Condition for transition to an unstable state (necking) of a specimen in tension

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Abstract. On the stress-strain curve obtained in simple tension experiments, there is a region of instability due to the neck formation. In the theory of plasticity, conditions for the transition to an unstable state and appearance of the maximum point on the stress-strain diagram are determined. These condition were derived under the assumption of the material incompressibility. However, this assumption cannot be justified, since in the neck region, there are numerous damages (pores, micro-cracks), i.e., the material is compressible. With the current coefficient of lateral deformation, the authors formulate a condition for transition to an unstable state in a compressible plastic medium.

1. Introduction
The basic mechanical properties of materials are determined, in particular, in simple tension experiments at room temperature and constant strain rate.

The results of measurements of the force, the current length and diameter of the specimen were used to calculate the values of the true stress \( \sigma = P/F = \sigma_0 F_0/F \), the relative \( \varepsilon_0 = (l - l_0)/l_0 \) and the logarithmic strains \( \varepsilon = \ln l/l_0 \) (\( P \) is force, \( \sigma_0 = P/F_0 \) is nominal stress, \( l_0 \) and \( F_0 \) are initial, \( l \) and \( F \) are the current length and the cross-section area of the specimen). A typical stress-strain curve received in a tension experiment is shown in figure 1.

If we neglect the change of the cross-section area of the specimen during the deformation, the stress-strain curves are constructed in the nominal stress–relative strain coordinates \( \sigma_0 - \varepsilon \).

In the case of large deformations (over twenty percent), the logarithmic deformation is used. At point \( M \) in figure 1, the nominal stress attains the maximum, a neck is formed on the specimen and the deformation becomes unstable. The drop-down part of the curve \( \sigma_0 - \varepsilon \) is a result of a sharp decrease in the cross-section area of the specimen due to the necking. In the neck, the stresses and strains become non-uniform. The drop-down part of the nominal stress-strain curve is a consequence of the decrease in the load and not in the stress. According to the true stress-strain curve, the true stress does continue to rise until the fracture is reached. The metal continues to work-harden.

2. Condition for attaining the maximum in the stress-strain curve for incompressible and compressible plastic media
In the literature [1–5], the conditions for attaining the maximum at point \( M \) (figure 1) are determined under the incompressibility condition \( l_0 F_0 = l F \), and then \( P = \sigma F = \sigma_0 F_0 e^{-\varepsilon} \).
Figure 1. Stress–strain diagrams. Solid line corresponds to tension diagram plotted in true stresses and the dotted line is the diagram plotted in engineering stresses.

Differentiating the last expression with respect to $\varepsilon$, we have

$$\frac{dP}{d\varepsilon} = F_0 e^{-\varepsilon} \left( \frac{d\sigma}{d\varepsilon} - \sigma \right).$$

(1)

At the maximum point on the nominal stress-strain curve, the condition $dP/d\varepsilon = 0$ is satisfied and (1) implies the ratio

$$\frac{d\sigma}{d\varepsilon} = \sigma,$$

(2)

which determines a condition for transition to an unstable state and attainment of maximum at point $M$ (figure 1).

Thus, relations (2) are derived under the incompressibility condition, which implies that the plastic deformation occurs by the viscous flow mechanism without formation of defect structures. At the same time, in real materials, for example, in metals, during the plastic deformation, particularly in the region of instability, there arise numerous defects (pores, cracks), and so the assumption of incompressibility cannot be considered reasonable in the general case.

We determine the compressibility condition using the current coefficient of the lateral deformation $\nu$: $\nu = -\varepsilon_y/\varepsilon_x = -\varepsilon_z/\varepsilon_x$ ($\varepsilon_x = \ln(l/l_0)$ is the longitudinal deformation and $\varepsilon_y = \varepsilon_z = \ln(R/R_0)$ are the transverse deformations of a cylindrical specimen, $R_0$ is the initial radius, and $R$ is the current radius of the specimen cross-section). In the elastic region, this coefficient is constant $\nu = \nu_0$ ($\nu_0$ is the Poisson ratio).

With the transition to the plastic state, the current coefficient of the lateral deformation tends to $\nu = 0.5$, so we can assume that the material comes into the state of incompressibility. With further loading, the value $\nu$ decreases, which indicates a sharp increase in the irreversible bulk strain (loosening [6]) or a decrease in the density [7], which is characteristic of the pre-failure stage. To get to this point, the following two properties occur simultaneously: the process of destruction and the process of plastic deformation.

If the increase in tensile stresses is accompanied by a decrease in the current coefficient of the lateral deformation, then it is possible to assume that the destruction processes prevail over the processes of flow. Therefore, in formulating the condition of the specimen transition to an unstable state, it is necessary to consider the processes of loosening and, consequently, the compressibility of the material. These processes can be taken into account with the current...
coefficient of the lateral deformation

\[ \nu = -\frac{\ln(R/R_0)}{\ln(l/l_0)}. \] (3)

From (3) it follows that \( R_0/R = (l/l_0)^{2\nu} \). Taking into account the obvious relation 
\[ \frac{F_0}{F} = (R_0/R)^2, \] we obtain [7]

\[ \frac{F_0}{F} = \left( \frac{l}{l_0} \right)^{2\nu}. \] (4)

Taking into account (4), we have

\[ P = \sigma F_0 e^{-2\nu \varepsilon}. \] (5)

Differentiating (5) with respect to \( \varepsilon \), we obtain

\[ \frac{dP}{d\varepsilon} = F_0 e^{-2\nu \varepsilon}\left( \frac{d\sigma}{d\varepsilon} - 2\sigma \nu \frac{d\varepsilon}{d\varepsilon} - 2\sigma \nu \right). \] (6)

At the maximum point on the stress-strain curve \( dP/d\varepsilon = 0 \), it follows from (6) that

\[ \frac{d\sigma}{d\varepsilon} = 2\sigma \nu + 2\sigma \varepsilon \frac{d\nu}{d\varepsilon}. \] (7)

For \( d\nu/d\varepsilon = 0 \), relation (7) implies the maximum condition

\[ \frac{d\sigma}{d\varepsilon} = 2\nu \sigma. \] (8)

For an incompressible material, \( \nu = 0.5 \) and relation (8) coincides with formula (2). According to (8), the position of the maximum value on the stress-strain curve varies depending on the state of the material. Thus, the transition to instability with necking corresponds to the transition of a material from an incompressible state to a compressible state at the maximum point \( K \) on the \( \nu-\varepsilon \) curve.

In the case of elastic-plastic media, \( d\varepsilon = d\varepsilon^e + d\varepsilon^p \) (\( \varepsilon^e \) and \( \varepsilon^p \) are components of the elastic and plastic deformation, \( \varepsilon^e = \sigma/E, \) and \( E \) is the Young modulus). The relation between stress and deformation can be determined by the equation

\[ \frac{dl}{l} = \frac{d\sigma}{E} + \varphi(\sigma) d\sigma. \] (9)

Integrating equation (9), we obtain

\[ \varepsilon = \frac{\sigma}{E} + \int_0^{\sigma} \varphi(\sigma) d\sigma. \] (10)

In the general case, substituting the relation \( \sigma = \sigma_0 e^{2\nu \varepsilon} \) in (9), we obtain an equation written in terms of the lateral deformation \( \nu \). For different values of \( \nu \), we can plot nonmonotone \( \sigma_0-\varepsilon \) diagrams, and thus, can describe the experimental curves in engineering stresses-strain coordinates for metallic materials. Further, this approach is applied for the case of rigid-plastic Ludwig medium with nonlinear hardening

\[ \sigma = \sigma_T + b \varepsilon^m, \] (11)

where \( \sigma_T \) is the yield stress and \( b, m \) are constants.

Let us write equation (11) in terms of \( \sigma_0 \) (see figure 2)

\[ \sigma_0 = (\sigma_T + b \varepsilon^m) e^{-2\nu \varepsilon}. \] (12)
Conclusions
In the mechanics of materials, the incompressibility condition (Hill, Nadai, Kachanov, etc.) is accepted in deriving conditions for a specimen transition to a state of instability and neck formation in tension with a constant speed. At the same time, under active loading, there are interrelated processes of plastic deformation and fracture. In the neck area, numerous defects (pores, cracks) are formed and the material passes into a state of compressibility. The compressibility effect is determined by the current coefficient of transverse deformation. The condition of transition to an unstable state is formulated with regard to the change in this coefficient. An analytical stress-strain relation for a rigid-plastic medium with nonlinear hardening is obtained and the corresponding theoretical curves are constructed for different values of the current coefficient of transverse deformation. In particular, it is shown that, for a compressible material, the maximum point is displaced in the region of instability. In the case of transition to an unstable state for an incompressible medium, there is a fixed maximum point on the stress-strain curve.

Acknowledgments
This work was supported by the Russian Scientific Foundation (Grant No. 17-79-30056).

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