Inverse problem of electro-seismic conversion

Jie Chen and Yang Yang

Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195-4350, USA

E-mail: chenjie@uw.edu and yangy5@uw.edu

Received 9 March 2013, in final form 2 September 2013
Published 30 September 2013
Online at stacks.iop.org/IP/29/115006

Abstract
When a porous rock is saturated with an electrolyte, electrical fields are coupled with seismic waves via the electro-seismic conversion. Pride (1994 Phys. Rev. B 50 15678–96) derived the governing models, in which Maxwell equations are coupled with Biot’s equations through the electro-kinetic mobility parameter. The inverse problem of the linearized electro-seismic conversion consists in two steps, namely the inversion of Biot’s equations and the inversion of Maxwell equations. We analyze the reconstruction of conductivity and electro-kinetic mobility parameter in Maxwell equations with internal measurements, while the internal measurements are provided by the results of the inversion of Biot’s equations. We show that knowledge of two internal data based on well-chosen boundary conditions uniquely determines these two parameters. Moreover, a Lipschitz-type stability is proved based on the same sets of well-chosen boundary conditions.

1. Introduction
When a porous rock is saturated with an electrolyte, an electric double layer is formed at the interface of the solid and the fluid. One side of the interface is negatively charged and the other side is positively charged. Such an electric double layer (EDL) system is also called the Debye layer. Due to the EDL system, electromagnetic (EM) fields and mechanical waves are coupled through the phenomenon of electro-kinetics. Precisely, electrical fields or EM waves acting on the EDL will move the charges, creating the relative movement of the fluid and solid. This is called electro-seismic conversion. Conversely, mechanical waves moving fluid and solid will generate EM fields. This is called seismo-electric conversion. Thompson and Gist [9] have made field measurements clearly demonstrating seismo-electric conversion in saturated sediments. Zhu et al [12–14] performed laboratory experiments and observed seismo-electric conversion in model wells, and their experimental results confirm that seismo-electric logging could be a new bore-hole logging technique.

The investigation of wave propagation in fluid-saturated porous media was early developed by Biot [2, 3]. The governing equations of the electro-seismic conversion were derived by
Pride [7] as follows:

\[ \nabla \times E = \omega \mu H, \]  
\[ \nabla \times H = (\sigma - i\epsilon \omega)E + L(-\nabla p + \omega^2 \rho_f u) + J_s, \]  
\[ -\omega^2 (\rho u + \rho_f w) = \nabla \cdot \tau, \]  
\[ -\omega w = LE + \frac{k}{\eta} (-\nabla p + \omega^2 \rho_f u), \]  
\[ \tau = (\lambda \nabla \cdot u + c \nabla \cdot w) I + G(\nabla u + \nabla u^T), \]  
\[ -p = c \nabla \cdot u + M \nabla \cdot w, \]

where the first two are Maxwell’s equations, and the remainder are Biot’s equations. The notation is as follows:

- \( E \) electric field
- \( H \) magnetizing field or magnetic field intensity
- \( \omega \) seismic wave frequency
- \( \sigma \) conductivity
- \( \epsilon \) dielectric constant or relative permittivity
- \( \mu \) magnetic permeability
- \( J_s \) source current
- \( p \) pore pressure
- \( \rho_f \) density of pore fluid
- \( L \) electro-kinetic mobility parameter
- \( \kappa \) fluid flow permeability
- \( u \) solid displacement
- \( w \) fluid displacement
- \( \tau \) bulk stress tensor
- \( \eta \) viscosity of pore fluid
- \( \lambda, G \) Lamé parameters of elasticity
- \( C, M \) Biot moduli parameters.

Pride and Haartsen [8] also analyzed the basic properties of seismo-electric waves.

Note that the coupling is nonlinear, namely electro-seismic and seismo-electric conversions occur simultaneously. Under the assumption that the coupling is so weak that multiple coupling is negligible, we can linearize the forward system in two steps. Particularly, we focus on electro-seismic conversion and ignore seismo-electric conversion. The first step in the forward system is modeled by the Maxwell equations without the effect of the seismic waves, i.e., \( L = 0 \) in (2). While electro-seismic conversion occurs, seismic waves are generated and modeled by Biot’s equations with the potential \( LE \) in (4).

In this paper, we mainly focus on the inverse problem of the linearized electro-seismic conversion, which is a hybrid problem and consists of two steps. The first step of the inverse problem is to invert Biot’s equations, i.e., to recover the potential \( LE \) in (4) from any measurements observed on the domain boundary. Williams [11] presented an approximation to Biot’s equations, which could be a useful tool to study the inverse problem.

Assuming the first step is implemented successfully, the second step of the inverse problem is to invert Maxwell’s equations, which consists of reconstructing the conductivity \( \sigma \) and the electro-kinetic mobility parameter or the coupling coefficient \( L \) from boundary measurements of the electrical fields and the internal data \( LE \) obtained in the first step.
The problem of interest in this paper is the second step of the inverse problem. We study the reconstruction of the conductivity $\sigma$ and the coupling coefficient $L$ and prove uniqueness and stability results of the reconstructions. Particularly, we show that $\sigma$ and $L$ are uniquely determined by two well-chosen electrical fields at the domain boundary. The explicit reconstruction procedure is presented. The stability of the reconstruction is established from either two measurements under geometrical conditions or from six well-chosen boundary conditions.

Mathematically, our proof relies on explicit solutions to Maxwell’s equations, namely complex geometrical optics (CGO) solutions, constructed by Colton and Päivärinta [5]. In our reconstruction procedure, the coupling coefficient $L$ satisfies a transport equation with the vector field $\beta$. With CGO solutions, we can prove that the integral curves of the vector field $\beta$ are close to straight lines and exit the domain in finite time. Therefore, $L$ can be uniquely and explicitly solved by the characteristic method. Stability follows the analysis of the method of characteristic.

The rest of the paper is structured as follows. Section 2 presents our main results. The CGO solutions are introduced in section 3.1. The inverse Maxwell’s equations and an explicit reconstruction algorithm are addressed in the rest of section 3, while section 3.2 focuses on the proof of the uniqueness result and sections 3.3 and 3.4 focus on the stability proof.

2. Main results

Let $\Omega$ be an open, bounded and connected domain in $\mathbb{R}^3$ with the boundary $\partial \Omega$ of class $C^2$. In the second step of the electro-seismic conversion, the propagation of the electrical fields is modeled by Maxwell’s equations in $\Omega$,

\[
\begin{align*}
\nabla \times E &= i\omega \mu H, \\
\nabla \times H &= (\sigma - i\epsilon \omega)E + J_s.
\end{align*}
\]

The measurements available for the inverse problem include the internal data from the first step

\[D := LE, \quad \text{in } \Omega\]

and the boundary illumination, i.e., the tangential boundary measurement of the electrical field

\[G := tE, \quad \text{on } \partial \Omega.\]

Define the operator

\[\Lambda_M(L, \sigma) := (J_s, D, G).\]

The problem now is to invert the operator $\Lambda_M$, or namely to reconstruct $(L, \sigma)$ from some measurements $(J_{s,j}, D_j, G_j)$ indexed by $j$, assuming that $\mu$ and $\epsilon$ are given.

The main purpose of this paper is to prove the uniqueness and stability of the coefficient reconstructions. For small $\iota > 0$, define the set of coefficients $(L, \sigma) \in \mathcal{M}$ as

\[\mathcal{M} = \{(L, \sigma) \in C^{d+1}(\overline{\Omega}) \times H^{1+1/2+\iota d+\iota}(\Omega): \text{and } 0 \text{ is not an eigenvalue of } \nabla \times \nabla \times \cdot - k^2n\},\]

where the wave number $k > 0$ and the refractive index $n$ are given by

\[k = \omega \sqrt{\epsilon_0 \mu_0}, \quad n = \frac{1}{\epsilon_0} \left(\epsilon + i \frac{\sigma}{\omega}\right).\]

The main results are as follows, where the measurements $G$ and $D$ are complex-valued.
Theorem 2.1. Let $\Omega$ be an open, bounded subset of $\mathbb{R}^3$ with the boundary $\partial\Omega$ of class $C^d$. Let $(L, \sigma)$ and $(\tilde L, \tilde \sigma)$ be two elements in $M$ with $L|_{\partial\Omega} = \tilde L|_{\partial\Omega}$. Let $D := (D_1, D_2)$ and $\tilde D := (\tilde D_1, \tilde D_2)$ be two sets of internal data on $\Omega$ for the coefficients $(L, \sigma)$ and $(\tilde L, \tilde \sigma)$, respectively, and with the boundary illuminations $G := (G_1, G_2)$.

Then, there is a non-empty open set of $G \in (C^{d+4}(\partial\Omega))^2$, such that if $D_j = \tilde D_j$, $j = 1, 2$, we have $(L, \sigma) = (\tilde L, \tilde \sigma)$.

Here and in the following, we shall abuse the notation and use $C^d(\overline{\Omega})$ to denote either set of complex-valued functions or set of vector-valued functions whose elements have up to $d$-order continuous derivatives. It should be clear from the context which one it is. The function space $(C^{d+4}(\partial\Omega))^2$ is an abbreviation of the product space $C^{d+4}(\partial\Omega) \times C^{d+4}(\partial\Omega)$.

To consider the stability of the reconstruction, we need to restrict ourselves to a subset of $\Omega$. Let $\zeta_0$ be a constant unit vector. Let $x_0 \in \partial\Omega$ be the tangent point of $\partial\Omega$ with respect to $\zeta_0$, i.e., the tangent line of $\partial\Omega$ at $x_0$ is parallel to $\zeta_0$. Define $\Omega_1$ to be the subset of $\Omega$ by removing a neighborhood of each tangent point $x_0 \in \partial\Omega$.

Theorem 2.2. Let $d \geq 2$. Let $\Omega$ be convex with the boundary $\partial\Omega$ of class $C^d$ and $\Omega_1$ is defined as above. Assume that $(L, \sigma)$ and $(\tilde L, \tilde \sigma)$ are two elements in $M$ with $L|_{\partial\Omega} = \tilde L|_{\partial\Omega}$. Let $D = (D_j)$ and $\tilde D = (\tilde D_j)$, $j = 1, 2$, be the internal data for the coefficients $(L, \sigma)$ and $(\tilde L, \tilde \sigma)$, respectively, with the boundary conditions $G = (G_j)$, $j = 1, 2$.

Then there exists a non-empty open set of illuminations $G \in (C^{d+4}(\partial\Omega))^2$, such that restricting ourselves to $\Omega_1$, we have
\[
\|L - \tilde L\|_{C^{d+1}(\partial\Omega)} + \|\sigma - \tilde \sigma\|_{C^{d+1}(\partial\Omega)} \leq C\|D - \tilde D\|_{(C^{d+1}(\partial\Omega))^2}.
\] (13)

The geometric conditions can be removed when more measurements are available. In particular, when six complex measurements are provided, we have the following stability result.

Theorem 2.3. Let $d \geq 3$. Let $\Omega$ be convex and $\Omega_1$ is defined as above. Assume that $(L, \sigma)$ and $(\tilde L, \tilde \sigma)$ are two elements in $M$ with $L|_{\partial\Omega} = \tilde L|_{\partial\Omega}$. Let $D = (D_j^1, D_j^2)$ and $\tilde D = (\tilde D_j^1, \tilde D_j^2)$, $j = 1, 2, 3$, be the internal data for the coefficients $(L, \sigma)$ and $(\tilde L, \tilde \sigma)$, respectively, with the boundary conditions $G = (G_j^1, G_j^2)$, $j = 1, 2, 3$.

Then there exists a non-empty open set of illuminations $G \in (C^{d+4}(\partial\Omega))^6$, such that
\[
\|L - \tilde L\|_{C^{d+1}(\partial\Omega)} + \|\sigma - \tilde \sigma\|_{C^{d+1}(\partial\Omega)} \leq C\|D - \tilde D\|_{(C^{d+1}(\partial\Omega))^6}.
\] (14)

Note that the above measurements are all complex-valued. We will need two real measurements to make up one complex datum.

3. Inversion of Maxwell’s equations with internal data

Let $\Omega$ be an open, bounded and connected domain in $\mathbb{R}^3$ with the boundary $\partial\Omega$ of class $C^2$. In the case when $\mu \equiv \mu_0$ is constant and $J_e = 0$ in $\Omega$, we can rewrite the system in (7) as
\[
\nabla \times \nabla \times E - k^2 n E = 0
\] (15)
and
\[
\nabla \cdot n E = 0,
\] (16)
where the wave number $k > 0$ and the refractive index $n$ are given by (12). We would like to consider equations (15) and (16) in the whole $\mathbb{R}^3$. For this purpose, we extend $n \in H^{1+3d+4}(\Omega)$ to be a function defined in the whole $\mathbb{R}^3$ in such a way that $1 - n \in H^1 + 3d+4(\mathbb{R}^3)$ is compactly supported. We denote this extension still by $n$. 
3.1. Complex geometrical optics solutions

Colton and Päivärinta [5] constructed explicit solutions, namely CGO solutions, to Maxwell’s equations (15) and (16). CGOs will be the main technique we will use to solve the inverse Maxwell problem. We follow the construction of CGOs in [5] and extend their properties from $L^2$ space to higher order Sobolev spaces. CGOs are of the form

$$E(x) = e^{ik·x}(\eta + R_ε(x)), \quad (17)$$

where $\xi \in \mathbb{C}^3 \setminus \mathbb{R}^3$ and $\eta \in \mathbb{C}^3$ are the constant vectors satisfying

$$\xi \cdot \xi = k^2, \quad \xi \cdot \eta = 0. \quad (18)$$

Substituting (17) into (15) and (16) gives

$$\tilde{\nabla} \times \tilde{\nabla} \times R_ε = k^2(n - 1)\eta + k^2nR_ε, \quad (19)$$

$$\tilde{\nabla} \cdot R_ε = -\alpha \cdot (\eta + R_ε), \quad (20)$$

where $\tilde{\nabla} := \nabla + i\xi$ and $\alpha := \nabla n(x)/n(x)$. We further define $\tilde{\Delta} := \Delta + 2i\xi \cdot \nabla - k^2$. By substituting the formula $\tilde{\nabla} \times \tilde{\nabla} \times R_ε = -\tilde{\Delta}R_ε + \tilde{\nabla} \cdot \nabla \cdot R_ε$ into (19) and (20), we see that $R_ε$ is a solution to

$$(\Delta + 2i\xi \cdot \nabla)R_ε = -\tilde{\nabla}(\alpha \cdot (\eta + R_ε)) + k^2(1 - n)\eta + R_ε. \quad (21)$$

It was proved in [5], the existence of $R_ε$ to (21) as a $C^2(\mathbb{R}^3)$ function. For our analysis, we need to extend the results of CGOs in [5] to smoother function spaces.

Let the space $L^2_δ$ for $δ \in \mathbb{R}$ be the completion of $C_0^∞(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{L^2_δ}$ defined as

$$\|u\|_{L^2_δ} = \left(\int_{\mathbb{R}^3} |\langle x \rangle|^{2\delta} |u|^2 \, dx \right)^{1/2}, \quad (22)$$

To obtain smoother CGOs than that constructed in [5], we introduce the space $H^δ$ for $s > 0$ as the completion of $C_0^∞(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{H^δ}$ defined as

$$\|u\|_{H^δ} = \left(\int_{\mathbb{R}^3} \langle \xi \rangle^{2\delta} (I - \tilde{\Delta})^δ |u|^2 \, d\xi \right)^{1/2}. \quad (23)$$

Here, $(I - \tilde{\Delta})^δ u$ is defined as the inverse Fourier transform of $\langle \xi \rangle^δ \hat{u}(\xi)$, where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$. When $δ = 0$, this is the standard Sobolev space of order $s$, which we denote by $H^s(\mathbb{R}^3)$.

We recall [10] for $|\xi| \geq c > 0$ and $u \in L^2_{s+1}$ with $-1 < \delta < 0$, the equation

$$(\Delta + 2i\xi \cdot \nabla)u = v \quad (24)$$

admits a unique weak solution $u \in L^2_{s+1}$ with

$$\|u\|_{L^2_δ} \leq C(\delta, c) \frac{\|v\|_{L^2_{s+1}}}{|\xi|}. \quad (25)$$

Since $(\Delta + 2i\xi \cdot \nabla)$ and $(I - \Delta)^δ$ are the constant coefficient operators and hence commute, we deduce that when $v \in H^s_{s+1}$, for $s > 0$, then

$$\|u\|_{H^δ} \leq C(\delta, c) \frac{\|v\|_{H^s_{s+1}}}{|\xi|}. \quad (26)$$

We define the integral operator $G_ε : H^s_{s+1}(\mathbb{R}^3) \rightarrow H^s_δ(\mathbb{R}^3)$ by

$$G_ε(v) := F^{-1} \left( \frac{\hat{\nu}}{|\xi|^2 + 2\xi \cdot \xi} \right), \quad (27)$$
where $F^{-1}$ is the inverse Fourier transform. We see that $G_\xi$ is bounded and there exists a positive constant $C(\delta)$, such that
\[
\|G_\xi\| \leq \frac{C}{|\xi|}. \tag{28}
\]

Before we can prove the existence of a unique solution to (21), we first prove the following lemma.

**Lemma 3.1.** For any $v \in H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$ and $|\xi|$ sufficiently large, the equation
\[
(\Delta + 2i\xi \cdot \nabla + \alpha \cdot \vec{V})u = v
\]
has a unique solution $u \in H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$ satisfying
\[
\|u + n^{-1/2}G_\xi(n^{-1/2}v)\|_{H^{\frac{1}{2}+d+\epsilon}_\delta} \leq \frac{C}{|\xi|}, \tag{29}
\]
for some positive constant $C$ independent of $\xi$.

Here, the complex-valued function $n^{1/2}$ is defined as $n^{1/2} := e^{\frac{1}{2}\text{Log}(n)}$ where $\text{Log}$ is the principal branch of the complex logarithmic function, i.e., a branch cut along the negative $x$-axis with $\text{Log}(1) = 0$. This is possible since $\text{Re} n = \frac{n}{\delta}$ is always positive. Lemma 3.1 in [5] proves the case when $s = 0$. We study $s = \frac{1}{2} + d + \epsilon$ here.

**Proof.** From the identity
\[
n^{-1/2}(\Delta + 2i\xi \cdot \nabla)(n^{1/2}u) = (\Delta + 2i\xi \cdot \nabla + \alpha \cdot \vec{V})u + qu,
\]
where $q := \Delta n^{1/2}/n^{1/2} \in H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$, we can rewrite (29) as
\[
(\Delta + 2i\xi \cdot \nabla - q)f = g,
\]
where $f := n^{1/2}u$ and $g := n^{1/2}v$. The assumption on $n$ ensures that $1 - n^{1/2} \in H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$ and is compactly supported, so $1 - n^{1/2} \in H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$. Therefore,
\[
g = n^{1/2}v = v - (1 - n^{1/2})v \in H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3).
\]
Applying the integral operator $-G_\xi$ gives
\[
f + G_\xi(qf) = -G_\xi(g).
\]
Since $q \in H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$ is compactly supported, multiplication by $q$ is a bounded operator mapping $H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$ into $H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$, so $I + G_\xi(q \cdot)$ is invertible on $H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$ for $|\xi|$ sufficiently large. This shows that (33) has a unique solution $f$ in $H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$; correspondingly $u = n^{-1/2}f \in H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$ is the unique solution of (29). Equation (28) also gives
\[
\|f + G_\xi(qf)\|_{H^{\frac{1}{2}+d+\epsilon}_\delta} = \|G_\xi(q(g(G_\xi(qf) + G_\xi(g))))\|_{H^{\frac{1}{2}+d+\epsilon}_\delta} \leq \frac{C}{|\xi|}. \tag{34}
\]
for some positive constant $C$ independent of $\xi$. This proves the lemma. □

**Proposition 3.2.** For $|\xi|$ sufficiently large, there is a unique solution $R_\xi \in H^{\frac{1}{2}+d+\epsilon}_\delta (\mathbb{R}^3)$ to (21). Thus, the CGO solution $E$ defined by (17) satisfies (15) and (16). Moreover, $R_\xi$ satisfies
\[
\|R - in^{-1/2}G_\xi(n^{1/2}\alpha \cdot \eta)\|_{H^{\frac{1}{2}+d+\epsilon}_\delta} = O\left(\frac{1}{|\xi|}\right). \tag{35}
\]
Proof. By applying the vector identity
\[ \nabla (A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A, \]
we see that
\[ \nabla (\alpha \cdot (\eta + R_\xi)) = \alpha \times (\nabla \times R_\xi) + (\alpha \cdot \nabla)R_\xi + (R_\xi \cdot \nabla)\alpha + \nabla (\alpha \cdot \eta). \]
The terms which are potentially troublesome are \( \alpha \times (\nabla \times R_\xi) \) and \( (\alpha \cdot \nabla)R_\xi \). The latter can be dealt with using lemma 3.1, so we need only consider the term \( \alpha \times (\nabla \times R_\xi) \). Denote \( Q := \nabla \times R_\xi \); by (19), we have that
\[ \nabla \times Q = k^2(n - 1)\eta + k^2nR_\xi \]
and hence
\[ \nabla \times \nabla \times Q = k^2\nabla n \times (\eta + R_\xi) + k^2(n - 1)\zeta \times \eta + k^2nQ. \]
Since \( \nabla \cdot Q = 0 \), we now have
\[ \Delta Q + 2i\zeta \cdot \nabla Q = k^2\nabla m \times (\eta + R_\xi) + k^2(1 - n)(i\zeta \times \eta + Q). \]
Rearrange the terms to obtain
\[ (\Delta + 2i\zeta \cdot \nabla - k^2(1 - n))Q = k^2\nabla (1 - n) \times (\eta + R_\xi) + k^2(1 - n)(i\zeta \times \eta). \]
Applying \( -G_\zeta \) to this identity yields
\[ Q = -(I + k^2G_\zeta (1 - n))^{-1}G_\zeta (k^2\nabla (1 - n) \times (\eta + R_\xi) + k^2(1 - n)(i\zeta \times \eta)) \]
for large \( |\zeta| \), since the operator \( I + k^2G_\zeta (1 - n) \) is invertible on \( H^{\frac{5}{2}+d+\epsilon}_{\delta}(\mathbb{R}^3) \) for large \( |\zeta| \). On the other hand, applying \( -G_\zeta \) to (21) and using (37) gives
\[ R_\xi = G_\zeta (\alpha \times Q) + G_\zeta [(\alpha \cdot \nabla)R_\xi] + G_\zeta [(R_\xi \cdot \nabla)\alpha] + G_\zeta [\nabla (\alpha \cdot \eta) - k^2G_\zeta (1 - n)(\eta + R_\xi)] \]
where \( Q \) is given by (42). The integral equations (42) and (43) have a unique solution in \( H^{\frac{5}{2}+d+\epsilon}_{\delta}(\mathbb{R}^3) \) due to (28) and lemma 3.1.
Finally, we use the unique solvability of (42) and (43) to deduce the unique solvability of (21). To do this, let \( B \subset \mathbb{R}^3 \) be a ball containing \( \Omega \). Applying \( G_\zeta \) to (21) yields
\[ R_\xi = G_\zeta [\nabla (\alpha \cdot (\eta + R_\xi))] - k^2G_\zeta [(1 - n)(\eta + R_\xi)]. \]
This integral equation is of Fredholm-type in \( H^{\frac{5}{2}+d+\epsilon}_{\delta}(B) \) as both \( G_\zeta (1 - n) \) and \( R_\xi \mapsto G_\zeta [\nabla (\alpha \cdot R_\xi)] \) are smoothing operators. Now, suppose \( R_\xi^b \) is a solution of the following homogeneous equation in \( H^{\frac{5}{2}+d+\epsilon}_{\delta}(B) \),
\[ R_\xi^b = G_\zeta [\nabla (\alpha \cdot R_\xi^b)] - k^2G_\zeta [(1 - n)R_\xi^b]. \]
Then \( R_\xi^b \) also satisfies the homogeneous equation corresponding to (42) and (43). Since these equations are uniquely solvable, we conclude that \( R_\xi^b \equiv 0 \). Therefore, by Fredholm alternative, we conclude that (44) admits a unique solution in \( H^{\frac{5}{2}+d+\epsilon}_{\delta}(B) \). Defining \( R_\xi (x) \) for \( x \in \mathbb{R}^3 \) by the right-hand side of (44) and recalling \( \Omega \subset B \) yields a solution of (44), which is defined in \( \mathbb{R}^3 \). This shows that (21) has a unique solution in \( H^{\frac{5}{2}+d+\epsilon}_{\delta}(\mathbb{R}^3) \).
Furthermore, by lemma 3.1, we see that
\[
\|R_\xi + n^{-1/2}G_\zeta (n^{1/2}[-\zeta \times (\nabla \times R_\xi) - (R_\xi \cdot \nabla)\alpha - \nabla (\alpha \cdot \eta) + k^2(1 - n)(\eta + R_\xi)])\|_{H^{\frac{5}{2}+\epsilon}_{\delta}} \leq O \left( \frac{1}{|\zeta|^2} \right).
\]
\[ = O \left( \frac{1}{|\zeta|^2} \right). \]
Substituting $\tilde{\nabla}(\alpha \cdot \eta) = (\nabla + i\zeta)(\alpha \cdot \eta)$, (46) implies that
\[
\|R_\xi + in^{-1/2}G_\xi(n^{1/2}\alpha \cdot \eta)\|_{H^s_{\xi}} \leq \|n^{-1/2}G_\xi(n^{1/2}[\alpha \times (\tilde{\nabla} \times R_\xi)] - (R_\xi \cdot \nabla)\alpha - \nabla \alpha \cdot \eta + k^2(1 - n)(\eta + R_\xi))\|_{H^s_{\xi}} + O\left(\frac{1}{|\zeta|^2}\right)
\]
(47)
\[
= O\left(\frac{1}{|\zeta|^2}\right).
\]
(48)
This completes the proof. □

By the Sobolev embedding theorem, we have the estimate in $C^{d+1}(\Omega)$.

**Corollary 3.3.** Let $\Omega$ be an open, bounded domain in $\mathbb{R}^3$. With the same hypotheses as the previous proposition, we then have
\[
\|R_\xi - in^{-1/2}G_\xi(n^{1/2}\alpha \cdot \eta)\|_{C^{d+1}(\Omega)} \leq \frac{C}{|\zeta|},
\]
for some positive constant $C$.

**Proposition 3.4.** Suppose $\zeta \in C^3 \setminus \mathbb{R}^3$, $\eta \in C^3$, satisfy $\zeta \cdot \xi = k^2$ and $\zeta \cdot \eta = 0$, such that as $|\xi| \to \infty$ the limits $\zeta/|\zeta|$ and $\eta$ exist and
\[
|\zeta/|\zeta| - \zeta_0| = O\left(\frac{1}{|\zeta|}\right), \quad |\eta - \eta_0| = O\left(\frac{1}{|\zeta|}\right).
\]
(50)
\[
R_\xi \in H^{d+1}_{\delta} (\mathbb{R}^3) \text{ is the unique solution to (21) in proposition 3.2. For } |\xi| \text{ large,}
\]
\[
\|R_\xi - i|\zeta|n^{-1/2}G_\xi(n^{1/2}\alpha \cdot \eta_0)\|_{C^{d+1}(\Omega)} = O\left(\frac{1}{|\zeta|}\right).
\]
(51)

**Proof.** The proof follows directly by substituting (50) into (49). □

We choose the specific sets of $\zeta$, $\eta$ as in [5]. Precisely, let $h$ be a small real parameter and choose arbitrary $a \in \mathbb{R}$. We define $\zeta_1$, $\zeta_2$ and $\eta_1$, $\eta_2$ by
\[
\zeta_1 = (a/2, i\sqrt{1/h^2 + a^2/4 - k^2}, 1/h),
\]
\[
\zeta_2 = (a/2, -i\sqrt{1/h^2 + a^2/4 - k^2}, -1/h),
\]
\[
\eta_1 = \frac{1}{\sqrt{1/h^2 + a^2}}(1/h, 0, -a/2),
\]
\[
\eta_2 = \frac{1}{\sqrt{1/h^2 + a^2}}(1/h, 0, a/2),
\]
(52)
and note that
\[
\lim_{h \to 0} \eta_j = \eta_0 := (1, 0, 0), \quad j = 1, 2,
\]
\[
\lim_{h \to 0} \zeta_1/|\zeta_1| = \zeta_0 := \frac{1}{\sqrt{2}}(0, i, 1),
\]
\[
\lim_{h \to 0} \zeta_2/|\zeta_2| = -\zeta_0
\]
(53)
and
\[
\zeta_1 + \zeta_2 = (a, 0, 0), \quad \zeta_0 \cdot \zeta_0 = 0, \quad \eta_0 \cdot \zeta_0 = 0.
\]
(54)

**Proposition 3.4** implies that
\[
(\eta_1 + R_{\xi_1}) \cdot (\eta_2 + R_{\xi_2}) = 1 + o(1)
\]
in $C^{d+1}(\Omega)$.
3.2. Construction of vector fields and uniqueness result

Let us now consider the reconstruction of \((L, \sigma)\). Assume \(E_j, j = 1, 2\), to be two complex solutions to

\[
\nabla \times \nabla \times E_j - k^2 n E_j = 0 \quad \text{in} \ \Omega,
\]

with the tangential boundary conditions

\[
t E_j = G_j \quad \text{on} \ \partial \Omega,
\]

with \(G_j\) being the well-chosen boundary values and \(j = 1, 2\). We will see that

\[
\nabla \times \nabla \times E_1 \cdot E_2 - \nabla \times \nabla \times E_2 \cdot E_1 = 0.
\]

Let \(D_j = L E_j, j = 1, 2\), be the internal complex-valued measurements. Assume \(L \in C^{d+1} (\Omega)\) is non-vanishing, then \(D_j = L E_j \in C^{d+1} (\Omega)\), since \(E_j \in C^{d+1} (\Omega)\). Substituting \(E_j = D_j / L\) into (58), we have, after some algebraic calculation,

\[
\beta \cdot \nabla L + \gamma L = 0,
\]

where

\[
\beta = \chi (x) \{ [\nabla D_1] D_2 - [\nabla D_2] D_1 \} + [\nabla \cdot D_1] D_2 - [\nabla \cdot D_2] D_1
\]

\[
= -2 [\nabla D_1]^T D_2 - [\nabla D_2] D_1],
\]

\[
\gamma = \chi (x) [\nabla (\nabla \cdot D_1) \cdot D_2 - \nabla (\nabla \cdot D_2) \cdot D_1] - [\nabla^2 D_1 \cdot D_2 - \nabla^2 D_2 \cdot D_1].
\]

Here, \(\chi (x)\) is a smooth known complex-valued function with \(|\chi (x)|\) uniformly bounded from below by a positive constant on \(\Omega\).

To show the transport equation (59) has a unique solution, it suffices to prove that the direction of the vector field \(\beta\) is close to a vector field which has a fixed direction and thus the integral curves of \(\beta\) connect every internal point to two boundary points.

Let \(\hat{E}_1\) and \(\hat{E}_2\) be two CGOs with the parameters \(\zeta_1, \zeta_2\) and \(\eta_1, \eta_2\) defined in (52), i.e.,

\[
\hat{E}_j = e^{i \zeta_j} (\eta_j + R_{\zeta_j}), \quad j = 1, 2.
\]

Let \(\tilde{D}_j = L \hat{E}_j, j = 1, 2\), be the corresponding internal data. By choosing \(\chi (x) = -e^{-i (\zeta_1 + \zeta_2)} \frac{h}{4 \sqrt{2}}\) and substituting \(\tilde{D}_j\) into (60), we can analyze the asymptotic behavior of the vector field \(\beta\) as \(|\zeta_j| \to \infty\), or equivalently, \(h \to 0\). Indeed, we have

\[
\nabla \tilde{D}_j = e^{i \zeta_j} [\nabla L \cdot (\eta_j + R_{\eta_j}) + \mu (\eta_j + R_{\eta_j}) \zeta_j^T + \mu \nabla (\eta_j + R_{\eta_j})], \quad j = 1, 2,
\]

and

\[
\chi (x) (\nabla \tilde{D}_j) \tilde{D}_j = -\frac{Lh}{4 \sqrt{2}} [(\eta_1 + R_{\eta_1}) (\nabla L)^T + i L (\eta_1 + R_{\eta_1}) \zeta_1^T + \mu \nabla (\eta_1 + R_{\eta_1})] (\eta_2 + R_{\eta_2})
\]

\[
= -\frac{Lh}{4 \sqrt{2}} [(\eta_1 + R_{\eta_1}) (\nabla L)^T (\eta_2 + R_{\eta_2}) + i L \zeta_1^T (\eta_2 + R_{\eta_2})]
\]

\[
+ \mu \nabla (\eta_1 + R_{\eta_1}) (\eta_2 + R_{\eta_2})]
\]

\[
= -\frac{L^2 h}{4 \sqrt{2}} (\eta_1 + R_{\eta_1}) (\eta_2 + R_{\eta_2}) + O(h).
\]

Therefore, by proposition 3.4, \(\chi (x) (\nabla \tilde{D}_j) \tilde{D}_j \to 0\) in \(C^{d} (\Omega)\) norm as \(h \to 0\). Similarly,

\[
\chi (x) (\nabla \tilde{D}_j)^T \tilde{D}_j = -\frac{Lh}{4 \sqrt{2}} [\nabla L (\eta_1 + R_{\eta_1})^T (\eta_2 + R_{\eta_2}) + i L \zeta_1 (\eta_1 + R_{\eta_1})^T (\eta_2 + R_{\eta_2})
\]

\[
+ L \nabla (\eta_1 + R_{\eta_1})^T (\eta_2 + R_{\eta_2})]
\]

\[
= -\frac{i L^2 \zeta_1 h}{4 \sqrt{2}} (\eta_1 + R_{\eta_1}) \cdot (\eta_2 + R_{\eta_2}) + O(h)
\]

\[
\to -\frac{i L^2 \zeta_0}{4} \quad \text{in} \ C^{d} (\Omega) \quad \text{as} \quad h \to 0.
\]
More calculation gives that
\[
\chi(x)(\nabla \cdot \hat{D}_1)\hat{D}_2 = -\frac{L}{4\sqrt{2c}}[(\nabla \mu \cdot (\eta_1 + R_{\xi_1}))((\eta_2 + R_{\xi_2}) + iL(\zeta_1 \cdot (\eta_1 + R_{\xi_1}))((\eta_2 + R_{\xi_2})
\]
\[
\rightarrow 0 \text{ in } C^d(\Omega) \text{ as } h \rightarrow 0.
\]
(66)

By substituting (64), (65) and (66) into (60), we have
\[
\lim_{h \rightarrow 0} \| \beta - iL^2 \xi_0 \|_{C^d(\Omega)} = 0,
\]
(67)
i.e., the vector fields have approximately constant directions for small \(h\) and their integral curves connect every internal point to two boundary points. Thus, the transport equation (59) admits a unique solution.

To see the dependence of vector fields on the boundary conditions, we need to introduce a regularity theorem of Maxwell’s equations. Let \(tE\) be the tangential boundary condition of \(E\). The following function spaces were introduced in [6].

Define the Div-spaces as
\[
H^s_{\text{Div}}(\Omega) = \{u \in H^s(\Omega) : \text{Div}(tu) \in H^{s-1/2}(\partial \Omega)\},
\]
(68)
\[
TH^s_{\text{Div}}(\partial \Omega) = \{g \in H^s(\partial \Omega) : \text{Div}(g) \in H^s(\partial \Omega)\},
\]
(69)
where \(H^s(\Omega)\) is a space of vector functions of which each component is in \(H^s(\Omega)\). These are Hilbert spaces with norms
\[
\|u\|_{H^s_{\text{Div}}(\Omega)} = \|u\|_{H^s(\Omega)} + \|\text{Div}(tu)\|_{H^{s-1/2}(\partial \Omega)},
\]
(70)
\[
\|g\|_{TH^s_{\text{Div}}(\partial \Omega)} = \|g\|_{H^s(\partial \Omega)} + \|\text{Div}(g)\|_{H^s(\partial \Omega)}.
\]
(71)
It is clear that \(t(H^s_{\text{Div}}(\Omega)) = TH^s_{\text{Div}}(\partial \Omega)\).

**Proposition 3.5** ([6]). Let \(\epsilon, \mu \in C^s, s > 2\), be positive functions. There is a discrete subset \(\Sigma \subset \mathbb{C}\), such that if \(\omega\) is outside this set, then one has a unique solution \(E \in H^s_{\text{Div}}\) to (15) given any tangential boundary condition \(G \in TH^s_{\text{Div}}(\partial \Omega)\). The solution satisfies
\[
\|E\|_{H^s_{\text{Div}}(\Omega)} \leq C\|G\|_{TH^s_{\text{Div}}(\partial \Omega)}
\]
(72)
with \(C\) independent of \(G\).

Note that when the tangential boundary condition is prescribed by CGOs, i.e., \(\tilde{G}_j = t\tilde{E}_j\), \(j = 1, 2\). By proposition 3.5, \(E_j\) is the unique solution to (56) and (57). Then the corresponding vector field \(\beta\) defined in (60) satisfies (67), which implies that the direction of \(\hat{\beta}\) is close to the constant direction and thus its integral curves connect every internal point to two boundary points. Therefore, (59) admits a unique solution.

Furthermore, proposition 3.5 also allows one to relax the boundary condition \(\tilde{G}_j = t\tilde{E}_j\) and still obtain the uniqueness of the solution to (59).

**Proposition 3.6.** Under the assumption of proposition 3.5, when \(G_j\) is in a neighborhood of \(\tilde{G}_j = t\tilde{E}_j\) in \(C^{s+1}(\partial \Omega)\), \(j = 1, 2\), the corresponding vector field \(\beta\) defined in (60) satisfies
\[
\|\beta - iL^2 \xi_0\|_{C^d(\Omega)} = O(h),
\]
(73)
for small \(h\).
Proof. By definition \(\|E\|_{H^{s}(\Omega)} \leq \|E\|_{H^{s}_{\text{loc}}(\Omega)}\) and \(\|G\|_{H^{s}_{\text{loc}}(\partial \Omega)} \leq \|G\|_{H^{s+1}(\partial \Omega)}\). In particular, when \(s = \frac{d}{2} + \delta\), from the Sobolev embedding theorem and proposition 3.5, we have that
\[
\|E\|_{C^{\delta}(\overline{\Omega})} \leq C\|E\|_{H^{\delta}(\Omega)} \leq C\|E\|_{H^{\delta}_{\text{loc}}(\Omega)} \leq C\|E\|_{H^{\delta}_{\text{loc}}(\partial \Omega)} \leq C\|G\|_{H^{\delta+1}_{\text{loc}}(\partial \Omega)} \leq C\|G\|_{C^{\delta+1}(\partial \Omega)},
\]
where various constants are all named ‘C’. Hence
\[
\|E\|_{C^{\delta}(\overline{\Omega})} \leq C\|G\|_{C^{\delta+1}(\partial \Omega)}.
\]
Let us now define the boundary conditions \(G_j \in C^{\delta+4}(\partial \Omega), j = 1, 2,\) such that
\[
\|G_j - \tilde{E}_j\|_{C^{\delta+1}(\partial \Omega)} \leq \varepsilon,
\]
for some \(\varepsilon > 0\) sufficiently small. Let \(E_j\) be the solution to (15) and (16) with \(tE_j = G_j\). By (75), we thus have
\[
\|E_j - \tilde{E}_j\|_{C^{\delta+1}(\overline{\Omega})} \leq \varepsilon,
\]
for some positive constant \(C\). Define the complex valued internal data \(D_j = LE_j\). We deduce that
\[
\|D_j - \tilde{D}_j\|_{C^{\delta+1}(\overline{\Omega})} \leq \varepsilon.
\]
Define \(\beta\) by (60). We can easily deduce (73) from (67) and (78). This finishes the proof. \(\square\)

Recall \(\mathcal{M}\) is the parameter space of \((L, \sigma)\) defined in (11) and \(h\) is the parameter in (52). We are in the place to prove theorem 2.1.

Proof of theorem 2.1. By proposition 3.6, we choose the set of illuminations as a neighborhood of \((\tilde{G}_j) = (t\tilde{E}_j)\) in \((C^{\delta+4}(\partial \Omega))^{2}\). Since the measurements \(D = \tilde{D}\), we have that \(L\) and \(\tilde{L}\) solve the same transport equation (59), while \(L = \tilde{L} = D/G\) on \(\partial \Omega\). As \(\beta\) satisfies (73), we deduce that \(L = \tilde{L}\), since the integral curves of \(\beta\) map any \(x \in \Omega\) to the boundary \(\partial \Omega\). More precisely, consider the flow \(\theta_s(t)\) associated with \(\beta\), i.e., the solution to
\[
\dot{\theta}_s(t) = \beta(\theta_s(t)), \quad \theta(0) = x \in \Omega.
\]
By the Picard–Lindelöf theorem, (79) admits a unique solution, since \(\beta\) is of class \(C^1(\Omega)\). For \(x \in \Omega\), let \(x_{\pm}(x) \in \partial \Omega\) and \(t_{\pm}(x) > 0\), such that
\[
\theta_{s}(t_{\pm}(x)) = x_{\pm}(x) \in \partial \Omega.
\]
By the method of characteristics, the solution \(L\) to the transport equation (59) is given by
\[
L(x) = L_{0}(x_{\pm}(x)) \exp \left( - \int_{0}^{t_{\pm}(x)} \gamma(\theta_s(s)) \, ds \right),
\]
where \(L_{0} := L|_{\partial \Omega}\) is the restriction of \(L\) on the boundary. The solution \(\tilde{L}\) is given by the same formula, since \(\theta_{s}(t) = \tilde{\theta}_{s}(t)\). This implies that \(E_j = \tilde{E}_j = D_j/L\), \(j = 1, 2\). By the choice of illuminations, we have \(|E_j| \neq 0\) due to (77) and \(|\tilde{E}_j| \neq 0\). Under the assumption that \(D_j = \tilde{D}_j\), we have \(E_j = \tilde{E}_j, j = 1, 2\). Therefore, \(k^2n = k^2\tilde{n}\) by (56) and thus \(\sigma = \tilde{\sigma}\). \(\square\)
3.3. Vector fields and stability result

Recall that \( \theta_s(t) \) is the flow associated with \( \beta \). From the equality
\[
\theta_s(t) - \bar{\theta}_s(t) = \int_0^t [\beta(\theta_s(s)) - \bar{\beta}(\bar{\theta}_s(s))] \, ds,
\] (82)
and using the Lipschitz continuity of \( \beta \) and Gronwall’s lemma, we deduce the existence of a constant \( C \), such that
\[
|\theta_s(t) - \bar{\theta}_s(t)| \leq Ct\|\beta - \bar{\beta}\|_{C^1(\bar{\Omega})},
\] (83)
when \( \theta_s(t) \) and \( \bar{\theta}_s(t) \) are in \( \bar{\Omega} \). Inequality (83) is uniform in \( t \) as all characteristics exit \( \bar{\Omega} \) in finite time.

To see higher order estimates, we define \( W := D_s \theta_s(t) \), which solves the equation, \( \dot{W} = D_s \beta(\theta_s(t)) W \), with \( W(0) = I \). Define \( \bar{W} \) similarly. By using Gronwall’s lemma again, we deduce that
\[
|W - \bar{W}| \leq Ct\|D_s \beta - D_s \bar{\beta}\|_{C^1(\bar{\Omega})}
\] (84)
when \( \theta_s(t) \) and \( \bar{\theta}_s(t) \) are in \( \bar{\Omega} \). Since \( \beta \) and \( \bar{\beta} \) are of class \( C^d(\bar{\Omega}) \), then we obtain iteratively that
\[
|D_s^{d-1} \theta_s(t) - D_s^{d-1} \bar{\theta}_s(t)| \leq Ct\|D_s \beta - D_s \bar{\beta}\|_{C^{d-1}(\bar{\Omega})},
\] (85)
when \( \theta_s(t) \) and \( \bar{\theta}_s(t) \) are in \( \bar{\Omega} \).

Recall that \( \Omega_1 \) is defined to be the subset of \( \Omega \) by removing a neighborhood of each tangent point of \( \partial \Omega \) with respect to \( \zeta_0 \).

**Lemma 3.7.** Let \( \Omega \) be an open bounded and convex subset in \( \mathbb{R}^3 \) with a boundary of class \( C^d \). Let \( d \geq 2 \) and \( \beta \) and \( \bar{\beta} \) are \( C^d(\bar{\Omega}) \) the vector fields which satisfy (73). Restricting ourselves to \( \Omega_1 \), we have that
\[
\|x - \bar{x}\|_{C^{d-1}(\bar{\Omega}_1)} + \|t - \bar{t}\|_{C^{d-1}(\bar{\Omega}_1)} \leq C\|\beta - \bar{\beta}\|_{C^{d-1}(\bar{\Omega}_1)},
\] (86)
where \( C \) is a constant depending on \( \Omega \).

This lemma is similar to lemma 3.8 in [1] and lemma 4.1 in [4], but uses a different proof.

**Proof.** For \( x \in \Omega_1 \), let \( \theta_s(t) \) and \( \bar{\theta}_s(t) \) be two flows associated with the vector fields \( \beta \) and \( \bar{\beta} \), respectively. Denote \( A := \theta_s(t_1(x)) \in \partial \Omega \) and \( B := \bar{\theta}_s(t_2(x)) \in \partial \Omega \). Without loss of generality, we assume \( t_1(x) \leq t_2(x) \). We also denote \( C := \bar{\theta}_s(t_2(x)) \in \Omega_1 \). As in figure 1, we connect points by line segments, which approximately indicate the integral curves of \( \beta \) and \( \bar{\beta} \). Also notice that point \( C \) does not necessarily lie on the line \( xB \) or in the plane \( AxB \). To simplify the writing, let \( \delta := \|\beta - \bar{\beta}\|_{C^{d-1}(\bar{\Omega}_1)} \).

We first want to show that the angle \( \angle AxB \) is controlled by
\[
\angle AxB \leq C_1 \delta + C_2 h,
\] (87)
for some \( C_1 \), \( C_2 \). Indeed, by applying (83) and sine theorem, we can see that \( \angle AxC \) is bounded by \( C_1 \delta \). Also notice that \( \bar{\beta} \) satisfies (73). Therefore, similar argument shows that, for any \( t_1, t_2 \), the angle between the vector from \( x \) to \( \bar{\theta}_s(t_1) \) and the vector from \( x \) to \( \bar{\theta}_s(t_2) \) is bounded by \( C_2 h \). Thus \( \angle AxB \leq C_2 h \). This proves (87).

By the definition of \( \Omega_1 \), a neighborhood of the boundary point at which the tangent plane of \( \partial \Omega \) is parallel to \( \zeta_0 \) is removed. Therefore, there exists a constant value \( \phi_0 > 0 \) depending only on \( \Omega_1 \), such that for any \( x \in \Omega_1 \), \( \phi_1 = \phi_0 \), where \( \phi_1 \) is the angle between the vector \( xA \) and the tangent plane of \( \partial \Omega \) at \( A \), as in figure 1. Then by (87), when \( \delta \) and \( h \) are so small that
\[ \phi_0' := \phi_0 - C_1 \delta - C_2 h > 0, \] the extension of \( C B \) will intersect the tangent plane of \( \partial \Omega \) at \( A \), with the intersection point \( D \). Then it is easy to check that

\[ \angle ABC > \angle ADx = \phi_1 - \angle AxB > \phi'_0 > 0. \] (88)

The sine theorem gives that

\[ |AB| = \frac{|AC|}{\sin(\angle ACB)} \sin(\angle ABC). \] (89)

Equation (83) directly implies \( |AB| = |x_+ - \tilde{x}_+| \leq C' \delta \).

Since \( \beta, \tilde{\beta} \in C^d(\Omega) \) and \( \partial \Omega \) is of class \( C^d \), it is clear that \( \angle ABC \) and \( \angle ACB \) are \( C^d \) functions with respect to \( x \in \Omega \). By differentiating (89) and applying (85), we obtain higher order estimates

\[ \|x_+ - \tilde{x}_+\|_{C^{d-1}(\Omega)} \leq C' \delta. \] (90)

To see the second part in (86), we have that

\[ |CB| = \int_{t_-(x)}^{\tilde{t}_-(x)} \tilde{\beta}(\tilde{\theta}(d)) \, ds = |\tilde{\beta}(\tilde{\theta}(\tau))| (\tilde{t}_+(x) - t_+(x)), \] (91)

for \( t_+(x) \leq \tau \leq \tilde{t}_+(x) \). Similar argument shows the estimate of \( t_+ - \tilde{t}_+ \) in (86). \( \square \)

**Proposition 3.8.** Let \( d \geq 1 \). Let \( L \) and \( \tilde{L} \) be the solutions to (59) corresponding to the coefficients \( (\beta, \gamma) \) and \( (\tilde{\beta}, \tilde{\gamma}) \), respectively, where (73) holds for both \( \beta \) and \( \tilde{\beta} \).

Let \( L_0 = L|_{\partial \Omega} \) and \( \tilde{L}_0 = \tilde{L}|_{\partial \Omega} \); thus \( L_0, \tilde{L}_0 \in C^d(\partial \Omega) \). We also assume that \( h \) is sufficiently small and \( \Omega \) is convex. Then, there is a constant \( C \) such that restricting ourselves to \( \Omega_1 \):

\[ \|L - \tilde{L}\|_{C^{d-1}(\Omega_1)} \leq C\|L_0\|_{C^{d-1}(\partial \Omega)}\|\beta - \tilde{\beta}\|_{C^{d-1}(\Omega_1)} + \|\gamma - \tilde{\gamma}\|_{C^{d-1}(\Omega_1)} + C\|L_0 - \tilde{L}_0\|_{C^{d-1}(\partial \Omega_1)}. \] (92)
The proof is omitted as it follows exactly the proof of proposition 4.2 in [4]. Now we can prove the main stability theorem.

**Proof of theorem 2.2.** From (60) and (61), it is easy to check that

$$\| \beta - \tilde{\beta} \|_{C^{r-1}(\Gamma)} \leq C \| D - \tilde{D} \|_{C^{r}(\Gamma)}, \quad \| \gamma - \tilde{\gamma} \|_{C^{r-1}(\Gamma)} \leq C \| D - \tilde{D} \|_{C^{r+1}(\Gamma)},$$

(93)

where $C > 0$ is a positive constant. The first part then follows directly from proposition 3.8. To estimate the difference between $\sigma$ and $\tilde{\sigma}$, notice that

$$E - \tilde{E} = \frac{D - \tilde{D}}{L} = \frac{L(D - \tilde{D}) - D(L - \tilde{L})}{LL}.$$

Since $L$ and $\tilde{L}$ are non-vanishing, by the stability result for $L$, we obtain

$$\| E - \tilde{E} \|_{C^{r-1}(\Gamma)} \leq C \| D - \tilde{D} \|_{C^{r+1}(\Gamma)}.$$

(94)

By choosing the boundary illuminations close to the boundary conditions of the CGO solutions, (76) and (77) imply that $E_j$ is non-vanishing, since the CGO solutions are non-vanishing. From (15), we have

$$n = \frac{1}{k^2} \frac{-\Delta E + \nabla \cdot E \cdot \hat{E}}{|E|^2}.$$

$$\tilde{n} = \frac{1}{k^2} \frac{-\Delta \tilde{E} + \nabla \cdot \tilde{E} \cdot \hat{\tilde{E}}}{|\tilde{E}|^2}.$$

By taking the difference and using (94), we derive

$$\| n - \tilde{n} \|_{C^{r-1}(\Gamma)} \leq \| D - \tilde{D} \|_{C^{r+1}(\Gamma)}.$$

Similar stability holds for $\sigma$, since $\sigma = \epsilon_0 w \text{Im} n$. \(\square\)

### 3.4. Stability with six complex internal data

Rather than applying the characteristics method to (59), we can rewrite (59) into a matrix form by introducing more internal measurements. We first construct proper CGO solutions. Let $j = 1, 2, 3$ in this section. We can choose the unit vectors $\xi_0^j$ and $\eta_0^j$, such that $\xi_0^j \cdot \eta_0^j = 0$, $\xi_0^j \cdot \eta_0^j = 0$ and $\{\xi_0^j\}$ are linearly independent. Also, choose $(\xi_1^j, \xi_2^j)$ and $(\eta_1^j, \eta_2^j)$, such that $\| \xi \| = |\xi_1^j| = |\xi_2^j|$, $\lim_{|\xi| \to \infty} \frac{\xi_1^j}{|\xi|} = \lim_{|\xi| \to \infty} \frac{\xi_2^j}{|\xi|} = \xi_0^j$ and $\lim_{|\xi| \to \infty} \eta_1^j = \eta_0^j$.

(95)

We construct CGO solutions $\tilde{E}_1^j$ and $\tilde{E}_2^j$ corresponding to $(\xi_1^j, \eta_1^j)$ and $(\xi_2^j, \eta_2^j)$. Let the boundary illuminations $G_1^j, G_2^j$ be chosen according to (76) for $\epsilon$ small enough. The measured internal data are then given by $D_1^j, D_2^j$. Proposition 3.5 shows that the vector field defined by (60) satisfies that

$$\| \beta^j - L^j \epsilon_0^j \|_{C^r(\Gamma)} \leq \frac{C}{|\xi|}.$$  

(96)

While $|\xi|$ is sufficiently large and $L \neq 0$ on $\tilde{\Omega}$, we obtain that the vectors $\{\beta^j(x)\}$ are linearly independent at every $x \in \Omega$. Thus the matrix $(\beta^j(x))$ is invertible with an inverse of class $C^d(\tilde{\Omega})$. By constructing the vector-valued function $\Gamma(x) \in (C^d(\tilde{\Omega}))^3$, the transport equation (59) now becomes the matrix equation

$$\nabla L + \Gamma(x)L = 0.$$  

(97)
Note that $\Gamma(x)$ is stable under small perturbations in the data $D := (D^1, D^2) \in (C^{d+1}(\Omega))^6$, i.e.,

$$\|\Gamma - \tilde{\Gamma}\|_{(C^{d+1}(\Omega))^6} \leq C\|D - \tilde{D}\|_{(C^{d+1}(\Omega))^6}. \quad (98)$$

Assume that $\Omega$ is connected and $L_0 = L|_{\partial \Omega}$ is known. Choose a smooth curve from $x \in \Omega$ to a point on the boundary. Restricting ourselves to the curve, (97) is a stable ordinary differential equation. Keep the curve fixed. Let $L$ and $\tilde{L}$ be the solutions to (97) with respect to $\Gamma$ and $\tilde{\Gamma}$, respectively. By solving the equation explicitly and (98), we find that

$$\|L - \tilde{L}\|_{(C^1(\Omega))} \leq C\|D - \tilde{D}\|_{(C^{d+1}(\Omega))^6}. \quad (99)$$

**Proof of theorem 2.3.** The first result in (14) is directly from (99). The proof of the stability of $\sigma$ is exactly the same as the proof of theorem 2.2. □

**Acknowledgments**

The authors thank Professor Gunther Uhlmann for suggesting this problem and for helpful discussions. The authors also thank Professor Maarten de Hoop for helpful discussions on the inverse problem models. The work of both authors was partly supported by NSF.

**References**

[1] Bal G and Uhlmann G 2010 Inverse diffusion theory of photo-acoustics Inverse Problems 26 085010

[2] Biot M A 1956 Theory of propagation of elastic waves in a fluid-saturated porous solid: I. Low-frequency range J. Acoust. Soc. Am. 28 168–78

[3] Biot M A 1956 Theory of propagation of elastic waves in a fluid-saturated porous solid: II. High-frequency range J. Acoust. Soc. Am. 28 179–91

[4] Chen J and Yang Y 2012 Quantitative photo-acoustic tomography with partial data Inverse Problems 28 115014

[5] Colton D and Päivärinta L 1992 The uniqueness of a solution to an inverse scattering problem for electromagnetic waves Arch. Ration. Mech. Anal. 119 369–419

[6] Penig C E, Salo M and Uhlmann G 2011 Inverse problems for the anisotropic Maxwell equations Duke Math. J. 157 369–419

[7] Pride S R 1994 Governing equations for the coupled electro-magnetics and acoustics of porous media Phys. Rev. B 50 15678–96

[8] Pride S R and Haartsen M W 1996 Electroseismic wave properties J. Acoust. Soc. Am. 100 1301–15

[9] Thompson A and Gist G 1993 Geophysical applications of electro-kinetic conversion Leading Edge 12 1169–73

[10] Sylvester J and Uhlmann G 1987 A global uniqueness theorem for an inverse boundary value problem Ann. Math. 125 153–69

[11] Williams K L 2001 An effective density fluid model for acoustic propagation in sediments derived from Biot theory J. Acoust. Soc. Am. 110 2276–81

[12] Zhu Z, Haartsen M W and Toksöz M N 1999 Experimental studies of electro-kinetic conversions in fluid-saturated bore-hole models Geophysics 64 1349–56

[13] Zhu Z and Toksöz M N 2003 Cross hole seismo-electric measurements in bore-hole models with fractures Geophysics 68 1519–24

[14] Zhu Z and Toksöz M N 2005 Seismo-electric and seismo-magnetic measurements in fractured bore-hole models Geophysics 70 F45–F51