THE ALGEBRA OF CONFORMAL BLOCKS

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Abstract. We construct a flat sheaf of algebras over the moduli stack of stable punctured curves with fiber over a given curve equal to the Cox ring of the moduli of quasiparabolic principal bundles associated to a simple complex reductive group, also known as the algebra of conformal blocks. This construction generalizes the connection between the Hilbert functions from phylogenetic algebraic geometry and the Verlinde formula, as recently discovered by Sturmfels and Xu, gives phylogenetic varieties as Gorenstein toric deformations of the universal torsor of the moduli of quasiparabolic \( SL_2(\mathbb{C}) \) principle bundles bundles over a curve, and answers a conjecture of Millson. We also study the relationship between these algebras and classical branching algebras of the associated simply connected reductive group in the general case, and speculate on a recipe for toric deformations of moduli of semistable quasiparabolic principal bundles for more general groups.

Contents

1. Introduction 1
2. The sheaf of conformal blocks 9
3. Filtrations of the algebra of conformal blocks over the stable locus 16
4. Branching algebras and the genus 0 case 22
5. The case \( \mathfrak{g} = sl_2(\mathbb{C}) \) 29
6. Acknowledgements 30

References 30

1. Introduction

Graphs and trees, labeled by various kinds of integral data, make numerous appearances across mathematics and its applications, in this note we will discuss instances from algebraic geometry, conformal field theory, representation theory, and mathematical biology. Along the way we will see how these instances are related by the same type of geometric structure: a flat family of algebras over the moduli of curves. A natural example of a family of algebras of labeled trees is the class of deformations of the Plücker embedding of the Grassmannian of 2-planes \( Gr_2(\mathbb{C}^n) \) parametrized by the tropical Grassmannian \( \mathbb{T}(2, n) \), introduced by Speyer and Sturmfels in [SpSt]. The toric members of this family have a very nice description as the affine semigroup algebras of the cones \( P_T \), where \( T \) is a finite trivalent tree with \( n \) ordered leaves. The cone \( P_T \) is the set of weightings of the edges of \( T \) by nonnegative real numbers, such that the triangle inequalities are satisfied for the the three weights incident on any internal vertex of \( T \). The lattice of integral points in this cone is taken to be integer weightings of \( T \) such that the sum of the weights about any internal vertex is even.

A similar family of semigroup algebras was constructed by Buczynska and Wiesniewski in [BW] as an algebraic analogue to the Jukes-Cantor statistical model of phylogenetics. These models are parametrized by the same class of trees \( T \), and aid in the study of ancestral relationships between a set of \( n \)-taxa. The polytopes \( P_T^* \) (author’s notation) of Buczynska and Wiesniewski are defined as the set of weightings of the tree \( T \) with the same conditions and lattice as \( P_T \), along with the additional condition that the sum of the weights about each internal vertex is equal to 2. This polytope is compact, and so defines a projective toric variety. Buczynska and Wiesniewski were able to show that the graded algebras \( \mathbb{C}[P_T^*] \) are deformation equivalent, and therefore share the same Hilbert function.

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Their visually appealing method was to construct pair-wise deformations between algebras associated to topologically similar trees, those related by a so-called flip-move.

Buczynska and Wiesniewski conjectured the existence of a global algebra that would have one toric deformation for each tree $\mathcal{T}$ to $\mathbb{C}[P_T]$, and serve in the same role as the Plücker algebra serves for the family $\mathbb{C}[P_T]$. This conjecture was answered by Sturmfels and Xu in [StX], where they showed that the Cox-Nagata ring $R_{n,n-3}$ of the blow-up $X_{n,n-3}$ of $\mathbb{P}^{n-3}$ at $n$ generically positioned points has precisely these properties. In this same paper, Sturmfels and Xu drew attention to the work of Bauer, [Ba] who showed that the moduli space of quasiparabolic rank 2 semistable bundles on the $n$-punctured projective line $\mathcal{N}_{0,n}$ is connected to $X_{n,n-3}$ by a series of flops. This moduli space carries line bundles $\mathcal{L}(\vec{r}, L)$, where $(\vec{r}, L)$ is an $n+1$ tuple of nonnegative integers. The global sections $H^0(\mathcal{N}_{0,n}, \mathcal{L}(\vec{r}, L))$ is known as the space of non-abelian theta functions, and its dimension is computed by the celebrated Verlinde formula from mathematical physics, for the case where the symmetries are given by $sl_2(\mathbb{C})$. Birational flops preserve the Picard group and the Cox ring, and so ensure that the phylogenetic Hilbert functions of Buczynska and Wiesniewski can also be computed by the Verlinde formula.

This intriguing connection was the starting point for this project, it suggests that the combinatorial structures related to the Verlinde formula, the "factorization rules," have a geometric underpinning. In this paper we interpret the factorization rules in terms of deformation relations among graded rings attached to moduli of principal bundles. We will also extend this structure for all genus and all simple lie algebras $g$ over $\mathbb{C}$. The "geometric factorization rules" constructed here give a way to recursively study linear series on moduli of principle bundles, in particular for $SL_2$ we obtain toric deformations of the moduli of semistable bundles. This will also allow us to answer a conjecture of Millson [MII] on the relationship between linear series on the moduli of bundles for $SL_2$ and linear series on weight varieties of the Grassmannian variety of 2-planes. Generalizing to arbitrary genus for $SL_2$ also extends the result of Sturmfels and Xu to varieties which define a generalization of the statistical models of Buczynska and Wiesniewski to equivalent varieties for "phylogenetic networks," graphs with positive first Betti number, studied by Buczynska in [Bu].
1.1. **Background.** We will begin by describing the genus 0, $sl_2(\mathbb{C})$ variant of our ideas and build up to the general case. The pairwise deformations of Buczynska and Wiesniewski along with the global deformation constructed by Sturmfels and Xu suggest an ambient deformation structured on a stratified space, with the combinatorics of the stratification coming from a "space" of trees. A natural stratified object from algebraic geometry which carries this combinatorial structure is $\bar{M}_{0,n}$, the moduli stack of $n$-pointed stable curves of genus 0. Naively, we are looking for a flat sheaf of algebras on this stack, with the toric algebras from [BW] showing up as the fibers over the lowest strata. We would also like to incorporate the mathematical physics of the Verlinde Formula, which comes ready made with a graph-based combinatorial structure via the factorization rules. A natural candidate satisfying both conditions comes from the work of Tsuchiya, Ueno and Yamada [TUY] on conformal field theory, in the form of the locally free sheaf of conformal blocks $V^+_{\vec{r}, L}$ on the moduli stack of stable $n$-punctured curves of genus 0. Here $\vec{r}$ is an $n$-vector of dominant $sl_2(\mathbb{C})$ weights, and $L$ is a positive integer called the level. Let $C$ be a smooth genus 0 $n$-punctured curve, then there are isomorphisms of vector spaces which establish that the rank of the sheaf of conformal blocks is equal to the dimension of the space of generalized theta functions.

\[
V^+_{C, \vec{p}}(\vec{r}, L) \cong H^0(\bar{M}_{0,n}, L(\vec{r}, L))
\]

![Figure 3. The planar part of the stratified structure of trees with 5 ordered leaves](image)

Tsuchiya, Ueno, and Yamada used this identification to prove the factorization rules for the Verlinde formula by studying what happens as the curve $C$ is allowed to degenerate to a punctured stable curve. Any such curve $(C, \vec{p})$ has a smooth normalization $(\tilde{C}, \vec{p}, \vec{q})$ where $\vec{q}$ are the doubled points of the normalization. In [TUY] the following is proved

\[
V_{C, \vec{p}}(\vec{r}, L) = \bigoplus_{m_1, m'_1 \leq L} V_{\tilde{C}, \vec{p}, \vec{q}}(\vec{r}, \vec{m}, \vec{m}', L)
\]

where $m_i = m'_i$ if the associated puncture points are identified by the normalization. This is known as the factorization rule for conformal blocks with $sl_2(\mathbb{C})$ symmetry. When $C$ is stable of genus 0, its stable types fall into a finite number of equivalence classes, indexed by labeled trees $T$. Each term in the sum on the right above then becomes a tensor product over the internal vertices of the tree $T$. The above isomorphism gives the following formula for dimensions. We will simplify to the case of two internal vertices, the reader can generalize the formula to larger trees.
Figure 4. Normalization of a 3-punctured stable genus 2 curve.

\[
(3) \quad h_4(r_1, r_2, r_3, r_4, L) = \sum_{m \leq L} h_3(r_1, r_2, m, L)h_3(m, r_3, r_4, L)
\]

Figure 5. Factorization theorem.

Figure 6. Combinatorial type of a stable punctured surface of genus 0.

The reader may notice that formula (3) is the counting formula of Buczyna and Wiesniewski from [BW] when all vertices of \( T \) are trivalent. All of the constructions above work for any genus \( g \) and any simple Lie algebra \( \mathfrak{g} \). This work was carried out in [TUY], but we refer the reader to the book by Shimizu and Ueno [SU], the book by Frenkel and Ben-Zwi [FBZ], and the paper by Looijenga [L] for helpful discussion and other points of view. In what follows objects are fibered over the moduli stack of stable curves with \( n \) punctures \( \mathcal{M}_{g,n} \).

Choose once and for all a Weyl chamber \( \Delta \), and for the longest root \( \omega \), let \( \Delta_L = \{ \lambda \in \Delta | \lambda(H_\omega) \leq L \} \), where \( H_\omega \) is the corresponding coroot. Let \( \mathcal{M}_{C,p}(\bar{\Lambda}) \) be the moduli stack of parabolic principal \( G \)-bundles on the punctured curve \((C, \bar{p})\) with parabolic structures \( \Lambda_i \subset G \) at the puncture points, for \( \mathfrak{g} = \text{Lie}(G) \) with \( G \) simply connected. These stacks carry line bundles \( \mathcal{L}(\bar{\lambda}, L) \), where \( \bar{\lambda} \) is a vector of dominant \( \mathfrak{g} \)-weights, and \( L \) is a non-negative integer. For a punctured curve of genus \( g \) the Verlinde formula calculates the dimension of the space of global sections of these line bundles with the following expression.

\[
(4) \quad \forall_{g,n}(\bar{\lambda}, L) = |T_L|^{g-1} \sum_{\mu \in \Delta_L} Tr_{\mathfrak{g}}(exp(2\pi i \frac{(\mu + \rho)}{L + h^\vee})) \prod_\alpha |2sin(\pi \frac{(\alpha | (\mu + \rho))}{L + h^\vee})|^{2-2g}
\]
Here $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, $\rho$ is the half sum of the positive roots, $Tr_\chi$ is a character associated to the tensor product of irreducible representations $V(\tilde{\lambda})$, and $|T_L|$ is the cardinality of a certain subgroup of a maximal torus of $G$. This impressive expression hides the deeper symmetry of the Verlinde formula given by the factorization rules, which allow one to calculate the formula for any genus and number of punctures with expressions like equation 3. In fact, the above equation can be derived from the factorization rules once one understands the characters of an associated combinatorial structure called the fusion algebra, see the second half of Beauville’s paper [B] for details on this beautiful calculation.

In their proof of the factorization rules, Tsuchiya, Ueno and Yamada constructed, for any tuple $\tilde{\lambda}$ of dominant weights, and any non-negative integer $L$ (still called the level), a coherent, locally free sheaf $V^+(\tilde{\lambda}, L)$ with a projectively flat connection on $\mathcal{M}_{g,n}$. The sheaf of conformal blocks geometrically ties together spaces of partition functions for the WZW model of conformal field theory as the complex curve $(C, p)$ varies. A different proof of the factorization rules was obtained by Faltings in [F]. Various authors [KNR], [LS], have shown that for a point $(C, p) \in \mathcal{M}_{n,g}$, there is an isomorphism of vector spaces

$$h_{C,p} : V^+_{C,p}(\tilde{\lambda}, L) \cong H^0(\mathcal{M}_{C,p}(\tilde{\lambda}), \mathcal{L}(\tilde{\lambda}, L)).$$

Conformal blocks have generated much interest, due in part to their relationship with the moduli of principal bundles $\mathcal{M}_{C,p}(\tilde{\lambda})$. They have also been useful in the representation theory of Kac-Moody algebras and quantum groups, and have served as a source of examples for the theory of modular tensor categories, see [BaK]. In addition, Conformal blocks have a $D$-module structure which have made them objects of interest in the geometric Langlands program. Conformal blocks are not unique to nice categories of Lie algebras, they can also be constructed from the representation theory of chiral vertex operator algebras, [NT]. As combinatorial structures associated to conformal blocks have already made appearances in phylogenetic and combinatorial algebraic geometry, it is our hope that the constructions can be suitably generalized to these other cases to provide further sources of interesting combinatorics.

1.2. **Statement of main results.** Our first order of business is to construct a global algebraic object which ties the algebras of each combinatorial type together. We do this with sheaves of conformal blocks.

**Theorem 1.1.** For any simple Lie algebra $\mathfrak{g}$ with associated simple, simply connected reductive group $G$, the direct sum of vector bundles,

$$V^+ = \bigoplus_{\tilde{\lambda}, L} V^+(\tilde{\lambda}, L)$$

forms a sheaf of algebras on $\mathcal{M}_{g,n}$. Here the sum is over all tuples $\tilde{\lambda}$ with dominant weight entries from $\Delta$, and all nonnegative integers $L$. Fiber-wise, multiplication on this sheaf agrees with multiplication of global sections of the corresponding line bundles on the moduli stacks $\mathcal{M}_{C,p}(\tilde{\lambda})$.

This is our global object, connecting the algebras over smooth curves with those over stable curves. Since each of these sheaves is a sum of vector bundles, the multigraded Hilbert functions are independent of base point. We can calculate the rank of the sheaves $V^+(\tilde{\lambda}, L)$ with the Verlinde formula,

$$\text{rank}(V^+(\tilde{\lambda}, L)) = V_{g,n}(\tilde{\lambda}, L)$$

There is one catch, we lack a “factorization” property on the ring structure. We desire this property in order to study the more complicated algebras attached to surfaces of higher genus with their smaller genus counterparts. We remedy this with the following theorem, first a definition.

**Definition 1.2.** Let $C$ be a stable curve with punctures at $\vec{p}$, and let $(\tilde{C}, \vec{p}, \vec{q})$ be a normalization with new punctures $\vec{q}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, with dual $\mathfrak{h}^\ast$. Consider the cone $\Lambda_{\Delta} \subset \mathfrak{h}$ of linear functionals $\theta : \mathfrak{h}^\ast \to \mathbb{R}$ which satisfy $\theta(\alpha) \leq 0$ for all negative roots. This is the cone of functionals...
which respects the ordering on the dominant weights in $\Delta$. Let $\omega$ be the longest element of the Weyl group. We call
\begin{equation}
\tilde{\theta} : \{\vec{p}, \vec{q}\} \rightarrow \Lambda_{\Delta}
\end{equation}
such that pairs $q_1$ and $q_2$ are weighted with functionals $\frac{1}{2} \theta_1 = -\frac{1}{2} \theta_2 \circ \omega, \in \Lambda_{\Delta}$ a $g$-weighting of $\tilde{C}$. Note that the functionals on $\vec{p}$ need not be restricted to $\Lambda_{\Delta}$.

The reason for the factor of $\frac{1}{2}$ will be made clear later. Fixing a stable curve type $\Gamma$, we have a map of stacks
\begin{equation}
\pi_{\Gamma} : M_{g_1,n_1} \times \cdots \times M_{g_k,n_k} \rightarrow \tilde{M}_{g,n}
\end{equation}
which glues the associated points of the normalization. The image of this map is the stratum of $\tilde{M}_{g,n}$ associated to the type $\Gamma$. We now have enough to state the second theorem.

**Theorem 1.3.** For each stability type $\Gamma$, and a $g$-weighting $\tilde{\theta}$, which is strictly negative on negative roots, there is a flat deformation defined by filtration,
\begin{equation}
\pi^\ast_{\Gamma}(V^+_{g,n}) \Rightarrow (\mathbb{E}_T V^+_{g,n})^T
\end{equation}
where the tensor product is over the connected components of the normalization of the type $\Gamma$, and the symbol $T$ denotes fiber-wise torus invariants by the action which identifies weights $\alpha$ on a puncture $q_1$ with the dual weight $\alpha^\ast$ on its partner $q_2$, and forces the levels $L$ to be equal.

Over a point $(C, \vec{p}) \in \text{img}(\pi_{\Gamma})$ this says that there is a filtration which defines a flat deformation,
\begin{equation}
V^+_{C,\vec{p}} \Rightarrow (V^+_{C,\vec{p}^\ast})^T
\end{equation}
where $\tilde{C}$ is the normalization of $C$. This is where the combinatorial data of moduli of curves comes into the picture. Multiplication over the stable locus "flattens" to multiplication over each connected component of the normalization. The algebras over different strata of $M_{g,n}$ are then connected by flat families, as in the case $g = 0$, $g = sl_2(\mathbb{C})$ in [BW]. The deformations described in this paper add torus symmetries and so define new polytopes which come with surjections onto the effective cone of $M_{C,\vec{p}}(\tilde{B})$. If enough torus symmetries are added, one obtains a toric algebra.

**Definition 1.4.** Let $\Gamma$ be a finite trivalent graph, and let $P^\ast_{\Gamma}$ be the polytope defined by integer weightings of $\Gamma$ which satisfy the same local conditions which define $P^+_{\Gamma}$.

The algebras associated to these polytopes $\mathbb{C}[P^\ast_{\Gamma}]$ have been studied by Buczynska in [Bu], and of course match the phylogenetic algebras of Buczynska and Wiesniewski when the first betti number of $\Gamma$ is 0. The next proposition follows from theorems 1.1 and 1.3.

**Proposition 1.5.** For $g = sl_2(\mathbb{C})$, the algebra of conformal blocks $V^+_{C,\vec{p}}$ for a curve of genus $g$ with $n$ marked points flatly deforms to $\mathbb{C}[P^\ast_{\Gamma}]$ for any graph $\Gamma$ with first Betti number equal to the genus of $C$ and number of leaves equal to the number of marked points $\vec{p}$ on $C$.

For fixed genus and number of punctures, the cones defined in definition 1.2 fit together into a stratified complex, like a tropical variety. We will remark later on why we believe this is the case. In the case $g = sl_2(\mathbb{C})$ these are the moduli of pseudo-tropical curves constructed by Brannetti, Melo, and Viviani in [BMV], for $g = sl_2(\mathbb{C})$ and $g = 0$ these are abstractly tropical Grassmannians, and can be realized as the tree spaces of Billera, Holmes and Vogtman, [BHV].

For genus 0 all graphs $\Gamma$ are trees with $n$ labeled leaves. There is a natural map relating the space of conformal blocks with labels $\vec{\lambda}$ and an associated space of invariants in $V(\vec{\lambda})^\ast$, an $n$-fold tensor product of $g$-representations.
\begin{equation}
F_{C,\vec{p}} : V_{C,\vec{p}}(\vec{\lambda}, L) \rightarrow \text{Hom}_C(V(\vec{\lambda})/gV(\vec{\lambda}), \mathbb{C}) \cong V(\vec{\lambda})^g
\end{equation}
We call \( V(\tilde{\lambda})/gV(\tilde{\lambda}) \) the space of covariants, and \( \text{Hom}_C(V(\tilde{\lambda}))/gV(\tilde{\lambda}), \mathbb{C} \cong V(\tilde{\lambda})^g \) the space of invariants. When the genus of the curve \( C \) is 0 this map is injective. In order to make use of this we will study the deformation theory of branching algebras,

\[
\mathfrak{A}(\Delta_{n-1}) = \bigoplus_{\tilde{\lambda} \in \Delta^n} V(\tilde{\lambda}^\ast)^g
\]

from the representation theory of \( g \), these will be defined later. Branching algebras are a way to introduce commutative algebra (and therefore geometry) into the study of branching rules along morphisms of reductive groups. The algebras of conformal blocks are a quantum analogue for representations of Kac-Moody algebras with respect to the “fusion” tensor product structure, and when formulated as morphism of rings, the map \( F_{C,\tilde{p}} \) allows one to show that the geometry underlying conformal blocks becomes the “classical” geometry of the branching rules in the limit \( L \to \infty \). The branching algebra above has the same relationship to the algebra of conformal blocks as the Plücker algebra has to the Cox-Nagata Ring \( R_{n,n-3} \), and in fact for \( sl_2(\mathbb{C}) \), there is an isomorphism \( \mathfrak{A}(\Delta_{n-1}) \cong \mathbb{C}[Gr_2(\mathbb{C}^n)] \). This was shown by Howard, Manon, and Millson as a step in studying the symplectic geometry of polygons in euclidean space, see [HMM] for more on this point. To generalize this relationship and make it precise, we define a new algebra \( \mathfrak{A}(\Delta_{n-1})[t] \), using the product from \( \mathfrak{A}(\Delta_{n-1}) \).

\[
\mathfrak{A}(\Delta_{n-1})[t] = \bigoplus [V(\tilde{\lambda}^\ast)^g]t^L
\]

There is then a well-defined injection of rings \( F_{C,\tilde{p}} : \mathcal{V}_{C,\tilde{p}}^+ \to \mathfrak{A}(\Delta_{n-1})[t] \). For each tree and assignment of functionals, \((T,\theta)\) there is a natural filtration of \( \mathfrak{A}(\Delta_{n-1})[t] \) which we will construct from the representation theory of \( g \), it will then be possible to prove the following theorem.

**Theorem 1.6.** Let \((C,\tilde{p})\) be a punctured stable genus 0 curve of type \( T \). Then the deformations associated to \((T,\theta)\) on \( \mathcal{V}_{C,\tilde{p}}^+ \) and \( \mathfrak{A}(\Delta_n) \) agree under the correlation morphism \( F_{C,\tilde{p}} \).

Returning now to the moduli \( M_{C,\tilde{p}}(\tilde{\lambda}) \), the Cox rings of these moduli stacks are in a natural way subrings of \( \mathcal{V}_{C,\tilde{p}}^+ \). They are picked out by the blocks with \( \tilde{\lambda} \) sitting in the product of faces of the Weyl chamber \( \Delta \) defined by \( \tilde{\lambda} \). These subrings are respected by the deformations defined by \((\Gamma,\theta)\), hence we obtain flat families of \( \mathbb{C} \)-algebras which connect the Cox rings of parabolic moduli stacks to torus invariants of tensor products of the algebras of conformal blocks associated to punctured copies of \( \mathbb{P}^1 \). This can be taken as a “ringification” of the factorization rules, and shows that factorization is geometric property of the moduli of bundles as well as a representation theoretic phenomenon.

For each \( \tilde{\lambda} \in \Delta^n \) there is a stability condition on \( M_{C,\tilde{p}}(\tilde{\lambda}) \) which picks out a substack of semistable points \( M_{C,\tilde{p}}^{ss}(\tilde{\lambda},L) \), which has a smooth, projective coarse moduli space \( M_{C,\tilde{p}}^{ss}(\tilde{\lambda},L) \), which comes with an embedding given by the graded ring \( \bigoplus H^0(M_{C,\tilde{p}}(\tilde{\lambda},L),\mathcal{L}(n\lambda,nL)) \). The theorems we have described above all descend to this ring by taking torus invariants with respect to the appropriate character. As a consequence of proposition 1.5 we obtain toric deformations of moduli spaces of quasiparabolic semistable principle bundles \( M_{C,\tilde{p}}^{ss}(\tilde{r},L) \) in the case \( g = sl_2(\mathbb{C}) \).
Definition 1.7. Let $\Gamma$ be a finite graph, and let $P_T(\vec{r}, L)$ be the subpolytope of $L \mathcal{P}_T^*$ defined by setting the leaf-weights to $\vec{r}$.

Proposition 1.8. The flat families defined in theorems 1.4 and 1.5 define a deformation of graded algebras.

\begin{equation}
\bigoplus H^0(\mathcal{M}_{C, \vec{p}}(\vec{A}), \mathcal{L}(n\lambda, nL)) \Rightarrow \mathbb{C}[P_T(\vec{r}, L)]
\end{equation}

where $\Gamma$ is any graph with first Betti number equal to the genus of $C$ with $n = |\vec{p}|$ leaves.

Letting $P_T(\vec{r})$ be the subpolytope of $P_T$ obtained by fixing the weights on the leaf edges to be $\vec{r}$ we see there is an obvious injection of graded rings.

\begin{equation}
\mathbb{C}[P_T^*(\vec{r}, L)] \rightarrow \mathbb{C}[P_T(\vec{r})]
\end{equation}

The algebra $\mathbb{C}[P_T(\vec{r})]$ makes an appearance in [HMM] and [HMSV] as a toric deformation of a weight variety of the Grassmannian of 2-planes. From theorem 1.6 we get an injection of graded rings,

\begin{equation}
\bigoplus H^0(\mathcal{M}_{P^1, \vec{p}}(\vec{A}), \mathcal{L}(n\lambda, nL)) \rightarrow \bigoplus V(n\vec{\lambda})^g
\end{equation}

which echoes the injection on toric rings. For $g = sl_2(\mathbb{C})$, the algebra on the right above is given by a linear series on a weight variety of the Grassmannian of 2-planes, where the character of the torus $\vec{r}$ is equivalent to choosing a vector of dominant $sl_2(\mathbb{C})$ weights as above. This led Millson to make the following conjecture.

Conjecture 1.9 (Millson). For a trivalent tree $T$ with $n$-leaves, give the ring $\bigoplus V(n\vec{r})^{sl_2(\mathbb{C})}$ the associated term-order filtration from [HMM] and [HMSV] with associated graded isomorphic to $C[P_T(\vec{r})]$. Then the associated graded ring corresponding to the induced filtration on $\bigoplus H^0(\mathcal{M}_{P^1, \vec{p}}(\vec{r}, L), \mathcal{L}(n\vec{r}, nL))$ is isomorphic to $\mathbb{C}[P_T^*(\vec{r}, L)]$.

We see from theorem 1.6 that this is not far from the truth.

Proposition 1.10. For a trivalent tree $T$ with $n$-leaves, give the ring $\bigoplus V(n\vec{r})^{sl_2(\mathbb{C})}$ the associated term-order filtration from [HMM] and [HMSV] with associated graded isomorphic to $C[P_T(\vec{r})]$. Then the associated graded ring corresponding to the induced filtration on $\bigoplus V^{C, \vec{p}}(n\vec{r}, nL)$ is isomorphic to $\mathbb{C}[P_T^*(\vec{r}, L)]$ precisely when $C$ is the stable curve of genus 0 of stability type $T$.

We would like to know how to make toric deformations for other simple Lie algebras, in order to realize the Verlinde formula for these algebras in terms of counting lattice points in polytopes. This can be carried out for $sl_2(\mathbb{C})$ using the factorization rules.

Corollary 1.11. Let $g = sl_2(\mathbb{C})$, and let $\Gamma$ be a trivalent graph with $n$ leaves of genus $g$. Then $\mathcal{V}_{g,n}(\vec{r}, L)$ is equal to the number of lattice points in the polytope and lattice defined by weightings of $\Gamma$ satisfying

1. The three weights about any internal vertex of $\Gamma$ satisfy the triangle inequalities.
2. The sum of the three weights about any internal vertex of $\Gamma$ is even and less than or equal to $2L$.
3. The $i$-th leaf edge of $\Gamma$ is weighted $r_i$.

A significant step in the understanding of algebras of conformal blocks and obtaining interesting polytope descriptions of the general Verlinde formula would be made by obtaining toric deformations of $\mathcal{V}_{g,n}^T$ for $C$ of genus 0 with 3 punctures, because by theorem 1.3 all algebras of conformal blocks can be “constructed” from this case. One way to approach to this problem is to find toric deformations of $\mathcal{A}(\Delta_2)$, and understand them with respect to the map $T_{\vec{p}}$. This is our main motivation for using branching algebras. So far, SAGBI degenerations of $\mathcal{A}(\Delta_2)$ have been constructed in the case $G = GL_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$ among others by Howe, Tan, and Willenbring in [HTW2]. We will also outline an approach using dual canonical basis which applies for large values of $L$.

Moduli of bundles also have a well-known symplectic structure, (for example see [TW]), and the effective cones of $\mathcal{M}_{C, \vec{p}}(\vec{A})$ are deeply related to the geometry of conjugacy classes in the compact Lie
group $K$, where $K^G = G$. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus of $K$, and let $C_\lambda \subset K$ be the conjugacy class of $exp(\lambda)$, then the rational points of the polytope,

\begin{equation}
\{ \bar{\lambda} \mid Id \in C_{\lambda_1}C_{\lambda_2} \cdots C_{\lambda_n} \}
\end{equation}

are exactly the set of weights $\tilde{\lambda} \in \Delta_1^{\mathfrak{g}}$ such that $V^{\lambda_1}_{\mathfrak{g}, \mathfrak{g}}(k\tilde{\lambda}, kL) \neq 0$ for some $L$ where $k$ is chosen so $k\lambda_i$ is dominant. One wonders if factorization can be “geometrified” with respect to the symplectic geometry of conjugacy classes in $K$. Ideas along these lines can be found for the $SL_2(\mathbb{C})$ case in the paper of Hurtubise and Jeffrey on moduli of $SU(2)$ connections on surfaces, [HJ]. We believe the spectrum of $V^\lambda_{G, \mathfrak{g}}$ for $SL_2(\mathbb{C})$ ought to be deeply related to the “master space” of parabolic sheaves constructed in [L]. See also [HMM] for a similar connection between the symplectic geometry of polygons and deformations of the moduli of points on the projective line.

The paper is organized as follows. We discuss and prove theorem 1.1 in section 2, and construct filtrations of the algebra of conformal blocks to prove theorem 1.3 in section 3. In section 4 we introduce branching algebras and their filtrations, we then look at the special case of the diagonal morphism $\Delta_n : G \to G^n$ for a reductive group and describe the relationship between this algebra and the algebra of conformal blocks with theorem 1.6. In section 5 we look at the case $\mathfrak{g} = sl_2(\mathbb{C})$, where the most is known.

## 2. The sheaf of conformal blocks

In this section we review the definition of the sheaf of conformal blocks. We will also construct the multiplication operation, and show that its specialization at stable punctured curve $(C, \vec{p})$ is equal to multiplication of global sections of line bundles on the corresponding moduli stack. Following [TUY], we work with families of curves over complex varieties. We also refer the reader to [L] for a more detailed discussion of the conformal block construction. We thank Eduard Looijenga for helpful conversations on his construction of the sheaf of conformal blocks.

### 2.1. Basics of affine Kac-Moody algebras

In this subsection we review the construction of the affine Kac-Moody algebra associated to a simple Lie algebra $\mathfrak{g}$. We also review the construction of some important representations of this algebra, the Verma modules and the integrable highest weight modules. For this section we refer the reader to any of the numerous introductions to this subject as associated with conformal blocks, such as [SU], [TUY], [B], [K], [KNR]. For each irreducible $\mathfrak{g}$-representation $V(\lambda)$ we choose a highest weight vector $v_\lambda$. As a vector space the affine lie algebra is

\begin{equation}
\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}(t) \oplus \mathbb{C}c
\end{equation}

The bracket is defined by the following properties.

1. $c$ is a central element of $\hat{\mathfrak{g}}$
2. $[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y) Res_0(gdf)c$ where $[-, -]$ is the bracket on $\mathfrak{g}$ and $(-, -)$ is the normalized Killing form on $\mathfrak{g}$.

This Lie algebra has a triangular decomposition defined by the following subalgebras.

\begin{align*}
\hat{\mathfrak{g}}_+ &= \mathfrak{g} \otimes \mathbb{C}[t]\frac{d}{dt} \\
\hat{\mathfrak{g}}_- &= \mathfrak{g} \otimes \mathbb{C}[t^{-1}]\frac{d}{dt} \\
\hat{\mathfrak{g}}_0 &= \mathfrak{g} \oplus \mathbb{C}c
\end{align*}

We have

\begin{equation}
\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+.
\end{equation}

This decomposition implies the existence of various highest weight representations. We fix a level $L$. Define an action of $\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$ on $V(\lambda)$ by letting $c$ act as multiplication by $L$ and $\hat{\mathfrak{g}}_+$ act trivially. The Verma module is defined as
Definition 2.1. By a stable curve of genus $g$ of $C, \vec{p}$ elements must therefore act the same on a tensor product of integrable highest weight modules is the Lie algebra $H(\lambda, L) = \bar{V}(\lambda)/\langle U(\hat{\mathfrak{g}})(X_0 \otimes t^{-1})^{L-\lambda(H_0)+1}V(\lambda) \rangle$

for our chosen highest weight vector $v_\lambda \in V(\lambda)$. We also make use of the related Lie algebra $\hat{\mathfrak{g}}_n = [\bigoplus \hat{\mathfrak{g}} \otimes \mathbb{C}((t_i)) \big] \oplus \mathbb{C}c$, with $c$ regarded as central and bracket defined by

$$\sum_{i=1}^n X_i \otimes f_i, \sum Y_i \otimes g_i = \sum [X_i, Y_i] \otimes f_ig_i + \sum \text{Res}(g_if_i)c$$

This is the Lie algebra $\sum_{i=1}^n \hat{\mathfrak{g}}$ with the central elements $c_i$ all identified. The fact that all central elements must therefore act the same on a tensor product of integrable highest weight modules is the reason why we will work with $H(\hat{\lambda}, L) = H(\lambda_1, L) \otimes \cdots \otimes H(\lambda_n, L)$ where all the levels are the same. If we fix a punctured curve $(C, \vec{p})$, there is an associated Lie algebra $\mathfrak{g} \otimes \mathbb{C}[C \setminus \vec{p}]$. There is no map $\mathfrak{g} \otimes \mathbb{C}[C \setminus \vec{p}] \to \bigoplus_{i=1}^n \hat{\mathfrak{g}}$. However, there is a map $\mathfrak{g} \otimes \mathbb{C}[C \setminus \vec{p}] \to \bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((t_i))$, and because of the Residue Theorem, this map extends to the central extension, so we obtain $H(\hat{\lambda}, L)$ as a representation of $\hat{\mathfrak{g}} \otimes \mathbb{C}[C \setminus \vec{p}]$.

2.2. Definition of sheaves of vacua. Let $S$ be a smooth variety over $\mathbb{C}$. Following [TUY] we will sheafify the representation theory of $\hat{\mathfrak{g}}_n$ over $S$.

Definition 2.1. By a stable curve of genus $g$ over $S$ with $n$ puncture points we mean a proper, flat map of varieties $\pi : C \to S$ where all fibers of $\pi$ are stable curves of genus $g$, and $n$ pairwise non-intersecting sections $s_1, \ldots, s_n$ which avoid the doubled points.

We will abuse notation slightly, referring to the data $(C, \vec{s})$ as $C$. We need a sheafified version of the affine Lie algebra attached to a simple complex Lie algebra $\mathfrak{g}$. Let $D_\vec{s}$ be the divisor given by the images of the sections $s_i$. The following can be found in [SU] and [TUY], and we refer the reader to [L] for a more algebraic approach.

Definition 2.2. We define the following sheaves of Lie algebras over $S$. Let $\hat{O}_{\mathcal{C}\setminus s_i(S)}$ be the formal completion of $O_C$ along $s_i(S)$.

$$\hat{\mathfrak{g}}_n(S) = \mathfrak{g} \otimes \bigoplus_{i=1}^n \hat{O}_{\mathcal{C}\setminus s_i(S)} \otimes O_S$$

$$\hat{\mathfrak{g}}(C) = \mathfrak{g} \otimes O(D_\vec{s})$$

$$\hat{\mathfrak{g}}(S) = \mathfrak{g} \otimes O_S$$

These are sheaves of Lie algebras over $S$. The algebra $\mathfrak{g}(S)$ can be realized as a sub Lie-algebra object of $\hat{\mathfrak{g}}(C)$, and the fiber of $\hat{\mathfrak{g}}(C)$ at $x$ is equal to $\mathfrak{g} \otimes \mathbb{C}[\pi^{-1}(x) \setminus \vec{s}(x)]$. Let $H(\hat{\lambda}, L)$ be an $n$-fold tensor product of integrable highest weight modules of $\hat{\mathfrak{g}}$, let $V(\hat{\lambda})$ be an $n$-fold tensor product of irreducible highest weight modules of $\mathfrak{g}$, we define

$$H_S(\hat{\lambda}, L) = H(\hat{\lambda}, L) \otimes O_S$$

$$V_S(\hat{\lambda}) = V(\hat{\lambda}) \otimes O_S,$$

By choosing an isomorphism $\hat{O}_{\mathcal{C}\setminus s_i(S)} \cong O_S((\zeta_i))$, the sheaf $H_S(\hat{\lambda}, L)$ is a $\hat{\mathfrak{g}}_n(S)$ module, and $V_S(\hat{\lambda})$ is a $\mathfrak{g}(S)$ module in a natural way, which is equivariant with respect to its inclusion in $H_S(\hat{\lambda}, L)$. In [TUY] and [SU] this isomorphism is shown to be equivalent to selecting tangent vectors at punctures, see also [FrBZ] for discussion of this point. In [L], Looijenga gives a construction of this representation that is
Definition 2.3. The sheaf of covacua \( V_\mathcal{C}(\bar{\lambda}, L) \) is defined to be the sheaf of coinvariants of the action of \( \hat{g}(\mathbb{C}) \) on \( \mathcal{H}_S(\bar{\lambda}, L) \). The sheaf of vacua or conformal blocks, \( V^+_\mathcal{C}(\bar{\lambda}, L) \) is defined to be the corresponding sheaf of invariants.

\[
V^+_\mathcal{C}(\bar{\lambda}, L) = \text{Hom}_{\mathcal{O}_S}(\mathcal{V}_\mathcal{C}(\bar{\lambda}, L), \mathcal{O}_S)
\]

The sheaf of conformal blocks \( V^+_C(\bar{\lambda}, L) \) is shown to be coherent and locally free in \( [TUY] \), with clarifications in \( [SU] \) and \( [L] \). Taking a single fiber \( \pi^{-1}(x) \) of \( \pi \) we may define the vector space of conformal blocks,

\[
V^+_x(\bar{\lambda}, L) = \text{Hom}_\mathbb{C}(\mathcal{H}(\bar{\lambda}, L)/\hat{g}(\pi^{-1}(x)))\mathcal{H}(\bar{\lambda}, L), \mathbb{C})
\]

It is not hard to prove

\[
V^+_S(\bar{\lambda}, L)|_x = V_{\pi^{-1}(x),\bar{\lambda}(x)}(\bar{\lambda}, L)
\]

Moreover, the definition of conformal blocks commutes with pullbacks of families of \( n \)-punctured curves. Hence there are well-defined coherent, locally free sheaves \( V^+_C(\bar{\lambda}, L) \) on \( \mathcal{M}_{g,n} \).

2.3. Multiplication of conformal blocks. Now we define the multiplication operation on sheaves of conformal blocks. Recall that we have chosen highest weight vectors \( v_\lambda \in V(\bar{\lambda}) \) for each irreducible highest weight representation of \( g \). Essential to our discussion are the following two commutative diagrams.

\[
\begin{array}{ccc}
\mathcal{H}(\bar{\lambda} + \bar{\gamma}, L + K) & \xrightarrow{C_{\bar{\lambda} + \bar{\gamma}}} & \mathcal{H}(\bar{\lambda}, L) \otimes \mathcal{H}(\bar{\gamma}, K) \\
\uparrow i & & \uparrow i \otimes i \\
V(\bar{\lambda} + \bar{\gamma}) & \xrightarrow{C_{\bar{\lambda} + \bar{\gamma}}} & V(\bar{\lambda}) \otimes V(\bar{\gamma}) \\
\mathcal{H}_S(\bar{\lambda} + \bar{\gamma}, L + K) & \xrightarrow{C_{\bar{\lambda} + \bar{\gamma}}} & \mathcal{H}_S(\bar{\lambda}, L) \otimes \mathcal{H}_S(\bar{\gamma}, K) \\
\uparrow i & & \uparrow i \otimes i \\
V_S(\bar{\lambda} + \bar{\gamma}) & \xrightarrow{C_{\bar{\lambda} + \bar{\gamma}}} & V_S(\bar{\lambda}) \otimes V_S(\bar{\gamma})
\end{array}
\]

The first is obtained by identifying the highest weight vectors \( v_{\lambda + \bar{\gamma}} \Rightarrow v_\lambda \otimes v_\bar{\gamma} \). The second diagram is obtained from the first diagram by tensoring with \( \mathcal{O}_S \). It is a commuting square of \( g(S) \) sheaves over \( S \). Since the level is always fixed across the tensor products, the top row is a map of \( \hat{g}(\mathbb{C}) \) modules. The first diagram is the localization of the second at a point \( x \in S \). The following lemma will be used throughout the paper.

Lemma 2.4. Dualizing: Let the following be a commutative diagram of sheaves.

\[
\begin{array}{ccc}
P_1 & \xrightarrow{f} & M_1 \otimes N_1 \\
e & & \uparrow g_1 \otimes g_2 \\
P_2 & \xrightarrow{h} & M_2 \otimes N_2
\end{array}
\]

Then the following diagram also commutes.
Proof. First, apply the duality functor to the top diagram.

\[ \begin{array}{ccc} P^*_1 & \xleftarrow{f^*} & M^*_1 \otimes N_1^* \\
& g_1^* \otimes g_2^* & \\
& \downarrow & \\
P^*_2 & \xleftarrow{h^*} & M^*_2 \otimes N_2^* \end{array} \]

Now paste this diagram to the following diagram, which follows from the binaturality property of duality.

\[ \begin{array}{ccc} (M_1 \otimes N_1)^* & \xleftarrow{f^*} & M^*_1 \otimes N_1^* \\
& (g_1 \otimes g_2)^* & \\
& \downarrow & \\
(M_2 \otimes N_2)^* & \xleftarrow{h^*} & M^*_2 \otimes N_2^* \end{array} \]

\[ \square \]

**Lemma 2.5.** The following diagram commutes.

\[ \begin{array}{ccc} V^+_g(\bar{\lambda} + \bar{\gamma}, K + L) & \xleftarrow{C^\gamma_{\bar{\lambda} + \bar{\gamma}}} & V^+_g(\bar{\lambda}, L) \otimes_{O_S} V^+_g(\bar{\gamma}, K) \\
& i^* & \\
& \downarrow & \\
V^+_g(S(V_S(\bar{\lambda} + \bar{\gamma}))) & \xleftarrow{C^\gamma_{\bar{\lambda} + \bar{\gamma}}} & V^+_g(S(V_S(\bar{\lambda})) \otimes_{O_S} V^+_g(S(V_S(\bar{\gamma}))) \end{array} \]

*Proof.* First we consider the bottom row of diagram \[\text{2,3} \] For a sheaf of representations $W$, let $g(S)W$ denote the image of the action map, that is, the sheaf of sections of the form $X \circ s \in W(U)$ for $s \in W(U)$ and $X \in g(S)(U)$. This defines the quotient sheaf $V_{g(S)}[W] = W/g(S)W$. Since our diagram is equivariant, we get

\[ \begin{array}{ccc} V_{g(S)}[\mathcal{H}_S(\bar{\lambda} + \bar{\gamma}, L + K)] & \xrightarrow{V[C^\gamma_{\bar{\lambda} + \bar{\gamma}}]} & V_{g(S)}[\mathcal{H}_S(\bar{\lambda}, L) \otimes \mathcal{H}_S(\bar{\gamma}, K)] \\
& \uparrow & \\
V_{g(S)}[V_S(\bar{\lambda} + \bar{\gamma})] & \xrightarrow{V[C^\gamma_{\bar{\lambda} + \bar{\gamma}}]} & V_{g(S)}[V_S(\bar{\lambda}) \otimes V_S(\bar{\gamma})] \end{array} \]

Now, consider any tensor product of sheaves of representations $M \otimes_{O_S} N$, and the map

\[ \text{(32)} \quad p_1 \otimes p_2 : M \otimes_{O_S} N \rightarrow V[M] \otimes_{O_S} V[N]. \]

Given $s \otimes t \in M(U) \otimes_{O_S(U)} N(U)$, $X \in g(S)(U)$ acts as $X(s \otimes t) = X(s) \otimes (t + s \otimes X(t))$. Clearly then we have $p_1 \otimes p_2(X(s \otimes t)) = 0$, and therefore a natural map

\[ \text{(33)} \quad V[M \otimes_{O_S} N] \rightarrow V[M] \otimes_{O_S} V[N]. \]

It is easy then to check that we get a diagram,

\[ \begin{array}{ccc} V_{g(S)}[\mathcal{H}_S(\bar{\lambda} + \bar{\gamma}, L + K)] & \xrightarrow{V_{g(S)}[C^\gamma_{\bar{\lambda} + \bar{\gamma}}]} & V_{g(S)}[\mathcal{H}_S(\bar{\lambda}, L)] \otimes V_{g(S)}[\mathcal{H}_S(\bar{\gamma}, K)] \\
& \uparrow V[i] & \\
V_{g(S)}[V_S(\bar{\lambda} + \bar{\gamma})] & \xrightarrow{V_{g(S)}[C^\gamma_{\bar{\lambda} + \bar{\gamma}}]} & V_{g(S)}[V_S(\bar{\lambda})] \otimes V[V_S(\bar{\gamma})] \end{array} \]
Nothing we have said above applies specially to $g(S)$, we could have used any sheaf of Lie algebras with representations, so we also get a map

$$V_{\hat{g}(C)}[H_S(\tilde{\lambda} + \tilde{\gamma}, L + K)] \xrightarrow{V_{\hat{g}(C)}[\mathcal{C}_{\tilde{\lambda} + \tilde{\gamma}}]} V_{\hat{g}(C)}[H_S(\tilde{\lambda}, L)] \otimes V_{\hat{g}(C)}[H_S(\tilde{\gamma}, K)].$$

Consider now the sheaf of representations $H_S(\tilde{\lambda}, L)$. We have an injection of sheaves of Lie algebras, $g(S) \to \hat{g}(C)$, and so by collapsing out by more symmetries we get a map of sheaves

$$(34) \quad \hat{i} : V_{\hat{g}(S)}[H_S(\tilde{\lambda}, L)] \to V_{\hat{g}(C)}[H_S(\tilde{\lambda}, L)].$$

One checks this is a surjection by noting that both are quotient sheaves, and the corresponding map on fibers is surjective - however, this is not important. This gives us two diagrams,

$$\begin{array}{c}
V_{\hat{g}(C)}[H_S(\tilde{\lambda} + \tilde{\gamma}, L + K)] \xrightarrow{V_{\hat{g}(C)}[\mathcal{C}_{\tilde{\lambda} + \tilde{\gamma}}]} V_{\hat{g}(C)}[H_S(\tilde{\lambda}, L) \otimes H_S(\tilde{\gamma}, K)] \\
\hat{i} \uparrow \quad \downarrow \hat{i} \\
V_{\hat{g}(S)}[H_S(\tilde{\lambda} + \tilde{\gamma}, L + K)] \xrightarrow{V_{\hat{g}(S)}[\mathcal{C}_{\tilde{\lambda} + \tilde{\gamma}}]} V_{\hat{g}(S)}[H_S(\tilde{\lambda}, L) \otimes H_S(\tilde{\gamma}, K)] \\
\hat{i} \downarrow \quad \downarrow \hat{i} \\
V_{\hat{g}(C)}[H(\tilde{\lambda}, L) \otimes H(\tilde{\gamma}, K)] \xrightarrow{i \otimes i} V_{\hat{g}(C)}[H(\tilde{\lambda}, L)] \otimes V_{\hat{g}(C)}[H(\tilde{\gamma}, K)]
\end{array}$$

A little bit of pasting then gives us the following diagram,

$$\begin{array}{c}
V_{\hat{g}(C)}[H_S(\tilde{\lambda} + \tilde{\gamma}, L + K)] \xrightarrow{V_{\hat{g}(C)}[\mathcal{C}_{\tilde{\lambda} + \tilde{\gamma}}]} V_{\hat{g}(C)}[H_S(\tilde{\lambda}, L) \otimes V_{\hat{g}(C)}[H_S(\tilde{\gamma}, K)] \\
\hat{i} \uparrow \quad \downarrow \hat{i} \\
V_{\hat{g}(S)}[H_S(\tilde{\lambda} + \tilde{\gamma}, L + K)] \xrightarrow{V_{\hat{g}(S)}[\mathcal{C}_{\tilde{\lambda} + \tilde{\gamma}}]} V_{\hat{g}(S)}[H_S(\tilde{\lambda}, L) \otimes V_{\hat{g}(S)}[H_S(\tilde{\gamma}, K)]
\end{array}$$

After more pasting we obtain

$$\begin{array}{c}
V_{\hat{g}(C)}[H_S(\tilde{\lambda} + \tilde{\gamma}, L + K)] \xrightarrow{V_{\hat{g}(C)}[\mathcal{C}_{\tilde{\lambda} + \tilde{\gamma}}]} V_{\hat{g}(C)}[H_S(\tilde{\lambda}, L) \otimes V_{\hat{g}(C)}[H_S(\tilde{\gamma}, K)] \\
\hat{i} \uparrow \quad \downarrow \hat{i} \\
V_{\hat{g}(S)}[\Sigma_S(\tilde{\lambda} + \tilde{\gamma})] \xrightarrow{V_{\hat{g}(S)}[\mathcal{C}_{\tilde{\lambda} + \tilde{\gamma}}]} V_{\hat{g}(S)}[\Sigma_S(\tilde{\lambda})] \otimes V_{\hat{g}(S)}[\Sigma_S(\tilde{\gamma})]
\end{array}$$

Now dualize the above diagram as in the previous lemma. □

The space $V^+_S(\tilde{\lambda}, L)$ can also be characterized as the subsheaf of $H_S(\tilde{\lambda}, L)^* = Hom_{\mathcal{O}_S}(H_S(\tilde{\lambda}), \mathcal{O}_S)$ of maps which are 0 on the image of the action map of $\hat{g}(C)$, so we have another commutative diagram.

$$\begin{array}{c}
H_S(\tilde{\lambda}, L)^* \otimes_{\mathcal{O}_S} H_S(\tilde{\gamma}, K)^* \xrightarrow{C^*} H_S(\tilde{\lambda} + \tilde{\gamma}, K + L)^* \\
\hat{i} \otimes \hat{i} \downarrow \quad \downarrow \hat{i} \\
V^+_S(\tilde{\lambda}, L) \otimes_{\mathcal{O}_S} V^+_S(\tilde{\gamma}, K) \xrightarrow{C^*} V^+_S(\tilde{\lambda} + \tilde{\gamma}, K + L)
\end{array}$$

Conformal blocks are preserved by pullbacks along maps of families of stable curves, and the top row of the above diagram is obviously preserved by pullbacks, since it is obtained by tensoring a map of representations with the structure sheaf. This implies that multiplication of conformal blocks is functorial in families of stable curves, and gives the following lemma as a corollary.

**Lemma 2.6.** The following diagram is obtained from the commutative square in the previous lemma by localizing at a point $x \in S$. 

\[\]
Corollary 2.8. When the genus defining a map of rings of sheaves \( F \) on individual blocks is a monomorphism of sheaves when \( g \) is even.

Proof. This follows from an observation of Tsuchiya, Ueno and Yamada in [TUY] that the corresponding maps commute and associate. These diagrams serve the purposes of showing the existence of the multiplication operation, and proving the following theorem.

Theorem 2.7. There is a commuting diagram of sheaves over \( \hat{\mathcal{M}}_{g,n} \),

\[
\begin{array}{c}
\mathcal{V}^+((\lambda + \gamma, L + K) \leftarrow \mathcal{V}^+((\lambda, L) \otimes \mathcal{V}^+((\gamma, K)) \leftarrow \\
\downarrow \quad \quad \quad \quad \downarrow \\
\mathcal{V}^+_g(V(\gamma + \bar{\lambda})) \leftarrow \quad \quad \quad \quad \mathcal{V}^+_g(V(\bar{\lambda})) \otimes \mathcal{V}^+_g(V(\gamma))
\end{array}
\]

This lemma will be important when we relate conformal blocks to line bundles on the moduli of principal bundles. These are our multiplication maps on graded components. Because they are defined by identifying our chosen highest weight vectors, and because the tensor product action \( \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g} \) co-commutes and co-associates, the maps commute and associate. These diagrams serve the purposes of showing the existence of the multiplication operation, and proving the following theorem.

Corollary 2.8. When the genus \( g = 0 \), the map \( F \) defined in the previous theorem is a monomorphism.

Proof. This follows from an observation of Tsuchiya, Ueno and Yamada in [TUY] that the corresponding map on individual blocks is a monomorphism of sheaves when \( g = 0 \).

Now fix an \( n + 1 \) punctured curve \( (C, \bar{p}, \bar{q}) \). By identifying highest weight vectors, we get the following diagram of \( \mathfrak{g} \) representations.

\[
\begin{array}{c}
\mathcal{V}(0) \otimes \mathcal{H}(\bar{\lambda}) \longrightarrow \mathcal{H}(0, L) \otimes \mathcal{H}(\bar{\lambda}, L) \leftarrow \mathcal{H}(0, L) \otimes \mathcal{V}(\bar{\lambda})
\end{array}
\]

The space on the left is a \( \mathfrak{g} \otimes \mathbb{C}(C \setminus \bar{p}) \) representation, the middle is a \( \mathfrak{g} \otimes \mathbb{C}(C \setminus \bar{p}, \bar{q}) \) representation, and Beauville shows in [B] that the space on the right is a \( \mathfrak{g} \otimes \mathbb{C}(C \setminus q) \) representation where the action on \( \mathcal{V}^+(\lambda_i) \) is by evaluation at \( p_i \). The following is a ringification of a theorem of Beauville, [B].

Proposition 2.9. Let \( C \) be a stable curve. The following are isomorphisms of algebras over \( \mathbb{C} \).

\[
\begin{array}{c}
\mathcal{V}^+_{\lambda, L}((\lambda, L) \leftarrow \mathcal{V}^+_{\lambda, L}(0, \bar{\lambda}, L) \leftarrow \mathcal{V}^+_{\lambda, L}(\lambda, L) \otimes \mathcal{V}(\bar{\lambda}))
\end{array}
\]

Proof. By a theorem of Beauville [B] the morphism on the right is an isomorphism of vector spaces, and by vacuum propagation (see [TUY], [SU], [B]) the morphism on the left is also an isomorphism of vector spaces. Both maps are defined by identifying highest weight vectors, then dualizing, this gives a diagram of rings, with graded components,

\[
\begin{array}{c}
|\mathcal{V}(0) \otimes \mathcal{H}(\bar{\lambda})|^* \leftarrow |\mathcal{H}(0, L) \otimes \mathcal{H}(\bar{\lambda}, L)|^* \leftarrow |\mathcal{H}(0, L) \otimes \mathcal{V}(\bar{\lambda})|^*
\end{array}
\]

Since taking Lie algebra invariants picks out subspaces of these spaces which are preserved by multiplication, these are isomorphisms of algebras.

Remark 2.10. This may be stated as an isomorphism of sheaves of algebras over \( \hat{\mathcal{M}}_{g,n+1} \), where the sheaf on the left is the pullback of the sheaf of vacua over \( \hat{\mathcal{M}}_{g,n} \) which forgets the \( n + 1 \)-st punctured point.

Also, implicitly we have an identification

\[
(35) \quad \mathcal{V}^+_g(\mathcal{H}(0, L) \otimes \mathcal{V}(\bar{\lambda})) \cong (\mathcal{H}(0, L)^* \otimes \mathcal{V}(\bar{\lambda})^*)^{\mathcal{O}(C \setminus q)}
\]

where \( \mathcal{H}(0, L)^* \) is the full vector space dual. The multiplication on the right hand side of this isomorphism of vector spaces is induced by our familiar map given by highest weight vectors. We will need this in the next subsection.
2.4. Moduli of principal parabolic bundles over a punctured curve. For this subsection we refer the reader to the work of Kumar, Kumar-Narasimhan-Ramanathan, and Lazlo-Sorger ([K], [KNR], [KN], [LS]). From the last subsection we have that the space of conformal blocks over a punctured curve \((C, \bar{p})\) of level \(L\) with markings \(\bar{\lambda}\) can be identified with the space

\[
[H(0, L)^* \otimes V(\bar{\lambda}^*)]_{\mathfrak{g}(C, q)}
\]

here \(q\) is another point of \(C\). The following can be found in [KNR] and [LS]. Let \(G\) be a simply connected group with \(\text{Lie}(G) = \mathfrak{g}\), a simple Lie algebra over \(\mathbb{C}\).

**Theorem 2.11.** The moduli stack of parabolic \(G\)-bundles on \(\mathcal{M}_{C, \bar{p}}(\bar{\Lambda})\) carries a line bundle \(\mathcal{L}(\bar{\lambda}, L)\) for \(\lambda_i\) a dominant weight in the face of \(\Delta\) associated to \(\Lambda_i\), such that

\[
H^0(\mathcal{M}_{C, \bar{p}}(\bar{\Lambda}), \mathcal{L}(\bar{\lambda}, L)) = [H(0, L)^* \otimes V(\bar{\lambda}^*)]_{\mathfrak{g}(C, q)}
\]

The stack \(\mathcal{M}_C(\bar{\Lambda})\) is obtained as a quotient of the \(\text{ind}\)-variety \(Q \times G/\Lambda_1 \times \ldots \times G/\Lambda_n\), by an \(\text{ind}\)-group \(\Gamma\), where \(Q\) is the affine Grassmannian.

\[
Q = L(G)/L^+(G)
\]

Here \(L(G)\) is the loop group of \(G\). For \(q \in C\) let \(\hat{O}_q\) be the formal completion of the local ring at \(q\), and let \(\mathfrak{t}_q\) be the quotient field of \(\hat{O}_q\). Then \(L(G) = G(\mathfrak{t}_q)\), and \(L^+(G) = G(\hat{O}_q)\). The group \(L^+(G)\) is called a maximal parahoric subgroup of \(L(G)\). The space \(Q \times G/\Lambda_1 \times \ldots \times G/\Lambda_n\) carries line bundles \(L(\hat{\bar{\lambda}}, \bar{\lambda})\) with global sections equal to \(H(0, L)^* \otimes V(\bar{\lambda}^*)\), multiplication of global sections is given by identifying the highest weight vectors of the representations \(H(0, L)\) and \(V(\lambda_i)\) and then dualizing. This is the generalized Borel-Bott-Weil theorem, of Kumar [K] for Kac-Moody algebras. Moreover, in [LS] Lazlo and Sorger identified the Picard group of \(\mathcal{M}_{C, \bar{p}}(\bar{\Lambda})\),

\[
\text{Pic}(\mathcal{M}_{C, \bar{p}}(\bar{\Lambda})) = \mathbb{Z} \oplus \mathcal{X}_{\Lambda_1} \oplus \ldots \oplus \mathcal{X}_{\Lambda_n}
\]

where \(\mathcal{X}_{\Lambda_i}\) is the character group of \(\Lambda_i\), equal to the Picard group of \(G/\Lambda_i\). This result was obtained by analyzing line bundles under the quotient by \(\Gamma\). In fact, the case with markings \(\mathcal{M}_{C, \bar{p}}(\bar{\Lambda})\) is a \(G/\Lambda_1 \times \ldots \times G/\Lambda_n\) bundle over the case without markings. The effective cone in \(\mathcal{X}_{\Lambda_i}\) is given by the dominant weights associated to \(\Lambda_i\), in particular if \(\Lambda_i\) is the Borel subgroup \(B\), then the effective cone is \(\Delta\). The effective cone in the Picard group of \(Q\) is given by the non-negative integers. For any line bundle \(\mathcal{L}\) on \(\mathcal{M}_{C, \bar{p}}(\bar{\Lambda})\) there is an isomorphism between the sections of \(\mathcal{L}\) and the \(\Gamma\)-equivariant sections of the pullback bundle on \(Q \times G/\Lambda_1 \times \ldots \times G/\Lambda_n\) by a standard theorem on quotient stacks, see [LS]. This proves the following theorem.

**Theorem 2.12.** There is a monomorphism of multigraded rings

\[
h_\bar{\lambda} : \text{Cox}(\mathcal{M}_{C, \bar{p}}(\bar{\Lambda})) \to V^+_{C, \bar{p}}
\]

The image of this monomorphism is the direct sum of conformal blocks \(V^+_{C, \bar{p}}(\bar{\lambda}, L)\) with \(\lambda_i\) a dominant weight in the face of \(\Delta\) associated to \(\Lambda_i\). This is an isomorphism when all \(\Lambda_i\) are Borel subgroups.

**Remark 2.13.** It seems that much of this story could be reformulated by building the appropriate \(\mathcal{M}_{g,n}\)-stack \(\mathcal{M}(\bar{\Lambda})\), with line bundle \(\mathcal{L}(\bar{\lambda}, L)\). The sheaf \(V(\bar{\lambda}, L)\) could then be constructed as a pushforward of this line bundle. This also suggests a way to axiomatize proofs of our main theorems in terms of line bundles on stacks over moduli of curves. The general structure of the geometric side of factorization involves a stack \(\mathcal{X}_{g,n}\) over \(\mathcal{M}_{g,n}\), and for each stratum \(\pi_T : \prod \mathcal{M}_{g,n} \to \mathcal{M}_{g,n}\), a way to relate the base change of \(\mathcal{X}_{g,n}\) to a torus quotient of a product of components over the lower stratum.

\[
\mathcal{X}_{g,n} \times_{\pi_T} [\prod \mathcal{M}_{g,n}] \sim T/\prod [\mathcal{X}_{g,n}]
\]
In this paper $X_{g,n}$ is the universal torsor of the moduli of parabolic principal bundles, and the relationship is a flat deformation over $\mathbb{C}$. We wonder if a similar theorem can be proved in the symplectic category, where the relationship is sharing a dense open Hamiltonian system. See [HMM] for this type of relationship with moduli of weighted points on the projective line and [H] for parabolic principal $SL_2(\mathbb{C})$ bundles.

Remark 2.14. To a general representation $M(L)$ of $\hat{\mathfrak{g}}$ of level $L$ one can associate a sheaf $\mathcal{V}^+(M(\tilde{\lambda}, L + K)) = V^+_{\mathfrak{g}(\mathcal{C})}(M(L) \otimes \mathcal{H}(0, K) \otimes V(\lambda))$ on the moduli of curves $M_{g,n}$. Taking a direct sum over all $(\tilde{\lambda}, K)$ then produces a multigraded module for the sheaf of algebras $\mathcal{V}^+$.

3. Filtrations of the algebra of conformal blocks over the stable locus

In this section we use the factorization map of Tsuchiya-Ueno-Yamada to define term order deformations of the sheaf of conformal blocks over the locus of stable curves. Taking a punctured stable curve $(C, \mathcal{P})$ there is a normalization $(\tilde{C}, \tilde{\mathcal{P}}, \tilde{\mathbf{q}})$, where $\tilde{\mathbf{q}}$ are the doubled points, which are identified by the map $\tilde{C} \to C$, any such $q$ has a natural partner $q'$. The following is the factorization theorem, it was proved in [TUY] with refinements in [SU], [L] and [F]. Recall that we have selected $v_\eta \in V(\eta)$ for each irreducible representation of $\mathfrak{g}$. Let $\omega \in \Delta$ be the longest root, and recall that $\Delta_L = \{ \lambda \in \Delta | H_\omega(\lambda) \leq L \}$.

Theorem 3.1. There is a canonical isomorphism of vector spaces,

$$\mathcal{V}^+_{\mathcal{C}, \mathbf{p}}(\tilde{\lambda}, L) \xleftarrow{\simeq} \bigoplus_{\tilde{\alpha} \in \Delta^m_L} \mathcal{V}^+_{\mathcal{C}, \mathbf{p}, \tilde{\alpha}}(\tilde{\lambda}, \tilde{\alpha}, \tilde{\alpha}^*, L)$$

where $m$ is the number of double points, and $q$ is always assigned the weight dual to its partner $q'$.

This isomorphism is constructed as follows. For each weight $\lambda \in \Delta$ and its dual $\lambda^*$ let

$$(42) \quad F_\lambda : V(\lambda) \otimes V(\lambda^*) \to \mathcal{C}$$

be the unique equivariant map such that $F_\lambda(v_\lambda \otimes \tilde{v}_{\lambda^*}) = 1$, where $\tilde{v}_{\lambda^*}$ is the lowest weight vector of $V(\lambda^*)$. This fixes an identification

$$\text{Hom}_\mathcal{C}(V(\lambda), V(\lambda)) = V(\lambda) \otimes V(\lambda^*)$$

by letting $\sum_i x_i \otimes y_i$ act on $v \in V(\lambda)$ as $\sum_i x_i \otimes F_\lambda(y_i \otimes v)$. We choose $O_{\lambda, \lambda^*} \in V(\lambda) \otimes V(\lambda^*)$ to represent the identity. We use this element to define a map

$$V(\tilde{\lambda}) \xrightarrow{\rho_\alpha} V(\tilde{\lambda}) \otimes V(\alpha) \otimes V(\alpha^*)$$

on $\mathfrak{g}$ representations which sends $X$ to $X \otimes O_{\alpha, \alpha^*}$. Another way to construct this map is to find the image of the identity under the isomorphism $\text{Hom}_\mathfrak{g}(V(\lambda) \otimes V(\alpha), V(\lambda) \otimes V(\alpha)) \cong \text{Hom}_\mathfrak{g}(V(\lambda), V(\lambda) \otimes V(\alpha) \otimes V(\alpha^*)).$ The map $\rho_\alpha$ makes sense for integrable highest weight representations of $\hat{\mathfrak{g}}$, we can define $\rho_\alpha : \mathcal{H}(\tilde{\lambda}, L) \to \mathcal{H}(\tilde{\lambda}, \alpha, \alpha^*, L) = \mathcal{H}(\tilde{\lambda}, L) \otimes \mathcal{H}(\alpha, \alpha^*, L)$ as the map which sends a vector $X$ to $X \otimes O_{\alpha, \alpha^*}$, with $O_{\alpha, \alpha^*} \in V(\alpha) \otimes V(\alpha^*) \subset \mathcal{H}(\alpha, \alpha^*, L).$ Taking conformal blocks yields the following map, which is shown to be injective in [TUY].

$$\mathcal{V}^+_{\mathcal{C}, \mathbf{p}}(\tilde{\lambda}, L) \xleftarrow{\rho_\alpha} \mathcal{V}^+_{\mathcal{C}, \mathbf{p}, \tilde{\alpha}, \tilde{\alpha}^*}(\tilde{\lambda}, \tilde{\alpha}, \tilde{\alpha}^*, L)$$

This map is computed on an element $\Phi \in \text{Hom}_\mathcal{C}(\mathcal{H}(\tilde{\lambda}, \tilde{\alpha}, \tilde{\alpha}^*, L), \mathcal{C})$ by sending it to the map $\hat{\Phi} \in \text{Hom}_\mathcal{C}(\mathcal{H}(\tilde{\lambda}, L), \mathcal{C})$, defined as follows,

$$(43) \quad \hat{\Phi}(Y) = \Phi(Y \otimes O_{\tilde{\alpha}, \tilde{\alpha}^*})$$

Summing over all $\tilde{\alpha} \in \Delta^m_L$ gives the factorization theorem. The nuts and bolts of this construction will be important in what follows. We will show the existence of a filtration on the ring of conformal blocks, obtained by understanding the multiplication operation with respect to the factorization theorem. We begin by studying tensor product decompositions
taking care that $v_{\alpha+\beta}$ is identified with $v_{\alpha} \otimes v_{\beta}$. From our choices in the last section we get the following identification

$$V(\alpha) \otimes V(\beta) = \bigoplus W_{\alpha,\beta}^\eta \otimes V(\eta),$$

The picture is completed by applying a theorem of Beauville \[B\], also found in \[NT\], which asserts the identification of lemma 2.5 that each morphism connects the appropriate spaces of invariants. We obtain the following proof.

Proof. We may dualize the diagram above, then it follows from the factorization theorem and the proof of proposition 3.2. The following diagram commutes.

$$\begin{array}{c}
\mathcal{H}(\bar{\lambda}, L) \otimes \mathcal{H}(\bar{\gamma}, K) \xrightarrow{\rho_\alpha \otimes \rho_\beta} \left[ \mathcal{H}(\bar{\lambda}, L) \otimes \mathcal{H}(\alpha, \alpha^*, K) \right] \otimes \left[ \mathcal{H}(\bar{\gamma}, K) \otimes \mathcal{H}(\beta, \beta^*, L) \right] \\
C_{\lambda+\gamma} \downarrow \quad \sum_\eta \mathcal{C}_{\lambda+\gamma} \otimes f_{\eta, \eta^*} \downarrow \\
\bigoplus_{\eta} \mathcal{H}(\bar{\lambda} + \bar{\gamma}, K + L) \otimes (W_{\alpha,\beta}^\eta \otimes W_{\alpha^*,\beta^*}^\eta \otimes \bar{V}(\eta) \otimes \bar{V}(\eta^*))
\end{array}$$

Here $f_{\eta, \eta^*} = f_\eta \otimes f_{\eta^*}$. The bottom map in the diagram sends a vector $Y$ to $\bigoplus_{\eta} \bigoplus_{\eta^*} \rho_\eta \otimes f_{\eta, \eta^*} \otimes t_{\eta, \eta^*}$, so we can go ahead and replace the map $f_{\eta, \eta^*}$ with $F_{\eta, \eta^*}(X) = f_{\eta, \eta^*}(t_{\eta, \eta^*} \otimes X)$. Also, for $\eta = \alpha + \beta$, we have by definition

$$f_{\alpha+\beta, \alpha^*+\beta^*} = F_{\alpha+\beta, \alpha^*+\beta^*} = C_{\alpha+\beta, \alpha^*+\beta^*}.$$
representing the stable type of the curve. Each internal edge is given a functional theorem 1.3.

Curve.

the graph correspond to pairs of points identified by the normalization. We consider graphs up to leaves correspond to punctures and the internal edges of genus- at each internal

takes in the algebra \( V_{\bar{C},\bar{p}}(\lambda, L) \) can be canonically identified with \( V_{\bar{C},\bar{p}}(\lambda, L) \). Note that in the equation above, \( \chi_1 \times \chi_2 \) and \( C^*_{\lambda+\bar{\gamma},\alpha+\beta}(\chi_1 \otimes \chi_2) \) always have the same filtration level, whereas the filtration level of \( \chi_\eta \) is always less or equal to this, and is generically less.

\( \chi_1 \times \chi_2 = C^*_{\lambda+\bar{\gamma}}(\chi_1 \otimes \chi_2) = C^*_{\lambda+\bar{\gamma},\alpha+\beta}(\chi_1 \otimes \chi_2) + \sum \chi_\eta, \)

This all follows from the commutative diagram above.

Proposition 3.3. The filtration \( F^*_\beta \) for \( (\Gamma, \bar{\theta}) \) respects multiplication on the ring \( V_{\bar{C},\bar{p}}^+ \). If the components of \( \bar{\theta} \) are strictly positive on all positive roots, then the image of the multiplication map,

\( V_{\bar{C},\bar{p}}^+(\lambda, L) \otimes V_{\bar{C},\bar{p}}^+(\gamma, K) \to F^*_\beta(\bar{\theta}(\lambda+\gamma, L) \otimes \bar{V}_{\bar{C},\bar{p}}(\gamma, K)) \)/\( F^*_\beta(\bar{\theta}(\lambda+\gamma, L) \otimes \bar{V}_{\bar{C},\bar{p}}(\gamma, K)) \)

can be canonically identified with \( V_{\bar{C},\bar{p}}^+(\lambda+\gamma, K+L) \). Moreover this map is equal to multiplication in the algebra \( V_{\bar{C},\bar{p}}^+ \).

Proof. This all follows from the commutative diagram above.

If we choose the functional \( (\Gamma, \theta) \) to be positive on positive roots, and non-negative integral on weights of \( g \), then we may form the Reese algebra \( R_{(\Gamma, \theta)}[V_{\bar{C},\bar{p}}^+] \subset V_{\bar{C},\bar{p}}^+ \), defined as

\( R_{(\Gamma, \theta)}[V_{\bar{C},\bar{p}}^+] = \bigoplus_{k \geq 0} F^*_\theta[V_{\bar{C},\bar{p}}^+] t^k. \)

This algebra is flat over \( \mathbb{C}[t] \), and is equal to the associated graded algebra over \( t = 0 \). This proves theorem 3.3.

We will now say something about the structure of the functionals \( (\Gamma, \bar{\theta}) \) which define these filtrations. The strata of \( \mathcal{M}_{g,n} \) are indexed by graphs, which we will also denote with \( \Gamma \), with numbers \( g_i \)-the genus- at each internal vertex \( i \). Each internal vertex corresponds to a component of the normalization of a representative curve of the stratum. Leaves correspond to punctures and the internal edges of the graph correspond to pairs of points identified by the normalization. We consider graphs up to homeomorphisms which preserve genus and leaf information. This fixes the topological type of a stable curve.

The deformations we’ve considered here can be interpreted as labelings of the edges of the graph \( \Gamma \) representing the stable type of the curve. Each internal edge is given a functional \( \frac{1}{2} \theta \) and its reflection by the longest element of the Weyl group, \(-\frac{1}{2} \theta \circ \omega\), each associated to a different endpoint of the edge. The stratification poset information of \( \mathcal{M}_{g,n} \) can be recovered by considering weightings which are 0
on some edges. We imagine these edges as collapsing, bringing the endpoints together. The resulting vertex is then given the sum of the genus of the endpoints. If both endpoints are the same, then weighting the edge 0 corresponds to the graph with that edge removed with the endpoint having the previous plus 1.

For each $\Gamma$ the set of weightings forms a cone, with boundary components given by the cones associated to ancestors of $\Gamma$. We may glue the cones for distinct $\Gamma$ together along common ancestors. This necessitates that we quotient the cone associated to $\Gamma$ by the action of $\text{Aut}(\Gamma)$. We say two graphs are related by a flip move if both have a common ancestor obtained by eliminating a single edge. For $g = sl_2(\mathbb{C})$ dual weights are equal, so we can consider the edges of a graph $\Gamma$ to be labeled with non-negative real numbers on the internal edges, and general real numbers on the leaf edges. For $n = 0$ our definition of conformal blocks no longer makes sense, we remedy this with "vacuum propagation" $\mathcal{V}^+_\mathbb{C}(0) \cong \mathcal{V}^+_{\mathbb{C},q}(0)$. In this case deformations are technically weightings of 1-pointed, genus $g$ graph $\Gamma$, but we forget the lone leaf. For $g = sl_2(\mathbb{C})$ deformations are then indexed weightings of 0-pointed, genus $g$ graphs by non-negative real numbers. These are the pseudo tropical curves recently studied by Brannetti, Melo, and Viviani in [BMV].

Proposition 3.4. The set of $sl_2(\mathbb{C})$-weightings for topological types of stable curves of genus $g$ is isomorphic to the moduli space $M^\text{tr}_g$ of pseudo-tropical curves introduced in [BMV].
We interpret the set of $\text{sl}_2(C)$ weightings of $n$-pointed stable genus $g$ curves as $M_{g,n}^r$. It is not hard to show that the maximal "cones" of this space all have the same dimension, and that the "flip moves" defined by common ancestors connect each pair of maximal cones via paths through strictly codimension 1 cones. Forgetting the cones associated to graphs with non-0 markings on vertices gives the space of tropical curves described by Mikhalkin in [M1]. If we restrict to the genus 0 case, then all internal vertices $i$ are given $g_i = 0$, and no loops are allowed. The set of weighted trivalent trees with assignments of $\{0, \ldots, n\}$ to the leaves up to label preserving homeomorphism can be identified with the space of phylogenetic trees studied by Billera, Holmes, and Vogtman in [BHV], and is abstractly homeomorphic to the tropical Grassmannian of 2 planes.

Remark 3.5. The tropical Grassmannian or rather $M_{g,n}^r$ is the only case known to the author to have an actual tropical structure - as the tropical variety of a homogeneous ideal. This structure comes from the work of Speyer and Sturmfels [SpSt], where this space is realized as the tropicalization of the variety associated to the Plücker ideal, and also from the work of Mikhalkin [M1]. It is possible to produce embeddings of the cone for the graph $\Gamma$ into the tropical variety associated to any presentation of the algebra $V_{\bar{C},\bar{q}}^+$ where $(C, \bar{p})$ has type $\Gamma$, we will see more on this in the next section. As we remarked above, the case of general genus with symmetry $\text{sl}_2(C)$ has the dimensionality and connectivity properties of a tropical variety, but the presence of automorphisms makes it unclear to the author how to ascribe to it a tropical structure in any known sense, we have been tentatively referring to these spaces as "tropical stacks," or "tropical orbifolds."

As a vector space, the algebra of conformal blocks $V_{\bar{C},\bar{q}}^+$ for a punctured stable curve $(C, \bar{p})$ is isomorphic to $(V_{\bar{C},\bar{p},\bar{q}}^+)^T$, where taking invariants by the torus $T$ ensures that weights assigned to paired puncture points are dual. By choosing a basis for each space of conformal blocks associated to components of $\tilde{C}$, we can assemble a basis for $V_{\bar{C},\bar{p},\bar{q}}^+$, each element of which can be associated to a labeling of the graph $\Gamma$ which gives the type of $(C, \bar{p}, \bar{q})$. In the general case, each internal edge is assigned two $g$ dominant weights which are dual to each other, and we remember the endpoint of the edge associated to each weight. Notice that $\theta(\alpha) = -\theta(\omega(-\omega(\alpha)))$. This is why we've added the factor of $\frac{1}{2}$ to internal functionals, so that the definition agrees with the deformations of the Plücker algebra defined by weighted trees. The filtration level is then computed by applying the $g$-weighting $(\Gamma, \bar{\alpha})$ to the labeling $(\tilde{\Gamma}, \tilde{\alpha})$.

\[(\Gamma, \bar{\alpha}) \circ (\tilde{\Gamma}, \tilde{\alpha}) = \sum_{e \in E, \tilde{\alpha}(e) = 1} \theta_e(\alpha_e)\]

If $(\Gamma, \bar{\alpha})$ is chosen so that each component is strictly positive on positive roots, the resulting filtration gives a flat algebra $R^\theta$ over $\mathbb{C}[t]$ called the Reese algebra with generic fiber $V_{\bar{C},\bar{p},\bar{q}}^+$ and special fiber the subalgebra $(V_{\bar{C},\bar{p},\bar{q}}^+)^T$ of $V_{\bar{C},\bar{p},\bar{q}}^+$ such that the markings on paired points of the normalization are dual. This proves theorem 1.4. We may restate this as

Theorem 3.6. Let $(C, \bar{p})$ be a stable punctured curve, and let $(\tilde{C}, \tilde{\bar{p}}, \tilde{\bar{q}})$ be its normalization. For $g$ weightings of $C$ which have each component positive on positive roots, there is a term order deformation,

\[\text{Cox}(M_{\tilde{C},\tilde{\bar{p}}}(B^n)) \Rightarrow \text{Cox}(M_{\tilde{C},\tilde{\bar{p}},\tilde{\bar{q}}}(B^n \times B^{2k}))^T\]

where $B$ is a Borel subgroup of $G$.

In the case where $\tilde{C} = C_1 \coprod C_2$ the associated graded algebra is isomorphic to a torus invariant subalgebra of the tensor product of the Cox rings of the moduli over the component curves. This means that any algebra of conformal blocks lies in a flat family with a subalgebra of the tensor product of algebras of conformal blocks associated to triple punctured copies of $\mathbb{P}^1$, in this case the curve type $\Gamma$ is a trivalent graph.

Example 3.7. We will work out what this means for a curve $C$ with $g = 2$, and $n = 0$. By theorem 1.4 there is a flat family connecting the algebra $V_{\bar{C}}^+$ to $V_{\bar{C}}^+$ for $C'$ a stable curve with genus 0 components. This curve then has the normalization $\tilde{C}'$, which is a disjoint union of triple punctured smooth genus 0 curves. By theorem 1.5 there is a term order deformation.
Figure 11. Curve of genus 2, stable curve of genus 2 and disjoint union of two triple punctured curves of genus 0.

\[ V^+_C \Rightarrow (V^+_C, \vec{q})^T, \]

where \( \tilde{C'} = \mathbb{P}^1 \coprod \mathbb{P}^1 \). Explicitly we have,

\[ (V^+_C, \vec{q})^T = (V^+_{\tilde{C}'}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3) \otimes V^+_{\tilde{C}'}, \tilde{q}_4, \tilde{q}_5, \tilde{q}_6). \]

Multiplication is computed component-wise over the tensor product. In the case \( \mathfrak{g} = sl_2(\mathbb{C}) \) this ring is the semigroup of weightings on the graph pictured in figure 12, where the middle edge is always weighted even, is less than or equal to twice either of the loop edges, and the sum of twice either loop edge and the middle edge is bounded by the level.

Figure 12. Weighted genus 2 graph, say of level 4.

To finish off this section we will say a word about the sense in which the complexes cones given by the functionals \( (\Gamma, \vec{\theta}) \) are related to tropical geometry. Each such functional can be extended to a function on the algebra of conformal blocks,

\[ (\Gamma, \vec{\theta}) : V^+_{C, \vec{p}} \rightarrow \mathbb{R}_{\geq 0} \]

which satisfies the following properties with respect to the grading of the underlying vector space given by the labellings from the factorization rules, \( (\Gamma, \vec{a}) \).

1. \( (\Gamma, \vec{\theta})(C) = 0 \) for \( 0 \neq C \in \mathbb{C} \).

2. \( (\Gamma, \vec{\theta})(a \times b) = (\Gamma, \vec{\theta})(a) + (\Gamma, \vec{\theta})(b) \) when \( a \) and \( b \) are homogeneous.

3. \( (\Gamma, \vec{\theta})(a + b) \leq \max\{(\Gamma, \vec{\theta})(a), (\Gamma, \vec{\theta})(b)\} \)

The functional is already defined on homogeneous elements, we extend it to sums of homogeneous elements \( \sum a_i \) by the Max convention \( (\Gamma, \vec{\theta})(\sum a_i) = \max\{\ldots (\Gamma, \vec{\theta})(a_i) \ldots \} \). In general, for an algebra \( A \) with a grading of the underlying vector space \( A = \bigoplus A_s \), we call a function which satisfies the above properties a graded valuation. The following is easy to prove using the above properties.
Proposition 3.8. Let $A$ be as above, and let $v : A \to \mathbb{R}_{\geq 0}$ be a graded valuation, then for any homogeneous presentation of a subalgebra of $A$,

\[
0 \longrightarrow I \longrightarrow \mathbb{C}[X] \overset{\phi}{\longrightarrow} A
\]

The point $(v(\phi(x_1)), \ldots, v(\phi(x_n)))$ in the Gröbner fan of $I$ lies in the tropical variety $\text{tr}(I)$.

So the functionals $(\Gamma, \tilde{\theta})$ define a type of “universal” tropical point, in the sense that they define a point on each tropical variety associated to ideals which present $V_{C, \tilde{\theta}}^+$ and nice subalgebras which respect the grading by the factorization labellings, such as $\bigoplus_n H^0(M_{C, \tilde{\theta}}(\tilde{A}), L(n\tilde{X}, nL))$. The same is true for all of the functionals discussed in this note, including those which give filtrations on branching algebras in the following section.

4. Branching algebras and the genus 0 case

In this section we define the branching algebra $\mathfrak{A}(\phi)$ associated to a map of complex connected reductive groups $\phi : H \to G$. These algebras were also studied by Howe, Tan, and Willenbring in [HTW], for $H$ and $G$ a symmetric pair. We define and investigate flat degenerations of these algebras associated to factorizations of $\phi$. We then study the case of the branching algebra associated to the diagonal morphism $\Delta_n : G \to G^n$, and show how it is related to our term order deformations of the ring of conformal blocks in the genus 0 case. For more on this construction see [M2].

4.1. Branching algebras. Choose maximal unipotent subgroups $U_G$ and $U_H$ for connected reductive groups $G$ and $H$ over $\mathbb{C}$. Let $\phi : H \to G$ be a map of reductive groups. Let $R(G) = \mathbb{C}[G]^{U_G}$, which carries a left action of $G$, and therefore $H$ by the equivariance of $\phi$.

Definition 4.1. We define the branching algebra $\mathfrak{A}(\phi)$ to be the invariant subalgebra $U_H R(G)$ with respect the the left action of $H$ on $R(G)$ through $\phi$.

As a representation of $G$, the algebra $R(G)$ is equal to $\bigoplus_{\lambda \in \Delta_G} V(\lambda^*)$ with multiplication induced by dualizing the maps $C_{\alpha+\beta} : V(\alpha + \beta) \to V(\alpha) \otimes V(\beta)$ given by identifying highest weight vectors. The vector space $\text{Res}_H^G(V(\lambda^*))^{U_H}$ is a direct sum over all the highest $H$-weight vectors in $\text{Res}_H^G(V(\lambda^*))$, so it can be canonically identified with $\bigoplus_{\gamma \in \Delta_H} \text{Hom}_H(W(\gamma), \text{Res}_H^G(V(\lambda^*)))$. Multiplication in $\mathfrak{A}(\phi)$ is given by the following diagrams.

\[
W(\gamma_1) \otimes W(\gamma_2) \xrightarrow{f_{\gamma_1,\gamma_2}} \text{Res}_H^G(V(\lambda_1^*)) \otimes \text{Res}_H^G(V(\lambda_2^*))
\]

The branching algebra $\mathfrak{A}(\phi)$ is multigraded by the product of Weyl chambers $\Delta_H \times \Delta_G$ specified by the chosen maximal unipotents. The grade of the component $\text{Hom}_H(W(\gamma), \text{Res}_H^G(V(\lambda^*)))$ is $(\gamma, \lambda)$. We will study a class of filtrations of branching algebras induced by factorizations of morphisms of connected reductive groups over $\mathbb{C}$. Let

\[
H \xrightarrow{\psi} K \xrightarrow{\theta} G
\]

be a factorization of $\phi$ in the category of reductive groups over $\mathbb{C}$. The vector space $\text{Hom}_H(W(\gamma), \text{Res}_H^G(V(\lambda^*)))$ has a direct sum decomposition along this factorization.

\[
\text{Hom}_H(W(\gamma), \text{Res}_H^G(V(\lambda^*))) = \bigoplus_{\eta \in \Delta_K} \text{Hom}_H(W(\gamma), \text{Res}_H^K(Y(\eta))) \otimes \text{Hom}_K(Y(\eta), \text{Res}_K^G(V(\lambda^*)))
\]

From now on we denote $\text{Res}_H^G(V)$ as $V$ when the meaning is clear. We will use this direct sum decomposition to define filtrations of $\mathfrak{A}(\phi)$ by proving an analogue of Tsuchiya, Ueno, and Yamada’s factorization of conformal blocks for factorizations of maps of complex connected reductive groups, this will be lemma 4.2 and an analogue of our description of conformal block multiplication [3.3]; this will be diagram [4.1]. There is an isomorphic description of branching algebras more amenable to our purposes, we can also take $\mathfrak{A}(\phi)$ to be
Writing this another way we get the following map of homomorphisms:

\[ \mathfrak{A}(\phi) = H(R(H) \otimes R(G)). \]

The graded pieces of this algebra are the vector spaces

\[ \text{Hom}_H(\mathbb{C}, W(\gamma^*) \otimes V(\lambda^*)). \]

Let \( g, k, \) and \( h \) be the Lie algebras of \( G, K, \) and \( H \) respectively. Recall that \( V_g(M) = M/gM, \) we have the following identification.

\[ \text{Hom}_C(\mathbb{C}, M^*) \cong \text{Hom}_g(\mathbb{C}, M^*) \cong \text{Hom}_C(V_g(M), \mathbb{C}). \]

This is because an element \( f \in M^* \) is fixed by \( G \) if and only if \( f \) annihilates \( gM. \) Using our choice of \( F_\lambda : V(\lambda) \otimes V(\lambda^*) \to \mathbb{C} \) we can identify \( V(\lambda^*) \) with \( V(\lambda)^* \) by letting \( w \in V(\lambda^*) \) act on \( V(\lambda) \) by \( v \to F_\lambda(v, w) \in \mathbb{C}. \) This gives us

\[ \mathfrak{A}(\phi) = \bigoplus \text{Hom}_H(\mathbb{C}, W(\gamma^*) \otimes V(\lambda^*)) \cong \bigoplus \text{Hom}_C(V_h(W(\gamma) \otimes V(\lambda)), \mathbb{C}). \]

One then checks easily that multiplication on the right is induced by dualizing the maps \( C_{\gamma+\eta} \otimes C_{\alpha+\beta} : W(\gamma) \otimes V(\alpha + \beta) \to [W(\gamma) \otimes V(\alpha)] \otimes [W(\eta) \otimes V(\beta)] \). We may now construct the same structure of a multifiltration on branching algebras as we did for algebras of conformal blocks. Let \( O_{\eta, \eta^*} \) and \( \rho_\eta \) be as in the previous section. Let \( Y(\eta, \eta^*) \) denote the \( K \)-representation \( Y(\eta) \otimes Y(\eta^*). \)

\[ W(\gamma) \otimes V(\lambda) \xrightarrow{\rho_\eta} W(\gamma) \otimes Y(\eta, \eta^*) \otimes V(\lambda) \]

As before, let \( f_\eta : \text{Hom}_K(Y(\eta), Y(\eta_1) \otimes Y(\eta_2)) \otimes Y(\eta) \to Y(\eta_1) \otimes Y(\eta_2). \) We have the identity \( O_{\eta_1} \otimes O_{\eta_2} = \sum I^n_{\eta_1, \eta_2} \otimes O_{\eta, \eta^*}. \) Define \( F_{\eta, \eta^*} \) as in proposition 3.2. From these definitions we get a commutative diagram.

\[ W(\gamma_1) \otimes W(\gamma_2) \otimes V(\lambda_1) \otimes V(\lambda_2) \xrightarrow{\rho_{\eta_1} \otimes \rho_{\eta_2}} W(\gamma_1) \otimes Y(\eta_1, \eta^*) \otimes V(\lambda_1) \otimes W(\gamma_2) \otimes Y(\eta_2, \eta^*) \otimes V(\lambda_2) \]

Here the sum is over all \( \eta \leq \eta_1 + \eta_2, \) as dominant weights.

**Lemma 4.2.** Let \( g, k, \) and \( h \) be the Lie algebras of \( G, H, \) and \( K \) respectively. The map

\[ W(\gamma) \otimes V(\lambda) \xrightarrow{\sum \rho_\eta} \bigoplus \eta W(\gamma) \otimes Y(\eta, \eta^*) \otimes V(\lambda) \]

Induces an isomorphism

\[ V^+_h(W(\gamma) \otimes V(\lambda)) \cong \bigoplus \eta V^+_h(W(\gamma) \otimes Y(\eta)) \otimes V^+_k(Y(\eta^*) \otimes V(\lambda)) \]

**Proof.** We can dualize each component of this map to get

\[ W(\gamma^*) \otimes V(\lambda^*) \xrightarrow{\sum \rho_\eta} W(\gamma^*) \otimes Y(\eta^*) \otimes V(\lambda^*) \]

Viewing each representation as a dual space, this map is calculated by sending \( f_w \otimes f_1 \otimes f_2 \otimes f_v \) to \( f_w \otimes [f_1 \otimes f_2](O_{\eta, \eta^*}) \otimes f_v. \) Let \( O_{\eta, \eta^*} = \sum x_i \otimes g_i, \) then \( f_1 \otimes f_2 \circ (\sum x_i \otimes g_i) = \sum f_1(x_i) \otimes F_{\eta}(f_2 \otimes g_i) = \sum f_1(F_{\eta}(f_2 \otimes g_i)) x_i = f_1(\sum F_{\eta}(f_2 \otimes g_i) x_i) = f_1(f_2) \) by definition of \( O_{\eta, \eta^*}. \) We may take invariants by Lie algebra actions to obtain a map

\[ [W(\gamma^*) \otimes V(\lambda^*)]^h \xrightarrow{\text{hom}} [W(\gamma^*) \otimes Y(\eta^*)]^h \otimes [Y(\eta) \otimes V(\lambda^*)]^h \]

Writing this another way we get the following map of homomorphisms.
\[ \text{Hom}_\mathfrak{h}(W(\gamma), V(\lambda^*)) \leftarrow \text{Hom}_\mathfrak{h}(W(\gamma), Y(\eta^*)) \otimes \text{Hom}_\mathfrak{e}(Y(\eta^*), V(\lambda^*)) \]

It is easy to show using the calculation above that this is the composition of morphisms \( \mathfrak{e} \). By the semisimplicity of the categories involved, this map is injective, and the sum over \( \eta \) produces an isomorphism, therefore the same is true for the corresponding map on invariants. \( \square \)

After dualizing diagram 4.1, we can take covariants by \( \mathfrak{h} \) on the left and \( \mathfrak{h} \oplus \mathfrak{t} \) on the right, where the action map \( \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{t} \to \mathfrak{h} \oplus \mathfrak{t} \) is defined by the diagonal morphism and \( d(\theta) \). The lemma above and diagram 4.1 imply that \( \mathfrak{A}(\phi) \) has a filtration by dominant weights of \( K \) in the fashion of proposition 3.3, with a similar lower-triangular multiplication property. We may act on these weights with functionals \( \theta \) to produce filtrations.

**Theorem 4.3.** Let \( \phi, \psi \) be morphisms in the category of reductive groups over \( \mathbb{C} \). Then for linear functionals \( \theta_H, \theta_K, \theta_G \) which are positive on positive roots we obtain a filtration which defines a flat deformation,

\[ \mathfrak{A}(\phi \circ \psi) \Rightarrow [\mathfrak{A}(\phi) \otimes \mathfrak{A}(\psi)]^{\Delta_K} \]

where \( \Delta_K \) is the maximal torus associated to \( \Delta_K \), and the action on \( \mathfrak{A}(\phi) \) is conjugated by the negative of the longest element in the Weyl group of \( K \). The torus action picks out the sub algebra of \( \mathfrak{A}(\phi) \otimes \mathfrak{A}(\psi) \) with conjugate dominant \( K \) weights.

**Proof.** This follows by the analogue of the argument for proposition 3.3. We define the filtration in the same way, acting on the \( K \)-weights \( \eta \) and \( \eta^* \) with \( \frac{1}{2} \theta_K \) and the reflection \( -\frac{1}{2} \theta_K \circ \omega \), as in the definition of a \( \mathfrak{g} \)-weighting of a curve. \( \square \)

The associated graded algebra \( [\mathfrak{A}(\phi) \otimes \mathfrak{A}(\psi)]^{\Delta_K} \) is then graded by \( \Delta_H \times \Delta_K \times \Delta_G \). As the deformation seems to add torus symmetries, one asks when it is toric. This happens when all component branching algebras \( \mathfrak{A}(\phi) \) are toric. These correspond to morphisms \( H \to G \) such that every \( G \) representation factors into \( H \) representations each with multiplicity one. Since the associated graded algebra is obtained by taking torus invariants of a tensor product, we can iterate these deformations, so a chain of morphisms

\[ G_0 \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_k} G_k \]

gives a family of flat deformations

\[ \mathfrak{A}(\phi_k \circ \ldots \circ \phi_1) \Rightarrow [\mathfrak{A}(\phi_k) \otimes \ldots \otimes \mathfrak{A}(\phi_1)]^{\Delta_{G_1} \times \ldots \times \Delta_{G_{k-1}}} \]

indexed by functionals on \( \text{Lie}(T_{G_1})^* \times \ldots \times \text{Lie}(T_{G_k})^* \). For a fixed morphism, different factorizations and functionals yield different deformations, and the sets of functionals form a stratified collection of cones.

**Example 4.4.** The map of reductive groups \( GL_{n-1}(\mathbb{C}) \to GL_n(\mathbb{C}) \) given as upper left block inclusion has multiplicity one. We can consider the tower of groups

\[ 1 \to GL_2(\mathbb{C}) \to GL_3(\mathbb{C}) \to \ldots \to GL_n(\mathbb{C}) \]

which factors the map \( 1 \to GL_n(\mathbb{C}) \), and defines the branching deformation of \( \mathfrak{A}(1 \to GL_n(\mathbb{C})) = R(GL_n(\mathbb{C})) \) as above.

\[ R(GL_n(\mathbb{C})) \Rightarrow (\mathfrak{A}(1 \to GL_2(\mathbb{C})) \otimes \ldots \otimes \mathfrak{A}(GL_i(\mathbb{C}) \to GL_{i+1}(\mathbb{C})) \otimes \ldots \otimes \mathfrak{A}(GL_{n-1}(\mathbb{C}) \to GL_n(\mathbb{C})))^T \]

Each piece of this tensor product is toric, so we obtain a toric deformation of \( R(GL_n(\mathbb{C})) \) to the monoid given by the Gel’fand-Tsetlin basis.

**Remark 4.5.** It is possible to show that the filtrations \( \vec{\theta} \) described in this section each satisfy the conditions of proposition 3.3 and also define "universal" tropical points.
4.2. The branching algebra of the diagonal morphism. We study the map $\Delta_n : G \to G^n$ sending $g$ to $(g, \ldots, g)$. Let $T$ be a tree with $n+1$ labeled leaves. To any such tree we associate a factorization of the diagonal morphism $\Delta_n : G \to G^n$ by placing the unique orientation on the edges of $T$ which has the 0 leaf as a source and all other leaves as sinks, and for each internal vertex allows exactly one in-flowing edge.

To each internal vertex we assign the morphism $\Delta_k : G \to G^k$ where $k+1$ is the valence of the vertex. We partition the vertices into sets defined by their distance to the 0 leaf, and from this we construct a factorization of $\Delta_n$ by multiplying together all elements of the same partition, adding in identities when necessary. To this data, and a choice of linear functionals as above, we may associate a flat deformation of $\mathfrak{A}(\Delta_n)$. This amounts to specifying a rooted tree $T$ and a labeling of edges $e \in \text{Edge}(T)$ by functionals $\theta_e$. This defines a direct sum decomposition of $\mathfrak{A}(\Delta_n)$ into components associated to labelings of the tree $T$, with a given set of representations at each edge. A general element of each component is a tree with the labeling of edges by representations and vertices labeled by intertwiners.

The functional $(T, \bar{\theta})$ acts edge-wise on these elements, in particular for an internal edge $e$, the functional $(\frac{1}{2} \theta_e, -\frac{1}{2} \theta_e \circ \omega)$, acts on $(\alpha_e, \alpha_e^*)$. If $\chi(\lambda_e, \rho_e)$ is an element of $\mathfrak{A}(\Delta_n)$, then

$$\bar{\theta}(\chi) = \sum_{e \in \text{Edge}(T)} \theta_e(\lambda_e)$$

For $G = SL_2(\mathbb{C})$ each such deformation is given by a labeling of $T$ by nonnegative real numbers, and the resulting structure is a cone of the space of phylogenetic trees. This is to be expected, since in this case $\mathfrak{A}(\Delta_n)$ is isomorphic to the Plücker algebra of the Grassmanian of 2-planes. Also in this case...
the associated algebra is toric when $\mathcal{T}$ is trivalent, since $\mathfrak{A}(\Delta_2)$ is toric for $SL_2(\mathbb{C})$. In general, if one exhibits a toric deformation of $\mathfrak{A}(\Delta_2)$ for a reductive group $G$, then we get toric deformations of any $\mathfrak{A}(\Delta_n)$ by composing with a branching deformation. We will outline such a deformation below.

4.3. The correlation morphism. Let $\omega$ be the longest root of $\mathfrak{g}$. Using the graded multiplication in $\mathfrak{A}(\Delta_n)$ we can build a new algebra

$$\mathfrak{A}(\Delta_n)_{t} = \bigoplus_{L} \bigoplus_{\lambda, (H_{\lambda}) \leq L} [V^+_\mathfrak{g}(\mathcal{V}(\lambda))]^t L$$

This algebra also comes with a family of filtrations for every labeling of a tree $\mathcal{T}$ with $n + 1$ leaves by functionals. Each graded component has a direct sum decomposition into components indexed by labellings of $\mathcal{T}$, because they are also components of the branching algebra $\mathfrak{A}(\Delta_n)$. We let $\mathfrak{A}(\Delta_n)_{t}$ denote the trivial sheaf of algebras on $\mathcal{M}_{g,n+1}$ which has $\mathfrak{A}(\Delta_n)_{t}$ as the fiber at each point. Tsuchiya, Ueno, and Yamada noted in [TUY] that the morphism of $\mathfrak{g}_S$ representations

$$V_S(\lambda) \rightarrow \mathcal{H}_S(\lambda, L)$$

gives a morphism of coherent sheaves

$$F : V^+(\lambda, L) \rightarrow V^+_\mathfrak{g}(S)(V_S(\lambda)).$$

Recall that there is an isomorphism $V^+_\mathfrak{g}(S)(V_S(\lambda)) \cong V^+_S(\lambda^s)^\mathfrak{g}(S)$, and the following commutative diagram by lemma 2.5.

$$V^+_{S}(\lambda, L) \otimes_{\mathcal{O}_S} V^+_{S}(\gamma, K) \xrightarrow{F \otimes F} V^+_S(\lambda^s)^\mathfrak{g}(S) \otimes_{\mathcal{O}_S} V^+_S(\gamma^s)^\mathfrak{g}(S)$$

The maps on the right and left are the multiplication operations on the sheaves $\mathfrak{A}(\Delta_n)_{t}$ and $V^+$ respectively, so there is a morphism of sheaves of rings,

$$F : V^+ \rightarrow \mathfrak{A}(\Delta_n)_{t}$$

which respects the multigrading by $\lambda$ and $L$. From now on we restrict ourselves to the case $g = 0$. We will show that over a stable curve of type $\mathcal{T}$ the morphism $F$ intertwines the factorization property of conformal blocks with the branching decomposition defined by $\mathcal{T}$ on $\mathfrak{A}(\Delta_n)_{t}$. We will show how this works for trees with one internal edge, the property for more general trees follows by induction. Let $C$ be a stable curve of type $\mathcal{T}$, with $\hat{C} = C_1 \cup C_2$. We have a commuting square of $\mathfrak{g}$ representations,

$$\begin{array}{c}
\mathcal{H}(\lambda^s, L) \xrightarrow{\rho^\alpha} \mathcal{H}(\lambda^s_1, \alpha, L) \otimes \mathcal{H}(\lambda^s_2, \alpha^*, L) \\
\uparrow \\
V(\lambda^s) \xrightarrow{\rho^\alpha} V(\lambda^s_1, \alpha) \otimes V(\lambda^s_2, \alpha^*)
\end{array}$$

where $\lambda^s_1 \cup \lambda^s_2 = \lambda$. Morphisms in this diagram above are equivariant with respect the following Lie algebra maps.

$$\begin{array}{c}
\widehat{\mathfrak{g}}(C) \\
\uparrow \\
\mathfrak{g}
\end{array}$$

See [TUY] for a description of the top horizontal map of Lie algebras. We may take the coinvariants of the diagram of modules above, then dualize to get a diagram of invariants.
of each space \((V^+_\lambda, L)\) with a monoidal associated graded ring, as this would allow the same for general algebras of conformal blocks.

The vertical arrows are the correlation morphism. The top horizontal arrow is the factorization map in genus 0, and the bottom horizontal arrow is the factorization map for the diagonal morphism from the previous section, see specifically lemma \[\text{[4.2]}\]. This shows that the filtrations on the branching algebras and the algebras of conformal blocks agree over the locus of stable curves. The properties of the tree deformations of both types of algebras only depend on the labels, which have now been shown to agree, so this proves theorem \[\text{[4.3]}\].

Remark 4.6. Any time \(\tilde{C}\) is a disjoint union \(C_1 \cup C_2\) the diagram above commutes. This implies a version of theorem \[\text{[1.6]}\] is true for general genus, except the correlation \(F\) is no longer a monomorphism.

For a genus 0, triple punctured curve there is no moduli, \(\mathcal{M}_{0,3} = \{\text{pt}\}\), so the algebra of conformal blocks is unique. In this case, conformal blocks have a nice description as a subspace of the space of invariants in terms of weight spaces by the action of the copy of \(SL_2(\mathbb{C})\) corresponding to the longest root of \(g\), see \[\text{[TTY]}\] for the following.

**Proposition 4.7.** \(V^\pm_{0,3}(\lambda, \gamma, \mu, L) \subset (V(\lambda^*) \otimes V(\gamma^*) \otimes V(\mu^*))^g\) has the following description. Consider the factorization of \(V(\lambda^*), V(\gamma^*)\) and \(V(\mu^*)\) as \(sl_2(\mathbb{C})\) representations with respect to the longest root \(\omega\) of \(g\).

\[
\begin{align*}
& V(\lambda^*) = \bigoplus W(\lambda^*, i) \otimes V(i) \\
& V(\gamma^*) = \bigoplus W(\gamma^*, j) \otimes V(j) \\
& V(\mu^*) = \bigoplus W(\mu^*, k) \otimes V(k)
\end{align*}
\]

Let \(W(\lambda^*, \gamma^*, \mu^*, L)\) be the subspace of \(V(\lambda^*) \otimes V(\gamma^*) \otimes V(\mu^*)\) of components \(V(i) \otimes V(j) \otimes V(k)\) with \(i + j + k \leq 2L\).

\[
V^+_\lambda(\lambda, \gamma, \mu, L) = W(\lambda^*, \gamma^*, \mu^*, L) \cap (V(\lambda^*) \otimes V(\gamma^*) \otimes V(\mu^*))^g
\]

4.4. **Filtrations from dual canonical bases.** We are interested in constructing filtrations of \(V^+_{0,3}\), with a monoidal associated graded ring, as this would allow the same for general algebras of conformal blocks by theorems \[\text{[4.1]}\] and \[\text{[4.3]}\]. A sufficient condition for such a filtration would be a basis \(B(\lambda, \gamma, \mu)\) of each space \((V(\lambda) \otimes V(\gamma) \otimes V(\mu))^g\) which has the following properties.

1. The bases \(B(\lambda, \gamma, \mu)\) have a "lower-triangular multiplication" property with respect to the multiplication in \(\mathfrak{A}(\Delta_2)\).

2. The intersection \(B(\lambda, \gamma, \mu) \cap W(\lambda, \gamma, \mu, L) \subset (V(\lambda) \otimes V(\gamma) \otimes V(\mu))^g\) is a basis for each \(L\).

The first property above would mean that there is some natural ordering on the set \(\bigsqcup B(\lambda, \gamma, \mu)\) such that a product of elements \(a \times b\) when expressed in the basis has a unique highest term with weight equal to the weight of \(a\) plus the weight of \(b\). The associated graded ring with respect to the ordering would then be monoidal. Such a basis exists for \(R(G)\), the dual canonical basis of Lusztig \[\text{[4]}\]. This basis was used by Alexeev and Brion in \[\text{[AB]}\] to construct toric deformations of spherical varieties. We will show how to adapt this basis to \(\bigsqcup \mathfrak{A}(\Delta_2)\). We include this discussion because dual canonical bases can at least give deformations of \(\mathcal{M}_{0,3}^{\pm} (\Lambda)\) when the level is very large, and because the theory of dual canonical bases is tantalizingly close to what we need to construct toric deformations of the Cox ring of the moduli of quasiparabolic principal bundles in the general case.
Following page 383 of [Zh], the space $(V(\lambda) \otimes V(\gamma) \otimes V(\mu))^g$ can be realized as the space $V_{\lambda^* - \mu, \mu}(\gamma) \otimes \mathbb{C} b_\mu \subset V(\gamma) \otimes \mathbb{C} b_\mu$, of vectors of weight $\lambda^* - \mu$ which are annihilated by the raising operators $E_{\lambda^*(H_{a_i})}^{\mu} > 1$, where $b_\mu \in V(\mu)$ is a highest weight vector and a basis member of $\mathbb{C}[T]$. We summarize these observations with the following proposition, see [M2].

**Proposition 4.8.** There is an injection of rings $\mathfrak{A}(\Delta_2) \rightarrow R(G) \otimes \mathbb{C}[T]$, in particular multiplication on tensor product invariants coincides with Cartan multiplication in $R(G)$.

The subspace of conformal blocks $\mathcal{V}_{0,\beta}^+(\lambda, \gamma^*, \mu^*) \subset [V(\lambda^* \otimes V(\gamma) \otimes V(\mu))]^g \cong V_{\lambda^* - \mu, \mu}(\gamma)$ is defined by the condition

$$F_\omega^{L+1-\lambda^*(H_\nu)} = 0,$$

where $\omega$ is the longest root of $\mathfrak{g}$, and $F_\omega$ is the lowering operator in $\mathfrak{g}$. In [AB], Alexeev and Brion use the dual canonical basis of $R(G)$ to define toric degenerations of $R(G)$ and in general the coordinate rings of spherical varieties. This is possible because the dual canonical basis $B \subset R(G)$ has a very nice labeling by so-called string parameters $(\gamma, \tilde{t}) \in \Delta \times \mathbb{Z}^N$, where $N$ is the length of the longest word $w_0$ in the Weyl group of $G$. There is such a parametrization for every decomposition $i$ of $w_0$, and the multiplication operation in $R_G$ is lower-triangular with respect to the lexicographic ordering on $\tilde{t}$.

$$b_{\lambda_1, \tilde{t}_1} \times b_{\lambda_2, \tilde{t}_2} = b_{\lambda_1 + \lambda_2, \tilde{t}_1 + \tilde{t}_2 + [\ldots]}$$

Note that the multiplication described by the main theorem of this work does not satisfy this property, indeed the space of elements with the same multilight in $\mathcal{V}_{C,\tilde{p}}^+$ is not one-dimensional. Dual canonical bases $B(\gamma) \subset V(\gamma)$ can be shown to be so-called "good bases" see e.g. [Lu], this means that the intersection $B(\gamma) \cap V_{\lambda^*, \mu^*}(\gamma)$ is also a basis, so the subalgebra $\mathfrak{A}(\Delta_2) \subset R(G) \times \mathbb{C}[T]$ inherits a basis from $B$. The dual canonical basis members which lie in $\mathfrak{A}(\Delta_2)$ are parametrized by string parameters which lie in a convex cone $C_3(i) \subset \Delta \times \mathbb{Z}^N \times \Delta$. See [BZ1], [BZ2] and [M2] for descriptions of this cone and the following proposition.

**Proposition 4.9.** The dual canonical basis defines a toric degeneration of $\mathfrak{A}(\Delta_2) \Rightarrow \mathbb{C}[C_3(i)]$, one for each string parametrization boldi.

**Corollary 4.10.** The dual canonical basis defines toric degenerations of $\mathfrak{A}(\Delta_2)_i \Rightarrow \mathbb{C}[C_3(i)^*]$ where $C_3(i)^* \subset C_3(i) \times \mathbb{R}_{\geq 0}$ is the subcone of elements $(\phi, L)$ such that the associated dominant weights lie in $\Delta_L$.

It is unknown to us whether or not the dual canonical basis restricts to a basis on the subspaces of conformal blocks in general, although this appears to work for $SL_3(\mathbb{C})$, see [KMSW]. It would still be interesting if one could prove that each conformal block, when expanded into its dual canonical components, has a highest term with respect to an ordering of the dual canonical basis which uniquely identifies the conformal block. It would suffice to find a "good basis" which further respects equation [74]. This would then define monoidal deformations of all $\mathcal{V}_{C,\tilde{p}}^+$ by composing the above degeneration with those defined by theorem [13].

### 4.5. Projective coordinate rings.

All of our techniques to study algebras of conformal blocks and branching algebras are carried out on graded pieces of these algebras. Because of this, much of what we say can be extended to nice graded subalgebras, in particular for the moduli space $\mathcal{M}_{C,\tilde{p}}(\tilde{A})$ and the GIT quotient $P^\times(G) = G^\times \backslash O(\tilde{A})$ of coadjoint orbits, we have a map of projective coordinate rings.

$$F^\times_{C,\tilde{p}} : \bigoplus_N H^0(\mathcal{M}_{C,\tilde{p}}(\tilde{A}), \mathcal{L}(N \tilde{A}, NL)) \rightarrow \bigoplus_N H^0(P^\times(G), \mathcal{L}(NL))$$

where $\mathcal{L}(\tilde{A})$ is the line bundle on $P^\times(G)$ with global sections equal to $V(\tilde{A}^\times)^g$. When $g = 0$, this map is an injection, in which case one can deduce

**Proposition 4.11.** For $g = 0$ and $L >> 0$ the map $F^\times_{C,\tilde{p}}$ above is an isomorphism.
This is comparable to remark 4.3 in [TW]. The ring $\bigoplus_N H^0(M_{C,\vec{p}}(\tilde{\Lambda}), L(N\tilde{\Lambda}, NL))$ is the projective coordinate ring of $\mathcal{M}_{C,\vec{p}}^{\G}(\tilde{\Lambda})$ where the semi-stability condition comes from $(\tilde{\Lambda}, L)$. Deformations associated to labeled trees carry over to this case as well. This implies that for large $L$, a toric deformation of $P_3(G)$ gives a toric deformation of the ring of generalized theta functions. Toric deformations of $\bigoplus_N H^0(P_3(G), L(N\tilde{\Lambda}))$ can be constructed from a toric deformation of $A(\Delta_n)$ by taking torus invariants, so these algebras have toric deformations coming from the dual canonical basis.

**Corollary 4.12.** Let $C$ be genus 0. For $L \gg 0$ the algebra $\bigoplus_N H^0(M_{C,\vec{p}}(\tilde{\Lambda}), L(N\tilde{\Lambda}, NL))$ has a toric deformation for each trivalent tree $\mathcal{T}$ with $n$ leaves and each choice of string parametrization at each internal vertex of $\mathcal{T}$.

See [M2] for a description of the polytopes associated to these toric deformations. For $\mathfrak{g} = sl_2(\mathbb{C})$ these deformations were studied in [HMSV] in order to provide commutative algebra information about the moduli of points on $\mathbb{P}^1$, in [HMM] in relation to the symplectic geometry of configuration spaces of Euclidean polygons, and in [M] to construct commutative algebra data for projective coordinate rings of Cox-Nagata rings and equivalently the moduli of rank 2 semistable vector bundles, $N_{0,n}$.

5. The case $\mathfrak{g} = sl_2(\mathbb{C})$

In the case of $\mathfrak{g} = sl_2(\mathbb{C})$ the situation simplifies considerably. Let $(C, \vec{p})$ be stable curve of type $\mathcal{T}$, a trivalent tree. An element of $A(\Delta_n)^T$ is a linear combination of labellings of $Edge(\mathcal{T})$ by nonnegative integers such that

1. The weights about each internal vertex satisfy the triangle inequalities.
2. The sum of the weights about each internal vertex is even.
3. The leaf edges are all labeled with an integer less than or equal to a level $L$.

The first two conditions above are the Clebsh-Gordon rules for $sl_2(\mathbb{C})$ which determine when a branching can exist for tensor products of $sl_2(\mathbb{C})$ representations. The kernel of $T_{\vec{p}}$ is spanned precisely by those weightings such that the sum of the labels around any internal vertex is less than or equal to $2L$. This is the quantum Clebsh-Gordon condition for $sl_2(\mathbb{C})$, see [Ka] for a discussion of this condition with respect to $sl_2(\mathbb{C})$ conformal blocks. Notice that these conditions are identical to the linear conditions which define the toric algebras of Speyer-Sturmfels and Buczynska-Wiesniewski, respectively. Each graded piece of $A(\Delta_2)^T$ is either empty or of dimension 1, so the multiplication operation is given purely by addition on the labels. The subalgebra of vacua $V_{C,\vec{p}}$ is given by a linear condition on these labels, and is therefore also toric. The corresponding polytope is discussed in [BW] and [M], for $L = 1$ it is the simplex defined by the condition that the sum of any two weights be either 0 or 2, and the associated toric variety is $\mathbb{P}^3$. Since the general case is a torus invariant subalgebra of a tensor product of toric algebras, it is also toric. Notice that the same observation applies to the genus $> 0$ case. By the same reasoning these algebras of vacua are the semigroup algebras given by the semigroup of weighted trivalent graphs $\Gamma$, which obey the above conditions at each internal vertex and leaf. One can directly verify that the associated toric varieties are those discussed in 4.4 of the paper of Hurtubise and Jeffrey [HJ]. The operation of taking invariants by the torus is the "gluing" construction of Hurtubise and Jeffrey, used in their study of the complex and symplectic geometry of moduli of framed parabolic bundles. We note that our work here allows one to obtain the Verlinde formula as a count of lattice points in the moment polytope of the Hurtubise-Jeffrey toric moduli space. We also mention that Buczynska has constructed pairwise deformations connecting these toric algebras, [Bu]. We do not know the relationship between these flat families and the flat families constructed here.

The commutative algebra of the genus 0 case was extensively studied by Buczynska and Wiesniewski in [BW], and also in [M], in the former paper it was proved that the deformed algebra of vacua are generated in degree 1 with quadratic relations. Little is known about the general non-trival genus case, although some good results have been obtained by Buczynska, [Bu], in particular a generating set for the genus 1 case.

**Proposition 5.1.** For $SL_2(\mathbb{C})$, the algebras $V^{+}_{C,\vec{p}}$ are Gorenstein.

**Proof.** These algebras are domains, so it suffices to prove they share the same Hilbert function with a Gorenstein domain. Choosing a stable curve with the same genus and puncture information as $(C, \vec{p})$, with genus 0 components, we reduce the problem to showing an affine semigroup algebra is Gorenstein.
Figure 15. A labelled graph of genus 1, say of level 4.

This is the semigroup of lattice points in a cone $P^*_\Gamma \subset \mathbb{R}^{\text{Edge}(\Gamma)} + 1$, for some trivalent graph $\Gamma$. The associated algebra is Gorenstein if and only if the module defined by the lattice points in the interior of this cone is principal. A point is interior if and only if all the triangle inequalities and the inequality defining the level condition are strict. Over a single trinode, one checks that this is the case if and only if the weighting has the element $(2, 2, 2)$ in level 4 as a factor. In general, a weighting is interior if and only if the weighting $\omega : \text{Edge}(\Gamma) \rightarrow \{2\}$ of level 4 is a factor. This implies that $\mathbb{C}[P^*_\Gamma]$ is a Gorenstein algebra, which gives the proposition.

The stack $\mathcal{M}_{p_1, p_2, p_3}(B^3)$ for $SL_2(\mathbb{C})$ has Cox ring isomorphic to the affine semigroup ring for the semigroup in $\mathbb{Z}^4 \subset \mathbb{R}^4$ generated by $(0, 0, 0, 1), (1, 1, 0, 1), (1, 0, 1, 1)$, and $(0, 1, 1, 1)$. This is a polynomial algebra. Like the genus 0 case, the toric algebras of general genus and $sl_2(\mathbb{C})$ symmetry are thought to have connections with phylogenetic algebraic geometry. They are the projective coordinate rings of generalizations of phylogenetic statistical models to networks of evolution which are not contractible.

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