A NOTE ON THE GAUSS CURVATURE FLOW

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Abstract. By means of polar convex bodies and the $C_0$-bounds of Guan and Ni [Entropy and a convergence theorem for Gauss curvature flow in high dimension, arXiv:1306.0625v1 (2013)], we obtain a uniform lower bound on the Gauss curvature of the normalized solution of the Gauss curvature flow without using Chow’s Harnack inequality [On Harnack’s inequality and entropy for the Gaussian curvature flow, Comm. Pure Appl. Math. 44(1991), 469–483].

1. Introduction

The Gauss curvature flow was introduced by Firey [6] to model the changing shape of a tumbling stone subjected to collisions from all directions with uniform frequency. Assuming the existence, uniqueness and regularity of the solutions (settled later by K.S. Chou (K. Tsu) [17]), he proved that if the initial convex surface is origin-symmetric, then the solution to the flow converges to the origin in finite time, and the normalized solution, having fixed volume and equal to the volume of the unit ball, converges in the Hausdorff distance to the unit ball. Furthermore, Firey conjectured that the same conclusion must hold if one starts the flow from any convex surface. Andrews gave an affirmative answer to this question by an elegant approach [3]. In [3], Andrews made use of the parabolic maximum principle applied to the difference of principal curvatures and a Harnack estimate to obtain regularity of the normalized solution and the asymptotic roundness. Recently, Guan and Ni, using Chow’s Harnack inequality and a beautiful trick, completely resolved the long-standing issue of regularity of normalized solutions in higher dimensions: The problem of obtaining lower bound on the Gauss curvature of the normalized solution without imposing any condition on initial smooth, strictly convex hypersurface. In this paper, we show that, by means of polar convex bodies, it is possible to avoid the Harnack inequality in the process of obtaining a uniform lower bound (The uniform upper bound on the Gauss curvature of the normalized solution, in view of Tso’s trick [17], is fairly easy and was known for a long time; see [7, Theorem 5.1] for details.). Our approach might prove useful for geometric flows that no improving pinching estimates or no Haranck inequalities are known to exist.

The idea is as follows: We consider the evolution of polar bodies and apply the maximum principle to the difference of an appropriate power of Euclidean norm of the “polar embedding” and the speed of the “dual flow.” We point out, in general, in the presence of an improving pinching estimate, one may circumvent the need of Harnack estimates by applying the method of Andrews and McCoy from [4]. See [7], for many useful comments on earlier works of Andrews, Chow, and Hamilton on the Gauss curvature flow [1, 3, 5, 9].

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2. Background material

The setting of this paper is the $n$-dimensional Euclidean space. A compact convex subset of $\mathbb{R}^n$ with non-empty interior is called a convex body. Write $\mathcal{F}^n$ for the set of smooth, strictly convex bodies in $\mathbb{R}^n$.

Let $K$ be a smooth, strictly convex body $\mathbb{R}^n$ and let $\varphi: \partial K \to \mathbb{R}^n$, be a smooth embedding of $\partial K$, the boundary of $K$. Write $\mathbb{S}^{n-1}$ for the unit sphere and write $\nu : \partial K \to \mathbb{S}^{n-1}$ for the Gauss map of $\partial K$; at each point $x \in \partial K$, $\nu(x)$ is the unit outward normal at $x$.

The support function of $K$ as a function on the unit sphere is defined by

$$s_K(\nu(x)) := \langle \varphi(x), \nu(x) \rangle$$

for each $x \in \partial K$. We denote the standard metric on $\mathbb{S}^{n-1}$ by $\bar{g}_{ij}$ and the standard Levi-Civita connection of $\mathbb{S}^{n-1}$ by $\bar{\nabla}$. We denote the Gauss curvature of $\partial K$ by $K$ and remark that, as a function on $\partial K$, it is related to the support function of the convex body by

$$\sigma_{n-1} := \frac{1}{K \circ \nu^{-1}} := \frac{\det (\bar{t}_{ij} := \bar{\nabla}_i \bar{\nabla}_j s + \bar{g}_{ij} s)}{\det \bar{g}_{ij}}.$$

Let $\varphi_0$ be a smooth, strictly convex embedding of $\partial K$. A family of convex bodies $\{K_t\}_t \subset \mathcal{F}^n$ defined by smooth embeddings $\varphi : \partial K \times [0,T) \to \mathbb{R}^n$ is said to be a solution to the Gauss curvature flow if $\varphi$ satisfies the initial value problem

$$(2.1) \quad \partial_t \varphi(x,t) = -K(x,t) \nu(x,t), \quad \varphi(\cdot,0) = \varphi_0(\cdot).$$

In this equation $K(x,t)$ is the Gauss curvature of $\partial K_t := \varphi(\partial K,t)$ at the point where the outer unit normal is $\nu(x,t)$, and $T$ is the maximal time that the flow exists. By Tso’s work [17] $K_t$ shrink to a point in a finite time $T$. Throughout this paper, we assume that this point is the origin of $\mathbb{R}^n$. Moreover, it is easy to see that support functions of $\{K_t\}_{t \in [0,T)}$ satisfy

$$(2.2) \quad \partial_t s(z,t) = -\frac{1}{\sigma_{n-1}(z,t)}\cdot, \quad s(\cdot,t) = s_{K_t}(\cdot).$$

3. Evolution equation of polar bodies

The polar body, $K^*$, of the convex body $K$ with the origin of $\mathbb{R}^n$ in its interior is the convex body that is defined as

$$K^* = \{x \in \mathbb{R}^n : \langle x,y \rangle \leq 1 \text{ for all } y \in K\}.$$ 

In what follows, we furnish all geometric quantities associated with $K^*$ with $^*$. We find the evolution equation of the support function of $K^*_t$ as the support function of $K_t$ evolves by (2.2). To this aim, we parameterize $\partial K_t$ over the unit sphere

$$\varphi = r(z(\cdot,t),t)z(\cdot,t) : \mathbb{S}^{n-1} \to \mathbb{R}^n,$$

where $r(z(\cdot,t),t)$ is the radial function of $K_t$ in direction $z(\cdot,t)$.

Let $K$ be a smooth, strictly convex body whose boundary is parameterized over the unit sphere with the radial function $r$. The metric $[g_{ij}]_{1 \leq i,j \leq n-1}$, unit normal $\nu$, support function $s$, and the second fundamental form $[h_{ij}]_{1 \leq i,j \leq n-1}$ of $\partial K$ can be written in terms of $r$ and its partial derivatives as follows:

$$a: \quad g_{ij} = r^2 \bar{g}_{ij} + \bar{\nabla}_i r \bar{\nabla}_j r,$$
b: \[ \nu = \frac{r \vec{z} - \vec{\nabla} r}{\sqrt{r^2 + \|\vec{\nabla} r\|^2}}, \]

c: \[ s = \frac{r^2}{\sqrt{r^2 + \|\vec{\nabla} r\|^2}}, \]

d: \[ h_{ij} = \frac{-r \vec{\nabla}_i \vec{\nabla}_j r + 2 \frac{\vec{\nabla}_i r \vec{\nabla}_j r}{r^3} + r^2 \tilde{g}_{ij}}{\sqrt{r^2 + \|\vec{\nabla} r\|^2}}. \]

Since \( \frac{1}{r} \) is the support function of \( \partial K^* \), we can find the entries of \( [r^*_ij]_{1 \leq i,j \leq n-1} \):

\[ r^*_ij = \vec{\nabla}_i \vec{\nabla}_j \frac{1}{r} + \frac{1}{r} \tilde{g}_{ij} = \frac{-r \vec{\nabla}_i^2 r + 2 \frac{\vec{\nabla}_i r \vec{\nabla}_j r + r^2 \tilde{g}_{ij}}{r^3}}{\sqrt{r^2 + \|\vec{\nabla} r\|^2}}. \]

Thus, using (d) we get

(3.1) \[ r^*_ij = \frac{\sqrt{r^2 + \|\vec{\nabla} r\|^2}}{r^3} h_{ij}. \]

Lemma 1. As \( K_t \) evolve according to flow (2.2), their radial functions evolve as follows

(3.2) \[ \partial_t r = -\frac{\sqrt{r^2 + \|\vec{\nabla} r\|^2}}{r} K. \]

Proof.

\[ \partial_t \phi = \partial_t (r(\cdot, t) z(\cdot, t)) = (\partial_t r) z + \langle \vec{\nabla} r, \partial_t z \rangle + r \partial_t z = -K \nu = -K \frac{r \vec{z} - \vec{\nabla} r}{\sqrt{r^2 + \|\vec{\nabla} r\|^2}}, \]

where from the third line to the fourth line we used (b). By comparing terms on the second line with those on the fourth line, we get

(3.3) \[ r \partial_t z = -\frac{K \vec{\nabla} r}{\sqrt{r^2 + \|\vec{\nabla} r\|^2}} \]

and

(3.4) \[ \partial_t r + \langle \vec{\nabla} r, \partial_t z \rangle = -\frac{K r}{\sqrt{r^2 + \|\vec{\nabla} r\|^2}} \]

Replacing \( \partial_t z \) in (3.4) by its equal expression from equation (3.3) completes the proof. \( \square \)

Theorem 2. As \( \{K_t\}_{t \in [0, T]} \) evolves according to the evolution equation (2.2), the family of convex bodies \( \{K^*_t := (K_t)^*\}_{t \in [0, T]} \) evolves by

\[ s^*: \mathbb{S}^{n-1} \times (0, T) \to \mathbb{R} \]
\[ \partial_t s^*(\cdot, t) = \frac{s^{n+2} + \| \nabla s^* \|^2}{(s^2 + \| \nabla s^* \|^2)^{\frac{3}{2}}}(\cdot, t), \quad s^*(\cdot, t) = s_{K^*_t}(\cdot). \]

Proof. By means of the identities

\[ K = \frac{\det h_{ij}}{\det g_{ij}}, \quad \frac{1}{\sigma_{n-1}^*} = \frac{\det \bar{g}_{ij}}{\det r_{ij}^*}, \quad \det g_{ij} = \frac{1}{r^{2n-4}(r^2 + \| \nabla r \|^2)}. \]

equation (3.1), and evolution equation (3.2), we calculate

\[ \partial_t s^* = \partial_t \frac{1}{r} \]

\[ = \sqrt{\frac{r^2 + \| \nabla r \|^2}{r^3}} \]

\[ = \sqrt{\frac{r^2 + 2 \| \nabla r \|^2}{r^3}} \]

\[ \frac{\det h_{ij}}{\det g_{ij}} \]

\[ = \sqrt{\frac{r^2 + 2 \| \nabla r \|^2}{r^3}} \]

\[ \frac{\det r_{ij}^*}{\det \bar{g}_{ij}} \]

\[ = \sqrt{\frac{r^2 + 2 \| \nabla r \|^2}{r^3}} \]

\[ \frac{\sigma_{n-1}^*}{\sigma_{n-1}^*} \left( r^{2n-4}(r^2 + \| \nabla r \|^2) \right)^{-1}. \]

Replacing \( r \) by \( \frac{1}{s^*} \) in the expression \( \left( \frac{r^2 + \| \nabla r \|^2}{r^3} \right)^{2-n} \left( r^{2n-4}(r^2 + \| \nabla r \|^2) \right)^{-1} \) completes the proof. \( \square \)

Theorem 2 implies that there is a unique solution \( \varphi^* : \partial K \times [0, T) \rightarrow \mathbb{R}^n \) to the equation

\[ \partial_t \varphi^*(\cdot, t) = \left( \frac{\langle \varphi^*, \nu^* \rangle_{n+2}}{\| \varphi^* \|^2 - \frac{1}{K^*_t}} \right)(\cdot, t) \nu^*(\cdot, t), \quad \varphi^*(\partial K, 0) = \partial K_0^* \]

such that \( \varphi^*(\partial K, t) = \partial K_t^* \). That is, the support function of \( \varphi^*(\partial K, t) \) is \( s_{K_t^*} \) (See Andrews [2, Lemma 1.2] for details.).

4. Uniform lower bound on the Gauss curvature

In what follows, for simplicity, we restrict our calculations to the Gauss curvature flow of convex surfaces, e.g., \( n = 3 \); see Remark 5 for the general case \( n \geq 2 \).

Lemma 3. We have the following evolution equations along flow (3.5):

\[ \partial_t \| \varphi^* \|^2 = \frac{\langle \varphi^*, \nu^* \rangle^5}{\| \varphi^* \|^2} \frac{K_s}{K_s^*} \| \nabla^2 \varphi^* \|^4 \]

\[ - 8 \frac{\langle \varphi^*, \nu^* \rangle^5}{\| \varphi^* \|^3} \frac{K_s}{K_s^*} \frac{\langle \varphi^*, \nu^* \rangle^3}{\langle \varphi^*, \nu^* \rangle^3} \]

\[ + 12 \frac{\langle \varphi^*, \nu^* \rangle^6}{\| \varphi^* \|^3} \frac{4H^*}{K_s^*} \frac{\langle \varphi^*, \nu^* \rangle^5}{\| \varphi^* \|^3}. \]
Proof. The evolution equation of \( \|\varphi^*\|^4 \) is straightforward. To obtain the second evolution equation, we need to take into account that along flow \((3.5)\) the metric \([g^i_j]_{1\leq i,j\leq 2}\), the Weingarten tensor \([h^i_j = h^k_{ij}g^{*k}]\)\(1\leq i,j\leq 2\), and the normal vector \(\nu^*\) evolve as

\[
\partial_t g^i_j = 2\left(\frac{\langle \varphi^*, \nu^* \rangle^5 h^i_j}{\|\varphi^*\|^3} \right),
\]

\[
\partial_t h^i_j = -\nabla_i \nabla^j \left( \frac{\langle \varphi^*, \nu^* \rangle^5}{\|\varphi^*\|^3} \frac{1}{K^*} \right) - \left( \frac{\langle \varphi^*, \nu^* \rangle^5}{\|\varphi^*\|^3} \right) h^k_m h^m_j,
\]

\[
\partial_t \nu^* = -T \varphi^* \left( \nabla \left( \frac{\langle \varphi^*, \nu^* \rangle^5}{\|\varphi^*\|^3} \frac{1}{K^*} \right) \right).
\]

See Huisken [10, Theorem 3.4] for details in the context of the mean curvature flow.

**Lemma 4.** If \( \gamma \|\varphi^*\| \leq \langle \varphi^*, \nu^* \rangle \) for some \( 0 < \gamma \leq 1 \) on \([0, T]\), then we can find \( \lambda > \left(\frac{35}{2\gamma^2}\right)^2 \) large enough such that \( \|\varphi^*\|^2 \leq \frac{\lambda}{K^*} \) on \([0, T]\).

Proof. We let \( \lambda > \left(\frac{35}{2\gamma^2}\right)^2 \) be a large number such that \( \chi := \|\varphi^*\|^4 - \lambda \frac{(\langle \varphi^*, \nu^* \rangle)^5}{K^*} \frac{1}{K^*} \) is negative at \( t = 0 \). Our aim is to show that \( \chi < 0 \). We calculate the evolution equation of \( \chi \) and apply the maximum principle to \( \chi \) on \([0, \tau]\), where \( \tau > 0 \) is the first time that for some \( y \in \partial K^* \), we have \( \chi(\tau, y) = 0 \). Note that at such a point where the maximum of \( \chi \) is achieved:

\[
\nabla \chi = 0 \Rightarrow \langle \varphi^*, T \varphi^* \left( \nabla \left( \lambda \frac{(\langle \varphi^*, \nu^* \rangle)^5}{\|\varphi^*\|^3} \frac{1}{K^*} \right) \right) \rangle = \langle \varphi^*, T \varphi^* \left( \nabla \|\varphi^*\|^4 \right) \rangle = 4\|\varphi^*\|^2 \|\varphi^*\|^7 \|
\]

and in view of the assumption \( \gamma \|\varphi^*\| \leq \langle \varphi^*, \nu^* \rangle \):

\[
\gamma^2 \frac{\sqrt{\lambda}}{K^*} \leq \|\varphi^*\| \leq \frac{\sqrt{\lambda}}{K^*}.
\]

Here \( \varphi^* \) is the tangential component of \( \varphi^*(\cdot, t) \) to \( \partial K^* \). Moreover, we always have

\[
-8 \frac{(\langle \varphi^*, \nu^* \rangle)^5}{\|\varphi^*\|^3} \frac{K^{*ij}}{K^*} \langle \varphi^*_i, \varphi^*_j \rangle \langle \varphi^*_i, \varphi^*_j \rangle = -2 \frac{(\langle \varphi^*, \nu^* \rangle)^5}{\|\varphi^*\|^3} \frac{K^{*ij}}{K^*} \|\varphi^*\|^2_i \|\varphi^*\|^2_j \leq 0.
\]
Thus, at \((\tau, y)\) we have
\[
0 \leq \partial_t \chi \leq \frac{12}{K^*} \frac{K^*}{\|\varphi^*\|^2} - \frac{4\mathcal{H}^*}{K^*} \frac{\|\varphi^*\|^5}{\|\varphi^*\|^2} + \frac{20}{K^*} \frac{\|\varphi^*\|^7}{\|\varphi^*\|^2} \\
- \frac{\mathcal{H}^*}{K^*} \frac{\|\varphi^*\|^6}{\|\varphi^*\|^2} - \frac{5}{K^*} \frac{\|\varphi^*\|^8}{\|\varphi^*\|^2} + \frac{3}{K^*} \frac{\|\varphi^*\|^8}{\|\varphi^*\|^2} \\
\leq \frac{12}{K^*} \frac{\|\varphi^*\|^6}{\|\varphi^*\|^2} - \frac{2}{K^*} \frac{\|\varphi^*\|^6}{\|\varphi^*\|^2} + \frac{3}{K^*} \frac{\|\varphi^*\|^8}{\|\varphi^*\|^2} \\
+ \frac{20}{K^*} \frac{\|\varphi^*\|^5}{\|\varphi^*\|^2} - \frac{8}{K^*} \frac{\|\varphi^*\|^8}{\|\varphi^*\|^2} \leq \frac{2}{K^*} \frac{\|\varphi^*\|^6}{\|\varphi^*\|^2} + \frac{3}{K^*} \frac{\|\varphi^*\|^8}{\|\varphi^*\|^2} \\
\leq \frac{35\lambda^2}{K^*} - \frac{2\gamma^2 \lambda^3}{K^*} < 0.
\]
\(\Box\)

**Theorem (Uniform lower bound).** There is a uniform lower bound on the Gauss curvature of the normalized solution of the Gauss curvature flow.

**Proof.** In view of the \(C_0\) estimates in [7, Estimates (5.8)], we have
\[
0 < C_1 \leq s^*(\cdot, t)(T-t)^{\frac{2}{n}} \leq C_2 < +\infty.
\]
Therefore, as \(\|\varphi^*\|^2 = s^2 + \|\nabla s^*\|^2\), Lemma 4 with \(\gamma = \frac{C_1}{C_2}\) implies that there is a uniform upper bound on \(K^*(T-t)^{-\frac{2}{n}}\). To complete the proof, we recall Kaltenbach’s identity: for every \(x \in \partial K\), there exists an \(x^* \in \partial K^*\) such that
\[
\left( \frac{K}{s^{n+1}} \right)(x) \left( \frac{K^*}{s^{n+1}} \right)(x^*) = 1,
\]
where \(x\) and \(x^*\) are related by \(\langle x, x^* \rangle = 1\) (Proof of this identity is given in [8, 14]).

Using the \(C_0\) estimates of Guan and Ni [7, Estimates (5.8)] once again, we conclude that \(K(T-t)^{\frac{2}{n}}\) is uniformly bounded from below. \(\Box\)

**Remark 5.** In general, \(n \geq 2\), we may apply the maximum principle to
\[
\chi := \|\varphi^*\|^{n+1} - \lambda \frac{\|\varphi^*\|^n}{\|\varphi^*\|^n} \cdot \frac{1}{K^*},
\]
for \(\lambda > 0\) large enough. Then, using concavity and inverse-concavity of \(K + \frac{\pi}{\lambda}\)
\[
\hat{K}^* \hat{h}_i^j \geq (n-1)K^* \frac{\pi}{\lambda} \& \hat{K}^* \hat{h}_i^j \hat{h}_k^l \geq (n-1)K^* \frac{\pi}{\lambda},
\]
and similar calculations as above, we can obtain a uniform lower bound on the Gauss curvature of the normalized solution.

**Remark 6.** To my knowledge, a quantity similar to \(\chi\) was first considered by O.C. Schniirer [16, Lemma 6.1] and then by Q.R. Li in [15, Lemma 5.1] to obtain regularity of solutions to the normalized inverse Gauss curvature flow (In fact, O.C. Schniirer considered several more curvature flows.). Therefore, in view of the dual
flow (3.5) and the earlier works of the author and A. Stancu [11, 12, 13], it is very natural to consider $\chi$ as a candidate for the purpose of this paper.

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