Differential cross sections for K-shell ionization by electron or positron impact

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We have investigated the universal scaling behavior of differential cross sections for the single K-shell ionization by electron or positron impact. The study is performed within the framework of non-relativistic perturbation theory, taking into account the one-photon exchange diagrams. In the case of low-energy positron scattering, the doubly differential cross section exhibits prominent interference oscillations. The results obtained are valid for arbitrary atomic targets with moderate values of nuclear charge number Z.

I. INTRODUCTION

The single ionization of inner-shell electrons by lepton impact is the fundamental atomic process of particular interest [1, 2, 3, 4, 5, 6]. In the present paper, which is a further extension of our previous works [7, 8], we deduce the universal scaling behavior of differential cross sections for the single K-shell ionization of hydrogen-like multicharged ions by electron or positron impact. Special emphasis is laid on the energy domain near the ionization threshold, where accurate description of the electron-electron and electron-nucleus interactions is extremely significant. The study is performed to leading order of non-relativistic impact. Accordingly, the Coulomb functions are employed as electron wave functions in a zeroth approximation.

II. IONIZATION OF HYDROGEN-LIKE IONS

A. Electron impact

Let us consider first the inelastic electron scattering on hydrogen-like ion in the ground state, which results in ionization of a K-shell bound electron. We shall derive formulas for the differential cross sections of the process. An incident particle can be characterized by the energy $E = p^2/(2m)$ and the momentum $p$ at asymptotically large distances from the nucleus. We focus on the non-relativistic energies $E$ within the range $I \lesssim E \ll m$, where $I = \eta^2/(2m)$ is the ionization potential, $\eta = m\alpha Z$ is the average momentum of a K shell electron, and $m$ is the electron mass.

The process under consideration is described by the Feynman diagrams depicted in Fig. 1. In the final continuum state, the electron wave functions are denoted as $\psi_{p_1}$ and $\psi_{p_2}$. The energy conservation implies $E - I = E_1 + E_2$, where $E_1 = p_1^2/(2m)$ and $E_2 = p_2^2/(2m)$ are the energies of scattered and ejected electrons. In the single ionization by low-energy particle impact, the emission of electrons occurs with arbitrary energy sharing. In this case, both diagrams depicted in Figs. 1(a) and (b) give comparable contributions to the ionization cross section and should be taken into account. Accordingly, the total amplitude of the process reads

$$A = A_d \delta_{\tau_{1,2}} \delta_{\bar{\tau}_{1,2}} - A_b \delta_{\tau_{1,2}} \delta_{\bar{\tau}_{1,2}},$$

$$A_d = \langle \psi_{p_1} \psi_{p_2} | V_C | \psi_{p_1} \psi_{1s} \rangle,$$

$$A_b = \langle \psi_{p_2} \psi_{p_1} | V_C | \psi_{p_1} \psi_{1s} \rangle.$$

Here $\tau_{1,2}$ and $\bar{\tau}_{1,2}$ denote the spin projections of the Pauli spinors in the initial and final states, respectively. The Coulomb interaction between two electrons is described by the operator $V_C(r_{1,2}) = \alpha |r_{1,2}|^{-1}$, where $r_{1,2}$ are the electron coordinates. The amplitude $A_d$ corresponds to the direct diagram, while the amplitude $A_b$ is due to the exchange diagram.

As the wave functions of initial particles, we shall take the corresponding solutions of the Schrödinger equation for an electron
in the external field of the Coulomb source \([9]\)

\[
\psi_{1s}(r) = N_{1s} e^{-\eta r},
\]

\[
\psi_{p}(r) = \frac{4\pi}{2p} \sum_{l=0}^{\infty} i^l e^{il(\pm)} R_{pl}^{(-)}(r) \sum_{m=-l}^{l} Y_{lm}(\hat{r}) Y^*_{lm}(\hat{p}).
\]

Here \(N_{1s} = \eta^3 / \pi, \eta = m\alpha Z\) \([10]\), \(Y_{lm}(\hat{r})\) are the spherical harmonics, which depend on the variable \(\hat{r} = r / r, \) and \(\delta_{pl}^{(-)}\) are the phase shifts of the radial functions \(R_{pl}^{(-)}\). The latter are orthogonal and normalized according to

\[
\int_0^\infty dr r^2 R_{p'l}^{(-)}(r) R_{pl}^{(-)}(r) = 2\pi \delta(p' - p).
\]

The asymptotical behavior of the wave function \(\psi_{p}(r)\) is “the sum of a plane wave and a spherically outgoing wave”. The functions \([5]\) are normalized by the condition

\[
\int dr \psi_{p}^* \psi_{p}(r) = (2\pi)^3 \delta(p' - p).
\]

For the Coulomb field of a point nucleus, one has \([9]\)

\[
R_{pl}^{(+)}(r) = \frac{C_{pl}^{(+)}(2l+1)!}{(2\pi)^l e^{-ipr}} \Phi(l + 1 \mp i\xi, 2l + 2, 2ipr),
\]

\[
C_{pl}^{(+)} = 2pe^{\mp \pi\xi / 2} |\Gamma(l + 1 \pm i\xi)|,
\]

\[
\delta_{pl}^{(+)} = \arg \Gamma(l + 1 \pm i\xi),
\]

where \(\xi = \eta / p, \Phi(x, y, z)\) is the confluent hypergeometric function, and \(\Gamma(z)\) is the Euler’s gamma function. In Eqs. \((8)-(10)\), the lower (upper) sign corresponds to the attraction (repulsion).

Let us assume that on experiment we are interested in the asymptotic momentum of one outgoing electron only (for example, \(p_1\)). Then the wave function of this electron can be represented in terms of the partial-wave decomposition

\[
\psi_{p_1}(r) = \frac{4\pi}{2p_1} \sum_{l_1, m_1} i^{l_1} e^{-i\delta_{p_1 l_1}^{(-)}} R_{p_1 l_1}^{(-)}(r) Y_{l_1 m_1}(\hat{r}) Y^*_{l_1 m_1}(\hat{p_1}),
\]

which behaves asymptotically as “a plane wave plus a spherically converging wave”. The functions \((11)\) are normalized by the same condition \((7)\).

As a wave function of another outgoing electron, we shall take the wave function of the stationary state characterized by the definite values of the energy \(E_2\), the angular momentum \(l_2\), and its projection \(m_2\), namely,

\[
\psi_{p_2}(r) = R_{E_2 l_2}^{(-)}(r) Y_{l_2 m_2}(\hat{r}).
\]

The radial functions \(R_{E_l}^{(+)}(r)\) are normalized to \(\delta\) function in the energy

\[
\int_0^\infty dr r^2 R_{E_l}^{(+)}(r) R_{E_l}^{(+)}(r) = \delta(E' - E),
\]

being related to \(R_{pl}^{(+)}(r)\) as follows

\[
R_{E_l}^{(+)}(r) = \frac{\sqrt{m}}{\sqrt{2\pi} p} R_{pl}^{(+)}(r).
\]

The partial-wave decomposition of the Coulomb interaction \(V_C\) is given by

\[
V_C(r_1, r_2) = \sum_{\lambda=0}^{\infty} \frac{4\pi\alpha}{(2\lambda + 1)!} r_{1>\lambda}^\lambda \sum_{\mu=-\lambda}^\lambda Y_{\lambda\mu}(\hat{r_1}) Y_{\lambda\mu}(\hat{r_2}),
\]
where \( r_\prec = \min\{r_1, r_2\} \) and \( r_\succ = \max\{r_1, r_2\} \).

Choosing the z-axis along the momentum \( \mathbf{p} \), we can perform the integrations over the angular variables \( \hat{r}_1 \) and \( \hat{r}_2 \) in the matrix elements (2) and (3). It yields

\[
\mathcal{A}_a = \frac{2\pi\alpha\sqrt{2\pi m}}{\eta^2} \sqrt{pp_1} \sum_{l_1} e^{i\Delta_{l_1}} W_{l_1l_2} C_{l_1m_1l_2m_2} Y_{l_1m_1}(\hat{p}_1),
\]

\[
\mathcal{A}_b = \frac{2\pi\alpha\sqrt{2\pi m}}{\eta^2} \sqrt{pp_1} \sum_{l_1} e^{i\Delta_{l_1}} V_{l_1l_2} C_{l_1m_1l_2m_2} Y_{l_1m_1}(\hat{p}_1),
\]

\[
W_{l_1l_2} = \frac{1}{\sqrt{\pi}kk_1k_2} \Pi_{l_1} C_{l_1l_2m_1m_2}^{l_1},
\]

\[
V_{l_1l_2} = \frac{1}{\sqrt{\pi}kk_1k_2} \Pi_{l_1} C_{l_1l_2m_1m_2}^{l_1},
\]

\[
I_{l_1l_2} = \int_0^{\infty} dx_1 x_1^2 R_{k_1l_1}^{-}(x_1) R_{k_2l_2}^{(-)}(x_1) \int_0^{\infty} dx_2 x_2^2 R_{k_2l_2}^{(2)}(x_2) \frac{x_2^2}{x_2^2 + 1} e^{-x_2},
\]

\[
J_{l_1l_2} = \int_0^{\infty} dx_1 x_1^2 R_{k_1l_1}^{(-)}(x_1) R_{k_1l_1}^{(2)}(x_1) \int_0^{\infty} dx_2 x_2^2 R_{k_2l_2}^{(2)}(x_2) \frac{x_2^2}{x_2^2 + 1} e^{-x_2},
\]

where \( \Delta_{l_1} = \delta_{pl_1} + \delta_{p_1l_1} + \pi (l_1 - l_2)/2, \Pi_{l_1} = \sqrt{2l_1 + 1}, \ x_\prec = \min\{x_1, x_2\}, \ x_\succ = \max\{x_1, x_2\} \), and \( C_{l_1m_1l_2m_2}^{l_1} \) denotes the Clebsch-Gordan coefficient. In Eqs. (18)–(21), we have introduced the dimensionless momenta \( k = p/\eta, k_1 = p_1/\eta \) and the dimensionless coordinates \( x_i = \eta r_i, (i = 1, 2) \). Accordingly, the radial functions (5) satisfy to the relation \( R_{l_1}^{(-)}(r) = \eta R_{k_1l_2}^{(-)}(x) \).

Due to the identity of electrons, the functions \( V_{l_1l_2} \) can be obtained from \( W_{l_1l_2} \) by simultaneous substitutions \( k_1 = k_2 \) and \( l_1 = l_2 \).

The differential cross section for ionization of a K-shell electron is related to the amplitude (1) as follows

\[
d\sigma_K^+ = \frac{2\pi}{v} \sum_{l_2, m_2} |\mathcal{A}|^2 \frac{dp_1}{(2\pi)^3} dE_2 \delta(E_1 + E_2 + I - E),
\]

where \( v = p/m \) is the absolute magnitude of velocity of the incident particle. The summations are performed over the angular moment of the second electron, because its state is not fixed. Equation (22) defines distributions over the energy and scattering angle. The element of the phase volume for electrons scattered into the solid angle \( d\Omega \) can be written as

\[
dp_1 = mp_1 dE_1 d\Omega.
\]

In the case of unpolarized particles, Eq. (22) should be averaged over polarizations of the initial electrons and summed over polarizations of the final electrons. This can be achieved by means of the following substitution

\[
|\mathcal{A}|^2 \rightarrow \overline{|\mathcal{A}|^2} = \frac{1}{4} \sum_{\tau_1, \tau_2} \sum_{\tau_3, \tau_4} |\mathcal{A}|^2.
\]

Then the differential cross section takes the form

\[
\frac{d\sigma_K^+}{dE_1 d\Omega} = \frac{m^2}{(2\pi)^2} \frac{p_1}{p} \sum_{l_2, m_2} \left( |\mathcal{A}_a|^2 + |\mathcal{A}_b|^2 - \frac{1}{2} (\mathcal{A}_a^* \mathcal{A}_b^* + \mathcal{A}_a \mathcal{A}_b^*) \right). \tag{25}
\]

Let us introduce the dimensionless energies \( \varepsilon = E/I \) and \( \varepsilon_i = E_i/I, (i = 1, 2) \), where \( I = \eta^2/(2m) \) is the ionization potential for the K-shell electron. The energy-conservation law now reads \( \varepsilon - 1 = \varepsilon_1 + \varepsilon_2 \), where \( \varepsilon = k^2 \) and \( \varepsilon_i = k_i^2, (i = 1, 2) \). Inserting Eqs. (16) and (17) into Eq. (25) yields

\[
\frac{d\sigma_K^+}{d\varepsilon_1 d\Omega} = \frac{\sigma_0}{Z^2} \sum_{l_1, l_2} \sum_{l_1, l_2, m_2} \cos(\Delta_{l_1}^{l_1}) T_{l_1l_2}^{l_1l_2} C_{l_1m_1l_2m_2} C_{l_1m_1l_2m_2}^* Y_{l_1m_1}(\hat{p}_1) Y_{l_2m_2}^*(\hat{p}_1), \tag{26}
\]

\[
T_{l_1l_2}^{l_1l_2} = W_{l_1l_2}^{l_1l_2} V_{l_1l_2}^{l_1l_2} + V_{l_1l_2}^{l_1l_2} V_{l_1l_2}^{l_1l_2} - \frac{1}{2} \left( W_{l_1l_2}^{l_1l_2} V_{l_2l_1}^{l_1l_2} + V_{l_2l_1}^{l_1l_2} W_{l_1l_2}^{l_1l_2} \right). \tag{27}
\]
Here \( \Delta_{\mu_1}^\prime = \Delta_{\mu_1} - \Delta_{\nu_1}^\prime \) and \( \sigma_0 = \pi a_0^2 \) is 87.974 Mb, where \( a_0 = 1/(m\alpha) \) is the Bohr radius.

In Eq. (26), the summation over the projection \( m_2 \) can be easily performed using the following relation [11]

\[
\sum_{m_2} C_{l_1 m_1 l_2 m_2}^{l_1}\dot{C}_{l_1 m_1 l_2 m_2}^{l_1} Y_{l_1 m_1}(\hat{p}_1) Y_{l_2 m_2}^{*}(\hat{p}_1) = \frac{1}{4\pi} (-1)^{l_2} \Pi_{l_1, l_2} \theta_{l_1, l_2} \left\{ l_1 \quad l_2 \quad l \right\}. \tag{28}
\]

Here \( \Pi_{l_1 l_2} = \sqrt{(2l_1 + 1)(2l_2 + 1)\cdots} \). \( P_L(x) \) denotes the Legendre polynomial of order \( L \), and the standard notation for the \( 6j \)-symbol is used. The scattering angle \( \theta_1 \) is enclosed by the vectors \( p \) and \( p_1 \). The formula (28) possesses the axial symmetry with respect to the direction \( p \) of incoming particles. Accordingly, the solid angle \( d\Omega_1 \) is given by \( d\Omega_1 = 2\pi \sin \theta_1 d\theta_1 \).

Taking into account Eq. (28), the differential cross section can be cast into the following form

\[
\frac{d\sigma_1^\pm}{d\varepsilon_1 d\Omega_1} = \frac{\sigma_0}{4\pi Z^4} G(\varepsilon, \varepsilon_1, \theta_1), \quad \tag{29}
\]

\[
G(\varepsilon, \varepsilon_1, \theta_1) = F(\varepsilon, \varepsilon_1) + \sum_{L \geq 1} F_L(\varepsilon, \varepsilon_1) P_L(\cos \theta_1), \quad \tag{30}
\]

\[
F(\varepsilon, \varepsilon_1) = \frac{1}{\varepsilon} \sum_{l_1, l_2} T_{l_1 l_2}^{l_1 l_2} = \frac{1}{\varepsilon} \sum_{l_1, l_2} \left\{ (W_{l_1 l_2}^{l_1 l_2})^2 + (V_{l_1 l_2}^{l_1 l_2})^2 - W_{l_1 l_2}^{l_1 l_2} V_{l_1 l_2}^{l_1 l_2} \right\}, \quad \tag{31}
\]

\[
F_L(\varepsilon, \varepsilon_1) = \frac{1}{\varepsilon} \sum_{l_1, l_2} \sum_{l_1, l_2} (-1)^L \cos(\Delta_{l_1}^{l_1 l_2} T_{l_1 l_2}^{l_1 l_2} \Pi_{l_1 l_2}^{l_1 l_2} C_{l_1 l_2}^{l_1 l_2} C_{l_1 l_2}^{l_1 l_2} \left\{ l_1 l_2 l \right\}). \quad \tag{32}
\]

The angular dependence of the cross section (29) is governed by the Legendre polynomials \( P_L(\cos \theta_1) \). The functions (30–32) are universal, being independent of the nuclear charge \( Z \). The energy \( \varepsilon_1 \) of outgoing electrons lies within the range \( 0 \leq \varepsilon_1 \leq \varepsilon - 1 \). The limiting values of \( \varepsilon_1 = 0 \) and \( \varepsilon_1 = \varepsilon - 1 \) correspond to the situation, when one of the electrons in the final state is infinitely slow. The function (32) coincides with the function (31) in the particular case, if \( L = 0 \).

In Figs. 2 and 3 the universal functions \( G(\varepsilon, \varepsilon_1, \theta_1) \) are calculated for different energies \( \varepsilon \) of incident electrons. One can observe several qualitative features in behavior of the universal curves within the near-threshold energy domain. For very slow collisions, the backward (\( \theta_1 \approx \pi \)) electron scattering is more probable than the forward (\( \theta_1 \approx 0 \)) scattering. In particular, for \( \varepsilon \lesssim 1.1 \), this occurs for both slow \( [\varepsilon_1 \leq (\varepsilon - 1)/2] \) and fast \( [\varepsilon_1 \geq (\varepsilon - 1)/2] \) electrons. For \( \varepsilon \approx 1.2 \), the cross sections for backward and forward scattering become to be of the comparable magnitude. The slowest electrons \( (\varepsilon_1 \approx 0) \) are scattered predominantly backward, while the fastest electrons \( (\varepsilon_1 \approx \varepsilon - 1) \) are scattered mainly forward. With increasing incident energies \( (\varepsilon \geq 1.5) \), the dominant contribution to the ionization cross section arises from the fast outgoing electrons, which are scattered forward at small angles \( \theta_1 \). The backward scattering turns out to be increasingly suppressed. In addition, the angular distribution of the slow electrons becomes to be more isotropic.

Integrating Eq. (29) over the solid angle \( d\Omega_1 \) yields the energy distribution for outgoing electrons

\[
\frac{d\sigma_1^\pm}{d\varepsilon_1} = \frac{\sigma_0}{4\pi} F(\varepsilon, \varepsilon_1), \quad \tag{33}
\]

where \( F(\varepsilon, \varepsilon_1) \) is given by Eq. (31). This universal function has been already studied in the entire non-relativistic energy domain [7].

B. Positron impact

Let us now consider the ionization of a K-shell bound electron due to the inelastic positron scattering. As in the case of the electron impact, the incident positron can be characterized by the energy \( E = p^2/(2m) \) and the asymptotic momentum \( p \), while the scattered positron is characterized by the energy \( E_1 = p_1^2/(2m) \) and the momentum \( p_1 \). The energy-conservation law keeps the same form, namely, \( E - E_1 = E_2 + E_1 \), where \( E_2 = p_2^2/(2m) \) denotes the energy of ejected electron. Since the interacting particles are not identical, the exchange effect is absent. Accordingly, the ionization process is represented by the diagram depicted in Fig. 1(a) only. The differential cross section, which describes the universal energy and angular distributions for outgoing positrons, is given by the same formulas (29)–(32), where the function \( T_{l_1 l_2}^{l_1 l_2} \) contains only the first term on the right-hand side of Eqs. (27) and (31). In Eqs. (3)–(10), which correspond to the Coulomb wave functions of the incident and
scattered positrons, one needs to employ the case of repulsive field of the atomic nucleus (upper signs). In particular, the phase shift \( \Delta_{t1}^{\ell_1} \) reads now as follows: \( \Delta_{t1}^{\ell_1} = \Delta_{t1} - \Delta_{t1}^{\ell_1} \), where \( \Delta_{t1} = \delta_{\text{p}_1}^{(+)\text{p}_1} + \delta_{\text{p}_1}^{(-)\text{p}_1} + \pi(l - l_1)/2 \).

In Figs. 4 and 5, the universal energy and angular distributions for outgoing positrons are calculated for few values of the dimensionless energy \( \varepsilon \). Although the scattered positron can have any energy within the range \( 0 \leq \varepsilon_1 \leq \varepsilon - 1 \), the dominant contribution to the ionization cross section arises from the fast positrons with the energies \( \varepsilon_1 \), which are close enough to the excess energy \( \varepsilon - 1 \). For slow collisions with \( \varepsilon \lesssim 1.7 \), the differential cross section contains three pronounced maxima at different scattering angles \( \theta_1 \). With increasing the incident energies up to \( \varepsilon \sim 2 \), the maxima coalesce near the zeroth angle, so that the positrons are preferably scattered in the forward cone (\( \theta_1 < \pi/2 \)). With further increasing the incident energies \( \varepsilon \), the angular distribution of the fast outgoing positrons exhibits weak interference oscillations with increasing frequency.

The differential cross section, which describes the energy and angular distributions of the electrons ejected by positron impact, is given by

\[
\frac{d\sigma^+_{\ell_1}}{d\varepsilon_2 d\Omega_2} = \frac{\sigma_0}{4\pi Z^2} G(\varepsilon, \varepsilon_2, \theta_2), \tag{34}
\]

\[
G(\varepsilon, \varepsilon_2, \theta_2) = F(\varepsilon, \varepsilon_2) + \sum_{L \geq 1} F_L(\varepsilon, \varepsilon_2) P_L(\cos \theta_2), \tag{35}
\]

\[
F(\varepsilon, \varepsilon_2) = \frac{1}{\varepsilon} \sum_{l, l_1, l_2} T_{\ell_1 \ell_2}^{\ell_2} = \frac{1}{\varepsilon} \sum_{l, l_1, l_2} (W^{l}_{l_1 l_2})^2, \tag{36}
\]

\[
F_L(\varepsilon, \varepsilon_2) = \frac{1}{\varepsilon} \sum_{l, l_1, l_2} \sum_{l', l_2} (-1)^{l_1} \cos(\Delta'_{l_1}^{l_2}) T_{l_1 l_2, l' l_2}^{\ell_2} \Pi_{l_1}^{l_2} \Pi_{l_1}^{\ell_2} C_{l_1 l_2 0}^{l_2 0} C_{l_1 l_2 0}^{l_2 0} \left\{ l_2 \quad l_1 \quad l \quad l' \right\}, \tag{37}
\]

\[
T_{l_1 l_2}^{\ell_2} = W_{l_1 l_2}^{l_2} W_{l_1 l_2}^{l_2}, \quad W_{l_1 l_2}^{l_2} = \frac{1}{\sqrt{\pi k_{k_1} k_2}} \sum_{l_1} \Pi_{l_1}^{l_2} C_{l_1 l_2 0}^{l_2 0} I_{l_1 l_2}, \tag{38}
\]

\[
I_{l_1 l_2} = \int_0^\infty dx_1 x_1^2 R^{l_1^{(+)\text{p}_1}}(x_1) R^{l_2^{(+)\text{p}_2}}(x_2) \int_0^\infty dx_2 x_2^2 R^{l_2^{(-)\text{p}_2}}(x_2) \frac{\delta_{x_2 - x_1}^{(l - l_2)}}{x_2^{l_2 + 1}}, \tag{39}
\]

Here \( \sigma_0 = \pi a_0^2, a_0 = 1/(m\alpha) \), and \( \Delta'_{l_1}^{l_2} = \Delta_{l_2} - \Delta_{l_1}^{l_2} \), where \( \Delta_{l_2} = \delta_{\text{p}_1}^{(+)\text{p}_1} + \delta_{\text{p}_1}^{(-)\text{p}_1} + \pi(l - l_2)/2 \). The solid angle reads \( d\Omega_2 = 2\pi \sin \theta_2 d\theta_2 \), where the ejection angle \( \theta_2 \) is enclosed by the asymptotic momenta \( p_1 \) and \( p_2 \). The function \( \Pi_0 \) is reduced to the function \( \Pi_1 \), if \( L = 0 \). The energy \( \varepsilon_2 \) of ejected electrons lies within the range \( 0 \leq \varepsilon_2 \leq \varepsilon - 1 \).

In Figs. 6 and 7, the universal function \( F(\varepsilon, \varepsilon_2) \) is calculated for different values of the dimensionless energy \( \varepsilon \) of incident positrons. Within the near-threshold energy domain, the electron emission occurs preferably at small angles \( \theta_2 \simeq 0 \), although for \( \varepsilon \lesssim 1.3 \), the differential cross section exhibits also a weak maximum at \( \theta_2 = \pi \). The total cross section is exhausted within the range of small energies \( \varepsilon_2 \) and small angles \( \theta_2 \). With increasing the incident energies \( \varepsilon \), the relative amount of slow electrons ejected at arbitrary angles \( \theta_2 \) is growing.

Integrating Eq. (34) over the solid angle \( d\Omega_2 \) yields the energy distribution for outgoing electrons

\[
\frac{d\sigma^+_{\ell_1}}{d\varepsilon_2} = \frac{\sigma_0}{4Z^2} F(\varepsilon, \varepsilon_2), \tag{40}
\]

where \( F(\varepsilon, \varepsilon_2) \) is given by Eq. (35). This universal function has been already studied in the entire non-relativistic energy domain \( \varepsilon \). Note also that, for any incident positron energy \( \varepsilon \), the universal function \( F(\varepsilon, \varepsilon_1) \), which describes the energy distribution for scattered positrons, is symmetrical to the function \( F(\varepsilon, \varepsilon_2) \) with respect to the vertical axis crossing the energy interval in the middle point \( \varepsilon_1 = \varepsilon_2 = (\varepsilon - 1)/2 \).

### III. GENERALIZATION TO ARBITRARY ATOMIC TARGET AND CONCLUSIONS

Equations (29), (33), (34), and (40) describe the single ionization of hydrogen-like ions in the ground state. However, due to universality of the scaling behavior, these formulas can be easily generalized on the case of arbitrary non-relativistic atomic targets, in which the K shell is completely occupied. Firstly, the ionization cross sections should be multiplied by a factor 2, taking into account the number of K-shell electrons. Secondly, one needs to simulate the screening effect of the passive electrons on the active K-shell electron, participating in the ionization process. This can be achieved by substitution of the true nuclear charge \( Z \) by the corresponding effective value \( Z_{\text{eff}} \), which is defined via (12)

\[
I_{\exp} = \frac{m}{2} (\alpha Z_{\text{eff}})^2, \tag{41}
\]
where $I_{\text{exp}}$ is the experimentally observable threshold for the single K-shell ionization. Accordingly, the energies of incident and outgoing particles should be calibrated in units of the experimental value $I_{\text{exp}}$, that is, $\varepsilon = E/I_{\text{exp}}$ and $\varepsilon_i = E_i/I_{\text{exp}}$, ($i = 1, 2$). The universal functions (30) and (35) depicted in Figs. 2–7 keep the same scaling behavior for non-relativistic atomic targets with arbitrary nuclear charge $Z \gg 1$.

Concluding, we have deduced the universal scaling behavior of differential cross sections for the single K-shell ionization by electron and positron impact. The results are obtained within the framework of non-relativistic perturbation theory, taking into account the one-photon exchange diagrams. The universal scaling laws can be applied for both multicharged ions and neutral atoms with moderate values of the nuclear charge number $Z$. The interference oscillations in doubly differential cross sections for inelastic scattering of low-energy positrons deserve further experimental verification.

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![Feynman diagrams for ionization of the K-shell electron by an electron impact. Solid lines denote electrons in the Coulomb field of the nucleus, while dashed line denotes the electron-electron Coulomb interaction.](image)
FIG. 2: The universal function (30) is calculated for different values of the dimensionless energy $\varepsilon$ of the incident electron. The variable $\varepsilon_1$ is the energy of the outgoing electron, which is detected at the angle $\theta_1$. The center point $\varepsilon_1 \approx (\varepsilon - 1)/2$ corresponds to the equal-energy sharing ($\varepsilon_1 = \varepsilon_2$).
FIG. 3: The universal function \( G(\varepsilon, \varepsilon_1, \theta_1) \) is calculated for different energies of incident and outgoing electrons: dotted line, \( \varepsilon_1 = 0 \); dashed line, \( \varepsilon_1 = 0.25(\varepsilon - 1) \); dash-dotted line, \( \varepsilon_1 = 0.5(\varepsilon - 1) \); dash-dot-dotted line, \( \varepsilon_1 = 0.75(\varepsilon - 1) \); solid line, \( \varepsilon_1 = \varepsilon - 1 \).
FIG. 4: The universal function \( G(\epsilon, \epsilon_1, \theta_1) \) is calculated for different energies \( \epsilon \) of incident positrons. The variable \( \epsilon_1 \) is the energy of outgoing positron, which is scattered at the angle \( \theta_1 \) with respect to direction of the asymptotic momentum \( p \).
FIG. 5: The universal function is calculated for different energies of incident and outgoing positrons: dotted line, $\varepsilon_1 = 0.8(\varepsilon - 1)$; dashed line, $\varepsilon_1 = 0.9(\varepsilon - 1)$; solid line, $\varepsilon_1 = \varepsilon - 1$. 
FIG. 6: The universal function \( G(\varepsilon, \varepsilon_2, \theta_2) \) is calculated for different energies \( \varepsilon \) of incident positrons. The variable \( \varepsilon_2 \) is the energy of outgoing electron, which is ejected at the angle \( \theta_2 \) with respect to direction of the asymptotic momentum \( p \).
FIG. 7: The universal function \( G(\varepsilon, \varepsilon_2, R) \) is calculated for different energies of incident positrons and outgoing electrons: dash-dotted line, \( \varepsilon_2 = 0.3(\varepsilon - 1) \); dotted line, \( \varepsilon_2 = 0.2(\varepsilon - 1) \); dashed line, \( \varepsilon_2 = 0.1(\varepsilon - 1) \); solid line, \( \varepsilon_2 = 0 \).
[1] Byron Jr F W and Joachain C J 1989 Phys. Rep. 179 211
[2] Ray H, Werner U and Roy A C 1991 Phys. Rev. A 44 7834
[3] Knudsen H and Reading J F 1992 Phys. Rep. 212 101
[4] Nagashima Y, Saito F, Itoh Y, Goto A and Hyodo T 2004 Phys. Rev. Lett. 92 223201
[5] Dürr M, Dimopoulou C, Najjari B, Dorn A and Ullrich J 2006 Phys. Rev. Lett. 96 243202
[6] DuBois R D, de Lucio O G and Gavin J 2008 Nucl. Instrum. Methods B 266 397
[7] Mikhailov A I, Nefiodov A V and Plunien G 2008 Phys. Lett. A 372 4451
[8] Mikhailov A I, Nefiodov A V and Plunien G 2008 Phys. Lett. A 372 5171
[9] Landau L D and Lifshitz E M 1991 Quantum Mechanics: Non-Relativistic Theory (Oxford: Pergamon Press)
[10] The use of the same notations for the electron mass and projection of the orbital angular momentum is not confusing, because the latter is related with the angular dependence of wave functions only.
[11] Varshalovich D A, Moskalev A N and Khersonskii V K 1988 Quantum Theory of Angular Momentum (Singapore: World Scientific)
[12] Mikhailov A I, Mikhailov I A, Nefiodov A V, Plunien G and Soff G 2003 Zh. Eksp. Teor. Fiz. Pis’ma 78 141
    Mikhailov A I, Mikhailov I A, Nefiodov A V, Plunien G and Soff G 2003 JETP Lett. 78 110