Kinetic theory of QED plasmas in a strong electromagnetic field
I. The covariant hyperplane formalism

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Abstract

We present a covariant density matrix approach to kinetic theory of QED plasmas subjected to a strong external electromagnetic field. A canonical quantization of the system on space-like hyperplanes in Minkowski space and a covariant generalization of the Coulomb gauge is used. The condensate mode associated with the mean electromagnetic field is separated from the photon degrees of freedom by a time-dependent unitary transformation of both, the dynamical variables and the nonequilibrium statistical operator. Therefore even in the case of strong external fields a perturbative expansion in orders of the fine structure constant for the correlation functions as well as the statistical operator is applicable. A general scheme for deriving kinetic equations in the hyperplane formalism is presented.

Key words: relativistic kinetic theory; QED plasma; hyperplane formalism

1 Introduction

In recent years the theoretical study of dense relativistic plasmas is of increasing interest. Such plasmas are not only limited to astrophysics, but can nowadays be produced by high-intense shortpulse lasers [1–3]. In view of the inertial confinement fusion, one has to consider a plasma under extreme conditions which is created by a strong external field. This new experimental

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progress needs a systematic approach based on quantum electrodynamics and methods of nonequilibrium statistical mechanics.

Considerable attention has been focused on a mean-field (Vlasov-type) kinetic equation for the fermionic Wigner function, which is an essential step towards transport theory of laser-induced QED plasmas. Using the Wigner operator defined in four-dimensional momentum space [4–6], a manifestly covariant mean-field kinetic equation can be derived from the Heisenberg equations of motion for the field operators. In this approach, however, it is difficult to formulate an initial value problem for the kinetic equation since the four-dimensional Fourier transformation in the covariant Wigner function includes integration of two-point correlation functions over time. This difficulty does not appear in the scheme based on the one-time fermionic Wigner function where the field operators are taken at the same time and only the spatial Fourier transformation is performed. In the context of QED, the one-time formulation was proposed by Bialynicki-Birula et al. [7] (referred to in the following as BGR) and used successfully in their study of the electron-positron vacuum. Within this approach one can explore a number of attractive features. The one-time Wigner function has a direct physical interpretation and allows to calculate local observables, such as the charge density and the current density. The description in terms of one-time quantities is quite natural in kinetic theory based on the von Neumann equation for the statistical operator and provides a consistent account of causality in collision integrals.

It should be noted, however, that the one-time Wigner function does not contain complete information about one-particle dynamics; the spectral properties of correlation functions can be described only in terms of two-point Green's functions which are closely related to the covariant Wigner function. Recently this aspect of relativistic kinetic theory was studied within the mean-field approximation [8,9]. The aforementioned incompleteness of the one-time description is well known in non-relativistic kinetic theory, where two-time correlation functions can, in principle, be reconstructed from the one-time Wigner function by solving integral equations which follow from the Dyson equation for nonequilibrium Green's functions [10]. The reconstruction problem in relativistic kinetic theory remains to be explored. The solution of this problem requires a further development of the relativistic density matrix method as well as the relativistic Green's function technique.

In this and subsequent works we develop a density matrix approach to kinetic theory of QED plasmas subjected to a strong electromagnetic field. From the conceptual point of view, our aim is to generalize the BGR scheme [7] in two aspects. First, we wish to present the one-time formalism in covariant form. This removes a drawback of the BGR theory which is not manifestly covariant. Second, we will develop a scheme which allows to go beyond the mean-field approximation, including dissipative processes in QED plasmas and the inter-
play between collisions and the mean-field effects. Whereas subsequent studies will be concerned with explicit kinetic equations, the present first part considers some general problems of the one-time covariant approach to relativistic kinetic theory. In comparison to QED where the main object is the $S$-matrix constructed from vacuum averages of the field operators, kinetic theory of QED deals with averages over a nonequilibrium ensemble describing a many-body system. Therefore we use the Hamiltonian formalism which is typical for the density matrix method. In this case, however, one meets with some fundamental problems which are considered in this paper. In order to make the theory manifestly covariant, canonical quantization of the system will be carried out in a covariant fashion. Another point is that in the presence of a strong electromagnetic field, perturbation expansions in the fine structure constant are not suitable. To overcome this difficulty, we will present a procedure which allows to separate the classical part of the electromagnetic (EM) field and the photon degrees of freedom at any time.

The paper is organized as follows. In Section 2 we briefly sketch a scheme of relativistic statistical mechanics in the form adapted to kinetic theory. In our approach we use a manifest covariant Schrödinger picture on space-like hyperplanes in Minkowski space. Analogous formulations of relativistic quantum mechanics and quantum field theory can be found in literature for various applications (see, e.g., [11–15]). In this way, “equal-time” correlation functions are defined with respect to the “invariant time” variable on a hyperplane. In Section 3 we perform canonical quantization of QED on space-like hyperplanes and derive the covariant quantum Hamiltonian. Section 4 deals with the condensate mode which corresponds to the electromagnetic field induced by the polarization in the system. The condensate mode is eliminated by a time-dependent unitary transformation of the statistical operator and dynamical variables. As a result, we obtain the effective Hamiltonian, where the interaction of fermions with the mean electromagnetic field is incorporated non-perturbatively at any time, while the interaction between fermions and photons is described by a term which can be taken into account within perturbation theory. It is shown how Maxwell equations for the mean electromagnetic field are recovered in our scheme. In Section 5 the covariant one-time Wigner function and the photon density matrix are introduced and a method for deriving kinetic equations in the hyperplane formalism is outlined. The paper is summerized with a short discussion of the results and an outlook to further applications.

We use the system of units with $c = \hbar = 1$. The signature of the metric tensor is $(+, −, −, −)$.
2 Nonequilibrium statistical operator in the hyperplane formalism

2.1 The relativistic von Neumann equation

It is well known that in the special theory of relativity a quantum state of a system is defined by a complete set of commuting observables which can be associated with a three-parameter space-like surface $\sigma$ in Minkowski space. Among these surfaces three-dimensional hyperplanes are especially simple to deal with [11–14]. Since the use of arbitrary space-like surfaces does not lead to new physics, in what follows we restrict our consideration to hyperplanes. A space-like hyperplane $\sigma \equiv \sigma_{n,\tau}$ is characterized by a unit time-like normal vector $n^\mu$ and a scalar parameter $\tau$ which may be interpreted as an “invariant time”. The equation of the hyperplane $\sigma_{n,\tau}$ reads

$$x \cdot n = \tau, \quad n^2 = n^\mu n_\mu = 1.$$  (2.1)

In the special Lorentz frame where $n^\mu = (1, 0, 0, 0)$ and consequently Eq. (2.1) reads $x^0 = \tau$ the parameter $\tau$ coincides with the time variable $t = x^0$. We will refer to this special frame as the “instant frame”, since only here observables are measured at the same instant of time $t$. By treating a state vector $|\Psi[\sigma_{n,\tau}]\rangle$ as a functional of $\sigma_{n,\tau}$, the covariant Schrödinger equation can be derived from the relation between the state vector on the hyperplane $\sigma$ and the state vector on the hyperplane $\sigma' = L\sigma$ which is obtained by an inhomogeneous Lorentz transformation $L = \{a, \Lambda\}$:

$$\sigma \rightarrow \sigma' = L\sigma : \quad x \rightarrow x' = \Lambda x + a.$$  (2.2)

The relation between the state vectors is [16]

$$U(L) |\Psi[L\sigma]\rangle = |\Psi[\sigma]\rangle,$$  (2.3)

where $U(L) = U(a, \Lambda)$ is a unitary representation of the inhomogeneous Lorentz group. The generators of this representation, $\hat{P}^\mu$ and $\hat{M}^{\mu\nu}$, are the energy-momentum vector and the angular momentum tensor, respectively. For our purpose, the only transformations of relevance are pure time-like translations which change the value of $\tau$. Recalling the form of $U(a, \Lambda)$ for pure translations

$$U(a, 1) = \exp \left\{ i \hat{P}_\mu a^\mu \right\}$$  (2.4)
and introducing the notation $|\Psi[\sigma_{n,\tau}]\rangle = |\Psi(n, \tau)\rangle$, Eq. (2.3) can be written for an infinitesimal time-like translation $a^\mu = n^\mu \delta\tau$ as

$$|\Psi(n, \tau + \delta\tau)\rangle + i\delta\tau \left(\hat{P}_\mu n^\mu\right) |\Psi(n, \tau)\rangle = |\Psi(n, \tau)\rangle,$$

from which we obtain the relativistic Schrödinger equation

$$i \frac{\partial}{\partial \tau} |\Psi(n, \tau)\rangle = \hat{H}(n) |\Psi(n, \tau)\rangle$$

with the Hamiltonian on the hyperplane given by

$$\hat{H}(n) = \hat{P}_\mu n^\mu.$$ (2.7)

In the presence of a prescribed external field, the energy-momentum vector and, consequently, the Hamiltonian $\hat{H}^\tau(n)$ can depend explicitly on $\tau$. Combining Eq. (2.6) with the adjoint equation for the bra-vector, one finds that the statistical operator $\varrho(n, \tau)$ for a mixed quantum ensemble obeys the equation

$$\frac{\partial \varrho(n, \tau)}{\partial \tau} - i \left[\varrho(n, \tau), \hat{H}^\tau(n)\right] = 0,$$

which is analogous to the non-relativistic von Neumann equation.

### 2.2 Schrödinger and Heisenberg pictures on hyperplanes

The evolution of a mixed ensemble on space-like hyperplanes can be represented in different pictures. The statistical operator in the Heisenberg picture does not depend on the parameter $\tau$ and is associated with some fixed hyperplane $\sigma_{n,\tau_0}$. Dynamical variables are represented by operators $\hat{O}_H([\sigma_{n,\tau}])$ which are functionals of the hyperplanes. Local operators $\hat{O}_H(x)$, which depend on the space-time point $x$, are of particular interest in quantum field theory. In what follows it will be convenient to treat such operators as functions of the parameter $\tau$. To define this dependence, we introduce the transverse projector with respect to the normal vector $n^\mu$,

$$\Delta^\mu_\nu = \delta^\mu_\nu - n^\mu n_\nu,$$ (2.9)

and notice that a space-time four-vector $x^\mu$ can be represented in the form

$$x^\mu = n^\mu \tau + x^\mu_\perp, \quad \tau = n \cdot x,$$ (2.10)
where
\[ x^\mu_\perp = \Delta^\mu_\nu x^\nu \]  
(2.11)

is the transverse (space-like) component of \( x \). Geometrically, Eq. (2.10) means that the space-like vector \( x^\mu_\perp \) lies on the hyperplane \( \sigma_{n,\tau} \) passing through the space-time point \( x \). Using the decomposition (2.10), a local Heisenberg operator \( \hat{O}_H(x) \) can be written as
\[ \hat{O}_H(x) = \hat{O}_H(n\tau + x_\perp) \equiv \hat{O}_H(\tau, x_\perp). \]  
(2.12)

Let us assume that \( \hat{P}^\mu \) does not depend explicitly on \( \tau \). Then, recalling the well-known equation of motion for Heisenberg operators
\[ \partial_\mu \hat{O}_H(x) = -i[\hat{O}_H(x), \hat{P}_\mu], \]  
(2.13)

one readily finds that the time-like evolution of such operators is described by the equation
\[ \hat{O}_H(\tau, x_\perp) = e^{i(\tau - \tau_0)\hat{H}(n)} \hat{O}_H(\tau_0, x_\perp) e^{-i(\tau - \tau_0)\hat{H}(n)} \]  
(2.14)

with the Hamiltonian (2.7). The generalization of Eq. (2.14) to situations in which the Hamiltonian \( \hat{H}^\tau \) depends explicitly on \( \tau \) is obvious. Defining the evolution operator \( U(\tau, \tau'; n) \) as the ordered exponent
\[ U(\tau, \tau'; n) = T_\tau \exp \left\{ -i \int_{\tau'}^{\tau} \hat{H}^\tau(n) d\tau \right\}, \]  
(2.15)

we have
\[ \hat{O}_H(\tau, x_\perp) = U(\tau, \tau_0; n) \hat{O}_H(\tau_0, x_\perp) U(\tau_0; n). \]  
(2.16)

In the Schrödinger picture, the statistical operator \( \varrho(n, \tau) \) is \( \tau \)-dependent and its time-like evolution is governed by Eq. (2.8), whereas operators \( \hat{O}_S \) are defined on a fixed hyperplane. Assuming the Heisenberg and Schrödinger pictures to coincide on the hyperplane \( \sigma_{n,\tau_0} \), Eq. (2.16) implies that the transition from the Schrödinger picture to the Heisenberg picture is given by
\[ \hat{O}_H(\tau, x_\perp) = U(\tau, \tau_0; n) \hat{O}_S(x_\perp) U(\tau_0; n). \]  
(2.17)
The mean values $O(x)$ of local dynamical variables can be calculated in both pictures. Using a formal solution of Eq. (2.8)

$$g(n, \tau) = U(\tau, \tau_0; n) \, g(n, \tau_0) \, U(\tau, \tau_0; n),$$

we find that

$$O(x) = \langle \hat{O}_H(\tau, x_\bot) \rangle^{\tau_0} = \langle \hat{O}_S(x_\bot) \rangle^\tau.$$  \hspace{1cm} (2.19)

Here and in what follows the symbol $\langle \cdots \rangle^\tau$ stands for averages calculated with the statistical operator $g(n, \tau)$. In many problems one is dealing with partial derivatives $\partial_\mu O(x)$ which enter the equations of motion for local observables. In the hyperplane formalism, it is convenient to express the partial derivatives in terms of the derivatives with respect to $\tau$ and $x_\bot$. Recalling Eqs. (2.10) and (2.11), we write

$$\partial_\mu = n_\mu \frac{\partial}{\partial \tau} + \nabla_\mu, \quad \nabla_\mu = \Delta_\mu^\nu \partial_\nu = \Delta_\mu^\nu \frac{\partial}{\partial x_\bot^\nu}.$$ \hspace{1cm} (2.20)

Then, in the Heisenberg picture, Eq. (2.19) yields the equation of motion

$$\partial_\mu O(x) = \langle \partial_\mu \hat{O}_H(\tau, x_\bot) \rangle^{\tau_0},$$

where

$$\partial_\mu \hat{O}_H(x) = n_\mu \frac{\partial}{\partial \tau} \hat{O}_H(\tau, x_\bot) + \nabla_\mu \hat{O}_H(\tau, x_\bot)
\equiv -in_\mu \left[ \hat{O}_H(\tau, x_\bot), \hat{H}_H(n, \tau) \right] + \nabla_\mu \hat{O}_H(\tau, x_\bot).$$ \hspace{1cm} (2.22)

In the Schrödinger picture, the $\tau$-dependence of the mean values appear through the statistical operator which obeys the von Neumann equation (2.8). In this picture the equation of motion, bearing a formal resemblance to Eq. (2.21), is obtained from Eq. (2.19)

$$\partial_\mu O(x) = \langle \partial_\mu \hat{O}_S(x_\bot) \rangle^\tau,$$ \hspace{1cm} (2.23)

with the analogous definition of the operator $\partial_\mu$ acting on local dynamical variables:

$$\partial_\mu \hat{O}_S(x_\bot) = -in_\mu \left[ \hat{O}_S(x_\bot), \hat{H}(n) \right] + \nabla_\mu \hat{O}_S(x_\bot).$$ \hspace{1cm} (2.24)
2.3 “Equal-time” correlation functions

Describing the evolution of the system in terms of hyperplanes, we can introduce “equal-time” correlation functions of local dynamical variables with respect to the invariant time $\tau$. Let $\hat{O}_1^H(x), \hat{O}_2^H(x), \ldots, \hat{O}_k^H(x)$ be some local Heisenberg operators. Then the “equal-time” correlation function for these operators can be defined as

$$F_{1\ldots k}(x_{1\perp}, \ldots, x_{k\perp}; n, \tau) = \langle \hat{O}_1^H(x_1) \cdots \hat{O}_k^H(x_k) \rangle^\tau_0,$$  \hspace{1cm} \text{(2.25)}

where $n \cdot x_1 = n \cdot x_2 = \ldots = n \cdot x_k = \tau$. In the Schrödinger picture this correlation function takes the form

$$F_{1\ldots k}(x_{1\perp}, \ldots, x_{k\perp}; n, \tau) = \langle \hat{O}_1^S(x_1) \cdots \hat{O}_k^S(x_k) \rangle^\tau.$$  \hspace{1cm} \text{(2.26)}

The covariant von Neumann equation (2.8) yields the equations

$$\frac{\partial}{\partial \tau} F_{1\ldots k}(x_{1\perp}, \ldots, x_{k\perp}; n, \tau) = -i \langle [\hat{O}_1^S(x_1) \cdots \hat{O}_k^S(x_k), \hat{H}^\tau(n)] \rangle^\tau$$  \hspace{1cm} \text{(2.27)}

which can serve as a starting point for constructing the quantum hierarchy for the “equal-time” correlation functions.

3 Hamiltonian of QED on hyperplanes

We will now apply the foregoing scheme to a relativistic system of charged fermions interacting through the EM field. For definiteness, we take these fermions to be electrons and positrons, so that protons will be treated as a positively charged background which ensures electric neutrality of the system. There is no difficulty in describing protons by an additional Dirac field. Having in mind applications to relativistic plasmas produced by high-intense short-pulse lasers, we assume the system to be subjected into a prescribed external EM field which is not necessarily weak.

3.1 The Lagrangian density

The first step in formulating the kinetic theory of QED plasmas is to construct the Hamiltonian $\hat{H}(n)$. We start with the classical Lagrange density

$$\mathcal{L}(x) = \mathcal{L}_D(x) + \mathcal{L}_{EM}(x) + \mathcal{L}_{\text{int}}(x) + \mathcal{L}_{\text{ext}}(x),$$  \hspace{1cm} \text{(3.1)}
where $\mathcal{L}_D(x)$ and $\mathcal{L}_{EM}(x)$ are the Lagrangian densities of free Dirac and EM fields respectively, $\mathcal{L}_{int}(x)$ is the interaction Lagrangian density, and the term $\mathcal{L}_{ext}(x)$ describes the interaction of fermions with the external electromagnetic field. In standard notation (see, e.g., [17]), we have

$$\mathcal{L}_D(x) = \bar{\psi}(x) \left( \frac{i}{2} \gamma^\mu \overleftrightarrow{\partial}_\mu - m \right) \psi(x), \quad (3.2)$$

$$\mathcal{L}_{EM}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x), \quad (3.3)$$

$$\mathcal{L}_{int}(x) = -j_\mu(x) A^\mu(x), \quad (3.4)$$

$$\mathcal{L}_{ext}(x) = -j_\mu(x) A^\mu_{ext}(x), \quad (3.5)$$

where $\overleftrightarrow{\partial}_\mu = \overleftarrow{\partial}_\mu - \overrightarrow{\partial}_\mu$. In the following the electromagnetic field tensor is taken in the form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The current density four-vector will be expressed as $j^\mu = e\bar{\psi}\gamma^\mu \psi$ with $e < 0$. We wish to remark that in our approach the four-potential of the EM field is decomposed into two terms. The variables $A_\mu(x)$ correspond to the EM field caused by charges and currents in the system, while $A^\mu_{ext}(x)$ is a prescribed external field. In what follows, only the dynamical field $A^\mu(x)$ will be quantized.

### 3.2 Canonical quantization on hyperplanes

A canonical quantization implies that some gauge fixing condition is imposed on $A^\mu$. For many-particle systems studied in statistical mechanics, the Coulomb gauge seems to be the most natural. However, the disadvantage of this gauge is that it is not manifestly covariant. Therefore we will use a generalization of the Coulomb gauge condition which is consistent with the covariant description of evolution in terms of space-like hyperplanes. To formulate this condition, we introduce for any four-vector $V^\mu$ the decomposition into the transverse and longitudinal parts by

$$V^\mu = n^\mu V_\parallel + V_\perp^\mu, \quad V_\parallel = n_\nu V^\nu, \quad V_\perp^\mu = \Delta^\mu_\nu V^\nu, \quad (3.6)$$

where $\Delta^\mu_\nu$ is the projector (2.9). Then a natural generalization of the Coulomb gauge condition reads

$$\nabla_\mu A_\perp^\mu = 0. \quad (3.7)$$

In the special frame where $n^\mu = (1, 0, 0, 0)$ and $A^\mu = (A^0, \mathbf{A})$, Eq. (3.7) reduces to $\nabla \cdot \mathbf{A} = 0$, which is the usual Coulomb gauge condition.
To define canonical variables for the electromagnetic field on a hyperplane \( \sigma_{n,\tau} \), we first perform the decomposition (3.6) of the field variables \( A^\mu \) and the decomposition (2.20) of the derivatives in the Euler-Lagrange equations

\[
\frac{\partial L}{\partial A^\mu} - \partial_\nu \frac{\partial L}{\partial (\partial_\nu A^\mu)} = 0. \tag{3.8}
\]

A simple algebra shows that these equations are equivalent to

\[
\frac{\partial L}{\partial A_\|} - \partial_\tau \left( \frac{\partial L}{\partial A_\|} \right) - \nabla_\nu \frac{\partial L}{\partial (\nabla_\nu A_\|)} = 0, \tag{3.9}
\]

\[
\Delta^{\mu\nu} \left[ \frac{\partial L}{\partial A_\|} - \partial_\tau \left( \frac{\partial L}{\partial A_\|} \right) - \nabla_\lambda \frac{\partial L}{\partial (\nabla_\lambda A_\|)} \right] = 0, \tag{3.10}
\]

where we use the notation \( \dot{f} \equiv \partial f / \partial \tau \) for derivatives with respect to \( \tau \). Equation (3.9) allows to eliminate the variable \( A_\| \) in the Lagrangian. First we rewrite expressions (3.3) and (3.4) in terms of \( A_\| \) and \( A_\perp^\mu \) using the decomposition procedure for the derivatives and the field \( A^\mu \). As a result we obtain the Lagrangian density in the form

\[
L = -\frac{1}{4} F_{\perp \mu\nu} F_{\perp}^{\mu\nu} - \frac{1}{2} \left( \nabla^\mu A_\| - \dot{A}_\|^{\mu} \right) \left( \nabla_\mu A_\| - \dot{A}_\|_\perp^{\mu} \right) - j_\| A_\| - j_\perp A_\perp^{\mu} + L_D + L_{\text{ext}}, \tag{3.11}
\]

where we have introduced the notation

\[
F_{\perp}^{\mu\nu} = \nabla^\mu A_\perp^{\nu} - \nabla^\nu A_\perp^{\mu}. \tag{3.12}
\]

Note that the last two terms in Eq. (3.11) do not contain \( A_\| \) and \( A_\perp^{\mu} \). Now using the expression (3.11) to calculate derivatives in Eq. (3.9) and taking into account that, according to the gauge condition (3.7), \( \nabla_\mu \dot{A}_\perp^{\mu} = 0 \), we get

\[
\nabla_\mu \nabla^\mu A_\| = j_\|. \tag{3.13}
\]

In the “instant frame”, where \( n^\mu = (1,0,0,0) \), this reduces to the Poisson equation for \( A^0 \). The solution of Eq. (3.13) is

\[
A_\|(\tau, x_\perp) = \int_{\sigma_n} d\sigma' G(x_\perp - x'_\perp) j_\| (\tau, x'_\perp), \tag{3.14}
\]
where the Green function $G(x_\perp)$ satisfies the equation

$$\nabla_\mu \nabla^\mu G(x_\perp) = \delta^3(x_\perp)$$

(3.15)

with the three-dimensional delta function on a hyperplane $\sigma_n$ defined as

$$\delta^3(x_\perp) = \int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot x} \delta(p \cdot n).$$

(3.16)

The solution of Eq. (3.15) for $G(x_\perp)$ is given by

$$G(x_\perp) = -\int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot x} \frac{1}{p^2_\perp} \delta(p \cdot n).$$

(3.17)

The variable $A_\parallel$ can now be eliminated in the Lagrange density (3.11) with the aid of Eq. (3.14). Terms like $\nabla_\nu (\cdots)$ can be dropped since they do not contribute to the Lagrangian $L = \int L d\sigma$ under appropriate boundary conditions. Then a straightforward algebra leads to

$$L = -\frac{1}{4} F_{\perp\mu\nu} F^{\perp\mu\nu} - \frac{1}{2} \dot{A}_\perp^\mu \dot{A}_\perp^\mu - j_\perp^\mu \dot{A}_\perp^\mu - \frac{1}{2} \int_{\sigma_n} d\sigma' j^\parallel_\parallel (\tau, x_\perp) G(x_\perp - x'_\perp) j^\parallel_\parallel (\tau, x'_\perp).$$

(3.18)

We will treat the fields $A_\perp^\mu$ as dynamical variables for the EM field and follow the Dirac version of canonical quantization of theories with constraints [18,19]. The gauge condition (3.7) is one of the constraint equation in this scheme. Another constraint equation follows directly from the definition of transverse four-vectors, Eq. (3.6), and reads

$$n_\mu A_\perp^\mu (x) = 0.$$ 

(3.19)

We now define canonical conjugates for the field variables $A_\perp^\mu$ by

$$\Pi_\perp^\mu = \frac{\partial L}{\partial \dot{A}_\perp^\mu} = -\dot{A}_\perp^\mu.$$ 

(3.20)

Obviously the $\Pi$’s are not independent variables since they satisfy the constraint equations $\nabla_\mu \Pi_\perp^\mu = 0$, and $n_\mu \Pi_\perp^\mu = 0$. Thus, we have four constraints imposed on the canonical variables. Following the standard quantization procedure [19], the commutation relations for the field operators $\hat{A}_\perp^\mu$ and $\hat{\Pi}_\perp^\mu$ can be derived. As shown in Appendix A, these commutation relations are
\[
\left[ \hat{A}_\perp^\mu (\tau, x_\perp), \hat{\Pi}_\perp^\nu (\tau, x'_\perp) \right] = i e^{\mu\nu} (x_\perp - x'_\perp), \tag{3.21}
\]
\[
\left[ \hat{A}_\perp^\mu (\tau, x_\perp), \hat{A}_\perp^\nu (\tau, x'_\perp) \right] = [\hat{\Pi}_\perp^\mu (\tau, x_\perp), \hat{\Pi}_\perp^\nu (\tau, x'_\perp)] = 0, \tag{3.22}
\]
where
\[
e^{\mu\nu} (x_\perp - x'_\perp) = \int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot (x-x')} \delta (p \cdot n) \left[ \Delta^{\mu\nu} - \frac{p_\mu p_\nu}{p_\perp^2} \right]. \tag{3.23}
\]

In Appendix B the anticommutation relations for the Dirac field operators on hyperplanes are derived. The result can be written as
\[
\left\{ \hat{\psi}_a (\tau, x_\perp), \hat{\bar{\psi}}_{a'} (\tau, x'_\perp) \right\} = \left[ \gamma_\parallel (n) \right]_{aa'} \delta^3 (x_\perp - x'_\perp), \tag{3.24}
\]
\[
\left\{ \hat{\psi}_a (\tau, x_\perp), \hat{\psi}_{a'} (\tau, x'_\perp) \right\} = \left\{ \hat{\bar{\psi}}_a (\tau, x_\perp), \hat{\bar{\psi}}_{a'} (\tau, x'_\perp) \right\} = 0, \tag{3.25}
\]
where \(a, a'\) are the spinor indices. The matrix \(\gamma_\parallel (n)\) is introduced through the following decomposition of the Dirac matrices \(\gamma^\mu\):
\[
\gamma^\mu = n^\mu \gamma_\parallel (n) + \gamma_\perp^\mu (n), \quad \gamma_\parallel (n) = n^\nu \gamma^\nu, \quad \gamma_\perp^\mu (n) = (\delta^\mu_\nu - n^\mu n_\nu) \gamma^\nu. \tag{3.26}
\]

In the special Lorentz frame where \(x^\mu = (t, \mathbf{r})\) and \(n^\mu = (1, 0, 0, 0)\), we have \(\gamma_\parallel = \gamma^0\) and \(\delta^3 (x_\perp - x'_\perp) = \delta (\mathbf{r} - \mathbf{r'})\), so that Eq. (3.24) reduces to the well-known anticommutation relation for the quantized Dirac field.

### 3.3 Derivation of the Hamiltonian

The classical Hamiltonian on the hyperplane \(\sigma_{n,\tau}\) can be derived in two ways. Following the canonical procedure, \(H^\tau (n)\) is obtained by the Legendre transformation
\[
H(n) = \int_{\sigma_{n,\tau}} d\sigma \left\{ \Pi_{\perp\mu} \hat{A}_\perp^\mu + \bar{\pi} \dot{\bar{\psi}} + \dot{\psi} \pi - \mathcal{L} \right\}, \tag{3.27}
\]
where \(\mathcal{L}\) is given by Eq. (3.18). To find explicit expressions for the variables \(\pi\) and \(\bar{\pi}\), which are conjugates to the fields \(\bar{\psi}\) and \(\psi\), we rewrite the Dirac Lagrangian density (3.2) using the decomposition (2.20) of derivatives:
\[
\mathcal{L}_D = \bar{\psi} \left[ \frac{i}{2} \left( \gamma_\parallel \frac{\partial}{\partial \tau} + \gamma_\perp^\nu \nabla_\mu \right) - m \right] \psi. \tag{3.28}
\]
Then we have
\[ \bar{\pi} \equiv \frac{\partial L_D}{\partial \dot{\psi}} = i \frac{\bar{\psi}}{2} \gamma_4, \quad \pi \equiv \frac{\partial L_D}{\partial \dot{\bar{\psi}}} = -i \frac{\bar{\psi}}{2} \gamma_4. \] (3.29)

Substituting expressions (3.28) and (3.29) into Eq. (3.27), we arrive at the classical Hamiltonian. Another way is to start from the classical analog of Eq. (2.7) which reads
\[ H(n) = P_\mu n^\mu \equiv \int_{\sigma_{n,\tau}} d\sigma n^\mu T_{\mu\nu}, \] (3.30)

where \( T_{\mu\nu}(x) \) is the energy-momentum tensor. In order that the quantized Hamiltonian be hermitian, the energy-momentum tensor must be real. For instance, one can use the so-called Belinfante tensor [20]. When applied to the Lagrangian (3.1), the standard derivation of the Belinfante tensor (see, e.g., [17]) gives
\[ T_{\mu\nu}(x) = -g_{\mu\nu} \left\{ \bar{\psi} \left( i \frac{\gamma^\lambda}{2} \frac{\partial}{\partial x^\lambda} \frac{\gamma^\mu}{\partial x^\mu} - m \right) \psi - j_\lambda \left( A^\lambda + A^\lambda_{\text{ext}} \right) - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right\} \]
\[ + F_{\mu\lambda} F^\lambda_{\nu} + \frac{i}{4} \bar{\psi} \left( \gamma_\nu \frac{\partial}{\partial x^\mu} + \gamma_\mu \frac{\partial}{\partial x^\nu} \right) \psi - \frac{1}{2} \left( j_\nu A^\mu + j_\mu A^\nu \right). \] (3.31)

Separating the longitudinal and transverse components with respect to the normal vector \( n^\mu \) and then eliminating the \( \tau \)-derivatives of the fields with the aid of Eqs. (3.20) and (3.29), the classical Hamiltonian on the hyperplane is obtained from Eq. (3.30). It can be verified that in both cases we have the same expression for \( H^r(n) \). The final step is to replace the canonical variables \( A^\mu, \Pi^\mu, \bar{\psi}, \psi \) by the corresponding quantum operators. As a result, we find the Hamiltonian in the form
\[ \hat{H}^r(n) = \hat{H}_D(n) + \hat{H}_E^M(n) + \hat{H}_\text{int}(n) + \hat{H}_\text{ext}^r(n), \] (3.32)

where \( \hat{H}_D(n) \) and \( \hat{H}_E^M(n) \) are the Hamiltonians for free fermions and the polarization EM field respectively, \( \hat{H}_\text{int}(n) \) is the interaction term, and \( \hat{H}_\text{ext}^r(n) \) describes the external EM field effects. In the Schrödinger picture the explicit expressions for these terms are
\[ \hat{H}_D(n) = \int_{\sigma_n} d\sigma \hat{\psi} \left( -\frac{i}{2} \gamma^\mu(n) \frac{\overset{\leftrightarrow}{\partial}}{\partial x^\mu} + m \right) \hat{\psi}, \] (3.33)
\[ \hat{H}_E^M(n) = \int_{\sigma_n} d\sigma \left( \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} - \frac{1}{2} \hat{\Pi}_{\mu\nu} \hat{\Pi}^{\mu\nu} \right), \] (3.34)
\begin{align}
\hat{H}_{\text{int}}(n) &= \int d\sigma \hat{j}_{\mu\perp} \hat{A}_\perp^\mu + \frac{1}{2} \int d\sigma \int d\sigma' \hat{j}_{\parallel}(x_{\perp}) G(x_{\perp} - x'_{\perp}) \hat{j}_{\parallel}(x'_{\perp}), \quad (3.35) \\
\hat{H}_{\text{ext}}^\tau(n) &= \int d\sigma \hat{j}_\mu(x_{\perp}) A_\mu^\mu_{\text{ext}}(\tau, x_{\perp}). \quad (3.36)
\end{align}

In the EM field Hamiltonian (3.34) the tensor operator

\begin{equation}
\hat{F}^{\mu\nu}_\perp = \nabla^\mu \hat{A}^\nu_\perp - \nabla^\nu \hat{A}^\mu_\perp \quad (3.37)
\end{equation}

contains only the transverse components of the field operators \( \hat{A}^\mu \) which are decomposed as

\begin{equation}
\hat{A}^\mu = n^\mu \hat{A}^\parallel + \hat{A}^\mu_\perp. \quad (3.38)
\end{equation}

The longitudinal part \( \hat{A}^\parallel \) has been eliminated in the interaction Hamiltonian (3.35) by the equation

\begin{equation}
\nabla_\mu \nabla^\mu \hat{A}^\parallel = \hat{j}^\parallel \quad (3.39)
\end{equation}

which is analogous to Eq. (3.13). As usual, in Eqs. (3.33)–(3.36) normal ordering in operators is implied. The self-energy contribution to the last term in Eq. (3.35) is omitted, so that the product : \( \hat{j}^\parallel(x_{\perp}) :: \hat{j}^\parallel(x'_{\perp}) : \) is understood. For simplicity, we have written the Hamiltonian for the case that the fermionic subsystem is described by one Dirac field. The generalization to a many-component case is obvious.

4 The condensate mode of the EM field

An essential feature of the dynamical evolution of QED plasmas in a strong external field is that the mean values of the canonical operators\(^4\), \( A^\mu_\perp = \langle \hat{A}^\mu_\perp \rangle \) and \( \Pi^\mu_\perp = \langle \hat{\Pi}^\mu_\perp \rangle \), just as the mean values of creation and annihilation bosonic operators \( \hat{a} \) and \( \hat{a}^\dagger \) related to the canonical operators by plane-wave expansions, are not zero. Furthermore, they are macroscopic quantities associated with the mean EM field induced by the polarization in the system. In the language of statistical mechanics, the variables \( A^\mu_\perp(x) \) and \( \Pi^\mu_\perp(x) \) describe a macroscopic \textit{condensate mode} of the EM field. This fact does not allow to apply perturbation theory directly to the Hamiltonian (3.32) because the interaction of fermions with the condensate mode or, what is the same, with the

\(^4\) From this point onwards symbols \( A^\mu_\perp, \Pi^\mu_\perp, j^\mu, \) etc. denote mean values of the corresponding operators.
mean EM field is not weak. So, we have to separate the condensate mode from
the photon degrees of freedom in the Hamiltonian and the statistical operator.

4.1 The time-dependent unitary transformation

The condensate mode is most easily isolated by introducing the \( \tau \)-dependent
unitary transformation

\[
\tilde{\rho}(n, \tau) = e^{i\tilde{C}(n, \tau)} \rho(n, \tau) e^{-i\tilde{C}(n, \tau)},
\]

(4.1)

where the operator \( \tilde{C}(n, \tau) \) is given by

\[
\tilde{C}(n, \tau) = \int d\sigma \left\{ A^\mu_\perp(x_\perp) \Pi^\mu_\perp(x_\perp) - \Pi^\mu_\perp(x_\perp) A^\mu_\perp(x_\perp) \right\}.
\]

(4.2)

Note that the unitary transformation (4.1) does not affect fermionic operators
and has the properties

\[
e^{i\tilde{C}(n, \tau)} A^\mu_\perp(x_\perp) e^{-i\tilde{C}(n, \tau)} = A^\mu_\perp(x_\perp) + A^\mu_\perp(x),
\]

\[
e^{i\tilde{C}(n, \tau)} \Pi^\mu_\perp(x_\perp) e^{-i\tilde{C}(n, \tau)} = \Pi^\mu_\perp(x_\perp) + \Pi^\mu_\perp(x).
\]

(4.3)

Taking now into account that, for any operator \( \hat{O} \),

\[
\langle \hat{O} \rangle^\tau = \text{Tr} \left\{ e^{i\tilde{C}(n, \tau)} \hat{O} e^{-i\tilde{C}(n, \tau)} \tilde{\rho}(n, \tau) \right\} \equiv \left\langle e^{i\tilde{C}(n, \tau)} \hat{O} e^{-i\tilde{C}(n, \tau)} \right\rangle^\tau_{\tilde{\rho}},
\]

(4.4)

we find that

\[
\left\langle A^\mu_\perp(x_\perp) \right\rangle^\tau_{\tilde{\rho}} = \left\langle \Pi^\mu_\perp(x_\perp) \right\rangle^\tau_{\tilde{\rho}} = 0.
\]

(4.5)

Thus, in the state described by the transformed statistical operator (4.1), the
canonical dynamical variables \( \hat{A}^\mu_\perp \) and \( \hat{\Pi}^\mu_\perp \) have zero mean values and, hence,
they are not related to the condensate mode. In other words, after the unitary
transformation the EM field operators correspond to the photon degrees of
freedom. Based on the above arguments, it is convenient to use \( \tilde{\rho}(n, \tau) \) as the
statistical operator of the system. It should be noted, however, that \( \tilde{\rho}(n, \tau) \)
does not satisfy the von Neumann equation (2.8) since the operator \( \tilde{C} \) depends
on \( \tau \). In order to derive the equation of motion for \( \tilde{\rho}(n, \tau) \), we differentiate
Eq. (4.1) with respect to $\tau$. After some algebra which we omit, we find that the transformed statistical operator satisfies the modified von Neumann equation

$$\frac{\partial \tilde{\rho}(n, \tau)}{\partial \tau} - i \left[ \tilde{\rho}(n, \tau), \hat{H}^\tau(n) \right] = 0 \quad (4.6)$$

with the effective Hamiltonian

$$\hat{H}^\tau(n) = e^{i\hat{C}(n, \tau)} \hat{H}^\tau(n) e^{-i\hat{C}(n, \tau)}$$

$$+ \int_{\sigma_n} d\sigma \left\{ \frac{\partial \Pi_{\perp\mu}(x)}{\partial \tau} \hat{A}_{\perp\mu}(x_{\perp}) - \frac{\partial A_{\perp\mu}(x)}{\partial \tau} \hat{\Pi}_{\perp\mu}(x_{\perp}) \right\}. \quad (4.7)$$

The transformation of $\hat{H}(n)$ in the first term is trivial due to Eqs. (4.3) and the fact that the transformation does not affect fermionic operators. It is convenient to eliminate the derivatives in the last term of Eq. (4.7) with the aid of the equations of motion for the condensate mode

$$\frac{\partial A_{\perp\mu}(x)}{\partial \tau} = -\Pi_{\perp\mu}(x), \quad \frac{\partial \Pi_{\perp\mu}(x)}{\partial \tau} = \nabla_\lambda F_{\perp\lambda\mu}(x_{\perp}) - j_{\perp\mu}(x), \quad (4.8)$$

which are easily derived using Eqs. (2.23), (2.24), and the canonical commutation relations (3.21). The tensor $F_{\mu\nu}^{\perp}$ in Eq. (4.8) is the mean value of the operator (3.37), and $j_{\perp\mu}(x)$ is the transverse part of the mean polarization current

$$j^\mu(x) = \langle \hat{j}^\mu(x_{\perp}) \rangle^\tau. \quad (4.9)$$

Inserting Eqs. (4.8) into Eq. (4.7), the effective Hamiltonian can be written as a sum

$$\hat{H}^\tau(n) = \hat{H}^\tau_0(n) + \hat{H}^\tau_{\text{int}}(n). \quad (4.10)$$

The main term

$$\hat{H}^\tau_0(n) = \hat{H}_D(n) + \hat{H}_{EM} + \int_{\sigma_n} d\sigma \hat{j}_\mu(x_{\perp}) A^\mu(x) \quad (4.11)$$

describes free photons and fermions interacting with the total electromagnetic field

$$A^\mu(x) = A^\mu_{\text{ext}}(x) + A^\mu(x). \quad (4.12)$$
where the mean polarization field $A^\mu(x)$ is given by
\[ A^\mu(x) = \langle \hat{A}^\mu(x_\perp) \rangle^\tau. \] (4.13)

The term $\hat{\mathcal{H}}_{\text{int}}^\tau(n)$ in Eq. (4.10) describes a weak interaction between photons and fermions. The explicit expression for this term is
\[ \hat{\mathcal{H}}_{\text{int}}^\tau(n) = \int_{\sigma_n} d\sigma \Delta \hat{j}_1^\mu(x_\perp; \tau) \hat{A}_1^\mu(x_\perp) \]
\[ + \frac{1}{2} \int_{\sigma_n} d\sigma \int_{\sigma_n} d\sigma' \Delta \hat{j}_1^\mu(x_\perp; \tau) G(x_\perp - x'_\perp) \Delta \hat{j}_1^\mu(x'_\perp; \tau), \] (4.14)

where the operators
\[ \Delta \hat{j}_1^\mu(x_\perp; \tau) = \hat{j}_1^\mu(x_\perp) - \langle \hat{j}_1^\mu(x_\perp) \rangle^\tau \] (4.15)

represent quantum fluctuations of the fermionic current. The essential point is that now the interaction term (4.14) does not contain a contribution from the condensate mode and, consequently, one can use perturbation expansions in the fine structure constant.

4.2 Maxwell equations

To complete our discussion of the condensate mode, we will show how the Maxwell equations for the total mean EM are derived in our approach. According to Eq. (4.12), the total field tensor can be written as
\[ F_{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = F_{\text{ext}}^{\mu\nu}(x) + F^{\mu\nu}(x). \] (4.16)

The external field tensor $F_{\text{ext}}^{\mu\nu}$ is assumed to satisfy the Maxwell equations
\[ \partial_\mu F^{\mu\nu}_{\text{ext}}(x) = j^\nu_{\text{ext}}(x) \] (4.17)

with some prescribed external current $j^\mu_{\text{ext}}$. On the other hand, the polarization field tensor $F^{\mu\nu}(x)$ is the mean value of the operator
\[ \hat{F}^{\mu\nu}(x_\perp) = \partial^\mu \hat{A}^\nu - \partial^\nu \hat{A}^\mu \]
\[ = \hat{F}_\perp^{\mu\nu} + n^\nu (\hat{\Pi}_\perp^{\mu} + \nabla^\mu \hat{A}_\parallel) - n^\mu (\hat{\Pi}_\perp^{\nu} + \nabla^\nu \hat{A}_\parallel). \] (4.18)
Recalling Eqs. (2.23) and (2.24), straightforward algebraic manipulations with the equations of motion for the field operators lead to the Maxwell equations for the polarization field tensor

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x). \quad (4.19)$$

Now Eqs. (4.17) and (4.19) can be combined into the Maxwell equations for the total field tensor

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x) + j_{\text{ext}}^\nu(x). \quad (4.20)$$

A solution of these equations gives the total mean field $A^\mu$ in terms of the total mean current.

5 Kinetic description of QED plasmas

5.1 The “one-time” Wigner function

Within the hyperplane formalism a natural way of describing kinetic processes in the fermion subsystem is by the “one-time” Wigner function which depends on the variable $\tau$. Since there is the gauge freedom for the mean field $A^\mu$, it is convenient to employ the gauge-invariant Wigner function on the hyperplane $\sigma_{n,\tau}$ defined as

$$W_{aa'}(x_\perp, p_\perp; \tau) = \int d^4y e^{ip\cdot y} \delta(y \cdot n) \times \exp \left\{ ie\Lambda(x_\perp + \frac{1}{2}y_\perp, x_\perp - \frac{1}{2}y_\perp; \tau) \right\} \rho_{aa'} \left( x_\perp + \frac{1}{2}y_\perp, x_\perp - \frac{1}{2}y_\perp, \tau \right) \quad (5.1)$$

with the gauge function

$$\Lambda(x_\perp, x_\perp'; \tau) = \int_{x_\perp'}^{x_\perp} \mathcal{A}_{\perp\mu}(\tau, \mathcal{R}_\perp) dR^\mu_\perp \equiv \int_0^1 ds (x^\mu_\perp - x'^\mu_\perp) \mathcal{A}_{\perp\mu}(\tau, x'_\perp + s(x_\perp - x'_\perp)). \quad (5.2)$$

In Eq. (5.1) the one-particle density matrix $\rho_{aa'}$ is the mean value

$$\rho_{aa'}(x_\perp, x_\perp'; \tau) = \langle \hat{\rho}_{aa'}(x_\perp, x_\perp') \rangle^\tau = \langle \hat{\rho}_{aa'}(x_\perp, x_\perp') \rangle^\tau_{\tilde{E}} \quad (5.3)$$
of some density operator $\hat{\rho}_{aa'}$. In the literature one can find different definitions for the fermionic density operator. The most often used definitions are

$$\hat{\rho}_{aa'}(x_\perp, x'_\perp) = -\frac{1}{2}[\hat{\psi}_a(x_\perp), \hat{\psi}_{a'}(x'_\perp)], \quad (5.4)$$

$$\hat{\rho}'_{aa'}(x_\perp, x'_\perp) = :\hat{\psi}_{a'}(x'_\perp) \hat{\psi}_a(x_\perp):. \quad (5.5)$$

These two operators are related by

$$\hat{\rho}_{aa'}(x_\perp, x'_\perp) = \hat{\rho}'_{aa'}(x_\perp, x'_\perp) + K_{aa'}(x_\perp, x'_\perp), \quad (5.6)$$

where the last c-number term represents the vacuum expectation value of $\hat{\rho}$ since the vacuum expectation value of $\hat{\rho}'$ is zero. It can be shown, however, that the vacuum term in Eq. (5.6) does not contribute to local observables like the mean current $j^\mu(x)$. The advantage of the definition (5.5) is that the mean values of one-particle dynamical variables (summation over repeated spinor indices)

$$\langle \hat{O} \rangle = \int d\sigma d\sigma' O_{aa'}(x'_\perp, x_\perp) :\hat{\psi}_{a'}(x'_\perp) \hat{\psi}_a(x_\perp):, \quad (5.7)$$

are conveniently expressed in terms of the density matrix $\rho' = \langle \hat{\rho}' \rangle^\tau$:

$$\langle \hat{O} \rangle^\tau = \int d\sigma d\sigma' O_{aa'}(x'_\perp, x_\perp) \rho'_{aa'}(x_\perp, x'_\perp; \tau). \quad (5.8)$$

Unfortunately, the equation of motion for the density operator (5.5) with the Hamiltonian (4.10) contains vacuum (divergent) terms. On the other hand, such terms do not appear in the equation of motion for the operator (5.4). For this reason, we shall take the operator (5.4) as the one-particle density operator in Eq. (5.3). An analogous definition was used previously for the phase-space description of the QED vacuum in a strong field [7].

The Wigner function (5.1) is defined on a given family of hyperplanes $\sigma_{n,\tau}$ and, hence, depends parametrically on the normal four-vector $n$. It should be noted, however, that local observables calculated from the Wigner function do not depend on the choice of $n$. As an important example, the mean polarization current (4.9) can be written in the form

$$j^\mu(x) = e \langle :\hat{\psi}(x_\perp) \gamma^\mu \hat{\psi}(x_\perp): \rangle^\tau$$

$$= e \int \frac{d^4p}{(2\pi)^3} \delta(p \cdot n) \text{tr} \left\{ \gamma^\mu W(x_\perp, p_\perp; \tau = x \cdot n) \right\}, \quad (5.9)$$
where the symbol “tr” stands for the trace over spinor indices. Geometrically, the above relation means that, in calculating the current, the invariant time \( \tau \) has a value such that the space-time point \( x \) lies on the hyperplane \( \sigma_{n,\tau} \).

### 5.2 The photon density matrix

To define the photon density matrix, we start from the plane wave expansion of the vector potential operator \( \hat{A}_\perp \) in terms of creation and annihilation operators. By analogy with the well-known representation for the free photon field in the special Lorentz frame where \( n^\mu = (1, 0, 0, 0) \), we write

\[
\hat{A}_\perp^\mu(\tau, x_\perp) = \int \frac{d^4q}{\sqrt{2\omega_n(q_\perp)(2\pi)^3}} \delta\left(q_\parallel - \omega_n(q_\perp)\right)
\times \sum_{l=1,2} e^{\mu}(q_\perp, l) \left\{ a_l(q_\perp) e^{-iq_\cdot x} + a_l^\dagger(q_\perp) e^{iq_\cdot x}\right\},
\]

(5.10)

where \( e^{\mu}(q_\perp, l) \) are real-valued polarization four-vectors and

\[
\omega_n(q_\perp) = \omega_n(-q_\perp) = \left(-q_\perp^\mu q_\perp^\mu\right)^{1/2}
\]

(5.11)

is the dispersion relation for free photons on the hyperplane. The conditions \( \nabla_\mu \hat{A}_\perp^\mu = n_\mu \hat{A}_\perp^\mu = 0 \) mean that the polarization vectors satisfy

\[
q_\perp^\mu e^{\mu}(q_\perp, l) = n_\mu e^{\mu}(q_\perp, l) = 0.
\]

(5.12)

The expansion of the operator \( \hat{\Pi}_\perp^\mu \) into plane waves is found from (5.10) by using \( \hat{\Pi}_\perp^\mu = -\hat{A}_\perp^\mu \): 

\[
\hat{\Pi}_\perp^\mu(\tau, x_\perp) = \int \frac{d^4q}{\sqrt{2\omega_n(q_\perp)(2\pi)^3}} i\omega_n(q_\perp) \delta\left(q_\parallel - \omega_n(q_\perp)\right)
\times \sum_{l=1,2} e^{\mu}(q_\perp, l) \left\{ a_l(q_\perp) e^{-iq_\cdot x} - a_l^\dagger(q_\perp) e^{iq_\cdot x}\right\}.
\]

(5.13)

Assuming the commutation relations for the creation and annihilation operators

\[
[\hat{a}_l(q_\perp), \hat{a}_l^\dagger(q'_\perp)] = \delta_{ll'} \delta^3(q_\perp - q'_\perp),
\]

\[
[\hat{a}_l(q_\perp), \hat{a}_l(q'_\perp)] = [\hat{a}_l^\dagger(q_\perp), \hat{a}_l^\dagger(q'_\perp)] = 0,
\]

(5.14)
and the completeness relation for the polarization vectors

\[ \sum_{l=1,2} e^{\mu}(q_\perp, l) e^{\nu}(q_\perp, l) = -\left( \Delta^{\mu\nu} - \frac{q_\perp^{\mu} q_\perp^{\nu}}{q_\perp^2} \right), \]  \hspace{1cm} (5.15)

the commutation relation (3.21) for the field operators is recovered. Note that Eqs. (5.10) and (5.13) give the field operators in the interaction picture. The corresponding expansions for the field operators in the Schrödinger picture are obtained by setting \( \tau = 0 \). In this case the delta-function \( \delta(q_\parallel - \omega_n q_\perp) \) can be replaced by \( \delta(q_\parallel) \).

The above considerations suggest that it is natural to define the photon density matrix in terms of the Schrödinger operators

\[ \hat{\phi}_l(x_\perp) = \int \frac{d^4q}{(2\pi)^3/2} \delta(q_\parallel) e^{-iq_\perp \cdot x} \hat{a}_l(q_\perp), \]
\[ \hat{\phi}_l^\dagger(x_\perp) = \int \frac{d^4q}{(2\pi)^3/2} \delta(q_\parallel) e^{iq_\perp \cdot x} \hat{a}_l^\dagger(q_\perp), \]  \hspace{1cm} (5.16)

which satisfy the commutation relations

\[ [\hat{\phi}_l(x_\perp), \hat{\phi}_{l'}(x'_\perp)] = \delta_{ll'} \delta^3(x_\perp - x'_\perp), \]
\[ [\hat{\phi}_l(x_\perp), \hat{\phi}_l(x'_\perp)] = [\hat{\phi}_{l'}(x_\perp), \hat{\phi}_{l'}(x'_\perp)] = 0. \]  \hspace{1cm} (5.17)

The photon density matrix is defined as

\[ N_{ll'}(x_\perp, x'_\perp; \tau) = \langle \hat{N}_{ll'}(x_\perp, x'_\perp) \rangle_\theta^\tau, \]  \hspace{1cm} (5.18)

where

\[ \hat{N}_{ll'}(x_\perp, x'_\perp) = \hat{\phi}_{l'}^\dagger(x'_\perp) \hat{\phi}_l(x_\perp) \]  \hspace{1cm} (5.19)

is the photon density operator. It should be emphasized that in Eq. (5.18) the average is calculated with the transformed statistical operator \( \hat{\theta}(n, \tau) \) in which the condensate mode of EM field has been eliminated. When written in terms of the average with the statistical operator \( \theta(n, \tau) \), the photon density matrix takes the form

\[ N_{ll'}(x_\perp, x'_\perp; \tau) = \langle \hat{N}_{ll'}(x_\perp, x'_\perp) \rangle_\theta^\tau - \langle \hat{\phi}_l(x_\perp) \rangle_\theta^\tau \langle \hat{\phi}_{l'}^\dagger(x'_\perp) \rangle_\theta^\tau, \]  \hspace{1cm} (5.20)

where the last term corresponds to the contribution from the condensate mode.
5.3 The covariant statistical operator in QED kinetics

The evolution of the fermionic Wigner function (5.1) and the photon density matrix (5.18) is governed by kinetic equations which can be derived from the equations of motions

\[
\frac{\partial}{\partial \tau} \rho_{aa'}(x_{\perp}, x'_{\perp}; \tau) = -i \text{Tr} \left\{ [\hat{\rho}_{aa'}(x_{\perp}, x'_{\perp}), \hat{H}_\delta^r(n) + \hat{H}_{\text{int}}^r(n)] \bar{\rho}(n, \tau) \right\}, \quad (5.21)
\]

\[
\frac{\partial}{\partial \tau} N_{ll'}(x_{\perp}, x'_{\perp}; \tau) = -i \text{Tr} \left\{ [\hat{N}_{ll'}(x_{\perp}, x'_{\perp}), \hat{H}_\delta^r(n) + \hat{H}_{\text{int}}^r(n)] \bar{\rho}(n, \tau) \right\}. \quad (5.22)
\]

There are two ways to express the right-hand sides of these equations in terms of the fermionic and photon density matrices using perturbation expansions in the fine structure constant. One method is by considering the hierarchy for correlation functions which appear through the commutators with the interaction Hamiltonian \( \hat{H}_{\text{int}}^r(n) \) and then employing some truncation procedure. Another method is to construct an approximate solution of Eq. (4.6) in terms of the density matrices \( \rho \) and \( N \). In both cases one has to impose some boundary conditions of the retarded type on the correlation functions or the statistical operator. The standard boundary condition in kinetic theory is Bogoliubov’s boundary condition of weakening of initial correlations which implies the uncoupling of all correlation functions to one-particle density matrices in the distant past, i.e., for \( \tau \to -\infty \). In the scheme developed by Zubarev (see, e.g., [21]), such boundary conditions can be included by using instead of Eq. (4.6) the equation with an infinitesimally small source term

\[
\frac{\partial \bar{\rho}(n, \tau)}{\partial \tau} - i \left[ \bar{\rho}(n, \tau), \hat{H}_\delta^r(n) \right] = -\varepsilon \left\{ \bar{\rho}(n, \tau) - \rho_{\text{rel}}(n, \tau) \right\}, \quad (5.23)
\]

where \( \varepsilon \to +0 \) after the calculation of averages. Here \( \rho_{\text{rel}}(n, \tau) \) is the so-called relevant statistical operator which describes a Gibbs state for some given nonequilibrium state variables. In QED kinetics these variables are the Wigner function (5.1) and the photon density matrix (5.18). Therefore, following the standard procedure [21], we obtain the relevant statistical operator in the form (with summation over spinor and polarization indices)

\[
\rho_{\text{rel}}(n, \tau) = \frac{1}{Z_{\text{rel}}(n, \tau)} \exp \left\{ - \int \frac{d\sigma}{\sigma_n} \int d\sigma' \left[ \lambda^{(f)}_{aa'}(x_{\perp}, x'_{\perp}; \tau) : \hat{\psi}_a(x_{\perp}) \hat{\psi}_a(x'_{\perp}) : 
\right.
\]

\[
+ \lambda^{(ph)}_{ll'}(x_{\perp}, x'_{\perp}; \tau) : \hat{\varphi}_l(x_{\perp}) \hat{\varphi}_l(x'_{\perp}) : \right\}, \quad (5.24)
\]
where $Z_{\text{rel}}(n, \tau)$ is the normalization constant (or the partition function in the relevant ensemble) and $\lambda^{(f)}(x, x'; \tau)$, $\lambda^{(ph)}(x, x'; \tau)$ are Lagrange multipliers which are determined by the self-consistency conditions

$$
\rho_{aa'}(x, x'; \tau) = \text{Tr} \left\{ \hat{\rho}_{aa'}(x, x') \hat{\rho}_{\text{rel}}(n, \tau) \right\},
$$

$$
N_{ll'}(x, x'; \tau) = \text{Tr} \left\{ \hat{N}_{ll'}(x, x') \hat{\rho}_{\text{rel}}(n, \tau) \right\}.
$$

Using Eq. (5.23) for the transformed statistical operator leads to the hierarchy

$$
\frac{\partial}{\partial \tau} \bar{F}_{1..k}(x_{1\perp}, \ldots, x_{k\perp}; n, \tau) = -i \left\langle [\hat{O}_1(x_{1\perp}), \ldots, \hat{O}_k(x_{k\perp}), \hat{H}^\tau(n)] \right\rangle_{\hat{\rho}}^\tau
$$

$$
- \varepsilon \left\{ \bar{F}_{1..k}(x_{1\perp}, \ldots, x_{k\perp}; n, \tau) - \left\langle \hat{O}_1(x_{1\perp}), \ldots, \hat{O}_k(x_{k\perp}) \right\rangle_{\hat{\rho}_{\text{rel}}}^\tau \right\},
$$

(5.26)

where $\hat{O}_i(x_{i\perp})$ are some Schrödinger operators which may depend on the fermion operators as well as on the EM operators, and

$$
\bar{F}_{1..k}(x_{1\perp}, \ldots, x_{k\perp}; n, \tau) = \left\langle \hat{O}_1(x_{1\perp}), \ldots, \hat{O}_k(x_{k\perp}) \right\rangle_{\hat{\rho}}^\tau
$$

(5.27)

are the “equal-time” correlation functions in which the condensate mode of the EM field is eliminated. Since the relevant statistical operator (5.24) admits Wick’s decomposition, the last term in Eq. (5.26) ensures the boundary condition of complete weakening of initial correlations. Note that the explicit knowledge of the statistical operator $\hat{\rho}(n, \tau)$ is not needed when considering the hierarchy for the correlation functions. Use of some truncation procedure is a standard practice in this case. The hierarchy for correlation functions will be discussed in subsequent papers in the context of the derivation of collision integrals.

Another method of handling Eq. (5.23) is by considering its formal solution

$$
\bar{\rho}(n, \tau) = \varepsilon \int_{-\infty}^\tau d\tau' e^{-\varepsilon(\tau-\tau')} U(\tau, \tau') \rho_{\text{rel}}(n, \tau') U^\dagger(\tau, \tau'),
$$

(5.28)

where the evolution operator can be written as the ordered exponent

$$
U(\tau, \tau') = T_{\tau} \exp \left\{ -i \int_{\tau'}^\tau \hat{H}^\tau(n) d\tau' \right\}.
$$

(5.29)
After partial integration, the expression (5.28) becomes
\[ \tilde{\varrho}(n, \tau) = \varrho_{\text{rel}}(n, \tau) + \Delta \varrho(n, \tau), \] (5.30)

where
\[ \Delta \varrho(n, \tau) = -\int_{-\infty}^{\tau} d\tau' e^{-\varepsilon(\tau-\tau')} \times U(\tau, \tau') \left\{ \frac{\partial \varrho_{\text{rel}}(n, \tau')}{\partial \tau'} - i \left[ \varrho_{\text{rel}}(n, \tau'), \hat{H}^\tau(n) \right] \right\} U^\dagger(\tau, \tau'). \] (5.31)

The representation (5.30) for the statistical operator allows to separate the mean-field terms and the collision terms in Eqs. (5.21) and (5.22). Taking into account the self-consistency conditions (5.25) and the fact that the Hamiltonian (4.11) is bilinear in the fermion and photon operators, we arrive at the equations
\[ \frac{\partial}{\partial \tau} \rho_{aa'}(x_\perp, x'_\perp; \tau) = -i \left\langle [\hat{\rho}_{aa'}(x_\perp, x'_\perp), \hat{H}^\tau_0(n)] \right\rangle_{\text{rel}}^{\tau} + I_{aa'}^{(f)}(x_\perp, x'_\perp; \tau), \] (5.32)
\[ \frac{\partial}{\partial \tau} N_{ll'}(x_\perp, x'_\perp; \tau) = -i \left\langle [\hat{N}_{ll'}(x_\perp, x'_\perp), \hat{H}_{\text{EM}}^\tau(n)] \right\rangle_{\text{rel}}^{\tau} + I_{ll'}^{(ph)}(x_\perp, x'_\perp; \tau), \] (5.33)

where the collision integrals for fermions and photons are given by
\[ I_{aa'}^{(f)}(x_\perp, x'_\perp; \tau) = -i \left\langle [\hat{\rho}_{aa'}(x_\perp, x'_\perp), \hat{H}_{\text{int}}^\tau(n)] \right\rangle_{\text{rel}}^{\tau} \]
\[ - i \text{Tr} \left\{ [\hat{\rho}_{aa'}(x_\perp, x'_\perp), \hat{H}_{\text{int}}^\tau(n)] \Delta \varrho(n, \tau) \right\}, \] (5.34)
\[ I_{ll'}^{(ph)}(x_\perp, x'_\perp; \tau) = -i \text{Tr} \left\{ [\hat{N}_{ll'}(x_\perp, x'_\perp), \hat{H}_{\text{int}}^\tau(n)] \Delta \varrho(n, \tau) \right\}. \] (5.35)

In the presence of a strong EM field, the evolution of the fermion subsystem is governed predominantly by its interaction with the mean EM field. Thus, the covariant mean-field kinetic equation for the Wigner function (5.1) can be derived from Eq. (5.32) neglecting the collision integral. This kinetic equation as well as the collision integrals (5.34) and (5.35) will be considered in subsequent papers.
6 Concluding remarks

We have shown that the hyperplane formalism can serve as the basis for kinetic theory of QED plasmas in the presence of a strong external field. The formalism has the advantage that it is manifestly covariant and therefore allows to introduce different approximations in covariant form. Only minor changes with respect to the non-relativistic density matrix method are introduced, so that many well-developed approaches can be directly applied to QED plasmas. For instance, the explicit construction of the statistical operator allows to incorporate many-particle correlations through the extension of the set of basic state parameters (see, e.g., [21]). Note also that, using the Heisenberg picture on hyperplanes, nonequilibrium Green’s functions can be introduced with respect to the invariant time parameter $\tau$. In such a way, the spectral properties of microscopic dynamics can be incorporated.

The scheme outlined in this paper is also applicable to other field theories, like QCD transport theory. In QCD, however, additional problems arise due to its non-Abelian structure, which needs further considerations.

Finally, we would like to emphasize once again two key problems in a covariant density matrix approach to relativistic kinetic theory in the presence of a strong mean field. First, it is necessary to perform canonical quantization of the system on a hyperplane in Minkowski space. Second, the condensate mode must be separated from the quantum degrees of freedom at any time. We have shown how these problems can be solved in the context of QED plasmas. As a result, a general form of kinetic equations for fermions and photons was given.

The scheme outlined in this paper is also applicable to some quantum field models used in QCD transport theory. In this case the non-Abelian algebra must be worked out to describe the quark-gluon plasma.

Appendix A

Commutation relations for electromagnetic field on hyperplanes

Let us write the constraint equations for the canonical variables $A^\mu_\perp$ and $\Pi^\mu_\perp$ in the form $\chi_N(x_\perp) = 0$, where

\begin{align}
\chi_1(x_\perp) &= \nabla_\mu A^\mu_\perp(x_\perp), \\
\chi_2(x_\perp) &= \nabla_\mu \Pi^\mu_\perp(x_\perp), \\
\chi_3(x_\perp) &= n_\mu A^\mu_\perp(x_\perp), \\
\chi_4(x_\perp) &= n_\mu \Pi^\mu_\perp(x_\perp).
\end{align}

(A.1)
For any functionals $\Phi_1$ and $\Phi_2$ of the field variables $A_\perp$ and $\Pi_\perp$, we define the Poisson bracket

$$[\Phi_1, \Phi_2]_P \equiv \int d\sigma \left\{ \frac{\delta \Phi_1}{\delta A_\mu^\perp(x_\perp)} \frac{\delta \Phi_2}{\delta \Pi_\mu(x_\perp)} - \frac{\delta \Phi_2}{\delta A_\mu^\perp(x_\perp)} \frac{\delta \Phi_1}{\delta \Pi_\mu(x_\perp)} \right\},$$

(A.2)

where the constraints are ignored in calculating the functional derivatives. Applying this formula to the canonical variables we obtain

$$[A_\mu^\perp(x_\perp), \Pi_\nu(x'_\perp)]_P = \delta^{\mu\nu} \delta^3(x_\perp - x'_\perp)$$

(A.3)

with the three-dimensional delta function (3.16). All other Poisson brackets for the canonical variables are equal to zero. In the Dirac terminology, functions (A.1) correspond to second class constraints since the matrix

$$C_{NN'}(x_\perp, x'_\perp) = [\chi_N(x_\perp), \chi_{N'}(x'_\perp)]_P$$

(A.4)

is non-singular. A straightforward calculation of the Poisson brackets shows that the non-zero elements of $C$ are

$$C_{12}(x_\perp, x'_\perp) = -C_{21}(x_\perp, x'_\perp) = -\nabla_\mu \nabla^\mu \delta^3(x_\perp - x'_\perp),$$

$$C_{34}(x_\perp, x'_\perp) = -C_{43}(x_\perp, x'_\perp) = \delta^3(x_\perp - x'_\perp).$$

(A.5)

According to the general quantization scheme [18,19], commutation relations for canonical operators are defined by the Dirac brackets for classical canonical variables. In our case the Dirac brackets are written as

$$[\Phi_1, \Phi_2]_D = [\Phi_1, \Phi_2]_P$$

$$- \int d\sigma \int d\sigma' [\Phi_1, \chi_N(x_\perp)]_P C^{-1}_{NN'}(x_\perp, x'_\perp) [\chi_{N'}(x'_\perp), \Phi_2]_P$$

(A.6)

(summation over repeated indices). The inverse matrix, $C^{-1}_{NN'}(x_\perp, x'_\perp)$, satisfies the equation

$$\int d\sigma'' C_{NN''}(x_\perp, x''_\perp) C^{-1}_{NN'}(x''_\perp, x'_\perp) = \delta_{NN'} \delta^3(x_\perp - x'_\perp).$$

(A.7)

Since the matrix elements (A.5) of $C$ depend on the difference $x_\perp - x'_\perp$, Eq. (A.7) can be solved for $C^{-1}$ using a Fourier transform on $\sigma_{n,\tau}$, which
is defined for any function $f(x)$ as

$$\tilde{f}(\tau, p_\perp) = \int d^4x \, e^{ip\cdot x} \delta(x \cdot n - \tau) f(x). \quad (A.8)$$

The inverse transform is

$$f(x) \equiv f(\tau, x_\perp) = \int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot x} \delta(p \cdot n) \tilde{f}(\tau, p_\perp). \quad (A.9)$$

If we perform the Fourier transformation in Eq. (A.7), we find by inserting (A.5) that the non-zero elements of $C^{-1}$ are

$$C_{12}^{-1}(x_\perp, x'_\perp) = -C_{21}^{-1}(x_\perp, x'_\perp) = -\int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot (x-x')} \delta(p \cdot n) \frac{1}{p^2_\perp},$$

$$C_{34}^{-1}(x_\perp, x'_\perp) = -C_{43}^{-1}(x_\perp, x'_\perp) = -\delta^3(x_\perp - x'_\perp). \quad (A.10)$$

Now the Dirac brackets (A.6) for the canonical variables are easily calculated and we obtain

$$[A^\mu_\perp(x_\perp), \Pi^\nu_\perp(x'_\perp)]_D = c^{\mu\nu}(x_\perp - x'_\perp), \quad (A.11)$$

$$[A^\mu_\perp(x_\perp), A^\nu_\perp(x'_\perp)]_D = [\Pi^\mu_\perp(x_\perp), \Pi^\nu_\perp(x'_\perp)]_D = 0, \quad (A.12)$$

where the functions $c^{\mu\nu}(x_\perp - x'_\perp)$ are given by Eq. (3.23). According to the general quantization rules, the commutation relations for canonical operators correspond to $[\ldots]_\mathrm{D}$. Thus, in the hyperplane formalism, the commutation relations for the operators of EM field are given by (3.21) and (3.22). Obviously these relations are valid in the Schrödinger and Heisenberg pictures.

**Appendix B**

**Anticommutation relations for the Dirac field on hyperplanes**

To find the anticommutation relations for the fermion operators on the hyperplane $\sigma_\sigma^{n, \tau}$, it is sufficient to consider a free Dirac field. Our starting point is the standard quantization scheme in the frame where $x^\mu = (t, r)$ and $n^\mu = (1, 0, 0, 0)$ (see, e.g., [17]). In that case the field operators $\hat{\psi}_a$ and $\hat{\bar{\psi}}_a$
can be written in terms of creation and annihilation operators according to

\[
\hat{\psi}_a(x) = \int \frac{d^4p}{(2\pi)^3/2} \frac{\delta(p^0 - \epsilon(p))}{\sqrt{2\epsilon(p)}} \sum_{s=\pm 1} \left[ \hat{b}_s(p)u_{as}(p) e^{-ip\cdot x} + \hat{d}_s^\dagger(p)\bar{v}_{as}(p) e^{ip\cdot x} \right],
\]

\[
\hat{\psi}_a^\dagger(x) = \int \frac{d^4p}{(2\pi)^3/2} \frac{\delta(p^0 - \epsilon(p))}{\sqrt{2\epsilon(p)}} \sum_{s=\pm 1} \left[ \hat{d}_s(p)\bar{v}_{as}(p) e^{-ip\cdot x} + \hat{b}_s^\dagger(p)u_{as}(p) e^{ip\cdot x} \right],
\]

where \( \epsilon(p) = \sqrt{p^2 + m^2} \) is the free fermion dispersion relation. Constructing the expression \( \{\hat{\psi}_a(x), \hat{\psi}_{a'}(x')\} \) for two arbitrary space-time points and recalling the anticommutation relations

\[
\{ \hat{b}_s(p), \hat{b}^\dagger_{s'}(p') \} = \{ \hat{d}_s(p), \hat{d}^\dagger_{s'}(p') \} = \delta_{ss'}\delta^3(\mathbf{p} - \mathbf{p}'),
\]

as well as polarization sums

\[
\sum_{s=\pm 1} u_{as}(p)\bar{u}_{a's}(p) = \left[ \gamma^\mu p_\mu + m \right]_{aa'}, \quad \sum_{s=\pm 1} v_{as}(p)\bar{v}_{a's}(p) = \left[ \gamma^\mu p_\mu - m \right]_{aa'},
\]

we arrive at

\[
\left\{ \hat{\psi}_a(x), \hat{\psi}_{a'}(x') \right\} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\epsilon(p)} \left\{ \left[ \gamma^\mu p_\mu + m \right]_{aa'} e^{-ip\cdot(x-x')} \right.
\]

\[
+ \left. \left[ \gamma^\mu p_\mu - m \right]_{aa'} e^{ip\cdot(x-x')} \right\},
\]

where \( p^0 = \sqrt{p^2 + m^2} \). Using

\[
\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\epsilon(p)} = \int \frac{d^4p}{(2\pi)^4} \frac{\delta(p^2 - m^2)}{\epsilon(p)} \bigg|_{p^0 > 0},
\]

Eq. (B.2) can be rewritten in a Lorentz invariant form

\[
\left\{ \hat{\psi}_a(x), \hat{\psi}_{a'}(x') \right\} = \int \frac{d^4p}{(2\pi)^4} \left\{ \left[ \gamma^\mu p_\mu + m \right]_{aa'} e^{-ip\cdot(x-x')} \delta(p^2 - m^2) \bigg|_{p^0 > 0} \right.
\]

\[
+ \left. \left[ \gamma^\mu p_\mu - m \right]_{aa'} e^{ip\cdot(x-x')} \delta(p^2 - m^2) \bigg|_{p^0 > 0} \right\}.
\]

The anticommutation relation on the hyperplane \( \sigma_{n,\tau} \) is now obtained by setting \( x = n\tau + x_\perp \) and \( x' = n\tau + x'_\perp \). In calculating the integrals, it is convenient to use the decomposition \( p^\mu = n^\mu p_\parallel + p_\parallel^\mu, (p_\parallel > 0) \). Then we get
\[
\left\{ \hat{\psi}_a(\tau, x_\perp), \hat{\psi}^{\dagger}_a(\tau, x'_\perp) \right\} = \int \frac{d^4p}{(2\pi)^3} \frac{\delta(p_\parallel - \epsilon(p_\perp))}{2\epsilon(p_\perp)} \times \left\{ \left[ \gamma_\parallel p_\parallel + \gamma_\perp^\mu p_\perp^\mu + m \right]_{a'\alpha} e^{-ip_\perp^\mu(x_\perp^\alpha - x'\perp^\alpha)} + \left[ \gamma_\parallel p_\parallel + \gamma_\perp^\mu p_\perp^\mu - m \right]_{\alpha'\alpha} e^{ip_\perp^\mu(x_\perp^\alpha - x'\perp^\alpha)} \right\} \tag{B.5}
\]

with the dispersion relation on the hyperplane
\[
\epsilon(p_\perp) = \sqrt{-p_\perp^\mu p_\perp^\mu + m^2}. \tag{B.6}
\]

Finally, changing the variable \(p_\perp \rightarrow -p_\perp\) in the second integral in Eq. (B.5), we obtain the anticommutation relation (3.24). The relations (3.25) can be derived by the same procedure.

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