A CLASS OF FOURTH-ORDER NONLINEAR PARABOLIC EQUATIONS MODELING THE EPITAXIAL GROWTH OF THIN FILMS

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ABSTRACT. In this paper, the initial-boundary value problem for a class of fourth-order nonlinear parabolic equations modeling the epitaxial growth of thin films is studied. By means of the theory of potential wells, the global existence, asymptotic behavior and finite time blow-up of weak solutions are obtained.

1. Introduction. Many processes occur mainly on the surfaces of materials such as crystal growth, catalytic reactions and production of nanostructures. The growth of crystal thin films from molecular or atomic beams is usually referred to as molecular beam epitaxy (MBE), a technology used to manufacture computer chips and other semiconductor devices. Other applications requiring thin films include solar cells, mechanical coatings, microelectromechanical systems and microfluidic devices. Growth conditions have a profound effect on the morphological quality of thin films and have received increasing interest in material science. A major reason for this interest is that some compositions are expected to be high-temperature super-conducting so that they could be used in the design of semiconductors (see [1] and the references therein). The complex process of constructing thin films on substrates by chemical vapor deposition has given rise to several descriptions and simulations by atomistic as well as continuum models (see [20, 22]). One of the outstanding challenges is to understand these growth processes qualitatively and quantitatively so that control laws can be formulated which optimize certain characteristics of thin films, for instance, flatness and conductivity.

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The continuum equation of motion for the evolution of the film surface profile \( u(x, t) \) can be written in the form (see [20, 30])
\[
u_t = f - \nabla \cdot j + \eta. \tag{1}
\]

Here \( f \) is the deposition flux, \( j \) is the surface mass current which takes account of all microscopic processes that move atoms along the surface, and \( \eta \) is the Gaussian random variable which describes the fluctuations in the average deposition flux. On a purely phenomenological basis, we can write
\[
\eta = A_1 \nabla u + A_2 \nabla (\nabla^2 u) + A_3 |\nabla u|^2 \nabla u + A_4 |\nabla u|^2 + \cdots, \tag{2}
\]
where \( j \) has been expanded in a power series involving the surface slope \( \nabla u \) and various power and derivatives thereof. Substitution of (2) into (1) gives
\[
u_t + A_1 \Delta u + A_2 \Delta^2 u + A_3 \text{div}(|\nabla u|^2 \nabla u) + A_4 \Delta |\nabla u|^2 = f + \eta. \tag{3}
\]

This is considered to be a general partial differential equation of up to fourth-order for the growth rate of thin films when there is no desorption of surface overhangs (see [9]). \( A_2 \Delta u \) accounts for diffusion due to evaporation-condensation (see [18]). \( A_2 \Delta^2 u \) models capillarity-driven surface diffusion (see [18]). \( A_3 \text{div}(|\nabla u|^2 \nabla u) \) represents hopping of atoms (see [6]). \( A_4 \Delta |\nabla u|^2 \) is related to the equilibration of the inhomogeneous concentration of the diffusing particles on the surface (see [1]).

In the case \( A_1, A_4 = 0, A_2 > 0 \) and \( A_3 < 0 \), Zhao and Liu [32] studied
\[
u_t + A_2 \Delta^2 u - \text{div}(|\nabla u|^2 \nabla u) = f, \quad x \in \Omega, \ t > 0,
\]
where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain. Under some assumptions on \( f \), they derived the existence and uniqueness of time-periodic weak solutions and time-periodic classical solutions.

In the case \( A_1, A_2 > 0, A_3 < 0 \) and \( A_4 = 0 \), Kohn and Yan [11] considered
\[
u_t + \Delta^2 u + \text{div}(2(1 - |\nabla u|^2) \nabla u) = 0, \quad x \in \Omega, \ t > 0,
\]
where \( \Omega \subset \mathbb{R}^2 \) is a square domain. They obtained the decay of energy in time. King et al. [10] investigated
\[
u_t + \Delta^2 u - \text{div}(\varphi(\nabla u)) = f, \quad x \in \Omega, \ t > 0,
\]
where \( \Omega \subset \mathbb{R}^N \) is a bounded domain of class \( C^{4+k} \) for some \( k > 0 \). Under certain assumptions on \( f \) and \( \varphi : \mathbb{R}^N \rightarrow \mathbb{R}^N \) that include \( \varphi(z) = |z|^{p-2}z - z \) with some constant \( p > 2 \) as a special case, they established the existence, uniqueness and regularity of global weak solutions in an appropriate function space. In the one-dimensional setting, they characterized the set of steady states and determined its unique asymptotically stable element.

In the case \( A_1, A_2, A_4 > 0 \) and \( A_3 = 0 \), Blöcker and Gugg [2] considered
\[
u_t + A_1 \Delta u + A_2 \Delta^2 u + A_3 \Delta |\nabla u|^2 = \eta, \quad x \in \Omega, \ t > 0,
\]
where \( \Omega \subset \mathbb{R} \) is a bounded interval. The global existence of weak solutions was addressed based on ideas of Capinski and Gaterek [4] for the Navier-Stokes equation. This result was extended by Blöcker et al. [3] to
\[
u_t + A_1 \Delta u + A_2 \Delta^2 u + A_4 \Delta |\nabla u|^2 = \nu|\nabla u|^2 + \eta, \quad x \in \Omega, \ t > 0,
\]
where \( \Omega \subset \mathbb{R} \) is a bounded interval and \( \nu > 0 \). Stein and Winkler [23] handled
\[
u_t + \Delta u + \Delta^2 u + \Delta |\nabla u|^2 = 0, \quad x \in \Omega, \ t > 0,
\]
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where $\Omega \subset \mathbb{R}$ is a bounded interval and $\alpha > 1$. They proved the global existence of weak solutions relying on the range of $\alpha$. Subsequently, Winkler [25] investigated

$$u_t + \mu \Delta u + \Delta^2 u + \lambda \Delta |\nabla u|^\alpha = f(x), \ x \in \Omega, \ t > 0,$$

(4)

where $\Omega \subset \mathbb{R}^N$ ($N \leq 3$) is a bounded convex domain with smooth boundary, $\mu \geq 0$ and $\lambda > 0$. Under appropriate assumptions on $f$, the global existence of weak solutions was obtained. Furthermore, under an additional smallness condition on $\mu$ and the size of $f$, it was shown that there exists a bounded set which is absorbing for (4) in some sense. In the presence of $A_3 \text{div}(|\nabla u|^2 \nabla u)$, Agélas [1] dealt with

$$u_t + \nu_1 \Delta u + \nu_2 \Delta^2 u - \nu_3 \text{div}(|\nabla u|^2 \nabla u) + \nu_4 \Delta |\nabla u|^2 = \nu_5 |\nabla u|^2, \ x \in \Omega, \ t > 0,$$

where $\Omega = \mathbb{R}^N$ ($N = 1, 2$) with solutions vanishing at infinity as $|x| \to \infty$ or $\Omega = \mathbb{R}^N \setminus \mathbb{Z}^N$. Under the condition that $\nu_2 \nu_3 > \nu_5^2$, he proved the existence and uniqueness of global strong solutions for sufficiently regular initial data. Moreover, he proved the existence, uniqueness and regularity of global weak solutions.

All the works mentioned above concentrate on the case $A_1 \geq 0$. In this paper, neglecting the Gaussian noise, we are mainly interested in a nonlinear version of Eq. (3) with $A_1, A_3 < 0$, $A_2 > 0$ and $A_4 = 0$. Without loss of generality, we consider

$$u_t - \Delta u + \Delta^2 u - \text{div}(|\nabla u|^{p-2} \nabla u) = f(u), \ x \in \Omega, \ t > 0,$$

(5)

with initial condition

$$u(x, 0) = u_0(x), \ x \in \Omega$$

(6)

and Neumann boundary condition

$$u = \frac{\partial u}{\partial \nu} = 0, \ x \in \partial \Omega, \ t \geq 0,$$

(7)

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary $\partial \Omega$, and $\frac{\partial u}{\partial \nu}$ is the exterior normal derivative of $u$ on $\partial \Omega$.

Our main purpose is to discuss the relationship between the global well-posedness of solutions and the initial data. The Nehari manifold plays a link role in the establishment of this relationship. Our main results not only provide the sufficient conditions on the global existence, asymptotic behavior and finite time blow-up of weak solutions, but also show exact descriptions of smallness conditions on the initial data. The effects of initial energy on these properties of solutions are also reflected to a certain extent. Our main technical tool is the theory of potential wells (see e.g. [5, 7, 8, 12, 13, 17, 21, 24, 26, 27, 29, 31]) with a slight modification, which plays an essential role in the proofs of main results.

This paper is organized as follows. In Section 2 some notations and assumptions on $p$ and $f$ are given. Some preliminary definitions related to problem (5)-(7) are stated. Moreover, our main results on problem (5)-(7) are presented. In Section 3 the local existence of solutions is obtained. In Sections 4-6, the global existence, asymptotic behavior and finite time blow-up of solutions are proved, respectively.

2. Preliminaries and main results. Throughout the paper, in order to simplify the notations, we denote

$$\| \cdot \|_p := \| \cdot \|_{L^p(\Omega)}, \ \| \cdot \| := \| \cdot \|_2, \ (u, v) := \int_{\Omega} uv \, dx.$$
We now give the following assumptions on $p$ and $f$.

\((A_p)\) \quad p > 2, \quad p < \infty \text{ if } N \leq 2, \quad p \leq \frac{2N}{N-2} \text{ if } N > 2.

\((A_f)\) \quad f \in C(\mathbb{R})$, and there exist positive constants $b$ and $q$ such that $|f(u)| \leq b|u|^{q-1}$. Moreover, there exists a constant $k \geq p$ such that $0 \leq kF(u) \leq uf(u)$, where

$$F(u) = \int_{0}^{u} f(s) \, ds.$$ 

In addition, $q$ satisfies

\((A_{q-1})\) \quad 2 < q < \frac{(N+2)p}{N}.

**Definition 2.1** (Weak solutions). A function $u \in L^\infty(0,T; H^2_0(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ is called a weak solution to problem (5)-(7) in $\Omega \times [0,T)$ if $u(0) = u_0$ in $H^2_0(\Omega)$ and

$$(u_t(t),v) + (\nabla u(t), \nabla v) + (\Delta u(t), \Delta v) + (|\nabla u(t)|^{p-2}\nabla u(t), \nabla v) = (f(u(t)),v) \quad (8)$$

for all $v \in H^2_0(\Omega)$ and a.e. $t \in (0,T)$.

Now we are in a position to define the energy functional associated with problem (5)-(7)

$$E(u) = \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\Delta u\|^2 + \frac{1}{p} \|\nabla u\|_p^p - \int_{\Omega} F(u) \, dx$$

and the Nehari functional

$$I(u) = \|\nabla u\|^2 + \|\Delta u\|^2 + \|\nabla u\|_p^p - (f(u),u).$$

Thus all nontrivial stationary solutions belong to the Nehari manifold defined by

$$\mathcal{N} = \{u \in H^2_0(\Omega) \setminus \{0\} | I(u) = 0\},$$

which separates two sets

$$\mathcal{N}_+ = \{u \in H^2_0(\Omega) | I(u) > 0\} \cup \{0\}$$

and

$$\mathcal{N}_- = \{u \in H^2_0(\Omega) | I(u) < 0\}.$$

We define the potential well

$$W = \{u \in H^2_0(\Omega) | I(u) > 0, E(u) < d\} \cup \{0\},$$

where the depth of potential well

$$d = \frac{k - 2}{2k} b^{-\frac{2}{p-2}} \mathcal{C}^{-\frac{2p}{p-2}},$$

and $\mathcal{C}$ is the best Sobolev constant for the embedding $H^2_0(\Omega) \hookrightarrow L^q(\Omega)$, i.e.,

$$\mathcal{C} = \sup_{u \in H^2_0(\Omega) \setminus \{0\}} \frac{\|u\|_q}{\|\Delta u\|_2}.$$ 

The main results of this paper are stated as follows.
Theorem 2.2 (Local existence). Let $(A_p)$, $(A_f)$ and $(A_{q-1})$ be fulfilled. Assume that $u_0 \in H_0^2(\Omega)$. Then there exists a constant $T > 0$ such that problem (5)-(7) admits a local solution $u$. Moreover, for all $t \in [0, T)$, there holds the energy inequality

$$E(u(t)) + \int_0^t \|u_\tau(\tau)\|^2 \, d\tau \leq E(u_0). \quad (9)$$

Theorem 2.3 (Global existence). Let $(A_p)$ and $(A_f)$ be fulfilled, and $q$ satisfy

$$(A_{q-2}) \quad q > 2, \quad q < \infty \text{ if } N \leq 4, \quad q < \frac{2N}{N-4} \text{ if } N > 4.$$ \nonumber

Assume that $u_0 \in N_+$ and $E(u_0) < d$. Then problem (5)-(7) admits a global solution $u(t) \in W$ satisfying (9) for all $t \in [0, \infty)$.

Theorem 2.4 (Asymptotic behavior). Let $(A_p)$, $(A_f)$ and $(A_{q-2})$ be fulfilled, and $q = k$. Assume that $u_0 \in N_+$ and $E(u_0) < d$. Then there exists a constant $\lambda > 0$ such that solutions to problem (5)-(7) satisfy $\|u(t)\|^2 \leq \|u_0\|^2 e^{-\lambda t}$ for all $t \in [0, \infty)$.

Theorem 2.5 ( Blow-up). Let $(A_p)$, $(A_f)$ and $(A_{q-1})$ be fulfilled. Assume that $u_0 \in H_0^2(\Omega)$ and either one of the following conditions is satisfied:

(i) $E(u_0) \leq 0$;

(ii) $0 < E(u_0) \leq d$ and $u_0 \in N_-.$

Then solutions to problem (5)-(7) blow up in finite time.

3. Proof of Theorem 2.2.

Lemma 3.1 ([29]). For any $u \in H_0^2(\Omega)$, $\|\Delta u\|$ is equivalent to $\|u\|_{H^2(\Omega)}$.

As in [15, 16, 19, 28], in the framework of the Galerkin approximations, we prove Theorems 2.2 and 2.3.

Proof of Theorem 2.2. Let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of $H_0^2(\Omega)$ and an orthonormal basis of $L^2(\Omega)$ given by eigenfunctions of $\Delta^2$ with the boundary condition (7). We construct the approximate solutions to problem (5)-(7)

$$u_n(t) = \sum_{j=1}^n \xi_{jn}(t)w_j, \quad n = 1, 2, \ldots,$$

which satisfy

$$(u_{nt}(t), w_j) + (\nabla u_n(t), \nabla w_j) + (\Delta u_n(t), \Delta w_j)
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + (|\nabla u_n(t)|^{p-2}\nabla u_n(t), \nabla w_j) = (f(u_n(t)), w_j), \quad j = 1, 2, \ldots, n, \quad (10)$$

$$u_n(0) = \sum_{j=1}^n \xi_{jn}(0)w_j \to u_0 \quad \text{in} \quad H_0^2(\Omega). \quad (11)$$

Multiplying (10) by $\xi_{jn}(t)$ and summing for $j$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_n(t)\|^2 + \|\nabla u_n(t)\|^2 + \|\Delta u_n(t)\|^2 + \|\nabla u_n(t)\|^p = (f(u_n(t)), u_n(t)). \quad (12)$$

Due to the fact that

$$(f(u_n(t)), u_n(t)) \leq b\|u_n(t)\|^q,$$

we deduce from the Gagliardo-Nirenberg interpolation inequality that there exists a constant $C_1 > 0$ independent of $n$ such that

$$(f(u_n(t)), u_n(t)) \leq C_1 \|\nabla u_n(t)\|_p^\alpha \|u_n(t)\|^{(1-\alpha)q},$$
where

\[ 0 < \alpha = \frac{Np(q - 2)}{q(Np + 2p - 2N)} < 1. \]

By Young’s inequality with \( \epsilon \), we further get

\[ (f(u_n(t)), u_n(t)) \leq C_1(\epsilon \| \nabla u_n(t) \|_p^p + C_2(\epsilon) \| u_n(t) \|^{\beta}), \]

where

\[ \beta = \frac{2Np + 2pq - 2Nq}{Np + 2p - Nq} > 2. \]

Consequently, by virtue of (12) and (13) with \( \epsilon = \frac{1}{C_1} \), we obtain

\[ \frac{d}{dt} \| u_n(t) \|^2 \leq C_3 \| u_n(t) \|^{\beta}. \]

Hence there exists a constant \( \| u_n(t) \| < C_4(T) \)

for all \( t \in [0, T) \), where \( C_4(T) \) stands for a positive constant which is independent of \( n \) but depends on \( T \).

Multiplying (10) by \( \xi_j^{\prime}(t) \) and summing for \( j \), we obtain

\[ \left\| u_{n_1}(t) \right\|^2 + \frac{1}{2} \frac{d}{dt} \| \nabla u_n(t) \|^2 + \frac{1}{2} \frac{d}{dt} \| \nabla u_n(t) \|^2 + \frac{1}{p} \| \nabla u_n(t) \|_p^p = \frac{d}{dt} \int_{\Omega} F(u_n(t)) \, dx. \]

Integrating both sides of this equality with respect to \( t \), we deduce that

\[ \int_0^t \left\| u_{n_1}(\tau) \right\|^2 d\tau + \frac{1}{2} \| \nabla u_n(t) \|^2 + \frac{1}{2} \| \nabla u_n(t) \|^2 + \frac{1}{p} \| \nabla u_n(t) \|_p^p \]

\[ = \int_{\Omega} F(u_n(t)) \, dx - \int_{\Omega} F(u_n(0)) \, dx \]

\[ + \frac{1}{2} \| \nabla u_n(0) \|^2 + \frac{1}{2} \| \nabla u_n(0) \|^2 + \frac{1}{p} \| \nabla u_n(0) \|_p^p. \]

It follows from (11), (13)-(15) that

\[ \left\| \Delta u_n(t) \right\| < C_5(T) \]

and

\[ \int_0^t \left\| u_{n_1}(\tau) \right\|^2 d\tau < C_6(T). \]

Moreover,

\[ \| f(u_n(t)) \|_\mu^\mu \leq b^\mu \| u_n(t) \|_\nu^\nu \leq b^\mu \| \nabla u_n(t) \|^q < C_7(T), \quad \mu = \frac{q}{q - 1}. \]

Hence there exist \( u, \chi_1, \chi_2 \) and a subsequence of \( \{ u_n \} \), always relabeled as the same and we shall not repeat, such that as \( n \to \infty \),

\[ u_n \rightharpoonup u \text{ weakly star in } L^\infty(0, T; H_0^2(\Omega)), \]

\[ u_{n_k} \rightharpoonup u \text{ weakly in } L^2(0, T; L^2(\Omega)), \]

\[ u_n \to u \text{ a.e. in } \Omega \times [0, T) \text{ and strongly in } L^q(\Omega) \text{ for each } t \in (0, T), \]

\[ |\nabla u_n|^{p-2} \nabla u_n \rightharpoonup \chi_1 \text{ weakly in } L^{p'}(\Omega \times [0, T)), \quad p' = \frac{p}{p - 1}, \]

\[ f(u_n) \rightharpoonup \chi_2 \text{ weakly star in } L^\infty(0, T; L^{p'}(\Omega)). \]

In terms of the theory of monotone operators, we have \( \chi_1 = |\nabla u|^{p-2} \nabla u \). According to [14, Chapter 1, Lemma 1.3], we have \( \chi_2 = f(u) \). Therefore, we can pass to the
limit in the approximate problem (10), (11). Thus \( u \) is a solution to problem (5)-(7) in the sense of Definition 2.1.

Next we prove (9). From

\[
\left| \int_{\Omega} F(u_n(t)) \,dx - \int_{\Omega} F(u(t)) \,dx \right| \leq \int_{\Omega} \|f(v_n(t))\| u_n(t) - u(t) \|dx \\
\leq \|f(v_n(t))\|_\mu \|u_n(t) - u(t)\|_q,
\]

\( \nu_n = \theta_n u_n + (1 - \theta_n) u, 0 < \theta_n < 1 \), it follows that

\[
\lim_{n \to \infty} \int_{\Omega} F(u_n(t)) \,dx = \int_{\Omega} F(u(t)) \,dx.
\]

Hence, from (16), (17), (15) and (11), we deduce that

\[
\int_0^t \|u_\tau(\tau)\|^2 \,d\tau + \frac{1}{2} \|\nabla u(\tau)\|^2 + \frac{1}{2} \|\Delta u(\tau)\|^2 + \frac{1}{p} \|\nabla u(\tau)\|_p^p \\
\leq \liminf_{n \to \infty} \left( \int_0^t \|u_n(\tau)\|^2 \,d\tau + \frac{1}{2} \|\nabla u_n(\tau)\|^2 + \frac{1}{2} \|\Delta u_n(\tau)\|^2 + \frac{1}{p} \|\nabla u_n(\tau)\|_p^p \right) \\
= \liminf_{n \to \infty} \left( \int_{\Omega} F(u_n(t)) \,dx + E(u_n(0)) \right) \\
= \int_{\Omega} F(u(t)) \,dx + E(u_0).
\]

Thus the proof of Theorem 2.2 is complete. \( \square \)

4. Proof of Theorem 2.3.

Lemma 4.1. Let \((A_p),(A_f)\) and \((A_{q-2})\) be fulfilled. If \( u \in \mathcal{N} \), then \( E(u) \geq d \).

Proof. From \( u \in \mathcal{N} \) we get

\[
\|\nabla u\|^2 + \|\Delta u\|^2 + \|\nabla u\|_p^p = (f(u),u),
\]

and so

\[
\|\Delta u\|^2 \leq b \|u\|_q^q \leq b \mathcal{C}^q \|\Delta u\|_q^q.
\]

A direct calculation yields

\[
\|\Delta u\| \geq b^{-\frac{1}{q-2}} \mathcal{C}^{-\frac{q}{q-2}}. \quad (18)
\]

Since

\[
E(u) \geq \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\Delta u\|^2 + \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{k}(f(u),u) \\
\geq \frac{k - 2}{2k} \|\Delta u\|^2 + \frac{1}{k} f(u),
\]

it follows from (18) and the definition of \( d \) that \( E(u) \geq d \). \( \square \)

Proof of Theorem 2.3. As in the proof of Theorem 2.2, we construct the approximate solutions \( u_n(t) \) to problem (5)-(7). Multiplying (10) by \( \xi_j'(t) \), summing for \( j \) and integrating with respect to \( t \), we obtain

\[
\int_0^t \|u_n(\tau)\|^2 \,d\tau + E(u_n(t)) = E(u_n(0)), \quad \forall t \in [0,\infty).
\]

From \( u_0 \in \mathcal{N}_+ \) and \( E(u_0) < d \), i.e., \( u_0 \in W \), we have \( u_n(0) \in W \) for sufficiently large \( n \).
We now claim that
\[ u_n(t) \in W \tag{20} \]
for sufficiently large \( n \) and \( t \in [0, \infty) \).

Indeed, if it is false, then there exists a time \( t_0 > 0 \) such that \( u_n(t_0) \in \partial W \), i.e.,
\[ u_n(t_0) \in \mathcal{N} \) or \( E(u_n(t_0)) = d \). From (19) we get
\[ E(u_n(t)) \leq E(u_n(0)) < d \tag{21} \]
for sufficiently large \( n \) and \( t \in [0, \infty) \). If \( u_n(t_0) \in \mathcal{N} \), then it follows from Lemma 4.1 that \( E(u_n(t_0)) \geq d \), which contradicts (21). On the other hand, it is easy to see from (21) that \( E(u_n(t_0)) = d \) is impossible. Thus assertion (20) follows as desired.

Since
\[
E(u_n(t)) \geq \frac{1}{2} \|\nabla u_n(t)\|^2 + \frac{1}{2} \|\Delta u_n(t)\|^2 + \frac{1}{p} \|\nabla u_n(t)\|_p^p - \frac{1}{k} f(u_n(t), u_n(t)) \\
\geq \frac{k - 2}{2k} \|\nabla u_n(t)\|^2 + \frac{k - 2}{2k} \|\Delta u_n(t)\|^2 + \frac{k - p}{pk} \|\nabla u_n(t)\|_p^p + \frac{1}{k} I(u_n(t)),
\]
we deduce from (19)-(21) that
\[ \|\Delta u_n(t)\|^2 < \frac{2kd}{k - 2}, \]
and
\[ \int_0^t \|u_n(\tau)\|^2 d\tau < d \]
for sufficiently large \( n \) and \( t \in [0, \infty) \). Furthermore,
\[ \|f(u(t))\|_p^p \leq b^\mu c^q \|\Delta u_n(t)\|_q^q < b^\mu c^q \left( \frac{2kd}{k - 2} \right)^\frac{q}{2}, \quad \mu = \frac{q}{q - 1}. \tag{22} \]

By a repetition of the arguments in the proof of Theorem 2.2, problem (5)-(7) admits a global solution \( u \). Moreover, by the similar arguments to the proof of assertion (20), we have \( u(t) \in W \) for all \( t \in [0, \infty) \).

5. **Proof of Theorem 2.4.** We introduce a family of potential wells
\[ W_\delta = \{ u \in H^2_0(\Omega) | I_\delta(u) > 0, E(u) < d(\delta) \} \cup \{ 0 \}, \quad 0 < \delta < \frac{q}{2}, \]
and their outside sets
\[ V_\delta = \{ u \in H^2_0(\Omega) | I_\delta(u) < 0, E(u) < d(\delta) \}, \]
where the \( \delta \)-Nehari functional
\[ I_\delta(u) = \delta \|\nabla u\|^2 + \delta \|\Delta u\|^2 + \delta \|\nabla u\|_p^p - (f(u), u) \]
and the depth function of potential wells
\[ d(\delta) = \frac{q - 2\delta}{2q} \delta^{-\frac{q}{2}} b^{-\frac{q}{q-2}} c^{-\frac{q}{q-2}}. \]
In addition,
\[ \mathcal{N}_\delta = \{ u \in H^2_0(\Omega) \setminus \{ 0 \} | I_\delta(u) = 0 \}. \]

It is easy to verify that \( d(\delta) > 0 \) is strictly increasing on \( (0, 1) \), strictly decreasing on \( \left( 1, \frac{q}{2} \right) \), and takes the maximum at \( \delta = 1 \).

**Lemma 5.1.** Let \( (A_p), (A_f) \) and \( (A_{q-2}) \) be fulfilled, and \( q = k \). If \( u \in \mathcal{N}_\delta \), then \( E(u) \geq d(\delta) \).
Proof. From $u \in \mathcal{N}_\delta$ we get
\[
\delta \| \nabla u \|^2 + \delta \| \Delta u \|^2 + \delta \| \nabla u \|^p_p = (f(u), u),
\]
and so
\[
\delta \| \Delta u \|^2 \leq b \| u \|_q^q \leq b C^q \| \Delta u \|^q.
\]
A direct calculation yields
\[
\| \Delta u \| \geq \delta^{-\frac{1}{2}} b^{-\frac{1}{2q} + \frac{1}{p}} c^{-\frac{2}{p-2}}.
\] (23)
Since
\[
E(u) \geq \frac{q - 2\delta}{2q} \| \Delta u \|^2 + \frac{1}{q} I_\delta(u),
\]
it follows from (23) and the definition of $d(\delta)$ that $E(u) \geq d(\delta)$. \hfill \Box

Lemma 5.2. Let $(A_p)$, $(A_f)$ and $(A_{q-2})$ be fulfilled, and $q = k$. Suppose that $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = e$, where $0 < e < d$. Then,

(i) solutions to problem (5)-(7) with $E(u_0) = e$ belong to $W_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided $u_0 \in \mathcal{N}_\delta$;

(ii) solutions to problem (5)-(7) with $E(u_0) = e$ belong to $V_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided $u_0 \in \mathcal{N}_\delta$.

Proof. (i) The sign of $I_\delta(u_0)$ is unchangeable for all $\delta \in (\delta_1, \delta_2)$. In fact, if it was not the case, there would exist a constant $\delta_0 < \delta_1 < \delta_2$ such that $I_{\delta_0}(u_0) = 0$. Note that $E(u_0) = e > 0$ implies $\| \Delta u_0 \| \neq 0$. In terms of Lemma 5.1, we have $E(u_0) \geq d(\delta_0)$, which contradicts
\[
E(u_0) = d(\delta_1) = d(\delta_2) < d(\delta_0).
\] Hence it follows from $u_0 \in \mathcal{N}_\delta$ that $I_\delta(u_0) > 0$ for all $\delta \in (\delta_1, \delta_2)$, which, together with $E(u_0) < d$, yields $u_0 \in W_\delta$ for all $\delta \in (\delta_1, \delta_2)$.

Let $u$ be a solution to problem (5)-(7). Next we prove $u(t) \in W_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$. If it was not true, there would exist a time $0 < t_0 < T$ such that $u(t_0) \in \partial W_\delta$ for some $\delta \in (\delta_1, \delta_2)$, i.e., $u(t_0) \in \mathcal{N}_\delta$ or $E(u(t_0)) = d(\delta)$. If $u(t_0) \in \mathcal{N}_\delta$, then we have $E(u(t_0)) \geq d(\delta)$ due to Lemma 5.1, which contradicts
\[
E(u(t)) \leq E(u_0) < d(\delta)
\] (24) for all $t \in [0, T)$. On the other hand, it is easy to see from (24) that $E(u(t_0)) \geq d(\delta)$ is impossible.

(ii) The conclusion of (ii) can be obtained by the similar arguments to the proof of (i). \hfill \Box

Proof of Theorem 2.4. Multiplying (8) by any $q \in C[0, \infty)$, we get
\[
(u_t(t), \varphi(t)v) + (\nabla u(t), \nabla(\varphi(t)v)) + (\Delta u(t), \Delta(\varphi(t)v))
+ (|\nabla u(t)|^{p-2} \nabla u(t), \nabla(\varphi(t)v)) = (f(u(t)), \varphi(t)v),
\] which can be written in the form
\[
(u_t(t), \omega(t)) + (\nabla u(t), \nabla \omega(t)) + (\Delta u(t), \Delta \omega(t))
+ (|\nabla u(t)|^{p-2} \nabla u(t), \nabla \omega(t)) = (f(u(t)), \omega(t))
\] (25)
for all $\omega \in L^\infty(0, \infty; H^2_0(\Omega))$. Taking $\omega(t) = u(t)$ in (25), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|^2 + I(u(t)) = 0.
\] (26)
According to Lemma 5.2, we have \( u(t) \in W_{\delta} \) for all \( \delta \in (\delta_1, \delta_2) \) and \( t \in [0, \infty) \). Hence \( I_\delta(u(t)) \geq 0 \) for all \( \delta \in (\delta_1, \delta_2) \) and \( t \in [0, \infty) \). We further get \( I_{\delta_1}(u(t)) \geq 0 \) for all \( t \in [0, \infty) \). Thus it follows from (26) that

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + (1 - \delta_1)(\|\nabla u(t)\|^2 + \|\Delta u(t)\|^2 + \|\nabla I_{\delta_1}(u(t))\|^2) \geq 0, \quad \forall t \in [0, \infty).
\]

Consequently, in view of \( \delta_1 < 1 \), there exists a constant \( C > 0 \) such that

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + C(1 - \delta_1)\|u(t)\|^2 \leq 0, \quad \forall t \in [0, \infty),
\]

which yields

\[
\|u(t)\|^2 \leq \|u_0\|^2 e^{-2C(1-\delta_1)t}, \quad \forall t \in [0, \infty).
\]

Thus the proof of Theorem 2.4 is complete. \( \Box \)

6. Proof of Theorem 2.5.

Lemma 6.1. Let \((A_p), (A_f)\) and \((A_{q-1})\) be fulfilled. Assume that \( u \) is a solution to problem (5)-(7) with \( u_0 \in N_- \) and \( 0 < E(u_0) \leq d \). Then \( u(t) \in N_- \) for all \( t \in [0, T) \).

Proof. We divide the proof of this lemma into two cases.

Case 1. \( 0 < E(u_0) < d \).

Argue by contradiction. If there exists a time \( t_0 > 0 \) such that \( u(t_0) \in N \), then by Lemma 4.1 we have \( E(u(t_0)) \geq d \), which contradicts \( E(u_0) < d \) due to (9).

Case 2. \( E(u_0) = d \).

Argue again by contradiction. If there exists the first time \( t_0 > 0 \) such that \( u(t_0) \in N \), then it is easy to see from Lemma 4.1 that \( E(u(t_0)) > d \). At the same time, since

\[
(u(t), u(t)) = -I(u(t)),
\]

we get \((u(t), u(t)) > 0\) for all \( t \in [0, t_0) \), which implies that \( \int_0^t \|u_\tau(\tau)\|^2 \sigma(\tau) d\tau \) is strictly increasing on \([0, t_0]\). Hence \( \int_0^{t_0} \|u_\tau(\tau)\|^2 \sigma(\tau) d\tau > 0 \), which, together with (9) and \( E(u_0) = d \), yields \( E(u(t_0)) < d \). This contradicts \( E(u(t_0)) \geq d \). \( \Box \)

Proof of Theorem 2.5. Let \( u \) be a solution to problem (5)-(7). Next we prove \( T < \infty \). If it is false, then \( T = \infty \).

(i) We divide the proof of (i) into two cases.

Case 1. \( E(u_0) < 0 \).

We consider the auxiliary function \( \phi: [0, T_1] \to \mathbb{R}^+ \) defined by

\[
\phi(t) = \int_0^t \|u(\tau)\|^2 d\tau + (T_1 - t)\|u_0\|^2 + \sigma(t + T_2)^2.
\]

Here \( T_1, \sigma \) and \( T_2 \) are positive constants to be determined later. Next we perform the proof of Case 1 along three steps.

Step 1. We first claim that

\[
\phi(t)\phi''(t) - \frac{k + 2}{4} \phi'(t)^2 \geq \phi(t)\phi(t)
\]

(27)
for a.e. \( t \in [0, T_1] \), where
\[
\varphi(t) = \phi''(t) - (k + 2) \left( \int_{0}^{t} \|u_{\tau}(\tau)\|^2 \, d\tau + \sigma \right). \tag{28}
\]

Indeed, a direct calculation gives
\[
\phi'(t) = \|u(t)\|^2 - \|u_0\|^2 + 2\sigma(t + T_2) = 2 \int_{0}^{t} (u(\tau), u_{\tau}(\tau)) \, d\tau + 2\sigma(t + T_2).
\]

By Schwarz’s inequality and Cauchy’s inequality, we further obtain
\[
\phi'(t)^2 \leq 4 \int_{0}^{t} \|u(\tau)\|^2 \, d\tau \int_{0}^{t} \|u_{\tau}(\tau)\|^2 \, d\tau + 4\sigma \int_{0}^{t} \|u(\tau)\|^2 \, d\tau \\
+ 4\sigma(t + T_2)^2 \int_{0}^{t} \|u_{\tau}(\tau)\|^2 \, d\tau + 4\sigma^2(t + T_2)^2 \\
= 4 \left( \int_{0}^{t} \|u(\tau)\|^2 \, d\tau + \sigma(t + T_2)^2 \right) \left( \int_{0}^{t} \|u_{\tau}(\tau)\|^2 \, d\tau + \sigma \right) \\
\leq 4\phi(t) \left( \int_{0}^{t} \|u_{\tau}(\tau)\|^2 \, d\tau + \sigma \right). \tag{29}
\]

Consequently, assertion (27) follows from (29).

**Step 2.** We will show that \( \varphi(t) \geq 0 \).

To prove this, we note that
\[
\phi''(t) = 2(u(t), u_{\tau}(t)) + 2\sigma \\
= -2\|\nabla u(t)\|^2 - 2\|\Delta u(t)\|^2 - 2\|\nabla u(t)\|_p^p + 2(f(u(t)), u(t)) + 2\sigma \\
= -2I(u(t)) + 2\sigma
\]
for a.e. \( t \in [0, T_1] \). Substituting this into (28), we get
\[
\varphi(t) = -2I(u(t)) + 2\sigma - (k + 2) \left( \int_{0}^{t} \|u_{\tau}(\tau)\|^2 \, d\tau + \sigma \right). \tag{30}
\]

From (9) and
\[
E(u(t)) \geq \frac{k - 2}{2k} \|\Delta u(t)\|^2 + \frac{1}{k} I(u(t)),
\]
we get
\[
-2I(u(t)) \geq (k - 2)\|\Delta u(t)\|^2 + 2k \int_{0}^{t} \|u_{\tau}(\tau)\|^2 \, d\tau - 2kE(u_0). \tag{31}
\]

It follows that from (30) and (31) that
\[
\varphi(t) \geq (k - 2)\|\Delta u(t)\|^2 - 2kE(u_0) - k\sigma. \tag{32}
\]
Choosing \( 0 < \sigma \leq -2E(u_0) \), we have \( \varphi(t) \geq 0 \).

**Step 3.** We claim that there exists a finite time \( T_3 \) such that
\[
\lim_{t \to T_3} \phi(t) = \infty. \tag{33}
\]

Indeed, by the definition of \( \phi(t) \) and the result of Step 2, assertion (27) allows us to derive
\[
\phi(t)\phi''(t) - \frac{k + 2}{4} \phi'(t)^2 \geq 0
\]
for a.e. $t \in [0, T_1]$. Moreover, it follows from (28) and the result of Step 2 that $\phi''(t) > 0$. We now choose $T_2$ satisfying

$$T_2 > \frac{2\|u_0\|^2}{(k-2)\sigma}.$$ 

Note that $\phi'(0) = 2\sigma T_2 > 0$. Hence $\phi'(t) > 0$. Thus

$$(\phi^{-\gamma}(t))' = -\frac{\gamma\phi'(t)}{\phi^{\gamma+1}(t)} < 0$$

and

$$(\phi^{-\gamma}(t))'' = \frac{-\gamma}{\phi^{\gamma+2}(t)} (\phi(t)\phi''(t) - (\gamma + 1)\phi'(t)^2) \leq 0,$$

where $\gamma = \frac{k-2}{4}$. Consequently, there exists a $T_3$ such that assertion (33) holds. According to [24, Lemma 15], we have

$$T_3 \leq \frac{\phi(0)}{\gamma \phi'(0)} = \frac{2(T_1\|u_0\|^2 + \sigma T_2^2)}{(k-2)\sigma T_2^2}.$$ 

Thus, for fixed $\sigma$ and $T_2$, we could choose the following finite time $T_1$ such that $T_3 < T_1$,

$$T_1 > \frac{2\sigma T_2^2}{(k-2)\sigma T_2^2 - 2\|u_0\|^2}.$$ 

Clearly, assertion (33) contradicts $T = \infty$. This completes the proof of Case 1 of (i) in Theorem 2.5.

**Case 2.** $E(u_0) = 0$.

We consider the auxiliary function

$$M(t) = \int_0^t \|u(\tau)\|^2 d\tau.$$ 

A direct calculation gives

$$M'(t) = \|u(t)\|^2$$

and

$$M''(t) = (u(t), u_t(t)) = -2I(u(t)). \quad (34)$$

In view of (34), (31) and $E(u_0) = 0$, we arrive at

$$M''(t) \geq C\|u(t)\|^2 + 2k \int_0^t \|u(\tau)\|^2 d\tau$$

with some constant $C > 0$. Consequently,

$$M(t)M'''(t) - \frac{k}{2} M'(t)^2 \geq CM(t)M'(t) + 2k \int_0^t \|u(\tau)\|^2 d\tau \int_0^t \|u(\tau)\|^2 d\tau - \frac{k}{2} M'(t)^2.$$ 

Applying Schwarz’s inequality to the second term on the right side of the above formula, we obtain

$$M(t)M'''(t) - \frac{k}{2} M'(t)^2 \geq CM(t)M'(t) + 2k \left( \int_0^t (u(\tau), u_\tau(\tau)) d\tau \right)^2 - \frac{k}{2} M'(t)^2. \quad (35)$$
Since
\[ \left( \int_0^t (u(\tau), u_\tau(\tau)) \, d\tau \right)^2 = \frac{1}{4} (\|u(t)\| - \|u_0\|)^2 \]
\[ \geq \frac{1}{4} M'(t)^2 - \frac{1}{2} \|u_0\|^2 M'(t), \]
we deduce from (36) that
\[ M(t)M''(t) - \frac{k}{2} M'(t)^2 \geq CM(t)M'(t) - k\|u_0\|^2 M'(t). \tag{37} \]

It is easy to see from (35) that
\[ M''(t) \geq 0 \text{ for all } t \in [0, \infty). \]
Hence there exists a \( t_0 > 0 \) such that
\[ M'(t) \geq M'(t_0) > 0, \]
and so
\[ M(t) \geq M(t_0)(t - t_0) + M(t_0) \text{ for all } t \in [t_0, \infty). \]
Consequently, for sufficiently large \( t \), we have
\[ CM(t) \geq k\|u_0\|^2. \tag{38} \]

Clearly, from (37) and (38), it follows that
\[ M(t)M''(t) - \frac{k}{2} M'(t)^2 \geq 0 \]
for sufficiently large \( t \). Hence \((M^{-\gamma}(t))' < 0 \) and \((M^{-\gamma}(t))'' \leq 0 \) for sufficiently large \( t \), where \( \gamma = \frac{k - 2}{2} \). Consequently, there exists a \( T_3 \) such that (33) holds. This contradicts \( T = \infty \).

(ii) We distinguish two cases.

Case 1. \( 0 < E(u_0) < d \).

By the same arguments as the proof of (i), there hold (27) and (32). Taking \( \sigma = 2(d - E(u_0)) \), we obtain
\[ \varphi(t) \geq (k - 2)\|\Delta u(t)\|^2 - 2kd. \]
Note that if
\[ \|\Delta u(t)\|^2 > \frac{2k}{k - 2} d, \tag{39} \]
then it is obvious that \( \varphi(t) > 0 \). Therefore, it suffices to confirm (39). Indeed, by Lemma 6.1, we have \( u(t) \in \mathcal{N}_- \) for all \( t \in [0, T_1] \), and so
\[ \|\Delta u(t)\|^2 < (f(u(t)), u(t)) \leq b\mathcal{E}^q\|\Delta u(t)\|^q. \]
We further get
\[ \|\Delta u(t)\|^2 > b^{-\frac{2}{q+2}}\mathcal{E}^{-\frac{2q}{q+2}}, \]
which, together with the definition of \( d \), tells us that there holds (39). The remainder of the proof of Case 1 can be finished by a repetition of Step 3 in the proof of Case 1 of (i).

Case 2. \( E(u_0) = d \).

By Lemma 6.1 and \((u_t(t), u(t)) = -I(u(t))\), we have
\[ \frac{d}{dt} \int_0^t \|u_\tau(\tau)\|^2 \, d\tau > 0. \]
Hence, from (9) and \( E(u_0) = d \), there exists a \( t_0 > 0 \) such that
\[ 0 < E(u(t_0)) = d - \int_0^{t_0} \|u_\tau(\tau)\|^2 \, d\tau < d, \]
Moreover, according to Lemma 6.1, we have $u(t_0) \in \mathcal{N}_-$. Therefore, the proof of this case is finished by the conclusion of Case 1.

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