Computable Analysis on the Space of Marked Groups

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Abstract. We investigate decision problems for groups described by word problem algorithms. This is equivalent to studying groups described by labelled Cayley graphs. We show that this corresponds to the study of computable analysis on the space of marked groups, and point out several results of computable analysis that can be directly applied to obtain group theoretical results. These results, used in conjunction with the version of Higman’s Embedding Theorem that preserves solvability of the word problem, provide powerful tools to build finitely presented groups with solvable word problem but with various undecidable properties. We also investigate the first levels of an effective Borel hierarchy on the space of marked groups, and show that on many group properties usually considered, this effective hierarchy corresponds sharply to the Borel hierarchy. Finally, we prove that the space of marked groups is a Polish space that is not effectively Polish. Because of this, many of the most important results of computable analysis cannot be applied to the space of marked groups. This includes the Kreisel-Lacombe-Schoenfield-Čeitin Theorem and a theorem of Moschovakis. The space of marked groups constitutes the first natural example of a Polish space that is not effectively Polish.

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Introduction

We begin the systematic study of decision problems for groups described by word problem algorithms. We show that this is equivalent to considering groups described by (computable) labelled Cayley graphs. We have explained in [Rau21] that the study of decision problems based on word problem algorithms is very natural from the point of view of effective mathematics, and the present work can be seen as resuming an old investigation started by Malcev in [Mal61] (translated in [Mal71]) and Rabin in [Rab60], and which has been mostly abandoned since.

If $G$ is a finitely generated group and $S$ a finite generating family for $G$, then the pair $(G, S)$ is called a marked group. A word problem algorithm $A_{WP}$ for $(G, S)$ is an algorithm (Turing machine, recursive function, etc) that decides, given a word whose letters are elements of $S \cup S^{-1}$, whether or not this word defines the identity element in $G$. Of course, this algorithms encodes all the relations that are satisfied by $(G, S)$, and thus the marked group $(G, S)$ is entirely described by the generating family $S$ together with the algorithm $A_{WP}$.

Such an algorithm can be encoded and given as input of other effective processes, the present article investigates which functions are computable when given as input groups described in this manner. We formalize precisely what it means for a function to be computable for groups described by word problem algorithms thanks to the theory of numberings, this is detailed later in this introduction and in Section 3.

When a group is described by a word problem algorithm, all the relations it satisfies and does not satisfy can be listed. However, when taking a decision based on a word problem algorithm, in a finite amount of time, only finitely many relations can be checked. Similarly, (and this is equivalent), if we consider that a word problem algorithm
gives access to the description of a labelled Cayley graph of a marked group, this algorithm will allow us to inspect arbitrarily large finite portions of the Cayley graph, but not to catch the whole graph in one glance.

Because of this, the description of \((G, S)\) by a word problem algorithm can be thought of as a description by successive approximations: \((G, S)\) is uniquely defined by its word problem algorithm, but in a finite amount of time, all we can see from \((G, S)\) is a certain approximation of it, say, a certain ball of its labelled Cayley graph.

It is this idea, that groups are described by successive approximations, that relates decision problems for groups described by word problem algorithms to computable analysis.

Computable analysis is a concept that goes back to Turing’s seminal 1936 paper ([Tur36]), where Turing, right after having defined Turing machines, defines those real numbers whose decimal expansion can be output by such machines, the computable real numbers, and discusses computability of several classical real valued functions.

Computable analysis was later on extended to computability on the Cantor and Baire spaces, and in each of those settings, the topologies and metric of the underlying sets play a very important role in the study of computable functions, because it can be proven that \textit{computable functions must be continuous}. In fact, it can even be shown that computable functions defined on those sets are effectively continuous, and the different continuity theorems that exist in computable analysis are some of the most fundamental results of this domain. We will quote in Section 4 the continuity results of Mazur ([Maz63]), Kreisel, Lacombe and Schoenfield ([KLS57]), Cei tin ([Cei67]) and Moschovakis ([Mos64]).

Just as in \(\mathbb{R}\) and on the Cantor space, the use of a topology on the set of marked groups is relevant in the study of computability for groups described by word problem algorithms. The relevant topology is the Chabauty-Grigorchuk topology ([Gri85]), that defines what is known as the space of marked groups, it is naturally generated by an ultrametric distance.

Thanks to those, we are able to anchor the study of decision problems for groups described by word problem algorithms into computable analysis: the space of marked groups is an effectively complete recursive metric space, to which several known results of computable analysis can be applied, in particular a lemma due to Markov ([Mar54]), English in [Mar63]) which we use to study effective descriptive set theory on the space of marked groups.

However, we prove that the space of marked groups is not an effectively Polish space, and that none of the known methods of computable analysis that are used to prove continuity of computable functions can be applied to it. Because of this, we propose the continuity problem on the space of marked groups as an interesting problem, that may foster research both in group theory and in computable analysis.

We now present the contents of this article in more details, this requires the introduction of several definitions.

\textit{The space of marked groups.} Fix a natural number \(k > 0\). A \(k\)-marked group is a finitely generated group \(G\) together with a tuple \(S\) of \(k\) elements that generate it. A morphism (resp. isomorphism) between two \(k\)-marked groups is a group morphism (resp. isomorphism) which maps the generating tuple of the first group to the one of the second group.

The set of isomorphisms classes of \(k\)-marked groups can be identified with the set of normal subgroups of a rank \(k\) free group \(F_k\), and thus endowed by the product topology of \(\{0,1\}^{F_k}\). This is the topology that we will be interested in thoroughly. It is easily seen to be metrizable, we will use mostly the following ultrametric distance: fix some bijection between \(F_k\) and \(\mathbb{N}\), which allows us to identify \(\{0,1\}^{F_k}\) with \(\{0,1\}^{\mathbb{N}}\), then, for \((u_n)_{n\in\mathbb{N}}\) and \((v_n)_{n\in\mathbb{N}}\) elements of \(\{0,1\}^{\mathbb{N}}\), we define \(n_0 = \inf\{n, u_n \neq v_n\} \in \mathbb{N} \cup \{+\infty\}\), and put \(d((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}) = 2^{-n_0}\).

We denote \(\mathcal{G}_k\) the topological space defined this way, and by \(\mathcal{G}\) the disjoint union of the \(\mathcal{G}_k\), \(k \geq 1\), equipped with the disjoint union topology. The topology of \(\mathcal{G}\) is also metrizable, as the distance \(d\) can be extended to \(\mathcal{G}\) by imposing that groups marked by families of different cardinalities be far away.

We call a group an \textit{abstract group} when we want to emphasize the fact that it is not a marked group.

\textit{Numberings associated to word problem algorithms.} Numberings are functions that allow one to transfer the notion of computable function, which is usually defined only on subsets of \(\mathbb{N}\) and of its cartesian powers, to computations with any set of mathematical objects that admit finite descriptions.

We include in Section 2 an introduction on numberings which includes all the basic definitions that are needed to express theorems of computable analysis in their general form, e.g. the definitions of product numberings, of numberings of computable functions, etc. In particular we describe the lattice operations on the set of numbering types of a given set \(X\).

A \textit{numbering} \(\nu\) of a set \(X\) is a map defined on a subset of \(\mathbb{N}\) and with co-domain \(X\). We denote this by \(\nu : \subseteq \mathbb{N} \to X\). The domain of \(\nu\) is denoted \(\text{dom}(\nu)\). If \(\nu(n) = x\), we say that \(n\) is a \(\nu\)-name or a \(\nu\)-description of \(x\).

A function \(f\) between two numbered sets \((X, \nu)\) and \((Y, \mu)\) is then called \(\nu, \mu\)-computable if there exists a partial recursive function \(F : \subseteq \mathbb{N} \to \mathbb{N}\) defined at least on \(\text{dom}(\nu)\) which satisfies that, for any natural number \(n\) in \(\text{dom}(\nu)\), one has \(f \circ \nu(n) = \mu \circ F(n)\).
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If id\(_N\) denotes the identity on \(\mathbb{N}\), we define a \(\nu\)-computable sequence to be a \((\text{id}_\mathbb{N}, \nu)\)-computable function.

Studying computability on groups, one of our main goals will be to determine which group properties can be recognized thanks to word problem algorithms. To study this, we use the following definitions.

If \((X, \nu)\) is a numbered set, a subset \(A\) of \(X\) is called \(\nu\)-decidable if there is a always halting procedure that, given a natural number \(n\) in \(\text{dom}(\nu)\), decides whether or not \(\nu(n)\) is an element of \(A\), i.e. if the characteristic function of \(A\) is \((\nu, \text{id}_\mathbb{N})\)-computable.

The set \(A\) is \(\nu\)-semi-decidable if there exists a procedure that, given a natural number \(n\) in \(\text{dom}(\nu)\), will stop if and only if \(\nu(n)\) belongs to \(A\), i.e. if the preimage \(\nu^{-1}(A)\) can be expressed as \(\text{dom}(\nu) \cap B\), where \(B\) is a recursively enumerable subset of \(\mathbb{N}\).

The set \(A\) is \(\nu\)-co-semi-decidable if its complement is \(\nu\)-semi-decidable.

Two numberings \(\nu\) and \(\mu\) of \(X\) are equivalent when the identity function on \(X\) is both \((\nu, \mu)\)-computable and \((\mu, \nu)\)-computable. We denote this by \(\nu \equiv \mu\). The equivalence classes of numberings are called numbering types.

The use of numbering types allows to define canonical objects that do not depend on arbitrary choices of encodings one could make. For instance, while there are many ways to define a numbering of \(\nu\) will be given in Section 3, where four equivalent definitions of \(\Lambda WP\) are given.

We now give a brief definition of the numbering type \(\Lambda WP\) associated to word problem algorithms, more details will be given in Section 3, for which the quotient map \(\mathcal{G} \to \mathcal{G}/\sim\) that maps a marked group to the isomorphism class of abstract groups it defines. In other words: since a word problem algorithm constitutes a description of a marked group, it also constitutes the description of an abstract group. Throughout, we will be interested in decision problems that concern both marked groups and abstract groups. To simplify notations, we also denote by \(\Lambda WP\) the numbering type of abstract groups associated to word problem algorithms.

**Recursive metric spaces.** A recursive metric space is a numbered metric space with a computable distance function. We include in Section 4 a detailed introduction on recursive metric spaces, effectively complete spaces, effectively Polish spaces, etc.

We define first a numbering \(\nu_\mathbb{R}\) of \(\mathbb{R}\).

For this, we must use a numbering \(\nu_\mathbb{Q}\) of the set of rational numbers. Define \(\nu_\mathbb{Q}\) as follows: given a natural number \(n\), decompose it as a product \(n = 2^a3^b5^c\), with \(\text{gcd}(n'\mathbb{,}30) = 1\), put \(\nu_\mathbb{Q}(n) = (-1)^{a+b+c}\). We still denote by \(\phi\) the \(i\)th recursive function in a fixed effective enumeration of the partial recursive functions.

We then define the numbering \(\nu_\mathbb{R} : \mathbb{N} \to \mathbb{R}\) by saying that \(\nu_\mathbb{R}(n) = x\) if the sequence \((\nu_\mathbb{Q}(\phi_n(i)))_{i\in\mathbb{N}} \in \mathbb{Q}^\mathbb{N}\) converges to \(x\) with exponential speed, i.e. if for all \(\epsilon > 0\), \(|\nu_\mathbb{Q}(\phi_n(i)) - x| < 2^{-i}\).

Thanks to the numbering \(\nu_\mathbb{R}\), we can define recursive metric spaces.

A recursive metric space is a triple \((X, d, \nu)\), where \((X, d)\) is a metric space, and \(\nu\) is a numbering of \(X\) for which the function \(d : X \times X \to \mathbb{R}\) is \((\nu \times \nu, \nu_\mathbb{R})\)-computable (here, \(\nu \times \nu\) designates the product of the numbering \(\nu\) by itself, we can define it as follows: if \(n = 2^a3^b5^c\)), then \(\nu \times \nu(n) = (\nu(n_1), \nu(n_2))\).

Whether \((X, d, \nu)\) is a recursive metric space depends only on the numbering type associated to \(\nu\), and thus recursive metric spaces can be defined with respect to numbering types.

A function \(f\) between computable metric spaces \((X, d, \nu)\) and \((Y, d, \mu)\) is called effectively continuous if there is a procedure that, given a \(\nu\)-name \(n\) of a point \(x\) in \(X\) and a \(\nu_\mathbb{R}\)-name of a real number \(\epsilon > 0\), computes the \(\nu_\mathbb{R}\)-name of a number \(\eta > 0\) such that

\[\forall y \in X; d(x, y) < \eta \implies d(f(x), f(y)) < \epsilon.\]
Denote by $d$ the ultrametric distance on $G$ described above. A recursive metric space is called effectively complete if there is an effective procedure that can compute the limit of a Cauchy sequence, given this sequence and a function that gives its convergence speed. The result that justifies the study of computable analysis on the space of marked groups is the following (Corollary 4.11 and Corollary 4.16):

Lemma 0.1. The triple $(G, d, \Lambda_{WP})$ is an effectively complete recursive metric space.

Negative results for $(G, d, \Lambda_{WP})$. The space $(G, d, \Lambda_{WP})$, although effectively complete, is in fact not well behaved from the point of view of computability theory. In Section 5, we gather our main results, which are all aimed at showing that the classical continuity results of computable analysis cannot be applied to $(G, d, \Lambda_{WP})$.

An effective Polish space is an effectively complete recursive metric space that has a computable and dense sequence.

Groups with solvable word problem are not dense in $G$ (Proposition 5.1, following results of [Mil81] and [dCGP07]), and so $G$ in fact has no chance of being an effective Polish space, but the adherence of the set of groups with solvable word problem, which we denote $G_{WP}$, could still be one. However we have the following result (Theorem 5.2), which is an easy reformulation of the Boone-Rogers Theorem ([BR66]):

Theorem 0.2. No $\Lambda_{WP}$-computable sequence is dense in $(G_{WP}, d, \Lambda_{WP})$, and thus $(G_{WP}, d, \Lambda_{WP})$ is not an effective Polish space.

This theorem shows in particular that Ceitin’s Theorem, which asserts that the computable functions defined on effectively Polish spaces are effectively continuous, cannot be applied to $(G_{WP}, d, \Lambda_{WP})$.

In [Mos64], Moschovakis proved a generalization of Ceitin’s Theorem, which gives, as of today, the weakest known conditions on a recursive metric space, that are sufficient in order for the functions defined on this space to be effectively continuous. The hypotheses of this theorem are detailed in SubSection 4.4.

We however prove:

Theorem 0.3. The hypotheses of Moschovakis’ Effective Continuity Theorem are not satisfied by the space of marked groups (by neither $(G, d, \Lambda_{WP})$ nor $(G_{WP}, d, \Lambda_{WP})$).

This theorem appears as Corollary 5.15 in the text, it is a consequence of the following result, which is our most important theorem (and which appears as Theorem 5.10 in the text):

Theorem 0.4 (Failure of an Effective Axiom of Choice for groups). There is no algorithm that can, given two finite sets $R_1$ and $R_2$ of relations, produce a word problem algorithm for a group in which the relations of $R_1$ hold, while those of $R_2$ fail.

This holds even under the assumption that the sets $R_1$ and $R_2$ given as input are always chosen so that there exists a group with solvable word problem that satisfies the relations of $R_1$, and in which the relations of $R_2$ fail.

Even though we are unable to prove that computable functions defined on the space of marked groups should be continuous, the recursive metric spaces where discontinuous computable functions exist that are known are very pathological objects (see SubSection 4.3.1 for an example), it would be very surprising if the space of marked groups was amongst them. Our main conjecture is thus the following.

Conjecture 0.5 (Main Conjecture). Any $\Lambda_{WP}$-computable function defined on the set of marked groups with solvable word problem is continuous.

This would have the following consequence on decision problems for groups described by word problem algorithms:

Conjecture 0.6. A $\Lambda_{WP}$-decidable property of groups with solvable word problem should be clopen.

The existence of a “natural” example of a Polish space which is not effectively Polish is remarkable. We can quote here a sentence of Moschovakis from [Mos80, Chapter 3], who, after defining a recursive presentation of a Polish space, states:

“No every Polish space admits a recursive presentation—but every interesting space certainly does.”

Recursive presentations of Polish spaces are defined in Definition 4.24, the following remark, which appears as Proposition 4.25 in the text, is sufficient to understand the sentence quoted above: a Polish space admits a recursive presentation if and only if it admits some numbering that makes of it an effectively Polish space.

We prove in Theorem 5.4:

Theorem 0.7. The metric space $(G, d)$ does not have a recursive presentation.
Moschovakis’ statement should be understood as the belief that the only examples of Polish spaces that do not admit recursive presentations will be artificially built counterexamples, while the Polish spaces that occur naturally in mathematics should always have those presentations. Thus the fact that a naturally arising Polish space exists for which this fails is very interesting, and raises the challenge of finding new ways to prove continuity theorems, that do not rely on the effective separability of the ambient space.

**Markov’s Lemma for groups.** Even though the more advanced results of computable analysis fail for the space of marked groups, the fact that \((G, d, \LambdaWP)\) is an effectively complete recursive metric space provides us with some basic results of computable analysis that can be used to obtain group theoretical results. The most useful result in this regard is Markov’s Lemma, as applied to the space of marked groups (Lemma 6.3).

**Lemma 0.8 (Markov’s Lemma for groups).** If \((G_n)_{n \in \mathbb{N}}\) is a \(\LambdaWP\)-computable sequence of marked groups that converges to a marked group \(H\) with solvable word problem, with \(G_n \neq H\) for each \(n\), then no algorithm taking word problem algorithms as input can tell \(H\) from the groups in \(\{G_n, n \in \mathbb{N}\}\), i.e. \(\{H\}\) is not a \(\LambdaWP\)-semi-decidable subset of \(\{H\} \cup \{G_n, n \in \mathbb{N}\}\).

This result is easy and elementary, and in fact one can find in the literature results that are obtained by “manual” applications of this lemma to different effectively converging sequences. See for instance [McC70], [Loc81].

Markov’s Lemma is our main tool to study decidability on the space of marked groups, we now describe some of its uses.

**First application of Markov’s Lemma: effective descriptive set theory.** In Section 6, we study effective descriptive set theory on the space of marked groups. The general results of effective descriptive set theory are set in the context of effective Polish spaces (\([Mos80]\)), and thus we do not have hierarchy theorems at our disposal for the space of marked groups. However can still study, ad hoc, the first levels of the arithmetical hierarchy for decision problems asked for groups described by word problem algorithms: properties are classified as \(\LambdaWP\)-decidable, \(\LambdaWP\)-semi-decidable, \(\LambdaWP\)-co-semi-decidable, or \(\LambdaWP\)-fully undecidable (which means none of the others). Those levels of the arithmetical hierarchy for decision problems asked for groups described by word problem algorithms are the only levels we will be interested in.

The effective hierarchy resembles the Borel hierarchy on \(\mathcal{G}\), which was for instance studied in [BK19]. We expect the following correspondence (\(P\) denotes a property of marked groups):

\[
P \text{ is Clopen} \iff P \text{ is Decidable}
\]

\[
P \text{ is Open} \iff P \text{ is Semi-Decidable}
\]

\[
P \text{ is Closed} \iff P \text{ is co-Semi-Decidable}
\]

Our main conjecture asks whether the top arrow between clopen and decidable can be replaced by the implication: \(P \text{ is Decidable} \implies P \text{ is Clopen}\). The reverse implication, \(P \text{ is Clopen} \implies P \text{ is Decidable}\), is known to be true in the compact case, if we fix bound on the number of generators of a marked group, but to fail in \(\mathcal{G}\).

All other arrows are known to fail as actual implications.

However, they have the following informal meaning:

- A natural \(\LambdaWP\)-semi-decidable property in \(\mathcal{G}\) is (very much) expected to be open, and a natural \(\LambdaWP\)-co-semi-decidable property in \(\mathcal{G}\) is (very much) expected to be closed.

Semi-decidable properties that are not open are built using Kolmogorov complexity, those are sets one should not expect to run into when dealing with properties defined by algebraic or geometric constructions. (See SubSection 4.3 for an example.)

- A natural open property in \(\mathcal{G}\) is (a little) expected to be \(\LambdaWP\)-semi-decidable, and a natural closed property in \(\mathcal{G}\) is (a little) expected to be \(\LambdaWP\)-co-semi-decidable.

This second fact is rather of an empirical nature, and is justified by the results given in SubSection 6.2.9.

Results that establish that properties belong to the same level in the Borel and Kleene-Mostowski hierarchies are called correspondence results.

Our main tool to proving that some properties are not decidable is the following version of Markov’s Lemma (Proposition 6.5):

**Proposition 0.9.** Let \(A\) be a subset of \(\mathcal{G}\). Suppose that there is a \(\LambdaWP\)-computable sequence of marked groups that do not belong to \(A\) which converges to a marked group in \(A\). Then \(A\) cannot be \(\LambdaWP\)-semi-decidable.
In SubSection 6.2, we present a table that contains a wide range of group properties for which the correspondence between the Borel and Kleene-Mostowski hierarchy holds perfectly.

We are unable to give a natural example of an open property which fails to be $\Lambda_{WP}$-semi-decidable, but we propose several candidates for which we conjecture that the topological classification does not capture the decidability status:

**Conjecture 0.10.** The set of LEF groups is closed but not $\Lambda_{WP}$-co-semi-decidable. The set of isolated groups is open but not $\Lambda_{WP}$-semi-decidable.

We detail this conjecture in Section 8, where we establish in particular that the part of this conjecture that concerns LEF groups would imply Slobodskoi’s Theorem about the undecidability of the universal theory of finite groups.

*Second application of Markov’s Lemma: around the isomorphism problem for $\Lambda_{WP}*. After having studied the decidability status of various group properties, we start in Section 7 the study of the isomorphism problem for groups described by word problem algorithms. This study is naturally preceded by the study of marked and abstract recognizability, which corresponds to the study of the problem of recognizing a fixed (abstract or marked) group, i.e. the problem of characterizing the $\Lambda_{WP}$-decidable singletons of $G$ and the $\Lambda_{WP}$-decidable singletons of the set $G/\sim$ of abstract finitely generated groups.

In a previous paper (Rau21), we stated that the most natural description of a (marked) group, in terms of developing the theory of global decision problems for groups, is a finite presentation of this group together with an algorithm that solves the word problem in it. This statement was in part motivated by a result of Daniel Groves and Henry Wilton from [GW09], which states that this description of finitely generated groups allows to recognize free groups.

**Theorem 0.11** (Groves, Wilton, [GW09], Corollary 4.3). There exists an algorithm that, given a finite presentation of a group $G$, together with an algorithm that solves the word problem in this group, decides whether or not this group is free.

This result is non-trivial, and its proof is elegant, (relying on advanced results in the study of limit groups), and it sheds new light on the fact that it is impossible to tell whether or not a given finite presentation defines a free group (a well known consequence of the Adian-Rabin Theorem): the impossibility of recognizing free groups (or even the trivial group) from finite presentations should not be considered as setting a baseline for what is computable from descriptions of groups, but as showing that the finite presentation description is inappropriately weak to develop a theory of global decision problems for groups.

This led us in [Rau21] to posing the following problem as a possible fruitful motivational problem in the study of decision problems for groups:

**Problem 0.12.** Describe the abstract groups that are recognizable in the class of finitely presented groups with solvable word problem, when we suppose that groups are described by pairs finite presentation-word problem algorithm.

The problem thus stated can be easily formalized thanks to the conjunction operation $\wedge$ of the lattice of numbering types, which is described in Section 2: if $\Lambda_{FP}$ denotes the numbering type associated to finite presentations, the numbering type $\Lambda_{WP} \wedge \Lambda_{FP}$ allows us to define what it means for a group to be described by a finite presentation together with a word problem algorithm. The above problem then asks to describe the $\Lambda_{WP} \wedge \Lambda_{FP}$-decidable singletons of $G/\sim$. Similarly, the above mentioned result, Theorem 0.11, exactly states that the set of free groups is $\Lambda_{WP} \wedge \Lambda_{FP}$-decidable.

While there exists an extensive literature on decision problems for groups described by finite presentations, the study of decision problems for groups described by finite presentations and word problem algorithms, in classes of groups where no uniform solution to the word problem exists, remains virtually untouched.

In particular, understanding which groups can be recognized thanks to either of $\Lambda_{WP}$ and $\Lambda_{FP}$ is a natural step towards understanding which groups can be recognized thanks to the numbering type $\Lambda_{WP} \wedge \Lambda_{FP}$, we here tackle the one of those two preliminary questions which is new.

Our results concerning the isomorphism problem for $\Lambda_{WP}$ are as follow.

We emphasize the role of isolated groups, which define $\Lambda_{WP}$-semi-decidable singletons.

We then relate group recognizability for $\Lambda_{WP}$ to a preorder relation on the set of finitely generated groups introduced by Bartholdi and Erschler in [BE15], called the “preform” relation. A group $G$ preforms a group $H$ if some (or, equivalently, any) marking of $H$ is adherent to the set $[G]$ of all markings of $G$ in $G$. In particular, we have the following easy result (Proposition 7.14):
Proposition 0.13. A finite family $\mathcal{D}$ of groups with solvable word problem has solvable isomorphism problem for $\Lambda_{WP}$ exactly when no group in $\mathcal{D}$ preforms another group in $\mathcal{D}$.

Call two abstract groups $G$ and $H$ $\Lambda_{WP}$-completely undistinguishable if neither the set $[G]$ of all markings of $G$ nor the set $[H]$ of all markings of $H$ is a $\Lambda_{WP}$-semi-decidable subset of $[G] \cup [H]$.

We use the results of [BE15] to establish the following theorem (Theorem 7.13):

Theorem 0.14. There exists an infinite set of non-isomorphic groups with solvable word problem which are pairwise $\Lambda_{WP}$-completely undistinguishable.

There remain however many open problems related to the recognition of groups from word problem algorithms. In particular, we ask several questions regarding the relation between the topology of $G$ and the topology that is seen looking only at groups with solvable word problem: can a marked group with solvable word problem be isolated from other marked groups with solvable word problem, while being the limit of a sequence of groups with unsolvable word problem? Any infinite group must preform a group that is non-isomorphic to it, but must an infinite group with solvable word problem preform another group with solvable word problem?

Third application of Markov’s Lemma: finitely presented groups with undecidable properties. In Section 9, we use our previous results to obtain finitely presented groups with various undecidable properties. In particular, we explain how decision problems for groups described by word problem algorithms correspond to decision problems about finitely generated subgroups of finitely presented groups with solvable word problem.

Markov’s Lemma can be used in conjunction with the Higman-Clapham-Valiev Theorem (the version of Higman’s Embedding Theorem which preserves solvability of the word problem) to obtain the following result (which appears as Theorem 9.3):

Theorem 0.15. Suppose that a $\Lambda_{WP}$-computable sequence $(G_n)_{n \in \mathbb{N}}$ of $k$-marked groups converges to a $k$-marked group $H$ with solvable word problem, and suppose that $H \notin \{G_n, n \in \mathbb{N}\}$. Then there exists a finitely presented group $\Gamma$, with solvable word problem, in which no algorithm can stop exactly on $k$-tuples of elements of $\Gamma$ that define $H$.

This is the first general theorem to provide finitely presented groups with solvable word problem that have various undecidable properties. In particular, such a result was asked for in [DI22], where a result that can be seen as an application of Theorem 0.15 to a sequence of marked finite groups that converges to a free group was obtained thanks to an application of results of the theory of intrinsically computable relations ([Ash98]).

The following is a nicely illustrative example: applying Theorem 0.15 to the sequence $((\mathbb{Z}/n\mathbb{Z}, 1))_{n \in \mathbb{N}}$, which converges to $(\mathbb{Z}, 1)$ in $G_1$, yields a finitely presented group with solvable word problem but unsolvable order problem.

A remark on Markovian computable analysis. Note that the expression “computable analysis”, which we have used already several times in this introduction, is ambiguous, because several approaches to computing with real numbers (and to computing in metric spaces) were considered and developed independently. The only notion which interests us here is what is known as Markov computability, or as the Russian approach, see [AB14] for some historical remarks. The reason for this is that Markov computability is the only notion of computability which corresponds to what the theory of decision problems for groups aims at studying: what information can be computed about a group given a finite description of it, description which can be manipulated by a computer.

The Markovian definition of computable analysis was in fact the definition considered by Turing himself (and even by Borel as early as 1912), and while it was studied in various countries by many mathematicians, an important body of work in Markovian computable analysis comes from Markov’s school of mathematics, which is a school of constructive mathematics.

The tenants of Markov’s school accept only a weak form of the excluded middle law, in the form of the following statement, known as Markov’s Principle: “a Turing machine either halts or does not halt”. They consider that objects exist only if their existence can be attested to by algorithmic means.

This fact renders a part of the literature that we refer to ([Abe80, Kus84, Cei67, Mar63, Mar54]) less accessible than it would be, had it been written in a classical language. This motivated us in writing a rather detailed exposition of computable analysis on metric spaces, which appears in Section 4, and which should be very accessible.

For instance, we include a simple proof of a theorem that says that computable functions defined on effective Polish spaces must be continuous. This simple proof cannot be found in the constructivists’ works, as in those works, the statement “the function $f$ is continuous” has to be interpreted as “the computable function $f$ is effectively continuous”, and Ceitin’s Theorem on the effective continuity of functions defined on effective Polish spaces is significantly more complicated than the corresponding non-effective continuity theorem.
Note also that in some settings, objects very similar to numberings are studied, called notation systems, see [Ce67, Kus84, Mos66]. Those use sets of word instead of natural numbers, the theory remains virtually unchanged.

Contents of this paper. In Section 1, we describe the space of marked groups, and give a few results related to computability: impossibility of deciding whether or not a basic clopen set is empty, etc.

In Section 2, we fix the vocabulary about numberings that is required to present the concepts of computable analysis.

In Section 3, we give several equivalent definitions that formalize the concept that a group is described by a word problem algorithm, or by its labeled Cayley graph.

In Section 4, we describe the main results of computable analysis, giving proofs for some of them and references for the others. We quote in particular Markov’s Lemma, Mazur’s Continuity Theorem, the Kreisel-Lacombe-Schoenfield-Ceitin Theorem, and Moschovakis’ extension of this theorem.

In Section 5, we start investigating the space of marked groups as a computable metric space. We prove our most important theorems; which state that none of the continuity results given in the previous section can be applied to the space of marked groups.

In Section 6, we start applying Markov’s Lemma on the space of marked groups. We give a wide range of examples of open or closed group properties which are partially recognizable for \( \Lambda_{WP} \).

In Section 7, we study the isomorphism problem for groups described by word problem algorithms, and the special problem of group recognizability.

In Section 8, we propose the sets of LEF groups and of isolated groups as sets for which the correspondence between the arithmetical hierarchy and the Borel hierarchy might fail, we motivate those conjectures.

In Section 9, we use Markov’s Lemma in conjunction with various versions of Higman’s Embedding Theorem, in order to give short and elegant proofs of some well known results, for instance of the existence of a group with solvable word problem but unsolvable order problem, etc.

1. THE TOPOLOGICAL SPACE OF MARKED GROUPS

1.1. Definitions. Let \( k \) be natural number. A \( k \)-marked group is a finitely generated group \( G \) together with a \( k \)-tuple \( S = (s_1, ..., s_k) \) of elements of \( G \) that generate it. We call \( S \) a generating family. Note that repetitions are allowed in \( S \), the order of the elements matters, and \( S \) could contain the identity element of \( G \). A morphism of marked groups between \( k \)-marked groups \((G, (s_1, ..., s_k))\) and \((H, (t_1, ..., t_k))\) is a group morphism \( \varphi \) between \( G \) and \( H \) that additionally satisfies \( \varphi(s_i) = t_i \). It is an isomorphism of marked groups if \( \varphi \) is a group isomorphism, and of course marked groups are considered up to isomorphism. We call a group an abstract group when we want to emphasize the fact that it is not a marked group.

It is in fact convenient, when studying \( k \)-marked groups, to fix a free group \( \mathbb{F}_k \) of rank \( k \), together with a basis \( S \) for \( \mathbb{F}_k \). A \( k \)-marking of a group \( G \) can then be seen as an epimorphism \( \varphi : \mathbb{F}_k \to G \), the image of \( S \) by \( \varphi \) defines a marking with respect to the previous definition. Two \( k \)-marked groups are then isomorphic if they are defined by morphisms with identical kernels: the isomorphism classes of \( k \)-marked groups are classified by the normal subgroups of a rank \( k \) free group. The set \( S \) can be thought of as a set of generating symbols, and we often consider that all groups are generated by those letters.

Remark that a word problem algorithm for a group \( G \) is thus a description of a marking of \( G \), and similarly, a presentation of a group defines a marked group.

We note \( \mathcal{G}_k \) the set of isomorphism classes of \( k \)-marked groups, and \( \mathcal{G} \) the disjoint union of the \( \mathcal{G}_k \).

Note that some authors consider that each set \( \mathcal{G}_k \) is embedded in the set \( \mathcal{G}_{k+1} \), identifying a \( k \)-marking \((G, (g_1, ..., g_k))\) with the same marking where the identity \( e_G \) of \( G \) is added as a last generator: the \( k \)-marking \((G, (g_1, ..., g_k, e_G))\) is identified with the \( k+1 \)-marking \((G, (g_1, ..., g_k, e_G))\). We do not adhere to this convention, for reasons that appear clearly in [Rau21]: adding generators that define the identity to a marking can change whether or not a marked group is recognizable from finite presentations. It is thus detrimental in the study of decision problems for groups to identify a marking to the markings obtained by adding trivial generators.

For an abstract group \( G \), we denote \([G]_k \) the set of all its markings in \( \mathcal{G}_k \), and \([G] \) the set of all its markings in \( \mathcal{G} \) (as in [CG05]). If \((G, S)\) is a marking of \( G \), we also define \([G, S]_k \) and \([G, S] \), those are identical to \([G]_k \) and \([G] \) respectively.

1.2. Topology on \( \mathcal{G} \). We define a topology on \( \mathcal{G} \) by equipping each separate space \( \mathcal{G}_k \) with a topology, the topology we then consider on \( \mathcal{G} \) is the disjoint union topology of the \( \mathcal{G}_k \).

For each \( k \), consider a finite set \([s_1, ..., s_k]\), choose arbitrarily an order on the set \([s_1, ..., s_k] \cup [s_1^{-1}, ..., s_k^{-1}]\), and enumerate by length and then lexicographically the elements of the free group \( \mathbb{F}_k \) over \( S \).

Denote by \( i_k(n) \) the \( n \)th element obtained in this enumeration, \( i_k \) is thus a bijection between \( \mathbb{N} \) and \( \mathbb{F}_k \).
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To a normal subgroup $N$ of $\mathbb{F}_k$ we can associate its characteristic function $\chi_N : \mathbb{F}_k \to \{0, 1\}$, and composing it with the bijection $i_k$, we obtain an element of the Cantor space $C = \{0, 1\}^\mathbb{N}$. This defines an embedding:

$$\Phi_k : \begin{cases} \mathcal{G}_k & \to \{0; 1\}^\mathbb{N} \\ N \in \mathbb{F}_k & \mapsto \chi_N \circ i_k \end{cases}$$

of the space of $k$-marked groups into the Cantor space. We call the image $\Phi_k(G)$ of a $k$-marked group $G$ the binary expansion of $G$.

With the Cantor set being equipped with its usual product topology, the topology we will study on $\mathcal{G}_k$ is precisely the topology induced by this embedding.

It is easy to see that $\Phi_k(\mathcal{G}_k)$ is a closed subset of $\{0; 1\}^\mathbb{N}$ of empty interior. It is thus compact.

The product topology on $\{0; 1\}^N$ admits a basis which consists of clopen sets: given any finite set $A \subseteq \mathbb{N}$ and any function $f : A \to \{0; 1\}$, define the set $\Omega_f$ by: $(u_n)_{n \in \mathbb{N}} \in \Omega_f \iff \forall n \in A; u_n = f(n)$. Sets of the form $\Omega_f$ are clopen and form a basis for the topology of $\{0; 1\}^\mathbb{N}$.

Sets of the form $\mathcal{G}_k \cap \Omega_f$ thus define a basis for the topology of $\mathcal{G}_k$.

The set $\mathcal{G}_k \cap \Omega_f$ is defined as a set of marked groups that must satisfy some number of imposed relations, while on the contrary a fixed set of elements must be different from the identity.

We fix the following notation. For $m$ and $m'$ natural numbers, and elements $r_1, \ldots, r_m; s_1, \ldots, s_{m'}$ of $\mathbb{F}_k$, we note $\Omega_{r_1, \ldots, r_m; s_1, \ldots, s_{m'}}^k$ the set of $k$-marked groups that satisfy the relations $r_1, \ldots, r_m$, while they do not satisfy $s_1, \ldots, s_{m'}$.

We call $s_1, \ldots, s_{m'}$ irrelations.

The sets $\Omega_{r_1, \ldots, r_m; s_1, \ldots, s_{m'}}^k$ are called the basic clopen sets.

In what follows, we call the set $r_1, \ldots, r_m; s_1, \ldots, s_{m'}$ of relations and irrelations coherent if $\Omega_{r_1, \ldots, r_m; s_1, \ldots, s_{m'}}^k$ is non-empty. The Boone-Novikov theorem, which implies that there exists a finitely presented group with unsolvable word problem, directly implies the following:

**Theorem 1.1** (Boone-Novikov reformulated). *No algorithm can decide whether or not a given finite set of relations and irrelations is coherent. More precisely, there is an algorithm that stops exactly on coherent sets of relations and irrelations, but no algorithm can stop exactly on coherent sets of relations and irrelations.*

**Proof.** A finite set of relations and irrelations $r_1, \ldots, r_m; s_1, \ldots, s_{m'}$ is coherent if and only if $\Omega_{r_1, \ldots, r_m; s_1, \ldots, s_{m'}}^k$ is non-empty, but $\Omega_{r_1, \ldots, r_m; s_1, \ldots, s_{m'}}^k$ is non-empty if and only if it contains the finitely presented group $\langle S \mid r_1, \ldots, r_m \rangle$, i.e. if and only if the relations $s_1, \ldots, s_{m'}$ are not satisfied by $\langle S \mid r_1, \ldots, r_m \rangle$.

It is always possible, given a finite set $r_1, \ldots, r_m$ of relations, to enumerate their consequences, thus it can be found out if one of the irrelations $s_1, \ldots, s_{m'}$ is a consequence of the relations $r_1, \ldots, r_m$, in which case $r_1, \ldots, r_m; s_1, \ldots, s_{m'}$ is incoherent.

On the other hand, suppose that $\langle S \mid r_1, \ldots, r_m \rangle$ defines a group $G$ with unsolvable word problem, as provided by the Boone-Novikov Theorem. In this case, no algorithm can stop of sets of irrelations of $G$. □

We will call a set $r_1, \ldots, r_m; s_1, \ldots, s_{m'}$ of relations and irrelations word problem coherent, or wp-coherent, if the basic clopen set $\Omega_{r_1, \ldots, r_m; s_1, \ldots, s_{m'}}^k$ contains a group with solvable word problem. The remarkable fact that the notions of coherence and wp-coherence differ follows from a theorem of Miller ([Mi92, Corollary 3.9]) which we expose in details in Subsection 5.3 (see Theorem 5.12).

The fact that coherence and wp-coherence differ can be equivalently formulated as: “groups with solvable word problem are not dense in $\mathcal{G}$”. This is stated in Proposition 5.1.

This remark calls for the following theorem:

**Theorem 1.2** (Boone-Rogers reformulated). *No algorithm can stop exactly on wp-coherent sets of relations and irrelations.*

**Proof.** This follows from the Boone-Rogers theorem ([BR66]) which states that there is no uniform solution to the word problem on the set of finitely presented groups with solvable word problem. Indeed, an effective way of recognizing wp-coherent sets of relations and irrelations would provide a uniform algorithm for the word problem on finitely presented groups with solvable word problem, by an argument similar to that of Theorem 1.1. □

The following theorem shows that wp-coherence is a property that is more complex than coherence.

**Theorem 1.3.** *No algorithm can stop exactly on those sets of relations and irrelations which are not wp-coherent.*

The proof of this theorem relies on the construction of Miller mentioned above (from [Mi92, Corollary 3.9]). We will use this construction to prove another important result, Theorem 5.10. Because of this, the proof of Theorem 1.3 is postponed until Subsection 5.3.
1.3. Different distances. The topology defined above on the space of marked groups is metrizable. We describe here two possible distances which generate this topology, and which we may most of the time use interchangeably throughout this paper.

Those distances are defined on each $\mathcal{G}_k$ separately, the distances between groups marked by generating families that have different cardinalities can be fixed arbitrarily, as long as there remains a strictly positive lower bound to those distances. For convenience, we will adopt throughout the convention that the distance between groups marked by families with different cardinalities is exactly 10.

1.3.1. Ultrametric distance. For sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ in $\{0;1\}^\mathbb{N}$ that are different, denote $n_0$ the least number for which $u_{n_0} \neq v_{n_0}$, and set $d((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}) = 2^{-n_0}$. If the sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ are equal, set $d((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}) = 0$.

This defines an ultrametric distance on $\{0;1\}^\mathbb{N}$ which generates its topology, and which induces a distance on $\mathcal{G}_k$ via the embedding $\mathcal{G}_k \rightarrow \{0;1\}^{\mathbb{N}}$.

1.3.2. Cayley Graph distance. Yet another way to define a distance on $\mathcal{G}_k$ is by using labeled Cayley graphs. A labeled Cayley graph defines uniquely a marked group. And a word problem algorithm can be seen as an algorithm that produces arbitrarily large (finite) portions of the labeled Cayley graph of a given group, this is explained precisely in Section 3. We can define a new distance $d_{\text{Cay}}$ as follows.

For two groups $G$ and $H$, generated by the same family $S$, consider the respective Cayley graphs of $G$ and $H$, $\Gamma_G$ and $\Gamma_H$. The balls centered at the identity in $\Gamma_G$ and $\Gamma_H$ agree up to a certain radius, call $r$ the radius for which the balls of radius $r$ of $\Gamma_G$ and $\Gamma_H$ are identical, while their balls of radius $r + 1$ differ. Then put $d_{\text{Cay}}((G,S), (H,S)) = 2^{-r}$. If $\Gamma_G$ and $\Gamma_H$ are identical, $r$ is infinite, and we put $d_{\text{Cay}}((G,S), (H,S)) = 0$. It is easy to check that $d_{\text{Cay}}$ is an ultrametric distance which induces the topology of the space of marked groups.

The distance $d_{\text{Cay}}$ could be preferred to $d$, as it brings a visual dimension to proofs, however in most cases it is just less precise than $d$: the only difference between $d$ and $d_{\text{Cay}}$ is that, in the computation of $d_{\text{Cay}}$, the relations are considered “in packs”, corresponding to their length as elements of the free groups, while they are considered each one by one when using $d$, the choice of an order of the free group being precisely what allows to give more or less importance to relations of the same length.

2. Vocabulary about numberings

2.1. Numberings and numbering types.

2.1.1. First definitions. We will now introduce numbered spaces and numbering types, which will be useful throughout this paper. For more details, see the chapter on numberings in Weihrauch’s book [Wei87]. The expression “numbering type” which designates equivalence classes of numberings is ours.

Definition 2.1. Let $X$ be a set. A numbering of $X$ is a function $\nu$ that maps a subset $A$ of $\mathbb{N}$ to $X$. We denote this by: $\nu : \subseteq \mathbb{N} \rightarrow X$.

It is important for us not to impose on numberings to be surjective, contrary to what is customary, as this allows for a much more natural approach to the study of the different numberings of $\mathcal{G}$.

The pair $(X, \nu)$ is a numbered set. The domain of $\nu$ is a subset of $\mathbb{N}$ denoted by $\text{dom}(\nu)$. The set of all numberings of $X$ is denoted $\mathcal{N}_X$.

The image $\nu(X)$ of $\nu$ is called the set of $\nu$-computable points of $X$, and denoted $X_\nu$. Given a point $x$ in $X$, an integer $n$ such that $\nu(n) = x$ is called a $\nu$-name, or a $\nu$-description, of $x$.

Definition 2.2. Let $(X, \nu)$ and $(Y, \mu)$ be numbered spaces. A function $f : X \rightarrow Y$ is called $(\nu, \mu)$-computable if there exists a partial recursive function $F : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n$ in the domain of $\nu$, $f \circ \nu(n) = \mu \circ F(n)$. That is to say, there exists $F$ which renders the following diagram commutative:

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\nu \downarrow & & \mu \downarrow \\
\mathbb{N} & \xrightarrow{F} & \mathbb{N}
\end{array}$$

Of course, whether a function $f$ between numbered spaces $(X, \nu)$ and $(Y, \mu)$ is computable only depends on its behavior on the set of $\nu$-computable points of $X$. One easily checks the following:

Lemma 2.3. If $(X, \nu)$, $(Y, \mu)$ and $(Z, \tau)$ are numbered sets, the composition of a $(\nu, \mu)$-computable function with a $(\mu, \tau)$-computable function is $(\nu, \tau)$-computable.

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The identity function \( \text{id}_\mathbb{N} \) on \( \mathbb{N} \) defines its most natural numbering.

**Definition 2.4.** If \((X, \nu)\) is a numbered set, a \(\nu\)-computable sequence is an \((\text{id}_\mathbb{N}, \nu)\)-computable function from \(\mathbb{N}\) to \(X\).

We consider a partial order on the numberings of a set \(X\):

**Definition 2.5.** A numbering \(\nu\) of a space \(X\) is stronger than a numbering \(\mu\) of this same space if the identity on \(X\) is \((\nu, \mu)\)-computable. We denote this \(\nu \succeq \mu\). Those numberings are equivalent if \(\nu \succeq \mu\) and \(\mu \succeq \nu\) both hold. We denote this by \(\nu \equiv \mu\).

The relation \(\nu \succeq \mu\) exactly means that there is an algorithm that, given a \(\nu\)-description of a point in \(X\), produces a \(\mu\)-description of it, and thus can be interpreted as: a \(\nu\)-description of a point \(x\) contains more information about it than a \(\mu\)-description of this same point.

The following lemma is a direct consequence of the fact that composition preserves computability of functions.

**Lemma 2.6.** The relation \(\succeq\) is transitive and reflexive. In particular, the relation \(\equiv\) is an equivalence relation.

**Definition 2.7.** The numbering types on \(X\) are the equivalence classes of \(\equiv\).

If \((X, \nu)\) and \((Y, \mu)\) are numbered spaces, and if \(f : X \to Y\) is a \((\nu, \mu)\)-computable function between \(X\) and \(Y\), then \(f\) will be computable with respect to any pair of numberings of \(X\) and \(Y\) which are \(\equiv\)-equivalent respectively to \(\nu\) and \(\mu\). Thus \(f\) can be considered computable with respect to the numbering types associated to \(\nu\) and \(\mu\).

Numbering types are in fact the objects that we want to be studying, rather than numberings.

We denote \(\mathcal{N}\mathcal{T}_X\) the set of numbering types on \(X\). Given a numbering, we denote \([\nu]\) its \(\equiv\)-equivalence class. We usually denote numbering types by capital greek letters, \(\Lambda, \Delta, \ldots\)

The relation \(\succeq\) defines an order on the set of numbering types, whose description is an important part of the study of the different numberings of a set.

**Definition 2.8.** Let \((X, \nu)\) be a numbered set. Then the equivalence relation defined on \(\mathbb{N}\) by \(n \sim m \iff \nu(n) = \nu(m)\) is called the numbering equivalence induced by \(\nu\) ([Ers99]), and denoted \(\eta_\nu\).

A numbering is called positive when there is a recursively enumerable set \(L \subseteq \mathbb{N} \times \mathbb{N}\) such that \(\eta_\nu = L \cap \text{dom}(\nu) \times \text{dom}(\nu)\).

A numbering is called negative when there is a co-recursively enumerable set \(L \subseteq \mathbb{N} \times \mathbb{N}\) such that \(\eta_\nu = L \cap \text{dom}(\nu) \times \text{dom}(\nu)\).

It is called decidable when it is both positive and negative.

**Proposition 2.9.** If \(\nu \succeq \mu\), and if \(\mu\) is any of negative, positive or decidable, then so is \(\nu\).

**Proof.** We leave it to the reader to write a formal proof. Conceptually, this just means that, as \(\nu\)-descriptions provide more information than \(\mu\)-descriptions, if \(\mu\)-descriptions allow to (partially) distinguish points, then so should \(\nu\)-descriptions. □

2.1.2. Constructions and examples. There are several useful constructions that allow one to build numberings of complicated sets using numberings of simpler sets.

**Definition 2.10.** Given a numbered set \((X, \nu)\) and a subset \(Y\) of \(X\), define the restriction of \(\nu\) to \(Y\) to be the numbering \(\nu|_Y\) defined by the following:

\[
\text{dom}(\nu|_Y) = \text{dom}(\nu) \cap \nu^{-1}(Y),
\forall n \in \text{dom}(\nu|_Y), \nu|_Y(n) = \nu(n).
\]

We define the product of two numberings.

We use a pairing function: for natural numbers, let \(\langle k_1, k_2\rangle\) designate the integer \(2^{k_1}3^{k_2}\), and denote by \(\text{val}_2\) and \(\text{val}_3\) the 2-adic and 3-adic valuations.

**Definition 2.11.** If \((X, \nu)\) and \((Y, \nu)\) are numbered sets, the product of the numberings \(\nu\) and \(\mu\) is the numbering \(\nu \times \mu\) of \(X \times Y\) defined by the following:

\[
\text{dom}(\nu \times \mu) = \{ n \in \mathbb{N}, \ \text{val}_2(n) \in \text{dom}(\nu), \ \text{val}_3(n) \in \text{dom}(\mu) \}\)
\[
\forall n \in \text{dom}(\nu \times \mu), \ \nu \times \mu(n) = (\nu(\text{val}_2(n)), \mu(\text{val}_3(n))).
\]

We do not prove the following easy proposition:

**Proposition 2.12.** If \(\nu_1 \equiv \nu_2\) and if \(\mu_1 \equiv \mu_2\) then \(\nu_1 \times \mu_1 \equiv \nu_2 \times \mu_2\).
Finally, we give the definition of the natural numbering of the set of computable functions between numbered sets. Denote by $\phi_0, \phi_1, \phi_2...$ an effective enumeration of all recursive functions, as were defined first by Church and Turing. (We do not describe here how to obtain such an enumeration: numberings only allow to translate any enumeration of the partial recursive functions to an enumeration of the computable functions between numbered sets. Effective enumerations of the partial recursive functions are constructed in all textbooks on computability, see for instance [Rog87].)

Recall that associated to a numbering $\nu$ is an equivalence relation $\eta_\nu$, in what follows we use it as a predicate: we denote $\eta_\nu(n, m)$ if $n$ and $m$ are equivalent for $\eta_\nu$.

**Definition 2.13.** If $(X, \nu)$ and $(Y, \mu)$ are numbered sets, we define a numbering $\mu^\nu$ of the set of functions that map $\nu$-computable points of $X$ to $\mu$-computable points of $Y$ as follows:

$$
\begin{align*}
\text{dom}(\mu^\nu) &= \{ i \in \mathbb{N} \mid \text{dom}(\nu) \subseteq \text{dom}(\phi_i), \\
\forall n, m \in \text{dom}(\nu), &\eta_\nu(n, m) \implies \eta_\mu(\phi_i(n), \phi_i(m)) \}, \\
\forall i \in \text{dom}(\mu^\nu), &\forall x \in X_\nu, \forall k \in \text{dom}(\nu), (x = \nu(k)) \implies (\mu^\nu(i)(x) = \mu(\phi_i(k))).
\end{align*}
$$

The following commutative diagram renders this definition clearer:

$$
\begin{array}{ccc}
X_\nu & \xrightarrow{\mu^\nu(i)} & Y_\mu \\
\downarrow{\nu} & & \downarrow{\mu} \\
\mathbb{N} & \xrightarrow{\phi_i} & \mathbb{N}
\end{array}
$$

Several examples follow from those constructions:

- The Baire space $\mathcal{N} = \mathbb{N}^\mathbb{N}$ is naturally equipped with the numbering $\text{id}_\mathbb{N}^\text{id}_\mathbb{N}$, which we usually denote $\nu_\mathcal{N}$.
- The Cantor space $\mathcal{C} = \{0, 1\}^\mathbb{N}$ admits a numbering induced by its natural embedding into $\mathcal{N}$, we denote it $\nu_\mathcal{C}$.

2.2. **Lattice operations on $\mathcal{N}/\mathcal{T}_X$.** We now introduce the lattice structure on $\mathcal{N}/\mathcal{T}_X$. Here, by lattice, we mean a partially ordered set that admits meet and join operations: we will thus show that any pair of numbering types in $\mathcal{N}/\mathcal{T}_X$ admits both a greatest lower bound and a least upper bound for the order $\supseteq$.

The lattice operations of $\mathcal{N}/\mathcal{T}_X$ are the conjunction and the disjunction. Given two numberings $\nu$ and $\mu$ of a set $X$, we will define new numberings $\nu \land \mu$ and $\nu \lor \mu$ by saying respectively that a $\nu \land \mu$-name for a point $x$ is a $\nu$-name for $x$ together with a $\mu$-name for it, and that a $\nu \lor \mu$-name for a point $y$ should be either a $\nu$-name for it, or a $\mu$-name for it. Those definitions are explained below.

**Definition 2.14.** Let $\nu$ and $\mu$ be numberings of $X$. Define a numbering $\nu \lor \mu$ (the *disjunction* of $\nu$ and $\mu$) by setting, for any natural number $k$, $\nu \lor \mu(2k) = \nu(k)$ and $\nu \lor \mu(2k + 1) = \mu(k)$. The domain of $\nu \lor \mu$ is the set $\{2k, k \in \text{dom}(\nu)\} \cup \{2k + 1, k \in \text{dom}(\mu)\}$.

Thus a $\nu \lor \mu$-name for a point $x$ of $X$ is either a $\nu$-name for it, or a $\mu$-name for it.

**Proposition 2.15.** Let $\nu$ and $\mu$ be numberings of $X$. Then $\nu \supseteq \nu \lor \mu$ and $\mu \supseteq \nu \lor \mu$, and for any $\tau$ in $\mathcal{N}_X$, if $\nu \supseteq \tau$ and $\mu \supseteq \tau$, then $\nu \lor \mu \supseteq \tau$.

**Proof.** Left to the reader. □

**Proposition 2.16.** Let $\nu$, $\mu$, and $\tau$ be numberings of $X$.

- If $\nu \equiv \mu$, then $\nu \lor \tau \equiv \mu \lor \tau$,
- $\nu \lor \mu \equiv \mu \lor \nu$,
- $\nu \lor \nu \equiv \nu$

**Proof.** Left to the reader. □

As a corollary to this proposition we can define the following:

**Definition 2.17.** Let $\Lambda$ and $\Theta$ be numbering types of $X$. The numbering type $\Lambda \lor \Theta$ is defined as being the $\equiv$-class of $\nu \lor \mu$ for any $\nu$ in $\Lambda$ and $\mu$ in $\Theta$.

We now define the numbering obtained by giving as a name for a point in $x$ both a $\nu$-name and a $\mu$-name for it, the “conjunction” of the numberings $\nu$ and $\mu$.

Denote again by $\text{val}_2$ and $\text{val}_3$ the 2-adic and 3-adic valuations, and by $\langle n, m \rangle$ the number $2^n3^m$. 

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Definition 2.18. Let $\nu$ and $\mu$ be numberings of $X$. Define a numbering $\nu \land \mu$ (the conjunction of $\nu$ and $\mu$) by the following:

$$\text{dom}(\nu \land \mu) = \{n \in \mathbb{N}, \text{val}_2(n) \in \text{dom}(\nu), \text{val}_3(n) \in \text{dom}(\mu), \nu(\text{val}_2(n)) = \mu(\text{val}_3(n))\},$$

$$\forall n \in \text{dom}(\nu \land \mu), \nu \land \mu(n) = \nu(\text{val}_2(n)).$$

As we have already said, a $\nu \land \mu$-name for a point $x$ of $X$ is constituted of both a $\nu$-name and a $\mu$-name for it.

Proposition 2.19. Let $\nu$ and $\mu$ be numberings of $X$. Then $\nu \land \mu \geq \nu$ and $\nu \land \mu \geq \mu$, and for any $\tau$ in $\mathcal{N}_X$, if $\tau \geq \nu$ and $\tau \geq \mu$, then $\tau \geq \nu \land \mu$.

Proof. We first show that $\nu \land \mu \geq \nu$ and $\nu \land \mu \geq \mu$. But given $x$ in $X$ and $n$ in $\mathbb{N}$ such that $\nu \land \mu(n) = x$, by definition of $\nu \land \mu$ one must have $x = \nu(\text{val}_2(n)) = \mu(\text{val}_3(n))$, and thus the functions val$_2$ and val$_3$ are recursive witnesses respectively for $\nu \land \mu \geq \nu$ and $\nu \land \mu \geq \mu$.

Suppose now that $\tau \in \mathcal{N}_X$ is such that $\tau \geq \nu$ and $\tau \geq \mu$. This means that there are recursive functions $F : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ and $G : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n \in \text{dom}(\tau), \tau(n) = \nu(F(n))$ and $\forall n \in \text{dom}(\tau), \tau(n) = \mu(G(n))$. Let $n$ be a $\tau$-name for a point $x$ in $X$. Then $(F(n), G(n))$ is a $\nu \land \mu$-name for $x$, since $\nu(\text{val}_2((F(n), G(n)))) = \nu(F(n)) = \tau(n)$ and $\mu(\text{val}_3((F(n), G(n)))) = \mu(G(n)) = \tau(n)$. Thus $\tau \geq \nu \land \mu$. □

The following proposition is straightforward.

Proposition 2.20. Let $\nu$, $\mu$, and $\tau$ be numberings of $X$.

- If $\nu \equiv \mu$, then $\nu \land \tau \equiv \mu \land \tau$;
- $\nu \land \mu \equiv \mu \land \nu$;
- $\nu \land \nu \equiv \nu$;

Proof. Left to the reader. □

This proposition allows us to define the conjunction of numbering types:

Definition 2.21. Let $\Lambda$ and $\Theta$ be numbering types of $X$. The numbering type $\Lambda \land \Theta$ is defined as being the $\equiv$-class of $\nu \land \mu$ for any $\nu$ in $\Lambda$ and $\mu$ in $\Theta$.

We thus obtain the following important result:

Theorem 2.22. $(\mathcal{N}_X, \geq, \land, \lor)$ is a lattice.

2.3. Recursively-enumerable and semi-decidable sets. Let $(X, \nu)$ be a numbered set.

Definition 2.23. A subset $Y$ of $X$ is called $\nu$-recursively enumerable ($\nu$-r.e.) if there is a recursively enumerable subset $A$ of $\mathbb{N}$, such that $A \subseteq \text{dom}(\nu)$ and $Y = \nu(A)$.

Remark that a code for the r.e. set $A$ that appears in this definition constitutes via $\nu$ a description of the $\nu$-r.e. set $Y$. This allows us to define a numbering $\nu_{r.e.} : \subseteq \mathbb{N} \rightarrow \mathcal{P}(X)$ of the powerset of $X$.

Recall that if $\phi_0, \phi_1, \phi_2, \ldots$ is an effective enumeration of the partial recursive functions, we obtain an effective enumeration $W_0, W_1, W_2, \ldots$ of recursively enumerable subsets of $\mathbb{N}$ by setting $W_i = \text{dom}(\phi_i)$. Define $\nu_{r.e.}$ by the following:

$$\text{dom}(\nu_{r.e.}) = \{i \in \mathbb{N}, W_i \subseteq \text{dom}(\nu)\};$$

$$\forall i \in \text{dom}(\nu_{r.e.}), \nu_{r.e.}(i) = \nu(W_i).$$

Of course we have the following lemma:

Lemma 2.24. If $\nu \equiv \mu$, then $\nu_{r.e.} \equiv \mu_{r.e.}$.

Proof. Straightforward. □

We now define decidable, semi-decidable and co-semi-decidable subsets.

Definition 2.25. A set $Y$ is $\nu$-semi-decidable if there is a recursively enumerable subset $A$ of $\mathbb{N}$ such that $A \cap \text{dom}(\nu) = \nu^{-1}(Y)$. The set $A$ is $\nu$-co-semi-decidable if there is a co-recursively enumerable subset $B$ of $\mathbb{N}$ such that $B \cap \text{dom}(\nu) = \nu^{-1}(Y)$.

A set is $\nu$-decidable if it is both $\nu$-semi-decidable and $\nu$-co-semi-decidable.
Of course, an equivalent formulation is that a set $Y$ is $\nu$-decidable if there exists a procedure that, given a $\nu$-description of an element in $X$, decides whether or not it belongs to $Y$. It is $\nu$-semi-decidable if there exists a procedure that stops exactly on the $\nu$-descriptions of elements of $Y$, and $\nu$-co-semi-decidable if there exists a procedure that stops exactly on the $\nu$-descriptions of elements that do not belong to $Y$.

Each of the definitions above allows to define numberings of the set $\mathcal{P}(X_\nu)$ of all subsets of the $\nu$-computable points of $X$.

We detail those definitions.

Associated to the numbering $\nu$ of $X$, we define three numberings $\nu_D : \subseteq \mathbb{N} \rightarrow \mathcal{P}(X_\nu)$, $\nu_{SD} : \subseteq \mathbb{N} \rightarrow \mathcal{P}(X_\nu)$ and $\nu_{co-SD} : \subseteq \mathbb{N} \rightarrow \mathcal{P}(X_\nu)$ associated respectively to $\nu$-decidable, $\nu$-semi-decidable and $\nu$-co-semi-decidable subsets of $X_\nu$.

As before, we denote $\eta_\nu(n, m)$ if and only if $\nu(n) = \nu(m)$, and denote $W_i = \text{dom}(\phi_i)$.

- Define $\nu_D$ by
  
  \[ \text{dom}(\nu_D) = \{ i \in \mathbb{N} | \text{dom}(\nu) \subseteq W_i, \phi_i(\text{dom}(\nu)) \subseteq \{0, 1\}, \forall n, m \in \text{dom}(\nu), \eta_\nu(n, m) \implies (\phi_i(n) = \phi_i(m))\}; \]
  
  \[ \forall i \in \text{dom}(\nu_D), \nu_D(i) = \nu(\phi_i^{-1}(\{1\})). \]

- Define $\nu_{SD}$ by
  
  \[ \text{dom}(\nu_{SD}) = \{ i \in \mathbb{N}| \forall n, m \in \text{dom}(\nu), \eta_\nu(n, m) \implies (n \in W_i \iff m \in W_i)\}; \]
  
  \[ \forall i \in \text{dom}(\nu_{SD}), \nu_{SD}(i) = \nu(\text{dom}(\nu) \cap W_i). \]

- Define $\nu_{co-SD}$ by
  
  \[ \text{dom}(\nu_{co-SD}) = \{ i \in \mathbb{N}| \forall n, m \in \text{dom}(\nu), \eta_\nu(n, m) \implies (n \notin W_i \iff m \notin W_i)\}; \]
  
  \[ \forall i \in \text{dom}(\nu_{co-SD}), \nu_{co-SD}(i) = \nu(\text{dom}(\nu) \setminus W_i). \]

We do not prove the following easy lemmas:

**Lemma 2.26.** If $\nu \equiv \mu$, then $\nu_D \equiv \mu_D$, $\nu_{SD} \equiv \mu_{SD}$ and $\nu_{co-SD} \equiv \mu_{co-SD}$.

**Lemma 2.27.** The numberings $\nu_D$, $\nu_{SD}$ and $\nu_{co-SD}$ satisfy $\nu_D \equiv \nu_{SD} \land \nu_{co-SD}$.

A $\nu_{SD}$-name of a $\nu$-semi-decidable set is usually simply called its “code”, the same goes for r.e., decidable and co-semi-decidable sets, and the four numberings $\nu_{r.e.}$, $\nu_D$, $\nu_{SD}$ and $\nu_{co-SD}$ defined here are usually not explicitly called upon: the sentence “given a decidable subset of $X$” is usually used instead of “given the $\nu_D$-name of a subset of $X$”.

### 2.4. Vocabulary remarks.

We include here a paragraph which aims at explaining some choices of nomenclature made in this paper. While none of our name choices are original, we sometimes follow authors whose vocabulary choices were not very influential.

Given a quantified statement $ST$ that concerns the elements of a numbered set $(X, \nu)$, one can construct an effective statement that corresponds to $ST$ by replacing all existential quantifiers by effective existential quantifiers, whose meaning is that the object whose existence is claimed can be constructed from the data that appears to the left of the effective existential quantifier in the quantified statement $ST$.

The obtained statement will usually be called “effective $ST$”, we thus say that $(X, \nu)$ satisfies $ST$ effectively, or on the contrary that $ST$ does not hold effectively in $(X, \nu)$.

An interesting aspect of this procedure, that allows one to associate effective statements to classical statements, is that it can turn classically equivalent notions into non equivalent effective notions. While this fact is the source of many interesting research problems and of many interesting results, it leads to many an ambiguity when it comes to naming properties.

Indeed, at times, the question of knowing which notion should be called “effectively $P$” is hard to decide, for instance when $P$ is defined by a statement of the form: “$x$ is said to have $P$ if it satisfies one of the following equivalent conditions: ...”, conditions which are not effectively equivalent.

Our definition of an effectively continuous function between metric spaces given in Section 4 corresponds to an effectivisation of the usual $\varepsilon$-$\delta$ definition (as in [Cel67]), whereas in computable analysis, the more usual definition is an effectivisation of the characterization of continuous functions in terms of preimage of open sets. Those definitions are equivalent in effective Polish spaces, but it is unclear whether this remains true for functions defined on generic recursive metric spaces.
Note also that what we call an effective Polish space, or effectively Polish space, (which is an effectively complete and effectively separable recursive metric space, see Section 4 for a precise definition), is usually called a computable metric space, or an effective metric space, see for instance [BP03, Wei00, Spr01].

On the contrary, what we call a recursive metric space is not necessarily complete nor separable, this follows the definitions of Ceiitin in [Ceit67], of Kushner in [Kus84] and of Moschovakis in [Mos64].

3. The word problem numbering type

We give four equivalent definitions of the word problem numbering type. One is the usual definition, one uses the notion of computable group of Malcev and Rabin, one uses labelled Cayley graphs, the last one uses the embedding of the space of marked group in a disjoint union of countably many Cantor spaces. These definitions and their equivalence are well known.

Before defining numberings of marked groups, we first include a paragraph on the numberings of the elements of fixed a marked group.

3.1. Numberings of the elements of a marked group. In a marked group \((G, S)\), it is customary to describe group elements by words on \(S \cup S^{-1}\). This description is in fact canonical, in a sense that can be made precise using numberings.

We denote by \(\Lambda_{(G,S)}\) the numbering type of \((G, S)\) associated to the idea that elements be described by words on \(S \cup S^{-1}\), we define it formally in the next definition. Denote by \((p_n)_{n \in \mathbb{N}}\) the sequence of prime numbers.

Definition 3.1. If \((G, S)\) is a marked group, and \(S = (s_0, s_2, ..., s_k-1)\), we define the numbering \(\nu_{(G,S)}\) on \(\mathbb{N}\) as follows. For \(i\) between \(k\) and \(2k - 1\), denote by \(s_i\) the element \(s_i^{-1}\) of \(G\). Given a natural number \(n\), decompose it as a product of primes \(n = p_0^{\nu_0}...p_m^{\nu_m}\). Then, for each \(i\) between 1 and \(m\), denote \(\alpha_i\) the remainder in the Euclidian division of \(\alpha_i\) by \(2k\). We then put:

\[
\nu_{(G,S)}(n) = s_{\alpha_1}s_{\alpha_2}...s_{\alpha_m} \in G.
\]

The numbering type \(\Lambda_{(G,S)}\) is the \(\equiv\)-equivalence class of \(\nu_{(G,S)}\).

The arbitrary choices that are made in this definition are of course unimportant, as is shown by the following proposition:

Proposition 3.2. The numbering type \(\Lambda_{(G,S)}\) is the greatest numbering type in the lattice \(\mathcal{N}T_G\), which satisfies the following conditions:

- All elements of \(G\) are \(\Lambda_{(G,S)}\)-computable;
- The group law and the inverse function on \(G\) are respectively \((\Lambda_{(G,S)} \times \Lambda_{(G,S)}), \Lambda_{(G,S)}\) and \((\Lambda_{(G,S)}, \Lambda_{(G,S)})\) computable.

(In particular, any numbering type which satisfies these conditions can be compared to \(\Lambda_{(G,S)}\) for the order \(\succeq\).)

Note that the first condition of this proposition could be replaced equivalently by: “The elements of the generating tuple \(S\) are \(\Lambda_{(G,S)}\)-computable”.

Proof. Suppose that \(\mu\) is any numbering which is surjective and for which the group law and the inverse function of \(G\) are computable.

We show that \(\nu_{(G,S)} \succeq \mu\), where \(\nu_{(G,S)}\) is the numbering which was used to define the numbering type \(\Lambda_{(G,S)}\). The generating set of \(G\) is denoted \(S = (s_0, s_2, ..., s_k-1)\). As \(\mu\) should be surjective, there are numbers \(u_0, ..., u_{k-1}\) such that \(\mu(u_i) = s_i\).

As the group law should be computable for \(\mu\), there is a recursive function \(F\) such that \(\mu(F(i, j)) = \mu(i)\mu(j) \in G\), and a recursive function \(I\) that computes the inverse for \(\mu\).

Given an integer \(n\), which we decompose as a product of primes, \(n = p_0^{\alpha_0}...p_m^{\alpha_m}\), recall that \(\nu_{(G,S)}(n) = s_{\alpha_1}s_{\alpha_2}...s_{\alpha_m}\). Consider the function \(H : \mathbb{N} \to \mathbb{N}\) defined as follows: map an integer \(\alpha\) to its reminder modulo \(2k\), we denote \(\tilde{\alpha}\), then, if \(\tilde{\alpha} \leq k - 1\), map it to \(u_{\tilde{\alpha}}\), otherwise, if \(\tilde{\alpha} \geq k\), map it to \(I(u_{\tilde{\alpha} - k})\). Then \(F(H(\alpha_1), F(H(\alpha_2), ...)))\) gives a \(\mu\)-name for \(\nu_{(G,S)}(n)\), and the procedure which produces this name from \(n\) is clearly recursive.

This proposition is in fact a simple application of a well known fact about numberings: if \((X, \Delta)\) is a numbered set, and if \(\{f_1, ..., f_n\}\) are functions defined on cartesian powers of \(X\) to \(X\), then there is a greatest numbering type \(\Lambda\), that is below \(\Delta\) (\(\Delta \succeq \Lambda\)), and for which all the functions \(f_i\) are computable. This was first detailed in [Wei87, Section 2.2]. The numbering type \(\Lambda_{(G,S)}\) is obtained following this principle, using for \(\Delta\) the equivalence class of the numbering \(\nu_0\) that describes only \(S\): \(\nu_0\) is defined on \(\{0, ..., k - 1\}\) and it maps \(i\) to \(s_i\).
Proposition 3.2 shows that describing group elements as products of the generators is the description that gives as much information as can be given on group elements if we want the group operations to be computable. The previous proposition thus has an easy corollary:

**Corollary 3.3.** The numbering type \( \Lambda_{(G,S)} \) is decidable if and only if there exists a numbering type \( \Delta \) of \( G \), which is surjective and decidable, and which makes the group operations of \( G \) computable.

**Proof.** Suppose that \( \Delta \) is as in the hypotheses of the corollary. Then, by Proposition 3.2, one should have \( \Lambda_{(G,S)} \succeq \Delta \). Thus if \( \Delta \) is decidable, then so should be \( \Lambda_{(G,S)} \). \( \square \)

We can finally define solvability of the word problem:

**Definition 3.4.** A marked group \((G,S)\) is said to have solvable word problem if \( \Lambda_{(G,S)} \) is decidable.

In this case, what we call a word problem algorithm is the recursive function that witnesses for the fact that \( \Lambda_{(G,S)} \) is decidable.

**3.2. First definition of \( \Lambda_{WP} \).** We will now define the numbering type associated to word problem algorithms. Our first definition is based on the numbering \( \nu_{(G,S)} \) defined above.

The numbering \( \nu_{(G,S)} \) is decidable if and only if there is a recursive function of two variables \( H \) such that \( H(i,j) = 0 \) if \( \nu_{(G,S)}(i) = \nu_{(G,S)}(j) \) and \( H(i,j) = 1 \) otherwise. In this case, \( H \) is said to witness for the fact that \( \nu_{(G,S)} \) is decidable.

Let \( \phi_0, \phi_1, ... \) be an effective enumeration of all partial recursive functions. We can consider that those functions depend on two variables using an encoding of pairs of natural numbers.

Define as follows a numbering \( \nu_{WP} \).

Pose \( \nu_{WP}(n) = (G,S) \) if and only if \( n \) can be decomposed as \( n = 2^k(2m + 1) \), \( S \) is a family with \( k \) elements, \( \nu_{(G,S)} \) is decidable, and \( \phi_m \) is a recursive function that witnesses for the fact that \( \nu_{(G,S)} \) is decidable.

To check that this is a correct definition, we must check that the marked group \((G,S)\) is uniquely defined by any of its \( \nu_{WP} \)-names. This is to say: we must check that a word problem algorithm defines uniquely a marked group.

But this is straightforward: if, in the definition above, \( n = 2^k(2m + 1) \) codes for two marked groups \((G,S)\) and \((H,S')\), first it must be that \( S \) and \( S' \) have the same cardinality \( k \), and, secondly, that \( \phi_m \) is a recursive function that witnesses for the decidability of both \( \nu_{(G,S)} \) and \( \nu_{(H,S')} \), but then \((G,S)\) and \((H,S')\) should satisfy exactly the same relations, and thus be isomorphic as marked groups.

We then define \( \Lambda_{WP} \) to be the \( \equiv\)-equivalence class of \( \nu_{WP} \).

**3.3. Labelled Cayley graph definition of \( \Lambda_{WP} \).** Describing a marked group by a word problem algorithm is equivalent to describing it by its labelled Cayley graph. And thus, as we investigate decision problems for groups described by word problem algorithms, we are also studying “what can be deduced about a marked group, given its labeled Cayley graph”. Of course, the graph should be suitably encoded into a finite amount of data. We detail this now.

**Definition 3.5.** If \((G,S = (s_1, ..., s_n))\) is a marked group, the labelled Cayley graph associated to it is the graph whose vertexes are elements of \( G \), and whose (directed) edges are defined as follows: there is an edge with label \( s_i \) from the vertex \( g_1 \in G \) to the vertex \( g_2 \in G \) if and only if \( g_1 s_i = g_2 \).

We can then use the standard way to encode infinite graphs (of course, computable graphs) to define a numbering of Cayley graphs. This again uses a standard enumeration \( \phi_0, \phi_1, \phi_2, ... \) of partial computable functions.

An oriented edge-labelled graph is a quadruple \((V,E,C,c : E \to C)\), where \( V \) is a set, the set of vertices, \( E \) is the set of oriented edges, i.e. a subset of \( V \times V \), \( C \) is a set of colors, which here we suppose finite, and \( c \) is a function which defines the color of a given edge.

Such a graph \( \Gamma = (V,E,C,c : E \to C) \) is called computable if it is isomorphic to a graph \( \Gamma_1 = (V_1,E_1,C_1,c_1 : E_1 \to C_1) \), which satisfies additionally that: \( V_1 \) is a recursive subset of \( \mathbb{N} \), \( E_1 \) is a recursive subset of \( \mathbb{N} \times \mathbb{N} \), \( C_1 = \{1, ..., k\} \) for some \( k \in \mathbb{N} \), and \( c_1 : E_1 \to C_1 \) is a recursive function. In this case, \( \Gamma_1 \) is called a computable model of \( \Gamma \).

Notice that each element of the tuple \((V_1,E_1,C_1,c_1 : E_1 \to C_1)\) is associated to some finite data that can be encoded: the characteristic function of \( V_1 \), denoted \( \chi_{V_1} \), which should be computable, the characteristic function of \( E_1 \), denoted \( \chi_{E_1} \), the natural number \( k \) such that \( C_1 = \{1, ..., k\} \), and the code of the function \( c_1 \).

We can thus define a numbering \( \nu_{\Gamma} \) of edge-labelled graphs by saying that a computable model \( \Gamma_1 = (V_1,E_1,C_1,c_1 : E_1 \to C_1) \) of a graph \( \Gamma \) is encoded by a tuple \((i,j,k,l)\), where \( \phi_i = \chi_{V_1}, \phi_j = \chi_{E_1}, C_1 = \{1, ..., k\}, \phi_l = c_1 \).

One easily checks that the tuple \((i,j,k,l)\) defines a unique graph, and thus the definition given above is sound.
We now have a numbering $\nu_{\Gamma}$ of edge-labelled graphs, we can restrict it to the set of Cayley graphs, and, because a labelled Cayley graph defines uniquely a marked group, we can consider that this new numbering is a numbering of the set of marked groups (we compose the numbering of graphs to the function that maps a labelled Cayley graph to the group it defines).

Note that, in the computable model of a Cayley graph, we can always suppose that there is a vertex at 0, and that it is associated to the identity element of the group whose graph it is.

This defines a numbering that we denote $\nu_{\text{Cay}}$, which is associated to the idea “a marked group is described by algorithms that describe its Cayley graph”.

We can now show:

**Theorem 3.6.** The numbering $\nu_{\text{Cay}}$ is $\equiv$-equivalent to $\nu_{WP}$.

**Proof.** Sketch.

Given a $\nu_{\text{Cay}}$-name for a marked group $(G, S)$, i.e. given a Cayley graph $\Gamma_1$ for it, we can solve the word problem in $(G, S)$ by following edges along a word: given a word $w = a_1 a_2 ... a_n$ on $S \cup S^{-1}$, starting from any vertex $v_1$ in $\Gamma_1$, we can find (by an exhaustive search) a sequence of vertices $v_2, ..., v_{n+1}$ such that $v_{i+1} = v_i a_i$. We then solve the word problem by checking whether $v_1 = v_{n+1}$, i.e. by checking whether the word $w$ defines a loop in the Cayley graph of $G$.

Conversely, given a word problem algorithm for $(G, S)$, we can build a recursive model $\Gamma_1 = (V_1, E_1, C_1, c_1 : E_1 \to C_1)$ of the Cayley graph of $(G, S)$ as follows:

- Consider an enumeration of all words on $S \cup S^{-1}$ following a given order, say by length and then lexicographically. We can then delete, using the word problem algorithm of $G$, any element that is redundant in this list. We obtain a list $w_0, w_1, w_2, ...$ which contains a single word on $S \cup S^{-1}$ for each element of $G$.
- Define a numbering $\mu$ of $G$ by saying that $\mu(i) = w_i$. We put $V_1 = \text{dom}(\mu)$.
- $V_1$ is $\mathbb{N}$ if $G$ is infinite, it is $\{0, ..., \text{card}(G) - 1\}$ otherwise. A computable characteristic function for $V_1$ can be obtained as follows: the list $w_0, w_1, ...$ can be enumerated, and thus, given some number $i$, if it was found that $G$ contains more than $i + 1$ elements, $i$ belongs to $V_1$. On the contrary, while the list $w_0, w_1, w_2, ...$ is enumerated, we can search for an initial segment of it of length less than $i$, and that is stable by multiplication by any generator. If such a segment is found, $G$ must be finite, and we know it has cardinality less than $i$. In this case, we can conclude that $i \notin V_1$.
- We define $E_1$ and $c_1$ by saying that $(i, j)$ is an edge labelled by $s \in S$ if and only if $w_i s = w_j$. This can be effectively checked thanks to the word problem algorithm for $(G, S)$, and thus we can produce the recursive functions that define $E_1$ and $c_1$.

3.4. Computer groups definition of $\Lambda_{WP}$. Another point of view on $\Lambda_{WP}$ follows (more or less) the point of view of Malcev in [Mal71, Chapter 18] and Rabin in [Rab66].

**Definition 3.7.** A countable group $G$ is computable if there are recursive function $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $I : \mathbb{N} \to \mathbb{N}$ such that either $(\mathbb{N}, P, I, 0)$ or $(\{0, ..., \text{card}(G) - 1\}, P, I, 0)$ is a group that is isomorphic to $(G, \cdot, \cdot^{-1}, e)$.

If $G$ is a finitely generated group, an isomorphism $\Theta : (G, \cdot, \cdot^{-1}, e) \to (\mathbb{N}, P, I, 0)$ can be described by giving the images of a generating family of $G$ in $\mathbb{N}$.

We define a new numbering of marked groups, denoted $\nu_{MR}$, as follows.

The description of a marked group $(G, S)$ for $\nu_{MR}$ is an encoded quintuple $(p, k, m, i, j)$, such that:

- The function $\phi_p$ has domain $\mathbb{N}$ if $G$ is infinite, $\{0, ..., \text{card}(G) - 1\}$ otherwise;  
- There exists a group isomorphism $\Theta : (G, \cdot, \cdot^{-1} e) \to (\text{dom}(\phi_p), P, I, 0)$, where $P$ and $I$ are recursive function;  
- $k$ gives the cardinality of $S$;  
- $m$ can be decoded as a $k$-tuple of elements of $\mathbb{N}$, which give the images of the elements of $S$ by $\Theta$;  
- $i$ and $j$ define the recursive functions $P$ and $I$, i.e. $\phi_i = P$ and $\phi_j = I$.

As before, one can easily check that this definition is sound by checking that there is no ambiguity as to which isomorphism $\Theta$ is encoded by a number $n$.

The following theorem is well known (for instance it is contained in Theorem 7.1 in [Can66], or it is Theorem 4 of [Rab60]), although it is rarely expressed in terms of numberings.

**Theorem 3.8.** The numberings $\nu_{WP}$ and $\nu_{MR}$ are $\equiv$-equivalent.

**Proof.** The proof is essentially the same as Theorem 3.6.  

\[ \square \]
3.5. Numberings of $G$ induced by numberings of the Cantor space. Recall that in Section 1 we defined an embedding $\Phi_k$ of the space $G_k$ of $k$-marked groups into the Cantor space.

The natural numbering $\nu_G$ of the Cantor space is obtained by seeing it as the set of functions from $\mathbb{N}$ to $\{0, 1\}$. The $\nu_G$-computable elements are just called computable.

We can extend the numberings $\nu_G$ to a countable disjoint union of Cantor spaces. Consider a countable set of Cantor spaces, denoted $C_i$, $i \in \mathbb{N}$.

Define a numbering $\tilde{\nu}_G$ of $\bigcup_{n \in \mathbb{N}} C_i$ by the following:

$$\tilde{\nu}_G(2^i(2j + 1)) \in C_i,$$

$$\tilde{\nu}_G(2^i(2j + 1)) = \nu_G(j).$$

The following proposition proves that this definition gives yet another way to introduce the numbering type $\Lambda_{WP}$.

**Proposition 3.9.** The restriction of $\tilde{\nu}_G$ to the space of marked groups (seen as a subset of a countable union of Cantor spaces via the maps $\Phi_k$) is equivalent to $\nu_{WP}$.

Of course, the embeddings $\Phi_k : G_k \rightarrow \{0; 1\}^N$ were chosen so that this proposition would hold: so that $\Phi_k$ would be $(\nu_{WP}, \nu_G)$-computable. Those embeddings were defined thanks to bijections $i_k : \mathbb{N} \rightarrow F_k$.

**Proof.** We show that $\nu_{WP}$ is equivalent to $\tilde{\nu}_G$ on $G$.

If $\nu_{WP}(n) = (G, S)$, then $n$ encodes both the cardinality of $S$ and the code for a function that, given two natural numbers, decides whether or not they encode the same element in $G$ with respect to the encoding of the elements of a marked group defined in SubSection 3.1. Denote here $c_k : \mathbb{N} \rightarrow F_k$ this encoding applied to the elements of the free groups.

If $\tilde{\nu}_G(n) = (G, S)$, then $n$ encodes the cardinality $k$ of $S$ and the code for a function that, given a natural number $m$, indicates whether the elements $i_k(m)$ of $F_k$ is a relation satisfied by $(G, S)$.

The result then follows from the easy fact that each function $c_k \circ i_k^{-1} : \mathbb{N} \rightarrow \mathbb{N}$ is a computable surjection which has a computable right inverse, that this holds uniformly in $k$, and from the standard fact that the code of the composition of two functions can be computed given the codes for those functions. 

**Remark 3.10.** The numberings associated to recursive presentations and to co-recursive presentations could be obtained similarly, replacing the usual numbering $\nu_G$ of the Cantor space with the numberings associated to upper and lower semi-computable sequences of $C$.

4. Introduction on some results in Markovian computable analysis

4.1. Effective Polish spaces. This introduction follows mostly Kushner ([Kus84]), but it is hopefully more accessible, since the constructivist setting adds technical complications. Note that SubSection 4.2 follows closely Hertling ([Her01]), who studies Banach-Mazur computable functions.

4.1.1. The computable reals. A precise definition of the set $\mathbb{R}_c$ of computable reals first appeared in Turing’s famous 1936 article ([Tur36]), but the numbering type of computable real numbers which is best suited to developing computable analysis was introduced one year later in the corrigendum [Tur37].

We recall the definition of the numbering $\nu_Q$ of the set of rationals. The numbering $\nu_Q$ is defined on $\mathbb{N}$. Given a natural number $n$, expressed as a product $n = 2^a 3^b n'$, with $\gcd(n', 30) = 1$, we put $\nu_Q(n) = (-1)^a \frac{b}{\epsilon + 1}$.

We now define the Cauchy numbering $\nu_R$ of $\mathbb{R}$. Fix an effective enumeration $\phi_0, \phi_1, \phi_2...$ of all partial recursive functions.

**Definition 4.1.** The Cauchy numbering of $\mathbb{R}$ is defined by the formulas:

$$\text{dom}(\nu_R) = \{i \in \mathbb{N}, \exists x \in \mathbb{R}, \forall n \in \mathbb{N}, |\nu_Q(\phi_i(n)) - x| < 2^{-n}\};$$

$$\forall i \in \text{dom}(\nu_R), \nu_R(i) = \lim_{n \rightarrow \infty} (\nu_Q(\phi_i(n))).$$

Thus the description of a real number $x$ is a Turing machines that produces a sequence $(u_n)_{n \in \mathbb{N}}$ of rationals with exponential convergence speed.

**Definition 4.2.** The set of $\nu_R$-computable real numbers is denoted $\mathbb{R}_c$, the $\nu_R$-computable reals are simply called the computable real numbers.
Several other definitions of the real numbers (decimal expansions, Dedekind cuts), when rendered effective, yield numbering types that define the same set of computable real numbers, but that are not $\equiv$-equivalent to the Cauchy numbering type—they are strictly stronger. See for instance \cite{Mos79}.

**Proposition 4.3** (Rice, \cite{Ric54}). Addition, multiplication and divisions are $(\nu_R \times \nu_R, \nu_R)$-computable functions defined respectively on $\mathbb{R}_c \times \mathbb{R}_c$, $\mathbb{R}_c \times \mathbb{R}_c$ and $\mathbb{R}_c \times (\mathbb{R}_c \setminus \{0\})$.

The following is a well known proposition which follows easily from Markov’s Lemma, see Lemma 4.26, this result is implicit in \cite{Tur30}.

**Proposition 4.4.** Equality is undecidable for computable reals. There is no algorithm that, given two computable reals $x$ and $y$, chooses one of $x \leq y$ or $y < x$ which is true.

4.1.2. **Recursive metric spaces.** We can now define what is a recursive metric space.

**Definition 4.5.** A recursive metric space (RMS) is a metric space $(X, d)$ equipped with a numbering $\nu : \subseteq \mathbb{N} \to X$, such that the distance function $d : X \times X \to \mathbb{R}$ is $(\nu \times \nu, \nu_R)$-computable.

Note that for convenience we do not impose that the set of computable points be dense in $X$. Of course, whether a triple $(X, d, \nu)$ is a RMS depends only on the $\equiv$-equivalence class of the numbering $\nu$, we can thus define a RMS to be a metric space equipped with a compatible number type.

There are different notions of “effective continuity”, as we have discussed in SubSection 2.4, we give here that of Kushner \cite{Kus84}.

**Definition 4.6.** A function $f$ between computable metric spaces $(X, d, \nu)$ and $(Y, d, \mu)$ is called effectively continuous if given a $\nu$-name $n$ of a point $x$ in $X$ and the $\nu_R$-name of a computable real number $\epsilon > 0$, it is possible to compute the $\nu_R$-name of a number $\eta > 0$ such that
\[
\forall y \in X ; d(x, y) < \eta \implies d(f(x), f(y)) < \epsilon.
\]

Note that the number $\eta$ is allowed to depend not only on $x$ and $\epsilon$, but also on the given names for those points. A program that computes the $\nu_R$-name for $\eta$ given $x$ and $\epsilon$ is said to witness for the effective continuity of $f$.

**Proposition 4.7.** If $(X, d, \nu)$ is a RMS, and if $Y \subseteq X$, then $(Y, d, \nu_Y)$ is also a RMS.

*Proof.* Obvious. \hfill $\square$

**Example 4.8.** The following spaces, equipped with their usual distances and numberings, are recursive metric spaces: $\mathbb{N}$, $\mathbb{R}_c$, the Cantor space $\{0, 1\}^\mathbb{N}$, the Baire Space $\mathbb{N}^\mathbb{N}$. Although details can be found in \cite[Chapter 9]{Kus84}, we still include a proof for the Cantor space.

**Proposition 4.9.** The Cantor space realized as $\{0, 1\}^\mathbb{N}$ equipped with its ultrametric distance $d$ and its usual numbering $\nu_C$ is a RMS.

*Proof.* Given the $\nu_C$ name of two sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$, we show that we can compute arbitrarily good approximations of their distance. To compute the distance between $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ with an error of at most $2^{-n}$, it suffices to enumerate the first $n$ digits of $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$. Then, if $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ agree on their first $n$ digits, it must be that $d((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}) < 2^{-n}$, and so 0 constitutes a good approximation of $d((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}})$. Otherwise, the first index after which the sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ differ can be computed exactly, and thus also the distance $d((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}})$.

In both cases, the desired approximation of $d((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}})$ can be computed. \hfill $\square$

**Corollary 4.10.** A disjoint union of Cantor spaces $\bigcup_{i \in \mathbb{N}} \{0, 1\}^\mathbb{N}$, equipped with a metric $d$ that puts different copies of $\{0, 1\}^\mathbb{N}$ at distance exactly 10 and with its natural numbering $\nu_C$ (which was defined in SubSection 3.5), is a RMS.

*Proof.* Given the $\nu_C$-names of two sequences, we decide whether or not they belong to the same copy of the Cantor space, if they don’t, we know that their distance is 10, otherwise apply Proposition 4.9. \hfill $\square$

The following proposition is fundamental for us.

**Corollary 4.11.** The space of marked groups with its ultrametric distance $d$ and with the numbering type $\Lambda_{WP}$ is a recursive metric space.
Proof. This follows directly from the previous corollary, together with Proposition 4.7, which states that a subset of a RMS with the induced numbering remains a RMS, and with the fact that the numbering type $\Lambda WP$ is the numbering induced on the space of marked groups by the numbering type $[\nu_C]$ defined on a disjoint union of Cantor spaces, as detailed in SubSection 3.5.

The space of marked groups equipped with the distance $d_{Cay}$ defined in Section 1 and with the numbering type $\Lambda WP$ is also a RMS. We leave it to the reader to prove this easy fact.

4.1.3. Effective completeness and effective separability. Since the space of marked groups is a Polish space, it is natural to ask whether it is an effective Polish space, that is, whether it is effectively complete and effectively separable.

Here, we define those two notions, and give some properties that follow from them. The importance of these notions lies in the facts that Ceitin’s Theorem is set on effectively Polish spaces.

Definition 4.12. A sequence $(u_n)_{n\in\mathbb{N}}$ of computable points in $X$ is called effectively convergent if it converges to a point $y \in X$, and if there exists a recursive function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that:

$$\forall(n,m) \in \mathbb{N}^2 : n \geq f(m) \implies d(u_n, y) \leq 2^{-m}.$$ 

A sequence $(u_n)_{n\in\mathbb{N}}$ of computable points in $X$ is called effectively Cauchy if there exists a recursive function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that:

$$\forall(p,q,m) \in \mathbb{N}^3 : p,q \geq f(m) \implies d(u_p, u_q) \leq 2^{-m}.$$ 

In both cases the function $f$ is called a regulator for the sequence $(u_n)_{n\in\mathbb{N}}$.

Definition 4.13. Let $(X,d,\nu)$ be a recursive metric space. An algorithm of passage to the limit (the name is from [Kus84]) is an algorithm that takes as input a computable Cauchy sequence together with a regulator for it, and produces the $\nu$-name of a point towards which this sequence converges.

A recursive metric space $(X,d,\nu)$ is effectively complete if it admits an algorithm of passage to the limit.

Note that even though we will in this paper focus exclusively on effectively complete spaces when stating continuity theorems, a weaker condition is sufficient to apply Markov’s Lemma and the theorems of Ceitin and Moschovakis: that there exist an algorithm of passage to the limit which works only on converging sequences. Thus for instance the interval $(0,1)$ admits such an algorithm for $\nu_\mathbb{R}$, even though it is not effectively complete. As the space of marked groups is effectively complete, we are not concerned here with theorems that do not rely on effectively completeness.

It is easy to see that any recursive metric space can be effectively completed into an effectively complete metric space.

Indeed, given a recursive metric space $(X,d,\nu)$, and denoting $(\overline{X},d)$ the abstract completion of $X$, we naturally obtain a numbering $\overline{\nu}$ of $\overline{X}$, which makes of $(\overline{X},d,\overline{\nu})$ a recursive metric space, by choosing, as $\overline{\nu}$-description of a point $x$ of $\overline{X}$, the code of a Turing machine that produces a $\nu$-computable Cauchy sequences of points of $X$, that admits the identity on $\mathbb{N}$ as a regulator.

This is just the construction of the completion of a metric space that uses Cauchy sequences, as rendered effective. See [Her01] or [Kus84] for more details.

We have the following easy proposition:

**Proposition 4.14.** Let $(X,d,\nu)$ be an effectively complete metric space. A closed subset $Y$ of $X$, together with the numbering induced by $\nu$, is also an effectively complete metric space.

**Proof.** It suffices to notice that algorithm of passage to the limit for $X$ also works for $Y$. □

The following proposition can be found in [Kus84]:

**Proposition 4.15.** The Cantor space is effectively complete.

This has the immediate corollary:

**Corollary 4.16.** The space of marked groups equipped with the word problem numbering is effectively complete.

This last fact could easily have been proved directly. We now describe the effective notion associated to separability.

**Definition 4.17.** A recursive metric space $(X,d,\nu)$ is called effectively separable if there exists a computable sequence $(u_n)_{n\in\mathbb{N}}$ of points in $X$ that is dense in $X$. 

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We can finally define effectively Polish spaces.

**Definition 4.18.** A recursive metric space \((X,d,\nu)\) which is both effectively complete and effectively separable is an **effective Polish space**.

There is not a single accepted term for this notion, which is sometimes referred to as a “recursive metric space”. We explained in SubSection 2.4 why we talk instead of effective Polish space.

**Example 4.19.** The following spaces, equipped with their usual distances and numberings, are effective Polish spaces: \(\mathbb{N}, \mathbb{R}_c\), the Cantor space \(\{0,1\}^\mathbb{N}\), the Baire Space \(\mathbb{N}^\mathbb{N}\).

The following proposition, while obvious, shall be very useful in its group theoretical version.

**Proposition 4.20.** Let \((X,d,\nu)\) be a recursive metric space, and \(Y\) be a \(\nu\)-r.e. set in \(X\). Then any computable point in the closure \(\overline{Y}\) of \(Y\) is the effective limit of a computable sequence of points of \(Y\).

This proposition could have been phrased: any computable point in the closure of \(Y\) is automatically in its “effective closure”.

**Proof.** This is straightforward: given a point \(x\) adherent to \(Y\), we can define a computable sequence \((v_n)_{n \in \mathbb{N}}\) by: \(v_n\) is the first element, in a fixed enumeration of \(Y\), which is proven to satisfy \(d(x,v_n) < 2^{-n}\).

This proposition has the immediate corollary:

**Corollary 4.21.** In an effective Polish space with a dense and computable sequence \((u_n)_{n \in \mathbb{N}}\), the computable points are exactly the effective limits of effectively Cauchy sequences extracted from \((u_n)_{n \in \mathbb{N}}\).

The following result shows one of the appeals of effective Polish spaces: the computable structure on an effective Polish space is entirely defined by the distance function between elements of its dense sequence.

**Theorem 4.22.** An effective Polish space is computably isometric to the effective completion of any of its computable and dense sequences.

**Proof.** Let \((X,d,\nu)\) be an effective Polish space, and \((u_n)_{n \in \mathbb{N}}\) any \(\nu\)-computable and dense sequence. Denote \(\phi_0, \phi_1, \phi_2, \ldots\) an effective enumeration of all partial recursive functions.

The effective completion of \((u_n)_{n \in \mathbb{N}}\) defines another numbering of \(X\), which we denote \(\mu\), and which is defined by the following: \(\mu(i) = x\) if the subsequence \((u_{\phi_i(n)})_{n \in \mathbb{N}}\) extracted from \((u_n)_{n \in \mathbb{N}}\) thanks to the function \(\phi_i\) converges to \(x\) with exponential speed: \(d(u_{\phi_i(n)}, x) < 2^{-n}\) holds for all \(n\).

Theorem 4.22 can then be formulated equivalently: the numberings \(\nu\) and \(\mu\) are equivalent, i.e. the identity on \(X\) is both \((\nu,\mu)\)-computable and \((\mu,\nu)\)-computable. This is what we show now.

By Corollary 4.21, the \(\nu\) and \(\mu\) computable points of \(X\) are identical, denote \(X_e\) this set.

A \(\mu\)-description of a point \(x\) in \(X_e\) is the description of a computable Cauchy sequence that converges to \(x\), with \(g : n \mapsto 2^{-n}\) being a regulator for this sequence. The algorithm of passage to the limit of \((X,d,\nu)\) can thus be applied to this description with \(g\) as regulator, and it yields precisely a \(\nu\)-description of \(x\). This shows that the identity on \(X\) is \((\mu,\nu)\)-computable.

To show that it is also \((\nu,\mu)\)-computable, one only has to notice that the procedure described in the proof of Proposition 4.20 is uniform, in that it allows, given a \(\nu\)-description of a point \(x\), to produce a computable sequence extracted from \((u_n)_{n \in \mathbb{N}}\) which converges to \(x\) with the desired speed. This is precisely a \(\mu\)-description for \(x\).

The following corollary to Theorem 4.22 shows that when a Polish space can be equipped with an effective Polish space structure, this structure is unique.

**Corollary 4.23.** Given a Polish space \((X,d)\) with a dense sequence \((u_n)_{n \in \mathbb{N}}\), there is at most one numbering type on \((X,d)\) which makes of it an effective Polish space with \((u_n)_{n \in \mathbb{N}}\) as a computable and dense sequence.

**Proof.** This follows directly from Theorem 4.22.

**Theorem 4.22** allows for a very simple definition of what is an effective Polish space, that relies only on the distance between the elements of a dense sequence. Weihrauch and Moschovakis both used such definitions. We give here a definition that mimics that of Moschovakis, see [GKP16] for the complete definitions of Weihrauch and Moschovakis, and their differences. Note however that the definition that we give is weaker than the ones given by Weihrauch and Moschovakis - a space that admits a recursive presentation in this sense also admits one following the definitions of Weihrauch and Moschovakis.
Definition 4.24. A recursive presentation of a Polish space \((X, d)\) is a dense sequence \((u_n)_{n \in \mathbb{N}}\) of points of \(X\) such that the function 
\[
\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \\
(n, m) \mapsto d(u_n, u_m)
\]
is \((\text{id}_\mathbb{N} \times \text{id}_\mathbb{N}, \nu_\mathbb{R})\)-computable.

The term presentation is from [Mos80], and has no relation the notion of a presentation for a group. The following proposition renders explicit the link between recursive presentations and effective Polish spaces.

Proposition 4.25. A Polish space \((X, d)\) admits a recursive presentation if and only if it admits a numbering that makes of it an effectively Polish space.

Proof. If \(\nu\) is a numbering of \(X\) that makes of \((X, d, \nu)\) an effective Polish space, it means that there exists a \(\nu\)-computable dense sequence \((u_n)_{n \in \mathbb{N}}\). The function \(\phi\) defined by \(\phi(n, m) = d(u_n, u_m)\) is then computable, and thus \((u_n)_{n \in \mathbb{N}}\) defines a recursive presentation of \((X, d)\).

Conversely, if \((u_n)_{n \in \mathbb{N}}\) is a sequence which is dense in \(X\), the condition that \((n, m) \mapsto d(u_n, u_m)\) be computable exactly asks that the function 
\[
\mu : \mathbb{N} \to X \\
\quad n \mapsto u_n
\]
defines a numbering of \(X\) which makes of \((X, d, \mu)\) a recursive metric space. Then, the effective completion of \(\mu\) defines a numbering \(\nu\) of \(X\), for which \((X, d, \nu)\) is an effectively Polish space. \(\square\)

We will prove the following result:

Theorem. The space of marked group, associated to the distance \(d\), does not have a recursive presentation.

This theorem appears in Section 5 as Theorem 5.4. We will also prove that the space of marked groups does not contain dense and computable sequences of groups described by word problem algorithms, but Theorem 5.4 is more general because we do not suppose a priori that a dense sequence should consist in groups described by word problem algorithms.

4.2. Markov’s Lemma and abstract continuity. We will give here a proof of the fact that computable functions on an effective Polish space are continuous, starting with Markov’s Lemma, which is both very useful and very simple to use, and which will remain our main tool in the space of marked groups, since the stronger theorems are not applicable there.

We fix a recursive metric space \((X, d, \nu)\) which we suppose effectively complete. Denote by \(A_{lim}\) an algorithm of passage to the limit for it.

Lemma 4.26 (Markov, [Mar54], English version: [Mar63]). Suppose that a \(\nu\)-computable sequence \((u_n)_{n \in \mathbb{N}}\) effectively converges in \(X\) to a \(\nu\)-computable point \(x\). Suppose additionally that for any \(n\), \(u_n \neq x\). Then there is a \(\nu\)-computable sequence \((w_p)_{p \in \mathbb{N}}\) of \(X^\mathbb{N}\) such that: for each \(p\), \(w_p \in \{u_n, n \in \mathbb{N}\} \cup \{x\}\), and the set \(\{p, w_p = x\} \subseteq \mathbb{N}\) is co-r.e. but not r.e..

Proof. Consider an enumeration of all Turing machines \(M_0, M_1, \ldots\). To the machine \(M_p\), we associate a computable sequence \((x^p_n)_{n \in \mathbb{N}}\) of points in \(X\). To define \((x^p_n)_{n \in \mathbb{N}}\), start a run of the machine \(M_p\) with no input. While it lasts, the sequence \((x^p_n)_{n \in \mathbb{N}}\) should be identical to the sequence \((u_n)_{n \in \mathbb{N}}\). If, at some point, the machine \(M_p\) stops, the sequence \((x^p_n)_{n \in \mathbb{N}}\) should become constant.

To sum this definition up, \((x^p_n)_{n \in \mathbb{N}}\) is defined as follows:

While \((M_p\) does not stop) enumerate \((u_n)_{n \in \mathbb{N}}\).

If \((M_p\) stops in \(k\) computation steps), set \(x^p_n = u_k\) for \(n \geq k\).

Each sequence \((x^p_n)_{n \in \mathbb{N}}\) is Cauchy, and in fact it converges at least as fast as the original sequence \((u_n)_{n \in \mathbb{N}}\).

Thus the algorithm of passage to the limit \(A_{lim}\) can be applied to any sequence \((x^p_n)_{n \in \mathbb{N}}\), using the regulator of convergence of \((u_n)_{n \in \mathbb{N}}\).

The sequence \((w_p)_{p \in \mathbb{N}}\) is the sequence obtained by using the algorithm of passage to the limit on each sequence \((x^p_n)_{n \in \mathbb{N}}\), for \(p \in \mathbb{N}\).

If follows directly from our definitions that if the machine \(M_p\) is non-halting, the sequence \((x^p_n)_{n \in \mathbb{N}}\) is identical to \((u_n)_{n \in \mathbb{N}}\), and thus \(w_p\), which is its limit, is equal to \(x\). On the other hand, if \(M_p\) halts in \(k\) computation steps, we have \(w_p = u_k\) and thus \(w_p\) is different from \(x\). \(\square\)
Corollary 4.27. Let $f$ be a computable function between recursive metric spaces $X$ and $Y$, suppose that $X$ is effectively complete, and let $(x_n)_{n \in \mathbb{N}}$ be a computable sequence that effectively converges to a point $x$ in $X$. Then the sequence $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x)$.

Proof. This proof is not effective, as we proceed by contradiction. Suppose that the sequence $(f(x_n))_{n \in \mathbb{N}}$ does not converge to $f(x)$. Then there must exist a subsequence $(f(x_{n_k}))_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ and a rational $r > 0$ such that

$$\forall n \in \mathbb{N}, d(f(x_{n_k}), f(x)) > r.$$ 

The existence of such a sequence, which need not a priori be computable, implies that there must also exist such a sequence where, additionally, the function $\phi : \mathbb{N} \to \mathbb{N}$ is computable.

This follows from Proposition 4.20, as the set of terms of the sequence $(x_n)_{n \in \mathbb{N}}$ for which $d(f(x_n), f(x)) > r$ holds is a r.e. set.

Finally, the function $f$ can be used to distinguish between the elements of the sequence $(x_{\phi(n)})_{n \in \mathbb{N}}$ and its limit $x$, as, given a computable point $u$ in $X$, it is possible to chose one which is true between $d(f(u), f(x)) > r$ and $d(f(u), f(x)) < r$, if we know a priori that $d(f(u), f(x))$ is not equal to $r$. This contradicts Markov’s Lemma, and thus $(f(x_n))_{n \in \mathbb{N}}$ must converge to $f(x)$. \qed

We can use the previous corollary to prove that, under the additional assumption that the space $X$ be an effective Polish space, any computable function $f : X \to Y$ is continuous -not necessarily effectively so. This corollary of Markov’s Lemma was first proven by Mazur for functions defined on intervals of $\mathbb{R}$ (see [Maz63]). Note that Mazur uses a notion of computability for functions defined on metric spaces that differs for the one we use throughout, and which is known as Banach-Mazur computability. However, the proof of his theorem is identical, when applied either to Markov computable functions, or to Banach-Mazur computable functions.

Corollary 4.28 (Mazur’s Continuity Theorem). Consider a computable function $f : X \to Y$ between an effective Polish space $X$ and a recursive metric space $Y$. Then $f$ is continuous.

Proof. Denote $(u_n)_{n \in \mathbb{N}}$ an effective and dense sequence of $X$.

Suppose that $f$ is not continuous at a point $x$ of $X$. This means that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ and a real number $r > 0$ such that $(x_n)_{n \in \mathbb{N}}$ converges to $x$, but for any $n \in \mathbb{N}, d(f(x_n), f(x)) > r$. Any point of $(x_n)_{n \in \mathbb{N}}$ is the limit of an effective subsequence of $(u_n)_{n \in \mathbb{N}}$, by Proposition 4.20. And thus, by Corollary 4.27, for each point $x_k$ of the sequence $(x_n)_{n \in \mathbb{N}}$, there must exist a point $u_{\phi(k)}$ in the sequence $(u_n)_{n \in \mathbb{N}}$, such that both inequalities $d(u_{\phi(k)}, x_k) < 2^{-k}$ and $d(f(u_{\phi(k)}), f(x)) > r$ hold.

Thus there exists a subsequence $(u_{\phi(n)})_{n \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$, which converges to $x$ and such that for any $n \in \mathbb{N}, d(f(u_{\phi(n)}), f(x)) > r$.

This subsequence is a priori not computable, but by Proposition 4.20, the abstract fact that such a sequence exists automatically implies that there must also exist such a subsequence that is, in addition, both computable and effectively converging to $x$.

Finally, we conclude by applying Corollary 4.27 again. \qed

4.3. Differences with the Borel hierarchy.

4.3.1. Computable but discontinuous function. We give here an example of a Markov computable function that is not continuous. This is done by considering a function defined on a peculiar domain. The fact that those exist is well known, we explain it here in terms of Kolmogorov complexity, following [HR16]. See Chapter 1 of [SVU17] for an introduction to Kolmogorov complexity.

We set ourselves in the Cantor space $\{0; 1\}^\mathbb{N}$, but this could be done in $\mathbb{R}$, as well.

Consider a sequence $(u_n)_{n \in \mathbb{N}}$ of finite strings of zeros and ones, such that the length of $u_n$ is $n$, and which has linear asymptotic Kolmogorov complexity: $K(u_n) \sim n$. It is well known that such a sequence exists, but cannot be effectively enumerated. Consider now the sequence $v_n = 0^n 1 u_n 00000 .....$ of elements of $\{0; 1\}^\mathbb{N}$. This sequence also has linear asymptotic Kolmogorov complexity: a single Turing Machine can transform any element $v_n$ into the corresponding $u_n$, this implies that asymptotically $K(v_n) \leq K(u_n)$. The other inequality is obvious as well.

We call $\mathcal{A}$ the subset of $\{0; 1\}^\mathbb{N}$ consisting of the null sequence (which we denote by $0^n$) and of the set $\{v_n, n \in \mathbb{N}\}$.

Proposition 4.29. The function $\delta_0 : \mathcal{A} \to \{0; 1\}$ which sends the null sequence to 1 and all other sequences to 0 is computable on $\mathcal{A}$. However, it is discontinuous.

Proof. Because $K(v_n) \sim n$, there must exist an integer $b \in \mathbb{Z}$ such that for all $n$, $K(v_n) > \frac{n}{2} + b$. 

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We now show how to compute $\delta_0$. Let $M_x$ be a Turing Machine that codes for an element $x$ in $A$. Denote by $k$ the number of states of this machine. The element $x$ has Kolmogorov complexity at most $k$, and thus either it is the null sequence, or, if it can be written $v_n$ for some $n$, we have $k > \frac{2}{n} + b$, and thus $n < 2(k - b)$.

But since the element $v_n$ agrees with the null sequence only on its first $n$ terms, this means that if $x$ is not the null sequence, one of its first $2(k - b)$ digits must be a one. This can be easily checked, using the Turing Machine $M_x$ until it has written the first $2(k - b)$ digits of $x$.

The following question however remains open:

**Problem 4.30.** Characterize those sequences $(u_n)_{n \in \mathbb{N}}$ converging to the null sequence in the Cantor space for which any Markov computable function defined on $\{u_n, n \in \mathbb{N}\} \cup \{0^\omega\}$ has to be continuous.

Note that peculiar instances of this problem naturally arise for groups: in [Rau20], the author has constructed a residually finite group $G$ with solvable word problem with the property that any sequence of its finite quotients that converges to it must be non computable for $\omega WP$. Thus one cannot apply Markov’s Lemma to prove that $G$ cannot be distinguished from its finite quotients. On the other hand, the finite quotients of this group can easily be described, and there is no reason to think that their asymptotic Kolmogorov complexity is maximal. Thus the problem “can $G$ be distinguished from its finite quotients?” probably falls in between the cases which we are able to deal with.

### 4.3.2. A semi-decidable set that is not open

The previous example was obtained by considering a function defined on a set with bad properties. On an effective Polish space, the decidable sets must be clopen, since their characteristic functions must be continuous. By Markov’s Lemma, the semi-decidable sets cannot be “effectively not-open”: if $x$ is a point of a semi-decidable set $Y$, and if $(u_n)_{n \in \mathbb{N}}$ is a computable sequence that effectively converges to $x$, then infinitely many elements of this sequence must belong to $Y$. One might wonder whether this result can be strengthened to: “the $n$-semi-decidable sets on an effective Polish space $(X, d, \nu)$ are open”. An example of Friedberg ([Fri58]) shows that this is not the case, we reproduce here the account of this result from [HR16], which renders explicit the role of Kolmogorov complexity in the construction of this example.

This example is set in the Cantor space $\{0, 1\}^\omega$. For $w$ an element of $\{0, 1\}^*$, denote by $[w]$ the clopen set of all sequences that start by $w$.

**Theorem 4.31** (Friedberg, see [HR16], Theorem 4.1). On the Cantor space, the set

$$A = \{0^\omega\} \cup \bigcup_{n : K(n) < \frac{\log(n)}{2}} [0^n1]$$

is semi-decidable but not open.

**Proof.** $A$ is not open, as it does not contain a neighborhood of $0^\omega$, because infinitely often in $n$ one has $K(n) > \frac{\log(n)}{2}$.

We now show that $A$ is semi-decidable.

There exists a program $T$ that maps any element $x$ of the Cantor space that is different from $0^\omega$ to the number of zeroes that appear at the beginning of $x$.

We are now given a computable point $x$ of $\{0, 1\}^\omega$.

The description of $x$ by a Turing machine that produces it gives an upper bound $K_0$ on the Kolmogorov complexity of $x$. Noting $l$ the length of the program $T$ defined above, one has that either $x$ is $0^\omega$, or, if $x$ can be decomposed as $x = 0^n1x'$, it must be that $K(n) \leq K(x) + l \leq K_0 + l$.

Start enumerating $x$. If $x$ starts with more than $2^{2(K_0 + l)}$ zeroes, then either it is the sequence $0^\omega$, or it can be written as $x = 0^n1x'$, with $\frac{\log(n)}{2} > K_0 + l$, and thus with $\frac{\log(n)}{2} > K(n)$. In any case, we know that $x$ belongs to $A$, without having to compute $n$.

If $x$ starts with less than $2^{2(K_0 + l)}$ zeroes, a number $n$ such that $x$ can be rewritten as $x = 0^n1x'$ can be effectively found. From this, to determine whether $x$ belongs to $A$, one only needs to check whether $K(n) < \frac{\log(n)}{2}$ holds, which can be proven whenever it holds, as the Kolmogorov complexity is upper-semi-computable.

Other examples of non-open but semi-decidable sets can be found in [HR16]. It is however clear that those examples are artificially built, and this justifies the heuristic which says that a natural semi-decidable property can be expected to be open.

Note finally that although we have just seen that a semi-decidable subset of an effectively Polish space does not have to be open, it must share the following property of open sets: it meets any computable and dense sequence.

**Proposition 4.32** (Moschovakis, [Mos64], Theorem 4). Let $(X, d, \nu, (u_n)_{n \in \mathbb{N}})$ be an effectively Polish space. A non-empty $\nu$-semi-decidable subset of $X$ must intersect the dense sequence $(u_n)_{n \in \mathbb{N}}$. 

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The proof of this result is in fact very close to that of Mazur’s Continuity Theorem, Corollary 4.28. A consequence of this fact, pointed out in [HR16], is the following:

**Corollary 4.33** (Hoyrup, Rojas, [HR16], Section 4, Proposition 3). In an effective Polish space \((X, d, \nu, (u_n)_{n \in \mathbb{N}})\), there is an algorithm that takes as input the code for a \(\nu\)-semi-decidable set \(D\) and stops if and only if this set is non-empty.

In case the set \(D\) is non-empty, this algorithm will produce the \(\nu\)-name for a point in it.

**Proof.** Just enumerate the sequence \((u_n)_{n \in \mathbb{N}}\) in search of a point in \(D\), by Proposition 4.32, \(D\) is non-empty if and only if it contains a point from \((u_n)_{n \in \mathbb{N}}\). \(\square\)

Note that the previous proposition contains an effective version of the Axiom of Choice for effectively Polish spaces: a possible formulation of AC is that for any set \(X\), there exists a choice function that maps a non-empty subset of \(X\) to a point in this subset. The previous result implies that a computable choice function exists for semi-decidable subsets of an effective Polish space. We will give in Theorem 5.10 a strong negation of such an effective Axiom of Choice for the space of marked groups: there does not exist a computable function that, given a non-empty basic clopen set \(\Omega^k_{r_1, \ldots, r_m; s_1, \ldots, s_{m'}}\), can produce a point in this set.

### 4.4. Kreisel, Lacombe and Schoenfield and Ceitin Theorems, Moschovakis’ addendum.

**4.4.1. Theorems of Kreisel, Lacombe, Schoenfield, Ceitin.** The following theorem is one of the most important theorems in computable analysis. It was first proved by Kreisel, Lacombe and Schoenfield in 1957 in [KLS57] in the case of functions defined on the Baire space \(\mathbb{N}^\mathbb{N}\), and obtained independently by Ceitin in 1962 (English translation in [Ceit67]), in the more general setting of effective Polish spaces.

**Theorem 4.34** (Kreisel-Lacombe-Schoenfield, Ceitin). A computable function defined on an effective Polish space (with images in any RMS) is effectively continuous.

Moreover, for each pair constituted of an effective Polish space and a RMS, there is an algorithm that takes as input the description of a computable function defined between those spaces, and produces a program that will attest for the effective continuity of this function.

**4.4.2. Moschovakis’ Theorem.** In 1964, Moschovakis gave a new proof of Ceitin’s Theorem, at the same time providing the only known effective continuity result for metric spaces set in a more general context than that of effective Polish spaces.

In what follows, \((X, d, \nu)\) denotes a recursive metric space.

**Definition 4.35.** We say that \((X, d, \nu)\) satisfies Moschovakis’ condition (B) if there is an algorithm that, given the code of a \(\nu\)-semi-decidable set \(A \subseteq X\), and an open ball \(B(x, r)\), described by a \(\nu\)-name for \(x\) and a \(\nu\)-name for \(r \in \mathbb{R}_+\), such that \(A \cap B(x, r) \neq \emptyset\), will produce the \(\nu\)-name of a point \(y\) in the intersection \(A \cap B(x, r)\).

Note that in this definition, the algorithm is always given as input a pair of intersecting sets, it then produces a point in the intersection. This algorithm is not supposed to be able to determine whether or not two given sets intersect. This definition asks for an effective Axiom of Choice, similar to the one described in Proposition 4.32. And an easy consequence of Proposition 4.32 is the following:

**Proposition 4.36.** An effectively Polish space satisfies Moschovakis’ condition (B).

We can now state Moschovakis’ Theorem on the effective continuity of computable functions.

**Theorem 4.37** (Moschovakis, [Mos64], Theorem 3). A computable function defined on an effectively complete RMS that satisfies Moschovakis’ condition (B) is effectively continuous.

Moreover, for each pair constituted of such a space and of any RMS, there is an algorithm that takes as input the description of a computable function defined between those spaces, and produces a program that will attest for the effective continuity of this function.

We will prove in Corollary 5.15 that the hypotheses of this theorem fail for the space of marked groups, leaving open the conjecture which says that computable functions on \(G\) should be effectively continuous.

Note that several other results that can be found in [Mos64] can give rise to conjectures for the space of marked groups, in particular the characterization of effective open sets as Lacombe sets, which is Theorem 11 of [Mos64]. We do not detail those results nor the corresponding conjectures here.
5. The space of marked groups as a recursive metric space

5.1. Non effective separability of \( G \). The following fact was remarked in [dCGP07].

**Proposition 5.1.** The computable points of \((G, d, \nu_{WP})\) are not dense in it.

*Proof.* This follows directly from Theorem 5.12, which is due to Miller, and which we have already quoted in a previous section: there exists a coherent set of relations and irrelations that is not wp-coherent. \(\Box\)

This shows that the study of computability on the space of marked groups could be more precisely set in the closure of the set of groups with solvable word problem, the structure of an open set in \( G \) which contains no group with solvable word problem has no bearing on the present study.

Denote \( G_{WP} \) the closure of the set of (markings of) groups with solvable word problem in \( G \). By definition, the computable points of \( G_{WP} \) are dense in it. However, we have the following theorem:

**Theorem 5.2.** No sequence of marked groups can be both computable and dense in \( G_{WP} \).

*Proof.* This is a simple application of Theorem 1.2, together with Corollary 4.33. Corollary 4.33 states that in an effectively Polish space, there is an algorithm that stops exactly on semi-decidable sets that are non-empty.

The basic clopen sets \( \Omega_{\lambda} \) are obviously semi-decidable in \( G_{WP} \), but a program that recognizes these basic clopen sets that are non-empty would allow one to recognize wp-coherent sets of relations and irrelations, contradicting Theorem 1.2. \(\Box\)

It is interesting to interpret this proof as a variation on McKinsey’s algorithm for finitely presented residually finite groups. Notice that if \( X \) is a set of marked groups which is dense in \( G_{WP} \), then every finitely presented group with solvable word problem is “residually-\( X \)”, and a proof similar to McKinsey’s would then contradict the Boone and Rogers Theorem ([BR66]).

This proposition directly implies the following:

**Corollary 5.3.** The recursive metric space \((G_{WP}, d, \nu_{WP})\) is a Polish space that is effectively complete but not effectively separable, and thus it is not an effective Polish space.

As we have already seen in the previous section, Mazur’s Continuity Theorem and Ceitin’s Effective Continuity Theorem both apply to effective Polish spaces. This corollary thus shows that those theorems cannot be directly applied to the space of marked groups.

We now prove the following result:

**Theorem 5.4.** The metric space \((G, d)\) does not have a recursive presentation in the sense of Definition 4.24.

*Proof.* Recall that a recursive presentation of \((G, d)\) would consist in a sequence \((u_n)_{n \in \mathbb{N}}\), dense in \( G \), and for which the distance between the \( n \)-th and \( m \)-th terms is computable.

If \( G \) admitted a recursive presentation, then so would \( G_k \) for any \( k \geq 1 \). Thus we only have to show that \( G_2 \) does not have a recursive presentation as a Polish space.

Recall that we have defined an embedding \( \Phi_2 : G_2 \rightarrow \{0, 1\}^\mathbb{N} \) by fixing a computable order on the rank two free group. Call a set \( r_1, \ldots, r_m; s_1, \ldots, s_{m'} \) of relations and irrelations *initial* if the \( m + m' \) elements of the free group it contains are exactly the first \( m + m' \) elements of this order. The number \( m + m' \) is called the length of this set of relations and irrelations.

We will prove that having a recursive presentation of \( G_2 \) would allow one to compute, given an integer \( n \), the number of initial coherent sets of relations and irrelations that contain \( n \) relations.

We first show that this is sufficient to obtain a contradiction.

There are exactly \( 2^n \) possible initial sets of relations and irrelations of length \( n \). Since the incoherent sets of relations and irrelations form a r.e. set, if we had access to the number of initial coherent sets of relations and irrelations of length \( n \), we would be able to compute exactly those sets, by starting with the \( 2^n \) possible initial sets, and deleting incoherent ones until the number of coherent sets is attained.

But being able to compute the *initial* coherent sets of relations and irrelations in fact also allows one to compute all coherent sets of relations and irrelations, because a set \( r_1, \ldots, r_m; s_1, \ldots, s_{m'} \) which is not initial, is coherent if and only if there is an initial and coherent set which contains the elements \( r_1, \ldots, r_m \) as relations, and the elements \( s_1, \ldots, s_{m'} \) as irrelations. Choosing \( n \) big enough, it would then suffice to construct all initial sets of relations and irrelations of length \( n \) to determine whether \( r_1, \ldots, r_m; s_1, \ldots, s_{m'} \) is coherent. And we have seen that this is impossible.

Suppose that \((u_n)_{n \in \mathbb{N}}\) defines a recursive presentation of \( G_2 \), we show how to compute the number of initial coherent sets of relations and irrelations that contain \( n \) relations. Denote by \( \lambda(n) \) this number. Again, because the
incoherent sets of relations and irrelations form a r.e. set, \( \lambda \) is an upper semi-computable function: there exists a computable function \( \lambda^\geq \) that, given \( n \), produces a computable and decreasing sequence of integers that converges to \( \lambda(n) \). What we show is that the existence of a recursive presentation of \( G_2 \) implies that \( \lambda \) is also lower semi-computable, meaning that there exists a computable function \( \lambda^\leq \) that, on input \( n \), produces an increasing sequence of integers which converges to \( \lambda(n) \).

Given \( i \) and \( n \) natural numbers, define \( x_i^n \) to be the maximal size of a subset of \( \{u_0, u_1, \ldots, u_i\} \) of which any two elements are at least \( 2^{-n} \) apart.

We claim that \( (i, n) \mapsto x_i^n \) is a computable function, and that, for any \( n \), \( i \mapsto x_i^n \) is an increasing function that converges to \( \lambda(n) \).

Setting \( \lambda^\leq(n) = (x_i^n)_{i \in \mathbb{N}} \) will then conclude the proof.

As the distance function \( d \) takes values only in \( \{0\} \cup \{2^{-n}, n \in \mathbb{N}\} \), given the description of the distance \( d(u_i, u_j) \) as a computable real, one can always effectively choose one of \( d(u_i, u_j) < 3 \times 2^{-n-2} \) and \( d(u_i, u_j) > 3 \times 2^{-n-2} \) which holds, and thus decide whether or not \( u_i \) and \( u_j \) are \( 2^{-n} \) apart. One can thus check every subset of \( \{u_0, u_1, \ldots, u_i\} \) to find one of maximal size, all the elements of which are \( 2^{-n} \) apart. Thus \( (i, n) \mapsto x_i^n \) is computable.

The function \( (i, n) \mapsto x_i^n \) is increasing in \( i \) by definition.

Finally, we show that \( x_i^n \) goes to \( \lambda(n) \) as \( i \) goes to infinity. Two points of \( G_2 \) are at least \( 2^{-n} \) apart if and only if their binary expansion differ on one of their first \( n \) digits: those groups must be associated to different initial sets of relations and irrelations of length \( n \). Because the sequence \( \{u_n\}_{n \in \mathbb{N}} \) is supposed to be dense in \( G_2 \), for any coherent initial set of relations and irrelations of length \( n \), there should be a point of this sequence which satisfies those relations and irrelations.

And thus there must indeed exist a set of \( \lambda(n) \) points in the dense sequence which are pairwise \( 2^{-n} \) apart, and this number is clearly maximal. \( \square \)

5.2. **Optimality of the numbering** \( \nu_{WP} \). Note that Theorem 5.4, which states that the space of marked groups does not have a recursive presentation, can be seen, through Proposition 4.25, as answering the question: “which numberings of \( G \) make of \((G, d)\) an effectively Polish space?” The answer to this question is that no numbering can make of \((G, d)\) an effectively Polish space.

We will now include some results that ask which numberings of \( G \) can make of it a recursive metric space.

The precise question we want to ask is: what are the numberings that contain the least possible amount of information, while still making \( G \) a RMS? And in particular, is \( \nu_{WP} \) optimal in this sense?

A formal statement that would express the optimality of \( \nu_{WP} \) could be: “any numbering \( \nu \) that makes of \((G, d, \nu)\) a recursive metric space satisfies \( \nu \succeq \nu_{WP} \)”. And one could even ask for such a result, replacing the distance \( d \) by any distance that generates the topology of \( G \).

Mind however that this statement is false.

Indeed, given a marked group \( G \) with unsolvable word problem and isolated from groups with solvable word problem, as provided by Proposition 5.1, we can define a new numbering \( \nu_G \) of \( G \), defined by \( \nu_G(0) = G \) and, for \( n > 0 \), \( \nu_G(n) = \nu_{WP}(n-1) \). Because \( G \) is isolated from groups with solvable word problem, there is a natural number \( n_G \) such that any group \( H \) that satisfies \( d(G, H) < 2^{-n_G} \) also has unsolvable word problem. Thus it suffices to know the first \( n_G \) terms in the binary expansion of \( n_G \) to be able to compute the distance \( d(G, H) \) from \( G \) to any marked group \( H \) with solvable word problem. And thus the new numbering \( \nu_G \) also makes of \((G, d, \nu_G)\) a recursive metric space, in which \( G \) is a computable point, and one has \( \nu_{WP} \succeq \nu_G \) and \( \nu_G \notin \nu_{WP} \).

Notice however that the numbering \( \nu_G \) described above is not saturated, in the sense that the marked group \( G \) is \( \nu_G \)-computable, but it is the only \( \nu_G \)-computable point in the set \([G]\). We thus ask:

**Problem 5.5.** Suppose that \( \nu \) is a saturated numbering of \( G \), and that \((G, d, \nu)\) is a RMS. Must one have \( \nu \succeq \nu_{WP} \)?

We however have a theorem that is close enough to the false statement given above.

**Theorem 5.6.** Suppose that \( \hat{d} \) is any distance on \( G \) that generates the same topology as \( d \), and that \( \mu \) is a numbering of \( G \) such that \((G, \hat{d}, \mu)\) is a RMS.

Suppose furthermore that there is an algorithm that takes as input a \( \mu \)-name for a \( k \)-marked group \( G \) and a \( \nu_G \)-name for a radius \( r > 0 \), and produces a basic clopen set \( \Omega_{RT}^k \) (described by a pair of tuples of elements of the free group) such that:

\[
G \in \Omega_{RT}^k;
\]
\[
\Omega_{RT}^k \subseteq B_{\hat{d}}(G, r).
\]

Then one has \( \mu \succeq \nu_{WP} \).
Before proving this theorem, let us note that the hypotheses of this theorem can be explained as follows: we ask not only that the distance \( d \) define the usual topology of \( G \), but also that it define its usual effective topology. What we suppose for Theorem 5.6 is precisely that any effective open set of \( (G, d, \mu) \) should also be effectively open for the topology defined by the basic clopen sets \( \Omega^R_G \).

A discriminating family of a group \( G \) is a subset of \( G \) which does not contain the identity element of \( G \), and which intersects any non-trivial normal subgroup of \( G \). We will use Theorem 3.4 from [dCGP07], which is an analysis of Kuznetsov’s method for solving the word problem in simple groups ([Kuz58]).

**Theorem 5.7** (Cornulier, Guyot, Pitsch, [dCGP07]). A group has solvable word problem if and only if it is both recursively presentable and recursively discriminating.

And this statement is uniform: there is an effective method that allows, given a recursive presentation and an algorithm that enumerates a discriminating family in a marked group \( G \), to find a word problem algorithm for \( G \).

We add the statement about the uniformity of this theorem, but it is easy to see that the proof given in [dCGP07] is uniform.

**Proof of Theorem 5.6.** Consider a \( \mu \)-computable marked group \( G \). We show, given a \( \mu \)-name for \( G \), how to obtain a word problem algorithm for it.

Using the algorithm given by the hypotheses of the theorem, consecutively on each ball \( B_\frac{1}{n}(G) \), we obtain a computable sequence \( (\Omega_{R_n:T_n})_{n \in \mathbb{N}} \) of basic clopen subsets, such that for each \( n \) we have:

\[
G \in \Omega_{R_n:T_n} \subseteq B_\frac{1}{n}(G).
\]

It follows that \( \bigcap \Omega_{R_n:T_n} = \{G\} \), and thus that the union \( \bigcup R_n \) defines a recursively enumerable set of relations that defines \( G \), and that the set \( \bigcup T_n \) defines a recursively enumerable discriminating family for \( G \).

We can thus apply Theorem 5.7, which indicates that a word problem algorithm for \( G \) can be obtained from this data. \( \square \)

**Remark 5.8.** Any numbering type that is stronger than \( \Lambda_{WP} \) also makes of the space of marked groups a recursive metric space. An example of particular importance is the numbering type \( \Lambda_{WP} \wedge \Lambda_{FP} \), associated to the description “finite presentation and word problem algorithm”, and whose study, as we stated in the introduction, is of foremost importance. Notice that \( \{(\mathbb{Z}, 1)\} \) is a \( \Lambda_{WP} \wedge \Lambda_{FP} \)-decidable singleton. This shows that Markov’s Lemma cannot be applied with respect to \( \Lambda_{WP} \wedge \Lambda_{FP} \): the sequence \( (\mathbb{Z}/n\mathbb{Z}, 1) \) is a \( \Lambda_{WP} \wedge \Lambda_{FP} \)-computable sequence that converges effectively to \( (\mathbb{Z}, 1) \), but \( (\mathbb{Z}, 1) \) can still be distinguished from finite groups with respect to \( \Lambda_{WP} \wedge \Lambda_{FP} \).

The space of marked groups associated to the numbering type \( \Lambda_{WP} \wedge \Lambda_{FP} \) is thus a recursive metric space that is not effectively complete. (An interesting problem would then be to describe the numbering of \( G \) obtained by taking the effective completion of \( \Lambda_{WP} \wedge \Lambda_{FP} \).

5.3. **Two applications of a construction of Miller, failure of Moschovakis’ (B) condition for the space of marked groups.** In this section, we prove two important theorems that use variations on Miller’s example of a finitely presented group that is isolated from groups with solvable word problem.

The following theorem was already stated, it is Theorem 1.3.

**Theorem 5.9.** No algorithm can stop exactly on those sets of relations and irreations which are not wp-coherent.

The following theorem is one of our most important results, we have already explained how it represents the negation of an effective Axiom of Choice for the space of marked groups.

**Theorem 5.10** (Failure of an Effective Axiom of Choice for groups). There is no algorithm that, given a wp-coherent set of relations and irreations, produces a word problem algorithm for a marked group that satisfies those relations and irreations.

The proofs for those results will be similar: they rely on Miller’s constructions of a family of groups \( L_{P,Q} \) indexed by two subsets \( P \) and \( Q \) of \( \mathbb{N} \). For each of those theorems, we will find some conditions on the sets \( P \) and \( Q \) that are sufficient for the groups \( L_{P,Q} \) to provide proofs for the Theorems 5.9 and 5.10, and then include a lemma to prove that such sets \( P \) and \( Q \) do exist.

We start by detailing Miller’s construction.

5.3.1. **Miller’s construction.** We detail the construction of Miller as it was exposed in [Mil92]. This construction was first introduced in [Mil81].
Step 1. Given two subsets $P$ and $Q$ of $\mathbb{N}$, we consider the group $L_{P,Q}^1$ given by the following presentation:

$$\langle \varepsilon_0, e_1, e_2, \ldots | \varepsilon_0 = e_i, i \in P, e_1 = e_j, j \in Q \rangle$$

For simplicity, we shall always assume that $P$ contains 0 and $Q$ contains 1.

Notice that $L_{P,Q}^1$ is recursively presented with respect to the family $(e_i)_{i \in \mathbb{N}}$ if and only if $P$ and $Q$ are r.e. sets, and that $L_{P,Q}^1$ has solvable word problem with respect to the family $(e_i)_{i \in \mathbb{N}}$ if and only if $P$ and $Q$ are recursive sets.

In what follows, the sets $P$ and $Q$ will always be recursively enumerable, and thus $L_{P,Q}^1$ is recursively presented.

Step 2. Embed the recursively presented group $L_{P,Q}^1$ in a finitely presented group $L_{P,Q}^2$ using some strengthening of Higman’s Embedding Theorem. For our purpose, we need to know that:

- A finite presentation of $L_{P,Q}^2$ can be obtained from the recursive presentation of $L_{P,Q}^1$.
- If the group $L_{P,Q}^1$ has solvable word problem with respect to the family $(e_i)_{i \in \mathbb{N}}$, then the group $L_{P,Q}^2$ also has solvable word problem.
- The embedding of $L_{P,Q}^1$ into $L_{P,Q}^2$ is effective, i.e. there exists a recursive function that maps a natural number $n$ to a way of expressing the element $e_n$ as a product of the generators of $L_{P,Q}^2$.

Clapham’s version of Higman’s Embedding Theorem ([Cla67]) satisfies the required conditions for this step of the construction. Clapham’s Theorem is quoted precisely in Subsection 9.1.1.

Step 3. Embed the group $L_{P,Q}^2$ into a finitely presented group $L_{P,Q}^3$ with the following property: in any non-trivial quotient of $L_{P,Q}^3$, the image of the element $e_0e_1^{-1}$ is a non-identity element.

This is done as follows.

Consider a presentation $(x_1, \ldots, x_k | r_1, \ldots, r_l)$ for $L_{P,Q}^2$, denote $w$ a word on $\{x_1, \ldots, x_k, x_1^{-1}, \ldots, x_k^{-1}\}$ that defines the element $e_0e_1^{-1}$ in $L_{P,Q}^2$. The group $L_{P,Q}^2$ is defined by adding to $L_{P,Q}^2$, in addition to the generators $x_1, \ldots, x_k$ that are still subject to the relations $r_1, \ldots, r_l$, three new generators $a$, $b$ and $c$, subject to the following relations:

1. $a^{-1}ba = c^{-1}b^{-1}cbc$
2. $a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2$
3. $a^{-3}[w, b]a^3 = c^{-3}bc^3$
4. $a^{-(3+i)}x_i ba^{(3+i)} = c^{-(3+i)}bc^{(3+i)}$, $i = 1, \ldots, k$

To use Miller’s construction, we need to check the following points:

- If $w \neq 1$ in $L_{P,Q}^2$, then $L_{P,Q}^2$ is embedded in $L_{P,Q}^3$ via the natural map $x_i \mapsto x_i$.
- The presentation of $L_{P,Q}^3$ can be computed from the presentation of $L_{P,Q}^2$ together with the word $w$.
- If $L_{P,Q}^2$ has solvable word problem, then so does the group $L_{P,Q}^3$.
- The element $e_0e_1^{-1}$ has a non-trivial image in any non-trivial quotient of $L_{P,Q}^3$.

The second point is obvious. The last point is easily proven: remark that the third written relation, together with $w = 1$, implies the relation $b = 1$. This in turn implies that $c = 1$ thanks to the first relation, that $a = 1$ thanks to the second relation, and then that all $x_i$ also define the identity element because of the relations of (4).

The first and third points are proven using the fact that the group $L_{P,Q}^3$ can be expressed as an amalgamated product.

Consider the free product $L_{P,Q}^2 \ast F_{a,b}$ of $L_{P,Q}^2$ with a free group generated by $a$ and $b$, and the free group $F_{b,c}$ generated by $b$ and $c$. Then, provided that $w \neq 1$ in $L_{P,Q}^2$, the subgroup of $L_{P,Q}^2 \ast F_{a,b}$ generated by $b$ and the elements that appear to the left hand side in the equations (1) – (4) is a free group on $4 + k$ generators, which we denote $A$, and so is the subgroup $B$ of $F_{b,c}$ generated by $b$ and the elements that appear to the right hand side in the equations (1) – (4).

Thus the given presentation of $L_{P,Q}^3$ shows that it is defined as an amalgamated product of the form:

$$L_{P,Q}^3 \ast F_{a,b} = F_{b,c}$$

This proves both the fact that $L_{P,Q}^2$ embeds in $L_{P,Q}^3$, and that the word problem is solvable in $L_{P,Q}^3$ as soon as it is in $L_{P,Q}^2$, since to solve the word problem in an amalgamated product such as $L_{P,Q}^3$, it suffices to be able to solve the membership problem for $A$ in $L_{P,Q}^2 \ast F_{a,b}$ and for $B$ in $F_{b,c}$, we leave it to the reader to see that this can be done as soon as the word problem is solvable in $L_{P,Q}^2$.

Finally, we designate by $\Pi_{P,Q}$ the finite set of relations and irrelations that is composed of the relations of $L_{P,Q}^3$ and of a unique irrelation $w \neq 1$, where $w$ is the word that defines the element $e_0e_1^{-1}$ in $L_{P,Q}^3$.

Note that the set $\Pi_{P,Q}$ can be effectively produced from the codes for the r.e. sets $P$ and $Q$. 


This ends Miller’s construction.

5.3.2. First application: Miller’s Theorem. We include here a proof of Miller’s Theorem.

A pair of disjoint subsets $P$ and $Q$ of $\mathbb{N}$ are said to be recursively inseparable if there cannot exist a recursive set $H$ such that $P \subseteq H$ and $Q \subseteq H^c$, where $H^c$ denotes the complement of $H$ in $\mathbb{N}$.

We will need the following well known result:

**Lemma 5.11.** There exists a pair $(P, Q)$ of disjoints subsets of $\mathbb{N}$ that are recursively enumerable and recursively inseparable.

**Proof.** Consider an effective enumeration $\phi_0, \phi_1, \phi_2, \ldots$ of all recursive functions. Consider the set $P = \{ n \in \mathbb{N}, \phi_n(n) = 0 \}$ and the set $Q = \{ n \in \mathbb{N}, \phi_n(n) = 1 \}$. Those sets are obviously recursively enumerable. Suppose now that some recursive set $H$ contains $P$ but does not intersect $H$. Consider an index $n_0$ such that $\phi_{n_0}$ is a total function that computes the characteristic function of $H$.

If $\phi_{n_0}(n_0) = 0$, $n_0$ does not belong to $H$, but it belongs to $P$, this is not possible. But if $\phi_{n_0}(n_0) = 1$, then $n_0$ belongs to $Q$ and to $H$, which is also impossible because $H$ should not meet $Q$.

This is a contradiction, and thus the sets $P$ and $Q$ are indeed recursively inseparable. □

**Theorem 5.12** (Miller, [Mil92]). Suppose that $P$ and $Q$ are disjoints subsets of $\mathbb{N}$ that are recursively enumerable and recursively inseparable. Then the set $\Pi_{P,Q}$ is coherent, but not wp-coherent.

**Proof.** Suppose that a group $K$ satisfies the relations and irrellations of $\Pi_{P,Q}$, and that is has solvable word problem.

Using the word problem algorithm for $K$, given an integer $i$ in $\mathbb{N}$, we can solve the questions “is $e_0 = e_i$ in $K$”, since, by the properties of Miller’s construction, an expression of the element $e_i$ in terms of the generators of $L^3_{P,Q}$, and thus of $K$, can be effectively found from $i$.

The set $\{ i \in \mathbb{N}, e_i = e_0 \}$ is thus a recursive set that contains $P$. And it is disjoint from $Q$, because we have assumed that $e_0 \neq e_1$ in $K$.

This contradicts the fact that $P$ and $Q$ are recursively inseparable. □

5.3.3. Proof of Theorem 5.9. We first prove Theorem 5.9:

**Theorem.** No algorithm can stop exactly on those sets of relations and irrellations which are not wp-coherent.

**Proof.** Given r.e. disjoints sets, we apply Miller’s construction to obtain the set $\Pi_{P,Q}$ of relations and irrellations.

By Theorem 5.12, if the sets $P$ and $Q$ are recursively enumerable, recursively inseparable sets, then $\Pi_{P,Q}$ is not wp-coherent.

On the contrary, if $P$ and $Q$ are both recursive sets, we have noted that $L^3_{P,Q}$ itself has solvable word problem, and thus $\Pi_{P,Q}$ is wp-coherent.

Because the set $\Pi_{P,Q}$ can be constructed from the codes for $P$ and $Q$, an algorithm that stops exactly on those sets of relations and irrellations which are not wp-coherent would produce, through Miller’s construction, an algorithm that, given a pair of r.e. sets $P$ and $Q$ that are either recursively inseparable or recursive, would stop if and only if those sets are recursively inseparable. We prove in the next lemma, Lemma 5.13, that such an algorithm cannot exit, this ends the proof of our theorem. □

**Lemma 5.13.** There is no algorithm that, given the code for two recursively enumerable and disjoint subsets of $\mathbb{N}$, that are either recursive or recursively inseparable, stops only if they are recursively inseparable.

**Proof.** Fix two recursively enumerable and recursively inseparable subsets $P$ and $Q$ of $\mathbb{N}$, that exist by Lemma 5.11.

Consider an effective enumeration $M_0, M_1, M_2, \ldots$ of all Turing machines. For each natural number $n$, define a pair of recursively enumerable sets $P_n$ and $Q_n$ defined as follows:

To enumerate $P_n$, start a run of $M_n$.

While this run lasts, an enumeration of $P$ gives the first elements of $P_n$. If $M_n$ halts after $k$ computation steps, stop the enumeration of $P$.

Thus if $M_n$ halts, the set $P_n$ is a finite set. On the contrary, if $M_n$ does not stop, $P_n$ is identical to $P$.

The set $Q_n$ is defined similarly, replacing $P$ by $Q$ in its definition.

One then easily sees that the sets $P_n$ and $Q_n$ are uniformly recursively enumerable, and that $P_n$ and $Q_n$ are recursively inseparable if and only if $M_n$ does not halt.

Since no algorithm can stop exactly on the indices of non-halting Turing machines, the lemma is proved. □
5.3.4. **Proof of Theorem 5.10.** We now prove Theorem 5.10:

**Theorem** (Failure of an effective Axiom of Choice for groups). There is no algorithm that, given a wp-coherent set of relations and irrelnations, produces a word problem algorithm for a marked group that satisfies those relations and irrelnations.

Note that, just as Miller’s Theorem (Theorem 5.12) was a strengthening of the Boone-Novikov Theorem on the existence of a finitely presented group with unsolvable word problem, this theorem strengthens the Boone-Rogers Theorem which states that the word problem does not have a uniform solution amongst finitely presented groups with solvable word problem.

**Proof.** We will build in Lemma 5.14 a pair of sequences \((P_n)_{n \in \mathbb{N}}\) and \((Q_n)_{n \in \mathbb{N}}\) such that:

- The sequences \((P_n)_{n \in \mathbb{N}}\) and \((Q_n)_{n \in \mathbb{N}}\) consist only of disjoint recursive sets;
- The sequences \((P_n)_{n \in \mathbb{N}}\) and \((Q_n)_{n \in \mathbb{N}}\) are uniformly r.e., but not uniformly recursive;
- For any sequence \((H_n)_{n \in \mathbb{N}}\) of uniformly recursive sets, there must be some index \(n_0\) such that either \(H_{n_0}\) does not contain \(P_{n_0}\), or \(H_{n_0}\) does not contain \(Q_{n_0}\).

We apply Miller’s construction to this sequence to obtain a computable sequence \((\hat{P}_n, \hat{Q}_n)_{n \in \mathbb{N}}\) of finite sets of relations and irrelnations.

Suppose by contradiction that there is an algorithm \(A\) as in the theorem. For each natural number \(n\), the set \(\Pi_{P_n, Q_n}\) is wp-coherent, because \(P_n\) and \(Q_n\) are recursive. Thus the algorithm \(A\) can be applied to \(\Pi_{P_n, Q_n}\), to produce the word problem algorithm for a group that satisfies the relations and irrelnations of \(\Pi_{P_n, Q_n}\). Denote \(G_n\) the group defined by this algorithm.

For each \(n\), the set \(H_n = \{i \in \mathbb{N}, e_i = e_0\text{ in }G_n\}\) is then a recursive set, and this in fact holds uniformly in \(n\).

But of course, one has the inclusions \(P_n \subseteq H_n\) and \(Q_n \subseteq H^c_n\). This contradicts the properties of the sequences \((P_n)_{n \in \mathbb{N}}\) and \((Q_n)_{n \in \mathbb{N}}\).

**Lemma 5.14.** There exists a pair of sequences \((P_n)_{n \in \mathbb{N}}\) and \((Q_n)_{n \in \mathbb{N}}\) such that:

- The sequences \((P_n)_{n \in \mathbb{N}}\) and \((Q_n)_{n \in \mathbb{N}}\) consist only of disjoint recursive sets;
- The sequences \((P_n)_{n \in \mathbb{N}}\) and \((Q_n)_{n \in \mathbb{N}}\) are uniformly r.e., but not uniformly recursive;
- For any sequence \((H_n)_{n \in \mathbb{N}}\) of uniformly recursive sets, there must be a some index \(n_0\) such that either \(H_{n_0}\) does not contain \(P_{n_0}\), or \(H_{n_0}\) does not contain \(Q_{n_0}\).

**Proof.** Fix a pair \((P, Q)\) of recursively enumerable but recursively inseparable sets.

Denote by \(f\) a recursive function that enumerates \(P\). Denote by \(\hat{f}\) an increasing function extracted from \(f\), defined as follows: \(\hat{f}(0) = f(0), \hat{f}(1) = f(\min\{k \in \mathbb{N}, f(k) > \hat{f}(0)\}), \hat{f}(2) = f(\min\{k \in \mathbb{N}, f(k) > \hat{f}(1)\})\), etc. It is clear that \(\hat{f}\) thus defined is recursive. It is well defined because \(P\) is necessarily infinite.

Consider an effective enumeration \(M_0, M_1, M_2...\) of all Turing machines.

We define the set \(P_n\) thanks to a run of the machine \(M_n\). While this run lasts, use the function \(\hat{f}\) to enumerate elements of \(P\) in increasing order. If the machine \(M_n\) halts in \(k\) steps, the last element of \(P\) that was added to \(P_n\) is \(\hat{f}(k-1)\). In this case, we chose that the set \(P_n\) should be the set

\[ P_n = P \cap \{0, 1, ..., \hat{f}(k-1)\}. \]

This set can be enumerated using the function \(f\), by keeping only the elements it produces that are below \(\hat{f}(k-1)\).

Because the process described above is effective, it is clear that given an integer \(n\), one can build a recursive enumeration of \(P_n\), and thus the sequence \((P_n)_{n \in \mathbb{N}}\) is uniformly r.e.

If the machine \(M_n\) never stops, the set \(P_n\) is exactly the image of the increasing and recursive function \(\hat{f}\), it is thus recursive. If the machine \(M_n\) stops, the set \(P_n\) is finite, it is then also recursive.

The sequence \((Q_n)_{n \in \mathbb{N}}\) is defined exactly as \((P_n)_{n \in \mathbb{N}}\), replacing the function \(f\) that enumerates \(P\) by a function \(g\) that enumerates \(Q\). Thus the sequence \((Q_n)_{n \in \mathbb{N}}\) is also a sequence of recursive sets, that are uniformly recursively enumerable.

For each \(n\), one has \(P_n \subseteq P\) and \(Q_n \subseteq Q\), and thus the sets \(P_n\) and \(Q_n\) are indeed disjoints. All that is left to show is that the sequences \((P_n)_{n \in \mathbb{N}}\) and \((Q_n)_{n \in \mathbb{N}}\) satisfy the last condition of the lemma: there cannot exist a sequence \((H_n)_{n \in \mathbb{N}}\) of uniformly recursive sets for which the following inclusions hold:

\[ \forall n \in \mathbb{N}, P_n \subseteq H_n \& Q_n \subseteq H^c_n \]

We proceed by contradiction, and suppose such a sequence exists.

Consider the set \(U\) of indices \(n\) for which \(P \not\subseteq H_n\).

**Claim:** The halting problem is solvable on the set of Turing machines whose index is in \(U\).
Given an integer \( n \) in \( U \), we can find an integer \( y \) such that \( y \in P \) but \( y \notin H_n \). Since we have assumed that \( P_n \subseteq H_n \), \( y \) is an element of \( P \), but not of \( P_n \).

Denote by \( k \) an integer for which \( f(k) = y \), this exists and can be computed, since \( f \) enumerates \( P \).

By construction of \( P_n \), the fact that \( y \notin P_n \) indicates either that the Turing machine \( M_n \) does not halt, or that, if it does stop, it must be in strictly less than \( k \) steps. This information is sufficient to solve the halting problem in \( U \). This proves the claim.

Denote by \( V \) the set of indices \( n \) for which \( Q \not subseteq H^n \). The definitions of the sets \( P_n \) and \( Q_n \) being symmetric, the halting problem is solvable on the set of Turing machines whose index is in \( V \).

Because the sets \( U \) and \( V \) are easily seen to be recursively enumerable, the previous results imply that the halting problem is solvable on \( U \cup V \). Indeed, given a point \( n \) of \( U \cup V \), one can find at least one of \( U \) and \( V \) to which \( n \) belongs, and apply the method of resolution of the halting problem there.

Because the halting problem is solvable on \( U \cup V \), its complement must be non-empty. But for any \( n \) in \((U \cup V)^c\), one has:

\[
\begin{align*}
P \subseteq H_n \\
Q \subseteq H^n \end{align*}
\]

Since the set \( H_n \) is recursive, this contradicts the fact that \( P \) and \( Q \) are recursively inseparable.

Theorem 5.10 allows us to prove that the space of marked groups does not satisfy Moschovakis' condition (B), and thus that Theorem 4.37 cannot be applied to the space of marked groups.

**Corollary 5.15.** The space of marked group equipped with the numbering \( \nu_{WP} \) does not satisfy Moschovakis' condition (B).

**Proof.** A RMS \((X,d,\nu)\) satisfies Moschovakis' condition (B) if there exists an algorithm \( A \) that takes as input the description of a \( \nu \)-semi-decidable set \( Y \) and the description of an open ball \( B(x,r) \) in \( X \), such that those set intersect, and produces a point that belongs to their intersection.

In \( G_k \), apply such an algorithm to a basic clopen subset \( \Omega_{r_i,s_i} \), and to an open ball that contains all of \( G_k \) -any open ball of radius \( r \geq 1 \). This yields a program that takes as input a set of relations and irrelations that is wp-coherent, and produces the \( \nu_{WP} \)-name of a point that belongs to it. The existence of such an algorithm was proven impossible in Theorem 5.10. \( \Box \)

6. Some results of effective descriptive set theory

We now apply the results of the previous sections to decision problems for groups described by word problem algorithms. The numbering type \( \Lambda_{WP} \) of \( G \) is most often used implicitly, a r.e. set of marked groups is thus a \( \Lambda_{WP} \)-r.e. set of marked groups, and so on.

6.1. Markov's Lemma for groups. In this subsection, we rephrase some results that are true of effectively complete recursive metric spaces to the special case of the space of marked groups.

**Proposition 6.1.** If a computable sequence \( (G_n)_{n \in \mathbb{N}} \) of marked groups effectively converge in \( G_k \), its limit has solvable word problem.

**Proof.** This follows directly from the fact that \( (G_{WP},d,\nu_{WP}) \) is effectively complete (Corollary 4.16). \( \Box \)

This proposition admits a converse.

**Proposition 6.2.** Suppose a \( \Lambda_{WP} \)-computable sequence \( (G_n)_{n \in \mathbb{N}} \) converges to a marked group \( H \) which has solvable word problem. Then a sequence \( (G_{\psi(n)})_{n \in \mathbb{N}} \) can be extracted from \( (G_n)_{n \in \mathbb{N}} \), such that: the extraction function \( \psi \) is recursive, and for all \( n \), \( d(G_{\psi(n)},H) \leq 2^{-n} \).

**Proof.** Define \( \psi(n) \) to be the least integer \( p \) such that \( G_p \) and \( H \) agree on the first \( k \) terms of their binary expansions. The hypotheses of the proposition ensure this defines a recursive function. \( \Box \)

We can now state Markov's Lemma applied to the space of marked groups.

**Lemma 6.3** (Markov's Lemma for groups). If \( (G_n)_{n \in \mathbb{N}} \) is a \( \Lambda_{WP} \)-computable sequence of marked groups that effectively converges to a marked group \( H \), with \( G_n \neq H \) for each \( n \), then there is a \( \Lambda_{WP} \)-computable sequence of marked groups \( (\Gamma_p)_{p \in \mathbb{N}} \), such that each \( \Gamma_p \) belongs to the set \( \{G_n; n \in \mathbb{N}\} \cup \{H\} \), and the set \( \{p \in \mathbb{N}; \Gamma_p = H\} \) is not recursively enumerable.
This result is a direct consequence of Markov’s Lemma (Lemma 4.26), since the space of marked groups is an effectively complete recursive metric space. We add here a direct proof of it, because it is very simple and does not require the notion of an “algorithm of passage to the limit”.

**Proof.** Let $f$ be a recursive modulus of convergence for $(G_n)_{n \in \mathbb{N}}$. Consider an effective enumeration $M_0, M_1, M_2, \ldots$ of all Turing Machines. Fix some natural number $l$, we define a word problem algorithm $A^l_{WP}$ as follows. To decide whether a word $w$ defines the identity, start a run of $M_l$. If, after $|w|$ steps, it still has not stopped, answer $A^l_{WP}(w)$ (by Proposition 6.1, $H$ has solvable word problem). If $M_l$ stops in $p$ steps, with $p < |w|$, find the first integer $n$ such that $f(n) < 2^{-p}$. In this case, we choose $A^l_{WP}$ to define the $n$-th group of the converging sequence, $G_n$, thus it should answer $A^l_{WP}(w)$.

It is easy to see that this definitions is indeed coherent and that $A^l_{WP}$ defines the group $H$ if and only if the $l$-th Turing Machine does not halt. □

In what follows, we will often use Markov’s Lemma for groups together with the following obvious proposition, which is a reformulation of Proposition 4.20.

**Proposition 6.4.** If a marked group $G$ with solvable word problem is adherent to a r.e. set $C$ of marked groups, then there is a $\Lambda_{WP}$-computable sequence of marked groups in $C$ that effectively converges to $G$.

This proposition, while very easy, is often useful, it can be used with $C$ being: the class of finite groups, of hyperbolic groups, of all the markings of a given abstract group $G$, etc.

**6.2. List of properties for which the Correspondence holds.** We will now proceed to list a series of group properties for which the correspondence between the first level of the Kleene–Mostowski hierarchy and that of the Borel hierarchy holds perfectly: each quoted clopen property is $\Lambda_{WP}$-decidable, each quoted open property is $\Lambda_{WP}$-semi-decidable but not $\Lambda_{WP}$-decidable, each closed property is $\Lambda_{WP}$-co-semi-decidable and not $\Lambda_{WP}$-decidable, and for properties that are neither closed not open, no partial algorithm exist.

This list is presented as a table which appears in SubSection 6.2.9.

Note that most affirmations which appear in this table are obvious, easy applications of Markov’s Lemma in each case provide the desired results. What we use is in fact the following immediate consequence of Markov’s Lemma:

**Proposition 6.5.** Suppose that a subset $A$ of $G$ is effectively not open, i.e. that there is a computable sequence of marked groups that do not belong to $A$, which converges to a marked group in $A$. Then $A$ cannot be $\Lambda_{WP}$-semi-decidable.

Those affirmations that appear in SubSection 6.2.9 which are not accompanied by either a reference, or for which a proof does not appear in the following paragraphs, are left to the reader.

6.2.1. **Residually finite groups.** It is known that the set of residually finite groups is not closed, because the adherence of the set of finite groups, which is the set LEF groups, strictly contains the set of residually finite groups. The semi-direct product $\mathbb{Z} \ltimes \mathbb{S}_\omega$ is the limit of the sequence of finite groups $\mathbb{Z}/n\mathbb{Z} \ltimes \mathbb{S}_n$, as $n$ goes to infinity; it is not residually finite since it contains an infinite simple group. Note that the described sequence effectively converges. (This example comes from [VG98]).

It is also well known that the set of residually finite groups is not open: non-abelian free groups are limits of non-residually finite groups. For instance, it follows from well known results on the Burnside problem that free Burnside groups of sufficiently large exponent are infinite groups that are not residually finite, and that, if $S$ denotes a basis of a non-abelian free group, the sequence $(\mathbb{F}/\mathbb{F}^n, S)$, starting with $n \gg 1$, is $\Lambda_{WP}$-computable and converges to $(\mathbb{F}, S)$ as $n$ goes to infinity.

6.2.2. **Amenable groups.** Amenable groups do not form a closed set, since free groups are limits of finite groups. They do not form an open set either. An interesting example that proves this comes from [BE15]: there exists a sequence of markings of $\mathbb{F}_2 \wr \mathbb{Z}$ that converges to a marking of $\mathbb{Z}^2 \wr \mathbb{Z}$. (See Example 7.4 in [BE15].) This sequence is $\Lambda_{WP}$-computable.

6.2.3. **Having sub-exponential growth.** The set of finitely generated groups with sub-exponential growth is neither closed nor open. This was proved by Grigorchuk in [Gri85]: there is constructed a family of groups $G_\omega$, $\omega \in \{0, 1, 2\}^\mathbb{N}$, for which it is explained when $G_\omega$ has intermediate or exponential growth, and for which it is proved that the convergence in the space of marked groups of a sequence of groups $(G_\omega_n)_{n \in \mathbb{N}}$ coincides with the convergence in $(\{0, 1, 2\}^\mathbb{N}$ (for the product topology) of the sequence $(\omega_n)_{n \in \mathbb{N}}$. The sequence $(G_\omega_n)_{n \in \mathbb{N}}$ is $\Lambda_{WP}$-computable when the sequence $(\omega_n)_{n \in \mathbb{N}}$ is computable for the usual numbering of the Cantor space.
6.2.4. **Orderable groups.** A finitely generated group $G$ is orderable if there exists a total order $\leq$ on $G$ which is compatible with the group operation:

$$\forall x, y, a, b \in G, \ a \leq b \implies xay \leq xby.$$ 

The following well known characterization of orderable groups was already used in [BCR19] in order to study the complexity of the property “being orderable”:

**Proposition 6.6.** *A group $G$ is orderable if and only if for any finite set $\{a_1, a_2, \ldots, a_n\}$ of non-identity elements of $G$, there are signs $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$, such that the sub-semi-group generated by $\{a_1^{\epsilon_1}, a_2^{\epsilon_2}, \ldots, a_n^{\epsilon_n}\}$ does not contain the identity of $G$.***

It is straightforward to notice that this characterization provides a way of recognizing word problem algorithms for non orderable groups, and that it shows that the set of orderable groups is closed.

**Proposition 6.7.** *The set of orderable groups is $\Lambda_{WP}$-co-semi-decidable and closed in $\mathcal{G}$.***

6.2.5. **Virtually cyclic groups; virtually nilpotent groups; polycyclic groups.** For the three properties of being virtually cyclic, virtually nilpotent or polycyclic, we use the following lemma. In what follows, denote by $\Lambda_{r.p.}$ the numbering type of $\mathcal{G}$ associated to recursive presentations. We leave it to the reader to define it, it is for instance the numbering induced by the embedding of $\mathcal{G}$ in a countable union of Cantor spaces, when this union is equipped with the well known numbering associated to lower semi-computable sequences.

**Lemma 6.8.** *Suppose that $P$ is a $\Lambda_{WP}$-semi-decidable (resp. $\Lambda_{r.p.}$-semi-decidable) subset of $\mathcal{G}$, and that $Q$ is a $\Lambda_{r.p.}$-semi-decidable subset of $\mathcal{G}$. Then the set of groups that have a finitely generated normal subgroup in $P$, and such that the quotient by this normal subgroup is in $Q$, is $\Lambda_{WP}$-semi-decidable (resp. $\Lambda_{r.p.}$-semi-decidable).***

Note that in this statement, the normal subgroup should be finitely generated as a group, and not only finitely generated as a normal subgroup.

**Proof.** Given a word problem algorithm for a group $G$ generated by a family $S$, we proceed as follows.

Enumerate all finite subsets of $G$.

There is an effective procedure that recognizes those finite subsets of $G$ that generate a normal subgroup. Indeed, consider a finite set $A$ in $G$, and the subgroup $H$ it generates. The subgroup $H$ is normal in $G$ if and only if for each $a$ in $A$ and each $s$ in $S$, the elements $s^{-1}as$ and $sas^{-1}$ both belong to $H$. An exhaustive search for ways of expressing $s^{-1}as$ and $sas^{-1}$ as products of elements of $A$ will terminate if indeed those elements belong to $H$.

For each finite subset $A$ which generates a normal subgroup $H$ of $G$, we can obtain a word problem algorithm (resp. a recursive presentation) for $H$ thanks to the word problem algorithm for $G$ (resp. thanks to a recursive presentation for $G$), and a recursive presentation for the quotient $G/H$, since an enumeration of the relations of $G$ together with an enumeration of the elements of $H$ yields a recursive presentation of $G/H$.

The hypotheses of the lemma then allow us to recognize when the group $H$ is in $P$ and the quotient $G/H$ is in $Q$.

This lemma can be used directly to show that the properties of being virtually nilpotent or virtually cyclic are $\Lambda_{WP}$-semi-decidable.

**Corollary 6.9.** *The set of virtually cyclic groups is open in $\mathcal{G}$ and $\Lambda_{WP}$-semi-decidable.***

**Proof.** Apply Lemma 6.8 with $P$ being the set of cyclic groups and $Q$ the set of finite groups, to prove that the set of virtually cyclic groups $\Lambda_{WP}$-semi-decidable.

Analyzing the way the algorithm that stops on word-problem algorithms for virtually cyclic groups thus obtained works, one sees that the proof it produces of the fact that a group is virtually cyclic consists in finitely many relations. Those relations define an open set that contains only virtually cyclic groups.

**Corollary 6.10.** *The set of virtually nilpotent groups is open in $\mathcal{G}$ and $\Lambda_{WP}$-semi-decidable.***

**Proof.** Apply Lemma 6.8 with $P$ being the set of nilpotent groups and $Q$ the set of finite groups.

Again, this shows that the set of virtually nilpotent groups is $\Lambda_{WP}$-semi-decidable, and analyzing the algorithm given by Lemma 6.8, we see that the set of virtually nilpotent groups is open in $\mathcal{G}$.

Note that we have chosen those two properties because they both admit equivalent definitions in terms of asymptotic geometry: by a well-known theorem of Gromov, a group is virtually nilpotent if and only if it has polynomial growth, and a group is virtually cyclic if and only if it has zero or two ends, in the sense of Stallings. It is thus remarkable that both of these properties can be recognized thanks to only finitely many relations.
Corollary 6.11. The set of polycyclic groups is open in $\mathcal{G}$ and $\Lambda_{WP}$-semi-decidable.

Proof. (Sketch) Iterate Lemma 6.8 with $Q$ being the set of cyclic groups, and $P$ being the set polycyclic groups with a subnormal series of length $n$, to obtain the result.

We leave it to the reader to prove that this proof also gives the fact that the set of polycyclic groups is open in $\mathcal{G}$. □

6.2.6. Groups with infinite conjugacy classes. A group $G$ has infinite conjugacy classes (ICC) if for each non identity element $g$ of $G$ the conjugacy class $\{xgx^{-1}, x \in G\}$ of $g$ is infinite.

Proposition 6.12. The set of ICC groups is $\Lambda_{WP}$-co-semi-decidable, and it is closed in $\mathcal{G}$.

Proof. Given a word problem algorithm for a group $G$ over a generating family $S$ and an element $g$ of $G$, it is possible to prove that the conjugacy class of $g$ is finite: define a sequence of sets $(A_n)_{n \in \mathbb{N}}$ by

$$A_0 = \{g\},$$

$$A_{n+1} = \{s^{-1}xs; s \in S \cup S^{-1}, x \in A_n\}.$$ 

The conjugacy class of $g$ is finite if and only if there exists an integer $n$ such that $A_n = A_{n+1}$. A blind search for such an integer will terminate if it exists.

This shows that the set of non-ICC groups is $\Lambda_{WP}$-semi-decidable, one easily checks that this also shows that this set is open. □

6.2.7. Hyperbolic groups. Let $\delta$ be a positive real number. A marked group $G$ is $\delta$-hyperbolic if the triangles (defined thanks to the word metric in $G$) are $\delta$-thin, that is to say if for any three elements $g_1$, $g_2$ and $g_3$ of $G$, the geodesic that joins $g_1$ to $g_2$ stays in a $\delta$-neighborhood of the geodesics that join respectively $g_2$ and $g_3$ and $g_1$ and $g_3$.

Proposition 6.13. Being $\delta$-hyperbolic is a closed and $\Lambda_{WP}$-co-semi-decidable property.

Proof. A group is not $\delta$-hyperbolic if it admits a triangle that is not $\delta$-thin. The fact that this triangle is not $\delta$-thin can be seen in a sufficiently large ball of the Cayley graph of $G$, and thus any group that corresponds to $G$ on this ball is also not $\delta$-hyperbolic. It is easy to see that this condition can be effectively checked. □

Remark that being $\delta$-hyperbolic is a marked group property, but not a group property, as can be seen from the fact that there exists a sequence of markings of $\mathbb{Z}$ that converges to a marking of $\mathbb{Z}^2$, which is not hyperbolic.

A group is Gromov hyperbolic if any of its marking is $\delta$-hyperbolic, for some $\delta$ that can depend on the marking.

The set of Gromov hyperbolic groups is neither open nor closed in $\mathcal{G}$, but the previous proposition implies that it is a union of closed sets.

Corollary 6.14. The set of hyperbolic groups is a $F_\sigma$ subset of $\mathcal{G}$, and it is effectively not closed and effectively not open.

6.2.8. Sofic groups. The set of sofic groups is known to be closed in $\mathcal{G}$. However, whether it is all of $\mathcal{G}$ or a strict subset of $\mathcal{G}$ is still an open problem. We ask:

Problem 6.15. Is there an algorithm that recognizes word problem algorithms for non-sofic groups?

6.2.9. Table of results. The following table gathers our examples. Remark that for each subset of $\mathcal{G}$ that appears in this table, four properties of this set are expressed: whether or not it is open, whether or not it is closed, whether or not it is $\Lambda_{WP}$-semi-decidable, and whether or not it is $\Lambda_{WP}$-co-semi-decidable.
7. Recognizing groups from word problem algorithms

We will now include some results that concern the study of the isomorphism problem for groups described by word problem algorithms. We proceed step by step, studying first which marked groups are recognizable when described by word problem algorithms, then proceeding to ask which abstract groups are recognizable when described by word problem algorithms, and ending with the study of the isomorphism problem.

A group (marked or abstract) is called recognizable for $\Lambda_{WP}$ if the property “being isomorphic to this group” is $\Lambda_{WP}$-decidable. Semi-recognizable or co-semi-recognizable groups are defined similarly.

As we will see, the finite sets of groups which have solvable isomorphism problem are characterized by the topology of $G$.

7.1. Marked recognizability. The study of marked recognizability corresponds to the study of properties which are singletons in the space $G$. A singleton is closed in $G$, and, given two marked groups $G$ and $H$, it is always possible to prove that they are different. This implies in particular that the isomorphism problem is always solvable for finite sets of marked groups, contrary to what happens for groups described by recursive presentations (see [Rau21]).

The singletons which are open in $G$ are its isolated points.

Those were studied by Cornulier, Guyot and Pitsch in [dCGP07]. We first quote a Lemma from [dCGP07] which shows that if a group admits an isolated marking, all of its markings are isolated.

Lemma 7.1. ([dCGP07], Lemma 1) Consider two marked groups $G_1 \in G_{m_1}$, $G_2 \in G_{m_2}$. Suppose that they are (abstractly) isomorphic. Then there are clopen neighborhoods $V_i$, $i = 1, 2$ of $G_i$ in $G_{m_i}$, and a homeomorphism $\phi : V_1 \to V_2$ mapping $G_1$ to $G_2$ and preserving isomorphism classes, i.e. such that, for every $H \in V_1$, $\phi(H)$ is isomorphic to $H$ (as abstract groups).

Because of this, being isolated can be seen as a group property. Say that a group $G$ admits a finite discriminating family if there is a finite set $X$ of non-identity elements such that any non-trivial normal subgroup of $G$ contains an element of $X$. A group which admits a finite discriminating family is called finitely discriminable.

Proposition 7.2. ([dCGP07], Proposition 2) A group is isolated if and only if it is finitely presented and finitely discriminable.

If a basic clopen neighborhood $\Omega_{r_1,\ldots,r_n; s_1,\ldots,s_n}$ of $G_k$ is a singleton $\{G\}$, a presentation of $G$ is given by $\langle S r_1,\ldots, r_n \rangle$, and the set $\{s_1,\ldots, s_n\}$ defines a finite discriminating family for $G$. Note that in the vocabulary of B. H. Neumann in [Neu73], the triple $(S; r_1,\ldots, r_n; s_1,\ldots, s_n)$ is an absolute presentation for $G$. Isolated groups have solvable word problem, by Theorem 5.7.

We have already remarked that the basic clopen sets $\Omega_{r_1,\ldots,r_n; s_1,\ldots,s_n}$ define decidable properties, we thus have:

Proposition 7.3. Isolated groups are recognizable as marked groups.
Obvious examples of isolated groups are the finite groups (for which the whole group forms a finite discriminating family), and the finitely presented simple groups (for which any non-trivial element is a discriminating family). More examples can be found in [dCPG07]. A nice example which is not finite and not simple is Thompson’s group F: it is not abelian, but all its proper quotients are, thus any non-identity element of its derived subgroup forms a finite discriminating family.

Now we conjecture:

**Conjecture 7.4.** Marked isolated groups are the only marked groups that can be recognized for \( \Lambda_{WP} \), i.e. the only marked groups that define \( \Lambda_{WP} \)-decidable singletons of \( G \).

This is an interesting conjecture because it raises problems of two different kinds. First of all, asking whether or not a decidable singleton must be clopen in \( G_{WP} \) is a peculiar instance of our Main Conjecture (Conjecture 0.5). On the other hand, it remains an open problem to know whether it is possible that a group with solvable word problem that is not isolated in \( G \) be isolated in \( G_{WP} \): this is an example of an instance where it is unclear how the topology of \( G \) changes when looking only at groups with solvable word problem.

### 7.2. Abstract Recognizability.

#### 7.2.1. First results.

For an abstract group \( G \), recall that \([G]_k\) designates the set of all its markings in \( G_k \), and \([G]_k\) the set of all its markings in \( G \). The study of abstract recognizability in \( G_k \) is the study of the decidability of the properties that can be written \([G]_k\) for some group \( G \). As before, we will first discuss where these properties lie in the Borel hierarchy, before trying to obtain decidability results.

The open and closed isomorphism classes of groups are completely described by the following:

**Proposition 7.5.** Let \( G \) be a finitely generated group. The set \([G]_k\) is open if and only if \( G \) is isolated, and \([G]_k\) is closed if and only if it is finite.

**Proof.** The first point states that \([G]_k\) is open if and only if all its points are open. Suppose that \([G]_k\) is open, while \( G \) is not isolated. It means that there exists a sequence of groups in \([G]_k\) that converge to some marking of \( G \). But then, by Lemma 7.1, all markings of \( G \) must be adherent to \([G]_k\). This implies that \([G]_k\) should have no isolated points. This should also hold in some clopen neighborhood of a marking of \( G \). But any perfect set (closed and without isolated points) in a Polish space is uncountable, by a well known result which is due to Cantor in the case of subsets of \( \mathbb{R} \). This would imply that \([G]_k\) should be uncountable, a contradiction.

Now suppose that \([G]_k\) is infinite. By compactness of \( G_k \), it must have an accumulation point. If \([G]_k\) is closed, we obtain a sequence of markings of \( G \) that converges in \([G]_k\), and the same contradiction as above arises. \( \Box \)

For any \( k \), finite groups have finitely many markings in \( G_k \). On the other hand, for \( G \) a group of rank \( k \), generated by a family \((g_1, \ldots, g_k)\), setting ourselves in \( G_{k+1} \), the set of generating families \((g_1, \ldots, g_k, w)\), where \( w \) ranges over all elements of \( G \), defines infinitely many markings of \( G \), as soon as \( G \) is infinite. This yields:

**Corollary 7.6.** \([G]_k\) is clopen for any \( k \) if and only if \( G \) is finite. And any finite group is abstractly recognizable.

Finitely generated free groups of Hopfian varieties provide examples of groups with finitely many markings in some \( G_k \), indeed, a free group of rank \( k \) in a Hopfian variety has a single \( k \)-marking. Those include the free groups of varieties of nilpotent groups, or of the variety of metabelian groups. However, we do not know whether an isolated group can have finitely many markings in \( G_k \), such a group would be abstractly recognizable amongst \( k \)-marked groups.

**Example 7.7.** Limits in \( G_3 \) of markings of the rank two free group \( F_2 \) consist in all HNN extensions of the form

\[
\langle a, b, t \mid t^{-1} wt = w \rangle
\]

where \( w \) is any element of the free group on \( a \) and \( b \) that is not a proper power (\( w \) can be trivial, in which case the defined group is free of rank three). This follows from the results of [FGM+98].

We finally introduce the “preform” relation.

By Proposition 7.1, if a marking of a group \( H \) is adherent to \([G] \), then all markings of \( H \) will be adherent to \([G] \). Define a binary relation \( \preceq \) on the set of finitely generated (abstract) groups by setting \( G \preceq H \) if and only if some marking of \( H \) is adherent to \([G] \) in \( G \). In this case, we say that \( G \) *preforms* \( H \). This relation was introduced, named and studied by Laurent Bartholdi and Anna Erschler in [BE15], some of their results will be useful here. In particular, they show that the relation \( \preceq \) is a pre-order on the set of finitely generated groups, i.e. it is transitive and reflexive, but it is not an order, because non-isomorphic groups \( G \) and \( H \) can satisfy both \( G \preceq H \) and \( H \preceq G \).
We now investigate decidability issues, using the previous topological results.

All possible generating families of an isolated group can be effectively listed, and this can be done while keeping track of a finite discriminating family (expressing its elements in terms of products of elements of each generating family). And more generally, if some marking of a group $G$ is isolated inside a class $C$ of abstract groups, then one can define the concept of a “discriminating family with respect to $C$”, and such discriminating families can be transposed from a generating set to another.

The following proposition directly follows from these considerations:

**Proposition 7.8.** Isolated groups are semi-recognizable as abstract groups, and more generally, if a marking of a group $G$ is open in $C$, for some class $C$ of abstract groups, then $[G]$ is open in $C$, and $G$ is semi-recognizable as an abstract group in $C$.

By the description of open classes of isomorphism in $G_k$, whether isolated groups are the only abstract groups that are $\Lambda_{WP}$-semi-recognizable depends on a conjecture that we already proposed when talking about marked recognizability, Conjecture 7.4.

To make use of the statement about the closeness of isomorphism classes in $G_k$, we must first obtain an effective version of this statement. Now, in $G_k$, if $G$ is a group with solvable word problem, all markings of $G$ can be enumerated, and thus the class $[G]_k$ satisfies the hypothesis of Proposition 6.4: a group which is adherent to $[G]_k$ is the effective limit of a sequence of groups in $[G]_k$ if and only if it has solvable word problem. Thus all we have to do is to prove that some point adherent to $[G]_k$ has solvable word problem to conclude that $G$ is not recognizable for $\Lambda_{WP}$.

**Proposition 7.9.** Let $G$ and $H$ be groups with solvable word problem. If $G \not\cong H$, then no algorithm can tell $H$ from $G$, in particular “being equal to $H$” cannot be semi-decidable.

*Proof.* This is a direct application of Markov’s Lemma, as we have stated that $[G]$ is a r.e. set, and thus Proposition 6.4 applies.

This proposition can be used with any of the many examples of groups $G$ and $H$ that satisfy $G \not\cong H$ that can be found in [BE15], for instance it is showed there that the Grigorchuk group preforms a free group.

We believe that the following statement holds in general, but are not able to prove it:

**Conjecture 7.10.** If $G$ is an infinite group with solvable word problem of rank $k$, then $G$ preforms a group $H$, non-isomorphic to $G$, with solvable word problem and of rank at most $k + 1$.

Some special instances of this question were studied in conjunction to the study of the elementary theory of free and hyperbolic groups. In [Sel09], Sela proved, using a variation of McKinsey’s result about finitely presented residually finite groups, the following (the slight difference with our statement is that in [Sel09] is used, instead of $[G]$, the set of all markings of $G$ and of its subgroups):

**Proposition 7.11.** If $G$ has solvable word problem, a finitely presented group which is adherent to $[G]$ also has solvable word problem.

*Proof.* This is a simple variation on McKinsey’s Theorem which states that finitely presented residually finite groups have solvable word problem ([McK43]). Such variations were investigated in details in [Rau21], we only sketch the proof here.

A finitely presented group $H$, marked by a family $S$, adherent to $[G]$, is residually-$G$, it has computable quotients in $[G]$, i.e. given a word problem algorithm for a marked group, it is possible to determine whether or not it is a quotient of $(H, S)$, and because $[G]$ is $\Lambda_{WP}$-r.e., McKinsey’s algorithm applies.

Note that this result can be used in conjunction with another result of Sela ([Sel01]), which states that all limit groups are finitely presented, to prove Conjecture 7.10 for limit groups: a limit group must always preform another group that has solvable word problem. The same techniques provide results for limits of hyperbolic groups, but this stays far from the degree of generality of Conjecture 7.10.

All this leaves the following unanswered:

**Conjecture 7.12.** Finite groups are the only abstractly recognizable groups in $(G, d, \nu_{WP})$.

7.2.2. A family of completely undistinguishable groups. We will now use a result of [BE15] to prove the following theorem, which shows how poorly suited the word problem numbering type is to solve the isomorphism problem for finitely generated groups.
Theorem 7.13. There exists an infinite set \( U = \{G_n, n \in \mathbb{N}\} \) of finitely generated word groups, such that for any pair \((G_i, G_j) \in U\), \(i \neq j\), \([G_i]\) and \([G_j]\) are completely indistinguishable: being a marking of \( G_i \) is neither \( \Lambda_{WP}\)-semi-decidable nor \( \Lambda_{WP}\)-co-semi-decidable in \([G_i] \cup [G_j]\).

The set \( U \) of groups contains infinitely many non-isomorphic groups, but any ball in a labeled Cayley graph of a group in \( U \) could in fact belong to any of the groups in \( U \). This is a situation drastically opposed to what happens for isolated groups, for which a ball of large enough radius defines the group uniquely.

Proof. We use the proof of Proposition 5.1. of [BE15], which is an elaborate variation of a well known construction of Hall. This proposition concerns the relation \( \lesssim \) as defined on the set of marked groups: say that a marked group \((G, S)\) preforms a marked group \((H, S')\) if \(G\) preforms \(H\).

We use the fact that is given a construction that associates to a pair of subsets \( C \) and \( X \) of \( \mathbb{N} \) a marked group \( H_{X,C} \), and that this construction satisfies the following: \( H_{X,C} \) preforms \( H_{Y,C'} \) if and only if \( Y \subseteq X \), and the marked groups \( H_{X,C} \) and \( H_{Y,C'} \) are isomorphic as marked groups if and only if \( Y = X \) and \( C = C' \).

To apply this construction in our setting, one can check that it additionally satisfies the two following features (this is straightforward):

1. If \( X \) and \( C \) are recursive sets, \( H_{X,C} \) has solvable word problem;
2. The word problem in \( H_{X,C} \) is at least as hard, in terms of time complexity, as the membership problem in \( C \), modulo an additive term which corresponds to a reduction from one problem to the other.

Using those properties, fixing \( X \) and having \( C \) vary while being recursive, one obtains infinitely many marked groups which are all \( \lesssim \)-equivalent to each other.

To justify that those marked groups actually correspond to infinitely many different abstract groups, we cannot directly use the argument used in [BE15], which goes back to Hall, and which says: as \( C \) varies in \( \mathcal{P}(\mathbb{N}) \), one obtains uncountably many marked groups \( H_{X,C} \), and since an abstract group only has countably many markings, those marked groups must in fact define uncountably many abstract groups. However, we can use the following argument, which can be seen as an effectivisation of Hall’s argument: replacing the set \( C \) by recursive sets for which the membership problem has an arbitrary high time complexity, using property (2) given above, and the fact that the time complexity of the word problem in a group is an isomorphism invariant (up to a coarse equivalence relation, see for instance [Sap11]), one must obtain infinitely many non isomorphic abstract groups.

\[ \square \]

7.3. Isomorphism Problem. Of course, the isomorphism problem is solvable for finite groups described by word problem algorithms, this can easily be seen, for finite groups the word problem and finite presentation numberings are equivalent.

Our previous results about abstract recognizability translate directly into a characterization of which finite families of groups can have solvable isomorphism problem from the word problem description, since for a finite family of groups to have solvable isomorphism problem it is necessary and sufficient that each of its groups be abstractly recognizable.

Proposition 7.14. A finite family \( \mathcal{D} \) of groups with solvable word problem has solvable isomorphism problem for \( \Lambda_{WP} \) exactly when no group in \( \mathcal{D} \) preforms another group in \( \mathcal{D} \).

Proof. This follows from Proposition 7.8 and Proposition 7.9. \( \square \)

In an infinite family of groups, to solve the isomorphism problem, once each group was proven abstractly recognizable, it is still left to show that this recognition can be made uniformly on all groups of the family.

In most families of groups which have solvable isomorphism problem when described by finite presentations, not only is the isomorphism problem unsolvable for the word problem algorithm description, but not all groups are recognizable. This is witnessed, for instance, by the pair \( \{\mathbb{Z}; \mathbb{Z}^2\} \) of abelian groups, and by the pair \( \{\mathbb{F}_2; \mathbb{F}_3\} \) of hyperbolic groups, since we have \( \mathbb{Z} \lesssim \mathbb{Z}^2 \) and \( \mathbb{F}_2 \lesssim \mathbb{F}_3 \).

Examples of infinite families of infinite groups with solvable isomorphism problem for \( \Lambda_{WP} \) can be constructed using examples of isolated groups from [dCGP07], taking for instance a sequence \( (G_n)_{n \in \mathbb{N}} \) of pairwise non-isomorphic isolated groups, for which absolute presentations, i.e. triples (generating set-defining relations-discriminating family) can be enumerated, since the algorithm that allows to partially recognize an isolated group for \( \Lambda_{WP} \) can be obtained from an absolute presentation. Note that all examples that arise this way define families for which the isomorphism problem is solvable from finite presentations as well. This is unfortunate, but it should be clear by now that finite presentations are in general better fitted descriptions than word problem algorithms, when it comes to solving the isomorphism problem in different classes of groups.

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8. Two candidates for failure of the correspondence

8.1. Isolated Groups. It does not seem possible to prove that a word problem algorithm belongs to an isolated group (even with a partial algorithm). From this, we conjecture that the set of isolated groups defines an open set which is not \( \Delta_{WP} \)-semi-decidable. However, we are unable to prove this undecidability result, which would require some techniques entirely independent of Markov’s Lemma.

Conjecture 8.1. The set of isolated group is open in \( \mathcal{G} \) but not semi-decidable.

Remark that the impossibility of partially recognizing isolated groups is also an open problem for groups described by finite presentations. While the Adian-Rabin theorem implies that no algorithm stops exactly on finite presentations of non-isolated groups (since being isolated is a Markov property, as any group with unsolvable word problem provides a negative witness for it), it fails to prove that no algorithm stops exactly on finite presentations of isolated groups. The problem of proving that the set of finite presentations of simple groups is not r.e. seems also to still be open. (The question appears for instance in [Mos73].)

It is also unknown (the question appears in [dCGP07]) whether isolated points are dense in the set of groups with solvable word problem of \( \mathcal{G} \). If they were, we would be able to tell that no sequence of word problem algorithms which contains each isolated group can be enumerated, by Proposition 5.2. We could still arrive to that conclusion if we knew that the word problem is not uniform on isolated groups, that is to say, since all isolated groups are finitely presented, if we knew that a solution to the word problem of an isolated group cannot be retrieved from a finite presentation for this group. For instance, it is well known that the word problem is uniform on the set of simple groups ([Kuz58]), however, Kuznetsov’s argument fails if we add the trivial group to the set of simple groups.

It would be very interesting to prove that the trivial group is unrecognizable from simple groups, from the finite presentation description, and this would prove both that the word problem is not uniform on all isolated groups, and that the set of finite presentations of simple groups is not r.e.. However, too few finitely presented infinite simple groups are known as of now to obtain such results.

8.2. LEF groups and the elementary theory of groups. In this paragraph, we use well known links between the elementary theories of groups and the space of marked groups to study limit groups and LEF groups.

Limit groups are groups that have markings in the adherence of the set of free groups, while LEF groups are groups that have markings in the adherence of the set of finite groups.

8.2.1. Introduction on universal and existential theories of groups. The space of marked groups was used by Champetier and Guirardel (in [CG05]) in order to study limit groups, which play an important role in the solution to Tarski’s problem on the elementary theory of free groups. We include here a paragraph that emphasizes the links between our present study and the study of the universal theories of various classes groups, and we point out some differences. This will be the occasion to propose the set of LEF groups as another candidate for the failure of the correspondence between the Borel and arithmetical hierarchies.

We do not want to include many definitions, and refer [CG05] for precise definitions, and references. A formula is obtained with variables, logical connectors (\& is “and”, \( \lor \) is “or”, and \( \neg \) is “not”), the equality symbol =, the group law \( \cdot \), the identity element 1, and the group inverse \( \cdot^{-1} \), and the two quantifiers \( \forall \) and \( \exists \). We use shortcuts where it is convenient (as the symbols \( \neq \) or \( \Rightarrow \) ), and always use implicitly all group axioms. A sentence is a formula with no free variables. A universal sentence is a sentence that uses only the universal quantifier, and an existential sentence uses only the existential quantifier.

For instance:

\[ \forall x \forall y, x = y \]
\[ \forall x \forall y \forall z, xy = yx \land yz = zy \land y \neq 1 \Rightarrow xz = zx \]
\[ \exists x, x \neq 1 \land x^2 = 1 \]

For a group \( G \), let \( T_\forall(G) \) denote the set of universal sentences that are true in \( G \), and \( T_\exists(G) \) the set of existential sentences that are true in \( G \). For a class \( \mathcal{C} \) of group we also write \( T_\forall(\mathcal{C}) \) and \( T_\exists(\mathcal{C}) \), meaning the set of universal (resp. existential) sentences that hold in all groups of \( \mathcal{C} \).

In the space of marked groups, a universal sentence defines a closed set, and the correspondence with the arithmetical hierarchy holds, i.e., from a word problem algorithm, it is possible to prove that a group does not satisfy a given universal sentence. Similarly, an existential sentence defines an open set and the correspondence holds for such sets. We will not be interested here in formulas with alternating quantifiers.

Remark 8.2. Some universal sentences define sets that are open, and thus clopen, (as the sentence that defines abelian groups), while other sentences define sets that are only closed, and not open (for instance the sentence that...
defines metabelian groups). It seems in fact very hard to determine whether or not a universal sentence defines a clopen set: indeed, notice that for sentences of the form
\[ \forall x, x^n = 1, \]
we know that if \( n \in \{1, 2, 3, 4, 6\} \), then the set defined by this sentence is clopen, by the negative solution to the Burnside problem for those exponents, while for \( n \gg 1 \), it is known that the set defined by the Burnside sentence is not open, as groups of large finite exponent can be constructed as limits of hyperbolic groups, which are not groups of finite exponent. (This is detailed in [Cha00].) And it is an open problem to determine whether or not for \( n = 5 \) the above sentence defines a clopen set.

The following proposition of [CG05] follows directly from the fact that universal sentences define closed sets:

**Proposition 8.3** ([CG05]: Proposition 5.2). If a sequence of marked groups \( (G_n)_{n \in \mathbb{N}} \) converges to a marked group \( G \), then \( \limsup T_\forall(G_n) \subseteq T_\forall(G) \).

This proposition admits a converse, also due to Champetier and Guirardel, which strengthens the relation between the space of marked groups and the study of the elementary theory of groups. We reproduce its proof here.

**Proposition 8.4** ([CG05]: Proposition 5.3). Suppose that two groups \( G \) and \( H \) satisfy \( T_\forall(H) \subseteq T_\forall(G) \). Then any marking of \( G \) is a limit of markings of subgroups of \( H \).

**Proof.** The proof in fact relies on the existential theories of the groups \( G \) and \( H \), which satisfy the reversed inclusion: \( T_\exists(G) \subseteq T_\exists(H) \). Fix a generating family \( S \) of \( G \), and a radius \( r \). Consider the set \( \{w_1, ..., w_k\} \) of reduced words of length at most \( r \) on the alphabet \( S \cup S^{-1} \). Consider the sets \( J_1 = \{(i, j); w_i =_G w_j\} \) (where \( =_G \) means that those words define identical elements of \( G \)) and \( J_2 = \{(i, j); w_i \neq_G w_j\} \). Then \( G \) satisfies the existential formula:
\[ \exists S, \bigwedge_{(i, j) \in J_1} w_i = w_j \wedge \bigwedge_{(i, j) \in J_2} w_i \neq w_j \]
By hypothesis, \( H \) must satisfy it as well, which means precisely that a subgroup of \( H \) must have the same ball of radius \( r \) as \( G \).

For a group \( H \), denote by \( S(H) \) the set of all markings of its subgroups.

**Corollary 8.5.** Let \( G \) and \( H \) be finitely generated groups. The following are equivalent:
- A marking of \( G \) is adherent to the set \( S(H) \);
- All markings of \( G \) are adherent to the set \( S(H) \);
- \( T_\forall(H) \subseteq T_\forall(G) \).

We end this paragraph by using Markov’s Lemma together with the above result.

**Lemma 8.6** (Markov’s Lemma for Elementary Theories). Suppose that two groups \( G \) and \( H \), with solvable word problem, satisfy \( T_\forall(H) \subseteq T_\forall(G) \).

Then \( [G] \) is not \( \Lambda_{WP} \)-semi-decidable inside the set \( [G] \cup \bigcup_{K \in S(H)} [K] \).

**Proof.** This follows from Corollary 8.5, Proposition 6.4 (the set \( S(H) \) is \( \Lambda_{WP} \)-r.e.), and from Markov’s Lemma for groups (Lemma 6.3).

8.2.2. **Limit groups and LEF Groups.** We will use the following definition for limit groups (those were named in [Sel01], see [CG05] for the equivalence with other definitions): a group \( G \) is a limit group if some (or all) of its markings are adherent to the set of marked free groups. Note that if \( G \) is a subgroup of a group \( H \), every universal sentence in \( G \) holds in \( H \). This implies that all non-abelian free groups have the same universal theory, since each non-abelian free group is a subgroup of each other non-abelian free group.

Thus by Corollary 8.5, a group \( G \) is a limit group if and only if it satisfies \( T_\forall(F_2) \subseteq T_\forall(G) \), where \( F_2 \) is the rank two free group. In fact, it is known that if a group \( G \) satisfies \( T_\forall(F_2) \subseteq T_\forall(G) \), then either it is abelian, and then it is free abelian, and \( T_\forall(Z) = T_\forall(G) \), or it has a free subgroup, which implies that \( T_\forall(F_2) = T_\forall(G) \).

The following proposition solves a decision problem for groups given by word problem algorithms, while relying heavily on the study of the elementary theory of groups.

**Proposition 8.7.** Being a limit group is \( \Lambda_{WP} \)-co-semi-decidable.
Proof. A group $G$ is a limit group if and only if it satisfies $T_\forall(\mathbb{F}_2) \subseteq T_\forall(G)$. A theorem of Makanin ([Mak85]) states that the universal theory of free groups is decidable, and thus that it is possible to enumerate all universal sentences that hold in free groups.

Since, given a word problem algorithm for a marked group $(G, S)$, it is always possible to prove that a given universal sentence is not satisfied in $G$, it is possible to detect groups that are not limit groups by testing in parallel all sentences of the universal theory of free groups.

This result is a slight improvement of a result in [GW09], where the same is obtained, but making use of both a finite presentation and a word problem algorithm.

This result calls to our attention a second example of a natural property for which the correspondence between the arithmetical hierarchy and the Borel hierarchy might fail. Indeed, this last proof relies heavily on Manakin’s theorem. While the universal theory of free groups is decidable, Slobodskoi proved in [Slo81] that the universal theory of finite groups is unsolvable.

From this we conjecture:

**Conjecture 8.8.** The set of LEF groups is not $\Lambda_{WP}$-co-semi-decidable.

Denote by $\mathcal{F}$ the set of marked finite groups, recall that its adherence $\overline{\mathcal{F}}$ is the set of LEF groups. (LEF groups were first defined in [VG98], in terms of partial homomorphisms onto finite group.) Note that, at first glance, Slobodskoi’s Theorem does not seem to be the sole thing preventing us from applying the proof of Proposition 8.7 to LEF groups. Indeed, this proof relied on the fact that a group $G$ is a limit group if and only if it satisfies $T_\forall(\mathbb{F}_2) \subseteq T_\forall(G)$, which in turn used the fact that the inclusion $T_\forall(\mathbb{F}_2) \subseteq T_\forall(G)$ is equivalent to the reverse inclusion $T_3(G) \subseteq T_3(\mathbb{F}_2)$ (see Proposition 8.4). This follows from the fact that the elementary theory of a single group is complete, i.e. every sentence or its negation is in it. But the theory of finite groups is of course not complete, as the existential theory $T_3(F)$ is empty, since the trivial group does not satisfy any existential sentence. However, the corresponding equivalence still holds.

**Proposition 8.9.** A group $G$ belongs to $\overline{\mathcal{F}}$ if and only if it satisfies $T_\forall(\mathcal{F}) \subseteq T_\forall(G)$.

Proof. We use the fact that there exists a group $K$ in $\overline{\mathcal{F}}$ that satisfies $T_\forall(K) = T_\forall(G)$. If a group $G$ satisfies $T_\forall(K) = T_\forall(G) \subseteq T_\forall(G)$, then it satisfies $T_3(K) \subseteq T_3(G)$, and by Corollary 8.5, $G$ is a limit of subgroups of $K$.

But $K$ and all its finitely generated subgroups are in $\overline{\mathcal{F}}$, thus $G$ must also be a limit of markings of finite groups.

The group $K$ can be taken as the semi-direct product $\mathbb{Z} \ltimes \mathcal{S}_\infty$, where $\mathcal{S}_\infty$ denotes the group of finitely supported permutations of $\mathbb{Z}$, on which $\mathbb{Z}$ acts by translation. This group is the limit of the finite groups $\mathbb{Z}/n\mathbb{Z} \ltimes \mathcal{S}_n$, as $n$ goes to infinity ($\mathcal{S}_n$ is the group of permutation over $\{1, \ldots, n\}$). Since $K$ is in $\overline{\mathcal{F}}$, $T_\forall(G) \subseteq T_\forall(K)$. However, because it contains a copy of every finite group, one also has the reversed inclusion.

Thanks to this proposition, we have:

**Proposition 8.10.** Conjecture 8.8 implies Slobodskoi’s Theorem.

Proof. Supposing that Slobodskoi’s Theorem fails, one can reproduce the proof of Proposition 8.7, and prove that Conjecture 8.8 fails.

Other conjectures can be obtained, that are similar to Conjecture 8.8: by a theorem of Kharlampovich ([Kha83]), the universal theory of finite nilpotent groups is also undecidable, and it is also known that the universal theory of hyperbolic groups is undecidable (as proven by Osin in [Osi09]).

**Problem 8.11.** Is the adherence $\mathcal{H}$ of the set of hyperbolic groups $\Lambda_{WP}$-co-semi-decidable? What of the adherence of the set of finite nilpotent groups?

8.2.3. Properties not characterized by universal theories. Note that not every decidable property for groups given by word problem algorithms can be solved by expressing the question that is to be solved as a problem about universal or existential theories, and applying techniques similar to the proof of Proposition 8.7.

We give here a simple example to illustrate this. Let $H$ be the group $\mathbb{Z} \ast \mathbb{Z}^3$. It is a non-abelian limit group, thus $H$ and the rank three free group $\mathbb{F}_3$ have the same universal theory. However, the property “being isomorphic to $H$” can be discerned from the property “being isomorphic to $\mathbb{F}_3$”. Indeed, no sequence of markings of $H$ can converge to a marking of $\mathbb{F}_3$, because “having rank at most three” is an open property in $\mathcal{G}$. On the other hand, suppose that a sequence of markings of $\mathbb{F}_3$ converges to a marking of $H$. By Lemma 7.1, this implies that the canonical marking of $\mathbb{Z} \ast \mathbb{Z}^3$ is a limit of 4-markings of $\mathbb{F}_3$. But this would imply that one can find a generating family $(a, b, c, d)$ of $\mathbb{F}_3$, such that $b$, $c$ and $d$ commute. This is impossible, because abelian subgroups of free groups are cyclic, and thus this would imply that $\mathbb{F}_3$ is of rank two. By Proposition 7.14, the isomorphism problem is solvable for the pair $\{\mathbb{F}_3, H\}$, while those two groups have identical existential and universal theories.
9. Subgroups of finitely presented groups with solvable word problem

9.1. Higman-Clapham-Valiev Theorem for groups with solvable word problem. We now remark how Higman’s Embedding Theorem gives further incentive for the study of algorithmic problems solved for groups described by word problem algorithms.

After Higman’s proof of his famous Embedding Theorem ([Hig61]), several theorems that resemble it were obtained.

In particular, it was remarked that the theorem is effective, meaning that it provides an algorithm that takes as input a recursive presentation for a group $G$, and outputs a finite presentation for a group $H$, together with a finite family of elements of $H$ that generate $G$. Note that in terms of numbering types, this implies that the numbering type $\Lambda_{\text{r.p.}}$, associated to recursive presentation, is equivalent to the numbering type associated to the idea “a marked group $(G, S)$ is described by a finite presentation of an overgroup of $G$ together with words that define the elements of $S$ in that overgroup”. We leave out the details.

We will be interested here in the version of Higman’s Theorem that preserves solvability of the word problem (see [Val75, Cla67]). This theorem is known as the Higman-Clapham-Valiev Theorem.

Historical remarks about these results can be found in [OS04]. The following formulation of the Higman-Clapham-Valiev Theorem can also be found in [BR66].

**Theorem 9.1** (Higman-Clapham-Valiev, I). There exists a procedure that, given a recursive presentation for a group $G$, produces a finite presentation for a group $H$, together with an embedding $G \hookrightarrow H$ described by the images of the generators of $G$, and such that if the word problem is solvable in $G$, then it is also solvable in $H$.

One can also check that if one has access to a word problem algorithm for the group given as input to this procedure, one can obtain a word problem algorithm for the constructed finitely presented group. This yields:

**Theorem 9.2** (Higman-Clapham-Valiev, II). There exists a procedure that, given a word problem algorithm for a finitely generated group, produces a finite presentation of a group in which it embeds, together with a word problem algorithm for this new group, and a set of elements that generate the first group.

This proves that, in general, the description of a group by its word problem algorithm, or by a finite generating family inside a finitely presented group with solvable word problem, are equivalent (we leave it to the reader to render this statement precise: define a numbering of $G$ associated to the idea “a group is given as a subgroup of a group described by a finite presentation together with a word problem algorithm”, thus using the numbering $\nu_{\text{WP}} \land \nu_{\text{FP}}$ to describe the overgroup, the Higman-Clapham-Valiev Theorem implies that this numbering is equivalent to $\nu_{\text{WP}}$.

Thus the study of algorithmic problems that can be solved from the word problem description is identical to the study of decision problems about subgroups of finitely presented groups with solvable word problem.

The following theorem is a joint application of the Higman-Clapham-Valiev Theorem and of Markov’s Lemma:

**Theorem 9.3.** Suppose that a $\Lambda_{\text{WP}}$-computable sequence $(G_n)_{n \in \mathbb{N}}$ of $k$-marked groups effectively converges to a $k$-marked group $H$, and suppose that $H \notin \{G_n, n \in \mathbb{N}\}$. Then there exists a finitely presented group $\Gamma$, with solvable word problem, in which no algorithm can, given a tuple of elements of $\Gamma$ that defines a marked group of $\{G_n, n \in \mathbb{N}\} \cup \{H\}$, stop if and only if this tuple defines $H$.

**Proof.** By Markov’s Lemma applied to groups, there exists a $\Lambda_{\text{WP}}$-computable sequence $(L_n)_{n \in \mathbb{N}}$ of marked groups, such that for each $p$, $L_p \in \{G_n, n \in \mathbb{N}\} \cup \{H\}$, and the set $\{n \in \mathbb{N} | L_n = H\}$ is co-r.e. but not r.e.. The direct sum of those groups can be embedded in a finitely generated group which has solvable word problem (using the well known construction of Higman, Neumann and Neumann [HNN49]), and in turn the Higman-Clapham-Valiev theorem can be applied to obtain a finitely presented group $\Gamma$ which contains the sequence $(L_n)_{n \in \mathbb{N}}$. Moreover, it is easy to see that there exists an algorithm that, given a natural number $n$, produces a $k$-tuple of elements of $\Gamma$ that generate $L_n$. (This comes from the fact that each of the described embeddings is effective.) This directly implies the claimed result.

Note that when the conjugacy problem is uniformly solvable for the groups in $(G_k)_{k \in \mathbb{N}}$, we may want to apply the version of Higman’s Theorem due to Alexander Olshanskii and Mark Sapir ([OS04], and [OS05] for non-finitely generated groups) that preserves its solvability.

9.2. Some examples. We now give some examples of possible applications of Theorem 9.3.

**Proposition 9.4.** There exists a finitely presented group with solvable word problem, but unsolvable order problem.

**Proof.** Apply Theorem 9.3 to a sequence of finite cyclic groups that converges to $\mathbb{Z}$. This yields a finitely presented group with solvable word problem in which one cannot decide whether a given element generates a subgroup of positive index.
isomorphic to \( \mathbb{Z} \) or to a finite cyclic group. This is precisely a finitely presented group with solvable word problem, but unsolvable order problem.

**Proposition 9.5.** There exists a finitely presented group with solvable word problem, but unsolvable power problem.

**Proof.** Apply Theorem 9.3 to the sequence of 2-markings of \( \mathbb{Z} \) defined by the generating families \((1, k), k \in \mathbb{N}^*\), which converges to (the only 2-marking of) \( \mathbb{Z}^2 \) when \( k \) goes to infinity (see [CG05]). This yields a finitely presented group with solvable word problem where, given a pair of commuting elements, one cannot decide whether they generate \( \mathbb{Z}^2 \), or if one of these elements is a power of the other: this is a group with unsolvable power problem.

We can also use this theorem to strengthen a result that was recently obtained in [DI22].

**Theorem 9.6.** There is a finitely presented group with solvable word problem in which the problem of deciding whether a given subgroup is amenable is neither semi-decidable nor co-semi-decidable.

**Proof.** This is proven by using both a sequence of marked amenable groups which converges to a non-amenable group and a sequence of non-amenable marked groups that converges to an amenable marked group. Such examples were given in SubSection 6.2.

The “not semi-decidable” half of this result is Theorem 6 in [DI22]. Of course, Theorem 9.3 can be applied to all the properties that appeared in SubSection 6.2 to produce results similar to this one.

We will stop multiplying examples, as all those results are well known, but it seems that explaining them in terms of convergence in the space of marked groups unifies several existing constructions.

**References**

[AB14] Jeremy Avigad and Vasco Brattka. Computability and analysis: the legacy of Alan Turing. In Rod Downey, editor, *Turing’s Legacy*, pages 1–47. Cambridge University Press, 2014.

[Abe80] Olivier Aheber. *Computable Analysis*. McGraw-Hill International Book Company, 1980.

[Ash98] Christopher John Ash. Isomorphic recursive structures. In Yu. L. Ershov, S.S. Goncharov, A. Nerode, J.B. Remmel, and V.W. Marek, editors, *Handbook of Recursive Mathematics*, volume 138 of *Studies in Logic and the Foundations of Mathematics*, pages 167–181. Elsevier, 1998.

[BKR19] Iva Bilanovic, Jennifer Chubb, and Sam Roven. Detecting properties from descriptions of groups. *Archive for Mathematical Logic*, 59(3-4):293–312, aug 2019.

[BE15] Laurent Bartholdi and Anna Erschler. Ordering the space of finitely generated groups. *Annales de l'Institut Fourier*, 65(5):2091–2144, 2015.

[BK19] Mustafa Gökhan Benli and Burak Kaya. Descriptive complexity of subsets of the space of finitely generated groups. 2019.

[BCR19] Iva Bilanovic, Jennifer Chubb, and Sam Roven. Detecting properties from descriptions of groups.

[BP03] Vasco Brattka and Gero Presser. Computability on subsets of metric spaces. *Theoretical Computer Science*, 305(1-3):43–76, aug 2003.

[BR66] William W. Boone and Hartley Jr. Rogers. On a problem of J.H.C. Whitehead and a problem of Alonzo Church. *Mathematica Scandinavica*, 19:185, jun 1966.

[Can66] Frank B. Cannonito. Hierarchies of computable groups and the word problem. *Journal of Symbolic Logic*, 31(3):376–392, sep 1966.

[Cei67] Gregory S. Cetin. Algorithmic operators in constructive metric spaces. *Trudy Matematicheskago Instituta Imeni V. A. Steklova*, pages 1–80, 1967.

[CG05] Christophe Champetier and Vincent Guirardel. Limit groups as limits of free groups. *Groups, Geometry, and Dynamics*, 5(4):2091–2144, 2011.

[Cha00] Christophe Champetier. L’espace des groupes de type fini. *Annales de l’institut Fourier*, vol. 50, 2000.

[Clah67] Christopher R. J. Clapham. An embedding theorem for finitely generated groups. *Proceedings of the London Mathematical Society*, s-3(17)(3):419–430, jul 1967.

[dCGP07] Yves de Cornulier, Luc Guyot, and Wolfgang Pitsch. On the isolated points in the space of groups. *Journal of Algebra*, 307(1):254–277, jan 2007.

[DI22] Karol Duda and Aleksander Ivanov. On decidability of amenability in computable groups. *Archive for Mathematical Logic*, mar 2022.

[Ers99] Yuri L. Ershov. Theory of numberings. In *Handbook of Computability Theory*, pages 473–503. Elsevier, 1999.

[FGM+98] Benjamin Fine, Anthony M. Gaglione, Alexei Myasnikov, Gerhard Rosenberger, and Dennis Spellman. A classification of fully residually free groups of rank three or less. *Journal of Algebra*, 200(2):571–605, feb 1998.

[Fri58] Richard Friedberg. Un contre-exemple relatif aux fonctionnelles récursives. *Comptes rendus hebdomadaires des séances de l’Académie des Sciences (Paris)*, vol. 24, 1958.

[GKP16] Vasileios Gregoriades, Tamás Kispéter, and Arno Pauly. A comparison of concepts from computable analysis and effective descriptive set theory. *Mathematical Structures in Computer Science*, 26(8):1414–1436, jun 2016.

[Gri85] Rostislav I. Grigorchuk. Degrees of growth of finitely generated groups, and the theory of invariant means. *Mathematics of the USSR-Izvestiya*, 25(2):259–300, apr 1985.

[GW09] Daniel Groves and Henry Wilton. Enumerating limit groups. *Groups, Geometry, and Dynamics*, pages 389–399, 2009.
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[Her01] Peter Hertling. Banach-Mazur computable functions on metric spaces. In Computability and Complexity in Analysis, pages 69–81. Springer Berlin Heidelberg, 2001.

[Hig61] Graham Higman. Subgroups of finitely presented groups. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 262(1311):455–475, aug 1961.

[HNN49] Graham Higman, Bernhard H. Neumann, and Hanna Neumann. Embedding theorems for groups. Journal of the London Mathematical Society, s1-24(4):247–254, oct 1949.

[HR16] Mathieu Hoyrup and Cristóbal Rojas. On the information carried by programs about the objects they compute. Theory of Computing Systems, 61(4):1214–1236, dec 2016.

[Kha83] Olga G. Kharrampovich. Universal theory of the class of finite nilpotent groups is undecidable. Mathematical Notes, 33:254–263, 1983.

[KLS57] Georg Kreisel, Daniel Lacombe, and Joseph R. Shoenfield. Partial recursive functionals and effective operations. Constructivity in mathematics, Proceedings of the colloquium held at Amsterdam:pp. 290âĂŞ297, 1957.

[Kus84] Boris A. Kushner. Lectures on Constructive Mathematical Analysis. American Mathematical Society, 1984.

[Kuz58] Alexander V. Kuznetsov. Algorithms as operations in algebraic systems. Izv. Akad. Nauk SSSR Ser. Mat., 1958.

[Loc81] Jody Lockhart. Decision problems in classes of group presentations with uniformly solvable word problem. Archiv der Mathematik, 37(1):1–6, dec 1981.

[Mal61] Anatolii I. Malcev. Constructive algebras I. Uspekhi Mat. Nauk, 1961.

[Mal71] Anatolii I. Malcev. The metamathematics of algebraic systems, collected papers: 1936-1967. North-Holland Pub. Co, Amsterdam, 1971.

[Mar54] Andrei Andreevich Markov. On the continuity of constructive functions. Uspekhi Matematicheskikh Nauk, No. 3(61), 226–230, 1954. (Russian).

[Mar63] Andrei Andreevich Markov. On constructive functions, chapter in Twelve papers on logic and differential equations, pages 163–195. American Mathematical Society Translations, 1963.

[Mos66] Yiannis Moschovakis. Notation systems and recursive ordered fields. Compositio Mathematica, 17:40–71, 1966.

[Mos73] Yiannis Moschovakis. Uniform algorithms for deciding group-theoretic problems. In Word Problems - Decision Problems and the Burnside Problem in Group Theory, pages 525–551. Elsevier, 1973.

[Mos79] Yiannis Moschovakis. On computable sequences. In Studies in Logic and the Foundations of Mathematics, pages 336–350. Elsevier, 1979.

[Mos80] Yiannis Moschovakis. Descriptive set theory. North-Holland, Amsterdam New York, 1980.

[Neu73] Bernhard H. Neumann. The isomorphism problem for algebraically closed groups. In Mathematical Sciences Research Institute Publications, pages 1–59. Springer New York, 1992.

[Sel00] Yehuda Shalom. Rigidity of commensurators and irreducible lattices. Inventiones mathematicae, 141(1):1–54, jul 2000.

[Slo81] Aleksandr M. Slobodskoi. Unsolvability of the universal theory of finite groups. Algebra and Logic, 20(2):139–156, mar 1981.

[Spr01] Dieter Spreen. Representations versus numberings: on the relationship of two computability notions. Theoretical Computer Science, 262(1-2):473–499, jul 2001.
[SVU17] Alexander Shen, Nikolai Konstantinovich Vereshchagin, and Vladimir Andreyevich Uspensky. Kolmogorov Complexity and Algorithmic Randomness. American Mathematical Society, Providence, 2017.

[Tur36] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. Proceedings of the London Mathematical Society, s2-42(1):230–265, 1936.

[Tur37] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. A correction. Proceedings of the London Mathematical Society, s2-43(1):544–546, 1937.

[Val75] Mars K. Valiev. On polynomial reducibility of word problem under embedding of recursively presented groups in finitely presented groups. In Mathematical Foundations of Computer Science 1975 4th Symposium, Mariánské Lázné, September 1–5, 1975, pages 432–438. Springer Berlin Heidelberg, 1975.

[VG98] Anatolii M. Vershik and Evgenii I. Gordon. Groups that are locally embeddable in the class of finite groups. St. Petersburg Math., 1998.

[Wei87] Klaus Weihrauch. Computability. Springer Berlin Heidelberg, 1987.

[Wei00] Klaus Weihrauch. Computable Analysis. Springer Berlin Heidelberg, 2000.