Primitive Central Idempotents of Rational Group Algebras

Geoffrey Janssens

Abstract

We give a description of the primitive central idempotents of the rational group algebra $\mathbb{Q}G$ of a finite group $G$. Such a description is already investigated by Jespers, Olteanu and del Río, but some unknown scalars are involved. Our description also gives answers to their questions.

Let $G$ be a finite group. The complex group algebra $\mathbb{C}G$ is semisimple and a description of its primitive central idempotents is well known. These are the elements $e(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(1) \chi(g^{-1})g$, where $\chi$ runs through the irreducible characters of $G$. Using Galois descent one obtains that the primitive central idempotents of the semisimple rational group algebra $\mathbb{Q}G$ are the elements $e_\mathbb{Q}(\chi) = \sum_{\sigma \in \mathbb{S}_\chi} \sigma(e(\chi))$, with $G_\chi = Gal(\mathbb{Q}(\chi)/\mathbb{Q})$. Rather recently, Olivieri et al. [3] obtained a character free method for describing the primitive central idempotents of $\mathbb{Q}G$ provided that $G$ is a finite monomial group. Their method relies on a theorem of Shoda on pairs of subgroups $(H, K)$ of $G$ with $K$ normal in $H$, $H/K$ abelian and so that an irreducible character of $H$ with kernel $K$ induces an irreducible character of $G$. Such pairs are called Shoda pairs. The main ingredient in this theory are the elements $\epsilon(H, K)$ of $\mathbb{Q}H$ with $K < H \leq G$. These are defined as $\prod_{M/K \in M(H/K)} (K-M)$ if $H \neq K$ and as $H$ if $H = K$, where $M(H/K)$ denotes the set of minimal non-trivial normal subgroups of $H/K$ and $K = \frac{1}{|K|} \sum_{k \in K} k$. Furthermore, $e(G, H, K)$ denotes the sum of the different $G$-conjugates of $\epsilon(H, K)$.

For arbitrary finite groups, Jespers, Olteanu and del Río obtained a description of $e_\mathbb{Q}(\chi)$ in [2]. It is shown that $e_\mathbb{Q}(\chi)$ is a $\mathbb{Q}$-linear combination of the elements $e(G, H_i, K_i)$, with $(H_i, K_i)$ Shoda pairs in some subgroups of $G$. They posed the question ([2, Remark 3.4]) whether one could determine the scalars and the Shoda pairs involved. In this paper we answer both questions by giving a full description of the primitive central idempotents of $\mathbb{Q}G$, for $G$ a finite group.

Throughout $G$ is a finite group. For $\chi$ an arbitrary complex character of $G$ we put: $e(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(1) \chi(g^{-1})g$ and $e_\mathbb{Q}(\chi) = \sum_{\sigma \in \mathbb{S}_\chi} \sigma(e(\chi))$. Note that in general these elements do not have to be idempotents. Recall that the Möbius $\mu$-function, $\mu : \mathbb{N} \rightarrow \{ -1, 0, 1 \}$, is the map defined by $\mu(1) = 1, \mu(n) = 0$ if $a^2 \mid n$ with $a > 1$ and $\mu(n) = (-1)^r$ if $n = p_1 p_2 \ldots p_r$ for different primes $p_1, \ldots, p_r$. The induction of a character $\phi$ of a subgroup $H$ to $G$ is defined as $\phi_H^G(g) = \frac{1}{|H|} \sum_{y \in G} \phi(y^{-1}gy)$, where $\phi(x) = \phi(x)$ if $x \in H$ and $\phi(x) = 0$ otherwise. By $1_G$ we denote the trivial character of $G$.

To prove our result we make use of the Artin Induction Theorem. Although this is probably well known, we state and prove it in the following specific form. Recall that for a rational valued character $\chi$ of a group $G$, $\chi(g) = \chi(g^i)$ for $(i, \sigma(g)) = 1$. 

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Proposition 1 (Artin) If \( \psi \) is a rational valued character of \( G \), then

\[
\psi = \sum_{i=1}^{r} d_{C_i} \chi_{G}^{(i)}
\]

where the sum runs through a set \( \{C_1, \ldots, C_r\} \) of representatives of conjugacy classes of cyclic subgroups of \( G \). Furthermore, if \( C_i = \langle c_i \rangle \) then

\[
d_{C_i} = \frac{|G : \text{Cent}(c_i)|}{|G : C_i|} \sum_{C_i^* \geq C_i} \mu([C_i^* : C_i]) \psi(z^*)
\]

where the sum runs through all the cyclic subgroups \( C_i^* \) of \( G \) containing \( C_i \) and \( C_i^* = \langle z^* \rangle \).

Proof. For every cyclic subgroup \( C = \langle c \rangle \) of \( G \), there exists exactly one \( i \in \{1, \ldots, r\} \) such that \( C \) is \( G \)-conjugated to \( C_i \). Say, \( C = C_i^{g^{-1}} \). Set \( a_C = \frac{|\text{Cent}(c)|}{|C_i|} d_C \). First we prove that \( a_C = a_{C_i} \) and \( 1_G^C = 1_G^{C_i} \). To prove the second equality, note that \( 1_G^C(g) = \frac{1}{|C_i|} \sum_{y \in G} 1_C(y^{-1}gy) \), where the function \( 1_C(y^{-1}gy) \) is defined as 1 if \( y^{-1}gy \in C \) and 0 otherwise. This combined with the fact that conjugation preserves the order of subgroups and that it is an automorphism of \( G \) we easily see that \( 1_G^C = 1_G^{C_i} \).

Now we prove that \( a_C = a_{C_i} \). Define the sets \( (C_i) \uparrow^g = \{ K \mid C_i \leq K \leq G \} \) and \( (C) \uparrow^g = \{ K \mid C \leq K \leq G \} \). There is a bijective correspondence between these sets. A map from \((C_i) \uparrow^g \to (C) \uparrow^g \) is given by conjugation with \( g^{-1} \) and the invers map is conjugation by \( g \). Along with the fact that \( C^g = \langle c^g \rangle \) if \( C = \langle c \rangle \) and \( |C| = |C^g| \), we see immediately that \( a_C = a_{C_i} \).

All this yields, \( \sum_C a_C 1_G^C = \sum_{i=1}^{r} k_i a_{C_i} 1_G^{C_i} = \sum_{i=1}^{r} d_{C_i} 1_G^{C_i} \), where \( k_i = \frac{|C_i|}{|\text{Cent}(c_i)|} \) (with \( C_G(c_i) \) the conjugacy class of \( c_i \) in \( G \)). The result now follows from Artin’s Induction Theorem, [1, page 489], which says that every rational valued character of \( G \) is of the form \( \sum_C a_C 1_G^C \), with \( a_C \) as above and the sum runs over all cyclic subgroups \( C \) of \( G \).

Recall that by \( \hat{C} \) we denote \( \epsilon(C_i, C_i) = \frac{1}{|C_i|} \sum_{c_i \in C_i} c_i \).

Theorem 2 Let \( G \) be a finite group and \( \chi \) an irreducible complex character of \( G \). Let \( C_i = \langle c_i \rangle \), then we denote

\[
b_{C_i} = \frac{|G : \text{Cent}(c_i)|}{|G : C_i|} \sum_{C_i^* \geq C_i} \mu([C_i^* : C_i]) \sum_{\sigma \in G} \sigma(\chi)(z^*)
\]

where the sum runs through all the cyclic subgroups \( C_i^* \) of \( G \) which contain \( C_i \) and \( z^* \) is a generator of \( C_i^* \). Then

\[
e_Q(\chi) = \sum_{i=1}^{r} b_{C_i} \chi(1) \frac{|G : \text{Cent}(G)|}{|G : C_i|} e(G, C_i, C_i) = \sum_{i=1}^{r} b_{C_i} \chi(1) \frac{|G : C_i|}{|G : C_i|} \sum_{k=1}^{m} \hat{C}_i g_{i,k},
\]

where the first sum runs through a set \( \{C_1, \ldots, C_r\} \) of representatives of conjugacy classes of cyclic subgroups of \( G \) and \( T_i = \{g_{i1}, \ldots, g_{im_i}\} \) a right transversal of \( C_i \) in \( G \).

Proof. Let \( \chi \) be an irreducible complex character of \( G \). First we suppose that \( \chi(G) \subseteq \mathbb{Q} \). Then \( G_\chi = \{1\} \) and then by Proposition 1, \( \chi = \sum_{i=1}^{r} b_{C_i} 1_G^{C_i} \).
We get
\[
e_Q(\chi) = e(\chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} (\sum_{i=1}^{r} bC_i, 1_G^G((y^{-1})g) \sum_{g \in G} 1_G^G((y^{-1})g)
= \frac{\chi(1)}{|G|} \sum_{i=1}^{r} \frac{bC_i}{|C_i|} \sum_{g \in G} 1_G^G((y^{-1})g)
= \frac{\chi(1)}{|G|} \sum_{i=1}^{r} \frac{bC_i}{|G:C_i|} |G| e(1_G^G)
= \sum_{i=1}^{r} \frac{bC_i}{|G:C_i|} e(1_G^G)
\]

Let } T_i = \{g_{i1}, \ldots, g_{im}\} \text{ be a right transversal of } C_i \text{ in } G. \text{ Then }
\[
e(1_G^G) = \frac{1}{|G|} \sum_{g \in G} 1_G^G((y^{-1})g)
= \frac{1}{|G|} \sum_{g \in G} \frac{|G|}{|C_i|} 1_C((y^{-1})g)
= \sum_{g \in G} \frac{|G|}{|C_i|} \sum_{g \in G} 1_C((y^{-1})g)
= \sum_{g \in G} \frac{|G|}{|C_i|} \sum_{j=1}^{m_i} 1_{C_i}(g_{ij} g^{-1} g_{ij})
= \sum_{j=1}^{m_i} g_{ji}
\]

With this expression for } e(1_G^G) \text{ we obtain one of the equalities in the statement of the result.

Obviously, the sum } \sum_{k=1}^{m_i} \tilde{C}^{qik}_i \text{ adds the elements of the } G \text{-orbit of } \tilde{C}_i = e(C_i, C_i) \text{ and each of them } [Cen_G(\tilde{C}_i) : C_i] \text{ times. So }
\[
\sum_{k=1}^{m_i} \tilde{C}^{qik}_i = [Cen_G(\tilde{C}_i) : C_i] e(G, C_i, C_i).
\]

A simple substitution in the earlier found expression for } e_Q(\chi) \text{ yields the theorem.

Assume now that } \chi \text{ is an arbitrary irreducible complex character of } G. \text{ Then it is clear and well known that } \sum_{\sigma \in G_x} \sigma \circ \chi \text{ is a rational valued character of } G. \text{ Hence, by the first part we get }
\[
e(\sum_{\sigma \in G_x} \sigma(\chi)) = \frac{1}{|G|} \sum_{g \in G} (\sum_{\sigma \in G_x} \sigma(1)) \sum_{\sigma \in G_x} \sigma(\chi(g^{-1}))
= \frac{|G_x| \chi(1)}{|G|} \sum_{\sigma \in G_x} \sigma(\chi(g^{-1}))
= \frac{|G_x| \chi(1)}{|G|} \sum_{\sigma \in G_x} \sigma(\chi(g^{-1}))
= |G_x| e_Q(\chi)
\]

Since } \sum_{\sigma \in G_x} \sigma(\chi(1)) = |G_x| \chi(1), \text{ the rational case yields the theorem. \hfill \Box}

We finish with some remarks. First note that the elements } e(G, C_i, C_i) \text{ are not necessarily idempotents. Second, the definition of } b_{C_i} \text{ is not character-free. However one easily obtains a character free upperbound:
\[
b_{C_i} \leq \frac{|G : Cen_G(c_i)|}{|G : C_i|} \sum_{c_i \geq C_i} \mu((C_i^* : C_i)) \phi(\nu(1)) \leq \frac{|G : Cen_G(c_i)||G : Z(G)|}{|G : C_i|} \sum_{c_i \geq C_i} \mu((C_i^* : C_i)) \phi(\nu(n)),
\]

where } \phi \text{ denotes the } \phi \text{-Euler function. Hence, we can obtain a finite algorithm, that easily can be implemented in for example GAP, to compute all primitive central idempotents of } \mathbb{Q}G. \text{ This answers one of the questions posed in [2, Remark 3.4]. Also the description of the idempotents only makes use of pairs of subgroups } (C_i, C_i) \text{, with } C_i \text{ cyclic. This answers the second question posed in [2, Remark 3.4].}
References

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G.Janssens
Departement of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, 1050, Brussels, Belgium
e-mail: geofjans@vub.ac.be