SEGAL SPACES, SPANS, AND SEMICATEGORIES

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ABSTRACT. We show that Segal spaces, and more generally category objects in an ∞-category ℂ, can be identified with associative algebras in the double ∞-category of spans in ℂ. We use this observation to prove that “having identities” is a property of a non-unital (∞, n)-category.

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1. Introduction

A “semicategory” or non-unital category is a category without identity morphisms. It is an easy exercise to show that “having identities” is a property of a semicategory, and “preserving identities” is a property of functors of semicategories. More precisely, the forgetful functor Cat → Semicat gives an equivalence between Cat and a subcategory of Semicat.

The analogues of this statement for higher categories turn out to be very useful: To define particular examples of higher categories or functors between them, it can be extremely convenient to first ignore the identities and then at the end check that the resulting non-unital structure has the required property. For (∞, 1)-categories (which we will refer to as ∞-categories), such a result is already known: it is due to Harpaz [Har15] in the context of Segal spaces, and for quasicategories it is a combination of work of Tanaka [Tan18] and Steimle [Ste18]. Similar results have also been proved for other higher-categorical structures, including A∞-categories (see [LM06] for a comparison of different notions of weak units in this setting) and monoidal 2-categories [JK13].

The goal of the present paper is to show that “having identities” is also a property of (∞, n)-categories for all n:

Theorem 1.1. Let Seg^n(𝕊) denote the ∞-category of n-fold Segal spaces and Seg^n_{nu}(𝕊) its non-unital analogue. Then the forgetful functor Seg^n(𝕊) → Seg^n_{qu}(𝕊) induces an equivalence

Seg^n(𝕊) ∼→ Seg^n_{qu}(𝕊)

where Seg^n_{qu}(𝕊) ⊆ Seg^n_{qu}(𝕊) is a subcategory of quasi-unital n-fold Segal spaces and quasi-unital functors between them.

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We will prove this in Section 4 by first proving the case $n = 1$ for category objects (or internal $\infty$-categories) in any $\infty$-category with finite limits; the general statement then follows easily by iterating this case.

In the case $n = 1$ we will deduce the theorem from the analogous statement for non-unital associative algebras in monoidal $\infty$-categories, which has been proved by Lurie [Lur17]. To do so, we must first identify category objects as certain associative algebras. For ordinary categories, it seems to have been first observed by Bénabou [Bén67] that a category can be viewed as an associative algebra (or monad) in a 2-category $\text{Span}^+$ of spans of sets; this has

- sets as objects,
- spans $I \leftarrow S \rightarrow J$ as 1-morphisms from $I$ to $J$, with composition given by taking pullbacks, i.e.
  \[ \begin{array}{c}
  S \\
  \downarrow \\
  J \\
  \circ \\
  K \\
  \downarrow \\
  I \\
  \downarrow \\
  T \\
  \downarrow \\
  J \\
  \downarrow \\
  I \\
  \end{array} \quad := \quad \begin{array}{c}
  T \times_J S \\
  \downarrow \\
  K, \\
  \end{array} \]
- and morphisms of spans
  \[ \begin{array}{c}
  S \\
  \downarrow \\
  I \\
  \downarrow \\
  J \\
  \end{array} \quad \Rightarrow \quad \begin{array}{c}
  S' \\
  \downarrow \\
  I \\
  \downarrow \\
  J \\
  \end{array} \]

as 2-morphisms, composing in the obvious way.

In particular, a category with $S$ as its set of objects is the same thing as an associative algebra in the “double slice” $\text{Set}/S/S$ with the tensor product defined by pullbacks over $S$. However, functors are not the same thing as morphisms of algebras in $\text{Span}^+$ (Set). To remedy this, we can upgrade to a double category $\text{SPAN}^+(\text{Set})$ (see e.g. [GP99, §3.2]) whose objects are sets, vertical morphisms are functions, horizontal morphisms are spans, and whose squares are diagrams of the form

\[ \begin{array}{c}
  \bullet \quad \leftarrow \quad \bullet \\
  \downarrow \\
  \bullet \quad \leftarrow \quad \bullet \\
  \end{array} \quad \Rightarrow \quad \begin{array}{c}
  \bullet \\
  \downarrow \\
  \bullet \\
  \end{array} \quad \leftarrow \quad \begin{array}{c}
  \bullet \\
  \downarrow \\
  \bullet \\
  \end{array} \]

Then $\text{Cat}$ is equivalent to the category of associative algebras in $\text{SPAN}^+(\text{Set})$. In Section 3 we prove an $\infty$-categorical version of this statement, using the double $\infty$-category of spans constructed in [Hau18]:

**Theorem 1.2.** Let $\mathcal{C}$ be an $\infty$-category with finite limits. There is an equivalence of $\infty$-categories

$\text{Cat}(\mathcal{C}) \simeq \text{Alg}(\text{SPAN}^+(\mathcal{C}))$

between category objects in $\mathcal{C}$ and associative algebras in the double $\infty$-category of spans in $\mathcal{C}$.

2. Preliminaries

In this section we briefly review the higher-algebraic structures we will make use of below.

**Notation 2.1.** We write $\Delta$ for the simplex category of ordered sets $[n] := \{0, \ldots, n\}$. A morphism $\phi : [m] \to [n]$ is called inert if it is the inclusion of a subinterval, i.e. $\phi(i) = \phi(0) + i$ for $i = 0, \ldots, n$. For $0 \leq i \leq j \leq n$ we write $\rho_{i,j} : [j - i] \to [n]$ for the inert morphism with $\rho_{i,j}(0) = i$, $\rho_{i,j}(j - i) = j$.

**Definition 2.2.** Let $\mathcal{C}$ be an $\infty$-category with pullbacks. A category object in $\mathcal{C}$ is a simplicial object $X : \Delta^{op} \to \mathcal{C}$ such that for all $[n] \in \Delta$ the morphism

$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$,
induced by the morphisms $ρ_{i,i+1}$ and $ρ_{i,i}$ is an equivalence. We write $\text{Cat}(\mathcal{C})$ for the full subcategory of $\text{Fun}(\Delta^{op}, \mathcal{C})$ spanned by the category objects. Category objects in the $\infty$-category $S$ of spaces are called Segal spaces [Rez01].

**Remark 2.3.** Category objects in $\mathcal{C}$ model the algebraic structure of a (homotopy-coherent) category internal to $\mathcal{C}$: If we think of $X_0$ as the objects of $X$ and $X_1$ as the morphisms then we have:

- $d_1, d_0 : X_1 \to X_0$, assigning source and target objects to morphisms,
- $s_0 : X_0 \to X_1$, assigning identity morphisms to objects,
- $X_1 \times_{X_0} X_1 \leftarrow X_2 \xrightarrow{d_1} X_1$, assigning composites to composable pairs of morphisms.

The remaining structure in the simplicial object $X$ ensures that the composition is homotopy-coherently associative and unital.

**Remark 2.4.** If $\mathcal{C}$ has a terminal object $*$, we can identify the category objects $X : \Delta^{op} \to \mathcal{C}$ such that $X_0 \simeq *$ with associative monoids in $\mathcal{C}$.

**Definition 2.5.** A double $\infty$-category is a cocartesian fibration $M \to \Delta^{op}$ such that the corresponding functor $\Delta^{op} \to \text{Cat}_\infty$ is a category object. A monoidal $\infty$-category is a double $\infty$-category $M$ such that $M_0$ is contractible, corresponding to an associative monoid in $\text{Cat}_\infty$.

A double $\infty$-category is thus an $\infty$-categorical analogue of a category internal to categories, or a double category. This notion has a useful generalization:

**Definition 2.6.** A generalized non-symmetric $\infty$-operad is a functor $p : \mathcal{O} \to \Delta^{op}$ such that:

(i) For every object $X$ in $\mathcal{O}_n := \mathcal{O} \times_{\Delta^{op}} \{n\}$, and every inert morphism $φ : [m] \to [n]$ in $\Delta$, there exists a $p$-cocartesian morphism $X \to φ_nX$ lying over $φ$.

(ii) For every object $[n] \in \Delta$, the functor

$$\mathcal{O}_n \to \mathcal{O}_1 \times_\mathcal{O}_0 \cdots \times_\mathcal{O}_0 \mathcal{O}_1$$

induced by the cocartesian morphisms over the maps $ρ_{i,i+1}$ and $ρ_{i,i}$, is an equivalence.

(iii) Given $X$ in $\mathcal{O}_n$, choose compatible cocartesian lifts $X \to X_{i,j}$ over $ρ_{i,j}$. Then for any $Y \in \mathcal{O}_m$, the commutative square

$$\begin{array}{ccc} 
\text{Map}_\mathcal{O}(Y,X) & \xrightarrow{\pi} & \text{Map}_\mathcal{O}(Y,0,1) \times_\text{Map}_\mathcal{O}(Y,X_{1,1}) \cdots \times_\text{Map}_\mathcal{O}(Y,X_{n-1,n-1}) \text{Map}_\mathcal{O}(Y,X_{n-1,n}) \\
\downarrow & & \downarrow \\
\text{Map}_\Delta^{op}([m],[n]) & \xrightarrow{\pi} & \text{Map}_\Delta^{op}([m],[1]) \times_\text{Map}_\Delta^{op}([m],[0]) \cdots \times_\text{Map}_\Delta^{op}([m],[0]) \text{Map}_\Delta^{op}([m],[1]) 
\end{array}$$

is cartesian.

**Remark 2.7.** Generalized non-symmetric $\infty$-operads are an $\infty$-categorical analogue of the virtual double categories of [CS10] or $\mathcal{F}_c$-multicategories of [Lei02]; see [GH15, §2] for further discussion and motivation. We can identify the double $\infty$-categories as the generalized non-symmetric $\infty$-operads that are cocartesian fibrations.

**Definition 2.8.** A non-symmetric $\infty$-operad is a generalized non-symmetric $\infty$-operad $\mathcal{O}$ such that $\mathcal{O}_0$ is contractible.

**Definition 2.9.** Let $\mathcal{O}$ be a generalized non-symmetric $\infty$-operad, and suppose $\mathcal{C}$ is an $\infty$-category with pullbacks. A Segal $\mathcal{O}$-object in $\mathcal{C}$ is a functor $Φ : \mathcal{O} \to \mathcal{C}$ such that for all $X \in \mathcal{O}_n$ the natural map

$$Φ(X) \to Φ(X_{0,1}) \times_Φ(Φ(X_{1,1}) \cdots \times_Φ(Φ(X_{n-1,n-1}) \ Φ(X_{n-1,n}))$$
is an equivalence, where \( X \to X_{i,j} \) is a cocartesian morphism over \( \rho_{i,j} \). We write \( \text{Seg}_O(\mathcal{C}) \) for the full subcategory of \( \text{Fun}(O, \mathcal{C}) \) spanned by the Segal \( O \)-objects.

**Proposition 2.10.** Let \( \mathcal{C} \) be an \( \infty \)-category with finite limits and \( O \) a generalized non-symmetric \( \infty \)-operad, and let \( i_O \) denote the inclusion \( O_0 \hookrightarrow O \). Then the functor

\[
i_O^*: \text{Seg}_O(\mathcal{C}) \to \text{Fun}(O_0, \mathcal{C})
\]

is a cartesian fibration. For \( \Phi \in \text{Seg}_O(\mathcal{C}) \) and \( \eta: F \to i_O^*\Phi \), the cartesian morphism over \( \eta \) is given by the pullback

\[
\eta^*\Phi \longrightarrow \Phi
\]

in \( \text{Fun}(O, \mathcal{C}) \), where \( i_{O,*} \) denotes right Kan extension.

**Proof.** We first check that \( i_{O,*}: \text{Fun}(O_0, \mathcal{C}) \to \text{Fun}(O, \mathcal{C}) \) factors through \( \text{Seg}_O(\mathcal{C}) \). For \( F: O_0 \to \mathcal{C} \) and \( X \in O_n \), we have

\[
i_{O,*}F(X) \simeq \lim_{(Y,X) \in O_0 \times O_X} F(Y).
\]

If \( X \to X_{i,i} \) is a cocartesian morphism over \( \rho_{i,i}: [0] \to [n] \) then the set \( \{X \to X_{i,i}\} \) is a coinitial subcategory of \( O_0 \times O_X \), and so we have

\[
i_{O,*}F(X) \simeq \prod_{i=0}^n F(X_{i,i}),
\]

which clearly gives a Segal object. The functor \( i_O^*: \text{Seg}_O(\mathcal{C}) \to \text{Fun}(O_0, \mathcal{C}) \) therefore has a right adjoint. To see that \( i_O^* \) is a cartesian fibration, we apply the criterion of [Hau17, Corollary 4.52]. We must check that for \( \Phi \in \text{Seg}_O(\mathcal{C}) \) and \( \eta: F \to i_O^*\Phi \), if we define \( \eta^*\Phi \) by the pullback square above, then the composite \( i_O^*\eta^*\Phi \to i_O^*i_{O,*}\Phi \to \Phi \) is an equivalence.

Since \( i_O \) is fully faithful we have \( i_O^*i_{O,*} \simeq \text{id} \) and as \( i_O^* \) preserves limits we see that \( i_O^*\eta^*\Phi \) is the pullback

\[
i_O^*\eta^*\Phi \longrightarrow i_O^*\Phi
\]

whence \( i_O^*\eta^*\Phi \to F \) is indeed an equivalence. The characterization of cartesian morphisms follows from [Hau17, Proposition 4.51]. \[ \Box \]

**Definition 2.11.** A morphism of generalized non-symmetric \( \infty \)-operads is a commutative triangle

\[
\Delta \longrightarrow \mathcal{O} \longrightarrow \mathcal{P}
\]

where \( f \) preserves cocartesian morphisms over inert maps in \( \Delta^{op} \). We write \( \text{Opd}^{\text{ns}}_\infty \) for the \( \infty \)-category of generalized non-symmetric \( \infty \)-operads, defined as a subcategory of \( \text{Cat}_\infty/\Delta^{op} \). A morphism of generalized non-symmetric \( \infty \)-operads from \( \mathcal{O} \) to \( \mathcal{P} \) is also called an \( \mathcal{O} \)-algebra in \( \mathcal{P} \), and we write \( \text{Alg}_\mathcal{O} (\mathcal{P}) \) for the \( \infty \)-category of \( \mathcal{O} \)-algebras in \( \mathcal{P} \), defined as a full subcategory of \( \text{Fun}_{/\Delta^{op}}(\mathcal{O}, \mathcal{P}) \).
Definition 2.12. For the terminal (generalized) non-symmetric ∞-operad $Δ_{\text{op}}$, we refer to $Δ_{\text{op}}$-algebras in a generalized non-symmetric ∞-operad $O$ as associative algebras, and write
\[ \text{Alg}(O) := \text{Alg}_{Δ_{\text{op}}}(O). \]

Definition 2.13. For $C$ an ∞-category, we write $Δ_{\text{op}}^C \to Δ_{\text{op}}$ for the cocartesian fibration corresponding to the right Kan extension $i_{Δ_{\text{op}}^C}: Δ_{\text{op}}^C \to \text{Cat}_\infty$. The proof of Proposition 2.10 shows that $Δ_{\text{op}}^C$ is a double ∞-category. Moreover, if $M \to Δ_{\text{op}}$ is any double ∞-category then there is a canonical morphism $M \to Δ_{\text{op}}^M$ over $Δ_{\text{op}}$, corresponding to the unit morphism $M \to i_{Δ_{\text{op}}^C,i_{Δ_{\text{op}}}}M$ where $M$ is the functor $Δ_{\text{op}} \to \text{Cat}_\infty$ associated to $M$. This preserves all cocartesian morphisms, and so is in particular a morphism of generalized non-symmetric ∞-operads.

Lemma 2.14. For any generalized non-symmetric ∞-operad $O$, we have a natural equivalence
\[ \text{Alg}_O(Δ_{\text{op}}^C) \simeq \text{Fun}(O_0, C), \]
given by restriction to the fibre over $[0]$.  

Proof. Since we can replace $O$ by $O \times Δ^n$, it suffices to show that we have a natural equivalence of mapping spaces
\[ \text{Map}_{\text{Opd}_{\text{gns}}} (O, Δ_{\text{op}}^C) \simeq \text{Map}_{\text{Cat}_{\infty}} (O_0, C). \]
The generalized non-symmetric ∞-operad $O$ has an enveloping double ∞-category (see [Hau17, §A.8]) $\text{Env}(O)$ with a morphism of generalized non-symmetric ∞-operads $O \to \text{Env}(O)$ such that composition with this gives an equivalence between $O$-algebras in a double ∞-category $M$ and functors $\text{Env}(O) \to M$ that preserve all cocartesian morphisms. Moreover, $\text{Env}(O)_0 \simeq O_0$. It therefore suffices to prove that the natural map
\[ \text{Map}_{\text{Cat}_{\infty}/Δ_{\text{op}}} (M, Δ_{\text{op}}) \to \text{Map}_{\text{Cat}_{\infty}} (M_0, C) \]
is an equivalence, where $M$ is now a double ∞-category. If $M$ is the corresponding functor $Δ_{\text{op}} \to \text{Cat}_{\infty}$, then we can rewrite this as
\[ \text{Map}_{\text{Cat}_{\infty}/Δ_{\text{op}}} (M, Δ_{\text{op}}) \simeq \text{Map}_{\text{Fun}(Δ_{\text{op}}, \text{Cat}_{\infty})} (M, i_*C) \]
\[ \simeq \text{Map}_{\text{Cat}_{\infty}} (i^*M, C) \]
\[ \simeq \text{Map}_{\text{Cat}_{\infty}} (M_0, C), \]
as required. □

Remark 2.15. Let $M$ be a double ∞-category and $X$ an object of $M_0$. Then $X$ induces a functor $Δ_{\text{op}} \to Δ_{\text{op}}$ over $Δ_{\text{op}}$ (corresponding to the morphism of right Kan extensions $i_{Δ_{\text{op}}}(X) \to i_{Δ_{\text{op}}}(M_0)$, which preserves cocartesian morphisms. We can then define a monoidal ∞-category $M_{\Delta}^X$ as the pullback
\[ \begin{array}{ccc} M_\Delta^X & \longrightarrow & M \\ \downarrow & & \downarrow \\ Δ_{\text{op}} & \longrightarrow & Δ_{\text{op}}^M_0 \end{array} \]
of ∞-categories, which is also a pullback of cocartesian fibrations over $Δ_{\text{op}}$. If we think of objects of $M_1$ as “horizontal morphisms” in the double ∞-category, then this gives a monoidal structure on the ∞-category $M_1(X, X) := M_1 \times M_0 \times M_0, \{(X, X)\}$ of horizontal endomorphisms of $X$, given by composition of horizontal morphisms.
3. Category Objects as Algebras in Spans

In this section we will prove that category objects in an ∞-category C can be identified with associative algebras in the double ∞-category of spans in C. We first recall the construction of this double ∞-category, following [Hau18] (see also [Bar17, DK12, GR17] for alternative approaches).

Definition 3.1. Let \( \Sigma^n \) denote the partially ordered set of pairs \((i, j)\) with \(0 \leq i \leq j \leq n\), where \((i, j) \leq (i', j')\) if \(i \leq i'\) and \(i' \leq j\). These give a functor \( \Sigma^n : \Delta \to \text{Cat} \), where for \(\phi : [n] \to [m]\) the functor \( \Sigma^n \to \Sigma^m \) takes \((i, j)\) to \((\phi(i), \phi(j))\). If C is an ∞-category, we write \( \text{SPAN}^n(C) \to \Delta^{op} \) for the cocartesian fibration corresponding to the functor

\[
\text{Fun}(\Sigma^n, C) : \Delta^{op} \to \text{Cat}_\infty.
\]

If C is an ∞-category with pullbacks, we write \( \text{SPAN}^+(C) \) for the full subcategory of \( \text{SPAN}^n(C) \) spanned by the objects \( F : \Sigma^n \to C \) such that the canonical morphism

\[
F(i, j) \to F(i, i + 1) \times_{F(i + 1, i + 1)} \cdots \times_{F(j - 1, j - 1)} F(j - 1, j)
\]

is an equivalence for all \(i, j\).

Proposition 3.2 ([Hau18, Proposition 5.14]). For any ∞-category C with pullbacks, the restricted functor \( \text{SPAN}^+(C) \to \Delta^{op} \) is a double ∞-category.

Definition 3.3. For \(X \in C\), we can define a monoidal ∞-category \( C_{/X, X}^\otimes \) := \( \text{SPAN}^+(C)_{/X, X} \) as in Remark 2.15. This gives a monoidal structure on the ∞-category \( \text{C}_{/X, X} \simeq \text{SPAN}^+(C)_{/X, X}\) of objects of C equipped with two maps to \(X\). The tensor product of \(X \leftarrow Y \to X\) and \(X \leftarrow Z \to X\) is defined by the pullback \(X \leftarrow Y \times_X Z \to X\), and the unit is \(X \leftarrow X \to X\).

Definition 3.4. Let \(p : \hat{\Sigma} \to \Delta^{op}\) denote the cartesian fibration for the functor \(\Sigma^\bullet : \Delta \to \text{Cat}\). We can identify objects of \(\hat{\Sigma}\) with pairs \([n], (i, j)\) where \([n] \in \Delta\) and \(0 \leq i \leq j \leq n\); a morphism \([n], (i, j) \to [m], (i', j')\) is given by a morphism \(\phi : [n] \to [m]\) in \(\Delta\) such that \((i, j) \leq (\phi(i'), \phi(j'))\) in the partially ordered set \(\Sigma^n\). Note that this morphism is cartesian precisely when \((i, j) = (\phi(i'), \phi(j'))\).

Proposition 3.5. For any ∞-category C over \(\Delta^{op}\), there is a natural equivalence

\[
\text{Fun}(\Delta^{op}, \text{SPAN}^+(C)) \simeq \text{Fun}(\Delta^{op} \times \hat{\Sigma}, C).
\]

A functor \(F : \Delta^{op} \to \text{SPAN}^+(C)\) takes a morphism \(\phi : x \to y\) in \(\Delta\) to a cocartesian morphism in \(\text{SPAN}^+(C)\) if and only if the corresponding functor \(F' : \Delta^{op} \to \hat{\Sigma} \to C\) (which satisfies \(F'(x, (i, j)) \simeq F(x, (i, j))\)) takes all morphisms \((\phi, \gamma)\) where \(\gamma\) in \(\hat{\Sigma}\) is cartesian to equivalences in C.

Proof. The equivalence follows from [GHN17, Proposition 7.3], which identifies \(\text{SPAN}^+(C)\) with the cartesian fibration defined in [Lur09, Corollary 3.2.2.13(1)] by this universal property. The second statement then follows from the description of the cartesian morphisms in [Lur09, Corollary 3.2.2.13(2)].

Definition 3.6. We define a functor \(\Pi : \hat{\Sigma} \to \Delta^{op}\) by setting \(\Pi([n], (i, j)) := [j - i]\) and sending a map \([n], (i, j) \to [m], (i', j')\) lying over \(\phi : [m] \to [n]\) to the map \([j' - i] \to [j - i]\) given by \(t \mapsto \phi(t + i') - i\). (In other words, we restrict \(\phi\) to a map \(\{i', i' + 1, \ldots, j'\} \to \{i, i + 1, \ldots, j\}\).)

Definition 3.7. We can define another functor \(\Psi : \Delta^{op} \to \hat{\Sigma}\) by sending \([n]\) to \([n], (0, n)\) and a morphism \(\phi : [m] \to [n]\) in \(\Delta\) to the morphism \([m], (0, m) \to [m], (0, n)\) lying over \(\phi\). Observe that we have \(\Pi \Psi \simeq \text{id}_{\Delta^{op}}\). We can also define a natural transformation \(\eta : \Pi \Omega\) by taking \(\eta([n], (i, j))\) to be the map \([n], (i, j) \to (j - i), (0, j - i)\) lying over \(\rho_{i, j}\). We also have \(p_\Psi = \text{id}_{\Delta^{op}}\), and
a natural transformation $\epsilon: \Psi p \to \text{id}_{\mathcal{E}}$ given by the natural maps $([n], (0, n)) \to ([n], (i, j))$.

**Proposition 3.8.** The functor $\Pi: \hat{\Sigma} \to \Delta^{\text{op}}$ exhibits $\Delta^{\text{op}}$ as the localization of $\hat{\Sigma}$ at the set $I$ of cartesian morphisms that lie over inert maps in $\Delta^{\text{op}}$.

**Proof.** Let $W$ be the set of morphisms in $\hat{\Sigma}$ that are mapped to isomorphisms (i.e., identity morphisms) by $\Pi$. A morphism $([n], (i, j)) \to ([m], (i', j'))$ over $\phi: [m] \to [n]$ is in $W$ if and only if we have $j' = j - i$ and $\phi(i_t + t) = i + t$ for $t = 0, \ldots, j' - i'$.

In this case we have a commutative triangle

$$
\begin{array}{ccc}
([n], (i, j)) & \longrightarrow & ([m], (i', j')) \\
\downarrow & & \downarrow \\
([j - i], (0, j - i)) & &
\end{array}
$$

where the two diagonal morphisms are in $I$. Thus by the 2-of-3 property for equivalences, any functor $\hat{\Sigma} \to \mathcal{E}$ that takes the maps in $I$ to equivalences takes all the maps in $W$ to equivalences. The localizations of $\hat{\Sigma}$ at $I$ and $W$ are therefore the same. On the other hand, the components of the natural transformation $\eta$ are all in $W$, so using $\eta$ we see that composition with $\Psi$ is an inverse to

$$
\Pi^*: \text{Fun}(\Delta^{\text{op}}, \mathcal{E}) \to \text{Fun}_W(\hat{\Sigma}, \mathcal{E}),
$$

for any $\infty$-category $\mathcal{E}$, where $\text{Fun}_W(\hat{\Sigma}, \mathcal{E})$ denotes the full subcategory of functors $\hat{\Sigma} \to \mathcal{E}$ that take the morphisms in $W$ to equivalences. In other words, $\Pi$ exhibits $\Delta^{\text{op}}$ as the localization of $\hat{\Sigma}$ at $W$.

**Proposition 3.9.** Suppose $f: \mathcal{I} \to \Delta^{\text{op}}$ is a functor such that $\mathcal{I}$ has $f$-cocartesian morphisms over inert maps in $\Delta^{\text{op}}$. Then there is a functor $\Pi: \mathcal{I} \times_{\Delta^{\text{op}}} \hat{\Sigma} \to \mathcal{I}$ (where the fiber product is over $p$) lying over $\Pi$, which exhibits $\hat{\Sigma}$ as the localization of $\mathcal{I} \times_{\Delta^{\text{op}}} \hat{\Sigma}$ at the set $I_f$ of morphisms $(x, (i, j)) \to (x', (i', j'))$ such that $f(x) \to f(x')$ is inert, $I \to I'$ is cocartesian, and $(f(I), (i, j)) \to (f(I'), (i', j'))$ is cartesian.

**Proof.** The functor $\Psi: \Delta^{\text{op}} \to \hat{\Sigma}$ satisfies $p \circ \Psi \equiv \text{id}_{\Delta^{\text{op}}}$, and so induces a functor $\overline{\Psi}: \mathcal{I} \to \mathcal{I} \times_{\Delta^{\text{op}}} \hat{\Sigma}$. If $\overline{\Psi}: \mathcal{I} \times_{\Delta^{\text{op}}} \hat{\Sigma} \to \mathcal{I}$ is the projection (which lies over $p: \hat{\Sigma} \to \Delta^{\text{op}}$), then we have an equivalence $\overline{\Psi}^* \simeq \text{id}$, and we also get a natural transformation $\tau: \overline{\Psi}^* \to \text{id}$ over $\epsilon$.

Since the components of $\eta$: $\hat{\Sigma} \times \Delta^1 \to \hat{\Sigma}$ lie over inert morphisms in $\Delta^{\text{op}}$, there is a unique cocartesian lift of $\eta$ to a natural transformation $\overline{\eta}: \mathcal{I} \times_{\Delta^{\text{op}}} \hat{\Sigma} \to \mathcal{I} \times_{\Delta^{\text{op}}} \hat{\Sigma}$, where $\overline{\eta}_0$ is the identity and $\overline{\eta}(x, (i, j)) = (x, (i, j)) \to (x_{i, j}, (0, j - i))$ where $x \to x_{i, j}$ is a cocartesian morphism over $\rho_{i, j}$. (This follows from the lifting property obtained by combining [Lur09, Propositions 3.1.1.6 and 3.1.2.3].) We define $\overline{\Pi}: \mathcal{I} \times_{\Delta^{\text{op}}} \hat{\Sigma} \to \mathcal{I}$ to be the composite $\overline{\eta}$. Then $\overline{\Pi}$ lies over $p \Psi \Pi$, which is $\Pi$ as $p \Psi = \text{id}_{\Delta^{\text{op}}}$. We can identify $\overline{\eta}_1$ with $\overline{\Pi}^\Pi$ since

$$
\overline{\Pi}^\Pi \simeq \overline{\Psi} \overline{\eta}_1 \overline{\Pi} \overset{\overline{\eta}_1}{\longrightarrow} \eta_1
$$

is an equivalence. Moreover, $\overline{\eta} \overline{\Pi}^\Pi: \overline{\Psi} \overline{\eta}_1 \overline{\Pi} \overline{\Psi} \simeq \text{id}$ is an equivalence, being given by cocartesian morphisms over identities, hence $\overline{\Pi}^\Pi \simeq \overline{\Psi} \overline{\eta}_1 \overline{\Pi} \simeq \text{id}$.

Let $W_f$ be the set of morphisms in $\mathcal{I} \times_{\Delta^{\text{op}}} \hat{\Sigma}$ that are sent to equivalences by $\Pi$. If $(x, (i, j)) \to (x', (i', j'))$ is such a morphism, then we have a commutative square

$$
\begin{array}{ccc}
(x, (i, j)) & \longrightarrow & (x', (i', j')) \\
\downarrow & & \downarrow \\
(x_{i, j}, (0, j - i)) & \longrightarrow & (x'_{i, j'}, (0, j' - i'))
\end{array}
$$
where the vertical maps are in $I_3$. By the 2-of-3 property for equivalences, this means that any functor that takes the morphisms in $I_3$ to equivalences must take all morphisms in $W_4$ to equivalences. Thus the localizations of $J \times_{Δ_0} Σ$ at $I_3$ and $W_4$ are the same. The same argument as in the proof of Proposition 3.8 now shows that $\hat{Π}$ is exhibits $J$ as the localization of $J \times_{Δ_0} Σ$ at $W_4$.

Combining Proposition 3.5 with Proposition 3.9, we get:

**Corollary 3.10.** Suppose $f : J \to Δ^{op}$ is a functor such that $J$ has $f$-cocartesian morphisms over inert maps in $Δ^{op}$. Then there is a fully faithful functor of $∞$-categories

$$Fun(J, C) \hookrightarrow Fun_{Δ^{op}}(J, SPAN^+(C))$$

that identifies $Fun(J, C)$ with the functors that preserve cocartesian morphisms over inert morphisms in $Δ^{op}$. $\square$

Under this equivalence, functors $J \to SPAN^+(C)$ over $Δ^{op}$ that preserve cocartesian morphisms over inert maps correspond to functors $F : J \to C$ such that for all $x \in J$ over $[n] \in Δ^{op}$, the morphism

$$F(x) \to F(x_{0,1}) \times F(x_{1,1}) \cdots \times F(x_{n-1,n-1}) F(x_{n-1,n})$$

is an equivalence, where $x \to x_{i,j}$ denotes the cocartesian morphism over $ρ_{i,j}$. In particular, we have:

**Corollary 3.11.** Let $O \to Δ^{op}$ be a (generalized) non-symmetric $∞$-operad and $C$ an $∞$-category with pullbacks. Then there is a natural equivalence of $∞$-categories

$$Seg_O(C) \simeq Alg_O(SPLAN^+(C)).$$

**Proposition 3.12.** For $X \in C$, the fibre of $ev_{[0]} : Seg_O(C) \to Fun(O_0, C)$ at the constant functor with value $X$ is naturally equivalent to $Alg_O(C_{X,X})$.

**Proof.** Combining the equivalences of Corollary 3.11 and Lemma 2.14, we can identify $Seg_O(C) \to Fun(O_0, C)$ with the functor $Alg_O(SPLAN^+(C)) \to Alg_O(Δ^{op})$; the constant functor $O_0 \to C$ corresponds to the composite $O \to Δ^{op} \to Δ^{op}_{e}$ where the second morphism is the associative algebra in $Δ^{op}_{e}$ associated to $X \in C$. Since $Alg_O(\cdot)$ preserves limits, the fibre we want is given by the pullback square

$$\begin{array}{ccc}
Alg_O(C_{X,X}) & \longrightarrow & Alg_O(SPLAN^+(C)) \\
\downarrow & & \downarrow \\
* \simeq Alg_O(Δ^{op}) & \longrightarrow & Alg_O(Δ^{op}_{e}),
\end{array}$$

as required. $\square$

4. QUASI-UNITAL CATEGORY OBJECTS

Our goal in this section is to prove that “having identities” is a property of a category object. We will prove this by using the results of the previous section to reduce to the case of associative algebras, which has already been proved by Lurie. We begin by recalling Lurie’s result, which requires introducing some notation:

**Definition 4.1.** Let $Δ_{inj}^{op}$ denote the subcategory of $Δ^{op}$ containing only the injective maps; this is a non-symmetric $∞$-operad, and its algebras are non-unital associative algebras. If $O$ is a generalized non-symmetric $∞$-operad, we write $Alg_{nu}(O)$ for the $∞$-category $Alg_{Δ_{inj}^{op}}(O)$ of non-unital associative algebras in $O$. The inclusion $j : Δ_{inj}^{op} \to Δ^{op}$ is a morphism of non-symmetric $∞$-operads, and induces the expected forgetful functor $j^* : Alg(O) \to Alg_{nu}(O)$.
Definition 4.2. Let $\mathbb{V}^{\otimes} \to \Delta^{\text{op}}$ be a monoidal $\infty$-category. If $A$ is a non-unital associative algebra in $\mathbb{V}$, a quasi-unit for $A$ is a morphism $u: I \to A$ such that the composite

$$A \simeq I \otimes A \xrightarrow{u \otimes A} A \otimes A \xrightarrow{m} A,$$

where $m$ is the algebra multiplication, is equivalent to $\text{id}_A$, and similarly for the map with $u$ on the other side. We say that a non-unital algebra $A$ is quasi-unital if there exists a quasi-unit for $A$. If $A$ and $B$ are quasi-unital algebras, we say that a morphism $f: A \to B$ in $\text{Alg}_\text{nu}(\mathbb{V})$ is quasi-unital if $f \circ u$ is a quasi-unit for $B$, where $u: I \to A$ is a quasi-unit for $A$.

Warning 4.3. We emphasize that being quasi-unital is a property of a non-unital algebra. In particular, the data of a quasi-unit is not part of the structure of a quasi-unital algebra, we are merely asserting that it is possible to choose one.

Definition 4.4. Let $\text{Alg}_\text{qu}(\mathbb{V})$ denote the subcategory of $\text{Alg}_\text{nu}(\mathbb{V})$ whose objects are the quasi-unital algebras, and whose morphisms are the quasi-unital ones.

Theorem 4.5 (Lurie, [Lur17, Theorem 5.4.3.5]). If $\mathbb{V}$ is a monoidal $\infty$-category, then the functor $j^*: \text{Alg}(\mathbb{V}) \to \text{Alg}_\text{nu}(\mathbb{V})$ induces an equivalence $\text{Alg}(\mathbb{V}) \simeq \text{Alg}_\text{qu}(\mathbb{V})$ onto the quasi-unital subcategory.

For the rest of this section we fix an $\infty$-category $\mathcal{C}$ with finite limits. We can then extend the definitions above to category objects in $\mathcal{C}$:

Definition 4.6. A non-unital category object in $\mathcal{C}$ is a Segal $\Delta^{\text{op}}_{\text{inj}}$-object, i.e. a functor $\Delta^{\text{op}}_{\text{inj}} \to \mathcal{C}$ satisfying the same limit condition as a category object. We write $\text{Cat}_\text{nu}(\mathcal{C})$ for the $\infty$-category $\text{Seg}_{\Delta^{\text{op}}_{\text{inj}}}(\mathcal{C})$ of non-unital category objects.

Definition 4.7. Let $X: \Delta^{\text{op}}_{\text{inj}} \to \mathcal{C}$ be a non-unital category object. A quasi-unit for $X$ is a commutative diagram

such that the composite

$$X_1 \simeq X_0 \times_{X_0} X_1 \xrightarrow{u \times_{X_0} X_1} X_1 \times_{X_0} X_1 \simeq X_2 \xrightarrow{d_1} X_1$$

is equivalent to the identity, and similarly for the morphism with $u$ on the other side. We say a non-unital category object $X$ is quasi-unital if there exists a quasi-unit for $X$.

Remark 4.8. A non-unital category object $X$ in $\mathcal{C}$ is quasi-unital if and only if $X$ is quasi-unital when viewed as a non-unital associative algebra in $\mathcal{C}^{\otimes}_{/X_0, X_0}$. Any two quasi-units are therefore equivalent, by [Lur17, Remark 5.4.3.2].

Remark 4.9. The functor $\text{ev}_{[0]}: \text{Cat}_\text{nu}(\mathcal{C}) \to \mathcal{C}$ is a cartesian fibration by Proposition 2.10. Suppose $X$ is a non-unital category object and $u: X_0 \to X_1$ is a quasi-unit for $X$. For $f: Y \to X_0$ in $\mathcal{C}$, let $f^* X \to X$ denote the cartesian morphism in
Cat_{nu}(\mathcal{C}) over f. Then we have a diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X_0 \\
\downarrow{f^*u} & & \downarrow{u} \\
(f^*X)_1 & \xrightarrow{} & X_1 \\
\downarrow & & \downarrow \\
Y_0 \times Y_0 & \xrightarrow{} & X_0 \times X_0,
\end{array}
\]
where the morphism \(f^*u\) exists since the bottom square is cartesian. The morphism \(f^*u\) is then a quasi-unit for \(f^*X\). In other words, if \(X\) is quasi-unital then so is \(f^*X\) for any \(f: Y \rightarrow X_0\).

**Definition 4.10.** Suppose \(X\) and \(Y \in \text{Cat}_{nu}(\mathcal{C})\) are quasi-unital. A morphism \(\phi: X \rightarrow Y\) is quasi-unital if there exists a commutative diagram
\[
\begin{array}{ccc}
X_0 & \xrightarrow{\phi_0} & Y_0 \\
\downarrow{\Delta} & & \downarrow{\Delta} \\
X_1 & \xrightarrow{\phi_1} & Y_1 \\
\downarrow{(d_0,d_1)} & & \downarrow{(d_0,d_1)} \\
X_0 \times X_0 & \xrightarrow{\phi_0 \times \phi_0} & Y_0 \times Y_0,
\end{array}
\]
where \(u\) and \(v\) are quasi-units for \(X\) and \(Y\), respectively. We write \(\text{Cat}_{qu}(\mathcal{C})\) for the subcategory of \(\text{Cat}_{nu}(\mathcal{C})\) containing the quasi-unital objects and the quasi-unital morphisms between them.

**Remark 4.11.** From Remark 4.9 and the uniqueness of quasi-units we see that a morphism \(\phi: X \rightarrow Y\) is quasi-unital if and only if \(X \rightarrow \phi_0^*Y\) is quasi-unital. Moreover, the latter is quasi-unital if and only if it corresponds to a quasi-unital morphism between non-unital algebras in \(\mathcal{C}^\otimes_{/X_0,X_0}\).

**Proposition 4.12.** Suppose \(p: \mathcal{E} \rightarrow \mathcal{B}\) is a cartesian fibration, and \(\mathcal{E}_0\) is a subcategory of \(\mathcal{E}\) such that

(i) for \(x \in \mathcal{E}_0\) and \(f: b \rightarrow p(x)\) in \(\mathcal{B}\), the cartesian morphism \(f^*x \rightarrow x\) lies in \(\mathcal{E}_0\).

(ii) if \(x\) and \(y\) are objects of \(\mathcal{E}_0\) then a morphism \(\phi: x \rightarrow y\) in \(\mathcal{E}\) lies in \(\mathcal{E}_0\) if and only if \(x \rightarrow p(\phi)^*y\) lies in \(\mathcal{E}_0\).

Then \(p|_{\mathcal{E}_0}: \mathcal{E}_0 \rightarrow \mathcal{B}\) is a cartesian fibration, and a morphism in \(\mathcal{E}_0\) is cartesian if and only if its image in \(\mathcal{E}\) is cartesian.

**Proof.** Given \(x \in \mathcal{E}_0\) and \(f: b \rightarrow p(x)\) we must show that the cartesian morphism \(f^*x \rightarrow x\) is cartesian when viewed as a morphism in \(\mathcal{E}_0\). For \(y \in \mathcal{E}_0\) we have a commutative diagram
\[
\begin{array}{ccc}
\text{Map}_{\mathcal{E}_0}(y,f^*x) & \xrightarrow{} & \text{Map}_{\mathcal{E}_0}(y,x) \\
\downarrow & & \downarrow \\
\text{Map}_{\mathcal{E}}(y,f^*x) & \xrightarrow{} & \text{Map}_{\mathcal{E}}(y,x) \\
\downarrow & & \downarrow \\
\text{Map}_{\mathcal{B}}(p(y),b) & \xrightarrow{} & \text{Map}_{\mathcal{B}}(p(y),p(x)).
\end{array}
\]
Here the top square is cartesian by assumption (ii) and the bottom square is cartesian since \(f^*x \rightarrow x\) is cartesian. It follows that the composite square is cartesian, which completes the proof. \(\square\)
Combined with Remark 4.11 and Proposition 2.10, this gives:

**Corollary 4.13.** The functor $ev_{[0]} : \text{Cat}_{\text{qu}}(\mathcal{C}) \to \mathcal{C}$ is a cartesian fibration; a morphism in $\text{Cat}_{\text{qu}}(\mathcal{C})$ is cartesian if and only if its image in $\text{Cat}_{\text{nu}}(\mathcal{C})$ is cartesian. □

**Theorem 4.14.** The functor $j^* : \text{Cat}(\mathcal{C}) \to \text{Cat}_{\text{nu}}(\mathcal{C})$ induces an equivalence

$$\text{Cat}(\mathcal{C}) \cong \text{Cat}_{\text{qu}}(\mathcal{C})$$

onto the quasi-unital subcategory.

**Proof.** We have a commutative triangle

$$\begin{array}{ccc}
\text{Cat}(\mathcal{C}) & \xrightarrow{j^*} & \text{Cat}_{\text{nu}}(\mathcal{C}) \\
\text{Cat}(\mathcal{C}) & \xrightarrow{j^*} & \text{Cat}_{\text{qu}}(\mathcal{C}) \\
i\Delta^{op} & \xrightarrow{i\Delta^{op}} & \mathcal{C}
\end{array}$$

Here the diagonal functors are both cartesian fibrations by Proposition 2.10. We claim that $j^*$ also preserves cartesian morphisms. Using the description of the cartesian morphisms in Proposition 2.10, this amounts to observing that the canonical natural transformation $j^*i\Delta^{op, \ast} \to i\Delta^{op, \ast}_{\text{nu}}$ is clearly an equivalence. The functor $j^*$ obviously factors through the subcategory $\text{Cat}_{\text{qu}}(\mathcal{C})$, so by Corollary 4.13 we get a commutative triangle

$$\begin{array}{ccc}
\text{Cat}(\mathcal{C}) & \xrightarrow{j_{\text{nu}}^*} & \text{Cat}_{\text{nu}}(\mathcal{C}) \\
\text{Cat}(\mathcal{C}) & \xrightarrow{j_{\text{qu}}^*} & \text{Cat}_{\text{qu}}(\mathcal{C}) \\
i\Delta^{op} & \xrightarrow{i\Delta^{op}_{\text{nu}, \ast}} & \mathcal{C}
\end{array}$$

where the diagonal functors are cartesian fibrations, and the horizontal functor preserves cartesian morphisms. To prove that $j_{\text{nu}}^*$ is an equivalence, it therefore suffices to prove that for every object $X \in \mathcal{C}$ the functor

$$\text{Cat}(\mathcal{C})_X \to \text{Cat}_{\text{nu}}(\mathcal{C})_X$$

on fibres over $X$ is an equivalence. But by Remark 4.11 and Proposition 3.12 we can identify this with the restriction of $j^* : \text{Alg}(\mathcal{C}^{\otimes}_{/X,X}) \to \text{Alg}_{\text{nu}}(\mathcal{C}^{\otimes}_{/X,X})$ to a functor

$$\text{Alg}(\mathcal{C}^{\otimes}_{/X,X}) \to \text{Alg}_{\text{qu}}(\mathcal{C}^{\otimes}_{/X,X}),$$

which is an equivalence by Theorem 4.5. □

We can inductively define the $\infty$-category of $n$-uple category objects in $\mathcal{C}$ as

$$\text{Cat}^n(\mathcal{C}) := \text{Cat}(\text{Cat}^{n-1}(\mathcal{C}));$$

this corresponds to a full subcategory of $\text{Fun}(\Delta^{n, \text{op}})$. Similarly, we can define $	ext{Cat}_{\text{nu}}^n(\mathcal{C})$ and $\text{Cat}_{\text{qu}}^n(\mathcal{C})$. Applying Theorem 4.14 inductively, we get:

**Corollary 4.15.** The restriction functor $\text{Cat}^n(\mathcal{C}) \to \text{Cat}_{\text{nu}}^n(\mathcal{C})$ factors through an equivalence

$$\text{Cat}^n(\mathcal{C}) \cong \text{Cat}_{\text{qu}}^n(\mathcal{C}).$$

The $n$-uple category objects in the $\infty$-category $\mathcal{S}$ of spaces are known as $n$-uple Segal spaces. By imposing constancy conditions, we can restrict to the class of $n$-fold Segal spaces [Bar05], which model (the algebraic structure of) $(\infty, n)$-categories. This notion makes sense more generally:

**Definition 4.16.** A 1-fold Segal object in $\mathcal{C}$ is a category object. For $n > 1$, an $n$-fold Segal object is an $n$-uple category object $X$ such that $X_0, \ldots, X_n : \Delta^{n-1, \text{op}} \to \mathcal{C}$ is constant and $X_i, \ldots, X_n$ is an $(n-1)$-fold Segal object for all $i > 0$. 
If we write \(\text{Seg}^n(C)\) for the full subcategory of \(\text{Cat}^n(C)\) spanned by the \(n\)-fold Segal objects, and \(\text{Seg}^n_{\text{nu}}(C)\) for the analogously defined non-unital \(n\)-fold Segal objects, restricting the equivalence of Corollary 4.15 gives:

**Corollary 4.17.** The restriction functor \(\text{Seg}^n(C) \to \text{Seg}^n_{\text{nu}}(C)\) factors through an equivalence

\[
\text{Seg}^n(C) \xrightarrow{\sim} \text{Seg}^n_{\text{qu}}(C),
\]

where \(\text{Seg}^n_{\text{qu}}(C)\) is the subcategory of non-unital \(n\)-fold Segal spaces that are quasi-unital (when viewed as \(n\)-uple category objects) and quasi-unital morphisms between them.

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