Categorification of Sign-Skew-Symmetric Cluster Algebras and Some Conjectures on $g$-Vectors

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Abstract
Using the unfolding method given in Huang and Li (Adv. Math. 340, 221–283 2018), we prove the conjectures on sign-coherence and a recurrence formula respectively of $g$-vectors for acyclic sign-skew-symmetric cluster algebras. As a (following) consequence, the conjecture is affirmed in the same case which states that the $g$-vectors of any cluster form a basis of $\mathbb{Z}^n$. Also, the additive categorification of an acyclic sign-skew-symmetric cluster algebra $\mathcal{A}(\Sigma)$ is given, which is realized as $(\mathcal{C}, \mathcal{C})$ for a Frobenius 2-Calabi-Yau category $\mathcal{C}$ constructed from an unfolding $(\mathcal{Q}, \mathcal{C})$ of the acyclic exchange matrix $B$ of $\mathcal{A}(\Sigma)$.

Keywords Acyclic sign-skew-symmetric cluster algebras · $g$-vectors · Sign-coherence

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1 Introduction
Cluster category was introduced by [2] for an acyclic quiver. In general, we view a Hom-finite 2-Calabi-Yau triangulated category $\mathcal{C}$ which has a cluster structure as a cluster category, see [1]. In fact, the mutation of a cluster-tilting object $T$ in $\mathcal{C}$ categorifies the mutation of a quiver $Q$, where the quiver $Q$ is the Gabriel quiver of the algebra $\text{End}_\mathcal{C}(T)$. Cluster
character gives an explicit correspondence between certain cluster objects of \( C \) and all the clusters of \( \mathcal{A}(\Sigma(Q)) \), where \( \Sigma(Q) \) means the seed associated with \( Q \). For details, see [10, 14, 15]. Thus, cluster category and cluster character are useful tools to study a cluster algebra.

Let \( \mathcal{A}(\Sigma) \) be a cluster algebra with principal coefficients at \( \Sigma = (X, Y, B) \), where \( B \) is an \( n \times n \) sign-skew-symmetric integer matrix, \( Y = (y_1, \ldots, y_n) \). The celebrated Laurent phenomenon says that \( \mathcal{A} \) is a subalgebra of \( \mathbb{Z}[y_{n+1}, \ldots, y_{2n}][X^{\pm 1}] \). Setting \( \deg(x_i) = e_i \), \( \deg(y_j) = -b_j \), then \( \mathbb{Z}[y_{n+1}, \ldots, y_{2n}][X^{\pm 1}] \) becomes to a graded algebra, where \( \{e_i \mid i = 1, \ldots, n\} \) is the standard basis of \( \mathbb{Z}^n \) and \( b_i \) is the \( i \)-th column of \( B \). Under such \( \mathbb{Z}^n \)-grading, the cluster algebra \( \mathcal{A} \) is a graded subalgebra, in which the degree of a homogenous element is called its \( g \)-vector; furthermore, each cluster variable \( x \) in \( \mathcal{A}(\Sigma) \) is homogenous, denoted its \( g \)-vector as \( g(x) \). For details, see [9].

It was conjectured that

**Conjecture 1.1** ([9], Conjecture 6.13) For any cluster \( X' \) of \( \mathcal{A}(\Sigma) \) and all \( x \in X' \), the vectors \( g(x) \) are sign-coherent, which means that the \( i \)-th coordinates of all these vectors are either all non-negative or all non-positive.

Such conjecture has been proved in the skew-symmetrizable case in ([11], Theorem 5.11).

**Conjecture 1.2** ([9], Conjecture 7.10(2)) For any cluster \( X' \) of \( \mathcal{A}(\Sigma) \), the vectors \( g(x), x \in X' \) form a \( \mathbb{Z} \)-basis of the lattice \( \mathbb{Z}^n \).

In terms of cluster pattern, fixed a regular tree \( \mathbb{T}_n \), for any vertices \( t \) and \( t_0 \) of \( \mathbb{T}_n \), let \( g^{B_1,B_0}_{1,t_0}, \ldots, g^{B_1,B_0}_{n,t_0} \) denote the \( g \)-vectors of the cluster variables in the seed \( \Sigma \), with respect to the principal coefficients seed \( \Sigma_{t_0} \). Under such different vertices to be principal coefficients seeds, it is conjectured that the \( g \)-vectors with respect to a fixed vertex \( t \) of \( \mathbb{T}_n \) have the following relation.

**Conjecture 1.3** ([9], Conjecture 7.12) Let \( t_1 \xrightarrow{k} t_2 \) \( \mathbb{T}_n \) and let \( B^2 = \mu_k(B^1) \). For \( a \in [1, n] \) and \( t \in \mathbb{T}_n \), assume \( g^{B_1,t_1}_{a,t} = (g_1^{t_1}, \ldots, g_n^{t_1}) \) and \( g^{B_2,t_2}_{a,t} = (g_1^{t_2}, \ldots, g_n^{t_2}) \), then

\[
g^{t_2}_{i,t} = \begin{cases} -g_1^{t_1} & \text{if } i = k; \\ g_1^{t_1} + b_{ik}^{t_1} + g_k^{t_1} - \mu_k min(g_1^{t_1}, 0) & \text{if } i \neq k. \end{cases}
\]

**Remark 1.4**
1. As is said in Remark 7.14 of [9], it is easy to see that Conjectures 1.1 and 1.3 imply Conjecture 1.2.
2. In the skew-symmetrizable case, the sign-coherence of the \( c \)-vectors can deduce Conjectures 1.1 and 1.3, see [13], where the given method strongly depends on the skew-symmetrizability. So far, the similar conclusion have not been given in the sign-skew-symmetric case. Further, the sign-coherence of the \( c \)-vectors has been proved in [12] for the acyclic sign-skew-symmetric case, because this is equivalent to \( F \)-polynomial has constant term 1, see [9]. Therefore, for the acyclic sign-skew-symmetric case, it is interesting to study directly Conjectures 1.1 and 1.3.

Unfolding of skew-symmetrizable matrices is introduced by Zelevinsky, whose aim is to study skew-symmetrizable cluster algebras using the version in skew-symmetric case. The second and third authors of this paper improved in [12] such method to arbitrary sign-skew-symmetric matrices. According to this previous work, there is an unfolding \( (\mathcal{Q}, F, \Gamma) \) of
any acyclic \(m \times n\) matrix \(\tilde{B}\), and also a 2-Calabi-Yau Frobenius category \(C_\tilde{Q}\) with \(\Gamma\) action constructed.

Our motivation and the main results of this paper are two-fold.

1. Give the \(\Gamma\)-equivariant cluster character for \(C_\tilde{Q}\), which can be regarded as the additive categorification of the cluster algebra \(A(\Sigma(Q))\). See Theorem 3.5 and Theorem 3.6.
2. Solve Conjectures 1.1 and 1.3 in the acyclic case. See Theorem 4.7 and Theorem 5.5. As a consequence, in a similar fashion, Conjecture 1.2 follows to be affirmed.

2 An Overview of Unfolding Method

In this section, we give a brief introduction of the concept of unfolding of totally sign-skew-symmetric cluster algebras and some necessary results in [12].

For any sign-skew-symmetric matrix \(B \in \text{Mat}_{n \times n}(\mathbb{Z})\), one defines a quiver as follows: the vertices are 1, \(\cdot\cdot\cdot\), \(n\) and there is an arrow from \(i\) to \(j\) if and only if \(b_{ij} > 0\). \(B\) is called acyclic if it is acyclic, a cluster algebra is called acyclic if it has an acyclic exchange matrix, see [3].

A locally finite ice quiver is a pair \((Q, F)\) where \(Q\) is a locally finite quiver without 2-cycles or loops and \(F \subseteq Q_0\) is a subset of vertices called frozen vertices such that there are no arrows among vertices of \(F\). For a locally finite ice quiver \((Q, F)\), we can associate an (infinite) skew-symmetric row and column finite (i.e. having at most finite nonzero entries in each row and column) matrix \((b_{ij})_{i \in Q_0, j \in Q_0 \setminus F}\), where \(b_{ij}\) equals to the number of arrows from \(i\) to \(j\) minus the number of arrows from \(j\) to \(i\). In case of no confusion, for convenience, we also denote the ice quiver \((Q, F)\) as \((b_{ij})_{i \in Q_0, j \in Q_0 \setminus F}\).

We say that an ice quiver \((Q, F)\) admits the action of a group \(\Gamma\) if \(\Gamma\) acts on \(Q\) such that \(\Gamma\) is stable under the action. Let \((Q, F)\) be a locally finite ice quiver with an action of a group \(\Gamma\) (maybe infinite). For a vertex \(i \in Q_0 \setminus F\), a \(\Gamma\)-loop at \(i\) is an arrow from \(i\) to \(h \cdot i\) for some \(h \in \Gamma\), a \(\Gamma\)-2-cycle at \(i\) is a pair of arrows \(i \rightarrow j\) and \(j \rightarrow h \cdot i\) for some \(j \notin \{h' \cdot i | h' \in \Gamma\}\) and \(h \in \Gamma\). Denote by \([i]\) the orbit set of \(i\) under the action of \(\Gamma\).

Say that \((Q, F)\) has no \(\Gamma\)-loops (respectively, \(\Gamma\)-2-cycles) at \([i]\) if \((Q, F)\) has no \(\Gamma\)-loops (\(\Gamma\)-2-cycles, respectively) at any \(i' \in [i]\).

**Definition 2.1** (Definition 2.1, [12]) Let \((Q, F) = (b_{ij})\) be a locally finite ice quiver with an action of a group \(\Gamma\) (maybe infinite). For a vertex \(i \in Q_0 \setminus F\), a \(\Gamma\)-loop at \(i\) is an arrow from \(i\) to \(h \cdot i\) for some \(h \in \Gamma\), a \(\Gamma\)-2-cycle at \(i\) is a pair of arrows \(i \rightarrow j\) and \(j \rightarrow h \cdot i\) for some \(j \notin \{h' \cdot i | h' \in \Gamma\}\) and \(h \in \Gamma\). Denote by \([i]\) the orbit set of \(i\) under the action of \(\Gamma\). Say that \((Q, F)\) has no \(\Gamma\)-loops (respectively, \(\Gamma\)-2-cycles) at \([i]\) if \((Q, F)\) has no \(\Gamma\)-loops (\(\Gamma\)-2-cycles, respectively) at any \(i' \in [i]\).

Denote \((Q', F)\) as \(\tilde{\mu}_{[i]}((Q, F))\) and call \(\tilde{\mu}_{[i]}\) the orbit mutation at direction \([i]\) or at \(i\) under the action \(\Gamma\). In this case, we say that \((Q, F)\) can do orbit mutation at \([i]\).

Note that if \(\Gamma\) is the trivial group \(\{e\}\), then the definition of orbit mutation of a quiver is the same as that of quiver mutation (see [7, 8]).
Definition 2.2 (Definition 2.4, [12]) (i) For a locally finite ice quiver \((Q, F) = (b_{ij})_{i, j \in Q_0 \setminus F}\) with a group \(\Gamma\) (maybe infinite) action, let \(\overline{Q}_0\) (respectively, \(\overline{F}\)) be the orbit sets of the vertex set \(Q_0\) (respectively, the frozen vertex set \(F\)) under the \(\Gamma\)-action. Assume that \(m = |\overline{Q}_0| < +\infty\), \(m - n = |\overline{F}|\) and \(Q\) has no \(\Gamma\)-loops and no \(\Gamma\)-2-cycles.

Define a sign-skew-symmetric matrix \(B(Q, F) = (b_{[i][j]}_{i \in \overline{Q}_0, j \in \overline{Q}_0 \setminus F})\) to \((Q, F)\) satisfying (1) the size of the matrix \(B(Q, F)\) is \(m \times n\); (2) \(b_{[i][j]} = \sum_{i' \in i} b_{i'j}\) for \([i] \in \overline{Q}_0\), \([j] \in \overline{Q}_0 \setminus F\).

(ii) For an \(m \times n\) sign-skew-symmetric matrix \(B\), if there is a locally finite ice quiver \((Q, F)\) with a group \(\Gamma\) such that \(B = B(Q, F)\) as constructed in (i), then we call \((Q, F, \Gamma)\) a covering of \(B\).

(iii) For an \(m \times n\) sign-skew-symmetric matrix \(B\), if there is a locally finite quiver \((Q, F)\) with an action of group \(\Gamma\) such that \((Q, F, \Gamma)\) is a covering of \(B\) and \((Q, F)\) can do arbitrary steps of orbit mutations, then \((Q, F, \Gamma)\) is called an unfolding of \(B\); or equivalently, \(B\) is called the folding of \((Q, F, \Gamma)\).

Remark 2.3 The definition of unfolding is slightly different from that in [12] where the definition was only applicable to square matrices.

By Lemma 2.5 of [12], we have the following consequence.

Lemma 2.4 If \((Q, F, \Gamma)\) is an unfolding of \(B\), for any sequence \([i_1], \ldots, [i_s]\) of orbits of \(Q_0 \setminus F\) under the action of \(\Gamma\), then \(\mu_{[i_1]} \cdots \mu_{[i_s]}(Q, F, \Gamma)\) is a covering of \(\mu_{[i_1]} \cdots \mu_{[i_s]} B\).

By Theorem 2.16 of [12], we have

Theorem 2.5 If \(\widetilde{B} \in \text{Mat}_{m \times n}(\mathbb{Z}) (m \geq n)\) is an acyclic sign-skew-symmetric matrix, then \(\widetilde{B}\) has an unfolding \((\widetilde{Q}, F, \Gamma)\), where \(\widetilde{Q}\) is given using of Construction 2.6 in [12].

Proof Assume \(\widetilde{B} = \begin{pmatrix} B & 0 \\ B' & I \end{pmatrix}\) with \(B \in \text{Mat}_{n \times n}(\mathbb{Z})\). Denote \(\widetilde{B}' = \begin{pmatrix} B & -B'^T \\ B' & 0 \end{pmatrix}\). Since \(\widetilde{B}\) is acyclic, \(\widetilde{B}' \in \text{Mat}_{m \times m}(\mathbb{Z})\) is acyclic. According to Construction 2.6 and Theorem 2.16 of [12], \(\widetilde{B}'\) has an unfolding \((\widetilde{Q}, \Gamma)\). Let \(F \subseteq \overline{Q}_0\) be the vertices of \(\overline{Q}_0\) corresponding to \(B'\). Thus, it is clear that \((\widetilde{Q}, F, \Gamma)\) is an unfolding of \(\widetilde{B}\).

Remark 2.6 In [12], we proved that \(\widetilde{Q}\) is strongly almost finite, that is, \(\widetilde{Q}\) is locally finite and has no path of infinite length.

This theorem means that an acyclic matrix is always totally sign-skew-symmetric. Thus, we can define a cluster algebra via an acyclic matrix.

For an acyclic matrix \(\widetilde{B} \in \text{Mat}_{m \times n}(\mathbb{Z}) (m \geq n)\), assume \((\widetilde{Q}, F, \Gamma)\) is an unfolding of \(\widetilde{B}\). Let \(\overline{Q}_0\) and \(\overline{F}\) be the orbits sets of vertices in \(\overline{Q}_0\) and \(F\). Let \(\Sigma = \Sigma(\overline{Q}) = (X, Y, \overline{Q})\) be the seed associated with \((\widetilde{Q}, F)\), where \(X = \{x_i \mid i \in \overline{Q}_0 \setminus F\}, Y = \{y_j \mid [j] \in \overline{F}\}\). Let \(\Sigma = \Sigma(\overline{B}) = (X, Y, \overline{B})\) be the seed associated with \(\overline{B}\), where \(X = \{x_i \mid [i] \in \overline{Q}_0 \setminus F\}, Y = \{y_j \mid [j] \in \overline{F}\}\). Then it is clear that there is a surjective algebra homomorphism:

\[
\pi: \mathbb{Q}[x_i^\pm 1, y_j \mid i \in \overline{Q}_0 \setminus F, j \in F] \to \mathbb{Q}[x_i^\pm 1, y_j \mid i \in \overline{Q}_0 \setminus F, [j] \in \overline{F}] \quad (2)
\]

such that \(\pi(x_i) = x_i^\pm 1\) and \(\pi(y_j) = y_j\).
For any cluster variable \( x_a \in \tilde{X} \), define \( \tilde{\mu}_{[i]}(x_a) = \mu_a(x_a) \) if \( u \in [i] \); otherwise, \( \tilde{\mu}_{[i]}(x_a) = x_a \) if \( u \not\in [i] \). Formally, write \( \tilde{\mu}_{[i]}(\tilde{X}) = \{ \tilde{\mu}_{[i]}(x) \mid x \in \tilde{X} \} \) and \( \tilde{\mu}_{[i]}(\tilde{X}^{\pm 1}) = \{ \tilde{\mu}_{[i]}(x)^{\pm 1} \mid x \in \tilde{X} \} \).

**Lemma 2.7** (Lemma 7.1, [12]) Keep the notations as above. Assume that \( B \) is acyclic. If \([i]\) is an orbit of vertices with \( i \in \tilde{Q}_0 \backslash F \), then

1. \( \tilde{\mu}_{[i]}(x_j) \) is a cluster variable of \( \mathcal{A}(\tilde{Q}) \) for any \( j \in \tilde{Q}_0 \backslash F \),
2. \( \tilde{\mu}_{[i]}(\tilde{X}) \) is algebraic independent over \( \mathbb{Q}[y_j \mid j \in F] \).

By Lemma 2.7, \( \tilde{\mu}_{[i]}(\tilde{X}) := (\tilde{\mu}_{[i]}(\tilde{X}), \tilde{Y}, \tilde{\mu}_{[i]}(\tilde{Q})) \) is a seed. Thus, we can define \( \tilde{\mu}_{[i]} \tilde{\mu}_{[i_{i-1}]} \cdots \tilde{\mu}_{[i]}(x) \) and \( \tilde{\mu}_{[i]} \tilde{\mu}_{[i_{i-1}]} \cdots \tilde{\mu}_{[i]}(\tilde{X}) \) and \( \tilde{\mu}_{[i]} \tilde{\mu}_{[i_{i-1}]} \cdots \tilde{\mu}_{[i]}(\tilde{\Sigma}) \) for any sequence \( ([i_1], [i_2], \ldots, [i_k]) \) of orbits in \( \tilde{Q}_0 \).

**Theorem 2.8** (Theorem 7.5, [12]) Keep the notations as above with an acyclic sign-skew-symmetric matrix \( B \) and \( \pi \) as defined in Eq. 2. Restricting \( \pi \) to \( \mathcal{A}(\tilde{\Sigma}) \), then \( \pi : \mathcal{A}(\tilde{\Sigma}) \to \mathcal{A}(\Sigma) \) is a surjective algebra morphism satisfying that \( \pi(\tilde{\mu}_{[j]}(x_a)) = \mu_{[j]}(x_a) \in \mathcal{A}(\Sigma) \) and \( \pi(\tilde{\mu}_{[j]}(\tilde{X})) = \mu_{[j]}(\tilde{X}) \) for any sequences of orbits \([j_1], \ldots, [j_k]\) and any \( a \in [i] \).

In case \( \mathcal{A}(\Sigma) \) with principal coefficients, from Lemma 2.4, we may assume that \( \mathcal{A}(\tilde{\Sigma}) \) is also with principal coefficients. Let \( \lambda : \bigoplus_{i \in \tilde{Q}_0 \backslash F} \mathbb{Z}e_i \to \bigoplus_{[i] \in \tilde{Q}_0 \backslash F} \mathbb{Z}e_{[i]} \) be the group homomorphism, where \( \bigoplus_{i \in \tilde{Q}_0 \backslash F} \mathbb{Z}e_i \) (resp. \( \bigoplus_{[i] \in \tilde{Q}_0 \backslash F} \mathbb{Z}e_{[i]} \)) is the free abelian group generated by \( \{e_i \mid i \in \tilde{Q}_0 \backslash F\} \) (resp. \( \{e_{[i]} \mid [i] \in \tilde{Q}_0 \backslash F\} \)). Under such group homomorphism, \( \mathcal{A}(\tilde{\Sigma}) \) becomes a \( \mathbb{Z}^n \)-graded algebra such that any cluster variable \( x \) is homogenous with degree \( \lambda(g(x)) \).

**Theorem 2.9** Keep the notations as in Theorem 2.8. If \( \mathcal{A}(\Sigma) \) with principal coefficients, then the restriction of \( \pi \) to \( \mathcal{A}(\tilde{\Sigma}) \) is a \( \mathbb{Z}^n \)-graded surjective homomorphism.

**Proof** Since \( \lambda(g(x_i)) = e_{[i]} = g(x_i) \) and \( \lambda(g(y_j)) = -b_{[j]} = g(y_{[j]}) \), where \( b_{[j]} \) is the \([j]\)-th column of \( B \). Further, because \( \{x_i^{\pm 1}, y_j \mid i \in \tilde{Q}_0 \backslash F, j \in \tilde{Q}_0\} \) is a generator of \( \mathbb{Q}[x_i^{\pm 1}, y_j \mid i \in \tilde{Q}_0 \backslash F, j \in \tilde{Q}_0] \), thus \( \pi \) is homogenous. Then our result follows by Theorem 2.8. \( \square \)

### 3 Cluster Character in Sign-Skew-Symmetric Case

Let \( \tilde{B} \in \text{Mat}_{m \times n}(\mathbb{Z}) \) be an acyclic sign-skew-symmetric matrix, \((\tilde{Q}, F, \Gamma)\) be an unfolding of \( B \) given in Theorem 2.5. Denote \( \tilde{\Sigma} = (\tilde{X}, \tilde{Y}, \tilde{Q}) \) and \( \Sigma = (X, Y, \tilde{B}) \) be the seeds corresponding to \( (\tilde{Q}, F) \) and \( \tilde{B} \).

From \((\tilde{Q}, F, \Gamma)\), we constructed a 2-Calabi-Yau Frobenius category \( \mathcal{C}^{\tilde{Q}} \) in [12] such that \( \Gamma \) acts on it exactly, i.e. each \( h \in \Gamma \) acts on \( \mathcal{C}^{\tilde{Q}} \) as an exact functor. Furthermore, there exists a cluster tilting subcategory \( \mathcal{T}_0 \) of \( \mathcal{C}^{\tilde{Q}} \) such that the Gabriel quiver of \( \mathcal{T}_0 \) is isomorphic to \( \tilde{Q} \), where \( \mathcal{T}_0 \) is the subcategory of the stable category \( \mathcal{C}^{\tilde{Q}} \) corresponding to \( T_0 \). For details, see Lemma 4.15 of [12]. Since \( \mathcal{C}^{\tilde{Q}} \) is a \( \text{Hom} \)-finite 2-Calabi-Yau Frobenius category, it follows...
that \( C_{\tilde{Q}} \) is a \( Hom\)-finite 2-Calabi-Yau triangulated category. Write \([1]\) as the shift functor in \( C_{\tilde{Q}} \). For any object \( X \) and subcategory \( \mathcal{X} \) of \( C_{\tilde{Q}} \), we denote by \( X \) and \( \mathcal{X} \) the corresponding object and subcategory of \( C_{\tilde{Q}} \) respectively. Since the action of the group \( \Gamma \) on \( C_{\tilde{Q}} \) is exact, \( C_{\tilde{Q}} \) also admits an exact \( \Gamma \)-action.

Since \( C_{\tilde{Q}} \) is a Frobenius category, by the standard result, see \([1]\), we have that

**Lemma 3.1** \( Ext^1_{C_{\tilde{Q}}}(Z_1, Z_2) \cong Ext^1_{C_{\tilde{Q}}}(Z_1, Z_2) \) for all \( Z_1, Z_2 \in C_{\tilde{Q}} \).

The category \( C_{\tilde{Q}} \) can be viewed as an additive categorification of the cluster algebra \( \mathcal{A}(\Sigma) \) given from \( Q \). For details, we refer readers to \([10]\) and \([15]\). Although the authors of \([10]\) and \([15]\) deal with the cluster algebras of finite ranks, it is easy to see that these results still hold in \( C_{\tilde{Q}} \) since \( \tilde{Q} \) is a strongly almost finite quiver.

Denote \( \mathcal{T}_0 = \text{add}(\mathcal{T}' \cup \mathcal{T}'' \cup \mathcal{T}_0 \cup \mathcal{T}'), \) where \( \mathcal{T}' \) and \( \mathcal{T}'' \) respectively consist of the indecomposable objects correspondent to cluster variables in the clusters \( \tilde{X} \) and \( \tilde{Y} \) of \( \Sigma \).

Let \( \mathcal{U} \) be the subcategory of \( C_{\tilde{Q}} \) generated by \( \{X \in C_{\tilde{Q}} \mid Hom_{C_{\tilde{Q}}}(\mathcal{T}[-1], X) = 0, \forall \mathcal{T} \in \mathcal{T}'' \} \).

For any \( X \in \mathcal{U} \), let \( \mathcal{T}_0 \to \mathcal{T}_0 \to X \to \mathcal{T}_1[1] \) be the triangle with \( f \) the minimal right \( \mathcal{T}_0 \)-approximation. By Lemma 3.1, \( \mathcal{T}_0 \) is a cluster tilting subcategory of \( C_{\tilde{Q}} \). Applying \( Hom_{C_{\tilde{Q}}}(\mathcal{T}, -) \) to the triangle for all \( \mathcal{T} \in \mathcal{T}_0 \), we have \( \mathcal{T}_1 \in \mathcal{T}_0 \). The index of \( X \) is defined as

\[
\text{ind}\mathcal{T}_0(X) = [\mathcal{T}_0] - [\mathcal{T}_1] \in K_0(\mathcal{T}_0) \cong \mathbb{Z}[\mathcal{Q}_0].
\]

Recall that the Gabriel quiver of \( \mathcal{T}_0 \) is isomorphic to \( \tilde{Q} \). We may assume that \( \{X_i \mid i \in \tilde{Q}_0\} \) is the complete set of the indecomposable objects of \( \mathcal{T}_0 \). For any \( L \in \text{mod}\mathcal{T}_0 \), we denote \( (\text{dim}_k(L(X_i)))_{i \in \tilde{Q}_0} = \bigoplus_{i \in \tilde{Q}_0} \mathbb{Z}^{e_i} \) as its dimensional vector.

After the preparations, we give the definition of cluster character \( CC(\cdot) \) on \( C_{\tilde{Q}} \). For any \( i \in \tilde{Q}_0 \setminus F \), since \( \tilde{Q} \) is strongly almost finite, we can set

\[
\tilde{y}_i = \prod_{j \in \tilde{Q}_0 \setminus F} x_{ji}^{n \mathcal{T}_0(X)} \prod_{j' \in F} y_{j'i}^{b_{ji}} \in \mathbb{Q}[x_i^\pm 1, y_j \mid i \in \tilde{Q}_0 \setminus F, j \in F].
\]

We say that an object \( X \in C_{\tilde{Q}} \) is rigid if \( Ext^1(X, X) = 0 \). For each rigid object \( X \in \mathcal{U} \), we define

\[
CC(X) = \tilde{x}^{\text{ind}\mathcal{T}_0(X)} \sum_{a \in \bigoplus_{i \in \tilde{Q}_0} \mathbb{Z} e_i} \chi(\text{Gr}_a(Hom_{C_{\tilde{Q}}}(\mathcal{T}, X[1]))) \prod_{j \in \tilde{Q}_0 \setminus F} \tilde{y}_j^{a_j},
\]

where \( \tilde{x}^{a} = \prod x_{ji}^{a_{ji}} \) for \( a = (a_i)_{i \in \tilde{Q}_0} = \bigoplus_{i \in \tilde{Q}_0} \mathbb{Z} e_i, \text{Gr}_a(Hom_{C_{\tilde{Q}}}(\mathcal{T}, X[1])) \) is the quiver Grassmannian whose points are corresponding to the sub-\( \mathcal{T}_0 \)-representations of \( Hom_{C_{\tilde{Q}}}(\mathcal{T}, X[1]) \) with dimension vector \( a \), and \( \chi \) is the Euler characteristic with respect to étale cohomology with proper support. It is easy to see that \( CC(X) \in \mathbb{Q}[x_i^\pm 1, y_j \mid i \in \tilde{Q}_0 \setminus F, j \in F] \).

**Theorem 3.2** Keep the notations as above. Then

1. \( CC(T_i) = x_i \) for all \( T_i \in \mathcal{T}' \).
2. \( CC(X \oplus X') = CC(X)CC(X') \) for any objects \( X, X' \in \mathcal{U} \).
(3) \( CC(X)CC(Y) = CC(Z) + CC(Z') \) for \( X, Y \in \mathcal{U} \) with \( \text{dimExt}_{C_0}^1(X, Y) = 1 \) and the two non-splitting triangles: \( Y \to Z \to X \to Y[1] \) and \( X \to Z' \to Y \to X[1] \).

**Proof** This theorem can be proved similarly as that of ([10], Theorem 3.3) using of local finiteness of \( \mathcal{Q} \) since it is strongly almost finite. \( \square \)

As in [5], for any \( X \in \mathcal{U} \), we define \( \text{ind}_{T_0}^X(X) = \lambda(\text{ind}_{T_0}^X(X)) \in \bigoplus_{[i] \in Q_0} \mathbb{Z}e_{[i]} \).

For each rigid object \( X \in \mathcal{U} \), using the definition of \( \pi \) in Eq. 2, we define the \( \Gamma \)-equivariant cluster character as follows:

\[
CC(X) = \pi (CC(X)) = x \sum_{a \in \bigoplus_{i \in Q_0} \mathbb{Z}e_i} \chi(Gr_aHom_{C_0}(-, X[i])) \prod_{[j] \in Q_0 \setminus \mathcal{F}} \gamma_{[j]}^{(a)_{[j]}},
\]

where \( x^a = \prod x_{[i]}^{a_{[i]}} \) for \( a = (a_{[i]})_{[i] \in Q_0 \setminus \mathcal{F}} \in \bigoplus_{[i] \in Q_0 \setminus \mathcal{F}} \mathbb{Z}e_{[i]} \) and \( \gamma_{[i]} = \pi(\gamma_{[i]}) = \prod_{[j] \in Q_0 \setminus \mathcal{F}} \gamma_{[j]}^{b_{[j]_{[i]}}} \prod_{([j] \in Q_0 \setminus \mathcal{F})} \gamma_{[j]_{[i]}}^{b_{[j]_{[i]}}} \).

Inspired by Definition 3.34 of [5], we give the following definition.

**Definition 3.3** Two objects \( X, X' \in C_0 \) are said to be **equivalent modulo** \( \Gamma \) if \( X \) and \( X' \) have decompositions into indecomposable objects \( X = \bigoplus_{k=1}^m X_k \), \( X' = \bigoplus_{k=1}^m X'_k \) such that the indecomposable objects \( X_k \) and \( X'_k \) are in the same \( \Gamma \)-orbit up to isomorphism for every \( k \).

Similar to Lemma 3.49 of [5], we have the following lemma.

**Lemma 3.4** The Laurent polynomial \( CC(X) \) depends only on the class of \( X \) under equivalent modulo \( \Gamma \) for and \( X \in C_0 \).

**Proof** We need only to prove that \( CC(X) = CC(h \cdot X) \) for \( h \in \Gamma \) by Theorem 3.2(2). It follows immediately from that \( \text{ind}_{T_0}^X(X) = \text{ind}_{T_0}^{h \cdot X}(X) \) and \( \chi(Gr_aHom_{C_0}(-, X[1])) = \chi(Gr_aHom_{C_0}(-, h \cdot X[1])) \).

Using the algebra homomorphism \( \pi \) and Theorem 3.2, we have the following theorem immediately.

**Theorem 3.5** Keep the notations as above. Then

1. \( CC'(T_i) = x_{[i]} \) for all \( T_i \in \mathcal{T}' \).
2. \( CC'(X \oplus X') = CC'(X)CC'(X') \) for any two objects \( X, X' \in \mathcal{U} \).
3. \( CC'(X)CC'(Y) = CC'(Z) + CC'(Z') \) for \( X, Y \in \mathcal{U} \) with \( \text{dimExt}_{C_0}^1(X, Y) = 1 \) satisfying two non-splitting triangles: \( Y \to Z \to X \to Y[1] \) and \( X \to Z' \to Y \to X[1] \).

Following [14], we say \( X \in C_0 \) to be **reachable** if it belongs to a cluster-tilting subcategory which can be obtained by a sequence of mutations from \( T_0 \) with the mutations do not take at the objects in \( \mathcal{T}' \). It is clear that any reachable object belongs to \( \mathcal{U} \).
Following Theorem 4.1 of [14], we have the following result:

**Theorem 3.6** Keep the notations as above. Then the cluster character $CC'(\ )$ gives a surjection from the set of equivalence classes of indecomposable reachable objects under equivalent modulo $\Gamma$ of $\mathcal{C}^\emptyset$ to the set of clusters variables of the cluster algebra $\mathcal{A}(\Sigma)$.

**Proof** By Theorem 3.5, the proof is similar as that of Theorem 4.1 in [14]. $\square$

### 4 Sign-coherence of $g$-vectors

Keep the notations in Section 3. In this section, we will prove the sign-coherence of $g$-vectors for acyclic sign-skew-symmetric cluster algebras. For convenience, suppose that $\mathcal{A}(\Sigma)$ is an acyclic sign-skew-symmetric cluster algebra with principal coefficients at $\Sigma$.

Since $B$ is acyclic, $(\tilde{Q}, F, \Gamma)$ is acyclic, too. We can construct an unfolding $(\tilde{Q}, F, \Gamma)$ according to Theorem 2.5. It is easy to check that the corresponding seed $\tilde{\Sigma}$ of $(\tilde{Q}, F, \Gamma)$ is with principal coefficients, where $\tilde{\Sigma}$ is the seed associate to $(Q, F)$.

For any object $X \in U$, by Theorem 3.2, we have $CC(X)$ is a cluster monomial of $\mathcal{A}(\Sigma)$ and thus $CC(X)$ is homogenous. We call its degree the $g$-vector of $CC(X)$. Similarly, by Theorem 3.5, $CC'(X)$ is a cluster monomial of $\mathcal{A}(\Sigma)$ and thus $CC'(X)$ is homogenous. We call its degree the $g$-vector of $CC'(X)$.

**Proposition 4.1** Assume that $X$ is an object of $U$ given in Section 3.

(i) The $g$-vector of $CC(X)$ is $(g_i)_{i \in \mathbb{Q}\setminus F}$, where $g_i = [ind_{T_0}(X) : T_i]$ for each $i$.

(ii) The $g$-vector of $CC'(X)$ is $(g_{[i]})([i] \in \mathbb{Q}\setminus F)$, where $g_{[i]} = \sum_{i' \in [i]} [ind_{T_0}(X) : T_i']$ for each $i$.

**Proof** (i) Since the quiver $\tilde{Q}$ is strongly almost finite, the proof is the same one as that of ([15], Proposition 3.6) using of the local finiteness of $\tilde{Q}$.

(ii) is obtained immediately from (i). $\square$

Let $h \in \Gamma$ be either of finite order or without fixed points. Define $\mathcal{C}^{\tilde{Q}}_h$ the $K$-linear category whose objects are the same as that of $\mathcal{C}^{\tilde{Q}}$, and whose morphisms consist of $Hom_{\mathcal{C}^{\tilde{Q}}_h}(X, Y) = \bigoplus_{h' \in \Gamma'} Hom_{\mathcal{C}^{\tilde{Q}}}(h' \cdot X, Y)$ for all objects $X, Y$. We view this category as a dual construction of the category $\mathcal{C}^{\tilde{Q}}_\emptyset$ in [12].

Denote by $\mathcal{C}^{\tilde{Q}}_h(T_0)$ the subcategory of $\mathcal{C}^{\tilde{Q}}_h$ consisting of all objects $T \in T_0$.

**Lemma 4.2** The category $\mathcal{C}^{\tilde{Q}}_h$ is Hom-finite if either (i) the order of $h$ is finite, or (ii) $Q$ has no fixed points under the action of $h \in \Gamma$.

**Proof** The proof is similar to that of Lemma 6.1 in [12]. $\square$

In the sequel, assume that $\mathcal{C}^{\tilde{Q}}_h(T_0)$ is Hom-finite. Let $F : \mathcal{C}^{\tilde{Q}} \to \text{mod}\mathcal{C}^{\tilde{Q}}_h(T_0)^{op}$ be the functor mapping an object $X$ to the restriction of $\mathcal{C}^{\tilde{Q}}_h(-, X)$ to $\mathcal{C}^{\tilde{Q}}_h(T_0)$. For each indecomposable object $T$ of $T_0$, denote by $S_T$ the simple quotient of $F(T)$ via $S_T(T') =$
\[ \text{End}_{C}(T)/J \text{ if } T' \cong T \text{ and } S_T(T') = 0 \text{ if } T' \not\cong T, \text{ for any object } T' \text{ of } C^\perp_h, \text{ where } J \text{ is the Jacobson radical of } \text{End}_{C^\perp_h}(T). \]

**Lemma 4.3** Keep the notations as above. For any morphism \( \tilde{f} : F(M) \to F(N) \) in \( \text{mod}C^\perp_h(T_0) \text{op} \) with \( M, N \in C^\perp \), there exists \( f : M \to N \) in \( C^\perp_h \) such that \( F(f) = \tilde{f} \).

**Proof** Let \( T_1 \xrightarrow{e} T_0 \xrightarrow{d} M \to T_1[1] \) be the triangle such that \( d \) is a minimal right \( T_0 \)-approximation. Since \( T_0 \) is a cluster tilting subcategory of \( C^\perp \), we have \( T_1 \in T_0 \). Applying \( F \) to the above triangle, we have \( F(T_1) \to F(T_0) \to F(M) \to 0 \). Similarly, there is a triangle \( T'_1 \xrightarrow{e'} T'_0 \xrightarrow{d'} N \to T'_1[1] \), and \( F(T'_1) \to F(T'_0) \to F(N) \to 0 \). Since \( T_0, T'_0, T_1, T'_1 \in T_0 \), it follows that \( F(T_0), F(T'_0), F(T_1), F(T'_1) \) are projective in \( \text{mod}C^\perp_h(T_0) \text{op} \). Thus, \( \tilde{f} \) can be lift to the following commutative diagram:

\[
\begin{array}{cccccc}
F(T_1) & \longrightarrow & F(T_0) & \longrightarrow & F(M) & \longrightarrow & 0 \\
\downarrow \tilde{f}_1 & & \downarrow \tilde{f}_0 & & \downarrow \tilde{f} & & \\
F(T'_1) & \longrightarrow & F(T'_0) & \longrightarrow & F(N) & \longrightarrow & 0,
\end{array}
\]

By the Yoneda Lemma, there exist \( f_1 : T_1 \to T'_1 \) and \( f_0 : T_0 \to T'_0 \) in \( C^\perp_h \) such that \( F(f_1) = \tilde{f}_1 \) and \( F(f_0) = \tilde{f}_0 \). Thus, \( f_1 \in \bigoplus_{h' \in \Gamma'} \text{Hom}_{C^\perp_h}(h' \cdot T_1, T'_1) \) and \( f_0 \in \bigoplus_{h' \in \Gamma'} \text{Hom}_{C^\perp_h}(h' \cdot T_0, T'_0) \) such that

\[
\begin{array}{cccccc}
\bigoplus_{h' \in \Gamma'} h' \cdot T_1 & \longrightarrow & \bigoplus_{h' \in \Gamma'} h' \cdot T_0 & \longrightarrow & \bigoplus_{h' \in \Gamma'} h' \cdot M & \longrightarrow & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow & & \\
T'_1 & \longrightarrow & T'_0 & \longrightarrow & N & \longrightarrow & 0,
\end{array}
\]

commutes. Since \( C^\perp_h(T_0) \) is \( \text{Hom} \)-finite, we may assume that only finite \( h' \in \Gamma' \) appear in the upper triangle, which means that there exists a finite subset \( I \) of \( \Gamma' \) such that

\[
\begin{array}{cccccc}
\bigoplus_{h' \in I} h' \cdot T_1 & \longrightarrow & \bigoplus_{h' \in I} h' \cdot T_0 & \longrightarrow & \bigoplus_{h' \in I} h' \cdot M & \longrightarrow & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow & & \\
T'_1 & \longrightarrow & T'_0 & \longrightarrow & N & \longrightarrow & 0,
\end{array}
\]

commutes. Thus, there exists \( f \in \bigoplus_{h' \in I} \text{Hom}_{C^\perp_h}(h' \cdot T_0, T'_0) \subseteq \bigoplus_{h' \in \Gamma'} \text{Hom}_{C^\perp_h}(h' \cdot T_0, T'_0) \) such that the above diagram commutes. This commutative diagram induces

\[
\begin{array}{cccccc}
F(T_1) & \longrightarrow & F(T_0) & \longrightarrow & F(M) & \longrightarrow & 0 \\
\downarrow \tilde{f}_1 & & \downarrow \tilde{f}_0 & & \downarrow F(f) & & \\
F(T'_1) & \longrightarrow & F(T'_0) & \longrightarrow & F(N) & \longrightarrow & 0.
\end{array}
\]

Therefore, we have \( F(f) = \tilde{f} \). \( \square \)
Lemma 4.4 Assume that for $X \in \mathcal{U}$, the category $\text{add}(\langle h \cdot X \mid h \in \Gamma \rangle)$ is rigid. Let $T_1 \xrightarrow{f'} T_0 \xrightarrow{f} X \to T_1[1]$ be a triangle in $\mathcal{C}^\mathcal{O}$ with $f$ a minimal right $T_0$-approximation. If $X$ has not any direct summand in $T_0[1]$, then

$$F(T_1) \xrightarrow{F(f')} F(T_0) \xrightarrow{F(f)} F(X) \to 0$$

is a minimal projective resolution of $F(X)$.

Proof Since $X$ does not have direct summand in $T_0[1]$, $f'$ is right minimal. Otherwise, if $f'$ is not right minimal, then $f'$ has a direct summand as $T' \to 0$, thus $T'[1]$ is a direct summand of $X$. This is a contradiction.

First, we prove that $F(f)$ is a projective cover of $F(X)$. For any projective representation $F(T)e$ of $\mathcal{C}^\mathcal{O}_h$ and surjective morphism $u : F(T)e \to F(X)$, where $T \in \mathcal{C}^\mathcal{O}_h$, $e \in \text{End}_{\mathcal{C}^\mathcal{O}_h}(T)$ is an idempotent. By Yoneda Lemma, there exists $g \in \text{Hom}_{\mathcal{C}^\mathcal{O}_h}(T, X)$ such that $F(g) = u$. Since $F(T)e$ and $F(T_0)$ are projective, there exist $v : F(T_0) \to F(T)e$ and $w : F(T) \to F(T_0)$ such that $F(f) = F(g)v$ and $F(g) = F(f)w$. Thus, $F(f) = F(f)vw$. Similarly, by Yoneda Lemma, there exist $g' \in \text{Hom}_{\mathcal{C}^\mathcal{O}_h}(T_0, T)$ and $g'' \in \text{Hom}_{\mathcal{C}^\mathcal{O}_h}(T_0, T_0)$ such that $F(g') = v, F(g'') = w$. Thus, $F(f) = F(f)vw$ is equivalent to $f = f \circ g'' \circ g'$. We may assume $g'' \circ g' = (g_{h'})_{h' \in \Gamma'}$, then $f = f \circ (g'' \circ g') = (fg_{h'})_{h' \in \Gamma'}$. Thus $f = fg_e$ and $0 = fg_{h'} = 0$ for any $h' \neq e$, where $e$ is the identity of $\Gamma'$. Further, since $f$ is a right minimal $T_0$-approximation and $h' T_0 \in T_0$. Therefore, for any $e \neq h' \in \Gamma$, $g_{h'} \in J(h' T_0, T_0)$ and $g_e$ is an isomorphism. Using Lemma 5.10 of [12], $g'' \circ g' = (g_{h'})_{h' \in \Gamma'}$ is an isomorphism. Thus, $F(T_0)$ is a direct summand of $F(T)e$.

Similarly, because $f'$ is right minimal, $F(f')$ induces a projective cover of $\ker(F(f))$. Our result follows. \qed

Inspired by (Proposition 2.1, [4]), (Lemma 3.58, [5]) and (Lemma 3.5, [15]), we have:

Lemma 4.5 Assume $X \in \mathcal{U}$ such that $\text{add}(\langle h \cdot X \mid h \in \Gamma \rangle)$ is rigid, and $T_1 \xrightarrow{f'} T_0 \xrightarrow{f} X \to T_1[1]$ is a triangle in $\mathcal{C}^\mathcal{O}$ and $f$ is a minimal right $T$-approximation. If $T$ is a direct summand of $T_0$ for indecomposable object $T \in \mathcal{C}^\mathcal{O}$, then $h \cdot T$ is not a direct summand of $T_1$ for any $h \in \Gamma$.

Proof By Lemma 2.3 and Lemma 2.4 of [12], we may assume that $h$ has no fixed points or finite order. By Lemma 4.2, $\mathcal{C}^\mathcal{O}_h$ is Hom-finite. For any $h' \in \Gamma'$ and $T' \in \mathcal{T}$, applying $\text{Hom}_{\mathcal{C}^\mathcal{O}_h}(h' \cdot T', -)$ to the triangle, we get

$$\text{Hom}_{\mathcal{C}^\mathcal{O}}(h' \cdot T', T_1) \to \text{Hom}_{\mathcal{C}^\mathcal{O}}(h' \cdot T', T_0) \xrightarrow{\text{Hom}_{\mathcal{C}^\mathcal{O}}(h' \cdot T', f)} \text{Hom}_{\mathcal{C}^\mathcal{O}}(h' \cdot T', X) \to 0.$$ 

Since $f$ is minimal, we get a minimal projective resolution of $F(X)$,

$$F(T_1) \to F(T_0) \to F(X) \to 0.$$ 

To prove that $h \cdot T$ is not a direct summand of $T_1$, it suffices to prove that $F(T)$ is not a direct summand of $F(T_1)$, or equivalently $\text{Ext}^1(F(X), S_T) = 0$, where $S_T$ is the simple quotient of $F(T)$. \qed
As \( F(T_0) \to F(X) \) is the projective cover of \( F(X) \) and \( F(T) \) is a direct summand of \( F(T_0) \), then there is a non-zero morphism \( \tilde{\rho} : F(X) \to S_T \). For any \( \tilde{g} : F(T_1) \to S_T \), since \( F(T_1) \) is projective, there exists \( \tilde{q} : F(T_1) \to F(X) \) such that \( \tilde{g} = \tilde{\rho} \tilde{q} \).

Let \( T \) be a lifting of \( T_1 \) by Lemma 3.1, we have \( T \in \mathcal{T} \) and \( T \) is non-projective. Moreover, since \( \mathcal{T} \) is indecomposable, we can choose \( T \) is indecomposable. By Lemma 5.3 of [12], there is an admissible short exact sequence \( 0 \to Y \to T' \to T \to 0 \). Since \( \mathcal{T} \) has no \( \Gamma \)-loop, by the dual version of Lemma 5.11 (2) of [12], and Lemma 3.1, we have \( S_T \cong F(Y[1]) \).

Thus, according to Lemma 4.3, lifting \( \tilde{q}, \tilde{g} \) and \( \tilde{\rho} \) as \( q, g, p \) in \( \mathcal{C} \), where \( q \in \bigoplus_{h' \in \Gamma'} \text{Hom}_{\mathcal{C}}(h' \cdot T_1, X) \), \( g \in \bigoplus_{h' \in \Gamma'} \text{Hom}_{\mathcal{C}}(h' \cdot T_1, Y[1]) \) and \( p \in \bigoplus_{h' \in \Gamma'} \text{Hom}_{\mathcal{C}}(h' \cdot X, Y[1]) \). Since \( \tilde{g} = \tilde{\rho} \tilde{q} \) and \( \text{Hom}(F(T), F(Y[1])) \cong \text{Hom}_{\mathcal{C}}(T, Y[1]) \), we obtain \( g = p \circ q \).

Since \( C_0 \) is \( \text{Hom} \)-finite, we may assume that \( g \in \bigoplus_{h' \in I} \text{Hom}_{\mathcal{C}}(h' \cdot T_1, Y[1]) \) and \( q \in \bigoplus_{h' \in I} \text{Hom}_{\mathcal{C}}(h' \cdot T_1, X) \) for a finite subset \( I \subseteq \Gamma' \). According to the composition of morphisms in \( C_0 \), \( g = p \circ q \) means \( g = p(\sum_{h' \in I} h' \cdot q) \), equivalently, we have the following commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{h' \in I} h' \cdot X[1] & \xrightarrow{\sum_{h' \in I} h' \cdot f'} & \bigoplus_{h' \in I} h' \cdot T_1 \xrightarrow{\sum_{h' \in I} h' \cdot f} & \bigoplus_{h' \in I} h' \cdot T_0 \\
& & \downarrow g & \\
\bigoplus_{h' \in I} h' \cdot X & \xrightarrow{p} & Y[1],
\end{array}
\]

Since \( \text{add}(\{ h \cdot X \mid h \in \Gamma \}) \) is rigid, we have \( g(\sum_{h' \cdot f} h') = 0 \). Therefore, \( g \) factor through \( \sum_{h' \cdot f} h' \cdot f \). Thus, in \( C_0 \), \( g \) factors through \( f \). By the arbitrary of \( g \), we get a surjective map \( \text{Hom}_{\mathcal{C}}(T_0, Y[1]) \to \text{Hom}_{\mathcal{C}}(T_1, Y[1]) \). Since \( \text{Hom}_{\mathcal{C}}(T_1, Y[1]) \cong \text{Hom}(F(T), F(Y[1])) = \text{Hom}(F(T), S_T) \) for \( i = 1, 2 \), we get \( \text{Hom}(F(T_0), S_T) \to \text{Hom}(F(T_0), S_T) \) is surjective. Therefore, we obtain \( \text{Ext}^1(F(X), S_T) = 0 \). Our result follows.

**Corollary 4.6** Let \( X \) be an object of \( \mathcal{C} \) such that \( \text{add}(\{ h \cdot X \mid h \in \Gamma \}) \) is rigid. If \( \text{ind}^\mathcal{T}_{T_0}(X) \) has no negative coordinates, then \( X \in \mathcal{T} \).

**Proof** Assume \( T_1 \xrightarrow{f'} T_0 \xrightarrow{f} X \to T_1[1] \) is a triangle and \( f \) is a minimal right \( \mathcal{T} \)-approximation. According to Lemma 4.5, \( T_0 \) and \( T_1 \) have no direct summands which have the same \( \Gamma \)-orbits. Further, since \( \text{ind}^\mathcal{T}_{T_0}(X) \) has no negative components. Therefore, by the definition of \( \text{ind}^\mathcal{T}_{T_0} \), we obtain \( T_1 = 0 \). Thus, \( X \cong T_0 \in \mathcal{T} \).

Using the above preparation, we now can prove Conjecture 1.1 for all acyclic sign-skew-symmetric cluster algebras. The method of the proof follows from that of Theorem 3.7 (i) in [15].

**Theorem 4.7** The conjecture 1.1 on sign-coherence holds for all acyclic sign-skew-symmetric cluster algebras.
Proof For any cluster $Z = (z_1, \ldots, z_n)$ of $A(\Sigma)$, by Theorem 3.6, we associate a cluster tilting subcategory $\mathcal{T}$ of $C_0$ which obtained by a series of mutations of $\mathcal{T}_0$ and the mutations do no take at $\mathcal{T}'$. Precisely, there are $n$ indecomposable objects $\{X_j \mid j = 1, \ldots, n\}$ such that $\mathcal{T} = add((h \cdot X_j \mid j = 1, \ldots, n) \cup \mathcal{T}'$) and $CC'(X_i) = z_i$ for $i = 1, \ldots, n$. By Proposition 4.1, the $g$-vector $g_{[1]}^j, \ldots, g_{[n]}^j$ of $z_j$ is given by $g_{[i]}^j = \sum_{i' \in [i]} [ind_{\mathcal{T}_0}(X_j) : T_{i'}].$

Suppose that there exist $s$ and $s'$ such that $g_{[1]}^s > 0$ and $g_{[i]}^s < 0$. Assume that $\mathcal{T}_{j_1} \to T_{j_0} \to X_j \to T_{j_1}[1]$ be the triangle with $f^j$ is a minimal right $\mathcal{T}_0$-approximation for $j = s, s'$. Thus, there exist $h, h' \in \Gamma$ such that $h \cdot T_i$ (respectively, $h' \cdot T_i$) is a direct summand of $T_{s_0}^i$ (respectively, $T_{s_1}^i$).

Furthermore, in the triangle

$$\bigoplus_{j = s, s'} T_{j_0}^j \to \bigoplus_{j = s, s'} T_{j_0}^j \to X_j \to \bigoplus_{j = s, s'} T_{j_1}^j[1],$$

the morphism $\bigoplus_{j = s, s'} f^j$ is a minimal right $\mathcal{T}_0$-approximation. According to Lemma 4.5, $h' \cdot T_i$ is not a direct summand of $T_{s_0}^i$ since $h \cdot T_i$ is a direct summand of $T_{s_0}^i$. This is a contradiction. Our result follows.

\[\Box\]

5 The Recurrence of g-vectors

Theorem 5.1 ([6]) Conjecture 1.3 holds true for all finite rank skew-symmetric cluster algebras.

It is easy to see that the above theorem can be extended to the situation of infinite rank skew-symmetric cluster algebras, that is, we have:

Theorem 5.2 Conjecture 1.3 holds true for all infinite rank skew-symmetric cluster algebras.

We first give the following easy lemma.

Lemma 5.3 Let $(Q, F, \Gamma)$ be the unfolding of a matrix $B$ and $A = A(\Sigma(Q, F)$). For any sequence of orbits $([i_1], \ldots, [i_s])$ and $a \in Q_0$, there exist finite subsets $S_j \subseteq [i_j]$, $j = 1, \ldots, s$ such that $\prod_{k \in V_j} \mu_k \cdot \prod_{k \in V_j} \mu_k(x_a) = \tilde{\mu}_{[i_1]} \cdots \tilde{\mu}_{[i_s]}(x_a)$ for all finite subsets $V_j$, $j = 1, \ldots, s$, satisfying $S_j \subseteq V_j \subseteq [i_j]$, $j = 1, \ldots, s$.

Proof Since $\tilde{\mu}_{[i_1]} \cdots \tilde{\mu}_{[i_s]}(x_a)$ is determined by finite vertices of $Q_0$, there exist finite subsets $S_j \subseteq [i_j]$, $j = 1, \ldots, s$ such that $\prod_{k \in S_j} \mu_k \cdot \prod_{k \in S_j} \mu_k(x_a) = \tilde{\mu}_{[i_1]} \cdots \tilde{\mu}_{[i_s]}(x_a)$. Then for all finite subsets $V_j$, $j = 1, \ldots, s$ satisfying $S_j \subseteq V_j \subseteq [i_j]$, $j = 1, \ldots, s$, we have $\prod_{k \in V_j} \mu_k \cdot \prod_{k \in V_j} \mu_k(x_a) = \tilde{\mu}_{[i_1]} \cdots \tilde{\mu}_{[i_s]}(x_a)$. \[\Box\]

Let $A_1$ (respectively, $A_2$) be the cluster algebra with principal coefficients at $\Sigma_1 = (\tilde{X}, \tilde{Y}, \tilde{Q})$ (respectively, $\Sigma_2 = (\tilde{X}', \tilde{Y}', \tilde{\mu}_{[k]}(Q))$. For any sequence $([i_1], \ldots, [i_s])$ of
orbits of \( Q_0 \) and \( a \in Q_0 = \tilde{\mu}_{[k]}(Q_0) \), denote \( g^{Q,a} = (g_i^{Q,a})_{i \in Q_0} \) (respectively, \( g^{\tilde{\mu}_{[k]}(Q),a} = (g_i^{\tilde{\mu}_{[k]}(Q),a})_{i \in Q_0} \)) be the \( g \)-vector of the cluster variable \( \tilde{\mu}_{[i_1]} \cdots \tilde{\mu}_{[i_l]}(x_a) \) (respectively, \( \tilde{\mu}_{[i_1]} \cdots \tilde{\mu}_{[i_l]}(x'_{a_j}) \)).

As a consequence of Theorem 5.2, we have the following property.

**Proposition 5.4** Keep the notations as above. The following recurrence holds:

\[
g_i^{\tilde{\mu}_{[k]}(Q)} = \begin{cases} 
g_i^Q & \text{if } i \in [k]; \\
 g_i^Q + \sum_{k' \in [k]} [b_{ik'}] + g_{k'}^Q - \sum_{k' \in [k]} b_{ik'} \min(g_{k'}^Q, 0) & \text{if } i \notin [k]. \end{cases}
\]

**Proof** By Lemma 5.3, for \( j = 1, \ldots, s \), there exist finite subsets \( S_j^1 \subseteq [i_j] \) (respectively, \( S_j^2 \subseteq [i_j] \)) such that \( g^{Q,a} \) (respectively, \( g^{\tilde{\mu}_{[k]}(Q),a} \)) is the \( g \)-vector of the cluster variable \( \prod_{i \in V_j^1} \mu_i \cdots \prod_{i \in V_j^2} \mu_i (x_a) \) (respectively, \( \prod_{i \in S_j^1} \mu_i \cdots \prod_{i \in S_j^1} \mu_i \tilde{\mu}_{[k]}(x'_{a_j}) \)) for all finite subsets \( V_j \) (respectively, \( V_j \)) satisfying that \( S_j^1 \subseteq V_j^1 \subseteq [i_j] \) (respectively, \( S_j^2 \subseteq V_j^2 \subseteq [i_j] \)). Choose \( S_j = S_j^1 \cup S_j^2 \), we have \( g^{Q,a} \) and \( g^{\tilde{\mu}_{[k]}(Q),a} \) as the \( g \)-vectors of the cluster variables \( \prod_{i \in S_j} \mu_i \cdots \prod_{i \in S_j} \mu_i \tilde{\mu}_{[k]}(x'_{a_j}) \) respectively.

For any finite subset \( T \subseteq [k] \), denote by \( \mathcal{A}^T \) the cluster algebra with principal coefficients at \( \Sigma^T = (\tilde{X}^T, \tilde{Y}^T, \prod_{i \in T} \mu_i k_i(Q)) \). Denote by \( g^{\mu_T(Q),a}_{i \in T} = (g_i^{\mu_T(Q)})_{i \in Q_0} \) the \( g \)-vector of the cluster variable \( \prod_{i \in T} \mu_i \cdots \prod_{i \in T} \mu_i \tilde{\mu}_{[k]}(x'_{a_j}) \).

Since the cluster variable \( \prod_{i \in S_j} \mu_i \cdots \prod_{i \in T} \mu_i \tilde{\mu}_{[k]}(x'_{a_j}) \) is only determined by finite vertices of \( Q_0 \), there exists a finite subset \( S \subseteq [k] \) such that \( g_i^{\tilde{\mu}_{[k]}(Q)} = \prod_{i \in T} \mu_i^{\mu_T(Q)}(Q) \) for any finite set \( T \) satisfying \( S \subseteq T \subseteq [k] \).

Furthermore, by Theorem 5.2, for any subset \( T \subseteq Q_0 \), we have

\[
g_{i \in T}^{\mu_T(Q)} = \begin{cases} 
g_i^Q & \text{if } i \in [k]; \\
 g_i^Q + \sum_{i \in T} [b_{ik'}] + g_{k'}^Q - \sum_{i \in T} b_{ik'} \min(g_{k'}^Q, 0) & \text{if } i \notin [k]. \end{cases}
\]

Therefore, the result holds. \( \square \)

**Theorem 5.5** Conjecture 1.3 holds true for all acyclic sign-skew-symmetric cluster algebras. That is, let \( B = (b_{[i][j]}) \in \text{Mat}_{n \times n} (\mathbb{Z}) \) be a sign-skew-symmetric matrix which is mutation equivalent to an acyclic matrix and let \( t_1 \cdots t_2 \in \mathbb{T}_n \) and \( B^2 = \mu_{[k]}(B^1) \). For \( [a] \in \{[1], \ldots, [n]\} \) and \( t \in \mathbb{T}_n \), assume \( g^{B^1,t}_{[a]} = (g_{[a]}^{[1]}, \ldots, g_{[a]}^{[n]}) \) and \( g^{B^2,t}_{[a]} = (g_{[a]}^{[1]}, \ldots, g_{[a]}^{[n]}) \), then

\[
g^{B^2}_{[i]} = \begin{cases} 
g^{B^2}_{[i]} & \text{if } [i] = [k]; \\
 g^{B^2}_{[i]} + [b_{[i][k]}] + g_{[k]}^t - b_{[i][k]} \min(g_{[k]}^t, 0) & \text{if } [i] \neq [k]. \end{cases}
\]

**Proof** By Lemma 2.4 and Theorem 2.5, let \( (Q, \Gamma) \) be an unfolding of \( B \). Assume \( t_1 \cdots t_{i_1} \cdots t_{i_l} \) and \( t \). By Theorem 2.9, we have \( g^{B^2}_{[i]} = \sum_{i' \in [i]} \tilde{g}_{i'}^{\tilde{\mu}_{[k]}(Q)}(Q) \) and \( g^{B^2}_{[i]} = \sum_{i' \in [i]} \tilde{g}_{i'}^{\tilde{\mu}_{[k]}(Q)}(Q) \). By
Proposition 4.1 and Lemma 4.5, for \( k, k'' \in [k] \), both \( g^Q_{k'} \) and \( g^Q_{k''} \) are non-negative or non-positive. Thus, \( \sum_{k' \in [k]} \min(g^Q_{k'}, 0) = \min(\sum_{k' \in [k']} g^Q_{k'}, 0) \). Using Proposition 5.4, if \([i] = [k]\), then

\[
g^Q_{[i]} = \sum_{i' \in [i]} g^Q_{i'i'} = -\sum_{i' \in [i]} g^Q_{i'i'} = -s^Q_{[i]}.
\]

if \([i] \neq [k]\), since \( \sum_{k' \in [k]} \min(g^Q_{k'}, 0) = \min(\sum_{k' \in [k]} g^Q_{k'}, 0) \) and \( b^Q_{i'k} \), then

\[
g^Q_{[i]} = \sum_{i' \in [i]} (g^Q_{i'i'} + \sum_{k' \in [k]} |b^Q_{i'k}'| + g^Q_{k'}) - \sum_{k' \in [k]} b^Q_{i'k'} \min(g^Q_{k'}, 0))
\]

\[
= g^Q_{[i]} + |b^Q_{[i]}|[k] + s^Q_{[k]} - b^Q_{[i][k]} \min(g^Q_{[k]}, 0).
\]

The result holds.

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