On the normal phase of 2D Fermi liquid with weak attraction between particles

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Abstract

Proceeding from the simplest field theoretical model of 2D metal, the normal phase Green functions of the weakly interacting fermions and the order parameter fluctuations (responsible for the attraction between fermions) are obtained. It is shown that taking into consideration the fluctuations mentioned leads to a considerable reduction of the fermion wave function renormalization constant (quasiparticle weight) as well as to a linear dependence of the quasiparticle damping on the temperature. A general dependence of 2D Fermi liquid properties on the fermion density is discussed. The relevance of the proposed model to the marginal behavior of the Fermi liquid of high-$T_c$ superconductors, in particular, to their linear temperature dependence of the resistivity is indicated.

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1 Introduction

Theoretical study of the normal properties of 2D (as well as quasi–2D) metals was strongly intensified by the discovery of high–$T_c$ superconductors (HTSCs). These compounds display a number of nonconventional features that make it difficult to use the Fermi liquid (FL) model to describe their main properties. The most striking features are: (i) an essentially reduced quasiparticle weight (QW), $Z$, of the excitations in the vicinity of the Fermi surface; (ii) a rather large damping of the corresponding quasiparticles; (iii) a linear dependence of the resistivity on the temperature above $T_c$, and many others (see, for example [1]). Now, as far as we know, there is no satisfactory understanding as to what makes these compounds behave so strangely\(^\text{1}\) and why they cannot be described by the theory of Landau FL. Thus, searching for theoretical models, being able to explain at least some of those odd properties of HTSCs, remains a very important problem. In this paper we shall make an attempt to show that one of the possible answers is that the observable properties of HTSCs can be the result of the extreme susceptibility of 2D attractive interaction FL with respect to a deep intrinsic structural rearrangement.

We recall that the most characteristic property of the ideal Fermi gas is the existence of the Fermi energy, $\epsilon_F$, which, in accordance with the Pauli principle, separates occupied and empty states. The distribution of particles over the energy (and momenta) at $T = 0$ can be represented by the step–function. Adding to the system a weak interaction between the particles corresponds to the transition from the ideal Fermi gas to the FL. When the interaction is weak and there is no danger of any kind of instability, the distribution function changes only a little bit: the QW slightly decreases ($Z \approx 1$) and a weak damping, $\gamma \sim (\epsilon - \epsilon_F)^2$, appears [2, 3]. In other words, despite the interaction between fermions, the Fermi surface, determined by the “jump” in the distribution function, does exist in the system (this is the direct manifestation of the famous Migdal theorem).

The situation changes drastically at lower dimensions and/or in the case of a very strong interaction between fermions. As for the low dimensions, the most convincing example is the 1D conductor, where even the weakest electron–electron interaction causes the disappearance of the jump in the distribution function [4]. Such a FL (called so Luttinger), strictly speaking cannot be referred to as FL since all its excitations belong to those of the bose–type [4, 5, 6].

The example, demonstrating the influence of an interaction, is the strongly correlated Hubbard model. On–site repulsion, $U$, in this model acts on all particles in the system in contrast to the case of a generally accepted FL where the interaction is actual only in the small vicinity of the Fermi surface. As to the result, such a repulsion leads to a considerable reduction of the QW. When the repulsion increases — the QW decreases [7, 8] and in the limiting case $U \to \infty$, the Fermi surface disappears completely ($Z = 0$) at any fixed density of fermions [9, 10].

As for the HTSCs, according to much experimental data, neither Landau no Luttinger FL can explicate them. What is interpreted as QW in the case of HTSCs is relatively small ($Z \approx 0.4 – 0.6$ [11, 12]), so the corresponding FL could not be considered as Landau FL.\(^\text{1}\)

\(^1\)Sometimes they are called the ”strange metals” [1].
On the other hand, it is not Luttinger FL since the QW is not small enough. In addition, quasiparticles themselves are less stable than in the case of ordinary Landau FL. Actually, the damping of them in a close vicinity of the Fermi surface at $T = 0$ looks like $\gamma \sim |\epsilon - \epsilon_F|^\alpha$, with $\alpha \approx 1$ \[12, 13\]; this differs from the usual FL power law with $\alpha = 2$. Such a FL with these exceptional characteristics and a qualitatively different “intermediate” behavior is now called “marginal FL” \[8, 14, 15, 16\].

If one assumes that the marginal FL is relevant to HTSCs, then a sufficient number of observable physical properties of HTSCs can be quite naturally explained. Up to the present, however, the latter has been achieved only in the framework of some phenomenological theory \[14\], containing a lot of additional assumptions, the physical meaning of which has yet to be clarified.

In what follows we consider the simplest 2D field theoretical model in the normal phase with a weak inter–fermion attraction (its nature is not specified here). As will be seen, the Green function of the order parameter fluctuations (OPF), responsible for the attraction between fermions in our model, turns out to be consistent with the very general requirements of the phenomenological theory \[14\], being capable of explaining many HTSC properties. Here we should note, however, that in the cited paper the interaction between fermions put down due to the exchange by charge or spin fluctuations. Though it is crucial for the interpretation of some experimental data, as well as for the calculation of the Green function itself, the latter does not play any important role. Therefore, below we shall restrict ourselves to studying only those properties of HTSCs which can be extracted from the Green function of interacting fermions. In order to be able to consider the other properties, as in Ref. \[14\], we would need to calculate corresponding charge and spin correlation functions. But that, in fact, is beyond the scope of this paper.

Below it will be shown that taking into account the interaction mediated by the OPF leads to a considerable reduction of the QW even at a very weak attraction between fermions. This, we believe, is the result of dimensionality lowering, what is accompanied by the enhancement of the role of any attractive interaction. Investigating the dependence of the QW on the density of fermions, we shall come to a quite natural conclusion that decreasing the density leads to decreasing the QW and increasing the damping. Note that just these two features: (i) a quasi–2D character and (ii) a relatively low density of carriers, are as a rule emphasized when properties of HTSCs are discussed \[1\].

Lastly, maybe the most interesting result of our paper is a linear temperature dependence of quasiparticle damping. It is just this feature that was in fact \textit{a priori} accepted in the phenomenological description \[14\] for explaining the surprise linear temperature dependence of resistivity observed in high–$T_c$ compounds.

Throughout the paper we use units in which $\hbar = 1$ and $k_B = 1$.  

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2 Model and general discussion

To solve the problem analytically as far as possible, we choose the simplest field theoretical model of fermions with attraction. The corresponding Hamiltonian is [3]:

\[ \hat{H} = \int d^2r \hat{\mathcal{H}}(r), \]

\[ \hat{\mathcal{H}}(r) = -\psi_\sigma^\dagger \left( \frac{\nabla^2}{2m} + \mu \right) \psi_\sigma - \frac{g}{2}(1 - \delta_{\sigma\sigma_1})\psi_\sigma^\dagger \psi_{\sigma_1} \psi_{\sigma_1}^\dagger \psi_\sigma, \] (2)

Here we use the standard notations: \( m \) — for the effective fermion mass, \( \mu \) — for the chemical potential, \( \sigma = \uparrow \) or \( \downarrow \) denotes a spin variable. The interacting constant, \( g \), is positive what corresponds to the attraction between fermions.

Introducing Nambu spinors for fermion field \( \Psi^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow) \), one can rewrite Eq.(2) in a more convenient form:

\[ \hat{\mathcal{H}}(r) = -\Psi^\dagger \tau_z \left( \frac{\nabla^2}{2m} + \mu \right) \Psi - g\Psi^\dagger \tau_+ \Psi \Psi^\dagger \tau_- \Psi, \]

(3)

where \( \tau_z, \tau_+ \equiv (\tau_x + i\tau_y)/2, \tau_- \equiv (\tau_x - i\tau_y)/2 \) are Pauli matrices.

Below we shall use the functional integral approach along with the Matsubara thermal technique. Thus, the partition function is expressed through the Hamiltonian as:

\[ Z(T) = \int[d\Psi^\dagger d\Psi] \exp \left[ -\int_0^\beta d\tau \int d^2r \left( \Psi^\dagger \partial_\tau \Psi + \hat{\mathcal{H}}(r) \right) \right], \]

(4)

where \( T \) is the temperature and \( [d\Psi^\dagger d\Psi] \) denotes the measure of the functional integration over the Grassmann variables \( \Psi \) and \( \Psi^\dagger \), satisfying the antiperiodic boundary conditions:

\( \Psi(\tau; r) = -\Psi(\tau + \beta; r) \) and \( \Psi^\dagger(\tau; r) = -\Psi^\dagger(\tau + \beta; r) \).

To be able to construct arbitrary Green functions one should add to (4) classical sources associated with the Grassmann fields in the Hamiltonian (3). However, we shall not write them down explicitly, even though it would facilitate the understanding of some key formulae below.

Using now an auxiliary Hubbard–Stratonovich complex scalar field in the usual way, one can represent the Eq.(3) in an equivalent form:

\[ Z(T) = \int[d\Psi^\dagger d\Psi d\Phi d\Phi^*] \exp \left[ -\int_0^\beta d\tau \int d^2r \left( \frac{|\Phi|^2}{g} + \Psi^\dagger \left[ \partial_\tau - \tau_z \left( \frac{\nabla^2}{2m} + \mu \right) + \tau_- \Phi + \tau_+ \Phi^* \right] \Psi \right) \right]. \]

(5)

The main virtue of this representation is a nonperturbative introduction of the composite fields: \( \Phi = -g\Psi^\dagger \tau_+ \Psi \) and \( \Phi^* = -g\Psi^\dagger \tau_+ \Psi \) and possibility to develop the consistent approach. Specifically, the expression (5) turns out to be rather convenient for studying such
a nonperturbative phenomenon as, for example, superconductivity. In this case a complex Hubbard–Stratonovich field naturally describes the order parameter arising due to formation of pairs (to some extent, the Cooper ones), or composite bosons. The average value of $|\Phi|$ is proportional to the density of these bosons, on one hand, and determines the gap in the one–particle fermi–spectrum, on the other. In the normal phase the homogeneous order parameter disappears, i.e. $|\Phi|$ equals zero, but the OPF inevitably take place even there. Intuition, however, suggests that these fluctuations should play the crucial role in a small neighbourhood of the corresponding phase transition. As it will be shown, this is in fact the case in our model. Moreover, OPF not only exist but also have an influence on different characteristics of normal phase on the whole.

Before going further, it may help to recall the most essential property of 2D systems. The question is the famous Mermin–Wagner–Hohenberg (MWH) theorem which forbids the appearance of a complex (in our case — charged) order parameter $\Phi$ at any finite temperature \[17,18\]. If we studied the problem in the mean field approximation we would obtain a finite critical temperature $T_c$, for instance, of the superconducting phase transition. Taking into account the next to leading order approximation immediately reduces $T_c$ to zero. Such a manifestation of the MWH theorem may make the study of pure 2D superconductors at $T \neq 0$ meaningless. However, as was shown recently, the situation here is much more subtle. Actually, 2D metals undergo two phase transitions \[19\]. The first one is connected with violating only some discrete symmetry and the appearance of a neutral order parameter (modulus of the Hubbard–Stratonovich field, $|\Phi|$). It is this transition that was usually interpreted in the mean field approximation as the appearance of superconductivity. Despite this modulus being different from zero, the field $\Phi$ on the average is zero, due to the temporal and spatial fluctuations of its phase factor. We would call such a state of the system by the term “molecular gas”, emphasizing by this notion that there is no phase coherence in the “Cooper pair” dynamics.\textsuperscript{2} The second phase transition is connected with the appearance of the so called Berezinsky–Kosterlitz–Thouless (BKT) phase \[21,22\]. Developing of such a phase is just accompanied also by the superconductivity \[19\].

In what follows we shall study only the normal phase of 2D metal, i.e. the phase where even the neutral homogeneous order parameter is absent. We believe that precisely this phase of pure 2D model represents the true normal phase of quasi–2D HTSCs more adequately.

Below we shall use a generally accepted assumption that the approximation taking into account only the quadratic (Gauss) fluctuations of $\Phi$ (around zero in our normal phase) describes the system quite well. Thus, integrating out the fermion fields in (5), one obtains:

\[
Z(T) = \exp(-S_0) \int [d\tilde{\Phi} d\tilde{\Phi}^*] \exp \left[ - \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \right.
\]
\[
\times \int d^2r \int d^2r' \tilde{\Phi}^*(\tau_1; r_1) \Gamma^{-1}(\tau_1 - \tau_2; r_1 - r_2) \tilde{\Phi}(\tau_2; r_2) \right],
\]

\[6\]

\textsuperscript{2}It must be noted that in Ref. \[20\] there was introduced the temperature $T_p$ which characterized the energy scale of composite boson formation.
where
\[ S_0 = TrLn[G_0] \] (7)
is the one–loop effective action, and
\[ \Gamma^{-1}(\tau; r) = \frac{1}{g} \delta(\tau) \delta(r) + tr \left[ G_0(\tau; r) \tau \tau_0 G_0(-\tau; -r) \tau \right] \] (8)
is the inverse Green function (propagator) for OPF. Both quantities (7) and (8) are expressed through the free fermion Green function which obeys the equation:
\[ \left[ -\partial_\tau + \tau_z \left( \frac{\nabla^2}{2m} + \mu \right) \right] G_0(\tau; r) = \delta(\tau) \delta(r) \] (9)with the following boundary condition:
\[ G_0(\tau + \beta; r) = -G_0(\tau; r). \] (10)
The integration in (6) can be formally performed. The result is
\[ Z(T) = \exp(-S_0 - TrLn[\Gamma^{-1}]). \] (11)
If we could have obtained the explicit expression for the partition function we could, in principle, obtain all thermodynamical functions but in reality it is usually impossible to do this. Therefore, we shall study more simple objects, namely, the Green functions which can also give the information about the system.

To avoid possible further misunderstanding, we shall write down the formulae which are used throughout the paper for Fourier transformations, connecting coordinate and momentum representations:
\[ F(i\omega_n, k) = \int_0^\beta d\tau \int d^2r F(\tau, r) \exp(i\omega_n \tau - ikr), \] (12)
\[ F(\tau, r) = T \sum_{n=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} F(i\omega_n, k) \exp(-i\omega_n \tau + ikr), \] (13)
where \( \omega_n = \pi T(2n+1) \) are fermion (odd) Matsubara frequencies. In the case of bosons they should be replaced by even ones: \( \Omega_n = 2\pi Tn. \)

3 Green function of OPF

The Green function for free fermions in the momentum representation (12) takes, as follows from (11), the very simple form:
\[ G_0(i\omega_n, k) = \frac{1}{i\omega_n - \tau_z \left( \frac{k^2}{2m} + \mu \right)}. \] (14)
Substitution of this expression into (8) leads to the explicit formula for the inverse Green function for OPF in the momentum representation:

$$\Gamma^{-1}(i\Omega_n; K) = \frac{1}{g} - \frac{1}{8\pi^2} \int d^2k \frac{\tanh(\beta \xi_+/2) + \tanh(\beta \xi_-/2)}{\xi_+ + \xi_- - i\Omega_n}$$  \hspace{1cm} (15)

where $\xi_{\pm} \equiv \frac{1}{2m}(k \pm K/2)^2 - \mu$; $k$ and $K$ are relative and total momenta of a pair, respectively. The divergence at large momenta, $k$, in (15) is regularized by the ultraviolet cutoff, $W$, which may be considered as the free fermion band width. Assuming that the cutoff is much bigger than all other characteristic scales of the problem, we can get rid of it after coupling constant renormalization and next turning $W$ to infinity (see, for example, [23]). So, expressing the coupling constant in the form

$$\frac{1}{g} = \frac{m}{4\pi} \ln \frac{2W}{|\epsilon_b|}.$$  \hspace{1cm} (16)

where $\epsilon_b$ is the two–particle bound state energy, and substituting it into Eq. (15), one can immediately take the limit $W \to \infty$. The detailed calculation can be found in Appendix A (formula (41)). As is easily seen, the expression obtained is rather complicated for use or even for qualitative analysis. However, there is one particular aspect of it which should be emphasized here — the asymptotics of the imaginary part of $\Gamma^{-1}(\Omega; K)$. The explicit expression for it is:

$$I m \Gamma^{-1}(\Omega; K) = -\frac{m}{4} \theta(\Omega + 2\mu - K^2/4m) \tanh(\beta \Omega/4) \times$$

$$\times [1 - f(\Omega + 2\mu - K^2/4m, K)],$$  \hspace{1cm} (17)

where the function $f(x, y)$ (see Appendix A) varies from zero to one and vanishes as $x, y \to 0$ or $x, y \to \infty$. It is easy to be convinced that the asymptotics of (17) at $\Omega \to 0$ and $\Omega \to \infty$ coincide exactly with those postulated in Ref. [14]. Recall that the phenomenological theory which was proposed in that paper explains in principle a lot of experimental data on HTSCs. Therefore, one can hope that our microscopic approach will also be able to describe correctly some observable properties of HTSCs.

In order to calculate the Green function of interacting fermions (see the next section) one still needs to find some rather simple approximation for OPF Green function. The most commonly used one is the so called the derivative expansion, which is valid at small frequencies and momenta. To obtain a consistent expression we need first to construct the retarded real–time Green function (by means of analytical continuation $i\Omega_n \to \Omega + i0$) and then to expand it at $K \to 0$ and $\Omega \to 0$. Using the explicit formula (11), we arrive at the following representation:

$$\Gamma^{-1}(\Omega; K) = a + \frac{K^2}{4m} - c\Omega,$$  \hspace{1cm} (18)

where

$$a = \frac{m}{4\pi} \left[ \ln \frac{\pi}{\beta |\epsilon_b| \gamma} - \frac{1}{\gamma} \int_0^\infty \frac{dx}{x} \tanh \frac{x\beta \mu}{2} \right],$$  \hspace{1cm} (19)
\[
\begin{align*}
  b &= \frac{m}{8\pi\mu} \left[ \tanh \frac{\beta\mu}{2} + \frac{\beta\mu}{2} + \frac{(\beta\mu)^2}{4} \int_{-\beta\mu/2}^{\infty} \frac{dx}{x^2 \tanh^2 x} \right], \\
  c &= \frac{m}{8\pi\mu} \left[ \int_{-\infty}^{1} \frac{dx}{x^2 \tanh x} \frac{x\beta\mu}{2} + \frac{\pi\beta\mu}{2} \theta(2\mu) \right],
\end{align*}
\]  

and \( \ln \gamma \simeq 0.577 \) is the Euler constant. Note that here and henceforth the same symbols are used for both the real–time and thermal Green functions.

The remarkable feature of the expansion (18) is the existence of a term that is linear in frequency (the next, quadratic, term is omitted in (18)). We stress this point here because sometimes the papers devoted to nonrelativistic systems appear (for example, [24]) where even the possibility of such a term in derivative expansion (or, what is the same, in the composite boson effective action) is not assumed. Perhaps this is the result of the uncritical use of some relativistic methods in nonrelativistic systems. In any case, there is no any reason for omitting this term in the model under consideration. Note that the importance of such a term has been recently emphasized also in Ref. [25, 26], where a similar model for 3D metal was investigated. Finally, the appearance of a term that is linear in frequency seems to be quite natural in Galilean–invariant models, and its absence should be regarded as being suspicious, if there is no any rather convincing reason for its absence.

We note here that the condition, indicating the appearance of instability in the system, reads as \( a(\mu, T) = 0 \), i.e. (see (19)) in our case\(^3\):

\[
\ln \frac{\pi}{\beta|\epsilon_b|\gamma} - \frac{1}{x} \int_{0}^{x} \tanh \frac{x\beta\mu}{2} = 0.
\]  

This equality defines a curve (below it will be called “the stability line”) which separates two different regions on the \( \mu–T \) plane (see Fig.1). The region “N” in Fig. 1 corresponds to the normal phase of the system where there is no order parameter. Inside the second region, denoted by “OP”, some nonzero order parameter appears. It has to be noted that the developing of the BKT phase also takes place somewhere inside the OP area.

As for the explicit dependence \( \mu = \mu(T) \) on the stability line determined by the equation (22), we can only give asymptotics at both low and high temperatures:

\[
\begin{align*}
  \mu &\simeq -\frac{|\epsilon_b|}{2} \quad \text{at} \quad T \ll |\epsilon_b|, \\
  \mu &\simeq \frac{\pi^2 T^2}{2\gamma^2 |\epsilon_b|} \quad \text{at} \quad T \gg |\epsilon_b|.
\end{align*}
\]

To understand the physical meaning of the stability line one needs to depict the curve which describes the physical dependence of the chemical potential (so far it was treated as

\(^3\)In Ref. [23] just such an equation was considered as one of the system of self–consistent equations for determining the values of \( T_c \) and \( \mu_c \equiv \mu(T_c) \) in mean–field approximation at constant density of particles.
an independent external parameter) on the temperature at fixed density of carriers in our system. The equation that determines this second curve is

\[ n_f(\mu, T) = \frac{T}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int d^2 p \text{ tr}[\tau_z G(i\omega_n, p)], \]

(25)

where \( G(i\omega_n, p) \) is the interacting fermion Green function expressed in its turn through the temperature and the chemical potential. Note that, after taking into account the expression (11) and the definition for the bare density:

\[ n_f = \frac{T}{V} \frac{\partial}{\partial \mu} \ln Z(T), \]

Eq. (25) can also be represented as the sum of two terms (see also [25, 27]). The first term, \( n_F \), is expressed through the free fermion Green function and the second one, \( n_B \), — through the Green function of OPF (or composite bosons). Thus, one may interpret such a situation as if a dynamical balance

\[ n_f(\mu, T) = n_F(\mu, T) + 2n_B(\mu, T) \]

in the distribution of the initial noninteracting particles over fermion and boson degrees of freedom is established in the system.

As the first approximation, we could substitute the free fermion Green function (14) into (25) what is equivalent to omitting the bosonic term in the expression for the distinctive “conservation law” \( n_f = n_F + 2n_B \). As a result, one can see that the curve which describes the dependence of the chemical potential on the temperature looks somewhat similar to the dashed line in Fig. 1. Increasing or decreasing the density of carriers in the system would lead to moving this curve up or down, respectively.

Turning back to the physical meaning of the stability line, one can be convinced that, changing the temperature in the system corresponds to removal to the right along the dashed line in Fig. 1. Thus, we can naturally conclude that our system is in its normal phase at temperatures that are high enough. On the contrary, decreasing the temperature corresponds to displacement to the left along the dashed line. When the system proves to be in the point where this line intersects the stability line a neutral order parameter appears. Subsequent removal, in OP region, will result in the transition into BKT phase (the second stability line, which must be calculated from another equation [19], is not depicted in Fig. 1).

4 Fermion Green function

The general form for the inverse thermal Green function of interacting fermions reads (compare with (14)):

\[ G^{-1}(i\omega_n, p) = i\omega_n - \tau_z \left[ \frac{p^2}{2m} - \mu + \Sigma(i\omega_n, p) \right], \]

(26)

where the self–energy operator for interacting fermions, \( \Sigma(i\omega_n, p) \), can be already expressed through the free Green function of fermions and OPF Green function by means of the following formula:

\[
\Sigma(i\omega_n, p) = \frac{\tau_z}{\beta} \sum_{n'=-\infty}^{\infty} \int \frac{d^2 k}{(2\pi)^2} \left[ \tau_+ G_0(i\omega_{n'}, k) \tau_- \Gamma(i\Omega_{n-n'}, p - k) + \tau_- G_0(i\omega_{n'}, k) \tau_+ \Gamma(i\Omega_{n'-n}, k - p) \right].
\]

(27)
To trace the complete derivation of this expression one needs to use the standard Feynman technique for the system given by (5) [28]. Now the substitution of the OPF Green function in its derivative approximation (18) into the last formula leads to the integral representation for the self–energy operator (27):

$$\Sigma(i\omega_n, \mathbf{p}) = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \coth \frac{\beta}{2c} \left( a + b \frac{(p-k)^2}{4m} \right) - \tanh \frac{\beta}{2} \left( \frac{k^2}{2m} - \mu \right) - ic\tau_z \omega_n,$$

where the summation over the Matsubara frequencies was performed. Assuming that the result of integration in the last expression does not strongly depend on the particular form of the smooth function in the numerator, we can obtain the following approximate self–energy operator:

$$\Sigma(i\omega_n, \mathbf{p}) = \int \frac{d^2 k}{(2\pi)^2} \frac{\theta(\mu - \frac{k^2}{2m})}{a + b \frac{(p-k)^2}{4m} - c \left( \frac{k^2}{2m} - \mu \right) - ic\tau_z \omega_n}.$$

To arrive at this expression, we have carried out some formal limit of zero temperature in (28). In other words, only the explicit dependence of this expression on the temperature through the arguments of the hyperbolic functions was subjected to this limit: $\coth \beta(\ldots) \rightarrow 1$, $\tanh \beta(\ldots) \rightarrow \text{sign}(\ldots)$. Of course, such a treatment introduces some uncertainty in the quantitative analysis. But the advantage of making further step by means of analytical manipulations is so attractive that we would like to expect the reliability of such an approximation. After simplification mentioned, the integration over momenta in Eq.(29) can be performed explicitly and the result is written down in Appendix B (formula (44)).

The real–time retarded fermion Green function in a close vicinity of its pole (as follows from (26)) has the standard form:

$$G(\omega, \mathbf{p}) = \frac{Z}{\omega - \tau_z (E_p - i\gamma_p)},$$

where $Z$ is the introduced above QW, $\gamma_p$ is the quasiparticle damping and $E_p$ is some function determining the low–energy one–particle spectrum. All these functions in (30) are expressed through the self–energy operator (29):

$$Z^{-1} = 1 - \tau_z \text{Re} \frac{\partial \Sigma(\omega; p_0)}{\partial \omega} \bigg|_{\omega=0},$$

$$E_p = Z \left( 1 + 2m \text{Re} \frac{\partial \Sigma(0; p_0)}{\partial p_0^2} \right) \left( \frac{p^2 - p_0^2}{2m} \right),$$

$$\gamma_p = -Z \text{Im} \left( \Sigma(0; p_0) + \frac{\partial \Sigma(0; p_0)}{\partial p_0^2} (p^2 - p_0^2) \right),$$

where $p_0$ is renormalized “Fermi” momentum, which is the solution to the equation $E_p = 0$, or

$$\frac{p_0^2}{2m} = \mu - \text{Re} \Sigma(0; p_0).$$
Note that this equation could be solved, in principle, at any chosen values of the chemical potential and the temperature. However, taking into account physical reasons, we restrict ourselves to solving the equation (34) for two sets of parameters \( \{T, \mu\} \) only. The first one corresponds to those values of the temperature and the chemical potential which lie on the stability line. From the physical point of view, this choice implies that the system is kept close to the critical region, but the density of carriers in the system increases monotonically at moving up along the stability line, or decreases in the opposite case. As a result, one will obtain the qualitative dependence of the near–critical properties of 2D metal with inter–fermion attraction on the density of carriers. The second set relates to the curve (the dashed line in Fig. 1), where the density of carriers is constant. This gives us the information about the dependence of 2D metal properties on the temperature.

5 Numerical results and discussion

We start from the analysis of near–critical properties at different carrier densities. As was noted above (see Section 3), the stability line is determined by the condition \( a(\mu, T) = 0 \). It is easy to check that being taken into account this condition simplifies the analysis essentially. Indeed, the only quantities which one needs to know for the calculation of \( Z, E_p \) and \( \gamma_{p_0} \) (defined in (30)) are the self–energy operator and its first derivative with respect to frequency at \( \omega = 0 \) (see (31)–(33)). Both these quantities are given in Appendix B. Performing the analysis, we come to the conclusion that in the case of parameters belonging to the stability line the relevant self–energy operator does not depend on the momentum and, as a result, the equation (34) transforms into expression for the momentum \( p_0 \). Substituting then this momentum into (31), (32) and (33) we would obtain complete information about the Green function for the interacting fermions. We, however, shall not write down corresponding cumbersome complex expressions, but instead present the graphs for \( \tilde{\xi} = (p_0^2/2m - \mu)/\mu \), \( Z \) and \( \gamma_{p_0} \) as functions of \( \beta\mu \) (Fig. 2–Fig. 4, respectively).

Note that only the region of positive values of the chemical potential is considered. Our investigation fails in the region near the point \( \mu = 0 \), since the derivative expansion (18) is not valid there. Though there are no obstacles for studying the region of negative values of the chemical potential separately, we shall not do that because we are interested in obtaining some qualitative information about the properties of the system proceeding just from the general monotonic dependence of the chemical potential on the particle density on the stability line. The singularity in Eq.(18) at \( \mu = 0 \) (because of \( c \to \infty \), see (21)) forbids us to go into the region of negative chemical potential continuously.

Since the parameter \( \beta\mu \) monotonically changes along the stability line, the graphs in Fig. 2–Fig. 4 can be thought of as a qualitative dependence of corresponding quantities on the density of carriers in the system. Thus, we conclude that when the carrier density is high the system naturally tends to become conventional FL with a rather small damping of the particles and QW practically equals one. In the opposite case, damping increases and QW reduces, i.e. marginal features of FL really appear.

Let us now turn to studying the properties of the system at fixed density of carriers. This
case turns out to be much more difficult than the previous one. In particular, the equation (34) cannot be solved so easily as before. In fact, one should consider the system of two self-consistent equations (25) and (34) at each chosen temperature. To simplify the system, one can use a quite natural approximation in which the additional equation (25) is taken in the mean field form, i.e. instead of interacting fermion Green function we shall use the free one. But even in this case the equation (34) cannot be solved explicitly. Solving it numerically and substituting into (31) and (33), we obtained the dependence of $\tilde{\xi} = \left[ p_0^2/2m - \mu \right] / \mu$, $Z$ and $\gamma_{p_0}$ on the temperature. The corresponding graphs are represented in Fig. 5–Fig. 7 for several different densities: (1) $\epsilon_F/|\epsilon_b| = 500$; (2) $\epsilon_F/|\epsilon_b| = 750$; (3) $\epsilon_F/|\epsilon_b| = 1000$, where $\epsilon_F \equiv \pi n/m$ by definition. All curves start at the corresponding critical temperatures. For example, it is seen that in the wide range of $T$’s the QW considerably differs from 1, essentially depending on $T$.

But the most amazing, to our mind, result of the numerical analysis is a linear temperature dependence of the quasiparticle damping within a rather wide range of temperatures (from $T_c$ up to about $4T_c$ and even higher). As was noted above, it is just such a linear dependence that was supposed in Ref. [14] for explaining the unusual temperature dependence of resistivity of HTSCs. Our calculations show also that the decreasing of the ratio $\epsilon_F/|\epsilon_b|$ rather quickly breaks the linear dependence of damping $\gamma_{p_0}$ on $T$ what can be probably explained by the sharp decreasing $\mu$ up to so small values where the derivative expansion fails.

Of course, numerical results presented here cannot, by themselves, give us a completely convincing answer concerning the physical reasons for an explanation of the peculiarities of the 2D metal model. Our hypothesis is quite simple and seems to be rather natural: the most important features which are responsible for the marginal FL behavior are lowered dimensionality and, of course, attractive character of inter-particle interaction. Indeed, the two-dimensionality has resulted in the asymptotics of the imaginary part of the inverse Green function (I7):

$$ Im\Gamma^{-1}(\Omega; K) \simeq -\frac{m\beta\Omega}{16}, \quad \text{at} \quad \Omega \to 0 $$  

$$ Im\Gamma^{-1}(\Omega; K) \simeq -\frac{m}{4}, \quad \text{at} \quad \Omega \to \infty, $$

which were required in Ref.[14] (pay attention that the Green functions for spin and charge fluctuations were used in the paper cited). As to the attraction, it may play the important role even in 3D models but in 2D case its effect proves to be decisive. While in the former case OPF are very important mainly in the superconducting phase, in the latter one they turn out to be of great importance in the normal phase too.

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This result does not seem to correspond to the marginal Fermi liquid definition (see Introduction). It must be kept in mind, however, that the definition mentioned refers to the case $T = 0$ which should be excluded from our consideration by physical reasons. As to $T \neq 0$ the quasiparticle damping proves to be nonzero (see (33)) even at Green function pole $p = p_0$. 

12
6 Conclusion

In this paper we studied a model of a 2D metal with a local weak attraction between fermions in the normal phase. Within the framework of rather natural approximations, we have shown that such a system displays properties which differ essentially (mainly in the vicinity of the stability line) from the conventional properties of Landau FL. In particular, the QW noticeably differs from 1 and the damping of quasiparticles is rather large. The manifestation of this effect becomes more essential as the density of particles decreases.

We also investigated the temperature dependence of the QW and the quasiparticle damping. The most important result, we believe, is the behavior of obtained curves for the quasiparticle damping, exhibiting features which can explain the linear temperature dependence of resistivity which is observed at all temperatures of HTCS normal phase existance.

Since through the paper we restricted our consideration of the 2D model with non-retarded inter–particle interaction only which, as is well known, is valid in the case of rather small Fermi energy, it seems to be very interesting and important to investigate the influence of an indirect interaction between fermions (mediated, for example, by the phonons, magnons or some other bosons and resulted in a retarded attraction) and its possible connection with the OPF in the normal phase too. One more rather urgent for HTCSs problem is to study the possible effects when the OPF have non–trivial (for example, d–wave) symmetry. At last, we would like also to note that since our approach involves some intuitive assumptions (even though they are commonly used in solid state physics), more consistent consideration would be useful.

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Appendix A

Here we derive the renormalized OPF Green function. Substituting the coupling constant (16) into (15), one obtains

$$\Gamma^{-1}(i\Omega_{n};K) = \frac{m}{4\pi} \left[ \ln \left| \frac{2W}{|\epsilon_b|} \right| - \int_0^{2W} dx \frac{tanh \left( \beta(x + \frac{K^2}{4m} - 2\mu)/4 \right)}{x + \frac{K^2}{4m} - 2\mu - i\Omega_n} + \int_0^{2W} dx \frac{tanh \left( \beta(x + \frac{K^2}{4m} - 2\mu)/4 \right)}{x + \frac{K^2}{4m} - 2\mu - i\Omega_n} f(x, K) \right], \quad (37)$$

where the function $f(x, K)$ is

$$f(x, y) = \frac{2}{\pi} \int_0^{\pi/2} d\phi \frac{cosh \left( \frac{\beta \sqrt{\frac{x^2}{2m} \cos \phi}}{2} \right)}{cosh \left( \beta(x + \frac{K^2}{4m} - 2\mu)/2 \right) + cosh \left( \frac{\beta \sqrt{\frac{x^2}{2m} \cos \phi}}{2} \right)} - 1. \quad (38)$$
As was pointed above and can be easily checked directly this function varies from zero to one and vanishes when \( x \) or \( y \) → 0 as well as when \( x \) or \( y \) → ∞.

Using identities:

\[
\frac{1}{x + \frac{K^2}{4m} - 2\mu - i\Omega_n} = \frac{1}{x + \frac{K^2}{4m} - 2\mu} + \frac{i\Omega_n}{(x + \frac{K^2}{4m} - 2\mu)(x + \frac{K^2}{4m} - 2\mu - i\Omega_n)},
\]

and

\[
\int_0^\alpha \frac{dx}{x} \tanh x = \ln \frac{4\alpha\gamma}{\pi} - \int_\alpha^\infty \frac{dx}{x}(\tanh x - 1), \quad \text{at} \quad \alpha > 0,
\]

(as earlier \( \ln \gamma \) is the Euler constant), one can take the limit \( W \to \infty \) in (37) and come to the final expression:

\[
\Gamma^{-1}(i\Omega_n; K) = \frac{m}{4\pi} \left[ \ln \frac{\pi}{\beta|\epsilon_b|\gamma} + \int_0^\infty \frac{dx}{x} \tanh \left( x\beta\left(\frac{K^2}{4m} - 2\mu\right)/4 \right) - i\Omega_n \int_0^\infty \frac{dx}{x + \left(\frac{K^2}{4m} - 2\mu\right)(x + \frac{K^2}{4m} - 2\mu - i\Omega_n)} \right. \\
\left. + \int_0^\infty \frac{dx}{x + \frac{K^2}{4m} - 2\mu - i\Omega_n} f(x, K) \right],
\]

Appendix B

To calculate the integral (29), one needs to use the following table integrals (see, for example, [29]):

\[
\int_0^\pi \frac{d\phi}{b + a \cos \phi} = \frac{\pi}{\sqrt{b^2 - a^2}} \quad \text{at} \quad |b| > |a|,
\]

and

\[
\int \frac{dx}{\sqrt{R(x)}} = \frac{1}{\sqrt{c}} \ln \left( 2\sqrt{cR(x)} + 2cx + b \right),
\]

where \( R(x) = a + bx + cx^2 \) and \( c > 0 \).

Thus, straightforward manipulations lead to the formal result for the self–energy operator expressed through the elementary functions:

\[
\tilde{\Sigma}(\tilde{\omega}; \tilde{\xi}) = \frac{4}{b - \tilde{\epsilon}} \times
\]
\[ \times \ln \frac{\sqrt{\left(\tilde{b} \xi\right)^2 + (\tilde{a} - \tilde{c} \tilde{\omega})^2 + b(\tilde{\xi} + 2)(\tilde{a} - \tilde{c} \tilde{\omega}) - \frac{\tilde{b} \xi + \xi^2}{2(b - \tilde{c})} + \tilde{a} - \tilde{\omega} \tilde{c}}}{2(\tilde{a} - \tilde{c} \tilde{\omega}) - \frac{\tilde{b} \xi + \xi^2}{b - \tilde{c}}}, \]  

where the dimensionless functions

\[ \tilde{a}(\beta | \epsilon_b, \beta \mu) = \frac{4\pi}{m} a, \quad \tilde{b}(\beta \mu) = \frac{4\pi}{m} b, \quad \tilde{c}(\beta \mu) = \frac{4\pi}{m} c, \]  

more convenient for numerical calculations, were introduced. Note that one of the parameters \((\tilde{c})\) in (44) is complex, so there is some uncertainty in the expression. To avoid it one needs to clarify which branch of the complex function is chosen. However, in the main text of the paper only the self–energy operator and its derivative with respect to frequency at \(\omega = 0\) and large value \(\beta \mu\) are used. Both required functions at those conditions are free of the mentioned uncertainty, so we shall not concern about the difficulty of choosing the needed branch.

When parameters of the system are chosen so that they obey the equation \(\tilde{a}(\beta | \epsilon_b, \beta \mu) = 0\), which identically coincides with (22) and determines the stability line, the self–energy operator and its derivative at \(\omega = 0\) become very simple:

\[ \tilde{\Sigma}(0; \tilde{\xi}) = -\frac{4}{b - \tilde{c}} \ln \frac{\tilde{b}}{\tilde{c}} \text{ at } \tilde{\xi} < 0; \]  

\[ \left. \frac{\partial \tilde{\Sigma}(\omega; \tilde{\xi})}{\partial \tau_z \omega} \right|_{\omega=0} = -\frac{4 \tilde{c}}{\xi b(\tilde{\xi} + \tilde{c})} - \frac{4 \tilde{b}}{(b - \tilde{c})(\tilde{\xi} b + \tilde{c})} \text{ at } \tilde{\xi} < 0. \]  

The expressions for \(\tilde{\xi} > 0\) can also be obtained from (44). We do not write down them since they were not used in our analysis in the main part of the paper because the equation \(\tilde{\xi} = -\tilde{\Sigma}\) (see (34) only has a solution with negative sign.

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