COHOMOLOGICAL INVARIANTS OF THE STACK OF HYPERELLiptIC CURVES OF ODD GENUS

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Abstract. We compute the cohomological invariants of \( \mathcal{H}_g \), the moduli stack of smooth hyperelliptic curves, for every odd \( g \).

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Introduction

In topology, an important notion is the one of characteristic classes, initially developed by Stiefel and Whitney in the first half of the twentieth century. Roughly, we can say that, once fixed a topological group \( G \) and a cohomology theory \( H \), characteristic classes are a functorial way to associate to every principal \( G \)-bundle over a topological space \( X \) a cohomology class in \( H(X) \). In other terms, characteristic classes are natural transformations from the functor

\[
\mathcal{B}G : \text{Top} \rightarrow \text{Set}, \quad X \mapsto \{ \text{\( G \)-bundles over} \ X \}\]

to the cohomology functor \( X \mapsto H(X) \). Cohomological invariants first appeared as the reformulation of this idea in an algebraic setting.

More precisely, fix a base field \( k_0 \), a prime \( p \) and an algebraic group \( G \). Then we replace the category Top with \( \text{Field}/k_0 \), the category of field extensions of \( k_0 \), and the cohomology theory \( H \) with

\[
H^* : \text{Field}/k_0 \rightarrow \text{Ring}, \quad K \mapsto \bigoplus_i H^i_{\text{et}}(\text{Spec}(K), \mu_p^{\otimes i})
\]

We also substitute \( \mathcal{B}G \) with its algebro-geometrical counterpart, namely the classifying stack \( \mathcal{B}G \) or, better, its functor of points:

\[
P_{\mathcal{B}G} : \text{Field}/k_0 \rightarrow \text{Set}, \quad K \mapsto \{ \text{\( G \)-torsors over} \ \text{Spec}(K) \}\]

Cohomological invariants are then defined by copying the definition of characteristic classes in topology:

Definition. A \textit{cohomological invariant of } \( \mathcal{B}G \) \textit{is a natural transformation } \( P_{\mathcal{B}G} \rightarrow H^* \).
Observe that the set of cohomological invariants, which is denoted $\text{Inv}^\bullet(BG)$, have a natural structure of graded-commutative ring.

Their first appearance, though not in this formulation, can be traced back to the seminal paper \[Wit37\] and since then they have been extensively studied (see for instance \[GMS03\]).

In the recent work \[Pir18\], Pirisi extended the notion of cohomological invariants from classifying stacks to smooth algebraic stacks over $k_0$:

**Definition.** Let $X$ be a smooth algebraic stack. Then a cohomological invariant of $X$ is a natural transformation $P_X \rightarrow H^\bullet$ from the functor of points of $X$ to $H^\bullet$ which satisfies a certain continuity condition (see \[Pir18, definition 1.1\]).

The graded-commutative ring of cohomological invariants of a smooth algebraic stack $X$ is denoted $\text{Inv}^\bullet(X)$.

In \[Pir18\] Pirisi also computed the cohomological invariants of $\mathcal{M}_{1,1}$, the moduli stack of smooth elliptic curves, and in the subsequent works \[Pir17\] and \[Pir\] he computed the cohomological invariants of $\mathcal{H}_g$, the moduli stack of smooth hyperelliptic curves, when $g$ is even or equal to 3. The goal of the present work is to compute the cohomological invariants of $\mathcal{H}_g$ for every $g$ odd and our main result is the following:

**Theorem.** Suppose $p = 2$. Then $\text{Inv}^\bullet(\mathcal{H}_g)$ is generated as an $\mathbb{F}_2$-module by the cohomological invariants $x_1, w_2, x_2, \ldots, x_{g+1}, x_{g+2}$ where the degree of each $x_i$ is $i$ and $w_2$ is the second Stiefel-Whitney class coming from $\text{Inv}^\bullet(B\text{PGL}_2)$.

Suppose $p \neq 2$. Then $\text{Inv}^\bullet(\mathcal{H}_g)$ is trivial unless $p$ divides $2g + 1$, in which case they are generated as $\mathbb{F}_p$-module by 1 and a single non-zero invariant of degree 1.

The computation of cohomological invariants is based on the isomorphism between $\text{Inv}^\bullet([X/G])$, where $X$ is a smooth scheme endowed with an action of an algebraic group $G$, and the equivariant Chow group with coefficients $A_0^G(X, H^\bullet)$ (see \[Pir18, section 4\]). The Chow groups with coefficients were first introduced in \[Ros96\] as a generalization of ordinary Chow groups with some new properties, in particular the existence of a certain long exact sequence induced by closed immersions of schemes, which is a tool of crucial importance for our purposes.

What enables us to extend the result of Pirisi for the stack $\mathcal{H}_3$ to $\mathcal{H}_g$ for every odd $g \geq 3$ is the notion of $\text{GL}_3$-counterpart of a $\text{PGL}_2$-scheme, first introduced in \[DL\]. More precisely we have:

**Definition.** Let $X$ be a scheme of finite type over a field, endowed with a $\text{PGL}_2$-action. Then a $\text{GL}_3$-counterpart of $X$ is a scheme $Y$ endowed with an action of $\text{GL}_3$ such that $[Y/\text{GL}_3] \simeq [X/\text{PGL}_2]$. If $Y$ is a $\text{GL}_3$-counterpart of a $\text{PGL}_2$-scheme, it follows almost immediately that $A_{\text{PGL}_2}(X, H^\bullet) \simeq A_{\text{GL}_3}(Y, H^\bullet)$.

The main obstruction in extending the result of Pirisi in genus 3 consists in proving that a certain morphism of $\text{PGL}_2$-equivariant Chow groups with coefficient is zero. More precisely, if $\mathbb{P}(1, 2n)$ denotes the projective space of binary forms of degree $2n$, endowed with the $\text{PGL}_2$-action $A \cdot f(x,y) := \det(A)^n f(A^{-1}(x,y))$.
and $\Delta_{1,2n} \subset \mathbb{P}(1,2n)$ is the closed subscheme parametrising singular forms, then to extend the result of Pirisi for $H_3$ is enough to prove that
\[ i_* : A^0_{\text{GL}_2}(\Delta_{1,2n}) \rightarrow A^1_{\text{GL}_2}(\mathbb{P}(1,2n)) \]
is zero.

We are able to prove this claim by finding a $GL_3$-counterpart of $\mathbb{P}(1,2n)$, which we denote $\mathbb{P}(V_n)_\text{sm}$, and of $\Delta_{1,2n}$, which we denote $D_{\text{sm}}$, and then using some new tools, which were not available in the $PGL_2$-equivariant setting, to study the morphism
\[ i_* : A^0_{\text{GL}_3}(D_{\text{sm}}, H^*) \rightarrow A^1_{\text{GL}_3}(\mathbb{P}(V_n)_\text{sm}, H^*) \]
and to prove that is actually zero.

**Structure of the paper.** In section 1 we recall some basic properties of equivariant Chow groups with coefficients. In section 2 we prove the main theorem of the paper, but without proving the key lemma which is only assumed to hold. The strategy of proof follows closely the one contained in [Pir]. The remainder of the paper is devoted to prove the key lemma.

In section 3 we introduce the notion of $GL_3$-counterpart of a $PGL_2$-scheme. This machinery is then applied to a $PGL_2$-scheme which is relevant for our purposes, namely $\mathbb{P}(1,2n)$, the projective space of binary forms of degree $2n$.

In section 4 we study the geometry of a central object, whom we refer to as the fundamental divisor $D$.

The observations made in this section are then applied in section 5 in order to do some intersection theoretical computations useful to prove the key lemma: the proof is completed in section 6.

For the convenience of the reader, a more detailed description of the contents can be found at the beginning of every section.

**Assumptions and notations.** We fix once and for all a prime $p$ and an algebraically closed field $k_0$ of characteristic not dividing $p$. Every scheme is assumed to be of finite type over $\text{Spec}(k_0)$. Every time we will refer to $H_0$, we will implicitly assume $g \geq 3$ and odd.

If $X$ is a variety, with the notation $H^\bullet(X)$ we will always mean the graded-commutative ring $\bigoplus_i H_i^\bullet(\xi_X, \mu^0_p)$, where $\xi_X$ is the generic point of $X$. Sometimes, we will write $H^\bullet(R)$, where $R$ is a finitely generated $k_0$-algebra, to indicate $H^\bullet(\text{Spec}(R))$. The Chow groups with coefficients in $H^\bullet$ will be denoted $A^i(-, H^\bullet)$ or $A^i(-, H^*)$, depending on the choice of the grading, by dimension or by codimension. At a certain point we will adopt the shorthand $A^i(-)$ to denote Chow groups with coefficients in $H^\bullet$ of codimension $i$, and we will drop the apex to refer to the direct sum of Chow groups with coefficients of every codimension. A Chow group with coefficient is said to be trivial when is isomorphic to $\mathbb{F}_p$.

The $G$-equivariant Chow groups with coefficients in $H^\bullet$ will be denoted $A^i_G(-, H^\bullet)$ and we will write $A^n_G$ to indicate $A^n_G(\text{Spec}(k_0))$. Similar notations will be adopted for Chow groups $CH^i(-)$. Also, we will write $CH^i(-) \otimes \mathbb{F}_p$ for the tensor product $CH^i(-) \otimes \mathbb{F}_p$.

Throughout the paper, a relevant role will be played by $A(2,2)$, the space of trinary forms of degree 2. The closed subscheme parametrising forms which are squares of linear forms will be denoted $A(2,2)_{\text{sq}}$. The closed subscheme parametrising singular forms will be denoted $A(2,2)_{\text{sing}}$, and we will refer to $A(2,2) \setminus A(2,2)_{\text{sing}}$ as $A(2,2)_{\text{sm}}$, and to $A(2,2)_{\text{sing}} \setminus A(2,2)_{\text{sq}}$ as $A(2,2)_{\text{red}}$. Finally, we will denote $A(2,2)_{\text{red}}$ the scheme $A(2,2) \setminus A(2,2)_{\text{sq}}$. At any rate, these definitions will be frequently repeated along the paper.
Acknowledgements. I wish to thank my advisor Angelo Vistoli for his constant support during this work and for introducing me to this subject. Also, I’m indebted with Roberto Pirisi, for his patience in answering my questions: I think that the relevance of his ideas can be easily detected all along the paper.

1. Equivariant Chow groups with coefficients

In this section we collect together some basic definitions and useful properties of the equivariant Chow groups with coefficients in $H^\bullet$. Our interest in these groups is due to [Pir18, theorem 4.9]:

Theorem 1.1. If $X$ is a smooth scheme endowed with an action of an algebraic group $G$, then we have

$$A^0_G(X, H^\bullet) \simeq \text{Inv}^\bullet([X/G])$$

First, let us sketch the construction of the standard Chow groups with coefficients in $H^\bullet$. The original definition can be found in [Ros96], and a survey on this and related subjects is [Gmi07]. Let $X$ be a scheme and define

$$Z_i(X, H^\bullet) = \oplus_{x \in X, \text{dim} x = i} H^\bullet(k(x))$$

where the sum is taken over all the points of $X$ having dimension equal to $i$. For every $i$ ranging from 0 to the dimension of $X$, there exists a differential

$$d_i : Z_i(X, H^\bullet) \rightarrow Z_{i-1}(X, H^\bullet)$$

The Chow groups with coefficients are then defined as

$$A_i(X, H^\bullet) := \ker(d_i)/\text{im}(d_{i+1})$$

As usual, the notation $A^i(X, H^\bullet)$ stands for $A_{n-i}(X, H^\bullet)$, where $n$ is the dimension of $X$.

There are two important things that have to be stressed. The first one is that the Chow groups with coefficients have two natural gradings, one given by codimension and the other given by the degree: an element $\alpha$ has codimension $i$ and degree $d$ if it is in $A^i(X, H^d)$. The second important fact, which is rather simple to check, is that we can recover from the Chow groups with coefficients in $H^\bullet$ the usual Chow groups tensorized with $\mathbb{F}_p$: indeed we have

$$\text{CH}^i(X) \otimes \mathbb{F}_p = A^i(X, H^0)$$

In other terms, the elements of degree 0 are the usual algebraic cycles, tensorized with $\mathbb{F}_p$.

When $X$ is smooth, the Chow groups with coefficients inherit the structure of a graded ring: the multiplication of an element of codimension $i$ and degree $d$ by an element of codimension $i'$ and degree $d'$ returns an element of codimension $i + i'$ and degree $d + d'$.

Just as for the usual Chow groups, for every $f : X \rightarrow Y$ proper there is a well defined morphism

$$f_* : A_i(X, H^\bullet) \rightarrow A_i(Y, H^\bullet)$$

and for every $f$ flat of relative constant dimension, or when $Y$ is smooth, there exists a well defined morphism

$$f^* : A^i(Y, H^\bullet) \rightarrow A^i(X, H^\bullet)$$

All the properties that hold for the usual Chow groups (see [Ful98, chapter 1]) actually are true also in the case of Chow groups with coefficients.
One of the main distinctive features of Chow groups with coefficients is that, given a closed subscheme \( Z \hookrightarrow X \) with complementary open subscheme \( U \hookrightarrow X \), there exists a long exact sequence

\[
\cdots \to A_i(X, \mathbb{H}^*) \xrightarrow{i^*} A_i(U, \mathbb{H}^*) \xrightarrow{\partial} A_{i-1}(Z, \mathbb{H}^*) \xrightarrow{\imath_*} A_{i-1}(X, \mathbb{H}^*) \to \cdots
\]

This naturally extends the usual localization exact sequence for Chow groups. Observe that a key role for the definition of the sequence above is played by the boundary morphism \( \partial \): it sends an element of codimension \( i \) and degree \( d \) to an element of codimension \( i - 1 \) and degree \( d - 1 \). Instead, the other morphisms appearing in the sequence above preserves both the codimension and the degree.

An important property is the following: if we have a cartesian square

\[
\begin{array}{ccc}
Y & \xrightarrow{i} & X \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{i'} & X'
\end{array}
\]

where all the morphisms are closed immersion, this induces a commutative square

\[
\begin{array}{ccc}
A_k(Y' \setminus Y, \mathbb{H}^*) & \xrightarrow{i''^*} & A_k(X' \setminus X, \mathbb{H}^*) \\
\downarrow & & \downarrow \\
A_k(Y, \mathbb{H}^*) & \xrightarrow{i''_*} & A_k(X, \mathbb{H}^*)
\end{array}
\]

where \( i'' \) is the restriction of \( i' \) to \( Y' \setminus Y \).

If \( \pi : E \to X \) is a vector bundle, then we have an isomorphism

\[
\pi^* : A^i(X, \mathbb{H}^*) \simeq A^i(E, \mathbb{H}^*)
\]

If \( \pi : \mathbb{P}(E) \to X \) is a projective bundle, then for \( i < \text{rk}(E) \) we have:

\[
A^i(\mathbb{P}(E), \mathbb{H}^*) \simeq \bigoplus_{j=0}^i A^j(X, \mathbb{H}^*)
\]

There is also a well defined theory of Chern classes for Chow groups with coefficients in \( \mathbb{H}^* \), first introduced in \cite{Pir17}, which resembles very much the theory of Chern classes for the usual Chow groups. In particular, for \( \pi : \mathbb{P}(E) \to X \) a projective bundle, there is a well defined element \( h = c_1(\mathcal{O}(1)) \) in \( A^1(\mathbb{P}(E), \mathbb{H}^*) \) and moreover if \( X \) is smooth there is an isomorphism of rings

\[
A(\mathbb{P}(E), \mathbb{H}^*) \simeq A(X, \mathbb{H}^*)[h]/(f)
\]

where \( f \) is an element of \( A(X, \mathbb{H}^*)[h] \) monic of degree equal to the rank of \( E \).

The whole theory of Chow groups with coefficients has an equivariant counterpart. Let \( G \) be an algebraic group acting on a scheme \( X \). Following the same ideas of \cite{EG88}, one can define the equivariant groups \( A^*_G(X, \mathbb{H}^*) \) as follows: take a representation \( V \) of \( G \) such that \( G \) acts freely on an open subscheme \( U \subset V \) whose complement has codimension greater than \( i + 1 \). Then we define

\[
A^*_G(X, \mathbb{H}^*) := A^i(X \times U/G, \mathbb{H}^*)
\]

This definition is independent of all the choice we have made, and all the properties that we stated for the Chow groups with coefficients hold in the equivariant setting.
2. Cohomological invariants of $\mathcal{H}_g$

A **family of hyperelliptic curves of genus $g$** is a pair $(C \to S, \iota)$ where $C \to S$ is a proper and smooth morphism whose fibres are curves of genus $g$, and $\iota \in \text{Aut}(C)$ is an involution such that the quotient $C/\langle \iota \rangle \to S$ is a proper and smooth morphism whose fibres are curves of genus 0.

Therefore, the stack of hyperelliptic curves of genus $g$ is defined as the stack in groupoids over the site $\text{Sch}/k_0$ whose objects are:

$$\mathcal{H}_g(S) = \{(C \to S, \iota)\}$$

In this section we will prove our main theorem, which is the following:

**Theorem 2.1.** Suppose $p = 2$. Then $\text{Inv}^\bullet(\mathcal{H}_g)$ is generated as an $\mathbb{F}_2$-module by $x_1, w_2, x_2, ..., x_{g+1}, x_{g+2}$, where the degree of each $x_i$ is $i$ and $w_2$ is the second Stiefel-Whitney class coming from $\text{Inv}^\bullet(\mathbb{B}\text{PGL}_2)$.

Suppose $p \neq 2$. Then $\text{Inv}^\bullet(\mathcal{H}_g)$ is trivial unless $p$ divides $2g + 1$, in which case they are generated as $\mathbb{F}_p$-module by 1 and a single non-zero invariant of degree 1.

The case $g = 3$ had already been proved in [Pir]. Actually, the only obstruction to generalize the result contained there to higher genus is given by [Pir, corollary 3.9], whose proof does not obviously extend to the other cases. Once one generalizes that corollary, the computation of the cohomological invariants is basically done. Therefore, what we present here is substantially a rewriting of the proof contained in [Pir]: the only difference is in lemma 2.4.(1), which is the extension of [Pir, corollary 3.9] that was missing. The proof of this key result, which is rather non-trivial, is postponed to section 6, because we need to develop more theory in order to complete it.

2.1. **Setup.** Let $A(1, n)$ be the affine space of binary forms of degree $n$ and let $X_n$ be the open subscheme parametrising forms with distinct roots. Then in [AY04] the authors gave the following presentation of $\mathcal{H}_g$ as a quotient stack, when $g \geq 3$ is an odd number:

$$\mathcal{H}_g \simeq [X_{2g+2}/\text{PGL}_2 \times \mathbb{G}_m]$$

where the action on an element $(A, \lambda)$ in $\text{PGL}_2$ on a form $f(x, y)$ is given by the formula

$$(A, \lambda) \cdot f(x, y) := \lambda^{-2} \det(A)^{g+1} f(A^{-1}(x, y))$$

Using theorem 1.1 we obtain

$$\text{Inv}^\bullet(\mathcal{H}_g) \simeq A^0_{\text{PGL}_2 \times \mathbb{G}_m}(X_{2g+2})$$

Therefore the computation of the cohomological invariants of $\mathcal{H}_g$ blows down to the computation of the codimension 0 part of an equivariant Chow ring with coefficients.

Let $\mathbb{P}(1, 2n)$ be the projective space of binary forms of degree $2n$, and denotes $\Delta_{1, 2n}$ the divisor parametrising singular forms. We are going to compute first $A^0_{\text{PGL}_2}(\mathbb{P}(1, 2g + 2) \setminus \Delta_{1, 2g+2})$, and then we will use the fact that

$$X_{2g+2} \longrightarrow \mathbb{P}(1, 2g + 2) \setminus \Delta_{1, 2g+2}$$

is a $\text{PGL}_2 \times \mathbb{G}_m$-equivariant $\mathbb{G}_m$-torsor to deduce a presentation of $A^0_{\text{PGL}_2 \times \mathbb{G}_m}(X_{2g+2})$.

2.2. **Proof of the main theorem.** Let us recall the following useful computation, which is [Pir17 prop. 2.11] and [Pir] prop.3.1, prop.3.2:

**Proposition 2.2.** We have:

1. $A_{\text{GL}_3} \simeq CH_{\text{GL}_3} \otimes \mathbb{F}_p$. 

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(2) Suppose $p = 2$. Then $A_{PGL_2}$ is freely generated over $CH_{PGL_2} \otimes \mathbb{F}_2$ by an element $w_2$ of codimension 0 and degree 2 and by an element $\tau_1$ of codimension 1 and degree 1. For $p \neq 2$, then $A_{PGL_2}(\text{Spec}(k))$ is equal to $CH_{PGL_2} \otimes \mathbb{F}_p$.

(3) We have $A^0_{PGL_2}(\Delta_{1,2}) \cong A^0_{PGL_2}(\mathbb{P}^1) \simeq k_0$.

The proposition below is the starting point for understanding $\text{Inv}^* (H_g)$:

**Proposition 2.3.** Suppose $p = 2$. Then the ring $A^0_{PGL_2}(\mathbb{P}(1,2n) \setminus \Delta_{1,2n})$ is freely generated as $\mathbb{F}_2$-module by $n + 1$ elements $x_1, \ldots, x_n, w_2$ where the degree of $x_i$ is $i$ and $w_2$ is the second Stiefel-Whitney class coming from the cohomological invariants of $BPGL_2$.

Suppose $p \neq 2$. Then the ring $A^0_{PGL_2}(\mathbb{P}(1,2n) \setminus \Delta_{1,2n})$ is trivial unless $p$ divides $n - 1$, in which case they are generated as a $\mathbb{F}_p$-module by 1 and a single non-zero invariant of degree 1.

The proof of this proposition is by induction on $n$. To set up the induction argument, we need the following technical lemma, which is of fundamental importance:

**Lemma 2.4.** We have:

1. for $n \geq 1$ the boundary morphism $\partial : A^0_{PGL_2}(\mathbb{P}(1,2n)) \to A^0_{PGL_2}(\Delta_{1,2n})$ is surjective.
2. for $n \geq 2$, there is an isomorphism
   $$A^0_{PGL_2}(\Delta_{1,2n}) \cong A^0_{PGL_2}((\mathbb{P}(1,2n - 2) \setminus \Delta_{1,2n-2}) \times \mathbb{P}^1)$$
3. for $n \geq 1$, the pullback morphism
   $$A^0_{PGL_2}(\mathbb{P}(1,2n) \setminus \Delta_{1,2n}) \to A^0_{PGL_2}((\mathbb{P}(1,2n) \setminus \Delta_{1,2n}) \times \mathbb{P}^1)$$
   is surjective with kernel generated by $w_2$, the Stiefel-Whitney class coming from the cohomological invariants of $PGL_2$.

As already stressed in the introduction of the current section, the lemma above is proved in [Pir] for every $n$ when $p \neq 2$, and for $n \leq 4$ when $p = 2$. Actually in this second case, assuming the first point for every $n$, the arguments used in [Pir] to prove the second and the third point work without any change. On the other hand, the proof of the first point does not extend in an obvious way when $n > 4$. In order to show the first point, we will use ideas quite different from the ones used in [Pir] and we will heavily rely on the theory developed in section 3 and on the computations of equivariant intersection theory made in section 5. We preferred to postpone the proof of the first point of lemma 2.4 to section 6 so to continue now the computation of the cohomological invariants of $H_g$.

**Proof of prop 2.3.** As already said, the proof of [Pir, corollary 3.10] can be generalized in an obvious way once one knows lemma 2.4 (1). Nevertheless, for the sake of completeness we rewrite here the proof, so to show how the technical lemma 2.4 is used. The case $p \neq 2$ is completely worked out in [Pir].

Suppose $p = 2$. Applying lemma 2.4 (1) to the long exact sequence of Chow groups with coefficients induced by the closed immersion $\Delta_{1,2n} \hookrightarrow \mathbb{P}(1,2n)$ we obtain the following short exact sequence of $\mathbb{F}_2$-modules:

$$0 \to A^0_{PGL_2}(\mathbb{P}(1,2n)) \to A^0_{PGL_2}(\mathbb{P}(1,2n) \setminus \Delta_{1,2n}) \xrightarrow{\partial} A^0_{PGL_2}(\Delta_{1,2n}) \to 0$$

The sequence is obviously split, from which we deduce

$$A^0_{PGL_2}(\mathbb{P}(1,2n) \setminus \Delta_{1,2n}) \cong A^0_{PGL_2}(\mathbb{P}(1,2n)) \oplus A^0_{PGL_2}(\Delta_{1,2n})[1]$$

where the notation $A_{PGL_2}(\Delta_{1,2n})[1]$ means that the elements of this group are degree shifted by one. We are in position to use an induction argument on $n$. The
base case \( n = 1 \) is handled using proposition \( \text{[2.2]}(1) \), plus the fact that \( \mathbb{P}(1, 2) \to \text{Spec}(k) \) is a \( \text{PGL}_2 \)-equivariant projective bundle, which permits us to compute \( A^0_{\text{PGL}_2}(\mathbb{P}(1, 2)) \) and proposition \( \text{[2.2]}(2) \) in order to compute \( A^0_{\text{PGL}_2}(\Delta_{1,2}) \). Then the inductive step is a consequence of proposition \( \text{[2.2]}(1) \) that, combined with the fact that \( \mathbb{P}(1, 2n) \to \text{Spec}(k) \) is a \( \text{PGL}_2 \)-equivariant projective bundle for every \( n \), allows us to compute \( A^0_{\text{PGL}_2}(\mathbb{P}(1, 2)) \), and lemma \( \text{[2.3]}(3) \), which permits us to use the inductive hypothesis to compute \( A^0_{\text{PGL}_2}(\Delta_{1,2n}) \). □

As announced at the beginning of the section, we will use proposition \( \text{[2.3]} \) to compute \( A_{\text{PGL}_2}(X_{2n}) \). First we have:

**Lemma 2.5.** Let \( Y \) be a scheme endowed with an action of \( \text{PGL}_2 \), and let \( \mathbb{G}_m \) acts trivially on it. Then

\[ A_{\text{PGL}_2 \times \mathbb{G}_m}(Y) \simeq A_{\text{PGL}_2}(Y)[t] \]

where \( t \) has codimension 1 and degree 0.

**Proof.** See \[ \text{[Pin]} \] proposition 2.3 □

We observed that \( X_{2n} \to \mathbb{P}(1, 2n) \setminus \Delta_{1,2n} \) is a \( \text{PGL}_2 \times \mathbb{G}_m \)-equivariant \( \mathbb{G}_m \)-torsor. Let \( \mathcal{L} \) be the associated \( \text{PGL}_2 \times \mathbb{G}_m \)-equivariant line bundle, so that if we embed \( \mathbb{P}(1, 2n) \setminus \Delta_{1,2n} \) into \( \mathcal{L} \) via the zero section, we have

\[ \mathcal{L} \setminus (\mathbb{P}(1, 2n) \setminus \Delta_{1,2n}) \simeq X_{2n} \]

Then from this and from the formula for the equivariant Chow ring with coefficients of a vector bundle applied to \( \mathcal{L} \to \mathbb{P}(1, 2n) \setminus \Delta_{1,2n} \), we deduce the following exact sequence:

\[ 0 \to A^0_{\text{PGL}_2 \times \mathbb{G}_m}(\mathbb{P}(1, 2n) \setminus \Delta_{1,2n}) \to A^0_{\text{PGL}_2 \times \mathbb{G}_m}(X_{2n}) \xrightarrow{\partial} \]

\[ \partial \to A^0_{\text{PGL}_2 \times \mathbb{G}_m}(\mathbb{P}(1, 2n) \setminus \Delta_{1,2n}) \xrightarrow{-c_1(\mathcal{L})} A^1_{\text{PGL}_2 \times \mathbb{G}_m}(\mathbb{P}(1, 2n) \setminus \Delta_{1,2n}) \]

The fact that the last arrow, after all the identifications we made, is the same as intersecting with \( c_1(\mathcal{L}) \) is basically the definition of the intersection with the first Chern class. This sequence plus theorem \[ \text{[1.1]} \] tells us that

\[ \text{Inv}^*([X_{2n}]/\text{PGL}_2 \times \mathbb{G}_m) \simeq \text{Inv}^*([(\mathbb{P}(1, 2n) \setminus \Delta_{1,2n})/\text{PGL}_2 \times \mathbb{G}_m]) \oplus \ker(\sim c_1(\mathcal{L}))[1] \]

Therefore, all we have to do is finding a presentation of the right addendum as an \( \mathbb{F}_2 \)-module.

**Lemma 2.6.** When \( p = 2 \) we have \( \ker(\sim c_1(\mathcal{L})) \simeq \mathbb{F}_2 \cdot x_n \).

**Proof.** The proof of \[ \text{[Pin]} \], once we know lemma \[ \text{[2.4]} \] for every \( n \), can be easily modified so to work also for all \( n \). □

We now have all the elements necessary to prove the main result of the paper.

**Proof of theorem \[ \text{[2.7]} \]** The case \( p \neq 2 \) is completely worked out in \[ \text{[Pin]} \]. When \( p = 2 \), it follows from proposition \[ \text{[2.3]} \] and lemma \[ \text{[2.6]} \]. □

### 3. GL\(_3\)-counterpart of PGL\(_2\)-schemes

In this section we start developing the theory which will be used in section \[ \text{[2]} \] to prove the key lemma \[ \text{[2.3]}(1) \]. The main new ingredient is the definition \[ \text{[6.1]} \] already introduced in \[ \text{[DL]} \], and the related proposition \[ \text{[5.5]} \]. This last result enable us, in order to prove lemma \[ \text{[2.3]}(1) \], to replace the \( \text{PGL}_2 \)-equivariant Chow groups with coefficients of \( \mathbb{P}(1, 2n) \) with the \( \text{GL}_3 \)-equivariant Chow groups with coefficients of a certain projective bundle \( \mathbb{P}(V_n)_{\text{sm}} \) defined over the affine space of smooth quadrics \( \mathbb{A}(2, 2)_{\text{sm}} \). What we gain in this way is basically more space: indeed, the projective bundles \( \mathbb{P}(V_n)_{\text{sm}} \) are naturally open subschemes of certain projective bundles \( \mathbb{P}(V_n) \)
defined over $\mathbb{A}(2,2) \setminus \{0\}$, and this observation opens the way to a new approach for doing computations with equivariant Chow ring of coefficients.

3.1. Basic definitions and some properties. We collect here some definitions and results from [DL] that will be needed in the remainder of this work.

**Definition 3.1.** Let $X$ be a scheme of finite type over $\text{Spec}(k)$ endowed with a $\text{PGL}_2$-action. Then a $\text{GL}_3$-counterpart of $X$ is a scheme $Y$ endowed with a $\text{GL}_3$-action such that $[Y/\text{GL}_3] \simeq [X/\text{PGL}_2]$.

**Definition 3.2.** Let $X$ and $X'$ be two schemes of finite type over $\text{Spec}(k)$ endowed with a $\text{PGL}_2$-action, and let $f : X \rightarrow X'$ be a proper $\text{PGL}_2$-equivariant morphism. Then a $\text{GL}_3$-counterpart of $f$ is a proper $\text{GL}_3$-equivariant morphism $g : Y \rightarrow Y'$ between two schemes endowed with a $\text{GL}_3$-action such that:

1. The scheme $Y$ (resp. $Y'$) is a $\text{GL}_3$-counterpart of $X$ (resp. $X'$).
2. The following diagram commutes:

$$
\begin{array}{ccc}
[X/\text{PGL}_2] & \xrightarrow{f} & [X'/\text{PGL}_2] \\
\downarrow \cong & & \downarrow \cong \\
[Y/\text{GL}_3] & \xrightarrow{g} & [Y'/\text{GL}_3]
\end{array}
$$

Let us briefly sketch how to construct a $\text{GL}_3$-counterpart. Let $X$ be a $\text{PGL}_2$-scheme, i.e. a scheme on which $\text{PGL}_2$ acts. Then we have the following cartesian square:

$$
\begin{array}{ccc}
X & \xrightarrow{} & [X/\text{PGL}_2] \\
\downarrow & & \downarrow \\
\text{Spec}(k_0) & \xrightarrow{} & \mathcal{B}\text{PGL}_2
\end{array}
$$

where $\mathcal{B}\text{PGL}_2$ denotes the classifying stack of $\text{PGL}_2$. In [DL] proposition 1.1] is proved that

$$
\mathcal{B}\text{PGL}_2 \simeq [S/\text{GL}_3]
$$

where $S = \mathbb{A}(2,2)_{\text{sm}}$, the scheme parametrising smooth ternary forms of degree 2. In other terms, the morphism $S \rightarrow \mathcal{B}\text{PGL}_2$ is a $\text{GL}_3$-torsor. Then we can form the cartesian square:

$$
\begin{array}{ccc}
Y & \xrightarrow{} & [X/\text{PGL}_2] \\
\downarrow & & \downarrow \\
S & \xrightarrow{} & \mathcal{M}_0
\end{array}
$$

It is immediate to check that the right vertical morphism is a representable morphism of algebraic stacks, which implies that $Y$ is a scheme. Moreover, we have that $Y \rightarrow [X/\text{PGL}_2]$ is a $\text{GL}_3$-torsor, so that

$$
[Y/\text{GL}_3] \simeq [X/\text{PGL}_2]
$$

We have proved:

**Proposition 3.3.** Let $X$ be a scheme endowed with a $\text{PGL}_2$-action. Then a $\text{GL}_3$-counterpart of $X$ is the scheme $Y := [X/\text{PGL}_2] \times_{\mathcal{B}\text{PGL}_2} S$.

Moreover, a $\text{PGL}_2$-equivariant morphism $X \rightarrow X'$ between $\text{PGL}_2$-schemes induces the morphism of stacks $[X/\text{PGL}_2] \rightarrow [X'/\text{PGL}_2]$. This can be pulled back along the $\text{GL}_3$-torsor $S \rightarrow \mathcal{B}\text{PGL}_2$, and we obtain a $\text{GL}_3$-equivariant morphism $Y \rightarrow Y'$ between the $\text{GL}_3$-counterparts of $X$ and $X'$.
Proposition 3.4. Let $X$ and $X'$ be two schemes of finite type over $\text{Spec}(k)$ endowed with a $\text{PGL}_2$-action, and let $f : X \to X'$ be a $\text{PGL}_2$-equivariant morphism. Then a $\text{GL}_3$-counterpart of $f$ is the induced morphism $Y \to Y'$ between the $\text{GL}_3$-equivariant counterparts of $X$ and $X'$.

The following proposition is also immediate:

Proposition 3.5. Let $f : X \to X'$ be a $\text{PGL}_2$-equivariant proper morphism between two $\text{PGL}_2$-schemes, and let $g : Y \to Y'$ be its $\text{GL}_3$-equivariant counterpart. Then we have:

1. a commutative diagram of equivariant Chow groups of the form

$$
\begin{array}{c}
\text{CH}^i_{\text{PGL}_2}(X) \xrightarrow{f_*} \text{CH}^i_{\text{PGL}_2}(X') \\
\downarrow \quad \downarrow \\
\text{CH}^i_{\text{GL}_3}(Y) \xrightarrow{g_*} \text{CH}^i_{\text{GL}_3}(Y')
\end{array}
$$

where the vertical arrows are isomorphisms.

2. a commutative diagram of equivariant Chow groups with coefficients of the form

$$
\begin{array}{c}
\text{A}^i_{\text{PGL}_2}(X) \xrightarrow{f_*} \text{A}^i_{\text{PGL}_2}(X') \\
\downarrow \quad \downarrow \\
\text{A}^i_{\text{GL}_3}(Y) \xrightarrow{g_*} \text{A}^i_{\text{GL}_3}(Y')
\end{array}
$$

where the vertical arrows are isomorphisms.

3.2. Applications. We apply now the machinery above to a particular case. Let $\mathbb{P}(1, 2n)$ be the projective space of binary forms of degree $2n$. This scheme has a natural action of $\text{PGL}_2$ given by $A : f(x, y) = f(A^{-1}(x, y))$, and we want to find a $\text{GL}_3$-counterpart for it.

For doing so, we need to introduce some particular vector bundles over $\mathbb{A}(2, 2) \setminus \{0\}$, where $\mathbb{A}(2, 2)$ denotes as usual the affine space of ternary forms of degree 2. In general, we will indicate with $\mathbb{A}(n, d)$ the affine space of forms in $n + 1$ variables of degree $d$.

Consider the following injective morphism of (trivial) vector bundles over $\mathbb{A}(2, 2) \setminus \{0\}$:

$$
\mathbb{A}(2, n - 2) \times (\mathbb{A}(2, 2) \setminus \{0\}) \hookrightarrow \mathbb{A}(2, n) \times (\mathbb{A}(2, 2) \setminus \{0\}), \quad (f, q) \mapsto (f, fq)
$$

Then the vector bundle $V_n$ is defined as the quotient of $\mathbb{A}(2, n) \times \mathbb{A}(2, 2) \setminus \{0\}$ by the image of the morphism above. We can restrict $V_n$ to $\mathcal{S} = \mathbb{A}(2, 2)_{\text{sm}}$, the open subscheme of smooth ternary forms of degree 2, and we can take its projectivization, which we denote $\mathbb{P}(V_n)_{\text{sm}}$.

Proposition 3.6. The $\text{GL}_3$-counterpart of $\mathbb{P}(1, 2n)$ is $\mathbb{P}(V_n)_{\text{sm}}$, endowed with the $\text{GL}_3$-action:

$$
A : (q, [f]) := (\det(A)q(A^{-1}(x, y, z), [f(A^{-1}(x, y, z))])
$$

where $q$ is a smooth ternary forms of degree 2 and $f$ is a representative of the equivalence class $[f]$ of a ternary forms of degree $n$.

Proof. See [DL, proposition 2.4] (what we call here $\mathbb{P}(V_n)_{\text{sm}}$ there is denoted $\mathbb{P}(V_n)$).
Remark 3.7. A good way to think of the points of \( \mathbb{P}(V_n)_{\text{sm}} \) is as pairs \((q, E)\), where \( E \) is an effective divisor of degree \( 2n \) of the plane conic \( Q \) defined by the equation \( q = 0 \). Indeed, let \( F \) and \( G \) be the plane curves respectively defined by \( f \) and \( g \), not containing \( Q \) as an irreducible component. By the classical Noether’s theorem \( AF + BG \), the intersection of \( F \) with \( Q \) is equal to the intersection of \( G \) with \( Q \) if and only if the difference \( f - g \) is divisible by \( q \), that is to say if and only if \( f - g \) is in the image of \( \mathbb{A}(2, n - 2) \times (\mathbb{A}(2, 2) \setminus \{0\}) \to \mathbb{A}(2, n) \times (\mathbb{A}(2, 2) \setminus \{0\}) \). From this we deduce that the points of \( \mathbb{P}(V_n)_{\text{sm}} \) are in bijection with the pairs \((q, E)\), where \( E \) is an effective divisor of degree \( 2n \).

Inside \( \mathbb{P}(1, 2n) \) there is the closed, \( \text{PGL}_2 \)-invariant subscheme \( \Delta_{1, 2n} \), that is the scheme parametrising forms with a multiple root or, from another point of view, the effective divisors of \( \mathbb{P}^1 \) of the form \( E + 2E' \) for some effective divisor \( E' \). The following corollary is also proved in [DL].

Corollary 3.8. The \( \text{GL}_3 \)-counterpart of \( \Delta_{1, 2n} \) is \( D_{\text{sm}} \), that is the closed subscheme of \( \mathbb{P}(V_n)_{\text{sm}} \) parametrising pairs \((q, [f])\) such that the subscheme \( V_+(q, f) \) inside \( \mathbb{P}^2 \) is not smooth. We also have commutative diagrams

\[
\begin{align*}
CH_1^{\text{PGL}_2}(\Delta_{1, 2n}) & \xrightarrow{i_*} CH_1^{\text{PGL}_2}(\mathbb{P}(1, 2n)) \\
CH_1^{\text{GL}_3}(D_{\text{sm}}) & \xrightarrow{i_*} CH_1^{\text{GL}_3}(\mathbb{P}(V_n)_{\text{sm}}) \\
A_1^{\text{PGL}_2}(\Delta_{1, 2n}) & \xrightarrow{i_*} A_1^{\text{PGL}_2}(\mathbb{P}(1, 2n)) \\
A_1^{\text{GL}_3}(D_{\text{sm}}) & \xrightarrow{i_*} A_1^{\text{GL}_3}(\mathbb{P}(V_n)_{\text{sm}})
\end{align*}
\]

where the vertical arrows are all isomorphisms.

4. The geometry of the fundamental divisor

In this section we will define the fundamental divisor \( D \subset \mathbb{P}(V_n) \), whose restriction over \( \mathbb{A}(2, 2)_{\text{sm}} \) coincides with \( D_{\text{sm}} \). Then we will focus on \( D_{\text{nod}} \), the restriction of \( D \) to \( \mathbb{P}(V_n)_{\text{nod}} \), and we will study its geometry (proposition 4.5) and its proper transform \( \tilde{D} \) in \( \mathbb{P}(V_n)_{\text{tal}} \), where this last scheme is the pullback of \( \mathbb{P}(V_n)_{\text{sing}} \) to the blow up \( \mathbb{A}(2, 2)_{\text{sing}} \) of \( \mathbb{A}(2, 2)_{\text{sing}} \) along \( \mathbb{A}(2, 2)_{\text{sq}} \) (proposition 4.5).

4.1. Basic definitions. Let us introduce some notation: if \( q \) and \( f \) are both ternary forms, we denote \( J(q, f) \) the associated jacobian matrix. This is a \( 3 \times 3 \) matrix, so that it makes sense to define \( \det J(q, f) \) as the determinant of the minor obtained by removing the \( i^{th} \)-column, for \( i = 1, 2, 3 \).

Consider inside \( \mathbb{A}(2, n) \times (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{P}^2 \) the closed subscheme \( \mathcal{D} \) defined as:

\[
\mathcal{D} = \left\{ (f, q, u) \text{ such that } f(u) = q(u) = \det_i J(q, f)(u) = 0 \text{ for } i = 1, 2, 3 \right\}
\]

Observe that the equations needed for defining \( \mathcal{D} \) are locally redundant. Indeed, if we restrict to the open subscheme \( U = \mathbb{A}(2, n) \times (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{A}^2 \) where the third homogeneous coordinate of \( \mathbb{P}^2 \) does not vanish, it is immediate to check that

\[
\mathcal{D}|_U = \left\{ (f, q, u) \text{ such that } f(u) = q(u) = \det_3 J(q, f)(u) = 0 \right\}
\]

It is also easy to verify that \( \mathcal{D}|_U \) is a complete intersection, and thus \( \mathcal{D}|_U \) has codimension 3. Clearly, a similar description holds if we restrict to the open subscheme
where the first or the second homogeneous coordinate of \( \mathbb{P}^2 \) does not vanish, implying that \( D \) has codimension 3. Moreover, from the Jacobian criterion of regularity we deduce that \( D \) parametrizes triples \((f, q, u)\) such that \( u \) is a singular point of the subscheme \( V_\ast (f, q) \) inside \( \mathbb{P}^2 \).

We can first project \( D \) on \( \mathbb{A}(2, n) \times (\mathbb{A}(2) \setminus \{0\}) \), and then we project it again on the quotient vector bundle \( V_\ast \). Finally, we can take the projectivization of this image inside \( \mathbb{P}(V_\ast) \). The resulting closed subscheme of \( \mathbb{P}(V_\ast) \) will be denoted \( D \). This is the fundamental divisor. Observe that it is \( \mathbb{G}_m \)-invariant and if we restrict \( D \) to \( \mathbb{P}(V_\ast)_{\text{sm}} \) we obtain exactly the closed subscheme \( D_{\text{sm}} \) defined before.

4.2. The geometry of \( D_{\text{nod}} \). Let \( \mathbb{A}(2, 2)_{\text{nod}} \) be the locally closed subscheme of \( \mathbb{A}(2, 2) \setminus \{0\} \) that parametrises the ternary forms of the type \( q = l_1 l_2 \), where \( l_1 \) and \( l_2 \) are two distinct ternary forms of degree 1.

In other terms, if we denote with \( \mathbb{A}(2, 2)_{\text{sing}} \) the closed subscheme of singular ternary forms of degree 2 and with \( \mathbb{A}(2, 2)_{\text{sq}} \) the closed subscheme of squares of ternary linear forms, then \( \mathbb{A}(2, 2)_{\text{nod}} = \mathbb{A}(2, 2)_{\text{sing}} \setminus \mathbb{A}(2, 2)_{\text{sq}} \).

Let us indicate as \( \mathbb{P}(V_\ast)_{\text{nod}} \) (resp. \( D_{\text{nod}} \)) the restriction of \( \mathbb{P}(V_\ast) \) (resp. \( D \)) to \( \mathbb{A}(2, 2)_{\text{nod}} \). We focus now on the geometry of \( D_{\text{nod}} \).

By definition, \( D_{\text{nod}} \) parametrises ternary forms of degree 2 of the form \( l_1 l_2 \), where \( l_i \) is a ternary form of degree 1, plus an equivalence class of a ternary form \( f \) of degree \( n \) with the property that \( V_\ast (l_1 l_2, f) \) is singular.

This property does not depend on the choice of the representative of the equivalence class of \( f \), because the ideal \((l_1 l_2, f)\) coincides with the ideal \((l_1 l_2, f + l_1 l_2 f')\).

Let us consider the subset \( D_{\text{nod}}^1 \) of \( \mathbb{P}(V_\ast)_{\text{nod}} \) defined as

\[
D_{\text{nod}}^1 := \{(q, f) \text{ such that } q = l_1 l_2 \text{ and } V_\ast (l_1, f) \text{ is singular in } \mathbb{P}^2 \text{ for some } i\}
\]

We can put a scheme structure on it as follows: consider the \( \mathbb{G}_m \)-invariant closed subscheme

\[
D' := \{(l, f) \text{ such that } V_\ast (l, f) \text{ is singular in } \mathbb{P}^2 \}
\]

of \( \mathbb{A}(2, 1) \times \mathbb{A}(2, n) \): the scheme structure of this set can be obtained just as we have done before for the subscheme \( D \) of \( \mathbb{P}(V_\ast) \).

Let \( D'' \) be the pullback of \( D' \) to \( \mathbb{A}(2, 1) \times \mathbb{A}(2, 1) \times \mathbb{A}(2, n) \) along the projection \( p_{13} \). Consider now the proper morphism

\[\psi : \mathbb{A}(2, 1) \times \mathbb{A}(2, 1) \times \mathbb{A}(2, n) \to \mathbb{A}(2, 2)_{\text{sing}} \times \mathbb{A}(2, n), \quad (l_1, l_2, f) \mapsto (l_1 l_2, f)\]

and let \( D''' \) be the restriction of \( \psi(D'') \) (to whom we give the image scheme structure) to the open subscheme \( \mathbb{A}(2, 2)_{\text{nod}} \times \mathbb{A}(2, n) \).

We project then \( D''' \) via the quotient morphism \( \mathbb{A}(2, 2)_{\text{nod}} \times \mathbb{A}(2, n) \to (V_\ast)_{\text{nod}} \), we restrict it to the open complement of the zero section (observe that the restriction of \( D''' \) is \( \mathbb{G}_m \)-invariant, where \( \mathbb{G}_m \) acts by multiplication on \( f \)) and finally we project it again on \( \mathbb{P}(V_\ast)_{\text{nod}} \).

What we obtain is exactly \( D_{\text{nod}}^1 \), and with this procedure it inherits a scheme structure. It is easy to check that \( D_{\text{nod}}^1 \) is \( \mathbb{G}_m \)-invariant.

Observe that \( D_{\text{nod}}^1 \) has codimension 1 in \( \mathbb{P}(V_\ast)_{\text{nod}} \), it is irreducible and it is contained in \( D_{\text{nod}} \), but it does not coincide with it. Indeed, let \( D_{\text{nod}}^2 \) be the subset of \( \mathbb{P}(V_\ast)_{\text{nod}} \) defined as follows:

\[
D_{\text{nod}}^2 := \{(l_1 l_2, f) \text{ such that } V_\ast (l_1, l_2, f) \text{ is not empty}\}
\]

In other terms, the subset \( D_{\text{nod}}^2 \) parametrises pairs \((l_1 l_2, f)\) such that \( F \), the plane curve of equation \( f = 0 \), passes through the node of \( Q \), the conic defined by the equation \( l_1 l_2 = 0 \) (we are always assuming now that \( l_1 \) and \( l_2 \) are distinct).
We can put a scheme structure on $D_\text{nod}^2$ by considering the closed subscheme $D$ of $\mathbb{P}^2_{(V_n)_{\text{nod}}} \times \mathbb{P}^2$ that is defined as
\[ D := \{(q, f, u) \text{ such that } q_x(u) = q_y(u) = q_z(u) = f(u) = 0\} \]
and then by taking its image along the projection on the first factor: what we obtain is exactly $D_\text{nod}^2$. Observe that $D_\text{nod}^2$ is $GL_3$-invariant.

Also $D_\text{nod}^2$ has codimension 1 in $\mathbb{P}((V_n)_{\text{nod}})$, it is irreducible and it is easy to check that it is contained in $D_\text{nod}$, but it does not coincide with it.

So far we have proved that $D_\text{nod}$ is not irreducible and it has at least two distinct components. Actually, these two are the only ones.

**Proposition 4.1.** The closed subscheme $D_\text{nod} \subset \mathbb{P}(V_n)_{\text{nod}}$ has two irreducible components, both $GL_3$-invariants and of codimension 1, which are $D_\text{nod}^1$ and $D_\text{nod}^2$.

**Proof.** The definition and the properties of $D_\text{nod}^1$ and $D_\text{nod}^2$ had already been showed in the discussion we made just before the proposition, thus we are only left to prove that there are no other irreducible components.

This is a consequence of the following observation: let $Q$ be a conic defined by the equation $l_1l_2 = 0$ and let $F$ be a plane curve of equation $f = 0$ such that the restriction of $F$ to $Q$ defines a singular subscheme.

Suppose that the restrictions of $F$ to the two irreducible components of $Q$ are both regular: then the only possibility left is that $F$ passes through the node of $Q$, as in this case the restriction of $F$ to $Q$ will be automatically singular, without affecting the regularity of the restriction of $F$ to the irreducible components of $Q$.

We can also consider the restriction of $\mathbb{P}(V_n)$ over $\mathbb{A}(2,2)_{\text{sq}} \backslash \{0\}$, which we denote $\mathbb{P}(V_n)_{\text{sq}}$, and the restriction of $D$ to the same subscheme, which we denote $D_{\text{sq}}$.

**Proposition 4.2.** We have $D_{\text{sq}} = \mathbb{P}(V_n)_{\text{sq}}$.

**Proof.** It easily follows from the definition of $D$. □

### 4.3. The geometry of $\tilde{D}$

Let $\mathbb{A}(2,2)_{\text{sing}}$ be the blow up of $\mathbb{A}(2,2)_{\text{sing}}$ along the closed subscheme $\mathbb{A}(2,2)_{\text{sq}}$. Then $\mathbb{A}(2,2)_{\text{sing}}$ has another description which is useful for our purposes.

Consider indeed the singular locus $Q_{\text{sing}}$ inside the tautological conic $Q \subset \mathbb{A}(2,2)_{\text{sing}} \times \mathbb{P}^2$. In other terms, we have
\[ Q_{\text{sing}} = \{(q, u) \text{ such that } q_x(u) = q_y(u) = q_z(u) = 0\} \]

Clearly, if we restrict $Q_{\text{sing}}$ over $\mathbb{A}(2,2)_{\text{nod}}$, the projection map from the restriction of $Q_{\text{sing}}$ on $\mathbb{A}(2,2)_{\text{nod}}$ is an isomorphism: roughly, the inverse is given by sending a trinary form $q = l_1l_2$ to the pair $(q, u)$ where $u$ is the nodal point of $Q$, the conic of equation $q = 0$.

Moreover, we see that the restriction of $Q_{\text{sing}}$ over $\mathbb{A}(2,2)_{\text{sq}}$ is a $\mathbb{P}^1$-bundle. This suggests the following result:

**Proposition 4.3.** We have $Q_{\text{sing}} \simeq \mathbb{A}(2,2)_{\text{sing}}$.

**Proof.** We have an obvious morphism $Q_{\text{sing}} \to \mathbb{A}(2,2)_{\text{sing}}$. Observe that the pull-back of $\mathbb{A}(2,2)_{\text{sq}}$ along this morphism is a Cartier divisor, thus for the universal property of the blow-up there exists a unique lifting $Q_{\text{sing}} \to \mathbb{A}(2,2)_{\text{sing}}$.

This morphism is birational, because once we restrict to the locus parametrizing quadrics of the form $q = l_1l_2$ with $l_1 \neq l_2$ there is a section $\mathbb{A}(2,2)_{\text{nod}} \to (Q_{\text{sing}})_{|\mathbb{A}(2,2)_{\text{nod}}}$ which maps the point corresponding to $l_1l_2$ to the node of the conic $l_1l_2 = 0$. 

Moreover, the morphism $Q_{\text{sing}} \to \tilde{A}(2,2)_{\text{sing}}$ is quasi-finite and surjective. Over $A(2,2)_{\text{nod}}$ this claim is obvious, so we only have to prove it over $A(2,2)_{\text{sq}}$, which means that we have to show that the morphism

$$(Q_{\text{sing}})_{|A(2,2)_{\text{sq}}} \longrightarrow \mathbb{P}(N)$$

is quasi-finite and surjective, where $N$ is the normal bundle of $A(2,2)_{\text{sq}}$ in $A(2,2)_{\text{sing}}$. Observe that this is a morphism of $\mathbb{P}^1$-bundles over $A(2,2)_{\text{sq}}$, thus we can equivalently show that the morphism

$$\mathbb{P}^1 \longrightarrow \mathbb{P}(N)_{|(\mathbb{P}^2)}$$

is quasi-finite and surjective for every $(l^2)$ in $A(2,2)_{\text{sq}}$, where $\mathbb{P}^1 \simeq L$, the line in $\mathbb{P}^2$ of equation $l = 0$. It is enough then to show that this morphism is non-constant.

Finally, it is easy to check that $Q_{\text{sing}} \to \tilde{A}(2,2)_{\text{sing}}$ is proper, because $Q_{\text{sing}}$ is proper over $A(2,2)_{\text{sing}}$, $A(2,2)_{\text{sing}}$ is of finite type and separated over $A(2,2)_{\text{sing}}$ and $Q_{\text{sing}} \to A(2,2)_{\text{sing}}$ is surjective.

Putting all together, we have proved that $Q_{\text{sing}} \to \tilde{A}(2,2)_{\text{sing}}$ is birational and finite. But $A(2,2)_{\text{sing}}$ is normal, thus that morphism should actually be an isomorphism. \hfill $\square$

**Corollary 4.4.** We have $A^1_{\text{GL}_3}(\tilde{A}(2,2)_{\text{sing}}) \simeq A^1_{\text{GL}_3}(\mathbb{P}^2)$.

**Proof.** From proposition 4.3 we know that $A^1_{\text{GL}_3}(\tilde{A}(2,2)_{\text{sing}}) \simeq A^1_{\text{GL}_3}(Q_{\text{sing}})$. Then the corollary follows from the easy observation that $Q_{\text{sing}} \to \mathbb{P}^2$ is a $\text{GL}_3$-equivariant vector subbundle of the (trivial) vector bundle $A(2,2) \times \mathbb{P}^2 \to \mathbb{P}^2$. \hfill $\square$

Let $\tilde{D}_{\text{sing}}$ be the proper transform of $D_{\text{sing}}$ inside $P(V_n)_{\text{ha}}$, the pullback of $P(V_n)_{\text{sing}}$ along $A(2,2)_{\text{sing}} \setminus \{0\}$. Obviously, if we restrict $\tilde{D}$ over the open subscheme $A(2,2)_{\text{nod}} \subset A(2,2)_{\text{sing}}$ we obtain $D_{\text{nod}}$, but what do we get if we restrict $\tilde{D}$ to the exceptional divisor $E$?

**Proposition 4.5.** The codimension 1 subscheme $\tilde{D}_E$ of $P(V_n)_E$ has two irreducible components $\tilde{D}^1_E$ and $\tilde{D}^2_E$.

**Proof.** Follows easily from proposition 4.4. \hfill $\square$

## 5. Some equivariant intersection theory

The main goal of this section is to compute the cycle classes of some schemes that we introduced in section 4, namely $D^i_{\text{nod}}$ and $D^i_E$ for $i = 1, 2$. These results will be used to give a proof of lemma 2.4(1) in section 6.

We assume the knowledge of the basic tools of equivariant intersection theory, first developed in [EGIS]. A brief introduction to the subject and to the techniques involved can be found in [FV18] section 2-4.

### 5.1. Cycle classes of $D^i_{\text{nod}}$

As before, we will denote $A(n, d)$ the affine space of forms in $n + 1$ variables of degree $d$. We start with an important remark:

**Remark 5.1.** Recall that $V_n$ is constructed as the coker of exact sequence of (trivial) vector bundles over $A(2,2) \setminus \{0\}$

$$0 \to (A(2,2) \setminus \{0\}) \times A(2, n - 2) \xrightarrow{\varphi} (A(2,2) \setminus \{0\}) \times A(2, n)$$

where the last arrow sends a pair $(q, f)$ to $(q, qf)$. Identifying $\text{im}(\varphi)$ with $(A(2,2) \setminus \{0\}) \times A(2, n - 2)$, we easily deduce that

$$(A(2,2) \setminus \{0\}) \times A(2, n) \to (A(2,2) \setminus \{0\}) \times A(2, n - 2) \longrightarrow V_n$$
is a vector bundle, and the same thing holds if we pass to the projectivizations, i.e.

\[ p : (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{P}(2, n) \setminus (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{P}(2, n - 2) \rightarrow \mathbb{P}(V_{n}) \]

is a vector bundle. This implies that, if \( Z \subset \mathbb{P}(V_{n}) \) is a \( GL_{3} \)-invariant subvariety, we can compute its class as follows: by the usual properties of Chow groups, it is equivalent to calculate \([p^{-1}(Z)]\), and to compute this class it is enough to compute the class of any \( GL_{3} \)-invariant subscheme \( Z' \subset (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{P}(2, n) \) such that its restriction to \((\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{P}(2, n) \setminus (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{P}(2, n - 2)\) coincides with \([p^{-1}(Z)]\).

Now observe that the morphism of Chow rings

\[ \text{pr}^{*} \otimes \text{id} : CH_{GL_{3}}(\mathbb{P}(2, 2) \times \mathbb{P}(2, n)) \rightarrow CH_{GL_{3}}((\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{P}(2, n)) \]

is surjective with kernel generated by \( s \). By standard results of equivariant intersection theory the product of the four cycle classes will coincide with the cycle class of any \( GL_{3} \)-invariant subscheme of \( \mathbb{P}(2, 2) \times \mathbb{P}(2, n) \) such that its restriction to \( (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{P}(2, n) \setminus (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{P}(2, n - 2) \) coincides with \([p^{-1}(Z)]\).

So, denoting \( Z'' \) the projectivization of \( Z' \), if we compute \([Z'']\), then we can easily obtain \([Z']\) by substituting \( s \) with \( c_{1} \). Usually, it happens that the computation of a cycle class inside the Chow ring of \( \mathbb{P}(2, 2) \times \mathbb{P}(2, n) \) can be easier than the computation of a cycle class inside \( \mathbb{P}(V_{n}) \).

Also, let us remark that the same arguments stay true if instead of \( \mathbb{P}(V_{n}) \) we consider its restriction over a \( G_{m} \)-invariant open subscheme of \( \mathbb{A}(2, 2) \setminus \{0\} \).

Our first computation is the following:

**Lemma 5.2.** We have \([D_{\text{red}}^{1}] = c_{1} h \) in \( CH^{2}_{GL_{3}}(\mathbb{P}(V_{n})_{\text{red}})^{\mathbb{P}_{2}} \).

**Proof.** From remark 1.2 we see that it is equivalent to compute the cycle class of the \( GL_{3} \)-invariant, closed subscheme

\[ D := \{(q, f)|Q \text{ is singular and } F \text{ intersects } Q \text{ in singular points}\} \]

of \( \mathbb{P}(2, 2) \times \mathbb{P}(2, n) \), and then substitute the hyperplane class \( s \) of \( \mathbb{P}(2, 2) \) with \( c_{1} \).

The scheme structure of \( D \) is obtained as follows: consider the proper morphism

\[ \text{pr}_{1} : \mathbb{P}(2, 2) \times \mathbb{P}(2, n) \times \mathbb{P}^{2} \rightarrow \mathbb{P}(2, 2) \times \mathbb{P}(2, n) \]

and the \( GL_{3} \)-invariant closed subscheme \( D \) of \( \mathbb{P}(2, 2) \times \mathbb{P}(2, n) \times \mathbb{P}^{2} \) defined as

\[ D := \{(q, f, u) \text{ such that } q_{x}(u) = q_{y}(u) = q_{z}(u) = f(u) = 0\} \]

Then we have that \( \text{pr}_{1}(D) = D, \) so that \( D \) inherit a scheme structure. Moreover \( \text{pr}_{1*}[D] = [D], \) thus it is enough to compute the cycle class of \( D \). More precisely, if we denote \( t \) the hyperplane class of \( \mathbb{P}^{2} \), there exist \( \xi_{0}, \xi_{1} \) and \( \xi_{2} \) in \( CH_{GL_{3}}(\mathbb{P}(2, 2) \times \mathbb{P}(2, n)) \) such that

\[ [D] = t^{2}\xi_{0} + t\xi_{1} + \xi_{2} \]

Then it is immediate to check that \( \text{pr}_{1*}[D] = \xi_{0} \). Now we are going to compute explicitly \([D]\) and thus \( \xi_{0} \).

Observe that \( D \) is a global complete intersection of four hypersurfaces, so that we only need to compute the cycle classes of these four hypersurfaces and then multiply them together.

Observe also that the hypersurfaces are not \( GL_{3} \)-invariant but only \( T \)-invariant, where \( T \) is the subtorus of \( GL_{3} \) of diagonal matrices, thus their cycle classes live in \( CH_{T}(\mathbb{P}(2, 2) \times \mathbb{P}(2, n) \times \mathbb{P}^{2}) \). Nevertheless, by standard results of equivariant intersection theory the product of the four cycle classes will coincide with the cycle class of \( D \) in \( CH_{GL_{3}}(\mathbb{P}(2, 2) \times \mathbb{P}(2, n) \times \mathbb{P}^{2}) \).

Denoting \( \lambda_{i} \), for \( i = 1, 2, 3 \), the generators of \( CH_{T} \), using [EF09] lemma 2.4 to compute the cycle classes of the four hypersurfaces, we obtain:

\[ [D] = (s + t - \lambda_{1})(s + t - \lambda_{2})(s + t - \lambda_{3})(h + nt) \]
Using the identities $t^3 + c_1 t^2 + c_2 t + c_3 = 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = c_1$, and tensoring with $\mathbb{P}^2$, in the end we obtain

$$[D] = t^2(s^2 + sh + sc_1) + t\xi_1 + \xi_2$$
for $n$ odd

$$[D] = t^2(hs) + t\xi_1 + \xi_2$$
for $n$ even

Substituting $s$ with $c_1$, we conclude the proof of the lemma. \qed

Now we want to compute the cycle class of the irreducible component $D^2_{\text{red}}$.

**Lemma 5.3.** We have $[D^2_{\text{red}}] = 0$ in both $CH^1_{\text{GL}_3}(\mathbb{P}(V_n)_{\text{nod}})$ and $CH^2_{\text{GL}_3}(\mathbb{P}(V_n)_{\text{red}})$.

**Proof.** Clearly, it is enough to show that $[D^2_{\text{red}}] = 0$ in $CH^2_{\text{GL}_3}(\mathbb{P}(V_n)_{\text{nod}})$. By remark 5.1, it is equivalent to compute the cycle class of the GL$_3$-invariant, closed subscheme

$$D := \{(q,f) \mid q = l_1 l_2 \text{ and } V_+(l_i,f) \text{ is singular in } \mathbb{P}^2 \text{ for some } i\}$$
of $\mathbb{P}(2,2)_{\text{nod}} \times \mathbb{P}(2,n)$. Let us show how to put a scheme structure on $D$: consider the GL$_3$-equivariant proper morphism

$$\psi : \mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}(2,1) \longrightarrow \mathbb{P}(2,2)_{\text{sing}} \times \mathbb{P}(2,n), \quad (l_1,f,l_2) \longmapsto (l_1 l_2,f)$$

Let $D'$ be the subset of $\mathbb{P}(2,1) \times \mathbb{P}(2,n)$ that is defined as follows:

$$D' := \{(l,f) \mid V_+(l,f) \text{ is singular in } \mathbb{P}^2\}$$

We claim that $D'$ is actually a closed subscheme.

If this is the case, then we can take the closed subscheme $D' \times \mathbb{P}(2,1)$ inside $\mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}(2,1)$ and we see that the image of this closed subscheme via $\psi$, once restricted to $\mathbb{P}(2,2)_{\text{nod}} \times \mathbb{P}(2,n)$, is exactly $D$.

This induces the scheme structure on $D$. Moreover, we have $\psi_*[D'] = [D]$, because $\psi$ restricted to $D'$ is generically one to one.

To give to $D'$ a scheme structure, consider the closed subscheme $D'$ of $\mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2$ defined as

$$D' := \{(l,f,u) \mid l(u) = f(u) = \det_i J(l,f)(u) = 0, i = 1,2,3\}$$

where $\det_i(l,f)$ denotes the determinant of the minor of the jacobian matrix obtained by eliminating the $i^{th}$ column. Then we have that the image of $D'$ via the projection on the first and second factor is exactly $D'$.

In this way we can define a scheme structure on $D'$. Moreover, we see that $\text{pr}_{234}[D'] = D'$. Let $\xi_0, \xi_1$ and $\xi_2$ be the cycles in the Chow ring of $\mathbb{P}(2,1) \times \mathbb{P}(2,n)$ such that

$$[D'] = t^2\xi_0 + t\xi_1 + \xi_2$$

where $t$ is the hyperplane class of $\mathbb{P}^2$. Then it is immediate to check that $\text{pr}_{234}[D'] = \xi_0$. Now we are going to compute $\xi_0$.

Observe that, just as in [DL, subsection 4.2], the scheme $D'$ is not a complete intersection but, if we restrict to the open subscheme of $\mathbb{P}^2$ where $u_2 \neq 0$, then we see that we need exactly three equations to describe the restriction of $D'$, namely $l(u) = f(u) = \det_3 J(l,f)(u) = 0$. Consider the $T$-invariant subscheme

$$D'' := \{(l,f,u) \mid l(u) = f(u) = \det_3 J(l,f)(u) = 0\}$$

where $T$ is the usual subtorus of GL$_3$ made of the diagonal matrices. Then we have that $D''$ has two irreducible components, which are $D'$ and the $T$-invariant subscheme

$$Z := \{(l,f,u) \mid u_2 = 0\}$$
From this we deduce that $[D'] = [D'] - [Z]$ in $CH_T(\mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2)$. By standard results of equivariant intersection theory, the $T$-equivariant cycle class of $[D']$ coincides with the $\text{GL}_3$-equivariant one.

The computations of $[D']$ and $[Z]$ works exactly as in the proof of lemma \[\text{5.2} \]

In the end we obtain:

\[
[D'] - [Z] = (s + t)(h + nt)(s + h + (n - 1)t - \lambda_1 - \lambda_2) - (s + t)(h + nt)(t + \lambda_3) = (s + t)(h + nt)(s + h + (n - 2)t - c_1)
\]

Expanding the expression above, using the identity $t^3 + c_1 t^2 + c_2 t + c_3 = 0$ and after tensoring with $F_2$, we obtain that

\[
[D'] = t\xi_1 + \xi_2
\]

thus $[D'] = 0$. This implies that $[D] = \psi_*[D'] = 0$ and concludes the proof of the lemma.

\[\square\]

5.2. Cycle classes of $\bar{D}_E$. Let $E$ be the exceptional divisor of $\mathbb{A}(2,2)_{\text{sing}}$, and let $\mathbb{P}(V_n)$ be the pullback of $\mathbb{P}(V_n)_{\text{sing}}$ along the morphism $\mathbb{A}(2,2)_{\text{sing}} \to \mathbb{A}(2,2)_{\text{sing}} \setminus \{0\}$.

**Lemma 5.4.** We have

\[
CH_{\text{GL}_3}(E) = \mathbb{Z}[c_1, c_2, c_3, s, h_E]/(2s - c_1, f_E, f_s)
\]

where $f_E$ is a polynomial of degree 2 monic in $h_E$ and $f_s$ is a polynomial of degree 3 monic in $s$ with coefficients in $CH_{\text{GL}_3}$.

**Proof.** First, recall from proposition \[\text{5.3} \] that we have

\[
E = \{(l^2, u) \text{ such that } l(u) = 0\} \subset \mathbb{A}(2,2)_{\text{sq}} \setminus \{0\} \times \mathbb{P}^2
\]

and it coincides with the projectivization of the normal bundle of $\mathbb{A}(2,2)_{\text{sq}} \setminus \{0\} \subset \mathbb{A}(2,2)_{\text{sing}} \setminus \{0\}$, which is a vector bundle of rank 2. From this we deduce:

\[
CH_{\text{GL}_3}(E) = CH_{\text{GL}_3}(\mathbb{A}(2,2)_{\text{sq}} \setminus \{0\})[h_E]/(f_E)
\]

with $f_E$ as in the thesis of the lemma.

To compute the Chow ring of $\mathbb{A}(2,2)_{\text{sq}} \setminus \{0\}$, consider its projectivization $\mathbb{P}(2,2)_{\text{sq}}$ and observe that we can identify $\mathbb{A}(2,2)_{\text{sq}} \setminus \{0\}$ with the $\mathbb{G}_m$-torsor over $\mathbb{P}(2,2)_{\text{sq}}$ associated to the $\text{GL}_3$-equivariant line bundle $O(-1) \otimes D$, where $O(-1)$ is the restriction to $\mathbb{P}(2,2)_{\text{sq}}$ of the tautological line bundle over $\mathbb{P}(2,2)$ and $D$ denotes the one dimensional representation of $\text{GL}_3$ given by the determinant.

From this we deduce, just as in \[\text{[Vis98]} \text{ pg. 638}, \] that the pullback morphism

\[
CH_{\text{GL}_3}(\mathbb{P}(2,2)_{\text{sq}}) \to CH_{\text{GL}_3}(\mathbb{A}(2,2)_{\text{sq}} \setminus \{0\})
\]

is surjective with kernel given by the top Chern class of $O(-1) \otimes D$.

There is an obvious equivariant isomorphism

\[
\varphi : \mathbb{P}(2,1) \to \mathbb{P}(2,2)_{\text{sq}}, \quad l \mapsto l^2
\]

In this way we can identify $CH_{\text{GL}_3}(\mathbb{P}(2,2)_{\text{sq}})$ with the equivariant Chow ring of $\mathbb{P}(2,1)$, so we have

\[
CH_{\text{GL}_3}(\mathbb{P}(2,2)_{\text{sq}}) \simeq \mathbb{Z}[c_1, c_2, c_3, s]/(f_s)
\]

with $f_s$ as in the thesis of the lemma.

Clearly, we also have an equivariant isomorphism $\varphi^*(\mathbb{A}(2,2)_{\text{sq}} \setminus \{0\}) \simeq (\mathbb{A}(2,2)_{\text{sq}} \setminus \{0\})$, and $\varphi^*(\mathbb{A}(2,2)_{\text{sq}} \setminus \{0\})$ is a $\mathbb{G}_m$-torsor over $\mathbb{P}(2,1)$. Therefore, we have

\[
CH_{\text{GL}_3}(\mathbb{A}(2,2)_{\text{sq}} \setminus \{0\}) \simeq CH_{\text{GL}_3}(\mathbb{P}(2,1))/(c_{\text{top}}(L))
\]
where $\mathcal{L}$ is the line bundle associated to $\varphi^*(\mathcal{A}(2,2)_{sq} \setminus \{0\})$, which is isomorphic to $\varphi^*(\mathcal{O}(-1) \otimes \mathbb{D})$. It is immediate to check that $\varphi^*(\mathcal{O}(-1)) \simeq \mathcal{O}(-2)$ and that $\varphi^*\mathbb{D} \simeq \mathbb{D}$, from which we easily deduce that $c_{op}(\mathcal{L}) = 2s - c_1$. This completes the proof of the lemma. □

Recall that the divisor $\tilde{D}_E \subset \mathbb{P}(V_n)_E$ has two irreducible components, denoted $\tilde{D}^1_E$ and $\tilde{D}^2_E$, defined as

$\tilde{D}^1_E = \{(l^2, f, u) \text{ such that } l(u) = f(u) = 0\}$

$\tilde{D}^2_E = \{(l^2, f, u) \text{ such that } l(u) = 0, V_+(l, f) \text{ is singular in } \mathbb{P}^2\}$

We want to compute the cycle classes of these two components.

**Lemma 5.5.** We have $[\tilde{D}^i_E] = h + ns$ in $CH_{GL_3}(\mathbb{P}(V_n)_E)$, where $h$ denotes the hyperplane class of the projective bundle $\mathbb{P}(V_n)_E \to E$.

**Proof.** By remark 5.1, it is completely equivalent to compute the class of $D := \{(l^2, f, u) \text{ such that } l(u) = f(u) = 0\}$ in the equivariant Chow ring of $\mathbb{P}(2, n) \times E$. Consider now the scheme $E' := \{(l^2, u) \text{ such that } l(u) = 0\} \subset \mathbb{P}(2, n) \times \mathbb{P}^2$.

Then we have the cartesian square

$\begin{array}{ccc}
\mathbb{A}(2, 2)_{sq} \setminus \{0\} & \xrightarrow{f} & \mathbb{P}(2, 2)_{sq} \\
\downarrow{\varphi'} & & \downarrow{p} \\
E' & \xrightarrow{p'} & E''
\end{array}$

from which it is almost immediate to deduce that the pullback morphism

$p'^* : CH_{GL_3}(E') \to CH_{GL_3}(E)$

is surjective with kernel equal to the kernel of $p^*$. As in the proof of lemma 5.4, we can identify $CH_{GL_3}(\mathbb{P}(2, n)_{sq})$ with the equivariant Chow ring of $\mathbb{P}(2, 1)$, therefore we have:

$CH_{GL_3}(E') \simeq \mathbb{Z}[c_1, c_2, c_3, h_{E'}, s] / (f_{E'}, s)$

and $p'^*$ sends $h_{E'}$ to $h_{E}$, and the kernel is generated, as an ideal, by $2s - c_1$ (this last claim follows from the proof of lemma 5.3). Define $D' := \{(f, l, u) \text{ such that } l(u) = f(u) = 0\} \subset \mathbb{P}(2, n) \times E'$ so that $p'^*[D'] = [D]$. In this way we have basically reduced the computation of $[D]$ to the computation of $[D']$. Indeed, once we know $[D']$, all we have to do is to compute $p^*[D']$, which blows down to substitute $c_1$ with $2s$.

Let $i : \mathbb{P}(2, n) \times E' \to \mathbb{P}(2, n) \times \mathbb{P}(2, 1) \times \mathbb{P}^2$ be the closed immersion: then $i$ is regular of codimension 1, i.e. $\mathbb{P}(2, n) \times E'$ is a Cartier divisor. Consider the Gysin homomorphism

$i^* : CH_{GL_3}(\mathbb{P}(2, n) \times E') \to CH_{GL_3}(\mathbb{P}(2, n) \times \mathbb{P}(2, 1) \times \mathbb{P}^2)$

and let $D''$ be the closed subscheme of $\mathbb{P}(2, n) \times \mathbb{P}(2, 1) \times \mathbb{P}^2$ defined as

$D'' := \{(f, l, u) \text{ such that } f(u) = 0\}$

We claim that $i^*[D''] = [D']$.

The class of $D''$ can be easily computed, and we obtain $[D''] = h + nt$. Observe that we have $i^* h = h$ and $i^* t = s$: from this we obtain that $i^*[D''] = h + ns$, and this concludes the proof. □
Corollary 5.6. The class \([\tilde{D}_E]\) is non-zero in \(CH^2_{GL_3}(\mathbb{P}(V_n)_{bl})\).

Proof. From lemma 5.5, we know that \([\tilde{D}_E]\) is singular in \(CH^1_{GL_3}(E)\). Let \(j : E \to A(2, 2)_{\text{sing}}\) be the closed immersion of the exceptional divisor in the blow up, then we have the cartesian square

\[
\begin{array}{ccc}
P(V_n)_E & \xrightarrow{j'} & P(V_n)_{\text{bl}} \\ \downarrow{p'} & & \downarrow{p} \\ E & \xrightarrow{j} & A(2, 2)_{\text{sing}}
\end{array}
\]

We want to prove that \(j'_* [\tilde{D}_E]\) is non-zero in \(CH^2_{GL_3}(\mathbb{P}(V_n)_{bl})\).

This last equivariant Chow group can be decomposed as follows:

\[
CH^2_{GL_3}(\mathbb{P}(V_n)_{bl}) \simeq p^* CH^2_{GL_3}(A(2, 2)_{\text{sing}}) \oplus p^* CH^1_{GL_3}(A(2, 2)_{\text{sing}}) \cdot h
\]

\[
\oplus p^* CH^0_{GL_3}(A(2, 2)_{\text{sing}}) \cdot h^2
\]

We have a similar picture for \(CH^1_{GL_3}(\mathbb{P}(V_n)_E)\), namely:

\[
CH^1_{GL_3}(\mathbb{P}(V_n)_E) \simeq \tilde{j}^* CH^1_{GL_3}(E) \oplus \tilde{j}^* CH^0_{GL_3}(E) \cdot h
\]

From the cartesianity of the diagram above and the fact that \(j^* h = h\) we deduce that the morphism \(j^*_*\) splits into two morphisms

\[
p^* j_1 : CH^1_{GL_3}(E) \to CH^2_{GL_3}(A(2, 2)_{\text{sing}}), \quad p^* j_0 : CH^0_{GL_3}(E) \to CH^1_{GL_3}(A(2, 2)_{\text{sing}})
\]

We can decompose \([\tilde{D}_E]\) as \(ns \oplus [E]\), and therefore \(j'_* [\tilde{D}_E]\) can be decomposed as \(p^* j_1 ns \oplus p^* j_0 [E]\). Observe that such decomposition remains valid also after tensoring with \(\mathbb{F}_2\). Thus, if we prove that \(p^* j_1 [E]\) is non-zero after tensoring with \(\mathbb{F}_2\), then we are done. The morphism \(p^* j_1\) is injective, so that it is enough to show that \(j_1[E]\) is non-zero in \(CH^1_{GL_3}(A(2, 2)_{\text{sing}})\), which is obvious. \(\square\)

Lemma 5.7. We have \([\tilde{D}_E] = 0\) in \(CH_{GL_3}(\mathbb{P}(V_n)_E)_{\mathbb{F}_2}\) and \(CH_{GL_3}(\mathbb{P}(V_n)_{bl})_{\mathbb{F}_2}\).

Proof. Clearly, the second assertion follows from the first one. As in the proof of lemma 5.5, we can equivalently compute the class of

\[
D := \{(f, l^2, u) \text{ such that } l(u) = 0 \text{ and } V_+(l, f) \text{ is singular in } \mathbb{P}^2\}
\]

in the equivariant Chow ring of \(\mathbb{P}(2, n) \times E\).

Using the same notation of the proof of lemma 5.5, we can actually reduce ourselves to show that the cycle class of

\[
D' := \{(f, l, u) \text{ such that } l(u) = 0 \text{ and } V_+(l, f) \text{ is singular in } \mathbb{P}^2\}
\]

inside the equivariant Chow ring of \(\mathbb{P}(2, n) \times E'\) is a multiple of 2.

Again, with the same arguments of the proof of lemma 5.5, it is enough to show that

\[
D'' := \{(f, l, u) \text{ such that } V_+(l, f) \text{ is singular in } \mathbb{P}^2\}
\]

has even cycle class, because \([D'] = i^*[D'']\), where

\[
i^* : CH_{GL_3}(\mathbb{P}(2, n) \times \mathbb{P}(2, 1) \times \mathbb{P}^2) \to CH_{GL_3}(\mathbb{P}(2, n) \times E')
\]

is the Gysin homomorphism.

Actually, we have that \([D''] = pr^*_{12}[D'']\), where

\[
pr_{12} : \mathbb{P}(2, n) \times \mathbb{P}(2, 1) \times \mathbb{P}^2 \to \mathbb{P}(2, n) \times \mathbb{P}(2, 1)
\]
is the projection on the first and second factor, and \( D'' \) is defined as
\[
D'' := \{(f, l) \text{ such that } V_\alpha(l, f) \text{ is singular in } \mathbb{P}^2\}
\]
Recall that the cycle class of \( D'' \) had already been computed in the proof of lemma 4.3 (there it was called \( D' \)), where we found that it was 2-divisible. This completes the proof of the lemma.

6. The key lemma

The goal of this section is to prove lemma 5.3 (1), which was the only missing ingredient for completing the computation of the cohomological invariants of \( \mathcal{H}_g \) done in section 5. Let us restate here what we are going to prove, which is now

**Lemma 6.1.** Assume \( p = 2 \). Then \( i_* : A^0_{\text{GL}_3}(\Delta_{1, 2n}) \rightarrow A^1_{\text{GL}_3}(\mathbb{P}(1, 2n)) \) is zero for every \( n \).

From now on, we will always assume \( p = 2 \). Using proposition 5.3 we see that the lemma above is equivalent to saying that the morphisms
\[
i_* : A^0_{\text{GL}_3}(D_{\text{sm}}) \rightarrow A^1_{\text{GL}_3}(\mathbb{P}(V_n)_\text{sm})
\]
are zero for every \( n \).

6.1. Strategy of proof. Before giving a detailed proof, let us sketch here, in an abstract setting, what will be our strategy. Suppose to have \( Y \subset X \) a closed subscheme of codimension 1, and suppose also to have \( X^o \subset X \) an open subscheme. Denote \( \partial X \) the closed subscheme of \( X \) defined as \( X \setminus X^o \), and assume it has codimension 1. Let \( Y^o \) (resp. \( \partial Y \)) be the pullback of \( X^o \) (resp. \( \partial X \)) along the closed immersion \( Y \hookrightarrow X \). Assume that \( \partial Y \) also has codimension 1 in \( Y \). In this way we obtain the following commutative diagram:

\[
\begin{array}{ccc}
A^0_{\text{GL}_3}(Y) & \xrightarrow{i_*} & A^1_{\text{GL}_3}(X) \\
\downarrow j^* & & \downarrow j^* \\
A^0_{\text{GL}_3}(Y^o) & \xrightarrow{i^o_*} & A^1_{\text{GL}_3}(X^o) \\
\downarrow \partial^* & & \downarrow \partial^* \\
A^0_{\text{GL}_3}(\partial Y) & \xrightarrow{i^o_*} & A^1_{\text{GL}_3}(\partial X)
\end{array}
\]

Suppose moreover that the group \( A^1_{\text{GL}_3}(X) \) has only elements of degree 0. Then if one wants to prove that \( i^o_*(\alpha_d) = 0 \) for an element \( \alpha_d \) of degree \( d \) greater than 0, thanks to the exactness of the vertical sequences of the diagram above, we can equivalently show that \( i^o_*(\partial \alpha_d) = 0 \). Indeed, if this the case, then \( \partial(i^o_*(\alpha_d)) \) must be zero, which implies that \( i^o_*(\alpha_d) = j^* \beta \) for some element \( \beta \). But \( j^* \) preserves the degrees, thus the hypothesis on \( A^1_{\text{GL}_3}(X) \) that we made implies that \( i^o_*(\alpha_d) \) must be zero. The upshot is that we reduced ourselves to show that \( i^o_*(\alpha_{d-1}) = 0 \), where \( \alpha_{d-1} = \partial \alpha_d \) is an element of degree \( d - 1 \).

Suppose now to have two set of schemes \( (X_k, X^o_k, \partial X_k) \) and \( (Y_k, Y^o_k, \partial Y_k) \), \( k = 0, \ldots, d - 1 \), with the same properties of the schemes above, and assume moreover that for each \( k \) we have \( \partial X_k = X^o_{k+1} \) and \( \partial Y_k = Y^o_{k+1} \). Then we can repeat the argument we used before, defining each time \( \alpha_{d-k-1} := \partial Y_k \alpha_{d-k} \). Then we deduce that proving the equation \( i^o_*(\alpha_d) = 0 \) is the same as proving \( i^o_*(\alpha_0) = 0 \) in \( A^1_{\text{GL}_3}(\partial X_{d-1}) \). The advantage is that now we have to deal with an element of degree 0, and the degree 0 part of the Chow groups with coefficients coincides with ordinary Chow groups tensored with \( \mathbb{F}_p \). Therefore \( \alpha_0 \) is an algebraic cycle, with whom is usually easier to do computations.
6.2. **Proof of the key lemma.** We know almost nothing about \( A^0_{GL_3}(D_{sm}) \) but, on the other side, we know a lot about \( A^1_{GL_3}(\mathbb{P}(V_n)_{sm}) \). Indeed, from the formula for equivariant Chow rings with coefficients of projective bundles, we have

\[
A^1_{GL_3}(\mathbb{P}(V_n)_{sm}) \simeq A^1_{GL_3}(\mathbb{A}(2,2)_{sm}) \oplus A^0_{GL_3}(\mathbb{A}(2,2)_{sm}) \cdot h
\]

where \( h \) is equal to \( c_1^{GL_3}(O(1)) \), which is an element of codimension 1 and degree 0. Applying again proposition 3.3 to the PGL2-scheme Spec(k\(3\)), whose GL3-counterpart is \( \mathbb{A}(2,2)_{sm} \), and using proposition 2.2 we readily deduce that

\[
A^1_{GL_3}(\mathbb{P}(V_n)_{sm}) = (CH^1_{GL_3}(\mathbb{P}(V_n)_{sm})_{F_2}) \oplus k \cdot \tau_1 \oplus k \cdot w_2 h
\]

where the first addend coincides with the elements of degree 0, the element \( \tau_1 \) has codimension and degree both equal to 1, and finally \( w_2 \) has codimension 0 and degree 2, so that \( w_2 h \) has codimension 1 and degree 2.

Thanks to the fact that \( i_* \) preserves the degree, every element in \( A^0_{GL_3}(D_{1,n}) \) of degree greater than 2 will be sent by \( i_* \) to 0. We need to find out if there is any element \( \alpha \) of degree smaller or equal to 2 such that \( i_* \alpha \) is not zero.

**Lemma 6.2.** The morphism \( i_* : CH^0_{GL_3}(D_{sm})_{F_2} \to CH^1_{GL_3}(\mathbb{P}(V_n)_{sm})_{F_2} \) is zero.

**Proof.** We have to show that the cycle class \([D_{sm}] = 0 \) in \( CH^1_{GL_3}(\mathbb{P}(V_n)_{sm})_{F_2} \). From [DL] proposition 4.2 we have that \([D_{sm}] = 4(n-2)h \) in \( CH^1_{GL_3}(\mathbb{P}(V_n)_{sm}) \). This implies the lemma.

The lemma above tells us that if \( i_* \alpha \neq 0 \), then the degree of \( \alpha \) can be 1 or 2.

We have a closed immersion of \( \mathbb{P}(V_n)_{nod} \) inside \( \mathbb{P}(V_n)_{red} \), whose open complement is \( \mathbb{P}(V_n)_{sm} \). Moreover, if we pullback \( D_{red} \) along this closed immersion, we obtain \( D_{nod} \). Using the compatibility formulas, we deduce the following commutative diagram:

\[
\begin{array}{ccc}
A^0_{GL_3}(D_{red}) & \xrightarrow{\iota_{nod}} & A^1_{GL_3}(\mathbb{P}(V_n)_{red}) \\
\downarrow j_D & & \downarrow j^* \\
A^0_{GL_3}(D_{sm}) & \xrightarrow{i_*} & A^1_{GL_3}(\mathbb{P}(V_n)_{sm}) \\
\downarrow \alpha_D & & \downarrow \beta \\
A^0_{GL_3}(D_{nod}) & \xrightarrow{i^*_{nod}} & A^1_{GL_3}(\mathbb{P}(V_n)_{nod}) \\
\downarrow f_D & & \downarrow f^* \\
A^1_{GL_3}(D_{red}) & \xrightarrow{\iota_{red}} & A^2_{GL_3}(\mathbb{P}(V_n)_{red})
\end{array}
\]

Observe that the vertical sequences are exact.

**Lemma 6.3.** Suppose that \( \alpha \) in \( A^1_{GL_3}(\mathbb{P}(V_n)_{sm}) \) is of degree greater than 0. Then \( \partial \alpha \neq 0 \) or \( \alpha = 0 \).

**Proof.** Suppose that \( \partial \alpha \neq 0 \). The exactness of the right vertical sequence of [1] implies that \( \alpha = j^* \beta \) for some \( \beta \) in \( A^1_{GL_3}(\mathbb{P}(V_n)_{red}) \). Observe that such \( \beta \) must have degree greater than 0 because \( j^* \) preserves the degrees. Using the formula for projective bundles we have

\[
A^1_{GL_3}(\mathbb{P}(V_n)_{red}) = A^1_{GL_3}(\mathbb{A}(2,2)_{red}) \oplus A^0_{GL_3}(\mathbb{A}(2,2)_{red})
\]

The scheme \( \mathbb{A}(2,2)_{red} \) is an open subscheme of \( \mathbb{A}(2,2) \) whose complement has codimension 3. This implies that

\[
A^1_{GL_3}(\mathbb{A}(2,2)_{red}) = A^i_{GL_3}(\mathbb{A}(2,2)) \text{ for } i = 0, 1
\]
We have that $A(2, 2)$ is a $GL_3$-equivariant vector bundle over $\text{Spec}(k_0)$, from which we deduce
\[ A_{GL_3}(A(2, 2)) = A_{GL_3} \]
By proposition 6.4, the only element of degree greater than 0 in this ring is 0, thus $\beta = 0$ and $\alpha = 0$.

**Corollary 6.4.** Let $\alpha$ be an element of $A^0_{GL_3}(D_{\text{red}})$ of degree greater than 0. Then $i_*\alpha = 0$ if and only if $i_*^{\alpha=0}(\partial\alpha) = 0$.

**Proof.** Follows from the commutativity of the diagram 1 and from lemma 6.3. $\square$

Suppose that $\alpha$ in $A^0_{GL_3}(D_{\text{red}})$ has degree 1. Then $\partial\alpha$ is a degree zero element of $A^0_{GL_3}(D_{\text{nod}})$: the degree zero part of this group can be identified with $CH^0_{GL_3}(D_{\text{nod}})$. From proposition 6.1 we deduce that
\[ CH^0_{GL_3}(D_{\text{nod}}) \cong CH^0_{GL_3}(D^1_{\text{nod}}) \oplus CH^0_{GL_3}(D^2_{\text{nod}}) \]
where $D^1_{\text{nod}}$ and $D^2_{\text{nod}}$ are the two irreducible components of $D_{\text{nod}}$. Recall that $D^1_{\text{nod}}$ is the locus of pairs $(q, f)$ inside $\text{Spec}(V'_n)$ such that $F$, the plane curve defined by the equation $f = 0$, passes through a singular point of $Q$, the conic of equation $q = 0$. Write $\partial\alpha = (n, m)$. From corollary 6.3 we have that $i_*\alpha = 0$ if and only if $0 = i_*^{\alpha=0}(n, m) = n[D^1_{\text{nod}}] + m[D^2_{\text{nod}}] \in CH^1_{GL_3}(\mathbb{P}(V_n))$

We need to understand if this is the case, or not.

**Lemma 6.5.** If $\alpha$ in $A^0_{GL_3}(D_{\text{sm}})$ has degree 1, then $\partial\alpha = (0, m)$.

**Proof.** Because of the exactness of the left vertical sequence of the diagram 1 we have $f_*(\partial\alpha) = 0$. This implies that
\[ 0 = i_*^{\alpha=0}f_*(\partial\alpha) = n[D^1_{\text{nod}}] + m[D^2_{\text{nod}}] \]
in $CH^2_{GL_3}(\mathbb{P}(V_n)) \cong CH^2_{GL_3}(\mathbb{P}(V_n))$. By lemma 5.2 and 5.3 we obtain that $0 = ac_1h$, thus $n = 0$. $\square$

We are ready to prove the first half of the main lemma 6.1 which is implied by the following proposition.

**Proposition 6.6.** If $\alpha$ in $A^0_{GL_3}(D_{\text{sm}})$ has degree 1, then $i_*\alpha = 0$.

**Proof.** By corollary 6.1, it is equivalent to prove that $i_*^{\alpha=0}\partial\alpha = 0$. By lemma 6.5 we deduce that $i_*^{\alpha=0}\partial\alpha = m[D^2_{\text{nod}}]$ in $CH^2_{GL_3}(\mathbb{P}(V_n))$, and by lemma 5.3 we know that this last term is zero, thus concluding the proof. $\square$

In order to prove lemma 6.1 we need to find out whether or not $i_*\alpha = 0$ when $\alpha$ has degree 2. Using corollary 6.1 this is the same as understanding if $i_*^{\alpha=0}(\partial\alpha) = 0$. Actually, we are going to prove that $i_*^{\alpha=0}\beta = 0$ for every $\beta$ of degree 1.

Let $\hat{A}(2, 2)_{\text{sing}}$ be the closed subscheme of $A(2, 2)$ parametrising singular forms, and as in section 4 let $\hat{A}(2, 2)_{\text{eq}}$ be the closed subscheme parametrising trinary forms of degree 2 which are the square of linear trinary forms. Let $\hat{A}(2, 2)_{\text{sing}}$ be the blow-up of $\hat{A}(2, 2)_{\text{eq}} \setminus \{0\}$ along $\hat{A}(2, 2)_{\text{eq}} \setminus \{0\}$ (pay attention to the fact that in section 4 the scheme $\hat{A}(2, 2)_{\text{sing}}$ was the blow up of $\hat{A}(2, 2)_{\text{sing}}$, not of $\hat{A}(2, 2)_{\text{sing}} \setminus \{0\}$). Let us denote $\mathbb{P}(V_n)_{\text{bl}}$ the pullback of $\mathbb{P}(V_n)_{\text{sing}}$ along the morphism $\hat{A}(2, 2)_{\text{sing}} \to \hat{A}(2, 2)_{\text{eq}} \setminus \{0\}$, and let $\mathbb{P}(V_n)_E$ be the restriction of $\mathbb{P}(V_n)_{\text{bl}}$ on the exceptional divisor $E \subset \hat{A}(2, 2)_{\text{sing}}$. Finally, let $\tilde{D}$ be the proper transform of $D_{\text{sing}}$
$A(2, 2)_{\text{sing}}$ and let $\tilde{D}_E$ indicate the restriction of $\tilde{D}$ over $E$. Then we have the following commutative diagram, whose vertical sequences are exact:

\[
\begin{array}{cccc}
A^0_{GL_3}(\tilde{D}) & \xrightarrow{\iota_*} & A^1_{GL_3}(\mathbb{P}(V_n)_{bl}) \\
\downarrow j^\dagger & & \downarrow j^* \\
A^0_{GL_3}(D_{nod}) & \xrightarrow{i^\mod} & A^1_{GL_3}(\mathbb{P}(V_n)_{nod}) \\
\downarrow \partial_D & & \downarrow \gamma \\
A^0_{GL_3}(\tilde{D}_E) & \xrightarrow{i^E} & A^1_{GL_3}(\mathbb{P}(V_n)_E) \\
\downarrow f_D & & \downarrow f_* \\
A^1_{GL_3}(\tilde{D}) & \xrightarrow{\iota_*} & A^2_{GL_3}(\mathbb{P}(V_n)_{bl})
\end{array}
\]

The diagram above looks very similar to diagram [1] the one we used before to prove that $\iota_* \partial \beta = 0$, when $\beta$ has degree 1, we are going to rely on arguments quite similar to the ones used before.

**Lemma 6.7.** Suppose that $\beta$ in $A^1_{GL_3}(\mathbb{P}(V_n)_{nod})$ is of degree greater than 0. Then $\partial \beta \neq 0$ or $\beta = 0$.

**Proof.** Suppose that $\partial \beta = 0$. Then it follows from the exactness of the left vertical sequence of [2] that $\beta = j^* \gamma$, with $\gamma$ of degree 1. We have $A^1_{GL_3}(\mathbb{P}(V_n)_{bl}) \simeq A^1_{GL_3}(\mathbb{A}(2, 2)_{\text{sing}})$, and by corollary [4] we have that $A^1_{GL_3}(\mathbb{A}(2, 2)_{\text{sing}}) \simeq A^1_{GL_3}(\mathbb{P}^2)$. Again, pay attention to the fact that in section [3] we defined $\mathbb{A}(2, 2)_{\text{sing}}$ as the blow up of $\mathbb{A}(2, 2)_{\text{sing}}$, whereas here we are taking the blow up of $\mathbb{A}(2, 2)_{\text{sing}} \setminus \{0\}$. Nevertheless, this does not affect the equivariant Chow groups with coefficients of codimension 1, because the closed subscheme that we are deleting has higher codimension. Now observe that $\mathbb{P}^2$ is a $GL_3$-equivariant projective bundle over $\text{Spec}(k_0)$, thus $A^1_{GL_3}(\mathbb{P}^2) \simeq A^1_{GL_3} \oplus A^0_{GL_3}$. Recall from proposition [2] that $GL_3$ has no non-zero element of degree greater than 0. From this follows that $\gamma = 0$ and $\beta = 0$. 

**Corollary 6.8.** Let $\beta$ be an element of $A^0_{GL_3}(D_{nod})$ of degree greater than 0. Then $i^\mod \beta = 0$ if and only if $i^E(\partial \beta) = 0$.

**Proof.** Follows from the commutativity of diagram [2] and from lemma [6.7].

In particular, in order to prove that $i^\mod \beta = 0$ for every $\beta$ of degree 1, we can equivalently prove that $i^E(\partial \beta) = 0$. Observe that $\partial \beta$ has degree 0, so that we have again reduced ourselves to make computations with equivariant Chow groups tensorized with $\mathbb{F}_2$ instead of equivariant Chow groups with coefficients. To show that $i^E(\partial \beta) = 0$ we will apply the same strategy used before.

Identify the subgroup of elements of degree 0 in $A^0_{GL_3}(\tilde{D}_E)$ with $CH^0_{GL_3}(\tilde{D}_E)_{\mathbb{F}_2}$, so that we have

$$CH^0_{GL_3}(\tilde{D}_E)_{\mathbb{F}_2} = CH^0_{GL_3}(\tilde{D}_E^1)_{\mathbb{F}_2} \oplus CH^0_{GL_3}(\tilde{D}_E^2)_{\mathbb{F}_2} \simeq \mathbb{F}_2 \oplus \mathbb{F}_2$$

Write $\partial \beta = (n, m)$. We need to understand what possible values of $n$ and $m$ can appear in this expression.

**Lemma 6.9.** If $\beta$ in $A^0_{GL_3}(D_{nod})$ has degree 1, then $\partial \beta = (0, m)$. 

Proof. By exactness of the left vertical sequence of diagram\textsuperscript{2}, we have that \( f_D(\partial \beta) = 0 \), thus \( \tilde{i}_E f_D(\partial \beta) = 0 \). If we write \( \partial \beta \) as \((n,m)\), then we are saying that \( n[\tilde{D}u_1] + m[\tilde{D}u_2] = 0 \) in \( CH^1_{GL_3}(\mathbb{P}(V_n)_{bl}F_2) \). By corollary \textsuperscript{5.6} and lemma \textsuperscript{5.7} this readily implies that \( n = 0 \). □

We are now in position to complete the proof of lemma \textsuperscript{6.1} thanks to the next proposition:

**Proposition 6.10.** If \( \beta \) is an element of degree 1 in \( A^0_{GL_3}(D_{nod}) \), then \( i_{E}^{\text{nod}} \beta = 0 \).

**Proof.** By corollary \textsuperscript{6.8} we can equivalently show that \( i^E \beta = 0 \). From lemma \textsuperscript{6.9} we know that \( \partial \beta = (0,m) \). This implies that \( i^E \beta = m[\tilde{D}_u] \), which is equal to zero by lemma \textsuperscript{5.7}. □

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