Radial symmetry and its breaking
in the Caffarelli-Kohn-Nirenberg type inequalities for \( p = 1 \)

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Abstract: The main purpose of this article is to study the Caffarelli-Kohn-Nirenberg type inequalities (1.2) with \( p = 1 \). We show that symmetry breaking of the best constants occurs provided that a parameter \(|\gamma|\) is large enough. In the argument we effectively employ equivalence between the Caffarelli-Kohn-Nirenberg type inequalities with \( p = 1 \) and the isoperimetric inequalities with weights.

Key words: CKN-type inequality; symmetry break; weighted Hardy-Sobolev inequality; best constant.

1. Introduction. The Caffarelli-Kohn-Nirenberg type inequalities (the CKN-type inequalities) were introduced in [4] originally as rather general multiplicative inequalities with weights being powers of distance from the origin, and they have been studied eagerly afterwards by many authors (see e.g. [1,2,5,6,8,10,12,17–20,23,24,26]).

Symmetry breaking of extremal functions for the CKN-type inequalities was introduced in [4] as a rather general multiplicative inequalities with weights. The full proofs will be given in the paper [7].

First we define a class of weighted function. For \( \alpha \in \mathbb{R} \) and \( n \geq 1 \) we set

\[
I_\alpha(x) = I_\alpha(|x|) = |x|^{\alpha-n} \quad \text{for } x \in \mathbb{R}^n \backslash \{0\}.
\]

When \( 0 < \alpha < n \) holds, \( I_\alpha \) is called a Riesz kernel of order \( \alpha \). Then the classical CKN-type inequalities with \( p \geq 1 \) are represented in the following way: For \( \gamma \in \mathbb{R} \backslash \{0\} \), there exists a positive number \( C \) such that we have

\[
I_\gamma^p \left( \int_{\mathbb{R}^n} |\nabla u(x)|^p I_\gamma^{(1+\gamma)}(x)dx \right)^{1/p} \geq C \left( \int_{\mathbb{R}^n} |u(x)|^q I_\gamma(x)dx \right)^{1/q}
\]

for any \( u \in C^\infty_0(\mathbb{R}^n \backslash \{0\}) \), where \( 1 \leq p \leq q < \infty \), \( 1/p - 1/q \leq 1/n \), and \( C \) depends only on \( p, q, \gamma \) and \( n \). As was mentioned before, when \( \gamma \geq 0 \) holds, the CKN-type inequalities were introduced in [4] as a part of multiplicative interpolation inequalities, and later in [22] the CKN-type inequalities were further investigated for all \( \gamma \in \mathbb{R} \). In the present paper, we shall give a simple proof of the CKN-type inequalities (1.2) for \( p = 1 \) with the best constant \( S^{1,q,\gamma} \) defined below, using equivalence between the CKN-type inequalities with \( p = 1 \) and the isoperimetric inequalities with weights. We note that in [19] a class of weighted Sobolev inequalities were studied by using isoperimetric inequalities with general weight functions. To give a precise definition of the best constant \( S^{1,q,\gamma} \) it is convenient to adopt the following function spaces and relating norms including the case where \( p \geq 1 \):

Definition 1.1. Let \( 1 \leq p \leq q < \infty \), \( \gamma \in \mathbb{R} \) and the isoperimetric inequalities with weights. We note that in [19] a class of weighted Sobolev inequalities were studied by using isoperimetric inequalities with general weight functions. To give a precise definition of the best constant \( S^{1,q,\gamma} \) it is convenient to adopt the following function spaces and relating norms including the case where \( p \geq 1 \):

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\]
The inequality (1.7) follows directly from Theorem 1.1 and the positivity of $S_{\gamma}^{1,q}$ in Definition 1.3.

Definition 1.4. An open set $M \subset \mathbb{R}^n$ is called admissible if $\overline{M}$ is bounded and $\partial M$ is a $C^\infty$ manifold.

In the next we state equivalence between the CKN-type inequalities with $p = 1$ and isoperimetric inequalities with weights (the proof is given in Section 3).

Theorem 1.2. Let $n \geq 1$ and $\gamma \in \mathbb{R}\setminus\{0\}$. Assume that $1 \leq q \leq n/(n-1)$ if $n > 1; 1 \leq q < \infty$ if $n = 1$. Assume that $M$ is an arbitrary admissible open set of $\mathbb{R}^n$. Then the CKN-type inequalities are equivalent to the corresponding weighted isoperimetric inequalities with the same best constants, which are given by the following

$$\int_{\partial M} I_{1+\gamma}(x) \, dS \geq S_{\gamma}^{1,q} \left( \int_M I_{\gamma}(x) \, dx \right)^{1/q},$$

where $S$ denotes the $(n-1)$-dimensional Lebesgue measure.

Remark 1.1. By $A_{\text{rad}}$ we denote a family of all admissible open sets of $\mathbb{R}^n$ which are radially symmetric with respect to the origin. From Theorem 1.2 and its proof we easily see that

$$S_{\text{rad}}^{1,q} = \inf_{M \in A_{\text{rad}}(\{0\})} \left( \int_M I_{\gamma}(x) \, dx \right)^{1/q}.$$

In the next we describe symmetric properties and important relations among $S_{\text{rad}}^{1,q}$ and $S_{\gamma}^{1,q}$.

Theorem 1.3 (Symmetry). Let $n \geq 1$. Assume that $1 \leq q \leq n/(n-1)$ if $n > 1; 1 \leq q < \infty$ if $n = 1$. Let $\tau_{n,q} = 1 - 1/q$ if $n \leq 1/(n-1)$. Then it holds that:

1. $S_{\gamma}^{1,q} = S_{1-q}^{1,q}$,
2. $S_{\text{rad}}^{1,q} = S_{\text{rad}}^{1-q}$,
3. $S_{\text{rad}}^{1,q} = S_{\text{rad}}^{1-q}$

for $\gamma \neq 0$.

Proof. The assertion 1 of this theorem is already established in Proposition 3.1; [22] and the assertions 2 and 3 are proved by direct calculations using Theorem 1.2 and its remark. □

Now we state main theorems in this article.

Theorem 1.4 (Symmetry breaking). Assume that $q$ is fixed such as $1 \leq q \leq n/(n-1)$ if $n > 1; 1 \leq q < \infty$ if $n = 1$. Then we have the following:

1. If $n > 1$, then we have $S_{\text{rad}}^{1,q} > S_{\gamma}^{1,q}$ for sufficiently large $|\gamma|$,
2. If $n = 1$, then we have $S_{\text{rad}}^{1,q} = 2^{1-1/q} S_{1-q}^{1,q}$ ($> S_{\gamma}^{1,q}$) for any $\gamma \in \mathbb{R}\setminus\{0\}$.

Remark 1.2. When $q = 1$ holds, the CKN-type inequality becomes the weighted Hardy inequality.
The best constant $S^{1,1;\gamma}$ coincides with the one restricted in $W^{1,1;\gamma}(\mathbb{R}^n)_{\text{rad}}$:

\begin{equation}
S^{1,1;\gamma} = S^{1,1;\gamma}_{\text{rad}} = |\gamma| \quad \text{for } \gamma \neq 0.
\end{equation}

2. For $n > 1$, the number $S^{1,1;n-1} = S_{\text{rad}}^{1,1;n-1} = n \left( \frac{\omega_n}{n} \right)^{1/n}$ coincides with the best constant of the Sobolev inequality (c.f. [25]) and $S^{1,1;n-1}$ is attained by characteristic functions of balls which do not belong to $W^{1,1}_n(\mathbb{R}^n)$ but to the space of functions of bounded variation. Therefore the space $W^{1,1}_n(\mathbb{R}^n)$ is too small to discuss existences of extremal functions of the best constants. In the future we shall study existences of extremal functions using suitable spaces of functions of bounded variation (c.f. [9]).

**Definition 1.5.** For $1 \leq q \leq n/(n-1)$ if $n > 1$; $1 \leq q < \infty$ if $n = 1$, we set

\begin{equation}
\Gamma_1(q) = \sup \{ \gamma > 0 : S^{1,q;\gamma} = S^{1,q;\gamma}_{\text{rad}} \}.
\end{equation}

It follows from Theorem 1.3; 3 and Theorem 1.4 that we have $n - 1 \leq \Gamma_1(q) < \infty$ ($1 \leq q \leq n/(n-1)$ if $n > 1$; $1 \leq q < \infty$ if $n = 1$). More precisely we have:

**Theorem 1.5.** Assume that $1 \leq q \leq n/(n-1)$ if $n > 1$; $1 \leq q < \infty$ if $n = 1$. Then we have the followings:

1. If $\gamma \in (0, \Gamma_1(q)]$, then $S^{1,q;\gamma} = S^{1,q;\gamma}_{\text{rad}}$.
2. If $\gamma \in (\Gamma_1(q), \infty)$, then $S^{1,q;\gamma} < S^{1,q;\gamma}_{\text{rad}}$.

Our article is organized in the following way. In Section 2 we introduce useful change of variables. In Section 3 we give sketch of proofs of Theorems 1.2, 1.4 and 1.5.

2. **Change of variables.** Here we see relations among the best constants by a method of change of variables.

**Definition 2.1.** For $\beta > 0$, we set $Y_{\beta}(y) = |y|^{\beta-1}y$ for $y \in \mathbb{R}^n$.

**Definition 2.2.** Let $\beta > 0$ and $u : \mathbb{R}^n \to \mathbb{R}$. Then we have the next lemma.

**Lemma 2.1.** Assume that $1 \leq p \leq q < \infty$, $\gamma > 0$, and $\beta > 0$. Then we have the followings:

\begin{align*}
\|u\|_{L^{p}(\mathbb{R}^n)} &= \beta^{\beta/q} \|T_{\beta}u\|_{L^{q}_{\text{rad}}}(\mathbb{R}^n) \\
\|\nabla u\|_{L_{\text{rad}}^{p,1}(\mathbb{R}^n)}
\end{align*}

for $u \in L^{p}_{\text{rad}}(\mathbb{R}^n)$.

As an application of Lemma 2.1, let us prepare some estimates of the best constants as a basic tool for the proofs of main results.

**Proposition 2.1.** Let $n \geq 1$. Assume that $1 \leq q \leq n/(n-1)$ if $n > 1$; $1 \leq q < \infty$ if $n = 1$. Let $\tau_{n; q} = 1 - 1/q (\leq 1/n)$. Then it holds that:

1. $\frac{\gamma}{\tau_{n; q}} S^{1,q;\gamma} \leq S^{1,q;\gamma} \leq \frac{\tau_{n; q}}{\gamma} S^{1,q;\gamma}$ for $0 < |\gamma| \leq \frac{\tau_{n; q}}{\gamma}$.
2. \[ S^{1,q;1-n} \leq S^{1,q;1} \leq S^{1,q;1-n} = S^{1,q;1-n}_{\text{rad}} \quad \text{for } |\gamma| \geq n-1 \text{ and } n > 1, \text{ where } 1^* = n/(n-1). \]

**Proof.** These estimates are simple variants of the assertions 4 and 5 of Theorem 2.2 in [22].

3. **Proofs of Theorems.** In the proofs of theorems, we may assume that $\gamma > 0$ by virtue of the assertion 1 of Theorem 1.3. Let us prepare two lemmas without proofs (see e.g. Theorem 1.24 and Lemma 1 in Section 1.35 in [24]).

**Lemma 3.1 (Coarea formula).** Assume that $\Omega$ is domain of $\mathbb{R}^n$ and $\Phi$ is a nonnegative Borel measurable function on $\Omega$. Then we have for $u \in C^{0,1}_{\text{rad}}(\Omega)$

\begin{equation}
\int_{\Omega} \Phi(x) |\nabla u(x)| \, dx = \int_0^\infty \int_{|u(x)|=t} \Phi(x) \, dS(x).
\end{equation}

Here $C^{0,1}_{\text{rad}}(\Omega)$ denotes the space of all Lipschitz continuous functions with compact support in $\Omega$, and $S$ denotes the $(n-1)$-dimensional Lebesgue measure.

**Lemma 3.2.** Assume that $f$ is a nonnegative non-increasing function on $(0, \infty)$ and $p \geq 1$. Then we have

\begin{equation}
\int_0^\infty (f(x))^p \, dx \leq \left( \int_0^\infty (f(x))^q \, dx \right)^{p/q}.
\end{equation}

**Proof of Theorem 1.2.** Since it is standard to show the implication (1.7) to (1.8), we assume that (1.8). Let us set $M_t = \{ |u(x)| > t \}$ for any $u \in C^\infty_\text{c}(\mathbb{R}^n \setminus \{0\})$. Since we may assume that $M_t$ is admissible for almost all $t$, by Lemma 3.1 and Lemma 3.2 we have
\[ \int_{\mathbb{R}^n} |\nabla u(x)| I_{1+\gamma}(x) \, dx \]
\[ = \int_0^\infty dt \int_{\partial M} I_{1+\gamma}(x) \, dS \]
\[ \geq S^{1-a} \int_0^\infty dt \left( \int_M I_{1+\gamma}(x) \, dx \right)^{1/q} \]
\[ \geq S^{1-a} \left( \int_0^\infty d(t^\gamma) \int_M I_{1+\gamma}(x) \, dx \right)^{1/q} \]
\[ = S^{1-a} \left( \int_{\mathbb{R}^n} |u(x)|^q I_{1+\gamma}(x) \, dx \right)^{1/q}. \]

Hence we have (1.7) with the same best constant \( S^{1-a} \).

**Proof of Theorem 1.4.** First we prove Theorem 1.4 when \( n > 1 \). For \( 0 < \delta < 1 \) we set \( A_\delta = \{ x \in \mathbb{R}^n : |x| < 1 \} \) and \( B_\delta = \{ x \in \mathbb{R}^n : |x| < 2\delta \} \). By \( C \) and \( C' \) we denote positive numbers depending only on the dimension of the space. Let \( e \in (1, 0, \ldots, 1) \) and set \( M = D \). We prove fundamental estimates involving \( A_\delta \) and \( D_\delta \). First we see that there is a \( \delta_0 > 0 \) and a positive number \( C \) such that we have

\[ \text{vol}(D_\delta)/\text{vol}(A_\delta) \geq C^\delta \quad (0 < \delta < \delta_0). \]

Then by a direct calculation we can prove

\[ \int_{\partial M} I_{1+\gamma}(x) \, dS \leq C^\delta \int_{\partial B_\delta} I_{1+\gamma}(x) \, dS \]
\[ (0 < \delta < \delta_0). \]

On the other hand, we see

\[ \int_{A_\delta} I_{1+\gamma}(x) \, dx = \int_{B_\delta} dS \int_{1-\delta}^{1} t^\gamma \, dt \]
\[ = (1 - (1 - \delta)^\gamma) \int_{B_\delta} I_{1+\gamma}(x) \, dx. \]

Let \( x = (x_1, x') \) and \( D_\delta' = A_\delta \cap \{ x' \in \mathbb{R}^{n-1} \} \). Then \( D_\delta' \subset D_\delta = M \) and \( \text{vol}(D_\delta')/\text{vol}(D_\delta) \geq C \) hold. Thus

\[ \int_{D_\delta'} I_{1+\gamma}(x) \, dx \geq \int_{D_\delta} I_{1+\gamma}(x) \, dx \]
\[ \geq \frac{\text{vol}(D_\delta')}{\text{vol}(A_\delta)} \int_{A_\delta} I_{1+\gamma}(x) \, dx \]
\[ \geq C^\delta \int_{A_\delta} I_{1+\gamma}(x) \, dx. \]

It follows from (3.3) and (3.4) that we have

\[ \int_{B_\delta} |u(x)|^q I_{1+\gamma}(x) \, dx \]
\[ \geq C^\delta \int_{A_\delta} I_{1+\gamma}(x) \, dx. \]

Thus we see that \( S^{1-a} \) is the best possible constant for \( n > 1 \) in the estimate (3.5).

**Remark 3.1.** In the estimate (3.5), it suffices to assume \( \gamma \in (q, n) \delta (n-1)/(q-1) \), where \( C(q, n) \) are some positive numbers depending only on \( q \) and \( n \).

**Proof of Theorem 1.5.**

1. We assume \( 0 < \gamma < \tilde{\gamma} \) and \( S^{1-a} \tilde{\gamma} = S^{1-a} \), then it follows from Theorem 1.3 and Proposition 2.1.1 that \( S^{1-a}_\text{rad} \leq S^{1-a} \), and this proves the assertion.

2. There exists a nonradial function \( u \in W^{1,1}_{\gamma,\beta}(\mathbb{R}^n) \) such that for any \( \varepsilon > 0 \)

\[ S^{1-a} - \varepsilon \leq S^{1-a}(u) \leq S^{1-a} + \varepsilon. \]

Assume that \( \gamma > \tilde{\gamma} \), then we have

\[ S^{1-a} \leq E^{1-a,\gamma}(T_{\gamma,\gamma}u) < (\tilde{\gamma}/\gamma)^{\gamma-1} E^{1-a,\gamma}(u) \]

(by Lemma 2.1 with \( \beta = \tilde{\gamma}/\gamma \))

\[ \leq (\tilde{\gamma}/\gamma)^{\gamma-1} S^{1-a,\gamma}_\text{rad} + (\tilde{\gamma}/\gamma)^{\gamma-1} \varepsilon \]

\[ \leq (\gamma/\gamma)^{\gamma-1} S^{1-a,\gamma}_\text{rad} + (\gamma/\gamma)^{\gamma-1} \varepsilon \]

\[ = S^{1-a,\gamma}_\text{rad} + (\gamma/\gamma)^{\gamma-1} \varepsilon. \]

Hence we see \( S^{1-a,\gamma} < S^{1-a,\gamma}_\text{rad} \) by \( \varepsilon \to 0 \).
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