Polarization time

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Abstract. The Poincaré vector, combined with the optical intensity, provides a geometric representation of the polarization state of a beam-like electromagnetic field. For fluctuating beam fields this vector normalized by the instantaneous intensity traces, as a function of time, a path on the Poincaré sphere. We introduce a measure for the average time period over which the Poincaré vector in a stationary beam remains substantially unchanged. This time interval defines the ‘polarization time’ and from it one deduces the polarization length of the beam. Our analysis is based on an intensity-normalized correlation function associated with the Poincaré vector. For fields obeying Gaussian statistics this correlation function takes on a simple form in terms of previously defined quantities that characterize the polarization and temporal coherence properties of the beam. We discuss beams of blackbody radiation as an illustrative example.

1. Introduction

While considerable attention recently has been paid towards new developments in the theories of the coherence of classical electromagnetic (vector-valued) waves and the polarization of three-dimensional (3D) fields [1, 2, 3], some simple and fundamental notions retain their importance. Among such old, useful concepts are the coherence time of fluctuating optical scalar waves and the state (and degree) of polarization of beam-like (2D) vector fields. The coherence time and the associated coherence length are characteristic measures over which the wave field remains capable of interference. The coherence length in combination of the transverse coherence area gives the coherence volume of a random scalar wave field [4]. For electromagnetic beam fields, in which the electric field vector only has two components (transverse to the propagation direction of the wave), the state of polarization traditionally is described using the four Stokes parameters [4, 5]. An elegant geometric representation then is obtained in terms of the Poincaré vector, whose length divided by the optical intensity specifies the degree of polarization. For a fully polarized beam the normalized Poincaré vector is located on the surface of the Poincaré sphere (of unit radius) and its direction identifies the state of polarization.

In this work we investigate the dynamics of the polarization fluctuations of statistically stationary, planar (2D) electromagnetic fields. We make use of the analogy to the traditional scalar-optical coherence time and coherence length, which are briefly recalled in Section 2, to introduce a normalized quantity that describes the temporal correlation of the polarization state of a 2D electromagnetic wave. For such beam fields we then find the corresponding polarization time and polarization length, as outlined in Section 3. When the 2D wave field obeys Gaussian statistics, as is often the case in nature, closed-form expressions in terms of previously used quantities can be found for the normalized polarization correlation function,
as shown in Section 4. In Section 5 we consider, as a practical example, a pencil of blackbody radiation. The main results and conclusions are summarized in Section 6.

### 2. Coherence time and length

Let \( V(t) \) be a complex analytic signal realization of a random, statistically stationary, beam-like scalar wave field. Then the quantity

\[
\Gamma(\tau) = \langle V^*(t) V(t + \tau) \rangle,
\]

where the asterisk denotes complex conjugation and the brackets ensemble averaging, is the associated mutual coherence function. The (optical) intensity of the beam can be obtained as \( I(t) = \langle |V(t)|^2 \rangle = \Gamma(0) \), being simply a constant. The normalized function

\[
\gamma(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)},
\]

known as the complex degree of coherence, then satisfies the conditions \( \gamma(0) = 1 \) and \( |\gamma(\tau)| \leq 1 \). Some typical forms of the absolute value \( |\gamma(\tau)| \) are illustrated in Fig. 1.

![Figure 1](image)

**Figure 1.** Definition of the characteristic coherence time \( \tau_0 \) by means of the degree of coherence \( |\gamma(\tau)| \) for statistically stationary scalar-optical beams. Two typical behaviors of coherence are illustrated.

For ergodic fields \( \gamma(\tau) \to 0 \), as \( \tau \to \infty \) [4]. The coherence time of the beam, \( \tau_0 \), can then be defined by the requirement that \( |\gamma(\tau_0)| = \gamma_0 \), where \( \gamma_0 \) is some small number (say, on the order of 0.1, but possibly as large as 0.5). In case there are many such \( \tau_0 \)s, the smallest one is picked. If \( |\gamma(\tau)| \) takes on the value zero for some \( \tau \), the smallest of such zero crossings may also be chosen as \( \tau_0 \) (in this case \( \gamma_0 = 0 \)). Typically, the coherence time \( \tau_0 \) and the spectral width \( \Delta \nu \) of the beam are related as \( \tau_0 \Delta \nu \approx 1 \) [4]. In terms of the coherence time one may introduce the coherence length as \( l_c = c \tau_0 \), where \( c \) is the speed of light. The coherence length represents the maximum separation (along the propagation direction), at which the scalar waves still interfere with each other.

Other definitions of the coherence time and related quantities, leading to useful relationships convenient for specific applications, are discussed in the literature [2, 4, 6].

### 3. Poincaré vector and polarization time

For a monochromatic field, the electric field vector traces an ellipse (including a line or a circle) as a function of time and the field is fully polarized. In this case, we may say that the polarization state is static, although it may vary from point to point. When the field is polychromatic, the electric vector is fluctuating and we may consider it as a (stationary) random process. For such a wave field, as a function of time, the polarization ellipse may undergo random rotations in the plane orthogonal to the propagation direction and the magnitudes of the minor and major axes can change. Hence, the instantaneous polarization state can be different at different instants of time, and we may say that the field exhibits polarization dynamics. When the polarization ellipse varies in time in a completely random manner, i.e., such that no preferred state of polarization exists when observed over a long period of time,
the field is fully unpolarized. However, the important point is that for any field at sufficiently short time separations no significant change in the polarization state takes place and, therefore, even a fully unpolarized field when considered within a short enough time interval is fully polarized.

For beam-like wave fields the instantaneous polarization state is fully determined by the time-dependent Stokes parameters, denoted by \( S_i(t) \), with \( i = 0 \ldots 3 \) \([4, 5]\). Using these parameters we can introduce the Poincaré vector as

\[
\mathbf{S}(t) = [S_1(t), S_2(t), S_3(t)],
\]

which is real and whose length equals \( S_0(t) \), i.e., the instantaneous intensity. In analogy with scalar-optical coherence theory, we may view the expectation

\[ \Gamma_S(\tau) = \langle \mathbf{S}(t) \cdot \mathbf{S}(t + \tau) \rangle \]

as the basic polarization correlation function. For \( \tau = 0 \), Eq. (3) takes on its maximum value of \( \Gamma_S(0) = \langle S_0^2(t) \rangle = S_0^2 \) (stationary field). As \( \tau \) increases, \( \mathbf{S}(t + \tau) \) typically deviates from \( \mathbf{S}(t) \) in some random way, since the fluctuations in the electric-field components influence both the polarization state and the intensity. However, at any instant of time, we have \( \mathbf{S}(t) = \mathbf{u}(t)S_0(t) \), where \( \mathbf{u}(t) \) is a unit vector. Hence \( \Gamma_S(\tau) = \langle [\mathbf{u}(t) \cdot \mathbf{u}(t + \tau)]S_0(t)S_0(t + \tau) \rangle \) and we may use the fact that \( |\mathbf{u}(t) \cdot \mathbf{u}(t + \tau)| \leq 1 \).

For quantifying the change of the polarization state in a time interval \( \tau \) we, therefore, introduce the normalized quantity

\[ \gamma_p(\tau) = \Gamma_S(\tau)/[\langle S_0(t)S_0(t + \tau) \rangle], \]

which obviously has the following two properties: \( \gamma_p(0) = 1 \) and \( |\gamma_p(\tau)| \leq 1 \). Again, some typical forms of the absolute value \( |\gamma_p(\tau)| \) are illustrated in Fig. 2.

In full analogy with the coherence time, the polarization time can be defined via the relation \( |\gamma_p(\tau_p)| = \gamma_{p0} \), where \( \gamma_{p0} \) is a small positive number (taken 1/2 in Fig. 2). For time separations \( \tau < \tau_p \) the beam’s state of polarization can be regarded, on average, as effectively unchanged. In Fig. 2, the two lower curves lead to finite polarization times and for durations \( \tau_{pj} \) \( (j = 1, 2) \) the beams are essentially polarized. The beams corresponding to the two upper curves have such a high degree of polarization that \( \tau_{pj} \) \( (j = 3, 4) \) can be considered to be infinite. Once the polarization time is established, the corresponding polarization length follows as \( l_p = c\tau_p \).

4. Beams obeying Gaussian statistics

Let us now assume that the beam’s electric field \( \mathbf{E}(t) \) is a (vector) Gaussian random process, i.e., the fluctuations of the field components \( E_x(t) \) and \( E_y(t) \) are governed by jointly Gaussian probability densities. Such a situation occurs frequently in nature, for instance in connection with chaotic light. The Poincaré vector correlation function \( \Gamma_S(\tau) \) in Eq. (3) and the intensity correlation \( \langle S_0(t)S_0(t + \tau) \rangle \) can readily be evaluated and we find for the normalized polarization correlation function in Eq. (4) the result

\[
\gamma_p(\tau) = \frac{P^2 - \gamma_{EM}^2(\tau) + 2|\gamma_W(\tau)|^2}{1 + \gamma_{EM}^2(\tau)},
\]

where \( \gamma_{EM}(\tau) = \gamma_1(\tau) + \gamma_2(\tau) \) and \( \gamma_W(\tau) = \gamma_3(\tau) + \gamma_4(\tau) \).
where $P$ is the degree of polarization of the beam [4]. $\gamma_{EM}(\tau)$ is the degree of electromagnetic coherence of the field [7, 8], and $\gamma_{W}(\tau)$ is a complex function (introduced by Wolf [9]; see also the work of Karczewski [10]) that characterizes the intensity modulation in interference of the field and its time-delayed version. Explicitly, these quantities are given by

$$P^2 = 1 - \frac{4\text{det}\Phi(0)}{\text{tr}^2\Phi(0)},$$

$$\gamma_{EM}^2(\tau) = \frac{\text{tr}[\Phi(\tau) \cdot \Phi(-\tau)]}{\text{tr}^2\Phi(0)},$$

$$\gamma_{W}(\tau) = \frac{\text{tr} \Phi(\tau)}{\text{tr} \Phi(0)},$$

where $\Phi_{ij}(\tau) = \langle E_i'(t)E_j'(t+\tau) \rangle$ ($i, j = x, y$) is the $2 \times 2$ temporal coherence matrix of the beam, and $\text{det}$ and $\text{tr}$ denote the determinant and trace of the matrix, respectively.

For $\tau = 0$ we readily find $\gamma_{EM}^2(0) = P^2/2 + 1/2$ and $\gamma_{W}(0) = 1$, and therefore $\gamma_p(0) = 1$, as was required. This result is a reflection of the fact that at any instant of time the field has a certain polarization state. In the opposite limit when $\tau \to \infty$, the correlations of the electric-field components vanish and both $\gamma_{EM} \to 0$ and $\gamma_{W} \to 0$, implying that $\lim_{\tau \to \infty} \gamma_p(\tau) = P^2$. This result means that for sufficiently long delays $\tau$, the transient polarization states (temporal correlations) do not play any role and $\gamma_p$ approaches the square of the (time-averaged) degree of polarization. This behavior is also qualitatively illustrated in Fig. 2. In particular, we see that curve 1 represents an unpolarized beam ($P_1 = 0$), whereas beams 2 and 3 have finite degrees of polarization $P_2$ and $P_3$, respectively. Beam 4 is fully polarized $P_4 = 1$.

5. Blackbody radiation beams

Let us assume that an opening (of dimensions much larger than the mean wavelength) is made into one of the walls of a blackbody cavity. The radiation emanating from such a hole obeys Lambert’s law and is unpolarized in all directions at all frequencies [11]. We now consider a pencil of such radiation propagating in some given direction. The spectrum of both orthogonal electric-field components is the same and given by Planck’s spectrum, i.e.,

$$F(\omega) \propto \frac{\omega^3 \exp(h\omega/k_BT)}{\exp(h\omega/k_BT) - 1},$$

where $h$ is Planck’s constant divided by $2\pi$, $k_B$ is Boltzmann’s constant, and $T$ is the cavity temperature. Making use of the chaotic-light result given by Eq. (5), the unpolarized nature of the beam ($P = 0$), and the Planck spectrum of Eq. (9), we find after some calculations that

$$\gamma_p(\tau) = \frac{3|90\zeta(4, 1 + i k_B T \tau/h)|/\pi^4|^2}{2 + |90\zeta(4, 1 + i k_B T \tau/h)|/\pi^4|^2},$$

where $\zeta(s, a)$ is the generalized Riemann (Hurwitz) zeta function [5]. The behavior of the polarization correlation function $\gamma_p(\tau)$ at temperatures $T = 10$ K, $T = 30$ K, and $T = 300$ K (room temperature) is depicted in Fig. 3. The polarization times for these temperatures are obtained with the criterion $\gamma_p(\tau_p) = \gamma_p(0) = 1/2$, as described before. These polarization times have, respectively, the values of $\tau_p = 3.8 \cdot 10^{-13}$ s, $\tau_p = 1.3 \cdot 10^{-13}$ s, and $\tau_p = 1.3 \cdot 10^{-14}$ s, and they are marked with the vertical dotted lines in Fig. 3. We observe that the polarization times become shorter as the temperature increases.

The polarization lengths associated with the beams of blackbody radiation can be computed from the relation $l_p = c\tau_p$, where $c$ is the speed of light. For the beams illustrated in Fig. 3 we then obtain $l_p = 114 \mu m$, $l_p = 39 \mu m$, and $l_p = 3.9 \mu m$, respectively.

The cosmic microwave background (CMB) radiation filling the entire universe has a thermal blackbody spectrum of temperature $T = 2.725$ K [12]. This corresponds to a polarization time of $\tau_{CMB} = 1.4$ ps and a polarization length of $l_{CMB} = 0.42$ mm.
6. Discussion and conclusions

We have considered statistically stationary beams of electromagnetic radiation and assessed the time interval it takes for the polarization state to change to a considerable degree. Our analysis is based on the geometric description of the polarization state in terms of the Poincaré vector. More specifically, we introduced a normalized temporal correlation function for the polarization state and defined the polarization time as the time duration in which the polarization correlation decays to a small value, in full analogy of the coherence time of fluctuating scalar-optical beams. The polarization length then is the distance the beam field propagates within a polarization time. For beams obeying Gaussian statistics, a simple expression is found for the polarization correlation function in terms of quantities that previously have been introduced to characterize stationary electromagnetic beams, with application to blackbody radiation beams.

In this work we described the evolution of the beam’s polarization state using the Poincaré vector $S(t)$ and the instantaneous intensity $S_0(t)$. The advantage of such a representation is its geometric simplicity and elegance, although in optical systems the $4 \times 4$ Mueller matrices need to be invoked. But there are also other possible ways to describe the polarization state. Among them are the Jones vectors (together with the associated $2 \times 2$ Jones matrixes), which may lead to a deeper physical understanding of the polarization state’s behavior and the energy distribution among the polarization states beyond a mere geometric description.

Acknowledgments

This work was supported by the Academy of Finland. A.T. Friberg also acknowledges funding from the Swedish Foundation for Strategic Research (SSF).

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