DEFORMATIONS OF $G_2$ AND Spin(7) STRUCTURES ON MANIFOLDS

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Abstract. We consider some deformations of $G_2$-structures on 7-manifolds. We discover a canonical way to deform a $G_2$-structure by a vector field in which the associated metric gets “twisted” in some way by the vector cross product. We present a system of partial differential equations for an unknown vector field $w$ whose solution would yield a manifold with holonomy $G_2$. Similarly we consider analogous constructions for Spin(7)-structures on 8-manifolds. Some of the results carry over directly, while others do not because of the increased complexity of the Spin(7) case.

Changes made after publication: There were sign errors in equations (2.26) and (2.34) and in the proof of Lemma 2.4.3 that have now been corrected in this arXiv version. Also Remark 3.3.7 was incorrect and has been removed.

Contents

1. Introduction 2
  1.1. Cross Product Structures 2
  1.2. Overview of New Results 4
  1.3. Notation and Conventions 6
  1.4. Acknowledgements 6
  2. Manifolds with a $G_2$-structure 6
    2.1. $G_2$-structures 6
    2.2. Decomposition of $\bigwedge^*(M)$ into irreducible $G_2$-representations 7
    2.3. The metric of a $G_2$-structure 9
    2.4. The cross product of a $G_2$-structure 11
    2.5. The 16 classes of $G_2$-structures 13
  3. Deformations of a fixed $G_2$-structure 16
    3.1. Conformal Deformations of $G_2$-structures 16
    3.2. Deforming $\varphi$ by an element of $\bigwedge^3_1$ 19
    3.3. Infinitesimal deformations in the $\bigwedge^3_1$ direction 24
  4. Manifolds with a Spin(7)-structure 28
    4.1. Spin(7)-structures 28
    4.2. Decomposition of $\bigwedge^*(M)$ into irreducible Spin(7)-representations 29
    4.3. The metric of a Spin(7)-structure 31
    4.4. The triple cross product of a Spin(7)-structure 34
    4.5. The 4 classes of Spin(7)-structures 39
  5. Deformations of a fixed Spin(7)-structure 40
    5.1. Conformal Deformations of Spin(7)-structures 40

Date: May 16, 2019.

1991 Mathematics Subject Classification. 53, 58.

Key words and phrases. $G_2$, $\text{Spin}(7)$, holonomy, metrics, cross product.
1. INTRODUCTION

1.1. Cross Product Structures. Special structures on manifolds in differential geometry are usually described by globally defined smooth sections of some tensor bundle, satisfying some pointwise algebraic conditions, and with potentially stronger global requirements. One such example of an additional structure that can be imposed on a smooth Riemannian manifold $M$ of dimension $n$ is that of an $r$-fold cross product. This is an alternating $r$-linear smooth map

$$B : TM \times \ldots \times TM \rightarrow TM$$

that is compatible with the metric in the sense that

$$|B(e_1, \ldots, e_r)|^2 = |e_1 \wedge \ldots \wedge e_r|^2$$
$$\langle B(e_1, \ldots, e_r), e_j \rangle = 0 \quad 1 \leq j \leq r$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric. Such a cross product also gives rise to an $(r + 1)$-form $\alpha$ given by

$$\alpha(e_1, \ldots, e_r, e_{r+1}) = \langle B(e_1, \ldots, e_r), e_{r+1} \rangle$$

Cross products on real vector spaces were classified by Brown and Gray in [2]. Global vector cross products on manifolds were first studied by Gray in [14]. They fall into four categories:

1. When $r = n - 1$ and $\alpha$ is the volume form of the manifold. Under the metric identification of vector fields and one forms, this cross product corresponds to the Hodge star operator on $(n - 1)$-forms. This is not an extra structure beyond that given by the metric.
2. When $n = 2m$ and $r = 1$, we can have a one-fold cross product $J : TM \rightarrow TM$. Such a map satisfies $J^2 = -I$ and is an almost complex structure. The associated 2-form is the Kähler form $\omega$.
3. The first of two exceptional cases is a 2-fold cross product on a 7-manifold. Such a structure is called a $G_2$-structure, and the associated 3-form $\varphi$ is called a $G_2$-form.
4. The second exceptional case is a 3-fold cross product on an 8-manifold. This is called a Spin(7)-structure, and the associated 4-form $\Phi$ is called a Spin(7)-form.

In cases 2–4 the existence of these structures is a topological condition on $M$ given in terms of characteristic classes (see [14, 24, 26]). One can also study the restricted sub-class of such manifolds where the associated differential form $\alpha$ is parallel with respect to the Levi-Civita connection $\nabla$. In case (1), the volume form is always parallel. For the almost complex structures $J$ of case (2), $\nabla J = 0$ if and only if the manifold is Kähler, which is equivalent to $d\omega = 0$ and the almost complex structure...
is integrable. In this case, the Riemannian holonomy of the manifold is a subgroup of $U(m)$. For cases (3) and (4), the condition that the differential form be parallel is a non-linear differential equation. Manifolds with parallel $G_2$-structures have holonomy a subgroup of $G_2$ and manifolds with parallel Spin(7)-structures have holonomy a subgroup of Spin(7), hence their names. One can also show (see [1]) that such manifolds are all Ricci-flat.

There is a subclass of the Kähler manifolds which are Ricci-flat. Such manifolds possess a global non-vanishing holomorphic volume form $\Omega$ in addition to the Kähler form $\omega$, and these two forms satisfy some relation. These manifolds are called Calabi-Yau manifolds as their existence was demonstrated by Yau’s proof of the Calabi conjecture [27]:

**Theorem 1.1.1** (Calabi-Yau, 1978). Let $M$ be a compact complex manifold with vanishing first Chern class $c_1 = 0$. Then if $\omega$ is a Kähler form on $M$, there exists a unique Ricci-flat Riemannian metric $g$ on $M$ whose associated Kähler form is in the same cohomology class as $\omega$.

This theorem characterises those manifolds admitting Calabi-Yau metrics in terms of certain topological information. The equivalence is demonstrated by writing the Ricci-flat condition as a partial differential equation and proving existence and uniqueness of solutions. Calabi-Yau manifolds have holonomy a subgroup of $SU(m)$ and are characterized by two parallel forms, $\omega$ and $\Omega$. In fact, they possess two parallel cross products: a 1-fold cross product $J$, and a complex analogue of case (1) above, where $\Omega$ plays the role of the volume form and the $(m - 1)$-fold cross product is a complex Hodge star.

Calabi-Yau manifolds (at least in complex dimension 3) have long been of interest in string theory. More recently, manifolds with holonomy $G_2$ and Spin(7) have also been studied. (See, for example, [4, 6, 21, 22, 23, 19, 20, 17]). It would be useful to have an analogue of the Calabi-Yau theorem, or something similar, in the $G_2$ and Spin(7) cases. There is a significant difference, however, which makes $G_2$ and Spin(7) manifolds much more difficult to study. An almost complex structure $J$ does not by itself determine a metric. If we also have a Riemannian metric, then together the compatibility requirement yields the Kähler form $\omega(u, v) = g(Ju, v)$. In contrast, a 2-fold or 3-fold cross product structure does determine the metric uniquely, and thus also determines the associated 3-form $\varphi$ or 4-form $\Phi$. Because the metric and complex structure are “uncoupled” in the Calabi-Yau case, we can start with a fixed integrable complex structure $J$, and then look for different metrics (which correspond to different Kähler forms for the same $J$) which are Ricci-flat and make $J$ parallel. As $J$ is already integrable, it is parallel precisely when $\omega$ is closed, so we can simply look at different metrics which all correspond to closed Kähler forms, and from that set look for a Ricci-flat metric. Hence we can restrict ourselves to starting with a Kähler manifold, and looking at other Kähler metrics which could be Ricci-flat. The Calabi-Yau theorem then says that there exists precisely one such metric in each cohomology class which contains at least one Kähler metric.

In the $G_2$ and Spin(7) cases, however, we cannot fix a cross product structure and then vary the metric to make it parallel. For a given cross product, the metric is determined. In the Calabi-Yau case, we can start with $U(m)$ holonomy and describe the conditions for being able to obtain $SU(m)$ holonomy. For $G_2$ and Spin(7), there is no intermediate starting class. A crucial ingredient in the proof of
the Calabi-Yau theorem is the $\partial \bar{\partial}$ lemma, which allows us to write the difference of any two Kähler forms in terms of an unknown function $f$. Therefore as a first step towards an analogous result in the $G_2$ and Spin(7) cases, we would like to determine the simplest data required to describe the relations between any two $G_2$ or Spin(7) forms.

1.2. Overview of New Results. If we start with only a $G_2$-structure, not necessarily parallel, this gives us a 3-form which satisfies some “positive-definiteness” property, since it determines a Riemannian metric. In [11], Fernández and Gray classified such manifolds by looking at the decomposition of $\nabla \varphi$ into $G_2$-irreducible components. There are 16 such classes, with various inclusion relations between them. There is a similar decomposition in [16] of almost complex manifolds into subclasses. Some of these classes are: integrable (complex), symplectic, almost Kähler, and nearly Kähler. Thus these 16 subclasses of manifolds with a $G_2$-structure are analogues of these “weaker than Kähler” conditions. Similar studies by Fernández in [9] of the Spin(7) case yield 4 subclasses of manifolds with a Spin(7)-structure.

As a first step in trying to determine an analogue for the Calabi conjecture in the $G_2$ case, we can study these various weaker subclasses and their deformations. If we start in one class, and change the 3-form $\varphi$ in some way (which changes the metric too) we would like to know under what conditions this subclass is preserved, or more generally what subclass the new $G_2$-structure now belongs to. The space of 3-forms on a manifold with a $G_2$-structure decomposes into a direct sum of irreducible $G_2$-representations:

$$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$$

where $\Lambda^3_k$ is a $k$-dimensional vector space at each point on $M$. This decomposition depends on our initial 3-form $\varphi_0$, however. This again is in stark contrast to the decomposition on a complex manifold into forms of type $(p, q)$, which depends only on the complex structure and does not change as we vary the Kähler (or metric) structure. We can consider a deformation $\tilde{\varphi} = \varphi_0 + \eta$ of the $G_2$-structure, for $\eta \in \Lambda^3_k$ and determine conditions on $\varphi_0$ and $\eta$ which preserve the subclass or change it in an interesting way.

If $\eta \in \Lambda^3_1$, this corresponds to a conformal scaling of the metric, and one can explicitly describe which of the 16 classes are conformally invariant. (These results were already known to Fernández and Gray but here they are reproduced in a different way.) A new result in this case is the following:

**Theorem 1.2.1.** Let $\theta_o = \ast_o (\ast_o d\varphi_o \wedge \varphi_o)$ be the canonical 1-form arising from a $G_2$-structure $\varphi_o$. Then if $\tilde{\varphi} = f^3 \varphi_o$ for some non-vanishing function $f$, the new canonical 1-form $\tilde{\theta}$ differs from the old $\theta_o$ by an exact form:

$$\tilde{\theta} = -12 d(\log(f)) + \theta_o$$

Thus in the classes where $\theta$ is closed, (there are some and they are conformally invariant classes), we get a well-defined cohomology class in $H^1(M)$, invariant under conformal changes of metric. A similar result also holds in the Spin(7) case.

If, however, we deform $\varphi_0$ by an element $\eta \in \Lambda^3_7$, then $\eta = w \ast_o \varphi_o$ for some vector field $w$, and in Section 3.2 we prove the following:
**Theorem 1.2.2.** Under such a deformation $\tilde{\varphi}$ is again a $G_2$-structure and the new metric on vector fields $v_1$ and $v_2$ is given by

$$\langle v_1, v_2 \rangle = \frac{1}{(1 + |w|^2_o)} \left( \langle v_1, v_2 \rangle_o + \langle w \times v_1, w \times v_2 \rangle_o \right)$$

where $\times$ is the vector cross product associated to the original $G_2$-structure $\varphi_o$.

From this one can write down non-trivial differential equations on the vector field $w$ for certain subclasses to be preserved. It would be interesting to solve some of these equations for the unknown vector field $w$. This would mean that there were certain distinguished vector fields on some classes of manifolds with $G_2$-structures.

The important result here, however, is that the new 3-form $\tilde{\varphi}$ is always positive-definite. That is, it always corresponds to a $G_2$-structure. This gives information about the structure of the open set $\wedge^3(M)$ of positive definite 3-forms on $M$.

If instead we deform $\varphi$ in the $\wedge^4_7$ direction infinitesmally by the flow equation

$$\frac{\partial}{\partial t} \varphi_t = w \lrcorner \varphi_t$$

then we show in Section 3.3 that the metric $g$ does not change and also:

**Theorem 1.2.3.** The solution is given by

$$\varphi(t) = \varphi_0 + \frac{1 - \cos(|w|t)}{|w|^2} (w \lrcorner (w \lrcorner \varphi_0)) + \frac{\sin(|w|t)}{|w|} (w \lrcorner \varphi_0)$$

Hence the solution exists for all time and is a closed path in $\wedge^3(M)$. Also, the path only depends on the unit vector field $\pm w/|w|$, and the norm $|w|$ only affects the speed of travel along this path.

In [6] the fact that the space of $G_2$-structures which correspond to the same metric as a fixed $G_2$-structure yields an $\mathbb{RP}^7$ bundle over $M$ is mentioned. This is the content of the above theorem, and we provide an explicit description of these $G_2$-structures in terms of vector fields on $M$. In addition, in the special cases of $M = N \times S^3$ or $M = L \times T^3$, where $N$ is a Calabi-Yau 3-fold and $L$ is a $K3$ surface, we show that this closed path of $G_2$-structures corresponds to the freedom of changing the phase of the holomorphic volume form $\Omega \mapsto e^{it} \Omega$ on $N$ or performing a hyperKähler rotation on $L$. Thus this theorem can be seen as a generalization of these two situations.

The same kind of analysis can be done in the Spin(7) case. Similar but more complicated results hold in this case and are presented in Section 5. Here there are only 4 subclasses but the decomposition of $\wedge^4$ into irreducible Spin(7)-representations is more complicated:

$$\wedge^4 = \wedge^4_1 \oplus \wedge^4_2 \oplus \wedge^4_{27} \oplus \wedge^4_{35}$$

In this case it is the space $\wedge^4_7$ which infinitesmally gives a closed path of Spin(7)-structures all corresponding to the same metric. However, perhaps initially somewhat surprisingly, this time non-infinitesimal deformations in the $\wedge^4_7$ direction do not yield a new Spin(7)-structure. This is explained in detail in Section 5.2. Much of the construction does indeed carry over, however, and it may be possible to alter it somehow to make it work.
1.3. Notation and Conventions. Many of the calculations that follow use various relations between the interior product $\iota$, the exterior product $\wedge$, and the Hodge star operator $\ast$. Readers unfamiliar with this can refer to Appendix A. The appendix also contains some preliminary results about determinants that are used repeatedly in many of the proofs that follow.

In much of the computations there are two metrics present: an old metric $g_o$ and a new metric $\tilde{g}$. Their associated volume forms, induced metrics on differential forms, and Hodge star operators are also identified by a subscript $o$ for old or a $\tilde{}$ for new. We also often use the metric isomorphism between vector fields and one-forms, and denote this isomorphism by $\flat$ for the one-form associated to the vector field $w$ and $\sharp$ for the vector field associated to the one-form $\alpha$. In the presence of two metrics, this isomorphism is always only used for the old metric $g_o$.

1.4. Acknowledgements. A significant portion of this paper was the author’s doctoral thesis at Harvard University under the supervision of S.T. Yau. For helpful discussions the author would like to thank R.L. Bryant, H.B. Chttp://hefin, S.L. Kong, C.C. Liu, J. Rasmussen, S.T. Yau, and X.W. Wang. The author’s research was partially funded by the Natural Sciences and Engineering Research Council of Canada and le Fonds pour la Formation de Chercheurs et l’Aide à la Recherche.

2. Manifolds with a $G_2$-structure

2.1. $G_2$-structures. Let $M$ be an oriented 7-manifold with a global 2-fold cross product structure. Such a structure will henceforth be called a $G_2$-structure. Its existence is a topological condition, given by the vanishing of the second Stiefel-Whitney class $w_2 = 0$. (See [14, 24, 26] for details.) This cross product $\times$ gives rise to an associated Riemannian metric $g$ and an alternating 3-form $\varphi$ which are related by:

\begin{equation}
\varphi(u, v, w) = g(u \times v, w).
\end{equation}

This should be compared to the relation between a Kähler metric $\omega$ and a compatible almost complex structure $J$:

\begin{equation}
\omega(u, v) = g(Ju, v).
\end{equation}

Note that in the Kähler case, the metric and the almost complex structure can be prescribed independently. This is not true in the case of manifolds with a $G_2$-structure, and this leads to some complications (and the inherit non-linearity of the problem). For a $G_2$-structure $\varphi$, near a point $p \in M$ we can choose local coordinates $x^1, \ldots, x^7$ so that at the point $p$, we have:

\begin{equation}
\varphi_p = dx^{123} - dx^{167} - dx^{527} - dx^{563} + dx^{415} + dx^{426} + dx^{437}
\end{equation}

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$. In these coordinates the metric at $p$ is the standard Euclidean metric

\begin{equation}
g_p = \sum_{k=1}^{7} dx^k \otimes dx^k
\end{equation}

and the Hodge star dual $\ast \varphi$ of $\varphi$ is

\begin{equation}
(\ast \varphi)_p = dx^{4567} - dx^{4523} - dx^{4163} - dx^{4127} + dx^{2637} + dx^{1537} + dx^{1526}
\end{equation}
Remark 2.1.1. Different conventions exist in the literature for (2.2) and (2.3), which may or may not differ from our choice by renumbering of coordinates and/or a change of orientation.

The 3-forms on $M$ that arise from a $G_2$-structure are called positive 3-forms or non-degenerate. We will denote this set by $\wedge^3_{\text{pos}}$. The subgroup of SO(7) that preserves $\varphi_p$ is the exceptional Lie group $G_2$. This can be found for example in [4, 18]. Hence at each point $p$, the set of $G_2$-structures at $p$ is isomorphic to $\text{GL}(7, \mathbb{R})/G_2$, which is $49 - 14 = 35$ dimensional. Since $\wedge^3(\mathbb{R}^7)$ is also 35 dimensional, the set $\wedge^3_{\text{pos}}(p)$ of positive 3-forms at $p$ is an open subset of $\wedge^3_{\text{pos}}$. We will determine some new information about the structure of $\wedge^3_{\text{pos}}$ in Section 3.2.

Remark 2.1.2. Note that in the Spin(7) case the situation is very different. The set of 4-forms on an 8-manifold $M$ that determine a Spin(7)-structure is not an open subset of $\wedge^4(M)$. This is discussed in Section 4.1.

2.2. Decomposition of $\wedge^*(M)$ into irreducible $G_2$-representations. All of the facts collected in this section are well known and more details can be found in [11, 24, 26].

The group $G_2$ acts on $\mathbb{R}^7$, and hence acts on the spaces $\wedge^*$ of differential forms on $M$. One can decompose each space $\wedge^k$ into irreducible $G_2$-representations. The results of this decomposition are presented below (see [11, 26]). The notation $\wedge^k_l$ refers to an $l$-dimensional irreducible $G_2$-representation which is a subspace of $\wedge^k$.

Also, “vol” will denote the volume form of $M$ (determined by the metric $g$), and $w$ is a vector field on $M$.

\[
\begin{align*}
\wedge^1_0 &= \{ f \in C^\infty(M) \} \\
\wedge^2 &= \wedge^2_7 \oplus \wedge^2_{14} \\
\wedge^4 &= \wedge_4^1 \oplus \wedge_4^3 \oplus \wedge^4_{27} \\
\wedge^6 &= \{ w \cdot \text{vol} \} \\
\wedge^3_7 &= \{ f \text{ vol}; f \in C^\infty(M) \}
\end{align*}
\]

Since $G_2 \subset \text{SO}(7)$, the decomposition respects the Hodge star $*$ operator, and $*\wedge^k_l = \wedge^{7-k}_l$. Nevertheless, we will still describe the remaining cases explicitly, in several ways, as all the descriptions will be useful for us.

Before we describe $\wedge^k_l$ for $k = 2, 3, 4, 5$, let us describe some isomorphisms between these subspaces:

Proposition 2.2.1. The map $\alpha \mapsto \varphi \wedge \alpha$ is an isomorphism between the following spaces:

\[
\begin{align*}
\wedge^0_1 &\cong \wedge^3_1 \\
\wedge^2_7 &\cong \wedge^4_7 \\
\wedge^1_1 &\cong \wedge^3_{14} \\
\wedge^1_4 &\cong \wedge^4_1 \\
\wedge^4_7 &\cong \wedge^4_7 \\
\wedge^6_7 &\cong \wedge^4_7 \\
\wedge^7_7 &\cong \wedge^4_1
\end{align*}
\]

The map $\alpha \mapsto * \varphi \wedge \alpha$ is an isomorphism between the following spaces:

\[
\begin{align*}
\wedge^0_1 &\cong \wedge^4_1 \\
\wedge^1_1 &\cong \wedge^5_1 \\
\wedge^2_7 &\cong \wedge^6_7 \\
\wedge^3_7 &\cong \wedge^7_7
\end{align*}
\]
In addition, if $\alpha$ is a 1-form, we have the following identities:

\begin{align}
\ast (\varphi \wedge (\varphi \wedge \alpha)) &= -4\alpha \\
\ast \varphi \wedge \ast (\varphi \wedge \alpha) &= 0 \\
\ast (\ast \varphi \wedge (\ast \varphi \wedge \alpha)) &= 3\alpha \\
\varphi \wedge \ast (\ast \varphi \wedge \alpha) &= 2 (\ast \varphi \wedge \alpha)
\end{align}

**Proof.** Since the statements are pointwise, it is enough to check them in local coordinates using (2.2) and (2.3). This is tedious but straightforward. \qed

From these identities, we can prove the following lemma:

**Lemma 2.2.2.** If $\alpha$ is a 1-form on $M$, then we have:

\begin{align}
|\varphi \wedge \alpha|^2 &= 4|\alpha|^2 \\
|\ast \varphi \wedge \alpha|^2 &= 3|\alpha|^2
\end{align}

**Proof.** From equation (2.4), we have:

\[
\varphi \wedge \ast (\varphi \wedge \alpha) = -4 \ast \alpha \\
\alpha \wedge \varphi \wedge \ast (\varphi \wedge \alpha) = -4\alpha \wedge \ast \alpha \\
-|\varphi \wedge \alpha|^2 \text{vol} = -4|\alpha|^2 \text{vol}
\]

which proves (2.7). An analogous calculation using (2.5) yields (2.8). \qed

We have the following relations between $\varphi$, $\ast \varphi$, and an arbitrary vector field $w$:

**Lemma 2.2.3.** The following relations hold for any vector field $w$, where $w^b$ is the associated 1-form (obtained from the metric isomorphism):

\begin{align}
\ast (\varphi \wedge w^b) &= \ast \varphi \\
\ast (\ast \varphi \wedge w^b) &= w \varphi \\
\varphi \wedge (w \ast \varphi) &= -4 \ast w^b \\
\ast \varphi \wedge (w \ast \varphi) &= 0 \\
\ast \varphi \wedge (w \varphi) &= 3 \ast w^b \\
\varphi \wedge (w \varphi) &= 2 (w \varphi)
\end{align}

**Proof.** Since on a 7-manifold $\ast^2 = 1$, these results follow from Lemma A.0.1 and Proposition 2.2.1. \qed

We now explicitly describe the decomposition beginning with $k = 2, 5$.

\begin{align}
\wedge_7^2 &= \{ w \varphi; w \in \Gamma(T(M)) \} \\
&= \{ \beta \in \wedge^2; \ast (\varphi \wedge \beta) = 2\beta \} \\
&= \{ \beta \in \wedge^2; \ast (\ast \varphi \wedge (\ast \varphi \wedge \beta)) = 3\beta \}
\end{align}

\begin{align}
\wedge_{14}^2 &= \{ \beta \in \wedge^2; \ast \varphi \wedge \beta = 0 \} \\
&= \{ \beta \in \wedge^2; (\varphi \wedge \beta) = -\beta \} \\
&= \{ \sum a_{ij} e^i \wedge e^j; (a_{ij}) \in g_2 \}
\end{align}
\[
\wedge_7^5 = \{ \alpha \wedge \varphi; \alpha \in \wedge_7^1 \}
\]
\[
= \{ \gamma \in \wedge^5; \varphi \wedge \gamma = 2\gamma \}
\]
\[
= \{ \gamma \in \wedge^5; \star \varphi \wedge (\star (\varphi \wedge \gamma)) = 3\gamma \}
\]

\[
\wedge_{14}^5 = \{ \gamma \in \wedge^5; \varphi \wedge \gamma = -\gamma \}
\]
\[
= \{ \gamma \in \wedge^5; \star \varphi \wedge \gamma = 0 \}
\]

Notice that these subspaces are +2 and −1 eigenspaces of the operators \( L(\beta) = \star (\varphi \wedge \beta) \) on \( \wedge^2 \) and \( M(\gamma) = \varphi \wedge \gamma \) on \( \wedge^5 \). From this fact we get the following useful formulas for the projections \( \pi_k \) onto the \( k \)-dimensional representations, for \( \beta \in \wedge^2 \) and \( \gamma \in \wedge^5 \):

\[
\star (\varphi \wedge \beta) = 2\pi_7(\beta) - \pi_{14}(\beta)
\]
\[
\pi_7(\beta) = \frac{\beta + \star (\varphi \wedge \beta)}{3}
\]
\[
\pi_{14}(\beta) = \frac{2\beta - \star (\varphi \wedge \beta)}{3}
\]

and

\[
\varphi \wedge \gamma = 2\pi_7(\gamma) - \pi_{14}(\gamma)
\]
\[
\pi_7(\gamma) = \frac{\gamma + \varphi \wedge \gamma}{3}
\]
\[
\pi_{14}(\gamma) = \frac{2\gamma - \varphi \wedge \gamma}{3}
\]

We now move on to the decompositions for \( k = 3, 4 \).

\[
\wedge_1^3 = \{ f\varphi; f \in C^\infty(M) \}
\]
\[
= \{ \eta \in \wedge^3; \varphi \wedge (\star (\varphi \wedge \eta)) = 7\eta \}
\]
\[
\wedge_7^3 = \{ \star (\varphi \wedge \alpha); \alpha \in \wedge_7^1 \}
\]
\[
= \{ w_{\alpha} \star \varphi; w \in \Gamma(T(M)) \}
\]
\[
= \{ \eta \in \wedge^3; \star (\varphi \wedge (\star (\varphi \wedge \eta))) = -4\eta \}
\]
\[
\wedge_{27}^3 = \{ \eta \in \wedge^3; \varphi \wedge \eta = 0 \text{ and } \star \varphi \wedge \eta = 0 \}
\]

\[
\wedge_1^4 = \{ f \star \varphi; f \in C^\infty(M) \}
\]
\[
= \{ \sigma \in \wedge^4; \varphi \wedge (\star (\varphi \wedge \sigma)) = 7\sigma \}
\]
\[
\wedge_7^4 = \{ \varphi \wedge \alpha; \alpha \in \wedge_7^1 \}
\]
\[
= \{ \sigma \in \wedge^4; (\varphi \wedge (\star (\varphi \wedge \sigma))) = -4\sigma \}
\]
\[
\wedge_{27}^4 = \{ \sigma \in \wedge^4; \varphi \wedge \sigma = 0 \text{ and } \varphi \wedge \sigma = 0 \}
\]

2.3. The metric of a \( G_2 \)-structure. From Lemma 2.2.2, we can obtain a formula for determining the metric \( g \) from the 3-form \( \varphi \):

**Proposition 2.3.1.** If \( v \) is a vector field on \( M \), then

\[
(v \wedge \varphi) \wedge (v \wedge \varphi) \wedge \varphi = 6|v|^2 \text{ vol}
\]
Proof. From Lemma A.0.1 and (2.6) we have

\[ v \lrcorner \varphi = * (v^b \wedge \varphi) \]

and

\[ (v \lrcorner \varphi) \wedge \varphi = 2(v^b \wedge \varphi) \]

Thus we obtain

\[ (v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = 2|v^b \wedge \varphi|^2 \text{vol} = 6|v|^2 \text{vol} \]

where we have used (2.8).

By polarizing (2.25) in \( v \), we obtain the relation:

\[ (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi = 6 \langle v, w \rangle \text{vol} \]

From this equation we can obtain the metric.

Lemma 2.3.2. Fix a vector field \( v = v^k e_k \), where \( e_1, e_2, \ldots, e_7 \) is an oriented local frame of vector fields. The expression obtained from \( v \) by

\[ ((v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi) (e_1, e_2, \ldots, e_7) \]

is homogeneous of order 2 in \( v \), and is independent of the choice of \( e_1, e_2, \ldots, e_7 \).

Proof. The homogeneity of (2.26) of order 2 in \( v \) is clear. Now suppose we choose a different oriented basis \( e'_i \), \( e'_2, \ldots, e'_7 \). Then we have

\[ e'_i = P_{ij} e_j \]

and hence

\[ (e'_i \lrcorner \varphi) \wedge (e'_j \lrcorner \varphi) \wedge \varphi = P_{ik} P_{jl} (e_k \lrcorner \varphi) \wedge (e_l \lrcorner \varphi) \wedge \varphi \]

Hence in the new basis the denominator of (2.26) changes by a factor of

\[ \left( \det(\det((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi) (e_1, e_2, \ldots, e_7))) \right)^\frac{1}{2} \]

and the numerator also changes by a factor of \( \det(P) \), leaving the quotient invariant.

We can now give the expression for the metric in terms of the 3-form \( \varphi \).

Theorem 2.3.3. Let \( v \) be a tangent vector at a point \( p \) and let \( e_1, e_2, \ldots, e_7 \) be any basis for \( T_p M \). Then the length \( |v| \) of \( v \) is given by

\[ |v|^2 = 6^{-\frac{2}{7}} \frac{((v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi) (e_1, e_2, \ldots, e_7)}{\left( \det(\((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi) (e_1, e_2, \ldots, e_7)) \right)^\frac{1}{2}} \]

Proof. We work in local coordinates at the point \( p \). In this notation \( g_{ij} = \langle e_i, e_j \rangle \) with \( 1 \leq i, j \leq 7 \). Let \( \det(g) \) denote the determinant of \( (g_{ij}) \). We have from (2.25) that

\[ ((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi) = 6g_{ij} \text{vol} \]

\[ = 6g_{ij} \sqrt{\det(g)} e^1 \wedge e^2 \wedge \ldots \wedge e^7 \]

\[ \det(\((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi) (e_1, e_2, \ldots, e_7)) = 6^7 \det(g) \det(g)^\frac{7}{2} \]

\[ = 6^7 \det(g) \]
and since
\[(v \cdot \varphi) \wedge (v \cdot \varphi) \wedge \varphi = 6|v|^2 \text{vol}
\]
\[= 6|v|^2 \sqrt{\det(g)} e^1 \wedge e^2 \wedge \ldots \wedge e^7\]
\[((v \cdot \varphi) \wedge (v \cdot \varphi) \wedge \varphi) (e_1, e_2, \ldots, e_7) = 6|v|^2 \det(g)^{\frac{1}{2}}\]
these two expressions can be combined to yield (2.27). \(\square\)

2.4. The cross product of a \(G_2\)-structure. In this section we will describe the cross product operation on a manifold with a \(G_2\)-structure in terms of the 3-form \(\varphi\), and present some useful relations.

**Definition 2.4.1.** Let \(u\) and \(v\) be vector fields on \(M\). The cross product, denoted \(u \times v\), is a vector field on \(M\) whose associated 1-form under the metric isomorphism satisfies:
\[(u \times v)^\flat = v \cdot u \cdot \varphi\]
Notice that this immediately yields the relation between \(\times\), \(\varphi\), and the metric \(g\):
\[g(u \times v, w) = (u \times v)^\flat (w) = w \cdot u \cdot \varphi = \varphi(u, v, w)\]

Another characterization of the cross product is obtained from this one using Lemma A.0.1:
\[(u \times v)^\flat = v \cdot u \cdot \varphi = -\ast(v^b \wedge \ast(u \cdot \varphi))\]
\[= -\ast(v^b \wedge u^b \wedge \ast \varphi)\]
\[= \ast(u^b \wedge v^b \wedge \ast \varphi)\]
Now since \(u^b \wedge v^b\) is a 2-form, we can write it as \(\beta_7 + \beta_{14}\), with \(\beta_j \in \Lambda_2^3\). Then we have, using (2.14) and (2.15):
\[(u \times v)^\flat \wedge \ast \varphi = 3\ast(\beta_7 \wedge \ast \varphi) \wedge \ast \varphi\]
\[= 3\ast \beta_7\]
Taking the norm of both sides, and using (2.8):
\[|(u \times v)^\flat \wedge \ast \varphi|^2 = 3|(u \times v)^\flat|^2 = 3|u \times v|^2 = 9|\beta_7|^2\]
from which we obtain
\[|\beta_7|^2 = \frac{1}{3}|u \times v|^2\]

**Lemma 2.4.2.** Let \(u\) and \(v\) be vector fields. Then
\[|u \times v|^2 = |u \wedge v|^2\]

**Proof.** With \(\beta = u^b \wedge v^b\), we have from (2.14) and (2.15):
\[\beta \wedge \varphi = 2\ast \beta_7 - \ast \beta_{14}\]
\[\beta \wedge \beta \wedge \varphi = 2|\beta_7|^2 \text{vol} - |\beta_{14}|^2 \text{vol}\]
\[= 0\]
since \(\beta = u^b \wedge v^b\) is decomposable. So \(|\beta_{14}|^2 = 2|\beta_7|^2\) and finally we obtain from (2.32):
\[|u \times v|^2 = 3|\beta_7|^2 = |\beta_7|^2 + |\beta_{14}|^2 = |\beta|^2 = |u \wedge v|^2\]
The next lemma describes the iteration of the cross product.

**Lemma 2.4.3.** Let $u$, $v$, and $w$ be vector fields. Then we have

\[(2.34) \quad u \times (v \times w) = -\langle u, v \rangle w + \langle u, w \rangle v + \langle u, v \wedge w \wedge \star \phi \rangle \] \]

**Proof.** From (2.30), we have

\[(u \times (v \times w))^\flat = * \left( u^b \wedge * \left( v^b \wedge w^b \wedge *\phi \right) \right) \wedge *\phi \]

Now since $\beta \wedge *\phi = 0$ for $\beta \in \Lambda^2_{14}$, we can replace $v^b \wedge w^b$ by $\pi_7(v^b \wedge w^b) = \frac{1}{3} (v^b \wedge w^b + * (\phi \wedge v^b \wedge w^b))$. Then using (2.14), we have

\[\star \left( \pi_7(v^b \wedge w^b) \wedge *\phi \right) \wedge *\phi = 3 \star \pi_7(v^b \wedge w^b) \]

\[= \star \left( v^b \wedge w^b + * (\phi \wedge v^b \wedge w^b) \right) \]

\[= \star \left( v^b \wedge w^b - v \wedge w \wedge \star \phi \right) \]

which we substitute back above to obtain

\[ (u \times (v \times w))^\flat = \star \left( u^b \wedge * \left( v^b \wedge w^b - v \wedge w \wedge \star \phi \right) \right) \]

\[= -u \wedge (v^b \wedge w^b - v \wedge w \wedge \star \phi) \]

\[= -\langle u, v \rangle w + \langle u, w \rangle v + u \wedge v \wedge w \wedge \star \phi \]

which completes the proof.

**Corollary 2.4.4.** For the special case $u = v$, we obtain the useful relation

\[(2.35) \quad u \times (u \times w) = -|w|^2 w + \langle u, w \rangle u \]

**Remark 2.4.5.** Note that if we fix a unit vector field $u$, then the operator $Jw = u \times w$ when restricted to the space of vector fields orthogonal to $u$, satisfies $J^2 = -I$, and hence is an almost complex structure on the orthogonal complement of $u$.

The following lemma will be used in Section 3.2 to determine how the metric changes under a deformation in the $\wedge^3_{14}$ direction.

**Lemma 2.4.6.** The following identity holds for $v$ and $w$ vector fields:

\[(2.36) \quad (v \wedge w \wedge \star \phi) \wedge (v \wedge w \wedge \star \phi) = 2|v \wedge w|^2 \text{vol} \]

**Proof.** We start with Lemma A.0.1 to rewrite

\[v \wedge w \wedge \star \phi = \star (v^b \wedge (w \wedge \star \phi)) \]

\[= -\star (v^b \wedge w^b \wedge \phi) \]

\[= -2\beta_7 + \beta_{14} \]

using the notation as above. From equations (2.28) and (2.31) we have

\[(v \wedge w \wedge \phi) \wedge \star \phi = -3 \star \beta_7 \]
Combining these two equations and (2.32),
\[
(v \lrcorner w \star \varphi) \wedge (v \lrcorner w \star \varphi) \wedge \varphi = (-2 \beta_7 + \beta_{14}) \wedge (-3 \star \beta_7)
\]
\[
= 6|\beta_7|^2 \text{ vol}
\]
\[
= 2|u \wedge v|^2 \text{ vol}
\]
which completes the proof. \qed

Finally, we prove a theorem which will be useful in Section 3.2 where we will use it to show that to first order, deforming a $G_2$-structure by an element of $\wedge^3_{14}$ does not change the metric.

**Theorem 2.4.7.** Let $u, v, w$ be vector fields. Then
\[
(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) = 0.
\]

Note that in terms of the decompositions in (2.14) and (2.20), this theorem says that the wedge product map
\[
\wedge^2_{14} \times \wedge^2_{14} \times \wedge^3_{27} \rightarrow \wedge^7_1
\]
is the zero map.

**Proof.** Since it is an 8-form,
\[
(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = 0.
\]
Taking the interior product with $w$ and rearranging,
\[
(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) = -(w \lrcorner u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi
\]
\[
- (u \lrcorner \varphi) \wedge (w \lrcorner v \lrcorner \varphi) \wedge \varphi
\]
Now using (2.12), we get
\[
(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) = -3 (w \lrcorner u \lrcorner \varphi) \wedge \star w - 3 (w \lrcorner v \lrcorner \varphi) \wedge \star u
\]
Finally, from (A.7), we have
\[
(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) = -3 (w \lrcorner u \lrcorner \varphi) \wedge \star \left(w \wedge v\right) - 3 (w \lrcorner v \lrcorner \varphi) \wedge \star \left(u \wedge v\right)
\]
\[
= -3 \varphi \wedge \star \left(w \wedge v\wedge v\right) - 3 \varphi \wedge \star \left(u \wedge w \wedge v\right)
\]
\[
= 0.
\]
\qed

**2.5. The 16 classes of $G_2$-structures.** According to the classification of Fernández and Gray in [11], a manifold with a $G_2$-structure has holonomy a subgroup of $G_2$ if and only if $\nabla \varphi = 0$, which they showed to be equivalent to
\[
d \varphi = 0 \quad \text{and} \quad d * \varphi = 0.
\]
They established this equivalence by decomposing the space $W$ that $\nabla \varphi$ belongs to into irreducible $G_2$-representations, and identifying the invariant subspaces of $W$ with isomorphic subspace of $\wedge^*(M)$. This space $W$ decomposes as
\[
W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}
\]
where the subscript $k$ denotes the dimension of the irreducible representation $W_k$. Now $d \varphi \in \wedge^1_1 \oplus \wedge^1_7 \oplus \wedge^1_{14}$ and $d * \varphi \in \wedge^3_1 \oplus \wedge^3_{14}$. Up to isomorphism, the projections $\pi_k(d \varphi)$ and $\pi_k(d * \varphi)$ are non-zero constant multiples of $\pi_k(\nabla \varphi)$. Therefore in the
following we will consider \( d\varphi \) and \( d*\varphi \) instead of \( \nabla \varphi \). Since both of these have a component in a \( 7 \)-dimensional representation, they are multiples:

**Lemma 2.5.1.** The following identity holds:

\[
(2.37) \quad \mu = *d\varphi \wedge \varphi = -*d*\varphi \wedge *\varphi
\]

where we have defined the 6-form \( \mu \) by the above two equal expressions. They are the components \( \pi_7(d\varphi) \) and \( \pi_7(d*\varphi) \) transferred to the isomorphic space \( \Lambda_7^0 \).

**Proof.** See [4] for a proof of this fact. \( \square \)

We prefer to work with the associated 1-form, \( \theta = *\mu \). We will see that in some subclasses this 1-form is closed or at least “partially closed.”

Now we say a \( G_2 \)-structure is in the class \( W_i \oplus W_j \oplus W_k \) with \( i, j, k \) distinct where \( \{i, j, k\} \subset \{1, 7, 14, 27\} \) if only the component of \( d\varphi \) or \( d*\varphi \) in the \( l \)-dimensional representation vanishes. Here \( \{l\} = \{1, 7, 14, 27\} \setminus \{i, j, k\} \). Similarly the \( G_2 \)-structure is in the class \( W_i \oplus W_j \) if the \( k \) and \( l \)-dimensional components vanish, and in the class \( W_i \) if the other three components are zero. In this way we arrive at 16 classes of \( G_2 \)-structures on a manifold. In Table 2.1 we describe the classes in terms of differential equations on the form \( \varphi \). This classification first appeared in [11] and then in essentially this form in [7].

| Class | Defining Equations | Name | \( d\theta \) |
|-------|--------------------|------|------------|
| \( W_1 \oplus W_7 \oplus W_{14} \oplus W_{27} \) | no relation on \( d\varphi \), \( d*\varphi \). | | |
| \( W_7 \oplus W_{14} \oplus W_{27} \) | \( d\varphi \wedge \varphi = 0 \) | \( d\theta =? \) | |
| \( W_1 \oplus W_{14} \oplus W_{27} \) | \( \theta = 0 \) | \( \theta = 0 \) | |
| \( W_1 \oplus W_7 \oplus W_{27} \) | \( d\varphi + \frac{4}{7}\theta \wedge *\varphi = 0 \) or \( \varphi \wedge (d*\varphi) = -2d*\varphi \) | “integrable” \( \pi_7(d\theta) = 0 \) | |
| \( W_1 \oplus W_7 \oplus W_{14} \) | \( d\varphi + \frac{4}{7}\theta \wedge \varphi - h*\varphi = 0 \) | \( d\theta =? \) | |
| \( W_{14} \oplus W_{27} \) | \( d\varphi \wedge \varphi = 0 \) and \( \theta = 0 \) | \( \theta = 0 \) | |
| \( W_7 \oplus W_{27} \) | \( d\varphi \wedge \varphi = 0 \) and \( d*\varphi + \frac{4}{7}\theta \wedge *\varphi = 0 \) | \( \pi_7(d\theta) = 0 \) | |
| \( W_7 \oplus W_{14} \) | \( d\varphi + \frac{4}{7}\theta \wedge \varphi = 0 \) | LC nearly \( G_2 \) \( d\theta = 0 \) | |
| \( W_1 \oplus W_{27} \) | \( d\varphi = 0 \) | semi-\( G_2 \) \( \theta = 0 \) | |
| \( W_1 \oplus W_{14} \) | \( d\varphi - h*\varphi = 0 \) | \( \theta = 0 \) | |
| \( W_1 \oplus W_7 \) | \( d\varphi + \frac{4}{7}\theta \wedge \varphi - h*\varphi = 0 \) and \( d\varphi + \frac{4}{7}\theta \wedge *\varphi = 0 \) | LC nearly \( G_2 \) \( d\theta = 0 \) | |
| \( W_{27} \) | \( d\varphi \wedge \varphi = 0 \) and \( d*\varphi = 0 \) | \( \theta = 0 \) | |
| \( W_{14} \) | \( d\varphi = 0 \) | almost \( G_2 \) \( \theta = 0 \) | |
| \( W_7 \) | \( d\varphi + \frac{4}{7}\theta \wedge *\varphi = 0 \) and \( d\varphi + \frac{4}{7}\theta \wedge \varphi = 0 \) | LC \( G_2 \) \( d\theta = 0 \) | |
| \( W_1 \) | \( d\varphi - h*\varphi = 0 \) and \( d*\varphi = 0 \) | nearly \( G_2 \) \( \theta = 0 \) | |
| \( \{0\} \) | \( d\varphi = 0 \) and \( d*\varphi = 0 \) | \( G_2 \) \( \theta = 0 \) | |

**Table 2.1.** The 16 classes of \( G_2 \)-structures

In Table 2.1, the function \( h = \frac{1}{4}*(\varphi \wedge d\varphi) \) is the image of \( \pi_1(d\varphi) \) in \( \Lambda^0 \) under the isomorphism \( \Lambda_4^1 \cong \Lambda^0_1 \). The abbreviation “LC” stands for locally conformal to and means that for those classes, we can (at least locally) conformally change the metric to enter a strictly smaller subclass. This will be explained in Section 3.1.
We now prove the closedness or partial closedness of $\theta$ in the various classes as given in the final column of Table 2.1. The closedness of $\theta$ in the classes $W_1 \oplus W_7$ and $W_7 \oplus W_{14}$ was originally shown using a different approach by Cabrera in [7].

**Lemma 2.5.2.** If $\varphi$ is in the classes $W_7$, $W_7 \oplus W_{14}$, or $W_1 \oplus W_7$, then $d\theta = 0$.

Furthermore, if $\varphi$ is in the classes $W_7 \oplus W_{27}$ or $W_1 \oplus W_7 \oplus W_{27}$ then $\pi_7(d\theta) = 0$.

**Proof.** We begin by showing that if $\varphi$ satisfies $d\varphi + \frac{1}{3} \theta \wedge \varphi = 0$, then $d\theta = 0$, and if $\varphi$ satisfies $d \ast \varphi + \frac{1}{3} \theta \wedge \ast \varphi = 0$, then $\pi_7(d\theta) = 0$.

Suppose $d\varphi + \frac{1}{3} \theta \wedge \varphi = 0$. We differentiate this equation to obtain:

$$d\theta \wedge \varphi = \theta \wedge d\varphi = \theta \wedge \left( -\frac{1}{4} \theta \wedge \varphi \right) = 0$$

But wedge product with $\varphi$ is an isomorphism from $\wedge^2$ to $\wedge^5$, so $d\theta = 0$. Now suppose $d \ast \varphi + \frac{1}{3} \theta \wedge \ast \varphi = 0$. Differentiating this equation yields

$$d\theta \wedge \ast \varphi = \theta \wedge d \ast \varphi = \theta \wedge \left( -\frac{1}{3} \theta \wedge \ast \varphi \right) = 0$$

But wedge product with $\ast \varphi$ is an isomorphism from $\wedge^2$ to $\wedge^5$, so $\pi_7(d\theta) = 0$.

Thus by comparing with Table 2.1, we have shown that in the classes $W_7 \oplus W_{14}$ and $W_7$, we have $d\theta = 0$. Also, in the classes $W_1 \oplus W_7 \oplus W_{27}$ and $W_7 \oplus W_{27}$ we have $\pi_7(d\theta) = 0$. We still have to show that $\theta$ is closed in the class $W_1 \oplus W_7$. We already have that $\pi_7(d\theta) = 0$, so we need only show that $\pi_{14}(d\theta) = 0$ in this case. We differentiate $d\varphi + \frac{1}{3} \theta \wedge \varphi - h \ast \varphi = 0$ to obtain

$$0 = \frac{1}{4} d\theta \wedge \varphi - \frac{1}{4} \theta \wedge d\varphi - dh \wedge \ast \varphi - h \ast \varphi$$

$$= \frac{1}{4} d\theta \wedge \varphi - \frac{1}{4} \theta \wedge \left( -\frac{1}{4} \theta \wedge \varphi + h \ast \varphi \right) - dh \wedge \ast \varphi - h \ast \left( -\frac{1}{3} \theta \wedge \ast \varphi \right)$$

$$= \frac{1}{4} d\theta \wedge \varphi + \alpha \wedge \ast \varphi$$

for some 1-form $\alpha$, where we have used the fact that $d \ast \varphi + \frac{1}{3} \theta \wedge \ast \varphi = 0$ in this class. But $\alpha \wedge \ast \varphi$ is in $\wedge^5$, and since wedge product with $\varphi$ is an isomorphism from $\wedge^2$ to $\wedge^5$ for $k = 7, 14$, this shows that $\pi_{14}(d\theta) = 0$. \qed

The inclusion relations among these various subclasses are analyzed in [11, 7, 8, 10, 4, 6, 21, 22, 26]. For all but one case, examples can be found of manifolds which are in a particular class but not in a strictly smaller subclass. For example, a manifold in the class $W_{14}$ which does not have holonomy $G_2$ appears in [10]. There is one case of an inclusion in Table 2.1 which is not strict. This is given by the following result, which first appeared in [7].

**Proposition 2.5.3.** The class $W_1 \oplus W_{14}$ equals $W_1 \cup W_{14}$ exactly.

**Proof.** In the class $W_1 \oplus W_{14}$, we have $d\varphi - h \ast \varphi = 0$ (and by consequence $\theta = 0$). Differentiating this equation,

$$dh \wedge \ast \varphi = -hd \ast \varphi$$

If $h \neq 0$, then by dividing by $h$ and using Proposition 2.2.1, we see that $d \ast \varphi \in \wedge^5$, so $\pi_{14}(d \ast \varphi) = 0$. But since we already have that $\theta = 0$, this means $d \ast \varphi = 0$ and hence $\varphi$ is actually of class $W_1$ (nearly $G_2$). If $h = 0$ then $d\varphi = 0$ and $\varphi$ is of class $W_{14}$ (almost $G_2$). \qed
Remark 2.5.4. Note that in the proof of the above proposition, we see that if \( \varphi \) is of class \( W_1 \) (nearly \( G_2 \)), then \( dh \wedge \ast \varphi = 0 \), and so \( dh = 0 \) by Proposition 2.2.1. Therefore in the nearly \( G_2 \) case, the function \( h \) is \text{locally constant}, or constant if the manifold \( M \) is connected. In [15] Gray showed that all nearly \( G_2 \) manifolds are actually \text{Einstein}.

In [12, 13], Fernández and Ugarte show that for manifolds with a \( G_2 \)-structure in the classes \( W_1 \oplus W_7 \oplus W_{27} \) ("integrable") or \( W_7 \oplus W_{14} \), there exists a subcomplex of the deRham complex. They then show how to define analogues of Dolbeault cohomology of complex manifolds in these two cases, including analogues of \( \bar{\partial} \)-harmonic forms. They derive properties of these cohomology theories and topological restrictions on the existence of \( G_2 \)-structures in some strictly smaller subclasses.

3. Deformations of a fixed \( G_2 \)-structure

Let us begin with a fixed \( G_2 \)-structure on a manifold \( M \) in a certain class. We are interested in how deforming the form \( \varphi \) affects the class. In other words, we are interested in what kinds of deformations preserve which classes of \( G_2 \)-structures. Now since \( \varphi \in \wedge^3 \oplus \wedge^3 v \oplus \wedge^3 w \), there are three canonical ways to deform \( \varphi \). For example, since \( \wedge^3 \) = \( \{ f \varphi \} \), adding to \( \varphi \) an element of \( \wedge^3 \) amounts to conformally scaling \( \varphi \). This preserves the decomposition into irreducible representations in this case. However, since the decomposition does depend on \( \varphi \) (unlike the decomposition of forms into \( (p, q) \) types on a Kähler manifold) in general if we add an element of \( \wedge^3 \) or \( \wedge^3 v \), the decomposition does change. So deforming in those two directions really only makes sense infinitesimally. However, we shall attempt to get as far as we can with an actual deformation before we restrict to infinitesimal deformations.

3.1. Conformal Deformations of \( G_2 \)-structures. Let \( f \) be a smooth, nowhere vanishing function on \( M \). For notational convenience, which will become evident, we will conformally scale \( \varphi \) by \( f^2 \). Let the new form \( \tilde{\varphi} = f^2 \varphi \). We first compute the new metric \( \tilde{g} \) and the new volume form \( \text{vol} \) in the following lemma.

Lemma 3.1.1. The metric \( g_\circ \) on vector fields, the metric \( g_\circ^{-1} \) on one forms, and the volume form \( \text{vol}_\circ \) transform as follows:

\[
\begin{align*}
\text{vol} &= f^7 \text{vol}_\circ \\
\tilde{g} &= f^2 g_\circ \\
\tilde{g}^{-1} &= f^{-2} g_\circ^{-1}
\end{align*}
\]

Proof. Using Proposition 2.3.1, we have in a local coordinate chart:

\[
\tilde{g}(u, v)\text{vol} = \frac{1}{6} (u \cdot \tilde{\varphi}) \wedge (v \cdot \tilde{\varphi}) \wedge \tilde{\varphi} = f^6 g_\circ(u, v) \text{vol}_\circ \\
\tilde{g}(u, v) \sqrt{\det(\tilde{g})} dx^1 \ldots dx^7 = f^6 g_\circ(u, v) \sqrt{\det(g_\circ)} dx^1 \ldots dx^7.
\]

Thus, taking determinants of the coefficients of both sides,

\[
\det(\tilde{g}) = f^{63} \det(g_\circ) \\
\sqrt{\det(\tilde{g})} = f^7 \sqrt{\det(g_\circ)}
\]

This gives \( \text{vol} = f^7 \text{vol}_\circ \), from which we can immediately see that \( \tilde{g} = f^2 g_\circ \) and \( \tilde{g}^{-1} = f^{-2} g_\circ \). \( \square \)
Notice that the new metric $\tilde{g}$ is always a positive definite metric as long as $f$ is non-vanishing, even if $f$ is negative. However, the orientation of $M$ changes for negative $f$ since the volume form changes sign. Now we can determine the new Hodge star $\tilde{*}$ in terms of the old $*_{\omega}$.

**Lemma 3.1.2.** If $\alpha$ is a $k$-form, then $\tilde{*}\alpha = f^{7-2k} *_{\omega} \alpha$.

*Proof.* Let $\alpha, \beta$ be $k$-forms. Then from Lemma 3.1.1 the new metric on $k$-forms is $\langle \cdot, \cdot \rangle = f^{-2k} \langle \cdot, \cdot \rangle_{\omega}$. From this we compute:

$$
\beta \wedge \tilde{*}\alpha = \langle \beta, \alpha \rangle_{\omega} \text{vol} \\
= f^{-2k} \langle \beta, \alpha \rangle_{\omega} f^7 \text{vol}_{\omega} \\
= f^{7-2k} \beta \wedge *_{\omega} \alpha.
$$

□

**Corollary 3.1.3.** The new form $\tilde{\varphi}$ satisfies $\tilde{*}\varphi = f^4 *_{\omega} \varphi$.

*Proof.* This follows immediately from Lemma 3.1.2. □

Combining all our results so far yields:

**Lemma 3.1.4.** We have the following relations:

- $d\tilde{\varphi} = 3f^2 df \wedge \varphi + f^3 d\varphi$.
- $d\tilde{*}\varphi = 4f^3 df \wedge *_{\omega} \varphi + f^4 d *_{\omega} \varphi$.
- $\tilde{*} d\tilde{\varphi} = 3f *_{\omega} (df \wedge \varphi) + f^2 *_{\omega} d\varphi$.
- $\tilde{*} d\tilde{*}\varphi = 4 *_{\omega} (df \wedge *_{\omega} \varphi) + f *_{\omega} (d *_{\omega} \varphi)$.

*Proof.* This follows from Lemma 3.1.2 and Corollary 3.1.3. □

Using these results, we can determine which classes of $G_2$-structures are conformally invariant. We can also determine what happens to the 6-form $\mu$ from equation (2.37) as well as the associated 1-form $\theta = *_{\mu} \mu$. This is all given in the following theorem:

**Theorem 3.1.5.** Under the conformal deformation $\tilde{\varphi} = f^3 \varphi$, we have:

\[
\begin{align*}
(3.1) \quad d\tilde{\varphi} + \frac{1}{3} \theta \wedge \tilde{*}\varphi &= f^4 \left( d *_{\omega} \varphi + \frac{1}{3} \theta_{\omega} \wedge *_{\omega} \varphi \right) \\
(3.2) \quad d\tilde{*}\varphi + \frac{1}{4} \theta \wedge \tilde{\varphi} &= f^3 \left( d\varphi + \frac{1}{4} \theta_{\omega} \wedge \varphi \right) \\
(3.3) \quad d\tilde{\varphi} \wedge \tilde{*}\varphi &= f^6 (d\varphi \wedge \varphi) \\
(3.4) \quad d\tilde{*}\varphi + \frac{1}{4} \theta \wedge \tilde{\varphi} - \tilde{h} \tilde{*}\varphi &= f^3 \left( d\varphi + \frac{1}{4} \theta_{\omega} \wedge \varphi - h_{\omega} *_{\omega} \varphi \right) \\
(3.5) \quad \tilde{\mu} &= -12f^4 *_{\omega} df + f^5 \mu_{\omega} \\
(3.6) \quad \tilde{\theta} &= -12d(\log(f)) + \theta_{\omega}.
\end{align*}
\]

Hence, we see (from Table 2.1) that the classes which are conformally invariant are exactly $W_7 \oplus W_{14} \oplus W_{27}$, $W_1 \oplus W_7 \oplus W_{27}$, $W_1 \oplus W_7 \oplus W_{14}$, $W_7 \oplus W_{27}$, $W_7 \oplus W_{14}$, $W_1 \oplus W_7$, and $W_7$. These are precisely the classes which have a $W_7$ component. (This conclusion was originally observed in [11] using a different method.)
Additionally, (3.6) shows that since $\theta$ changes by an exact form, in the classes where $d\theta = 0$, we have a well defined cohomology class $[\theta]$ which is unchanged under a conformal scaling. These are the classes $W_7 \oplus W_{14}$, $W_1 \oplus W_7$, and $W_7$.

Proof. We begin by using Lemma 3.1.4 and (2.37) to compute $\tilde{\mu}$ and $\tilde{\theta}$:

$$\tilde{\mu} = \hat{\mu} d\tilde{\varphi} = 3f^4 \varphi_0 f^2 + f^4 \varphi_0 + 3f^4 \varphi_0 + f^3 \tilde{\varphi}_0$$

where we have used (2.4) in the last step. Now from Lemma 3.1.2, we get:

$$\tilde{\theta} = \hat{\mu} = -12 f^{-1} df + \theta_\circ = -12 d(\log(f)) + \theta_\circ.$$

Now using the above expression for $\tilde{\theta}$, we have:

$$d\hat{\varphi} + \frac{1}{3} \hat{\varphi} = 4f^3 df \wedge \varphi_0 + f^4 df \wedge \varphi_0 + \frac{1}{3} (12 f^{-1} df + \theta_\circ) \wedge f^4 \varphi_0$$

and finally, since $\varphi_0 \wedge \varphi_0 = 0$,

$$d\hat{\varphi} \wedge \hat{\varphi} = (3f^2 df \wedge \varphi_0 + f^3 d\varphi_0) \wedge f^3 \varphi_0 = f^6 (d\varphi_0 \wedge \varphi_0).$$

Finally, since $h = \frac{1}{f} \ast (\varphi \wedge d\varphi)$, we have

$$h \hat{\varphi} = \frac{1}{f} \hat{\varphi} \wedge d\hat{\varphi} \wedge f^4 \varphi_0$$

which yields (3.4) when combined with (3.2). This completes the proof. □

These results now enable us to give necessary and sufficient conditions for obtaining a closed or co-closed $\tilde{\varphi}$ by conformally scaling the original $\varphi_0$.

**Theorem 3.1.6.** Let $\varphi_0$ be a positive 3-form (associated to a G$_2$-structure). Under the conformal deformation $\varphi = f^3 \varphi_0$, the new 3-form $\varphi$ satisfies

- $d\varphi = 0 \iff \varphi_0$ is at least class $W_7 \oplus W_{14}$ and $12 d \log(f) = \theta_\circ$.
- $d \ast_0 \varphi_0 = 0 \iff \varphi_0$ is at least class $W_1 \oplus W_7 \oplus W_{27}$ and $12 d \log(f) = \theta_\circ$.

Note that in both cases, in order to have $\varphi$ be closed or co-closed after conformal scaling, the original 1-form $\theta_\circ$ has to be exact. In particular if the manifold is simply-connected or more generally $H^1(M) = 0$ then this will always be the case if $\varphi_0$ is in the classes $W_7 \oplus W_{14}$, $W_1 \oplus W_7$, or $W_7$, where $d\theta_\circ = 0$.

Proof. From Lemma 3.1.4, for $d\varphi = 0$, we need

$$d\varphi = 3f^2 df \wedge \varphi_0 + f^3 d\varphi_0 = 0$$

$$\Rightarrow d\varphi_0 = -3 d \log(f) \wedge \varphi_0.$$
which says that $d\varphi_0 \in \wedge^4 \mathcal{T}$ by Proposition 2.2.1. Hence $\pi_1 (d\varphi_0)$ and $\pi_{27} (d\varphi_0)$ both vanish and $\varphi_0$ must be already at least of class $W_7 \oplus W_{14}$. Then to make $d\tilde{\varphi} = 0$, we need to eliminate the $W_7$ component, which requires $12d\log (f) = \theta_5$ by Theorem 3.1.5. Similarly, to make $d^* \tilde{\varphi} = 0$, Lemma 3.1.4 gives

$$d^* \tilde{\varphi} = 4 f^3 df \wedge \star_w \varphi_0 + f^4 d \star_w \varphi_0 = 0$$

which says $d \star_w \varphi_0 \in \wedge^3 \mathcal{T}$ and $\pi_{14} (d \star_w \varphi_0) = 0$ by Proposition 2.2.1. Thus $\varphi_0$ must already be at least class $W_1 \oplus W_7 \oplus W_{27}$ and we need to choose $f$ by $12 d \log (f) = \theta_5$ to scale away the $W_7$ component.

**Remark 3.1.7.** We have shown that the transformation $\tilde{\varphi} = f^3 \varphi_0$ stays in a particular subclass as long as there is a $W_7$ component to that class. If there is, and the original $\theta_5$ is exact, then we can choose $f$ to scale away the $W_7$ component and enter a stricter subclass. Conversely, Theorem 3.1.5 shows that a conformal scaling by a non-constant $f$ will always generate a non-zero $W_7$ component if we started with none. Hence, if we are trying to construct metrics of holonomy $G_2$ on a simply-connected manifold, it is enough to construct a metric in the class $W_7$, since we can then conformally scale (uniquely) to obtain a metric of holonomy $G_2$. This is why the class $W_7$ is called *locally conformal* $G_2$.

### 3.2. Deforming $\varphi$ by an element of $\wedge^3 \mathcal{T}$

The type of deformation of $\varphi$ that is next in line in terms of increasing complexity is to add an element of $\wedge^3 \mathcal{T}$. This space is isomorphic to $\wedge^3 \mathcal{T} \cong \Gamma (T (M))$, so we can think of this process as deforming $\varphi$ by a vector field. In fact, an element $\eta \in \wedge^3 \mathcal{T}$ is of the form $\iota_w \star \varphi$ for some vector field $w$, by (2.20). Let $\tilde{\varphi} = \varphi_0 + tw \iota_w \varphi_0$, for $t \in \mathbb{R}$. We will develop formulas for the new metric $\tilde{g}$, the new Hodge star $\tilde{\star}$, and other expressions entirely in terms of the old $\varphi_0$, the old $\iota_w \varphi_0$, and the vector field $w$. Note in this case the background decomposition into irreducible $G_2$-representations changes, and in Section 3.3 we will linearize by taking $\frac{d}{dt} \big|_{t=0}$ of our results.

**Lemma 3.2.1.** In the expression

$$6 [v \wedge \iota_w \varphi_0 \wedge (v \iota_w \tilde{\varphi}) \wedge \tilde{\varphi}]$$

which is a cubic polynomial in $t$, the linear and cubic terms both vanish, and the coefficient of the quadratic term is

$$6 [v \wedge \iota_w \varphi_0 \wedge \varphi_0]$$

**Proof.** The coefficient of $t^3$ is:

$$(v \iota_w \varphi_0) \wedge (v \iota_w \iota_w \varphi_0) \wedge (w \iota_w \iota_w \varphi_0)$$

This expression is zero because it arises by taking the interior product with $w$ of the 8-form

$$(v \iota_w \varphi_0) \wedge (v \iota_w \varphi_0) \wedge \iota_w \varphi_0 = 0.$$  

The coefficient of $t$ is:

$$(v \iota_w \varphi_0) \wedge (v \iota_w \varphi_0) \wedge (w \iota_w \varphi_0) + 2 (v \iota_w \varphi_0) \wedge (v \iota_w \varphi_0) \wedge \varphi_0$$

Using (A.8) on the second term and rearranging, this coefficient becomes

$$3 (v \iota_w \varphi_0) \wedge (v \iota_w \varphi_0) \wedge (w \iota_w \varphi_0)$$

which vanishes by Theorem 2.4.7.
The coefficient of $t^2$ is:

\[(3.7) \quad (v, w, \ast_7 \varphi_6) \wedge ((v, w, \ast_7 \varphi_6) \wedge \varphi_6 + 2(v, \ast_7 \varphi_6) \wedge (w, \ast_7 \varphi_6))\]

Applying (A.8) twice and rearranging, this coefficient becomes

\[3(v, w, \ast_7 \varphi_6) \wedge (v, w, \ast_7 \varphi_6) \wedge \varphi_6\]

The statement now follows from Lemma 2.4.6. This completes the proof. \hfill \Box

Before we can use Lemma 3.2.1 to obtain the new metric, we have to extract the new volume form.

**Proposition 3.2.2.** With $\tilde{\phi} = \varphi_6 + w, \ast_7 \varphi_6$, the new volume form is

\[(3.8) \quad \text{vol.} = (1 + |w|^2_7)^\frac{7}{2} \text{vol}_6\]

**Proof.** We work in local coordinates. Let $e_1, e_2, \ldots, e_7$ be a basis for the tangent space, with $w = w^i e_i$, $g_{ij} = \langle e_i, e_j \rangle_6$ and $\tilde{g}_{ij} = \langle e_i, e_j \rangle$. Then Lemma 3.2.1 says that

\[|v|^2 \sqrt{\det(g)} = \left( |v|^2 + |v \wedge w|^2_6 \right) \sqrt{\det(g)}\]

Polarizing this equation, we have:

\[\langle v_1, v_2 \rangle \sqrt{\det(g)} = \langle v_1, v_2 \rangle_6 + \langle v_1, v_2 \rangle_6 |w|^2_6 - \langle v_1, w \rangle_6 \langle v_2, w \rangle_6 \sqrt{\det(g)}\]

\[\tilde{g}_{ij} \sqrt{\det(g)} = \left( g_{ij} + \langle e_i \wedge w, e_j \wedge w \rangle_6 \right) \sqrt{\det(g)}\]

with $v_1 = e_i$ and $v_2 = e_j$. Now substituting $w = w^k e_k$ in the second term,

\[\langle e_i \wedge w, e_j \wedge w \rangle_6 = |w|^2_6 g_{ij} - w_i w_j\]

Thus we have

\[\tilde{g}_{ij} \sqrt{\det(g)} = \left( g_{ij} (1 + |w|^2_6) - w_i w_j \right) \sqrt{\det(g)}\]

We take determinants of both sides of this equation, and use the fact that they are 7 × 7 matrices, to obtain

\[(3.9) \quad (\det(\tilde{g}))^\frac{7}{2} = (\det(g))^\frac{7}{2} \det(g_{ij} (1 + |w|^2_6) - w_i w_j).\]

Using Lemma A.0.4, the determinant on the right is

\[(3.10) \quad (1 + |w|^2_6)^7 \det(g) - |w|^2_6 (1 + |w|^2_6)^6 \det(g) = (1 + |w|^2_6)^6 \det(g)\]

Substituting this result into equation (3.9), we obtain

\[\frac{(\det(\tilde{g}))^\frac{7}{2}}{\sqrt{\det(g)}} = \left( 1 + |w|^2_6 \right)^\frac{7}{2} \det(g)\]

which completes the proof. \hfill \Box

Now letting $t = 1$, with $\tilde{\phi} = \varphi_6 + w, \ast_7 \varphi_6$, Lemma 3.2.1 and Proposition 3.2.2 yield

\[|v|^2 \text{vol.} = \left( |v|^2 + |v \wedge w|^2_6 \right) \text{vol}_6\]

\[\langle v, v \rangle = \frac{1}{(1 + |w|^2_6)^6} \left( \langle v, v \rangle_6 + |v|^2_6 |w|^2_6 - \langle v, w \rangle^2_6 \right)\]
Polarizing this equation, we obtain:

\[
\langle v_1, v_2 \rangle = \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} \left( \langle v_1, v_2 \rangle_o + \langle v_1, v_2 \rangle |w|_o^2 - \langle v_1, w \rangle \langle v_2, w \rangle_o \right) 
\]

which by Lemma 2.4.2 can also be written as

\[
\langle v_1, v_2 \rangle = \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} \left( \langle v_1, v_2 \rangle_o + \langle w \times v_1, w \times v_2 \rangle \right) 
\]

Note that in the above expression \( \times \) refers to the vector cross product associated to the initial \( G_2 \)-structure \( \varphi_0 \). Later we will describe this metric geometrically. In local coordinates with \( w = w^i e_i \), \( g_{ij} = \langle e_i, e_j \rangle_o \), and \( w^b = w_{1} e_i \), we see that

\[
g_{ij} = \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} (g_{ij}(1 + |w|_o^2) - w_i w_j) 
\]

**Proposition 3.2.3.** In local coordinates, the metric \( \tilde{g}^{ij} \) on 1-forms is given by:

\[
\tilde{g}^{ij} = \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} (g^{ij} + w^i w^j) 
\]

**Proof.** We compute:

\[
g_{ij} \tilde{g}^{jk} = \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} (g_{ij}(1 + |w|_o^2) - w_i w_j) \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} (g^{jk} + w^j w^k) 
\]

\[
= \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} \left( g_{ij} g^{jk} + g_{ij} w^j w^k \right) (1 + |w|_o^2) - g^{jk} w_i w_j - w_i w_j w^j w^k 
\]

\[
= \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} \left( (\delta_i^k + w_i w^k) (1 + |w|_o^2) - |w|_o^2 w_i w^k - w_i w^k \right) 
\]

which completes the proof. \( \square \)

Now with \( \alpha = \alpha_i e^i \) and \( \beta = \beta_j e^j \) two 1-forms, their new inner product is

\[
\langle \alpha, \beta \rangle = \alpha_i \beta_j \tilde{g}^{ij} = \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} (\alpha_i \beta_j g^{ij} + \alpha_i w^i \beta_j w^j) 
\]

\[
(3.14) \quad = \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} (\langle \alpha, \beta \rangle_o + \langle w_{\alpha}, w_{\beta} \rangle_o) 
\]

From this expression we can derive a formula for the new metric \( \langle \ , \ \rangle \) on \( k \)-forms:

**Theorem 3.2.4.** Let \( \alpha, \beta \) be \( k \)-forms. Then

\[
\langle \alpha, \beta \rangle = \frac{1}{(1 + |w|_o^2)^{\frac{3}{2}}} (\langle \alpha, \beta \rangle_o + \langle w_{\alpha}, w_{\beta} \rangle_o) 
\]

**Proof.** We have already established it for the case \( k = 1 \) in (3.14), and the case \( k = 0 \) is trivial. For the general case, we will prove the statement on decomposable
forms and it follows in general by linearity. Let \( \alpha = e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k} \) and \( \beta = e^{j_1} \wedge e^{j_2} \wedge \ldots \wedge e^{j_k} \). Then by the definition of the metric on \( k \)-forms,

\[
\langle \alpha, \beta \rangle = \det \begin{pmatrix}
\langle e^{i_1}, e^{j_1} \rangle & \langle e^{i_1}, e^{j_2} \rangle & \cdots & \langle e^{i_1}, e^{j_k} \rangle \\
\langle e^{i_2}, e^{j_1} \rangle & \langle e^{i_2}, e^{j_2} \rangle & \cdots & \langle e^{i_2}, e^{j_k} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle e^{i_k}, e^{j_1} \rangle & \langle e^{i_k}, e^{j_2} \rangle & \cdots & \langle e^{i_k}, e^{j_k} \rangle
\end{pmatrix}
\]

Now from equation \((3.14)\) each entry in the above matrix is of the form

\[
\langle e^{i_a}, e^{j_b} \rangle = \frac{1}{(1 + |w|_o^2)^3} (g^{i_a j_b} + w^{i_a} w^{j_b})
\]

and we have

\[
\langle \alpha, \beta \rangle = \frac{1}{(1 + |w|_o^2)^3} \det \begin{pmatrix}
g^{i_1 j_1} + w^{i_1} w^{j_1} & \cdots & g^{i_1 j_k} + w^{i_1} w^{j_k} \\
\vdots & \ddots & \vdots \\
g^{i_k j_1} + w^{i_k} w^{j_1} & \cdots & g^{i_k j_k} + w^{i_k} w^{j_k}
\end{pmatrix}
\]

Now we apply Lemma A.0.3 to obtain

\[
\langle \alpha, \beta \rangle_o + \sum_{l,m=1}^k (-1)^{l+m} \langle e^{i_1} \wedge \ldots \wedge e^{i_l} \wedge e^{j_1} \wedge \ldots \wedge e^{j_m} \rangle_o \langle e^{i_k} \rangle \wedge \ldots \wedge \langle e^{j_k} \rangle
\]

for the determinant above. Now with \( w = w^i e_i \), we can take the interior product with both \( \alpha \) and \( \beta \):

\[
w_{\cdot \alpha} = \sum_{l=1}^k (-1)^{l-1} w^l e^{i_1} \wedge \ldots \wedge \hat{e}^l \wedge \ldots \wedge e^{i_k}
\]

\[
w_{\cdot \beta} = \sum_{m=1}^k (-1)^{m-1} w^m e^{j_1} \wedge \ldots \wedge \hat{e}^m \wedge \ldots \wedge e^{j_k}
\]

and hence the sum over \( l \) and \( m \) above is just \( \langle w_{\cdot \alpha}, w_{\cdot \beta} \rangle_o \). Putting everything together, we arrive at \((3.15)\):

\[
\langle \alpha, \beta \rangle = \frac{1}{(1 + |w|_o^2)^3} ((\langle \alpha, \beta \rangle_o + \langle w_{\cdot \alpha}, w_{\cdot \beta} \rangle_o))
\]

To continue our analysis of the new G\(_2\)-structure \( \tilde{\varphi} \), we now need to compute the new Hodge star \( \tilde{\ast} \).

**Theorem 3.2.5.** The Hodge star for the new metric on a \( k \)-form \( \alpha \) is given by:

\[
\tilde{\ast} \alpha = (1 + |w|_o^2)^{\frac{2-k}{3}} \left( \ast_o \alpha + (-1)^{k-1} w_{\cdot \alpha} \left( \ast_o (w_{\cdot \alpha}) \right) \right)
\]

\[
= (1 + |w|_o^2)^{\frac{2-k}{3}} \left( \ast_o \alpha + w_{\cdot \alpha} (w^x \wedge \ast_o \alpha) \right)
\]

**Proof.** The second form follows from the first from \((A.2)\). Although it looks a little more cluttered, we prefer to use the first form for \( \tilde{\ast} \). Notice that up to a scaling factor, the new star is given by ‘twisting by \( w \)’, taking the old star, then
‘untwisting by \( w \), and adding this to the old star. To establish this formula, let \( \beta \) be an arbitrary \( k \)-form and compute:

\[
\beta \wedge \tilde{\ast} \alpha = \langle \beta, \alpha \rangle \text{vol.} \\
= \frac{1}{(1 + |w|_o^2)^{\frac{2-k}{2}}} \left( (\alpha, \beta)_o + (w, \alpha, w, \beta)_o \right) (1 + |w|_o^2)^{-\frac{2-k}{2}} \text{vol.} \\
= (1 + |w|_o^2)^{\frac{2-k}{2}} (\beta \wedge \ast \alpha + (w, \beta) \wedge \ast (w, \alpha))
\]

Now if we take the interior product with \( \ast \) of the \( 8 \)-form

\[\beta \wedge \ast \alpha \wedge (\ast \beta) \wedge \ast (\ast \beta) = 0\]

we obtain

\[(w, \beta) \wedge \ast (w, \alpha) = (-1)^{k-1} \beta \wedge (w, \ast (w, \alpha))\]

and this completes the proof, since \( \beta \) is arbitrary.

At this point before continuing it is instructive to observe directly that \( \tilde{\ast}^2 = 1 \), as of course it should on a 7-manifold. It clarifies the necessity of all the factors of \((1 + |w|_o^2)\). Let \( \alpha \) be a \( k \)-form:

\[
\tilde{\ast} \alpha = (1 + |w|_o^2)^{\frac{2-k}{2}} \left( \ast \alpha + (-1)^{k-1} w, \ast (w, \alpha) \right)
\]

\[
\tilde{\ast}(\tilde{\ast} \alpha) = (1 + |w|_o^2)^{\frac{2-(7-k)}{2}} \left( \ast (\ast \alpha) + (-1)^{(7-k)-1} w, \ast (w, \ast (\ast \alpha)) \right)
\]

Now the last term is zero because of the two successive interior products with \( w \).

Using \( \ast \alpha = 1 \) we now have:

\[
\tilde{\ast}(\ast \alpha) = (1 + |w|_o^2)^{-1} \left[ \alpha + (-1)^{k-1} \ast (w, \ast (\ast \alpha)) \right] + (-1)^k w, \ast (w, \ast (\ast \alpha))\]

Now using equation (A.10), (with \( n = 7 \)) we get

\[
\tilde{\ast}(\ast \alpha) = (1 + |w|_o^2)^{-1} \left( \alpha + |w|_o^2 \ast \alpha \right) = \alpha
\]

We now give a geometric interpretation of the transformation \( \varphi_o \mapsto \varphi_o + w, \ast \varphi_o \).

From (3.11) for the new metric \( \tilde{g} \), with \( v_1 = v \) and \( v_2 = w \), we have

\[
\langle v, w \rangle = \frac{1}{(1 + |w|_o^2)^{\frac{2-k}{2}}} (\langle v, w \rangle_o + \langle v, w \rangle_o |w|_o^2 - \langle v, w \rangle_o <w, w>_o) \\
= \frac{1}{(1 + |w|_o^2)^{\frac{2-k}{2}}} (\langle v, w \rangle_o)
\]

Hence we see that all the distances are shrunk by a factor of \( (1 + |w|_o^2)^{-\frac{2-k}{2}} \) in the direction of the vector field \( w \). On the other hand, if either \( v_1 \) or \( v_2 \) is orthogonal to \( w \) in the old metric, then (3.11) gives

\[
\langle v_1, v_2 \rangle = \frac{1}{(1 + |w|_o^2)^{\frac{2-k}{2}}} (\langle v_1, v_2 \rangle_o + \langle v_1, v_2 \rangle_o |w|_o^2 - 0) \\
= (1 + |w|_o^2)^{\frac{2-k}{2}} \langle v_1, v_2 \rangle_o
\]
Thus in the directions perpendicular to the vector field \( w \), the distances are \( \text{stretched} \) by a factor of \( (1 + |w|^2_o)^{\frac{1}{3}} \). Therefore this new metric is expanded in the 6 directions perpendicular to \( w \) and is compressed in the direction parallel to \( w \). Of course, the situation is more complicated if neither \( v_1 \) nor \( v_2 \) is parallel or perpendicular to \( w \). This produces a \( \text{tubular} \) manifold. For example in the case of \( M = N \times S^1 \), where \( N \) is a Calabi-Yau 3-fold and the metric on \( M \) is the product metric, if we take \( w = \frac{\partial}{\partial \theta} \) where \( \theta \) is a coordinate on \( S^1 \), then the Calabi-Yau manifold \( N \) is expanded and the circle factor \( S^1 \) is compressed under \( \phi_o \mapsto \phi_o + w \downarrow_{\phi_o} \phi_o \). By replacing \( w \) by \( tw \) and letting \( t \to \infty \), we can make this “tube” as long and thin as we want. The total volume, however, always increases by \( (1 + |w|^2_o)^{\frac{2}{3}} \) by Proposition 3.2.2.

In general, determining the class of \( G_2 \)-structure that \( \tilde{\phi} \) belongs to for \( \tilde{\phi} = \phi_o + w \downarrow_{\phi_o} \phi_o \) involves some very complicated differential equations on the vector field \( w \). However, since \( \tilde{\phi} \) is always a positive 3-form for any \( w \), it may be interesting to study some of these differential equations in the simplest cases to determine if one can choose \( w \) to produce a \( \tilde{\phi} \) in a strictly smaller subclass. From Theorem 3.2.5, we have

\[
\tilde{\ast} \tilde{\phi} = (1 + |w|^2_o)^{\frac{1}{3}} (\ast_o \tilde{\phi} + w \downarrow_{\ast_o} (w \downarrow \ast_o \tilde{\phi})) \tag{3.17}
\]

For example this transformation will yield a manifold of holonomy \( G_2 \) if \( w \) satisfies

\[
0 = d(\phi_o + w \downarrow_{\phi_o} \phi_o)
\]

\[
0 = d \left( (1 + |w|^2_o)^{\frac{1}{3}} (\ast_o \phi_o + \ast_o (w \downarrow \ast_o \phi_o)) + w \downarrow_{\ast_o} (w \downarrow \phi_o) \right)
\]

The ellipticity and other properties of this system under certain hypotheses is currently being investigated [25].

### 3.3. Infinitesimal deformations in the \( \wedge^3 \) direction.

Now since the decomposition of the space of differential forms corresponding to the \( G_2 \)-structure \( \varphi \) \emph{changes} when we add something in \( \wedge^3 \), we could also consider a one-parameter family \( \varphi_t \) of \( G_2 \)-structures satisfying

\[
\frac{\partial}{\partial t} \varphi_t = w \downarrow_{\ast_t} \varphi_t \tag{3.18}
\]

for a fixed vector field \( w \). That is, at each time \( t \), we move in the direction \( w \downarrow_{\ast_t} \varphi_t \) which is a 3-form in \( \wedge^3 \), the decomposition depending on \( t \). Since the Hodge star \( \ast_t \) is also changing in time, this is \emph{a priori} a nonlinear equation. However, our first observation is that this is in fact not the case:

**Proposition 3.3.1.** Under the flow described by equation (3.18), the metric \( g \) does not change. Hence the volume form and Hodge star are also constant.

**Proof.** From (2.25) which gives the metric from the 3-form, we have:

\[
g_t(u, v) \operatorname{vol}_t = \frac{1}{6} (u \downarrow \varphi_t) \wedge (v \downarrow \varphi_t) \wedge \varphi_t
\]
Differentiating with respect to $t$, and using the differential equation (3.18),

$$6 \frac{\partial}{\partial t} (g_t(u, v) \text{vol}_t) = (u \wedge w \wedge * \varphi_t) \wedge \varphi_t + (v \wedge w \wedge * \varphi_t) \wedge \varphi_t$$

$$+ (u \wedge * \varphi_t) \wedge (v \wedge w \wedge * \varphi_t) \wedge \varphi_t$$

Now from the proof of Lemma 3.2.1 (the linear term) we see that this expression is zero, by polarizing. From this it follows easily by taking determinants that $\text{vol}_t$ is constant and thus so is $g_t$ and $*_{t}$. □

Therefore we can replace $*_{t}$ by $*_{0} = *$ and equation (3.18) is actually linear. Moreover, the flow determined by this linear equation gives a one-parameter family of $G_2$-structures each yielding the same metric $g$. Our equation is now

$$\frac{\partial}{\partial t} \varphi_t = w \wedge * \varphi_t = A \varphi_t$$

where $A$ is the linear operator $\alpha \mapsto A \alpha = w \wedge * \alpha$ on $\wedge^3$.

**Proposition 3.3.2.** The operator $A$ is skew-symmetric. Furthermore, the eigenvalues $\lambda$ of $A$ and their multiplicities $N_{\lambda}$ are:

$$\begin{align*}
\lambda &= 0 \quad N_{\lambda} = 21 \\
\lambda &= i|w| \quad N_{\lambda} = 7 \\
\lambda &= -i|w| \quad N_{\lambda} = 7
\end{align*}$$

**Proof.** Let $e^i, e^2, \ldots, e^{35}$ be a basis of $\wedge^3$. Then

$$A_{ij} \text{vol} = \langle e^i, Ae^j \rangle \text{vol}$$

$$= e^i \wedge *(w \wedge * e^j)$$

$$= -e^i \wedge w^b \wedge e^j$$

$$= w^b \wedge e^i \wedge e^j$$

$$= -A_{ji} \text{vol}$$

since 3-forms anti-commute. Therefore $A$ is diagonalizable over $\mathbb{C}$. Suppose now that $\alpha \in \wedge^3$ is an eigenvector with eigenvalue $\lambda = 0$. Then

$$A \alpha = w \wedge * \alpha$$

$$= -*(w^b \wedge \alpha)$$

$$= 0$$

so $w^b \wedge \alpha = 0$ and hence $\alpha = w^b \wedge \beta$ for some $\beta \in \wedge^2$. Therefore the multiplicity of $\lambda = 0$ is $\dim(\wedge^2) = 21$. If $A \alpha = \lambda \alpha$ for $\lambda \neq 0$, then $\alpha = \frac{1}{\lambda}(w \wedge * \alpha)$ and $w \wedge \alpha = 0$. Then we can write (A.10) as

$$|w|^2 \alpha = -w \wedge * (w \wedge * \alpha) = -A^2 \alpha = -\lambda^2 \alpha$$

and hence $\lambda = \pm i|w|$. Since the eigenvalues come in complex conjugate pairs and there are $35 - 21 = 14$ remaining, there must be 7 of each. This completes the proof. □

Now if $\alpha$ is an eigenvector for $\frac{\partial}{\partial t} \alpha_t = A \alpha_t = \lambda \alpha_t$, then $\alpha(t) = e^{\lambda t} \alpha(0)$. Let $u_1, u_2, \ldots, u_{21}$ be a basis for the $\lambda = 0$ eigenspace, and $v_1, \ldots, v_7$ and $\bar{v_1}, \ldots, \bar{v_7}$.
be bases of complex eigenvectors corresponding to the $\lambda = +i|w|$ and $\lambda = -i|w|$ eigenspaces, respectively. We can write

$$\phi_0 = \sum_{k=1}^{7} c_k v_k + \sum_{k=1}^{7} \bar{c}_k \bar{v}_k + \sum_{k=1}^{21} h_k u_k$$

where $\eta_0$ as defined by the above equation is the part of $\phi_0$ in the kernel of $A$. Then the solution is given by

$$\phi_t = \sum_{k=1}^{7} c_k e^{i|w|t} v_k + \sum_{k=1}^{7} \bar{c}_k e^{-i|w|t} \bar{v}_k + \eta_0$$

$$\varphi_t = \cos(|w|t) \sum_{k=1}^{7} (c_k v_k + \bar{c}_k \bar{v}_k) + \sin(|w|t) \sum_{k=1}^{7} i(c_k v_k - \bar{c}_k \bar{v}_k) + \eta_0$$

(3.19)

All that remains is to determine $\beta_0$, $\gamma_0$, and $\eta_0$ in terms of the initial condition $\varphi_0$. Substituting $t = 0$ into (3.19), we have

$$\varphi_0 = \beta_0 + \eta_0$$

Differentiating, we have

$$\frac{\partial}{\partial t} \varphi_t = -|w| \sin(|w|t) \beta_0 + |w| \cos(|w|t) \gamma_0$$

$$A \varphi_t = \cos(|w|t) A \beta_0 + \sin(|w|t) A \gamma_0 + A \eta_0$$

Comparing coefficients, we have

$$A \beta_0 = |w| \gamma_0$$

$$A \gamma_0 = -|w| \beta_0$$

$$A \eta_0 = 0$$

From $\beta_0 = \varphi_0 - \eta_0$ and the equations above, we get

$$\gamma_0 = \frac{1}{|w|} (A \varphi_0)$$

and substituting this into the second equation, we obtain

$$\beta_0 = -\frac{1}{|w|^2} (A^2 \varphi_0)$$

Finally, we can state the general solution:

**Theorem 3.3.3.** The solution to the differential equation

$$\frac{\partial}{\partial t} \varphi_t = w \cdot \ast_t \varphi_t$$

is given by

$$\varphi(t) = \varphi_0 + \frac{1 - \cos(|w|t)}{|w|^2} (w \cdot \ast (w \cdot \ast \varphi_0)) + \frac{\sin(|w|t)}{|w|} (w \cdot \ast \varphi_0)$$

(3.20)

The solution exists for all time and is closed curve in $\wedge^3$. Also, the path only depends on $\pm \frac{w}{|w|}$, and the norm $|w|$ only affects the speed of travel along this curve.
Proof. This is all immediate from the above discussion. □

Remark 3.3.4. In [6], it is shown that the set of $G_2$-structures on $M$ which correspond to the same metric as that of a fixed $G_2$-structure $\varphi_0$ is an $\mathbb{RP}^7$-bundle over the manifold $M$. The above theorem gives an explicit formula (3.20) for a path of $G_2$-structures all corresponding to the same metric $g$ starting from an arbitrary vector field $w$ on $M$.

Remark 3.3.5. This can also be compared to the Kähler case. Since the metric and the almost complex structure $J$ are independent in this case, for a fixed metric $g$, the family of 2-forms $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ for varying $J$’s are all Kähler forms corresponding to the same metric.

Remark 3.3.6. Even though the metric is unchanged under an infinitesimal deformation in the $\wedge^3_7$ direction, the class of $G_2$-structure can change. Therefore simply knowing that a metric on a 7-manifold arises from a $G_2$-structure and knowing the metric explicitly does not determine the class.

Remark 3.3.7. Removed after publication. This was incorrect.

We now apply this theorem to two specific examples, where we will reproduce known results.

Example 3.3.8. Let $N$ be a Calabi-Yau threefold, with Kähler form $\omega$ and holomorphic $(3,0)$ form $\Omega$. The complex coordinates will be denoted by $z^j = x^j + iy^j$. Then there is a natural $G_2$-structure $\varphi$ on the product $N \times S^1$ given by

$$\varphi = \text{Re}(\Omega) + d\theta \wedge \omega$$

where $\theta$ is the coordinate on the circle $S^1$. This induces the product metric on $N \times S^1$, with the flat metric on $S^1$. With the orientation on $N \times S^1$ given by $(x^1, x^2, x^3, \theta, y^1, y^2, y^3)$, it is easy to check that

$$\ast \varphi = -d\theta \wedge \text{Im}(\Omega) + \frac{\omega^2}{2}$$

Now let $w = \frac{\partial}{\partial \theta}$ be a globally defined non-vanishing vector field on $S^1$ with $|w| = 1$. Then we have

$$w \ast \varphi = -\text{Im}(\Omega)$$
$$\ast (w \ast \varphi) = -d\theta \wedge \text{Re}(\Omega)$$
$$w \ast (w \ast \varphi) = -\text{Re}(\Omega)$$

Thus for this choice of vector field $w$, the flow in (3.20) is given by

$$\varphi_t = \text{Re}(\Omega) + d\theta \wedge \omega - (1 - \cos(t)) \text{Re}(\Omega) - \sin(t) \text{Im}(\Omega)$$

$$= \cos(t) \text{Re}(\Omega) - \sin(t) \text{Im}(\Omega) + d\theta \wedge \omega$$

which is the canonical $G_2$ form on $N \times S^1$ where now the Calabi-Yau structure on $N$ is given by $e^{it}\Omega$ and $\omega$. It is well-known that we have this freedom of changing the holomorphic volume form $\Omega$ by a phase and preserving the Ricci-flat metric. Here it arises naturally using the flow described by (3.20) and the canonical vector field $w = \frac{\partial}{\partial \theta}.$
Example 3.3.9. As a second example, let $W$ be a $K3$ surface, which is hyperKähler with hyperKähler triple $\omega_1, \omega_2,$ and $\omega_3$. With local complex coordinates $z^1 = y^0 + iy^1$ and $z^2 = y^2 + iy^3$ these forms can be written as:

$$\begin{align*}
\omega_1 &= dy^0 \wedge dy^1 + dy^2 \wedge dy^3 \\
\omega_2 &= dy^0 \wedge dy^2 + dy^3 \wedge dy^1 \\
\omega_3 &= dy^0 \wedge dy^3 + dy^1 \wedge dy^2
\end{align*}$$

The volume form $\text{vol}_W$ on $W$ is given by $\frac{\omega^2}{2}$ for any $j = 1, 2, 3$. There is a natural $G_2$-structure $\varphi$ on the product $W \times T^3$ given by

$$\varphi = d\theta^1 \wedge d\theta^2 \wedge d\theta^3 - d\theta^1 \wedge \omega_1 - d\theta^2 \wedge \omega_2 - d\theta^3 \wedge \omega_3$$

where $\theta^1, \theta^2, \theta^3$ are coordinates on the torus $T^3$. This induces the product metric on $W \times T^3$, with the flat metric on $T^3$. With the orientation on $W \times T^3$ given by $(\theta^1, \theta^2, \theta^3, y^0, y^1, y^2, y^3)$, it is easy to check that

$$*\varphi = \text{vol}_W - d\theta^2 \wedge d\omega^3 \wedge \omega_1 - d\theta^3 \wedge d\omega^1 \wedge \omega_2 - d\theta^1 \wedge d\omega^2 \wedge \omega_3$$

Now let $w = \frac{\partial}{\partial \theta^j}$ be one of the globally defined non-vanishing vector fields on $T^3$ with $|w| = 1$. Then we have

$$\begin{align*}
w_j * \varphi &= d\theta^3 \wedge \omega_2 - d\theta^2 \wedge \omega_3 \\
* (w_j * \varphi) &= d\theta^1 \wedge d\theta^2 \wedge \omega_2 - d\theta^1 \wedge d\omega^3 \wedge \omega_3 \\
w_j * (w_j * \varphi) &= d\theta^2 \wedge \omega_2 + d\theta^3 \wedge \omega_3
\end{align*}$$

Thus for this choice of vector field $w$, the flow in (3.20) is given by

$$\begin{align*}
\varphi_t &= d\theta^1 \wedge d\theta^2 \wedge d\theta^3 - d\theta^1 \wedge \omega_1 - d\theta^2 \wedge \omega_2 - d\theta^3 \wedge \omega_3 \\
&+ (1 - \cos(t)) (d\theta^2 \wedge \omega_2 + d\theta^3 \wedge \omega_3) + \sin(t) (d\theta^3 \wedge \omega_2 - d\theta^2 \wedge \omega_3) \\
&= d\theta^1 \wedge d\theta^2 \wedge d\theta^3 - d\theta^1 \wedge \omega_1 \\
&- d\theta^2 \wedge (\cos(t) \omega_2 + \sin(t) \omega_3) - d\theta^3 \wedge (- \sin(t) \omega_2 + \cos(t) \omega_3)
\end{align*}$$

which is the canonical $G_2$ form on $W \times T^3$ where now the hyperKähler triple on $W$ is given by $\tilde{\omega}_1 = \omega_1, \tilde{\omega}_2 = \cos(t) \omega_2 + \sin(t) \omega_3,$ and $\tilde{\omega}_3 = - \sin(t) \omega_2 + \cos(t) \omega_3$. This is just a restatement of the fact that on a hyperKähler manifold, we have an $S^2$ worth of complex structures, and we can choose any triple $I, J, K$ such that $IJ = K$ to obtain the three Kähler forms. The above construction corresponds to a hyperKähler rotation where $J \mapsto J \cos(t) + K \sin(t)$ and $K \mapsto -J \sin(t) + K \cos(t)$. This is rotation by an angle $t$ around the axis in $S^2$ that represents the complex structure $I$. It is clear that we can hyperKähler rotate around any axis by taking $w = a^1 \frac{\partial}{\partial \theta^1} + a^2 \frac{\partial}{\partial \theta^2} + a^3 \frac{\partial}{\partial \theta^3}$, where $(a^1)^2 + (a^2)^2 + (a^3)^2 = 1$ in the flow described by (3.20). All these hyperKähler structures on $W$ yield the same metric, and hence determine the same metric on $W \times T^3$, as expected.

4. Manifolds with a Spin(7)-structure

4.1. Spin(7)-structures. Let $M$ be an oriented 8-manifold with a global 3-fold cross product structure. Such a structure will henceforth be called a Spin(7)-structure. Its existence is also given by (this time more complicated) topological conditions. (Again see [14, 24, 26] for details.) Similarly to the $G_2$ case, this cross
product $X(\cdot, \cdot)$ gives rise to an associated *Riemannian* metric $g$ and an alternating 4-form $\Phi$ which are related by:

$$\Phi(a, b, c, d) = g(X(a, b, c), d).$$

As in the $G_2$ case, the metric and the cross product structure cannot be prescribed independently. We will see in Section 4.3 how the 4-form $\Phi$ determines the metric $g(\cdot, \cdot)$. For a (7)-structure $\Phi$, near a point $p \in M$ we can choose local coordinates $x^0, x^1, \ldots, x^7$ so that at the point $p$, we have:

$$\Phi_p = dx^{0123} - dx^{0167} - dx^{0527} - dx^{0563} + dx^{0415} + dx^{0426} + dx^{0437}
+ dx^{4567} - dx^{4523} - dx^{4163} - dx^{4127} + dx^{2637} + dx^{1537} + dx^{1526}$$

(4.2) where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$. In these coordinates the metric at $p$ is the standard Euclidean metric

$$g_p = \sum_{k=1}^{8} dx^k \otimes dx^k$$

and $*\Phi = \Phi$, so $\Phi$ is *self-dual*.

The 4-forms that arise from a (7)-structure are called *positive* or *non-degenerate*, and this set is denoted $\wedge^4_{\text{pos}}$. The subgroup of $SO(8)$ that preserves $\Phi_p$ is (7). (see [4].) Hence at each point $p$, the set of (7)-structures at $p$ is isomorphic to $GL(8, \mathbb{R})/Spin(7)$, which is $64 - 21 = 43$ dimensional. This time, however, in contrast to the $G_2$ case, since $\wedge^4_{\text{pos}}(\mathbb{R}^8)$ is $70$ dimensional, the set $\wedge^4_{\text{pos}}(p)$ of positive 4-forms at $p$ is *not* an open subset of $\wedge^4_p$. One of the consequences of this is that the analogous non-infinitesimal deformation in the (7) case will *not* work. This is discussed in Section 5.2.

### 4.2. Decomposition of $\wedge^*(M)$ into irreducible Spin(7)-representations.

The facts collected in this section about the decomposition of the space of forms in the (7) case can also be found in [9, 24, 26].

There is an action of the group Spin(7) on $\mathbb{R}^8$, and hence on the spaces $\wedge^*$ of differential forms on $M$. We can decompose each space $\wedge^k$ into irreducible Spin(7)-representations. The results of this decomposition are presented below. As before, the notation $\wedge^k_l$ refers to an $l$-dimensional irreducible Spin(7)-representation which is a subspace of $\wedge^k$, $w$ is a vector field on $M$ and vol is the volume form.

\[
\begin{align*}
\wedge^0 &= \{ f \in C^\infty(M) \} & \wedge^1 &= \{ \alpha \in \Gamma(\wedge^1(M)) \} \\
\wedge^2 &= \wedge^2_7 \oplus \wedge^2_{21} & \wedge^3 &= \wedge^3_8 \oplus \wedge^3_{48} \\
\wedge^4 &= \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27} \oplus \wedge^4_{35} & \wedge^5 &= \wedge^5_8 \oplus \wedge^5_{48} \\
\wedge^6 &= \wedge^6_7 \oplus \wedge^6_{21} & \wedge^7 &= \{ w \cdot \text{vol} \} \\
\wedge^8 &= \{ \text{vol; } f \in C^\infty(M) \} \\
\end{align*}
\]

This decomposition respects the *Hodge star* $*$ operator since Spin(7) $\subset SO(8)$, so $*\wedge^k_l = \wedge^{8-k}_l$. Again, we will give explicit descriptions of the remaining cases.

Before we describe $\wedge^k_l$ for $k = 2, 3, 4, 5, 6$, there are some isomorphisms between these subspaces:
Proposition 4.2.1. The map $\alpha \mapsto \Phi \wedge \alpha$ is an isomorphism between the following spaces:
\[
\begin{align*}
\wedge^0_1 & \cong \wedge^4_1 & \wedge^1_8 & \cong \wedge^5_8 \\
\wedge^2_7 & \cong \wedge^6_7 & \wedge^2_{21} & \cong \wedge^6_{21} \\
\wedge^3_8 & \cong \wedge^7_8 & \wedge^4_1 & \cong \wedge^8_1
\end{align*}
\]
In addition, if $\alpha$ is a 1-form, we have the following identity:
\[(4.3) \quad * (\Phi \wedge * (\Phi \wedge \alpha)) = -7 \alpha\]

Proof. These statements can be easily checked pointwise using local coordinates and (4.2). □

From this identity, we can prove the following lemma:

Lemma 4.2.2. If $\alpha$ is a 1-form on $M$, then we have:
\[(4.4) \quad |\Phi \wedge \alpha|^2 = 7 |\alpha|^2\]

Proof. From (4.3), we have:
\[
\begin{align*}
\Phi \wedge * (\Phi \wedge \alpha) &= 7 * \alpha \\
\alpha \wedge \Phi \wedge * (\Phi \wedge \alpha) &= 7 \alpha \wedge * \alpha \\
|\Phi \wedge \alpha|^2 \text{vol} &= 7 |\alpha|^2 \text{vol}
\end{align*}
\]
which proves (4.4). □

We have some relations between $\Phi$ and an arbitrary vector field $w$:

Lemma 4.2.3. The following relations hold for any vector field $w$, where $w^\flat$ is the associated 1-form:
\[(4.5) \quad * \left( \Phi \wedge w^\flat \right) = w \cdot \Phi \]
\[(4.6) \quad \Phi \wedge (w \cdot \Phi) = 7 * w^\flat\]

Proof. Since on an 8-manifold $*^2 = (-1)^k$ on $k$-forms, these results follow from Lemma A.0.1 and Proposition 4.2.1. □

We now explicitly describe the decomposition of the space of forms, beginning with $k = 2, 6$. These should be compared to the $G_2$ case which were given in (2.14) – (2.17).
\[(4.7) \quad \wedge^2_7 = \{ \beta \in \wedge^2; \, * (\Phi \wedge \beta) = 3 \beta \} \]
\[(4.8) \quad \wedge^2_{21} = \{ \beta \in \wedge^2; \, * (\Phi \wedge \beta) = -\beta \} \]
\[= \{ \sum a_{ij} e^i \wedge e^j; \, (a_{ij}) \in \text{spin}(7) \} \]
\[(4.9) \quad \wedge^6_7 = \{ \mu \in \wedge^6; \, \Phi \wedge * \mu = 3 \mu \} \]
\[(4.10) \quad \wedge^6_{21} = \{ \mu \in \wedge^6; \, \Phi \wedge * \mu = -\mu \} \]

Note that these subspaces are +3 and −1 eigenspaces of the operators $G(\beta) = * (\Phi \wedge \beta)$ on $\wedge^2$ and $M(\mu) = \Phi \wedge * \mu$ on $\wedge^6$. From this fact we get the following useful formulas for the projections $\pi_k$ onto the $k$-dimensional representations, for $\beta \in \wedge^2$ and $\mu \in \wedge^6$:
(4.11) \[ * (\Phi \wedge \beta) = 3\pi_7(\beta) - \pi_{21}(\beta) \]
\[ \pi_7(\beta) = \frac{\beta + * (\Phi \wedge \beta)}{4} \]
\[ \pi_{21}(\beta) = \frac{3\beta - * (\Phi \wedge \beta)}{4} \]

and

\[ \Phi \wedge * \mu = 3\pi_7(\mu) - \pi_{21}(\mu) \]
\[ \pi_7(\mu) = \frac{\mu + \Phi \wedge * \mu}{4} \]
\[ \pi_{21}(\mu) = \frac{3\mu - \Phi \wedge * \mu}{4} \]

We now move on to the decompositions for \( k = 3, 4, 5 \). For \( k = 3 \), we have:

\[ \wedge^3 8 = \{ * (\Phi \wedge \alpha); \alpha \in \wedge^1 8 \} \]
\[ = \{ w \wedge \Phi; w \in \Gamma(T(M)) \} \]
\[ = \{ \eta \in \wedge^3; * (\Phi \wedge * (\Phi \wedge \eta)) = -7\eta \} \]

(4.12)

(4.13)

For \( k = 5 \), the decomposition is:

\[ \wedge^5 8 = \{ \Phi \wedge \alpha; \alpha \in \wedge^1 8 \} \]
(4.14)

\[ \wedge^5 48 = \{ \mu \in \wedge^1; \Phi \wedge * \mu = 0 \} \]
(4.15)

And finally, the middle dimension \( k = 4 \) decomposes as:

\[ \wedge^4 4 = \{ f \Phi; f \in C^\infty(M) \} \]
\[ = \{ \eta \in \wedge^4; \Phi \wedge * (\Phi \wedge \eta) = 14\eta \} \]
(4.16)

\[ \wedge^4 17 = \{ \beta_i e^i \wedge (e_j, \Phi) - \beta_j e^i \wedge (e_i, \Phi); \beta_{ij} e^i \wedge e^j \in \wedge^2 7 \} \]
(4.17)

\[ \wedge^4 27 = \{ \sigma \in \wedge^4; * \sigma = \sigma, \sigma \wedge \Phi = 0, \sigma \wedge \tau = 0 \ \forall \tau \in \wedge^2 7 \} \]
(4.18)

\[ \wedge^4 35 = \{ \sigma \in \wedge^4; * \sigma = -\sigma \} \]
(4.19)

4.3. The metric of a Spin(7)-structure. Here the situation differs significantly from the \( G_2 \) case. Because \( \Phi \) is self-dual equation (4.3) gives us only one useful identity rather than the four identities in equations (2.4) – (2.6). In particular it was equation (2.6) which enabled us to prove Proposition 2.3.1 to obtain a formula for the metric from the 3-form \( \phi \) in the \( G_2 \) case.

The prescription for obtaining the metric from the 4-form \( \Phi \) in the Spin(7) case is much more complicated. Before we can do this, we need to collect some facts about various 2-forms which can be constructed from pairs of vector fields, as these facts will be used both to determine the metric and later to analyze how it changes under a \( \wedge^4 4 \) deformation in Section 5.2.

Proposition 4.3.1. Let \( a, b, c, d \) be vector fields. Define the 2-forms \( \beta = a^\flat \wedge b^\flat = \beta_7 + \beta_{21} \) and \( \mu = c^\flat \wedge d^\flat = \mu_7 + \mu_{21} \). Then we can construct other 2-forms \( a \wedge b, \Phi \) and \( * ((a \wedge \Phi) \wedge (b \wedge \Phi)) \) from \( a \) and \( b \) and these are related to \( \beta \) by

\[ a \wedge b \wedge \Phi = -3\beta_7 + \beta_{21} \]
(4.20)

\[ * ((a \wedge \Phi) \wedge (b \wedge \Phi)) = 2\beta_7 - 6\beta_{21} \]
(4.21)
Furthermore, if we define
\[
A = \langle a \wedge b, c \wedge d \rangle = \langle a, c \rangle \langle b, d \rangle - \langle a, d \rangle \langle b, c \rangle
\]
\[
B = \Phi(a, b, c, d)
\]
then the following relations hold between these 2-forms:
\[
(a \wedge b) \wedge (c \wedge d) \wedge \Phi = (-3A - 2B) \text{vol}
\]
\[
(a^* \wedge b^*) \wedge (c^* \wedge d^*) \wedge \Phi = (-4A + 2B) \text{vol}
\]
\[
(a \wedge b) \wedge (c \wedge d) \wedge \Phi = (6A + 7B) \text{vol}
\]

Proof. Let \( \beta = a^* \wedge b^* = \beta_7 + \beta_{21} \) using the decompositions in (4.7) and (4.8). From Lemma A.0.1 we can write
\[
(a \wedge b) \wedge \Phi = *((a^* \wedge *((b, \Phi)))
\]
\[
= - *[a^* \wedge b^* \wedge \Phi]
\]
\[
= -3\beta_7 + \beta_{21}
\]
where we have used the self-duality \(*\Phi = \Phi\) and the characterizations of \(\wedge^2\) and \(\wedge^2_{21}\). Now since \(\Phi \wedge \Phi = 14 \text{vol}\), we have
\[
(w \wedge \Phi) \wedge \Phi = 7w \wedge \text{vol} = 7 * w^b
\]
where we have used (A.6). Taking the interior product on both sides with \(v\),
\[
(v \wedge w) \wedge \Phi - (w \wedge \Phi) \wedge (v \wedge \Phi) = 7v \wedge \Phi
\]
\[
= -7 * (v^b \wedge w^b)
\]
\[
(-3\beta_7 + \beta_{21}) \wedge \Phi + (v \wedge \Phi) \wedge (w \wedge \Phi) = -7 * \beta_7 - 7 * \beta_{21}
\]
\[
-9 * \beta_7 - *\beta_{21} + (v \wedge \Phi) \wedge (w \wedge \Phi) = -7 * \beta_7 - 7 * \beta_{21}
\]
which can be rearranged to give (4.21). We also have
\[
B \text{vol} = \Phi(a, b, c, d) \text{vol} = a^* \wedge b^* \wedge c^* \wedge d^* \wedge \Phi
\]
\[
= (\beta_7 + \beta_{21}) \wedge (3 * \mu_7 - *\mu_{21})
\]
\[
= (3(\beta_7, \mu_7) - \langle \beta_{21}, \mu_{21} \rangle) \text{vol}
\]
and
\[
A = \langle \beta, \mu \rangle = \langle \beta_7, \mu_7 \rangle + \langle \beta_{21}, \mu_{21} \rangle
\]
which together give that
\[
\langle \beta_7, \mu_7 \rangle = A + B \quad \langle \beta_{21}, \mu_{21} \rangle = 3A - B
\]
Hence, for example
\[
(a \wedge b) \wedge (c \wedge d) \wedge \Phi = (-3\beta_7 + \beta_{21}) \wedge (-9 * \mu_7 - *\mu_{21})
\]
\[
= 27 \left( \frac{A + B}{4} \right) \text{vol} - \left( \frac{3A - B}{4} \right) \text{vol}
\]
\[
= (6A + 7B) \text{vol}
\]
which is (4.26). The other two are obtained similarly. \(\square\)

Proposition 4.3.1 immediately yields the following corollary, which is analogous to Proposition 2.3.1 in the G_2 case.
Corollary 4.3.2. The following identity holds for $v$ and $w$ vector fields:

\[ (v \wedge w) \wedge (v \wedge w) \wedge \Phi = 6|v \wedge w|^2 \text{vol} \]

Proof. This follows from (4.26). \qed

If we polarize (4.27) in $w$, we obtain the useful equation:

\[
(v \wedge w_1) \wedge (v \wedge w_2) \wedge \Phi = 6(v \wedge w_1, v \wedge w_2) \text{vol}
\]

(4.28)

From this equation we can obtain the metric.

Lemma 4.3.3. Fix a non-zero vector field $v = v^k e_k$, and extend to $v, e_1, e_2, \ldots, e_7$ an oriented local frame of vector fields. The expression obtained from $v$ by

\[
(4.29) \quad \frac{(\det(((e_i \wedge v \wedge \Phi) \wedge (v \wedge \Phi)) (e_1, e_2, \ldots, e_7)))^\frac{1}{3}}{((v \wedge \Phi) \wedge \Phi)(e_1, e_2, \ldots, e_7))^{\frac{1}{3}}}
\]

is homogeneous of order 4 in $v$, and is independent of the choice of extension to a basis $e_1, \ldots, e_7$. We will see in the next theorem that up to a constant, this is $|v|^4$.

Proof. The expression $(e_i \wedge v \wedge \Phi) \wedge (e_j \wedge v \wedge \Phi) \wedge (v \wedge \Phi)$ is cubic in $v$, so after taking the $7 \times 7$ determinant and the cube root, the numerator of (4.29) is of order 7 in $v$. Since the denominator is cubic in $v$, the whole expression is homogeneous of degree 4 in $v$. Now suppose we extend $v$ to an oriented basis $v, e'_1, e'_2, \ldots, e'_7$ in a different way. Then

\[ e'_i = P_{ij} e_j + Q_i v \]

and we have

\[
(e'_i \wedge v \wedge \Phi) \wedge (e'_j \wedge v \wedge \Phi) \wedge (v \wedge \Phi) = P_{ik} P_{jl} (e_k \wedge v \wedge \Phi) \wedge (e_j \wedge v \wedge \Phi) \wedge (v \wedge \Phi)
\]

Hence in the new basis the numerator of (4.29) changes by a factor of

\[
(\det(P)^2 \det(P)^7)^\frac{1}{3} = \det(P)^3
\]

and the denominator also changes by a factor of $\det(P)^3$, so the quotient is invariant. \qed

Remark 4.3.4. Note how different from Lemma 2.26 this is. In this case, we have a determinant in the numerator rather than the denominator. Also, the numerator and denominator are totally different in the Spin(7) case whereas one is the determinant of the other in the $G_2$ case.

We now derive the expression for the metric in terms of the 4-form $\Phi$ in the Spin(7) case.

Theorem 4.3.5. Let $v$ be a non-zero tangent vector at a point $p$ and let $e_0, e_1, \ldots, e_7$ be any oriented basis for $T_p M$, so that $\text{vol}(e_0, e_1, \ldots, e_7) > 0$. Assume without loss of generality that $v^0 \neq 0$. Then the length $|v|$ of $v$ is given by

\[
|v|^4 = \frac{(7)^3 (\det(((e_1 \wedge v \wedge \Phi) \wedge (e_j \wedge v \wedge \Phi) \wedge (v \wedge \Phi)) (e_1, e_2, \ldots, e_7)))^{\frac{1}{3}}}{(6)^{\frac{1}{3}} (((v \wedge \Phi) \wedge \Phi)(e_1, e_2, \ldots, e_7))^{\frac{1}{3}}}
\]
Proof. We work in local coordinates at the point \( p \). In this notation \( g_{ij} = \langle e_i, e_j \rangle \) with \( 0 \leq i, j \leq 7 \). Let \( \text{det}_8(g) \) denote the \( 8 \times 8 \) determinant of \( (g_{ij}) \) and let \( \text{det}_7(g) \) denote the \( 7 \times 7 \) determinant of the submatrix where \( 1 \leq i, j \leq 7 \). Using the fact that \( \Phi^2 = 14 \text{ vol} = 14 \sqrt{\text{det}_8(g)} e^0 \wedge e^1 \ldots \wedge e^7 \), and writing \( v = v^ke_k \), we compute

\[
A(v) = ((v^i\Phi) \wedge \Phi)(e_1, e_2, \ldots, e_7)
\]

\[
= 7v^0 \sqrt{\text{det}_8(g)}
\]

Now \( \langle v, e_j \rangle = v^kg_{kj} = v_j \). We also have the \( 7 \times 7 \) matrix (for \( 1 \leq i, j \leq 7 \))

\[
B_{ij}(v) = ((e_i, v^j\Phi) \wedge (e_j, v^i\Phi) \wedge (v^3\Phi)) (e_1, e_2, \ldots, e_7)
\]

\[
= 6 \left( |v|^2 g_{ij} - v_iv_j \right) v^0 \sqrt{\text{det}_8(g)}
\]

where we have used Corollary 4.3.2. Now consider the \( 7 \times 7 \) matrix \((|v|^2 g_{ij} - v_iv_j)\). By examining the proof of Lemma A.0.4, we see that its determinant is

\[
|v|^{14} \text{det}_7(g) - |v|^{12} |v^b \wedge e^0|^2 \text{det}_8(g)
\]

Now from Cramer’s rule \( \text{det}_7(g) = g^{00} \text{det}_8(g) \) and we also have \( |v^b \wedge e^0|^2 = |v|^2 g^{00} - v^0v^0 \). Hence (4.32) becomes

\[
|v|^{12} |v^0v^0 \text{det}_8(g)
\]

Returning to (4.31), we have now shown that

\[
\text{det} B_{ij}(v) = 6^7 |v|^{12} (v^0)^2 \text{det}_8(g) (v^0)^7 (\text{det}_8(g))^{\frac{7}{2}}
\]

\[
= 6^7 |v|^{12} (v^0)^9 (\text{det}_8(g))^3
\]

and hence

\[
(\text{det} B_{ij}(v))^3 = 6^7 |v|^{12} (v^0)^3 (\text{det}_8(g))^\frac{7}{2}.
\]

Finally, since from (4.30) we have

\[
(A(v))^3 = (7)^3 (v^0)^3 (\text{det}_8(g))^{\frac{7}{2}}
\]

these two expressions can be combined to yield

\[
|v|^{4} = \left( \frac{(7)^3 (\text{det} B_{ij}(v))^3}{(6)^{\frac{7}{2}} (A(v))^3} \right)^{\frac{1}{3}}
\]

which completes the proof. \( \square \)

4.4. The triple cross product of a Spin(7)-structure. In this section we will describe the triple cross product operation on a manifold with a Spin(7)-structure in terms of the 4-form \( \Phi \), and present some useful relations.

Definition 4.4.1. Let \( u, v, \) and \( w \) be vector fields on \( M \). The triple cross product, denoted \( X(u, v, w) \), is a vector field on \( M \) whose associated 1-form under the metric isomorphism satisfies:

\[
(X(u, v, w))^k = w \wedge u \wedge v \Phi
\]

This immediately yields the relation between \( X, \Phi, \) and the metric \( g \):

\[
g(X(u, v, w), y) = (X(u, v, w))^k(y) = y \wedge w \wedge u \wedge \Phi = \Phi(u, v, w, y).
\]
We can obtain another useful characterization of the triple cross product from this one using Lemma A.0.1:

\[
(X(u, v, w))^b = w \wedge v \wedge u \Phi
\]

\[
= * (w^b \wedge (v \wedge u \Phi))
\]

\[
= * (w^b \wedge v^b \wedge * (u \wedge \Phi))
\]

\[
= - * (w^b \wedge v^b \wedge u^b \wedge * \Phi)
\]

\[
= * (u^b \wedge v^b \wedge w^b \wedge \Phi)
\]

Note the similarity to the G2 case given by (2.30).

Since \(u^b \wedge v^b \wedge w^b\) is a 3-form, we can write it as \(\gamma_8 + \gamma_{48}\), with \(\gamma_j \in \Lambda_j^3\). Using (4.12) and (4.13) we see that:

\[
(X(u, v, w))^b \wedge \Phi = * (\gamma_7 \wedge \Phi) \wedge \Phi
\]

\[
= 7 * \gamma_7
\]

Taking the norm of both sides, and using (4.4):

\[
|(X(u, v, w))^b \wedge \Phi|^2 = 7|(X(u, v, w))^b|^2 = 7|X(u, v, w)|^2 = 49|\gamma_7|^2
\]

from which we obtain

\[
|\gamma_7|^2 = \frac{1}{7}|X(u, v, w)|^2
\]

We can now establish the following Lemma.

**Lemma 4.4.2.** Let \(u, v, \) and \(w\) be vector fields. Then

\[
|X(u, v, w)|^2 = |u \wedge v \wedge w|^2
\]

**Proof.** First we note that

\[
|u \wedge v \wedge w|^2 = \det \begin{pmatrix}
|u|^2 & \langle u, v \rangle & \langle u, w \rangle \\
\langle u, v \rangle & |v|^2 & \langle v, w \rangle \\
\langle u, w \rangle & \langle v, w \rangle & |w|^2
\end{pmatrix}
\]

\[
= |u|^2 |v|^2 |w|^2 + 2 \langle u, v \rangle \langle v, w \rangle \langle u, w \rangle
\]

\[
- |u|^2 \langle v, w \rangle^2 - |v|^2 \langle u, w \rangle^2 - |w|^2 \langle u, v \rangle^2
\]

as we will have to identify an expression of this form in what follows. Starting from \(\Phi^2 = 14 \text{vol}\) and taking the interior product with \(u, v, \) and \(w,\) we obtain

\[
(w \wedge v \wedge u \Phi) \wedge \Phi = (w \wedge v \wedge u \Phi) \wedge (w \wedge \Phi)
\]

\[
- (w \wedge v \wedge u \Phi) \wedge (v \wedge \Phi)
\]

\[
- (v \wedge w \wedge \Phi) \wedge (u \wedge \Phi) = 7 * (u^b \wedge v^b \wedge w^b)
\]

where we have interchanged interior and exterior multiplication through the * operator using the identities in Appendix A. Now from (4.34) and (4.36), if we take the wedge product of the above expression with \(u^b \wedge v^b \wedge w^b\), we have

\[
|X(u, v, w)|^2 \text{vol} = \frac{1}{7}|u \wedge v \wedge w|^2 \text{vol}
\]
Consider the second term:

\[ u^b \wedge v^b \wedge w^b \wedge (u_\wedge v_\wedge \Phi) \wedge (w_\wedge \Phi) \]

This can be rewritten as

\[
\Phi \wedge \left( w_\wedge \left( u^b \wedge v^b \wedge w^b \wedge (u_\wedge v_\wedge \Phi) \right) \right) \\
= \langle w, u \rangle \Phi \wedge v^b \wedge w^b \wedge (u_\wedge v_\wedge \Phi) - \langle w, v \rangle \Phi \wedge u^b \wedge w^b \wedge (u_\wedge v_\wedge \Phi) \\
+ |w|^2 \Phi \wedge u^b \wedge v^b \wedge (u_\wedge v_\wedge \Phi) - \Phi \wedge u^b \wedge v^b \wedge w^b \wedge (w_\wedge u_\wedge v_\wedge \Phi)
\]

Using (4.24), this is equal to

\[
\left( 3\langle w, u \rangle \langle v \wedge w, v \wedge u \rangle + 3\langle w, v \rangle \langle u \wedge w, u \wedge v \rangle - 3|w|^2|u \wedge v|^2 \right) \vol + |X(u, v, w)|^2 \vol \\
= -3|u \wedge v \wedge w|^2 \vol + |X(u, v, w)|^2 \vol
\]

The other two terms are identical by cyclically permuting \( u, v, \) and \( w. \) Thus we have

\[
|X(u, v, w)|^2 \vol + 9|u \wedge v \wedge w|^2 \vol - 3|X(u, v, w)|^2 \vol = 7|u \wedge v \wedge w|^2
\]

and the lemma is proved. \( \square \)

Iterating the 3-fold cross product is described in the next lemma.

**Lemma 4.4.3.** Let \( a, b, u, v, \) and \( w \) be vector fields. Then we have

\[
X(a, b, X(u, v, w)) = \\
- \langle a \wedge b, u \wedge v \rangle w - \Phi(a, b, u, v)w + \langle b, w \rangle X(a, u, v) - \langle a, w \rangle X(b, u, v) \\
- \langle a \wedge b, v \wedge w \rangle u - \Phi(a, b, v, w)u + \langle b, u \rangle X(a, v, w) - \langle a, u \rangle X(b, v, w) \\
- \langle a \wedge b, w \wedge u \rangle v - \Phi(a, b, w, u)v + \langle b, v \rangle X(a, w, u) - \langle a, v \rangle X(b, w, u)
\]

**Proof.** We begin with (4.40),

\[
X(u, v, w)^b \wedge \Phi = (w_\wedge u_\wedge v_\wedge \Phi) \wedge \Phi = (u_\wedge v_\wedge \Phi) \wedge (w_\wedge \Phi) + (w_\wedge u_\wedge \Phi) \wedge (v_\wedge \Phi) + (v_\wedge w_\wedge \Phi) \wedge (u_\wedge \Phi) + 7 * (w^b \wedge v^b \wedge w^b)
\]

We take the wedge product of both sides of this equation with \( a^b \wedge b^b, \) and obtain

\[
a^b \wedge b^b \wedge X(u, v, w)^b \wedge \Phi = a^b \wedge b^b \wedge (u_\wedge v_\wedge \Phi) \wedge (w_\wedge \Phi) \\
+ a^b \wedge b^b \wedge (v_\wedge w_\wedge \Phi) \wedge (u_\wedge \Phi) \\
+ a^b \wedge b^b \wedge (w_\wedge u_\wedge \Phi) \wedge (v_\wedge \Phi) \\
+ 7a^b \wedge b^b \wedge (w^b \wedge v^b \wedge w^b)
\]

Note that up to the metric isomorphism, the desired iterated cross product we are looking for is the Hodge star of the above expression. It is easy to check that the last term can be simplified to

\[
7 \left( \langle a \wedge b, v \wedge w \rangle \star u^b + \langle a \wedge b, w \wedge u \rangle \star v^b + \langle a \wedge b, u \wedge v \rangle \star w^b \right)
\]
The first three terms are identical under cyclic permutation of \(u, v, w\). Let us consider the first term. From (4.24) we have
\[
a^b \wedge b^b \wedge (u \wedge \nu \Phi) \wedge \Phi = -3 \langle a \wedge b, u \wedge v \rangle \, \text{vol} - 2 \Phi(a, b, u, v) \, \text{vol}
\]
Taking the interior product with \(w\) and rearranging,
\[
a^b \wedge b^b \wedge (u \wedge \nu \Phi) \wedge (w \wedge \Phi) = -3 \langle a \wedge b, u \wedge v \rangle \, \ast w^b - 2 \Phi(a, b, u, v) \ast w^b
\]
\[
+ \langle w, b \rangle a^b \wedge (u \wedge \nu \Phi) \wedge \Phi - \langle w, a \rangle b^b \wedge (u \wedge \nu \Phi) \wedge \Phi
\]
\[
+ a^b \wedge b^b \wedge X(u, v, w)^b \wedge \Phi
\]
Adding up the four expressions in (4.41), we have
\[
-2a^b \wedge b^b \wedge X(u, v, w)^b \wedge \Phi = 4 \langle a \wedge b, u \wedge v \rangle \ast w^b - 2 \Phi(a, b, u, v) \ast w^b
\]
\[
+ 4 \langle a \wedge b, v \wedge w \rangle \ast u^b - 2 \Phi(a, b, v, w) \ast u^b
\]
\[
+ 4 \langle a \wedge b, w \wedge u \rangle \ast v^b - 2 \Phi(a, b, w, u) \ast v^b
\]
\[
+ \langle w, b \rangle a^b \wedge (u \wedge \nu \Phi) \wedge \Phi - \langle w, a \rangle b^b \wedge (u \wedge \nu \Phi) \wedge \Phi
\]
\[
+ \langle u, b \rangle a^b \wedge (v \wedge \nu \Phi) \wedge \Phi - \langle u, a \rangle b^b \wedge (v \wedge \nu \Phi) \wedge \Phi
\]
\[
+ \langle v, b \rangle a^b \wedge (w \wedge \nu \Phi) \wedge \Phi - \langle v, a \rangle b^b \wedge (w \wedge \nu \Phi) \wedge \Phi
\]
Now consider the first of the last six terms:
\[
\langle w, b \rangle a^b \wedge (u \wedge \nu \Phi) \wedge \Phi
\]
with \(\beta = \beta_7 + \beta_{21} = u^b \wedge v^b\), exactly as in the derivation of (4.24), one can show easily that
\[
b^b \wedge (u \wedge \nu \Phi) \wedge \Phi = -3A - 2B
\]
where \(A = b^b \wedge \ast (u^b \wedge v^b) = \langle b, v \rangle \ast u^b - \langle b, u \rangle \ast v^b\) and \(B = b^b \wedge u^b \wedge v^b \wedge \Phi = - \ast X(b, u, v)\). Substituting this expression in the last six terms above, simplifying, and taking the Hodge star of both sides yields the desired result. \(\square\)

**Corollary 4.4.4.** For the special case \(v = b\), we obtain the relation
\[
X(a, b, X(u, b, w)) = -\langle a \wedge b, u \wedge b \rangle w - \langle b, w \rangle X(a, b, u) + \langle a \wedge b, w \wedge b \rangle u + \langle b, w \rangle X(a, b, w)
\]
\[
- \langle a \wedge b, w \wedge u \rangle b - \Phi(a, b, w, u)b + |b|^2 X(a, w, u) - \langle a, b \rangle X(b, w, w)
\]
This expression will be used in Section 5.2 to study the \(\wedge^4_7\) deformation. Finally, if we also have \(u = a\) we have the special case
\[
X(a, b, X(a, b, w)) = -|a \wedge b|^2 w + \langle a \wedge b, w \wedge b \rangle a + \langle a \wedge b, a \wedge w \rangle b
\]
which is analogous to (2.35) in the \(G_2\) case.

Finally, the corollary to the next theorem is analogous to Theorem 2.4.7 and we will use it in Section 5.2 to show that to first order, deforming a Spin(7)-structure by an element of \(\wedge^4_7\) does not change the metric. Note first that by (2.23), one can check (by computing explicitly) that if we start with two vector fields \(v\) and \(w\), take the projection \(\pi_7(v^b \wedge w^b)\), and transfer this to \(\wedge^4_7\), we get the 4-form \(v^b \wedge (w \wedge \Phi) - w^b \wedge (v \wedge \Phi)\) in \(\wedge^4_7\). Of course, this is not the most general element of \(\wedge^4_7\), but we will work with elements of this form.
Theorem 4.4.5. Let a, b, c, and d be vector fields. Let \( \sigma \in \wedge_4^2 \) be given by
\[
\sigma = v^b \wedge (w, \Phi) - w^d \wedge (v, \Phi)
\]
for two other vector fields v and w. Then
\[
(a \wedge b \wedge c) \wedge (d \wedge \Phi) \wedge \sigma = (3\langle w, c \rangle \Phi(a, b, v, d) - 3\langle w, d \rangle \Phi(a, b, v, c) + 3\langle w, a \rangle \Phi(c, d, v, b) - 3\langle w, b \rangle \Phi(c, d, v, a) - 3\langle v, c \rangle \Phi(a, b, w, d) + 3\langle v, d \rangle \Phi(a, b, w, c) - 3\langle v, a \rangle \Phi(c, d, w, b) + 3\langle v, b \rangle \Phi(c, d, w, a)) \text{ vol}
\]

Proof. We start with the 9-form
\[
(a \wedge b \wedge \Phi) \wedge (c \wedge d \wedge \Phi) \wedge w^b \wedge \Phi = 0
\]
and take the interior product with v:
\[
(a \wedge b \wedge \Phi) \wedge (c \wedge d \wedge \Phi) \wedge v^d (w^b \wedge \Phi) = -(a \wedge b \wedge \Phi) \wedge (c \wedge d \wedge \Phi) \wedge w^b \wedge \Phi
\]
Now interchanging the roles of v and w and subtracting, and using (4.34), we obtain
\[
(a \wedge b \wedge \Phi) \wedge (c \wedge d \wedge \Phi) \wedge \sigma = -w^b \wedge \Phi(a, b, v) \wedge (c \wedge d \wedge \Phi) \wedge \Phi - w^b \wedge \Phi(c, d, v) \wedge (a \wedge b \wedge \Phi) \wedge \Phi + v^b \wedge \Phi(a, b, w) \wedge (c \wedge d \wedge \Phi) \wedge \Phi + v^b \wedge \Phi(c, d, w) \wedge (a \wedge b \wedge \Phi) \wedge \Phi
\]
Now from (4.24), this becomes
\[
(a \wedge b \wedge \Phi) \wedge (c \wedge d \wedge \Phi) \wedge \sigma = \left(3\langle w, X(a, b, v)^b, c \wedge d \rangle + 2\Phi(w, X(a, b, v)^b, c, d)\right) \text{ vol} + \left(3\langle w, X(c, d, v)^b, a \wedge b \rangle + 2\Phi(w, X(c, d, v)^b, a, b)\right) \text{ vol} - \left(3\langle v, X(a, b, w)^b, c \wedge d \rangle + 2\Phi(v, X(a, b, w)^b, c, d)\right) \text{ vol} - \left(3\langle v, X(c, d, w)^b, a \wedge b \rangle + 2\Phi(v, X(c, d, w)^b, a, b)\right) \text{ vol}
\]
The right hand side can be further simplified to (up to a factor of vol)
\[
3\langle w, c \rangle \Phi(a, b, v, d) - 3\langle w, d \rangle \Phi(a, b, v, c) + 2\langle X(a, b, v)^b, X(c, d, w)^b \rangle + 3\langle w, a \rangle \Phi(c, d, v, b) - 3\langle w, b \rangle \Phi(c, d, v, a) + 2\langle X(c, d, v)^b, X(a, b, w)^b \rangle - 3\langle v, c \rangle \Phi(a, b, w, d) + 3\langle v, d \rangle \Phi(a, b, w, c) - 2\langle X(a, b, w)^b, X(c, d, v)^b \rangle - 3\langle v, a \rangle \Phi(c, d, w, b) + 3\langle v, b \rangle \Phi(c, d, w, a) - 2\langle X(c, d, w)^b, X(a, b, v)^b \rangle
\]
The last terms in each line cancel and the theorem is proved. \( \square \)

Corollary 4.4.6. Let h, u_1, and u_2 be vector fields. Let \( \sigma \in \wedge_4^2 \) be given by
\[
\sigma = v^b \wedge (w, \Phi) - w^d \wedge (v, \Phi)
\]
for two other vector fields v and w. Then
\[
(h \wedge u_1 \wedge \Phi) \wedge (h \wedge u_2 \wedge \Phi) \wedge \sigma = 0
\]
Proof. Letting \( a = c \) in Theorem 4.4.5 yields this result. \( \square \)

Remark 4.4.7. We can actually show the stronger result that in terms of the decompositions in (4.7) and (4.17), the wedge product map
\[
\wedge_7^2 \times \wedge_7^2 \times \wedge_7^4 \to \wedge_7^8
\]
is the zero map. This is a direct analogy with Theorem 2.4.7. However, we will not have occasion to use this fact.

4.5. The 4 classes of Spin(7)-structures. Similar to the classification of G₂-structures by Fernández and Gray in [11], Fernández studied Spin(7)-structures in [9]. In this case, the results are slightly different because a 4-form Φ which determines a Spin(7)-structure is self-dual. Such a manifold has holonomy a subgroup of Spin(7) if and only if ∇Φ = 0, which Fernández showed to be equivalent to
\[ dΦ = 0. \]

Again this equivalence was established by decomposing the space \( W \) that \( ∇Φ \) belongs to into irreducible Spin(7)-representations, and comparing the invariant subspaces of \( W \) to the isomorphic spaces in \( \wedge^*(M) \). In the Spin(7) case this space \( W \) decomposes as
\[ W = W_8 \oplus W_{48} \]
where again the subscript \( k \) denotes the dimension of the irreducible representation \( W_k \). Again in analogy with the G₂ case, we have a canonically defined 7-form \( ζ \) and 1-form \( θ \), given by
\begin{align*}
ζ &= *dΦ ∧ Φ \quad (4.42) \\
θ &= *ζ = *(dΦ ∧ Φ) \quad (4.43)
\end{align*}

Note that \( θ = 0 \) when the manifold has holonomy contained in Spin(7), and more generally \( θ \) vanishes if \( π_8(dΦ) = 0 \). We will see below that in the case \( π_{48}(dΦ) = 0 \) the form \( θ \) is closed.

This time we have only 4 classes of Spin(7)-structures: the classes \{0\}, \( W_8 \), \( W_{48} \), and \( W = W_8 \oplus W_{48} \). Table 4.1 describes the classes in terms of differential equations on the form \( Φ \). Unlike the G₂ case, the inclusions between these classes are all strict, and this is discussed in [9].

| Class       | Defining Equations | Name            | \( dθ \) |
|-------------|--------------------|-----------------|---------|
| \( W_8 \oplus W_{48} \) | no relation on \( dΦ \). | | |
| \( W_8 \)  | \( dΦ + \frac{1}{2}θ ∧ Φ = 0 \) | LC Spin(7) | \( dθ = 0 \) |
| \( W_{48} \) | \( θ = 0 \) | | \( θ = 0 \) |
| \{0\}      | \( dΦ = 0 \) | Spin(7) | \( θ = 0 \) |

\textbf{Table 4.1. The 4 classes of Spin(7)-structures}

Remark 4.5.1. Note that in the Spin(7) case, there is no analogue of an “integrable” structure, nor are there analogues of almost or nearly Spin(7)-structure as there are in the G₂ case. An almost Spin(7) manifold \( (dΦ = 0) \) automatically has holonomy Spin(7). And \( dΦ \) does not have a one-dimensional component which would give us the analogue of a nearly G₂-structure.

We now prove the closedness of \( θ \) in the class \( W_8 \) as given in the final column of Table 4.1.

\textbf{Lemma 4.5.2.} If \( Φ \) satisfies \( dΦ + \frac{1}{2}θ ∧ Φ = 0 \), then \( dθ = 0 \).
**Proof.** Suppose \( d\Phi + \frac{1}{7} \theta \wedge \Phi = 0 \). We differentiate this equation to obtain:

\[
d\theta \wedge \Phi = \theta \wedge d\Phi = \theta \wedge \left( -\frac{1}{7} \theta \wedge \Phi \right) = 0
\]

But wedge product with \( \Phi \) is an isomorphism from \( \wedge^2 \) to \( \wedge^6 \), so \( d\theta = 0 \). \( \square \)

5. **Deformations of a fixed Spin(7)-structure**

We begin with a fixed Spin(7)-structure on a manifold \( M \) in a certain class. We will deform the form \( \Phi \) and see how this affects the class. This time there are only 4 classes, and only two intermediate classes. However, the ways we can deform \( \Phi \) in the Spin(7) case are more complicated. Since \( \Phi \in \wedge^4 \mathbb{R}^7 \oplus \wedge^4 \mathbb{R}^{27} \oplus \wedge^4 \mathbb{R}^{35} \), there are now four canonical ways to deform the 4-form \( \Phi \). Again, since \( \wedge^4 \mathbb{R}^7 = \{ f\Phi \} \), adding to \( \Phi \) an element of \( \wedge^4 \mathbb{R}^7 \) amounts to conformally scaling \( \Phi \). This preserves the decomposition into irreducible representations. In all other case, however, since the decomposition does depend on \( \Phi \) the decomposition will change for the other kinds of deformations.

We will see that analogously to the G\(_2\) case, flowing in the \( \wedge^4 \mathbb{R}^7 \) direction gives us a path in the space of positive 4-forms, all corresponding to the same metric. However, this time simply deforming non-infinitesmally by an element of \( \wedge^4 \mathbb{R}^7 \) will not yield a positive 4-form, in fact we can show that it never does. We will explain how much of the construction does carry over and some reasons why it should not be a surprise that discovering an analogous construction in the Spin(7) case that works should be considerably more complicated.

5.1. **Conformal Deformations of Spin(7)-structures.** Let \( f \) be a smooth, nowhere vanishing function on \( M \). We conformally scale \( \Phi \) by \( f^4 \), for notational convenience. Denote the new form by \( \tilde{\Phi} = f^4 \Phi \). We first compute the new metric \( \tilde{g} \) and the new volume form \( \text{vol} \) in the following lemma.

**Lemma 5.1.1.** The metric \( g_o \) on vector fields, the metric \( g_o^{-1} \) on one forms, and the volume form \( \text{vol}_o \) transform as follows:

\[
\begin{align*}
\tilde{g} & = f^2 g_o \\
\tilde{g}^{-1} & = f^{-2} g_o^{-1} \\
\text{vol} & = f^8 \text{vol}_o
\end{align*}
\]

**Proof.** We substitute \( \tilde{\Phi} = f^4 \Phi \) into equations (4.30) and (4.31) to obtain

\[
\begin{align*}
\tilde{A}(v) & = \left( (v \wedge f^4 \Phi_o) \wedge f^4 \Phi_o \right) (e_1, e_2, \ldots, e_7) \\
& = f^8 A_o(v) \\
\tilde{B}_{ij}(v) & = \left( (e_i \wedge v \wedge f^4 \Phi_o) \wedge (e_j \wedge v \wedge f^4 \Phi_o) \wedge (v \wedge f^4 \Phi_o) \right) (e_1, e_2, \ldots, e_7) \\
& = f^{12} (B_o)_{ij}(v)
\end{align*}
\]

Substituting these expressions into (4.33) we compute

\[
|v|^4 = \frac{(7)^{1/2} (f^{12})^{1/2} (\det (B_o)_{ij}(v))^{1/2}}{(6)^{1/2} (f^8)^{1/2} (A_o(v))^{1/2}} = f^4 |v|_o^4
\]

from which we have \( |v|^2 = f^2 |v|_o^2 \) and the remaining conclusions now follow. \( \square \)
We now determine the new Hodge star \( \tilde{\star} \) in terms of the old \( \star_o \).

**Lemma 5.1.2.** If \( \alpha \) is a \( k \)-form, then \( \tilde{\star} \alpha = f^{8-2k} \star_o \alpha \).

**Proof.** Let \( \alpha, \beta \) be \( k \)-forms. Then from Lemma 5.1.1 the new metric on \( k \)-forms is \( \langle \cdot, \cdot \rangle = f^{8-2k} \langle \cdot, \cdot \rangle_o \). From this we compute:

\[
\tilde{\alpha} \wedge \tilde{\star} \alpha &= \langle \beta, \alpha \rangle \text{vol} \\
&= f^{8-2k} \langle \beta, \alpha \rangle_o f^8 \text{vol}_o \\
&= f^{8-2k} \beta \wedge \star_o \alpha.
\]

\( \square \)

From this we obtain the following:

**Lemma 5.1.3.** The exterior derivative of the new 4-form \( \tilde{d} \phi \) and \( \tilde{\star} d \phi \) are

\[
\tilde{d} \phi = 4f^3 df \wedge \phi_o + f^4 d \phi_o \\
\tilde{\star} d \phi = 4f \star_o df \wedge \phi_o + f^2 \star_o d \phi_o
\]

**Proof.** This is immediate from \( \tilde{\phi} = f^4 \phi_o \) and Lemma 5.1.2. \( \square \)

Using these results, we can determine which classes of Spin(7)-structures are conformally invariant. We can also determine what happens to the 7-form \( \zeta \) and the associated 1-form \( \theta = \star \zeta \). This is all given in the following theorem:

**Theorem 5.1.4.** Under the conformal deformation \( \tilde{\phi} = f^4 \phi_o \), we have:

\[
(5.1) \quad \tilde{d} \phi + \frac{1}{7} \tilde{\theta} \wedge \tilde{\phi} = f^4 \left( d \phi_o + \frac{1}{f} \theta_o \wedge \phi_o \right)
\]

\[
(5.2) \quad \tilde{\zeta} = -28f^5 \star_o df + f^6 \zeta_o
\]

\[
(5.3) \quad \tilde{\theta} = -28d(\log(f)) + \theta_o
\]

Hence, we see from Table 4.1 and equation (5.1) that only the class \( W_8 \) is preserved under a conformal deformation of \( \phi \). (This part was originally proved in [9] using a different method.) Also, (5.3) shows that \( \theta \) changes by an exact form, so in the class \( W_8 \), where \( \theta \) is closed, we have a well defined cohomology class \([\theta]\) which is unchanged under a conformal scaling.

**Proof.** We begin by using Lemma 5.1.3 and (4.42) to compute \( \tilde{\zeta} \) and \( \tilde{\theta} \):

\[
\tilde{\zeta} = \tilde{\star} d \phi \wedge \tilde{\phi} \\
= (4f \star_o (df \wedge \phi_o) + f^2 \star_o d \phi_o) \wedge f^4 \phi_o \\
= 4f^5 \phi_o \wedge \star_o (\phi_o \wedge df) + f^6 \zeta_o \\
= -28f^5 \star_o df + f^6 \zeta_o
\]

where we have used (4.3) in the last step. Now from Lemma 5.1.2, we get:

\[
\tilde{\theta} = \tilde{\star} \tilde{\zeta} = -28f^{-1} df + \theta_o = -28d(\log(f)) + \theta_o.
\]

Now using the above expression for \( \tilde{\theta} \), we have:

\[
d \tilde{\phi} + \frac{1}{7} \tilde{\theta} \wedge \tilde{\phi} = 4f^3 df \wedge \phi_o + f^4 d \phi_o + \frac{1}{f} \left( -28f^{-1} df + \theta_o \right) \wedge f^4 \phi_o
\]

\[
= f^4 \left( d \phi_o + \frac{1}{f} \theta_o \wedge \phi_o \right)
\]

which completes the proof. \( \square \)
The next result gives necessary and sufficient conditions for being able to achieve holonomy Spin(7) by conformally scaling.

**Theorem 5.1.5.** Let $\Phi_0$ be a positive 4-form (associated to a Spin(7)-structure). Under the conformal deformation $\Phi = f^4\Phi_0$, the new 4-form $\Phi$ satisfies $d\Phi = 0$ if and only if $\Phi_0$ is already at least class $W_8$ and $28d\log(f) = \theta_0$. Hence in order to have $\Phi$ be closed (and hence correspond to holonomy Spin(7)), the original 1-form $\theta_0$ has to be exact. In particular if the manifold is simply-connected or more generally $H^1(M) = 0$ then this will always be the case if $\Phi_0$ is in the class $W_8$, since $d\theta_0 = 0$.

**Proof.** From Lemma 5.1.3, for $d\tilde{\phi} = 0$, we need
\[
d\tilde{\Phi} = 4f^3df \wedge \Phi_0 + f^4d\Phi_0 = 0
\]
which says that $d\Phi_0 \in \wedge^5_8$ by Proposition 4.2.1. Hence $\pi_{48}(d\Phi_0) = 0$ so $\Phi_0$ must be already of class $W_8$. Then to make $d\Phi = 0$, we need to eliminate the $W_8$ component, which requires $28d\log(f) = \theta_0$ by Theorem 5.1.4. \qed

**Remark 5.1.6.** Note that if we start with a Spin(7)-structure $\Phi_0$ that is already holonomy Spin(7), then Theorem 5.1.4 shows that a conformal scaling by a non-constant $f$ will always generate a non-zero $W_8$ component.

5.2. **Deforming $\Phi$ by an element of $\wedge^4_7$.** We continue our analogy with the $G_2$ case and now try to deform the Spin(7) 4-form $\Phi$ by an element of $\wedge^4_7$. Using (4.17), one can check that if we start with two vector fields $v$ and $w$, we can construct a special kind of element $\sigma_7 \in \wedge^4_7$ by $\sigma_7 = v^\flat \wedge (w \lrcorner \Phi_0) - w^\flat \wedge (v \lrcorner \Phi_0)$. We will consider this type since at least locally every element in $\wedge^4_7$ is a linear combination of elements of this type. Now let $\tilde{\Phi} = \Phi_0 + t(v^\flat \wedge (w \lrcorner \Phi_0) - w^\flat \wedge (v \lrcorner \Phi_0))$, for $t \in \mathbb{R}$. Using the notation of Theorem 4.3.5, we first prove the following.

**Proposition 5.2.1.** Let $\sigma_7 = (v^\flat \wedge (w \lrcorner \Phi_0) - w^\flat \wedge (v \lrcorner \Phi_0))$. Under the transformation $\tilde{\Phi} = \Phi_0 + \sigma_7$, we have
\[
(5.4) \quad \tilde{\Phi}^2 = \left(1 + \frac{4}{7}|v \wedge w|^2_6\right)\Phi_0^2
\]

**Proof.** We compute
\[
\tilde{\Phi}^2 = \left(\Phi_0 + v^\flat \wedge (w \lrcorner \Phi_0) - w^\flat \wedge (v \lrcorner \Phi_0)\right)^2
\]
\[
= \Phi_0^2 + 2v^\flat \wedge (w \lrcorner \Phi_0) \wedge \Phi_0 - 2w^\flat \wedge (v \lrcorner \Phi_0) \wedge \Phi_0 - 2v^\flat \wedge (w \lrcorner \Phi_0) \wedge w^\flat \wedge (v \lrcorner \Phi_0)
\]
\[
= \Phi_0^2 + 14v^\flat \wedge \ast_6 w^\flat - 14w^\flat \wedge \ast_6 v^\flat - 2v^\flat \wedge w^\flat \wedge (v \lrcorner \Phi_0) \wedge (w \lrcorner \Phi_0)
\]
\[
= \Phi_0^2 + 8|v \wedge w|^2_6 \text{vol}_o
\]
where we have used both (4.6) and (4.25). Now since $\psi_0^2 = 14 \text{vol}_o$, we have
\[
\tilde{\Phi}^2 = 14 \text{vol}_o + 14 \left(\frac{4}{7}|v \wedge w|^2_6\right) \text{vol}_o = \left(1 + \frac{4}{7}|v \wedge w|^2_6\right)\Phi_0^2
\]
which completes the proof. \qed
Corollary 5.2.2. Let \( h \) be a vector field. Under the transformation \( \tilde{\Phi} = \Phi_o + \sigma_t \), the expression \( A_o(h) = ((h, \Phi_o) \wedge \Phi_o) (e_1, e_2, \ldots, e_7) \) changes by

\[
\tilde{A}(h) = \left( \langle h, \Phi \rangle \wedge \tilde{\Phi} \right) (e_1, e_2, \ldots, e_7) = \left( 1 + \frac{4}{7} |v \wedge w|_o^2 \right) A_o(h)
\]

Proof. This follows from Proposition 5.2.1 by taking the interior product of both sides with \( h \). \( \square \)

We continue the computation of the expressions needed to determine if \( \tilde{\Phi} \) is indeed a Spin(7)-structure with the following lemma.

Lemma 5.2.3. With \( \tilde{\Phi} = \Phi_o + t\sigma \), in the expression

\[
\left( e_i \wedge u \wedge \tilde{\Phi} \right) \wedge \left( e_i \wedge u \wedge \tilde{\Phi} \right)
\]

which is a cubic polynomial in \( t \), the linear and cubic terms both vanish, and the coefficient of the quadratic term is

\[
6 \left( -\Phi_o(v, w, h, e_i)^2 + |v \wedge w \wedge h \wedge e_i|^2 - 2 \langle h \wedge e_i, v \wedge w \rangle \Phi_o(v, w, h, e_i) \right) \text{vol}_o
\]

\[
+ 6 \left( \langle w \wedge e_i, w \wedge v \rangle \langle h \wedge e_i, h \wedge v \rangle + \langle e_i \wedge h, e_i \wedge v \rangle \langle w \wedge h, w \wedge v \rangle \right) \text{vol}_o
\]

\[
+ 6 \left( \langle h \wedge e_i, h \wedge w \rangle \langle v \wedge e_i, v \wedge w \rangle + \langle e_i \wedge h, e_i \wedge w \rangle \langle v \wedge h, v \wedge w \rangle \right) \text{vol}_o
\]

\[
- 12 \langle h \wedge e_i, v \wedge w \rangle^2 \text{vol}_o
\]

Proof. We begin with the linear term. The coefficient of \( t \) is

\[
2(e_i \wedge u \wedge \sigma) \wedge (e_i \wedge u \wedge \Phi_o) + (e_i \wedge u \wedge \Phi_o) \wedge (e_i \wedge u \wedge \Phi_o) \wedge \sigma
\]

Two applications of (A.8) and collecting terms shows that this coefficient is

\[
3(e_i \wedge u \wedge \Phi_o) \wedge (e_i \wedge u \wedge \Phi_o) \wedge \sigma
\]

which vanishes by Corollary 4.4.6.

Next we consider the cubic term:

\[
(e_i \wedge u \wedge \sigma) \wedge (e_i \wedge u \wedge \sigma) \wedge \sigma
\]

This is a lengthy computation. We will sketch the steps involved and the interested reader can reproduce the details if desired. Let \( e_i = h \) for notational simplicity. First, one can compute that

\[
h \wedge u \wedge \sigma = \langle u, v \rangle \langle h, w \wedge \Phi_o \rangle - \langle h, v \rangle \langle u, w \wedge \Phi_o \rangle - \langle u, w \rangle \langle h, w \wedge \Phi_o \rangle + \langle h, u \rangle \langle w, \Phi_o \rangle + v^b \wedge (h \wedge u \wedge w \wedge \Phi_o) - w^b \wedge (h \wedge u \wedge w \wedge \Phi_o)
\]

and also

\[
v \wedge h \wedge u \wedge \sigma = \langle u, v \rangle \langle v, h \wedge w \wedge \Phi_o \rangle - \langle h, v \rangle \langle v, u \wedge w \wedge \Phi_o \rangle + |v|^2 \langle h, u \wedge w \wedge \Phi_o \rangle
\]

\[
- \langle v, w \rangle \langle h, u \wedge w \wedge \Phi_o \rangle - \Phi_o(w, u, h, v) w^b
\]

Now starting from the 9-form \( (h \wedge u \wedge \sigma) \wedge (h \wedge u \wedge \sigma) \wedge \Phi_o = 0 \), taking the interior product with \( v \), and rearranging, we obtain

\[
(h \wedge u \wedge \sigma) \wedge (h \wedge u \wedge \sigma) \wedge \sigma = 2 (v \wedge h \wedge u \wedge \sigma) \wedge w^b \wedge (h \wedge u \wedge \sigma) \wedge \Phi_o
\]

\[
- 2 (w \wedge h \wedge u \wedge \sigma) \wedge v^b \wedge (h \wedge u \wedge \sigma) \wedge \Phi_o
\]
Since $\sigma \mapsto -\sigma$ if we interchange $v$ and $w$, we see that the cubic term will vanish provided we can check that the first term on the right hand side above is symmetric in $v$ and $w$. Now one substitutes the expressions above for $h_{ij\omega\sigma}$ and $v, h_{ij\omega\sigma}$ and uses (4.24) many times, as well as $a^i \wedge b^j \wedge c^k \wedge d^l \wedge \Phi_0 = \Phi_0(a, b, c, d) \text{vol}_o$ to verify that this is indeed the case.

We move on to the coefficient of $t^2$, which is

$$2(e_{ij\omega\sigma} \wedge \Phi_0) \wedge \sigma + (e_{ij\omega\sigma}) \wedge (e_{ij\omega\sigma}) \wedge \Phi_0$$

As with the linear term, this can be rewritten as

$$3(e_{ij\omega\sigma} \wedge (e_{ij\omega\sigma})) \wedge \Phi_0$$

As in the case of the cubic term, the interested reader can now apply Proposition 4.3.1 and Lemma 4.4.3 many times to obtain the stated expression. We will not provide the details since we will now see that this deformation will not yield a Spin(7)-structure anyway. □

If we now polarize the expression $(h_{ij\omega\sigma} \Phi) \wedge (h_{ij\omega\sigma} \Phi) \wedge \Phi$, take the interior product with $h$, and apply this to a basis extension $e_1, e_2, \ldots, e_7$, as required by Theorem 4.3.5, one can check that

$$\frac{1}{6} \vec{B}_{ij} = |h|^2_o \left(1 + |v \wedge w|^2_o\right) g_{ij} - \langle w, h \rangle^2 v_i v_j - \langle v, h \rangle^2 w_i w_j$$

$$+ \langle v, h \rangle \langle w, h \rangle (v_i w_j + w_i v_j) - \left(1 + |v \wedge w|^2_o\right) h_i h_j - X_i X_j$$

$$- \langle w, h \rangle (v_i X_j + X_i v_j) + \langle v, h \rangle (w_i X_j + X_i w_j)$$

where $X$ is the vector field $X(v, w, h)$. From this expression the determinant of $\vec{B}_{ij}$ can be computed as

$$\det(\vec{B}_{ij}) = 6^7 |h|^2_o \left(1 + |v \wedge w|^2_o\right)^6$$

Now if this was indeed a Spin(7)-structure then Theorem 4.3.5 would imply that

$$|h|^4_o = \frac{\left(1 + |v \wedge w|^2_o\right)^2}{\left(1 + \frac{4}{7} |v \wedge w|^2_o\right)^3} |h|^4_o$$

This would mean the metric changes conformally, but the conformal factor is not compatible with what would be the new volume form $\text{vol} = \frac{1}{h^3} \Phi^2$ from Proposition 5.2.1. Hence this is never a Spin(7)-structure. Note that the construction very closely parallels the $G_2$ case. Even though the deformation does not yield a Spin(7)-structure, it is nevertheless true that $\det(\vec{B}_{ij})$ turns out to be a positive definite quadratic form.

Recall now one major difference between the $G_2$ and Spin(7) cases: the space $\wedge^3_{\text{pos}}$ of $G_2$-structures at a point is an open subset of the space $\wedge^3$ of 3-forms at that point. In constrast, the space $\wedge^4_{\text{pos}}$ of Spin(7)-structures at a point is a 43-dimensional submanifold of the 70-dimensional vector space $\wedge^4$ of 4-forms at that point, and this submanifold is not linearly embedded. So we should not expect that moving linearly in $\wedge^4$ would keep us on this submanifold, in general [5]. In effect, the Spin(7) case is more non-linear than the $G_2$ case. There may still exist, however, some non-linear way of deforming $\Phi_0$ by an element of $\wedge^4$ to obtain a Spin(7)-structure. In Section 6 we present an argument as to why this may be.
5.3. Infinitesimal deformations in the $\wedge^4$ direction. Even though the non-infinitesimal $\wedge^4$ deformation did not produce a Spin(7)-structure, we will see that in analogy with the $G_2$ case, we can get a family of Spin(7)-structures all corresponding to the same metric by taking infinitesimal deformations in the $\wedge^4$ direction. Consider a one-parameter family $\Phi_t$ of Spin(7)-structures, satisfying

$$
\frac{\partial}{\partial t} \Phi_t = w_\alpha *_t (v_\beta \Phi_t) - v_\alpha *_t (w_\beta \Phi_t)
$$

for a pair of vector fields $v$ and $w$. That is, at each time $t$, we move in the direction of a 4-form in $\wedge^4$, since the decomposition of $\wedge^4$ depends on $\Phi_t$ and hence is changing in time. Since the Hodge star $*_t$ is also changing in time, this is again a priori a nonlinear equation. However, just like in the $G_2$ case, it is actually linear:

**Proposition 5.3.1.** Under the flow described by equation (5.6), the metric $g$ does not change. Hence the volume form and Hodge star are also constant.

**Proof.** From Theorem 4.3.5, Corollary 5.2.2, and Lemma 5.2.3 we see that if we expand the expression for $|h|^4$ for some vector field $h$, as a power series in $t$, there is no linear term and hence to first order the metric does not change. □

Therefore we can replace $*_t$ by $*_0 = *$ and equation (5.6) is actually linear. Moreover, the flow determined by this linear equation gives a one-parameter family of Spin(7)-structures each yielding the same metric $g$. Our equation is now

$$
\frac{\partial}{\partial t} \Phi_t = w_\alpha * (v_\beta \Phi_t) - v_\alpha * (w_\beta \Phi_t) = B \Phi_t
$$

where $B$ is the linear operator $\alpha \mapsto B\alpha = w_\alpha * (v_\beta \alpha) - v_\alpha * (w_\beta \alpha)$ on $\wedge^4$.

**Proposition 5.3.2.** The operator $B$ is skew-symmetric. Furthermore, the eigenvalues $\lambda$ of $B$ are $\lambda = 0, \pm |v \wedge w|$.

**Proof.** Let $e^1, e^2, \ldots, e^{70}$ be a basis of $\wedge^4$. Then

$$
B_{ij} \text{vol} = \langle e^i, B e^j \rangle \text{vol}
$$

and hence $B$ is diagonalizable over $\mathbb{C}$. In order to find the eigenvalues of $B$, we first compute some powers of $B$, using the fact that $B\alpha$ can also be written as $B\alpha = v^b \wedge (w_\beta \alpha) - w^b \wedge (v_\beta \alpha)$:

$$
B^2 \alpha = v_\beta (w^b \wedge B\alpha) - w_\beta (v^b \wedge B\alpha)
$$

$$
= v_\beta \left( w^b \wedge w^b \wedge (w_\alpha) + w^b \wedge (v_\beta \alpha) \right) + w_\beta \left( v^b \wedge v^b \wedge (w_\alpha) + v^b \wedge (v_\beta \alpha) \right)
$$

$$
= \langle v, w \rangle \left( v^b \wedge (w_\alpha) + w^b \wedge (v_\alpha) \right) - |v|^2 v^b \wedge (w_\alpha) + |w|^2 w^b \wedge (v_\alpha)
$$

$$
- 2 v^b \wedge w^b \wedge (v_\beta \alpha)
$$
and thus

\[ B^3 \alpha = v_{\wedge} (w^b \wedge B^2 \alpha) - w_{\wedge} (v^b \wedge B^2 \alpha) \]

\[ = v_{\wedge} \left( (v, w) w^b \wedge v^b \wedge (w_{\wedge} \alpha) - v^2 w^b \wedge v^b \wedge (v_{\wedge} \alpha) \right) - w_{\wedge} \left( (v, w) v^b \wedge w^b \wedge (v_{\wedge} \alpha) - v^2 v^b \wedge w^b \wedge (w_{\wedge} \alpha) \right) \]

\[ = \langle v, w \rangle^2 B \alpha - |v|^2 |w|^2 B \alpha \]

after some simplification. Thus we have \( \lambda^3 = \left( \langle v, w \rangle^2 - |v|^2 |w|^2 \right) \lambda \). Therefore the non-zero eigenvalues are \( \lambda = \pm i |v \wedge w| \). This completes the proof. \( \square \)

Now we proceed exactly as in the \( G_2 \) case. If we replace \( A \) by \( B \) and the non-zero eigenvalues by \( \pm i |v \wedge w| \), then all the remaining calculations of Section 3.3 carry through. Therefore we have

\[ \Phi_t = -\frac{1}{|v \wedge w|^2} \cos(|v \wedge w| t) B^2 \Phi_0 + \frac{1}{|v \wedge w|} \sin(|v \wedge w| t) B \Phi_0 + \frac{1}{|v \wedge w|^2} B^2 \Phi_0 \]

which we summarize as the following theorem.

**Theorem 5.3.3.** The solution to the differential equation

\[ \frac{\partial}{\partial t} \Phi_t = w_{\wedge} *(v_{\wedge} \Phi_t) - v_{\wedge} *(w_{\wedge} \Phi_t) \]

is given by

\[ (5.7) \Phi(t) = \Phi_0 + \frac{1 - \cos(|v \wedge w| t)}{|v \wedge w|^2} B^2 \Phi_0 + \frac{\sin(|v \wedge w| t)}{|v \wedge w|} B \Phi_0 \]

where \( B \alpha = v_{\wedge} (w^b \wedge \alpha) - w_{\wedge} (v^b \wedge \alpha) \). The solution exists for all time and is closed curve in \( \wedge^4 \).

**Proof.** This follows from the above discussion. \( \square \)

**Remark 5.3.4.** In [6], it is shown that the set of Spin(7)s on \( M \) which correspond to the same metric as that of a fixed Spin(7)-structure \( \Phi_0 \) is an \( O(8) / \text{Spin}(7) \)-bundle (which is rank 7) over the manifold \( M \). The above theorem gives an explicit formula \( (5.7) \) for a path of Spin(7)-structures all corresponding to the same metric \( g \) starting from two vector fields \( v \) and \( w \) on \( M \).

**Remark 5.3.5.** Again, even though the metric is unchanged under an infinitesimal deformation in the \( \wedge^4 \) direction, the class of Spin(7)-structure can change.

We now apply this theorem to two examples, where we will again reproduce known results.

**Example 5.3.6.** Let \( N \) be a Calabi-Yau fourfold, with Kähler form \( \omega \) and holomorphic \((4,0)\) form \( \Omega \). The complex coordinates will be denoted by \( z^i = x^i + iy^i \). Then \( N \) has a natural Spin(7)-structure \( \Phi \) on it given by

\[ (5.8) \Phi = \text{Re}(\Omega) + \frac{\omega^2}{2} \]

It is easy to check in local coordinates that \( \omega \in \wedge^2 \) in the Spin(7) decomposition. Since we are computing pointwise, if we take two tangent vectors \( v \) and \( w \) for
which \( \pi^7(v^\flat \wedge w^\flat) = \omega \), then one can compute that

\[
B \Phi = - \text{Im}(\Omega) \\
B^2 \Phi = - \text{Re}(\Omega)
\]

Thus for the element of \( \wedge^4 \mathbb{C} \), which corresponds to \( \omega \), the flow in (5.7) is given by

\[
\Phi_t = \text{Re}(\Omega) + \frac{\omega^2}{2} - (1 - \cos(t)) \text{Re}(\Omega) - \sin(t) \text{Im}(\Omega) \\
\Phi_t = \frac{\omega^2}{2} \text{Re}(e^{it\Omega}) + \frac{\omega^2}{2}
\]

which is the canonical Spin(7) form on \( N \) where now the Calabi-Yau structure is given by \( e^{it\Omega} \) and \( \omega \). Thus we arrive at the phase freedom for Calabi-Yau fourfolds.

**Example 5.3.7.** Consider a 7-manifold \( M \) with a \( G_2 \)-structure \( \varphi \). We can put a Spin(7)-structure \( \Phi \) on the product \( M \times S^1 \) given by

\[
\Phi = d\theta \wedge \varphi + \ast_7 \varphi \nabla
\]

where \( \ast_7 \varphi \) is the 4-form dual to \( \varphi \) on \( M \). This induces the product metric on \( M \times S^1 \), with the flat metric on \( S^1 \). Now let \( v = \frac{\partial}{\partial \theta} \) be a globally defined non-vanishing vector field on \( S^1 \) with \( |v| = 1 \). Choose another vector field \( w \) on \( M \). Then one computes

\[
w \lrcorner (v \lrcorner \Phi) - v \lrcorner (w \lrcorner \Phi) = B \Phi = d\theta \wedge (w \lrcorner \ast_7 \varphi) + \ast_7 (w \lrcorner \ast_7 \varphi) \\
B^2 \Phi = d\theta \wedge (w \lrcorner \ast_7 (w \lrcorner \ast_7 \varphi)) + \ast_7 (w \lrcorner \ast_7 (w \lrcorner \ast_7 \varphi))
\]

The flow in (5.7) gives

\[
\Phi_t = d\theta \wedge \varphi_t + \ast_7 \varphi_t
\]

where \( \varphi_t \) is the flow given by (3.20) for the vector field \( w \). Thus in the product case \( M \times S^1 \) we recover the results of Section 3.3.

### 6. Conclusion

In the construction of Calabi-Yau manifolds, we start from a Kähler manifold and we reduce the holonomy from \( \text{U}(n) \) to \( \text{SU}(n) \), which is a drop of 1 in dimension and hence it might be expected that it would involve the solution of an equation for one function. In going from \( \text{SO}(7) \) to \( G_2 \) we have a drop of 7 in dimension, so we might expect to need 7 conditions, which could involve an equation for a vector field (or equivalently an element of \( \wedge^3_7 \)). Similarly the difference in dimension between \( \text{SO}(8) \) and \( \text{Spin}(7) \) is also 7, which could be related to an element \( \wedge^3_7 \).

Note that in the \( G_2 \) case, since elements of \( \wedge^3_7 \) are canonically identified with vector fields, they are intrinsic to the manifold without reference to a \( G_2 \)-structure. For Spin(7)-structures, we need the 4-form \( \Phi \) to define \( \wedge^4_7 \) and this introduces more non-linearity. To maintain the analogy with the \( G_2 \) case, there should be some (non-linear) way of transforming a Spin(7)-structure \( \Phi \) using an element of \( \wedge^3_7 \) so that we get a new Spin(7)-structure whose new metric is related to the old one by

\[
\langle u_1, u_2 \rangle = f \left( \langle u_1, u_2 \rangle + \langle X(v, w, u_1), X(v, w, u_2) \rangle \right)
\]

where \( v \) and \( w \) are vector fields which determine the corresponding element of \( \wedge^3_7 \) and \( f \) is some positive function of \( |v \wedge w|^2 \).
APPENDIX A. SOME LINEAR ALGEBRA

Here we collect together various identities involving the exterior and interior products and the Hodge star operator. Also some identities involving determinants are proved. We establish the results in the general case of a Riemannian manifold $M$ of dimension $n$, although only the cases $n = 7, 8$ are used in the text. Let $\langle \cdot, \cdot \rangle$ denote the metric on $M$, as well as the induced metric on forms. In all that follows, $\alpha$ and $\gamma$ are $k$-forms, $\beta$ is a $(k-1)$-form, $w$ is a vector field, and $w^\flat$ is the 1-form dual to $w$ in the given metric. That is,

$$|w|^2 = \langle w, w \rangle = w^b(w) = \langle w^b, w^b \rangle$$

Now $*$ takes $k$-forms to $(n-k)$-forms, and is defined by

$$\langle \alpha, \gamma \rangle \text{vol} = \alpha \wedge * \gamma = \gamma \wedge * \alpha$$

We also have

$$*^2 = (-1)^{k(n-k)}$$

on $k$-forms.

**Lemma A.0.1.** We have the following four identities:

(A.2) \[ * (w \lrcorner \alpha) = (-1)^{k+1} (w^b \wedge * \alpha) \]

(A.3) \[ (w \lrcorner \alpha) = (-1)^{nk+n} (w^\flat \wedge * \alpha) \]

(A.4) \[ *(w \lrcorner * \alpha) = (-1)^{nk+n+1} (w^b \wedge \alpha) \]

(A.5) \[ (w \lrcorner * \alpha) = (-1)^k (w^b \wedge \alpha) \]

and when $\alpha = \text{vol}$, the special case

(A.6) \[ w \lrcorner \text{vol} = *w^b \]

**Proof.** We compute:

$$\langle \beta, w \lrcorner \alpha \rangle \text{vol} = \beta \wedge *(w \lrcorner \alpha)$$

$$= \langle w \lrcorner \alpha \rangle (\beta^\flat) \text{vol}$$

$$= \alpha (w \wedge \beta^\flat) \text{vol}$$

$$= \langle \alpha, w^b \wedge \beta \rangle \text{vol}$$

$$= \langle w^b \wedge \beta \wedge * \alpha \rangle$$

$$= (-1)^{k-1} \beta \wedge (w^b \wedge * \alpha)$$

Since $\beta$ is arbitrary, (A.2) follows. Substituting $* \alpha$ for $\alpha$, and using (A.1), we obtain (A.4). The other two are obtained by taking $*$ of both sides of the first two identities. \[ \square \]

Note that from the above proof we have also the useful relation

(A.7) \[ (X \lrcorner \alpha) \wedge * \beta = \alpha \wedge * \left( X^\flat \wedge \beta \right) \]

for any $k$-form $\alpha$, $(k-1)$-form $\beta$, and vector field $X$. If $\alpha$ is a $k$-form, and $\beta$ is an $(n+1-k)$-form, then $\alpha \wedge \beta = 0$ and taking the interior product with a vector field $X$ gives

(A.8) \[ (X \lrcorner \alpha) \wedge \beta = (-1)^{k+1} \alpha \wedge (X \lrcorner \beta) \]

which is also used very often.
The next lemma gives further relations between a $k$-form $\alpha$ and a vector field $w$.

**Lemma A.0.2.** With the same notation as above, we have the following three identities:

(A.9) $|w|^2 |\alpha| = w^b \wedge (w_\alpha) + w_j (w^b \wedge \alpha)$

(A.10) $|w|^2 |\alpha| = (-1)^{nk+1} (w_\alpha (w_j (w^b \wedge \alpha))) + (-1)^{nk+n+1} (w_\alpha (w_j \alpha))$

(A.11) $|w|^2 |\alpha|^2 = |w_j \alpha|^2 + |w_j \alpha|^2$

**Proof.** Using the fact that the interior product is an anti-derivation, we have

$$w_j (w^b \wedge \alpha) = (w_j w^b) \wedge \alpha - w^b \wedge (w_j \alpha)$$

which is equation (A.9) since $w_j w^b = |w|^2$. Now (A.10) follows from this one using the identities of Lemma A.0.1. The last identity can also be obtained this way, but it is faster to proceed as follows:

$$0 = w^b \wedge \alpha \wedge *\alpha$$

$$0 = w_j (w^b \wedge \alpha \wedge *\alpha)$$

$$= |w|^2 \alpha \wedge *\alpha - w^b \wedge (w_\alpha) \wedge *\alpha + (-1)^{k+1} w^b \wedge \alpha \wedge (w_j \alpha)$$

$$= |w|^2 |\alpha|^2 \text{vol} - (w_\alpha) \wedge *\alpha - (w_j * \alpha) \wedge *\alpha$$

using Lemma A.0.1, which can be rearranged to give (A.11). □

The following lemma about determinants is used many times in the computation of the metrics and volume forms arising from $G_2$ and Spin(7)-structures.

**Lemma A.0.3.** Let $g_{ij}$ be an $n \times n$ matrix, $v_i$ and $w_j$ be two $n \times 1$ vectors, and $C, K$ constants. Consider the matrix $B_{ij} = C g_{ij} + K v_i w_j$

Its determinant is given by

(A.12) $\det(B) = C^n \det(g) + \sum_{k,l=1}^n (-1)^{k+l} v_k w_l C^{n-1} K G_{kl}$

where $G_{kl}$ is the $(k,l)^{th}$ minor of the matrix $g_{ij}$. That is, it is the determinant of $g_{ij}$ with the $k^{th}$ row and $l^{th}$ column removed.

**Proof.** The determinant of $B_{ij}$ is

$$\det \begin{pmatrix} C g_{11} + K v_1 w_1 & \ldots & C g_{1n} + K v_1 w_n \\ \vdots & \ddots & \vdots \\ C g_{n1} + K v_n w_1 & \ldots & C g_{nn} + K v_n w_n \end{pmatrix}$$

Since the determinant is a linear function of the columns of a matrix, we can write the above determinant as a sum of determinants where each column is of one of these two forms:

$$\begin{pmatrix} C g_{1k} \\ C g_{2k} \\ \vdots \\ C g_{nk} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} K v_1 w_k \\ K v_2 w_k \\ \vdots \\ K v_n w_k \end{pmatrix}$$

Each of the determinants which has at least two columns of $K v_i w_k$’s will have at least two proportional columns and hence will vanish. So the only non-zero
contributions come from the determinant with all \( C_{kl} \)'s and the \( n \) determinants with only one column of \( K_{vl} \)'s. Therefore we are left with:

\[
\text{(A.13)} \quad \det(C_{kl}) + \sum_{k,l=1}^{n} (-1)^{k+l} (K_{vl}w_l) C^{n-1}(G_{kl})
\]

In equation (A.13) the first term is the determinant with all columns of \( C_{kl} \)'s, the sum over \( l \) is the sum over the \( n \) different determinants with a single column of \( K_{vl} \)'s, in the \( l \)th column, and in the sum over \( k \) we expand each of those determinants along the \( l \)th column. This completes the proof. \( \square \)

We can obtain a special case of Lemma A.0.3 when the matrix \( g_{ij} \) is a metric. It is used several times in the text, most notably in the derivation of the metric from the 4-form \( \Phi \) in the Spin(7) case in Theorem 4.3.5.

**Lemma A.0.4.** Let \( g_{ij} = \langle e_i, e_j \rangle \) be a Riemannian metric in local coordinates, and \( v^b = v_i e^i \) and \( w^b = w_j e^j \) be two one forms dual to the vector fields \( v \) and \( w \). Then if we define

\[
B_{ij} = C g_{ij} + K v_i w_j
\]

we have

\[
\det(B) = C^n \det(g) + C^{n-1} K \langle v, w \rangle \det(g)
\]

**Proof.** In local coordinates, the volume form is \( \text{vol} = \sqrt{\det(g)} e^1 \wedge e^2 \wedge \ldots \wedge e^n \).

Then its dual \( (n-1) \)-vector is

\[
\text{vol}^\sharp = \frac{1}{\sqrt{\det(g)}} e^1 \wedge e^2 \wedge \ldots \wedge e_n
\]

The induced inner product on the two \((n-1)\)-vectors \( v^\flat \wedge \text{vol}^\sharp \) and \( w^\flat \wedge \text{vol}^\sharp \) is

\[
\langle v^\flat \wedge \text{vol}^\sharp, w^\flat \wedge \text{vol}^\sharp \rangle
\]

\[
= \frac{1}{\det(g)} \sum_{k,l=1}^{n} (-1)^{k+l} v_k w_l \langle e_1 \wedge \ldots \wedge \hat{e}_k \wedge \ldots \wedge e_n, e_1 \wedge \ldots \wedge \hat{e}_l \wedge \ldots \wedge e_n \rangle
\]

\[
= \frac{1}{\det(g)} \sum_{k,l=1}^{n} (-1)^{k+l} v_k w_l G_{kl}
\]

Now comparing with (A.13), we have

\[
\det(B) = C^n \det(g) + C^{n-1} K \langle v^\flat \wedge \text{vol}^\sharp, w^\flat \wedge \text{vol}^\sharp \rangle \det(g)
\]

The result now follows since \( v^\flat \wedge \text{vol}^\sharp = *v \) and the Hodge star is an isomorphism. (This is the star operator on the space of \( k \)-vectors.) \( \square \)

**References**

[1] E. Bonan, *Sur des Variétés riemanniennes à groupe d’holonomie \( G_2 \) ou Spin(7)*, C.R. Acad. Sci. Paris 262 (1966), 127-129.

[2] R.B. Brown and A. Gray, *Vector Cross Products*, Comment. Math. Helv. 42 (1967), 222-236.

[3] R.L. Bryant, *Submanifolds and Special Structures on the Octonions*, J. Differential Geometry 17 (1982), 185-232.

[4] R.L. Bryant, *Metrics with Exceptional Holonomy*, Ann. of Math. 126 (1987), 525-576.

[5] R.L. Bryant, *private communication*.

[6] R.L. Bryant and S.M. Salamon, *On the Construction of Some Complete Metrics with Exceptional Holonomy*, Duke Math. J. 58 (1989), 829-850.
[7] F.M. Cabrera, *On Riemannian Manifolds with $G_2$-Structure*, Bolletino U.M.I. (7) **10-A** (1996), 99-112.
[8] F.M. Cabrera, M.D. Monar and A.F. Swann, *Classification of $G_2$-Structures*, J. London Math. Soc (2) **53** (1996), 407-416.
[9] M. Fernández, *A Classification of Riemannian Manifolds with Structure Group $Spin(7)$*, Ann. Mat. Pura Appl. (IV) **143** (1986), 101-122.
[10] M. Fernández, *An Example of a Compact Calibrated Manifold Associated with the Exceptional Lie Group $G_2$*, J. Differential Geometry **26** (1987), 367-370.
[11] M. Fernández and A. Gray, *Riemannian Manifolds with Structure Group $G_2$*, Ann. Mat. Pura Appl (IV) **32** (1982), 19-45.
[12] M. Fernández and L. Ugarte, *Dolbeault Cohomology for $G_2$-Manifolds*, Geometriae Dedicata **70** (1998), 57-86.
[13] M. Fernández and L. Ugarte, *A Differential Complex for Locally Conformal Calibrated $G_2$-Manifolds*, Illinois J. Math. **44** (2000), 363-391.
[14] A. Gray, *Vector Cross Products on Manifolds*, Trans. Amer. Math. Soc. **141** (1969), 463-504; correction **148** (1970), 625.
[15] A. Gray, *Weak Holonomy Groups*, Math. Z. **123** (1971), 290-300.
[16] A. Gray and L.M. Hervella, *The Sixteen Classes of Almost Hermitian Manifolds and Their Linear Invariants*, Ann. Mat. Pura Appl. (IV) **123** (1980), 35-58.
[17] S. Gukov, S.T. Yau, and E. Zaslow, *Duality and Fibrations on $G_2$-Manifolds*, Turkish J. Math. **27** (2003), 61-97.
[18] R. Harvey, *Spinors and Calibrations*, Academic Press, San Diego, 1990.
[19] N. Hitchin, *The Geometry of Three-Forms in Six and Seven Dimensions*, J. Differential Geometry **55** (2000), 547-576.
[20] N. Hitchin, *Stable Forms and Special Metrics*, Global Differential Geometry: The Mathematical Legacy of Alfred Gray, Contemp. Math. **288** (2001).
[21] D.D. Joyce, *Compact Riemannian 7-Manifolds with Holonomy $G_2$. I*, J. Differential Geometry **43** (1996), 291-328.
[22] D.D. Joyce, *Compact Riemannian 7-Manifolds with Holonomy $G_2$. II*, J. Differential Geometry **43** (1996), 329-375.
[23] D.D. Joyce, *Compact 8-Manifolds with Holonomy $Spin(7)$*, Invent. Math. **123** (1996), 507-552.
[24] D.D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford University Press, Oxford, 2000.
[25] S. Karigiannis, *On a system of PDE’s related to $G_2$ holonomy*, in preparation.
[26] S.M. Salamon, *Riemannian Geometry and Holonomy Groups*, Longman Group UK Limited, Harlow, 1989.
[27] S.T. Yau, *On the Ricci Curvature of a Compact Kähler Manifold and the Complex Monge-Ampère Equation I.*, Comm. Pure Appl. Math. **31** (1978), 339-411.

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