Consistency, Acyclicity, and Positive Semirings

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Local Consistency vs. Global Consistency
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Locally Consistent
Globally Inconsistent
MC Escher
Local Consistency vs. Global Consistency

Locally Consistent
Globally Inconsistent
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Locally Consistent
Globally Consistent
Santorini, Greece
Local Consistency vs. Global Consistency

Fact:

• In several different settings, the objects of study are “locally consistent” but they may or may not be “globally consistent”. Such settings include:
  – Quantum Mechanics, Probability Theory,
  – Constraint Satisfaction, Database Theory, …
Local Consistency vs. Global Consistency

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Research Program:
Study the structural aspects of global consistency
• Can we unveil the “intelligible structure” of global consistency?
• When is local consistency equivalent to global consistency?
Local Consistency vs. Global Consistency

Earlier Work:

• Vorob’ev – 1962
  Characterized when a family of probability distributions defined on overlapping sets of variables has a joint distribution.

• Beeri, Fagin, Maier, Yannakakis – 1983
  Characterized when a family of database relations with overlapping sets of attributes has a universal relation.
Local Consistency vs. Global Consistency

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Goal of this work:
- A common generalization of the results of Vorob’ev and of Beeri et al.
- A unifying framework for studying local vs. global consistency that uses K-relations, where K is a positive semiring.
Local vs. Global Consistency in Databases

Basic Concepts:
- **Attribute**: a symbol $A$ with an associated set $\text{dom}(A)$ of values.
- $R(X)$: relation $R$ with $X$ as its set of attributes (names of columns)
- $R[Z]$ with $Z \subseteq X$: the **projection** of $R$ on $Z$

Example:

| A | B | C |
|---|---|---|
| 1 | 2 | 3 |
| 1 | 2 | 5 |
| 2 | 4 | 6 |

$R(A,B,C)$

| A | B |
|---|---|
| 1 | 2 |
| 2 | 4 |

$R[A,B]$
Local vs. Global Consistency in Databases

Definition:

• Two relations R(X) and S(Y) are consistent if there is a relation T over X ∪ Y such that T[X] = R and T[Y] = S.
• R₁(X₁), …, Rₙ(Xₙ) are globally consistent if there is a relation T over X₁ ∪ ⋯ ∪ Xₙ such that T[X₁] = R₁, ⋯, T[Xₙ] = Rₙ.
Local vs. Global Consistency in Databases

Definition:
• Two relations $R(X)$ and $S(Y)$ are consistent if there is a relation $T$ over $X \cup Y$ such that $T[X] = R$ and $T[Y] = S$.
• $R_1(X_1), \ldots, R_n(X_n)$ are globally consistent if there is a relation $T$ over $X_1 \cup \cdots \cup X_n$ such that $T[X_1] = R_1, \ldots, T[X_n] = R_n$.

Basic Facts:
• If $R_1(X_1), \ldots, R_n(X_n)$ are globally consistent, then they are pairwise consistent, i.e., $R_i$ and $R_j$ are consistent for all $i$ and $j$.
• The converse is not always true, i.e., there are relations $R_1(X_1), \ldots, R_n(X_n)$ that are pairwise consistent but are not globally consistent.
Hardy’s Paradox

| A₁ | B₁ |
|----|----|
| 0  | 0  |
| 0  | 1  |
| 1  | 0  |
| 1  | 1  |

R(A₁, B₁)

| A₁ | B₂ |
|----|----|
| 0  | 1  |
| 1  | 0  |
| 1  | 1  |

S(A₁, B₂)

| A₂ | B₁ |
|----|----|
| 0  | 1  |
| 1  | 0  |
| 1  | 1  |

T(A₂, B₁)

| A₂ | B₂ |
|----|----|
| 0  | 0  |
| 0  | 1  |
| 1  | 0  |

U(A₂, B₂)
Local-to-Global Consistency for Relations

**Definition:** Schema \( H = (X_1, \ldots, X_n) \) of sets of attributes. \( H \) has the local-to-global consistency property for relations if every pairwise consistent collection \( R_1(X_1), \ldots, R_n(X_n) \) of relations is globally consistent.
Local-to-Global Consistency for Relations

Definition: Schema \( H = (X_1, \ldots, X_n) \) of sets of attributes. \( H \) has the **local-to-global consistency property for relations** if every pairwise consistent collection \( R_1(X_1), \ldots, R_n(X_n) \) of relations is globally consistent.

Example 1: The schema

\[ H = (\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_4\}) \]

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**Example 2:** The schema

$$H = (\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_4\}, \{A_4, A_1\})$$

does not have the local-to-global consistency property for relations.
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Theorem (Beeri, Fagin, Maier, Yannakakis – 1983):
The following are equivalent for a schema $H = (X_1, \ldots, X_n)$
1. $H$ is an acyclic hypergraph.
2. $H$ is a conformal and chordal hypergraph.
3. $H$ has the running intersection property.
4. $H$ has a join tree.
5. $H$ has the local-to-global consistency property for relations.
Definition: Let \( H \) be a hypergraph

- The\textit{ primal graph} of \( H \) is the undirected graph whose edges are pairs of nodes that appear together in at least one hyperedge of \( H \).
- \( H \) is\textit{ conformal} if every clique of the primal graph of \( H \) is contained in some hyperedge of \( H \).
- \( H \) is\textit{ chordal} if its primal graph is chordal (i.e., every cycle of length at least four has a chord).
Conformal and Chordal Hypergraphs

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Example 1: \( H = (\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_4\}) \) is conformal and chordal.
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**Example 2:** H = (\{A_1,A_2\}, \{A_2,A_3\}, \{A_3,A_4\}, \{A_4,A_1\}) is conformal but not chordal.
Conformal and Chordal Hypergraphs

Definition: Let $H$ be a hypergraph
- The **primal graph** of $H$ is the undirected graph whose edges are pairs of nodes that appear together in at least one hyperedge of $H$.
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**Example 2:** $H = (\{A_1,A_2\}, \{A_2,A_3\}, \{A_3,A_4\}, \{A_4,A_1\})$ is conformal but not chordal.

**Example 3:** $H = (\{A_1,A_2\}, \{A_2,A_3\}, \{A_3,A_1\})$ is chordal but not conformal.
The Running Intersection Property

**Definition:** A hypergraph $H$ has the **running intersection property** if there is an ordering $X_1, \ldots, X_n$ of its hyperedges such that for every $i \leq n$, there is a $j < i$ such that

$$X_i \cap (X_1 \cup \cdots \cup X_{i-1}) \subseteq X_j.$$
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Example 1: $H = (\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_4\})$ has the running intersection property.

Example 2: $H = (\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_4\}, \{A_4, A_1\})$ does not have the running intersection property.
Local-to-Global Consistency for Relations

Definition: Schema $H = (X_1, \ldots, X_n)$ of sets of attributes. $H$ has the local-to-global consistency property for relations if every pairwise consistent collection $R_1(X_1), \ldots, R_n(X_n)$ of relations is globally consistent.

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Structural Notions: $\quad$ Semantic Notion
Positive Semirings

Definition: A positive semiring is a structure $\mathbf{K} = (\mathbf{K}, +, \times, 0, 1)$ such that

- $+$ and $\times$ are binary operations that are commutative and associative and have 0 and 1 as their identity elements;
- $0 \neq 1$;
- $\times$ distributes over $+$, i.e., $a \times (b + c) = (a \times b) + (a \times c)$, for all $a, b, c$;
- 0 annihilates $\mathbf{K}$, i.e., $0 \times a = 0$, for all $a$;
- $a + b = 0$ implies $a = 0$ and $b = 0$, for all $a, b$ (i.e., $\mathbf{K}$ is plus-positive);
- $a \times b = 0$ implies $a = 0$ or $b = 0$, for all $a, b$ (i.e., $\mathbf{K}$ has no zero divisors).

Note:

- Plus-positivity ensures that the sum of non-zero elements is non-zero.
- No zero divisors ensures that if a product is 0, then at least one factor is 0.
Positive Semirings Are Everywhere

- **Boolean semiring**: \( B = (\{0, 1\}, \land, \lor, 0, 1) \)
- **Bag semiring**: \( N = (N, +, \times, 0, 1) \), where \( N = \{0, 1, 2, \ldots\} \)
  (SQL semantics of database queries via multisets)
- **Non-negative reals**: \( R^+ = ([0, \infty), +, \times, 0, 1) \)
- **Tropical semiring** \( T = ([0, \infty], \min, +, \infty, 0) \)
  (shortest paths in graphs)
- **Viterbi semiring** \( V = ([0,1], \max, \times, 0, 1) \)
  (confidence scores – isomorphic to \( T \) via \( h(x) = e^{-x} \))
- **Fuzzy semiring**: \( F = ([0,1], \max, \min, 0, 1) \)
  (fuzzy logic semantics)
- **Polynomial semiring**: \( N[X] = (N[X], +, \times, 0, 1) \) with \( X \) a set of variables,
  \( N[X] \) all polynomials with variables from \( X \) and coefficients from \( N \)
  (database provenance – where the answers come from and how)
**K-Relations**

- Attribute A with \( \text{dom}(A) \) as its set of values
- \( X = \{A_1, \ldots, A_k\} \) set of attributes
- \( \text{Tup}(X) = \text{dom}(A_1) \times \ldots \times \text{dom}(A_k) \)

**Definition:** \( K = (K, +, \times, 0, 1) \) positive semiring, \( X \) be a set of attributes. A **K-relation over** \( X \) is a function \( R: \text{Tup}(X) \rightarrow K \) having finite support \( R' \), i.e., \( R' = \{ t \in \text{Tup}(X): R(t) \neq 0 \} \) is finite.
**K-Relations**

- Attribute $A$ with $\text{dom}(A)$ as its set of values
- $X = \{A_1, \ldots, A_k\}$ set of attributes
- $\text{Tup}(X) = \text{dom}(A_1) \times \ldots \times \text{dom}(A_k)$

**Definition:** $K = (K, +, \times, 0, 1)$ positive semiring, $X$ be a set of attributes. A $K$-relation over $X$ is a function $R: \text{Tup}(X) \rightarrow K$ having finite support $R'$, i.e., $R' = \{ t \in \text{Tup}(X): R(t) \neq 0 \}$ is finite.

**Examples:**
- **Relations** are $\mathcal{B}$-relations, where $\mathcal{B} = (\{0,1\}, \land, \lor, 0, 1)$.
- **Bags** are $\mathcal{N}$-relations, where $\mathcal{N} = (\mathbb{N}, +, \times, 0, 1)$ (each tuple has a non-negative integer as multiplicity).
- **Probability distributions of finite support** are $\mathcal{R}^+$-relations $P$ such that $\Sigma_{t \in \text{Tup}(X)} P(t) = 1$, where $\mathcal{R}^+ = ([0, \infty), +, \times, 0, 1)$. 
Equivalence of \( K \)-relations

**Definition:** Let \( R(X) \) and \( S(Y) \) be two \( K \)-relations.
\( R \equiv S \) if there are non-zero elements \( a,b \) in \( K \) such that \( aR = bS \), where \( aR : \text{Tup}(X) \to K \) with \( (aR)(t) = a \times R(t) \), and similarly for \( bS \).

**Fact:** \( \equiv \) is an equivalence relation on the collection of all \( K \)-relations.
Equivalence of $K$-relations

**Definition:** Let $R(X)$ and $S(Y)$ be two $K$-relations. $R \equiv S$ if there are non-zero elements $a,b$ in $K$ such that $aR = bS$, where $aR : \text{Tup}(X) \rightarrow K$ with $(aR)(t) = a \times R(t)$, and similarly for $bS$.

**Fact:** $\equiv$ is an equivalence relation on the collection of all $K$-relations.

**Note:**
- If $R$ and $S$ are $\mathcal{B}$-relations, then $R \equiv S$ if and only if $R = S$.
- There are $\mathcal{N}$-relations (bags) $R(X)$ and $S(Y)$ such that $R \equiv S$ but $R \neq S$.
- If $R$ and $S$ are probability distributions of finite support, then $R \equiv S$ if and only if $R = S$.
- For every $\mathcal{R}^+$-relation $R$, there is a probability distribution $P$ of finite support such that $R \equiv P$ (normalize $R$ to get $P$).
Local vs. Global Consistency for $K$-Relations

Definition: Let $R_1(X_1), \ldots, R_n(X_n)$ be $K$-relations
• $R_1(X_1), \ldots, R_n(X_n)$ are globally consistent if there is a $K$-relation $T$ over $X_1 \cup \cdots \cup X_n$ such that $T[X_1] \equiv R_1$, $\ldots$, $T[X_n] \equiv R_n$.

Basic Facts:
• If $R_1(X_1), \ldots, R_n(X_n)$ are globally consistent $K$-relations, then they are pairwise consistent, i.e., $R_i$ and $R_j$ are consistent for all $i$ and $j$.
• The converse is not always true, i.e., there are relations $R_1(X_1), \ldots, R_n(X_n)$ that are pairwise consistent but are not globally consistent.
Local-to-Global Consistency for $K$-relations

**Definition:** $K$ positive semiring, Schema $H = (X_1, \ldots, X_n)$ of sets of attributes. $H$ has the local-to-global consistency property for $K$-relations if every pairwise consistent collection $R_1(X_1), \ldots, R_n(X_n)$ of $K$-relations is globally consistent.

**Main Theorem:** Let $K$ be a positive semiring. The following are equivalent for a schema $H = (X_1, \ldots, X_n)$

1. $H$ is an acyclic hypergraph.
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Proof Hint: Different proof architecture than the BFMY Theorem.

Step 1: If $H$ has the running intersection property, then $H$ has the local-to-global consistency property for $K$-relations.

Step 2: If $H$ is not conformal or $H$ is not chordal, then $H$ does not have the local-to-global consistency property for $K$-relations.
Local-to-Global Consistency for $K$-relations

Step 1: If $H$ has the running intersection property, then $H$ has the local-to-global consistency property for $K$-relations.

Proof Outline: Let $X_1, \ldots, X_n$ be an ordering of the hyperedges of $H$ such that for every $i \leq n$, there is a $j < i$ such that $X_i \cap (X_1 \cup \cdots \cup X_{i-1}) \subseteq X_j$.

- Assume that $R_1(X_1), \ldots, R_n(X_n)$ are pairwise consistent $K$-relations. By induction on $i \leq n$, show that $R_1(X_1), \ldots, R_i(X_i)$ are globally consistent.

- Assume that $R_1(X_1), \ldots, R_{i-1}(X_{i-1})$ are globally consistent and let $W(X_1 \cup \cdots \cup X_{i-1})$ be a $K$-relation witnessing their global consistency.

- Define the notion of the join $R \bowtie S$ of two $K$-relations and show that $W \bowtie R_i$ witnesses the consistency of $W$ and $R_i$.

Note: The definition of $R \bowtie S$ is rather delicate (and is not the obvious one).
Local-to-Global Consistency for $K$-relations

Step 2: If $H$ is not conformal or $H$ is not chordal, then $H$ does not have the local-to-global consistency property for $K$-relations.
Local-to-Global Consistency for $K$-relations

Step 2: If $H$ is not conformal or $H$ is not chordal, then $H$ does not have the local-to-global consistency property for $K$-relations.

Proof Outline: Show the following intermediate results:

- If $H$ is not conformal or $H$ is not chordal, then $H$ contains a “simple” induced hypergraph $H^*$ with hyperedges of one of the forms:
  - $\{ V \setminus A : A \in V \}$, for some set $V$ with $|V| \geq 3$.
  - $\{ \{ A_1, A_2 \}, \ldots, \{ A_{n-1}, A_n \}, \{ A_n, A_1 \} \}$ with $n \geq 4$.
- If $H$ has the local-to-global consistency property for $K$-relations, then so do the above “simple” induced hypergraphs.
- The “simple” induced hypergraphs $H^*$ do not have the local-to-global consistency property for $K$-relations.

Explicit construction of $K$-relations that are pairwise consistent but not globally consistent; inspired by Tseitin’s hard-to-prove tautologies.
Local-to-Global Consistency

**Main Theorem:** Let \( K \) be a positive semiring. The following are equivalent for a schema \( H = (X_1, \ldots, X_n) \)

1. \( H \) is an acyclic hypergraph.
2. \( H \) has the local-to-global consistency property for \( K \)-relations.

**Corollary:** The following are equivalent for a schema \( H = (X_1, \ldots, X_n) \)

1. \( H \) is an acyclic hypergraph.
2. \( H \) has the local-to-global consistency property for relations.
3. \( H \) has the local-to-global consistency property for probability distributions of finite support.
Local-to-Global Consistency

Main Theorem: Let $K$ be a positive semiring. The following are equivalent for a schema $H = (X_1, \ldots, X_n)$

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3. $H$ has the local-to-global consistency property for probability distributions of finite support.

Note:

- The equivalence between 1. and 2. is the BFMY result.
- How is the equivalence between 1. and 3. related to Vorob’ev’s work?
Vorob’ev’s Theorem and Related Work

Vorob’ev’s Theorem - 1962:
The following are equivalent for a schema $H = (X_1, \ldots, X_n)$

- The hypergraph $H = (X_1, \ldots, X_n)$ is regular.
- $H$ has the local-to-global consistency property for probability distributions of finite support.

Note:
- In the paper, we give a direct proof that $H$ is regular iff $H$ is acyclic.
- Thus, Vorob’ev’s Theorem and the BFMY Theorem are instances of a single unifying result.
Consistency over Positive Monoids

Observations: Let $K = (K, +, \times, 0, 1)$ be a positive semiring.

- The definition of the projection $R[Z]$ uses only addition $+$.
- Multiplication $\times$ was used to define
  - the equivalence relation $R \equiv S$ (there are $a, b$ such that $aR = bS$) and
  - the join operation $R \Join S$. 
Consistency over Positive Monoids

**Observations:** Let $\mathbf{K} = (\mathbb{K}, +, \times, 0, 1)$ be a positive semiring.
- The definition of the projection $R[Z]$ uses only addition $+$
- Multiplication $\times$ is used to define
  - the equivalence relation $R \equiv S$ (there are $a$, $b$ such that $aR = bS$) and
  - the join operation $R \bowtie S$.

**Definitions:**
- A positive monoid is a commutative monoid $\mathbf{K} = (\mathbb{K}, +, 0)$ such that $a + b = 0$ implies $a = 0$ and $b = 0$, for all $a, b \in \mathbb{K}$.
- Two $\mathbb{K}$-relations $R(X)$ and $S(Y)$ are strictly consistent if there is a $\mathbb{K}$-relation $T(X \cup Y)$ such that $T[X] = R$ and $T[Y] = S$.
- Define analogously the notions of strict global consistency property and strict local-to-global consistency property for a hypergraph $H$. 
Consistency over Positive Monoids

Results (work in progress):

• Let $K = (K, +, 0)$ be a positive monoid and let $H$ be a hypergraph. If $H$ has the strict local-to-global consistency property for $K$-relations, then $H$ is acyclic.

• There are positive monoids $K$ and acyclic hypergraphs $H$ such that $H$ does not have the strict local-to-global consistency property for $K$-relations.

• We characterize the positive monoids $K$ for which every acyclic hypergraph $H$ has the strict local-to-global consistency property for $K$-relations.

A new expanded framework for local vs. global consistency
Thank you for your attention!
Backup Slides
The Join of two $K$-relations

If $R(X)$ and $S(Y)$ are $K$-relations, then the join of $R$ and $S$ is the $K$-relation $R \bowtie S$ over $X \cup Y$ such that for every $(X \cup Y)$-tuple $t$, we have

$$(R \bowtie S)(t) = R(t[X]) \times S(t[Y]) \times \prod_{u \neq t[X \cap Y]} S[X \cap Y](u)$$