Cosmology in Six Dimensions

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We discuss cosmological models in six-dimensional spacetime. For codimension-1 branes, we consider a (4+1) braneworld model and discuss its cosmological evolution. For codimension-2 branes, we consider an infinitely thin conical braneworld model in the presence of an induced gravity term on the brane and a Gauss-Bonnet term in the bulk. We discuss the cosmological evolution of isotropic and anisotropic matter on the brane. We also briefly discuss cosmological models in six-dimensional supergravity.

1. INTRODUCTION

Recently, there have been many observational and theoretical motivations for the study of theories with extra spacetime dimensions and in particular the braneworld scenario. From the observational side, the current paradigm, supported by many recent observations like the cosmic microwave background anisotropies [1], large scale galaxy surveys [2] and type IA supernovae [3, 4] suggest that most of the energy content of our universe is in the form of dark matter and dark energy. Although there have been many plausible explanations for these dark components, it is challenging to try to explain these exotic ingredients of the universe using alternative gravity theories as such of the braneworlds. From the theoretical side, such extra-dimensional braneworld models are ubiquitous in theories like string or M-theory. Since these theories claim to give us a fundamental description of nature, it is important to study what kind of gravity dynamics they predict. The hope is to propose such modified gravity theories, which share many common features with general relativity, but at the same time give alternative non-conventional cosmology.

The essence of the braneworld scenario is that the Standard Model, with its matter and gauge interactions, is localized on a three-dimensional hypersurface (called brane) in a higher-dimensional spacetime. Gravity propagates in all spacetime (called bulk) and thus connects the Standard Model sector with the internal space dynamics. This idea, although quite old [5], gained momentum the last years [6, 7] because of its connection with string theory (for a review on braneworld dynamics see [8]).

Cosmology in theories with branes embedded in extra dimensions has been the subject of intense investigation during the last years. The most detailed analysis has been done for braneworld models in five-dimensional space [9]. The effect of the extra dimension can modify the cosmological evolution, depending on the model, both at early and late times. The cosmology of this and other related models with one transverse to the brane extra dimension (codimension-1 brane models) is well understood (for a review see [10]). In the cosmological generalization of [7], the early times (high energy limit) cosmological evolution is modified by the square of the matter density on the brane, while the bulk leaves its imprints on the brane by the “dark radiation” term [9, 11]. The presence of a bulk cosmological constant in [7] gives conventional cosmology at late times (low energy limit) [11]. The early time modification for example, can be interesting phenomenologically because may require less fine-tuned inflationary parameters [12].

In the above models, there are strong theoretical arguments for including in the gravitational action extra curvature terms apart from the higher dimensional Einstein-Hilbert term. The localized matter fields on the brane, which couple to bulk gravitons, can generate via quantum loops a localized four-dimensional kinetic term for gravitons [13]. The latter comes in the gravitational action as a four-dimensional scalar curvature term localized at the position of the brane (induced gravity) [14]. In addition, curvature square terms in the bulk, in the Gauss-Bonnet combination, give the most general action with second-order field equations in five dimensions [16]. This correction is also motivated by string theory, where the Gauss-Bonnet term corresponds to the leading order quantum correction to gravity, and its presence guaran-

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*Plenary talk at 100 Years of Relativity: International Conference on Classical and Quantum Aspects of Gravity and Cosmology, Sao Paulo, Brazil, 22-24 August 2005.

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1 There has been a lot of discussion about the potential problem of the extra polarization states of the massive gravitons regarding phenomenology (discontinuity problem), but the particular model seems to be consistent in a non-trivial way [15].
tees a ghost-free action [17]. Let us note, however, that if the curvature squared terms are to play an important role in the low energy dynamics, one may face a difficulty when interpreting gravity as an effective field theory (in the sense that even higher dimensional operators would seem to be also relevant).

If curvature corrections are included in [7], the presence of the 3-brane gives similar modifications to the standard cosmology. In the case of a pure Gauss-Bonnet term in the bulk, the early time cosmology is modified by a term proportional to the matter density on the brane to the power two thirds [18]. If an induced gravity term is included, the conventional cosmology is modified by the square root of the matter density on the brane at low energies [19]. Finally, if both curvature corrections are present, at early time the effect of both the Gauss-Bonnet and the induced gravity terms make the universe to start with a finite temperature [20] while at late times the effect of induced gravity make the universe to accelerate without the need of dark energy [21].

Braneworld models can also be extended in higher than five-dimensions. We can consider a \((n + 1)\) brane embedded in \((n + 2)\) spacetime or a \((3 + 1)\) brane embedded in \((n + 1)\) spacetime. In this talk we will discuss \((4 + 1)\) cosmological braneworld models embedded in six-dimensional spacetime (codimension-1 models) and \((3+1)\) cosmological braneworld models embedded in six-dimensional spacetime (codimension-2 models).

Six or higher-dimensional braneworld models of codimension-1 are considered as generalizations of the Randall-Sundrum model. In [22] static and non-static solutions in a six-dimensional bulk as well as the stability of the radion field were discussed, while in [23] the evolution of the extra dimensions transverse to the brane in a Kasner-like metric was studied.

In braneworld scenarios, contrary to the Kaluza-Klein theories, the extra dimensions can be large if the geometry is non trivial. If the hierarchy problem is addressed in the braneworld scenarios for example, the extra dimensions should be large [6]. In a cosmological context however, these extra dimensions are observationally much smaller than the size of our perceived universe. Initially the universe could have started with the sizes of all dimensions at the Planck length. Then, a successful cosmological model should accommodate in a natural way a mechanism by which the extra dimensions remained comparatively small during cosmological evolution.

Such a mechanism was proposed in [24, 25]. The basic idea is that strings dominate the dynamics of the early universe and can see each other most efficiently in \(2(p+1)\) \((p=1\) for strings) dimensions. Therefore, strings can only interact in three spatial dimensions, while strings moving in higher dimensions eventually cease to interact efficiently and their winding modes will prevent them from further expanding. If branes are included, it was shown in [26] that strings will still dominate the evolution of the universe at late times so the mechanism of [24] still survives.

In six dimensions and for codimension-2 braneworlds, the gravity dynamics appear even more radical and still a good understanding of cosmology and more generally gravity in such theories is missing. The most attractive feature of codimension-2 braneworlds is that the vacuum energy (tension) of the brane instead of curving the brane world-volume, merely induces a deficit angle in the bulk solution around the brane [27] (see also [28] for string-like defects in six dimensions). This looks very promising in relation to the cosmological constant problem, although in existing models, nearby curved solutions cannot be excluded [29], unless one allows for singularities more severe than conical in particular supersymmetric models [30]. It was soon realized [31] that one can only find nonsingular solutions if the brane energy momentum tensor is proportional to its induced metric, which means simply that it is pure tension. A non-trivial energy momentum tensor on the brane causes singularities in the metric around the braneworld which necessitates the introduction of a cut-off (brane thickness) [32–34].

An alternative approach to study the gravitational dynamics of matter on infinitely thin branes is to modify the gravitational action as discussed previously. Indeed, it was shown in [35] that the inclusion of a Gauss-Bonnet term in the gravitational action allows a non-trivial energy momentum tensor on the brane, and in the thin brane limit, four dimensional gravity is recovered as the dynamics of the induced metric on the brane. The peculiar characteristic of this way to obtain four dimensional gravity for codimension-2 branes, is that, apart from the inclusion of a (deficit angle independent) cosmological constant term, there appear to be no corrections to the Einstein equations coming from the extra dimensions in the purely conical case. Another possibility, discussed in [36], is to study (instead of conical 3-branes) codimension-2 branes sitting at the intersection of codimension-1 branes in the presence again of a bulk Gauss-Bonnet term.

Much less has been done, however, for the cosmology of theories in six or higher dimensions with branes of codimension greater than one. This is because, unlike the codimension one case, these branes exhibit bulk curvature singularities which are worse than \(\delta\)-function singularities. They then need some regularization (introduction of brane thickness) which makes the study of cosmology on them rather complicated [37]. An alternative way to study cosmologies of branes of higher codimension would be to consider corrections to the gravitational action, such as an induced curvature term on the brane [14] and a Gauss-Bonnet term in the bulk [17], which allow the brane to have a mild singularity structure (see also [39]). These thin brane cosmologies would have the additional advantage that the internal structure of the brane does not influence the macroscopic cosmological evolution.

Six-dimensional models can also be obtained in supergravity theories. In six-dimensional supergravity theory there is a vacuum solution which has the structure of...
Minkowski \times S^2 \ [40, 41]. Cosmological applications of this solution were presented in \[42\]. These theories are interesting because many of the parameters of the models are fixed from the theory and in particular the form of the scalar potential is dictated from the structure of the supergravity theory. The drawbacks of these models however is that they are plagued from anomalies. The reason is that they are chiral and therefore a gauging of some symmetries is required for cancellation of these anomalies \[41\].

In this talk we will discuss braneworld models of codimension-1 and codimension-2 in six dimensions analyzing their cosmology. We will also briefly discuss cosmological models in six-dimensional supergravity. First we will discuss a \((4 + 1)\) braneworld model in a six-dimensional spacetime bulk with a cosmological constant. We will consider a 4-brane fixed at some position in the sixth dimension and we derive the dynamical six-dimensional bulk equations in normal gaussian coordinates and then we have the usual scale factor dependent solutions allowing for two scale factors, the nates with a cosmological constant in the bulk and con-
dimensional bulk equations in normal gaussian coordi-
the sixth dimension and we derive the dynamical six-
We will consider a 4-brane fixed at some position in
the fourth dimension, we make a systematic numerical
ation and assuming that the brane matter has to obey a tuning relation. The physical reason of the existence of such a relation between \(a\) and \(b\) is that the requirement of having a cosmological evolution on the brane introduces a kind of compactification on it and the relation between \(a\) and \(b\) acts as a constraint of the brane motion in the bulk.

Using the six-dimensional generalized Friedmann equa-
tion and assuming that \(p \neq \tilde{p}\), where \(p\) is the pressure of the physical three dimensions and \(\tilde{p}\) corresponds to the fourth dimension, we make a systematic numerical study of the cosmological evolution of the scale factors \(a(t)\) and \(b(t)\) for several values of the parameters of the model, \(\Lambda_6\) the six-dimensional cosmological constant, \(k\) the brane spatial curvature and \(w\) \(\tilde{w}\) parameterizing the form of the brane energy-matter content of the three dimensions and of the extra fourth dimension respectively. We find that [46], in order the fourth dimen-
sion to be small relatively to the other three dimensions and to remain constant during cosmological evolution, \(\tilde{w}\) must be negative, indicating the presence of a dark form of energy in the extra fourth dimension. We find this result for all cases considered, (A)dS or Minkowski bulk, open, closed or flat universe, radiation, dust, cosmologi-
cal constant and dark energy dominated universe and the specific value of \(\tilde{w}\) depends on the energy-matter content of the other three dimensions.

A codimension-2 braneworld model in six dimensions will be discussed next. Results similar to the Gauss-
Bonnet case [35], i.e. four dimensional gravity for an arbitrary energy momentum tensor, can be obtained if we include in the action an induced gravity correction term instead. Again, in the purely conical case, there appear to be no corrections to the Einstein equations coming from the extra dimensions. The most important observation is that the brane and bulk energy momentum ten-
sors are strongly related and any cosmological evolution on the brane is dictated by the bulk content [48]. We also see how this correlation is relaxed in the case where bulk Gauss-Bonnet terms and brane induced gravity terms are combined. Thus, the necessary presence of extra curva-
ture terms in the gravitational action in order to give non-
trivial gravitational dynamics on a codimension-2 brane, leads to a realistic cosmological evolution on the brane in the thin brane limit, only if a Gauss-Bonnet term is included. However, let us note that the most physical way to investigate the dynamics of codimension-2 branes, is by giving thickness to the brane [33, 34].

We will discuss in details the cosmological evolution of a conical brane with both an induced gravity and a Gauss-Bonnet term added in the higher dimensional gravity action [49]. For simplicity we will assume that the only matter in the bulk is cosmological constant \(\Lambda_B\). We will solve the equations of motion evaluated on the brane and assume that the integration of them in the bulk does not give rise to pathologies \(i.e.\) singularities).

Firstly, we will discuss the isotropic cosmology, in which the brane matter has to obey a tuning relation. The physical meaning of this relation is that for any matter we put on the brane its "image" should be present in the bulk. We will see that the evolution of the system for \(\Lambda_B = 0\) tends to a fixed point with \(w = 1/3\) and for \(\Lambda_B > 0\) to a fixed point with \(w = -1\). For \(\Lambda_B < 0\) the system has a runaway behaviour to \(w \rightarrow \infty\).

We will then relax the isotropy requirement for the metric (keeping however, the matter distribution isotropic) in order to find whether the above matter tuning is an attractor or not. The matter on the brane need not now satisfy the previous tuning relation and the allowed regions of initial values of the energy density and pressure are significantly larger. The analysis of the dynamics of the system shows that line of isotropic tuning is an attractor for \(\Lambda_B \geq 0\) and thus the system isotropises towards it. However, for values of \(\Lambda_B\) which give a realistic cosmological evolution of the equation of state, the attractor property of the previous line is very weak and
fine tuning of the initial conditions is necessary in order to have an evolution with acceptably small anisotropy. For \( \Lambda_B < 0 \) the system shows, as in the isotropic case, a runaway behaviour.

Finally we will discuss in brief, six-dimensional cosmological models coming from supergravity theories. These models are interesting because they give conventional four-dimensional cosmology with less arbitrariness in cosmological parameters and less fine-tuning.

2. A (4 + 1) COSMOLOGICAL BRANEWORLD MODEL IN SIX-DIMENSIONAL SPACETIME

In this section we describe a braneworld cosmological model of codimension-1 in six dimensions. This model is a generalization of the existing braneworld models in five-dimensions. We study the model following two different approaches. First we use normal gaussian coordinates to describe a static 4-brane at fixed position in a six-dimensional spacetime bulk. The six-dimensional Einstein equations are derived and with the use of the appropriate junction conditions the generalized Friedmann equation is given. The cosmological evolution described by this generalized Friedmann equation involves the usual three-dimensional scale factor \( a(t) \) and the scale factor \( b(t) \) describing the cosmological evolution of the extra fourth dimension.

We consider next a dynamical brane moving in a bulk described by six-dimensional static “coordinates”. In this case the dynamical brane is moving on a geodesic which is given by the junctions conditions. We derive the equations of motion of the brane which for a brane observer describe the cosmological evolution on the brane. We discuss the connection between the static and dynamical brane models and the physical information that can be extracted from these approaches.

2-1. Static Brane in a Dynamical Bulk

We look for a solution to the Einstein equations in six-dimensional spacetime with a metric of the form

\[
\begin{align*}
ds^2 &= -n^2(t, y, z)dt^2 + a^2(t, y, z)\Sigma_k^2 + b^2(t, y, z)dy^2 + d^2(t, y, z)dz^2, \\
&= \frac{\delta(z - z_0)}{d} \text{diag}(-\rho, p, p, p, \hat{p}, 0),
\end{align*}
\]

where the energy-momentum tensor on the brane is

\[
T_{\Sigma}^{M(b)} = \frac{\delta(z - z_0)}{d} \text{diag}(-\rho, p, p, p, \hat{p}, 0),
\]

where \( \hat{p} \) is the pressure in the extra brane dimension. We assume that there is no matter in the bulk and the energy-momentum tensor of the bulk is proportional to the six-dimensional cosmological constant.

The presence of the brane in \( z_0 \) imposes boundary conditions on the metric: it must be continuous through the brane, while its derivatives with respect to \( z \) can be discontinuous at the brane position. This means that the generated Dirac delta function in the metric second derivatives with respect to \( z \) must be matched with the energy-momentum tensor components (2.3) to satisfy the Einstein equations. The Darmois-Israel conditions are [46],

\[
\left[ \frac{\partial \alpha}{\alpha_0} \right] = -\frac{\kappa^2}{4} (p - \hat{p} + \rho),
\]

\[
\left[ \frac{\partial \beta}{\beta_0} \right] = -\frac{\kappa^2}{4} \{\rho - 3(p - \hat{p})\},
\]

\[
\left[ \frac{\partial \gamma}{\gamma_0} \right] = \frac{\kappa^2}{4} \{\hat{p} + 3(p + \rho)\},
\]

where the subscript \((0)\) indicates quantities on the brane. The energy conservation equation on the brane can be derived taking the jump of the \( (06) \) component of the Einstein equations and using the junction conditions (2.4) and the corresponding time derivatives. We obtain

\[
\dot{\rho} + 3(p + \rho)\frac{\dot{a}_0}{a_0} + (\hat{p} + \rho)\frac{\dot{b}_0}{b_0} = 0.
\]

To find the Friedmann equation we take the jump of the \( (66) \) component of the Einstein equations and we use the fact that

\[
[\partial_z f \partial_z g] = \#\partial_z f \# [\partial_z g] + [\partial_z f] \#\partial_z g,\]

where

\[
\# f(y) \# = \frac{f(0^+) + f(0^-)}{2},
\]

is the mean value of the function \( f \) through \( y = 0 \), and we arrive to the following equation

\[
\frac{\#\partial_z a \#}{a_0} \rho = \frac{1}{3} \frac{\#\partial_z n \#}{n_0} - \frac{1}{3} \frac{\#\partial_z b \#}{b_0}.
\]

Taking the mean value of the \( (66) \) component of the Einstein equations and using the junction equations, (2.7) (we have assumed a \( Z_2 \) symmetry), we arrive to the generalized Friedmann equation in six-dimensions

\[
\left( \frac{\dot{a}_0}{a_0} + \frac{1}{3} \frac{\dot{b}_0}{b_0} - \frac{\dot{a}_0 b_0}{a_0 b_0} - \frac{\dot{a}_0}{a_0} \right) =
\]

\[
-\frac{\kappa^2}{32} \left( \rho(\rho + 2p + \frac{2}{3} \hat{p}) + (p - \hat{p})^2 \right) -
\]

\[
-\frac{2}{a_0^2} - \frac{\kappa^2}{3a_0^2} T_{66},
\]
where we have assumed \((\partial_{\mu}a)_{\alpha} = 0\), \((\partial_{\mu}^2a)_{\alpha} = 0\) and we have chosen \(n_0 = 1\).

In the case of a 3-brane in a five-dimensional bulk, the first integral of the space-space component of the Einstein equations, with the help of the other equations can be done analytically [9] and this results in the Friedmann equation on the brane with the dark radiation term as an integration constant [50]. In our case the Einstein equations cannot be integrated analytically and therefore, the usual form of the Friedmann equation on the brane cannot be extracted from (2.8). Nevertheless, if \(a(t)\) and \(b(t)\) are related, this equation will give the cosmological evolution of the scale factor \(a(t)\).

If \(a(t) = b(t) = R(t)\) then (2.8) becomes

\[
\frac{2}{R} + \frac{3}{R^2} \left( \frac{\dot{R}}{R} \right)^2 = -3 \frac{k^4}{64} \rho^2 - \frac{k^4}{8} \rho p - 3 \frac{k}{R^2} - \frac{k^2}{2d^2} \dot{R}^2 .
\]

This equation is the generalization of the Randall-Sundrum Friedmann-like equation in six dimensions, and it has, as expected, the \(\rho^2\) term with a coefficient adjusted to six dimensions. Note that this equation can easily be generalized to D dimensions.

### 2-2. Dynamical Brane in a Static Bulk

We consider a 4-brane moving in a six-dimensional Schwarzschild-AdS spacetime. The metric in the “Schwarzschild” coordinates can be written as

\[
ds^2 = -h(z)dt^2 + \frac{z^2}{l^2} d\Sigma_4^2 + h^{-1}(z)dz^2 ,
\]

where

\[
d\Sigma_4^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2_5 + (1 - kr^2)dy^2 ,
\]

and

\[
h(z) = k + \frac{z^2}{l^2} \frac{M}{r^3} .
\]

Comparing the metric (2.1) with the metric (2.10) we can make the following identifications

\[
n(z) = \sqrt{h(z)} , \\
a(z) = b(z) = z/l , \\
d(z) = \sqrt{k^{-1}(z)} .
\]

The two approaches of a static brane in a dynamical bulk described in Sec. 2.2-1 and of a moving 4-brane in a static bulk are equivalent [51]. To prove this equivalence, consider the Darmois-Israel conditions for a moving brane, \([K^\mu_\nu] = -\kappa_{(6)}^2 \left( T^\mu_\nu - \frac{1}{4} T h^\mu_\nu \right)\), with \(h^\mu_\nu\) being the induced metric on the brane; analogously to the case in 5 dimensions one can obtain

\[
\frac{-h^2 - 2\dot{R}}{\sqrt{h} + \dot{R}^2} = -\kappa_{(6)}^2 \left( \frac{3}{4} \rho - p \right) ,
\]

where we have defined \(z = R(t)\). Combining these two equations and using (2.12) we find that

\[
\frac{2}{R} + 3 \left( \frac{\dot{R}}{R} \right)^2 = -3 \kappa_{(6)}^4 \rho^2 - \frac{k^4}{8} \rho p - 3 \frac{k}{R^2} - \frac{k^2}{2d^2} \dot{R}^2 .
\]

Comparing with (2.9) we see that

\[
\kappa_{(6)}^2 \dot{R} = \kappa_{(6)}^2 \dot{R} = \frac{10}{l^2} ,
\]

with \(\dot{T}_{\alpha}^\beta = -\Lambda_{\alpha}\) the bulk cosmological constant, and \(l\) the size of the AdS space. Therefore, a brane observer describes the cosmological evolution of a four-dimensional universe with the Friedmann equation (2.16) with \(R\) parameterizing the motion of the brane in the \(z\) direction.

To describe the motion of the 4-brane with \(a(z) \neq b(z)\) we can write the metric (2.10) as

\[
ds^2 = -n(z)^2 dt^2 + \alpha(z) d\Sigma_3^2 + b(z)^2 dy^2 + d^2(z) dz^2 .
\]

We denote the position of the brane at any bulk time \(t\) by \(z = R(t)\) as before. Then, an observer on the brane defines the proper time from the relation

\[
n^2(t, R(t)) \dot{t}^2 - d^2(t, R(t)) \dot{R}^2 = 1 ,
\]

which ensures that the induced metric on the brane will be in FRW form

\[
\frac{ds^2}{d_{\text{induced}}} = - \left[ n^2(t, R(t)) \dot{t}^2 - d^2(t, R(t)) \dot{R}^2 \right] + a^2(t, R(t)) d\Sigma_3^2 + b^2(t, R(t)) dy^2
\]

and

\[
\dot{t}^2 + \alpha^2(t, R(t)) d\Sigma_3^2 + b^2(t, R(t)) dy^2 ,
\]

where the dot indicates derivative with respect to the brane time \(\tau\).

Introducing an energy-momentum tensor on the brane

\[
\hat{T}^\mu_\nu = h_{\alpha\beta} T^\alpha_\mu - \frac{1}{4} T h^\mu_\nu ,
\]

where \(T^\alpha_\mu = \text{diag}(\rho, p, p, p, p, p)\), the Darmois-Israel conditions become

\[
[K^\mu_\nu] = -\kappa_{(6)}^2 \hat{T}^\mu_\nu ,
\]

where \(K^\mu_\nu\) is the extrinsic curvature tensor. These give the equations of motion of the brane

\[
\frac{d^2 \dot{R}^3 - d\dot{R}}{\sqrt{1 + d^2 \dot{R}^2}} - \frac{\sqrt{1 + d^2 \dot{R}^2}}{n} (\ddot{n} \dot{R} + \frac{\ddot{n}}{d} n) .
\]
3. THE COSMOLOGICAL EVOLUTION OF A (4+1) BRANE-UNIVERSE

To study the cosmological evolution of the four-dimensional brane-universe, we made the following assumptions for the initial conditions and the matter distribution on the brane. We assume that the universe started as a four-dimensional one at the Planck scale, all the dimensions were of the Planck length and the matter was isotropically distributed. In this case the cosmological evolution is described by the generalized Friedmann equation (2.9). Then an anisotropy was developed in the sense that \( \ddot{p} = Qp \) with \( Q \neq 1 \). The cosmological evolution is now described by (2.8) supplemented with the constrained equation (2.26) and the matter distribution on the brane is given by the equations of state

\[
\begin{align*}
p & = w \rho , \quad (3.1) \\
\dot{\rho} & = \dot{w} \rho. \quad (3.2)
\end{align*}
\]

Using (2.26) and relations (3.1) and (3.2) the generalized Friedmann equation (2.8) becomes

\[
\begin{align*}
\ddot{w} - w & \frac{1}{1 + \dot{w}} \ddot{H}_a + \frac{(1 + \dot{w})(-3w + 2\dot{w} - 1) + 3(1 + \dot{w})^2}{(1 + w)^2} H_a^2 \\
& + \frac{2w - \dot{w} + 1}{(1 + \dot{w})^2} \rho H_a + \frac{2 + \dot{w}}{3(1 + \dot{w})^2} \frac{\dot{\rho}^2}{\rho^2} \\
& - \frac{\dot{\rho}}{3(1 + \dot{w})\rho} \\
& = - \frac{k^2_{(6)}}{32} \left\{ 1 + 2w + \frac{2}{3} \dot{w} + (w - \dot{w})^2 \right\} \rho^2 \\
& - 2 \frac{k}{a^2} + \frac{k^2_{(6)}}{3} \Lambda_6 .
\end{align*}
\]

Using the conservation equation (2.5) to eliminate \( \rho \) and its derivatives from the above equation, the cosmological evolution of the three-dimensional scale factor \( a(t) \) is given by the equation

\[
\begin{align*}
\frac{\ddot{a}}{a} & = 2C + 1 + \frac{2B}{3} + 1 \frac{a^2}{a^2} \int_{a}^{\infty} \frac{1}{a^2} \, da^{2C} \\
& \quad + \frac{k^2_{(6)}}{32} a^2 \left[ 1 + 2w + \frac{2}{3} \dot{w} + (w - \dot{w})^2 \right] \\
& - a^{2C} \left[ a^2 \frac{k^2_{(6)}}{3} \Lambda_6 - 2k \right] = 0 , \quad (3.4)
\end{align*}
\]

where the constants \( B \) and \( C \) are given by

\[
\begin{align*}
B & = 1 - 3w + 3\dot{w} \\
\frac{1}{1 + w - \dot{w}} \quad , \quad (3.5) \\
C & = 3(1 + w) + B(\dot{w} + 1) , \quad (3.6)
\end{align*}
\]

while the \( b(t) \) scale factor is

\[
\begin{align*}
b(t) & = a(t)^B . \quad (3.7)
\end{align*}
\]

We made a numerical analysis of equations (3.4) and (3.7) and studied the time evolution of the two scale factors for
different backgrounds and spatial brane-curvature. We allowed for all possible forms of energy-matter on the physical three dimensions \((w=0, 1/3, -1/3)\) and also for the possibility of dark energy \((w=-1)\), leaving \(\hat{w}\) as a free parameter.

For the scale factor \(b(t)\) to be small compared to the scale factor \(a(t)\), the constant \(B\) in (3.7) should be negative. In Table 1 we give the allowed range of values of \(\hat{w}\) for various values of \(w\). These values in turn were used to plot the time evolution of the scale factors \(a(t)\) and \(b(t)\) using (3.4) and (3.7) respectively. The criterion for the acceptance of a solution is to give a growing evolution of \(a(t)\) and a decaying and freezing out evolution for \(b(t)\). The results for various choices of the parameters of the model are presented in the following table.

| \(w\) | \(\hat{w}\) |
|------|------|
| -1   | \(> 0\) or \(< -4/3\) |
| 0    | \(> 1\) or \(< -1/3\) |
| 1/3  | \(> 4/3\) or \(< 0\) |
| -1/3 | \(> 2/3\) or \(< -2/3\) |

TABLE I: The allowed values of \(\hat{w}\) for \(B\) to be negative.

The numerical analysis showed \([46]\) that in all the cases considered, \(\hat{w}\) is negative in the range of values given in Table 1 for all acceptable solutions, indicating the need of dark energy to suppress the extra fourth dimension compared to the three other dimensions.

4. CODIMENSION-2 BRANEWORLD MODEL WITH INDUCED GRAVITY IN SIX DIMENSIONS

We will next discuss a codimension-2 braneworld model in six dimensions. We will first consider a six-dimensional theory with general bulk dynamics encoded in a Lagrangian \(\mathcal{L}_{\text{Bulk}}\) and a 3-brane at some point \(r = 0\) of the two-dimensional internal space with general dynamics \(\mathcal{L}_{\text{brane}}\) in its world-volume. If we include an induced curvature term localized at the position of the brane, the total action is written as:

\[
\mathcal{S} = \frac{M_6^4}{2} \left[ \int d^6x \sqrt{g} R^{(6)} + r_c^2 \int d^4x \sqrt{g} R^{(4)} \frac{\delta(r)}{2\pi L} \right] + \int d^5x \mathcal{L}_{\text{Bulk}} + \int d^4x \mathcal{L}_{\text{brane}} \frac{\delta(r)}{2\pi L} .
\]

(4.1)

In the above action, \(M_6\) is the six-dimensional Planck mass, \(M_4\) is the four-dimensional one and \(r_c = M_4/M_6^2\) the cross over scale between four-dimensional and six-dimensional gravity. The above induced term has been written in the particular coordinate system in which the metric is

\[
d^2s_6^2 = g_{\mu\nu}(x,r)dx^\mu dx^\nu + dr^2 + L^2(x,r)d\theta^2 ,
\]

(4.2)

where \(g_{\mu\nu}(x,r)\) is the braneworld metric and \(x^\mu\) denote four non-compact dimensions, \(\mu = 0, ..., 3\), whereas \(r, \theta\) denote the radial and angular coordinates of the two extra dimensions (the \(r\) direction may or may not be compact and the \(\theta\) coordinate ranges form \(0\) to \(2\pi\)). Capital \(M, N\) indices will take values in the six-dimensional space. Note, that we have assumed that there exists an azimuthal symmetry in the system, so that both the induced four-dimensional metric and the function \(L\) do not depend on \(\theta\). The normalization of the \(\delta\)-function is the one discussed in \([52]\).

To obtain the braneworld equations we expand the metric around the brane as

\[
L(x,r) = \beta(x)r + O(r^2) .
\]

(4.3)

At the boundary of the internal two-dimensional space where the 3-brane is situated, the function \(L\) behaves as \(L'(x,0) = \beta(x)\), where a prime denotes derivative with respect to \(r\). As we will see in the following, the demand that the space in the vicinity of the conical singularity is regular, imposes the supplementary conditions that \(\partial_\beta \beta = 0\) and \(\partial_\beta g_{\mu\nu}(x,0) = 0\).

The Einstein equations which are derived from the above action in the presence of the 3-brane are

\[
G^{(6)\mu\nu}_M + r_c^2 G^{(4)\nu}_\mu \delta_M \delta_N \frac{\delta(r)}{2\pi L} = \frac{1}{M_6^4} \left[ T^{(B)\nu}_M + T^{(br)\nu}_\mu \delta_M \delta_N \frac{\delta(r)}{2\pi L} \right] ,
\]

(4.4)

with \(G^{(6)\mu\nu}_M\) and \(G^{(4)\nu}_\mu\) the six-dimensional and the four-dimensional Einstein tensors respectively, \(T^{(B)\nu}_M\) the bulk energy momentum tensor and \(T^{(br)\nu}_\mu\) the brane one.

We will now use the fact that the second derivatives of the metric functions contain \(\delta\)-function singularities at the position of the brane. The nature of the singularity then gives the following relations \([35]\)

\[
\frac{L^\nu}{L} = -(1 - L') \frac{\delta(r)}{L} + \text{non - singular terms} \quad (4.5)
\]

\[
K^\nu_{\mu\nu} = K_\mu \frac{\delta(r)}{L} + \text{non - singular terms} .
\]

(4.6)

From the above singularity expressions we can match the singular parts of the Einstein equations (4.4) and get the following “boundary” Einstein equations

\[
G^{(4)\nu}_\mu |_0 = \frac{1}{r_c^2 M_6^2} T^{(br)\nu}_\mu + \frac{2\pi}{r_c^2} (1 - \beta) \delta^\nu\mu + \frac{2\pi L}{2r_c^2} (K^\nu_\mu - \delta^\nu_\mu K) |_0 ,
\]

(4.7)

where \(K_\mu\) is the extrinsic curvature and we denote by \(|_0\) the value of the corresponding function at \(r = 0\).
We will now make the assumption that the singularity is purely conical. In the opposite case, there would be curvature singularities $R^{(6)} \propto 1/r$, because in the Ricci tensor $R^{(6)}_{\mu\nu}$ there are terms of the form [35]

$$R^{(6)}_{\mu\nu} = -\frac{1}{2} \frac{L'}{L} \partial_r g_{\mu\nu} + ... = -\frac{\partial_r g_{\mu\nu}}{2r} + O(1) ,$$

which in the vicinity of $r = 0$ are singular if $\partial_r g_{\mu\nu}(x,0) \neq 0$. The absence of this type of singularities imposes the requirement that $K_{\mu\nu}|_0 = 0$. Then, the Einstein equations (4.7) reduce to

$$G^{(4)}_{\mu\nu}|_0 = \frac{1}{r_c^2 M_6^4} T^{(br)}_{\mu\nu} + \frac{2\pi}{r_c^2}(1 - \beta) g_{\mu\nu}|_0 . \tag{4.9}$$

The four-dimensional Einstein equations (4.9) describe the gravitational dynamics on the brane [48]. The effective four-dimensional Planck mass and cosmological constant are simply

$$M_{pl}^2 = M_4^2 = r_c^2 M_6^4 , \tag{4.10}$$

$$\Lambda_4 = \lambda - 2\pi M_6^4(1 - \beta) , \tag{4.11}$$

where $\lambda$ is the contribution of the vacuum energy of the brane fields. The normalization of $\Lambda_4$ is defined by the convention that the four dimensional Einstein equation reads $G_{\mu\nu} = \frac{1}{M_{pl}^2} (T_{\mu\nu} - \Lambda_4 g_{\mu\nu})$. Note that in contrast to the case of [35], the four dimensional Planck mass is independent of the deficit angle.

Furthermore, it is interesting to see that contrary to the five-dimensional case, the induced gravity term in six dimensions does not introduce any correction terms, apart from a cosmological term, in the four-dimensional Einstein equations on the brane, unless singularities of other type than conical are allowed in the theory, and a regularization scheme is employed. In the latter case, the last term of the right hand side of (4.7) would provide information of the bulk physcis. This absence of corrections in the purely conical case, is exactly what happens also in the case of the bulk Gauss-Bonnet theory [35].

What is important to note at this point, is that although we have found a “boundary” Einstein equation, there is more information about the dynamics of the theory contained in the full six-dimensional Einstein equations. In [48] it was showed that the energy momentum tensor of the bulk is strongly related to the energy momentum tensor of the brane via the relation

$$T^{(b)}_{\mu\nu}|_0 = \frac{1}{2r_c} \left[ T^{(br)}_{\mu\nu} + 8\pi M_6^4 (1 - \beta) \right] . \tag{4.12}$$

This equation constitutes a very strong tuning relation between brane ($T^{(br)}_{\mu\nu}$) and bulk ($T^{(B)}_{\mu\nu}|_0$) matter. It shows that, in order to have some cosmological evolution on the brane (i.e. time dependent $T^{(br)}_{\mu\nu}$) and since $\beta$ is constant, the bulk content should evolve as well in a precisely tuned way.

We can compare the above result with what is happening in five dimensions. The action of a general five-dimensional theory with a 3-brane at the point $r = 0$ of the extra dimension, and with an induced curvature term localized on it, is

$$S = \frac{M_5^2}{2} \left[ \int d^5 x \sqrt{g} R^{(5)} + r_c \int d^4 x \sqrt{g} R^{(4)}(r) \right]$$

$$+ \int d^5 x L_{\text{Bulk}} + \int d^4 x L_{\text{brane}} \delta(r) . \tag{4.13}$$

In the above action, $M_5$ is the five-dimensional Planck mass, $M_4$ is the four dimensional one and $r_c = M_5^2/M_6^3$ the cross over scale of the five-dimensional theory. The above induced term has been written in the particular coordinate system in which the metric is written as

$$ds^2 = g_{\mu\nu}(x,r) dx^\mu dx^\nu + dr^2 ,$$

where the $x^\mu$ denote the usual four non-compact dimensions, $\mu = 0, ..., 3$, whereas $r$ denotes the radial extra coordinate and capital $M, N$ indices will now take values in the five-dimensional space.

The Einstein equations which are derived from the action (4.13) in the presence of the 3-brane are

$$G^{(5)}_{MN} + r_c G^{(4)\nu}_{\mu} \delta^\nu_M \delta^\nu_N \delta(r) = \frac{1}{M_5^2} \left[ T^{(B)N}_{M} + T^{(br)\nu}_{\mu} \delta^\nu_M \delta^\nu_N \delta(r) \right] , \tag{4.15}$$

with $T^{(B)N}_{M}$ the bulk energy momentum tensor and $T^{(br)\nu}_{\mu}$ the brane one. Performing a similar analysis like the six-dimensional case [48] we get

$$R^{(4)}|_0 + \frac{1}{4} (K^\mu_{\nu} K^\nu_{\mu} - K^2)|_0 = -\frac{2}{M_5^2} T^{(br)}_{\mu\nu}|_0 . \tag{4.16}$$

From the above equation we see that the bulk matter content does not necessarily dictate the brane cosmological evolution. This is because the extrinsic curvature on the brane $K_{\mu\nu}$ can be non-trivial and it is this one which plays the most crucial role in the cosmology. In other words, in five dimensions it is the freedom of the brane to bend in the extra dimension which makes the evolution not tuned to the bulk matter content. The absence of such bending in six dimensions (imposed by singularity arguments) gives the bulk the crucial role for how the brane evolves.

We can also introduce a Gauss-Bonnet term in the six-dimensional action (4.1) and see how the above results are modified. In this case the action (4.1) is augmented by the term

$$S_{GB} = \frac{M_6^4}{2} \int d^6 x (R^{(6)} - 4 R^{(MN)}_{MN} + R^{(6)}_{MN\lambda\kappa}) . \tag{4.17}$$

Then the variation of the above action introduces an extra term in the left hand side of the Einstein equations
(4.4),
\[ H_M^N = -\alpha \left[ \frac{1}{2} R_M^N (R^{(6)}_\kappa \lambda^2 - 4R^{(6)}_{\kappa\lambda\lambda} + R^{(4)}_{\kappa\lambda\lambda}) \right. \]
\[ \left. - 2R^{(6)}_M R^{(4)}_M + 4R^{(6)}_M R^{(6)}_N \right] + 4R^{(6)}_K P^{(6)}_K - 2R^{(6)}_M K A P^N R^{KN} K A P \].  

Equating the singular terms of the Einstein equations by the standard procedure of section 2, and demanding that the singularity is purely conical, we obtain the following “boundary” Einstein equations
\[ G_{(4)\mu\nu} = \frac{1}{M_6^2} \left[ r^2 + 8\pi(1 - \beta)\alpha \right] T_{\mu\nu}^{(br)} \]
\[ + \frac{2\pi(1 - \beta)}{r^2 + 8\pi(1 - \beta)\alpha} g_{\mu\nu} |_{r = 0} \].  

Equation (4.19) describes the gravitational dynamics on a codimension-2 brane when both induced gravity and Gauss-Bonnet correction terms are present. The effective four-dimensional Planck mass and cosmological constant are simply
\[ M_6^2 = M_6^2 \left( r^2 + 8\pi(1 - \beta)\alpha \right) \],  
\[ \Lambda_4 = \lambda - 2\pi M_6^2 (1 - \beta) \],

where \( \lambda \) is the brane tension. Note that the Planck mass this time can depend on the deficit angle. This is an effect of solely the bulk Gauss-Bonnet term.

Evaluating the \((rr)\) component of the Einstein equations at the position of the brane \( r = 0 \) we obtain the following relation [48]
\[ R^{(4)} |_{r = 0} + \alpha \left( R^{(4)}_\kappa \lambda^2 - 4R^{(4)}_{\kappa\lambda\lambda} + R^{(4)}_{\kappa\lambda\lambda} \right) |_{r = 0} = -\frac{2}{M_6^2} T_{(br)}^{(br)} |_{r = 0} \].  

From (4.22) we see that \( \alpha \) cannot be no relation between the extra dimensional component \( T_{(br)}^{(br)} |_{r = 0} \) of the bulk energy momentum tensor at the position of the brane with the brane energy momentum tensor \( T_{(br)}^{(br)} \). This is due to the appearance of the Riemann curvature which cannot be evaluated from previous equations (the Ricci tensor and scalar can be substituted from (4.19)). Instead, using (4.19) and (4.22), one can solve for \( R^{(4)}_{\kappa\lambda\lambda} |_{r = 0} \) as a function of the brane and bulk matter at the position of the brane.

5. COSMOLOGICAL EVOLUTION OF A CONICAL CODIMENSION-2 BRANEWORLD MODEL

We consider a six-dimensional theory with general bulk dynamics encoded in a Lagrangian \( \mathcal{L}_{\text{Bulk}} \) and a 3-brane at some point \( r = 0 \) of the two-dimensional internal space with general dynamics \( \mathcal{L}_{\text{brane}} \) in its world-volume. The gravitational dynamics is described as we discuss in the previous sections by a Gauss-Bonnet term in the bulk and an induced four-dimensional curvature term localized at the position of the brane. Then the total action is written as
\[ S = \frac{M_6^2}{2} \left\{ \int d^4 x \sqrt{-g^{(6)}} R^{(6)} + \alpha R^{(4)}_\kappa \lambda^2 \right. \]
\[ - 4R^{(6)}_{MN} R^{(6)}_{MN} + R^{(6)}_N R^{(6)}_N \right. \]
\[ \left. + \int d^4 x \mathcal{L}_{\text{brane}} - \frac{\alpha}{2\pi L} \right\} \].  

The full equations of motion that are derived from the above action using the metric (4.2) are
\[ G_{(4)\mu\nu} = \frac{1}{M_6^2} \left[ T_{(br)}^{(br)} - \alpha H_M^N \right] - \frac{2\pi(1 - \beta)}{r^2 + 8\pi(1 - \beta)\alpha} g_{\mu\nu} |_{r = 0} \].

with \( G_{(4)\mu\nu} = R_{(4)\mu\nu} - \frac{1}{2} R_{(4)}^N g_{\mu\nu} \) the six-dimensional Einstein tensor, \( G_{(4)\mu\nu} = R_{(4)\mu\nu} - \frac{1}{2} R_{(4)}^N g_{\mu\nu} \) the four-dimensional Einstein tensor and \( H_M^N \) is given by (4.18).

In order that there are no curvature singularities more severe than conical, we will impose certain conditions on the value of the extrinsic curvature \( K_{\mu\nu} \) on the brane, where the prime denotes derivative with respect to \( r \), and on the expansion coefficients of the function \( L \)
\[ L = \beta_1 (x) r + \beta_2 (x) r^2 + \beta_3 (x) r^3 + \ldots \]

These conditions read [35]
\[ K_{\mu\nu} |_{r = 0} = 0 \],  
\[ \beta_1 = \text{const.} \] and \( \beta_2 = 0 \).

Imposing these conditions and keeping only the finite part in \( L''/L \), the Einstein equations (5.2) can be evaluated at \( r = 0 \). The effective Einstein equations on the brane (obtained by equating the \( \delta \)-function parts of the Einstein equations) are
\[ G_{(4)\mu\nu} = \frac{1}{M_6^2} \left[ T_{(br)}^{(br)} - \Lambda_4 \delta_{\mu\nu} \right] \],

with \( M_6^2 = M_6^2 [r^2 + 8\pi(1 - \beta_1)\alpha] \) and \( \Lambda_4 = -2\pi M_6^2 (1 - \beta_1) \). The various components of the bulk Einstein equations evaluated at \( r = 0 \) are given in the following:

The \((\mu\nu)\) component
\[ G_{(4)\mu\nu} = \frac{1}{2} R_{(4)\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \left( K_{(4)} + 2 \frac{L''}{L} \right) \]
\[ - \alpha \frac{1}{2} \delta_{\mu\nu} \left( R_{(4)} - 4R_{(4)}^2 + R_{(4)}^N \right) \]
\[ + 4K_{(4)} R_{(4)}^N - 2K_{(4)} R - 4 \frac{L''}{L} R_{(4)} \]
\[ - 2RR_{(4)} + 4R_{(4)} R_{(4)} + 4R_{(4)} R_{(4)} R_{(4)} \]
\[-2R_{\mu\nu\lambda\rho}R^{\nu\lambda\rho} + \frac{L'}{L}R'_{\mu} + 2K'R_{\mu} + K^{\nu}R_{\mu} - 2K^{\nu}R'_{\mu} - 2g_{\mu\lambda}K^{\nu}R_{\mu} = \frac{1}{M_{0}^{2}}T^{(B)\nu}_{\mu}. \quad (5.7)\]

The \((rr)\) component
\[-\frac{1}{2}R - \frac{1}{2}(R^{2} - 4R_{\mu}^{\nu} + R_{\mu}^{2}) = \frac{1}{M_{0}^{2}}T^{(B)r}_{r}. \quad (5.8)\]

The \((\theta\theta) - (rr)\) component
\[\frac{1}{2}K' + \alpha \left( K'R_{\mu} - 2K_{\mu}R'_{\mu} \right) = \frac{1}{M_{0}^{2}}(T^{(B)\theta}_{\theta} - T^{(B)r}_{r}). \quad (5.9)\]

The \((\mu, r)\) component
\[T_{(\mu)}^{(B)r} = 0. \quad (5.10)\]

In the following, we will study the above equations in a time dependent background, assuming that the bulk consists of a pure cosmological constant \(T_{N}^{(B)N} = -\Lambda_{B}\delta_{N}^{M}\) and that the matter content of the brane is an isotropic fluid with \(T_{\mu}^{(B)\nu} = \delta_{\mu}^{\nu} \text{diag}(-\rho_{b}, P_{b}, P_{b}, P_{b})\).

### 5-1. The Constrained Isotropic Case

We are interested in the cosmological evolution of a flat isotropic brane-universe, therefore we will consider the following time dependent form of the metric (4.2)
\[ds^{2} = -N^{2}(t, r)dt^{2} + A^{2}(t, r)d\bar{x}^{2} + dr^{2} + L^{2}(x, r)d\theta^{2}. \quad (5.11)\]

We can use the gauge freedom to fix \(N(t, 0) = 1\), while we define \(A(t, 0) = a(t)\). The curvature singularity avoidance condition (5.4) we imposed, dictates that \(N'(t, 0) = A'(t, 0) = 0\), while the second derivatives of these metric functions are unconstrained.

For this ansatz, the Einstein equations (5.6) and (5.8) give
\[\frac{3}{a^{2}}(\ddot{a} + \frac{\dot{a}^{2}}{a^{2}}) = \frac{\rho_{b} + \Lambda_{4}}{M_{P}^{2}}, \quad (5.12)\]
\[2\frac{\ddot{a}}{a} + \frac{\dot{a}^{2}}{a^{2}} = \frac{-P_{b} + \Lambda_{4}}{M_{P}^{2}}, \quad (5.13)\]
\[3 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^{2}}{a^{2}} \right) + 12\alpha \frac{\dot{a}^{2}}{a^{3}} = \frac{\Lambda_{B}}{M_{0}^{4}}. \quad (5.14)\]

The equations (5.12) and (5.13) are the usual Friedmann and Raychaudhuri equations of a four-dimensional universe with a scale factor \(a\), while the third equation (5.14) appears because of the presence of the bulk and acts as a constraint between the matter density and pressure on the brane. To see this, a simple manipulation of the above equations gives
\[-\frac{\Lambda_{B}}{M_{0}^{4}} = \left( \frac{1}{2} + \frac{2}{3} \frac{\Lambda_{4}}{M_{P}^{4}} \right) 3P_{b} - \rho_{b} - 2\frac{\Lambda_{4}}{M_{P}^{4}} \left( 1 + \frac{2}{3} \frac{\Lambda_{4}}{M_{P}^{4}} \right) + \frac{2}{3} \frac{1}{M_{P}^{2}} \rho_{b}(3P_{b} + \rho_{b}), \quad (5.15)\]

which shows a precise relation between \(\Lambda_{B}, \rho_{b}\) and \(P_{b}\).

To simplify the equations, we assume that the vacuum energy (tension) of the brane cancels the contribution \(\Lambda_{4}\) induced by the deficit angle, i.e.
\[\rho_{b} = -\Lambda_{4} + \rho_{m} \quad \text{and} \quad P_{b} = \Lambda_{4} + P_{m}, \quad (5.16)\]

with \(P_{m} = w_{c}\rho_{m}\). Then the above equations read
\[3\frac{\dot{a}^{2}}{a^{2}} = \frac{\rho_{m}}{M_{P}^{2}}, \quad (5.17)\]
\[2\frac{\ddot{a}}{a} + \frac{\dot{a}^{2}}{a^{2}} = -w_{c}\frac{\rho_{m}}{M_{P}^{2}}, \quad (5.18)\]

while the constraint equation becomes
\[-\frac{\Lambda_{B}}{M_{0}^{4}} = \frac{\rho_{m}}{M_{P}^{2}} \frac{1}{2} \left( 3w_{c} - 1 + \frac{3}{2}(w_{c} + 1)\alpha \frac{\rho_{m}}{M_{P}^{2}} \right), \quad (5.19)\]

Using the metric (5.11), the Einstein equations (5.7) and (5.9) can be solved for the second \(r\)-derivatives of the metric, as functions of the matter content on the brane, and in principle they can give us information about the structure of the bulk at \(r = 0\).

\[\frac{A''}{A} = \frac{\frac{1}{4} \left( 1 + 4\alpha \frac{\Lambda_{B}}{M_{0}^{4}} \right) (w_{c} + 1) + \frac{\rho_{m}}{M_{P}^{2}}}{1 - 2\alpha \frac{\rho_{m}}{M_{P}^{2}} - w_{c} - 1 + 2\alpha \frac{\rho_{m}}{M_{P}^{2}}(w_{c} + 1)(3w_{c} - 1)}\]
\[\frac{N''}{N} = \frac{3\frac{A''}{A}}{4\alpha \frac{\rho_{m}}{M_{P}^{2}} - 1}, \quad (5.20)\]
\[\frac{L''}{L} = \frac{1}{1 + 4\alpha \frac{\rho_{m}}{M_{P}^{2}} \frac{\Lambda_{B}}{M_{0}^{4}} + \frac{\rho_{m}}{M_{P}^{2}} \frac{\Lambda_{B}}{M_{0}^{4}}} - 3\frac{A''}{A} \left( 1 + 2\alpha(w_{c} + 1) \frac{\rho_{m}}{M_{P}^{2}} \right). \quad (5.21)\]

A potential problem in the cosmology of the system would be, if the denominator of (5.20) is equal to zero, i.e. when \(w_{c} = w_{s}\) with
\[w_{s}^{\pm} = \frac{-1 + 4\alpha \frac{\rho_{m}}{M_{P}^{2}}}{64\alpha^{2} \frac{\rho_{m}}{M_{P}^{2}} + 32\alpha \frac{\rho_{m}}{M_{P}^{2}} + 13} \pm 12\alpha \frac{\rho_{m}}{M_{P}^{2}}. \quad (5.21)\]

When this happens, the six-dimensional curvature invariant will diverge close to the brane. Thus, after discussing the cosmological evolution of the brane world-volume, we should always check that it does not pass through a point which satisfies the above relation.
5-2. Cosmological Evolution of an Isotropic Brane-Universe

From the constraint relation (5.19), we can solve for \(w_c\), the allowed equation of state of the matter on the brane. It should satisfy the following equation

\[
w_c = -2 \frac{\dot{\rho}_m}{\rho_m} + \frac{\dot{\rho}_m}{\rho_m} - \frac{\dot{M}^4_{Pl}}{3 M^4_{Pl}} (1 + \frac{4}{3} \frac{\dot{\rho}_m}{\rho_m}) = 0 ,
\]

(5.22)

with \(\rho_m > 0\) so that the Hubble parameter is real.

Before analyzing the system, let us note a first important difference between the system of the pure four dimensional dynamics and the one with the extra constraint added because of the presence of the bulk. In the four dimensional system, a constant \(w\) is allowed and its value is preserved during the evolution of the system. On the other hand, the evolution of the system with the extra constraint forbids any evolution with constant \(w\). Indeed, by differentiating (5.22) and using (5.17) and the conservation equation \(\dot{\rho}_m + 3(1 + w_c)\rho_m \frac{\dot{a}}{a} = 0\), we can find a differential equation for \(w_c\)

\[
\dot{w_c} + 3(1 + w_c)\rho_m \frac{\partial w_c \dot{a}}{\partial \rho_m \dot{a}} = 0 .
\]

(5.23)

Imposing a further condition of keeping \(w_c\) constant, would result to a constant \(\rho_m\), related to \(w_c\) in a specific way, and by the conservation equation, to zero Hubble for \(a\). Thus, an a priori fixing of \(w_c\) would result to an inconsistent system [49].

To study the cosmological evolution, we look at the system

\[
\dot{H} = \frac{1}{2}(1 + w_c) \frac{\rho_m}{M^2_{Pl}} ,
\]

(5.24)

\[
\dot{\rho}_m = -3(1 + w_c)\rho_m H ,
\]

(5.25)

where \(H = \dot{a}/a\) is the Hubble parameter. We will analyze the above system of the isotropic case for \(\Lambda_B = 0\) and \(\Lambda_B \neq 0\), because of the different features that arise in the two choices of this parameter.

5-2-1. Evolution of the System for \(\Lambda_B = 0\)

From the above dynamical system, taking into account (5.22), we find that there is only one fixed point in the evolution, the one with

\[
(\rho_m, H^2, w_c) = (0, 0, 1/3) .
\]

(5.26)

Linear perturbation around this point reveals that it is an attractor. From (5.23) and the conservation equation (5.25) we find for the Hubble parameter

\[
\frac{\dot{a}}{a} = \frac{2\dot{w}_c}{(1 + 3w_c)(1 - 3w_c)(1 + w_c)} .
\]

(5.27)

The above equation can easily be integrated and solved for \(w_c\) (the "−" sign in the solution of \(w_c\) is rejected because it gives imaginary Hubble parameter)

\[
w_c = -\frac{1}{3} + \frac{2}{3} \frac{a^4}{\sqrt{3 + a^2}} .
\]

(5.28)

From this equation, we see that from any initial condition along the line of tuning (5.22), the expansion of the universe drives the equation of state to \(w_c \rightarrow 1/3\), i.e., radiation. We have also verified this by integrating the system numerically. During this cosmological evolution, it can never happen that \(w_c = w^+_s\) (compare (5.21) with (5.22)) and thus the whole system is regular.

5-2-2. Evolution of the System for \(\Lambda_B \neq 0\)

From the dynamical system, as written in the previous subsection, taking into account (5.22) with \(\Lambda_B \neq 0\), we find that there is a fixed point in the cosmological evolution

\[
(\rho_m, H^2, w_c) = \left( \frac{\rho_f}{3 M^2_{Pl}}, -1 \right) ,
\]

(5.29)

with \(\frac{\alpha \rho_f M^2_{Pl}}{3 M^2_{Pl}} = -\frac{4}{3} + \frac{2}{3} \sqrt{1 + \frac{4 \sqrt{M^4_{Pl}}}{3 M^6_{Pl}}}\). Since we should have \(\rho_m > 0\), this fixed point exists only for \(\Lambda_B > 0\) and corresponds to a de Sitter vacuum. Linear perturbation around this point reveals that it is an attractor. Thus, any matter density on the brane eventually evolves to a state of vacuum energy.

A potentially interesting case, with a cosmological evolution resembling that of our universe, would be the one where \(0 < \alpha \Lambda_B/M^6_{Pl} \ll 1\). Then, the line of isotropic tuning, as it is given by relation (5.22), has a maximum close to \(w_c \sim 1/3\). The evolution of an initial energy density larger than the one corresponding to that maximum, will evolve towards \(w_c \sim 1/3\), pass from \(w_c = 0\) and asymptote to \(w_c \rightarrow -1\) (see Fig. 1 for an example). The asymptotic value of the effective cosmological constant at the fixed point will be \(\frac{\alpha \rho_f M^2_{Pl}}{3 M^2_{Pl}} \sim \alpha \frac{M^4_{Pl}}{M^6_{Pl}} \ll 1\) and should be rather small to match with observations (when performing such a comparison, it is reasonable to assume that all the dimensionful scales of the theory are roughly of the same order, i.e., \(\alpha^{-1/2} \sim M^2_{Pl} \sim M_6\)). The latter requirement of extremely small \(\alpha \Lambda_B/M^6_{Pl}\) is the usual cosmological constant problem. Although the standard cosmological evolution is described by piecewise constant equations of state with \(w \sim 1/3, 0, -1\), in the present theory the equation of state has always time dependence.

For \(\Lambda_B < 0\), there does not exist any fixed point. The evolution of this system has a runaway behaviour and flows to \(w_c \rightarrow \infty\) while \(\rho_m \rightarrow 0^+\). During this evolution, the singular point \(w_c = w^+_s\) can be encountered only if \(\Lambda_B/\Lambda^6_6 \lesssim -0.326\) (equating the expressions (5.21) and (5.22) the energy density \(\rho_m\) can be real and positive only for this range of \(\Lambda_B\)). It is easily verified that for these
“dangerous” values of $\Lambda_B$, it is $w^+_B > 1/3$. Therefore even in the case in which $\Lambda_B$ takes these values, the dangerous point $w^+_B$ is reached only if initially $w_c < w^+_B$.

We finally note that, for $\Lambda_B \neq 0$, there does not exist a fixed point with $(\rho_m, H^2) = (0, 0)$ because the equation of state $w_c$ diverges at that point.

6. THE UNCONSTRAINED ANISOTROPIC CASE

In the previous section we found some interesting cosmological evolutions of an isotropic brane-universe if $\Lambda_B > 0$. We should keep in mind however, that in all cases studied, the energy density on the brane is tuned to the equation of state in a specific way, which seems at first sight artificial. Therefore, it would be worth studying some cosmological evolution, in a codimension-2 brane-world model, in which this tuning is not required. If the system then evolves towards the previously studied line of isotropic tuning, we will conclude that this tuning is an attractor and thus not artificial.

To do so, we have to consider geometries which are not isotropic and in which the Riemann tensor cannot be expressed in terms of the Ricci tensor and the curvature scalar [48]. Then, the constraint equation (5.8) will not give rise to a brane-bulk matter tuning, but rather to a dynamical equation for the anisotropy of the space. For this purpose, let us consider the following anisotropic ansatz where the metric functions depend only on the time $t$ and the radial coordinate $r$ (i.e. keeping the azimuthal symmetry)

$$ds^2 = -N^2(t, r)dt^2 + \sum_{i=1}^{3} A_i^2(t, r)(dx^i)^2 + dr^2 + L^2(x, r)d\theta^2.$$  

We can again use the gauge freedom to fix $N(t, 0) = 1$.

while we define $A_i(t, 0) \equiv a_i(t)$. The singularity conditions dictate as before that $N'(t, 0) = A_i'(t, 0) = 0$, while the second derivatives of these metric functions are unconstrained. The most general anisotropic evolution scale factors with the above property can be written as

$$A_1(t, r) = a(t)b(t)c(t) + \xi_1(t)r^2 + \ldots,$$  

$$A_2(t, r) = \frac{a(t)}{b(t)} + \xi_2(t)r^2 + \ldots,$$  

$$A_3(t, r) = \frac{a(t)}{c(t)} + \xi_3(t)r^2 + \ldots,$$

where $a = (a_1 a_2 a_3)^{1/3}$ represents the “mean” scale factor and $b, c$ represent two degrees of anisotropy. To simplify further the analysis of the system, we will choose $c = \text{const.}$ (by a coordinate redefinition then we can always set $c = 1$). The dynamics of this particular choice can help us to understand the qualitative features of the general case.

The purely four-dimensional brane Einstein equations take the form

$$\frac{3}{a^2} \dddot{a} - \ddot{b}^2 = \frac{\rho_b + \Lambda_4}{M_{Pl}^2},$$  

$$2 \dddot{a} - 4 \ddot{a} + \frac{\ddot{a}^2}{a} = \frac{\rho_b - P_b + 2\Lambda_4}{M_{Pl}^2},$$  

$$\ddot{b} - \frac{\dot{b}^2}{b^2} + 3 \frac{\dot{a}b}{ab} = 0,$$

while the equation which is coming from the extra dimensions is now dynamical, providing a Hubble equation for $b$

$$\ddot{b} - \frac{\dot{b}^2}{b^2} = \frac{\rho_b + \Lambda_4}{4M_{Pl}^2} \pm \frac{\sqrt{3}}{16a^3} \left( -16a \frac{\Lambda_4}{M_{Pl}^2} \left( 2 + \frac{\Lambda_4}{M_{Pl}^2} \right) \right)^{1/2} \left( 1 - 2a \frac{\rho_b}{M_{Pl}^2} \right)$$  

$$+ \sqrt{16a \frac{\Lambda_B}{M_{Pl}^2} + \frac{8a^2 \rho_b}{M_{Pl}^2} \left( -1 + 2a \frac{\rho_b}{M_{Pl}^2} \right)} \left( \frac{3}{2} + 2a \frac{\rho_b + \Lambda_4}{M_{Pl}^2} \right).$$

We will now assume, as before, that the vacuum energy (tension) of the brane cancels the contribution $\Lambda_4$ induced my the deficit angle as in (5.16). After this simplification, the above equations read

$$\frac{3a^2}{\dot{a}^2} - \frac{\dot{b}^2}{b^2} = \frac{\rho_m}{M_{Pl}^2},$$  

$$2 \dddot{a} - 4 \ddot{a} + \frac{\ddot{a}^2}{a} = (1 - w) \frac{\rho_m}{M_{Pl}^2},$$  

$$\ddot{b} - \frac{\dot{b}^2}{b^2} + 3 \frac{\dot{a}b}{ab} = 0,$$

while the equation coming from the extra dimensions becomes

$$\ddot{b} = -\frac{\rho_m}{4M_{Pl}^2} + \sqrt{\frac{3}{32a}}.$$
\begin{equation}
\sqrt{\frac{2\Lambda_B}{M_6^4} + \frac{\rho_m}{M_{Pl}^2} \left[ (3w - 1) + 2(2w + 1)\frac{\rho_m}{M_{Pl}^2} \right]} \equiv f(\rho_m, w). \tag{6.12}
\end{equation}

It is interesting to observe that the Hubble equation (6.9) for the “mean” scale factor \(a\), after substitution of (6.12), has apart from the usual linear term in \(\rho\) (of the conventional four-dimensional cosmology), additional correction terms in \(\rho\). This is similar to what happens also to five-dimensional brane-world models [9] and is due to the presence of extra dimensions. This modification happens only in the anisotropic case. In the pure isotropic case the four-dimensional brane-universe feels the extra dimensions by only adjusting its energy density to its equation of state, but without any modification in the structure of the Friedmann equation.

### 6.1. Cosmological Evolution of an Anisotropic Brane-Universe

Before analyzing the system, let us note again that in contrast to pure four dimensional anisotropic dynamics, where a constant \(w\) is allowed, in the present case, where there is an extra dynamical equation because of the presence of the bulk, \(w\) has to evolve. Indeed, by differentiating (6.12) and then using the four-dimensional equations of motion and the conservation equation \(\dot{\rho}_m + 3(1 + w)\dot{\rho}_m \frac{\dot{a}}{a} = 0\) we can find a differential equation for \(w\):

\begin{equation}
\frac{\partial f}{\partial w} \dot{w} + 3 \left[ 2f - (1 + w)\rho_m \frac{\partial f}{\partial \rho_m} \right] \frac{\dot{a}}{a} = 0. \tag{6.13}
\end{equation}

Imposing a further condition to keep \(w\) constant, would result to a constant \(\rho_m\) related to \(w\) in a specific way and by the conservation equation, to zero Hubble for \(a\). Thus, an \(a\) priori fixing of \(w\) would result to an inconsistent system [38, 49].

To have real Hubble parameters for \(a\) and \(b\), \(\rho_m\) and \(w\) have to lie in specific regions for which the following inequalities are satisfied

\begin{equation}
f > 0, \quad f + \frac{\rho_m}{M_{Pl}^2} > 0. \tag{6.14}
\end{equation}

Define the following boundaries of the allowed regions

\begin{align*}
w_1 &= \frac{-2\Lambda_B}{M_6^4} + \frac{\rho_m}{M_{Pl}^2} \left( 1 - \frac{4}{3}\frac{\rho_m}{M_{Pl}^2} \right), \tag{6.15} \\
w_2 &= \frac{-2\Lambda_B}{M_6^4} + \frac{\rho_m}{M_{Pl}^2} \left( 1 + 4\frac{\rho_m}{M_{Pl}^2} \right). \tag{6.16}
\end{align*}

The quantity \(w_1\) coincides with the line of isotropic tuning \(w_c\) given in (5.22). The inequalities (6.14) can then be re-expressed as conditions for \(\rho_m\) and \(w\):

- For \(\rho_m > 0\), we should have \(w > w_1\).
- For \(\rho_m < 0\), we distinguish two cases
  - for \(-\frac{2}{3} < \frac{\rho_m}{M_{Pl}^2} < 0\), we should have \(w < w_2\).
  - for \(-\frac{2}{3} < \frac{\rho_m}{M_{Pl}^2} < -\frac{2}{3}\), we should have \(w > w_2\).

To study the cosmological evolution, we look at the system

\begin{align*}
\dot{H}_a &= -\frac{1}{2}(1 + w)\frac{\rho_m}{M_{Pl}^2} - f, \tag{6.17} \\
\dot{H}_b &= -3H_a H_b, \tag{6.18} \\
\dot{\rho}_m &= -3(1 + w)\rho_m H_a, \tag{6.19}
\end{align*}

where \(H_a = \dot{a}/a\) and \(H_b = \dot{b}/b\) are the Hubble parameters for \(a\) and \(b\) respectively. The third equation is the energy conservation equation in which only the Hubble parameter \(H_a\) for the “mean” scale factor \(a\) appears. Again we will analyze the anisotropic case for \(\Lambda_B = 0\) and \(\Lambda_B \neq 0\) and we will compare the cosmological evolution with the cosmological evolution of the tuned isotropic case.

#### 6.1.1. Evolution of the System with \(\Lambda_B = 0\)

There are three different regions in the \((w, \rho_m)\) plane where these inequalities are satisfied. This relative freedom to choose the matter on the brane should be compared to the tuning that happens in the isotropic case. Relaxing the isotropy condition, the system can have initial conditions in a vast region of the parametric space. From the dynamical system (6.17), (6.18), (6.19) we see that there is only one fixed point and that it is the same with that of the isotropic evolution, i.e.

\begin{equation}
(\rho_m, H_a^2, H_b^2, w) = (0, 0, 0, 1/3). \tag{6.20}
\end{equation}

Linear perturbation around this point reveals again that it is an attractor.

The presence of the previous attractor fixed point will drive the system towards a final isotropic state of radiation. The way in which this fixed point is approached from an arbitrary initial energy density, can tell us whether the line of isotropic tuning is an attractor or not. If the anisotropy monotonically decreases during the evolution, it means that the line of isotropic tuning is an attractor.

In order to analyze the features of the anisotropic evolutions, we proceed numerically. We solve the system of the two second order equations (6.10), (6.11) and the two first order equations (6.13), (6.19) for the four functions \(a, b, \rho_m, w\). To understand how the anisotropy involves we define the mean anisotropy by the following quantity

\begin{equation}
A = \sqrt{\frac{\sum_{i=1}^{3} (\langle H \rangle - H_i)^2}{3\langle H \rangle^2}} = \sqrt{\frac{2}{3} \left| \frac{\dot{a}b}{\dot{a}b} \right|}, \tag{6.21}
\end{equation}

where \( H_i = \dot{a}_i / a_i \) (with \( a_i \) defined after (6.1)) and \( \langle H \rangle = \frac{1}{3} \sum_{i=1}^{3} H_i = \dot{a} / a \).

Our analysis shows that, the relaxation of the tuning relation between \( w \) and \( \rho_m \), has as a consequence the appearance of new branches of brane world evolution, while the system tends quickly to the isotropic fixed point attractor. Furthermore, we observe that the anisotropy in all cases decreases much more quickly than in the four dimensional case with the same initial conditions but without the extra dimensional constraint.

6.1.2. Evolution of the System with \( \Lambda_B \neq 0 \)

Studying the asymptotics for \( w_1 \) and \( w_2 \) from (6.15), (6.16) we find that for \( \Lambda_B \neq 0 \), there are three intervals of \( \Lambda_B \) with different shape of the allowed regions in the \((\rho_m, w)\) plane.

From the dynamical system (6.17), (6.18), (6.19) we see that there are two fixed points in general. The first one is the same with that of the isotropic evolution, i.e.

\[
(\rho_m, H^2, H^6, w) = \left( \rho_f, \frac{\rho_f}{M^4_{Pl}}, 0, -1 \right),
\]

with \( \frac{\alpha \rho_f}{M^4_{Pl}} = -\frac{3}{2} + \frac{1}{2} \sqrt{1 + \frac{2 \alpha \Lambda_B}{3 M^4_{Pl}}} \) and it exists only for \( \Lambda_B > 0 \). Linear perturbation around this point reveals again that it is an attractor.

The second fixed point that we find is

\[
(\rho_m, H^2, H^6, w) = \left( -\Lambda_B \frac{M^4_{Pl}}{M^6_{Pl}}, 0, \frac{\Lambda_B}{M^6_{Pl}}, 1 \right).
\]

and it exists only for \( \Lambda_B > 0 \). Linear perturbation around this point reveals that it is a repeler.

Whenever the previous attractor fixed point exists, the system will be driven towards a final isotropic de Sitter state. The way in which this fixed point is approached from an arbitrary initial energy density, can tell us whether the line of isotropic tuning is an attractor or not. We analyzed the anisotropic system numerically and we found [49] that, the relaxation of the tuning between \( w \) and \( \rho_m \), has as a consequence the appearance of new branches of brane world evolution, while for choice of the parameters the system tends to the attractor fixed points, whenever they exist. For the cases when \( \Lambda_B > 0 \) the lines of isotropic tuning are attractors, with strength depending on the value of \( \Lambda_B \).

Furthermore, we observe that the anisotropy in all cases, apart for \( \Lambda_B < 0 \), decreases much more quickly than in the four dimensional case, with the same initial conditions, but without the extra dimensional constraint. For \( \Lambda_B < 0 \), the anisotropy increases and is larger than the one of the purely four dimensional case.

Let us now examine again the interesting possibility of \( 0 < \alpha \Lambda_B / M^4_{Pl} \ll 1 \) with the transition between \( w = 1/3 \) to \( w = 0 \) and finally to \( w = -1 \). As it can be inferred from the previous discussion, the system tracks the line of isotropic tuning and evolves towards the attractor fixed point with \( w = -1 \). However, due to the small value of \( \alpha \Lambda_B / M^4_{Pl} \), the line of isotropic tuning is a very weak attractor. Most of the evolution is rather anisotropic with \( A \sim \mathcal{O}(1) - \mathcal{O}(10^{-1}) \) until the fixed point is approached, in which region it drops to zero.

This large anisotropy makes the cosmological evolution phenomenologically problematic. In order that the anisotropy is acceptably small, the initial conditions for the energy density and the equation of state should be fine tuned to lie very close to the line of isotropic tuning initially.

In conclusion, by analyzing the anisotropic dynamics of the system we have seen that the lines of isotropic tuning are attractors for \( \Lambda_B > 0 \), with \( \Lambda_B \)-dependent attracting strength. The most phenomenologically accepted evolutions with \( 0 < \alpha \Lambda_B / M^4_{Pl} \ll 1 \) do not isotropise quickly enough and thus need a fine tuning in order to evolve with acceptable anisotropy.

7. SIX-DIMENSIONAL SUPERGRAVITY COSMOLOGICAL MODELS

There are also cosmological models in six dimensions coming from supergravity theories [53]. In some early works cosmological solutions were found in D=6, N=2 supergravity theories. These solutions describe vacuum state or radiation four-dimensional universe in a Minkowski \( \times S^2 \) space [42].

Chaotic inflation was studied in an analysis-free, gauged (1,0) supergravity model in six dimensions in which the unique vacuum state is Minkowski spacetime cross an internal \( S^2 \) which leaves half of the six-dimensional supersymmetries unbroken [54]. It was shown that inflationary dynamics consistent with the cosmological constraints can be realized provided that the radius of the internal space \( S^2 \) satisfies a constraint. In this model, the inflaton field \( \phi \) originates from the complex scalar fields in the D=6 scalar hyper-multiplet. The mass and the self couplings of the scalar field are dictated by the D=6 Lagrangian. The scalar potential has an absolute minimum at \( \phi = 0 \) with no undetermined moduli fields.

The model is based in a Lagrangian which have been constructed in [55, 56]. It is chiral, hence potentially inconsistent due to the presence of gauge, gravity and mixed anomalies. In [41], an anomaly free model with a gauge group \( E_6 \times E_7 \times U(1)_R \) has been constructed which involves a hyper-scalar multiplet transforming in a 912-dimensional pseudo-real representation of \( E_7 \). In a convenient parameterization, the potential for the scalars take a simple form

\[
V = \frac{g^2}{\kappa^2} e^{-\kappa \sigma} [(1 + |\phi|^2)^2 + \frac{g^2}{\kappa^2} (\phi T^a \phi)^2] \quad (7.1)
\]
where $\kappa$ is the D=6 gravitational coupling and has a dimension of square length. The $g_1$ and $g_7$ are the coupling constants of the $U(1)_R$ and $E_7$, respectively, and they have the dimensions of a length, the $T^a$ are the Hermitian generators of $E_7$ in its 912-dimensional representation. The most important property of this potential is that it has a unique minimum at $\phi = 0$. Chaotic inflation is realized for the zero KK mode with a potential (7.1) without the constant term.

8. CONCLUSIONS

In this talk we discussed cosmological models in six dimensions. First we presented a (4+1)-braneworld cosmological model in a six-dimensional bulk. If $a(t) = b(t)$, with $a(t)$ the usual scale factor of the three physical dimensions, and $b(t)$ the scale factor of the extra fourth dimension, we found the generalized Friedmann equation in six-dimensions of the Randall-Sundrum model describing the cosmological evolution of a four-dimensional braneworld. If $a(t) \neq b(t)$ the four-dimensional universe evolves with two scale factors. However, for an observer in the moving brane, $a$ and $b$ are static depending only on the coordinate on which the 4-brane is moving. Then, demanding to have an effective Friedmann-like equation on the brane, we showed that the motion of the 4-brane in the static bulk is constrained by Darmois-Israel bound.

In the static bulk is constrained by Darmois-Israel bound on the brane, we showed that the motion of the 4-brane demanding to have an effective Friedmann-like equation on the coordinate on which the 4-brane is moving. Then, induced gravity term, the higher dimensional Einstein equation, which is exactly four-dimensional and bears no information of the internal space (modulo a cosmological constant contribution). In the case of a pure brane induced gravity term, the higher dimensional Einstein equations evaluated at the position of the brane, give a very precise and strong relation between the matter on the brane and the matter in the bulk in the vicinity of the brane. In other words, the bulk energy content is the primary factor for the cosmological evolution on the brane. Alternatively, for a static matter distribution on the brane to be possible, there should exist its bulk matter “image”.

This strong relation, that we have noted, can be avoided with the inclusion of a bulk Gauss-Bonnet term. An even more natural way to achieve this is to relax the requirement of purely conical branes and admit general brane solutions with an appropriate regularization (thickening of the brane), so that the singularities are smoothened. In view of the difficulties related to the Gauss-Bonnet term in the context of effective field theory, we pointed out that the thickening of the brane is the most physical direction that one should follow in order to discuss the dynamics of codimension-2 branes.

We next discussed the gravitational dynamics of conical codimension-2 braneworlds in a theory with a Gauss-Bonnet term in the six-dimensional bulk and an induced gravity term on the three-brane. For simplicity, we considered that the bulk matter consists only of a cosmological constant $\Lambda_B$ but the brane matter is general and isotropic. We then analyzed in detail the Einstein equations evaluated on the boundary.

We studied the system first for an isotropic metric ansatz. In the pure induced gravity dynamics there is a tuning between the matter allowed on the brane and in the bulk. This tuning, when the matter in the bulk is only a cosmological constant, gives a precise relation between the matter density on the brane and its equation of state. If a Gauss-Bonnet term is added in the bulk, the constraint equation giving the previous tuning is modified by the addition of a Riemann squared term. However, since for isotropic evolutions the Riemann tensor can be expressed in terms of the Ricci tensor and the scalar curvature, the conclusion about the presence of the tuning remains the same as in the induced gravity case. We found that if $\Lambda_B > 0$ the system has a fixed point with equation of state $w = -1$ and corresponds to a de Sitter vacuum. If $\Lambda_B = 0$, the system has a fixed point with equation of state $w = 1/3$. Both of the previous fixed points are attractors. If $\Lambda_B < 0$ the system has no fixed point and the evolution exhibits a runaway behaviour to $w \rightarrow +\infty$.

We then looked on the cosmological evolution of the system for a particular anisotropy. If the system starts its evolution in the region of parametric space which has as a boundary the line of isotropic tuning, it tracks the latter line and isotropises towards the attractor fixed point with $w = -1$ for $\Lambda_B > 0$, or the one with $w = 1/3$ for $\Lambda_B = 0$. In the two other regions the system has a runaway behaviour $w \rightarrow \infty$ and does not isotropise. For $\Lambda_B < 0$, the system has always a runaway behaviour. The important result of this analysis is that the line of isotropic tuning, is an attractor. However, for values of
Λ_{d}$ which give acceptable cosmological evolutions, a fine tuning is unavoidable because of the weak strength of the above-mentioned attractor.

Six-dimensional supergravity models have a much richer structure. There are many scalar fields with a scalar potential fixed by the supergravity theory. There are also gauge fluxes which guarantee the stability of the theory. Recently, they are six-dimensional warped braneworld solutions in supergravity theories [57] and it would be interesting to find also codimension-1 or codimension-2 cosmological braneworld solutions in six-dimensional supergravity theories.

Acknowledgements

Most of the work reported in this talk was done in collaboration with Bertha Cuadros-Melgar and Antónios Papazoglou and it is supported by (EPEAEK II)-Pythagoras (co-funded by the European Social Fund and National Resources).

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