The Nambu-Jona-Lasinio Chiral Soliton with Constrained Baryon Number †

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Abstract

A regularization for the baryon number consistent with the energy in the Nambu-Jona-Lasinio model is introduced. The soliton solution is constructed with the regularized baryon number constrained to unity. It is furthermore demonstrated that this constraint prevents the soliton from collapsing when scalar fields are allowed to be space dependent. In this scheme the scalar fields actually vanish at the origin reflecting a partial restoration of chiral symmetry. Also the influence of this constraint on some static properties of baryons is discussed.

† Supported by the Deutsche Forschungsgemeinschaft (DFG) under contract Re 856/2-1.
Introduction

In recent years the investigation of the solitons of the Nambu-Jona-Lasinio (NJL) model [1] have experienced steady progress [2, 3, 4, 5, 6]. While in the Skyrme model [7, 8] the baryon number is assumed to be given by the topological charge [9] the chiral soliton of the NJL model is not strictly a topological soliton. On the contrary, the baryon number is given more directly in terms of the quark fields and, in general, contains contributions from the valence quarks as well as from the Dirac sea [10]. In leading order of the derivative expansion the latter is given by the topological charge [11]. Whether the baryon number is dominantly carried by the sea or the valence quarks depends on the specific features of the model [5].

Since the NJL action is not renormalizable regularization is required and more importantly the resulting energy functional, baryon number density, charge radii, etc. will depend on the regularization scheme employed [12]. Regularization is commonly applied to the Wick-rotated Euclidean NJL action. A special feature of this Euclidean action is the fact that only its real part is ultraviolet divergent while the imaginary part stays finite. Accordingly in most treatments only the real part undergoes regularization. As long as time components of vector and axialvector fields are ignored only the real part contributes to the static energy functional, falsely pretending that the imaginary part is of no importance for soliton solutions. The imaginary part is indeed relevant because it completely determines the baryon number current. Since we intend to explore the unit baryon number solutions the problem of regularizing the imaginary part plays a central role. Actually, a regularized imaginary part will not \textit{a priori} yield an integer baryon number.

The aim of this letter is to investigate the soliton solutions of the NJL model with a regularized imaginary part, \textit{i.e.} a regularized baryon number. Unit baryon number will be enforced by adding an appropriate constraint to the energy functional. We will demonstrate that this constraint cannot be satisfied by considering only the chiral angle field. However, a unit regularized baryon number is attainable when additionally scalar degrees of freedom are allowed to vary in space. This feature is also related to the recently observed collapse of the NJL soliton [13]. It is unstable against building up a narrow and infinitely high peak of the scalar field at the origin. The valence quark is thereby joining the Dirac sea such that infinitely many avoided crossings occur. This effectively corresponds to a situation where a level stemming from the positive part of the spectrum acquires an infinitely large negative energy eigenvalue. Without regularizing the baryon number such a configuration would correspond to $B = 1$. Obviously it should be clear that cutoff models like the NJL model cannot be trusted when infinitely large energies play a significant role. On the other hand, the regularized baryon number tends to zero during the collapse, \textit{i.e.} baryon number is leaking out of the soliton. In this letter we will demonstrate that this leaking is prevented and the collapse is avoided by fixing the regularized baryon number to unity. Then stable solutions exist with scalar and pseudoscalar mesons included. Furthermore we will discuss the influence of fixing the regularized baryon number on several baryon properties.

The model

The starting point of our calculations is the two-flavor NJL action $\mathcal{A}_{NJL}$ which, after bosonization [11], may be expressed as the sum $\mathcal{A}_{NJL} = \mathcal{A}_F + \mathcal{A}_m$ of a fermion determinant and a purely mesonic part

\footnote{The soliton with scalar and pseudoscalar mesons is known to be also stabilized by adding a four meson interaction [14].}
\[ \mathcal{A}_F = \text{Tr} \log(iD) = \text{Tr} \log(i\partial - (P_R M + P_L M^\dagger)), \]
\[ \mathcal{A}_m = \int d^4x \left( -\frac{1}{4g} \text{tr}(M^\dagger M - m_0(M + M^\dagger) + m_0^2) \right). \] (1)

Here, \( P_{R,L} \) are the projection operators on the right- and left-handed quark fields, respectively, and \( m_0 = \text{diag}(m_u, m_d) \) denotes the current quark mass matrix. We will restrict ourselves to the two flavor case and assume isospin symmetry: \( m^u = m^d = m \). The coupling constant \( g \) will be determined from meson properties. The complex field \( M = S + iP \) describes the scalar and pseudoscalar meson fields which can be parametrized by \( M = \Phi U \), \( \Phi \) and \( U \) being hermitian and unitary matrices, respectively. We will refer to \( \Phi \) as the chiral radius while the chiral angle \( \Theta \) is introduced via \( U = \exp(i\Theta) \).

The quark determinant \( \mathcal{A}_F \) diverges and must therefore be regularized. For the regularization procedure it is necessary to continue to Euclidean space. This yields a complex Euclidean action and we consider its real \( (\mathcal{A}_R) \) and imaginary \( (\mathcal{A}_I) \) parts separately:

\[ \mathcal{A}_R = \frac{1}{2} \text{Tr} \log(\mathcal{D}_E^\dagger \mathcal{D}_E), \]
\[ \mathcal{A}_I = \frac{1}{2} \text{Tr} \log((\mathcal{D}_E^\dagger)^{-1} \mathcal{D}_E). \] (2)

To keep the \( O(4) \)-invariance, we use Schwinger’s proper time regularization \([15]\). For the real part this prescription consists of replacing the logarithm by a parameter integral

\[ \mathcal{A}_R \rightarrow -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} ds \frac{ds}{s} \text{Tr} \exp \left( -s \mathcal{D}_E^\dagger \mathcal{D}_E \right), \] (3)

which for \( \Lambda \to \infty \) reproduces the logarithm up to an irrelevant constant.

Corresponding to the action \([\text{1}]\) the energy of the static soliton \( E^{\text{sol}} = E^f + E^m \) splits into a fermionic and a mesonic part, where \( E^f = E^{\text{val}} + E^{\text{vac}} \) contains valence and vacuum parts. To calculate \( E^f \) we express the Euclidean Dirac operator in its Hamiltonian form:

\[ i\mathcal{D}_E = \beta(-\partial_\tau - h), \] (4)

wherein \( \tau \) denotes the Euclidean time coordinate. Substituting the hedgehog ansatz \( U(r) = \exp\{i\hat{r} \cdot \hat{r} \Theta(r)\} \) for the chiral field and a radial function \( \Phi = \Phi(r) \) for the chiral radius the static Hamiltonian reads:

\[ h = \alpha \cdot p + \beta \Phi \left( \cos \Theta + i \hat{r} \cdot \tau \gamma_5 \sin \Theta \right). \] (5)

Denoting the associated eigenvalues by \( \epsilon_\mu \) the fermionic contribution to the static energy is given by \([10]\):

\[ E^{\text{vac}}[\Theta, \Phi] = \frac{N_c}{2} \frac{1}{\sqrt{4\pi}} \int_{1/\Lambda^2}^{\infty} ds s^{-3/2} \sum_\mu \left( \exp \left( -s\epsilon_\mu^2 \right) - \exp \left( -s(\epsilon_\mu^0)^2 \right) \right) \]
\[ E^{\text{val}}[\Theta, \Phi] = N_c \sum_\mu \eta_\mu |\epsilon_\mu| \] (6)

where \( \eta_\mu \) denote the (anti-) quark occupation numbers and \( \epsilon_\mu^0 \) are the eigenvalues of \( h \) for the mesonic vacuum \( \mathcal{M} = \langle \mathcal{M} \rangle = M \mathbf{1} \) with \( M \) being the constituent quark mass. Note that
the vacuum energy has been subtracted in (3). The only free parameter is \( M \) since \( \Lambda \) is fixed by fitting the pion decay constant \( f_\pi = 93\text{MeV} \). The gap equation relates the current quark mass \( m \) to the constituent quark mass \( M \). The coupling constant \( g \) is eliminated by fitting the pion mass \( m_\pi = 135\text{MeV} \): 

\[
g = \frac{mM}{m_\pi^2 f_\pi^2}[m_\pi^2 f_\pi^2] \tag{11}.
\]

Thus the mesonic part of the soliton energy may be expressed as

\[
E^m[\Theta, \Phi] = \frac{2\pi m_\pi^2 f_\pi^2}{mM} \int dr r^2 \left\{ \Phi^2(r) - M^2 - 2m (\Phi(r) \cos \Theta(r) - M) \right\}. \tag{7}
\]

Again the contribution of the vacuum configuration has been subtracted.

Until now, we did not take into account regularization of \( A_I \). Since this part is finite, the question arises whether it has to be regularized or not. Here we argue that regularization is necessary in order to have a consistent appearance of the one particle eigenenergies \( \epsilon_\mu \) \cite{16, 17, 6, 18}. \( A_I \) does not contribute to the soliton mass, however, its regularization has drastic consequences when we regard the sea contribution to baryon number

\[
B_{\text{vac}} = -\frac{1}{N_c} \lim_{T \to \infty} \frac{1}{T} \text{Tr} \left\{ h (-\partial_\tau^2 + h^2)^{-1} \right\}, \tag{8}
\]

where \( T \) is the Euclidean time interval under consideration. We introduce regularization of the baryon number by replacing \cite{3, 18}

\[
(-\partial_\tau^2 + h^2)^{-1} \to \int_1^{\infty} ds e^{-s(-\partial_\tau^2 + h^2)}, \tag{9}
\]

which again is an identity for \( \Lambda \to \infty \). The regularized baryon number \( B_\Lambda \) then reads:

\[
B_\Lambda = \sum \text{sign}(\epsilon_\mu) N_\mu + \eta_{\text{val}}, \tag{10}
\]

wherein \( N_\mu = -\frac{1}{2} \text{erfc} \left( \frac{\epsilon_\mu}{\Lambda} \right) \) denote the vacuum occupation numbers of the single quark orbits \( \mu \) in the proper-time regularization \cite{2}. In the limit \( \Lambda \to \infty \), \( B_\Lambda \) is obviously integer, however, this is no longer the case for finite \( \Lambda \). The main purpose of this paper is to investigate self-consistent solitons constrained to the baryon number \( B_\Lambda = 1 \). Accordingly, the energy functional is replaced by

\[
E = E^\text{val} + E^\text{vac} + E^m + M \left[ \lambda (B_\Lambda - 1)^2 - \frac{1}{2} a \lambda^2 \right]. \tag{11}
\]

The constituent quark mass \( M \) is included such that the Lagrange multiplier \( \lambda \) and the auxiliary parameter \( a \) are dimensionless. Varying the total energy functional \( E \) with respect to \( \lambda \) yields

\[
\lambda = \frac{(B_\Lambda - 1)^2}{a}. \tag{12}
\]

The limit \( a \to 0 \) has to be assumed in order to achieve \( B_\Lambda = 1 \). Variation of \( E \) with respect to the meson fields provides the equations of motion

\[
P(r) \cos \Theta(r) = \left[ S(r) - \frac{4\pi m_\pi^2 f_\pi^2}{N_C M} \right] \sin \Theta(r)
\]

\[
\Phi(r) = m \cos \Theta(r) - \frac{mN_C M}{4\pi m_\pi^2 f_\pi^2} \left[ S(r) \cos \Theta(r) + P(r) \sin \Theta(r) \right] \tag{13}
\]

\[4\]
where we have denoted

\[ S(r) = \sum_{\mu} \left[ (N_{\mu} + \eta_{\mu}) \text{sign}(\epsilon_{\mu}) + C_{\mu} \right] \int d\Omega \bar{\Psi}_{\mu} \Psi_{\mu}, \]

\[ P(r) = \sum_{\mu} \left[ (N_{\mu} + \eta_{\mu}) \text{sign}(\epsilon_{\mu}) + C_{\mu} \right] \int d\Omega \bar{\Psi}_{\mu} \left[ i \tau \cdot r \Theta(r) \gamma_{5} \right] \Psi_{\mu}. \]  

(14)

The derivative of the constraint with respect to the one particle eigenenergy \( \epsilon_{\mu} \) is proportional to

\[ C_{\mu} = \frac{2M\lambda(B_{\Lambda} - 1)}{N_{C} \sqrt{\pi \Lambda}} \exp \left\{ -\frac{\epsilon_{\mu}^{2}}{\Lambda^{2}} \right\}. \]  

(15)

**Numerical Results**

The self-consistent soliton is constructed by iterating the constraint (12) and the equations of motion (13) together with the corresponding eigenvalue problem associated with the Hamiltonian (5). The latter is solved by discretizing the momentum eigenvalues of the free Hamiltonian. This is achieved by putting the system into a large spherical box of radius \( D \). Our calculations are performed with a constituent quark mass \( M = 400 \) MeV. We consider this to be sufficient in order to demonstrate that the soliton is stabilized by constraining the regularized baryon number to unity.

Restricting the meson profiles to the chiral circle and omitting the constraint (11) we find \( B_{\Lambda} = 0.966 \) for the regularized baryon number. This number cannot be augmented considerably by including (11) and staying on the chiral circle. *i.e.* we do not find stable solutions in the limit \( a \rightarrow 0 \) when the chiral angle \( \Theta \) is taken to be the only space dependent meson field. Allowing, in addition, the chiral radius to be space dependent as well, we face the well-known problem that the soliton of scalar and pseudoscalar fields collapses. As discussed in the introduction this collapse is associated with the leakage of baryon number \( B_{\Lambda} \rightarrow 0 \) and as we will see, including the constraint (11) stabilizes the soliton. Indeed we find stable solutions with scalar and pseudoscalar fields when the regularized baryon number is constrained to unity. As can be observed from table 1, \( B_{\Lambda} = 1 \) cannot exactly be fulfilled for finite \( D \) and we conjecture that \( B_{\Lambda} = 1 \) is only attainable in the continuum limit, \( D \rightarrow \infty \).

In table 2 we display the energies of the self-consistent soliton solutions for various strengths of the constraint. These strengths are given in terms of the auxiliary parameter \( a \) and can be transferred to different regularized baryon numbers \( B_{\Lambda} \). One sees that the valence quark eigenenergy increases as \( B_{\Lambda} \) tends to unity. The resulting meson profiles are still localized as is obvious from figs. 1 and 2 where we display the meson profiles \( \Theta(r) \) and \( \Phi(r) \) corresponding to the regularized baryon numbers \( B_{\Lambda} \) of table 2. The chiral angle \( \Theta(r) \) exhibits a strong squeezing at the origin with growing baryon number, while for \( r > 1 \) fm all profile functions show the same spatial dependence. Thus, the constraint effects the meson profiles only at small \( r \). This can also be observed from the profile function \( \Phi(r) \) which almost vanishes at the origin, reflecting a local partial restoration of chiral symmetry.

In order to further investigate whether the solutions are localized objects we have considered the baryon densities (*cf.* fig. 3). Here, the sea quark part of the baryon density is of particular interest because it almost vanishes, whereas the energy of the Dirac sea turns out to be sizable. In order to understand this result, we split in fig. 4 the sea quark density into two parts stemming from the intrinsic positive and negative parity eigenfunctions of (5), respectively. It can clearly be seen that the contributions of these parts to the baryon density
Fig. 1. The chiral angle $\Theta(r)$ for the non-linear NJL soliton (short dashed line) and different strengths of the constraint.

Fig. 2. Same as fig. 1 for the chiral radius $\Phi(r)$. 
Table 1. The maximal baryon number obtained for different box radii $D$.

| D/fm | $B_\Lambda$ |
|------|-------------|
| 4    | 0.9970      |
| 5    | 0.9976      |
| 6    | 0.9977      |
| 7    | 0.9979      |
| 8    | 0.9980      |

Table 2. The soliton energy $E$ for constituent mass $M = 400$ MeV with its valence quark, sea and mesonic contributions $E^{\text{val}}$, $E^{\text{vac}}$ and $E^{\text{m}}$ for different baryon numbers $B_\Lambda$. The first line refers to the non-linear case. The small deviation of $E$ from $E^{\text{val}} + E^{\text{vac}} + E^{\text{m}}$ is due to the constraint in (11). All energies are given in MeV.

| $B_\Lambda$ | $E^{\text{val}}$ | $E^{\text{vac}}$ | $E^{\text{m}}$ | $E$  |
|-------------|------------------|------------------|----------------|------|
| 0.9662      | 632              | 572              | 34             | 1238 |
| 0.9932      | 914              | 753              | -387           | 1285 |
| 0.9939      | 925              | 755              | -398           | 1286 |
| 0.9946      | 941              | 758              | -426           | 1287 |
| 0.9965      | 996              | 770              | -475           | 1293 |
| 0.9973      | 1030             | 776              | -510           | 1294 |
| 0.9978      | 1043             | 780              | -529           | 1296 |

cancel. For the static energy, however, they sum up coherently resulting in a relatively large value.

Let us next turn to the discussion of a few static properties resulting from our soliton solution. In table 3 we display the isoscalar radius $\langle r_{I=0}^2 \rangle^{1/2}$ which is associated to the baryon charge distribution, the axial charge of the nucleon, $g_A$, as well as the moment of inertia $\alpha^2$ for collective rotations in isospace. The latter measures the $\Delta$-nucleon mass difference $M_\Delta - M_n = 3/2\alpha^2$. We should mention that the analytical form of these quantities in terms of sums involving matrix elements of the eigenstates of (5) do not differ from the original treatment of the pure pseudoscalar case. We thus may employ the relevant formulas from refs. [10, 19]. In agreement with the results for the static energy we observe for these quantities a dominating valence quark contribution as $B_\Lambda$ tends towards unity. Considering $e.g.$ the moment of inertia, $\alpha^2$, the Dirac sea contributions reduces from 23% to 4% when

Table 3. Baryon radius $\langle r_{I=0}^2 \rangle^{1/2}$, axial charge $g_A$ and the moment of inertia $\alpha^2$ for different baryon numbers. The first line denotes the data from the unconstrained soliton on the chiral circle ($\Phi \equiv M$).

| $B_\Lambda$ | $\langle r_{I=0}^2 \rangle^{1/2}$/fm | $\alpha^2/(\text{GeV})^{-1}$ | $g_A$ |
|-------------|------------------------------------|-----------------------------|-------|
| 0.9662      | 0.77                               | 5.52                        | 0.72  |
| 0.9933      | 0.93                               | 9.11                        | 0.83  |
| 0.9946      | 0.96                               | 9.87                        | 0.84  |
| 0.9965      | 1.03                               | 11.94                       | 0.86  |
| 0.9973      | 1.08                               | 13.54                       | 0.87  |
| 0.9978      | 1.11                               | 14.62                       | 0.89  |
Fig. 3. The baryon density $\rho(r)$ (full line) and its valence (short dashed line) and sea quark contribution (long dashed line) for the soliton with $B_\Lambda = 0.9978$.

Fig. 4. The sea quark contribution to the baryon density (full line) and its contributions from states with intrinsic positive parity (short dashed line) and negative intrinsic parity (long dashed line) for the soliton with $B_\Lambda = 0.9978$. 

\[ \rho = \rho_{\text{vac}} + \rho_{\text{val}} \]
we go from the pure pseudoscalar to \( B_\Lambda = 0.9978 \). This scheme may easily be understood by remarking that the quantities under consideration are rather sensitive to the large \( r \)-behavior of the fields: The valence quarks get enhanced of large \( r \) while the meson fields are squeezed at the origin. It is remarkable that the inclusions of the constraint improves our predictions for \( \langle r_{z=0}^2 \rangle^{1/2} \) and \( g_A \) while \( \alpha^2 \approx 15 \text{ GeV}^{-1} \) is far off the empirical value \( \alpha^2 = 3/2(M_\Delta - M_n) = 5 \text{ GeV}^{-1} \).

**Conclusions**

The main result of this letter is that the collapse of the NJL soliton is prevented when constraining the regularized baryon number to unity. This constraint can only be fulfilled when the scalar meson field is allowed to be space dependent. Then it displays a partial restoration of chiral symmetry close to the origin. Despite of the fact that the valence quark energy eigenvalue approaches the one without soliton the valence quark wave function is still quite well localized, and the Dirac sea is still polarized. The vacuum contribution to the baryon number vanishes, however, the vacuum contribution to the soliton energy is sizeable. Whereas some static properties of the baryons, the isoscalar mean square radius and the axial coupling \( g_A \), are improved when considering the \( B_\Lambda = 1 \) soliton, the moment of inertia is too large and therefore the predicted nucleon-\( \Delta \) splitting becomes too small.

We have seen that constraining the regularized baryon number leads to a valence quark dominated picture of the soliton. It would therefore be interesting to include the constraint also in the case when vector and axial vector fields are present. As is well known, these models support the Skyrmion picture of the baryon, *i.e.* the valence quark joins the Dirac sea [5]. First results indeed show that the inclusion of the constraint leads again to an increased valence quark eigenenergy which may even be positive.

In this letter we used the proper time regularization which already “damps” the contribution of low lying levels to physical quantities, especially to the baryon number. From this point of view it is easily understandable that levels are pushed “out of the gap”. The question remains whether a regularization procedure which does only influence high-lying levels will lead to a different qualitative behavior. This idea can probably be tested using dimensional regularization which is known to even enhance contributions from low-lying levels [20].
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