Backward Deep BSDE Methods and Applications to Nonlinear Problems
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Abstract
In this paper, we present a backward deep BSDE method applied to Forward Backward Stochastic Differential Equations (FBSDE) with given terminal condition at maturity that time-steps the BSDE backwards. We present an application of this method to a nonlinear pricing problem - the differential rates problem. To time-step the BSDE backward, one needs to solve a nonlinear problem. For the differential rates problem, we derive an exact solution of this time-step problem and a Taylor-based approximation. Previously backward deep BSDE methods only treated zero or linear generators. While a Taylor approach for nonlinear generators was previously mentioned, it had not been implemented or applied, while we apply our method to nonlinear generators and derive details and present results. Likewise, previously backward deep BSDE methods were presented for fixed initial risk factor values $X_0$ only, while we present a version with random $X_0$ and a version that learns portfolio values at intermediate times as well. The method is able to solve nonlinear FBSDE problems in high dimensions.

1 Introduction
As proposed in E, Han and Jentzen [EHJ17], deep learning (DL) and deep neural networks (DNN) method can be used to solve high dimensional nonlinear PDEs by converting them to Forward Backward Stochastic Differential Equations (FBSDE) and building neural networks to learn the control and initial value of the corresponding stochastic control problem. One example in that paper uses their proposed forward deep BSDE method to price a combination of two call options with differential rates (different borrowing and lending interest rates) as studied in Mercurio [Mer15]. Hientzsch [Hie19] also gives an overview of pricing different instruments in quantitative finance via deep BSDE and FBSDE. Ganesan, Yu and Hientzsch [GYH20] show how to price Barrier options with deep BSDE and FBSDE.

Han, Jentzen and E [HJE18] propose time-stepping both forward and backward SDE forward in time and transform the final value problem to a stochastic control problem in which the objective function measures how well the given final value has been approximated. We call their method "forward deep BSDE" method since it time-steps the BSDE forward. Wang et al [WCS+18] consider a BSDE with zero drift term which can be trivially time-stepped backwards and propose and demonstrate forward and backward methods with fixed $X_0$, describing the first backward deep BSDE method. Liang, Xu and Li [LXL19] solve BSDEs with linear generators with both forward and backward methods in their examples. They indicate that a nonlinear generator could be handled with a Taylor expansion approach, but do not work out nor implement
the case of the nonlinear generator in the backward method. In this paper, we will describe both
the general approach as well as the application to the differential rates setting for two variants
of the backward method, demonstrating to the best of our knowledge the first application of the
backward method to nonlinear problems.

The main idea of the backward method is that the BSDE is started at maturity with the
given final value and then time-stepped backwards until a given initial time $t_0$. In the case of the
dynamics of $X$ being started at $t_0$ at a fixed value $X_0$, with the right trading strategy, for the
time-continuous case, all realizations of $Y_{t_0}$ (spot price of derivative at initial time) should have
the same value at time $t_0$. Thus, a measure of the size of the range - variance in this particular
case - of $Y_{t_0}$ is picked as the objective function and the variance of the mini-batch is chosen
in mini-batch stochastic gradient descent. For the case of random $X_0$, we minimize the square
distance from an also to-be-determined function $y_{\text{init}}(X_0)$ represented by a DNN. This function
is also the predictable adapted $L^2$-projection of the values obtained from the pathwise roll-back.

The particular nonlinear pricing problem that we consider is the case of differential rates
together with Black-Scholes forward dynamics for European option pricing problem involving, for
example, a linear combination of two calls with coefficients with opposite signs. Differential rates
mean that positive cash balances in the trading strategy accrue interest at a lower lending rate
while negative cash balances (debts/loans) accrue interest at a higher borrowing rate. Standard
self financing trading strategy arguments lead to a nonlinear BSDE.

For the differential rates problem, E, Han and Jentzen [EHJ17] present a nonlinear PDE
which can be solved by appropriate nonlinear PDE solvers in small dimensions (see, for in-
stance, Forsyth and Labahn [FL07]). For a more general setting, Mercurio [Mer15] presents
PDEs and proposes PDE solution or binomial tree methods. None of these methods works in
higher dimensions due to the curse of dimensionality. All these methods require problem specific
implementation of nonlinear PDE or tree solver.

Standard Monte-Carlo approaches that simulate, discount, and average can not handle non-
linear pricing or control problems that depend on the solution or its gradient.

There are some other approaches to such nonlinear problems in high dimensions such as
Warin [War18] or Huré, Pham and Warin [HPW19]. However, they use nested Monte-Carlo or
more elaborate methods rather than a path-wise approach (and they do not treat the differential
rates problem as an example).

In this paper, we first introduce FBSDE for general nonlinear problems, with particular
details for the differential rates problem, time-discretize them, and then derive exact and Taylor
approximations for the backward step. We then quickly describe the forward and backward deep
BSDE approaches that we consider - both the batch-variance variant already described in the
literature but also the novel initial variable and network versions, the last one for random $X_0$,
together with the computational graphs for the implementations. Then we apply these methods
to the differential rates problem for the call combination case from Han, Jentzen and E [HJE18]
and for the straddle case from Forsyth and Labahn [FL07]. We compare the results for a case
with fixed $X_0$ and for a case with varying $X_0$ with the results from Forsyth and Labahn [FL07]
and see that they agree well. We visualize and discuss some of the results. Finally, we conclude.

2 FBSDE for Nonlinear Problems

One type of nonlinear PDEs that we are interested in solving has the general form:

$$u_t(t, x) + \mathcal{L}_t u(t, x) + f(t, x, u(t, x), \nabla u(t, x)) = 0,$$

(1)
with
\[
\mathcal{L}_t u(t, x) := \frac{1}{2} \text{Tr} (\sigma_N \sigma_N^T (t, x) (\text{Hess}_x u) (t, x)) + \mu (t, x) \nabla u (t, x),
\]
where Hess$_x u$ is the Hessian matrix, with terminal condition at maturity given as:
\[
u(T, x) = g(x).
\]
A nonlinear Feynman-Kac theorem shows the solution of above PDE also satisfies the following FBSDE system under appropriate assumptions:

The forward SDE (FSDE) for the underlying assets:
\[
dX_t = \mu (t, X_t) dt + \sigma (t, X_t) dW_t,
\]
and the backward SDE (BSDE) in terms of the coefficient of the Brownian $Z_t$:
\[
-dY_t = f (t, X_t, Y_t, Z_t) dt - Z_t^T dW_t,
\]
or in terms of values $\Pi_t$:
\[
-dY_t = f (t, X_t, Y_t, \Pi_t) dt - \Pi_t^T \sigma (t, X_t) dW_t,
\]
with terminal condition
\[
Y_T = g(X_T),
\]
where
\[
Y_t = u(t, X_t), \Pi_t = \nabla X u(t, X_t), Z_t = \sigma (t, X_t) \Pi_t.
\]

In terms of pricing applications in finance, $g(X_T)$ is the final payoff of the European option that one tries to replicate with a self-financing portfolio in the underlying asset(s) $X_t$ and a remaining cash position. That portfolio will contain $\pi_i(t)$ worth of $X_i(t)$ ($\Pi_t$ being the vector of $\pi_i(t)$). The portfolio (including cash position) is worth $Y_t$ at time $t$.

Now $Y_t$ or equivalently $u(t, X_t)$ represent the needed wealth at $t$ to exactly or approximately replicate the payoff when starting at $X_t$ at time $t$. This gives one of the possible ways to define price (pricing by replication): $\text{price}(t, X_t; X_T \mapsto g(X_T))$ as the solution of the FBSDE and/or the nonlinear PDE. Linear pricing satisfies (among other things)
\[
\text{price}(t, X_t; X_T \mapsto g(X_T)) = -\text{price}(t, X_t; X_T \mapsto -g(X_T)).
\]

In nonlinear pricing in general (as for instance for differential rates, as we will see in examples later), these two prices are no longer necessarily the same but will give an upper and a lower price.

\section{2.1 Time-discretizing Time-continuous FBSDE}

Using Euler-Maruyama method to discretize time direction forward for both $X_t$ and $Y_t$, we have
\[
X_{t_{i+1}} = X_{t_i} + \mu (t_i, X_{t_i}) \Delta t_i + \sigma (t_i, X_{t_i}) \Delta W^t_i
\]
and
\[
Y_{t_{i+1}} = Y_{t_i} - f (t_i, X_{t_i}, Y_{t_i}, \Pi_{t_i}) \Delta t_i + \Pi_{t_i}^T \sigma (t_i, X_{t_i}) \Delta W^t_i.
\]

\footnote{If $\Pi_t$ measures the hedging delta in the portfolio, it would be $\sigma_N$ rather than $\sigma_{LN}$ in the stochastic term of the $Y$ BSDE, where $\sigma_N(t, X) = \sigma_{LN}(t, X) X$.}

\footnote{If measured by value, or $\pi(t)X_i(t)$ worth of $X_i(t)$ if measured by delta/size.}
2.2 Backward Time-stepping

2.2.1 Analytical Solution

To backward step in time direction, we rewrite (11) as:

\[ Y_{t_i} - f(t_i, X_{t_i}, Y_{t_i}, \Pi_{t_i}) \Delta t_i = Y_{t_{i+1}} - \Pi_{t_i}^T \sigma(t_i, X_{t_i}) \Delta W^i \]  \hspace{1cm} (12)

and solve for \( Y_{t_i} \).

For a differential rates setup in a risk neutral measure, the \( f \) generator function in the BSDE is:

\[ f(t, X_t, Y_t, \Pi_t) = -r^l(t) Y_t + (r^b(t) - r^l(t)) \left( \sum_{i=1}^{n} \pi_i(t) - Y_t \right)^+ . \]  \hspace{1cm} (13)

This driver expresses that all assets \( X_i(t) \) and positive cash balances grow at a risk-neutral rate \( r^l(t) \) unless the cash position \( Y_t - \sum_{i=1}^{n} \pi_i(t) \) is negative, and that negative cash balance will grow at a rate \( r^b(t) \) corresponding to the higher borrowing rate as compared to the lower or equal lending rate.

There are two cases for equation (13):

1. If \( \sum_{i=1}^{n} \pi_i(t) > Y(t) \):
   \[ f(t, X_t, Y_t, \Pi_t) = -r^l(t) Y_t + (r^b(t) - r^l(t)) \left( \sum_{i=1}^{n} \pi_i(t) - Y_t \right) . \]  \hspace{1cm} (14)

Inserting this into equation (12) and solving, we obtain:

\[ Y_{t_i} = \frac{Y_{t_{i+1}} + (r^b(t_i) - r^l(t_i)) \left( \sum_{j=1}^{n} \pi_j(t_i) \right) \Delta t_i - \Pi_{t_i}^T \sigma(t_i, X_{t_i}) \Delta W^i}{1 + r^b(t_i) \Delta t_i} . \]  \hspace{1cm} (15)

2. If \( \sum_{i=1}^{n} \pi_i(t) \leq Y(t) \):
   \[ f(t, X_t, Y_t, \Pi_t) = -r^l(t) Y_t . \]  \hspace{1cm} (16)

Inserting this into equation (12) and solving, we obtain:

\[ Y_{t_i} = \frac{Y_{t_{i+1}} - \Pi_{t_i}^T \sigma(t_i, X_{t_i}) \Delta W^i}{1 + r^l(t_i) \Delta t_i} . \]  \hspace{1cm} (17)

However, we do not know \( Y_{t_i} \) before solving the nonlinear equation (12) for it. From (15) and (17) and the conditions involving \( Y_{t_i} \), we obtain that \( Y_{t_i} < \sum_{j=1}^{n} \pi_j(t_i) \) is equivalent to

\[ Y_{t_{i+1}} < \left( \sum_{j=1}^{n} \pi_j(t_i) \right) \{ \sigma(t_i, X_{t_i}) \Delta W^i + (1 + r^l(t_i)) \Delta t_i \} \]  \hspace{1cm} (18)

and the same for the relation with \( \geq \). Thus, if (18) is satisfied, we use (15), otherwise (17).

2.2.2 Taylor Expansion

For first order Taylor expansion, we have:

\[ f \left( t_i, X_{t_i}, Y_{t_i}, \Pi_{t_i}^T \sigma(t_i, X_{t_i}) \right) \approx f \left( t_i, X_{t_i}, Y_{t_{i+1}}, \Pi_{t_i}^T \sigma(t_i, X_{t_i}) \right) - \frac{\partial f}{\partial Y} \left( t_i, X_{t_i}, Y_{t_{i+1}}, \Pi_{t_i}^T \sigma(t_i, X_{t_i}) \right) \Delta Y^i . \]  \hspace{1cm} (19)
Inserting this into equation (12) and solving for $Y_t$, we have the following:

$$Y_t = Y_{t+1} + \frac{f(t_i, X_{t_i}, Y_{t+1}, \Pi^T_t \sigma(t_i, X_{t_i})) \Delta t - \Pi^T_t \sigma(t_i, X_{t_i}) \Delta W_i}{1 - \frac{\partial f}{\partial Y}(t_i, X_{t_i}, Y_{t+1}, \Pi^T_t \sigma(t_i, X_{t_i})) \Delta t}.$$  \hspace{1cm} (20)

Note that $f$ and $\frac{\partial f}{\partial Y}$ are evaluated at $Y_{t+1}$.

With the same setup for the differential rates problem, it is clear that there are only two possible forms for $f$:

1). If $\sum_{j=1}^{n} \pi_j(t_i) > Y_{t+1}$:

$$f(t_i, X_{t_i}, Y_{t+1}, \Pi_t) = -r^l(t_i)Y(t_i) + (r^b(t_i) - r^l(t_i)) \left( \sum_{j=1}^{n} \pi_j(t_i) - Y_{t+1} \right)$$  \hspace{1cm} (21)

and

$$\frac{\partial f}{\partial Y} = -r^b(t_i).$$  \hspace{1cm} (22)

Inserting this into equation (20), we obtain:

$$Y_{t_i} = \frac{Y_{t+1} + (r^b(t_i) - r^l(t_i)) \left( \sum_{j=1}^{n} \pi_j(t_i) \right) \Delta t - \Pi^T_t \sigma(t_i, X_{t_i}) \Delta W_i}{1 + r^b(t_i) \Delta t}. $$  \hspace{1cm} (23)

2). If $\sum_{j=1}^{n} \pi_j(t_i) \leq Y_{t+1}$:

$$f(t_i, X_{t_i}, Y_{t+1}, \Pi_t) = -r^l(t_i)Y_{t+1}$$  \hspace{1cm} (24)

and

$$\frac{\partial f}{\partial Y} = -r^l(t_i).$$  \hspace{1cm} (25)

Inserting this into equation (20), we have:

$$Y_{t_i} = \frac{Y_{t+1} - \Pi^T_t \sigma(t_i, X_{t_i}) \Delta W_i}{1 + r^l(t_i) \Delta t}.$$  \hspace{1cm} (26)

Notice that (15) and (23) are the same and that (17) and (26) are the same. The only difference lies in the conditions when they are applied.

3 Deep BSDE Approach

3.1 Forward Approach

As introduced in E, Han and Jentzen [EHJ17], with forward time-stepped equations (10) and (11), one minimizes the loss function

$$E(||Y_T - g(X_N)||^2).$$  \hspace{1cm} (27)

The initial portfolio value $Y_0$ is a parameter of the minimization problem as are all the parameters of the DNN functions $\pi_0(t_i, X_{t_i})$ treated as functions of $X_{t_i}$ (that give the stochastic vector process $\Pi_t$ as value or size of the holdings of the risky underlying securities in the portfolio). Since $X_0$ is fixed, instead of learning a function $\pi_0(X_0)$, one learns a parameter $\pi_0$. Alternatively,
one can learn a single function \( \pi(t_i, X_{t_i}) \) as function of \( t_i \) and \( X_{t_i} \), which means that all the parts of the computational graph that represent the evaluation of \( \pi(t, x) \) share the same DNN parameters\(^3\). The minimization problem is then solved with standard deep learning approaches such as mini-batch stochastic gradient methods, using approaches such as Adam, pre-scaling and/or batch-normalization, etc.

For the case of random \( X_0 \), one also learns the initial value of \( Y_0 \) as a function \( \text{Yinit}(X_0) \) of \( X_0 \), using the same loss function. Han, Jentzen, and E \( [HJE18] \) mention this approach for random \( X_0 \) on page 8509. We are not aware of any publication presenting results or implementations of the random \( X_0 \) approach except in our own work.

### 3.2 Backward Approach

In the backward approach, one time-steps equations \([10]\) forward but time-steps equations \([11]\) backward, starting from \( Y_T = g(X_T) \). As discussed in the previous section, one can use an analytical solution of \([12]\) or some Taylor expansion approach. Using either approach, one will obtain an expression or implementation

\[
Y_{t_i} = \text{ybackstep}(t_i, Y_{t_i + 1}, X_{t_i}, \Pi_{t_i}, \Delta W^i).
\]

For fixed \( X_0 \), the loss function will be

\[
\text{var}(Y_0).
\]

For the mini-batch stochastic gradient step, the loss function will be the mini-batch variance

\[
E(||Y_0 - \bar{Y}_0||^2),
\]

where \( \bar{Y}_0 \) will be the mean over the mini-batch. For MC (Monte Carlo) estimates for \( Y_0 \), one can use the last mini-batch mean or one can compute the mean of \( Y_0 \) over a larger sample of paths (but fixing the trading strategy).

Instead of using the mini-batch mean in the loss function, one can learn \( \bar{Y}_0 \) as a parameter/variable (resulting in the same loss function but with different meaning of \( Y_0 = \text{Yinit} \)).

Once \( X_0 \) is random, one can no longer use batch variance in a straightforward way. Instead (and inspired by the parameter version just discussed), one uses a loss function

\[
E(||Y_0 - \text{Yinit}(X_0)||^2),
\]

where the \( \text{Yinit}(X_0) \) is a function represented by a DNN which is learned as part of the DL approaches.

Similarly, one can introduce additional terms

\[
E(||Y_{t_i} - \text{Ylearned}_i(X_{t_i})||^2)
\]

at some (or all) intermediate times \( t_i \) to learn some approximations for the solution function \( \text{Ylearned}_i(X_{t_i}) = u(t_i, X_{t_i}) \) (or one could learn the trading strategy and intermediate solution functions in stages in a roll-back fashion).

All the methods except the one using batch variance are novel, to the best of our knowledge.

For these different backward approaches, the first time step translates into the three different computational graphs shown in figures \([1][2][3]\). The general time step for all three methods is shown in figure \([4]\) while the last time step is shown in figure \([5]\). First, one would simulate \( X \)

\(^3\)There are many introductions into DL, DNN, and common forms of DNN. For a minimal one geared towards deep BSDE, see Hientzsch \([Hic19]\).
forward, starting at the first time step, proceeding through intermediate time steps, and reaching the final time step. At the final time step, $Y_T$ is set to $g(X_T)$, and backward steps $y\text{bstep}$ are taken, proceeding through intermediate steps, until one reaches back at the first time step. In the figures, gray boxes are given implementations/operations that do not change, pink boxes (circles) are networks (variables/parameters) to be trained, blue circles are random and green circles are input parameters.

4 Results

We present results on two financial derivatives treated in the literature so that we can compare our results more easily with those of others. The two financial derivatives are a call combination (long a call on the maximum across assets with strike 120.0 and short two calls on the maximum with strike 150.0, with maturity 0.5 years) as in E, Han and Jentzen [EHJ17] and a straddle on the maximum (both long and short a straddle with strike 100.0 with maturity 1 year) as in Forsyth and Labahn [FL07].

For ease of visualization, testing, and presentation, we present results for the one-dimensional case (which is also the case treated in Forsyth and Labahn [FL07]).

4.1 Call Combination

For the E, Han, and Jentzen example, we picked $\sigma = 0.2$, $\mu = 0.06$, $r_l = 0.04$, and $r_b = 0.06$. We used 50 time steps.

For the call combination example, for the fixed $X_0$ case, we picked $X_0 = 120.0$. For the random/varying $X_0$ case, we picked a uniform distribution in $[70, 170]$. We used a batch-size of 512, pre-scaling, two hidden layers with $dim + 10$ (dim is the dimension of PDE we are trying to solve) neurons and activation functions ELU for the first two layers and then identity on the output layer.

We first show results for the fixed $X_0$ case for the call combination. Figure 6 shows how the loss function values behave over the number of minibatches.

Figures 7 and 8 show the trading strategy and the portfolio value $Y$. The batch variance method uses shared parameters for the risky portfolio vector functions $\pi(t,x)$ while the initial parameter method uses separate DNNs with separate parameters for different times.

We next show results for the random $X_0$ case for the call combination. Figure 9 shows the loss functions. Figure 10 shows initial $Y_{\text{init}}$ network and rollback and the $Z_0$ network. Figure 11 shows the strategy DNN outputs and the path $Y$ values. This example uses separate DNNs.

4.2 Straddle

Forsyth and Labahn picked the settings $\sigma = 0.3$, $\mu = r_b = 0.05$, and $r_l = 0.03$ [FL07, Table 1 on page 28 in hjb.pdf]. We used 100 time steps (one of the numbers of time steps for which results are given in tables in Forsyth and Labahn [FL07]). The strike for the straddle is 100.0.

We first consider the fixed $X_0$ case. Like Forsyth and Labahn [FL07], we pick initial spot $X_0$ to be 100.0.

We plot the $Y_0$ estimates or parameters for different backward and forward methods in figure 12 for exact and figure 13 for Taylor backward step, for both long and short straddles, together with a more detail view. We see that the method that learns the $y_0$ parameter initially converges more slowly than the batch variance methods and that computing the mean over 100 mini-batches rather than one leads to a faster and more smooth convergence. (And seemingly the learning
Figure 1: First time step in backward method when using MC mean for $Y_0$ for a fixed $X_0$

Figure 2: First time step in backward method when learning $Y_0$ for a fixed $X_0$
Figure 3: First time step in backward method when learning $Y_0$ as a network from a random $X_0$

Figure 4: General step in backward method
The $y_0$ parameter method converges faster for upper price than for lower price. However, once the learning $y_0$ method gets close, its convergence is smoother and better than the batch-variance methods.

It would be interesting to see whether a method that updates the $y_0$ parameter based on a weighted average of the batch mean and the DL update would combine the advantages of both methods.

We compare our results against the results given in Forsyth and Labahn [FL07, Tables 2 and 3] in tables 2 and 1 for the lower and upper price, respectively.

We see that we are close to the values given in Forsyth and Labahn [FL07, Tables 2 and 3]. Forsyth and Labahn [FL07] Tables 2 and 3 have numbers for higher number of time steps and space steps as well which are even closer to our results so it could be that if the Forsyth and Labahn method would be run and reported for more space steps but same number of time steps, it would give values even closer to ours.

For random $X_0$ case, we pick $X_0$ uniformly within [50, 150] but plot results within [80, 120].

We extracted the curves from Figure 1 from Forsyth and Labahn [FL07, Figure 1] (the hjb PDF version) and plotted them as background in all the figures (curves shown in black). Figure 14 shows different backward methods with exact backward step for different batch sizes, figure 15 shows different backward methods with Taylor backward step for different batch sizes, and figure 16 shows forward method with random $X_0$ for different batch sizes, all for long and short positions. We can see that for increasing batch sizes, the agreement is improving.

It can be seen that the results for exact backward step and Taylor backward step are very close.

Lastly, for the exact backward step for batch size 1024, we show RiskyPortfolio size (delta), RiskyPortfolio value (delta times stock price), cash position value for long and short straddle, and the regions with borrowing (red) and lending (black) in figures 17 and 18 respectively.

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4 Notice that [FL07, Figure 1] do not give the number of space or time steps used for the plot.
Figure 6: Loss function over 20000 mini-batches for different backward methods (fixed $X_0$) for the call combination example

(a) Batch variance method - exact
(b) Batch variance method - Taylor
(c) Method with $Y_0$ as a parameter - exact
(d) Method with $Y_0$ as a parameter - Taylor
Figure 7: Strategy DNN (delta) outputs at 20000 mini-batches for different backward methods (fixed $X_0$) for the call combination example

(a) Batch variance - exact

(b) Batch variance - Taylor

(c) Method with $Y_0$ as a parameter - exact

(d) Method with $Y_0$ as a parameter - Taylor
Figure 8: $Y$ path values at 20000 mini-batches for different backward methods (fixed $X_0$) for the call combination example

(a) Initial network - exact
(b) Initial network - Taylor
(c) Method with $Y_0$ as a parameter - exact
(d) Method with $Y_0$ as a parameter - Taylor

Figure 9: Loss function over 20000 mini-batches for different backward methods (random $X_0$) for the call combination example

(a) Initial network - exact
(b) Initial network - Taylor
Figure 10: $Y_0$ initial network, roll-back, and $\pi_0$ initial network at 20000 mini-batches for different backward methods (random $X_0$) for the call combination example.
Figure 11: Strategies deltas and \( Y \) path values at 20000 mini-batches for different backward methods (random \( X_0 \)) for the call combination example
Figure 12: $Y_0$ estimates or parameters for the straddle case - over 20000 mini-batches and detail for 5000-1000 minibatches - exact backward step - batch size 512
Figure 13: Y$_0$ estimates or parameters for the straddle case - over 20000 mini-batches and detail for 5000-1000 minibatches - Taylor backward step - batch size 256
| Method                                                                 | Result                              |
|-----------------------------------------------------------------------|-------------------------------------|
| Results from Forsyth and Labahn - 101 nodes                           |                                     |
| Fully Implicit HJB PDE (implicit control)                             | 24.02047                            |
| Crank-Nicolson HJB PDE (implicit control)                             | 24.0512                             |
| Fully Implicit HJB PDE (pwc policy)                                   | 24.01163                            |
| Crank-Nicolson HJB PDE (pwc policy)                                   | 24.0652                             |
| Forward deep BSDE - 20000 batches, size 256                          |                                     |
| Learned y0                                                            | 24.078833 (24.044819-24.098291)     |
| Backward deep BSDE (exact) - 20000 batches, size 256                 |                                     |
| Batch variance, 1 mini-batch mean                                     | 24.119225 (23.912037-24.306263)     |
| Batch variance, 100 mini-batch mean                                   | 24.061472 (24.022112-24.132414)     |
| Learned y0                                                            | 24.072815 (24.043901-24.095972)     |
| Backward deep BSDE (Taylor) - 20000 batches, size 256                |                                     |
| Batch variance, 1 mini-batch mean                                     | 24.119202 (23.911976-24.30629)      |
| Batch variance, 100 mini-batch mean                                   | 24.061443 (24.020626-24.132645)     |
| Learned y0                                                            | 24.072783 (24.043854-24.09594)      |
| Difference over all batches, exact-Taylor                             |                                     |
| Batch variance, 1 mini-batch mean                                     | 7.0242124e-05 (0.0040798187-0.0010948181) |
| Batch variance, 100 mini-batch mean                                   | 7.2354465e-05 (-0.00023078918-0.0051631927) |
| Learned y0                                                            | 2.8257615e-05 (-0.00012207031- 0.00032234192) |

Table 1: Upper Price

| Method                                                                 | Result                              |
|-----------------------------------------------------------------------|-------------------------------------|
| Results from Forsyth and Labahn - 101 nodes                           |                                     |
| Fully Implicit HJB PDE (implicit control)                             | 23.05854                            |
| Crank-Nicolson HJB PDE (implicit control)                             | 23.08893                            |
| Fully Implicit HJB PDE (pwc policy)                                   | 23.06752                            |
| Crank-Nicolson HJB PDE (pwc policy)                                   | 23.09371                            |
| Forward deep BSDE - 20000 batches, size 256                          |                                     |
| Learned y0                                                            | 23.127728 (23.09221-23.139273)      |
| Backward deep BSDE (exact) - 20000 batches, size 256                 |                                     |
| Batch variance, 1 mini-batch mean                                     | 23.1515 (22.834194-23.39135)        |
| Batch variance, 100 mini-batch mean                                   | 23.100569 (23.083847-23.139555)     |
| Learned y0                                                            | 23.126017 (23.091948-23.138735)     |
| Backward deep BSDE (Taylor) - 20000 batches, size 256                |                                     |
| Batch variance, 1 mini-batch mean                                     | 23.151543 (22.834248-23.391354)     |
| Batch variance, 100 mini-batch mean                                   | 23.100618 (23.083895-23.133991)     |
| Learned y0                                                            | 23.12606 (23.091982-23.138788)      |
| Difference over all batches, exact-Taylor                             |                                     |
| Batch variance, 1 mini-batch mean                                     | -3.9644332e-05 (-0.0004576367-0.00018501282) |
| Batch variance, 100 mini-batch mean                                   | -3.406489e-05 (-0.00035476685-9.536743e-05) |
| Learned y0                                                            | -2.3744791e-05 (-0.00030136108-0.00034713745) |

Table 2: Lower Price
Figure 14: $Y_{\text{init}}(X_0)$ for various backward methods with exact backward step plotted over Forsyth and Labahn curves
Figure 15: $Y_{init}(X_0)$ for various backward methods with Taylor backward step plotted over Forsyth and Labahn curves
Figure 16: $Y_{\text{init}}(X_0)$ for various forward methods plotted over Forsyth and Labahn curves
Figure 17: Risky portfolio size, risky portfolio value, cash position value, and locations where strategy borrows (red) or lends (black) for random $X_0$, exact backward step, batch size 1024, long position/upper price.
Figure 18: Risky portfolio size, risky portfolio value, cash position value, and locations where strategy borrows (red) or lends (black) for random $X_0$, exact backward step, batch size 1024, short position/lower price.
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6 Conclusion

In this paper, we first introduced FBSDE for general nonlinear problems, with particular details for the differential rates problem, time-discretize them, and then derived exact and Taylor approximations for the backward step. We then quickly described the forward and backward deep BSDE approaches that we consider - both the batch-variance variant already described in the literature but also the novel initial variable and network versions, the last one for random $X_0$. Then we applied these methods for the differential rates problem for the call combination case from Han, Jentzen and E [HJE18] and for the straddle case from Forsyth and Labahn [FL07]. We compare the results for a case with fixed $X_0$ and for a case with varying $X_0$ with the results from Forsyth and Labahn [FL07] and see that they agree well. We also compare methods for the exact backward step and the Taylor backward step and in the straddle and call combination examples that we ran, the results seem to be very close. We also visualized some of the results to show what they mean in terms of trading strategy and borrowing and lending.

The deepBSDE methods described in this paper are using a very different approach from the PDE methods by Forsyth and Labahn [FL07], but they give results very close to those published there. That makes us confident that these methods can be used to generically and efficiently approximate solutions to such nonlinear pricing problems, even with relatively small batch-size such as 512 or 1024.

7 Disclaimer

Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of Wells Fargo Bank, N.A., its parent company, affiliates and subsidiaries.

References

[EHJ17] Weinan E, Jiequn Han, and Arnulf Jentzen. Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. *Communications in Mathematics and Statistics*, 5(4):349–380, 2017. arXiv:1706.04702.

[FL07] Peter A Forsyth and George Labahn. Numerical methods for controlled hamilton-jacobi-bellman pdes in finance. *Journal of Computational Finance*, 11(2):1–44, 2007. preprint version available online for instance at: [https://cs.uwaterloo.ca/~paforsyt/hjb.pdf](https://cs.uwaterloo.ca/~paforsyt/hjb.pdf).

[GYH20] Narayan Ganesan, Yajie Yu, and Bernhard Hientzsch. Pricing barrier options with deepBSDEs. arXiv preprint arXiv:2005.10966, May 2020.
Bernhard Hientzsch. Introduction to solving quant finance problems with time-stepped FBSDE and deep learning. *arXiv preprint arXiv:1911.12231*, Nov 2019. Also available at SSRN: https://ssrn.com/abstract=3494359 or http://dx.doi.org/10.2139/ssrn.3494359.

Jiequn Han, Arnulf Jentzen, and Weinan E. Solving high-dimensional partial differential equations using deep learning. *Proceedings of the National Academy of Sciences, 115*(34):8505–8510, 2018.

Côme Huré, Huyën Pham, and Xavier Warin. Some machine learning schemes for high-dimensional nonlinear PDEs. *arXiv preprint arXiv:1902.01599*, 2019.

Jian Liang, Zhe Xu, and Peter Li. Deep learning-based least square forward-backward stochastic differential equation solver for high-dimensional derivative pricing. *arXiv preprint arXiv:1907.10578*, 2019. Also available at SSRN: https://ssrn.com/abstract=3381794 or http://dx.doi.org/10.2139/ssrn.3381794.

Fabio Mercurio. Bergman, Piterbarg, and beyond: pricing derivatives under collateralization and differential rates. In *Actuarial Sciences and Quantitative Finance*, pages 65–95. Springer, 2015. Also available at SSRN: https://ssrn.com/abstract=2326581 or http://dx.doi.org/10.2139/ssrn.2326581.

Xavier Warin. Nesting monte carlo for high-dimensional non linear PDEs. *arXiv preprint arXiv:1804.08432*, 2018.

Haojie Wang, Han Chen, Agus Sudjianto, Richard Liu, and Qi Shen. Deep learning-based BSDE solver for LIBOR market model with application to bermudan swaption pricing and hedging. *arXiv preprint arXiv:1807.06622*, 2018. Also available at SSRN: https://ssrn.com/abstract=3214596 or http://dx.doi.org/10.2139/ssrn.3214596.