REGULAR MEASURABLE DYNAMICS FOR REACTION-DIFFUSION EQUATIONS ON NARROW DOMAINS WITH ROUGH NOISE

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Abstract. This paper is concerned with the regular random dynamics for the reaction-diffusion equation defined on a thin domain and perturbed by rough noise, where the usual Wiener process is replaced by a general stochastic process satisfied the basic convergence. A bi-spatial attractor is obtained when the non-initial space is $p$-times Lebesgue space or Sobolev space. The measurability of the solution operator is proved, which leads to the measurability of the attractor in both state spaces. Finally, the upper semi-continuity of attractors under the $p$-norm is established when the narrow domain degenerates onto a lower dimensional set. Both methods of symbolical truncation and spectral decomposition provide all required uniform estimates in both Lebesgue and Sobolev spaces.

1. Introduction. Recently, D. Li et al [18, 19] had investigated the random dynamics in the state space $L^2$ for the stochastic reaction-diffusion equations on a thin domain:

$$
\begin{align*}
\frac{d\tilde{u}^\epsilon}{dt} - \Delta \tilde{u}^\epsilon dt + \lambda \tilde{u}^\epsilon dt &= (f(t, x, \tilde{u}^\epsilon) + G(t, x))dt + \tilde{u}^\epsilon dW, \quad t \geq \tau, \\
\frac{\partial \tilde{u}^\epsilon}{\partial \nu_\epsilon} &= 0, \quad \text{on } \partial \Omega_\epsilon, \quad \tilde{u}^\epsilon(\tau, x) = \tilde{u}^\epsilon(\tau, x), \quad x \in \Omega_\epsilon, \tau \in \mathbb{R},
\end{align*}
$$

where $\lambda > 0, \nu_\epsilon$ is the unit outward normal vector on $\partial \Omega_\epsilon$ for $\epsilon \in (0, 1]$. The $n + 1$-dimensional thin domain $\Omega_\epsilon$ is given by

$$
\Omega_\epsilon = \{x = (x^*, x_{n+1})|x^* = (x_1, \ldots, x_n) \in Q, \ 0 < x_{n+1} < g(x^*)\},
$$

where $Q$ is a bounded smooth domain in $\mathbb{R}^n$ and $g \in C^2(\overline{Q}, (0, +\infty))$. The nonlinearity $f$ and the body force $G$ will be specified later.

Such thin-domain problems for non-random equations was first investigated by Hale, Raugel [15, 16] and Raugel, Sell [30]. More investigation from different points of view can be found in many literatures (see e.g. [2, 5, 10, 11, 27, 28, 29]).
In this paper, we assume that $W$ is a general stochastic process defined on a general probability space $(\Omega, \mathcal{F}, P)$ (see Hypothesis W in section 2), which contains the special case of usual Wiener process used in \cite{18, 19}.

The first purpose is to prove $\mathcal{F}$-measurability of the solution map for Eq. (1) in three spaces: $L^2$, $L^p$ and $H^1$. This measurability (even for Wiener process when the state space is $L^2$) seems not to be proved although claimed in the literatures, but it is a foundation to prove $\mathcal{F}$-measurability of an attractor.

The second purpose is to discuss strong attraction of the $(L^2, L^2)$-attractors (obtained by \cite{19}) for Eq. (1) when the terminate space becomes more regular such as $L^p$, $H^1$. More precisely, we will prove the existence of a bi-spatial $(L^2, L^p \cap H^1)$ random attractor $A_\epsilon$ for Eq. (1) by using the theory of bi-spatial random attractors developed by Li et al \cite{21}. Although the theory of bi-spatial attractors aimed at a non-thin domain (see \cite{12, 34, 35}), it is possible to adapt the thin-domain problem as done in section 2.

The third purpose is to prove the upper semi-continuity of $A_\epsilon$ as $\epsilon \to 0$, not only in $L^2$ but also in $L^p$. More precisely, as $\epsilon \to 0$, the $n+1$-dimensional thin domain $\Omega_\epsilon$ degenerates onto the $n$-dimensional domain $Q$. We consider the limiting equation defined on $Q$ as follows.

\[
\begin{align*}
\frac{d u^0}{dt} - \frac{1}{g} \sum_{i=1}^{n} (g u^0_{y_i}) u_i dt + \lambda u^0 dt &= (f_0(t, y^*, u^0) + G_0(t, y^*)) dt + u^0 \circ dW, \\
\frac{\partial u^0}{\partial \nu_0} &= 0 \quad \text{on} \quad \partial Q, \quad u^0(t, y^*) = u^0_\nu(y^*), \quad y^* \in Q, \ t \geq \tau, \ \tau \in \mathbb{R},
\end{align*}
\]

(2)

where $f_0(t, y^*, u^0) = f(t, (y^*, 0, u^0)$, $G_0(t, y^*) = G(t, (y^*, 0))$ and $\nu_0$ is the unit outward normal vector on $\partial Q$. Similarly, we can prove that Eq. (2) possesses a bi-spatial random attractor $A_0$ in $(L^2(Q), L^p(Q))$. Then, our main result is that $A_\epsilon \to A_0$ according to the Hausdorff semi-metric of $L^p$.

Notice that $L^p$ ($p \neq 2$) is only a Banach space (it is not a Hilbert space). In such a Banach space, the subject of random invariant manifolds had been investigated by Lian and Lu \cite{25}, while the subject of random attractors had been studied in \cite{20, 26, 33, 36, 38, 39}. However, for the best of our knowledge, there is no paper discussing random dynamics for thin domain problem when the state space is a Banach space.

When providing some uniform estimates in $L^p$ and $H^1$ respectively, our techniques are to impose symbolic truncation of the solution and decomposition of the spectrum.

2. Random cocycle from the equation.

2.1. Hypotheses. Since $g$ is continuous, there are $\gamma_2 > \gamma_1 > 0$ such that $\gamma_1 \leq g(x^*) \leq \gamma_2$ for all $x^* \in \bar{Q}$.

Let $\bar{\Omega} = Q \times (0, \gamma_2)$ and $\hat{\Omega} = Q \times [0, \gamma_2)$. We have $\Omega_\epsilon \subset \bar{\Omega} \subset \hat{\Omega}$ for each $\epsilon \in (0, 1]$.

**Hypothesis G.** The force $G : \mathbb{R} \times \hat{\Omega} \to \mathbb{R}$ is continuous such that

\[
G \in L^2_{loc}(\mathbb{R}, L^\infty(\hat{\Omega})) \quad \text{and} \quad G_0 \in L^2_{loc}(\mathbb{R}, L^\infty(Q)).
\]

(3)

**Hypothesis F.** The nonlinearity $f : \mathbb{R} \times \hat{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions: for all $x \in \hat{\Omega}$ and $t, s \in \mathbb{R},$

\[
f(t, x, s) s \leq -\alpha_1 |s|^p + \psi_1(t, x), \quad \psi_1 \in L^2_{loc} \cap L^2_{loc} \cap L^2_{loc}(\mathbb{R}, L^\infty(\hat{\Omega})),
\]

(4)
\[ |f(t, x, s)| \leq \alpha_2 |s|^{p-1} + \psi_2(t, x), \quad \psi_2 \in L^2_{loc}(\mathbb{R}, L^\infty(\hat{O})), \] (5)
\[ \frac{\partial f(t, x, s)}{\partial s} \leq \beta, \quad \frac{\partial f(t, x, s)}{\partial s} \leq \alpha_3 |s|^{p-2} + \psi_3(t, x), \quad \psi_3 \in L^2_{loc}(\mathbb{R}, L^\infty(\hat{O})), \] (6)
\[ \left| \frac{\partial f(t, x, s)}{\partial x} \right| \leq \psi_4(t, x), \quad \psi_4 \in L^2_{loc}(\mathbb{R}, L^\infty(\hat{O})), \] (7)
where \( p > 2, \alpha_i, \beta > 0 \). Moreover, the restrictions \( \psi_{j,0} \) of \( \psi_j \) \( (j = 1, 2, 3, 4) \) on \( Q \times \{0\} \) satisfy the same assumption in (4)-(7) with \( Q \) instead of \( \hat{O} \).

**Hypothesis T.** The above functions satisfy some tempered conditions:
\[ \int_0^\infty e^{\frac{1}{2} \lambda s} (\|G(s)\|_2^2 + \|\psi_1(s)\|_\infty + \|\psi_1(s)\|_\infty^2 + \|\psi_4(s)\|_\infty^2) ds < \infty, \] (8)
\[ \lim_{r \to -\infty} e^{\sigma r} \int_{-\infty}^0 e^{\frac{1}{2} \lambda s} (\|G(s + r)\|_2^2 + \|\psi_1(s + r)\|_\infty + \|\psi_4(s + r)\|_\infty^2) ds = 0, \] (9)
for any \( r \in \mathbb{R} \) and \( \sigma > 0 \), where we use \( \| \cdot \|_\infty \) to denote the norm in \( L^\infty(\hat{O}) \).

**Remark 1.** Comparing with \([18, 19]\), we have expanded the defining fields of \( f \) and \( G \) from \( \hat{O} \) to \( \hat{O} = \hat{O} \cup (Q \times \{0\}) \). This is necessary to ensure that the restrictions \( f_0, G_0 \) are well-defined, and satisfy some growth conditions on \( Q \). In this case, the limiting equation (2) is well-defined and solvable.

On the other hand, \( u \in L^\infty(\hat{O}) \) if and only if \( u \in L^\infty(\hat{O}) \) with the same norm. However, we do not have \( u_0 \in L^\infty(Q) \) even if \( u \in L^\infty(\hat{O}) \), where \( u_0(y^*) = u(y^*, 0) \).

### 2.2. Transformation of the thin domain.

Let \( O = Q \times (0, 1) \). We introduce a transformation \( T_\epsilon : \mathcal{O}_\epsilon \to \mathcal{O} \) defined by
\[ (y^*, y_{n+1}) = T_\epsilon(x^*, x_{n+1}) = (x^*, \frac{y_{n+1}}{\epsilon g(x^*)}) \quad \text{for all } x = (x^*, x_{n+1}) \in \mathcal{O}_\epsilon. \]

It is easy to see that \( T_\epsilon \) is bijective with the Jacobian matrix:
\[ J = \frac{\partial (y_1, \ldots, y_{n+1})}{\partial (x_1, \ldots, x_{n+1})} = \begin{pmatrix} \frac{1}{\epsilon g(y^*)} & 0 \\ -\frac{y_{n+1}}{\epsilon g(y^*)^2} & 1 \end{pmatrix} \]

where \( I \) is the \( n \)-dimensional unit matrix, and the determinant \( |J| = \frac{1}{\epsilon g(y^*)^2} \). Let \( \nabla_x, \nabla_y, \Delta_x \) and \( \text{div}_y \) be the gradient, Laplace and divergence operators in \( x \in \mathcal{O}_\epsilon \) or \( y \in \mathcal{O} \). Then they are related by (see \([16, 19]\)): \( \nabla_x \tilde{u}(x) = J^* \nabla_y u(y) \) and \( \Delta_x \tilde{u}(x) = |J| \text{div}_y(|J|^{-1} J^* \nabla_y u(y)) = \frac{1}{\epsilon g(y^*)} \text{div}_y(\Upsilon_\epsilon u(y)), \)

where we denote by \( u(y) = \tilde{u}(x) \) (\( y = T_\epsilon x \in \mathcal{O} \)), \( J^* \) is the transport of \( J \) and \( \Upsilon_\epsilon \) is the operator given by
\[ \Upsilon_\epsilon u(y) = \begin{pmatrix} g u_{y_1} - g_{y_1} y_{n+1} u_{y_{n+1}} \\ \vdots \\ g u_{y_n} - g_{y_n} y_{n+1} u_{y_{n+1}} \\ -\sum_{i=1}^{n} y_{n+1} g_{y_i} u_{y_i} + \frac{1}{\epsilon g} (1 + \sum_{i=1}^{n} (g_{y_{n+1}} g_{y_i})^2) u_{y_{n+1}} \end{pmatrix} \] (10)

Now, we rewrite \( f(t, x, u), G(t, x) \) as some functions in \( y = (y^*, y_{n+1}) \in \mathcal{O} \):
\[ G_\epsilon(t, y^*, y_{n+1}) = G(t, y^*, \epsilon g(y^*) y_{n+1}), \quad f_\epsilon(t, y^*, y_{n+1}, u) = f(t, y^*, \epsilon g(y^*) y_{n+1}, u). \]
Hypothesis W

Corresponding process is the usual Wiener process, which is widely used in literatures.

In the above construction, different probability measures determine different stochastic processes. By [3, 7, 9], if $P$ is a Wiener measure, then, the corresponding process is the usual Wiener process, which is widely used in literatures (see [1, 6, 8, 13, 32]).
Lemma 2.4. Let \( z(t, \omega) = e^{-\omega t} \) and \( v^\epsilon(t, \tau, \omega, v_r) = z(t, \omega)u^\epsilon(t, \tau, \omega, u_r) \), where \( u^\epsilon \) is a solution of problem (12), then we can obtain that
\[
\begin{cases}
\frac{dv^\epsilon}{dt} + A_\epsilon v^\epsilon + \lambda v^\epsilon = z(t, \omega)f_\epsilon(t, y, z^{-1}(t, \omega)v^\epsilon) + z(t, \omega)G_\epsilon(t, y), \\
v^\epsilon(\tau, \tau, \omega, v_r) = v_r \quad y \in \mathcal{O}, t \geq \tau.
\end{cases}
\] (14)
By employing Galerkin method, one can prove the well-posedness of problem (14) (cf. [19]).

Lemma 2.3. Suppose \( \omega_k \to \omega_0 \) on the Fréchet space \( \Omega, g \). Then for each \( T > 0 \) we have
\[
\sup_{t \in [\tau, \tau + T]} (|e^{-\omega_k(t)} - e^{-\omega(t)}| + |e^{\omega_k(t)} - e^{\omega(t)}|) \to 0,
\] (16)
where \( c \) and \( C \) are positive constants which are independent of \( k \) and \( t \), but depend on \( \tau, T, \omega_0 \).

Proof. Suppose \([\tau, \tau + T] \subset [-n_0, n_0] \) with \( n_0 \in \mathbb{N} \). Then, by the continuity of \( t \to \omega_0(t) \),
\[
\sup_{t \in [-n_0, n_0]} |\omega_k(t)| \leq \sup_{t \in [-n_0, n_0]} |\omega_0(t)| + \sup_{t \in [-n_0, n_0]} |\omega_k(t) - \omega_0(t)|
\] \[
\leq C + g_0(\omega_k, \omega_0) \leq C,
\]
uniformly in \( k \in \mathbb{N} \), where we have used the fact: \( g(\omega_k, \omega_0) \to 0 \) if and only if \( g_n(\omega_k, \omega_0) \to 0 \) for all \( n \in \mathbb{N} \). Then (17) follows from the above estimate immediately. Note that \( (e^{-t})' = -e^{-t} \), by the mean valued theorem, we have, for all \( t \in [-n_0, n_0] \),
\[
|e^{-\omega_k(t)} - e^{-\omega_0(t)}| \leq \sup_{s \in [-n_0, n_0]} (e|\omega_k(s)| + e|\omega_0(s)|)|\omega_k(t) - \omega_0(t)| \leq C g_0(\omega_k, \omega_0),
\]
which tends to zero as \( k \to \infty \). We have proved the first part and similarly the second part in (16).

Lemma 2.4. The mapping \( \omega \to v^\epsilon(t, \tau, \omega, v_r) \) is continuous from \( \Omega, g \) to \( (X, \|\cdot\|_g) \), where \( v \) is the solution of Equ.(14) with the initial value \( v_r \in X \).

Proof. We omit the superscript * when there is no ambiguity. Let \( \omega_0 \in \Omega \) be fixed and suppose \( \omega_k \in \Omega \) such that \( \rho(\omega_k, \omega_0) \to 0 \) as \( k \to \infty \). We denote by \( v_k := v(t, \tau, \omega_k, v_r) \), \( v_0 := v(t, \tau, \omega_0, v_r) \) and \( V_k := v_k - v_0 \), where \( t \in [\tau, \tau + T] \) with \( T > 0 \). By Equ.(14) and the linearity of \( A_\epsilon \), we have
\[
\frac{dV_k}{dt} + \lambda V_k + A_\epsilon V_k = e^{-\omega_k(t)}f_\epsilon(t, y, e^{\omega_k(t)}v_k) - e^{-\omega_0(t)}f_\epsilon(t, y, e^{\omega_0(t)}v_0)
\]
\[
+ (e^{-\omega_k(t)} - e^{-\omega_0(t)})G_\epsilon(t, y)
\] (18)
with the initial data $V_k(\tau) = v_\tau - v_\tau = 0$. We multiply (18) with $gV_k$ and then integrate over $\mathcal{O}$ to obtain
\[
\frac{1}{2} \frac{d}{dt} \|V_k\|_g^2 + \lambda \|V_k\|_g^2 + a_\epsilon(V_k, V_k) = I(f) + I(G),
\] (19)
where $I(f)$ and $I(G)$ are defined and estimated as follows.
\[
I(G) := ((e^{-\omega_k(t)} - e^{-\omega_0(t)})G(t, y), V_k)_g
\]
\[
\leq \|V_k\|_g^2 + C|e^{-\omega_k(t)} - e^{-\omega_0(t)}|^2 \int_\Omega G^2(t, y)dy
\]
\[
\leq \|V_k\|_g^2 + C|e^{-\omega_k(t)} - e^{-\omega_0(t)}|^2 \|G(t)\|_{\infty}^2,
\]
where we recall that $\| \cdot \|_{\infty}$ denotes the norm in $L^\infty(\mathcal{O})$. We then split the nonlinear term into three terms
\[
I(f) := (e^{-\omega_k(t)}f_\epsilon(t, y, e^{\omega_k(t)}v_k) - e^{-\omega_0(t)}f_\epsilon(t, y, e^{\omega_0(t)}v_0), V_k)_g
\]
\[
= (e^{-\omega_k(t)}f_\epsilon(t, y, e^{\omega_k(t)}v_k) - f_\epsilon(t, y, e^{\omega_k(t)}v_0), V_k)_g
\]
\[
+ (e^{-\omega_k(t)}f_\epsilon(t, y, e^{\omega_k(t)}v_0) - f_\epsilon(t, y, e^{\omega_0(t)}v_0), V_k)_g
\]
\[
+ ((e^{-\omega_k(t)} - e^{-\omega_0(t)})f_\epsilon(t, y, e^{\omega_0(t)}v_0), V_k)_g = I_1 + I_2 + I_3.
\]
By the first condition in (6), it follows from the mean valued theorem that $I_1 \leq \beta \|V_k\|_g^2$. By the second condition in (6), we have,
\[
I_2 = e^{-\omega_k(t)} \int_\Omega \frac{\partial f_\epsilon}{\partial s} |e^{\omega_k(t)} - e^{\omega_0(t)}|v_0|V_k|gdy
\]
\[
\leq C|e^{-\omega_k(t)} - e^{-\omega_0(t)}|\left( \int_\Omega (e^{\omega_k(t)} + e^{\omega_0(t)})^p - 2|v_0|^p - 2|\psi_3(t, y, e^{\omega_0(t)}v_0)|dy
\]
\[
+ \int_\Omega |\psi_3(t, y, e^{\omega_0(t)}v_0)|V_k|dy\right)
\]
\[
\leq C|e^{-\omega_k(t)} - e^{-\omega_0(t)}|(|v_k|_p^p + |v_0|_p^p) + C|e^{-\omega_k(t)} - e^{-\omega_0(t)}|^2 \|\psi_3(t)\|_{\infty}^2 + \|V_k\|_g^2,
\]
where we have used (17) in Lemma 2.3. By the condition (5) we have
\[
I_3 \leq |e^{-\omega_k(t)} - e^{-\omega_0(t)}|\left( \int_\Omega e^{(p-1)\omega_0(t)}|v_0|^{p-1}|v_k - v_0|gdy
\]
\[
+ \int_\Omega |\psi_2(t, y, e^{\omega_0(t)}v_n)|V_k|gdy\right)
\]
\[
\leq C|e^{-\omega_k(t)} - e^{-\omega_0(t)}|(|v_k|_p^p + |v_0|_p^p) + C|e^{-\omega_k(t)} - e^{-\omega_0(t)}|^2 \|\psi_2(t)\|_{\infty}^2 + \|V_k\|_g^2.
\]
Let $F_k = \sup_{t\in[\tau, \tau + T]}(|e^{-\omega_k(t)} - e^{-\omega_0(t)}| + |e^{\omega_k(t)} - e^{\omega_0(t)}|)$, then, by Lemma 2.3, $F_k \to 0$ as $k \to \infty$. We substitute all above estimates into (19) to find, for all $t \in [\tau, \tau + T]$,
\[
\frac{d}{dt} \|V_k\|_g^2 \leq C\|V_k\|_g^2 + CF_k(|v_k|_p^p + |v_0|_p^p)
\]
\[
+ CF_k^2(|v_0|_g^2 + \|G(t)\|_{\infty}^2 + \|\psi_2(t)\|_{\infty}^2 + \|\psi_3(t)\|_{\infty}^2).
\] (20)
We then apply the Gronwall inequality on (20) over $[\tau, t]$ to find
\[
\|V_k(t)\|_g^2 \leq Ce^{CT}F_k \int_\tau^{t+T} (|v_k(s)|_p^p + |v_0(s)|_p^p)ds
\]
\[ + Ce^{CT} F_k^2 \int_\tau^{\tau+T} (\|v_0(s)\|_g^2 + \|G(s)\|_\infty^2 + \|\psi_2(s)\|_\infty^2 + \|\psi_3(s)\|_\infty^2) \]
\[ \leq CF_k(1 + \int_\tau^{\tau+T} \|v_k(s)\|_p^p ds) + CF_k^2, \]

where we have used (15) in Lemma 2.2 and Hypotheses \( G, F \) in the last step.

Therefore, in order to prove \( \|V_k(t)\|_g^2 \to 0 \), it suffices to prove

\[ \sup_{k \in \mathbb{N}} \int_\tau^{\tau+T} \|v_k(s)\|_p^p ds \leq C. \tag{21} \]

For this end, we use the following energy inequality (cf. [19, (3.6)]):

\[ \frac{d}{dt} \|v_k\|_g^2 + \lambda \|v_k\|_g^2 + c_1 z^{2-p}(t, \omega_k) \|v_k\|_p^p \leq C z^2(t, \omega_k)(\|G(t)\|_\infty^2 + \|\psi_1(t)\|_\infty^2). \tag{22} \]

Since \( t \) lies in a finite interval, it follows from (17) in Lemma 2.3 that (22) can be rewritten as

\[ \frac{d}{dt} \|v_k\|_g^2 + \lambda \|v_k\|_g^2 + c_2 \|v_k\|_p^p \leq C(\|G(t)\|_\infty^2 + \|\psi_1(t)\|_\infty^2). \]

Note that \( v_k(\tau) \equiv v_\tau \). Then the Gronwall inequality yields

\[ \|v_k(\tau + T)\|_g^2 + c_2 \int_\tau^{\tau+T} e^{\lambda(s-\tau-T)} \|v_k(s)\|_p^p ds \]
\[ \leq e^{-\lambda T} \|v_\tau\|_g^2 + C \int_\tau^{\tau+T} e^{\lambda(s-\tau-T)} \|G(s)\|_\infty^2 + \|\psi_1(s)\|_\infty^2 ds \leq C(T, \tau, v_\tau). \]

Since \( e^{\lambda(s-\tau-T)} \geq e^{-\lambda T} \) for \( s \in [\tau, \tau + T] \), (21) follows immediately and the whole proof is complete. \( \square \)

As a conclusion of Lemma 2.4, the solution mapping is \((\mathcal{F}, \mathcal{B}(X))\) measurable.

We need to prove that it is \((\mathcal{F}, \mathcal{B}(Y))\) and \((\mathcal{F}, \mathcal{B}(Z))\) measurable. In fact, this measurability (in more regular spaces) can be deduced from some abstract results on the concept of quasi-continuity, which seems to be first introduced by Li and Guo [22]. While the relationship between measurability and quasi-continuity has been discussed recently by Cui, Langa and Li [14].

**Lemma 2.5.** For \( t > \tau \), the solution mapping \( \omega \to v(t, \tau, \omega, v_\tau) \) is \((\mathcal{F}, \mathcal{B}(Y))\) measurable and \((\mathcal{F}, \mathcal{B}(Z))\) measurable.

**Proof.** By Lemma 2.4, the mapping is continuous (thus quasi-continuous) from \((\Omega, \varrho)\) to \( X \). By Lemma 2.2, \( v(t, \tau, \omega, v_\tau) \in Y \) for \( t > \tau \) and \( v_\tau \in X \). Since \( Y \hookrightarrow X \) and \( X^* \hookrightarrow Y^* \) densely, it follows from inheritability of the quasi-continuity (see [14, Prop.3]) that the mapping is quasi-continuous from \((\Omega, \varrho)\) to \( Y \). Then, by the measurability of a quasi-continuous mapping (see [14, Prop.5]), the solution mapping is \((\mathcal{F}, \mathcal{B}(Y))\) measurable as required. It is similar to show that the solution mapping is \((\mathcal{F}, \mathcal{B}(Z))\) measurable. \( \square \)

By Lemma 2.2, we can define a family of mappings \( \phi_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X \) by

\[ \phi_\varepsilon(t, \tau, \omega, u_\tau) = u^\varepsilon(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau) = \frac{1}{z(t + \tau, \theta_{-\tau}\omega)} v^\varepsilon(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau), \]

where \( v_\tau = z(\tau, \theta_{-\tau}\omega)u_\tau \). Recall that a concept of random cocycle is given by Wang [31].
Definition 2.6. A mapping $\phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X$ is called a random cocycle on $X$ if

(i) $\phi$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ measurable;
(ii) it holds the cocycle property: for all $t, s \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\phi(t + s, \tau, \omega) = \phi(t, \tau + s, \theta_s \omega)\phi(s, \tau, \omega), \quad \phi(0, \tau, \omega) = \text{id}_X.$$ 

Applying Lemmas 2.2, 2.4, 2.5, we have proved the following result.

Theorem 2.7. For each $\epsilon \in (0, \epsilon_0]$, $\phi_\epsilon$ is a continuous random cocycle on $X$. Its restriction on $Y$ is a quasi-continuous random cocycle on $Y$, and its restriction on $Z$ is a quasi-continuous random cocycle on $Z$.

In order to study random attractors, we take some universes $D_i$, $i = \epsilon, 0, 1$, which are consisted of all set-valued mappings $D_i : \mathbb{R} \times \Omega \rightarrow 2^{X_i} \setminus \emptyset$ satisfying for any $\gamma > 0$,

$$\lim_{t \to +\infty} e^{-\gamma t}\|D_i(t, \theta_{-t} \omega)\|_{X_i}^2 = 0, \quad \tau \in \mathbb{R}, \omega \in \Omega,$$

where $\|D\|$ denote the supremum of norms for all elements, and $X_\epsilon = L^2(\Omega), X_0 = L^2(O), X_1 = X = L^2(O)$.

3. Random attractors in $p$-times Lebesgue space. We need the following basic estimates for the solution $v^\epsilon(s, \tau - t, \theta_{-\tau} \omega, v_0)$ in $X$. When $P$ is a Winner measure, the following lemma was proved in [19] only by using the convergence: $\omega(t)/t \to 0$ as $t \to \pm \infty$. However, this convergence is assumed in Hypothesis $W$, and so the following lemma holds true.

Lemma 3.1. [19] For any $D \in D_1$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exist $T = T(D, \tau, \omega) \geq 4$ such that for all $t \geq T$, $v_0 \in D(\tau - t, \theta_{-\tau} \omega)$ and $\epsilon \in (0, \epsilon_0)$ (with $\epsilon_0$ given in Lemma 2.1),

$$\|v^\epsilon(\tau, \tau - t, \theta_{-\tau} \omega, v_0)\|_{H^1(O)}^2 \leq c\rho_0(\tau, \omega) \quad (23)$$

$$\int_{\tau-t}^{\tau} e^{\frac{2}{\lambda} s(\tau - s)} z^2(s, \theta_{-\tau} \omega)\|u^\epsilon(s)\|_{L^p}^p ds \leq c\rho_0(\tau, \omega), \quad (24)$$

where $\rho_0$ is tempered and given by

$$\rho_0(\tau, \omega) = z^{-2}(\tau, \omega) \int_{-\infty}^{0} e^{\frac{4}{\lambda} s} z^2(s, \omega)(1 + \Psi(s + \tau)) ds$$

with $\Psi(s) = \|G(s)\|_\infty^2 + \|\psi_1(s)\|_\infty^2 + \|\psi_4(s)\|_\infty^2$.

Slightly generalizing (23), we have the following estimates. The proof is standard and so omitted.

$$\sup_{s \in [\tau - 3, \tau]} \sup_{t \geq T} \sup_{\epsilon \in (0, \epsilon_0)} \|v^\epsilon(s, \tau - t, \theta_{-\tau} \omega, v_0)\|_{H^1(O)}^2 \leq c\rho_0(\tau, \omega). \quad (25)$$

Now, we intend to provide further estimates in more regular spaces. The following Gronwall-type lemma will be used frequently, which can be founded in [23, 24, 37].

Lemma 3.2. Let $z, z_1$ and $z_2$ be nonnegative, locally integrable such that $dz/ds$ is locally integrable and

$$\frac{dz}{ds} + az(s) + z_1(s) \leq z_2(s), \quad s \in \mathbb{R},$$
where $a > 0$. If $\tau \in \mathbb{R}$ and $\mu > 0$, then

$$z(\tau) \leq \frac{1}{\mu} \int_{\tau-\mu}^{\tau} e^{a(r-\tau)} z(r) dr + \int_{\tau-\mu}^{\tau} e^{a(r-\tau)} z_2(r) dr,$$

(26)

$$\int_{\tau-\mu}^{\tau} z_1(r) dr \leq \frac{e^{-a\mu}}{\mu} \int_{\tau-3\mu}^{\tau} z(r) dr + \int_{\tau-3\mu}^{\tau} z_2(r) dr.$$

(27)

**Lemma 3.3.** For any $\mathcal{D} \in \mathcal{D}_1$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exist $T = T(\mathcal{D}, \tau, \omega) \geq 4$ such that

$$\sup_{s \in [-T, T]} \sup_{\tau \in (0, r_0)} \|v^\prime(s, \tau - t, \theta - \tau \omega, v_0)\|_p \leq c\rho_1(\tau, \omega),$$

(28)

whenever $v_0 \in D(\tau - t, \theta - \omega)$, where $c > 0$ and $\rho_1$ is a tempered variable given by

$$\rho_1(\tau, \omega) = \rho_0(\tau, \omega) + z^{-2}(\tau, \omega) \int_{-\infty}^{0} e^{\frac{z}{2}} \lambda \alpha z^2(s, \omega) ||\psi_1(s + \tau)||_{L^\infty} ds.$$

**Proof.** We drop the superscript $\prime$ for convenience. Multiply (14) with $g|v|^{p-2}v$ and integrating over $\mathcal{O}$ to obtain

$$\frac{1}{\mu} \int_{\tau-\mu}^{\tau} \int_{\mathcal{O}} gA(x) v \cdot |v|^{p-2} v dy \leq z(t, \omega)(f(x(t, y, u), |v|^{p-2}v)_g + z(t, \omega)(G_\epsilon(t, y), |v|^{p-2}v^\prime)_g).$$

(29)

We first prove non-negativity of the Laplace term. Indeed,

$$\int_{\mathcal{O}} gA(x) v \cdot |v|^{p-2} v dy = \frac{1}{\epsilon} \int_{\mathcal{O}} |v|^{p-2} \Delta v dy = \frac{1}{\epsilon} \int_{\mathcal{O}} \nabla v \cdot \nabla (|v|^{p-2} v) dy$$

$$+ \frac{p-1}{\epsilon} \int_{\mathcal{O}} |v|^{p-2} \nabla v \cdot \nabla (|v|^{p-2} v) dy$$

which implies that

$$z(t, \omega) \int_{\mathcal{O}} g\epsilon f(x(t, y, u), v|v|^{p-2} dy$$

$$\leq -\alpha_1 \gamma_1 z^{-2-p} \int_{\mathcal{O}} |v|^{2p-2} dy + \gamma \int_{\mathcal{O}} z^2 \psi(t, y, \epsilon g(y^*) y_n + 1) ||v|^{p-2} dy$$

$$\leq -\alpha_1 \gamma_1 z^{-2-p} ||v||_{2p-2}^2 + \frac{\lambda}{2} ||v||_p^2 + cz^p \psi(t) ||v||_{L^\infty}^2.$$
Lemma 3.4. in Lemma 2.1. $v$ is the largest time for the absorption in $\epsilon$.

All above estimates implies that there are constants $c_1 > 0$ such that

\[
\frac{d}{dt} \|v\|_p^p + \lambda \|v\|_p^p + c_1 z^{2-p} \|v\|_{2p-2}^{2p-2} \leq c_2 z^p \left( \|\psi_1(t)\|_\infty^2 + \|G(t)\|_\infty^2 \right).
\]  

(30)

For each $s \in [\tau - 3, \tau]$, applying the Gronwall-type inequality (26) with $\mu = s - (\tau - 4) \geq 1$ and replacing $\omega$ by $\theta_{\omega}$ in (30), we obtain that

\[
\|v(s, \tau - t, \theta_{\omega}, v_0)\|_p^p \leq \int_{\tau-4}^s e^{\lambda(s-t)} \|v(\sigma, \tau - t, \theta_{\omega}, v_0)\|_p^p d\sigma \\
+ c \int_{\tau-4}^s e^{\lambda(s-t)} z^p(\sigma, \theta_{\omega})(\|\psi_1(\sigma)\|_\infty^2 + \|G(\sigma)\|_\infty^2) d\sigma \\
\leq c \int_{\tau-4}^s e^{\lambda(s-t)} \|v(\sigma, \tau - t, \theta_{\omega}, v_0)\|_p^p d\sigma \\
+ c \int_{-\infty}^\tau e^{\lambda(s-t)} z^p(\sigma, \theta_{\omega})(\|\psi_1(\sigma)\|_\infty^2 + \|G(\sigma)\|_\infty^2) d\sigma.
\]

Since $\omega(t)/t \to 0$ as $t \to \pm\infty$, there is $\rho_2 = \rho_2(\tau, \omega) > 0$ such that

\[
0 < \sup_{-\infty < \sigma \leq 0} \left( e^{\frac{1}{2} \lambda(\sigma - \tau)} z^{p-2}(\sigma, \theta_{\omega}) \right) \leq \rho_2
\]

which yields

\[
\int_{\tau-4}^\tau e^{\lambda(s-t)} \|v(\sigma, \tau - t, \theta_{\omega}, v_0)\|_p^p d\sigma \\
= \int_{\tau-4}^\tau (e^{\frac{1}{2} \lambda(\sigma - \tau)} z^{p-2}(\sigma, \theta_{\omega})) e^{\frac{1}{2} \lambda(\sigma - \tau)} z^{2}(\sigma, \theta_{\omega}) \|u(\sigma, \tau - t, \theta_{\omega}, u_0)\|_p^p d\sigma \\
\leq \rho_2 e^{-\frac{1}{2} \lambda \tau} \int_{\tau-4}^\tau e^{\frac{1}{2} \lambda(\sigma - \tau)} z^{2}(\sigma, \theta_{\omega}) \|u(\sigma, \tau - t, \theta_{\omega}, u_0)\|_p^p d\sigma.
\]

Hence, by (42) in Lemma 3.1, we obtain (28) as required.

\[\square\]

Remark 3. We say that $T > 0$ is an entry time for absorption if $T = T(D, \tau, \omega)$ is the largest time for the absorption in $X, Y$ and $Z$, given in Lemmas 3.1, 3.3 and (25). Notice that any entry time $T$ is independent of $\epsilon \in (0, \epsilon_0)$, where $\epsilon_0$ is given in Lemma 2.1.

Lemma 3.4. For each $(D, \tau, \omega) \in \mathcal{D}_1 \times \mathbb{R} \times \Omega$, let $T := T(D, \tau, \omega) \geq 4$ be an entry time, independent of $\epsilon \in (0, \epsilon_0)$. Then,

\[
\lim_{K \to \infty} \sup_{\epsilon \in (0, \epsilon_0)} \left( \int_{\mathcal{O}(\epsilon)} \int_{t \geq T \wedge \mathcal{O}(\epsilon)} |v^\epsilon(\tau, t, \theta_{\omega}, v_0)|^p \, dy \, ds \right) = 0,
\]  

(31)

uniformly in $v_0 \in \mathcal{D}(\tau - t, \theta_{\omega})$, where $\mathcal{O}(|v^\epsilon| \geq K) = \mathcal{O}_K \cup \mathcal{O}_{-K}$ with

\[
\mathcal{O}_K = \mathcal{O}_K(s, \tau - t, v_0) = \{ y \in \mathcal{O}; v^\epsilon(s, \tau - t, \theta_{\omega}, v_0)(y) \geq K \},
\]

\[
\mathcal{O}_{-K} = \mathcal{O}_{-K}(s, \tau - t, v_0) = \{ y \in \mathcal{O}; v^\epsilon(s, \tau - t, \theta_{\omega}, v_0)(y) \leq -K \}.
\]

Proof. Let $\mathcal{D}, \tau, \omega$ be fixed, and so the entry time $T$ is fixed. We claim that

\[
\lim_{K \to \infty} \sup_{s \in [\tau - 3, \tau] \epsilon \in (0, \epsilon_0)} \sup_{t \geq T \wedge \mathcal{O}(\epsilon)} \left( |\mathcal{O}_K| \right) = 0,
\]

(32)

where the Lebesgue measure $|\mathcal{O}_K|$ decreases as $K$ increases. Indeed, by Lemma 3.3, we know that

\[
|\mathcal{O}_K(s, \tau - t, v_0)| \leq \int_{\mathcal{O}_K} |v^\epsilon(s, \tau - t)|^p \, dy \leq \int_{\mathcal{O}} |v^\epsilon(s, \tau - t)|^p \, dy \leq C < +\infty,
\]

as required.
where and below, we denote by $C = C(\tau, \omega)$ and denote by $c$ a constant. Note that $C$ is independent of $\epsilon \in (0, \epsilon_0)$ and $t \geq T$. Letting $K \to +\infty$ in the above inequality yields (32).

On the other hand, by the continuity of $s \to z(s, \omega)$, we can find $C_1 = C_1(\tau, \omega) > 0$ and $K_1 = K_1(\tau, \omega) > 0$ such that

$$C_1 \leq z(s, \theta_{-\tau} \omega) \leq K_1, \forall s \in [\tau - 3, \tau].$$

By the condition (4), we can take $K_2 > 0$ such that

$$f(s, x, u) \leq -\alpha_1 u^{p-1} + \psi_1(s, x) u^{-1}, \text{ if } u > K_2. \tag{33}$$

Suppose $K$ is large enough such that $K \geq K_1 K_2 + 1$, and take the inner product of (14) with $g(v - K)_+^{p-1}$ in $L^2(\Omega)$, where $w_+ := \max\{w, 0\}$. The result is

$$\frac{1}{p} \frac{d}{ds} \| (v - K)_+^p \|_p^p + \lambda(v, (v - K)_+^{p-1})_g + (A_v, (v - K)_+^{p-1})_g = z(s, \theta_{-\tau} \omega) f(s, y, u, (v - K)_+^{p-1})_g + z(s, \theta_{-\tau} \omega) (G_v(s, y), (v - K)_+^{p-1})_g \tag{34}$$

for all $s \in [\tau - 3, \tau]$. After straightforward calculations, we have

$$\lambda \int_{\Omega} gv(v - K)_+^{p-1} dy \geq \lambda\| (v - K)_+^p \|_p^p. \tag{35}$$

If $v \geq K$, then $u = z^{-1}(s, \theta_{-\tau} \omega) u \geq K_1 K \geq K_2$. By (33),

$$f(s, x, u) \leq -\alpha_1 u^{p-1} + \psi_1(s, x) u^{-1} = -\alpha_1 z^{1-p}(s, \theta_{-\tau} \omega) u^{p-1} + z(s, \theta_{-\tau} \omega) \psi_1(s, x) u^{-1}.$$

Hence, the nonlinear term $z(s, \theta_{-\tau} \omega) f(s, y, u, (v - K)_+^{p-1})_g$ is equal to

$$z(s, \theta_{-\tau} \omega) \int_{\Omega_K} g f(s, y^*, \epsilon g(y^*) y_{n+1}, u) (v - K)_+^{p-1} dy$$

$$\leq -\alpha_1 \gamma_1 z^{2-p}(s, \theta_{-\tau} \omega) \int_{\Omega_K} u^{p-1} (v - K)_+^{p-1} dy$$

$$+ z^2(s, \theta_{-\tau} \omega) \gamma_2 \int_{\Omega_K} |\psi_1(s, y^*, \epsilon g(y^*) y_{n+1})| v^{-1} (v - K)_+^{p-1} dy$$

$$\leq -\frac{\alpha_1 \gamma_1}{2} z^{2-p}(s, \theta_{-\tau} \omega) \int_{\Omega_K} u^{p-1} (v - K)_+^{p-1} dy + C \| \psi_1(s) \|_{\infty}^2 |\Omega_K|, \tag{36}$$

where we have used the fact $v^{-1} \leq 1/K \leq 1$ and $a(v - K)_+^{p-1} \leq C(\delta) a^2 + \delta v^{-1} (v - K)_+^{p-1}$ on $\Omega_K$. Similarly,

$$z(s, \theta_{-\tau} \omega) (G_v(s, y), (v - K)_+^{p-1})_g = z(s, \theta_{-\tau} \omega) \int_{\Omega_K} g G_v(s, y) (v - K)_+^{p-1} dy$$

$$\leq \frac{\alpha_1 \gamma_1}{4} z^{2-p}(s, \theta_{-\tau} \omega) \int_{\Omega_K} u^{p-1} (v - K)_+^{p-1} dy + C \| G(s) \|_{\infty}^2 |\Omega_K|. \tag{37}$$

By (34)-(37) and by the continuity of $z(\cdot, \theta_{-\tau} \omega)$, we have

$$\frac{d}{ds} \| (v - K)_+^p \|_p^p + 2C_2 \int_{\Omega_K} u^{p-1} (v - K)_+^{p-1} dy$$

$$\leq C_3 (\| \psi_1(s) \|_{\infty}^2 + \| G(s) \|_{\infty}^2) |\Omega_K|, \tag{38}$$
where \( C_2, C_3 \) are independent of \( K \) and \( \epsilon \). Notice that
\[
\int_{\mathcal{O}_K} v^{p-1}(v - K)^{p-1}_+ dy \geq \int_{\mathcal{O}_K} v^{p-2}(v - K)_+^p dy \geq K^{p-2}\|v - K\|_p^p,
\]
then, (38) can be rewritten as follows:
\[
\frac{d}{ds}\|v - K\|_p^p + C_2 K^{p-2}\|v - K\|_p^p + C_2 \int_{\mathcal{O}_K} v^{p-1}(v - K)^{p-1}_+ dy \leq C_3(\|\psi_1(s)\|_\infty^2 + \|G(s)\|_\infty^2)|\mathcal{O}_K|.
\]
By the Gronwall-type inequality (26) in Lemma 3.2 with \( \mu = 1 \), we have
\[
\|v(\tau) - K\|_p^p \leq \int_0^\tau e^{C_2 K^{p-2}(s - \tau)}\|v(s) - K\|_p^p ds + C_3|\mathcal{O}_K| \int_0^\tau (\|\psi_1(s)\|_\infty^2 + \|G(s)\|_\infty^2) ds.
\]
The last integral is finite in view of \( \psi_1, G \in L^2_{\text{loc}}(\mathbb{R}, L^\infty(\tilde{\mathcal{O}})) \). Since \( \|v - K\|_p \leq \|v\|_p \), it follows from Lemma 3.3 that
\[
\sup_{s \in [\tau - \eta, \tau]} \sup_{t \geq T} \sup_{\epsilon \in (0, \epsilon_0]} \|\psi_1(s, \tau - t, \theta_{-t} \omega) - K\|_p \leq C_4.
\]
Therefore, by (32), as \( K \to \infty \),
\[
\|v(\tau, \tau - t, \theta_{-t} \omega, v_0) - K\|_p \leq C_4 e^{-C_2 K^{p-2}} + C|\mathcal{O}_K| \to 0,
\]
uniformly in \( \epsilon \in (0, \epsilon_0], t \geq T \) and \( v_0 \in \mathcal{D}(\tau - t, \theta_{-t} \omega) \). Note that \( v \leq 2(v - K) \) if \( v \geq 2K \). We have
\[
\int_{\mathcal{O}_K} |v(\tau, \tau - t, \theta_{-t} \omega, v_0)|^p dy \leq 2^p |\psi_1|^{-1}(v - K)_+^p \to 0,
\]
as \( K \to +\infty \), uniformly in \( \epsilon \in (0, \epsilon_0], t \geq T \) and \( v_0 \in \mathcal{D}(\tau - t, \theta_{-t} \omega) \). Similarly, the above uniform convergence holds true on \( \mathcal{O} - 2K \) if we replace \( (v - K)_+ \) by the function \( (v + K)_- := \max\{0, -v - K\} \). The proof is complete. \( \square \)

**Proposition 1.** For each \( \epsilon \in (0, \epsilon_0] \), the random cocycles \( \phi_\epsilon \) (given in Theorem 2.7) is eventually compact in \( L^p(\tilde{\mathcal{O}}) \), that is, for each \( \mathcal{D}, \tau, \omega \in \mathcal{D}_1 \times \mathbb{R} \times \Omega \), there is a \( T_\epsilon = T_\epsilon(\mathcal{D}, \tau, \omega) \) such that \( B_\epsilon(T) \) is a pre-compact set in \( L^p \) for all \( T \geq T_\epsilon \), where,
\[
B_\epsilon(T) := \bigcup_{t \geq T} \phi_\epsilon(t, \tau - t, \theta_{-t} \omega) \mathcal{D}(\tau - t, \theta_{-t} \omega).
\]

**Proof.** Let \( T_\epsilon = T_\epsilon(\mathcal{D}, \tau, \omega) \) be an entry time and let \( \epsilon \in (0, \epsilon_0] \) be fixed. By Lemma 3.1, \( B_\epsilon(T_\epsilon) \) is bounded in \( H^1(\mathcal{O}) \), and so \( B_\epsilon(T_\epsilon) \) is pre-compact in \( L^2(\mathcal{O}) \). Given \( \eta > 0 \), it follows from Lemma 3.4 that there is a \( K > 0 \) such that
\[
\int_{\mathcal{O}(|v| \geq K)} |v(y)|^p dy < \frac{\eta^p}{2^{p+3}}, \quad \text{for all } v \in B_\epsilon(T_\epsilon).
\]
As \( B_\epsilon(T_\epsilon) \) is pre-compact in \( L^2(\mathcal{O}) \), it has a finite \( (2^{-p} K^{2-p} \eta^p)^{1/2} \)-net in \( L^2 \) such that the centers \( v_i \in B_\epsilon(T), \ i = 1, 2, \ldots, m \). Then, for any \( v \in B_\epsilon(T_\epsilon) \), we can
find a center \( v_0 \) such that \( \| v - v_0 \|^2 \leq 2^{-p} K^{2-p} \eta^p \). We will prove \( \| v - v_0 \|_p \leq \eta, \) by dividing the domain \( \mathcal{O} = \bigcup_{j=1}^{d} \mathcal{O}_j \), where,

\[
\mathcal{O}_1 = \mathcal{O}(|v| \geq K) \cap \mathcal{O}(|v_i| \leq K), \quad \mathcal{O}_2 = \mathcal{O}(|v| \leq K) \cap \mathcal{O}(|v_i| \geq K), \\
\mathcal{O}_3 = \mathcal{O}(|v| \geq K) \cap \mathcal{O}(|v_i| \geq K), \quad \mathcal{O}_4 = \mathcal{O}(|v| \leq K) \cap \mathcal{O}(|v_i| \leq K).
\]

It is easy to see that \( |v| \geq K \geq |v_i| \) on \( \mathcal{O}_1 \), and \( |v| \leq K \leq |v_i| \) on \( \mathcal{O}_2 \), then, by (41),

\[
\int_{\mathcal{O}_1} |v - v_i|^p \leq 2^p \int_{\mathcal{O}_1} (|v|^p + |v_i|^p) \leq 2^{p+1} \int_{\mathcal{O}(|v| \geq K)} |v|^p \leq \frac{\eta^p}{4},
\]

\[
\int_{\mathcal{O}_2} |v - v_i|^p \leq 2^{p+1} \int_{\mathcal{O}(|v_i| \geq K)} |v_i|^p \leq \frac{\eta^p}{4}.
\]

On the other hand,

\[
\int_{\mathcal{O}_3} |v - v_i|^p \leq 2^p \left( \int_{\mathcal{O}(|v| \geq K)} |v|^p + \int_{\mathcal{O}(|v_i| \geq K)} |v_i|^p \right) \leq \frac{\eta^p}{4},
\]

\[
\int_{\mathcal{O}_4} |v - v_i|^p \leq (2K)^{p-2} \int_{\mathcal{O}} |v - v_i|^2 \leq (2K)^{p-2} \| v - v_i \|^2 \leq \frac{\eta^p}{4}.
\]

Therefore, \( \| v - v_i \|_p \leq \eta \), which implies that, for any \( \eta > 0 \), \( B_i(T_\eta) \) has a finite \( \eta \)-net in \( L^p(\mathcal{O}) \), and thus it is pre-compact in \( L^p(\mathcal{O}) \). It is obvious that \( B_i(T) \) decreases as \( T \) increase, then \( B_i(T) \) is pre-compact in \( L^p \) for all \( T \geq T_0 \). \( \square \)

**Theorem 3.5.** For each \( \epsilon \in (0, \epsilon_0] \), the cocycle \( \phi_\epsilon \) generated by the problem (14) has a unique \((X,Y)\) random attractor \( \mathcal{A}_\epsilon = \{ \mathcal{A}_\epsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \), where \( X = L^2(\mathcal{O}) \) and \( Y = L^p(\mathcal{O}) \). That is,

1. For each \( \tau \in \mathbb{R}, \omega \to \mathcal{A}_\epsilon(\tau, \omega) \) is \( \mathcal{F} \)-measurable in \( X \) and in \( Y \) respectively;
2. \( \mathcal{A}_\epsilon \in \mathcal{D}_1 \), and \( \mathcal{A}_\epsilon(\tau, \omega) \) is compact in \( X \cap Y \);
3. \( \mathcal{A}_\epsilon \) is invariant, i.e. \( \phi_\epsilon(s, \tau, \omega) \mathcal{A}_\epsilon(\tau, \omega) = \mathcal{A}_\epsilon(\tau + s, \theta_s \omega) \) for \( s \geq 0 \);
4. \( \mathcal{A}_\epsilon \) is pullback attracting in \( Y \), i.e. for every \( \mathcal{D} \in \mathcal{D}_1 \),

\[
\lim_{t \to \infty} \text{dist}_Y (\phi_\epsilon(t, \tau - t, \theta_{-t} \omega) \mathcal{D} - \tau, \theta_{-\tau} \omega), \mathcal{A}_\epsilon(\tau, \omega)) = 0.
\]

**Proof.** By Lemma 3.1, \( \phi_\epsilon \) has a random absorbing set \( \mathcal{E} \) from \( \mathcal{D}_1 \), which is defined by

\[
\mathcal{E}(\tau, \omega) = \{ u \in L^2(\mathcal{O}) : \| u \|^2 \leq c\rho_0(\tau, \omega) \}, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.
\]

By Proposition 1, \( \phi_\epsilon \) is \((X,Y)\)-eventually compact and thus \((X,Y)\)-omega-limit compact. Then, by an abstract result on bi-spatial attractors given in [21, 23], it is easy to prove the existence of a \((X,Y)\)-attractor \( \mathcal{A}_\epsilon \) (except for \( \mathcal{F} \)-measurability).

By Lemma 2.4, the cocycle \( \phi_\epsilon \) is \( \mathcal{F} \)-measurable in \( X \), which, together with an abstract result given in [31], implies \( \mathcal{F} \)-measurability of \( \mathcal{A}_\epsilon \) in \( X \). By Lemma 2.5, the cocycle \( \phi_\epsilon \) is \( \mathcal{F} \)-measurable in \( Y \). By Lemma (3.3), \( \phi_\epsilon \) has a random absorbing set \( \mathcal{E}_p \) in \( L^p(\mathcal{O}) \) given by

\[
\mathcal{E}_p(\tau, \omega) = \{ u \in L^2 \cap L^p(\mathcal{O}) : u \in \mathcal{E}(\tau, \omega), \| u \|_p^p \leq c\rho_1(\tau, \omega) \}, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.
\]

Therefore, it follows from [14, Theorem 19] that the attractor \( \mathcal{A}_\epsilon \) is \( \mathcal{F} \)-measurable in the initial space only, and the bi-spatial

**Remark 4.** In the definition of a bi-spatial random attractor given above, we require the \( \mathcal{F} \)-measurability holds true in both state spaces, while the bi-spatial attractor given in [21] is \( \mathcal{F} \)-measurable in the initial space only, and the bi-spatial
attractor given in [14] is \( \mathcal{F} \)-measurable in the terminate space only. The above definition may be more reasonable.

4. Random attractors in Sobolev space. We need to consider the eigenvalue problem:

\[-\Delta_x \tilde{u}(x) = \lambda \tilde{u}(x), \ x \in \mathcal{O}_\epsilon, \ \frac{\partial \tilde{u}}{\partial \nu_\epsilon} = 0, \text{ on } \partial \mathcal{O}_\epsilon,\]

which is equivalent to

\[A_\epsilon u(y) = \lambda u(y), \ y \in \mathcal{O}, \ \Upsilon_\epsilon u \cdot \nu = 0, \text{ on } \partial \mathcal{O}.\]

Then, it is well known that each unbounded operator \( A_\epsilon \) on \( X \) has countable eigenvalues \( \lambda_j^\epsilon, j = 1, 2, \cdots \) such that

\[0 < \lambda_1^\epsilon \leq \cdots \leq \lambda_j^\epsilon \to \infty \quad \text{as } j \to \infty \quad \text{for each } \epsilon \in (0, \epsilon_0).\]

By using the method of symbolical truncation given in Lemma 3.4, we can provide an auxiliary estimate.

**Lemma 4.1.** For each \((\mathcal{D}, \tau, \omega) \in \mathcal{D}_1 \times \mathbb{R} \times \Omega \) and \( \epsilon \in (0, \epsilon_0) \), we have

\[
\limsup_{j \to \infty} \sup_{t \geq T} \int_{\tau - 1}^{\tau} e^{(\lambda_j^\epsilon + 2\lambda)(s - \tau)} \int_\mathcal{D} |v'(s, \tau - t, \theta_{-\tau} \omega, v_0)|^{2p - 2} dy \, ds = 0, \tag{42}
\]

uniformly in \( v_0 \in \mathcal{D}(\tau - t, \theta_{-t} \omega) \), where \( T := T(\mathcal{D}, \tau, \omega) \geq 4 \) is an entry time, independent of \( \epsilon \) and \( j \).

**Proof.** By applying the Gronwall-type inequality (27) to the energy inequality (38), we have

\[
C_2 \int_{\tau - 1}^{\tau} \int_{\mathcal{O}_K} v^{p - 1}(s)(v - K)^{p - 1} dy \, ds \leq e^{-C_2 K^{p - 2}} \int_{\tau - 3}^{\tau} \|(v - K)^+\|_p^p \, ds + C_3 |\mathcal{O}'_K| \int_{\tau - 3}^{\tau} (\|v_1(s)\|_\infty^2 + \|G(s)\|_\infty^2) \, ds.
\]

By Lemma 3.3, we know that for all \( \epsilon \in (0, \epsilon_0) \), \( t \geq T \) and \( v_0 \in \mathcal{D}(\tau - t, \theta_{-t} \omega) \),

\[
\int_{\tau - 3}^{\tau} \|(v - K)^+\|_p^p \, ds \leq \int_{\tau - 3}^{\tau} \|v'(s, \tau - t, \theta_{-t} \omega, v_0)\|_p^p \, ds \leq C \int_{\tau - 3}^{\tau} ds = C_{4}.
\]

By the assumptions \( \mathbf{G} \) and \( \mathbf{F} \), \( \int_{\tau - 3}^{\tau} (\|v_1(s)\|_\infty^2 + \|G(s)\|_\infty^2) \, ds < C_5(\tau) < +\infty \), which implies, as \( K \to \infty \),

\[
\int_{\tau - 3}^{\tau} \int_{\mathcal{O}'_K} v^{p - 1}(s, \tau - t)(v - K)^{p - 1} dy \, ds \leq C_4 e^{-C_2 K^{p - 2}} + C_3 C_5 |\mathcal{O}'_K| \to 0, \tag{43}
\]

uniformly in \( t \geq T \) and \( v_0 \in \mathcal{D}(\tau - t, \theta_{-t} \omega) \). Similarly, as \( K \to \infty \),

\[
\int_{\tau - 1}^{\tau} \int_{\mathcal{O}'_{-K}} |v|^{p - 1}(s, \tau - t)(v + K)^{p - 1} dy \, ds \to 0. \tag{44}
\]

Next, we prove the main convergence (42). Let \( \delta_j^\epsilon := \lambda_j^\epsilon + 2\lambda \to +\infty \). We make the decomposition: \( \mathcal{O} = \mathcal{O}'_K \cap \mathcal{O}'_{-K} \cap \mathcal{O}(|v'| < K) \) and

\[
I(j) := \int_{\tau - 1}^{\tau} e^{\delta_j^\epsilon(s - \tau)} \int_\mathcal{O} |v(s, \tau - t, \theta_{-t} \omega)|^{2p - 2} = I_1(j, K) + I_2(j, K) + I_3(j, K),
\]
where $I_1, I_2, I_3$ are defined and estimated as follows.

$$I_1(j, K) := \int_{\tau-1}^{\tau} e^{\delta_j(s-\tau)} \int_{\Omega_K} |v(s, \tau - t, \theta-\tau \omega)|^{2p-2}$$

$$= \int_{\tau-1}^{\tau} e^{\delta_j(s-\tau)} \int_{\Omega_K} |(v - K) + K|^{p-1} |v|^{p-1}$$

$$\leq c \int_{\tau-1}^{\tau} e^{\delta_j(s-\tau)} \int_{\Omega_K} (v^{p-1}(v - K)_{+}^{p-1} + K^{p-1}v^{p-1})$$

$$\leq c \int_{\tau-1}^{\tau} \int_{\Omega_K} v^{p-1}(v - K)_{+}^{p-1} + cK^{p-2} \int_{\tau-1}^{\tau} e^{\delta_j(s-\tau)} \int_{\Omega} |v|^p$$

Similarly, on $\Omega_{c, K}$, we can rewrite

$$|v|^{2p-2} = |(v + K) - K|^{p-1} |v|^{p-1} \leq 2^p |(v + K)|^{p-1} |v|^{p-1} + K^{p-2} |v|^p,$$

which implies that

$$I_2(j, K) := \int_{\tau-1}^{\tau} e^{\delta_j(s-\tau)} \int_{\Omega_{c, K}} |v(s, \tau - t, \theta-\tau \omega)|^{2p-2}$$

$$\leq c \int_{\tau-1}^{\tau} \int_{\Omega_{c, K}} |v|^{p-1}(v + K)_{+}^{p-1} + cK^{p-2} \int_{\tau-1}^{\tau} e^{\delta_j(s-\tau)} \|v(s)\|^p ds.$$  

By (28) in Lemma 3.3, we have

$$\sup_{\epsilon \in [0, \epsilon_1]} \sup_{\tau \in [\tau-1, \tau]} \sup_{t \geq T} \sup_{\epsilon} \|v(\tau - t, \theta-\tau \omega)\|^p \leq C.$$  

Let $\eta > 0$. Then, by (43) and (44), we can choose a large $K$ such that for all $j \in \mathbb{N}$,

$$I_1(j, K) + I_2(j, K) \leq C \eta + C K^{p-2} \int_{\tau-1}^{\tau} e^{\delta_j(s-\tau)} ds \leq C \eta + \frac{C K^{p-2}}{\delta_j}.$$  

Finally, $I_3$ is given by

$$I_3(j, K) := \int_{\tau-1}^{\tau} e^{\delta_j(s-\tau)} \int_{\Omega(|v| \leq K)} |v(s, \tau - t, \theta-\tau \omega)|^{2p-2}$$

$$\leq |\Omega| K^{2p-2} \int_{\tau-1}^{\tau} e^{\delta_j(s-\tau)} ds \leq \frac{c K^{2p-2}}{\delta_j}.$$  

Since $\delta_j \to \infty$, we can take $j_0$ large enough such that $\delta_j \geq \eta^{-1}(K^{p-2} + K^{2p-2})$ for all $j \geq j_0$, which implies that for all $j \geq j_0$,

$$I(j) = I_1(j, K) + I_2(j, K) + I_3(j, K) \leq C \eta + C(K^{p-2} + K^{2p-2})\delta_j^{-1} \leq C \eta,$$

uniformly in $t \geq T$. The proof is complete. $\square$

It is known that the eigenvectors $\{e_j^\epsilon\}_{j=1}^\infty$ consist of a complete orthonormal basis of $X$ and $\{e_j^\epsilon\}_{j=1}^\infty \subset H^1(\Omega) := Z$. Let $H_m^\epsilon = \text{span}\{e_1^\epsilon, e_2^\epsilon, \ldots, e_m^\epsilon\} \subset Z$ and $P_m^\epsilon : Z \to H_m^\epsilon$ be the canonical projector and $I$ be the identity. Then for every $v \in Y$ there exists a unique decomposition

$$v = v_1^\epsilon + v_2^\epsilon, \quad v_1^\epsilon = P_m^\epsilon v \in H_m^\epsilon, \quad v_2^\epsilon = (I - P_m^\epsilon)v \in H_m^{\epsilon, \perp},$$

where $H_m^{\epsilon, \perp}$ is the orthogonal complement of $H_m^\epsilon$. 

\[ \]
Lemma 4.2. For each \((\mathcal{D}, \tau, \omega) \in \mathcal{D}_1 \times \mathbb{R} \times \Omega\) and \(\epsilon \in (0, \epsilon_0]\), we have
\[
\lim_{m \to \infty} \sup_{t \geq T} \| (I - \frac{P_m}{m}) v^\epsilon(\tau, t - \theta, \omega, v_0) \|^2_{H^1(\Omega)} = 0,
\]
whenever \(v_0 \in \mathcal{D}(\tau - t, \theta_\omega)\), where \(T := T(\mathcal{D}, \tau, \omega)\) is the entry time, independent of \(\epsilon\) and \(m\).

Proof. We drop the superscript \(\epsilon\) for convenience. Taking the inner product of (14) with \(A, v_2\) in \((L^2(\Omega), \| \cdot \|_g)\), we find that,
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} a_v(v_2, v_2) + \| A_v v_2 \|^2_g + \lambda a_v(v_2, v_2) & = z(t, \omega)(f_e(t, y, u), A_v v_2) + z(t, \omega)(G_e(t, y), A_v v_2).
\end{align*}
\]
(46)

We first estimate the nonlinear term. By the condition (5), we have
\[
\begin{align*}
z(t, \omega)(f_e(t, y, u), A_v v_2) & \leq \gamma_2 z(t, \omega) \int_{\Omega} f_e(t, y, u) A_v v_2 dy \\
& \leq \frac{1}{4} \| A_v v_2 \|^2_g + \gamma_2^2 z^2(t, \omega) \int_{\Omega} |f_e(t, y, u)|^2 dy \\
& \leq \frac{1}{4} \| A_v v_2 \|^2_g + c z^2(t, \omega) \| z \|^2_{L^2} + c z^2(t, \omega) \| \psi_2(t) \|^2_{\infty}.
\end{align*}
\]
(47)

For the last term in (46), we have
\[
\begin{align*}
z(t, \omega)(G_e(t, y), A_v v_2) & \leq \frac{1}{4} \| A_v v_2 \|^2_g + z^2(t, \omega) \| G_e(t, y) \|^2_g \\
& \leq \frac{1}{4} \| A_v v_2 \|^2_g + c z^2(t, \omega) \| G(t) \|^2_{\infty}.
\end{align*}
\]

Since \(\lambda_{m+1}^e\) is the first eigenvalue of \(A_v\) restricted on the subspace \(H^1_{\mathcal{D}_1}\), we have
\[
\begin{align*}
(A_v v_2, A_v v_2) & \geq \lambda_{m+1}^e (A_v v_2, v_2) = \lambda_{m+1}^e a_v(v_2, v_2).
\end{align*}
\]
(48)

Therefore, (46)-(48) imply a differential inequality of \(v_2(s, \tau - t, \theta_\omega, v_2, m)\) as follows.
\[
\begin{align*}
\frac{d}{ds} a_v(v_2, v_2) + (\lambda_{m+1}^e + 2\lambda) a_v(v_2, v_2) & \leq c z^{1-2p}(s, \theta_{-\omega}) \| v \|^2_{L^{2p-2}} + c z^2(s, \theta_{-\omega}) (\| \psi_2(s) \|^2_{\infty} + \| G(s) \|^2_{\infty}).
\end{align*}
\]
(49)

By applying the Gronwall-type inequality (26) to (49), we have
\[
\begin{align*}
a_v(v_2(s, \tau - t, \theta_{-\omega}, v_0), v_2(s, \tau - t, \theta_{-\omega}, v_0)) & \leq \int_{\tau - 1}^{\tau} e^{(\lambda_{m+1}^e + 2\lambda)(s-\tau)} a_v(v_2(s, \tau - t, \theta_{-\omega}, v_0), v_2(s, \tau - t)) ds \\
& + C \int_{\tau - 1}^{\tau} e^{(\lambda_{m+1}^e + 2\lambda)(s-\tau)} \int_{\Omega} |v(s, \tau - t, \theta_{-\omega}, v_0)|^{2p-2} dy ds \\
& + C \int_{\tau - 1}^{\tau} e^{(\lambda_{m+1}^e + 2\lambda)(s-\tau)} (\| \psi_2(s) \|^2_{\infty} + \| G(s) \|^2_{\infty}) ds
\end{align*}
\]
\[
= J_1(m, t, v_0) + J_2(m, t, v_0) + J_3(m).
\]

We first consider the term of order \(2p - 2\). By Lemma 4.1, \(J_2(m, t, v_0) \to 0\) as \(m \to \infty\), uniformly in \(t \geq T\) and \(v_0 \in \mathcal{D}(\tau - t, \theta_\omega)\).
We then estimate $J_1$. It is easy to see that $a_{\epsilon}(v_2, v_2) \leq \|v_2\|^2_{H_t^1} = \|(I - P_m^\infty)v\|^2_{H_t^1}$, which is bounded from (23). Hence, as $m \to \infty$,

$$J_1(m, t, v_0) \leq \int_{\tau-1}^\tau e^{(\lambda_{m+1} + 2\lambda)(s-\tau)}c\rho_0(\tau, \omega)ds \leq \epsilon \rho_0 \frac{1}{\lambda_{m+1} + 2\lambda} \to 0,$$

uniformly in $t \geq T$ and $v_0 \in D(\tau - t, \theta - \omega)$.

Finally, by Hypotheses F, G, we know $\psi_2, G \in L^2_{\text{local}}(\mathbb{R}, L^\infty(\hat{O})))$, and so

$$J_3(m) \leq \int_{\tau-1}^\tau (\|\psi_2(s)\|^2_{\infty} + \|G(s)\|^2_{\infty})ds \to +\infty.$$

As $e^{(\lambda_{m+1} + 2\lambda)(s-\tau)} \to 0$ for all $s < \tau$, the Lebesgue controlled convergence theorem implies that

$$J_3(m) = \int_{\tau-1}^\tau e^{(\lambda_{m+1} + 2\lambda)(s-\tau)}(\|\psi_2(s)\|^2_{\infty} + \|G(s)\|^2_{\infty})ds \to 0 \text{ as } m \to \infty.$$

The whole proof is complete. \qed

**Theorem 4.3.** For each $\epsilon \in (0, \epsilon_0]$, the cocycle $\phi_{\epsilon}$ generated by the problem (14) has a unique $(X, Z)$ random attractor $A_{\epsilon}$, this attractor is the same set as given in Theorem 3.5, where $X = L^2(\hat{O})$ and $Z = H^1(\hat{O})$.

**Proof.** Let $B_{\epsilon}(T) := \bigcup_{\tau \geq T} \phi_{\epsilon}(t, \tau, t, \theta - \omega)D(\tau - t, \theta - \omega)$ for $D \in D_1$ and $\epsilon \in (0, \epsilon_0]$. Let $T_0$ be an entry time for absorption. We will prove $\kappa_ZB_{\epsilon}(T_0) = 0$, where the Kuratowski measure $\kappa(\cdot)$ denotes the minimal diameter of all sets constituted a finite cover. Indeed, by Lemma 4.2, for each $\eta > 0$, there is an $m = m(\eta) \in \mathbb{N}$ such that

$$\|(I - P_m^\infty)B_{\epsilon}(T_0)\|_{H_t^1} < \eta.$$

By Lemma 3.1, we have

$$\|P_m^\infty B_{\epsilon}(T_0)\|_{H_t^1} \leq c\|B_{\epsilon}(T_0)\|_{H_t^1} \leq c\rho_0(\tau, \omega),$$

which means that $P_m^\infty B_{\epsilon}(T_0)$ is bounded in the finitely dimensional subspace of $Z$, and thus it is pre-compact in $Z$ with the Kuratowski measure zero. Hence,

$$\kappa_Z B_{\epsilon}(T_0) \leq \kappa_Z P_m^\infty B_{\epsilon}(T_0) + \kappa_Z(I - P_m^\infty)B_{\epsilon}(T_0) = \kappa_Z(I - P_m^\infty)B_{\epsilon}(T_0) < \eta.$$

We have, $\kappa_Z B_{\epsilon}(T_0) = 0$, which implies that $B_{\epsilon}(T_0)$ is pre-compact in $Z$, and so $\phi_{\epsilon}$ is eventually compact.

Therefore, by an abstract result on bi-spatial attractors given in [21], the random cocycle $\phi_{\epsilon}$ has a $(X, Z)$ attractor $A_{\epsilon}$, which is the same set as an attractor in $X$ or in $Y$ given in Theorem 3.5. By Lemma 2.5, the cocycle $\phi_{\epsilon}$ is $\mathcal{F}$-measurable in $Z$. By Lemma 3.1, $\phi_{\epsilon}$ has a random absorbing set $\mathcal{E}_\epsilon$ in $H^1(\hat{O})$ given by

$$\mathcal{E}_\epsilon(\tau, \omega) = \{u \in H^1(\hat{O}) : \; u \in \mathcal{E}_\theta(\tau, \omega), \; \|u\|^2_{H_t^1} \leq \epsilon \rho_0(\tau, \omega)\}, \; \forall \tau \in \mathbb{R}, \; \omega \in \Omega.$$

Therefore, it follows from [14, Theorem 19] that the attractor $A_{\epsilon}$ is $\mathcal{F}$-measurable in $Z$. \qed

In order to consider the limiting equation (2) on $Q$, we define an operator $A_0$ by $D(A_0) = \{u \in H^2(Q), \frac{\partial u}{\partial n} = 0 \text{ on } \partial Q\}$, and for $u \in D(A_0)$

$$A_0u = -\frac{1}{\gamma} \sum_{i=1}^n (g_{a_{i\gamma}}) u_i, \; (A_0u, v)_g = a_0(u, v) = \int_Q g \nabla u \cdot \nabla v \; dy^*.$$
Lemma 5.3. \[19, \text{Theorem 5.2}\].

Under the more strong topology.

We denote by \(v^0(t, \tau, \omega, \nu^0_\tau) = z(t, \omega)u^0(t, \tau, \omega, \nu^0_\tau)\), where \(u^0\) is a solution of problem (2). Then \(v^0\) satisfies

\[
\frac{dv^0}{dt} + A_0v^0 + \lambda v^0 = z(t, \omega)f_0(t, y^*, z^{-1}(t, \omega)v^0) + z(t, \omega)G_0(t, y^*),
\]

(50)

The solution determines a continuous random cocycle \(\phi_0(t, \tau, \omega, u^0_\tau)\) on \(L^2(Q)\). Analogous results on random attractors for this cocycle are easily obtained when the \(n\)-dimensional domain \(Q\) replaces the \(n + 1\)-dimensional domain \(O\).

**Proposition 2.** The cocycle \(\phi_0\) generated by the problem (50) has a unique \(\mathcal{D}_0\)-pullback \((L^2(Q), L^p(Q))\) random attractor \(\mathcal{A}_0 = \{\mathcal{A}_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}\) \(\mathcal{D}_0\). Moreover, \(\mathcal{A}_0\) is a \((L^2(Q), H^1(Q))\) random attractor.

5. **Upper semicontinuity of bi-spatial random attractors.** In order to consider convergence for random attractors, we assume some convergence for both source and force.

**Hypothesis C.** There exist two functions \(\mu_1(\cdot), \mu_2(\cdot) \in L^2_{\text{loc}}(\mathbb{R})\) such that

\[
\|f(t, \tau, s) - f_0(t, \tau, s)\|_{L^2(\mathcal{O})} \leq \mu_1(t)\epsilon, \quad \text{for all } t, s \in \mathbb{R},
\]

\[
\|G(t, \cdot) - G_0(t, \cdot)\|_{L^2(\mathcal{O})} \leq \mu_2(t)\epsilon, \quad \text{for all } t \in \mathbb{R}.
\]

In the above hypothesis, we regard that a function \(u\) defined on \(Q\) is identical to the function \(\hat{u}(y^*, y_{n+1}) = u(y^*), (y^*, y_{n+1}) \in \mathcal{O}\). It is easy to see that \(u \in L^2(Q)\) if and only if \(\hat{u} \in L^2(\mathcal{O})\).

Conversely, for a function defined on \(\mathcal{O}\), we consider its average function on \(Q\) by using the average operator \(\mathcal{M} : L^2(\mathcal{O}) \mapsto L^2(Q)\),

\[
(\mathcal{M}u)(y^*) = \int_0^1 u(y^*, y_{n+1})dy_{n+1}.
\]

**Lemma 5.1.** \[17\] If \(u \in H^1(\mathcal{O})\), then \(\mathcal{M}u \in H^1(Q)\) and

\[
\|u - \mathcal{M}u\|_{L^2(\mathcal{O})} \leq c\|u\|_{H^1(\mathcal{O})},
\]

where \(c\) is a constant independent of \(\epsilon\).

Under the hypothesis C, the following convergence of the cocycle \(\phi_\epsilon\) can be found in [19, Theorem 5.1].

**Lemma 5.2.** \[19\] Suppose \(\{v^0_\epsilon, \epsilon \in (0, \epsilon_0]\} \subset H^1(\mathcal{O})\) such that \(\sup_{\epsilon \in (0, \epsilon_0]} \|v^0_\epsilon\|_{H^1(\mathcal{O})} \leq c\). Then,

\[
\lim_{\epsilon \to 0} \|\phi_\epsilon(t, \tau, \omega)v^0_\epsilon - \phi_0(t, \tau, \omega)v^0_0\|_{L^2(\mathcal{O})} = 0,
\]

(51)

for each \(t \geq 0, \tau \in \mathbb{R}\) and \(\omega \in \Omega\).

The following convergence of the random attractor \(\mathcal{A}_\epsilon\) in \(L^2(\mathcal{O})\) can be found in [19, Theorem 5.2].

**Lemma 5.3.** \[19\] The random attractor \(\mathcal{A}_\epsilon\) is upper semicontinuous in \(L^2(\mathcal{O})\) at \(\epsilon = 0\), that is, for every \(\tau \in \mathbb{R}, \omega \in \Omega\),

\[
\lim_{\epsilon \to 0} \text{dist}_{L^2(\mathcal{O})}(\mathcal{A}_\epsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0.
\]

(52)

Next, we show that upper semicontinuity of the random attractor holds true under the more strong topology.
Theorem 5.4. The random attractor $\mathcal{A}_\epsilon$ is upper semicontinuous in $L^p(\mathcal{O})$ at $\epsilon = 0$, that is,

$$\lim_{\epsilon \to 0} \text{dist}_{L^p(\mathcal{O})}(\mathcal{A}_\epsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0, \forall \tau \in \mathbb{R}, \omega \in \Omega.$$  \hfill (53)

Proof. We divided the proof into three parts.

**Step 1.** We show that any sequence $\{z_k\}_{k=1}^\infty$ is pre-compact in $L^2(\mathcal{O})$, where $z_k \in \mathcal{A}_{\epsilon_k}(\tau, \omega)$, $\epsilon_k \to 0$, and $((\tau, \omega)) \in \mathbb{R} \times \Omega$ is fixed. Indeed, by the convergence of the attractors given in Lemma 5.3, we have

$$\text{dist}_{L^2(\mathcal{O})}(z_k, \mathcal{A}_0(\tau, \omega)) \leq \text{dist}_{L^2(\mathcal{O})}(\mathcal{A}_{\epsilon_k}(\tau, \omega), \mathcal{A}_0(\tau, \omega)) \to 0$$

as $k \to \infty$. For each $k \in \mathbb{N}$, we can take $\tilde{z}_k \in \mathcal{A}_0(\tau, \omega)$ such that

$$\|z_k - \tilde{z}_k\|_{L^2(\mathcal{O})} \leq \text{dist}_{L^2(\mathcal{O})}(z_k, \mathcal{A}_0(\tau, \omega)) + \frac{1}{k}.$$

As $\mathcal{A}_0(\tau, \omega)$ is a compact set in $L^2(Q)$, passing to a subsequence, we have $\|\tilde{z}_k - \tilde{z}\|_{L^2(\mathcal{O})} \to 0$ for some $\tilde{z} \in L^2(Q)$. Therefore, as $k \to \infty$,

$$\|z_k - \tilde{z}\|_{L^2(\mathcal{O})} \leq \|\tilde{z}_k - \tilde{z}\|_{L^2(\mathcal{O})} + \|z_k - \tilde{z}_k\|_{L^2(\mathcal{O})} \leq \|\tilde{z}_k - \tilde{z}\|_{L^2(\mathcal{O})} + \text{dist}_{L^2(\mathcal{O})}(z_k, \mathcal{A}_0(\tau, \omega)) + \frac{1}{k} \to 0.$$

**Step 2.** We prove that any sequence $z_k \in \mathcal{A}_{\epsilon_k}(\tau, \omega)$ is pre-compact in $L^p(\mathcal{O})$, where $\epsilon_k \to 0$, and we assume without lose of generality that $\epsilon_k \in (0, \epsilon_0]$ for all $k \in \mathbb{N}$. By Lemma 3.1, each cocycle $\phi_{\epsilon_k}$ has a collective absorbing set $\mathcal{E}$ defined by

$$\mathcal{E}(\tau, \omega) := \{u \in L^2(\mathcal{O}) : \|u\|^2 \leq c\rho_0(\tau, \omega)\}.$$  \hfill (54)

Then, the invariance of $\mathcal{A}_{\epsilon_k}$ and the absorption of $\mathcal{E}$ implies that

$$\bigcup_{k \in \mathbb{N}} \mathcal{A}_{\epsilon_k}(\tau, \omega) \subset \mathcal{E}(\tau, \omega).$$

Let $T$ be an entry time when $\mathcal{E} \in \mathcal{D}_1$ is absorbed by itself. By the invariance of $\mathcal{A}_{\epsilon_k}$ again, we know that for each $k \in \mathbb{N}$,

$$z_k \in \phi_{\epsilon_k}(T, \tau - T, \theta - T \omega) \mathcal{A}_{\epsilon_k}(\tau - T, \theta - T \omega) \subset \phi_{\epsilon_k}(T, \tau - T, \theta - T \omega) \mathcal{E}(\tau - T, \theta - T \omega).$$

Since $\mathcal{E} \in \mathcal{D}_1$ and $\{\epsilon_k\} \subset (0, \epsilon_0]$, it follows from (31) in Lemma 3.4 that for each $\eta > 0$ there is a $K = K(\eta)$ such that

$$\sup_{k \in \mathbb{N}} \int_{\mathcal{O}(|z_k| \geq K)} |z_k|^p \leq \eta^p.$$  \hfill (55)

By Step 1, $\{z_k\}_{k=1}^\infty$ is pre-compact in $L^2(\mathcal{O})$, and so it has a finite $(K^{2-p} \eta^p)^{1/2}$-net in $L^2(\mathcal{O})$ such that the finite centers are taken from the sequence $\{z_k\}$. Then, for each $z_k$, there is a center $z^i$ such that

$$\|z_k - z^i\|_{L^2(\mathcal{O})} \leq K^{2-p} \eta^p.$$  \hfill (56)

It is similar to the proof of Proposition 2 to split the domain $\mathcal{O} = \bigcup_{j=1}^4 \mathcal{O}_j$ by

$$\mathcal{O}_1 = \mathcal{O}(|z_k| \geq K) \cap \mathcal{O}(|z^i| \leq K), \mathcal{O}_2 = \mathcal{O}(|z_k| \leq K) \cap \mathcal{O}(|z^i| \geq K),$$

$$\mathcal{O}_3 = \mathcal{O}(|z_k| \geq K) \cap \mathcal{O}(|z^i| \geq K), \mathcal{O}_4 = \mathcal{O}(|z_k| \leq K) \cap \mathcal{O}(|z^i| \leq K).$$

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By Step 2, the sequence that Step 3 δ > is pre-compact in such that, passing to a subsequence, ∥H

By (55), we have E

By (56), we have 3682 FUZHI LI, YANGRONG LI AND RENHAI WANG E

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Hence, ∥z_k − z^i∥_p ≤ 2^p + 3∥z||_p and so ∥z_k − z^i||_p ≤ 8∥z||_p. Therefore, the sequence \{z_k\}_{k=1}^\infty has a finite \eta-net in L^p(\Omega) for any \eta > 0, which implies that the sequence \{z_k\}_{k=1}^\infty is pre-compact in L^p(\Omega) as required.

**Step 3.** We argue the convergence of random attractors in L^p(\Omega) by contradiction. Suppose (53) is not true, then, there exist \delta > 0, \epsilon_k → 0 and z_k ∈ A_{\epsilon_k}(\tau, \omega) such that

\[
\text{dist}_{L^p(\Omega)}(z_k, A_0(\tau, \omega)) ≥ \delta, \quad \forall k ∈ \mathbb{N}.
\]

By Step 2, the sequence \{z_k\}_{k=1}^\infty is pre-compact in L^p(\Omega). So, there is z ∈ L^p(\Omega) such that, passing to a subsequence,

\[
\lim_{k \to \infty} \|z_k - z\|_{L^p(\Omega)} = 0, \quad \text{and dist}_{L^p(\Omega)}(z, A_0(\tau, \omega)) ≥ \delta. \tag{57}
\]

For the random absorbing set E given by (54), we define a bi-parametric set E_0 on L^2(\Omega) by

\[
E_0(\tau, \omega) = \text{cl}_{L^2(\Omega)}\{Mu : u ∈ E(\tau, \omega)\},
\]

where cl_{L^2(\Omega)} denote the closure in L^2(\Omega). It is easy to prove that E_0 is a closed tempered set in L^2(\Omega) and so E_0 ∈ D_0 in view of E ∈ D_1. By the attraction of the random attractor A_0 in L^p(\Omega), there is a T_0 > 0 such that for all t ≥ T_0,

\[
\text{dist}_{L^p(\Omega)}(\phi_0(t, \tau - t, \theta - t\omega)E_0(\tau - t, \theta - t\omega), A_0(\tau, \omega)) < \delta. \tag{58}
\]

Let T = T(\mathcal{E}) ≥ T_0 be an entry time when \mathcal{E} ∈ D_1 is absorbed by itself, where T is independent of \epsilon_k. For each k ∈ \mathbb{N}, by the invariance of A_{\epsilon_k}, there are \hat{z}_k ∈ A_{\epsilon_k}(\tau - T, \theta - T\omega) ⊂ E(\tau - T, \theta - T\omega) such that

\[
z_k = \phi_{\epsilon_k}(T, \tau - T, \theta - T\omega)\hat{z}_k.
\]

By Lemma 3.1 and by invariance again, there exist another entry time \hat{T} = \hat{T}(\mathcal{E}, \tau - T, \theta - T\omega) such that for all t ≥ \hat{T} and k ∈ \mathbb{N},

\[
\|\hat{z}_k\|_{H^1_{\epsilon_k}} ≤ \|\phi_{\epsilon_k}(t, \tau - T - t, \theta - t\theta - T\omega)A_{\epsilon_k}(\tau - T - t, \theta - t\theta - T\omega)\|_{H^1_{\epsilon_k}}
\leq \|\phi_{\epsilon_k}(t, \tau - T - t, \theta - t\theta - T\omega)E(\tau - T - t, \theta - T\omega)\|_{H^1_{\epsilon_k}}
\leq c_\epsilon(\tau - T, \theta - T\omega), \tag{59}
\]

which means that \|\hat{z}_k\|_{H^1_{\epsilon_k}} is bounded in k, and satisfies the assumption of Lemma 5.2. Hence, Lemma 5.2 gives

\[
\|\phi_{\epsilon_k}(T, \tau - T, \theta - T\omega)\hat{z}_k - \phi_0(T, \tau - T, \theta - T\omega)M\hat{z}_k\|_{L^2(\Omega)} → 0, \quad \text{as } k → \infty.
\]
Remark 5. The upper continuity of random attractors in $H^1(Q)$ remains open, although we have shown the existence of an attractor in $H^1$. The main difficulty arises from that the equivalence between $\| \cdot \|_{H^1}$ and $\| \cdot \|_{H^1}$ is not uniform in $\epsilon$ (see Lemma 2.1), and that the eigenvalue $\lambda'_m$ depends on $\epsilon$. Maybe, the spectral continuity given by Arrieta and Carvalho [4] can provide some new insights for this question.

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