A CONSTRUCTION OF GENERALIZED
HARISH-CHANDRA MODULES FOR LOCALLY
REDUCTIVE LIE ALGEBRAS

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Abstract

We study cohomological induction for a pair \((\mathfrak{g}, \mathfrak{k})\), \(\mathfrak{g}\) being an infinite dimensional locally reductive Lie algebra and \(\mathfrak{k} \subset \mathfrak{g}\) being of the form \(\mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)\), where \(\mathfrak{k}_0 \subset \mathfrak{g}\) is a finite dimensional reductive in \(\mathfrak{g}\) subalgebra and \(C_{\mathfrak{g}}(\mathfrak{k}_0)\) is the centralizer of \(\mathfrak{k}_0\) in \(\mathfrak{g}\).
We prove a general non-vanishing and \(\mathfrak{k}\)-finiteness theorem for the output. This yields in particular simple \((\mathfrak{g}, \mathfrak{k})\)-modules of finite type over \(\mathfrak{k}\) which are analogs of the fundamental series of generalized Harish-Chandra modules constructed in \([PZ1]\) and \([PZ2]\). We study explicit versions of the construction when \(\mathfrak{g}\) is a root-reductive or diagonal locally simple Lie algebra.

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1 Introduction

A locally reductive Lie algebra is defined as a union \(\bigcup_{n \in \mathbb{Z}_{>0}} \mathfrak{g}_n\) of nested finite dimensional reductive Lie algebras \(\mathfrak{g}_n \subset \mathfrak{g}_{n+1}\) such that each \(\mathfrak{g}_n\) is reductive in \(\mathfrak{g}_{n+1}\). The class of locally reductive Lie algebras is a very natural and interesting class of infinite dimensional Lie algebras, and no classification is known. There are two (intersecting) subclasses of locally reductive Lie algebras which are relatively well-understood, see Subsection 2.3: the root-reductive Lie algebras, \([DP]\), \([B]\), and the locally simple diagonal Lie algebras, \([BZh]\).
For instance, the Lie algebra \(g\ell(\infty)\) of infinite matrices with only finitely many non-zero entries is root-reductive, and the Lie algebra \(g\ell(2^\infty)\), defined as the union \(\bigcup_{n \in \mathbb{Z}_{>0}} g\ell(2^n)\) via the injections

\[
\begin{align*}
\mathfrak{gl}(2^n) & \subset \mathfrak{gl}(2^{n+1}) \\
A & \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},
\end{align*}
\]

is diagonal. Both of the above classes of Lie algebras yield explicit examples of the general construction of this paper.

Representations of direct limit Lie groups have been studied for quite a considerable time now, \([Ha]\), \([Ne]\), \([O1]\), \([O2]\), \([NO]\), \([W]\), \([NRW]\), however the theory of direct limit
group representations has not been related in a systematic way to modules over the direct limit Lie algebra. In our opinion, this problem deserves further investigation.

In this paper we restrict ourselves to representations of locally reductive Lie algebras \( g \) and we initiate the study of \((g,\mathfrak{t})\)-modules of finite type over \( \mathfrak{t} \). More specifically, we provide a construction of such modules when \( \mathfrak{t} \) is the form \( \mathfrak{t}_0 + C_{g} (\mathfrak{t}_0) \) for a finite-dimensional reductive in \( g \) subalgebra \( \mathfrak{t}_0 \) \( (C_{g}(\cdot) \) denotes centralizer in \( g \). If \( g \) is root-reductive, such subalgebras \( \mathfrak{t} \) may equal the fixed vectors of an involution on \( g \), hence \((g,\mathfrak{t})\)-modules of finite type generalize Harish-Chandra modules. Our main construction is a generalization of the fundamental series for subalgebras \( \mathfrak{t} \subset g \) of the form \( \mathfrak{t} = \mathfrak{t}_0 + C_{g}(\mathfrak{t}_0) \), cf. [PZ2]. We use the derived functor of the functor of locally finite \( g \)-output provided the input is \( g \)-finite, \( m \) being the reductive part of the compatible parabolic subalgebra. A main technical observation of this paper is that one can construct reasonably large classes of parabolically induced modules which are \( g \)-finite, \( m \) being the reductive part of the compatible parabolic subalgebra. A main technical observation of this paper is that one can construct reasonably large classes of parabolically induced modules which are \( \mathfrak{t} \cap m \)-finite, both when \( g \) is root-reductive and when \( g \) is a diagonal. This is based on the stabilization of the branching multiplicities of certain tensor representations of classical Lie algebras of increasing rank.

Our main interest is in constructing simple \((g,\mathfrak{t})\)-modules \( M \) which in addition to being of finite type are also strict, i.e. for which \( \mathfrak{t} \) coincides with the subalgebra of \( g \) consisting of all elements \( g \in g \) which act locally finitely on \( M \) (the Fernando-Kac subalgebra of \( M \)). In particular, we provide sufficient conditions for strictness of the modules constructed.

The theory of \((g,\mathfrak{t})\)-modules for locally reductive Lie algebras \( g \) is still in its infancy and many questions remain off limits for this paper. This concerns for instance the problem of unitarizability of the \((g,\mathfrak{t})\)-modules we construct. Another very interesting problem is to describe the locally reductive subalgebras \( \mathfrak{t} \subset g \) which admit strict simple \((g,\mathfrak{t})\)-modules of finite type. Our paper deals with subalgebras of the form \( \mathfrak{t}_0 + C_{g}(\mathfrak{t}_0) \), and hence not with the case when \( \mathfrak{t} = \mathfrak{h} \) is a splitting Cartan subalgebra of \( \mathfrak{sl}(\infty) \), \( \mathfrak{so}(\infty) \) and \( \mathfrak{sp}(\infty) \). In fact, strict simple \((g,\mathfrak{h})\)-modules of finite type exist only for \( \mathfrak{sl}(\infty) \) and \( \mathfrak{sp}(\infty) \), and for \( \mathfrak{sl}(\infty) \), and I. Dimitrov has been working on their classification, [Di]. Finally, we would like to point out that the idea of studying direct limits of cohomologically induced modules was first suggested by A. Habib in [Ha], and that this idea has been an inspiration for us.

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2 Preliminaries

2.1 Conventions

All vector spaces and Lie algebras are defined over \( \mathbb{C} \). If \( p \) is a positive integer and \( W \) is a vector space or a Lie algebra, we set \( W^p := \bigoplus_{k=0}^{p-1} W \). \( \bigwedge^p W \) is the tensor algebra of \( W \). The superscript * indicates dual space, and \( \bigotimes = \bigotimes_{\mathbb{C}} \). If \( g \) is a Lie algebra, \( Z_g \) stands for the center of \( g \), \( C_{g}(\alpha) \) stands for the centralizer in \( g \) of a subset \( \alpha \subset g \), \( U(g) \) stands for the enveloping algebra and \( Z_{U(g)} \) stands for the center of \( U(g) \). The sign \( \oplus \) denotes semidirect sum of Lie algebras. A subalgebra \( \mathfrak{t} \subset g \) is reductive in
\( \mathfrak{g} \) if under the adjoint action of \( \mathfrak{k}, \mathfrak{g} \) is a semisimple \( \mathfrak{k} \)-module. If \( \mathfrak{l} \) is any subalgebra of \( \mathfrak{g} \) and \( M \) is an \( \mathfrak{l} \)-module, we denote the induced module \( U(\mathfrak{g}) \otimes U(\mathfrak{l}) M \) by \( \text{Ind}_{\mathfrak{l}}^{\mathfrak{k}} M \). If \( \mathfrak{l}' \) is a finite dimensional Lie algebra, by \( V_{\mathfrak{l}'}(\lambda) \) we denote the simple finite dimensional \( \mathfrak{l}' \)-module with highest weight \( \lambda \). When we write a vector space \( W \) as \( \bigcup_{n \in \mathbb{Z}_{>0}} W_n \) we automatically assume that \( W_n \subset W_{n+1} \) for \( n \in \mathbb{Z}_{>0} \).

### 2.2 A stabilization result

**Proposition 2.1** Let \( \mathfrak{s}_n \) be a sequence of classical finite dimensional simple Lie algebras of rank \( n \) and of fixed type \( A, B, C \) or \( D \). Denote by \( V_n \) the natural \( \mathfrak{s}_n \)-module. Then, for any fixed \( a, b, c, k \in \mathbb{Z}_{>0} \) the length of the \( \mathfrak{s}_n \)-module \( T^k(V_n^a \oplus (V_n^b \otimes \mathbb{C}_c) \) stabilizes when \( n \to \infty \) (here \( \mathbb{C} \) stands for the trivial 1-dimensional \( \mathfrak{s}_n \)-module).

**Proof.** This result is a relatively straightforward corollary of the results in [HTW], and we describe the argument only very briefly. Assume that \( \mathfrak{s}_n = \mathfrak{s}(n+1) \), let \( \mathfrak{h}_n \) be the diagonal subalgebra, \( \mathfrak{b}_n \) be the upper-triangular subalgebra, and \( \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_n - \varepsilon_{n+1} \) be the standard basis in \( \mathfrak{h}_n^* \). We will view any \( \mathfrak{b}_n \)-dominant weight \( \lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i \) of \( \mathfrak{s}_n \), \( \lambda_1 \geq \cdots \geq \lambda_n, \lambda_i \in \mathbb{Z} \) as a \( \mathfrak{b}_{n+k} \)-dominant weight of \( \mathfrak{s}_{n+k} \) by inserting \( k \) zeroes in the non-increasing sequence \( \lambda_1 \geq \cdots \geq \lambda_{n+1} \) so that the remaining sequence remains non-increasing. Therefore, for a fixed \( n_0 \) and a \( \mathfrak{b}_{n_0} \)-dominant weight \( \lambda \) as above, the \( \mathfrak{s}_n \)-module \( V_{\mathfrak{s}_n}(\lambda) \) is well defined for \( n \geq n_0 \). The first fact needed in the proof of Proposition 2.1 is that for fixed \( a, b, c, k \), there is an integer \( n_0 \) such that all simple constituents of \( X_n := T^k(V_n^a \oplus (V_n^b \otimes \mathbb{C}_c) \) are of the form \( V_{\mathfrak{s}_n}(\lambda) \) for \( n \geq n_0 \), where \( \lambda \) runs over a finite set of \( \mathfrak{b}_{n_0} \)-dominant weights of \( \mathfrak{s}_{n_0} \). This is proved by a straightforward induction on \( k \).

All that remains to show now is that for each \( V_{\mathfrak{s}_n}(\lambda) \) with \( \lambda \) as above, \( \dim \text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n) \) stabilizes when \( n \to \infty \). This can also be done by induction on \( k \). The case \( k = 1 \) is obvious, so we can assume that the statement is true for \( 1, 2, \ldots, k \). Then, in order to prove the Proposition for \( k+1 \), it suffices to show that \( \dim \text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n \otimes V_n) \) and \( \dim \text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n \otimes V_n^*) \) stabilize for \( n \to \infty \). Note that

\[
\dim \text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n \otimes V_n) = \dim \text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda) \otimes V_n^*, X_n),
\]

\[
\dim \text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n \otimes V_n^*) = \dim \text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda) \otimes V_n, X_n).
\]

The statement follows now from the induction assumption and from the key formula 1.2.1 in [HTW] which implies that

\[
\text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda'), V_{\mathfrak{s}_n}(\lambda) \otimes V_n^*) \neq 0
\]

for an independent on \( n \) finite set of weights \( \lambda' \) only (respectively,

\[
\text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda''), V_{\mathfrak{s}_n}(\lambda) \otimes V_n) \neq 0
\]

for an independent on \( n \) finite set \( \lambda'' \) only), and that \( \dim \text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda'), V_{\mathfrak{s}_n}(\lambda) \otimes V_n^*) \) (resp., \( \dim \text{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda''), V_{\mathfrak{s}_n}(\lambda) \otimes V_n) \)) stabilizes for \( n \to \infty \). The reader will easily fill in the details.

For \( \mathfrak{s}_n \) of types \( B, C, D \) the argument is essentially the same and uses formulas 1.2.2 and 1.2.3 in [HTW]. \( \Box \)
2.3 Locally reductive Lie algebras

We defined locally reductive Lie algebras in the Introduction. In the rest of this paper, when writing \( g = \cup_{n \in \mathbb{Z}_{>0}} g_n \) for a locally reductive Lie algebra \( g \), we will always assume that the \( g_n \)'s form a chain

\[
g_1 \subset g_2 \subset \cdots \subset g_n \subset g_{n+1} \subset \cdots
\]

of finite dimensional reductive Lie algebras such that each \( g_n \) is reductive in \( g_{n+1} \).

An important but quite restrictive class of locally reductive Lie algebras are the root-reductive Lie algebras. They have the form \( \cup_{n \in \mathbb{Z}_{>0}} g_n \), where the chain (1) satisfies the requirement that each inclusion \( g_n \subset g_{n+1} \) is a root homomorphism, i.e. maps a Cartan subalgebra of \( g_n \) into a Cartan subalgebra of \( g_{n+1} \) and any root space of \( g_n \) into a root space of \( g_{n+1} \). A most natural example of a root-reductive Lie algebra is the Lie algebra \( gl(\infty) \), defined via the chain \( gl(i) \subset gl(i+1) \) of upper left-hand corner embeddings.

A Lie algebra \( s \) is locally simple if \( s = \cup_{n \in \mathbb{Z}_{>0}} s_n \) where \( s_n \) are simple Lie algebras (in this case \( s_n \) is automatically reductive in \( s_{n+1} \)), in particular a locally simple Lie algebra is locally reductive. Up to isomorphism, there are three simple infinite dimensional locally simple root-reductive Lie algebras: \( sl(\infty), so(\infty) \) and \( sp(\infty) \). They are defined by obvious chains of inclusions which are root-homomorphisms (in the case of \( so(\infty) \) there are two natural choices: \( \cdots \subset so(2i) \subset so(2i+2) \subset \cdots \) and \( \cdots \subset so(2i+1) \subset so(2i+3) \subset \cdots \), however these yield isomorphic locally simple Lie algebras). The following structure theorem has been proved in [DN].

**Theorem 2.2** Let \( g \) be a root-reductive Lie algebra.

(a) The exact sequence

\[
0 \rightarrow [g, g] \rightarrow g \rightarrow a := g/[g, g] \rightarrow 0
\]

splits, hence \( g \) is isomorphic to the semidirect sum \([g, g] \ltimes a \) (\( a \) being an abelian Lie algebra).

(b) \([g, g] \) is isomorphic to a direct sum of at most countably many copies of \( sl(\infty), so(\infty), sp(\infty) \), as well as of simple finite dimensional Lie algebras.

A more general and very interesting class of locally reductive Lie algebras which are not necessarily root-reductive are the diagonal Lie algebras. By definition, a chain (1) of classical finite dimensional Lie algebras is diagonal, if for any \( n \), the natural representation of \( g_{n+1} \) is isomorphic to a direct sum of copies of the natural representation of \( g_n \), of its dual and of the trivial representation. Locally simple diagonal Lie algebras have been classified up to isomorphism in [BZh]. In the present paper, we will restrict ourselves to the simplest subclass of diagonal Lie algebras \( gl(p\Theta) \) defined below, however our results should extend without significant difficulty to general diagonal Lie algebras. Let \( \theta_1, \theta_2, \ldots \) be an infinite sequence of integers greater than 1. We denote by \( \Theta \) the formal product \( \theta_1\theta_2 \ldots \) and, for each \( p \in \mathbb{Z}_{\geq 1} \), we define the Lie algebra \( gl(p\Theta) \) (for \( p = 1 \) we write simply \( gl(\Theta) \)) as the union of the following diagonal chain

\[
gl(p) \subset gl(p\theta_1) \subset gl(p\theta_1\theta_2) \subset \cdots
\]

where, for \( n \in \mathbb{Z}_{\geq 0}, gl(p\theta_1\theta_2 \ldots \theta_{n-1}) \) is embedded into \( gl(p\theta_1 \ldots \theta_n) \) by repeating a matrix \( A \in gl(p\theta_1 \ldots \theta_{n-1}) \) \( \theta_n \) times along the main diagonal in \( gl(p\theta_1 \ldots \theta_n) \). The locally simple diagonal Lie algebra \( sl(p\Theta) \) is defined in the same way with \( gl(p\theta_1 \ldots \theta_n) \) replaced by
union of natural \( k \). The reader will check immediately that \( g(\ell(p\Theta)) = Z_{g(\ell(p\Theta))} \oplus s(\ell(p\Theta)) \), the center \( Z_{g(\ell(p\Theta))} \) being 1-dimensional. The Lie algebra \( g(\ell(2\infty)) \) (see the Introduction) is the simplest example of a Lie algebra of the form \( g(\ell(p\Theta)) \) (here \( p = 2 = \theta_n, n \in \mathbb{Z}_{>0} \)).

2.4 \((g, \ell)-modules\)

If \( g \) is a locally reductive Lie algebra and \( M \) is a \( g \)-module, the Fernando-Kac subalgebra \( g[M] \subset g \) consists of all elements \( g \in g \) which act locally finitely on \( M \), see [F], [DMP] and the references therein.

If \( g \) is locally reductive and \( \ell \subset g \) is a Lie subalgebra, we call a \( g \)-module \( M \) a \((g, \ell)-module\) if \( \ell \subset g[M] \). In other words, \( M \) is a \((g, \ell)-module\) if for any \( m \in M \) and any \( n \in \mathbb{Z}_{>0} \) the \( \ell_n \)-submodule of \( M \) generated by \( m \) is finite-dimensional. We call a \((g, \ell)-module \) \( M \) strict if \( \ell = g[M] \). Sometimes we use the term \( \ell \)-integrable \( g \)-module as an equivalent to \((g, \ell)-module\).

Furthermore, we define a \((g, \ell)-module \) \( M \) to be of finite type if the following two conditions hold:
- every finitely generated \( \ell \)-submodule \( M' \) of \( M \) has finite length as a \( \ell \)-module;
- for every fixed simple integrable \( \ell \)-module \( L \), the multiplicity of \( L \) as a subquotient of \( M' \) is bounded when \( M' \) runs over all finitely generated \( \ell \)-submodules of \( M \). If a \((g, \ell)-module \) \( M \) is not of finite type, we say that \( M \) is of infinite type. A generalized Harish-Chandra module is a finitely generated \( g \)-module \( M \) such that \( M \) is a \((g, \ell)-module \) of finite type for some Lie subalgebra \( \ell \subset g \).

Note that given any integrable \( \ell \)-module \( E \), the induced \( g \)-module \( \text{ind}_\ell^g E \) is a strict \((g, \ell)-module \), however in general (and more specifically, for \( \ell = \ell_0 + C_\theta(\ell_0) \) as in Section 3 below) \( \text{ind}_\ell^g E \) has infinite type\(^1\). Therefore for the construction of strict simple \((g, \ell)-module \)s of finite type, one needs more sophisticated techniques than ordinary induction. As we show below, cohomological induction is an ideal tool for this purpose.

Here are two examples illustrating the notions of a \((g, \ell)-module \) of finite and of infinite type in the extreme case of an integrable \( g \)-module.

**Proposition 2.3** Let \( s = \cup_{n \in \mathbb{Z}_{>0}} s_n \) be any infinite dimensional locally simple Lie algebra and \( \ell_0 \subset s_1 \) be a finite dimensional subalgebra of \( s_1 \). Let \( M \) be any non-trivial integrable \( s \)-module. Then \( M \) is an \((s, \ell_0)-module \) of infinite type.

**Proof.** Note first that \( \dim M = \infty \). This follows from the fact that all \( s_n \) have no non-trivial common finite dimensional module since \( \dim s_n \) tends to \( \infty \) when \( n \to \infty \). Now, assume to the contrary that \( M \) is an \((s, \ell_0)-module \) of finite type. Then \( M \) is a \((s, s_n)-module \) of finite type for any \( s_n \). We claim that this contradicts a result of Willenbring and Zuckerman. Indeed, Theorem 4.0.11 in [WZ] implies that if the difference of dimensions \( \dim s_n - \dim s_1 \) is sufficiently large, then there is a finite number of simple finite dimensional \( s_1 \)-modules \( W_1, \ldots, W_x \) such that any simple finite dimensional \( s_n \) module contains some \( W_j \) as a \( s_1 \)-submodule. It is an immediate consequence of this fact that any infinite dimensional \((s, s_n)-module \) of finite type is an \((s, s_1)-module \) of infinite type as some \( W_j \)

\(^1\)An interesting case when \( \text{ind}_\ell^g E \) has finite \( \ell \)-type is as follows. Using results of [NP] it is easy to construct an embedding \( g(\ell(\infty)) \simeq \ell \subset g \simeq g(\ell(\infty)) \), so that \( g(\ell) \) is isomorphic as a \( \ell \)-module to natural \( \ell \)-module \( V \) (i.e. to the union of natural \( \ell_n \)-modules \( V_n \), where \( \ell_n \simeq g(\ell(n)) \)). Then \( \text{ind}_\ell^g C \simeq S((g/\ell) \simeq S(V) \), and it is easy to see that the symmetric algebra is a multiplicity free \( \ell \)-module, i.e., in particular, \( \text{ind}_\ell^g \) has finite type as a \((g, \ell)-module \).
will appear with infinite multiplicity. This contradiction shows that our assumption was false, i.e. \( M \) is an \((s, t_0)\)-module of infinite type. \( \Box \)

Let now \( g = g_\ell(p\Theta) \) where \( \Theta = \theta_1\theta_2 \ldots \) with \( \theta_n > 1 \) for all \( n \in \mathbb{Z}_{>0} \), and let \( t_0 := g_1 = g_\ell(p) \). Set \( t_n := t_0 + C_{g_n}(t_0) \) for \( g_n = g_\ell(p\theta_1 \ldots \theta_{n-1}) \), and let \( t := \bigcup_{n\in\mathbb{Z}_{>0}} t_n \). Then, as it is easy to check, \( C_{g_n}(t_0) = g_\ell(\theta_1 \ldots \theta_{n-1}) \), and the inclusion \( C_{g_n}(t_0) \subset C_{g_{n+1}}(t_0) \) is nothing but the \( \theta_n \)-diagonal inclusion. Hence \( t \simeq g_\ell(p) + g_\ell(\Theta) \).

**Proposition 2.4** The adjoint representation of \( g_\ell(p\Theta) \) is a \( C_g(t_0) \)-module of finite length and thus, in particular, a \((g_\ell(p\Theta), t)\)-module of finite type.

**Proof.** The statement follows from the observation that for each \( n \), the adjoint representation of \( g_\ell(p\theta_1 \ldots \theta_{n-1}) \) considered as a \( C_{g_n}(t_0) = g_\ell(\theta_1 \ldots \theta_{n-1}) \)-module is a submodule of \( T^2(V_n^\ast \oplus \{V_n^\ast\}) \), where \( V_n \) is the natural \( g_\ell(\theta_1 \ldots \theta_{n-1}) \)-module. By Proposition 2.11 the length of \( T^2(V_n^\ast \oplus \{V_n^\ast\}) \) as an \( \mathfrak{s}\ell(\theta_1 \ldots \theta_{n-1}) \)-module stabilizes for \( n \to \infty \), hence the length of \( g_\ell(p\theta_1 \ldots \theta_{n-1}) \) considered as a \( C_{g_n}(t_0) \)-module is bounded for \( n \to \infty \). The reader will check immediately that this implies that the adjoint module of \( g_\ell(p\Theta) \) has finite length as a \( C_g(t_0) \)-module. \( \Box \)

### 2.5 The Zuckerman functor

In this Subsection \( g \) is any Lie algebra and \( \mathfrak{t}' \subset g \) is a finite dimensional subalgebra which acts locally finitely and semisimply on \( g \). For instance, if \( g = \bigcup_n g_n \) is locally reductive and \( \mathfrak{t}' \subset g_n \) is a reductive in \( g_n \) subalgebra for some \( n \), the above condition is satisfied.

By \( C(g, \mathfrak{t}') \) we denote the category of all \((g, \mathfrak{t}')\)-modules which are semisimple over \( \mathfrak{t}' \). For any reductive in \( \mathfrak{t}' \) subalgebra \( \mathfrak{m}' \subset \mathfrak{t}' \), we consider the left exact functor

\[
\Gamma_{\mathfrak{t}', \mathfrak{m}'} : C(g, \mathfrak{m}') \to C(g, \mathfrak{t}')
\]

\[
M \mapsto \Gamma_{\mathfrak{t}', \mathfrak{m}'}(M) := \sum_{X \subseteq M, X \in \text{Ob}(C(g, \mathfrak{t}'))} X.
\]

The category \( C(g, \mathfrak{m}') \) has sufficiently many injectives and hence one can introduce the right derived functor \( R \Gamma_{\mathfrak{t}', \mathfrak{m}'} \). This functor is known as the Zuckerman functor.

A well known property of the Zuckerman functor which we use below is that if \( Z_{U(g)} \) acts via a fixed character on \( M \), then \( Z_{U(g)} \) acts via the same character on \( R \Gamma_{\mathfrak{t}', \mathfrak{m}'}(M) \). The following two propositions discuss some further fundamental properties of the functor \( R \Gamma_{\mathfrak{t}', \mathfrak{m}'} \).

**Proposition 2.5**

(a) (restriction principle). Let \( g' \subset g \) be an arbitrary Lie subalgebra of \( g \) such that \( \mathfrak{t}' \subset g' \). Then the diagram of functors

\[
\begin{array}{ccc}
C(g, \mathfrak{m}') & \xrightarrow{R \Gamma_{\mathfrak{t}', \mathfrak{m}'}} & C(g, \mathfrak{t}') \\
\downarrow & & \downarrow \\
C(g', \mathfrak{m}') & \xrightarrow{R \Gamma_{\mathfrak{t}', \mathfrak{m}'}} & C(g', \mathfrak{t}'),
\end{array}
\]

whose vertical arrows are restriction functors, is commutative.
(b) Let $U^0(\mathfrak{t}') := \Gamma_{\mathfrak{t}',\mathfrak{m}'}(\text{Hom}_{\mathfrak{g}}(U(\mathfrak{t}'), \mathbb{C}))$. Then $U^0(\mathfrak{t}')$ is a $U(\mathfrak{t}')$-bimodule, and for any $M$ in $\mathcal{C}(\mathfrak{t}', \mathfrak{m}')$ there is a natural isomorphism of $\mathfrak{t}'$-modules

$$R \Gamma_{\mathfrak{t}',\mathfrak{m}'}(M) \cong H^*(\mathfrak{t}', \mathfrak{m}', M \otimes U^0(\mathfrak{t}'))$$

(here we apply $R \Gamma_{\mathfrak{t}',\mathfrak{m}'}$ to objects of $\mathcal{C}(\mathfrak{t}', \mathfrak{m}')$ by setting $\mathfrak{g}' = \mathfrak{t}'$, see (a)).

(c) Let $M$ be an inductive limit $\varinjlim M_i$ of modules $M_i$ in $\mathcal{C}(\mathfrak{t}', \mathfrak{m}')$. Then

$$R \Gamma_{\mathfrak{t}',\mathfrak{m}'}(M) \cong \varinjlim R \Gamma_{\mathfrak{t}',\mathfrak{m}'}(M_i).$$

**Proof.**

(a) It suffices to show that an injective object $I$ in $\mathcal{C}(\mathfrak{g}, \mathfrak{m}')$ is also injective in $\mathcal{C}(\mathfrak{g}', \mathfrak{m}')$. If $Q$ is an arbitrary object in $\mathcal{C}(\mathfrak{g}', \mathfrak{m}')$, then $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}')} Q$ is an object in $\mathcal{C}(\mathfrak{g}, \mathfrak{m}')$, and the functor

$$Q \mapsto U(\mathfrak{g}) \otimes_{U(\mathfrak{g}')} Q$$

is exact. The natural isomorphism $\text{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{g}')} Q, I) = \text{Hom}_{\mathfrak{g}}(Q, I)$ shows that $I$ represents an exact functor in $\mathcal{C}(\mathfrak{g}', \mathfrak{m}')$. Therefore $I$ is injective in $\mathcal{C}(\mathfrak{g}', \mathfrak{m}')$, and (a) follows.

(b) This statement is a rephrasing of the isomorphism (4.5) in [EW].

(c) For any $M$ in $\mathcal{C}(\mathfrak{t}', \mathfrak{m}')$, we use the standard complex for relative Lie algebra cohomology:

$$C^*(\mathfrak{t}', \mathfrak{m}', M \otimes U^0(\mathfrak{t}')) = \text{Hom}_{\mathfrak{m}'}(\Lambda^*(\mathfrak{t}'/\mathfrak{m}'), M \otimes U^0(\mathfrak{t}')).$$

As $\mathfrak{t}'$ is finite-dimensional, we have an isomorphism

$$C^*(\mathfrak{t}', \mathfrak{m}', M \otimes U^0(\mathfrak{t}')) \cong \varinjlim C^*(\mathfrak{t}', \mathfrak{m}', M_i \otimes U^0(\mathfrak{t}')),$$

and the fact that cohomology commutes with inductive limits implies (c). □

**Proposition 2.6** (comparison principle). Suppose $\mathfrak{t}' = \mathfrak{t}'' \oplus \mathfrak{t}'''$ is a decomposition into two ideals, and let $\mathfrak{m}''$ be a reductive in $\mathfrak{t}''$ subalgebra. Set $\mathfrak{m}' := \mathfrak{m}'' \oplus \mathfrak{t}'''$. Then for any $(\mathfrak{g}, \mathfrak{m}')$-module $M$, there is a natural isomorphism of $\mathfrak{g}$-modules

$$R \Gamma_{\mathfrak{t}',\mathfrak{m}'}(M) \cong R \Gamma_{\mathfrak{t}',\mathfrak{m}''}(M).$$

(2)

**Lemma 2.7** Under the assumptions of Proposition 2.6 let $I$ be an injective object in $\mathcal{C}(\mathfrak{g}, \mathfrak{m}')$. Then

$$R^t \Gamma_{\mathfrak{t}',\mathfrak{m}''}(I) = 0 \text{ for } t > 0.$$

**Proof of Lemma 2.7** As a $\mathfrak{t}'$-module $I$ can be decomposed as $\oplus_{\lambda}(J_{\lambda} \otimes V_{\mathfrak{t}''}(\lambda))$, where $\lambda$ runs over all dominant integral weights of $\mathfrak{t}'''$ and where the $J_{\lambda}$'s are $(\mathfrak{t}'', \mathfrak{m}'')$-modules. We claim that each $J_{\lambda}$ is injective in $\mathcal{C}(\mathfrak{t}'', \mathfrak{m}'')$. Indeed, by the proof of the restriction principle (Proposition 2.5(a)) $I$ is injective in $\mathcal{C}(\mathfrak{t}', \mathfrak{m}')$, hence for each $\lambda$, $J_{\lambda} \otimes V_{\mathfrak{t}''}(\lambda)$ is injective in $\mathcal{C}(\mathfrak{t}', \mathfrak{m}')$. Therefore $J_{\lambda}$ is injective in $\mathcal{C}(\mathfrak{t}'', \mathfrak{m}'').$

By Proposition 2.5(b)

$$R \Gamma_{\mathfrak{t}',\mathfrak{m}''}(I) \cong H^*(\mathfrak{t}'', \mathfrak{m}'', I \otimes U^0(\mathfrak{t}'')),$$

and thus (since relative Lie algebra cohomology commutes with direct sums), it suffices to show that

$$H^*(\mathfrak{t}'', \mathfrak{m}'', (J_{\lambda} \otimes V_{\mathfrak{t}''}(\lambda)) \otimes U^0(\mathfrak{t}'')) = 0$$

(3)
for \( t > 0 \). However,
\[
H^t(t'', m'', (J_\lambda \boxtimes V''_{t''}(\lambda)) \otimes U^0(t'')) = \\
= H^t(t'', m'', J_\lambda \boxtimes U^0(t'')) \boxtimes V''_{t''}(\lambda) = \\
= R^t\Gamma_{t', m'}(J_\lambda) \boxtimes V''_{t''}(\lambda) = 0
\]
since \( J_\lambda \) is injective in \( C(t'', m'') \), and the Lemma follows. \( \square \)

**Proof of Proposition 2.6** By Lemma 2.7 any \( C(g, m') \)-injective resolution of \( M \) is \( \Gamma_{t', m'} \)-acyclic hence it can be used both for the computation of \( R\Gamma_{t', m'}(M) \) and of \( R\Gamma_{t', m'}(M) \). This yields the natural isomorphism \( [2] \). \( \square \)

### 3 The Construction

Let \( g = \cup_n g_n \) be a locally reductive Lie algebra and \( t_0 \subseteq g_1 \) be a finite dimensional subalgebra reductive in \( g \) (equivalently, in \( g_1 \)). Fix a Cartan subalgebra \( t_0 \in t_0 \). For any \( g_n \) we have the notion of a \( t_0 \)-compatible parabolic subalgebra of \( g_n \); by definition this is a parabolic subalgebra \( p_n \subseteq g_n \) of the form \( \bigoplus_{\lambda, \text{Re} \sigma \geq 0} (g_n)_{\lambda, \sigma} \), where \( h_n \) is a semisimple element of \( t_0 \), \( \sigma \) runs over the eigenvalues of \( h_n \) in \( g_n \), and \( (g_n)_{\lambda, \sigma} \) are the corresponding eigenspaces. We call a subalgebra \( p \subseteq g \) a \( t_0 \)-compatible parabolic subalgebra if, for all \( n \), \( p \cap g_n \) is a \( t_0 \)-compatible parabolic subalgebra of \( g_n \) and \( n_n = n_{n+1} \cap g_n \), where \( n_n \) is the nilradical of \( p_n \). It is possible (but not required) that there is a semisimple element \( h \in t_0 \) such that \( p = \bigoplus_{\sigma, \text{Re} \sigma \geq 0} g_h^\sigma \).

One can always choose decompositions \( p_n = m_n \oplus n_n \) where, for each \( n \), \( m_n \) is a reductive in \( g_n \) subalgebra such that \( m_{n+1} \cap g_n = m_n \). This yields a decomposition \( p = m \oplus n \), where \( m = \cup_n m_n \) and \( \cup_n n_n \). By definition, \( n \) is the nilradical of \( p \) and \( m \) is a locally reductive subalgebra of \( g \). In what follows, we consider the decomposition \( p = m \oplus n \) fixed and define \( n \) as the union \( \cup_n n_n \), where for each \( n \), \( g_n = n_n \oplus m_n \oplus n_n \) is the canonical \( m_n \)-module decomposition. In this way, \( n \) is of course an integrable \( m \)-module.

Let \( \ell := t_0 + C_g(t_0) \). Then \( \ell_n = \ell_0 + C_g(t_0) \) is reductive in \( g \) for each \( n \). Note that \( \ell \cap m = m_0 + C_g(t_0) \), where \( m_0 := \ell_0 \cap m \). Our goal is to construct nontrivial \((g, \ell)\)-modules by starting with a nontrivial \((m, \ell \cap m)\)-module \( E \) and then applying a functor of cohomological induction type. We first extend \( E \) to a \( m \)-module by setting \( n \cdot E = 0 \). We then consider the induced module \( M(p, E) := \text{ind}_{p_0}^p E \). This is an integrable \( m \cap \ell \)-module. Indeed, the equality of \( m \)-modules \( g = n \oplus m \oplus n \) implies via the Poincaré-Birkhoff-Witt theorem that \( M(p, E) \) has an \( m \)-module filtration with associated graded equal to \( S(n) \otimes E \). Both \( S(n) \) and \( E \) are integrable \( m \cap \ell \)-modules, thus \( M(p, E) \) is also \( m \cap \ell \)-integrable.

We now set \( A(p, E) := R^s\Gamma_{t_0, m_0}(M(p, E)) \), where \( s := \frac{1}{2} \dim(\ell_0/m_0) \). By definition \( A(p, E) \) is a \((g, t_0)\)-module, but as we show below \( A(p, E) \) is in fact a \((g, \ell)\)-module. We also set \( A(p_0, E) := R^s\Gamma_{t_0, m_0}(\text{ind}_{p_0}^p E) \), where \( p_0 := t_0 \cap p \) and we regard \( E \) as a module over \( m_0 + C_g(t_0) \) and \( \text{ind}_{p_0}^p E \) as \( t_0 + C_g(t_0) \) module. By Proposition 2.5(a) there is a functorial morphism of \( t_0 \)-modules
\[
\Psi_E : A(p_0, E) \rightarrow A(p, E).
\]

Knapp and Vogan [KV] call \( \Psi_E \) the bottom layer map. In the present paper, we call any \( g \)-subquotient of \( A(p, E) \) generated by vectors in \( \text{im} \Psi_E \) a bottom layer subquotient of \( A(p, E) \).
Note that \( m_0 \cap C_\mathfrak{g}(\mathfrak{t}_0) = Z_{\mathfrak{t}_0} \). Therefore, if \( \mathfrak{b}_{m_0} \) is a fixed Borel subalgebra of \( m_0 \), we can decompose \( E \) as
\[
\bigoplus_{\nu} V_{m_0}(\nu) \boxtimes U(Z_{\mathfrak{t}_0}) E''_{\nu},
\]
where we consider \( E''_{\nu} := \text{Hom}_{m_0}(V_{m_0}(\nu), E) \) as a \( C_\mathfrak{g}(\mathfrak{t}_0) \)-module and \( \nu \) runs over all \( \mathfrak{b}_{m_0} \)-dominant integral weights of \( m_0 \).

Fix now a Borel subalgebra \( \mathfrak{b}_0 \) of \( \mathfrak{t}_0 \) such that \( \mathfrak{b}_0 \cap m_0 = \mathfrak{b}_{m_0} \). This defines two Weyl group elements: the element \( w_{\mathfrak{b}_0} \in W_{\mathfrak{b}_0} \) of maximal length with respect to \( \mathfrak{b}_0 \), and the element \( w_{m_0} \in W_{m_0} \) of maximal length with respect to \( \mathfrak{b}_0 \cap m_0 \). For any \( \mathfrak{b}_{m_0} \)-dominant \( \mathfrak{t}_0 \)-integral weight \( \nu \), we set
\[
\nu' := w_{\mathfrak{b}_0} \circ w_{m_0}^{-1}(\nu + \rho_{\mathfrak{b}_0}) - \rho_{\mathfrak{b}_0},
\]
where \( \rho_{\mathfrak{b}_0} \) is the half-sum of the \( \mathfrak{b}_0 \)-positive roots of \( \mathfrak{t}_0 \).

**Lemma 3.1** The \( \mathfrak{t} \)-module \( A(\mathfrak{p}_0, E) \) is \( \mathfrak{t} \)-integrable and is isomorphic to
\[
\bigoplus_{\nu} V_{\mathfrak{b}_0}(\nu') \boxtimes U(Z_{\mathfrak{t}_0}) E''_{\nu},
\]
where as above \( \nu \) runs over all dominant integral weights of \( m_0 \), and where \( V_{\mathfrak{b}_0}(\nu') := 0 \) whenever \( \nu' \) is not \( \mathfrak{b}_0 \)-dominant and integral for \( \mathfrak{t}_0 \).

**Proof.** This statement is a direct corollary of the Bott-Borel-Weil theorem proved in [EW], see [EW] Proposition 6.3. \( \square \)

The following theorem is our main result.

**Theorem 3.2**

(a) \( A(\mathfrak{p}, E) \) is a \( (g, \mathfrak{k}) \)-module.

(b) If \( M(\mathfrak{p}, E) \) is an \( (m, \mathfrak{t} \cap m) \)-module of finite type, then \( A(\mathfrak{p}, E) \) is a \( (g, \mathfrak{t}) \)-module of finite type.

(c) Assume \( E = \bigcup_n E_n \) where each \( E_n \) is an \( (m_n, \mathfrak{t} \cap m_n) \)-module on which \( Z_{m_n} \) acts via a \( 1 \)-dimensional representation. Then the bottom layer map \( \Psi_E \) is an injection. Assume that for some \( \nu \), \( E''_{\nu} \neq 0 \) and \( \nu' \) is dominant integral for \( \mathfrak{t}_0 \). Then \( \text{Hom}_{\mathfrak{b}_0}(V_{\mathfrak{b}_0}(\nu'), A(\mathfrak{p}, E)) = E''_{\nu} \). Hence \( A(\mathfrak{p}, E) \) has a simple bottom layer subquotient.

(d) Assume \( E = \bigcup_n E_n \) where each \( E_n \) is an \( (m_n, \mathfrak{t} \cap m_n) \)-module with \( Z_{U(m_n)} \)-character, that \( A(\mathfrak{p}, E) \neq 0 \), and that for some \( N \) the \( Z_{U(m_N)} \)-character of \( \text{ind}^{\mathfrak{b}_0}_m E_N \) is not regular integral. Then some bottom layer subquotient of \( A(\mathfrak{p}, E) \) is not an integrable \( g \)-module. If in addition, \( \mathfrak{t} \) is a maximal subalgebra of \( g \), then some simple bottom layer subquotient of \( A(\mathfrak{p}, E) \) is a strict \( (g, \mathfrak{t}) \)-module.

(e) Under the assumptions of (c) assume further that \( m = C_\mathfrak{g}(\mathfrak{t}_0) \) and that \( E \) is simple. Then \( \mathfrak{t}_0 \) acts via weight \( \mu \in \mathfrak{t}_0^{*} \) on \( E \), \( \mu' \) is dominant integral for \( \mathfrak{t}_0 \), and there is an isomorphism of \( \mathfrak{t} = \mathfrak{t}_0 + C_\mathfrak{g}(\mathfrak{t}_0) \)-modules
\[
A(\mathfrak{p}_0, E) \simeq V_{\mathfrak{b}_0}(\mu') \boxtimes U(Z_{\mathfrak{t}_0}) E''_{\nu},
\]
where \( E''_{\nu} \) equals \( E \) considered as a \( C_\mathfrak{g}(\mathfrak{t}_0) \)-module. Furthermore, \( \Psi_E \) yields an isomorphism between the \( \mathfrak{t} \)-modules \( A(\mathfrak{p}_0, E) \) and \( V_{\mathfrak{b}_0}(\mu') \otimes \text{Hom}_{\mathfrak{b}_0}(V_{\mathfrak{b}_0}(\mu'), A(\mathfrak{p}, E)) \).

(f) If, under the assumptions of (e), \( \text{im} \Psi_E \) is a simple \( \mathfrak{t} \)-submodule of \( A(\mathfrak{p}, E) \), then \( A(\mathfrak{p}, E) \) has a unique simple bottom layer subquotient. A sufficient condition for the simplicity of \( \text{im} \Psi_E \) is the inclusion \( m \subset \mathfrak{t} \).
(a) By construction, $M(p, E)$ is a $(g, \mathfrak{k} \cap m)$-module. Since $m \cap \mathfrak{k} \supset C_g(\mathfrak{k}_0)$, $M(p, E)$ is an integrable $C_g(\mathfrak{k}_0)$-module. Let $\tilde{M}$ denote the restriction of $M(p, E)$ to $\mathfrak{k}$: by Proposition 2.5(a) $A(p, E)$ is isomorphic as a $\mathfrak{k}$-module to $R^k \Gamma_{\mathfrak{k}_0, m_0}(M)$. By the Poincaré-Birkhoff-Witt Theorem, the $\mathfrak{k}$-module $\tilde{M}$ has an increasing filtration with associated graded

$$\text{Gr} \tilde{M} = \bigoplus_{t \in \mathbb{Z}_{\geq 0}} \text{ind}_{p_0}^{\mathfrak{k}_0}(S^t(\mathfrak{k}_0 \cap \hat{n}) \otimes E),$$

where $\mathfrak{k}_0$ is a fixed $\mathfrak{k}$-module complement of $\mathfrak{k}_0$ in $g$. □

Lemma 3.3 $R \Gamma_{\mathfrak{k}_0, m_0} (\text{Gr} \tilde{M})$ is a graded integrable $\mathfrak{k}$-module.

Proof of Lemma 3.3. Decompose the $m_0 + C_g(\mathfrak{k}_0)$-module $S^t(\mathfrak{k}_0 \cap \hat{n}) \otimes E$ as

$$\bigoplus_{\nu} V_{m_0}(\nu) \boxtimes_{U(z_{\mathfrak{k}_0})} X_{\nu, t}$$

for some $C_g(\mathfrak{k}_0)$-modules $X_{\nu, t}$. Observe that each $X_{\nu, t}$ is an integrable $C_g(\mathfrak{k}_0)$-module. We obtain a $\mathfrak{k}$-module isomorphism

$$\text{Gr} \tilde{M} \cong \bigoplus_{\nu, t} \text{ind}_{p_0}^{\mathfrak{k}_0} \left( V_{m_0}(\nu) \boxtimes_{U(z_{\mathfrak{k}_0})} X_{\nu, t} \right).$$

For each $\nu$, let $G_{\nu}$ be a resolution of $\text{ind}_{p_0}^{\mathfrak{k}_0} V_{m_0}(\nu)$ by $\Gamma_{\mathfrak{k}_0, m_0}$-acyclic $(\mathfrak{k}_0, m_0)$-modules. We can compute $R \Gamma_{\mathfrak{k}_0, m_0} (\text{Gr} \tilde{M})$ as

$$H^\cdot (\Gamma_{\mathfrak{k}_0, m_0} (\bigoplus_{\nu, t} G_{\nu} \boxtimes_{U(z_{\mathfrak{k}_0})} X_{\nu, t})),
$$

which is isomorphic as a $\mathfrak{k}$-module to

$$\bigoplus_{\nu, t} H^\cdot (\Gamma_{\mathfrak{k}_0, m_0} (G_{\nu, t})) \boxtimes_{U(z_{\mathfrak{k}_0})} X_{\nu, t},$$

and hence to

$$\bigoplus_{\nu, t} R \Gamma_{\mathfrak{k}_0, m_0} (V_{m_0}(\nu)) \boxtimes_{U(z_{\mathfrak{k}_0})} X_{\nu, t}.$$

Therefore $R \Gamma_{\mathfrak{k}_0, m_0} (\text{Gr} \tilde{M})$ is an integrable $\mathfrak{k}$-module. This proves the Lemma. □

To complete the proof of (a) note that, by Proposition 2.5(c), $R \Gamma_{\mathfrak{k}_0, m_0}$ commutes with inductive limits. Since furthermore, $C_g(\mathfrak{k}_0)$ acts by $\mathfrak{k}_0$-endomorphisms on $\tilde{M}$, $R \Gamma_{\mathfrak{k}_0, m_0} (\tilde{M})$ has an increasing filtration of $\mathfrak{k}_0 + C_g(\mathfrak{k}_0)$-modules induced by the filtration on $\tilde{M}$. An obvious induction argument using the fact that $R \Gamma_{\mathfrak{k}_0, m_0} (\text{Gr} \tilde{M})$ is a $\mathfrak{k}$-integrable module (Lemma 3.3) implies that $R \Gamma_{\mathfrak{k}_0, m_0} (\tilde{M})$ is filtered by $\mathfrak{k}$-integrable modules, and hence is itself $\mathfrak{k}$-integrable. This proves (a).

(b) Suppose $M(p, E)$ is of finite type over $\mathfrak{k} \cap m = m_0 + C_g(\mathfrak{k}_0)$. We can rewrite (4) as

$$\text{Gr} \tilde{M} = \bigoplus_{\nu} (\text{ind}_{p_0}^{\mathfrak{k}_0} V_{m_0}(\nu)) \boxtimes_{U(z_{\mathfrak{k}_0})} Y_{\nu}.
with each \( Y_\nu = \oplus_t X_{\nu,t} \) an integrable \( C_0(t_0) \)-module. Since \( \text{ind}_{p_0}^{p_0} V_{m_0}(\nu) \) is a \( (t_0, m_0) \)-module, we conclude that every \( Y_\nu \) is of finite type over \( C_0(t_0) \). Combining (5) with Lemma 3.3 we obtain

\[
R^s\Gamma_{t_0, m_0}(\text{Gr} \tilde{M}) \cong \bigoplus_{\nu} V_{t_0}(\nu') \boxtimes_{U(\mathbb{Z}t_0)} Y_\nu.
\]

(6)

The right hand side of (6) is of finite type over \( \mathfrak{g} \) as each \( Y_\nu \) is of finite type over \( C_0(t_0) \) and \( V_{t_0}(\nu') \not\cong V_{t_0}(\nu'') \) for \( \nu' \neq \nu'' \). Finally, the fact that \( R^s\Gamma_{t_0, m_0}(\text{Gr} \tilde{M}) \) is of finite type over \( \mathfrak{g} \) implies that \( R^s\Gamma_{t_0, m_0}(M) \) is of finite type over \( \mathfrak{g} \). Indeed, this follows from the observation, that since \( R^s\Gamma_{t_0, m_0} \) commutes with inductive limits,

\[
\text{Gr}(R^s\Gamma_{t_0, m_0}(\tilde{M})) \cong R^s\Gamma_{t_0, m_0}(\text{Gr} \tilde{M}),
\]

(7)

where the left hand side of (7) refers to the filtration of \( R^s\Gamma_{t_0, m_0}(\tilde{M}) \) induced by the filtration on \( \tilde{M} \). This proves (b).

(c) The theory of the bottom layer map in the finite dimensional case is elaborated by Knapp and Vogan in [KV] Ch. \( \Sigma \), Sec.6. There the authors assume that they are working with a symmetric pair. However, a careful examination of Theorem 5.80 in [KV] reveals that the assumption that \( \mathfrak{f}_0 \) is symmetric in \( \mathfrak{g}_0 \) is not needed; hence our hypothesis on \( E_0 \) implies that \( \Psi_{E_0} \) is an injection from \( A(p_0, E_0) \) to \( A(p_n, E_n) = R^s\Gamma_{t_0, m_0}(\text{ind}_{p_0}^{p_n} E_n) \) for each \( n \). Furthermore, we have an injection of \( \text{ind}_{p_0}^{p_n} E_n \) to \( \text{ind}_{p_0}^{p_{n+1}} E_{n+1} \) which induces a \( \mathfrak{g}_n \)-module homomorphism \( \varphi_n : A(p_n, E_n) \to A(p_{n+1}, E_{n+1}) \).

On the other hand, we have a canonical \( \mathfrak{f}_0 \)-module homomorphism \( \chi_n : A(p_0, E_0) \to A(p_0, E_{n+1}) \) induced by the inclusion of \( E_n \) into \( E_{n+1} \). Moreover, the diagram

\[
\begin{array}{ccc}
A(p_0, E_{n+1}) & \xrightarrow{\Psi_{E_{n+1}}} & A(p_{n+1}, E_{n+1}) \\
\downarrow \chi_n & & \downarrow \varphi_n \\
A(p_0, E_n) & \xrightarrow{\Psi_{E_n}} & A(p_n, E_n)
\end{array}
\]

(8)

is commutative, and \( \Psi_{E_n} \) and \( \Psi_{E_{n+1}} \) are injections. Consider the inductive limit homomorphism

\[
\lim \Psi_{E_n} : \lim A(p_0, E_n) \to \lim A(p_n, E_n).
\]

By Proposition 2.5 (c) \( \Psi_E = \lim \Psi_{E_n} \) is an injection.

Assume now that for some \( \nu, E''_\nu \neq 0 \) and \( \nu' \) is dominant integral for \( \mathfrak{f}_0 \). For sufficiently large \( n, E''_{n, \nu} := \text{Hom}_{\mathfrak{g}_n}(V_{m_0}(\nu), E_n) \) is always nonzero. The fact that \( \text{Hom}_{\mathfrak{g}_0}(V_{t_0}(\nu'), A(p_n, E_n)) \cong \text{Hom}_{\mathfrak{g}_0}(V_{t_0}(\nu'), A(p_0, E_n)) \) ([KV] Theorem 5.80), together with the fact that \( \Psi_E = \lim \Psi_{E_n} \), implies

\[
\text{Hom}_{\mathfrak{g}_0}(V_{t_0}(\nu'), A(p, E)) = E''_{\nu}
\]

as required. In particular, the bottom layer \( \text{Im} \Psi_E \subset A(p, E) \) is non-zero. Finally, to construct a simple bottom layer quotient of \( A(p, E) \) it suffices to consider a simple quotient of a cyclic module \( U(\mathfrak{g}) \cdot v \), where \( v \in \text{Im} \Psi_E \). This proves (c).

For the proof of (d) we need the following lemma.

**Lemma 3.4** Suppose \( F \) is an integrable \( m_0 \)-module. Extend \( F \) to a \( p_0 \)-module so that \( n_0 \cdot F = 0 \). Then if \( i < s \), \( R^i\Gamma_{t_0, m_0}(\text{ind}_{p_0}^{p_0} F) = 0 \).
Consider the short exact sequence $m$

Note that, under our assumptions, this implies that one and the same character on $H$

It yields a long exact sequence for $R\Gamma_{t_0,m_0}$. Lemma 3.4 implies that each $\chi_n$ is an injection. Therefore, by the commutativity of diagram (8), $\varphi_n \circ \Psi_{E_n}$ is an injection for each $n$, and hence the maps $\varphi_n \circ \Psi_{E_n}$ induce an injection

for each $n$.

Fix a value of $N$ so that $A(p_0, E_N) \neq 0$, and so that the $Z_{U(g_N)}$-character of $\text{ind}_{P_N}^{P_N} E_N$ is not regular integral. Fix a nonzero vector $v \in A(p_0, E_N)$, let $A_v$ be the $g$-submodule generated by $\tilde{v} := \Psi_E(\varphi_n(v))$ (note that $\tilde{v} \neq 0$), and let $A'_v$ be a simple quotient of $A_v$. We claim that $A'_v$ is not $g$-integrable. To see this consider the image $A'_{v,N}$ in $A'_{v}$ of the $g_N$-submodule $U(g_N) \cdot \tilde{v} \subset A(p,E)$. The commutativity of the diagram

implies that $A'_{v,N}$ is isomorphic to a subquotient of $A(p_N, E_n)$. Since $Z_{U(g_N)}$ acts by one and the same character on $\text{ind}_{P_N}^{P_N} E_N$ and on $A(p_N, E_N)$, $A'_{v,N}$ is a $g_N$-module with a central character which is not regular integral, and is thus not an integrable $g_N$-module. This implies that $A'_{v}$ itself is not an integrable $g$-module.

Note that, under our assumptions, $m_0 = t_0$. As $t_0 \subset Z_m$, $t_0$ acts via weight $\mu$ on $E$, and moreover, $E = \mathbb{C}_\mu \otimes_{U(Z_{t_0})} E''$ where $\mathbb{C}_\mu$ is the 1-dimensional $t_0$-module corresponding to $\mu$. Lemma 3.1 yields now (3), and (c) implies that $\Psi_E$ is an isomorphism between $A(p_0, E)$ and $V_{t_0}(\mu^\vee) \otimes \text{Hom}_{t_0}(V_{t_0}(\mu^\vee), A(p, E))$. 

Proof of Lemma 3.4. According to Proposition 2.5(b) we need to show that

for $i < s$. Since $U^0(t_0)$ is a semisimple integrable $t_0$-module, it is enough to show that $H^i(t_0, m_0, V \otimes \text{ind}_{P_0}^{P_0} F) = 0$ for $i < s$ and for any simple finite-dimensional $t_0$-module $V$. By Poincaré duality for relative Lie algebra cohomology we must show that

for $i < s$. It is well known that

So we must show that

for $i < s$. But Shapiro’s Lemma implies that the above homology is isomorphic to $H_{2s-i}(p_0, m_0, V \otimes F)$, and the latter vanishes for $i < s$ because $\dim(p_0/m_0) = s$. The Lemma follows. □

(d) Consider the short exact sequence

It yields a long exact sequence for $R\Gamma_{t_0,m_0}$. Lemma 3.4 implies that each $\chi_n$ is an injection. Therefore, by the commutativity of diagram (8), $\varphi_n \circ \Psi_{E_n}$ is an injection for each $n$, and hence the maps $\varphi_n \circ \Psi_{E_n}$ induce an injection

for each $n$. 

Fix a value of $N$ so that $A(p_0, E_N) \neq 0$, and so that the $Z_{U(g_N)}$-character of $\text{ind}_{P_N}^{P_N} E_N$ is not regular integral. Fix a nonzero vector $v \in A(p_0, E_N)$, let $A_v$ be the $g$-submodule generated by $\tilde{v} := \Psi_E(\varphi_n(v))$ (note that $\tilde{v} \neq 0$), and let $A'_v$ be a simple quotient of $A_v$. We claim that $A'_v$ is not $g$-integrable. To see this consider the image $A'_{v,N}$ in $A'_{v}$ of the $g_N$-submodule $U(g_N) \cdot \tilde{v} \subset A(p,E)$. The commutativity of the diagram

implies that $A'_{v,N}$ is isomorphic to a subquotient of $A(p_N, E_n)$. Since $Z_{U(g_N)}$ acts by one and the same character on $\text{ind}_{P_N}^{P_N} E_N$ and on $A(p_N, E_N)$, $A'_{v,N}$ is a $g_N$-module with a central character which is not regular integral, and is thus not an integrable $g_N$-module. This implies that $A'_v$ itself is not an integrable $g$-module.
(f) Assume in addition that $\text{im}\Psi_E$ is a simple $\mathfrak{h}$-module. Let $A^\#$ denote the $\mathfrak{g}$-submodule of $A(\mathfrak{p}, E)$ generated by $\text{im}\Psi_E$, and let $A^\delta$ be the sum of all $\mathfrak{g}$-submodules $X$ of $A^\#$ with $\text{Hom}_{\mathfrak{t}_0}(V_\mu(\mathfrak{p}^\nu), X) = 0$. Then (e) together with the $\mathfrak{t}_0$-semisimplicity of $A(\mathfrak{p}, E)$ imply that $A^\delta$ is a maximal proper $\mathfrak{g}$-submodule of $A^\#$, and hence $A^\# / A^\delta$ is the unique bottom layer subquotient of $A(\mathfrak{p}, E)$.

Finally, the inclusion $\mathfrak{m} \subset \mathfrak{t}$ yields $\mathfrak{m} = C_\mathfrak{g}(\mathfrak{t}_0) \subset \mathfrak{t}_0 + C_\mathfrak{g}(\mathfrak{t}_0)$ which implies that $\mathfrak{m} = \mathfrak{t}_0 + C_\mathfrak{g}(\mathfrak{t}_0)$. As $\mathfrak{t}_0$ is abelian, $E''$ is a simple $C_\mathfrak{g}(\mathfrak{t}_0)$-module, and the isomorphism $[3]$ of (e) implies that $A(\mathfrak{p}_0, E)$ is a simple $\mathfrak{t}$-module. Therefore (by (c)) $\text{im}\Psi_E$ is isomorphic to $A(\mathfrak{p}_0, E)$, and is thus a simple $\mathfrak{t}$-module. $\square$

In the spirit of [PSZ] we call a locally reductive subalgebra $\mathfrak{l} \subset \mathfrak{g}$ of a locally reductive Lie algebra $\mathfrak{g}$ primal, if there exists a simple strict $(\mathfrak{g}, \mathfrak{l})$-module $M$ such that $\mathfrak{l}$ is a maximal locally reductive subalgebra of $\mathfrak{g}[M]$. Using Theorem 3.2 one can prove that certain subalgebras $\mathfrak{l}$ are primal, for instance a subalgebra $\mathfrak{t} = \mathfrak{t}_0 + C_\mathfrak{g}(\mathfrak{t}_0)$ is primal whenever there exists an $\mathfrak{m}$-module $E$ satisfying the assumption of Theorem 3.2(d). Below we show the primality of $\mathfrak{t}$ in some special cases.

4 The case $\mathfrak{g} = \mathfrak{g} \ell(\mathfrak{p} \Theta)$

To illustrate our main result in the specific case of $\mathfrak{g} = \mathfrak{g} \ell(\mathfrak{p} \Theta)$, fix the exhaustion $\mathfrak{g} = \cup_n \mathfrak{g} \ell(\mathfrak{p} \theta_1 \ldots \theta_{n-1})$ as in Subsection 2.5. Let $\mathfrak{t}_0 \subset \mathfrak{g}_1 = \mathfrak{g} \ell(\mathfrak{p})$ be any reductive in $\mathfrak{g}_1$ subalgebra which contains a $\mathfrak{g}_1$-regular element $h$, and such that the $p$-dimensional natural $\mathfrak{g} \ell(\mathfrak{p})$-module $\mathcal{C}$ is simple as a $\mathfrak{t}_0$-module. For instance, $\mathfrak{t}_0$ may equal $\mathfrak{g} \ell(\mathfrak{p})$, $\mathfrak{s} \ell(\mathfrak{p})$ or a principal $\mathfrak{s} \ell(2)$-subalgebra of $\mathfrak{s} \ell(\mathfrak{p})$. Let $\mathfrak{t}_0 := C_{\mathfrak{t}_0}(h)$. We define $\mathfrak{p}$ as the $\mathfrak{t}_0$-compatible parabolic subalgebra $\bigoplus_{\sigma, \Theta \geq 0} \mathfrak{g}_h^\sigma$.

Lemma 4.1

(a) $\mathfrak{m} \cap \mathfrak{g}_n \simeq \mathfrak{g} \ell(\theta_1 \ldots \theta_{n-1})^p$.
(b) $C_{\mathfrak{g}_n}(\mathfrak{t}_0) \simeq \mathfrak{g} \ell(\theta_1 \ldots \theta_{n-1})$ is the diagonal subalgebra in $\mathfrak{g} \ell(\theta_1 \ldots \theta_{n-1})^p$.

Proof. As an $C_{\mathfrak{g}_n}(\mathfrak{h})$-module, the natural representation $V_n$ of $\mathfrak{g} \ell(\mathfrak{p} \theta_1 \ldots \theta_{n-1})$ decomposes as a direct sum of $p$ isotypic components each of dimension $\theta_1 \ldots \theta_{n-1}$. This yields (a).

As a $\mathfrak{t}_0$-module $V_n$ decomposes as a direct sum of $\theta_1 \ldots \theta_{n-1}$ copies of the simple $\mathfrak{t}_0$-module $\mathcal{C}^p$. This implies (b). $\square$

Corollary 4.2

(a) $\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{t}_0) = \mathfrak{g} \ell(\Theta)^p$;
(b) $\mathfrak{t} \simeq \mathfrak{t}_0 + \mathfrak{g} \ell(\Theta)$, $\mathfrak{t}_0 \cap \mathfrak{g} \ell(\Theta) \subset \mathbb{Z}_\mathfrak{g} \ell(\Theta)$;
(c) if $\mathfrak{t}_0 = \mathfrak{g} \ell(\mathfrak{p})$, then $\mathfrak{t} \simeq \mathfrak{g} \ell(\mathfrak{p}) + \mathfrak{g} \ell(\Theta)$ is a maximal proper subalgebra of $\mathfrak{g} \ell(\mathfrak{p} \Theta)$.

We now construct a class of simple $\mathfrak{g} \ell(\Theta)$-modules. Let $V_n$ denote the natural representation of $\mathfrak{g} \ell(\theta_1 \ldots \theta_{n-1})$. Fix $n_0 > 1$ and let $V(\lambda_{n_0})$ be the simple finite dimensional $\mathfrak{g} \ell(\theta_1 \ldots \theta_{n_0-1})$-module with highest weight $\lambda_{n_0} = (\lambda^1, \ldots, \lambda^{n_0-1}), \lambda^i \geq \lambda^{i+1}$. Define $n' = n'(\lambda_{n_0-1})$ as the largest index for which the entry $\lambda^{n'}$ is non-negative; if $\lambda^1 < 0$, we put $n' = 0$. To $\lambda_{n_0}$ we assign the following highest weight of $\mathfrak{g} \ell(\theta_1 \ldots \theta_{n_0})$:

$$\lambda_{n_0+1} := (\lambda^1, \ldots, \lambda^{n'}, 0, 0, \ldots, 0, \lambda^{n'+1}, \ldots, \lambda_{\theta_1 \ldots \theta_{n_0}(\theta_{n_0+1-1})\text{times}}).$$
Lemma 4.3 There is a natural injection of \( \mathfrak{g}\ell(\theta_1 \ldots \theta_{n_0-1})^{\theta_{n_0}} \)-modules
\[
V(\lambda_{n_0})^{\theta_{n_0}} \to V(\lambda_{n_0+1}),
\]
and hence a diagonal injection of \( \mathfrak{g}\ell(\theta_1 \ldots \theta_{n-1}) \)-modules
\[
V(\lambda_n) \to V(\lambda_{n+1})
\]
for any \( n > n_0 \).

Proof. The natural injection \( V_{n_0}^{\theta_{n_0}} \to V_{n_0+1} \) induces a natural injection of \( \mathfrak{g}\ell(\theta_1 \ldots \theta_{n_0})^{\theta_{n_0+1}} \)-modules
\[
T^r(V_{n_0} \oplus V_{n_0}^*)^{\theta_{n_0+1}} \to T^r(V_{n_0+1} \oplus V_{n_0+1}^*)
\]
which in turn induces an injection
\[
V(\lambda_{n_0})^{\theta_{n_0}} \to V(\lambda_{n_0+1})
\]
as required. \( \square \)

Corollary 4.4 For every \( n_0 \) and any dominant integral weight \( \lambda_{n_0} \) of \( \mathfrak{g}\ell(\theta_1 \ldots \theta_{n_0-1}) \), \( \tilde{V}(\lambda_{n_0}) \) is a simple \( \mathfrak{g}\ell(\Theta) \)-module defined as the direct limit \( \varinjlim_{n \geq n_0} V(\lambda_n) \), where \( V(\lambda_n) \) is embedded diagonally into \( V(\lambda_{n+1}) \) according to Lemma 4.3.

Let now \( \lambda_{n_1} \ldots \lambda_{n_p} \) be \( p \) dominant weights as in Corollary 4.4. Assume that the ordering of the weights is compatible with \( \mathfrak{n} \), i.e. that the \( h \) value of any root \( \varepsilon_i - \varepsilon_j, i < j \), of \( \mathfrak{g}_1 = \mathfrak{g}\ell(p) \) has non-negative real part. Define \( E \) as \( V(\lambda_{n_1}) \otimes \ldots \otimes \tilde{V}(\lambda_{n_p}) \) with trivial action of \( \mathfrak{n} \).

Proposition 4.5 \( M(\mathfrak{p}, E) = \text{ind}_{\mathfrak{p}}^\mathfrak{m} E \) is an \( (\mathfrak{m}, \mathfrak{t} \cap \mathfrak{m}) \)-module of finite type.

Proof. It suffices to show that \( \text{Gr} M(\mathfrak{p}, E) \) is an \( (\mathfrak{m}, \mathfrak{t} \cap \mathfrak{m}) \)-module of finite type. As a \( \mathfrak{m} \)-module \( \text{Gr} M(\mathfrak{p}, E) \) is isomorphic to \( S'(\tilde{\mathfrak{n}}) \otimes E \), and is in particular a weight module over the Cartan subalgebra \( \mathfrak{t}_0 \) of \( \mathfrak{t}_0 \). This subalgebra acts via a single weight on \( E \) and via arbitrary sums of \( \mathfrak{p} \)-negative \( \mathfrak{t}_0 \)-weights on \( S'(\tilde{\mathfrak{n}}) \). Since each \( \mathfrak{t}_0 \)-weight of \( S'(\tilde{\mathfrak{n}}) \) occurs only in finitely many symmetric powers of \( \tilde{\mathfrak{n}} \), it suffices to show that each fixed tensor product \( S'(\tilde{\mathfrak{n}}) \otimes E \) is a \( \mathfrak{t} \cap \mathfrak{m} \)-module of finite length. Notice that \( E \) is a direct limit of \( E_\mathfrak{n} \) such that each \( E_\mathfrak{n} \) is a \( C_{\mathfrak{g}_0}(\mathfrak{t}_0) \simeq \mathfrak{g}\ell(\theta_1 \ldots \theta_{n-1}) \)-submodule of a fixed tensor power \( T^k(V_n^p \oplus (V_n^*)^p) \). Hence \( S'((\tilde{\mathfrak{n}}_n) \otimes E_\mathfrak{n} \) is also contained in a fixed tensor power \( T^k(V_n^p \oplus (V_n^*)^p) \). Proposition 2.7 now implies that, for each \( n \), \( S'((\tilde{\mathfrak{n}}_n) \otimes E_\mathfrak{n} \) is a \( C_{\mathfrak{g}_0}(\mathfrak{t}_0) \cap \mathfrak{g}_{n_0} \)-module of finite length, hence \( S'((\tilde{\mathfrak{n}}) \otimes E \) is a \( \mathfrak{t} \cap \mathfrak{m} \)-module of finite length. The Proposition follows. \( \square \)

Note that the assumptions of Theorem 3.8(e) apply to the case we consider. Therefore, to ensure that \( A(\mathfrak{p}, E) \) is non-zero, it suffices to ensure that the weight \( \mu^\vee \) is integral \( \mathfrak{t}_0 \)-dominant. An easy computation shows that the weight \( \mu \) is nothing but the weight \( \sum_i \lambda^i_{n_0} \sum_i \lambda^i_{n_0} \ldots \sum_i \lambda^i_{n_0} \) of \( \mathfrak{g}_1 \), restricted to \( \mathfrak{t}_0 \). Let \( \mathfrak{t}_0 = \mathfrak{g}\ell(p) \). Then the regularity and \( \mathfrak{t}_0 \)-dominancy condition on \( \mu^\vee \) are equivalent to the condition
\[
\sum_i \lambda^i_{n_0} \leq \sum_i \lambda^i_{n_0} \leq \ldots \leq \sum_i \lambda^i_{n_0}.
\]
Note furthermore, that our choice of weights \( \lambda_{n_0}, \ldots, \lambda_{n_p} \) allows for the possibility the \( Z_{U(g_n)} \)-character of \( \text{ind}_{\mathfrak{p}_N}^{\mathfrak{g}_N}E_N \) to be non-regular for some \( N \), and hence in the latter case, no irreducible bottom layer quotient of \( A(\mathfrak{p}, E) \) is \( \mathfrak{g} \)-integrable. Since \( \mathfrak{t}_0 = g\ell(p) \), \( \mathfrak{t} \) is a maximal proper subalgebra of \( g\ell(p\Theta) \). This implies (via Theorem 3.2(d)) that whenever \( A(\mathfrak{p}, E) \) is not integrable, any irreducible bottom layer quotient of \( A(\mathfrak{p}, E) \) is a strict \((\mathfrak{g}, \mathfrak{t})\)-module. In particular, \( \mathfrak{t} = g\ell(p) + g\ell(\Theta) \) is a proper subalgebra of \( g\ell(p\Theta) \).

Finally, Lemma 4.1(a) and (b) imply that the condition \( \mathfrak{m} \subset \mathfrak{t} \) from Theorem 3.2(f) holds only when \( p = 1 \). However, in this case \( s = 0 \), hence the claim of (f) is trivial. Nevertheless, there is an interesting non-trivial case in which Theorem 3.2(f) applies: this is when \( \lambda_{n_0} = \ldots = \lambda_{n_p-1} = 0 \) and \( \lambda_{n_p} \neq 0 \). In this latter case \( E'' \) is clearly a simple \( C_{\mathfrak{g}}(\mathfrak{t}_0) \)-module. Furthermore, as it is easy to see, for large \( n \) the \( Z_{U(g_n)} \)-character of \( \text{ind}_{\mathfrak{p}_N}^{\mathfrak{g}_N}E_N \) is integral but not regular, hence the \((\mathfrak{g}, \mathfrak{t})\)-module \( A(\mathfrak{p}, E) \) has a unique strict simple subquotient.

\section{The root-reductive case}

Let now \( \mathfrak{g} \) be a simple infinite dimensional root-reductive Lie algebra, i.e. \( \mathfrak{g} \cong s\ell(\infty), so(\infty), sp(\infty) \). Fix an exhaustion \( \mathfrak{g} = \cup_n \mathfrak{g}_n \), where \( \mathfrak{g}_n \subset \mathfrak{g}_{n+1} \) is a root injection of the form \( s\ell(i) \subset s\ell(i+1) \), \( so(i) \subset so(i+2) \), or \( sp(2i) \subset sp(2i+2) \), for \( \mathfrak{g} \) isomorphic respectively to \( s\ell(\infty) \), \( so(\infty) \) or \( sp(\infty) \). Then each \( \mathfrak{g}_n \) is reductive in \( \mathfrak{g} \) and \( C_{\mathfrak{g}}(\mathfrak{g}_n) \cong \mathfrak{g} \) for \( \mathfrak{g} \cong so(\infty), sp(\infty) \), and \( C_{\mathfrak{g}}(\mathfrak{g}_n) \cong g\ell(\infty) \) for \( \mathfrak{g} = s\ell(\infty) \). Moreover, for a fixed \( n \), the subalgebra \( \mathfrak{g}_n + C_{\mathfrak{g}}(\mathfrak{g}_n) \) has the property that its intersections with \( \mathfrak{g}_n' \) for all \( n' > n \) are symmetric subalgebras.

We fix next a reductive in \( \mathfrak{g}_1 \) subalgebra \( \mathfrak{t}_0 \subset \mathfrak{g}_1 \), a Cartan subalgebra \( \mathfrak{t}_0 \subset \mathfrak{t}_0 \) and a \( \mathfrak{t}_0 \)-compatible parabolic subalgebra \( \mathfrak{p} = \mathfrak{m} \supset \mathfrak{n} \), and let \( \mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{t}_0 \). For instance, for \( \mathfrak{g} \cong s\ell(\infty) \), \( \mathfrak{p} \) can be a maximal proper subalgebra of \( \mathfrak{g} \), whose intersection with \( \mathfrak{g}_n \) for \( n > 1 \) equals a maximal parabolic subalgebra of \( \mathfrak{g}_n \) containing \( C_{\mathfrak{g}_n}(\mathfrak{g}_1) \). Note that

\begin{equation}
\mathfrak{m}_0 + C_{\mathfrak{g}}(\mathfrak{g}_1) \subset \mathfrak{t} \cap \mathfrak{m}.
\end{equation}

Let \( E = \cup_n E_n \), where, for \( n \) large enough, each \( E_n \) is a simple \( \mathfrak{m}_n \)-submodule of a tensor power \( T^k(V_{n}^a \oplus (V_n^b)^c) \) for fixed \( k, a, b, c \) (when \( \mathfrak{g} \cong so(\infty), sp(\infty) \), there is an isomorphism \( V_n \cong V_n^* \)).

\textbf{Proposition 5.1} \( M(\mathfrak{p}, E) \) is an \((\mathfrak{m}, \mathfrak{t} \cap \mathfrak{m})\)-module of finite type.

\textbf{Proof.} According to (9), it suffices to show that \( M(\mathfrak{p}, E) \) is an \( \mathfrak{m}_0 + C_{\mathfrak{g}}(\mathfrak{g}_1) \)-module of finite type. The argument is very similar to that in the proof of Proposition 4.1. Consider \( \text{Gr}M(\mathfrak{p}, E) \cong S'(\bar{\mathfrak{n}}) \otimes E \) and note that only finitely many \( \mathfrak{t}_0 \)-weights occur in \( E \), and that each \( \mathfrak{t}_0 \)-weight of \( S'(\bar{\mathfrak{n}}) \) will occur only in finitely many symmetric powers of \( \bar{\mathfrak{n}} \). Hence it suffices to show that each fixed tensor product \( S^t(\bar{\mathfrak{n}}) \otimes E \) is a \( C_{\mathfrak{g}}(\mathfrak{g}_1) \)-module of finite length. However, a direct verification based on the definition of \( \mathfrak{g}_1 \) shows that for each \( n > 1, \bar{\mathfrak{n}} \cap \mathfrak{g}_n \) is a \( C_{\mathfrak{g}}(\mathfrak{g}_1) \cap \mathfrak{g}_n \)-submodule of a fixed tensor power \( T^k(V_n^a \oplus (V_n^b)^c) \), where \( V_n \) is the natural representation of \( C_{\mathfrak{g}}(\mathfrak{g}_1) \cap \mathfrak{g}_n \), and \( a, b, c \in Z_{>0} \). Hence, for each fixed \( t \), \( S^t(\bar{\mathfrak{n}} \cap \mathfrak{g}_n) \otimes E_n \) is a submodule of an analogous fixed tensor power, and by Proposition 2.1 \( S^t(\bar{\mathfrak{n}}) \otimes E \) is a \( C_{\mathfrak{g}}(\mathfrak{g}_1) \)-module of finite length. \( \square \)

In the remainder of this section we concentrate on the case \( \mathfrak{t}_0 = \mathfrak{g}_1 \), assuming that \( \mathfrak{g}_1 \) is non-abelian. In this case \( \mathfrak{t}_n = (\mathfrak{g}_1 + C_{\mathfrak{g}}(\mathfrak{g}_1)) \cap \mathfrak{g}_n \) is a symmetric subalgebra of \( \mathfrak{g}_n \).
for \( n \geq 2 \) and the existing literature on Harish-Chandra modules enables us to prove a
stronger version of our main result under slightly different conditions on the compatible
parabolic subalgebra \( \mathfrak{p} \) and the \( \mathfrak{p} \)-module \( E \). More precisely, let \( \mathfrak{p} \) equal \( \bigoplus_{\sigma \geq 0} \mathfrak{g}_{\sigma}^\sigma \) for some
real diagonal matrix \( h \in \mathfrak{t}_0 \), and \( \mathfrak{m} := C_{\mathfrak{g}}(h) \). Then \( \mathfrak{m} \) is the direct sum of a reductive in
\( \mathfrak{t}_0 \) subalgebra \( \mathfrak{m}' \) and an infinite dimensional subalgebra \( \mathfrak{m}'' \) isomorphic to \( \mathfrak{gl}(\infty), \mathfrak{so}(\infty) \) or \( \mathfrak{sp}(\infty) \). Note that \( \mathfrak{m}'' \cong C_{\mathfrak{g}}(0) \) and that \( (m_n, \mathfrak{t}_n \cap \mathfrak{m}_n) \) is a symmetric pair for each \( n \).

**Theorem 5.2** For \( \mathfrak{g} \) and \( \mathfrak{k} \) as above, let the \( \mathfrak{p} \)-module \( E \) satisfy the condition of Theorem
3.2(c). In addition, assume that, for some \( N \in \mathbb{Z}_{\geq 0} \), \( E_N \) is a simple finite dimensional
\( \mathfrak{m}_N \)-module such that \( A(\mathfrak{p}_N, E_N) \) is a simple strict \( (\mathfrak{g}_N, \mathfrak{t}_N) \)-module with non-zero bottom
layer. Let \( v \in A(\mathfrak{p}, E) \) be a non-zero vector in the image of the bottom layer of \( A(\mathfrak{p}_N, E_N) \)
(the existence of \( v \) follows from Theorem 3.2(c)) and let \( X_v \) be a simple quotient of \( U(\mathfrak{g}) \cdot v \).
Then
(a) \( X_v \) is a strict \( (\mathfrak{g}, \mathfrak{t}) \)-module;
(b) if, for all \( n \), \( E_n \) has finite length as a \( (\mathfrak{t}_n \cap \mathfrak{m}_n) \)-module, \( X_v = \cup_n (X_v)_n \) where each
\( (X_v)_n \) is a Harish-Chandra \( (\mathfrak{g}_n, \mathfrak{t}_n) \)-module.

**Proof.**

(a) Let \( \pi : U(\mathfrak{g}) \cdot v \to X_v \) be the projection which defines \( X_v \), and let \( \kappa : A(\mathfrak{p}_N, E_N) \to
A(\mathfrak{p}, E) \) be the functorially induced map of \( (\mathfrak{g}_N, \mathfrak{t}_N) \)-modules. By our assumptions, \( (\pi \circ \kappa)(v) \neq 0 \) and as \( A(\mathfrak{p}_N, E_N) \) is simple, \( \pi \circ \kappa \neq 0 \) is injective. It follows that \( \mathfrak{g}_N[A(\mathfrak{p}_N, E_N)] 
\subseteq \mathfrak{g}[X] \cap \mathfrak{g}_N \). Since \( \mathfrak{g}_N[A(\mathfrak{p}_N, E_N)] = \mathfrak{t}_N \) and is a \( (\mathfrak{g}, \mathfrak{t}) \)-module we conclude that \( \mathfrak{g}[X] \cap \mathfrak{g}_N = \mathfrak{t}_N \).

The inclusion \( \mathfrak{g}[X] \supset \mathfrak{t} \) implies the following possibilities for \( \mathfrak{g}[X] \). If \( \mathfrak{g} = \mathfrak{so}(\infty), \mathfrak{sp}(\infty) \)
\( \mathfrak{g}[X] \) equals \( \mathfrak{t} \) or \( \mathfrak{g} \) as \( \mathfrak{t} \) is a maximal subalgebra of \( \mathfrak{g} \), and if \( \mathfrak{g} = \mathfrak{sl}(\infty) \) there are four
possibilities for \( \mathfrak{g}[X] \): \( \mathfrak{g} \), the two opposite parabolic subalgebras \( \mathfrak{q}^\pm \) containing \( \mathfrak{t} \), and the
subalgebra \( \mathfrak{t} \). However, in all cases the only possibility compatible with the equality
\( \mathfrak{g}[X] \cap \mathfrak{g}_N = \mathfrak{t}_N \) is \( \mathfrak{g}[X] = \mathfrak{t} \). This proves (a).

(b) Define \( X_n \) as the image of the functorial map of \( A(\mathfrak{p}_n, E_n) \) to \( X \). We have \( A(\mathfrak{p}_n, E_n) = R^\Gamma_{\mathfrak{t}_n, \mathfrak{m}_n}(\text{ind}_{\mathfrak{t}_n, \mathfrak{m}_n}^\mathfrak{p}_n E_n) \), \( \mathfrak{t}_n = \mathfrak{t}_0 + C_{\mathfrak{g}_n}(\mathfrak{t}_0) \), and \( \mathfrak{t}_n \cap \mathfrak{m}_n = \mathfrak{m}_0 + C_{\mathfrak{g}_n}(\mathfrak{t}_0) \).

The comparison principle yields an isomorphism of \( (\mathfrak{g}_n, \mathfrak{t}_n) \)-modules
\[ A(\mathfrak{p}_n, E_n) \cong R^\Gamma_{\mathfrak{t}_n, \mathfrak{t}_n \cap \mathfrak{m}_n}(\text{ind}_{\mathfrak{t}_n, \mathfrak{m}_n}^\mathfrak{p}_n E_n). \]

Since \( (\mathfrak{m}_n, \mathfrak{t}_n \cap \mathfrak{m}_n) \) and \( (\mathfrak{g}_n, \mathfrak{t}_n) \) are finite dimensional symmetric pairs, any \( (\mathfrak{g}_n, \mathfrak{t}_n) \)-module
(respectively \( (\mathfrak{m}_n, \mathfrak{t}_n \cap \mathfrak{m}_n) \)-module) of finite length is also of finite type, and hence is a
Harish-Chandra module. Moreover, results in [KV] Ch. V imply that if \( E_n \) has finite
length, then \( A(\mathfrak{p}_n, E_n) \) likewise has finite length. Hence \( X_n \) itself has finite length, i.e. is
a Harish-Chandra module. □

It is easy to construct \( (\mathfrak{m}, \mathfrak{t} \cap \mathfrak{m}) \)-modules \( E \) which satisfy both the assumptions of
Proposition 5.1 and Theorem 5.2. To satisfy the assumption of Theorem 5.2 we can take
\( E \) to be the union \( \cup_n E_n \) of finite dimensional simple \( \mathfrak{m}_n \)-modules under appropriate
inclusions of \( \mathfrak{m}_n \)-modules \( E_n \hookrightarrow E_{n+1} \). For a fixed \( N \), we can take \( E_N \) (for instance \( E_N = \mathbb{C}_{\mathfrak{p}_N}, \) see Theorem 6.1 below) so that \( A(\mathfrak{p}_N, E_N) \) is simple with non-zero bottom layer. It is
also clear that each \( E_n \) can be chosen to be a simple submodule of \( T^k(V^a_n \oplus (V^*_N)^b \oplus \mathbb{C}^c) \)
for some fixed \( a, b, c, k \in \mathbb{Z}_{\geq 0} \). Indeed, one can fix \( a, b, c, k \) so that the already chosen
\( \mathfrak{m}_N \)-module \( E_N \) be a submodule of \( T^k(V^a_N \oplus (V^*_N)^b \oplus \mathbb{C}^c) \) and then, for \( n \geq N \), recursively
choose $E_n$ as a simple submodule of $T^k(V_0^a \oplus (V_0^b)^b \oplus C^c)$ for which there is an injection of $m_{n-1}$-modules $E_{n-1} \to E_n$. Such a module $E_n$ clearly exists.

**Corollary 5.3** If $g = sl(\infty), so(\infty), sp(\infty)$ and $\mathfrak{e}_0 = g_1$ where $g_1$ is not abelian, then $\mathfrak{e} = \mathfrak{e}_0 \subseteq C_{g}(\mathfrak{e}_0)$ is a primal subalgebra of $g$, and moreover there exists a simple strict $(g, \mathfrak{e})$-module $X$ of finite type such that $X = \cup_{n} X_n$ where $X_n$ are Harish-Chandra $(g_n, \mathfrak{e} \cap \mathfrak{g}_n)$-modules.

### 6 Appendix: The Fernando-Kac subalgebra of a Vogan-Zuckerman module

Our aim in this appendix is to relate some of the basic literature on applications of cohomological induction with Section 5 of this paper. More precisely, we recall the definition of a class of Harish-Chandra modules known as the Vogan-Zuckerman modules, [VZ], and compute the Fernando-Kac subalgebra of a Vogan-Zuckerman module.

Let $g$ be a finite dimensional reductive Lie algebra (over $\mathbb{C}$), $\mathfrak{t}$ be a symmetric subalgebra of maximal rank, $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{t}$ and let $\mathfrak{p}$ be a $\mathfrak{t}$-compatible parabolic subalgebra of $g$. Fix a Levi decomposition $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ of $\mathfrak{p}$ with $\mathfrak{t} \subseteq \mathfrak{m}$, and also a $\mathfrak{t}$-compatible Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{p}$. Then $\mathfrak{b} \cap \mathfrak{t}$ is a Borel subalgebra of $\mathfrak{t}$ and $\mathfrak{b} \cap \mathfrak{m}$ is a Borel subalgebra of $\mathfrak{m}$. Relative to $\mathfrak{b}$, let $w_0$ be the longest element in the Weyl group of $\mathfrak{t}$ in $g$; relative to $\mathfrak{b} \cap \mathfrak{m}$ let $w_m$ be the longest element in the Weyl group of $\mathfrak{t}$ in $\mathfrak{m}$. Finally, let $\lambda_p := w_0 \circ w_m^{-1}(\rho_b) - \rho_b$. Note that $\lambda_p|_{[\mathfrak{m}, \mathfrak{m}]} = 0$, so that $\lambda_p$ defines a one-dimensional $\mathfrak{p}$-module $\mathbb{C}_{\lambda_p}$.

The induced $g$-module $\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}C_{\lambda_p}$ and the $(g, \mathfrak{t})$-module $A_p := R^s\Gamma_{e, \mathfrak{t}, \mathfrak{m}}(\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}C_{\lambda_p})$ have the same central character as the trivial $g$-module. (Here, as usual, $s = \frac{1}{2} \dim(\mathfrak{t}/\mathfrak{t} \cap \mathfrak{m})$.)

More generally, if $F := V_{\tilde{\mathfrak{g}}}(\tilde{\lambda})$ and $\tilde{\lambda} := w_0 \circ w_m^{-1}(\lambda + \rho_b) - \rho_b$, then the induced $g$-module $\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V_m(\tilde{\lambda}))$ and the $(g, \mathfrak{t})$-module $A_p(F) := R^s\Gamma_{e, \mathfrak{t}, \mathfrak{m}}(\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V_m(\tilde{\lambda})))$ have the same central character as $F$. We call $A_p(F)$ the Vogan-Zuckerman module attached to the pair $(\mathfrak{p}, F)$.

(This definition can be extended to the case rank $\mathfrak{t} < \text{rank } g$, but we do not consider this generalization here.)

**Theorem 6.1**

(a) The bottom layer of $A_p$ is simple, in particular non-zero.

(b) $A_p(F)$ is a simple $(g, \mathfrak{t})$-module, which is infinite dimensional if $\mathfrak{p}$ is proper in $g$.

**Proof.**

(a) By Lemma 3.1, the bottom layer of $A_p$ is isomorphic to $V_{\mathfrak{t}}(\lambda_p^\vee)$. This implies that the bottom layer of $A_p$ is simple if non-zero. To ensure that it is indeed non-zero, we need to verify that $\lambda_p^\vee$ is dominant with respect to $\mathfrak{t}$. This follows from [VZ Section 3], where it is established that $V_{\mathfrak{t}}(\lambda_p^\vee)$ is a non-zero constituent of the $\mathfrak{t}$-module $\Lambda(\mathfrak{t}^\perp)$.

(b) For the simplicity of $A_p(F)$ see Theorem 8.2 on p. 550 in [KV]. When $\mathfrak{p}$ is proper, it is shown in [VZ Section 2] that $A_p$ has a non-trivial $\mathfrak{t}$-submodule. Since $A_p$ has the central character of the trivial $g$-module, $\dim A_p = \infty$. By using the translation functor one shows that $A_p(F)$ is likewise infinite dimensional. \(\square\)

From now on we assume that $[g, g]$ is simple and that $\mathfrak{p}$ is proper in $g$. We want a formula for the Fernando-Kac subalgebra associated to $A_p(F)$. If $\mathfrak{t}$ is maximal in $g$,
clearly $A_p(F)$ is a strict $(\mathfrak{g}, \mathfrak{t})$-module under our assumptions. If $\mathfrak{t}$ is not maximal, then its orthogonal complement $\mathfrak{t}^\perp \subset \mathfrak{g}$ is reducible as a $\mathfrak{t}$-module: $\mathfrak{t}^\perp = \mathfrak{r} \oplus \bar{\mathfrak{r}}$, where $\mathfrak{r}$ and $\bar{\mathfrak{r}}$ are abelian subalgebras of $\mathfrak{g}$, and $\mathfrak{t} \supset \mathfrak{r}$ and $\mathfrak{t} \supset \bar{\mathfrak{r}}$ are parabolic subalgebras of $\mathfrak{g}$. Moreover, there are precisely four subalgebras of $\mathfrak{g}$ containing $\mathfrak{t}$: $\mathfrak{t}, \mathfrak{t} \supset \mathfrak{r}, \mathfrak{t} \supset \bar{\mathfrak{r}}, \mathfrak{g}$.

**Theorem 6.2** Assume $[\mathfrak{g}, \mathfrak{g}]$ is simple, $\mathfrak{t}$ is not maximal and $\mathfrak{p}$ is proper in $\mathfrak{g}$.

(a) $\mathfrak{g}[A_p(F)] = \mathfrak{t} \supset \mathfrak{r}$ if $\mathfrak{r} \cap \mathfrak{n} = 0$.

(b) $\mathfrak{g}[A_p(F)] = \mathfrak{t} \supset \bar{\mathfrak{r}}$ if $\mathfrak{r} \cap \mathfrak{n} = 0$.

(c) $\mathfrak{g}[A_p(F)] = \mathfrak{t}$ if $\mathfrak{r} \cap \mathfrak{n}$ and $\bar{\mathfrak{r}} \cap \mathfrak{n}$ are both nonzero.

The proof of Theorem 6.2 is based on a lemma relating $\mathfrak{g}[A_p]$ with $\text{Hom}_k(\Lambda^\cdot(\mathfrak{r} \oplus \bar{\mathfrak{r}}), A_p)$, where $\Lambda^\cdot$ stands for a bigraded exterior algebra. Set $a := \dim \mathfrak{r} \cap \mathfrak{n}$ and $b := \dim \mathfrak{r} \cap \mathfrak{n}$. Then, according to the key Proposition 6.19 of [VZ], $\text{Hom}_k(\Lambda^\cdot(\mathfrak{r} \oplus \bar{\mathfrak{r}}), A_p)$ is concentrated in bidegrees of the form $(a + j, b + j)$.

**Lemma 6.3**

(a) $\mathfrak{g}[A_p] = \mathfrak{t} \supset \mathfrak{r} \iff a = 0$.

(b) $\mathfrak{g}[A_p] = \mathfrak{t} \supset \bar{\mathfrak{r}} \iff b = 0$.

(c) $\mathfrak{g}[A_p] = \mathfrak{t} \iff a \neq 0$ and $b \neq 0$.

**Proof of Lemma 6.3**

(a) $\mathfrak{g}[A_p] = \mathfrak{t} \supset \mathfrak{r}$ if and only if there exists a simple finite dimensional $\mathfrak{t}$-module $V$ such that $A_p$ is isomorphic to the unique irreducible quotient $L(\mathfrak{t} \supset \mathfrak{r}, V)$ of $\text{ind}_{\mathfrak{t} \supset \mathfrak{r}} \mathfrak{g}$. But the central character of $A_p$ is trivial and this constrains $V$ to a finite set: $\mathfrak{r}$ must be a $\mathfrak{t}$-type in $\Lambda^\cdot(\bar{\mathfrak{r}})$. Hence, $\mathfrak{g}[A_p] = \mathfrak{t} \supset \mathfrak{r}$ implies $\text{Hom}_k(\Lambda^\cdot(\bar{\mathfrak{r}}), A_p) \neq 0$ which in turn implies $a = 0$.

Conversely, suppose $a = 0$. Let, for some simple finite dimensional $\mathfrak{t}$-module $V$, the $V$-isotypic subspace $A_p[V]$ of $A_p$ be in the bottom layer of $A_p$. Theorem 2.5 in [VZ] gives a necessary condition for a simple $\mathfrak{t}$-module $V$ to occur in the restriction of $A_p$ to $\mathfrak{t}$. This condition implies that $\mathfrak{r} \cdot A_p[V] = 0$. Hence $A_p \cong L(\mathfrak{t} \supset \mathfrak{r}, V)$.

(b) Repeat proof of (a) but substitute $\bar{\mathfrak{r}}$ for $\mathfrak{r}$.

(c) Follows from the combination of (a) and (b) and the statement above about $\text{Hom}_k(\Lambda^\cdot(\mathfrak{r} \oplus \bar{\mathfrak{r}}), A_p)$. □

**Proof of Theorem 6.2** First we reduce to the case $F = \mathbb{C}$, $\lambda = 0$: for any $F$ we have a pair of translation functors $\varphi_\Lambda$ and $\psi_\Lambda$ such that $A_p(F) \cong \varphi_\Lambda(A_p)$ and $A_p \cong \psi_\Lambda(A_p(F))$ (see [KV, Ch. VII, Thm. 7.237]). Since $\varphi_\Lambda(A_p)$ is a direct summand of $F \otimes A_p$, we have $\mathfrak{g}[A_p(F)] \supset \mathfrak{g}[A_p]$. Likewise, $\psi_\Lambda(A_p(F))$ is a direct summand of $F^* \otimes A_p(F)$. Hence, $\mathfrak{g}[A_p] \supset \mathfrak{g}[A_p(F)]$. Thus, $\mathfrak{g}[A_p(F)] = \mathfrak{g}[A_p]$. □

**Example.** Let $\mathfrak{g} = sl(n)$ with $n = p + q$, $p > 1$ and $q > 0$, and $\mathfrak{t} = s(gl(p) \oplus gl(q))$, the traceless matrices in the subalgebra $gl(p) \oplus gl(q)$ embedded in the standard fashion in $gl(n)$. We have $\mathfrak{t} = sl(p) \oplus gl(q)$, where $gl(q)$ is embedded as the centralizer of $sl(p)$ in $\mathfrak{g}$. Let $\mathfrak{t} \subseteq \mathfrak{t}$ be the diagonal matrices; $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and of $\mathfrak{g}$. Choose any real nonzero matrix $h \in \mathfrak{t} \cap sl(p)$ and let $\mathfrak{p}$ be the $\mathfrak{t}$-compatible parabolic subalgebra associated to $h \in \mathfrak{t}$. The subalgebra $\mathfrak{t}$ is not maximal and we have a triangular decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t} \oplus \bar{\mathfrak{t}}$, where $\mathfrak{t}$ and $\bar{\mathfrak{t}}$ are nonzero simple $\mathfrak{t}$-submodules of $\mathfrak{g}$. Furthermore, since $h$ has both positive and negative diagonal values, $\mathfrak{p} \cap \mathfrak{t} \neq 0$ and $\mathfrak{p} \cap \bar{\mathfrak{t}} \neq 0$. Therefore, for any simple finite dimensional $\mathfrak{g}$-module $F$, Theorem 6.2(c) implies that $A_p(F)$ is a strict simple $(\mathfrak{g}, \mathfrak{t})$-module.
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