LIFTING ENDO-\(p\)-PERMUTATION MODULES

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Abstract. We prove that all endo-\(p\)-permutation modules for a finite group are liftable from characteristic \(p > 0\) to characteristic 0.

1. Introduction

Throughout we let \(p\) be a prime number and \(G\) be a finite group of order divisible by \(p\). We let \(\mathcal{O}\) denote a complete discrete valuation ring of characteristic 0 with a residue field \(k := \mathcal{O}/p\) of positive characteristic \(p\), where \(p = J(\mathcal{O})\) is the unique maximal ideal of \(\mathcal{O}\). Moreover, we assume that \(\mathcal{O}\) is large enough in the sense that it contains a root of unity of order \(\exp(G)\), the exponent of \(G\), and for \(R \in \{\mathcal{O}, k\}\) we consider only finitely generated \(RG\)-lattices.

Amongst finitely generated \(kG\)-modules very few classes of modules are known to be liftable to \(\mathcal{O}G\)-lattices. Projective \(kG\)-modules are known to lift uniquely, and more generally, so do \(p\)-permutation \(kG\)-modules (see e.g. [Ben84, §2.6]). In the special case where the group \(G\) is a \(p\)-group, Alperin [Alp01] proved that endo-trivial \(kG\)-modules are liftable, and Bouc [Bou06, Corollary 8.5] observed that so are endo-permutation \(kG\)-modules as a consequence of their classification.

Passing to arbitrary groups, it is proved in [LMS16] that Alperin’s result extends to endo-trivial modules over arbitrary groups. It is therefore legitimate to ask whether Bouc’s result may be extended to arbitrary groups. A natural candidate for such a generalisation is the class of so-called endo-\(p\)-permutation \(kG\)-modules introduced by Urfer [Urf07], which are \(kG\)-modules whose \(k\)-endomorphism algebra is a \(p\)-permutation \(kG\)-module. We extend this definition to \(\mathcal{O}G\)-lattices and prove that any indecomposable endo-\(p\)-permutation \(kG\)-module lifts to an endo-\(p\)-permutation \(\mathcal{O}G\)-lattice with the same vertices.

We emphasise that our proof relies on a nontrivial result, namely the lifting of endo-permutation modules, which is a consequence of their classification. Moreover, there are two crucial points to our argument: the first one is the fact that reduction modulo \(p\) applied to the class of endo-\(p\)-permutation \(\mathcal{O}G\)-lattices preserves both indecomposability and vertices, while the second one relies on properties of the \(G\)-algebra structure of the endomorphism ring of endo-permutation \(RG\)-lattices.

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2. ENDO-\(p\)-PERMUTATION LATTICES

Recall that an \(OG\)-lattice is an \(OG\)-module which is free as an \(O\)-module. For \(R \in \{O, k\}\) an \(RG\)-lattice \(L\) is called a \(p\)-permutation lattice if \(\text{Res}_p^G(L)\) is a permutation \(RP\)-lattice for every \(p\)-subgroup \(P\) of \(G\), or equivalently, if \(L\) is isomorphic to a direct summand of a permutation \(RG\)-lattice.

Following Urfer [Urf07], we call an \(RG\)-lattice \(L\) an endo-\(p\)-permutation \(RG\)-lattice if its endomorphism algebra \(\text{End}_R(L)\) is a \(p\)-permutation \(RG\)-lattice, where \(\text{End}_R(L)\) is endowed with its natural \(RG\)-module structure via the action of \(G\) by conjugation:

\[
\phi(m) = g \cdot \phi(g^{-1} \cdot m) \quad \forall g \in G, \forall \phi \in \text{End}_R(L) \text{ and } \forall m \in L.
\]

Equivalently, \(L\) is an endo-\(p\)-permutation \(RG\)-lattice if and only if \(\text{Res}_p^G(L)\) is an endo-permutation \(RP\)-lattice for a Sylow \(p\)-subgroup \(P \in \text{Syl}_p(G)\), or also if \(\text{Res}_Q^G(L)\) is an endo-permutation \(RQ\)-lattice for every \(p\)-subgroup \(Q\) of \(G\).

This generalises the notion of an endo-permutation \(RP\)-lattice over a \(p\)-group \(P\), introduced by Dade in [Dad78a, Dad78b]. In fact an \(RP\)-lattice is an endo-\(p\)-permutation \(RP\)-lattice if and only if it is an endo-permutation lattice. An endo-permutation \(RP\)-lattice \(M\) is said to be capped if it has at least one indecomposable direct summand with vertex \(P\), and in this case there is in fact a unique isomorphism class of indecomposable direct summands of \(M\) with vertex \(P\), called the cap of \(M\). Moreover, considering an equivalence relation called compatibility on the class of capped endo-permutation \(RP\)-lattices gives rise to a finitely generated abelian group \(D_R(P)\), called the Dade group of \(P\), whose multiplication is induced by the tensor product \(\otimes R\). For details, we refer the reader to [Dad78a] or [The95, §27-29].

If \(P \leq G\) is a \(p\)-subgroup, we write \(D_R(P)^{G-st}\) for the set of \(G\)-stable elements of \(D_R(P)\), i.e. the set of equivalence classes \([L] \in D_R(P)\) such that

\[
\text{Res}_{xP \cap P}^P([L]) = \text{Res}_{xP \cap P}^P \circ c_x([L]) \in D_R(\cap P), \quad \forall x \in G,
\]

where \(c_x\) denotes conjugation by \(x\).

The following results can be found in Urfer [Urf07] for the case \(R = k\), under the additional assumption that \(k\) is algebraically closed. However, it is straightforward to prove that they hold for an arbitrary field \(k\) of characteristic \(p\), and also in case \(R = O\).

**Remark 2.1.** It follows easily from the definitions that the class of endo-\(p\)-permutation \(RG\)-lattices is closed under taking direct summands, \(R\)-duals, tensor products over \(R\), (relative) Heller translates, restriction to a subgroup, and tensor induction to an overgroup. However, this class is not closed under induction, nor under direct sums.

Two endo-\(p\)-permutation \(RG\)-lattices are called compatible if their direct sum is an endo-\(p\)-permutation \(RG\)-lattice.

**Lemma 2.2** ([Urf07, Lemma 1.3]). Let \(H \leq G\) and \(L\) be an endo-\(p\)-permutation \(RH\)-lattice. Then \(\text{Ind}_H^G(L)\) is an endo-\(p\)-permutation \(RG\)-lattice if and only if \(\text{Res}_H^G(L)\) and \(\text{Res}^{*H}_{H \cap H}(*L)\) are compatible for each \(x \in G\).
Theorem 2.3 ([Urf07, Theorem 1.5]). An indecomposable \(RG\)-lattice \(L\) with vertex \(P\) and \(RP\)-source \(S\) is an endo-\(p\)-permutation \(RG\)-lattice if and only if \(S\) is a capped endo-permutation \(RP\)-lattice such that \([S] \in DR(P)^{G-\text{st}}\). Moreover, in this case \(\text{Ind}^G_P(S)\) is an endo-\(p\)-permutation \(RG\)-lattice.

3. Preserving indecomposability and vertices by reduction modulo \(p\)

For an \(OG\)-lattice \(L\), the reduction modulo \(p\) of \(L\) is

\[
L/pL \cong k \otimes_O L.
\]

Note that \(k \otimes_O \text{End}_O(L) \cong \text{End}_k(L/pL)\). A \(kG\)-module \(M\) is said to be liftable if there exists an \(OG\)-lattice \(\hat{M}\) such that \(M \cong \hat{M}/p\hat{M}\).

Lemma 3.1. Let \(L\) be an endo-\(p\)-permutation \(OG\)-lattice and \(A := \text{End}_O(L)\). Then the natural homomorphism \(k \otimes_O A^G \rightarrow (k \otimes_O A)^G\) is an isomorphism of \(k\)-algebras.

Proof. Consider first a transitive permutation \(OG\)-lattice \(U = \text{Ind}_G^Q(O)\). Then \(Q \leq G\) is the stabiliser of \(x = 1 \otimes 1\), so that \(\{gx \mid g \in [G/Q]\}\) is a \(G\)-invariant \(O\)-basis of \(U\) and \(U^G \cong O(\sum_{g \in [G/Q]} gx)\). It follows that \(\{1_k \otimes gx \mid g \in [G/Q]\}\) is a \(G\)-invariant \(k\)-basis of \(k \otimes_O U^G\). Hence the canonical homomorphism \(k \otimes_O U^G \rightarrow (k \otimes_O U)^G\) is an isomorphism. Because taking fixed points commutes with direct sums, the latter isomorphism holds as well for every \(p\)-permutation \(OG\)-lattice \(U\). Therefore, writing \(A = \bigoplus_{i=1}^m U_i\) as a direct sum of indecomposable \(p\)-permutation \(OG\)-lattices, we obtain that the canonical homomorphism

\[
k \otimes_O A^G \cong \bigoplus_{i=1}^m k \otimes_O U_i^G \rightarrow \bigoplus_{i=1}^m (k \otimes_O U_i)^G \cong (k \otimes_O A)^G
\]

is an isomorphism. The following characterisation of vertices is well-known, but we include a proof for completeness.

Lemma 3.2. Let \(R \in \{O, k\}\) and let \(L\) be an indecomposable \(RG\)-lattice. Let \(L^\vee = \text{Hom}_R(L, R)\) denote the \(R\)-dual of \(L\) and let

\[
\text{End}_R(L) \cong L \otimes_R L^\vee \cong U_1 \oplus \cdots \oplus U_n
\]

be a decomposition of \(L \otimes_R L^\vee\) into indecomposable summands. Then a \(p\)-subgroup \(P\) of \(G\) is a vertex of \(L\) if and only if every \(U_i\) has a vertex contained in \(P\) and one of them has vertex \(P\).
Proof. Suppose \( L \) has vertex \( P \). Then \( L \) is projective relative to \( P \) and, by tensoring with \( L^\vee \), we see that \( L \otimes_R L^\vee \) is projective relative to \( P \), and therefore so are \( U_1, \ldots, U_n \).

In other words, \( P \) contains a vertex of \( U_i \) for each \( 1 \leq i \leq n \). Now \( L \) is isomorphic to a direct summand of \( L \otimes_R L^\vee \otimes_R L \) because the evaluation map

\[
L \otimes_R L^\vee \otimes_R L \to L, \quad x \otimes \psi \otimes y \mapsto \psi(x)y
\]
splits via \( y \mapsto \sum_{i=1}^n y \otimes v_i^\vee \otimes v_i \), where \( \{v_1, \ldots, v_n\} \) is an \( R \)-basis of \( L \) and \( \{v_1^\vee, \ldots, v_n^\vee\} \) is the dual basis. Therefore \( L \) is isomorphic to a direct summand of some \( U_i \otimes_R L \) (by the Krull-Schmidt theorem). If, for each \( 1 \leq i \leq n \), a vertex of \( U_i \) was strictly contained in \( P \), then \( U_i \otimes_R L \) would be projective relative to a proper subgroup of \( P \), hence the direct summand \( L \) would also be projective relative to a proper subgroup of \( P \), a contradiction.

This proves that, for some \( i \), a vertex of \( U_i \) is equal to \( P \).

Suppose conversely that every \( U_i \) has a vertex contained in \( P \) and one of them has vertex \( P \). Let \( Q \) be a vertex of \( L \). By the first part of the proof, every \( U_i \) has a vertex contained in \( Q \) and one of them has vertex \( Q \). This forces \( Q \) to be equal to \( P \) up to conjugation. \( \square \)

**Proposition 3.3.** If \( L \) is an indecomposable endo-p-permutation \( \mathcal{O}G \)-lattice with vertex \( P \leq G \), then \( L/pL \) is an indecomposable endo-p-permutation \( kG \)-module with vertex \( P \).

**Proof.** Set \( A := \text{End}_\mathcal{O}(L) \), so that \( A^G = \text{End}_{\mathcal{O}G}(L) \). First we prove that \( \text{End}_{kG}(L/pL) = (k \otimes \mathcal{O} A)^G \) is a local algebra. Write \( \psi : A^G \to A^G/pA^G \) for the canonical homomorphism. By Nakayama’s Lemma \( pA^G \subseteq J(A^G) \), so that any maximal left ideal of \( A^G \) contains \( pA^G \).

Therefore

\[
\psi^{-1}(J(A^G/pA^G)) = \psi^{-1}\left( \bigcap_{a \in \text{Maxl}(A^G)} m \right) = \bigcap_{a \in \text{Maxl}(A^G)} a = J(A^G),
\]

where \( \text{Maxl} \) denotes the set of maximal left ideals of the considered ring. Thus \( \psi \) induces an isomorphism \( A^G/J(A^G) \cong (k \otimes \mathcal{O} A^G)/J((k \otimes \mathcal{O} A^G)) \). Now \( k \otimes A^G \cong (k \otimes A)^G \) as \( k \)-algebras, by Lemma 3.1. Therefore it follows that

\[
\text{End}_{kG}(L/pL)/J(\text{End}_{kG}(L/pL)) \cong (k \otimes \mathcal{O} A^G)/J((k \otimes \mathcal{O} A^G)) \cong A^G/J(A^G) .
\]

This is a skew-field since we assume that \( L \) is indecomposable. Hence \( L/pL \) is indecomposable.

For the second claim, let \( P \) be a vertex of \( L \). Let \( L^\vee \) denote the \( \mathcal{O} \)-dual of \( L \) and consider a decomposition of \( \text{End}_\mathcal{O}(L) \) into indecomposable summands

\[
\text{End}_\mathcal{O}(L) \cong L \otimes_\mathcal{O} L^\vee \cong U_1 \oplus \cdots \oplus U_n .
\]

Then there is also a decomposition

\[
\text{End}_{k}(L/pL) \cong k \otimes \text{End}_\mathcal{O}(L) \cong U_1/pU_1 \oplus \cdots \oplus U_n/pU_n .
\]

Since \( L \) is an endo-p-permutation \( \mathcal{O}G \)-lattice, \( U_i \) is a \( p \)-permutation module for each \( 1 \leq i \leq n \). Therefore the module \( U_i/pU_i \) is indecomposable and the vertices of \( U_i \) and \( U_i/pU_i \) are the same (see [The95, Proposition 27.11]). By Lemma 3.2, every \( U_i \) has a vertex contained in \( P \) and one of them has vertex \( P \). Therefore every \( U_i/pU_i \) has a
vertex contained in $P$ and one of them has vertex $P$. By Lemma 3.2 again, $P$ is a vertex of $L/pL$.

\[ \qed \]

4. LIFTING ENDO-$p$-PERMUTATION $kG$-MODULES

We are going to use the fact that the sources of endo-$p$-permutation $kG$-modules are liftable. However, a random lift of the sources will not suffice and our next lemma deals with this question.

**Lemma 4.1.** Let $P$ be a $p$-subgroup of $G$. If $S$ is an indecomposable endo-permutation $kP$-module with vertex $P$ such that $[S] \in D_k(P)^{G-st}$, then there exists an endo-permutation $OP$-lattice $\hat{S}$ lifting $S$ such that $[\hat{S}] = [P] \in D_k(P)^{G-st}$.

**Proof.** Let $S' = \text{Res}_P^G \text{Ind}_P^G(S)$. Since $[S] \in D_k(P)^{G-st}$, the induced module $\text{Ind}_P^G(S)$ is an endo-$p$-permutation $kG$-module (by Theorem 2.3), hence $S'$ is an endo-permutation $kP$-module. Since $S$ is isomorphic to a direct summand of $S'$, it is the cap of $S'$ and they must have the same class $[S] = [S'] \in D_k(P)$. We now show that the module $S'$ is $G$-stable (not only its class in the Dade group). For any $x \in G$, we have $c_x(\text{Ind}_P^G(S)) \cong \text{Ind}_P^G(S)$ and therefore

$$\text{Res}_{xP \cap P}^P(S') \cong \text{Res}_{xP \cap P}^P \text{Ind}_P^G(S) \cong \text{Res}_{xP \cap P}^P \text{Res}_P^G c_x \text{Ind}_P^G(S) \cong \text{Res}_{xP \cap P}^P c_x \text{Res}_P^G \text{Ind}_P^G(S) \cong \text{Res}_{xP \cap P}^P c_x(S').$$

As a consequence of the classification of endo-permutation modules, Bouc proved that every endo-permutation $kP$-module is liftable [Bou06, Corollary 8.5] (without any indecomposability assumption, see [The07, Theorem 14.2]). Therefore $S'$ is liftable to an endo-permutation $OP$-lattice $\hat{S}'$, i.e. $\hat{S}'/p\hat{S}' \cong S'$. Note that $\hat{S}'$ is not unique because $\hat{S}' \otimes \mathcal{O} L$ also lifts $S'$ for any one-dimensional $OP$-lattice $L$. This is because $L/pL \cong k$ since the trivial module $k$ is the only one-dimensional $kP$-module up to isomorphism. However, the lifted $P$-algebra $\text{End}_\mathcal{O}(\hat{S}')$ is unique up to isomorphism and we can choose $\hat{S}'$ to be the unique $OP$-lattice with determinant 1 which lifts $S'$ (see [The95, Lemma 28.1]). This choice of an $OP$-lattice with determinant 1 is made possible because the dimension of $S$ is prime to $p$ (see [The95, Corollary 28.11]), hence that of $S'$ as well.

In order to prove that the module $\hat{S}'$ is $G$-stable, we note that the determinant 1 is preserved by conjugation and by restriction. Therefore, the isomorphism

$$\text{Res}_{xP \cap P}^P(S') \cong \text{Res}_{xP \cap P}^P \circ c_x(S') \quad \forall x \in G$$

implies an isomorphism for the unique lifts with determinant 1

$$\text{Res}_{xP \cap P}^P(\hat{S}') \cong \text{Res}_{xP \cap P}^P \circ c_x(\hat{S}') \quad \forall x \in G.$$

Note that we need here actual modules rather than classes, because the determinant 1 may not be preserved by taking the cap of an endo-permutation lattice (this problem arises in characteristic 2), and this is why we work with $S'$ rather than $S$. Now let $\hat{S}$ be the cap of the endo-permutation $OP$-lattice $\hat{S}'$. Then $[\hat{S}] = [\hat{S}']$ and therefore $[\hat{S}] \in D_\mathcal{O}(P)^{G-st}$ since the module $\hat{S}'$ is $G$-stable. Moreover $\hat{S}/p\hat{S}$ is a cap of $\hat{S}'/p\hat{S}' \cong S'$, hence $\hat{S}/p\hat{S} \cong S$ because $S$ is the cap of $S'$. This completes the proof. \[ \qed \]
Theorem 4.2. Let $M$ be an indecomposable endo-$p$-permutation $kG$-module, and let $P \leq G$ be a vertex of $M$. Then there exists an indecomposable endo-$p$-permutation $OG$-lattice $\hat{M}$ with vertex $P$ such that $\hat{M}/p\hat{M} \cong M$.

Proof. Let $S$ be a $kP$-source of $M$. By Theorem 2.3, $S$ is a capped endo-permutation $kP$-module such that $[S] \in D_k(P)^{G-st}$. By Lemma 4.1, $S$ lifts to an endo-permutation $OP$-lattice $\hat{S}$ such that $[\hat{S}] \in D_O(P)^{G-st}$. Moreover $\text{Ind}_P^G(\hat{S})$ is an endo-$p$-permutation $OG$-lattice, by Lemma 2.2 and the fact that $[\hat{S}]$ is $G$-stable. Now consider a decomposition of $\text{Ind}_P^G(\hat{S})$ into indecomposable summands

\[ \text{Ind}_P^G(\hat{S}) = L_1 \oplus \cdots \oplus L_s \quad (s \in \mathbb{N}). \]

By Remark 2.1, each of the lattices $L_i$ ($1 \leq i \leq s$) is an endo-$p$-permutation $OG$-lattice. Then, by Proposition 3.3,

\[ \text{Ind}_P^G(S) \cong \text{Ind}_P^G(\hat{S})/p \text{Ind}_P^G(\hat{S}) \cong L_1/pL_1 \oplus \cdots \oplus L_s/pL_s \]

is a decomposition of $\text{Ind}_P^G(S)$ into indecomposable summands which preserves the vertices of the indecomposable summands. Because $S$ is a source of $M$, there exists an index $1 \leq i \leq s$ such that $M \cong L_i/pL_i$. Then $\hat{M} := L_i$ lifts $M$. \qed

Remark 4.3. In [BK06], Boltje and Külschammer consider the class of modules with an endo-permutation source, which also play a role in the study of Morita equivalences, as observed by Puig [Pui99]. In recent work of Kessar and Linckelmann [KL17], it is proved that in odd characteristic any Morita equivalence with an endo-permutation source is liftable from $k$ to $O$, under the assumption that $k$ is algebraically closed.

As a typical example, we remark that simple modules for $p$-soluble groups are known to be instances of modules with an endo-permutation source (see [The95, Theorem 30.5]) and they are also known to be liftable to characteristic zero (Fong-Swan Theorem). Urfer proved in his Ph.D. thesis [Urf06] that such simple modules are endo-$p$-permutation modules in case they are not induced from proper subgroups, but in general they need not be endo-$p$-permutation.

One may ask whether our result extends to $kG$-modules with an endo-permutation source, i.e. whose class in the Dade group is not necessarily $G$-stable. We do not have an answer to this question. Our proof that endo-$p$-permutation modules are liftable to characteristic zero does not seem to extend to this larger class of modules, because it relies on the fact that the endomorphism algebra is a $p$-permutation module.

Remark 4.4. Finally, we note that the lifts produced by Theorem 4.2 are not uniquely determined in general. Indeed, if $M$ is an endo-$p$-permutation $kG$-module and $\hat{M}$ is an $OG$-lattice lifting $M$, then $\hat{M} \otimes_O X$ is again a lift of $M$ for each one-dimensional $OG$-lattice $X$ lifting the trivial $kG$-module $k$.

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