Evaluating the impact of increasing temperatures on changes in Soil Organic Carbon stocks: sensitivity analysis and non-standard discrete approximation

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Abstract A novel model is here introduced for the SOC change index defined as the normalized difference between the actual Soil Organic Carbon and the value assumed at an initial reference year. It is tailored on the RothC carbon model dynamics and assumes as baseline the value of the SOC equilibrium under constant environmental conditions. A sensitivity analysis is performed to evaluate the response of the model to changes of temperature, Net Primary Production (NPP), and land use soil class (forest, grassland, arable). A non-standard monthly time-stepping procedure has been proposed to approximate the SOC change index in the Alta Murgia National Park, a protected area in the Italian Apulia region, selected as test site. In the case of arable class, the SOC change index exhibits a negative trend which can be inverted by a suitable organic fertilization program here proposed.

Keywords Soil Organic Carbon model · sensitivity analysis · non-standard discrete approximation

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1 Introduction

For reporting on Target 15.1, one of the seventeen Sustainable Development Goal (SDG) adopted by the United Nations [11] in 2015, the Good practice guidance [19] indicates how to calculate the extent of land degradation. It recommends the development and the use of analytical methods for measuring
the three indicators which address the key aspects of land-based natural capital: trends in land cover, trends in land productivity and trends in soil organic carbon (SOC) stocks. These indicators can assess the quantity and the quality of land-based natural capital and most of the associated ecosystem services. Roughly speaking, SOC stock is the carbon captured by plants through photosynthesis which remains in the soil after decomposition of soil organic matter. A decrease in SOC stocks is among the significant universal indicators for land and soil degradation and can compromise all the efforts to achieve the SDGs especially those with reference to food, health, water, climate, and land management [10].

Well-validated models which take into account the interactions among climate, soil and land use management can be used to predict SOC changes under the different management and climatic conditions. The Rothamsted carbon model (RothC, [1, 15]) is one of the most commonly used tool to simulate soil organic carbon dynamics in arable, grassland and forest systems. Although it does not place the action of bacteria at the hearth of the mechanisms of decomposition as required by current theories [9,7], it is widely used because it captures the general principles of soil organic dynamics, it is relatively simple and general, it requires relatively few parameters and can be easily applied at scales from regional [5], to global [13].

In this paper, for making a scenario analysis of SOC changes, we propose a novel model tailored on RothC dynamics, which describes the evolution of the so-called SOC change index. It is defined as the difference between the SOC values at the last and the first year (as in [12]), here normalized by the carbon inputs generated by the total plant and the farmyard manure, both evaluated at the initial baseline year. As test example, we evaluate the impact of changes in temperature on the achievement of land degradation neutrality for the SOC indicator in the Alta Murgia National Park, a protected area in the Apulia region located in the south of Italy. It is known that the increase or decrease of the SOC stocks under climate change will depend upon which process, in the future and in a given location, dominates between increased plant inputs through increases in net primary production (NPP), and increased decomposition rates [6]. With the aim of detecting factors which determine the size and the direction of change in the considered protected area, a sensitivity analysis, based on the direct method described in [2], is performed. The sensitivity analysis is applied to a modified version of the SOC change index model, based on time averaged values, and provides local information on the impact of parameters change on the behavior of the system solution. In particular, we evaluate the impact on the SOC change index of the variation of three representative parameters: mean annual temperature, NPP annual values with respect to reference values and degree of decomposability of plant material (the so-called DPM/RPM ratio), which in turn is related to the class of land use (forest, grassland and arable).

Trends in SOC changes from 2005, taken as baseline year, to 2019, the final year, are simulated by means of a monthly discrete non-standard approximation of the continuous model for forest, grassland and arable systems. It
is based on the discrete non-standard monthly time stepping procedure provided in [3] for solving the carbon dynamics in all of the compartments. Given the linearity of the RothC model, the SOC change can be discretized with the same matrix function of the monthly stepsize. Results obtained indicate positive trends for SOC change in case of both forest and grassland systems. When the arable class is considered without including the input of farm fertilizers, our model predicts a negative trend of the introduced normalized SOC change variable. As a final result, we evaluate the optimal organic fertilization program to invert the trend and keep positive the SOC change. When used with predicted climate and NPP data, the optimal fertilization program may guarantee the achievement of land degradation neutrality for the SOC indicator.

The paper is organized as follows. In Section 2 we briefly describe the original RothC model and define the SOC indicator for the continuous counterpart of the original model. Moreover we introduce a more realistic representation of the density function of the plant carbon input which can be proven to be periodic. Input data and parameters are then identified and described. In Section 3 we explain how the issue of determining the initial carbon input is solved in the proposed formulation and we define a novel SOC change index which overcome the problem. Then, in Section 4 we analyze the model assuming that there is no carbon input due to the organic fertilization and determine the sensitivity of the model to the variation of the above mentioned parameters: temperature, NPP and land use class. The issue of a possible positive contribution of organic fertilization is faced in Section 5 where we propose to consider the farmyard manure input as a control variable to reach neutrality, and modify the model accordingly. To perform the simulations, we apply a numerical non-standard technique which preserves the equilibrium state of the continuous dynamics and is described in Section 6. In Section 7 we present a test case illustrating the trends of SOC change in a protected area, in the years 2005-2019, as a function of the measured changes of temperature and NPP for the three land use classes analyzed (forest, grassland, arable). Finally, in Section 8 we draw our conclusions.

2 The RothC model

Within the RothC model, soil organic carbon is divided into the five carbon pools noted: \( c_{dpm}, c_{rpm}, c_{hum}, c_{bio} \) and \( c_{iom} \) (see Figure 1). The already decomposed plant material is regarded as \( c_{hum} \), whereas the total carbon mass of microbial organisms is represented by the \( c_{bio} \) pool. All non decomposable or inert material is defined as \( c_{iom} \). In general, all pools \( c_i \) will decompose and form \( CO_2, c_{bio} \) and \( c_{hum} \). The four active compartments \( c_{dpm}, c_{rpm}, c_{hum} \) and \( c_{bio} \) undergo decomposition as a function of different rate constants which correspond to the entries of the vector \( k = [k_{dpm}, k_{rpm}, k_{bio}, k_{hum}]^T \), and of the rate modifier \( \rho(t) \) which depends on the clay content of the soil, on climate variables (rainfall, temperature, open pan evaporation) and land cover. The
fraction $\alpha + \beta$ of metabolised carbon incorporated into the sum of compartments $c_{bio}(t) + c_{hum}(t)$ is determined by the clay content of the soil, while the remaining part $\delta := 1 - \alpha - \beta$ is released as $CO_2$ and lost by the system.

For the aim of what follows we denote with $T > 0$ the length of a reference time interval (generally one year) and we formulate the RothC model as:

$$\frac{dc}{dt} = \rho(t) A c + b(t), \quad t \in [t_0 + nT, t_0 + (n+1)T], \quad n = 0, \ldots, \quad (1)$$

where $c(t) = [c_{dpm}(t), c_{rpm}(t), c_{bio}(t), c_{hum}(t)]^\top$ and $c(t_0) = c_0 \geq 0$ denotes the vector of the initial concentrations. The matrix $A$ is given by

$$A = \begin{pmatrix}
- k_{dpm} & 0 & 0 & 0 \\
0 & - k_{rpm} & 0 & 0 \\
\alpha k_{dpm} & \alpha k_{rpm} & (\alpha - 1) k_{bio} & \alpha k_{hum} \\
\beta k_{dpm} & \beta k_{rpm} & \beta k_{bio} & (\beta - 1) k_{hum}
\end{pmatrix}.$$  

The vector $b(t)$ represents the carbon amount entering the system at time $t$. It takes into account both the input of plant residues $g(t) a^{(2)}$ and the input...
of farmyard manure (FYM) \( f(t) a(t) \), so that
\[
\mathbf{b}(t) := g(t) \mathbf{a}^{(g)} + f(t) \mathbf{a}^{(f)}.
\]
The entries of vectors \( \mathbf{a}^{(g)} := [\gamma, 1 - \gamma, 0, 0]^{\top} \) and \( \mathbf{a}^{(f)} := [\eta, \eta, 0, 1 - 2\eta]^{\top} \) are the fraction inputs \( 0 \leq \gamma \leq 1, 0 \leq \eta \leq 1/2 \), which sum up to 1.

**Definition 1** We define as SOC indicator of the continuous RothC model (1) the function
\[
\text{SOC}(t) = c_{iom}(t) + c_{dpm}(t) + c_{rpm}(t) + c_{bio}(t) + c_{hum}(t)
\]
for \( t \geq t_0 \), where \( c_{iom} \) denotes the constant carbon content in the inactive compartment IOM.

Although different approaches can be adopted for calculating the size of IOM, [14], [16], here we use the classical equation given by Falloon et al. in [4]:
\[
c_{iom}(t) = 0.049 \text{SOC}^{1.139}(t)
\]
so that the SOC indicator is obtained by solving the equation
\[
0.049 \text{SOC}^{1.139}(t) - \text{SOC}(t) + \text{soc}(t) = 0,
\]
where
\[
\text{soc}(t) := c_{dpm}(t) + c_{rpm}(t) + c_{bio}(t) + c_{hum}(t)
\]
satisfies the differential equation
\[
\frac{d\text{soc}}{dt}(t) = \mathbf{1}^{\top} \frac{d\mathbf{c}}{dt}(t) = \rho(t) \mathbf{1}^{\top} \mathbf{A} \mathbf{c} + g(t) + f(t)
\]
\[
= -\rho(t) \delta \mathbf{k}^{\top} \mathbf{c} + g(t) + f(t).
\]

2.1 A realistic representation of \( g(t) \)

Towards a realistic analytic representation of the density function \( g(t) \) of plant carbon input, we consider that \( g(t) \) can be represented as follows
\[
g(t) = P(t_0 + nT) \hat{g}(t) \quad \forall t \in [t_0 + nT, t_0 + (n+1)T], \quad n = 0, \ldots,
\]
where
\[
\hat{g}(t) := \frac{g(t)}{\int_{t_0 + nT}^{t_0 + (n+1)T} g(s) \, ds}.
\]
The function \( \hat{g} \) represents the density distribution of plant carbon inputs into the soil expressed as a proportion of the total
\[
P(t_0 + nT) := \int_{t_0 + nT}^{t_0 + (n+1)T} g(s) \, ds,
\]
in each time interval \([t_0 + nT, t_0 + (n+1)T]\) of length \( T \), for \( n = 0, 1, \ldots \). In real applications the function \( \hat{g}(t) \) is known and, as it depends only on seasonality, it is well represented by an annual periodic function. We have the following result.
**Theorem 1** Set $T > 0$ and suppose that $g(t)$ is a positive function which satisfies the following property

$$
g(t + T) = g(t) \frac{\int_{t_0 + (n+1)T}^{t_0 + (n+2)T} g(s) \, ds}{\int_{t_0 + nT}^{t_0 + (n+1)T} g(s) \, ds},
$$

for all $t \in [t_0 + nT, t_0 + (n + 1)T]$, and $n = 0, 1, \ldots$. Then, the function $\hat{g}(t)$, defined in (5), satisfies $0 < \hat{g}(t) < 1$, results periodic with period $T$ and

$$
\hat{g}(s) \, ds = \int_{t_0 + nT}^{t_0 + (n+1)T} g(s) \, ds = 1, \text{ for all } n = 0, 1 \ldots.
$$

**Proof** The result trivially follows by observing that if $t \in [t_0 + nT, t_0 + (n + 1)T]$, then $t + T \in [t_0 + (n + 1)T, t_0 + (n + 2)T]$. Consequently,

$$
\hat{g}(t + T) = \frac{g(t + T)}{\int_{t_0 + (n+1)T}^{t_0 + (n+2)T} g(s) \, ds} = \hat{g}(t),
$$

for all $t \in [t_0 + nT, t_0 + (n + 1)T]$ and $n = 0, 1 \ldots$.

2.2 Input data and parameters

Let us identify all the input data necessary to the RothC dynamics.

- Input per unit time (month) of plant residues $g(t) [\text{tC ha}^{-1} \text{month}^{-1}]$ and farmyard manure $f(t) [\text{tC ha}^{-1} \text{month}^{-1}]$, if any. The function $g(t)$ is supposed to be expressed as in (4). By means of Net Primary Production (NPP), it is possible to estimate

$$
P(t_0 + nT) = P(t_0 + (n - 1)T) \frac{NPP(t_0 + nT)}{NPP(t_0 + (n - 1)T)}
$$

the total plant carbon input in the year $[t_0 + nT, t_0 + (n + 1)T]$, where $N_p^{(n)} = \frac{NPP(t_0 + nT)}{NPP(t_0)}$. The function $\hat{g}(t) = \hat{g}_r(t)$ is supposed annual periodic and assuming different known shapes according to the land use.

- clay content of the soil $cly$ (as a percentage);
- $r$ the degree of decomposability of incoming plant material, i.e. the $DPM$ over $RPM$ ratio;
- air temperature $Temp(t) [^\circ C]$, rainfall $\text{rain}(t) [mm]$, potential evapotranspiration $pet(t)$. In our tests $pet(t)$ is estimated from weather data by means of Thornthwaite’s formula (see Appendix).

1. The original model uses open pan evaporation; here the model is used in a modified version which makes use of potential evapotranspiration.
\(- c(t_0) [tC ha^{-1}]\) the vector of the initial concentrations sampled at a soil layer of depth \(d [cm]\).

Let us identify all the parameters involved in the RothC dynamics.

\(- A = A(\alpha, \beta, k) \). From the clay content, we can evaluate the Soil Texture Factor according to \(x = 1.67(1.85 + 1.60 e^{-0.0736 \cdot cl_y})\), and consequently \(\alpha = \frac{0.46}{x + 1} \) and \(\beta = \frac{1}{x + 1} - \alpha\); the entries of \(k\) are given by \(k_{dp} = 10/T [time^{-1}]\), \(k_{pm} = 0.3/T [time^{-1}]\), \(k_{bio} = 0.66/T [time^{-1}]\), \(k_{hum} = 0.02/T [time^{-1}]\).

\(- b(t) = b(t, \gamma, \eta) \). Here \(\eta = 0.49\) while \(\gamma(r) = \frac{r}{r + 1}\) varies according to the land use. Values \(0 < r < 0.5\) of DPM over RPM ratio are associated to the forest class, \(0.5 \leq r < 1\) to the grassland class, \(r \geq 1\) to the arable class.

\(- \rho(t) = k_a(Temp(t)) k_b(Acc(rain(t), M(cl_y, d)) k_c(t, r)\).

The modifying factor related to the temperature is generalized with respect to the original given in \([1]\), in order to assume value equal to 1 in correspondence of the mean annual temperature \(Temp(0)\) in the interval \([t_0, t_0 + T]\), i.e.

\[
k_a(Temp(t)) := \frac{47.91}{1 + e^{Temp(t) + (106.06/\log(46.91) - Temp(0))}}
\]

so that \(k_a(Temp(0)) = 1\).

The factor \(k_c(t, r)\), associated to the soil cover,

\[
k_c(t, r) = \begin{cases} 
0.6 & 0 < r < 1 \\
S_r(t) & r \geq 1,
\end{cases}
\]

with \(S_r(t) = S_r(t + T)\) assuming values between 0.6 in the periods of the year when soil is vegetated and the maximum value 1, when bare.

The maximum soil moisture deficit \(M\) and the point at which respiration (i.e. microorganism activity) begins to slow \(M_b\), are defined as \(M := M(cl_y, d) = -(20 + 1.3 cl_y - 0.01 cl_y^2) \frac{d}{23}\) and \(M_b = 0.444 M\). The accumulated soil moisture deficit \(Acc(t, M)\) is calculated from the first time in \([t_0 + nT, t_0 + (n + 1)T]\) where evaporation \(pet(t)\) exceeds rainfall the maximum soil moisture deficit \(M\). When there is more rainfall than evaporation, the soil will start to wet up.

The rate modifying factor for moisture varies between 0.2 and 1 as follows

\[
k_b(Acc(t, M)) := \begin{cases} 
0.2 + (1 - 0.2) \frac{M - Acc(t, M)}{M - M_b} & Acc(t, M) < M_b \\
1 & \text{otherwise}
\end{cases}
\]
3 Determining the initial plant inputs

In all practical applications, RothC is run in ‘reverse mode’ to calculate the initial plant inputs to the soil for the given environmental conditions. The underlying hypothesis is that the observed carbon stocks correspond to a stable constant or annual periodically varying long-term solution for their dynamics. Once the initial plant inputs have been established in this way, in order to simulate future scenarios, the time changes in carbon inputs to the soil, associated with changes in NPP (Smith et al., 2005), changes in climate conditions, or change in land use are implemented.

Under the hypothesis that the observed carbon stocks correspond to their values at a stable equilibrium, we are going to illustrate how it is possible to avoid the first run in ‘reverse mode’ to calculate the initial plant inputs.

Once a monitoring temporal interval \([t_0 + T, T_f]\) is set, by following the approach indicated in [11], the baseline of SOC indicator against which Land Degradation Neutrality is to be achieved, is supposed to correspond to the carbon stocks equilibrium for averaged values of temperature, accumulated soil moisture deficit, and soil cover in a period \([t_0, t_0 + T]\) immediately prior the monitoring time interval.

As concerns the average value for the factor \(k_c(t, r)\) associated to the soil cover, it can be approximated as follows:

\[
\overline{k_c(r)} = \begin{cases} 
0.6 & 0 \leq r < 1 \\
\int_{t_0}^{t_0+T} S_r(s) \, ds \approx 0.6 + \frac{N_b}{30} & r \geq 1,
\end{cases}
\]

where \(0 \leq N_b \leq 12\) (generally \(N_b = 4\), see e.g. [20]) is the number of months per year of bare soil for arable class. In order to have a smooth dependence on \(r\), we approximate \(k_c(r)\) with the \(C^\infty\)-function

\[
k_c := 0.6 + \frac{N_b}{30} \frac{e^{x(r)}}{1 + e^{x(r)}}, \quad x(r) := \frac{30(r - 1)}{r} \quad r > 0.
\]

The function \(k_c(r)\) for a generic crop related to \(N_b = 4\) bare months per year, is illustrated in Figure 2.

Denoting with \(Temp(0)\) and \(Acc(0)\) the averaged values for temperature and accumulated soil deficit on the period \([t_0, t_0 + T]\) assumed as reference interval, then the modifying factor \(\rho(t)\) is approximated by \(\rho^{(0)}(r) := k_a(Acc(0)) k_c(r)\), as \(k_a(Temp(0)) = 1\).

Setting \(F(t_0) = \int_{t_0}^{t_0+T} f(s) \, ds\), then the model (1), can be written as

\[
\frac{dc}{dt}(t) = \rho^{(0)}(r) A c + \frac{P(t_0)}{T} a^{(g)} + \frac{F(t_0)}{T} a^{(f)}, \quad t \in [t_0, t_0 + T].
\]

Suppose that \(c(t_0)\) i.e. the distribution of the measured SOC\((t_0)\) among compartments is known and satisfies

\[
0.049 \text{SOC}^{1.139}(t_0) - \text{SOC}(t_0) + \mathbb{1}^T c(t_0) = 0.
\]
We assume that $c(t_0)$ is equal to the equilibrium of the dynamical system \([8]\), i.e.

$$c(t_0) = -\frac{1}{T \rho^{(0)}(r)} A^{-1} \left( P(t_0) a^{(g)} + F(t_0) a^{(f)} \right). \tag{9}$$

Consequently,

$$P(t_0) a^{(g)} = - T \rho^{(0)}(r) A c(t_0) - F(t_0) a^{(f)} \tag{10}$$

$$P(t_0) + F(t_0) = - T \rho^{(0)}(r) 1^T A c(t_0) = T \rho^{(0)}(r) \delta k^T c(t_0).$$

Under the hypothesis that $F(t_0)$ is known (i.e. the amount of the total farmyard manure used in the interval $[t_0, t_0 + T]$), it follows that the initial plant inputs to the soil is given by

$$P(t_0) = T \rho^{(0)}(r) \delta (k_{dpm} c_{dpm}(t_0) + k_{rpm} c_{rpm}(t_0)$$

$$+ k_{bio} c_{bio}(t_0) + k_{hum} c_{hum}(t_0)) - F(t_0) \tag{11}$$

Then, for all $n = 1, 2 \ldots$ the system

$$\frac{dc}{dt}(t) = \rho(t) A c + P(t_0 + nT) \tilde{g}_r(t) a^{(g)} + f(t) a^{(f)}$$

$$P(t_0 + nT) = P(t_0) N^{(n)}_P \tag{12}$$
is solved for $t \in [t_0 + nT, t_0 + (n + 1)T]$ starting from $c(t_0 + T) = c(t_0)$ given in (9) and $P(t_0)$ given in (11), until $t_0 + (n + 1)T \leq T_f$.

For making a scenario analysis of SOC change, which does not depend on the specific initial measured SOC value but only on the hypothesis of an initial environmental equilibrium, a useful tool is given by the SOC change index defined as the variable of change of carbon stocks normalized as follows.

**Definition 2** We indicate with $\Delta soc^{(0)}(r)(t)$ the SOC change index defined as $\Delta soc^{(0)}(r)(t) := \frac{soc(t) - soc(t_0)}{P(t_0) + F(t_0)}$ with $soc(t) := c^T(t)$, where $c(t)$ solves (12) and $P(t_0) + F(t_0)$ is given in (10).

Notice that the sign of the index $\Delta soc^{(0)}(r)(t)$ detects if at the time $t$ the sum of soil carbon contained in compartments is greater than its initial value. In what follows we firstly consider the dynamics of SOC index when no farmyard manure input the system that is, generally the case of (not improved) grassland and forest classes.

### 4 A model for SOC changes without farmyard manure input

Soil organic carbon dynamics are driven by changes in climate and land cover or land use. In natural ecosystems, the balance of SOC is determined by gains, through plant and other organic inputs, and losses, due to the organic matter turnover [20]. Globally, under a warming climate, increases are seen both in carbon inputs to the soil due to higher NPP, and in SOC losses due to increased decomposition. The balance between these processes defines the change in SOC stock. In some regions the processes balance, but in others, one process is affected by climate more than the other.

In order to test the effectiveness of SOC change index defined in (2) for detecting changes in SOC stock in a specific area, we deduce its temporal dynamics in Corollary 1, preceded by the following theorem.

**Theorem 2** In case of no farmyard manure input, the dynamics of the variable

$$\Delta c^{(0)}(r)(t) := \frac{c(t) - c(t_0)}{P(t_0)}, \quad t \in [t_0 + nT, t_0 + (n + 1)T], \quad n = 1, 2, \ldots,$$

is governed by the equation

$$\frac{d\Delta c^{(0)}(r)}{dt}(t) = \rho(t) A \Delta c^{(0)}(r) + \left(N_p^{(n)} \hat{g}_r(t) - \frac{\rho(t)}{T_{\rho(0)}} \right) a^{(g)}, \quad (13)$$

where $\Delta c^{(0)}(r)(t_0 + T) = \Delta c^{(0)}(r)(t_0) = 0$ and $N_p^{(n)} = \frac{NPP(t_0 + nT)}{NPP(t_0)}$. 

Proof In case of no farmyard manure input, by plugging the expression of \( P(t_0 + nT) \) into the equation for \( \frac{dc}{dt} \), the equation (12) becomes
\[
\frac{dc}{dt} = \rho(t) A c + P(t_0) N_P^{(n)} \tilde{g}_r(t) a^{(g)}, \quad t \in [t_0 + nT, t_0 + (n + 1)T].
\]
Thus,
\[
d\Delta c\rho(0)(r) dt(t) = 1 P(t_0) A c(t_0) + P(t_0) N_P^{(n)} \tilde{g}_r(t) a^{(g)}
\]
Recalling the relation between \( P(t_0) \) and \( c(t_0) \) in (9) that yields
\[
- P(t_0) T \rho(0)(r) a^{(g)} = A c(t_0),
\]
the result follows.

Corollary 1 In case of no farmyard manure input, the dynamics of the SOC change index \( \Delta soc\rho(0)(r)(t) \) for \( t \in [t_0 + nT, t_0 + (n + 1)T] \), for \( n = 1, 2, \ldots \), is governed by the equation
\[
\frac{d\Delta soc\rho(0)(r)(t)}{dt} = - \rho(t) \delta k^T \Delta c\rho(0)(r) + \left( N_P^{(n)} \tilde{g}_r(t) - \frac{\rho(t) A c(t_0)}{P(t_0)} \right)
\]
where \( \Delta c\rho(0)(r)(t) \) solves (13) and \( \Delta soc\rho(0)(r)(t_0 + T) = \Delta soc\rho(0)(r)(t_0) = 0 \).

Proof The dynamics for \( \Delta soc\rho(0)(r)(t) \) can be immediately deduced from the dynamics of \( \Delta c\rho(0)(r)(t) \) as \( \frac{d\Delta soc\rho(0)(r)(t)}{dt} = \mathbf{1}^T \frac{d\Delta c\rho(0)(r)}{dt} \) and observing that \( \mathbf{1}^T A = - \delta k^T \).

4.1 Sensitivity of the SOC change index to parameters

In this section, we want to study the relative importance of the different factors responsible for change in SOC stock. This will be done throughout a sensitivity analysis of SOC change index related to the dependence on the temperature, on NPP and on the class of land use, here restricted to forest and grassland classes. We will make use of the direct method in [2] where the analysis of sensitivity is local and described by first-order derivatives.

In this setting, \( \phi \in \mathbb{R} \) is a parameter affecting the dynamics \( \frac{dy}{dt} = f(y(t), \phi) \) of the \( n \) dimensional variable \( y(t) \). The direct method requires the integration
of an additional set of differential equations, together with the original system, to obtain the vector of sensitivities $s_{y,\phi}(t)$, whose components are defined as $\frac{\partial y_i(t, \phi)}{\partial \phi}$, i.e.

$$\frac{dy}{dt} = f(y(t, \phi), \phi), \quad y(t_0, \phi) = y_0(\phi)$$

$$\frac{ds_{y,\phi}(t, \phi)}{dt} = \frac{\partial f}{\partial \phi}(y(t, \phi), \phi) + \frac{\partial f}{\partial y}(y(t, \phi), \phi) s_{y,\phi}(t), \quad (15)$$

$$s_{y,\phi}(t_0) = \frac{\partial y_0(\phi)}{\partial \phi},$$

where $\frac{\partial f}{\partial y}$ denotes the Jacobian matrix.

In order to apply the above described direct method, we need to replace the non-autonomous dynamics described in Theorem 2 and Corollary 1, with an autonomous one. Let us come back to the equation for $\Delta c \rho(0)(r)$ in (13). At first, we replace $\text{Temp}(t)$ and $\text{Acc}(t)$ with their averaged values, say $\text{Temp}^{(n)}$ and $\text{Acc}^{(n)}$, in each interval $[t_0 + nT, t_0 + (n+1)T]$ so that $\rho(t)$ can be approximated by $\rho^{(n)}(r) := k_a(\text{Temp}^{(n)}) k_b(\text{Acc}^{(n)}) k_c(r)$, where $k_c(r)$ given in (7). As $\int_{t_0 + nT}^{t_0 + (n+1)T} \dot{\gamma}_r(s) \, ds = 1$, we define the autonomous counterpart of the model (13) as follows:

$$\frac{d\Delta \text{c} \rho^{(n)}(r)}{dt} = \rho^{(n)}(r) A \Delta \text{c} \rho^{(n)}(r) + \dot{\gamma}^{(n)} \text{a}(g), \quad (16)$$

$$\Delta \text{c} \rho^{(n)}(r)(t_0 + T) = 0,$$

for $t \in [t_0 + nT, t_0 + (n+1)T], \quad n = 1, 2, \ldots$, where $2$.

$$\dot{\gamma}^{(n)} := \frac{1}{T} \left( N_F^{(n)} - \frac{\rho^{(n)}(r)}{\rho^{(n)}(r)} \right), \quad (17)$$

With the previous notations, we define

**Definition 3** The sensitivity of the SOC change index to the parameter $\phi$ is defined as the sum of the entries of the vector $s_{\Delta \text{c},\phi}$, which is the sensitivity to the parameter $\phi$ of the variable $\Delta \text{c} \rho^{(n)}(r)(t)$, whose dynamics is described in (16).

2 Let us observe that $\dot{\gamma}^{(n)}$ does not depend on $r$, in fact $\dot{\gamma}^{(n)} = \frac{1}{T} \left( N_F^{(n)} - \frac{k_a(\text{Temp}^{(n)}) k_b(\text{Acc}^{(n)})}{k_b(\text{Acc}^{(n)})} \right)$. 

In the following we are going to analyze the sensitivity of SOC change index to three different parameters: \( Temp^{(1)} \) representing the annual averaged temperature, \( N_{P}^{(1)} := NPP(t_{0}+T)/NPP(t_{0}) \) representing the NPP input normalized by the value at the reference year, and \( r \) related to change of land use, from forest (lowest values of \( r \)) to arable (highest value of \( r \)).

Before proceeding we provide the following result useful for the sensitivity analysis of the SOC change index to parameters \( Temp^{(1)} \) and \( r \) in the time interval \( [t_{0} + T, t_{0} + 2T] \).

**Theorem 3** The solution of the initial value problem \([16]\) in the time interval \( [t_{0} + T, t_{0} + 2T] \) is given by

\[
\Delta \rho^{(0)}(t) = (t - t_{0} - T) \varphi^{(1)} \left( (t - t_{0} - T) \rho^{(1)}(r) A \right) a^{(g)},
\]

where \( \varphi(z) := z^{-1}(e^{z} - 1) \).

**Proof** Since in each interval equation \([16]\) corresponds to an autonomous, non homogeneous and linear differential system, the initial value problem in correspondence of \( n = 1 \), has a unique solution given by

\[
\Delta \rho^{(0)}(t) = \varphi^{(1)}(r) A(t - t_{0} - T) \Delta \rho^{(0)}(t_{0} + T)
\]

\[
+ \varphi^{(1)}(r) A(t - t_{0} - T) \int_{t_{0}+T}^{t} e^{-\rho^{(1)}(r)A\tau} \varphi^{(1)}(\rho^{(1)}(r) A) d\tau
\]

\[= \varphi^{(1)}(r) A(t - t_{0} - T) \left( \int_{t_{0}+T}^{t} e^{-\rho^{(1)}(r)A\tau} d\tau \right) a^{(g)}\]

\[= \varphi^{(1)}(r) A(t - t_{0} - T) A^{-1} \rho^{(1)}(r) \left( I - e^{-\rho^{(1)}(r)A(t - t_{0} - T)} \right) a^{(g)}.
\]

By observing that the matrices \( e^{\rho^{(1)}(r)A(t - t_{0} - T)} \) and \( A^{-1} \) commute, we have that

\[
\Delta \rho^{(0)}(t) = \varphi^{(1)}(r) A^{-1} \rho^{(1)}(r) A(t - t_{0} - T) - I \right) a^{(g)}.
\]

### 4.2 Sensitivity of the SOC change index to the parameter \( Temp^{(1)} \)

Accordingly to Definition \([3]\) we define the sensitivity of the SOC change index to \( Temp^{(1)} \) the quantity \( s_{\Delta soc,Temp^{(1)}} := \Delta \rho^{(0)}(r) \Delta \rho^{(0)}(r) \). The following theorem holds.
Theorem 4 The sensitivity of the SOC change index to $\text{Temp}^{(1)}$ satisfies the following differential equation

$$\frac{ds_{\Delta \text{soc}, \text{Temp}^{(1)}}}{dt} = -\rho^{(1)}(r)\delta k^T s_{\Delta \text{Temp}^{(1)}}$$

for $t \in [t_0 + T, t_0 + 2T]$, with the initial condition

$$s_{\Delta \text{soc}, \text{Temp}^{(1)}}(t_0 + T) = 0.$$

Moreover, there exists an $\epsilon > 0$ such that for all $t \in [t_0 + T, t_0 + T + \epsilon]$

$$s_{\Delta \text{soc}, \text{Temp}^{(1)}}(t) \leq 0.$$

Proof Since the sensitivity of $\Delta \text{soc}_{\rho^{(0)}(r)}$ to $\text{Temp}^{(1)}$ is defined as $s_{\Delta \text{soc}, \text{Temp}^{(1)}} = \Gamma^T s_{\Delta \text{Temp}^{(1)}}$, let us begin by obtaining the initial value problem for $s_{\Delta \text{Temp}^{(1)}}$. According to equations (15), applied to equations (16) for $t \in [t_0 + T, t_0 + 2T]$ (i.e. $n = 1$), we have that

$$\frac{ds_{\Delta \text{Temp}^{(1)}}}{dt} = \rho^{(1)}(r) A s_{\Delta \text{Temp}^{(1)}} + \frac{\partial}{\partial \text{Temp}^{(1)}} \left( \rho^{(1)}(r) A \Delta \text{Temp}^{(0)} + \vartheta^{(1)} \text{a}^{(g)} \right),$$

where

$$\frac{\partial}{\partial \text{Temp}^{(1)}} \left( \rho^{(1)}(r) A \Delta \text{Temp}^{(0)} + \vartheta^{(1)} \text{a}^{(g)} \right) =$$

$$= \frac{\partial}{\partial \text{Temp}^{(1)}} \left( \rho^{(1)}(r) A \Delta \text{Temp}^{(0)} \right) + \frac{\partial}{\partial \text{Temp}^{(1)}} \left( \vartheta^{(1)} \text{a}^{(g)} \right) - \frac{\partial}{\partial \text{Temp}^{(1)}} \left( \vartheta^{(1)} \right).$$

Thus, for all $t \in [t_0 + T, t_0 + 2T]$

$$\frac{ds_{\Delta \text{Temp}^{(1)}}}{dt} = \rho^{(1)}(r) A s_{\Delta \text{Temp}^{(1)}} + \frac{\partial}{\partial \text{Temp}^{(1)}} \left( A \Delta \text{Temp}^{(0)} \right) + \frac{\vartheta^{(1)} a^{(g)}}{T \rho^{(0)}(r)}.$$
By setting \( \psi(t) := k^\top \varphi (\rho^{(1)}(r) A (t - t_0 - T)) a^{(9)} \), we have that
\[
k^\top \Delta \rho(0) (t) = (t - t_0 - T) \psi(t) \vartheta^{(1)},
\]
and, by replacing \( \vartheta^{(1)} \) with Definition 17, equation (19) becomes
\[
\frac{ds_{\Delta soc,Temp^{(1)}}}{dt} = -\rho^{(1)}(r) \delta k^\top s_{\Delta\rho,Temp^{(1)}}
\]
\[
- \frac{\partial \rho^{(1)}(r)}{\partial Temp^{(1)}} \left[ \delta(t - t_0 - T) \psi(t) \left( N_p^{(1)} - \rho^{(1)}(r) \right) + \frac{1}{T \rho^{(0)}(r)} \right].
\]
Consider that \( \psi(t_0 + T) = k^\top a^{(9)} > 0 \), then, by continuity, there exists an \( \epsilon > 0 \) such that \( \psi(t) > 0 \) for all \( t \in [t_0 + T, t_0 + T + \epsilon] \). By defining \( k_{min} := \min_i k_i \), then \( k^\top s_{\Delta\rho,Temp^{(1)}} \leq k_{min} \Gamma s_{\Delta\rho,Temp^{(1)}} \), and \( \delta(t - t_0 - T) \psi(t) N_p^{(1)} > 0 \) for all \( t \in [t_0 + T, t_0 + T + \epsilon] \). It follows that
\[
\frac{ds_{\Delta soc,Temp^{(1)}}}{dt} \leq -\rho^{(1)}(r) \delta k_{min} \Gamma s_{\Delta\rho,Temp^{(1)}}
\]
\[
- \frac{\partial \rho^{(1)}(r)}{\partial Temp^{(1)}} \frac{1 - \delta(t - t_0 - T) \psi(t) \rho^{(1)}(r)}{T \rho^{(0)}(r)}
\]
for all \( t \in [t_0 + T, t_0 + T + \epsilon] \). By continuity, the function \((t - t_0 - T) \psi(t)\) is positive for all \( t \in [t_0 + T, t_0 + T + \epsilon] \) and it is equal to zero at \( t = t_0 + T \). Since \( \frac{1}{\delta \rho^{(1)}(r)} > 0 \), there exists an \( \epsilon > 0 \) such that \( (t - t_0 - T) \psi(t) \leq \frac{1}{\delta \rho^{(1)}(r)} \) for all \( t \in [t_0 + T, t_0 + T + \epsilon] \). Thus, exploiting the positivity of \( \partial \rho^{(1)}(r) / \partial Temp^{(1)} \), we have that
\[
\frac{ds_{\Delta soc,Temp^{(1)}}}{dt} \leq -\rho^{(1)}(r) \delta k_{min} \Gamma s_{\Delta\rho,Temp^{(1)}} \quad \forall t \in [t_0 + T, t_0 + T + \epsilon]
\]
\[
s_{\Delta soc,Temp^{(1)}}(t_0 + T) = 0.
\]
The solution of the Cauchy problem \( \frac{dx}{dt} = -\rho^{(1)}(r) \delta k_{min} x \), with \( x(t_0 + T) = 0 \), is the function \( x(t) \equiv 0 \), for all \( t \in [t_0 + T, t_0 + T + \epsilon] \). Since \( s_{\Delta soc,Temp^{(1)}}(t_0 + T) \leq x(t_0 + T) = 0 \), we have that \( s_{\Delta soc,Temp^{(1)}} \leq x(t) = 0 \), for all \( t \in [t_0 + T, t_0 + T + \epsilon] \).

---

3 \( \frac{\partial \rho^{(1)}(r)}{\partial Temp^{(1)}} = \frac{106.06}{47.91} (k_a (Temp^{(1)}))^2 k_b (Acc^{(1)}) k_c (r) \frac{Temp^{(1)} + 106.06}{Temp^{(1)} + 106.06} \frac{Temp^{(1)}}{Temp^{(1)} + 106.06} > 0. \)
Remark 1 For sufficiently small values of $t$, the sensitivity of SOC change index to $Temp^{(1)}$ is a negative function of time. Consequently, an initial increase in annual averaged temperature $Temp^{(1)}$ decreases the null initial value of $\Delta soc^{(0)}(r)$. Recalling that the sign of the index $\Delta soc^{(0)}(r)(t)$ detects if at the time $t$ the sum of soil carbon contained in compartments is greater than its initial value, we conclude that an initial increase in annual averaged temperature $Temp^{(1)}$ has a negative effect on the achievement of land degradation neutrality.

4.3 Sensitivity of the SOC change index to the $N^{(1)}_P$ ratio

According to Definition 3, the sensitivity of SOC change index to $N^{(1)}_P$ is given by $s_{\Delta soc,N^{(1)}_P} := \frac{d}{dt} s_{\Delta c,N^{(1)}_P} + \frac{1}{T}$, $t \in [t_0 + T, t_0 + 2T]$

$$s_{\Delta soc,N^{(1)}_P}(t_0 + T) = 0.$$ (21)

Moreover, $s_{\Delta soc}(t) \geq 0$ for all $t \in [t_0 + T, t_0 + 2T]$.

Proof At first, let us consider the sensitivity of $\Delta c^{(0)}(r)$ to $N^{(1)}_P$, which satisfies the following initial value problem

$$\frac{d}{dt} s_{\Delta c,N^{(1)}_P} = \rho^{(1)}(r) k^T s_{\Delta c,N^{(1)}_P} + \frac{1}{T}, \quad t \in [t_0 + T, t_0 + 2T]$$

$$s_{\Delta c,N^{(1)}_P}(t_0 + T) = 0.$$ (22)

according to equations (15) applied to equations (16).

By recalling that $k^T A = -\delta k^T$ and $k^T a^{(0)} = 1$ it is easy to see that $s_{\Delta c,N^{(1)}_P}$ satisfies the initial value problem (21).

To complete the proof, let us define $k_{\text{max}} := \max_i k_i$. Thus,

$$\frac{d}{dt} s_{\Delta c,N^{(1)}_P} \geq -\rho^{(1)}(r) \delta k_{\text{max}} s_{\Delta c,N^{(1)}_P},$$

for all $t \in [t_0 + T, t_0 + 2T]$. Since $s_{\Delta c,N^{(1)}_P}(t_0 + T) = 0$, we have that $s_{\Delta c,N^{(1)}_P}(t) \geq 0$ for all $t \in [t_0 + T, t_0 + 2T]$.
Remark 2 The sensitivity of the SOC change index to $N_{P}^{(1)}$ is positive, consequently an increase of the $N_{P}^{(1)}$ ratio increases the null initial value of $\Delta \text{soc}_{p^{(0)}}$. Recalling that the sign of the index $\Delta \text{soc}_{p^{(0)}}(t)$ detects if at the time $t$ the sum of soil carbon contained in compartments is greater than its initial value, we conclude that an increase in annual NPP values has a positive effect on the achievement of land degradation neutrality.

4.4 Sensitivity of the SOC change index to the parameter $r$

According to Definition 3, the sensitivity of the SOC change index to $r$ is given by

$$s_{\Delta \text{soc},r} := \frac{1}{\text{T}} s_{\Delta \text{c},r}.$$  

The following theorem holds.

Theorem 6 The sensitivity of the SOC change index to $r$ satisfies the following initial value problem

$$\begin{align*}
\frac{ds_{\Delta \text{soc},r}}{dt} &= -\rho^{(n)}(r) \delta k^{T} s_{\Delta \text{c},r} - \frac{\partial \rho^{(n)}(r)}{\partial r} \delta k^{T} \Delta \text{c}_{p^{(0)}(r)} \\
\text{s}_{\Delta \text{soc},r}(t_{0} + T) &= 0,
\end{align*}$$

(23)

for $t \in [t_{0} + nT, t_{0} + (n + 1)T]$, $n = 1, 2, \ldots$.

Moreover, if $\vartheta^{(1)}$ is positive, then there exists an $\epsilon > 0$ such that $s_{\Delta \text{soc},r}(t) \leq 0$ for all $t \in [t_{0} + T, t_{0} + T + \epsilon]$. Conversely, if $\vartheta^{(1)}$ is negative, then there exists an $\epsilon > 0$ such that $s_{\Delta \text{soc},r}(t) \geq 0$ for all $t \in [t_{0} + T, t_{0} + T + \epsilon]$.

Proof Let us begin by obtaining the initial value problem for $s_{\Delta \text{c},r}$. According to equations (15) applied to equations (16), we have that

$$\begin{align*}
\frac{ds_{\Delta \text{c},r}}{dt} &= \rho^{(n)}(r) A s_{\Delta \text{c},r} + \frac{\partial \rho^{(n)}(r)}{\partial r} \left( \rho^{(n)}(r) A \Delta \text{c}_{p^{(0)}(r)} + \vartheta^{(n)} a^{(g)} \right) \\
s_{\Delta \text{c},r}(t_{0} + T) &= \frac{\partial \Delta \text{c}_{p^{(0)}(r)}}{\partial r}(t_{0} + T) = 0,
\end{align*}$$

(24)

where

$$\begin{align*}
\frac{\partial}{\partial r} \left( \rho^{(n)}(r) A \Delta \text{c}_{p^{(0)}(r)} + \vartheta^{(n)} a^{(g)} \right) &= \\
&= \left( \frac{\partial \rho^{(n)}(r)}{\partial r} \right) A \Delta \text{c}_{p^{(0)}(r)} + \vartheta^{(n)} \frac{\partial a^{(g)}}{\partial r} \\
&= \left( \frac{\partial \rho^{(n)}(r)}{\partial r} \right) A \Delta \text{c}_{p^{(0)}(r)} + \frac{\vartheta^{(n)}}{(r + 1)^{2}} v,
\end{align*}$$

and $v := [1, -1, 0, 0]^{T}$. Thus, we have that

$$\frac{ds_{\Delta \text{c},r}}{dt} = \rho^{(n)}(r) A s_{\Delta \text{c},r} + \left( \frac{\partial \rho^{(n)}(r)}{\partial r} \right) A \Delta \text{c}_{p^{(0)}(r)} + \frac{\vartheta^{(n)}}{(r + 1)^{2}} v.$$
By multiplying both sides of the above equation by $\mathbf{1}^\top$, and recalling that $\mathbf{1}^\top \mathbf{A} = -\delta \mathbf{k}^\top$, and $\mathbf{1}^\top \mathbf{v} = 0$, equation (23) is proved.

For the second part of the proof, let us consider $n = 1$. We have that

$$
\frac{ds_{\Delta \text{soc},r}}{dt} = -\rho^{(1)}(r) \frac{\partial \rho^{(1)}(T)}{\partial r} \mathbf{k}^\top \mathbf{s}_{\Delta \text{soc},r} - \frac{\partial \rho^{(1)}(r)}{\partial r} \mathbf{k}^\top \Delta \mathbf{c}_{\rho^{(0)}(r)} \\
\mathbf{s}_{\Delta \text{soc},r}(t_0 + T) = 0,
$$

for all $t \in [t_0 + T, t_0 + 2T]$. As in the proof of Theorem 3, there exists an $\epsilon > 0$ such that for all $t \in [t_0 + T, t_0 + T + \epsilon]$ the sign of the function

$$
\mathbf{k}^\top \Delta \mathbf{c}_{\rho^{(0)}(r)}(t) = \vartheta^{(1)}(t - t_0 - T) \mathbf{k}^\top \varphi \left( \rho^{(1)}(r) A(T, t_0 - T), \Delta \mathbf{c}(r) \right) \mathbf{a}^{(q)}
$$

is the same as the sign of $\vartheta^{(1)}$. For this reason, we distinguish the two cases: $\vartheta^{(1)} \geq 0$ and $\vartheta^{(1)} < 0$. Let us observe that $\frac{\partial \rho^{(1)}(r)}{\partial r} > 1$ so that, when $\vartheta^{(1)} \geq 0$, it results

$$
\frac{ds_{\Delta \text{soc},r}}{dt} = -\rho^{(1)}(r) \frac{\partial \rho^{(1)}(r)}{\partial r} \mathbf{k}^\top \mathbf{s}_{\Delta \text{soc},r} - \frac{\partial \rho^{(1)}(r)}{\partial r} \mathbf{k}^\top \Delta \mathbf{c}_{\rho^{(0)}(r)} \\
\leq -\rho^{(1)}(r) \delta \min \mathbf{s}_{\Delta \text{soc},r}.
$$

Since $s_{\Delta \text{soc},r}(t_0 + T) = 0$, we have that $s_{\Delta \text{soc},r}(t) \leq 0$ for all $t \in [t_0 + T, t_0 + T + \epsilon]$. If $\vartheta^{(1)} < 0$, then $\frac{ds_{\Delta \text{soc},r}}{dt} \geq -\rho^{(1)}(r) \delta \max \mathbf{s}_{\Delta \text{soc},r}$ so that, as $s_{\Delta \text{soc},r}(t_0 + T) = 0$, then $s_{\Delta \text{soc},r}(t_0 + T) = 0$, for all $t \in [t_0 + T, t_0 + T + \epsilon]$ and this completes the proof.

**Remark 3** For sufficiently small values of $t$, the sensitivity of the SOC change index to $r$ has opposite sign of $\vartheta^{(1)}$. This means that an initial increase in the parameter $r$ increases or decreases the null initial value of $\Delta \text{soc}_{\rho^{(0)}(r)}$ accordingly to negative or positive values of $\vartheta^{(1)}$. More in details, when changes in temperature increase the annual value NPP more then the modifying factor $\rho^{(1)}(r)$, both with respect to their initial values i.e. $\frac{NPP(t_0 + T)}{NPP(t_0)} \leq \frac{\rho^{(1)}(r)}{\rho^{(0)}(r)}$, this positively impacts all land use classes; viceversa, when changes in temperature increase the modifying factor $\rho^{(1)}(r)$ more then the annual value NPP with respect to their initial value i.e. $\frac{NPP(t_0 + T)}{NPP(t_0)} > \frac{\rho^{(1)}(r)}{\rho^{(0)}(r)}$, then SOC change negatively impacts all the land use class. In both positive and negative case the arable land use class results the most affected.

\[
\frac{\partial \rho}{\partial r}(\text{Temp}^{(n)}, r) = k_a(\text{Temp}^{(n)}) k_b(\text{Acc}^{(n)}) N_b \frac{e^x(r)}{r^2 (1 + e^x(r))^2}, \quad x(r) = \frac{30(r - 1)}{r}
\]
5 A model for SOC changes with farmyard input as control variable

In case of farmyard manure input, Theorem 2 is modified as follows.

Theorem 7 Under the hypothesis \( F(t_0) \neq 0 \), the dynamics of the variable \( \Delta c_{\rho^{(0)}(r)}(t) := \frac{c(t) - c(t_0)}{P(t_0) + F(t_0)} \) for \( t \in [t_0 + nT, t_0 + (n+1)T] \), for \( n = 1, 2, \ldots \), is governed by the equation

\[
\frac{d\Delta c_{\rho^{(0)}(r)}(t)}{dt} = \rho(t) A \Delta c_{\rho^{(0)}(r)}(t) + \left( N_p^{(n)} \hat{g}_r(t) - \frac{\rho(t)}{T \rho^{(0)}(r)} \right) \epsilon a^{(g)}
\]

\[
+ \left( \frac{f(t)}{F(t_0)} - \frac{\rho(t)}{T \rho^{(0)}(r)} \right) (1 - \epsilon) a^{(f)}, \quad \Delta c_{\rho^{(0)}(r)}(t_0 + T) = 0, \tag{25}
\]

where \( 0 \leq \epsilon := \frac{P(t_0)}{P(t_0) + F(t_0)} < 1 \).

Proof By plugging the expression of \( P(t_0 + nT) \) into the equation (12), for all \( t \in [t_0 + nT, t_0 + (n+1)T] \), we have

\[
\frac{dc}{dt} = \rho(t) A c + P(t_0) N_p^{(n)} \hat{g}_r(t) a^{(g)} + f(t) a^{(f)}.
\]

Thus,

\[
\frac{d\Delta c_{\rho^{(0)}(r)}(t)}{dt} = \frac{1}{P(t_0) + F(t_0)} \left( \rho(t) A c + P(t_0) N_p^{(n)} \hat{g}_r(t) a^{(g)} + f(t) a^{(f)} \right)
\]

\[
= \rho(t) A \Delta c_{\rho^{(0)}(r)} + \frac{1}{P(t_0) + F(t_0)} \left( \rho(t) A c(t_0) + P(t_0) N_p^{(n)} \hat{g}_r(t) a^{(g)} + f(t) a^{(f)} \right)
\]

\[
= \rho(t) A \Delta c_{\rho^{(0)}(r)} + \frac{P(t_0)}{P(t_0) + F(t_0)} N_p^{(n)} \hat{g}_r(t) a^{(g)} + \frac{\rho(t) A c(t_0)}{P(t_0) + F(t_0)} + \frac{f(t)}{P(t_0) + F(t_0)} a^{(f)}.
\]

Recalling the relation between \( P(t_0) \) and \( c(t_0) \) in (9) that yields

\[
Ac(t_0) = -\frac{1}{T \rho^{(0)}(r)} \left( P(t_0) a^{(g)} + F(t_0) a^{(f)} \right),
\]

we have

\[
\frac{d\Delta c_{\rho^{(0)}(r)}(t)}{dt} = \rho(t) A \Delta c_{\rho^{(0)}(r)}(t) + \left( N_p^{(n)} \hat{g}_r(t) - \frac{\rho(t)}{T \rho^{(0)}(r)} \right) \epsilon a^{(g)}
\]

\[
+ \left( \frac{f(t)}{F(t_0)} - \frac{\rho(t)}{T \rho^{(0)}(r)} \right) (1 - \epsilon) a^{(f)}.
\]

The dynamics for \( \Delta soc_{\rho^{(0)}(r)}(t) \) can be immediately deduced from the dynamics of \( \Delta c_{\rho^{(0)}(r)}(t) \) as follows.
Corollary 2 In case of farmyard manure input, the dynamics of the SOC change index \( \Delta \text{soc}_{\rho^{(0)}(r)}(t) \) for \( t \in [t_0 + nT, t_0 + (n+1)T] \), for \( n = 1, 2, \ldots \), is governed by the equation
\[
\frac{d \Delta \text{soc}_{\rho^{(0)}(r)}(t)}{dt} = -\delta(t) k^T \Delta \text{c}_{\rho^{(0)}(r)} + \epsilon \left( N_{\rho^{(0)}(r)} \gamma_{\rho^{(0)}(r)} - \frac{\rho(t)}{\epsilon T \rho^{(0)}(r)} \right) + (1 - \epsilon) \frac{f(t)}{F(t_0)}.
\] (26)
where \( \Delta \text{c}_{\rho^{(0)}(r)}(t) \) solves \( \hat{\text{P}} \) and \( \Delta \text{soc}_{\rho^{(0)}(r)}(t_0 + T) = \Delta \text{soc}_{\rho^{(0)}(r)}(t_0) = 0 \).

Proof The result trivially arises recalling the definition of \( \Delta \text{soc}_{\rho^{(0)}(r)} \) which gives that \( \frac{d \Delta \text{soc}_{\rho^{(0)}(r)}(t)}{dt} = 1^T \frac{d \Delta \text{c}_{\rho^{(0)}(r)}}{dt} \) and by observing that \( 1^T \text{A} = -\delta k^T \).

In view of Theorem 8, we introduce the following definition.

Definition 4 Set
\[
r_0(t) := \rho(t) \frac{1}{1 - \epsilon} \left[ \delta k^T \Delta \text{c}_{\rho^{(0)}(r)}(t) + \frac{1}{T \rho^{(0)}(r)} \right] - \frac{\epsilon}{1 - \epsilon} N_{\rho^{(0)}(r)} \gamma_{\rho^{(0)}(r)}(t).
\]
We define the modifying factor of the farmyard manure as the quantity
\[
f_0(t) := \max \{0, r_0(t)\}.
\]

Theorem 8 The density function of farmyard manure defined as \( f(t) := f_0(t) F(t_0) \) assumes that \( \text{soc}_{\rho^{(0)}(r)}(t) \geq \text{soc}_{\rho^{(0)}(r)}(t_0) \) for all \( t \in [t_0 + nT, t_0 + (n+1)T] \) and \( n = 1, 2, \ldots \).

Proof Notice that \( \frac{1}{1 - \epsilon} \frac{d \Delta \text{soc}_{\rho^{(0)}(r)}(t)}{dt} = -r_0(t) + \frac{f(t)}{F(t_0)} \). Suppose \( r_0(t) \geq 0 \). By plugging the expression of \( f(t) = f_0(t) F(t_0) \) in the equation \( \hat{\text{P}} \), we have that \( \frac{d \Delta \text{soc}_{\rho^{(0)}(r)}(t)}{dt} = 0 \). Hence \( \Delta \text{soc}_{\rho^{(0)}(r)}(t) = 0 \) and consequently \( \text{soc}_{\rho^{(0)}(r)}(t) = \text{soc}_{\rho^{(0)}(r)}(t_0) \) for all \( t \in [t_0 + nT, t_0 + (n+1)T] \), \( n = 1, 2, \ldots \).

When \( r_0(t) < 0 \) then \( \frac{d \Delta \text{soc}_{\rho^{(0)}(r)}(t)}{dt} > 0 \) and \( \text{soc}_{\rho^{(0)}(r)}(t) > \text{soc}_{\rho^{(0)}(r)}(t_0) \) for all \( t \in [t_0 + nT, t_0 + (n+1)T] \), \( n = 1, 2, \ldots \).

Remark 4 Notice that the value \( \epsilon = 0 \), which corresponds to \( P(t_0) = 0 \) (or \( F(t_0) >> P(t_0) \)), gives \( r_0(t) := \rho(t) \delta k^T \Delta \text{c}_{\rho^{(0)}(r)}(t) + \frac{1}{T \rho^{(0)}(r)} > 0 \) then \( \text{soc}_{\rho^{(0)}(r)}(t) = \text{soc}_{\rho^{(0)}(r)}(t_0) \) for all \( t \in [t_0 + nT, t_0 + (n+1)T] \), \( n = 1, 2, \ldots \). By increasing values of the parameter \( \epsilon \), the value of \( \Delta \text{soc}_{\rho^{(0)}(r)}(t) \) increases. For \( \epsilon = 1 \) (which holds for \( F(t_0) = 0 \)), equation \( \hat{\text{P}} \) corresponds to the case with no farmyard manure input. Hence, by increasing \( \epsilon \) from 0 to 1, we explore all the cases from only farmyard manure input to only plant input.
6 A non-standard approximation of SOC changes

In [15] the author proved that the original discrete RothC model in [1] can be thought as one-step, first-order in time, discretization of the continuous model [1]. In light of this interpretation, a novel non-standard first-order approximation which inherits the discrete decomposition process of the original model and has the same equilibrium state of the continuous dynamics [1], was proposed in [3]. When applied as a monthly time-stepping procedure, it can be considered a suitable alternative to the original discrete RothC model.

In monthly units the annual length corresponds to $T = 12$ and the interval $[t_0 + nT, t_0 + (n + 1)T]$ is discretized in the set of instants $t_{m+1}^{(n)} := t_m^{(n)} + \Delta t_m^{(n)}$, with $m = 0, \ldots, 11$ and $t_0^{(n)} = t_0 + nT$. The step lengths are set as $\Delta t_m := \frac{T}{N_m} \approx 1$, where $N_m$ is the number of days of the $m^{th}$ month of the $n^{th}$ year. By denoting with $I$ the 4 dimensional identity matrix, and setting $f(c; t) := \rho(t) A c + b(t)$ and $\tilde{A} := A (I - A)^{-1} = -(I - A) D (I - A)^{-1}$, with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & \beta & \beta \end{pmatrix},$$

the approximated values $c_{m+1}^{(n)} \approx c(t_{m+1}^{(n)})$ of the solution of (12), are given by

$$c_{m+1}^{(n)} = c_m^{(n)} + \Delta t_m \varphi(\Delta t_m^{(n)} \rho(t_m^{(n)}) \tilde{A}) f(c_m^{(n)}; t_m^{(n)})$$

or, equivalently,

$$c_{m+1}^{(n)} = F(\Delta t_m^{(n)} \rho(t_m^{(n)})) c_m^{(n)} + \Delta t_m \varphi(\Delta t_m^{(n)} \rho(t_m^{(n)}) \tilde{A}) b(t_m^{(n)}),$$

where $F(t) := A + (I - A) e^{-t D}$ and $\Delta t_m^{(n)} \varphi(\Delta t_m^{(n)} \rho(t_m^{(n)}) \tilde{A}) = O(diag(\Delta t_m^{(n)})) [3]$, the function $\varphi$ being defined as in Theorem 3. The formulation (27) emphasizes the sharing of the stationary equilibria of the continuous autonomous model $\frac{dc}{dt} = f(c)$ in case when the explicit temporal dependence is neglected and temporal averaged quantities are exploited. Formulation (28) highlights the similarity with the discrete original RothC model which proceeds according to

$$c_{m+1}^{(n)} = F(\Delta t_m^{(n)} \rho(t_m^{(n)})) c_m^{(n)} + \Delta t_m b(t_m^{(n)}).$$

In this paper, we are interested in finding an analogous monthly time-stepping procedure for approximating the changes of $c(t)$ provided by the evolution of the variable $\Delta c^{(n)}(t)$. From the observation that the homogeneous systems for $c(t)$ and $\Delta c^{(n)}(t)$ are both governed by the matrix $\rho(t) A$, it makes sense to use the non standard procedure described above. Consequently, the approximated values $\Delta c_{m+1}^{(n)} \approx \Delta c^{(n)}(t_{m+1}^{(n)})$ of the solution of (13), are given by

$$\Delta c_{m+1}^{(n)} = \Delta c_m^{(n)} + \Delta t_m \varphi(\Delta t_m^{(n)} \rho(t_m^{(n)}) \tilde{A}) f(\Delta c_m^{(n)}; t_m^{(n)})$$

or, equivalently,

$$\Delta c_{m+1}^{(n)} = F(\Delta t_m^{(n)} \rho(t_m^{(n)})) \Delta c_m^{(n)} + \Delta t_m \varphi(\Delta t_m^{(n)} \rho(t_m^{(n)}) \tilde{A}) b(t_m^{(n)}).$$
or, equivalently,
\[ \Delta c_{m+1}^{(n)} = F(\Delta t_{m}^{(n)} \rho(t_{m}^{(n)})) \Delta c_{m}^{(n)} + \Delta t_{m}^{(n)} \varphi(\Delta t_{m}^{(n)} \rho(t_{m}^{(n)})) A \ b(t_{m}^{(n)}), \] (31)
where, with abuse of notation, 
\[ f(\Delta c_{\rho(0)}(r); t) = \rho(t) A \Delta c_{\rho(0)}(r) + b(t) \]
\[ b(t) = \left( N_{p}^{(n)} \hat{g}_{t}(t) - \frac{\rho(t)}{T \rho(0)(r)} \right) a^{(g)} \]
in case of no farmyard manure input, while
\[ b(t) = \left( N_{p}^{(n)} \hat{g}_{t}(t) - \frac{\rho(t)}{T \rho(0)(r)} \right) \epsilon a^{(g)} + \left( \frac{f(t)}{F(t_{0})} - \frac{\rho(t)}{T \rho(0)(r)} \right) (1 - \epsilon) a^{(f)}, \]
where \( 0 < \epsilon := \frac{P(t_{0})}{P(t_{0}) + F(t_{0})} < 1 \), in the opposite case.

Finally, \( \Delta soc_{\rho(0)}(t_{m}^{(n)}) \) are approximated by \( \Delta soc_{m}^{(n)} := 1^{T} \Delta c_{m}^{(n)} \), for \( m = 1, \ldots, 12 \) and \( n = 1, 2 \ldots \).

7 A test case: trends of SOC changes in Alta Murgia National Park.

As an application of the illustrated procedure, we analyze the change of SOC in Alta Murgia National Park, a protected area in Italian Apulia region, southern Italy, established in 2004 (see Figure 3). Two parameters are fixed for all the land surface area of 68077 ha, i.e. the depth layer is fixed at \( d = 23 cm \) and the clay content is set at the percentage \( cly = 50 \), i.e. the value used in [5] for experiments at the experimental farm of the CRA-Cereal Research Centre (41°C 27' N, 15°C 30' E) in Foggia. Temperature, rainfall, diurnal temperature range from 2005 to 2019 at (40°C 75' N, 16°C 75' E,) are extracted from the CRU TS 4.04 grid-box dataset [8] of the Climatic Research Unit (University of East Anglia) and NCAS (see Figure 4). Potential evapotranspiration is calculated from the available climate data according to the Thornthwaite’s
formula given in the Appendix. Estimates of Net Primary Production across Earth’s entire vegetated land surface are taken from MOD17 project\(^5\) part of the NASA Earth Observation System (EOS) program, which is the first satellite-driven dataset \(^18\) to monitor vegetation productivity on a global scale. We have extracted NPP data in the temporal range from 2005 to 2019 by means of the Application for Extracting and Exploring Analysis Ready Samples (AppEEARS) \(^21\) in a polygon containing the boundary of Alta Murgia Park (see Figure 5).

In Figure 5 we report the annual NPP values and the averaged annual temperatures with respect to their reference values set at \(t_0 = 2005\), extracted by the above dataset. As expected, to increasing temperatures correspond increasing values for NPP.

Three different formulations are used for modelling the periodic function \(\hat{g}_r(t)\). For values of \(r \in r(a) := \{r \geq 1\}\) corresponding to the arable class, we set \(\hat{g}_r(t) = \hat{g}_{r(a)}(t)\); for \(r \in r(g) := \{0.5 \leq r < 1\}\) associated to the grassland class, \(\hat{g}_r(t) = \hat{g}_{r(g)}(t)\) and we set \(\hat{g}_r = \hat{g}_{r(f)}(t)\) in correspondence of the forest class described by values \(r \in r(f) := \{0 \leq r \leq 0.5\}\). The monthly values at \(t = t_{m}^{(n)}\) for \(m = 1, \ldots, 12\), of the three main land use distributions \(\hat{g}_{r(a)}, \hat{g}_{r(g)}, \hat{g}_{r(f)}\) are reported in Table 1. The reported values are assumed equal to the distribution of plant carbon inputs given in \(^6\) which mimics the dynamics of typical crop rotations and of permanent grassland or forest in Europe. Finally, in Table 1 we report also the values for \(k_c(t, r)\) at \(t = t_{m}^{(n)}\), for the three main land use, i.e \(k_c(t_{m}^{(n)}, r(a)), k_c(t_{m}^{(n)}, r(g)), k_c(t_{m}^{(n)}, r(f))\), assuming that the soil cover function \(S_r(t)\) is periodic. Plant cover was assumed to occur in months 1-7 and 12 for the arable (croplands) class \(^20\).

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\(^5\) https://www.ntsg.umt.edu/project/modis/mod17.php

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| \(t\) | \(g_{r(a)}(t)\) | \(k_c(t_r, r(a))\) | \(g_{r(g)}(t)\) | \(k_c(t_r, r(g))\) | \(g_{r(f)}(t)\) | \(k_c(t_r, r(f))\) |
|-------|------------------|------------------|------------------|------------------|------------------|------------------|
| \(t_{1}^{(1)}\) (Jan, 31) | 0.0 | 0.6 | 0.05 | 0.6 | 0.025 | 0.6 |
| \(t_{1}^{(2)}\) (Febr, 28) | 0.0 | 0.6 | 0.05 | 0.6 | 0.025 | 0.6 |
| \(t_{1}^{(3)}\) (Mar, 31) | 0.0 | 0.6 | 0.05 | 0.6 | 0.025 | 0.6 |
| \(t_{1}^{(4)}\) (Apr, 30) | 1/6 | 0.6 | 0.05 | 0.6 | 0.025 | 0.6 |
| \(t_{1}^{(5)}\) (May, 31) | 1/6 | 0.6 | 0.10 | 0.6 | 0.05 | 0.6 |
| \(t_{1}^{(6)}\) (Jun, 30) | 1/6 | 0.6 | 0.15 | 0.6 | 0.05 | 0.6 |
| \(t_{1}^{(7)}\) (Jul, 31) | 0.5 | 0.6 | 0.15 | 0.6 | 0.05 | 0.6 |
| \(t_{1}^{(8)}\) (Aug, 31) | 0.0 | 1 | 0.10 | 0.6 | 0.05 | 0.6 |
| \(t_{1}^{(9)}\) (Sept, 30) | 0.0 | 1 | 0.10 | 0.6 | 0.20 | 0.6 |
| \(t_{1}^{(10)}\) (Oct, 31) | 0.0 | 1 | 0.10 | 0.6 | 0.20 | 0.6 |
| \(t_{1}^{(11)}\) (Nov, 30) | 0.0 | 1 | 0.05 | 0.6 | 0.20 | 0.6 |
| \(t_{1}^{(12)}\) (Dec, 31) | 0.0 | 0.6 | 0.05 | 0.6 | 0.10 | 0.6 |

Table 1 Monthly \((t = t_m^{(n)}, n=0, 1, 2, \ldots\) distribution of plant carbon inputs into the soil expressed as a proportion of the total \(\hat{g}_r(t)\) and rate modifying factor \(k_c(t_r, r)\) related to soil cover. Data from \(^6\) and \(^20\).
Fig. 4 Climate data at (40°C 75' N, 16°C 75' E) from CRU TS 4.04 grid-box dataset of the Climatic Research Unit (University of East Anglia).

Fig. 5 Selected layer and temporal values of NPP from MOD17 project of NASA EOS program.
7.1 Numerical trends of sensitivity from 2005 to 2007

In this section, using the Alta Murgia National Park data in the period 2005-2007 we want to show the behaviour of the sensitivities of the SOC change index to average annual temperature, to the relative value of NPP and to \( r = \frac{DPM}{RPM} \) ratio. We chose \( t_0 = 2005 \), with \( T = 12 \), thus \( Temp^{(1)} = 14.27^\circ C \) is the average temperature of 2006 and \( N_P^{(1)} = 1.08 \) is the ratio between the Net Primary Production of 2006 and the Net Primary Production of 2005. Once we have computed the numerical solution of the Cauchy problem (16) for \( n = 1 \), we obtain the sensitivities by summing up the four components of the numerical solution of the initial value problems (20), (22) and (24), for \( n = 1 \).

The numerical approximation of the sensitivity to the average temperature in 2006, depicted in Figure 7.a, is a negative function of time, consistently with Theorem 4. Thus, an increase in the average temperature of 2006 would have reduced \( \Delta soc_p^{(0)}(r) \) during the year and, consequently, the sum of the soil carbon contained in compartments would have decreased too. Moreover, since the sensitivity of \( \Delta soc_p^{(0)}(r) \) to \( Temp^{(1)} \) is a decreasing function of time, we can deduce that the perturbation in the average temperature of 2006 would have affected the rate of decomposition at every month, and this effect would
Fig. 7 Numerical non-standard approximation of the temporal evolution of $s_{\Delta soc,\text{Temp}(1)}$, $s_{\Delta soc,N_{P}^{(1)}}$ and $s_{\Delta soc,r}$ in 2006, with time-step $\Delta t = 0.01$. Parameters: $r = 0.25$ for the forest class, $r = 0.67$ for the grassland class and $r = 1.44$ for the arable class.

have been amplified over time.

Analogously, we can observe that the numerical approximation of the sensitivity of $\Delta soc_{\rho^{(0)}(r)}$ to $N_{P}^{(1)}$ is consistent with Theorem 5. In fact in Figure 7.b it is depicted as a positive (and increasing) function of time. This means that an increase in the Net Primary Production in 2006 with respect to the Net Primary Production in 2005, would have increased $\Delta soc_{\rho^{(0)}(r)}$, and consequently the sum of the soil carbon contained in compartments, during the year. Moreover, the perturbation in $N_{P}^{(1)}$ would have affected the rate of decomposition at every month with this effect amplified over time although at a decreasing pace.

Finally, let us focus on the sensitivity of $\Delta soc_{\rho^{(0)}(r)}$. According to our data, $\vartheta^{(1)} = 4.3620 \cdot 10^{-4}$. Thus, since $\vartheta^{(1)}$ is positive, by Theorem 6 we have that
Fig. 8 Numerical non-standard approximation of the temporal evolution of $\Delta_{\text{soc}, r}$ over 14 years, with time-step $\Delta t = 0.01$. Parameters: $r = 0.25$ for the forest class, $r = 0.67$ for the grassland class and $r = 1.44$ for the arable class.

The sensitivity is a negative function of time and this is consistent with Figure 7c. Thus, an increase in the parameter $r$ at the beginning of 2006, i.e., a transition from forest to grassland and to arable classes, would have caused a decrease in $\Delta_{\text{soc}, \rho(0)}(r)$ and the sum of the soil carbon over the compartments during that year. Also in this case, the perturbation in $r$ would have affected the rate of decomposition at every month, again with an amplification of the effect over time.

Notice that the numerical approximation of the sensitivity of $\Delta_{\text{soc}, \rho(0)}(r)$ to $r$ can be computed not only on the first time interval but also on the following years, by integrating the initial value problem (24) together with the Cauchy problem (16), for $t \in [t_0 + nT, t_0 + (n + 1)T]$, $n = 1, \ldots, 14$ (see Figure 8).

7.2 SOC changes scenarios in years 2005-2019

We are going to illustrate the evolution of SOC changes in Alta Murgia National Park in the period 2005-2019 taking as baseline its distribution in 2005 ($t_0 = 2005T$ with $T = 12$). The approximated values $\Delta_{\text{soc}, \rho(0)}(t_n^{(m)})$ of the solution of (23) for $t_n^{(m)} \in [t_0 + nT, t_0 + (n + 1)T]$ with $n = 1, \ldots, 14$, provided by means of the non-standard discrete procedure described in (31), are evaluated for the three main land use classes: forest, grassland and arable.

For the arable case, we also show the farmyard manure program which would
be able to assure the achievement of land degradation neutrality in 2019 with respect to 2005 taken as reference year.

7.2.1 Forest class

For the forest class, the evolution of $\Delta soc_{\rho(0)}(t_m)$, together with its averaged annual values, is given in Figure 9. We set $r = 10^{-4}$, $r = 0.25$, $r = 0.5$ in order to span all the values corresponding to this class. We notice that, for $r$ spanning the reference set $r(f)$, the trends do not differ much. However, even if it is still negative at the end of the interval, the general behaviour of $\Delta soc_{\rho(0)}(t_m)$ suggests that a positive value can be achieved by 2030.

7.2.2 Grassland class

For the grassland class, the evolution of $\Delta soc_{\rho(0)}(t_m)$, together with its averaged annual values, is given in Figure 10. We set $r = 0.67$, $r = 0.9$ and $r = 0.95$ in order to span all the values corresponding to this class. As for the forest class, the general trend of $\Delta soc_{\rho(0)}(t_m)$ seems to be increasing even for a grassland scenario. Notice however that this class is much influenced by the value of $r$. 

Fig. 9 The temporal evolution of $\Delta soc_{\rho(0)}(t_m)$, together with its averaged annual values for forest class. Parameters $r = 10^{-4}$, $r = 0.25$, $r = 0.5$. 

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### Diagram Description

- **Forest Class**: The diagram illustrates the temporal evolution of $\Delta soc_{\rho(0)}(t_m)$ for the forest class, showing how its values change over time with different monthly values and averaged annual values for parameters $r = 10^{-4}$, $r = 0.25$, and $r = 0.5$. The trend suggests a possible achievement of land degradation neutrality in 2030.

- **Grassland Class**: Similar to the forest class, the diagram for the grassland class shows the evolution of $\Delta soc_{\rho(0)}(t_m)$, with parameters $r = 0.67$, $r = 0.9$, and $r = 0.95$. The trend indicates an increasing trend even for this class, though it is influenced significantly by the value of $r$. 

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Fig. 10 The temporal evolution of $\Delta \text{soc}_{\rho(0)}(r(t_m^{(n)}))$, together with its averaged annual values for grassland class. Parameters $r = 0.67, r = 0.9, r = 0.95$.

For value $r = 0.95$, close to the value which bounds from above the class $r(g)$, the curve reaches positive values at 2011 and, although oscillating, it remains positive till the end of 2019. In correspondence of the value $r = 0.67$ which is the one adopted in the literature for this class, the final value is negative; however the general trend seems to be increasing so that a positive value might be envisaged by 2030.

7.2.3 Arable class

For the arable class, we firstly assume that no farmyard manure enter the system so that the evolution of $\Delta \text{soc}_{\rho(0)}(r(t_m^{(n)}))$, together with its averaged annual values, is given in Figure 11. We set $r = 1, r = 1.44$ (i.e. the value used in case of forest class in literature [1]), and $r = 100$ in order to span all the values corresponding to this class. This case is the most critical one: the dynamics, even quantitatively different according to the values of $r \in r(a)$, is decreasing with this denoting a general trend departing from the baseline of positive values. For this class, in order to reach positive quantities, it is necessary to intensify the organic carbon input. To this aim we can apply the findings of Theorem 8 in order to detect the optimal farmyard manure program to enforce positive values of $\Delta \text{soc}_{\rho(0)}(r)$. In Figure 12 we report the temporal evolution of the modifying factor for farmyard manure $f_0(t)$, as defined in
Definition 4 for several values of $\epsilon$ spanning the interval $[0, 1]$. The effects of the fertilization process are shown in Figure 13.

Fig. 11 The temporal evolution of $\Delta soc_{\rho(0)}(t_m^{(n)})$, together with its averaged annual values for the arable class. Parameters $r = 1, r = 1.44, r = 100$.

8 Comments and conclusion

Soil carbon models (e.g. RothC [1], Century [17]) which take into account the interactions between climate and land use management, are widely used to predict SOC changes under future climate scenarios. Warmer temperatures positively affect SOC stocks since they reduce decomposition, as an effect of a decreased soil moisture, and also increase Net Primary Production thus augmenting carbon inputs to the soil. On the other hand, increasing temperatures negatively affect the SOC stocks as they increase the decomposition rate of soil organic matter. Hence, whether soils gain or lose SOC, depends upon how balanced the competing gain and loss processes are, with subtle interacting changes in temperature, moisture, soil type and land use [6].

With the aim of improving the prediction of the factors that determine the size and direction of change, we have introduced the so-called SOC change index and we have described its evolution based on the RothC carbon model. Under the hypothesis of constant environmental and organic fertilization conditions,
it does not require to evaluate or measure the specific initial value of SOC, as it describes the deviation from the assumed initial equilibrium.

The effectiveness of the novel index has been tested for evaluating the impact of warming temperatures on the achievement of land degradation neutrality for the SOC indicator in Alta Murgia National Park, a protected area in the Apulia region located in the south of Italy. The performed sensitivity analysis, based on time averaged parameter values, has provided local information on the impact of change in mean annual temperature, of deviations of the mean annual NPP from its reference value and of the degree of decomposability of plant material. The results of the sensitivity analysis is in accordance with the experimental results, as we found that the SOC change index is negatively affected by increasing mean annual temperature and positively by increasing deviation of NPP. Changes in DPM/RPM ratio $r$, which in turn are related to land use change, indicate that all land use classes are positively affected when deviation of NPP prevails on deviation in decomposition and negatively in the opposite case. In both cases the arable class results the most affected. The simulated dynamics of the SOC change index in the Alta Murgia National Park in years [2005, 2019] with climate data of CRU (University of East Anglia) and estimates of NPP taken from MOD17 project4, indicate positive trends for forest and grassland classes. The arable class which is most affected by changes in NPP and temperature, as suggested by our sensitivity analysis, shows a negative trend. The dynamics of the SOC change index under the hy-

![Fig. 12 The temporal evolution of modifying factor $f_0(t)$, for the arable class with $r = 1$. Parameter $\epsilon = 0.8, 0.5, 0.2, 0$.](image)
Fig. 13 The temporal evolution of $\Delta \text{soc}_{\rho(0)}(t_m)$, together with its averaged annual values for the arable class with $r = 1$ controlled by farmyard manure. Increasing values of $\Delta \text{soc}_{\rho(0)}(t_m)$ for parameters $\epsilon = 0$, (no plant input), $\epsilon = 0.2$, $0.5$, $0.8$. and $\epsilon = 1$ (no farmyard manure).

Hypothesis of farmyard manure input has revealed a powerful tool for predicting the optimal land fertilization practice to implement for enhancing the SOC stocks in the arable soil of Alta Murgia Park and invert the negative trend. The construction of the SOC change index can be tailored on different soil carbon model dynamics. In particular, a future research direction is represented by the description of SOC change index under a suitable carbon model dynamics which places the action of bacteria at the hearth of the mechanisms of decomposition process as indicated in [9,7].

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9 Appendix

9.1 Thornthwaite’s formula for estimating the potential evapotranspiration

We need to estimate the potential evapotranspiration $pet(t)$, $[mm/month]$, estimated by means of the Thornthwaite’s formula which is expressed, for the $n^{th}$ year, on a monthly basis at the instants $t^{(n)}_m := t_0 + nT + \frac{T}{365} \sum_{i=1}^{m} N_i$ with $m = 1, \ldots, 12$ and $N_i$ denoting the number of days of the $i^{th}$ month of the $n^{th}$ year, as follows:

$$pet(t^{(n)}_m) := 16 \frac{L^{(n)}_{d,m}}{12} N_m \frac{10 Temp^{(n)}_{d,m}}{I_n} a.$$  

In the above formula, $L^{(n)}_{d,m}$ and $Temp^{(n)}_{d,m}$ represent the average day length (hours) and the average daily temperature of the $m^{th}$ month of the $n^{th}$ year, respectively. Finally, $I_n$ is the heat index for the $n^{th}$ year given by

$$I_n = \sum_{k=1}^{12} \left( \frac{Temp^{(n)}_k}{5} \right)^{1.5}$$

where $Temp^{(n)}_k := \int_{t^{(n)}_{k-1}}^{t^{(n)}_k} Temp(s) ds$ is the $k^{th}$ monthly mean temperature, for $k = 1, \ldots, 12$. Finally,

$$a = 6.7 \times 10^{-7} I_n^3 - 7.7 \times 10^{-5} I_n^2 + 1.8 \times 10^{-2} I_n + 0.49.$$  

9.2 Estimation of the accumulate soil moisture deficit

The accumulate soil moisture deficit in the $n^{th}$ year, is also estimated on a monthly basis at the instants $t^{(n)}_m := t_0 + nT + \frac{T}{366} \sum_{i=1}^{m} N_i$ with $m = 1, \ldots, 12$.

Then $Acc(t^{(n)}_m, M) = 0$ for all $m = 1, \ldots, \bar{m}$ such that $pet(t^{(n)}_m) \leq rain(t^{(n)}_m)$, while

$$Acc(t^{(n)}_m, M) = \min \left( \max \left( M, Acc(t^{(n)}_{m-1}, M) + rain(t^{(n)}_m) - pet(t^{(n)}_m) \right), 0 \right)$$

for $m = \bar{m} + 1, \ldots, T$.

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$^6$ In a leap year $t^{(n)}_m := t_0 + nT + \frac{T}{366} \sum_{i=1}^{m} N_i$ and $N_2 = 29$. 