gamma_a \omega_a} \cdot J_a \, d\mathbf{r}. \quad (48)
\]

The Maximum Entropy Production Principle (MEPP) consists in choosing the currents \( J_a \) which maximize the rate of entropy production \( \dot{S} \) respecting the constraints \( \dot{E} = 0, \dot{L} = 0, \dot{\mathbf{P}} = 0 \), and \( \sum_a J_a^2/(2\omega_a \gamma_a) \leq C(r,t) \). The last constraint expresses a bound (unknown) on the value of the diffusion currents. Convexity arguments justify that this bound is always reached so that the inequality can be replaced by an equality. The corresponding condition on first variations can be written at each time \( t \) as:

\[
\delta \dot{S} - \beta(t)\delta \dot{E} - \beta(t)\frac{\Omega(t)}{2}\delta \dot{L} + \beta(t)\mathbf{U}(t) \cdot \delta \mathbf{P} = -\sum_a \frac{1}{D_a(r,t)} \delta J_a^2 \omega_a \gamma_a \, d\mathbf{r} = 0, \quad (49)
\]

where \( \beta(t), \Omega(t), \mathbf{U}(t), \) and \( D(r,t) \) are time-dependent Lagrange multipliers. The MEPP leads to optimal currents of the form

\[
J_a = -D(r,t) \left\{ \nabla \omega_a + \beta(t) \gamma_a \omega_a [\nabla \psi + \Omega(t) \mathbf{r} - \mathbf{U}_\perp(t)] \right\}. \quad (50)
\]

We can also write

\[
J_a = -D(r,t) \left\{ \nabla \omega_a + \beta(t) \gamma_a \omega_a [\mathbf{u} - \Omega(t) \times \mathbf{r} - \mathbf{U}_\perp(t)] \right\}. \quad (51)
\]

Under this form, we see that the flow structures rotate with angular velocity \( \Omega(t) \) and/or translate with velocity \( \mathbf{U}(t) \). The total current is:

\[
\mathbf{J} = -D(r,t) \left\{ \nabla \omega_a + \beta(t) \gamma_a \omega_a [\nabla \psi + \Omega(t) \mathbf{r} - \mathbf{U}_\perp(t)] \right\}, \quad (52)
\]

where \( \omega_a = \sum_a \omega_a \gamma_a \) is the local microscopic entrophy. Introducing the time-dependent relative stream function

\[
\psi_{\text{eff}}(r,t) = \psi(r,t) + \frac{\Omega(t)}{2}r^2 - \mathbf{U}_\perp(t) \cdot \mathbf{r}, \quad (53)
\]

we obtain the relaxation equations

\[
\frac{\partial \omega_a}{\partial t} + u \cdot \nabla \omega_a = \nabla \cdot \{ D(r,t) \left\{ \nabla \omega_a + \beta(t) \gamma_a \omega_a \nabla \psi_{\text{eff}} \right\} \}. \quad (54)
\]

The equation for the total vorticity is:

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nabla \cdot \{ D(r,t) \left\{ \nabla \omega + \beta(t) \omega_a \nabla \psi_{\text{eff}} \right\} \}. \quad (55)
\]

The time evolution of the Lagrange multipliers \( \beta(t), \Omega(t), \) and \( \mathbf{U}(t) \) is determined by introducing equation (52) in the constraints (45)–(47). This yields

\[
\int D \nabla \psi \cdot \{ \nabla \omega + \beta(t) \omega_a [\nabla \psi + \Omega(t) \mathbf{r} - \mathbf{U}_\perp(t)] \} \, d\mathbf{r} = 0, \quad (56)
\]

\[
\int D \mathbf{r} \cdot \{ \nabla \omega + \beta(t) \omega_a [\nabla \psi + \Omega(t) \mathbf{r} - \mathbf{U}_\perp(t)] \} \, d\mathbf{r} = 0, \quad (57)
\]

\[
\int D \left\{ \nabla \omega + \beta(t) \omega_a [\nabla \psi + \Omega(t) \mathbf{r} - \mathbf{U}_\perp(t)] \right\} \, d\mathbf{r} = 0. \quad (58)
\]

Equations (56)–(58) form a system of linear equations that determines the evolution of \( \beta(t), \Omega(t), \) and \( \mathbf{U}(t) \). In a domain without special symmetry (\( \Omega = \mathbf{U} = 0 \)), they reduce to:

\[
\beta(t) = \frac{\int D \nabla \psi \cdot \nabla \omega \, d\mathbf{r}}{\int D \omega_a (\nabla \psi)^2 \, d\mathbf{r}}. \quad (59)
\]
The evolution of the inverse temperature $\beta(t)$ is determined by the conservation of energy.

The Boltzmann entropy (7) is the Lyapunov functional of the relaxation equations (54). It satisfies an $H$-theorem provided that $D(r,t) \geq 0$. Indeed, using the expression (50) of the currents, the entropy production (48) can be rewritten as:

$$\dot{S} = \sum_a \int \frac{J_a^2}{D\gamma_a\omega_a} \, dr + \beta(t) \int \mathbf{J} \cdot [\nabla \psi + \Omega(t)r - \mathbf{U}_\perp(t)] \, dr.$$  

(60)

Using the conservation laws (45)–(47), the second term is seen to vanish giving

$$\dot{S} = \sum_a \int \frac{J_a^2}{D\gamma_a\omega_a} \, dr \geq 0.$$  

(61)

A stationary solution of equation (54) satisfies $\dot{S} = 0$ implying $\mathbf{J}_a = 0$ for each species. Using equation (50), we recover the mean field Boltzmann distribution (13) where $A_a$ appears as a constant of integration.

Because of the $H$-theorem, the relaxation equations converge, for $t \to +\infty$, towards a mean field Boltzmann distribution that is a (local) maximum of entropy at fixed energy, circulations, angular momentum, and linear impulse\footnote{The steady states of the relaxation equations are the critical points (maxima, minima, saddle points) of the constrained entropy. It can be shown\cite{15} that a critical point of entropy is dynamically stable with respect to the relaxation equations if, and only if, it is a (local) maximum. Minima are unstable for all perturbations so they cannot be reached by the relaxation equations. Saddle points are unstable only for certain perturbations so they can be reached if the system does not spontaneously generate these dangerous perturbations. We note that these properties are valid whatever the form of the diffusion coefficient $D(r,t) \geq 0$.}. If several maxima exist for the same values of the constraints, the selection depends on a notion of basin of attraction.

The relaxation equations adapted to the canonical ensemble can be obtained by maximizing the rate of free energy production $\dot{J}$ with the constraint $\sum_a J_a^2/(2\omega_a\gamma_a) \leq C(r,t)$. This leads to equation (54) where $\beta$, $\Omega$ and $\mathbf{U}$ are fixed. These relaxation equations increase the free energy (23) monotonically (canonical $H$-theorem) and they relax towards a (local) maximum of free energy at fixed circulations.

Remark. We stress that the relaxation equations (54) do not describe the real dynamics of point vortices in the microcanonical ensemble. The kinetic theory of point vortices developed in reference\cite{8} (and references therein) leads to very different equations. Therefore, the relaxation equations (54) have probably no physical justification\footnote{We note that the relaxation equations (54) and the MEPP are closely related to the linear thermodynamics of Onsager\cite{48}. The linear thermodynamics of Onsager is expected to be valid close to equilibrium. However, the present remark shows that this is not always the case since the rigorous kinetic theory of 2D point vortices\cite{8} does not reduce to Onsager’s linear thermodynamics even close to equilibrium. The reason may be related to the fact that Onsager’s theory is valid in the canonical ensemble (see\cite{49} for more details) while the kinetic theory of point vortices has to be developed in the microcanonical ensemble. If we consider the kinetic theory of Brownian point vortices in the canonical ensemble\cite{40}, we obtain Fokker-Planck equations (see Sect. 3.7.2) that are equivalent to Onsager’s linear thermodynamics\cite{49}.}.

However, since they relax towards a (local) maximum of constrained entropy, they can be used as a numerical algorithm to construct the statistical equilibrium state of the multi-species point vortex gas for given values of $E$, $\Gamma_a$, $L$ and $\mathbf{P}$. This confers them some practical interest since it is not easy to directly solve the mean field Boltzmann-Poisson equation (19) and be sure that the solution is stable (entropy maximum).

### 3.2 The importance of the drift term

The current $\mathbf{J}_a$ arising in the relaxation equation (54) is the sum of two terms. A diffusion term $\mathbf{J}_{a\text{diff}} = -D\nabla \omega_a$ and a drift term $\mathbf{V}_{a\text{drift}} = -D\beta \gamma_a \nabla \psi_{a\text{eff}}$ that is perpendicular to the relative mean field velocity $\mathbf{u}_{a\text{eff}} = -\mathbf{z} \times \nabla \psi_{a\text{eff}}$. The drift coefficient $\mu_a = D \gamma_a$ (mobility) depends on the sign of the point vortices and is given by a sort of Einstein relation. The diffusion term dissipates the energy ($\dot{E}_\text{diff} = \int \mathbf{J}_{a\text{diff}} \cdot \nabla \psi \, dr = -\int D \nabla \omega_a \cdot \nabla \psi \, dr = -D \int \omega^2 \, dr \leq 0$) while the drift term acts in the opposite direction ($\dot{E}_{a\text{drift}} = -\dot{E}_{a\text{diff}}$) in order to conserve the energy ($\dot{E} = 0$). On the other hand, the diffusion term increases the entropy ($\dot{S}_{a\text{diff}} = \int \mathbf{J}_{a\text{diff}} \cdot \nabla \psi_{a\text{eff}} \, dr \geq 0$) while the drift term usually decreases the entropy ($\dot{S}_{a\text{drift}} \leq 0$). However, as a whole, the entropy increases ($\dot{S} \geq 0$). As emphasized in previous papers\cite{13,15,40,50}, the diffusion term tends to disperse the vortices while the drift term is responsible for the clustering of point vortices of the same sign at negative temperatures. It is therefore a fundamental feature of two-dimensional vortex dynamics. We can also relate these results to the phenomenon of inverse cascade in 2D turbulence. With only the diffusion term (which can be large since it represents a turbulent viscosity), we have a direct cascade of energy towards smaller and smaller scales. When the drift term is taken into account, we get an inverse cascade of energy towards the large scales. By contrast, the neg-entropy $-S$ (and more generally the generalized entropies $\Gamma_a = \int \omega_a \, dr$) always cascade towards the small scales and are dissipated (see Sect. 4.2.2).

### 3.3 The relaxation equations for two species of point vortices with opposite circulation

If we consider a collection of $N/2$ vortices with circulation $+\gamma$ and $N/2$ vortices with circulation $-\gamma$, the relaxation equations of motion (54) and the MEPP are closely related to the linear thermodynamics of Onsager\cite{48}. The linear thermodynamics of Onsager is expected to be valid close to equilibrium.
The evolution of the Lagrange multipliers is given by equations (65)–(68) with $\omega_2 = (\omega_+ - \omega_-) \gamma = n \gamma^2$. The equilibrium state is:

$$\omega_\pm = \pm A \pm \gamma \epsilon^{\pm \beta \gamma \psi \sigma}. \quad (64)$$

When $A_+ = A_-$, this leads to the sinh-Poisson equation (42) but, as indicated previously, this is not the general case.

### 3.4 The relaxation equations for a single species of point vortices

If the point vortices have the same circulation $\gamma$ (single species system), we find that the relaxation equation for the smooth vorticity field is:

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nabla \cdot \{D(r, t) [\nabla \omega + \beta(t) \gamma \omega \nabla \psi_{\text{eff}}]\}, \quad (65)$$

$$-\Delta \psi = \omega. \quad (66)$$

The time evolution of the Lagrange multipliers $\beta(t)$, $\Omega(t)$, and $U(t)$ is given by:

$$\int D \omega \cdot \{\nabla \omega + \beta(t) \gamma \nabla \psi_{\text{eff}}\} \, dr = 0, \quad (67)$$

$$\int D r \cdot \{\nabla \omega + \beta(t) \gamma \nabla \psi_{\text{eff}}\} \, dr = 0, \quad (68)$$

$$\int D \{\nabla \omega + \beta(t) \gamma \nabla \psi_{\text{eff}}\} \, dr = 0. \quad (69)$$

In a domain without special symmetry, they reduce to:

$$\beta(t) = -\frac{\int D \omega \cdot \nabla \omega \, dr}{\int D \gamma \omega [\nabla \omega]^2 \, dr}. \quad (70)$$

The conservation of energy determines the evolution of the inverse temperature $\beta(t)$. At equilibrium, we obtain the Boltzmann distribution

$$\omega = Ae^{-\beta \gamma \psi_{\text{eff}}} \quad (71)$$

that maximizes the Boltzmann entropy

$$S = -\int \frac{\omega}{\gamma} \ln \frac{\omega}{N \gamma} \, dr \quad (72)$$

at fixed energy, circulation, angular momentum, and linear impulse.

### 3.5 Simplification of the constraints in an infinite domain

We consider a single species of point vortices and we assume that the diffusion coefficient $D$ is constant in order to simplify the expressions of the constraints. In an infinite domain, we have the identities

$$\int \nabla \omega \cdot \nabla \psi \, dr = \Gamma_2, \quad Q \equiv \int \omega (\nabla \psi)^2 \, dr,$$

$$\int \omega r \cdot \nabla \psi \, dr = V, \quad \int \omega \nabla \psi \, dr = 0,$$

$$\int r \cdot \nabla \omega \, dr = -2 \Gamma, \quad \int \nabla \omega \, dr = 0, \quad (73)$$

where

$$\Gamma_2 = \int \omega^2 \, dr \quad (74)$$

is the macroscopic enstrophy and the Virial $V$ is given by equation (34). With these expressions, the constraints (67)–(69) determining the evolution of the Lagrange multipliers $\beta(t)$, $\Omega(t)$ and $U(t)$ become

$$\frac{\Gamma_2}{\beta \gamma} + Q + \Omega V = 0, \quad (75)$$

$$-\frac{2 \Gamma}{\beta \gamma} + V + \Omega L = 0, \quad (76)$$

$$\Omega P - \Gamma \Omega U = 0. \quad (77)$$

We can always impose $P = 0$ by taking the origin at the center of vorticity. Then, equation (77) implies $U = (\Omega / \Gamma) P = 0$. The remaining constraints reduce to

$$\frac{\Gamma_2}{\beta \gamma} + Q + \Omega V = 0, \quad -\frac{2 \Gamma}{\beta \gamma} + V + \Omega L = 0. \quad (78)$$

Solving these equations, we find that the Lagrange multipliers evolve according to:

$$\beta(t) = \frac{\Gamma_2 L + 2 \Gamma V}{\gamma (\gamma^2 - Q L)} \quad (79)$$

Using the expression (34) of the Virial, equation (78b) may be rewritten as:

$$\Omega(t) = -\frac{2 \Gamma Q + \Omega \Gamma_2}{\Gamma_2 L + 2 \Gamma V} \quad (80)$$

where $\beta_\epsilon = -8 \pi / (\gamma L)$ is the critical inverse temperature (37). At equilibrium, we recover the Virial theorem of the point vortex gas (39) in an unbounded domain. This relation remains valid out-of-equilibrium in the framework of the relaxation equations. We also note that $L = 0$ does not imply $\Omega = 0$.

In the canonical ensemble, the relaxation equations are given by equations (65), (66) where $\beta$, $\Omega$, and $U$ are fixed. We take $U = 0$ for simplicity and we again assume that
D is constant. In that case, proceeding as in Section 2.8, we can establish the out-of-equilibrium Virial theorem

$$\frac{1}{4\gamma D\beta} \dot{L} + \frac{1}{2} \Omega L = \frac{\Gamma}{\gamma} \left( \frac{1}{\beta} - \frac{1}{\beta_c} \right).$$

(81)

At equilibrium, we recover the Virial theorem of the point vortex gas (39) in an unbounded domain. Additional results and discussion are given in reference [51].

### 3.6 Simplification of the constraints for an axisymmetric flow in a disk

In a disk, the constraints are given by equations (67), (68) with $U = 0$ since the linear impulse is not conserved. We again assume that $D$ is constant for simplicity. Solving these equations, we find that the Lagrange multipliers $\beta(t)$ and $\Omega(t)$ evolve according to:

$$\beta(t) = \frac{L \int \nabla \omega \cdot \nabla \psi \, dr - \frac{V}{\gamma} \int \nabla \omega \cdot \nabla \psi \, dr}{\gamma V^2 - \gamma L \int \omega(\nabla \psi)^2 \, dr},$$

(82)

$$\Omega(t) = \frac{\int \nabla \omega \cdot \nabla \psi \, dr + \frac{V}{\gamma} \int \omega(\nabla \psi)^2 \, dr - \frac{V}{\gamma} \int \nabla \omega \cdot \nabla \psi \, dr}{\frac{1}{2} \Gamma L + 2 \Gamma V}.$$  

(83)

For an axisymmetric flow, we have the identities

$$\int \nabla \omega \cdot \nabla \psi \, dr = \Gamma_2 - \Gamma \omega_b,$$

(84)

$$\int \nabla \omega \cdot \nabla \psi \, dr = 2\pi R^2 \omega_b - 2\Gamma,$$

(85)

where $\omega_b = \omega(R, t)$ is the vorticity on the boundary of the disk. On the other hand, the Virial $V$ is given by equation (34). With these expressions, the constraints (67), (68) reduce to:

$$\frac{\Gamma_2 - \Gamma \omega_b}{\beta \gamma} + Q + \Omega V = 0, \quad \frac{2\pi R^2 \omega_b - 2\Gamma}{\beta \gamma} + V + \Omega L = 0,$$

(86)

and the Lagrange multipliers $\beta(t)$ and $\Omega(t)$ evolve according to:

$$\beta(t) = \frac{(\Gamma_2 - \Gamma \omega_b) L + 2(\Gamma - \pi R^2 \omega_b) V}{\gamma (V^2 - Q L)},$$

(87)

$$\Omega(t) = \frac{2(\Gamma - \pi R^2 \omega_b) V + V(\Gamma_2 - \Gamma \omega_b)}{(\Gamma_2 - \Gamma \omega_b) L + 2(\Gamma - \pi R^2 \omega_b) V}.$$  

(88)

Equation (86b) may be rewritten as:

$$\frac{\omega_b(t)}{\beta(t) \gamma} \pi R^2 = N \left( \frac{1}{\beta(t)} - \frac{1}{\beta_c} \right) - \frac{1}{2} \Omega(t) L,$$

(89)

where $\beta_c$ is the critical inverse temperature (37). At equilibrium, we recover the Virial theorem (35) of the axisymmetric point vortex gas in a disk. This relation remains valid out-of-equilibrium in the framework of the relaxation equations.

In the canonical ensemble, the relaxation equations are given by equations (65), (66) where $\beta$ and $\Omega$ are fixed and $U = 0$. In that case, proceeding as in Section 2.8, we can establish the out-of-equilibrium Virial theorem

$$\frac{1}{2\gamma D\beta} \dot{L} = \frac{\Gamma}{\gamma} \left( \frac{1}{\beta} - \frac{1}{\beta_c} \right) - \frac{1}{2} \Omega L - 2PV.$$

(90)

For an axisymmetric flow, using the expression (34) of the Virial, we get:

$$\frac{1}{4\gamma D\beta} \dot{L} = \frac{\Gamma}{\gamma} \left( \frac{1}{\beta} - \frac{1}{\beta_c} \right) - \frac{1}{2} \Omega L - \frac{\omega_b(t)}{\gamma \beta} \pi R^2.$$

(91)

At equilibrium, we recover the Virial theorem (35) of the axisymmetric point vortex gas in a disk. Additional results and discussion are given in reference [51].

### 3.7 Connection with other relaxation equations

In this section, we mention some connections (but also stress important differences) between the relaxation equations derived in the previous sections and other equations of the same kind. To simplify the discussion, we take $\Omega = 0$ and $U = 0$.

#### 3.7.1 Relaxation of a test vortex in a thermal bath

The relaxation equation (65) shares some analogies with the Fokker-Planck equation

$$\frac{\partial P}{\partial t} + u_{\text{bath}} \cdot \nabla P = \nabla \cdot \{ D(r) [ \nabla P + \beta \gamma P \nabla \psi_{\text{bath}} ] \}$$

(92)

developed in the kinetic theory of point vortices in the thermal bath approximation [8]. The Fokker-Planck equation (92) describes the evolution of the probability density $P(r, t)$ of the position $r$ of a test vortex evolving in a “bath” of field vortices at statistical equilibrium. The stream function $\psi_{\text{bath}}(r)$ in equation (92) is fixed. It is determined by the equilibrium distribution $\omega_{\text{bath}}(r) = A e^{-\beta \gamma \psi_{\text{bath}}(r)}$ of the field vortices forming the bath. The probability current in equation (92) is the sum of two terms. A term $D \nabla P$ corresponding to a pure diffusion with a diffusion coefficient $D$ (its expression is given in [8]) and an additional term $D \beta \gamma \nabla \psi_{\text{bath}}$ that can be interpreted as a drift velocity $^{13}$.

The drift is perpendicular to the mean velocity $u_{\text{bath}} = -z \times \nabla \psi_{\text{bath}}$. The drift coefficient $\mu = D \beta \gamma$ (mobility) is given by a sort of Einstein relation. At equilibrium, we get the Boltzmann distribution

$$P_{eq}(r) = A e^{-\beta \gamma \psi_{\text{bath}}(r)}.$$

(93)

$^{13}$ There exist numerous analogies between the kinetic theory of 2D vortices and the kinetic theory of stellar systems [1]. In these analogies, the drift of a point vortex [50] is the counterpart of the dynamical friction experienced by a star [52].
Therefore, the distribution of the test vortex relaxes towards the distribution of the field vortices (bath). As a result, the Fokker-Planck equation (92) describes a process of thermalization.

Despite some obvious analogies, the relaxation equations (65) and (92) are very different (physically and mathematically). In equation (65) the stream function $\psi(r,t)$ is determined self-consistently by the vorticity $\omega(r,t)$ using the Poisson equation (66). As a result, it relaxes towards the Boltzmann distribution (71) coupled self-consistently to the Poisson equation (66). By contrast, in equations (92) and (93) the stream function $\psi_{\text{bath}}(r)$ is given.

### 3.7.2 Brownian vortices

In the canonical ensemble (fixed $\beta$), the relaxation equations (54) coincide with the Fokker-Planck equations

$$\frac{\partial \omega_a}{\partial t} + \mathbf{u} \cdot \nabla \omega_a = \nabla \cdot \left[ D \left( \nabla \omega_a + \beta \gamma_a \omega_a \nabla \psi \right) \right],$$

$$- \Delta \psi = \omega = \sum_a \omega_a$$

(94)

(95)

describing a multi-species system of “Brownian point vortices” in the mean field approximation [40]. At equilibrium, we get the Boltzmann-Poisson equation

$$- \Delta \psi = \sum_a A_a \gamma_a e^{-\beta \gamma_a \psi}.$$  

(96)

The temperature may be positive or negative. At negative temperatures, like-sign Brownian vortices attract each other and at positive temperatures like-sign Brownian vortices repel each other (in this model, the temperature is the relevant control parameter).

### 3.7.3 Smoluchowski-Poisson system and Keller-Segel model

The relaxation equations (94), (95) with $\gamma_a > 0$ and $\beta < 0$ are similar to the mean field Fokker-Planck equations

$$\frac{\partial \rho_a}{\partial t} = \nabla \cdot \left[ D \left( \nabla \rho_a + \beta \rho_a \nabla \Phi \right) \right],$$

$$\Delta \Phi = S_d G \rho = S_d G \sum_a \rho_a = S_d G \sum_a n_a m_a$$

(97)

(98)

describing a multi-species system of self-gravitating Brownian particles in the strong friction limit [30]. These equations form the so-called Smoluchowski-Poisson system. Here, $m_a$ is the mass of particles of species a, $\rho_a$ is the density of mass of species a, $\Phi$ is the gravitational potential, and $\beta = 1/(k_B T) > 0$ is the inverse temperature ($S_d$ is the surface of a unit sphere in $d$ dimensions). At equilibrium, we get the Boltzmann-Poisson equation

$$\Delta \Phi = S_d G \sum_a A_a m_a e^{-\beta m_a \Phi}.$$  

(99)

Of course, the mass of the particles is always positive and the gravitational force is attractive. As discussed in reference [30], the relaxation equations (97), (98) also provide a multi-species Keller-Segel [31] model of chemotaxis in biology. With the notations of biology, it writes

$$\frac{\partial \rho_a}{\partial t} = D_a \Delta \rho_a - \gamma_a \nabla (\rho_a \nabla c),$$

$$\Delta c = -\lambda \rho = -\lambda \sum_a \rho_a,\quad (100)$$

where $\rho_a$ is the density of particles of species a and c is the concentration of the secreted chemical (pheromone). The particles diffuse with a diffusion coefficient $D_a$ and they also move in the direction of the gradient of the secreted chemical (chemotactic drift). The chemotactic sensitivity $\gamma_a$ is a measure of the strength of the influence of the chemical gradient on the flow of particles. Finally, the coefficient $\lambda$ measures the quantity of chemical produced by the particles. At equilibrium, we get the Boltzmann-Poisson equation

$$\Delta c = -\lambda \sum_a A_a e^{\frac{\Delta c}{\beta \gamma_a}}.$$  

(102)

that is equivalent to equation (99) up to a change of notations. In this analogy, the concentration $-c$ of the chemical plays the role of the gravitational potential $\Phi$.

### 3.7.4 Debye-Hückel equations

The relaxation equations (94)–(95) with $\beta > 0$ are similar to the mean field Fokker-Planck equations

$$\frac{\partial \rho_a}{\partial t} = \nabla \cdot \left[ D \left( \nabla \rho_a + \beta \rho_a \nabla \Phi \right) \right],$$

$$\Delta \Phi = -S_d \rho = -S_d \sum_a \rho_a = -S_d \sum_a n_a e_a$$

(103)

(104)

derived by Debye and Hückel [7] in their theory of electrolytes. Here, $e_a$ is the charge of particles of species a, $\rho_a$ is the density of charge of species a, $\Phi$ is the electric potential, and $\beta = 1/(k_B T) > 0$ is the inverse temperature. At equilibrium, we get the Boltzmann-Poisson equation

$$\Delta \Phi = -S_d \sum_a A_a e_a e^{-\beta e_a \Phi}.$$  

(105)

Particles of opposite sign attract each other while particles of the same sign repel each other. If we consider a globally neutral system of charges with $N/2$ charges $+e$ and $N/2$ charges $-e$, and assume $A_+ = A_- = A$, we find that the Boltzmann-Poisson equation (105) takes the form $\Delta \Phi = 2S_d A e \sinh(\beta e \Phi)$ similar to the sinh-Poisson equation (42) but with positive temperature. This sinh-Poisson equation explicitly appears in the paper of Debye and Hückel [6].

14 These equations were previously derived by Nernst [53,54] and Planck [55] so they are also called the Nernst-Planck equations.
In conclusion, the relaxation equations (54) for the point vortex gas at negative temperatures share some analogies with the Smoluchowski-Poisson system and with the Keller-Segel model, while the relaxation equations (54) for the point vortex gas at positive temperatures share some analogies with the Debye-Hückel equations. However, we note that in equation (54) the energy is conserved thanks to a time dependent inverse temperature $\beta(t)$ (microcanonical ensemble) while the inverse temperature $\beta$ is fixed in the other equations (canonical ensemble).

4 The strong mixing limit

In this section, we consider the case where $\beta \gamma_t \psi \ll 1$ so that the equations of the statistical theory can be expanded as a function of this small parameter. This corresponds to a limit of strong mixing ($\beta$ small) or low energy ($\psi$ small) where the vorticity is almost uniform. This is also similar to the Debye-Hückel approximation in plasma physics, except that in the present context the temperature may be positive or negative. This expansion was introduced by Chavanis and Sommeria [34] in the context of the MRS statistical theory of continuous vorticity fields. We adapt it here to the case of point vortices and discuss the similarities and the differences with the case of continuous vorticity flows. In this section, we present the main results of the expansion. The details of the derivation, and some complements, are given in Appendices D and E.

4.1 Statistical equilibrium state

We consider a multi-species system of point vortices in a bounded domain. We introduce appropriate length and time scales such that the area of the domain is $V = 1$ and the microscopic enstrophy is $\Gamma_2^m = \sum_a N_a \gamma_a = \sum_a N_a \gamma_a^2 = 1$. At statistical equilibrium, the relation between the vorticity and the stream function is given by equation (13). Assuming $\beta \gamma_t \psi \ll 1$, we obtain to second order

$$\omega_a = A_a \gamma_a \left(1 - \beta \gamma_t \psi + \frac{1}{2} \beta^2 \gamma_a^2 \psi^2 + \ldots\right). \quad (106)$$

Integrating over the domain, we get

$$\Gamma_a = A_a \gamma_a \left(1 - \beta \gamma_t \langle \psi \rangle + \frac{1}{2} \beta^2 \gamma_a^2 \langle \psi^2 \rangle + \ldots\right). \quad (107)$$

This equation can be reversed to give

$$A_a = N_a \left(1 + \beta \gamma_t \langle \psi \rangle - \frac{1}{2} \beta^2 \gamma_a^2 \langle \psi^2 \rangle + \beta^2 \gamma_a^2 \langle \psi^2 \rangle^2 + \ldots\right). \quad (108)$$

At first order, the relation between the vorticity and the stream function is linear. Combining equations (106) and (108), we get:

$$\omega_a = \Gamma_a - \beta \gamma_a \Gamma_a \langle \psi - \langle \psi \rangle \rangle. \quad (109)$$

The total vorticity is:

$$\omega = \Gamma - \beta \langle \psi - \langle \psi \rangle \rangle. \quad (110)$$

Substituting this relation in the Poisson equation (3), we obtain the Helmholtz-type mean field equation

$$- \Delta \psi = \Gamma - \beta \langle \psi - \langle \psi \rangle \rangle. \quad (111)$$

The inverse temperature $\beta$ is determined by the energy constraint. Using equations (4a) and (110), we get:

$$E = \frac{1}{2} \Gamma \langle \psi \rangle - \frac{1}{2} \beta \langle \psi^2 \rangle - \langle \psi \rangle^2. \quad (112)$$

The mean field equation (111) with the energy constraint (112) may have several solutions with the same values of $\Gamma$ and $E$. To compare these solutions, we need to determine the entropy at second order. Substituting equation (13) in equation (7), we obtain

$$S = 2 \beta E + \sum_a \alpha_a \Gamma_a + \sum_a \sum_b \sum_c N_a \ln N_a + N \quad (113)$$

Using the expression of $A_a$ at second order given by equation (108), we find that

$$S = 2 \beta E - \beta \Gamma \langle \psi \rangle + \frac{1}{2} \beta^2 \left(\langle \psi^2 \rangle - \langle \psi \rangle^2\right). \quad (114)$$

Using equation (112) to simplify the expression, we get

$$S = -\frac{1}{2} \beta^2 \left(\langle \psi^2 \rangle - \langle \psi \rangle^2 \right) \quad (115)$$

On the other hand, using equation (110), the macroscopic enstrophy (74) is given at second order by

$$\Gamma_2 = \Gamma^2 + \beta^2 \left(\langle \psi^2 \rangle - \langle \psi \rangle^2 \right) = \Gamma^2 + \beta \Gamma \langle \psi \rangle - 2 \beta E. \quad (116)$$

Comparing equations (115) and (116), we obtain the following relation between the entropy and the macroscopic enstrophy

$$S = -\frac{1}{2} (\Gamma_2 - \Gamma^2). \quad (117)$$

Therefore, in the strong mixing limit, a maximum of entropy is equivalent to a minimum of enstrophy. At leading order, these results are exactly the same as those obtained by Chavanis and Sommeria [34] for continuous vorticity fields in the MRS theory.

The expansion can be pushed to higher orders by a similar method (see Appendix D). We then obtain

$$\omega = -\Delta \psi = B_0 + B_1 \beta \langle \psi \rangle + B_2 \beta^2 \langle \psi^2 \rangle + B_3 \beta^3 \langle \psi^3 \rangle \quad (118)$$

In order to establish the expression of the enstrophy at second order, we need to write the $\omega - \psi$ relationship at second order. However, it is shown in Appendix D that the term of second order in the $\omega - \psi$ relationship does not contribute to the enstrophy at second order leaving equation (116) unchanged.
with
\[
B_2 = \frac{1}{2} \Gamma_m^\text{m} + \frac{1}{2} \Gamma_m^\text{m} \beta \langle \psi \rangle
\]  
and
\[
B_3 = -\frac{1}{6} \Gamma_m^\text{m},
\]
where \( \Gamma_m^\text{n} = \sum_n N_a \gamma_a^n \) denotes the microscopic moment of order \( n \) of the vorticity. The coefficients \( B_0 \) and \( B_1 \) can be obtained from the general results given in Appendix D. In the case of a symmetric distribution of point vortices with positive and negative circulations, we have \( \Gamma_{2n+1} = 0 \) and equation (118) reduces to:
\[
\omega = B_0 + B_1 \beta \psi + \frac{1}{2} \Gamma_m^\text{m} \beta \langle \psi \rangle \psi^2 - \frac{1}{6} \Gamma_m^\text{m} \beta^3 \psi^3.
\]  
If in addition \( \langle \psi^{2n+1} \rangle = 0 \), the even terms in \( \psi \) disappear, and we obtain (using the results of Appendix D):
\[
\omega = -\left( \Gamma_m^\text{m} - \frac{1}{2} \Gamma_m^\text{m} \beta^2 \langle \psi^2 \rangle \right) \beta \psi - \frac{1}{6} \Gamma_m^\text{m} \beta^3 \psi^3.
\]

These expressions differ from those derived by Chavanis and Sommeria [34] in the framework of the MRS theory (see Appendix A of [34] and Appendix E). In particular, the prefactor \( \Gamma_m^\text{m} \) of the cubic term in equations (121) and (122) is replaced, for continuous vorticity fields, by \( \Gamma_4^{\text{m}} - 3(\Gamma_2^{\text{m}})^2 \) where \( \Gamma_n^{\text{m}} = \int \omega_a^m \text{d}r \) are the microscopic moments of the vorticity (see [34] and Eq. (E.23)). It is then found that the Kurtosis \( K_u = \Gamma_4^{\text{m}} / (\Gamma_2^{\text{m}})^2 \) controls the deviation to the linear \( \omega - \psi \) relationship [34]. Consider for simplicity the case \( \langle \psi^{2n+1} \rangle = 0 \). For continuous vorticity fields, the \( \omega - \psi \) relationship behaves like a tanh function for \( K_u < 3 \) and like a sinh function for \( K_u > 3 \). In the case of point vortices, there is no such transition and the \( \omega - \psi \) relationship always behaves like a sinh function. It is also interesting to consider the case of a system with \( N/2 \) point vortices of circulation \( \gamma \) and \( N/2 \) point vortices of circulation \( -\gamma \). In that case \( \Gamma_2^m = N\gamma^2 \) and \( \Gamma_4^m = N\gamma^4 \), and equation (121) becomes
\[
\omega = B_0 + B_1 \beta \psi + \frac{1}{2} N\gamma^4 \beta \langle \psi \rangle \psi^2 - \frac{1}{6} N\gamma^4 \beta^3 \psi^3.
\]  
When \( \langle \psi^{2n+1} \rangle = 0 \), we get from equation (122):
\[
\omega = -N\gamma^2 \left(1 - \frac{1}{2} \gamma^2 \beta^2 \langle \psi^2 \rangle \right) \beta \psi - \frac{1}{6} N\gamma^4 \beta^3 \psi^3.
\]  
This expansion can be directly obtained from the sinh-Poisson equation (42). However, equation (123) shows the deviations to the sinh-Poisson equation induced by a non-vanishing value of \( \langle \psi \rangle \). Actually, equation (111) shows that the effect of \( \langle \psi \rangle \) is already present at the linear order (see Sect. 4.3).

4.2 Connection with other approaches

4.2.1 The Debye-Hückel approximation

At leading order, the expansion of the equations of the statistical theory as a function of \( \beta \gamma_a \psi \ll 1 \) leads to an equation of the form:
\[
\Delta \psi - \frac{\beta \Gamma_m^\text{m}}{V} \psi = -\frac{\Gamma}{V} - \frac{\beta \Gamma_m^\text{m}}{V} \langle \psi \rangle,
\]
where we have restored the domain area \( V \) and the microscopic enstrophy \( \Gamma_m^\text{m} \). This expansion is similar to the Debye-Hückel approximation in their theory of electrolytes [6] except that in the present situation the temperature may be positive or negative. At positive temperatures, and for a neutral system, like-sign vortices “repel” each other and opposite-sign vortices “attract” each other. A point vortex of a given sign is surrounded by a cloud of point vortices of the opposite sign so the interaction between two vortices is shielded on a length scale \( \lambda_D = (\beta \sum_n n_a \gamma_a^n)^{-1/2} \) equivalent to the Debye length. As a result of this shielding, the system is spatially homogeneous. At negative temperatures, like-sign vortices “attract” each other and opposite-sign vortices “repel” each other. They form large scale structures at the system length scale in order to maximize their entropy (see [34] and Sect. 4.3). This corresponds to the phenomenon of “condensation” discovered by Kraichnan [33] with his statistical theory of 2D turbulence in spectral space. This is also a direct consequence of the general arguments of statistical mechanics given by Onsager [5].

4.2.2 The minimum enstrophy principle

The strong mixing (or low energy) limit allows us to make a connection between the rigorous maximum entropy principle of statistical mechanics [14] and the phenomenological minimum enstrophy principle [32] introduced for a slightly viscous flow. Indeed, in this limit, the statistical theory leads to a linear \( \omega - \psi \) relationship (110), like when we minimize the enstrophy at fixed energy and circulation, writing the first variations as:
\[
\delta \Gamma_2 + 2 \beta \delta \psi + \alpha \delta \Gamma = 0,
\]

16 In their original paper, Debye and Hückel [6] explicitly write the multi-species Boltzmann-Poisson equation (19) and the sinh-Poisson equation (42) with \( \beta > 0 \) and \( A_a = n_a \). They linearize these equations at high temperature leading to equation (125) with the right hand side equal to zero. This is how the Debye screening length appears in their theory. A more precise justification of the Debye length is given in Appendix C.

17 For slightly viscous flows, the enstrophy cascades towards the small scales where it is dissipated while the energy remains blocked in the largest scales (inverse cascade) and is well-conserved. This selective decay has motivated the minimum enstrophy principle where it is argued that the flow tends to minimize enstrophy at fixed energy (and circulation).
where $\beta$ and $\alpha$ are Lagrange multipliers. Furthermore, equation (117) shows that a maximum entropy state corresponds to a minimum enstrophy state. In the case of point vortices there is no viscosity but the macroscopic enstrophy is not conserved and can decrease. The same connection between maximum entropy states and minimum enstrophy states is found for continuous vorticity fields in the limit of strong mixing [34]. In the inviscid MRS theory, the macroscopic enstrophy is conserved but the macroscopic enstrophy calculated with the coarse-grained vorticity $\mathcal{E}(r)$ is not conserved and can decrease.

4.2.3 The energy-enstrophy ensemble

In the context of the MRS theory, a linear $\omega - \psi$ relationship can be obtained by maximizing the MRS entropy at fixed circulation, energy, and macroscopic enstrophy (neglecting the conservation of the higher order moments of the vorticity). This approach has been developed in reference [56]. In that case, the vorticity distribution is Gaussian and the mean flow is characterized by a linear $\omega - \psi$ relationship. Furthermore, it can be proven that a maximum entropy state is equivalent to a minimum enstrophy state (see Appendix A of [56]). This approach may be seen as a formulation of the statistical mechanics of Kraichnan [33] based on the “energy-enstrophy” ensemble in terms of the more modern MRS theory.

4.3 Geometry induced phase transitions

The linear mean field equation (111) with the quadratic energy constraint (112) has been solved in reference [34] by expanding the stream function on the eigenmodes of the Laplacian. For small values of the control parameter $\Lambda = \Gamma/\sqrt{2E}$, one finds several solutions. In that case, we have to select the solution with the highest entropy. The general problem has been solved in reference [34]. If we consider the particular case $\Gamma = 0$, one obtains two types of solution: a “dipole” ($\langle \psi \rangle = 0$) which is an eigenfunction of the Laplacian and a “monopole” ($\langle \psi \rangle \neq 0$). Note that if we use the sinh-Poisson equation (42), we only get the dipole solution ($\langle \psi \rangle = 0$) and the monopole solution ($\langle \psi \rangle \neq 0$) is forgotten. This shows the limitation of this equation. Chavanis and Sommeria [34] have found that the selection of the maximum entropy state depends on the geometry of the domain. For example, in a rectangular domain, there is a critical aspect ratio $\tau_c = 1.12$. For $\tau < \tau_c$ the maximum entropy state is the “monopole” while for $\tau > \tau_c$ the maximum entropy state is the “dipole” (it can be shown furthermore that the state with the lower value of the entropy is a saddle point). This gives rise to geometry induced phase transition between “monopolos” and “dipoles”.

In a more recent paper, Taylor et al. [57] have remarked that the solutions with $\langle \psi \rangle = 0$ have zero angular momentum while the solutions with $\langle \psi \rangle \neq 0$ have nonzero angular momentum, even though their circulation is zero. Therefore, the “dipoles” are “zero-momentum” states and the “monopolos” are “nonzero-momentum” states or “spin-up” states. As a result, as discussed by Taylor et al. [57], the geometry induced phase transition between “monopolos” and “dipoles” found by Chavanis and Sommeria [34] can explain the “spin-up” phenomenon observed experimentally by Clercx et al. [58].

The methodology of Chavanis and Sommeria [34] has been generalized recently to several systems: axisymmetric flows [59,60], Fofonoff flows in oceanic basins [61,62], and barotropic flows on a sphere [63,64]. On the other hand, Venaille and Bouchet [65] have shown that the phase transitions between “monopolos” and “dipoles” found by Chavanis and Sommeria [34] could be interpreted in terms of a “bicritical point” and a “second order azeotropy”.

4.4 Relaxation equations

It is interesting to see how the relaxation equations (54) of the point vortices can be simplified in the limit of strong mixing (or low energy). At leading order, it suffices to make the approximation $\omega_0 \approx \Gamma_0$ in the drift term. This yields

$$\frac{\partial \psi_0}{\partial t} + \mathbf{u} \cdot \nabla \psi_0 = \nabla \cdot \{D(r, t) [\nabla \psi_0 + \beta(t) \Gamma_0 \nabla \psi]\}. \tag{127}$$

The equation for the total vorticity is therefore

$$\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla \psi = \nabla \cdot \{D(r, t) [\nabla \omega + \beta(t) \nabla \psi]\} \tag{128}$$

corresponding to the vorticity current

$$\mathbf{J} = -D(r, t) [\nabla \omega + \beta(t) \nabla \psi]. \tag{129}$$

We note that this equation is closed contrary to equation (55) when we are not in the strong mixing limit. The conservation of energy determines the evolution of the inverse temperature $\beta(t)$. Substituting equation (129) in equation (45), we get

$$\beta(t) = -\frac{\int D \nabla \psi \cdot \nabla \omega \, dr}{\int D (\nabla \psi)^2 \, dr}. \tag{130}$$

Finally, we can show that the macroscopic enstrophy (74) is the Lyapunov functional of the relaxation equation (128). Indeed, it satisfies an $H$-theorem provided that $D \geq 0$. Using equation (44), the rate of enstrophy dissipation is given by:

$$\dot{\Gamma}_2 = 2 \int \mathbf{J} \cdot \nabla \omega \, dr. \tag{131}$$

Using the expression (129) of the current, it can be rewritten as:

$$\dot{\Gamma}_2 = -2 \int \frac{J^2}{D} \, dr - 2\beta(t) \int \mathbf{J} \cdot \nabla \psi \, dr. \tag{132}$$

Using the conservation of energy (45), the second term is seen to vanish giving

$$\dot{\Gamma}_2 = -2 \int \frac{J^2}{D} \, dr \leq 0. \tag{133}$$
A stationary solution of equation (128) satisfies $\dot{F}_2 = 0$ implying $J = 0$. Using equation (129), we recover the minimum enstrophy state (110). Because of the $H$-theorem, the relaxation equation (128) converges, for $t \to +\infty$, towards a (local) minimum of enstrophy at fixed energy and circulation. If several minima exist for the same values of the constraints, the selection depends on a notion of basin of attraction.

Remark. The relaxation equation (128) with (130) can also be obtained by maximizing the rate of dissipation of enstrophy at fixed circulation and energy [66]. This kind of relaxation equations has been numerically solved in references [56,67] in order to illustrate the phase transitions discussed in Section 4.3.

5 Conclusion

We have complemented the literature on the statistical mechanics of point vortices in two-dimensional hydrodynamics. We have derived the Boltzmann-Poisson equation for a multi-species gas of point vortices by using the maximum entropy principle that we have justified from the theory of large deviations. We have also derived relaxation equations towards the maximum entropy state by using a maximum entropy production principle. These equations do not describe the true dynamics of the point vortex gas (this requires to develop a more complicated kinetic theory of point vortices [8]) but they can be used as a numerical algorithm to determine the maximum entropy state. This can be very useful in practice. Furthermore, these relaxation equations share interesting analogies with the Debye-Hückel equations for electrolytes [7], the Keller-Segel model of chemotaxis [31], and the Smoluchowski-Poisson system describing self-gravitating Brownian particles [30]. We have considered a limit of strong mixing (or low energy) where the maximum entropy state becomes equivalent to a minimum enstrophy state. This limit is similar to the Debye-Hückel [6] approximation for electrolytes, except that the temperature is negative instead of positive so that the effective interaction between like-signed point vortices is attractive instead of being repulsive. This results in a self-organization of the system at the largest scales (condensation), instead of a shielding of the interaction. We have stressed the limitations of the sinh-Poisson equation that is not the most general result of the statistical theory. It can miss important solutions such as the “monopolar” or “spin-up” state with zero circulation and nonzero angular momentum.

Point vortices constitute a nice example of systems with long-range interactions that shares numerous analogies with stellar systems, plasmas, and the HMF model [1]. However, real hydrodynamic flows are not made of point vortices contrary to galaxies and plasmas that are basically made of point mass stars or point charges. Therefore, for physical applications [68], the point vortex model remains a crude model and one should rather consider the evolution of continuous vorticity fields governed by the 2D Euler equation\(^\text{\textsuperscript{19}}\). The 2D Euler equation can reach a statistical equilibrium state as a result of a violent relaxation. This is described by the MRS theory. However, the equilibrium state depends on an infinity of constraints (the Casimirs) in addition to energy. To simplify the problem, Chavanis and Sommeria [34] have considered a strong mixing (or low energy) limit which makes a hierarchy between these constraints. This amounts to expanding the equations of the MRS theory in powers of $\beta \sigma \psi \ll 1$ (where $\sigma$ denotes the vorticity levels). As the lowest order, only the circulation and the microscopic enstrophy matter, in addition to the energy. When we go to higher orders in the expansion, more and more vorticity moments must be taken into account. For example, at the third order, the Kurtosis plays a crucial role. By contrast, the Kurtosis does not appear in the analogous expansion performed in this paper for point vortices\(^\text{\textsuperscript{20}}\). Therefore, the limit of strong mixing can be used to study the differences between the equilibrium state of point vortices [14] and the equilibrium state of continuous vorticity fields [10,11]. This is also a good approach to describe phase transitions in 2D flows with a reduced number of constraints.

Appendix A: Density of states and partition function

In this Appendix, we justify the variational problems (8) and (24) from the theory of large deviations (without going into technical details) and show their connection to the density of states in the microcanonical ensemble and to the partition function in the canonical ensemble. For the sake of simplicity, we consider a single species gas of point vortices and ignore the conservation of angular momentum and linear impulse (the generalization is straightforward).

A.1 The microcanonical ensemble

For an isolated Hamiltonian system of point vortices at statistical equilibrium, the $N$-body distribution is given by the microcanonical distribution

$$P_N(r_1, \ldots, r_N) = \frac{1}{g(E)} \delta[E - H(r_1, \ldots, r_N)] \quad (A.1)$$

expressing the equiprobability of the accessible microscopic configurations. This distribution gives the probability density of the microstate $(r_1, \ldots, r_N)$. This distribution is based on the postulate that all the accessible microscopic configurations (that have energy $E$) are equiprobable. Therefore, it assumes a uniform probability

\(^{19}\) As explained in the introduction, the 2D Euler equation of continuous vorticity fields also describes the collisionless evolution of the point vortex gas in the $N \to +\infty$ limit with $\gamma \sim 1/N$.

\(^{20}\) In this analogy, the microscopic moments $I_m = \sum_n N_a \gamma_n^m \gamma$ of the vorticity of point vortices are the counterparts of the moments $I_a^m = \int \omega^m \, dt$ of the fine-grained vorticity.
distribution for the phase space variables $\mathbf{r}_1, \mathbf{r}_2, \ldots$ over the energy shell at $H = E$. The normalization factor is the density of states with energy $E$. It is given by:

$$g(E) = \int \delta(E - H(\mathbf{r}_1, \ldots, \mathbf{r}_N)) \prod_i d\mathbf{r}_i. \quad (A.2)$$

The number of microstates with energy between $E$ and $E + dE$ is $g(E)dE$. The microcanonical entropy of the system is defined by $S(E) = \ln g(E)$ and the microcanonical temperature by $1/T(E) = \partial S/E$. The entropy is defined up to an additive constant.

We introduce the smooth (coarse-grained) density of point vortices $n(\mathbf{r})$. A macrostate is determined by the specification of the exact positions $\{\mathbf{r}_i\}$ of the $N$ point vortices. A macrostate is determined by the specification of the smooth density $n(\mathbf{r})$ of point vortices in each cell $[\mathbf{r}, \mathbf{r} + d\mathbf{r}]$ irrespectively of their precise position in the cell. Let us call $W[\omega]$ the unconditional number of microstates $\{\mathbf{r}_i\}$ corresponding to the macrostate $\omega(\mathbf{r}) = \gamma n(\mathbf{r})$. The entropy of the macrostate $\omega$ is defined by the Boltzmann formula

$$S[\omega] = \ln W[\omega]. \quad (A.3)$$

The unconditional probability density of the distribution $\omega$ is therefore $P_0[\omega] \propto W[\omega] \propto e^{S[\omega]}$. The number of complexes $W[\omega]$ can be obtained by a standard combinatorial analysis. For $N \gg 1$, we find that the Boltzmann entropy is given by:

$$S[\omega] = -\int \frac{\omega}{\gamma} \ln \left( \frac{\omega}{N\gamma} \right) d\mathbf{r}. \quad (A.4)$$

Instead of integrating over the microstates $\{\mathbf{r}_i\}$ in equation (A.2), we can integrate over the macrostates $\{\omega(\mathbf{r})\}$. For $N \gg 1$, using a mean field approximation, we find that the density of states (A.2) can be written as:

$$g(E) \approx \int W[\omega] \delta(E[\omega] - E) \delta(\Gamma[\omega] - \Gamma) \, d\omega, \quad (A.5)$$

where $E[\omega] = 1/2 \int \omega \psi d\mathbf{r}$ is the mean field energy, $\Gamma[\omega] = \int \omega d\mathbf{r}$ is the circulation, and $W[\omega]$ gives the number of microstates corresponding to the macrostate $\omega$. Since $W[\omega] = e^{S[\omega]}$, the foregoing expression may be rewritten as:

$$g(E) \approx \int e^{S[\omega]} \delta(E[\omega] - E) \delta(\Gamma[\omega] - \Gamma) \, d\omega. \quad (A.6)$$

The microcanonical density probability of the distribution $\omega$ is therefore

$$P[\omega] = \frac{1}{g(E)} e^{S[\omega]} \delta(E[\omega] - E) \delta(\Gamma[\omega] - \Gamma). \quad (A.7)$$

This distribution can be obtained directly by stating that $P[\omega] \propto W[\omega] \delta(E[\omega] - E) \delta(\Gamma[\omega] - \Gamma)$ since all the accessible microstates are equiprobable.

### A.2 Canonical ensemble

For a system of Brownian point vortices in contact with a thermal bath at statistical equilibrium [40], the $N$-body distribution is given by the canonical distribution

$$P_N(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \frac{1}{Z(\beta)} e^{-\beta H(\mathbf{r}_1, \ldots, \mathbf{r}_N)}. \quad (A.8)$$

This distribution gives the probability density of the microstate $\{\mathbf{r}_i\}$. The normalization factor is the partition function. It is given by:

$$Z(\beta) = \int e^{-\beta H(\mathbf{r}_1, \ldots, \mathbf{r}_N)} d\mathbf{r}_1 \ldots d\mathbf{r}_N. \quad (A.9)$$

The free energy is defined by $J(\beta) = \ln Z(\beta)$. The average energy $E = \langle H \rangle$ is given by $E = -\partial J/\partial \beta$. The fluctuations of energy are given by $\langle (H^2) - \langle H \rangle^2 \rangle = T^2 C$ where $C = dE/dT$ is the specific heat. This relation implies that the specific heat is always positive in the canonical ensemble.

Instead of integrating over the microstates $\{\mathbf{r}_i\}$ in equation (A.9), we can integrate over the macrostates $\{\omega(\mathbf{r})\}$. Introducing the unconditional number of microstates $W[\omega]$ corresponding to the macrostate $\omega$, and using a mean field approximation valid for $N \gg 1$, we find that the partition function can be written as:

$$Z(\beta) \approx \int e^{-\beta E}[\omega] W[\omega] \delta(\Gamma[\omega] - \Gamma) \, d\omega \quad \approx \int e^{S[\omega] - \beta E[\omega]} \delta(\Gamma[\omega] - \Gamma) \, d\omega \quad \approx \int e^{J[\omega]} \delta(\Gamma[\omega] - \Gamma) \, d\omega, \quad (A.10)$$

where $J[\omega] = S[\omega] - \beta S[\omega]$ is the mean field free energy. The canonical density probability of the distribution $\omega$ is therefore:

$$P[\omega] = \frac{1}{Z(\beta)} e^{J[\omega]} \delta(\Gamma[\omega] - \Gamma). \quad (A.11)$$

This distribution can be obtained directly by stating that $P[\omega] \propto W[\omega] e^{-\beta E[\omega]} \delta(\Gamma[\omega] - \Gamma)$ since the microstates with energy $E$ have a probability $\propto e^{-\beta E}$.

### A.3 Variational principles

In the thermodynamic limit $N \to +\infty$ defined in Section 2.3, we have the extensive scalings $S \sim E/T \sim N$ and $J \sim N$, so we can define $S[\omega] = N S[\omega]$ and $J[\omega] = N J[\omega]$ with $s \sim 1$ and $j \sim 1$. Accordingly, the preceding results may be rewritten as:

$$g(E) \approx \int e^{N S[\omega]} \delta(E[\omega] - E) \delta(\Gamma[\omega] - \Gamma) \, d\omega \quad (A.12)$$

and

$$Z(\beta) \approx \int e^{N J[\omega]} \delta(\Gamma[\omega] - \Gamma) \, d\omega. \quad (A.13)$$
where the δ-functions take into account the constraints in the microcanonical and canonical ensembles.

The microcanonical probability of the macrostate ω is

\[ P[\omega] = \frac{1}{g(E)} e^{N_s[\omega]} \delta(E[\omega] - E)\delta(\Gamma[\omega] - \Gamma) \]  

(A.14)

and the canonical probability of the macrostate ω is

\[ P[\omega] = \frac{1}{Z(\beta)} e^{N_j[\omega]} \delta(\Gamma[\omega] - \Gamma). \]  

(A.15)

For \( N \to +\infty \), we can make the saddle point approximation. In the microcanonical ensemble, we obtain

\[ g(E) = e^{S(E)} \approx e^{N_s[\omega]}, \]  

(A.16)

i.e.

\[ \lim_{N \to +\infty} \frac{1}{N} S(E) = s[\omega_*], \]  

(A.17)

where \( \omega_* \) is the global maximum of constrained entropy. We are led therefore to the maximization problem

\[ S(E) = \max_\omega \{ S[\omega] | E[\omega] = E, \Gamma[\omega] = \Gamma \}. \]  

(A.18)

At equilibrium, in the \( N \to +\infty \) limit, we have \( S(E) = S[\omega_*] = N s[\omega_*] \). In the canonical ensemble, we obtain

\[ Z(\beta) = e^{J(\beta)} \approx e^{N_j[\omega_*]}, \]  

(A.19)

i.e.

\[ \lim_{N \to +\infty} \frac{1}{N} J(\beta) = j[\omega_*], \]  

(A.20)

where \( \omega_* \) is the global maximum of constrained free energy. We are led therefore to the maximization problem

\[ J(\beta) = \max_\omega \{ J[\omega] | \Gamma[\omega] = \Gamma \}. \]  

(A.21)

At equilibrium, in the \( N \to +\infty \) limit, we have \( J(\beta) = J[\omega_*] = N j[\omega_*]. \)

These results can be considered as results of large deviations \([69,70]\) and they can be made mathematically rigorous \([23–28]\).

**Appendix B: Uniformly rotating or translating steady states of the 2D Euler equation**

In the \( N \to +\infty \) limit with \( \gamma \sim 1/N \), the average vorticity \( \omega(r,t) = \langle \sum \gamma_i \delta(r - r_i(t)) \rangle \) of the point vortex gas is governed by the 2D Euler equation

\[ \frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0, \quad \omega = -\Delta \psi. \]  

(B.1)

If the flow is steady in a frame rotating with angular velocity \( \Omega \), then \( \omega(r, \theta, t) = \omega(r, \theta - \Omega t) \) where \( (r, \theta) \) is a polar system of coordinates. This implies that \( \partial_t \omega = -\Omega \partial_\theta \omega \).

Substituting this relation in the 2D Euler equation we obtain:

\[ \frac{\partial}{\partial \theta} \left( \psi + \frac{\Omega^2}{2} \right) \frac{\partial \omega}{\partial r} - \frac{\partial}{\partial r} \left( \psi + \frac{\Omega^2}{2} \right) \frac{\partial \omega}{\partial \theta} = 0. \]  

(B.2)

This is the general equation determining a steady state in a rotating frame. This equation is satisfied by any relation of the form \( \omega = f(\psi + \Omega r^2/2) \).

If the flow is steady in a frame translating with linear velocity \( U = U e_x \), then \( \omega(x, y, t) = \omega(x - Ut, y) \) where \( (x, y) \) is a cartesian system of coordinates. This implies that \( \partial_t \omega = -U \partial_x \omega \). Substituting this relation in the 2D Euler equation we obtain

\[ \frac{\partial}{\partial y} (\psi - U y) \frac{\partial \omega}{\partial x} - \frac{\partial}{\partial x} (\psi - U y) \frac{\partial \omega}{\partial y} = 0. \]  

(B.3)

This is the general equation determining a steady state in a translating frame. It is satisfied by any relation of the form \( \omega = f(\psi - U y) \).

**Appendix C: The Debye length**

We consider a neutral plasma at equilibrium with uniform charge density \( (\rho = \sum a n_a e_a = 0) \). We introduce a test charge \(+e\) in the system at \( r = 0 \). This charge produces a “naked” electric potential \( \Phi_0(r) \). This potential modifies the distribution of the field charges around the test charge. The resulting change of density \( \tilde{\rho}(r) \) in turn produces an extra-potential which adds to the original one \( \Phi_0(r) \). The effective, or “dressed”, potential created by the test charge is therefore the solution of the equation

\[ \Delta \Phi_{\text{eff}} = -S_d[\tilde{\rho}(r) + e \delta(r)]. \]  

(C.1)

The slightly perturbed distribution of the field charges is given by the Boltzmann statistics

\[ \tilde{\rho}(r) = \sum_a n_a e_a e^{-\beta e_a \Phi_{\text{eff}}(r)}, \]  

(C.2)

where \( n_a \) is the uniform distribution of species \( a \). Substituting equation (C.2) in equation (C.1) we obtain the self-consistency equation

\[ \Delta \Phi_{\text{eff}} = -S_d \sum_a n_a e_a e^{-\beta e_a \Phi_{\text{eff}} - S_d e \delta(r)} \]  

(C.3)

determining the effective (dressed) potential. In the weak coupling approximation \( \beta e_a \Phi_{\text{eff}} \ll 1 \), we can expand the exponential term in equation (C.3) leading to the equation

\[ \Delta \Phi_{\text{eff}} - S_d \beta \left( \sum_a n_a e_a^2 \right) \Phi_{\text{eff}} = -S_d e \delta(r). \]  

(C.4)

This equation shows that the potential created by the test charge is shielded on a typical length \( \lambda_D = (k_B T/S_d \sum_a n_a e_a^2)^{1/2} \) called the Debye length. This length can also be obtained in a less rigorous manner by expanding equation (105) for \( \beta e_a \Phi \ll 1 \).
Appendix D: The $\beta\gamma a \psi \ll 1$ expansion for point vortices

In this Appendix, we expand the equations of the statistical theory of point vortices in terms of $\beta\gamma a \psi \ll 1$ to third order. This generalizes the study of Section 4.1 that was limited to second order.

It is convenient to work in terms of the variable $\phi = \psi - \langle \psi \rangle$, satisfying $\langle \phi \rangle = 0$, instead of $\psi$. Equation (13) is thus rewritten as:

$$\omega_a = A_a \gamma a e^{-\beta \gamma a \phi}, \quad A_a = \frac{N_a}{\int e^{-\beta \gamma a \phi} \, dr}. \quad (D.1)$$

To third order, we obtain

$$e^{-\beta \gamma a \phi} = 1 - \beta \gamma a \phi + \frac{1}{2} \beta^2 \gamma a^2 \phi^2 - \frac{1}{6} \beta^3 \gamma a^3 \phi^3 \quad (D.2)$$

and

$$A_a = N_a \left( 1 - \frac{1}{2} \beta^2 \gamma a^2 \langle \phi^2 \rangle + \frac{1}{6} \beta^3 \gamma a^3 \langle \phi^3 \rangle \right), \quad (D.3)$$

where we have assumed that the domain area is unity ($V = 1$). Therefore

$$\omega_a = N_a \gamma a \left[ 1 - \beta \gamma a \phi + \frac{1}{2} \beta^2 \gamma a^2 \langle \phi^2 \rangle - \frac{1}{6} \beta^3 \gamma a^3 \langle \phi^3 \rangle \right]. \quad (D.4)$$

Summing over the species, we obtain the total vorticity

$$\omega = \Gamma + C_1 \beta \phi + C_2 \beta^2 \langle \phi^2 \rangle + C_3 \beta^3 \langle \phi^3 \rangle \quad (D.5)$$

with

$$C_1 = -\frac{\Gamma_m}{2} + \frac{1}{2} \frac{\Gamma_m}{4} \beta^2 \langle \phi^2 \rangle, \quad C_2 = \frac{1}{2} \frac{\Gamma_m}{3}, \quad C_3 = -\frac{1}{6} \frac{\Gamma_m}{4}. \quad (D.6)$$

where $\Gamma_m^m = \sum_n N_n \gamma_n^m$ denotes the microscopic moment of order $n$ of the vorticity. Using equation (D.4), the entropy (7) is given to third order by:

$$S = S_0 - \frac{1}{2} \frac{\Gamma_m^m}{2} \beta^2 \langle \phi^2 \rangle + \frac{1}{3} \frac{\Gamma_m^m}{3} \beta^3 \langle \phi^3 \rangle, \quad (D.8)$$

where $S_0 = \sum_n N_n \ln N_n$.

We can use this expansion to show the relation between maximum entropy states and minimum entropy states in the strong mixing limit. At second order, we get

$$\omega = \Gamma - \frac{\Gamma_m^m}{2} \beta \phi + \frac{1}{2} \frac{\Gamma_m^m}{3} \beta^2 \langle \phi^2 \rangle - \langle \phi^2 \rangle \quad (D.9)$$

and

$$S = S_0 - \frac{1}{2} \frac{\Gamma_m^m}{2} \beta^2 \langle \phi^2 \rangle. \quad (D.10)$$

Introducing the macroscopic entrophy $I_2 = \int \omega^2 \, dr$, and using equation (D.9), we obtain at second order

$$I_2 = I^2 + \frac{1}{2} \frac{\Gamma_m^m}{2} \beta^2 \langle \phi^2 \rangle. \quad (D.11)$$

Comparing equation (D.10) with equation (D.11) we get

$$S = S_0 - \frac{1}{2} \frac{I_2 - I^2}{I_2}. \quad (D.12)$$

This relation shows that, at second order in the strong mixing limit, a maximum entropy state is equivalent to a minimum entropy state.

In the case of a symmetric distribution of point vortices with positive and negative circulations, we have $\Gamma_m^{m+1} = 0$, so equations (D.5)-(D.7) reduce to:

$$\omega = \left( -\frac{\Gamma_m^m}{2} + \frac{1}{2} \frac{\Gamma_m^m}{2} \beta \langle \phi^2 \rangle \right) \beta \phi - \frac{1}{6} \frac{\Gamma_m^m}{2} \beta^2 \langle \phi^3 \rangle$$

$$-\langle \phi^3 \rangle \phi + \frac{1}{6} \frac{\Gamma_m^m}{2} \beta^2 \langle \phi^3 \rangle \phi - \langle \phi^3 \rangle \phi, \quad (D.13)$$

where $Ku = \Gamma_m^m/(\Gamma_m^m)^2$ is the Kurtosis of the microscopic vorticity distribution. On the other hand, the entropy (D.8) reduces to:

$$S = S_0 - \frac{1}{2} \beta^2 \frac{\Gamma_m^m}{2} \langle \phi^2 \rangle. \quad (D.14)$$

We now consider the low energy limit $E \ll 1$ (still assuming $\Gamma_m^{m+1} = 0$) and use standard perturbation theory [57,71]. We write $\phi = E^{1/2}(\phi_0 + E\phi_1 + \ldots)$ and $\beta = \beta_0 + E\beta_0 + \ldots$ and substitute these expansions in equation (D.13). At order $E^{1/2}$ we get

$$-\Delta \phi_0 + \Gamma_m \beta_0 \phi_0 = 0, \quad (D.15)$$

and at order $E^{3/2}$ we obtain

$$-\phi_1 + \Gamma_m \beta_0 \phi_1 = -\Gamma_m \beta_1 \phi_0 + \frac{1}{2} \frac{\Gamma_m^m}{2} \beta_0 \langle \phi_0^2 \rangle, \quad (D.16)$$

where $Ku = \Gamma_m^m/(\Gamma_m^m)^2$ is the Kurtosis of the microscopic vorticity distribution. On the other hand, the entropy (D.8) reduces to:

$$S = S_0 - \frac{1}{2} \beta^2 \frac{\Gamma_m^m}{2} \langle \phi^2 \rangle. \quad (D.14)$$

The linear problem (D.15) has been studied in reference [34] in terms of $\psi$. The inverse temperature $\beta_0$ is “quantized” and can take only discrete values $\beta_0 < 0$ labeled by $n = 1, 2, \ldots$. When $\langle \psi_0 \rangle = 0$ (type I solutions), $\Gamma_m \beta_0$ is an eigenvalue of the Laplacian $\Delta$ with zero mean and with the boundary condition $\psi_0 = 0$ on $(\partial D)$. When $\langle \psi_0 \rangle \neq 0$ (type II solutions), $\Gamma_m \beta_0$ is a root of the function $F(\beta)$ defined by equation (3.8) of [34] (it is constructed with the eigenvalues of the Laplacian with non-zero mean and with the boundary condition $\psi_0 = 0$ on $(\partial D)$). The respective values of $\beta_0$ for type I and type II solutions depend on the shape of the domain. It is the competition between these two types of solutions that yields the geometry-induced phase transitions found in reference [34]. Coming back to the variable $\phi$, we call $\Psi_0$ the ortho-normalized basis of eigenfunctions associated with the eigenvalues $\Gamma_m \beta_n$ regrouping the two
types of solutions described above. They are defined by
\(-\Delta \Psi^0 + \Gamma_m^0 \beta_m \Psi^0 = 0\) with \(\Psi^0 = \text{cst.}\) on the boundary (not
necessarily zero), \(\Psi^0 = 0\), and \(\Psi^0 |_{\partial}\) where \(\langle f | g \rangle = \int f g \, dr\). The solution of equation (D.15) satisfying
the energy constraint \(E = -\frac{1}{2} \int \omega \phi \, dr = \frac{1}{2} \int \omega \phi \, dr = -\frac{1}{2} \int \phi \Delta \phi \, dr\) (since \(\Gamma = 0\) can be obtained as:
\[
\phi^{(0)}_n = \left( \frac{-2}{\beta_n^2 \Gamma_m^2} \right)^{1/2} \Psi_n, \quad \beta^{(0)}_n = \beta_n. \tag{D.17}
\]
This defines the starting point (at \(E = 0\)) of the branch \(\beta^{(0)}(E)\) of order \(n\). In order to get the first order correction to the temperature, we multiply equation (D.16) by \(\phi^{(n)}\) and integrate over the domain. Using the identity
\[
\langle \phi \rangle - \Delta \phi + \Gamma_m \beta \phi = (\langle \Delta \phi \rangle + \Gamma_m \beta \phi) = 0 \tag{D.18}
\]
which results from a simple integration by parts (we have omitted the superscript \((n)\) on the variables to simplify the expression), we obtain
\[
\beta^{(n)}_1 = Ku \beta^{(2)}_n \left( \frac{1}{3} \Psi_n^4 \right) - 1. \tag{D.19}
\]
This result, which can be viewed as a solvability condition, generalizes the perturbative result obtained by Taylor et al. [57] (see also [71]) for a two species system of point vortices \((N/2, \gamma)\) and \((N/2, -\gamma)\) for which \(Ku = \Gamma_m^0 / \Gamma_m^2 = N \gamma^4 / (N \gamma^2)^2 = 1 / N\). On the other hand, the first order correction to the stream function can be expanded on the basis of eigenfunctions as:
\[
\phi^{(1)}_1 = \sum_k c_k^{(1)} \Psi_k, \quad \psi_k^{(1)}(\phi^{(1)}_n). \tag{D.20}
\]
Multiplying equation (D.16) by \(\Psi_k\) with \(k \neq n\), we find
\[
c_k^{(n)} = \frac{1}{3} \beta_n^2 \left( \frac{-2}{\beta_n \Gamma_m^2} \right)^{1/2} \Psi_k \left( \frac{1}{\beta_n} \right). \tag{D.21}
\]
The coefficient \(c_n^{(n)}\) can be taken equal to zero.

The specific heat is defined by \(C = dE / dT = -\beta^2 dE / d\beta\). For \(E = 0\) we obtain \(C = -\beta^2 / \beta_1^{(1)}\) where \(\beta_1^{(1)}\) is given by equation (D.19). The specific heat is positive when \(\Psi_n^4 < 3\), negative when \(\Psi_n^4 > 3\), and infinite when \(\Psi_n^4 = 3\). Depending on the respective values of \(\beta_1^{(1)}\), the different branches \(n = 1, 2, \ldots\) may cross each other at some energy \(E_{n > 0}\) leading to energy-induced phase transitions (their crossing may also be due to nonlinear effects not captured by the perturbative expansion) [57,71].

At order \(E^{3/2}\), using equation (D.14) and \(E = -\frac{1}{2} \Gamma_m^2 \beta (\phi^2)\) we obtain
\[
S^{(n)} = S_0 + \beta_n E. \tag{D.22}
\]
At \(E = 0\), all the solutions \(n = 1, 2, \ldots\) have the same entropy \(S_0\). For \(0 < E \ll 1\), the entropies \(S^{(n)}\) of the different solutions are just proportional to their inverse temperature \(\beta_n\). Therefore, the maximum entropy state is the mode \(n = 1\), i.e. the one with the highest \(\beta_n\) [34]. If two modes have the same inverse temperature \((\beta_n = \beta_m\) with \(n \neq m\)), there is a degeneracy that has to be raised by expanding the entropy at order \(E^2\).

**Appendix E: The \(\beta \sigma \psi \ll 1\) expansion for continuous vorticity fields**

In this Appendix, we expand the equations of the MRS statistical theory for continuous vorticity fields in terms of \(\beta \sigma \psi \ll 1\) to third order. This complements the study of Chavanis and Sommeria [34] where this expansion was developed in detail to second order and extended (without giving detail) to third order. This Appendix also clarifies the analogies and the differences between the statistical mechanics of point vortices and continuous vorticity fields. We refer to [34] for a detailed presentation of the MRS theory and for the notations. Below, we just recall the basic formulae that are needed for our study.

As in Appendix D, it is convenient to work in terms of the variable \(\phi = \psi - \langle \psi \rangle\), satisfying \(\langle \phi \rangle = 0\), instead of \(\psi\). According to the MRS theory, the probability density of finding the vorticity level \(\sigma\) in \(r\) is given by:
\[
\rho(r, \sigma) = \frac{1}{Z(r)} g(\sigma) e^{-\beta \sigma \phi} \tag{E.1}
\]
with the normalization condition \(\int \rho(r, \sigma) \, d\sigma = 1\) leading to:
\[
Z(r) = \int g(\sigma) e^{-\beta \sigma \phi} \, d\sigma. \tag{E.2}
\]
The function \(g(\sigma)\) can be viewed as a Lagrange multiplier determined by the total area \(\gamma(\sigma) = \int \rho(r, \sigma) \, d\sigma\) of each vorticity level \(\sigma\), which is a conserved quantity. The coarse-grained vorticity field is given by \(\bar{\Omega}(r) = \int \rho(r, \sigma) \, d\sigma\).

The entropy is:
\[
S = -\int \rho(r, \sigma) \ln \rho(r, \sigma) \, d\sigma \, d\sigma. \tag{E.3}
\]
To third order, we obtain
\[
e^{-\beta \phi} = 1 - \beta \phi + \frac{1}{2} \beta^2 \phi^2 - \frac{1}{6} \beta^3 \phi^3. \tag{E.4}
\]
and
\[
Z = 1 - \beta A_1 \phi + \frac{1}{2} \beta^2 A_2 \phi^2 - \frac{1}{6} \beta^3 A_3 \phi^3, \tag{E.5}
\]
where we have defined \(A_n = \int g(\sigma) \sigma^n \, d\sigma\) and taken \(A_0 = 1\) without loss of generality. Therefore,
\[
\rho(r, \sigma) = g(\sigma) \left[ 1 + \beta \phi (A_1 - \sigma) + \beta^2 \phi^2 \times \left( \frac{1}{2} \sigma^2 - A_1 \sigma - \frac{1}{2} A_2 + A_1^2 \right) + \beta^3 \phi^3 \left( \frac{1}{6} \sigma^3 + \frac{1}{2} A_1 \sigma^2 + \frac{1}{2} A_2 \sigma ight) + \frac{1}{6} A_3 - A_1^2 \sigma - A_1 A_2 + A_1^3 \right]. \tag{E.6}
\]
Integrating equation (E.6) over the whole domain, we obtain $\gamma(\sigma)$ as a function of $g(\sigma)$. Reversing this relation, we obtain

$$g(\sigma) = \gamma(\sigma) \left[ 1 - \beta^2 \langle \phi^2 \rangle \left( \frac{1}{2} \sigma^2 - A_1 \sigma - \frac{1}{2} A_2 + A_1^2 \right) - \beta^2 \langle \phi^3 \rangle \left( -\frac{1}{6} \sigma^3 + \frac{1}{2} A_1 \sigma^2 + \frac{1}{2} A_2 \sigma \right) + \frac{1}{6} A_3 - A_1^2 \sigma - A_1 A_2 + A_1^3 \right],$$

(E.7)

where we have assumed that the domain area is unity ($V = 1$). We note that $g(\sigma)$ still appears implicitly in the coefficients $A_n$. Multiplying equation (E.7) by $\sigma^n$ and integrating over $\sigma$, we get

$$A_n = \Gamma_n^{f,g} - \beta^2 \langle \phi^2 \rangle \left( \frac{1}{2} \Gamma_n^{f,g}_{n+2} - A_1 \Gamma_n^{f,g}_{n+1} - \frac{1}{2} A_2 \Gamma_n^{f,g} + A_1^2 \Gamma_n^{f,g} \right) - \beta^3 \langle \phi^3 \rangle \left( -\frac{1}{6} \Gamma_n^{f,g}_{n+3} + \frac{1}{2} A_1 \Gamma_n^{f,g}_{n+2} + \frac{1}{2} A_2 \Gamma_n^{f,g} + \frac{1}{6} A_3 \Gamma_n^{f,g} - A_1^2 \Gamma_n^{f,g} - A_1 A_2 \Gamma_n^{f,g} + A_1^3 \Gamma_n^{f,g} \right),$$

(E.8)

where $\Gamma_n^{f,g} = \int \overline{\omega r} \, dx = \int \gamma(\sigma) \sigma^n \, d\sigma$ is the moment of order $n$ of the fine-grained vorticity field (conserved quantity). Identifying terms of equal order in equation (E.8) we obtain at second order

$$A_n = \Gamma_n^{f,g} - \beta^2 \langle \phi^2 \rangle \left( \frac{1}{2} \Gamma_n^{f,g}_{n+2} - \Gamma_n^{f,g}_{n+1} \right) - \frac{1}{2} \Gamma_n^{f,g} \Gamma_n^{f,g} + \Gamma^2 \Gamma_n^{f,g} \right).$$

(E.9)

Combining equations (E.6), (E.7), and (E.9) we get

$$\rho(r, \sigma) = \gamma(\sigma) \left[ 1 + B_1 \beta \phi + B_2 \beta^2 \langle \phi^2 - \langle \phi^2 \rangle \right] + B_3 \beta^3 \langle \phi^3 - \langle \phi^3 \rangle \right],$$

(E.10)

with

$$B_1 = \Gamma - \sigma - \beta^2 \langle \phi^2 \rangle \left( \frac{1}{2} \Gamma_3^{f,g} - \frac{3}{2} \Gamma_2^{f,g} + 2 \Gamma^3 \right) + \frac{3}{2} \Gamma \sigma^2 - 2 \Gamma^2 \sigma - \frac{1}{2} \Gamma \Gamma_2^{f,g} - \frac{1}{2} \sigma^3 + \frac{1}{2} \Gamma_2^{f,g} \sigma \right),$$

(E.11)

$$B_2 = \frac{1}{2} \sigma^2 - \Gamma \sigma - \frac{1}{2} \Gamma_2^{f,g} + \Gamma^2,$$

(E.12)

$$B_3 = -\frac{1}{6} \sigma^3 + \frac{1}{2} \Gamma \sigma^2 + \frac{1}{2} \Gamma_2^{f,g} \sigma + \frac{1}{6} \Gamma_3^{f,g} - \Gamma^2 \sigma - \Gamma \Gamma_2^{f,g} + \Gamma^3.$$

(E.13)

The coarse-grained vorticity is:

$$\overline{\omega} = \Gamma + C_1 \beta \phi + C_2 \beta^2 (\phi^2 - \langle \phi^2 \rangle) + C_3 \beta^3 (\phi^3 - \langle \phi^3 \rangle)$$

(E.14)

with

$$C_1 = \Gamma^2 - \Gamma_2^{f,g} - \beta^2 \langle \phi^2 \rangle \left[ 2 \Gamma \Gamma_2^{f,g} - 4 \Gamma^2 \Gamma_2^{f,g} + 2 \Gamma_3^{f,g} \right],$$

(E.15)

$$C_2 = \frac{1}{2} \left( \Gamma_2^{f,g} - 3 \Gamma \Gamma_2^{f,g} + 2 \Gamma^3 \right),$$

(E.16)

$$C_3 = -\frac{1}{6} \left( \Gamma_2^{f,g} - 3 \Gamma \Gamma_2^{f,g} + 2 \Gamma^3 \right) \Gamma + 12 \Gamma^2 \Gamma_2^{f,g} - 6 \Gamma^4 \right).$$

(E.17)

These coefficients can be related to the cumulants of the generating function $\ln Z(\phi)$. Using equation (E.10), the entropy (E.3) is given to third order by:

$$S = S_0 - \frac{1}{2} \left( \Gamma_2^{f,g} - \Gamma^2 \right) \beta^2 \langle \phi^2 \rangle + \frac{1}{6} \left( 4 \Gamma^3 - 6 \Gamma \Gamma_2^{f,g} + 2 \Gamma^3 \right) \beta^3 \langle \phi^3 \rangle,$$

(E.18)

where $S_0 = -\int \gamma(\sigma) \ln(\gamma(\sigma)) \, d\sigma$.

We can use this expansion to show the relation between maximum entropy states and minimum enstrophy states in the strong mixing limit. At second order, we get:

$$\overline{\omega} = \Gamma \left( \Gamma^2 - \Gamma_2^{f,g} \right) \beta \phi + \frac{1}{2} \left( \Gamma_3^{f,g} - 3 \Gamma \Gamma_2^{f,g} + 2 \Gamma^3 \right) \beta^2 (\phi^2 - \langle \phi^2 \rangle)$$

(E.19)

and

$$S = S_0 - \frac{1}{2} \left( \Gamma_2^{f,g} - \Gamma^2 \right) \beta^2 (\phi^2).$$

(E.20)

Introducing the enstrophy of the coarse-grained field $\Gamma_2 = \int \overline{\omega}^2 \, dx$, and using equation (E.19), we obtain at second order

$$\Gamma_2 = \Gamma^2 + \left( \Gamma^2 - \Gamma_2^{f,g} \right)^2 \beta^2 (\phi^2).$$

(E.21)

Comparing equation (E.20) with equation (E.21) we obtain:

$$S = S_0 - \frac{1}{2} \Gamma_2^{f,g} - \Gamma^2.$$
where \( \text{Ku} = \Gamma_2 f \rho / (\Gamma_2 f \rho)^2 \) is the Kurtosis of the microscopic vorticity distribution. On the other hand, the entropy (E.18) reduces to:

\[
S = S_0 - \frac{1}{2} \Gamma_2 f \rho \beta^2 \langle \phi^2 \rangle. \tag{E.24}
\]

We now consider the low energy limit \( E \ll 1 \) (still assuming \( \Gamma_{2+n} f = 0 \)) and use standard perturbation theory. We write \( \phi = E^{1/2}(\phi_0 + E\phi_1 + \ldots) \) and \( \beta = \beta_0 + E\beta_1 + \ldots \) and substitute these expansions in equation (E.23). At order \( E^{1/2} \) we get

\[
- \Delta \phi_0 + \Gamma_2 f \rho \beta_0 \phi_0 = 0, \tag{E.25}
\]

and at order \( E^{3/2} \) we obtain

\[
- \Delta \phi_1 + \Gamma_2 f \rho \beta_2 \phi_1 = -E^{1/2} \beta_1 \phi_0
+ \frac{1}{2} I_2 f \rho \left( \Gamma_2 f \rho \right)^2 (\text{Ku} - 1) \beta_0^2 \langle \phi_0^2 \rangle \phi_0
- \frac{1}{6} \left( \Gamma_2 f \rho \right)^2 (\text{Ku} - 3) \beta_0^2 \phi_0^2 - \langle \phi_0^2 \rangle. \tag{E.26}
\]

Proceeding as in Appendix D, the solution of equation (E.25) satisfying the energy constraint is:

\[
\phi_0^{(n)} = \left( \frac{-2}{\beta_0 \Gamma_2 f \rho} \right)^{1/2} \langle \Psi_n \rangle, \quad \beta_0^{(n)} = \beta_n. \tag{E.27}
\]

The solvability condition gives the first correction to the inverse temperature

\[
\beta_1^{(n)} = \beta_2 \left[ \frac{1}{3} (\text{Ku} - 3) \langle \Psi_4 \rangle - (\text{Ku} - 1) \right]. \tag{E.28}
\]

On the other hand, the first correction to the stream function is given by:

\[
\phi_1^{(n)} = \sum_k c_k^{(n)} \psi_k, \quad c_k^{(n)} = \langle \psi_k | \phi_1^{(n)} \rangle, \tag{E.29}
\]

with

\[
c_k^{(n)} = \frac{1}{3} \beta_0 \left( \frac{-2}{\beta_0 \Gamma_2 f \rho} \right)^{1/2} (\text{Ku} - 3) \langle \Psi_4 \rangle / \beta_n - \beta_k \tag{E.30}
\]

for \( k \neq n \) and \( c_n^{(n)} = 0 \).

The specific heat is defined by \( C = dE/dT = -\beta^2 dE/d\beta \). For \( E \rightarrow 0 \) we obtain \( C = -\beta_n^2 / \beta_1^{(n)} \) where \( \beta_1^{(n)} \) is given by equation (E.28). There is a critical value of the Kurtosis given by:

\[
(\text{Ku})^{(n)}_c = \frac{\langle \Psi_4 \rangle - 1}{4 \langle \Psi_4 \rangle - 1}. \tag{E.31}
\]

If \( \langle \Psi_4 \rangle < 1 \), the specific heat is negative for \( \text{Ku} < (\text{Ku})_c \) and positive for \( \text{Ku} > (\text{Ku})_c \). If \( 1 < \langle \Psi_4 \rangle < 3 \), the specific heat is always positive. If \( \langle \Psi_4 \rangle > 3 \), the specific heat is positive for \( \text{Ku} < (\text{Ku})_c \) and negative for \( \text{Ku} > (\text{Ku})_c \).

We note that the results obtained for point vortices (see Appendix D) are recovered from the present results obtained for continuous vorticity fields if we keep only the moment of highest order at each level of the expansion. This implies, in particular, that the terms Ku = 3 and Ku = 1 for continuous vorticity fields are replaced by Ku for point vortices\(^{21}\). On the other hand, the discussion of the entropy and of the possible degeneracy of the solutions for continuous vorticity fields is the same as in the case of point vortices (see the last paragraph of Appendix D where it suffices to replace \( \Gamma_{2+n}^m \) by \( \Gamma_{2+n}^{f} \)).

The relaxation equations for the probability density \( \rho(r, \sigma, t) \) associated with the MRS theory are [13,29]:

\[
\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = \nabla \cdot \{ -D(r, t) [\nabla \rho + \beta(t)(\sigma - \bar{\sigma}) \rho \nabla \psi] \}, \tag{E.32}
\]

\[
\beta(t) = -\frac{\int D \nabla \bar{\sigma} \cdot \nabla \psi \, dr}{\int D \omega_2 \nabla \psi \, (\nabla \psi)^2 \, dr}. \tag{E.33}
\]

where \( \omega_2 = (\bar{\sigma} - \bar{\sigma})^2 = \int \rho (\sigma - \bar{\sigma})^2 \, d\sigma \) is the local centered variance of the vorticity. They can be simplified in the limit of strong mixing (or low energy). At leading order, it suffices to make the approximation \( \rho(r, \sigma, t) \approx \gamma(\sigma) \) and \( \bar{\sigma}(r, t) \approx \Gamma \) in the drift term. This yields

\[
\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = \nabla \cdot \{ D(r, t) [\nabla \rho + \beta(t)(\sigma - \Gamma) \gamma(\sigma) \nabla \psi] \}, \tag{E.34}
\]

\[
\beta(t) = -\frac{\int D \nabla \bar{\sigma} \cdot \nabla \psi \, dr}{(\Gamma (\nabla \psi)^2 - \Gamma^2) \int D \nabla \psi \, (\nabla \psi)^2 \, dr}. \tag{E.35}
\]

The equation for the total vorticity is therefore

\[
\frac{\partial \bar{\sigma}}{\partial t} + u \cdot \nabla \bar{\sigma} = \nabla \cdot \{ D(r, t) [\nabla \phi + (\Gamma (\nabla \psi)^2 - \Gamma^2) \beta(t) \nabla \psi] \}. \tag{E.36}
\]

This equation coincides with equation (128) obtained for point vortices. We can also check that the equilibrium states of equations (E.34) and (E.36) return equations (E.10) and (E.14) at leading order.

### Appendix F: Difference between the Joyce-Montgomery theory and the MRS theory

It may be useful to re-emphasize the physical difference between the statistical theory of point vortices [14] and the statistical theory of continuous vorticity fields [10,11].

Let us first consider a continuous vorticity field \( \omega(r, t) \) whose evolution is described by the 2D Euler equation. If\(^{21}\) the equations of the statistical theory of point vortices can be recovered from the MRS theory in a dilute limit. It corresponds, for example, to an initial condition involving compact vortices, thus having high Kurtosis [34].
the initial state is unsteady or dynamically unstable, the fine-grained vorticity \( \omega(r,t) \) will undergo a complicated mixing process during which the system generates filaments at smaller and smaller scales. In this sense, there is no relaxation (the Euler equation is reversible). However, if we locally average over the filaments, the coarse-grained vorticity \( \mathcal{W}(r,t) \) will reach a steady state \( \mathcal{W}(r) \). This is because the evolution continues at a scale smaller than the resolution scale. The coarse-grained field \( \mathcal{W}(r) \) is expected to be a stable steady state of the 2D Euler equation. If mixing is efficient, so that the ergodicity assumption is fulfilled, the coarse-grained field \( \mathcal{W}(r) \) can be predicted by the MRS statistical theory (this theory also gives the fluctuations around this averaged field). It corresponds to the statistical equilibrium state of the 2D Euler equation. Since this equilibrium state is established very rapidly, after a few dynamical times, this process is called violent relaxation [12]. In reality, there is always some viscosity. If viscous effects are weak, the system first undergoes a violent relaxation towards a quasi stationary state, then decays on a longer timescale because of viscous effects.

We now consider a system of \( N \) point vortices with circulation \( \gamma \sim 1/N \) (for simplicity we assume that the point vortices have the same circulation but the following arguments can be extended to more general cases). The **discrete** vorticity field \( \omega_d(r,t) = \sum_i \gamma \delta(r - r_i(t)) \) is a sum of \( \delta \)-functions. We note that \( \omega_d(r,t) \) satisfies the 2D Euler equation (called the Klimontovich equation in plasma physics) for any \( N \). If \( N \gg 1 \), then, in a first regime (collisionless regime), the correlations between point vortices can be neglected and the **smooth** vorticity field \( \omega(r,t) = \omega_d(r,t) = \sum_i \gamma \delta(r - r_i) \) also satisfies the 2D Euler equation (called the Vlasov equation in plasma physics) [8,15]. In the limit \( N \to +\infty \), the smooth vorticity field \( \omega(r,t) \) satisfies the 2D Euler equation for all times [9]. Since the point vortex gas is (first) described by the 2D Euler equation, it may undergo a process of violent relaxation towards a quasi stationary state described by the MRS theory (assuming ergodicity). This **quasi stationary state** corresponds to the statistical equilibrium state of the 2D Euler equation for a continuous vorticity field. The relaxation time towards that state is of the order of a few dynamical times, independent of \( N \). Then, in a second regime (collisional regime), and since \( N \) is necessarily finite, correlations (also called distant collisions between point vortices) come into play and the smooth vorticity field \( \omega(r,t) = \omega_d(r,t) = \sum_i \gamma \delta(r - r_i) \) is not a solution of the 2D Euler equation anymore. It is the solution of a kinetic equation which includes a sort of collision term [8,15]. This kinetic equation is expected to relax towards the Boltzmann distribution corresponding to the Joyce-Montgomery theory. This distribution corresponds to the statistical equilibrium state of the discrete point vortex gas. The relaxation time towards that state depends on the number of point vortices and increases rapidly with \( N \). The scaling of the relaxation time with \( N \) is not firmly established [8]. It is not even clear whether the statistical equilibrium state is reached in all cases. Indeed, the evolution may be non-ergodic. When \( N \to +\infty \), the collisional regime and the establishment of the Boltzmann distribution are pushed to infinitely long times so the domain of validity of the 2D Euler equation (collisionless regime) becomes huge. In that case, only the QSS described by the MRS theory can be observed.

These considerations of kinetic theory [8,15] make clear that the MRS theory and the Joyce-Montgomery theory are fundamentally different (despite their mathematical similarities) since they describe two very different regimes with very different timescales: violent collisionless relaxation vs slow collisional relaxation. It is not always clear in numerical simulations of point vortices whether the observed “equilibrium state” corresponds to the QSS distribution predicted by the MRS theory or to the Boltzmann distribution predicted by Joyce and Montgomery. This important remark will be given more consideration in future works. Note, finally, that for “small” \( N \), the point vortex gas may reach the Boltzmann distribution without passing through the collisionless regime of violent relaxation.

**Remark.** This two-stages process for point vortices (violent collisionless relaxation followed by slow collisional relaxation) is similar to what is known for stellar systems [1,3,18]. There is, however, a fundamental difference between stellar systems and 2D vortices. Galaxies are basically made of a finite number of stars and the Vlasov equation is an approximation of the discrete dynamics of stars for large \( N \). Inversely, 2D flows are basically described by the 2D Euler equation for a continuous vorticity field and the point vortex model with large \( N \) is an approximation of the continuous dynamics. Therefore, the physical problematic is, in some sense, reversed.

### References

1. P.H. Chavanis, Statistical Mechanics of Two-dimensional vortices and stellar systems, in *Dynamics and Thermodynamics of Systems with Long Range Interactions*, edited by T. Dauxois, S. Ruffo, E. Arimondo, and M. Wulkens, Lecture Notes in Physics (Springer, 2002), Vol. 602

2. R. Balescu, *Statistical Mechanics of Charged Particles* (Wiley, 1963)

3. J. Binney, S. Tremaine, *Galactic Dynamics* (Princeton Series in Astrophysics, 1987)

4. A. Campa, T. Dauxois, S. Ruffo, Phys. Rep. **480**, 57 (2009)

5. L. Onsager, Nuovo Cimento, Suppl. **6**, 279 (1949)

6. P. Debye, H. Hückel, Phys. Z. **24**, 185 (1923)

7. P. Debye, H. Hückel, Phys. Z. **24**, 305 (1923)

8. P.H. Chavanis, Physica A **391**, 3657 (2012)

9. C. Marchioro, M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids* (Springer, New York, 1994)

10. J. Miller, Phys. Rev. Lett. **65**, 2137 (1990)

11. R. Robert, J. Sommeria, J. Fluid Mech. **229**, 291 (1991)

12. D. Lynden-Bell, Mon. Not. R. Astron. Soc. **136**, 101 (1967)

13. P.H. Chavanis, J. Sommeria, R. Robert, ApJ **471**, 385 (1996)

14. G. Joyce, D. Montgomery, J. Plasma Phys. **10**, 107 (1973)

15. P.H. Chavanis, Phys. Rev. E **64**, 026309 (2001)
16. R. Kawahara, H. Nakanishi, J. Phys. Soc. Jpn 75, 054001 (2006)
17. R. Kawahara, H. Nakanishi, J. Phys. Soc. Jpn 76, 074001 (2007)
18. T. Padmanabhan, Phys. Rep. 188, 285 (1990)
19. D. Montgomery, G. Joyce, Phys. Fluids 17, 1139 (1974)
20. S. Kida, J. Phys. Soc. Jpn 39, 1395 (1975)
21. Y.B. Pointin, T.S. Lundgren, Phys. Fluids 19, 1459 (1976)
22. T.S. Lundgren, Y.B. Pointin, J. Stat. Phys. 17, 323 (1977)
23. E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, Commun. Math. Phys. 143, 501 (1992)
24. M. Kiessling, Commun. Pure Appl. Math. 47, 27 (1993)
25. G.L. Eyink, H. Spohn, J. Stat. Phys. 70, 833 (1993)
26. E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, Commun. Math. Phys. 174, 229 (1996)
27. K. Sawada, T. Suzuki, Theoret. Appl. Mech. Jpn 56, 285 (2008)
28. R. Robert, J. Sommeria, Phys. Rev. Lett. 69, 2776 (1992)
29. J. Sopik, C. Sire, P.H. Chavanis, Phys. Rev. E 72, 026105 (2005)
30. E. Keller, L.A. Segel, J. Theor. Biol. 26, 399 (1970)
31. C.E. Leith, Phys. Fluids 27, 1388 (1984)
32. R.H. Kraichnan, J. Fluid Mech. 67, 155 (1975)
33. P.H. Chavanis, J. Sommeria, J. Fluid Mech. 314, 267 (1996)
34. P.K. Newton, The N-Vortex Problem: Analytical Techniques, in *Applied Mathematical Sciences* (Springer-Verlag, Berlin, 2001), Vol. 145
35. G. Kirchhoff, in *Lectures in Mathematical Physics, Mechanics* (Teubner, Leipzig, 1877)
36. J. Fröhlich, D. Ruelle, Commun. Math. Phys. 87, 1 (1982)
37. D. Ruelle, J. Stat. Phys. 61, 865 (1990)
38. S.F. Edwards, J.B. Taylor, Proc. R. Soc. Lond. A 336, 257 (1974)
39. P.H. Chavanis, Physica A 387, 6917 (2008)
40. P.H. Chavanis, Eur. Phys. J. Plus 127, 159 (2012)
41. J.G. Esler, T.L. Ashbee, N.R. McDonald, Phys. Rev. E 88, 012109 (2013)
42. P.H. Chavanis, J. Sommeria, J. Fluid Mech. 356, 259 (1998)
43. P.H. Chavanis, Eur. Phys. J. B 70, 73 (2009)
44. P.H. Chavanis, J. Stat. Phys. 101, 999 (2000)
45. R. Ellis, K. Haven, B. Turkington, J. Stat. Phys. 20, 3113 (2006)
46. M. Kiessling, Lett. Math. Phys. 34, 49 (1995)
47. L. Onsager, Phys. Rev. 37, 405 (1931)
48. P.H. Chavanis, Eur. Phys. J. B 62, 179 (2008)
49. P.H. Chavanis, Phys. Rev. E 58, R1199 (1998)
50. P.H. Chavanis, Int. J. Mod. Phys. B 26, 1241002 (2012)
51. S. Chandrasekhar, ApJ 97, 255 (1943)
52. W. Nernst, Z. Phys. Chem. 2, 613 (1888)
53. W. Nernst, Z. Phys. Chem. 4, 129 (1889)
54. M. Planck, Ann. Phys. 39, 161 (1890)
55. A. Naso, P.H. Chavanis, B. Dubrulle, Eur. Phys. J. B 77, 187 (2010)
56. J.B. Taylor, M. Borchardt, P. Helander, Phys. Rev. Lett. 102, 124505 (2009)
57. H.J.H. Clercx, S.R. Maassen, G.J.F. van Heijst, Phys. Rev. Lett. 80, 5129 (1998)
58. A. Naso, S. Thalabard, G. Collette, P.H. Chavanis, B. Dubrulle, J. Stat. Mech. 06019 (2010)
59. S. Thalabard, B. Dubrulle, F. Bouchet, arXiv:1306.1081 (2013)
60. A. Venaille, F. Bouchet, J. Stat. Phys. 143, 346 (2011)
61. A. Naso, P.H. Chavanis, B. Dubrulle, Eur. Phys. J. B 80, 493 (2011)
62. C. Herbert, B. Dubrulle, P.H. Chavanis, D. Paillard, Phys. Rev. E 85, 056304 (2012)
63. C. Herbert, B. Dubrulle, P.H. Chavanis, D. Paillard, J. Stat. Mech. 05023 (2012)
64. A. Venaille, F. Bouchet, Phys. Rev. Lett. 102, 104501 (2009)
65. R.S. Ellis, *Large Deviations and Statistical Mechanics* (Springer, New York, 1985)
66. D. Ruelle, Phys. Rep. 515, 227 (2012)
67. H. Touchette, Phys. Rep. 478, 1 (2009)
68. T. Ashbee, Ph.D. Thesis, University College London, 2014