DISCONTINUOUS GALERKIN METHODS FOR NONVARIATIONAL PROBLEMS

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Abstract. We extend the finite element method introduced by Lakkis and Pryer [2011] to approximate the solution of second order elliptic problems in nonvariational form to incorporate the discontinuous Galerkin (DG) framework. This is done by viewing the NVFEM as a mixed method whereby the “finite element Hessian” is an auxiliary variable in the formulation. Representing the finite element Hessian in a discontinuous setting yields a linear system of the same size and having the same sparsity pattern of the compact DG methods for variational elliptic problems. Furthermore, the system matrix is very easy to assemble. Thus this approach greatly reduces the computational complexity of the discretisation compared to the continuous approach.

We conduct a stability and consistency analysis making use of the unified framework set out in Arnold et. al. [2001]. We also give an apriori analysis of the method. The analysis applies to any consistent representation of the finite element Hessian, thus is applicable to the previous works making use of continuous Galerkin approximations.

1. Introduction

Nonvariational partial differential equations (PDEs) are those which are given in the form

\[ -\mathbf{A} : \mathbf{D}^2 \mathbf{u} = \mathbf{f}, \]

where \( \mathbf{X} : \mathbf{Y} = \text{trace} (\mathbf{X}^\top \mathbf{Y}) \) is the Frobenious inner product between matrices. If the matrix \( \mathbf{A} \) is differentiable then there is an equivalence between this problem and its variational sibling

\[ -\text{div} (\mathbf{A} \nabla \mathbf{u}) + \mathbf{D} \mathbf{A} \nabla \mathbf{u} = \mathbf{f}, \]

where

\[ \mathbf{D} \mathbf{A} = \left( \sum_{i=1}^{d} \partial_i a_{i,1}(\mathbf{x}), \ldots, \sum_{i=1}^{d} \partial_i a_{i,d}(\mathbf{x}) \right). \]

Rewriting in this form is sometimes undesirable. For example, if the coefficient matrix \( \mathbf{A} \) has near singular derivatives the problem will become advection dominated and possibly unstable for conforming finite element methods. There is a wealth of material on the treatment of advection dominated problems [EG04, ESW05, c.f.]. If \( \mathbf{A} \) is not differentiable then the problem has no variational structure. In this case standard finite element methods cannot be applied.

In a previous work [LP11] a finite element method for the approximation of the nonvariational problem (1.1) was introduced. This involved the introduction of a finite element Hessian represented in the same finite element space as the solution.
(modulo boundary conditions). The applications of the discrete representation of a Hessian of a piecewise function are becoming broader, for example, it can be used to drive anisotropic adaptive algorithms [AV02, VMD+07], as a notion of discrete convexity [AM09] and in the design of finite element methods for nonlinear fourth order problems [Pry12]. We are particularly interested in nonvariational PDEs due to their relation to general fully nonlinear PDEs

\( F(D^2 u) = 0 \),

which are of significant current research. There have been finite element methods presented for this general class of problem for example in [Böhm08] the author presents a \( C^1 \) finite element method shows stability and consistency (hence convergence) of the scheme which requires a high degree of smoothness on the exact solution. In [FN09b, FN09a] the authors give a method in which they approximate the general second order fully nonlinear PDE by a sequence of fourth order quasilinear PDEs. This is reminiscent of the vanishing viscosity method introduced for classically studying first order fully nonlinear PDEs. Efficiency of any method used to approximate a problem such as this is key. Each of the methods are computationally costly due to their reliance on \( C^1 \) finite elements [Böhm08, FN09b] or mixed methods [FN09a].

In [BS91] a generic framework was set up to prove convergence of numerical approximations to the solutions of fully nonlinear PDEs. This involved constructing monotone sequences of approximations which are typically applied to finite difference approximations of the nonlinear problem [Obe06], c.f. The assumption of consistency made in the [BS91] framework are incompatible with finite element methods, however, an extremely important observation made in [JS11] is that the consistency condition may be weakened to incorporate the finite element case using a localisation argument (in the case of isotropic diffusion).

In this contribution we present a method for the discontinuous approximation of the linear nonvariational problem (1.1). We also present convergence analysis for a certain subclass of the nonvariational problems, those which are coercive. This allows us to use variational techniques to analyse the problem. We prove optimal convergence rates for the finite element solution in broken Sobolev norms. Note that the results presented here are immediately applicable to the method derived for the continuous case given in [LP11].

The algebraic formulation of the continuous approximation of the nonvariational problem requires the solution of large sparse \((d + 1)^2 \times N^2\) linear system [LP11, Lem 3.3], where \( d \) is the dimension of the problem and \( N \) the number of degrees of freedom. Equivalently, using a Schür complement argument, this can be reduced to an \( N^2 \) full linear system. The reason that this system is full is due to the global nature of the \( L_2(\Omega) \) projection operator into a continuous finite element space. The motivation for extending the nonvariational finite element method into the discontinuous setting is the massive gain in computational efficiency over the continuous case. Indeed, due to the local representation of the projection operators in these discontinuous spaces we are able to make massive computational savings, in that the system matrix will become sparse and is the same size as that of a standard discontinuous Galerkin stiffness matrix.

To test the method numerically we make use of the finite element package DUNE [BBD+08a, BBD+08b]. In this work we are interested in the asymptotic behaviour
of the discontinuous approximation. In a subsequent work we will study the computational gains using the discontinuous framework presented over the continuous one given in [LP11], as well as exploit the powerful parallelisation capabilities of the package.

The rest of the paper is set out as follows: In §2 we formally introduce the model problem and give a brief review of known classical facts about nonvariational PDEs. In §3 we examine the discretisation of the nonvariational method in the discontinuous Galerkin framework, making use of the unified framework set out in [ABCM02] to derive a very general formulation of the finite element Hessian represented as a discontinuous object. We present some examples and examine the natural question of what happens when we try to eliminate the finite element Hessian from the formulation. In §4 we look at the consistency and stability of the finite element Hessian and present our main analytical results of convergence. Finally, in §5 we detail a summary of extensive numerical experiments aimed at examining convergence and computational speed of the method presented.

2. Problem formulation

In this section we formulate the model problem, fix notation and give some basic assumptions. In addition we review the existence and uniqueness of the nonvariational problems. We begin by introducing the Lebesgue spaces

\[ L^2(\Omega) = \{ \phi : \int_\Omega |\phi(x)|^2 \, dx < \infty \} \quad \text{and} \quad L_\infty(\Omega) = \{ \phi : \sup_{x \in \Omega} |\phi(x)| < \infty \}, \]

and the Sobolev and Hilbert spaces

\[ W^k_p(\Omega) = \{ \phi \in L^p(\Omega) : D^\alpha \phi \in L^p(\Omega) \}, \quad \text{for} \quad |\alpha| \leq k \quad \text{and} \quad H^k(\Omega) := W^k_2(\Omega). \]

These are equipped with the norms

\[ \|\phi\|_{L^2(\Omega)}^2 = \int_\Omega |\phi|^2 \, dx, \quad \|\phi\|_{L_\infty(\Omega)} = \sup_{x \in \Omega} |\phi(x)|, \]

\[ \|v\|_{W^k_p(\Omega)}^p = \sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^p(\Omega)}^p \quad \text{and} \quad \|v\|_{W^k_p(\Omega)}^p = \sum_{|\alpha| = k} \|D^\alpha v\|_{L^p(\Omega)}^p. \]

where \( \alpha = \{\alpha_1, \ldots, \alpha_d\} \) is a multi-index, \( |\alpha| = \sum_{i=1}^d \alpha_i \) and derivatives \( D^\alpha \) are understood in a weak sense. We pay particular attention to the cases \( k = 1, 2 \) and

\[ H^1_0(\Omega) := \text{closure of} \ C_0^\infty(\Omega) \text{ in} \ H^1(\Omega). \]

The model problem in strong form is: Find \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) such that

\[ \langle \mathcal{L} u, \phi \rangle = \langle f, \phi \rangle \quad \forall \phi \in H^1_0(\Omega), \]

where the data \( f \in L^2(\Omega) \) is prescribed and \( \mathcal{L} \) is a general linear, second order, uniformly elliptic partial differential operator. Let \( A \in L_\infty(\Omega)^{d \times d} \), we then define

\[ \mathcal{L} : \quad H^2(\Omega) \cap H^1_0(\Omega) \rightarrow L^2(\Omega), \quad u \mapsto \mathcal{L} u := -A : D^2 u. \]

We assume that \( A \) is uniformly positive definite, i.e., there exists a \( \gamma > 0 \) such that for all \( x \)

\[ y^T A(x) y \geq \gamma |y|^2 \quad \forall y \in \mathbb{R}^d, \]

and we call \( \gamma \) the ellipticity constant.
Nonvariational PDEs are not as well studied as their variational brethren from a numerical analysis viewpoint. For the benefit of the reader we present a concise account of known results for strong solutions of this class of problem.

2.1. Definition (strong solution). A strong solution of (1.1) is a function \( u \in H^2(\Omega) \cap H^1_0(\Omega) \), that is a twice weakly differentiable function, which satisfies the problem almost everywhere.

2.2. Theorem (existence and regularity of a strong solution of (1.1) [GT83, Thm 9.15]). Let \( \Omega \subset \mathbb{R}^d \) be a \( C^{1,1} \) domain. Suppose also that \( A \in C^0(\Omega)^{d \times d} \) and \( f \in L^2(\Omega) \) such that the problem

\[
-A \cdot D^2 u = f \quad \text{in } \Omega
\]
\[
u = 0 \quad \text{on } \partial \Omega
\]

is uniformly elliptic. Then (2.9) has a unique strong solution. There also exists a constant independent of \( u \) such that

\[
\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.
\]

2.3. Remark (less regular solutions). Note that the theory of viscosity solutions has been developed for non classical solutions of (2.6) if the problem data does not satisfy the regularity assumed above see [GT83].

2.4. Assumption (regularity of \( A \)). From hereon in we will assume that the problem data is sufficiently smooth such that solutions exist and belong to at least \( H^2(\Omega) \cap H^1_0(\Omega) \).

2.5. Remark (regularity of \( \Omega \)). Theorem 2.2 specifies that \( \Omega \) must be a \( C^{1,1} \) domain. We will be approximating such a domain with one which is only \( C^{0,1} \) (i.e., a polyhedral one). We thus assume that the model problem admits a unique strong solution even when \( \Omega \) is only \( C^{0,1} \). To circumvent this assumption curved finite elements could be used to fit the boundary exactly [Ber89]. For simplicity we will not present this case here, although we believe our analysis can be extended to this case.

3. Discretisation

Let \( \mathcal{T} \) be a conforming, shape regular triangulation of \( \Omega \), namely, \( \mathcal{T} \) is a finite family of sets such that

1. \( K \in \mathcal{T} \) implies \( K \) is an open simplex (segment for \( d = 1 \), triangle for \( d = 2 \), tetrahedron for \( d = 3 \)),
2. for any \( K, J \in \mathcal{T} \) we have that \( K \cap J \) is a full subsimplex (i.e., it is either \( \emptyset \), a vertex, an edge, a face, or the whole of \( K \) and \( J \) of both \( K \) and \( J \) and

\[
\bigcup_{K \in \mathcal{T}} K = \Omega.
\]

We use the convention where \( h : \Omega \rightarrow \mathbb{R} \) denotes the meshsize function of \( \mathcal{T} \), i.e.,

\[
h(x) := \max_{K \ni x} h_K,
\]

where \( h_K \) is the diameter of \( K \). We let \( \mathcal{E} \) be the skeleton (set of common interfaces) of the triangulation \( \mathcal{T} \) and say \( e \in \mathcal{E} \) if \( e \) is on the interior of \( \Omega \) and \( e \in \partial \Omega \) if \( e \) lies on the boundary \( \partial \Omega \). The assumptions on the tessellation made here are typical in the finite element analysis. For the presentation of the method and its analysis, some assumption could be relaxed (e.g. the form of the elements or the assumption
on a conforming triangulation) but this would lead to an unnecessary increase in
the complexity of the presentation.

Let $P^k(T)$ denote the space of piecewise polynomials of degree $k$ over the
triangulation $T$, i.e.,

$$P^k(T) = \{ \phi : \phi|_K \in P^k(K) \}$$

and introduce the finite element spaces

$$\hat{V}_D = V_D(T,k) := P^k(T) \cap H_0^1(\Omega)$$
$$V_D = V_D(T,k) := P^k(T)$$

to be the usual spaces of discontinuous piecewise polynomial functions which are
compactly and non compactly supported respectively.

3.1. Remark (generalised Hessian). Assume a function $v \in H^2(\Omega)$, let $n : \partial \Omega \to \mathbb{R}^d$ be the outward pointing normal of $\Omega$ then the Hessian $D^2v$ of $v$, satisfies the
following identity:

$$\int_{\Omega} D^2v \phi \, dx = -\int_{\Omega} \nabla v \otimes \nabla \phi \, dx + \int_{\partial \Omega} \nabla v \otimes n \phi \, ds \quad \forall \phi \in H^1(\Omega).$$

If $v \in H^1(\Omega)$ \cite{3.5} is still well defined in view of duality, in this case we set

$$\langle D^2v \mid \phi \rangle = -\int_{\Omega} \nabla v \otimes \nabla \phi \, dx + \int_{\partial \Omega} \nabla v \otimes n \phi \, ds \quad \forall \phi \in H^1(\Omega),$$

where the last term is understood as a pairing between $H^{-1/2}(\Omega)$ and $H^{1/2}(\Omega)$.

3.2. Definition (broken Sobolev spaces, trace spaces). We introduce the broken
Sobolev space

$$H^k(T) := \{ \phi : \phi|_K \in H^k(K), \text{ for each } K \in T \}.$$

We also make use of functions defined in these broken spaces restricted to the
skeleton of the triangulation. This requires an appropriate trace space

$$\mathcal{T}(\mathcal{E}) := \prod_{K \in \mathcal{F}} L_2(\partial K) = \prod_{K \in \mathcal{F}} H^1_0(K).$$

3.3. Definition (jumps, averages and tensor jumps). We define average, jump and
tensor jump operators for arbitrary scalar functions $v \in \mathcal{T}(\mathcal{E})$, vectors $\mathbf{v} \in \mathcal{T}(\mathcal{E})^d$
and matrices $V \in \mathcal{T}(\mathcal{E})^{d \times d}$ as

$$\mathbf{\{ v \}} = \frac{1}{2}(v|_{K_1} + v|_{K_2}), \quad \| v \| = \frac{1}{2}(v|_{K_1} + v|_{K_2}),$$
$$\mathbf{[ v]} = v|_{K_1} n_{K_1} + v|_{K_2} n_{K_2}, \quad \mathbf{[ v]} = (v|_{K_1})^T n_{K_1} + (v|_{K_2})^T n_{K_2},$$
$$\mathbf{[ V]} = V|_{K_1} n_{K_1} + V|_{K_2} n_{K_2}, \quad \mathbf{[ v]} = v|_{K_1} \otimes n_{K_1} + v|_{K_2} \otimes n_{K_2}.$$
Note that on the boundary of the domain $\partial\Omega$ the jump and average operators are defined as
\begin{equation}
\langle v \rangle \bigg|_{\partial\Omega} := v, \quad \langle v \rangle \bigg|_{\partial\Omega} := v,
\end{equation}
\begin{equation}
[v] \bigg|_{\partial\Omega} := vn, \quad [v] \bigg|_{\partial\Omega} := v^Tn,
\end{equation}
\begin{equation}
\langle [V] \rangle \bigg|_{\partial\Omega} := Vn, \quad [v] \otimes n \bigg|_{\partial\Omega} := v \otimes n.
\end{equation}

We will often use the following Proposition which we state in full for clarity but whose proof is merely using the identities in Definition 3.3.

3.4. **Proposition** (elementwise integration). For a generic vector valued function $p$ and scalar valued function $\phi$ we have
\begin{equation}
\sum_{K \in \mathcal{T}} \int_K \text{div}(p) \phi \, dx = \sum_{K \in \mathcal{T}} \left( - \int_K p^T \nabla_h \phi \, dx + \int_{\partial K} \phi p^T n_K \, ds \right),
\end{equation}
where $\nabla_h = (D_h)^T$ is the elementwise spatial gradient. Furthermore, if we have
\begin{equation}
\sum_{K \in \mathcal{T}} \int_{\partial K} \phi p^T n_K \, ds = \int_{\partial \Omega} [p] \otimes \phi \, ds + \int_{E \cup \partial \Omega} [\phi] \otimes [p] \, ds = \int_{E \cup \partial \Omega} [p\phi] \, ds,
\end{equation}
An equivalent tensor formulation of (3.15)–(3.16) is
\begin{equation}
\sum_{K \in \mathcal{T}} \int_K D_h p \phi \, dx = \sum_{K \in \mathcal{T}} \left( - \int_K p \otimes \nabla_h \phi \, dx + \int_{\partial K} \phi p \otimes n_K \, ds \right),
\end{equation}
where
\begin{equation}
\sum_{K \in \mathcal{T}} \int_{\partial K} \phi p \otimes n_K \, ds = \int_{\partial \Omega} [p] \otimes \phi \, ds + \int_{E \cup \partial \Omega} [\phi] \otimes [p] \, ds = \int_{E \cup \partial \Omega} [p\phi] \, ds.
\end{equation}
In addition for matrix valued $V$ we have that
\begin{equation}
\sum_{K \in \mathcal{T}} \int_K (D_h p) : V \, dx = \sum_{K \in \mathcal{T}} \left( - \int_K p : D_h V \, dx + \int_{\partial K} (V p)^T n \, ds \right)
\end{equation}
and
\begin{equation}
\sum_{K \in \mathcal{T}} \int_{\partial K} (V p)^T n \, ds = \int_{\partial \Omega} [V] \otimes [p] \, ds + \int_{E \cup \partial \Omega} [p] \otimes [V] \, ds = \int_{E \cup \partial \Omega} [V p] \, ds.
\end{equation}

3.5. **Construction of an appropriate discrete Hessian.** We now use the framework set out in [ABCM02] to construct a general notion of discrete Hessian. We first give a definition using a flux formulation:

3.6. **Definition** (generalised finite element Hessian: flux formulation). Let $u \in H^2(\mathcal{T})$ and $\hat{U} : H^1(\mathcal{T}) \to \mathcal{T}(\mathcal{E} \cup \partial \Omega)$ be a linear form and $\hat{p} : H^2(\mathcal{T}) \times H^1(\mathcal{T})^d \to \mathcal{T}(\mathcal{E} \cup \partial \Omega)^d$ a bilinear form representing approximations to $u$ and $\nabla u$ over the skeleton of the triangulation. Then we define the generalised finite element Hessian
\(H[u]\) as the solution of
\[
\int_{\Omega} H[u] \Phi \, dx = -\int_{\Omega} p \otimes \nabla_h \Phi \, dx + \int_{\partial K} \hat{p}_K \otimes n \, \Phi \, ds \quad \forall \Phi \in H^1(\mathcal{T})
\]
(3.21) \(\int_{\Omega} H[u] \Phi \, dx = -\int_{\Omega} p \otimes q \, dx + \int_{\partial K} \hat{p}_K \otimes n \, \hat{q} \, ds \quad \forall q \in (H^1(\mathcal{T}))^d\),
(3.22) for all \(\Phi \in \mathbb{V}_D\).

We now present the primal formulation for the generalized finite element Hessian:

3.7. **Theorem** (generalised finite element Hessian: primal). Let \(u \in H^2(\mathcal{T})\) and let \(\hat{U}\) and \(\hat{p}\) be defined as in Definition 3.6 then the generalised finite element Hessian \(H[u]\) is given for each \(\Phi \in \mathbb{V}_D\) as
\[
\int_{\Omega} H[u] \Phi \, dx = -\int_{\Omega} \nabla_h u \otimes \nabla_h \Phi \, dx + \int_{\partial \Omega} [\Phi] \otimes [\hat{p}] \, ds + \int_{\mathcal{E}} [\Phi] \otimes [\hat{p}] \, ds
\]
(3.23)
\[
-\int_{\mathcal{E}} [\hat{U} - u] \otimes [\nabla_h \Phi] \, ds - \int_{\partial \Omega} [\hat{U} - u] \otimes [\nabla_h \Phi] \, ds.
\]
Then summing (3.21) over \(K \in \mathcal{T}\) and making use of the identity (3.24) we see
\[
\frac{1}{2} \sum_{K \in \mathcal{T}} \int_{\partial K} v q \otimes n \, ds = \int_{\partial \Omega} [v] \otimes [q] \, ds + \int_{\mathcal{E}} [v] \otimes [q] \, ds.
\]
(3.24)
Then summing (3.21) over \(K \in \mathcal{T}\) and making use of the identity (3.24) we see
\[
\int_{\Omega} H[u] \Phi \, dx = \int_{\Omega} H[u] \Phi \, dx = \sum_{K \in \mathcal{T}} \left( \int_{\Omega} p \otimes \nabla_h \Phi \, dx + \int_{\partial K} \hat{p}_K \otimes n \, \Phi \, ds \right)
\]
(3.25)
\[
-\int_{\Omega} p \otimes q \, dx + \int_{\partial \Omega} [\Phi] \otimes [\hat{p}_K] \, ds + \int_{\mathcal{E}} [\Phi] \otimes [\hat{p}] \, ds.
\]
(3.26)
Using the same argument for (3.22) we have for each \(q \in H^1(\mathcal{T})^d\) and \(v \in H^1(\mathcal{T})\) that
\[
\int_{\Omega} q \otimes \nabla_h v \, dx = -\int_{\Omega} D_h q v \, dx + \int_{\partial \Omega} [q] \otimes [v] \, ds + \int_{\mathcal{E}} [q] \otimes [v] \, ds.
\]
(3.27)
Taking \(v = u\) in (3.27) and substituting into (3.22) we see
\[
\int_{\Omega} p \otimes q \, dx = \int_{\Omega} q \otimes \nabla_h u \, dx + \int_{\partial \Omega} [\hat{U} - u] \otimes [q] \, ds + \int_{\mathcal{E}} [\hat{U} - u] \otimes [q] \, ds.
\]
(3.28)
Now choosing \(q = \nabla_h \Phi\) and substituting (3.28) into (3.21) we arrive at the fully generalised finite element Hessian given by (3.23). \(\square\)
3.8. **Remark** (consistent representations of the gradient operator). If one were interested in consistent representations of other derivatives, for example the gradient operator, one would need to modify the proof of Theorem 3.7. Examples of consistent gradient representations can be found in [ABCM02]. See also [BO99, DPE10, BE08]. Using this methodology it should be possible to construct an entire hierarchy of derivatives.

3.9. **Example.** An example of a dG formulation for the approximation to the Hessian, $D^2 u$, can be derived by taking the fluxes in the following way

\begin{equation}
\hat{U} = \begin{cases} 
\theta \{ \{ u_h \} \} & \text{over } \mathcal{E} \\
0 & \text{on } \partial \Omega
\end{cases}
\end{equation}

\begin{equation}
\hat{p} = \{ \{ \nabla_h u_h \} \} \text{ on } \mathcal{E} \cup \partial \Omega,
\end{equation}

where $\theta \in \{-1, 1\}$. The result is a discrete representation of the Hessian $H[u_h]$ as unique element of $V^d_d \times d$ such that

\begin{equation}
\int_{\Omega} H[u_h] \Phi \, dx = -\int_{\Omega} \nabla_h u_h \otimes \nabla_h \Phi \, dx \\
+ \int_{\mathcal{E} \cup \partial \Omega} \theta \{ \{ u_h \} \} \otimes \{ \{ \Phi \} \} + \{ \{ \Phi \} \} \otimes \{ \{ \nabla_h u_h \} \} \, ds
\end{equation}

\begin{equation}
= \int_{\Omega} D^2_h u_h \Phi \, dx - \int_{\mathcal{E}_{\text{rel}}} \{ \{ \nabla_h u_h \} \} \otimes \{ \{ \Phi \} \} \, ds \\
+ \int_{\mathcal{E} \cup \partial \Omega} \theta \{ \{ u_h \} \} \otimes \{ \{ \nabla_h \Phi \} \} \, ds \quad \forall \Phi \in V_D.
\end{equation}

3.10. **The discontinuous nonvariational finite element method.** We are now in a position to state the numerical method for the approximation of (1.1). We look to find $u_h \in V_D$ together with $H[u_h] \in V^d_d \times d$ such that

\begin{equation}
\mathcal{A}_h(u_h, \Psi) = l(\Psi) \quad \forall \Psi \in V_D
\end{equation}

with

\begin{equation}
\mathcal{A}_h(u_h, \Psi) := \int_{\Omega} -A : H[u_h] \Psi \, dx + \int_{\mathcal{E} \cup \partial \Omega} \sigma h^{-1} \{ \{ u_h \} \} \otimes \{ \{ \Psi \} \} \, ds
\end{equation}

\begin{equation}
l(\Psi) := \int_{\Omega} f \Psi \, dx,
\end{equation}

where the *penalisation parameter* $\sigma > 0$ is to be chosen sufficiently large to guarantee coercivity.

Using the $L_2$ projection operator $P_V : L_2(\Omega) \rightarrow V_D$ defined for $v \in L_2(\Omega)$ through

\begin{equation}
\int_{\Omega} P_V(v) \, \Psi \, dx = \int_{\Omega} v \Psi \, dx \quad \forall \Psi \in V_D
\end{equation}

it is possible to eliminate the finite element Hessian from the bilinear form for sufficiently smooth $A$:
3.11. Lemma (elimination of the finite element Hessian in a general setting). If $A \in \left[ W^{k+1}_\infty(\Omega) \right]^{d \times d}$ and the fluxes are chosen as in Example 3.9 then

$$
\mathcal{A}_h(u_h, \Psi) = \int_\Omega D_h(P_V(\Psi A)) \nabla_h u_h \, dx - \int_{\partial \Omega} \theta \| u_h \|^T \left\{ D_h(P_V(\Psi A)) \right\} \, ds \\
- \int_{\partial \Omega} \left\{ P_V(\Psi A) \right\}^T \left\{ \nabla_h u_h \right\} \, ds + \int_{\partial \Omega} \sigma h^{-1} \| u_h \|^T \left\{ \Psi \right\} \, ds.
$$

Proof. This follows from the following identity

$$
\int_\Omega -A : H[u_h] \Psi \, dx = \int_\Omega -H[u_h] : (\Psi A) \, dx = \int_\Omega -H[u_h] : P_V(\Psi A) \, dx \\
= \int_\Omega D_h(P_V(\Psi A)) \nabla_h u_h \, dx - \int_{\partial \Omega} \theta \| u_h \|^T \left\{ D_h(P_V(\Psi A)) \right\} \, ds \\
- \int_{\partial \Omega} \left\{ P_V(\Psi A) \right\}^T \left\{ \nabla_h u_h \right\} \, ds.
$$

3.12. Remark. The solution of the problem in this form is nontrivial due to the global $L^2(\Omega)$ projection appearing in the formulation. However, in the discontinuous setting the global $L^2(\Omega)$ projection is in fact computable locally. We may actually exploit this fact to optimise our schemes efficiency. We will discuss this further in the sequel.

3.13. Example (Laplacian formulation). Note that if in (1.1) we have that $A = I$ then we have that

$$f = -A : D^2 u = -\Delta u$$

and our bilinear form reduces to

$$
\mathcal{A}_h(u_h, \Psi) = \int_\Omega \left( \nabla_h \Psi \right)^T \nabla_h u_h \, dx - \int_{\partial \Omega} \theta \| u_h \|^T \left\{ \nabla_h \Psi \right\} \, ds \\
- \int_{\partial \Omega} \left\{ \Psi \right\}^T \left\{ \nabla_h u_h \right\} \, ds + \int_{\partial \Omega} \sigma h^{-1} \| u_h \|^T \left\{ \Psi \right\} \, ds.
$$

since $P_V(\Psi A) = \Psi I$.

The nonvariational finite element method thus coincides with the classical (symmetric) interior penalty method for the Laplacian [DD76].

3.14. Remark (relation to standard dG methods). It is not difficult to prove that choosing to numerical fluxes in the same way as presented in [ABCM02, Table 3.2] results in the same correlation to the dG methods summarised in the aforementioned paper for the case that $A$ is constant. For brevity we will not prove this here.

Note that when $A$ is not constant we have that the nonvariational finite element method does not coincide with its standard variational finite element counterpart. There is an extra stability property which allows the method to successfully cope with advection dominated problems [LP11, §4.2] which is illustrated by the result of Lemma 3.11.

We conclude this section with a proof consistency of the method and then show that Galerkin orthogonality holds.
3.15. **Lemma** (consistency). Let $u \in H^2(\mathcal{T})$ and assume that the numerical fluxes are chosen in a consistent fashion in the sense of [ABCM02, §3.1], that is,

\begin{align}
\hat{U} &= u|_{\mathcal{E}\cup\partial \Omega} \\
\hat{p} &= \nabla u|_{\mathcal{E}\cup\partial \Omega}
\end{align}

Then for $\Phi \in V_D$

\begin{equation}
\int_\Omega H[u] \Phi \, dx = \int_\Omega D^2 u \Phi \, dx
\end{equation}

Therefore we have that $H[u] = P_V(D^2 u)$.

**Proof** Applying Proposition 3.4 to the first term in the definition of $H[u]$ yields

\begin{align}
\int_\Omega H[u] \Phi \, dx &= \int_\Omega D^2 u \Phi \, dx + \int_{\mathcal{E}} \left[ \hat{p} - \nabla u \right]_{\mathcal{E}} \mathbb{S} \left\{ \Phi \right\} \, ds + \int_{\mathcal{E}\cup\partial \Omega} \left[ \hat{p} - \nabla u \right] \mathbb{S} \left\{ \nabla \Phi \right\} \, ds \\
&\quad - \int_{\mathcal{E}} \left[ \hat{U} - u \right] \mathbb{S} \left\{ \nabla \Phi \right\} \, ds - \int_{\mathcal{E}\cup\partial \Omega} \left[ \hat{U} - u \right] \mathbb{S} \left\{ \nabla \Phi \right\} \, ds \\
&= \int_\Omega D^2 u \Phi \, dx \quad \forall \Phi \in V_D.
\end{align}

which proves the results under the consistency conditions on the fluxes. \hfill \Box

3.16. **Lemma** (Galerkin orthogonality). Let $u \in H^2(\Omega) \cap H^1_0(\Omega)$ be a strong solution to the problem (1.1) and let $u_h \in \tilde{V}_D$ be its nonvariational finite element approximation. Assume that the numerical fluxes $\hat{U}$ and $\hat{p}$ are consistent then we have the following orthogonality result:

\begin{equation}
A_h(u_h - u, \Psi) = J(\Psi) \quad \forall \Psi \in \tilde{V}_D,
\end{equation}

with the error functional given by

\begin{equation}
J(\Psi) = \int_\Omega (D^2 u - H[u]) : (A \Psi) \, dx.
\end{equation}

**Proof** Using the consistency result and that $\|u\| = 0$ we conclude

\begin{align}
A_h(u_h - u, \Psi) &= A_h(u_h, \Psi) + \int_\Omega A : H[u] \Psi \, dx = l(\Psi) + \int_\Omega H[u] : (A \Psi) \, dx \\
&= - \int_\Omega A : D^2 u \Psi - H[u] : (A \Psi) \, dx = J(\Psi),
\end{align}

concluding the proof. \hfill \Box

3.17. **Remark.** If $A$ is piecewise constant then since $H[u] = P_V(D^2 u)$ we have $J(\Psi) = 0$ and we recover the usual Galerkin orthogonality $A_h(u_h - u, \Psi) = 0$. We will show in the next section that in general the error functional $J$ is of higher order for smooth enough $u$.

4. **Coercivity, continuity and convergence**

In this section we examine the coercivity, continuity and convergence of the method. We will focus on the fluxes given in Example 3.9 to simplify the presentation. Furthermore we make the following additional assumption on the problem data.
4.1. **Assumption** (coercivity of the nonvariational problem). For the rest of this section we will assume that the nonvariational operator is coercive, that is $A \in W^{k+1}_\infty(\Omega)^{d \times d}$ and that $\text{div}(DA) \leq 0$.

4.2. **Remark** (variational nature of the coercive problem). Under Assumption 4.1 the problem can be written variationally. The solution to the nonvariational problem is the minimiser to the (degenerate) second order variational problem: Find $u \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

\[ J[u] = \inf_{v \in H^2(\Omega) \cap H^1_0(\Omega)} J[v], \]

where

\[ J[v] := \int_{\Omega} (A : D^2u + f - D^2 : A) \, u \, dx. \]

4.3. **Definition** ($H^1(T)$, $H^2(T)$ and $H^{-1}(T)$ norms). We introduce the broken $H^1(T)$ and $H^2(T)$ norms as

\[ \|u_h\|_{dG, 1} := \|\nabla_h u_h\|_{L^2(\Omega)}^2 + h^{-1} \|\nabla_h u_h\|_{L^2(\mathcal{T})}^2, \]

\[ \|u_h\|_{dG, 2} := \|D^2_A u_h\|_{L^2(\Omega)}^2 + h^{-1} \|\nabla_h u_h\|_{L^2(\mathcal{T})}^2 + h^{-3} \|\nabla_h u_h\|_{L^2(\mathcal{T})}^2, \]

and the $H^{-1}(T)$ norm as

\[ \|u_h\|_{dG, -1} := \sup_{v_h \in \mathcal{V}_D} \frac{\int_{\Omega} u_h v_h \, dx}{\|v_h\|_{dG, 1}}. \]

These are equivalent to their continuous equivalent norms for functions in $\mathcal{V}_D$.

4.4. **Proposition** (projection approximation in $\mathcal{V}_D$). Let $P_V : L^2(\Omega) \rightarrow \mathcal{V}_D$ be the $L^2(\Omega)$ orthogonal projection operator defined by (3.35). Using standard approximation arguments we have that

\[ \|v - P_V v\|_{dG, 1} \leq C h^k \|v\|_{H^{k+1}(\Omega)} \text{ and } \|v - P_V v\|_{L^2(\Omega)} \leq C h^{k+1} \|v\|_{H^{k+1}(\Omega)}. \]

In particular, let $A_h$ denote the $L^2$ orthogonal projection of $A$ into the space of piecewise constant functions, then we have

\[ \|A - A_h\|_{W^1_{\infty}(\Omega)} \leq (1 + C_1 h) \|DA\|_{W^1_{\infty}(\Omega)}. \]

4.5. **Theorem** (stability of $H$ [Pry12, Theorem 4.10]). Let $H$ be defined as in Example 3.3 then the $dG$ Hessian is stable in the sense that

\[ \|D^2_H[v_h] - H[v_h]\|_{L^2(\Omega)}^2 \leq C \left( \int_{\mathcal{T}} h^{-1} \|\nabla_h v_h\|^2 + h^{-3} \|v_h\|^2 \, ds \right). \]

Consequently we have

\[ \|H[v_h]\|_{L^2(\Omega)}^2 \leq C \|v_h\|_{dG, 2}^2. \]

We now state the following technical Lemmata.
4.6. Lemma (upper bound on \( \| v_h A - P_V(v_h A) \|_{L^2(\Omega)} \)). Let \( A \in [W^{k+1}_\infty(\Omega)]^{d \times d} \) and \( v_h \in V \) then it holds that

\[
\| D_h(v_h A - P_V(v_h A)) \|_{L^2(\Omega)} \leq C_2 h \left( 1 + C_1 h \right) \| DA \|_{L^\infty(\Omega)} + \sum_{i=2}^{k+1} |A|_{W^{i+1}_\infty(\Omega)} \| \nabla_h v_h \|_{L^2(\Omega)}. \tag{4.10}
\]

**Proof** Let \( A_h \) denote the \( L^2 \) orthogonal projection of \( A \) into the space of piecewise constant functions. Then adding and subtracting appropriate terms we see

\[
\| D_h(v_h A - P_V(v_h A)) \|_{L^2(\Omega)} = \| D_h(v_h(A - A_h) - P_V(v_h A - A_h)) \|_{L^2(\Omega)}. \tag{4.11}
\]

Using the approximation properties of the \( L^2(\Omega) \) projection given in Proposition [4.4] we see that

\[
\| D_h(v_h A - P_V(v_h A)) \|_{L^2(\Omega)} \leq C h^k \| v_h(A - A_h) \|_{H^{k+1}(\Omega)},
\]

\[
\leq C h^k \| A - A_h \|_{W^{k+1}_\infty(\Omega)} \| v_h \|_{H^{k+1}(\Omega)}. \tag{4.12}
\]

Now using the properties of \( A_h \) [4.7] and inverse inequalities we have

\[
\| D_h(v_h A - P_V(v_h A)) \|_{L^2(\Omega)} \leq C h \| A - A_h \|_{W^{k+1}_\infty(\Omega)} \| \nabla_h v_h \|_{L^2(\Omega)}
\]

\[
\leq C h \left( 1 + C_1 h \right) \| DA \|_{L^\infty(\Omega)} + \sum_{i=2}^{k+1} |A|_{W^{i+1}_\infty(\Omega)} \| \nabla_h v_h \|_{L^2(\Omega)},
\]

as required. \( \square \)

4.7. Lemma (upper bound on some skeletal terms). Let \( v_h \in V_D, B \in W^{1}_\infty(\Omega)^{d \times d} \) and \( B_h \in \mathcal{V}_D^{d \times d} \) be the \( L^2 \) orthogonal projection of \( B \), then in view of trace and inverse inequalities we have the following:

\[
\| D_h P_V(v_h B) \|_{L^2(\partial \Omega)} \leq C h^{-1/2} \| D_h(P_V(v_h B)) \|_{L^2(\Omega)}
\]

\[
\leq C_3 h^{-1/2} \| B \|_{W^{1}_\infty(\Omega)} \| v_h \|_{dG,1},
\]

\[
\| P_V(v_h B) \|_{L^2(\Omega)} \leq C h^{1/2} \| D_h(P_V(v_h B)) \|_{L^2(\Omega)}
\]

\[
\leq C_4 h^{1/2} \| B \|_{W^{1}_\infty(\Omega)} \| v_h \|_{dG,1},
\]

\[
\| \nabla_h v_h \|_{L^2(\Omega)} \leq C_5 h^{-1/2} \| \nabla_h v_h \|_{L^2(\Omega)}
\]

\[
\| v_h \|_{L^2(\Omega)} \leq C_6 h^{1/2} \| \nabla_h v_h \|_{L^2(\Omega)}
\]

\[
\| B_h \|_{L^\infty(\partial \Omega)} \leq C h \| DB \|_{L^\infty(\Omega)}. \tag{4.14}
\]

**Proof** For brevity we prove only the first inequality, the second and third follow similar arguments. In view of the definition of the average operator [3.9] it follows that

\[
\| D_h P_V(v_h B) \|_{L^2(\partial \Omega)} \leq \frac{1}{2} \sum_{K \in \mathcal{K}} \| D_h P_V(v_h B) \|_{L^2(\partial K)}. \tag{4.15}
\]

Now by a trace inequality we see that

\[
\| D_h P_V(v_h B) \|_{L^2(\partial \Omega)} \leq C \sum_{K \in \mathcal{K}} h^{-1/2} \| D_h P_V(v_h B) \|_{L^2(\Omega)}. \tag{4.16}
\]
Using the stability of the $L_2(\Omega)$ projection operator in $H^1(\Omega)$ we have
\begin{equation}
\|D_h P_\gamma(v_h B)\|_{L_2(\Omega)} \leq C \sum_{K \in \mathcal{T}} h^{-1/2} \|D_h(v_h B)\|_{L_2(K)} \leq C_2 h^{-1/2} \|B\|_{W^{-1}_\infty(\Omega)} \|v_h\|_{dG,1},
\end{equation}
as required.

4.8. **Theorem** (discrete continuity and coercivity). Let the conditions in Assumption 4.7 hold. Suppose also that $\mathbf{DA}$ is sufficiently small such that
\begin{equation}
\gamma - 2\epsilon - \left( C_4 C_5 + C_1 C_4 h + \frac{C_6 C_7}{4} \right) \left( C_4 C_5 (1 + C_1 h) \right) \|\mathbf{DA}\|_{L_\infty(\Omega)} - C_2 h \sum_{i=2}^{k+1} |\mathbf{A}|_{W^{-1}_\infty(\Omega)} > 0,
\end{equation}
where $\gamma$ is the ellipticity constant, $C_1$ is a constant appearing in Proposition 4.4 with $k = 0$, $C_2$, $C_4$ and $C_5$ are the constants appearing in Lemmata 4.6 and 4.7 and $\epsilon > 0$ is some parameter. In addition assume $\sigma$ the penalisation term is sufficiently large, specifically
\begin{equation}
\sigma - \frac{C_2^2 (1 + C_1 h)^2}{4 \epsilon} \|\mathbf{DA}\|_{L_\infty(\Omega)}^2 + 4\epsilon^2 \left( \frac{C_4 C_5 (1 + C_1 h)}{2} + C_7 h \right) \|\mathbf{DA}\|_{L_\infty(\Omega)} - \frac{(\theta + 1)^2 C_2^2 \|\mathbf{A}_h\|_{L_\infty(\Omega)}}{4 \epsilon} > 0.
\end{equation}
where $\mathbf{A}_h$ is some piecewise constant approximation to $\mathbf{A}$.

Then there exist positive constants $C_B$ and $C_C$ such that
\begin{equation}
|\mathcal{R}_h(u_h, v_h)| \leq C_B \|u_h\|_{dG,1} \|v_h\|_{dG,1} \quad \text{and}
\end{equation}
\begin{equation}
|\mathcal{R}_h(u_h, u_h)| \geq C_C \|u_h\|_{dG,1}^2 \quad \forall u_h, v_h \in \tilde{V}_D.
\end{equation}

We postpone the proof of this theorem to the end of this section and first prove the error estimates for our discrete solution.

The results of Theorem 4.8 allow us to invoke Strang’s second Lemma.

4.9. **Corollary** (Strang [EGL04] c.f.). There exists a $C > 0$ such that
\begin{equation}
\|u - u_h\|_{dG,1} \leq C \left( \inf_{v_h \in \tilde{V}_D} \|u - v_h\|_{dG,1} + \sup_{w_h \in \tilde{V}_D} \left\| \mathcal{R}_h(u, w_h) - \mathcal{I}(w_h) \right\|/\|w_h\|_{dG,1} \right).
\end{equation}

4.10. **Lemma** (discrete negative norm convergence). Let $\mathbf{A} \in L_\infty(\Omega)$ and $u \in H^{k+3}(\Omega)$. Then we have that there exists a constant $C > 0$ such that
\begin{equation}
\|\mathbf{A} : (\mathbf{D}^2 u - \mathbf{H}[u])\|_{dG,-1} \leq C h^{k+1} \|\mathbf{A}\|_{\infty} \|u\|_{H^{k+3}(\Omega)}.
\end{equation}

**Proof** We have, in view of Cauchy–Schwarz inequality, that for $\Psi \in \tilde{V}_D$
\begin{equation}
\int_{\Omega} \mathbf{A} : (\mathbf{D}^2 u - \mathbf{H}[u]) \Psi \, dx \leq \|\mathbf{A}\|_{L_\infty(\Omega)} \left( \|\mathbf{D}^2 u - \mathbf{H}[u]\|_{L_2(\Omega)} \|\Psi\|_{L_2(\Omega)} \right) \leq C h^{k+1} \|\mathbf{A}\|_{L_\infty(\Omega)} \|u\|_{H^{k+3}(\Omega)} \|\Psi\|_{dG,1},
\end{equation}
Adding and subtracting appropriate terms we have that
\[ (4.28) \]
\[ \| J \|_{dG, -1} = O(h^{k+1}). \]

4.11. Remark. Noting the definition of the error functional in the Galerkin orthogonality we deduce that
\[ (4.29) \]
\[ H_{14 ANDREAS DEDNER AND TRISTAN PRYER} \]

4.12. Theorem (convergence of the nonvariational method). Let \( u \) solve the nonvariational problem \( (4.1) \) and \( (u_h, H [u_h]) \) solve the nonvariational finite element approximation \( (3.32) \) where \( H [u_h] \) is a consistent approximation of \( D^2 u \) (for example that given in Example 3.9). Then the following error bound holds:
\[ (4.30) \]
\[ \| u - u_h \|_{dG, 1} \leq C \left( h^k |u|_{H^{k+1}(\Omega)} + h^{k+1} |u|_{H^{k+2}(\Omega)} \right). \]

**Proof** The proof of \( (4.26) \) is immediate from applying Proposition 4.4 to Corollary 4.9 with \( v_h = P_V u \) and noting that the bound for the consistency error nothing but the result of Lemma 4.10 concluding the proof.

To conclude this section we prove Theorem 4.8

**Proof** Theorem 4.8 Let \( u, v_h \in V_D \), then we have
\[ (4.27) \]
\[ \frac{\partial}{\partial t} (u, v_h) = - \int_{\Omega} A : H [u_h] v_h \, dx + \sigma h^{-1} \int_{\partial \Omega} \| u_h \| \| v_h \| \, ds \]
\[ - \int_{\Omega} H [u_h] : (v_h A) \, dx + \sigma h^{-1} \int_{\partial \Omega} \| u_h \| \| v_h \| \, ds \]
\[ = - \int_{\Omega} H [u_h] : P_V (v_h A) \, dx + \sigma h^{-1} \int_{\partial \Omega} \| u_h \| \| v_h \| \, ds. \]

Now making use of the definition of \( H \) from Example 3.9 we see
\[ (4.28) \]
\[ \frac{\partial}{\partial t} (u, v_h) = - \int_{\Omega} D_h^2 u_h : P_V (v_h A) \, dx + \int_{\partial \Omega} \| \nabla_h u_h \|_{\partial \Omega} : \| P_V (v_h A) \| \, ds \]
\[ - \int_{\partial \Omega} \| u_h \| \, ds + \sigma h^{-1} \int_{\partial \Omega} \| u_h \| \| v_h \| \, ds. \]

Adding and subtracting appropriate terms we have that
\[ (4.29) \]
\[ \frac{\partial}{\partial t} (u, v_h) = - \int_{\Omega} A : D_h^2 u_h v_h + D_h^2 u_h : (v_h A - P_V (v_h A)) \, dx \]
\[ + \int_{\partial \Omega} \| \nabla_h u_h \|_{\partial \Omega} : \| P_V (v_h A) \| \, ds - \int_{\partial \Omega} \| u_h \| \, ds \]
\[ + \sigma h^{-1} \int_{\partial \Omega} \| u_h \| \| v_h \| \, ds, \]

which rewriting variationally gives
\[ (4.30) \]
\[ \frac{\partial}{\partial t} (u, v_h) = - \int_{\Omega} D_h (A \nabla_h u_h) v_h + D_h A \nabla_h u_h v_h + D_h^2 u_h : (v_h A - P_V (v_h A)) \, dx \]
\[ + \int_{\partial \Omega} \| \nabla_h u_h \|_{\partial \Omega} : \| P_V (v_h A) \| \, ds - \int_{\partial \Omega} \| u_h \| \, ds \]
\[ + \sigma h^{-1} \int_{\partial \Omega} \| u_h \| \| v_h \| \, ds. \]
Note that
\begin{equation}
- \int_{\Omega} D_h (A \nabla_h u_h) v_h + D_h^2 u_h : (v_h A - P \nu (v_h A)) \, dx \\
= \sum_{K \in \mathcal{T}} \left[ \int_K (A \nabla_h u_h) \nabla_h v_h - D_h (v_h A - P \nu (v_h A)) \nabla_h u_h \, dx \\
+ \int_{\partial K} - (A \nabla_h u_h) v_h n + (v_h A - P \nu (v_h A)) \nabla_h u_h \, n \, ds \right] \\
= \sum_{K \in \mathcal{T}} \left[ \int_K (A \nabla_h u_h) \nabla_h v_h - D_h (v_h A - P \nu (v_h A)) \nabla_h u_h \, dx \\
- \int_{\partial K} (P \nu (v_h A)) \nabla_h u_h \, n \, ds \right] \\
= \int_{\Omega} (A \nabla_h u_h) \nabla_h v_h - D_h (v_h A - P \nu (v_h A)) \nabla_h u_h \, dx \\
- \int_{\mathcal{E} \cup \partial \Omega} [P \nu (v_h A)]^T \{ \nabla_h u_h \} \, ds - \int_{\mathcal{E}} [\nabla_h u_h]_{\mathcal{E}} : \{ P \nu (v_h A) \} \, ds,
\end{equation}
and hence we see that upon substituting \((4.31)\) into \((4.30)\) that
\begin{equation}
\omega_h (u_h, v_h) = \int_{\Omega} (A \nabla_h u_h) \nabla_h v_h + D A \nabla_h u_h v_h - D_h (v_h A - P \nu (v_h A)) \nabla_h u_h \, dx \\
- \int_{\mathcal{E} \cup \partial \Omega} \theta [u_h]^T \{ D_h (P \nu (v_h A)) \} + [P \nu (v_h A)]^T \{ \nabla_h u_h \} \, ds \\
+ \sigma h^{-1} \int_{\mathcal{E} \cup \partial \Omega} [u_h]^T [v_h] \, ds.
\end{equation}
We proceed by applying Cauchy–Schwarz componentwise to \((4.32)\) and estimating \(\theta\) by 1 arriving at
\begin{equation}
\omega_h (u_h, v_h) \leq \| \nabla_h u_h \|_{L^2(\Omega)} \left( \| A \|_{L^\infty(\Omega)} \| \nabla_h v_h \|_{L^2(\Omega)} + \| D A \|_{L^\infty(\Omega)} \| v_h \|_{L^2(\Omega)} \right) \\
+ \| D_h (v_h A - P \nu (v_h A)) \|_{L^2(\Omega)} \| \nabla_h u_h \|_{L^2(\Omega)} \\
+ \| u_h \|_{L^2(\mathcal{E})} \| D_h (P \nu (v_h A)) \|_{L^2(\mathcal{E})} \\
+ \| P \nu v_h A \|_{L^2(\mathcal{E})} \| \nabla_h u_h \|_{L^2(\mathcal{E})} + \sigma h^{-1} \| u_h \|_{L^2(\mathcal{E})} \| v_h \|_{L^2(\mathcal{E})}.
\end{equation}
In view of Lemma 4.6 and the Poincaré inequality we have
\begin{equation}
\omega_h (u_h, v_h) \leq \left( \| A \|_{L^\infty(\Omega)} + C_P \| D A \|_{L^\infty(\Omega)} + C_2 h \| A \|_{W^{1,1}_h(\Omega)} \right) \| \nabla_h u_h \|_{L^2(\Omega)} \| \nabla_h v_h \|_{L^2(\Omega)} \\
+ \| u_h \|_{L^2(\mathcal{E})} \| D_h (P \nu (v_h A)) \|_{L^2(\mathcal{E})} \\
+ \| P \nu v_h A \|_{L^2(\mathcal{E})} \| \nabla_h u_h \|_{L^2(\mathcal{E})} + \sigma h^{-1} \| u_h \|_{L^2(\mathcal{E})} \| v_h \|_{L^2(\mathcal{E})}.
\end{equation}
For the skeletal terms we apply the result of Lemma 4.7 which upon substituting these into \((4.34)\) we see that
\begin{equation}
| \omega_h (u_h, v_h) | \leq C_B \| u_h \|_{dG, 1} \| v_h \|_{dG, 1}
\end{equation}
as required.
For coercivity we use the equality given in [4.32] with \( v_h = u_h \) to find

\[
\mathcal{J}_h(u_h, u_h) = \int_{\Omega} (A \nabla_h u_h) \nabla_h u_h + D \nabla_h u_h u_h - D_h(u_h A - P_\gamma(u_h A)) \nabla_h u_h \, dx
\]

\[
- \int_{\partial \Omega} \theta \| u_h \|^T \left\langle D_h(P_\gamma(u_h A)) \right\rangle + \| P_\gamma(u_h A) \|^T \left\langle \nabla_h u_h \right\rangle
+ \sigma h^{-1} \| u_h \|^T \| u_h \| \, ds
\]

\[
= \sum_{i=1}^{6} \mathcal{J}_i.
\]

We proceed by bounding each term individually. By the ellipticity of the problem we have that

\[
\mathcal{J}_1 = \int_{\Omega} (A \nabla_h u_h) \nabla_h u_h \, dx \geq \gamma \| \nabla_h u_h \|_{L^2(\Omega)}^2.
\]

By the coercivity of the problem we have

\[
\mathcal{J}_2 = \int_{\Omega} D \nabla_h u_h u_h = \int_{\Omega} D \frac{1}{2} \nabla_h (u_h^2) \, dx = -\frac{1}{2} \int_{\Omega} \text{div}(D A) u_h^2 \, dx > 0.
\]

By the Cauchy–Schwartz inequality and making use of Lemma 4.6

\[
-\mathcal{J}_3 = \int_{\Omega} D_h(u_h A - P_\gamma(u_h A)) \nabla_h u_h \, dx
\]

\[
\leq \| D_h(u_h A - P_\gamma(u_h A)) \|_{L^2(\Omega)} \| \nabla_h u_h \|_{L^2(\Omega)}
\]

\[
\leq C_2 h \left( 1 + C_1 h \right) \| D A \|_{L^\infty(\Omega)} + \sum_{i=2}^{k+1} |A|_{W^i(\Omega)} \| \nabla_h u_h \|_{L^2(\Omega)}^2.
\]

We combine the fourth and fifth terms and let \( A_h \) denote the \( L_2 \) orthogonal projection of \( A \) onto the space of piecewise constant functions. Upon adding and subtracting appropriate terms

\[
-\mathcal{J}_4 - \mathcal{J}_5 = \int_{\partial \Omega} \theta \| u_h \|^T \left\langle D_h(P_\gamma(u_h A)) \right\rangle + \| P_\gamma(u_h A) \|^T \left\langle \nabla_h u_h \right\rangle \, ds
\]

\[
= \int_{\partial \Omega} \theta \left\langle [u_h] \right\rangle \left\langle D_h(P_\gamma(u_h A) - u_h A_h) \right\rangle + \theta \| [u_h] \|^T \left\langle D_h(u_h A_h) \right\rangle
+ \| P_\gamma(u_h A - u_h A_h) \|^T \left\langle \nabla_h u_h \right\rangle + \| u_h A_h \|^T \left\langle \nabla_h u_h \right\rangle \, ds.
\]

Using the identities

\[
\int_{\partial \Omega} \| u_h A_h \| \, ds = \int_{\partial \Omega} \| A_h \| \| u_h \| \, ds + \int_{\partial \Omega} \| A_h \|^T \| u_h \| \, ds
\]

\[
\int_{\partial \Omega} \| A_h \| \| \nabla_h u_h \| \, ds = \int_{\partial \Omega} \frac{1}{4} \| A_h \| \| \nabla_h u_h \| \, ds + \int_{\partial \Omega} \| A_h \|^T \| \nabla_h u_h \| \, ds
\]

\[
\int_{\partial \Omega} \| A_h \| \| \nabla_h u_h \| \, ds = \int_{\partial \Omega} \frac{1}{4} \| A_h \| \| \nabla_h u_h \| \, ds + \int_{\partial \Omega} \| A_h \|^T \| \nabla_h u_h \| \, ds
\]

\[
\int_{\partial \Omega} \| A_h \| \| \nabla_h u_h \| \, ds = \int_{\partial \Omega} \frac{1}{4} \| A_h \| \| \nabla_h u_h \| \, ds + \int_{\partial \Omega} \| A_h \|^T \| \nabla_h u_h \| \, ds
\]
we have that
\begin{equation}
-\mathcal{J}_4 - \mathcal{J}_5 = \int_{\partial \Omega} \theta \|u_h\|^T \{ D_h (P_V (u_h A - u_h A_h)) \} + \{ P_V (u_h A - u_h A_h) \} + (\theta + 1) \{ \nabla_h u_h \} \{ \nabla_h u_h \} + \{ A_h \} \{ \nabla_h u_h \} + \frac{\theta}{4} \{ A_h \}^T \{ u_h \} \{ \nabla_h u_h \} \, ds.
\end{equation}

Using Cauchy–Schwartz we see
\begin{equation}
-\mathcal{J}_4 - \mathcal{J}_5 \leq \|u_h\|_{L_2(\Omega)} \|D_h (P_V (u_h A - u_h A_h))\|_{L_2(\Omega)} + \|P_V (u_h A - u_h A_h)\|_{L_2(\Omega)} + \|\nabla_h u_h\|_{L_2(\Omega)} + (\theta + 1) \|\nabla_h u_h\|_{L_2(\Omega)} + \|A_h\|_{L_\infty(\Omega)} \|\nabla_h u_h\|_{L_2(\Omega)} + \frac{\theta}{4} \|A_h\|_{L_\infty(\Omega)} \|u_h\|_{L_2(\Omega)}.
\end{equation}

Making use of the various bounds from Lemma 4.7 we have
\begin{equation}
-\mathcal{J}_4 - \mathcal{J}_5 \leq C_3 h^{-1/2} \|A - A_h\|_{W^{1_\infty}(\Omega)} \|u_h\|_{dG,1} \|u_h\|_{L_2(\Omega)} + C_4 C_5 \|A - A_h\|_{W^{1_\infty}(\Omega)} \|u_h\|_{dG,1} \|\nabla_h u_h\|_{L_2(\Omega)} + (\theta + 1) C_5 h^{-1/2} \|A_h\|_{L_\infty(\Omega)} \|u_h\|_{L_2(\Omega)} \|\nabla_h u_h\|_{L_2(\Omega)} + \frac{C_5 C_6 C_7 h}{4} \|DA\|_{L_\infty(\Omega)} \|\nabla_h u_h\|_{L_2(\Omega)} + C_7 \|DA\|_{L_\infty(\Omega)} \|u_h\|_{L_2(\Omega)}^2.
\end{equation}

We now apply a Cauchy inequality and use the approximation properties of $A_h$ from Proposition 4.4 to find for any $\epsilon > 0$ that
\begin{equation}
\mathcal{K}_1 = C_3 h^{-1/2} \|A - A_h\|_{W^{1_\infty}(\Omega)} \|u_h\|_{dG,1} \|u_h\|_{L_2(\Omega)} \leq \frac{C_3^2 \|A - A_h\|_{W^{1_\infty}(\Omega)}^2 h^{-1} \|u_h\|_{L_2(\Omega)}^2 + \epsilon \|u_h\|_{dG,1}^2}{4\epsilon}.
\end{equation}

The other terms are bounded similarly in that
\begin{equation}
\mathcal{K}_2 = C_4 C_5 \|A - A_h\|_{W^{1_\infty}(\Omega)} \|u_h\|_{dG,1} \|\nabla_h u_h\|_{L_2(\Omega)} \leq \frac{C_4 C_5 \|A - A_h\|_{W^{1_\infty}(\Omega)}^2 (\|u_h\|_{dG,1}^2 + \|\nabla_h u_h\|_{L_2(\Omega)}^2)}{2} \leq \frac{C_4 C_5 (1 + C_1 h) \|DA\|_{L_\infty(\Omega)} \|\nabla_h u_h\|_{L_2(\Omega)}^2 + C_4 C_5 (1 + C_1 h) \|DA\|_{L_\infty(\Omega)} h^{-1} \|u_h\|_{L_2(\Omega)}^2}{4\epsilon}.
\end{equation}
and

\[
\mathcal{X}_3 = (\theta + 1) C_5 h^{-1/2} \left\| \begin{bmatrix} \mathbf{A}_h \end{bmatrix}_{L_\infty(\mathcal{E})} \right\|_{L_2(\mathcal{E})} \| \nabla_h u_h \|_{L_2(\Omega)}
\]
\[
\leq \epsilon \| \nabla_h u_h \|_{L_2(\Omega)}^2 + \frac{(\theta + 1)^2 C_5^2 \left\| \begin{bmatrix} \mathbf{A}_h \end{bmatrix}_{L_\infty(\mathcal{E})} \right\|^2_{L_2(\mathcal{E})}}{4\epsilon} h^{-1} \| u_h \|_{L_2(\mathcal{E})}^2.
\]

Note that the final two terms are already in their desired form since

\[
\mathcal{X}_4 = \frac{C_5 C_6 C_7 h}{4} \| \mathbf{D} \mathbf{A} \|_{L_\infty(\Omega)} \| \nabla_h u_h \|_{L_2(\Omega)}^2
\]
\[
\mathcal{X}_5 = C_7 \| \mathbf{D} \mathbf{A} \|_{L_\infty(\Omega)} \| u_h \|_{L_2(\mathcal{E})}^2.
\]

Collecting the bounds from (4.46)–(4.50) shows (4.51)

\[-\mathcal{X}_4 - \mathcal{X}_5 \leq \left( 2\epsilon + C_5 \left( C_1 + h \left( C_1 C_4 + \frac{C_6 C_7}{4} \right) \right) \right) \| \mathbf{D} \mathbf{A} \|_{L_\infty(\Omega)} \| \nabla_h u_h \|_{L_2(\Omega)}^2
\]
\[+ \left( \frac{C_5^2 (1 + C_1) h^2}{4\epsilon} \right) \| \mathbf{D} \mathbf{A} \|_{L_\infty(\Omega)}^2 + 4\epsilon^2
\]
\[+ \left( \frac{C_4 C_5 (1 + C_1 h)}{2} + C_7 h \right) \| \mathbf{D} \mathbf{A} \|_{L_\infty(\Omega)}
\]
\[+ \left( \frac{(\theta + 1)^2 C_5^2 \left\| \begin{bmatrix} \mathbf{A}_h \end{bmatrix}_{L_\infty(\mathcal{E})} \right\|^2_{L_2(\mathcal{E})}}{4\epsilon} \right) h^{-1} \| u_h \|_{L_2(\mathcal{E})}^2.
\]

The final term in (4.36) is given by

\[
\mathcal{X}_6 = \int_{\mathcal{E} \cup \partial \Omega} \sigma h^{-1} \| u_h \|_{T}^2 \| u_h \| \, ds = \sigma h^{-1} \| u_h \|_{L_2(\mathcal{E})}^2.
\]

Finally, collecting the bounds from (4.37), (4.38), (4.39), (4.51) and (4.52) shows

\[
\mathcal{X}_h(u_h, u_h) \geq \gamma - 2\epsilon - C_2 h \left( 1 + C_1 h \right) \| \mathbf{D} \mathbf{A} \|_{L_\infty(\Omega)}^2 + \sum_{i=2}^{k+1} \| \mathbf{A} \|_{W_{\infty}(\Omega)}
\]
\[+ \left( \frac{C_5 (C_1 + h) \left( C_1 C_4 + \frac{C_6 C_7}{4} \right) \| \mathbf{D} \mathbf{A} \|_{L_\infty(\Omega)}^2 + 4\epsilon^2}{4\epsilon} \right) \| \nabla_h u_h \|_{L_2(\Omega)}^2
\]
\[+ \left( \frac{C_5^2 (1 + C_1) h^2}{4\epsilon} \| \mathbf{D} \mathbf{A} \|_{L_\infty(\Omega)}^2 + 4\epsilon^2 \right)
\]
\[+ \left( \frac{C_4 C_5 (1 + C_1 h)}{2} + C_7 h \right) \| \mathbf{D} \mathbf{A} \|_{L_\infty(\Omega)}
\]
\[+ \left( \frac{(\theta + 1)^2 C_5^2 \left\| \begin{bmatrix} \mathbf{A}_h \end{bmatrix}_{L_\infty(\mathcal{E})} \right\|^2_{L_2(\mathcal{E})}}{4\epsilon} \right) h^{-1} \| u_h \|_{L_2(\mathcal{E})}^2.
\]

Coercivity of the discrete bilinear form follows using the assumption in Theorem 4.8 by choosing \( \epsilon \) sufficiently small and the penalisation parameter \( \sigma \) sufficiently large for small enough \( h \).  \( \square \)
4.13. Remark (the coercivity bound). We note that the coercivity bound relies on
the term $DA$ not becoming too large, as specified in Theorem 4.8. If it is we view
this as an advection dominated problem. Our numerical experiments suggest that
there is no condition on the size of this term.

If the coefficient matrix $A$ is divergence free, i.e., $DA = 0$ then the bound
simplifies considerably. For example, in the case that $A$ is constant we regain the
same theoretical results as for the method given in Example 3.13.

5. Numerical experiments

In this section we detail numerical experiments carried out in the finite element
package DUNE-Fem [DKNO10] which is based on the DUNE software framework
[BBD+08a, BBD+08b]. The code used to test the method will be made freely
available within the DUNE-Fem-Howto in a future release.

We present some benchmark problems designed such that the exact solution is
known. In each of the experiments the domain $\Omega = [0, 1]^2$ and we consider the
coefficient matrix to be

$$
A(x) = \begin{bmatrix}
1 & b(x) \\
b(x) & a(x)
\end{bmatrix}
$$

(5.1)

varying $a(x)$ and $b(x)$.

In each of the numerical experiments we make use a stabilised conjugate gradient
solver taken from the DUNE-Istl module [BB07] preconditioned with an incomplete
LU factorisation. We choose the penalty parameter $\sigma = 20$.

5.1. Test 1: a coercive operator. In this test we take the components of $A$
such that the operator is coercive, fitting into the analytical framework presented
in [4]. With $x = (x_1, x_2)$, we set

$$
a(x) = -\ln((x_1 - 1/2)^2 + 10^{-10}) + 1
$$

(5.2)

$$
b(x) = 0.
$$

(5.3)

We choose the problem data such that the exact solution is given by

$$
u(x) = \sin(\pi x_1) \sin(\pi x_2)
$$

(5.4)

and approximate this using the formulation (3.32). In Tables 1a–1b we present the
results for the cases $k = 1, 2$, numerically demonstrating that the analytical rates
of convergence are achieved in the dG energy norm, moreover, optimal convergence
is achieved in $L^2(\Omega)$.

5.2. Test 2: nondifferentiable operator [LP11 §4.4]. In this test we take $A$
such that it is comparable to [LP11 §4.4]. We take

$$
a(x) = 2
$$

(5.5)

$$
b(x) = (x_1^2 x_2^2)^{1/3}.
$$

(5.6)

We choose the exact solution as in 5.1 and conduct the same tests. Tables 2a–2b
detail the results. Note that this is not a coercive operator and as such, does not fit
into the analytical framework presented in [4] we do however still achieve optimal
convergence in $\|\cdot\|$ and $\|\cdot\|_{dG, 1}$. 
Table 1. Test 1. We present errors and convergence rates of the approximation given by solving (3.32).

(a) Piecewise linears, $k = 1$.

| #elements | $\| u - u_h \|$ | EOC | $\| u - u_h \|_{DG, 1}$ | EOC |
|----------|----------------|-----|----------------|-----|
| 128      | 0.0196123      | 1.86116 | 0.414643 | 0.953491 |
| 512      | 0.00506166     | 1.95408 | 0.209225 | 0.986817 |
| 2048     | 0.00128044     | 1.98298 | 0.104907 | 0.995937 |
| 8192     | 0.000321803    | 1.99238 | 0.0525047 | 0.998597 |
| 32768    | 8.06862e-05    | 1.99578 | 0.0262623 | 0.999456 |

(b) Piecewise quadratics, $k = 2$.

| #elements | $\| u - u_h \|$ | EOC | $\| u - u_h \|_{DG, 1}$ | EOC |
|----------|----------------|-----|----------------|-----|
| 128      | 0.000475513    | 2.96865 | 0.0308463 | 1.95 |
| 512      | 5.99935e-05    | 2.98661 | 0.00779373 | 1.98471 |
| 2048     | 7.52887e-06    | 2.9943  | 0.0015531 | 1.99492 |
| 8192     | 9.42737e-07    | 2.99751 | 0.000489487 | 1.99805 |
| 32768    | 1.17929e-07    | 2.99893 | 0.000122443 | 1.99916 |

Table 2. Test 2. We present errors and convergence rates of the approximation given by solving (3.32).

(a) Piecewise linears, $k = 1$.

| #elements | $\| u - u_h \|$ | EOC | $\| u - u_h \|_{DG, 1}$ | EOC |
|----------|----------------|-----|----------------|-----|
| 128      | 0.0172648      | 1.89433 | 0.41799 | 0.955709 |
| 512      | 0.00441656     | 1.96684 | 0.210818 | 0.987469 |
| 2048     | 0.00111269     | 1.98887 | 0.105688 | 0.996186 |
| 8192     | 0.00278969     | 1.99588 | 0.0528915 | 0.998707 |
| 32768    | 6.98234e-05    | 1.99832 | 0.0264548 | 0.999507 |

(b) Piecewise quadratics, $k = 2$.

| #elements | $\| u - u_h \|$ | EOC | $\| u - u_h \|_{DG, 1}$ | EOC |
|----------|----------------|-----|----------------|-----|
| 128      | 0.00047216     | 2.9554  | 0.0309416 | 1.9514 |
| 512      | 5.98325e-05    | 2.98028 | 0.00782197 | 1.98394 |
| 2048     | 7.52118e-06    | 2.9919  | 0.00196325 | 1.99429 |
| 8192     | 9.42426e-07    | 2.99651 | 0.000491575 | 1.99776 |
| 32768    | 1.17933e-07    | 2.99904 | 0.000122975 | 1.99904 |

5.3. Test 3: irregular solutions. In this test we consider the case the exact solution does not satisfy the regularity requirements presented in the analytical framework of §4, i.e., $u \not\in H^{k+3}(\Omega)$. In addition we consider the case that $u \not\in H^2(\Omega)$, demonstrating the method converges even for viscosity solutions of the problem.

We consider the coercive operator from §5.1 and choose the problem data such that

\begin{equation}
\begin{aligned}
u(x) = \begin{cases} 
\frac{1}{4} \left( \cos \left( 8\pi \left| x - \frac{1}{2} \right| \right) + 1 \right) & \text{if } \left| x - \frac{1}{2} \right| \leq \frac{1}{8} \\
0 & \text{otherwise}
\end{cases}
\end{aligned}
\end{equation}
Note that this function is $H^2(\Omega)$ but not $H^3(\Omega)$. We also take the problem data such that

$$(5.8) \quad u(x) = \frac{100x_1(1-x_1)x_2(1-x_2)}{|x|}.$$ 

This function is $H^1(\Omega)$ but not $H^2(\Omega)$. The results are given in Tables 3a–3b.

In the case $u$ is given by (5.7) the scheme converges with optimal rate in the $\|\cdot\|_{dG,1}$ norm even if the solution is not in $H^3$. The convergence in the $L_2$ is more erratic, but we observe the same behavior testing the standard IP FEM taking $A$ to be the identity.

In the case $u$ is given by (5.8) the convergence rates are suboptimal since the solution is not $H^2$.

### Table 3. 5.3 - Test 3. We present errors and convergence rates of the approximation given by solving (3.32). In both cases we consider $k = 1$.

| #elements | $\|u - u_h\|$ | EOC  | $\|u - u_h\|_{dG,1}$ | EOC   |
|-----------|----------------|------|-----------------------|-------|
| 128       | 0.0362651      | 2.47943 | 0.837082               | 0.939619 |
| 512       | 0.0267684      | 0.43805 | 0.406003               | 1.04388 |
| 2048      | 0.0179914      | 0.573227 | 0.253977               | 0.67679 |
| 8192      | 0.00292257     | 2.6215  | 0.103168               | 1.29971 |
| 32768     | 0.00174473     | 0.744729 | 0.0541648              | 0.929566 |
| 131072    | 0.000421749    | 2.04854 | 0.0258935              | 1.06476 |

| #elements | $\|u - u_h\|$ | EOC  | $\|u - u_h\|_{dG,1}$ | EOC   |
|-----------|----------------|------|-----------------------|-------|
| 128       | 0.223469       | 1.80378 | 6.42181                | 0.843123 |
| 512       | 0.0616572      | 1.85773 | 3.49469                | 0.877816 |
| 2048      | 0.017159       | 1.84531 | 1.87984                | 0.894556 |
| 8192      | 0.00509901     | 1.75067 | 1.00295                | 0.906363 |
| 32768     | 0.00177874     | 1.51936 | 0.531521               | 0.916047 |
| 131072    | 0.00076433     | 1.21859 | 0.280092               | 0.924224 |

### 6. Conclusions and outlook

In this work we have extended the framework from [LP11] for linear nonvariational problems to incorporate discontinuous approximations.

We have shown the method presented (and subsequently that of the continuous case from [LP11]) is well posed and converges optimally under coercivity assumptions on the coefficient matrix $A$.

In the numerical experiments we note the the method is well posed and converges optimally even for $A$ which do not satisfy the coercivity assumptions or $u$ which do not satisfy the regularity needed in the analytical framework. This motivates that another analytical approach needs to be developed. This approach can not be
variational in nature as such will be completely non standard. This is the topic of ongoing research.

References

[ABCM02] Douglas N. Arnold, Franco Brezzi, Bernardo Cockburn, and L. Donatella Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2001/02.

[AM09] Néstor E. Aguilera and Pedro Morin. On convex functions and the finite element method. *SIAM J. Numer. Anal.*, 47(4):3139–3157, 2009.

[AV02] A. Agouzal and Yu. Vassilevski. On a discrete Hessian recovery for $P_1$ finite elements. *J. Numer. Math.*, 10(1):1–12, 2002.

[BB07] M. Blatt and P. Bastian. The iterative solver template library. In B. Kagström, E. Elmroth, J. Dongarra, and J. Wasniewski, editors, *Applied Parallel Computing – State of the Art in Scientific Computing*, pages 666–675, Berlin/Heidelberg, 2007. Springer.

[BBD+08a] P. Bastian, M. Blatt, A. Dedner, C. Engwer, R. Klöfkorn, R. Kornhuber, M. Ohlberger, and O. Sander. A generic grid interface for parallel and adaptive scientific computing. II. Implementation and tests in DUNE. *Computing*, 82(2-3):121–138, 2008.

[BBD+08b] P. Bastian, M. Blatt, A. Dedner, C. Engwer, R. Klöfkorn, M. Ohlberger, and O. Sander. A generic grid interface for parallel and adaptive scientific computing. I. Abstract framework. *Computing*, 82(2-3):103–119, 2008.

[BE08] Erik Burman and Alexandre Ern. Discontinuous Galerkin approximation with discrete variational principle for the nonlinear Laplacian. *C. R. Math. Acad. Sci. Paris*, 346(17-18):1013–1016, 2008.

[Ber89] Christine Bernardi. Optimal finite-element interpolation on curved domains. *SIAM J. Numer. Anal.*, 26(5):1212–1240, 1989.

[BO09] AnnaLisa Buffa and Christoph Ortner. Compact embeddings of broken Sobolev spaces and applications. *IMA J. Numer. Anal.*, 29(4):827–855, 2009.

[Böh08] Klaus Böhmer. On finite element methods for fully nonlinear elliptic equations of second order. *SIAM J. Numer. Anal.*, 46(3):1212–1249, 2008.

[BS91] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4(3):271–283, 1991.

[DD76] Jim Douglas, Jr. and Todd Dupont. Interior penalty procedures for elliptic and parabolic Galerkin methods. In *Computing methods in applied sciences (Second Internat. Sympos., Versailles, 1975)*, pages 207–216. Lecture Notes in Phys., Vol. 58. Springer, Berlin, 1976.

[DKNO10] A. Dedner, R. Klöfkorn, M. Nolte, and M. Ohlberger. A generic interface for parallel and adaptive scientific computing: Abstraction principles and the DUNE-FEM module. *Computing*, 89(1), 2010.

[DPE10] Daniele A. Di Pietro and Alexandre Ern. Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations. *Math. Comp.*, 79(271):1303–1330, 2010.

[EG04] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[ESW05] Howard C. Elman, David J. Silvester, and Andrew J. Wathen. *Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2005.

[FN09a] Xiaobing Feng and Michael Neilan. Mixed finite element methods for the fully nonlinear Monge-Ampère equation based on the vanishing moment method. *SIAM J. Numer. Anal.*, 47(2):1226–1250, 2009.

[FN09b] Xiaobing Feng and Michael Neilan. Vanishing moment method and moment solutions for fully nonlinear second order partial differential equations. *J. Sci. Comput.*, 38(1):74–98, 2009.

[GT83] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, second edition, 1983.
Max Jensen and Iain Smears. On the convergence of finite element methods for Hamilton-Jacobi-Bellman equations. Technical report, 01 2011.

Omar Lakkis and Tristan Pryer. A finite element method for second order nonvariational elliptic problems. *SIAM J. Sci. Comput.*, 33(2):786–801, 2011.

Adam M. Oberman. Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problems. *SIAM J. Numer. Anal.*, 44(2):879–895 (electronic), 2006.

Tristan Pryer. A discontinuous galerkin method for the $p$-biharmonic equation from a discrete variational perspective. *Submitted - tech report available on ArXiV http://arxiv.org/abs/1209.4002*, 2012.

M.-G. Vallet, C.-M. Manole, J. Dompierre, S. Dufour, and F. Guibault. Numerical comparison of some Hessian recovery techniques. *Internat. J. Numer. Methods Engrg.*, 72(8):987–1007, 2007.

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