Diagrammatic evaluation of conformal weights in the
\[ U_q[SU(2)] \] symmetric Heisenberg chain

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Abstract

We consider the \[ U_q[SU(2)] \] symmetric Heisenberg chain when \( q = e^{i\pi/(m+1)} \) and \( m \) is integer. We consider the cases \( m = 3 \) and \( m = 5 \) which correspond to the Ising and 3-state Potts models. We study the finite size scaling (FSS) of the ground states in different quantum spin sectors and restricting to highest weights of type-II representations. We compute the levels by a diagrammatic technique which needs only the commutation relations of the underlying Temperley-Lieb algebra. The results match the FSS predictions which hold for the Bethe levels.

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I. INTRODUCTION

The XXZ quantum chain on a finite lattice has a non trivial quantum symmetry when suitable boundary conditions are considered. Here, we are interested in the case of a chain whose symmetry is that of the $U_q[SU(2)]$ quantum group with $q$ a root of unity:

$$q = e^{i\pi/(m+1)}.$$  \hspace{1cm} (1.1)

The chain is massless and the central charge is

$$c = 1 - \frac{6}{m(m+1)}$$ \hspace{1cm} (1.2)

and for integer $m$ the the minimal unitary series emerges.

It is very important to understand the structure of the configuration space of a $N$ sites chain under the action of the quantum group. For a chain of $1/2$ spins, $C^{2N}$ decomposes into large indecomposable but reducible (type-I) representations and irreducible (type-II) representations.

The energy levels can be determined by a Bethe-Ansatz calculation. The resulting Bethe states are highest weights with respect to the raising operator $S^+$. The Kac conformal weight associated to the finite size scaling between $U_q[SU(2)]$ spin 0 and spin $j$ ground states is predicted to be the entry $h_{1,1+2j}$ in the Kac table.

It is a conjecture that the highest weights of type-II representations exhaust all the Bethe states with spin $0 \leq s < m/2$ (see also [3,4]).

In this letter we study explicitly the finite size scaling of energies and the corresponding conformal weights in the space of type-II highest weights. These are associated to the paths in the Bratteli diagram. In this path space we can represent the underlying Temperley-Lieb algebra of the XXZ chain and compute the levels by purely algebraic manipulations.

The states of the path space admits convenient binary representations which allows for simple and compact numerical codes for their manipulation. Relatively large lattices may be studied with small computing resources.
The interest of this work is also in showing how the diagrammatic approach of [6] is an effective tool for heuristic investigations, for instance, we obtain strong evidence that the above mentioned conjecture holds true.

II. THE XXZ CHAIN AND ITS QUANTUM SYMMETRY

The hamiltonian is

\[ H = \sum_{i=1}^{N-1} e_i \]  

(2.1)

where \( \{e_1, \ldots, e_{N-1}\} \) are the generators of the Temperley-Lieb algebra

\[ e_i^2 = x e_i \quad x = q + q^{-1} \]  

(2.2)

\[ e_i e_{i \pm 1} e_i = e_i \]  

(2.3)

\[ [e_i, e_j] = 0 \quad \text{if} \quad |i - j| > 1 \]  

(2.4)

All the following algebraic manipulations are based on the above relations only, but we shall keep in mind the following specific quantum chain representation

\[ e_i = \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q + q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z + \frac{q - q^{-1}}{2} (\sigma_i^z - \sigma_{i+1}^z) \]  

(2.5)

The \( U_q[SU(2)] \) quantum group is generated by \( \{S^\pm, q^{S_z}\} \) with the relations

\[ [S^+, S^-] = [2S_z]_q \]  

(2.6)

\[ q^{S_z} S^\pm q^{-S_z} = q^{\pm 1} S^\pm \]  

(2.7)

where

\[ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \]  

(2.8)

Its spin 1/2 representation coincides with the classical one \( (q = 1) \) and the tensored representations are built as usual by mean of the Hopf-coproduct

\[ \Delta(q^{\pm S_z}) = q^{\pm S_z} \otimes q^{\pm S_z} \]  

(2.9)

\[ \Delta(S^\pm) = q^{\pm S_z} \otimes S^\pm + S^\pm \otimes q^{\mp S_z} \]  

(2.10)
The Hamiltonian is exactly invariant under $U_q[SU(2)]$ also at finite $N$

\[ [H, S^\pm] = [H, S_z] = 0 \]  

(2.11)

According to [5,7] we can introduce orthonormal vectors $V_k$ labelled by

\[ k = (k_0, \cdots, k_N) \]  

(2.12)

with

\[ k_0 = 0 \quad k_i \geq 0 \quad k_{i+1} = k_i \pm 1/2 \]  

(2.13)

If $q$ is a root of unity and $q^p = \pm 1$ then we have the additional constraint

\[ k_i \leq \frac{p}{2} - 1 \]  

(2.14)

related to $[p]_q = 0$. These vectors are in one-to-one correspondence with the highest weights of type-II representations. The action of the Temperley-Lieb algebra on these vectors is

\[ e_i V_k = \delta_{k_{i-1}, k_{i+1}} \sum_{k'_{i-1} = k_{i-1} \pm 1/2} \sqrt{\frac{[2k_i + 1]_q [2k'_i + 1]_q}{[2k_{i-1} + 1]_q [2k_{i+1} + 1]_q}} V_{k'} \]  

(2.15)

where

\[ k' = (k_0, \cdots, k_{i-1}, k'_i, k_{i+1}, \cdots, k_N) \]  

(2.16)

Every weight $k$ corresponds to a representation with spin $k_N$.

III. BINARY REPRESENTATION

Let us call $S_m^{(j)}$ the path subspace with $q = e^{i\pi/(m+1)}$ and $k_N = j$. We shall study the cases $m = 3$ and $m = 5$ corresponding to the Ising model and to the 3-state Potts model. We shall restrict to chains with an even number of sites.

Every state $s \in S_m^{(j)}$ is determined by the signs $\sigma_i \in \{-1, 1\}$ defined by

\[ \sigma_i = 2(k_{i+1} - k_i) \]  

(3.1)
Let $W_N$ denote the set of $2^{N-1}$ binary words with $N$ bit: every state $s \in S_m^{(j)}$ is mapped into a word $w \in W_N$, the $i$-th bit $b_i$ being $(\sigma_i + 1)/2$. This binary representation is interesting since, in a numerical study, allows us to write states in terms of integer numbers. Of course, the $2^{N-1}$ words in $W_N$ must satisfy the constraints Eqs. (2.13,2.14) imposed over the multilabel $k$. This reduces greatly the dimension of the state space. Let us turn to some specific examples.

A. $m = 3$

This case corresponds to the Ising model. The structure of the generic path is well understood by looking at the following grid which hosts $S_3^{(0)}$ with $N = 4$.

\[
\begin{array}{cccc}
1 & 1 & 1 & \\
\uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \\
1/2 & 1/2 & 1/2 & 1/2 & \\
\uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \\
0 & 0 & 0 & 0 & 0 & \\
\end{array}
\]

Moreover, $S_3^{(1)}$ is in $1 - 1$ correspondence with $S_3^{(0)}$ as the following grid shows.

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \\
1/2 & 1/2 & 1/2 & 1/2 & \\
\uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \\
0 & 0 & 0 & 0 & \\
\end{array}
\]

We readily see that the relevant signs $\sigma_1$ are only $(N - 2)/2$. Thus

\[
\dim S_3^{(0)} = \dim S_3^{(1)} = 2^{N/2 - 1}
\]
B. \( m = 5 \)

In this case the paths are more complicated due to the weaker constraint Eq.(2.14). An example with \( N = 10 \) could be

\[
\begin{array}{c}
2 \\
\uparrow \quad \downarrow \\
3/2 \\
\uparrow \\
1 \\
\uparrow \\
1/2 \\
\uparrow \\
0
\end{array}
\]

\[
\begin{array}{c}
3/2 \\
\downarrow \\
3/2 \\
\downarrow \\
1 \\
\downarrow \\
1/2 \\
\downarrow \\
0
\end{array}
\]

\[
\begin{array}{c}
3/2 \\
\downarrow \\
3/2 \\
\downarrow \\
1 \\
\downarrow \\
1/2 \\
\downarrow \\
0
\end{array}
\]

\[
\begin{array}{c}
3/2 \\
\downarrow \\
3/2 \\
\downarrow \\
1 \\
\downarrow \\
1/2 \\
\downarrow \\
0
\end{array}
\]

(3.5)

In the \( j = 0 \) sector every path starts and ends in 0. Therefore the first and last bits in \( \mathcal{W}_N \) are fixed. The remaining \( N - 2 \) bits are divided equally into \( (N - 2)/2 \) bits set to 0 and \( (N - 2)/2 \) bits set to 1. The resulting

\[
\begin{pmatrix}
N-2 \\
(N-2)/2
\end{pmatrix}
\]

(3.6)

words must satisfy the additional requirement that no spins higher than 2 do appear in the Bratteli diagram. Defining

\[
\Gamma_k^{(N)} = \begin{pmatrix}
N \\
N/2 - k
\end{pmatrix} - \begin{pmatrix}
N \\
N/2 + k + 1
\end{pmatrix}
\]

(3.7)

the dimension of the state space turns out to be

\[
\dim \mathcal{S}_5^0 = \Gamma_0^{(N)} + \sum_{k \geq 1} (\Gamma_{6k}^{(N)} - \Gamma_{6k-1}^{(N)}) = \frac{1}{2} (1 + 3^{N/2-1})
\]

(3.8)

The paths in the sector \( j = 1 \) have only the first bit fixed. At the right end, \( k_N = 1 \) implies \( k_{N-1} = 1/2 \) or \( k_{N-1} = 3/2 \). We find the dimension
\[
\dim S_5^{(1)} = 3^{N/2-1}
\] (3.9)

For the paths in the sector \(j = 2\), two bits are fixed and the total number of states is

\[
\dim S_5^{(2)} = \frac{1}{2}(3^{N/2-1} - 1)
\] (3.10)

We utilized \(N \leq 36\) for the Ising model and \(N \leq 24\) for the Potts model. At the largest \(N\) we have \(\dim S_3^{(0)} = \dim S_3^{(4)} = 131072\), \(\dim S_5^{(0)} = 88574\), \(\dim S_5^{(1)} = 177147\) and \(\dim S_5^{(2)} = 88573\).

\section*{IV. NUMERICAL RESULTS}

Let us define the rescaled energies

\[
\hat{E} = \frac{E}{2\pi (m+1) \sin \left( \frac{\pi}{m+1} \right)}
\] (4.1)

The finite size scaling predictions for the Bethe energy eigenvalues are \cite{2} the following. Let us consider a chain with \(N\) sites and let \(E_j\) be the ground state energy in the space of spin \(j\) type-II highest weights. Then

\[
N(\hat{E}_0 - \hat{E}_j) = h_{1,1+2j} + \cdots
\] (4.3)

where

\[
c = 1 - \frac{6}{m(m+1)}
\] (4.4)

\[
h_{1,1+2j} = \frac{j(mj - 1)}{m+1}
\] (4.5)

We have obtained a numerical estimate of \(c\), \(h_{1,3}\) and \(h_{1,5}\) at \(m = 3\) and \(m = 5\). As in \cite{3} we used the power method in order to determine iteratively the eigenvector with greatest absolute eigenvalue. We stopped the iteration when the double precision value did not change in the next iteration. In the determination of \(c\) we used the values of \(e_\infty\) and \(f_\infty\).
quoted in [9]. We used a fitting procedure in order to eliminate the residual size dependence in Eq. (4.2). The energy eigenvalues are shown in Figs (I-II). The extrapolation is shown in Table I where one can see the very good agreement with conformal theory.

V. CONCLUSIONS

In this letter we have shown that the diagrammatic technique prompted out in [8] is an effective tool. The quantum group symmetry was utilized in [7] in order to build two-point $U_q[SU(2)]$ scalar operators associated to conformal operators of the corresponding critical theory. Here, we have used the quantum symmetry in order to restrict the analysis of the spectrum to a subset of states with well defined symmetry properties. The FSS of the energy levels found in the type-II highest weights subspace coincides with that of the Bethe states and suggests that all of them are indeed isolated highest weights under the action of the quantum group.
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CAPTIONS

Fig. I : Ising model: Energy levels as functions of the lattice size.

Fig. II : Potts model: Energy levels as functions of the lattice size.
TABLE I. Determination of $c$ and $h_{1,1+2j}$

| $m$ | $c$     | $c^{(exact)}$ | $h_{1,3}$ | $h_{1,3}^{(exact)}$ | $h_{1,5}$ | $h_{1,5}^{(exact)}$ |
|-----|---------|---------------|-----------|---------------------|-----------|---------------------|
| 3   | 0.501(5)| 1/2          | 0.500(5)  | 1/2                 | –         | –                   |
| 5   | 0.794(5)| 4/5          | 0.661(5)  | 2/3                 | 2.993(5)  | 3                   |