DIVERGENCE OF THE $\frac{1}{N_f}$ - SERIES EXPANSION IN QED

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Abstract

The perturbative expansion series in coupling constant in QED is divergent. It is either an asymptotic series or an arrangement of a conditionally convergent series. The sums of these types of series depend on the way we arrange the partial sums for successive approximations. The $1/N_f$ series expansion in QED, where $N_f$ is the number of flavours, defines a rearrangement of the perturbative series in coupling constant, and therefore, its convergence would serve as a proof that the perturbative series is, in fact, conditionally convergent. Unfortunately, the $1/N_f$ series also diverges. We proof this using arguments similar to those of Dyson.

We expect that some of the ideas and techniques discussed in our paper will find some use in finding the true nature of the perturbation series in coupling constant as well as the $1/N_f$ expansion series.

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It is more than half a century since Dyson proved that the perturbation theory in the coupling constant in Quantum Electrodynamics (QED) is divergent [1]. In this paper, we consider QED with large number of flavours (i.e., large number of species) of fermions. Large flavour limit is used in QED to argue for the existence of Landau singularity in the leading order in $\frac{1}{N_f}$ [2,3]. Large flavour expansion is also used in other relativistic as well as non-relativistic field theories. In particular, it has been very successfully used to proof many seminal results in Landau Fermi liquid theory, again, in the leading order in $\frac{1}{N_f}$ [4–6]. Therefore, if the leading order results are to be taken on their face values, it is imperative to know whether the series obtained by $\frac{1}{N_f}$ expansion is convergent.

There is another important reason why one should look for expansion series in parameters other than the coupling constant. We know that, even though the ordinary perturbation theory diverges, order by order summation of the series gives excellent agreement with experiments. However, we do believe that, at some order, this is going to fail. This opens up two possibilities: either series is conditionally convergent or it is an asymptotic power series. We explain the difference between the two types of series. Let us consider the series given by,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - .... .... ....$$ (1)

Term by term summation of the series up to a certain number of terms, gives a good approximation to $\ln 2$. However, after that, it starts diverging. Note that this series is an arrangement of a conditionally convergent series (CCS). An infinite real series is called conditionally convergent if it converges but not absolutely. The convergence is associated with an arrangement of the series. In other words, one can always arrange the partial sums in such a that the series converges. Note that for a convergent series the sum does not dependent on the arrangement of the partial sums. Let a CCS be given by, $S = \sum_n a_n$, then the following property holds: (1) $\lim_{n \to \infty} a_n \to 0$ (for divergent series appearing in physical problems, it is always hard to decide which is a generic term because terms can always be regrouped to define a completely different generic term), (2) the absolute series $\sum_n |a_n|$ diverges, and
(3) the negative and positive series diverge independently. The most important property of CCS in the present context is its behaviour under rearrangement. It is well known that the sum of a CCS crucially depends on the way partial sums are arranged. For example, by suitably arranging the partial sums, the sum of series given by Eq.(1) can be made zero. A remarkable theorem due to Riemann [7,8] brings out this property.

Theorem: For any given number on the real line (including $-\infty$ and $+\infty$), there exists an arrangement of a CCS such that the sum of the series converges to it.

There is another type of series which behaves like a convergent series up to a certain number of terms but after that it behaves like a divergent series. This type of series is called asymptotic series [9] and is generally defined through a power series representation of a function. Function $f(x)$ is said to have an asymptotic power series representation if for all $n$,

$$\lim_{x \to 0} \left| \frac{f(x) - \sum_{i=0}^{n} a_i x^i}{x^n} \right| = 0$$

In other words,

$$f(x) = \sum_{i=0}^{n} a_i x^i + O(x^n)$$

This means that the error in estimating the function is of the same order as the last term in the series. To explain, let us consider the following function,

$$F(x) = \int_0^\infty \frac{e^{-t}}{1 + xt} dt$$

for real positive $x$ and $x \to 0$. Since,

$$\frac{1}{1 + xt} = 1 - xt + x^2 t^2 + ... + \frac{(-xt)^k}{1 + xt}$$

we have,

$$F(x) = \sum_{k=0}^{N} (-1)^k x^k k! + R_{N+1}(x) ; \quad |R_N(x)| = N!x^N$$

The ratio of the two successive terms is
\[
\frac{x^k k!}{x^{(k-1)}(k-1)!} = x^k
\]
This shows that the terms first decrease (since by assumption \(0 < x \ll 1\)) and then increases (when \(k > \frac{1}{2}\)). From this it follows that for a given value of \(x\), there exists a best approximation. In other words, for a fixed value of \(x\), only a definite accuracy can be achieved. These properties are found to hold for most asymptotic expansions which appear in physical problems. It is clear from above that under rearrangement this type of series would be drastically altered.

In this paper, we consider QED with large number of flavours, \(N_f\), of fermions. Coupling constant \(e\) is small, \(N_f\) is large and \(e^2 N_f\) is also small. This is what we represent by saying that \(e^2 N_f = \text{small constant}\) when \(N_f \to \infty\). To calculate the value any physical observable, we can, in principle, carry out the usual perturbative analysis. The results would clearly depend on coupling constant as well as \(N_f\). On the other hand, we may choose to carry out expansion in \(1/N_f\). This, \(1/N_f\)-series expansion is based on the regrouping of the parameters \((e^2 N_f = \text{constant})\) and rearrangement of the perturbative series in coupling constant. Instead of summing the perturbative series order by order, we sum it loop wise. First, the one loop diagrams are summed which gives the leading term in \(1/N_f\), and then the two loop diagrams are summed which gives the next-to-leading term and so on. If the original perturbative series were conditionally convergent or an asymptotic power series, then this rearranged series could in principle yield a different sum of the series. We, obviously, are not in a position to calculate the sum of the series. However, if we could argue that this series in \(1/N_f\) is convergent, then this could serve as a proof that the perturbation series in coupling constant is conditionally convergent. Unfortunately, it turns out that the \(1/N_f\)-series is divergent. We proof this using arguments similar to those of Dyson.

There are claims in literature regarding the proof of the divergence of perturbative theory based on the fact that the number of Feynman diagrams increases factorially with the order in the large orders of the perturbation theory. Note that these proofs would pertain to the absolute convergence of the series, and the positive and negative series independently (for
a recent survey of these results see [11] and references there in).

Dyson’s arguments for the divergence of perturbation theory in QED is elegant in its simplicity. Since we will be using similar arguments for the divergence of $\frac{1}{N_f}$ - expansion series in QED, we quote the following paragraphs from his paper: ".... let

$$F(e^2) = a_0 + a_1 e^2 + a_2 e^4 + ...$$

be a physical quantity which is calculated as a formal power series in $e^2$ by integrating the equations of motion of the theory over a finite or infinite time. Suppose, if possible, that the series... converges for some positive value of $e^2$; this implies that $F(e^2)$ is an analytic function of $e$ at $e = 0$. Then for sufficiently small value of $e$, $F(-e^2)$ will also be a well-behaved analytic function with a convergent power series expansion.

But for $F(-e^2)$ we can also make a physical interpretation. ... In the fictitious world, like charges attract each other. The potential between static charges, in the classical limit of large distances and large number of elementary charges, will be just the Coulomb potential with the sign reversed. But it is clear that in the fictitious world the vacuum state as ordinarily defined is not the state of lowest energy. By creating a large number $N$ of electron-positron pairs, bringing the electrons in one region of space and the positrons in another separate region, it is easy to construct a pathological state in which the negative potential energy of the Coulomb forces is much greater than the total rest energy and the kinetic energy of the particles. ...... . Suppose that in the fictitious world the state of the system is known at a certain time to be an ordinary physical state with only a few particles present. There is a high potential barrier separating the physical state from the pathological state of equal energy; to overcome the barrier it is necessary to supply the rest energy for creation of many particles. Nevertheless, because of the quantum-mechanical tunnel effect, there will always be a finite probability that in any finite time-interval the system will find itself in a pathological state. Thus every physical state is unstable against the spontaneous creation of many particles. Further, a system once in a pathological state will not remain steady; there will be rapid creation of more and more particles, an explosive disintegration of the vacuum...
by spontaneous polarization. In these circumstances it is impossible that the integration of
the equation of motion of the theory over any finite or infinite time interval, starting from
a given state of the fictitious world, should lead to well-defined analytic functions. Therefore
\( F(-e^2) \) can not be analytic and the series \( \cdots \) can not be convergent.”

The central idea in Dyson’s proof of the divergence of perturbation theory in coupling
constant, as is evident from the lengthy quotation above, is that the convergence of the
perturbation theory in coupling constant would lead to the existence of pathological states to
which the normal states of QED would decay. These pathological states correspond to states
of a quantum field theory whose ground state or vacuum state is unstable. In the case of \( \frac{1}{N_f} \)
expansion series of QED, we will prove that its’ convergence also leads to the existence of
pathological states to which normal states of large flavour QED would decay. We explicitly
construct the field theory to which these pathological states correspond. This field theory
with unstable vacuum state turns out to be different from the one constructed by Dyson. It
is a quantum field theory with commuting fermions.

Before we discuss the divergence of \( \frac{1}{N_f} \)-expansion series in QED we would like to make some
remarks. Dyson’s description of the instability of ground state in QED through spontane-
ous particle production, gives the impression that his arguments regarding divergence of
perturbative series applies only to relativistic quantum field theories. Subsequently, it has
been clarified by Arkady Veinshstein [10] and others that the arguments applies equally well
to perturbation series in quantum quantum mechanics such as anharmonic oscillators with
quartic interaction terms ( see [11] for details ). The second point concerns the relevance
of large flavour expansion in physics. Our discussion of large flavour expansion is centered
around QED . In this case it may look a bit artificial because the number flavours in QED
is very small. Bellow we demonstrate that there are physical situations where large number
of flavours appear very naturally and our arguments can be trivially carried over to these
cases. We have in mind the the Feldman Model [4] of weakly interacting electron gas (see
ref. [5,6] for details, Sec-II of ref. [2] for a brief description of the model and summary of
main results). This model describes a condensed matter Fermi system in thermal equilib-
rium at some temperature $T$ (for simplicity, assume $T = 0$) and chemical potential $\mu$. On microscopic scale ($\approx 10^{-8}$ cm), it can be described approximately in terms of non-relativistic electrons with short range two body interactions. The thermodynamic quantities such as conductivity depend only on physical properties of the system at mesoscopic length scale ($\approx 10^{-4}$ cm), and therefore, are determined from processes involving momenta of the order of $\frac{k_F}{\lambda}$ around the Fermi surface, where the parameter, $\lambda >> 1$, should be thought of as a ratio of meso-to-microscopic length scale. This is generally referred to as the scaling limit (large $\lambda$, low frequencies) of the system. The most important observation of Feldman et. al. is that in the scaling limit, systems of non-relativistic (free) electrons in $d$ spatial dimensions behave like a system of multi-flavoured relativistic chiral Dirac fermions in 1 + 1 dimensions. The number of flavours $N_f \approx \text{const.} \lambda^{d-1}$.

Consider a system of non-relativistic free electrons in $d$ spatial dimensions with the Euclidean action,

$$S_0(\psi^*, \psi) = \sum_\sigma \int d^{d+1}x \psi^*_\sigma(x)(i\partial_0 - \frac{1}{2m}\Delta - \mu)\psi_\sigma(x)$$  \hspace{1cm} (2)

The Euclidean free fermion Green’s function, $G^0_{\sigma\sigma'}(x - y)$, where $\sigma$ and $\sigma'$ are the spin indices, $x = (t, \vec{x})$ and $y = (s, \vec{y})$, $t$ and $s$ are imaginary times, $t > s$, is given by,

$$G^0_{\sigma\sigma'}(x - y) = \langle \psi^*_\sigma(x)\psi_\sigma(y) \rangle_\mu = -\delta_{\sigma\sigma'} \int (dk) \frac{e^{-ik_0(t-s)+i\vec{k}(\vec{x}-\vec{y})}}{ik_0 - (\frac{k^2}{2m} - \mu)}$$  \hspace{1cm} (3)

In the scaling limit, the leading contributions to $G^0_{\sigma\sigma'}(x - y)$ come from modes whose momenta are contained in a shell $S_F^{(\lambda)}$ of thickness $\frac{k_F}{\lambda}$ around the Fermi surface $S_F$. Let us introduce the new variables $\vec{\omega}$, $p_\parallel$, $p_0$ such that $k_F\vec{\omega} \in S_F$, $p_0 = k_0$ and $\vec{k} = (k_F + p_\parallel)\vec{\omega}$. If $\vec{k} \in S_F^{(\lambda)}$, then $p_\parallel << k_F$, and we can approximate the integrand of Eq.(3), by dropping $p_\parallel^2$ term in the denominator. In other words,

$$G^0_{\sigma\sigma'}(x - y) = \delta_{\sigma\sigma'} \int \frac{d\sigma(\vec{\omega})}{(2\pi)^{d-1}} k_F^{d-1} e^{ik_F\vec{\omega}(\vec{x}-\vec{y})} G_c(t - s, \vec{\omega}(\vec{x}-\vec{y}))$$  \hspace{1cm} (4)

where $d\sigma(\vec{\omega})$ is the uniform measure on unit sphere and

$$G_c(t - s, \vec{\omega}(\vec{x}-\vec{y})) = -\int \frac{dp_0 dp_\parallel}{2\pi 2\pi} \frac{e^{-ik_0(t-s)+ip_\parallel\vec{\omega}(\vec{x}-\vec{y})}}{ip_0 - v_Fp_\parallel}$$  \hspace{1cm} (5)
is the Green’s function of chiral Dirac fermion in 1 + 1 dimension. \( v_F = \frac{k_F}{m} \) is the Fermi velocity. The \( \vec{\omega} \)-integration in Eq.(4) can be further approximated by replacing it with summation over discrete directions \( \vec{\omega}_j \) by dividing the shell \( S_F^{(A)} \) into \( N \) small boxes \( B_{\vec{\omega}_j}, j = 1, \ldots, N \) of roughly cubical shape. The box, \( B_{\vec{\omega}_j} \), is centered at \( \vec{\omega}_j \in S_F \) and has an approximate side length \( \frac{k_F}{\lambda} \). The number of boxes \( N \) is given by,

\[
N = \frac{\text{Volume of the shell } S_F^{(A)}}{\text{Volume of the cubical boxes } B_{\vec{\omega}_j}} = \frac{\Omega_{d-1} k_F^{d-1} \times \frac{k_F}{\lambda}}{(k_F \lambda)^d} = \Omega_{d-1} \lambda^{d-1}
\]

where \( \Omega_{d-1} \) is the surface volume of unit sphere in \( d \) spatial dimensions. The Green’s function is, now, given by

\[
G^0_{\sigma\sigma'}(x - y) = -\delta_{\sigma\sigma'} \sum_{\vec{\omega}_j} \int \frac{dp_0 dp_{\parallel} p_{\perp}}{2\pi 2\pi 2\pi} \frac{e^{-ip_0(t-s) + ip_{\parallel}(x-y)}}{ip_0 - v_F p_{\parallel}} \tag{6}
\]

where \( \vec{p} = p_{\parallel} \vec{\omega} + p_{\perp} \) is a vector in \( B_{\vec{\omega}_j} - k_F \vec{\omega}_j \) and \( p_0 \in R \). Thus in the scaling limit, the behaviour of a \( d \)-dimensional non-relativistic free electron gas is described by \( (N_f = ) N = \Omega_{d-1} \lambda^{d-1} \) flavours of free chiral Dirac fermions in 1 + 1 dimensional space-time. The weakly interacting system electrons can be described as interacting \((1+1)\)-dimensional chiral fermions with large number of flavours. In the concluding section, we provide arguments to show that our method has some interesting consequences for this model.

Now we describe the central theme of the paper. The Langrangian of QED with number of flavours, \( N_f \), is given by,

\[
\mathcal{L} = \sum_{j=1}^{N_f} \bar{\psi}^j \left( i \gamma^\mu \partial_\mu + m - e \gamma^\mu A_\mu \right) \psi^j + \frac{1}{4} F_{\mu\nu}^2 \tag{7}
\]

where \( \psi^j \) and \( \bar{\psi}^j \) are the Dirac field and its’ conjugate, \( j \) is the flavour index, and \( A_\mu \) and \( F_{\mu\nu} \) are the electromagnetic potential and the field strength respectively. We will, ultimately, be considering cases with both the positive and negative sign of \( N_f \), and therefore, we introduce the notation \( |N_f| = \text{sign}(N_f) \times N_f \) for latter convinience. The \( \frac{1}{N_f} \)-expansion is introduced by assuming that, in the limit \( |N_f| \to \infty \), \( e^2 |N_f| = \text{constant} = \alpha^2 \) (say). Alternatively, instead of the Lagrangian given by Eq.(7), we may consider the following Lagrangian,

\[
\mathcal{L} = \sum_{j=1}^{N_f} \bar{\psi}^j \left( i \gamma^\mu \partial_\mu + m - \frac{e}{\sqrt{|N_f|}} \gamma^\mu A_\mu \right) \psi^j + \frac{1}{4} F_{\mu\nu}^2 \tag{8}
\]
With this form of the Langrangian, it is easy to set up Feynman diagram technique. To each photon and fermion line corresponds their usual propagator. Each vertex contributes a factor of $\frac{e}{\sqrt{|N_f|}}$, each fermion loop contributes a factor of $(-1)$ for anticommuting fermions and a factor of $N_f$ because of summation over fermion flavours. Using these rules, it is easy to set up $1/N_f$ expansion series for any physical observable. Just as in the case of perturbation theory in the coupling constant, the expansion in $1/N_f$ allows us to express an observable $F$ in the form,

$$F\left(\frac{1}{N_f}\right) = Q_0 + \frac{1}{N_f} Q_1 + \frac{1}{N_f^2} Q_2 + \ldots \quad (9)$$

$Q_0$, $Q_1$, $Q_2$, .... are some functions of the coupling constant. Now suppose that the series converges for some small value of $\frac{1}{N_f}$ (large value of $N_f$), then the observable function $F(\frac{1}{N_f})$ is analytic for $\frac{1}{N_f} = 0$ ($N_f = \infty$). Therefore, we can consider a small negative value of $\frac{1}{N_f}$ (large negative value of $N_f$) for which the function is analytic and convergent. In other words, the function $F(\frac{1}{N_f})$ can be analytically continued to small negative value of $1/N_f$ and the series thus obtained will be convergent.

What is meaning of negative $N_f$? Before we look for an answer to this question, let us calculate the effective coupling constant for positive as well as negative $N_f$ using the formal $1/N_f$-expansion series for the two point Green’s function. The series is assumed to be convergent, and therefore, for sufficient large $N_f$, one can restrict to the leading order term. The leading order term is given by the one-loop diagrams which can easily be evaluated to obtain the polarization from which one can read off the effective coupling constant. It is given by,

$$e_{eff}^2(\Lambda^2) = \frac{e^2}{1 - \frac{e^2 N_f}{3\pi |N_f|} ln \frac{\Lambda^2}{m^2}} \quad (10)$$

If $N_f$ is negative,

$$e_{eff}^2(\Lambda^2) = \frac{e^2}{1 + \frac{e^2}{3\pi} ln \frac{\Lambda^2}{m^2}} \quad (11)$$

From the equation above, we find that in the limit $\Lambda \rightarrow \infty$, $e_{eff}^2 \rightarrow 0$, when $N_f$ is negative. Therefore, the formal theory that we obtain from the analytical continuation of $1/N_f$ (
for large $N_f$) to the small negative value of $1/N_f$, is (at least formally) asymptotically free. This seems to suggest that the physical meaning of the negative sign of $N_f$ could possibly be traced in the free theory without the interaction term.

We will argue that the choice of negative $N_f$ for anticommuting fermions amounts to considering commuting fermions with positive $N_f$. Let us consider the Lagrangian given by Eq.(7) in four dimensional Euclidean space. The partition function is given by the following functional integral,

$$Z_{ac} = \int DA(x)D\bar{\psi}(x)D\psi(x)exp(-\int d^4xL)$$ (12)

We carry out functional integration with respect to the anticommuting fermion fields (grassman variables) and obtain,

$$Z_{ac} = \int DA(x) det^{N_f}(i\gamma^\mu\partial_\mu + m - e\gamma^\mu A_\mu)exp(-\frac{1}{4}\int d^4xF_{\mu\nu}^2)$$ (13)

However, if one considers fermion fields to be commuting variables, then it turns out to be functional integration over complex fields, and we obtain,

$$Z_c = \int DA(x) det^{-N_f}(i\gamma^\mu\partial_\mu + m - e\gamma^\mu A_\mu)exp(-\frac{1}{4}\int d^4xF_{\mu\nu}^2)$$ (14)

This expression could be obtained from the previous expression, simply by assuming that $N_f$ is negative. Therefore, anticommuting fermions with negative $N_f$ has the same partition function as the commuting fermions with positive $N_f$. Since, physically interesting observables can be calculated from the partition function, our claim is that the negative flavour anticommuting fermion is equivalent to positive flavour commuting fermions.

This can also be argued using formal perturbation theory. Consider the two point Green’s functions for the photons using Lagrangian given by Eq.(8). First, we consider just one loop diagram and show how the contribution due to flavours appears in the calculations. There are two vertices and a fermion loop, each vertex contributes a factor of $e^{\frac{\epsilon}{\sqrt{|N_f|}}}$, the fermion loop contributes a multiplicative factor of $(-1)$ because the fermions anticommute and a multiplicative factor of $N_f$ because of summation over flavours of the internal fermion.
Now, if $N_f$ happens to be negative, the factor $(-1)$ and the factor $N_f$, combines to give the factor $|N_f|$. This is also the contribution if the fermions commute and the flavour is positive (the factor $(-1)$ is absent for commuting fermions). The same procedure applies for the multiloop diagrams. Calculation of any observable in QED, essentially amounts to calculating a set of Feynman diagrams. The information regarding the anticommuting nature of the fermions in the calculation of the Feynman diagrams enters through the multiplicative factor of $(-1)$ for each fermion loop that appear in the diagram. Each such loop, as explained above also contributes a factor of $N_f$. Therefore, when $N_f$ is taken to be negative, the over all multiplicative factor becomes $|N_f|$. As explained above, we would obtain the same multiplicative factor if we treat the fermions as commuting fields and consider $N_f$ to be positive. This shows that the choice of negative $N_f$ for anticommuting fermions ammounts to considering commuting fermions with positive $N_f$. We argued above, on formal grounds, that QED with anticommuting fermions and negative value of $N_f$ is asymptotically free. Therefore, formally speaking, QED with commuting fermions and positive $N_f$ is asymptotically free.

It is well known that the free field theory of commuting fermions does not have a stable vacuum or ground state [12–14]. From the arguments above, it then follows that the interacting theory also can not have a stable vacuum state. All states in this theory are pathological. All these results follow from the single assumption that the $1/N_f$-expansion series in QED is convergent. Therefore, the convergence of the series in QED with anticommuting fermions would leads to the decay of normal states to the pathological states of QED with commuting fermions via the process of quantum mechanical tunnelling. Therefore, for QED to be meaningful, the series in $\frac{1}{N_f}$ expansion must diverge.

Our main aim in this work was to find the nature of the divergent perturbation series. We considered the $1/N_f$ expansion series which, as explained in the text, is a rearrangement of the perturbation series in coupling constant. Had the series converged, we could have concluded that the perturbative series is conditionally convergent. However, we proved that this series also diverges. We are also unable to say anything about the nature this divergent
series in $1/N_f$. However, our method provides some insight into nature of $1/N_f$ expansion series in the case of Feldman model which describes very well the properties of weakly interacting system of electrons. We will describe the case when the interaction among the electrons in the condensed matter medium is attractive. We have argued earlier that such a system is equivalent to a system of $(1 + 1)$-dimensional Dirac fermions with number of flavour $N_f = \text{const.} \, \lambda^d$. Using large flavour expasion technique in the renormalization group analysis of such systems it has been proven that the perturbative ground state of such a system is unstable [2,6]. This is the BCS instability. The proof is based on the assumption that $\frac{1}{N_f}$ expansion series converges and one can restrict to the leading order term. Formal analytical continuation to the negative flavour in the renormalization group analysis, leads to effective coupling constant, $g_{eff} = 0$. Therefore, the analytically continued theory is equivalent to a theory of free commuting $(1 + 1)$-dimensional Dirac fermions. The ground state of this theory is also unstable. Instability of ground state of the theory at both ends of analytically continued domains is consistent with the convergence of $\frac{1}{N_f}$-expansion series in case of weakly interacting electron systems with attractive interactions. In the case of repulsive interaction among the electrons in the condensed matter medium, it is not hard to prove that the $1/N_f$ series expansion is divergent. However, the arguments involved in the proof is slightly different from the one discussed in this paper.
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