PERIODIC BILLIARD TRAJECTORIES IN SMOOTH
CONVEX BODIES

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Abstract. We consider billiard trajectories in a smooth convex
body in \( \mathbb{R}^d \) and estimate the number of distinct periodic trajec-
tories that make exactly \( p \) reflections per period at the boundary
of the body. In the case of prime \( p \) we obtain the lower bound
\((d - 2)(p - 1) + 2\), which is much better than the previous esti-
mates.

1. Introduction

First, we give several definitions on billiards in convex bodies.

Definition. A billiard trajectory in a smooth convex body \( T \in \mathbb{R}^d \) is a
polygon \( P \subset T \), with all its vertices on the boundary of \( T \), and at each
vertex the direction of line changes according to the elastic reflection
rule.

Definition. Let \( P \) be a periodic billiard trajectory. Its number of
vertices is called its length.

From here on \( d \) will denote the dimension of convex body under
consideration. We encode periodic trajectories by the sequence of their
vertices \((x_1, x_2, \ldots, x_l)\). The indices are considered \( \mod l \). We do
not allow coincidences \( x_i = x_{i+1} \), but \( x_i \) may equal \( x_j \) if \( i - j \neq 0, \pm 1 \)
\( \mod l \). Of course, the polygon segments can have mutual intersections.

The problem of finding lower bounds on the number of distinct tra-
jectories of a given length has quite a long history. We mean by “dis-
tinct trajectories” the orbits of the dihedral group, acting naturally on
trajectories.

The first lower bound was obtained in \([1]\) for the two-dimensional
case.

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Federation grants No. MK-5724.2006.1 and No. MK-1005.2008.1.
In the case of length 2 and dimension $d$ the paper [2] gives the lower bound $d$. This bound is tight, it may be seen by considering a generic ellipsoid.

The case of dimension 3 was considered in [5], but later an error was found, as noted in [8, 9].

In the book [7] the following problem was formulated (Problem 1.7):

**Problem.** Let $T \subset \mathbb{R}^d$ be a smooth convex body. Prove that it has at least $d$ periodic billiard trajectories of length 3.

In the papers [8, 9] some lower bounds were obtained for length $n$. There were two cases: odd $n$ without other assumptions, and odd prime $n$. In the latter case the bounds were better. Here we give another lower bound for prime lengths.

**Theorem 1.** Let $T \subset \mathbb{R}^d$ be a smooth convex body, $d \geq 3$, let $p > 2$ be a prime. Then there are at least $(d - 2)(p - 1) + 2$ distinct periodic billiard trajectories of length $p$ in $T$.

The proof of this theorem is mainly based on results from [8, 9], calculating the cohomology ring of relevant configuration spaces. The case of non-prime length is not considered here.

2. COHOMOLOGICAL INDEX OF $Z_p$-SPACES

In the sequel, if we want to consider $Z_p$ as a group of transformations, we denote it $G$. If it is used as a ring of coefficients for cohomology it is denoted $Z_p$.

Consider the cohomology algebra $A_G = H^*_G(pt, Z_p) = H^*(BG, Z_p)$. It is well known [3] that $A_G$ has two generators $v, y$ of dimensions 1 and 2 respectively, the relations being

$$v^2 = 0, \quad \beta(v) = y.$$ 

We denote $\beta(x)$ the Bockstein homomorphism. We see that $A_G$ is $Z_p$ in every dimension.

**Definition.** Let $X$ be a $G$-space. The maximal $n$ such that the natural map

$$H^n_G(pt, Z_p) \to H^n_G(X, Z_p)$$

is nontrivial, is called the **cohomological index** of $X$ and denoted $\text{hind} X$.

Note that if the map $H^n_G(pt, Z_p) \to H^n_G(X, Z_p)$ is trivial for some $n$, it should be trivial for all larger $n$, since every element of $A_G$ can be obtained from any nonzero element of less dimension by multiplications by $v, y$, elements of $Z_p^*$, and by applying the Bockstein homomorphism.
Every $G$-space $X$ gives a fibration $$(X \times EG)/G \to BG.$$ 

If $G$ acts on cohomology of $X$ trivially, there exists a spectral sequence with $E_2$-term equal to $H^*(X, \mathbb{Z}_p) \otimes A_G$, converging to $H^*_G(X, \mathbb{Z}_p)$. If $G$ has a nontrivial action on cohomology, $E_2$-term is $E_2^{x, y} = H^*(G, H^y(X, \mathbb{Z}_p))$, where $H^*(G, M) = \text{Ext}^{\ast}_{Z_G[M]}(M, \mathbb{Z}_p)$ is the cohomology of a $G$-module $M$.

If $G$ acts freely on $X$, we have $H^*_G(X, \mathbb{Z}_p) = H^*(X/G, \mathbb{Z}_p)$. In this case $	ext{hind} X$ does not exceed the dimension of $X$.

The following property is obvious by the definition of index:

**Lemma 1 (Monotonicity of index).** *If there is a $G$-equivariant map $X \to Y$, then* $$\text{hind} X \leq \text{hind} Y.$$ 

### 3. Lyusternik-Schnirelmann theory

We recall the definition of the Lyusternik-Schnirelmann category of a topological space.

**Definition.** The relative category of a pair $Y \subseteq X$ is the minimal cardinality of a family of contractible in $X$ open subsets of $X$, covering $Y$. It is denoted $\text{cat}_X Y$. The number $\text{cat}_X X = \text{cat} X$ is called the category of $X$.

Here we state the Lyusternik-Schnirelmann theorem on the number of critical points in the form it is used in [8] for manifolds with boundary.

**Theorem 2.** *Let $X$ be a compact smooth manifold with boundary. Let $f : X \to \mathbb{R}$ be a smooth function, its gradient at $\partial X$ always having strictly outward (inward) direction. Then $f$ has at least $\text{cat}_X X$ critical points.*

We will need the following lemma, that is a special case of the results of [6].

**Lemma 2.** *Let a space $X$ have $k$ cohomology classes $x_1, \ldots, x_k \in H^1(X, \mathbb{Z}_p)$ and $l$ classes of arbitrary positive dimension $y_1, \ldots, y_l \in H^*(X, \mathbb{Z}_p)$. If* $$\beta(x_1)\beta(x_2)\ldots\beta(x_k)y_1\ldots y_l \neq 0,$$
*then* $\text{cat} X \geq 2k + l + 1$.

Actually, we use the following corollary of the previous lemma.
Lemma 3. Let a finite group $G$ act freely on a space $X$, assume that for some subgroup of prime order $G' \subseteq G$ we have $\text{hind}_{G'} X \geq n$. Then $\text{cat } X/G \geq n + 1$.

Proof. As it was noted in [9] $\text{cat } X/G \geq \text{cat } X/G'$.

$G'$ acts freely on $X$, hence $H^*(X/G', Z_p) = H^*_G(X, Z_p)$. The latter algebra has a nonzero product of $A_G$ elements $y^{\frac{n}{2}}$ or $vy^{\frac{n+1}{2}}$, for even or odd $n$ respectively. So we can apply Lemma 2.

4. The configuration space

Following the papers [8, 9] we describe the configuration spaces that arise naturally in the billiard problems. For a topological space $X$ denote $G(X, p) = \{(x_1, \ldots, x_p) \in X^p : x_1 \neq x_2, x_2 \neq x_3, \ldots, x_{p-1} \neq x_p, x_p \neq x_1\}$.

The space $G(X, p)$ has an action of the dihedral group $D_p$, which has a subgroup of even permutations, isomorphic to $Z_p = G$.

Proposition 4.1 of [8] claims that $G(\partial T, p)$ contains a $D_p$-invariant compact manifold with boundary $G_\varepsilon(\partial T, p)$, which is $D_p$-equivariantly homotopy equivalent to $G(\partial T, p)$.

In this section we find the cohomological $G$-index of $G(\partial T, p)$, obviously equal to the index of $G_\varepsilon(\partial T, p)$. The space $\partial T$ is homeomorphic to a $d - 1$-sphere, so we write $G(S^{d-1}, p)$.

The first lower bound on the index of $G(S^{d-1}, p)$ will be obtained by considering another space: $G(\mathbb{R}^d, p)$. Recall a special case of Proposition 2.2 from [8].

Theorem 3. Let $d \geq 2$. The cohomology algebra $H^*(G(\mathbb{R}^d, p), Z_p)$ is multiplicatively generated by $d - 1$-dimensional classes $s_1, \ldots, s_p$ and relations

$s_1^2 = s_2^2 = \cdots = s_p^2 = 0, \quad s_1 s_2 \cdots s_{p-1} + s_2 s_3 \cdots s_p + s_3 s_4 \cdots s_p s_1 + s_p s_1 \cdots s_{p-2} = 0.$

The group $G$ acts on $s_1, \ldots, s_p$ by cyclic permutations.

Considering the action of $G$ on $G(\mathbb{R}^d, p)$, we deduce a corollary.

Theorem 4. If $d \geq 2$, then $\text{hind } G(\mathbb{R}^d, p) = (d - 1)(p - 1)$.

Proof. Denote $Y = G(\mathbb{R}^d, p)$.

Consider a spectral sequence with $E_2^{x,y} = H^x(Y, Z_p)$. Note that $H^*(Y, Z_p)$ are free $Z_p[G]$-modules in dimensions between 0 and $(d - 1)(p - 1)$. In dimension 0 it is $Z_p$, in dimension $(d - 1)(p - 1)$ it is $Z_p[G]/Z_p$.

Let $I_G^k$ be the ideal of $A_G$, consisting of elements of dimension at least $k$. 
In $E_2$-term the bottom row is $A_G$, the top row is $I_G^1$ with shifted by $-1$ grading. The latter claim is deduced from the cohomology exact sequence for $0 \to Z_p \to Z_p[G] \to Z_p[G]/Z_p \to 0$.

All other rows of $E_2$ have nontrivial groups in 0-th column only. Every differential of the spectral sequence is a homomorphism of $A_G$-modules. Hence, the intermediate rows cannot be mapped non-trivially to the bottom row. The only nonzero differential can map the top row to a part of the bottom row, isomorphic to $I_G^{(d-1)(p-1)+1}$. This map has to be nontrivial because the index of $Y$ is finite. \hfill \Box

We see that if $X$ contains $\mathbb{R}^d$, then $G(X, p) \supseteq G(\mathbb{R}^d, p)$. By the monotonicity of index we have $\operatorname{hind}(G(X, p)) \geq (d-1)(p-1)$, in particular $G(S^{d-1}, p) \geq (d-1)(p-1)$.

Then we need more precise estimates on $\operatorname{hind}(G(S^{d-1}, p))$. Recall two theorems from \cite{9} (Theorem 18, Theorem 19), describing the algebra $H^\ast(G(S^{d-1}, p), Z_p)$. We use the notation $[n] = \{1, 2, \ldots, n\}$.

\textbf{Theorem 5.} Let $d \geq 4$ be even. Then $H^\ast(G(S^{d-1}, p), Z_p)$ is generated by the elements

$$u \in H^{d-1}(G(S^{d-1}, p), Z_p), \quad s_i \in H^{i(d-2)}(G(S^{d-1}, p), Z_p), i \in [p-2]$$

and relations

$$u^2 = 0, \quad s_is_j = \frac{(i+j)!}{i!j!}s_{i+j}, \text{ if } i + j \leq p-2, \text{ or otherwise } s_is_j = 0.$$

\textbf{Theorem 6.} Let $d \geq 3$ be odd. Then $H^\ast(G(S^{d-1}, p), Z_p)$ is generated by the elements

$$w \in H^{2d-3}(G(S^{d-1}, p), Z_p), \quad t_i \in H^{i(2d-4)}(G(S^{d-1}, p), Z_p), i \in \left[\frac{p-3}{2}\right]$$

and relations

$$w^2 = 0, \quad t_it_j = \frac{(i+j)!}{i!j!}t_{i+j}, \text{ if } i + j \leq \frac{p-3}{2}, \text{ or otherwise } t_it_j = 0.$$

Theorem 5 is formulated in \cite{9} for the cohomology coefficients $\mathbb{Q}$. But the proof is valid without change for $Z_p$, since we only have to divide by $i!$, where $i \leq p-2$.

Note that $G$ acts on the cohomology in these theorems trivially, since the cohomology is either $Z_p$ or 0 in every dimension.

Finally, let us give another lower bound for $\operatorname{hind}(G(S^{d-1}, p))$. We have a $D_p$-equivariant map from the Stiefel variety of 2-frames in $\mathbb{R}^d$ $V_d^2 \to G(S^{d-1}, p)$, defined as follows. Take some regular $p$-gon in the plane. Every frame $(e_1, e_2) \in V_d^2$ gives an embedding of this $p$-gon to
$S^{d-1}$, which is $D_{p}$-equivariant. So hind $G(S^{d-1}, p) \geq$ hind $V_d^2$, the latter index being equal to (see [4]) $2d - 3$.

**Theorem 7.** If $d \geq 3$, then hind $G(S^{d-1}, p) = (d - 2)(p - 1) + 1$.

**Proof.** First, as it was mentioned in [5] [8], the Morse theory shows that $G(S^{d-1}, p)$ is homotopy equivalent to a CW-complex of dimension $(d - 2)(p - 1) + 1$. So the index is naturally bounded from above and we prove the lower bound only.

If $p = 3 (p - 1)(d - 2) + 1 = 2d - 3$ and the lower bound is already done. So we consider the case $p \geq 5$.

Note that $H^*(G(S^{d-1}, p), Z_p)$ is multiplicatively generated by $(u, s_1)$ and $(w, t_1)$ for even and odd $d$ respectively.

First consider the case of even $d$.

Consider the spectral sequence with $E_2 = H^*(G(S^{d-1}, p), Z_p) \otimes A_G$. In this case the multiplicative generators of $H^*(G(S^{d-1}, p), Z_p)$ have dimension at most $d - 1$. If the differentials $d_m$ were trivial for $m \leq d$, then they should be trivial for $m > d$ from the dimension of generators. But in this case hind $G(S^{d-1}, p) = +\infty$, which is false. So some of $d_m$ is nontrivial for $m \leq d$.

Note that in $E_2$ the bottom row is $A_G$, the next from bottom is $s_1 A_G$, then $u A_G$. Let $d_m$ be the first nontrivial differential. The generators $u$ and $s_1$ cannot be mapped non-trivially to the bottom row, since in this case hind $G(S^{d-1}, p)$ would be at most $d - 1$. The only possibility for nontrivial $d_m$ is $d_2$ that has $d_2(u) = as_1 y (a \in Z_p^*)$. In this case $E_3$ will be multiplicatively generated by $v, y \in A_G$, the image of $s_1$ and of $us^{k-2}$, denote the latter by $z$. The relations would be (besides those of $A_G$):

$$s_1^{p-1} = 0, \quad s_1 y = 0, \quad z^2 = 0.$$ 

That means that $E_3$ has two full rows in the top and in the bottom, and several rows in between contain $s_1^k$ and $s_1^k v$ for $k \in [p - 2]$.

The next $d_m$ cannot map $z$ non-trivially if $m \leq (p - 1)(d - 2) + 1 = \dim z$. They also cannot map $s_1$ non-trivially from the dimension considerations and the lower bound for index. The latter statement can also be deduced from the fact that all differentials are homomorphisms of $A_G$-modules. So finally $d_{(p-1)(d-2)+1}$ has to map $z$ to the bottom row and we have hind $G(S^{d-1}, p) = (p - 1)(d - 2) + 1$.

Now consider the case of odd $d$. Everything is quite the same, but besides the alternative that gives hind $G(S^{d-1}, p) = (p - 1)(d - 2) + 1$ we have another alternative. It could happen that $d_2$ is trivial and the first nontrivial $d_{2d-2}$ maps $w$ non-trivially to the bottom row. But in this case hind $G(S^{d-1}, p) = 2d - 3$, which is less then the bound $(d - 2)(p - 1)$ for $p \geq 5$. □
5. Proof of Theorem 1

Consider the function on $G(\partial T, p)$

$$f : (x_1, \ldots, x_p) \mapsto \sum_{i=1}^{p} |x_i x_{i+1}|.$$

If $\frac{\partial f}{\partial x_i} = 0$, then at $x_i$ the polyline $x_1 x_2 \ldots x_p x_1$ reflects by the elastic reflection rule. $f$ can be considered as a function on $G(\partial T, p)/D_p$, its critical points being in one-to-one correspondence with distinct periodic billiard trajectories of length $p$.

If $T$ is not strictly convex, some line segments of the trajectory can lie on $\partial T$. But in this case all the segments must lie on the same line, which is impossible.

Following [8] we consider $G_{\varepsilon}(\partial T, p)$ instead of $G(\partial T, p)$ and apply Theorem 2 to $f$.

Then we estimate $\text{cat} G(\partial T, p)/D_p \geq (d-2)(p-1) + 2$ by Theorem 7 and Lemma 3.

6. Conclusion

For an arbitrary $d$-manifold $M$ ($d \geq 2$) hind $G(M, p)$ is between $(d-1)(p-1)$ (Theorem 4) and $(d-1)(p-1) + 1$ (by the dimension considerations). It seems probable that for closed $M$ hind $G(M, p) = (d-1)(p-1) + 1$, as it is for spheres (Theorem 7).

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