Fateev-Zamolodchikov spin chain: excitation spectrum,
completeness and thermodynamics

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Abstract

The sector of zero $Z_N$-charge is studied for the ferromagnetic (FM) and antiferromagnetic (AFM) version of the $Z_N \times Z_2$ invariant Fateev-Zamolodchikov quantum spin chain. We conjecture that the relevant Bethe ansatz equations should admit, beside the usual string-like solutions, exceptional multiplets, and a number of non-physical solutions. Once the physical ones are identified, we show how to get completeness and the gapless excitation spectrum. The central charge is computed from the specific heat and found to be $c = 2\frac{N-1}{N+2}$ (FM) and $c = 1$ (AFM).

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1 Introduction

It is of interest to study the Fateev-Zamolodchikov spin chain

\[ H = - \sum_{k=1}^{M} \sum_{n=1}^{N-1} \frac{1}{\sin(n\pi/N)} (X_k^n + Z_k^n Z_{k+1}^{-n}) \]  

(1)

where the operators \( X_k \) and \( Z_k \) act on the \( N^M \)-dimensional vector space spanned by the basis \( |n_1, n_2, \ldots, n_M \rangle, n_k = 0, 1 \ldots N - 1 \)

\[ X_k|n_1 \ldots n_k \ldots n_M \rangle = |n_1 \ldots n_k + 1 \ldots n_M \rangle \mod N \]
\[ Z_k|n_1 \ldots n_k \ldots n_M \rangle = \omega^{n_k}|n_1 \ldots n_k \ldots n_M \rangle \quad \omega = \exp(2\pi i / N) \]

in its ferromagnetic (i.e. \( H \) itself) and its antiferromagnetic version (i.e. \( -H \)). The Hamiltonian (1) is exactly integrable since it can be derived from the family of commuting transfer matrices \( T(u) \) of an integrable 2-dimensional spin model

\[
[T(u), T(u')] = 0 \quad \forall u, u' \in C
\]

\[ T(u) = 1_{id} - Mu \sum_{n=1}^{N-1} \frac{1}{\sin(n\pi/N)} - uH + O(u^2) \]

As it was conjectured in the paper [2], and subsequently proved through the calculations of the critical exponents [3], the model is critical and, in the scaling limit, it gives a \( Z_N \) invariant conformal field theory with parafermion currents.

While this leaves little to be discovered about the ferromagnetic (FM) version of (1), little is still known about the antiferromagnetic version (AFM). The case \( N = 3 \), corresponding to the three-state critical Potts chain, has been studied in great detail, leading to the calculation of the excitation spectrum [4], the central charge and the spectrum of conformal dimensions of the corresponding field theory [5, 6] and finally the characters of the relevant representations of the Virasoro algebra as well as the full modular invariant partition function [7]. On one hand, these results have led to recognize that the \( N = 3 \) AFM spin chain is critical and in the scaling limit it describes, rather surprisingly, a conformal field theory with \( Z_4 \) parafermions. On the other hand, even the results for \( N = 3 \) FM have proven fruitful because they
have provided a new way, directly related to lattice models, to express characters of the representations of the Virasoro algebra and affine Lie algebras [3].

In this paper we extend some of these results to arbitrary (but odd) $N$. We summarize here what is known about the exact diagonalization of (1) [9]. By means of the analytic Bethe ansatz, the whole spectrum of (1) can be expressed in terms of a set of variables $\{\lambda_k\}$, related in a simple way to the zeroes of the transfer matrix eigenvalues of the two dimensional model. The set $\{\lambda_k\}$ obeys a system of transcendental equations whose appearance is commonplace in integrable models

$$\prod_{k=1}^{L} \frac{\sinh(\lambda_j - \lambda_k + i\gamma)}{\sinh(\lambda_j - \lambda_k - i\gamma)} = (-)^{M+1}[\frac{\sinh(\lambda_j + is\gamma)}{\sinh(\lambda_j - is\gamma)}]^{2M}$$ (2)

$$\gamma = \frac{(N-1)\pi}{2N} \quad s = \frac{1}{2(N-1)}$$

$$L = (N-1)M - 2Q \quad Q = 0, 1 \ldots (N-1)/2$$

Here $Q$ is the $Z_N$ charge, defined mod $N$, corresponding to the eigenvalue of the conserved operator

$$\exp(2\pi Q/N) = \prod_{k=1}^{M} X_k$$

Sectors of charge $Q$ and $N-Q$ are mapped into each other by the (conserved) charge conjugation operator

$$C|n_1, n_2, \ldots n_M > = |N - n_1, N - n_2, \ldots N - n_M >$$

so that the symmetry group of (1) is $Z_N \times Z_2$. Energy and momentum of each state are related to a solution of (2)

$$E = \sum_{k=1}^{L} \cot(i\lambda_k + \frac{\pi}{4N}) - 2M \sum_{k=1}^{(N-1)/2} \cot(\pi k/N) \quad (3)$$

$$\exp(iP) = \prod_{k=1}^{L} \frac{\sinh(\lambda_k + \frac{i\pi}{4N})}{\sinh(\lambda_k - \frac{i\pi}{4N})}$$ (4)

In the following sections we consider the sector $Q = 0$ and show that the excitation spectrum is massless and made up of one species of “quasiparticle” (FM) and $N$
species of “quasiparticles” (AFM) with linear dispersion relations at small momenta

\[
\begin{align*}
\text{FM} & \quad \epsilon(p) \approx Np \\
\text{AFM} & \quad \epsilon(p) \approx \frac{N}{N-1}p
\end{align*}
\]

(5) (6)

We also determine a set of completeness rules that allow to classify all physical solutions of (2), i.e those solutions that actually correspond to an eigenstate of (1), and show how to count them to obtain the correct dimension of the sector \( Q = 0 \) in the vector space spanned by the basis \( n \). Finally, the set of nonlinear integral equations [10] that describe the thermodynamics of (1), are used to determine the low temperature asymptotics of the specific heat, related to the central charge of the conformal field theory by [11]

\[
C \approx \frac{\pi c}{3v} T
\]

where \( v \) is the group velocity of massless excitation, from which we find

\[
\begin{align*}
\text{FM} & \quad c = 2 \frac{N - 1}{N + 2} \\
\text{AFM} & \quad c = 1
\end{align*}
\]

(7) (8)

The value (7) is of course the one predicted in [2].

\section{Strings and multiplets}

The traditional hypothesis on the solutions of (2) is that, in the limit \( M \to \infty \), they group into strings

\[
\lambda_{l,\alpha}^{(n,v)} = \lambda_\alpha^{(n,v)} + \frac{2i\gamma}{2}(n + 1 - 2l) + \frac{i\pi(1 - v)}{4} \quad l = 1, 2 \ldots n
\]

(9)

where \( \lambda_\alpha^{(n,v)} \) is the real center of the string, \( n \) its length, and \( v = \pm 1 \) its parity. It’s been long known [12], and the numerical analysis of the case \( N = 3 \) confirms it [13], that complex pairs with imaginary part different from the one given in (9) can also appear. We formulate the following conjecture: for fixed \( N = 2p + 1 \), the solutions (roots) of (2) fall into three classes
1. 1-strings with both parities: \((1, v), \quad v = \pm 1\)

2. Even length strings: \((n, v), \quad n = 2, 4, \ldots N - 1, \quad v = \pm 1\)

3. \(p\) multiplets, to be denoted \((M, m)\), of length \(4m + 2, \quad m = 0, 1, \ldots p - 1\)

\[
\lambda_{l,\alpha}^{(M,m)} = \lambda_{\alpha}^{(M,m)} \pm i(\frac{\pi}{4} + \frac{l\pi}{2N}) \quad l = -m, -m + 1, \ldots m - 1, m
\]

with \(\lambda_{\alpha}^{(M,m)}\) real. So, for \(N = 3\) \((p = 1)\) we have only a pair \((m = 0)\), for \(N = 5\) \((p = 2)\) we have a pair \((m = 0)\) and a sextet \((m = 1)\), etc.. This conjecture is partly motivated by a numerical diagonalization of the transfer matrix and partly warranted \textit{a posteriori} by the fact that it produces the correct counting of states (see Section 4). To diagonalize \(T(u)\) numerically, one fixes the spectral parameter \(u\) at some conveniently chosen complex value, finds the eigenvectors numerically and then applies \(T(u)\) to them. The eigenvalues are then polynomials in \(e^{2iu}\) and it is easy to locate numerically their zeroes \[13\].

Equations for the real centers are obtained by taking the product of (2) over members of a string (multiplet), so that all factors are in the form

\[
G(\lambda, \alpha, v) = \frac{\sinh(i\alpha + \lambda + i(1 - v)\frac{\pi}{4})}{\sinh(i\alpha - \lambda + i(1 - v)\frac{\pi}{4})}
\]

\(\lambda \in \mathbb{R} \quad \alpha \in (-\pi/2, \pi/2]\)

In (3) strings have been assigned their limiting “perfect” value, and one may have \(\alpha = 0\) or \(\alpha = \pi/2\) with \(G(\lambda, 0, v) = -v\) and \(G(\lambda, \pi/2, v) = v\). We group these exceptional factors to yield an overall \(\pm\) sign, and then take the Log choosing the branch

\[
i \text{Log}G(\lambda, \alpha, v) = \phi(\lambda, \alpha, v) = 2v \arctan(\cot(\alpha)^v \tanh(\lambda))
\]

\(\alpha \in (-\pi/2, 0) \cup (0, \pi/2]\)

so that \(\phi(\lambda, \alpha, v)\) is a monotonical continuous function with values in \((-\pi, \pi)\). The new equations are written with the help of \(Z\)-functions

\[
Z_j(\lambda_{\alpha}^{(j)}) = I_{\alpha}^{(j)}/M
\]
\[ Z_j(\lambda) = \frac{1}{2\pi}t_j(\lambda) - \frac{1}{2\pi M} \sum_{k} \sum_{\beta=1}^{M_k} \Theta_{j,k}(\lambda - \lambda^{(k)}_{\beta}) \] (13)

Here \( j, k \) denote the type of string (multiplet), \( M_k \) is their number, \( I^{(j)}_a \) are integral or half-odd, and the functions in (13) are defined

\[ t_{(1,v)}(\lambda) = 2\phi(\lambda, \frac{\pi}{4N}, v) \] (14)
\[ t_{(n,v)}(\lambda) = \sum_{l=1}^{n} 2\phi(\lambda, \frac{\gamma}{2}(n + 2s + 1 - 2l), v) \] (15)
\[ t_{(M,m)}(\lambda) = \sum_{\epsilon=\pm1} 2\epsilon\phi(\lambda, \frac{\pi}{4} + \epsilon\frac{2m + 1}{4N}, +) \] (16)

\[ \Theta_{(n,v),(n',w)}(\lambda) = \phi(\lambda, \frac{\gamma}{2}(n + n'), vw) + \phi(\lambda, \frac{\gamma}{2}|n - n'|, vw) + \sum_{l=1}^{\min(n,n')-1} 2\phi(\lambda, \frac{\gamma}{2}|n - n' + 2l|, vw) \] (17)

(including the case \( n = n' = 1 \))

\[ \Theta_{(1,v),(M,m)}(\lambda) = \sum_{\epsilon=\pm1} [\epsilon\phi(\lambda, \frac{\pi}{4} - \epsilon\frac{\pi m}{2N}, v) + \epsilon\phi(\lambda, \frac{\pi}{4} - \epsilon\frac{\pi(m+1)}{2N}, v)] \] (18)

\[ \Theta_{(n,v),(M,m)}(\lambda) = \sum_{\epsilon=\pm1} \sum_{l=1}^{2m} 2\phi(\lambda, \frac{1 + \epsilon}{4}\pi - \frac{\pi}{4N}(n - 2m - 1 + 2l), v) + \] (19)
\[ \sum_{\epsilon=\pm1} \frac{\pi}{4N}(n - 2m - 1), v) + \phi(\lambda, \frac{1 + \epsilon}{4}\pi - \frac{\pi}{4N}(n + 2m + 1), v)] \] (n > 2m)

\[ \Theta_{(n,v),(M,m)}(\lambda) = \sum_{\epsilon=\pm1} \sum_{l=1}^{n-1} 2\phi(\lambda, \frac{1 + \epsilon}{4}\pi - \frac{\pi}{4N}(2m + 2l + 1 - n), v) + \] (20)
\[ \sum_{\epsilon=\pm1} \frac{\pi}{4N}(2m - n + 1), v) + \phi(\lambda, \frac{1 + \epsilon}{4}\pi - \frac{\pi}{4N}(2m + n + 1), v)] \] (n ≤ 2m)

\[ \Theta_{(M,m),(M,m')}(\lambda) = 2\sum_{\epsilon=\pm1} [\phi(\lambda, \frac{1 + \epsilon}{4}\pi - \frac{\pi}{2N}|m - m'|, +) + \phi(\lambda, \frac{1 + \epsilon}{4}\pi - \frac{\pi}{2N}(m + m' + 1), +)] \]
\[ + 4\sum_{\epsilon=\pm1} \sum_{l=1}^{2\min(m,m')} \phi(\lambda, \frac{1 + \epsilon}{4}\pi - \frac{\pi}{2N}(|m - m'| + l), +) \] (21)

The notation in ([14, 21]) has been used for the sake of brevity but it is understood, in agreement with the remarks before ([I]), that in \( \phi(\lambda, \alpha, v) \) \( \alpha \) is shifted by periodicity.
to be in \((-\frac{\pi}{2}, \frac{\pi}{2}]\) and \(\phi(\lambda, 0, v) = \phi(\lambda, \pi/2, v) = 0\). These exceptional values of \(\alpha\) only change the oddness of \(I^{(j)}\) in (12) and keeping track of them one finds

\[
I^{(1,+)} = \text{integer(half - odd) if } 1 + M + M^{(1,+)} + M^{(N-1,(-p))} = \text{even(odd)}
\]

\[
I^{(1,-)} = \text{integer(half - odd) if } 1 + M + M^{(1,-)} + M^{(N-1,(-p+1))} = \text{even(odd)}
\]

\[
I^{(N-1,(-p+1))} = \text{integer(half - odd) if } 1 + M^{(1,-)} + M^{(N-1,(-p+1))} = \text{even(odd)}
\]

\[
I^{(N-1,(-p))} = \text{integer(half - odd) if } 1 + M^{(1,+)} + M^{(N-1,(-p))} = \text{even(odd)} \quad (22)
\]

\[
I^{(n,+)} = \text{integer(half - odd) if } 1 + M^{(n,+)} = \text{even(odd)}
\]

\[
I^{(n,-)} = \text{integer(half - odd) if } 1 + M^{(n,-)} = \text{even(odd)}
\]

\[
I^{(M,m)} = \text{always integer}
\]

Energy (3) and momentum (4) become functions of the string centers. Real part of the energy and momentum are simply related to the \(t\)-functions in (14-16)

\[
\text{Re}E_j(\lambda) = \frac{1}{4} t'(\lambda) \quad \forall j
\]

\[
p_j(\lambda) = -\frac{1}{2} t_j \quad \forall j \neq (1,+) \quad \text{(mod}2\pi)
\]

\[
p_{(1,+)}(\lambda) = -\frac{1}{2} t_{(1,+)}(\lambda) + \pi \quad \text{(mod}2\pi)
\]

but even after summation over string (multiplet) members, the energies retain an imaginary part

\[
\text{Im}e_{(n,v)} = -\sum_{k=1}^{n} \frac{\sinh(2\lambda)}{\cosh(2\lambda) - \cos\left(\frac{N-1}{2N}\pi(n + 1 - 2k) + \frac{(1-v)}{2} \pi - \frac{\pi}{2N}\right)} \quad (23)
\]

including \((1, \pm)\)-strings and

\[
\text{Im}e_{(M,m)} = -\sum_{k=-m}^{m} \left[ \frac{\sinh(2\lambda)}{\cosh(2\lambda) + \sin\left(\frac{\pi(2k+1)}{2N}\right)} + \frac{\sinh(2\lambda)}{\cosh(2\lambda) + \sin\left(\frac{\pi(2k-1)}{2N}\right)} \right] \quad (24)
\]

It will be shown, at least in the thermodynamic limit, how these imaginary parts can be removed assuming suitable correlations between the quantum numbers \(I^{(j)}_{\alpha}\) in (12).
3 Ground state and excitations. Correlation of rapidities.

The survival of an imaginary part in the bare energies (23-24) signals that correlations between the rapidities \( \{ \lambda_\alpha^{(j)} \} \) must exist in order to ensure reality of the total energy. In fact, the detailed investigation carried out in the simplest \( N = 3 \) case [13] clearly shows that (12) contain several spurious solutions and only a subset of the possible choices of quantum numbers \( \{ I^{(j)}_\alpha \} \) in (12) reproduces the correct physical solutions, i.e. the spectrum of \( H \). This peculiarity, together with the massive appearance of non-string multiplets, is probably a consequence of the “unorthodox” approach used here to diagonalize the transfer matrix [9]. In fact, the unknowns of (2) are, up to a change of variables, the zeroes of the transfer matrix eigenvalues themselves, rather than the zeroes of an auxiliary \( Q \)-matrix satisfying a “\( T-Q \) recursion relation” [14, 15]. Therefore some details of the conventional method of dealing with (12) must be modified. In particular, densities of rapidities and holes that describe the solutions of (12) in the thermodynamic limit are expected to be related by a set of constraints [4] and this should be taken into account if one were searching for a minimum of the free energy functional of the gas of rapidities and holes [10].

Instead of assuming from the start a set of selection rules on \( \{ I^{(j)}_\alpha \} \), and consequently on the rapidities \( \{ \lambda^{(j)}_\alpha \} \), we prefer to make a working assumption, essentially based on small chain observations, on the ground state in the infinite chain limit. We will check the validity of the assumption by performing small variations around the ground state configuration to ensure that it is indeed a minimum, at least locally, of the energy. The case \( N = 3 \) will provide an heuristic guidance. Since \((N - 1, \pm)\) and \((1, \pm)\) strings play a preeminent role in the following, we adopt the shortened, \( N \)-dependent notation: \((1, +) \doteq (a)\), \((1, -) \doteq (b)\), \((N - 1, (-)^p + 1) \doteq (c)\), \((N - 1, (-)^p) \doteq (d)\).

The FM ground state is a band of \((N - 1, (-)^p + 1) = (c)\) strings [9], whose
\(I_{\alpha}^{(N-1,(-)^{p+1})}\) form a closely packed sequence symmetric around 0, and whose centers fill the real axis with density

\[
\rho_c^{(0)}(\lambda) = \frac{2N}{\pi \cosh(2N\lambda)}
\]  

(25)
solution of \((\) here \(*\) means convolution) \[16\]

\[
\rho_c^{0}(\lambda) = -Z'_c(\lambda) = -\frac{1}{2\pi}t'_c(\lambda) + \frac{1}{2\pi}\Theta_{c,c} * \rho_c^{(0)}(\lambda)
\]  

(26)

The (real) density of ground state energy is

\[
e_0 = \lim_{M \to \infty} \frac{E_0}{M} = \int d\lambda \rho_c^{0}(\lambda)e_c(\lambda) - 2\sum_{k=1}^{(N-1)/2} \cot(\pi k/N)
\]

In \(26\) we have taken into account that \(Z_c\) is a decreasing function of \(\lambda\). Observables in the excited states are computed from dressing equations \([4, 17]\) that incorporate the effect of the backflow in the ground state distribution \(25\) when holes are pinched in it and other types of roots are added. For a state described by a density of \(c\)-strings \(\rho_c\) containing a finite set of \(M_c^{(h)}\) holes \(\{\lambda_{\beta}^{(h,c)}\}\), and by a finite set \(\{\lambda_{\beta}^{(k)}\}\) of other roots, we have

\[
Z'_c(\lambda) \equiv -\sigma_c(\lambda) = -\rho_c(\lambda) - \frac{1}{M}\sum_{\beta=1}^{M_c^{(h)}} \delta(\lambda - \lambda_{\beta}^{(h,c)})
\]

From the definition of \(Z\)-function

\[
\sigma_c(\lambda) = \frac{1}{2\pi}\Theta_{c,c} * \sigma_c(\lambda) = -\frac{1}{2\pi}t'_c(\lambda)
\]

\[
= \frac{1}{2M\pi} \sum_{\beta=1}^{M_c^{(h)}} \Theta_{c,c}(\lambda - \lambda_{\beta}^{(h,c)}) + \frac{1}{2M\pi} \sum_{k \neq c}^{M_k} \sum_{\beta=1}^{M_k} \Theta_{c,k}(\lambda - \lambda_{\beta}^{(k)})
\]

(27)

where the sum \(\sum_{k \neq c}\) is taken over all roots other than \(c\). Even without solving \(27\) explicitly, one sees that the energy

\[
E = \sum_{k=1}^{M_k} e_k(\lambda_{\beta}^{(k)}) = \int d\lambda \sigma_c(\lambda)e_c(\lambda)
\]

\[
= \sum_{\beta=1}^{M_c^{(h)}} e_c(\lambda_{\beta}^{(h,c)}) + \sum_{k \neq c}^{M_k} \sum_{\beta=1}^{M_k} e_k(\lambda_{\beta}^{(k)})
\]
can be written as

\[ E = E_0 - \sum_{\beta=1}^{M_c^{(h)}} \epsilon_c(\lambda^{(h,c)}_{\beta}) + \sum_{k \neq c} \sum_{\beta=1}^{M_k} \epsilon_k(\lambda^{(k)}_{\beta}) \]  

(28)

where \( E_0 \) is the ground state energy and the dressed energies are defined as solutions of

\[ \epsilon_k - \frac{1}{2\pi} \Theta^{(c)}_{k,c} * \epsilon_c = \epsilon_k \]  

(29)

In (29) \( k \) runs over all strings and multiplets. The solution of (29) is easily found by Fourier transform method

\[ \text{Re} \epsilon_a(\lambda) = -\text{Re} \epsilon_c(\lambda) = \frac{N}{\cosh(2N\lambda)} \]  

(30)

\[ \text{Im} \epsilon_a(\lambda) = \text{Im} \epsilon_c(\lambda) = -N \tanh(2N\lambda) \]  

(31)

all other \( \epsilon_k \) being identically zero. The simplest way to cancel the imaginary part in (28) is to assume that pairwise

\[ \{ \lambda^{(h,c)}_{\beta} \} = \{ \lambda^{(a)}_{\beta} \} \quad \beta = 1, 2 \ldots M_a \]  

(32)

Then (28) is non-negative, thereby proving that (28) is indeed the ground state distribution (strictly speaking, it has been shown that it is a local minimum of the energy functional). It would seem that, due to the vanishing of all other dressed energies, one could add strings (or multiplets) other than \((a)\) without increasing \( E \), but it will be proved in the next section that this is not the case.

The calculation of momentum is identical. The ground state momentum vanishes since \( \rho_c^{(0)}(\lambda) \) is an even function of \( \lambda \) and in an excited state

\[ P = M_a \pi + \sum_{k \neq c} \sum_{\beta=1}^{M_k} \pi_k(\lambda^{(k)}_{\beta}) - \sum_{\beta=1}^{M_h} \pi_c(\lambda^{(h,c)}_{\beta}) \]  

(33)

where the dressed momenta \( \pi_k \) solve

\[ \pi_k - \frac{1}{2\pi} \Theta^{(c)}_{k,c} * \pi_c = \tilde{p}_k \]  

(34)
It is convenient to work with a subtracted (odd in $\lambda$) bare momentum $\tilde{p}$ defined by $\tilde{p}_k = p_k \forall k \neq (a)$ and $p_a = \tilde{p}_a + \pi$, which accounts for the term $M_a\pi$ in (33). Then $\pi'_k = -2\text{Re}e_k$ which is easily integrated

$$\pi_a(\lambda) = -\pi_c(\lambda) = -2 \arctan(\tanh(N\lambda))$$

and $\pi_k(\lambda) = 0$ for all other strings and multiplets. From (28), (30) and (32) it is clear that the excitation spectrum is made up of one kind of quasiparticle appearing as a “bound state” (not to be understood as a physical bound state of elementary particles): $(1, +)$ paired to a hole in the $(N - 1, (-)^{p+1})$ band

$$E = E_0 + \sum_{\beta=1}^{M_a} \frac{2N}{\cosh(2N\lambda_{\beta})}$$

$$P = \sum_{\beta=1}^{M_a} [\pi - 4 \arctan(\tanh(N\lambda_{\beta}))]$$

The dispersion curve for each quasiparticle is

$$\epsilon(p) = 2N \sin\left(\frac{p}{2}\right) \quad p \in (0, 2\pi)$$

which yields (3) at small momenta.

The AFM case has a richer spectrum. We proceed as in the FM case, assuming a ground state, namely a filled band of $(a)$ and a filled band of $(b)$ corresponding to a closely packed sequence of $I^{(a)}$ and $I^{(b)}$ in (12). Consequently, their densities are simply the derivatives of the relevant (increasing) $Z$-functions

$$\rho_j^{(0)}(\lambda) = Z_j'(\lambda) = \frac{1}{2\pi} t_j'(\lambda) - \frac{1}{2\pi} \sum_{t=a,b} \Theta_{j,t}^{(0)} \rho_t^{(0)}$$

where $j$ takes values in $\{a, b\}$. The solution is

$$\rho_a^{(0)}(\lambda) = \frac{1}{2\pi} \int d\lambda e^{-i\lambda} \frac{\sinh\left(\frac{q\pi}{4N}\right)}{\sinh\left(\frac{q\pi}{2N}\right) \cosh\left(\frac{2\pi(N-1)}{4N}\right)}$$

$$\rho_b^{(0)}(\lambda) = \frac{1}{2\pi} \int d\lambda e^{-i\lambda} \frac{\sinh\left(\frac{q\pi(N-2)}{4N}\right)}{\sinh\left(\frac{q\pi}{2N}\right) \cosh\left(\frac{2\pi(N-1)}{4N}\right)}$$

The ground state energy density is

$$e_0 = - \sum_{j=a,b} \int d\lambda \rho_j^{(0)}(\lambda)e_j(\lambda) + 2 \sum_{k=1}^{(N-1)/2} \cot(k\pi/N)$$
We perturb these distributions by pinching two sets of holes \( \{ \lambda^{(h,a)}_\beta \} \) and \( \{ \lambda^{(h,b)}_\beta \} \) and adding extra roots of type \( k \neq a, b \)

\[
Z'_j(\lambda) = \sigma_j(\lambda) = \rho_j(\lambda) + \frac{1}{M} \sum_{\beta=1}^{M_j^{(h)}} \delta(\lambda - \lambda^{(h,j)}_\beta)
\]

where \( j \in \{a, b\} \). The following steps are a straightforward generalization of the previous case. Densities of vacancies are found from the system of two integral equations

\[
\sigma_j(\lambda) = \frac{1}{2\pi} t'_j(\lambda) - \frac{1}{2\pi} \sum_{l=a,b} \Theta'_{j,l} * \sigma_l
\]

\[
+ \frac{1}{2M\pi} \sum_{l=a,b} M_j^{(h)} \sum_{\beta=1}^{M_l} \Theta'_{j,l}(\lambda - \lambda^{(h,l)}_\beta) - \frac{1}{2M\pi} \sum_{k \neq a,b} M_k \sum_{\beta=1}^{M_k} \Theta'_{j,k}(\lambda - \lambda^{(k)}_\beta)
\]

(38)

where \( j \in \{a, b\} \). Owing to (38), energy and momentum in the thermodynamic limit reduce to

\[
E = E_0 - \sum_{j=a,b} \sum_{\beta=1}^{M_j^{(h)}} \epsilon_j(\lambda^{(h,j)}_\beta) + \sum_{k \neq a,b} \sum_{\beta=1}^{M_k} \epsilon_k(\lambda^{(k)}_\beta)
\]

(39)

\[
P = P_0 + M_a \pi - \sum_{j=a,b} \sum_{\beta=1}^{M_j^{(h)}} \pi_j(\lambda^{(h,j)}_\beta) + \sum_{k \neq a,b} \sum_{\beta=1}^{M_k} \pi_k(\lambda^{(k)}_\beta)
\]

(40)

provided that the dressed observables \( \epsilon \) and \( \pi \) are defined as solutions of

\[
\epsilon_k + \frac{1}{2\pi} \sum_{j=a,b} \Theta'_{k,j} * \epsilon_j = -\epsilon_k
\]

(41)

\[
\pi_k + \frac{1}{2\pi} \sum_{j=a,b} \Theta'_{k,j} * \pi_j = \tilde{\pi}_k
\]

(42)

where \( k \) runs over all types of roots. The solution reveals that all imaginary parts are odd functions of \( \lambda \) and \( \text{Im} \epsilon_{(M,k)} = 0 \), \( \text{Im} \epsilon_{(n,+)} = -\text{Im} \epsilon_{(n,-)} \), \( \text{Im} \epsilon_a = \text{Im} \epsilon_c \), \( \text{Im} \epsilon_b = \text{Im} \epsilon_d \), suggesting that, again, a pairing of rapidities must be in effect. One is led to assume that \( M_{(n,+)} = M_{(n,-)} = M_n \), \( M_c = M^{(h)}_a \), \( M_d = M^{(h)}_b \) and

\[
\{ \lambda^{(n,+)}_\beta \} = \{ \lambda^{(n,-)}_\beta \} \quad \beta = 1, 2, \ldots M_n
\]

(43)

\[
\{ \lambda^{(c)}_\beta \} = \{ \lambda^{(h,a)}_\beta \} \quad \beta = 1, 2, \ldots M_c
\]

(44)

\[
\{ \lambda^{(d)}_\beta \} = \{ \lambda^{(h,b)}_\beta \} \quad \beta = 1, 2, \ldots M_b
\]

(45)
As to the real parts we have

\[
\text{Re} \epsilon_a(\lambda) = -\text{Re} \epsilon_c(\lambda) = - \frac{N}{2 \cosh(2N\lambda)} \left( 2N \sum_{i=0}^{p-1} \frac{\cosh(\frac{2N\lambda}{N-1}) \cos\left(\frac{(2j+1)\pi}{2(N-1)}\right)}{\cosh\left(\frac{4N\lambda}{N-1}\right) + \cos\left(\frac{(2j+1)\pi}{N-1}\right)} \right)
\]

\[
\text{Re} \epsilon_b(\lambda) = -\text{Re} \epsilon_d(\lambda) = \text{Re} \epsilon_a(\lambda) + \frac{N}{\cosh(2N\lambda)}
\]

\[
\text{Re} \epsilon_{(n,\pm)}(\lambda) = \frac{4N}{N-1} \sum_{j=1}^{n/2} \frac{\cosh(\frac{2N\lambda}{N-1}) \cos\left(\frac{(n-2j+1)\pi}{2(N-1)}\right)}{\cosh\left(\frac{4N\lambda}{N-1}\right) + \cos\left(\frac{(n-2j+1)\pi}{N-1}\right)} \quad 2 \leq n \leq N-3
\]

\[
\text{Re} \epsilon_{(M,m)}(\lambda) = \frac{4N}{N-1} \sum_{j=-m}^{m} \frac{\cosh(\frac{2N\lambda}{N-1}) \cos\left(\frac{j\pi}{2(N-1)}\right)}{\cosh\left(\frac{4N\lambda}{N-1}\right) + \cos\left(\frac{j\pi}{N-1}\right)}
\]

It can be checked that \( \text{Re} \epsilon_j \) with \( j \in \{c, d, (n, v), (M, m)\} \) is positive definite, confirming that the conjecture about the ground state is correct. Clearly \( N \) different massless excitations are present: \( p = \frac{N-1}{2} \) multiplets, plus \( \frac{N-3}{2} \) “bound states” \( \{(n,+), (n,-)\} \) correlated through (43) and finally the “bound states” \( \{c, \text{hole in } a\} \) and \( \{d, \text{hole in } b\} \) correlated through (44-45). The equations for the momenta are easily integrated noticing that \( \pi_k' = 2\text{Re} \epsilon_k \)

\[
\pi_a(\lambda) = -\pi_c(\lambda) = - \arctan(\tanh(N\lambda)) - \sum_{j=0}^{p-1} \arctan\left(\frac{\sinh(\frac{2N\lambda}{N-1})}{\cos\left(\frac{(2j+1)\pi}{2(N-1)}\right)}\right)
\]

\[
\pi_b(\lambda) = -\pi_d(\lambda) = 2 \arctan(\tanh(N\lambda)) + \pi_a(\lambda)
\]

\[
\pi_{(n,\pm)}(\lambda) = 2 \sum_{j=1}^{n/2} \arctan\left(\frac{\sinh\left(\frac{2N\lambda}{N-1}\right)}{\cos\left(\frac{(n-2j+1)\pi}{2(N-1)}\right)}\right) \quad 2 \leq n \leq N-3
\]

\[
\pi_{(M,m)}(\lambda) = 2 \sum_{j=-m}^{m} \arctan\left(\frac{\sinh\left(\frac{2N\lambda}{N-1}\right)}{\cos\left(\frac{j\pi}{2(N-1)}\right)}\right)
\]

These expressions for the dressed momenta are not sufficient to fix completely the dispersion curve for each quasiparticle since we have to deal with the contribution \( \pi M_a \) in (40). This will be done in the next section after a rule on the string content of each physical solution of (2) is derived.
4 Completeness. Correlation of quantum numbers.

The task of counting, let alone finding, all solutions of the system of equations (4) is very difficult. Nevertheless, in the framework of the string hypothesis, one can replace the original equations with (12) and hope that, to each set of distinct numbers \( \{I^{(j)}_{\alpha}\} \) there corresponds one and only one solution. Counting solutions becomes then a problem of combinatorics, and the result should equal the dimension of the vector space on which the Hamiltonian is defined.

The sets \( \{I^{(j)}_{\alpha}\} \) cannot be chosen arbitrarily. An obvious constraint is that the total number of roots in the sector \( Q = 0 \) must be

\[
(N-1)M = M_a + M_b + (N-1)(M_c + M_d) + \sum_{n=2}^{N-3} nM_{(n,v)} + \sum_{m=0}^{p+1} (4m+2)M_{(M,m)} \tag{46}
\]

because \( L = (N-1)M \) in (4). Secondly, since the \( Z \)-functions are supposed to behave monotonically, the \( \{I^{(j)}_{\alpha}\} \) must be chosen within the limits \( MZ_j(\pm\infty) \). In principle, for a, say, increasing \( Z \)-function and a given string content of a state, one should determine the largest available vacancy \( I_{\max}^{(j)} \) as the largest integer (or half-odd, according to (22)) strictly smaller than \( MZ_j(\pm\infty) \) and count the integers (or half-odd) contained in \([−I_{\max}^{(j)}, I_{\max}^{(j)}]\). This gives the set of available vacancies for \( \{I^{(j)}_{\alpha}\} \), that we denote \( \text{vac}(j) \). It is a remarkable property of the sector \( Q = 0 \), and this is proven in Appendix B, that

\[
\frac{\text{vac}(j)}{M} = \pm(Z_j(\infty) - Z_j(-\infty)) \tag{47}
\]

where \( (+) \) is used if \( Z_j \) is increasing (decreasing).

An additional feature of (12) is that, as it was extensively shown by the numerical study for the case \( N = 3 \), only a subset of the possible choices \( \{I^{(j)}_{\alpha}\} \), even when constrained by (13) and (17), gives an eigenvalue of \( H \): the string content of an eigenvalue cannot be chosen arbitrarily and correlations are in effect between the
\{I^{(j)}_\alpha\}. Motivated by these observations, and by (43-45), we are led to assume that, for any state
\[ M_{(n,+)} = M_{(n,-)} = M_n \]
(48)
and denoting by \{I^{(h,j)}_\alpha\} the empty vacancies (holes) and by \(M^{(h)}_j\) their number
\[ M^{(h)}_a = M_c \quad M^{(h)}_b = M_d \]
(49)
Since \(\text{vac}(j) = M_j + M^{(h)}_j\), imposing (49) means
\[ Z_a(+\infty) - Z_a(-\infty) = \frac{M_a + M_c}{M} \]
\[ Z_b(+\infty) - Z_b(-\infty) = \frac{M_b + M_d}{M} \]
Either one yields, when (44) and (47) are taken into account, the content rule
\[ (N - 2)M_a = NM_b + 2(N - 1)M_d + \sum_{n=2}^{N-3} 2nM_n + \sum_{m=0}^{p-1} (4m + 2)M_{(M,m)} \]
which generalizes the content rule (3.1) of [4], and it is in agreement with all numerical tests performed for \(N = 5, 7\). It has to be understood as the statement that for any eigenvalue the numbers \(M_j\) must satisfy (50) as well as (46). From (46) and (50) we choose to take \(M_b, M_d, M_n\) and \(M_{(M,m)}\) as independent variables. We then find, after observing that all \(Z\)-functions except \(Z_a\) and \(Z_b\) are decreasing
\[ \text{vac}(a) = \text{vac}(c) = M + M_b + M_d \]
\[ \text{vac}(b) = \text{vac}(d) = M_b + M_d \]
\[ \text{vac}(n, +) = \text{vac}(n, -) = \frac{2n}{N - 2} (M_b + M_d) + \sum_{n' \neq n} \left( \frac{2nn'}{N - 2} - 2\min(n, n') \right) M_{n'} \]
\[ \quad + \left( \frac{2n^2}{N - 2} - 2n + 1 \right) M_n + \sum_{m \leq \frac{n}{2}} (4m + 2) \left( \frac{n}{N - 2} - 1 \right) M_{(M,m)} + \sum_{m \geq \frac{n}{2}} 2n \left( \frac{2m + 1}{N - 2} - 1 \right) M_{(M,m)} \]
\[ \text{vac}(M, m) = \frac{4m + 2}{N - 2} (M_b + M_d) + \sum_{n \leq 2m} n \left( \frac{4m + 2}{N - 2} - 2 \right) M_n + \sum_{n > 2m} (4m + 2) \left( \frac{n}{N - 2} - 1 \right) M_n + \sum_{m' \neq m} \left( \frac{(4m + 2)(4m' + 2)}{2(N - 2)} - 4\min(m, m') - 2 \right) M_{(M,m')} + \left( \frac{(4m + 2)^2}{2(N - 2)} - 4m - 1 \right) M_{(M,m)} \]
The reduction by $1/2$ of the vacancies of multiplets [13] has been adopted here too (see Appendix B). Of course, the number of vacancies must be, by definition, an integer. The content rule (54), implying that the right hand side must be divisible by $N - 2$, guarantees that that is the case for $\text{vac}(n)$ and $\text{vac}(M, k)$.

As to the correlations between $\{I^{(j)}_\alpha\}$, from (32) and (44-45), and noticing that (46) affects the oddness table (22), we are led to assume

$$I^{(a)}_\alpha = -I^{(h,c)}_\alpha \quad \alpha = 1, 2 \ldots M_a$$

$$I^{(b)}_\alpha = -I^{(h,d)}_\alpha \quad \alpha = 1, 2 \ldots M_b$$

and, necessarily because of (51)

$$I^{(c)}_\alpha = -I^{(h,a)}_\alpha \quad \alpha = 1, 2 \ldots M_c$$

$$I^{(d)}_\alpha = -I^{(h,b)}_\alpha \quad \alpha = 1, 2 \ldots M_d$$

and from (13)

$$I^{(n,+)}_\alpha = I^{(n,-)}_\alpha$$

which generalize in a rather obvious way the selection rules of [13]. It is shown in Appendix A that these conditions on the integers do imply the correlations (32) and (43-45).

The number of physical solutions of (12) is therefore

$$\sum_{M_b, M_c, M_n, M(M, m)} \left( \frac{\text{vac}(M_c)}{M_c} \right) \left( \frac{\text{vac}(M_b)}{M_b} \right) \prod_{n=2}^{N-3} \left( \frac{\text{vac}(n)}{M_n} \right) \prod_{m=0}^{p-1} \left( \frac{\text{vac}(M, m)}{M(M, m)} \right)$$

where

$$M_c = M - \frac{1}{N - 2} \left( N M_d + 2 M_b + \sum_{n=2}^{N-3} 2 n M_n + \sum_{m=0}^{p-1} (4 m + 2) M(M, m) \right)$$

and the sum is subjected to the constraints

1. the right hand side of (46) must be divisible by $N - 2$

2. $M_c \geq 0$
3. \( \text{vac}(j) \geq M_j \ \forall j \)

The result of (57) is expected to be \( N^{M-1} \), the dimension of the sector \( Q = 0 \). This has been checked numerically for \( N = 7, 9 \) and it will be proven analytically for \( N = 5 \). The proof for \( N = 3 \) has already appeared in [13].

It should be noticed that the number of vacancies is a piecewise continuous function of the imaginary parts of the roots belonging to a multiplet and it exhibits jumps at some special values. The conjecture (10) on the structure of the multiplets is based on the observation that it yields the right counting in (57). Therefore, the fact that (57) gives \( N^{M-1} \) should be regarded as a check of the various assumptions made so far.

When \( N = 5 \), (46) reduces to

\[
M_a = \frac{5M_b + 8M_d + 4M_2 + 2M_{(M,0)}}{3} + 2M_{(M,1)}
\]

Hence, there must be an integer \( k \), not necessarily positive, such that

\[
-M_b - M_d + M_2 + 2M_{(M,0)} = -3k
\]

which is used to eliminate \( M_2 \) in favor of \( k \). Then (57) becomes

\[
\sum \left( \begin{array}{l}
M + M_b + M_d \\
M - 3M_d - 2M_b + 2M_{(M,0)} + 4k - 2M_{(M,1)} \\
M_b + M_d + k \\
M_b + M_d - 2M_{(M,0)} - 3k
\end{array} \right) \left( \begin{array}{c}
M_b + M_d \\
M_{(M,0)} + 2k \\
M_{(M,0)} \\
M_{(M,1)}
\end{array} \right) = \sum_{n=0}^{2n \leq A} \binom{C + n}{n} \frac{B}{A - 2n}
\]

where the sum is over \( M_b, M_d, M_{(M,0)}, M_{(M,1)}, k \) and the constraints 2. and 3. apply. We use the integral representation

\[
\frac{1}{2\pi i} \oint dz \frac{(1 - z^2)^{-1-C}(1 + z)^B}{z^{1+A}} = \sum_{n=0}^{2n \leq A} \binom{C + n}{n} \frac{B}{A - 2n}
\]

\[
\frac{1}{2\pi i} \oint dz \frac{(1 + z)^A}{z^{1+B}} = \binom{A}{B}
\]

to reduce (58) to

\[
(\frac{1}{2\pi i})^2 \oint dz_1 dz_2 S_1(z_1, z_2) S_2(z_1, z_2) \frac{(1 + z_1)^M}{z_2 z_1^{M+1}}
\]
where

\[ S_1 = \sum_{k,M(M,0) \geq 0} \frac{(1 + z_2)^k z_2^{3k + 2M(M,0)}}{z_1^{4k + 2M(M,0)}} \left( \frac{2k + M(M,0)}{M(M,0)} \right) \]

\[ S_2 = \sum_{M_b,M_d \geq 0} \left( \frac{M_b + M_d}{M_d} \right) \frac{[(1 + z_2)(1 + z_1)]^{M_b + M_d} z_1^{3M_d + 2M_b}}{(1 - z_2^2)^{2M_b + 2M_d} z_2^{2M_b + M_d}} \]

Convergence of all series is guaranteed provided that we take \(|z_1| \sim \epsilon^a\), \(|z_2| \sim \epsilon^\beta\) with \(\epsilon \ll 1\) and \(\beta/2 < \alpha < 3\beta/4\). Then one finds

\[ S_1(z_1, z_2) = \frac{z_1^2(z_1^2 - z_2^2)}{z_1^3 - z_2^3 - 2z_1^2 z_2^2} \]

\[ S_2(z_1, z_2) = \frac{z_2(1 - z_1)^2(1 + z_1)}{z_2 - z_1 z_2^2 - 2z_2 z_1^2 - z_2^2 - z_1^2} \]

The integration over \(z_2\) is performed first. Only the simple pole of \(S_2(z_1, z_2)\)

\[ z_2 = \frac{z_1^2}{1 - 2z_1} \]

is encircled by the contour, while the poles of \(S_1\) lie outside the path of integration if \(\epsilon\) is sufficiently small. Finally, after some algebra, (59) reduces to

\[ \frac{1}{2\pi i} \oint dz \frac{(1 + z)^{M-1}(1 - 3z)}{z^{M+1}(1 - 4z)} = 5^{M-1} \]

The last equality has been obtained by deforming the contour of integration around the simple pole at \(z = 1/4\). The proof given here is just the upgrading of an old method [18]. Unfortunately we haven’t been able to extend the proof to \(N > 5\).

What seems to make higher \(N\) harder is not so much the constraint (50), which can be taken care of by a double contour integration like for \(N = 5\), as the complicated form of \(\text{vac}(n)\) and \(\text{vac}(M,m)\).

Clearly, the content rule (54) has some consequences on the nature of the excitation spectrum in the thermodynamic limit, as computed in the previous section. In the FM case, it has been shown that only \(\epsilon_a\) and \(\epsilon_c\) are nonvanishing, but (50) entails that roots other than (a) can be added only by increasing \(M_a\) and with it the energy. Hence, the ground state is not degenerate, and moreover no one particle excitation can exist (at least in this sector).
As to the AFM case, we solve (46) and (50) for $M_a, M_b$

$$M_a = \frac{NM}{2} - \frac{NM_c}{2} - \frac{(N-2)M_d}{2} - \sum_{n=2}^{N-3} nM_n - \sum_{m=0}^{p-1} (2m+1)M_{(M,m)}$$

(60)

$$M_b = \frac{(N-2)M}{2} - \frac{(N-2)M_c}{2} - \frac{NM_d}{2} - \sum_{n=2}^{N-3} nM_n - \sum_{m=0}^{p-1} (2m+1)M_{(M,m)}$$

(61)

We restrict ourselves to the simpler situation of even number of lattice sites. Then (60-61) show that multiplets and $n$-strings, $n < N-1$, can appear as one-particle excitations, whereas $M_c + M_d$ is bound to be even. The ground state has $M_a = \frac{NM}{2}$, $M_b = \frac{(N-2)M}{2}$, which exactly fill the available vacancies (51). Since the distributions (83-86) are even in $\lambda$, the only contribution to the ground state momentum is

$$P_0 = \pi M_a^{(0)} = \frac{\pi NM}{2}$$

As to the excited states, $\Delta P$ picks a contribution $(M_a - M_a^{(0)})\pi$. Owing to the pairing (43-45) and the arbitrariness of $2\pi$ in the definition of momentum, we have the following dispersion relations for the elementary excitation

1. **“Bound state” (c) plus hole in (a)**

   $$E(\lambda) = 2\epsilon_c(\lambda) \quad P(\lambda) = 2\pi_c(\lambda) + \frac{N\pi}{2} \quad P \in (0, N\pi)$$

2. **“Bound state” (d) plus hole in (a)**

   $$E(\lambda) = 2\epsilon_d(\lambda) \quad P(\lambda) = 2\pi_d(\lambda) + \frac{(N-2)\pi}{2} \quad P \in (0, (N-2)\pi)$$

3. **(n, +)-string paired with (n, −)-string, $n = 2, 4, \ldots N-3$**

   $$E(\lambda) = 2\epsilon_{(n,\pm)}(\lambda) \quad P(\lambda) = 2\pi_{(n,\pm)}(\lambda) + n\pi \quad P \in (0, 2n\pi)$$

4. **Multiplets $m = 0, 1, \ldots p - 1$**

   $$E(\lambda) = \epsilon_{(M,m)}(\lambda) \quad P(\lambda) = \pi_{(M,m)}(\lambda) + \pi(2m+1) \quad P \in (0, (4m+2)\pi)$$
Calling $P$ the upper limit of the momentum, we have in all four cases

$$E \approx \frac{N}{N-1} P \quad P \to 0^+$$

and

$$E \approx \frac{N}{N-1}(P - P) \quad P \to P^-$$

so that the group velocity is always $|v| = \frac{N}{N-1}$.

## 5 Thermodynamics and central charge

In the thermodynamic limit, the states of the spin chain are described by the density of rapidities $\rho_j$ and the density of holes $\rho_j^{(h)}$ (alternatively the density of vacancies $\sigma_j = \rho_j + \rho_j^{(h)}$) related by

$$ (\pm)^{r(j)} \sigma_j = Z'_j = \frac{1}{2\pi} t'_j - \frac{1}{2\pi} \sum_k \Theta'_{j,k} * \rho_k $$

(62)

The factor $(\pm)^{r(j)}$ has been included because $Z(\lambda)$ can be increasing or decreasing, so $r(a) = r(b) = 0$ and $r(j) = 1$ if $j \neq a, b$. It is well known [10, 15, 19] that, at finite temperature, the equilibrium state is determined by a system of nonlinear integral equations obtained by minimizing the free energy functional. The standard method, though, has to be suitably generalized to the present situation because not all densities $\rho_j$, $\rho_j^{(h)}$ are independent. The following discussion is closely patterned after [5] where the $N = 3$ case has been dealt with.

We assume that, not only for low lying states over the FM and AFM vacua, but for all states of the chain, the following constraints hold

$$\rho_c = \rho_a^{(h)} \quad \text{and} \quad \rho_d = \rho_b^{(h)}$$

(63)

$$\rho_a = \rho_c^{(h)} \quad \text{and} \quad \rho_b = \rho_d^{(h)}$$

(64)

$$\rho_{(n,+)} = \rho_{(n,-)} = \rho_n$$

(65)

Notice that (63) and (64) imply $\sigma_c = \sigma_a$ and $\sigma_d = \sigma_b$. It is convenient to treat FM and AFM separately [5]. We begin with FM and eliminate $\rho_c$, $\rho_d$ from the equations
We define the Fourier transform by
\[ \hat{f}(q) = \frac{1}{2\pi} \int d\lambda e^{iq\lambda} f(\lambda) \]
and introduce the compact notation \([n] = \sinh(\frac{qn\pi}{4N})\) and \(\{n\} = \cosh(\frac{qn\pi}{4N})\). We then find
\[ \sigma_j = a_j - \sum_k \delta T_{j,k} * \rho_k \tag{66} \]
where \(\sum'\) is defined to extend to all types of roots other than \(c, d\) and
\[ \begin{align*}
\hat{a}_j(q) &= \frac{1}{\{1\}} \delta_{j,a} \\
\hat{T}_{j,k}(q) &= -\frac{[N-2]}{2\{1\}[N-1]} \quad j = a, b \quad k = a, b \\
\hat{T}_{j,n}(q) &= -\hat{T}_{n,j}(q) = \frac{[n]}{[N-1]} \quad j = a, b \\
\hat{T}_{j,(M,m)}(q) &= -\hat{T}_{(M,m),j}(q) = \frac{[2m+1]}{[N-1]} \quad j = a, b \\
\hat{T}_{n,n'}(q) &= \frac{2\{1\}[\min(n,n')] [N-1-\max(n,n')] \{1\}[N-1]}{[N-1]} \quad (n \neq n') \\
\hat{T}_{n,(M,m)}(q) &= \hat{T}_{(M,m),n}(q) = \frac{2\{1\}[\min(n,2m+1)] [N-1-\max(n,2m+1)] \{1\}[N-1]}{[N-1]} \\
\hat{T}_{(M,m),(M,m')}((q) &= \frac{2\{1\}[2\min(m,m') + 1] [N-2-2\max(m,m')] \{1\}[N-1]}{[N-1]} \quad (m \neq m') \\
\hat{T}_{(M,m),(M,m)}((q) &= \frac{2m [N-2m-2] + [2m+1][N-2m-3] \{1\}[N-1]}{[N-1]} 
\end{align*} \]

Note that the parities of the \(n\)-strings have disappeared because we find \(\sigma_{(n,+)} = \sigma_{(n,-)}\) that implies, with (63), \(\rho^{(h)}_{(n,+)} = \rho^{(h)}_{(n,-)}\), so that only one of the two parities is independent and needs to be kept. Instead, replacing (63) in (62) with \(j = c, d\) yields \(\sigma_c = \sigma_a\) and \(\sigma_d = \sigma_b\), showing that (64) actually follows from (62) and (63).

The energy density functional
\[ E(\rho) = \sum_j \int d\lambda \rho_j(\lambda) e_j(\lambda) \]
is cast in the effective form by eliminating $\rho_c$ and $\rho_d$ by means of (63) and then using (66)

$$E(\rho) = \pi \sum_j' \int d\lambda \rho_j(\lambda) a_j(\lambda) + e_0$$

where, like in (66), $\sum_j'$ extends to $(a), (b), (n), (M, m)$ and $e_0$ is the ground state energy density. Note that the functions $a_j$ (actually, in the case at hand, only one is nonvanishing) are closely related to the dressed energies and that, even in the effective form (67), the energy functional depends on $\rho_j$ only and not on $\rho_j^{(h)}$. This is the reason why we have chosen to eliminate $\rho_c$ and $\rho_d$. In the entropy density functional [10], the degrees of freedom are also reduced by (63-65), so the effective expression is

$$S(\rho) = \sum_j' \int d\lambda [\sigma_j \log \sigma_j - \rho_j \log \rho_j - \rho_j^{(h)} \log \rho_j^{(h)}]$$

The minimum condition for the free energy density functional $F(\rho) = E(\rho) - TS(\rho)$ is the system of integral equations

$$\epsilon_j = \pi a_j + T \sum_k' T_{k,j} \log(1 + e^{-\epsilon_k/T})$$

where $e^{-\epsilon_j/T} = \rho_j^{(h)}$. The free energy itself is

$$F = -T \sum_j' \int d\lambda a_j(\lambda) \log(1 + e^{-\epsilon_j/T}) (\lambda) + e_0$$

It can be seen that the finite temperature dressed energies $\epsilon_j$ as defined by (69) reduce, at $T = 0$ to the actual energies of the excitations, rather than to the dressed energies as defined by (29). This is because the pairing “$a$-hole in $c$” has been included in the calculation imposing the constraint (63-64). We are particularly interested in the behavior of the specific heat at $T \to 0$. The regions at $|\lambda| \sim -\frac{1}{2N} \log T$ give the leading contribution to the $T \ll 1$ asymptotics. We define shifted functions $f^{*}(\lambda) = f(\lambda - \frac{1}{2N} \log T)$ and compare (66) and (69) in the limit $T \to 0$, finding

$$\rho_j^{*} \simeq (-)^{(j) + 1} \frac{1}{2N\pi} \frac{d\epsilon_j^{*}}{T} \frac{1}{d\lambda}$$

$$\rho_j^{(h)*} \simeq (-)^{(j) + 1} \frac{1}{2N\pi} (1 - f(\frac{e_j^{*}}{T})) \frac{d\epsilon_j^{*}}{d\lambda}$$

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where \( f(x) = (1 + e^x)^{-1} \). These relations can be inserted in the entropy, so that the leading term at \( T \to 0 \) can be written solely in terms of scaled dressed energies \( \phi(\lambda) \)

\[
S = \sum_j' \frac{T}{N \pi} (-1)^{r(j)} \int_{-\infty}^{+\infty} d\lambda \frac{d\phi_j}{d\lambda} \left[ f(\phi_j) \log(1 + e^{\phi_j}) + (1 - f(\phi_j)) \log(1 + e^{-\phi_j}) \right]
\] (73)

\[
\phi(\lambda) = \lim_{T \to 0} \frac{1}{T} (\lambda - \frac{1}{2N} \log T)
\] (74)

From their definition and (69), the scaled dressed energies solve

\[
\phi_j(\lambda) = 4Ne^{-2N\lambda} \delta_{j,a} + \sum_k' T_{k,j} \log(1 + e^{-\phi_k}(\lambda))
\] (75)

We perform a change of integration variables in (73). When \( j = a, b \) \( \phi_j \to g(\phi_j) = (1 + e^{-\phi_j})^{-1} \), while for \( j \neq a, b \) \( \phi_j \to f(\phi_j) \). Then

\[
S = -\frac{2T}{N \pi} \left\{ \sum_{j=a,b} [L(g(\phi_j(+\infty)))-L(g(\phi_j(-\infty)))] + \sum_{j \neq a,b} \left[ L(f(\phi_j(+\infty)))-L(f(\phi_j(-\infty))) \right] \right\}
\]

where \( L(x) \) is the dilogarithmic Rogers function [21]

\[
L(x) = -\frac{1}{2} \int_0^x dy \left( \frac{\log y}{1 - y} + \frac{\log(1 - y)}{y} \right)
\]

The limiting values \( \phi(\pm\infty) \) can be obtained from the system of nonlinear (ordinary) equations to which (75) reduces when \( \lambda \to \pm\infty \). The solutions are, at \( \lambda \to +\infty \)

\[
\phi_a(+\infty) = \phi_b(+\infty) = -\log(N - 1)
\]
\[
\phi_n(+\infty) = \log((n + 1)^2 - 1)
\]
\[
\phi_{(M,m)}(+\infty) = \log((2m + 2)^2 - 1)
\]

and at \( \lambda \to -\infty \)

\[
\phi_a(-\infty) = +\infty \quad \phi_b(-\infty) = -\log\left( \frac{\sin^2\left( \frac{(n+1)\pi}{N+2} \right)}{\sin^2\left( \frac{\pi}{N+2} \right)} - 1 \right)
\]
\[
\phi_n(-\infty) = \log\left( \frac{\sin^2\left( \frac{(n+1)\pi}{N+2} \right)}{\sin^2\left( \frac{\pi}{N+2} \right)} - 1 \right)
\]
\[
\phi_{(M,m)}(-\infty) = \log\left( \frac{\sin^2\left( \frac{(2m+2)\pi}{N+2} \right)}{\sin^2\left( \frac{\pi}{N+2} \right)} - 1 \right)
\]
By means of the identities
\[
\sum_{k=1}^{N-2} L \left( \frac{1}{(k+1)^2} \right) + 2L \left( \frac{1}{N} \right) = L(1) = \frac{\pi^2}{6} 
\]
and
\[
\sum_{k=2}^{N} L \left( \frac{\sin^2 \left( \frac{\pi}{N+2} \right)}{\sin^2 \left( \frac{k\pi}{N+2} \right)} \right) = \frac{2(N-1)}{N+2} L(1) 
\]
the leading term of the specific heat \( C = T \frac{\partial S}{\partial T} \) turns out to be
\[
C = \frac{2T\pi}{3N} \left( \frac{N-1}{N+2} \right)
\]
from which we find the central charge (7).

The AFM case differs in the fact that, in order to avoid having hole densities in the effective energy functional, we prefer to eliminate \( \rho_a \) and \( \rho_b \) from (62). The result has the same form as (66) but now \( T_{j,k} \) is symmetric and
\[
\hat{a}_c(q) = \frac{[N]}{2\{N-1\}} \quad \hat{a}_d(q) = \frac{[N-2]}{2\{N-1\}} \\
\hat{a}_n(q) = \frac{[n]}{1\{N-1\}} \quad \hat{a}_{(M,m)} = \frac{[2m+1]}{1\{N-1\}} \\
\hat{T}_{c,c}(q) = \hat{T}_{c,d}(q) = \hat{T}_{d,d}(q) = \frac{[N-2]}{2[1\{N-1\]} \\
\hat{T}_{c,n}(q) = \hat{T}_{d,n}(q) = \frac{\{1\}[n]}{1\{N-1\}} \quad \hat{T}_{c,(M,m)}(q) = \hat{T}_{d,(M,m)}(q) = \frac{\{1\}[2m+1]}{1\{N-1\}} \\
\hat{T}_{n,n'}(q) = \frac{2[\min(n,n')][1\{N-1\}]}{[1\{N-1\}]} \quad (n \neq n') \\
\hat{T}_{n,n}(q) = \frac{[n\{N-n-2\} + [n-1\{N-n-1\}]}{1\{N-1\}} \\
\hat{T}_{n,(M,m)}(q) = \frac{2[\min(n,2m+1)][1\{N-1\}]}{[1\{N-1\}]} \quad (m \neq m') \\
\hat{T}_{(M,m),(M,m')}(q) = \frac{2[\min(m,m') + 1][1\{N-2 - 2\max(m,m')\}]}{[1\{N-1\}]} \\
\hat{T}_{(M,m),(M,m)}(q) = \frac{[2m\{N-2m-2\} + [2m+1\{N-2m-3\}]}{[1\{N-1\}]
\]
As to the equations with \( \sigma_a \) and \( \sigma_b \) in (62), once (64) is replaced we find \( \sigma_a = \sigma_c \) and \( \sigma_b = \sigma_d \), i.e. either one of (63) and (64) implies the other when inserted in (62).
With the new range of $\Sigma'$ and the new definitions of $a_j$ and $T_{j,k}$, (67), (68), (69), and (70) are formally the same (of course $e_0$ in (67) is now the AFM ground state energy density, as given in (37)). Again $T = 0$ in (69) yields the energies of the zero temperature excitations, i.e. the zero temperature dressed energies after they have been paired by the correlations (43-45).

In the $T \ll 1$ limit, the relevant region is $|\lambda| \sim -\frac{N-1}{2N} \log T$, so the shifted functions will read $f^*(\lambda) = f(\lambda - \frac{N-1}{2N} \log T)$ and (71-72) is replaced by ($f(x)$ is still $(1 + e^x)^{-1}$)

$$
\rho^*_j \simeq -\frac{N-1}{2N\pi} f\left(\frac{\epsilon^*_j}{T}\right) \frac{d\epsilon^*_j}{d\lambda} \quad (78)
$$

$$
\rho^{(b)*}_j \simeq -\frac{N-1}{2N\pi} (1 - f\left(\frac{\epsilon^*_j}{T}\right)) \frac{d\epsilon^*_j}{d\lambda} \quad (79)
$$

Defining scaled dressed energies $\phi(\lambda)$ as in (75), we see that they take on the values at $\lambda \to +\infty$

$$
\phi_c(+\infty) = \phi_d(+\infty) = \log(N-1)
$$

$$
\phi_n(+\infty) = \log((n+1)^2 - 1)
$$

$$
\phi(M,m)(+\infty) = \log((2m+2)^2 - 1)
$$

and at $\lambda \to -\infty$

$$
\phi_j(-\infty) = +\infty \quad \forall j
$$

Using (78-79), the entropy is now reduced to

$$
S = \frac{(N-1)T}{N\pi} \sum_j \int_{-\infty}^{+\infty} d\lambda \frac{d\phi_j}{d\lambda} [f(\phi_j) \log(f(\phi_j)) + (1 - f(\phi_j)) \log(1 - f(\phi_j))]
$$

$$
= \frac{2T(N-1)}{N\pi} \sum_j \left[ L(f(\phi_j(+\infty))) - L(f(\phi(-\infty))) \right]
$$

and the identity (76) is sufficient to determine the specific heat leading term

$$
C = \frac{\pi T(N-1)}{3N}
$$

from which, since the velocity of the excitations is $\frac{N}{N-1}$, $c = 1$ follows.
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Appendix A

In this appendix we prove that the rules (52-56) imply the correlations of rapidities, at least for finite energy excited states over FM and AFM vacua. We begin with FM and consider states having an arbitrarily high but finite number of holes in the \((c)\) distribution and an arbitrarily high but finite number of other roots. We then have, neglecting terms of order \(O(1/M)\)

\[
Z_c(\lambda) = \frac{1}{2\pi} t_c(\lambda) - \frac{1}{2\pi} \Theta_{c,c} \ast \rho_c^{(0)}(\lambda) \tag{80}
\]

\[
Z_a(\lambda) = \frac{1}{2\pi} t_a(\lambda) - \frac{1}{2\pi} \Theta_{a,c} \ast \rho_c^{(0)}(\lambda) \tag{81}
\]

It's easy to see, from their Fourier transform, that \(Z'_c(\lambda) = -Z'_a(\lambda)\) and therefore

\[
Z_c(\lambda) = -Z_a(\lambda)
\]

since they are both odd in \(\lambda\). Furthermore, (80) and (81) imply that they are both monotonic. But, by definition,

\[
Z_j(\lambda_{j}(\alpha)) = \frac{I_{\beta}^{(j)}}{M} \quad Z_j(\lambda_{h,j}(\alpha)) = \frac{I_{\beta}^{(h,j)}}{M}
\]

so from (82) we have

\[
Z_a(\lambda_{\beta}^{(a)}) = -Z_c(\lambda_{\beta}^{(h,c)}) = Z_a(\lambda_{\beta}^{(h,c)}) \tag{82}
\]

and since \(Z_a\) is monotonic, (32) follows.
In the AFM case, one considers, again discarding terms \(O(1/M)\),

\[
Z_j(\lambda) = \frac{1}{2\pi} t_j(\lambda) - \frac{1}{2\pi} \Theta_{j,a} \ast \rho_a^{(0)}(\lambda) - \frac{1}{2\pi} \Theta_{j,b} \ast \rho_b^{(0)}(\lambda)
\]

with \(j = a, b, (n, +), (n, -)\). In this approximation

\[
Z_a(\lambda) = -Z_c(\lambda)
\]

\[
Z_b(\lambda) = -Z_d(\lambda)
\]

\[
Z_{(n,+)}(\lambda) = Z_{(n,-)}(\lambda)
\]

and all Z-functions are monotonic, so that, arguing as in (82), the rules \(54-56\) imply the pairings \(43-45\).

**Appendix B**

Let’s suppose that the content rule \((50)\) and assumption \((48)\) hold. We want to prove that the number of vacancies as defined by \((47)\) coincide with the more precise definition. Using \((46)\) and \((50)\) we see that the oddness table \((22)\) is modified

\[
I^{(a)}, I^{(c)} = \text{integer(half - odd) if } M + M_b + M_d = \text{odd(even)}
\]

\[
I^{(b)}, I^{(d)} = \text{integer(half - odd) if } M_b + M_d = \text{odd(even)}
\]

\[
I^{(n,+)}, I^{(n,-)} = \text{integer(half - odd) if } M_n = \text{odd(even)}
\]

Furthermore, with \((46)\) and \((50)\) we find

\[
MZ_a(\infty) = \frac{M + M_b + M_d}{2}
\]

\[
MZ_c(\infty) = -\frac{M + M_b + M_d}{2}
\]

\[
MZ_b(\infty) = \frac{M_b + M_d}{2}
\]

\[
MZ_d(\infty) = -\frac{M_b + M_d}{2}
\]

Consider the case \(M + M_b + M_d = \text{odd}\). Then \(I^{(a)}\) are integers and

\[
I_{\text{max}}^{(a)} = \frac{M + M_b + M_d - 1}{2}
\]

(83)

It is easy to see that the number of integers in \([-I_{\text{max}}^{(a)}, I_{\text{max}}^{(a)}]\) is \(M + M_b + M_d\). On the other hand, if \(M + M_b + M_d = \text{even}\), \(I^{(a)}\) are half-odd. Again \((83)\) holds, and
again, the number of half-odd numbers in $[-I_{\text{max}}^{(n)}, I_{\text{max}}^{(n)}]$ is $M + M_b + M_d$. This is exactly the number of vacancies found from $M(Z_a(+\infty) - Z_a(-\infty))$. The same argument applies, with some obvious changes, to $Z_b, Z_c, Z_d$, noticing that the last two are decreasing and $I_{\text{max}}$ is found from $Z(-\infty) = -Z(+\infty)$.

As to $n$-strings, the result is the same for $(n, +)$ and $(n, -)$ so we consider only one. We find that the limit at $\lambda \to -\infty$ can be expressed

$$MZ_n(-\infty) = \frac{n(M - M_c - M_d)}{2} - \sum_{n' \neq n} \min(n', n)M_{n'} - \frac{(2n - 1)M_n}{2}$$

$$- \sum_{m} \min(n, 2m + 1)M_{(M,m)} = I_0^{(n)} + \frac{M_n}{2}$$

where $I_0^{(n)}$ is an integer. Again, we inspect the two possible cases. If $M_n$ is even, $I^{(n)}$ is half-odd and

$$I_{\text{max}}^{(n)} = I_0^{(n)} + \frac{M_n - 1}{2}$$

(84)

and the number of vacancies in $[-I_{\text{max}}^{(n)}, I_{\text{max}}^{(n)}]$ is $2I_0^{(n)} + M_n$, the same as $M(Z_n(-\infty) - Z_n(+\infty))$. Likewise, when $M_n$ is odd, $I^{(n)}$ are integers and (84) still holds, giving the same number of vacancies.

Finally, with multiplets, the $Z$-function is decreasing and from its definition, (46) and (50) we have

$$MZ_{(M,m)}(-\infty) = (2m + 1)(M - M_c - M_d) - 2 \sum_{n} \min(n, 2m + 1)M_n$$

$$- \sum_{m' \neq m} (4\min(m, m') + 2)M_{(M,m')} - (4m + 1)M_{(M,m)}$$

(85)

From (60) and (61) we see that $M - M_c - M_d$ is necessarily an even integer, so we can rewrite (83) as

$$MZ_{(M,m)}(-\infty) = 2I_0^{(M,m)} - M_{(M,m)}$$

$I_0^{(M,m)}$ being integer. As shown in (22), $I^{(M,m)}$ must always be integers, still in the case $N = 3$ (13), where only pairs $(m = 0)$ are present, it was observed that $I^{(M,0)}$ is even(odd) when $M_{(M,0)}$ is odd(even). Let’s assume this restriction hold for any $N$ and any $m$, and inspect the two possibilities. If $M_{(M,m)}$ is even, clearly

$$I_{\text{max}}^{(M,m)} = 2I_0 - M_{(M,m)} - 1$$

(86)
Counting the half-odd numbers in \([-I_{max}^{(M,m)}, I_{max}^{(M,m)}]\) we find \(2I_0^{(M,m)} = M_{(M,m)}\). If \(M_{(M,m)}\) is odd, \((86)\) still holds, but now \(I_{max}^{(M,m)}\) is even and the number of even integers in the interval is the same. We conclude

\[
\text{vac}(M, m) = 2I_0 - M_{(M,m)} = \frac{M}{2}(Z_{(M,m)}(-\infty) - Z_{(M,m)}(+\infty))
\]

which accounts for the factor \(1/2\) used in \((51)\)

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