Finite Square Well and Quantum State Discrimination

Songtai You*
School of Science, Rensselaer Polytechnic Institute, Troy, NY 12180, United States
*Corresponding author’s e-mail: yous3@rpi.edu

Abstract. In this paper, the finite square well and its application are investigated, namely the quantum state discrimination. The finite square well is treated in all standard textbooks on introductory quantum mechanics. It is used as a simple ‘model of departure’ in many areas of physics. In atomic and molecular physics, it may be used as a model of an electron moving in the mean field of a linear molecule. It also arises as the partial wave radial equation for a spherically symmetric, finite square-well potential. The Schrodinger equation for finite square well is solved. The domain is divided into three regions by the existing potential $V_0$, so for convenience, those three regions are named Region I, Region II, and Region III, respectively. Specifically, the potential of Region I and III are $V_0$, and that of Region II is 0. Also, a constant is needed to make this wave function normalized. Then since it is a wave function, it is necessary to make sure the function is continuous and differentiable everywhere within the domain. Besides, the wave function needs to be either odd or even just like infinite square well. After that, the wave functions for the three intervals can be obtained, and the exact quantum state is distinguished from a group of different quantum states. In the end, two wave functions are obtained; one for even form, and one for odd form. Then all the energy level of the corresponding function with different wavelength needs to be found and listed. Local discrimination of orthogonal quantum states has attracted much attention during the last twenty years. The results are applied to the quantum-information task of state discrimination, by using the obtained six states in finite square well. It is assumed that all the quantum states are locally distinguishable, and the six states are distinguished using the hypothesis of quantum measurements.

1. Introduction
The finite square well is considered as standard and the basis of all quantum physics textbooks. It has been found useful in various applications in recent years. Also, it is the basic model of a theoretical model called “Gas in a Box”, which can be used to describe some famous gases such as ideal gas, Bose gas, Fermi gas, photon gas (Black Body Radiation), and so on. Furthermore, the finite square well model is great in the industry of semiconductor. Theoretically, the study of the analytical solution of the finite quantum square-well problem has been studied recently [1]. Next, the finite depth square well model with its application and limitations have been studied using both theory and experiments [2]. The finite square well is also an excellent model for the computation of the resonance spectrum of flat microwave cavities loaded with dielectric inserts, which were used recently to verify the existence of a ray-splitting correction to the Weyl formula [3]. The finite depth square well has also been adopted for effective mass approximation [4-6].

The above results are applied to the quantum state discrimination. Quantum state discrimination is the basic building block of various applications in quantum information processing tasks [7]. It essentially describes the distinguishability of quantum systems in different states, and the general
process of extracting classical information from quantum systems. Also, quantum states' local
distinguishability can be used to design quantum protocols, such as quantum cryptography [8-12].

2. Result of Finite Square Well
After introducing the general definition of the finite square well, the result of the experiment and the
whole process will be explained in detail in this section. First, the setting of a finite square well is

\[ V(x) = \begin{cases} V_0 & |x| > a/2 \\ 0 & |x| < a/2 \end{cases} \]  

(1)

The graph of finite square well is shown below

![Graph of finite square well](image)

Figure 1. The domain is divided into three regions by the existing potential \( V_0 \)

2.1 The domain of three regions
The domain is divided into three regions by the existing potential \( V_0 \), so for convenience, we name
those three regions Region I, Region II, and Region III, respectively.

After setting all the factors we need, we need to investigate the wave function that is bounded in this
well. In order to find that, we need to set up the Schrodinger Equations for all three regions, since this
situation is unlike the infinite square well, which we could confirm that the probability of finding the
photon in Region I and Region III is definitely 0.

For Region II, the potential is 0, namely \( V=0 \), so the Schrodinger Equation will look like this

\[-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi \]  

(2a)

Then since Region I and III have a constant potential \( V_0 \), the equation should be the two shown below.
And because Region I and Region III are symmetric with each other, their Schrodinger Equation should
be the same when we first set them up.

\[-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0\psi = E\psi \]  

(2b)

Then we can solve the equations to finish the first step. In order to solve them, we need to make all
the constant to the right-hand side. In order to represent the function in a simpler way, assuming \( V_0 > E > 0 \), we define

\[ k = \sqrt{\frac{2mE}{\hbar}} \]  

(3a)

and

\[ \kappa = \sqrt{\frac{2m(V_0-E)}{\hbar}} > 0 \]  

(3b)

Then we substitute \( k \) and \( \kappa \) into (2a) and (2b) & (2c), we will obtain \n
\[ \frac{d^2\psi}{dx^2} = -k^2\psi \]  

(2a)

\[ \frac{d^2\psi}{dx^2} = \kappa^2\psi \]  

(2b) & (2c)

Finally, we just simply solve these two second-order differential equations. As a result, we will obtain
the general solution of Region II from equation (2a)

\[ \psi(x) = A e^{ikx} + B e^{-ikx} \quad |x| < a/2 \]  

(4a)

The general solution of Region I and Region III from equation (2b) and (2c) is shown as

\[ \psi(x) = C e^{\kappa x} + D e^{-\kappa x} \quad |x| > a/2 \]  

(5)
Even though the general solution of Region I and Region III are both solutions (5), we can obtain two different solutions for Region I and Region III if we think about the physical principle. The domain of Region I is \((-\infty, -a/2)\), and because the potential field is \(V_0\) is constantly consuming the energy of the photon, so it is impossible to find the photon when \(x \to -\infty\), which is \(\lim_{x \to -\infty} \psi(x) = 0\), so it cannot have \(D e^{-\kappa x}\) in the wave function for Region I. A similar method works for Region III. When \(x \to \infty\), it is impossible to find the photon and have any energy, which is \(\lim_{x \to \infty} \psi(x) = 0\), so \(C e^{\kappa x}\) does not exist in the wave function of Region III. And the functions should look like

\[
\psi(x) = Ce^{\kappa x} \quad x < a/2
\]

and

\[
\psi(x) = De^{-\kappa x} \quad x > a/2
\]

2.2 The wave functions

Therefore, now we have the general solution of this wave function in the whole domain.

\[
\psi(x) = \begin{cases} 
Ce^{\kappa x} & x < a/2 \\
A e^{i\kappa x} + Be^{-i\kappa x} & |x| < a/2 \\
De^{-\kappa x} & x > a/2
\end{cases}
\]

(6)

After obtaining the general solution, the next step is to solve for the constants A, B, C, D. In order to do that, there are two boundary conditions required. One is at \(x=a/2\), and the other is at \(x=-a/2\). And since it is a wave function, we need to make sure the function is continuous and differentiable everywhere within the domain. Therefore, there are four equations that can be obtained.

These two equations make the functions continuous as follows:

\[
Ce^{-ka/2} = Ae^{-ika/2} + Be^{ika/2}
\]

(7a)

\[
De^{-ka/2} = Ae^{ika/2} + Be^{-ika/2}
\]

(7b)

The rest two equations make the function differentiable everywhere

\[
kCe^{-ka/2} = i k(Ae^{-ika/2} - Be^{ika/2})
\]

(7c)

\[
-kDe^{-ka/2} = i k(Ae^{ika/2} - Be^{-ika/2})
\]

(7d)

Figure 2. Infinite Square Well (Different Energy Level)

The picture above is the solution of the Infinite Square Well, and as we observed, the final wave function is always either odds or even function within the Region that does not have a potential field, which is Region II in the Finite Square Well experiment. And this means that

\[
\psi(-a/2) = \psi(a/2)
\]

(8a)
or
\[ \psi(-a/2) = -\psi(a/2) \] (8b)

for the Finite Square Well. Furthermore, because of the differentiability and the continuity of the wave function, we obtain

or\[ Ce^{-\kappa a/2} = De^{-\kappa a/2} \] (9a)

or\[ Ce^{-\kappa a/2} = -De^{-\kappa a/2} \] (9b)

and
\[ Ae^{-ika/2} + Be^{ika/2} = Ae^{ika/2} + Be^{-ika/2} \] (9c)

or\[ Ae^{-ika/2} + Be^{ika/2} = -(Ae^{ika/2} + Be^{-ika/2}) \] (9d)

The solutions are
\[ A = B \text{ and } C = D \]

or\[ A = -B \text{ and } C = -D \]

According to these solutions, we can obtain 2 solutions by using the Euler’s Law.

\[ \cos kx = \frac{e^{ikx} + e^{-ikx}}{2} \]
\[ \sin kx = \frac{e^{ikx} - e^{-ikx}}{2i} \]

First solution, when \( A = B \text{ and } C = D \),

\[ \psi(x) = \begin{cases} 
Ce^{kx} & x < a/2 \\
2A \cos kx & |x| < a/2 \\
Ce^{-kx} & x > a/2 
\end{cases} \] (10)

Then for \( A = -B \text{ and } C = -D \), we obtain

\[ \psi(x) = \begin{cases} 
Ce^{kx} & x < a/2 \\
2iA \sin kx & |x| < a/2 \\
-Ce^{-kx} & x > a/2 
\end{cases} \] (11)

2.3 The energy

If we substitute equation (7a) into equation (7c), we will get the equation below (For \( A = B \text{ and } C = D \))

\[ \frac{ik}{\kappa} = \frac{e^{-ika/2} + e^{ika/2}}{e^{-ika/2} - e^{ika/2}} = \frac{(e^{-ika/2} + e^{ika/2})/2}{(e^{-ika/2} - e^{ika/2})/2i} = \frac{\cos ka/2}{\sin ka/2} = \tan \left( \frac{ka}{2} \right) \] (12)

Then we can define
\[ \xi = \frac{ka}{2}, \] (14)
\[ \xi_0 = \frac{a}{\sqrt{\frac{m}{2}}} \] (15)

After substitute (14) and (15) into (13), we will obtain

\[ \tan(\xi) = \frac{\sqrt{\xi^2 - \xi_0^2}}{\xi} \] (16)

We also can use the same method to solve for \( A = -B \text{ and } C = -D \) situation

\[ -\cot(\xi) = \frac{\sqrt{\xi^2 - \xi_0^2}}{\xi} \] (17)

The following are graphs of equations (16) and (17).
Since this is similar to Infinity Square Well, the energy level function should be the same as the Infinity Square Well.

\[ E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \]  

and the what shows below is the Energy Level Graph for the wave function.

3. Result on quantum state discrimination

Our next step is to talk about how quantum state discrimination works. In quantum communication, we need to distinguish a specific quantum state from a combination of the quantum system. There are some prerequisite points we need to discuss beforehand.

There are some properties of the quantum system we got from the other side of the communication. First, here is some property of a quantum state in superposition mode.

\[ |\psi\rangle = \sum_{i=1}^{m} a_i |i\rangle \]  

and

\[ \sum_{i=1}^{m} |a_i|^2 = 1 \]  

The vector \(|i\rangle\) represent the quantum states within this superposition system. Also, the most significant thing is that those quantum states are orthogonal to one another, which means they have to be the basis of this quantum vector space. \(|a_i|^2\) represent the probability of obtaining the corresponding pure quantum state after some kind of measurement since the measurement is also a type of operation. Furthermore, this indirectly indicates that equation (20) is correct because the probability of obtaining all the possible quantum states is 1. Although our quantum system is not in a superposition state, they still have orthogonality between each state. For example, from function (10) and (11), we obtain 6 distinct wave functions, and we can set these states orthogonal to each other. Then we can distinguish
which states they are. We need to confirm that there are no other possibilities. Therefore, we need to distinguish the states $|1\rangle$, $|2\rangle$, ..., $|m\rangle$ in (19), which all are orthogonal to one another.

After talking about the object of the measurement, it is also essential to mention the method of measuring. We assume that there are $k$ possible quantum states as a result after the measurement (since we need to figure out which state does $k$ corresponds to in this superposed system), so there has to be the same number of operators corresponding to those quantum states. Therefore, we have a group of operators:

$$\{M_m, m = 1, \ldots, k\}$$

with all the operators is an $d \times n$ matrix. If these operators are Hermitian operators, we have

$$M_m^\dagger = M_m$$

(22)

We will use this property of the Hermitian operator later.

Also, we need to give this group of measurement operators a constrain, so we assume that the operator satisfies the completeness equation (23) as shown below.

$$\sum_{m=1}^{k} M_m^\dagger M_m = I_n$$

(23)

The probability of obtaining the corresponding quantum state $|\psi_m\rangle$ after exerting the corresponding operator, $p(m)$, in matrix form, is

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$$

(24)

After the measurement, the quantum state will become

$$|\psi\rangle = \sum_{m=1}^{k} \langle \psi | M_m^\dagger M_m | \psi \rangle = \sum_{m=1}^{k} |\psi_m\rangle$$

(25)

Then we connect equation (23) and equation (24), we will obtain

$$\sum_{m=1}^{k} \langle \psi | M_m^\dagger M_m | \psi \rangle = \sum_{m=1}^{k} |\langle \psi | I_n | \psi \rangle| = \sum_{m=1}^{k} |\langle \psi | \psi_m\rangle| = 1$$

(26)

Equation (25) indicate that all the possible result of operator $M_m$ is within the all system $|\psi\rangle$. In other words, we don’t know which $m$ it is, but we can confirm that $m \in \{1, k\}$.

Since $m \in \{1, k\}$, there has to be a $|\psi_m\rangle = |\psi_k\rangle$ when $m=k$. In order to get that, we can set up a $p(m)$ to obtain the probability of getting an $|\psi_m\rangle$ out of $|\psi_k\rangle$. Therefore, we can make $M_m = |\psi_m\rangle\langle\psi_m|$. Since $|\psi_m\rangle$ is belong to a Hilbert space, $M_m$ should also belong to Hilbert space, which means that $M_m$ is a Hermitian operator. As a result, we will obtain the following.

$$p_k(m) = \langle \psi_k | M_m^\dagger M_m | \psi_k \rangle$$

(27)

For ease of explanation, we name three steps of equation (27). First, according to equation (23), we can obtain (27.1), which means the dagger sign doesn’t matter. Then since $\langle \psi_m | \psi_m\rangle = 1$, so we can obtain (27.3) from (27.2). At last, we have the inner product of $\psi_m$ and $\psi_k$. Then we name it $\delta_{mk}$. Recall that $\psi_m$ and $\psi_k$ are quantum states of the quantum system we got before the measurement, and all the states are orthogonal to one another. So

$$\delta_{mk} = \begin{cases} 1 & m = k \\ 0 & m \neq k \end{cases}$$

(28)

Since only when $m=k$, the probability of obtaining $|\psi_k\rangle$ will be 1. Also, we know all the possible numbers of $m$, so we are able to know which state $|\psi_k\rangle$ is.

Now we show the connection between this section and the results in the last section. We refer to the six states in (10) and (11) as six orthogonal states $|1\rangle$, ..., $|6\rangle$. Using the aforementioned techniques, we can distinguish the six states perfectly.
4. Conclusion
This paper has investigated the finite square well and its application, namely the quantum state discrimination. This article has also solved the Schrödinger equation for finite square well, obtained the wave functions for the three intervals, and distinguished the exact quantum state from a group of different quantum states. In the end, all the energy levels of the corresponding function with different wavelengths are obtained. The six states using the hypothesis of quantum measurements have also been distinguished. Furthermore, the results are applied to the quantum-information task of state discrimination, by using the obtained six states in finite square well. We have managed to distinguish the six states from finite square well using quantum measurements as an application. The next step is to find out further connections between finite square well and other quantum-information tasks.

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