Tunneling spectroscopy of Luttinger-liquid structures far from equilibrium

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We develop a theory of tunneling spectroscopy of interacting electrons in a non-equilibrium quantum wire coupled to reservoirs. The problem is modelled as an out-of-equilibrium Luttinger liquid with spatially dependent interaction. The interaction leads to the renormalization of the tunneling density of states, as well as to the redistribution of electrons over energies. Energy relaxation is controlled by plasmon scattering at the boundaries between regions with different interaction strength, and affects the distribution function of electrons in the wire as well as that of electrons emitted from the interacting regions into non-interacting electrodes.

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\section{I. INTRODUCTION}

One-dimensional (1D) interacting fermionic systems show remarkable physical properties and are promising elements for future nanoelectronics. The electron-electron interaction manifests itself in a particularly dramatic way in 1D systems, inducing a strongly correlated electronic state – Luttinger liquid (LL)\textsuperscript{1-4}. A paradigmatic experimental realization of quantum wires are carbon nanotubes\textsuperscript{5}, for a recent review see Ref.\textsuperscript{6}. Further realizations encompass semiconductor, metallic and polymer nanowires, as well as quantum Hall edges.

There is currently a growing interest in non-equilibrium phenomena on nanoscales. A tunneling spectroscopy (TS) technique for non-equilibrium nanostructures was developed in Ref.\textsuperscript{7}. Employing a superconducting tunneling electrode allows one to explore not only the tunneling density of states (TDOS) but also the energy distribution function. The energy relaxation found in this way provides information about inelastic scattering in the system. In a very recent experiment\textsuperscript{8} this TS method was applied to a carbon nanotube under strongly non-equilibrium conditions.

In this paper, we develop a theory of TS of a LL out of equilibrium. Specifically, we consider a LL conductor connected, via non-interacting leads, to reservoirs with different electrochemical potentials, $\mu_L - \mu_R = eV$ and different temperatures $T_L, T_R$ (where the indices $L, R$ stand for left- and right-movers). It is assumed that the coupling to the leads is adiabatic on the scale of the Fermi wave length, so that no backscattering of electrons takes place. We model the leads as non-interacting 1D wires, so that the electron-electron interaction is turned on at the vicinity of the points $x = \pm L/2$, see Fig.\textsuperscript{1}. This model is quite generic to properly describe the problem at hand, independently of the actual geometry of the leads. Note also that the 1D setup with strongly non-uniform interaction may be experimentally realized by using external screening gates.

It is known that energy relaxation is absent in a uniform clean LL. Within the golden-rule framework, the lack of energy relaxation for forward scattering processes results from 1D kinematic constraints that do not allow to satisfy the energy and momentum conservation laws simultaneously. On a more formal level, the conservation of energies of individual particles in a spatially uniform LL is protected by the integrability of the system, which implies an infinite number of conservation laws\textsuperscript{9,10}. Inclusion of spatial dependence into the model violates these laws and leads to energy relaxation that takes place at the regions where the interaction varies in space\textsuperscript{11}.

The fact that inhomogeneous interaction induces energy relaxation of electrons has been pointed out for the first time in Ref.\textsuperscript{12} in the context of interacting quantum Hall edges but a detailed analysis of this effect has been missing until now. On the other hand, one may expect this to be a dominant effect on the electron distribution function in experiments done on modern high-quality quantum wires (such as ultraclean carbon nanotubes\textsuperscript{13}), under non-equilibrium conditions. There is thus a clear need in the theory of TS in non-equilibrium LL.

It is worth noting that we assume the absence of backscattering due to impurities in the wire. While present, such impurities strongly affect the electronic properties of a LL wire: they induce diffusive dynamics at sufficiently high temperature $T$ and localization phenomena proliferating with lowering $T$ (Ref.\textsuperscript{14}), as well as inelastic processes\textsuperscript{15,16}. We also neglect the nonlinearity of the electron dispersion whose influence on spectral and kinetic properties of 1D electrons was recently studied in Refs.\textsuperscript{17,18}.

\section{II. FORMALISM}

Within the LL model, the electron field is decoupled in a sum of right- and left-moving terms, $\psi(x,t) = \psi_L(x,t)e^{ipFx} + \psi_R(x,t)e^{-ipFx}$, where $p_F$ is the Fermi mo-
As a result, Gmation and bosonic fields can be eliminated by a gauge transformation term via a bosonic field Stratonovich transformation, one decouples the interactions of tunnel probes. The solid curve in the lower part of the figure shows a spatially dependent LL interaction parameter K(x). The dashed line corresponds to the limit of a sharp variation of K(x) at the boundaries.

The Hamiltonian of the system reads
\[ H = H_0 + H_{\text{int}}, \]
\[ H_0 = -iv \int dx \left( \psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L \right), \]
\[ H_{\text{int}} = \frac{1}{2} \int dx g(x)(\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L)^2, \]

where \( v \) is the electron velocity and \( g(x) \) is the spatially dependent electron-electron interaction constant.

We will proceed by following the lines of the functional bosonization approach in the non-equilibrium (Keldysh) formulation. Performing the Hubbard-Stratonovich transformation, one decouples the interaction term via a bosonic field \( \phi \) and gets the action
\[ S[\psi, \phi] = i \sum_{\eta=R,L} \psi_\eta^\dagger (\partial_x - \phi) \psi_\eta - \frac{1}{2} \int dx g(x)^2, \]

where \( \partial_{R,L} = \partial_t \pm v \partial_x \) and the fields are defined on the Keldysh time contour. The information about physical observables is contained in Keldysh Green functions and \( G^\leq \) and \( G^\geq ; \) see, in particular, Appendix A where we express tunneling current in terms of functions \( G^\geq \) and discuss how its measurement allows to determine \( G^\geq \) experimentally. The Green functions \( G^\geq \) can be presented in the form
\[ G^\geq_{\eta}(x, t; x', t') = \int d\phi \ Z[\phi] \ e^{-\frac{i}{\hbar} \phi^\dagger \phi} \times G^{\geq}_{\eta}[\phi](x, t; x', t'), \]

where we introduced the Green function in a given field configuration, \( G^{\geq}_{\eta}[\phi] \), and the sum of vacuum loops, \( Z[\phi] \).

In 1D geometry the coupling between the fermionic and bosonic fields can be eliminated by a gauge transformation \( \psi_\eta(x, t) \rightarrow \psi_\eta(x, t) e^{i\theta_\eta(x, t)} \), if we require
\[ i \partial_t \theta_\eta = \phi. \]

As a result, \( G^{\geq}_{\eta}[\phi] \) can be cast in the form
\[ G^{\geq}_{\eta}[\phi](x, t; x', t') = \frac{G^{\geq}_{\eta,0}(x - x'; t - t') e^{-i\eta V(t-t')/2}}{2} \times e^{\phi^\dagger_{\eta}(x, t; x', t')} \cdot \]

Here
\[ \Phi^{\geq}_{\eta}(x, t; x', t') = i\theta_{\pm, \eta}(x, t) - i\theta_{\mp, \eta}(x', t'), \]
\[ G^{\geq}_{\eta,0}(\xi) = \frac{1}{2\hbar \sinh \pi \xi_{\eta} \pm i0}, \]

the coordinate \( \xi_{R/L} = x/v \mp t \) labels the trajectory of a particle, and we use the convention that in formulas \( \eta \) should be understood as \( \eta = \pm 1 \) for right/left moving electrons.

It is convenient to perform a rotation in Keldysh space, thus decomposing fields into classical and quantum components, \( \phi_1, \phi_2 = (\phi_+ \pm \phi_-)/\sqrt{2} \), where the indices + and - refer to the fields on two branches of the Keldysh contour. Further, we introduce vector notations by combining \( \phi_1 \) and \( \phi_2 \) in a 2-vector \( \phi \). To proceed further, we resolve Eq. (5) and express \( \theta_\eta \) through \( \phi \) as
\[ \theta_\eta = G_{\eta,0} \sigma_1 \phi, \]

where \( G_{\eta,0} \) is the Green function of free bosons,
\[ G_{\eta,0} = \begin{pmatrix} G^K_{\eta} & G^r_{\eta} \\ G^a_{\eta} & 0 \end{pmatrix}. \]

Its retarded and advanced components are given by
\[ G^{r,a}_{\eta,0} = \frac{1}{\omega - \eta v \sigma_1 \pm i0} \]

The Keldysh component of \( G_{\eta,0} \) is given by \( G^K_{\eta,0} = (G_{\eta,0} - G^a_{\eta,0}) B^{(0)}_{\eta}(\omega) \), where \( B^{(0)}_{\eta}(\omega) \) is determined by the temperature \( T_{\eta} \) of the reservoir from which the electrons moving in direction \( \eta \) emerge,
\[ B^{(0)}_{\eta}(\omega) = \coth \omega/2T_{\eta}. \]

Using Eqs. (8) and (10) and performing a transformation to the coordinate space, we express the exponent \( \Phi^{\geq}_{\eta}(x, t; x', t') \) through the bosonic field \( \phi(y) \),
\[ \Phi^{\geq}_{\eta}[\phi](x, t; x', t') = \int \frac{d\omega}{2\pi} dy \Phi^{\geq}_{\omega}(y) J^{\geq}_{\eta,0}(y; x, t; x', t'). \]

The components of \( J \) are found as
\[ J^{\geq}_{1,\eta,0}(y) = e^{i\pi \delta y} \left( \theta[\eta(x - y)] e^{-\omega \xi_{\eta}} - \theta[\eta(x' - y)] e^{-\omega \xi_{\eta}} \right), \]
\[ J^{\leq}_{2,\eta,0}(y) = e^{i\pi \delta y} \left( e^{i\omega \xi_{\eta}} - e^{-i\omega \xi_{\eta}} \right) B^{(0)}_{\eta}(\omega) \]
\[ \mp e^{i\pi \delta y} \left( \theta[\eta(y - x)] e^{-i\omega \xi_{\eta}} + \theta[\eta(y - x')] e^{-i\omega \xi_{\eta}} \right) \]

where \( \theta(x) \) is the Heaviside \( \theta \)-function. The vacuum loop factor in Eq. (5) is given by
\[ Z[\phi] = \exp \left( -\frac{i}{2} \phi^T \Gamma \phi \right), \]
where \( \Pi \) is the polarization operator,

\[
\Pi = \begin{pmatrix} 0 & \Pi^a \\ \Pi^a & \Pi^K \end{pmatrix}.
\]

It can be decomposed into left and right moving parts, \( \Pi = \Pi_R + \Pi_L \), with

\[
\Pi^R_{\eta,L} = -\frac{1}{2\pi} \frac{q}{\omega_+ \mp vF q}, \quad \Pi^L_{\eta,R} = -\frac{1}{2\pi} \frac{q}{\omega_- \pm vF q},
\]

\[
\Pi^K_\eta = (\Pi^R_\eta - \Pi^L_\eta)B^{(0)}_\eta(\omega), \tag{17}
\]

where \( \omega_\pm = \omega \pm i0 \). Performing the averaging over the auxiliary field \( \phi \), we get

\[
G_\eta^{R}(x, t; x', t') = G_{\eta,0}^{R}(\xi_\eta - \xi_\eta')e^{-i\nu V(t-t')/2}e^{x_\eta^R(x')}, \tag{18}
\]

where the effect of the interaction is represented by the “Debye-Waller factor” \( e^{x_\eta^R(x')} \) with

\[
F_\eta^{R}(x, t; x', t') = -\frac{i}{2} \int \frac{d\omega}{2\pi}dy_1dy_2 x_\eta^{R}(y_1)\gamma(\omega, y_1, y_2)J_\omega^{R}(y_2). \tag{19}
\]

Here

\[
\gamma = (\Pi + g^{-1}\sigma_1)^{-1} \tag{20}
\]

is the screened electron-electron interaction potential. Its retarded component is given by

\[
\gamma^{R}(y, y') = g(y) [\delta(y - y') + \frac{vg(y')}{\pi} \partial_{y'} \gamma^{R}(y, y')]. \tag{21}
\]

where the function \( \gamma^{R} \) is determined by the following differential equation

\[
(\omega^2 + \partial_y u^2(y)\partial_y)\gamma^{R}(y, y') = \delta(y - y'), \tag{22}
\]

which describes the plasmon propagation in a medium with spatially dependent sound velocity \( u(x) = v(1 + g(x)/\pi v)^{1/2} \). The Keldysh component of the interaction propagator is obtained as

\[
\gamma^{K}(y_1, y_2) = -\frac{i\omega}{2\pi v^2} \sum_{\eta = \pm} B_{\eta}(\omega)P_{\eta}(y_1)P_{\eta}(y_2), \tag{23}
\]

where

\[
P_{\eta}(y) = \int dy' \delta(\eta y - y') \gamma^{R}(y, y'). \tag{24}
\]

At equilibrium, \( B_R(\omega) = B_L(\omega) \equiv B(\omega) \), this reduces to

\[
\gamma^{K} = \left[ \gamma^{R} - \gamma^{R} \right] B(\omega), \tag{25}
\]

in agreement with the fluctuation-dissipation theorem.

### III. SHARP BOUNDARIES

So far we made no restriction on the way the interaction changes in space. Let us consider first the case when the interaction turns on and off sharply on the scale set by the temperatures, \( T_R \sim v/\text{max}(T_L, T_R) \). This limit can be modelled via a stepwise interaction as represented by the dashed line in Fig. 1. Equation (22) for \( \gamma^{R} \) can be then straightforwardly solved by using the fact that the velocity \( u \) is constant in each of three regions and employing the proper boundary conditions [continuity of \( \gamma^{R}(y, y') \) and of \( u^2(y)\partial_y \gamma^{R}(y, y') \) at \( y = \pm L/2 \).

In the TS context, we are interested in the Green functions \( G^{R} \) with coinciding spatial arguments, \( x = x' \). Assuming \( x \) to be in the interacting part of the wire (and not too close to the boundaries) and setting \( t' = 0 \), we find

\[
F_\eta^{R}(\omega) = -\gamma \int_0^{\infty} \frac{d\omega}{\omega} \left[ (1-K)^2 B^{(0)}(\omega) + (1+K)^2 B^{(0)}_L(\omega) \right] \frac{1}{2(1+K^2)} \times (1 - \cos \omega t) \pm i \sin \omega t, \tag{26}
\]

where

\[
K = v/u \equiv (1 + g/\pi v)^{-1/2} \tag{27}
\]

is the conventional dimensionless parameter characterizing the interaction strength in a LL and

\[
\gamma = \frac{(K-1)^2}{2K}. \tag{28}
\]

The integral in Eq. (26) and in analogous formulas below is logarithmically divergent at large frequencies and requires an ultraviolet regularization. Specifically, these integrals are understood as regularized by a factor \( e^{-\omega/\Lambda} \), where \( \Lambda \) is an ultraviolet cutoff. Deriving Eq. (26), we have neglected terms of the form \( e^{i\omega t/\pi v} \) (with non-zero integer \( n \)) that arise due to the Fabry-Perot-type interference of plasmon modes reflected at the boundaries. Keeping these terms would lead to an additional oscillatory structure in energy with the scale \( \pi v/L \). Since we are interested in TS of long wires, we assume that this scale is much less than \( \text{max}(T_R, T_L) \), so that oscillations are suppressed.

Substituting Eq. (26) into Eq. (15), we finally get the Green functions:

\[
G^{R}(t) = (2\pi v)\gamma \left[ G^{R,0}(t) \right]^{1+\alpha} \left[ G^{R,0}(t) \right]^\beta e^{-i\nu Vt/2}, \tag{29}
\]

where

\[
\alpha = \frac{(K-1)^4}{4K(1+K^2)}, \quad \beta = \frac{(K^2 - 1)^2}{4K(1+K^2)}. \tag{30}
\]

The Green functions \( G^{R}(t) \) can be determined experimentally from TS measurements, see Appendix A. Their difference determines the TDOS \( \nu(\epsilon) \):

\[
G^{R}(\epsilon, x, x) - G^{R}(\epsilon, x, x) = -2\pi i \nu(\epsilon), \tag{31}
\]
while each of them separately (or their sum) contains also information about the distribution function, as discussed below. The results for the TDOS have been found in Ref. 21.

Next we consider the non-interacting parts of the wire, and discuss, e.g., the right moving electrons. In the region I (see Fig. 1), x, x’ < −L/2, we find from Eqs. (19), (22) that $F_{R}^{\equiv} = 0$, so that the Green functions of the right movers are not modified by interaction. Physically this is quite transparent: the right-moving electrons in this part of the system are just coming from the reservoir and are not yet “aware” of the interaction with the left-movers.

The situation is distinctly different in the region III, x, x’ > L/2. Assuming x = x’, we find

$$F_{R}^{\equiv} = \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{(1-K')}{1+K'} (1 - \cos \omega t) \times \left[ B_{R}^{(0)}(\omega) - B_{L}^{(0)}(\omega) \right].$$

(32)

Substituting Eq. (32) into Eq. (15), one gets

$$G_{R}^{\equiv}(t) = \left[ G_{R,0}^{\equiv}(t) \right]^{T} \left[ G_{L,0}^{\equiv}(t) \right]^{R} e^{-i\nu Vt/2},$$

(33)

where

$$T = \frac{2K}{1+K'}, \quad \mathcal{R} = \frac{(1-K')}{1+K'},$$

(34)

Since $F_{R}^{\equiv}$ in Eq. (32) is real, the TDOS is not affected by the interaction, $\nu R(\epsilon) = \nu_0 \equiv \pi \omega v$, as expected. The modification of the functions $G_{R}^{\equiv}$ as compared to that of incoming electrons, $G_{R,0}^{\equiv}$, implies therefore the change in the distribution function $n_{R}(\epsilon)$ of right-movers. Indeed, for non-interacting particles $G_{R}^{\equiv} = 2\pi i\nu_0 n_{R}(\epsilon)$ and $G_{R}^{\equiv} = -2\pi i\nu_0 [1-n_{R}(\epsilon)]$. We thus see that the electrons ejected from the interacting part of the wire into the lead are affected by the interaction: their distribution function has changed.

The left-moving electrons can be analyzed in the same way; the corresponding results are obtained by replacing $R \leftrightarrow L, \nu \leftrightarrow -\nu$ in Eqs. (20), (32). Clearly, the role of the regions I and III is interchanged in this case. It is also worth mentioning that in the non-interacting parts of the wire the Green functions are both Galilean and translationally invariant, depending on coordinates and times via $\xi - \xi'$ only.

IV. ARBITRARY BOUNDARIES

We turn now to generalization of these results for the case of an arbitrary shape of $g(x)$ in the contact region between the interacting part of the wire and the non-interacting leads. The contact regions are in general characterized by some reflection coefficients $r_{i}(\omega)$ for the plasmon amplitude, yielding reflection coefficients $\mathcal{R}_{i} = |r_{i}|^2$ for the plasmon intensity ($i = 1, 2$ for the left and right contact, respectively). The corresponding transmission coefficients are $T_{i} = 1 - \mathcal{R}_{i}$. It is instructive in this context to compare our present approach with that developed in Ref. 21, where we analyzed the tunneling density of states and focussed on the case of smooth variation of $g(x)$ in the contact regions. As we are going to show, the method of Ref. 24 can be generalized to the case of arbitrary contacts (this was briefly discussed at the end of Ref. 20) and is also useful for the analysis of the electron distribution function. Within that approach, the propagator of bosons is calculated in momentum space (rather than in real space as in the above calculation).

The Keldysh component of the propagator is then characterized by distribution function distribution functions $B_{n}(\omega)$ and $B_{\eta}(\omega)$ associated with poles at $q = \eta \omega / v$ and $q = \eta \omega / u$ and describing “ghosts” (free electron-hole pairs) and plasmons, respectively. While the distribution function of ghosts is simply determined by that of incoming electrons, the plasmons experience general reflection at the boundaries. We have for the left boundary (see Fig. 2)

$$B_{R}^{w} = T_{1} B_{R}^{(0)} + \mathcal{R}_{1} B_{L}^{w}, \quad B_{R}^{out} = \mathcal{R}_{1} B_{R}^{(0)} + T_{1} B_{L}^{w},$$

(35)

and similarly at the right boundary. Here we have introduced the notation $B_{w}^{\eta}$ for plasmon distributions in the interacting region of the wire and $B_{out}^{\eta}$ for out-going channels. Solving these equations, we find the plasmon distribution functions of right-movers in the interacting part of the wire, as well as in the outgoing channel (in the right lead):

$$B_{R}^{w} = \frac{T_{1}}{1 - \mathcal{R}_{1} \mathcal{R}_{2}} B_{R}^{(0)} + \frac{T_{2} \mathcal{R}_{1}}{1 - \mathcal{R}_{1} \mathcal{R}_{2}} B_{L}^{w},$$

$$B_{R}^{out} = \frac{T_{1} T_{2}}{1 - \mathcal{R}_{1} \mathcal{R}_{2}} B_{R}^{(0)} + \frac{T_{1} + T_{2} - 2 T_{1} T_{2}}{1 - \mathcal{R}_{1} \mathcal{R}_{2}} B_{L}^{w}.$$  

(37)

The corresponding results for left movers are obtained by exchanging the indices $R \leftrightarrow L$ and $1 \leftrightarrow 2$.

FIG. 2: Distribution functions of plasmons $B_{\eta}$ in different parts of the wire. The distributions of incoming plasmons are determined by respective leads, $B_{in}^{\eta} = B_{R}^{(0)}$.

The method of Ref. 20 allows us to express the exponents $F_{R}^{\equiv}$ in terms of these distribution functions. For the interacting part of the wire, we get

$$F_{R}^{\equiv} = -\int_{0}^{\infty} \frac{d\omega}{\omega} \left[ B_{R}^{w} - B_{R}^{(0)} \right] (1 - \cos \omega t) + \gamma \left( B_{R}^{w} + B_{R}^{\eta} \right) (1 - \cos \omega t) \pm i \sin \omega t.$$  

(38)
The result for the tunneling into the non-interacting region III of Fig. 1 can be obtained from Eq. (38) by using the distribution functions $B_{\eta}^{\text{out}}$ corresponding to this region and replacing the interaction constant $\gamma$ by zero,

$$F_{\text{R}}^x = F_{\text{R}} = - \int_0^\infty d\omega \omega (B_{\text{R}}^{\text{out}} - B_{\text{R}}^{(0)}) (1 - \cos \omega t)$$

$$= \int_0^\infty d\omega R(1 - \cos \omega t)[B_{\text{R}}^{(0)}(\omega) - B_{\text{L}}^{(0)}(\omega)],$$

(39)

where $R$ is the total reflection coefficient on a double-step structure, $R = 1 - T_L T_2/(1 - R_1 R_2)$.

For the case of sharp boundaries the reflection and transmission coefficients are given by the Fresnel law, $R_{1,2} = (1 - K)^2/(1 + K)^2$ and $T_{1,2} = 4K/(1 + K)^2$, so that Eqs. (38) and (39) reduce to the earlier results (26) and (32). The total reflection and transmission coefficients $R$ and $T$ take in this case the values (43) (which explains the notations introduced there). Clearly, the general formulas (38) and (39) can also be obtained in the framework of a real-space calculation that was presented above for sharp boundaries. To do this, one has to modify the boundary conditions for the Green function in $G_\omega$ in Eq. (22) by including the appropriate reflection and transmission amplitudes $r_i(\omega)$ and $t_i(\omega)$ at two boundaries and then proceeding in the same way as in course of the derivation of Eqs. (20) and (22). The two methods (real space and k space) are thus in full agreement with each other.

The formal results obtained thus far can be implemented to obtain physical observables. Consider first the non-interacting part of the setup, region III of Fig. 4. The effect of the interaction there amounts to modification of the distribution function of outgoing particles (right-movers), which has (in time domain) the form

$$n_R(t) = n_{R,0}(t)e^{F_R(t)},$$

(40)

where $F_R$ is given by Eq. (39). This yields

$$n_R(t) = \frac{i}{2}e^{-ieVt/2} \left( \frac{T_R}{\sinh \pi T_Rt + i0} \right)^T \times \left( \frac{T_L}{\sinh \pi T_Lt + i0} \right)^R.$$  

(41)

The way in which the electron distribution function is modified depends on the kinetics of the plasmons inside the interacting region. For adiabatic switching of interaction, there is essentially no plasmon scattering. Therefore, the total reflection coefficient $R$ and, consequently, the exponent $F_R$ in the region III vanish. In this case the fermions retain their distribution function: the right-movers going out into the right lead have the same distribution as the right-movers injected into the interacting region from the left lead. (The same applies to the left-movers, of course.) Let us now discuss the opposite limit of strong reflection, $R \rightarrow 1$. For a structure with a sharp boundary, this is the case provided the interaction is strong, $K \rightarrow 0$. Alternatively, this limit may be realized if the boundary regions are sufficiently extended and characterized by random $K(x)$ such that plasmons with relevant frequencies are localized. Regardless of the cause, in the limit $R \rightarrow 1$ the left- and right-moving electrons exchange their distribution functions, except for keeping their total flux (i.e. the chemical potential).

Next, we consider the interacting part of the wire. Analyzing the result (38), we see that two terms in square brackets have distinctly different physical origin. The second term, which is proportional to the local strength of the interaction $\gamma$ at the measurement point is responsible for creation of the zero-bias anomaly (ZBA) as well as for its dephasing smearing, with the non-equilibrium dephasing rate

$$\tau_0^{-1} = \pi \gamma \left[ \frac{(1 - R_1)(1 + R_2)}{1 - R_1 R_2} T_R + \frac{(1 + R_1)(1 - R_2)}{1 - R_1 R_2} T_L \right].$$

(42)

On the other hand, the first term in the integrand of (38), which is governed by the difference between the incoming and local distribution of plasmons, is fully analogous to the expression for $F_R^x$ in the non-interacting region, Eq. (39), and describes the modification of the distribution function inside the wire,

$$n_\eta(t) = n_{\eta,0}(t) \exp \left\{ - \int_0^\infty \frac{d\omega}{\omega} [B_\eta^{\text{out}}(\omega) - B_\eta^{(0)}(\omega)] (1 - \cos \omega t) \right\}$$

$$= \frac{i}{2\pi} \frac{1}{t + i0} \exp \left\{ - \int_0^\infty \frac{d\omega}{\omega} [B_\eta^{\text{out}}(\omega) - 1] (1 - \cos \omega t) \right\}.$$  

(43)

As is clear from Eq. (43), the “ghost” term with $B_\eta^{(0)}$ essentially serves to cancel the bare distribution function $n_{\eta,0}$, so that the distribution function $n(t)$ is determined only by the plasmonic distribution $B_\eta^{\text{out}}(\omega)$ in the wire. This is in fact a manifestation of a general relation between the functional and full bosonization approaches, as
will be discussed in detail elsewhere. In the right lead (region I in Fig. 1) sharp variation of the interaction at the boundaries (as in Sec. III) is assumed. The distribution function of free electrons in non-interacting parts of the wire, \( G(n) \), is given by [Eq. (40)]. Representative results are shown in Fig. 3. In Fig. 3 we present distribution functions for non-interacting parts of the wire. Temperatures of the leads are \( T_L = 0.2 \) and \( T_R = 0.001 \); the bias voltage is \( eV = 0.25 \).

Fourier transformation of our results into the energy representation can be done numerically (for analytic calculation at equilibrium see Appendix B); representative results are shown in Figs. 3-4. In Fig. 3 we present distribution functions for non-interacting parts of the wire. Temperatures are set to \( T_L = 0.2 \) and \( T_R = 0.001 \) (in arbitrary units), the applied voltage is \( eV = 0.25 \), and a sharp variation of the interaction at the boundaries (as in Sec. III) is assumed. The distribution function of free fermions \( (K = 1) \), plotted by a dashed line, is the same on both ends of the wire. For interacting electrons we choose the interaction parameter to be \( K = 0.2 \), which is in the range of characteristic values reported for carbon nanotubes, see, e.g., Ref. 5. The distribution functions in two leads are different. In particular, the distribution function in the left lead (region I in Fig. 1) has a sharp edge at the energy \( \epsilon = \mu + eV/2 \), which corresponds to cold right-moving electrons. In the right lead (region II), this edge is broadened due to interaction with hot left-moving electrons. The situation is opposite for left-moving particles. The distribution in the right lead has a broad edge at \( \epsilon = \mu - eV/2 \) that corresponds to hot left-moving electrons. Due to interaction inside the wire this edge in the region I sharpens.

In Fig. 4 we present the results for the distribution functions of left- and right-moving quasiparticles in the central (interacting) part of the wire, Eq. (44). For \( K = 0.2 \) the plasmon reflection at the boundaries is strong. In a symmetric structure this leads to almost equal distribution functions of both types of carriers inside the wire.

In the upper panel of Fig. 5 we show the results for TDOS for \( K = 0.8 \). The minima of TDOS are reached at energies \( \epsilon = \mu \pm eV/2 \). The broadening of the ZBA dips has two origins: smearing of the distribution function and dephasing. While the dephasing broadening [cf. second term in Eq. (38)] is the same for both chiral branches, the distribution functions [cf. first term in Eq. (38)] are different. A deeper minimum at \( \epsilon = \mu + eV/2 \) reflects the fact that right-moving electrons in the wire have a much narrower distribution function. This is because at \( K = 0.8 \) the energy relaxation at the boundaries is quite weak, so that the distribution functions of cold right-movers and hot left-movers are only slightly modified. The situation is different for \( K = 0.2 \), when distribution functions \( n_R \) and \( n_L \) are nearly identical (up to a shift by \( eV \)), see Fig. 3. As a result, the structure of the TDOS also becomes symmetric. In fact, for the chosen value of the voltage, two broad ZBA dips merge together.

V. THERMAL CONDUCTIVITY AND ELECTRONIC DISTRIBUTION FUNCTION

We discuss now a relation between our results for the electron distribution function and previous findings on the electric and thermal conductance of a LL wire. In the absence of backscattering the number of left and right moving particles is separately preserved. As a result, the electric current is linear in the voltage \( V \),

\[
I = ev(N_R - N_L) = \frac{e^2}{h} V
\]

with unrenormalized Landauer conductance \( G = e^2/h \), Ref. 24. In our formalism, this relation immediately follows from Eq. (44) and the condition \( \mathcal{F}_R(t \to 0) \to 0 \). This ensures that the modification of the distribution function of right (or left) movers by a spatially varying interaction does not affect the integral of the distribution function over energy, i.e. the total number of carriers of each type.

We turn now to the thermal conductance. The energy current is easily found from the Green functions of electrons in non-interacting parts of the wire,

\[
I_E = v \partial_t \left[ G^R_R(t, t') - G^L_L(t, t') \right]_{t = t'},
\]
which can be rewritten in terms of the electron distribution functions,

$$I_E = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} [n_L(\epsilon) - n_R(\epsilon)].$$

Substituting the result Eq. (40), (39) for the distribution functions, we get the expression of the thermal current in terms of distribution functions of incoming electron-hole pairs,

$$I_E = \frac{1}{4\pi} \int_0^\infty d\omega \omega T(\omega) [B_{L}^{(0)}(\omega) - B_{R}^{(0)}(\omega)].$$

According to Eq. (47), the thermal conductance is affected by the interaction [through the reflection coefficient $T(\omega)$], as was first found in Ref. [27]. Note that due to the particle-hole symmetry of LL model, the applied voltage drops out of Eq. (47). For the case of sufficiently sharp boundaries, when $T(\omega)$ can be considered as $\omega$-independent for relevant frequencies, Eq. (46) reduces to

$$I_E = \frac{\pi}{12} T(T_R^2 - T_L^2).$$

Deviation of the transmission coefficient $T(\omega)$ from unity leads to the violation of the Wiedemann-Franz law[22]. As is seen from our analysis, this deviation is a manifestation of a microscopic phenomenon: energy relaxation of electrons due to non-uniform interaction.

The heat current (47) can be equivalently represented in terms of plasmonic distributions in the wire

$$I_E = \frac{1}{4\pi} \int_0^\infty d\omega \omega [B_{L}^{\infty}(\omega) - B_{R}^{\infty}(\omega)].$$

This implies that the presentation of the heat current in the form Eq. (46) is also valid in the interacting part of the wire, with the electronic distribution functions $n_{\eta}(\epsilon)$ given by Eq. (43). Thus, also in the interacting part of the wire, the energy current can be understood as carried by properly defined quasi-particle excitation. This is a remarkable result, which demonstrates that the concept of fermionic quasiparticles remains meaningful in a strongly interacting 1D system (LL) despite its non-Fermi-liquid features.
VI. SUMMARY AND OUTLOOK

To summarize, we have developed a theory of tunneling spectroscopy of LL conductor connected to reservoirs away from equilibrium. In the specific setup considered here, each branch originates from a source which is at equilibrium. However, the right and the left sources have different temperatures and different chemical potentials. We have modeled the system as a LL with spatially non-uniform interaction, and calculated the single-electron Green functions $G^<\approx$ that carry information about the TDOS and the fermionic distribution functions in different parts of the wire. The interaction affects the tunneling characteristics in three distinct ways. First, it induces a power-law ZBA in the TDOS $\nu(\epsilon)$ (with two dips split by the voltage) in the interacting part of the wire. Second, it leads to broadening of ZBA singularities due to dephasing, with the dephasing rate governed by the interaction strength and the plasmon distribution inside the wire. Both the ZBA and the dephasing effects are encoded in the second term of Eq. (38).

The third effect of the interaction—which is specifically at the focus of the present work—is the inelastic scattering of electrons, leading to their redistribution over energies. This effect takes place in those regions where the interaction strength varies in space (near the wire boundaries in our model), inducing backscattering of plasmons (but not of electrons). This leads to relaxation of the electron distribution functions: left and right moving fermions “partly exchange” their distributions, see Eqs. (41), (43) and Figs. 3, 4, 5. For slowly varying interaction, when the plasmons with relevant frequencies go through essentially without reflection, the energy relaxation of electrons is negligible. In the opposite limit, when the plasmons are almost entirely reflected (due to strong and sharply switched interaction or, else, due to disordered boundary regions inducing the plasmon localization), the left- and right-movers essentially exchange their distribution functions (but not their total density). We have also discussed a connection between these results and earlier findings on the thermal conductivity of LL structures.

Our results are important for the analysis of TS experiments on strongly correlated 1D structures (in particular, carbon nanotubes) out of equilibrium. In this connection, let us emphasize the following important point. What can actually be measured in experiment are Green functions, $G^>\approx$ and $G^<\approx$. The TDOS $\nu(\epsilon)$ in the interacting part of the wire, as well as the distribution function $n(\epsilon)$ in the non-interacting regions are related to $G^>$ and $G^<$ in a simple way. On the other hand, in order to extract the distributions $n_R(\epsilon)$ and $n_L(\epsilon)$ from $G^<\approx$ in the interacting part of the wire, a non-trivial deconvolution procedure is necessary. The broadening of (split) Fermi-edge structures in $G^<\approx$ in the interacting part of the wire is governed by both the distribution function and the dephasing. The dephasing contributes to the smearing of Fermi-edge singularities also in higher-dimensional (diffusive) systems, and should be taken into account for the accurate interpretation of corresponding experiment. In the 1D case the role of dephasing becomes particularly dramatic (if the interaction is sufficiently strong). This is very well illustrated by Fig. 3 two Fermi-edge singularities almost (middle panel) or even completely (lower panel) merge, despite the fact that the Fermi edges in the distribution functions remain well separated (Fig. 4).

A comment of a more general nature is in order here. Our results illustrate the fact that there is no unique answer to the question: “How much is a LL different from a Fermi liquid?” On one hand, the strong, power-law ZBA in TDOS of a LL clearly distinguishes it from the Fermi liquid. In more formal terms, the single-particle residue $Z$, which is finite in the Fermi liquid, vanishes in a power-law fashion at the Fermi level of the LL. Also the dephasing rate determining the broadening of ZBA, Eq. (42), is linear in temperature, contrary to the Fermi-liquid $T^2$ behavior. One could think that it makes little sense to speak about fermionic excitations in this situation, but this is not the case. First, the power-law vanishing of TDOS has little importance (like the value of $Z$ in the Fermi liquid) for kinetic properties of the system. Second, the dephasing rate (42) is governed by processes with zero energy transfer and do not lead to any energy relaxation. As a result, the distribution function of fermionic excitations, $n_\mu(\epsilon)$, is a fully meaningful concept even in the case of a strong interaction. It stays preserved as long as the interaction is spatially constant (or varies adiabatically slow with $x$). Furthermore, both the charge and the energy current in the interacting part of the wire can be understood as carried by these fermionic quasiparticles. From this point of view, the LL is a perfect Fermi liquid.

We conclude the paper by reviewing some future research prospects; the work in those directions is currently underway. First, one may consider a more general non-equilibrium situation where the distribution functions “injected” into the interacting part of the wire are of non-equilibrium (e.g., double-step) form by themselves, see setups b, c in Fig.1 of Ref. 20; the first of these setups is close to the experimental situation of Ref. 8. This requires a generalization of the bosonization technique that will be presented elsewhere. Second, it is interesting to study correlations between outgoing left- and right-movers. In a general situation, one finds that their density matrices are not decoupled, i.e. they are entangled, which manifests itself, in particular, in current cross-correlations. Third, one may study the effect of a random variation of the interaction strength $K(x)$ in the wire. If the wire is sufficiently long, plasmons with not too low frequencies get localized. Using our general results, one concludes that in the left (right) half of the wire both distributions $n_R$, $n_L$ are determined by that of the left (respectively, right) reservoir, with a transition region which extends over the localization length of the middle section. To refine this picture, one has to include into consideration also plasmons with low fre-
On the other hand, if the density of states in the tunneling probe is strongly energy dependent (as for superconducting electrodes), the first term in Eq. (A3) survives. Unlike TDOS (which is determined by the difference \( G_w^\geq - G_w^\leq \)), this term contains also the information about \( G_w^\geq + G_w^\leq \). Therefore, measurement of the tunneling current with two different types of tunneling probes (normal and superconducting) allows one to find functions \( G_w^\geq \) and \( G_w^\leq \) separately. The idea to use superconducting electrodes for the tunneling spectroscopy was introduced in Ref. [7] and more recently employed in Ref. [8].
APPENDIX B: GREEN FUNCTIONS $G^\pm$ AT THERMAL EQUILIBRIUM

At thermal equilibrium the Green functions in the energy domain can be calculated explicitly. Using Eq. (B3) and $P^R_R = P^L_L = P^0_R = P^0_L = \coth \frac{\gamma}{2}$, we find the exponent $\mathcal{F}_\eta(t)$ for the Green functions in interacting part of the wire,

$$
\mathcal{F}(t) = \gamma \log \frac{\pi T}{\Lambda \sinh \pi T(t - i/\Lambda)}, \quad (B1)
$$

where we drop the chirality index $\eta$, as it is immaterial for $x = x'$ in equilibrium. Using Eq. (13) and performing a Fourier transform from the time into the energy domain, one finds

$$
G^>(\epsilon) = -\frac{(\pi T)^{1+\gamma}}{2\pi v(i\Lambda)^\gamma} \int_{-\infty}^{\infty} dt e^{it\epsilon} \frac{1}{\sinh^{1+\gamma} \pi T(t - i/\Lambda)}. \quad (B2)
$$

After calculating an auxiliary integral

$$
\int_{-\infty}^{\infty} dt \frac{e^{itz}}{\sinh^{1+\gamma} (t - i0)} = \frac{i^{1+\gamma} 2^\gamma}{\Gamma(1+\gamma)} \frac{\pi z/2}{\Gamma[(1+\gamma)/2]} |\Gamma[(1+\gamma)+iz]/2|^2,
$$

one obtains

$$
G^>(\epsilon) = -\frac{i}{2\pi v} \frac{2^\gamma}{\Gamma(1+\gamma)} \frac{(\pi T)^\gamma}{\Lambda} e^{\pm i\frac{2\gamma}{\Lambda} \pi |\epsilon| (1+\gamma)/2} |\Gamma[(1+\gamma)+iz]/2|^2, \quad (B3)
$$

where $z = \epsilon/\pi T$. Similarly, one finds the function $G^<$,

$$
G^<(\epsilon) = \frac{i}{2\pi v} \frac{2^\gamma}{\Gamma(1+\gamma)} \frac{(\pi T)^\gamma}{\Lambda} e^{-\pm i\frac{2\gamma}{\Lambda} \pi |\epsilon| (1+\gamma)/2} |\Gamma[(1+\gamma)+iz]/2|^2. \quad (B4)
$$

This yields the following asymptotic behavior of the Green function at low temperatures ($|\epsilon| \gg T$),

$$
G^>(\epsilon) = -\frac{i}{v\Gamma(1+\gamma)} e^{\pi (\epsilon - |\epsilon|)/2} \left( \frac{|\epsilon|}{\Lambda} \right)^\gamma, \quad (B5)
$$

and high temperatures ($|\epsilon| \ll T$),

$$
G^>(\epsilon) = -\frac{i}{2\pi v} \frac{2^\gamma}{\Gamma(1+\gamma)} \Gamma^2[(1+\gamma)/2] \left( \frac{\pi T}{\Lambda} \right)^\gamma. \quad (B6)
$$

Using Eqs. (31), (33), and (34), one obtains TDOS at equilibrium,

$$
\nu(\epsilon, T) = \frac{2^\gamma-1}{\pi^2 v \Gamma(1+\gamma)} \frac{(\pi T)^\gamma}{\Lambda} \times |\Gamma[(1+\gamma)+iz]/2|^2 \cosh \frac{\pi z}{2}. \quad (B7)
$$

Equation (B7) describes the well-known ZBA in TDOS, $\nu(\epsilon) \propto |\epsilon|^\gamma$, smeared at the scale $\epsilon \sim 2\pi T(1+\gamma)$. This smearing results from a combined effect of (i) the thermal broadening of the distribution function and (ii) the dephasing rate $\gamma = 2\pi T / \tau$. It is straightforward to check that the Fermi-Dirac distribution function is recovered from the ratio

$$
\frac{G^>(\epsilon) + G^<(\epsilon)}{G^>(\epsilon) - G^<(\epsilon)} = \tanh \frac{\epsilon}{2T} = 1 - 2n_0(\epsilon), \quad (B8)
$$

in agreement with the fluctuation-dissipation theorem.
18 H. C. Fogedby, J. Phys. C 9, 3757 (1976); D. K. Lee and Y. Chen, J. Phys. A 21, 4155 (1988); C.M. Naon, M.C. von Reichenbach, and M.L. Trobo, Nucl. Phys. B 435, 567 (1995); C.M. Naon, M.J. Salvay, and M.L. Trobo, Int. J. Mod. Phys. A 19, 4953 (2004); I. V. Yurkevich, in Strongly Correlated Fermions and Bosons in Low-Dimensional Disordered Systems, edited by I.V. Lerner, B.L. Altshuler, V.I. Fal’ko, and T. Giamarchi; A. Grishin, I.V. Yurkevich and I.V. Lerner, Phys. Rev. B. 69, 165108 (2004).
19 I.V. Lerner and I.V. Yurkevich, in Nanophysics: Coherence and Transport (Elsevier, 2005), edited by H. Bouchiat, Y. Gefen, G. Montambaux, and J. Dalibard, p.109.
20 D.B. Gutman, Y. Gefen and A.D. Mirlin, Phys. Rev. Lett. 101, 126802 (2008).
21 for review of the Keldysh technique see, e.g., J. Rammer and H. Smith, Rev. Mod. Phys. 58, 323 (1986); A. Kamenev, in Nanophysics: Coherence and Transport (Elsevier, 2005), edited by H. Bouchiat, Y. Gefen, G. Montambaux, and J. Dalibard, p. 177.
22 Y.V. Nazarov, A.A. Odintsov, and D.A. Averin, Europhys. Lett. 37, 213 (1997).
23 in Ref. 24 we used notations $B_{i}^{\nu} (\omega)$ and $B_{i}^{\nu} (\omega)$ for these distribution functions.
24 ZBA splitting in a similar setup was studied in S.G. Jakobs, V. Meden, and H. Schoeller, Phys. Rev. Lett. 99, 150603 (2007).
25 D. Gutman, Y. Gefen, and A.D. Mirlin, in preparation.
26 D.L. Maslov and M. Stone, Phys. Rev. B 52, R5539 (1995); V. Ponomarenko, Phys. Rev. B 52, R8666 (1995); I. Safi and H.J. Schulz, Phys. Rev. B 52, R17040 (1995); Y. Oreg and A.M. Finkel’stein, Phys. Rev. B 54, R14265 (1996).
27 R. Fazio, F.W.J. Hekking, and D.E. Khmelnitskii, Phys. Rev. Lett. 80, 5611 (1998); I.V. Krive, Low Temp. Phys. 24, 498 (1998).
28 D.B. Gutman, Y. Gefen, and A.D. Mirlin, Phys. Rev. Lett. 100, 086801 (2008).
29 A. Anthore, F. Pierre, H. Pothier, and D. Esteve, Phys. Rev. Lett. 90, 076806 (2003).
30 Y. Ji, Y. Chung, D. Sprinzak, M. Heiblum, D. Mahalu, and H. Shtrikman, Nature 422, 415 (2003); I. Neder, F. Marquardt, M. Heiblum, D. Mahalu, and V. Umansky, Nature Phys. 3, 534 (2007); I.P. Levkivsky and E.V. Sukhorukov, Phys. Rev. B 78, 045322 (2008).