Some properties of q-Gaussian distributions

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Abstract

The q-Gaussian is a probability distribution generalizing the Gaussian one. In spite of a q-normal distribution is popular, there is a problem when calculating an expectation value with a corresponding normalized distribution and not a q-normal distribution itself. In this paper, two q-moments types called normalized and unnormalized q-moments are introduced in details. Some properties of q-moments are given, and several relationships between them are established, and some results related to q-moments are also obtained. Moreover, we show that these new q-moments may be regarded as a generalization of the classical case for $q = 1$. Firstly, we determine the q-moments of q-Gaussian distribution. Especially, we give explicitly the kurtosis parameters. Secondly, we compute the expression of the q-Laplace transform of the q-Gaussian distribution. Finally, we study the distribution of sum of q-independent Gaussian distributions.

Keywords: q-Laplace transform; q-Gaussian distribution; q-moments; q-estimator.

1. Introduction and Preliminaries

Several q-analogues of certain probability distributions have been recently investigated by many authors \cite{16, 9, 10, 17}. The q-distributions have been introduced in statistical physics for the characterization of chaos and multifractals. These distributions $f_q$ are a simple one parameter transformation of an original density function $f$ according to

$$f_q(x) = \frac{f(x)^q}{\int f(x)^q dx}$$

The parameter $q$ behaves as a microscope for exploring different regions of the measure $p$: for $q > 1$, the more singular regions are amplified, while for $q < 1$ the less singular regions are accentuated.

Tsallis distributions (q-distributions) have encountered a large success because of their remarkable agreement with experimental data, see \cite{6, 7, 3, 11, 12}, and references therein.

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In particular, the q-Gaussian distribution is also well-known as a generalisation of the Gaussian, or the Normal distribution. This distribution can also represent the heavy tailed distribution such as the Student-distribution or the distribution with bounded support such as the semicircle of Wigner. For these reasons, the q-Gaussian distribution has been applied in the fields of statistical mechanics, geology, finance, and machine learning. Admitting the q-normal distribution is in demand as above, there exists a problem to calculate the expectation value with a corresponding q-distribution not a q-normal distribution itself. But we have an amazing property such that an escort distribution obtained by a q-normal distribution with a parameter $q$ and a variance is another q-normal distribution with a different value of $q$ and a scaled variance. Then calculating an expectation value with an escort distribution corresponds to calculating the expectation value with another associated q-normal distribution, but it gets even the question why an expectation value should be calculated by another q-normal distribution. We call the procedure to get another q-normal distribution from a given q-normal distribution through an escort distribution proportion.

Furthermore, we target attention on q-Gaussians, an essential tool of q-statistics [14], that was not discussed in [2]. The q-Gaussian behavior is often detected in quite distinct settings [14].

It is well known that, in the literature, there are two types of q-Laplace transforms, and they are studied in detail by several authors ([20, 15], etc.). Recently Tsallis et al. have been interested in calculating the Fourier transform of q-Gaussian and have proved a generalization of the central limit theorem for $1 \leq q < 3$. The case $q < 1$ requires essentially different technique, therefore we leave it for a separate paper. In this paper, we propose new definitions of the q-laplace transform of some probability distributions. These results are motivated by recent developments in the calculation of Fourier transforms, where new formulas have been defined [17].

In this article, we develop our results into four sections. In Section 2, we recall some known definitions and notations from the q-theory.

In Section 3, we give definitions of some q-analogues of mean and variance. In Section 4, we introduce the q-Gaussian distribution includes some properties. In Section 5, we give the news formula of Laplace transform and we treat kurtosis both in its standard definition and in q statistics, namely q-kurtosis. In Section 6, we estimate the q-mean and q-variance.

We start with definitions and facts from the q-calculus.

2. q-theory calculus

Assume that $q$ be a fixed number satisfying $q \in [0, 1]$. If is a classical object, say, its q-version is defined by $[x]_q = \frac{1 - q^x}{1 - q}$. As is well know, the q-exponential and the q-logarithm, which are denoted by $e_q(x)$ and $\ln_q(x)$, are respectively defined as $e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$ and $\ln_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{[n]_q}$.
\[(1+(1-q)x)^{\frac{1}{1-q}} \text{ and } \ln_q(x) = \frac{x^{1-q}-1}{1-q}, (x > 0). \] For q-exponential, the relations \(e_q^{x \otimes_q y} = e_q^x e_q^y\) and \(e_q^{x+y} = e_q^x \otimes_q e_q^y\) hold true. These relations can be rewritten equivalently as follows: \(\ln_q(x \otimes_q y) = \ln_q(x) + \ln_q(y)\), and \(\ln_q(xy) = \ln_q(x) \otimes_q \ln_q(y)\).

A q-algebra can also be defined in [16] by applying the generalized operation for sum and product:

\[
x \oplus_q y = x + y + (1 - q)xy,
\]

\[
x \otimes_q y = [x^{1-q} + y^{1-q} - 1]^{1-q},
\]

with the following neutral and inverse elements:

\[
x \otimes_q (x)_q = 0, \text{ with } (x_q) = x[1 + (1 - q)x]^{-1}
\]

\[
x \otimes_q (x^{-1})_q = 1, \text{ with } : (x^{-1})_q = [2 - x^{1-q}]^{\frac{1}{1-q}}.
\]

For the new algebraic operation, q-exponential and q-logarithm have the following properties:

**Properties 2.1.**

1. \(e_q^x e_q^y = e_q^{x \otimes_q y}\)
2. \(e_q^x \otimes_q e_q^y = e_q^{x+y}\)
3. \(\log_q(xy) = \log_q(x) \otimes_q \log_q(y)\)
4. \(\log_q(x \otimes_q y) = \log_q(x) + \log_q(y)\)

It can be easily proved that the operation \(\otimes_q\) and \(\oplus_q\) satisfy commutativity and associativity. For the operator \(\oplus_q\), the identity additive is 0, while for the operator \(\otimes_q\) the identity multiplicative is 1 [1]. Two distinct mathematical tools appears in the study of physical phenomena in the complex media which is characterized by singularities in a compact space.

From the associativity of \(\oplus_q\) and \(\otimes_q\), we have the following formula:

\[
t \oplus_q t \oplus_q \ldots \oplus_q t = \frac{1}{1-q}\{(1 + (1 - q)t)^n - 1\}
\]

\[
t \otimes_q^n = t \otimes_q t \otimes_q \ldots \otimes_q t = nt^{1-q} - (n - 1)^{\frac{1}{1-q}}.
\]

The real space vector with regular sum and product operations \(\mathbb{R}(+, \times)\) is a field, and the \(\mathbb{R}(\oplus_q, \otimes_q)\) defines a quasi-field.
3. q-mean and q-variance values

Let $q$ be a real number and $f$ be a properly normalized probability density with $\text{supp}f \subseteq \mathbb{R}$ of some random variable $X$ such that the quantity

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$  

The mean $m$ is defined, of a given $X$, as follows

$$E(X) = m = \int_{-\infty}^{+\infty} xf(x)dx.$$ 

The variance $V$ is defined, of a given $X$, as follows

$$V(X) = \int_{-\infty}^{+\infty} (x - m)^2 f(x)dx.$$ 

The unnormalized q-moments, of a given $X$, is defined as

$$E_q(X) = m_q = \int_{-\infty}^{+\infty} x[f(x)]^q dx.$$ 

Similarly, the unnormalized q-variance, $\sigma_{2q-1}^2$ is defined analogously to the usual second order central moment, as

$$V_{2q-1}(X) = \sigma_{2q-1}^2 = \int_{-\infty}^{+\infty} (x - m_{2q-1})^2[f(x)]^{2q-1} dx.$$ 

On the other hand, we denote by $f_q(x)$ the normalized density (see e.g. [13]) and defined as

$$f_q(x) = \frac{[f(x)]^q}{\nu_q(f)}$$

where

$$\nu_q(f) = \int_{-\infty}^{+\infty} [f(x)]^q dx < \infty.$$ 

The normalized q-mean values, of a given $X$, is

$$\overline{E}_q(X) = \overline{m}_q = \int_{-\infty}^{+\infty} x f_q(x)dx$$

The normalized q-variance values, of a given $X$, is

$$\overline{V}_{2q-1}(X) = \overline{\sigma}_{2q-1}^2 = \int_{-\infty}^{+\infty} (x - m_{2q-1})^2 f_{2q-1}(x)dx.$$
4. q-Gaussian distribution

In this section, we review the q-Gaussian distribution, or the q-normal distribution according to Furuichi [16] and Suyari [8]. Let $\beta$ be a positive number. We call the q-Gaussian $N_q(m, \sigma^2)$ with parameters $m$ and $\sigma^2 > 0$ if its density function is defined by

$$f(x) = \frac{\sqrt{\beta}}{\sigma C_q} \frac{\beta(x - m)^2}{\sigma^2}, \quad x \in \mathbb{R}$$

with $q < 3, q \neq 1$; and $C_q$ is the normalizing constant, namely

$$C_q = \int_{-\infty}^{\infty} e^{-x^2} d\mu = \begin{cases} \left( \frac{1}{q - 1} \right)^{\frac{1}{2}} B\left( \frac{3 - q}{2(q - 1)}, \frac{1}{2} \right) & 1 < q < 3 \\ \sqrt{\pi}, & q = 1 \\ \left( \frac{1}{1 - q} \right)^{\frac{1}{2}} B\left( \frac{2 - q}{1 - q}, \frac{1}{2} \right) & -\infty < q < 1 \end{cases}$$

and $B(a, b)$ denotes the beta function. The width parameter of the distribution is characterized by

$$\beta = \frac{1}{3 - q}.$$

Denote a general q-Gaussian random variable $X$ with parameters $m$ and $\sigma^2$ as $X \sim N_q(m, \sigma^2)$, and call the special case of $m = 0$ and $\sigma^2 = 1$ a standard q-Gaussian $Y \sim N_q(0, 1)$. The density of the standard q-Gaussian distribution may then be written as

$$N_q(0, 1)(y) = \frac{\sqrt{\beta}}{C_q} e^{-\beta y^2}.$$

Note that, if

$$Y \sim N_q(0, 1) \text{ then } X = m + \sigma Y \sim N_q(m, \sigma^2).$$

If we change the value of $q$, we can represent various types of distributions. The q-Gaussian distribution represents the usual Gaussian distribution when $q = 1$, has compact support for $q < 1$, and turns asymptotically as a power law for $1 \leq q < 3$. For $3 \leq q$, the form given is not normalizable. The usual variance (second order moment) is finite for $q < \frac{5}{3}$, and, for the standard q-Gaussian $N_q(0, 1)$, is given by $V(Y) = \frac{3 - q}{5 - 3q}$.

The usual variance of the q-Gaussian diverges for $\frac{5}{3} \leq q < 3$, however the q-variance remains finite for the full range $\infty < q < 3$, equal to unity for the standard q-Gaussian. Finally, we can easily check that there are relationships between different values of $q$. For example,

$$e_q^{-y^2} = \left(e_{2 - \frac{q}{3}}^{-qy^2}\right)^{\frac{1}{q}}.$$

In this section, we consider the q-analogues of the Laplace transform, which we call the q-Laplace transform, and investigate some of its properties.
5. New q-Laplace transforms

From now, we assume that $1 \leq q < 3$. For these values of $q$ we introduce the q-Laplace transform $L_q$ as an operator, which coincides with the Laplace transform if $q = 1$. Note that the q-Laplace transform is defined on the basis of the q-product and the q-exponential, and, in contrast to the usual Laplace transform, is a nonlinear transform for $q \in (1, 3)$.

The q-Laplace transform of a random variable $X$ with density function $f$ is defined by the formula

$$L_q(X)(\theta) = \int_{\text{supp}f} e^{\theta x} \otimes_q f(x)dx,$$

where the integral is understood in the Lebesgue sense.

The following lemma establishes the expression of the q-Laplace transform in terms of the standard product, instead of the q-product.

**Lemma 5.1.** The q-Laplace transform of a random variable $X$ with density $f$ is expressed as

$$L_q(X)(\theta) = \int_{-\infty}^{\infty} f(x)e^{\theta x (f(x))^{q-1}}dx. \quad (3)$$

**Proof.** For $x \in \text{supp}f$, we have

$$e^{i\theta x} \otimes_q f(x) = [1 + (1-q)\theta x + (f(x))^{1-q} - 1]^{\frac{1}{1-q}} = f(x)[1 + (1-q)\theta x (f(x))^{q-1}]^{\frac{1}{1-q}} \quad (4)$$

Integrating both sides of Eq. (4) we obtain (3).

Let $X$ be a random variable defined on the probability space $(\Omega, F, P)$ with density function $f \in L^q$. It can be verified that the derivatives of the q-Laplace transform $L_q(X)(\theta)$ are closely related to an appropriate set of unnormalized q-moments of the original probability density. Assume that $L_q(X)(\theta) < +\infty$ in a neighborhood of 0.

Indeed, the first few low-order derivatives (including the zeroth order) are given by

$$L_q(X)(0) = 1$$

$$\left. \frac{\partial L_q(X)(\theta)}{\partial \theta} \right|_{\theta=0} = \int_{-\infty}^{\infty} x(f(x))^qdx = E_q(X)$$

$$\left. \frac{\partial^2 L_q(X)(\theta^2)}{\partial \theta} \right|_{\theta=0} = q \int_{-\infty}^{\infty} x^2(f(x))^{2q-1}dx = qE_{2q-1}(X^2)$$

$$\left. \frac{\partial^3 L_q(X)(\theta^3)}{\partial \theta} \right|_{\theta=0} = q(2q-1) \int_{-\infty}^{\infty} x^3(f(x))^{3q-2}dx = q(2q-1)E_{3q-2}(X^3).$$
\[ \frac{\partial^4 L_q(X)(\theta^4)}{\partial \theta} \bigg|_{\theta=0} = q(2q-1)(3q-2) \int_{-\infty}^{\infty} x^4(f(x))^{4q-3} dx = q(2q-1)(3q-2)E_{4q-3}(X^4). \]

The general n-derivative is
\[ \frac{\partial^n L_q(X)(\theta^n)}{\partial \theta} \bigg|_{\theta=0} = \prod_{m=0}^{n-1} (1 + m(q - 1)) \int_{-\infty}^{\infty} x^n(f(x))^{1+n(q-1)} dx, \quad n = 1, 2, 3, \ldots. \]

Note that, in the case \( n = 1 \) the first derivative of the Laplace transform corresponds to \( E_q(X) \).

**Proposition 5.2.** Let \( 1 \leq q < 3 \) and let \( X \) be a random variable following a \( q \)-Gaussian distribution \( N_q(m, \sigma^2) \) then \( E(X) = m \) and \( V(X) = \frac{3-q}{5-3q} \sigma^2 \) with \( 1 \leq q < \frac{5}{3} \).

**Proof.**  
1. The first moment, of a given \( X \), is
\[ E(X) = \frac{\sqrt{\beta}}{\sigma C_q} \int_{-\infty}^{\infty} xe^{-\frac{(x-m)^2}{\sigma^2}} dx = \frac{\sqrt{\beta}}{C_q} \int_{-\infty}^{\infty} (\sigma y + m) e^{-\beta y^2} dy = \sigma \frac{\sqrt{\beta}}{C_q} \int_{-\infty}^{\infty} ye^{-\beta y^2} dy + \frac{m \sqrt{\beta}}{C_q} \int_{-\infty}^{\infty} e^{-\beta y^2} dy = m \]

2. The second order moment of the standard Gaussian \( N_q(0, 1) \) is computed as
\[ E(Y^2) = \frac{\sqrt{\beta}}{C_q} \int_{-\infty}^{\infty} y^2 e^{-\beta y^2} dy = \frac{1}{2\sqrt{\beta}(2-q)C_q} \int_{-\infty}^{\infty} (e^{-\beta y^2})^{2-q} dy = \frac{1}{2\sqrt{\beta}(2-q)C_q} \int_{-\infty}^{\infty} e_{q_1}^{-\beta(2-q)y^2} dy, \quad q_1 = \frac{1}{2-q} \]

The substitution \( \beta_1 z^2 = \beta(2-q)y^2, \beta_1 = \frac{1}{3-q_1} \)
\[ E(Y^2) = \frac{\sqrt{\beta_1}}{2\beta(2-q)C_q} \int_{-\infty}^{\infty} e_{q_1}^{-\beta_1 z^2} dz = \frac{C_{q_1}}{2\beta(2-q)\frac{1}{2}C_q} \int_{-\infty}^{\infty} \sqrt{\beta_1} e_{q_1}^{-\beta_1 z^2} dz = \frac{1}{2\beta_1(2-q)\frac{1}{2} C_q} C_{q_1} \]
The condition $1 \leq q < 3$ implies that $1 \leq q_1 < \frac{5}{3}$. By using the identity $B(x+1, y) = \frac{x}{x+y}B(x, y)$ we obtain the ratio between $C_q$ and $C_{q_1}$ as

$$\frac{C_{q_1}}{C_q} = \frac{3}{2(2-q)^2} \frac{5}{5-3q} \quad (5)$$

By applying the formula $V(X) = E(X^2) - (E(X))^2$, we obtain the result

In this theorem, we give the average of the fourth power of the standardized deviations from the q-mean.

In this theorem, we determine the q-kurtosis of q-Gaussian distribution $(N_q(0,1))$.

**Theorem 5.3.** Let $1 \leq q < 3$ and let $X$ be a random variable following a q-central Gaussian distribution $N_q(0,1)$, then the coefficient of kurtosis is

$$Kurt[Y] = \frac{E(Y^4)}{(E(Y^2))^2} = \frac{3(5-3q)}{(7-5q)} \quad 1 \leq q < \frac{7}{5}$$

**Proof.**

1. The fourth central moment moment of the standard Gaussian $N_q(0,1)$ is computed as

$$E(Y^4) = \frac{\sqrt{\beta}}{C_q} \int_{-\infty}^{\infty} y^4 e^{-\beta y^2} dy$$

$$= \frac{3}{2\sqrt{\beta}(2-q)C_q(1)} \int_{-\infty}^{\infty} y^2 (e^{-\beta y^2})^{2-q} dy$$

The substitution $\beta_1 z^2 = \beta(2-q)y^2$

$$= \frac{3\beta_1}{2\beta^2(2-q)^2} \frac{C_q}{C_{q_1}} \int_{-\infty}^{\infty} \sqrt{\beta_1} z^2 e^{-\beta_1 z^2} dz$$

with $1 \leq q_1 = \frac{1}{2-q} < \frac{5}{3}$

$$= \frac{3\beta_1}{2\beta^2(2-q)^2} \frac{C_{q_1}}{C_q} E_{q_1}(Z^2)$$

Using equation 5, we obtain

$$= \frac{3(3-q)^2}{(5-3q)(7-5q)} \quad 1 \leq q < \frac{7}{5}$$

According to Proposition 5.2, we obtain the result.

For $1 \leq q < \frac{7}{5}$ a value greater than $\frac{3(3-q)^2}{(5-3q)(7-5q)}$ indicates a leptokurtic distribution; a values less than $\frac{3(3-q)^2}{(5-3q)(7-5q)}$ indicates a platykurtic distribution. For the sample estimate $X$, $\frac{3(3-q)^2}{(5-3q)(7-5q)}$ is subtracted so that a positive value indicates leptokurtosis and a negative value indicates platykurtosis.
Theorem 5.4. Let $1 \leq q < 3$ and let $X$ be a random variable following a $q$-Gaussian distribution $N_q(m, \sigma^2)$, then

1. $L_q(X)(\theta) = \left( e^{\theta ma - \frac{\theta^2 a^2}{4\beta} - \frac{\theta^2}{2} \frac{\sigma^2}{\beta}} \right)^{\frac{3-q}{2}}$, with $a = \sqrt{\beta} C_q$ and $\theta \in \mathbb{R}$

2. $E_q(X) = \int_{-\infty}^{\infty} x(f(x))^q dx = \frac{m(3-q)}{2(\sigma C_q)^{q-1}} \frac{3-q}{2}$

3. $E_{2q-1}(X^2) = \int_{-\infty}^{\infty} x^2(f(x))^{2q-1} dx = \frac{1}{4q(3-q)^{q-2}(\sigma C_q)^{2q-2}} [(3-q)\sigma^2 + (q+1)m^2]$
Proof. 1. From definition of q-Laplace transform, it following by denote \( a = \frac{\sqrt{\beta}}{\sigma C_q} \)

\[
L_q(X)(\theta) = \int_{-\infty}^{\infty} e_{q}^{\theta x} \otimes_q ae_q \frac{-(\beta(x-m))^2}{\sigma^2} \, dx
\]

(by appliying lemma [5.1])

\[
= a \int_{-\infty}^{\infty} e_{q}^{\theta x a q^{-1}} \otimes_q e_q \frac{-(\beta(x-m))^2}{\sigma^2} \, dx
\]

\[
= a \int_{-\infty}^{\infty} e_q^{-\beta(x-m)^2/\sigma^2 + \theta x a q^{-1}} \, dx
\]

\[
= a \int_{-\infty}^{\infty} e_q^{-\beta(x-m)^2/\sigma^2 + \theta x a q^{-1}} - \frac{\beta}{\sigma^2} \theta a q^{-1} - \theta a q^{-1} m \, dx
\]

\[
= ae_q \frac{\beta}{\sigma^2} \left( \frac{\sigma^2 \theta a q^{-1}}{2\beta} \right)^2 + \theta a q^{-1} m
\]

\[
= \sigma_1 a C_q \frac{\beta}{\sigma^2} \left( \frac{\sigma^2 \theta a q^{-1}}{2\beta} \right)^2 + \theta a q^{-1} m
\]

\[
= \frac{\beta}{\sigma^2} \left( \frac{\sigma^2 \theta a q^{-1}}{2\beta} \right)^2 + \theta a q^{-1} m
\]

\[
= \sqrt{\frac{(q-1)(\frac{\beta}{\sigma^2} \left( \frac{\sigma^2 \theta a q^{-1}}{2\beta} \right)^2 + \theta a q^{-1} m)}{\theta a q^{-1} m}}
\]

where \( \gamma = e_1(\frac{\beta}{\sigma^2} \left( \frac{\sigma^2 \theta a q^{-1}}{2\beta} \right)^2 + \theta a q^{-1} m)^{q-1} \) and \( \sigma_1^2 = \frac{\sigma^2}{\gamma} \).
Hence, applying again Lemma 5.1 we have

\[ L_q(X)(\theta) = e_q \left( \frac{\theta m(3 - q)}{2} a^{q-1} - \frac{\theta^2 a^{2q-2} \sigma^2 (3 - q)}{8 \beta} \right) \]

\[ = e_{q_1} \]

2. We compute the first and the second derivative of \( L_q(X) \), with respect to \( \theta \), we obtain

\[ E_q(X) = L'_q(X)(0) \]
\[ E_{2q-1}(X^2) = \frac{1}{q} L''_q(0) \]

Intending to interpret these moments, we consider a wonderful property such that an escort distribution obtained by a q-normal distribution with variance \( \sigma^2 \) is equivalent to another q-normal distribution with \( q_1 = 2 - \frac{1}{q} \) and a variance \( \frac{3 - q}{q + 1} \sigma^2 \) with \( 1 \leq q < 3 \).

**Proposition 5.5.** Let \( X \) be a random variable following a q-Gaussian distribution \( N_q(m, \sigma^2) \), then

1. \( \overline{E}_q(X) = \int_{-\infty}^{+\infty} x \frac{[f(x)]^q}{\nu_q(f)} dx = m \)

2. \( \overline{V}_{2q-1}(X) = \sigma^2_{2q-1} = \int_{-\infty}^{+\infty} (x - m)^2 \frac{[f(x)]^{2q-1}}{\nu_{2q-1}(f)} dx = \frac{3 - q}{q + 1} \sigma^2 \)

**Proof.**

1. Let’s begin with observing a following proportion on a given q-Normal distribution:

\[ q_3 = \frac{2q - 1}{q}, \sigma^2_3 = \frac{\beta_3}{(q)\beta} \sigma^2 = \frac{3 - q}{q + 1} \sigma^2 \]

Under this relations, we have

\[ \sqrt{\beta} \left( \frac{e_q}{\sigma C_q} \right)^q \frac{-\beta(x - m)^2}{\sigma^2} \]
\[ \propto e_{q_3} \frac{-\beta(x - m)^2 q \beta}{\sigma^2} \]
\[ \propto N_{q_3}(m, \sigma^2_3) \]

Therefore,

\[ \overline{E}_q(X) = \int_{-\infty}^{+\infty} x \frac{[f(x)]^q}{\nu_q(f)} dx = \int_{-\infty}^{+\infty} x N_{q_3}(m, \sigma^2_3)(x) dx \]

By applying proposition 5.2 we obtain the result.
2. The escort function is proportional to the q-gaussian $N_q(m, \sigma^2)$

$$q_4 = \frac{3q - 2}{2q - 1}, \sigma^2_4 = \frac{\beta_4}{(2q - 1)\beta} \sigma^2 = \frac{3 - q}{3q - 1} \sigma^2.$$  

Hence

$$\frac{\beta(x - m)^2}{\sqrt{\beta} \left(\frac{\sigma^2}{\nu_{2q-1}}\right)^q} \sigma_{C_q} \propto e^{-\frac{(x-m)^2}{\beta_4}} \sigma_{\nu_2}$$

Then

$$\overline{V}_{2q-1}(X) = \int_{-\infty}^{+\infty} (x-m)^2 \frac{\nu_2^{-1}(f)}{\nu_{2q-1}(f)} dx = \int_{-\infty}^{+\infty} (x-m)^2 N_q(m, \sigma^2_4)(x) dx = \frac{3 - q_4}{5 - 3q_4} \sigma^2_4$$

By applying proposition 5.2 we obtain the result.

In this theorem, we prove that for $q < \frac{3}{5}$ a value greater than $\frac{3(q+1)^2}{(5q-3)(3q-1)}$ indicates a leptokurtic distribution; a values less than $\frac{3(q+1)^2}{(5q-3)(3q-1)}$ indicates a platykurtic distribution. For the sample estimate $X$, $\frac{3(q+1)^2}{(5q-3)(3q-1)}$ is subtracted so that a positive value indicates leptokurtosis and a negative value indicates platykurtosis.

**Theorem 5.6.** Let $Y$ be a random variable following a q-Central Gaussian distribution $N_q(0,1)$ then the coefficient of normalized kurtosis is

$$\overline{Kurt}[Y] = \frac{\overline{E}(Y^4)}{(\overline{E}(Y^2))^2} = \frac{3(q + 1)^2}{(5q - 3)(3q - 1)}, 1 \leq q < \frac{3}{5}$$

**Proof.** Checking a following proportion on a given q-Normal distribution:

$$q_1 = \frac{5q - 4}{4q - 3}, \sigma^2_1 = \frac{\beta_1}{(4q - 3)\beta'}$$

$$\sqrt{\beta} \frac{\sigma^2}{\nu_{4q-3}} e^{-\frac{(x-m)^2}{\beta_1}} \sigma_{\nu_{2q-1}} \sigma_{\nu_{4q-3}} \propto e^{-\frac{\nu_2}{\sigma_{\nu_{2q-1}}}}$$

$$\propto N_{q_1}(0, \sigma^2_1)$$
Then \( \overline{E}(Z^4) = \int_{-\infty}^{+\infty} z^4 \left( \frac{\sqrt{\beta}}{\sigma C_q} \right)^{4q-3} e_q^{\nu q-3} = \int_{-\infty}^{+\infty} z^4 N_{q_1}(0, \sigma^2_{1}) (z) dz. \)

Hence, according to Theorem 5.3, we obtain \( \overline{E}(Z^4) = 3(3 - q_1)^2 (5 - 3q_1)(7 - 5q_1) \sigma^2_{1}; Z \sim N_{q_1}(0, \sigma^2_{1}) \).

Hence, according to Theorem 5.3, we obtain
\[
\overline{E}(Z^4) = \frac{3(3 - q)^2}{(5q - 3)(3q - 1)}.
\]

Furthermore by observing a following proportion on a given q-Normal distribution:
\[
q_2 = \frac{3q - 2}{2q - 1}, \sigma^2_{2} = \frac{\beta_2}{(2q - 1)\beta}
\]

\[
\left( \frac{\sqrt{\beta}}{\sigma C_q} \right)^{2q-1} e_q^{\nu q-1} \propto e_{q_2}^{-q_2 (2q-1)\beta \sigma^2_{2}}
\]

\[
\propto N_{q_2}(0, \sigma^2_{2})
\]

Then
\[
\overline{E}(Z^2) = \int_{-\infty}^{+\infty} z^2 \left( \frac{\sqrt{\beta}}{\sigma C_q} \right)^{2q-1} e_q^{\nu q-1} = \int_{-\infty}^{+\infty} z^2 N_{q_2}(0, \sigma^2_{2}) (z) dz.
\]

Hence, according to proposition 5.2, we obtain
\[
\overline{E}(Z^2) = \frac{(3 - q_2)\sigma^2_{2}}{5 - 3q_2}; Z \sim N_{q_2}(0, \sigma^2_{2})
\]

q-estimator for random variables are arising from non-extensive statistical mechanics. In this section, we will estimate the q-mean and q-variance using the notions of q-Laplace transform, q-independence.

**6. Estimator of q-Mean and q-variance**

**Definition 6.1.** Two random variables \( X_1 \) and \( X_2 \) are said to be q-independent if
\[
L_q(X_1 + X_2)(\theta) = L_q(X_1)(\theta) \otimes_q L_q(X_2)(\theta).
\]
Definition 6.2. Let \( X_n \) be a sequence of identically distributed random variables and 
\[ m = E(X_1) \]. Denote \( S_n = \sum_{k=1}^{n} X_k \). By definition \( X_k, k = 1, 2, 3, \ldots \), is said to be \( q \)-independent of the first type (or \( q \)-i.i.d.) if for all \( n = 2, 3, 4, \ldots \), the relations 
\[ L_q[S_n - nm](\theta) = L_q[X_1 - m](\theta) \otimes_q \ldots \otimes_q L_q[X_n - m](\theta) \]
hold.

Proposition 6.3. Let \( X_1 \) and \( X_2 \) be tow \( q \)-independent random variables following respectively \( N_q(m_1, \sigma_1^2) \) and \( N_q(m_2, \sigma_2^2) \). Then 
\[ X_1 + X_2 \sim N_q(m_1 + m_2, \sigma_1^2 + \sigma_2^2) \]

Proof. 
\[
L_q(X_1 + X_2)(\theta) = L_q(X_1)(\theta) \otimes_q L_q(X_2)(\theta) \\
= e_q \left( 3 - \frac{q}{2} (\sigma_1^2 + \sigma_2^2) \frac{\theta^2 a^{2q-2}}{4\beta} + \frac{3 - q}{2} (\sigma_1^2 + \sigma_2^2) \theta a^{q-1} \right) \\
= \left( e_q \frac{\theta^2 a^{2q-2}}{4\beta} + (\sigma_1^2 + \sigma_2^2) \theta a^{q-1} \right) \frac{3 - q}{2}
\]

Note that if \( X_1 \) and \( X_2 \) are \( q \)-Gaussian and \( q \)-independent random variables with distributions \( N_q(m_1, \sigma_1^2) \) and \( N_q(m_2, \sigma_2^2) \) respectively then 
\[ V(X_1 + X_2) = \frac{3 - q}{5 - 3q} (\sigma_1^2 + \sigma_2^2); \quad 1 \leq q < \frac{5}{3} \]
and 
\[ V_{2q-1}(X_1 + X_2) = \frac{3 - q}{q+1} (\sigma_1^2 + \sigma_2^2) = V_{2q-1}(X_1) + V_{2q-1}(X_2); \]
In this case \( \text{cov}(X_1, X_2) = 0 \), because \( V(X_1 + X_2) = V(X_1) + V(X_2) + 2\text{cov}(X_1, X_2) \).

Corollary 6.4. Let \( X_1, X_2, \ldots, X_n \) be \( n \) \( q \)-independent random variables with same \( q \)-Gaussian distribution \( N_q(m, \sigma^2) \), then \( \overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \) follows the \( q \)-Gaussian \( N_q(m, \frac{\sigma^2}{n}) \).

Observe that \( V(\overline{X}) \to 0 \) as \( n \to \infty \). Since \( E(\overline{X}) = m \), then the estimates of \( m \) becomes increasingly concentrated around the true population parameter. Such an
estimate is said to be consistent. The empirical variance \( S_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \) is not an unbiased estimate of \( \sigma^2 \). Indeed

\[
E(S_n^2) = \frac{n-1}{n} V(X_1) = \frac{n-1}{n} 3 - q \sigma^2
\]

Therefore

\[
\hat{\sigma}^2 = \frac{n}{n-1} \frac{5 - 3q}{3 - q} S_n^2
\]

is an unbiased estimate of \( \sigma^2 \).

**Proposition 6.5. Law of Large Numbers (LLN):** If the distribution of the i.i.d. \( q \)-independent \( X_1, \ldots, X_n \) is such that \( X_1 \) has finite \( q \)-expected value, i.e. \( |E_q(X_1)| < \infty \), then the sample average

\[
\bar{X}_n = \frac{X_1 + \ldots + X_n}{n} \rightarrow E_q(X_1)
\]

converges to its expectation in probability.

**Theorem 6.6. Central Limit Theorem (CLT):** For \( q \in (1, 2) \), if \( X_1, X_2, \ldots, X_n \) are \( q \)-independent and identically distributed with \( q \)-mean \( m_q \) and a finite second \((2q - 1)\)-moment \( \sigma^2_{2q-1} \), then

\[
Z_n = \frac{X_1 + \ldots + X_n - nm_q}{C_{q,n,\sigma}}
\]

\( q \)-converges to \( N_{q-1}(0, 1) \) Gaussian distribution.

Let \( X \) be an arbitrary random variable with known variance \( \frac{3 - q}{5 - 3q} \sigma^2 \), and let \( X_1, X_2, \ldots, X_n \) be \( n \)-\( q \)-independent random variables with common Gaussian distribution \( N_q(m, \sigma^2) \).

According to central limit theorem, the confidence interval for \( m \) with level \( 1 - \alpha \) for arbitrary data and known \( \sigma^2 \) is defined as

\[
m \in (\bar{X}_n \pm \frac{z_{1 - \frac{\alpha}{2}} \sigma}{C_{q,N,\sqrt{n}}}).
\]

Where \( z_{1 - \frac{\alpha}{2}} \) is the quantile for \( N_q(0, 1) \) with level \( 1 - \frac{\alpha}{2} \):

\[
\int_{-\alpha}^{\alpha} N_q(0, 1)(x)dx = 1 - \alpha, \alpha \in [0, 1].
\]
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