General Boundary Formulation for $n$–dimensional classical abelian Yang-Mills theory with corners

Homero G. Díaz-Marín *

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Abstract

We propose a general reduction procedure for classical field theories provided with abelian gauge symmetries in a Lagrangian setting. These ideas come from an axiomatic presentation of the General Boundary Formulation (GBF) of field theories, mostly inspired in Topological Quantum Field Theories (TQFT). We exemplify the consistency of this procedure for the abelian Yang Mills theories. We treat the case for space-time manifolds with smooth boundary components but also the case of manifolds with corners. This treatment is the GBF analogous of extended TQFTs. The aim for developing this classical formalism is to accomplish, in a future work, geometric quantization at least for the abelian case.

1 Introduction

In the variational formulation of classical mechanics, time evolution from an "initial" to a "final" state in a symplectic phase space $(A,\omega)$ is given by a relation defined by a lagrangian space $L$ contained in the symplectic product $(A \oplus \bar{A},\omega \oplus -\omega)$. Similarly classical field theories can be formalized rigorously in a symplectic framework. The evolution relation associates "incoming" to "outgoing" Cauchy boundary data for the case where $M$ has incoming and outgoing boundary components, $\partial M = \partial M_{in} \cup \partial M_{out}$. Fields are valued along the boundary altogether with their derivatives. This relation defines an isotropic space of boundary conditions that extend to solutions in the interior of $M$, $L_M \subset A_{\partial M} = A_{\partial M_{in}} \times A_{\partial M_{out}}$, where the symplectic

*On leave from: Centro de Ciencias Matemáticas Universidad Nacional Autónoma de México, C.P. 58190, Morelia, Michoacán, México, Email: homero@matmor.unam.mx currently at Universidad Michoacana, Email: hdiaz@umich.mx
structure, $\omega_{\partial M} = \omega_{in} \oplus \omega_{out}$, is formed by certain symplectic structures $\omega_{in}$ and $\omega_{out}$ defined in $A_{\partial M_{in}}$ and $A_{\partial M_{out}}$, respectively. For recent progress from a categorical point of view of this classical formalism in the case of linear symplectic spaces, see for instance [We]. In some cases, degeneracies of the lagrangian density, yield degeneracies of $\omega_{\partial M}$, defining just a the presymplectic structure for the Cauchy data $A_{\partial M}$.

A wise observation appearing for the first time in [KT], is that it is possible to formulate a symplectic framework for field theories in general spacetime regions $M$. Here general boundaries $\partial M$, are composed of general hypersurfaces, which do not necessarily correspond to "in" and "out" space-like boundary components. A local expression for the symplectic structure, $\omega_{in}$, is given in [KT], for the space $A_{\partial M}$ of 1−jets arising from Cauchy data, namely, Dirichlet and Neumann boundary data. This derivation of the symplectic formalism, was independently rediscovered in a General Boundary Formulation (GBF) for classical theories in [O3], [O], arising from their quantum counterparts. Here the definition of a (pre)symplectic structure is given for the space $\tilde{A}_{\partial M}$ of germs of solutions of a cylinder of the boundary $\partial M_{\varepsilon} := \partial M \times [0,\varepsilon]$. Axiomatic frameworks incorporating this symplectic formalisms appeared in [O3], [O], for linear field theories, whereas for the case of affine field theories it appeared in [O1]. This formalism in another point of view, appeared independently in [CMR], where it is related to the BFV and BV formalism. Here also appears explicitly the distinction for the (pre)symplectic structure for 1−jets and for germs.

The space of germs $\tilde{A}_{\partial M}$ contains much more information than the field and its first derivatives in $\partial M$, as a consequence, instead of the symplectic structure $\omega_{\partial M}$, we have a presymplectic structure $\tilde{\omega}_{\partial M}$ and a coisotropic space $\tilde{A}_{\partial M}$. Thus $\tilde{A}_{\partial M}$ needs to be reduced in order to obtain a symplectic space.

In either case, whether we use the space of germs $\tilde{A}_{\partial M}$ or the space of Cauchy boundary data, we require a reduction procedure in order to get a symplectic structure. That is why, we suppose that all kinds of degeneracies: those due to germ extension and those due to lagrangian density, may be incorporated altogether in the kernel of a presymplectic structure $\tilde{\omega}_{\partial M}$ in $\tilde{A}_{\partial M}$. So that the reduced space $A_{\partial M}$ is a symplectic space.

The description of the dynamics in the interior of the space-time region $M$, is given by the boundary data $\tilde{A}_{\partial M} \subset A_{\partial M}$ of solutions of Euler-Lagrange equations. For infinite dimensional symplectic spaces, isotropic spaces are required to be coisotropic in order to be lagrangian. Isotropy is always verified [KT], but the general proof for the coisotropic embedding of $\tilde{A}_{\partial M} \subset A_{\partial M}$ does not hold, see counterexamples in [CMR].
From the quantum side the axiomatic setting for GBF is inspired on the axiomatic setting of Topological Quantum Field Theories (TQFT), see [At] and the formulation of G. Segal. We consider objects in the category of \((n-1)\)-manifolds, i.e. closed boundary components or hypersurfaces \(\Sigma\), provided with additional normal structure required by germs of solutions: for instance for field theories without metric dependence we consider gluings by diffeomorphisms of tubular neighborhoods of \(\Sigma\), [Mi], meanwhile for field theories depending on the metric we consider gluing by isometries of \(\Sigma, \Sigma'\) leaving invariant the metric tensor germ along \(\Sigma\). The gluing of two regions \(M_1, M_2\) can be performed along hypersurfaces \(\Sigma \subset M_1, \Sigma' \subset M_2\), both diffeomorphic and oriented manifolds, \(\Sigma \cong \Sigma'\), and \(\Sigma'\) means reversed orientation. The precise axiomatic setting for quantum field theories along with their classical counterpart appears in [O3] and for affine theories in [O1].

**Corners.** This TQFT-inspired approach requires a classification of the basic regions or building blocks used to reconstruct the whole space-time region \(M_1 \cup_\Sigma M_2\), by gluing the pieces \(M_1, M_2\), along the boundary hypersurface \(\Sigma \cong \Sigma'\). This classification from the topological point of view can be achieved at least for the case of two dimensional surfaces. In higher dimensions, it would be appealing to avoid such classification issues, by considering simpler building blocks, such as \(n\)-balls. Unfortunately, the price to pay is that we should allow gluings of regions along hypersurfaces \(\Sigma\) with nonempty boundaries \(\partial \Sigma\). For instance we can consider the gluing of two \(n\)-balls \(M_1, M_2\) along \((n-1)\)-balls contained in their boundaries \(\Sigma, \Sigma'\). This means that we should allow non differentiability and lack of normal derivatives of fields along the \((n-2)\)-dimensional corners contained in the boundaries \(\partial \Sigma\), of boundary faces, \(\Sigma \subset \partial M_1\). A well suited language to describe such phenomena, consists in treating regions \(M_i\) as manifolds with corners. For TQFT the attempt to deal with the case of corners gives rise to Extended Topological Quantum Field Theories, one possible approach for two dimensional theories is given for instance in [LP]. There is also a specific formulation for 2-dimensional Yang-Mills with corners in [O2]. Our aim is to extend this last approach to higher dimensions.

**Gauge field theories.** When we consider principal connections on a principal bundle \(P \to M\), with structure compact Lie group \(G\), they are represented by sections of the quotient affine 1–jet bundle \(J^1P/G \to M\). In this case the space of sections \(K_M\) is an affine space. Furthermore for quadratic Lagrangian densities we will have that the space of solutions, \(A_M\), is an affine space. This enable us to consider a GBF formalism for affine spaces such as is described in [O1].

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The novel issue with respect to \[O1\], is to consider gauge symmetries, \(G_M\), acting on \(A_M\). These symmetries are vertical automorphisms of the bundle \(P\), that in turn yield vertical automorphisms of the bundle \(J^1P/G\). Infinitesimal gauge symmetries should preserve the action, \(S_M : K_M \to \mathbb{R}\). These can be identified with vertical \(G\)–invariant vector fields \(\vec{X}\) on \(P\), and also with sections of \(VP/G \to M\), where \(VP\) is the vertical tangent bundle of \(P \to M\). These vertical vector fields act on \(J^1P/G\) preserving the lagrangian density.

When we consider germs of solutions of the boundary, we also have symmetries \(\tilde{G}_{\partial M}\), and quotienting by degeneracies we obtain a gauge group action \(G_{\partial M}\) acting by symplectomorphisms on \((A_{\partial M}, \omega_{\partial M})\). The main problem is to give sense to the quotient space \(A_M/G_M\) of solutions and how relate the reduced boundary conditions contained in the symplectic reduction \(A_{\partial M}/G_{\partial M}\). The issues of gluing solutions need also to be clarified.

**Main results.** Our aim is to give an axiomatic GBF formulation for gauge field theories in the case of space-time regions with corners. For the classical theory we will consider the following simplifications: Abelian structure groups and affine structure for the space of solutions of Euler-Lagrange equations. The test example will be Yang-Mills action. The most general setting of nonabelian structure groups remains as a conjecture even in the classical case, see \([CMR1]\). Along this program we study the case without corners and then we focus our attention on the case with corners.

The important result for \(n\)–dimensional field theories without corners is the symplectic reduction theorem \([1]\). For the case with corners this theorem replicates as theorem \([3]\). One of the most important ingredients comes from establishing suitable local Fermi type coordinates, this in turn comes from a volume transport argument by Moser \([Mo]\) in lemma \([1]\). Dynamics as lagrangian relation is particularly interesting and it is established in theorem \([2]\) whose statement for the case with corners remains the same.

As we were finishing writing this article we realized that Lagrangian embedding in the abelian case had been shown independently in \([CMR1]\) for the case without corners. Nevertheless there are some differences in our approach: Whereas we used axial gauge fixing and used Hodge star in the boundary in order to describe a Hodge decomposition for the space of boundary conditions that extend to solutions the interior. On the other hand, in appendix C of \([CMR1]\), the authors used Lorentz fixing and independently give the space of boundary conditions \(H^1(M, \partial M)\) this Hodge structure by considering such decompositions for any subspace of Dirichlet boundary conditions \(L \subset \Omega^1(\partial M)\).

**Description of sections.** Section \([2]\) consists in a review of the sym-
plectic formalism for classical field theories altogether with an exposition of the axiomatic setting that we propose to abelian gauge field theories. We also divide the exposition of the axioms into two cases: the case where regions are considered as manifolds and the case where regions are manifolds with corners. In section 3 we exemplify the application of the concepts and axioms described in section 2. Here we address the simpler case without corners and propose the abelian Yang-Mills theory. For local arguments Moser’s argument on the transport flow for volume forms will be relevant. We describe the symplectic reduction of the space of boundary conditions and emphasize the proofs of the lagrangian embedding of solutions once reduction is achieved. The Friedrichs-Morrey-Hodge theories are the main source of our results. For the case with corners, in section 4 a complete description of the reduction can be achieved. We review dimension 2 as an example in section 4.1.

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2 Axioms for classical abelian gauge field theories

2.1 The symplectic setting for classical lagrangian field theories

For the sake of completeness, we resume the symplectic formalism for lagrangian field theories in the following paragraphs. Local descriptions for the case of the space of Dirichlet-Neumann conditions appear in [KT], on the other hand discussion on the space of germ of solutions in the axiomatic setting appears in [O], [O1], parallel developments appear in [CMR]. We adopt an abstract coordinate-free description of the (pre-)symplectic structure for boundary data, by means of a suitable a cohomological point of view for the presentation of this formalism.

Classical field theories assume that over an $n$–dimensional space-time region $M$, there exist a "configuration space", $K_M$, of fields $\varphi \in K_M$. The word "space" used for referring to $K_M$ usually denote infinite dimensional Fréchet manifolds, defined as a space of sections of a smooth bundle $E$ over $M$. It also assumes the existence of a Lagrangian density, $\Lambda \in \Omega^n(J^1M)$, depending on the first-jet $j^1\varphi \in J^1M$, i.e. on the first order derivatives $\partial \varphi$ and on the values of the fields $\varphi$. The action corresponding to the lagrangian
density is then defined as

\[ S_M(\varphi) = \int_M j^1(\varphi)^* \Lambda. \]

On the other hand we consider the factorization of the space of \( k \)-forms over the \( l \)-jet manifold \( J^l M \) as

\[ \Omega^k(J^l M) = \bigoplus_{r=0}^k \Omega^r_H(J^l M) \otimes \Omega^{k-r}_V(J^l M) \]

where the complex \( \Omega^k_H(J^l K_M) \) (resp. \( \Omega^k_V(J^l K_M) \)) corresponds to horizontal (resp. vertical) \( k \)-forms. For instance, using local coordinates \( x^i, i = 1, \ldots, n \), for the manifold \( M \), take \((x^i; u^a; u^a_i)\) as local coordinates for \( J^1 M \), horizontal forms have basis the exterior product of the basis \( dx^i \). Meanwhile for vertical forms say in \( J^1 M \), we have as basis the exterior product of \( du^a, du^a_i \).

The horizontal (resp. vertical) differential is

\[ d_H : \Omega^k_H(J^1 M) \to \Omega^{k+1}_H(J^{l+1} M) \]

(resp. \( d_V := d - d_H \)). For instance, for horizontal 0-forms we have

\[ d_H := \sum_{i=1}^n \left( \frac{\partial}{\partial x_i} + \sum_{a=1}^r u^a_i \frac{\partial}{\partial u^a} \right) dx^i : \Omega^0_H(J^1 M) \to \Omega^1_H(J^1 M), \]

where \( r \) equals the dimension of each fiber of the bundle \( E \). Thus, vertical \( k \)-forms vanish on horizontal vector fields \( \vec{X} \) such that \( d_V(\vec{X}) = 0 \). This decomposition yields a variational bicomplex, see for instance [GMS].

\[ \begin{array}{cccc}
0 & 0 & \ldots \\
\Omega^0_H(J^1 M) & \Omega^1_H(J^1 M) \otimes \Omega^1_V(J^2 M) & \ldots \\
\Omega^1_H(J^0 M) & \Omega^0_H(J^0 M) \otimes \Omega^1_V(J^1 M) & \ldots \\
\ldots & \ldots & \ldots
\end{array} \]

Denote the space of solutions of the Euler Lagrange equations as

\[ A_M = \{ \varphi \in K_M \mid (j^2 \varphi)^* (d_V \Lambda) = 0 \} . \]
In the case where we are dealing with connections $A_M$ is an affine space corresponding to a linear space that we call $L_M$.

Consider the image $d_V\Lambda \in \Omega_H^n(J^1M) \otimes \Omega_V^1(J^2M)$, of the lagrangian density, $\Lambda \in \Omega_H^n(J^1M)$. Take a preimage

$$\theta_\Lambda \in d_H^{-1} \circ d_V \Lambda \in \Omega_H^{n-1}(J^0M) \otimes \Omega_V^1(J^1M).$$

Of course the representative $\theta_\Lambda \in d_H^{-1} \circ d_V \Lambda$ depends just on the $d_H$–cohomology class of the lagrangian density. On the other hand, by integration by parts, the differential

$$dS_M(\delta \varphi) = (dS_M)_\varphi(X) = \int_M (j^2\varphi)^* (t_{(j^2\bar{X})}d_V\Lambda) + \int_{\partial M} (j^1\varphi)^* (t_{\bar{X}}\theta_\Lambda)$$

evaluated on variations $\delta \varphi = X \in T_xK_M$. Locally each variation $\delta \varphi$ is identified with a vector field, $\bar{X}$, along the section $\varphi$ in $J^1M$. This $\bar{X}$ in turn induces a vector field $j^2\bar{X}$, the 2–jet prolongation of the vector field $\bar{X}$, along $j^2\varphi$, on the 2–jet manifold $J^2M$, both vanishing on horizontal 1–forms. This shows that total variations consist of two contributions: One due to the variation on the bulk of the fields corresponding to Euler-Lagrange equations, but also a contribution coming from the values of the field and its normal derivatives on the boundary $\partial M$.

Let us concentrate on the boundary term of the variation. The calculus on the 1–jet total space, $J^1M$, translates onto the calculus on the infinite dimensional space, $K_M$, so that $\theta_\Lambda$ induces a 1–form

$$(dS_M)_\varphi(X) = \int_{\partial M} (j^1\varphi)^* (t_{\bar{X}}\theta_\Lambda) \tag{1}$$

for variations of 1–jets of solutions restricted to the boundary $X \in T_xA_M$. This enables us to consider a 1–form $dS_\varphi$, for variations $X \in T_xA_M$.

For a $(n - 1)$–dimensional boundary manifold $\Sigma$, the boundary conditions for solutions on a tubular neighborhood $\Sigma_\varepsilon \cong \Sigma \times [0,\varepsilon]$, of the cylinder $\Sigma \times [0,1]$, can be described as germs of solutions.

The affine space of germs of solutions on the boundary, and the corresponding linear space are defined as the injective limit

$$\tilde{A}_\Sigma := \lim \rightarrow \; A_{\Sigma_\varepsilon}, \quad \tilde{L}_\Sigma := \lim \rightarrow \; L_{\Sigma_\varepsilon}$$

where the inclusion $\Sigma_\varepsilon \subset \Sigma_{\varepsilon'}$, for $\varepsilon < \varepsilon'$, induces an inclusion $A_{\Sigma_{\varepsilon'}} \subset A_{\Sigma_\varepsilon}$. Similarly for the linear spaces $L_{\Sigma_{\varepsilon'}} \subset L_{\Sigma_\varepsilon}$. We consider spaces of solutions on cylinders $\Sigma \times [0,\varepsilon]$, $\varepsilon > 0$, embedded as tubular neighborhoods $\Sigma_\varepsilon$ of $\Sigma$.
The submersion of variations of germs $\tilde{X} \in T \tilde{A}_\Sigma$, onto variations of jets $\tilde{X} \in TA_\Sigma$, leads to the definition of the $1$–form on $\tilde{A}_{\partial M}$,

\[
\left( \tilde{\theta}_\Sigma \right)_\varphi (\tilde{X}) := (dS_M)_\varphi (X)
\]

Ultimately, our purpose is to consider the presymplectic structure on $\tilde{A}_\Sigma$,

\[
\tilde{\omega}_\Sigma = d\tilde{\theta}_\Sigma.
\]

There are degeneracies of the presymplectic structure $\tilde{\omega}_\Sigma$ due to the degeneracy of the lagrangian density and the degeneracies arising from considering arbitrary order derivatives for the germs of solutions. We suppose that these degeneracies altogether can be eliminated by quotienting $K_{\omega_\Sigma} : = \ker \omega_\Sigma$, then we obtain a symplectic space $A_{\Sigma, \omega_\Sigma}$, we will prescribe this condition as an axiom.

Consider an action map $S_M(\varphi)$ defined for connections $\varphi$ of a principal bundle $P$ over $M$ with compact abelian structure group $G$. We denote as $A_M$, the space of solutions Euler-Lagrange equations in the interior of the region $M$. In general, we suppose that $\partial M$ is not empty. Hence when we restrict the action functional $S_M$, from the configuration field space $K_M$ to the space of solutions $A_M$, it induces a non-constant map

\[
S_M : A_M \to \mathbb{R}.
\]

On the other hand we have the groups, $G_M$, of gauge symmetries on regions acting on the solutions on the bulk $A_M$ that come from the Euler-Lagrange variational symmetries of lagrangian density, see definition 2.3.1 of [GMS]. Infinitesimal symmetries can be identified with $G$–invariant vertical vector fields on $P$ that can be identified with vertical vector fields acting on $J^1 P/G$ and preserving the lagrangian density.

By taking the cylinder $\Sigma \times [0, \varepsilon]$ as $M$, those symmetries by the group $G_\Sigma \times [0, \varepsilon]$ act on germs of solutions in $A_{\Sigma, \varepsilon}$ hence in $\tilde{A}_\Sigma$. By taking the quotient by the stabilizer of the $\tilde{A}_\Sigma$, we obtain a group of gauge symmetries on hypersurfaces,

\[
\tilde{G}_\Sigma := \lim_{\rightarrow} G_\Sigma \times [0, \varepsilon]
\]

acting on $\tilde{A}_\Sigma$.

Once we have taken the quotient of the space of germs $\tilde{A}_\Sigma$, and its corresponding linear space $\tilde{L}_\Sigma$, by the degeneracy space $K_{\omega_\Sigma}$, we get a space $A_\Sigma$, and a gauge group $G_\Sigma$ acting on $A_\Sigma$. This group $\tilde{G}_\Sigma$ decomposes into two kind of symmetries: those coming from the degeneracy of the presymplectic
structure and those preserving the symplectic structure coming from vector fields preserving the lagrangian density. This means that there is a normal subgroup that takes into account all degeneracies altogether, $K_{\omega_\Sigma} \subset \tilde{G}_\Sigma$, and whose orbits on $\tilde{A}_\Sigma$ consist of the integral leaves of the characteristic distribution generated by the kernel of the presymplectic structure $\tilde{\omega}_\Sigma$. Meanwhile $G_\Sigma$ act by symplectomorphisms on $\tilde{A}_\Sigma$ with respect to the symplectic structure $\omega_\Sigma$.

2.2 Regions with and without corners

In the following presentation of the axiomatic for classical lagrangian field theories, we will consider regions and hypersurfaces as manifolds with corners.

Hypersurfaces are $(n-1)$-dimensional topological manifolds $\Sigma$, decomposing as a union of $(n-1)$-dimensional manifolds with corners, 

$$\Sigma = \bigcup_{i=1}^{m} \Sigma^i \equiv \bigsqcup_{i=1}^{m} \tilde{\Sigma}^i / \sim_P.$$ 

This union in turn is obtained by gluing of $(n-1)$-dimensional manifolds with corners $\tilde{\Sigma}^i, \tilde{\Sigma}^j$, along pairs of $(n-2)$-faces. This can be done by means of an equivalence relation $\sim_P$, defined by certain set $P$ of pairs $(i, j)$, $i \neq j$.

More precisely, non trivial equivalence identifications take place at the set

$$\bigcup_{(i,j) \in P} \tilde{\Sigma}^{ij} := \bigcup_{(i,j) \in P} \Sigma^i \cap \Sigma^j.$$ 

This means that gluings of the faces $\Sigma^i, \Sigma^j$, take place at $(n-2)$-faces $\Sigma^{ij} \subset \partial \Sigma^i, \Sigma^{ji} \subset \partial \Sigma^j, \Sigma^{ij} \approx \Sigma^{ji}$.

A region is an $n$-dimensional manifold with corners $M$. Its boundary $\partial M$, is a topological manifold. Each hypersurface $\Sigma \subset \partial M$ consists of faces $\Sigma^i \subset \partial M$, which are manifolds with corners. For an abstract hypersurface, not necessarily related to a region $M$, each $\Sigma^i$ may be consider as a face of the $n$-dimensional manifold with corners given by the cylinder $\Sigma \times [0, \varepsilon]$. In general $\partial \Sigma, \partial \Sigma^i$ may be nonempty.

The cylinder $\Sigma \times [0, \varepsilon]$ for closed smooth hypersurfaces $\Sigma, \partial \Sigma = \emptyset$, can be generalized for each manifold with corners $\Sigma^i, \partial \Sigma^i \neq \emptyset$. A regular cylinder consist of the image $W(\Sigma^i \times [0, \varepsilon]) \subset \Sigma^i \times [0, \varepsilon]$ of smooth maps

$$W : \Sigma^i \times [0, \varepsilon] \rightarrow \Sigma^i \times [0, \varepsilon],$$

such that $W(s, 0) = s, \forall s \in \Sigma^i$, and $W(s, t) = s \forall s \in \partial \Sigma^i, t \in [0, \varepsilon]$.

The gluing of a region $M$ along two nonintersecting faces $\Sigma_0, \Sigma'_0$, can be defined. The more general gluing along two nonintersecting hypersurfaces
\[ \Sigma, \Sigma', \] may also be defined. Nonetheless, when we consider, for instance, the gluing of riemannian metrics, this gluing may be problematic. For if we glue faces with non intersecting boundaries \( \partial \Sigma_0 \cap \partial \Sigma_0' = \emptyset \), then conic singularities of the metric along the corners may arise in the resulting spacetime region. Therefore gluings should be restricted to nonintersecting faces.

For smooth hypersurfaces \( \Sigma \subset \partial M \) we consider tubular neighborhoods \( \{ M_\epsilon \} \), \( \Sigma_\epsilon \subset M \) with diffeomorphisms
\[
X : \Sigma \times [0, \epsilon] \to \Sigma_\epsilon
\]
We will also have regular tubular neighborhoods of faces \( \Sigma_i \), consisting of images
\[
\hat{\Sigma}_i^\epsilon := X(W(\Sigma \times [0, \epsilon])).
\]
The corners of the region \( M \) lie on the union of the \( (n-2) \)-dimensional submanifolds, \( \cup_{(i,j) \in P} \Sigma^{ij} \).

For a more detailed description of faces and corners, see for instance \[\text{AGO}].\]

### 2.3 GBF axioms

Now we give a detailed description of the axiomatic setting for classical gauge field theories. Axioms \([\text{A1}]\) to \([\text{A9}]\) describe the kinematics of the classical theory, while axioms \([\text{A10}]\) to \([\text{A12}]\) describe the dynamics for gauge fields.

\begin{enumerate}
\item **Affine structure:** We suppose that for regions \( M \) and hypersurfaces \( \Sigma \) we have \( A_M, A_\Sigma \), affine spaces with associated linear spaces \( L_M, \tilde{L}_\Sigma \), respectively with real linear and affine maps \( \tilde{r}_M : L_M \to \tilde{L}_\partial M \) and \( \tilde{a}_M : A_M \to \tilde{A}_\partial M \), respectively.

\item **Presymplectic structure:** For every hypersurface \( \Sigma \subset \partial M \), there is a presymplectic structure \( \tilde{\omega}_\Sigma \) on \( \tilde{A}_\Sigma \) invariant under \( \tilde{L}_\Sigma \) actions. Equivalently we can consider a \( \tilde{L}_\Sigma \) as a presymplectic vector space with presymplectic structure that we can also denote as \( \tilde{\omega}_\Sigma \).

\item **Symplectic structure:** There is a group \( K_{\omega_\Sigma} \) acting freely by translations on \( \tilde{A}_\Sigma \), such that \( K_{\omega_\Sigma} \) is isomorphic to a closed linear subspace \( \ker \tilde{\omega}_\Sigma \subset \tilde{L}_\Sigma \), so that \( \tilde{\omega}_\Sigma \) induces a symplectic structure, \( \omega_\Sigma \), on the orbit space
\[
A_\Sigma := \tilde{A}_\Sigma / K_{\omega_\Sigma}
\]
This space is an affine space modeled on the linear space \( L_\Sigma := \tilde{L}_\Sigma / K_{\omega_\Sigma} \).
\end{enumerate}
(A4) **Symplectic potential:** There is a symplectic potential i.e. a $L_\Sigma$-valued 1-form $\theta_\Sigma(\cdot, \cdot)$ for each $\varphi \in A_\Sigma$, identified with a linear map $\theta_\Sigma(\cdot, \cdot) : L_\Sigma \rightarrow \mathbb{R}$. There is a bilinear map $[\cdot, \cdot]_\Sigma : L_\Sigma \times L_\Sigma \rightarrow \mathbb{R}$ such that

$$[\phi, \phi']_\Sigma + \theta_\Sigma(\eta, \phi') = \theta_\Sigma(\phi + \eta, \phi'), \quad \eta \in A_\Sigma, \quad \phi, \phi' \in L_\Sigma \quad (2)$$

and

$$\omega_\Sigma(\phi, \phi') = \frac{1}{2} [\phi, \phi']_\Sigma - \frac{1}{2} [\phi', \phi]_\Sigma, \quad \phi, \phi' \in L_\Sigma. \quad (3)$$

There exists an action map $S_M : A_M \rightarrow \mathbb{R}$, such that

$$S_M(\eta) = S_M(\eta') - \frac{1}{2} \theta_M(\eta, \eta' - \eta') - \frac{1}{2} \theta_M(\eta', \eta - \eta') \quad (4)$$

and also $S_M(\eta) = S_M(\eta')$ for $a_M(\eta) = a_M(\eta')$.

(A5) **Involution:** For each hypersurface $\Sigma$ there exists an involution $A_\Sigma \rightarrow A_\Sigma$, where $\Sigma$ is the hypersurface with reversed orientation. There is also a linear involution $L_\Sigma \rightarrow L_\Sigma^\ast$. For the linear forms we have: $\theta_\Sigma(\eta, \phi) = -\theta_\Sigma(\eta, \phi)$ and $[\phi, \phi']_\Sigma = -[\phi', \phi]_\Sigma$.

(A6) **Disjoint regions:** For a disjoint union, $M = M_1 \sqcup M_2$, there is a bijection $A_{M_1} \times A_{M_2} \rightarrow A_M$ an compatible linear isomorphisms $L_{M_1} \times L_{M_2} \rightarrow L_M$, such that $a_M = a_{M_1} \times a_{M_2}$ and $r_M = r_{M_1} \times r_{M_2}$, satisfy associative conditions. For the action map $S_M = S_{M_1} + S_{M_2}$.

(A7) **Factorization of fields on hypersurfaces:** For a hypersurface $\Sigma$ obtained as the quotient $\tilde{\Sigma}^1 \sqcup \cdots \sqcup \tilde{\Sigma}^k$ by an equivalence relation $\sim_\varphi$, define $A_{[\Sigma]^{n-1}} := A_{\tilde{\Sigma}^1} \times \cdots \times A_{\tilde{\Sigma}^m}$, $L_{[\Sigma]^{n-1}} := L_{\tilde{\Sigma}^1} \oplus \cdots \oplus L_{\tilde{\Sigma}^m}$. Then there are affine gluing maps $a_{[\Sigma]^{n-1}} : A_\Sigma \rightarrow A_{[\Sigma]^{n-1}}$, and compatible linear maps $r_{[\Sigma]^{n-1}} : L_\Sigma \rightarrow L_{[\Sigma]^{n-1}}$ with commuting diagrams

$$A_\Sigma \xrightarrow{a_{[\Sigma]^{n-1}}} A_{[\Sigma]^{n-1}} \quad L_\Sigma \xrightarrow{r_{[\Sigma]^{n-1}}} L_{[\Sigma]^{n-1}}$$

We also have the relation

$$[\cdot, \cdot]_\Sigma = r_{[\Sigma]^{n-1}}^\ast ([\cdot, \cdot]_{\tilde{\Sigma}^1} + \cdots + [\cdot, \cdot]_{\tilde{\Sigma}^m}), \quad (5)$$

$$\theta_\Sigma = r_{[\Sigma]^{n-1}}^\ast (\theta_{\tilde{\Sigma}^1} + \cdots + \theta_{\tilde{\Sigma}^m})$$

we denote $A_{[\Sigma]}$ as the image of $A_\Sigma$ into $A_{[\Sigma]}$, and similarly $L_{[\Sigma]}$.  

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(A8) **Gauge action:** There are groups $\tilde{G}_\Sigma$ acting on $\tilde{A}_\Sigma$ preserving the affine structure and the presymplectic structure $\tilde{\omega}_\Sigma$ such that $K_{\omega_\Sigma} \triangleleft \tilde{G}_\Sigma$. The quotient group

$$G_\Sigma := \tilde{G}_\Sigma / K_{\omega_\Sigma}.$$  

acts on $A_\Sigma$, preserving the symplectic structure $\omega_\Sigma$. There is a group, $G_M$, of symmetries of variational symmetries $S_M$ acting on the space of solutions $A_M$. There is a restriction map $a_M : A_M \to A_{\partial M}$ and group homomorphisms $h_M : G_M \to G_{\partial M}$. There is a compatibility of gauge actions given by the commuting diagram

$\begin{array}{ccc}
A_M \times G_M & \longrightarrow & A_{\partial M} \times G_{\partial M} \\
\downarrow & & \downarrow \\
A_M & \longrightarrow & A_{\partial M}
\end{array}$

There is also a compatible action on the corresponding linear spaces $r_M : L_M \to L_{\partial M}$

$\begin{array}{ccc}
L_M \times G_M & \longrightarrow & L_{\partial M} \times G_{\partial M} \\
\downarrow & & \downarrow \\
L_M & \longrightarrow & L_{\partial M}
\end{array}$

(A9) **Factorization of gauge actions on hypersurfaces:** For the case with corners there is a homomorphism $h_{\Sigma;[\Sigma]^{n-1}} : G_\Sigma \to G_{[\Sigma]^{n-1}}$ form the direct product group $G_{[\Sigma]^{n-1}} := G_{\Sigma} \times \cdots \times G_{\Sigma}$ onto $G_\Sigma$ coming from homomorphisms

$$h_{[\Sigma]^{n-1};\Sigma_j} : G_{[\Sigma]^{n-1}} \to G_{\Sigma_j}$$

and commuting diagrams

$\begin{array}{ccc}
G_{[\Sigma]^{n-1}} & \longrightarrow & G_{\Sigma_j} \\
\downarrow & & \downarrow \\
G_\Sigma & \longrightarrow & G_{\Sigma_j}
\end{array}$
There is an involution of the gauge groups and $G_\Sigma \to G_{\bar{\Sigma}i}$, compatible with the action.

We denote $G_\Sigma$ as the image $h|_{\Sigma|^{n-1},\Sigma}(G_\Sigma) \subset G_{\bar{\Sigma}i}$.

(A10) **Lagrangian relation modulo gauge:** Let $A_M$ be image in $A_{|\partial M}$ of the space $a_M(A_M) \subset \tilde{A}_{\partial M}$ of boundary conditions on $\partial M$, that extend to solutions on the bulk $M$. Let $r_M : L_M \to L_{\partial M}$ be the compatible linear map. The subspace $L_{\bar{\tilde{M}}} = \tilde{r}_M(L_M) \subset L_{\partial M}$ is lagrangian.

The zero component of the $G_{\partial M}$–orbit is isomorphic to $C_{\partial M}^\perp$, the symplectic orthogonal complement of a coisotropic subspace $C_{\partial M} \subset L_{\partial M}$.

There is a lagrangian reduced subspace $a_M(A_M) \subset \tilde{A}_{\partial M}$ isomorphic to

$$L_{\tilde{M}} \cap C_{\partial M}/L_{\tilde{M}} \cap C_{\partial M}^\perp$$

of the symplectic reduced space $C_{\partial M}/C_{\partial M}^\perp$.

(A11) **Locality of gauge fields:** Let $M_1$ be the region that can be obtained by the gluing of $M$ along the disjoint faces, $\Sigma_0, \overline{\Sigma_0} \subset \partial M$, where $\Sigma_0 \cong \Sigma_0$.

Then there is an injective affine maps, $a_{M,\Sigma_0}\overline{\Sigma_0} : A_{M_1} \leftrightarrow A_{M}$,
compatible linear map, $r_{M_1;\Sigma_0,\Sigma_0} : L_{M_1} \leftrightarrow L_M$, and a homomorphism $h_{M_1;\Sigma_0,\Sigma_0} : G_{M_1} \leftrightarrow G_M$, with exact sequences

$$A_{M_1} \leftrightarrow A_M \rightarrow A_{\Sigma_0}, \quad L_{M_1} \leftrightarrow L_M \rightarrow L_{\Sigma_0},$$

$$G_{M_1} \leftrightarrow G_M \rightarrow G_{\Sigma_0}$$

where we consider the involution, $A_{\Sigma_0} \rightarrow A_{\Sigma_0}$, for the second arrow on the double map. Recall that $A_{\Sigma_0}$ is the image in $A_{\hat{\Sigma}_0}$. We consider the gluing of the actions

$$A_{M_1} \times G_{M_1} \rightarrow A_M \times G_M \rightarrow A_{\Sigma_0} \times G_{\Sigma_0}$$

compatible with the actions on linear spaces

$$L_{M_1} \times G_{M_1} \rightarrow L_M \times G_M \rightarrow L_{\Sigma_0} \times G_{\Sigma_0}$$

and also $S_{M_1} = S_M \circ a_{M_1;\Sigma_0,\Sigma_0}$.

(A12) **Gluing of gauge fields**: Let $M_1, M$ be regions with corners $M_1$ is obtained by gluing $M$ along hypersurfaces $\Sigma, \Sigma \subset \partial M$. The following diagrams commute

$$A_{M_1} \rightarrow A_M, \quad G_{M_1} \rightarrow G_M,$$

$$A_{\partial M_1} \rightarrow A_{\partial M}, \quad G_{\partial M_1} \rightarrow G_{\partial M}.$$

Finally, we have

$$A_{|\partial M_1|^{n-1}} \rightarrow A_{|\partial M|^{n-1}}, \quad G_{|\partial M_1|^{n-1}} \rightarrow G_{|\partial M|^{n-1}}.$$
where if $|\partial M|^{n-1} = \hat{\Sigma} \sqcup \bar{\Sigma}' \sqcup (\hat{\Sigma}_1 \sqcup \cdots \sqcup \hat{\Sigma}_r)$, and $|\partial M_1|^{n-1} = \hat{\Sigma}_1 \sqcup \cdots \sqcup \hat{\Sigma}_r$, then the map $a_{|\partial M|^{n-1},|\partial M_1|^{n-1}} : A_{|\partial M|^{n-1}} \to A_{|\partial M_1|^{n-1}}$ equals the canonical inclusion

$$A_{\hat{\Sigma}_1} \times \cdots \times A_{\hat{\Sigma}_r} \subset A_{\hat{\Sigma}} \times A_{\bar{\Sigma}'} \times (A_{\hat{\Sigma}_1} \times \cdots \times A_{\hat{\Sigma}_r})$$

analogously for the inclusions

$$r_{|\partial M|^{n-1},|\partial M_1|^{n-1}} : L_{|\partial M|^{n-1}} \to L_{|\partial M_1|^{n-1}},$$

$$h_{|\partial M|^{n-1},|\partial M_1|^{n-1}} : G_{|\partial M|^{n-1}} \to G_{|\partial M_1|^{n-1}}.$$

There is also a compatibility for the gluing of the actions of the gauge groups

\[
\begin{array}{ccc}
A_{M_1} & \xrightarrow{A_{\partial M_1} \times G_{M_1}} & A_M \\
\downarrow & & \downarrow \\
A_{\partial M} & \xrightarrow{A_{\partial M} \times G_{\partial M}} & A_{\partial M}
\end{array}
\]

\[
\begin{array}{ccc}
A_{|\partial M_1|^{n-1}} \times G_{|\partial M_1|^{n-1}} & \xrightarrow{A_{|\partial M|^{n-1}} \times G_{|\partial M|^{n-1}}} & A_{|\partial M|^{n-1}} \times G_{|\partial M|^{n-1}} \\
\downarrow & & \downarrow \\
A_{|\partial M_1|^{n-1}} & \xrightarrow{A_{|\partial M_1|^{n-1}} \times G_{|\partial M_1|^{n-1}}} & A_{|\partial M|^{n-1}}
\end{array}
\]

\[
\begin{array}{ccc}
L_{M_1} & \xrightarrow{L_{\partial M_1} \times G_{M_1}} & L_M \\
\downarrow & & \downarrow \\
L_{\partial M} & \xrightarrow{L_{\partial M} \times G_{\partial M}} & L_{\partial M}
\end{array}
\]

\[
\begin{array}{ccc}
L_{|\partial M_1|^{n-1}} \times G_{|\partial M_1|^{n-1}} & \xrightarrow{L_{|\partial M|^{n-1}} \times G_{|\partial M_1|^{n-1}}} & L_{|\partial M|^{n-1}} \times G_{|\partial M|^{n-1}} \\
\downarrow & & \downarrow \\
L_{|\partial M_1|^{n-1}} & \xrightarrow{L_{|\partial M_1|^{n-1}} \times G_{|\partial M_1|^{n-1}}} & L_{|\partial M|^{n-1}}
\end{array}
\]
2.4 Further explanations on the axioms

Axioms [A1] to [A7] are just a restatement of axioms (C 1) to (C 6) for a classical setting of affine (linear) field theories as stated in [O1]. Some clarifications are added: in [A2] we consider presymplectic spaces of connections instead of symplectic spaces. We do not consider Hilbert space structures since we are not yet introducing a prequantization scenario for field theories.

Some comments can be said about postulate [A4]. The translation rule of the $1$–form $\theta M$ can be deduced from translation rule for the differential $dS_M$ of an action action map. This in turn can be deduced from (4). This last relation could be stated as a primordial property and arises from taking a quadratic lagragian density $\Lambda$. Affine structure for the space of solutions $A_M$ can also be deduced from this condition on $\Lambda$.

In (A7) we adapt the decomposition stated in (C 3) for the corners case. Here we are using the definition of stratified spaces, [AGO]. $|\Sigma|^{(n-1)}$ denote the structure of $\Sigma$ as $(n-1)$–dimensional stratified space

$$\Sigma = \Sigma^1 \cup \cdots \cup \Sigma^m \cong |\Sigma|^{(n-1)}$$

This in turn is the quotient of the disjoint union of faces $|\Sigma|^{n-1} := \Sigma^1 \cup \cdots \cup \Sigma^k$ by an equivalence relation $\sim_P$, defined by certain identification of pairs $(i, j)$ along connected $(n-2)$–dimensional faces $\Sigma^{ij} \subset \partial \Sigma^i, \Sigma^{ji} \subset \partial \Sigma^j$, for certain set $P$ of pairs $(i, j)$. The set of corners correspond to the $(n-2)$–dimensional faces $\Sigma^{ij} := \Sigma^i \cap \Sigma^j, (i, j) \in P$. The $(n-2)$–dimensional skeleton of corners is the stratified space

$$|\Sigma|^{(n-2)} \cong \bigcup_{(i, j) \in P} \Sigma^{ij}$$

The lack of surjectivity for dotted arrows in (A7) comes from the non differentiability of the hypersurface $\Sigma$ along the corners $|\Sigma|^{(n-2)}$ in the intersections $\Sigma^i \cap \Sigma^j, (i, j) \in P$.

Axiom (A8) introduce the gauge symmetries. Axiom (A9) presents the decomposition and involution properties for gauge actions on the boundary. Finally axioms (A11) and (A12) are derivations for the locality and gluing rule of gauge fields arising from the gluing axiom (C 7).

Locality arguments for gauge fields is implicitly exploited in (A8), (A11) and (A12) deserve further clarification. For instance in (A11) the existence of the exact sequence is not trivial and it is derived from locality for connections in $A_M$ and gauge actions in $G_M$. Thanks to the inclusions $\partial M \subset M$
of tubular neighborhoods we get exact sequences

\[
A_{M_1-(\Sigma \cup \Sigma')^c} \to A_M \to A_{\Sigma^c} \quad \text{and} \quad A_{\Sigma'^c} \to A_M \to A_{\Sigma^c},
\]

followed by the maps

\[
A_{\Sigma^c} \leftarrow A_\Sigma \to A_{\Sigma^c}
\]

induce the sequence proposed, when \( \varepsilon \to 0 \). Recall that \( A_\Sigma \) is an inductive limit, and \( A_{\Sigma^c} \) is a quotient of \( A_\Sigma \). For (A8) similar arguments using the following commutative diagrams

\[
\begin{array}{ccc}
A_M \times G_M & \to & A_{\partial M \varepsilon} \times G_{\partial M \varepsilon} \\
\downarrow & & \downarrow \\
A_M & \to & A_{\partial M \varepsilon}
\end{array}
\]

The compatibility stated here as axiom (A8) arises from locality: the embedding of gauge symmetries in \( M \) as local gauge symmetries in a tubular neighborhood \( \partial M \varepsilon \) and then in \( G_{\partial M \varepsilon} \subset \tilde{G}_{\partial M} \), then symmetries for germs hence symmetries in the quotient group \( G_{\partial M} \).

The axiom (A10) encodes the dynamics a gauge fields since it is an adapted version of the lagrangian embedding onto the symplectic space \( A_{\partial M} \) considered in (C 5). In (A10) we use the notion of reduced lagrangian space, see [We]. We could also postulate this axiom as follows. In fact this will be the approach that will be used along this work. There exists a symplectic closed subspace \( \Phi_{A_{\partial M}} \subset L_{\partial M} \), such that \( L_{\tilde{M}} \) intersects transversally the space \( \Phi_{A_{\partial M}} \). Hence \( L_{\tilde{M}} \cap \Phi_{A_{\partial M}} \subset \Phi_{A_{\partial M}} \) is lagrangian. Furthermore every \( G_{\partial M} \)-orbit intersects \( \Phi_{A_{\partial M}} \) in a discrete set. We call \( \Phi_{A_{\partial M}} \) a gauge-fixing space for the gauge symmetries \( G_{\partial M} \).

2.5 Simplifications in the absence of corners

As we mentioned previously for some axioms, namely (A7), (A9) and (A12), we will consider separately two cases: regions with and without corners.
We write down explicitly these axioms in the case where regions $M$ and hypersurfaces $\Sigma$ are smooth manifolds. Here $\partial \Sigma = \emptyset$.

(A7)’ Suppose that and $(n - 1)$–dimensional hypersurface $\Sigma$ decomposes as a disjoint union

$$\Sigma := \Sigma^1 \sqcup \cdots \sqcup \Sigma^m$$

of connected components $\Sigma^1, \ldots, \Sigma^m$. Define $A_{|\Sigma|^{n-1}} := A_{\Sigma^1} \times \cdots \times A_{\Sigma^m}, L_{|\Sigma|^{n-1}} := L_{\Sigma^1} \oplus \cdots \oplus L_{\Sigma^m}$. Then there are linear and affine isomorphisms respectively

$$r_{\Sigma ;|\Sigma|^{n-1}} : L_{\Sigma} \to L_{|\Sigma|^{n-1}}, \quad a_{\Sigma ;|\Sigma|^{n-1}} : A_{\Sigma} \to A_{|\Sigma|^{n-1}}$$

such that (5) holds.

(A9)’ For the case without corners $|\Sigma|^{n-1} \cong \Sigma$ and the direct product group $G_{|\Sigma|^{n-1}} := G_{\Sigma^1} \times \cdots \times G_{\Sigma^m}$ is isomorphic to $G_{\Sigma}$ with a gluing homomorphisms $h_{\Sigma ;|\Sigma|^{n-1}} : G_{\Sigma} \to G_{|\Sigma|^{n-1}}$ with compatibility commuting diagrams

and analogous compatibility diagrams for actions on linear spaces $L_{\partial M}, L_{|\partial M|^{n-1}}$.

(A12)’ Let $M_1, M$ be regions without corners as above with gluing along hypersurfaces $\Sigma, \Sigma' \subset \partial M$

There is also a compatibility for the gluing of the actions of the gauge
For section 3 we will consider the case without corners hence we will consider the simplified version (A7)', (A9)', (A12)'. Meanwhile for section 4 we will consider the corners version (A7), (A9), (A12).

3 The $n$–dimensional case without corners

In this section we consider the case without corners. Another assumption is to consider linear (or affine) field theories, see the comment of axiom (A4) that are well suited for the axiomatic scenario presented in the previous section.

The examples that will be useful as a test case is Yang-Mills lagrangian density. We consider gauge principal bundles on a compact manifold $M$ provided with a riemanian metric $h$, nonempty boundary $\partial M$ and compact abelian fiber group $G$. For the sake of simplicity we assume the following.

We suppose that regions $M$ are smooth manifolds $\dim M \geq 2$, provided with a trivial principal bundle $P$ with abelian structure.
group $G = U(1)$. Hypersurfaces $\Sigma$ are also smooth manifolds, thus $\partial \Sigma = \emptyset$.

Under these assumptions we will prove the consistency of the axioms for the case without corners. In section 4 we will show the case of regions with corners.

3.1 Classical abelian Yang-Mills action

Since the bundle is trivial, the space of connections $A_M$ has a linear structure and can be identified with $L_M$. We consider the Yang-Mills action

$$S_M(\varphi) = \int_M d\varphi \wedge *d\varphi$$

where $\varphi \in A_M$ is a connection that is a solution of the Euler-Lagrange equations in the bulk, i.e. $d^*d\varphi = 0$. The corresponding linear space is

$$L_M \subset \left\{ \varphi \in \Omega^1(M) \mid d^*d\varphi = 0 \right\},$$

here $\Omega^1(M, g)$ denotes $g$-valued 1-forms on $M$ that can be identified with 1-forms in $M$, $\Omega^1(M)$. This objects fulfill (A1).

The identity component of gauge symmetries can be identified with certain $f \in \Omega^0(M)$ acting by $\varphi \mapsto \varphi + df$, thus $G^0_M \cong \Omega^0(M)/\mathbb{R}^{b_0}$, where $\mathbb{R}^{b_0}$ denote the locally constant functions on $M$. In addition we will suppose that there are no corners. Hence we will consider hypersurfaces as closed submanifolds $\Sigma \subset \partial M$. Since $G^0_M$ preserves Yang-Mills action on $A_M$ this requirement mentioned in (A8) is satisfied.

We will describe an embedding

$$X : \Sigma \times [0, \varepsilon] \rightarrow \Sigma_\varepsilon$$

and a normal vector field $\partial_\tau$ on $\Sigma_\varepsilon$, whose flow lines are the trajectories $X(\cdot, \tau) \in \Sigma_\varepsilon$, $0 \leq \tau \leq \varepsilon$, that are normal to the boundary. This embedding arises from the solution of the volume preserving evolution problem on $\Sigma$, solved by Moser's trick, see [Mo].

**Lemma 1.** Let $\Sigma$ be a compact closed $(n-1)$-manifold that is a component of the boundary of a riemannian manifold $\Sigma_\varepsilon$ diffeomorphic to a cylinder $\Sigma \times [0,1]$, provided with a riemannian metric $h$. Then there exists an embedding $X : \Sigma \times [0,\varepsilon] \rightarrow \Sigma_\varepsilon$ such that:

1. The vector field $\partial_\tau$ is normal to $\Sigma$. The flow lines through $s \in \Sigma$ correspond to trajectories $X(s, \tau) \in \Sigma_\varepsilon$, $0 \leq \tau \leq \varepsilon$, transverse to $\Sigma$. 

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2. If \( \ast_{\Sigma} \) denotes the Hodge operator defined in \( \Sigma \), and if \( X_{\Sigma} : \Sigma \to \Sigma_{\varepsilon} \) stands for the inclusion \( X_{\Sigma}(\cdot) := X(\cdot, 0) \) then
\[
\ast_{\Sigma} X_{\Sigma}^{*}(\varphi) = X_{\Sigma}^{*}(\ast \varphi), \quad \forall \varphi \in \Omega^{k}(\Sigma_{\varepsilon}).
\]

3. If \( \mathcal{L} \) denotes the Lie derivative, then
\[
X_{\Sigma}^{*}(\mathcal{L}_{\partial_{t}} \ast \cdot) = X_{\Sigma}^{*}(\ast \mathcal{L}_{\partial_{t}} \cdot) = \ast_{\Sigma} X_{\Sigma}^{*}(\mathcal{L}_{\partial_{t}} \cdot).
\]

4. \( X_{\Sigma}^{*}(\mathcal{L}_{\partial_{t}} (d^{*} \varphi)) = X_{\Sigma}^{*}(d^{*}(\mathcal{L}_{\partial_{t}} \varphi)) \), for any \( \varphi \in \Omega^{k}(\Sigma_{\varepsilon}) \).

5. Suppose that \( \varphi \in \Omega^{1}(\Sigma_{\varepsilon}) \) satisfies \( \iota_{\partial_{t}} \varphi = 0 \) then
\[
\ast_{\Sigma} X_{\Sigma}^{*}(\mathcal{L}_{\partial_{t}} \varphi) = X_{\Sigma}^{*}(d^{*} \varphi).
\]

6. If \( X_{\Sigma}^{*}(d^{*} \varphi) = 0 \), then \( X_{\Sigma}^{*}(d^{*}(\mathcal{L}_{\partial_{t}} \varphi)) = 0 \), for any \( \varphi \in \Omega^{k}(\Sigma_{\varepsilon}) \).

Proof. Consider the exponential map \( Y : \Sigma \times [0, \varepsilon] \to \Sigma_{\varepsilon}, Y^{t}(\cdot) := Y(\cdot, t), \) on a tubular neighborhood \( \Sigma_{\varepsilon} \) of \( \Sigma \) (see for instance \[\text{Mi}\]). This means that for every initial condition \( s \in \Sigma \) and \( t \in [0, \varepsilon] \), \( Y^{t}(s) \in \Sigma_{\varepsilon} \), is a geodesic passing through \( s = Y^{0}(s) \) whose arc-length is \( t \). The initial velocity vector field \( \frac{\partial Y^{t}(s)}{\partial t}|_{t=0} = \frac{\partial Y^{0}(s)}{\partial t}, s \in \Sigma \) is a vector field \( \partial_{t} \), normal to \( \Sigma \subset \Sigma_{\varepsilon} \).

Let \( \lambda \in \Omega^{n-1}(\Sigma_{\varepsilon}) \) be the \((n-1)\)-volume form associated to the riemannian metric in \( \Sigma_{\varepsilon} \), recall that \( \text{dim} \Sigma_{\varepsilon} = n \). Define \( \lambda^{t} := (Y^{t})^{\ast} \lambda \) as the form induced by the restriction of the \((n-1)\)-volume form on the embedded \((n-1)\)-hypersurface \( Y^{t}(\Sigma) \subset \Sigma_{\varepsilon} \). Now take the differentiable function \( c(t) := \int_{\Sigma} \lambda^{0}/\int_{\Sigma} \lambda^{t} \in \mathbb{R}^{+}, \forall t \in [0, \varepsilon] \). Notice that \( c(0) = 1 \). Then by the compactness of \( \Sigma \subset \partial M \), \( [c(\tau) \lambda^{t}] = [\lambda^{0}] \in H^{n-1}_{\partial M}(\Sigma) \), for every fixed \( \tau \in [0, \varepsilon] \). Hence by Moser’s trick, see \[\text{MF}\], there exists an isotopy of the identity, \( Z : \Sigma \times [0, \tau] \to \Sigma \) such that \( (Z^{\tau})^{\ast} (c(\tau) \cdot \lambda^{t}) = \lambda^{0}, Z^{0}(s) = s, \forall s \in \Sigma, \) where \( Z^{t}(s) := Z(s, t) \).

We define
\[
X(s, \tau) := Z^{\tau} \circ Y^{t}(s), \quad \forall (s, \tau) \in \Sigma \times [0, \varepsilon] \tag{6}
\]
also \( X^{t}(\cdot) := X(\cdot, t), X_{\Sigma}(\cdot) := X(\cdot, 0) = X^{0}(\cdot) \).

Consider the explicit form of the Hodge star operator, \( \ast \), for the riemannian metric \( h \) on \( \Sigma_{\varepsilon} \), and the star operator, \( \ast_{\Sigma} \), for the induced metric \( \overline{h} := X_{\Sigma}^{\ast} h \) on \( \Sigma \). For \( k \)-forms \( \varphi \in \Omega^{k}(\Sigma_{\varepsilon}) \), we have that \( X_{\Sigma}^{\ast}(\ast \varphi) \) locally equals the pullback of
\[
\ast \left( \sum_{I} a_{I} dx^{i_{1}} \wedge \cdots \wedge dx^{i_{k}} + \sum_{I'} b_{I'} dx^{i'_{1}} \wedge \cdots \wedge dx^{i'_{k-1}} \wedge dx^{\tau} \right) = ...
\]

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\[
= \sqrt{|\det(h_{ij})| \left( \sum_j h^{i,j_1} \ldots h^{i,j_k} a_{i} dx^{j_1} \wedge \ldots \wedge dx^{j_{n-k}+1} + \right.}
\]
\[
+ \sqrt{|\det(h_{ij})| \left( \sum_{j'} h^{i,j'_1} \ldots h^{i,j'_{n-k}} b_{j'} dx^{j'_1} \wedge \ldots \wedge dx^{j'_{n-k}} \right.}
\]
\]
where \((x_1, \ldots, x_{n-1})\) denotes a coordinate chart in \(\Sigma\), meanwhile the ordered index sets \(I = \{i_1 < \cdots < i_k\}\), \(J = \{j_1 < \cdots < j_{n-k-1}\}\), in such a way that their union \(I \cup J\), as ordered set, correspond to an oriented basis \((dx_1, \ldots, dx_{n-1})\) of 1-forms on \(\Sigma\). Similarly for the ordered sets \(I' = \{i'_1 < \cdots < i'_{k-1}\}, J = \{j'_1 < \cdots < j'_{n-k}\}\). Thus
\[
X_{\Sigma}^*(*\varphi) = \sqrt{|\det(h_{ij})|} \left( \sum_{j'} h^{i,j'_1} \ldots h^{i,j'_{n-k}} b_{j'} dx^{j'_1} \wedge \ldots \wedge dx^{j'_{n-k}} \right)
\]
Meanwhile
\[
*_{\Sigma} X_{\Sigma}^*(\varphi) = \sqrt{|\det(h_{ij})|} \left( \sum_{\{j'_1 < \cdots < j'_{n-k}\}} h^{i,j'_1} \ldots h^{i,j'_{n-k}} b_{j'} dx^{j'_1} \wedge \ldots \wedge dx^{j'_{n-k}} \right)
\]
But \(h^{ij} = h^{ij}\) for \(i, j \in \{1, \ldots, n-1\}\), and also \(h^{in} = \delta_{i,n}\), since \(\partial_\tau\) is normal to \(\Sigma\). Hence \(\sqrt{|\det(h_{ij})|} = \sqrt{|\det(h_{ij})|}\), and \(*_{\Sigma} X_{\Sigma}^*(\varphi) = X_{\Sigma}^*(*\varphi)\). This proves assertion 2.

If the volume form on \(\Sigma\) in local coordinates can be descibed as \(|\det(h_{ij})|^{1/2} dx^1 \wedge dx^2 \cdots dx^{n-1}\), then \((X^\tau)^*(c(\tau)\lambda) = \lambda^0\), implies
\[
c(\tau) \sqrt{|\det(h \circ X^\tau)_{ij}|} \cdot dx^1 \wedge \cdots dx^{n-1} = |\det(h_{ij})|^{1/2} dx^1 \wedge dx^2 \cdots dx^{n-1}
\]
Furthermore \((X^\tau)^*[\mathcal{L}_{\partial_\tau}(c(\tau)\lambda)] = \frac{\partial}{\partial \tau} (X^\tau)^*(c(\tau)\lambda)\), then
\[
X_{\Sigma}^*[\mathcal{L}_{\partial_\tau}(c(\tau)\lambda)] = \frac{\partial}{\partial \tau} (X^\tau)^*(c(\tau)\lambda) \big|_{\tau=0} = \frac{\partial}{\partial \tau} (\lambda^0) \big|_{\tau=0} = 0
\]
hence,
\[
\frac{\partial}{\partial \tau} \left(c(\tau) \sqrt{|\det(h \circ X^\tau)_{ij}|} dx^1 \wedge \cdots dx^{n-1}\right) \big|_{\tau=0} =
\]
\[
\left[ \frac{\partial}{\partial \tau} \left|\det(h \circ X^\tau)_{ij}\right|^{1/2} c(t) + \left|\det(h \circ X^\tau)_{ij}\right|^{1/2} \frac{\partial c(\tau)}{\partial \tau} \right] \big|_{\tau=0} = dx^1 \wedge \cdots dx^{n-1} = 0
\]
Recall that $c(\tau) = |\det(h^{ij})|^{1/2}/|\det(h \circ X^\tau)|^{1/2}$; hence
\[
\frac{\partial c(\tau)}{\partial \tau} \bigg|_{\tau=0} = \frac{-3}{2} \left| \det(h_{ij}) \right|^{1/2} \left| \det(h \circ X^\tau)_{ij} \right|^{3/2} \frac{\partial}{\partial \tau} \left| \det(h \circ X^\tau)_{ij} \right| \bigg|_{\tau=0}
\]
therefore
\[
\frac{\partial}{\partial \tau} \left| \det(h \circ X^\tau)_{ij} \right|^{1/2} \bigg|_{\tau=0} = \frac{\partial c(\tau)}{\partial \tau} \bigg|_{\tau=0} = 0.
\]
Hence the derivative of $Z^0$ at $\Sigma$ equals $Z^0_* = \text{Id}$, since $\frac{\partial c(\tau)}{\partial \tau} \bigg|_{\tau=0} = 0$. Therefore
\[
\frac{\partial X^\tau}{\partial \tau} \bigg|_{\tau=0} = Z^0_* \left( \frac{\partial Y^0}{\partial \tau} \right) = \partial_*.
\]
This proves assertion II.

Now, since $\partial_\tau$ is normal to $\Sigma$,
\[
\frac{\partial}{\partial \tau} \left| \det(h_{ij}) \right|^{1/2} \bigg|_{\tau=0} = \frac{\partial}{\partial \tau} \left| \det(h_{ij}) \right|^{1/2} \bigg|_{\tau=0} = 0
\]
then $X^*_\Sigma (L_{\partial_\tau} (\ast\varphi))$ equals
\[
\sqrt{|\det(h_{ij})|} \left( \sum_{\{j'_1 < \ldots < j'_{n-k}\}} \frac{\partial}{\partial \tau} \left( h^{i'_{j'_1}} \ldots h^{i'_{n-k}j'_{n-k}} b_{I'} \right) \bigg|_{\tau=0} \right) \cdot dx^{j'_1} \wedge \ldots \wedge dx^{j'_{n-k}}.
\]

Recall that the derivative of $Y^t$ the exponential map $Y^t$ at $s \in \Sigma$, $Y^0_* : T_0(T_s \Sigma) \simeq T_s \Sigma \rightarrow T_s \Sigma$, equals the identity, $Y^0_* = \text{Id}$. This in turn implies that
\[
\frac{\partial}{\partial \tau} \left( h^{i'_{j'_1}} \ldots h^{i'_{n-k}j'_{n-k}} \right) \bigg|_{\tau=0} = 0.
\]
Therefore $X^*_\Sigma L_{\partial_\tau} (\ast\varphi)$ equals
\[
\sqrt{|\det(h_{ij})|} \left( \sum_{\{j'_1 < \ldots < j'_{n-k}\}} h^{i'_{j'_1}} \ldots h^{i'_{n-k}j'_{n-k}} \frac{\partial}{\partial \tau} (b_{I'}) \bigg|_{\tau=0} \right) dx^{j'_1} \wedge \ldots \wedge dx^{j'_{n-k}} = \]
\[
X^*_\Sigma (\ast L_{\partial_\tau} (\varphi)).
\]
This proves assertion III. Assertion IV is an immediate consequence of assertion III and assertion VI is in turn a consequence of assertion IV.
Part 5 is a direct calculation for if $\iota_{\partial_{\tau}}\varphi = 0$, $\varphi \in \Omega^1(\Sigma_{\varepsilon})$, then locally $\varphi = \sum_{i=1}^{n-1} f_i(x, \tau) dx^i$, thus $X^\#(\ast d\varphi)$ equals

$$X^\#(\ast d\varphi) = |\det h_{ij}|^{1/2} \sum_{i=1}^{n-1} (-1)^i h_{i,1} \cdots h_{i,1} f_i(x, \tau) dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^{n-1}$$

where $\hat{h}^i, \hat{d}^i$ denote missing terms. This last expression corresponds to $\ast \Sigma X^\#(L_{\partial_{\tau}} \varphi)$, therefore assertion 5 holds.

**Definition 1.** The following expression corresponds to the presymplectic structure in $\tilde{L}_\Sigma$, for the Yang Mills action, see for instance [Wo],

$$\tilde{\omega}_\Sigma(\tilde{\eta}, \tilde{\xi}) = \frac{1}{2} \int_{\Sigma} X^\#(\eta \wedge d^* \xi - \xi \wedge d^* \eta),$$

(7)

for all $\tilde{\xi}, \tilde{\eta} \in \tilde{L}_\Sigma$ with representatives $\xi, \eta \in L_{\Sigma_{\varepsilon}}$.

In addition, the degeneracy subspace of the presymplectic form is

$$K_{\omega_{\Sigma}} := \{ \tilde{\eta} \in \tilde{L}_\Sigma \mid \eta = df, f(s, 0) = 0, f \in \Omega^0(\Sigma_{\varepsilon}), \forall s \in \Sigma \}.$$

From this very definition we have that the degeneracy gauge symmetries group $K_{\omega_{\Sigma}}$ is a (normal) subgroup of the identity component group $\tilde{G}_{\Sigma}^0 \leq \tilde{G}_{\Sigma}$ of the gauge symmetries,

$$K_{\omega_{\Sigma}} \leq \tilde{G}_{\Sigma}^0 \leq \tilde{G}_{\Sigma}.$$

Let

$$\Phi_{\tilde{A}_\Sigma} := \{ \tilde{\eta} \in \tilde{L}_\Sigma \mid \iota_{\partial_{\tau}} \eta = 0, \eta \in L_{\Sigma_{\varepsilon}} \text{ representive of } \tilde{\eta} \}$$

(8)

be the axial gauge fixing subspace of $\tilde{L}_\Sigma$. The following statement leads to a simpler expression for the presymplectic structure.

**Lemma 2.** For every $\varphi \in L_{\Sigma_{\varepsilon}}$ corresponding to a Yang-Mills solution, there is

$$\varphi = \varphi + df,$$

(9)

the gauge orbit representative such that $\iota_{\partial_{\tau}} \varphi = 0$, and $f |_{\Sigma} = 0$.

a) Every $K_{\omega_{\Sigma}}$–orbit in $\tilde{L}_\Sigma$ intersects in just one point the subspace $\Phi_{\tilde{A}_\Sigma}$. 

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b) The presymplectic form \( \hat{\omega}_\Sigma \) restricted to the subspace \( \Phi_{\tilde{A}_\Sigma} \) may be written as

\[
\hat{\omega}_\Sigma (\tilde{\eta}, \tilde{\xi}) = \frac{1}{2} \int_{\Sigma} X_\Sigma^x (\tilde{\eta} \wedge \star \mathcal{L}_{\partial_\tau} \tilde{\xi} - \tilde{\xi} \wedge \star \mathcal{L}_{\partial_\tau} \tilde{\eta}),
\]

for every \( \tilde{\xi}, \tilde{\eta} \in \tilde{L}_\Sigma \) with representatives \( \xi, \eta \in L_{\Sigma^e} \). Hence \( \hat{\omega}_\Sigma \) is a non-degenerated 2-form when restricted to the gauge fixing subspace \( \Phi_{\tilde{A}_\Sigma} \subset \tilde{L}_\Sigma \).

Proof of a). Let \( \left( \sum_{i=1}^{n-1} \eta^i dx^i \right) + \eta^\tau d\tau \) be a local expression for a solution \( \eta \in L_{\Sigma^e} \). Let us apply a gauge symmetry

\[
X_\Sigma^x (\eta + df) = \sum_{i=1}^{n-1} (\eta^i + \partial_i f) dx^i + (\eta^\tau + \partial_\tau f) d\tau
\]

in such a way that \( \eta^\tau + \partial_\tau f = 0 \). We can solve the corresponding ODE for \( f(s, \tau) \) once we fix an initial condition \( f(s, 0) = g(s) \). If we take this initial condition \( g(s) \) as a constant, then we get a gauge symmetry in \( K_{\omega_\Sigma} \). The remaining part is a straightforward calculation. This proves \( \Box \) The other assertion may be inferred from lemma 2.

With this statement we satisfy \((A2)\) and \((A3)\). Let

\[
A_\Sigma := \tilde{A}_\Sigma / K_{\omega_\Sigma}, L_\Sigma = \tilde{L}_\Sigma / K_{\omega_\Sigma}
\]

be the quotient by the linear space \( K_{\omega_\Sigma} \) corresponding to degenerate gauge symmetries. And also let \( G^0_\Sigma := \tilde{G}^0_\Sigma / K_{\omega_\Sigma} \) be the quotient by the normal subgroup. By the previous lemma, when we restrict the quotient class \( \tilde{A}_\Sigma \rightarrow A_\Sigma \) to \( \Phi_{\tilde{A}_\Sigma} \), then we get an isomorphism of affine spaces. Let \( \omega_\Sigma \) the corresponding symplectic structure on \( A_\Sigma \) induced by the restriction of \( \hat{\omega}_\Sigma \) to the subspace \( \Phi_{\tilde{A}_\Sigma} \subset \tilde{L}_\Sigma \).

We now proceed to give a precise description of the symplectic space \( L_\Sigma \).

Lemma 1 implies that

\[
X_\Sigma^x (\xi \wedge \star \mathcal{L}_{\partial_\tau} \eta) = X_\Sigma^x (\xi) \wedge \star_\Sigma X_\Sigma^x (\mathcal{L}_{\partial_\tau} \eta)
\]

where \( \star_\Sigma \) stands for the Hodge star on \( \Sigma \). Since \( \iota_{\partial_\tau} (\mathcal{L}_{\partial_\tau} \eta) = \iota_{\partial_\tau} (\iota_{\partial_\tau} d\eta) = 0 \), then we have a linear map \( L_{\Sigma^e} \rightarrow \Omega^1(\Sigma) \times \Omega^1(\Sigma) \), where

\[
\eta \mapsto \left( \phi^\eta, \dot{\phi}^\eta \right) := (X_\Sigma^x (\eta), X_\Sigma^x (\mathcal{L}_{\partial_\tau} \eta)),
\]

(13)
for every $\bar{\eta} \in \bar{L}_\Sigma$ with representative $\xi, \eta \in L_{\Sigma^c}$ and $\bar{\eta}$ defined in (9) that leads to a map

$$L_\Sigma \to \Omega^1(\Sigma) \times \Omega^1(\Sigma) \simeq T(\Omega^1(\Sigma)),$$

where we consider the identification with the tangent space $T(\Omega^1(\Sigma))$.

Notice that $i_{\partial_\tau} \bar{\eta} = 0$ implies that $\eta \in L_{\Sigma^c}$ corresponds to a $1$--form $\phi^\eta$ on $\Sigma$. Notice also that $\partial_\tau \nu = 0$ implies $\nu \in L_{\Sigma^c}$,

hence $d^* (\nu \nu^\nu) = 0$ implies $\nu \nu^\nu \in \ker d^* \nu$.

We have the following expression for the symplectic structure on $L_\Sigma$

$$\omega_\Sigma \left((\phi^\eta, \phi^\nu), (\phi^\xi, \phi^\xi)\right) = \frac{1}{2} \int_{\Sigma} \left(\phi^\eta \wedge \star_\Sigma \phi^\xi - \phi^\xi \wedge \star_\Sigma \phi^\eta\right),$$

for every $(\phi^\xi, \phi^\xi), (\phi^\nu, \phi^\nu) \in L_{\Sigma^c}$, with representatives $\bar{\xi}, \bar{\eta} \in L_{\Sigma^c}$. Form this very definition we can verify (A4), i.e. translation invariance and also relation (3) where

$$\int_{\Sigma} \left((\phi^\eta, \phi^\nu), (\phi^\xi, \phi^\xi)\right) := \int_{\Sigma} \phi^\eta \wedge \star_\Sigma \phi^\xi.$$

Furthermore (A6) is easily verified and the claims from (A5) can be inferred from the relation $\star_\Sigma = -\star_\Sigma$.

Let us consider a hypersurface $\Sigma$ as a disjoint union of oriented hypersurfaces $\Sigma = \Sigma^1 \sqcup \cdots \sqcup \Sigma^m$. Then there is a linear map

$$\Omega^1(\Sigma) \to \Omega^1(\Sigma^1) \oplus \cdots \oplus \Omega^1(\Sigma^m)$$

given by $\eta \mapsto X^\eta_{\Sigma^1}(\eta) \oplus \cdots \oplus X^\eta_{\Sigma^m}(\eta)$. This map induces the isomorphism $r_{\Sigma | \Sigma^1 | \cdots | \Sigma^m} : L_\Sigma \to L_{\Sigma^1} \oplus \cdots \oplus L_{\Sigma^m}$. Furthermore, the chain decomposition

$$\int_{\Sigma} \cdot = \int_{\Sigma^1} \cdot + \cdots + \int_{\Sigma^m} \cdot$$

verifies (A7).

Let $G^0_M = \{ df \mid f \in \Omega^0(M) \}$ be the identity component of the bulk gauge symmetry group $G_M$. Each bulk symmetry $df \in G^0_M$ induces a symmetry $X^*_{\Sigma^1}(df) \in G^0_{\partial M^c}$ in the boundary cylinder $\partial M^c$, and also a symmetry $h_M(df) \in G^0_{\partial M^c}$ in the boundary conditions. This was mentioned in the locality arguments [2.4] This is basically part of the content of (A8).

Till now we have validated axioms (A1) to (A8) which describe kinematic information of gauge fields. In the following paragraphs we consider gauge equivalence.
3.2 Symplectic reduction

We still need to describe the quotient for the symplectic action of the gauge group $G_0^0$ on $L^\Sigma$. The suitable gauge fixing space $\Phi^\Sigma$ in $A^\Sigma$ for this action will be the space of divergence free $1-$forms, i.e. if we define

$$\Phi_{A^\Sigma} := \left\{ (\phi, \dot{\phi}) \in L^\Sigma \mid d^*\phi = 0 = d^*\dot{\phi} \right\}.$$

The following task is the detailed description of the symplectic quotient space

$$A^\Sigma/G_0^0 \simeq L^\Sigma/G_0^0 \simeq \Phi_{A^\Sigma}.$$

According to Hodge theory [Sc], associated with the inner product

$$\int^\Sigma \phi \wedge \ast^\Sigma \phi', \quad \phi, \phi' \in \Omega^1(\Sigma) \quad (16)$$

we have an orthogonal decomposition

$$\phi = \phi_h + d^*\alpha, \quad \{ \phi \in \Omega^1(\Sigma) \mid d^*\phi = 0 \} = \mathcal{H}^1(\Sigma) \oplus d^*\Omega^2(\Sigma) \simeq H^1_{dR}(\Sigma) \oplus d^*\Omega^2(\Sigma)$$

where the space $\mathcal{H}^1(\Sigma)$ of harmonic $1-$forms is isomorphic to the de Rham cohomology $H^1_{dR}(\Sigma)$. The following lemma fulfills [A8] and shows the gauge-fixing space definition required in (A10).

**Lemma 3.** For $\eta \in L^\Sigma$, take $(\phi^0, \dot{\phi}^0) \in T\Omega^1(\Sigma)$ as defined in (13), with gauge transformation group $G_0^0$.

a) The gauge group action of $G_0^0$ on $L^\Sigma$ is induced in the tangent space $T\Omega^1(\Sigma)$ by the translation action $\phi \mapsto \phi + df, f \in \Omega^0(\Sigma)$ on $\Omega^1(\Sigma)$.

b) Every $G_0^0-$orbit in $L^\Sigma$ intersects in just one point the subspace $\Phi_{A^\Sigma}$.

c) The symplectic form $\omega^\Sigma$ is preserved under the $G_0^0-$action.

**Proof of b).** Consider $X^\Sigma_\eta = \sum_{i=1}^{n-1} \eta^i dx^i$, a local expression for a solution $\eta \in L^\Sigma \cap \Phi_{A^\Sigma} \subset L^\Sigma$. Consider $\overline{f} : \Sigma \to \mathbb{R}$, then $d^*\omega^\Sigma(X^\Sigma_\eta + d\overline{f}) = 0$ implies

$$\sum_{i=1}^{n-1} \partial_i \left[ |\det(h)|^{1/2} \sum_{i=1}^{n-1} \left( \eta^i + \partial_i \overline{f} \right)(-1)^i h^{1,i} \ldots \hat{h}^{i,i} \ldots h^{i,n-1} \right] = 0 \quad (17)$$

The existence and regularity of a solution, $\overline{f}(s)$, for this PDE on $\Sigma$ is warranted precisely by Hodge theory. Since

$$X^\Sigma_\eta(\eta) \in \Omega^1(\Sigma) \simeq d\Omega^0(\Sigma) \oplus \mathcal{H}^1(\Sigma) \oplus d^*\Omega^2(\Sigma)$$
then there exists \( \mathbf{f} \in \Omega^0(\Sigma) \), such that \( X^*_\Sigma(\eta) + d\mathbf{f} \) is the orthogonal projection of \( X^*_\Sigma(\eta) \) onto \( \ker d^{*\Sigma} \simeq \mathfrak{h}^1(\Sigma) \oplus d^{*\Sigma}\Omega^2(\Sigma) \). Define

\[
\phi^0 := X^*_\Sigma(\eta) + d\mathbf{f} \in \ker d^{*\Sigma}.
\]

On the other hand

\[
d^*\mathcal{L}_{\partial_\tau} (\eta + df) = 0 \tag{18}
\]

implies

\[
\sum_{i=1}^{n-1} \partial_i \left[ |\det(h)|^{1/2} \sum_{i=1}^{n-1} (\partial_\tau \eta^i + \partial_\tau \partial_i f) (-1)^i h^{1,i} \cdots h^{i,i} \cdots h^{1,n-1} \right] = 0. \tag{19}
\]

When we substitute \( \partial_\tau \eta^i + \partial_\tau \partial_i f \) by the coefficients \( \phi^i_\tau \), of a time dependent 1-form in \( \Sigma \), \( \phi_\tau \in \Omega^1(\Sigma) \), equation (19) has a solution \( \phi_\tau \). This leads to an ODE for \( g_i(s, \tau) := \partial_i f \)

\[
\partial_\tau \eta^i + \partial_\tau \partial_i f = \phi^i_\tau \tag{20}
\]

Equation (20) can be solved, once we can fix the boundary condition \( \partial_i f(x^i, 0) = \partial_i \mathbf{f}(x^i) \). This boundary condition, in turn, has been obtained by solving (17) in \( \Sigma \).

We conclude that \( \sum_{i=1}^{n-1} g_i(s, \tau) dx^i \) is an exact form on \( \Sigma \), so that there exists \( f(s, \tau) \in \Omega^0(\Sigma_x) \) such that (18) holds.

To conclude define \( \phi^0 := X^*_\Sigma(\mathcal{L}_{\partial_\tau} (\eta + df)) \), notice that \( (\phi^0, \phi^0) \in \Phi_{A\Sigma} \).

Remark that form the very form of the solution \( \phi_\tau = X^*_\Sigma(\eta_\tau) + d\mathbf{f}_\tau \), \( \phi_\tau \) and \( \partial_\tau \eta \) have the same integrals along closed cycles, hence they have the same cohomology class in \( H^1_{dR}(M) \simeq \mathfrak{h}^1(\Sigma) \).

There is extension of local gauge actions: In the particular case of trivial principal bundle local gauge symmetries in \( G_{\partial M} \) extend via partitions of unity to symmetries in the bulk \( G_M \). This means that we can define sections \( \sigma : G^0_{\partial M} \to G_M \). Hence there is a well defined (set-theoretic) orbit map of the homomorphism \( G_M \partial G_{\partial M} \),

\[
\tau_M : L_M/G^0_M \to L_{\partial M}/G^0_{\partial M}
\]

Furthermore by linearity of the actions \( L_M/G^0_M \), \( A_M/G^0_M \) have linear and affine structures respectively.

The corresponding axial gauge fixing space can be described with the isomorphism, given by Hodge theory

\[
\Phi_{A\Sigma} \simeq T \left[ H^1_{dR}(\Sigma) \oplus d^{*\Sigma}\Omega^2(\Sigma) \right] \simeq \left[ TH^1_{dR}(\Sigma) \right] \oplus \left[ T(d^{*\Sigma}\Omega^2(\Sigma)) \right]
\]
where in the r.h.s. we take tangent spaces. In the abelian case holonomy 
\[ \text{hol}_\gamma(\phi) = \exp \oint \gamma \phi \in G \]\nalong a closed trajectory \( \gamma \), can be defined up to cohomology class of \( \gamma \). Recall that for \( G = U(1) \), \( \int_\gamma \phi \in \sqrt{-1} \mathbb{R} \). Thus by considering independent generators \( \{ \gamma_1, \ldots, \gamma_{b_1} \} \) of the homology \( H_1(\Sigma, \mathbb{Z}) \), and a dual harmonic basis \( \phi^1_h, \ldots, \phi^b_h \), we have the exact sequence

\[
0 \to \bigoplus_{i=1}^{b_1} \mathbb{Z} \cdot \left[ \phi^i_h \right] \to H^1(\Sigma) \otimes \mathbb{C} \to \mathbb{C} \to 1
\]

Hence a surjective map from the derivative \( D \text{hol}_\Gamma : TH^1_{dR}(\Sigma) \to TG^{b_1} \).

Now we consider the reduction of \( \Phi_{A\Sigma} \) under the action of the discrete group \( G_{\Sigma}/G^0_{\Sigma} \).

**Theorem 1.** We have the quotient space

\[
A_{\Sigma}/G_{\Sigma} \simeq \Phi_{A\Sigma}/(G_{\Sigma}/G^0_{\Sigma}) \simeq T\left( G^{b_1} \right) \times T(d^*\Sigma \Omega^2(\Sigma))
\]

with reduced symplectic structure \( \omega_{\Sigma} \) given in (15).

With this result we end up the kinematical part of the axiomatic description, i.e. axioms \([A1]\) to \([A9]\).

### 3.3 Dynamics modulo gauge

These paragraphs are aimed to verify axioms \([A10]\) to \([A12]\) where dynamics of gauge fields is constructed. We discuss the behavior of the solutions near the boundary in more detail. Recall that here is a map \( \tilde{\tau}_M : L_M \to \tilde{L}_{\partial M} \) coming from the restriction of the solutions to germs on the boundary, and composing we the quotient class map we have a map \( \tau_M : L_M \to L_{\partial M} \).

Let \( L_\tilde{M} \subset L_{\Sigma} \) be the image under this map. The aim is to describe the image \( L_{\tilde{M}} \subset L_{\partial M} \) of the space of solutions as a Lagrangian subspace once we have taken gauge quotient. The aim of this part is to verify the dynamics postulate \([A10]\).

We recall some useful facts of Hodge-Morrey-Friedrich theory for manifolds with boundary, see for instance \([SG]\), \([AM]\) and \([GMS]\). We can consider both Neumann and Dirichlet boundary conditions in order to define \( k \)-forms on \( M \), i.e.

\[
\Omega^k_N(M) := \left\{ \varphi \in \Omega^k(M) \mid X^*_{\partial M}(\ast \varphi) = 0 \right\},
\]

\[
\Omega^k_D(M) := \left\{ \varphi \in \Omega^k(M) \mid X^*_{\partial M}(\varphi) = 0 \right\}.
\]
The differential $d$ preserves the Dirichlet complex $\Omega^k_D(M)$ and on the other hand, the codifferential $d^*$ preserves the Neumann complex $\Omega^k_N(M)$. In addition, the space $\mathcal{H}^k(M)$ of harmonic fields $d\varphi = 0 = d^*\varphi$, turns out to be infinite dimensional, nevertheless finite dimensional spaces arise when we restrict to Dirichlet or Neumann boundary conditions $\mathcal{H}^k_N(M), \mathcal{H}^k_D(M)$.

**Lemma 4** (Sc). 1. There is an orthogonal decomposition

$$\Omega^k(M) = \mathcal{H}^k_N(M) \oplus \left( \mathcal{H}^k(M) \cap d\Omega^{k-1}(M) \right) \oplus d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1}_N(M).$$

2. In particular there is an orthogonal decomposition for divergence-free fields

$$\ker \left[ d^* : \Omega^k_N(M) \to \Omega^{k-1}_N(M) \right] = \mathcal{H}^k_N(M) \oplus d^*\Omega^{k+1}_N(M).$$

3. Each de Rham cohomology class can be represented by a unique harmonic field without normal component, i.e. there is an isomorphism

$$H^k_{dR}(M) \simeq \mathcal{H}^k_N(M).$$

4. Each de Rham relative cohomology class can be represented by a harmonic field null at the boundary, i.e. there is an isomorphism

$$H^k_{dR}(M, \partial M) \simeq \mathcal{H}^k_D(M).$$

Define

$$\tilde{\Phi}_A_M := L_M \cap \Omega^1_N(M) \subset L_M$$

(22)

Notice that $r_M \left( \tilde{\Phi}_A_M \right) \subset L_{\partial M}$. If $\varphi \in \Omega^k(\partial M)$ satisfies the Neumann condition $X_{\partial M}^*(\alpha \varphi) = 0$, then it also satisfies $X_{\partial M}^* (\mu_{\partial} \varphi) = 0$ with $\mu_{\partial} \varphi \in \Omega^{k-1}(\partial M_e)$.

Let us consider the divergence free fields which, according to lemma 4, have an orthogonal decomposition of th axial gauge fixing space of solutions

$$\Phi_{A M} := \{ \varphi_h + d^*\alpha \mid \varphi_h \in \mathcal{H}^1_N(M), \alpha \in \Omega^2_N(M), d^*dd^*\alpha = 0 = dd^*d\alpha \}$$

(23)

If $\varphi \in \Phi_{A M}$, then $\varphi$ satisfies the equation $d^*\varphi = 0$, it also satisfies Euler-Lagrange equation $d^*d\varphi = 0$, since $d^*dd^*\alpha = 0 = dd^*d\alpha$. Hence $\Phi_{AM} \subset L_M$, furthermore $\Phi_{A M} \subset \Phi_{A M}$. This space

$$\Phi_{A M} \subset \mathcal{H}^1_N(M) \oplus d^*\Omega^2_N(M),$$

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constitutes the orthogonal projection of the space of solutions $L_M$, according to the decomposition
\[
\Omega^1(M) = \mathcal{N}^N_M(M) \oplus d^*\Omega^2_N(M) \oplus d\Omega^0_D(M) \oplus (\mathcal{N}^1(M) \cap d\Omega^0(M)).
\]
It also coincides with the orthogonal projection of $\tilde{\Phi}_{AM}$. From this orthogonal decomposition it can be shown that every solution $\varphi \in L_M$ can be transformed, modulo the bulk gauge transformation,
\[
\varphi \mapsto \tilde{\varphi} = \varphi + df
\]
on onto a field belonging to the space $\Phi_{AM}$. Thus the following statement can be proven.

**Lemma 5.**
1. Every $G^0_0 M$-orbit intersects $\Phi_{AM} \subset L_M$ in exactly one point, i.e. for every $\varphi \in L_M$ there exists $f \Omega^0(M)$, such that $\tilde{\varphi} = \varphi + df$.
2. $r_M : \Phi_{AM} \to L_M \cap \Phi_{A\partial M}$ is a linear surjection.

**Corollary 1.** $L_M/G^0_0 \partial M = r_M(\Phi_{AM}) / G^0_0 \partial M \subset L_{\partial M} / G^0_0 \partial M$.

Consider the identification $\mathcal{N}^1_N(M) \simeq \mathcal{N}^1(\partial M)$, and $d^* N \Omega^2_N(M) \simeq d^* N^2(\partial M)$.

We have the following statement.

**Lemma 6.**
1. There is a well defined restriction map
\[
X^{\ast}_{\partial M} : \Phi_{AM} \to \mathcal{N}^1(\partial M) \oplus d^* \Omega^2(\partial M)
\]
2. If we adopt the identification given in theorem \[1\]
\[
L_{\partial M} / G^0_0 \partial M \simeq T(\mathcal{N}^1(\partial M)) \oplus T(d^* \Omega^2(\partial M))
\]
then the map $r_M$ coincides with the first jet of the pullback, i.e. we have a commutative diagram of linear mappings
\[
\begin{array}{ccc}
\mathcal{N}^1_N(M) \oplus d^* \Omega^2_N(M) & \xrightarrow{\mathcal{N}^1_M} & L_{\partial M} / G^0_0 \partial M \\
& & \downarrow r_M \\
& & L_{\partial M} / G^0_0 \partial M \\
\end{array}
\]
\[
\begin{array}{ccc}
\mathcal{N}^1_N(M) \oplus d^* \Omega^2_N(M) & \xrightarrow{T} & T(\mathcal{N}^1(\partial M)) \oplus T(d^* \Omega^2(\partial M)) \\
& & \downarrow j^1_{X_{\partial M}} \\
& & \mathcal{N}^1_N(M) \oplus d^* \Omega^2_N(M) \\
\end{array}
\]
Now we are in position to prove that the image of solutions modulo gauge onto the space of boundary conditions modulo gauge, stated in proposition \[\[\[\]]\] is in fact a lagrangian space. The following statement completes the dynamical picture described in \[\[\[\]]\].

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Theorem 2. Let $L_{\tilde{M}} = r_M(L_M)$ be the boundary conditions that can be extended to solutions in the interior $L_M$. Then for the symplectic vector space $L_{\partial M}/G^0_{\partial M}$

1. $L_{\tilde{M}}/G^0_{\partial M}$ is an isotropic subspace.
2. $L_{\tilde{M}}/G^0_{\partial M}$ is a coisotropic subspace.

In other words $L_{\tilde{M}} \cap \Phi_{A_{\partial M}}$ is a lagrangian subspace of the symplectic space $\Phi_{A_{\partial M}}$.

As we mentioned in the introduction for $L_{\tilde{M}}/G^0_{\partial M}$ isotropy is always true, see [KT]. For the sake of completeness we give a proof that is a straightforward calculation. Take $\varphi, \varphi' \in L_M$ and consider its image $(j^1 X^*_{\partial M}) (\varphi) = (X^*_{\partial M}(\varphi), X^*_{\partial M}(\mathcal{L}_{\partial M}(\varphi))) = (\phi^\varphi, \dot{\phi}^\varphi) \in L_{\partial M} \cap \Phi_{A_{\partial M}}$,

where $\mathcal{L}$ was defined in lemma 2 and where $d^* \phi = d^* \dot{\phi} = 0$. We also consider $\varphi \in \Phi_{A_M}$. Then

$$\omega_{\partial M}(\varphi, \dot{\varphi}) = \frac{1}{2} \int_{\partial M} \phi^\varphi \wedge *_{\partial M} \dot{\phi}^\varphi - \dot{\phi}^\varphi \wedge *_{\partial M} \phi^\varphi =$$

$$\frac{1}{2} \int_{\partial M} X^*_{\partial M}(\varphi) \wedge *_{\partial M} X^*_{\partial M}(\mathcal{L}_{\partial M}(\varphi)) = \frac{1}{2} \int_{\partial M} X^*_{\partial M}(\overline{\varphi}) \wedge *_{\partial M} X^*_{\partial M}(\mathcal{L}_{\partial M}(\overline{\varphi}))$$

From a property shown in lemma 1 we have that the last expression equals

$$\frac{1}{2} \int_{\partial M} X^*_{\partial M}(\varphi) \wedge *d\varphi = \frac{1}{2} \int_{\partial M} X^*_{\partial M}(\varphi) \wedge *d\varphi,$$

Recall that $\varphi, \varphi'$ are global solutions in the interior $d^*d\varphi = d^*d\varphi'$, hence by applying Stokes' Theorem we have

$$\omega_{\partial M}(\varphi, \dot{\varphi}) = \int_{M} d\varphi \wedge *d\varphi = 0.$$

Proof for coisotropic embedding. Take $\varphi \in \Phi_{A_M}$, as indicated in lemma 5 take $(\phi^\varphi, \dot{\phi}^\varphi) := r_M(\overline{\varphi})$ and suppose that $\omega_{\partial M}(\varphi, \dot{\varphi}) = 0$ for every $\overline{\varphi} \in L_{\partial M}$ with $(\phi^\varphi, \dot{\phi}^\varphi) \in L_{\partial M}$ corresponding to $\varphi \in \Phi_{A_M}$. Recall that $d^* \phi = 0 = d^* \dot{\phi}$. Then thanks to the representative (12) we have

$$\int_{\partial M} \phi^\varphi \wedge *_{\partial M} \dot{\phi}^\varphi = \int_{\partial M} \dot{\phi}^\varphi \wedge *_{\partial M} \phi^\varphi, \quad \forall (\phi^\varphi, \dot{\phi}^\varphi) \in L_{\partial M} \quad (24)$$
According to the orthogonal decomposition described in lemma 4, we have
\[ \phi^\omega = \phi^h + d^{*\alpha}X^*_\partial M(\alpha), \quad \dot{\phi}^\omega = \dot{\phi}^h + d^{*\dot{\alpha}}X^*_\partial M(\dot{\alpha}), \]
where
\[ \phi^h, \dot{\phi}^h \in \mathcal{H}^1(\partial M), \quad \alpha, \dot{\alpha} \in \Omega^2(\partial M), \dot{\alpha} = L_{\partial \nu} \alpha, d^*dd^* \alpha = 0 \]
Hence equation (24) implies \( \forall \left( \phi', \dot{\phi}' \right) \in L_{\partial M} \)
\[ \int_{\partial M} \phi^h \wedge \ast_{\partial M} \dot{\phi}' + \int_{\partial M} d^{*\alpha}X^*_\partial M(\alpha) \wedge \ast_{\partial M} \dot{\phi}' = \]
\[ \int_{\partial M} \phi' \wedge \ast_{\partial M} \dot{\phi}^h + \int_{\partial M} \phi' \wedge \ast_{\partial M} d^{*\dot{\alpha}}X^*_\partial M(\dot{\alpha}), \]
We calculate in more detail the first summand of the r.h.s. of equation (25). According to lemma 6, \( \dot{\phi}^h = X^*_\partial M(\mathcal{L}_{\partial \tau} \overline{\varphi}_h) \), where we consider the orthogonal decomposition
\[ \overline{\varphi} = \overline{\varphi}_h + d^*\alpha \in \Phi_{A_M} \subset \mathcal{H}^1_N(M) \oplus d^*\Omega^2_N(M) \]
with \( \overline{\varphi}_h \in \mathcal{H}^1_N(M), \alpha \in \Omega^2_N(M) \). Hence
\[ \int_{\partial M} \phi' \wedge \ast_{\partial M} \dot{\phi}^h = \int_{\partial M} \phi' \wedge \ast_{\partial M} X^*_\partial M(\mathcal{L}_{\partial \tau} \overline{\varphi}_h) = \int_{\partial M} \phi' \wedge \ast_{\partial M} d^{*\dot{\alpha}}X^*_\partial M(\dot{\alpha}), \]
In the last line we have used the properties described for \( X^*_\partial M \) given in lemma 1 and \( d\overline{\varphi}_h = 0 \).
Now consider the first summand of the l.h.s. of equation (25), the extension \( \tilde{\varphi} := \psi \cdot \overline{\varphi} \in \Omega^1_N(M) \) of \( \overline{\varphi} \in \Omega^1(\partial M) \), given by a partition of unity
\[ \psi : M \rightarrow [0, 1], \text{ such that } \partial M = \psi^{-1}(1). \]
Then
\[ \int_{\partial M} \phi^h \wedge \ast_{\partial M} \dot{\phi}' = \int_{\partial M} \phi^h \wedge \ast_{\partial M} X^*_\partial M(\mathcal{L}_{\partial \tau} \tilde{\varphi}) = \int_{\partial M} \phi^h \wedge X^*_\partial M(*d\tilde{\varphi}). \]
Furthermore, lemma 6 claims that there exists \( \overline{\varphi}_h \in \mathcal{H}^1_N(M) \) such that \( \phi^h = X^*_\partial M(\overline{\varphi}_h) \). Therefore by Stokes' Theorem
\[ \int_{\partial M} \phi^h \wedge \ast_{\partial M} \dot{\phi}' = \int_{\partial M} X^*_\partial M(\overline{\varphi}_h \wedge *d\dot{\varphi}) = \int_M d(\overline{\varphi}_h \wedge *d\dot{\varphi}) = 0 \]
Therefore equation (25) yields

\[ \int_{\partial M} d^{*\partial M} X^*_\partial M(\alpha) \wedge \ast_{\partial M} \phi' = \int_{\partial M} \phi' \wedge \ast_{\partial M} d^{*\partial M} X^*_\partial M(\dot{\alpha}), \]  

(27)

According to the isomorphism \( H^1(M) \simeq \tilde{H}_N^1(M) \) in lemma 4, \([\tilde{\varphi}] \in H^1(M)\) corresponds to a harmonic field

\[ \phi' \mapsto \varphi_h \in \tilde{H}_N^1(M), \]

(28)

and this in turn corresponds to the harmonic component \( \phi'_h \in H^1(\partial M) \) of

\[ X^*_\partial M(\varphi) = \phi' = \phi'_h + d^{*\partial M} \beta \]

with \( \beta \in \Omega^2(\partial M) \), as is stated in lemma 5. Thus, for the r.h.s. of equation (27) we have

\[ \int_{\partial M} \phi'_h \wedge \ast_{\partial M} d^{*\partial M} X^*_\partial M(\dot{\alpha}) + \int_{\partial M} d^{*\partial M} \beta \wedge \ast_{\partial M} d^{*\partial M} X^*_\partial M(\dot{\alpha}), \]

notice that \( \partial \partial M = 0 \), therefore the last expression equals

\[ \int_{\partial M} d\phi' \wedge \ast_{\partial M} d^{*\partial M} X^*_\partial M(\dot{\alpha}) + \int_{\partial M} d^{*\partial M} \beta \wedge \ast_{\partial M} d^{*\partial M} X^*_\partial M(\dot{\alpha}) = \]

\[ \int_{\partial M} d^{*\partial M} \beta \wedge \ast_{\partial M} d^{*\partial M} X^*_\partial M(\dot{\alpha}) \]

Similarly for \( \dot{\phi}' = \dot{\phi}_h + d^{*\partial M} \dot{\beta} \) with \( \dot{\phi}_h \in \tilde{H}_1^1(\partial M) \), \( \dot{\beta} \in \Omega^2(\partial M) \) and therefore the l.h.s. of equation (27) equals \( \int_{\partial M} d^{*\partial M} \dot{\beta} \wedge \ast_{\partial M} d^{*\partial M} X^*_\partial M(\alpha) \).

\[ \int_{\partial M} \beta \wedge \ast_{\partial M} dd^{*\partial M} X^*_\partial M(\dot{\alpha}) = \int_{\partial M} \dot{\beta} \wedge \ast_{\partial M} dd^{*\partial M} X^*_\partial M(\alpha). \]

(29)

Finally this equation describes a condition on pairs \( \beta, \dot{\beta} \in \Omega^2(\partial M), \forall \alpha, \dot{\alpha} \in \Omega^2_N(\partial M) \subset \Omega^2_N(M) \). Again by Stokes’ Theorem applied to the r.h.s. of the previous expression (29) we have

\[ \int_{\partial M} \beta \wedge \ast_{\partial M} dd^{*\partial M} X^*_\partial M(\dot{\alpha}) = \int_M d\hat{\beta} \wedge \ast dd^*(\alpha), \forall \alpha \in \Omega^2_N(M), \]

(30)

where \( \hat{\beta} = \psi \cdot \beta \in \Omega^2_N(M) \) is an extension of a 2-form in the cylinder \( \beta \in \Omega^2(\partial M) \) to the interior of \( M \), given by a partition of unity \( \psi \), (26).
Recall that since $\varphi \in \Phi_{A_M}$, then $d^*d^*\alpha = 0$. From the orthogonal decomposition
\[
\Omega^3(M) = d\Omega^2_D(M) \oplus \mathcal{I}_N^3(M) \oplus (\mathcal{I}_N^3(M) \cap d\Omega^2(M)) \oplus d^*\Omega^4_N(M)
\]
we have $d^*\alpha \in \mathcal{I}_N^3(M) \cap d\Omega^2(M)$. By the non-degeneracy of the Hodge inner product in $M$, there is a well defined exact harmonic field $d\tilde{\beta} \in \mathcal{I}_N^3(M) \cap d\Omega^2$, that is the projection of $d\tilde{\beta}$, such that $d^*d\tilde{\beta} = 0$, and the r.h.s. of (30) reads as
\[
\int_M d\tilde{\beta} \wedge *d^*\alpha
\]
Therefore
\[
\int_{\partial M} \beta \wedge *\partial M d^*\alpha X^*_{\partial M}(\hat{\alpha}) = \int_{\partial M} X^*_{\partial M}(\tilde{\beta}) \wedge *\partial M d^*\alpha X^*_{\partial M}(\alpha). \quad (31)
\]
On the other hand consider the l.h.s. of (30). Recall that $\beta \in \Omega^2(\partial M) = \mathcal{I}_N^2(\partial M) \oplus d\Omega^1(\partial M) \oplus d^*\Omega^2(\partial M)$, in fact we can take
\[
\beta = \beta_h + d\gamma \in \mathcal{I}_N^2(\partial M) \oplus d\Omega^1(\partial M).
\]
On the other hand, consider extension $\bar{\beta} := \psi \beta$,
\[
\bar{\beta} = \bar{\beta}_h + d\bar{\gamma} \in \mathcal{I}_N^2(M) \oplus (\mathcal{I}_N^2(M) \cap d\Omega^1(M)) \oplus d\Omega^1_D(M) \oplus d^*\Omega^2_N(M).
\]
If we take the orthogonal projection of $\bar{\beta}$,
\[
\bar{\beta} := \bar{\beta}_h + d\gamma \in \mathcal{I}_N^2(M) \oplus (\mathcal{I}_N^2(M) \cap d\Omega^1(M)) \oplus d\Omega^1_D(M), \quad \gamma = \bar{\gamma} + \gamma_D \quad (32)
\]
then $X^*_{\partial M}(\bar{\beta}_h) = \bar{\beta}_h$ and $X^*_{\partial M}(\bar{\gamma}) = \gamma$. Also for 3–forms as arguments of $dd^*\alpha_M$ we have the functionals
\[
\int_{\partial M} X^*_{\partial M}(\bar{\beta}) \wedge *\partial M dd^*\alpha_M = \int_{\partial M} X^*_{\partial M}(\bar{\beta}_h + d\bar{\gamma}) \wedge *\partial M dd^*\alpha_M.
\]
If we look more carefully the l.h.s. of expression (31), then
\[
\int_{\partial M} X^*_{\partial M}(\bar{\beta}_h) \wedge *\partial M dd^*\alpha_M X^*_{\partial M}(L_{\partial \gamma}(\alpha)) = \int_{\partial M} X^*_{\partial M}(\bar{\beta}_h) \wedge X^*_{\partial M}(L_{\partial \gamma}*(dd^*\alpha)) =
\]

\[ \int_{\partial M} X_{\partial M} \mathcal{L}_{\partial r} \left( \hat{\beta} \wedge \ast (d^* \alpha) \right) + \int_{\partial M} X_{\partial M} \mathcal{L}_{\partial r} \left( \hat{\beta} \right) \wedge X_{\partial M} \left( \ast (d^* \alpha) \right) = \]

\[ \mathcal{L}_{\partial r} \left( \int_M d \left( \hat{\beta} \wedge \ast (d^* \alpha) \right) \right) + \int_{\partial M} X_{\partial M} \mathcal{L}_{\partial r} \left( \hat{\beta} \right) \wedge X_{\partial M} \left( \ast (d^* \alpha) \right) = \]

\[ \int_{\partial M} \hat{\beta} \wedge \ast \partial_M d^* \alpha X_{\partial M}^* (\alpha) \].

Where in the last equality we used \( X_{\partial M}^* \mathcal{L}_{\partial r} \left( \mathcal{L}_{\partial r} \hat{\beta} \right) = X_{\partial M}^* (\mathcal{L}_{\partial r} \beta) = \hat{\beta} \) and that \( \mathcal{L}_{\partial r} \int_M \hat{\beta} \wedge \ast d^* \alpha = 0 \). Hence

\[ \int_{\partial M} X_{\partial M}^* (\hat{\beta}) \wedge \ast \partial_M d^* \alpha X_{\partial M}^* (\alpha) = \int_{\partial M} \hat{\beta} \wedge \ast \partial_M d^* \alpha X_{\partial M}^* (\alpha) \] (33)

Looking back again at expression (31) and (29) we have

\[ \int_{\partial M} \hat{\beta} \wedge \ast \partial_M d^* \alpha X_{\partial M}^* (\alpha) = \int_{\partial M} \beta \wedge \ast \partial_M d^* \alpha X_{\partial M}^* (\alpha) = \]

\[ X_{\partial M}^* (\beta) \wedge \ast \partial_M d^* \alpha X_{\partial M}^* (\alpha) \]

Hence for every \( \alpha \) we have

\[ \int_{\partial M} X_{\partial M}^* \left( \mathcal{L}_{\partial r} \hat{\beta} \right) \wedge X_{\partial M}^* \left( \ast (d^* \alpha) \right) = \int_{\partial M} X_{\partial M}^* (\hat{\beta}) \wedge \ast \partial_M d^* \alpha X_{\partial M}^* (\alpha) \]

this implies that \( X_{\partial M}^* \left( \mathcal{L}_{\partial r} \hat{\beta} \right) = X_{\partial M}^* (\hat{\beta}) \).

Finally we can extend the solution \( \overline{\phi'} \) in the cylinder \( \partial M \varepsilon \) to a solution in the interior \( M \), by means of

\[ \overline{\phi'} := \overline{\phi'_h} + d^* \left( \hat{\beta} \right) \]

where \( \overline{\phi'_h} \) was defined in (28) and \( \hat{\beta} \) is defined in (32). Notice that \( d^* dd^* \gamma = d^* dd^* d^* \gamma = 0 \), therefore \( \overline{\phi'} \in \Phi_A \). Furthermore

\[ \phi' = X_{\partial M}^* (\overline{\phi'}) \], \[ \phi' = X_{\partial M}^* \left( \mathcal{L}_{\partial r} \overline{\phi'} \right) \].

This ends up the validity of (A10).

As we mentioned in previous section, locality follows for Yang-Mills fields and actions, in particular (A11) hold. Gluing axiom (A12) also follows from locality arguments. This completes the dynamical description for this gauge field theory.
4 n–dimensional case with corners

We will focus on the case of regions $M$, that are manifolds with corners provided with Yang-Mills fields.

We suppose that every hypersurface

$$ \Sigma = \Sigma^1 \cup \cdots \cup \Sigma^m = \hat{\Sigma}^1 \cup \cdots \cup \hat{\Sigma}^m / \sim_P $$

is a topological $(n-1)$–dimensional topological manifold that decomposes according to [A7] as with each $\hat{\Sigma}^i$ a manifold with corners.

Consider a hypersurface $\Sigma$ as a stratified space consisting of a union $\bigcup_{i=1}^m \Sigma^i$ of manifolds with corners $\hat{\Sigma}^i$ identified by their their faces $\partial \hat{\Sigma}^i$. Denote the structure of stratified spaces, as $|\Sigma|$ respectively. For a stratified space $|\Sigma|$ we denote the $k$–dimensional skeleton as $|\Sigma|^{(k)}$, $k = 0, 1, 2, \ldots, n-1$ notice that $|\Sigma|^{(n-1)} \cong \Sigma$ and

$$ |\Sigma|^{(n-2)} = \bigcup_{(i,j) \in P} \Sigma^{ij} \subset \Sigma $$

corresponds to the corners set. We adopt the notation for the set of $k$–dimensional faces as $|\Sigma|^k$, thus

$$ |\Sigma|^{n-1} = \{ \hat{\Sigma}^1, \ldots, \hat{\Sigma}^m \} $$

is the set of $(n-1)$–dimensional faces and

$$ |\Sigma|^{n-2} = \{ \hat{\Sigma}^{ij} \mid (i, j) \in P \} $$

is the set of $(n-2)$–faces. Here $\hat{\Sigma}^{ij} \subset \hat{\Sigma}^i$ is the preimage of the corner $\Sigma^{ij} = \Sigma^i \cap \Sigma^j \subset \Sigma$, $(i, j) \in P$. We adopt the definitions of stratified spaces in [ACO].

The $r$–forms in the $k$–skeleton, $r \leq k, k = 0, 1, 2, \ldots, n-1$ is the set of restrictions of these $r$– forms to its faces:

$$ \Omega^r \left( |\Sigma|^{(n-1)} \right) = \bigcup_{i=1}^m \left\{ \varphi^i \in \Omega^k(\Sigma^i) \mid \varphi^i |_{\Sigma^j} = \varphi^j \forall (i, j) \in P \right\} $$

$$ \Omega^r (|\Sigma|^{(n-2)}) = \bigcup \left\{ \varphi^I \Omega^k \Sigma^I \mid \varphi^I |_{\Sigma^J} = \varphi^J, I, J \in P \right\} $$

Notice that we have the inclusion of $k$–forms on stratified spaces, for $r = 0, 1$, given by the pullbacks

$$ \Omega^r (\Sigma) = \Omega^r \left( |\Sigma|^{(n-1)} \right) \subset \Omega^r \left( |\Sigma|^{(n-2)} \right). $$
Each $\Sigma^i$ is embedded in a regular tubular neighborhood $\hat{\Sigma}_i^\varepsilon$, $\varepsilon > 0$. The analogous of lemma [1] can be established for a regular tubular neighborhood of $\Sigma^i$. We can proof ibidem the existence of embedding $W$ that have similar properties as those described for $X$. It needs an adaptation of Moser’s argument to volume forms on open manifolds $\Sigma^i$ and diffeomorphisms vanishing on the boundary $\partial \Sigma^i$, see [DaM].

We describe the embedding by considering a smooth function $\varepsilon : \Sigma^i \rightarrow [0, \varepsilon]$, such that for $s \in \partial \Sigma^i$, $\varepsilon(s) = 0$, meanwhile for $s \in \Sigma^i - \partial \Sigma^i$, $Y(s, \tau)$, is the geodesic through $Y(s, 0) = s$ with length $\tau \in [0, \varepsilon(s))$ normal to $\Sigma^i$. As in the previous discussion $\partial \tau = \partial Y(s, 0)/\partial \tau$ is the normal vector field on $\Sigma^i - \partial \Sigma^i$. In this case the "cylinder" is the image

$$
\hat{\Sigma}_i^\varepsilon := \{ Y(s, \tau) \mid s \in \Sigma^i, 0 \leq \tau < \varepsilon(s) \} \subset M = W(\Sigma^i \times [0, \varepsilon])
$$

where $W(s, \tau) = Y(s, \tau\varepsilon(s))$.

To give a detailed description of the space of divergence-free fields on $\Sigma$, let us first consider harmonic fields.

If $\phi_h \in H^1(\Sigma^i)$, then the restriction, $\phi_h^i$, over every face closure $\Sigma^i \subset \Sigma$, contained in $\Sigma^i$, is harmonic as stated in the following lemma.

**Lemma 7** (Theorems 7 and 8 in [AGO]). Let $\Sigma^i$ a riemannian manifold with corners homeomorphic to the cylinder $\Sigma \times [0, \varepsilon]$. For every harmonic form $\varphi \in \mathcal{F}^r(\Sigma^i)$,

1. It is closed and coclosed, $d\varphi = 0 = d^*\varphi$, i.e. $\varphi \in \mathcal{F}^r(\Sigma^i)$

2. $\varphi^i := \varphi \mid_{\Sigma^i} \in \mathcal{F}^r(\Sigma^i)$, where $\Sigma^i \subset |\Sigma|^{(n-1)}$ are the $(n-1)$-dimensional faces.

3. The boundary and coboundary operators satisfy,

$$
((d\varphi)^i) = (d(\varphi^i)), \quad (((d^*\varphi)^i) = (d^*\varphi^i))
$$

so that they define complexes $(\Omega(|\Sigma|^{(n-2)}), d), (\Omega(|\Sigma|^{(n-2)}), d^*)$.

4. $\varphi \mid_{\Sigma^i} \in \mathcal{F}^r_N(\Sigma^i)$, if and only if $\iota_{\partial^r} \varphi = 0$, where $\partial^r$ is a vector field normal to $\Sigma^i$.

Let $|\Sigma|^{(n-1)} = |\Sigma|$ be an $(n-1)$-dimensional stratified space homeomorphic to an $(n-1)$-dimensional manifold. For every harmonic $r$-form $\phi \in \mathcal{F}^r(|\Sigma|)$

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1. It is closed and coclosed, \(d\phi = 0 = d^\ast \Sigma \phi\), i.e. \(\phi^i := \phi|_{\Sigma^i} \in \mathcal{H}^r(\Sigma^i)\), for each \((n - 1)\)-dimensional closed stratum \(\Sigma^i\).

2. \(\phi^I := \phi|_{\Sigma^I} \in \mathcal{H}^r(\Sigma^I)\), where \(\Sigma^I \subset \mid \Sigma \mid^{(n-2)}\) are the \((n-2)\)-dimensional faces.

3. The boundary and coboundary operators satisfy,
   \[
   (d\phi^I) = (d\phi^I), \quad \left( (d^\ast \phi^I) \right) = \left( d^\ast (\phi^I) \right)
   \]
   so that they define complexes \((\Omega(\mid \Sigma \mid^{(n-2)}), d)\), \((\Omega(\mid \Sigma \mid^{(n-2)}), d^\ast)\).

This finishes the description of harmonic forms on the stratified space \(\mid \Sigma \mid\). Notice that when \(\Sigma^i\) are balls then the harmonic forms \(\phi \in \Omega^r(\Sigma^i)\) are completely defined by their Dirichlet boundary conditions on \(\partial \Sigma^i\).

We also have an analogous of hodge decomposition for stratified spaces, that follows also from theorems 7 and 8 in \([AGO]\):

**Corollary 2.** 1. There is an orthogonal decomposition
   \[
   \Omega^r(\mid \Sigma \mid) = \mathcal{H}^r_N(\mid \Sigma \mid) \oplus \left( \mathcal{H}^r(\mid \Sigma \mid) \cap d\Omega^{r-1}(\mid \Sigma \mid) \right) \oplus \\
   \oplus d\Omega^{r-1}_D(\mid \Sigma \mid) \oplus d^\ast \Omega^{r+1}_N(\mid \Sigma \mid)
   \]

2. In particular there is an orthogonal decomposition for divergence-free fields
   \[
   \ker \left[ d^\ast : \Omega^r(\mid \Sigma \mid) \to \Omega^{r-1}(\mid \Sigma \mid) \right] = \\
   \mathcal{H}^r_N(\mid \Sigma \mid) \oplus d^\ast \Omega^{r+1}_N(\mid \Sigma \mid)
   \]

Therefore the divergence free \(1\)-forms on \(\mid \Sigma \mid\) are described as
\[
\mathcal{H}^1(\mid \Sigma \mid) \oplus d^\ast \Omega^2(\mid \Sigma \mid)
\]

Thus we could define the gauge-fixing space as
\[
\Phi_{\Delta_S} := T \left( \mathcal{H}^1(\mid \Sigma \mid) \right) \oplus T \left( \star \Sigma d\Omega^0(\mid \Sigma \mid) \right).
\]

(34)

Form the projections \(\Sigma^i \to \Sigma^i \subset \Sigma\) we obtain the linear maps
\[
\mathcal{H}^1(\mid \Sigma \mid) \subset \bigoplus_{i=1}^r \mathcal{H}^1(\Sigma^i)
\]

and
\[
\Omega^0(\mid \Sigma \mid) \subset \bigoplus_{i=1}^m \Omega^0(\Sigma^i).
\]
If $\Phi_{A\Sigma} := T (\Omega^1 (\Sigma^i)) \oplus T (\ast\Sigma d\Omega^0 (\Sigma^i))$, then we get the injective maps

$$\Phi_{A\Sigma} \to \oplus_{i=1}^r \Phi_{A\Sigma^i} =: \Phi_{A|\Sigma^i}.$$  

This inclusion is the restriction of an inclusion referred in axiom (A7) as is described in the following commuting diagram

Diagram 1

Similarly, the inclusions in the affine spaces $A\Sigma \to \prod_{i=1}^m A\Sigma^i$ can be described. Also for the gauge symmetries we have exact sequences

$$H^0_{dR}(|\Sigma|) \to \cdots \to \Omega^0 (|\Sigma|) \to \oplus_{i=1}^r \Omega^0 (\Sigma^i) \to G_\Sigma^0 \to \cdots \to \oplus_{i=1}^r G_{\Sigma^i} =: G_{|\Sigma|^2}$$

where $G_{\Sigma^i} \simeq \Omega^0 (\Sigma^i)/R$ (since $\Sigma^i$ is simply connected), and $G_\Sigma^0$ stands for the identity component of the gauge symmetries group $G_\Sigma$.

The analogous result of theorem 1 can be established for surfaces with corners.

**Theorem 3.** We have the gauge fixing space

$$\Phi_{A\Sigma} = \Phi_{A\Sigma}/G_\Sigma^0 \simeq T (\Omega^1 (|\Sigma|)) \oplus T (\ast\Sigma d\Omega^0 (|\Sigma|)) \subset \oplus_{i=1}^r \Phi_{A\Sigma^i}$$

with symplectic structure $\omega_{\Sigma^i}$ induced by the pullback of $\omega_{\Sigma^1} \oplus \cdots \oplus \omega_{\Sigma^r}$. We also have the quotient space

$$A\Sigma/G_\Sigma \simeq T (G^0) \times T (d^\ast \Omega^2 (|\Sigma|))$$

where $b$ is the Betti number of $\Sigma$, $\dim H_{n-2} (\Sigma; \partial \Sigma) = \dim H_1 (\Sigma)$, $n - 1 = \dim \Sigma$. 

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According to [Du], [DuS], the space of harmonic forms on a smooth manifold $M$ with smooth boundary $\partial M$, have the Betti number as rank and equals $\dim \mathfrak{H}^1_N(M) = \dim H_1(M) = \dim H_{n-1}(M, \partial M)$. Similarly for smooth closed hypersurfaces $\Sigma$,

$$\dim \mathfrak{H}^1_N(\Sigma) = \dim H_1(\Sigma) = \dim H_{n-2}(\Sigma, \partial \Sigma).$$

For manifolds with corners $\Sigma$, the space of harmonic forms has the same description. Take a homeomorphism $F : \Sigma' \rightarrow \Sigma$, that defines a diffeomorphism, with lack of differentiability on the corners on $\partial \Sigma'$. If $\phi \in \mathfrak{H}^1_N(\Sigma)$ is harmonic form with null normal component, then $F^* (\phi) |_{\partial \Sigma'} \in \mathfrak{H}^1_N(\Sigma')$ is also well defined harmonic form on $\partial \Sigma'$. Hence, for stratified spaces homeomorphic to manifolds, such as $|\Sigma|$ harmonic forms have also rank given by the Betti number.

We use the decomposition of the integration chain $\int_{\Sigma'}$, as a sum of $(n-1)$-dimensional integration cells, $\int_{\Sigma_1} + \cdots + \int_{\Sigma_m}$. See for instance [AGO] lemma 5. Thus obtaining [A7].

Dynamics described in [A10] is also valid in the context of corners. The statement of theorem [2] remains the same as in the case without corners. Isotropic embedding of $L_M \cap \Phi_{\Lambda M} \subset \Phi_{\Lambda M}$, goes ibidem as in the case without corners.

The proof of the coisotropic embedding in the case without corners depends entirely on the orthogonal decompositions and isomorphisms described in lemma [4]. Explicitly we require the isomorphism

$$H^1(M) \simeq \mathfrak{H}^1_N(|M|)$$

an orthogonal decomposition

$$\Omega^k(M) = d\Omega^{k-1}(|M|) \oplus \mathfrak{H}^k_N(M) \oplus (\mathfrak{H}^k(|M|) \cap d\Omega^{k-1}(|M|)) \oplus d^* \Omega^{k+1}(M)$$

And also

$$\Omega^2(|\partial M|) = \mathfrak{H}^2(|\partial M|) \oplus d\Omega^1(|\partial M|) \oplus d^* \Omega^3(|\partial M|),$$

which may be verified by using the results stated in [AGO] theorems 7 and 8.

### 4.1 Example: 2–dimensional case

For a better understanding of our model, we review our constructions in a more down to earth examples namely the 2–dimensional case. We provide
this presentation as comparison tool with other known procedures for quantization of two dimensional Yang-Mills theories, see for instance [Wi], [DH], [La]. This will suggest the following steps that are necessary in quantization for general dimensions in further research [D].

Recall that we are supposing that we have a trivial gauge principal bundles on a compact surface $M$, with structure group $G = U(1)$. The following lemma will lead to a description of the presymplectic structure $\tilde{\omega}_\Sigma$, on $A_\Sigma$, for a proof see [Il]. Lemma 1 in this case can be simplified as the following statement.

**Lemma 8** (Fermi). *Given a cylinder $\Sigma \times [0,1]$, there exists an embedding* \[ X : \Sigma \times [0,\varepsilon] \to M \] *of the cylinder $\Sigma \times [0,\varepsilon]$ into a tubular neighborhood $\Sigma_\varepsilon$ of $\Sigma$, such that if $(s,\tau)$ are local coordinates, then $\partial/\partial s, \partial/\partial \tau$ are orthonormal vector fields along $\Sigma$. Here $s$ corresponds to arc length along $\Sigma$ with respect to the riemannian metric $h$ on $M$. Furthermore $h|_\Sigma$ is locally described as the identity matrix.*

The presymplectic structure can be written by using these local coordinate as in (7) as \[ \tilde{\omega}_\Sigma (\tilde{\eta}, \tilde{\xi}) = \frac{1}{2} \int_\Sigma [\eta^s (\partial_s \xi^\tau - \partial_\tau \xi^s) - \xi^s (\partial_s \eta^\tau - \partial_\tau \eta^s)] ds, \] (37) where $X_\Sigma^s (\eta) = \eta^s ds + \eta^\tau d\tau, X_\Sigma^\tau (\xi) = \xi^s ds + \xi^\tau d\tau$ are 1–forms corresponding to solutions in the cylinder i.e. $\xi, \eta \in \Omega^1 (\Sigma_\varepsilon)$ satisfying the Euler-Lagrange equations. We can also describe the gauge group $G_\Sigma$ on $A_\Sigma$, by describing the action corresponding to the identity component gauge group of germs: $\tilde{G}_\Sigma^0 := \lim \to G_\Sigma^0\varepsilon$, where $G_\Sigma^0 := \Omega^0 (\Sigma_\varepsilon)/\mathbb{R}b_0$ is acting by the translations $\eta \mapsto \eta + df$ and inducing the corresponding action $\tilde{\eta} \mapsto \tilde{\eta} + d\tilde{f}$ on germs $\tilde{\eta} \in \tilde{L}_\Sigma$.

In addition, the degeneracy subspace of the symplectic form is \[ K_{\omega_\Sigma} := \{ \tilde{\eta} \in \tilde{L}_\Sigma \mid \eta = \partial_\tau f d\tau, \partial_s f (s,0) = 0, f \in \Omega^0 (\Sigma_\varepsilon), \} \] From this very definition we have that the degeneracy gauge symmetry group $K_{\omega_\Sigma}$ is a (normal) subgroup of the abelian group $\tilde{G}_\Sigma^0$.

By considering an axial gauge fixing, as in [S], let \[ \Phi_{A_\Sigma} := \{ \tilde{\eta} \in \tilde{L}_\Sigma \mid \iota_{\partial_s^\tau} \tilde{\eta} = 0 \} \]
be a subspace of $\tilde{L}_\Sigma$. As we did in lemma \[\text{we have that every } K_{\omega^\Sigma} - \text{orbit in } \tilde{L}_\Sigma \text{ intersects in just one point the subspace } \Phi_{A^\Sigma}. \] The presymplectic form $\tilde{\omega}^\Sigma$ restricted to the subspace $\Phi_{A^\Sigma}$ may be written as

$$\tilde{\omega}^\Sigma(\tilde{\eta}, \tilde{\xi}) = \frac{1}{2} \int_{\Sigma} [-\eta^s \partial_\tau \xi^s + \xi^s \partial_\tau \eta^s] \, ds, \quad \tilde{\eta}, \tilde{\xi} \in \tilde{L}_\Sigma.$$  

(38)

Hence $\tilde{\omega}^\Sigma$ is non-degenerated when we restrict it to the subspace $\Phi_{A^\Sigma} \subset \tilde{L}_\Sigma$.

Let $\omega^\Sigma$ the corresponding symplectic structure on $A^\Sigma$ induced by the restriction of $\tilde{\omega}^\Sigma$ to the subspace $\Phi_{A^\Sigma} \subset \tilde{L}_\Sigma$.

Let us consider hypersurfaces $\Sigma := \Sigma^1 \cup \cdots \cup \Sigma^m \subset \partial M$ where $\Sigma^i$ are homeomorphic to $S^1$ in the case without corners, or to intervals possibly identified in some pairs by their boundaries in the case with corners.

Then there is a linear map

$$\Omega^1_{\partial M} \to \Omega^1 (\Sigma^1) \oplus \cdots \oplus \Omega^1 (\Sigma^m)$$

where $\eta \mapsto \left( X^\Sigma_0 \right)^* (\eta) \oplus \cdots \oplus \left( X^\Sigma_m \right)^* (\eta)$. This map induces $r_{\partial M, \Sigma} : L_{\partial M} \to L_{\Sigma^1} \oplus \cdots \oplus L_{\Sigma^m}$. Furthermore, we have the chain decomposition $\int_{\partial M} \cdot = \int_{\Sigma^1} \cdot + \cdots + \int_{\Sigma^m} \cdot$, induces axioms (A7) and (A7).

Recall that here is map, $\tilde{r}_M : L_M \to \tilde{L}_{\partial M}$, coming from the restriction of the solutions to germs on the boundary, and composing we the quotient class map we have a map $r_M : L_M \to L_{\partial M}$. Let $L_{\tilde{M}}$ be the image $L_{\tilde{M}} = r_M(L_M)$ under this map. The aim is to describe the image $L_{\tilde{M}} = r_M(L_M) \subset L_{\Sigma}$ of the space of solutions as a lagrangian subspace modulo gauge.

Take $\varphi \in L_M$ so that $d^* d\varphi = 0$, then $d\varphi$ is constant scalar multiple of the $h$–area form $\mu$, i.e. $d\varphi = \dot{c}_{\varphi} \mu$, for a constant $\dot{c}_{\varphi}$. Suppose that $\overline{\varphi} \in L_M$ is such that $\iota_{\partial_r \overline{\varphi}} = 0$, then $\overline{\varphi} = 0$. Hence $\partial_s \overline{\varphi} - \partial_r \overline{\varphi} = \dot{c}_{\varphi}$ is constant. That is, $-\partial_r \overline{\varphi} = \dot{c}_{\varphi}$. Therefore if $\phi := r_M(\overline{\varphi})$, $\phi' := r_M(\varphi') \in A_{\partial M}$, by substituting in (38) we obtain

$$\omega_{\partial M} \left( \varphi, \varphi' \right) = \int_{\partial M} \left( \varphi^* \dot{c}_{\varphi'} - \left( \varphi'^* \dot{c}_{\varphi} \right) \right) \, ds, \forall \varphi, \varphi' \in L_{\partial M}$$

Were we recall that $\varphi, \varphi' \in L_{\partial M}$. By Stokes' theorem

$$\omega_{\partial M} \left( \varphi, \varphi' \right) = \dot{c}_{\varphi} \int_M d\varphi - \dot{c}_{\varphi'} \int_M d\varphi' = \left( \dot{c}_{\varphi'} \dot{c}_{\varphi} - \dot{c}_{\varphi} \dot{c}_{\varphi'} \right) \cdot \text{area}(M) = 0. \quad (39)$$

where $\text{area}(M) := \int_M \mu$.  

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Let us now consider the orbit space for gauge orbits. We consider the unity component subgroup \( G_0^\Sigma \leq G_\Sigma \). Recall the map (14), let us take a gauge fixing subspace

\[
\Phi_{A\Sigma} := \{(\eta_0, \dot{\eta}_0) \in A_\Sigma \mid \partial_s \eta_0 = 0 = \partial_s \dot{\eta}_0\} = \{(c \, ds, \dot{c} \, ds) \in A_\Sigma \mid (c, \dot{c}) \in \mathbb{R}^2\}
\]

We can see the proof of lemma 3 for this context. Let \((c ds, \dot{c} ds)\) be a point in \( \Phi_{A\Sigma} \sim = \mathbb{R}^2 \). Consider

\[
X^{*}_\Sigma(\eta) = \eta^s ds + \eta^\tau d\tau = \eta^s ds,
\]

a local expression for a solution \( \eta \in A_\Sigma \cap \Phi_{A\Sigma} \). By considering a gauge symmetry we can get an ODE for \( f : A_\Sigma \rightarrow \mathbb{R} \),

\[
\eta^s + \partial_s f = c
\]

(40)

\[
\partial_\tau \eta^s + \partial_\tau \partial_s f = \dot{c}
\]

(41)

Equation (41) can be solved for \( g(s, \tau) := \partial_s f \), once we can fix the boundary condition \( \partial_s f (s, 0) = g(s, 0) \). This boundary condition in turn can be obtained by solving (40) in \( \Sigma \). The holonomy along \( \Sigma \),

\[
\text{hol}_\Sigma(\eta) = \exp \sqrt{-1} \int_\Sigma \eta \in G = U(1)
\]

(42)

remains the same for \( c \) and for \( \eta \), furthermore since they are in the same component, \( \int_\Sigma c \, ds \) equals \( \int_\Sigma \eta^s ds \mod 2\pi \mathbb{Z} \). \( \eta \) belongs to the \( G_0^\Sigma \)-orbit of \( c \), therefore there is a homotopy between both evaluations. Hence

\[
c \cdot \text{length}(\Sigma) = \int_\Sigma c \, ds = \int_\Sigma \eta^s ds
\]

this implies that equation (40) can be solved.

We also have lemma 5. It follows that \( L_{\tilde{M}} = r_M(\Phi_{A\Sigma}) \). The isotropic embedding described in theorem 2 is proven in (39). The corresponding coisotropic embedding in the 2-dimensional version goes as follows:

Take \( \varphi \in \Phi_{A\Sigma}, \phi = r_M(\varphi) \) and suppose that \( \omega_{\partial M}(\phi, \phi') = 0 \) for every \( \phi' \in L_{\partial M} \), with \( \phi' \in L_{\partial M} \) corresponding to \( \varphi' \). Then

\[
\dot{\varphi} \int_{\partial M} (\varphi')^s ds = \int_{\partial M} (\varphi^s \partial_{\tau}(\varphi')^s) ds.
\]

Since \( \varphi' \) is a solution in a tubular neighborhood \( \partial M_\varepsilon \) then \( \partial_{\tau}(\varphi')^s |_{\Sigma} = \dot{\varphi} \varphi' \). Hence

\[
\dot{\varphi} \int_{\partial M} (\varphi')^s ds = \dot{\varphi} \int_{\partial M} \varphi^s ds = \dot{\varphi} \int_{\partial M} \varphi^s \int_{\partial M} \mu
\]

Therefore

\[
\int_{\partial M} \varphi' = \dot{\varphi} \cdot \text{area}(M).
\]

(43)
We claim that this is a sufficient condition so that $\varphi' \in L_{\partial M_\varepsilon}$ can be extended to the interior of $M$, i.e., there exists a solution $\tilde{\varphi} \in L_M$ such that $\varphi' = r_M(\tilde{\varphi})$. This will be an exercise of calculus of differential forms.

The first step is to construct an extension $\theta = \psi \varphi \in \Omega^1(M)$, where we take a partition of unity $\psi$ whose value on $\partial M_\varepsilon$ is 1 and is 0 outside an open neighborhood $V \subset M$ of $\partial M_\varepsilon$. We see that $\hat{\varphi}' d\theta$ is closed and also has the same relative de Rham cohomology class in $H^2_{\text{dR}}(M, \partial M; \mathbb{R})$ as $\hat{\varphi}' \mu$, thus $\hat{\varphi}' (d\theta - \mu) = \hat{\varphi}' d\beta$, for a 1-form $\beta$ such that $\beta|_{\partial M_\varepsilon} = 0$. Therefore we can define $\tilde{\varphi} := \theta - \beta$ such that it is a solution, $d\tilde{\varphi} = \hat{\varphi}' \mu$ and it is also an extension, $\tilde{\varphi}|_{M} = \varphi'|_{M}$.

This proves again in the 2-dimensional case the lagrangian embedding of theorem 2.

Notice that in this case the bilinear form $[,]_\Sigma$ used in axiom (A4) corresponds to $[\phi^\eta, \phi^\xi]_\Sigma := -\int_\Sigma (\eta^s \partial_s \xi^s) \, ds$.

Here $(c \, ds, \hat{c} \, ds) \in \Phi_{A_{\Sigma}}$ can be identified with $c + \sqrt{-1} \hat{c} \in \mathbb{C}$ provided with the Kahler structure: $\text{length}(\Sigma) \cdot dc \wedge d\hat{c}$. The holonomy $\text{hol}_{\Sigma} : \Omega^1(\Sigma) \to U(1)$ induces the derivative map $D \text{hol}_{\Sigma} : \Phi_{A_{\Sigma}} \to T U(1)$. We have the following commutative diagram

$$
\begin{array}{ccc}
\Phi_{A_{\Sigma}} & \xrightarrow{D \text{hol}_{\Sigma}} & T U(1) \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times
\end{array}
$$

We can finally define the reduced space as the topological cylinder

$$
A_{\Sigma}/G_{\Sigma} := \Phi_{A_{\Sigma}} / G_{\Sigma} \cong \mathbb{C}^\times
$$

(44)

where $G_{\Sigma} := G_{\Sigma}/G^0_{\Sigma} \cong \mathbb{Z}$. We can get the symplectic structure $\overline{\omega}_{\Sigma}$ on $A_{\Sigma}/G_{\Sigma}$. This $\overline{\omega}_{\Sigma}$ is length$(\Sigma)$ times the area form on the cylinder $T U(1)$.

The reduced symplectic structure: $\overline{\omega}_{\Sigma}$ on $A_{\Sigma}/G_{\Sigma}$, is length$(\Sigma)$ times the area form on the cylinder $T U(1)$. For $\partial M = \Sigma_1 \cup \cdots \cup \Sigma_m$:

$$
A_M/G_M \to A_{\partial M}/G_{\partial M} = T U(1) \times \cdots \times T U(1)
$$

has lagrangian image whose image in each factor is the quotiented line:

$$
(A/G)_{\Sigma} := \{(c, \hat{c}) \in \mathbb{R}^2 \mid c \cdot \text{length}(\partial M) = \hat{c} \cdot \text{area}(M)\} / \mathbb{Z} \subset T U(1)
$$

As a consequence the map $A_M/G_M \to A_{\partial M}/G_{\partial M}$ does depend on global data of the metric, such as area$(M)$ and length$(\partial M)$. Recall that the same
global dependence of dynamics holds for the quantum version, i.e. the quantum TQFT version of Yang-Mills gauge fields.

Once we have completed reduction, the picture of quantization on this finite dimensional space can be specified cfr. [DH], [W1], [La]. For a complete description of the quantization in 2−dimensions in general non abelian case with corners see [O2].

5 Outlook: quantization in higher dimensions

The geometric quantization program with corners will be treated elsewhere [D]. Once the reduction - quantization procedure is completed, the next task is the formulation of the quantization - reduction process and the equivalence of both procedures. See the discussion of these issues in dimension two for instance in [W1], [DH] and [La]. In order to administrate the geometric quantization program [Wo] for the reduced space we need to describe a suitable hermitian structure in $\Phi_{A_\Sigma}$ to be used as a prequantization ingredient.

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