AVERAGES OF ARITHMETIC FUNCTIONS OVER PRINCIPAL IDEALS

T. D. BROWNING AND E. SOFOS

ABSTRACT. For a general class of non-negative functions defined on integral ideals of number fields, upper bounds are established for their average over the values of certain principal ideals that are associated to irreducible binary forms with integer coefficients.

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1. INTRODUCTION

The study of averages of non-negative multiplicative arithmetic functions $f$ over the values of polynomials has a long and venerable history in number theory. From the point of view of upper bounds, this topic goes back to work of Nair [7], which has since been substantially generalised by Nair–Tenenbaum [8] and Henriot [5]. For suitable expanding regions $B \subset \mathbb{Z}^2$, several authors have extended these results to cover averages of the shape

$$\sum_{(s,t) \in B} f(|F(s,t)|),$$

where $F \in \mathbb{Z}[s,t]$ is an irreducible binary form. This is the object of work by la Bretèche–Browning [1] and la Bretèche–Tenenbaum [2], for example. These estimates have since had many applications to a range of problems, most notably in the quantitative arithmetic of Châtelet surfaces [3].

Assuming for the moment that $F(x,1)$ is monic and irreducible, any root $\theta$ of the polynomial generates a number field $K = \mathbb{Q}(\theta)$ whose degree is equal to the degree of $F$. In this paper we instead consider a variant in which we...
take a general non-negative multiplicative function \( f \) defined on the ideals of \( K \), and ask to bound the size of the sum

\[
\sum_{(s,t) \in \mathcal{B}} f(s - \theta t),
\]

where we view \((s - \theta t)\) as an ideal in the ring of integers \( \mathfrak{o}_K \) of \( K \). Our primary motivation for considering this sum is the crucial role that it plays in work of the authors \([1]\) on Manin’s conjecture for smooth quartic del Pezzo surfaces.

In order to present our main result we require some notation and definitions. Let \( K/\mathbb{Q} \) be a number field with ring of integers \( \mathfrak{o}_K \). Denote by \( \mathcal{I}_K \) the set of ideals in \( \mathfrak{o}_K \). We say that a function \( f : \mathcal{I}_K \to \mathbb{R}_{\geq 0} \) is pseudomultiplicative if there exist strictly positive constants \( A, B, \varepsilon \) such that

\[
f(a b) \leq f(a) \min \{ A^{\Omega_K(b)}, B (N_K b)^\varepsilon \},
\]

for all coprime ideals \( a, b \in \mathcal{I}_K \), where \( \Omega_K(b) = \sum_{p | b} \nu_p(b) \) and \( N_K b = \| \mathfrak{o}_K / b \| \) is the ideal norm. We denote the class of all pseudomultiplicative functions associated to \( A, B, \varepsilon \) by \( \mathcal{M}_K \). When \( K = \mathbb{Q} \), this class contains the class of submultiplicative functions that arose in pioneering work of Shiu \([9]\). Note that any \( f \in \mathcal{M}_K \) satisfies \( f(a) \ll A^{\Omega_K(a)} \) and \( f(a) \ll (N_K a)^\varepsilon \), for any \( a \in \mathcal{I}_K \), where the second implied constant depends on \( B \).

We will need to work with functions supported away from ideals of small norm. To facilitate this, for any ideal \( a \in \mathcal{I}_K \) and \( W \in \mathbb{N} \), we set

\[
a_W = \prod_{p^\nu | a \atop \gcd(N_K p, W) = 1} p^\nu.
\]

We extend this to rational integers in the obvious way. Next, for any \( f \in \mathcal{M}_K \), we define \( f_W(a) = f(a_W) \). We will always assume that \( W \) is of the form

\[
W = \prod_{p \leq w} p,
\]

for some \( w > 0 \). Thus \( \gcd(N_K p, W) = 1 \) if and only if \( p > w \), if \( N_K p = p^{f_p} \) for some \( f_p \in \mathbb{N} \). Let

\[
\mathcal{P}_K = \{ a \subset \mathfrak{o}_K : p | a \Rightarrow f_p = 1 \}
\]

be the multiplicative span of all prime ideals \( p \subset \mathfrak{o}_K \) with residue degree \( f_p = 1 \). For any \( x > 0 \) and \( f \in \mathcal{M}_K \) we set

\[
E_f(x; W) = \exp \left( \sum_{p \in \mathcal{P}_K \text{ prime} \atop w < N_K p \leq x \atop f_p = 1} \frac{f(p)}{N_K p} \right),
\]
if $f$ is submultiplicative, and
\[
E_f(x; W) = \sum_{a \in \mathcal{P}_K \text{ square-free}} \frac{f(a)}{N_K a},
\]
otherwise.

Suppose now that we are given irreducible binary forms $F_1, \ldots, F_N \in \mathbb{Z}[x, y]$, which we assume to be pairwise coprime. Let $i \in \{1, \ldots, N\}$. Suppose that $F_i$ has degree $d_i$ and that it is not proportional to $y$, so that $b_i = F_i(1, 0)$ is a non-zero integer. It will be convenient to form the homogeneous polynomial
\[
\tilde{F}_i(x, y) = b_i^{d_i-1} F(b_i^{-1} x, y).
\]
This has integer coefficients and satisfies $\tilde{F}_i(1, 0) = 1$. We let $\theta_i$ be a root of the monic polynomial $\tilde{F}(x, 1)$. Then $\theta_i$ is an algebraic integer and we denote the associated number field of degree $d_i$ by $K_i = \mathbb{Q}(\theta_i)$. Moreover,
\[
N_{K_i/\mathbb{Q}}(b_is - \theta_i t) = \tilde{F}_i(b_is, t) = b_i^{d_i-1} F_i(s, t),
\]
for any $(s, t) \in \mathbb{Z}^2$. (If $b_i = 0$, so that $F_i(x, y) = cy$ for some non-zero $c \in \mathbb{Z}$, we take $\theta_i = -c$ and $K_i = \mathbb{Q}$ in this discussion.) Our main result is a tight upper bound for averages of $f_1w((b_1s - \theta_1t)) \ldots f_Nw((b_Ns - \theta_Nt))$, over primitive vectors $(s, t) \in \mathbb{Z}^2$, for general pseudomultiplicative functions $f_i \in \mathcal{M}_{K_i}$ and suitably large values of $w$.

Next, for any $k \in \mathbb{N}$ and any polynomial $P \in \mathbb{Z}[x]$, we set
\[
\rho_P(k) = \sharp \{x \text{ (mod } k) : P(x) \equiv 0 \text{ (mod } k)\}.
\]
We put
\[
\overline{\rho}_i(k) = \begin{cases} \rho_{F_i(x, 1)}(k) & \text{if } F_i(1, 0) \neq 0, \\ 1 & \text{if } F_i(1, 0) = 0, \end{cases}
\]
and
\[
h^*(k) = \prod_{p | k} \left(1 - \overline{\rho}_1(p) + \ldots + \overline{\rho}_N(p)\right)^{-1}. \tag{1.6}
\]
To any non-empty bounded measurable region $\mathcal{R} \subset \mathbb{R}^2$, we associate
\[
K_{\mathcal{R}} = 1 + \|\mathcal{R}\|_\infty + \partial(\mathcal{R}) \log(1 + \|\mathcal{R}\|_\infty) + \frac{\text{vol}(\mathcal{R})}{1 + \|\mathcal{R}\|_\infty}, \tag{1.7}
\]
where $\|\mathcal{R}\|_\infty = \sup_{(x, y) \in \mathcal{R}} \{|x|, |y|\}$. We say that such a region $\mathcal{R}$ is regular if its boundary is piecewise differentiable, $\mathcal{R}$ contains no zeros of $F_1 \ldots F_N$ and there exists $c_1 > 0$ such that $\text{vol}(\mathcal{R}) \geq K_{\mathcal{R}}^{c_1}$. Note that we then have
\[
\text{vol}(\mathcal{R}) \leq 4\|\mathcal{R}\|^2_\infty \leq (1 + \|\mathcal{R}\|_\infty)^4 \leq K_{\mathcal{R}}^4.
\]
Bearing all of this in mind, we may now record our main result.
Theorem 1.1. Let \( \mathcal{R} \subset \mathbb{R}^2 \) be a regular region, let \( V = \text{vol}(\mathcal{R}) \) and let \( G \subset \mathbb{Z}^2 \) be a lattice of full rank, with determinant \( q_G \) and first successive minimum \( \lambda_G \). Assume that \( q_G \leq V^{c_2} \) for some \( c_2 > 0 \). Let \( f_i \in \mathcal{M}_K(A_i, B_i, \varepsilon_i) \), for \( 1 \leq i \leq N \) and let

\[
\varepsilon_0 = \max \left\{ 1 + \frac{4}{c_1}, \frac{4(5 + 3 \max \{\varepsilon_1, \ldots, \varepsilon_N\})}{c_1} \right\} \left( \sum_{i=1}^{N} d_i \varepsilon_i \right).
\]

Then, for any \( \varepsilon > 0 \) and \( w > w_0(f_i, F_i, N) \), we have

\[
\sum_{(s, t) \in \mathbb{Z}_p^{2 \times \text{prim}} \cap \mathcal{R} \cap G} \prod_{i=1}^{N} f_i(q_G W((b_i s - \theta_i t))} \ll \frac{V}{(\log V)^N} \frac{h^*_W(q_G)}{q_G} \prod_{i=1}^{N} F_i(V; W) + \frac{K^{1+\varepsilon_0+\varepsilon}}{\lambda_G},
\]

where the implied constant depends at most on \( c_1, c_2, A_i, B_i, F_i, \varepsilon, \varepsilon_i, N, W \).

We may compare this estimate with the principal result in work of la Bretèche–Tenenbaum \[2\]. Take \( G = \mathbb{Z}^2 \) and \( d_1 = \cdots = d_N = 1 \). Then \( b_i s - \theta_i t = F_i(s, t) \), for \( 1 \leq i \leq N \). In this setting Theorem 1.1 can be deduced from the special case of \[2\] Thm. 1.1, in which all of the binary forms are linear.

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2. Technical results

2.1. Lattice point counting. We will need general results about counting lattice points in an expanding region. Let \( A \in \text{Mat}_{2 \times 2}(\mathbb{Z}) \) be a non-singular upper triangular matrix and consider the lattice given by \( G = \{ Ay : y \in \mathbb{Z}^2 \} \). Recall that \( G \) is said primitive if the only integers \( m \) fulfilling \( G \subset m \mathbb{Z}^2 \) are \( m = \pm 1 \). We denote its determinant and first successive minimum by \( \det(G) \) and \( \lambda_G \), respectively. Assume that \( \mathcal{R} \subset \mathbb{R}^2 \) is a regular region, in the sense of Theorem 1.1. Then, for any \( x_0 \in \mathbb{Z}^2 \) and \( q \in \mathbb{N} \) such that \( \gcd(\det(G)x_0, q) = 1 \), we will require an asymptotic estimate for the counting function

\[
N(\mathcal{R}) = \# \{ x \in \mathbb{Z}_p^{2 \times \text{prim}} \cap \mathcal{R} \cap G : x \equiv x_0 (\text{mod } q) \}.
\]

The following estimate follows from work of Sofos \[10, \text{Lemma 5.3}\].
Lemma 2.1. Assume that $G$ is primitive. Then
\[ N(\mathcal{R}) = \frac{\nu(\mathcal{R})}{\zeta(2) \det(G)q^2} \prod_{p|\det(G)} \left(1 + \frac{1}{p}\right)^{-1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{-1} + O\left(\frac{\tau(\det(G))K_N}{\lambda_G}\right), \]
where $K_N$ is given by (1.7) and the implied constant is absolute.

2.2. Restriction to square-free support. For a given number field $K/\mathbb{Q}$ of degree $d$ and given $f \in \mathcal{M}_K$, it will sometimes be useful to bound sums of the shape
\[ \sum_{N_K a \leq x} \frac{f(a)}{N_K a} \]
by a sum restricted to square-free integral ideals supported away from ideals of small norm. This is encapsulated in the following result.

Lemma 2.2. Let $f \in \mathcal{M}_K(A, B, \varepsilon)$. Assume that $f^\dagger$ is multiplicative and that there exists $M > 0$ such that
\[ |f^\dagger(p^\nu) - 1| \leq \frac{M}{N_K p}, \]
for all prime ideals $p$ and $\nu \in \mathbb{N}$. Assume that $W$ is given by (1.1), with $w > 2(A + M)$. Then
\[ \sum_{N_K a \leq x} \frac{f(a_W)f^\dagger(a_W)}{N_K a} \leq_{A,M,W} \sum_{N_K b \leq x} \frac{f(b)}{N_K b}, \]
where $b$ is square-free and $\gcd(N_K b, W) = 1$.

If $f$ is submultiplicative then the right hand side can be replaced by
\[ \exp \left( \sum_{w \leq N_K p \leq x} \frac{f(p)}{N_K p} \right). \]

Proof. The final part of the lemma is obvious. To see the first part we note that there is a unique factorisation $a = qa_W$, where $N_K q \mid W^\infty$. Here, and throughout this paper, for $a, b \in \mathbb{N}$ the notation $a \mid b^\infty$ is taken to mean that every prime divisor of $a$ is a divisor of $b$ as well. Next, we decompose uniquely $a_W = bc$ where $b, c$ are coprime integral ideals such that $b$ is square-free and $c$ is square-full. The sum in the lemma is at most
\[ \sum_{N_K q \leq x} \frac{1}{N_K q} \sum_{N_K b \leq x} \frac{f(b)f^\dagger(b)}{N_K b} \sum_{N_K c \leq x} \frac{A\Omega_K(c)f^\dagger(c)}{N_K c}. \]
For prime ideals with $N_K \mathfrak{p} > 2(A + M)$, we have

$$1 + \sum_{d=2}^{\infty} \frac{A_{\Omega_K}(p^d)}{(N_K \mathfrak{p})^d} f^\dagger(p^d) \leq 1 + \sum_{d=2}^{\infty} \frac{A_d}{(N_K \mathfrak{p})^d} \left( 1 + \frac{M}{N_K \mathfrak{p}} \right) \leq \left( 1 - \frac{2A^2}{(N_K \mathfrak{p})^2} \right)^{-1}.$$  

Thus the sum over $\mathfrak{c}$ converges absolutely. Defining the multiplicative function $g : \mathcal{A}_K \to \mathbb{R}$ via $g(p^d) = (f^\dagger(p) - 1) N_K \mathfrak{p}$, for $d \in \mathbb{N}$, we clearly have

$$f^\dagger(b) = \sum_{\mathfrak{d} \mid b} \frac{g(\mathfrak{d})}{N_K \mathfrak{d}},$$

with $|g(\mathfrak{d})| \leq M^{\Omega_K(\mathfrak{d})}$, for any square-free ideal $\mathfrak{b}$. Therefore

$$\sum_{\substack{N_K b \leq x \\
\gcd(N_K b, W) = 1}} \frac{f(b) f^\dagger(b)}{N_K b} = \sum_{\substack{N_K \mathfrak{c} \leq x \\
\gcd(N_K \mathfrak{c}, W) = 1}} \frac{f(\mathfrak{c})}{N_K \mathfrak{c}} \sum_{\substack{N_K \mathfrak{d} \leq x/N_K \mathfrak{c} \\
\gcd(N_K \mathfrak{d}, W) = 1}} \frac{g(\mathfrak{d})}{(N_K \mathfrak{d})^2}.$$  

The sum over $\mathfrak{d}$ is absolutely convergent, whence

$$\sum_{\substack{N_K a \leq x \\
\gcd(N_K a, W) = 1}} \frac{f(a_W) f^\dagger(a_W)}{N_K a} \ll_{A,M,W} \sum_{\substack{N_K \mathfrak{c} \leq x \\
\gcd(N_K \mathfrak{c}, W) = 1}} \frac{f(\mathfrak{c})}{N_K \mathfrak{c}} \sum_{\substack{N_K q \leq x \\
q \mid N_K W}} \frac{1}{N_K q^2}.$$  

The inner sum over $q$ is at most $\prod_{N_K \mathfrak{p} \mid W^d} (1 - \frac{1}{N_K \mathfrak{p}})^{-1} \ll_{A,M,W} 1$, which thereby completes the proof of the lemma. \hfill \square

2.3. The relevance of $\mathcal{P}_K$. Let $F \in \mathbb{Z}[x, y]$ be an irreducible non-zero binary form of degree $d$, which is not proportional to $y$. In particular $b = F(1, 0)$ is a non-zero integer. We recall from (1.3) the associated binary form $\tilde{F}(x, y) = b^{d-1}F(b^{-1}x, y)$, with integer coefficients and $\tilde{F}(1, 0) = 1$. We let $\theta$ be a root of the polynomial $f(x) = \tilde{F}(x, 1)$. Then $\theta$ is an algebraic integer and $K = \mathbb{Q}(\theta)$ is a number field of degree $d$ over $\mathbb{Q}$. It follows that $\mathbb{Z}[\theta]$ is an order of $K$ with discriminant $\Delta_\theta = |\det(\sigma_i(\omega^d))|^2$, where $\sigma_1, \ldots, \sigma_d : K \hookrightarrow \mathbb{C}$ are the associated embeddings. As is well-known, we have

$$\Delta_\theta = [\mathfrak{o} : \mathbb{Z}[\theta]]^2 D_K, \quad (2.1)$$

where $D_K$ is the discriminant of $K$. Recall the definition (2.2) of $\mathcal{P}_K^0$ and define

$$\mathcal{P}_K = \{ \mathfrak{a} \subset \mathcal{P}_K^0 : p_1 p_2 \mid \mathfrak{a} \Rightarrow N_K p_1 \neq N_K p_2 \text{ or } p_1 = p_2 \}, \quad (2.2)$$

which is the subset of $\mathcal{P}_K^0$ that is cut out by ideals divisible by at most one prime ideal above each rational prime. The following result is crucial in our analysis and will frequently allow us to restrict attention to ideals supported on $\mathcal{P}_K$. 

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Lemma 2.3. Let \((s, t) \in \mathbb{Z}_{\text{prim}}^2\) such that \(F(s, t) \neq 0\). Then \(a \in \mathcal{P}_K\) for any integral ideal \(a \mid (bs - \theta t)\) such that gcd\((N_K a, 2b\Delta_\theta) = 1\).

Proof. Let \(D = 2b\Delta_\theta\) and let \((s, t) \in \mathbb{Z}_{\text{prim}}^2\) such that \(F(s, t) \neq 0\). We form the integral ideal \(n = (bs - \theta t)\). This has norm \(N_K n = |\tilde{F}(bs, t)|\). Let \(k \mid \tilde{F}(bs, t)\) with gcd\((k, D) = 1\). In particular gcd\((k, \Delta_\theta) = 1\).

Now let \(p \mid k\). Then \(p \nmid t\) since gcd\((s, t) = 1\) and \(p \nmid b\). We choose \(\tilde{t} \in \mathbb{Z}\) such that \(tt \equiv 1 \pmod{p}\). Let \(p\) be any prime ideal such that \(p \mid (p)\) and \(p \mid n\).

Consider the group homomorphism

\[\pi : \mathbb{Z}/p\mathbb{Z} \rightarrow (\mathbb{Z}[\theta] + p)/p,\]

given by \(m \mapsto m + p\). Suppose that \(\pi(m_1) = \pi(m_2)\) for \(m_1, m_2 \in \mathbb{Z}/p\mathbb{Z}\). Then \(m_1 - m_2 \in p\), whence \(N_K p \mid (m_1 - m_2)^d\). But this implies that \(p \mid m_1 - m_2,\) since \(p \mid (p),\) and so \(\pi\) is injective. Next suppose that \(P(\theta) + p \in (\mathbb{Z}[\theta] + p)/p,\)

where \(P(\theta) = \sum c_i\theta^i\) for \(c_i \in \mathbb{Z}\). Since \(p \mid n\) and \(p \mid \tilde{t}\), we have \(bs\tilde{t} \equiv \theta \pmod{p}\).

Thus \(P(\theta) - P(bs\tilde{t}) \equiv 0 \pmod{p}\). It then follows that \(\pi(m) = P(\theta) + p\). Thus \(\pi\) is surjective and so it is an isomorphism. Hence \([\mathbb{Z}[\theta] + p : p] = p\). In view of (2.1), we also have

\[D_K[\mathbb{Z}[\theta] + p]/p = \mathbb{Z}[\theta]/p,\]

This implies that \(p \nmid [\mathbb{Z}[\theta] + p]/p\). Since \(N_p\) is power of \(p\), we readily conclude that

\[N_p = [\mathbb{Z}[\theta] + p][\mathbb{Z}[\theta] + p] = p.\]

This therefore establishes that \(a \in \mathcal{P}_K\).

To finish the proof it remains to show that there are no distinct prime ideals \(p_1, p_2\) with \(p_1 \mid p_2\) and \(N_p_1 = N_p_2\). Suppose for a contradiction that there exist such primes \(p_1 \neq p_2\). Letting \(p = N_p_1 = N_p_2\) and noting that \(p \nmid \Delta_\theta\), an application of Dedekind’s theorem on factorisation of prime ideals supplies us with distinct \(n_1, n_2 \in \mathbb{Z}/p\mathbb{Z}\) such that

\[f(x) \equiv (x - n_1)(x - n_2)\Upsilon(x) \pmod{p},\]

for a polynomial \(\Upsilon \in (\mathbb{Z}/p\mathbb{Z})[x]\) of degree \([K : \mathbb{Q}] - 2\), with \(p_1 = (p, \theta - n_1)\) and \(p_2 = (p, \theta - n_2)\).

We conclude from this that \(bs\tilde{t} - \theta \in p_1\) and \(\theta - n_1 \in p_1\), whence \(bs\tilde{t} - n_1 \in p_1\). Similarly, we have \(bs\tilde{t} - n_2 \in p_2\). But then it follows that \(p = N_p_1 \mid bs\tilde{t} - n_1\) and \(p = N_p_2 \mid bs\tilde{t} - n_2\). This implies that \(n_1 \equiv n_2 \pmod{p}\), which is a contradiction. \(\square\)

We close this section with an observation about the condition \(a \mid (bs - \theta t)\) that appears in Lemma 2.3.

Lemma 2.4. Let \(a \in \mathcal{P}_K\) such that gcd\((N_K a, D_K) = 1\). Then there exists \(k = k(a) \in \mathbb{Z}\) such that

\[a \mid (bs - \theta t) \iff bs \equiv kt \pmod{N_K a}\]
for all \((s, t) \in \mathbb{Z}^2\),

**Proof.** It suffices to check this when \(a = p^a\), for some \(p \in \mathcal{P}_K\) such that \(p = N_K p\) is unramified, by the Chinese remainder theorem. This is because the definition of \(\mathcal{P}_K\) implies that for every rational unramified prime \(p\) there is at most one prime ideal \(p\) above \(p\) such that \(p \mid a\). To continue with the proof we note that since \(p \in \mathcal{P}_K\), there exists \(k' \in \mathbb{Z}\) satisfying \(k' \equiv \theta \pmod{p}\), whence there exists \(k \in \mathbb{Z}\) such that \(k \equiv \theta \pmod{p^a}\). Therefore

\[bs \equiv \theta t \pmod{p^a} \iff bs - kt \in \mathbb{Z} \cap p^a.\]

We claim that the latter condition is equivalent to \(bs \equiv kt \pmod{N_K p^a}\). The reverse implication is obvious since \(N_K a \equiv a\) for any integral ideal \(a\). The forward implication follows on noting that \(\nu_p((bs - kt)) \geq a\).

\[\square\]

### 3. The main argument

This section is devoted to the proof of Theorem 1.1 following an approach that is inspired by work of Shiu [9]. We assume familiarity with the notation introduced in [11]. Since \(F_1, \ldots, F_N\) are pairwise coprime it follows that the resultants \(\text{Res}(F_i, F_j)\) are all non-zero integers for \(i \neq j\). Along the way, at certain stages of the argument, we will need to enlarge the size of \(W\) in (1.1).

For now we assume that \(w > \max_{i \neq j}\{|D_i|, \lfloor \text{Res}(F_i, F_j)\rfloor\}\), where \(D_i = 2b_i \Delta_{\theta_i}\) and \(\text{Res}\) denotes the resultant of two polynomials. We let \(N_i = N_{K_i}\) and write \(F = \prod_{i=1}^N F_i\). Let

\[z = V^\omega,\]

where \(V = \text{vol}(\mathcal{R})\), for a small constant \(\omega > 0\) that will be chosen in due course. (In particular, it will need to be sufficiently small in terms of \(\varepsilon_1, \ldots, \varepsilon_N\).)

For each \((s, t) \in \mathbb{Z}_{\text{prim}}^2 \cap \mathcal{R} \cap G\), it follows from (1.1) that we have a factorisation

\[
\prod_{i=1}^N |N_i(b_is - \theta_it)_{qGW}| = \prod_{i=1}^N |F_i(s, t)|_{qGW} = |F(s, t)|_{qGW} = p_1^{\alpha_1} \cdots p_l^{\alpha_l},
\]

with \(w < p_1 < \cdots < p_l\). We define \(a_{s,t}\) to be the greatest integer of the form \(p_1^{\alpha_1} \cdots p_j^{\alpha_j}\) which is bounded by \(z\) and we define \(b_{s,t} = F(s, t)_{qGW}/a_{s,t}\). We have \(\gcd(a_{s,t}, b_{s,t}) = 1\) and \(P^-(b_{s,t}) > P^+(a_{s,t})\). Our lower bound for \(w\) ensures that

\[
\gcd(N_i a_i, N_j a_j) = 1,
\]

for any \(a_i \mid (b_is - \theta_it)_{qGW}\) and \(a_j \mid (b_is - \theta_jt)_{qGW}\), with \(i \neq j\). Lemma 2.3 implies that \(a_i \in \mathcal{P}_i = \mathcal{P}_{K_i}\), for \(1 \leq i \leq N\).

The sum in which we are interested,

\[
\sum_{(s, t) \in \mathbb{Z}_{\text{prim}}^2 \cap \mathcal{R} \cap G} \prod_{i=1}^N f_i((b_is - \theta_it)),
\]

is not a trivial one.
is sorted into four distinct contributions $E^{(I)}(\mathcal{R}), \ldots, E^{(IV)}(\mathcal{R})$. For an appropriate small parameter $\eta > 0$, these sums are determined by which of the following attributes are satisfied by $(s, t)$:

(I) $P^-(b_{s,t}) \geq z^{\frac{\eta}{2}}$;

(II) $P^-(b_{s,t}) < z^{\frac{\eta}{2}}$ and $a_{s,t} \leq z^{1-\eta}$;

(III) $P^-(b_{s,t}) \leq \log z \log \log z$ and $z^{1-\eta} < a_{s,t} \leq z$;

(IV) $\log z \log \log z < P^-(b_{s,t}) < z^{\frac{\eta}{2}}$ and $z^{1-\eta} < a_{s,t} \leq z$.

In what follows, we will allow all of our implied constants to depend on $c_1, c_2, A_i, B_i, F_i, \varepsilon, \varepsilon_i, N, W$, as in the statement of Theorem 1.1, as well as on $\omega$ and $\eta$, whose values will be indicated during the course of the proof. Any further dependence will be indicated by an explicit subscript.

We let $\Omega_i = \Omega_{K_i}$ be the number of prime ideal divisors (counted with multiplicity) and note that $\Omega_i(a) = \Omega(N_i a)$ when $a \in \mathcal{R}_i$. For given $(s, t)$, the choice of $a_{s,t}, b_{s,t}$ that we have made uniquely determines coprime ideals $a_{s,t}^{(i)}, b_{s,t}^{(i)} \subset \mathfrak{o}_i$, with $(b_i s - \theta_i t)_{qG W} = a_{s,t}^{(i)} b_{s,t}^{(i)}$, such that

$$\prod_{i=1}^{N} N_i a_{s,t}^{(i)} = a_{s,t} \quad \text{and} \quad \Omega_i(a_{s,t}^{(i)}) = \Omega(N_i a_{s,t}^{(i)}).$$

In particular we emphasise that $a_{s,t}^{(i)}, b_{s,t}^{(i)}$ are supported on prime ideals whose norms are coprime to $qG W$. We now have everything in place to start estimating the various contributions. Our main tools will be the geometry of numbers and the fundamental lemma of sieve theory.

**Case I.** We begin by considering the case $P^-(b_{s,t}) \geq z^{\frac{\eta}{2}}$. Recalling that $f_i \in M_{K_i}$, we have $f_i q_G W((b_i s - \theta_i t)) \leq f_i (a_{s,t}^{(i)})^{\Omega_i(b_{s,t}^{(i)})}$, by the coprimality of $a_{s,t}^{(i)}, b_{s,t}^{(i)}$. Hence

$$E^{(I)}(\mathcal{R}) = \sum_{(s,t) \in \mathbb{Z}_{\text{prim}}^{2} \cap \mathcal{R} \cap G} \prod_{i=1}^{N} f_i q_G W((b_i s - \theta_i t))$$

$$\ll \sum_{a_i \notin \mathcal{R}_i} \mathcal{H}(a_1, \ldots, a_N) \prod_{i=1}^{N} f_i(a_i),$$

$$\prod_{i=1}^{N} N_i a_i \leq z$$

(3.2)
where
\[
\mathcal{U}(a_1, \ldots, a_N) = \sum_{(s,t) \in \mathbb{Z}_{\text{prim}}^2 \cap \mathcal{A} \cap G} \prod_{i=1}^{N} A^{\Omega_i \left( \frac{(b_i s - \theta_i t) qGW}{a_i} \right)},
\]
\[
p|F(s,t)qGW/\prod_{i=1}^{N} a_i = |F(s,t)|^{\deg(F)} \leq K_{\mathcal{A}}^{\deg(F)}.
\]
Here, the condition \(\prod_{i=1}^{N} a_i \leq z\) comes from the fact that \(a_{s,t} \leq z\). Moreover, we write \(a_i = 1\) if and only if the ideals \(a_i, a_i \mathcal{A}\) are coprime.

Defining \(b\) through \(b \prod_{i=1}^{N} a_i = \prod_{p|qGW} p^{p_{\mathcal{A}}(F(s,t))}\), we see that \((z^{\frac{\gamma}{2}})^{\Omega(b)} \leq P(b)^{\Omega(b)} \leq |F(s,t)| \ll \|\mathcal{A}\|_{\infty}^{\deg(F)} \leq K_{\mathcal{A}}^{\deg(F)}\).

In view of (3.1) and the inequality \(K_{\mathcal{A}}^{\gamma} \leq V\) that is assumed in Theorem 1.1, this shows that \(\Omega(b) \ll 1\). Noting that
\[
\sum_{i=1}^{N} \Omega_i \left( \frac{(b_i s - \theta_i t) qGW}{a_i} \right) = \Omega(b),
\]
we may therefore conclude that
\[
\mathcal{U}(a_1, \ldots, a_N) \ll \mathcal{U}_{\frac{\gamma}{2}}(a_1, \ldots, a_N),
\]
where for any \(\gamma > 0\) we define \(\mathcal{U}_{\gamma}(a_1, \ldots, a_N)\) to be the cardinality of \((s, t)\) appearing in the definition of \(\mathcal{U}(a_1, \ldots, a_N)\), with the lower bound \(z^{\frac{\gamma}{2}}\) replaced by \(z^{\gamma}\). Our next concern is with an upper bound for this quantity.

Before revealing our estimate for \(\mathcal{U}_{\gamma}(a_1, \ldots, a_N)\), recall the definition of \(h^*\) from (1.6) and set
\[
h^*(k) = \prod_{p|k} \left( 1 - \frac{d_1 + \cdots + d_N}{1 + p} \right)^{-1}.
\]
Then we have the following result.

**Lemma 3.1.** Let \(\delta, \gamma > 0\) and let \(a_i \in \mathcal{A}_i\), for \(1 \leq i \leq N\), with
\[
gcd(N_i a_i, qGW N_j a_j) = 1 \quad \text{and} \quad \prod_{i=1}^{N} N_i a_i \leq z.
\]
Then
\[
\mathcal{U}_{\gamma}(a_1, \ldots, a_N) \ll_{\delta} \frac{V}{\gamma^N (\log z)^N} h^*_W(qGW) \prod_{i=1}^{N} \frac{h^*(N_i a_i)}{N_i a_i} + \frac{K_{\mathcal{A}} z^{2\gamma + \delta}}{\lambda_G},
\]
uniformly in \(\gamma\).
We defer the proof of this result, temporarily, and show how it can be used to complete the treatment of $E^{(I)}(\mathfrak{R})$, via (3.2) and (3.3). We apply Lemma 3.1 with $\gamma = \frac{n}{2}$. We also note that since $f_i \in M_{K_i}(A_i, B_i, \varepsilon_i)$, we have

$$\prod_{i=1}^{N} f_i(a_i) \ll \prod_{i=1}^{N} N_i(a_i)^{\varepsilon_i} \leq \prod_{i=1}^{N} N_i(a_i)^{\tilde{\varepsilon}},$$

where $\tilde{\varepsilon} = \max\{\varepsilon_1, \ldots, \varepsilon_N\}$. The overall contribution from the second term is therefore found to be

$$\ll \frac{K_{\mathfrak{R}} \varepsilon + \ell}{\lambda_G} \#\{a_1, \ldots, a_N : N_1 a_1 \cdots N_N a_N \leq z\} \leq \frac{K_{\mathfrak{R}} \varepsilon + \ell + 2\varepsilon}{\lambda_G} \leq \frac{K_{\mathfrak{R}} \varepsilon + 2\varepsilon}{\lambda_G},$$

where we used the bound $V \leq K_{\mathfrak{R}}^4$, as well as $z = V^\omega$. In view of (3.2) and (3.3), the first term in Lemma 3.1 makes the overall contribution

$$\ll \frac{V}{(\log z)^N} h_W^*(q_G) \prod_{i=1}^{N} \sum_{\substack{a_i \in \mathcal{P}_i \ \text{gcd}(N_i a_i, q_G) = 1 \ \text{square-free} \ \text{when} \ N_i a_i \leq z}} f_i(a_i) h^*(N_i a_i) \frac{1}{N_i a_i}.$$

Since $a_i \in \mathcal{P}_i$, (3.4) implies that

$$h^*(N_i a_i) \leq \prod_{p|a_i} \left(1 - \frac{d_1 + \cdots + d_N}{1 + N_i p}\right)^{-1} = h_i^*(a_i),$$

say, where we recall that $d_i = [K_i : \mathbb{Q}]$. We enlarge $w$ in order to use Lemma 2.2 and thereby obtain the overall contribution

$$\ll \frac{V}{(\log z)^N} h_W^*(q_G) \prod_{i=1}^{N} \sum_{\substack{a_i \in \mathcal{P}_i \ \text{gcd}(N_i a_i, q_G) = 1 \ \text{square-free} \ \text{when} \ N_i a_i \leq z}} f_i(a_i) \frac{1}{N_i a_i}.$$

We have therefore proved that for every $\delta > 0$ we have the bound

$$E^{(I)}(\mathfrak{R}) \ll \frac{V}{(\log z)^N} h_W^*(q_G) \prod_{i=1}^{N} \sum_{\substack{a_i \in \mathcal{P}_i \ \text{gcd}(N_i a_i, q_G) = 1 \ \text{square-free} \ \text{when} \ N_i a_i \leq z}} f_i(a_i) \frac{K_{\mathfrak{R}} \varepsilon + 2\varepsilon}{\lambda_G}, \ (3.5)$$

where we recall that $\tilde{\varepsilon} = \max\{\varepsilon_1, \ldots, \varepsilon_N\}$. 

Proof of Lemma 3.1. Let \( c_i \in \mathcal{P}_i \) be given, with \( \gcd(N_i c_i, qGW N_j c_j) = 1 \).
Define the set

\[ \Lambda(c_1, \ldots, c_N) = \{ (s, t) \in \mathbb{Z}^2 : b_i s \equiv \theta_i t \pmod{c_i}, \text{ for } i = 1, \ldots, N \} . \]

Since \( \gcd(qG, \prod_{i=1}^{N} N_i c_i) = 1 \), it follows from Lemma [22] that this defines a lattice in \( \mathbb{Z}^2 \) of rank 2 and determinant

\[ \det(\Lambda(c_1, \ldots, c_N)) = qG \prod_{i=1}^{N} N_i c_i . \]

Write \( P(z_0) = \prod_{p < z_0} P \), for any \( z_0 > 0 \), with the usual convention that \( P(z_0) = 1 \) if \( z_0 < 2 \). This allows us to write

\[ \mathcal{U}_r(a_1, \ldots, a_N) \leq \sum_{(s, t) \in S} \frac{1}{\left| \frac{F(s, t)}{P(s, t)} \right|} \frac{\mu(d)}{d} \prod_{i=1}^{N} N_i a_i \]

where \( S = \mathbb{Z}_{\text{prim}}^2 \cap \mathcal{R} \cap \Lambda(a_1, \ldots, a_N) \). We shall use the fundamental lemma of sieve theory, as presented by Iwaniec and Kowalski [8] \( \S 6.4 \). This provides us with a sieve sequence \( \lambda_d^+ \) supported on square-free integers in the interval \([1, 2z^\gamma]\), with \( \lambda_1^+ = 1 \) and \( |\lambda_d^+| \leq 1 \), such that

\[ \mathcal{U}_r(a_1, \ldots, a_N) \leq \sum_{(s, t) \in S} \frac{1}{\left| \frac{F(s, t)}{P(s, t)} \right|} \frac{\lambda_d^+}{d} \prod_{i=1}^{N} N_i a_i \]

Since \( \gcd(a_i, b_i, t) = 1 \), we note that only \( d \) coprime to \( \prod_{i=1}^{N} N_i a_i \) appear in the inner sum. Interchanging the order of summation, we find that

\[ \mathcal{U}_r(a_1, \ldots, a_N) \leq \sum_{\epsilon_1 | a_1} \mu_1(\epsilon_1) \cdots \mu_N(\epsilon_N) \sum_{\frac{1}{\gcd(d, qGW N_1 a_1) = 1}}^{\frac{1}{\gcd(d, qGW N_j a_j) = 1}} \lambda_d^+ \sum_{d_1 \cdots d_N \in \mathbb{N}} 1 \]

where if \( d = d_1 \cdots d_N \), then

\[ S(d) = \sum_{\frac{(s, t) \in \mathbb{Z}_{\text{prim}}^2 \cap \mathcal{R} \cap \Lambda(a_1, \ldots, a_N \epsilon_N)}{F_1(s, t) = 1 \pmod{d_1}}} 1 \]

If \( F_i(x, y) = cy \) for some \( i \), then the condition \( b_i s \equiv \theta_i t \pmod{a_i \epsilon_i} \) should be replaced by \( t \equiv 0 \pmod{a_i \epsilon_i} \).
Recall the definition (1.5) of $\rho_i$ and let

$$h(d) = \prod_{p | d} \left(1 + \frac{1}{p}\right)^{-1}.$$ 

The number of possible $(\sigma, \tau)$ (mod $d$) is equal to $\varphi(d)\overline{\gamma}_1(d_1)\cdots\overline{\gamma}_N(d_N)$. In $S(d)$ the inner sum can be estimated using the geometry of numbers. Calling upon Lemma 2.1, we deduce that

$$S(d) = \frac{V}{\zeta(2)} \frac{h(q_G)}{q_G} \prod_{i=1}^N \frac{\overline{\gamma}_i(d_i)h(N_i a_i)}{d_i N_i \epsilon_i N_i a_i} + O_\delta \left(\frac{K \gamma^{\gamma+\frac{4}{\gamma}}}{\lambda_G}\right),$$

for any $\delta > 0$. We emphasise that the implied constant in this estimate does not depend on any of $R, d_i, a_i$ or $c_i$. Since $|\lambda_i^+| \leq 1$ and $\tau_{K_i}(a_i) \ll (N_i a_i)^{\frac{1}{N_i}}$, we find that the overall contribution to $\mathcal{M}_i(a_1, \ldots, a_N)$ from the error term is $\mathcal{O}_\delta(K \gamma^{2\gamma+\delta}/\lambda_G)$, on summing trivially over $c_1, \ldots, c_N$ and $d_1, \ldots, d_N$. This is plainly satisfactory for Lemma 3.1.

Turning to the contribution from the main term, we set

$$g(d) = \frac{1}{d, q_G W} \prod_{i=1}^N \frac{h(d)}{d} = \sum_{d_1 \cdots d_N = d} \prod_{i=1}^N \overline{\gamma}_i(d_i),$$

where $1(d, a) = 1$ if $\gcd(d, a) = 1$ and $1(d, a) = 0$, otherwise. Since $h(d) \leq 1$ and $\varphi_i(a_i) \leq N_i a_i$, the main term contributes

$$\ll \frac{V}{q_G} \prod_{i=1}^N \frac{1}{N_i a_i} \left|\sum_{1 \leq d \leq 2\gamma^\gamma \atop d| \overline{\gamma}_i} \lambda_i^+ g(d)\right|.$$ 

We may clearly assume without loss of generality that $w < 2\gamma^{\max\{\gamma, \frac{4}{\gamma}\}}$. For any prime $p \nmid W$, let

$$c_p = 1 - \frac{h(p)}{p} \sum_{i=1}^N \overline{\gamma}_i(p) - \frac{\overline{\gamma}_1(p) + \cdots + \overline{\gamma}_N(p)}{p + 1}.$$ 

Recalling that $\deg(F_i) = d_i$ for $1 \leq i \leq N$, we see that

$$c_p \geq 1 - \frac{d_1 + \cdots + d_N}{p + 1},$$

for $p \nmid W$. Next, for $y \geq 0$, define

$$\Pi(y) = \prod_{p \leq y} c_p, \quad \Pi_1 = \prod_{p \leq 2\gamma^\gamma} c_p, \quad \Pi_2 = \prod_{p \leq 2\gamma^\gamma} c_p.$$
By the fundamental lemma of sieve theory [6, Lemma 6.3], we find that
\[
\sum_{1 \leq d \leq 2z^{\gamma}} \frac{\lambda_{d}^{+} g(d)}{d | P(z^{\gamma})} < \prod_{p < 2z^{\gamma}} (1 - g(p)) = \Pi(2z^{\gamma}) \Pi_{i} \Pi_{1} \Pi_{2} h_{i}^{+} (q_{G}) \prod_{i=1}^{N} h_{i}^{+} (N_{i} a_{i}),
\]
in the notation of (1.6) and (3.4). It is clear that \(\Pi_{i} \leq 1\) for \(i = 1, 2\). Noting that \(\Pi(y) \ll (\log y)^{-N}\), this therefore concludes the proof of the lemma. \(\square\)

**Cases II and III.** We now estimate \(E^{II}(\mathcal{R})\) and \(E^{III}(\mathcal{R})\). For any \((s, t) \in \mathbb{Z}_{\text{prim}}^{2} \cap \mathcal{R}\), we take the trivial bound
\[
\prod_{i=1}^{N} f_{i,q_{G}} ((b_{i}s - \theta_{i} t)) < \prod_{i=1}^{N} (N_{i}(b_{i}s - \theta_{i} t))^{\varepsilon_{i}} \ll \|\varepsilon\|_{\mathcal{X}_{\ell}}^{d_{i} \varepsilon_{i}} \ll K_{\varepsilon}^{d_{i} \varepsilon_{i}}.
\]
In Case II the relevant extra constraints are \(P^{-}(b_{s,t}) < z^{2} \) and \(a_{s,t} \leq z^{1-\eta}\). Let \(p = P^{-}(b_{s,t}) < z^{2}\) and let \(\nu\) be such that \(p^{\nu} \| F(s, t)\). We must have \(p^{\nu} \geq z^{\eta}\), since otherwise \(z < p^{\nu} a_{s,t} < z^{\eta} z^{1-\eta} = z\), which is a contradiction. For each prime \(p \nmid q_{G} W\) with \(p < z^{2}\), we define
\[
l_{p} = \min \{ l \in \mathbb{Z}_{\geq 0} : p^{l} \geq z^{\eta} \}.
\]
Clearly \(z^{\eta} \leq p^{l} < z^{2} \cdot p^{l}\), whence \(l_{p} \geq 2\) for every prime \(p\). Therefore
\[
\sum_{p < z^{2}} \frac{1}{p^{\nu}} \leq \sum_{p \leq z^{2}} \min \left\{ \frac{1}{z^{\eta}}, \frac{1}{p^{2}} \right\} \leq \sum_{p \leq z^{2}} \frac{1}{z^{\eta}} \leq z^{-\frac{\eta}{2}}.
\]
(3.6)
The number of elements \((s, t)\) satisfying the constraints of Case II is at most
\[
\sum_{i=1}^{N} \sum_{p < z^{2}} \sum_{p \nmid q_{G} W} (s, t)_{\text{prim}}^{\mathcal{R}} \sum_{p^{l} | F_{i}(s, t)} 1 \ll \sum_{i=1}^{N} \sum_{p < z^{2}} \sum_{p \nmid q_{G} W} \mathcal{P}_{i}(p^{l}) \left( \frac{h(q_{G}) V}{q_{G} p^{l}} + \tau(q_{G}) \frac{K_{\delta}}{\lambda_{G}} \right).
\]
Here we have split the inner sum into \(\mathcal{P}_{i}(p^{l})\) different lattices of the form \(\{(s, t) \in G : s \equiv xt \pmod{p^{l}}\}\), where \(x\) ranges over solutions of the congruence \(F_{i}(x, 1) \equiv 0 \pmod{p^{l}}\), before applying Lemma 2.1 with \(q = 1\). Hensel’s lemma implies that \(\mathcal{P}_{i}(p^{l}) = \mathcal{P}_{i}(p) \ll d_{i}\) for each prime \(p \nmid W\) and \(l \in \mathbb{N}\). Let \(\delta > 0\) be arbitrary. Taking \(h(q_{G}) \leq 1\) and \(\tau(q_{G}) \ll_{\delta} \frac{q_{G}^{\frac{\delta}{2}}}{q_{G}^{\frac{3}{2}}} \ll V^{\delta} \frac{q_{G}^{\frac{\delta}{2}}}{q_{G}^{\frac{3}{2}}} \ll K_{\delta}^{\frac{\delta}{2}}\), this therefore reveals that
\[
\sum_{i=1}^{N} \sum_{p < z^{2}} \sum_{p \nmid q_{G} W} \sum_{p^{l} | F_{i}(s, t)} 1 \ll_{\delta} V \sum_{p < z^{2}} \sum_{p \nmid q_{G} W} \frac{1}{p^{l}} + \frac{K_{\delta}^{1+\frac{\delta}{2}} z^{2}}{\lambda_{G}} \ll_{\delta} V \frac{1}{q_{G} z^{2}} + \frac{K_{\delta}^{1+\frac{\delta}{2}} z^{2}}{\lambda_{G}},
\]
by (3.6). Recalling (3.1), we see that $z^{\frac{\omega}{2}} = V^{-\beta}$, with $\beta = \frac{\eta \omega}{2}$. Likewise, $z^{\frac{w}{2}} = V^{\frac{\omega}{2}} \leq K^{2p\omega}$. Noting that

$$K_{\omega} \sum_{\ell} d_{\ell}^{\epsilon_{\ell}} \leq V^{\frac{1}{c_1} \sum_{\ell} d_{\ell} \epsilon_{\ell}},$$

we have therefore proved that for every $\delta > 0$ we have the bound

$$E^{(II)}(\mathcal{R}) \ll_{\delta} \frac{V^{1+\frac{\omega}{2} - \frac{1}{c_1} \lambda^\delta}}{qG} + \frac{K_{\omega}^{1+\frac{\omega}{2} + 2p\omega + \sum_{\ell} d_{\ell} \epsilon_{\ell}}}{\lambda_G}. \quad (3.7)$$

We now turn to the contribution from Case III, for which the defining constraints are $P^{-}(b_{s,t}) \leq \log z \log \log z$ and $z^{1-\eta} < a_{s,t} \leq z$. We assume that $w > \max_{i \neq j} |\text{Res}(F_i, F_j)|$ in the definition (1.1) of $W$. For any $(s, t) \in \mathbb{Z}_{\text{prim}}^{2}$ it follows that the integer factors of $F_i(s, t)_W$ are necessarily coprime to the factors of $F_j(s, t)_W$ for all $i \neq j$. Hence the number of elements $(s, t)$ satisfying the constraints of Case III is at most

$$\sum_{(s, t) \in \mathbb{Z}_{\text{prim}}^{2} \cap \mathcal{R} \cap G} 1 \leq \sum_{\substack{z^{1-\eta} < a \leq z \\ \text{gcd}(a, W) = 1}} \sum_{\substack{(s, t) \in \mathbb{Z}_{\text{prim}}^{2} \cap \mathcal{R} \cap G \\ a | F(s, t)}} 1 \leq \sum_{\substack{z^{1-\eta} < a_1 \cdots a_N \leq z \\ \text{gcd}(a_1, W) = 1}} \sum_{\substack{(s, t) \in \mathbb{Z}_{\text{prim}}^{2} \cap \mathcal{R} \cap G \\ a | F(s, t)}} 1.$$

As before, the final sum can be split into at most $\prod_{i=1}^{N} p_i(a_i) = O_{\delta}(z^\delta)$ lattices, for any $\delta > 0$, each of determinant $qG \prod_{i=1}^{N} a_i$. Thus the right hand side is

$$\ll_{\delta} z^{\delta} \sum_{\substack{z^{1-\eta} < a_1 \cdots a_N \leq z \\ \text{gcd}(a, W) = 1}} \left( \frac{V}{qGa_1 \cdots a_N} + \frac{K_{\omega}}{\lambda_G} \right),$$

whence [9] Lemma 1] yields

$$\sum_{\substack{(s, t) \in \mathbb{Z}_{\text{prim}}^{2} \cap \mathcal{R} \cap G \\ P^{-}(b_{s,t}) \leq \log z \log \log z \leq z^{1-\eta} < a_{s,t} \leq z}} 1 \ll_{\delta} z^{3\delta} \left( \frac{V}{qGz^{1-\eta}} + \frac{K_{\omega}}{\lambda_G} \right).$$

Hence we have shown that for every $\delta > 0$ one has

$$E^{(III)}(\mathcal{R}) \ll_{\delta} \frac{V^{1-(1-\eta)\omega + 3p\omega + \frac{1}{c_1} \sum_{\ell} d_{\ell} \epsilon_{\ell}}}{qG} + \frac{K_{\omega}^{1+3p\omega + \sum_{\ell} d_{\ell} \epsilon_{\ell}}}{\lambda_G}. \quad (3.8)$$
Case IV. The final case to consider is characterised by the constraints
\[ \log z \log \log z < P^-(b_{s,t}) < z^{1-\eta} \quad \text{and} \quad z^{1-\eta} < a_{s,t} \leq z. \]
Arguing as in (3.2) in the treatment of Case I, we find that
\[ E^{(IV)}(R) \ll \sum_{a_i \in \mathcal{P}_i} \mathcal{U}^+(a_1, \ldots, a_N) \prod_{i=1}^N f_i(a_i), \]
where \( \mathcal{U}^+(a_1, \ldots, a_N) \) is as in the definition of \( \mathcal{U}(a_1, \ldots, a_N) \) after (3.2), but with the condition \( P^-(F(s, t)_{qG}W/ \prod_{i=1}^N N_i a_i) \geq z^{1/2} \) replaced by
\[ \log z \log \log z < P^-(F(s, t)_{qG}W/ \prod_{i=1}^N N_i a_i) < z^{1/2}. \]
In particular, in view of the coprimality of \( a_{s,t} \) and \( b_{s,t} \), we see that
\[ P^+\left(\prod_{i=1}^N N_i a_i\right) < P^-\left(\frac{F(s, t)_{qG}W}{\prod_{i=1}^N N_i a_i}\right). \]
We will find it convenient to enlarge the sum slightly, replacing the condition \( a_i \in \mathcal{P}_i \) by the condition that each \( a_i \) belongs to the multiplicative span of degree 1 prime ideals in \( a_i \).
We may assume without loss of generality that \( \frac{2}{\eta} \in \mathbb{Z}_{\geq 2} \). Thus
\[ (\log z \log \log z, z^{1/2}) \subset \bigcup_{k=0}^{k_0+1} \left( \frac{1}{k+1}, \frac{1}{k^2} \right), \]
where \( k_0 = \lfloor \log z / \log(\log z \log \log z) \rfloor \) satisfies \( k_0 \leq \log z / \log \log z \). Notice that for any integer \( b \) satisfying \( \log b \ll \log z \) and \( z^{1/2} < P^-(b) \leq z^{1/2} \) we must have \( \Omega(b) \ll k \). Applying this with \( b = F(s, t)_{qG}W/ \prod_{i=1}^N N_i a_i \), for any \( (s, t) \in \mathbb{Z}_{\text{prim}} \cap R \), we deduce that
\[ \prod_{i=1}^N \Omega_i \left( \frac{F(s, t)_{qG}W}{a_i} \right) \ll \max_{1 \leq i \leq N} A_i^{\Omega(b)} \ll A^k, \]
for a suitable constant \( A \gg \max_{1 \leq i \leq N} A_i \), where \( A_i \) is the constant appearing in the definition of \( M_{K_i} = M_{K_i}(A_i, B_i, \varepsilon_i) \). Hence
\[ E^{(IV)}(R) \ll \sum_{k=0}^{k_0+1} A^k \sum_{a_i \in \mathcal{P}_{K_i}} \mathcal{U}^+_{\Omega(b)}(a_1, \ldots, a_N) \prod_{i=1}^N f_i(a_i), \]
where
\[ \mathcal{U}^+_{\Omega(b)}(a_1, \ldots, a_N) = \sum_{a_i \in \mathcal{P}_{K_i}} \mathcal{U}^+(a_1, \ldots, a_N) \prod_{i=1}^N f_i(a_i), \]
in the notation of Lemma 3.1, which we now use to estimate \( \mathcal{U}_{\frac{1}{\#}}(a_1, \ldots, a_N) \).

The overall contribution from the second term is

\[
\ll \frac{K}{\lambda G} \sum_{k=\frac{2}{\delta}}^{k_0+1} A^k z^{1+\frac{2\delta}{k+1}} \ll \frac{K}{\lambda G} \sum_{k=\frac{2}{\delta}}^{k_0+1} A^k \ll \frac{K}{\lambda G} \cdot \frac{K_1+4\omega(\frac{2}{\delta}+3\delta)}{\lambda G},
\]

since \( 2 \leq 2/\eta \leq k \leq k_0 \ll \log z/\log \log z \) and \( z = V^\omega \leq K_1^\omega \).

It remains to consider the contribution from the main term in Lemma 3.1. This is

\[
\ll \frac{V}{(\log z)^N} h_{w}^*(q_G) \sum_{k=\frac{2}{\eta}}^{k_0+1} A^k(k+1)^N E(z^{1-\eta}, z^{\frac{3}{2}}), \tag{3.9}
\]

where

\[
E(S, T) = \sum_{a_i \in \mathcal{P}_{K_i}^*} \prod_{i=1}^{N} \frac{f_i(a_i) h_i^*(a_i)}{N_i a_i}.
\]

Note that we have dropped the condition \( \gcd(\prod_{i=1}^{N} N_i a_i, q_G) = 1 \).

Let us define the multiplicative function \( u : \mathbb{N} \to \mathbb{R}_{\geq 0} \) via

\[
u(a) = \sum_{a_i \in \mathcal{P}_{K_i}^*} \prod_{i=1}^{N} f_i(a_i) h_i^*(a_i). \tag{3.10}
\]

Note that

\[
u(p^k) = \sum_{i=1}^{N} \sum_{p_i < a_i \text{ prime}} f_i(p^k) h_i^*(p_i^k) \leq C^k, \tag{3.11}
\]

for an appropriate constant \( C > 1 \) depending on \( A_i, d_i \) and \( N \). We may therefore write

\[
E(S, T) = \sum_{\substack{\gcd(a,w) = 1 \\ a > S \\ P^+(a) < T}} \frac{u(a)}{a}.
\]

Drawing inspiration from the proof of [8, Lemma 2], we shall find an upper bound for \( E(S, T) \) in terms of partial sums involving \( u(a) \). This is the object of the following result.
Lemma 3.2. Assume that \( T > e^{\frac{G}{10}} \) and let \( \kappa \in (\frac{1}{10}, C^{-1} \log T) \). Then

\[
E(S, T) \ll \kappa e^{-\kappa \log T} \sum_{\gcd(b, W) = 1 \atop b \leq T} \frac{u(b)}{b}.
\]

Taking this result on faith for the moment, we return to (3.9) and apply it with \( \kappa \) satisfying \( e^{\kappa(1-\eta)} > 2A \). (Note that the implied constant in Lemma 3.2 depends on \( \kappa \) and so the choice \( \kappa = (\log 2A)/(1-\eta) + 1 \) is acceptable.) This produces the overall contribution

\[
\ll \frac{V}{(\log z)^N} \sum_{k=\frac{2}{\eta}}^{k_{\eta}+1} \frac{A^k(k+1)^N}{2^k} \sum_{\gcd(b, W) = 1 \atop b \leq z} \frac{u(b)}{b}.
\]

Recalling (3.10) and enlarging \( w \) to enable the use of Lemma 2.2, shows that the last quantity is

\[
\ll \frac{V}{(\log z)^N} \sum_{\gcd(b, W) = 1 \atop b \leq z} \frac{u(b)}{b}.
\]

which shows that for every \( \delta > 0 \) we have

\[
E^{(IV)}(\mathcal{R}) \ll \frac{V}{(\log z)^N} \sum_{i=1}^{N} \sum_{a_i \in \mathcal{P}_{K_i}^* \text{square-free} \atop \gcd(N, a_1, W) = 1} \frac{f_i(a_i)}{N_i a_i} + \frac{K^{1+4\omega(\frac{G}{\delta}+3\delta)}}{\lambda_G}. \tag{3.12}
\]

Proof of Lemma 2.2. Let \( \beta = \frac{\kappa}{\log T} \). Then

\[
E(S, T) \ll S^{-\beta} \sum_{\gcd(a, W) = 1 \atop P^k(a) < T} \frac{u(a)}{a^\beta}.
\]

Define the multiplicative function \( \psi_\beta \) via \( a^\beta = \sum_{c \mid a} \psi_\beta(c) \), for \( a \in \mathbb{N} \). We observe that \( \psi_\beta(p^k) = p^{\beta k} - p^{\beta(k-1)} \), for any \( k \in \mathbb{N} \), whence \( 0 < \psi_\beta(a) < a^\beta \) for
all \( a \in \mathbb{N} \). We now have
\[
E(S, T) \leq S^{-\beta} \sum_{\substack{\gcd(c, W) = 1 \\ P^+(c) < T}} \frac{\psi_\beta(c)}{c} \sum_{\substack{\gcd(d, W) = 1 \\ P^+(d) < T}} \frac{u(cd)}{d}.
\]

Writing \( d = j d' \), with \( \gcd(d', c) = 1 \) and \( j \mid c^\infty \), shows that
\[
E(S, T) \leq S^{-\beta} \sum_{\substack{\gcd(d', W) = 1 \\ P^+(d') < T}} \frac{u(d')}{d'} \sum_{\substack{\gcd(c, d' W) = 1 \\ P^+(c) < T}} \frac{\psi_\beta(c) u(cj)}{cj}.
\]

After possibly enlarging \( w \), it follows from (3.11) that the sum over \( c \) is
\[
\leq \prod_{p < T \atop p \mid d'W} \left( 1 + \sum_{k \geq 1 \atop j \geq 0} \psi_\beta(p^k) u(p^{k+j}) \right) \leq \prod_{p < T \atop p \mid d'W} \left( 1 + \sum_{k \geq 1 \atop j \geq 0} \frac{(p^{\beta k} - p^{\beta(k-1)}) C^{k+j}}{p^{k+j}} \right)
\leq \exp \left( O \left( \sum_{p < T \atop p \mid d'W} \frac{\beta - 1}{p} \right) \right).
\]

Writing \( p^\beta = \exp(\frac{\kappa \log p}{\log T}) = 1 + O(\frac{\kappa \log p}{\log T}) \), this is found to be at most
\[
\exp \left( O \left( \frac{\kappa}{\log T} \sum_{p \leq T} \frac{\log p}{p} \right) \right) \ll \kappa \cdot 1.
\]

Our argument so far shows that
\[
E(S, T) \ll \kappa e^{-\kappa \frac{\log S}{\log T}} \sum_{\substack{\gcd(d, W) = 1 \\ P^+(d) < T}} \frac{u(d)}{d}. \quad (3.13)
\]

Let \( \xi \in (0, 1) \). Observe that each \( d \) with \( P^+(d) < T \) can be written uniquely in the form \( d = d_- d_+ \) for coprime \( d_-, d_+ \in \mathbb{N} \) such that \( P^+(d_-) \leq T^\xi \) and \( P^-(d_+) > T^\xi \). We clearly have \( P^+(d_+) < T \). Thus
\[
\sum_{\substack{\gcd(d, W) = 1 \\ P^+(d) < T}} \frac{u(d)}{d} \leq \sum_{\substack{\gcd(d_-, W) = 1 \\ P^+(d_-) \leq T^\xi}} \frac{u(d_-)}{d_-} \sum_{\substack{\gcd(d_+, W) = 1 \\ P^-(d_+) > T^\xi}} \frac{u(d_+)}{d_+}.
\]
By (3.11), the inner sum is at most
\[ \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-} \leq C, \]
Thus, once combined with (3.13), we deduce that
\[ E(S, T) \ll \kappa e^{-\kappa \log \frac{S}{N}} \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-}. \]
In order to complete the proof of the lemma, it remains to show that the
\[ \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-} \ll \sum_{d < T} \frac{u(d)}{d}. \]
This is trivial when \( T^\xi < 2 \). Suppose now that \( T^\xi > 2 \). Taking \( \kappa = 2 \) and \( (T, T^\xi) \) in place of \( (S, T) \), it follows from (3.13) that
\[ \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-} \ll e^{-2} \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-}. \]
Taking \( \xi \) suitably small, we conclude that
\[ \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-} = \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-} + \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-} \]
\[ \leq \sum_{d_0 < T} \frac{u(d_-)}{d_-} + \frac{1}{2} \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-}. \]
so that
\[ \sum_{\gcd(d_-, W) = 1}^{\gcd(d_-, T^\xi)} \frac{u(d_-)}{d_-} \leq 2 \sum_{d_0 < T} \frac{u(d_-)}{d_-}, \]
as claimed. \( \square \)

**Proof of Theorem 1.1.** Let us define
\[ \eta = \frac{2}{3} \quad \text{and} \quad \omega = \frac{(3 + \delta)}{c_1} \sum_{i=1}^{N} d_i \varepsilon_i, \]
where \( \delta > 0 \) is to be determined. Now let \( \varepsilon \) be any positive constant. Taking \( \delta \)
sufficiently small compared to \( \varepsilon \) we see that the exponent of \( K_{\delta \varepsilon} \) in the second
term of (3.5) is

\[ 1 + 4\omega(1 + \eta + 2\delta + \varepsilon) \leq 1 + \varepsilon + \frac{4}{c_1} \left( \sum_{i=1}^{N} d_i \varepsilon_i \right)(5 + 3\varepsilon), \]

where \( \hat{\varepsilon} = \max\{\varepsilon_1, \ldots, \varepsilon_N\} \). Thus \( E^{(l)}(\mathcal{R}) \) makes a satisfactory contribution for Theorem 1.1. Taking \( \delta \) sufficiently small allows us to check that

\[
\frac{\eta \omega}{2} > \frac{1}{c_1} \sum_{i=1}^{N} d_i \varepsilon_i \quad \text{and} \quad (1 - \eta) \omega > 3\delta \omega + \frac{1}{c_1} \sum_{i=1}^{N} d_i \varepsilon_i,
\]

for our choice of \( \eta \) and \( \omega \). This therefore shows that the first term in the right hand side of (3.7) and (3.8) is

\[
\ll \frac{V}{(\log V)^{N+1} q_G},
\]

which, owing to \( h_{W}^{q_G}(q_G) \geq 1 \) and \( E_f(V; W) \geq 1 \), is satisfactory for Theorem 1.1. A straightforward calculation now shows that for sufficiently small \( \delta \) the contribution of the second terms on the right of (3.7), (3.8) and (3.12) is also satisfactory for Theorem 1.1. \( \square \)

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School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK
E-mail address: t.d.browning@bristol.ac.uk

Max Planck Institute for Mathematics, Bonn, Vivatgasse 7, 53111, Germany
E-mail address: sofos@mpim-bonn.mpg.de