Constructions of homotopy 4-spheres by pochette surgery

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Abstract
The boundary sum of the product of a circle with a 3-ball and the product of a disk with a 2-sphere is called a pochette. Pochette surgery, which was discovered by Iwase and Matsumoto, is a generalization of Gluck surgery and a special case of torus surgery. For a pochette \( P \) embedded in a 4-manifold \( X \), a pochette surgery on \( X \) is the operation of removing the interior of \( P \) and gluing \( P \) by a diffeomorphism of the boundary of \( P \). We present an explicit diffeomorphism of the boundary of \( P \) for constructing a 4-manifold after any pochette surgery. We also describe a necessary and sufficient condition for some pochette surgeries on any simply-connected closed 4-manifold create a 4-manifold with the same homotopy type of the original 4-manifold. In this paper we construct infinitely many embeddings of a pochette into the 4-sphere and prove that homotopy 4-spheres obtained from surgeries along these embedded pochettes are all diffeomorphic to the 4-sphere by some explicit handle calculus and relative handle calculus.

Keywords 4-Manifolds · Gluck surgery · Pochette surgery · Handle calculus · Relative handle calculus

Mathematics Subject Classification 57R65 · 57K40

1 Introduction
One of famous conjectures in 4-manifold topology is the 4-dimensional smooth Poincaré conjecture, which states that every homotopy 4-sphere is diffeomorphic to the 4-sphere. A Gluck surgery on a 4-manifold \( X \) is an operation of removing the interior of a tubular neighborhood of a 2-sphere in \( X \) with trivial normal bundle from \( X \) and gluing \( D^2 \times S^2 \) by a non-trivial diffeomorphism of the boundary \( S^1 \times S^2 \). All Gluck surgeries on the 4-sphere create homotopy 4-spheres. Whether these manifolds are diffeomorphic to the 4-sphere is a well-known unsolved problem. The homotopy 4-sphere obtained by the Gluck surgery along a twist spun 2-knot, a 0-slice 2-knot or \( m \)-twist \( n \)-roll spin of an unknotted number one 2-knot is diffeomorphic to the 4-sphere (see [3, 7] and [9]).
In 2004, Iwase and Matsumoto [5] introduced a generalization of Gluck surgery called pochette surgery. A pochette is the boundary sum $P = S^1 \times D^3 \# D^2 \times S^2$ of $S^1 \times D^3$ and $D^2 \times S^2$. A pochette surgery on a 4-manifold $X$ is an operation of removing the interior of $P$ embedded in $X$ and gluing $P$ to $X - \text{int } P$ by a diffeomorphism of $\partial P$. The diffeomorphism type of the manifold $X'$ obtained by a pochette surgery along $P$ embedded in $X$ is determined by the embedding $e: P \to X$, an element $p/q$ of $\mathbb{Q} \cup \{\infty\}$ called the slope, and an element $\varepsilon$ of $\{0, 1\}$ called the mod 2 framing because the isotopy class of the gluing diffeomorphism of $\partial P$ is characterized by $p/q$ and $\varepsilon$. We denote $X'$ by $X(e, p/q, \varepsilon)$ and call it the pochette surgery on $X$ for $e$, $p/q$, $\varepsilon$. The manifold $X(e, 1/0, 1)$ is nothing but the Gluck surgery on $X$ for the embedded 2-sphere $e(\ast) \times S^2$. A torus surgery on a 4-manifold $X$ is an operation of removing the interior of a tubular neighborhood of a torus in $X$ with trivial normal bundle from $X$ and gluing $S^1 \times S^1 \times D^2$ by a diffeomorphism of the boundary $S^1 \times S^1 \times S^1$. The manifold $X(e, p/q, \varepsilon)$ is a special case of a torus surgery on $X$.

Let $DP = P \cup (-P)$ be the double of $P$ and $i_P: P \to DP$ the inclusion map. Kashiwagi [6] found an algorithm for drawing handle diagrams of pochette surgeries of $DP$ and showed that $DP(i_P, 1/q, \varepsilon)$ is diffeomorphic to the Pao manifold $L(q; 0, 1; \varepsilon)$ (see [11]). Murase [8] constructed handle diagrams of all pochette surgeries of $DP$ and proved that $DP(i_P, p/q, \varepsilon)$ is diffeomorphic to $L(q; 0, 1; \varepsilon)$.

We can consider $P$ as $h^0 \cup h^1 \cup h^2$, where $h^i$ is an $i$-handle for $i = 0, 1, 2$. Okawa [10] proved that if the pochette surgery $S^4(e, p/q, \varepsilon)$ is a homology 4-sphere and the core of $e(h^1)$ is ‘trivial’ in $S^4 = \text{int } e(h^0) \cup h^2$, then $p$ must be 1. He also showed that $S^4(e, 1/q, \varepsilon)$ is diffeomorphic to the 4-sphere if $e(\ast) \times S^2$ is a ribbon 2-knot.

In this paper we construct infinitely many embeddings of $P$ into the 4-sphere and prove that homotopy 4-spheres obtained from surgeries along these embedded pockettes are all diffeomorphic to the 4-sphere.

The diagram depicted in Fig. 1 is a handle diagram for the 4-sphere, where $k$ is an integer greater than one, $K = (n_1, \ldots, n_{k^2-1})$ is a $(k^2 - 1)$-tuple of integers, and the sign of $\pm 1$ can be taken arbitrarily. Let $e_{k,K}: P \to S^4$ be the inclusion map from the pochette $P$ which consists of the $0$-handle, the $1$-handle presented by the leftmost dotted circle, and the $2$-handle presented by the rightmost $0$-framed unknot in Fig. 1.

Theorem 1 The pochette surgery $S^4(e_{k,K}, 1/q, \varepsilon)$ on the 4-sphere $S^4$ for $e_{k,K}$, $1/q$, $\varepsilon$ is diffeomorphic to $S^4$ for every $k$, $K$, $q$ and $\varepsilon$ (see also Fig. 2).

The diagram depicted in Fig. 3 is also a handle diagram for the 4-sphere, where $s$, $t$ are positive integers, $M = (m_1, \ldots, m_s)$ is a $s$-tuple of integers such that $\sum_{i=1}^{s} m_i = 0$, and $N = (n_1, \ldots, n_{st+1})$ is a $(st + 1)$-tuple of integers. Let $e_{M,N}: P \to S^4$ be the inclusion map from $P$ which consists of the $0$-handle, the $1$-handle presented by the leftmost dotted circle, and the $2$-handle presented by the rightmost $0$-framed unknot in Fig. 3.

Theorem 2 The pochette surgery $S^4(e_{M,N}, 1/q, \varepsilon)$ on the 4-sphere $S^4$ for $e_{M,N}$, $1/q$, $\varepsilon$ is diffeomorphic to $S^4$ for every $M$, $N$, $q$ and $\varepsilon$ (see also Fig. 4).

In Sect. 2 we review a precise definition and known properties of pochette surgery. In Sect. 3 we give a construction of handle diagrams for pochette surgeries. In Sect. 4 we compute the homology of pochette surgeries on any simply-connected closed 4-manifold. In Sect. 5 we give proofs of the main results. We assume that all manifolds are smooth, compact and oriented and all maps are smooth.
Fig. 1 A handle diagram of the 4-sphere

Fig. 2 The handle diagram of $S^4(\epsilon_k, K \cdot 1/q, \epsilon)$
2 Preliminaries

Let $X$ be a 4-manifold and $E(A)$ the exterior $X - \text{int} A$ of a subset $A$ of $X$. Let $Q_e$ be the image $e(Q)$ of a subset $Q$ of $P$, $e : P \to X$ an embedding and $g : \partial P \to \partial E(P_e)$ a diffeomorphism. We call the curves $l := S^1 \times \{\ast\}$ and $m := \partial D^2 \times \{\ast\}$ on $\partial P$ a longitude and a meridian of $P$, respectively. First, we define the pochette surgery.

**Definition 1** (Iwase-Matsumoto [5]) A pochette surgery on $X$ is an operation of removing $\text{int} P_e$ and gluing in $P$ by $g : \partial P \to \partial E(P_e)$. The 4-manifold $E(P_e) \cup g P$ obtained by the pochette surgery on $X$ using $e$ and $g$ is denoted by $X(e, g)$. The manifold $X(e, g)$ is also called the pochette surgery on $X$ for $e$ and $g$.

In pochette surgery on a 4-manifold, after attaching $D^2 \times S^2$ to $P$ along $g(m)$, the method of attaching $S^1 \times D^3$ is unique. Therefore, when gluing $P$, it is sufficient to consider an identification between neighborhoods of $m$ and $g(m)$ via $g$.

Fix an identification between $\partial P$ and $S^1 \times \partial D^3 \# \partial D^2 \times S^2 = S^1 \times S^2 \# S^1 \times S^2$. The meridian $m$ of $P$ has the natural product framing. By embedding $e$, we get identification $\iota : \partial E(P_e) \to S^1 \times S^2 \# S^1 \times S^2$. Then, $S^1 \times S^2 \# S^1 \times S^2$ can be expressed as the 2-component unlink which consists of 2 0-framed knots. Therefore, $g$ maps the natural framing on $m$ of $\partial P$ to a framing on $g(m)$. This framing on $g(m)$ is represented by some integer determined...
by \( \ell \). The pochette can be regarded as \( S^1 \times D^3 \) attaching a 2-handle with the cocore \( m \).

Let \( g_1, g_2 : \partial P \to \partial E(P_e) \) be two gluing maps. If \( g_1(m) \) and \( g_2(m) \) are the same and a difference between the framing on \( g_1(m) \) and that of \( g_2(m) \) is even, the map \( g_1^{-1}g_2|\partial(m) \) can be extended to the inside of the 2-handle. Here, \( N(A) \) is the open tubular neighborhood for a submanifold \( A \) of \( P \). Therefore, when considering the diffeomorphism type of the pochette surgery, we should consider an integer modulo 2 as the framing on submanifold \( X_i \):

Then \( g \) is extended to the inside of the 2-handle. Here, \( g \) is the meridian homomorphism of the composite map of the gluing map \( g = \partial P \to \partial E(P_e) \) as introduced in [5, First paragraph in p.162].

By [5, Lemma 4], the diffeomorphism type of \( X(e, g) \) is determined by the embedding \( e : P \to X \), the isotopy class of a simple closed curve \( g(m) \) and the mod 2 framing around \( g(m) \). For orientation preserving self-diffeomorphisms \( g, g' \) of \( \partial P \), if \( g_*([m]) \) is equal to \( g'_*([m]) \), then \( g(m) \) is isotopic to \( g'(m) \) (see [5, Lemma 5]). Hence, the diffeomorphism type of \( X(e, g) \) is determined by an embedding \( e : P \to X \), a homology class \( g_*([m]) \) in \( H_1(\partial E(P_e)) \cong \mathbb{Z}[m] \oplus \mathbb{Z}[l] \) and the mod 2 framing around \( g(m) \).

Let \( p, q \) be coprime integers and \( \epsilon \) an element of \( \{0, 1\} \). By [5, the seventh paragraph in p.163], the homology class \( p[m] + q[l] \in \mathbb{Z}[m] \oplus \mathbb{Z}[l] \) is determined by \( p/q \in \mathbb{Q} \cup \{\infty\} \) up to sign of \( p \). The next theorem immediately follows from observations here (see [5, Theorem 2]).

**Theorem 3** (Iwase-Matsumoto [5]) The diffeomorphism type of \( X(e, g) \) is determined by the following data:

1. an embedding \( e : P \to X \),
2. a slope \( p/q \),
3. a mod 2 framing around \( g(m) \).

Let \( g_{p/q, \epsilon} : \partial P \to \partial E(P_e) \) be a diffeomorphism which satisfies \( g_{p/q, \epsilon}([m]) = p[m] + q[l] \) and the mod 2 framing of \( g_{p/q, \epsilon}(m) \) is \( \epsilon \in \{0, 1\} \). We can define \( X(e, p/q, \epsilon) = X(e, g_{p/q, \epsilon}) \). From the construction, any pochette surgery for \( e, 1/0, \epsilon \) is nothing but the Gluck surgery along \( S \), where \( S \) is the subset \( \{\} \times S^2 \) of \( P \).

We define \( ST := S^1 \times D^2 \). Let \( e_0 : S^1 \times ST \to X \) be an embedding and \( g_0 : \partial(S^1 \times ST) \to \partial E(S^1 \times ST) \) a diffeomorphism. The diffeomorphism type of the manifold \( E((S^1 \times ST)_{e_0}) \cup_{g_0} (S^1 \times ST) \) obtained by a torus surgery on \( X \) is determined by \( e_0 \) and \( (g_0)_*([\{\} \times S^1 \times \{\}]) \) in \( H_1(S^1 \times \partial ST) \). Let \( H \) be a 0-framed 2-handle attached to \( S^1 \times ST \) along \( S^1 \times \{\} \times \{\} \). Fix an identification between \( S^1 \times ST \cup H \) and \( P \). Then \( \{\} \times S^1 \times \{\} \) is the meridian \( m \) of \( P \), \( \{\} \times \{\} \times \partial D^2 \) is the longitude \( l \) of \( P \). We define \( s := S^1 \times \{\} \times \{\} \times \{\} \). Then \( [m], [l], [s] \) are a basis of \( H_1(S^1 \times \partial ST) \). If \( e_0 = e|_{S^1 \times ST} \), then we have the manifold \( X(e, p/q, \epsilon) \) is the torus surgery with \( e_0 \) and \( (g_0)_*([m]) = p[m] + q[l] + \epsilon p[s] \) (see [5, Section 3]). This means that any pochette surgery on \( X \) is nothing but a torus surgery on \( X \).

Suppose \( X \) is a homology 4-sphere, and \( i_{11} : H_1(\partial P) \to H_1(E(P_e)) \) is the induced homomorphism of the composite map of the gluing map \( g : \partial P \to \partial E(P_e) \) and the inclusion map \( i_{0E(P_e)} : \partial E(P_e) \to E(P_e) \). Okawa calculated some homology groups of the pochette surgery \( X(e, p/q, \epsilon) \):

**Theorem 4** (Okawa [10, Theorem 1.1]) If \( i_{11}(l) = 0 \), then \( H_1(X(e, p/q, \epsilon)) \cong \mathbb{Z}_p \) and \( H_2(X(e, p/q, \epsilon)) = 0 \). Moreover, if \( \mid p \mid \) is equal to 1, then \( X(e, p/q, \epsilon) \) is a homology 4-sphere.

Homology groups of pochette surgeries on any simply-connected closed 4-manifold will be calculated in Sect.4.
Recall $P$ can be interpreted as $h^0 \cup h^1 \cup h^2$. We call the core of $(h^1)_e$ a cord. A cord is trivial if it is boundary parallel. Okawa also showed that $S^4(e, 1/q, \varepsilon)$ is diffeomorphic to the 4-sphere if the cord $(h^1)_e$ is trivial in $E((h^0 \cup h^2)_e)$ and $S_e$ is a ribbon 2-knot.

**Theorem 5** (Okawa [10, Theorem 1.2]) If $(h^1)_e$ is the trivial cord and $S_e$ is a ribbon 2-knot, then the manifold $S^4(e, 1/q, \varepsilon)$ is diffeomorphic to the 4-sphere.

**Proof** This is a variation of [4, Exercise 6.2.11(b)].

### 3 Handle diagram for pochette surgery

In this section we give a construction of handle diagrams for pochette surgeries under special conditions. Let $X$ be a 4-manifold and $e : P \rightarrow X$ an embedding from a pochette $P$ into $X$. Let $p, q$ be coprime integers and $\varepsilon$ an element of $\{0, 1\}$. Suppose that the diagram depicted in Fig. 5 is a part of a handle diagram for $X$, where all the curves partially drawn in Fig. 5 are framed knots, and any framed knot entwined with the dotted circle in Fig. 5 has a 0-framed meridian. The pochette $P_e$ consists of the 0-handle, the 1-handle presented by the leftmost 0-framed unknot and the 2-handle presented by the rightmost 0-framed unknot in Fig. 5.

**Proposition 1** A handle diagram of $X(e, p/q, \varepsilon)$ is depicted in Fig. 6, where $p'$ is 1 if $p$ is equal to 0 and $p$ otherwise.

**Proof** Here we will consider the case where only a framed knot is entwined with the 0-framed knot on the right side exactly once. The case where framed knots are entwined with the 0-framed knot on the right side can be proved in the same way. If $|p|$ and $|q|$ are coprime positive integers, then there exist a positive integer $n$, a non-negative integer $a_0$ and positive integers $a_1, \ldots, a_n$ such that

$$\frac{|p|}{|q|} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}}.$$

We define the diffeomorphism $E_0, E_1, E_2, E_3, E_4$ and $E_5 : \partial P \rightarrow \partial P$ as the 1-Rolfsen twist for the leftmost (0)-framed knot, the handle slide in Figs. 7, 8, 9 and 10, the operation changing the direction of the meridian $m$, respectively.
Then we have

\[
E_{i*}([m]) = \begin{cases} 
[m] & (i = 0, 2, 4), \\
[m] + [l] & (i = 1), \\
[m] - [l] & (i = 3), \\
-[m] & (i = 5), 
\end{cases}
\]

\[
E_{i*}([-l]) = \begin{cases} 
[l] & (i = 0, 1, 3, 5), \\
-[m] + [l] & (i = 2), 
\end{cases}
\]

We define \( E_{p/q, \varepsilon} \) to be

\[
\begin{align*}
E_0^e & \quad (p = 1, q = 0), \\
E_4E_1E_0^e & \quad (p = 0, q = 1), \\
E_5E_0^e & \quad (p = -1, q = 0), \\
E_4E_1E_5E_0^e & \quad (p = 0, q = -1), \\
E_2^{a_0}E_1^{a_1}E_2^{a_2} \cdots E_2^{a_{n-1}}E_1E_0^e & \quad (p > 0, q > 0, n \in 2\mathbb{N}), \\
E_2^{a_0}E_1^{a_1}E_2^{a_2} \cdots E_2^{a_{n-1}}E_1^{a_0}E_0^e & \quad (p > 0, q > 0, n \in 2\mathbb{N} - 1), \\
E_4^{a_0}E_3^{a_1}E_4^{a_2} \cdots E_4^{a_{n-1}}E_3E_0^e & \quad (p > 0, q < 0, n \in 2\mathbb{N}), \\
E_4^{a_0}E_3^{a_1}E_4^{a_2} \cdots E_4^{a_{n-1}}E_3^{a_0}E_0^e & \quad (p > 0, q < 0, n \in 2\mathbb{N} - 1), \\
E_4^{a_0}E_3^{a_1}E_4^{a_2} \cdots E_4^{a_{n-1}}E_3E_5E_0^e & \quad (p < 0, q > 0, n \in 2\mathbb{N}), \\
E_4^{a_0}E_3^{a_1}E_4^{a_2} \cdots E_4^{a_{n-1}}E_3^{a_0}E_5E_0^e & \quad (p < 0, q > 0, n \in 2\mathbb{N} - 1), \\
E_2^{a_0}E_1^{a_1}E_2^{a_2} \cdots E_2^{a_{n-1}}E_1E_5E_0^e & \quad (p < 0, q < 0, n \in 2\mathbb{N}), \\
E_2^{a_0}E_1^{a_1}E_2^{a_2} \cdots E_2^{a_{n-1}}E_1^{a_0}E_5E_0^e & \quad (p < 0, q < 0, n \in 2\mathbb{N} - 1).
\end{align*}
\]

and \( g_{p/q, \varepsilon} = eE_{p/q, \varepsilon} \). Then we have \( E_{p/q, \varepsilon}([m]) = p[m] + q[l] \) for any \( p/q \in \mathbb{Q} \cup \{\infty\} \) and \( \varepsilon \in \{0, 1\} \). By Theorem 3, the pochette surgery \( X(e, p/q, \varepsilon) \) is diffeomorphic to \( X(e, g_{p/q, \varepsilon}) \) for any \( p/q \in \mathbb{Q} \cup \{\infty\} \) and \( \varepsilon \in \{0, 1\} \). A part of a handle diagram of \( X \) is depicted in Fig. 11. By several handle slides on the 0-framed meridians of the framed knots entwined with the dotted circle, we obtain a part of a handle diagram of \( X(e, p/q, \varepsilon) \) depicted in Fig. 12.
Concretely, the homotopy class of \( g_{p/q, \varepsilon}(m) \) is the natural lift defined in [5]:

\[
\begin{align*}
&\left\{ m' l'^q \prod_{k=1}^{[p]} l'^q / q (|k|q / |l| - |(k-1)q / |l|) m' p / q \right\} (pq = 0), \\
&\prod_{k=1}^{[p]} l'^q / q (|k|q / |l| - |(k-1)q / |l|) m' p / q \right\} (pq \neq 0).
\end{align*}
\]

Here, \( l', m' \) are the images on \( \pi_1(\partial E(P_e)) \) of based, oriented longitude and meridian in \( \partial P \) via \( e \).

If \( q = 0 \), then we reach the desired result.

If \( q \neq 0 \), we obtain Fig. 13 by creating a 2-handle/3-handle pair in Fig. 12. By the handle slide in Fig. 13 and several handle slides on the 0-framed meridians of the framed knots entwined with the dotted circle, we obtain the handle diagram depicted in Fig. 14. By the handle slides between the leftmost 0-framed knot and the rightmost 0-framed knot in Fig. 14:

\[
\begin{align*}
&E^p E^{-1}_{p/q, \varepsilon} (p > 0 \text{ or } (p, q) = (0, 1)), \\
&E^p E^{-1}_{p/q, \varepsilon} (p < 0 \text{ or } (p, q) = (0, -1)),
\end{align*}
\]

we obtain the handle diagram depicted in Fig. 15. Changing the self-intersection of the framed knot in Fig. 15 by several handle slides on the 0-framed meridian and canceling the 2-handle/3-handle pair, we obtain the handle diagram depicted in Fig. 16. Therefore, we also obtain the conclusion in the case of \( q \neq 0 \).  

\( \square \)

Remark 1 For any 4-manifold \( X \), an embedding \( e : P \rightarrow X \), coprime integers \( p, q \) and an element \( \varepsilon \) of \( \{0, 1\} \), we give the explicit diffeomorphism \( e E_{p/q, \varepsilon} : \partial P \rightarrow \partial E(P_e) \) for constructing the pochette surgery \( X(e, p/q, \varepsilon) \).
Remark 2 If $p = 1$, Proposition 1 holds even without the 0-framed meridians, so any pochette surgery $X(e, 1/q, \varepsilon)$ is given by Fig. 6.

Remark 3 Proposition 1 is a generalization of [6, Theorem 1.2] and [8, Theorem 1.1, 1.2]. The curve $g_{p/q, r}(m)$ in $\partial P$ is depicted in Fig. 17 in the case of $|p| = 1, q = 0$, in Fig. 18 in the case of $p = 0, |q| = 1$, in Fig. 19 in the case of $|q| > |p| > 0, pq > 0$, in Fig. 20 in the case of $|q| > |p| > 0, pq < 0$, in Fig. 21 in the case of $|p| > |q| > 0, pq > 0$, in Fig. 22 in the case of $|p| > |q| > 0, pq < 0$. We perform a Rolfsen twist just before or just after performing each handle slide $E_1, E_2, E_3, E_4$ by using either of two $\langle 0 \rangle$-framed knots in $\partial P_e$. Then we can obtain the curves depicted in Figs. 19, 20, 21 and 22. The curve depicted in Fig. 19 was first discovered by Murase [8].
4 Homology of pochette surgery

Let \( p, q \) be coprime integers and \( \varepsilon \) an element of \{0, 1\}. Let \( B \) be a subset \{\ast\} \times \partial D^3 of \( \partial P \) and \( S \) a subset \{\ast\} \times \partial D^3 of \( \partial P \). Let \( X \) be a simply-connected closed 4-manifold and \( e : P \to X \) an embedding from a pochette \( P \) into \( X \). Here, we prove lemmas needed later.

**Lemma 1** If the homomorphism \( t_2 : H_2(X) \to H_2(X, E(P_e)) \) induced by the inclusion map \((X, \emptyset) \to (X, E(P_e))\) is a zero map, the homology groups of the exterior \( E(P_e) \) of \( P_e \) are calculated as follows:

\[
H_n(E(P_e)) = \begin{cases} 
\mathbb{Z}[x_e] & (n = 0), \\
\mathbb{Z}[m_e] & (n = 1), \\
\mathbb{Z}[B_e] \oplus H_2(X) & (n = 2), \\
0 & (\text{otherwise}).
\end{cases}
\]

Here \( x \) is a point in \( \partial P \).

**Proof** By the long exact sequence of the pair \((P, \partial P)\):

\[
\cdots \xrightarrow{\partial_{n+1}} H_n(\partial P) \xrightarrow{s_n} H_n(P) \xrightarrow{t_n} H_n(P, \partial P) \xrightarrow{\partial_n} \cdots,
\]

we have

\[
H_n(P, \partial P) = \begin{cases} 
\mathbb{Z}[D^2 \times \{\ast\}] & (n = 2), \\
\mathbb{Z}[\{\ast\} \times D^3] & (n = 3), \\
\mathbb{Z}[P] & (n = 4), \\
0 & (\text{otherwise}).
\end{cases}
\]

By the Excision Theorem, we obtain

\[ H_n(X, E(P_e)) \cong H_n(P, \partial P) \text{ for any } n \in \mathbb{Z}. \]
Let $\partial_n : H_n(X) \to H_n(X, E(P_e))$, $i_n : H_n(\partial P) \to H_n(E(P_e))$,

$i_{n_2} : H_n(\partial P) \to H_n(P)$ be homomorphisms induced by inclusion maps. If $t_2 = 0$, $i_{11}([l]) = 0$ and $i_{21}(H_2(\partial P))$ is included in $\mathbb{Z}[B_e] \oplus \mathbb{Z}[S]$, then we have $i_{21}([B]) = \pm p[B_e]$ and $i_{21}([S]) = 0$.

**Proof** By the definitions of $E_0, E_1, E_2, E_3, E_4$ and $E_5$ in the proof of Proposition 1, we obtain

\[
E_i^s([m]) = \begin{cases} 
[m] & (i = 0, 2, 4), \\
[m] + [l] & (i = 1), \\
[m] - [l] & (i = 3), \\
-[m] & (i = 5),
\end{cases}
E_i^s([l]) = \begin{cases} 
[l] & (i = 0, 1, 3, 5), \\
[m] + [l] & (i = 2), \\
-[m] + [l] & (i = 4),
\end{cases}
\]

\[
E_i^s([B]) = \begin{cases} 
[B] & (i = 0, 2, 4), \\
[B] \mp [S] & (i = 1), \\
[B] \pm [S] & (i = 3), \\
\pm [B] & (i = 5),
\end{cases}
E_i^s([S]) = \begin{cases} 
[S] & (i = 0, 1, 3), \\
\mp [B] + [S] & (i = 2), \\
\pm [B] + [S] & (i = 4), \\
\mp[S] & (i = 5)
\end{cases}
\]

(double-sign corresponds). Then, there exist some integers $r, s$ such that

\[
g_{p/q,s^*}([m]) = p[m_e] + q[l_e], g_{p/q,s^*}([l]) = r[m_e] + s[l_e], g_{p/q,s^*}([B]) = \pm p[B_e] \pm q[S_e]
\]

and $g_{p/q,s^*}([S]) = \mp r[B_e] \pm s[S_e]$. Let $i_{\delta E(P_e)} : \partial E(P_e) \to E(P_e)$ be the inclusion map. Then we have $i_{n_1} = (i_{\delta E(P_e)} g_{p/q,e})_{n^*}$ for any $n \in \mathbb{Z}$. Here $f_{n^*} : H_n(A) \to H_n(B)$ is the $n$-th induced homomorphism on homology of a continuous map $f : A \to B$. If $t_2 = 0$, by Lemma 1, $H_1(E(P_e)) = \mathbb{Z}[m_e]$ and $H_2(E(P_e)) = \mathbb{Z}[B_e] \oplus H_2(X)$. If $i_{11}([l]) = 0$ and $i_{21}(H_2(\partial P))$ is included in $\mathbb{Z}[B_e] \oplus \mathbb{Z}[S]$, then we have $i_{21}([S]) = 0$. By $i_{11}([m]) = p[m_e]$, we also have $i_{21}([B]) = \pm p[B_e]$. Therefore, we obtain the desired result above.

For a simply-connected closed 4-manifold $X$, we give a necessary condition for a pochette surgery of $X$ to have the same homology as $X$.

**Proposition 2** Let $t_n : H_n(X) \to H_n(X, E(P_e))$, $i_n : H_n(\partial P) \to H_n(E(P_e))$,

$i_{n_2} : H_n(\partial P) \to H_n(P)$ be homomorphisms induced by inclusion maps. If $t_2 = 0$, $i_{11}([l]) = 0$ and $i_{21}(H_2(\partial P))$ is included in $\mathbb{Z}[B_e] \oplus \mathbb{Z}[S]$, then the homology groups of the pochette surgery $X(e, p/q, \varepsilon)$ are calculated as follows:

\[
H_n(X(e, p/q, \varepsilon)) = \begin{cases} 
\mathbb{Z} & (n = 0, 4), \\
\mathbb{Z}_p & (n = 1).
\end{cases}
\]

Moreover, if $|p|$ is equal to 1, then $X(e, p/q, \varepsilon)$ has the same homology groups as $X$.

**Proof** We define $H_n := H_n(X(e, p/q, \varepsilon))$ for $n = 0, \ldots, 4$. Since $X$ is connected and oriented, $H_n \cong \mathbb{Z}$ for any $n = 0, 4$. We compute $H_1$ here. By Lemma 1 and the Mayer-Vietoris sequence

\[
\cdots \xrightarrow{\partial_{n+1}} H_n(\partial P) \xrightarrow{i_n \oplus i_{n_2}} H_n(E(P_e)) \oplus H_n(P) \xrightarrow{j_n} H_n \xrightarrow{\partial_n} \cdots
\]
we obtain the following:

\[ \begin{array}{cccccc}
\longrightarrow & 0 & \longrightarrow & H_3 & \xrightarrow{\partial_3} & \mathbb{Z}[B] \oplus \mathbb{Z}[S] \\
\xrightarrow{i_{11} \oplus i_{12}} & \mathbb{Z}[m_e] \oplus \mathbb{Z}[l] & \xrightarrow{j_1} & H_1 & \xrightarrow{\partial_1} & H_0(\partial P).
\end{array} \]

Then, we have \( i_{11}([m]) = p[m_e], i_{12}([m]) = 0, i_{12}([l]) = [l], i_{22}([B]) = 0 \) and \( i_{22}([S]) = [S]. \) If \( i_{11}([l]) = 0, \) then we have

\[ \text{Ker} \partial_1 = \text{Im} j_1 \cong \mathbb{Z}[m_e] \oplus \mathbb{Z}[l]/\text{Im} (i_{11} \oplus i_{12}) \cong \mathbb{Z}[m_e]/p\mathbb{Z}[m_e] \cong \mathbb{Z}/p \]

and \( \text{Im} \partial_1 = 0. \) Thus \( H_1 \cong \mathbb{Z}/p. \) If \( |p| = 1, \) then we have \( H_n \cong H_n(X) \) for any \( n = 0, 1, 4. \)

By Lemma 2, we have \( i_{21}([B]) = \pm p[B_e], i_{21}([S]) = 0. \) Therefore, we have \( \text{Ker} \partial_3 = 0, \text{Im} \partial_3 = \text{Ker} (i_{21} \oplus i_{22}) = 0, \) and

\[ \text{Ker} \partial_2 = \text{Im} j_2 \cong (\mathbb{Z}[B_e] \oplus H_2(X)) \oplus \mathbb{Z}[S]/\text{Im} (i_{21} \oplus i_{22}) = H_2(X), \]

\( \text{Im} \partial_2 = \text{Ker} (i_{11} \oplus i_{12}) = 0. \) Then we have \( H_n \cong H_n(X) \) for any \( n = 2, 3. \) \( \square \)

**Remark 4** Proposition 2 is a generalization of Theorem 4.

The next corollary follows from Proposition 2 and the Freedman theorem (see [1] and [2]).

**Corollary 1** If \( i_2 = 0, i_{11}([l]) = 0 \) and \( i_{21}(H_2(\partial P)) \) is included in \( \mathbb{Z}[B_e] \oplus \mathbb{Z}[S], \) then \( X(e, p/q, \varepsilon) \) is homeomorphic to \( X \) if and only if \( X(e, p/q, \varepsilon) \) is a simply-connected 4-manifold and \( |p| = 1. \)

**Proof** If \( X(e, p/q, \varepsilon) \) is homeomorphic to \( X, \) then \( X(e, p/q, \varepsilon) \) has the same homology groups as \( X. \) By Proposition 2, \( X(e, p/q, \varepsilon) \) is a simply-connected 4-manifold and \( |p| = 1. \) Conversely, if \( X(e, p/q, \varepsilon) \) is a simply-connected 4-manifold and \( |p| = 1, \) we obtain a natural isomorphism \( H_2(X(e, p/q, \varepsilon)) \cong H_2(X) \) by the proof of Proposition 2. Hence, \( Q_{X(e, p/q, \varepsilon)} \cong Q_X. \) Here, \( Q_Y \) is the intersection form of the 4-manifold \( Y. \) Since \( X(e, p/q, \varepsilon) \) and \( X \) are simply-connected 4-dimensional closed manifolds with differential structures, \( X(e, p/q, \varepsilon) \times \mathbb{R} \) and \( X \times \mathbb{R} \) have differential structures. Therefore, we obtain \( kS(X(e, p/q, \varepsilon)) = 0 = kS(X). \) Here, \( kS(Y) \) is the Kirby-Siebenmann invariant of \( Y. \) By the Freedman theorem, \( X(e, p/q, \varepsilon) \) is homeomorphic to \( X. \) Therefore, we obtain the desired result above. \( \square \)

### 5 Proofs of main theorems

Canceling the 1-handle/2-handle pairs and the 2-handle/3-handle pair in the handle diagrams depicted in Figs. 1 and 2, we obtain the standard handle diagram of the 4-sphere which consists of a 0-handle and a 4-handle. Therefore both of the handle diagrams depicted in Figs. 1 and 2 are those of the 4-sphere. By Proposition 1 and Remark 2, the handle diagrams depicted in Figs. 3 and 4 are those of the manifold \( S^4(e_{k,K}, 1/q, \varepsilon) \) and the manifold \( S^4(e_{M,N}, 1/q, \varepsilon) \), respectively. Let \( H(n) \) be the union of \( n \) 3-handles and a 4-handle.

**Proof of Theorem 1** We remove the 3-handle and the 4-handle in the handle diagram depicted in Fig. 2. Taking the double of the obtained handle diagram and removing all the 1-handles and
2-handles that has existed since the handle diagram depicted in Fig. 2, we obtain the handle diagram depicted in Fig. 23. By several handle slides on 0-framed meridians and \( \langle 0 \rangle \)-framed knots in Fig. 23, \( 4k - 7 \) 0-framed knots can be changed to \( 4k - 7 \) 0-framed unknots. Canceling the \( 4k - 7 \) 2-handle/3-handle pairs, we obtain the handle diagram depicted in Fig. 24. By the handle slides in Fig. 24, we obtain the handle diagram depicted in Fig. 25. By several handle slides on the \( 2 \langle 0 \rangle \)-framed meridians in Fig. 25, we obtain the handle diagram depicted in Fig. 26. We can cancel the 2 Hopf links which consists of 2 \( \langle \cdot \rangle \)-framed knots in Fig. 26. By the handle slides in Fig. 26 and several handle slides on the \( 3 \langle 0 \rangle \)-framed meridians, we obtain the handle diagram depicted in Fig. 27. By the handle slides in Fig. 27, we obtain a Hopf link and a 0-framed meridian. By several handle slides on 0-framed meridians and \( \langle 0 \rangle \)-framed knots in Fig. 27, we obtain the handle diagram depicted in Fig. 28. Repeating the handle calculus in Figs. 24, 25, 26, 27 and 28 in the same way, we obtain the handle diagram depicted in Fig. 29. Here, \( L = \sum_{i=3}^{k^2 - 1} n_i \).

By the handle slide in Fig. 29, we obtain the handle diagram depicted in Fig. 30. By several handle slides on the moved \( \langle 0 \rangle \)-framed knot, we obtain the handle diagram depicted in Fig. 31.
Canceling the 2 2-handle/3-handle pairs and the 3 Hopf links, we obtain the handle diagrams depicted in Figs. 32, 33 and 34 in order. Changing the $\langle 0 \rangle$-framed knot in Fig. 34 to a dotted circle, the upside down of the handle diagram depicted in Fig. 1 is completed. Canceling the 1-handle/2-handle pair, we obtain the standard handle diagram of the 4-sphere which consists of a 0-handle and a 4-handle.

Therefore, $S^4(e_k, K, 1/q, \epsilon)$ is diffeomorphic to $S^4$. □
Proof of Theorem 2  By the handle slides in Fig. 35, several handle slides on dotted circles and canceling the $s(t - 1)$ 1-handle/2-handle pairs, we obtain the handle diagram depicted in Fig. 36. Here, $x_i = \sum_{j=1}^{t} n_{s(j-1)+i}$ for any $i = 1, \ldots, s$. By the handle slides in Figs. 36 and 37, we obtain the handle diagram depicted in Fig. 38. Here, $a, b_1$ are some integers. Repeating the method of the handle calculus in Figs. 37 and 38, we obtain the handle diagram depicted in Fig. 39. Here, $b_2, \ldots, b_{s-2}$ are some integers.

By the handle slide in Fig. 40 or 41, we obtain the handle diagram depicted in Fig. 42. By the handle slide in Fig. 42, we obtain the handle diagram depicted in Fig. 43. By several handle slides on the 0-framed meridian, the handle diagram depicted in Fig. 43 can be changed.
to that depicted in Fig. 44. By the handle calculus in Figs. 42 and 43 or 44 $|m_s|$ times, we obtain the handle diagram depicted in Fig. 45. By the handle slide in Fig. 45 and the method of the handle calculus in Figs. 42 and 43 or 44 $|m_{s-1} + m_s|$ times, we obtain the handle diagram depicted in Fig. 46. Here, $\epsilon = m_{s-1} + m_s$. By repeating the method of the handle calculus in Figs. 42, 43, 44, 45 and 46 and $\sum_{i=1}^s m_i = 0$, we can the $(x_1 + \epsilon)$-framed knot in Fig. 39 away from the dotted circles entwined with the $n_{st+1}$-framed knot. Thus, we obtain the handle diagram depicted in Fig. 47.

By the handle slide in Fig. 40 (Fig. 41) and the handle calculus in Figs. 42, 43, 44, 45, 46 and 47, we obtain the handle diagram depicted in Fig. 48 (Fig. 49) (double sign corresponds).
Therefore, we have the handle diagram depicted in Fig. 39 with $q = 0$ by the handle slide in Fig. 40 or 41 $|q|$ times. By canceling the $s + 1$ 1-handle/2-handle pairs and the 2-handle/3-handle pair in the handle diagrams depicted in Fig. 39 with $q = 0$, we obtain the standard handle diagram of the 4-sphere which consists of a 0-handle and a 4-handle.

Therefore, $S^4(e_{M,N}, 1/q, \varepsilon)$ is diffeomorphic to $S^4$. \qed

We end this section by raising a conjecture about pochette surgery of the 4-sphere.
Fig. 39 The fifth handle diagram in the proof of Theorem 2

Fig. 40 The sixth handle diagram in the proof of Theorem 2

Fig. 41 The seventh handle diagram in the proof of Theorem 2
Fig. 42 The eighth handle diagram in the proof of Theorem 2

Fig. 43 The ninth handle diagram in the proof of Theorem 2

Fig. 44 The tenth handle diagram in the proof of Theorem 2
Fig. 45  The eleventh handle diagram in the proof of Theorem 2

Fig. 46  The twelfth handle diagram in the proof of Theorem 2

Fig. 47  The thirteenth handle diagram in the proof of Theorem 2
Conjecture 1 If the pochette surgery $S^4(e, p/q, \varepsilon)$ is homotopy equivalent to the 4-sphere $S^4$, then $S^4(e, p/q, 0)$ is diffeomorphic to $S^4$ and $S^4(e, p/q, 1)$ is the Gluck surgery along $S_e$. Especially when $S_e$ is a twist spun 2-knot, a 0-slice 2-knot or $m$-twist $n$-roll spin of an unknotted number one 2-knot, $S^4(e, p/q, \varepsilon)$ is diffeomorphic to $S^4$ for any element $\varepsilon$ of $\{0, 1\}$.

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