Naturalness and Naturalness Criteria

Su Yang

Department of Physics and Astronomy, Northwestern University, Evanston, IL 60201

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We analyse how to describe the level of naturalness and pointed out that Barbieri and Giudice’s the widely adopted sensitivity criteria of naturalness can not reflect the level of naturalness correctly, we analyse the problems of the sensitivity criteria and proposed a new criterion that can solve these problems, and also give a clear physical meaning to the naturalness cut-off level.

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The physical principle of naturalness introduced by Wilson and ‘t Hooft requires that in order to get a small observable parameters at the weak scale, we do not need to extremely fine-tune the lagrangian parameters at the grand unification scale. For example, the renormalization of $\Phi^4$ model:

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\Phi)^2 - m_0^2\Phi^2 - \frac{g}{4!}\Phi^4$$

the scalar mass $m^2$ can be written as:

$$m^2 = m_0^2 - g^2\Lambda^2$$

Where $m_0^2$ is the bare mass, and $\Lambda$ is the cut-off scale. Because both bare mass and the cut-off scale are around $10^{18}\text{GeV}$, in order to have a small weak scale renormalized mass $m^2$, we need a fine-tuning mechanism.

Similar cases are widely existed in renormalizations and various mixing mechanisms. The naturalness principle requires that, any realistic model won’t need too much fine-tuning, it also requires that the lagrangian parameters can not choose the values that will result in excessive fine-tuning. It is one of the main reasons that we prefer the supersymmetric standard model, and it is also the main consideration when we build a neutrino mass mechanism. These tasks requires a naturalness criteria that can reflect the level of naturalness correctly.

The sensitivity criteria proposed by R. Barbirei and G.F. Giudice et al. is the first widely adopted quantitative indicator of the naturalness level. Its idea is quite simple. If $x_0$ is a input lagrangian parameter at the grand unification scale, and $y$ is a computed observable output parameter like masses, Yukawa couplings etc at the weak scale, if we varies lagrangian parameter $x_0$ at the grand unification scale, the corresponding computed weak scale observable parameter $y$ will be varied, Barbieri and Giudice’s sensitivity criteria $c$ is defined as:

$$c = \frac{\Delta y/y}{\Delta x_0/x_0} = |\frac{\partial \ln y}{\partial \ln x_0}|$$

Here need to emphasis that we usually need to consider the effect of a parameter on another parameter that has different canonical dimension. We prefer a dimensionless $c$, thus Barbieri and Giudice chooses $\Delta y/y$ as the basis of comparison.

Barbieri and Giudice set $c \approx 10$ as the naturalness cut-off, any $c$ much greater than 10 will be classified as fine-tuned. Since then hundreds authors apply this criteria to various problems, from setting a naturalness contour for SUSY particle search, to the fine-tuning problem of the neutrino seesaw mechanism. Barbieri and Giudice’s sensitivity criteria has been widely adopted as the doctrine of naturalness judgment.

But Barbieri and Giudice’s sensitivity criteria is not reliable, many examples show that it fails for certain cases, because the sensitivity criteria plays an important role in new model building, it is worth to investigate the relationship between the naturalness and the sensitivity, and find a correct and reliable criteria.

In mathematics, naturalness can be classified as a type of initial condition sensitivity problem. Find how large the probability is for a output parameter in a certain range of values is the best way to describe the initial condition sensitivity. If we have a system with an input parameter $x_0$ and a corresponding output parameter $y$, If the probability of the input parameter around the value $x_0$ is much smaller than the probability of the output parameter around the value $y$, then the system is initial condition sensitive, which means need fine-tuning in physics.

If we assume the input parameter $x_0$ has a uniform probability distribution, then the output parameter $y$ will have a probability distribution of $|\partial x_0/\partial y|$, if we choose $|\partial y/\partial x_0|$, the inverse function of the output parameter probability distribution as a criteria to judge fine-tuning, then those parameter regions that will result in too small probability distribution $|\partial x_0/\partial y|$, or too big its inverse function $|\partial y/\partial x_0|$ can be classified as the fine-tuned region.

Because we need to consider the fine-tuning property with two different types of parameters, for example, compare masses with the gauge couplings, and we also want a dimensionless fine-tuning indicator that does not depend on the parameters we are comparing. So Barbieri and Giudice’s sensitivity criteria chooses the logarithmic function $|\partial \ln y/\partial \ln x_0|$ as a fine-tuning criterion. Al-
though $c$ becomes dimensionless, while changing from $|\partial y/\partial x_0|$ to $|\partial \ln y/\partial \ln x_0|$ means we have changed the assumption that the probability distribution of the input parameter $x$ from the uniform distribution to $y(x_0)/x_0$ distribution, which means we assumed that lagrangian parameters tend to choose certain values at the grand unification scale. Obviously this will greatly influenced the naturalness judgment.

For example, if we apply Barbieri and Giudice’s formula (Eq. (4)) to the $\Phi^4$ model (Eq. (11)). Clearly from Eq. (2) we know the scalar mass is highly fine-tuned. But on the other hand, if we integrate the mass renormalization group equation, we have:

$$m^2 = m_0^2 \exp(\int_0^{\tau} (g^2/16\pi^2 - 1)dt)$$  \hspace{1cm} (4)

Apply Barbieri and Giudice’s sensitivity definition to Eq. (4), it gives a result of sensitivity $\partial \ln m^2/\partial \ln m_0^2$ equals to one, and it is not fine-tuned.

Obviously, the origin of this problem is whatever energy scale it is, the ratio $\Delta m^2/m^2$ is always fixed, Chosen the logarithmic function of the parameter rather than the parameter itself as a basis of comparison means we assume the variation ratio of the langrangian parameter rather than the parameter itself as even probability distributed at the grand unification scale. Thus even $\Delta m^2$ is only around hundreds GeV$^2$s while $\Delta m_0^2$ is around $10^{36}$GeV$^2$, we still think they are equivalent. Generally, the consequence of this is, if any parameter runs as an Exponent function $y = y_0 \exp(f)$ as energy scale $t$ changes, if the exponent $f = f(t)$ happens to be a function of the energy scale $t$ only, then even $f$ is very large, and $y$ blows up so quickly as $t$ increased Barbieri and Giudice’s sensitivity criteria will still give a sensitivity equals to 1 result, which tells you that the pure scalar field is not fine-tuned. Only when the index $f$ is not only the function of $t$, but also the function of $y_0$, then will Barbieri and Giudice’s sensitivity criteria give a not equal to one sensitivity.

Return to the renormalization, normally the exponent $f$ is consists of a constant part, several anonymous dimension parts $\gamma_i$, some of the anonymous dimension parts are depend on the lanrangian parameters $x_0$, some are not, although those parts of exponent that not depended on $x_0$ will contribute the fine-tuning, it will still be ignored by Barbieri and Giudice’s sensitivity criteria. The anonymous dimension part reflects the relative changing of various parameters, and will depend on the initial conditions like masses, coupling constants, it is obvious, if we adopted Barbieri and Giudice’s criteria to calculate the sensitivity, this part will give a not equal to one result, so in the fact, What Barbieri and Giudice et al.’s sensitivity criteria judged is not the initial condition sensitivity, but the anonymous dimension sensitivity, which reflects how much a parameter depends on the relative changes of the various parameters. So we’d better call Barbieri and Giudice et al.’s sensitivity “anonymous sensitivity” rather than the initial condition sensitivity.

Besides the above mentioned problem, the sensitivity criteria have many other problems. After Barbieri and Giudice’s naturalness criteria has been proposed, G. Anderson et al. first pointed out that under certain circumstances Barbieri and Giudice’s naturalness criteria failed to give a correct result that consistent with known phenomena. P. Ciafaloni et al. also gave examples show that Barbieri and Giudice’s naturalness judgment is not valid under certain circumstances.

The example given by G. Anderson et al is regarding the high sensitivity of $\Lambda_{QCD}$ to the strong coupling constant $g$.

$$\Lambda_{QCD} = M_P \exp \left(-\frac{(4\pi)^2}{bg^2(M_P)}\right)$$  \hspace{1cm} (5)

Apply Barbieri and Giudice et al’s definition of naturalness indicator we can calculate the sensitivity of $\Lambda_{QCD}$ to the strong coupling constant $g$ at the grand unification scale:

$$C(g) = \frac{4\pi}{b} \frac{1}{\alpha_s(M_P)}$$  \hspace{1cm} (6)

This value is greater than 100, much larger than the naturalness upper bound set by Barbieri and Giudice. but actually it is protected by gauge symmetry, and is not fine tuned.

Carefully examined examples similar to the large sensitivity of $\Lambda_{QCD}$, we found all these examples occur when comparing parameters with different canonical dimensions. Mathematically, comparing two parameters with different canonical dimensions is difficult, Barbieri and Giudice did aware this difficulty, and introduced the logarithmic function to rescale each parameter to a dimensionless formation, they thought this could be sufficient to eliminated the effects of the scale difference and dimensional difference, but these examples show that, this method can not cancel these effects.

We know generally, there’s a Gaussian fixed point at the origin of the parameter space for renormalization group equations. around the origin, If we rescale the momenta by a factor of $\Lambda$, then two different parameters $\tau$ and $h$ can be expanded as:

$$\tau \approx \Lambda^\alpha \tau_0$$  \hspace{1cm} (7)
$$h \approx \Lambda^\beta h_0$$  \hspace{1cm} (8)

Here $\alpha$ and $\beta$ are corresponding canonical dimensions. Because of the renormalization, the scale $\Lambda$ will link these two parameters together. even they may not have any
other relations. If we calculate the sensitivity of \( h \) to the variation of \( \tau \), approximately, it would be:

\[
\frac{\partial h}{\partial \tau} \approx -\frac{\beta h}{\alpha \tau}
\]  

(9)

This effect was known as scaling effect in statistical physics, which exists anywhere when two parameters have different canonical dimensions, obviously, it has nothing to do with the fine-tuning, but when we convert both parameters to dimensionless parameters by Barbieri and Giudice’s technique, the factor of different canonical dimension \(-\beta/\alpha\) is still there, this is because the scaling phenomena is not linear, it can not be eliminated by Barbieri and Giudice’s technique.

If one parameter \( \tau \) has a marginal canonical dimension, then we can not use the above argument, we must consider the higher order term. take various couplings renormalization as an example:

\[\tau \approx (1 + \alpha \tau_0 \ln \frac{1}{\Lambda})\tau_0\]  

(10)

Similarly, the result would be:

\[
\frac{\partial h}{\partial \tau} \approx \frac{\beta h}{\alpha \tau \tau}
\]  

(11)

We can define a dimensional effect factor \( \Delta = \beta h/\alpha \tau^2 \) or \( \beta h/\alpha \tau \) for later reference. In the parameter space if two parameters have the same dimension then \( \Delta \) becomes one, otherwise this factor may become significant. A schematic diagram is shown in Fig. 1.

If we look the naturalness problem from the phase diagram spanned by all the parameters (Fig. 1), there is a small area in this phase diagram which represents the weak scale, and there is also an area represents the grand union scale, the renormalization flows run from the grand unification scale area and go to the weak scale area, the “naturalness” requires that, whatever initial condition we choose, the weak scale area is always smaller or, maybe a little large than the grand unification scale area, even the ratio \( \Delta h_0/\Delta \tau_0 \) is big. So it is better to understand the naturalness principle as the weak scale stability rather than the small sensitivity. High sensitivity doesn’t mean unnatural.

The effect of different canonical dimensions is widely existed, for example, at low temperature, the Plank radiation law becomes \( E_\nu = \frac{8\pi h^3 c^8}{c^2} e^{-h\nu/kT} \), If we calculate the sensitivity of \( E_\nu \) to the variation of the temperature, consider \( T \approx 4k \), and \( \nu \approx 10^{15} Hz \), then you will have an extremely large sensitivity \( c \approx 10^4 \), but we never doubt the correctness of the Plank radiation law.

Besides these problems, The sensitivity criteria also implied the relationship between the input and the output is monotonic, only the input parameter \( x \) can lead to the output parameter \( y \), there’s no such circumstance that both input parameters \( x_1 \) and \( x_2 \) will eventually lead to a same output parameter \( y \).

Although the relationships between parameters linked by most renormalization group equations are monotonic, while most mixing cases are not, one output parameter is usually corresponding to two input parameters. For example, mixing of \( M_z \) mass and \( M_W \) mass, mixing of CP-even Higgs masses in Supersymmetric Standard Models, and mixing of fermionic masses etc.

Take the mixing of \( M_z \) mass in MSSM model as an example, calculate the \( M_z \) mass at the initial condition (at grand unification scale) \( m = 200, M = 40, \tan \beta = 18 \), and gradually reduced mass \( m \), then we will find the non-monotonic relationship between grand unification scale variable \( m \) and weak scale variable \( M_z \).

Obviously, it is not a monotonic function of \( m \), for example, if weak scale mass \( M_z \) is around 80GeV, there are two grand unification scale parameter regions that can contribute this result, one is around \( m \) is 80GeV, the other region is around \( m \) is 150GeV. Barbieri and Giudice’s definition only counted one region’s contribution, it will overestimate the naturalness level. We should count all possible GUT scale parameters contributions.

A good definition of naturalness criteria should be able to solve all the problems listed above. Because the problems listed above, we can not choose the logarithmic function, instead, we need to use \( \partial y/\partial x_0 \) directly. refers to the definition of Lyapunov exponent, which used to define the initial condition sensitivity in dynamical systems, we can write down the \( t \) evolution of the probability distribution to the variation of \( x_0 \):

\[
\frac{\delta y}{\delta x_0} = \Delta_0 e^{\lambda t}
\]  

(12)

The dimensionless factor \( \lambda \) reflects the shrinkage of the
probability, here $\Delta_0$ is a kind of background probability density at the grand unification scale which need to be subtracted, mathematically, $\lambda$ reflects the level of $y$ fine-tuning when $x_0$ changes, when the parameters $x$ and $y$ have the same canonical dimension, $\Delta_0$ becomes one, also considering the non-monotonic propriety, finally we can define a Lyapunov exponent like index $\lambda$ for the naturalness criteria:

$$\lambda = \frac{1}{t} \ln \sum \frac{1}{\Delta_0} \frac{\delta y}{\delta x_0}$$  \hspace{1cm} (13)$$

If it is not monotonic, we divided and sum over all monotonic regions.

Although the fine-tuning criteria problem in high energy physics is somewhat similar to the problem of using Lyapunov exponent to judge whether a nonlinear system is chaos or not, these two situations also have important difference. In nonlinear physics, if the Lyapunov exponent is negative, then the phase space shrinks while time increases, it is obviously not initial condition sensitive, thus it is not chaos, if the Lyapunov exponent is greater than zero, then the system is initial condition sensitive and will be classified as chaos. Similarly, when we consider the fine-tuning problem in high energy physics, if $\lambda < 0$, for the same reason, we can easily classify the system as not initial condition sensitive of not fine-tuning, but for the cases that have $\lambda > 0$, the situation is a little more complex, this is because for systems in nonlinear physics, the time variable $t$ can go to infinity while in high energy physics, the running parameter $t$ can not go beyond the grand unification scale, which is around 38. so for the situations that have small positive $\lambda$, even the range of grand unification parameter space is a little bigger than the weak scale parameter space, it still can be thought as not initial condition sensitive, or not fine-tuned. So we should define a reasonable positive fine-tuning upper limit for $\lambda$.

In probability theory people usually define the probability $p < 0.05$ as small probability event and can be considered as hard to happen, although this is a more strict condition than Barbieri and Giudice’s sensitivity less than 10 criteria, we still adopt this doctrine, and define a upper limit for the fine-tuning. Suppose fine-tuning occurs when $p < 0.05$, that means $\exp(\lambda t) = 1/0.05$, we immediately have the upper limit of the $\lambda$ index is 0.08. According to this definition, all parameters with $\lambda < 0.08$ will be safe and not fine-tuned, and if $\lambda > 0.08$ we learn that it is less than 5% of chance to have this weak scale value thus quite impossible. Not like Barbieri and Giudice’s naturalness $c \approx 10$ cut-off, which doesn’t have any physical meaning, our method gives a clear physical meaning of the naturalness cut-off.

Numerical calculations with both Barbieri and Giudice’s sensitivity criteria and $\lambda$ show that, for the cases with zero engineering dimension $4 - \frac{3}{2} n_f - n_b$ and both parameters have the same canonical dimension, and the relations are monotonic, the difference is not significant, this is because the effect of anonymous dimensions $\gamma_i$ are similar. for MSSM model, when $M = 40 GeV, m = 83.5 GeV$ and $\tan \beta = 18$, sensitivity of $m_h$ to the variation of $m$ equals to 1, while new criteria when $m = 68.5 GeV \lambda$ becomes positive. But for $\Phi^4$ model mass, which has engineering dimension equals to 1, sensitivity $c = 1 < 10$, while $\lambda = 1 + \frac{65}{89}$, greater than 0.08. For the large sensitivity of $\Lambda_{QCD}$, it is not difficult to calculate that $\lambda = (\ln \frac{\Lambda_{QCD}}{g M_F})/t$, which is far less than 0.08. for non-monotonic case $M_e$, if $M_e = 89.05 GeV$, which corresponding to $m = 188.5 GeV$ and $m = 65.1 GeV$ at the grand unification scale, we calculated that sensitivity $c = 0.499$ and $c = 0.910$ respectively. while $\lambda$ for $M_e = 89.05 GeV$ is $-0.046$.

In this paper we have investigated the widely adopted sensitivity criteria of naturalness, we found when comparing parameters with different canonical dimensions the sensitivity usually will be very big, this should be understood as the scaling effect rather than the fine-tuning, under these circumstances the sensitivity is larger than the level of naturalness fine-tuning. When comparing parameters with the same canonical dimensions what we get is a type of “anonymous sensitivity”, the result may be larger or may be smaller than the true level of naturalness fine-tuning. All the calculations based on sensitivity criteria become unreliable. In summary, the widely adopted sensitivity criteria is not reliable, can not truly reflect the naturalness properties. We defined a new criteria to solve all the problems the sensitivity criteria has, and also gives a clear physical meaning to the naturalness cut-off value.

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