On the number of rational points close to a compact manifold under a less restrictive curvature condition

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Abstract. Let $\mathcal{M}$ be a compact submanifold of $\mathbb{R}^M$. In this article we establish an asymptotic formula for the number of rational points within a given distance to $\mathcal{M}$ and with bounded denominators under the assumption that $\mathcal{M}$ fulfills a certain curvature condition. Our result generalizes earlier work from Schindler and Yamagishi [25], as our curvature condition is a relaxation of that used by them. We are able to recover a similar result concerning a conjecture by Huang and a slightly weaker analogue of Serre’s dimension growth conjecture for compact submanifolds of $\mathbb{R}^M$.

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1. Introduction

Let $\mathcal{M}$ be a compact immersed submanifold of $\mathbb{R}^M$ and $R = M - \dim \mathcal{M}$ its codimension. Given an integer $Q \in \mathbb{N}$ and $\delta \geq 0$, we study the number

$$N(\mathcal{M}; Q, \delta) := \# \left\{ (a, q) \in \mathbb{Z}^M \times \mathbb{N} \bigg| 1 \leq q \leq Q, \text{dist} \left( \frac{a}{q}, \mathcal{M} \right) \leq \frac{\delta}{q} \right\}$$

of rational points with denominators bounded by $Q$ and $L^\infty$-distance to $\mathcal{M}$ bounded by $\delta$. The study of rational points lying \textquote{close} to manifolds is an approach to the fundamental study of rational points on algebraic varieties. As such it has relevant applications to a
number of problems within Diophantine geometry, besides being an interesting problem in its own right. See for example [11], [15] §2-§5, [16], [19] or [24].

We can readily state the trivial estimate

\[ N(\mathcal{M}; Q, \delta) \lesssim Q^{\dim \mathcal{M} + 1}, \]

and employing a probabilistic heuristic yields

\[ \delta^R Q^{\dim \mathcal{M} + 1} \lesssim N(\mathcal{M}; Q, \delta) \lesssim \delta^R Q^{\dim \mathcal{M} + 1}. \]

It is known that this heuristic does not hold unconditionally. For example, if \( M \) is a rational hyperplane in \( \mathbb{R}^M \) and \( \delta \leq 1 \), then we find that

\[ Q^{\dim \mathcal{M} + 1} \lesssim N(\mathcal{M}; Q, \delta) \lesssim Q^{\dim \mathcal{M} + 1}. \]

We additionally see from that example, that in order to establish non-trivial bounds we may be inclined to study manifolds with a 'proper curvature condition'. Huang proposed the following conjecture in his groundbreaking work [15].

**Conjecture 1.1.** Let \( \mathcal{M} \) be a bounded immersed submanifold of \( \mathbb{R}^M \) with boundary. Let \( R = M - \dim \mathcal{M} \) and suppose \( \mathcal{M} \) satisfies a 'proper curvature condition'. Then there exists a constant \( c_{\mathcal{M}} > 0 \) depending only on \( \mathcal{M} \) such that

\[ N(\mathcal{M}; Q, \delta) \sim c_{\mathcal{M}} \delta^R Q^{\dim \mathcal{M} + 1} \]

when \( \delta \geq Q^{-\frac{1}{R} + \epsilon} \) for some \( \epsilon > 0 \) and \( Q \to \infty \).

It is not made explicit what 'proper curvature conditions' means in the given context.

The first non-trivial case that has been studied extensively is that of a compact curve in \( \mathbb{R}^2 \) with curvature bounded away from zero. In this setting, Huxley [17] was the first to obtain a notable upper bound for a \( C^2 \) curve \( C \), which has later been given in the version

\[ N(\mathcal{C}; Q, \delta) \lesssim Q^{1-\epsilon} Q^2 + Q \log Q \]

for any \( \delta \) and \( Q \) in [26]. In fact, Vaughan and Velani [26] showed that

\[ N(\mathcal{C}; Q, \delta) \lesssim Q^2 + Q^{1+\epsilon} \]

for a \( C^3 \) curve \( \mathcal{C} \), which is the upper bound that Conjecture 1 predicts.

Conversely, a sharp lower bound has been established by Beresnevich, Dickinson and Velani [1]

\[ \delta Q^2 \lesssim N(\mathcal{C}; Q, \delta) \]

with \( \delta \gg Q^{-1} \) and \( \delta Q \to \infty \), for a \( C^3 \) curve \( \mathcal{C} \) admitting at least one point with non-vanishing curvature. Further work by Huang [14] established an asymptotic formula for \( C^3 \) curves. The interested reader may find more details on the case for planar curves in [14].

For the case of general manifolds Beresnevich established the sharp lower bound

\[ \delta^k Q^{\dim \mathcal{M} + 1} \lesssim N(\mathcal{M}; Q, \delta) \]

for any \( \delta \gg Q^{-\frac{1}{R}} \), assuming \( \mathcal{M} \) is an analytic submanifold of \( \mathbb{R}^M \) which admits at least one non-degenerate point, in his spectacular work [2].
Huang established Conjecture \[1\] in the case when \(\mathcal{M}\) is a hypersurface with Gaussian curvature bounded away from zero in \(\mathbb{R}^M\) in [13].

A recent generalization of this result is due to Schindler and Yamagishi [25], who established the Conjecture \[1\] in the case of a compact immersed submanifold of \(\mathbb{R}^M\) in codimension \(R\) with a curvature condition that reduces to Huang’s case for \(R = 1\). In particular, if the manifold \(\mathcal{M}\) is locally parametrized by the functions \(f_1, ..., f_R\), the required curvature condition is as follows.

**Condition 1.2.** Given any \(t \in \mathbb{R}^R \setminus \{0\}\), we have

\[
\det H_{t_1 f_1 + \cdots + t_R f_R}(x_0) \neq 0,
\]

where \(H_f\) denotes the Hessian matrix of the function \(f\).

We continue by presenting the details of our main result. By the compact nature of \(\mathcal{M}\), the argument reduces to a finite number of local arguments, hence we may assume without loss of generality that

\[
\mathcal{M} := \{(x, f_1(x), ..., f_R(x)) \in \mathbb{R}^M \mid x = (x_1, ..., x_n) \in B_{\varepsilon_0}(x_0)\},
\]

where \(x_0 \in \mathbb{R}^n, \varepsilon_0 > 0\) and \(f_r \in C^\ell(\mathbb{R}^n)\) for \(1 \leq r \leq R\) and some \(\ell \geq 2\). Note that this specifically means \(\dim \mathcal{M} = n\).

Let \(\omega \in C^\infty_0(\mathbb{R}^n)\) be a non-negative weight function that is compactly supported in a sufficiently small neighbourhood of \(x_0\) and define

\[
N_\omega(Q, \delta) = \sum_{\begin{array}{c} \mathbf{a} \in \mathbb{Z}^n \\ \|qf_r(a/q)\| \leq \delta \\ 1 \leq r \leq R \end{array}} \omega \left(\frac{a}{q}\right),
\]

where \(\|\cdot\|\) denotes the distance to the closest integer. Obviously \(\|x\| \leq 1/2\) for any \(x \in \mathbb{R}\), hence we only consider \(0 \leq \delta \leq 1/2\). Let

\[
N_0 := \sum_{\begin{array}{c} \mathbf{a} \in \mathbb{Z}^n \\ q \leq Q \end{array}} \omega \left(\frac{a}{q}\right).
\]

For a given function \(f \in C^2(\mathbb{R}^n)\) we denote by \(H_f(x)\) the Hessian matrix of \(f\) evaluated at \(x\), i.e. the \(n \times n\)-matrix whose entries are \(\frac{\partial^2 f}{\partial x_\mu \partial x_\nu}(x)\) for \(1 \leq \mu, \nu \leq n\).

We use the following relaxed curvature condition throughout this article.

**Condition 1.3.** Given any \(t \in \mathbb{R}^R \setminus \{0\}\), we have

\[
\text{rank} \ H_{t_1 f_1 + \cdots + t_R f_R}(x_0) \geq n - 1.
\]

With these notations we have the following result.

**Theorem 1.4.** Let \(n \geq 3\) and \(\ell > \max\{n + 1, \frac{n}{2} + 4\}\). Suppose \(\text{[3]}\) holds and that \(\varepsilon_0 > 0\) is sufficiently small. Then we have

\[
|N_\omega(Q, \delta) - (2\delta)^R N_0| \leq \begin{cases} 
\delta^\frac{(R-1)(n-1)}{n+1} Q^{n+\frac{2}{n+1} \delta n^{-1}(Q)} & \text{if } \delta \geq Q^{\frac{n-1}{n+2-2R}}, \\
Q^{n+\frac{2}{n+1} \delta n^{-1}(Q)} \delta n^{-1}(Q) & \text{if } \delta < Q^{\frac{n-1}{n+2-2R}}, 
\end{cases}
\]
where
\[ E_{n-1}(Q) = \begin{cases} \exp(\epsilon_1 \sqrt{\log Q}) & \text{if } n = 3, \\ \exp(\log Q \epsilon_2) & \text{if } n \geq 4, \end{cases} \]
for some positive constants \( \epsilon_1 \) and \( \epsilon_2 \). These constants as well as the implicit constants only depend on \( \mathcal{M} \) and \( \omega \).

Comparing our exponents to those obtained in [25, Theorem 1.2], we have \( n + 2n + 1 \) instead of \( n \) in the first case and \( n - \frac{(R-1)(n+1-3R)}{n+2R+1} \) instead of \( n - \frac{(n-2)(R-1)}{n+2R-2} \) in the second one. Note that the cases in [25] are distinguished by the comparison of \( \epsilon \) against \( Q^{-\frac{n+1}{n+2R-1}} \). As expected the bounds under the less restrictive curvature condition are worse, yet they have similar growth for large \( n \).

By Poisson summation formula we find that \( N_0 = \sigma Q^{n+1} + O(Q^n) \) for some positive constant \( \sigma \) depending only on \( \omega \) (compare [15, (6.2)]). Combining this with 1.4 yields
\[ N_{\omega}(Q, \delta) = \frac{(2\epsilon)^R \sigma Q^{n+1} + O\left(\epsilon^{\frac{3R-1}{n+1}} Q^n + \frac{\log Q}{Q^{n+1}} E_{n-1}(Q)\right)}{\delta^R Q^{n+1}}, \]
when \( \delta \geq Q^{\frac{n+1}{n+2R-1}+\epsilon} \) for any \( \epsilon > 0 \) sufficiently small. Following the arguments in [15], Section 7, we can approximate the characteristic function of \( B_{\epsilon_0}(x_0) \) by smooth weight functions and obtain:

**Corollary 1.5.** Let \( \mathcal{M} \) be as in (1.1), \( n \geq 3 \) and \( \ell > \max\{n+1, \frac{3}{2}+4\} \). Suppose Condition 1.3 holds and that \( \epsilon_0 > 0 \) is sufficiently small. Then there exists a constant \( c_{\mathcal{M}} > 0 \) depending only on \( \mathcal{M} \) such that
\[ N(\mathcal{M}; Q, \delta) \sim c_{\mathcal{M}} \delta^R Q^{\dim \mathcal{M} + 1} \]
when \( \delta \geq Q^{-\frac{n+1}{n+2R-1}+\epsilon} \) for any \( \epsilon > 0 \) sufficiently small and \( Q \to \infty \).

Note that
\[ Q^{-\frac{3}{2}+4} \geq Q^{-\frac{n+1}{n+2R-1}} \]
only holds for \( R > 1 \) and \( n \geq \frac{3R-1}{R-1} \), hence Conjecture 1.1 holds in those cases. In fact, the asymptotic formula is obtained beyond the range of \( \delta \) that was conjectured in those cases.

If we let \( \delta = 0 \), then our (weighted) counting function gives the (weighted) number of rational points with bounded denominators that lie on the manifold \( \mathcal{M} \). Applying the arguments from [15, pp. 2047] to Conjecture 1.1, we obtain
\[ N(\mathcal{M}; Q, 0) \ll Q^{\dim \mathcal{M} + \epsilon}, \]
for any \( \epsilon > 0 \) sufficiently small with a generally sharp upper bound. This can be interpreted as an analogue of Serre’s famous dimension growth conjecture for projective varieties in the context of smooth submanifolds of \( \mathbb{R}^M \).

**Conjecture 1.6 (Serre’s dimension growth conjecture).** Let \( X \subseteq \mathbb{P}^{M-1}_Q \) be an irreducible projective variety of degree at least two defined over \( Q \). Let \( N_X(B) \) be the number of rational points on \( X \) with naive height bounded by \( B \). Then
\[ N_X(B) \ll B^\dim X (\log B)^c \]
for some constant \( c > 0 \).
The reader may be interested in the large amount of literature concerning the dimension growth conjecture and find [10] to be a nice introduction to the topic. For further reading we may refer to several examples, such as [2], [11], [12], [13], [15], [16], [17], [18], [21], [22], [23], [24], [27]. Our analogue result for smooth submanifolds of $\mathbb{R}^N$ is the following estimate.

**Corollary 1.7.** Let $\mathcal{M}$ be as in (1.1), $n \geq 4$ and $\ell > \max\{n + 1, \frac{n}{2} + 4\}$. Suppose Condition 1.3 holds and that $\epsilon_0 > 0$ is sufficiently small. Then

$$N(\mathcal{M}; Q, 0) \ll Q^{n - \frac{(R - 1)n + 1 - 3\pi}{n + 2\pi + 1}}(\log Q)^c$$

for some constant $c > 0$.

Note that in contrast to the situation in [25] we do not unconditionally break the dim $\mathcal{M}$ barrier here, only if $n > \frac{4\pi}{\log 2}$ and $R > 1$.

We adapt the strategy for proving Theorem 1.4 established in [25], which relies on the methods developed by Huang in [15] and fibration arguments. In particular, Schindler and Yamagishi reduced the problem to that for one function, such that the main result of [15] can be used. This is achieved by a more complicated version of the procedure developed by Huang in [15], which relates the counting problem of a function to that of its Legendre transform, for a family of functions satisfying the curvature condition 1.2 and applying it twice. For our relaxed curvature condition 1.3 we can use a similar approach with some necessary adjustments to accommodate an additional degree of freedom. After collecting some preliminary results in Section 2, we discuss the setup of our proof for Theorem 1.4 in Section 3. Section 4 is dedicated to establishing some auxiliary bounds, one of which depending on a result which is proven in Section 5. Lastly, we combine our findings to prove Theorem 1.4 in Section 6.

## 2. Preliminaries

In this article we denote by $C^\ell(V)$ the set of $\ell$-times continuously differentiable functions defined on an open set $V \subseteq \mathbb{R}^n$. Analogously $C^\infty(V)$ denotes the set of smooth functions defined on $V$ and $C^\infty_c(V)$ the set of smooth functions defined on $V$ that have a compact support. Given any $f \in C^1(\mathbb{R}^n)$ we let $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ be the gradient of $f$. For a subset $X \subseteq \mathbb{R}^n$ we denote the boundary of $X$ by $\partial X = \overline{X} \setminus X^o$, where $\overline{X}$ denotes the closure of $X$ and $X^o$ denotes the interior of $X$. For any $z \in \mathbb{R}$ we let $e(z) = e^{2\pi i z}$ and $||z||$ denotes the distance to the closest integer. For $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ let $|z| = \max_{1 \leq i \leq n} |z_i|$ denote the $L^\infty$-Norm and given any $\epsilon > 0$ we let

$$B^\epsilon_c(z) = \{x \in \mathbb{R}^n \mid |x - z| < \epsilon\} = (z_1 - \epsilon, z_1 + \epsilon) \times \cdots \times (z_n - \epsilon, z_n + \epsilon).$$

We may write $B^\epsilon(z)$ instead of $B^\epsilon_c(z)$ if the dimension is clear from context. By the notation $f(x) \ll g(x)$ or $f = O(g(x))$ we mean that there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all $x$ in consideration.

Given $F \in C^\ell(\mathbb{R}^n)$ let $U \subseteq \mathbb{R}^n$ be an open subset such that $\nabla F$ is invertible on $U$. We define the Legendre transform $F^* : \nabla F(U) \to \mathbb{R}$ of $F$ by

$$F^*(z) = z \cdot (\nabla F)^{-1}(z) - (F \circ (\nabla F)^{-1})(z).$$
It can be verified that $F^*$ is $\ell$-times continuously differentiable, $F^{**} = F$ and $\nabla F^* = (\nabla F)^{-1}$. If $x = \nabla F(z)$ we obviously have

$$F^*(x) = x \cdot z - F(z)$$  \hspace{1cm} (2.1)

and furthermore

$$H_{F^*}(x) = H_F(z)^{-1}.$$  \hspace{1cm} (2.2)

For the following results on oscillatory integrals, we refer the reader to see for example [13, Theorem 7.71 and Theorem 7.7.5].

**Lemma 2.1 (non-stationary phase).** Let $\ell \in \mathbb{N}$ and $U_+ \subseteq \mathbb{R}^d$ a bounded open set. Let $\omega \in \mathcal{C}^{\ell-1} \mathcal{C}^{\ell}(\mathbb{R}^d)$ with supp $\omega \subseteq U_+$ and $\varphi \in \mathcal{C}^{\ell}(U_+)$ with $\nabla \varphi(x) \neq 0$ for all $x \in \text{supp}\omega$. Then for any $\lambda > 0$

$$\left| \int_{\mathbb{R}^d} \omega(x)e(\lambda \varphi(x))dx \right| \leq c_{\ell} \lambda^{-\ell+1},$$

where the constant $c_{\ell}$ only depends on $\ell,d$, upper bounds for the absolute values of finitely many derivatives of $\omega$ and $\varphi$ on $U_+$, and a lower bound for $|\nabla \varphi|$ on supp $\omega$.

Recall given a symmetric matrix we define its signature to be the number of positive eigenvalues minus the number of negative eigenvalues.

**Lemma 2.2 (stationary phase).** Let $\ell > \frac{d}{2} + 4$ and $\mathcal{D}, \mathcal{D}_+ \subseteq \mathbb{R}^d$ bounded open sets such that $\mathcal{D} \subseteq \mathcal{D}_+$. Let $\omega \in \mathcal{C}^{\ell-1}(\mathbb{R}^d)$ with $\text{supp}\omega \subseteq \mathcal{D}$ and $\varphi \in \mathcal{C}^{\ell}(\mathcal{D})$. Suppose $\nabla \varphi(v_0) = 0$ and $H_{\varphi}(v_0) \neq 0$ for some $v_0 \in \mathcal{D}$. Let $\sigma$ be the signature of $H_{\varphi}(v_0)$ and $\Delta = |\det H_{\varphi}(v_0)|$. Suppose further that $\nabla \varphi(x) \neq 0$ for all $x \in \mathcal{D} \setminus \{v_0\}$. Then for any $\lambda > 0$

$$\int_{\mathbb{R}^d} \omega(x)e(\lambda \varphi(x))dx = c \left( \lambda \varphi(v_0) + \frac{\sigma}{8} \right) \Delta^{-\frac{1}{2}} \lambda^{-\frac{d}{2}} (\omega(v_0) + O(\lambda^{-1})), $$

where the implicit constant only depends on $\ell,d$, upper bounds for the absolute values of finitely many derivatives of $\omega$ and $\varphi$ on $\mathcal{D}_+$, an upper bound for $|x - v_0|/|\nabla \varphi(v_0)|$ on $\mathcal{D}_+$, and a lower bound for $\Delta$.

Note that this is a simplified version of [13, Theorem 7.7.5], where the assumption on $\ell$ can be deduced from [13, pp. 222, Remark].

We consider two compactness result. Let $m \in \mathbb{N}_0$ and $G = G_1 \times G_2 \subseteq \mathbb{R}^{n+m}$, where $G_1 \subseteq \mathbb{R}^n$ and $G_2 \subseteq \mathbb{R}^m$ are bounded connected open sets. Let $x_0$ be a fixed point in $G_1$.

**Lemma 2.3.** Let $G \in \mathcal{C}^\ell(G)$, $\ell \geq 2$ and assume $H_{G_t}(x) \neq 0$ for every $t \in G_2$, where $G_t$ is the real-valued map on $G_1$ given by $x \mapsto G(x,t)$. Let $F_2$ be a compact set contained in $G_2$, then there exist a real number $\tau > 0$ and constants $c_1,c_2 > 0$ such that

$$c_1 \leq |\det H_{G_t}(x)| \leq c_2$$

for all $x \in B_\tau(x_0)$ and $t \in F_2$. Moreover the map $x \mapsto \nabla G_t(x)$ is a $C^{\ell-1}$-diffeomorphism on $B_\tau(x_0)$ for all $t \in F_2$.

**Lemma 2.4.** Let $G \in \mathcal{C}^\ell(G)$, $\ell \geq 2$ and assume $H_{G_t}(x) \neq 0$ for every $t \in G_2$. Let $F_2$ and $\tau$ be as in Lemma 2.3. Then for any $0 < \kappa < \tau$ sufficiently small, there exists $\rho > 0$ such that

$$\text{dist}(\partial(\nabla G_t(B_\tau(x_0))), \partial(\nabla G_t(B_\kappa(x_0)))) \geq 2\rho$$

for all $t \in F_2$.

For proofs of 2.3 and 2.4 we refer the reader to [25, Lemmas 3.4 and 3.5].
3. Setting up the proof of [1.4]

By virtue of the characteristic functions

\[ \chi_\delta(\theta) = \begin{cases} 1 & \text{if } ||\theta|| \leq \delta, \\ 0 & \text{else.} \end{cases} \]  

(3.1)

for \(0 < \delta \leq 1/2\) we can rewrite

\[ N_\omega(Q, \delta) = \sum_{\substack{a \in \mathbb{Z}^n \\cap (q \leq Q)} \omega\left(\frac{a}{q}\right) \prod_{r=1}^{R} \chi_\delta\left(q f_r\left(\frac{a}{q}\right)\right). \]

Consider the Selberg magic functions as described in [20] for the interval \([-\delta, \delta] \subseteq \mathbb{R}/\mathbb{Z}\) and a parameter \(J \in \mathbb{N}\)

\[ S_J^\pm(x) = \sum_{|j| \leq J} \hat{S}_J^\pm(j)e(jx). \]

They obey the properties

\[ S_J^-(y) \leq \chi_\delta(y) \leq S_J^+(y) \]

and

\[ \hat{S}_J^+(0) = 2\delta \pm \frac{1}{J+1} \]

and are bounded by

\[ |\hat{S}_J^\pm(j)| \leq \frac{1}{J+1} + \min\left(2\delta, \frac{1}{\pi |j|}\right) \]

(3.2)

for all \(y \in \mathbb{R}/\mathbb{Z}\) and \(0 \leq |j| \leq J\). Hence we can bound the characteristic functions from above by the Selberg magic functions and obtain

\[ N_\omega(Q, \delta) \leq \sum_{\substack{a \in \mathbb{Z}^n \\cap (q \leq Q)} \omega\left(\frac{a}{q}\right) \prod_{r=1}^{R} \hat{S}^+_J\left(q f_r\left(\frac{a}{q}\right)\right) \]

\[ = \sum_{\substack{a \in \mathbb{Z}^n \\cap (q \leq Q)} \omega\left(\frac{a}{q}\right) \prod_{r=1}^{R} \left( \sum_{j_r = -J}^{J} \hat{S}^+_J(j_r)e\left(j_r q f_r\left(\frac{a}{q}\right)\right) \right) \]

\[ = \sum_{\substack{a \in \mathbb{Z}^n \\cap (q \leq Q)} \omega\left(\frac{a}{q}\right) \prod_{r=1}^{R} \left( \sum_{0 \leq |j_r| \leq J} \sum_{1 \leq i \leq R} \hat{S}^+_J(j_r) e\left(\sum_{r=1}^{R} j_r q f_r\left(\frac{a}{q}\right)\right) \right) \]

The terms with \(j_i = 0\) for all \(i = 1, \ldots, R\) contribute

\[ \left(2\delta + \frac{1}{J+1}\right)^R N_0 = (2\delta)^R N_0 + O\left(\delta^{R-1} Q^{n+1} \frac{1}{J} + \frac{Q^{n+1}}{J^R}\right), \]

with the implicit constant possibly depending on \(R\) and an upper bound for the diameter of \(\text{supp } w\). Bounding the characteristic function from below by the Selberg magic functions
yields a similar result, such that we conclude

$$|N_\omega(Q, \delta) - (2\delta)^R N_0| \ll \delta^{R-1} \frac{Q^{n+1}}{J} + \frac{Q^{n+1}}{JR}$$

(3.3)

$$+ \sum_{1 \leq |j| \leq J, \frac{1}{2} \leq r \leq R} \left( \prod_{r=1}^{R} b_{jr} \right) \left| \sum_{a \in \mathbb{Z}^n \atop q \leq Q} \omega \left( \frac{a}{q} \right) e \left( \sum_{r=1}^{R} j_r q f_r \left( \frac{a}{q} \right) \right) \right|,$$

where

$$b_{jr} := \frac{1}{J+1} + \min \left( \frac{2\delta}{\pi |J_r|} \right)$$

is the bound for the Selberg magic functions given in (3.2). Via Poisson summation formula we can rewrite

$$\sum_{a \in \mathbb{Z}^n} \omega \left( \frac{a}{q} \right) e \left( \sum_{r=1}^{R} j_r q f_r \left( \frac{a}{q} \right) \right)$$

(3.4)

$$= \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \omega \left( \frac{x}{q} \right) e \left( \sum_{r=1}^{R} j_r q f_r \left( \frac{x}{q} \right) - k \cdot x \right) \, dx$$

$$= q^n \sum_{k \in \mathbb{Z}^n} I(q; j; k)$$

with

$$I(q; j; k) = \int_{\mathbb{R}^n} \omega(x) e \left( \sum_{r=1}^{R} q j_r f_r(x) - q k \cdot x \right) \, dx.$$

We now make use of (3.3). By assumption the Hessian matrix $H_{t_1, t_2, \ldots, t_r}(x_0)$ has a non-vanishing minor of size $n - 1$, i.e. we can find $1 \leq i, j \leq n$ such that after deleting the $i$-th row and the $j$-th column of $H_{t_1, t_2, \ldots, t_r}(x_0)$, the resulting matrix is invertible. Since the Hessian matrix is symmetric, we necessarily have $i = j$ in this case. Consider the functions

$$\vartheta_i : \mathbb{R}^R \setminus \{0\} \to \mathbb{R}, \, t \mapsto \frac{\partial^2 t_1 f_1 + \cdots + t_R f_R}{\partial x_\nu \partial x_\mu} |_{1 \leq \nu, \mu \leq n} (x_0)$$

for $1 \leq i \leq n$. Given (3.3) the preimages $\{ \vartheta_i^{-1}(\mathbb{R} \setminus \{0\}) \}_{1 \leq i \leq n}$ form an open cover of $\mathbb{R}^R \setminus \{0\}$. For any fixed $t$ the rank condition is invariant under scaling with a linear factor $a \in \mathbb{R}_+$, i.e. if $t$ belongs to $\vartheta_i^{-1}(\mathbb{R} \setminus \{0\})$ so does $at = (at_1, \ldots, at_R)$. Therefore we can assume $t$ to be normalized in the sense $|t| = 1$. For every $1 \leq i \leq n$ let $\tilde{T}_i = \vartheta_i^{-1}(\mathbb{R} \setminus \{0\}) \cap B_1^R(0)$, then $\{\tilde{T}_1, \ldots, \tilde{T}_n\}$ is an open cover of $B_1^R(0)$. Since $B_1^R(0)$ is compact and Hausdorff, it is also normal, hence the open cover $\{\tilde{T}_1, \ldots, \tilde{T}_n\}$ admits a shrinkage. That is an open cover $\{T'_1, \ldots, T'_n\}$ such that $T_i := T_i' \subseteq \tilde{T}_i$ for $1 \leq i \leq n$.

To find a bound for the last term in (3.3) it suffices to find an upper bound for

$$N^{(r; \epsilon_\nu)}(Q, \delta) = \sum_{1 \leq j \leq J, \frac{1}{2} \leq r \leq R} \left( \prod_{r=1}^{R} b_{jr} \right) \left| \sum_{q \leq Q} \sum_{k \in \mathbb{Z}^n} I(q; (\epsilon_1 j_1, \ldots, \epsilon_R j_R); k) \right|$$

(3.5)
for each $1 \leq r \leq R$, $\epsilon \in \{-1,1\}^R$ and $1 \leq \nu \leq n$. The arguments turn out to be identical for all $(r; \epsilon; \nu)$, since different choices of $r$ or $\epsilon$ admit only to relabeling and the choice of $\nu$ is merely an exercise in notation. Therefore we only present the details for $N^{(1;\ldots;1,n)}(Q, \delta)$. Note that the same upper bound in fact holds for all $N^{(r;\epsilon;\mu)}$.

To $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $\hat{x} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and define functions $f_{j,y}: \mathbb{R}^{n-1} \to \mathbb{R}, \hat{x} \mapsto f_j(\hat{x}, y)$ for $y$ in a sufficiently small neighborhood $U(x_0, n)$ of $x_{0,n}$. Define further $G_y(\hat{x}, t) = f_{1,y}(\hat{x}) + \sum_{r=2}^{R} t_r f_{r,y}(\hat{x})$ and consider the continuous function $\psi: U(x_0, n) \to \mathbb{R}, y \mapsto \det \left( \frac{\partial^2 (f_1 + t_2 f_2 + \cdots + t_R f_R)}{\partial x_r \partial x_\mu} \right)_{1 \leq r, \mu \leq n-1}(\hat{x}, y)$.

For a suitable $\epsilon' > 0$ we have $\psi(y) \neq 0$ for $y \in B_\epsilon(x_{0,n})$, since $\psi(x_{0,n}) \neq 0$ by construction. Take $0 < \varepsilon < \varepsilon'$ sufficiently small, then on the compact set $\overline{\mathcal{W}}$ with $\mathcal{W} = B_{\varepsilon}(x_{0,n})$ we have

$$c_1' \leq \left| \det \left( \frac{\partial^2 (f_1 + t_2 f_2 + \cdots + t_R f_R)}{\partial x_r \partial x_\mu} \right)_{1 \leq r, \mu \leq n-1}(\mathbf{0}, y) \right| \leq c_2'$$

with constants $0 < c_1', c_2'$ for all $t \in T_n$. Now $G_y$ satisfies the conditions of lemmas 2.3 and 2.4 for $\mathcal{F}_2 = \overline{\mathcal{W}} \times T_n$, i.e. there are constants $\tau(1;\ldots;1,n) > 0$ and $c_1, c_2 > 0$ such that

$$c_1 \leq \left| \det \left( \frac{\partial^2 (f_1 + t_2 f_2 + \cdots + t_R f_R)}{\partial x_r \partial x_\mu} \right)_{1 \leq r, \mu \leq n-1}(\hat{x}, y) \right| \leq c_2$$

(3.6)

for all $t \in T_n$, $y \in \overline{\mathcal{W}}$ and $x \in B_{2\tau(1;\ldots;1,n)}(\mathbf{0})$. Moreover, the map

$$\hat{x} \mapsto \left( f_{1,y} + \sum_{r=2}^{R} t_r f_{r,y} \right)(\hat{x})$$

is a $C^{\ell-1}$ diffeomorphism on $B_{2\tau(1;\ldots;1,n)}(\mathbf{0})$ for all $t \in T_n$ and $y \in \overline{\mathcal{W}}$. Define $\tau(r;\epsilon;\mu)$ in the same way for $1 \leq r \leq R$, $\epsilon \in \{-1,1\}^R$ and $1 \leq \mu \leq n$ and let

$$0 \leq \tau = \min_{1 \leq r \leq R} \tau(r;\epsilon;\mu) \leq \min_{\epsilon \in \{-1,1\}^R, 1 \leq \nu \leq n} \tau(r;\epsilon;\nu)$$

be sufficiently small such that (4.3) is applicable. For this choice of $\tau$ with (2.4) we find constants $0 < \kappa < \tau$ and $\rho$ such that

$$\text{dist} \left( \partial \left( \nabla \left( f_{1,y} + \sum_{r=2}^{R} t_r f_{r,y} \right) (B_\epsilon(\mathbf{0})) \right), \partial \left( \nabla \left( f_{1,y} + \sum_{r=2}^{R} t_r f_{r,y} \right) (B_\kappa(\mathbf{0})) \right) \right) \geq 2\rho$$

(3.7)
for all $t \in T_n$ and $y \in \mathcal{Y}$. Note that $\varepsilon_0 < 2\tau$ is a sufficient choice in [1,4].

Let $\mathcal{D} = B_r(\hat{x}_0)$ and let $\omega \in C^\infty_c(\mathbb{R}^n)$ be a non-negative weight function such that for any $y \in \mathcal{Y}$ the closure of

$$U_y := \{ x \in \mathbb{R}^{n-1} \mid \omega(x, y) \neq 0 \}$$

is contained in $B_n(\hat{x}_0)$. Define $\hat{F}_{y,j} = \hat{f}_{1,y} + (j_2/j_1)\hat{f}_{2,y} + \cdots + (j_R/j_1)\hat{f}_{R,y}$ and $V_{y,j} = \nabla \hat{F}_{y,j}(U_y)$. Since $0 \leq j_r/j_1 \leq 1$ for $2 \leq r \leq R$ we know that $\nabla \hat{F}_{y,j}$ is a diffeomorphism on $U_y$ and $\mathcal{D}$.

**Lemma 3.1.** The functions $\hat{f}_{r,y}$ for $1 \leq r \leq R$ are bounded on $\overline{B_{2\tau}(\hat{x}_0)}$ for all $y \in \mathcal{Y}$ and the bounds are independent of $y$. Additionally, there is $L \in \mathbb{N}$ such that for all $t \in T_n$ and all $i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}$ with $\sum_{j=1}^{n-1} i_j \leq \ell$ we have

$$\left| \frac{\partial^{i_1+i_2+\cdots+i_{n-1}} t_1 f_1 + \cdots + t_R f_R}{\partial x_1^{i_1} \cdots \partial x_{n-1}^{i_{n-1}}} (x) \right| \leq L$$

on $\overline{B_{2\tau}(\hat{x}_0)} \times \mathcal{Y}$ and $\rho \leq L$ for $\rho$ in (3.7).

**Proof.** By assumption $f_r(x)$ is smooth, hence on the compact domain $\overline{B_{2\tau}(\hat{x}_0)} \times \mathcal{Y}$ it attains a maximum $M_r$. Now by definition $\hat{f}_{r,y}(\hat{x}) = f_r(\hat{x}, y)$, hence for any $y \in \mathcal{Y}$

$$|\hat{f}_{r,y}(\hat{x})| < M_r$$

on $\overline{B_{2\tau}(\hat{x}_0)}$. For the derivatives note that all domains of definition, i.e. $\overline{B_{2\tau}(\hat{x}_0)}$, $\mathcal{Y}$ and $T_n$ are compact and all the relevant functions depend at least continuously on $x = (\hat{x}, y)$ and $t$, hence for any given suitable $(i_1, \ldots, i_{n-1})$ there exists a maximum

$$M_{(i_1, \ldots, i_{n-1})} = \max_{x = (\hat{x}, y)} \left| \frac{\partial^{i_1+i_2+\cdots+i_{n-1}} t_1 f_1 + \cdots + t_R f_R}{\partial x_1^{i_1} \cdots \partial x_{n-1}^{i_{n-1}}} (x) \right|.$$ 

Since there are only finitely many suitable choices for $(i_1, \ldots, i_{n-1})$ we can also take the maximum over them and define $L$ to be the smallest natural number such that

$$L \geq \max_{(i_1, \ldots, i_{n-1})} M_{(i_1, \ldots, i_{n-1})}$$

and

$$L \geq \rho.$$ 

Specifically we have that $V_{y,j} \subseteq [-L, L]^{n-1}$ independently of $y$ and $j$.

We split the set of $k \in \mathbb{Z}^n$ into three disjoint subsets. Let $\hat{k} = (k_1, \ldots, k_{n-1})$ and

$$D(\hat{k}, j) = \min_{y \in \mathcal{Y}} \text{dist} \left( \frac{\hat{k}}{j_1}, \frac{V_{y,j}}{j_1} \right).$$

Now define

$$\mathcal{K}_{j,1} = \left\{ k \in \mathbb{Z}^n \mid \frac{\hat{k}}{j_1} \in \bigcup_{y \in \mathcal{Y}} V_{y,j}, |k_n| \leq 2j_1 L \right\},$$

and

$$\mathcal{A}_{j,1} = \frac{\mathbb{R}^n}{j_1} \setminus \mathcal{K}_{j,1}. $$

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\[ \mathcal{K}_j = \left\{ k \in \mathbb{Z}^n \bigg| D(\hat{k}, j) \geq \rho \right\} \cup \left\{ k \in \mathbb{Z}^n \mid |k_n| > 2j_1L \right\} \]
and
\[ \mathcal{K}_j = \left\{ k \in \mathbb{Z}^n \bigg| D(\hat{k}, j) < \rho, \frac{\hat{k}}{j_1} \notin \bigcup_{y \in \mathcal{Y}} V_{y,j}, |k_n| \leq 2j_1L \right\}. \]

For each \( 1 \leq i \leq 3 \), we let
\[ N_i = \sum_{1 \leq j_1 \leq j} \left( \prod_{r=1}^{R} b_{j_r} \right) \left| \sum_{q \leq Q} \sum_{k \in \mathcal{X}_{j,i}} I(q; j; k) \right| \tag{3.8} \]
such that
\[ N_1 = \sum_{0 \leq j_2, \ldots, j_R \leq j_1} \left( \prod_{r=1}^{R} b_{j_r} \right) \left| \sum_{q \leq Q} \sum_{k \in \mathcal{X}_{j,i}} I(q; j; k) \right| \]

such that
\[ N_1 \leq N_2 + N_3 \tag{3.9} \]
and proceed to bound each \( N_i \) separately.

4. Bounds for \( N_1, N_2 \) and \( N_3 \)

Lemma 4.1. For any \( K > 0 \) we have that
\[ \left\{ \frac{\hat{k}}{j_1} \mid \hat{k} \in \mathbb{Z}^{n-1}, D(\hat{k}, j) < K \right\} \subseteq [-L - K, L + K]^{n-1}, \]
where \( L \) is defined as in (3.1).

Proof. For \( (\hat{k}/j_1) \in [-L, L]^{n-1} \) the inclusion is obvious, so let \( (\hat{k}/j_1) \not\in [-L, L]^{n-1} \). We have
\[ K > D(\hat{k}, j) = \min_{y \in \mathcal{Y}} \inf_{z \in V_{y,j}} \left| \frac{\hat{k}}{j_1} - z \right| \geq \min_{z \in [-L, L]^{n-1}} \left| \frac{\hat{k}}{j_1} - z \right| \]
\[ \geq \min_{z \in [-L, L]^{n-1}} \left| \frac{|k|}{j_1} - |z| \right| = \frac{|k|}{j_1} - L, \]
hence \( K + L > |\hat{k}/j_1| \) as desired. \( \square \)

Case \( k \in \mathcal{X}_{j,2} \). Let
\[ D_1(\hat{k}, j) = j_1 D(\hat{k}, j) = \min_{y \in \mathcal{Y}} \text{dist}(\hat{k}, j_1V_{y,j}). \]

For a fixed \( k_n \in \mathbb{Z} \), consider the integral
\[ \int_{\mathbb{R}^{n-1}} \omega(\bar{x}, y)e \left( q_{j_1} \left( \frac{F_{y,j}(\bar{x})}{j_1} - \frac{\hat{k} \cdot \bar{x} + k_n y}{j_1} \right) \right) d\bar{x}. \]
with \((\hat{k}, k_n) \in \mathcal{X}_{j;2}\) and \(D(\hat{k}, j) \geq \rho\). Let

\[
\varphi_{y,1}(\hat{x}) = \frac{j_1 \hat{f}_{1,y}(\hat{x}) + \cdots + j_R \hat{f}_{R,y}(\hat{x}) - \hat{k} \cdot \hat{x} - k_n y}{D_1(\hat{k}, j)}
\]

and \(\lambda_1 = qD_1(\hat{k}, j)\). Then by definition of \(V_{y,j}\)

\[
|\nabla \varphi_{y,1}(\hat{x})| = \frac{|j_1 \nabla \hat{f}_{1,y} + \cdots + j_R \nabla \hat{f}_{R,y}(\hat{x}) - \hat{k}|}{D_1(\hat{k}, j)} \geq 1
\]

for \(\hat{x} \in U_y\). Let \(U^+_y \subseteq \mathbb{R}^{n-1}\) be an open set such that \(U_y \subseteq U_y^+ \subseteq \mathcal{D}, V_{y,j}^+ = \nabla F_j(U_y^+) \subseteq [-2L, 2L]^{n-1}\),

\[
\min_{\hat{x} \in U_y} < \frac{\text{dist}(\partial \mathcal{D}, \partial U_y)}{4}
\]

(4.1)

for any \(\hat{x} \in U_y^+\) and

\[
\min \max_{\hat{x} \in U_y^+} |\nabla (t_1 f_{1,y} + \cdots + t_R f_{R,y})(\hat{x}) - \nabla (t_1 f_{1,y} + \cdots + t_R f_{R,y})(x)| < \frac{\rho}{2}
\]

for any \(\hat{x} \in U_y^+\). Then we have

\[
\nabla \varphi_{y,1}(\hat{x}) \geq \frac{1}{2}
\]

for all \(\hat{x} \in U_y^+\).

Now assume that \(|k_n| \leq D_1(\hat{k}, j)\) or \(|k_n| \leq 2j_1 L\).

**Lemma 4.2.** Let \(i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}\) with \(\sum_{\mu=1}^{n-1} i_\mu \leq \ell\). Then for all \(\hat{x} \in U_y^+\) we have

\[
\left| \frac{\partial^{i_1 + \cdots + i_{n-1}} \varphi_{y,1}}{\partial x_1^{i_1} \cdots \partial x_{n-1}^{i_{n-1}}}(\hat{x}) \right| \ll 1,
\]

where the implicit constant depends only on \((i_1, \ldots, i_{n-1}), \rho, \tau, \varepsilon_1\) and upper bounds for (the absolute values of) finitely many derivatives of \(f_r\) on \(\mathcal{D} \times \mathcal{Y}\) for \(1 \leq r \leq R\).

**Proof.** Choose \(C > 0\) such that

\[
\frac{1}{C} \max_{t \in T_n, y \in \mathcal{Y}, \hat{x} \in U_y} |\nabla (t_1 f_{1,y} + \cdots + t_R f_{R,y})(\hat{x})| < \frac{1}{2}.
\]

and assume \(j_1 C \leq \|k\|\). Then we have

\[
\left| \frac{\hat{k}}{|k|} - \frac{j_1 z}{|k|} \right| \geq 1 - \frac{j_1 |z|}{|k|} > \frac{1}{2}
\]

for all \(z \in \bigcup_{y \in \mathcal{Y}} V_{y,j}\) and hence

\[
\frac{1}{|k|} D_1(\hat{k}, j) = \min_{y \in \mathcal{Y}} \left( \frac{\hat{k}}{|k|}, \frac{j_1 V_{y,j}}{|k|} \right) \geq \frac{1}{2}.
\]
Therefore

\[ |\varphi_{y,1}(\hat{x})| \leq \left| \frac{j_{1} \hat{f}_{1,Y}(\hat{x}) + \cdots + j_{R} \hat{f}_{R,Y}(\hat{x}) - k_{y}}{|k|} | \frac{k_{n,y}}{D_{1}(k,j)} \right| + \left| \frac{k_{n,y}}{D_{1}(k,j)} \right| \]

\[ \leq 2 \left( \frac{1}{|k|} |f_{1,Y}(\hat{x})| + \cdots + \frac{j_{R}}{|k|} |f_{R,Y}(\hat{x})| + \frac{k_{y}}{|k|} \cdot |\hat{x}| \right) + \left| \frac{k_{n,y}}{D_{1}(k,j)} \right| |y|. \]

Since \(|\hat{k}| > j_{1}C\), we have \(\frac{j_{r}}{|k|} \leq 1\) for \(1 \leq r \leq R\). Let \(M_{r} > 0\) be the bound for \(|f_{r,Y}|\) established in \(3.1\) on \(U_{y}^{+}\) for \(1 \leq r \leq R\) and \(S = M_{1} + \cdots + M_{R}\). Now by assumption we either have

\[ \left| \frac{k_{n,y}}{D_{1}(k,j)} \right| < 1 \quad \text{or} \quad \left| \frac{k_{n,y}}{D_{1}(k,j)} \right| \leq \frac{2L}{\rho}, \]

hence the final term is also bounded independently of \(j\). Then

\[ |\varphi_{y,1}(\hat{x})| \leq 2(S + \tau) + \varepsilon_{1} \quad \text{or} \quad |\varphi_{y,1}(\hat{x})| \leq 2(S + \tau) + \frac{2L}{\rho} \varepsilon_{1} \]

as desired. If \(|\hat{k}| < C_{1}k\) we immediately conclude

\[ \varphi_{y,1}(\hat{x})| \leq \left| \frac{j_{1} \frac{\partial \hat{f}_{1,Y}}{\partial x_{1}}(\hat{x}) + \cdots + j_{R} \frac{\partial \hat{f}_{R,Y}}{\partial x_{1}}(\hat{x}) - k_{y}}{\rho} \right| \leq 1 \]

for all \(\hat{x} \in U_{y}^{+}\) with the same argument for the final term as above. For the first partial derivatives consider

\[ \left| \frac{\partial \varphi_{y,1}}{\partial x_{1}}(\hat{x}) \right| = \left| \frac{j_{1} \frac{\partial \hat{f}_{1,Y}}{\partial x_{1}}(\hat{x}) + \cdots + j_{R} \frac{\partial \hat{f}_{R,Y}}{\partial x_{1}}(\hat{x}) - k_{y}}{\rho} \right| \]

which can be treated with a similar argument. For higher partial derivatives the terms \(\hat{k} \cdot \hat{x}\) vanish and the desired result follows easily with

\[ \left| \frac{\partial^{i_{1} + \cdots + i_{n-1}} \varphi_{y,1}}{\partial x_{1}^{i_{1}} \cdots \partial x_{n-1}^{i_{n-1}}}(\hat{x}) \right| \]

\[ \leq \frac{1}{\rho} \left( \left| \frac{\partial^{i_{1} + \cdots + i_{n-1}} \hat{f}_{1,Y}(\hat{x}) + \cdots + j_{R} \frac{\partial^{i_{1} + \cdots + i_{n-1}} \hat{f}_{R,Y}(\hat{x})}{\partial x_{1}^{i_{1}} \cdots \partial x_{n-1}^{i_{n-1}}}(\hat{x})}{\partial x_{1}^{i_{1}} \cdots \partial x_{n-1}^{i_{n-1}}}(\hat{x}) \right| \right) \]

Therefore with Lemma \(2.1\) for \(\varphi = \varphi_{y,1}\) and \(\lambda = \lambda_{1}\) as chosen above we have

\[ \int_{\mathbb{R}^{n-1}} \omega(\hat{x}, y) e \left( qj_{1} f_{y,j}(\hat{x}) \right) \frac{k_{y} + k_{n,y}}{j_{1}} d\hat{x} \ll \left( qD_{1}(k,j) \right)^{-\ell+1} \quad (4.2) \]
and hence
\[
I(q; j; k) \ll \int_y (qD_1(\hat{k}, j))^{-\ell + 1} dy \ll \epsilon_1 (qD_1(\hat{k}, j))^{-\ell + 1}.
\] (4.3)

Now assume that \(|k_n| > D_1(\hat{k}, j)| and \(|k_n| > 2j_1L\). Consider
\[
\int_R \omega(\xi, y) e \left( q_j \left( \hat{F}_{y,j}(\xi) - \frac{\hat{k} \cdot \xi + k_n y}{j_1} \right) \right) dy.
\]

Let \(\lambda_2 = qk_n\) and
\[
\varphi_{\xi, 2}(y) = \frac{j_1 f_1(\xi, y) + \cdots + j_R f_R(\xi, y) - \hat{k} \cdot \xi - k_n y}{k_n}.
\]

Observe that
\[
|\varphi_{\xi, 2}'(y)| = \left| \frac{j_1 \frac{\partial f_1}{\partial y}(\xi, y) + \cdots + j_R \frac{\partial f_R}{\partial y}(\xi, y) - k_n}{k_n} \right|
\geq \frac{1}{2}.
\]

Consider further that
\[
|\varphi_{\xi, 2}(y)| = \left| \frac{j_1 f_1(\xi, y) + \cdots + j_R f_R(\xi, y) - \hat{k} \cdot \xi - k_n y}{k_n} \right|
\leq \frac{|f_1(\xi, y)| + \frac{j_1}{|k_n|} |f_2(\xi, y)| + \cdots + \frac{j_1}{|k_n|} |f_R(\xi, y)| + |\hat{k}| \cdot |\xi|}{|k_n|} + |y|
\leq \frac{|f_1(\xi, y)| + \frac{j_1}{|k_n|} |f_2(\xi, y)| + \cdots + \frac{j_1}{|k_n|} |f_R(\xi, y)| + (L + |k_n|) \cdot |\xi|}{|k_n|} + |y|
\leq \frac{|f_1(\xi, y)| + \frac{j_1}{|k_n|} |f_2(\xi, y)| + \cdots + \frac{j_1}{|k_n|} |f_R(\xi, y)| + L |\xi|}{2L} + |\xi| + |y|,
\]

and
\[
|\varphi_{\xi, 2}'(y)| = \left| \frac{\partial f_1(\xi, y) + \frac{j_1}{|k_n|} \frac{\partial f_2(\xi, y)}{\partial y} + \cdots + \frac{j_1}{|k_n|} \frac{\partial f_R(\xi, y)}{\partial y} - \frac{k_n y}{j_1}}{k_n} \right|
\leq \frac{j_1}{|k_n|} \left| \frac{\partial f_1(\xi, y)}{\partial y} \right| + \cdots + \frac{j_R}{|k_n|} \left| \frac{\partial f_R(\xi, y)}{\partial y} \right| + 1, \]

hence with analogous arguments as above we conclude that
\[
\left| \frac{\partial^2 \varphi_{\xi, 2}}{\partial y^2}(\xi, y) \right| \ll 1,
\]

\[14\]
where the implicit constants only depend on \(i, \rho, \tau, \varepsilon_1\) and upper bounds for (the absolute values of) finitely many derivatives of \(f_r\) on \(\mathcal{D} \times \mathcal{Y}\). So Lemma 2.1 applies in the one dimensional case, hence

\[
\int_{\mathbb{R}} \omega(x, y) e \left( q j_1 \left( \hat{F}_{y,j}(x) - \frac{\hat{k} \cdot \hat{x} + k_n y}{j_1} \right) \right) dy \ll (q k_n)^{-\ell + 1} \tag{4.4}
\]

and therefore

\[
I(q; j; k) \ll \int_{U_y} (q k_n)^{-\ell + 1} d\hat{x} \ll_k (q k_n)^{-\ell + 1}. \tag{4.5}
\]

Given these estimates we can split up the sum

\[
\sum_{k \in \mathcal{X}(j;2)} I(q; j; k) = \sum_{k \in \mathcal{X}(j;2)} \int_{\mathbb{R}^n} \omega(x, y) e \left( q j_1 \left( \hat{F}_{y,j}(x) - \frac{\hat{k} \cdot \hat{x} + k_n y}{j_1} \right) \right) d\hat{x}dy
\]
as follows

\[
\sum_{(k, k_n) \in \mathcal{X}(j;2)} I(q; j; (k, k_n)) = \sum_{D(k, j) \geq \rho} \sum_{D_1(k, j) \geq |k_n|} I(q; j; (k, k_n))+ \sum_{\substack{D(k, j) \geq \rho \\ \ \ \ \ \ D_1(k, j) \geq |k_n| \\ 2j_1 L \geq |k_n| \geq D_1(k, j)}} I(q; j; (k, k_n))
\]

\[
+ \sum_{|k_n| > D_1(k, j)} I(q; j; (k, k_n)) \]

With (4.3) and (4.5) we obtain

\[
\sum_{\substack{D(k, j) \geq \rho \\ \ \ \ \ D_1(k, j) \geq |k_n| \\ \ \ \ \ \ D_1(k, j) \geq |k_n| \\ \ \ \ \ \ 2^d j_1 \rho \geq D_1(k, j) < 2^d + j_1 \rho}} D_1(k, j)^{-\ell + 1}
\]

\[
= q^{-\ell + 1} \sum_{d=0}^{\infty} \sum_{\substack{D(k, j) \geq \rho \\ \ \ \ \ D_1(k, j) \geq |k_n| \\ \ \ \ \ \ 2^d j_1 \rho \geq D_1(k, j) < 2^d + j_1 \rho}} 1 \left( 2^d j_1 \rho \right)^{d-1}
\]

\[
\leq q^{-\ell + 1} \sum_{d=0}^{\infty} \sum_{\substack{D(k, j) \geq \rho \\ \ \ \ \ D_1(k, j) \geq |k_n| \\ \ \ \ \ \ 2^d j_1 \rho \geq D_1(k, j) < 2^d + j_1 \rho}} \frac{1}{(2^d j_1 \rho)^{d-1}}
\]

\[
\ll q^{-\ell + 1} \sum_{d=0}^{\infty} \frac{(j_1 L + 2^d + 1 j_1 \rho)^n}{(2^d j_1 \rho)^{d-1}} \ll_{L, n} q^{-\ell + 1}
\]
and

$$\sum_{|\hat{k}_n| > D_1(\hat{k}_j)} I(q; j, (\hat{k}, k_n)) \ll q^{-\ell+1} \sum_{|\hat{k}_n| > D_1(\hat{k}_j)} D_1(\hat{k}_j)^{-\ell+1} \quad (4.7)$$

$$\leq q^{-\ell+1} \sum_{d=0}^{\infty} \sum_{2j_1 L \geq |k_n| \geq 2^{d+1} j_1 \rho} D_1(\hat{k}_j)^{-\ell+1}$$
$$\leq q^{-\ell+1} \sum_{d=0}^{\infty} \sum_{2j_1 L \geq |k_n| \geq 2^{d+1} j_1 \rho} \frac{1}{(2^d j_1 \rho)^{\ell-1}}$$
$$\ll q^{-\ell+1} \sum_{d=0}^{\infty} 2j_1 L \left( j_1 L + 2^{d+1} j_1 \rho \right)^{n-1} \frac{1}{(2^d j_1 \rho)^{\ell-1}}$$
$$\ll L, n \ q^{-\ell+1}$$

and

$$\sum_{|k_n| > 2j_1 L \ \text{and} \ |k_n| > D_1(\hat{k}_j)} I(q; k; j) \ll q^{-\ell+1} \sum_{|k_n| > 2j_1 L \ \text{and} \ |k_n| > D_1(\hat{k}_j)} |k_n|^{-\ell+1} \quad (4.8)$$
$$\ll q^{-\ell+1} \sum_{|k_n| > 2j_1 L \ \text{and} \ |k_n| > D_1(\hat{k}_j)} \frac{(j_1 L + |k_n|)^{n-1}}{|k_n|^{\ell-1}}$$
$$\ll q^{-\ell+1} 2^{n-1} \sum_{|k_n| > 2j_1 L} |k_n|^{n-\ell}$$
$$\ll q^{-\ell+1} \int_{2j_1 L}^{\infty} \frac{1}{t^{\ell-1}} \ dt$$
$$\ll q^{-\ell+1} j_1^{n-\ell+1}.$$

Here the implicit constants only depend on $L, \rho, n, \ell, \varepsilon_1, \kappa$ and upper bounds for (the absolute values of) finitely many derivatives of $\omega$ and $f_r$ for $1 \leq r \leq R$ on $\mathcal{D} \times \mathcal{Y}$. Consequently we
obtain
\[ N_2 \ll \sum_{1 \leq j_1 \leq J} \left( \prod_{r=1}^{R} \left( \frac{1}{J} + \min \left( \delta, \frac{1}{J_r} \right) \right) \right) J_1^{n-\ell+1} \sum_{q \leq Q} q^{n-\ell+1} \]
\[ \ll \sum_{1 \leq j_1 \leq J} \left( \prod_{r=1}^{R} \left( \frac{1}{J} + \min \left( \delta, \frac{1}{J_r} \right) \right) \right) \int_{1}^{Q} q^{n-\ell+1} dq \]
\[ \ll \sum_{1 \leq j_1 \leq J} \left( \prod_{r=1}^{R} \left( \frac{1}{J} + \min \left( \delta, \frac{1}{J_r} \right) \right) \right) \int_{1}^{Q} \frac{1}{q} dq \]
\[ = \log Q \sum_{1 \leq j_1 \leq J} \left( \prod_{r=1}^{R} \left( \frac{1}{J} + \min \left( \delta, \frac{1}{J_r} \right) \right) \right) \sum_{0 \leq j_R \leq J_1} \left( \frac{1}{J} + \min \left( \delta, \frac{1}{J_r} \right) \right) \]
\[ \leq \log Q \prod_{r=1}^{R} \sum_{0 \leq j_r \leq J} \left( \frac{1}{J} + \min \left( \delta, \frac{1}{J_r} \right) \right) \]
\[ = \log Q \left( \sum_{0 \leq j \leq J} \left( \frac{1}{J} + \min \left( \delta, \frac{1}{J} \right) \right) \right)^R \]
\[ \ll \log Q (1 + \log J)^R. \]

Case $k \in \mathcal{X}_{j;3}$. Let $\lambda = qj_1$ and
\[ \varphi_y(\bar{x}) = \hat{F}_{y,j}(\bar{x}) - \frac{k}{j_1} \hat{x} - \frac{k_n}{j_1} y. \]

For each $y$ we know that $\nabla \hat{F}_{y,j}$ is a diffeomorphism on $\mathscr{D}$, hence for any fixed $j$ and any $\hat{k}$ with $(\hat{k}/j_1) \in \nabla \hat{F}_{y,j}(\mathscr{D})$ we have a unique preimage
\[ \bar{x}_{j;\hat{k}} = (\nabla \hat{F}_{y,j})^{-1}(\hat{k}/j_1). \]

This defines a critical point for $\varphi_y$, since
\[ \nabla \varphi_y(\bar{x}_{j;\hat{k}}) = \nabla \hat{F}_{y,j}(\bar{x}_{j;\hat{k}}) - \frac{\hat{k}}{j_1} = 0. \]

Let $\mathscr{D}_+ \subseteq \mathscr{D}$ be an open set such that $\mathscr{D}_+ \subseteq B_{3\tau/2}(\bar{x}_0)$.

Lemma 4.3. Let $\bar{x} \in \mathscr{D}_+ \setminus \{\bar{x}_{j;\hat{k}}\}$. Then
\[ \frac{|\bar{x} - \bar{x}_{j;\hat{k}}|}{|\nabla \varphi_y(\bar{x})|} \ll 1. \]
where the implicit constant is independant of \( j \) and \( \hat{k} \).

**Proof.** By definition of \( \hat{x}_{j;\hat{k}} \) as a preimage of \( \nabla \hat{F}_{y,j} \) we have

\[
\frac{|\hat{x} - \hat{x}_{j;\hat{k}}|}{|\nabla \varphi_y(\hat{x})|} = \frac{|\hat{x} - \hat{x}_{j;\hat{k}}|}{|\nabla \hat{F}_{y,j}(\hat{x}) - \nabla \hat{F}_{y,j}(\hat{x}_{j;\hat{k}})|},
\]

Now for any distinct \( \hat{x}, \hat{z} \in \mathcal{D}_+ \) we know by Taylor’s theorem that

\[
\nabla \hat{F}_{y,j}(\hat{x}) - \nabla \hat{F}_{y,j}(\hat{z}) = H_{\hat{F}_{y,j}}(\hat{z})(\hat{x} - \hat{z}) + O(|\hat{x} - \hat{z}|^2),
\]

where the implicit constant does not depend on \( j \). Considering the eigenvalues of the invertible real symmetric matrix \( H_{\hat{F}_{y,j}} \) we have that

\[
\lambda_{\min}|\hat{x} - \hat{z}| \ll |H_{\hat{F}_{y,j}}(\hat{z})(\hat{x} - \hat{z})|,
\]

where \( \lambda_{\min} \) is the minimum of the absolute values of the eigenvalues and the implicit constant depends only on \( n \). We already showed that \( |\det H_{\hat{F}_{y,j}}| \) is bounded away from zero on \( \mathcal{D}_+ \) hence by virtue of the eigenvalues being continuous in the coefficients of the matrix we find constants \( C, \eta > 0 \) such that

\[
|\hat{x} - \hat{z}| \leq C|\nabla \hat{F}_{y,j}(\hat{x}) - \nabla \hat{F}_{y,j}(\hat{z})|,
\]

for all \( |\hat{x} - \hat{z}| \leq \eta \). Here \( C, \eta \) are independend of \( j \). Now in particular choosing \( \hat{z} = \hat{x}_{j;\hat{k}} \) yields the desired result.

Following similar arguments as in (12) we find that for \( i_1, ..., i_{n-1} \in \mathbb{Z}_{\geq 0} \) with \( \sum_{i=1}^{n-1} i_\mu \leq \ell \). Then for all \( \hat{x} \in \mathcal{D}_+ \) we have

\[
\left| \frac{\partial^{i_1 + \cdots + i_{n-1}} \varphi_y}{\partial x_1^{i_1} \cdots \partial x_{n-1}^{i_{n-1}}}(\hat{x}) \right| \ll 1,
\]

where the implicit constant depends only on \( (i_1, ..., i_{n-1}), \rho, \tau, \varepsilon_1 \) and upper bounds for (the absolute values of) finitely many derivatives of \( f_r \) on \( \mathcal{D}_+ \times \mathcal{Y} \) for \( 1 \leq r \leq R \). Notice that scaling with \( D_{j}(\hat{k}, j) \) is unnecessary here since for \( k \in \mathcal{K}_{j;3} \) this distance is bounded from above by \( \rho \) and that the bounds are all independant of \( j \) and \( \hat{k} \). By definition \( H_{\varphi_y} = H_{\hat{F}_{y,j}} \), hence we can apply (2.2) for \( \varphi = \varphi_y \), and \( \lambda \) as above together with (3.6) to obtain

\[
I(q; j, \hat{k}) \ll \int_{\mathcal{Y}} \left| \det H_{\hat{F}_{y,j}}(\hat{x}_{j;\hat{k}}) \right|^{-\frac{1}{2}} (q_{j_1})^{-\frac{n-1}{2}} \frac{1}{r^{n-1}} dy \ll_{c_1, \varepsilon_1} (q_{j_1})^{-\frac{n}{2} - \frac{1}{2}},
\]

where the implicit constant only depends on \( \ell, n, \varepsilon_1 \), upper bounds for the absolute values of finitely many derivatives of \( \omega \) and \( \varphi_y \) on \( \mathcal{D}_+ \), an upper bound for \( |x - v_0|/|\nabla \varphi_y(v_0)| \) on \( \mathcal{D}_+ \), and a lower bound for \( \det H_{\hat{F}_{y,j}}(\hat{x}_{j;\hat{k}}) \), all of which are independant of \( j \) and \( \hat{k} \). We
obtain
\[
\sum_{(\hat{k},k_n) \in X(j,3)} I(q; j, (\hat{k}, k_n)) \ll \sum_{(\hat{k},k_n) \in X(j,3)} (qj_1)^{-\frac{\nu}{2} - \frac{1}{2}} \ll 2j_1L \sum_{\hat{k} \in \mathbb{Z}^{n-1}} (qj_1)^{-\frac{\nu}{2} - \frac{1}{2}}
\]

where the implicit constant only depends additionally on \(L, \rho\) and \(n\). Arguing similarly to (4.9) we obtain
\[
N_3 \ll \sum_{1 \leq j_1 \leq J} \sum_{0 \leq j_2, \ldots, j_R \leq j_1} \left( \prod_{r=1}^{R} \left( \frac{1}{j} + \min \left( \frac{1}{j}, \frac{1}{j_r} \right) \right) \right) \left( qj_1 \right)^{-\frac{\nu}{2} - \frac{1}{2}} Q^{\frac{\nu}{2} + \frac{1}{2} + \frac{1}{2}} R. \tag{4.10}
\]

**Case \(k \in X_{j_1}\).** Choose \(\varphi_y\) and \(\lambda\) as in the previous case, such that we still have
\[
\nabla \varphi_y(\tilde{x}_{j_k}) = \nabla \tilde{F}_{y,j}(\tilde{x}_{j_k}) - \frac{\hat{k}}{j_1}
\]
and (4.3) still applies. The signature \(\sigma\) of the matrix \(H_{\varphi_y}(\tilde{x}_{j,k}) = H_{\tilde{F}_{y,j}}(\tilde{x}_{j,k})\) is constant for all \(j, k\) and \(y\) in consideration, since the determinant is bounded away from zero, the eigenvalues of a matrix depend continuously on its coefficients and the coefficients depend continuously on \(y\). Applying Lemma 2.2 again yields
\[
I(q; j, k) \ll \left| \int_{\mathbb{R}} (qj_1)^{-\frac{\nu}{2} - \frac{1}{2}} \frac{\omega(\tilde{x}_{j_k})}{\det H_{\tilde{F}_{y,j}}(\tilde{x}_{j_k})} \epsilon \left( qj_1 \varphi_y(\tilde{x}_{j,k}) + \frac{\sigma}{8} \right) dy \right| + (qj_1)^{-\frac{\nu}{2} - \frac{1}{2}}. \tag{4.11}
\]
Since all of $\omega, \varphi_y$ and $\hat{x}_{j,k}$ depend on $y$, we delay evaluating the integral for now and consider the terms

$$N_{1,y,j} = \left| \sum_{k \in \mathcal{X}_{j,1}} \sum_{q \leq Q} q^n \left( (q_{j1})^{-\frac{n}{2} + \frac{1}{2}} \frac{\omega(\hat{x}_{j,k}, y)}{|\det H_{F_{y,j}}(\hat{x}_{j,k})|^{\frac{1}{2}}} e \left( q_{j1} \varphi_y(\hat{x}_{j,k}) + \frac{\sigma}{8} \right) \right) \right| + O \left( \sum_{k \in \mathcal{X}_{j,1}} \sum_{q \leq Q} q^n (q_{j1})^{-\frac{n}{2} - \frac{1}{2}} \right)$$

$$= \left| \sum_{k \in \mathcal{X}_{j,1}} \sum_{q \leq Q} q^n \left( (q_{j1})^{-\frac{n}{2} + \frac{1}{2}} \frac{\omega(\hat{x}_{j,k}, y)}{|\det H_{F_{y,j}}(\hat{x}_{j,k})|^{\frac{1}{2}}} e \left( q_{j1} \varphi_y(\hat{x}_{j,k}) + \frac{\sigma}{8} \right) \right) \right| + O \left( j_1^{\frac{n}{2} - \frac{1}{2}} Q^{\frac{n}{2} + \frac{1}{2}} \right)$$

and

$$N_{1,y} = \sum_{1 \leq j \leq J} \left( \prod_{r=1}^{R} b_{j_r} \right) N_{1,y,j}.\tag{4.12}$$

We start with the innermost sum

$$\left| \sum_{q \leq Q} q^n \left( (q_{j1})^{-\frac{n}{2} + \frac{1}{2}} \frac{\omega(\hat{x}_{j,k}, y)}{|\det H_{F_{y,j}}(\hat{x}_{j,k})|^{\frac{1}{2}}} e \left( q_{j1} \varphi_y(\hat{x}_{j,k}) + \frac{\sigma}{8} \right) \right) \right| \leq \frac{\omega(\hat{x}_{j,k}, y)}{|\det H_{F_{y,j}}(\hat{x}_{j,k})|^{\frac{1}{2}}} \sum_{q \leq Q} \left( q_{j1}^{\frac{n}{2} + \frac{1}{2}} e \left( q_{j1} \varphi_y(\hat{x}_{j,k}) \right) \right) \leq \frac{\omega(\hat{x}_{j,k}, y)}{|\det H_{F_{y,j}}(\hat{x}_{j,k})|^{\frac{1}{2}}} \sum_{q \leq Q} \left( q_{j1}^{\frac{n}{2} + \frac{1}{2}} e \left( q_{j1} \varphi_y(\hat{x}_{j,k}) \right) \right).$$

The remaining sum over $q$ can be dealt with by means of partial summation

$$\sum_{q \leq Q} q^{\frac{n}{2} + \frac{1}{2}} e(q_{j1} \varphi_y(\hat{x}_{j,k})) = Q^{\frac{n}{2} + \frac{1}{2}} \sum_{q \leq Q} e(q_{j1} \varphi_y(\hat{x}_{j,k}))$$

$$= \int_1^Q \sum_{q \leq \xi} e(q_{j1} \varphi_y(\hat{x}_{j,k})) \left( \frac{n}{2} + \frac{1}{2} \right) \xi^{\frac{n}{2} - \frac{1}{2}} d\xi$$

Note that we have the bound

$$\left| \sum_{q \leq Q} e(q_{j1} \varphi_y(\hat{x}_{j,k})) \right| \leq \min \{ Q, \| j_1 \varphi_y(\hat{x}_{j,k}) \|^{-1} \},$$

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so we distinguish two cases. First if \( ||j_1 \varphi_y(\hat{x}_{j,k})|| \geq Q^{-1} \) we obtain

\[
\sum_{q \leq Q} q^{\frac{n}{2} + \frac{1}{2}} e(qj_1 \varphi_y(\hat{x}_{j,k})) \ll \frac{Q^{\frac{n}{2} + \frac{1}{2}}}{||j_1 \varphi_y(\hat{x}_{j,k})||} + \frac{1}{||j_1 \varphi_y(\hat{x}_{j,k})||} \int_1^Q \left( \frac{n}{2} + \frac{1}{2} \right) \xi^{\frac{n}{2} - \frac{1}{2}} d\xi \quad (4.13)
\]

On the other hand if \( ||j_1 \varphi_y(\hat{x}_{j,k})|| < Q^{-1} \) we have

\[
\sum_{q \leq Q} q^{\frac{n}{2} + \frac{1}{2}} e(qj_1 \varphi_y(\hat{x}_{j,k})) \ll Q^{\frac{n}{2} + \frac{1}{2}} + \left( \frac{n}{2} + \frac{1}{2} \right) \int_1^Q \xi^{\frac{n}{2} + \frac{1}{2}} d\xi \quad (4.14)
\]

Hence we obtain

\[
N_{1,\nu} \ll \sum_{1 \leq j_1 \leq J} \sum_{0 \leq j_2, \ldots, j_N \leq j_1} \left( \prod_{r=1}^R b_{j_r} \right) \sum_{k \in \mathcal{X}_{j_1}} \omega(\hat{x}_{j,k}) j_1^{\frac{n}{2} + \frac{1}{2}} Q^{\frac{n}{2} + \frac{1}{2}} \frac{||j_1 \varphi_y(\hat{x}_{j,k})||^{-1}}{||j_1 \varphi_y(\hat{x}_{j,k})||} \quad (4.15)
\]

\[
+ \sum_{0 \leq j_2, \ldots, j_N \leq j_1} \left( \prod_{r=1}^R b_{j_r} \right) \sum_{k \in \mathcal{X}_{j_1}} \omega(\hat{x}_{j,k}) j_1^{\frac{n}{2} + \frac{1}{2}} Q^{\frac{n}{2} + \frac{1}{2}} \quad (4.15)
\]

The last term can be bounded similarly to (4.9) and (4.10). We have the following essential result to be proven in Section 5.

**Proposition 4.4.** Let \( T > 0 \) and \( J_2, \ldots, J_R \in [1, J] \). Then with the notations of this section and for all \( y \in \mathcal{W} \), we have

\[
\sum_{1 \leq j_1 \leq J} \sum_{k \in \mathcal{X}_{j,k}} \omega(\hat{x}_{j,k}) \ll \left( \prod_{r=2}^R J_r \right) (J^{n+1}T^{-1} + J^n \delta_{n-1}(J)),
\]
where
\[ \mathcal{E}_{n-1}(J) = \mathcal{E}_{n-1}(c_1, c_2) = \begin{cases} \exp(c_1 \sqrt{\log J}) & \text{if } n = 3 \\ (\log J)^{c_2} & \text{if } n \geq 4 \end{cases} \]

for some positive constants \( c_1 \) and \( c_2 \). Here the implicit constant as well as \( c_1 \) and \( c_2 \) only depend on \( n, R, c_1 \) and \( c_2 \) in (3.6), \( \rho \) in (3.7), \( \rho' \) in (3.10) and upper bounds for (the absolute value) of finitely many derivatives of \( \omega \) and \( f_r \) for \( 2 \leq r \leq R \) on \( \mathcal{D} \times \mathcal{K} \). In particular, they are independent of \( T, J_2, ..., J_R \) and \( y \).

Recall that \( b_j \ll \frac{1}{j} \) for \( 1 \leq j \leq J \). Hence with \( \mathcal{J}_0 = \{0\} \) and \( \mathcal{J}_s = [2^{s-1}, 2^s] \) it follows that
\[
\sum_{1 \leq j_1 \leq J} \prod_{r=1}^{R} b_r \left( \sum_{k \in \mathbb{Z}^1} \omega(\mathcal{X}_{j_1,k}) \right) j_1^{-\frac{n}{2} + \frac{n}{2}} \sum_{1 \leq j_1 \leq J} \prod_{2 \leq r \leq R} j_1^{-\frac{n}{2} - \frac{n}{2}} \sum_{k \in \mathbb{Z}^1} \omega(\mathcal{X}_{j_1,k}) \ll \sum_{0 \leq s_2, ..., s_R \leq \frac{\log J}{\log 2} + 1} \prod_{r=2}^{R} (2^{s_r} - 1) J_1^{-\frac{n}{2} - \frac{n}{2}} \sum_{k \in \mathbb{Z}^1} \omega(\mathcal{X}_{j_1,k}).
\]

Now using partial summation and Proposition 4.4 we find that for all \( s_r \) in consideration
\[
\sum_{0 \leq j_r \leq \min\{2^{s_r}, J_1\}} \prod_{2 \leq r \leq R} \omega(\mathcal{X}_{j_1,k}) \ll J^{-\frac{n}{2} - \frac{n}{2}} \left( \prod_{r=2}^{R} 2^{s_r} \right) (J^{n+1} T^{-1} + J^n \mathcal{E}_{n-1}(J)).
\]

Therefore
\[
\sum_{1 \leq j_1 \leq J} \prod_{r=1}^{R} b_r \left( \sum_{k \in \mathbb{Z}^1} \omega(\mathcal{X}_{j_1,k}) \right) j_1^{-\frac{n}{2} + \frac{n}{2}} \sum_{k \in \mathbb{Z}^1} \omega(\mathcal{X}_{j_1,k}) \ll (1 + \log J) R J^{-\frac{n}{2} - \frac{n}{2}} (T^{-1} J^{n+1} + J^n \mathcal{E}_{n-1}(J)).
\]

Now the second term in (4.13) can be estimated by taking \( T = Q \) in (4.18). For the first sum in (4.13) we split the interval \([Q^{-1}, 1/2]\) into dyadic intervals. We may assume \( Q \geq 2 \),
i.e. \( Q^{-1} \leq 1/2 \), and conclude

\[
\sum_{0 \leq j_2, \ldots, j_R \leq j_1} \left( \prod_{r=1}^{R} b_{j_r} \right) \sum_{k \in \mathcal{X}_{j_1}} \omega(\hat{x}_{j_1}) j_1^{-\frac{\nu}{2} + \frac{1}{2} \sum_{k \in \mathcal{X}_{j_1}} ||j_1 \varphi_{y}(\hat{x}_{j_1})||^{-1}} (4.19)
\]

\[
\leq \sum_{1 \leq i \leq \log_2 Q+1} (1 + \log J)^R \sum_{0 \leq j_2, \ldots, j_R \leq j_1} \left( \prod_{r=1}^{R} b_{j_r} \right) \sum_{k \in \mathcal{X}_{j_1}} \omega(\hat{x}_{j_1}) j_1^{-\frac{\nu}{2} + \frac{1}{2} \sum_{k \in \mathcal{X}_{j_1}} ||j_1 \varphi_{y}(\hat{x}_{j_1})||^{-1}}
\]

\[
\ll (1 + \log J)^R((\log Q)J^\frac{\nu}{2} + QJ^\frac{\nu}{2} - \frac{1}{2} \frac{\nu}{2} \epsilon_{n-1}(J))
\]

using (4.18) again. Combining (4.18), (4.19) and (4.18), we obtain

\[
N_{1,y} \ll Q^{\frac{\nu}{2} + \frac{1}{2} \sum_{k \in \mathcal{X}_{j_1}} ||j_1 \varphi_{y}(\hat{x}_{j_1})||^{-1}} (1 + \log J)^R((\log Q)J^\frac{\nu}{2} + QJ^\frac{\nu}{2} - \frac{1}{2} \frac{\nu}{2} \epsilon_{n-1}(J)) + (1 + \log J)^RQ^{\frac{\nu}{2} + \frac{1}{2} \sum_{k \in \mathcal{X}_{j_1}} ||j_1 \varphi_{y}(\hat{x}_{j_1})||^{-1}}
\]

\[
\ll (1 + \log J)^R((\log Q)Q^{\frac{\nu}{2} + \frac{1}{2} \sum_{k \in \mathcal{X}_{j_1}} ||j_1 \varphi_{y}(\hat{x}_{j_1})||^{-1}} + Q^{\frac{\nu}{2} + \frac{1}{2} \sum_{k \in \mathcal{X}_{j_1}} ||j_1 \varphi_{y}(\hat{x}_{j_1})||^{-1}} \epsilon_{n-1}(J)).
\]

Consequently we have

\[
N_1 \ll \epsilon_1 (1 + \log J)^R((\log Q)Q^{\frac{\nu}{2} + \frac{1}{2} \sum_{k \in \mathcal{X}_{j_1}} ||j_1 \varphi_{y}(\hat{x}_{j_1})||^{-1}} + Q^{\frac{\nu}{2} + \frac{1}{2} \sum_{k \in \mathcal{X}_{j_1}} ||j_1 \varphi_{y}(\hat{x}_{j_1})||^{-1}} \epsilon_{n-1}(J)).
\]

5. PROOF OF PROPOSITION 4.4

Recall that we defined functions

\[
\hat{f}_{j,y} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}, \hat{x} \mapsto f_{j}(\hat{x}, y)
\]

for \( y \in \mathcal{Y}, \varepsilon_1 \) as in (3.6), and

\[
\hat{F}_{y,j} = \hat{f}_{1,y} + \frac{j_2}{j_1} \hat{f}_{2,y} + \cdots + \frac{j_R}{j_1} \hat{f}_{R,y}
\]

for \( j \in \mathbb{R} \setminus \{0\} \). For a non-negative weight function \( \omega \in C^\infty_c(\mathbb{R}^n) \) we defined

\[
U_y = \{ \hat{x} \in \mathbb{R}^{n-1} \mid \omega(\hat{x}, y) \neq 0 \}
\]

and \( V_{y,j} = \nabla \hat{F}_{y,j} \). Note that \( \nabla \hat{F}_{y,j} \) is a diffeomorphism on \( U_y \). Now let \( \omega^x_j = \omega \circ (\nabla \hat{F}_{y,j})^{-1} \), and for \( T \geq 2 \) and \( j_2, \ldots, j_R \in [1, J] \), define

\[
\mathcal{M}(J, T^{-1}) = \sum_{1 \leq j_1 \leq J} \sum_{0 \leq j_2, \ldots, j_R \leq j_1} \omega^x_j \left( \hat{a}_{j_1} \right) (5.1)
\]

\[
= \sum_{1 \leq j_1 \leq J} \sum_{0 \leq j_2, \ldots, j_R \leq j_1} \omega^x_j \left( \hat{a}_{j_1} \right) (5.1)
\]

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Note that \(4.4\) for \(0 < T < 2\) immediately follows from the case \(T = 2\). We consider the Fejér kernel
\[
\mathcal{F}_D(\theta) = D^{-2} \left| \sum_{d=1}^{D} e(d\theta) \right|^2 = \left( \frac{\sin(\pi D\theta)}{D \sin(\pi \theta)} \right)^2 = \sum_{d=-D}^{D} \frac{D - |d|}{D^2} e(d\theta)
\]
for \(D = |T/2|\). Let \(\theta \in \mathbb{R}\) with \(0 < ||\theta|| < T^{-1}\), then by the concave property of the sine function on \([0, \pi/2]\) we have
\[
\left( \frac{\sin(D\pi\theta)}{D \sin(\pi \theta)} \right)^2 \geq \left( \frac{2\pi^{-1}D\pi||\theta||}{D \pi ||\theta||} \right) \geq \frac{4}{\pi^2},
\]
Therefore, it follows that
\[
\chi_{T^{-1}}(\theta) \leq \frac{\pi^2}{4} \mathcal{F}_D(\theta),
\]
with \(\chi_{T^{-1}}\) as in \((5.1)\). Combining \((5.1)\), \((5.2)\) and \((5.3)\) we obtain
\[
\mathcal{M}(J, T^{-1}) \leq \frac{\pi^2}{4} \sum_{1 \leq j_1 \leq J} \sum_{0 \leq j_r \leq \min\{J_r, j_1\}} \sum_{|a_n| \leq 2J_1 L} \sum_{d=-D}^{D} \frac{D - |d|}{D^2} \omega^*_j \left( \frac{\hat{a}}{j_1} \right) e(dj_1 \varphi_y(\hat{x}_{j_1}))
\]
By definition \(\omega^*_j\) vanishes outside of \(\bigcup_{y \in \mathcal{F}} V_{y,j} \subseteq [-L, L]^{n}\), hence the contribution of terms with \(d = 0\) in \((5.4)\) is
\[
\frac{\pi^2}{4D} \sum_{1 \leq j_1 \leq J} \sum_{0 \leq j_r \leq \min\{J_r, j_1\}} \sum_{|a_n| \leq 2J_1 L} \sum_{0 \leq j_r \leq \min\{J_r, j_1\}} |a_n| \leq 2J_1 L \omega^*_j \left( \frac{\hat{a}}{j_1} \right) e(0)
\]
where the implicit constants only depend on \(n\) and \(L\). Let \(\tilde{F}^*_y\) be the Legendre transform of \(F^*_y\). Then with \((2.1)\) and since \(\tilde{x}_{j_1, \hat{a}} = (\nabla \tilde{F}^*_y)\left(\frac{\hat{a}}{j_1}\right)\) we have
\[
\tilde{F}^*_y \left( \frac{\hat{a}}{j_1} \right) = \tilde{x}_{j_1, \hat{a}} \cdot \tilde{F}^*_y(j_{1}) = -\varphi_y(\tilde{x}_{j_1, \hat{a}}) - \frac{a_n y}{j_1}.
\]
Now we can rewrite
\[
\sum_{\hat{a} \in \mathbb{Z}^n} \sum_{|a_n| \leq 2j_1 L} \sum_{d=-D}^{D} \frac{D - |d|}{D^2} \omega^*_j \left( \frac{\hat{a}}{j_1} \right) e(dj_1 \varphi_y(\tilde{x}_{j_1, \hat{a}}))
\]
\[
= \sum_{\hat{a} \in \mathbb{Z}^n} \sum_{|a_n| \leq 2j_1 L} \sum_{d=-D}^{D} \frac{D - |d|}{D^2} \omega^*_j \left( \frac{\hat{a}}{j_1} \right) e \left( -dj_1 \left( \frac{a_n y}{j_1} + \tilde{F}^*_y(j_{1}) \right) \right)
\]
\[
= \sum_{\hat{a} \in \mathbb{Z}^n} \sum_{|a_n| \leq 2j_1 L} \sum_{d=-D}^{D} \frac{D - |d|}{D^2} \omega^*_j \left( \frac{\hat{a}}{j_1} \right) e(-da_n y) e \left( -dj_1 \tilde{F}^*_y(j_{1}) \right)
\]
\[
= \sum_{|a_n| \leq 2j_1 L} \sum_{d=-D}^{D} \frac{D - |d|}{D^2} e(-da_n y) \sum_{\hat{a} \in \mathbb{Z}^{n-1}} \omega^*_j \left( \frac{\hat{a}}{j_1} \right) e \left( -dj_1 \tilde{F}^*_y(j_{1}) \right),
\]
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and since
\[ e\left(-dj_1\hat{F}_{y,j}\left(\frac{\hat{a}}{j_1}\right)\right) = e\left(dj_1\hat{F}_{y,j}\left(\frac{\hat{a}}{j_1}\right)\right), \quad e(-da_ny) = e(da_ny) \]
we have
\[
\left| \sum_{|a_n| \leq 2j_1, L \leq |d| \leq D} \sum_{\hat{a} \in \mathbb{Z}^{n-1}} \frac{D - |d|}{D^2} e(da_ny) \sum_{\hat{a} \in \mathbb{Z}^{n-1}} \omega_j^* \left(\frac{\hat{a}}{j_1}\right) e\left(dj_1\hat{F}_{y,j}\left(\frac{\hat{a}}{j_1}\right)\right) \right| \leq 2 \left| \sum_{|a_n| \leq 2j_1, L \leq |d| \leq D} \frac{D - d}{D^2} e(da_ny) \sum_{\hat{a} \in \mathbb{Z}^{n-1}} \omega_j^* \left(\frac{\hat{a}}{j_1}\right) e\left(dj_1\hat{F}_{y,j}\left(\frac{\hat{a}}{j_1}\right)\right) \right|. \tag{5.7}
\]
Applying the \( n \)-dimensional Poisson summation formula to the inner most sum yields
\[
\sum_{\hat{a} \in \mathbb{Z}^{n-1}} \omega_j^* \left(\frac{\hat{a}}{j_1}\right) e\left(dj_1\hat{F}_{y,j}\left(\frac{\hat{a}}{j_1}\right)\right) \tag{5.8}
\]
\[
= \sum_{\hat{k} \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}^{n-1}} \omega_j^* \left(\frac{\hat{z}}{j_1}\right) e\left(dj_1\hat{F}_{y,j}\left(\frac{\hat{z}}{j_1}\right) - j_1\hat{k} \cdot \hat{z}\right) d\hat{z}
\]
\[
=j_1^{n-1} \sum_{\hat{k} \in \mathbb{Z}^{n-1}} I_0(d; j_1\hat{k})
\]
where
\[
I_0(d; j_1\hat{k}) = \int_{\mathbb{R}^{n-1}} \omega_j^* (\hat{x}) e\left(dj_1\hat{F}_{y,j}(\hat{x}) - j_1\hat{k} \cdot \hat{x}\right) d\hat{x}.
\]
Therefore to obtain a bound for \( \mu(J, T^{-1}) \) it is sufficient to provide a bound for
\[
\left| \sum_{0 \leq j_1 \leq J} \sum_{\hat{k} \in \mathbb{Z}^{n-1}} \frac{D - d}{D^2} e(da_ny) j_1^{n-1} \sum_{\hat{a} \in \mathbb{Z}^{n-1}} I_0(d; j_1\hat{k}) \right| \tag{5.9}
\]
Since \( \nabla \hat{F}_{y,j} = (\nabla \hat{F}_{y,j})^{-1} \) and \( \nabla \hat{F}_{y,j} \) is a diffeomorphism on \( \mathcal{D} \) we have that \( \nabla \hat{F}_{y,j}^* \) is a diffeomorphism on \( \nabla \hat{F}_{y,j}(\mathcal{D}) \) and \( \nabla \hat{F}_{y,j}^*(V_{y,j}^+) = U_y^+ \). Let
\[
\rho' = \frac{\text{dist}(\partial \mathcal{D}, \partial U_y)}{2}. \tag{5.10}
\]
We repeat the technique of Section 4 and split the set \( \mathbb{Z}^{n-1} \) into three disjoint subsets. Let
\[
\mathcal{K}_1 = \left\{ \hat{k} \in \mathbb{Z}^{n-1} \mid \frac{\hat{k}}{d} \in U_y \right\},
\]
\[
\mathcal{K}_2 = \left\{ \hat{k} \in \mathbb{Z}^{n-1} \mid \text{dist} \left(\frac{\hat{k}}{d}, U_y\right) \geq \rho' \right\}
\]
\[ K_3 = \mathbb{Z}^{n-1} \setminus (K_1 \cup K_2). \]

For each \( 1 \leq i \leq 3 \) we define
\[
M_i = \sum_{d=1}^{D} \frac{D-d}{D^2} \sum_{0 \leq j_1 \leq J} \sum_{2 \leq r \leq R} e(\hat{d}_n y)j_1^{n-1} \sum_{k \in K_i} I_0(d; j; \hat{k}) \quad (5.11)
\]
\[
\ll L \sum_{0 \leq j_1 \leq J} \sum_{2 \leq r \leq R} D \left| j_1^{n-1} \sum_{k \in K_i} \mu(d, j; \hat{k}) \right|,
\]
such that
\[
\mathcal{M}(J, T^{-1}) \ll \left( \prod_{r=2}^{R} J_r \right) \sum_{0 \leq j_1 \leq J} \sum_{2 \leq r \leq R} D j_1^{n-1} \sum_{k \in K_i} I_0(d; j; \hat{k}) \quad (5.12)
\]
and seek to bound each \( M_i \) separately.

**Remark.** Note that we deliberately choose to estimate the term \( e(\hat{d}_n y) \) in (5.11) trivially. Considering its contribution could possibly lead to further results.

**Case \( \hat{k} \in K_2. \)** Define
\[
\varphi_1(\hat{x}) = \frac{d\hat{F}_{y,j}(\hat{x}) - \hat{k} \cdot \hat{x}}{\text{dist}(k, dU_y)}
\]
and
\[
\lambda_1 = j_1 \text{dist}(\hat{k}, dU_y).
\]

Then for all \( \hat{x} \in V_{y,j} \)
\[
\left| \nabla \varphi_1(\hat{x}) \right| = \frac{|d\hat{F}_{y,j}(\hat{x}) - \hat{k}|}{\text{dist}(k, U)} \geq 1
\]
and like in (4.1) we conclude
\[
\left| \nabla \varphi_1(\hat{x}) \right| \geq \frac{1}{2}
\]
for \( \hat{x} \in V_{y,j}^+. \) Next we establish upper bounds for the derivatives of \( \varphi_1 \) and \( \omega_j^*. \) In order to do so, we establish bounds for the derivatives of \( \hat{F}_{y,j}^* \) first.

**Lemma 5.1.** Let \( i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0} \) with \( \sum_{\mu=1}^{n-1} i_\mu \leq \ell. \) Then for all \( \hat{x} \in U_y^+ \) we have
\[
\left| \frac{\partial^{i_1 + \cdots + i_{n-1}} \hat{F}_{y,j}^*(\hat{x})}{\partial x_1^{i_1} \cdots \partial x_{n-1}^{i_{n-1}}} \right| \ll 1,
\]
where the implicit constant depends only on \( (i_1, \ldots, i_{n-1}), \rho', \tau \) and upper bounds for (the absolute values of) finitely many derivatives of \( f_r \) on \( U_y^+ \times \mathcal{Y} \) for \( 1 \leq r \leq R. \)
Proof. For $\mathbf{x} \in V_{y,j}^+$ and $\mathbf{z} \in U_{y,j}^+$ we have

$$|\mathbf{x} \cdot \mathbf{z}| + |\tilde{F}_{y,j}| \ll 1,$$

hence we easily deduce $|\tilde{F}_{y,j}^*| \ll 1$ with (5.11). Recall that $\nabla \tilde{F}_{y,j}^* = (\nabla \tilde{F}_{y,j})^{-1}$ and $U_y^+ = (\nabla \tilde{F}_{y,j})^{-1}(V_{y,j}^+)$. Hence $|\nabla \tilde{F}_{y,j}^*(\mathbf{x})| \ll 1$ for $\mathbf{x} \in V_{y,j}^+$. For $\mathbf{x} = \nabla \tilde{F}_{y,j}(\mathbf{z})$ with $\mathbf{z} \in U_{y,j}^+$ we have

$$\text{Jac}_{\tilde{F}_{y,j}^*} (\mathbf{x}) = \text{Jac}_{\nabla \tilde{F}_{y,j}} (\mathbf{z}) = (\text{Jac}_{\nabla \tilde{F}_{y,j}} (\mathbf{z}))^{-1}, \quad (5.13)$$

where $\text{Jac}_f$ denotes the Jacobian matrix of the function $f$ and we used the chain rule. Consequently every second partial derivative of $\tilde{F}_{y,j}^*$ can be written as

$$P$$

where the implicit constant depends only on $(|\mathbf{h}| = \text{max}_i |h_i|)$. Hence we easily deduce the desired bounds for the second derivatives follows directly with (5.14) and (3.6). Essentially the same idea will be used to argue for higher partial derivatives. Note that for any $i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}$ with $\sum_{\mu=1}^{n-1} i_\mu = \ell$ and $\mathbf{z} \in U_{y,j}^+$, and $\text{Jac}_{\nabla \tilde{F}_{y,j}^*} = H_{\tilde{F}_{y,j}}$, the entries of the Jacobian matrix, can be expressed as

$$P$$

where $P$ is a polynomial expression in the terms of the entries of $\text{Jac}_{\nabla \tilde{F}_{y,j}} (\mathbf{z})$. Note that $P$ has degree $(n-1)$ and each coefficient can only be $\pm 1$ or $0$. Since

$$\left| \frac{\partial^{i_1+\cdots+i_{n-1}}}{\partial x_1 \cdots \partial x_{n-1}} \tilde{F}_{y,j}^* (\mathbf{z}) \right| \ll 1$$

is obvious for any $i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}$ with $\sum_{\mu=1}^{n-1} i_\mu = \ell$ and $\mathbf{z} \in U_{y,j}^+$, and $\text{Jac}_{\nabla \tilde{F}_{y,j}^*} = H_{\tilde{F}_{y,j}}$, the desired bounds for the second derivatives follows directly with (5.14) and (3.6). Essentially the same idea will be used to argue for higher partial derivatives. Note that for any $k \in \mathbb{N}$ we can express the $k$-th partial derivative with respect to the $\mathbf{x}$-variables of an entry in $\text{Jac}_{\nabla \tilde{F}_{y,j}^*} (\mathbf{x})$ as a real polynomial with coefficients independant of $\mathbf{j}$ in terms of:

(i) $(\text{Jac}_{\nabla \tilde{F}_{y,j}} (\mathbf{z}))^{-m}$, where $m \leq k + 1$;

(ii) entries of $\text{Jac}_{\nabla \tilde{F}_{y,j}} (\mathbf{z})$;

(iii) $m$-th partial derivatives with respect to the $\mathbf{z}$-variables of entries in $\text{Jac}_{\nabla \tilde{F}_{y,j}}$, where $m \geq k$;

(iv) $m$-th partial derivatives with respect to the $\mathbf{x}$-variables of entries in $\nabla \tilde{F}_{y,j}^* = (\nabla \tilde{F}_{y,j})^{-1}(\mathbf{x})$,

where $m \geq k$.

Now, using (3.6) again, the desired result follows inductively.

Now with similar arguments as in (4.2) we can deduce the following.

Corollary 5.2. Let $i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}$ with $\sum_{\mu=1}^{n-1} i_\mu = \ell$. Then for all $\mathbf{x} \in U_{y,j}^+$ we have

$$\left| \frac{\partial^{i_1+\cdots+i_{n-1}}}{\partial x_1 \cdots \partial x_{n-1}} \varphi_1 (\mathbf{x}) \right| \ll 1,$$

where the implicit constant depends only on $(i_1, \ldots, i_{n-1})$, $\rho'$, $\tau$ and upper bounds for (the absolute values of) finitely many derivatives of $f_{\rho}$ on $U_{y,j}^+ \times \mathbb{R}$ for $1 \leq r \leq R$.

Recall that $\omega_j^* = \omega \circ \nabla \tilde{F}_{y,j}^*$, hence we also obtain the following.

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Corollary 5.3. Let $i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}$ with $\sum_{\mu=1}^{n-1} i_\mu \leq \ell - 1$. Then for all $\hat{\mathbf{x}} \in U_y^+$ we have
\[
\left| \frac{\partial^{i_1 + \cdots + i_{n-1}} \omega_2^*}{\partial x_1 \cdots \partial x_{n-1}} (\hat{\mathbf{x}}) \right| \ll 1,
\]
where the implicit constant depends only on $(i_1, \ldots, i_{n-1})$, $\rho'$, $\tau$ and upper bounds for (the absolute values of) finitely many derivatives of $f$ on $U_y^+ \times \mathcal{W}$ for $1 \leq r \leq R$.

Applying 2.2 with $\phi = \varphi_1$ and $\lambda = \lambda_1$ as defined above now yields
\[
I_0(d; j; \hat{k}) \ll \lambda^{-\ell+1} = (j_1 \text{dist}(\hat{k}, dU_y))^{-\ell+1},
\]
where the implicit constant is independent of $y$, $j$ and $\hat{k}$. Now since $\ell - n \geq 1$ we find similarly to (4.6)
\[
\sum_{\hat{k} \in \mathcal{K}_2} I_0(d; j; \hat{k}) \ll j_1^{-\ell+1} \sum_{\hat{k} \in \mathcal{K}_2} \text{dist}(\hat{k}, dU_y)^{-\ell+1}
\]
(5.15)
\[
\ll j_1^{-\ell+1} \sum_{m=0}^{\infty} \sum_{\hat{k} \in \mathcal{K}_2} \frac{1}{(2^m j_1 \rho')^{\ell+1}},
\]
\[
\ll j_1^{-\ell+1} \sum_{m=0}^{\infty} \frac{(L_j + 2^{m+1} j_1 \rho')^{n-1}}{2^m j_1 \rho'}
\]
\[
\ll j_1^{-\ell+1},
\]
where the implicit constant is independent of $y$ and $d$. Consequently we obtain
\[
M_2 \leq \sum_{0 \leq j_1 \leq \min \{J_r, j_1\}} \sum_{d=1}^{D \text{min}} \frac{D - d}{D^2} j_1^{n-1} \sum_{\hat{k} \in \mathcal{K}_2} I_0(d; j; \hat{k})
\]
(5.16)
\[
\ll \frac{D - 1}{2D} \sum_{0 \leq j_1 \leq \min \{J_r, j_1\}} \sum_{2 \leq r \leq R} j_1^{n-\ell+1}
\]
\[
\ll \left( \prod_{r=2}^{R} J_r \right) \log J.
\]

Case $\hat{k} \in \mathcal{K}_3$. Let $\lambda = j_1 d$ and
\[
\varphi(\hat{\mathbf{x}}) = \hat{\mathbf{F}}^*_{y,j} (\hat{\mathbf{x}}) - \frac{\hat{k}}{d} \cdot \hat{\mathbf{x}}.
\]
By definition, for each fixed $d$ we have that $\hat{k} \in d\mathcal{W}$ determines a unique preimage
\[
\hat{\mathbf{x}}_{d;j;\hat{k}} = (\nabla \hat{\mathbf{F}}^*_{y,j})^{-1} \left( \frac{\hat{k}}{d} \right) = \nabla \hat{\mathbf{F}}^*_{y,j} \left( \frac{\hat{k}}{d} \right)
\]
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that is also a critical point for $\varphi$ in the sense that

$$\nabla \varphi(\hat{x}_{d,j;k}) = \nabla \hat{F}_{y,j}^{*}(\hat{x}_{d,j;k}) - \frac{\hat{k}}{d} = 0.$$  

**Lemma 5.4.** Let $\hat{x} \in \nabla \hat{F}_{y,j}(D_{+}) \setminus \{\hat{x}_{d,j;k}\}$. Then

$$\frac{|\hat{x} - \hat{x}_{d,j;k}|}{|\nabla \varphi(\hat{x})|} \ll 1$$

where the implicit constant is independant of $d$, $j$ and $\hat{k}$.

**Proof.** Observe that for $\frac{\hat{k}}{d} \in D$ we have

$$\frac{|\hat{x} - \hat{x}_{d,j;k}|}{|\nabla \varphi(\hat{x})|} = \frac{|\hat{x} - (\nabla \hat{F}_{y,j}^{*})^{-1}(\hat{k}/d)|}{|\nabla \hat{F}_{y,j}^{*}(\hat{x}) - \frac{\hat{k}}{d}|},$$

hence it is sufficient to prove

$$\frac{|\hat{x} - \hat{z}|}{|\nabla \hat{F}_{y,j}^{*}(\hat{x}) - \nabla \hat{F}_{y,j}^{*}(\hat{z})|} \ll 1$$

for $\hat{x}, \hat{z} \in \nabla \hat{F}_{y,j}(D_{+})$ and $\hat{x} \neq \hat{z}$. Taking $\hat{x}', \hat{z}' \in D_{+}$ with $\hat{x}' \neq \hat{z}'$ such that $\hat{x} = \nabla \hat{F}_{y,j}(\hat{x}') = (\nabla \hat{F}_{y,j}^{*})^{-1}$ (and the same for $\hat{z}$) the inequality is equivalent to

$$\frac{|\nabla \hat{F}_{y,j}(\hat{x}') - \nabla \hat{F}_{y,j}(\hat{z}')|}{|\hat{x}' - \hat{z}'|} \ll 1,$$

or alternatively

$$1 \ll \frac{|\hat{x}' - \hat{z}'|}{|\nabla \hat{F}_{y,j}(\hat{x}') - \nabla \hat{F}_{y,j}(\hat{z}')|}.$$  

We have already established in the proof of [4.3] that

$$\nabla \hat{F}_{y,j}(\hat{x}') - \nabla \hat{F}_{y,j}(\hat{z}') = H_{\hat{F}_{y,j}}(\hat{z}')(|\hat{x}' - \hat{z}'| + O(|\hat{x}' - \hat{z}'|^2),$$

which yields the desired lower bound immediately.

Note that because of [5.1] we can deduce the same result from [5.2] for $\varphi$ in this case, i.e. for given $i_1, ..., i_{n-1} \in \mathbb{Z}_{\geq 0}$ with $\sum_{\mu=1}^{n-1} i_\mu \leq \ell$ and $\hat{x} \in V_{y,j}^{+}$ we have

$$\left|\frac{\partial^{i_{1}+\ldots+i_{n-1}} \varphi}{\partial^{i_{1}} \partial^{i_{2}} \ldots \partial^{i_{n-1}}} (\hat{x})\right| \ll 1.$$  

(5.17)

The implicit constant is again independent of $y$, $d$, $j$ and such $\hat{k}$ that satisfy this case. By construction we have $H_{\varphi} = H_{\hat{F}_{y,j}}$ so with (2.2) and (3.0) we have $H_{\varphi}(\hat{x}_{d,j;k}) \neq 0$ and consequently Lemma 2.2 yields

$$I_0(d; j; \hat{k}) \ll \lambda^{-\frac{n-\ell}{2}} = j_1^{-\frac{\ell}{2}} d^{-\frac{\ell}{2} - \frac{1}{2}}.$$  

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Note that we chose $\mathbf{k}/d \not\in U_y$, hence $\mathbf{y}_{d;\mathbf{j};\mathbf{k}} \not\in \nabla F_{y,x}(U_y)$, i.e. $\omega^*_j(\mathbf{y}_{d;\mathbf{j};\mathbf{k}}) = 0$. With a similar argument as in (4.1) we obtain

$$\sum_{\mathbf{k} \in \mathcal{X}_d} I_0(d;\mathbf{j};\mathbf{k}) \ll d^{n-1} j_1^{-\frac{3}{2}} d^{-\frac{n}{2} - \frac{3}{8}} = j_1^{-\frac{3}{8}} d^{\frac{n}{2} - \frac{3}{8}}.$$ 

Hence we have

$$M_3 \leq \sum_{0 \leq j_r \leq \min(J_r, j_1)} \left| \sum_{d=1}^D \frac{D-d}{D^2} \sum_{\mathbf{k} \in \mathcal{X}_d} I_0(d;\mathbf{j};\mathbf{k}) \right| \left( j_1 \sum_{\mathbf{k} \in \mathcal{X}_d} I_0(d;\mathbf{j};\mathbf{k}) \right) \ll (\prod_{j=2}^R J_j) d^{\frac{n}{2} + \frac{3}{8}} D^{-\frac{n}{2} - \frac{3}{8}}.$$ 

**Case $\mathbf{k} \in \mathcal{X}_1$.** Let $\lambda$ and $\varphi$ be as in the previous case, specifically maintaining Lemma 5.14 and (5.17). Then we have

$$\varphi(\mathbf{y}_{d;\mathbf{j};\mathbf{k}}) = \mathbf{F}_{y,j}(\mathbf{y}_{d;\mathbf{j};\mathbf{k}}) - \frac{\mathbf{k}}{d} \cdot \mathbf{x}_{d;\mathbf{j};\mathbf{k}} = -\mathbf{F}_{y,j} \left( \frac{\mathbf{k}}{d} \right)$$

and by (2.2)

$$H_{\mathbf{F}_{y,j}}(\mathbf{y}_{d;\mathbf{j};\mathbf{k}}) = H_{\mathbf{F}_{y,j}} \left( \frac{\mathbf{k}}{d} \right)^{-1}.$$ 

Similar to the arguments right before (4.11) we find that the signature $\sigma$ of $H_{\mathbf{F}_{y,j}}(\mathbf{y}_{d;\mathbf{j};\mathbf{k}}) = H_{\mathbf{F}_{y,j}}(\mathbf{y}_{d;\mathbf{j};\mathbf{k}})$ is constant for all $d$, $\mathbf{j}$ and $\mathbf{k}$ in consideration, hence with Lemma 2.2 (5.6) and (5.19) we obtain

$$I_0(d;\mathbf{j};\mathbf{k}) \ll \omega^*_j(\mathbf{y}_{d;\mathbf{j};\mathbf{k}}) \left| \frac{\mathbf{k}}{d} \right| \left( j_1 d \right)^{-\frac{3}{8} + \frac{3}{8}} e \left( -j_1 d \mathbf{F}_{y,j} \left( \frac{\mathbf{k}}{d} \right) + \frac{\sigma}{8} \right) + O \left( (j_1 d)^{-\frac{3}{8} - \frac{3}{8}} \right)$$

$$= \omega^*_j(\mathbf{y}_{d;\mathbf{j};\mathbf{k}}) \left| \frac{\mathbf{k}}{d} \right| \det H_{\mathbf{F}_{y,j}}(\mathbf{k}/d)^{\frac{3}{8}} \left( j_1 d \right)^{-\frac{3}{8} + \frac{3}{8}} e \left( -d(j_1 f_{1,y} + \cdots + j_R f_{R,y}) \left( \frac{\mathbf{k}}{d} \right) + \frac{\sigma}{8} \right)$$

$$+ O \left( (j_1 d)^{-\frac{3}{8} - \frac{3}{8}} \right),$$

where the implicit constant is independant of $d$, $\mathbf{j}$ and $\mathbf{k}$. For $(u_1, ..., u_R) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}^{R-1}$ we consider the function

$$\Psi_{k,d}(u_1, ..., u_R) = u_1^{\frac{n}{2} + \frac{3}{8}} \left| \det H_{\mathbf{F}_{y,j} + \frac{w_1 f_{1,y} + \cdots + w_R f_{R,y}}{d} \left( \frac{\mathbf{k}}{d} \right)} \right|^{\frac{3}{8}}$$

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and obtain

\[
\sum_{0 \leq r \leq \min\{J_r, J_1\}} \left| \sum_{1 \leq j < J, 2 \leq r \leq R} j_1^n f_0(d; j; \hat{\kappa}) \right| (5.21)
\]

\[
\ll \omega \left( \frac{\hat{k}}{d} \right) d^{-\frac{\delta}{2} + \frac{1}{2}} \sum_{0 \leq r \leq J, 2 \leq r \leq R} \max\{1, j_2, \ldots, j_R\} \leq j_1 \leq J
\]

\[
\sum_{\max\{1, j_2, \ldots, j_R\} \leq j_1 \leq J} \Psi_{k;d}(j_1, \ldots, j_R) e(-d_1 \hat{f}_{1,y}(\hat{\kappa}/d))
\]

\[
+ O \left( \left( \prod_{r=2}^{R} J_r \right) J^\frac{\delta}{2} + \frac{1}{2} d^{-\frac{\delta}{2} - \frac{1}{2}} \right).
\]

Given any fixed \( u_2, \ldots, u_R \in \mathbb{R}_{\geq 0} \) we find that \( \Psi(\cdot, u_2, \ldots, u_R) \) is a smooth function on the set \{ \( u_1 \in \mathbb{R}_{>0} \mid u_1 \geq u_r, 2 \leq r \leq R \) \}. Let \( \Psi^{(1)}_{k;d} \) denote the partial derivative of \( \Psi_{k;d} \) in \( u_1 \)-direction. Then by partial summation we have for the innermost sum

\[
\sum_{\max\{1, j_2, \ldots, j_R\} \leq j_1 \leq J} \Psi_{k;d}(j_1, \ldots, j_R) e(-d_1 \hat{f}_{1,y}(\hat{\kappa}/d))
\]

\[
\leq \Psi_{k;d}(J, j_2, \ldots, j_R) \sum_{j_1=1}^{J} e(-j_1 d_1 \hat{f}_{1,y}(\hat{\kappa}/d)) - \int_{j_1=1}^{J} e(-j_1 d_1 \hat{f}_{1,y}(\hat{\kappa}/d)) \Psi^{(1)}_{k;d}(\xi, j_2, \ldots, j_R) d\xi.
\]

We distinguish two cases. First if \(|d\hat{f}_{1,y}(\hat{\kappa}/d)|| \geq J^{-1} \) we obtain

\[
\sum_{\max\{1, j_2, \ldots, j_R\} \leq j_1 \leq J} \Psi_{k;d}(j_1, \ldots, j_R) e(-d_1 \hat{f}_{1,y}(\hat{\kappa}/d))
\]

\[
\ll \frac{\Psi_{k;d}(J, j_2, \ldots, j_R)}{|d\hat{f}_{1,y}(\hat{\kappa}/d)||} + \frac{1}{|d\hat{f}_{1,y}(\hat{\kappa}/d)||} \int_{1}^{J} \Psi^{(1)}_{k;d}(\xi, j_2, \ldots, j_R) d\xi
\]

\[
\ll \frac{\Psi_{k;d}(J, j_2, \ldots, j_R)}{|d\hat{f}_{1,y}(\hat{\kappa}/d)||}.
\]

On the other hand if \(|d\hat{f}_{1,y}(\hat{\kappa}/d)|| < J^{-1} \) we have

\[
\sum_{\max\{1, j_2, \ldots, j_R\} \leq j_1 \leq J} \Psi_{k;d}(j_1, \ldots, j_R) e(-d_1 \hat{f}_{1,y}(\hat{\kappa}/d))
\]

\[
\ll \Psi_{k;d}(J, j_2, \ldots, j_R) J + \int_{1}^{J} \xi \Psi^{(1)}_{k;d}(\xi, j_2, \ldots, j_R) d\xi.
\]

To simplify further, we need estimates for \( \Psi_{k;d} \) and \( \Psi^{(1)}_{k;d} \) respectively.

**Lemma 5.5.** Let \((u_1, \ldots, u_R) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}^{R-1}\) be such that \( u_r \leq u_1 \) for \( 2 \leq r \leq R \). Then for any \( \hat{\kappa} \in \mathcal{K}_1 \) we have

\[
|\Psi_{k;d}(u_1, \ldots, u_R)| \ll u_1^{\frac{\delta}{2} + \frac{1}{2}}
\]
and

\[ |\Psi_{k,d}^{(1)}(u_1, ..., u_R)| \ll u_1^{\frac{3}{2}} \]

where the respective implicit constants are independant of \( y, d \) and \( \hat{k} \).

**Proof.** The first estimate is an obvious consequence of (3.6). Write

\[ \det H_{\tilde{f}_{1,y} + \alpha_2 \tilde{f}_{2,y} + \ldots + \alpha_R \tilde{f}_{R,y}} = \sum_{0 \leq \nu_2 + 1 \leq \nu_R \leq n-1} A_{\nu_2} \ldots \nu_R \alpha_2^{\nu_2} \ldots \alpha_R^{\nu_R} \]

Then since \( \hat{k}/d \in U_y \) we have \( |A_{\nu_2} \ldots \nu_R| \ll 1 \), where the implicit constant is independant of \( y, d \) and \( \hat{k} \). Now by product and chain rule

\[
\begin{align*}
|\Psi_{k,d}(u_1, ..., u_R)| & \ll \frac{n + 1}{2} u_1^{\frac{n}{2} - \frac{1}{2}} |\det H_{\tilde{f}_{1,y} + \frac{u_2}{u_1} \tilde{f}_{2,y} + \ldots + \frac{u_R}{u_1} \tilde{f}_{R,y}}(k/d)|^\frac{1}{n} \\
& \quad + \frac{2}{2} |\det H_{\tilde{f}_{1,y} + \frac{u_2}{u_1} \tilde{f}_{2,y} + \ldots + \frac{u_R}{u_1} \tilde{f}_{R,y}}(k/d)|^\frac{1}{n} \sum_{0 \leq \nu_2 + 1 \leq \nu_R \leq n-1} |A_{\nu_2} \ldots \nu_R| \left( \frac{u_2}{u_1} \right)^{\nu_2} \ldots \left( \frac{u_R}{u_1} \right)^{\nu_R} \\
& \ll u_1^{\frac{n}{2} - \frac{1}{2}},
\end{align*}
\]

since \( 0 \leq u_2, ..., u_R \leq u_1 \) and \( u_1 > 0 \) by assumption.

Therefore we obtain

\[ M_1 \ll \frac{1}{D} \sum_{d=1}^{D} \sum_{k \in \mathbb{Z}^{n-1}} \omega \left( \frac{k}{d} \right) d^{-\frac{n}{2} + \frac{1}{2}} \left( \prod_{r=2}^{R} J_r \right) J_2^{\frac{1}{2} + \frac{1}{2}} \]  

\[ \ll \left( \prod_{r=2}^{R} J_r \right) J_2^{\frac{1}{2} + \frac{1}{2}} d^{-\frac{n}{2} - \frac{1}{2}}. \]

With a similar argument as in (4.14) we can bound the last term in (5.25) by

\[ \frac{1}{D} \sum_{d=1}^{D} \sum_{k \in \mathbb{K}_1} \left( \prod_{r=2}^{R} J_r \right) J_2^{\frac{1}{2} + \frac{1}{2}} d^{-\frac{n}{2} - \frac{1}{2}} \ll \left( \prod_{r=2}^{R} J_r \right) \frac{J_2^{\frac{1}{2} + \frac{1}{2}}}{D} \sum_{d=1}^{D} d^{-\frac{n}{2} - \frac{1}{2}} \]

\[ \ll \left( \prod_{r=2}^{R} J_r \right) J_2^{\frac{1}{2} + \frac{1}{2}} D^{\frac{n}{2} - \frac{1}{2}}. \]
In order to estimate the second term in (5.25) we want to sum dyadically. Note that since $||d\hat{f}_{1,y}(\hat{k}/d)|| \leq \frac{1}{2}$ we can assume $J^{-1} \leq \frac{1}{2}$ and obtain

$$\frac{1}{D} \sum_{d=1}^{D} \sum_{\hat{k} \in \mathbb{Z}^{n-1}} \omega \left( \frac{\hat{k}}{d} \right) d^{-\frac{\rho}{2} + \frac{1}{2}} \left( \prod_{r=2}^{R} J_r \right) \left( J^\frac{-\rho}{2} + \frac{1}{2} \right)^{-1} \sum_{J^{-1} < ||d\hat{f}_{1,y}(\hat{k}/d)||} \omega \left( \frac{\hat{k}}{d} \right)$$

(5.27)

Now for both the first term in (5.25) and (5.27) we utilize the following result from [15, Theorem 2]: For any $X > 0$ we have

$$\sum_{d=1}^{D} \sum_{\hat{k} \in \mathbb{Z}^{n-1}} \omega \left( \frac{\hat{k}}{d} \right) d^{-\frac{\rho}{2} + \frac{1}{2}} \ll X^{-1} D^n + D^{n-1} \varepsilon_{n-1}(D),$$

(5.28)

where

$$\varepsilon_m(D) = \varepsilon_m(c_3;c_4)(D) = \begin{cases} \exp(c_1 \log D) & \text{if } m = 2 \\ \log D & \text{if } m \geq 3 \end{cases}$$

for some positive constants $c_1$ and $c_4$. Here the implicit constants as well as $c_1$ and $c_4$ only depend on $n$, $c_1$ and $c_2$ in (3.6), $\rho$ in (3.7), $\rho'$ in (5.10) and upper bounds for (the absolute values) of finitely many derivatives of $\omega$ and $f_1$ on $\mathcal{D}_\pm \times \mathcal{Y}$. In particular, they are independent of $y$.

By partial summation we find that

$$\sum_{d=1}^{D} \sum_{\hat{k} \in \mathbb{Z}^{n-1}} \omega \left( \frac{\hat{k}}{d} \right) d^{-\frac{\rho}{2} + \frac{1}{2}} \ll D^{-\frac{\rho}{2} + \frac{1}{2}} (X^{-1} D^n + D^{n-1} \varepsilon_{n-1}(D)).$$

(5.29)

Therefore we can estimate the first term in (5.25) by

$$\frac{1}{D} \sum_{d=1}^{D} \sum_{\hat{k} \in \mathbb{Z}^{n-1}} \omega \left( \frac{\hat{k}}{d} \right) d^{-\frac{\rho}{2} + \frac{1}{2}} \left( \prod_{r=2}^{R} J_r \right) J^\frac{-\rho}{2} + \frac{1}{2}$$

(5.30)

and obtain

$$\ll \left( \prod_{r=2}^{R} J_r \right) \frac{J^{\frac{-\rho}{2} + \frac{1}{2}}}{D} (J^{-1} D^n + D^{n-1} \varepsilon_{n-1}(D))$$

and

$$\ll \left( \prod_{r=2}^{R} J_r \right) \left( J^{\frac{-\rho}{2} + \frac{1}{2}} D^{\frac{1}{2} - \frac{1}{2}} + J^{\frac{-\rho}{2} + \frac{1}{2}} D^{\frac{1}{2} - \frac{1}{2}} \varepsilon_{n-1}(D) \right)$$
Putting together (5.26), (5.30) and (5.31) yields

\[
\left( \prod_{r=2}^{R} J_r \right) \frac{J^{n+\frac{1}{2}}}{D} \sum_{i=1}^{\log_{\frac{1}{2}}(\frac{D}{T})+1} J^{2^{1-i}} \sum_{d=1}^{D} d^{-\frac{1}{2}+\frac{i}{2}} \sum_{k \in \mathbb{Z}^{n-1}} \omega\left( \frac{k}{d} \right) (5.31)
\]

\[
\ll \left( \prod_{r=2}^{R} J_r \right) \frac{J^{n+\frac{1}{2}}}{D} \sum_{i=1}^{\log_{\frac{1}{2}}(\frac{D}{T})+1} J^{2^{1-i}} D^{-\frac{1}{2}+\frac{i}{2}} (2^{i} J^{-1} D^{n} + D^{n-1} \mathcal{E}_{n-1}(D))
\]

\[
\ll \left( \prod_{r=2}^{R} J_r \right) \frac{J^{n+\frac{1}{2}}}{D} ((\log J) D^{\frac{n}{2}+\frac{1}{2}} + J D^{\frac{n}{2}+\frac{1}{2}} \mathcal{E}_{n-1}(D))
\]

Putting together (5.26), (5.30) and (5.31) yields

\[
M_1 \ll \left( \prod_{r=2}^{R} J_r \right) (J^{n+\frac{1}{2}} D^{\frac{n}{2}+\frac{1}{2}} + J^{n+\frac{1}{2}} D^{\frac{n}{2}+\frac{1}{2}} \mathcal{E}_{n-1}(D))
\]  

(5.32)

\[
+ \left( \prod_{r=2}^{R} J_r \right) ((\log J) D^{\frac{n}{2}+\frac{1}{2}} + J D^{\frac{n}{2}+\frac{1}{2}} \mathcal{E}_{n-1}(D))
\]

\[
+ \left( \prod_{r=2}^{R} J_r \right) J^{n+\frac{1}{2}} D^{\frac{n}{2}-\frac{1}{2}}
\]

\[
\ll \left( \prod_{r=2}^{R} J_r \right) ((\log J) D^{\frac{n}{2}+\frac{1}{2}} + J D^{\frac{n}{2}+\frac{1}{2}} \mathcal{E}_{n-1}(D)).
\]

Final estimate. Recall \( D = \lfloor T/2 \rfloor \) and \( T \geq 2 \). By combining (5.12), (5.16), (5.18) and (5.32) we obtain

\[
\mathcal{M}(J, T^{-1}) \ll \left( \prod_{r=2}^{R} J_r \right) \frac{J^{n+1}}{T} + \left( \prod_{r=2}^{R} J_r \right) ((\log J) D^{\frac{n}{2}+\frac{1}{2}} + J^{\frac{n}{2}+\frac{1}{2}} T^{\frac{n}{2}-\frac{1}{2}} \mathcal{E}_{n-1}(T))
\]  

(5.33)

\[
+ \left( \prod_{r=2}^{R} J_r \right) \log J + \left( \prod_{r=2}^{R} J_r \right) J^{\frac{n}{2}+\frac{1}{2}} T^{\frac{n}{2}-\frac{1}{2}}
\]

\[
\ll \left( \prod_{r=2}^{R} J_r \right) (J^{n+1} T^{-1} + ((\log J) D^{\frac{n}{2}+\frac{1}{2}} T^{\frac{n}{2}-\frac{1}{2}} + J^{\frac{n}{2}+\frac{1}{2}} T^{\frac{n}{2}-\frac{1}{2}} \mathcal{E}_{n-1}(T))
\]

We distinguish two cases. First if \( T^{-1} \leq J^{-1} \) we have

\[
\mathcal{M}(J, T^{-1}) \leq \mathcal{M}(J, J^{-1}) \ll \left( \prod_{r=2}^{R} J_r \right) J^{n} ((\log J) + \mathcal{E}_{n-1}(J)).
\]  

(5.34)
On the other hand, if \( T^{-1} > J \), i.e. \( J > T \), then
\[
\mathcal{M}(J, T^{-1}) \ll \left( \prod_{r=2}^{R} J_r \right) (J^{n+1}T^{-1} + J^n(\log J) + \mathcal{E}_{n-1}(J)).
\] (5.35)

Therefore we conclude
\[
\sum_{1 \leq j_1 \leq J} \sum_{k \in 2^n \atop 0 \leq j_r \leq \min(J, j_1) \atop 2 \leq r \leq R} \omega_j^\ast \left( \frac{k}{j_1} \right) = \mathcal{M}(J, T^{-1}) \ll \left( \prod_{r=2}^{R} J_r \right) (J^{n+1}T^{-1} + J^n\mathcal{E}_{n-1}(J)).
\]

Recall that \( \omega_j^\ast \left( \frac{k}{j_1} \right) = \omega \circ (\nabla F_{y_3})^{-1}(\frac{k}{j_1}) = \omega(\tilde{x}_j^\ast k) \), hence \[4.4\] follows.

6. Proof of Theorem \[1.4\]

Recall \[3.9\], hence with the bounds obtained for \( N_1, N_2 \) and \( N_3 \) in Section 4, i.e. \[4.21\], \[4.9\] and \[4.10\], we have
\[
N^{(1; \ldots, 1), n)}(Q, \delta) \ll N_1 + N_2 + N_3
\] (6.1)
\[
\ll (1 + \log J)^R \left( (\log Q)Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{2}{3} + Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{1}{2} - \frac{1}{2} \mathcal{E}_{n-1}(J) \right)
\]
\[
+ \log Q(1 + \log J)^R + J^\frac{2}{3}Q^{\frac{n}{2} + \frac{1}{3}}(1 + \log J)^R.
\]
\[
\ll (1 + \log J)^R \left( (\log Q)Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{2}{3} + Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{1}{2} - \frac{1}{2} \mathcal{E}_{n-1}(J) \right)
\]

Now with \[3.3\], \[3.5\] and the remark made right after \[3.5\] we obtain
\[
|N_\omega(Q, \delta) - (2\delta)^R N_0| \ll \delta^{R-1} \frac{Q^{n+1}}{J} + \frac{Q^{n+1}}{J^R} + (1 + \log J)^R (\log Q)Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{1}{2} + (1 + \log J)^R Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{2}{3} \mathcal{E}_{n-1}(J).
\]

Note that the constants \( c_1' \) and \( c_2' \) in \( \mathcal{E}_{n-1}(J) = \mathcal{E}_{n-1}^{(c_1', c_2')}(J) \) as well as the implicit constants only depend on \( n, R, c_1 \) and \( c_2 \) in \[3.9\], \( \rho \) in \[3.7\], \( \rho' \) in \[5.10\] for each choice of \((r; \epsilon; \nu) \) \((1 \leq r \leq R, \epsilon \in \{\pm 1\}^R, 1 \leq \nu \leq n)\) and upper bounds for (the absolute values) of finitely many derivatives of \( \omega \) and \( f_1 \) on \( \mathcal{D} \times \mathcal{Y} \). Recall that while we only addressed the case \((r; \epsilon; \nu) \) the bounds are identical in each case. Since we can still choose the parameter \( J \geq 1 \), consider the equivalences
\[
Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{2}{3} < Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{1}{2} - \frac{1}{2} \iff J < Q
\] (6.3)
\[
\delta^{R-1} \frac{Q^{n+1}}{J} < \frac{Q^{n+1}}{J^R} \iff J < \delta^{-1}
\]
\[
\frac{Q^{n+1}}{J^R} \leq Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{1}{2} - \frac{1}{2} \iff Q^\frac{n+1}{R+1} \leq J
\]
\[
\delta^{R-1} \frac{Q^{n+1}}{J} \leq Q^{\frac{n}{2} + \frac{1}{3}}J^\frac{2}{3} - \frac{1}{2} \iff \delta^{\frac{2(R-1)}{R+1}} Q^\frac{n-1}{R+1} \leq J
\]
and distinguish two cases. If \( \delta^{-1} > Q^{n-1} \) then let \( J = Q^{n-1} \), so by the first, second and third equivalence in [6,3] we have

\[
|N_\omega(Q, \delta) - (2\delta)^RN_0| \ll (\log Q)^R Q^{\frac{n^2 + 2R + 1}{n+1}} \epsilon_{n-1}(Q).
\]

If \( \delta^{-1} \leq Q^{n-1} \) then let \( J = \delta^{2(R-1)} Q^{n-1} \geq \delta^{-1} \), so by the second, third and fourth equivalence in [6,3] we have

\[
|N_\omega(Q, \delta) - (2\delta)^RN_0| \ll \delta^{\frac{(R-1)(n-1)}{n+1}} (\log Q)^R Q^{\frac{n^2 + 2}{n+1}} \epsilon_{n-1}(Q).
\]

Note that

\[
(\log Q)^R \epsilon_{n-1}(Q) = (\log Q)^R \epsilon_n^{(\epsilon'_1; \epsilon'_2)}(Q) = \begin{cases} 
\exp(c'_1 \sqrt{\log Q + R \log \log Q}) & \text{if } n = 3, \\
(\log Q)^{\epsilon'_2 + R} & \text{if } n \geq 4,
\end{cases}
\]

hence for some absolute constant \( c_0 \) we can choose \( c_1 = c'_1 + c_0 R \) and \( c_2 = c'_2 + R \) and obtain

\[
(\log Q)^R \epsilon_n^{(\epsilon'_1; \epsilon'_2)}(Q) \ll \epsilon_n^{(c_1; c_2)}(Q).
\]

This completes the proof of theorem [1.4].

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