Novel properties of wave propagation in biaxially anisotropic left-handed materials

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Some physically interesting properties and effects of wave propagation in biaxially anisotropic left-handed materials are investigated in this paper. We show that in the biaxially gyrotropic left-handed material, the left-right coupling of circularly polarized light arises due to the negative indices in permittivity and permeability tensors of gyrotropic media. It is well known that the geometric phases of photons inside a curved fiber in previous experiments often depend on the cone angles of solid angles subtended by a curve traced by the direction of wave vector of light, at the center of photon momentum space. Here, however, for the light propagating inside certain anisotropic left-handed media we will present a different geometric phase that is independent of the cone angles. The extra phases of electromagnetic wave resulting from the instantaneous helicity inversion at the interfaces between left- and right- handed (LRH) media is also studied in detail by using the Lewis-Riesenfeld invariant theory. Some interesting applications (e.g., controllable position-dependent frequency shift, detection of quantum-vacuum geometric phases and helicity reversals at the LRH interfaces etc.) of above effects and phenomena in left-handed media is briefly discussed.

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I. INTRODUCTION

More recently, a kind of artificial composite metamaterials (the so-called left-handed media) having a frequency band where the effective permittivity and the effective permeability are simultaneously negative attracts considerable attention of many authors both experimentally and theoretically [1–7]. In 1967, Veselago first considered this peculiar medium and showed from Maxwellian equations that such media having negative simultaneously negative $\varepsilon$ and $\mu$ exhibit a negative index of refraction, i.e., $n = -\sqrt{\mu/\varepsilon}$ [8]. It follows from the Maxwell’s curl equations that the phase velocity of light wave propagating inside this medium is pointed opposite to the direction of energy flow, that is, the Poynting vector and wave vector of electromagnetic wave would be antiparallel, i.e., the vector $k$, the electric field $E$ and the magnetic field $H$ form a left-handed system; thus Veselago referred to such materials as “left-handed” media, and correspondingly, the ordinary medium in which $k$, $E$ and $H$ form a right-handed system may be termed the “right-handed” one. Other authors call this class of materials “negative-index media (NIM)” [9], “double negative media (DNM)” [4] and Veselago’s media. It is readily verified that in such media having both $\varepsilon$ and $\mu$ negative, there exist a number of peculiar electromagnetic and optical properties, for instance, many dramatically different propagation characteristics stem from the sign change of the optical refractive index and phase velocity, including reversal of both the Doppler shift and Cherenkov radiation, anomalous refraction, modified spontaneous emission rates and even reversals of radiation pressure to radiation tension [2]. In experiments, this artificial negative electric permittivity media may be obtained by using the array of long metallic wires (ALMWs) [10], which simulates the plasma behavior at microwave frequencies, and the artificial negative magnetic permeability media may be built up by using small resonant metallic particles, e.g., the split ring resonators (SRRs), with very high magnetic polarizability.

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1 Note that, in the literature, some authors [5,6] mentioned the year when Veselago suggested the left-handed media by mistake. They claimed that Veselago proposed or introduced the concept of left-handed media in 1968 or 1964. On the contrary, the true history is as follows: Veselago’s excellent paper was first published in Russian in July, 1967 [Usp. Fiz. Nauk 92, 517-526 (1967)]. This original paper was translated into English by W.H. Furry and published again in 1968 in the journal of Sov. Phys. Usp. [8]. Unfortunately, Furry stated erroneously in his English translation that the original version of Veselago’ work was first published in 1964.
A combination of the two structures yields a left-handed medium. Recently, Shelby et al. reported their first experimental realization of this artificial composite medium, the permittivity and permeability of which have negative real parts [3]. One of the potential applications of negative refractive index materials is to fabricate the so-called "superlenses" (perfect lenses): specifically, a slab of such materials may have the power to focus all Fourier components of a 2D image, even those that do not propagate in a radiative manner [14,15].

In the present paper, we take into consideration the physical phenomena and effects of circularly polarized photons in biaxially anisotropic (gyrotropic) left-handed media. Veselago’s original paper and most of the recent theoretical works discussed mainly the characteristics of electromagnetic wave propagation through isotropic left-handed media, but up to now, the left-handed media that have been prepared successfully experimentally are actually anisotropic in nature, and it may be very difficult to prepare an isotropic left-handed medium [1,16,17]. Hu and Chui presented a detailed investigation on the characteristics of electromagnetic wave propagation in uniaxially anisotropic left-handed media [17], but they concentrated primarily on the classical properties of wave propagation from the point of view of classical wave optics and applied electromagnetism. During the last three years, many researchers in various fields such as materials science, condensed matter physics, optics and classical applied electromagnetism [1–4,12] investigated many peculiar optical and electromagnetic properties in left-handed media. However, to the best of our knowledge, some physical properties (particularly the purely quantum-mechanical effects) of polarized photons in left-handed media have not been considered yet. We think that, in the literature, these problems get less attention and interest than it deserves. For this reason, in this paper, these physically interesting properties, phenomena and effects of wave propagation (polarized photons) in biaxially anisotropic left-handed materials are investigated in detail.

This paper is organized as follows: in Sec.II, we consider the left-right (L-R) coupling of circularly polarized light in biaxially gyrotropic left-handed media; in Sec.III, it is shown that the geometric phase of photons in a noncoplanarly curved optical fiber fabricated from biaxially anisotropic left-handed media is independent of the cone angle of solid angles subtended by a curve traced by the direction of wave vector of light at the center of photon momentum space. A scheme of testing quantum-vacuum geometric phases by using certain biaxially anisotropic left-handed media is also briefly discussed in this section; an additional phase acquired by the incident electromagnetic wave near the interfaces between left- and right-handed (LRH) media is investigated in detail in Sec.IV, where we argue that, if, for example, the photons propagating inside a fiber that is composed periodically of left- and right-handed (LRH) media, then an additional phase acquired by the incident electromagnetic wave near the interfaces between left- and right-handed (LRH) media will arise. In Sec.V, we conclude with some remarks.

II. LEFT-RIGHT COUPLING OF CIRCULARLY POLARIZED LIGHT IN BIAXIALLY GYROTROPIC LEFT-HANDED MEDIA

A. Wave propagation in biaxially gyrotropic left-handed media

It is well known that in ordinary uniaxially gyrotropic media with the electric permittivity and the magnetic permeability tensors being of the form

\[
(\hat{\varepsilon})_{ik} = \begin{pmatrix} \varepsilon_1 & i\varepsilon_2 & 0 \\ -i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}, \quad (\hat{\mu})_{ik} = \begin{pmatrix} \mu_1 & i\mu_2 & 0 \\ -i\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix},
\]

(2.1)

the wave equations of left- and right-handed circularly polarized light are respectively written

\[
\nabla^2 E_L = \frac{n_L^2}{c^2} \frac{\partial^2 E_L}{\partial t^2}, \quad \nabla^2 E_R = \frac{n_R^2}{c^2} \frac{\partial^2 E_R}{\partial t^2}
\]

(2.2)

with the optical refractive indices squared \(n_L^2 = (\varepsilon_1 - \varepsilon_2)(\mu_1 - \mu_2)\) and \(n_R^2 = (\varepsilon_1 + \varepsilon_2)(\mu_1 + \mu_2)\), respectively [8]. It is seen that the two circular modes have different refractive indices. This is referred to as double circular refraction or circular birefringence. Note that here both left- and right-handed circularly polarized light are the eigenmodes of the permittivity and permeability tensors (2.1). So, \(E_L\) and \(E_R\) propagate independently in the above conventional uniaxially gyrotropic media, i.e., no interaction between them exists in the wave propagation process.

Now we study a completely different case compared with the one discussed above, namely, it is shown that an interaction between left- and right-handed circularly polarized light is present in the wave propagation inside biaxially gyrotropic materials with left-handed media involved. If, for example, the permittivity and permeability tensors of the medium considered are written in the form
where \( \epsilon_2, \mu_2 \) are real numbers, and \( \epsilon_1 > 0, \mu_1 > 0 \), then with the help of Maxwell’s equations, one can arrive at

\[
\begin{align*}
\nabla^2 E_1 &= \left( \frac{\epsilon_1 \mu_1 + \epsilon_2 \mu_2}{c^2} \right) \frac{\partial^2 E_1}{\partial t^2} - i \left( \frac{\epsilon_1 \mu_2 - \epsilon_2 \mu_1}{c^2} \right) \frac{\partial^2 E_2}{\partial t^2}, \\
\nabla^2 E_2 &= \left( \frac{\epsilon_1 \mu_1 + \epsilon_2 \mu_2}{c^2} \right) \frac{\partial^2 E_2}{\partial t^2} - i \left( \frac{\epsilon_1 \mu_2 - \epsilon_2 \mu_1}{c^2} \right) \frac{\partial^2 E_1}{\partial t^2},
\end{align*}
\]

(2.4)

For simplicity, without loss of generality, we assume that the two mutually perpendicular real unit polarization vectors \( \vec{\varepsilon}(k, 1) \) and \( \vec{\varepsilon}(k, 2) \) are taken to be as follows: \( \varepsilon_1(k, 1) = \varepsilon_2(k, 2) = 1 \), \( \varepsilon_1(k, 2) = \varepsilon_2(k, 1) = 0 \) and \( \varepsilon_3(k, 1) = \varepsilon_3(k, 2) = 0 \).

It is readily verified that the electric field vectors corresponding to left- and right-handed circularly polarized light are respectively expressed by \( E_L = \frac{E_1 - iE_2}{\sqrt{2}} \) and \( E_R = \frac{E_1 + iE_2}{\sqrt{2}} \) [18]. It should be noted that since here \( E_1 \) and \( E_2 \) are complex, \( E_L \) cannot be considered the complete conjugate of \( E_R \). From Eq. (2.4) it follows that the wave equations of left- and right-handed polarized light propagating inside the biaxially gyrotropic left-handed materials are obtained as follows

\[
\begin{align*}
\nabla^2 E_L &= \frac{n^2}{c^2} \frac{\partial^2 E_L}{\partial t^2} + \frac{\zeta^2}{c^2} \frac{\partial^2 E_R}{\partial t^2}, \\
\nabla^2 E_R &= \frac{n^2}{c^2} \frac{\partial^2 E_R}{\partial t^2} - \frac{\zeta^2}{c^2} \frac{\partial^2 E_L}{\partial t^2},
\end{align*}
\]

(2.5)

where \( n^2 \) and \( \zeta^2 \) are taken \( n^2 = \epsilon_1 \mu_1 + \epsilon_2 \mu_2, \zeta^2 = \epsilon_2 \mu_1 - \epsilon_1 \mu_2 \). Thus the above equations show that the coupling of left-handed light to the right-handed one arises from both the gyrotropic and left-handed properties of materials. This interaction can be treated by introducing two frequency shifts \( (\Omega_L \text{ and } \Omega_R) \) to the electromagnetic wave amplitudes, namely, \( E_L \) and \( E_R \) can be written in the following form [19]

\[
E_L \sim \exp \left\{ \frac{1}{i} \left[ (\omega + \Omega_L) t - \frac{n \omega}{c} z \right] \right\}, \quad E_R \sim \exp \left\{ \frac{1}{i} \left[ (\omega + \Omega_R) t - \frac{n \omega}{c} z \right] \right\},
\]

(2.6)

where it has been assumed that the waves propagate along the \( z \)-direction in Cartesian coordinate system.

It is worth noticing that, differing from the double circular refraction expressed in (2.2), where \( E_L \) and \( E_R \) have the same frequency but different optical refractive indices, here, on the contrary, the two modes in (2.6) have the same refractive index but different frequencies. The latter case may therefore be considered a time analogue to the former one.

Substitution of these two expressions (2.6) into the wave equations (2.5) yields

\[
n^2(2\omega + \Omega_L)(2\omega + \Omega_R)\Omega_L\Omega_R + \zeta^4(\omega + \Omega_L)^2(\omega + \Omega_R)^2 = 0,
\]

(2.7)

which is a restriction imposed on the two frequency shifts \( \Omega_L \) and \( \Omega_R \).

It should be noted that the above-presented treatment of wave propagation in generalized gyrotropic media may be applicable to the uniaxially anisotropic left-handed materials and this kind of media will also give rise to the left-right coupling of circularly polarized light. The permittivity and permeability tensors of uniaxially gyrotropic left-handed media, which can be actually fabricated by current technology [17], are written

\[
(\hat{\varepsilon})_{ik} = \begin{pmatrix}
\epsilon & i\epsilon_2 & 0 \\
-i\epsilon_2 & -\epsilon' & 0 \\
0 & 0 & \epsilon
\end{pmatrix}, \quad (\hat{\mu})_{ik} = \begin{pmatrix}
-\mu' & i\mu_2 & 0 \\
-i\mu_2 & \mu & 0 \\
0 & 0 & \mu
\end{pmatrix},
\]

(2.8)

where the parameters (i.e., \( \epsilon, \epsilon', \mu \) and \( \mu' \)) in the tensors are all positive. It is apparently seen that the L-R coupling of polarized light is present in this type of materials.

Additionally, note that in both biaxially gyroelectric (with \( \mu_2 = 0 \)) and gyromagnetic (with \( \epsilon_2 = 0 \) left-handed media, the above left-right coupling of polarized light will also occur.

**B. Discussions: nonlocal effects and potential applications**

What is the physical origin of left-right coupling of polarized light in biaxially gyrotropic left-handed media? Here we offer an interpretation as to why this effect might occur via nonlocal polarization effect and left-handed properties
(negative index) of media: specifically, neither left- nor right-handed circularly polarized light is the eigenmode of the permittivity and permeability tensors (2.3). For this reason, the interaction between left- and right-handed circularly polarized light may arise in this type of media.

The electric-dipole approximation method shows that if the optical wavelength $\lambda$ is large compared with the dimension $\ell$ of the polarizable units (e.g., electric dipoles), then the induced polarization at a point will only be a spatially local function of the electric field at the same point. This phenomenon often appears in isotropic media, where certain optical effects associated with the small ratio $\frac{\ell}{\lambda}$ can be ignored. In anisotropic media, however, the spatial variations of the fields over the polarizable units (i.e., the magnetic dipoles and electric quadrupoles that are induced by the applied external fields) should also be taken into account. This generalized polarization, the physical quantity of which is no longer denoted by a local function of the electromagnetic fields, will also oscillate and emit radiation which adds to the contribution from the oscillating electric dipoles. It is known that the spatial nonlocality (spatial dispersion) of generalized polarization in electromagnetic media leads to the natural optical activity (natural gyrotropy), which results from the lowest-higher nonlocal terms in the expansion series of induced polarization, and some electro-optical effects and magnetic-field induced gyrotropies, which arises from the terms with a higher-order nonlocality [20]. In the electro-optical effects (e.g., the Pockels effect and Kerr effect) and magnetic-field induced gyrotropy (i.e., the Faraday rotation effect), the anisotropism of permittivity and permeability is often considered the nonlinear optical phenomenon (nonlinear polarization effect), since the electric and magnetic field strengths occur more than once in the nonlocal terms of expansion of induced polarization.

We think that the L-R coupling of polarized light can be regarded as the nonlocal optical effect in character. But this coupling does not appear in regular gyrotropic media, the electromagnetic response of which is described by (2.1), where both the left- and the right-handed circularly polarized light are the eigenmodes of the permittivity and permeability tensors and no interaction between these eigenmodes occurs. Only the gyrotropic anisotropism of permittivity and permeability is combined with left-handed features of media (where neither the left- nor the right-handed circularly polarized light is the eigenmode of the permittivity and permeability tensors) will the L-R coupling of polarized light be realized. In brief, the physical origin of L-R coupling of polarized light lies in the nonlocal polarization and left-handed properties of media.

Generally speaking, it follows from the Kramers-Kronig relation that the permittivity and permeability of composite materials are often both dispersive and absorptive. If the imaginary parts of optical constants $\epsilon_{1,2}$ and $\mu_{1,2}$ in (2.3) are not small in comparison with their real parts, then $\zeta^2$ is a complex number and consequently both $\Omega_L$ and $\Omega_R$ are complex also. This, therefore, means that the electromagnetic wave amplitudes $E_L$ and $E_R$ will exponentially decrease in the time evolution process, rather than only along the path.

In what follows we consider a potential application of L-R coupling to controlling the behavior of lightwave, i.e., the so-called controllable position-dependent frequency shift in certain inhomogeneous media, e.g., photonic crystals composed of such anisotropic left-handed media. Photonic crystals are artificial materials patterned with a periodicity in dielectric constant, which can create a range of forbidden frequencies called a photonic band gap [21,22]. Such dielectric structure of crystals offers the possibility of molding the flow of light. It follows from (2.5) that if both $\frac{\partial n}{\partial z}$ and $\frac{\partial \varepsilon}{\partial z}$ are negligibly small, then the frequency shifts $\Omega_L$ and $\Omega_R$ of LRH polarized light due to the left-right coupling depend upon the spatial position in photonic crystals. This enables us to control the frequencies of circularly polarized light by making use of the spatial structures (or distribution) of optical refractive indices in crystals.

In addition, it is believed that the influence of L-R coupling of polarized light on the photonic band gap structure is of physical interest and therefore deserves investigation, since both the wave vector and the frequency of light are dependent on the spatial positions in this kind of biaxially gyrotrropic left-handed photonic crystals.

III. GEOMETRIC PHASES INDEPENDENT OF CONE ANGLE

Since Berry discovered that a topological (geometric) phase exists in quantum mechanical wave function of time-dependent systems [23], geometric phase problems have captured intensive attention of researchers in a variety of fields such as quantum mechanics [24], differential geometry [25], gravity theory [26,27], atomic and molecular physics [28–30], nuclear physics [31], quantum optics [32], condensed matter physics [33,34], molecular structures and molecular chemical reaction [28] as well. More recently, many authors concentrated on their special attention on the potential applications of geometric phases to the geometric (topological) quantum computation, quantum decoherence and related topics [35,36]. One of the most interesting realizations of Berry’s phase (i.e., cyclic adiabatic geometric phase) is the propagation of photons inside a helically curved optical fiber, which was first proposed by Chiao and Wu [37], and performed experimentally by Tomita and Chiao [38]. Afterwards, a large number of investigators treated this geometric phase problem by making use of the classical Maxwell’s electrodynamics, differential geometry method
(parallel transport) and quantum adiabatic theory [23] both theoretically and experimentally [39–46]. Based on the above investigations, we studied the nonadiabatic noncyclic geometric phases of photons propagating inside a noncoplanarly curved optical fiber by using the Lewis-Riesenfeld invariant theory and the invariant-related unitary transformation formulation [47–49]. In the published paper [50], we considered the photon helicity inversion in the curved fiber and its potential applications to information science and, on the basis of the second-quantized spin model, we calculated the quantum-vacuum geometric phases of electromagnetic fields, which results from the vacuum zero-point fluctuation.

A. Nonadiabatic noncyclic geometric phases of photons in the noncoplanar optical fiber

In what follows we consider a potential vacuum effect in a time-dependent quantum system, i.e., time evolution of photon wave function and rotations of polarization planes in a noncoplanar fiber. The spin angular momentum operators of photon fields read (in the natural units $\hbar = c = 1$)

$$S_{ij} = -\int (\dot{A}_i A_j - \dot{A}_j A_i) d^3 x$$  \hspace{1cm} (3.1)

with the three-dimensional magnetic vector potentials $A(x, t)$ being expanded as a Fourier series [18,51]

$$A(x, t) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \varepsilon(k, \lambda)[a(k, \lambda) \exp(-i k \cdot x) + a^\dagger(k, \lambda) \exp(i k \cdot x)],$$  \hspace{1cm} (3.2)

where the frequencies $\omega_k = |k|$, and $\varepsilon(k, 1)$ and $\varepsilon(k, 2)$ are the two mutually perpendicular real unit polarization vectors, which are also orthogonal to the wave vector $k$ of the time harmonic electromagnetic wave.

Consider a noncoplanarly curved optical fiber that is wound smoothly on a large enough diameter [38], the effective Hamiltonian that describes the time evolution of photon wavefunction in the curved fiber is [50]

$$H_{\text{eff}}(t) = \frac{k(t) \times \dot{k}(t)}{k^2} \cdot S$$  \hspace{1cm} (3.3)

with the wave vector being defined to be $k(t) = k(\sin \lambda \cos \gamma, \sin \lambda \sin \gamma, \cos \lambda)$, where $\dot{k}(t)$ denotes the derivative of $k(t)$ with respect to time $t$. Note that here the wave vector $k(t)$ of a photon propagating inside the fiber is always along the tangent to the curved fiber at each point at arbitrary time. Readers may be referred to the Appendices to this paper for the derivation of effective Hamiltonian (3.3). According to the Lewis-Riesenfeld invariant theory [47] and the invariant-related unitary transformation formulation [48], the exact particular solution to the time-dependent Schrödinger equation

$$i \frac{\partial |\sigma, k(t)\rangle}{\partial t} = \frac{k(t) \times \dot{k}(t)}{k^2} \cdot S |\sigma, k(t)\rangle$$  \hspace{1cm} (3.4)

governing the propagation of photons in the fiber is given by

$$|\sigma, k(t)\rangle = \exp \left[ \frac{i}{\hbar} \phi^{(g)}(t) \right] V(t) |\sigma, k\rangle,$$  \hspace{1cm} (3.5)

where $|\sigma, k\rangle \equiv |\sigma, k(t = 0)\rangle$ is the initial photon polarized state, and $V(t) = \exp[\beta(t) S_+ - \beta^*(t) S_-]$ [48,52] with the time-dependent parameters $\beta(t) = -\frac{\Omega(t)}{2} \exp[-i \gamma(t)], \beta^*(t) = -\frac{\Omega(t)}{2} \exp[i \gamma(t)]$. The geometric phase of photons whose initial helicity eigenvalue is $\sigma$ can be expressed by [50,53,54]

$$\phi^{(g)}_{\sigma}(t) = \left\{ \int_0^t \dot{\gamma}(t') \left[ 1 - \cos \lambda(t') \right] dt' \right\} \langle \sigma, k | S_3 | \sigma, k \rangle.$$  \hspace{1cm} (3.6)

In the adiabatic process where both the precessional frequency $\dot{\gamma}$ (expressed by $\Omega$) and $\lambda (k$ deviating from the third axis in the fixed frame by an angle $\lambda$) can be regarded as constants (which can be realized in a helically coiled fiber [37,38]), the adiabatic geometric phase (i.e., Berry’s topological phase) in a cycle ($T = \frac{2\pi}{\Omega}$) in the photon momentum $k$ space is written

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\begin{equation}
\phi_s^{(g)}(T) = 2\pi(1 - \cos \lambda) \langle \sigma, k | S_3 | \sigma, k \rangle,
\end{equation}

where \(2\pi(1 - \cos \lambda)\) is equal to a solid angle subtended at the origin of momentum \(k\) space. This fact thus means that the geometric phase \(3.6\) or \(3.7\) carries information on the global and topological properties of time evolution of quantum systems. It should be emphasized that here the quantum-vacuum geometric phase may be involved in \(\langle \sigma, k | S_3 | \sigma, k \rangle\) if the third component \(S_3\) of photon spin operator \(S\) is of a non-normal-order form, which will be taken into account in the following.

Substituting the Fourier expansion series \(3.2\) of \(A(x,t)\) into the expression \((3.1)\) for photon spin operator, one can obtain the non-normal-order photon \(S\) [50,53], i.e.,

\begin{equation}
S_3 = \frac{i}{2}[a(k,1)a^\dagger(k,2) - a^\dagger(k,1)a(k,2) - a(k,2)a^\dagger(k,1) + a^\dagger(k,2)a(k,1)].
\end{equation}

In what follows we define the creation and annihilation operators, \(a^\dagger_R(k), a^\dagger_L(k), a_R(k), a_L(k)\), of right- and left-handed circularly polarized light [18]

\begin{align}
a^\dagger_R(k) &= \frac{1}{\sqrt{2}}[a^\dagger(k,1) + ia^\dagger(k,2)], \\
a_R(k) &= \frac{1}{\sqrt{2}}[a(k,1) - ia(k,2)], \\
a^\dagger_L(k) &= \frac{1}{\sqrt{2}}[a^\dagger(k,1) - ia^\dagger(k,2)], \\
a_L(k) &= \frac{1}{\sqrt{2}}[a(k,1) + ia(k,2)].
\end{align}

So, the third component of monomode-photon spin operator can be rewritten

\begin{equation}
S_3 = \frac{1}{2}\{[a_R(k)a^\dagger_R(k) + a^\dagger_R(k)a_R(k)] - [a_L(k)a^\dagger_L(k) + a^\dagger_L(k)a_L(k)]\}.
\end{equation}

The monomode multi-photon states of left- and right-handed (LRH) circularly polarized light (at \(t = 0\)) can be defined

\begin{align}
|\sigma = -1, k, n_L\rangle &= \frac{[a^\dagger_L(k)]^n}{\sqrt{n!}}|0_L\rangle, \\
|\sigma = +1, k, n_R\rangle &= \frac{[a^\dagger_R(k)]^n}{\sqrt{n!}}|0_R\rangle
\end{align}

with \(n_L\) and \(n_R\) being the LRH polarized photon occupation numbers, respectively. Now we calculate the geometric phases of multi-photon states

\begin{equation}
|\sigma = +1, k, n_R; \sigma = -1, k, n_L\rangle \equiv |\sigma = +1, k, n_R\rangle \otimes |\sigma = -1, k, n_L\rangle
\end{equation}

in the fiber. Substitution of \((3.12)\) into \((3.6)\) yields

\begin{equation}
\phi^{(g)}(t) = \left\{ \int_0^t \dot{\gamma}(t') \left[1 - \cos \lambda(t')\right] \, dt' \right\} \langle \sigma = +1, k, n_R; \sigma = -1, k, n_L | S_3 | \sigma = +1, k, n_R; \sigma = -1, k, n_L \rangle.
\end{equation}

and the final result is given

\begin{equation}
\phi^{(g)}(t) = \langle n_R - n_L \rangle \left\{ \int_0^t \dot{\gamma}(t') \left[1 - \cos \lambda(t')\right] \, dt' \right\},
\end{equation}

which is independent of \(k\) (the magnitude of \(k\)) but dependent on the geometric nature of the pathway (expressed in terms of \(\lambda\) and \(\gamma\)) along which the light wave propagates. This fact indicates that geometric phases possesses the topological and global properties of time evolution of quantum systems. It is emphasized that the phases of multi-photon states \((3.14)\) associated with the photonic occupation numbers \(n_R\) and \(n_L\) are quantal in character [48]. Gao has shown why \(\phi^{(g)}(t) = \langle n_R - n_L \rangle \left\{ \int_0^t \dot{\gamma}(t') \left[1 - \cos \lambda(t')\right] \, dt' \right\}\) is referred to as the quantal geometric phases [48] by taking into consideration the uncertainty relation between the operators \(\frac{i}{2}[a^\dagger_R(k) + a_{R,L}(k)]\) and \(\frac{i}{2}[a^\dagger_R(k) - a_{R,L}(k)]\). Although the phases \(\phi^{(g)}(t)\) in \((3.14)\) are quantal geometric phases of photons, they do not belong to the geometric phases at quantum-vacuum level which arise, however, from the zero-point electromagnetic energy of quantum vacuum fluctuation. Since the cyclic adiabatic cases of \((3.14)\) have been measured experimentally by Tomita and Chiao et al. [38–40,43], we will not consider them further. In the following we will study instead the geometric phases at quantum vacuum level, which has not been tested experimentally yet. The reason for why it cannot be easily tested will be given at the end of this section.
According to the expression (3.10) for $S_3$, both the geometric phases of left- and right-handed circularly polarized photon states, i.e., $|\sigma = -1, k, n_L \rangle$ and $|\sigma = +1, k, n_R \rangle$, are respectively of the form

$$\phi^{(L)}_{\sigma}(t) = -(n_L + \frac{1}{2}) \left\{ \int_0^t \gamma(t') \left[ 1 - \cos \lambda(t') \right] \, dt' \right\}, \quad \phi^{(R)}_{\sigma}(t) = +(n_R + \frac{1}{2}) \left\{ \int_0^t \gamma(t') \left[ 1 - \cos \lambda(t') \right] \, dt' \right\}. \quad (3.15)$$

It follows that the time-dependent zero-point energy possesses physical meanings and therefore contributes to geometric phases of photon fields. Thus the noncyclic nonadiabatic geometric phases of left- and right-handed polarized states at quantum-vacuum level are given

$$\phi^{(\text{vac})}_{\sigma = \pm 1}(t) = \pm \frac{1}{2} \left\{ \int_0^t \gamma(t') \left[ 1 - \cos \lambda(t') \right] \, dt' \right\}. \quad (3.16)$$

Investigation of quantum-vacuum geometric phases due to vacuum fluctuation energies possesses theoretical significance: in conventional time-independent quantum field theory the infinite zero-point energy of vacuum is harmless and can be easily removed by the normal-order procedure [55]. However, for the time-dependent quantum field systems, (e.g., photon fields propagating inside a helically curved fiber, and quantum fields in an expanding universe or time-dependent gravitational backgrounds), the time-dependent vacuum zero-point fields may also participate in the time evolution process and therefore cannot be regarded merely as an inactive onlooker (i.e., a simple passive background). According to the formulation applied to the time-independent field theory, these physically interesting vacuum effects would unfortunately have been deducted by the second-quantization normal-order technique. For this reason, we think that perhaps it is necessary to consider the validity problem of normal-order procedure in time-dependent quantum field theory.

Since quantum-vacuum geometric phases has an important connection with vacuum quantum fluctuation, its experimental realization deserves consideration, which will be briefly discussed at the end of this section.

### B. Wave propagation in biaxially anisotropic left-handed materials

It is well known that the geometric phases of photons inside a curved fiber in previous experiments [38] often depend on the cone angles of solid angles subtended by a curve traced by the direction of wave vector of light at the center of the photon momentum $k$ space. Here, however, by taking into account the peculiar properties of wave propagation in certain biaxially anisotropic left-handed media, we will present a physically interesting geometric phase that is independent of the cone angles. First we consider the wave propagation in this type of left-handed materials with the permittivity and permeability tensors as follows

$$\varepsilon_{ik} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}, \quad \mu_{ik} = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}. \quad (3.17)$$

If the propagation vector of time-harmonic electromagnetic wave is $k = (0, 0, k)$, then according to the Maxwellian Equations, one can arrive at $k \times E = (-k E_2, k E_1, 0)$, $\{[(\mu)_{ik} H_k] = (-\mu H_1, \mu H_2, 0)$. It follows from the Faraday’s electromagnetic induction law $\nabla \times E = -\frac{\partial B}{\partial t}$ that $H_1 = \frac{k E_2}{\mu k \omega}$ and $H_2 = \frac{k E_1}{\mu k \omega}$. Thus, the third component of Poynting vector of this time-harmonic wave is obtained

$$S_3 = E_1 H_2 - E_2 H_1 = \frac{k}{\mu k \omega} (E_1^2 - E_2^2), \quad (3.18)$$

which implies that the Poynting vectors corresponding to the $E_1$- and $E_2$- fields are of the form

$$S^{(1)} = \frac{E_1^2}{\mu k \omega} k, \quad S^{(2)} = -\frac{E_2^2}{\mu k \omega} k, \quad (3.19)$$

respectively. It is apparently seen from (3.19) that the direction of wave vector $S^{(2)}$ is opposite to that of $S^{(1)}$. This, therefore, means that if $\mu > 0$, then for the $E_1$ field, this biaxially anisotropic medium characterized by (3.17) is like a right-handed material (regular material) whereas for the $E_2$ field, it serves as a left-handed one.

In view of above discussions, it is concluded that inside the above biaxially anisotropic medium, the wave vectors of $E_1$- and $E_2$- fields of propagating planar wave are opposite to each other. Consider a hypothetical optical fiber
that is made of this biaxially anisotropic left-handed medium, inside which the wave vector of $E_1$ field propagating is assumed to be $k(t) = k(\sin \lambda \cos \gamma, \sin \lambda \sin \gamma, \cos \lambda)$. If both $\lambda$ and $\gamma$ are nonvanishing, then this fiber is noncoplanarly curved and in consequence the geometric phases of light will arise. It is readily verified from (3.19) that the wave vector of $E_2$-field is $-k(t) = k(\sin \lambda' \cos \gamma', \sin \lambda' \sin \gamma', \cos \lambda')$ with $\lambda' = \pi - \lambda$ and $\gamma' = \gamma + \pi$. Note that for the latter case (i.e., $E_2$-field), the expression for the time-dependent coefficient in (3.6) changes from $\int_0^t \gamma(t') \left[ 1 - \cos \lambda(t') \right] dt'$ to $\int_0^t \gamma(t') \left[ 1 + \cos \lambda(t') \right] dt'$ (because of $\lambda \to \pi - \lambda$ and $\gamma \to \gamma + \pi$). In the next subsection these results will be useful in calculating the cone angle independent geometric phases of circularly polarized light in biaxially anisotropic left-handed media.

C. Cone angle independent geometric phases in biaxially anisotropic left-handed media

The creation operators of left- and right-handed circularly polarized light are $a_L^\dagger = \frac{a_1^\dagger + ia_2^\dagger}{\sqrt{2}}$ and $a_R^\dagger = \frac{a_1^\dagger - ia_2^\dagger}{\sqrt{2}}$, respectively [18]. The photon states corresponding to right- and left-handed polarized light with the photon occupation numbers being $n_R$ and $n_L$ are respectively defined to be $|n_R\rangle = \left(\frac{a_1^\dagger}{\sqrt{n_R+1}}\right)^{n_R} |0_R\rangle$ and $|n_L\rangle = \left(\frac{ia_1^\dagger}{\sqrt{n_L+1}}\right)^{n_L} |0_L\rangle$. Since according to the discussion in the previous subsection the wave vector of $E_1$-field in such anisotropic left-handed media is antiparallel to that of $E_2$-field, we should first calculate the following expectation value $\langle n_R|a_L^\dagger a_1|n_R\rangle$, $\langle n_R|a_L^\dagger a_2|n_R\rangle$, $\langle n_L|a_1^\dagger a_1|n_L\rangle$ and $\langle n_L|a_2^\dagger a_2|n_L\rangle$ in order to obtain the expressions for geometric phases of left- and right-handed circularly polarized light in this peculiar biaxially anisotropic left-handed medium. By the aid of $a_1|n_R\rangle = \frac{n_R!}{\sqrt{n_R!}} \sum_{l=0}^{n_R} \frac{(-i)^l}{l!(n_R-l)!} (a_1^\dagger)^l (ia_2^\dagger)^{n_R-l} |0_R\rangle$, one can arrive at

$$a_1|n_R\rangle = \frac{n_R!}{\sqrt{n_R!} \sqrt{n_R+1}} \sum_{l=1}^{n_R} \frac{(n_R-1)!}{(l-1)!(n_R-l)!} (a_1^\dagger)^{l-1} (ia_2^\dagger)^{n_R-l} |0_R\rangle = \sqrt{n_R \over 2} |n_R-1\rangle,$$

where use is made of the formula $a_1 (a_1^\dagger)^l = l (a_1^\dagger)^{l-1} + (a_1^\dagger)^l a_1$. Thus, we obtain $a_1|n_R\rangle = \sqrt{n_R \over 2} |n_R-1\rangle$ and consequently $\langle n_R|a_1^\dagger a_1 |n_R\rangle = {n_R \over 2}$.

In the similar manner, one can obtain

$$a_2|n_R\rangle = i \sqrt{n_R \over 2} |n_R-1\rangle, \quad \langle n_R|a_2^\dagger a_2 |n_R\rangle = {n_R \over 2}.$$  

Hence the nonadiabatic noncyclic geometric phases of right-handed polarized photons corresponding to $E_1$- and $E_2$-fields are

$$\phi_R^{(1)}(t) = \frac{n_R}{2} \left\{ \int_0^t \gamma(t') \left[ 1 - \cos \lambda(t') \right] dt' \right\}, \quad \phi_R^{(2)}(t) = \frac{n_R}{2} \left\{ \int_0^t \gamma(t') \left[ 1 + \cos \lambda(t') \right] dt' \right\},$$

respectively, and their sum is

$$\phi_R(t) = \phi_R^{(1)}(t) + \phi_R^{(2)}(t) = n_R \int_0^t \gamma(t') dt',$$

which is independent of the cone angle $\lambda(t)$ of photon momentum $k$ space.

In the same fashion, we obtain

$$a_1|n_L\rangle = \sqrt{n_L \over 2} |n_L-1\rangle, \quad \langle n_L|a_1^\dagger a_1 |n_L\rangle = {n_L \over 2}$$

and

$$a_2|n_L\rangle = -i \sqrt{n_L \over 2} |n_L-1\rangle, \quad \langle n_L|a_2^\dagger a_2 |n_L\rangle = {n_L \over 2}.$$
Hence the nonadiabatic noncyclic geometric phases of left-handed polarized photons corresponding to $E_1$- and $E_2$-fields are

$$
\phi_L^{(1)}(t) = -\frac{n_L}{2} \left\{ \int_0^t \dot{\gamma}(t') \left[ 1 - \cos \lambda(t') \right] dt' \right\}, \quad \phi_L^{(2)}(t) = -\frac{n_L}{2} \left\{ \int_0^t \dot{\gamma}(t') \left[ 1 + \cos \lambda(t') \right] dt' \right\},
$$
(3.26)

respectively, and their sum is

$$
\phi_L(t) = \phi_L^{(1)}(t) + \phi_L^{(2)}(t) = -n_L \int_0^t \dot{\gamma}(t') dt',
$$
(3.27)

which is also independent of the cone angle $\lambda(t)$.

Thus the total geometric phases of left- and right-handed polarized photons is given by

$$
\phi^{(g)}(t) = \phi_R(t) + \phi_L(t) = (n_R - n_L) \int_0^t \dot{\gamma}(t') dt',
$$
(3.28)

which differs from (3.14) only by a cone angle $\lambda(t)$ of photon momentum $k$ space.

Since the geometric phases of both left- and right-handed polarized light propagating in the above biaxially anisotropic left-handed materials depend no longer on the cone angle, someone may argue that the geometric phases presented here lose their topological and global nature. This is not the true case. Geometric phases presents the topological properties of quantum systems in time-evolution process. Differing from dynamical phase that depends on dynamical quantities of systems such as energy, frequency, velocity as well as coupling coefficients, geometric phase is independent of these dynamical quantities. Instead, it is only related to the geometric nature of the pathway along which quantum systems evolve. It follows that here $\phi_L^{(g)}(t)$ is related only to the precessional frequency $\dot{\gamma}$ of photon propagation in the curved fiber, which is not of dynamical nature and therefore cannot be considered the dynamical quantity. This precessional frequency depends upon the geometric shape of curved fiber. For example, in the helically curved fiber, which was used first in Tomita and Chiao’s experiment to produce photon cyclic Berry’s phase [38], the precessional frequency equals $\frac{2\pi}{\sqrt{d^2 + (na)^2}}$ [50], where $d$ and $a$ respectively denote the pitch length and the radius of the helix, and $c$ is the speed of light in a vacuum. For this reason, we think that $\phi_L^{(g)}(t)$ still possesses the geometric nature of time evolution of photon wave function and can be regarded as the geometric phase.

D. Brief discussion: testing quantum-vacuum geometric phases

As is stated above, the photon geometric phases at quantum-vacuum level originates from the zero-point electromagnetic fluctuations. Since geometric phases indicates topological and global properties of quantum systems in time-evolution processes, the quantum-vacuum geometric phases of electromagnetic fields in the helically wound fiber may contain the information on the global properties of time evolution of vacuum fluctuation fields. Moreover, we should make it clear whether the normal-order procedure is valid or not in the time-dependent quantum field theory and it is therefore essential to detect the quantum-vacuum geometric phases (3.16) in experiments.

However, it should be pointed out that, unfortunately, even at the quantum level, this observable quantum-vacuum geometric phases $\phi^{(vac)}(t)$ is absent in the fiber experiment, since it follows from (3.15) and (3.16) that the signs of quantal geometric phases of left- and right-handed circularly polarized photons are just opposite to one another, and so that the quantum-vacuum geometric phases would have been counteracted by each other. Hence the observed geometric phases are only those expressed by (3.14), the adiabatic case of which, as stated above, has been measured in the optical fiber experiments performed by Tomita and Chiao et al. [38–40,43]. It is impossible for physicists to detect the quantum-vacuum geometric phases, which has been eliminated, in these fiber experiments.

Since the quantum-vacuum geometric phases is so important but unfortunately cancelled by each other (hence the total vacuum geometric phases vanishes), we must ask such question: how can we detect the quantum-vacuum geometric phases corresponding to only one of the circularly polarized light? The studies in the previous subsection enlighten us on this subject. If one can design and fabricate a kind of such artificial composite metamaterials, where for the left-handed polarized light the material serves as a left-handed medium, while for the right-handed polarized light it behaves like an ordinary material, i.e., the right-handed medium, then in this medium the quantum-vacuum geometric phases corresponding to the right- and left-handed polarized light are

$$
\phi_R^{(vac)}(t) = \frac{1}{2} \left\{ \int_0^t \dot{\gamma}(t') \left[ 1 - \cos \lambda(t') \right] dt' \right\}, \quad \phi_L^{(vac)}(t) = -\frac{1}{2} \left\{ \int_0^t \dot{\gamma}(t') \left[ 1 + \cos \lambda(t') \right] dt' \right\},
$$
(3.29)
respectively. Here, the integrand in $\phi^{(\text{vac})}_L(t)$ is obtained from $\phi^{(\text{vac})}_R(t)$ via the parameter replacements $\lambda \to \pi - \lambda$ and $\gamma \to \gamma + \pi$.

Thus according to the expression (3.29) that the total quantum-vacuum geometric phases is $\phi^{(\text{tot})}_R(t) = \phi^{(\text{vac})}_L(t) + \phi^{(\text{vac})}_R(t) = -\int^t_0 \gamma(t') \cos(\lambda(t'))\,dt'$, which is no longer vanishing in this artificial composite medium. Although it is of physically interest to take into account this topic, designing such materials is very complicated and is not the main subject in this paper, so here we will not discuss further this problem. It is under consideration and will be published elsewhere. Here we only emphasize that it is truly possible for us to detect quantum-vacuum geometric phases by using certain anisotropic left-handed media. For instance, in section II, we stated that the indices squared of left- and right-handed circularly polarized light in gyrotropic media are respectively $n_1^2 = (\epsilon_1 - \epsilon_2)(\mu_1 - \mu_2)$ and $n_2^2 = (\epsilon_1 + \epsilon_2)(\mu_1 + \mu_2)$, respectively [8]. If, for example, by taking some certain values of $\epsilon_1$, $\epsilon_2$, $\mu_1$ and $\mu_2$, then $n_1^2 < 0$ while $n_2^2 > 0$ and consequently the left-handed polarized light cannot be propagated in this medium, and in the meanwhile the quantum vacuum fluctuation corresponding to the left-handed polarized light will also be absorbed (e.g., the wave amplitude exponentially decreases because of the imaginary part of the refractive index) in this anisotropic absorptive medium (i.e., the vacuum-fluctuation electromagnetic field alters its mode structures in the absorptive medium). For this reason, the only retained geometric phases is that of right-handed polarized light, which we can test experimentally.

IV. EXTRA PHASES OF LIGHT AT THE INTERFACES BETWEEN LEFT- AND RIGHT- HANDED MEDIA

In this section, we consider the effects of light appearing at the interfaces between left- and right-handed media. In order to treat this problem conveniently, we study the wave propagation inside an optical fiber which is periodically modulated by altering regular and negative media. Although it is doubtful whether such periodically modulated fibers could be designed and realized or not in experiments at the optical scale, it could be argued that the work presented here can be considered only a speculative one. But the method and results obtained via the use of this optical fiber system composed by such sequences of right- and left-handed materials can also be applied to the light propagation at the interfaces between left- and right-handed media in arbitrary geometric shapes of optical materials. In this periodical optical fiber, helicity inversion (or the transitions between helicity states) of photons may be easily caused by the interaction of light field with media near both sides of the interfaces between LRH materials. Since photon helicity inversion at the interfaces mentioned above is a time-dependent process, this new geometric phase arises during the light propagates through the interfaces (in the following we will call them the LRH interfaces) between left- and right-handed media, where the anomalous refraction occurs when the incident lightwave travels to the LRH interfaces. We think that, in the literature it gets less attention than it deserves. In what follows we calculate the photon wavefunction and corresponding extra phases (including the geometric phases) in this physical process, and emphasize that we should attach importance to this geometric phases when considering the wave propagation near the LRH interfaces.

A. Model Hamiltonian

We now treat the helicity reversal problem of light wave adjacent to the interfaces of left- and right-handed media. For convenience, let us consider a hypothetical optical fiber that is fabricated periodically from both left- and right-handed (LRH) media with the optical refractive indices being $-n$ and $n$, respectively. Thus the wave vector of photon moving along the fiber is respectively $-n\frac{2\pi}{b}$ in left-handed (LH) section and $n\frac{2\pi}{b}$ in right-handed (RH) section, where $\omega$ and $c$ respectively denote the frequency and the speed of light in a vacuum. For simplicity, we assume that the periodical length, $b$, of LH is equal to that of RH in the fiber. If the eigenvalue of photon helicity is $\sigma$ in right-handed sections, then, according to the definition of helicity, $h = \frac{k}{\hbar} \cdot \mathbf{J}$ with $\mathbf{J}$ denoting the total angular momentum of the photon, the eigenvalue of helicity acquires a minus sign in left-handed sections. We assume that at $t = 0$ the light propagates in the right-handed section and the initial eigenvalue of photon helicity is $\sigma$. So, in the wave propagation inside the LRH-periodical optical fiber, the helicity eigenvalue of $h$ is then $(-)^m \sigma$ with $m = \left[\frac{\omega}{\omega_0}\right]$, where $\left[\frac{\omega}{\omega_0}\right]$ represents the integer part of $\frac{\omega}{\omega_0}$. It is clearly seen that $(-)^m$ stands for the switching on and off of the helicity reversal, i.e., the positive and negative value of $(-)^m$ alternate in different time intervals. This, therefore, means that if $2k \left(\frac{2\pi}{b}\right) < t \leq (2k + 1) \left(\frac{2\pi}{b}\right)$, then $(-)^m = +1$, and if $(2k + 1) \left(\frac{2\pi}{b}\right) < t \leq (2k + 2) \left(\frac{2\pi}{b}\right)$, then $(-)^m = -1$, where $k$ is zero or a positive integer. It follows that the incidence of lightwave on the LRH interfaces in the fiber gives rise
to the transitions between the photon helicity states \(|+\rangle\) and \(|-\rangle\). This enables us to construct a time-dependent effective Hamiltonian

\[ H(t) = \frac{1}{2} \omega(t) (S_+ + S_-) \tag{4.1} \]

in terms of \(|+\rangle\) and \(|-\rangle\) to describes this instantaneous transition process of helicity states at the LRH interfaces, where \(S_+ = |+\rangle \langle -|, S_- = |-\rangle \langle +|, S_3 = \frac{1}{2} (|+\rangle \langle +| - |-\rangle \langle -|)\) satisfying the following SU(2) Lie algebraic commuting relations \([S_+, S_-] = 2S_3\) and \([S_3, S_\pm] = \pm S_\pm\). The time-dependent frequency parameter \(\omega(t)\) may be taken to be \(\omega(t) = \zeta \frac{4}{c^2} p(t)\), where \(p(t) = (-)^m\) with \(m = \lfloor \frac{2m}{n_b}\rfloor\), and \(\zeta\) is the coupling coefficient, which can, in principle, be determined by the physical mechanism of interaction between light fields and media. Since \(p(t)\) is a periodical function, by using the analytical continuation procedure, it can be rewritten as the following linear combinations of analytical functions

\[ p(t) = \sum_{k=1}^{\infty} \frac{2}{k\pi} \left[ 1 - (-)^k \right] \sin \left( \frac{k\pi c}{n_b} t \right). \tag{4.2} \]

In what follows, we solve the time-dependent Schrödinger equation (in the unit \(\hbar = 1\))

\[ H(t) |\Psi_\sigma(t)\rangle = \frac{\partial}{\partial t} |\Psi_\sigma(t)\rangle \tag{4.3} \]

governing the propagation of light in the LRH-periodical fiber. According to the Lewis-Riesenfeld invariant theory \[47\], the exact particular solution \(|\Psi_\sigma(t)\rangle\) of the time-dependent Schrödinger equation (4.3) is different from the eigenstate of the invariant \(I(t)\) only by a time-dependent \(c\)-number factor \(\exp \left[ \frac{i}{\hbar} \phi_\sigma(t) \right]\), where

\[ \phi_\sigma(t) = \int_0^t \langle \Phi_\sigma(t') | [H(t') - i \frac{\partial}{\partial t}] | \Phi_\sigma(t') \rangle \, dt'. \tag{4.4} \]

with \(|\Phi_\sigma(t)\rangle\) being the eigenstate of the invariant \(I(t)\) (corresponding to the particular eigenvalue \(\sigma\)) and satisfying the eigenvalue equation \(I(t) |\Phi_\sigma(t)\rangle = \sigma |\Phi_\sigma(t)\rangle\), where the eigenvalue \(\sigma\) of the invariant \(I(t)\) is time-independent. Thus we have

\[ |\Psi_\sigma(t)\rangle = \exp \left[ \frac{i}{\hbar} \phi_\sigma(t) \right] |\Phi_\sigma(t)\rangle. \tag{4.5} \]

In order to obtain \(|\Psi_\sigma(t)\rangle\), we should first obtain the eigenstate \(|\Phi_\sigma(t)\rangle\) of the invariant \(I(t)\).

### B. Photon geometric phases due to helicity inversions inside a periodical fiber made of Left-handed media

Here we investigate the time evolution of photon wavefunctions and extra phases due to photon helicity inversion at the LRH interfaces. As has been stated above, for convenience, we consider the wave propagation inside a hypothetical optical fiber which is composed periodically of left- and right-handed media. Now we solve the time-dependent Schrödinger equation. In accordance with the Lewis-Riesenfeld theory, the invariant \(I(t)\) is a conserved operator (i.e., it possesses time-independent eigenvalues) and agrees with the following Liouville-Von Neumann equation

\[ \frac{\partial I(t)}{\partial t} + \frac{1}{i} [I(t), H(t)] = 0. \tag{4.6} \]

It follows from Eq.(4.6) that the invariant \(I(t)\) may also be constructed in terms of \(S_\pm\) and \(S_3\), i.e.,

\[ I(t) = 2 \left\{ \frac{1}{2} \sin \theta(t) \exp [-i \varphi(t)] S_+ + \frac{1}{2} \sin \theta(t) \exp [i \varphi(t)] S_- + \cos \theta(t) S_3 \right\}. \tag{4.7} \]

Inserting Eq.(4.1) and (4.7) into Eq.(4.6), one can arrive at a set of auxiliary equations

\[ \exp [-i \varphi] \left( \dot{\theta} \cos \theta - i \dot{\varphi} \sin \theta \right) - i \omega \cos \theta = 0, \]

\[ \dot{\theta} + \omega \sin \varphi = 0, \tag{4.8} \]

11
which are used to determine the time-dependent parameters, \( \theta(t) \) and \( \varphi(t) \), of the invariant \( I(t) \) [47].

It should be noted that we cannot easily solve the eigenvalue equation \( I(t)|\Phi_\sigma(t)\rangle = \sigma|\Phi_\sigma(t)\rangle \), for the time-dependent parameters \( \theta(t) \) and \( \varphi(t) \) are involved in the invariant (4.7). If, however, we could find (or construct) a unitary transformation operator \( V(t) \) to make \( V(t)I(t)V(t)^\dagger \) be \textit{time-independent}, then the eigenvalue equation problem of \( I(t) \) is therefore easily resolved. According to our experience for utilizing the invariant-related unitary transformation formulation [27], we suggest a following unitary transformation operator

\[
V(t) = \exp[\beta(t) S_+ - \beta^*(t) S_-],
\]

where \( \beta(t) \) and \( \beta^*(t) \) will be determined by calculating \( I_V = V(t)I(t)V(t)^\dagger \) in what follows.

Calculation of \( I_V = V(t)I(t)V(t)^\dagger \) yields

\[
I_V = V(t)I(t)V(t)^\dagger = 2S_3,
\]

if \( \beta \) and \( \beta^* \) are chosen to be \( \beta(t) = -\frac{\theta(t)}{2} \exp[-i\varphi(t)] \), \( \beta^*(t) = -\frac{\theta(t)}{2} \exp[i\varphi(t)] \). This, therefore, means that we can change the \textit{time-dependent} \( I(t) \) into a \textit{time-independent} \( I_V \), and the result is \( I_V = 2S_3 \). Thus, the eigenvalue equation of \( I_V \) is \( I_V|\sigma\rangle = \sigma|\sigma\rangle \) with \( \sigma = \pm 1 \), and consequently the eigenvalue equation of \( I(t) \) is written \( I(t)V(t)|\sigma\rangle = \sigma V(t)|\sigma\rangle \). So, we obtain the eigenstate \( |\Phi_\sigma(t)\rangle \) of \( I(t) \), i.e., \( |\Phi_\sigma(t)\rangle = V(t)|\sigma\rangle \).

Correspondingly, \( H(t) \) is transformed into

\[
H_V(t) = V(t)^\dagger \left[ H(t) - i\frac{\partial}{\partial t} \right] V(t) \]

and the time-dependent Schrödinger equation (4.3) is rewritten

\[
H_V(t)|\Psi_\sigma(t)\rangle_V = i\frac{\partial}{\partial t} |\Psi_\sigma(t)\rangle_V
\]

under the unitary transformation \( V(t) \), where \( |\Psi_\sigma(t)\rangle_V = V(t)|\Psi_\sigma(t)\rangle \).

Further analysis shows that the exact particular solution \( |\Psi_\sigma(t)\rangle_V \) of the time-dependent Schrödinger equation (4.12) is different from the eigenstate \( |\sigma\rangle \) of the \textit{time-independent} invariant \( I_V \) only by a time-dependent \( \sigma \)-number factor \( \exp \left\{ \frac{1}{i}\phi_\sigma(t) \right\} [47] \), which is now rewritten as \( \exp \left\{ \frac{1}{i} \left[ \phi^{(d)}_\sigma(t) + \phi^{(g)}_\sigma(t) \right] \right\} \).

By using the auxiliary equations (4.8), the Glauber formula and the Baker-Campbell-Hausdorff formula [56], it is verified that \( H_V(t) \) depends only on the operator \( S_3 \), i.e.,

\[
H_V(t) = \{ \omega(t) \sin \theta(t) \cos \varphi(t) + \varphi(t) [1 - \cos \theta(t)] \} S_3
\]

and the time-dependent \( \sigma \)-number factor \( \exp \left\{ \frac{1}{i}\phi_\sigma(t) \right\} \) is therefore \( \exp \left\{ \frac{1}{i} \left[ \phi^{(d)}_\sigma(t) + \phi^{(g)}_\sigma(t) \right] \right\} \),

where the dynamical phase is

\[
\phi^{(d)}_\sigma(t) = \sigma \int_0^t \omega(t') \sin \theta(t') \cos \varphi(t') dt'
\]

and the geometric phase is

\[
\phi^{(g)}_\sigma(t) = \sigma \int_0^t \varphi(t') [1 - \cos \theta(t')] dt'.
\]

Hence the particular exact solution of the time-dependent Schrödinger equation (4.3) corresponding to the particular eigenvalue, \( \sigma \), of the invariant \( I(t) \) is of the form

\[
|\Psi_\sigma(t)\rangle = \exp \left\{ \frac{1}{i} \left[ \phi^{(d)}_\sigma(t) + \phi^{(g)}_\sigma(t) \right] \right\} V(t)|\sigma\rangle.
\]

It follows from the obtained expression (4.15) for geometric phase of photons that, if the frequency parameter \( \omega \) is small (i.e., the adiabatic quantum process) and then according to the auxiliary equations (4.8), \( \theta \approx 0 \), the Berry phase (adiabatic geometric phase) in a cycle (i.e., one round trip, \( T \approx \frac{2\pi}{\omega} \)) of parameter space of invariant \( I(t) \) is

\[
\phi^{(g)}_\sigma(T) = 2\pi \sigma (1 - \cos \theta),
\]

where \( 2\pi (1 - \cos \theta) \) is a solid angle over the parameter space of the invariant \( I(t) \), which means that the geometric phase is related only to the geometric nature of the pathway along which quantum systems evolve. Expression (4.17) is analogous to the magnetic flux produced by a monopole of strength \( \sigma \) existing at the origin of the parameter space. This, therefore, implies that geometric phases differ from dynamical phases and involve the global and topological properties of the time evolution of quantum systems.
C. In biaxially anisotropic left-handed media

Note that in the previous subsection, we treat the helicity reversals of single photon in electromagnetic media made of isotropic left- and right-handed materials. Now we consider this problem in a system composed by a sequence of right-handed (isotropic) and biaxially anisotropic left-handed media, the permittivity and permeability of the latter is given in (3.17). It has been verified in Sec.III that for the $E_1$-field this biaxially anisotropic medium can be regarded as a right-handed material while for the $E_2$-field it can be considered a left-handed one. This, therefore, implies that only the $E_2$-field propagating through this biaxially anisotropic left-handed medium will acquire an additional phase due to its helicity inversion at the LRH interfaces. But since the $E_2$-field is not the eigenmode of the photon helicity, we cannot obtain the extra phases immediately by using the formula (4.4). According to the treatment in Sec.III, one can arrive at the expression for the total extra phases of circularly polarized light (with the occupation numbers of polarized photons being $n_L$ and $n_R$) propagating through the system made of a sequence of isotropic regular media and biaxially anisotropic left-handed one, and the result is written as

$$\phi_{\text{tot}}(t) = \frac{n_R - n_L}{2} \left[ \phi^{(d)}(t) + \phi^{(s)}(t) \right] \quad (4.18)$$

with $\phi^{(d)}(t) + \phi^{(s)}(t) = \int_0^t \{ \omega(t') \sin \theta(t') \cos \varphi(t') + \varphi(t') (1 - \cos \theta(t')) \} dt'$. It follows that if $n_R = n_L$, the added phase due to photon helicity reversal on the interfaces between left- and right-handed media vanishes.

Based on the restriction Eq.(2.7) imposed on the frequency shifts of circularly polarized light, we now consider a possibility that only one of the polarized light, say $E_R$, can be propagated in biaxially gyrotropic left-handed media. If the frequency shift $\Omega_L$ of left-handed polarized light is $-\omega - i\Gamma$, i.e., $\omega + \Omega_L = -i\Gamma$, where $\Gamma$ is a positive real number, then according to Eq.(2.7), one can arrive at

$$n^4(\Gamma^2 + \omega^2)(\omega + \Omega_R)\Omega_R + \zeta^4\Gamma^2(\omega + \Omega_R)^2 = 0. \quad (4.19)$$

It is easy to obtain $\Omega_R$ from Eq.(4.19), and the result is given $\Omega_R = -\omega + \gamma^2(1 + \gamma)^{-\frac{1}{2}}\omega$ with $\gamma = \frac{n^4(\Gamma^2 + \omega^2)}{\zeta^4\Gamma^2}$. Thus, the frequency of right-handed polarized light is

$$\omega + \Omega_R = \gamma^2(1 + \gamma)^{-\frac{1}{2}}\omega, \quad (4.20)$$

which means that the frequency of $E_R$ is modified by a factor $\gamma^2(1 + \gamma)^{-\frac{1}{2}}$. If no L-R coupling exists, i.e., $\zeta = 0$, then $\gamma$ tends to infinity and the factor $\gamma^2(1 + \gamma)^{-\frac{1}{2}}$ approaches unity, and then the frequency shift $\Omega_R$ of right-handed polarized light is vanishing, which can be easily seen in the expression (4.20).

Note that in the case discussed above, the left-handed polarized light in this type of media exponentially decreases (due to the imaginary frequency $\omega + \Omega_L$, which is $-i\Gamma$) while the right-handed one can be propagated, i.e., only one wave can be present in this media. So, in this case the additional phase acquired by photon wavefunction due to helicity inversions on LRH interfaces is

$$\phi_{\text{tot}}(t) = \frac{n_R}{2} \left[ \phi^{(d)}(t) + \phi^{(s)}(t) \right]. \quad (4.21)$$

Additionally, it is of interest to show that the above scheme is applicable to the detection of quantum-vacuum geometric phases expressed by (3.15). Since the left-handed polarized light (including the quantum vacuum fluctuation corresponding to the left-handed polarized light) cannot be propagated in this media [2,8], the quantum-vacuum geometric phase of right-handed polarized light will not be cancelled by that of left-handed one, namely, the only retained quantum-vacuum geometric phase is that of right-handed circularly polarized light and therefore it is possible for the nonvanishing quantum-vacuum geometric phases to be detected in experiments.

D. Discussion: physical significance and potential applications

It is worthwhile to point out that the geometric phase of photons due to helicity inversion presented here is of quantum level. However, whether the Chiao-Wu geometric phase due to the spatial geometric shape of fiber is of quantum level or not is not apparent (see, for example, the arguments between Haldane and Chiao et al. about this problem [44,45]), since the expression for the Chiao-Wu geometric phase can be derived by using both the classical Maxwell’s electromagnetic theory, differential geometry and quantum mechanics [39,41,43–45]. However, the geometric phase in this paper can be considered only by Berry’s adiabatic quantum theory and Lewis-Riesenfeld invariant theory,
namely, the classical electrodynamics cannot predict this geometric phase. Although many investigators have taken into account the boundary condition problem and anomalous refraction in left-handed media by using the classical Maxwell’s theory [3,8], less attention is paid to this geometric phase due to helicity inversion. It is believed that this geometric phase originates at the quantum level, but survives the correspondence-principle limit into the classical level. So, We emphasize that it may be essential to take into consideration this geometric phase in investigating the anomalous refraction at the LRH interfaces.

It is well known that geometric phases arise only in time-dependent quantum systems. In the present problem, the transitions between helicity states on the LRH interfaces, which is a time-dependent process, results in the geometric phase of photons. This may be viewed from two aspects: (i) it is apparently seen in Eq.(4.8) that if the frequency parameter $\omega$ in the Hamiltonian (4.1) vanishes, then $\dot{\varphi} = 0$ and the geometric phase (4.15) is therefore vanishing; (ii) it follows from (4.2) that the frequency coefficient $\omega (t)$ of Hamiltonian (4.1) is

$$\omega (t) = \frac{d}{dt} p (t) = \frac{2c}{nb} \sum_{k=1}^{\infty} \left[ 1 - (-)^{k} \right] \cos \left( \frac{k \pi c}{nb} t \right).$$

(4.22)

Since $|\cos \left( \frac{k \pi c}{nb} t \right)| \leq 1$, the frequency coefficient, the transition rates between helicity states, and the consequent time-dependent phase $(\varphi^{(g)} (t) + \varphi^{(d)} (t))$ greatly decrease correspondingly as the periodical optical path $nb$ increases. Thus we can conclude that the interaction of light fields with media near the LRH interfaces gives rise to this topological quantum phase.

In addition to obtaining the expression (4.15) for geometric phase, we obtain the wavefunction (4.16) of photons in the LRH-optical fiber by solving the time-dependent Schrödinger equation (4.3) based on the Lewis-Riesenfeld invariant theory [47] and the invariant-related unitary transformation formulation [48,49]. We believe that this would enable us to consider the propagation of light fields inside the optical fiber in more detail.

In the above treatment, we constructed an effective Hamiltonian (4.1) to describe the time evolution of helicity states of photons. It should be noted that the method presented here is only a phenomenological description of propagation of lightwave in the LRH-periodical fiber. This phenomenological description is based on the assumption that the direction of wave vector $k$ becomes opposite nearly instantaneously on the LRH interfaces. This assumption holds true so long as the periodical length $b$ is much larger than the wavelength of lightwave in the fiber.

To close this section, we conclude with some remarks on the potential significance of the subject in this section:

(i) The obtained geometric phase itself is physically interesting. Moreover, it is necessary to consider this geometric phase in discussing the anomalous refraction and wave propagation in left-handed media (adjacent to the LRH interfaces).

(ii) Helicity inversion of photons, which is in exact analogy with the transition operation between 0 and 1 in digital circuit, can be caused due to the electromagnetic interactions at LRH interfaces, and the time evolution of helicity states is governed by (4.16). It is of essential significance to control and utilize the degrees of freedom of photons (photon number, polarization, helicity, geometric phase, etc.) in information science and technology [57]. In the curved optical fiber, the interaction of the photon spin with the wave vector causes the helicity inversion of the photon, some authors have considered its application to information theory [57,58]. Likewise, here we think the instantaneous process of photon helicity reversals may also have some potential applications in information technology and therefore deserve further investigation.

(iii) It is of physical interest to consider the quantum effects such as propagation of photons field, polarization of photon states (time evolution of photon wavefunction) and spontaneous emission decay rate of atoms [2] in left-handed media. In this section, an illustrative example of quantum effects of photons field resulting from helicity inversion caused by the LRH interfaces is presented. We hope the present consideration in this paper would open up new opportunities for investigating more quantum mechanical properties, phenomena and effects in left-handed media.

V. CONCLUDING REMARKS

We consider the bianisotropic structures in optical “constants” of left-handed media whereby the left-right coupling of circularly polarized light and cone angle independent geometric phases arises.

Nonadiabatic noncyclic geometric phases of photons in a noncoplanarly curved optical fiber fabricated from biaxially anisotropic left-handed media is considered in this paper. It is well known that the geometric phases of photons
inside a curved fiber in previous experiments often relate to the cone angles of solid angles subtended at the center. Here, however, we present a new geometric phase that is independent of the cone angles, by taking into account the peculiar properties of wave propagation in some anisotropic left-handed media. Some related topics (i.e., experimental realizations of quantum vacuum geometric phases of photons by using certain anisotropic (e.g., gyrotropic) left-handed materials, and photon geometric phases due to instantaneous helicity inversion on the interfaces between left- and right-handed media) are also discussed.

Geometric phases of circularly polarized light has some possible applications to quantum computation [35,36], since the wave propagation of coiled light is somewhat analogous to the behavior of nuclear magnetic resonance system [36]. Realizing quantum computation by means of geometric origin of adiabatic cyclic Berry’s phase is now receiving attention due to its intrinsic tolerance to noise. Recently, a conditional geometric phase shift gate, which is fault tolerant to certain types of errors due to its geometric nature, was realized via nuclear magnetic resonance (NMR) under adiabatic conditions [59]. However, the adiabatic conditions makes any fast conditional Berry’s phase shift impossible. So, more recently, Wang and Keiji suggested a scheme of nonadiabatic conditional geometric phase shift with NMR [36], the mathematical treatment of which is just the same as (3.4)-(3.6), so long as both the cone angle $\lambda$ and the precessional frequency $\dot{\gamma}$ (denoted by $\Omega$) are taken to be constant.

To summarize, in this paper we focus on some interesting properties of wave propagation of polarized photons in biaxially anisotropic left-handed materials, which has not been considered yet by the authors in the fields of classical optics, condensed matter physics and materials science. Since it is possible for current technology to fabricate uniaxially and biaxially anisotropic left-handed media in experiments, we hope the optical effects and phenomena discussed above would be investigated experimentally in the near future.

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Appendices

Appendix I. Three derivations of effective Hamiltonian of photons inside a noncoplanarly curved fiber

The effective Hamiltonian describing the light wave propagation in a curved optical fiber is helpful in considering the nonadiabatic noncyclic time evolution process of photon wavefunction in the fiber. We have three methods to derive this effective Hamiltonian (3.3).

Method i By using the infinitesimal rotation operator of wavefunction

The photon wavefunction $|\sigma, \mathbf{k}(t)\rangle$ varies as it rotates by an infinitesimal angle, say $\vec{\vartheta}$, namely, it obeys the following transformation rule

$$|\sigma, \mathbf{k}'(t)\rangle = \exp \left[ -i \vec{\vartheta} \cdot \mathbf{J} \right] |\sigma, \mathbf{k}(t)\rangle,$$  \hspace{1cm} (A1)

where $\exp \left[ -i \vec{\vartheta} \cdot \mathbf{J} \right] \approx 1 - i \vec{\vartheta} \cdot \mathbf{J}$ with $\mathbf{J}$ being the total angular momentum operator of photon and $\mathbf{k}'(t) = \mathbf{k}(t) + \Delta \mathbf{k}(t)$ with $\Delta \mathbf{k}(t) = \dot{\mathbf{k}} \Delta t$. Here $|\vec{\vartheta}\rangle$ is the angle between $\mathbf{k}(t)$ and $\mathbf{k}'(t)$, and the direction of $\vec{\vartheta}$ is parallel to that of $\mathbf{k}(t) \times \dot{\mathbf{k}}(t)$. One can therefore arrive at

$$\vec{\vartheta} = \frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2} = \frac{\mathbf{k}(t) \times \dot{\mathbf{k}}}{k^2} \Delta t.$$ \hspace{1cm} (A2)

Thus it follows from Eq.(A1) and (A2) that

$$\frac{\partial}{\partial t} |\sigma, \mathbf{k}(t)\rangle = \frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2} \cdot \mathbf{J} |\sigma, \mathbf{k}(t)\rangle,$$ \hspace{1cm} (A3)

by calculating the time derivative of $|\sigma, \mathbf{k}'(t)\rangle$. The total angular momentum is $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where the orbital angular momentum $\mathbf{L}$ is orthogonal to the linear momentum $\mathbf{k}$ for the photon. So, $\frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2} \cdot \mathbf{L} = 0$ and the only retained term
in $\frac{k(t) \times \dot{k}(t)}{k^2} \cdot J$ is $\frac{k(t) \times \dot{k}(t)}{k^2} \cdot S$. This, therefore, means that if we think of Eq.(A3) as the time-dependent Schrödinger equation governing the propagation of photons in the noncoplanar fiber, then we can obtain the effective Hamiltonian (3.3).

**Method ii**  By using the equation of motion of a photon

If the momentum squared $k^2$ of a photon moving in a noncoplanarly curved optical fiber is conserved, then we can derive the following identity

$$\dot{k} + k \times \left( \frac{k \times \dot{k}}{k^2} \right) = 0,$$

(A4)

which can be regarded as the equation of motion of a photon in the fiber. Since Eq.(A4) is exactly analogous to the equations of motion of a charged particle moving in a magnetic field or a spinning particle moving in a rotating frame of reference, $-\frac{k \times \dot{k}}{k^2}$ can be considered a “magnetic field” or “gravitomagnetic field” (thus $-k \times (\frac{k \times \dot{k}}{k^2})$ can be thought of as a “Lorentz magnetic force” or “Coriolis force”). Similar to the Mashhoon et al.’s work (i.e., the derivation of the interaction Hamiltonian of gravitomagnetic dipole moment in a gravitomagnetic field) [27,60], one can also readily write the Hamiltonian describing the coupling of the photon “gravitomagnetic moment” (i.e., photon spin $S$) [27] to the “gravitomagnetic field” as follows

$$H = \frac{k \times \dot{k}}{k^2} \cdot S,$$

(A5)

which is just the expression (3.3).

**Method iii**  By using the Liouville-Von Neumann equation

If a photon is moving inside a noncoplanarly curved optical fiber that is wound smoothly on a large enough diameter [38], then its helicity reversal does not easily take place [57] and the photon helicity $\frac{\hbar}{2} S$ is therefore conserved [37] and can thus be considered a Lewis-Riesenfeld invariant $I(t)$ [47], which agrees with the Liouville-Von Neumann equation (4.6). With the help of the spin operator commuting relations $S \times S = iS$, one can solve the Liouville-Von Neumann equation (4.6), namely, if the effective Hamiltonian is written as $H(t) = h \cdot S$, then according to the Liouville-Von Neumann equation, one can arrive at $[I(t), H(t)] = (\frac{\hbar}{2} \times h) \cdot iS$, and readily obtain the expression for the effective Hamiltonian (3.3) of photons in the curved optical fiber, i.e., the coefficients of the effective Hamiltonian is $h = \frac{k \times \dot{k}}{k^2}$.

**Appendix II. The invariant-related unitary transformation formulation**

In this appendix we briefly review the invariant-related unitary transformation formulation [48,49], which is important in investigating the nonadiabatic evolution process of time-dependent quantum systems (in Sec.IV). It has been shown in Lewis and Riesenfeld’s work [47] that the particular solution to the time-dependent Schrödinger equation, the Hamiltonian generators of which form a certain Lie algebra, is different from the eigenstate $|\lambda, t\rangle$ of the invariant $I(t)$ (which satisfies the Liouville-Von Neumann equation (4.6)) only by a time-dependent $c$-number factor $\exp\left[\frac{1}{i} \phi_\lambda(t)\right]$, where the time-dependent phase is [47]

$$\phi_\lambda(t) = \int_0^t \langle \lambda, t' | H(t') - i \frac{\partial}{\partial t'} \rangle |\lambda, t'\rangle dt'.$$

(A6)

It follows that we can obtain the solution to the time-dependent Schrödinger equation via solving the eigenstates of the eigenvalue equation $I(t)|\lambda, t\rangle = \lambda|\lambda, t\rangle$. But it is difficult to solve the eigenvalue equation of invariant $I(t)$ immediately, for the invariant is often the linear combination of certain Lie algebraic generators and, moreover, the coefficient factors in $I(t)$ are time-dependent. If we can find a unitary transformation $V(t)$ that transforms the eigenvalue equation $I(t)|\lambda, t\rangle = \lambda|\lambda, t\rangle$ into as follows

$$[V^\dagger(t) I(t) V(t)]|\lambda, t\rangle = \lambda V^\dagger(t)|\lambda, t\rangle,$$

(A7)

where $V^\dagger(t) I(t) V(t)$ is time-independent (see, for example, Eq.(4.10)), then the eigenstate of Eq.(A7) corresponding to the eigenvalue $\lambda$ can be easily obtained, which, say $|\lambda\rangle$, is also time-independent, namely, we have $|\lambda\rangle = V^\dagger(t)|\lambda, t\rangle$ or $|\lambda, t\rangle = V(t)|\lambda\rangle$. Correspondingly, the time-dependent phase (A7) is rewritten
\[
\phi_\lambda(t) = \int_0^t \langle \lambda | \left[ V^\dagger(t')H(t)V(t') - iV^\dagger(t') \frac{\partial}{\partial t'} V(t') \right] | \lambda \rangle dt'.
\]  

(A8)

Thus, by making use of the above unitary transformation method, we can obtain the exact solution to the time-dependent Schrödinger equation of quantum systems, the Hamiltonians of which possess some certain Lie algebraic structures.

Appendix III. The Baker-Campbell-Hausdorff formula

The Baker-Campbell-Hausdorff formula is given as follows

\[
V^\dagger(t) \frac{\partial}{\partial t} V(t) = \frac{\partial}{\partial t} L + \frac{1}{2!} \left[ \frac{\partial}{\partial t} L, L \right] + \frac{1}{3!} \left[ \left[ \frac{\partial}{\partial t} L, L \right], L \right] + \frac{1}{4!} \left[ \left[ \left[ \frac{\partial}{\partial t} L, L \right], L \right], L \right] + \cdots
\]  

(A9)

with \( V(t) = \exp[L(t)] \), \( V^\dagger(t) = \exp[-L(t)] \), which is of great importance for the calculation of nonadiabatic noncyclic geometric phases of time-dependent systems.

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