Sensitivity of Uncertainty Propagation for the Elliptic Diffusion Equation

Oliver G. Ernst∗ Alois Pichler∗† Björn Sprungk‡

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Abstract

For elliptic diffusion equations with random coefficient and source term, the probability measure of the solution random field is shown to be Lipschitz-continuous in both total variation and Wasserstein distance as a function of the input probability measure. These results extend to Lipschitz continuous quantities of interest of the solution as well as to coherent risk functionals of those applied to evaluate their uncertainty. Our analysis is based on the sensitivity of risk functionals and pushforward measures for locally Lipschitz mappings with respect to the Wasserstein distance of perturbed input distributions. The established results particularly apply to the case of lognormal diffusions and the truncation of series representations of input random fields.

Keywords: Uncertainty propagation, forward UQ, risk measure, risk functional, Wasserstein distance, total variation distance, diffusion equation, sensitivity, robustness

1 Introduction

A fundamental task in uncertainty quantification (UQ) for models in the physical sciences is the solution of differential equations with random inputs. These account in a probabilistic fashion for uncertainty in the data of a differential equation potentially arising in coefficient functions, source terms, initial and boundary data as well as the domain on which the problem is posed. Within the broadening discipline of UQ this task is referred to as uncertainty propagation, or simply forward UQ. Once probability laws for the uncertain data have been identified, the solution is given by a random function, random field or stochastic process, and often functionals of the solution, known as quantities of interest (QoI) and their statistics are of primary interest in the analysis. The statistical post-processing of (random) QoI aims to extract useful information from the results of the computation for objectives such as optimization or decision support. Besides statistical moments or the probability of specific events, statistics known as risk measures or risk

∗Fakultät für Mathematik, TU Chemnitz, Germany
†DFG, German Research Foundation – Project-ID 416228727 – SFB 1410
‡Faculty of Mathematics and Computer Science, TU Bergakademie Freiberg, Germany
Functionals have been in use for decades in mathematical finance (see Artzner et al. [2, 3]) to quantify the (typically negative) impact of random outcomes.

This paper investigates sensitivity of the results of an uncertainty propagation calculation with respect to perturbation of input probability measures. We follow established practice in considering this question for the model problem of an elliptic diffusion equation on a (fixed) bounded domain with uncertain coefficient and source term, which are modeled as random fields. The dependence of the solution of a random diffusion equation on these data has been considered previously, e.g., by Babuška et al. [4, Section 2.3], [5, Section 2.3] and Charrier [9, Section 4]. In these analyses of sensitivity, the random inputs and outputs are treated as (function-valued) random variables and their perturbations are measured by Bochner space norms. In this setting, the random PDE is equivalent to a PDE with a (high- or countably infinite-dimensional) parameter, which is represented by a vector or sequence of independent basic random variables serving as a coordinate system in the random dimensions. Their variation is naturally measured in weighted $L^p$ norms weighted by the densities of the basic random variables. In this way, the uncertainty propagation problem reduces to the numerical solution of parameterized PDEs, resulting in an entirely deterministic problem formulation.

In UQ analysis it is typically the probability distribution of the random outcomes that is of primary interest. At the same time, the precise probability distribution of the random inputs is generally unavailable and possibly guessed, elicited from expert opinion or the result of conditioning a prior probability distribution on observations as in the Bayesian formulation of inverse problems. There is, therefore, considerable uncertainty associated with the very probability measures used to model uncertainty in the inputs.

This paper derives basic sensitivity results of Lipschitz type for (i) the distribution of the random solution of an elliptic problem, (ii) the distribution of Lipschitz continuous QoI of this solution, and (iii) general risk functionals of the former—in each case with respect to perturbations of the underlying probability measure of the random inputs. The results further extend to QoI, which are only locally Lipschitz.

Among the vast variety of distances and divergences for probability measures (see, e.g., Gibbs and Su [17] or Rachev [30]), we focus on the Wasserstein metric for measuring perturbations of distributions for the following reasons:

(i) The Wasserstein distance metrizes weak convergence; that is, convergence of measures is ensured by testing convergence for appropriate test functions.

(ii) The Wasserstein metric provides a sensible distance also for mutually singular probability measures (as opposed to, e.g., the total variation or Hellinger metrics). This is important for the considered UQ setting since probability measures on infinite-dimensional function spaces tend to be mutually singular.

(iii) By convex duality, the Wasserstein distance allows sharp lower and upper bounds for the problem in mind. Here we develop upper bounds in detail.

(iv) In particular, the $p$-Wasserstein distance of two probability measures (on a normed space) is given by the infimum of the $L^p$-distance between any two random variables following these distributions. Thus, to bound the Wasserstein distance for perturbed random field
models, existing results on the Lebesgue norm distance of such random variables (as in Babuška et al. [4, 5], Charrier [9]) can be used.

(v) Finite convex combinations of Dirac measures form a dense subset of all probability measures with respect to Wasserstein distance (cf. Bolley [6]). This yields, in particular, the convergence of empirical approximations to the true distribution in the large sample limit which is important for applications in, for instance, finance and insurance.

In the context of optimization under uncertainty, risk functionals have also been investigated earlier by Kouri and Surowiec [19, 20] and Geiersbach and Wollner [15] as well as Dupuis et al. [14], Chowdhary and Dupuis [10]. We enhance the contribution of these authors by a sensitivity analysis regarding the probability measure and a new mathematical setting, which seems more natural to us by extending the technical interpretation of the results.

Outline of the paper. The following section provides the mathematical setting of the elliptic problem, discusses its stability and its extension to the probabilistic setting. Section 3 introduces risk functionals. To analyze risk functionals under changing probability measures we investigate them on product spaces and establish new continuity results with respect to the Wasserstein distance in Section 3.2. Section 4 presents the sensitivity analysis for uncertainty propagation for the elliptic problem in full detail and presents the main results of the paper. Section 5 concludes.

2 The Random Elliptic Diffusion Problem

The elliptic diffusion problem formulated on a bounded Lipschitz domain $D \subset \mathbb{R}^d$, $d = 2, 3$, requires input data including the diffusion coefficient and the source term. The problem is well-posed, thoroughly understood and the solution depends continuously on these problem data. In real-world situations, however, the model parameters are not known precisely and can often only be estimated up to some remaining uncertainty.

2.1 Dependence on Data

The standard model problem of uncertainty propagation is the elliptic diffusion equation with Dirichlet boundary conditions

$$-
abla \cdot (a \nabla u) = f \quad \text{on } D \subset \mathbb{R}^d \quad \text{and} \quad u = 0 \quad \text{along } \partial D,$$

with scalar diffusion coefficient $a$ and source term $f$. The solution $u$ is understood in the weak sense and lies in the Sobolev space $H^1_0(D)$, uniquely determined by the variational equation

$$(a \nabla u, \nabla v)_{L^2(D)} = (f, v)_{L^2(D)} \quad \text{for all } v \in H^1_0(D); \quad \text{(BVP)}$$

here, $(\cdot, \cdot)_{L^2(D)}$ denotes the inner product in $L^2(D)$. Assuming $a \in L^\infty(D)$ with uniform ellipticity

$$a(x) \geq a_{\text{min}} > 0 \quad \text{for } x \in D \text{ a.e.}$$

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and \( f \in L^2(D) \) there exists a unique solution to (BVP) such that (cf. [8, Theorem 2.7.7], [24, Theorem 6.6.2])

\[
\|u\|_{H^1_0(D)} \leq \frac{c}{\alpha_{\text{min}}} \|f\|_{L^2(D)},
\]

where \( c \) is the Poincaré constant of the domain \( D \). Denoting the set of admissible diffusion coefficients by \( \mathcal{L}_a(D) := \{ a \in L^\infty(D) : \text{ess inf } a > 0 \} \),

we recall that the (nonlinear) solution operator

\[
\mathcal{S} : L^\infty_2(D) \times L^2(D) \to H^1_0(D)
\]

mapping the data \((a, f)\) to the corresponding solution \( u \) of (BVP) is \textit{locally Lipschitz}. More precisely (cf. [7, Lemma 2.1]),

\[
\|\mathcal{S}(a_2, f_2) - \mathcal{S}(a_1, f_1)\|_{H^1_0(D)} \leq \frac{c}{a_{2,\text{min}}} \|f_2 - f_1\|_{L^2(D)} + \frac{\|\mathcal{S}(a_1, f_1)\|_{H^1_0(D)}}{a_{1,\text{min}}} \|a_2 - a_1\|_{L^\infty(D)},
\]

where \( a_{i,\text{min}} \) denotes the essential infimum of the diffusion coefficient, \( a_i(x) \geq a_{i,\text{min}} \) (\( i = 1, 2 \)). Employing the bound (1) for \( \|\mathcal{S}(a_1, f_1)\|_{H^1_0(D)} \) we get

\[
\|\mathcal{S}(a_2, f_2) - \mathcal{S}(a_1, f_1)\|_{H^1_0(D)} \leq \frac{c}{a_{2,\text{min}}} \|f_2 - f_1\|_{L^2(D)} + \frac{c}{a_{1,\text{min}}} \frac{\|f_i\|_{L^2(D)}}{a_{2,\text{min}}} \|a_2 - a_1\|_{L^\infty(D)},
\]

which is the basis for the following continuity result (cf. [22, Theorem 2.46]).

**Proposition 1** (Local Lipschitz continuity). Let \( r_i, r_f \in (0, \infty) \) be given radii. Then it holds for any \( a_1, a_2 \in L^\infty_2(D) \) with \( \|\log(a_i)\|_{L^\infty} \leq r_i \), \( i = 1, 2 \), and any \( f_1, f_2 \in L^2(D) \) with \( \|f_i\|_{L^2} \leq r_f \), \( i = 1, 2 \), that

\[
\|\mathcal{S}(a_2, f_2) - \mathcal{S}(a_1, f_1)\|_{H^1_0(D)} \leq c_a \|a_2 - a_1\|_{L^\infty(D)} + c_f \|f_2 - f_1\|_{L^2(D)},
\]

where \( c_a := c r_f e^{2r_a} \) and \( c_f := ce^{r_a} \), as well as

\[
\|\mathcal{S}(a_2, f_2) - \mathcal{S}(a_1, f_1)\|_{H^1_0(D)} \leq \tilde{c}_a \|\log(a_2) - \log(a_1)\|_{L^\infty(D)} + c_f \|f_2 - f_1\|_{L^2(D)},
\]

where \( \tilde{c}_a := c r_f e^{3r_a} \) and \( c_f \) as above.

**Proof.** The estimate (5) follows immediately by (4) using \( a_{\text{min}} := \exp(-r_a) \leq \exp(-\|\log(a_i)\|_{L^\infty}) \leq \text{ess inf } a_i \). Estimate (6) follows by (5) combined with

\[
\|a_1 - a_2\|_{L^\infty(D)} = \|\exp(\log(a_2)) - \exp(\log(a_1))\|_{L^\infty(D)} \\
\leq \exp(\max(\|\log(a_1)\|_{L^\infty(D)}, \|\log(a_2)\|_{L^\infty(D)})) \|\log(a_2) - \log(a_1)\|_{L^\infty(D)} \\
\leq \exp(r_a) \|\log(a_2) - \log(a_1)\|_{L^\infty(D)},
\]

where we have used local Lipschitz continuity of the exponential map. \( \square \)

In what follows we will denote by \( \mathcal{S} \) also the solution operator on \( L^\infty_2(D) \times L^2(D) \) mapping \((\log a, f)\) to the solution \( u \) of (BVP). It will be clear from the context which version of \( \mathcal{S} \) is meant. We believe this slight abuse of notation will avoid cumbersome distinctions throughout the paper.
2.2 Random Data

In practice, the diffusion coefficient \( a \in L^\infty(D) \) in \( \text{(BVP)} \), or its logarithm \( \log a \in L^\infty(D) \), respectively, is often not known precisely, and the same is the case for the source term \( f \in L^2(D) \). This limited knowledge of \( a \) (\( \log a \), resp.) and \( f \) is modeled probabilistically, i.e., as random variables with realizations

\[
\log a(\cdot, \omega) \in L^\infty(D) \quad \text{and} \quad f(\cdot, \omega) \in L^2(D), \quad \omega \in \Omega,
\]

with respect to some reference probability space \( (\Omega, \mathcal{F}, P) \). As the Banach space \( L^\infty \) is non-separable we may occasionally prefer

\[
\log a(\cdot, \omega) \in C(D)
\]
as the space of outcomes for the log diffusion coefficient instead.

The random \( \log a(\cdot, \omega) \) and \( f(\cdot, \omega) \) now result in a random solution \( u(\cdot, \omega) \) of \( \text{(BVP)} \), i.e.,

\[
\begin{align*}
-\nabla \cdot (a(\cdot, \omega) \nabla u(\cdot, \omega)) &= f(\cdot, \omega) \quad \text{on } D \subset \mathbb{R}^d \quad \text{and} \\
u(\cdot, \omega) &= 0 \quad \text{on } \partial D,
\end{align*}
\]

with \( u(\cdot, \omega) \in H^1_0(D) \) with probability 1, i.e., almost surely. In this setting, the solution

\[
\omega \mapsto u(\cdot, \omega) = S(a(\cdot, \omega), f(\cdot, \omega))
\]
of \( \text{(7)} \) is a random variable taking values in \( H^1_0(D) \).

The focus in practice is typically less on the complete random solution \( u \) than on specific aspects of it, collectively termed quantities of interest (QoI). We model these as functionals

\[
\phi: H^1_0(D) \to \mathbb{R}.
\]

Simple but important examples include point evaluations \( \phi(u) := u(x_0) \) (if defined) or \( \phi(u) := \int_{D'} u(x) \, dx \) for some fixed domain \( D' \subset D \). The nonlinear QoI

\[
\phi(u) := \int_{D'} |u_0 - u(x)|^2 \, dx, \quad u_0 \in L^2(D),
\]

and

\[
\phi(u) := \int_{D'} |\nabla u_0 - \nabla u(x)|^2 \, dx \quad \text{or} \quad \phi(u) := \|u_0 - u\|_{H^1(D)} \quad u_0 \in H^1(D),
\]

are all Lipschitz in \( u \) in a ball in \( H^1_0(D) \). The composition

\[
\omega \mapsto \phi(u(\cdot, \omega)) = (\phi \circ S)(a(\cdot, \omega), f(\cdot, \omega))
\]
is then a real-valued random variable, for which we are interested in statistics such as, for example, \( \mathbb{E}[\phi(u(\cdot, \omega))] \) or \( \text{Var} \phi(u(\cdot, \omega)) \).
3 Risk Functionals for Uncertainty Analysis

The focus of an uncertainty propagation analysis is typically a QoI associated with the solution of a random PDE, from which useful information may be extracted by statistical post-processing. Risk functionals can be viewed as a specific type of post-processing, as they condense the probability distribution of a random variable into a number reflecting the impact of these fluctuations for a particular QoI.

3.1 Risk Functionals

Risk functionals have gained significant importance in the recent past, particularly driven by mathematical finance. Initially, they were considered in insurance (cf. Denneberg [11] and Deprez and Gerber [12]). Risk functionals assign real numbers to random variables in such a way that these values constitute a measure of the risk associated with their random outcomes. As such they are defined on a vector space $L$ of real-valued random variables.

**Definition 2** (Risk functional, cf. Artzner et al. [2]). Assume an abstract probability space $(\Omega, \mathcal{F}, P)$ and a vector space $L$ of real-valued random variables $X : \Omega \to \mathbb{R}$. A mapping $\rho : L \to \mathbb{R} \cup \{\infty\}$ is a risk functional if it satisfies the following axiomatic properties for $X, Y \in L$

\[
\begin{align*}
\rho(X) &\leq \rho(Y), \text{ if } X \leq Y \text{ a.s.} & \text{(monotonicity),} \\
\rho(X + c) &\leq \rho(X) + c \text{ for } c \in \mathbb{R} & \text{(translation equivariance),} \\
\rho(X + c) &\leq \rho(X) + \rho(Y) & \text{(subadditivity),} \\
\rho(\lambda \cdot X) &\leq \lambda \cdot \rho(X) \text{ for } \lambda > 0 & \text{(positive homogeneity).}
\end{align*}
\]

These properties possess natural interpretations in the context of risk management and insurance. Occasionally in the literature the term risk functional can also be found for mappings $\rho$ satisfying only (9a)–(9c), while risk functionals satisfying also (9d) are then referred to as coherent risk functionals.

**Remark 3** (Domain of risk functionals). Ruszczyński and Shapiro [34] discuss risk functionals on $L^p$ spaces, $p \in [1, \infty]$. They conclude that $\rho$ is either continuous on $L^p$ or the set $\{X \in L^p : \rho(X) = \infty\}$ is dense in $L^p$. To specify the largest class of tractable random variables one may associate a norm and a domain $L$ with a risk functional $\rho$ in a natural way. To this end define

\[
\|X\|_\rho := \rho(|X|) \quad \text{for } X \in L := \{X : \Omega \to \mathbb{R} \text{ measurable with } \|X\|_\rho < \infty\}.
\]

The pair $(L, \|\cdot\|_\rho)$ is then a Banach space, the largest possible for which $\rho$ is finite on $L$. The spaces $L$ is most typically a Lorentz rearrangement space, cf. Pichler [27] and Kalmes and Pichler [18].

A convenient way to construct risk functionals when $L$ is a Banach space is via duality. Let $L^*$ denote the dual (or pre-dual) of $L$ with duality pairing

\[
\langle X, Z \rangle = \mathbb{E}[XZ] = \int_{\Omega} X(\omega) Z(\omega) \mathbb{P}(d\omega), \quad X \in L, Z \in L^*.
\]
Introducing a subset
\[ \mathcal{A} \subset \{ Z \in L^* : \mathbb{E}[Z] = 1 \text{ and } Z \geq 0 \}, \]
(11)
a risk functional satisfying all properties (9a)–(9d) above is given by
\[ \rho(X) := \sup \{ \mathbb{E}[XZ] : Z \in \mathcal{A} \}. \]
(12)
We then call \( \mathcal{A} \) the support set of \( \rho \) in (12).

**Remark 4 (Representation by convex duality).** By (9c) and (9d), any risk functional \( \rho(\cdot) \) is convex. It follows from convex duality (the Fenchel–Moreau theorem, cf. Rockafellar [31]) that
\[ \rho(X) = \sup \{ \mathbb{E}[XZ] - \rho^*(Z) : Z \in L^* \}, \]
where for \( Z \in L^* \)
\[ \rho^*(Z) := \sup \{ \mathbb{E}[XZ] - \rho(X) : X \in L \} \]
is the convex dual function to \( \rho \) (cf. Shapiro et al. [35]). The setting in (12) is actually the most typical situation. It follows from (9b) and (9a) that
\[ \rho^*(Z) = \begin{cases} 0 & \text{if } \mathbb{E}[Z] = 1 \text{ and } Z \geq 0, \\ +\infty & \text{otherwise}. \end{cases} \]

We may thus define the support set of any risk functional \( \rho \) by
\[ \mathcal{A} := \{ Z \in L^* : \rho^*(Z) < \infty \}, \]
(13)
so that (12) applies.

The random variables \( Z \in \mathcal{A} \) are densities with respect to the probability measure \( \mathbb{P} \): they are nonnegative and they satisfy \( \mathbb{E}[Z] = 1 \). The support set defined in (13) thus consists of densities (cf. (11)) and we may assume without loss of generality that \( \mathcal{A} \) is weak* closed.

**Remark 5 (Assessment and quantification of risk).** The random variables \( Z \in \mathcal{A} \) provide a useful interpretation of the risk functional. To this end suppose that \( Z^* \) is optimal in (12) so that
\[ \rho(X) = \sup_{Z \in \mathcal{A}} \mathbb{E}[XZ] = \mathbb{E}[XZ^*]. \]
(14)
Then \( Z^* \) acts as a weight, as it is nonnegative (\( Z \geq 0 \)) and as it sums to 1 (\( \mathbb{E}[Z] = 1 \)). The weighted expectation \( \mathbb{E}[XZ^*] \) in (14) weights every outcome \( X(\omega) \) with \( Z^*(\omega) \) in the worst possible way (\( Z^* \) attains the supremum): unfavorable outcomes \( X(\omega) \) will be overvalued and assigned a high weight \( Z^*(\omega) > 1 \), while favorable outcomes \( X(\omega) \) will be assigned a lower weight \( Z^*(\omega) \leq 1 \). In this interpretation, the random variable \( Z^* \) is an individual assessment of risk for the particular random variable \( X \).

**Example 6 (Risk neutrality and total risk aversion).** The risk functional
\[ \rho(X) := \mathbb{E}[X] \]
is the simplest functional satisfying all axioms above, it is called the risk neutral risk functional, as it ignores fluctuations around the mean completely. Its support set $\mathcal{A} = \{1\}$ consists of a single element, the constant density $1(\cdot) \equiv 1$.

By contrast, the functional
\[
\rho(X) := \text{ess sup } X
\]
is the most conservative risk functional. It indicates total risk aversion, as it represents the risk associated with the random outcome $X$ by its largest possible outcome. It has the maximal support set $\mathcal{A} = \{Z : Z \geq 0 \text{ and } E[Z] = 1\}$.

**Example 7** (Average value-at-risk). The average value-at-risk is the most prominent example of a risk functional. At risk level $\alpha \in [0, 1)$, this functional is given by
\[
\text{AV@R}_\alpha(X) := \frac{1}{1-\alpha} \int_{\alpha}^{1} F_X^{-1}(u) \, du,
\]
where
\[
F_X^{-1}(u) := \text{V@R}_\alpha(X) := \inf \{ x \in \mathbb{R} : P(X \leq x) \geq \alpha \}
\]
is the quantile function, or value-at-risk at level $\alpha$. Note that the value-at-risk itself is not a risk functional in the sense of Definition 2, since it does not satisfy subadditivity (9c). The support set
\[
\mathcal{A} = \left\{ Z : E[Z] = 1 \text{ and } 0 \leq Z \leq \frac{1}{1-\alpha} \text{ a.s.} \right\}
\]
allows a representation of $\text{AV@R}$ as a supremum as in (12). By contrast, the representation
\[
\text{AV@R}_\alpha(X) := \inf_{q \in \mathbb{R}} q + \frac{1}{1-\alpha} E[(X - q)_+],\quad \text{where } x_+ := \max(0, x),
\]
as an infimum derives from convex duality, cf. Ogryczak and Ruszczyński [25], Rockafellar and Uryasev [32] and Pflug [26]. The average value-at-risk is also known as conditional value-at-risk. Actuaries, however, prefer the terms expected shortfall or conditional tail expectation.

**Example 8** (Semideviation). The semideviation risk functional selectively penalizes deviations above the mean and is defined by
\[
\rho(X) := E[X + \beta \cdot E[(X - E[X])_+]],
\]
where $\beta \in [0, 1]$ is a coefficient of risk aversion. More generally, for $p \geq 1$ the $p$-semideviation is
\[
\rho(X) := E[X + \beta \cdot \|X - E[X]\|_p].
\]
The semideviation has the alternative representation in terms of average value-at-risk
\[
\rho(X) = \sup_{\kappa \in (0,1)} (1 - \beta \kappa) E[X + \beta \kappa \text{ AV@R}_{1-\kappa}(X)].
\]
A slightly more complicated version is available for the $p$-semideviation as well and given in Pichler and Shapiro [29, Corollary 6.1].
**Example 9** (Spectral risk functionals). Spectral risk functionals, defined in terms of a spectral function $\sigma : [0, 1) \rightarrow \mathbb{R}_0^+$ satisfying $\int_0^1 \sigma(u) \, du = 1$ by

$$
\rho_{\sigma}(X) := \int_0^1 \sigma(u) F_X^{-1}(u) \, du,
$$

are another way of quantifying risk, which for $\sigma = \frac{1}{1-\alpha} \mathbb{1}_{[\alpha, 1]}$ recovers average value-at-risk. Here the support set is given by (cf. Pichler [28])

$$
\mathcal{A}_{\sigma} := \left\{ Z : Z \geq 0, \ AV@R_{\alpha}(Z) \leq \frac{1}{1-\alpha} \int_\alpha^1 \sigma(u) \, du \text{ for all } \alpha < 1 \right\}.
$$

**Example 10** (Entropic value-at-risk). The entropic value-at-risk is getting increased attention. It is defined as

$$
EV@R_{\alpha}(X) := \inf_{t > 0} \frac{1}{t} \log \frac{1}{1-\alpha} \mathbb{E} \left[ e^{tX} \right].
$$

Its support set is

$$
\mathcal{A} := \left\{ Z \geq 0 : \mathbb{E} [Z] = 1 \text{ and } \mathbb{E} [Z \log Z] \leq \log \frac{1}{1-\alpha} \right\},
$$

where $H(Z) := \mathbb{E} [Z \log Z]$ is the entropy of the density $Z$.

### 3.2 Sensitivity Analysis for Risk Functionals

All risk functionals discussed above depend in different ways on the underlying probability measure $P$ of the probability space $(\Omega, \mathcal{F}, P)$. In an uncertain setting, however, $P$ is not known precisely. It may be approximated by statistical estimation of parameters defining a family of probability distributions; alternatively, only an empirical measure $P_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ may be available. To discuss uncertainty it is thus essential to account for variations of the probability measure $P$ and investigate the impact resulting from such imprecise knowledge.

In the following discussion we consider probability measures $P$ on a metric space $(X, d)$ equipped with the associated Borel $\sigma$-algebra. Such metric spaces can be admissible function spaces for the data log $a$ and $f$ or for the solution $u$ of the elliptic problem (BVP). We denote the set of all probability measures on $X$ by $\mathcal{P}(X)$ and further define the subsets

$$
\mathcal{P}_p(X) := \left\{ P \in \mathcal{P}(X) : \int_X d(x, x_0)^p \, P(dx) < \infty \text{ for some } x_0 \in X \right\}, \quad p \geq 1.
$$

In order to measure perturbations of a $P \in \mathcal{P}(X)$ we require a suitable distance for probability measures. As motivated in the Introduction we focus on the $p$-Wasserstein distance, which for $P, Q \in \mathcal{P}_p(X)$ is given by

$$
d_p(P, Q) := \inf_{\pi \in \Pi(P, Q)} \left( \int_{X \times X} d(x, y)^p \, \pi(dx, dy) \right)^{\frac{1}{p}},
$$

where $\Pi(P, Q)$ is the set of all probability measures on $X \times X$.
where \( \Pi(P,Q) \) denotes the set of all measures \( \pi \in \mathcal{P}(X \times X) \) with marginals \( \pi(A \times X) = P(A) \) and \( \pi(X \times B) = Q(B) \) for all Borel sets \( A, B \subset X \). Any such measure \( \pi \in \mathcal{P}(X \times X) \) with these marginals is called a *coupling* of \( P \) and \( Q \). We recall a few basic properties of the Wasserstein distance:

- If \((X, d)\) is complete and separable, then there always exists an optimal coupling \( \pi^* \in \Pi(P, Q) \) for which the infimum in the definition of \( d_p(P, Q) \) is attained, see [39, Chapter 4].
- If \((X, d)\) is complete and separable, then so is \((\mathcal{P}_p(X), d_p)\), \( p \geq 1 \), see [39], and a dense subset is given by the convex hull of \( \{\delta_x : x \in X\} \).
- By Jensen’s inequality we have that \( d_q(P, Q) \leq d_p(P, Q) \) for any \( P, Q \in \mathcal{P}_p(X) \) and \( 1 \leq q \leq p \).
- \((X, d)\) embeds isometrically in \((\mathcal{P}_p(X), d_p)\) via the embedding \( x \mapsto \delta_x \), since \( d_p(\delta_x, \delta_y) = d(x, y) \) for all \( p \geq 1 \).

We now study the sensitivity of risk functionals with respect to the underlying probability measure \( P \), measuring perturbations of the latter in Wasserstein distance. To this end, consider \( P, Q \in \mathcal{P}(X) \). Evaluations with respect to different probability measures are made explicit by writing the probability measure as a subscript, e.g., \( E_P [X] = \int_X X \, dP \) and \( E_Q [X] = \int_X X \, dQ \).

In order to analyze the effect of the probability measure \( P \) on the value \( \rho(X) \) we consider an associated risk functional \( \rho_\pi \) for random variables \( X' : X \times X \to \mathbb{R} \) on the product space equipped with a coupling \( \pi \in \Pi(P, Q) \) as the underlying probability measure. That is, we consider

\[
\rho_\pi(X') := \sup \{E_\pi [X' \, Z] : Z \in \mathcal{A}_\pi \},
\]

where

\[
\mathcal{A}_\pi \subset \{Z : X \times X \to [0, \infty) \text{ such that } E_\pi [Z] = 1\}
\]
denotes the support set of \( \rho_\pi \) on the product space. Furthermore, let \( p_i : X \times X \to X, i = 1, 2 \), with \( p_i((x_1, x_2)) := x_i \) denote the canonical projections so that \( P = \pi \circ p_1^{-1} \) and \( Q = \pi \circ p_2^{-1} \). We thus may define \( \rho(X) \) for a \( X : X \to \mathbb{R} \) given either \( P \) or \( Q \), by

\[
\rho_P(X) := \rho_\pi(X \circ p_1) = \sup \{E_\pi [Z \cdot (X \circ p_1)] : Z \in \mathcal{A}_\pi \}
\]

and

\[
\rho_Q(X) := \rho_\pi(X \circ p_2) = \sup \{E_\pi [Z \cdot (X \circ p_2)] : Z \in \mathcal{A}_\pi \}.
\]

**Remark 11.** The converse is possible as well. Appendix B explicitly constructs a risk functional \( \rho_\pi \) from its marginals \( \rho_P \) and \( \rho_Q \) as in (21) and (22).

**Remark 12.** *Law-invariant* risk functionals \( \rho \) depend on the cumulative distribution function of \( X \) only (see (18), e.g.). For these risk functionals it is obvious that (21) and (22) are consistent definitions: consider, for instance, the average value-at-risk and a coupling \( \pi \in \Pi(P, Q) \) for two arbitrary probability measures on \( X \). Then, for any random variable \( X : X \to \mathbb{R} \), we have

\[
\text{AV}_@R_{\alpha,\pi}(X \circ p_1) = \inf_{q \in \mathbb{R}} q + \frac{1}{1 - \alpha} E_\pi [(X \circ p_1 - q)^+] = \text{AV}_@R_{\alpha,\pi}(X)
\]

\[
= \inf_{q \in \mathbb{R}} q + \frac{1}{1 - \alpha} E_P [(X - q)^+] = \text{AV}_@R_{\alpha,\pi}(X)
\]
and analogously that \( \text{AV@R}_{\rho,\pi}(X \circ p_2) = \text{AV@R}_{\rho,Q}(X) \).

We note that all examples in the preceding subsection are law-invariant risk functionals. However, the setting outlined here also allows the analysis of risk functional which are not law-invariant. Such risk functionals appear, for example, in insurance when the cause of the loss \( X(\omega) \) is also important: damage caused by floods or natural disasters can pose a higher risk to insurance companies (due to cross-correlations) than damage by accidents.

The following theorem states that it suffices to consider risk functionals on the marginals only.

**Theorem 13.** Let \( \rho_{\pi} \) be a risk measure on the product space \( X \times X \) and \( X : X \to \mathbb{R} \) a random variable. Then there is a marginal support set \( \mathcal{A}_p \) such that

\[
\rho_p(X) = \sup \{ E_p[Z_1 \cdot X] : Z_1 \in \mathcal{A}_p \}.
\]

The marginal support set is

\[
\mathcal{A}_p = \{ E_p[Z \mid p_1 = \cdot] : Z \in \mathcal{A}_\pi \},
\]

where we view \( E_p[Z \mid p_1 = \cdot] \) as a real-valued random variable on \( X = \text{range}(p_1) \). The assertion holds analogously for \( \rho_Q \) with \( \mathcal{A}_Q = \{ E_p[Z \mid p_2 = \cdot] : Z \in \mathcal{A}_\pi \} \).

**Proof.** Let \( Z \in \mathcal{A}_\pi \). By the Doob-Dynkin lemma there exists a measurable function \( \psi : X \to \mathbb{R} \) such that \( E_\pi[Z \mid p_1] = \psi \circ p_1 \) almost surely. Thus, we set \( Z_1(x) := E_\pi[Z \mid p_1 = x] = \psi(x) \) which is a real-valued random variable on \( X \) whereas \( E_\pi[Z \mid p_1] \) is a real-valued random variable on \( X \times X \). By \( \mathbb{P} = \pi \circ p_1^{-1} \) and the very definition of conditional expectation we have

\[
E_p[Z_1] = E_{(p_1),\pi} [\psi] = E_\pi[\psi(p_1)] = E_\pi[E_\pi[Z \mid p_1]] = E_\pi[Z] = 1.
\]

Moreover, \( Z_1 = E_\pi[Z \mid p_1 = \cdot] \geq 0 \) almost surely, since \( Z \geq 0 \) almost surely. Furthermore, as \( X \circ p_1 \) is measurable with respect to \( \sigma(p_1) \), we obtain

\[
E_p[X \cdot Z_1] = E_{(p_1),\pi}[X \cdot \psi] = E_\pi[(X \circ p_1) \cdot E_\pi[Z \mid p_1]] = E_\pi[E_\pi[(X \circ p_1) \cdot Z \mid p_1]]
\]

Set \( \mathcal{A}_p = \{ E_\pi[Z \mid p_1 = \cdot] : Z \in \mathcal{A}_\pi \} \) to obtain the assertion of the theorem. \( \square \)

Exploiting the connection of \( \rho_p \) and \( \rho_Q \) via \( \rho_\pi \), we obtain the following sensitivity result.

**Theorem 14** (Sensitivity with respect to the probability measure). Let \( \mathbb{P}, \mathbb{Q} \in \mathcal{P}_p(X) \) and \( \pi \in \Pi(\mathbb{P}, \mathbb{Q}) \). Further, assume that the mapping \( X : X \to \mathbb{R} \) is Hölder-continuous with exponent \( \beta \in (0, 1] \), i.e., \( |X(x) - X(y)| \leq C \cdot d(x,y)^\beta \). Then

\[
|\rho_p(X) - \rho_Q(X)| \leq C \cdot \sup_{Z \in \mathcal{A}_\pi} E_\pi[Z^{p_B}]^{1/p_B} \cdot E_\pi[d^{p_B}]^{1/p_B},
\]

where \( p_B := \frac{p}{p_B} \). In particular, if \( (X, d) \) is complete and separable, choosing the optimal coupling \( \pi^* \in \Pi(\mathbb{P}, \mathbb{Q}) \) for \( d_p(\mathbb{P}, \mathbb{Q}) \), yields

\[
|\rho_p(X) - \rho_Q(X)| \leq C \cdot \sup_{Z \in \mathcal{A}_{\pi^*}} E_{\pi^*}[Z^{p_B}]^{1/p_B} \cdot d_p(\mathbb{P}, \mathbb{Q})^\beta.
\]
Proof. Let $Z \in \mathcal{A}_\pi$ be fixed. Note that
\[
\mathbb{E} [Z \cdot X \circ p_1] - \mathbb{E} [Z \cdot X \circ p_2] = \int_{X \times X} Z(x, y) \cdot X(x) - Z(x, y) \cdot X(y) \pi(dx, dy)
= \int_{X \times X} Z(x, y) \cdot (X(x) - X(y)) \pi(dx, dy)
\leq C \int_{X \times X} d(x, y)^\beta \cdot Z(x, y) \pi(dx, dy),
\]
as $Z \geq 0 \pi$-a.s. Taking the supremum among all $Z \in \mathcal{A}_\pi$ on the right hand side gives
\[
\mathbb{E} [Z \cdot X \circ p_1] - \mathbb{E} [Z \cdot X \circ p_2] \leq C \cdot \rho_\pi (d^\beta).
\]
Employing the definition (21) and the optimal $Z$ there it follows that
\[
\rho_p(X) - \mathbb{E} [Z \cdot X \circ p_2] \leq C \cdot \rho_\pi (d^\beta),
\]
and now with (22) further that
\[
\rho_p(X) - \rho_Q(X) \leq C \cdot \rho_\pi (d^\beta). \tag{25}
\]
Furthermore, note that $p/\beta$ and $p_{\beta}$ are Hölder conjugate exponents, $1/p + 1/p_{\beta} = 1$. With Hölder’s inequality we thus obtain
\[
\rho_\pi (d^\beta) = \sup_{Z \in \mathcal{A}_\pi} \mathbb{E}_\pi [Z \cdot d^\beta] \leq \sup_{Z \in \mathcal{A}_\pi} \mathbb{E}_\pi [Z^{p_{\beta}}]^{1/p_{\beta}} \mathbb{E}_\pi [d^{\beta p} ]^{1/p}
\]
Interchanging the roles of $P$ and $Q$ yields the absolute value in (25). □

Theorem 14 involves the supremum $\sup_{Z \in \mathcal{A}_\pi} \|Z\|_{\pi,p_{\beta}}$. This quantity is indeed finite for many important risk measures and in what follows we give explicit expressions for this bound:

- The optimal random variable $Z^\pi$ for the average value-at-risk satisfies $\pi \left( Z^\pi = \frac{1}{1-\alpha} \right) = 1-\alpha$ and $\pi(Z = 0) = \alpha$, so that $\mathbb{E}_\pi [Z^q]^{1/q} \leq \left( \frac{1}{1-\alpha} \right)^{1-\frac{q}{\alpha}} \leq \frac{1}{1-\alpha}$ for any $q \geq 1$.

- For the spectral risk functional it follows from (19) that it is enough to require $\sigma \in L^p(\{0,1\})$, as $\mathbb{E}_\pi [Z^q]^{1/q} = \|\sigma\|_{L^q([0,1])}$.

- Bounds for the entropic risk functional are elaborated in Ahmadi-Javid and Pichler [1] so that $\mathbb{E}_\pi [Z^q]^{1/q} \leq \max \left( 1, \frac{q-1}{\log \frac{q}{q-1}} \right)$ in this case.

Thus, for these risk measures and Hölder-continuous $X : X \to \mathbb{R}$ defined on a general metric space $(X, d)$ with Hölder-exponent $\beta \in (0,1]$ there exists a constant $C = C(\rho, \beta, \rho) < \infty$ such that
\[
|\rho_p(X) - \rho_Q(X)| \leq C \cdot d_p(P, Q)^\beta. \tag{26}
\]
4 Sensitivity of Uncertainty Propagation

Returning to the uncertainty propagation task for the model random PDE (BVP), this section discusses the sensitivity of the distribution of a quantity of interest \( \phi: H^1_0(D) \rightarrow \mathbb{R} \) resulting from different random models for the uncertain coefficients \( a, \log a, \) resp.) and \( f. \) For the sensitivity analysis we focus on the Wasserstein distance of probability measures for the reasons outlined in the Introduction. However, as an initial consideration and due to its simplicity we first present a sensitivity result in total variation distance. This metric is a common one in probability theory and uncertainty quantification and relates to the Hellinger distance—they are topologically equivalent—which is often used in the analysis of Bayesian inverse problems \(^{36}\) as well as to the Wasserstein distance—they coincide for the trivial metric \( d(x, y) = \mathbb{1}_{\{x\}}(y) \) on \( X. \)

4.1 Sensitivity in Total Variation Distance

The total variation (TV) distance of measures \( P, Q \in \mathcal{P}(X) \) is given by

\[
\text{d}_{TV}(P, Q) := \sup_{A \subset X} |P(A) - Q(A)|, \quad P, Q \in \mathcal{P}(X),
\]

where the supremum taken is over all measurable subsets of a measurable (e.g., Polish) space \( X. \) Let \( G: X \rightarrow \mathcal{Y} \) denote a measurable mapping to another measurable space \( \mathcal{Y}. \) We mention the following simple result concerning the sensitivity in TV distance of the general pushforward measures \( G^*P := P \circ G^{-1} \) and \( G^*Q := Q \circ G^{-1}. \)

**Proposition 15.** Let \( P, Q \in \mathcal{P}(X) \) and \( G: X \rightarrow \mathcal{Y} \) be measurable. Then

\[
\text{d}_{TV}(G^*P, G^*Q) \leq \text{d}_{TV}(P, Q).
\]

Moreover, if \( X = X_1 \times X_2 \) and \( P \) the independent product of measures \( P := P_1 \otimes P_2 \) as well as \( Q = Q_1 \otimes Q_2 \) with \( P_i, Q_i \in \mathcal{P}(X_i), \) then

\[
\text{d}_{TV}(P, Q) \leq \text{d}_{TV}(P_1, Q_1) + \text{d}_{TV}(P_2, Q_2).
\]

**Proof.** The first statement is immediate from

\[
\text{d}_{TV}(G^*P, G^*Q) = \sup_{B \subset \mathcal{Y}} |P(G^{-1}(B)) - Q(G^{-1}(B))| = \sup_{A \in \sigma(G)} |P(A) - Q(A)| \leq \text{d}_{TV}(P, Q),
\]

where \( \sigma(G) = \{G^{-1}(B): B \subset \mathcal{Y} \text{ measurable}\} \) denotes the \( \sigma \)-algebra induced by \( G. \) The second statement follows from

\[
\text{d}_{TV}(P_1 \otimes P_2, Q_1 \otimes Q_2) = \sup_{A \times B \subset X_1 \times X_2} |P_1(A)P_2(B) - Q_1(A)Q_2(B)|
\]

\[
\leq \sup_{A \subset X_1} |P_1(A) - Q_1(A)| \sup_{B \subset X_2} |P_2(B) - Q_2(B)|
\]

\[
= \text{d}_{TV}(P_1, Q_1) + \text{d}_{TV}(P_2, Q_2).
\]

\[\square\]
Regarding the elliptic problem (BVP) and its corresponding solution map \( S : L^\infty(D) \times L^2(D) \rightarrow H^1_0(D) \) we obtain for any \( P, Q \in \mathcal{P}(L^\infty(D) \times L^2(D)) \) and any measurable quantity of interest \( \phi : H^1_0(D) \rightarrow \mathbb{R} \)

\[
d_{TV}(S,P,S,Q) \leq d_{TV}(P,Q), \quad \text{and} \quad d_{TV}((\phi \circ S),P,(\phi \circ S),Q) \leq d_{TV}(P,Q).
\]

Given that \( P, Q \) are product measures, i.e., \( P = P_{loga} \otimes P_f \) and \( Q = Q_{loga} \otimes Q_f \), then

\[
d_{TV}(S,P,S,Q) \leq d_{TV}(P_{loga},Q_{loga}) + d_{TV}(P_f,Q_f)
\]

analogously for \( d_{TV}((\phi \circ S),P,(\phi \circ S),Q) \). Of course, the same statement holds analogously if we consider distributions \( P, Q \in \mathcal{P}(L^\infty(D) \times L^2(D)) \) for the variables \((a,f)\) instead of \((loga,f)\). Thus, the distribution of any measurable quantity of interest of the solution of (BVP) depends Lipschitz continuously on the input distributions for \((a,f)\), or \((loga,f)\), respectively, with Lipschitz constant one.

Probability measures on infinite dimensional spaces such as \( L^\infty(D) \) or \( L^2(D) \) tend to be mutually singular, resulting in a maximal total variation distance of one. This is undesirable for sensitivity analysis and we therefore next consider sensitivity in the Wasserstein distance, which does not rely on absolute continuity of the probability measures.

### 4.2 Sensitivity in Wasserstein Distance

The following discussion considers a general setting with forward maps \( G : \mathcal{X} \rightarrow \mathcal{Y} \) between complete metric spaces \((\mathcal{X},d_X)\) and \((\mathcal{Y},d_Y)\), equipped with their Borel \( \sigma \)-algebras, and then specializes the results to the elliptic problem (BVP). We begin the discussion with globally Hölder continuous forward maps \( G \).

**Proposition 16.** Let \( P, Q \in \mathcal{P}_p(\mathcal{X}), p \geq 1, \) and \( G : \mathcal{X} \rightarrow \mathcal{Y} \) be Hölder continuous with exponent \( \beta \in (0,1] \) and constant \( C_G < \infty \), i.e., \( d_Y(G(x),G(y)) \leq C_G d_X(x,y)^\beta \). Then we have for the pushforward measures \( G_* P, G_* Q \in \mathcal{P}(\mathcal{Y}) \) of \( P, Q \) that

\[
d_p(G_* P, G_* Q) \leq C_G d_{\mathcal{P}_p}(P,Q)^\beta \leq C_G d_p(P,Q)^\beta
\]

with \( d_p \) denoting the \( p \)-Wasserstein distance on \( \mathcal{P}_p(\mathcal{X}) \) and \( \mathcal{P}_p(\mathcal{Y}) \), respectively. In particular, we have \( G_* P, G_* Q \in \mathcal{P}_{\mathcal{P}_p}(\mathcal{Y}) \).

**Proof.** Let \( \pi \in \Pi(P,Q) \), then \( \pi^G(A \times B) := \pi(G^{-1}(A) \times G^{-1}(B)) \) satisfies \( \pi^G \in \Pi(G_* P, G_* Q) \) and we obtain by a change of variables

\[
\iint_{\mathcal{Y} \times \mathcal{Y}} d_Y(y_1,y_2)^p \pi^G(dy_1,dy_2) = \iint_{\mathcal{X} \times \mathcal{X}} d_Y(G(x_1),G(x_2))^p \pi(dx_1,dx_2)
\]

\[
\leq C_G^p \iint_{\mathcal{X} \times \mathcal{X}} d_X(x_1,x_2)^{p\beta} \pi(dx_1,dx_2).
\]

The assertion follows by taking the infimum over all \( \pi \in \Pi(P,Q) \) on both sides and noting that \( \{\pi^G : \pi \in \Pi(P,Q)\} \subset \Pi(G_* P, G_* Q) \). \qed
Proposition 16 establishes that Hölder continuity of the forward map $G$ carries over to the pushforward mapping of measures $P_{|g|}(X) \equiv P \mapsto G_* P \in P_p(Y)$. Applying this result to the elliptic problem gives, in combination with Proposition 1, Theorem 17 below. To formulate the theorem we equip the product space $L^\infty(D) \times L^2(D)$ with the metric

$$d_{L^\infty(D) \times L^2(D)}((\log a_1, f_1), (\log a_2, f_2)) := \| \log a_1 - \log a_2 \|_{L^\infty(D)} + \| f_1 - f_2 \|_{L^2(D)}.$$ 

We know from Proposition 1 that the solution map $S$ of (BVP) is Lipschitz on bounded subsets of $L^\infty(D) \times L^2(D)$ with respect to this metric. In particular, for a given radius $r < \infty$ we conclude that

$$\| S(a_1, f_1) - S(a_2, f_2) \|_{H^1_0(D)} \leq \left( c(1 + r)e^{3r} \right) d_{L^\infty(D) \times L^2(D)}((\log a_1, f_1), (\log a_2, f_2))$$

for all pairs $(\log a_i, f_i), i = 1, 2$, provided that $d_{L^\infty(D) \times L^2(D)}((\log a_i, f_i), (0, 0)) \leq r$.

**Theorem 17** (Wasserstein sensitivity for measures with bounded support). Let $r < \infty$ and $P, Q$ be probability measures for $a \in L^\infty(D)$ and $f \in L^2(D)$ with supports

$$\text{supp } P, \text{ supp } Q \subseteq \{(\log a, f) \in L^\infty(D) \times L^2(D) : \| \log a \|_{L^\infty(D)}, \| f \|_{L^2(D)} \leq r \}.$$

Then, the pushforward measures $S_* P, S_* Q \in P_p(H^1_0(D))$ of the random solutions to (BVP) satisfy

$$d_p(S_* P, S_* Q) \leq c(1 + r)e^{3r} d_p(P, Q).$$

The same statement holds for $P, Q \in P(L^\infty(D) \times L^2(D))$ with

$$\text{supp } P, \text{ supp } Q \subseteq \{(a, f) \in L^\infty(D) \times L^2(D) : \| \log a \|_{L^\infty(D)}, \| f \|_{L^2(D)} \leq r \}.$$

Then constant $e^{3r}$ in the estimate above can be replaced by $e^{2r}$.

### 4.3 Wasserstein Sensitivity for Locally Lipschitz Forward Maps

Proposition 16 states that the global Lipschitz constant of a pushforward map carries over to the mapping of the probability measures. This statement relates to the Kantorovich–Rubinstein duality theorem (cf. Villani [39]). For forward maps, which are only locally Lipschitz continuous, we can, in general, not expect global Lipschitz continuity for the pushforward measures in the $p$-Wasserstein distance. The following example elaborates on this issue and points out a particular situation, where one can conclude at least local Lipschitz continuity for the probability measures for a forward map which is not globally Lipschitz.

**Example 18** (Locally versus globally Lipschitz forward maps). Consider the Gaussian measures $P = \mathcal{N}(0, 1)$ and $Q_m = \mathcal{N}(m, 1)$ on the real line with mean $m \in \mathbb{R}$ and unit variance. The 2-Wasserstein distance of these Gaussian measures is (cf. Dowson and Landau [13])

$$d_2(P, Q_m) = |m|.$$
The exponential function $G(x) := \exp(x)$ on $X = \mathbb{R}$ is locally, but not globally Lipschitz continuous. By employing the dual representation [38, Chapter 5]

$$d_1(P, Q) = \sup_{\phi : \text{Lip}(\phi) \leq 1} \left| \int_X \phi(x) \, P(dx) - \int_X \phi(x) \, Q(dx) \right|$$

(the supremum is taken over all Lipschitz continuous functions $\phi : X \to \mathbb{R}$ with Lipschitz constant $\text{Lip}(\phi) \leq 1$) we obtain with $\phi(x) = x$ (and the formula for the mean of a lognormal distribution) that

$$\sqrt{\varepsilon} |1 - \exp(m)| \leq d_1(G_*P, G_*Q_m) \leq d_2(G_*P, G_*Q_m).$$

Thus, there is no constant $C < \infty$, independent of $m > 0$, such that

$$\sqrt{\varepsilon} |1 - \exp(m)| \leq d_2(G_*P, G_*Q_m) \leq C d_2(P, Q_m) = C|m|$$

since $\frac{|1-\exp(m)|}{m}$ is unbounded. However, restricting $|m| \leq M$ we have for $X \sim \mathcal{N}(0, 1)$ that

$$d_2(G_*P, G_*Q_m) \leq \mathbb{E} \left[ |G(X) - G(m + X)|^2 \right]^{1/2} = |1 - \exp(m)| \mathbb{E} [\exp(2X)]^{1/2} \leq C_M d_2(P, Q_m),$$

where $C_M := \frac{\varepsilon}{2\mathbb{E} \exp(M)} < \infty$.

The example demonstrates that local Lipschitz forwards can yield (at best) local Lipschitz continuity in Wasserstein distance for the pushforward measures.

By employing the Cauchy–Schwarz inequality we now derive a Lipschitz bound in $p$-Wasserstein distance of the pushforwards by the $2p$-Wasserstein distance of the input measures for local Lipschitz forwards. We state the result in a slightly more general form for locally Hölder continuous mappings, thus obtaining a Hölder bound in the corresponding Wasserstein distances.

**Proposition 19** (Continuity for locally Hölder pushforwards). Let $G : X \to Y$ be locally Hölder continuous with exponent $\beta \in (0, 1]$, i.e., there exists an $x_0 \in X$ and a nondecreasing function $C_G : [0, \infty) \to [0, \infty)$ such that for any radius $r < \infty$ we have

$$d_Y(G(x), G(y)) \leq C_G(r) \cdot d_X(x, y)^{\beta} \quad \text{for all } x, y \in X \text{ with } d_X(x, x_0) \leq r \text{ and } d_X(y, x_0) \leq r.$$

Furthermore, let $p \in [1, \infty)$ and $P, Q \in \mathcal{P}_{2\beta p}(X)$ satisfy

$$\int_X C_G(d(x, x_0))^{2p} \, P(dx) \leq C \quad \text{and} \quad \int_X C_G(d(x, x_0))^{2p} \, Q(dx) \leq C \quad (27)$$

for a constant $C < \infty$. Then, the pushforward measures $G_*P, G_*Q \in \mathcal{P}_p(Y)$ satisfy

$$d_p(G_*P, G_*Q) \leq 2C^{1/(2p)} \cdot d_{2\beta p}(P, Q)^{\beta}.$$

**Proof.** Employing the setting as in the proof of Proposition 16 and local Lipschitz continuity we have that

$$\int_{y \times y} d_Y(y_1, y_2)^p \pi^G(dy_1, dy_2) = \int_{X \times X} d_Y(G(x_1), G(x_2))^p \pi(dx_1, dx_2) \leq \int_{X \times X} C_G(d(x_1, x_0) \vee d(x_2, x_0))^p \cdot d_X(x_1, x_2)^{\beta p} \pi(dx_1, dx_2),$$
where \(a \lor b := \max(a, b)\). Applying the Cauchy–Schwarz inequality reveals that
\[
E_{\pi^G} \left[ d_y^p \right] \leq \left( \int_{X^2} C_G(d(x_1, x_0) \lor d(x_2, x_0))^{2p} \pi(dx_1, dx_2) \right)^{\frac{1}{2}} \left( \int_{X^2} d(x_1, x_2)^{2p} \pi(dx_1, dx_2) \right)^{\frac{1}{2}}
\]
and, since \(C_G(a \lor b)^{2p} \leq (C_G(a) + C_G(b))^{2p} \leq 2^{2p-1}(C_G(a)^{2p} + C_G(b)^{2p})\), we obtain
\[
E_{\pi^G} \left[ d_y^p \right] \leq \left( 2^{2p} C \right)^{\frac{1}{2}} \int_{X \times X} d(x_1, x_2)^{\beta p} \pi(dx_1, dx_2),
\]
where we have used the fact that the marginals of \(\pi\) are \(P\) and \(Q\). Taking the infimum over all \(\pi \in \Pi(P, Q)\) yields the statement. \(\square\)

Proposition 19 considers a local Hölder constant \(C_G(\cdot)\), which is integrable with respect to \(P\) and \(Q\) as detailed in (27). We then also obtain local Hölder continuity for the pushforward measures \(G, P\) and \(G, Q\) from \(\mathcal{P}_P\) to \(\mathcal{P}_{2P}\). That is, the tails of \(P\) and \(Q\) decay faster than the local Hölder constant \(C_G\) grows. We remark on two generalizations of this proposition before applying it to the locally Lipschitz solution operator \(S\) of (BVP).

Remark 20. The statement of Proposition 19 can be generalized by applying Hölder’s inequality instead of the Cauchy–Schwarz inequality in the proof. This allows to consider probability measures \(P\) and \(Q\) on \(X\) which satisfy, for an arbitrary \(q > 1\) and \(x_0 \in X\),
\[
\int_X C_G(d(x, x_0))^q \ P(dx), \int_X C_G(d(x, x_0))^q \ Q(dx) \leq C_q < \infty
\]
and which belong to \(\mathcal{P}_{\beta p, \frac{q}{q-p}}(X)\). We then obtain for \(p > q\)
\[
d_p(G, P, G, Q) \leq (2C_q^{\frac{1}{q}})^p \cdot d_{\beta p, \frac{q}{q-p}}(P, Q)^\beta.
\]
This generalization of Proposition 19 can be used in two ways: (i) in order to relax the conditions on \(P, Q\) or (ii) in order to get \(\frac{q}{q-p}\) close to 1, i.e., obtaining an almost Hölder estimate in the \(p\)-Wasserstein distance. However, for the purposes of this paper, we will only work with Proposition 19 in what follows.

Remark 21. The main assumption of Proposition 19 can be refined for the case of a product space \(X = X_1 \times X_2\) equipped with the metric \(d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)\), where \(d_i\) denotes a metric on \(X_i, i = 1, 2\) by assuming that \(G: X \to Y\) is locally Hölder continuous in the following way: there exists a nondecreasing \(C_G: [0, \infty) \times [0, \infty] \to [0, \infty)\) such that
\[
d_Y(G(x_1, x_2), G(y_1, y_2)) \leq C_G(r_1, r_2) d(x, y)^\beta,
\]
for all \(x = (x_1, x_2)\) and \(y = (y_1, y_2) \in X\) with \(x_i, y_i \in X_i\) belonging to a ball with radius \(r_i\) with respect to \(d_i\) around a center \(z_i \in X_i\). We then obtain the same result as in Proposition 19 provided that
\[
\int_X C_G(d_1(x_1, z_1), d_2(x_2, z_2))^{2p} \ P(dx) \leq C,
\]
and analogously for \(Q\).
The following theorem on the elliptic problem results from combining the preceding proposition with Proposition 1 and Remark 21. Recall that Proposition 1 yields
\[
\left\| S(a_2, f_2) - S(a_1, f_1) \right\|_{H^1_0(D)} \leq c(1 + r_f) e^{3r_a} \left( \| \log a_2 - \log a_1 \|_{L^\infty(D)} + \| f_2 - f_1 \|_{L^2(D)} \right),
\]
for all \( \| \log a_i \|_{L^\infty(D)} \leq r_a \) and \( \| f_i \|_{L^2(D)} \leq r_f \).

**Theorem 22** (Wasserstein sensitivity). Let \( P, Q \) be probability measures for \((\log a, f) \in L^\infty(D) \times L^2(D)\) with
\[
\int (1 + \| f \|_{L^2(D)})^{2p} \exp(6p \| \log a \|_{L^\infty(D)}) \, dP \leq C < \infty
\]
and
\[
\int (1 + \| f \|_{L^2(D)})^{2p} \exp(6p \| \log a \|_{L^\infty(D)}) \, dQ \leq C < \infty.
\]
Then the pushforward measures \( S_* P, S_* Q \in \mathcal{P}(H^1_0(D)) \) of the resulting random solutions to (BVP) satisfy
\[
d_P(S_* P, S_* Q) \leq 2 c^2 C^{(2p)} d_{2p}(P, Q),
\]
where \( c \) denotes the constant in (1).

**Discussion of the lognormal case.** We discuss sufficient conditions on \( P, Q \) such that the conditions of Theorem 22 are satisfied. To this end consider product measures \( P = P_a \otimes P_f \) and \( Q = Q_a \otimes Q_f \), where \( P_a, Q_a \in \mathcal{P}(L^\infty(D)) \) describe measures for \( \log a \) and \( P_f, Q_f \in \mathcal{P}(L^2(D)) \) describe measures for \( f \) in (BVP). It is natural to require \( P_f, Q_f \in \mathcal{P}_{2p}(L^2(D)) \) in order to apply Theorem 22. Concerning the measures \( P_a, Q_a \) for the log diffusion coefficient we consider the popular choice of Gaussian measures on \( C(D) \) given by Gaussian random field models on \( D \). These models are characterized by a mean function \( m \in C(D) \) and a continuous covariance function \( c \in C(D \times D) \) which then describe the finite dimensional distribution of \((\log a(x_1), \ldots, \log a(x_n)), n \in \mathbb{N} \) and \( x_i \in D \), by a Gaussian distribution \( N(m, C) \), where \( m = (m(x_1), \ldots, m(x_n)) \in \mathbb{R}^n \) and \( C \in \mathbb{R}^{n \times n} \) has entries \( c_{ij} = c(x_i, x_j) \). We denote the resulting Gaussian measures on \( C(D) \) by \( P = N(m, c) \). The question arises for which classes of mean functions \( m \) and covariance functions \( c \) can we ensure uniform integrability of \( \exp(6p \| \log a \|_{L^\infty(D)}) \), as required in Theorem 22? In more detail: for which sets \( M \subset C(D) \) of mean functions and \( C \subset C(D \times D) \) of covariance functions does there exists a finite constant \( C < \infty \) such that
\[
\int \exp(6p \| \log a \|_{L^\infty(D)}) \, dP \leq C \quad \text{for all } P \in \{ N(m, c) : m \in M, c \in C \}?
\]

**Fernique’s theorem** ensures finiteness of the above integral for a Gaussian measure \( N(m, c) \). However, deriving a uniform bound \( C < \infty \) for all Gaussians \( N(m, c) \) where \( m \) and \( c \) are allowed to vary within classes \( M \) and \( C \), respectively, is not trivial.

By intuition \( M, C \) have to be bounded, otherwise the mean and variance of \( \log a(x) \) would be unbounded. Here, we further discuss a particular widely-used subclass of covariance functions: the family of Matérn covariance functions. This family is parametrized by three scalar parameters:
the pointwise variance $\sigma^2$, the correlation length $\rho > 0$ and a smoothness parameter $\nu$. For $\nu$ being half integer, i.e., $\nu = k + \frac{1}{2}$ for $k \in \mathbb{N}$, the corresponding covariance function is

$$c_{\sigma^2,\rho,k+\frac{1}{2}}(x, y) := \sigma^2 \frac{k!}{(2k)!} \sum_{i=0}^{k} \frac{(k+i)!}{i!(k-i)!} \left( \frac{2\sqrt{2k+1}}{\rho} \right)^i \exp \left( -\frac{2\sqrt{2k+1}}{\rho} |x - y| \right).$$

(28)

For this class of covariance functions we mention the following result.

**Corollary 23.** Let $r_f, r_m < \infty$ be radii and let $\sigma_{\text{max}} < \infty$, $k_{\text{max}} \in \mathbb{N}$ and $\rho_{\text{min}} > 0$. Suppose the product measures $P = P_a \otimes P_f$, $Q = Q_a \otimes Q_f$ on $L^\infty(D) \times L^2(D)$ satisfy

- $P_f, Q_f \in \mathcal{P}_{2p}(L^2(D))$, where
  $$d_{2p}(P_f, \delta_0), d_{2p}(Q_f, \delta_0) \leq r_f$$
- $P_a = N(m, c)$, $Q_a = N(\tilde{m}, \tilde{c})$, where
  $$\|m\|_{C(D)}, \|\tilde{m}\|_{C(D)} \leq r_m$$
  and
  $$c, \tilde{c} \in \left\{ c_{\sigma^2,\rho,k+\frac{1}{2}} : \sigma \leq \sigma_{\text{max}}, \rho \geq \rho_{\text{min}}, k \in \{0, \ldots, k_{\text{max}}\} \right\}.$$

Then the solution operator $S$ of \textbf{(BVP)} mapping $(\log a, f)$ to $u$ is bounded, i.e., there exists a finite constant $C < \infty$ such that

$$d_p(S, P, S, Q) \leq C \ d_{2p}(P, Q).$$

**Proof.** We have to verify the assumptions of Theorem 22 for $P$ and $Q$. Notice that, due to the product structure, we have

$$\int (1 + \|f\|_{L^2(D)})^{2p} \exp(6p \|\log a\|_{L^\infty(D)}) \ dP = \int (1 + \|f\|_{L^2(D)})^{2p} \ dP_f \int \exp(6p \|\log a\|_{C(D)}) \ dP_a,$$

analogously for $Q$. The first term is uniformly bounded by $2^{2p}(1 + r_f^{2p})$ given the assumptions on $P_f$ and $Q_f$. To show that also the second term is uniformly bounded for all admissible Gaussian $P_a$ and $Q_a$ requires some powerful tools from Gaussian process theory. This is detailed in Appendix ??.

**Remark 24.** Regarding Remark 20, we can modify Corollary 23 in the following way: let $P_f = Q_f = \delta_{\tilde{f}_0}$, $\tilde{f}_0 \in L^2(D)$ and let $P_a, Q_a$ be as in Corollary 23. Then, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon) < \infty$ such that for $P = P_a \otimes \delta_{\tilde{f}_0}$ and $Q = Q_a \otimes \delta_{\tilde{f}_0}$ we have

$$d_p(S, P, S, Q) \leq C \ d_{p+\varepsilon}(P, Q).$$

Hence, we almost obtain Lipschitz continuity in the $p$-Wasserstein distance. However, the constant depends on $\varepsilon$ and $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$.  

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4.4 Sensitivity of Risk Functionals

We can use the results above for Wasserstein sensitivity of the solution operator $S$ of the elliptic problem (BVP) in order to obtain sensitivity results for risk functionals of Lipschitz continuous quantities of interest for the solution $u$ of (BVP). Since the formulation of Theorem 14 relies on the existence of an optimal coupling, and this is assured only for probability measures on complete and separable spaces, we assume for the remainder of this subsection that $\log a \in C(\overline{D})$ rather than $L^\infty(D)$. Applying Theorem 14 with $\phi: H^1_0(D) \to \mathbb{R}$ Hölder with Hölder exponent $\beta > 0$ and constant $C_\phi$ as well as $P, Q \in \mathcal{P}(C(\overline{D}) \times L^2(D))$, we obtain

$$\left| \rho_p(\phi \circ S) - \rho_q(\phi \circ S) \right| \leq C_\phi \cdot \sup_{Z \in \mathcal{A}_{\pi^*}} E_{\pi^*} \left[ Z^p \right]^{\frac{1}{p\beta}} \cdot d_p(S,P,S,Q)\beta,$$

where $p_\beta \geq \frac{p}{p-1}$. Thus, by controlling $d_p(S,P,S,Q)$, we can bound the difference of the risk functionals $| \rho_p(\phi \circ S) - \rho_q(\phi \circ S) |$. By means of Theorem 17 (and Theorem 22, resp.), we state the following result.

**Corollary 25.** Let $P, Q \in \mathcal{P}(C(\overline{D}) \times L^2(D))$ and $\phi: H^1_0(D) \to \mathbb{R}$ have Lipschitz constant $C_\phi$ (cf. (8)).

(i) If $P, Q$ satisfy the assumptions of Theorem 17, i.e., their support is contained in a ball in $C(\overline{D}) \times L^2(D)$ with radius $r$, then there exists a constant $C_r < \infty$ such that

$$\left| \rho_p(\phi \circ S) - \rho_q(\phi \circ S) \right| \leq C_\phi \cdot \sup_{Z \in \mathcal{A}_{\pi^*}} E_{\pi^*} \left[ Z^p \right]^{\frac{1}{p\beta}} \cdot d_p(P,Q)\beta,$$

where $q = \frac{p}{p-1}$ is the Hölder conjugate exponent to $p \geq 1$ and where $\pi^* \in \Pi(P,Q)$ denotes the optimal coupling for $d_p(P,Q)$.

(ii) If $P, Q$ satisfy the assumptions of Theorem 22, i.e., the local Lipschitz constant of $S$ belongs to $L^2_P$ and $L^2_Q$ with a uniform bound on its corresponding norm, then there exists a constant $C < \infty$ such that

$$\left| \rho_p(\phi \circ S) - \rho_q(\phi \circ S) \right| \leq C_\phi \cdot \sup_{Z \in \mathcal{A}_{\pi^*}} E_{\pi^*} \left[ Z^p \right]^{\frac{1}{p\beta}} \cdot d_{2p}(P,Q)\beta,$$

where $q = \frac{p}{p-1}$ is the Hölder conjugate exponent to $p \geq 1$ and where $\pi^* \in \Pi(P,Q)$ denotes the optimal coupling for $d_{2p}(P,Q)$.

**Remark 26.** For many common risk functionals such as the average value-at-risk, spectral risk functionals and the entropic risk functional the restriction log $a \in C(\overline{D})$ is not necessary. In particular, for these risk functionals we obtain for $P, Q \in \mathcal{P}(L^\infty(D) \times L^2(D))$ and $\phi: H^1_0(D) \to \mathbb{R}$ having Lipschitz constant $C_\phi$ that

$$\left| \rho_p(\phi \circ S) - \rho_q(\phi \circ S) \right| \leq C_\phi \cdot C \cdot d_p(P,Q)\beta$$

with $q = p$ and $q = 2p$ in case of assumption (i) and (ii) of Corollary 25, respectively. Here, the constant $C$ depends on $p, \beta$ and parameters related to the particular risk functional, e.g., the level $\alpha$ for the average value-at-risk—cf. the discussion following Theorem 14.
Remark 27 (Locally Hölder-continuous QoI). We note that Corollary 25 extends to only locally Hölder QoI \( \phi : H^1_0 \rightarrow \mathbb{R} \), i.e., \( \phi \) for which there exists a nondecreasing \( C_\phi : [0, \infty) \rightarrow [0, \infty) \) such that \( |\phi(u_1) - \phi(u_2)| \leq C_\phi(r) \|u_1 - u_2\|_{H^1(D)}^\beta \) for all \( \|u_1\|_{H^1(D)} \leq r \). Then, \( \phi \circ S \) is also locally Hölder continuous with exponent \( \beta \) and local Hölder constant \( C_{\phi \circ S}(r) = C_\phi(c(r) \cdot (c(1 + r)e^{3r})^\beta \) and case (i) of Corollary 25 holds with replacing \( C_\phi C_\beta \) by \( C_{\phi \circ S}(r) \). Extending case (ii) of Corollary 25 can be done by modifying Theorem 14 for locally Hölder maps following the main idea of Proposition 19.

4.5 Sensitivity with respect to truncation

A common and convenient representation of random functions or random fields on a bounded domain \( D \subset \mathbb{R}^d \), such as \( \log a \) or \( f \), are series expansions with random coefficients, e.g.,

\[
f(x, \omega) = f_0(x) + \sum_{k=0}^\infty \sigma_k f_k(x) \xi_k(\omega), \quad x \in D, \tag{29}\]

where \( \xi_k \) are mutually uncorrelated mean-zero real-valued random variables with unit variance; \( \{f_k\}_{k \in \mathbb{N}} \) is a suitable system of basis functions with unit norm and \( f_0(\cdot) \) represents the mean function of the random field, i.e., \( f_0(x) = \mathbb{E}[f(x, \cdot)] \). The probably most common one of such expansions is the Karhunen–Loève expansion (KLE): let \( f \) be a random field with continuous mean \( f_0 \) and continuous covariance function \( c(x, y) = \text{cov}(f(x), f(y)) \) and let \( C : L^2(D) \rightarrow L^2(D) \) denote its (trace-class) covariance operator in \( L^2(D) \) given by \( C\varphi(x) := \int_D c(x, y) \varphi(y) \, dy \). Then, the KLE of \( f \) is a representation as in (29), where \( (\sigma_k^2, f_k) \) are the eigenpairs of the operator \( C \). However, the eigensystem of the associated covariance operator is not the only suitable system for expanding random fields. In general, any Parseval frame of \( L^2(D) \) will yield a similar expansion with uncorrelated random coefficients [23].

In numerical simulations we can then truncate an expansion (29) after sufficiently many \( (K \in \mathbb{N}, \text{say}) \) terms and work with the resulting random function

\[
f_K(x, \omega) := f_0(x) + \sum_{k=0}^K \sigma_k \xi_k(\omega) f_k(x) \tag{30}\]

as uncertain coefficient in (BVP). In order to study the Wasserstein distance of the resulting distributions \( P, P_K \in \mathcal{P}(L^2(D)) \) of \( f \) and \( f_K \), respectively, we can use

\[
d_p(P, P_K) \leq \mathbb{E}\left[\|f - f_K\|_{L^p(D)}^p\right]^{1/p},
\]

since the distribution of \( (f, f_K) \) is obviously a coupling of \( P \) and \( P_K \). For \( p = 2 \) the right-hand side is explicitly

\[
d_2(P, P_K) \leq \sqrt{\sum_{k=K+1}^\infty \sigma_k^2}.
\]

In case of Gaussian random fields \( f \) and their truncated KLE we even obtain equality.
We have that

In the case of bounded random coefficients, i.e., constant $C$, their eigenpairs converge almost surely in $L^2$ as desired.

By construction $P = N(f_0, C)$ and $P_K = N(f_0, C_K)$ are Gaussian distributions on the Hilbert space $L^2(D)$ where $C_K \phi := \int_D c_K(x,y) \phi dy$ with $c_K(x,y) := \text{cov}(f_K(x), f_K(y)) = \sum_{k=1}^{K} \sigma_k^2 f_k(x) f_k(y)$. For Gaussian measures on Hilbert spaces there exists an exact formula for their 2-Wasserstein distance Gelbrich [16] which in this case is

$$d_2(P, P_K)^2 = d_2(N(f_0, C), N(f_0, C_K))^2 = \text{tr}(C) + \text{tr}(C_K) - 2 \text{tr} \left( \sqrt{C_K^{1/2} C C_K^{1/2}} \right).$$

We have that $\text{tr}(C) = \sum_{k=1}^{\infty} \sigma_k^2$ and $\text{tr}(C_K) = \sum_{k=1}^{K} \sigma_k^2$. Moreover, $C_K$ and $C$ share the same eigensystem and the null space of $C_K$ is the closure of the span of $\{f_k : k > K\}$. Thus, $C_K^{1/2} C C_K^{1/2}$ has the eigenpairs $(\tilde{\sigma}_k^2, \tilde{f}_k)$ with $\tilde{\sigma}_k^2 = \sigma_k^2$ for $k = 1, \ldots, K$ and $\tilde{\sigma}_k^2 = 0$ for $k > K$. This leads to,

$$\text{tr}(C) + \text{tr}(C_K) - 2 \text{tr} \left( \sqrt{C_K^{1/2} C C_K^{1/2}} \right) = 2 \sum_{k=1}^{K} \sigma_k^2 + \sum_{k>K} \sigma_k^2 - 2 \sum_{k=1}^{K} \sigma_k^2 = \sum_{k>K} \sigma_k^2,$$

as desired.

A similar statement holds for the Wasserstein distance of truncated random fields in Banach space norms.

**Proposition 29.** Let

$$\log a(x, \omega) = a_0(x) + \sum_{k=1}^{\infty} \sigma_k \xi_k(\omega) a_k(x), \quad x \in D,$$  \hspace{1cm} (33)

converge almost surely in $L^\infty(D)$ with uncorrelated mean-zero $\xi_k$ where $\text{Var}[\xi_k] = 1$. Let $\log a_k$ denote the random field resulting from truncating the series in (33) after $K$ terms and let $P, P_K \in \mathcal{P}(L^\infty(D))$ denote the distributions of $\log a$ and $\log a_K$, respectively. If there exists a constant $C < \infty$ such that $E \|\xi_k\| < C$ for all $k \in \mathbb{N}$, then

$$d_1(P, P_K) \leq C \sum_{k=K+1}^{\infty} \|\sigma_k\| a_k \|L^\infty(D), \quad \hspace{1cm} (34)$$

In the case of bounded random coefficients, i.e., $|\xi_k| \leq C$ almost surely for all $k \in \mathbb{N}$, we also have

$$d_p(P, P_K) \leq C \sum_{k=K+1}^{\infty} |\sigma_k| a_k \|L^\infty(D), \quad \hspace{1cm} (35)$$

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Proof. Let \( \pi \in \Pi(P, P_K) \) denote the distribution of the pair \((\log a, \log a_K)\) on \( L^\infty(D) \times L^\infty(D) \). Then

\[
d_p(P, P_K) \leq E \left[ \| \log a - \log a_K \|_{L^\infty(D)}^p \right]^{1/p} \leq E \left[ \left( \sum_{k \geq N} |\sigma_k| |\xi_k| \|a_k\|_{L^\infty(D)} \right)^p \right]^{1/p}.
\]

For the bounded case, \(|\xi_k| \leq C\) almost surely for all \( k \in \mathbb{N} \), we obtain the second statement immediately. And for \( p = 1 \) the first statement follows easily by taking the expectation into the series. \( \square \)

Remark 30. We can also use further existing results on the truncation error in the \(L^p(\Omega; L^\infty(D))\)-norm, see [9] in order to bound the Wasserstein distance. This yields in the case of truncated Karhunen–Loève expansions, i.e., \((\sigma_k^2, a_k)\) are eigenpairs of the covariance operator of \( \log a \) on \( L^2(D) \), and given appropriate assumptions [9], that

\[
d_p(P, P_K) \leq C_p \max \left( \sum_{k=K+1}^{\infty} \sigma_k^2 \|a_k\|_{L^\infty(D)}^2, \sum_{k=K+1}^{\infty} \sigma_k^2 \|\nabla a_k\|_{L^\infty(D)}^2 \right)^{1/2},
\]

where \( \alpha \in (0, 1) \) is such that the second series converges.

One may combine the previous two results and Corollary 25 to obtain an explicit bound for the QoI. Let \( \log a \) be as in Proposition 29 with almost surely bounded \( \xi_k \) and

\[ f(x) = f_0(x) + \sum_{k=1}^{\infty} \tilde{\xi}_k f_k(x) \]

with uncorrelated mean-zero \( \tilde{\xi}_k \), \( \text{Var}[\tilde{\xi}_k] = 1 \). Then

\[ |\rho_p(\phi \circ S) - \rho_{P_K}(\phi \circ S)| \leq C_\phi \sqrt{\left( \sum_{k=K}^{\infty} |\sigma_k| \|a_k\|_{L^\infty(D)} \right)^2 + \sum_{k=K}^{\infty} \sigma_k^2 \|f_k\|_{L^2(D)}^2}, \quad (36) \]

where \( P, P_K \in \mathcal{P}(L^\infty(D) \times L^2(D)) \) are the distributions of \((\log a, f)\) and \((\log a_K, f_K)\), respectively, and \( \phi \) denotes a Lipschitz continuous quantity of interest \( \phi : H^1_0(D) \to \mathbb{R} \). This truncation result could be easily extended to unbounded random coefficients \( \xi_k \), provided that results as addressed in Remark 30 are available.

5 Summary

We have derived stability results for uncertainty propagation for a stationary diffusion problem with random coefficient and source functions against perturbation of their probability distribution. Quantities of interest deriving from the solution of such a random PDE inherit its randomness, and we have employed risk functionals to quantify their random behavior. In order to capture deviations in (input) probability measures we used the Wasserstein distance which has a natural relation to Lipschitz continuous forward maps in view of the Kantorovich–Rubinstein theorem.
However, the solution operator of the problem considered is merely locally, not globally Lipschitz. Employing new results on locally Lipschitz forward maps we were able to bound the deviations in the random solution and in risk functionals of derived quantities of interest. We applied our analysis to the common case of lognormal diffusion with a Matérn covariance kernel exploiting some classical boundedness results on Gaussian processes as well as to the usual approximation of input random fields by truncated series expansions.

Having established these basic stability results for uncertainty propagation, one could extend the presented analysis to other PDE models with locally Lipschitz solution operators using the general results of Section 4.2. Moreover, for practical purposes it is helpful to have estimates on the Wasserstein distance of the input distributions, e.g., to be able to bound the distance of Gaussian random fields with different Matérn covariance kernels by the difference in the Matérn parameters. In this way, the effects of the statistical estimation of these parameters on the outcome of uncertainty propagation for lognormal diffusion could be evaluated using our results.

A Proof of Corollary 23

In what follows we provide several lemmas which combine to complete the proof of Corollary 23, see the very end of this appendix. However, the argumentation and the statements of the lemmas are deliberately presented for arbitrary continuous Gaussian random fields, and we focus on Matérn covariance functions only in the last part. In particular, we consider pathwise continuous Gaussian random fields \( g : D \times \Omega \rightarrow \mathbb{R} \) with continuous mean function \( m(\cdot) := \mathbb{E}[g(\cdot)] \in C(D) \) and continuous covariance function \( c \in C(D \times D) \), \( c(x, y) := \text{cov}(g(x), g(y)) \). We denote the resulting Gaussian distribution on \( C(D) \) of such a Gaussian random field by \( N(m, c) \in \mathcal{P}(C(D)) \).

We study now the following question: for which sets \( M \subset C(D) \) and \( C \subset C(D \times D) \) of continuous mean and covariance functions can we ensure that for a given \( \beta > 0 \) we have

\[
\sup_{x \in \mathcal{G}(M, C)} \int \exp(\beta \|g\|_{C(D)}) \, dP < \infty, \quad \mathcal{G}(M, C) := \{N(m, c) : m \in M, c \in C\}.
\]

By Fernique’s theorem we know that each single exponential moment above exists for arbitrary \( \beta > 0 \). However, the uniform boundedness over a given set \( \mathcal{G}(M, C) \) is harder to ensure. To this end, we apply the well-known Borell-TIS inequality for Gaussian measures on Banach spaces [21, Chapter 3]: let \( g \) be a centered Gaussian process on a compact set \( D \subset \mathbb{R}^d \) which is pathwise continuous, then \( \mathbb{E}[\|g\|_{C(D)}] < \infty \) and with \( P \) denoting its distribution on \( C(D) \) we have

\[
P\left(\|g\|_{C(D)} - \mathbb{E}[\|g\|_{C(D)}] \geq r\right) \leq 2 \exp(-r^2/2\sigma^2), \tag{37}
\]

where \( \sigma^2 := \sup_{x \in D} \mathbb{E}[g^2(x)] = \sup_{x \in D} \text{Var}(g(x)) \). This yields the following result.

**Lemma 31.** Let \( \mathcal{G} = \mathcal{G}(M, C) \), where \( M \subset C(D) \) and \( C \subset C(D \times D) \), satisfy the following conditions:

(i) The sets \( M \) and \( C \) are bounded, i.e., there exist radii \( r_M, r_C < \infty \) such that \( \|m\|_{C(D)} \leq r_M \) for all \( m \in M \) and \( \|c\|_{C(D \times D)} \leq r_C \) for all \( c \in C \).

(ii) There exists a constant \( s < \infty \) such that \( \mathbb{E}_P[\|g\|_{C(D)}] \leq s \) for all \( P \in \mathcal{G}(\{0\}, C) \).
Then, for any $0 < \beta < \infty$, there exists a constant $K_\beta = K_\beta(r_m, r_c, s) < \infty$ such that

$$\mathbb{E}_P \left[ \exp(\beta \|g\|_{C(D)}) \right] \leq K_\beta, \quad \forall P \in \mathcal{G}(M, C).$$

**Proof.** Consider first an arbitrary $P = \mathcal{N}(m, c) \in \mathcal{G}(M, C)$. Then we have

$$\mathbb{E}_P \left[ \exp(\beta \|g\|_{C(D)}) \right] \leq \exp(\beta r_m) \mathbb{E}_P \left[ \exp(\beta \|b - m\|_{C(D)}) \right].$$

Thus, we may restrict ourselves in the following to an arbitrary centered $P$. For Gaussian processes such a metric is, for instance, given by

$$\|g\|_{C(D)} = \max_{x \in D} |g(x)| = \max_{x \in D} \|g(x) - m(x)\|_{C(D)}.$$

Studying $\mathbb{E}_P \left[ \exp(\beta \|g\|_{C(D)}) \right]$ and obtain by the second assumption and the Borell-TIS inequality

$$\mathbb{P} \left( \{g \in C(D) : \|g\|_{C(D)} \geq M_p + r\} \right) \leq 2 \exp(-r^2/2r_c).$$

This implies that

$$\mathbb{P} \left( \{g \in C(D) : \|g\|_{C(D)} \geq M_p + r\} \right) \leq 2 \exp(-r^2/2r_c)$$

which can now be used as follows:

$$\mathbb{E}_P \left[ \exp(\beta \|g\|_{C(D)}) \right] \leq \exp(\beta M_p) + \int_{g : \|g\|_{C(D)} \geq M_p} \exp(\beta \|g\|_{C(D)}) \, dP$$

$$\leq \exp(\beta M_p) + \sum_{n=0}^{\infty} e^{\beta(M_p+n)} \mathbb{P}(\{g \in C(D) : n \leq \|g\|_{C(D)} - M_p < n + 1\})$$

$$\leq \exp(\beta M_p) \left( 1 + \sum_{n=0}^{\infty} e^{\beta(n+1)} \mathbb{P}(\{g \in C(D) : M_p + n \leq \|g\|_{C(D)}\}) \right)$$

$$\leq e^{\beta s} \left( 1 + 2 \sum_{n=0}^{\infty} e^{\beta(n+1) - n^2/2r_c} \right) < \infty,$$

which proves the assertion. \qed

Actually, Fernique’s theorem can be proven similarly to Lemma 31. It remains to ensure a uniform bound of $\mathbb{E}_P \left[ \|g(x)\|_{C(D)} \right]$. $P \in \mathcal{G}(\{0\}, C)$ by imposing suitable conditions on $C$. Studying $\mathbb{E} \left[ \sup_{x \in D} g(x) \right]$ has a long history in probability theory. We refer to Talagrand [37] for a comprehensive discussion and exploit Dudley’s entropy bound, a classical result.

Consider a centered Gaussian process $g$ with distribution $P = \mathcal{N}(0, c)$ on $C(D)$ and assume a metric $d : D \times D \to [0, \infty)$ satisfying

$$\forall x, y \in D \forall \tau > 0: P(|g(x) - g(y)| \geq \tau) \leq 2 \exp \left( -\frac{\tau^2}{2d^2(x, y)} \right). \quad (38)$$

For Gaussian processes such a metric is, for instance, given by

$$d_c(x, y) := \left( \mathbb{E} \left[ |g(x) - g(y)|^2 \right] \right)^{1/2} = \sqrt{\text{Var}(g(x) - g(y))} = \sqrt{c(x, x) + c(y, y) - 2c(x, y)}.$$
Now, Dudley’s entropy bound [37] for Gaussian processes \( g \) with distribution \( \mathcal{P} = \mathcal{N}(0, c) \in \mathcal{P}(C(D)) \) states that if (38) holds for a metric \( d \), then we have
\[
\mathbb{E}_P \left[ \|g\|_{C(D)} \right] \leq K \int_0^\infty \sqrt{\log N(D, d, r)} \, dr,
\]
where \( K < \infty \) denotes a (universal) constant and \( N(D, d, r) \) are the covering numbers of \( D \) with respect to the metric \( d \), i.e.,
\[
N(D, d, r) := \inf \left\{ n \in \mathbb{N} : \exists x_1, \ldots, x_n \in D \text{ such that } D \subseteq \bigcup_{j=1}^n B_d^r(x_j) \right\},
\]
where \( B_d^r(x) := \{ y \in D : d(x, y) \leq r \} \) are balls in \( D \) of radius \( r \) with respect to \( d \). This gives rise to the following lemma.

**Lemma 32.** Consider \( \mathcal{G}(\{0\}, C) \subseteq C(D \times D) \). If there exists a metric \( d : D \times D \to [0, \infty) \) such that
\[
c(x, x) + c(y, y) - 2c(x, y) \leq d^2(x, y) \quad \forall x, y \in D \quad \forall c \in C
\]
and
\[
\int_0^\infty \sqrt{\log N(D, d, r)} \, dr < \infty,
\]
then
\[
\sup_{P \in \mathcal{G}(\{0\}, C)} \mathbb{E}_P \left[ \|g\|_{C(D)} \right] < \infty.
\]

We now state a result for subsets of Matérn covariance functions, as introduced in Subsection 4.2, which in combination with the previous two lemmas completes the proof of Corollary 23 in Subsection 4.2.

**Lemma 33.** Consider the following class of Matérn covariance functions:
\[
C = C(\sigma_{\text{max}}, \rho_{\text{min}}, k_{\text{max}}) := \left\{ c_{\sigma^2, \rho, k+\frac{1}{2}} : \sigma \leq \sigma_{\text{max}}, \rho \geq \rho_{\text{min}}, k \in \{0, \ldots, k_{\text{max}}\} \right\},
\]
where \( \sigma_{\text{max}} < \infty, k_{\text{max}} \in \mathbb{N} \) and \( \rho_{\text{min}} > 0 \). Then, we have
\[
c(x, x) + c(y, y) - 2c(x, y) \leq d^2(x, y) \quad \forall c \in C,
\]
where
\[
d^2(x, y) := 2\sigma_{\text{max}}^2 \left( 1 - \exp \left( -\frac{\sqrt{2k_{\text{max}} + 1}}{\rho_{\text{min}}} |x - y| \right) \right),
\]
which is the associated metric \( d_{c_*} \) for \( c_* := c_{\sigma_{\text{max}}, \rho_{\text{min}}, k_{\text{max}} + \frac{1}{2}} \) with \( \rho := \frac{\rho_{\text{min}}}{\sqrt{2k_{\text{max}} + 1}} \). Moreover, for any bounded domain \( D \subseteq \mathbb{R}^k \) we have for this metric that
\[
\int_0^\infty \sqrt{\log N(D, d, r)} \, dr < \infty.
\]
Thus, by a change of variables argument, we get
\[ c_{\sigma^2, \rho, k+\frac{1}{4}}(x, x) = \sigma^2. \]

Thus, the first assertion follows by
\[ c_{1, \rho, k+\frac{1}{4}}(x, y) = \exp\left(-\frac{\sqrt{2k+1}}{\rho} |x - y|\right) \geq \exp\left(-\frac{\sqrt{2k_{\max} + 1}}{\rho_{\min}} |x - y|\right). \]

For the second assertion we note that \( N(D, d, r) = N(D, \tilde{d}, r/2\sigma_{\max}^2) \) where
\[ \tilde{d}(x, y) := 1 - \exp\left(-\frac{|x - y|}{\hat{\rho}}\right). \]

Thus, by a change of variables argument, we get
\[ \int_0^\infty \log N(D, \tilde{d}, r) dr = 2\sigma_{\max}^2 \int_0^\infty \log N(D, \tilde{d}, r) dr. \]

Besides that we have
\[ N(D, \tilde{d}, r) = N(D, | \cdot |, \hat{\rho} \log((1 - r)^{-1})) \]
and, thus, need to estimate \( N(D, | \cdot |, r) \). We do so quite crudely, embed \( D \) in a cube \( \hat{D} \) of edge length \( \text{diam}(D) := \sup_{x, y \in D} |x - y| \) and simply estimate \( N(D, | \cdot |, r) \leq N(\hat{D}, | \cdot |, r) \). Moreover, since in every (Euclidean) ball of radius \( r \) in \( \mathbb{R}^k \) we can insert a smaller cube of edge length \( a = 2r/\sqrt{k} \), we can bound the covering number \( N(\hat{D}, | \cdot |, r) \) by the number of cubes of edge length \( a \) covering \( \hat{D} \), i.e.,
\[ N(D, | \cdot |, r) \leq N(\hat{D}, | \cdot |, r) \leq \left(\frac{\text{diam}(D)}{a}\right)^k = \left(\frac{\sqrt{k} \text{diam}(D)}{2r}\right)^k. \]

Thus,
\[ N(D, \tilde{d}, r) \leq \left[\frac{\sqrt{k} \text{diam}(D)}{2\hat{\rho} \log((1 - r)^{-1})}\right]^k = \kappa \left[\frac{1}{\log((1 - r)^{-1})}\right]^k, \quad \kappa := \left[\frac{\sqrt{k} \text{diam}(D)}{2\hat{\rho}}\right]^k. \]

and, since \( \text{diam}_{\tilde{d}}(D) := \sup_{x, y \in D} \tilde{d}(x, y) = 1 \) and \( N(D, \tilde{d}, r) = 1 \) for all \( r \geq \text{diam}_{\tilde{d}}(D) \), we have
\[ \int_0^\infty \log N(D, \tilde{d}, r) dr = 2\sigma_{\max}^2 \int_0^1 \log N(D, \tilde{d}, r) dr \leq 2\sigma_{\max}^2 \left(\log \kappa + \sqrt{k} \int_0^1 \log ((1/\log(1/r))) dr\right). \]

It remains to prove that \( \int_0^1 \frac{1}{\log((1/\log(1/r)))} dr < \infty \). To this end, we observe that
\[ |1/\log(1/r)| = 1 \quad \text{iff} \quad r \in (0, e^{-1}]. \]
and for $2 \leq i \in \mathbb{N}$

$$[1/\log(1/r)] = i \quad \text{iff} \quad r \in \left(\exp(-1/(i-1)), \exp(-1/i)\right].$$

We obtain

$$\int_{0}^{1} \sqrt{\log \left([1/\log(1/r)]\right)} dr \leq e^{-1} + \sum_{i=2}^{\infty} \log(i) \exp \left(-\frac{1}{i}\right) - \exp \left(-\frac{1}{i-1}\right) < \infty. \quad \Box$$

Remaining proof of Corollary 23. In order to complete the proof of Corollary 23 in Section 4.3, we need to show that the class of Gaussian measures specified in Corollary 23 satisfies the assumption of Lemma 31. Now, condition (i) of Lemma 31 is ensured by the assumptions of Corollary 23, since $\|e_{\sigma_{X}}.k+1\|_{D(D)} = \sigma^{2}$. Condition (ii) of Lemma 31 follows by combining Lemma 32 and Lemma 33. This concludes the proof. \quad \Box

B Explicit Construction of (Coupled) Risk Measures

Let $\pi$ be a coupling with marginals $P$ and $Q$. In what follows we explicitly construct a risk functional $\rho_{\pi}$ so that

$$\rho_{\pi}(X \circ p_{1}) = \mathcal{P}_{p}(X) \quad \text{and} \quad \rho_{\pi}(X \circ p_{2}) = \mathcal{P}_{Q}(X).$$

For ease of exposition we define the measure $Z_{1}P$ ($Z_{2}Q$, resp.) with density $Z_{1}$ ($Z_{2}$, resp.) by

$$Z_{1}P(B) := \int_{B} Z \, dP \quad (Z_{2}Q(B) := \int_{B} Z_{2} \, dQ, \text{resp.}).$$

**Proposition 34.** Let the risk functionals $\rho_{p}$ and $\rho_{Q}$ have support sets $\mathcal{A}_{p}$ and $\mathcal{A}_{Q}$. Then the set

$$\mathcal{A}_{\pi} := \left\{ Z = \frac{dy}{dx} : \gamma \in \Pi(Z_{1}P, Z_{2}Q), \, Z_{1} \in \mathcal{A}_{p}, \, Z_{2} \in \mathcal{A}_{Q} \right\}$$

is the support set of a risk functional $\rho_{\pi}$ satisfying $\rho_{\pi}(X \circ p_{1}) = \rho_{p}(X)$ and $\rho_{\pi}(X \circ p_{2}) = \rho_{Q}(X)$.

**Proof.** Let $Z \in \mathcal{A}_{\pi}$ be arbitrary and denote by $Z_{1} \in \mathcal{A}_{p}$ and $Z_{2} \in \mathcal{A}_{Q}$ the two random variables such that $Z_{\pi} \in \Pi(Z_{1}P, Z_{2}Q)$, where $Z_{\pi}(A \times B) := \int_{A \times B} Z_{\pi}(x_{1}, x_{2}) (dx_{1}, dx_{2})$ is defined in analogy to $Z_{1}P$ and $Z_{2}Q$. We then have

$$\mathbb{E}_{\pi} [Z(X \circ p_{1})] = \int_{X \times X} (X \circ p_{1})(x_{1}, x_{2}) (Z\pi)(dx_{1}, dx_{2}) = \int_{X} X(x_{1}) Z_{1}P(dx_{1}) = \mathbb{E}_{P} [Z_{1}X].$$

Further, for any $Z_{1} \in \mathcal{A}_{p}$ it holds that $Z(x_{1}, x_{2}) := Z_{1}(x) \in \mathcal{A}_{\pi}$, as $Z_{2}(x_{2}) = 1 \in \mathcal{A}_{Q}$. We thus obtain that

$$\rho_{\pi}(X \circ p_{1}) = \sup_{Z \in \mathcal{A}_{\pi}} \mathbb{E}_{\pi} [Z \cdot (X \circ p_{1})] = \sup_{Z_{1} \in \mathcal{A}_{p}} \mathbb{E}_{P} [Z_{1}X] = \rho_{P}(X).$$

The statement for $\rho_{Q}$ is proven analogously. \quad \Box
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