A note on the antimagic orientation problems for the Mycielski construction of graphs

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Abstract

A simple graph $G$ is said to admit an antimagic orientation if there exist an orientation on the edges of $G$ and a bijection from $E(G)$ to $\{1, 2, \ldots, |E(G)|\}$ such that the vertex sums of vertices are pairwise distinct, where the vertex sum of a vertex is defined to be the sum of the labels of the in-edges minus the that of the out-edges incident to the vertex. It was conjectured by Hefetz, Mütze, and Schwartz [5] in 2010 that every connected simple graph admits an antimagic orientation. In this note, we prove that the Mycielski construction of any simple graph with at most one isolated vertex admits an antimagic orientations, regardless of whether the original graph admits an antimagic orientation.

Keywords: antimagic labeling, antimagic orientation, Euler circuit, Mycielski construction.

1 Introduction

In this note, unless we particularly mention, all graphs are simple. Most of the notation and terminology follow from [21]. For a graph $G$, an antimagic labeling of $G$ is a bijection from the edge set $E(G)$ to the set $\{1, 2, \ldots, |E(G)|\}$ such that the vertex sums are pairwise distinct, where the vertex sum of a vertex $v \in V(G)$ is defined as the sum of the labels of all edges incident to $v$. It was conjectured by Hartsfied and Ringel [8] in 90’s that every connected graph other than $K_2$ has an antimagic labeling. The conjecture has been verified to be true for various graphs [1, 4, 16, 3, 12, 10, 2], but is still widely open. For more details, we refer the readers to the survey [6].

The study of antimagic labeling on directed graphs began much later. It was initiated by Hefetz, Mütze, and Schwartz [5] in 2010. For a directed graph $G$, the vertex sum of

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a vertex $v$ is defined as the sum of the labels of all edges entering $v$ minus the sum of the labels of all edges leaving $v$. As before, We say $G$ is antimagic if the vertex sums are pairwise distinct. Hefetz et al. pointed out that the directed $C_3$ and $P_3$ are not antimagic. On the other hand, it is not hard to show among all of the directed graphs whose underlying graph is $P_3$ (or $C_3$), there exists a directed graph that is antimagic. So they proposed two questions:

**Question 1.1** Is every connected directed graph with at least 4 vertices antimagic?

**Conjecture 1.2** Every connected graph admits an antimagic orientation.

While Question 1.1 seems to be very difficult, Conjecture 1.2 has been studied intensively recently. We say a graph $G$ admits an antimagic orientation, if there exists an orientation $D$ on $E(G)$ and a bijection $\tau$ from $E(G)$ to $\{1, 2, \ldots, |E(G)|\}$ such that the vertex sums $s_{(D, \tau)}(v)$, defined earlier in this paragraph, of all vertices are pairwise distinct.

There are several families of graphs proved to satisfy Conjecture 1.2 by different groups of researchers: $K_n$ with $n \geq 4$, $S_n$ with $n \geq 3$, $W_n$ with $n \geq 3$, the $(2d+1)$-regular graphs with $d \geq 0$ by Hefetz et al [5]; the $2d$-regular graphs with $d \geq 2$ by Li, Song, Wang, Yang, and Zhang [9], by Yang [22]; the biregular graphs by Shan and Yu [15]; the Halin graphs by Yu, Chang, and Zhou [24]; the caterpillars by Shan and Yu [15]; the lobsters by Gao and Shan [7]; and the complete $k$-ary trees by Song and Zhang [19]. In addition to the above special graphs, Yang, Carson, Owens, Perry, Singh, Song, Zhang, and Xhang [23], proved that every connected graph with at least $n \geq 9$ vertices and maximum degree at least $n - 5$ admits an antimagic labeling, and Song, Yang, and Zhang [19] proved that every graph $G$ with independent number at most 4 or least $|V(G)|/2$ admits an antimagic orientation. Some of the above results are also true for disconnected graphs.

Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$. The Mycielski construction of $G$, $M(G)$ is a new graph with $V(M(G)) = \{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_n\} \cup \{w\}$ and $E(M(G)) = E(G) \cup \{v_iu_j \mid v_i, v_j \in E(G)\} \cup \{wu_i \mid 1 \leq i \leq n\}$. Moreover, we call $\{u_1, \ldots, u_n\}$ the image vertices of $V(G)$. An example is $M(K_2) = C_5$. This graph operation was introduced by Mycielski [13] to construct the graphs having the large chromatic number but the small clique number. In this note, we study the antimagic orientation problem for the Mycielski construction of simple graphs. We manage to prove that the Mycielski construction of any graph with at most one isolated vertex admits an antimagic orientation. The proof of this result will be presented in the next section.

## 2 Main result

To prove our result, we need a technical lemma which is used frequently for constructing the antimagic orientations in recent papers. A closed walk in a multiple graph $G$ is an Euler circuit if it traverses every edge in $G$ exactly once. The following theorem is classical:
**Theorem 2.1 (Euler, 1736)** A connected multiple graph has an Euler circuit if and only if the degree of every vertex is even.

Using the Euler circuit, one can estimate the vertex sum of each vertex induced by the circuit. Our first lemma below is almost the same as Lemma 2.1 in [9], Lemma 2.2 in [19], and Lemma 7 in [14]. For the sake of completeness, we present the proof in the note.

**Lemma 2.2** Let \( p \geq 0 \) be an integer, and \( G \) be a graph with \( |E(G)| = m \geq 1 \). Then there exist an orientation \( D \) of \( G \) and a bijection \( \tau : E(G) \to \{p + 1, p + 2, \ldots, p + m\} \) such that for any \( v \in V(G) \),

\[-(2m + p - 2) \leq s_{(D, \tau)}(v) \leq p + m + (d_G(v) - 1)/2.\]

**Proof.** It suffices to prove the lemma for connected graphs. Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( V_1 = \{v \in V(G) \mid d_G(v) \text{ is odd}\} \). By the handshaking theorem, we see that \( |V_1| \) is even. If \( V_1 \neq \emptyset \), we may assume, without loss of generality, \( V_1 = \{v_1, v_2, \ldots, v_{2t}\} \). Define \( G^* = G \) if \( t = 0 \), else \( G^* \) is obtained by adding \( t \) new edges \( e_1^*, e_2^*, \ldots, e_t^* \) to \( G \), where \( e_i^* \) is an edge connecting \( v_i \) and \( v_{i+1} \). Thus, \( d_{G^*}(v_i) \) is even for every vertex \( v_i \in G^* \).

Pick an Euler circuit \( C \) in \( G^* \) as following:

\[C : u_1, e_1, u_2, e_2, \ldots, u_{m+t}, e_{m+t}, u_1,\]

where the \( u_i \)'s are vertices in \( V(G) \) may have repetition, and \( e_1, e_2, \ldots, e_{m+t} \) are exactly the edges in \( E(G^*) \). Let \( e_{i_1}, e_{i_2}, \ldots, e_{i_m} \) be the edges in \( E(G) \). By orienting each edge a direction the same as it is in \( C \), we obtain an orientation \( D \) of \( G \). Observe that \( |d^+_D(v_i) - d^-_D(v_i)| \leq 1 \).

Define \( \tau : E \to \{p + 1, p + 2, \ldots, p + m\} \) with \( \tau(e_{i_j}) = p + m + 1 - j \) for all \( 1 \leq j \leq m \). For any vertex \( v_i \) with \( d^+_D(v_i) = d^-_D(v_i) = d_G(v_i)/2 \), either \( S_{(D, \tau)}(v_i) = -m + d_G(v_i)/2 \) if \( v_i \) is incident to both the edges labeled with \( p + m \) and \( p + 1 \), or \( S_{(D, \tau)}(v_i) = d_G(v_i)/2 \) otherwise. For any vertex \( v_i \) with \( d^+_D(v_i) - d^-_D(v_i) = 1 \), let \( e \) be the edge leaving \( v_i \), whose predecessor in \( C \) is some \( e_i^* \). We have

\[S_{(D, \tau)}(v_i) = \begin{cases} -m + (d_G(v_i) - 1)/2 - \tau(e), & v_i \text{ is incident to both the edges labeled with } p + m \text{ and } p + 1; \\ (d_G(v_i) - 1)/2 - \tau(e), & \text{otherwise}. \end{cases}\]

Similarly, for any vertex \( v_i \) with \( d^+_D(v_i) - d^-_D(v_i) = -1 \), we have

\[S_{(D, \tau)}(v_i) = \begin{cases} -m + (d_G(v_i) - 1)/2 + \tau(e), & v_i \text{ is incident to both the edges labeled with } p + m \text{ and } p + 1; \\ (d_G(v_i) - 1)/2 + \tau(e), & \text{otherwise}, \end{cases}\]

where \( e \) is the edge entering \( v_i \), whose successor in \( C \) is some \( e_j^* \). Now the lower and upper bounds of the vertex sums are obtained from the first and last cases of a vertex of odd degree. \( \square \)

Conventionally, we define the vertex sum of an isolated vertex to be zero. Clearly, a graph with more than one isolated vertex does not admit an antimagic orientation. The next is our main result in the note.
**Theorem 2.3** Let $G$ be a graph with at most one isolate vertex. Then the Mycielski construction of $G$, $M(G)$ admits an antimagic orientation.

**Proof.** First suppose that $G$ contains no isolated vertex. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Define $V(M(G)) = \{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_n\} \cup \{w\}$ and $E(M(G)) = E(G) \cup \{v_i u_j \mid v_i, v_j \in E(G)\} \cup \{wu_i \mid 1 \leq i \leq n\}$. Let $|E(G)| = m$. Then $|E(M(G))| = 3m + n$. We partition the label set $\{1, 2, \ldots, 3m + n\}$ into $\{1, \ldots, m\}$, $\{m + 1, \ldots, 3m - n\}$, $\{3m - n + 1, \ldots, 3m\}$, and $\{3m + 1, \ldots, 3m + n\}$.

First apply Lemma 2.2 to obtain an orientation $D_1$ on $E(G)$ and $\tau$ from $E(G)$ to $\{1, \ldots, m\}$ so that $s_{(D_1, \tau)}(v_i) \leq m + (d_G(v_i) - 1)/2$ for each $v_i \in V(G)$. We extend the orientation $D_1$ to $D$ on $E(M(G))$. Since every edge in $E(M(G)) - E(G)$ is incident to some image vertex $u_i$, we define the direction of it to be entering the $u_i$’s. For each vertex $v_i$, we select an edge in $E(M(G)) - E(G)$ incident to it in $e_{v_i}$. We arbitrary label the edges of the form $v_i u_j$, except for the $e_{v_i}$’s, with labels in $\{m + 1, \ldots, 3m - n\}$. Let $s'_{(D, \tau)}(v_i)$ be the partial (lacking of the label of $e_{v_i}$) vertex sum of $v_i$. Without loss of generality, we may assume

\[
s'_{(D, \tau)}(v_1) \leq s'_{(D, \tau)}(v_2) \leq \cdots \leq s'_{(D, \tau)}(v_n).
\]

Then we label $e_{v_i}$ with $3m + 1 - i$ for $1 \leq i \leq n$. Since $e_{v_i}$ is leaving $v_i$, we have $s_{(D, \tau)}(v_i) = s'_{(D, \tau)}(v_i) - 3m - 1 + i$ for $1 \leq i \leq n$, which implies $s_{(D, \tau)}(v_i) \neq s_{(D, \tau)}(v_j)$ for $i \neq j$. Finally, for the image vertices $u_1, u_2, \ldots, u_n$, suppose we have the partial (lacking of the label of $wu_i$) vertex sums satisfying the following:

\[
s'_{(D, \tau)}(u_1) \leq s'_{(D, \tau)}(u_2) \leq \cdots \leq s'_{(D, \tau)}(u_n).
\]

Then we label $wu_i$ with $3m + j$ for $1 \leq j \leq n$. Since $wu_i$ is entering $u_i$, we have $s_{(D, \tau)}(u_i) = s'_{(D, \tau)}(u_i) + 3m + j$ for $1 \leq j \leq n$, which implies $s_{(D, \tau)}(u_i) \neq s_{(D, \tau)}(u_k)$ for $j \neq k$.

Our orientation $D$ gives $s_{(D, \tau)}(u_i) > 0$ for $1 \leq i \leq n$. Let us see $s_{(D, \tau)}(v_i) < 0$ for $1 \leq i \leq n$. By the fact that every edge of the form $v_i u_j$ is leaving $v_i$ and by Lemma 2.2, we have

\[
s_{(D, \tau)}(v_i) \leq s_{(D_1, \tau)}(v_i) - d_G(v_i)(m + 1)
\leq m + (d_G(v_i) - 1)/2 - d_G(v_i)(m + 1)
\leq 0.
\]

It remains to show $s_{(D, \tau)}(v_i) \neq s_{(D, \tau)}(w)$ for $1 \leq i \leq n$. For each $v_i$, there are $d_G(v_i)$ edges leaving $v_i$ and entering the image vertices, and one of them is $e_{v_i}$. The largest possible sum of labels of these edge is $(\sum_{j=1}^{d_G(v_i)} 3m - n + 1 - j) + 3m$. By Lemma 2.2, the smallest
\( s(D_1, \tau)(v_i) \) is at least \(-2m + 2\). Thus,

\[
\begin{align*}
    s(D, \tau)(v_i) &\geq s(D_1, \tau)(v_i) - \left( \sum_{j=1}^{d_G(v_i)-1} 3m - n + 1 - j \right) - 3m \\
                 &\geq -2m + 2 - \left( \sum_{j=1}^{n-2} 3m - n + 1 - j \right) - 3m \\
                 &> - \sum_{j=1}^{n} 3m + j = s(D, \tau)(w).
\end{align*}
\]

Now if \( G \) contains an isolated vertex, we first apply the previous arguments to obtain an orientation \( D \) and a bijection \( \tau \) for \( G - v \). It is clear \( V(M(G)) = V(M(G-v)) \cup \{v, u\} \), where \( u \) is the image vertex of \( v \), and \( E(M(G)) = E(M(G-v)) \cup \{wu\} \). Now let the direction of \( wu \) be leaving \( w \), label \( wu \) with the smallest \( \tau(wu_i) \) for all \( u_i \in V(M(G))-\{u\} \), and replace the label of \( wu_i \) with \( \tau(wu_i) + 1 \) for \( u_i \in V(M(G)) - \{u\} \). Clearly, all the vertex sums of the image vertices are still distinct and positive, the vertex sums of every vertex in \( V(G) - \{v\} \) remains unchanged, and the vertex sum of \( w \) decreases. Therefore the proof is completed. \( \square \)

### 3 Conclusions

If \( G \) contains exactly an isolated vertex, then one can show the independence number of \( M(G) \) is \((|V(M(G))| + 1)/2\), and our result follows from the result in [19]. However, the independence number of \( M(G) \) is \((|V(M(G))| - 1)/2\) in general, and thus we cannot use the result in [19] to show Theorem 2.3.

In an early paper, Wang and Hsiao [20] studied the Cartesian product and the lexicographic product of graphs, and proved that for some types of graphs the above products of the graphs will produce graphs that are antimagic. In addition to the Mycielski construction, it should be interesting to study whether of not a graph operation can produce graphs that also admit an antimagic orientation.

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