Abstract. In this survey, we discuss some basic problems concerning random matrices with discrete distributions. Several new results, tools and conjectures will be presented.

1. Introduction

Random matrices is an important area of mathematics, with strong connections to many other areas (mathematical physics, combinatorics, theoretical computer science, to mention a few).

There are two types of random matrices: continuous and discrete. The continuous models have an established theory (see [38], for instance). On the other hand, the discrete models are still not very well understood. In this survey, we discuss a few basic problems concerning these models. The topics to be discussed are:

- The limiting distribution of the spectrum (Section 3).
- The spectral norm and the second largest eigenvalue (Sections 4, 5).
- Determinant (Section 6).
- Rank and Singular probability (Sections 7, 8).
- The condition number (Section 9).
- Tools from additive combinatorics (Sections 10, 11, 12).

Notations. We denote by $M_n$ the $n$ by $n$ random matrix whose entries are i.i.d Bernoulli random variables (taking values 1 and $-1$ with probability $1/2$). This matrix is not symmetric. Symmetric matrices often come from graphs. We denote by $Q(n, p)$ the adjacency matrix of the Erdős-Rényi random graph $G(n, p)$. Thus $Q(n, p)$ is a random symmetric matrix whose upper diagonal entries are i.i.d random variables taking value 1 with probability $p$ and 0 with probability $q = 1 - p$. Another popular model for random graphs is that of random regular graphs. A random regular graph $G_{n,d}$ is obtained by sampling uniformly over the set of all simple $d$-regular graphs on the vertex set $\{1, \ldots, n\}$. The adjacency matrix of this graph is denoted by $Q_{n,d}$.

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In the whole paper, we assume that $n$ is large. The asymptotic notation is used under the assumption that $n \to \infty$. We write $A \ll B$ if $A = o(B)$. $c$ denotes a universal constant. All logarithms have natural base, if not specified otherwise.

2. The universality principle

Intuitively, one would expect a universal behavior among random models of the same object. For random matrices in particular, one would expect the distributions of specific eigenvalues be the same (after a proper normalization), regardless the model. Thus, given a theorem for continuous models, it is often simple to come up with a reasonable conjecture for discrete ones. For instance, there are fairly accurate tail estimates for the smallest singular value of a random matrix whose entries are i.i.d Gaussians (see for example Theorem 9.2 in Section 9). It would be natural to try to prove similar estimates for a random matrix whose entries are i.i.d Bernoulli. However, this kind of task is usually challenging, as the tools developed for continuous models are typically not applicable in a discrete setting. In the last few sections (Sections 10, 11, 12) of this survey we will present new tools developed recently in order to treat the discrete models. These tools, among others, reveal an intriguing connection between the theory of random matrices and additive combinatorics.

For random graphs, there is a specific conjecture which establishes the universality between the two models $G(n, p)$ and $G_{n,d}$ (Erdős-Rényi graphs and random regular graphs).

**Conjecture 2.1. (Sandwich Conjecture) [28]** For $d \gg \log n$, there is a joint distribution (or coupling) on random graphs $H, G_{n,d}, G$ such that

- $H$ has the same distribution as $G(n, p_1)$ where $p_1 = \frac{d}{n}(1 - \frac{c\sqrt{d \log n}}{n})$ and $G$ has the same distribution as $G(n, p_2)$ where $p_2 = \frac{d}{n}(1 + \frac{c\sqrt{d \log n}}{n})$.
- $P(H \subset G_d) = 1 - o(1)$.
- $P(G_{n,d} \subset G) = 1 - o(1)$.

The conjecture asserts that a random regular graph can be approximated from both below and above by Erdős-Rényi graphs of approximately the same densities. The conjecture has been proved for $d \leq n^{1/3-o(1)}$ [28].

**Theorem 2.2.** The sandwich conjecture holds for $\log n \ll d \ll n^{1/3} / \log^2 n$.

The main difficulty when dealing with the random regular graph $G_{n,d}$ is that the (upper diagonal) entries of its adjacency matrix are not independent variables. But using Theorem 2.2, one can often deduce information about the spectrum of $G_{n,d}$ using information about the spectrum of $G(n, p)$.
3. Limiting distribution

One of the cornerstones of the theory of random matrices is Wigner’s semi-circle law, which established the limiting distribution of a certain class of random symmetric matrices [58]. We present here a more general version, due to Arnold [5].

Let $a_{ij}$, $1 \leq i \leq j \leq n$, be i.i.d random variables with common variance one and distribution function $F(x)$ such that $\int_0^{\infty} |x|^k dF < \infty$ for all $k = 1, 2, \ldots$. Let $A_n$ be the random symmetric matrix of size $n$ whose upper diagonal entries are $\xi_{ij} = a_{ij}/\sqrt{n}$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the (real) eigenvalues of $A_n$. Define

$$ W_n(x) := \frac{1}{n} |\{i| \lambda_i \leq x\}|. $$

Let $W(x)$ denote the semi-circle density function

$$ W(x) := \frac{2}{\pi} \sqrt{1 - x^2} $$

for $|x| \leq 1$ and $W(x) := 0$ otherwise.

**Theorem 3.1.** *(Semi-circle law)* With probability one,

$$ \lim_{n \to \infty} W_n(x) = W(x). $$

In order to prove the semi-circle law, Wigner introduced the so-called trace method, the heart of which is the calculation of the expectation of $\text{Trace}(A_n^k)$ for $k = 1, 2, \ldots$. This method is useful for many other problems (see Section 4 for example).

Let us now turn to the special matrix $Q(n, p)$. The entries of $Q(n, p)$ have variance $\sigma^2 = p(1 - p)$. Dividing each entry of $Q(n, p)$ by $\sigma$, we obtain a matrix $Q'(n, p)$ whose entries have common variance one. However, one cannot apply Theorem 3.1 directly as the entries of $Q'(n, p)$ do not have bounded moments when $p$ tends to zero with $n$. On the other hand, by applying Wigner’s trace method, one can prove the following theorem

**Theorem 3.2.** [21, 57] There is a constant $c$ such that the following holds. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $Q(n, p)$ where $p \geq n^{-1} \log^c n$ and define

$$ R_n^1(x) := \frac{1}{n} |\{i| \lambda_i \leq 2x \sqrt{np(1 - p)}\}|. $$

Then with probability one,

$$ \lim_{n \to \infty} R_n^1(x) = W(x). $$
A corollary of a general theorem by Guionet and Zeitouni [24] shows that $R_1^1(x)$ and many other quantities concerning the spectrum of $Q(n, p)$ are highly concentrated.

Next we discuss the situation with the random regular graph $G_{n,d}$. Define

$$R_n^2(x) := \frac{1}{n} |\{ i | \lambda_i \leq 2x\sqrt{d-1}\}|$$

and

$$W(d, x) := \frac{d^2 - d}{d^2 - 4(d - 1)x^2} \frac{2}{\pi} \sqrt{1-x^2}$$

for $|x| \leq 1$ and $W(d, x) := 0$ otherwise. A theorem of McKay [39] on the spectrum of regular graphs (not necessarily random) implies

**Theorem 3.3.** (Distribution of the eigenvalues in random regular graphs with fixed degree) For any fixed $d$ the following holds with probability one

$$\lim_{n \to \infty} R_n^2(x) = W(d, x).$$

Observe that the limiting distribution $W(d, x)$ in this theorem is not semi-circular because of the extra term $\frac{d^2 - d}{d^2 - 4(d - 1)x^2}$. On the other hand, it becomes arbitrarily close to the semi-circle distribution if $d$ is sufficiently large. Thus it is reasonable to conjecture that if $d$ tends to infinity with $n$, $Q_{n,d}$ follows the semi-circle law. However, McKay’s proof used Wigner’s trace method and relied on the crucial fact that the graph has few small cycles. Theorem 3.3 still holds for $d = n^{c(1)}$. But for $d = n^\epsilon$ with any constant $c > 0$ the graph has too many small cycles and it seems very hard to apply this method. On the other hand, using the sandwicking theorem (Theorem 2.2), Vu and Wu [57] proved that if $\log n \ll d \ll n^{1/3}/\log^2 n$ then with probability one

$$\lim_{n \to \infty} R_n^2(x) = W(x).$$

If the sandwich conjecture holds for all $d \gg \log n$, then this statement can be extended for all $d \gg \log n$. Recently, Zeitouni (private communication) suggested to the author another approach that also seems to work for a wide range of $d$. The details will appear in [57]. For results concerning more general models of random graphs, see [11, 57].

To conclude this section, let us briefly discuss the case when $A_n$ is not symmetric. In this case, the eigenvalues are complex numbers. Let $a_{ij}, 1 \leq i, j \leq n$, be i.i.d complex random variables with mean zero and variance one. Let $A_n$ be the
random matrix with entries \( \xi_{ij} = a_{ij}/\sqrt{n} \) and let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( A_n \). Consider the two-dimensional empirical distribution

\[
\mu_n(x, y) := \frac{1}{n} |\{i|Re(\lambda_i) \leq x, Im(\lambda_i) \leq y\}|.
\]

It was proved by Girko [22] and Bai [6] that under some weak conditions, \( \mu_n(x, y) \) tends to the uniform distribution over the unit disc.

**Theorem 3.4.** (Circular law) Assume that the \( \xi_{ij} \) have finite sixth moment and the joint distribution of the real and imaginary part has bounded density. Then with probability one \( \mu_n(x, y) \) tends to the uniform distribution over the unit disc in \( \mathbb{R}^2 \).

4. The spectral norm

The spectral norm of an \( n \) by \( n \) matrix \( A \) is defined as

\[
\|A\| = \sup_{v \in \mathbb{R}^n, \|v\| = 1} |Av|.
\]

If \( A \) is symmetric, then \( \|A\| \) is the largest eigenvalue of \( A \) (in absolute value).

We consider the following general model of random symmetric matrices. Let \( \xi_{ij}, 1 \leq i \leq j \leq n \), be independent (but not necessarily identical) random variables with the following properties

- \( |\xi_{ij}| \leq K \) for all \( 1 \leq i \leq j \leq n \).
- \( \mathbb{E}(\xi_{ij}) = 0 \), for all \( 1 \leq i < j \leq n \).
- \( \text{Var}(\xi_{ij}) = \sigma^2 \), for all \( 1 \leq i < j \leq n \).

For a moment, let us assume that \( \sigma \) and \( K \) are positive constants.

Define \( \xi_{ji} = \xi_{ij} \) and consider the symmetric random matrix \( A_n = (\xi_{ij})^n \). Notice that for any matrix \( A \),

\[
\|A\| = \lim_{k \to \infty} Tr(A^k)^{1/k}.
\]

This suggests that the trace method would be an effective tool for bounding \( \|A_n\| \).

Indeed, all upper bounds mentioned below are based on this method. Füredi and Komlós [21] proved that a.s.

\[
\|A_n\| \leq 2\sigma \sqrt{n} + cn^{1/3} \ln n.
\]
The error term $cn^{1/3} \ln n$ was recently improved to $cn^{1/4} \ln n$ by Vu [56]. From below, Alon, Krivelevich and Vu [4] showed that a.s.

$$2\sigma \sqrt{n} - c \ln n \leq \|A_n\|,$$

for some constant $c$. Putting these bounds together, we have

**Theorem 4.1.** For a random matrix $A$ as above there is a positive constant $c = c(\delta, K)$ such that

$$2\sigma \sqrt{n} - c \ln n \leq \|A\| \leq 2\sigma \sqrt{n} + cn^{1/4} \ln n,$$

holds almost surely.

In many situations $\sigma$ and $K$ may depend on $n$. A typical example is when $A$ is the "normalized" adjacency matrix of $G(n, p)$ (1 and 0 are replaced by $1 - p$ and $-p$, respectively; this forces all entries to have mean zero) where $p$ is decreasing with $n$ ($p = n^{-\epsilon}$, say). In this case, by following the proof of the upper bound in the previous theorem, one can obtain

**Theorem 4.2.** [56] There are constants $c$ and $c'$ such that the following holds. Let $\xi_{ij}$, $1 \leq i \leq j \leq n$ be independent random variables, each of which has mean 0 and variance at most $\sigma^2$ and is bounded in absolute value by $K$, where $\sigma \geq c' n^{-1/2} K \ln^2 n$. Then almost surely

$$\|A_n\| \leq 2\sigma \sqrt{n} + c(K\sigma)^{1/2} n^{1/4} \ln n.$$

One can also obtain a somewhat weaker bound by following the proof from [21].

If one assumes more about the distributions of the entries $\xi_{ij}$, one can obtain a sharper bound. Soshnikov and Sinai [46] proved the following

**Theorem 4.3.** (Spectral bound for random matrices with symmetric entries) Let $A_n$ be a random symmetric matrix whose upper diagonal entries $\xi_{ij}$, $1 \leq i \leq j \leq n$ are independent random variables satisfying

- $\xi_{ij}$ have symmetric distribution.
- $E(\xi_{ij}^2) = 1$ and $E(\xi_{ii}^2) = O(1)$.
- For all $m \geq 1$, $E(\xi_{ij}^{2m}) = O(m^m)$.

Then a.s. $\|A_n\| = 2\sqrt{n} + O(n^{-1/6})$.

In certain algorithmic applications, it is useful to have a tail distribution for the spectral norm (see [1, 32], for instance). Using Talagrand's inequality, one can show
Theorem 4.4. [32, 4] (Concentration of the spectral norm) There is a positive constant $c = c(K)$ such that for any $t > 0$

$$
P\left(\|A_n\| - E(\|A_n\|) \geq ct\right) \leq 4e^{-t^2/32}.
$$

Similar results hold for larger classes of random matrices and also for other eigenvalues (see [4, 40]).

In the case of $Q(n, p)$, if $p$ is sufficiently large, then all rows have a.s. roughly $np$ ones and the norm of $Q(n, p)$ is (a.s.) $(1 + o(1))np$. But if $p$ is relatively small, this is no longer true. Krivelevich and Sudakov [30] proved that $\|Q(n, p)\|$ is almost surely

$$(1 + o(1)) \max\{np, \sqrt{D}\}$$

where $D$ denotes the maximum degree of the (random) graph. See [47, 26] for more results of this type.

5. The second eigenvalue of random regular graphs

Let $G$ be a graph on $n$ points and $A$ its adjacency matrix. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A$. If $G$ is $d$-regular, then $\lambda_1 = d$. In this case, a critical parameter of the graph is

$$\lambda(G) := \max\{|\lambda_2|, |\lambda_n|\}.$$ 

In the literature, $\lambda(G)$ is frequently called the second eigenvalue of $G$. (This name is inaccurate but somewhat convenient.) The good way to think of $\lambda(G)$ is

$$\lambda(G) = \max_{\|v\|=1, v \cdot 1 = 0} |v^T \cdot Av|,$$

where $\mathbf{1}$ is the all ones vector. One can also think of $\lambda(G)$ as the spectral norm of the "normalized" adjacency matrix of $G$, where 1 and 0 are replaced by $(n - d)/n$ and $-d/n$, respectively.

One can derive many interesting properties of the graph $G$ from the value of $\lambda(G)$. The general phenomenon here is that if $\lambda(G)$ is significantly less than $d$, then the edges of $G$ distribute like those of a random graph with edge density $d/n$ [3, 49, 10]. One can use this information to derive various properties of the graphs (see [31] for many results of this kind). The whole concept can be generalized for non-regular
graphs. In this case, one needs to consider the Laplacian rather than the adjacency matrix (see, for example, [9]).

Estimating $\lambda(G)$ for a random regular graph is a well known problem in the discrete math/theoretical computer science community. A consequence of the well known Alon-Boppana bound [2] asserts that if $d$ is fixed and $n$ tends to infinity, a.s.

$$\lambda_2(G_{n,d}) \geq 2\sqrt{d - 1} - o(1).$$

Since $\lambda(G) \geq |\lambda_2(G)|$ it follows that a.s.

$$\lambda(G_{n,d}) \geq 2\sqrt{d - 1} - o(1).$$

Alon [2] conjectured that for any fixed $d$, a.s.

$$\lambda_2(G_{n,d}) = 2\sqrt{d - 1} + o(1).$$

Friedman [18] and Kahn and Szemerédi [20] showed that if $d$ is fixed and $n$ tends to infinity, then a.s. $\lambda(G_{n,d}) = O(\sqrt{d})$. Recently, Friedman, in a highly technical paper [19], proved Alon’s conjecture. In fact, he proved the stronger statement that a.s. $\lambda(G_{n,d}) = 2\sqrt{d - 1} + o(1)$. This, together with the lower bound above, determines the asymptotic of $\lambda(G_{n,d})$.

**Theorem 5.1.** [19] (Second eigenvalue of random regular graphs with fixed degree) For any fixed $d$ and $n$ tending to infinity, a.s.

$$\lambda(G_{n,d}) = (2 + o(1))\sqrt{d - 1}.$$ 

A $d$-regular graph $G$ is Ramanujan if $\lambda(G) \leq 2\sqrt{d - 1}$. Explicit constructions of Ramanujan graphs are highly non-trivial and usually come from deep results in number theory (see [35] or [36], for example). On the other hand, the following conjecture has been circulated in the last few years (mentioned to the author by Sarnak)

**Conjecture 5.2.** For $d$ fixed and $n$ tends to infinity, $G_{n,d}$ is Ramanujan with positive constant probability.

So far, we discussed the case when $d$ is a constant. What happens if $d$ also tends to infinity with $n$? It is not clear (at least to the author) that Friedman’s proof of Alon’s conjecture in [19] can be extended to this case. On the other hand, it is not hard to show that $\lambda(G(n,p))$, where $G(n,p)$ is the Erdős-Rényi random graph, is $(2 + o(1))\sqrt{np(1 - p)}$ for sufficiently large $p$ (e.g., $p \geq n^{-1+\epsilon}$ for any fixed $0 < \epsilon < 1$). Motivated by the universality principle, we make the following conjecture
Conjecture 5.3. Assume that $d \leq n/2$ and both $d$ and $n$ tend to infinity. Then a.s

$$\lambda(G_{n,d}) = (2 + o(1))\sqrt{d(1 - d/n)}.$$ 

Nilli [41] showed that for any $d$-regular graph $G$ having two edges with distance at least $2k + 2$ between them $\lambda_2(G) \geq 2\sqrt{d - 1 - 2\sqrt{d - 1}/(k + 1)}$. If $d = n^{o(1)}$ then $G_{n,d}$ has diameter $\omega(1)$ with probability $1 - o(1)$. Thus in this case

$$\lambda(G_{n,d}) \geq \lambda_2(G_{n,d}) \geq (2 + o(1))\sqrt{d}$$

with probability a.s. This proves the lower bound in Conjecture 5.3. For a general $d$, it is easy to show (by computing the trace of the square of the adjacency matrix) that any $d$-regular graph $G$ on $n$ vertices satisfies

$$\lambda(G) \geq \sqrt{d(n - d)/(n - 1)} \approx \sqrt{d(1 - d/n)}.$$ 

(We would like to thank N. Alon for pointing out this bound.)

Let us now turn to the upper bound. For $d = o(n^{1/2})$, one can follow the Kahn-Szemerédi approach to show that $\lambda(G_{n,d}) = O(\sqrt{d})$ a.s. For larger $d$, there is a weaker bound $o(d)$ [33, Theorem 2.8] proved by the trace method. The following two approaches look promising:

- (Suggested by Krivelevich) Combine the sharp concentration result in the previous section with the probability that a random graph is regular. Using this, one can show for example that $\lambda(G_{n,d}) = O(\sqrt{d \log n})$ for $d$ close to $n$ ($d = n/2$, for instance).

- The Sandwich Theorem (Theorem 2.2) implies

$$\lambda(G_{n,d}) = \lambda(G(n, d/n)) + O(\sqrt{d \log n}).$$

For most values of $d$, $\lambda(G(n, d/n)) = O(\sqrt{d})$. Thus, if the Sandwich conjecture holds, it would imply a upper bound of $O(\sqrt{d \log n})$ for most values of $d$.

The author feels confident that one can prove that $\lambda(G_{n,d}) = O(\sqrt{d \log n})$ for all $d$ using these approaches. However, removing the log term seems non-trivial. In fact, even the following special and weakened case already looks challenging

**Problem.** Prove that $\lambda(G_{n,n/2}) = O(\sqrt{n})$ almost surely.
6. Determinant

The problem of determining the determinant of $M_n$ has been considered by various researchers for at least 40 years. It was proved by Komlós in 1967 [29] that almost surely $\det M_n$ is not zero. In fact, it is easy to see that $\det M_n$ is divisible by $2^{n-1}$, thus it follows that a.s. $|\det M_n| \geq 2^{n-1}$. From above, Hadamard’s inequality implies that $|\det M_n| \leq n^{n/2}$ (notice that all row vectors of $M_n$ have length $\sqrt{n}$.

It was often conjectured that with probability close to 1, $|\det M_n|$ is close to this upper bound.

**Conjecture 6.1.** Almost surely $|\det M_n| = n^{(1/2-o(1))n}$.

This conjecture is supported by the following observation of Turán, whose proof is a simple application of the linearity of expectation.

**Fact 6.2.**

$$E((\det M_n)^2) = n!.$$ It follows immediately by Markov’s bound that for any function $\omega(n)$ tending to infinity with $n$, almost surely

$$|\det M_n| \leq \omega(n)\sqrt{n!}.$$ Tao and Vu [50] established the matching lower bound, which confirms Conjecture 6.1.

**Theorem 6.3.** Almost surely

$$|\det M_n| \geq \sqrt{n!} \exp(-29\sqrt{n \log n}).$$ We are going to sketch the proof very briefly as it contains a useful lemma. For a more detailed proof, we refer to [50].

**Proof.** We view $|\det M_n|$ as the volume of the parallelepiped spanned by $n$ random $\{-1, 1\}$ vectors. This volume is the product of the distances from the $(d+1)$st vector to the subspace spanned by the first $d$ vectors, where $d$ runs from 0 to $n-1$. We are able to obtain very tight control of this distance (as a random variable), thanks to the following lemma.

**Lemma 6.4.** Let $W$ be a fixed subspace of dimension $1 \leq d \leq n-4$ and $X$ a random $\pm 1$ vector. Then

$$E(\text{dist}(X, W)^2) = n - d.$$ Furthermore, for any $t > 0$

$$P(\left|\text{dist}(X, W) - \sqrt{n - d}\right| \geq t + 1) \leq 4 \exp(-t^2/16).$$
Observe that in this lemma, we do not need to assume that $W$ is spanned by random vectors. The lemma, however, is not applicable when $d$ is very close to $n$ as it does not imply that the distance is positive almost surely. In this case, we do need to use the assumption that $W$ is random. This assumption allows us to derive information about the normal vector of $W$, which, combined with Erdős-Littlewood-Offord bound (see Theorem 10.1), provides control of the last few distances.

**Remark 6.5.** After having written [50], Tao and the author discovered that Girko (Section 6 of [23]) claimed a very general theorem which implies Theorem 6.3. We are not able to understand his proof and have not found anyone who does.

Theorem 6.3 can be extended for much more general models of random matrices. Let $\xi_{ij}, 1 \leq i, j \leq n$, be independent (but not necessarily i.i.d.) r.v.’s with the following two properties:

- Each $\xi_{ij}$ has mean zero and variance one.
- There is a constant $K$ that $|\xi_{ij}| \leq K$ with probability one.

**Theorem 6.6.** Consider the random matrix $M'_n$ with entries $\xi_{ij}$ as above. Let $\epsilon$ be an arbitrary positive constant. With probability $1 - o(1)$,

$$|\det M'_n| \geq \sqrt{n!} \exp(-n^{1/2+\epsilon}).$$

In certain situations, the assumption that $|\xi_{ij}|$ are bounded from above by a constant is too strong. We are going to consider the following less restricted model. Let $\xi_{ij}, 1 \leq i, j \leq n$ be i.i.d. random variables with mean zero and variance one. Assume furthermore that their fourth moment is finite. Consider the random matrix $M''_n$ with $\xi_{ij}$ as its entries.

**Theorem 6.7.** [50] We have, with probability $1 - o(1)$, that

$$|\det M''_n| \geq n^{(1/2-o(1))n}.$$

An open problem concerning determinants is to extend Theorem 6.3 to symmetric matrices. Let $Q_n$ denote the random symmetric matrix whose upper diagonal entries are i.i.d. Bernoulli random variables.

**Conjecture 6.8.** Almost surely, $|\det Q_n| = n^{(1/2+o(1))n}$.

It was proved only very recently [13] that a.s $|\det Q_n|$ is positive (which can be seen as the symmetric version of Komlós’ theorem mentioned above). The main difficulty here is that the row vectors of $Q_n$ are no longer independent and so Lemma 6.4 is not applicable as there is a correlation between the subspace $W$ and the vector $X$.

Finally, let us briefly discuss the situation with the permanent. Notice that the estimate in Fact 6.2 is still valid for the permanent of $M_n$. Thus, one would expect
that, like the determinant, the permanent of $M_n$ is typically of order $n^{(1/2+o(1))n}$ (in absolute value). However, the following problem is still open.

**Problem.** Show that the permanent of $M_n$ is a.s. not zero.

7. **Rank and singular probability: non-symmetric models**

Let us consider the basic model $M_n$ and let $p_n$ be the probability that $M_n$ is singular. Estimating $p_n$ is well known problem in discrete probability. From below it is clear that $p_n \geq (1/2 + o(1))^n$, as a matrix is singular if it has two equal rows. A famous conjecture in the field asserts that this trivial lower bound is sharp.

**Conjecture 7.1.** $p_n = (1/2 + o(1))^n$.

There is a refined version of the above conjecture where the right hand side is more precise (see [27]). However, Conjecture 7.1, as formulated, is still open.

It is already non-trivial to show that $p_n = o(1)$. As mentioned in the previous section, this was done by Komlós almost forty years ago [29]. The bound on $p_n$ in his original proof tends very slowly to zero with $n$. Later, he found a new proof which showed $p_n = O(n^{-1/2})$. In 1995, a breakthrough by Kahn, Komlós and Szemerédi [27] yielded the first exponential bound $p_n = O(0.999^n)$. Their arguments were simplified by Tao and Vu in 2004 [50], resulting in a slightly better bound $O(0.958^n)$ and a somewhat simpler proof. Shortly afterwards, Tao and Vu [51] combined the approach from [27] with ideas from additive combinatorics to obtained the following more significant improvement

**Theorem 7.2.** [51] $p(n) \leq (3/4 + o(1))^n$.

The proof in [51] is highly technical and requires many tools from discrete Fourier analysis, additive combinatorics and the geometry of numbers. On the other hand, the proving scheme is flexible and can be adapted to other models, sometime yielding (surprisingly) sharp bounds. Let us present one such result. Instead of $M_n$ we consider the following "lazy" model $M_n^{\text{lazy}}$. The entries of $M_n^{\text{lazy}}$ are i.i.d random variables which equal zero with probability one half and 1 and $-1$ with probability one quarter. (If one thinks of the entries of $M_n$ as fair coin flips, then in the "lazy" model about half of the time we are lazy and simply write zero instead flipping a coin.) It is clear that for the lazy model the singular probability $p_n^{\text{lazy}}$ is again at least $(1/2 + o(1))^n$ (which is the probability that there is a zero row). We are able to show that this bound is actually sharp

**Theorem 7.3.** [55] $p_n^{\text{lazy}} = (1/2 + o(1))^n$.

Let us conclude this section by two conjectures motivated by studies from random graphs. These questions concern the resilience of a structure, introduced in [28, 48]. Roughly speaking, the resilience of a structure $S$ with respect to a property $\mathcal{P}$ measures how much we have to change $S$ in order to destroy $\mathcal{P}$. We would like to measure the resilience of $M_n$ with respect to the property of being non-singular.
Given \(\{-1,1\}\) matrix \(M\), we denote by \(\text{Res}(M)\) the minimum number of entries we need to switch (from 1 to \(-1\) and vice versa) in order to make \(M\) singular. If \(M\) is a sample of \(M_n\), it is easy to show that \(\text{Res}(M)\) is, a.s., at most \((1/2 + o(1))n\), as we can, a.s., change that many entries in the first row to make the first two rows equal. We conjecture that this is the best one can do.

**Conjecture 7.4.** Almost surely \(\text{Res}(M_n) = (1/2 + o(1))n\).

A closely related question (motivated by the notion of local resilience from [48]) is the following. Call a \(\{-1,1\}\) \((n \times n)\) matrix \(M\) **good** if all matrices obtained by switching (from 1 to \(-1\) and vice versa) the diagonal entries of \(M\) are non-singular (there are \(2^n\) such matrices).

**Conjecture 7.5.** Almost surely \(M_n\) is good.

8. **Rank and singular probability: symmetric models**

Let us now consider symmetric matrices. The symmetric counterpart of \(M_n\) is \(Q_n\). In fact it is more convenient to consider \(Q(n,1/2)\) instead of \(Q_n\), as the graph terminology is more convenient and leads to natural extensions. (It is easy to show that if \(Q(n,1/2)\) is a.s. non-singular then \(Q_n\) is and vice versa.)

While the non-singularity of \(M_n\) has been known for forty years since [29], that of \(G(n,1/2)\) was established only recently by Costello, Tao and Vu [13]. This confirmed a conjecture of B. Weiss (this conjecture was communicated to the author by G. Kalai and N. Linial).

**Theorem 8.1.** \(Q(n,1/2)\) is a.s non-singular.

As pointed out earlier in Section 6, the main difficulty in going from the non-symmetric setting to the symmetric one is that the row vectors in a random symmetric matrix are not independent. The key tool that helped us to overcome this difficulty was the so-called quadratic Littlewood-Offord inequality (Theorem 10.2), discussed in Section 10.

It is natural to ask if Theorem 8.1 still holds for a smaller density \(p\). The answer is negative after a certain threshold. Indeed, if \(p < (1 - \epsilon) \log n/n\) for some positive constant \(\epsilon\), then \(G(n,p)\) has a.s. isolated vertices which means that its adjacency matrix has all zero rows and so is singular. Costello and Vu proved that \(\log n/n\) is the right threshold.

**Theorem 8.2.** [12] For any constant \(\epsilon > 0\), \(Q(n, (1 + \epsilon) \log n/n)\) is a.s. non-singular.

It remains an interesting problem to estimate the rank of \(Q(n,p)\) for \(p < \log n/n\). The answer here is not yet conclusive, but some partial results are known. For instance, it was shown in [12] that if \(p > (1 + \epsilon) \log n/2n\), then the rank of \(Q(n,p)\)
is a.s. equal $n$ minus the number of isolated vertices in the graph. (The upper bound is trivial.)

Let us conclude this section by stating two conjectures. The first is a variant of Conjecture 7.1. We denote by $p^\text{sym}_n$ the probability that $Q(n, 1/2)$ is singular. It is easy to show that this probability is at least $(1/2 + o(1))^n$.

**Conjecture 8.3.** $p^\text{sym}_n = (1/2 + o(1))^n$.

This conjecture is perhaps very hard. The current best upper bound on $p^\text{sym}_n$ is $n^{-1/4 + o(1)}$. It seems already non-trivial to replace $1/4$ by an arbitrary constant $C$.

The second conjecture concerns random regular graphs. If $d = 2$, $G_{n, d}$ is a union of disjoint cycles and it is easy to show that its adjacency matrix $Q_{n, d}$ is a.s. singular, as many of these cycles have length divisible by 4. We conjecture that this is the only case.

**Conjecture 8.4.** For all $d \geq 3$, $Q_{n, d}$ is a.s. non-singular.

### 9. The condition number

For a matrix $M$ the condition number $c(M)$ is defined as

$$c(M) := \|M\| \cdot \|(M^{-1})\| .$$

We adopt the convention that $c(M)$ is infinite if $M$ is not invertible.

The condition number plays a crucial role in applied linear algebra. In particular, the complexity of any algorithm which requires solving a system of linear equations usually involves the condition number of a matrix [7]. Another area of mathematics where this parameter is important is the theory of probability in Banach spaces (see [43] and the references therein).

The condition number of a random matrix is a well-studied object (see [14] and the references therein). In the case when the entries of $M$ are i.i.d Gaussian random variables (with mean zero and variance one), Edelman [14] proved

**Theorem 9.1.** Let $N_n$ be a $n \times n$ random matrix, whose entries are i.i.d Gaussian random variables (with mean zero and variance one). Then $\mathbf{E}(\ln c(N_n)) = \ln n + c + o(1)$, where $c > 0$ is an explicit constant.

In applications, it is usually useful to have a tail estimate. It was shown by Edelman and Sutton [15] that

**Theorem 9.2.** Let $N_n$ be a $n$ by $n$ random matrix, whose entries are i.i.d Gaussian random variables (with mean zero and variance one). Then for any constant $A > 0$,

$$\mathbf{P}(c(N_n) \geq n^{A+1}) = O_A(n^{-A}).$$
With the universality principle, it is reasonable to conjecture that this estimate holds for the random Bernoulli matrix $M_n$ as well (see [45] for an even more precise conjecture). However, this seems very hard to prove. On the other hand, the following was obtained recently by Tao and Vu [52]

**Theorem 9.3.** For any positive constant $A$, there is a positive constant $B$ such that

$$\mathbb{P}(c(M_n) \geq n^B) \leq n^{-A}.$$  

It is well known that there is a constant $C$ such that the norm of $M_n$ is at most $Cn^{1/2}$ with exponential probability $1 - \exp(-\Omega_n(n))$ (in fact, one can prove this using the results in Section 4). Thus, Theorem 9.3 reduces to the following lower tail estimate for the norm of $M_n^{-1}$:

**Theorem 9.4.** For any positive constant $A$, there is a positive constant $B$ such that

$$\mathbb{P}(\|M_n^{-1}\| \geq n^B) \leq n^{-A}.$$  

Shortly prior to Theorem 9.3, Rudelson [43] proved the following result.

**Theorem 9.5.** There are positive constants $c_1, c_2$ such that the following holds. For any $\epsilon \geq c_1 n^{-1/2}$

$$\mathbb{P}(\|M_n^{-1}\| \geq c_2 \epsilon n^{3/2}) \leq \epsilon.$$  

Both theorems can be generalized considerably (see [52, 53] and [43]). Theorem 9.3 in particular still holds if we replace $M_n$ by $M + M_n$ where $M$ is an arbitrary matrix with polynomially bounded norm.

**Theorem 9.6.** [53] For any positive constants $A$ and $C$, there is a positive constant $B$ such that the following holds. For any $n$ by $n$ matrix $M$ where $\|M\| \leq n^C$,

$$\mathbb{P}(c(M + M_n) \geq n^B) \leq n^{-A}.$$  

The point here is that $M$ itself can have very large condition number. (In fact if $M$ is singular then its condition number is infinity.) Theorem 9.6 asserts that a Bernoulli perturbation of $M$ has small condition number with high probability. The Gaussian version of Theorem 9.6 was proved by Spielman and Teng in [45]. For the connection of these theorems to numerical linear algebra and theoretical computer science, we refer to [45] and [53].

10. Littlewood-Offord and quadratic Littlewood-Offord

Let $v = \{v_1, \ldots, v_n\}$ be a set of $n$ integers and $\xi_1, \ldots, \xi_n$ be i.i.d random Bernoulli variables. Define $S := \sum_{i=1}^n \xi_i v_i$ and $p_v(a) := \mathbb{P}(S = a)$ and $p_v := \sup_{a \in \mathbb{Z}} p_v(a)$. 


Erdős, answering a question of Littlewood and Offord, proved the following theorem, which we are referring to as the Erdős-Littlewood-Offord inequality.

**Theorem 10.1.** Let \( v_1, \ldots, v_n \) be non-zero numbers and \( \xi_i \) be i.i.d Bernoulli random variables. Then

\[
pv \leq \left( \frac{n}{|n/2|} \right)^2 = O(\sqrt{n}).
\]

Theorem 10.1 is a classical result in combinatorics and has many non-trivial extensions (see [54, Chapter 7] or [25] and the references therein).

The random sum \( S := \sum_{i=1}^n \xi_i v_i \) plays a central role in the study of random Bernoulli matrices. In many problems (such as that of the determinant, rank or condition number), the critical parameter is the distance from a random Bernoulli vector to the hyperplane spanned by another \( n - 1 \) random Bernoulli vectors (these are the row vectors of \( M_n \)). If \( (v_1, \ldots, v_n) \) is the (unit) normal vector of the hyperplane, then this distance is exactly the absolute value of the random sum \( S = \sum_{i=1}^n \xi_i v_i \).

Bounding the above mentioned distance is one of the main difficulties when one goes from Gaussian matrices to Bernoulli matrices. If the entries of the matrix are Gaussian, then the distance in question is a simple object. Thanks to symmetry, the position of the hyperplane does not really matter and so we can condition on it. Furthermore, the distribution of the distance from a random Gaussian vector to a fixed hyperplane is well understood. The situation in the Bernoulli case is very different. In this case, the random vectors are chosen from the vertices of the \( n \)-dimensional \( \{-1,1\} \) cube. Very little is known about the hyperplanes spanned by \( n - 1 \) such vectors. Let us point out, however, that there are planes that contain a constant fraction of the vertices of the \( \{-1,1\} \) cube and in this case the distance in question is zero with constant probability.

In order to treat random symmetric matrices (in particular \( Q(n, 1/2) \)), Costello, Tao and Vu [13] introduced the following quadratic version of Theorem 10.1

**Theorem 10.2.** Let \( c_{ij}, 1 \leq i, j \leq n \) be non-zero numbers. Consider the quadratic form \( F = \sum_{1 \leq i \leq j \leq n} c_{ij} \xi_i \xi_j \), where \( \xi_i, 1 \leq i \leq n \) are i.i.d Bernoulli random variables. Then for any \( a \)

\[
\mathbb{P}(F = a) = O(n^{-1/8}).
\]

The exponent \(-1/8\) was improved to \(-1/4\) (see [12]). It is conjectured [12] that the sharp exponent would be \(-1/2\). The lower bound is given by the quadratic form \( F = (\sum_{i=1}^n \xi_i)^2 \).
11. Random walks and Lazy Random Walks

Let \( v = \{v_1, v_1, \ldots, v_n\} \) be the set of \( n \) non-zero numbers and consider the random walk \( W \) on the real line (starting at 0) which at step \( i \) goes to the left by \( v_i \) with probability one half and to the right by \( v_i \) with probability one half. The probability \( p_v(0) = P(\sum_{i=1}^{n} \xi_i v_i = 0) \) is exactly the probability that \( W \) returns to the origin after \( n \) steps.

Let \( \mu \) be a constant between 0 and 1 and consider the "lazy" random walk \( W^\mu \) (starting at 0) which at step \( i \) stays with probability \( 1 - \mu \) and goes to the left by \( v_i \) with probability \( \mu/2 \) and to the right by \( v_i \) with probability \( \mu/2 \). Let \( p_{v^\mu}(0) \) be the probability that the lazy walk return to zero after \( n \) steps.

Intuitively, one would expect that \( p_{v^\mu}(0) \) is larger than \( p_v(0) \) (especially when \( \mu \) is small), as the lazy walk has a stronger tendency to stay near the starting point. Quantitatively, one can show (using Fourier analysis and the elementary fact that \( |\cos x| \leq 3/4 + 1/4 \cos 2x \)) that for any \( v \)

\[ p_v(0) \leq p_v^{1/4}(0). \]

The next question is: Can one improve this to

\[ p_v(0) \leq \epsilon p_v^{1/4}(0), \quad (3) \]

for any positive constant \( \epsilon \)? The answer is negative. If we take \( v = \{1, 1, \ldots, 1\} \), then it is easy to show that

\[ p_v(0) = (c + o(1)) p_v^{1/4}(0) \]

where \( c \) is a positive constant (depends on 1/4). However, it is possible to classify all sets \( v \) where (3) fails (under some slight assumptions). This classification is the heart of the proof of Theorem 7.2. The precise statement is somewhat technical (we refer to [51] for details), but roughly it says that if (3) fails then \( v \) is contained in a generalized arithmetic progression with constant rank and small volume.

12. Inverse Littlewood-Offord theorems

A set

\[ P = \{c + m_1a_1 + \cdots + m_da_d | M_i \leq m_i \leq M_i'\} \]

is called a generalized arithmetic progression (GAP) of rank \( d \). It is convenient to think of \( P \) as the image of an integer box \( B := \{(m_1, \ldots, m_d) | M_i \leq m_i \leq M_i'\} \) in
\( \mathbb{Z}^d \) under the linear map
\[
\Phi : (m_1, \ldots, m_d) \mapsto c + m_1 a_1 + \cdots + m_d a_d.
\]
The numbers \( a_i \) are the *generators* of \( P \). For a set \( A \) of reals and a positive integer \( k \), we define the iterated sumset
\[
kA := \{ a_1 + \cdots + a_k | a_i \in A \}.
\]

Let us take another look at Theorem 10.1. This theorem is sharp, as is shown by taking \( v_1 = v_2 = \cdots = v_n = 1 \). However, the bound changes significantly if one forbid this special case. Erdős and Moser [17] showed that under the stronger assumption that the \( v_i \) are non-zero and different,
\[
p_v = O\left(n^{-3/2} \ln n\right).
\]
They conjectured that the logarithmic term is not necessary and this was confirmed by Sárközy and Szemerédi [46] (see also [25]). Again, the bound is sharp (up to a constant factor), as can be seen by taking \( v_1, \ldots, v_n \) to be a proper arithmetic progression such as \( 1, \ldots, n \).

In the above two examples, we observe that in order to make \( p_v \) large, we have to impose a very strong additive structure on \( v \) (in one case we set the \( v_i \)'s to be the same, while in the other we set them to be elements of an arithmetic progression). A more general example is the following

*Example.* Let \( P \) be a GAP of rank \( d \) and volume \( V \). Let \( v_1, \ldots, v_n \) be (not necessarily different) elements of \( P \). Then the random variable \( S = \sum_{i=1}^n \xi_i v_i \) takes values in the GAP \( nP \) which has volume \( n^d V \). From the pigeonhole principle and the definition of \( p_v \), it follows that
\[
p_v \geq n^{-d} V^{-1}.
\]
This example shows that if the elements of \( v \) belong to a GAP with small rank and small volume then \( p_v \) is large. One may conjecture that the inverse also holds, namely,

*If \( p_v \) is large, then (most of) the elements of \( v \) belong to a GAP with small rank and small volume.*

Tao and Vu [52] have managed to quantify this statement.

**Theorem 12.1.** [52] Let \( A, \alpha > 0 \) be arbitrary constants. There are constants \( d \) and \( B \) depending on \( A \) and \( \alpha \) such that the following holds. Assume that \( v = \{v_1, \ldots, v_n\} \) is a multiset of integers satisfying \( \Pr(S = 0) \geq n^{-A} \). Then there is a GAP \( Q \) of rank at most \( d \) and volume at most \( n^B \) which contains all but at most \( n^\alpha \) elements of \( v \) (counting multiplicity).
Notice that the small set of exceptional elements is not avoidable. For instance, one can add $O(\log n)$ completely arbitrary elements to $v$, and only decrease $p_v$ by a factor of $n^{-O(1)}$ at worst.

Theorem 12.1 is one of the main ingredients in the proofs of Theorems 9.3 and 9.6. For many other theorems of this type, see [52].

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References

[1] D. Achlioptas and F. McSherry, Fast computation of low rank matrix approximations, Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing, 611–618 (electronic), ACM, New York, 2001.

[2] N. Alon, Eigenvalues and expanders, Combinatorica 6 (1986), 83-96.

[3] N. Alon and V. Milman, $\lambda_1$, isoperimetric inequalities for graphs, and superconcentrators, J. Combin. Theory Ser. B 38 (1985), no. 1, 73–88.

[4] N. Alon, M. Krevelevich and V. Vu, On the concentration of eigenvalues of random symmetric matrices, Israel J. Math. 131 (2002), 259–267.

[5] L. Arnold, On the asymptotic distribution of the eigenvalues of random matrices, J. Math. Anal. Appl. 20 (1967) 262–268.

[6] Z. D. Bai, Circular law, Ann. Probab. 25 (1997), no. 1, 494–529

[7] D. Bau and L. Trefethen, Numerical linear algebra. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[8] B. Bollobás, Random graphs. Second edition, Cambridge Studies in Advanced Mathematics, 73. Cambridge University Press, Cambridge, 2001.

[9] F. Chung, Spectral graph theory, CBMS series, no. 92 (1997).

[10] F. Chung, R. Graham and R. Wilson, Quasi-random graphs, Combinatorica 9 (1989), no. 4, 345–362.

[11] F. Chung, L. Lu and V. Vu, The spectra of random graphs with expected degrees, Proceedings of National Academy of Sciences, 100, no. 11, (2003).

[12] K. Costello and V. Vu, The ranks of random graphs, submitted.

[13] K. Costello, T. Tao and V. Vu, Random symmetric matrices are almost surely non-singular, to appear in Duke Math. Journal.

[14] A. Edelman, Eigenvalues and condition numbers of random matrices. SIAM J. Matrix Anal. Appl. 9 (1988), no. 4, 543–560.

[15] A. Edelman and B. Sutton, Tails of condition number distributions, SIAM J. Matrix Anal. Appl. 27 (2005), no. 2

[16] P. Erdös, On a lemma of Littlewood and Offord, Bull. Amer. Math. Soc. 51 (1945), 898–902.

[17] P. Erdös, Extremal problems in number theory. 1965 Proc. Sympos. Pure Math., Vol. VIII pp. 181–189 Amer. Math. Soc., Providence, R.I.

[18] J. Friedman, On the second eigenvalue and random walks in random $d$-regular graphs, Combinatorica 11 (1991), no. 4, 331–362.

[19] J. Friedman, A proof of Alon’s second eigenvalue conjecture, Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing, 720–724, ACM, New York, 2003.

[20] J. Friedman, J. Kahn, and E. Szemeredi, On the second eigenvalue in random regular graphs, Proc of 21th ACM STOC (1989), 587–598.

[21] Z. Füredi and J. Komlós, The eigenvalues of random symmetric matrices. Combinatorica 1 (1981), no. 3, 233–241.

[22] V. Girko, Circle law, (Russian) Teor. Veroyatnost. i Primenen. 29 (1984), no. 4, 669–679.

[23] V. Girko, Theory of random determinants. Translated from the Russian. Mathematics and its Applications (Soviet Series), 45. Kluwer Academic Publishers Group, Dordrecht, 1990.

[24] A. Guionnet and O. Zeitouni, Concentration of the spectral measure for large matrices, Electron. Comm. Probab. 5 (2000), 119–136 (electronic).
G. Halász, Estimates for the concentration function of combinatorial number theory and probability, *Period. Math. Hungar.* 8 (1977), no. 3-4, 197–211.

S. Janson, The first eigenvalue of random graphs, *Combin. Probab. Comput.* 14 (2005), no. 5-6, 815–828.

J. Kahn, J. Komlós, E. Szemerédi, On the probability that a random ±1 matrix is singular, *J. Amer. Math. Soc.* 8 (1995), 223–240.

J. H. Kim and V. Vu, Sandwiching random graphs: Universality between random models, *Advances in Mathematics* 188 (2004) 444-469.

J. Komlós, On the determinant of (0, 1) matrices, *Studia Sci. Math. Hungar.* 2 (1967) 7-22.

M. Krivelevich and B. Sudakov, The largest eigenvalue of sparse random graphs, *Combin. Probab. Comput.* 12 (2003), no. 1, 61–72.

M. Krivelevich and B. Sudakov, Pseudo-random graphs, More sets, graphs and numbers, 199–262. Bolyai Soc. Math. Stud., 15, Springer, Berlin, 2006.

M. Krivelevich and V. Vu, Approximating the independence number and the chromatic number in expected polynomial time, *J. Comb. Optim.* 6 (2002), no. 2, 143–155.

M. Krivelevich, B. Sudakov, V. Vu and N. Wormald, Random regular graphs of high degree, *Random Structures and Algorithms* 18 (2001), no. 4, 346–363.

J. E. Littlewood and A. C. Offord, On the number of real roots of a random algebraic equation. III. *Rec. Math. [Mat. Sbornik] N.S.* 12, (1943). 277–286.

A. Lubotsky, R. Phillips and P. Sarnak, Ramanujan graphs, *Combinatorica* 8 (1988), no. 3, 261–277.

G. Margulis, Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators (Russian) *Problemy Peredachi Informatsii* 24 (1988), no. 1, 51–60; translation in *Problems Inform. Transmission* 24 (1988), no. 1, 39–46.

J. E. Littlewood and A. C. Offord, On the number of real roots of a random algebraic equation. III. *Rec. Math. [Mat. Sbornik] N.S.* 12, (1943). 277–286.

M. L. Mehta, Random matrices, Third edition, Pure and Applied Mathematics, 142, Elsevier/Academic Press, Amsterdam, 2004.

B. McKay, The expected eigenvalue distribution of a large regular graph, *Linear Algebra Appl.* 40 (1981), 203–216.

M. Meckes, Concentration of norms and eigenvalues of random matrices, *J. Funct. Anal.* 211 (2004), no. 2, 508–524.

A. Nilli, On the second eigenvalue of a graph, *Discrete Mathematics* 91 (1991), 207-210.

A. Nilli, Tight estimates for eigenvalues of regular graphs, *Electronic J. Combinatorics* 11 (2004), N9, 4pp.

M. Rudelson, Invertibility of random matrices: Norm of the inverse. *submitted.*

A. Sárközy and E. Szemerédi, Uber ein Problem von Erdős und Moser, *Acta Arithmetica* 11 (1965) 205-208.

D. Spielman and S. H. Teng, Smoothed analysis of algorithms, *Proceedings of the International Congress of Mathematicians, Vol. I* (Beijing, 2002), 597–606, Higher Ed. Press, Beijing, 2002.

A. Soshnikov and Y. Sinai, A refinement of Wigner’s semicircle law in a neighborhood of the spectrum edge for random symmetric matrices, (Russian) *Funktsional. Anal. i Prilozhen.* 32 (1998), no. 2, 56–79; 96; translation in *Funct. Anal. Appl.* 32 (1998), no. 2, 114–131.

A. Soshnikov and B. Sudakov, On the largest eigenvalue of a random subgraph of the hypercube, *Comm. Math. Phys.* 239 (2003), no. 1-2, 53–63.

B. Sudakov and V. Vu, Resilience of graphs, *submitted.*

A. Thomason, Pseudorandom graphs, Random graphs ’85 (Poznań, 1985), 307–331, North-Holland Math. Stud., 144, North-Holland, Amsterdam, 1987.

T. Tao and V. Vu, On random ±1 matrices: singularity and determinant, *Random Structures Algorithms* 28 (2006), no. 1, 1–23.

T. Tao and V. Vu, On the singularity probability of random Bernoulli matrices, *to appear in J. A. M. S.*

T. Tao and V. Vu, Inverse Littlewood-Offord theorems and the condition number of random matrices, *submitted.*

T. Tao and V. Vu, The condition number of a randomly perturbed matrix, *submitted.*

T. Tao and V. Vu, Additive Combinatorics, Cambridge Univ. Press, 2006.
[55] T. Tao, V. Vu and P. Wood, *paper in preparation.*
[56] V. Vu, Spectral Norm of Random Matrices, *to appear in Combinatorica* (extended abstract appeared in STOC 2005).
[57] V. Vu and L. Wu, *paper in preparation.*
[58] Wigner, On the distribution of the roots of certain symmetric matrices, *Ann. of Math.* (2) 67 1958 325–327.

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