SMOOTH DENSITIES OF STOCHASTIC DIFFERENTIAL EQUATIONS
FORCED BY DEGENERATE STABLE TYPE NOISES

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ABSTRACT. Using the Bismut’s approach to Malliavin calculus, we introduce a simplified Malliavin matrix ([11]) for stochastic differential equations (SDEs) forced by degenerate stable like noises. For the degenerate SDEs driven by Wiener noises, one can derive a Norris type lemma and use it iteratively to prove the smoothness of density functions. Unfortunately, Norris type lemma is very hard to be iteratively applied to SDEs with stable like noises. In this paper, we derive a simple inequality as a replacement and use it to show that two families of degenerate SDEs with stable like noises admit smooth density functions. One family is the linear SDEs studied by Priola and Zabczyk ([13]), under some additional assumption we can iteratively use the inequality to get the smoothness of the density. The other family is the general SDEs with stable like noises, we can apply this inequality only once and thus derive that the SDEs admit smooth density if the first order Lie brackets span \( \mathbb{R}^d \). The crucial step in this paper is estimating the smallest eigenvalue of the simplified Malliavin matrix, which only uses some elementary facts of Poisson processes and undergraduate level ordinary differential equations.

1. INTRODUCTION

We are concerned with smooth densities for the degenerate stochastic differential equations forced by stable like noises as follows:

\[
\begin{align*}
\text{d}X_t &= a(X_t)\text{d}t + B\text{d}L_t, \\
X_0 &= x,
\end{align*}
\]

where \( X_t \in \mathbb{R}^d \) for each \( t \geq 0 \), \( x \in \mathbb{R}^d \) and the hypotheses of \( A, B, L_t \) will be stated below. We shall introduce a simplified Malliavin matrix associated to Eq. (1.1) and use it to study the smoothness of the associated transition probability densities.

As \( a(x) \) is linear and the classical Kalman rank condition holds, Priola and Zabczyk proved by Fourier analysis that transition probabilities associated to Eq. (1.1) admit smooth densities ([13]) for a large family of \( L_t \). Under some additional assumptions on \( L_t \), our results give a new proof for theirs. When \( a(x) \) is a general bounded smooth function, we show that Eq. (1.1) admits smooth density functions as long as the first order Lie brackets span \( \mathbb{R}^d \). Our results seem to be completely new.

Let us also compare our results with some known results on Malliavin calculus on SDEs with jump processes. [1] studied integration by parts for the jump processes with their jumps depending on the particle positions. [9, 18] also studied the density smoothness of the transition probabilities of a family of SDEs forced by jump processes, which seems not to cover our results.
[19, 20] studied the same problems as ours for degenerate SDEs forced by symmetric $\alpha$-stable noises. For more research in this direction, we refer to [2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16].

Let us specify our method with more details as below. For the degenerate SDEs driven by Wiener noises, one can derive a Norris type lemma and use it iteratively to prove the smoothness of density functions. Unfortunately, Norris type lemma is very hard to be iteratively applied to SDEs with stable like noises. In this paper, we derive a simple (coercive) inequality as a replacement and use it to estimate the smallest eigenvalue of our simplified Malliavin matrix. For the linear SDEs studied by Priola and Zabczyk, under some additional assumption we can use this inequality iteratively to get the smoothness of the densities. For the general SDEs with stable like noises, we can apply this inequality only one time and thus derive that the SDEs admit smooth density if the first order Lie brackets span $\mathbb{R}^d$. The crucial step in this paper is estimating the smallest eigenvalue of the simplified Malliavin matrix, which only uses some elementary facts of Poisson processes and undergraduate level ordinary differential equations.

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2. Some preliminary of Lévy processes and main results

Denote $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$. Let $L_t$ be a pure jump process with càdlàg trajectories, it is well known that there exist a Poisson random measure $N$ on $(\mathbb{R}_0^d \times \mathbb{R}^+, \mathcal{B}(\mathbb{R}_0^d \times \mathbb{R}^+))$ and a Lévy intensity measure $\nu$ on $(\mathbb{R}_0^d, \mathcal{B}(\mathbb{R}_0^d))$ associated to $L_t$, such that

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}_0^d} (1 \wedge |z|^2) \nu(dz) < \infty;$$

(2.1) $$L_t = \int_0^t \int_{|z| \leq 1} z \tilde{N}(dz, ds) + \int_0^t \int_{|z| > 1} z N(dz, ds);$$

where $\tilde{N}(dz, ds) = N(dz, ds) - \nu(dz)ds$. It is well known that the random measure $N$ can be defined by: for all $A \in \mathcal{B}(\mathbb{R}_0^d)$

$$N(A \times [0, t]) = \sum_{0 \leq s \leq t} \# \{L_s - L_{s-} : L_s - L_{s-} \in A\}.$$}

Moreover, $N(A \times [0, t])$ satisfies a Poisson distribution with the intensity $\nu(A)t$, more precisely,

$$\mathbb{P}(N(A \times [0, t]) = k) = \frac{(\nu(A)t)^k}{k!} e^{-\nu(A)t} \quad k = 0, 1, 2, ....$$

We shall use this easy relation frequently in the proof of our crucial Lemma 4.3 below.

Throughout this paper we assume that

$$(H1) \quad \nu$$ has a density function $\rho \in C^1(\mathbb{R}_0^d, \mathbb{R}^+)$ and there exists some $\alpha \in (0, 2)$ such that

$$\rho(z) = \frac{\vartheta(z)}{|z|^{d+\alpha}} \quad \forall z \in B_1 \setminus \{0\},$$
where $B_1 \setminus \{0\} = \{z \in \mathbb{R}^d : |z| < 1\}$ and $\vartheta : B_1 \setminus \{0\} \to \mathbb{R}^+$ is a $C^1$ bounded function such that for all $z \in B_0 \setminus \{0\}$

$$c \leq |\nabla \vartheta(z)| \leq C, \quad c \leq \vartheta(z) \leq C$$

with some constants $C > c > 0$.

(H2) $a \in C^\infty_c(\mathbb{R}^d, \mathbb{R}^d)$ is a nonzero smooth function whose all derivatives are bounded. 

(H3) $B \in \mathbb{R}^{d \times d}$ is a constant matrix and $B_i$ is the $i$-th column vector of $B$ ($i = 1, \ldots, d$).

Our main results are the following two theorems.

**Theorem 2.1.** Let (H1) – (H3) all hold. Assume that there exists a nonzero matrix $A \in \mathbb{R}^{d \times d}$ such that

$$a(x) = Ax \quad \forall x \in \mathbb{R}^d.$$ 

Further assume that there exists some $n \in \mathbb{N}$ such that

$$\text{rank}[B, AB, \ldots, A^n B] = d.$$ 

Then, for all $t > 0$ the transition probability $P_t(\cdot, \cdot)$ associated to the solution of Eq. (1.1) $X_t(x)$ has a smooth density function.

**Theorem 2.2.** Let (H1) – (H3) all hold. Assume that the following uniform Hörmander condition holds:

$$\inf_{x \in \mathbb{R}^d} \inf_{|u| = 1} \sum_{i=1}^d \left( |\langle \nabla a(x) B_i, u \rangle|^2 + |\langle B_i, u \rangle|^2 \right) > 0.$$ 

Then, for all $t > 0$ the transition probability $P_t(\cdot, \cdot)$ associated to the solution of Eq. (1.1) $X_t(x)$ has a smooth density function.

Comparing with [13], our assumption in (H1) is more strict than the one therein:

$$\inf_{|h| = 1} \int_{|\langle \xi, h \rangle| \leq r} |\langle \xi, h \rangle| \nu(d\xi) \geq r^{2-\alpha}$$

for some sufficiently small $r > 0$.

Because the Skorohod integral (3.6) below includes some gradient, it seems the differentiability assumption in (H1) is needed. Our second theorem seems to be completely new comparing with the known results. We shall denote

$$|B| = \max_{1 \leq i \leq d} |B_i|.$$

3. **Integration by parts formula and simplified Malliavin matrix for jump Lévy processes**

Denote the solution of Eq. (1.1) by $(X_t(x, L))_{t \geq 0}$, it is a functional of $x$ and $L$. For any $\xi \in \mathbb{R}^d$ it is well known that the derivative $\nabla_\xi X_t$ satisfies

$$d\nabla_\xi X_t = \nabla a(X_t) \nabla_\xi X_t dt, \quad \nabla_\xi X_0 = \xi.$$ 

There exists a Jacobi flow $J_t$ associated to Eq. (1.1) such that

$$dJ_t = \nabla a(X_t) J_t dt, \quad J_0 = I.$$ 

Clearly we have

$$\nabla_\xi X_t = J_t \xi.$$
For every \( t \geq 0 \), \( J_t \) has an inverse. We denote \( K_t = J_t^{-1} \) for each \( t \geq 0 \) and \( K_t \) satisfies

\[
(3.2) \quad dK_t = -K_t \nabla a(X_t)dt, \quad K_0 = I.
\]

Denote \( \Omega = D(\mathbb{R}^+, \mathbb{R}^d) \) the collection of function \( \omega : \mathbb{R}^+ \rightarrow \mathbb{R}^d \) which is right continuous and has left limit. In our situation, it is convenient for us to take \( \Omega = D(\mathbb{R}^+, \mathbb{R}^d) \). Let \( (\mathcal{F}_t)_{t \geq 0} \) be the canonical filtration of \( \Omega \) and \( \mathcal{P} \) be the predictable \( \sigma \)-field on \( \mathbb{R}^+ \times \Omega \). Let \( v : \mathbb{R}_0^d \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R} \) be a \( \mathcal{B}(\mathbb{R}_0^d) \times \mathcal{P} \)-measurable function such that

\[
\mathbb{E} \int_0^t \int_{\mathbb{R}_0^d} |v(z, s)| \nu(dz) ds < \infty \quad \forall \ t > 0.
\]

Define

\[
V(t) = \int_0^t \int_{\mathbb{R}_0^d} v(z, s) N(dz, ds),
\]

and

\[
D_V X_t = \lim_{\varepsilon \to 0} \frac{X_t(x, L + \varepsilon V) - X_t(x, L)}{\varepsilon},
\]

the above limit exists in \( L^1((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d) \) for each \( t \geq 0 \) ([1]). The \( D_V X_t \) satisfies

\[
dD_V X_t = \nabla a(X_t) D_V X_t dt + BdV_t, \quad D_V X_0 = 0,
\]

which is solved by

\[
(3.3) \quad D_V X_t = J_t \int_0^t \int_{\mathbb{R}_0^d} K_s B v(z, s) N(dz, ds).
\]

**Lemma 3.1.** Let \( \xi(t) \) be an adapted process valued on \( \mathbb{R}^d \) such that there exist some \( C_1, C_2 > 0 \) such that

\[
\sup_{\omega \in \Omega} |\xi(t, \omega)| \leq C_2 e^{C_1 t} \quad \forall \ t \geq 0.
\]

Let

\[
h(z) = \varphi(z)|z|^4
\]

where \( \varphi : \mathbb{R}^d \rightarrow \mathbb{R}^+ \) is a smooth function such that \( h(z) = 1 \) for \( |z| \leq 1 \) and \( h(z) = 0 \) for \( |z| \geq 2 \).

Take \( v(z, t) = h(z) \xi(t) \) and \( V(z, t) = \int_0^t v(z, s) ds \), then, for all \( f \in C_b^1(\mathbb{R}^d) \) the following relation holds:

\[
(3.5) \quad \mathbb{E} (D_V f(X_t)) = \mathbb{E} (f(X_t) \delta(V)) \quad \forall t \in [0, T],
\]

where

\[
\delta(V) = \int_0^t \int_{\mathbb{R}_0^d} \frac{\text{div}(\rho(z) h(z) \xi(s))}{\rho(z)} \tilde{N}(dz, ds)
\]

Moreover, for all \( \lambda > 0 \) we have

\[
(3.7) \quad \mathbb{E} e^{\lambda |\delta(V)|} < C,
\]

\[
(3.8) \quad \mathbb{E} e^{\lambda \int_0^t \int_{\mathbb{R}_0^d} h(z) N(dz, ds)} \leq C,
\]

where \( C \) depends on \( \lambda, \xi \) and \( t \).
Proof. (3.4) is not new, we shall give a fast sketchy proof in the appendix for the completeness. For more details, one can refer to [4, 5, 3]. Let us prove (3.7). It is easy to check that
\[
\sup_{0 \leq s \leq t} |\nabla (\rho(z)h(z)\xi(s))| \leq c|z|^3 \quad \forall |z| \leq 2
\]
\[
\sup_{0 \leq s \leq t} |\nabla (\rho(z)h(z)\xi(s))| = 0 \quad \forall |z| \geq 2
\]
where \( c \) is some constant depending on \( \alpha \) and \( \xi \). By [17, Theorem 25.3], we immediately get the desired bound (3.7). (3.8) follows from [17, Theorem 25.3] again. \( \Box \)

Let \( \{ e_1, \ldots, e_d \} \) be the standard basis of \( \mathbb{R}^d \), for \( i = 1, \ldots, d \) define
\[
\xi_i(t) = B^* K^*_i e_i, \quad v_i(z, t) = h(z)\xi_i(t),
\]
by (3.3) we have
\[
D_{V_i} X_t = J_t \int_0^t \int_{\mathbb{R}^d} K_s BB^* K^*_i e_i h(z) N(dz, ds) \quad \forall t > 0
\]
with \( V_i(t) = \int_0^t \int_{\mathbb{R}^d} h(z)\xi_i(s) N(dz, ds) \) for \( i = 1, \ldots, d \). Therefore,
\[
(D_{V_1} X_t, \ldots, D_{V_d} X_t) = J_t \int_0^t \int_{\mathbb{R}^d} K_s BB^* K^*_i h(z) N(dz, ds).
\]
Write
\[
\mathcal{M}_t = \int_0^t \int_{\mathbb{R}^d} K_s BB^* K^*_i h(z) N(dz, ds),
\]
it is called simplified Malliavin matrix ([11]). \( \mathcal{M}_t \) is a symmetric \( d \times d \) matrix whose smallest eigenvalue \( \lambda_{\min}(t) \) is
\[
\lambda_{\min}(t) = \inf_{u \in \mathbb{R}^d : |u| = 1} \langle \mathcal{M}_t u, u \rangle.
\]
A straightforward computation gives
\[
\lambda_{\min}(t) = \inf_{|u| = 1} \int_0^t \int_{\mathbb{R}^d} \sum_{i=1}^d |(K_s B_i, u)|^2 h(z) N(dz, ds).
\]
To prove the smoothness of densities, we need the following auxiliary lemmas.

**Lemma 3.2.** The following statements hold

1. We have \(|J_t|, |K_t| \leq e^{\|\nabla\|_{\infty} t} \forall t \geq 0\). In particular, \(|J_t|, |K_t| \leq e^{\|A\| t} \forall t \geq 0\) when the condition in Theorem 2.1 holds.

2. Let \( V_1, \ldots, V_d \) be as above. For all \( p > 0, m \geq 1, T > 0 \) and any \( (i_1, \ldots, i_m) \in \{1, \ldots, d\}^m \), we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} |D_{V_{i_1}, \ldots, V_{i_m}} X_t|^p < \infty,
\]
\[
\mathbb{E} \sup_{0 \leq t \leq T} |D_{V_{i_1}, \ldots, V_{i_m}} K_t|^p < \infty,
\]
\[
\mathbb{E} \sup_{0 \leq t \leq T} |D_{V_{i_1}, \ldots, V_{i_m}} \mathcal{M}_t|^p < \infty.
\]
Proof. It is very easy to get (1) from Eq. (3.2) and (3.1). By (1) and (3.3), for all \( i \in \{1, \ldots, d\} \) we have

\[
|D_{V_i} X_t| \leq \int_0^t \int_{\mathbb{R}^d_0} |J_i| |K_s||B|^2 e^{\|\nabla a\|_{\infty} s} h(z) N(dz, ds)
\]

(3.14)

\[
\leq e^{3\|\nabla a\|_{\infty} t} |B|^2 \int_0^t \int_{\mathbb{R}^d_0} h(z) N(dz, ds),
\]

thus,

\[
\sup_{0 \leq t \leq T} |D_{V_i} X_t| \leq e^{3\|\nabla a\|_{\infty} T} |B|^2 \int_0^T \int_{\mathbb{R}^d_0} h(z) N(dz, ds).
\]

(3.15)

This, together with (3.8), implies

\[
\mathbb{E} e^{\lambda \sup_{0 \leq t \leq T} |D_{V_i} X_t|} < \infty \quad \forall \lambda > 0,
\]

from which the first inequality in (2) for \( m = 1 \) follows immediately.

A straightforward computation gives

\[
dD_{V_i V_j} X_t = \nabla a(X_t) D_{V_i V_j}^2 X_t dt + \nabla^2 a(X_t) D_{V_i} X_t D_{V_j} X_t dt
\]

\[
+ \int_{\mathbb{R}^d_0} (BK_i)^* e_i \nabla h(z)(BK_i)^* e_j h(z) N(dz, dt) \quad \forall (i, j) \in \{1, \ldots, d\}^2
\]

with \( D_{V_i V_j}^2 X_t = 0 \), from which it is easy to see

\[
|D_{V_i V_j}^2 X_t| \leq \left| \int_0^t e^{\int_r^t \nabla a(X_r) dr} \nabla^2 a(X_s) D_{V_i} X_s D_{V_j} X_s ds \right|
\]

\[
+ \left| \int_0^t e^{\int_r^t \nabla a(X_r) dr} \int_{\mathbb{R}^d_0} (BK_s)^* e_i \nabla h(z)(BK_s)^* e_j h(z) N(dz, ds) \right|
\]

\[
\leq I_1 + I_2,
\]

where

\[
I_1(t) = \int_0^t e^{\|\nabla a\|_{\infty} (t-s)} \|\nabla^2 a\|_{\infty} |D_{V_i} X_s||D_{V_j} X_s| ds,
\]

(3.17)

\[
I_2(t) = \int_0^t e^{\|\nabla a\|_{\infty} (t-s)} |B|^2 e^{2\|\lambda\|_{\infty} s} \int_{\mathbb{R}^d} |\nabla h(z)| h(z) N(dz, ds).
\]

Thanks to (3.11) for \( m = 1 \), for all \( p > 0 \) we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} |I_1(t)|^p \leq e^{\|\nabla a\|_{\infty} T} \|\nabla^2 a\|_{\infty} \int_0^T (\mathbb{E}|D_{V_i} X_s|^{2p})^{\frac{1}{2}} (\mathbb{E}|D_{V_j} X_s|^{2p})^{\frac{1}{2}} ds < \infty.
\]

Observe

\[
\sup_{0 \leq t \leq T} I_2(t) \leq e^{2\|\nabla a\|_{\infty} T} |B|^2 \int_0^T \int_{\mathbb{R}^d_0} |\nabla h(z)| h(z) N(dz, ds),
\]

in view of (3.4), we have \( \int_{\mathbb{R}^d_0} |\nabla h(z)| h(z) \nu(dz) < \infty \), thus

\[
\mathbb{E} e^{\lambda \int_0^T \int_{\mathbb{R}^d_0} |\nabla h(z)| h(z) N(dz, ds)} < \infty \quad \forall \lambda > 0,
\]

\[6\]
which, together with (3.18), implies

\begin{equation}
E e^{\lambda \sup_{0 \leq t \leq T} |I_2(t)|} < \infty \quad \forall \lambda > 0.
\end{equation}

The estimates about $I_1$ and $I_2$ immediately give (3.11) for $m = 2$. By a similar (but more tedious) argument we get (3.11) for $m = 3, 4, \ldots$.

For (3.12), we can prove it by a similar argument as for (3.11). It remains to prove (3.13). An easy computation gives

$$D_V M_t = J_1(t) + J_2(t) + J_3(t),$$

where

$$J_1(t) = \int_0^t \int_{\mathbb{R}_d} D_V K_s BB^* K_s^* h(z) N(dz, ds),$$

$$J_2(t) = \int_0^t \int_{\mathbb{R}_d} K_s BB^*(D_V K_s)^* h(z) N(dz, ds),$$

$$J_3(t) = \int_0^t \int_{\mathbb{R}_d} K_s BB^* K_s^* \nabla h(z) h(z) B^* K_s^* e_i N(dz, ds).$$

It is easy to see that for all $t \in (0, T]$

$$|J_1(t)| \leq \int_0^t \int_{\mathbb{R}_d} |D_V K_s| |B|^2 e^{|\nabla a|} |h(z)| N(dz, ds)$$

$$\leq |B|^2 e^{|\nabla a|} T \sup_{0 \leq t \leq T} |D_V K_t| \int_0^T \int_{\mathbb{R}_d} |h(z)| N(dz, ds),$$

combining the above inequality with (3.11) and (3.8), by Hölder inequality we immediately get

\begin{equation}
E \sup_{0 \leq t \leq T} |J_1(t)|^p < \infty \quad \forall p > 0.
\end{equation}

By the same method, we have

\begin{equation}
E \sup_{0 \leq t \leq T} |J_2(t)|^p < \infty \quad \forall p > 0.
\end{equation}

For $J_3$, by a similar argument as above we have for all $t \in (0, T]$

$$|J_3(t)| \leq |B|^3 e^{|\nabla a|} T \int_0^T \int_{\mathbb{R}_d} |\nabla h(z)||h(z)| N(dz, ds),$$

which, together with (3.19), immediately gives

\begin{equation}
E \sup_{0 \leq t \leq T} |J_3(t)|^p < \infty \quad \forall p > 0.
\end{equation}

Collecting the estimates for $J_1, J_2, J_3$, we immediately get (3.13) for $m = 1$. By a similar (but more tedious) argument we get the inequalities in (3) for $m = 2, 3, \ldots, \Box$

The next lemma is a criterion for the smoothness of the density, which will be used to prove our main results.
Lemma 3.3. If $\mathcal{M}_t$ is invertible a.s. for all $t > 0$ and further satisfies
\[ \mathbb{E}|\mathcal{M}_t^{-1}|^p < \infty \quad \forall p > 0. \]
Then, for all $t > 0$ the transition probability $P_t(x, \cdot)$ associated to the solution of Eq. (1.1) $X_t(x)$ has a smooth density function.

Proof. To prove the smoothness of the density, it suffices to show that for all $f \in C_b^\infty(\mathbb{R}^d)$ we have
\[ (3.24) \quad \left| \mathbb{E} \left( \nabla_{i_1, \ldots, i_m}^m f(X_t) \right) \right| \leq C \| f \|_\infty \quad \forall m \geq 1 \quad \forall (i_1, \ldots, i_m) \in \{1, \ldots, d\}^m, \]
where $\nabla_{i_1, \ldots, i_m}^m = \frac{\partial^m}{\partial x_{i_1} \cdots \partial x_{i_m}}$ and $C$ depends on $t$ and $(i_1, \ldots, i_m)$.

For the notational simplicity, write
\[ V(t) = [V_1(t), \ldots, V_d(t)], \quad D_V X_t = [D_{V_1} X_t, \ldots, D_{V_d} X_t], \]
they are both $d \times d$ matrices. It is clear to see from (3.9)
\[ D_V X_t = J_t \mathcal{M}_t. \]
By the relation $D_V f(X_t) = \nabla f(X_t) D_V X_t$, we get
\[ \nabla f(X_t) = D_V f(X_t) \mathcal{M}_t^{-1} K_t \]
and thus
\[ \nabla_i f(X_t) = \sum_{j=1}^d D_{V_j} f(X_t) (\mathcal{M}_t^{-1} K_t)_{ji} \quad i = 1, \ldots, d. \]
It is easy to see that
\[ (3.25) \quad \mathbb{E} \left( \nabla_i f(X_t) \right) = \sum_{j=1}^d \left\{ \mathbb{E} \left[ D_{V_j} \left( f(X_t) (\mathcal{M}_t^{-1} K_t)_{ji} \right) \right] - \mathbb{E} \left[ f(X_t) D_{V_j} \left( (\mathcal{M}_t^{-1} K_t)_{ji} \right) \right] \right\}. \]
Using integration by parts (3.5) and Hölder inequality we have
\[ (3.26) \quad \left| \mathbb{E} \left[ D_{V_j} \left( f(X_t) (\mathcal{M}_t^{-1} K_t)_{ji} \right) \right] \right| \leq \| f \|_\infty \| K_t \|_\infty \left( \mathbb{E}|\mathcal{M}_t^{-1}|^2 \right)^{\frac{1}{2}} \left( \mathbb{E}|\delta(V_j)|^2 \right)^{\frac{1}{2}} \]
Moreover, we have
\[ D_{V_j} \left( (\mathcal{M}_t^{-1} K_t)_{ji} \right) = (D_{V_j} \mathcal{M}_t^{-1} K_t)_{ji} + (\mathcal{M}_t^{-1} D_{V_j} K_t)_{ji} = (\mathcal{M}_t^{-1} D_{V_j} \mathcal{M}_t \mathcal{M}_t^{-1} K_t)_{ji} + (\mathcal{M}_t^{-1} D_{V_j} K_t)_{ji}, \]
this, together with Hölder inequality, implies
\[ (3.27) \quad \left| \mathbb{E} \left[ f(X_t) D_{V_j} \left( (\mathcal{M}_t^{-1} K_t)_{ji} \right) \right] \right| \leq \| f \|_\infty \| K_t \|_\infty \left( \mathbb{E}|\mathcal{M}_t^{-1}|^4 \right)^{\frac{1}{4}} \left( \mathbb{E}|D_{V_j} \mathcal{M}_t|^2 \right)^{\frac{1}{4}} \]
\[ + \| f \|_\infty \left( \mathbb{E}|\mathcal{M}_t^{-1}|^2 \right)^{\frac{1}{4}} \left( \mathbb{E}|D_{V_j} K_t|^2 \right)^{\frac{1}{4}} \]
Combining (3.25)-(3.27), by Lemma 3.2 and the assumption we have
\[ \left| \mathbb{E} \left( \nabla_i f(X_t) \right) \right| \leq C \| f \|_\infty \quad \forall i \in \{1, \ldots, d\}, \]
where $C$ depends on $t, i$. 

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A straightforward computation gives
\[
\nabla^2 f(X_t) = \mathcal{M}_t^{-1} K_t (D^2_V f(X_t) - \nabla f(X_t) D^2_V X_t) \mathcal{M}_t^{-1} K_t,
\]
using integration by parts and Hölder inequality, by Lemma 3.2 and Corollary ?? we get
\[
\left| \mathbb{E} \left( \nabla^2 f(X_t) \right) \right| \leq C \|f\|_\infty \quad \forall (i,j) \in \{1, \ldots, d\}^2,
\]
where \( C \) depends on \( i, j \) and \( t \).

Iteratively using the same argument as above, we finally get the desired (3.24). \( \square \)

4. Proof of Theorem 2.1

When \( a(x) := Ax \) is linear, we have
\[
J_t = e^{At}, \quad K_t = e^{-At}.
\]

Lemma 4.1. Let \( u, v \in \mathbb{R}^d \) both be nonzero vectors with some \( p > 0 \) such that
\[
\langle v, u \rangle \geq p \quad \text{or} \quad \langle u, v \rangle \leq -p.
\]
Then there exist some \( \theta = \frac{1}{2 |u||v| |A|} e^{-|A|} \) and \( \delta = (\theta p) \land 1 \) such that for all \( t \in (0, \delta) \),
\[
\langle K_t v, u \rangle \geq p/2 \quad \text{(respectively} \quad \langle K_t v, u \rangle \leq -p/2).\]
Moreover, for all \( v \in \mathbb{R}^d \) the following relation holds: for all \( l \geq 1 \),
\[
K_t v = \sum_{j=0}^{l-1} \frac{(-t)^j}{j!} A^j v + (-1)^l \int_0^t \int_0^{s_1} \cdots \int_0^{s_{l-1}} K_{s_l} A^l v ds_k \ldots ds_1.
\]

Proof. Differentiating \( K_t \) with respect to \( t \), we get
\[
\frac{dK_t}{dt} = -K_t A,
\]
thus for all \( t \in (0, 1) \),
\[
|\langle K_t v, u \rangle - \langle v, u \rangle| \leq \int_0^t |A| e^{A|s|}|u||v| ds \leq t |u||v||A| e^{A|s|}.
\]
Therefore, we get
\[
|\langle K_t u, v \rangle - \langle u, v \rangle| \leq p/2 \quad \forall t \in (0, \delta).
\]
This immediately implies the first inequality.

For each \( j \geq 0 \), differentiating \( K_t A^j v \) with respect to \( t \), we obtain
\[
\frac{d}{dt} K_t A^j v = -K_t A^{j+1} v.
\]
Iteratively applying above equation gives (4.3). \( \square \)

Remark 4.2. The inequality (4.2) is a replacement of Norris Lemma in our special situation.
Thanks to (4.3), we can use this inequality (4.2) iteratively.
Let us now prove the following crucial lemma.

**Lemma 4.3.** Assume the conditions in Theorem 2.1 hold. For any $\gamma > 0$ and $\ell \in (0, 1/4)$, there exist some $\varepsilon_0 > 0$ depending on $\gamma$, $\ell$ and some $t_0 \in (0, 1)$ depending on $\varepsilon_0$ such that $\lim_{\varepsilon \to 0} t_0 = 0$ and that for all $\varepsilon \in (0, \varepsilon_0)$ and $t \geq t_0$ we have

$$P \left( \Lambda_{\min}(t) \leq \varepsilon \right) \leq Ce^{c(\varepsilon/\log \varepsilon^\gamma)^{-1}}. \tag{4.6}$$

where $c$ only depends on $|A|$, $|B|$ and $C$ depends on $|A|$, $|B|$, $t$.

**Proof.** Our proof follows the spirit in [11]. Write

$$\Lambda(t, u, \varepsilon^\ell) = \int_0^t \int_{|z| \geq \varepsilon^\ell} \sum_{i=1}^d |\langle K_s B_i, u \rangle|^2 h(z) N(dz, ds),$$

by (3.10), to prove the desired inequality, it suffices to show that there exist some $\varepsilon_0 > 0$ depending on $\ell, \gamma$ and some $t_0$ depending on $\varepsilon_0$ such that $\lim_{\varepsilon \to 0} t_0 = 0$ and that for all $\varepsilon \in (0, \varepsilon_0)$ and $t \geq t_0$,

$$P \left( \inf_{|u| = 1} \Lambda(t, u, \varepsilon^\ell) \leq \varepsilon \right) \leq Ce^{c(\varepsilon/\log \varepsilon^\gamma)^{-1}}. \tag{4.7}$$

Since $\Lambda(t, u, \varepsilon^\ell)$ is increasing with respect to $t$, it suffices to prove

$$P \left( \inf_{|u| = 1} \Lambda(t, u, \varepsilon^\ell) \leq \varepsilon \right) \leq Ce^{c(\varepsilon/\log \varepsilon^\gamma)^{-1}} \quad \forall \varepsilon \in (0, \varepsilon_0) \forall t \in [t_0, 1].$$

Let us prove (4.7) in the following three steps.

**Step 1:** Write

$$N_{t,h} = \int_0^t \int_{|z| \geq \varepsilon^\ell} h(z) N(dz, ds),$$

$$N_{t,\varepsilon^\ell,h} = \int_0^t \int_{|z| \geq \varepsilon^\ell} h(z) N(dz, ds),$$

it is clear $N_{t,\varepsilon^\ell,h} \leq N_{t,h}$. By (3.8) and Chebyshev inequality we have

$$P \left( N_{t,\varepsilon^\ell,h} > M \right) \leq P \left( N_{t,h} > M \right) \leq Ce^{-M} \quad \forall M > 0, \tag{4.8}$$

where $C$ depends on $t$.

Taking $\eta = \frac{e^{-|2A|/2M^2}}{2d|B|^2M^2}$, by (1) of Lemma 3.2, we easily get that for all $u, v \in \mathbb{S}^{d-1}$ with $|u - v| \leq \eta$,

$$\left| \sum_{i=1}^d |\langle K_s B_i, u \rangle|^2 - \sum_{i=1}^d |\langle K_s B_i, v \rangle|^2 \right| \leq \frac{\varepsilon}{M} \quad \forall s \in [0, 1].$$

Hence, as $N_{t,\varepsilon^\ell,h} \leq M$ we have

$$|\Lambda(t, u, \varepsilon^\ell) - \Lambda(t, v, \varepsilon^\ell)| \leq \varepsilon \quad \forall t \in [0, 1]. \tag{4.9}$$

By the compactness, $\mathbb{S}^{d-1}$ has a finite open sets cover $(U_k)_{1 \leq k \leq W}$ such that $W \leq C_d (M \varepsilon^{-1})^{d-1}$ with $C_d$ only depending on $d$ and that the diameter of each open set $U_k$ is $\eta$. 


Take any \( u_k \in U_k \) for all \( k \), it is easy to see from (4.9) that for all \( t \in [0, 1] \) we have
\[
\left\{ \inf_{|u|=1} \Lambda(t, u, \varepsilon^t) \leq \varepsilon, N_{t, \varepsilon^t, h} \leq M \right\} \subset \bigcup_{k=1}^{W} \left\{ \Lambda(t, u_k, \varepsilon^t) \leq 2\varepsilon, N_{t, \varepsilon^t, h} \leq M \right\}
\]
and thus
\[
P\left( \inf_{|u|=1} \Lambda(t, u, \varepsilon^t) \leq \varepsilon, N_{t, \varepsilon^t, h} \leq M \right) \leq \sum_{k=1}^{W} P\left( \Lambda(t, u_k, \varepsilon^t) \leq 2\varepsilon, N_{t, \varepsilon^t, h} \leq M \right)
\leq C_d(M \varepsilon^{-d}) \sup_{u \in \mathbb{S}^{d-1}} P\left( \Lambda(t, u, \varepsilon^t) \leq 2\varepsilon \right).
\]
(4.10)

Step 2: We shall prove in the step 3 below that for any \( \gamma > 0 \) and \( \ell \in (0, 1/4) \), there exist some \( \varepsilon_0 > 0 \) depending on \( \gamma \), \( \ell \) and some \( t_0 \in (0, 1) \) depending on \( \varepsilon_0 \) such that \( \lim_{\varepsilon_0 \to 0} t_0 = 0 \) and that for all \( \varepsilon \in (0, \varepsilon_0) \) and \( t \geq t_0 \) we have
\[
P\left( \Lambda(t, u, \varepsilon^t) \leq 2\varepsilon \right) \leq e^{-c|\log \varepsilon|^{-\gamma} \nu(\varepsilon^t\leq |\varepsilon| \leq 1)}
\]
for all \( u \in \mathbb{S}^{d-1} \), where \( c > 0 \) only depends on \( |A| \) and \( |B| \).

Now we use the inequalities in the step 1 and (4.11) to prove the desired (4.7). By (4.8) with \( M = \frac{1}{\varepsilon^2} \) therein, we get
\[
P\left( N_{t, \varepsilon^t, h} > \frac{1}{\varepsilon^2} \right) \leq C e^{-1/\varepsilon^2}.
\]
(4.12)

This, together with (4.11) and (4.10), implies
\[
P\left( \inf_{|u|=1} \Lambda(t, u, \varepsilon^t) \leq \varepsilon \right) \leq P\left( \inf_{|u|=1} \Lambda(t, u, \varepsilon^t) \leq \varepsilon, N_{t, \varepsilon^t, h} \leq \frac{1}{\varepsilon^2} \right) + P\left( N_{t, \varepsilon^t, h} > \frac{1}{\varepsilon^2} \right)
\leq C e^{-1/\varepsilon^2} + C_d e^{-3(\ell-1) \varepsilon^{-\gamma} \nu(\varepsilon^t\leq |\varepsilon| \leq 1)}.
\]
(4.13)

Tuning the number \( c \) to be smaller and using the assumption (H1), we immediately obtain the desired inequality (4.7).

Step 3: It remains to show (4.11). From the rank condition in Theorem 2.1, there exist some \( j_0 \leq n \), \( i_0 \leq d \) and some constant \( \kappa_0 > 0 \) such that
\[
|\langle A^{j_0} B_{i_0}, u \rangle| \geq \kappa_0.
\]
(4.14)

Without loss of generality, we assume that \( j_0 \geq 1 \). Denote \( \theta = \frac{e^{-|A|}}{2|A|} \) and choose a small number \( \varepsilon_0 \in (0, 1/4) \) satisfying the following conditions:
\[
|\log \varepsilon_0|^{-\gamma(4n)^{-n}} < \min\{1/\theta, \kappa_0, 1/2\},
\]
(4.15)
\[
|\log \varepsilon_0|^{-2j} h(\varepsilon_0^t) > 8\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_0],
\]
(4.16)
\[
|\log \varepsilon_0|^{-2j} h(\varepsilon_0^t) > 8\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_0],
\]
(4.17)
\[
\inf_{1 \leq j \leq n} \left( \frac{\theta^j}{2j+1} \right)^2 |\log \varepsilon_0|^{-2j(4n)^{-n}} h(\varepsilon_0^t) > 8\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_0],
\]
(4.18)
It is easy to check that as \( \varepsilon_0 \) is sufficiently small the conditions (4.15) and (4.17) both hold. (4.16) and (4.18) follow from (3.4) and the assumption \( \ell \in (0, 1/4) \) for sufficiently small \( \varepsilon_0 \).

We choose \( t_0 = \max \{ \delta, \tilde{\delta} \} \) with \( \delta \) and \( \tilde{\delta} \) defined by (4.19) and (4.24) respectively. It is clear that \( \lim_{\varepsilon_0 \to 0} t_0 = 0 \). Now we prove (4.11) by considering the following two cases. The conditions (4.15) and (4.16) will be used in the Case 1 below, while (4.17) and (4.18) will be used in Case 2.

**Case 1:** \( |\langle B_{t_0}, u \rangle| \geq |\log \varepsilon_0|^{-\gamma} \). Choose

(4.19)

\[ \delta = \theta |\log \varepsilon_0|^{-\gamma}, \]

thanks to (4.15) we have \( \delta < 1 \). By Lemma 4.1 we get

\[ |\langle K_\delta B_{t_0}, u \rangle| \geq \frac{1}{2} |\log \varepsilon_0|^{-\gamma} \quad \forall s \in (0, \delta). \]

Write \( N_{t, \varepsilon^\ell} = \int_0^t \int_{|s|\leq 1} N(ds, ds) \), it follows from the above inequality and (3.4) that for all \( \varepsilon \in (0, \varepsilon_0] \)

\[ \int_0^\delta \int_{|s|\leq 1} |\langle K_\delta B_{t_0}, u \rangle|^2 h(z)N(ds, ds) \geq \frac{1}{4} \int_0^\delta \int_{|s|\leq 1} |\log \varepsilon_0|^{-2\gamma} h(z)N(ds, ds) \]

\[ \geq \frac{1}{4} |\log \varepsilon_0|^{-2\gamma} h(\varepsilon^\ell) N_{\delta, \varepsilon^\ell}. \]

A straightforward computation gives

(4.21)

\[ \mathbb{P}(N_{\delta, \varepsilon^\ell} = 0) = e^{-\delta \nu(\varepsilon^\ell \leq |s|\leq 1)}. \]

As \( N_{\delta, \varepsilon^\ell} \geq 1, (4.20) \) and (4.16) imply

\[ \int_0^\delta \int_{|s|\leq 1} |\langle K_\delta B_{t_0}, v \rangle|^2 h(z)N(ds, ds) \geq \frac{1}{4} |\log \varepsilon_0|^{-2\gamma} h(\varepsilon^\ell) > 2\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0]. \]

Hence,

(4.22)

\[ \mathbb{P} \left( \int_0^\delta \int_{|s|\leq 1} |\langle K_\delta B_{t_0}, u \rangle|^2 h(z)N(ds, ds) \leq 2\varepsilon, N_{\delta, \varepsilon^\ell} \geq 1 \right) = 0. \]

By (4.21), (4.22) and the fact \( t_0 > \delta \), we have that for all \( t \geq t_0 \),

(4.23)

\[ \mathbb{P} \left( \int_0^\delta \int_{|s|\leq 1} |\langle K_\delta B_{t_0}, u \rangle|^2 h(z)N(ds, ds) \leq 2\varepsilon \right) \leq \mathbb{P} \left( \int_0^\delta \int_{|s|\leq 1} |\langle K_\delta B_{t_0}, u \rangle|^2 h(z)N(ds, ds) \leq 2\varepsilon \right) \leq e^{-\delta \nu(\varepsilon^\ell \leq |s|\leq 1)}. \]

By the definition of \( \Lambda(t, u, \varepsilon^\ell) \) and \( \theta |\log \varepsilon|^{-\gamma} \leq \delta \), the above inequality immediately implies the desired inequality (4.11).

**Case 2:** \( |\langle B_{t_0}, u \rangle| < |\log \varepsilon_0|^{-\gamma} \). Define

\[ l = \inf \{ j \geq 1 : |\langle A^j B_{t_0}, u \rangle| < |\log \varepsilon_0|^{-\gamma(4n/k)}, 0 \leq k \leq j - 1; \]

\[ |\langle A^j B_{t_0}, u \rangle| \geq |\log \varepsilon_0|^{-\gamma(4n/k)} \}, \]

where \( n \) is the constant in Theorem 2.1. By (4.14) and (4.15), as \( \varepsilon_0 \) is sufficiently small we have

\[ |\log \varepsilon_0|^{-\gamma(4n/k)} \leq |\log \varepsilon_0|^{-\gamma(4n/k)} \leq \kappa_0 \leq |\langle A^j B_{t_0}, u \rangle|. \]
This and the condition \(|(B_{i_0}, u)| < |\log \varepsilon_0|^{-\gamma}\) imply 
\[1 \leq l \leq j_0.\]

Choose
\[\tilde{\delta} = \theta |\log \varepsilon_0|^{-\gamma(4n)^{-t}}\]
it is obvious \(\tilde{\delta} \leq t_0\) by the definition of \(t_0\) above. By Lemma 4.1 and the definition of \(l\), we have for all \(s \in (0, \tilde{\delta})\)
\[\langle K_s A^l B_{i_0}, u \rangle \geq \frac{1}{2} |\log \varepsilon_0|^{-\gamma(4n)^{-t}} \quad \text{or} \quad \langle K_s A^l B_{i_0}, u \rangle \leq -\frac{1}{2} |\log \varepsilon_0|^{-\gamma(4n)^{-t}}.
\]
The above two inequalities imply
\[\left| \int_0^t \int_0^{s_1} \cdots \int_0^{s_{l-1}} \langle K_s A^l B_{i_0}, u \rangle ds_l \cdots ds_1 \right| \geq \frac{\theta^l}{2l!} |\log \varepsilon_0|^{-\gamma(4n)^{-t}} \quad \forall t \in (0, \tilde{\delta}).\]

From the definition of \(l\) again, we have
\[\left| (-t)^j \frac{j!}{j!} \langle A^l B_{i_0}, u \rangle \right| < \frac{\theta^j}{j!} |\log \varepsilon_0|^{-\gamma(4n)^{-j}} \quad \forall 0 \leq j \leq l - 1.
\]
Applying (4.3), by (4.25) and (4.26) we get
\[\left| \langle K_l B_{i_0}, u \rangle \right| \geq \frac{\theta^l}{2l!} |\log \varepsilon_0|^{-\gamma(4n)^{-1}} - \sum_{j=0}^{l-1} \frac{\theta^j}{j!} |\log \varepsilon_0|^{-\gamma(4n)^{-j}} \quad \forall t \in (0, \tilde{\delta}).\]

For all \(t \in \tilde{\delta}/2, \tilde{\delta}\) we have
\[\left| \langle K_l B_{i_0}, u \rangle \right| \geq \frac{\theta^l}{2l!} |\log \varepsilon_0|^{-\gamma(4n)^{-1}} - \sum_{j=0}^{l-1} \frac{\theta^j}{j!} |\log \varepsilon_0|^{-\gamma(4n)^{-j}} - \sum_{j=0}^{l-1} \frac{\theta^j}{j!} |\log \varepsilon_0|^{-\gamma(4n)^{-j}}.
\]
Observe
\[\sum_{j=0}^{l-1} \frac{\theta^j}{j!} |\log \varepsilon_0|^{-\gamma(4n)^{-j}} \leq (\theta + 1)^l |\log \varepsilon_0|^{-\gamma(4n)^{(l-j)-1}},\]
and
\[(4n)^{l-j} - (l-j) - 1 \geq (l-j) \quad \forall j < l,
\]
we get
\[\sum_{j=0}^{l-1} \frac{\theta^j}{j!} |\log \varepsilon_0|^{-\gamma(4n)^{(l-j)-1}} \leq (\theta + 1)^l |\log \varepsilon_0|^{-\gamma(4n)^{(l-j)-1}} \sum_{j=0}^{l-1} |\log \varepsilon_0|^{-\gamma(4n)^{(l-j)-1}}
\]
\[\leq 2(\theta + 1)^l |\log \varepsilon_0|^{-\gamma(4n)^{l-1}},\]
where the last inequality is by (4.15). It follows from (4.17) that
\[\frac{\theta^l}{2l+1} - 2(\theta + 1)^l |\log \varepsilon_0|^{-\gamma(4n)^{l-1}} \geq \frac{\theta^l}{2l+1},\]
which, together with (4.28) and (4.27), gives

\[ |\langle K_t B_{t_0}, u \rangle| \geq \frac{\theta^l}{2^{l+2}} |\log \varepsilon_0|^{-\frac{l+1}{(4n)^7}} \quad \forall t \in [\delta/2, \delta]. \]

By the same argument as in the case 1, we have

\[ \mathbb{P} \left( \int_{t_0}^{\delta/2} \int_{|z| \leq 1} |\langle K_t B_{t_0}, u \rangle|^2 h(z) N(dz, ds) \leq 2\varepsilon \right) \leq e^{-\frac{1}{2}\nu(|\varepsilon| \leq 1)} \quad \forall \varepsilon \in (0, \varepsilon_0], \]

hence, for all \( t \geq t_0 \) (recall \( t_0 \geq \delta \)) we have

\[ \mathbb{P} \left( \int_{0}^{t} \int_{|z| \leq 1} |\langle K_t B_{t_0}, u \rangle|^2 h(z) N(dz, ds) \leq 2\varepsilon \right) \leq e^{-\frac{2}{\nu}(\varepsilon \geq 0)} \quad \forall \varepsilon \in (0, \varepsilon_0]. \]

In view of \( \delta \geq \delta \) and \( \delta = \theta |\log \varepsilon_0|^{-\gamma} \), it follows from the previous inequality that for \( \forall \varepsilon \in (0, \varepsilon_0] \)

\[ \mathbb{P} \left( \int_{0}^{t} \int_{|z| \leq 1} |\langle K_t B_{t_0}, u \rangle|^2 h(z) N(dz, ds) \leq 2\varepsilon \right) \leq e^{-\frac{1}{2}\nu(|\varepsilon| \leq 1)} \leq e^{-\frac{1}{2}\theta |\log \varepsilon|^{-\gamma} \nu(|\varepsilon| \leq 1)} \]

and thus the desired (4.11).

**Proof.** By Lemma 3.3, it suffices to show \( M_t \) is invertible a.s. and

\[ \mathbb{E}|M_t^{-1}|^p < \infty \quad \forall p > 0. \]

Take any \( t_1 > 0 \) and fix it. From Lemma 4.3 we can choose \( \varepsilon_0 > 0 \) sufficiently small so that

\[ \delta \leq t_1/2 \]

and that (4.6) holds for \( t > \delta \) (in particular for \( t = t_1 \)). Taking \( \varepsilon = 1/n \) in (4.6) and writing \( E_n = \{ \lambda_{\min}(t_1) \leq 1/n \} \), we have

\[ \sum_{n=n_0}^{\infty} \mathbb{P}(E_n) \leq \sum_{n=n_0}^{\infty} C e^{-c(n^{\alpha(1/|\log n|^{1/\gamma})})} < \infty, \]

where \( n_0 = [1/\varepsilon_0] + 1 \). By Borell-Cantelli Lemma we have \( \lambda_{\min}(t_1) > 0 \) a.s. and thus \( M_{t_1} \) is invertible a.s.. We take the largest eigenvalue of \( M_{t_1}^{-1} \) i.e. \( (\lambda_{\min}(t_1))^{-1} \) as \( |M_{t_1}^{-1}| \) (recall all the norms of a finite dimension space are equivalent), (4.6) implies

\[ \mathbb{P}(|M_{t_1}^{-1}| \geq 1/\varepsilon) \leq Ce^{-c(e^{\alpha(1/|\log \varepsilon|^{1/\gamma})})^{-1}} \quad \forall \varepsilon \in (0, \varepsilon_0], \]

which immediately implies the desired inequality for \( t = t_1 \). Since \( t_1 > 0 \) is arbitrary, the proof is completed.

\[ \square \]

5. PROOF OF THEOREM 2.2

To prove Theorem 2.2, we shall use the same procedure as proving Theorem 2.1. The crucial step is Lemma 5.1 below, which plays the same role as Lemma 4.3 in the proof of Theorem 2.2. With this lemma, we can prove Theorem 2.2 by the same argument as showing Theorem 2.1. So, in this section we only prove the crucial lemma but omit how to apply it to prove the theorem.
Lemma 5.1. Assume that the conditions in Theorem 2.2 hold. For any $\gamma > 0$ and $\ell \in (0, 1/4)$, there exist some $\varepsilon_0 > 0$ depending on $\gamma$, $\ell$ and some $t_0 \in (0, 1)$ depending on $\varepsilon_0$ such that $\lim_{\varepsilon_0 \to 0} t_0 = 0$ and that for all $\varepsilon \in (0, \varepsilon_0)$ and $t \geq t_0$ we have
\begin{equation}
P(\lambda_{\min}(t) \leq \varepsilon) \leq C e^{-c(\gamma t \log \varepsilon)^{-1}}.
\end{equation}
where $c$ only depends on $|A|$, $|B|$ and $C$ depends on $|A|$, $|B|$, $t$.

To prove the above lemma, we need the following auxiliary lemma, which can be shown by an argument similar to proving Lemma 4.1.

Lemma 5.2. Let $u, v \in \mathbb{R}^d$ both be nonzero vectors with some $p > 0$ such that
\begin{equation}
\langle v, u \rangle \geq p \ (\text{or } \langle v, u \rangle \leq -p).
\end{equation}
Then there exist some $\theta = \frac{1}{2\|u\|\|v\|} e^{-\|\nabla a\|_{\infty}}$
\[
\delta = (\theta p) \wedge 1
\]
such that for all $t \in (0, \delta)$ and $x \in \mathbb{R}^d$
\begin{equation}
\langle K_{\ell}(x)v, u \rangle \geq p/2 \ (\text{respectively } \langle K_{\ell}(x)v, u \rangle \leq -p/2).
\end{equation}

Proof of Lemma 5.1. We first repeat exactly the steps 1 and 2 in the proof of Lemma 5.1. To complete the proof, we only proceed to prove the step 3.

Recall that the step 3 is to show that for any $\gamma > 0$ and $\ell \in (0, 1/4)$, there exist some $\varepsilon_0 > 0$ depending on $\gamma$, $\ell$ and some $t_0 \in (0, 1)$ depending on $\varepsilon_0$ such that $\lim_{\varepsilon_0 \to 0} t_0 = 0$ and that for all $\varepsilon \in (0, \varepsilon_0)$ and $t \geq t_0$ we have
\begin{equation}
P\left(\Lambda(t, u, \varepsilon^\ell) \leq 2\varepsilon\right) \leq e^{-c|\log \varepsilon|^{-\gamma} v(\varepsilon^\ell)}|\varepsilon| \leq 1)
\end{equation}
for all $u \in \mathbb{S}^{d-1}$, where $c > 0$ only depends on $|A|$ and $|B|$.

By the uniform Hörmander condition, we have some $\kappa_0 > 0$ such that
\begin{equation}
\inf_{|a| = 1} \sum_{i=1}^d \left(\|\langle a(x)B_i, u \rangle\| + \|\langle B_i, u \rangle\|\right) \geq 2d\kappa_0^2.
\end{equation}
Write $\theta = \frac{e^{-\|\nabla a\|_{\infty}}}{2\|\nabla a\|_{\infty}|B|}$, we choose an $\varepsilon_0 \in (0, 1/4)$ such that
\begin{equation}|\log \varepsilon_0|^{-\gamma} \leq \kappa_0,
\end{equation}
\begin{equation}|\log \varepsilon_0|^{-2\gamma} h(\varepsilon^\ell) > 8\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_0],
\end{equation}
\begin{equation}\frac{8}{\kappa_0} |\log \varepsilon_0|^{-\gamma} < (\theta \kappa_0) \wedge 1.
\end{equation}
As $\varepsilon_0 > 0$ is sufficiently large, the above four conditions clearly hold. Choosing
\[t_0 = \max\{\theta |\log \varepsilon_0|^{-\gamma}, \frac{4}{\kappa_0} |\log \varepsilon_0|^{-\gamma}\},\]
we have $t_0 < 1$ as $\varepsilon_0$ is sufficiently small. We shall prove (5.4) by considering the following two cases.

Case 1: If there exists some $i_0 \in \{1, \ldots, d\}$ such that
\[|\langle B_{i_0}, u \rangle| \geq |\log \varepsilon_0|^{-\gamma},\]
choose
\begin{equation}
(5.9) \quad \delta = \theta | \log \varepsilon_0 |^{-\gamma},
\end{equation}
it is clear that \(\delta < t_0 < 1\) by the definition of \(t_0\). By Lemma 5.2, for all \(x \in \mathbb{R}^d\) the following relation holds:
\begin{equation}
|\langle K_s(x)B_{i_0}, u \rangle| \geq \frac{1}{2} | \log \varepsilon_0 |^{-\gamma} \quad \forall s \in (0, \delta).
\end{equation}
Write \(N_{t, \varepsilon^f} = \int_0^t \int_{|z| \leq 1} N(dz, ds)\), it follows from the above inequality and (3.4) that for all \(\varepsilon \in (0, \varepsilon_0]\)
\begin{equation}
(5.10) \quad \int_0^\delta \int_{|z| \leq 1} |\langle K_s(x)B_{i_0}, u \rangle|^2 h(z)N(dz, ds) \geq \frac{1}{4} \int_0^\delta \int_{|z| \leq 1} | \log \varepsilon_0 |^{-2\gamma} h(z)N(dz, ds) \geq \frac{1}{4} | \log \varepsilon_0 |^{-2\gamma} h(\varepsilon^f)N_{\delta, \varepsilon^f}.
\end{equation}
A straightforward computation gives
\begin{equation}
(5.11) \quad \mathbb{P}(N_{\delta, \varepsilon^f} = 0) = e^{-\delta \nu(\varepsilon^f \leq |z| \leq 1)}.
\end{equation}
As \(N_{\delta, \varepsilon^f} \geq 1\), (5.10) and (5.7) imply
\begin{equation}
\int_0^\delta \int_{|z| \leq 1} |\langle K_sB_{i_0}, u \rangle|^2 h(z)N(dz, ds) \geq \frac{1}{4} | \log \varepsilon_0 |^{-2\gamma} h(\varepsilon^f) > 2\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0].
\end{equation}
Hence,
\begin{equation}
(5.12) \quad \mathbb{P} \left( \int_0^\delta \int_{|z| \leq 1} |\langle K_sB_{i_0}, u \rangle|^2 h(z)N(dz, ds) \leq 2\varepsilon, N_{\delta, \varepsilon^f} \geq 1 \right) = 0.
\end{equation}
By (5.11), (5.12) and the fact \(t_0 > \delta\) we have that for all \(t \geq t_0\),
\begin{equation}
(5.13) \quad \mathbb{P} \left( \int_0^t \int_{|z| \leq 1} |\langle K_sB_{i_0}, u \rangle|^2 h(z)N(dz, ds) \leq 2\varepsilon \right) \leq \mathbb{P} \left( \int_0^\delta \int_{|z| \leq 1} |\langle K_sB_{i_0}, u \rangle|^2 h(z)N(dz, ds) \leq 2\varepsilon \right) \leq e^{-\delta \nu(\varepsilon^f \leq |z| \leq 1)}.
\end{equation}
By the definition of \(\Lambda(t, u, \varepsilon^f)\) and \(\theta | \log \varepsilon |^{-\gamma} \leq \delta\), the above inequality immediately implies the desired inequality (5.4).
\begin{itemize}
\item **Case 2:** If \(|\langle B_i, u \rangle| < | \log \varepsilon_0 |^{-\gamma}\) for all \(i \in \{1, \ldots, d\}\), by (5.5) and (5.6), there exists some \(i_1 \in \{1, \ldots, d\}\) and some \(\kappa_0 > 0\) so that
\[ |\langle \nabla a(x)B_{i_1}, u \rangle| \geq \kappa_0 \quad \forall x \in \mathbb{R}^d. \]
By Lemma 5.2, as \(t \leq (\theta \kappa_0) \wedge 1\), for all \(x \in \mathbb{R}^d\) the following relation holds:
\[ \langle K_t \nabla a(x)B_{i_1}, u \rangle \geq \kappa_0 / 2 \quad \text{or} \quad \langle K_t \nabla a(x)B_{i_1}, u \rangle \leq -\kappa_0 / 2. \]
Therefore,
\[ \left| \int_0^t \langle K_s \nabla a(x)B_{i_1}, u \rangle ds \right| \geq \kappa_0 t / 2 \quad \forall t \leq (\theta \kappa_0) \wedge 1. \]
Choose
\[ \tilde{\delta} = \frac{8}{\kappa_0} | \log \varepsilon_0 |^{-\gamma}. \]
thanks to (5.8), the previous inequality, together with the easy relation
\[ K_t(x)B_{i1} = B_{i1} - \int_0^t K_s(x)\nabla a(X_s)B_{i1} ds, \quad \forall x \in \mathbb{R}^d \]
implies
\[ |\langle K_t(x)B_{i1}, u \rangle| \geq \kappa_0 t/2 - |\log \varepsilon_0|^{-\gamma} \geq |\log \varepsilon_0|^{-\gamma} \quad \forall t \in [\tilde{\delta}/2, \tilde{\delta}] \quad \forall x \in \mathbb{R}^d. \]
By the same argument as in Case 1, we have
\[ \mathbb{P} \left( \int_{\delta/2}^{\tilde{\delta}} \int_{|z| \leq 1} |\langle K_t B_{k0}, u \rangle|^2 h(z)(dz, ds) \leq 2\varepsilon \right) \leq e^{-\frac{\delta}{2} \nu(\varepsilon \leq |z| \leq 1)} \quad \forall \varepsilon \in (0, \varepsilon_0], \]
By the same arguments as those below (4.29), we get the desired inequality. □

6. Appendix: The sketchy proof of (3.5)

Step 1: Define \( v^\varepsilon(z, t) = z + \varepsilon v(z, t) \), as \( \varepsilon > 0 \) is sufficiently small \( v^\varepsilon(z, t) \) as a function from \( \mathbb{R}^d \) to \( \mathbb{R}^d \) has a unique inverse. We denote this inverse by \( u^\varepsilon(z, t) \). Under our assumption it is easy to see that as \( \varepsilon > 0 \) is sufficiently small,
\[ |u^\varepsilon(z, t) - z| \leq C\varepsilon |z|^{1+1/2} |z| \leq 2. \]
Further define \( N^\varepsilon(\Gamma \times [0, t]) = \int_0^t \int_{\mathbb{R}^d} 1_\Gamma(v^\varepsilon(z, s)) N(dz, ds) \), it is easy to check that \( N^\varepsilon \) has an intensity measure \( \nu^\varepsilon \) satisfying
\[ \nu^\varepsilon(\Gamma \times [0, t]) = \int_0^t \int_{\mathbb{R}^d} 1_\Gamma(v^\varepsilon(z, s)) \nu(dz)ds. \]
As \( \varepsilon > 0 \) is sufficiently small, the following Radon-Nikodym derivative always exists under our assumptions. A straightforward calculation gives
\[ \frac{d\nu^\varepsilon}{d\nu}(z, t) = \frac{\rho(u^\varepsilon(z, t))}{\rho(z)} =: \varphi^\varepsilon(z, t), \]
where \( \rho \) is the density function of \( \nu \) defined in (H1).

Step 2: Consider the following SDEs,
\[ dZ^\varepsilon_t = (\varphi^\varepsilon(z, t) - 1)\tilde{N}(dz, dt), \quad Z^\varepsilon_0 = 1, \]
by Itô formula we have
\[ Z^\varepsilon_t = \exp\left\{ \int_0^t \log \varphi^\varepsilon(z, s) N(dz, ds) - \int_0^t (\varphi^\varepsilon(z, s) - 1)\nu(dz)ds \right\}. \]
It is easy to check that \( Z^\varepsilon_t \) is a martingale under our assumptions. Define a measure \( \mathbb{P}^\varepsilon \) which is determined by
\[ \frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = Z^\varepsilon_t, \quad t > 0, \]
Thanks to (6.1), we have
\[ \lim_{\varepsilon \to 0} \mathbb{E}|Z^\varepsilon_t|^{\tilde{\delta}} < \infty, \]
(6.6) \[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \left( \frac{(Z^\varepsilon_t)^{-1} - 1}{\varepsilon} - \delta(V) \right)^2 \right] = 0,
\]
where \(\delta(V)\) is defined by (3.6). Define a process
\[
L^\varepsilon_t = L_t + \int_0^t \int_{\mathbb{R}^d} \varepsilon v(z, s) N(dz, ds).
\]
The crucial Girsanov type lemma holds:

**Lemma 6.1.** The law of the process \((L^\varepsilon_t)_{t \geq 0}\) under \(\mathbb{P}^\varepsilon\) is the same as that of the process \((L_t)_{t \geq 0}\) under \(\mathbb{P}\).

**Proof.** By checking the characteristic functions of the arbitrary finite discretization of \(L_t\) under \(\mathbb{P}\) and that of \(L^\varepsilon_t\) under \(\mathbb{P}^\varepsilon\) or by referring to [3, Theorem 6.16]. \(\square\)

**Step 3:** Consider the SDEs
\[
\begin{align*}
dX^\varepsilon_t &= a(X^\varepsilon_t)dt + BdL^\varepsilon_t, \\
X^\varepsilon_0 &= x,
\end{align*}
\]
where \(L^\varepsilon_t = L_t + \varepsilon \int_0^t \int_{\mathbb{R}^d} v(z, s) N(dz, ds)\). By Lemma 6.1, for all \(t > 0\) the law of \(X^\varepsilon_t\) under \(\mathbb{P}^\varepsilon\) and the law of \(X_t\) under \(\mathbb{P}\) are the same. Hence, for all \(f \in C^1_b\),
\[
\mathbb{E} [D_V f(X_t)] = \lim_{\varepsilon \to 0} \mathbb{E} \left( \frac{f(X^\varepsilon_t) - f(X_t)}{\varepsilon} \right)
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \mathbb{E} f(X^\varepsilon_t) - \mathbb{E}^\varepsilon f(X^\varepsilon_t) \right)
= \lim_{\varepsilon \to 0} \mathbb{E}^\varepsilon \left[ f(X^\varepsilon_t) \frac{1}{\varepsilon} \left( \frac{d\mathbb{P}}{d\mathbb{P}^\varepsilon} - 1 \right) \right]
= \lim_{\varepsilon \to 0} \mathbb{E} \left[ f(X^\varepsilon_t)(Z^\varepsilon_t)^{-1} - 1 \right].
\]
where the first equality is thanks to \(f \in C^1_b(\mathbb{R}^d)\) and the fact that \(\frac{X^\varepsilon_t - X_t}{\varepsilon}\) is uniformly integrable.

By (6.5) and (6.6), the above relation immediately gives the desired formula (3.5). By a classical extension procedure, we can show that the formula (3.5) also holds for \(f \in C_b(\mathbb{R}^d)\).

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