Privacy Amplification by Bernoulli Sampling

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Abstract

Balancing privacy and accuracy is a major challenge in designing differentially private machine learning algorithms. To improve this tradeoff, prior work has looked at privacy amplification methods which analyze how common training operations such as iteration and subsampling the data can lead to higher privacy. In this paper, we analyze privacy amplification properties of a new operation — sampling from the posterior — that is used in Bayesian inference. In particular, we look at Bernoulli sampling from a posterior that is described by a differentially private parameter. We provide an algorithm to compute the amplification factor in this setting, and establish upper and lower bounds on this factor. Finally, we look at what happens when we draw \( k \) posterior samples instead of one.

1 Introduction

Differential privacy \[12\] has emerged as the gold standard for privacy in machine learning. Differential privacy provides privacy by ensuring that the participation of a single person in the dataset does not change the probability of any outcome by much. This is achieved by adding noise to a function, such as a classifier or a generative model, computed on the sensitive data. Many private machine learning algorithms require noise addition at every step of an iterative process, which in turn, results in a substantial loss in accuracy or quality of the resulting output. Consequently, building a model from sensitive training data while maintaining a good privacy-accuracy tradeoff is a central challenge in differentially private machine learning.

To improve the privacy-accuracy tradeoff, a recent line of work has explored the idea of privacy amplification \[14\]. The key observation is that typical training methods involve complex yet stochastic operations such as randomly sampling a mini-batch or noisy iterations. If we can analyze and account for the inherent privacy gained through these operations, then we can improve the privacy-accuracy tradeoff by obtaining a tighter upper bound on the privacy loss of the entire training process. This has motivated analysis of the privacy amplification offered by a number of common operations such as subsampling \[1, 4–7, 17, 18, 20, 23, 25\] and stochastic gradient iterations \[2, 14\].

In this work, we initiate the study of privacy amplification properties of an entirely different operation: releasing a sample from a posterior distribution whose parameters have been obtained from sensitive data in a differentially private manner. This kind of sampling may be used, for example, in private Bayesian inference. We consider a specific kind of sampling called Bernoulli sampling that is simple enough to analyze, and yet captures the essence of the problem. Specifically, we are given a \( d \)-dimensional parameter vector \( \theta \in [0, 1]^d \), and would like to release a sample \( v \in \{0, 1\}^d \) where \( v \) is distributed as a product of Bernoullis — \( v_i = 1 \) with probability \( \theta_i \), 0 otherwise. The question we ask is:

If \( \theta \) is \((\alpha, \epsilon)\)-Rényi differentially private, then what is the privacy risk of releasing the sample \( v \)?
By post-processing invariance \( [1] [19] \), this risk is at most \( \epsilon \). In interesting situations, we expect the risk to be less, and this risk is our desired amplification factor. Observe that if \( \theta \) lies in \( \{0, 1\}^d \), then \( v = \theta \), and the amplification factor is exactly \( \epsilon \). To prevent this corner case, we restrict \( \theta \) to lie in \( [c, 1 - c]^d \) for a prespecified constant \( c \). This restriction can easily be satisfied by any algorithm which projects its output \( \theta \) to \([c, 1 - c]^d\). Other than this one restriction, we do not make any other assumption on the algorithm generating \( \theta \).

Even in this simple case, we find that there appears to be no closed form expression for the privacy amplification factor, and we provide an algorithm to directly compute it. Our algorithm takes as input parameters \( \epsilon, \alpha, c \) and \( d \), and outputs a factor \( \epsilon' \); it ensures that \( \epsilon' \) is an upper bound on the Rényi differential privacy risk of releasing \( v \), no matter what the private algorithm generating \( \theta \) is. Additionally, \( \epsilon' \) is optimal — there is at least one \( (\alpha, \epsilon) \) differentially private algorithm that generated \( \theta \) for which the risk of releasing \( v \) is exactly \( \epsilon' \). We characterize how much amplification our algorithm provides by providing upper and lower bounds on the amplification factor as a function of the constant \( c \), the dimension \( d \) and parameters \( \alpha \) and \( \epsilon \).

Many practical Bayesian inference algorithms use multiple posterior samples. We next consider what happens when instead of one, \( k \) Bernoulli samples \( v_1, \ldots, v_k \) are drawn and released from \( \theta \). We again provide an optimal algorithm for calculating the privacy amplification factor, and we provide upper and lower bounds on the risk obtained. As expected, as \( k \to \infty \), the privacy risk tends to \( \epsilon \), but we show that better risks can be obtained for smaller \( k \).

Finally, we validate our results by plotting the factor output by our algorithm as a function of the parameters \( c, \alpha, \epsilon, d \) and \( k \), and compare it with existing bounds. Our plots show that while there are regions where the amplification risk is close to \( \epsilon \), a number of parameter values do lead to interesting privacy amplification.

## 2 Preliminaries

Analyzing the privacy amplification when the output of an algorithm \( A \) is post-processed via Bernoulli sampling allows us to improve the differential privacy guarantee of \( A \). Here we define what we mean by privacy amplification. We introduce basic definitions and quantities, rehash the definition of Rényi differential privacy, define post-processing with Bernoulli sampling, and then define amplification.

### 2.1 Basic Definitions

We consider algorithms which have range in \( \Theta \subseteq \mathbb{R}^d \). \( \Theta \) is a set of parameters which will parametrize a Bernoulli distribution when we do Bernoulli post-processing.

We assume throughout the rest of this paper that \( \alpha \) is a real number bigger than 1 and \( \epsilon \) is a real number bigger than 0, even if we don’t state this explicitly.

**Definition 1.** \((R_\alpha(P, Q))\): For distributions \( P, Q \) on \( \Theta \), the Rényi divergence of order \( \alpha \) between \( P \) and \( Q \) denoted \( R_\alpha(P, Q) \), is

\[
R_\alpha(P, Q) = \frac{1}{\alpha - 1} \log \left( \int_{\Theta} dP^\alpha dQ^{1-\alpha} \right)
\]

The Rényi divergence enjoys many important, useful properties, such as quasi-convexity, independent composition, and being related to \( f \)-divergences \( [9] [22] \). One property is critical for our paper: the data processing inequality states that if two distribution \( P, Q \) on \( \Theta \) are post-processed by a randomized function \( M : \Theta \rightarrow \Theta' \), then \( R_\alpha(P, Q) \leq R_\alpha(MP, MQ) \) \( [22] \).

We can state some results of this paper cleanly using the Rényi Divergence between symmetric distributions supported on just two points \( \{x_1, x_2\} \subseteq \Theta \).

**Definition 2.** \((r_\alpha(p))\): Let \( \{x_1, x_2\} \subseteq \Theta \). Let \( P, Q \) be random variables with support on \( \{x_1, x_2\} \). Suppose \( p = \Pr[P = x_1] \) and \( 1 - p = \Pr[Q = x_1] \). Define the binary, symmetric Rényi divergence function \( r_\alpha(p) \) as follows:

\[
r_\alpha(p) = r_\alpha(P, Q) = \frac{1}{\alpha - 1} \log \left( p^\alpha (1 - p)^{1-\alpha} + (1 - p)^\alpha p^{1-\alpha} \right)
\]
2.2 Rényi Differential Privacy

Rényi differential privacy (RDP) \cite{RDP} is an alternative definition of differential privacy \cite{DP} that has enjoyed much recent interest because of tighter analyses for common operations such as subsampling and composition \cite{CompositionRDP}. RDP is defined using a binary symmetric relation on databases $D \sim D'$ which indicates whether $D, D'$ differ in the contribution of just one person.

**Definition 3.** (($\alpha, \varepsilon$)-RDP): Let $A$ be a randomized algorithm with range on $\Theta$ and $\alpha$ be a real number bigger than 1. $A$ satisfies $(\alpha, \varepsilon)$-RDP if $\sup_{D \sim D'} R_\alpha(A(D), A(D')) \leq \varepsilon$.

For a given $A$, one might want to know the smallest value of $\varepsilon$ such that $A$ satisfies $(\alpha, \varepsilon)$-RDP. The smallest possible value of $\varepsilon$ is the strongest RDP guarantee that $A$ can provide.

**Definition 4.** ($\varepsilon_A(\alpha)$): Let $\varepsilon_A(\alpha)$ be the smallest $\varepsilon$ for which $A$ satisfies $(\alpha, \varepsilon)$-RDP. This quantity is given by

$$\varepsilon_A(\alpha) = \sup_{D \sim D'} R_\alpha(A(D), A(D'))$$

Rényi differential privacy enjoys a crucial property that we will use later, as it directly relates to post-processing via Bernoulli sampling. Post-processing invariance \cite{RDP} states that if we apply a randomized function $M : \Theta \rightarrow \Theta'$ to the output of algorithm $A$, then $\varepsilon_{MA}(\alpha) \leq \varepsilon_A(\alpha)$. This follows directly from the data processing inequality.

2.3 Final Definitions and Problem Setup

In real-world deployments of privacy, data analysts have global constants $\varepsilon, \alpha$ and must deploy a private algorithm $A$ such that $\varepsilon_A(\alpha) \leq \varepsilon$. The type of amplification we study occurs when the analyst uses the output of $A$ as the parameter of a Bernoulli distribution and releases $k$ independent samples from the distribution as opposed to the output of $A$. We define the process that takes the output of $A$ and produces the $k$ samples as:

**Definition 5.** (Bernoulli process) Let $\text{Bern}(x)$ denote the randomized process which, for an input $x = (x_1, \ldots, x_d) \in [0, 1]^d$, flips $d$ coins with biases $(x_1, \ldots, x_d)$. Formally, this means for $b_1, \ldots, b_d \in \{0, 1\}^d$, $\Pr[\text{Bern}(x) = b_1, \ldots, b_d] = \prod_{i=1}^d x_i^{b_i}(1 - x_i)^{1 - b_i}$. Define $\text{Bern}_k(x)$ to be $k$ independent runs of $\text{Bern}(x)$.

We model the data analyst as an entity who chooses a private algorithm $A$ from a set of possible algorithms $\mathcal{A}$. Each algorithm in $\mathcal{A}$ must have range on $\Theta = [0, 1]^d$ to be compatible with the Bernoulli process. We define the Bernoulli post-processing amplification to be the worst-case value of $\varepsilon_{\text{Bern}}(A)(\alpha)$ out of all $A \in \mathcal{A}$, subject to the constraint that $\varepsilon_A(\alpha)$ is at most the global privacy constant $\varepsilon$.

**Definition 6.** ($\text{Post}_{A, \alpha, k}(\varepsilon)$): Given a family of algorithms $\mathcal{A}$ such that each algorithm has range on $\Theta = [0, 1]^d$, the The Bernoulli post-processing amplification function $\text{Post}_{A, \alpha, k}(\varepsilon)$ is given by

$$\text{Post}_{A, \alpha, k}(\varepsilon) = \sup_{A \in \mathcal{A}, \varepsilon_A(\alpha) \leq \varepsilon} \varepsilon_{\text{Bern}_k}(A)(\alpha)$$

We abbreviate $\text{Post}_{A, \alpha, 1}(\varepsilon)$ to $\text{Post}_{A, \alpha}(\varepsilon)$.

**Problem Setup** Using post-processing invariance, we can derive that $\text{Post}_{A, \alpha, k}(\varepsilon) \leq \varepsilon$. The goal of this work is to classify when $\text{Post}_{A, \alpha, k}(\varepsilon)$ is near and far from $\varepsilon$ making minimal assumptions about $\mathcal{A}$.

3 Single Sample Amplification Under Minimal Assumptions

We first study the simpler case $k = 1$, as many of the results will carry over. With the goal of comparing $\text{Post}_{A, \alpha}(\varepsilon)$ and $\varepsilon$ while making as few assumptions on $\mathcal{A}$ as possible, our first question is whether any assumptions are necessary at all. We find that there are algorithms with no amplification so we must make an assumption that excludes them. We make the smallest natural assumption we can think of, that $\mathcal{A}$ consists of all algorithms with range on $[c, 1 - c]^d$. Next, we partially characterize the subset of algorithms in $\mathcal{A}$ that have the worst amplification. This gives a method for computing a tight upper bound on $\text{Post}_{A, \alpha}(\varepsilon)$: search over all algorithms in the characterization, a much smaller search space than the whole of $\mathcal{A}$.
3.1 An Assumption on $\mathcal{A}$ is Necessary

Suppose $\mathcal{A}$ consists of all algorithms ranging over $\Theta = [0, 1]^d$. Unfortunately, some algorithms in $\mathcal{A}$ have no amplification. Specifically, let $A \in \mathcal{A}$ output just values in $\{0, 1\}^d$. Then, $\text{Bern}(A) = A$, so $\varepsilon_{\text{Bern}(A)}(\alpha) = \varepsilon_A(\alpha)$. At the very least, we must exclude from $\mathcal{A}$ algorithms that release distributions on $\{0, 1\}^d$. For $d = 1$, the natural step is to assume that $\Theta = [c, 1 - c]$ for a small constant $c < \frac{1}{2}$. For higher dimensions, we assume $\Theta = [c, 1 - c]^d$ to make our assumption symmetric on each coordinate. Both assumptions are quite light.

3.2 Amplification when Algorithms have Restricted Range

We consider the problem of computing $\text{Post}_{\mathcal{A}, \alpha}(\varepsilon)$ when $\mathcal{A}$ consists of algorithms with range on $\Theta = [c, 1 - c]^d$ and $c$ is a constant in the interval $[0, \frac{1}{2}]$. Naively, to compute $\text{Post}_{\mathcal{A}, \alpha}(\varepsilon)$, one would recognize that an $A \in \mathcal{A}$ can release any distributions on $\Theta$ on neighboring datasets. One would search over all distributions $P, Q$ on $\Theta$, subject to the constraints that $R_\alpha(P, Q) \leq \varepsilon$ and that $R_\alpha(Q, P) \leq \varepsilon$, to find the maximal value of $R_\alpha(\text{Bern}(P), \text{Bern}(Q))$. One could search this space by discretizing $\Theta$ into bins of width $\delta$ and then using $\Omega((\frac{1 - 2c}{\delta})^d)$ real-valued variables to represent the mass of $P, Q$ in each bin.

Instead, we make the observation that if $P, Q$ have mass on a point $x \in \Theta$ which is not $\{c, 1 - c\}^d$, we can move the mass to a point on $\{c, 1 - c\}^d$, and the following will hold: First, $R_\alpha(P, Q)$ will not increase due to post-processing invariance. Second, $R_\alpha(\text{Bern}(P), \text{Bern}(Q))$ will increase because out of all points in $\{c, 1 - c\}^d$, the points $\{c, 1 - c\}^d$ are the least random parameters for the Bernoulli distribution. The second statement requires proof. We define the following distributions:

**Definition 7.** $(c$-corner distributions): Let $C_d$ be the set of distributions supported on $\{c, 1 - c\}^d$.

The following theorem establishes that the $P, Q$ which maximize $R_\alpha(\text{Bern}(P), \text{Bern}(Q))$ subject to $R_\alpha(P, Q), R_\alpha(Q, P) \leq \varepsilon$ are a subset of the $c$-corner distributions.

**Theorem 1.** Let $\mathcal{A}$ be the set of all algorithms with output on $\{c, 1 - c\}^d$. Then, for all $\alpha > 0, \varepsilon \geq 0$,

$$\text{Post}_{\mathcal{A}, \alpha}(\varepsilon) = \sup\limits_{P, Q \in C_d} \sup\limits_{R_\alpha(P, Q), R_\alpha(Q, P) \leq \varepsilon} R_\alpha(\text{Bern}(P), \text{Bern}(Q))$$

**Remarks.** This theorem gives the exact value of $\text{Post}_{\mathcal{A}, \alpha}(\varepsilon)$; it is not an upper bound. An explicit $A \in \mathcal{A}$ such that $\varepsilon_{\text{Bern}(A)}(\alpha) = \text{Post}_{\mathcal{A}, \alpha}(\varepsilon_A(\alpha))$ is the $A$ such that $A(D) = P; A(D') = Q; P, Q \in C$ maximize the sup of the theorem statement, and $D, D'$ are arbitrary neighboring databases. However, the $c$-corner distributions are a partial characterization of the worst-case distributions since not all will have the worst-case value of $R_\alpha(\text{Bern}(P), \text{Bern}(Q))$.

Computing the optimization problem of Theorem 1 requires searching over a much smaller space than the naive method. As opposed to using $\Omega((\frac{1 - 2c}{\delta})^d)$ variables as the naive method does, we can write the problem using $2^{O(d)}$ variables. We can evaluate the constraints and the objective in $2^{O(d)}$ time. Finally, each constraint and the objective are convex, so we can use an iterative or convex method to solve the optimization problem for smaller $d$. The proofs of these claims appear in Appendix A.

4 Single Sample Amplification Lower and Upper Bounds

While Algorithm 1 is exact, it does not help us intuitively understand how far away $\text{Post}_{\mathcal{A}, \alpha}(\varepsilon)$ is from $\varepsilon$ due to the conceptually and computationally difficult optimization problem. To remedy these matters, we derive one lower and two upper bounds on $\text{Post}_{\mathcal{A}, \alpha}(\varepsilon)$ that are much easier to compute and to understand. By computing the lower bounds and the upper bounds, we can find a range on which $\text{Post}_{\mathcal{A}, \alpha}(\varepsilon)$ lies. Our hope is that this range is small. As a first step to proving this, we present evidence that at reasonable values of $c, d, \alpha$, the first upper bound is close for high values of $\varepsilon$ and the second is close for low to moderate values of $\varepsilon$.

4.1 Upper and Lower Bounds for $\text{Post}_{\mathcal{A}, \alpha}(\varepsilon)$

We outline two upper bounds and one lower bound on $\text{Post}_{\mathcal{A}, \alpha}(\varepsilon)$ that are easier to compute and understand than Algorithm 1. Our first upper bound follows from the post-processing inequality,
Algorithm 1: Algorithm for computing Post$_{A,\alpha}(\varepsilon)$ when $A$ consists of algorithms with range on $[c, 1-c]^d$. $\Delta(x, y)$ is the Hamming distance between $x, y$.

**Input:** Constant $c$, privacy parameter $\varepsilon$, order $\alpha$, dimension $d$.

**Output:** Post$_{A,\alpha}(\varepsilon)$

1. $\text{Constraint}_1 \leftarrow \sum_{i \in \{0,1\}}^d (x_i/y_i)^\alpha y_i$;
2. $\text{Constraint}_2 \leftarrow \sum_{i \in \{0,1\}}^d (y_i/x_i)^\alpha x_i$;

   for $b \in \{0,1\}^d$ do
   
   $\mathfrak{p}_b \leftarrow \sum_{i \in \{0,1\}}^d x_i c^{\Delta(i,b)}(1-c)^{d-\Delta(i,b)}$;
   $\mathfrak{y}_b \leftarrow \sum_{i \in \{0,1\}}^d y_i c^{\Delta(i,b)}(1-c)^{d-\Delta(i,b)}$;

end

3. $\text{Objective} \leftarrow \sum_{b \in \{0,1\}^d} (\mathfrak{p}_b/\mathfrak{y}_b)^\alpha \mathfrak{y}_b$;
4. $\text{MaxVal} \leftarrow \text{maximize}(\text{Objective}) \text{ subject to } \text{Constraint}_1 \leq e^{(\alpha-1)\varepsilon}, \text{Constraint}_2 \leq e^{(\alpha-1)\varepsilon}$, $\{x_i \geq 0\}, \{y_i \geq 0\}, \sum_{i=0}^{2^d-1} x_i = 1, \sum_{i=0}^{2^d-1} y_i = 1$;
5. return $\frac{1}{\alpha-1} \log(\text{MaxVal})$;

Post$_{A,\alpha}(\varepsilon) \leq \varepsilon$. We refer to this bound as the *post processing upper bound*. Our second upper bound follows from the observation, immediate from (1), that Post$_{A,\alpha}(\varepsilon)$ is an increasing function of $\varepsilon$. This means, as $\varepsilon \to \infty$, Post$_{A,\alpha}(\varepsilon)$ will either increase without bound or approach an asymptote that is also an upper bound. Looking at Theorem 1 even if $R_\alpha(P,Q) = \infty$ for $P, Q \in \mathcal{C}_d$, $R_\alpha(\text{Bern}(P), \text{Bern}(Q))$ is still finite because both $\text{Bern}(P)$ and $\text{Bern}(Q)$ will have at least $c$ mass on both 0 and 1. Therefore, Post$_{A,\alpha}(\varepsilon)$ approaches an asymptote:

$$\lim_{\varepsilon \to \infty} \text{Post}_{A,\alpha}(\varepsilon) = R(\text{Bern}(\{c\}^d), \text{Bern}((1-c)^d)) = d r_\alpha(c) \tag{2}$$

The proof of this appears in Appendix B Theorem 5. For the lower bound, we simply plug two distributions $P, Q \in \mathcal{C}_d$ into Theorem 1 where

$$P = p \mathbb{I}[X = \{c\}^d] + (1-p) \mathbb{I}[X = \{1-c\}^d] \tag{3}$$

$$Q = (1-p) \mathbb{I}[X = \{c\}^d] + p \mathbb{I}[X = \{1-c\}^d] \tag{4}$$

We choose $p \leq \frac{1}{2}$; the other case is symmetric. Notice $R_\alpha(P,Q) = R_\alpha(Q,P) = r_\alpha(p)$. This results in the lower bound:

$$R_\alpha(\text{Bern}(P), \text{Bern}(Q)) \leq \text{Post}_{A,\alpha}(r_\alpha(p)) \tag{5}$$

Furthermore, there is a way to compute $R_\alpha(\text{Bern}(P), \text{Bern}(Q))$ efficiently in all parameters including $d$ (Appendix B Theorem 4). While we can’t prove it here, we conjecture that (5) is actually an equality for all $p \in (0, \frac{1}{2}]$. Proving this would give a full characterization of distributions $P, Q$ which provide the worst amplification (see Theorem 1 remarks).

### 4.2 Cases Where Upper and Lower Bounds are Close

Computing the upper and lower bounds of last section is an easy way to give a range on Post$_{A,\alpha}(\varepsilon)$. As $\varepsilon$ gets large, the lower bound (5) approaches the upper bound (2) since (2) is an asymptote. Another observation is that if the dimension $d$ is large, then using $P, Q$ described in (3) and (4), Bern$(P)$ and Bern$(Q)$ will barely mix $P$ and $Q$, so $R_\alpha(\text{Bern}(P), \text{Bern}(Q))$ will be close to $R_\alpha(P,Q)$.

**Theorem 2.** Let $p \in (0, \frac{1}{2}]$, $P, Q$ be the distributions in (3) and (4). Let $K = e^{-2(1/2-c)^2d}$. If $p + K \leq \frac{1}{2}$, then $R_\alpha(\text{Bern}(P), \text{Bern}(Q)) \geq r_\alpha(p + K)$.

**Remarks.** The crux of the proof is to apply a Hoeffding-style argument to show that Bern$(P)$ and Bern$(Q)$ hardly mix the two points in the support of $P, Q$.

Thus, for all $p \in (0, \frac{1}{2}]$, if $K$ exists, we have

$$r_\alpha(p + K) \leq R_\alpha(\text{Bern}(P), \text{Bern}(Q)) \leq \text{Post}_{A,\alpha}(r_\alpha(p)) \leq r_\alpha(p)$$
We interpret this theorem in the following natural way: We can always make $2(\frac{1}{2} - c)^2d$ large enough so that $p + K < \frac{1}{2}$ and so that $r_{\alpha}(p + K) \approx r_{\alpha}(p)$. If $p$ is "not too close" to 0 and from $\frac{1}{2}$, then the conditions will be met for a rather moderate value of $2(\frac{1}{2} - c)^2d$ due to the negative exponential dependence of $K$. For instance, for all $p \in [0.01, 0.49]$, at $\alpha = 50$, if $2(\frac{1}{2} - c)^2d > 20$, then $K < 10^{-8}$ and $r_{\alpha}(p + K), r_{\alpha}(p)$ will be very close. These values of $p$ correspond to all $\varepsilon \in [0.026, 4.59]$, so $Post_{A,\alpha}(\varepsilon) \approx \varepsilon$ for all $\varepsilon$ in that interval.

5 The Amplification Picture for Multiple Samples

Many Bayesian inference algorithms release more than one sample from the posterior. This motivates studying $Post_{A,\alpha,k}(\varepsilon)$, as it describes the amplification when $A$ is a family of posterior updating algorithms for the Bernoulli posterior. There is no obvious way to extend the single-sample results to $k$ samples. One may be tempted to apply composition of RDP [19], but here the private algorithm is run only once — its output is used to release $k$ samples. In this section, we give $k$ sample analogues to results in Sections 3 and 4. We give an algorithm that computes $Post_{A,\alpha,k}(\varepsilon)$ exactly. Then, we derive upper and lower bounds on $Post_{A,\alpha,k}(\varepsilon)$ that are efficiently computable.

While it is more difficult to visualize than when $k = 1$, the $c$-corner distributions are once again the worst distributions that $A$ can output, for the same reason: if $P, Q$ have mass on a point not in $\{c, 1 - c\}^d$, then moving the mass at that point to a point in $\{c, 1 - c\}^d$ can only decrease $R_\alpha(P, Q)$ but increases $R_\alpha(\text{Bern}(P), \text{Bern}(Q))$ (see Section 3 for more details). Because of this, the following theorem, while a generalization to Theorem [1] is not conceptually more complicated.

**Theorem 3.** Let $A$ be the set of all algorithms with output on $[c, 1 - c]^d$. Then, for all $\alpha > 1$, $\varepsilon \geq 0$,

$$Post_{A,\alpha,k}(\varepsilon) = \sup_{P, Q \in \mathcal{C}_d} R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q))$$

This implies we can compute $Post_{A,\alpha}(\varepsilon)$ by just changing the optimization function of Algorithm 1.

The fully general algorithm and the proof of correctness and convexity appear in Appendix A. We note the optimization function uses $2O(d)$ variables and takes $2O(kd)$ time to compute, so it is not efficient in either $d$ or $k$. When $d = 1$, we can take advantage of independence and symmetry to reduce the runtime to $O(k)$. This result is Algorithm 2 and a full proof of correctness and convexity lies in Appendix A.

**Algorithm 2:** Algorithm for computing $Post_{A,\alpha,k}(\varepsilon)$ when $A$ consists of algorithms with range on $[c, 1 - c]$ and $k$ is general.

**Input:** Constant $c$, privacy parameter $\varepsilon$, order $\alpha$, no. samples $k$

**Output:** $Post_{A,\alpha,k}(\varepsilon)$

1. $Constraint_1 \leftarrow r_\alpha(x, y)$;
2. $Constraint_2 \leftarrow r_\alpha(y, x)$;
3. for $0 \leq j \leq k$
   - $\pi_j \leftarrow x_0^j + (1 - x)(1 - c)^{k-j}$;
   - $\eta_j \leftarrow y_0^j + (1 - y)(1 - c)^{k-j}$;
4. $Objective \leftarrow \sum_{j=0}^{k} \left( \binom{k}{j} \pi_j^{\alpha} \eta_j^{\alpha-\varepsilon} \right)$;
5. $MaxVal \leftarrow \text{maximize}(Objective) \text{ subject to } Constraint_1 \leq e^{\alpha-1} \varepsilon, Constraint_2 \leq e^{\alpha-1} \varepsilon, 0 \leq x \leq 1, 0 \leq y \leq 1$;
6. return $\frac{1}{\alpha-1} \log(MaxVal)$

To upper bound $Post_{A,\alpha,k}(\varepsilon)$, we can use the post-processing upper bound and the following asymptotic value of $Post_{A,\alpha,k}(\varepsilon)$, which is similar to that in [2] (proof in Appendix B, Theorem 5).

$$\lim_{\varepsilon \to \infty} Post_{A,\alpha,k}(\varepsilon) = R(\text{Bern}_k([c]^d), \text{Bern}_k([1 - c]^d)) = dkr_\alpha(c)$$

We use the same $P, Q$ defined in [3] and [4] for our $k$ sample lower bound:

$$R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q)) \leq Post_{A,\alpha,k}(r_\alpha(p))$$

(7)
In the \( k \)-sample setting, using these specific \( P, Q \) for our lower bound is especially nice because there is an efficient way to compute \( R_k(\text{Bern}_k(P), \text{Bern}_k(Q)) \) (Appendix \( B \) Theorem \( 3 \)) which is not immediately obvious due to the increased complexity that \( k \) samples brings. Once again, we conjecture that equality actually holds in (7). We can compute a range for \( \text{Post}_{A,\alpha,k}(\epsilon) \) efficiently using (6) and (7). While we do not generalize Theorem \( 2 \), we note that the values of \( \epsilon, c, d, \) and \( \alpha \) that result in \( \text{Post}_{A,\alpha}(\epsilon) \approx \epsilon \) also result in \( \text{Post}_{A,\alpha,k}(\epsilon) \approx \epsilon \) because \( \epsilon \geq \text{Post}_{A,\alpha,k}(\epsilon) \geq \text{Post}_{A,\alpha}(\epsilon) \).

6 Validation

In the last three sections, we derived an algorithm that computes \( \text{Post}_{A,\alpha,k}(\epsilon) \) that is inefficient in \( k, d \). We’ve explored two upper bounds—the post-processing bound \( \text{Post}_{A,\alpha,k}(\epsilon) \leq \epsilon \) and the asymptotes (2) (6)—and a lower bound, (7) (5). All bounds are efficiently computable. These bounds give us a range on \( \text{Post}_{A,\alpha,k}(\epsilon) \) which we hope is small. We’ve also seen that at reasonable values of \( c, d, \) and \( \alpha \), if \( \epsilon \) is small to moderate, then \( \text{Post}_{A,\alpha,k}(\epsilon) \) is very close to \( \epsilon \), and if \( \epsilon \) is high, then \( \text{Post}_{A,\alpha,k}(\epsilon) \) is very close to (2). In this section, we explore through plots whether the range between the upper and lower bounds is always small and when the two upper bounds are close to \( \text{Post}_{A,\alpha,k}(\epsilon) \). We then answer the related question of at what values of \( \epsilon \) is \( \text{Post}_{A,\alpha,k}(\epsilon) \) far from \( \epsilon \). Finally, we find no cases where the lower bound (7) and \( \text{Post}_{A,\alpha,k}(\epsilon) \) are not equal, so our conjecture that they are equal has plausibility.

Setup. For \( k = 1 \), we plot \( \text{Post}_{A,\alpha}(\epsilon) \), the lower bound (5), the upper bound \( d \rho(\epsilon) \) (2), and the post-processing upper bound as functions of \( \epsilon \). We produce a plot for \( d \in \{1, 2, 3, 5, 15\} \), \( c \in \{0.01, 0.10, 0.30\} \), and \( \alpha \in \{5, 50\} \). For \( d > 3 \), we do not plot \( \text{Post}_{A,\alpha}(\epsilon) \) due to the runtime of Algorithm 1. The plots for \( \alpha = 50 \), \( d \in \{1, 2, 3\} \) appear in Figures 2a and the plots for \( \alpha = 50 \), \( d \in \{5, 15\} \) appear in 2b. The plots for \( \alpha = 5 \) are in Appendix \( C \). For \( k > 1 \), we plot \( \text{Post}_{A,\alpha,k}(\epsilon) \), the lower bound (7), the upper bound \( d k \rho(\epsilon) \) (6), and the post-processing upper bound. We fix \( c = 0.1 \) and \( \alpha = 50 \) and vary \( k \in \{1, 2, 4\} \) and \( d \in \{1, 3, 5\} \). These plots appear in Appendix \( C \). Due to numerical instability, some of the plotted values do not extend across the whole \( \epsilon \) domain.

Results. The plots for \( \alpha = 5 \) and \( \alpha = 50 \) look almost identical, so the following conclusions hold for both values. For all plots, there are three notable Regimes of \( \epsilon \): In Regime I, \( \text{Post}_{A,\alpha,k}(\epsilon) \) is close to \( \epsilon \). As predicted by Theorem \( 2 \), Regime I lasts from a very small value of \( \epsilon \) to a moderate value, and lasts longer with smaller \( c \) or larger \( d \). Often Regime I lasts until \( \epsilon \) is unreasonably large. Only when \( (d, k, c) \in \{(\leq 5, 1, 0.3),(\leq 2, 1, 0.1),(1, 1, 0.01),(1, 2, 0.1)\} \) does Regime I end before \( \epsilon = 5 \) which we consider to be an upper limit for a reasonable \( \epsilon \). In Regime II, \( \text{Post}_{A,\alpha,k}(\epsilon) \) is transitioning between the two upper bounds. The range between the upper and lower bounds is highest out of all \( \epsilon \) but is never higher than about 1.5. The larger range is not an inaccurate estimate of \( \text{Post}_{A,\alpha,k}(\epsilon) \) unless Regime I ends at a reasonable value of \( \epsilon \), at which point an uncertainty of 1.5 is huge compared to \( \epsilon \). Regime II doesn’t last very long, but appears to last longer with higher \( k \). In Regime III, \( \text{Post}_{A,\alpha,k}(\epsilon) \) has converged to its asymptote of \( d k \rho(\epsilon) \) (2). Regime III starts soon after Regime I ends due to Regime II being relatively short.

For most plots, \( \text{Post}_{A,\alpha,k}(\epsilon) \) is much smaller than \( \epsilon \) only well into Regime III, at which point \( \epsilon \) is quite large. When \( (d, k, c) \in \{(\leq 3, 1, 0.3),(1, 1, 0.1)\} \), Regime I doesn’t last long, and \( \text{Post}_{A,\alpha,k}(\epsilon) \) is much smaller than \( \epsilon \) for nearly all values of \( \epsilon \). These cases many not include most of the \( (d, k, c) \) values we consider, but still encompass many nontrivial algorithms.

7 Related Work

The area of differential privacy amplification is relatively new. The literature on privacy amplification can be organized by the type of post-processing mechanism under consideration. First, amplification by iteration [14] or diffusion [2] occurs through repeated applications of Markov operators to the output of a private algorithm. While Bernoulli post-processing is a Markov process, the fact that we focus on this specific problem allows us to obtain a tighter result. Second, amplification by subsampling [1,14,17,18,20,23,25] occurs when a sample from the dataset is input to the private algorithm, rather than the whole dataset itself. Clearly, this problem setup is different from ours. Third, privacy amplification by shuffling [3,8,13] occurs when differential privacy guarantees are strengthened under a different privacy model, the shuffle model. We do not consider the shuffle model here, but it would be an interesting area of future research.
As Bernoulli post-processing is related to posterior sampling, the Bayesian machine learning literature has considered problems related to ours. Specifically, there are results about the differential privacy guarantees of posterior sampling under different classes of prior-posterior distribution families [10, 15, 16, 24]. Our method takes these results one step further by considering the amplification in when samples from the private posterior, rather the private posterior itself, are released.

Our work is also related to the literature on strong data processing inequalities in the information theory community [21]. However, the difference is that those results apply to $f$-divergences and general processes while we focus on Rényi divergences and the Bernoulli post-processing mechanism.

8 Conclusion

We initiate the study of privacy amplification for a new setup: when the output of a private algorithm is used as the parameter for a distribution, and only samples from the distribution are released. Focusing on the Bernoulli distribution, we characterize the amount of amplification that occurs at different parameters—notably, the dimension of the private algorithm, the strength of an assumption we make about the private algorithm, and the number of samples released. To do this, we find an algorithm which enables us to compute the amplification for some parameters. We prove bounds on the amplification, showing there is an algorithm which has almost no amplification for many high-privacy settings, but on the other hand, there are low-privacy settings where all algorithms have high amplification. Our bounds give a narrow range for how much amplification occurs. Finally, we validate these results by computing the amplification and the bounds for a selection of the parameters.
9 Broader Impacts

As more and more sensitive data are collected from us, the need to protect user privacy becomes a central issue. Failure to protect privacy can expose people, often from vulnerable groups, to serious repercussions such as insurance discrimination, identity theft, and personal distress. The most promising privacy definition, differential privacy, prevents many types of privacy breaches but has a notable downside. There is a necessary tradeoff between data utility and strength of the privacy guarantee. Privacy amplification can tell us when a program written using private algorithms has a stronger privacy guarantee than the individual algorithms. This allows for a better tradeoff between privacy and accuracy. We provide the first result in privacy amplification via sampling with a private parameter. This and future results in this area will give many new situations in which the privacy guarantee can be strengthened.

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we have left is to express the objective function $R$.

Algorithm 3:

\[
\text{Algorithm for computing Post}\_\alpha,\alpha,k(\varepsilon)\text{ when }\mathcal{A}\text{ consists of algorithms with range on }[c, 1 - c]^d. \Delta(x, y)\text{ is the Hamming distance.}
\]

**Input:** Constant $c$, privacy parameter $\varepsilon$, order $\alpha$, dimension $d$, no. samples $k$.

**Output:** $\text{Post}_{\mathcal{A},\alpha,k}(\varepsilon)$

\[
\begin{align*}
\text{Constraint}_1 & \leftarrow \sum_{i \in \{0,1\}^d} (x_i / y_i)\alpha y_i; \\
\text{Constraint}_2 & \leftarrow \sum_{i \in \{0,1\}^d} (y_i / x_i)\alpha x_i; \\
\text{for} & \ (b_1, \ldots, b_k) \in \{\{0,1\}^d\}^k \text{ do} \\
\quad & \quad \quad \text{if } \sum_{i \in \{0,1\}^d} (x_i / b_i) \prod_{j=1}^d (1 - c)^{\Delta(i, b_j)} (1 - c)^{\Delta(i, b_j)}; \\
\quad & \quad \quad \text{end} \\
\text{Objective} & \leftarrow \sum_{(b_1, \ldots, b_k) \in ((0,1)^d)^k} (\prod_{i \in \{0,1\}^d} (x_i / b_i) \prod_{j=1}^d (1 - c)^{\Delta(i, b_j)} (1 - c)^{\Delta(i, b_j)}; \\
\text{MaxVal} & \leftarrow \text{maximize(Ojective) subject to Constraint}_1 \leq e^{(\alpha - 1)}e, \text{Constraint}_2 \leq e^{(\alpha - 1)}e, \{x_i \geq 0\}, \{y_i \geq 0\}, \sum_{i=0}^{d-1} x_i = 1, \sum_{i=0}^{d-1} y_i = 1; \\
\text{return} & \frac{1}{\alpha - 1} \log(\text{MaxVal}); \\
\end{align*}
\]

Recall that $\Delta(x, y)$ is the Hamming distance between binary strings $x, y$. Algorithm 3 shows the fully general algorithm computing $\text{Post}_{\mathcal{A},\alpha}(\varepsilon)$ given $d, k, c, \varepsilon,$ and $\alpha$. We prove the algorithm computes

\[
\sup_{P, Q \in \mathcal{L}_d} R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q)) \\
R_\alpha(P, Q) \leq \varepsilon \\
R_\alpha(Q, P) \leq \varepsilon
\]

which is equal to $\text{Post}_{\mathcal{A},\alpha,k}(\varepsilon)$, by Theorem 3. Let $z_i$ for $i = (i_1, \ldots, i_d) \in \{0,1\}^d$ be the point $\{c^{i_1} (1 - c)^{1-i_1}, \ldots, c^{i_d} (1 - c)^{1-i_d}\}$. Let $\{x_i : i \in \{0,1\}^d\}$ represent the mass that $P$ places at $z_i$, and let $\{y_i : i \in \{0,1\}^d\}$ be defined similarly for $Q$. Plugging directly into the definition of definition of Rényi divergence, $\text{Constraint}_1$ and $\text{Constraint}_2$ represent the constraints $R_\alpha(P, Q) \leq \varepsilon$ and $R_\alpha(Q, P) \leq \varepsilon$. The function

\[
\sum_{i \in \{0,1\}^d} \left( \frac{x_i}{y_i} \right)^\alpha y_i
\]

is an $f$-divergence, and is therefore a convex function [9]. The constraints that the $x_i$ and $y_i$ form two probability distributions are obvious and are included in the last line of the algorithm. The only step we have left is to express the objective function $R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q))$. For $b_1, \ldots, b_k$, each in
Algorithm 2 follows by using the variable $k$.
Algorithm 1 follows from specializing Algorithm 3 to $0$. This only depends on the number of $b$ constraints slightly.

The algorithm has these equations exactly, but it uses the variables $\{0,1\}$. It suffices to optimize

$$e^{(\alpha-1)R_\alpha(\text{Bern}_k(P),\text{Bern}_k(Q))} = \sum_{b_1,\ldots,b_k \in \{0,1\}} \Pr[\text{Bern}_k(P) = (b_1,\ldots,b_k)]^\alpha \Pr[\text{Bern}_k(Q) = (b_1,\ldots,b_k)]^{1-\alpha} \quad (8)$$

The algorithm has these equations exactly, but it uses the variables

$$\bar{x}_{b_1,\ldots,b_k} = \Pr[\text{Bern}_k(P) = (b_1,\ldots,b_k)]$$
$$\bar{y}_{b_1,\ldots,b_k} = \Pr[\text{Bern}_k(Q) = (b_1,\ldots,b_k)]$$

The objective function is convex in the variables $\bar{x}_{b_1,\ldots,b_k}$ and $\bar{y}_{b_1,\ldots,b_k}$, and each $\bar{x}_{b_1,\ldots,b_k}$ and $\bar{y}_{b_1,\ldots,b_k}$ is an affine transformation of the input variables $x_i$ and $y_i$. Therefore, Algorithm 3 is a convex optimization problem. The runtime bottleneck is in evaluating the objective function which takes $2O(dk)$ time.

Algorithm 1 follows from specializing Algorithm 3 to $k = 1$. When $d = 1$, there is more simplification. For $b_1,\ldots,b_k$, each in $\{0,1\}$,

$$\Pr[\text{Bern}_k(P) = (b_1,\ldots,b_k)] = \sum_{i \in \{0,1\}} x_i \prod_{j=1}^k c^{\Delta(i,b_j)} (1-c)^{1-\Delta(i,b_j)}$$

This only depends on the number of 0s in $(b_1,\ldots,b_k)$. Summing (8) over the number of possible zeros in $b_1,\ldots,b_k$, we get

$$e^{(\alpha-1)R_\alpha(\text{Bern}_k(P),\text{Bern}_k(Q))} = \sum_{j=0}^k \binom{k}{j} (x_0 c^j (1-c)^{k-j} + x_1 c^{k-j} (1-c)^j)^\alpha (y_0 c^j (1-c)^{k-j} + y_1 c^{k-j} (1-c)^j)^{1-\alpha}$$

Algorithm 2 follows by using the variable $(x,1-x)$ in place of $x_0,x_1$ and changing the other constraints slightly.

**B Proofs**

**Lemma 1.** The function $r_\alpha(p) = \frac{1}{\alpha-1} \log (p^\alpha (1-p)^{1-\alpha} + (1-p)^\alpha p^{1-\alpha})$ for $0 \leq p \leq 1$ is convex.

**Proof.** We can write

$$r_\alpha(p) = \frac{1}{\alpha-1} \log (p^\alpha (1-p)^{1-\alpha}) + \frac{1}{\alpha-1} \log (1 + (1-p)^{2\alpha-1} p^{1-2\alpha})$$
Proof. (Of Theorem 1) Recall that

\[
\text{Post}_{A, \alpha, k}(\varepsilon) = \sup_{A \in \mathcal{A}, \varepsilon(A) \leq \varepsilon} \epsilon_{\text{Bern}_k(A)}(\alpha)
\]

For a fixed \(A\) such that \(\varepsilon_A(\alpha) \leq \varepsilon\), we have, where \(\text{supp}(P)\) is the support of distribution \(P\),

\[
\epsilon_{\text{Bern}_k(A)}(\alpha) \leq \sup_{\text{supp}(P'), \text{supp}(Q) \subseteq \Theta} R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q))
\]

This is because an \(A \in \mathcal{A}\) always releases \(P\) and \(Q\) supported on \(\Theta\) such that \(R_\alpha(P, Q)\) and \(R_\alpha(Q, P)\) are less than \(\varepsilon\), meaning that \(\epsilon_{\text{Bern}_k(A)}(\alpha)\) is less than the sup above.

A weakening of Lemma 2 tells us that for any \(P, Q\), we can actually find \(c\)-corner distributions \(P', Q'\) such that \(R_\alpha(P', Q') \leq R_\alpha(P, Q)\). Therefore,

\[
\sup_{\text{supp}(P'), \text{supp}(Q) \subseteq \Theta} R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q)) = \sup_{P, Q \in \mathcal{C}_d} R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q))
\]

This means \(\epsilon_{\text{Bern}_k(A)}(\alpha)\) is upper bounded by the sup above for all \(A \in \mathcal{A}\). However, there is an \(A \in \mathcal{A}\) for which equality holds: If \(P, Q \in \mathcal{C}_d\) maximize the sup above, then \(A\) just releases \(A(D) = P\) and \(A(D') = Q\) for two neighboring databases \(D, D'\).

Lemma 2. Let the space \(\Theta = \prod_{i=1}^d (c_i, d_i)\) for \(0 < c_i < d_i < 1\). Let \(\Delta = \prod_{i=1}^d \{c_i, d_i\}\). Let \(P, Q\) be distributions on \(\Theta\). Then, there exist distributions \(P', Q'\) on \(\Delta\) such that \(R_\alpha(P', Q') \leq R_\alpha(P, Q)\), \(R_\alpha(Q', P') \leq R_\alpha(Q, P)\), and \(R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q)) = R_\alpha(\text{Bern}_k(P'), \text{Bern}_k(Q'))\).

Proof. We will prove this theorem on arbitrarily fine discretizations of \(P\) and \(Q\). We now assume \(P\) and \(Q\) are discrete. Suppose \(P, Q\) place mass on a point \(x \notin \Delta\). By Lemma 3, we can write, for coefficients \(\{\lambda_z\}_{z \in \Delta}\) of a convex combination,

\[
\text{Bern}_k(x) = \sum_{z \in \Delta} \lambda_z \text{Bern}_k(z)
\]

For any \(x \in \Theta\), we have \(\text{Bern}_k(x) = \text{Bern}_k(1[X = x])\). Thus,

\[
\text{Bern}_k(1[X = x]) = \sum_{z \in \Delta} \lambda_z \text{Bern}_k(1[X = z])
\]

Notice \(\text{Bern}_k\) is a Markov operator, so it factors across sums:

\[
\sum_{z \in \Delta} \lambda_z \text{Bern}_k(1[X = z]) = \text{Bern}_k\left(\sum_{z \in \Delta} \lambda_z 1[X = z]\right) = \text{Bern}_k(m)
\]

where \(m\) is the probability distribution that takes value \(z \in \Delta\) w.p. \(\lambda_z\). Thus, we conclude \(\text{Bern}_k(m) = \text{Bern}_k(1[X = x]) = \text{Bern}_k(x)\). Let

\[
P' = P - \text{Pr}[P = x]1[X = x] + \text{Pr}[P = x]m
\]

\[
Q' = Q - \text{Pr}[Q = x]1[X = x] + \text{Pr}[Q = x]m
\]

Then, using the Markov property of \(\text{Bern}_k\) again,

\[
\text{Bern}_k(P') = \text{Bern}_k(P) - \text{Pr}[P = x]\text{Bern}_k(1[X = x]) + \text{Pr}[P = x]\text{Bern}_k(m)
\]

\[
\text{Bern}_k(Q') = \text{Bern}_k(Q) - \text{Pr}[Q = x]\text{Bern}_k(1[X = x]) + \text{Pr}[Q = x]\text{Bern}_k(m)
\]

Because \(\text{Bern}_k(m) = \text{Bern}_k(1[X = x])\), the above equations simplify to \(\text{Bern}_k(P') = \text{Bern}_k(P)\) and \(\text{Bern}_k(Q') = \text{Bern}_k(Q)\). Hence, \(R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q)) = R_\alpha(\text{Bern}_k(P'), \text{Bern}_k(Q'))\). However, \(P', Q'\) are a post-processing of \(P, Q\); if we observe \(X = x\), then we sample from \(m\) instead. By the data processing inequality, \(R_\alpha(P', Q') \leq R_\alpha(P, Q)\) and \(R_\alpha(Q', P') \leq R_\alpha(Q, P)\).
Lemma 3. Let $\Theta, \Delta$ be defined as in Lemma \ref{lem:convex_combination}. For a point $x \in \Theta$, we can write $\text{Bern}_k(x)$ as the convex combination $\sum_{z \in \Delta} \lambda_z \text{Bern}_k(z)$ for coefficients $\{\lambda_z : z \in \Delta\}$.

Proof. $\text{Bern}_k(z)$ has the same distribution on each of its $k$ samples for any $z \in \Theta$. Thus, it suffices to prove the theorem for $k = 1$. Let $A \otimes B$ be the product distribution of $A$ and $B$. By definition,

$$\text{Bern}_1(x) = \bigotimes_{i=1}^d \text{Bern}(x_i)$$

Because $c_i \leq x_i \leq d_i$, there is some $\lambda_i$ such that $\text{Bern}(x_i) = \lambda_i \text{Bern}(c_i) + (1 - \lambda_i) \text{Bern}(d_i)$. Therefore, we can write

$$\text{Bern}_k(x) = \bigotimes_{i=1}^d \lambda_i \text{Bern}(c_i) + (1 - \lambda_i) \text{Bern}(d_i)$$

It is well known that the product of two convex combinations of distribution is itself a convex distribution. Each term of this convex combination will be $\text{Bern}(z_1) \otimes \cdots \otimes \text{Bern}(z_d)$ for $z_i \in \{c_i, d_i\}$. This is equal to $\text{Bern}((z_1, \ldots, z_d))$, and $(z_1, \ldots, z_d) \in \Delta$. \hfill \square

Proof. (Of Theorem \ref{thm:main}): Recall $P = p1[X = \{c\}^d] + (1 - p)1[X = \{1 - c\}^d]$ and $Q = (1 - p)1[X = \{c\}^d] + p1[X = \{1 - c\}^d]$. $P, Q$ and $\text{Bern}(P)$, $\text{Bern}(Q)$ are isomorphic pairs of distributions under flipping by flipping the 0s and 1s in their domains. Thus, $R_\alpha(P, Q) = R_\alpha(Q, P)$ and $R_\alpha(\text{Bern}(P), \text{Bern}(Q)) = R_\alpha(\text{Bern}(Q), \text{Bern}(P))$.

We let $\#_0(x)$ be the number of 0s appearing in binary vector $x$ and $\overline{p} = 1 - p$. We write the following lower and upper bound on the probability that $\text{Bern}(P)$ has many 1s:

$$\Pr[\#_1(\text{Bern}(P)) \geq d/2] \geq \Pr[P = \{1 - c\}^d] \Pr[\#_1(\text{Bern}(\{1 - c\}^d)) \geq d/2]$$

$$\geq \overline{p}(1 - \Pr[\#_1(\text{Bern}(\{1 - c\}^d)) < d/2])$$

$$\Pr[\#_1(\text{Bern}(P)) \geq d/2] \leq \Pr[P = \{1 - c\}^d] + \Pr[P = \{c\}^d] \Pr[\#_1(\text{Bern}(\{c\}^d)) > d/2]$$

$$\leq \overline{p} + p \Pr[\#_1(\text{Bern}(\{c\}^d)) > d/2]$$

Hoeffding’s inequality tells us that

$$\Pr[\#_1(\text{Bern}(\{1 - c\}^d)) < d/2] \leq e^{-2(1/2 - c)^2d}$$

$$\Pr[\#_1(\text{Bern}(\{c\}^d)) > d/2] \leq e^{-2(1/2 - c)^2d}$$

Therefore, with $K = e^{-2(1/2 - c)^2d}$,

$$\overline{p}(1 - K) \leq \Pr[\#_1(\text{Bern}(P)) \geq d/2] \leq \overline{p} + pK$$

Let Maj be the majority function, so that

$$\text{Maj}(\text{Bern}(P)) = \Pr[\#_1(\text{Bern}(P)) \geq d/2] 1[X = 1] + (1 - \Pr[\#_1(\text{Bern}(P)) \geq d/2]) 1[X = 0]$$

$$= \text{Bern}(\Pr[\#_1(\text{Bern}(P)) \geq d/2])$$

From our above analysis, $\text{Maj}(\text{Bern}(P)) = \text{Bern}(p')$ for some $p' \in [\overline{p}(1 - K), \overline{p} + pK]$. Relax the interval to $p' \in \overline{p} \pm K$. Because $\text{Bern}(Q)$ is isomorphic to $\text{Bern}(P)$ by flipping the 0s and 1s in the domain, we also have $\text{Maj}(\text{Bern}(Q)) = \text{Bern}(1 - p')$. Therefore,

$$R_\alpha(\text{Bern}(P), \text{Bern}(Q)) \geq R_\alpha(\text{Maj}(\text{Bern}(P)), \text{Maj}(\text{Bern}(Q))) = r_\alpha(p')$$

Since $r_\alpha(p)$ has one local minimum at $p = \frac{1}{2}$, the minimum value it achieves on the interval $[a, b]$ assuming $\frac{1}{2} \leq a \leq b$ is is achieved at $a$. The lower edge of the interval $\overline{p} \pm K$ is bigger than $\frac{1}{2}$ since $p + K < \frac{1}{2}$. Thus,

$$r_\alpha(p') \geq \inf_{x \in \overline{p} \pm [K, K]} r_\alpha(x) \geq r_\alpha(\overline{p} - K) = r_\alpha(p + K)$$

\hfill \square
Theorem 4. Let
\[
P = pI[X = \{c\}^d] + (1 - p)I[X = \{1 - c\}^d]
\]
\[
Q = (1 - p)I[X = \{c\}^d] + pI[X = \{1 - c\}^d]
\]
be distributions. Let \( P_j = pe^j(1 - c)^{dk-j} + (1 - p)c^{dk-j}(1 - c)^j \) and \( Q_j = (1 - p)c^j(1 - c)^{dk-j} + pc^{dk-j}(1 - c)^j \). Then,
\[
R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q)) = \frac{1}{\alpha - 1} \log \left( \sum_{j=0}^{dk} \binom{dk}{j} P_j^\alpha Q_j^{1-\alpha} \right)
\]

Proof. We have, for \( x_1, \ldots, x_k \) each in \( \{0,1\}^d \),
\[
\Pr[\text{Bern}_k(P) = (x_1, \ldots, x_k)] = p \Pr[\text{Bern}_k(\{c\}^d) = (x_1, \ldots, x_k)]
+ (1 - p) \Pr[\text{Bern}_k(\{1 - c\}^d) = (x_1, \ldots, x_k)]
= pe \sum \#_i(x_i) (1 - c)^{dk} \sum \#_i(x_i)
+ (1 - p)c^{dk} \sum \#_i(x_i) (1 - c) \sum \#_i(x_i)
\]

Where the first equality follows from conditioning on \( P \) and the second from independence of \( \text{Bern}_k \) across coordinates and of \( \text{Bern}(\{c\}^d), \text{Bern}(\{1 - c\}^d) \) across their \( d \) dimensions. Thus, the mass of \( \text{Bern}_k(P) \) at a point in \( \{0,1\}^{dk} \) with Hamming weight \( j \) is \( P_j \). For \( \text{Bern}_k(Q) \), it is \( Q_j \). There are \( \binom{dk}{j} \) points in \( \{0,1\}^{dk} \) with Hamming weight \( j \), so the result follows by the definition of Rényi divergence.

Theorem 5. \( \text{Post}_{A,\alpha,k}(\varepsilon) \leq R(\text{Bern}_k(\{c\}^d), \text{Bern}_k(\{1 - c\}^d)) = dkr_{\alpha}(c) \).

Proof. We get rid of some constraints of the \( \sup \) of Theorem 4 getting
\[
\text{Post}_{A,\alpha,k}(\varepsilon) \leq \sup_{P, Q \in \mathcal{C}_d} R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q))
\]
A general \( P, Q \in \mathcal{C}_d \) can be written as
\[
P = \sum_{i \in \{c, 1 - c\}^d} p_i 1[X = i]
\]
\[
Q = \sum_{i \in \{c, 1 - c\}^d} q_i 1[X = i]
\]
By the Bernoulli process is a Markov process, applying Bern to both sides gives
\[
\text{Bern}_k(P) = \sum_{i \in \{c, 1 - c\}^d} p_i \text{Bern}_k(i)
\]
\[
\text{Bern}_k(Q) = \sum_{i \in \{c, 1 - c\}^d} q_i \text{Bern}_k(i)
\]
Quasi-convexity of the Rényi Divergence \cite{22} states that for distributions \( P_1, \ldots, P_n \) and \( Q_1, \ldots, Q_n \), and a convex combination \( \lambda_1, \ldots, \lambda_n \),
\[
R_\alpha(\lambda_1 P_1 + \cdots + \lambda_n P_n, \lambda_1 Q_1 + \cdots + \lambda_n Q_n) \leq \max_{i=1}^n \lambda_i R_\alpha(P_i, Q_i)
\]
Here, our convex combinations \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_n \) are different, but we can pair them up as follows: if \( p_i \) the smallest nonzero coefficient out of all \( p_i \) and \( q_i \), then pair it with an arbitrary nonzero \( q_j \). Set \( p_i = 0 \) and \( q_j = q_j - p_i \). This process will terminate eventually, and we will be left with the equality
\[
\left( \sum_{i \in \{c, 1 - c\}^d} p_i \text{Bern}_k(i), \sum_{i \in \{c, 1 - c\}^d} q_i \text{Bern}_k(i) \right) = \sum_{i \in \{c, 1 - c\}^d} \lambda_i (\text{Bern}_k(i), \text{Bern}_k(i))
\]
By quasi-convexity,
\[
R_\alpha(\text{Bern}_k(P), \text{Bern}_k(Q)) \leq \max_{i,j \in [c,1-c]^d} R_\alpha(\text{Bern}_k(i), \text{Bern}_k(j))
\]

Because \(\text{Bern}_k(i)\) are independent across the \(d\) coordinates of \(i\), for \(i = (i_1, \ldots, i_d)\) and \(j = (j_1, \ldots, j_d)\), and by the additive property of Rényi divergence across product distributions [22],

\[
R_\alpha(\text{Bern}_k(i), \text{Bern}_k(j)) = \sum_{\ell=1}^d R_\alpha(\text{Bern}_k(i_\ell), \text{Bern}_k(j_\ell))
\]

Each \(R_\alpha(\text{Bern}_k(i_\ell), \text{Bern}_k(j_\ell))\) is zero when \(i_\ell = j_\ell\) and equal to \(kr_\alpha(c)\) otherwise, again using the additive property of Rényi divergence across the \(k\) product distributions. Therefore,

\[
\sum_{\ell=1}^d R_\alpha(\text{Bern}_k(i_\ell), \text{Bern}_k(j_\ell)) \leq dkr_\alpha(c)
\]

\(\square\)

C Supplementary Graphs

![Supplementary Graphs](image)

(a) Comparison of \(\text{Post}_{A,\alpha}(\varepsilon)\), lower bound (5) (LB), post-process. u.b. (PPI), and asymptote u.b. (2).

(b) Comparison of lower bound (5) (LB), post-processing u.b. (PPI), and asymptote u.b. (2).

Figure 2: \(\text{Post}_{A,\alpha}(\varepsilon)\), upper bounds, and lower bound as functions of \(\varepsilon\) when \(A\) consists of algorithms whose range is contained in \([c, 1-c]^d\), \(k = 1\), and \(\alpha = 5\).
Figure 3: Comparison of $\text{Post}_{A,\alpha,k}(\varepsilon)$ (only for $d = 1$), lower bound on $\text{Post}_{A,\alpha}(\varepsilon)$ \cite{7} (LB), the post-processing upper bound (PPI), and asymptote upper bound \cite{6} (Asympt.) when $A$ consists of algorithms whose range is contained in $[c, 1 - c]^d$, $\alpha = 50$, $k \in \{1, 2, 4\}$. 