ELLiptic CURves OVER $\mathbb{Q}$ AND 2-ADIC IMages OF galois

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abstract. we give a classification of all possible 2-adic images of galois representations associated to elliptic curves over $\mathbb{Q}$. to this end, we compute the ‘arithmetically maximal’ tower of 2-power level modular curves, develop techniques to compute their equations, and classify the rational points on these curves.

1. introduction

Serre proved in [Ser72] that, for an elliptic curve $E$ over a number field $K$ without complex multiplication, the index of the mod $n$ galois representation $\rho_{E,n}$ associated to $E$ is bounded – there is an integer $N_E$ such that for any $n$, the index of $\rho_{E,n}(G_K)$ in $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$ is at most $N_E$ (equivalently, the mod $\ell$ representation is surj ective for large $\ell$). Serre’s proof is ineffective in the sense that it does not compute $N_E$ explicitly; in fact one conjectures that for $\ell > 37$, $\rho_{E,\ell}$ is surjective. The early progress on this problem [Maz78] has recently been vastly extended [BPR11], but a proof in the remaining case – to show that the image cannot be contained in the normalizer of a non-split Cartan – is elusive and inaccessible through refinements of Mazur’s method.

Mazur’s [Maz77, Program B] (given an open subgroup $H \subset \text{GL}_2(\hat{\mathbb{Z}})$, classify all elliptic curves $E/K$ such that the image of $\rho_E = \varprojlim_n \rho_{E,n}$ is contained in $H$) suggests a more general uniformity conjecture – one expects that for every number field $K$, there exists a constant $B(K)$ such that for every elliptic curve $E/K$ without complex multiplication, the index of $\rho_E(G_K)$ in $\text{GL}_2(\hat{\mathbb{Z}})$ is bounded by $B(K)$.

Computational evidence supports the uniformity conjecture – for any given $E$, [Zyw11b] gives an algorithm (implemented in Sage) to compute the set of primes $\ell$ such that $\rho_{E,\ell}$ is not surjective, and verifies for non-CM $E$ with $j(E), N_E \leq 140000$ that $\rho_{E,\ell}$ is surjective for $\ell > 37$. Similarly, for small $\ell$ one can compute im $\rho_{E,\ell}$ directly; [Sut13] has computed im $\rho_{E,\ell}$ for every elliptic curve in the Cremona and Stein-Watkins databases for all primes $\ell < 80$. This is a total of 139 million curves, and Sutherland’s results are now listed in Cremona’s tables. In Appendix A we describe a method using [DD10] that can often provably compute the mod $n$ image of galois for any elliptic curve.

Complementing this are various results (going as far back as Fricke, possibly earlier; see [Maz77, Footnote 1]) computing equations for the modular curve $X_H$ parameterizing $E$ with $\rho_E(G_K) \subset H$ (see Section 2 for a definition). For instance, [BLS10] have extended the range of $\ell$ such that one can compute the modular polynomial $\Phi_\ell(X,Y)$ to $\ell \approx 10,000$ and Sutherland now maintains tables of equations for modular curves (see e.g. [Sut, Sut12]). Recently [DD12] (inspired by the earlier 3-adic analogue [Elk06]) computed equations for the modular curves necessary to compute whether the mod 8, and thus the 2-adic, image
of Galois is surjective (i.e. equations for $X_H$ with reduction $H(8) \subset \text{GL}_2(\mathbb{Z}/8\mathbb{Z})$ a maximal subgroup). (See Remark 4.1 for more such examples.)

In many cases these equations have been used to compute the rational points on the corresponding curves; see Remark 4.1 for some examples. Applications abound. In addition to verifying low level cases of known classification theorems such as [Maz78] (in this spirit we note the outstanding case of the “cursed” genus 3 curve $X^+_n(13)$ [BPR11], Remark 4.10, [Bar11]) and verifying special cases of the uniformity problem, various authors have used the link between integral points on modular curves and the class number one problem to give new solutions to the class number one problem; see [Ser97, A.5], and more recently [Bar10], [Bar09], [Che99], [ST12], and [Ken85].

**Main theorem.** In the spirit of Mazur’s ‘Program B’, we consider a “vertical” variant of the uniformity problem. For any prime $\ell$ and number field $K$, it follows from Faltings’ Theorem and a short argument (see [Zyw11a, Lemma 5.1]) that there is a bound $N_{\ell,K}$ on the index of the image of the $\ell$-adic representation associated to any elliptic curve over $K$. The uniformity conjecture implies that for $\ell > 37$, $N_{\ell,Q} = 1$, but $N_{\ell}$ can of course be larger for $\ell \leq 37$. Actually even more is true – the uniformity conjecture would imply the existence of a universal constant $N$ bounding the index of $\rho_{E,n}(\text{GL}_2(\mathbb{Z}_2))$ for every $n$ (equivalently, bounding the index of $\rho_{E}(\text{GL}_2(\mathbb{Z}_2))$; see [Zyw11a, Theorem 1.1(iv)]).

In this spirit, we give a complete classification of the possible 2-adic images of Galois representations associated to non-CM elliptic curves over $\mathbb{Q}$ and, in particular, compute $N_{2,Q}$.

**Theorem 1.1.** Let $E$ be an elliptic curve over $\mathbb{Q}$ without complex multiplication. Then there are exactly 1208 possibilities for the 2-adic image $\rho_{E,2}\sim(G_Q)$ (up to conjugacy in $\text{GL}_2(\mathbb{Z}_2)$).

See [RZB] for a complete list of these subgroups.

**Corollary 1.2.** Let $E$ be an elliptic curve over $\mathbb{Q}$ without complex multiplication. Then the index of $\rho_{E,2}\sim(G_Q)$ divides 64 or 96; all such indices occur. Moreover, the image of $\rho_{E,2}\sim(G_Q)$ is the inverse image in $\text{GL}_2(\mathbb{Z}_2)$ of the image of $\rho_{E,32}(G_Q)$.

**Remark 1.3.** All indices dividing 96 occur for infinitely many elliptic curves. For the $j$-invariants

$$j = 2^{11}, 2^4 \cdot 17^3, 4097^3, 257^3, \frac{2^8}{2^8}, \frac{857985^3}{628^8}, \text{ and } \frac{919425^3}{496^4};$$

the index of the image is 96, and for these, $-I \in H$ and this index occurs for all quadratic twists; moreover, the six mod 32 images for these six $j$-invariants are distinct. For the last two $j$-invariants listed above, the index of the mod 32 image is larger than the index of the mod 16 image. Additionally, there are several subgroups $H$ with $-I \notin H$ and $X_H \cong \mathbb{P}^1$, so that the there are infinitely many $j$-invariants such that the index is 96. Index 64 only occurs for the two $j$-invariants

$$j = -3 \cdot 2^{18}, 5^3 \cdot 13^3, 3^3 \cdot 17^{16} \text{ and } j = -7 \cdot 2^{21}, 3^3 \cdot 5^3, 13^3, 23^3, 41^3, 179^3, 409^3, 79^{-16}$$

which occur as the two non-cuspidal non-CM rational points on the genus 2 curve $X^+_n(16)$ ($X_{441}$ on our list; see the analysis of Subsection 8.3), which classifies $E$ whose mod 16 image is contained in the normalizer of a non-split Cartan. (The second $j$-invariant was missed in
Baran, because the map from \(X^+_n(16)\) to the \(j\)-line was not correctly computed. In this computation, Baran relied on earlier computations of Heegner, and the error could be due to either of them.) The smallest conductor of an elliptic curve with this second \(j\)-invariant is \(7^2 \cdot 79 \cdot 106123^2\) (which is greater than \(4 \cdot 10^{13}\)).

**Remark 1.4 (Failure of Hilbert irreducibility for a non-rational base).** A surprising fact is that not every subgroup \(H\) such that \(X_H(\mathbb{Q})\) is infinite occurs as the image of Galois of an elliptic curve over \(\mathbb{Q}\); see Section 6.

In preparation by other authors is a related result [SZ] – for every subgroup \(H \subset \text{GL}_2(\mathbb{Z}_\ell)\) such that \(-I \in H\), \(\det(H) = \hat{\mathbb{Z}}^\times\), and \(X_H\) has genus 0, they compute equations for \(X_H\), whether \(X_H(\mathbb{Q}) = \emptyset\) and, if not, equations for the map \(X_H \to X(1)\).

**Proof of Theorem 1.1.** For a subgroup \(H\) of \(\text{GL}_2(\mathbb{Z}_2)\) of finite index, there is some \(k\) such that \(\Gamma(2^k) \subset H\). The non-cuspidal points of the modular curve \(X_H := X(2^k)/H\) then roughly classify elliptic curves whose 2-adic image of Galois is contained in \(H\); see Section 2 for a more precise definition.

The idea of this paper is to find all of the rational points on the “tower” of 2-power level modular curves (see Figure 1). We only consider subgroups \(H\) such that \(H\) has surjective determinant and contains an element with determinant \(-1\) and trace zero (these conditions are necessary for \(X_H(\mathbb{Q})\) to be non-empty). In our proof, we will handle the case \(-I \in H\) first; see Subsection 2.1 for a discussion of \(X_H\) and the distinction between the cases \(-I \in H\) and \(-I \not\in H\).

**Proof of Theorem 1.1.** The proof naturally breaks into the following steps.

(1) (Section 3) First we compute a collection \(\mathcal{C}\) of open subgroups \(H \subset \text{GL}_2(\mathbb{Z}_2)\) such that every open \(K \subset \text{GL}_2(\mathbb{Z}_2)\) which satisfies the above necessary conditions and which is not in \(\mathcal{C}\) is contained in some \(H \in \mathcal{C}\) such that \(X_H(\mathbb{Q})\) is finite. (See Figure 1 for those with \(-I \in H\).)

(2) (Section 4) Next, we compute, for each \(H \in \mathcal{C}\) equations for (the coarse space of) \(X_H\) and, for any \(K\) such that \(H \subset K\), the corresponding map \(X_H \to X_K\).

(3) (Section 5) Then, for \(H \in \mathcal{C}\) such that \(-I \not\in H\) we compute equations for the universal curve \(E \to U\), where \(U \subset X_H\) is the locus of points with \(j \neq 0, 1728\) or \(\infty\).

(4) (Remainder of paper.) Finally, with the equations in hand, we determine \(X_H(\mathbb{Q})\) for each \(H \subset \mathcal{C}\). The genus of \(X_H\) can be as large as 7.

(5) (Appendix.) If we find a non-cuspidal, non-CM rational point on a curve \(X_H\) with genus \(\geq 2\), we use computations of resolvent polynomials (as described in [DD10]) to prove that the 2-adic image for the corresponding elliptic curve \(E\) is \(H\).

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2. The Modular curves $X_H$

For an integer $N$, we define the modular curve $Y(N)/\mathbb{Q}$ to be the moduli space parameterizing pairs $(E/S, \iota)$, where $E$ is an elliptic curve over some base scheme $S/\mathbb{Q}$ and $\iota$ is an isomorphism $(\mathbb{Z}/N\mathbb{Z})^2_\iota \cong E[N]$, and define $X(N)$ to be its smooth compactification (see [DR73 II] for a modular interpretation of the cusps). Note that $X(N)$ is connected but not geometrically connected (so differs from the geometrically connected variant of [Maz77 Section 2] where $\iota$ is “canonical” in that it respects the Weil pairing), and that $X(N)$ has an action of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ given by pre-composition with $\iota$.

Define $\Gamma(N) \subseteq \text{GL}_2(\mathbb{Z})$ to be the set of $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (mod $N$). (This differs from the usual definition, where $\Gamma(N)$ is defined to be a subgroup of $\text{SL}_2(\mathbb{Z})$.) Following [DR73], for a finite index subgroup $H$ of $\text{GL}_2(\mathbb{Z})$ and an integer $N$ such that $\Gamma(N) \subset H$, we define $X_H$ to be the quotient of the modular curve $X(N)$ by the image $H(N)$ of $H$ in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. This quotient is independent of $N$, is geometrically connected if $\text{det}(H) = \mathbb{Z}_x$, and roughly classifies elliptic curves whose adelic image of Galois is contained in $H$. More precisely, non-cuspidal $K$-rational points of $X_H$ correspond to $G_K$-stable $H$-orbits of pairs $(E/K, \iota)$; moreover, given an elliptic curve $E/K$, there exists an $\iota$ such that $(E, \iota) \in X_H(K)$ if and only if $\text{im} \rho_{E,n}$ is contained in a subgroup conjugate to $H$. (For a proof of this equivalence see [Bar10 Section 4].)

In general $X_H$ is a stack, and if $-I \notin H$, then the stabilizer of every point contains $\mathbb{Z}/2\mathbb{Z}$. (Some, but not all, CM points will have larger stabilizers.) In contrast, when $-I \notin H$, $X_H$ no longer has a generic stabilizer, but is generally still a stack since the CM points may have stabilizers. When $-I \in H$, quadratic twisting preserves the property that $\text{im} \rho_{E,n} \subset H$; in contrast, when $-I \notin H$, given a non-CM elliptic curve $E/K$ such that $j(E)$ is in the image of the map $j : X_H(K) \rightarrow \mathbb{P}^1(K)$, there is a unique quadratic twist $E_d$ of $E$ such that $\text{im} \rho_{E,n} \subset H$ (see Lemma 5.1 below).

There exists a coarse space morphism, i.e. a morphism $\pi : X_H \rightarrow X$, where $X$ is a scheme, the map $X_H(\overline{\mathbb{Q}}) \rightarrow X(\overline{\mathbb{Q}})$ is a bijection, and any map from $X_H$ to a scheme uniquely factors through this morphism. We compute equations for the coarse space of $X_H$ (and with no confusion will use the same notation $X_H$ for the coarse space). The coarse space has the following moduli interpretation – given a $K$-point $t$ of the coarse space, any elliptic curve with $j$-invariant $j(t)$ (where $j$ is the map $X_H \rightarrow X(1)$) is going to satisfy $\text{im} \rho_{E,n} \subset H$, and conversely, for any $E/K$ such that $\text{im} \rho_{E,n} \subset H$, there exists a $K$-point $t$ of the coarse space of $X_H$ such that $j(t) = j(E)$.

For more details see [DR73 IV-3]; alternatively, for a shorter discussion see [Bar10 Section 3], [Ser97 A.5], or [Maz77 Section 2].

2.1. Universal curves. Suppose that $-I \notin H$. Since we are not interested in the CM points anyway, we consider the complement $U \subset X_H$ of the cusps and preimages on $X_H$ of $j = 0$ and $j = 1728$. Then $U$ is a scheme, so there exists a universal curve $E \rightarrow U$; i.e. a surface $E$ with a map $E \rightarrow U$ such that for every $t \in U(K)$, the fiber $E_t$ is an elliptic curve over $K$ without CM such that $\text{im} \rho_{E,n} \subset H$, and conversely for any elliptic curve $E$ over a field $K$ such that $\text{im} \rho_{E,n} \subset H$ there exists a (non-unique) $t \in U(K)$ such that the $E \cong E_t$. 
In preparation for Section 5 (where we compute equations for \( \mathcal{E} \to U \)), we prove a preliminary lemma on the shape of the defining equations of \( \mathcal{E} \).

**Lemma 2.2.** Let \( f: \mathcal{E} \to U \) be as above and assume that \( U \subset \mathbb{A}^1 \). Then there exists a closed immersion \( \mathcal{E} \to \mathbb{P}^2_U \) given by a homogeneous polynomial

\[
YZ^2 - X^3 - aXZ^2 - bZ^3
\]

where \( a, b \in \mathbb{Z}[t] \).

**Proof.** The identity section \( e: U \to \mathcal{E} \) is a closed immersion whose image \( e(U) \) is thus a divisor on \( \mathcal{E} \) isomorphic to \( U \). By Riemann-Roch, the fibers of the pushforward \( f_* \mathcal{O}(3e(U)) \) are all 3-dimensional, so by the theorem on cohomology and base change \( f_* \mathcal{O}(3e(U)) \) is a rank 3 vector bundle on \( U \). Since \( U \subset \mathbb{A}^1 \), \( U \) has no non-trivial vector bundles and so \( f_* \mathcal{O}(3e(U)) \) is trivial. Let \( \mathcal{O}_U^{\oplus 3} \cong f_* \mathcal{O}(3e(U)) \) be a trivialization given by sections \( 1, x, y \), where \( 1 \) is the constant section \( 1 \) (given by adjunction), \( x \) has order 2 along \( e(U) \), and \( y \) has order 3. These sections determine a surjection \( f^* f_* \mathcal{O}(3e(U)) \to \mathcal{O}(3e(U)) \) and thus a morphism \( \mathcal{E} \to \mathbb{P}^2_U \), which, since the fibers over \( U \) are closed immersions, is also a closed immersion; \( 1, x, y \) satisfy a cubic equation (this is true over the generic point, so true globally) and, since we are working in characteristic 0, can be simplified to short Weierstrass form as desired. \( \square \)

### 3. Subgroups of \( \text{GL}_2(\mathbb{Z}_2) \)

**Definition 3.1.** Define a subgroup \( H \subset \text{GL}_2(\mathbb{Z}_2) \) to be **arithmetic maximal** if

1. \( \det: H \to \mathbb{Z}_2^\times \) is surjective,
2. there is an \( M \in H \) with determinant \(-1\) and trace zero, and
3. there is no subgroup \( K \) with \( H \subseteq K \) so that \( X_K \) has genus \( \geq 2 \).

If \( E/\mathbb{Q} \) is an elliptic curve and \( H = \rho_{E,2\infty}(G_{\mathbb{Q}}) \), then the properties of the Weil pairing prove that \( \det: H \to \mathbb{Z}_2^\times \) is surjective. Also, the image of complex conjugation in \( H \) must be a matrix \( M \) with \( M^2 = I \) and \( \det(M) = -1 \). This implies that the trace of \( M \) equals zero.

**Remark 3.2.** After the subgroup and model computations were complete, David Zywina and Andrew Sutherland pointed out that if \( E/\mathbb{Q} \) is an elliptic curve, complex conjugation fixes an element of \( E[n] \). This gives further conditions on a matrix \( M \) that could be the image of complex conjugation, and rules out a handful of other subgroups.

We enumerate all of the arithmetically maximal subgroups of \( \text{GL}_2(\mathbb{Z}_2) \) by initializing a queue containing only \( H = \text{GL}_2(\mathbb{Z}_2) \). We then remove a subgroup \( H \) from the queue, compute all of the open maximal subgroups \( M \subset H \). We add \( M \) to our list of potential subgroups if (i) \( \det: M \to \mathbb{Z}_2^\times \) is surjective, (ii) \(-I \in M \), (iii) \( M \) contains a matrix with determinant \(-1\) and trace zero, and (iv) if \( M \) is not conjugate in \( \text{GL}_2(\mathbb{Z}_2) \) to a subgroup already in our list. If the genus of \( X_M \) is zero or one, we also add \( M \) to the queue. We proceed until the queue is empty.

To enumerate the maximal subgroups, we use the following results. Recall that if \( G \) is a profinite group, then \( \Phi(G) \), the Frattini subgroup of \( G \), is the intersection of all open maximal subgroups of \( G \). Proposition 2.5.1(c) of [Wil98] states that if \( K \leq G, H \leq G \) and \( K \subset \Phi(H) \), then \( K \subset \Phi(G) \). Applying this with \( H = N \leq G \) and \( K = \Phi(N) \), we see that \( \Phi(N) \subset \Phi(G) \).
Lemma 3.3. Suppose that \( \Gamma(2^k) \subseteq H \subseteq G \) and \( k \geq 2 \). If \( K \) is a maximal subgroup of \( H \), then \( \Gamma(2^{k+1}) \subseteq K \).

Proof. We have that \( \Gamma(2^k) \leq H \) and by the above argument, we have
\[
\Phi(\Gamma(2^k)) \subseteq \Phi(H).
\]
Now, \( \Gamma(2^k) \) is a pro-2 group and this implies that every open maximal subgroup of \( \Gamma(2^k) \) has index 2. Hence,
\[
\Phi(\Gamma(2^k)) \supseteq \Gamma(2^k)^2.
\]
If \( g \in \Gamma(2^k) \), \( g = I + 2^kM \) for some \( M \in M_2(\mathbb{Z}_2) \). Then,
\[
g^2 = I + 2^{k+1}M + 2^{2k}M^2 \equiv I + 2^{k+1}M \pmod{2^{k+2}}
\]
provided \( k \geq 2 \). Hence, the squaring map gives a surjective homomorphism \( \Gamma(2^k)/\Gamma(2^{k+1}) \to \Gamma(2^{k+1})/\Gamma(2^{k+2}) \) for all \( k \geq 2 \). It follows that an element in \( \Gamma(2^{k+1}) \) can be written as a product of squares in every quotient \( \Gamma(2^k)/\Gamma(2^{n+k}) \) and since the \( \Gamma(2^{n+k}) \) form a base for the open neighborhoods of the identity in \( G \), we have that \( \Gamma(2^{k+1}) \subseteq \Phi(\Gamma(2^k)) \). This yields the desired result. \( \square \)

The enumeration of the subgroups is accomplished using Magma. The initial enumeration produces 1619 conjugacy classes of subgroups. The computation of the lattice of such subgroups finds that many of these are contained in subgroups \( H \) where the genus of \( X_H \) is \( \geq 2 \). These are then removed, resulting in 727 arithmetically maximal subgroups. The arithmetically maximal subgroups can have genus as large as 7 and index as large as 192.

4. Computing equations for \( X_H \) with \(-I \in H\)

Here we discuss the computation of equations for \( X_H \) as \( H \) ranges over the arithmetically maximal subgroups of \( \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \).

Remark 4.1. Equations for some of these curves already appear in the literature; see [Sut], [Sut12], [GJJ03], [Hee52], [Kna92, Table 12.1], [Shi95], [DD12], [Mom84, Proof of Lemma 3.2], [Bar10], [Hee52] [McM], [Zyw11b, 3.2] for equations of \( X_0(N) \) for \( N = 2, 4, 8, 16, 32, 64 \), \( X_1(N) \) for \( N = 2, 4, 8, 16 \), \( X_H \) with \( H \subset \text{GL}_2(\mathbb{Z}/8\mathbb{Z}) \) maximal, \( X_{n+}^+(N) \) for \( N = 2, 4, 8, 16 \), and various other small genus modular curves.

We first assume that \(-I \in H\). Let \( H_n \) be the \( n \)th subgroup in our list of 727 (as given in the file \text{gl2data.txt} \), and let \( X_n = X_{H_n} \). Instead of constructing the coverings \( X_n \to X_1 \) directly, we will instead construct coverings \( X_n \to X_m \) so that \( H_n \) is a maximal subgroup of \( H_m \) and compose to get \( X_n \to X_1 \). In almost all cases the degree of the covering \( X_n \to X_m \) is 2. (The exceptions are \( X_6 \to X_1 \), which has degree 3, and \( X_7 \to X_1, X_{55} \to X_7, \) and \( X_{441} \to X_{55} \) which all have degree 4. The curves \( X_1, X_7, X_{55} \) and \( X_{441} \) are the curves \( X_{ns}^+(2^k) \) for \( 1 \leq k \leq 4 \).)

In this process, if we find that \( X_n \) is a pointless conic, a pointless genus one curve, or an elliptic curve of rank zero, we do not compute any further coverings of \( X_n \). For this reason, it is only necessary for us to compute models of \( X_n \) for 345 choices of \( n \).
Figure 1. The tower of arithmetically maximal subgroups $H \subset \text{GL}_2(\mathbb{Z}_2)$ with $-I \in H$. 
In Section 6.2 of [Shi71], Shimura shows that the field of modular functions on $X(N)$ whose Fourier coefficients at the cusp at infinity are contained in $\mathbb{Q}(\zeta_N)$ is generated by

$$f_{\vec{a}}(z) = \frac{9}{\pi^2} \frac{E_4(z) E_6(z)}{\Delta(z)} \varphi_{\vec{a}} \left( \frac{cz + d}{N} \right)$$

where $\vec{a} = (c, d)$ and $(c, d) \in (\mathbb{Z}/N\mathbb{Z})^2$ has order $N$. Here, $\varphi_{\vec{a}}(\tau)$ is the classical Weierstrass $\varphi$-function attached to the lattice $\langle 1, z \rangle$. Shimura shows that the action of $GL_2(\mathbb{Z}/N\mathbb{Z})$ given by $f_{\vec{a}}|M = f_{\vec{a}M}$ defines gives an automorphism of the function field of $X(N)/\mathbb{Q}(j)$. To compute a model of $X_n$ we will compute a modular function $f$ in the function field of $X(N)$ fixed by the action of $H_n$ so that $\mathbb{Q}(X_n) = \mathbb{Q}(X_m)(f)$.

Suppose that we have already computed a model of $X_m$. If $X_m$ has genus zero, then we will only need to compute a covering $X_n \to X_m$ if $X_m \cong \mathbb{P}^1$. In this case we have already computed and stored a Fourier expansion for a function $f(z)$ so that $\mathbb{Q}(X_m) = \mathbb{Q}(f)$. If $X_m$ has genus one, we only need to compute a covering $X_n \to X_m$ if $X_n$ is an elliptic curve with positive rank. We use a minimal Weierstrass equation for $X_m$ and compute Fourier expansions for the modular functions $x$ and $y$.

We will construct the modular function $f$ by using a linear combination of products of weight 2 Eisenstein series, and dividing this by a modular form for $SL_2(\mathbb{Z})$. For a subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$, the dimension of the space of weight 2 Eisenstein series for $\Gamma$ is equal to the number of cusps of $X_\Gamma$ minus one. Therefore, given a subgroup $H_n$, we will find a subgroup $K \subseteq H_n$ so that $X_K$ has more cusps than $X_\Gamma$ for any subgroup $K < L \subseteq H_n$. Frequently, we can take $K = H_n$, but sometimes it is necessary for $[H_m : K]$ to be as large 24. Then, we will construct a weight 2 Eisenstein series for $K$, compute its images under the cosets of $K$ in $H_n$ and plug these modular forms into an elementary symmetric polynomial (of degree 1, 2, or 3) to get a weight 2, 4 or 6 modular form for $H_n$. In the weight 2 case, we divide this by a weight 2 Eisenstein series with rational coefficients for $\Gamma_0(2)$. In the weight 4 and weight 6 cases, we divide by $E_4(z)$ and $E_6(z)$, the usual Eisenstein series for $SL_2(\mathbb{Z})$.

Suppose that $N = 2^k$ and $\vec{a} = (c, d)$ is a vector with $c, d \in (\mathbb{Z}/N\mathbb{Z})^2$ with at least one of $c$ or $d$ odd. Let $g_{\vec{a}}(z) = \frac{9}{\pi^2} \varphi_{\vec{a}} \left( \frac{cz + d}{N} \right)$. This form is a weight 2 Eisenstein series for $\Gamma(N)$ with coefficients in $\mathbb{Q}(\zeta_N)$, and moreover, there is an action of $GL_2(\mathbb{Z}/N\mathbb{Z})$ on the forms $g_{\vec{a}}(z)$. We say that a weight $k$ modular form for $H \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$ is a homogeneous element of the graded ring generated by the $g_{\vec{a}}(z)$ that is fixed by each element $M \in H$. There are $\frac{3N^2}{8}$ forms $g_{\vec{a}}$ and they span the space of Eisenstein series for $\Gamma(N)$, which has dimension $\frac{3N^2}{8} - 1$. The single relation is $\sum_{\vec{a}} g_{\vec{a}} = 0$. To compute Eisenstein series for $K \cap SL_2(\mathbb{Z})$, we compute the subspace of this $\frac{3N^2}{8} - 1$ dimensional space (over $\mathbb{Q}(\zeta_N)$) fixed by $K \cap SL_2(\mathbb{Z})$. We then view this space as a vector space over $\mathbb{Q}$ with an action of $K$, noting that if

$$c = \sum_{i=0}^{r} c_i \zeta_N^i \in \mathbb{Q}(\zeta_N), \quad c \mid M = \sum_{i=0}^{r} c_i \zeta_N^{i \det M}.$$

We then compute a basis for the subspace fixed by $K$, say $v_1, \ldots, v_d$. We then pick a particular element $v$ of this subspace (we use $v = \sum_{i=1}^{d} iv_i$) and use this as our weight 2 Eisenstein series for $K$. We check that $v$ has $[H_m : K] \leq 5$ images under the action of $H_m$ and compute Fourier expansions of these weight 2 Eisenstein series. For each coset of $H_n$ in $H_m$, we compute the images of $v$ under the element in that coset and plug these into the appropriate symmetric polynomial. Dividing all of these forms by the appropriate denominator modular
We partition these into two sets, \(q\) the matrices and \(H\). The matrices \(H\) subspace fixed by the action of \(H\). We will consider the example of the covering Example 4.3. The covering map \(X\) computed and stored the Fourier expansion of a function \(f\) to \(GL_2\). Series for \(\Gamma(16)\) is a \(Q\) \(H\) 4. When the covering \(X\) be a degree \(m\) be an elliptic curve and \(g: E \to \mathbb{P}^1\) be a degree \(k\) morphism. Then,

\[
g = \frac{P(x) + yQ(x)}{R(x)}
\]

where \(P, Q\) and \(R\) are polynomials with \(\deg P \leq 3k - 3\), \(\deg Q \leq 3k - 5\) and \(\deg R \leq 3k - 3\).

Identifying the coefficients of \(P(x)\) as elements of \(Q(X_m)\) yields a model for \(X_n\) as well as the covering map \(X_n \to X_m\).

**Example 4.3.** We will consider the example of the covering \(X_{57} \to X_{22}\). The subgroup \(H_{22}\) is an index 8, level 8 subgroup of \(GL_2(\mathbb{Z})\). It is one of three maximal subgroups (up to \(GL_2(\mathbb{Z})\) conjugacy) of \(H_7\), which is the unique maximal subgroup of \(GL_2(\mathbb{Z})\) of index 4. When the covering \(X_{22} \to X_7\) was computed, we determined that \(X_{22} \cong \mathbb{P}^1\) and we computed and stored the Fourier expansion of a function \(f_{22}\) with \(Q(X_{22}) = Q(f_{22}, j)\). The subgroup \(H_{57}\) is an index 2 subgroup of \(H_{22}\), and \(H_{57} \supseteq \Gamma(16)\). It is generated by \(\Gamma(16)\) and the matrices

\[
\begin{bmatrix}
11 & 4 \\
8 & 3
\end{bmatrix}, \begin{bmatrix}
15 & 11 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
7 & 2 \\
2 & 1
\end{bmatrix}, \text{ and } \begin{bmatrix}
15 & 15 \\
1 & 0
\end{bmatrix}.
\]

Both \(H_{22}\) and \(H_{57}\) have two cusps. We choose \(K\) to be the subgroup generated by \(\Gamma(16)\) and the matrices

\[
\begin{bmatrix}
13 & 2 \\
14 & 11
\end{bmatrix}, \begin{bmatrix}
1 & 1 \\
15 & 0
\end{bmatrix}, \text{ and } \begin{bmatrix}
1 & 0 \\
7 & 7
\end{bmatrix}.
\]

We have \([H_{57} : K]\) = 4. The modular curve \(X_K\) has 8 cusps. The space of Eisenstein series for \(\Gamma(16)\) is a \(Q(\zeta_{16})\)-vector space of dimension 95 spanned by the forms \(f_{\vec{a}}\), where \(\vec{a} = (c, d) \in (\mathbb{Z}/16\mathbb{Z})^2\) and at least one of \(c\) or \(d\) is odd. The subspace fixed by \(K \cap SL_2(\mathbb{Z})\) has dimension 7. Let \(g_1, g_2, \ldots, g_7\) be a basis for this space. We consider the 56-dimensional \(Q\)-vector space spanned by \(\{\zeta_{16}^i g_j : 0 \leq i \leq 7, 1 \leq j \leq 7\}\) and we find the 7-dimensional subspace fixed by the action of \(K\). We take a linear combination of these 7 forms to obtain a weight 2 Eisenstein series \(h(z)\) for \(K\).

This \(h(z)\) is still represented as a linear combination of the forms \(f_{\vec{a}}\). We now compute the \(q\)-expansions of \(h(z)|_\gamma\), where \(\gamma\) ranges over representatives of the 8 right cosets of \(K\) in \(H_{22}\). We partition these into two sets, \(\{h_1(z), h_2(z), h_3(z), h_4(z)\}\) and \(\{h_1(z)|_\delta, h_2(z)|_\delta, h_3(z)|_\delta, h_4(z)|_\delta\}\) where the \(h_i(z)\) are the images of \(h(z)\) under cosets of \(K\) contained in \(H_{57}\), and \(\delta \in H_{22}\) but \(\delta \not\in H_{57}\).
We plug the $h_i(z)$ into the second elementary symmetric polynomial to obtain a weight 4 modular form $F_1(z)$ for $H_{57}$. Its image $F_2(z)$ under the action of $\delta$ is obtained from the $h_i(z)\delta$. Finally, a generator for $\mathbb{Q}(X_{57})/\mathbb{Q}(X_{22})$ is obtained as a root of the polynomial

$$x^2 - \frac{F_1(z) + F_2(z)}{E_4(z)} x + \frac{F_1(z)F_2(z)}{E_4(z)^2}.$$  

The function $f_{22}(z)$ with $\mathbb{Q}(X_{22}) = \mathbb{Q}(f_{22})$ has Fourier expansion

$$f_{22}(z) = 3\sqrt{2} + (36 + 24\sqrt{2})(1 + i)q^{1/4} + (288 + 216\sqrt{2})i q^{1/2} - (480\sqrt{2} + 720)(1 - i)q^{3/4} - 96\sqrt{2}q + \cdots.$$  

The function $\frac{F_1(z) + F_2(z)}{E_4(z)}$ has degree at most 3, and in fact we find that

$$\frac{F_1(z) + F_2(z)}{E_4(z)} = \frac{2^{11} \cdot 3^3 \cdot (155f_{22}^2 - 5946f_{22} - 26784)}{f_{22}^2 + 12f_{22} + 30}.$$  

Similarly, we find that

$$\frac{F_1(z)F_2(z)}{E_4(z)^2} = \frac{2^{20} \cdot 3^6 \cdot (174569f_{22}^4 - 739788f_{22}^3 + 26364168f_{22}^2 + 298652832f_{22} + 680985144)}{(f_{22}^2 + 12f_{22} + 30)^2}.$$  

These equations show that there is a modular function $g$ for $X_{57}$ so that $g^2 = 18 - f_{22}^2$. This equation for $X_{57}$ is a conic. Finding an isomorphism between this conic and $\mathbb{P}^1$ yields a function $f_{57}$ for which $\mathbb{Q}(X_{57}) = \mathbb{Q}(f_{57})$. This $f_{57}$ satisfies

$$f_{22} = \frac{3f_{57}^2 + 6f_{57} - 3}{f_{57}^2 + 1},$$

which gives the covering map $X_{57} \to X_{22}$. The entire calculation takes 26 seconds on a 64-bit 3.2 GHz Intel Xeon W3565 processor.

Taking, for example, $f_{57} = 0$ gives $f_{22} = -3$. Mapping from $X_{22} \to X_7 \to X_1$ gives $j = -320$. The smallest conductor elliptic curve with this $j$-invariant is

$$E: y^2 = x^3 - x^2 - 3x + 7,$$

and the 2-adic image for this curve is $H_{57}$.

5. The cases with $-I \notin H$

In this section we describe how to compute, for subgroups such that $-I \notin H$ and $g(X_{H}) = 0$, a family of curves $E_t$ over an open subset $U \subset \mathbb{P}^1$ such that an elliptic curve $E/K$ without CM has 2-adic image of Galois contained in a subgroup conjugate to $H$ if and only if there exists $t \in U(K)$ such that $E_t \cong E$.

When $-I \in H$, the 2-adic image for $E$ is contained in $H$ if and only if the same is true of the quadratic twists $E_D$ of $E$. For this reason, knowing equations for the covering map $X_H \to X_1$ is sufficient to check whether a given elliptic curve has 2-adic image contained in $H$.

When $-I \notin H$, more information is required. First, observe that if $-I \notin H$, then $\tilde{H} = \langle -I, H \rangle$ is a subgroup with $[\tilde{H} : H] = 2$ that contains $H$. In order for there to be non-trivial rational points on $X_H$, it must be the case that $X_{\tilde{H}}(\mathbb{Q})$ contains non-cuspidal, non-CM rational points. A detailed inspection of the rational points in the cases that $-I \in H$
| Type                                                      | Number |
|-----------------------------------------------------------|--------|
| $X_H \cong \mathbb{P}^1$                                  | 175    |
| Pointless conics                                          | 10     |
| Elliptic curves with positive rank                        | 27     |
| Elliptic curves with rank zero                            | 25     |
| Genus 1 curves computed with no points                    | 6      |
| Genus 1 curves whose models are not necessary             | 165    |
| Genus 2 models computed                                   | 57     |
| Genus 2 curves whose models are not necessary             | 40     |
| Genus 3 models computed                                   | 22     |
| Genus 3 curves whose models are not necessary             | 142    |
| Genus 5 models computed                                   | 20     |
| Genus 5 curves whose models are not necessary             | 24     |
| Genus 7 models computed                                   | 4      |
| Genus 7 curves whose models are not necessary             | 10     |

Figure 2. Summary of the computation of the 727 models.

shows that this only occurs if $X_H$ has genus zero. There are 1006 subgroups $H$ that must be considered.

Since we are not interested in the cases of elliptic curves with CM, we will remove the points of $X_H$ lying over $j = 0$ and $j = 1728$. Let $\pi: X_H \to \mathbb{P}^1$ be the map to the $j$-line and $U = \pi^{-1}(\mathbb{P}^1 - \{0, 12^3, \infty\}) \subset X_H$. Then points of $U$ have no non-trivial automorphisms and as a consequence, $U$ is fine moduli space (see Section 2). We let $E_H \to U$ denote the universal family of (non-CM) elliptic curves with 2-adic image contained in $H$. By Lemma 2.2 there is a model for $E_H$ of the form

$$E_H: y^2 = x^3 + A(t)x + B(t)$$

where $A(t), B(t) \in \mathbb{Z}[t]$. Knowing that such a model exists, we will now describe how to find it.

Let $K$ be any field of characteristic zero suppose that $E/K$ is an elliptic curve with $j(E) \notin \{0, 1728\}$ given by

$$E: y^2 = x^3 + Ax + B.$$  

Now, if

$$E_d: dy^2 = x^3 + Ax + B$$

is a quadratic twist of $E$, then $E$ and $E_d$ are isomorphic over $K(\sqrt{d})$ with the isomorphism sending $(x, y) \mapsto (x, y/\sqrt{d})$. Fix a basis for the 2-power torsion points on $E$ and let $\rho_E: \text{Gal}(\overline{K}/K) \to \text{GL}_2(\mathbb{Z}_2)$ be the corresponding Galois representation. Taking the image of the fixed basis on $E$ under this isomorphism gives a basis on $E_d$, and with this choice of basis, we have

$$\rho_{E_d} = \rho_E \cdot \chi_d$$

where $\chi_d$ is the natural isomorphism $\text{Gal}(K(\sqrt{d})/K) \to \{\pm I\}$. We can now state our next result.
Lemma 5.1. Assume the notation above. Let $\widetilde{H}$ be the subgroup generated by the image of $\rho$ and $-I$. Suppose $H \subset \widetilde{H}$ is a subgroup of index 2 with $-I \not\in H$. Then there is a unique quadratic twist $E_d$ so that the image of $\rho_{E_d}$ (computed with respect to the fixed basis coming from $E$) lies in $H$.

Remark 5.2. Without the chosen basis for the 2-power torsion on $E_d$, the statement is false. Indeed, it is possible for two different index two (and hence normal) subgroups $N_1$ and $N_2$ of $\widetilde{H}$ to be conjugate in $\text{GL}_2(\mathbb{Z}_2)$. The choice of a different basis for the 2-power torsion on $E_d$ would allow the image of $\rho_{E_d}$ to be either $N_1$ or $N_2$.

Proof. Observe that $j(E) \not\in \{0, 1728\}$ implies that $E \cong E_d$ if and only if $d \in (K^\times)^2$. Recall that $\rho_{E_d} = \rho_E \cdot \chi_d$.

Let $L$ be the fixed field of $\{\sigma \in \text{Gal}(\overline{K}/K) : \rho_E(\sigma) \in H\}$. Then since $H$ is a subgroup of $\widetilde{H}$ of index at most 2, $[L : K] \leq 2$. If $\rho_E(\sigma) \not\in H$, then $\rho_E(\sigma) \in (-I)H$. Thus, the image of $\rho_{E_d}$ is contained in $H$ if and only if $\chi_d(\sigma) = -1 \iff \sigma \not\in \text{Gal}(\overline{K}/L)$. Thus, the image of $\rho_{E_d}$ is contained in $H$ if and only if $L = K(\sqrt{d})$. This proves the claim. \hfill $\Box$

We start by constructing a model for an elliptic curve $E_t: y^2 = x^3 + A(t)x + B(t)$ where $A(t), B(t) \in \mathbb{Z}[t]$ and $j(E_t) = p(t)$, where $p : X_{\overline{K}} \to X_1$ is the covering map from $X_{\overline{K}}$ to the $j$-line. By the above lemma, the desired model of $E_H$ will be a quadratic twist of $E_t$, so $E_H: y^2 = x^3 + A(t)f(t)^2 + B(t)f(t)^3$ for some squarefree polynomial $f(t) \in \mathbb{Z}[t]$.

Theorem 5.3. Let $f(t) \in \mathbb{Z}[t]$ and $d \in \mathbb{Z}$ be squarefree and such that $E_H$ is isomorphic to the twist $E_{t,df(t)}$ of $E_t$ by $df(t)$ and let $D(t)$ be the discriminant of the model $E_t$ given above. Assume moreover that for every prime $p$, $f(t) \not\equiv 0 \pmod{p}$. Then $f(t)|D(t)$ in $\mathbb{Q}[t]$ and for every $p|d$, $D(t) \equiv 0 \pmod{p}$.

Proof. By the proof of Lemma 5.1, $\mathbb{Q}(t, \sqrt{df(t)}) \subset \mathbb{Q}(t)(E_t[2\infty])$. By Neron-Ogg-Shafarevich [VII Theorem 7.1], $\mathbb{Q}(t)(E_t[2\infty])$ is unramified at the places of $\mathbb{P}_Q^1$ of good reduction for $E_t$ and at the rational primes of good reduction for $E_t$; in particular, each factor of $f(t)$ must be a place of bad reduction for $E_t$ and thus must divide $D(t)$, and each rational prime $p$ dividing $d$ must satisfy $D(t) \equiv 0 \pmod{p}$. \hfill $\Box$

Here is a summary of the algorithm we apply to compute the polynomial $df(t)$.

1. We pick an integral model for $E_t$ and repeatedly choose integer values for $t$ for which $E_t$ is non-singular and does not have complex multiplication.
2. For each such $t$, we compute a family of resolvent polynomials, one for each conjugacy class of $\widetilde{H}$, that will allow us to determine the conjugacy class of $\rho_{E_t,2^k}(\text{Frob}_p)$. (See Appendix A for a procedure to do this.)
3. We make a list of the quadratic characters corresponding to $\mathbb{Q}(\sqrt{d})$ for each squarefree divisor $d$ of $2N(E_t)$. All twists of $E_t$ with 2-adic image contained in $H$ must be from this set.
(4) We compute the $GL_2(\mathbb{Z}_2)$-conjugates of $H$ inside $\tilde{H}$. (For the $\tilde{H}$ that we consider, computation reveals that there can be 1, 2, or 4 of these.)

(5) We use the resolvent polynomials to compute the image of $Frob_p$ for several primes $p$. Once enough primes have been used, it is possible to identify which twist of $E_t$ has its 2-adic image contained in each $GL_2(\mathbb{Z}_2)$-conjugate of $H$.

(6) The desired model of $E_t$ will be a twist by $cd(t)$ for some divisor $d(t)$ of the discriminant. We keep a list of a candidate values for $c$ for each divisor $d(t)$ that work for all of the $t$-values tested so far, and eliminate choices of $d(t)$.

(7) We go back to the first step and repeat until the number of options remaining for pairs $(c, d(t))$ is equal to the number of $GL_2(\mathbb{Z}_2)$-conjugates of $H$ in $\tilde{H}$. Each of these pairs $(c, d(t))$ gives a model for $E_H$. We output the simplest model found.

Remark 5.4. The algorithm above (step 2 in particular) sometimes requires a lot of decimal precision (in some cases as much as 8500 digits), and is in general fairly slow. Computing the equation for the universal curve over $X_H$ is thus much slower than computing equations for $X_H$ when $-I \in H$.

Example 5.5. There are two index 2 subgroups of $H_{57}$ that do not contain $-I$. One of these, which we call $H_{57a}$, contains $\Gamma(32)$, and is generated by

$$\begin{bmatrix} 10 & 21 \\ 3 & 13 \end{bmatrix}, \begin{bmatrix} 15 & 1 \\ 27 & 2 \end{bmatrix}, \begin{bmatrix} 7 & 7 \\ 0 & 1 \end{bmatrix}.\]$$

We will compute $E_{H_{57a}}$, the universal elliptic curve over $H_{57a}$. We let

$$E_t : y^2 = x^3 + A(t)x + B(t),$$

where

$$A(t) = -6(725t^8 + 1544t^7 + 2324t^6 + 2792t^5 + 2286t^4 + 1336t^3 + 500t^2 + 88t + 5),$$

$$B(t) = -32(3451t^{12} + 11022t^{11} + 22476t^{10} + 35462t^9 + 43239t^8 + 41484t^7 + 32256t^6 + 19596t^5 + 8601t^4 + 2630t^3 + 564t^2 + 78t + 5).$$

These polynomials were chosen so that

$$j(E_t) = \frac{2^6(25t^4 + 36t^3 + 26t^2 + 12t + 1)^3(29t^4 + 20t^3 + 34t^2 + 28t + 5)}{(t^2 - 2t - 1)^8} = p(t),$$

where $p : X_{57} \to X_1$ is the map to the $j$-line. There are four squarefree factors of the discriminant of $E_t$ in $Q[t]$: $1, t^2 - 2t - 1, t^4 + \frac{20}{29}t^3 + \frac{34}{29}t^2 + \frac{28}{29}t + \frac{5}{29}$, and $t^6 - \frac{38}{29}t^5 - \frac{35}{29}t^4 - \frac{60}{29}t^3 - \frac{85}{29}t^2 - \frac{38}{29}t - \frac{5}{29}$.

We specialize $E_t$ by taking $t = 1$, giving

$$E_t : y^2 = x^3 - 69600x + 7067648.$$

Considering $H_{57}$ as a subgroup of $GL_2(\mathbb{Z}/32\mathbb{Z})$, it has 416 conjugacy classes. We compute the resolvent polynomials for each of these conjugacy classes and verify that they have no common factors. Since $E_t$ has conductor $2^8 \cdot 3^2 \cdot 29^2$, the fixed field of $H_{57a}$ inside $Q(E_t[32])$ is a quadratic extension ramified only at 2, 3 and 29. There are sixteen such fields.

There are two index 2 subgroups of $H_{57}$ that are $GL_2(\mathbb{Z}_2)$-conjugate to $H_{57a}$. As a consequence, there are two quadratic twists of $E_t$ whose 2-adic image will be contained in some
The subgroup $H_{57a}$. By computing the conjugacy class of $\rho(\text{Frob}_p)$ for $p = 53, 157, 179$ and 193, we are able to determine that those are the $-87$ twist and the $174$ twist. This gives us a total of 8 possibilities for pairs $(c, d(t))$ (two for each $d(t)$).

Next, we test $t = 2$. This gives the curve

$$E_t: y^2 = x^3 - 4024542x + 3107583520.$$  

This time, we find that the $-4926$ and $2463$ twists are the ones whose 2-adic image is contained in $H_{57a}$ (up to conjugacy). This rules out all the possibilities for the pairs $(c, d(t))$ except for two. These are $c = 174$ and $c = -87$ and $d(t) = t^6 - \frac{38}{29}t^5 - \frac{35}{29}t^4 - \frac{69}{29}t^3 - \frac{85}{29}t^2 - \frac{38}{29}t - \frac{5}{29}$. This gives the model

$$E_{H_{57a}}: y^2 = x^3 + \tilde{A}(t)x + \tilde{B}(t),$$

where

$$\tilde{A}(t) = 2 \cdot 3^3 \cdot (t^2 - 2t - 1)^2 \cdot (25t^4 + 36t^3 + 26t^2 + 12t + 1)(29t^4 + 20t^3 + 34t^2 + 28t + 5)^3$$

$$\tilde{B}(t) = 2^5 \cdot 3^3 \cdot (t^2 - 2t - 1)^3 \cdot (t^2 + 1) \cdot (7t^2 + 6t + 1) \cdot (17t^4 + 28t^3 + 18t^2 + 4t + 1)$$

$$\cdot (29t^4 + 20t^3 + 34t^2 + 28t + 5)^4.$$

In total, this calculation takes 2 hours and 40 minutes.

The smallest conductor that occurs in this family is 6400. The curve $E: y^2 = x^3 + x^2 - 83x + 713$ and its $-2$-quadratic twist $E': y^2 = x^3 + x^2 - 333x - 6037$ both have conductor 6400 and 2-adic image $H_{57a}$.

6. A CURIOUS EXAMPLE

Before our exhaustive analysis of the rational points on the various $X_H$, we pause to discuss the following curious example, which demonstrates that Hilbert’s irreducibility theorem does not necessarily hold when the base is an elliptic curve with positive rank.

One expects that if $X_H(\mathbb{Q})$ is infinite then there exist infinitely many elliptic curves $E/\mathbb{Q}$ such that $\rho_E(G_\mathbb{Q})$ is actually equal to $H$. The following example shows that this is not necessarily true.

Example 6.1. The subgroup $H_{155}$ is an index 24 subgroup of $\Gamma(16)$ generated by

$$\begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 12 & 3 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 1 \\ 12 & 7 \end{bmatrix}.$$  

The curve $X_{155}$ is an elliptic curve

$$X_{155}: y^2 = x^3 - 2x$$

and $X_{155}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ and is generated by $(0, 0)$ and $(-1, -1)$. The map from $X_{155}$ to the $j$-line is given by $j(x, y) = \frac{256(x^4 - 1)^3}{x^4}$.

Since the two-torsion subgroup $\tilde{X}_{155}(\mathbb{Q})[2]$ is non-trivial (as imposed by Remark [7.2]), $X_{155}$ has an étale double cover $\phi: E \to X_{155}$ defined over $\mathbb{Q}$ and such that $E$ has good reduction away from 2. By the Riemann-Hurwitz formula, $E$ has genus 1; the map is thus a 2-isogeny and $E(\mathbb{Q})$ thus has rank one. By étale descent (see Subsection [7.5]), since $X_{155}$ and $E$ have good reduction outside of 2, every point of $X_{155}(\mathbb{Q})$ lifts to $E_d(\mathbb{Q})$ for $d \in \{-1, +2\}$. It turns
out that for each such \( d \), there is an index 2 subgroup \( H_d \subset H_{155} \) such that \( E_d \cong X_{H_d} \). (These are \( X_{284}, X_{318}, X_{328} \) and \( X_{350} \), respectively.) It follows that

\[
\bigcup_{d \in \{\pm 1, \pm 2\}} \phi_d(E_d(\mathbb{Q})) = X_{H_{155}}(\mathbb{Q}).
\]

In particular, for every point in \( X_{H_{155}}(\mathbb{Q}) \), the 2-adic image of Galois of the corresponding elliptic curve is contained in one of the four index two subgroups \( H_d \).

**Remark 6.2.** We note that if \( X_H \cong \mathbb{P}^1 \), then since \( \mathbb{P}^1 \) has no étale covers and since there is a finite collection of subgroups \( H_1, \ldots, H_n \) such that any \( K \) properly contained in \( H \) is a subgroup of some \( H_i \), the image

\[
\bigcup_{K \subset H} \phi_K(X_K(\mathbb{Q})) = \bigcup_{i=1}^n \phi_{H_i}(X_{H_i}(\mathbb{Q})) \subset X_H(\mathbb{Q})
\]

(where \( \phi_K \) is the map \( X_K \to X_H \) induced by the inclusion \( K \subset H \)) is a thin set, and in particular most (i.e. a density one set) of the points of \( X_H(\mathbb{Q}) \) correspond to \( E/\mathbb{Q} \) such that \( \rho_E(G_\mathbb{Q}) = H \).

** Remark 6.3.** There are seven genus one curves \( X_H \) that are elliptic curves of positive rank where the corresponding subgroup \( H \) has index 24. In all seven cases, all of the rational points lift to modular double covers (although it is not always the case that all four twists have local points). In fact, every one of the 20 modular curves \( X_H \), where \( H \) has index 48 and for which \( X_H(\mathbb{Q}) \) is a positive rank elliptic curve is a double cover of one of these seven curves.

This example is more than just a curiosity; it inspired the technique of Subsection 7.6 which allows us to determine the rational points on most of the genus 5 and 7 curves.

This example also raises the following question.

**Question 6.4.** Do there exist infinite unramified towers of modular curves such that each twist necessary for étale descent is modular?

If so, this would imply that none of the curves in such a tower have non-cuspidal non-CM points. A potential example is the following: the Cummins/Pauli database [CP03] reveals that there might be such a tower starting with \( 16A^2, 16B^3, 16B^5, 16B^9, 16A^{17} \). There is then a level 32, index 2 subgroup of \( 16A^{17} \) that has genus 33.

7. **Analysis of Rational Points - theory**

The curves whose models we computed above have genera either 0,1,2,3,5,7; see Table 2

For the genus 0 curves, we determine whether the curve has a rational point, and if so we compute an explicit isomorphism with \( \mathbb{P}^1 \). For the genus 1 curves, we determine whether the curve has a rational point, and if so compute a model for the resulting elliptic curve and determine its rank and torsion subgroup. This is straightforward: all covering maps except 4 have degree 2, so we end up with a model of the form \( y^2 = p(t) \), where \( p(t) \) is a polynomial, and the desired technique is implemented in Magma. The remaining 4 cases are handled via a brute force search for points.
In the higher genus cases, we determine the complete set of rational points. Each of the following techniques play a role:

1. local methods,
2. Chabauty for genus 2 curves,
3. elliptic curve Chabauty,
4. étale descent,
5. “modular” étale double covers of genus 5 and 7 curves, and
6. an improved algorithm for computing automorphisms of curves.

In this section we describe in detail the theory behind the techniques used to analyze the rational points on the higher genus curves. The remainder of the paper is a case by case analysis of the rational points on the various $X_H$.

**Remark 7.1 (Facts about rational points on $X_H$).**

1. Every rational point on a curve $X_H$ of genus one that has rank zero is a cusp or a CM point.
2. The only genus 2 curve with non-cuspidal, non-CM rational points is $X_{441}^+$, also known as $X_{na}^+(16)$. This curve has two non-cuspidal, non-CM rational points, with distinct j-invariants.
3. The only genus 3 curves with non-cuspidal, non-CM rational points are $X_{556}^+$, $X_{558}^+$, $X_{653}^+$, $X_{566}^+$, $X_{619}^+$, $X_{649}^+$. Each of these gives rise to a single, distinct j-invariant.
4. All the rational points on the genus 5 and 7 curves are either cusps or CM points.

**Remark 7.2.** The following observation powers many of these approaches – since Jacobians of 2-power level modular curves have good reduction outside of 2, each Jacobian is “forced” to have a non-trivial two torsion point (and more generally forced to have small mod 2 image of Galois). Indeed, the two division field $\mathbb{Q}(J[2])$ is unramified outside of 2, and there are few such extensions of small degree. In [Jon10], it is shown that if $[K : \mathbb{Q}] ≤ 16$ and $K/\mathbb{Q}$ is ramified only at 2, then $[K : \mathbb{Q}]$ is a power of 2. In particular, there are no degree 3 or 6 extensions of $\mathbb{Q}$ ramified only at 2, so an elliptic curve with conductor a power of 2 has a rational 2-torsion point. (In practice of course one can often compute directly the torsion subgroup of the Jacobian, by computing the torsion mod several primes, and then explicitly finding generators.) We remark that there is, however, a degree 17 extension of $\mathbb{Q}$ ramified only at 2, arising from the fact that the class number of $\mathbb{Q}(\zeta_{64})$ is 17.

### 7.3. Chabauty

See [MP07] for a survey. The practical output is that if $\text{rk} \text{Jac}_X(\mathbb{Q}) < \text{dim} \text{Jac}_X = g(X)$, then $p$-adic integration produces explicit 1-variable power series $f \in \mathbb{Q}_p[t]$ whose set of $\mathbb{Z}_p$-solutions contains all of the rational points. This is all implemented in Magma for genus 2 curves over number fields, which will turn out to be the only case needed. See the section below on genus 2 curves for a complete discussion.

### 7.4. Elliptic Chabauty

Given an elliptic curve $E$ over a number field $K$ of degree $d > 1$ over $\mathbb{Q}$ and a map $E \xrightarrow{\pi} \mathbb{P}^1_K$, one would like to determine the subset of $E(K)$ mapping to $\mathbb{P}^1(\mathbb{Q})$ under $\pi$. A method analogous to Chabauty’s method provides a partial solution to this problem under the additional hypothesis that $\text{rank } E(K) < d$ (and has been completely implemented in Magma). The idea is to expand the map $E \to \mathbb{P}^1_K$ in $p$-adic power series and analyze the resulting system of equations using Newton polygons or similar tools. See [Bru03], [Bru06] for a succinct description of the method and instructions for use of its Magma implementation.
A typical setup for applications is the following.

\[
\begin{array}{c}
C \\
\downarrow \phi \\
E \\
\downarrow \psi \\
\mathbb{P}^1
\end{array}
\]

We have a higher genus curve \( C \) whose rational points we want to determine, and we have a particular map \( C \to \mathbb{P}^1 \) which is defined over \( \mathbb{Q} \) and which factors through an elliptic curve \( E \) over a number field \( K \) (but does not necessarily factor over \( \mathbb{Q} \)). Then any \( K \)-point of \( E \) which is the image of a \( \mathbb{Q} \)-point of \( C \) has rational image under \( E \to \mathbb{P}^1 \), exactly the setup of elliptic curve Chabauty. (Finding the factorization \( C \to E \) can be quite tricky; see Subsection 9.4 for an example.)

7.5. Étale descent. Étale descent is a “going up” style technique, first studied in [CG89] and [Wet97] and developed as a full theory (especially the non-abelian case) in [Sko01]. It is now a standard technique for resolving the rational points on curves (see e.g. [Bru03], [FW01]) and lies at the heart of the modular approach to Fermat’s last theorem (see [Poo02, 5.6]).

Let \( \pi: X \to Y \) be an étale cover defined over a number field \( K \) such that \( Y \) is the quotient of some free action of a group \( G \) on \( X \). Then there exists a finite collection \( \pi_1: X_1 \to Y, \ldots, \pi_n: X_n \to Y \) of twists of \( X \to Y \) such that

\[
\bigcup_{i=1}^n \pi_i(X_i(K)) = Y(K).
\]

Moreover, if we let \( S \) be the union of the set of primes of bad reduction of \( X \) and \( Y \) and of the primes of \( \mathcal{O}_K \) over the primes dividing \( \#G \), then the cocycles corresponding to the twists are unramified outside of \( S \). (See e.g. [Sko01, 5.3].)

We will use this procedure only in the case of étale double covers. In this case, \( G = \mathbb{Z}/2\mathbb{Z} \) and, since the twists are consequently quadratic, we will instead denote twists of a double cover \( X \to Y \) by \( X_d \to Y \), where \( d \in K^\times/(K^\times)^2 \), and the above discussion gives that, for any point \( P \) of \( Y(K) \), there will exist \( d \in \mathcal{O}_{K,S}^\times/(\mathcal{O}_{K,S}^\times)^2 \) such that \( P \) lifts to a point of \( X_d(K) \).

7.6. Étale descent via double covers with modular twists. The following variant of Example 6.1 will allow us to resolve the rational points on some of the high genus curves.

We will occasionally be in the following setup: \( K \subset H \subset \text{GL}_2(\mathbb{Z}_2) \) are a pair of open subgroups such that \( g(X_H) > 1 \) and the corresponding map \( X_K \to X_H \) is an étale double cover. By étale descent (see Subsection 7.5), since \( X_H \) and \( X_K \) have good reduction outside of 2, every point of \( X_H(\mathbb{Q}) \) lifts to a rational point on quadratic twist \( X_{K,d}(\mathbb{Q}) \) for \( d \in \{\pm 1, \pm 2\} \), so that

\[
\bigcup_{d \in \{\pm 1, \pm 2\}} \phi_d(X_{K,d}(\mathbb{Q})) = X_H(\mathbb{Q}),
\]

where \( X_K \to X_H \) is induced by the inclusion \( K \subset H \) and \( \phi_{K,d} \) is the twist of this by \( d \).
It turns out that, additionally, for each such $d$ there is an index 2 subgroup $K_d \subset H$ such that $X_{K_d} \cong X_{K_d,d}$; i.e. each of the quadratic twists are also modular. Finally, a third accident occurs: each of the subgroups $K_d$ is contained in a subgroup $L_d$ such that $X_{L_d}$ either has genus 1 and has no rational point, is an elliptic curve of rank zero, or is a genus zero with no rational points. In particular, since the inclusion of subgroups $K_d \subset L_d$ induces a map $X_{K_d} \to X_{L_d}$, this determines all of the rational points on each twist $X_{K_d}$, and thus on $X_H$.

This phenomenon occurs for 16 of the 20 subgroups $H$ for which $X_H$ has genus 5, and all four of the cases when $X_H$ has genus 7. See Subsection 10.3 for details.

### 7.7. Constructing automorphisms of curves over number fields.

If $C$ is a curve of genus $g$ and $D \to C$ is a degree $n$ étale cover of $C$, then the genus of $D$ is $ng - (n - 1)$. In order to analyze rational points on $D$, it is very helpful to be able to find maps from $D$ to curves of lower genus. In this context, it is helpful to compute the group $G$ of automorphisms of $D$ and consider quotients $D/H$ for subgroups $H \subseteq G$.

Magma’s algebraic function field machinery is able to compute automorphism groups of curves. However, the performance of these routines varies quite significantly based on the complexity of the base field. The routines work quickly over finite fields, but are often quite slow over number fields, especially when working with curves that have complicated models.

For our purposes, we are interested in quickly constructing automorphisms (defined over $\overline{\mathbb{Q}}$) of non-hyperelliptic curves $C/\mathbb{Q}$ with genus $\geq 3$. (Magma has efficient, specialized routines for genus 2 and genus 3 hyperelliptic curves.) Our goal is not to provably compute the automorphism group, but to efficiently construct all the automorphisms that likely exist. The procedure we use is the following.

1. Given a curve $C/\mathbb{Q}$, use Magma’s routines to compute $\text{Aut}(C/\mathbb{F}_p)$ for several different choices of primes $p$. This data suggests what the order of $\text{Aut}(C)$ is, and the field $K$ over which the automorphisms are defined.
2. Consider the canonical embedding of $C \subseteq \mathbb{P}^{g-1}$. Any automorphism of $C$ can be realized as a linear automorphism of $\mathbb{P}^{g-1}$ that fixes the canonical image of $C$.
3. Construct the “automorphism scheme” $X/\mathbb{Q}$ of linear automorphisms from $\mathbb{P}^{g-1}$ that map $C$ to itself. Let $I(C) \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_g]$ denote the ideal of polynomials that vanish on the canonical image of $C$. For each homogeneous generator $f_i$ of $I(C)$ of degree $d_i$, we construct a basis $v_1, v_2, \ldots, v_{e_i}$ for the degree $d_i$ graded piece of $I(C)$. If $\phi: C \to C$ is an automorphism, then

$$\phi(f_i) = \sum_{j=1}^{e_i} c_{i,j} v_j.$$  

We construct the automorphism scheme as a subscheme of $\mathbb{A}^d$, where $d = g^2 + \sum_i d_i + 1$. We use $g^2$ variables for the linear transformation, $\sum_i d_i$ variables for the constants $c_{i,j}$ in the above equation, and one further variable to encode the multiplicative inverse of the determinant of the linear transformation. (This scheme actually has dimension 1 since an arbitrary scaling of the matrix is allowed.) We will extend $X$ to a scheme over $\text{Spec} \mathbb{Z}$ (which we also call $X$).
(4) Choose a prime $p$ that splits completely in $K$ and a prime ideal $\mathfrak{p}$ of norm $p$ in $\mathcal{O}_K$, the ring of integers in $K$. Use Magma’s routines to compute $\text{Aut}(C/\mathbb{F}_p)$ and represent these automorphisms as points in $X(\mathbb{F}_p)$.

(5) Use Hensel’s lemma to lift the points on $X(\mathbb{F}_p)$ to points on $X(\mathbb{Z}/p^r\mathbb{Z})$ for some modestly sized integer $r$. (We frequently use $r = 60$.) Hensel’s lemma is already implemented in Magma via `LiftPoint`.

(6) Scale the lifted points so that one nonzero coordinate is equal to 1. Then use lattice reduction to find points in $K$ of small height that reduce to the points in $X(\mathbb{Z}/p^r\mathbb{Z})$ modulo $p^r$. Use these to construct points in $X(K)$, i.e., automorphisms of $C$ defined over $K$.

The above algorithm runs very quickly in practice for curves of reasonably small genus. For example, the genus 5 curve given by

\[-2705a^2 + 1681b^2 - 1967bc + 2048c^2 - 2d^2 = 0\]
\[73a^2 - 41b^2 + 64bc - 64c^2 - 2de = 0\]
\[-2a^2 + b^2 - 2bc + 2c^2 - 2e^2 = 0\]

is one of the étale double covers of $X_{619}$. This curve has (at least) 16 automorphisms defined over $\mathbb{Q}(\sqrt{2} + \sqrt{2})$ which are found by the above algorithm in 25.6 seconds. However, Magma’s built in routines require a long time to determine the automorphism group (the routine did not finish after running it for 3 and 1/2 days).

7.8. Fast computation of checking isomorphism of curves. A related problem to computing automorphisms is proving that two curves are isomorphic. There are many instances of non-conjugate subgroups $H$ and $K$ with $X_H \cong X_K$. Within the 22 genus three curves, there are at most 7 isomorphism classes. Within the 20 genus five curves, there are at most 10 isomorphism classes. The 4 genus seven curves fall into two isomorphism classes.

Magma’s built-in command `IsIsomorphic` suffices for hyperelliptic curves and a few higher genus curves that happen to have nice models. The simplest way to determine if two non-hyperelliptic genus 3 curves are isomorphic is to compute their canonical models and apply `MinimizeReducePlaneQuartic` and inspect the resulting simplified polynomials - at this point the isomorphisms can be seen by inspection.

In the genus 5 case, we use a variant of the approach described for automorphisms, and, given two curves $C_1$ and $C_2$, we construct an “isomorphism scheme” in a similar way to the automorphism scheme above. Again, we use Magma’s internal commands to find isomorphisms mod $p$, and lift these to characteristic zero isomorphisms. In the genus 7 case, Magma’s built-in commands are the most efficient.

7.9. Probable computation of ranks. It is straightforward to compute the rank of a curve of genus at most 2 using Magma’s preexisting commands (e.g. via `RankBound`, an implementation of [Sto01]); computation of the rank of the Jacobian of a genus 3 plane curve has recently been worked out [BPS12], but is often impractical [BPS12, Remark 1.1] and moreover has not been implemented in a publicly available way. For genus $\geq 3$ little is known in general (though special cases such as cyclic covers of $\mathbb{P}^1$ are known [PS97], [SvL11]).

For the determination of the rational points on each $X_H$, we will only need a rigorous computation of rank for genus at most 2. Nonetheless, in many cases we can compute
“probable” ranks, and mention this in the discussion as an indication of why we chose a particular direction of analysis. By “probable”, we mean the following: each factor $A$ of $\text{Jac} X_H$ is likely modular, and we can often match up a candidate for the corresponding modular form $f$ (e.g. by comparing traces) and compute a guess for the analytic rank, but we cannot prove that $A \cong A_f$ or that the algebraic and analytic ranks of $A_f$ agree.

8. Analysis of Rational Points - Genus 2

In the remaining sections we provably compute all of the rational points on each modular curve. Magma code verifying the below claims is available at [RZB] and additionally at the Arxiv page of this paper.

There are 57 arithmetically maximal genus 2 curves. Among these, 46 have Jacobians with rank 0, 3 with rank 1, and 8 with rank 2. We will use étale descent on the rank 2 cases and Chabauty on the others. In each case, the rank of the Jacobian is computed with Magma’s `RankBound` command. See the transcript of computations for full details, and see [BS08] for a detailed discussion of all practical techniques for determining the rational points on a genus 2 curve.

8.1. Rank 0. If $\text{rk} \text{Jac}_X(\mathbb{Q}) = 0$ then $\text{Jac}_X(\mathbb{Q})$ is torsion. To find all of the rational points on $X$ it thus suffices to compute the torsion subgroup of $\text{Jac}_X(\mathbb{Q})$ and compute preimages of these under an inclusion $X \hookrightarrow \text{Jac}_X$. This is implemented in Magma as the `Chabauty0(J)` command, and in each case Magma computes that the only rational points are the known points.

8.2. Rank 1. If $\text{rk} \text{Jac}_X(\mathbb{Q}) = 1$ then one can attempt Chabauty’s method. This is implemented in Magma as the `Chabauty(ptJ)` command, and in each case Magma computes that the only rational points are the known points.

8.3. Rank 2. If $\text{rk} \text{Jac}_X(\mathbb{Q}) = 2$ then Chabauty’s method doesn’t apply and the analysis is more involved; instead we proceed by étale descent. In each case, the Jacobian of $X$ has a rational 2-torsion point. Thus, given a model

$$X : y^2 = f(x)$$

of $X$, $f$ factors as $f_1f_2$, where both are polynomials of positive degree (and both of even degree if $f$ has even degree), and $X$ admits étale double covers $C_d \rightarrow X$, where the curve $C_d$ is given by

$$C_d : \begin{align*}
\frac{dy_1^2}{y_1} &= f_1(x) \\
\frac{dy_2^2}{y_2} &= f_2(x)
\end{align*}$$

Since $X$ has good reduction outside of 2 and the 2-cover $C_1 \rightarrow X$ is étale away from 2 (since it is the pullback of a 2-isogeny $A \rightarrow \text{Jac}_X$, and such an isogeny is étale away from 2), by étale descent (see [7.5] above) every rational point on $X$ lifts to a rational point on $C_d(\mathbb{Q})$ for $d \in \{\pm 1, \pm 2\}$. The Jacobian of $C_d$ is isogenous to $\text{Jac}_X \times E_d$, where $E_d$ is the Jacobian of the (possibly pointless) genus one curve $dy_2^2 = f_2(x)$ (where we assume that $\deg f_2 \geq \deg f_1$, so that $\deg f_2 \geq 3$).

There are 4 isomorphism classes of genus 2 curves in our list with Jacobian of rank 2 ($X_{395}, X_{402}, X_{441}, X_{520}$). In two cases ($X_{395}$ and $X_{402}$), each twist $C_d$ maps to a rank 0
elliptic curve. For example, $X_{395}$ is the hyperelliptic curve $y^2 = x^6 - 5x^4 - 5x^2 + 1 = (x^2 - 2x - 1)(x^2 + 1)(x^2 + 2x - 1)$. This admits étale covers by the genus 3 curves

$$C_d: \begin{align*}
  dy_1^2 &= x^2 + 1 \\
  dy_2^2 &= (x^2 - 2x - 1)(x^2 + 2x - 1)
\end{align*}$$

each of which in turn maps to the genus 1 curve $dy_3^2 = (x^2 - 2x - 1)(x^2 + 2x - 1)$, and for $d \in \{\pm 1, \pm 2\}$, $\text{rk} \text{Jac}_{E_d} = 0$, allowing the determination the rational points on each $C_d$ and thus on $X_{395}$.

For the remaining genus 2 curves, three of the twists map to a rank 0 elliptic curve, but the twist by $-2$ maps to a rank 1 elliptic curve. Here one may apply étale descent again, but over a quadratic extension. For example, $X_{441}$ is the hyperelliptic curve $y^2 = x^6 - 3x^4 + x^2 + 1 = (x - 1)(x + 1)(x^4 - 2x^2 - 1)$. (This is the curve $X_{18}^+(16)$ whose non-cuspidal points classify elliptic curves whose mod 16 image of Galois is contained in the normalizer of a non-split Cartan subgroup. The rational points on this curve are resolved in \cite{Bar10} via elliptic Chabauty; we give an independent determination of the rational points on this curve.) This admits étale covers by the genus 3 curves

$$C_d: \begin{align*}
  dy_1^2 &= (x - 1)(x + 1) \\
  dy_2^2 &= (x^4 - 2x^2 - 1)
\end{align*}$$

The Jacobian of $dy_2^2 = x^4 - 2x^2 - 1$ has rank 0 for $d = \pm 1, 2$. For $d = -2$, we note that since $x^4 - 2x^2 - 1$ factors over $\mathbb{Q}(\sqrt{2})$ as $((x - 1)^2 - \sqrt{2})((x - 1)^2 + \sqrt{2})$, $C_{-2}$ admits a further étale double cover over $\mathbb{Q}(\sqrt{2})$ by

$$X_{-2, d'}: \begin{align*}
  -2y_1^2 &= (x - 1)(x + 1) \\
  -2d'y_2^2 &= (x - 1)^2 - \sqrt{2} \\
  d'y_3^2 &= (x - 1)^2 + \sqrt{2}
\end{align*}$$

(Note that a priori one expects this factorization to occur over a small field by Remark \cite{7.2}. By descent theory, every rational point on $C_{-2}$ lifts to a $K := \mathbb{Q}(\sqrt{2})$ point on $X_{-2, d'}$ for some $d' \in \mathcal{O}^*_K/S/(\mathcal{O}^*_K/S)^2$. These each map to the two genus 1 curves $y^2 = (x - 1)(x + 1)((x - 1)^2 \pm \sqrt{2})$. For 6 of the 8 such $d'$, the Jacobian of one of these curves has rank 0, and for 2 both have rank 1. Any point coming from a rational point on $X_{441}$ has rational $x$-coordinate, and elliptic Chabauty (as described in Subsection \cite{7.4}) successfully resolves the rational points on the remaining two curves.

### 9. Analysis of Rational Points - Genus 3

There are 18 genus 3 curves (and at most 7 isomorphism classes). Of the isomorphism classes, $X_{556}, X_{558}$ are hyperelliptic and handled by étale descent; $X_{618}$ admits a map to a rank zero elliptic curve defined over $\mathbb{Q}(\sqrt{2})$; $X_{628}, X_{641}, X_{650}$ have nice models and can be handled in a direct, ad hoc manner. Finally, $X_{619}$ is the most difficult case – it has six rational points and its Jacobian has (probable) analytic rank 3; we are nonetheless able to handle this curve via an elliptic Chabauty argument whose setup is non-trivial. All other genus 3 curves on our list are isomorphic to one of these.

**Remark 9.1.** Unfortunately, consideration of Prym varieties (see \cite{Bru08} for a discussion) do not simplify analysis of any of the above curves; for instance, $X_{619}$ admits an étale double cover, but one of the twists of the associated Prym varieties has rank 2.
9.2. Genus 3 hyperelliptic. The genus 3 curves $X_{556}, X_{558}, X_{563}, X_{566}$ are hyperelliptic. The last two curves are isomorphic to the first two, which are given by

$X_{556}$: \[ y^2 = x^7 + 4x^6 - 7x^5 - 8x^4 + 7x^3 + 4x^2 - x \]

$X_{558}$: \[ y^2 = x^8 - 4x^7 - 12x^6 + 28x^5 + 38x^4 - 28x^3 - 12x^2 + 4x + 1 \]

Their Jacobians have rank 1, but unfortunately much of the machinery necessary to do Chabauty on curves of genus $g > 2$ is not implemented in Magma (e.g., a simple search did not reveal generators for the Jacobian of $X_{556}$; for a genus 2 curve one can efficiently search on the associated Kummer surface, but the analogous computation for abelian threefolds is not implemented).

Instead, we proceed by descent. The hyperelliptic polynomials both factor, so each $X$ admits an étale double cover which itself admits a map to a genus 2 curve. Rational points on the associated Kummer surface, but the analogous computation for abelian threefolds is not implemented). Each of these four hyperelliptic curves has four non-cuspidal, non-CM rational points that all have the same image on the $j$-line. For $X_{556}$ we obtain $j = 2^4 \cdot 17^4$, for $X_{558}$ we obtain $j = \frac{4097^3}{16}$, for $X_{563}$ we obtain $j = 2^{11}$, and for $X_{566}$ we obtain $j = \frac{257^3}{256}$.

9.3. Analysis of $X_{618}$. The curve $X_{618}$ has two visible rational points. Over the field $\mathbb{Q}(\sqrt{2})$, $X_{618}$ maps to the elliptic curve

$E$: \[ y^2 = x^3 + (\sqrt{2} + 1)x^2 + (-3\sqrt{2} - 5)x + (-2\sqrt{2} - 3) \]

which has rank 0 over $\mathbb{Q}(\sqrt{2})$ and has four $\mathbb{Q}(\sqrt{2})$-rational points, two of which lift to rational points of $X_{618}$.

We found this cover by computing $\text{Aut}_{X_{618}, \mathbb{Q}(\sqrt{2})}$ (which has order 8) and computing $E$ as the quotient of $X_{618, \mathbb{Q}(\sqrt{2})}$ by one of these automorphisms. (See Subsection 7.7 for a description of this computation.)

9.4. Analysis of $X_{619}$. The above techniques do not work on $X_{619}$; its Jacobian has (probable) analytic rank 3 and, while it admits an étale double cover $D$, a twist of $D$ has rational points and associated Prym variety of rank 2.

A bit of work reduces this to an elliptic Chabauty computation. Over the quartic field $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$, $D_8$ has automorphism group $D_8 \times \mathbb{Z}/2\mathbb{Z}$. Let $H$ be the subgroup $\langle t_1, t_2 \rangle$ (given explicitly in the transcript of computations). The twist $D_{-2}$ has no $\mathbb{Q}_2$ points. When $\delta = 1$ or 2, the quotient $D_8/H$ is isomorphic to the elliptic curve

$E_+: \delta y^2 = x^3 + (a^3 + 1)x^2 + (194a^3 + 153a^2 - 660a - 509)x + (-1815a^3 - 1389a^2 + 6202a + 4747)$
and the quotient \( D_{-1} / H \) is isomorphic to the elliptic curve

\[
E_\pm : \delta y^2 = x^3 + (a^3 + a^2 + a + 1)x^2 + (4a^3 + 8a^2 + 6a - 11)x + (-3a^3 + 29a^2 + 11a - 27)
\]

where in both cases \( a = \sqrt{2 + \sqrt{2}} \). The quotient of \( D_\delta \) by \( \text{Aut} \, D_\delta \) is \( \mathbb{P}^1 \); the quotient map \( \phi_\delta : D_\delta \to \mathbb{P}^1 \) is defined over \( \mathbb{Q} \) and factors through the map \( D_\delta \to E_\pm \):

\[
\begin{array}{ccc}
D_\delta & \xrightarrow{\phi_\delta} & E_\pm \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{\psi_\delta} & \mathbb{P}^1
\end{array}
\]

We are thus in the situation of elliptic Chabauty – by construction, any \( K \)-point of \( E_\pm \) that is the image of a \( \mathbb{Q} \)-point of \( D_\delta \) maps to \( \mathbb{P}^1(\overline{K}) \) under \( \psi_\delta \); \( K \) has degree 4 and \( E_\pm(\overline{K}) \) has rank 2. Magma computes that the only \( K \)-rational points of \( E_\pm \) that map to \( \mathbb{P}^1(\mathbb{Q}) \) are the known ones coming from \( D_\delta \).

It takes a bit of work to compute explicitly the map \( \psi_\delta : E_\pm \to \mathbb{P}^1 \). The group \( H \) is not normal, so \( \psi_\delta \) is not given by the quotient of a group of automorphisms. We proceed by brute force. We know the degree of \( \psi_\delta \) and thus the general form of its equations (by Lemma 4.2). We construct points on \( D_\delta \) over various number fields; we can map them on the one hand to \( E_\pm \) and on the other hand to \( \mathbb{P}^1 \), giving a collection of pairs \( (P \in E_\pm(\overline{K}), \psi_\delta(P)) \). Sufficiently many such pairs will allow us to compute equations for \( \psi_\delta \).

See the transcript of computations for code verifying these claims. We find that there are six rational points on \( X_{619} \). Two of these are cusps, two of these are CM points, corresponding to \( j = 16581375 \) (CM curves with discriminant \(-28\)), and two of these correspond to \( j = \frac{8579853}{628} \). Three other curves in our list are isomorphic to \( X_{619} \). One of these, \( X_{649} \) also has non-CM rational points corresponding to \( j = \frac{914925}{496} \).

9.5. Analysis of \( X_{628} \). The (probable) analytic rank of the Jacobian of \( X_{628} \) is 3, ruling out the possibility of a direct Chabauty argument. While it admits an étale double cover, the Prym variety associated to each twist has rank 1 and Chabauty on the double cover is thus possible but tedious to implement. Alternatively, each étale double cover maps to a rank 0 elliptic curve. This map is not explicit and would require a moderate amount of ad hoc work to exploit.

Instead, we exploit the nice model \( y^4 = 4xz(x^2 - 2z^2) \) of this curve via the following direct argument. (This is equivalent to étale descent, but the simplicity of the model motivates a direct presentation.) An elementary argument shows that, for \( xyz \neq 0 \), there exist integers \( u, v, w \) such that either \( x = \pm u^4, y = \pm v^4 \), and \( x^2 - 2y^2 = \pm w^4 \), giving \( u^8 - 32v^8 = \pm w^4 \), or that \( x = \pm 2u^4, y = \pm v^4 \), and \( x^2 - 2y^2 = \pm 2w^4 \), giving \( 2u^8 - v^8 = \pm w^4 \). It follows from \([\text{Coh07}, \text{Exercise 6.24, Proposition 6.5.4}\]) that the only solution is to the latter equation with \( u = v = w = 1 \). It follows that the only points on \( y^4 = 4xz(x^2 - 2z^2) \) are \((0 : 0 : 1)\), \((1 : 0 : 0)\), \((2 : -2 : 1)\) and \((2 : 2 : 1)\).
9.6. Analysis of \( X_{641} \) and \( X_{650} \). Each of \( X_{641} \) and \( X_{650} \) have Jacobians of (probable) analytic rank 3, but admit various étale double covers. Each double cover has a twist with local points and such that the associated Prym variety has rank 1. This suggests a Chabauty argument via the Prym, but the details of such an implementation would be complicated. Instead we exploit the nice plane quartic models of these curves.

\( X_{641} \) has an affine model \((x^2 - 2y^2 - 2z^2)^2 = (y^2 - 2yz + 3z^2)(y^2 + z^2)\) and thus admits an étale double cover by the curve

\[
D_\delta: \begin{align*}
y^2 - 2yz + 3z^2 &= \delta u^2 \\
x^2 - 2y^2 - 2z^2 &= \delta uv \\
y^2 + z^2 &= \delta v^2.
\end{align*}
\]

The only twist with 2-adic points is \( \delta = 1 \). The quotient by the automorphism \([x : y : z : r : s] \mapsto [-x : y : z : -r : -s]\) is the genus 3 hyperelliptic curve \( y^2 = -x^8 + 8x^6 - 20x^4 + 16x^2 - 2 \). This curve is an unramified double cover of \( H: y^2 = -x^5 + 8x^4 - 20x^3 + 16x^2 - 2x \). The Jacobian of \( H \) has rank 1, and Chabauty successfully determines the rational points on \( H \); computing the preimages of these points on \( D \) allows us to conclude that only rational points on \( X_{641} \) are the known ones.

Similarly, \( X_{650} \) has a model \( y^4 = (x^2 - 2xz - z^2)(x^2 + z^2) \) and thus admits an étale double cover by the curve

\[
D_\delta: \begin{align*}
x^2 - 2xz - z^2 &= \delta u^2 \\
y^2 &= \delta uv \\
x^2 + z^2 &= \delta v^2
\end{align*}
\]

The only twist with 2-adic points is \( \delta = 1 \). This genus 5 curve has four automorphisms over \( \mathbb{Q} \), and the quotient of \( D_1 \) by one of the involutions is the genus 3 hyperelliptic curve \( y^2 = -x^8 + 2 \), which maps to the genus 2 curve \( H: y^2 = -x^5 + 2x \). The rank of the Jacobian of \( H \) is 1, and Chabauty again proves that the only rational points on \( X_{650} \) are the known points.

10. Analysis of Rational Points - Genus 5 and 7

There are 20 genus 5 curves (at most 10 isomorphism classes) and 4 genus 7 curves. The genus 5 curves \( X_{686} \) and \( X_{689} \) are handled in an ad hoc manner by explicit étale descent. The remaining genus 5 curves and all of the genus 7 curves are handled by the modular double cover method (see Subsection 10.3) or are isomorphic to one of \( X_{686} \) or \( X_{689} \).

10.1. Analysis of \( X_{689} \). The curve \( X_{689} \) has a model

\[
X_{672}: y^2 = x^3 + x^2 - 3x + 1 \\
w^2 = 2(y^2 + y(-x + 1))(x^2 - 2x - 1)
\]

The curve \( D_\delta \)

\[
\begin{align*}
y^2 &= x^3 + x^2 - 3x + 1 \\
\delta w_1^2 &= (x^2 - 2x - 1) \\
\delta w_2^2 &= 2(y^2 + y(-x + 1))
\end{align*}
\]

is an étale double cover of \( X_{689} \). (Magma computes that \( g(D) = 9 \), so this follows from Riemann-Hurwitz.) The cover is unramified outside of 2, so every rational point on \( X_{689} \)
lifts to a rational point on $D_d$ for some $d \in \{\pm 1, \pm 2\}$. The curve $D_\delta$ maps to the curve $H_\delta$ given by

\[
y^2 - (x^3 + x^2 - 3x + 1) = 0
\]
\[
\delta w_1^2 - (x^2 - 2x - 1) = 0
\]

which Magma computes is a genus 3 hyperelliptic curve. Each of these hyperelliptic curves has Jacobian of rank 1 or 2, with four visible automorphisms. Taking the quotient by a non-hyperelliptic involution gives a genus 2 hyperelliptic curve, the Jacobians of which have rank at most 1; Chabauty applied to the genus 2 curves thus proves that the only rational points on $X_{672}$ are the known points.

See the transcript of computations for Magma code verifying these claims.

10.2. **Analysis of $X_{686}$**. Similarly, the curve $X_{686}$ has a model

\[
X_{686}: y^2 = x^3 + x^2 - 3x + 1
\]
\[
w^2 = 2(y^2 - y(-x + 1))(x^2 - 2x - 1)
\]

and étale double covers $D_\delta \to X_{686}$ from the curves

\[
y^2 = x^3 + x^2 - 3x + 1
\]
\[
\delta w_1^2 = x^2 - 2x - 1
\]
\[
\delta w_2^2 = 2(y^2 - y(-x + 1))
\]

The curve $D_\delta$ maps to the genus 3 hyperelliptic curve $H_\delta$ given by

\[
y^2 - (x^3 + x^2 - 3x + 1) = 0
\]
\[
\delta w_1^2 - (x^2 - 2x - 1) = 0
\]

These are the same curves as in the analysis of $X_{689}$, and we conclude in the same way that the only rational points on $X_{686}$ are the known points.

10.3. **Non-explicit, modular double covers**. The remaining genus 5 curves and the genus 7 curves are inaccessible via other methods and will be handled by the modular double cover method described in subsection 7.6. We describe this method in more detail here.

Let $S = \{1, 2, -1, -2\}$ and for $\delta \in S$ define $\chi_\delta$ to be the Kronecker character associated to $\mathbb{Q}(\sqrt{\delta})$. Suppose that $X$ is one of these 20 such curves, with corresponding subgroup $H$. In each case, we can find four index 2 subgroups $K_\delta$ with $\delta \in S$ so that for all $g \in K_\delta$,

\[
g \in K_1 \text{ if and only if } \chi_\delta(\det g) = 1.
\]

Choose a modular function $h(z)$ for $K_1$ so that if $m$ is an element of the non-identity coset for $K_1$ in $H$, then $h|m = -h$. A model for $X_{K_1}$ is then given by $h^2 = r$, where $r \in \mathbb{Q}(X_H)$. Moreover, the condition on elements of $K_\delta$ implies that $\sqrt{\delta} h$ is a modular function for $K_\delta$. This implies that the curves $X_{K_\delta}$ are the twists (by the elements of $S$) of $K_1$, and hence every rational point on $X_{K_1}$ lifts to one of the $X_{K_\delta}$. In each case, the $X_{K_\delta}$ maps to a curve $X_n$ whose model we have computed that has finitely many rational points (namely a pointless conic, a pointless genus 1 curve, or an elliptic curve with rank zero).

Note that the group theory alone provides the properties we need for the curves $X_{K_\delta}$, and we do not construct models for them.
Example 10.4. The curve $X_{695}$ is a genus 5 curve that has two visible rational points corresponding to elliptic curves with $j$-invariant 54000. In this case, $X_{K_1}$ and $X_{K_{-1}}$ map to the rank zero elliptic curve $X_{285}: y^2 = x^3 + x$ (whose two rational points map to $j = 54000$). The curves $X_{K_2}$ and $X_{K_{-2}}$ map to $X_{283}$, a genus 1 curve with no 2-adic points.

See the transcript of computations for further details.

Appendix A. Proving the mod $N$ representation is surjective

Given a Galois extension $K/\mathbb{Q}$ with Galois group $G$, [DD10] gives an algorithm that will allow one to determine, for a given unramified prime $p$, the Frobenius conjugacy class $\text{Frob}_p$. Applied to the case $K = \mathbb{Q}(E[N])$, and given initial knowledge that $G$ is a subgroup of some particular $H$ (e.g. $E$ could arise from a rational point on $X_H$), this gives an algorithm to prove that $\text{im } \rho_{E,N} = H$.

Remark A.1. When $H = S_n$ or $\text{GL}_2(\mathbb{F}_\ell)$ this is well understood (e.g. in the latter case, if $\ell > 5$ and $G$ contains three elements with particular properties then $G = H$ [Ser72, Prop. 19]). For subgroups of $\text{GL}_2(\mathbb{F}_\ell)$, [Sut13] recently proved that if two subgroups $H,K$ of $\text{GL}_2(\mathbb{F}_\ell)$ have the same signature, defined to be
$$s_H := \{(\det A, \text{tr } A, \text{rank fix } A) : A \in H\},$$
then $H$ and $K$ are conjugate. (Note that the extra data of fix $A$ is necessary to distinguish the trivial and order 2 subgroups of $\text{GL}_2(\mathbb{F}_2)$. Already for $G \subset \text{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z})$ with $\ell > 2$, the additional data of fix $A$ does not suffice – for instance, the order $\ell$ subgroups generated by $\begin{bmatrix} 1 - \ell & \ell \\ 0 & 1 + \ell \end{bmatrix}$ and $\begin{bmatrix} 1 - \ell & -\ell \\ 0 & 1 + \ell \end{bmatrix}$ have the same signature.)

Remark A.2. It is in principal completely straight-forward to provably determine the image of $\rho_{E,n}$. Indeed, Magma can compute, for any $n$, the corresponding division polynomial, and compute the Galois group of the corresponding field. In practice though, as the degree of $\mathbb{Q}(E[n])$ grows, a direct computation of the Galois group using Magma’s built in commands quickly becomes infeasible.

We now describe the algorithm. Suppose that $K$ is the splitting field of
$$F(x) = \prod_{i=1}^{n} (x - a_i).$$

Given some fixed polynomial $h$ and a conjugacy class $C \subseteq G$, construct the resolvent polynomial
$$\Gamma_C(X) = \prod_{\sigma \in C} \left( X - \sum_{i=1}^{n} h(a_i) \sigma(a_i) \right).$$

Theorem 5.3 of [DD10] states the following (specializing to extensions of $\mathbb{Q}$).

Theorem. Assume the notation above.

(1) For each conjugacy class $C \subseteq G$, $\Gamma_C(X)$ has coefficients in $\mathbb{Q}$. 

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If \( p \) is a prime that does not divide the denominators of \( F(x) \), \( h(x) \) and the resolvents of \( \Gamma_C \) and \( \Gamma_{C'} \) for different \( C \) and \( C' \), then

\[
\text{Frob}_p = C \iff \Gamma_C \left( \text{Tr}_{F_p(x)/F_p}(h(x)x^p) \right) \equiv 0 \pmod{p}.
\]

We wish to apply this theorem in the case that \( G = H \) and when the Galois group of \( K/\mathbb{Q} \) may not necessarily be \( G \). An examination of the proof shows that the theorem remains true even if \( \text{Gal}(K/\mathbb{Q}) \) is a proper subgroup of \( G \).

Our setup is the following. Suppose that \( E/\mathbb{Q} \) is an elliptic curve with a model chosen that has integer coefficients. Suppose also that we know, a priori, that the image of the mod \( N \) Galois representation is contained in \( H \subseteq \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \). The following algorithm gives a method to prove that the mod \( N \) image is equal to \( H \). Define

\[
s_1(N) = \begin{cases} 4 & \text{if } N = 2 \\ p & \text{if } N > 2 \text{ is a power of the prime } p \\ 1 & \text{otherwise}, \end{cases}
\]

\[
s_2(N) = \begin{cases} 8 & \text{if } N = 2 \\ 9 & \text{if } N = 3 \\ p & \text{if } N > 3 \text{ is a power of the prime } p \\ 1 & \text{otherwise}. \end{cases}
\]

1. We fix an isomorphism \( \phi: (\mathbb{Z}/N\mathbb{Z})^2 \to E[N] \) and pre-compute decimal expansions of \( f(P) = s_1(N)x(P) + s_2(N)y(P) \) for all torsion points of \( P \) of order \( N \) on \( E \). By Theorem VIII.7.1 of [Sil09], these numbers are algebraic integers.

2. The action of Galois on the numbers \( s_1(N)x(P) + s_2(N)y(P) \) is given by some conjugate of \( H \). We attempt to identify a unique conjugate of \( H \) in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) that gives this action. We do this by numerically computing

\[
\sum_{k \in K} f(\phi(k(1,0)))f(\phi(k(0,1))) + f(\phi(k(1,0)))f(\phi(k(1,1))) + f(\phi(k(0,1)))f(\phi(k(1,1)))
\]

for each conjugate \( K \) of \( H \) inside \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \). If the image of the mod \( N \) representation is contained in \( K \), then the sum above will be an integer.

3. We compute the polynomial \( F(x) \) with integer coefficients whose roots are the numbers \( f(P) = s_1(N)x(P) + s_2(N)y(P) \).

4. We compute the resolvent polynomials for all of the conjugacy classes of \( H \) and check that these have no common factor. (In practice, we use \( h(x) = x^3 \) to construct these polynomials.)

5. Using the resolvent polynomials, we compute the conjugacy class of \( \rho_{E,N}(\text{Frob}_p) \subseteq H \) for lots of different primes \( p \).

6. We enumerate the maximal subgroups of \( H \) and determine which conjugacy classes they intersect. We check to see if the conjugacy classes found in the previous step all lie in some proper maximal subgroup of \( H \). If not, then the image of \( \rho_{E,N} \) is equal to \( H \).

Note that it is not possible for a maximal subgroup \( M \subseteq H \) to intersect all of the conjugacy classes of \( H \).
Example A.3. Let $E : y^2 = x^3 + x^2 - 28x + 48$. This elliptic curve has $j$-invariant 78608, which corresponds to a non-CM rational point on $X_{556}$ and hence, the 2-adic image for $E$ is contained in $H_{556}$, an index 96 subgroup of $GL_2(\mathbb{Z}_2)$ that contains $\Gamma(16)$. We must show that the 2-adic image equals $H_{556}$. Every maximal subgroup of $H_{556}$ also contains $\Gamma(16)$, so it suffices to compute the image of the mod 16 Galois representation attached to $E$. To do this, we fix an isomorphism $E[16] \cong (\mathbb{Z}/16\mathbb{Z})^2$, and precompute decimal expansions of $2x(P) + 2y(P)$ for all $P \in E[16]$, using 1000 digits of decimal precision. There are 24 conjugates of $H_{556}$ in $GL_2(\mathbb{Z}_2)$, and we find that the expression in step 2 above is an integer only for one of the conjugates of $H_{556}$.

The image of $H_{556}$ under the map $GL_2(\mathbb{Z}_2) \to GL_2(\mathbb{Z}/16\mathbb{Z})$ has 46 conjugates classes, and we compute the polynomial $F(x)$ whose roots are the 192 numbers $2x(P) + 2y(P)$. We then compute the resolvent polynomials for the 46 conjugacy classes. Then, for each prime $p \leq 30000$, we compute $\text{Tr}_{\mathbb{F}_p} (x^{p+3})$ and check which resolvent polynomial has this number as a root in $\mathbb{F}_p$. Using this, we can determine which conjugacy class is the image of $\text{Frob}_p$. We find that all 46 conjugacy classes are in the image of $\text{Frob}_p$ for some $p$. (For example, the smallest prime $p$ which splits completely in $Q(E[16])$ is $p = 5441$.) As a consequence the image of the mod 16 Galois representation of $E$ is $H_{556}$.

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