CHAINED QUANTUM ARNOLD TRANSFORMATIONS

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Abstract
We put forward the concatenation of Quantum Arnold Transformations as a tool to obtain the wave function of a particle subjected to a harmonic potential which is switched on and off successively. This simulates the capture and release process of an ion in a trap and provides a mathematical picture of this physical process.

Keywords: quantum Arnold transformation, quadratic Hamiltonians, ion traps

1 Introduction

In a recent paper \cite{1} the authors proposed a transformation that maps states and operators from quantum systems in one dimension described by generic, time-dependent, quadratic Hamiltonians to the system of the free particle. This transformation is the quantum version of the Arnold transform (QAT), which in its original classical version \cite{2} maps solutions of a certain type of classical equation of motion, a non-homogeneous linear second order ordinary differential equation (LSODE), to solutions of the classical equation for the free particle. Through this transformation, the class of quantum systems with quadratic, time-dependent Hamiltonian can be connected and its solutions and relevant operators related by using the free particle system as an intermediate point. This way, the symmetry properties of the free particle system are transferred to this whole class of quantum systems.

In fact, the QAT proves to be useful in the construction of wave functions, the quantum propagator or the evolution operator, taking advantage of the simple properties of the free particle quantum system. Even more, some insight in the physics of the free particle can also be obtained by importing to this system features that are, in principle, naturally attributed to the quantum harmonic oscillator. This is the case of coherent and squeezed states in one spatial dimension, as well as Hermite-Gauss and Laguerre-Gauss states in higher dimensions, as it was shown in \cite{3}.

There are some related strategies in the literature employed to solve the Schrödinger equation with a general quadratic, time-dependent Hamiltonian. We could mention the method of looking for integrals of motion showed, for instance, in \cite{4} and \cite{5,6}. Some other references may be found in \cite{1,3}.
A particular case of time-dependent potential is that of a harmonic potential which is turned on and off successively, preserving or not the original frequency $\omega$ of the oscillator. This could be useful to simulate the processes of capturing or releasing ions in traps. Usually, ion traps, like the Penning of the Paul trap, have time varying, periodic frequencies (for instance, for the Paul trap the classical equation of motion is a Mathieu equation), but we shall restrict ourselves to constant frequency to focus on the idea to turning on and off the potential without introducing unnecessary technicalities. The purpose of this paper is to take advantage of the QAT that relates the harmonic oscillator and the free particle to obtain a method to compute analytically the wave function and the evolution operator for such a system.

The complete time evolution operator along a lapse of time in which a particle is released and captured several times in a harmonic potential will be a product of evolution operators for the free particle and the harmonic oscillator. This is a consequence of the sudden or diabatic approximation if the processes of switching on (and off) are fast enough. The harmonic-oscillator evolution operation on certain states (namely eigenstates of the harmonic oscillator Hamiltonian) may be simpler than the free one. It will be shown that it is possible to split each free time evolution operator in two operations: a harmonic evolution and a quantum Arnold transformation, in such a way that the complete evolution can be written in terms of harmonic evolutions and QATs only. This might not be always the easiest way to compute the resulting wave function. We believe nevertheless that it is interesting to have the possibility that we present here at our disposal. This technique has also the advantage of being easily generalizable to generic quadratic potentials with abrupt jumps in its time-dependence. However, we will keep the simplicity of the harmonic potential to keep the idea as neat as possible.

The paper is organized as follows. In Sec. 2 we give an overview of the Quantum Arnold Transformation for general quadratic Hamiltonians. In Sec. 3 we provide the particularization for the case of the harmonic oscillator and fix some notation. In Sec. 4 we obtain the alternative expression for the time evolution operator in terms of QATs. A summary and some comments are given in Sec. 5.

## 2 The Quantum Arnold Transformation

The quantum Arnold transformation (or the inverse) relates the Hilbert space $\mathcal{H}_t$ of solutions of the free Schrödinger equation

$$i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2},$$

(1)

associated with the corresponding classical equation

$$\ddot{x} = 0,$$

(2)

to that space $\mathcal{H}_t'$ of solutions of the Schrödinger equation

$$i\hbar \frac{\partial \phi'}{\partial t'} = -\frac{\hbar^2}{2m} e^{-f} \frac{\partial^2 \phi'}{\partial x'^2} + \left(\frac{1}{2}m\omega^2 x'^2\right)e^f \phi',$$

(3)

where the quantum theory of a generic LSODE

$$\ddot{x}' + f\dot{x}' + \omega^2 x' = 0$$

(4)

\footnote{We consider here the case of homogeneous classical equation and the corresponding quantum theory. For the presence of an external force, see [4].}
is realized. The quantities $f$ and $\omega$ are, in general, dependent on time $t'$. The classical equation \[ (\hat{H}^2 - m^2\omega^2x^2)u'' + 2m\omega^2fx' = 0, \] can be derived from a Hamiltonian function

$$H = \frac{p^2}{2m} - f,$$

(5)

and, according to the standard canonical prescriptions, it leads to the Schrödinger equation \[ (\hat{H}^2 - m^2\omega^2x^2)\psi = E\psi. \] For $f$ linear in time and constant $\omega$ this equation is commonly known as Caldirola-Kanai equation for the damped harmonic oscillator \[7, 8].

Being both spaces of solutions of \[1\] and \[3\], $\mathcal{H}_t$ and $\mathcal{H}'_{t'}$, respectively, related, we will also have the basic, constant of motion quantum operators associated with the classical functions position and momentum connected and realized as well-defined, constant of motion operators on both $\mathcal{H}_t$ and $\mathcal{H}'_{t'}$. However, it is important to note that the Hamiltonian of one of the systems is not transformed into the Hamiltonian of the other: the generators of the symmetry of the free particle, the Schrödinger group, are mapped into a set of operators closing the same Lie algebra in the system corresponding to the generic LSODE. The Hamiltonian operator of the LSODE-system is not the same abstract element in this algebra or, even, it does not necessarily belongs to this symmetry algebra.

This is a generalization of the classical Arnold transformation $A$, and is obtained by completing $A$ with a change of the wave function. While the classical Arnold transformation $A$ is given by:

$$A : \mathbb{R} \times T' \longrightarrow \mathbb{R} \times T$$

$$(x', t') \longmapsto (x, t) = A((x', t')) = (\frac{x'}{u_2}, \frac{u_1}{u_2}),$$

(6)

where $T'$ and $T$ are open intervals of the real line containing $t' = 0$ and $t = 0$, respectively, and $u_1(t')$ and $u_2(t')$ are independent solutions of the classical equation of motion in $(x', t')$ \[\|\], the QAT is written:

$$\hat{A} : \mathcal{H}'_{t'} \longrightarrow \mathcal{H}_t$$

$$\phi'(x', t') \longmapsto \phi(x, t) = \hat{A}(\phi'(x', t')) = A^*(\sqrt{u_2(t')} e^{-\frac{i}{\hbar} \int_{t'}^{t} \frac{1}{u_2(s')} W(s')^{-\frac{1}{2}} \frac{u_2(s')}{u_2(t')} x'^2} \phi'(x', t')),$$

(7)

where $A^*$ denotes the pullback operation corresponding to $A$, and $W(t') \equiv \dot{u}_1u_2 - u_1\dot{u}_2 = e^{-f}$. It is straightforward to check that by this transformation the Schrödinger equation of the free particle is transformed into \[3\] up to a multiplicative factor which depends on the particular choice of the classical solutions $u_1$ and $u_2$ (partial derivatives must be changed by the classical part of the transformation while wave functions are shifted by the quantum part).

We impose on $u_1$ and $u_2$ the condition that they preserve the identity of $t$ and $x$, i.e., that $(x, t)$ coincide with $(x', t')$ at an initial point $t'_0$, arbitrarily taken to be $t'_0 = 0$:

$$u_1(0) = 0, \quad u_2(0) = 1, \quad \dot{u}_1(0) = 1, \quad \dot{u}_2(0) = 0.$$

(8)

This fixes a unique form of $A$ for a given “target” LSODE-type physical system. However, the quantum Arnold transformation is still valid if solutions $u_1$ and $u_2$ do not satisfy \[3\] (see \[\|\] for details).

It will be useful to have a pictorial representation of the situation. There will be a common Hilbert space $\mathcal{H}$ of wave functions, which plays the role of initial values for both the solutions $\phi'(x', t') \in \mathcal{H}'_{t'}$ and
\( \varphi(x,t) \in \mathcal{H}_t \); evolution operators, \( \hat{U}(t,0) \equiv \hat{U}(t) \) and \( \hat{U}'(t',0) \equiv \hat{U}'(t') \), acting on the initial wave functions; and the quantum Arnold transformation \( \hat{A} \) that relates Schrödinger equations and basic operators:

\[
\begin{align*}
\mathcal{H}_t & \quad \xleftarrow{\hat{A}} \quad \mathcal{H}'_t' \\
\hat{U}(t) & \quad \uparrow \quad \hat{U}'(t') \\
\mathcal{H}_0 & \equiv \mathcal{H} \quad \xrightarrow{1} \quad \mathcal{H} \equiv \mathcal{H}_0'
\end{align*}
\]  

(9)

where \( \hat{I} \) is the identity operator. Had the solutions \( u_1 \) and \( u_2 \) not been chosen satisfying \( \mathbb{N} \), the price would have been that the relation in the lower part of the diagram \( \mathbb{I} \) above would no longer be the identity and basic position and momentum operators would then be mixed (see \( \mathbb{I} \)).

Let us stress that QAT can be useful to quickly perform some calculations, avoiding tedious, direct evaluations which can become extremely involved in the system under study. For example, it can be used to compute the quantum propagator for any LSODE-type quantum system, following the idea of Takagi in \( \mathbb{G} \) for the simple case of the harmonic oscillator, or even the evolution operator \( \hat{U}'(t') \), which becomes very difficult to evaluate exactly when the Hamiltonian is time-dependent and does not commute with itself at different times.

Actually, the evolution operator of a LSODE system can be related with the free evolution operator. Having in mind the diagram \( \mathbb{I} \), we write:

\[
\hat{A}(\hat{U}(t')\phi'(x')) = \hat{U}(t)\varphi(x). 
\]  

(10)

Here \( \phi' \) and \( \varphi \) are the same function of only one argument \( (x \text{ or } x') \) and we will denote \( \varphi = \phi' = \psi \). Then,

\[
\hat{U}'(t')\psi(x') = \hat{A}^{-1}(\hat{U}(t)\psi(x)) = \frac{1}{\sqrt{u_2}} e^{\frac{i}{\hbar} \frac{m}{2} \frac{u_2}{u_2} x'^2} A_{s-1}(\hat{U}(t)) A_{s-1}(\psi(x)).
\]

(11)

To factorize the function \( \psi \) and single out the general action of \( \hat{U}'(t') \), we compute

\[
A_{s-1}(\psi(x)) = \psi(\frac{x'}{u_2}) = e^{\log(1/u_2)x' \frac{\partial}{\partial x'} } \psi(x'),
\]

(12)

where \( e^{\log(1/u_2)x' \frac{\partial}{\partial x'}} \) is a dilation operator which is not unitary. To unitarize this operator, the generator must be shifted (symmetrized) from \( x' \frac{\partial}{\partial x'} \) to \( x' \frac{\partial}{\partial x'} + \frac{1}{2} \), so that the true unitary operator is then

\[
\hat{U}_D(\frac{1}{u_2}) = e^{\log(1/u_2)(x' \frac{\partial}{\partial x'} + \frac{1}{2})} = \frac{1}{\sqrt{u_2}} e^{\log(1/u_2)x' \frac{\partial}{\partial x'} }. 
\]

(13)

But the factor \( \frac{1}{\sqrt{u_2}} \) is already present in the previous expression of \( \hat{U}'(t') \). Therefore, it now reads

\[
\hat{U}'(t') = e^{\frac{i}{\hbar} \frac{m}{2} \frac{u_2}{u_2} x'^2} A_{s-1}(\hat{U}(t)) \hat{U}_D(\frac{1}{u_2}) = \\
= \frac{1}{\sqrt{u_2}} e^{\frac{i}{\hbar} \frac{m}{2} \frac{u_2}{u_2} x'^2} e^{\frac{\hbar}{2m} u_1 u_2 \frac{\partial^2}{\partial x'^2}} e^{\log(1/u_2)x' \frac{\partial}{\partial x'} }.
\]

(14)
It is remarkable that we have been able to write an exact expression for the evolution operator as a product of operators. No perturbative approximation method, which could become cumbersome in some cases, is needed for any LSODE-related quantum system to obtain the evolution operator.

3 A particular case: the harmonic oscillator and the free particle

Now let us give the specific expressions for the harmonic oscillator, that is to say, \( \dot{f} = 0 \) and \( \omega(t') = \omega \), constant. The two independent classical solutions can be chosen as

\[
\begin{align*}
    u_1(t') &= \frac{1}{\omega} \sin(\omega t') \\
    u_2(t') &= \cos(\omega t')
\end{align*}
\]

with \( W(t') = 1 \). It can be checked that the change of variables in the classical Arnold transformation results in:

\[
\begin{align*}
    t' &= \frac{1}{\omega} \tan^{-1}(\omega t) \\
    x' &= \cos(\tan^{-1}(\omega t)) x = \frac{x}{\sqrt{1 + \omega^2 t^2}}
\end{align*}
\]

The solutions \( u_1 \) and \( u_2 \) satisfy the conditions (8). Therefore, the diagram (9) can be simplified to:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\hat{A}} & \mathcal{H}_{\text{HO}} \\
\mathcal{U} & \xleftarrow{\hat{\mathcal{U}}_{\text{HO}}} & \mathcal{U} \mathcal{H}_0
\end{array}
\]

where \( \hat{U} \) and \( \hat{\mathcal{U}}_{\text{HO}} \) stand for the evolution operators of the free particle and the harmonic oscillator, respectively, while \( \mathcal{H}_0 \) is the Hilbert space, either for the free particle or the harmonic oscillator, of solutions of their respective Schrödinger equations at \( t = 0 \).

It is possible to apply the QAT, for instance, to the time-dependent eigenstates of the harmonic oscillator Hamiltonian \( \hat{H}'_{\text{HO}} \), with energy \( E_n = \hbar \omega (n + \frac{1}{2}) \) and belonging to \( \mathcal{H}_{\text{HO}} \),

\[
\psi_n'(x',t') = \mathcal{N}_n e^{-i\omega (n+\frac{1}{2}) t'} e^{-\frac{m\omega}{\hbar} x'^2} H_n(\sqrt{\frac{m\omega}{\hbar}} x'),
\]

(16)

where \( \mathcal{N}_n = \left( \frac{m\omega}{\hbar \pi} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \). We obtain the following set of states, solutions of the Schrödinger equation for the free particle:

\[
\psi_n(x,t) = \mathcal{N}_n \frac{1}{\sqrt{|\delta|}} e^{-\frac{x^2 \delta^*}{4L^2 |\delta|^2}} \left( \frac{\delta^*}{|\delta|} \right)^{n+\frac{1}{2}} H_n\left(\frac{x}{\sqrt{2L|\delta|}}\right),
\]

(17)
where, in order to obtain a more compact notation, we have introduced the quantities \( L = \sqrt{\frac{\hbar}{2m\omega}} \), with dimensions of length, and \( \tau = \frac{2mL^2}{\hbar} = \omega^{-1} \), with dimensions of time. We also denote by \( \delta \) the complex, time dependent, dimensionless expression \( \delta = 1 + i\omega t = 1 + i\frac{mL^2}{2m}\tau = 1 + i\frac{\tau}{\tau} \). We have also used the fact that \( e^{-i\omega t} = e^{-i\tan^{-1}(\omega t)} = \frac{\delta^*}{|\delta|} \). Note that with these definitions the normalization factor \( \mathcal{N}_n \) can be written as \( \mathcal{N}_n = \left(\frac{2\pi}{2\pi n!}\right)^{\frac{1}{4}} \sqrt{\frac{L}{|\delta|}} \). 

This set of states constitutes a basis for the space of solutions of the free Schrödinger equation, since it is mapped from a basis for the harmonic oscillator through \( \hat{A} \), which is unitary. The first state of this basis, the one mapped from the harmonic oscillator vacuum state, is given by:

\[
\psi_0(x,t) = \left(\frac{2\pi}{L}\right)^{-\frac{1}{4}} \delta^* \frac{1}{|\delta|} e^{-\frac{x^2\delta^*}{4L^2|\delta|^2}} = \left(\frac{2\pi}{L}\right)^{-\frac{1}{4}} \frac{x^2}{4L^2\delta} e^{-\frac{x^2}{4L^2\delta}},
\]

which is nothing other than a Gaussian wave packet with center at the origin and width \( L \). The parameter \( \tau \) is the dispersion time of the Gaussian wave packet (see, for instance, [10]). The \( n \)-th state represents a wave packet with \( n + 1 \) humps (see [3]).

The family of wave functions (17) has been known in the literature as Hermite-Gauss wave packets [11], and they have been widely used, in their two dimensional version, in paraxial wave optics [12]. Note that, making use of the classical solutions only, and through the QAT, we have been able to import the time evolution from the stationary states of the harmonic oscillator, \( \psi'_n(x',t') \), to the non-stationary ones, \( \psi_n(x,t) \), without solving the time-dependent Schrödinger equation. This will be the motivation for the next section.

### 4 Switching on and off the harmonic potential

Time evolution of an eigenstate of a given Hamiltonian, and in particular the harmonic oscillator, is encoded in just a time-dependent phase factor in the wave function. We have seen that if a physical system \( S \), defined by another Hamiltonian, can be linked to the harmonic oscillator through an Arnold transformation, then the quantum time evolution of \( S \) can be found performing a change of variables in the wave function of the harmonic oscillator and multiplying by a specific phase and factor, that is, carrying out the QAT, provided the necessary classical solutions are known. This way, it is possible to avoid solving the Schrödinger equation or computing the evolution operator whenever the initial state of the system \( S \) is an eigenstate of the harmonic oscillator Hamiltonian. The same observation can be made if we have knowledge of the time evolution of a certain state of the harmonic oscillator and we use it as a initial state of \( S \). To illustrate the following reasoning, we are going to ascribe to the free particle system the role of \( S \) and use the results of the previous Section.

The physical situation we intend to describe is that of a particle in a harmonic potential which is switched off at a given time \( T_0 \). After that, the particle will evolve freely and then, at time \( T_1 \), is captured again by another (in particular, the same) harmonic potential. Time evolution will be harmonic again until it is released once more at \( T_2 \). Finally, it will be detected at time \( T_3 \).

This kind of process is of practical relevance, for instance in the preparation of the 1-dimensional Hermite-Gauss free states (17). This idea was proposed in [13], and was named “Quantum Sling”. The vacuum state of the harmonic oscillator, when switched off, will provide the “vacuum” Gaussian wave packet.
packet with width $L = \sqrt{\frac{\hbar}{2m\omega}}$, where $m$ is the mass of the particle and $\omega$ the frequency of the oscillator. Note that the dispersion time $\tau$ coincides with the inverse of the frequency of the oscillator. If the harmonic oscillator is in the $n$-th excited state, then a state with $(n+1)$ “humps” is obtained.

If we “capture” one of these traveling states at time $T_1$ switching on a harmonic oscillator potential with an appropriate frequency $\omega_1$, it would “freeze” in a harmonic oscillator state (without dispersion), until the potential is switched off again at a time $T_2$ and the wave packet is released, traveling again as a free wave packet that disperses in time. This way, information might be stored temporally in this “oscillator traps”, which can also be used to further manipulating them or even measuring the resulting state by means of adequate lasers. The frequency $\omega_1$ required to capture the dispersed wave packet might be fine tuned in such a way that the wave packet, at the time $T_1$, matches an appropriate eigenstate (with the same $n$) of the harmonic oscillator with frequency $\omega_1$ up to a phase (see below).

Which is the resulting wave function $\psi(T_3)$ after the whole process, given a state prepared for the initial harmonic oscillator $\psi'(T_0)$? Obviously, the formal solution will be a product of three evolution operators, describing free evolution from $T_0$ to $T_1$, harmonic with new frequency $\omega_1$ from $T_1$ to $T_2$ and free from $T_2$ to $T_3$:

$$\psi(T_3) = \hat{U}(T_3, T_2)\hat{U}_{\omega_1}(T_2, T_1)\hat{U}(T_1, T_0)\psi'(T_0).$$  \hspace{1cm} (19)

Before proceeding, let us discuss a little bit of notation. In the previous sections, we have used lowercase, unprimed letters for quantities referring to the free particle and lowercase, primed ones for those of the harmonic oscillator. Now we are indicating physical or true time with capital $T$’s. As the Arnold transformation includes a diffeomorphism in time, if we are going to use it as a tool to perform calculations, we have to be very careful of not confusing the physical time with that used in the Arnold transformations. Recovering the notation of lowercase letters, fix $T_0 = t_0 = t'_0$. Then $T_1 = t_1 = \frac{u_1(t'_1)}{u_2(t'_1)}$, with $u_1$ and $u_2$ satisfying $\Box$ at $t'_0$, and $t'_2 = T_2$, for each evolution to be the initial condition of the next one. Let us also call $t = T_3$.

Regarding the transformations we are going to use, denote by $\hat{A}_{\omega, t_0}(t)$ the QAT from a harmonic oscillator of frequency $\omega$ performed at time $t$, and by $\hat{U}_\omega$ the unitary time evolution operator for that harmonic oscillator. With this notation and the previous choice of classical solutions $u_1(t')$ and $u_2(t')$, for $t = t_0 \ A_{\omega, t_0}(t_0)$ is the identity.

Now, the product (19) can be decomposed by a sequence of QAT’s and harmonic oscillator evolution operators splitting each free evolution operator as the figure (15) in Sec. 3 suggests:

$$\psi(t) = \hat{A}_{\omega_1, t_2}(t)\hat{U}_{\omega_1}(t', t_2)\hat{U}_{\omega_1}(t_2, t_1)\hat{A}_{\omega, t_0}(t_1)\hat{U}_{\omega}(t'_1, t_0)\psi'(t_0) =$$

$$= \hat{A}_{\omega_1, t_2}(t)\hat{U}_{\omega_1}(t', t_1)\hat{A}_{\omega, t_0}(t_1)\hat{U}_{\omega}(t'_1, t_0)\psi'(t_0),$$  \hspace{1cm} (20)

where $t = \frac{\tilde{u}_1(t'_1)}{\tilde{u}_2(t'_1)}$, with $\tilde{u}_1$ and $\tilde{u}_2$ satisfying $\Box$ at $t_2$. This way, in the last expression only evolution operators for harmonic oscillators appear.

The following diagram helps to visualize the setup:
Here, the solid line indicates the physical process followed and the dashed line the “computational” timeline we propose here. The line on the right side denotes the physical timeline.

An obvious generalization of the proposed method consists in increasing the number of captures and releases. On the other hand, depending on the particular initial state, it may be a better choice a path of calculation along purely free evolutions and inverse Arnold transformations.

Note that the evolution will be particularly simple if the frequency $\omega_1$ is such that the width of the wave packet fits a natural width of this second harmonic potential. When the frequency of the harmonic oscillator is not modified, and the same frequency $\omega$ is used to capture the state in the harmonic oscillator trap, the resulting state will be a squeezed state (up to a phase) with squeezing parameter $r$ given by $r = -\log(|\delta_1|)$ (where $\delta_1 = 1 + i\omega t_1$), which is negative. This can be seen as a feasible way of producing squeezing in trapped states, simply switching off-switching on the trap for a lapse of time $t$, resulting in a squeezing parameter $r = -\frac{1}{2} \log(1 + \omega^2 t^2)$. In fact, a similar way of producing squeezed states in Bose-Einstein Condensates was reported in [14].

\[2\] Intuitively, one might say that, in this situation, the particle would be captured in an eigenstate of the oscillator Hamiltonian with frequency $\omega_1$. However, this is not the case: it can be checked that an extra, position-dependent phase appears. This fact and the relationship with the “quasistationary” or “pseudostationary” states [4] will be analyzed elsewhere.
5 Summary and comments

We have reviewed the formulation of the quantum Arnold transformation, particularized it for the relationship between the free particle and harmonic oscillator and suggested to use a series of QATs and pure harmonic oscillator evolutions to perform the calculations to obtain the resulting wave function of a particle that is subjected to the action of turning on and off a harmonic potential.

It must be emphasized that this method can be easily generalized to different quadratic, even time-dependent, potentials and external forces, both changing their time dependence abruptly. Even in this case, only classical solutions of each “phase” of the potential and external force are required to construct the resulting wave function and evolution operator. We believe that this tool may be useful for analytical computations.

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