Black hole complementarity from AdS/CFT

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Abstract

We study a simple version of the AdS/CFT (anti-de Sitter spacetime/Conformal Field Theory) correspondence, where operators have integer conformal dimensions. In this model, bulk causality follows from boundary analyticity, even in nontrivial black hole backgrounds that break the underlying conformal symmetry. This allows a natural set of quasi-local bulk observables to be constructed. Estimates of finite central charge corrections to semiclassical correlators are made. These corrections are used to determine the regime of validity of effective field theory in the bulk spacetime. The results are consistent with black hole complementarity.

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I. INTRODUCTION

The Anti-de Sitter/Conformal Field Theory correspondence [1] provides us with one of the most promising approaches to developing a nonperturbative formulation of string theory. In the present work, our goal is to use known properties of conformal field theory to learn about quantum gravity in an asymptotically anti-de Sitter background. One of the necessary steps toward this goal is to learn as much as possible about the mapping between conformal field theory operators and bulk observables. In particular, we would like to understand how bulk causality follows from properties of the boundary theory in a semiclassical limit, and how quantum effects correct these results.

The black hole information paradox [2] arises as an incompatibility between unitary quantum theory, and the purely thermal emission of Hawking radiation that appears in a semiclassical approximation. With the advent of AdS/CFT, we believe that the underlying quantum theory is a unitary conformal field theory [3, 4]. The essence of the paradox then becomes understanding why effective field theory breaks down in a region of spacetime where curvatures are small [5] (for example a region that includes points inside the horizon, and points outside in a region with a significant amount of Hawking radiation). Progress on the questions raised above can thus lead to a resolution of the paradox.

In a series of papers, the mapping from boundary CFT correlators to on-shell bulk correlators has been constructed via a kind of inverse LSZ [6] method [7, 8, 9, 10, 11] for Lorentzian signature anti-de Sitter spacetime, and the results have been generalized to the three-dimensional black hole of BTZ [12] in [9, 10]. It is clear from this construction that the large central charge limit of the boundary correlators reproduces the expected semiclassical bulk correlators. This was shown in the free limit in [9]. CFT correlators have been extracted from the interacting supergravity theory in [13, 14] (i.e. perturbatively in $1/N$ in the case of 4d $SU(N)$ CFT).

In the present work, the first goal is to build on the results of [7, 8, 9, 10] and develop a model where bulk causality can be derived for a more general class of states in the CFT, or asymptotically AdS backgrounds on the gravity side. To make progress on this, we make the simplifying assumption that the conformal weights in the CFT take integer values. Holomorphic CFT’s and the extremal CFT’s described in [15] provide examples of this type. These CFT’s have retarded boundary propagators that are non-vanishing only at
light-like separations. Following the prescription of \[7, 8, 9, 10\] we extract a straightforward geometric picture that determines when a commutator of quasi-local bulk observables is non-vanishing. At large central charge this matches with bulk causality, even for the BTZ black hole background.

The next goal is to take some steps toward understanding nonperturbative corrections from the bulk perspective. This immediately raises a number of questions of principle. A finite central charge CFT is not expected to have an exact continuum bulk spacetime interpretation. However for large central charge, \(c\), we expect to have a host of corrections perturbative in \(1/c\) that will modify the boundary-bulk map of \[7, 8, 9, 10\]. In general we do not expect these type of corrections to drastically modify the picture bulk causality that emerges from the above considerations.

A qualitative change comes when we consider effects nonperturbative in \(1/c\). A useful discussion of these effects on the boundary CFT correlators arising from a black hole background can be found in \[16, 17, 18, 19, 20, 21\]. The upshot of this discussion is that a thermal correlator decays exponentially, with a timescale of order the inverse temperature, until it reaches a relative magnitude of order \(e^{-S_{bh}}\), where \(S_{bh}\) is the Bekenstein-Hawking entropy of the black hole. Beyond this time, nonperturbative effects dominate the boundary two-point function. This can be incorporated into the boundary-bulk mapping, as described in preliminary form in \[10\]. The idea is simply to place a time cutoff on the boundary time integrals needed to construct quasi-local bulk operators. The regime of validity of an effective field theory based on these operators may then be extracted simply by asking when the results start to significantly differ from the semiclassical results.

One can imagine extending this construction to states more general than black holes. An issue that arises is defining when a family of states in the CFT corresponds to a smooth region of bulk spacetime. Certainly we expect one needs to include corrections to the “smearing functions” that appear in the mapping to take into account back-reaction of the state on the bulk spacetime. However finite \(c\) corrections can still lead to no semiclassical bulk description. It seems the natural way to proceed is to try to optimize corrections to the construction, and try to find finite regions of bulk spacetime for which a large set of low energy observables can be obtained, and their correlators reproduced to reasonable accuracy by a local bulk effective action. While we can expect the local effective action to respect general covariance, we do not expect this of the finite \(c\) corrections. In this sense we define
continuous bulk spacetime by a large set of correlators that can be extracted from the CFT, and reproduced to some reasonable accuracy by a local effective action. The local effective action then gives a complete summary of the, necessarily approximate, bulk interpretation of the physics. In general, one may need a number of distinct effective actions defined on overlapping regions of bulk spacetime to maximally extend the bulk description. However there is no guarantee that this set of effective actions can be replaced by a single local action, as would be expected by general covariance.

II. REVIEW OF BULK-BOUNDARY MAPPING FOR PURE ADS

A. SUGRA two-point function of massive scalar field

For simplicity, we will restrict our considerations to bulk observables corresponding to massive scalar fields. In general we have for the bulk Wightman function in $AdS_d$ [14, 22, 23]

$$G(x, x') = \frac{R^{2-d}\Gamma(\Delta)}{2^{\Delta+1}\pi^{(d-1)/2}\Gamma(\Delta - \frac{d-3}{2})}\Gamma(\Delta + \frac{1}{2})^{-\Delta} 2F_1(\frac{\Delta}{2}, \frac{\Delta + 1}{2}, \Delta; \frac{1}{\sigma^2}),$$

where

$$\Delta = \frac{d-1}{2} + \sqrt{\left(\frac{d-1}{2}\right)^2 + m^2 R^2},$$

and

$$\sigma = \frac{X_\mu Y_\nu \eta^{\mu\nu}}{R^2},$$

where $\eta^{\mu\nu}$ is the $d + 1$-dimensional Minkowski metric. The AdS spacetime is realized as an embedding where

$$X_\mu X_\nu \eta^{\mu\nu} = (\vec{X})^2 - (X^0)^2 - (X^d)^2 = -R^2.$$

In global coordinates

$$ds^2 = \frac{R^2}{\cos^2 \rho} (-d\tau^2 + d\rho^2 + \sin^2 \rho d\Omega_{d-2}^2),$$

we have the isometry invariant distance functions

$$\sigma(x, x') = \frac{\cos(\tau - \tau') - \sin \rho \sin \rho' \cos(\Omega - \Omega')}{\cos \rho \cos \rho'},$$

with $\Omega - \Omega'$ the angular separation on the sphere. The operator ordering is taken care of with a $\tau \rightarrow \tau - i\epsilon$ prescription. This expression is correct for global AdS. To generalize it
to the covering space of $\text{AdS}$, we need to take into account that $\sigma$ changes sign as one moves in the time direction. To do this we introduce the idea of a winding number, defined as

- $n(x, y) = 0$ if $x$ can be continuously deformed to $y$ without changing the sign of $\sigma$.
- $\Delta n(x, y) = 1$ every time $\sigma$ changes sign, for $x$ to the future of $y$.

Then the full expression, valid on the covering space of $\text{AdS}$ is

$$G = e^{i\pi n}G_{n=0},$$

which is obtained by analytically continuing $\Box$

This can be simplified for $\text{AdS}_3$ to this algebraic function of the invariant distance

$$G(x, x') = \frac{\sigma^{\Delta-2}(1 - \sqrt{1 - \sigma^2})^{\Delta-1}}{4\pi R \sqrt{1 - \sigma^2}}.$$

(1)

**B. Reproducing using smearing functions**

In this section we will show how (1) can be reproduced, following the prescription of [7, 8, 9, 10]. We will use this as an example to more fully explain the impact of the $i\epsilon$ prescription on this construction. It is more convenient to work in Poincare coordinates

$$ds^2 = \frac{R^2}{Z^2} (-dT^2 + dX^2 + dZ^2),$$

and we assume both bulk points are in the same coordinate patch. In these coordinates, the invariant distance function takes the form

$$\sigma(x, x') = \frac{-T^2 + X^2 + Z_1^2 + Z_2^2}{2Z_1Z_2}.$$

We begin with eqn. (16) of [9]. This expresses an on-shell local bulk operator as a boundary operator, smeared over an analytic continuation of the boundary manifold

$$\phi(T, X, Z) = \frac{\Delta - 1}{\pi} \int_{T'^2 + Y'^2 < Z^2} dT' dY' \left( \frac{Z^2 - T'^2 - Y'^2}{Z} \right)^{\Delta - 2} \phi_0(T + T', X + iY').$$

(2)

With the boundary correlator (correcting a typo in [9])

$$\langle \phi_0(T, X)\phi_0(0, 0) \rangle_{\text{CFT}} = \frac{1}{2\pi R} \frac{1}{(X^2 - T^2)\Delta},$$

(3)

we obtain
\( \langle \phi(T, X, Z_1)\phi(0, 0, Z_2) \rangle = -\frac{(\Delta - 1)^2}{2\pi^3 R} \int_0^{Z_1} dr_1 \int_0^{Z_2} dr_2 \oint_{C_1} dz_1 \oint_{C_2} dz_2 \)

\[
\frac{r_1 r_2}{z_1 z_2} \left( \frac{Z_1^2 - r_1^2}{Z_1} \right)^{-2} \left( \frac{Z_2^2 - r_2^2}{Z_2} \right)^{-2} \frac{1}{((X - T + i\epsilon - r_1/z_1 + r_2/z_2) (X + T - i\epsilon + r_1 z_1 - r_2 z_2))^\Delta},
\]

having performed a change of variables to \( T' + iY' = rz \).

We take \( \Delta \) to be integer valued, to avoid issues with branch cuts of the integrand. For scalar fields, integer conformal dimensions are guaranteed with sufficient supersymmetry. Without supersymmetry purely integer conformal dimensions have arisen in the proposal for pure gravity based on an extremal conformal field theory [15]. At the end of the day, one might try to analytically continue in \( \Delta \), as has been discussed in [13, 14, 24]. However we will not consider this approach in the present work.

The contours of integration \( C_1, C_2 \) must be handled with some care, and are defined as follows. The equation (2) can be used as is, provided the two integration patches on the boundary are non-overlapping. This means the contours \( C_1 \) and \( C_2 \) follow the unit circle. The \( i\epsilon \) prescription then ensures the resulting integrals are well-defined for all choices of bulk point via analytic continuation in \( T, X, Z_1 \) or \( Z_2 \). This implies the contours must be deformed as the singularities of the integrand move, to avoid crossing [41].

With this prescription, the integral (4) reproduces the expression (1), as we see from the following argument. First we note that it is straightforward to check for particular values of \( \Delta \) and general bulk points, which we have checked for \( \Delta \) ranging from 2 through 10. For general integer values of \( \Delta \) we have also checked agreement when one bulk point approaches the boundary (generalizing a calculation of [9]) and when two bulk points coincide. These calculations are shown in appendix A and B. With the \( i\epsilon \) prescription, the integral is analytic in the coordinates, so because it agrees at the singular points, and at the boundary, it will agree for general bulk points.
III. BULK CAUSALITY FROM BOUNDARY ANALYTICITY

A. Pure AdS Spacetime

The vacuum expectation value of the commutator of boundary fields takes a particularly simple form when $\Delta$ is integer-valued

$$\langle [\phi_0(x), \phi_0(x')] \rangle = 0,$$

unless $x$ and $x'$ are light-like separated on the boundary. This follows straightforwardly from the form (3), which is exact due to conformal invariance. This leads to a simple geometric picture of the commutator of two on-shell bulk operators when we use the boundary representation (2).

Consider the expectation value of the commutator of two fields in pure $AdS_3$ expressed as a difference between two integrals of the form (4), with $i\epsilon$’s of differing signs. Nonvanishing contributions will arise only from the singularities of the integrand, which typically lead to branch cuts in the integral. This may be analyzed using the standard method of Landau equations, as explained for example [25]. This boils down to finding the regions in parameter space where the singularities of the integrand pinch the contour of integration, as $\epsilon \to 0$.

To proceed, one defines a comparison function with singularities at the same position as the integral under study (4)

$$F(X, T, Z_1, Z_2) = \int_{H} \prod_{i=1}^{4} \frac{1}{S_1 S_2}$$

with analytic functions

$$S_1 = X + T + r_1 z_1 - r_2 z_2, \quad S_2 = X - T - r_1 z_1 + r_2 z_2$$

and

$$w_1 = z_1, \quad w_2 = z_2, \quad w_3 = r_1, \quad w_4 = r_2$$

and the integration hypersurface $H$ defined by the product of the two disc regions. For the purposes of this analysis the fixed singularities at $r_1 = 0$, $r_2 = 0$, $r_1 = Z_1$, $r_2 = Z_2$ can be neglected, provided $\Delta \geq 2$. Using a Feynman parameter we can then write

$$F(X, T, Z_1, Z_2) = \int_{H} dz_1 dz_2 dr_1 dr_2 \int d\alpha_1 d\alpha_2 \frac{\delta(\alpha_1 + \alpha_2 - 1)}{(\alpha_1 S_1 + \alpha_2 S_2)^2}.$$
The conditions for a generic singularity can then be expressed as

\[ S_1 = S_2 = 0, \quad \text{and} \quad \alpha_1 \frac{\partial S_1}{\partial w_i} + \alpha_2 \frac{\partial S_2}{\partial w_i} = 0 \quad \forall i \quad \text{and for some} \quad \alpha_1, \alpha_2. \]

There exist no solutions for both \( \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0 \) so pinching cannot happen at some generic point on the interior of \( H \).

Another possibility is that the surface defined by \( S_1 = 0 \) develops a conical singularity, which then traps the integration surface \( H \). This happens when

\[ S_1 = 0, \quad \frac{\partial S_1}{\partial w_i} = 0, \quad \forall i. \]

However this only has solutions at the endpoints when \( r_1 = 0 \) and \( r_2 = 0 \). We will come to this case next. A similar analysis for \( S_2 \) shows conical trapping does not occur in this case.

To treat the case when singularities appear on the edge of the integration region \( H \), we introduce additional analytic functions \( \tilde{S}_r \) so that \( \tilde{S}_r = 0 \) defines the boundary. In the case at hand, we have a rectangle in the \((r_1, r_2)\)-plane. Let us begin by considering the case that a singularity appears on an edge, away from a corner. Take for example \( \tilde{S}_1 = r_1 - Z_1 \).

The comparison function may be generalized to include an additional \( \tilde{\alpha}_1 \) factor, along with an additional Feynman parameter \( \tilde{\alpha}_1 \). The different cases considered above may then be reduced to solving the equations \[25\]

\[ \alpha_k S_k = 0, \quad \forall k \quad \text{and} \quad \tilde{\alpha}_1 \tilde{S}_1 = 0, \quad \text{and} \quad \frac{\partial}{\partial w_i} \left( \tilde{\alpha}_1 \tilde{S}_1 + \sum_k \alpha_k S_k \right) = 0 \quad \forall i. \quad (5) \]

It is then easy to see that no solution is possible, except at the corners of the rectangle.

To study the corners, we will use two \( \tilde{S}_r \) factors in our comparison function. At first sight, it appears that singularities might appear from the \( r_1 \rightarrow 0 \) or \( r_2 \rightarrow 0 \) region. However this is an artifact of our coordinate choice (see below (4)), as can be easily checked by switching to Cartesian coordinates. Therefore we need only examine the corner \( r_1 = Z_1 \) and \( r_2 = Z_2 \).

Solving the equations \[4\] yields two nontrivial solutions

\[ T^2 - X^2 = (Z_1 - Z_2)^2, \quad z_1 = z_2 \]
\[ T^2 - X^2 = (Z_1 + Z_2)^2, \quad z_1 = -z_2 \]

The first case occurs when a point on the edge of disc 1 becomes light-like separated from the edge of disc 2 as shown in figure 1. The second case occurs after disc 1 has moved outside
Figure 1: Regions of integration on the analytic continuation of the boundary. The commutator becomes non-trivial as soon as one disc edge touches (is light-like separated from) the other edge. This corresponds to bulk time-like separation, with $-1 < \sigma < 1$.

Having identified the possible positions of the singularities, we need to check whether these actually correspond to pinches of the hypersurface $H$ under the $T \to T - i\epsilon$ deformation, or whether the singularities harmlessly coalesce as $\epsilon \to 0$. For both cases we find indeed the contour is pinched as $\epsilon \to 0$ and is responsible for a branch cut in the integral (4).

This then gives a simple geometric picture of how the boundary theory encodes bulk causality. Boundary causality guarantees a vanishing commutator at spacelike boundary separations. Moreover for integer $\Delta$ the commutator will also vanish at timelike separations, and will only be non-vanishing for light-like separations. The bulk radial coordinate is encoded in the size of the boundary disc. Because the only non-vanishing contribution comes from a disc edge we reproduce the expected non-vanishing of the commutator at bulk timelike separations.
Figure 2: The commutator vanishes as one disc moves outside the light-cone of the other. This corresponds to bulk points that cannot be connected by geodesics, $\sigma < -1$.

One might be puzzled by the vanishing of the commutator at bulk timelike separations, when the discs no longer intersect. This corresponds to the case $\sigma < -1$. As is clear from the Wightman function (1) the commutator will vanish in this case. Geometrically, this corresponds to bulk points that cannot be connected by timelike geodesics. These regions appear because the negative cosmological constant “repels” timelike geodesics from spacelike infinity. The commutator is only non-vanishing at timelike separations within a sequence of causal diamonds, corresponding to $-1 \leq \sigma \leq 1$, as shown in figure 3.
Figure 3: The white regions of the AdS Penrose diagram indicate where the commutator is non-vanishing.

B. Finite temperature and the BTZ Black Hole

Now we wish to consider the case of the BTZ black hole. It is most convenient to now switch to Rindler coordinates to describe $AdS_3$,

$$ds^2 = \frac{R^2}{r^2 - r_+^2} dr^2 - \frac{r^2 - r_+^2}{R^2} dt^2 + r^2 d\phi^2,$$

where $\phi \in \mathbb{R}$ for pure AdS. We obtain the BTZ black hole (with vanishing angular momentum) simply by periodically identifying $\phi \sim \phi + 2\pi$. The parameter $R$ is the radius of curvature of the AdS space, while $r_+$ represents the position of the Rindler (or BTZ) horizon. Correlation functions in the Hartle-Hawking vacuum are obtained by considering the Euclidean geometry periodically identifying in imaginary time $t \sim t + i\beta$, with $\beta = 1/T_H = 2\pi R^2/r_+$ the inverse Hawking temperature.

Let us briefly recall the results of [9] on the bulk-boundary mapping in this case. For spacetime points in the right Rindler wedge (i.e. the right triangle in figure 3), the map
from boundary to bulk operators can be written in the form

$$\phi(t, r, \phi) = \frac{(\Delta - 1)2^{\Delta - 2}}{\pi R^3} \int_{\text{spacelike}} dx \, dy \lim_{r' \to \infty} (\sigma/r')^{\Delta - 2} \phi_0^R(t + x, \phi + iy)$$

(7)

where $\phi$ is the bulk operator, $\phi_0^R$ is the boundary operator associated with the right Rindler patch, and $\sigma$ is the analytic continuation of the invariant distance

$$\sigma(t, r, \phi|t + x, r', \phi + iy) = \frac{rr'}{r_+^2} \left[ \cos \frac{r_+ y}{R} + \left( \frac{r_+^2}{r^2} - 1 \right)^{1/2} \sinh \frac{r_+ x}{R^2} \right],$$

with signs determined by the right/left Rindler patch. The relation (7) may also be generalized to points inside the horizon, where now a piece coming from the boundary of the left Rindler patch is also needed

$$\phi(t, r, \phi) = \frac{(\Delta - 1)2^{\Delta - 2}}{\pi R^3} \left[ \int_{\sigma > 0} dx \, dy \lim_{r' \to \infty} (\sigma/r')^{\Delta - 2} \phi_0^R(t + x, \phi + iy) \\
+ \int_{\sigma < 0} dx \, dy \lim_{r' \to \infty} (-\sigma/r')^{\Delta - 2} (-1)^{\Delta} \phi_0^L(t + x, \phi + iy) \right]$$

(8)

One can choose to represent the contribution from the left Rindler patch as a contour integral in the complex $t$-plane of the CFT corresponding to the right boundary, as discussed in [9, 26]. This is achieved using the relation

$$\phi_0^L(t, \phi) = \phi_0^R(t + i\pi R^2/r_+, \phi)$$

(9)

relating boundary fields in the left and right Rindler patches.
The identifications used in constructing the BTZ geometry break the Lorentz invariance of the boundary conformal field theory. Nevertheless, since the correlators may be defined as weighted averages in the original conformal field theory

$$\langle \phi(x)\phi(x') \rangle_T = \sum_M \langle M|\phi(x)\phi(x')|M \rangle \frac{e^{-\beta E_M}}{Z(T)}$$  \hspace{1cm} (10)$$

with

$$Z(T) = \sum_M e^{-\beta E_M}$$

they inherit its causal structure. Thus it is guaranteed that commutators will vanish at spacelike separations on the boundary. In the large $c$ limit, likewise the bulk commutator will vanish at spacelike separations.

To extend the results of the previous section to the finite temperature case the periodicity in the $\phi$ direction must be taken into account. The supergravity correlators (and their boundary limits) may be obtained via an image sum

$$\langle \phi(x)\phi(x') \rangle_{BTZ} = \sum_{n=-\infty}^{\infty} \langle \phi(x)\phi(x' + 2\pi n e_\phi) \rangle_{AdS}$$

where $e_\phi$ is a unit vector in the $\phi$ direction. Working on the covering space, we see that whenever an image is light-like separated from another point, there will be in general a non-vanishing contribution to the commutator. In the range $|\delta t| < 2\pi R$ the commutator will vanish at timelike separations on the boundary as in the previous section.

An exact CFT that exhibits the same behavior is simply a free massless boson in two dimensions. The basic chiral correlators are functions of $t/R + \phi$, so are not only periodic in $\phi$ and in imaginary time, but also under $t \rightarrow t + 2\pi R$. Therefore the corresponding commutators will also be periodic in time. These will vanish for the range of times $|\delta t| < 2\pi R$, since the only dependence is through $t/R + \phi$. This idea may be extended to non-chiral correlators built out of products of chiral and anti-chiral factors, since each factor behaves as above.

In the following we will assume there exist CFT’s dual to gravity in $AdS_3$ that satisfy this criterion, that commutators vanish at spacelike separations, and have the periodic structure in the timelike direction noted above.\[42\] With this in mind, the results of Section III A carry over to the finite temperature case.

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IV. FINITE CENTRAL CHARGE

We have seen that at $c = \infty$ there is a simple relation between the analytic and causal properties of the boundary correlators and bulk causality, even in a nontrivial BTZ black hole background. Now we wish to incorporate the effect of finite $c$ corrections. A prescription for estimating the magnitude of such corrections was given in [10]. The essential idea was to follow the above prescription to determine approximate local bulk observables, but put a cutoff on the range of the time integral, restricting to $\text{Re}|\delta t| < t_c$. In the present work, we realize this as a cutoff on the radius of the disc in the $(t, \text{Im}\phi)$ plane, to preserve the isometries of the analytically continued geometry. The timescale $t_c$ is determined by examining when finite $c$ corrections can become of comparable magnitude to the semiclassical boundary correlator.

A useful discussion of these effects can be found in [17, 18, 19]. Initially the boundary correlator decays exponentially as $\exp(-\Gamma t)$ with $\Gamma \sim T_H$, the Hawking temperature. Non-perturbative effects, which we cannot hope to describe by some straightforward modification of our quasi-local operators [43], become of comparable magnitude when

$$e^{-\Gamma t_c} \sim e^{-S_{bh}}.$$  \hspace{2cm} (11)

With this cutoff prescription, the difference between using exact finite $c$ CFT correlators and semiclassical large $c$ correlators will be of order $e^{-S_{bh}}$. Moreover the correlators will have the same analytic structure for light-like separations on the boundary, which is the relevant property for determining when the commutator of bulk operators is nontrivial.

The entropy of the BTZ black hole is

$$S_{bh} = \frac{2\pi r_+}{4G} = \frac{\pi r_+ c}{3R},$$

using the identification between Newton’s constant and CFT central charge $c = 3R/2G$ [27, 28]. For the BTZ black hole, this yields the time scale

$$t_c = \frac{2\pi^2 c R}{3}.$$  \hspace{2cm} (12)

We can now investigate regions of the spacetime, as shown in figure 4, where the cutoff finite $c$ correlators can be reproduced to good accuracy by some local gravity action. Let us begin with point 1 in figure 4 a point outside the horizon of the BTZ black hole. A bulk
operator located as such a point is represented by integration over a finite size disc on the right CFT, according to (7). The cutoff is irrelevant provided

$$r > r_c = r_+ + 2r_+ e^{-4\pi^2 c r_+/3R}.$$  \hspace{1cm} (13)

This provides us with an indication that AdS/CFT is reproducing a stretched horizon as in the membrane paradigm picture of black hole evaporation \cite{29, 30}. This radius corresponds to a proper distance

$$ds = R e^{-2\pi cr_+/3R}$$

from the horizon (in the radial direction). For a large black hole, this is shorter than a Planck length. In this case, we would expect perturbative gravity interactions to prevent us probing such short length scales with probes built out of gravitational fields. Hence the effective thickness where one would expect to probe non-perturbative effects should extend out to Planck scales. The cross-over, when the free two-point computation yields a Planck scale stretched horizon, happens when $r_+ \sim G \log c$.

For points satisfying the bound (13) the analyticity results of the previous section will carry over, regardless of whether we use the exact finite $c$ CFT correlator, or the semiclassical boundary correlator. Therefore even at finite $c$ bulk correlators built to the right of the light-sheet emanating from the points $r = r_c$ will display commutators exactly vanishing at spacelike separations. The correlators will generically differ from the semiclassical correlators by terms of order $e^{-S_{bh}}$, since for $t < t_c$ this is the expected order of magnitude of the difference between semiclassical boundary correlators and the exact finite $c$ correlators. This also agrees with Page’s results on the expected initial information outflow from a black hole \cite{31, 32}. This picture is what one expects from black hole complementarity \cite{30} - the region outside the stretched horizon behaves as if the black hole was replaced by a hot membrane along the stretched horizon, with physics in this region causal as usual. It is also worth noting effective field theory formulated in the region $r > r_c$ (with appropriate boundary conditions) will have a discrete spectrum, like that of the CFT. If the region up to the horizon is included, the spectrum in the bulk becomes continuous.

Now let us investigate what happens for bulk operators with large time separations $t > t_c$, such as a correlator between points 1 and 3 in figure 4. Provided the points are radii $r > r_c$, the cutoff will not be relevant, and the commutator will vanish at spacelike separations as expected. However once one point, e.g. point 1 enters the stretched horizon region.
(r < r_c) this will no longer be the case. Now we expect a region of bulk spacetime when
the disc regions on the boundary intersect (supposing for simplicity δφ = 0) near t = t_c.
In this region the commutator will be nontrivial, even when the bulk points are spacelike
separated. The magnitude of the commutator can be estimated by using the semiclassical
boundary correlator, and computing the bulk correlator with and without the cutoff. This
generically yields a magnitude of order e^{-S_{bh}}. Again, this is consistent with Page’s estimate
of initial information outflow in black hole evaporation [31]. Since a large BTZ black hole is
an eternal black hole supported by an incoming flux of radiation, this is also of the correct
order of magnitude to describe unitary evolution of the black hole [4]. We conclude effective
field theory will give accurate results for correlators with small numbers of local operators,
however if the number becomes of order e^{S_{bh}}, corresponding to a measurement of the majority
of the radiation outside the black hole, then we expect to find special correlators where the
errors add coherently, and the expected error will be of order 1.

One can play the same game with one point outside the black hole and one point inside,
for example points 1 and 2 of figure 4. Now the cutoff prescription must be applied to
expressions of the form (8), with the left CFT mapped into the right via the antipodal
map (9). The analysis is essentially the same as that of points 1 and 3 discussed above,
when one point sits inside the stretched horizon, except now the region of integration on
the boundary includes a pair of disc regions for the operator inside, separated in imaginary
time. The results above can be easily extended to this case, and again we generically expect
commutators of order e^{-S_{bh}}. We do not expect all correlators with large numbers of operators
near point 3 to be correctly reproduced by effective field theory due to expected errors of
order 1.

Finally we can consider the correlator between two points inside the horizon, such as
points 2 and 4 of figure 4. When the points are spacelike separated, the boundary disc
regions can intersect, leading to a nontrivial commutator of magnitude e^{-S_{bh}}. As discussed
in [33] effective field theory can still be usefully formulated inside the horizon, since such
small effects are not operationally observable. This relies on the fact that measurements
inside the horizon have an intrinsic accuracy since they must be performed before timelike
geodesics hit the singularity (which happens in proper time less than πR/2, shorter than
the light-crossing time for a large black hole).

To sum up, the cutoff prescription can be used to determine in what regions of spacetime,
or for what class of correlators, effective field theory can be expected to break down. The estimates of the magnitudes of the finite $c$ effects lead to a bulk picture compatible with unitary evolution of the black hole as described by the conformal field theory. The essential new point is that effective field theory cannot simply be maximally extended over a spacetime region as general covariance would suggest. Rather effective field theory can only be properly formulated on patches of spacetime where finite $c$ effects are small, with some given intrinsic accuracy.

V. SPECULATIONS ON SMALL BLACK HOLES

To achieve a satisfying resolution of the black hole information problem, one would like to see that results similar to those described above could be generalized to the case of small black holes in higher-dimensional AdS, that actually evaporate away completely. One of the main difficulties here is finding a simple way to identify these states in the CFT. Nevertheless, let us assume the basic ideas carry over, and see what picture emerges.

The bulk to boundary mapping for higher dimensional pure AdS spacetime has been studied in [7, 8, 9]. Generalizing to eternal black hole states is expected to be conceptually straightforward, but technically more complicated, with fewer explicit expressions available. In higher dimensions, the cutoff prescription will involve integrals over a sphere on the analytic continuation of the boundary. Nevertheless, the basic analyticity arguments presented above should generalize.

For low temperatures, thermal AdS is the geometry giving rise to the largest entropy. In this background, the boundary correlator will oscillate with time, rather than having exponential falloff. Small black holes will be a minority subset of this low temperature canonical ensemble. Nevertheless, let us suppose we can construct suitable “chemical potentials” to filter out thermal AdS, and allow a set of small black holes to dominate the modified ensemble. We will assume the boundary correlator in this ensemble of small black holes exhibits thermal behavior, namely

$$\langle \phi_0(t) \phi_0(0) \rangle \sim e^{-T_H t}$$

which should be reasonable if a quasistatic approximation can be applied to the ensemble of CFT states allowing us to assume approximate ergodicity. The timescale at which the semiclassical correlator receives corrections of relative order 1 can be estimated via the same
argument as before. This leads to

\[ t_c \sim S_{bh}/T_H \sim 1/T_H^3 \text{ in } 4d \]  \hspace{1cm} (14)

and is of order the information retention time [31, 34], as discussed in [33] for general dimensions.

When considering the expectation value of the commutator of local operators, the same picture as described above will emerge. In particular, the commutator of an operator behind the horizon and one outside can become non-zero, and of order \( e^{-S_{bh}} \) when the time separation approaches the information retention time \( t_c \). Likewise we expect the commutator between an operator inside the horizon and a large number of local operators outside at \( t > t_c \) can have deviations from the semiclassical result of order 1, if the operators are chosen so that the errors add coherently. This corresponds to determining the internal state of the black hole by measuring the Hawking radiation in a manner compatible with unitarity.

The key difference with small versus large black holes is that they are not supported by a flux of infalling thermal radiation, so will eventually evaporate away to the dominant entropy state, that of thermal AdS [35]. The time-dependent geometry can be treated by working in an adiabatic approximation. In low-dimensional examples, the bulk-boundary map for time-dependent geometries has been studied in [36]. However time dependence means the \( t_c \) that enters into the cutoff prescription will depend on what region of the spacetime you are trying to describe with effective field theory. Clearly far from the endpoint of evaporation, the cutoff time will diverge, as expected for thermal AdS. On the other hand \( t_c \) should be chosen according to (14) to yield a description of bulk physics in the vicinity of the black hole. In general \( t_c \) will need to vary in some position dependent way to optimize the regime of validity of a patch of effective field theory.

As emphasized above, the new feature of this construction is a prediction of when effective field theory breaks down. This does not coincide with the standard view that effective field theory should be valid away from regions with large curvature invariants. Rather the region of validity is determined by the class of observables and the region of spacetime under consideration in accord with the cutoff prescription.
VI. CONCLUSIONS

A method for constructing approximate local bulk operators from a finite $c$ CFT has been studied. This enables exact CFT results to be turned into predictions for experiments conducted in the bulk spacetime. The analysis of the two-point function in pure AdS, and in a black hole background indicate the results agree well with semiclassical effective field theory when expected, and disagree in a manner compatible with unitarity of quantum gravity. This provides an example of a direct derivation from AdS/CFT of the information theoretic implications of unitarity on effective field theory considered in [33]. This is an important step toward resolving the black hole information paradox.

The cutoff procedure is necessarily ad hoc, and one might try to find a better cutoff procedure which maximizes the region of validity of effective field theory. We believe this is simply a way to parametrize the inherent imprecision of quasi-local observables in a theory of quantum gravity. A general discussion of such observables can be found in [37]. A nice example that illustrates this point is the quantum geometric generalization of de Sitter space (and its associated CFT) considered in [38]. In this example the continuous de Sitter space is replaced by a quantum geometry that for many purposes behaves as a lattice of points. Clearly any attempt to replace the exact observables on quantum geometry via effective field theory on a continuous classical geometry will necessarily involve some ad hoc approximation of the fundamental observables. We regard the cutoff procedure to be a success if a large class of observables can be reconstructed to a good approximation in some finite region of spacetime, and the prescription described here appears to satisfy that criterion.

An important open question is to go beyond the two-point functions studied in the present work, and consider the effect of interactions. These can be studied perturbatively in $1/c$. The expectation is these corrections can be captured by local terms in the effective action, and that the cutoff prescription will not be modified in a substantive way, until back-reaction on the geometry becomes significant.

The difference between the cutoff correlators and the semiclassical correlators is typically very small, of relative order $e^{-S_{bh}}$. As we have seen, these effects preserve causality for regions outside the black hole, but nevertheless they appear to violate general covariance, as they prevent us from maximally extending our effective field theory over all regions of low curvature. It would be very interesting to develop a set of observables (presumably
non-local) sensitive to these violations of general covariance in the vicinity of a black hole. This can lead to new experimental probes of quantum gravity.

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**Appendix A: COINCIDENCE LIMIT OF TWO-POINT CORRELATOR**

In this appendix we will examine the limit of (4) when \( X \to 0, T \to 0 \) so that the smeared operators are centered at the same point on the boundary, but still retain general radial positions \( Z_1, Z_2 \). The \( i\epsilon \) prescription renders the integrals well-defined, but it will prove convenient to take the limit \( \epsilon \to 0 \) first, and analytically continue in \( Z_1 \) and \( Z_2 \) to avoid the singularities that appear when \( r_1 = r_2 \) at \( \epsilon = 0 \). First the integral over \( z_1 \) is performed, by evaluating the residue at

\[
z_1 = \frac{r_1 z_2}{r_2},
\]

(note the residue at \( z_1 = 0 \) vanishes for \( \Delta \geq 1 \)) and using the definition of the Jacobi polynomial

\[
P_{\Delta-1}(\beta,\gamma)(x) = \frac{(-1)^{\Delta-1}}{2^{\Delta-1}(\Delta-1)!} (1-x)^{\beta}(1+x)^{-\gamma} \frac{d^{\Delta-1}}{dx^{\Delta-1}} \left( (1-x)^{\Delta-1} (1+x)^{\Delta-1} \right).
\]

This gives the residue

\[
Res = \frac{(\Delta-1)^2(-1)^{\Delta+1}}{2\pi^3 R} \frac{r_1}{z_2} \left( \frac{r_2^2 - r_1^2}{r_2} \right)^{1-2\Delta} \left( \frac{Z_1^2 - r_1^2}{Z_1} \right)^{\Delta-2} \left( \frac{Z_2^2 - r_2^2}{Z_2} \right)^{\Delta-2} P_{\Delta-1}^{(0,1-2\Delta)} \left( 1 - \frac{2r_1^2}{r_2^2} \right).
\]

This has a simple pole at \( z_2 = 0 \), so the integral over \( z_2 \) may be easily done. To perform the integral over \( r_1 \) we apply a formula rediscovered by Askey in 1975 [39]

\[
\begin{align*}
2F_1(a, b; c; x) &= \frac{\Gamma(c)}{\Gamma(\mu)\Gamma(c-\mu)} \int_0^1 dt \int_{-1}^{1} (1-t)^{\mu-1}(1-t)^{c-\mu-1} \left( \frac{1}{(1-t)x} \right)^{\lambda-\mu} 2F_1(\lambda - a, \lambda - b; \mu; tx) \\
&\quad \times 2F_1(a + b - \lambda, \lambda - \mu; c - \mu; (1-t)x/(1-tx)),
\end{align*}
\]

(A1)
with $\text{Re} c > \text{Re} \mu > 0$ and $|x| < 1$. Note the left-hand side is independent of $\mu$ and $\lambda$. We also need the relation between the Jacobi polynomial and the Hypergeometric function

$$P^{(\beta,\gamma)}_n(x) = \frac{\Gamma(n + 1 + \beta)}{n!\Gamma(1 + \beta)} \binom{n + \beta + \gamma + 1}{-n; 1 + \beta; \frac{1 - x}{2}}.$$ 

Therefore after integrating over $z_1$, $z_2$ and $r_1$ we arrive at

$$\langle \phi(0, 0, Z_1)\phi(0, 0, Z_2) \rangle = \frac{\Delta - 1}{\pi R} \int_0^{Z_2} dr_2 r_2 Z_1^\Delta (Z_1^2 - r_2^2)^{-\Delta}(Z_2^2 - r_2^2)^{\Delta - 2}Z_2^{2-\Delta}. \quad \text{(A2)}$$

Here we have simplified the resulting Hypergeometric function to an algebraic function. Finally the integral over $r_2$ may be performed, assuming $Z_1 > Z_2$

$$\langle \phi(0, 0, Z_1)\phi(0, 0, Z_2) \rangle = \frac{Z_2^{2-\Delta}Z_1^\Delta}{2\pi R(Z_1^2 - Z_2^2)}.$$ 

This agrees exactly with (1) in the same limit.

**Appendix B: TWO-POINT CORRELATOR WITH ONE LEG ON THE BOUNDARY**

Here we will compute (1) in the limit $Z_2 \to 0$. Without loss of generality, we can also set $T = 0$. The integrals over $r_2$ and $z_2$ are straightforward in this limit, giving

$$\langle \phi(0, X, Z_1)\phi(0, 0, Z_2) \rangle = \frac{\Delta - 1}{2\pi^2 Ri} \int_{C_1} dz_1 \int_0^{Z_1} dz_1 \int_0^{Z_2} dr_1 \frac{r_1}{z_1} \left( \frac{Z_1^2 - r_1^2}{Z_1} \right)^{\Delta - 2} \times \left( (X - i\epsilon + r_1z_1)(X + i\epsilon - r_1/z_1) \right)^{\Delta} ,$$

and the limit $\epsilon \to 0$ may be taken immediately. The analysis proceeds in a similar way to the preceding appendix. As discussed above, the contour $C_1$ encloses the point $z_1 = r_1/X$ and the origin. The residue vanishes at the origin, and at the point $z_1 = r_1/X$ the residue may be written using a Jacobi polynomial. The integral over $r_1$ may be performed using the formula (A1) to give

$$\langle \phi(0, X, Z_1)\phi(0, 0, Z_2) \rangle = \frac{Z_1^\Delta Z_2^\Delta}{2\pi R(X^2 + Z_1^2)^{\Delta}}.$$ 

This agrees exactly with (1).

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[41] With one point on the boundary, a simple prescription was given in [9] involving excising poles in the upper-half z-plane. For general points, this is no longer applicable.

[42] Note we do not expect the dual CFT to gravity in the BTZ background obey the strict periodicity in the time direction found in chiral conformal field theories. As we will see, timescales of interest for us are typically much larger than $2\pi R$, and would not appear in such theories.

[43] Note that one might try to view CFT operators as corresponding to bulk operators inserted in some kind of linear superposition of geometries, represented as a sum over bulk topologies [4,40]. Here we have in mind trying to reproduce correlators of such operators via a local effective field theory on a background of fixed topology, as advocated in [33], so we will not pursue these other interpretations here.

[44] The integral is valid for $|Z_1/r_2| < 1$, however using the $i\epsilon$ prescription, we can continue $Z_1$
away from the real axis, and use $A_2$ to define the integral throughout the complex $Z_1$ plane.