ON THE DIRICHLET PROBLEM FOR GENERAL AUGMENTED HESSIAN EQUATIONS

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Abstract. In this paper we apply various first and second derivative estimates and barrier constructions from our treatment of oblique boundary value problems for augmented Hessian equations, to the case of Dirichlet boundary conditions. As a result we extend our previous results on the Monge-Ampère and $k$-Hessian cases to general classes of augmented Hessian equations in Euclidean space.

1. Introduction

In this paper we apply various first and second derivative estimates and barrier constructions from our treatment of oblique boundary value problems in [14,15] to the classical Dirichlet problem for general classes of augmented Hessian equations, thereby extending our previous results in [18,19] on the Monge-Ampère and $k$-Hessian cases.

We consider general augmented Hessian equations in the form,

\begin{equation}
F[u] := F[D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du), \quad \text{in } \Omega,
\end{equation}

where the scalar function $F$ is defined on an open cone $\Gamma$ in $\mathbb{S}^n$, the linear space of $n \times n$ real symmetric matrices, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{S}^n$ is a symmetric matrix function and $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a scalar function. Our Dirichlet boundary conditions have the form

\begin{equation}
G[u] := u - \varphi = 0, \quad \text{on } \partial \Omega,
\end{equation}

where $\varphi$ is a smooth function on $\partial \Omega$. As usual, $Du$ and $D^2u$ denote respectively the gradient vector and the Hessian matrix of the unknown function $u \in C^2(\Omega)$ and we use $x, z, p$ and $r$ to denote points in $\Omega, \mathbb{R}, \mathbb{R}^n$ and $\mathbb{S}^n$, respectively.

Following [14,15] we assume further that cone $\Gamma$ in $\mathbb{S}^n$ is convex, with vertex at 0, containing the positive cone $K^+$, and that $F \in C^2(\Gamma)$ satisfies the basic conditions:

**F1:** $F$ is strictly increasing in $\Gamma$, that is

\begin{equation}
F_r := F_{r_{ij}} = \left\{ \frac{\partial F}{\partial r_{ij}} \right\} > 0, \quad \text{in } \Gamma.
\end{equation}

**F2:** $F$ is concave in $\Gamma$, that is

\begin{equation}
\sum_{i,j,k,l=1}^{n} \eta_{ij} \eta_{kl} \frac{\partial^2 F}{\partial r_{ij} \partial r_{kl}} \leq 0, \quad \text{in } \Gamma,
\end{equation}

for all symmetric matrices $\{\eta_{ij}\} \in \mathbb{S}^n$. 

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**F3:** $F(\Gamma) = (a_0, \infty)$ for a constant $a_0 \geq -\infty$ with

$$\sup_{r_0 \in \partial \Gamma} \limsup_{r \to r_0} F(r) \leq a_0.$$  

We say that an operator $\mathcal{F}$ satisfies the above properties if the corresponding function $F$ satisfies them. Note that we can take the constant $a_0$ in F3 to be 0 or $-\infty$. We also say that $\mathcal{F}$ is orthogonally invariant if $F$ is given as a symmetric function $f$ of the eigenvalues $\lambda_1, \cdots, \lambda_n$ of the matrix $r$, with $\Gamma$ closed under orthogonal transformations. While it was not essential for our study of oblique boundary conditions in [14], the orthogonal invariance property of $\mathcal{F}$ is critical for our study of the Dirichlet problem (1.1)-(1.2); see the $A = 0$ case in [2][27] for example. In the orthogonally invariant case, we use

$$\tilde{\Gamma} = \lambda(\Gamma) = \{\lambda \in \mathbb{R}^n \mid \lambda = (\lambda_1, \cdots, \lambda_n) \text{ are eigenvalues of some } r \in \Gamma\}$$

to denote the corresponding cone to $\Gamma$ in $\mathbb{R}^n$. For convenience of later usage, we define for $k = 1, \cdots, n$, the $k$-cone

$$\Gamma_k = \{r \in S^n \mid S_j [r] > 0, \forall j = 1, \cdots, k\},$$

where $S_k$ denotes the $k$-th order elementary symmetric function defined by

$$S_k [r] := S_k (\lambda (r)) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \cdots, n.$$  

We call $M[u] := D^2 u - A(\cdot, u, Du)$ the augmented Hessian matrix, which is the standard Hessian matrix adjusted by subtraction of a lower order symmetric matrix function. A $C^2$ function $u$ is admissible in $\Omega$ ($\tilde{\Omega}$), if

$$M[u] \in \Gamma, \text{ in } \Omega, (\tilde{\Omega}),$$

so that the operator $\mathcal{F}$ satisfying F1 is elliptic with respect to $u$ in $\Omega$ ($\tilde{\Omega}$) when (1.9) holds. If an admissible function $u$ satisfies equation (1.1), we call $u$ an admissible solution of equation (1.1). Since $\mathcal{F}$ satisfies F3, the requirement $B > a_0$ in $\Omega$ ($\tilde{\Omega}$) is necessary for an admissible solution of equation (1.1). A function $\underline{u} \in C^2(\tilde{\Omega})$ is said to be admissible with respect to $u$ in $\Omega$ ($\tilde{\Omega}$), if

$$M_u [\underline{u}] := D^2 \underline{u} - A(\cdot, u, Du) \in \Gamma, \text{ in } \Omega, (\tilde{\Omega}).$$

Clearly if $A$ is independent of $z$, then $M_u [\underline{u}] = M[u]$ so that $\underline{u}$ is admissible with respect to $u$ if and only if $u$ is admissible. While if $A$ is non-decreasing in $z$, (non-increasing in $z$), then $M_u [\underline{u}] \geq M[u]$ and $\underline{u}$ is admissible with respect to $u$, if $u$ is admissible and $u \geq u, (\leq u)$. If a function $\underline{u} (\bar{u})$ satisfies

$$F(M_u [\underline{u}]) \geq B(\cdot, u, Du), \quad (F(M_u [\bar{u}]) \leq B(\cdot, u, Du)),$$

at points in $\Omega$, we call $\underline{u} (\bar{u})$ a subsolution (supersolution) of equation (1.1). Moreover, we call $\underline{u} (\bar{u})$ an admissible subsolution (supersolution) of equation (1.1) if $\underline{u} (\bar{u})$ is admissible with respect to $u$.

The matrix $A$ is called regular (strictly regular), if

$$\sum_{i,j,k,l} A_{ij}^{kl}(x, z, p) \xi_i \xi_j \eta_k \eta_l \geq 0, \quad (> 0),$$

for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$ and $\xi \cdot \eta = 0$, where $A_{ij}^{kl} = D^2_{p_ip_k A_{ij}}$. The regular condition (1.12) was first introduced for the interior regularity in the context of optimal transportation in [24] in its strict form, and subsequently used for the global regularity in [29] in its weak form. If (1.12) holds without the restriction $\xi \cdot \eta = 0$, the matrix $A$ is called regular without orthogonality. Note that the case when $A = A(x, z)$, and in particular the basic Hessian case $A \equiv 0$, satisfies the regular condition without orthogonality.

We now begin to formulate the main theorems of this paper.
Theorem 1.1. Let \( u \in C^4(\Omega) \cap C^2(\overline{\Omega}) \) be an admissible solution of Dirichlet problem (1.1)-(1.2). where \( F \) is orthogonally invariant and satisfies F1-F3 in \( \Gamma \subset \Gamma_1 \), \( A \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \) is regular in \( \Omega \), \( B > a_0, \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \) is convex with respect to \( p \). Assume there exists an admissible subsolution \( u \in C^2(\overline{\Omega}) \) satisfying (1.11) in \( \Omega \) and \( u = \varphi \) on \( \partial \Omega \), with \( \varphi \in C^4(\partial \Omega) \), \( \partial \Omega \in C^4 \). Then we have the estimate

\[
\sup_{\overline{\Omega}} |D^2 u| \leq C, \tag{1.13}
\]

where the constant \( C \) depends on \( n, A, B, \Omega, \varphi, u \), and \( |u|_{1; \Omega} \).

Note that for the second derivative estimate in Theorem 1.1, \( F \) is only assumed to satisfy F1, F2 and F3, where \( a_0 \) can be either finite or infinite. For gradient estimates, there are a range of conditions on \( F, A \) and \( B \). In particular we recall a further condition for \( F \) from [14], which along with F1-F3 is satisfied by our examples in Section 4 of [14].

**F7:** For a given constant \( a > a_0 \), there exists constants \( \delta_0, \delta_1 > 0 \) such that

\[
F_{ij} \xi_i \xi_j \geq \delta_0 + \delta_1 \mathcal{F},
\]

if \( a \leq F(r) \) and \( \xi \) is a unit eigenvector of \( r \) corresponding to a negative eigenvalue, where \( \mathcal{F} = \text{trace}(F_r) \).

To apply F7, apart from orthogonal invariance, we also need (almost quadratic) structure conditions on \( A \) and \( B \), which we write here in a fairly general form:

\[
A = o(|p|^2)I, \quad p \cdot D_p A \leq O(|p|^2)I, \quad p \cdot D_p B \leq O(|p|^2), \tag{1.14}
\]

\[
p \cdot D_p A + |p|^2 D_p A \geq o(|p|^4)I, \quad p \cdot D_p B + |p|^2 D_p B \geq o(|p|^4), \tag{1.15}
\]

as \( |p| \to \infty \), uniformly for \( x \in \Omega, |z| \leq M \) for any \( M > 0 \), where \( I \) denotes the \( n \times n \) identity matrix. A global gradient estimate then follows from our proof of case (ii) of Theorem 1.3 in Section 3 of [14], while for a local gradient estimate we need to strengthen the last two inequalities in (1.14), (at least when \( a_0 \) is finite), with (1.14) replaced by (1.16).

\[
D_p A, D_p B = O(|p|). \tag{1.16}
\]

Note that in the special case when \( \Gamma = K^+ \), we only need the one-sided quadratic structure \( A = O(|p|^2)I \), as \( |p| \to \infty \), uniformly for \( x \in \Omega, |z| \leq M \) for any \( M > 0 \), [18], while from [14], if \( \Gamma = \Gamma_k \) for \( n/2 < k < n \), we can weaken \( o \) to \( O \) in (1.15), (at least when \( a_0 \) is finite), with (1.14) replaced by (1.16), that is a quadratic structure is sufficient. By a slight modification of our arguments in Section 3 of [14], we can use F2 instead of F7 in the global gradient bound, under some slight strengthening of our conditions on \( F, A \) and \( B \) which, for example, would still embrace the basic examples of functions \( F \) which are positive homogeneous of degree one and involve replacing \( O \) by \( o \) throughout (1.14). These alternative conditions are also discussed in the case of oblique boundary conditions in Section 3 of [16].

Further conditions for gradient bounds for strictly regular \( A \) are given in [14]. These global gradient estimates reduce the full gradient bound to the gradient bound on the boundary, which is readily deduced under our assumptions; see Section 3.

The maximum modulus estimate for solution \( u \) of the Dirichlet problem (1.1)-(1.2) is guaranteed by assuming the existence of an admissible subsolution and a supersolution of the problem. Since we assume the admissible subsolution \( u \in C^2(\overline{\Omega}) \) satisfies \( u = \varphi \) on \( \partial \Omega \), we already have the lower solution bound \( u \geq u \) in \( \overline{\Omega} \) by the comparison principle. For the upper solution bound, we can assume \(-A(x, z, 0) \notin \Gamma \) for all \( x \in \Omega \) and \( z \in \mathbb{R} \), which implies large constant functions are supersolutions. More generally we can assume that there exists a bounded viscosity supersolution \( \bar{u} \) as in [18], so that \( u \leq \bar{u} \) on \( \overline{\Omega} \).
We now formulate the following existence theorem for classical admissible solutions, where $A$ and $B$ are independent of $z$.

**Theorem 1.2.** Assume that $F$ is orthogonally invariant and satisfies $F1$-$F3$, $F7$ in $\Gamma \subset \Gamma_0$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with $\partial \Omega \in C^4$, $A \in C^2(\bar{\Omega} \times \mathbb{R}^n)$ is regular in $\bar{\Omega}$, $B \in C^2(\bar{\Omega} \times \mathbb{R}^n)$ is convex with respect to $p$. Assume there exist a bounded viscosity supersolution $\bar{u}$ and a subsolution $u \in C^2(\bar{\Omega})$ satisfying $\bar{u} = \varphi$ on $\partial \Omega$ with $\varphi \in C^4(\partial \Omega)$. Assume also either \((1.14)\) and \((1.15)\) hold or $\Gamma = K^+$, and $A(x,p) \geq O(|p|^2)I$ as $|p| \to \infty$, uniformly for $x \in \Omega$. Then there exists a unique admissible solution $u \in C^3(\bar{\Omega})$ of the Dirichlet problem \((1.1)-(1.2)\). Moreover, if $\Gamma = \Gamma_k$ for $n/2 < k < n$, the conclusion still holds by replacing \((1.14)\) by \((1.16)\) and weakening \(o\) to \(O\) in \((1.15)\).

Corresponding to our remarks above, we can relax condition $F7$, at least for finite $a_0$, in the above hypotheses for general $\Gamma$, provided $O$ is strengthened to $o$ in \((1.14)\), see Corollary 3.1. Since $A$ and $B$ are independent of $z$, it is convenient to call $u$ a subsolution as usual in Theorem 1.2, rather than an admissible subsolution.

Historically, the Dirichlet problem of the standard Hessian equations for general operators has been studied extensively in \[2, 4, 27\] and our conditions $F1$-$F3$ correspond to the basic conditions in these works. Second derivative estimates and the existence results are established under an associated uniform convexity of the domain or the existence of an admissible subsolution. Both the domain convexity and the subsolution are used to construct barrier functions, which are then used in the derivation of boundary second derivative estimates. For the Dirichlet problem of the augmented Hessian equations, we have treated the Monge-Ampère case in \[18\] and the $k$-Hessian case in \[19\], for regular matrices $A$, under the existence of a subsolution, which is also used to obtain the global second derivative bounds. There are also recent studies of the Dirichlet problem \((1.1)-(1.2)\) on Riemannian manifolds, under more restrictive conditions on the matrix function $A$, \[5-8\], where the existence of a subsolution is also critical for such bounds. These stem from the basic Hessian case in \[5\], where such a technique is developed independently of our discovery through the Monge-Ampère case in \[18\]. We also remark that our treatment here will also extend to the more general Riemannian manifold case and as well the condition $F3$ can be weakened as for example in \[12\]; (see also \[16\]).

The essential ingredients in this paper are already in our papers \[14, 15\]. These are the global second derivative estimates in Section 3 of \[15\] and the global gradient estimates in Section 3 of \[14\], in particular Remark 3.1. In Section 2 of this paper, we obtain the second derivative estimates on the boundary following the methods already established \[2, 4, 27\] and thus complete the proof of Theorem 1.1. A strengthened technical barrier construction, already invoked for the basic Hessian case in \[5\], is also discussed, which provides an alternative approach to the estimates of the mixed tangential-normal derivatives and pure normal derivatives on the boundary. In Section 3 we consider alternative gradient estimate hypotheses and in particular derive the gradient estimate, with $F7$ replaced by $F2$, by modification of our argument in case (ii) of Theorem 1.3 in \[14\]. Finally we prove the existence of classical admissible solutions in Theorem 1.2 by the method of continuity.

2. Boundary estimates for second derivatives

In this section, we shall make full use of the admissible subsolution $u \in C^2(\bar{\Omega})$ of equation \((1.1)\) to establish the second derivative estimate $|D^2u| \leq C$ on $\partial \Omega$. Together with the global second derivative bound in terms of its boundary bound in Theorem 3.1 in \[15\], we can get full second derivative estimate \((1.13)\) in Theorem 1.1 based on the boundary estimate in this section. We also discuss a new barrier
construction, which provides a more direct approach in both the mixed tangential-normal derivative estimate and the pure normal derivative estimate on the boundary.

By a standard perturbation argument, we can make a non-strict admissible subsolution \( u \in C^2(\bar{\Omega}) \) of equation (1.1) to be a strict admissible \( C^2 \) subsolution of equation (1.1). Similarly, for the admissible subsolution \( u \), if we restrict it in a neighbourhood of \( \partial \Omega \), we can modify it to be a strict admissible subsolution satisfying the same boundary condition. It is also readily checked that the form of the equation (1.1) and the regularity condition (1.12) can be preserved under translation and rotation of coordinates.

We now proceed to the boundary estimates. For any given point \( x_0 \in \partial \Omega \), by a translation and a rotation of the coordinates, we may take \( x_0 \) as the origin and \( x_n \) axis to be the inner normal of \( \partial \Omega \) at the origin. Near the origin, \( \partial \Omega \) can be represented as a graph

\[
x_n = \rho(x'),
\]

such that \( D'\rho(0) = 0 \), where \( D' = (D_1, \ldots, D_{n-1}) \) and \( x' = (x_1, \ldots, x_{n-1}) \). By tangentially differentiating (1.2) twice, we have

\[
D_{\alpha\beta}(u - \varphi)(0) = -D_n(u - \varphi)(0)\rho_{\alpha\beta}(0), \quad \alpha, \beta = 1, \ldots, n - 1,
\]

which leads to the pure tangential estimate,

\[
|D_{\alpha\beta}u(0)| \leq C, \quad \alpha, \beta = 1, \ldots, n - 1,
\]

where the constant \( C \) depends on \( \Omega, \varphi \) and \( |u|_{1;\Omega} \).

We then estimate the mixed tangential-normal derivatives \( |D_{\alpha n}u(0)| \) for \( \alpha = 1, \ldots, n - 1 \), by using barrier argument. For this estimate, we consider the following operator

\[
T_\alpha := \partial_\alpha + \sum_{\beta < n} \rho_{\alpha\beta}(0)(x_\beta \partial_n - x_n \partial_\beta), \quad \text{for fixed } \alpha < n.
\]

By calculations, we have for \( \alpha < n \),

\[
\mathcal{L}(T_\alpha u) = \mathcal{L}u_\alpha + \sum_{\beta < n} \rho_{\alpha\beta}(0)(x_\beta \mathcal{L}u_n - x_n \mathcal{L}u_\beta)
\]

\[
+ \sum_{\beta < n} \rho_{\alpha\beta}(0)[2(F^{\beta j}u_{nj} - F^{nj}u_{\beta j})
\]

\[
- F^{ij}(A_{ij}^\alpha u_n - A_n^\beta u_\beta) - (u_n D_{p\beta} B - u_\beta D_{pn} B)],
\]

where \( \mathcal{L} \) is the linearized operator defined by

\[
\mathcal{L} := F^{ij}D_{ij} - (F^{ij}A_{ij}^k - D_{pk} B)D_k,
\]

with

\[
F^{ij} := \frac{\partial F}{\partial w_{ij}}, \quad \text{and} \quad A_{ij}^k := D_{pk} A_{ij},
\]

where \( w_{ij} := u_{ij} - A_{ij} \). By differentiation equation (1.1) with respect to \( x_k \), we have

\[
\mathcal{L}u_k = F^{ij}(A_{ij}^k + u_k D_x A_{ij}) + (D_{x_k} B + u_k D_x B), \quad k = 1, \ldots, n.
\]

Since \( F \) is orthogonal invariant, we can derive, for \( \alpha < n \),

\[
\sum_{\beta < n} \rho_{\alpha\beta}(0)(F^{\beta j}w_{nj} - F^{nj}w_{\beta j}) = 0,
\]

so that

\[
\sum_{\beta < n} \rho_{\alpha\beta}(0)(F^{\beta j}u_{nj} - F^{nj}u_{\beta j}) = \sum_{\beta < n} \rho_{\alpha\beta}(0)(F^{\beta j}A_{nj} - F^{nj}A_{\beta j}).
\]
From (2.4), (2.8) and (2.11), we obtain
\begin{equation}
|\mathcal{L}(T_{\alpha}(u - \varphi))| \leq C(1 + \mathcal{T}), \quad \text{in} \ \Omega,
\end{equation}
for \(\alpha < n\), where the constant \(C\) depends on \(\Omega, A, B, \varphi\) and \(|u|_{1;\Omega}\). For \(\alpha < n\), we also have
\begin{equation}
|T_{\alpha}(u - \varphi)| \leq C|x|^2, \quad \text{on} \ \partial \Omega.
\end{equation}

We are now in a position to employ an appropriate barrier function. We present the following lemma without proof, which is a restatement of the general barrier construction in Lemma 2.1(ii) in [15].

**Lemma 2.1.** Let \(u \in C^2(\bar{\Omega})\) be an admissible solution of equation (1.1), \(u \in C^2(\bar{\Omega})\) be an admissible strict subsolution of equation (1.1) satisfying
\begin{equation}
F(M_u[u]) > B(\cdot, u, Du), \quad \text{in} \ \Omega.
\end{equation}
Assume \(F\) satisfies F1-F3, \(A \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)\) is regular in \(\Omega\), \(B > a_0, \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)\) is convex in \(p\). Then there exist positive constants \(K\) and \(\varepsilon_1\), depending on \(\Omega, A, B, u\) and \(|u|_{1;\Omega}\), such that
\begin{equation}
\mathcal{L}\left[e^K(u-u)\right] \geq \varepsilon_1(1 + \mathcal{T}), \quad \text{in} \ \Omega,
\end{equation}
where \(\mathcal{L}\) is the linearized operator defined in (2.9), and \(\mathcal{T} = \text{trace}(\mathcal{F}_r)\).

By applying Lemma 2.1 to our strict subsolution \(u\) satisfying (2.11) in the neighbourhood of the boundary, we then have
\begin{equation}
\mathcal{L}\eta \geq \varepsilon_1(1 + \mathcal{T}), \quad \text{in} \ \Omega_{\rho},
\end{equation}
with \(\eta := \exp[K(u-u)], \ u = u = \varphi\ \text{on} \ \partial \Omega\), and \(\Omega_{\rho} := \{x \in \Omega \mid d(x) < \rho\}\). Then the function \(\hat{\eta} = 1 - \eta\) satisfies
\begin{equation}
\mathcal{L}\hat{\eta} \leq \varepsilon_1(1 + \mathcal{T}), \quad \text{in} \ \Omega, \quad \hat{\eta} = 0, \quad \text{on} \ \partial \Omega, \quad \hat{\eta} > 0, \quad \text{on} \ \partial \Omega_{\rho} \cap \Omega.
\end{equation}
Letting
\begin{equation}
\tilde{\eta} = a\hat{\eta} + b|x|^2,
\end{equation}
with positive constants \(a \gg b \gg 1\), we then have for \(\alpha < n\),
\begin{equation}
\mathcal{L}\tilde{\eta} + |\mathcal{L}(T_{\alpha}(u - \varphi))| \leq 0, \quad \text{in} \ \Omega_{\rho},
\end{equation}
\begin{equation}
\tilde{\eta} + |T_{\alpha}(u - \varphi)| \geq 0, \quad \text{on} \ \partial \Omega_{\rho}.
\end{equation}
By the maximum principle, we derive the mixed tangential-normal derivative estimate
\begin{equation}
|D_{\alpha n} u(0)| \leq C, \quad \text{for} \ \alpha = 1, \cdots, n - 1,
\end{equation}
where the constant \(C\) depends on \(\Omega, A, B, \varphi, u\) and \(|u|_{1;\Omega}\).

Up to now, from (2.2) and (2.17), the following estimates on the boundary are already under control,
\begin{equation}
|D_{ij} u(x)| \leq M_1', \quad \text{for} \ i + j < 2n, \ x \in \partial \Omega,
\end{equation}
where the constant \(M_1'\) depends on \(\Omega, A, B, \varphi, u\) and \(|u|_{1;\Omega}\). In (2.18), the coordinate system is chosen so that the positive axis is directed along the inner normal at the point \(x \in \partial \Omega\).

The remaining estimate is the pure normal second order derivative estimate on the boundary. For this estimation, we shall use the idea in [27]. Since \(\Gamma \subset \Gamma_1\), the lower bound for \(u_{nn}\) is direct from \(\text{trace}(M[u]) > 0\) and (2.18). We need to derive an upper bound for \(u_{nn}\) on \(\partial \Omega\).

For any boundary point \(x \in \partial \Omega\), fixing a principal coordinate system at the point \(x\) and a corresponding neighbourhood \(\mathcal{N}\) of \(x\) with \(\gamma_n \in [-1, -1/2]\) in \(\mathcal{N} \cap \partial \Omega\), we let \(\xi^{(1)}, \cdots, \xi^{(n-1)}\) be an orthogonal
vector field on $N \cap \partial \Omega$, which is tangential to $N \cap \partial \Omega$, namely $\xi^{(j)} \cdot \gamma = 0$ for $j = 1, \ldots, n - 1$, where $\gamma$ is the unit outer normal vector field on $\partial \Omega$. Note that the vector field $\xi^{(1)}, \ldots, \xi^{(n-1)}$ agrees with the coordinate system at $x$, namely $\xi^{(\beta)}_\alpha(x) = \delta_{\alpha \beta}$, $\alpha, \beta = 1, \ldots, n - 1$. We introduce the following notations
\[
\nabla u = \xi^m D_m u, \quad \nabla u = \xi^m D_m u, \quad \xi^{(\beta)}_m = \xi^m \xi^{(\beta)}_l D_m \gamma_l, \quad 1 \leq \alpha, \beta \leq n - 1, \\
\nabla u = (\nabla u, \ldots, \nabla_n u), \quad D_\gamma u = \gamma^m D_m u, \\
\mathcal{A}_{\alpha \beta}(\cdot, u, \nabla u, -D_\gamma u) = \xi^m(\xi^{(\beta)}_l A_{ml}(\cdot, u, \nabla u, -D_\gamma u), \quad 1 \leq \alpha, \beta \leq n - 1, \\
\omega_{\alpha \beta} = \xi^m(\xi^{(\beta)}_l w_{ml} = \xi^m(\xi^{(\beta)}_l (D_m u - A_{ml}), \quad 1 \leq \alpha, \beta \leq n - 1, \\
\omega_{-\gamma} = -\gamma^m \xi^m l \omega_{ml}, \quad \omega_{-\gamma, \alpha} = \omega_{-\gamma, \alpha} = \omega_{-\gamma}(\cdot, \omega_{-\gamma, n-1})^T, \quad 1 \leq \alpha, \beta \leq n - 1, \\
\text{and} \\
\nabla^2 u = \{\nabla_{\alpha \beta} u\}_{1 \leq \alpha, \beta \leq n-1}, \quad \mathcal{E} = \{\xi^{(\beta)}_\alpha\}_{1 \leq \alpha, \beta \leq n-1}, \quad \mathcal{A} = \{\mathcal{A}_{\alpha \beta}\}_{1 \leq \alpha, \beta \leq n-1},
\]
Since $u = \varphi$ on $\partial \Omega$, we have for $x \in \partial \Omega$,
\[
\mathcal{A}_{\alpha \beta}(x, u(x), \nabla u(x), -D_\gamma u(x)) = \mathcal{A}_{\alpha \beta}(x, \varphi(x), D' \varphi(x), D_n u(x)),
\]
for $1 \leq \alpha, \beta \leq n - 1$, and
\[
B(x, u(x), \nabla u(x), -D_\gamma u(x)) = B(x, \varphi(x), D' \varphi(x), D_n u(x)),
\]
where $D' = (D_1, \ldots, D_{n-1})$. The augmented Hessian matrix under the orthogonal vector field $\xi^{(1)}, \ldots, \xi^{(n-1)}$ and $\gamma$, can be written as
\[
M[u] := \left( \begin{array}{cc} M'[u], & \omega_{-\gamma, \alpha} \\ \omega_{-\gamma, \alpha}^T, & \omega_{-\gamma, \gamma} \end{array} \right),
\]
where $M'[u] := \nabla^2 u - \mathcal{A}$, and $\gamma = -\gamma$. From the boundary condition $u = \varphi$ on $\partial \Omega$, we have
\[
\nabla^2 (u - \varphi) = D_\gamma (u - \varphi) \mathcal{E}
\]
on $\partial \Omega$, which agrees with $\bar{\Gamma}$ at $x_0$. We then have, on the boundary $\partial \Omega$,
\[
M'[u] = \{D_\gamma (u - \varphi) \mathcal{E} + \nabla_{\alpha \beta} \varphi - \mathcal{A}_{\alpha \beta}(x, \varphi, D' \varphi, -D_\gamma u)\}_{\alpha, \beta = 1, \ldots, n-1},
\]
For a sufficiently large constant $R$ satisfying $\langle \lambda'(M'[u]), R \rangle \in \bar{\Gamma}$, we define
\[
f_R(\lambda'(M'[u])) := F(M'[u] + R \gamma \otimes \gamma),
\]
where $\lambda' = \lambda'_x = (\lambda_1, \ldots, \lambda_{n-1})$ are the eigenvalues of $M'[u] := \{\omega_{\alpha \beta}\}_{\alpha, \beta = 1, \ldots, n-1}$, and
\[
\bar{M}[u] := \left( \begin{array}{cc} M'[u], & 0 \\ 0, & 0 \end{array} \right),
\]
We now fix a point $x_0 \in \partial \Omega$, where the function $g$ defined by
\[
g(x) := f_R(\lambda'(M'[u])) - B[u]
\]
is minimized over $\partial \Omega$, where $M'[u] := \{\omega_{\alpha \beta}\}_{\alpha, \beta = 1, \ldots, n-1}$, $B[u] := B(\cdot, u, Du)$, $R$ is a sufficiently large constant satisfying $\langle \lambda'(M'[u]), R \rangle \in \bar{\Gamma}$. In order to derive an upper bound for $u_m$ on $\partial \Omega$, we aim to get a positive lower bound for the function $g(x)$ in (2.21) on $\partial \Omega$.
We assume that the function $h$ defined by
\[
h(x) := f_R(\lambda'(M'[u])) - B[u]
\]
is minimized over $\partial \Omega$ at a point $y \in \partial \Omega$, where $M'[u] := \{D_{\alpha \beta}u - A_{\alpha \beta}(\cdot, u, Du)\}_{\alpha, \beta = 1, \ldots, n-1}$, and $B[u] := B(\cdot, u, Du)$. It is obvious that $h(x_0) \geq h(y) > 0$. Since $u = \varphi$ on $\partial \Omega$, similarly to (2.20) and (2.21), we also have
\[
\nabla^2 (u - \varphi) = D_\gamma (u - \varphi) \mathcal{E}
\]
\[
\gamma
\]
on \(\partial \Omega\), and
\[
M'_{u}[u] := \nabla^2 u - A(x, u, Du)
\]
\[
= \{D_{\gamma}(u - \varphi)\gamma + \nabla_{\alpha\beta}\varphi - A_{\alpha\beta}(x, \varphi, D'\varphi, -D_{\alpha}(u))\}_{\alpha,\beta = 1, \ldots, n-1}.
\]

Similar to (2.19), we can write a symmetric matrix \(r \in S^n\) as
\[
r := \left( \begin{array}{c}
    r' \\
    r_{\alpha,n} \\
    r_{n,n}
\end{array} \right),
\]
where \(r' \in S^{n-1}\), \(r_{\alpha,n} = \{r_{1n}, \ldots, r_{(n-1)n}\}^T\). Let \(\Gamma' := \{r' \in S^{n-1} | r \in \Gamma\}\) be the projection cone of the cone \(\Gamma \subset S^n\) onto \(S^{n-1}\), and \(\Gamma'\) be the corresponding cone to \(\Gamma'\) in \(\mathbb{R}^{n-1}\). For any matrix \(r' \in \Gamma'\), with eigenvalues \(\lambda_1, \ldots, \lambda_{n-1}\), let us now define
\[
G(r') := f_{\lambda}(1, \ldots, \lambda_{n-1}),
\]
and
\[
G^\alpha_{\alpha\beta} = \frac{\partial G}{\partial_{\alpha\beta}}, \quad G^\alpha_{\alpha\beta}(M'[u](x_0)),
\]
for \(1 \leq \alpha, \beta \leq n - 1\). Since the function \(f_{\lambda}\) is non-decreasing and concave in the cone \(\Gamma'\) from F1 and F2, then the function \(G\) is non-decreasing and concave in the cone \(\Gamma'\), see (2.27). From (2.21), we have
\[
g(x) \geq g(x_0), \quad \text{for} \ x \in \partial \Omega,
\]
namely,
\[
f_{\lambda}(\lambda'[M[u](x))] - B[u](x) \geq f_{\lambda}(\lambda[M[u](x_0)]) - B[u](x_0),
\]
on \(\partial \Omega\). From (2.29) and (2.32), we have, on \(\partial \Omega\),
\[
G(M'[u](x)) - B[u](x) \geq G(M'[u](x_0)) - B[u](x_0).
\]
From the concavity of \(G\), we then have, on \(\partial \Omega\),
\[
G^\alpha_{\alpha\beta}(\omega_{\alpha\beta}(x) - \omega_{\alpha\beta}(x_0)) - B[u](x) + B[u](x_0) \geq 0.
\]

We consider the two possible cases:

**Case 1.** \(g(x_0) \geq h(x_0)/2\). Since this inequality can provide a positive lower bound for \(g(x)\), we are done.

**Case 2.** \(g(x_0) < h(x_0)/2\). By successively using (2.24), (2.25), (2.29), the concavity of \(G\), (2.21) and (2.27), we have
\[
g(x_0) - h(x_0)
\]
\[
= \{G(M'[u](x_0)) - B[u](x_0)\} - \{G(M'[u](x_0)) - B[u](x_0)\} - [G\alpha_{\alpha\beta}(M'[u](x_0)) - B[u](x_0) + B[u](x_0)]
\]
\[
\geq G^\alpha_{\alpha\beta} \{\{M'[u](x_0)\} - \{M'[u](x_0)\} - [G\alpha_{\alpha\beta}(M'[u](x_0)) - B[u](x_0) + B[u](x_0)]
\]
\[
\geq -D_{\alpha}(u - u)(x_0) \{G\alpha_{\alpha\beta}[A_{\alpha\beta}(x_0, \varphi(x_0), D'\varphi(x_0), D_{n}u(x_0))]
\]
\[
+ B(x_0, \varphi(x_0), D'\varphi(x_0), D_{n}u(x_0))] - B(x_0, \varphi(x_0), D'\varphi(x_0), D_{n}u(x_0)) + B(x_0, \varphi(x_0), D'\varphi(x_0), D_{n}u(x_0))
\]
where the regularity of \(A\) and the convexity of \(B\) with respect to \(p\) are used in the last inequality. Assume that \(\sigma := h(y) = \min_{x \in \partial \Omega} h(x)\), then \(\sigma\) is a positive constant. Since \(g(x_0) < h(x_0)/2\), we then have
\[
g(x_0) - h(x_0) < -\frac{1}{2} \sigma.
\]
Since \(u\) can be regarded as a strictly subsolution near the boundary, we have
\[
0 < D_{n}(u - u)(x_0) \leq \kappa,
\]
for a positive constant $\kappa$ depending on $\sup |Du|$ and $\sup |Du|$. From (2.33), (2.34) and (2.37), we derive
\begin{equation}
G_{x_0}^{\alpha\beta}D_{\alpha\gamma}(x_0) + [G_{x_0}^{\alpha\beta}D_{p\alpha}A_{\alpha\beta} + D_{p\alpha}B](x_0, \varphi(x_0), D'\varphi(x_0), D_n u(x_0)) \geq \frac{\sigma}{2\kappa} > 0.
\end{equation}
Let
\begin{equation}
\vartheta(x) := G_{x_0}^{\alpha\beta}c_{\alpha\beta}(x) + [G_{x_0}^{\alpha\beta}D_{p\alpha}A_{\alpha\beta} + D_{p\alpha}B](x, \varphi(x), \nabla\varphi(x), -D_{\gamma}u(x_0)),
\end{equation}
then from (2.38), we have $\vartheta(x_0) \geq \frac{\vartheta}{\vartheta} > 0$. Since $\vartheta(x)$ is smooth near $\partial\Omega$, we can have
\begin{equation}
\vartheta(x) \geq c > 0, \quad \text{on } N \cap \partial\Omega,
\end{equation}
for some small positive constant $c$. From the regularity condition of $A$, we observe that $A_{\alpha\beta}$ is convex with respect to $p_\alpha$, for $1 \leq \alpha, \beta \leq n - 1$. Therefore, we have
\begin{equation}
A_{\alpha\beta}(x, \varphi(x), \nabla\varphi(x), -D_{\gamma}u(x_0)) - A_{\alpha\beta}(x, \varphi(x), \nabla\varphi(x), -D_{\gamma}u(x_0)) \leq D_{p\alpha}A_{\alpha\beta}(x, \varphi(x), \nabla\varphi(x), -D_{\gamma}u(x_0))(D_{p\alpha}u(x) - D_{\gamma}u(x_0)),
\end{equation}
on $\partial\Omega$, for $1 \leq \alpha, \beta \leq n - 1$. From (2.20), (2.34), (2.41) and the convexity of $B$ in $p$, we have, on $\partial\Omega$, \begin{equation}
\vartheta(x) |D_{\gamma}(u - \varphi) - D_{\gamma}(u - \varphi)(x_0)| \geq \vartheta^{-1}(x)\Theta(x), \quad \text{on } N \cap \partial\Omega,
\end{equation}
namely,
\begin{equation}
\gamma_{n}(x)D_{n}(u - \varphi)(x) + D_{n}(u - \varphi)(x_0) \geq \vartheta^{-1}(x)\Theta(x) - \sum_{\alpha=1}^{n-1} \gamma_{n}(x)D_{\alpha}(u - \varphi)(x), \quad \text{on } N \cap \partial\Omega.
\end{equation}
Since $\gamma_{n} \in [-1, -1/2]$ in $N \cap \partial\Omega$, we have
\begin{equation}
D_{n}(u - \varphi)(x) + \gamma_{n}^{-1}(x)D_{n}(u - \varphi)(x_0) \leq \gamma_{n}^{-1}(x)\vartheta^{-1}(x)\Theta(x) - \sum_{\alpha=1}^{n-1} \gamma_{n}(x)D_{\alpha}(u - \varphi)(x), \quad \text{on } N \cap \partial\Omega.
\end{equation}
From the form of the function $\Theta(x)$ in (2.42), since $\Theta(x_0) = 0$, we have
\begin{equation}
\gamma_{n}^{-1}(x)\vartheta^{-1}(x)\Theta(x) \leq \ell(x - x_0) + C|x - x_0|^2, \quad \text{on } N \cap \partial\Omega,
\end{equation}
where $\ell$ is a linear function of $x - x_0$ with $\ell(0) = 0$, and the constant $C$ depends on $\Omega, A, B, \varphi$ and $|u|_{1;\Omega}$. Since $-\gamma_{n}^{-1} \in [1, 2]$, $\gamma_{n}(x_0) = 0$ and $D_{n}(u - \varphi)(x_0) = 0$ for $\alpha = 1, \cdots, n - 1$, we have
\begin{equation}
-\gamma_{n}^{-1}(x)\sum_{\alpha=1}^{n-1} \gamma_{n}(x)D_{\alpha}(u - \varphi)(x)
\end{equation}
\begin{equation}
= -\gamma_{n}^{-1}(x)\sum_{\alpha=1}^{n-1} [\gamma_{n}(x) - \gamma_{n}(x_0)] [D_{\alpha}(u - \varphi)(x) - D_{\alpha}(u - \varphi)(x_0)]
\end{equation}
\begin{equation}
\leq C|x - x_0|^2, \quad \text{on } N \cap \partial\Omega,
\end{equation}
where the constant $C$ depends on $\Omega, \varphi$ and $M^{2}_{\Omega}$. From (2.35), (2.36) and (2.47), we have
\begin{equation}
v(x) := D_{n}(u - \varphi)(x) + \gamma_{n}^{-1}(x)D_{n}(u - \varphi)(x_0) - \ell(x - x_0) \leq C|x - x_0|^2, \quad \text{on } N \cap \partial\Omega,
\end{equation}
where the constant $C$ depends on $\Omega, A, B, \varphi, |u|_{1,\Omega}$ and $M'_2$. By extending $\varphi$ and $\gamma$ smoothly to the interior near the boundary to be constant in the normal direction, the function $v$ in (2.48) is extended to $\bar{\Omega} \cap B_\delta(x_0)$ for some small $\delta$ such that
\begin{equation}
B_\delta(x_0) \cap \partial \Omega \subset N \cap \partial \Omega.
\end{equation}
By calculation, we have
\begin{equation}
|\mathcal{L}v| \leq C(1 + \mathcal{F}), \text{ in } \Omega \cap B_\delta(x_0),
\end{equation}
where the differentiated equation (2.50) for $k = n$ is used. Recalling the barrier function $\tilde{\eta}$ in (2.50) with $a \gg b \gg 1$ and $|x|^2$ replaced by $|x - x_0|^2$, we have
\begin{equation}
\begin{aligned}
\mathcal{L}\tilde{\eta} &\leq \frac{ag}{2}(1 + \mathcal{F}), \quad \text{ in } \Omega \cap B_\delta(x_0),
\tilde{\eta} &\geq \frac{ag}{2}(1 + \mathcal{F}), \quad \text{ on } \partial \Omega \cap B_\delta(x_0),
\tilde{\eta} &> b\delta^2, \quad \text{ on } \Omega \cap \partial B_\delta(x_0).
\end{aligned}
\end{equation}
Therefore, for $a \gg b \gg 1$, we have
\begin{equation}
\begin{aligned}
\mathcal{L}v &\geq \mathcal{L}\tilde{\eta}, \quad \text{ in } \Omega \cap B_\delta(x_0),
v &\leq \tilde{\eta}, \quad \text{ on } \partial(\Omega \cap B_\delta(x_0)),
v &\equiv \tilde{\eta} = 0, \quad \text{ at } x_0.
\end{aligned}
\end{equation}
Then the maximum principle leads to
\begin{equation}
D_n v \leq D_n \tilde{\eta}, \quad \text{ at } x_0,
\end{equation}
namely
\begin{equation}
D_{nn} u(x_0) \leq C,
\end{equation}
where the constant $C$ depends on $\Omega, A, B, \varphi, u, |u|_{1,\Omega}$ and $M'_2$. Therefore, we have obtained the upper bound for all eigenvalues of $M[u](x_0)$. Then by F3, $\lambda(M[u](x_0))$ is contained in a compact subset of $\Gamma$. Hence for sufficiently large $R$, we have
\begin{equation}
g(x_0) = \min_{x \in \partial \Omega} g(x) > 0.
\end{equation}
Overall, from cases 1 and 2, we have obtained a positive lower bound $c_0 > 0$ for the function $g(x)$ defined in (2.24) on $\partial \Omega$. By Lemma 1.2 in [2] and using (2.18), there exists a constant $R_0 \geq R$ such that if $w_{\gamma\gamma}(x_0) > R_0$, we have
\begin{equation}
f(\lambda(M[u](x_0))) \geq f_{w_{\gamma\gamma}(x_0)}(\lambda'(M'[u](x_0))) - \frac{c_0}{2},
\end{equation}
where
\begin{equation}
f(\lambda(M[u])) := F(M[u]).
\end{equation}
Note that since $F$ is orthogonally invariant, $f_R(\lambda'(M'[u]))$ and $f(\lambda(M[u](x_0)))$ in (2.22) and (2.57) are well defined. Then if $w_{\gamma\gamma}(x_0) > R_0 \geq R$, from (2.22), (2.24), (2.56), (2.57) and $g(x_0) \geq c_0 > 0$, we have
\begin{equation}
F(M[u](x_0)) - B[u](x_0) \geq \frac{c_0}{2} > 0,
\end{equation}
which leads to a contradiction with (1.1). Consequently, we have $w_{\gamma\gamma}(x_0) \leq R_0$, which leads to
\begin{equation}
D_{\gamma\gamma} u(x_0) \leq C,
\end{equation}
for some constant $C$.

Since $x_0 \in \partial \Omega$ is a point where the function $g$ in (2.24) is minimized over $\partial \Omega$, we can repeat the argument from (2.46) to (2.49) at any boundary point $x \in \partial \Omega$ to get $D_{\gamma\gamma} u \leq C$ on $\partial \Omega$ for some
constant $C$. Then together with the lower bound (from the ellipticity), we finally get the pure normal second derivative estimate on the boundary,

$$|D_{\gamma\gamma}u| \leq C, \quad \text{on } \partial\Omega,$$

where the constant $C$ depends on $\Omega, A, B, \varphi, \underline{w}, |w|_{1;\Omega}$ and $M'_2$.

**Remark 2.1.** The *a priori* pure normal second derivative estimate (2.60) on $\partial\Omega$ is treated using the idea in [27]. The proof in [27] is divided into the bounded case and the unbounded case, which include concrete examples of the Hessian quotient operator and $k$-Hessian operator respectively. In [27], the bounded case is proved by using a limit function, namely replacing $g(x)$ in (2.24) by $g(x) := \lim_{R \to +\infty} f_R(\lambda'(M'[u])) - B[u]$. In this paper, the estimate (2.60) is proved in a uniform package.

Combining the estimates (2.18) and (2.60), we now have obtained the second derivative bound on the boundary

$$\sup_{\partial\Omega} |D^2u| \leq C,$$

where the constant $C$ depends on $\Omega, A, B, \varphi, \underline{w}$ and $|u|_{1;\Omega}$.

With the boundary estimate (2.61), we can now give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Since $F$ satisfies F1-F3, $A$ is regular, $B$ is convex in $p$, and $\underline{w}$ is an admissible subsolution, from case (ii) of Theorem 3.1 in [15], we have the global second derivative estimate

$$\sup_{\Omega} |D^2u| \leq C(1 + \sup_{\partial\Omega} |D^2u|),$$

where the constant $C$ depends on $A, B, F, \Omega, \underline{w}$ and $|u|_{1;\Omega}$. The full second derivative estimate (1.13) then follows from the estimates (2.61) and (2.62). We now complete the proof of Theorem 1.1.

**A strengthened barrier and its applications.** To end this section, we shall strengthen the key barrier construction in (2.12) in Lemma 2.1, which makes the proof of the pure normal derivative estimate on the boundary a bit simpler. This barrier also provides an alternative proof of the mixed tangential-normal derivative estimate on the boundary. Such a barrier is achieved by using $|\delta u|^2$, where $\delta u$ is the tangential gradient. The idea has already been used in the uniformly elliptic case in [25], in the case of curvature equations in [11, 22], and in the case of general fully nonlinear equations on Riemannian manifolds in [5].

In the proof of the pure normal second derivative estimate on $\partial\Omega$, immediately after (2.12), we define

$$v(x) := \delta(x)[D_\gamma(u - \varphi)(x) - D_\gamma(u - \varphi)(x_0)] - \Theta(x),$$

where the functions $\delta(x)$ and $\Theta(x)$ are defined in (2.39) and (2.42), respectively. Then by (2.42), we have

$$v(x) \geq 0, \quad \text{on } \partial\Omega.$$

By extending $\varphi$ and $\gamma$ smoothly to $\bar{\Omega}$ such that $|\gamma| = 1$, using (2.6) and the orthogonal invariance of $\mathcal{F}$, we have

$$|\mathcal{L}v| \leq C(1 + \mathcal{F}^*), \quad \text{in } \Omega,$$

where

$$\mathcal{F}^* = \sum_{i=1}^n F^{ii}(1 + |w_{ii}|).$$
Comparing (2.68) with the standard inequality of the form (2.10), there is an additional term $\sum_{i=1}^n F^{ij}|w_{ij}|$. We need to modify the barrier function in Lemma 2.1 to derive a barrier inequality which can control the additional term. For this purpose, we assume in addition that $\mathcal{F}$ is orthogonally invariant, and satisfies

\begin{equation}
|r \cdot F_r| \leq O(1)(\mathcal{T}(r) + |F(r)|),
\end{equation}

as $|r| \to \infty$, uniformly for $F(r) > a$, for any $a > a_0$. The condition (2.67) is a combination of the conditions (3.24) and (3.54) in [14]. Note that (2.67) is satisfied if F2 holds and either $a_0$ is finite or F4 in [14] holds, (or trivially if $\mathcal{F}$ is homogeneous). We now formulate the following lemma.

**Lemma 2.2.** Under the assumptions of Lemma 2.1, assume also that $\mathcal{F}$ is orthogonally invariant and satisfies (2.67). Then there exist positive constants $K$ and $\epsilon_2$, depending on $\Omega, A, B, u$ and $|u|_{1;\Omega}$, such that

\begin{equation}
\mathcal{L} \left[ e^{K(u-u)} + \frac{\epsilon_2}{2} |\delta u|^2 \right] \geq \epsilon_2 (1 + \mathcal{T}^*), \quad \text{in } \Omega,
\end{equation}

where $\mathcal{L}$ is the linearized operator defined in (2.3), $\delta u = Du - (D_\gamma u) \gamma$ denotes the tangential gradient of $u$, and $\mathcal{T}^*$ is defined in (2.66).

Here the unit vector field $\gamma$ in Lemma 2.2 is extended smoothly from the unit normal vector $\gamma$ on $\partial \Omega$.

**Proof of Lemma 2.2.** In view of the estimate (2.12) in Lemma 2.1, we only need to estimate $\frac{1}{2} \mathcal{L} |\delta u|^2$.

By calculations, we have

\begin{equation}
\frac{1}{2} \mathcal{L} |\delta u|^2 = \mathcal{F}^{ij} \left( a_{kl} u_{ik} u_{jl} + \beta_{ik} u_{jk} \right) + \delta_k u \mathcal{L} u_k - u_k u_l \left[ \gamma_i \mathcal{L} \gamma_k + \mathcal{F}^{ij} (D_i \gamma_k)(D_j \gamma_l) \right] \\
\geq \mathcal{F}^{ij} \left( a_{kl} u_{ik} u_{jl} + \beta_{ik} u_{jk} \right) - C(1 + \mathcal{T}),
\end{equation}

where $a_{kl} = \delta_{kl} - \gamma_k \gamma_l$, $\beta_{ik} = -2u_i (\gamma_k \gamma_j + \gamma_l \gamma_j)$, $C$ is a constant depending on $\Omega, A, B, |u|_{1;\Omega}$ and $|\gamma|_{1;\Omega}$, and (2.6) is used to obtain the inequality. Note that the estimate (2.69) can also be obtained directly from (3.8) in [14]. At any fixed point $x \in \Omega$, by choosing coordinates so that $M[u] = \{w_{ij}\}$ is diagonal at the point $x$. From the orthogonal invariance of $\mathcal{F}$, we can estimate the first term on the right hand side of (2.69),

\begin{equation}
\mathcal{F}^{ij} \left( a_{kl} u_{ik} u_{jl} + \beta_{ik} u_{jk} \right) = \mathcal{F}^{ij} \left[ a_{kl} (w_{ik} + A_{ik})(w_{jl} + A_{jl}) + \beta_{ik} (w_{jk} + A_{jk}) \right] \\
= \mathcal{F}^{ij} \left[ a_{kl} w_{ik} w_{jl} + (\beta_{ik} + 2a_{kl} A_{ik} - A_{kl}) w_{jk} + (a_{kl} A_{ik} A_{jl} + \beta_{ik} A_{jk}) \right] \\
\geq \mathcal{F}^{ij} (1 - \gamma_i^2) w_{ij}^2 - C \left( \mathcal{T} + \mathcal{F}^{ij} |w_{ij}| \right),
\end{equation}

where the constant $C$ depends on $\Omega, A, |u|_{1;\Omega}$ and $|\gamma|_{1;\Omega}$. Since $\gamma$ is a unit vector field, we can fix $k$ so that $\gamma_k^2 = \max_i \gamma_i^2 \geq \frac{1}{n}$. Then we have

\begin{equation}
1 - \gamma_i^2 \geq \frac{n-1}{n}, \quad \text{for } i \neq k.
\end{equation}
By successively using the reverse triangle inequality and the triangle inequality, we have
\[
\left| \sum_{i=1}^{n} F^{ii} w_{ii} \right| = \left| F^{kk} w_{kk} + \sum_{i \neq k} F^{ii} w_{ii} \right|
\geq \left| F^{kk} w_{kk} \right| - \left| \sum_{i \neq k} F^{ii} w_{ii} \right|
\geq F^{kk} |w_{kk}| - \left| \sum_{i \neq k} F^{ii} w_{ii} \right|
= \sum_{i=1}^{n} F^{ii} |w_{ii}| - 2 \sum_{i \neq k} F^{ii} |w_{ii}|.
\tag{2.72}
\]
From (2.67), (2.71), (2.72) and Cauchy’s inequality, we have
\[
\left| \sum_{i=1}^{n} F^{ii} |w_{ii}| \right| \leq \frac{(n-1)\epsilon}{n} \sum_{i \neq k} F^{ii} w_{ii}^2 + \frac{n}{(n-1)\epsilon} T + \mu(1 + T + |F(M[u])|)
\leq \epsilon \sum_{i \neq k} F^{ii} (1 - \gamma_i^2) w_{ii}^2 + \frac{2}{\epsilon} T + \mu(1 + T + |B|)
\leq \epsilon \sum_{i=1}^{n} F^{ii} (1 - \gamma_i^2) w_{ii}^2 + \frac{2}{\epsilon} T + \mu(1 + T + |B|),
\tag{2.73}
\]
for any constant \(\epsilon > 0\), and some positive constant \(\mu\). Namely,
\[
\sum_{i=1}^{n} F^{ii} (1 - \gamma_i^2) w_{ii}^2 \geq \frac{1}{\epsilon} \sum_{i=1}^{n} F^{ii} |w_{ii}| - \frac{1}{\epsilon} \left[ \mu + (\mu + \frac{2}{\epsilon}) T + \mu |B| \right]
\tag{2.74}
\]
holds for any constant \(\epsilon > 0\). Combining (2.69), (2.70) and (2.74), we have
\[
\frac{1}{2} L |\delta u|^2 \geq \left( \frac{1}{\epsilon} - C \right) \sum_{i=1}^{n} F^{ii} |w_{ii}| - C(\mathcal{T} + 1) - \frac{1}{\epsilon} \left[ \mu + (\mu + \frac{2}{\epsilon}) T + \mu |B| \right]
\geq \left( \epsilon_1 - \epsilon_2 C' \right) (1 + \mathcal{T}) + \epsilon_2 \sum_{i=1}^{n} F^{ii} |w_{ii}|,
\tag{2.75}
\]
by fixing \(\epsilon = \frac{1}{1 + C'}\), where the constant \(C'\) depends on \(\mu, \Omega, A, B, |u|_{1;\Omega}\) and \(|\gamma|_{1;\Omega}\). By choosing \(\epsilon_2 = \frac{\epsilon}{2 \max\{C, 1\}}\) in (2.73), we get the desired estimate (2.68) and complete the proof of Lemma 2.2.

To apply Lemma 2.2 for the pure normal second derivative estimate on \(\partial \Omega\), we need to make a slight modification of the function in (2.68). Let
\[
\Phi := e^{K(u - u)} - 1 + \frac{\epsilon_2}{2} |\delta (u - u)|^2,
\tag{2.77}
\]
where the constants \(K\) and \(\epsilon_2\) are the same as in (2.68). By directly using (3.8) in [14], we can also obtain an estimate
\[
\frac{1}{2} L |\delta (u - u)|^2 \geq F^{ij} \left( a_{kl} u_{ik} u_{jl} + \beta_{ik} u_{jk} \right) - C (1 + \mathcal{T}),
\tag{2.78}
\]
where $a_{kl}$ and $\tilde{\beta}_{jk}$ are the same as in $(2.69)$, $C$ is a further constant depending on $\Omega, A, B, |u|_{1, \Omega}$ and $|\gamma|_{1, \Omega}$. Therefore, following the steps in the proof of $(2.68)$, it is readily checked that

$$(2.79) \quad \mathcal{L}\Phi \geq \epsilon_2 (1 + \mathcal{J}^*) \quad \text{in } \Omega.$$  

Moreover, it is obvious that

$$(2.80) \quad \Phi = 0 \quad \text{on } \partial \Omega.$$  

From $(2.64)$, $(2.65)$, $(2.79)$ and $(2.80)$, we have

$$(2.81) \quad \mathcal{L}(v - \tau \Phi) \leq 0 \quad \text{in } \Omega,$$

$v - \tau \Phi \geq 0 \quad \text{on } \partial \Omega,$

for sufficiently large positive constant $\tau$, which leads to

$$(2.82) \quad v - \tau \Phi \geq 0 \quad \text{in } \Omega.$$  

Since $v - \tau \Phi = 0$ at $x_0 \in \partial \Omega$, we have

$$(2.83) \quad \pm \delta_i (u - \bar{u}) \leq \sqrt{2 \left( 1 - e^{K(u - \bar{u})} \right)} / \epsilon_2 \quad \text{in } \Omega,$$

$$\pm \delta_i (u - \bar{u}) = 0 \quad \text{on } \partial \Omega,$$

for $i = 1, \cdots, n$. Then we have

$$(2.84) \quad \pm D_{\gamma} \delta_i (u - \bar{u}) \geq D_{\gamma} \sqrt{2 \left( 1 - e^{K(u - \bar{u})} \right)} / \epsilon_2 \quad \text{on } \partial \Omega,$$

for $i = 1, \cdots, n$, where $\gamma$ is the unit outer normal vector field on $\partial \Omega$. Hence, from $(2.86)$ we have

$$(2.87) \quad |D_{\tau\gamma} u| \leq C \quad \text{on } \partial \Omega,$$

for any unit tangential vector field $\tau$ on $\partial \Omega$.

**Remark 2.2.** Under the additional assumptions that $\mathcal{F}$ is orthogonally invariant and $(2.67)$ holds, we derive the strengthened barrier inequality $(2.68)$, and further provide alternative proofs of the mixed tangential-normal derivatives and the pure normal derivatives on $\partial \Omega$. When $F_2$ holds and $a_0$ is finite, condition $(2.67)$ is automatically satisfied, (see $(1.10)$ in [14]). Note that in Theorem 1.1 we already assumed that $\mathcal{F}$ is orthogonally invariant and $\mathcal{F}_2$ holds. Therefore, when $a_0$ is finite, the pure normal
derivative estimate (2.84) and the mixed tangential-normal derivative estimate (2.87) can be used directly to obtain the full second order derivative estimate (1.13) in Theorem 1.1.

Remark 2.3. Note that in the Riemannian manifold case, we would encounter this type of estimate (2.65) in the course of estimating the mixed tangential-normal derivatives and the pure normal derivatives on the boundary, where the additional term $\sum_{i=1}^{n} F_{ii}|w_{ii}|$ can not be avoided. Therefore, such kind of barrier in (2.68) is useful in the mixed tangential-normal derivative estimate and the pure normal derivative estimate on the boundary for the Riemannian manifold case. For the Dirichlet problem (1.1)-(1.2) on Riemannian manifold, we refer the reader to [5–8] for more detailed discussions.

3. Gradient estimates and existence theorem

In this section, we discuss the gradient estimates for admissible solutions under appropriate growth conditions of $A$ and $B$ with respect to $p$, and then combine all the derivative estimates to prove the existence result, Theorem 1.2.

When $F$ satisfies F7, we have the global gradient estimate under the growth conditions (1.14) and (1.15) for $A$ and $B$.

**Theorem 3.1.** Assume that $F$ is orthogonally invariant satisfying F1, F3 and F7, $A, B \in C^{1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ satisfying (1.14) and (1.15), $B > a_0$, $u \in C^{3}(\Omega) \cap C^{2}(\bar{\Omega})$ is an admissible solution of equation (1.1). Then we have the gradient estimate

\[
\sup_{\Omega} |Du| \leq C(1 + \sup_{\partial \Omega} |Du|),
\]

where the constant $C$ depends on $F, A, B, \Omega$ and $|u|_{0,\Omega}$.

The proof of Theorem 3.1 here follows directly from the proof of Theorem 1.3(ii) in Section 3 of [14] and the remark of the case when discarding the boundary condition in Remark 3.1 in [14]. However it should be noted that our condition (1.14) is written more generally than the corresponding conditions (3.31) and (3.33) in Remark 3.1(ii’) in [14]. The replacement of (3.31) by the last two inequalities in (1.14) is immediate from (3.32) in [14] while the replacement of (3.33) by the corresponding inequality in (1.14) follows by examination the derivation of (3.42), in the case $g = |Du|^2$, and is readily seen by multiplying through inequality (3.38), (in the general case), by $u_i - \varphi_i$.

If we replace F7 by F2 in Theorem 3.1, we still need to assume, when $B$ is unbounded, condition F5 in [14] with $b = \infty$, that is

**F5(\infty):** For a given constant $a > a_0$, there exists a constant $\delta_0 > 0$ such that $\mathcal{J}(r) \geq \delta_0$ if $a < F(r)$, which is implied by F7.

We also need some control from below on $r \cdot F_r$, as in condition (3.54) in [14], namely

\[
r \cdot F_r \geq o(|\lambda_0(r)|) \mathcal{J}(r),
\]

as $\lambda_0(r) \to -\infty$, uniformly for $F(r) > a$, for any $a > a_0$, where $\lambda_0(r)$ denotes the minimum eigenvalue of $r$. Note that if F1-F3 hold with $a_0$ finite, we have $r \cdot F_r \geq 0$, so (3.2) is trivially satisfied.

We then have as an alternative to Theorem 3.1.
Theorem 3.2. Assume that $F$ is orthogonally invariant satisfying $F1$-$F3$, $F5(\infty)$ and (3.2), $A, B \in C^4(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfying (1.14) and (1.15), with “$O$” replaced by “$o$” in (1.14), $B > a_0$, $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ is an admissible solution of equation (1.1). Then we have the gradient estimate (3.1), where the constant $C$ depends on $F, A, B, \Omega$ and $|u|_{\Omega}^2$.

Proof of Theorem 3.2. The proof of Theorem 3.2 here follows from a slight modification of the proof of Theorem 3.1. The technical details are somewhat simpler as we can employ an auxiliary function of the form,

\begin{equation}
(3.3) \quad v := \left| Du \right|^2 + \alpha M_1^2 \eta,
\end{equation}

in $\Omega$, where $\eta = u - u_0$, $M_1 = \sup \limits_{\Omega} \left| Du \right|$, $u_0 = \inf \limits_{\Omega} u$ and $\alpha$ is a positive constant satisfying

\begin{equation}
(3.4) \quad \alpha \leq \frac{1}{2 \text{ osc}_{\Omega} \eta}.
\end{equation}

In place of (3.32) in (14), we now obtain from our strengthening of (1.14),

\begin{equation}
(3.5) \quad \mathcal{L} \eta \geq F^{ij} w_{ij} - C(1 + \mathcal{G})\left(\omega |Du|^2 + 1\right),
\end{equation}

where $C$ is a positive constant and $\omega = \omega(|Du|)$ a positive decreasing function on $[0, \infty)$ tending to 0 at infinity, depending on $A, B$ and $\Omega$. We consider the case that the maximum of $v$ occurs at a point $x_0 \in \Omega$. Following the proof of case (ii) of Theorem 1.3 in (14) with $g = |Du|^2$ and our simpler $\eta$, we obtain, in place of inequality (3.42) in (14),

\begin{equation}
(3.6) \quad w_{11} \leq -\frac{1}{2} \alpha M_1^2 + C(\omega |Du|^2 + 1).
\end{equation}

Now we observe that the estimate (3.6) is clearly applicable to the minimum eigenvalue $w_{kk}$ of $M[u]$ and moreover by F2 we must have $F^{kk} \geq \mathcal{G}/n$; (see (30) and Remark 3.1 below). Retaining the term $\mathcal{E}_2 = F^{ij} u_{ik} u_{jk}$ in (3.9) in (14), instead of using (3.43) in (14), we now obtain at $x_0$, in place of inequality (3.45) in (14), using $F(\infty)$, (3.2) and (3.4),

\begin{equation}
(3.7) \quad 0 \geq \mathcal{L} v \geq \mathcal{E}_2' - C(1 + \mathcal{G})(\omega |Du|^4 + 1) + \alpha M_1^2 [F^{ij} w_{ij} - C(1 + \mathcal{G})(\omega |Du|^2 + 1)] \\
\geq \frac{1}{n} \alpha^2 M_1^4 - C(\omega |Du|^4 + 1)|\mathcal{G}|,
\end{equation}

and we conclude $M_1 \leq C$ as desired. \hfill \Box

Remark 3.1. The concavity F2 and orthogonal invariance of $F$ to imply that if $F(r) = f(\lambda)$, where $\lambda = (\lambda_1, \cdots, \lambda_n)$ denote the eigenvalues of $r \in \Gamma$, then $D_i f \leq D_j f$ at any fixed point $\lambda$, where $\lambda_i \geq \lambda_j$.

Indeed, by applying the mean value theorem to the function $g = D_i f - D_j f$ at the points $\lambda$ and $\lambda^*$ where $\lambda^*$ is given by exchanging $\lambda_i$ and $\lambda_j$ in $\lambda$, we have

\begin{equation}
(3.8) \quad g(\lambda) - g(\lambda^*) = Dg(\hat{\lambda})(\lambda - \lambda^*) = [D_{ii} f(\hat{\lambda}) + D_{jj} f(\hat{\lambda}) - 2D_{ij} f(\hat{\lambda})](\lambda_i - \lambda_j) \leq 0,
\end{equation}

where $\hat{\lambda} = \theta \lambda + (1 - \theta) \lambda^*$ for some constant $\theta \in (0, 1)$, F2 and $\lambda_i \geq \lambda_j$ are used to obtain the inequality.

Since $g(\lambda^*) = -g(\lambda)$ holds by symmetry of $f$, (3.8) implies $g(\lambda) \leq 0$, and hence $D_i f(\lambda) \leq D_j f(\lambda)$. 

\textbf{16}
With these a priori derivative estimates in Theorems 1.1 and 3.1, we can now give the proof of the existence result, Theorem 1.2, using method of continuity.

Proof of Theorem 1.2. First, we need to establish the solution bound and the full gradient bound. The bounded viscosity solution \( \bar{u} \) and the subsolution \( u \) can provide the solution bound, namely
\[
(3.9) \quad u \leq \bar{u} \leq u, \text{ in } \bar{\Omega}.
\]
For the gradient estimate on \( \partial \Omega \), the tangential derivatives of \( u \) are given by the Dirichlet boundary condition and the inner normal derivative bound from below is controlled by using the subsolution \( u \). From the admissibility of \( u \) and \( \Gamma \subset \Gamma_1 \), we have \( \Delta u \geq \text{trace}(A) \), which leads to an inner normal derivative estimate of \( u \) from above on \( \partial \Omega \), under quadratic structure conditions of \( \text{trace}(A) \) with respect to \( p \), see proof of Theorem 14.1 in [3]. We then obtain the gradient estimate of \( u \) on \( \partial \Omega \),
\[
(3.10) \sup_{\Omega} |Du| \leq C,
\]
where the constant \( C \) depends on \( \Omega, \varphi \) and \( |u|_{1,\Omega} \). If (1.14) and (1.15) hold, the global gradient estimate (3.1) holds in Theorem 3.1. If \( \Gamma = K^+ \) and \( A(x,p) \geq O(|p|^2)I \) as \( |p| \to \infty \), uniformly for \( x \in \Omega \), the global gradient estimate (3.1) holds in Section 4 in [18]. Combining the solution estimates (3.9), global gradient estimate (3.1), and the boundary gradient estimate (3.10), we obtain
\[
(3.11) \quad \sup_{\Omega} |u| + \sup_{\partial \Omega} |Du| \leq C,
\]
where the constant \( C \) depends on \( F, A, B, \Omega, \varphi, \bar{u} \) and \( |u|_{1,\Omega} \).

Then from the lower order estimate (3.11) and the second derivative estimate (1.13), we have uniform estimates in \( C^2(\bar{\Omega}) \) for classical admissible solutions of the Dirichlet problems
\[
(3.12) \quad F[u] = tB(\cdot, Du) + (1-t)F[u], \quad \text{in } \Omega,
\]
\[
(3.13) \quad u = \varphi, \quad \text{on } \partial \Omega,
\]
for \( 0 \leq t \leq 1 \), where \( u \) is a subsolution. From the Evans-Krylov theorem, (Theorem 17.26’ in [3]), we have the Hölder estimate for second derivatives of the admissible solution to the Dirichlet problem (3.12)-(3.13). Then the existence follows from the method of continuity, (Theorem 17.8 in [3]), and the uniqueness follows from the maximum principle.

Moreover, if \( \Gamma = \Gamma_k \) for \( k > n/2 \), we have the continuity estimate \( |u(x)-u(y)| \leq C|x-y|^a(R^{-a} \text{osc}_{\Omega \cap B_R} u + 1) \) in (i), (iii) of Lemma 3.1 in [14]. By combining this continuity estimate and the local gradient estimates, we can still obtain the gradient estimate by replacing (1.14) by (1.16) and extending “o” to “O” in (1.15), (see the last part of Theorem 3.1 in [14]). We then obtain the existence and uniqueness of a classical admissible solution and complete the proof.

With the alternative gradient estimate in Theorem 3.2, we state the following existence result as a corollary of Theorem 1.2.
Corollary 3.1. Assume that \( F \) is orthogonally invariant and satisfies F1-F3, F5(\( \infty \)) and (3.2) in \( \Gamma \subset \Gamma_1 \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with \( \partial \Omega \in C^4 \), \( A \in C^2(\overline{\Omega} \times \mathbb{R}^n) \) is regular in \( \overline{\Omega} \), \( B > a_0, \in C^2(\overline{\Omega} \times \mathbb{R}^n) \) is convex with respect to \( p \). Assume there exist a bounded viscosity supersolution \( \bar{u} \) and a subsolution \( u \in C^2(\bar{\Omega}) \) satisfying \( u = \phi \) on \( \partial \Omega \) with \( \phi \in C^4(\partial \Omega) \). Assume also (1.14) and (1.15) hold, with “\( O \)” replaced by “\( o \)” in (1.14). Then there exists a unique admissible solution \( u \in C^3(\bar{\Omega}) \) of the Dirichlet problem (1.1)-(1.2).

If \( a_0 \) is finite, condition (3.2) in Corollary 3.1 can be dispensed with as it is automatically satisfied. If \( B \) is bounded, F5(\( \infty \)) in Corollary 3.1 can be replaced by F5. Recalling that F5 is implied by F1, F2 and F3 when \( a_0 \) is finite (see Section 4.2 in [14]), hence we can replace “F1-F3, F5(\( \infty \)) and (3.2)” by “F1-F3” in Corollary 3.1 in the case when \( a_0 \) is finite and \( B \) is bounded.

Remark 3.2. Note that Theorem 1.2 and Corollary 3.1 embrace many examples of matrices \( A \) and operators \( F \). One can refer to [14,23,24,29] for the examples of the matrices \( A \) and [14] for the examples of the operators \( F \). When the operator \( F \) is given by \( \log \det \), (or \( \det^{1/n} \)), and \( A \) is a matrix generated by the cost function in the optimal transportation problem, the global Schauder estimate for Dirichlet problem (1.1)-(1.2) was obtained in [9] under the strictly regular condition. It will be interesting to study the global Schauder estimate for the Dirichlet problem (1.1)-(1.2), when \( F \) satisfies F1-F3 and is neither the Laplacian operator \( \Delta \) nor the Monge-Ampère operator \( \log \det \), (or \( \det^{1/n} \)), and \( A \) satisfies the regular (or strictly regular) condition.

Remark 3.3. As in [14], the main examples of the admissible cones \( \Gamma \) in Theorem 1.1 and Theorem 1.2 are the Gårding’s cones \( \Gamma_k \) and the \( k \)-convex cones \( P_k \), which are defined by (1.7) and

\[
(3.14) \quad P_k := \left\{ r \in S^n \mid \sum_{i=1}^{k} \lambda_{i_k} > 0 \right\},
\]

where \( i_1, \cdots, i_k \subset \{ 1, \cdots, n \} \), \( \lambda(r) = (\lambda_1(r), \cdots, \lambda_n(r)) \) denote the eigenvalues of the matrix \( r \in S^n \).

Note that these two kinds of cones \( \Gamma_k \) and \( P_k \) satisfy \( \Gamma_n \subset \Gamma_k \subset \Gamma_1 \) and \( \Gamma_n \subset P_k \subset \Gamma_1 \) for \( k = 1, \cdots, n \).

For the background and inclusion relations of the cones \( \Gamma_k \) and \( P_k \), one can refer to [21].

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