On the inverse problem of calculus of variations for fourth-order equations

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Fels’ conditions (Fels, M. E. 1996 Trans. Amer. Math. Soc. 348, 5007–5029. (doi:10.1090/S0002-9947-96-01720-5)) ensure the existence and uniqueness of the Lagrangian in the case of a fourth-order equation. We show that when Fels’ conditions are satisfied, the Lagrangian can be derived from the Jacobi last multiplier, as in the case of a second-order equation. Indeed, we prove that if a Lagrangian exists for an equation of any even order, then it can be derived from the Jacobi last multiplier. Two equations from a Number Theory paper by Hall (Hall, R. R. 2002 J. Number Theory 93, 235–245. (doi:10.1006/jnth.2001.2719)), one of the second and one of the fourth order, will be used to exemplify the method. The known link between Jacobi last multiplier and Lie symmetries is also exploited. Finally, the Lagrangians of two fourth-order equations drawn from Physics are determined with the same method.

Keywords: fourth-order ordinary differential equations; inverse problem of calculus of variations; Lagrangian; Jacobi last multiplier

1. Introduction

It is well known that a Lagrangian always exists for any second-order ordinary differential equation (Whittaker 1999). What seems less known is that the key is the Jacobi last multiplier (Jacobi 1844, 1845, 1886; Whittaker 1999), which has many interesting properties, a list of which can be found in Nucci (2005).

It is a matter for historians alike to find out why Darboux (1894), Helmholtz (1887), Koenigsberger himself† and many other successive authors, e.g. Douglas (1941) and Havas (1957), never acknowledged the use of the Jacobi last multiplier in order to find Lagrangians of a second-order equation.

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In 1902–1903, Koenigsberger wrote Helmholtz’s biography (Koenigsberger 2001)—which, in 1906, was (abridged) translated into English with a Preface by Lord Kelvin (Koenigsberger 1906)—after he wrote his 1901 book on Mechanics (Koenigsberger 1901). Neither book cites the connection between Jacobi last multiplier and Lagrangians. In 1904, Koenigsberger wrote Jacobi’s biography (Koenigsberger 1904), where the Jacobi last multiplier is extensively described.

Havas even cites the 1937 edition of the book by Whittaker (1999), but only in connection with the formulation of Lagrangian equations.
The Norwegian Sophus Lie, who carefully studied Jacobi’s work (Hawkins 1991), found a connection between his groups of transformations and the Jacobi last multiplier (Lie 1874, 1912). The Italian Bianchi presented Lie’s work in his lectures on finite continuous groups of transformations, and described quite clearly the Jacobi last multiplier and its properties (Bianchi 1918). Neither of them cited the connection with Lagrangians.

Numerous papers have been dedicated to the solution of the inverse problem of calculus of variations, namely finding a Lagrangian of differential equations. Some of these acknowledge the seminal work of Jacobi (see references in Nucci (2005) and Nucci & Leach (2007)), but most failed to do so.

Fels (1996) derived the necessary and sufficient conditions under which a fourth-order equation

\[ u^{(iv)} = F(t, u, u', u'', u''') \] (1.1)

admits a unique Lagrangian, namely

\[ \frac{\partial^3 F}{\partial (u'')^3} = 0 \] (1.2)

and

\[ \frac{\partial F}{\partial u'} + \frac{1}{2} \frac{d^2}{dt^2} \left( \frac{\partial F}{\partial u''} \right) - \frac{d}{dt} \left( \frac{\partial F}{\partial u''} \right) - \frac{3}{4} \frac{\partial F}{\partial u'''} \frac{d}{dt} \left( \frac{\partial F}{\partial u'''} \right) + \frac{1}{2} \frac{\partial F}{\partial u''} \frac{\partial F}{\partial u'''} + \frac{1}{8} \left( \frac{\partial F}{\partial u''} \right)^3 = 0. \] (1.3)

Here, we propose to extend the use of the Jacobi last multiplier in order to find the Lagrangian for ordinary differential equations of order four satisfying Fels’ conditions (1.2) and (1.3).

The paper is organized in the following way. In §2, we present the properties of the Jacobi last multiplier and its connection to Lie symmetries; then we show the connection of the Jacobi last multiplier with Lagrangians for any second-order equation, and how and when this connection can be extended to fourth-order (and higher) equations. In §3, two equations from a Number Theory paper by Hall (2002), one of second and one of fourth order, are used to exemplify the method. The link between Jacobi last multiplier and Lie symmetries (Lie 1874, 1912) is also exploited. In §4, we consider two fourth-order equations drawn from Physics and use the method of the Jacobi last multiplier to find their respective Lagrangians, and finally we provide some of the examples where the method does not work. The last section contains some final remarks.

2. Jacobi last multiplier

The method of the Jacobi last multiplier (Jacobi 1844, 1845, 1886) provides means to determine all the solutions of the partial differential equation

\[ A f = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} = 0 \] (2.1)
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or its equivalent associated Lagrange’s system
\[
\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \cdots = \frac{dx_n}{a_n}.
\]

(2.2)

In fact, if one knows the Jacobi last multiplier and all but one of the solutions, then the last solution can be obtained by a quadrature. The Jacobi last multiplier \(M\) is given by
\[
\frac{\partial (f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial (x_1, x_2, \ldots, x_n)} = M Af,
\]

(2.3)

where
\[
\frac{\partial (f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial (x_1, x_2, \ldots, x_n)} = \det \begin{bmatrix}
\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\
\frac{\partial f}{\partial \omega_1} & \cdots & \frac{\partial f}{\partial \omega_1} \\
\cdots & \cdots & \cdots \\
\frac{\partial f}{\partial \omega_{n-1}} & \cdots & \frac{\partial f}{\partial \omega_{n-1}}
\end{bmatrix} = 0,
\]

(2.4)

and \(\omega_1, \ldots, \omega_{n-1}\) are \(n-1\) solutions of equation (2.1) or, equivalently, first integrals of equation (2.2) independent of each other. This means that \(M\) is a function of the variables \((x_1, \ldots, x_n)\) and depends on the chosen \(n-1\) solutions, in the sense that it varies as they vary. The essential properties of the Jacobi last multiplier are as follows:

— If one selects a different set of \(n-1\) independent solutions \(\eta_1, \ldots, \eta_{n-1}\) of equation (2.1), then the corresponding last multiplier \(N\) is linked to \(M\) by the relationship
\[
N = M \frac{\partial (\eta_1, \ldots, \eta_{n-1})}{\partial (\omega_1, \ldots, \omega_{n-1})}.
\]

— Given a non-singular transformation of variables
\[
\tau: \ (x_1, x_2, \ldots, x_n) \longrightarrow (x'_1, x'_2, \ldots, x'_n),
\]

then the last multiplier \(M'\) of \(A'F = 0\) is given by
\[
M' = M \frac{\partial (x_1, x_2, \ldots, x_n)}{\partial (x'_1, x'_2, \ldots, x'_n)},
\]

where \(M\) obviously comes from the \(n-1\) solutions of \(AF = 0\), which correspond to those chosen for \(A'F = 0\) through the inverse transformation, \(\tau^{-1}\).

— One can prove that each multiplier \(M\) is a solution of the following linear partial differential equation:
\[
\sum_{i=1}^{n} \frac{\partial (Ma_i)}{\partial x_i} = 0;
\]  

(2.5)

vice versa every solution \(M\) of this equation is a Jacobi last multiplier.

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If one knows two Jacobi last multipliers $M_1$ and $M_2$ of equation (2.1), then their ratio is a solution $\omega$ of equation (2.1), or, equivalently, a first integral of equation (2.2). Naturally, the ratio may be quite trivial, namely a constant. Vice versa, the product of a multiplier $M_1$ and any solution $\omega$ yields another last multiplier, $M_2 = M_1 \omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries was provided by Lie (1874, 1912). A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi (1918). If we know $n - 1$ symmetries of equations (2.1)/(2.2), say

$$
\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \ldots, x_n) \partial_{x_j}, \quad i = 1, n - 1,
$$

(2.6)
a Jacobi last multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$
\Delta = \det \begin{bmatrix}
  a_1 & \cdots & a_n \\
  \xi_{1,1} & \cdots & \xi_{1,n} \\
  \vdots & & \vdots \\
  \xi_{n-1,1} & \cdots & \xi_{n-1,n}
\end{bmatrix}.
$$

(2.7)

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. In particular, if each component of the vector field of the equation of motion has the variable associated with that component absent, i.e. $\partial a_i / \partial x_i = 0$, the last multiplier is a constant and any other Jacobi last multiplier is a first integral.

Another property of the Jacobi last multiplier is its (almost forgotten) relationship with the Lagrangian, $L = L(t, u, u')$, for any second-order equation

$$
u'' = F(t, u, u'),
$$

(2.8)
namely (Whittaker 1999)

$$
M = \frac{\partial^2 L}{\partial u'^2},
$$

(2.9)

where $M = M(t, u, u')$ is a Jacobi last multiplier of equation (2.8) that satisfies the following equation:

$$
\frac{dM}{dt} + M \frac{\partial F}{\partial u'} = 0.
$$

(2.10)

Then equation (2.8) becomes the Euler–Lagrange equation:

$$
- \frac{d}{dt} \left( \frac{\partial L}{\partial u'} \right) + \frac{\partial L}{\partial u} = 0.
$$

(2.11)

The proof is given by taking the derivative of equation (2.11) by $u'$ and showing that this yields equation (2.10) (Whittaker 1999). Thus, if one knows a Jacobi
last multiplier, then $L$ can be obtained by a double integration, i.e.

$$L = \int \left( \int M \, du' \right) \, du' + f_1(t, u) \, u' + f_2(t, u),$$

(2.12)

where $f_1$ and $f_2$ are functions of $t$ and $u$ that have to satisfy a single partial differential equation related to equation (2.8) (Nucci & Leach 2008). As was shown in Nucci & Leach (2008), $f_1, f_2$ are related to the gauge function $g = g(t, u, u')$. In fact, we may assume

$$\left\{ \begin{array}{l}
f_1 = \frac{\partial g}{\partial u} \\
f_2 = \frac{\partial g}{\partial t} + f_3(t, u),
\end{array} \right.$$  

(2.13)

where $f_3$ has to satisfy the mentioned partial differential equation and $g$ is obviously arbitrary. The importance of the gauge function should be stressed.

In order to apply Noether’s theorem correctly, one should not assume $g \equiv \text{const}$; otherwise, some first integrals may not be found (see Nucci & Leach (2008) and the second-order equation in §3).

We now consider a fourth-order equation (1.1). In this case, the Jacobi last multiplier satisfies the following equation:

$$\frac{d}{dt} \log M + \frac{\partial F}{\partial u'''} = 0.$$  

(2.14)

We assume that Fels’ conditions (1.2) and (1.3) are satisfied, and hence the Lagrangian of equation (1.1) exists. If the Lagrangian $L = L(t, u, u', u'')$ is taken such that

$$M^{1/2} = \frac{\partial^2 L}{\partial u'''^2},$$

(2.15)

then equation (1.1) becomes the Euler–Lagrange equation

$$\frac{d^2}{dt^2} \left( \frac{\partial L}{\partial u''} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial u'} \right) + \frac{\partial L}{\partial u} = 0.$$  

(2.16)

The proof is given by taking the derivative of equation (2.16) by $u'''$ and showing that this yields equation (2.14), i.e.

$$\frac{\partial F}{\partial u'''} \frac{\partial^2 L}{\partial u''^2} + 2 \frac{\partial^3 L}{\partial u u''^2} u' + 2 \frac{\partial^3 L}{\partial u' u''} u'' + 2 \frac{\partial^3 L}{\partial u'''^3} u''' + 2 \frac{\partial^3 L}{\partial u'' u''} = 0.$$  

(2.17)

Therefore, $L$ can be obtained by a double integration, i.e.

$$L = \int \left( \int \sqrt{M} \, du'' \right) \, du'' + f_1(t, u, u') \, u'' + f_2(t, u, u'),$$

(2.18)

where $f_1$ and $f_2$ are functions of $t, u, u'$ that have to satisfy some partial differential equations related to equation (1.1). We can relate $f_1$ and $f_2$ to the gauge function,
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\[ g = g(t, u, u'). \] In fact, we may assume

\[
\begin{align*}
   f_1 &= \frac{\partial g}{\partial u'} \\
   f_2 &= \frac{\partial g}{\partial u} + \frac{\partial g}{\partial t} + f_3(t, u, u'), 
\end{align*}
\]

where \( f_3 \) has to satisfy the mentioned partial differential equations and \( g \) is obviously arbitrary. Again we stress the importance of the gauge function. In order to apply Noether’s theorem correctly, one should not assume \( g \equiv \text{const} \), otherwise, some first integrals may not be found as in the examples given in §3.

We have also looked at higher order equations, and from our findings, we can show that if a Lagrangian, \( L = L(t, u, u', \ldots, u^{(n)}) \) exists for an equation of order \( 2n \)

\[ u^{(2n)} = F(t, u, u', \ldots, u^{(2n-1)}), \tag{2.20} \]

then it can be derived from the following formula:

\[ M^{1/n} = \frac{\partial^2 L}{\partial (u^{(n)})^2}, \tag{2.21} \]

where \( M \) is the Jacobi last multiplier of equation (2.20), i.e. \( M \) satisfies

\[ \frac{d \log M}{dt} + \frac{\partial F}{\partial u^{(2n-1)}} = 0. \tag{2.22} \]

Thus, equation (2.20) becomes the Euler–Lagrange equation

\[ \sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k} \left( \frac{\partial L}{\partial u^{(k)}} \right) = 0. \tag{2.23} \]

The proof is given by taking the derivative of equation (2.23) by \( u^{(2n-1)} \) and showing that this yields equation (2.22). Conditions for the existence of a Lagrangian in the cases \( n = 3 \) and \( n = 4 \) have been given in Juráš (2001).

3. Two examples from number theory

In this section, we show in detail the application of the Jacobi last multiplier along with its connection to Lie symmetries. We apply the method of the Jacobi last multiplier to two equations derived by Hall (2002).

(a) A second-order equation

In Hall (2002), the following functional was introduced:

\[ \int_{0}^{\pi} y'^4 + 6\nu y^2 y'^2 \, dx, \tag{3.1} \]

where \( y = y(x) \in C^2[0, \pi], \ y(0) = y(\pi) = 0 \) and \( \nu \geq 0 \). The corresponding Euler–Lagrange equation is

\[ y'^2 y'' + \nu y^2 y'' + \nu y y'^2 = 0. \tag{3.2} \]
If we apply Lie group analysis to this equation, we find, by using ad hoc REDUCE programs (Nucci 1996), that it admits a two-dimensional Abelian transitive Lie symmetry algebra (type I) generated by the following two operators:

$$\Gamma_1 = \partial_x \quad \text{and} \quad \Gamma_2 = y \partial_y.$$  \hspace{1cm} (3.3)

Then we can integrate equation (3.2). First, we introduce a basis of differential invariants of $\Gamma_1$, i.e.

$$u = \frac{y'}{y} \quad \text{and} \quad v = x.$$  \hspace{1cm} (3.4)

Then equation (3.2) reduces to the following first-order equation:

$$\frac{du}{dv} = \frac{-nuv}{nv^2 + u^2},$$  \hspace{1cm} (3.5)

which admits the operator $\Gamma_2$ in the space of variables $u, v$, i.e.

$$\Gamma_2 = v \partial_v + u \partial_u.$$  \hspace{1cm} (3.6)

Then its general solution is implicitly given by

$$\sqrt{u}(2nv^2 + u^2)^{1/4} = \text{const.},$$  \hspace{1cm} (3.7)

and in the original variables$^3$

$$\sqrt{y'(2vy^2 + y'^2)^{1/4} = \text{const.}}$$  \hspace{1cm} (3.8)

viz.

$$y' = \frac{\sqrt{(vy^2 a_1 + \sqrt{v^2 y^4 a_1^2 + 1}) y^2 a_1}}{ya_1 (vy^2 a_1 + \sqrt{v^2 y^4 a_1^2 + 1})}$$  \hspace{1cm} (3.9)

with $a_1$ an arbitrary constant. Finally, the general solution of equation (3.2) is given implicitly by

$$\int \frac{ya_1 (vy^2 a_1 + \sqrt{v^2 y^4 a_1^2 + 1})}{\sqrt{(vy^2 a_1 + \sqrt{v^2 y^4 a_1^2 + 1}) y^2 a_1}} dy = x + a_2.$$  \hspace{1cm} (3.10)

We note that if $y$ is positive and $a_1 = 1$, the integral on the left-hand side could be integrated in terms of a hypergeometric function $\mathcal{H}$, namely

$$\frac{1}{2} \sqrt{2vy^2} \mathcal{H}\left(\left[\begin{array}{ccc} -\frac{1}{2}, 1, & -\frac{1}{4} \\ \frac{1}{2}, 1, & 1 \end{array}\right]; \frac{1}{y^4 v^2} \right)$$  \hspace{1cm} (3.11)

$^3$The same first integral can be obtained by using the Noether’s theorem (see below).
Let us try to find a Lagrangian for equation (3.2) by using the Jacobi last multiplier, namely through equation (2.9). The two Lie point symmetries (3.3) yield a Jacobi last multiplier. In fact, the following matrix (Lie 1874, 1912)

\[
\begin{pmatrix}
1 & y' - \frac{vyy'^2}{y} & y^2 + y'^2 \\
1 & 0 & 0 \\
0 & y & y'
\end{pmatrix}
\]

(3.12)

has determinant different from zero and its inverse is a Jacobi last multiplier, i.e.

\[
M_1 = -\frac{2vyy^2 + y'^2}{y^2(vyy^2 + y'^2)}.
\]

(3.13)

The corresponding Lagrangian is

\[
L_1 = \frac{1}{4vy} \left( -\sqrt{2vy'} \arctan \left( \frac{y'}{\sqrt{2vy}} \right) + \log(2vyy^2 + 2y'^2)v + 2 \log(y'vv) \right) + f_1(x, y)y' + f_2(x, y),
\]

(3.14)

where \( f_1 \) and \( f_2 \) are solutions of

\[
\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} = 0.
\]

(3.15)

If we impose the link between \( f_1 \) and \( f_2 \) with the gauge function \( g(x, y) \), namely equation (2.13), then \( f_3(x, y) \) becomes just \( f_3(x) \), an arbitrary function of the independent variable \( x \). The Lagrangian (3.14) may appear ungainly. Nevertheless, the corresponding variational problem admits two Noether symmetries; namely both Lie symmetries given in equation (3.3) are Noether symmetries. Consequently, the following two first integrals of equation (3.2) can be found by applying Noether’s theorem:\footnote{Also the corresponding gauge function \( g \) is given. It is important to remark that in the case of the first integral (3.17), the gauge function \( g \) cannot be constant, while it can be a constant in the case of the first integral (3.16). Naturally, we have left out any inessential additive constants.}

\[
\Gamma_1 \Rightarrow I_1 = (2vy^2 + y'^2)y^2 \quad \text{and} \quad \left[ g = e^x \left( \int \frac{f_3(x)}{e^x} \, dx + a_2 \right) \right],
\]

(3.16)

and

\[
\Gamma_2 \Rightarrow I_2 = \frac{1}{4vy'} \left( -\sqrt{2vy'} \arctan \left( \frac{y'}{\sqrt{2vy}} \right) + 2vy - 4vy' \right) \quad \text{and} \quad \left[ g = s(x)y + x \right].
\]

(3.17)

with \( s(x) \) an arbitrary function of \( x \). We note that the first integral \( I_1 \) in (3.16) was already derived in (3.8).

At this point, one would like to know if it is possible to obtain the original Lagrangian given in equation (3.1), i.e.

\[
L_H = y'^4 + 6vy^2y'^2.
\]

(3.18)
A property of the Jacobi last multiplier is that if one knows a Jacobi last multiplier and a first integral, then their product gives another multiplier (Nucci 2005). If we take the product of the first integral $I_1$ (3.16) and the multiplier $M_1$ (3.13), then we obtain another Jacobi last multiplier of equation (3.2), i.e.

$$M_2 = -y'^2 - \nu y^2,$$

which can be integrated twice with respect to $y'$ in order to yield the following Lagrangian:

$$L_2 = -\frac{1}{12}(y'^4 + 6\nu y^2 y'^2) + f_1 y' + f_2,$$

where $f_1$ and $f_2$ are solutions of equation (3.15). It is interesting to emphasize that this Lagrangian (namely Hall’s Lagrangian) is such that the Lie operator $G_2$ in (3.1) does not generate a Noether symmetry for the corresponding variational problem. In fact, only $G_1$ is the generator of a Noether symmetry for Hall’s Lagrangian.

(b) A fourth-order equation

Another functional in the study of Hall (2002) is the following:

$$\int_0^\pi y'^4 + \mu y^2 y'^2 \, dx.$$ (3.21)

The corresponding Euler–Lagrange equation is

$$4\mu yy'y''' + \mu y^2 y'' + 2\mu y'^2 y'' + 3\mu yy''^2 - 6y^2 y''' = 0.$$ (3.22)

If we apply Lie group analysis to this equation, we find that it admits a three-dimensional Lie symmetry algebra generated by the following three operators:

$$X_1 = \partial_x, \quad X_2 = y \partial_y \quad \text{and} \quad X_3 = x \partial_x,$$ (3.23)

which means that we can reduce equation (3.22) to a first-order equation, i.e.

$$\frac{d\tilde{u}}{d\tilde{x}} = \frac{-7\mu \tilde{u} \tilde{x} - 6\mu \tilde{x}^3 - 4\mu \tilde{x}^2 + 6\tilde{x}}{\mu \tilde{u}},$$ (3.24)

with

$$\tilde{u} = \frac{y'' y^2}{y'^3} - 2\frac{y'^2 y^2}{y'^4} + \frac{y'' y}{y'^2} \quad \text{and} \quad \tilde{x} = \frac{y'' y}{y'^2}. $$ (3.25)

If $\mu = 3$, then equation (3.22) admits an eight-dimensional Lie symmetry algebra $\mathcal{L}$ generated by the following eight operators:

$$A_1 = x^2 \partial_x + \frac{3}{2} xy \partial_y, \quad A_2 = x \partial_x, \quad A_3 = \partial_x, \quad A_4 = y \partial_y, \quad A_5 = \frac{x^3}{y} \partial_y,$$

$$A_6 = \frac{x^2}{y} \partial_y, \quad A_7 = \frac{x}{y} \partial_y \quad \text{and} \quad A_8 = \frac{1}{y} \partial_y.$$ (3.26)

$^5$Note the inessential multiplicative constant.
This means that equation (3.22), i.e.

\[ 4y' y''' + yy'' + 3y''^2 = 0, \]  

(3.27)
is linearizable by means of a point transformation (Lie 1912). In order to find the linearizable transformation, we have to find an Abelian intransitive two-dimensional subalgebra of \( \mathcal{L} \) and, following Lie’s classification of two-dimensional algebras in the real plane (Lie 1912), we have to transform it into the canonical form

\[
\partial_u \quad \text{and} \quad t \partial_u, 
\]  

(3.28)

with \( u \) and \( t \) being the new dependent and independent variables, respectively. We found that one such subalgebra is that generated by \( A_7 \) and \( A_8 \). Then, we have to solve the following four linear partial differential equations of first order:

\[
A_7(t) = 0, \quad A_8(t) = 0, \quad A_7(u) = t \quad \text{and} \quad A_8(u) = 1. 
\]  

(3.29)

It is readily shown that the linearizable transformation is

\[
t = x \quad \text{and} \quad u = y^2, 
\]  

(3.30)

and equation (3.27) becomes

\[ u'' = 0. \]  

(3.31)

Finally, the general solution of equation (3.27) is

\[
y = \sqrt{a_1 + a_2 x + a_3 x^2 + a_4 x^3}, \]  

(3.32)

with \( a_i (i = 1, 4) \) being the arbitrary constants.

We note that if we apply the transformation (3.30) to equation (3.22) in the case of any \( \mu \), then the following equation is obtained:

\[
\frac{d}{dx} \ln M - \frac{4y'}{y} = 0 \quad \Rightarrow \quad M = y^4, 
\]  

(3.35)
which yields the following Lagrangian for equation (3.22):

\[ L = \frac{1}{2\mu} (\mu y'^2 + y^4) + \frac{dg}{dx}. \]  

(3.36)

This, apart from an inessential multiplicative constant and the presence of the gauge function \( g = g(x, y, y') \), is Hall’s Lagrangian in (3.21).

If we consider the Lagrangian (3.36) and apply Noether’s theorem, we find that the following two first integrals of equation (3.22) can be obtained:

\[ \frac{3}{4} X_3 + X_1 \Rightarrow I_1 = 3\mu u^3 u''' + 7\mu u^2 u' u'' - 4\mu u^2 u' u'' x + 2\mu u^2 u'' x \]
\[ - 8\mu u u'^2 u'' x - 6uu'^3 + 6u^4 x, \]
\[ X_2 \Rightarrow I_2 = -2\mu u^2 u' u''' + \mu u^2 u'' - 4\mu uu'^2 u'' + 3u^4. \]  

(3.37)

Moreover, if we apply the Noether theorem to the linearizable equation (3.27), which has Lagrangian (3.36), i.e.

\[ L_H = 3y^2 y'^2 + y^4, \]  

(3.38)

we obtain the following seven first integrals:

\[ A_1 \Rightarrow \text{Im}_1 = -3y^3 y''' + 3y^3 y'' x + y^2 y'^2 + 7y^2 y'' x \]
\[ - 2y^2 y' y'' x^2 + y^2 y'' x^2 - 2yy'^2 x - 4yy'^2 x^2 + y^4 x^2, \]
\[ \frac{3}{4} A_4 + A_2 \Rightarrow \text{Im}_2 = 9y^3 y''' + 21y^2 y' y'' - 12y^2 y' y'' x + 6y^2 y'' x \]
\[ - 6yy'^3 - 24yy'^2 y' x + 6y^4 x, \]
\[ A_3 \Rightarrow \text{Im}_3 = -2y^2 y' y''' + 2y'^2 y'' - 4yy'^2 y'' + y^4, \]
\[ A_5 \Rightarrow \text{Im}_5 = -3y^2 + 6yy' x - 3yy'' x^2 + yy''' x^3 - 3y^2 x^2 + 3y' y'' x^3, \]
\[ A_6 \Rightarrow \text{Im}_6 = 2yy' - 2yy'' x + yy''' x^2 - 2y'^2 x + 3y' y'' x, \]
\[ A_7 \Rightarrow \text{Im}_7 = -yy'' + yy''' x - y'^2 + 3y' y'' x \]
\[ A_8 \Rightarrow \text{Im}_8 = yy''' + 3y'y''. \]  

(3.39)

Although we do not write down the corresponding expressions of the gauge function, we emphasize that it cannot always be set equal to a constant; otherwise, none of \( \text{Im}_1, \text{Im}_5, \text{Im}_6, \text{Im}_7 \), and \( \text{Im}_8 \) could be obtained.

All seven first integrals (and even more) may be obtained without Noether’s theorem. In fact, we just need to find the Jacobi last multipliers of equation (3.27) that are obtained by inverting the non-zero determinants of the 70 possible matrices made out of the eight Lie symmetries (3.26). Then the ratio of any two

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\(^6\)We have multiplied the Lagrangian (3.36) by 6 and substituted \( \mu = 3 \).

\(^7\)Naturally, they are not all independent of each other.
multipliers is a first integral of equation (3.27). For example

\[
C_{1234} = \begin{pmatrix}
1 & y' & y'' & y''' & -4y'y''' + 3y''^2 \\
x^2 & \frac{3}{2}yx & \frac{3}{2}y - \frac{1}{2}xy' & y' - \frac{5}{2}xy'' & -\frac{3}{2}y'' - \frac{9}{2}xy'' \\
x & 0 & -y' & -2y'' & -3y'' \\
1 & 0 & 0 & 0 & 0 \\
0 & y & y' & y'' & y'''
\end{pmatrix},
\]

is the matrix obtained by considering the symmetries generated by operators \(A_1, A_2, A_3\) and \(A_4\) in (3.26); its determinant is

\[
\Delta_{1234} = 9y''yy'y''' - 2y'^3y''' - \frac{3}{2}y'^2y''^2 + 6y''^3y + \frac{9}{2}y''^2y^2,
\]

and the corresponding Jacobi last multiplier is

\[
M_{1234} = \frac{1}{\Delta_{1234}} = \frac{1}{18y'y'y''' - 4y'^3y''' - 3y'^2y''^2 + 12y''^3y + 9y''^2y^2}.
\]

Similarly, the matrix \(C_{5678}\) yields the determinant \(\Delta_{5678} = 12/y^4\), i.e. the Jacobi last multiplier,\(^8\)

\[
M_{5678} = y^4,
\]

which we have already found in equation (3.35) as an obvious solution of equation (2.14). Note that

\[
\frac{M_{5678}}{M_{1234}} = y^4(18y'y'y''' - 4y'^3y''' - 3y'^2y''^2 + 12y''^3y + 9y''^2y^2)
\]

is ‘another’ first integral of equation (3.27).

### 4. Two fourth-order equations from physics

A similarity reduction of the fifth-order Korteweg–De Vries equation is the following fourth-order equation (Littlewood 1999; Cosgrove 2000):

\[
u^{iv} = 20uu'' + 10u'^2 - 40u^3 + \alpha u + t.
\]

This equation satisfies Fels’ conditions (1.2) and (1.3) and \(M = 1\) is obviously a Jacobi last multiplier. Thus, formula (2.15) yields the following Lagrangian of equation (4.1):

\[
L = \frac{1}{2}(u'^2 + t^2u') + \alpha uu' - 40tu^3u' + 10uu'^2 + \frac{d}{dt}g(t, u, u'),
\]

where \(g = g(t, u, u')\) is the gauge function.

\(^8\)The multiplicative constant is inessential.
In a paper on wave caustics, Kitaev (1994) derived the following equation:

\[ u'' = \frac{1}{8u^2} (16 uu' u''' - 12\alpha u^5 + 8\beta u^3 - 20u^7 + 40u^4 u'' + 8u^4 t + 20u^3 u'^2 - 8uu') \]

\[ + 12uu'^2 - 8uu'' t + uy^2 - 16u^2 u'' + 8u'^2 t), \quad (4.3) \]

which satisfies Fels’ conditions (1.2) and (1.3), and admits \( M = u^{-2} \) as a Jacobi last multiplier. Therefore, formula (2.15) yields the following Lagrangian for equation (4.3), i.e.

\[ L = \frac{1}{2u} u'^2 + \frac{1}{8u^2} (20u^3 u'^2 - 4tu'^2 + 4\alpha u^5 - 8\beta u^3 + 4u^7 - 4tu^4) + \frac{d}{dt} g(t, u, u'), \quad (4.4) \]

where \( g = g(t, u, u') \) is the gauge function.

### 5. Final points

When one deals with a second-order differential equation, the following remarks should be kept in mind:

— The most efficient Lagrangian, namely that which allows the largest number of Noether symmetries, may not be the Lagrangian with the simplest form.

— Lie symmetries are the key tool for finding Jacobi last multipliers and therefore Lagrangians.

In Fels (1996), the necessary and sufficient conditions under which a fourth-order equation (1.1) admits a unique Lagrangian were determined. In this paper, we show that the Jacobi last multiplier yields that unique Lagrangian.

Of course, there are many fourth-order equations that do not satisfy Fels’ conditions. For example, in Martini et al. (2009), seven fourth-order equations were derived as similarity reductions of a mathematical model for thin liquid films (Ruckenstein & Jain 1973) and its corresponding heir equations (Nucci 1994). None of the seven equations satisfies condition (1.3), although they all satisfy condition (1.2).

Nevertheless, if one rewrites a fourth-order equation as either a suitable system of two second-order equations (Douglas 1941), or a system of four first-order equations (Kerner 1971; Havas 1973), the challenge of solving the inverse problem of calculus of variations may still be open.

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