Reasoning about Unreliable Actions

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Abstract

We analyse the philosopher Davidson’s semantics of actions, using a strongly typed logic with contexts given by sets of partial equations between the outcomes of actions. This provides a perspicuous and elegant treatment of reasoning about action, analogous to Reiter’s work on artificial intelligence. We define a sequent calculus for this logic, prove cut elimination, and give a semantics based on fibrations over partial cartesian categories: we give a structure theory for such fibrations. The existence of lax comma objects is necessary for the proof of cut elimination, and we give conditions on the domain fibration of a partial cartesian category for such comma objects to exist.

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1 Introduction

1.1 Background

In this paper we describe a logical system for reasoning about unreliable actions, or, to be precise, actions which can succeed or fail: it continues the programme, begun in [White, 2008], of developing strongly typed logical systems for reasoning about actions. As well as the motivations for the project as a whole, there are several purely technical reasons why this system in particular might be worth investigating: the notion of success or failure of actions means that, in our fibred categorical semantics, the base category is order-enriched, and this makes the proof theory quite interesting and, so far, somewhat unexplored. For
example, the Beck-Chevalley condition — which we will need for cut elimination — applies to comma squares rather than, as with the non-enriched case, to Cartesian squares.

There are, however, also non-technical grounds which make such a project interesting. The first is that it bears on the semantics of adverbs: adding an adverb to a verb modifies the success conditions of the action denoted by the verb (singing in tune, for example, has more restrictive success conditions than merely singing), and a logic which can handle these success conditions directly would seem to be important for the semantics of adverbs. Adverbs (or, more generally, verbal adjuncts) are a large and disparate class (Ernst, 2008), and the logic studied here can only handle a small subclass of these (for example, it can only handle adjuncts whose success conditions are a subset of the success conditions of the unmodified verb: it could not deal with an adjunct like ‘apparently’, for example). However, it is, at least, a start, and it gives some idea of what a more adequate theory might look like.

There are also reasons specific to the case of reasoning about action which make the success or failure of actions an interesting concept. Saying that actions succeed or fail is an example of normativity, that is, of dividing a set of entities into normal and deviant examples. Normative contexts can typically not be defined using purely physical vocabulary, and, for this reason, there has recently been a great deal of philosophical interest in the commonsense use of normative concepts: see (McDowell 1996, 1982) and the author’s own papers on normativity in the philosophy of computer science (White, 2011a,b). It is normativity that we are aiming at in the concept of unreliability of actions: it does not necessarily entail nondeterminacy, merely some notion of normativity.

Indeed, the concept of success or failure of actions has been recognised as important from the early days of artificial intelligence: it is usually referred to, using McCarthy’s terminology, as the problem of determining action qualifications (McCarthy, 1962, 1980, 1977, 1986, 1980) and (Reiter, 2001, Appendix B). However, although this concept has been much discussed in the AI community, the technical results have not been very illuminating: for example, in Reiter’s treatment (2001), the success of actions is represented as a first-order predicate EXECUTABLE(·) of sequences of actions, and the special logical role of success and failure does not really come to the fore.

When we do develop a formalism in which success and failure play their appropriate role, we discover important connections with other issues. There is a long-standing argument, due to Davidson, about the importance of equality in reasoning about actions (Davidson, 1980d). Our formalism supports equational reasoning in the appropriate way: this was almost apparent in our previous paper (White, 2008), but in this paper the role of equality becomes more perspicuous. Indeed, one can define an equality predicate using merely the order-enrichment together with an appropriately structured category of types (that of a partial Cartesian category or bicategory of partial maps (Carboni, 1987). Equalities between actions, then, are implicit in the normative concept of success or failure of actions, together with appropriate and plausible structure in our type theory.
There is a final and more technical reason for this research, which we alluded to above. A locally posetal 2-category with comma objects and final object is, in fact, a locally posetal 2-category closed under PIE limits. These limits (Lack, 2007, §6.6) are in many ways the natural 2-categorical generalisation of finite limits: just as we showed in White (2008) that analogous fibrations over 1-categories with finite limits have cut elimination, so too we can prove here that 2-fibrations over locally posetal 2-categories with PIE limits have cut elimination. One could conjecture, then, that 2-fibrations over general 2-categories have cut elimination: a proof of this, however, would require a certain amount of additional machinery.

This final reason may be technical, but it is not merely technical. As argues, if we regard proof theory as specifying the meaning of the connectives (that is, if we regard its left and right rules as a description of the meaning of a connective), then cut elimination says gives a sort of closure property for these specifications: it says that no more components of meaning will emerge if we compose the connectives with cut. Our results say that, provided the base category (i.e. the category of actions) is closed under certain limits, then we have cut elimination. So it says that our logic of actions will have nice closure properties provided that the actions themselves have suitable closure properties.

1.2 States and Possible Worlds

As we have said, our logic will be strongly typed: propositions will have types, and the types will be the objects of a category, with the category of propositions fibred over it.

In this section we describe the intuitive meaning of our fibrations. As in White (2008), we will start with Reiter’s treatment of action (2001). His work can be regarded as a phenomenology of reasoning about action, together with a logical formulation of that phenomenology: we will retain his phenomenology, but develop a formalisation of our own.

Reasoning about action has two sides, which we will, following philosophical terminology, call the intensional and the extensional. The intensional side is the agent’s view of actions: what actions are performed, in what sequence, and so on. It is this view of actions which is sometimes referred to as the “knowledge level” (Newell, 1982). We can think of this view as giving us a labelled transition system: the nodes of the system will be called states (in AI terminology, situations), the arrows will be, in philosophical terminology, action tokens, and the arrows will be labelled with action types (for the type/token distinction, see Davidson, 1980c,b, Hornsby, 1999, 1998, and Wetzel, 1998). Our actions will be deterministic – that is, there will be at most one action token of a given action type starting from a given state.

However, as well as their intensional aspect, actions also have an effect on the world. This is the extensional side of action and it will be important also to talk about it: we are concerned about what actually happens when we act, not merely about the actions that we performed, and so we need to represent the gap between the intensional and the extensional. We will represent the extensional
side of actions by propositional assertions about states. If, like Reiter, we use classical logic, “the way the world is” can be described by assigning truth values to propositions: that is, by what is called, in logical jargon, a possible world, and we can, therefore, think of the effect of an action as a function from possible worlds to possible worlds.

Extensions and intensions will be related as follows. States encode intensional information, and such information will, in general, only yield partial knowledge of the world: thus, each (intensional) state will, in general, correspond to several different possible worlds. However, the agent’s epistemic state will be part of the world, so that each possible world will correspond to a unique such state. So, each state will have, associated to it, a set of possible worlds, and these sets of possible worlds will be disjoint.

Pulling back predicates by these functions will give us a weakest preconditions map: this is what Reiter calls regression. Reiter also requires that there should be a Reiter also requires that regression should have a left adjoint, which he calls progression. It solves the problem: given a transition $s \xrightarrow{\alpha} t$ between situations $s$ and $t$, and given a proposition $P$ at $t$ — what Reiter would describe as a fluent — the regression problem is to find a proposition $P'$ at $s$ which which will be true iff $P$ is true at $t$.

Reiter also requires a solution to the following problem, which he calls progression: given an action $s \xrightarrow{\alpha} t$, and given a theory $P$ describing the state $s$, find the theory $Q$ describing $t$. Regression turns out to be a left adjoint to regression; we will, then, require that our substitution (or regression) operators should have left adjoints.

Now actions, as we have said and as Reiter (2001) emphasises, are not usually performable in all circumstances: furthermore, whether an action is performable or not will, in general, depend on circumstances unknown to the agent (for example, I may try to open a door, not knowing whether it is locked or not, or I may try to unlock a door not knowing that it is not locked). So whether an action is performable or not is a matter of the extensional side of things, in which we are representing actions as functions from possible worlds to possible worlds: and we can conveniently represent this by having these functions be partially defined. An action will be performable in precisely those worlds in which the corresponding function is defined. We should notice that partiality gives us a partial order on functions, namely the order given by extension ($f \sqsubseteq g$ iff $fg$ is defined whenever $f$ is, and, where both defined, $f$ and $g$ agree: think of the relation between murdering and murdering elegantly). It is this partial order that we will work with in the remainder of this paper.

This concept of success or failure can, it turns out, be internalised in our logic: an action will be performable in a situation provide that $\neg f^* \bot$ is true. Equality between actions can, likewise, be given a similar internalisation. We should note, here, that this definition of equality assumes classical logic: constructively, we do not get equality between actions, but apartness (and the corresponding logic in the fibres is given by co-Heyting semilattices).
2 Cartesian Bicategories and Comma Objects

2.1 Outline

The ultimate goal of this paper is to define a logic whose types and substitutions come from the objects and 1-cells of a locally posetal base category, or category of contexts. The semantics of this logic will be a category fibred over our category of contexts: thus, we will be to investigate such fibred categories. First, however, we investigate the structure in the base.

The appropriate structure on the category of contexts for the case where actions always succeed seems to be that of a cartesian category, i.e. a category with finite limits (White, 2008): we can construct from this a locally posetal bicategory by taking its bicategory of partial maps (Robinson & Rosolini, 1988), and we have argued above that the partial order on such a bicategory will give an appropriate notion of success or failure of actions. We can characterise these bicategories more abstractly: Carboni (1987) gives conditions for a locally posetal bicategory to be the category of partial maps in a cartesian category.

So we have two descriptions of a possible base category, one 2-categorical – as a functionally complete partial Cartesian category, in Carboni’s sense (1987) – and one categorical, as a finite limit category. The two are naturally related: the finite limit category, $C_{\text{tot}}$, is the category of total morphisms of the partial Cartesian category, $C$, and this induces an equivalence of 2-categories between, on the one hand, the 2-category of finite limit categories, functors, and natural transformations, and, on the other hand, the 2-category of partial cartesian categories, 2-functors, and natural transformations whose components are total. All of these results are well-known in the literature: I summarise them in Section 2.2.

Consider now a partial cartesian category $C$ (or, alternatively, its category of total morphisms $C_{\text{tot}}$). We can (following Hermida, 1999) define a notion of 2-fibration over a partial cartesian category: the restriction of a 2-fibration to the subcategory of total morphisms yields a fibration in the normal sense, and this gives an equivalence of categories between 2-fibrations over $C$ and fibrations over $C_{\text{tot}}$. So we can use the theory of fibrations over $C_{\text{tot}}$ to guide our investigations of 2-fibrations over $C$. In particular, we can show that the Frobenius properties correspond under the equivalence, and that a Beck-Chevalley condition over $C_{\text{tot}}$ corresponds to a somewhat modified Beck-Chevalley condition over $C$. So this will give us enough category theory to be able to define our logic and prove soundness, completeness, and cut elimination.

2.1.1 Notation

I have made a few unorthodox choices of notation. Comma objects I write with $\otimes$, because it has an analogous role to $\otimes$: furthermore, it is probably superior to the standard notations (it is asymmetric, unlike $\downarrow$, and it is legible, unlike the comma, and it can also be reversed easily, unlike the comma). We need a Heyting operation on the (distinguished) subobjects of an object of our
categories, and for this I have used $^B \! A$: it is not a wonderful choice, but it can be distinguished from, for example, $\rightarrow$, which we will also use, but with a different meaning.

2.2 Correspondences between Categories

We first describe the correspondences between categories of partial morphisms in cartesian categories and suitable locally posetal bicategories, known as partial cartesian categories. We will also describe what are known as restriction categories (or, more precisely, restriction categories with weak products): partial cartesian categories are equivalent to restriction categories together with appropriately defined finite products (Cockett & Lack, 2007, § 4.2). We will need, in addition to finite products, comma objects: restriction categories with comma objects can be defined in an analogous way.

We should note that, in this framework, concepts of two different sorts are represented. The first is the representation of partiality, and the corresponding partial order between one-cells: the second is the existence of finite limits of various sorts. Restriction categories enable a conceptually clean distinction between the two: a restriction category per se only represents partiality, and we can add suitable limits to it if we wish. We outline the restriction category framework, and the various equivalences between categories, in this section.

The usual category-theoretic treatment of partiality is in terms of spans whose left legs belong to a distinguished class of monos, closed under pullback. The relation between these and restriction categories is as follows.

Definition 1 (Cockett & Lack 2002, §3.2). The 2-category $\mathcal{MC}_{\text{art}}$ is defined as follows:

**Objects** are categories, together with systems $\mathcal{M}$ of monos containing the identity and closed under composition and pullbacks

**1-Cells** are Cartesian functors which respect $\mathcal{M}$

**2-Cells** are natural transformations $\alpha : F \rightarrow G$ such that, for every $m : A \rightarrow B$ in $\mathcal{M}$, the following square is Cartesian:

\[
\begin{array}{ccc}
FA & \xrightarrow{Fm} & FB \\
\downarrow^{\alpha A} & & \downarrow^{\alpha B} \\
GA & \xrightarrow{Gm} & GB
\end{array}
\]

$\mathcal{MC}_{\text{art}}$, then, defines categories with a distinguished class of monos.

Definition 2 (Cockett & Lack 2002, §2.1.1). A restriction category is a category together with the assignment, to each morphism $f : A \rightarrow B$, of a morphism $\bar{f} : A \rightarrow A$ such that
1. \( f \overline{f} = f \) for all \( f \),

2. \( \overline{fg} = g \overline{f} \) whenever \( \text{dom}(f) = \text{dom}(g) \) (i.e. whenever the composites make sense),

3. \( \overline{gf} = \overline{g} \overline{f} \) whenever \( \text{dom}(f) = \text{dom}(g) \), and

4. \( \overline{gf} = f \overline{g} \) whenever \( \text{dom}(g) = \text{cod}(f) \).

A morphism \( f \) in a restriction category is \textit{total} if \( \overline{f} = \text{Id} \).

**Definition 3** (Cockett & Lack 2002, §2.2.1). A \textit{restriction functor} is a functor between restriction categories which commutes with restrictions.

**Definition 4** (Cockett & Lack 2002, §2.2.2). The 2-category \( \text{rCat} \) is defined as follows:

- **Objects** are restriction categories
- **1-cells** are restriction functors
- **2-cells** are natural transformations whose components are total

**Definition 5** (Cockett & Lack 2002, §2.3.3). A morphism \( f : A \to A \) in a restriction category is a \textit{restriction idempotent} if \( f = \overline{f} \).

A restriction idempotent \( f \) is \textit{split} if there are \( r : A \to A_0 \) and \( i : A_0 \to A \) with \( f = ri \) (in this case \( f = \overline{r} \)).

A restriction category is \textit{split} if all of its restriction idempotents split.

**Definition 6** (Cockett & Lack 2002, §2.3.3). The 2-category \( \text{rCat}_s \) is the full sub-2-category of \( \text{rCat} \) whose objects are split restriction categories.

**Theorem 1** (Cockett & Lack 2002, Theorem 3.4). \( \text{rCat}_s \) and \( \text{MCart} \) are 2-equivalent.

**Proof.** Define functors

\[
\text{Par} : \text{MCart} \to \text{rCat}_s \quad (1)
\]

\[
\text{MTotal} : \text{rCat}_s \to \text{MCart} \quad (2)
\]

as follows.

Given a category \( \mathbb{C} \) together with a stable class of monos \( \mathcal{M} \), define a restriction category \( \text{Par}(\mathbb{C}) \) with the same objects as \( \mathbb{C} \), whose morphisms are spans whose left legs are in \( \mathcal{M} \) up to commuting isomorphism, and whose restriction sends the span \( \langle m, f \rangle \) to the span \( \langle m, m \rangle \). This assignment can easily be extended to a 2-functor \( \text{MTotal} \) from \( \text{MCart} \) to \( \text{rCat}_s \).

Conversely, given a split restriction category \( \mathcal{E} \), consider the category \( \mathcal{E}_{\text{tot}} \) whose objects are the same as those of \( \mathcal{E} \) and whose morphisms are the total morphisms of \( \mathcal{E} \). The sections of the restrictions of \( \mathcal{E} \) are total, and can be shown to form a stable system of monics in \( \mathcal{E}_{\text{tot}} \): this can be shown to extend to a 2-functor \( \text{MTotal} \) from \( \text{rCat}_s \) to \( \text{MCart} \). These 2-functors yield the desired equivalence. \( \square \)
So far, we have very minimal product structure: only pullbacks of a suitable class of monos. Next we shall discuss partial cartesian categories, which have more product structure.

2.2.1 Partial Cartesian Categories

Definition 7 (Carboni 1987). A partial cartesian category \( \mathcal{C} \) is a locally posetal symmetric monoidal bicategory such that:

1. every object \( A \) has a unique cocommutative comonoid structure
   \[
   \Delta_A : A \to A \otimes A \quad !_A : A \to I
   \]
   where \( I \) is the monoidal unit, and where \( \Delta_A \) is strict natural and \( !_A \) lax natural in \( A \).

2. \( \Delta_A \) has a right adjoint \( \nabla_A \) such that, for any \( A, B \) and any \( f, g : A \to B \),
   \[
   \Delta_A \nabla_A = (\nabla_A \otimes I)(I \otimes \Delta_A)
   \]
   \[
   \nabla_B(f \otimes g)\Delta_A \sqsubseteq f
   \]
   where \( \sqsubseteq \) is the partial order on the homsets of the category.

Remark 1. The operator on pairs of 1-cells \( f, g : A \to B \) defined by (5) is, in fact, the meet in the poset \( \text{Hom}_\mathcal{C}(A, B) \).

Definition 8. A 1-cell \( f : A \to B \) in a partial cartesian category is total if
\[
!_B f = !_A.
\]

Example 1. Let \( \mathcal{C} \) be a cartesian category, and let \( \mathcal{M} \) be a stable class of monics in \( \mathcal{C} \) which contains the diagonal morphisms \( \Delta_A : A \to A \times A \) for all \( A \). Then we define the partial cartesian category \( \text{bPar}(\mathcal{C}) \) as follows:

Objects are those of \( \mathcal{C} \)

1-cells are spans in \( \mathcal{C} \) whose left legs are in \( \mathcal{M} \), with composition defined in the usual way

The monoidal structure is given by \( \times \)

The comonoid structure on an object \( A \) is defined as follows:
\[
\Delta_A = \langle A, A \rangle : A \to A \times A
\]
\( I \) is the nullary product in \( \mathcal{C} \), and \( !_A \) is the unique total morphism \( A \to I \)

\( \nabla \) is defined by the following span:
Definition 9. The 2-category BPM is defined as follows:

**Objects** are partial cartesian categories

**1-cells** are monoidal functors (note that because of the uniqueness condition such functors preserve the comonoid structure on objects)

**2-cells** are natural transformations whose 1-cells are total

**Definition 10** (Carboni 1987, Def. 2.2). A partial cartesian category is *functionally complete* if coreflexives split: that is, if we have \( d \sqsubseteq \text{Id}_A : A \to A \) with \( d^2 = d \), then \( d = ij \) with \( i : A_0 \to A, j : A \to A_0 \), and \( i \vdash j \).

**Definition 11.** A 1-cell \( f : A \to B \) in a partial cartesian category is *total* if \( !_B f = !_A \).

Total maps contain the identities and are closed under composition, so we have a subcategory, \( \mathcal{C}_{\text{tot}} \), of a partial cartesian category \( \mathcal{C} \).

**Lemma 1** (Carboni 1987, Lemma 2.3.i). If \( \mathcal{C} \) is a functionally complete bicategory of partial maps, then \( \mathcal{C}_{\text{tot}} \) is cartesian.

Because \( \mathcal{C}_{\text{tot}} \) is cartesian, we can form its bicategory of partial maps with respect to the class \( \mathcal{M} \) of all monos: call this (slightly abusing notation) \( \text{BPar}(\mathcal{C}_{\text{tot}}) \); and we have

**Lemma 2** (Carboni 1987, Lemma 2.3.ii). If \( \mathcal{C}_{\text{tot}} \) is a functionally complete bicategory of partial maps, then the natural identity-on-objects functor

\[
\mathcal{C} \to \text{BPar}(\mathcal{C}_{\text{tot}})
\]

is strictly monoidal and faithful.

We can (subject to further conditions) prove that this functor is full: for this we need some more definitions.

**Definition 12** (Carboni 1987, Def. 2.4). A 1-cell \( f \) in a partial cartesian category is *monic* if \( \nabla(f \otimes f) = f \nabla \).

**Lemma 3.** A 1-cell \( f : A \to B \) is monic iff, for any \( g, h : S \to A \),

\[
f(g \sqcap h) = (fg) \sqcap (fh).
\]

**Proof.** Suppose first that \( f \) is monic. We have

\[
(fg) \sqcap (fh) = \nabla(fg) \otimes (fh) \Delta \\
= \nabla(f \otimes f)(g \otimes h) \Delta \\
= f \nabla(g \otimes h) \Delta \\
= f(g \sqcap h).
\]

10
Conversely, suppose that \( f \) satisfies (6). Define the projections \( p_1 : A \otimes B \to A \) and \( p_2 : A \otimes B \to B \) by

\[
\begin{array}{c}
A \otimes B \\
\downarrow \sim \\
A \\
\downarrow p_1 \\
P_1 : A \otimes I \\
\downarrow \sim \\
A \otimes I \\
\downarrow \sim \\
B
\end{array}
\]

Easy calculations show that \( p_1 \Delta = \text{id} \), and that \( p_1 \sqcap p_2 = \nabla : A \otimes A \to A \). We have

\[
\nabla(f \otimes f) = \nabla(f \otimes f)(p_1 \otimes p_2)
\]

\[
= \nabla(f \otimes f)(p_1 \otimes p_2)\Delta_A \otimes \Delta_A
\]

\[
= \nabla(f \otimes f)(p_1 \otimes p_2)(\text{id}_A \otimes \sigma_A \otimes \text{id}_A)\Delta_{A \otimes A}
\]

\[
= \nabla(f \otimes f)\sigma_A(p_1 \otimes p_2)(\text{id}_A \otimes \sigma_A \otimes \text{id}_A)\Delta_{A \otimes A}
\]

by symmetry of \( \nabla \)

\[
= \nabla(f \otimes f)(p_2 \otimes p_1)\sigma_A(\text{id}_A \otimes \sigma_A \otimes \text{id}_A)\Delta_{A \otimes A}
\]

by symmetry of \( \Delta \)

\[
= (f p_2) \sqcap (f p_1)\sigma_A
\]

by assumption

\[
= f(p_2 \sqcap p_1)\sigma_A
\]

by assumption

\[
= f(p_1 \sqcap p_2)
\]

\[
= f \nabla
\]

\[
\square
\]

**Corollary 1.** \( f : A \to B \) is monic in \( \mathcal{C} \) iff, for any \( C \), the postcomposition morphism

\[
f \circ \cdot : \text{Hom}_\mathcal{C}(C, A) \to \text{Hom}_\mathcal{C}(C, B)
\]

is an inclusion of posets.

**Lemma 4** ([Carboni 1987], Lemma 2.5). If \( \mathcal{C} \) is a partial cartesian category, and if \( \mathcal{C}_{\text{tot}} \) is its subcategory of total morphisms, then a 1-cell in \( \mathcal{C}_{\text{tot}} \) is a mono in \( \mathcal{C} \) iff it is monic in \( \mathcal{C} \).

**Definition 13** ([Carboni 1987], Def. 2.4). A quasi-inverse for a monic \( f \) is a 1-cell \( f^\dagger \) such that

\[
\text{dom}(f) = f^\dagger f \quad \text{and} \quad \text{dom}(f^\dagger) = ff^\dagger
\]

**Lemma 5** ([Carboni 1987], Lemma 2.5). Quasi-inverses are unique, and, if \( i \dashv j \) is the splitting of a coreflexive in \( \mathcal{C} \), then \( j = i^\dagger \).
Definition 14. The two-category $\mathbf{BPM}_s$ is the full sub-two-category of $\mathbf{BPM}$ given by partial cartesian categories all of whose coreflexives split.

Proposition 1. The two-category $\mathbf{BPM}_s$ is 2-equivalent to $\mathbf{M Cart}$.

Proof. Define 2-functors in both directions as follows.

Given a partial cartesian category $\mathcal{C}$ with split coreflexives the 1-category $\mathcal{C}_{\text{tot}}$ is cartesian, and the class of morphisms

\[ \{ j | i \vdash j \text{ the splitting of a coreflexive} \} \]

is a class of monos of $\mathcal{C}_{\text{tot}}$ closed under pullback and containing the identities. We have then an object of $\mathbf{M Cart}$: we can check that this assignment is, in fact, 2-functorial. Call this 2-functor

\[ \mathbf{bTot} : \mathbf{BPM}_s \to \mathbf{M Cart} \]

Given an object $\langle \mathcal{C}, \mathcal{M} \rangle$ of $\mathbf{M Cart}$, we define an object of $\mathbf{BPM}_s$ as follows:

- **Objects** are objects of $\mathcal{C}$,
- **1-cells** are spans in $\mathcal{C}$ whose left legs are in $\mathcal{M}$, up to the usual equivalence relation, and
- **2-cells** are defined by inclusion of subobjects in $\mathcal{C}$: that is, $\langle j, f \rangle \subseteq \langle j', f' \rangle$ iff there is a commuting diagram

\[
\begin{array}{ccc}
A & \xleftarrow{j} & A_0 \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & B
\end{array}
\]

The monoidal structure on this 2-category is given by $\times$ on objects (we need stability of $\mathcal{M}$ under pullbacks to make it functorial). The conditions on the tensor product are readily checked, as is the 2-functoriality of this assignment of an object in $\mathbf{BPM}_s$ to an object in $\mathbf{M Cart}$. Call this 2-functor

\[ \mathbf{bPar} : \mathbf{M Cart} \to \mathbf{BPM}_s \]

Finally we need to check that these two 2-functors give a 2-equivalence of categories between $\mathbf{BPM}_s$ and $\mathbf{M Cart}$. \qed

2.2.2 Restriction Products

Finally we have a characterisation of partial cartesian categories in terms of restriction categories and suitably defined products.

Definition 15 (Cockett & Lack 2007, § 4.1). Define the two-category $\mathbf{rCATL}$ as follows:
Objects are restriction categories

One-cells are restriction functors

Two-cells are lax natural transformations with total components: that is, a natural transformation from \( F : \mathcal{C} \to \mathcal{C}' \) to \( G : \mathcal{C} \to \mathcal{C}' \) is a family of total 1-cells \( \alpha_X : F(X) \to G(X) \) such that, for \( f : X \to Y \), we have

\[
\begin{array}{c}
F(X) \xrightarrow{f} F(Y) \\
\downarrow \alpha_X & \equiv & \downarrow \alpha_Y \\
G(X) \xrightarrow{Gf} G(Y)
\end{array}
\]

Definition 16. 1. A binary restriction product on a restriction category \( \mathcal{C} \) is a functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) right adjoint to \( \Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C} \) in \( \text{rCat}_l \).

2. A restriction terminal object on a restriction category \( \mathcal{C} \) is an object \( I \) (i.e. a functor from the terminal restriction category \( \mathcal{I} \) to \( \mathcal{C} \)) which is right adjoint to the unique functor \( \mathcal{C} \to \mathcal{I} \).

3. A restriction category has restriction products if it has binary restriction products and a restriction terminal object.

Then we have:

Theorem 2. (Cockett, Lack, Robinson et al.) A partial cartesian category is a restriction category with restriction products.

Proof. See Cockett & Lack (2007, § 4.2).

2.3 Weak Comma Objects

We can now start on the material specific to this paper. In the total case (i.e. when the base is a category rather than a locally posetal two-category) we need conditions on the base – namely the existence of fibred products – in order to prove cut elimination, together with conditions conditions on the fibration, known as the Beck-Chevalley conditions (see White, 2008). In the locally posetal case, we again need a Beck-Chevalley condition, and, as Hermida (2004) shows, in the bicategorical case we need to formulate these conditions with comma objects rather than fibre products.

In our case, we define comma objects as follows.

Definition 17. A bicategory of partial maps with weak comma objects is a bicategory of partial maps such that any diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow f & & \downarrow g \\
B & \xrightarrow{g} & C
\end{array}
\]
can be completed to a diagram

\[
\begin{tikzcd}
A & C \\
& B
\end{tikzcd}
\]

with \(\hat{f}\) and \(\hat{g}\) total, and such that, for any diagram

\[
\begin{tikzcd}
S \\
A & C \\
& B
\end{tikzcd}
\]

there is a unique mediating arrow \(\langle \phi, \psi \rangle : S \to f \otimes g\) such that \(\hat{g} \langle \phi, \psi \rangle = \phi \overline{\psi}\) and \(\hat{f} \langle \phi \psi \rangle = \psi \overline{\phi}\).

The following is immediate:

**Definition 18.** A *restriction category with weak comma objects* is a restriction category such that, for any morphisms \(f : A \to B\) and \(g : C \to B\), there is an object \(f \otimes g\) with morphisms \(\hat{g} : f \otimes g \to A\) and \(\hat{f} : f \otimes g \to C\) such that:

1. \(\hat{g}\) and \(\hat{f}\) are total
2. \(f \hat{g} = g \hat{f}\)
3. if we have \(\phi : S \to A\) and \(\psi : S \to C\), then there is a unique \(\langle \phi, \psi \rangle : S \to f \otimes g\) such that

\[
\hat{g} \langle \phi, \psi \rangle = \phi \overline{\psi}
\]
\[
\hat{f} \langle \phi \psi \rangle = \psi \overline{\phi}
\]

The proof of the following is elementary:

**Proposition 2.**

1. Comma objects are unique up to canonical isomorphism
2. Given pairs \(\phi, \psi\) and \(\phi', \psi'\) as above, we have \(\langle \phi, \psi \rangle = \langle \phi', \psi' \rangle\) iff

\[
\phi \overline{\psi} = \phi' \overline{\psi'}
\]
\[
\psi \overline{\phi} = \psi' \overline{\phi'}
\]

We should note also the following (which is likewise elementary):

**Proposition 3.** The following are equivalent:
1. \( f \otimes g \) is a comma object of \( f \) and \( g \)

2. For any total \( \phi : S \to A \) and \( \psi : S \to B \) such that \( f \phi \sqsubseteq g \psi \), there is a unique \( \langle \phi, \psi \rangle : S \to f \otimes g \) with the usual properties.

This entails that comma objects are, indeed, comma objects in the usual sense (that is, comma objects defined by weighted limits [Lack, 2007]: more precisely,

**Proposition 4.** \( f \otimes g \) is the weighted limit of the diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow f & & \downarrow g \\
C & \rightarrow & 0 \rightrightarrows 1
\end{array}
\]

\[
\begin{array}{ccc}
A & \rightarrow & C \\
\downarrow f & \downarrow g & \downarrow 1 \\
B & \rightarrow & 1
\end{array}
\]

2.4 Local Conditions for Comma Objects

We can give local conditions for a bicategory of partial maps to have comma objects; for convenience, we will do this in the restriction category case. First we need to fix some vocabulary.

**Definition 19.** Let \( A \) be an object of a restriction category. We say that \( A \) has a stable Heyting operation if, for any two restriction idempotents \( \alpha, \beta \) of \( A \), there is a restriction idempotent \( \beta \alpha \) such that, for any \( f : S \to A \), and any restriction idempotents \( \alpha, \beta \) and \( \gamma \) on \( A \), we have

\[
\alpha f \sqsubseteq \beta f \quad \text{iff} \quad f \sqsubseteq \beta \alpha f
\]

**Lemma 6.** Let \( f : A \to B, g : C \to B \). The canonical morphism

\[
f \otimes g \rightarrow A \otimes B
\]

is monic.

**Proof.** Apply Lemma 3 to the universal characterisations of \( A \otimes B \) and \( f \otimes g \).

**Theorem 3.** Let \( \mathcal{C} \) be a restriction category with restriction products and terminal object whose restriction idempotents split and whose monics have quasi-inverses. The following are equivalent:

1. \( \mathcal{C} \) has weak comma objects

2. the objects of \( \mathcal{C} \) have stable Heyting operations on their restriction idempotents

**Proof.** We first show that \( 1 \Rightarrow 2 \). Let \( \alpha \) and \( \beta \) be restriction idempotents on \( A \): consider the comma object \( \alpha \otimes \beta \). The natural morphism \( i : \alpha \otimes \beta \to \alpha \otimes \beta \) is monic, by Lemma 6 let \( j \) be its quasi-inverse. Then \( ij : A \otimes A \to A \otimes A \) is a restriction idempotent: let \( ^\alpha \beta = \nabla_A (ij \Delta_A) : A \to A \). We can verify that it is a
restriction idempotent. We can also verify that, for any \( f : S \to A \) for which \( \langle f, f \rangle \) is defined, \( j \Delta_A f = \langle f, f \rangle \).

Now let \( f \) be any morphism \( S \to A \). We have

\[
\alpha f \sqsubseteq \beta f \quad \text{iff} \\
\langle f, f \rangle : S \to \alpha \otimes \beta \quad \text{is defined, iff} \\
\alpha \beta f = \nabla ij \Delta f \\
= \nabla i(f, f) \\
= \nabla \Delta f \quad \text{by the naturality of } i \\
= f \quad \text{that is, } f^\alpha \beta = f
\]

and this is the property defining a stable Heyting operation.

To show that \( 2 \Rightarrow 1 \), we proceed as follows. If the objects of \( \mathcal{C} \) have stable Heyting operations, consider \( f : A \to B, g : C \to B \). Define the following restriction idempotents on \( A \otimes C \):

\[
\alpha = f \pi_1 \\
\beta = \nabla_C f \otimes g
\]

Consider the restriction idempotent \( \alpha \beta \): choose a splitting of the form

\[
f \otimes g \xrightarrow{i} A \otimes B
\]

Then \( f \otimes g \) will be our comma object. So we have, for any \( \phi : S \to A, \psi : S \to C \),

\[
\langle \phi, \psi \rangle : S \to f \otimes g \\
\phi \otimes \psi \Delta_S : S \to A \otimes C \\
\phi \otimes \psi \Delta_S = \alpha \beta \phi \otimes \psi \Delta_S \\
\alpha \phi \otimes \psi \Delta_S \sqsubseteq \beta \phi \otimes \psi \Delta_S, \\
\text{that is} \\
\nabla_C f \phi \otimes \psi \Delta_S \sqsubseteq \nabla_C f \otimes g \phi \otimes \psi \Delta_S \\
\phi \otimes \psi \Delta_S f \pi_1 \phi \otimes \psi \Delta_S \sqsubseteq \phi \otimes \psi \Delta_S \nabla_C f \phi \otimes g \psi \Delta_S \\
\phi \otimes \psi \Delta_S f \phi \otimes \psi \Delta_S \sqsubseteq \phi \otimes \psi \Delta_S f \phi \otimes g \psi \\
(\phi f \phi \psi) \otimes (\psi f \phi \psi) \Delta_S \sqsubseteq (\phi f \phi \otimes g \psi) \otimes (\psi f \phi \otimes g \psi) \Delta_S
\]

It is easy to prove that this last inequality holds if \( f \phi \sqsubseteq g \psi \); to prove the converse, we proceed as follows. Applying \( \pi_1 \) and \( \pi_2 \) to both sides, and cancelling a number of factors of the form \( \psi, \psi \) have that the last line holds only if

\[
\phi f \phi \psi \sqsubseteq \phi f \phi \sqcup g \psi \\
\psi f \phi \psi \sqsubseteq \psi f \phi \sqcup g \psi
\]

but the converse containments clearly hold, so we have equalities. These equalities yield, after composition with \( f \) and \( g \) as appropriate, and using the identity \( f \phi f \phi \otimes g \psi f \phi \otimes g \psi \),

\[
f \phi f \phi \psi = f \phi f \phi \sqcap g \psi = g \psi f \phi \psi
\]
This finally yields, as required,

\[ f \phi \psi = g \psi \bar{f} \phi. \]

**Corollary 2.** Let \( \mathcal{C} \) be a restriction category with a restriction final object whose monics have quasi-inverses and whose restriction idempotents split. The following are equivalent:

1. \( \mathcal{C} \) has weak comma objects
2. \( \mathcal{C} \) has weak comma objects and a binary restriction product
3. the objects of \( \mathcal{C} \) have stable Heyting operations on their restriction idempotents

**Proof.** The equivalence of 2 and 3 follows from Theorem 3. The equivalence of 1 and 2 follows from the fact that, in the presence of a restriction final object, \( !A \circ !B \) is the binary restriction product of \( A \) and \( B \). \( \square \)

### 2.5 Pasting for Comma Squares

We will need to paste comma squares: the following lemmas say that we can do so. Note that, because of the asymmetry of a comma square, there are two cases.

**Lemma 7.** Consider the following comma square:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow{g} & & \downarrow{h} \\
B & \xrightarrow{j} & C
\end{array}
\]

Let \( S \) be an object: then, in this diagram of posets and poset morphisms,

\[
\text{Hom}(S, f \circ g) \xrightarrow{\text{unpair}} \{ \phi : S \to A, \psi : S \to C \mid f\phi \subseteq g\psi \}
\]

where

\[
\text{unpair}(\chi) = (\hat{g}\chi, \hat{f}\chi) \quad \text{and} \quad \text{pair}(\phi, \psi) = \langle \phi, \psi, \rangle
\]

we have:

1. \( \text{unpair} \dashv \text{pair} \), with \( \text{pair} \text{unpair} = \text{Id} \) and \( \text{unpair} \text{pair} \subseteq \text{Id} \)
2. \( \text{Hom}(S, f \circ g) \cong \text{Im}(\text{unpair}) = \{ \phi, \psi \mid f\phi = g\psi, \phi = \psi \} \)
Proof. The first part is an immediate consequence of the definition of comma objects: the second part follows from the first.

Lemma 8. In the following diagram the outer rectangle is a comma square.

\[
\begin{array}{ccc}
  h \otimes \hat{g} & \xrightarrow{h} & f \otimes g \xrightarrow{f} C \\
  \downarrow g & & \downarrow g \\
  D & \xrightarrow{\hat{g} \otimes \hat{g}} & C \\
  \downarrow h & & \downarrow h \\
  A & \xrightarrow{f \otimes g} & B
\end{array}
\]

Proof. We use Lemma 7.

\[
\text{Hom}(S, h \otimes \hat{g}) \cong \{ \phi, \psi \mid h \phi \sqsubseteq \hat{g} \psi, \bar{\phi} = \bar{\psi} \}
\]
\[
= \{ \phi, \psi_1, \psi_2 \mid f \psi_1 \sqsubseteq g \psi_2, \bar{\psi_1} = \bar{\psi_2}, h \phi \sqsubseteq \hat{g} \langle \psi_1, \psi_2 \rangle, \bar{\phi} = \langle \bar{\psi_1}, \bar{\psi_2} \rangle \}
\]
\[
= \{ \phi, \psi_1, \psi_2 \mid \bar{\phi} = \bar{\psi} = \bar{\psi_1} = \bar{\psi_2}, f \psi_1 \sqsubseteq g \psi_2, h \phi \sqsubseteq \psi_1 \}
\]

since, for any \( \psi_1, \psi_2, \langle \psi_1, \psi_2 \rangle = \bar{\psi_1} \bar{\psi_2} \), and since \( \hat{g} \langle \psi_1, \psi_2 \rangle = \psi_1 \)
\[
= \{ \phi, \psi_1, \psi_2 \mid \bar{\phi} = \bar{\psi} = \bar{\psi_1} = \bar{\psi_2}, f \psi_1 \sqsubseteq g \psi_2, h \phi = \psi_1 \}
\]

since we have \( \bar{\psi_1} = \bar{\phi} \sqsubseteq h \phi \sqsubseteq \bar{\psi_1} \), and so \( h \phi = \bar{\psi_1} \), so, since \( h \phi \sqsubseteq \psi_1, h \phi = \phi_1 \)
\[
= \{ \phi, \psi_2 \mid \bar{\phi} = \bar{\psi_2}, f h \phi \sqsubseteq g \psi_2 \}
\]
\[
= \text{Hom}(S, (fh) \otimes g)
\]

and so (by a Yoneda argument) the natural morphism
\[
h \otimes \hat{g} \rightarrow (fh) \otimes g
\]
is an isomorphism (we have to check that the natural morphism induces the isomorphism of homsets which the argument above yields, but this is trivial).

Lemma 9. In the following diagram, the outer rectangle is a comma square.

\[
\begin{array}{ccc}
  \hat{f} \otimes h & \xrightarrow{\hat{f}} & D \\
  \downarrow \hat{h} & & \downarrow h \\
  f \otimes g & \xrightarrow{f} & C \\
  \downarrow \hat{g} & & \downarrow g \\
  A & \xrightarrow{f} & B
\end{array}
\]
Proof. Note first that, since \( \hat{f} \) and \( \hat{h} \) are total, the top square is commutative. We use, again, Lemma 7.

\[
\text{Hom}(S, \hat{f} \otimes h) \cong \{ \phi, \psi | \bar{\phi} = \bar{\psi}, \hat{f} \phi \sqsubseteq h \psi \}
\]

Since \( \hat{f} \) is total and \( \bar{\phi} = \bar{\psi} \)

\[
\begin{align*}
&= \{ \phi, \psi | \bar{\phi} = \bar{\psi}, \hat{f} \phi = h \psi \} \\
&\cong \text{Hom}(S, f \otimes (gh))
\end{align*}
\]

and the result follows as in the previous lemma. \( \square \)

Corollary 3. Consider the following diagram, where \( f \) is total and where \( i_C : C_0 \to C \) is the inclusion of the domain of \( g \).

\[
\begin{array}{ccc}
(f \otimes g) \times_C C_0 & \xrightarrow{f} & C_0 \\
\downarrow i_C & & \downarrow i_C \\
f \otimes g & \xrightarrow{\hat{f}} & C \\
\downarrow g & & \downarrow g \\
A & \xrightarrow{f} & B
\end{array}
\]

Then the top left hand object is, in fact, \( A \times_B C_0 \), and \( \hat{i}_C \) is an isomorphism

Proof. Since \( f \) and \( \hat{g} \) are total, the bottom square is commutative. By pasting, the outer rectangle is a comma square: since \( f \) and \( gi_C \) are both total, this square is, in fact, Cartesian. Consequently, \( (f \otimes g) \times_C C_0 \) is, in fact, \( A \times_B C_0 \), and thus classifies pairs of maps \( \phi, \psi \) such that \( \text{dom}(\phi) = \text{dom}(\psi) \) and \( f \phi = gi_C \psi \); but this is exactly what \( f \otimes g \) classifies, since, if we have a pair \( \phi, \psi' \) such that \( \text{dom}(\phi) = \text{dom}(\psi') \) and \( f \phi = g \psi' \), then, since \( f \) is total, \( \psi' \) must factor through \( i_C \) (uniquely, since \( i_C \) is monic). So we have the result by Yoneda. \( \square \)

Corollary 4. Let \( f : A \to B \), and let \( i : A_0 \to A \) be the inclusion of the domain of \( f \). Let \( g : C \to B \), and let \( j : C_0 \to C \) be the inclusion of the domain of \( g \) in \( C \). Then the following diagram commutes: the left square is Cartesian, the right square is comma, the enclosing rectangle is Cartesian, and \( i_A \) is the
natural morphism $A_0 \times_B C_0 \rightarrow f \otimes g$: 

$$
\begin{array}{ccc}
A_0 \times_B C_0 & \xrightarrow{i_A} & f \otimes g & \xrightarrow{j} & C \\
\tilde{g} & = & \tilde{g} & \subseteq & g \\
A_0 & \xrightarrow{i} & A & \xrightarrow{f} & B
\end{array}
$$

Proof. Construct the diagram as follows:

$$
\begin{array}{ccc}
A_0 \times_A (f \otimes g) & \xrightarrow{i} & f \otimes g & \xrightarrow{j} & C \\
\tilde{g} & = & \tilde{g} & \subseteq & g \\
A_0 & \xrightarrow{i} & A & \xrightarrow{f} & B
\end{array}
$$

where the right hand square is a comma square and the left hand square is Cartesian ($i$ and $\tilde{g}$ are total, so this makes sense). By pasting, and since $fi_A$ is total, we can apply the previous corollary and identify the top left object with $A_0 \times_B C_0$. 

3 Fibrations over Cartesian Bicategories

Having defined our base categories (that is, partial cartesian categories with weak comma objects), we will now define the notion of a 2-fibration.

Definition 20. An posetal 2-fibration is a 2-functor

$$
\pi : \mathcal{E} \rightarrow \mathcal{C}
$$

such that

1. $\mathcal{E}$ and $\mathcal{C}$ are locally posetal 2-categories (i.e. 2-categories such that the homsets between objects are partial orders)
2. the fibres of $\pi$ are posets, with trivial two-cells
3. the 1-cells of $\mathcal{E}$ are fibred over the 1-cells of $\mathcal{C}$ in the standard sense, and
4. if $P, Q \in \text{Ob}(\mathcal{E})$, then, for all 1-cells $f, g : P \rightarrow Q$, $f \subseteq g$ iff $\pi(f) \subseteq \pi(g)$, and
5. if $P, Q \in \text{Ob}(\mathcal{E})$, $g : P \rightarrow Q$, and if $\alpha \subseteq \pi(f)$, then there is $f' \subseteq g$ with $\pi(f') = \alpha$.

Remark 2. The fibrational conditions of this definition come from [Hermida 1996, Theorem 2.8 (iii)], with considerable simplifications because of our posetal case.

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3.1 Correspondences between Fibrations

We have described, in Section 2.2, the basic correspondences between partial cartesian categories and cartesian 1-categories. We will now show how this correspondence yields correspondences between 1-fibrations in posets over cartesian categories and posetal 2-fibrations over partial cartesian categories. This correspondence will have two ingredients: firstly a Grothendieck correspondence for posetal 2-fibrations, and secondly a result of Hermida which gives a universal characterisation of the construction of partial cartesian categories from cartesian 1-categories.

3.1.1 The Grothendieck correspondence

**Lemma 10.** Let \( \pi : \mathcal{E} \rightarrow \mathcal{C} \) be a posetal 2-fibration, and let \( Q \in \text{Ob}(\mathcal{E}) \), \( \alpha \sqsubseteq \beta : A \rightarrow \pi(Q) \). Then, in the poset \( \mathcal{E} \), \( \beta^*Q \leq \alpha^*Q \).

**Proof.** Because of the fibration of 1-cells, there is a map \( \hat{\beta} : \beta^*Q \rightarrow Q \) over \( \beta \). By condition 5 above, there is a morphism \( f : \beta^*Q \rightarrow Q \) over \( \alpha \): a vertical-horizontal factorisation of \( f \) gives the result. \( \square \)

**Lemma 11.** Let \( \mathcal{C} \) be a bicategory of partial maps. The 2-category of 2-fibrations in posets over \( \mathcal{C} \) is equivalent to the 2-category of strict functors \( \mathcal{C} \rightarrow \text{Poset}^{\text{coop}} \).

**Proof.** This is basically the Grothendieck correspondence. Consider first a fibration \( \pi : \mathcal{E} \rightarrow \mathcal{C} \). Choose a cleavage of \( \pi \): a 1-cell \( f : A \rightarrow B \) of \( \mathcal{C} \) then gives a poset morphism \( f^* : \mathcal{E}_B \rightarrow \mathcal{E}_A \). Composition is strict (i.e. \( f^*g^* = (gf)^* \)) because the vertical structure in the fibres is posetal. Lemma 10 gives us the 2-cells. So, given a fibration, we have a functor.

Conversely, given a functor \( \mathcal{F} \), we define a bicategory as follows:

**Objects** are pairs \( \langle A, P \rangle \), where \( A \) is an object of \( \mathcal{C} \) and \( P \) is an element of the poset \( \mathcal{F}(A) \)

**1-cells** between \( \langle A, P \rangle \) and \( \langle B, Q \rangle \) are 1-cells \( f : A \rightarrow B \) such that \( P \leq f^*Q \)

**2-cells** \( f \sqsubseteq g : \langle A, P \rangle \rightarrow \langle B, Q \rangle \) iff \( f \sqsubseteq g : A \rightarrow B \).

It is straightforward to check that this gives an equivalence of 2-categories. \( \square \)

3.1.2 The Hermida characterisation

Let \( \mathcal{C} \) be a bicategory of partial maps, and consider a 2-fibration (in posets, let us say) \( \mathcal{E} \rightarrow \mathcal{C} \) over it: then it is easy to check that \( \mathcal{E} \) restricts to a fibration over \( \mathcal{C}_{\text{tot}} \), the subcategory of total morphisms of \( \mathcal{C} \). The goal of this section is to show that, subject to mild conditions, this process can be inverted. We prove this using a result of Hermida (2002). First some lemmas:
Lemma 12. Let $\mathcal{C}$ and $\mathcal{C}_{\text{tot}}$ be as above, and suppose that coreflexives in $\mathcal{C}$ split: let $\mathcal{M}$ be the set of monos in $\mathcal{C}_{\text{tot}}$ which split coreflexives in $\mathcal{C}$. Let $\alpha$ be a coreflexive, and let $i \vdash j$ split $\alpha$. Let

$$
\begin{array}{ccc}
A_0 & \xrightarrow{f} & B_0 \\
\downarrow{i} & & \downarrow{i}
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{j} & & \downarrow{j}
\end{array}
$$

be a pullback in $\mathcal{C}_{\text{tot}}$: $i \in \mathcal{M}$, and so $\hat{i} \in \mathcal{M}$ and thus it has a right adjoint $\hat{j}$. Then

1. the following diagram commutes in $\mathcal{C}$:

$$
\begin{array}{ccc}
A_0 & \xrightarrow{f} & B_0 \\
\downarrow{j} & & \downarrow{j}
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\hat{j}} & & \downarrow{\hat{j}}
\end{array}
$$

and

2. we have

$$ijf = \hat{i}\hat{j}.$$

Proof. First note that we can rewrite the universal property of the Cartesian square as follows:

Let $f : S \to A$ be total: then, if $ijf\phi = f\phi$, $\hat{i}\hat{j}\phi = \phi$.

Now consider the reflection idempotent $ijf$; let $i' \vdash j'$ split it. By properties of restriction idempotents, we have $ijfi' = i'$, which is total, so that

$$
\begin{align*}
ffijf' &= ijfi' \\
&= ijf
\end{align*}
$$

is also total: but $ijfi' \subset fi'$, so

$$ijfi' = fi'$$

and thus, by the universal property

$$
\begin{align*}
\hat{i}j' &= i', \\
\hat{i}j'j' &= i'j', \text{i.e.} \\
i'j' &\subset \hat{i}j.
\end{align*}
$$
On the other hand,

\[
f \sqsubseteq ijf = i\hat{f}\hat{j} = i\hat{f}j = \hat{f}ij
\]

so, since \(i \dashv j\),

\[
ijf \sqsubseteq \hat{f}j,
\]

and thus

\[
ijf \sqsubseteq i\hat{f}j,
\]

so

\[
i'j' = i\hat{j} = i\hat{f}j
\]

= \hat{j} \quad \text{since } \hat{f} \text{ and } i \text{ are total}

= ij

Consequently, \(\hat{i}j = i'j' = i\hat{j}\), which proves the second part.

To prove the first part, notice that, since \(ij\) and \(ij\) are restriction idempotents,

\[
ijf = fijf
\]

= \hat{f}ij

= i\hat{f}j \quad \text{and so, precomposing with } j,

\[
iji = ji\hat{j} \quad \text{i.e.}
\]

\[
jj = \hat{j}
\]

which is the first part. \(\square\)

**Lemma 13.** Suppose that, in \(\mathcal{C}\), we have \(f : A \to B\) and \(g : B \to A\) with \(f \dashv g\). Then, as poset morphisms between \(\mathcal{E}_A\) and \(\mathcal{E}_B\), \(f^\ast \dashv g^\ast\).

**Proof.** Because \(f \dashv g\) in \(\mathcal{C}\), we have the unit and counit

\[
\text{Id} \sqsubseteq gf \quad \text{and} \quad fg \sqsubseteq \text{Id}.
\]

Then, by Lemma 10, we have

\[
(gf)^\ast P \leq P \quad \text{and} \quad Q \leq (fg)^\ast Q,
\]

for any \(P \in \text{Ob}(\mathcal{E}_A)\) and \(Q \in \text{Ob}(\mathcal{E}_B)\). By the contravariance of \((\cdot)^\ast\), we have

\[
f^\ast g^\ast P \leq P \quad \text{and} \quad Q \leq f^\ast g^\ast Q,
\]

which are, respectively, the counit and unit of \(f^\ast \dashv g^\ast\). The triangle equalities are, since we are working with posets, trivial. \(\square\)

**Lemma 14.** Let \(\mathcal{C} \to \mathcal{E}\) be a 2-fibration in posets over a partial cartesian category \(\mathcal{C}\), suppose that monics have quasi-inverses in \(\mathcal{C}\), and let \(i : A \to B\) be monic in \(\mathcal{C}\). Then \(i^\ast : \mathcal{E}_B \to \mathcal{E}_A\) has a right adjoint \(\prod_i\) satisfying Beck-Chevalley.

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Proof. By hypothesis, \( i \) has a right adjoint \( j \), and so, by Lemma [13], \( i^* \dashv j^* \): so we can identify \( \prod_i \) with \( j^* \). Furthermore, these right adjoints satisfy Beck-Chevalley by Lemma [12].

**Theorem 4.** Let \( \mathcal{C}_{\text{tot}} \) be the category of total morphisms of \( \mathcal{C} \), a split restriction category, and let \( \mathcal{M}(\mathcal{C}) \) be the class of monos in \( \mathcal{C}_{\text{tot}} \) which split restriction idempotents in \( \mathcal{C} \). Then there is an equivalence of bicategories between, on the one hand, fibrations in posets over \( \mathcal{C}_{\text{tot}} \) such that, for every \( i \in \mathcal{M}(\mathcal{C}) \), \( i^* \) has a right adjoint satisfying \( \triangleright \), and, on the other hand, 2-fibrations in posets over \( \mathcal{C} \).

Proof. By Lemma [11] we can establish the equivalence on the level of functors to \( \text{Poset} \). By Hermida (2002), the functor \( \text{bPar} \) is universal among functors to bicategories \( K \) which send monos in \( \mathcal{M}(\mathcal{C}) \) to 1-cells with right adjoints satisfying \( \triangleright \). We apply this with \( K = \text{Poset} \).

The 2-category of fibrations in posets over \( \mathcal{C}_{\text{tot}} \) such that, for every \( i \in \mathcal{M}(\mathcal{C}) \), \( i^* \) has a right adjoint satisfying \( \triangleright \) is equivalent to the 2-category \( \text{Hom}(\mathcal{C}^{\text{op}}_{\text{tot}}, \text{Poset}) \), which, by Hermida (2002), is equivalent to the 2-category \( \text{Hom}(\mathcal{C}^{\text{op}}, \text{Poset}) \), which in turn equivalent to the 2-category of 2-fibrations in in posets over \( \mathcal{C} \).

**Definition 21.** Under such circumstances, if fibrations \( \pi : \mathcal{E} \to \mathcal{C} \) and \( \pi_{\text{tot}} : \mathcal{E}_{\text{tot}} \to \mathcal{C}_{\text{tot}} \) correspond, we say that \( \pi_{\text{tot}} \) is a restriction of \( \pi \).

**Proposition 5.** Suppose that \( \pi_{\text{tot}} \) is a restriction of \( \pi \). Then \( \pi \) has left adjoints to the pullbacks iff \( \pi_{\text{tot}} \) does.

Proof. The direction from \( \pi \) to \( \pi_{\text{tot}} \) is clear, since every pullback in \( \pi_{\text{tot}} \) is a pullback in \( \pi \). Suppose, on the other hand, that \( f \) is a 1-cell in \( \mathcal{C}_{\text{tot}} \): then, because \( \mathcal{C} \) is equivalent to a category of partial morphisms of \( \mathcal{C}_{\text{tot}} \), we may suppose that \( f = f_0 j \), where \( j \) is the right adjoint of a mono \( i \) and \( f_0 \) is total. Because \( f_0 \) is total, it has, by hypothesis, a left adjoint \( \prod_{f_0} \), and \( \prod_{f_0} i \) is then the required left adjoint for \( f \).

### 3.2 The Total Category of a Fibration

We now consider the structure of the total category \( \mathcal{E} \).

**Lemma 15.** Let \( \pi : \mathcal{E} \to \mathcal{C} \) be a 2-fibration, and let \( \mathcal{C} \) be a restriction category. Then there is a unique restriction structure on \( \mathcal{E} \) which makes \( \pi \) into a restriction homomorphism.

Proof. Note first that, if \( \alpha : A \to A \) is a coreflection in the base, and if \( \pi(P) = A \), then there is a unique \( \vartheta \sqsubseteq \text{Id}_P \) with \( \pi(\vartheta) = \alpha \): the fibrational conditions on \( \sqsubseteq \) give us existence, and the same conditions give us equality of any two candidates. \( \vartheta \) and \( \vartheta \vartheta \) are both lifts of \( \alpha \), so that, by uniqueness of lifting, they are equal, which establishes idempotence. We can now define a restriction structure on \( \mathcal{E} \) by letting \( \overline{f} \) be the unique lift of \( \overline{\pi(f)} \): similar uniqueness arguments give us the restriction axioms.

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Proposition 6. Suppose that $\mathcal{E}$ is 2-fibred in posets over $\mathcal{C}$, that $\mathcal{C}$ is a bicategory of partial maps and that the fibres of $\mathcal{E}$ have finite joins. Then $\mathcal{E}$ is itself a bicategory of partial maps, and the projection $\pi : \mathcal{E} \to \mathcal{C}$ is compatible with the structure.

Proof. By Lemma 15, we already have a restriction category structure on $\mathcal{E}$; we now only need to show that we have a restriction final object and binary restriction products. The restriction final object will be the top element of the fibre over $I$, the restriction final object of $\mathcal{C}$: to define the binary restriction product, let $P$ and $Q$ be objects of $\mathcal{E}$, with $\pi(P) = A$ and $\pi(Q) = B$. Let $p_A$ and $p_B$ be the two projections of $\pi(P) \otimes \pi(Q)$. Then let

$$P \otimes Q = p_A^*P \land p_B^*Q$$

The universal property is easily verified. 

Theorem 5. Let $\pi : \mathcal{E} \to \mathcal{C}$ be a 2-fibration in posets, and suppose that $\mathcal{C}$ has comma objects: suppose also that the fibres of $\pi$ have finite joins. Then $\mathcal{E}$ has comma objects.

Proof. As above, we can show that $\mathcal{E}$ has a restriction structure and a restriction final object: we only have to show that it has weak comma objects. So, consider morphisms $f : P \to Q$, $g : R \to Q$ (that is, we have $P \leq \pi(f)^*Q$ and $R \leq \pi(g)^*Q$. Construct the comma object of $\pi(f)$ and $\pi(g)$:

$$\pi(f) \otimes \pi(g) \xrightarrow{\pi(f)} \pi(R)$$

$$\pi(P) \xrightarrow{\pi(g)} \pi(Q)$$

We now define

$$f \otimes g = \pi(g)^*P \land \pi(f)^*Q$$

and the universal property is easy to verify.

3.3 Frobenius and its Consequences

We now discuss Frobenius laws for these fibrations: they are important in themselves, but they also have useful consequences. In particular, they will give us correspondence theorems between 2-fibrations (in Boolean algebras or in co-Heyting semilattices) over bicategories of partial maps and 1-fibrations over their categories of total maps.

We first define Heyting and coHeyting semilattices: the latter are important because we are concerned, in this article, with equational reasoning. We will mainly investigate a classical system, but the constructive variant will be based on apartness, and the appropriate structure in the fibres will be coHeyting.
We need strong and weak morphisms for both objects: this is because, over \( \mathcal{C} \), pullbacks along partial morphisms of coHeyting semilattices will preserve \( \lor \), but will not, in general, preserve \( \bot \) and will only preserve the coHeyting operation in a rather weak sense. For the general correspondence theory, we will need both sorts of morphism, because only pullbacks along total morphisms will, in general, preserve \( \bot \) and the coHeyting operation.

**Definition 22.** A **Heyting semilattice** is a poset with all finitary meets, the binary meet being written \( \land \) and the nullary meeting being written \( \top \), together with a binary operation \( Q \) such that

\[
P \land Q \leq R \quad \Rightarrow \quad P \leq R Q
\]

**Definition 23.** A **morphism of Heyting semilattices** is a poset morphism which preserves \( \land \), \( \top \), and the Heyting operation.

**Definition 24.** A **weak morphism of Heyting semilattices** is a poset morphism \( \phi \) which preserves \( \land \), and for which \( \phi(QP) = \phi(Q) \phi(P) \land \phi(\top) \)

**Definition 25.** A **coHeyting semilattice** is a poset with all finitary joins, the binary join being written \( \lor \) and the nullary meet being written \( \bot \), together with a binary operation \( P \) such that

\[
P \leq Q \lor R \quad \Rightarrow \quad P Q \leq R
\]

**Definition 26.** A **morphism of coHeyting semilattices** is a poset morphism which preserves \( \lor \), \( \bot \), and the coHeyting operation.

**Definition 27.** A **weak morphism of coHeyting semilattices** is a poset morphism \( \phi \) which preserves \( \lor \), and for which

\[
\phi(P Q) = (\phi(P) \phi(Q)) \lor \phi(\bot)
\]  

(8)

**Remark 3.** The definitions of weak morphisms can be motivated as follows. For a given \( a \), the downward closure of \( a \) can be given a Heyting structure in a natural way: the new \( \top \) is \( a \), \( \land \) is as before, and the new Heyting operation is \((\wp x) \land a \). Then a weak Heyting morphism is just a Heyting morphism with codomain the downward closure of \( a \). The situation for coHeyting morphisms is dual.

Now we can start on Frobenius laws. The following is standard:

**Proposition 7.** Let \( \pi : \mathcal{E} \to \mathcal{C} \) be a (1- or 2-)fibration in Heyting semilattices, and suppose that, for a 1-cell \( f \) in the base, its pullback \( f^* \) has a left adjoint and commutes with the Heyting operation. Then the following Frobenius property holds:

\[
(\bigsqcup_f P) \land Q = \bigsqcup_f (P \land f^* Q).
\]
Proof. We have to establish lattice inequalities in both directions. The direction $(\prod_f P) \land Q \geq \prod_f (P \land f^*Q)$ is easy, and only requires the adjunction $\prod\dashv f^*$; for the other direction, we use the Heyting operation (see Jacobs, 1999, Lemma 1.9.12, p. 102), (White, 2008), noting that, since $f^*$ has a left adjoint, it preserves $\top$, and so the notions of weak Heyting and Heyting coincide.  

Dually, we have the following (this will also be useful to us, since the domain fibration, which we will study in Section 3.5, is a fibration in coHeyting semilattices.

**Proposition 8.** Let $f$ be a fibration in coHeyting semilattices, and suppose that, for a 1-cell $f$ in the base, its pullback $f^*$ has a right adjoint and commutes with the coHeyting operation. Then the following Frobenius property holds:

$$ (\prod_f P) \land Q = \prod_f (P \land f^*Q). \quad (9) $$

**Corollary 5.** Suppose that we have a 2-fibration in coHeyting semilattices $\pi : \mathcal{E} \to \mathcal{C}$, where $\mathcal{C}$ is a bicategory of partial maps. Let $i \dashv j$ in $\mathcal{C}$: then

$$ j^* P \land Q = j^*(P \land i^*Q). \quad (10) $$

Proof. $i \dashv j$, so $i^* \dashv j^*$: $i^*$ satisfies Frobenius by Proposition 8. (10) follows from this, writing $j^*$ instead of $\forall_i$.

**Corollary 6.** If we have a fibration in coHeyting semilattices, and if $i \dashv j$, with $P$ and $Q$ in the fibre over the codomain of $i$,

$$ i^*P \leq i^*Q \iff P \leq Q \land j^*\bot $$

Proof. Right to left is a straightforward calculation, since $i^*$ (having a right adjoint) preserves $\lor$ and since $i^*j^* = \text{Id}$. For left to right, we argue as follows:

$$ P \leq j^*i^*Q \quad i \dashv j \quad \frac{i^*P \leq i^*Q}{P \leq j^*i^*Q} \quad \frac{P \leq j^*(i^*Q \land \bot)}{P \leq Q \land j^*\bot} \quad \text{Frobenius} \quad \square $$

The following result will be important for our sequent calculus:

**Corollary 7.** Suppose that we have a fibration in coHeyting semilattices, and that we have, in the base, $f \subseteq g : A \to B$. Then, for any $P$ over $B$, we have

$$ f^*P = g^*P \lor f^*\bot. $$
Proof. We can assume, wlog, that \( f = gij \), with \( i \vdash j \). But now

\[
g^*P \lor f^*\bot = g^*P \lor j^*i^*g^*\bot \\
= j^*(i^*g^*P \lor i^*g^*\bot) \quad \text{by Frobenius}
\]

\[
= j^*i^*g^*P \quad \text{by monotonicity of } i^*g^*
\]

\[
= f^*P \quad \text{QED}
\]

We can now apply these results to correspondence results between fibrations in coHeyting semilattices over \( C \) and those over \( C_{\text{tot}} \): the following example shows that, as we claimed, we cannot have a fibration in coHeyting semilattices and strict morphism over a bicategory of partial maps.

**Example 2.** Suppose that we have a fibration in coHeyting algebras over a bicategory of partial maps, that we have \( i \vdash j \) in the base, and that \( i^* \) is a strict Heyting algebra morphism. \( j^* \) is a right adjoint, so it preserves \( \top \): furthermore, for any \( P, P\top = \bot \). So we have \( j^*(P\top) = j^*\bot \), but \( j^*P\top = j^*P\bot = \bot \). However, \( j^*\bot \) will not be equal to \( \bot \) in general: the subobject fibration of a category of sets and partial maps shows that.

**Proposition 9.** Suppose that \( C \) is a bicategory of partial maps. Then the following are equivalent:

1. 2-fibrations in coHeyting semilattices and weak coHeyting semilattice morphisms \( \mathcal{E} \rightarrow \mathcal{C} \) such that
   
   (a) for all objects \( A \), pullbacks along \( !_A \) preserve \( \bot \)
   
   (b) pullbacks have left adjoints

2. fibrations in coHeyting semilattices and coHeyting semilattice morphisms \( \mathcal{C}_{\text{tot}} \rightarrow \mathcal{C}_{\text{tot}} \) such that
   
   (a) pullbacks have left adjoints
   
   (b) pullbacks along monos have right adjoints which satisfy Frobenius

Proof. We first show that 1 \( \Rightarrow \) 2: the pullbacks preserve finite joins in \( \mathcal{C}_{\text{tot}} \) because they do so in \( \mathcal{C} \). Similarly, the pullbacks in \( \mathcal{C}_{\text{tot}} \) have left adjoints. The existence of right adjoints satisfying Frobenius for monos follows from the fact that, if \( i \) is a mono in \( \mathcal{C}_{\text{tot}} \), it has a right adjoint \( j \) in \( \mathcal{C} \): but then \( j^* \) is the desired right adjoint to \( i^* \), and the Frobenius properties correspond. Finally we need to show that pullbacks along total morphisms are strict coHeyting morphisms. Firstly, they preserve \( \bot \) because, if \( f : A \rightarrow B \) is total, then, by definition, \( !_Bf = !_A \), and so \( \bot_A = !_A\bot_I = f^*!_B\bot_I = \bot_B \). If they preserve \( \bot \), they, by the definition of weak coHeyting morphisms, they are strict coHeyting morphisms.

For the other direction we argue as follows. Proposition 5 gives us an extension of the fibration and left adjoints to the pullbacks: the Frobenius properties then correspond. We now have to show that pullbacks along 1-cells preserve
binary joins: this is true by assumption for total 1-cells, and we have to show that it holds for 1-cells which are right adjoint to monos. So, let \( i \dashv j \): \( ji = \text{Id} \), and so

\[
j^*(P \lor Q) = j^*(i^*j^*P \lor Q) = j^*P \lor j^*Q \quad \text{by Frobenius}.
\]

We have now to show that the pullbacks preserve the coHeyting operation in the required weak sense. Firstly, an easy calculation shows that \( \langle S \rangle \) is preserved under composition: so it suffices to show that it holds for total morphism and for right adjoints to monos. It holds for total morphisms because they preserve \( \bot \), and so \( \langle S \rangle \) requires, in this case, strict preservation of the coHeyting operation, which we have by assumption. So we have to show that \( \langle S \rangle \) holds for pullbacks along right adjoints to monos. Note first that, in any coHeyting semilattice, \( P \lor Q \) is the infimum of the \( X \) such that \( P \leq Q \lor X \). So we have, for any \( S \),

\[
\begin{align*}
S & \leq j^*(PQ) \\
i^*S & \leq PQ \\
i^*S & \leq T^* \quad \text{for any } T \text{ such that } P \leq Q \lor T \\
S & \leq j^*iT^* \quad \text{for any } T^* \text{ such that } P \leq Q \lor i^*T^* \text{ (} i^* \text{ is surjective)} \\
S & \leq j^*iT^* \quad \text{for any } T^* \text{ such that } j^*P \leq j^*Q \lor j^*i^*T^* \text{ (} j^* \text{ is injective)} \\
S & \leq j^*iT^* \quad \text{for any } T^* \text{ such that } j^*P \leq j^*Q \lor T^* \lor j^*\bot \text{ (Frobenius)} \\
S & \leq j^*iT^* \quad \text{for any } T^* \text{ such that } (j^*P)(j^*Q) \leq T^* \\
S & \leq T^* \lor j^*\bot \quad \text{for any } T^* \text{ such that } (j^*P)(j^*Q) \leq T^*, \text{ by Frobenius}
\end{align*}
\]

and so we have the result by Yoneda.

**Corollary 8.** Let \( \mathcal{C}_{\text{tot}} \) be the subcategory of total maps of \( \mathcal{C} \). The following are equivalent:

1. Fibrations in boolean algebras and \( \land, \lor, \top \)-preserving poset morphisms over \( \mathcal{C} \) such that pullbacks along \( !_A \), for any \( A \), preserve \( \bot \).

2. Fibrations in boolean algebras and boolean algebra morphisms over \( \mathcal{C}_{\text{tot}} \).

**Proof.** We use the obvious coHeyting structure on a boolean algebra, and apply the previous proposition. For 1 \( \Rightarrow \) 2, we use the fact that a lattice homomorphism of a boolean algebra is a boolean algebra morphism. For 2 \( \Rightarrow \) 1 we extend the fibration from \( \mathcal{C}_{\text{tot}} \) to \( \mathcal{C} \) in the usual way: we factorise a given 1-cell \( f \) of \( \mathcal{C} \) as \( f_0j \), with \( i \dashv j \), and express \( f^* \) as \( \prod_j f_0^* \). Now \( \prod_j \) preserves \( \land \), because it is a right adjoint, and \( f_0^* \) does by assumption. Preservation of \( \lor \) follows from the above proposition.  \( \square \)
Remark 4. We can motivate the results of this section as follows. We have a general structure theory for bicategories of partial maps which describes morphisms in these categories as total morphisms precomposed with partial morphisms of a special form: so, every \( f \) is of the form \( f_0 j \), with \( f_0 \) total and \( j \) being the quasi-inverse of a monic. \( j \) prevents \( f_0 j \) from being defined everywhere. Now if we look at a pullback along \( f \), then, because of contravariance, we find that \( f^* = j^* f_0^* \). If we have a fibration in, let us say, coHeyting algebras, then, if the pullbacks were coHeyting algebra morphisms, then they would, by Example 2, have to have \( \bot \) as a value, at least in some plausible cases. This may well be possible for pullbacks along total morphisms, but in general we will have pullbacks of the form \( j^* f_0^* \), and here \( j^* \) is an obstruction to the pullback having the required values: if \( j^* \) cannot hit \( \bot \), then neither can \( f^* \). So the best we can do is to have a morphism whose codomain is the segment \( [f^* \bot, \top] \), i.e. (by Remark 3) a weak coHeyting morphism.

3.4 Beck-Chevalley

Theorem 6. Suppose that we have a fibration over \( \mathcal{C} \) whose pullbacks have left adjoints, and suppose also that pullbacks along monos have right adjoints which satisfy Beck-Chevalley. Then the left adjoints of the fibration over \( \mathcal{C}_{\text{tot}} \) satisfy Beck-Chevalley with respect to pullback squares iff the left adjoints of the fibration over \( \mathcal{C} \) satisfy Beck-Chevalley with respect to comma squares: that is, if we have

\[
\begin{array}{ccc}
\hat{f} & \xrightarrow{g} & \hat{C} \\
\downarrow \hat{g} & & \downarrow \hat{g} \\
\hat{A} & \xrightarrow{\hat{f}} & \hat{B}
\end{array}
\]

and if we have \( P \) over \( A \), then we have

\[
g^* \prod_f P = \prod_{\hat{f}} \hat{g}^* P.
\]

Proof. The if direction is trivial: we have to prove that, if the fibration over \( \mathcal{C} \) satisfies Beck-Chevalley, then the fibration over \( \mathcal{C}_{\text{tot}} \) does.

We prove the only if direction by the usual pasting argument as follows. In the diagram of Proposition 4,

\[
\begin{array}{ccc}
A_0 \times_A (f \otimes g) & \xrightarrow{i} & f \otimes g \\
\downarrow \hat{g} & & \downarrow \hat{g} \\
A_0 & \xrightarrow{i} & A & \xrightarrow{f} & B
\end{array}
\]
note that $\hat{i}$ is a mono (because it is a pullback of a mono): thus, both $i$ and $\hat{i}$ have right adjoints $j$ and $\hat{j}$. Furthermore, we can factorise $f$ as $f_0j$ and $\hat{f}$ as $\hat{f}_0\hat{j}$, where $f_0 = fj$ and $\hat{f}_0 = \hat{f}\hat{j}$ are total. So we have a diagram

$$\begin{array}{ccc}
    f \otimes g & \otimes A & (f \otimes g) \Rightarrow C \\
    \downarrow & & \downarrow \\
    A & \otimes A & \Rightarrow A
\end{array}$$

The right hand square is a diagram in $\mathcal{C}_{tot}$, and we have Beck-Chevalley for that by assumption: because $i \dashv j$, Beck-Chevalley for the left hand square follows from the Beck-Chevalley condition of the left hand square of the previous diagram. So we have the result by pasting. 

\[\square\]

3.5 The Domain Fibration

The domain subobject fibration is defined for a broad range of 1-categories: the fibre over an object $A$ is the set of subobjects of $A$, with substitution defined by pullback.

In the case of partial cartesian categories, we have a particular class of subobjects of $A$, namely those given by the domains of definition of 1-cells from $A$. The corresponding fibration is called the domain fibration; it is a fibration in $\land$-semilattices. However, there are subtleties to do with the variance of the fibration thus defined. The fibrations that we have so far studied arise from 2-functors $\mathcal{C}^{coop} \to \text{Poset}$, where $\text{Poset}$ is the two-category of posets, the morphisms ordered pointwise. Thus, the substitution morphisms are contravariant on 1-cells and 2-cells. It is also possible to define fibrations with substitutions contravariant on 1-cells but covariant on 2-cells, that is, fibrations corresponding to 2-functors $\mathcal{C}^{op} \to \text{Poset}$.

This comes about as follows. Hermida’s correspondence, described in Section 3.1.2, shows how 1-fibrations over the total 1-cells of a split partial cartesian category $\mathcal{C}$ can be extended to 2-fibrations over $\mathcal{C}$. Let $\mathcal{M}$ be the class of monos in $\mathcal{C}$ which split restriction idempotents: then a fibration $\pi: \mathcal{C}_{tot} \to \mathcal{C}_{tot}$, contravariant on 1-cells, extends to a fibration over $\mathcal{C}$ contravariant on 1- and 2-cells iff for each $i$ in $\mathcal{M}$, $i^*$ has a right adjoint satisfying Beck-Chevalley with respect to pullbacks along any 1-cell in $\mathcal{C}_{tot}$.

But the same construction also yields, in exactly the same way, a result with different variance:

**Proposition 10.** A fibration over $\mathcal{C}_{tot}$, contravariant on 1-cells, can be extended to a fibration over $\mathcal{C}$, contravariant on 1-cells and covariant on 2-cells, iff, for each $i$ in $\mathcal{M}$, $i^*$ has a left adjoint satisfying Beck-Chevalley.

So we can, depending on the existence of appropriate adjoints, have either sort of domain fibration. We will consider each case separately: first, though, we show what the existence of either sort of adjoint amounts to.
Lemma 16. Let \( \mathcal{C} \) be a split restriction category, and let \( \mathcal{M} \) be the class of monos which split restriction idempotents in \( \mathcal{C} \).

1. The domain fibration of \( \mathcal{C} \) has left adjoints, satisfying Beck-Chevalley with respect to fibred products with total morphisms, to pullbacks along monos in \( \mathcal{M} \).

2. The domain fibration of \( \mathcal{C} \) has right adjoints to pullbacks along monos in \( \mathcal{M} \), satisfying Beck-Chevalley with respect to fibred products with total morphisms, iff \( \mathcal{C}_{\text{tot}} \) has stable Heyting operations.

Proof. This is mostly a reformulation of standard results. Observe that monos are in \( \mathcal{C}_{\text{tot}} \), and so left or right adjoints in \( \mathcal{C} \) are left or right adjoints, respectively, in \( \mathcal{C}_{\text{tot}} \). We can then apply [Jacobs (1999), pp. 256ff.], which shows that the domain fibration has left adjoints to pullbacks along monos iff it has meets: but it does have meets. Beck-Chevalley corresponds to the fact that meets are stable (i.e. that \( f^*(\alpha \land \beta) = f^*\alpha \land f^*\beta \)), which follows from [Cockett & Lack (2002), p. 254]. This establishes the first part.

For the second part, [Jacobs (1999), pp. 256ff.] shows that the domain fibration has right adjoints to pullbacks along monos iff it has Heyting operations on the posets of domains: Beck-Chevalley then corresponds to stability of the Heyting operations (we can show, by suitably factoring morphisms, that it suffices to verify the stable Heyting condition with total morphisms).

Remark 5. Terminology for this sort of thing is a disaster. “co” can either mean “in the same direction” (as in covariant), or “in the opposite direction” (as in counit). I shall abbreviate the names of these fibrations to “2-covariant” and “2-contravariant”, which is clumsy, but I can’t see any better solution.

3.5.1 The 2-Covariant Domain Fibration

Definition 28. Let \( \mathcal{C} \) be a restriction category. Define the 2-covariant domain fibration of \( \mathcal{C} \), \( \text{dom}^{\text{co}}(\mathcal{C}) \), as follows:

The fibre over an object \( A \) of \( \mathcal{C} \), \( \text{dom}(\mathcal{C})_A \), is the \( \land \)-semilattice \( \{ \overline{f} \mid f : A \to B \} \), where

- \( \alpha \sqsubseteq \beta \) iff \( \alpha \beta = \alpha \),
- \( \alpha \land \beta \) is \( \alpha \beta \), and
- \( \top \) is \( \text{Id}_A \).

The pullback of \( \alpha \) along \( f \) is \( \overline{\alpha f} \).

Proposition 11. The domain fibration is a two-fibration which is contravariant on 1-cells and covariant on 2-cells (that is, it corresponds to a 2-functor \( \mathcal{C}^{\text{op}} \to \land \text{Sela} \), where \( \land \text{Sela} \) is the 2-category of \( \land \)-semilattices).
Proof. Routine calculation: we use the fact that the above definition when restricted to \( C_{\text{tot}} \) gives a 1-fibration, and then, to show that it corresponds to a 2-fibration, we use the Hermida correspondence with the appropriate variance. For this we need to show that monos \( i \) in \( M \) have left adjoints \( \exists_i \) satisfying Beck-Chevalley with respect to fibred products with total morphisms. This adjoint is given by the first part of Lemma 16. Finally we show that the pullbacks constructed by the Hermida construction coincide with those given by the definition above, which is a routine calculation using Lemma 12. \( \square \)

### 3.5.2 The 2-Contravariant Domain Fibration

We can, as remarked above, define this when the domain posets have stable Heyting operations: as well as the general argument given there, we can define the fibration explicitly as follows:

**Definition 29.** Let \( \mathcal{C} \) be a split bicategory of partial maps, and let \( \mathcal{M} \) be the class of monos which split restriction idempotents. Suppose that \( C_{\text{tot}} \) is a fibration in Heyting semilattices and weak Heyting semilattice morphisms: then define the 2-contravariant category of domains, \( \mathcal{D} \).

**The fibre** over an object \( A \) of \( \mathcal{C} \), \( \text{dom}(\mathcal{C})_A \), is the Heyting-semilattice \( \{ f: A \to B \} \), where

\[
\alpha \sqsubseteq \beta \text{ iff } \alpha \beta = \alpha,
\]

\[
\alpha \land \beta \text{ is } \alpha \beta, \text{ and}
\]

\[
\top \text{ is } \text{Id}_A.
\]

**The pullback** of \( \alpha \) along \( f \) is \( \alpha f \downarrow f \).

**Theorem 7.** Let \( \mathcal{C} \) be a restriction category with a restriction final object where the subobject fibration has left adjoints to the pullbacks. TFAE:

1. \( \mathcal{C} \) has weak comma objects
2. the subobject fibration of \( C_{\text{tot}} \) has stable Heyting operations
3. the subobject fibration of \( C_{\text{tot}} \) has right adjoints to pullbacks along monics
4. the 2-contravariant domain fibration is defined

**Proof.** This is a combination of the results of Section 2.3 together with the previous proposition. \( \square \)

**Example 3.** Consider the category of sets and partial maps. The two domain fibrations are defined as follows: domains are, in both cases, simply subsets, but the pullbacks are as follows.

\( \text{dom}^{\text{co}} \) Let \( f: A \to B \) be a 1-cell, and let \( V \subseteq B \). Then, for \( x \in A \),

\[
x \in f^*V \text{ iff } (f(x) \downarrow) \land f(x) \in V
\]
\textbf{Remark 6.} If our category of total morphisms had stable sums and epi-mono factorisations in addition to the above conditions, then it would be a \textit{logos}.

\section{The Sequent Calculus}

We can now define a sequent calculus. We fix a functionally complete category of partial maps for the base. Our calculus will be typed: propositions are typed by objects of the base category, and, for each one-cell of the base, we have substitution operators on propositions of the appropriate types. Formation rules for propositions and sequents are given in Table 1: note the substitution rules for sets of propositions $\Gamma$ on the left, and $\Delta$ on the right.

\subsection{Notational Conventions}

As we have seen, the notation for tensors of objects and arrows tends to become rather cumbersome. We will frequently abbreviate it by leaving out the names of objects, and writing a diagram of the form

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A \otimes C$};
  \node (B) at (3,0) {$C$};
  \node (C) at (0,-3) {$A \otimes g$};
  \node (D) at (3,-3) {$g$};
  \draw[->] (A) -- (B) node[midway,above] {$f \otimes C$};
  \draw[->] (C) -- (D) node[midway,above] {$g$};
  \draw[->] (A) -- (C) node[midway,left] {$f$};
  \draw[->] (B) -- (D) node[midway,left] {$g$};
\end{tikzpicture}
\end{center}

We will hardly ever need the names of objects in our sequent calculus, and we will only use the superscript $\hat{\cdot}$ in diagrams of the above form (or those constructed from them): with these conventions, the diagrams should be unambiguous.

\subsection{The Rules}

As we have said, the logic in the fibres will be classical, and we use the standard sequent calculus rules for the sentential connectives and for cut: our primitives are $\lor$ and $\land$, and $\rightarrow$ will be a defined connective. These rules are given in Table 2. The rules specific to the bicategorical system are given in Table 3. Note that we define $f^\ast$ as follows.

\textbf{Definition 30.} Define $f^\ast \Gamma$, for a set of formulae $\Gamma$, as follows:

If $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ is on the left, then

$$f^\ast \Gamma = \{ f^\ast \gamma_1, \ldots, f^\ast \gamma_m \}$$
Table 1: Formation Rules for Formulae and Sequents

| Formulas | $\top : A$ | $\bot : A$ | $\neg P : A$ |
|----------|------------|------------|-------------|
| $P : A$  | $Q : A$    |            | $P : A$     |
| $P \land Q : A$ | $P \lor Q : A$ | $f : A \to B$ | $f : A \to B$ |
| $\Pi f : B$ | $\Pi f : B$ | $f : A \to B$ | $Q : B$ |
| $f^*(Q) : A$ | $\Delta = \{Q_1, \ldots, Q_n\}$ | $f^*\Gamma = \{f^*P_1, \ldots, f^*P_n\}$ | $f^*\Delta = \{f^*Q_1, \ldots, f^*Q_n, f^*\bot\}$ |

If $\Delta = \{\delta_1, \ldots, \delta_n\}$ is on the right, then

$$f^*\Delta = \begin{cases} 
\{f^*\delta_1, \ldots, f^*\delta_m\} & \text{if } \Delta \text{ is nonempty} \\
\{f^*\bot\} & \text{otherwise.}
\end{cases}$$

The rules for $\Pi$ incorporate the Beck-Chevalley condition: this will make the proof of cut elimination much easier.

4.2 Cut Elimination

Proving cut elimination for systems like these faces the following problem (see Goré et al. (2009)): if we have a cut such as

$$\Gamma \vdash A, B, \Delta \\
\Gamma \vdash A \lor B, \Delta \\
f^*\Gamma', f^*(A \lor B) \vdash Q, f^*\Delta'$$

then it is not obvious how to move the cut upwards. We can deal with this difficulty in two ways: we can either use a system with deep inference, as we did in White (2008), or we can, as we do here, use a more conventional syntax (with rules which make the deep inference rules admissible) and prove an inversion lemma together with an auxiliary result for the cases where the inversion lemma does not work. Both strategies cost about the same amount of work: the deep inference strategy relies on unfamiliar syntax and is, as it were, more high level, whereas the inversion lemma strategy relies on familiar syntax but is low level.
Table 2: The Rules for the Sentential Connectives and Cut
Table 3: The Bicategorical Rules
But the inversion lemma also makes clear the role of Beck-Chevalley in the proof of cut elimination. (Hermida, 2004).

First we prove some lemmas.

**Lemma 17.** The following rules are admissible in the cut-free system:

\[
\begin{align*}
&\frac{\Gamma, P \vdash \Delta}{f^*\Gamma, g^*P \vdash g^*\Delta} \\
&\frac{\Gamma \vdash Q\Delta}{g^*\Gamma \vdash f^*Q, g^*\Delta} \\
&\frac{f^*\Gamma, P \vdash f^*\Delta}{\Gamma, \prod_f P \vdash \Delta} \\
&\frac{\Gamma \vdash P, \Delta}{\Gamma \vdash f^* \prod_f P, \Delta} \\
&\frac{\Gamma \vdash P, \Delta}{\Gamma \vdash f^* \prod_f P, \Delta} \\
\end{align*}
\]

The diagram for \( h^* \prod L \) is as follows:

\[
\begin{array}{c}
\hline
h \ast \hat{g} \\
\hline
\hline
f \ast \hat{g} \\
\hline
\hline
f \\
\hline
\end{array}
\]

Proof. The \( \prod' \) rule is obtained by following the standard \( \prod \) rule with applications of \( \text{Id} \) and \( \text{Id}^{-1} \). The proof for \( h^* \prod L \) goes as follows:

\[
\begin{align*}
&\frac{\hat{h}^*\hat{g}^*\Gamma, \hat{h}^*f^* \prod gP \vdash \hat{h}^*\hat{g}^*\Delta}{\hline R} \\
&\frac{\hat{g}^*\hat{h}^*\Gamma, \hat{h}^*f^* \prod gP \vdash \hat{g}^*\hat{h}^*\Delta}{\hline L}
\end{align*}
\]

We also need to define a notion of the **height** of a proof:

**Definition 31.** The height of a proof is defined as follows:

1. The height of a proof consisting of a single application of an axiom rule is zero
2. If we have a proof of the form

\[
\Pi
\]

\[
\frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'} \quad R
\]

and if \( R \) is one of the \( \sqsubseteq, \circ, \circ^{-1}, \text{Id}, \) or \( \text{Id}^{-1} \) rules, then the height of this proof is equal to the height of \( \Pi \)

3. Otherwise the height of a proof of this form is equal to \( \text{height}(\Pi) + 1 \).
Similarly, the *complexity* of a formula is defined as follows:

**Definition 32.** The complexity of a formula is defined inductively as follows:

**Axiom** The complexity of an atomic formula $P$ is 1

$f^*$ The complexity of $f^*P$ is the complexity of $P$

**Unary Connectives** Other unary connectives ($\neg$, $\prod$) increase the complexity by 1

**Binary Connectives** The complexity of $P \land Q$ is one more than the maximum of the complexities of $P$ and $Q$.

We are, in this sequent calculus, not dealing with inference in the base category (that is, we treat data such as $f \sqsubseteq g$ as simply given), and this is consistent with our policy of treating $f^*$ as having no effect on the complexity of formula, and treating the $f^*$ rule as having no effect on the height of proofs.

We can now prove our inversion lemma: this will say that, for example, we have a proof of $\Gamma, f^*(P \land Q) \vdash \Delta$, then we also have proofs of $\Gamma, f^*P, f^*Q \vdash \Delta$. We can prove such a lemma in all cases except one: that case is when we have an existential quantifier on the right. In this case, directly inverting the $\prod R$ rule would amount to giving a witness for the existential quantifier, which, for well-known reasons, is impossible: we cannot permute the $\prod R$ rule below $\prod L$ or $\land R$.

Note also that, apart from the missing case, the remaining classification is, for rather trivial reasons, not exhaustive: we do not, for example, consider formulae like $f^*g^*(P \land Q)$, and we also consider formulae of the form $\text{Id}^*P$ rather than those of the form $P$. However, we will need the inversion lemma to prove cut elimination, and, because of the way that we have defined the height of proofs, if we have a proof where the cutformula is $f^*g^*(P \land Q)$, we can find another one with subproofs of the same height where the cutformula is $(gf)^*(P \land Q)$, and we can apply the inversion lemma to this proof.

**Lemma 18** (Inversion).

1. If $\Gamma, P \land Q \vdash \Delta$ is provable with a proof of height $n$, then so is $\Gamma, P, Q \vdash \Delta$, and similarly for $\Gamma \vdash P \lor Q, \Delta$, $\Gamma, \neg P \vdash \Delta$ and $\Gamma \vdash \neg P, \Delta$.

2. If $\Gamma, f^*(P \land Q) \vdash \Delta$ is provable with a proof of height $n$, then so is $\Gamma, f^*P, f^*Q \vdash \Delta$, and similarly for $\Gamma \vdash f^*(P \lor Q), \Delta$.

3. If $\Gamma \vdash \neg P, \Delta$ is provable with a proof of height $n$, then so is $\Gamma, P \vdash \Delta$

4. If $\Gamma \vdash f^*(\neg P), \Delta$ is provable with a proof of height $n$, then so is $\Gamma, f^*P \vdash f^*\bot, \Delta$.

5. If $\Gamma, \neg P \vdash \Delta$ is provable with a proof of height $n$, then so is $\Gamma \vdash P, \Delta$.

6. If $\Gamma, f^*(\neg P) \vdash \Delta$ is provable with a proof of height $n$, then so is $\Gamma \vdash f^*P, \Delta$.
7. If $\Gamma \vdash P \land Q, \Delta$ is provable with a proof of height $n$, then so are $\Gamma \vdash P, \Delta$ and $\Gamma \vdash Q, \Delta$, and similarly for $\Gamma, P \lor Q \vdash \Delta$.

8. If $\Gamma \vdash f^*(P \land Q), \Delta$ is provable with a proof of height $n$, then so are $\Gamma \vdash f^*P, \Delta$ and $\Gamma \vdash f^*Q, \Delta$.

9. If $\Gamma, f^* \prod \not P \vdash \Delta$ is provable with a proof of height $n$, and if $fh \sqsubseteq gk$, for some $h, k$, then

$$h^*\Gamma, k^*P \vdash h^*\Delta$$

is also provable with a proof of height $n$.

10. If $\Gamma, f^*P \vdash \Delta$ is provable with a proof of height $n$, then so is $\Gamma, f^*\bot \vdash \Delta$.

Proof. This is more or less standard, with a few modifications because of $f^*$: we will prove some illustrative cases. We first note that the base case – that is, axioms, $\bot L$, and $\top R$ – is trivial: none of the formulae in question can, in this case, be principal, and so we can make what modifications we please. So we can concentrate on the inductive step.

Case 1 The special cases are those in which the previous rule application is:

- $\land L$ with $P \land Q$ principal: we omit the last rule application.
- MWL with $P \land Q$ principal: we weaken with the multiset $P, Q \cup \Gamma'$, where $\Gamma'$ is the multiset involved in the original weakening, apart from $P \land Q$.
- $\prod_f L'$ Here $P \land Q$ cannot be principal, so we have a proof of the form

$$\vdots$$

$$\frac{f^*\Gamma, f^* (P \land Q), R \vdash f^*\Delta}{\Gamma, P \land Q, \prod_f R \vdash \Delta} \quad \prod L'$$

We apply Case 2 of the inductive hypothesis to the premise.

$\circ^{-1}$ We apply Case 2 of the inductive hypothesis to the premise

Otherwise, $P \land Q$ persists unchanged from the premise(s), and we can apply the inductive hypothesis to the premises.

Case 2 Here the special cases are weakening, $f^*$, both of the $\sqsubseteq$ rules, $\prod L$, and $\prod R$: in the first five cases we apply the relevant inductive hypothesis to the premises, and then the rule application. If we have $\prod L$, then its premise must be of the form

$$\dot{g}^*\Gamma, \dot{g}^* h^* (P \land Q), \dot{f}^* R \vdash \dot{g}^*\Delta;$$

we can apply the inductive hypothesis to the premise (together with $\circ$ and $\circ^{-1}$) to get

$$\dot{g}^*\Gamma, \dot{g}^* h^* P, \dot{g}^* h^* Q, \dot{f}^* R \vdash \dot{g}^*\Delta;$$
an application of $\prod L$ gives us the conclusion. $\prod R$ is very similar.

Otherwise, $f^* (P \land Q)$ persists unchanged, so we are done by induction.

**Case 3** As above.

**Case 4** Here we need some care. The special cases are $f^*$, both cases of $\sqsubseteq$, and $\prod L$. In the case of $f^*$ we need care when $\Delta$ is empty. In this case the inductive hypotheses, followed by an application of $f^*$, gives us

\[
\vdash \Gamma, P \vdash f^* \perp
\]

and so we need the $f^* \perp$ on the right.

With $\sqsubseteq L$, the premise must be of the form

\[
\Gamma \vdash g^*(\neg Q), \Delta;
\]

with $f \sqsubseteq g$. Inductively, we have

\[
\Gamma, g^* Q \vdash g^* \perp, \Delta.
\]

and now we can apply $\sqsubseteq R$ to conclude

\[
\Gamma, f^* Q \vdash f^* \perp, \Delta.
\]

$\sqsubseteq R$ is similar. $\prod L$ is handled similarly to Case 2

Cases 5 to 8 are similar to the above.

**Case 9** The special cases are $f^*$, $\sqsubseteq$, $\text{Id}$, $\circ$, and $\prod L$ (with $f^* \prod g P$ both principal and non-principal). We discuss each of them in turn.

**$f^*$** By hypothesis, the last inference of the proof looks as follows:

\[
\frac{\Gamma, g^* \prod_h P \vdash \Delta}{f^* \Gamma, f^* g^* \prod_h P \vdash f^* \Delta}
\]

What we have to show is that, if, for some $k, l$, we have $g fk \sqsubseteq hl$, then we have a proof of $k^* f^* \Gamma, l^* P \vdash k^* f^* \Delta$; but this follows immediately from the inductive hypothesis.

**$\sqsubseteq$** Here the last step is

\[
\frac{f \sqsubseteq g, \Gamma, f^* \prod_h P \vdash \Delta}{\Gamma, g^* \prod_h P \vdash \Delta}
\]
where \( f_1 \subseteq f_2 \). Suppose that \( gk \subseteq hl \): we want a proof of \( k^* \Gamma, l^* P \vdash k^* \Delta \). However, if \( gh \subseteq hl \), then \( fh \subseteq hl \), and, by induction, we have the needed proof directly.

**ld** Here we have a proof ending

\[
\frac{\Gamma, \prod_g P \vdash \Delta}{\Gamma, \text{ld}^* \prod_g P \vdash \Delta} \quad \text{ld}
\]

Suppose that \( \text{ld} h \subseteq gk \): then, since \( h \subseteq gk \), we can use Case 9 inductively to get a proof of \( h^* \Gamma, k^* P \vdash h^* \Delta \), which is what we require.

**\( \prod \text{L} \)** There are two cases, depending on whether \( f^* \prod_g P \) is principal in the last inference or not.

If \( f^* \prod_g P \) is principal in this rule application, then the premise must be \( \hat{g}^* \Gamma, f^* P \vdash \hat{g}^* \Delta \). Suppose now that \( fh \subseteq gk \): then, by the universal property of the comma object, \( \hat{g}(h, k) = h \Gamma k \) and \( \hat{f}(h, k) = k h \Gamma \).

So, pulling back by \( \langle h, k \rangle \), we have

\[
\overline{h} \Gamma \overline{k}^* h^* \Gamma, \overline{h} \Gamma \overline{k}^* k^* P \vdash \overline{h} \Gamma \overline{k}^* h^* \Delta
\]

from which we can derive \( h^* \Gamma, k^* P \vdash h^* \Delta \) by \( \sqsubseteq \text{LR} \), since \( \overline{hk} \sqsubseteq \text{ld} \).

Otherwise, another formula – say \( f^* \prod_{g_1} P_1 \) is principal, and so we have

\[
\frac{\hat{g}_1^* \Gamma, \hat{f}_1^* P_1, \hat{g}_1^* f^* \prod_g P \vdash \hat{g}_1^* \Delta}{\Gamma, f^* \prod_{g_1} P_1, f^* \prod_g P \vdash \Delta} \quad \text{\( \prod \text{L} \)}
\]

(11)

What we have to show is that, if \( fh \subseteq gk \), then

\[
h^* \Gamma, h^* f \prod_{g_1}^* P_1, k^* P \vdash h^* \Delta
\]

Consider the diagram

\[
\begin{array}{c}
D_0 \otimes_A (f_1 \otimes g_1) \xrightarrow{\pi'} f_1 \otimes g_1 \xrightarrow{\hat{f}_1} C_1 \\
\pi \downarrow \quad = \quad \hat{g}_1 \sqsubseteq g_1 \downarrow \quad \text{\( h \sqsupseteq \hat{g}_1 \)} \downarrow \quad g_1
\\
\downarrow \quad h_0 \quad \Pi \quad \downarrow \quad \text{\( h \sqsupseteq \hat{g}_1 \)} \\
D_0 \quad h \quad A \quad f_1 \quad B_1 \\
\downarrow \quad \Pi \quad \downarrow \quad \hat{f} \quad \downarrow \quad \Pi \quad \text{\( h \sqsupseteq \hat{g}_1 \)} \\
D \quad k \quad B \quad \Pi \quad \text{\( h \sqsupseteq \hat{g}_1 \)}
\end{array}
\]
Here we have factorised \( h \) as \( h_0 j \), with \( h_0 \) total; let \( i \vdash j \), so that \( hi = h_0 ji = h_0 \). The top right hand rectangle is a comma rectangle, generated by \( f_1 \) and \( g_1 \); the top left rectangle is cartesian. Pasting in the diagram gives \( f g_1 \pi' \subseteq g ki \pi \). Consequently we can assume, by the inductive hypothesis, that

\[
\pi'^* g_1 \Gamma, \pi'^* f_1 P_1, \pi'^* k^* P \vdash \pi'^* g_1 \Delta
\]

and so, by the commutativity of the top left rectangle,

\[
\pi^* h_0^* \Gamma, \pi^* f_1 P_1, \pi^* k^* P \vdash \pi^* h_0^* \Delta
\]

whence, by \( \coprod L \) applied to \( P_1 \), since the top rectangle is a comma square,

\[
h_0^* \Gamma, h_0^* f_1 \coprod_{g_1} P_1, i^* k^* P \vdash h_0^* \Delta
\]

and, when we pull back by \( j^* \), we get

\[
h^* \Gamma, h^* f_1 \coprod_{g_1} P_1, k^* P \vdash h^* \Delta
\]

which was what we had to prove.

**Case 10** Trivial induction.

For the proof of cut elimination, we need the following lemma:

**Lemma 19.** If \( \Gamma \vdash \Delta \) has a cut free proof of depth \( n \), then, for any appropriate \( \phi \), there is a cut free proof of \( \phi^* \Gamma \vdash \phi^* \Delta \).

**Proof.** Trivial: all of the rules are stable under pullback.

**Theorem 8.** The system allows cut elimination

**Proof.** We proceed by induction on the depth of the proof and the degree of the formula. So we assume first that we have a cut of the form

\[
\Pi \quad \Pi'
\]

\[
\vdots \quad \vdots
\]

\[
\Gamma \vdash P; \Delta \quad \Gamma'; P \vdash \Delta'
\]

\[
\Gamma, \Gamma' \vdash \Delta', \Delta
\]
Using Lemma 19 we can assume that $\phi^*$ is not used in either $\Pi$ or $\Pi'$. Using Lemma 18 we can also assume that, except in the case where $P = f^* \coprod_g Q$, $P$ is principal on both sides or an axiom: in these cases, the cut can be replaced with one of lower degree or eliminated.

So we are left with the case where the cut formula is $f^* \coprod_g Q$: we argue by cases on the bottom inference on the left. In the cases where the cut formula is unchanged by the inference (including contraction), we simply move the cut upwards. So we are left with the following cases:

**Axiom** We can directly eliminate the cut in the usual way

**L** In this case the bottom inference on the left is

$$
\frac{\hat{g}^* \Gamma, \hat{f}^* P_1 \vdash \hat{g}^* \hat{f}^* \coprod_g Q, \hat{g}^* \Delta}{\Gamma, f^* \coprod_g P_1 \vdash f^* \coprod_g Q, \Delta}
$$

Here we can apply Lemma 19 to the proof on the right and move the cut upwards.

**R** Here the final inference on the left is of the form

$$
\frac{\Gamma \vdash \tau^* Q, \Delta}{\Gamma \vdash f^* \coprod_g Q, \Delta}
$$

and we can apply Lemma 18 to the cut formula on the right to replace the cut with a cut on $Q$, which has lower degree.

\[\square\]

## 5 Semantics

### 5.1 Definitions

The semantics of this logic should be as follows: Let $\mathcal{C}$ be a partial cartesian category with comma objects. Consider a category $\mathcal{E}$ 2-fibred by Boolean algebras and $\top, \wedge, \lor$-preserving poset morphisms over $\mathcal{C}$: the reindexing functors $f^*$ should have left adjoints $\coprod_f$, which should satisfy the Beck-Chevalley conditions with respect to comma squares in Section 3.4. Furthermore, if we have $f \subseteq g$ for 1-cells $f$ and $g$ in $\mathcal{C}$, we should have $g^* \leq f^*$ in the pointwise order on poset morphisms. Now let $\mathcal{L}$ be a logic as described above, typed by the objects and 1-cells of $\mathcal{C}$.

Note that the poset morphisms of Boolean algebras which we consider are, when considered with the usual coHeytng structure on those algebras, weak coHeyting morphisms: this will allow us to use the results of Section 3.3.

**Definition 33.** An assignment is an choice, for every $t \in \text{Ob}(\mathcal{C})$ and every atomic $P \in \mathcal{L}$, of an element $[P]_t \in \text{Ob}(\mathcal{C}_t)$. 

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Given an assignment, we can define, for each \( P : t \), its semantic value \([P]_t\) by induction on its syntactic complexity: the sentential operators are interpreted in the usual way, \( f^* \) is interpreted as the reindexing functor (also written \( f^* \)), and \( \llbracket \bigwedge \rrbracket_t \) is interpreted as the left adjoint to reindexing.

We also define \([\Gamma]_t\) for contexts: here we have to make a distinction between left and right contexts, since the comma is interpreted differently on the left and on the right.

**Definition 34.** The semantic value of a left context is given by the clauses

- \([\Gamma]_t = [P]_t\) if \( \Gamma = P \)
- \([\Gamma, \Gamma']_t = [\Gamma]_t \land [\Gamma']_t\)

The semantic value of a right context is given by the clauses

- \([\Delta]_t = [P]_t\) if \( \Delta = P \)
- \([\Delta, \Delta']_t = [\Delta]_t \lor [\Delta']_t\)

And, finally, a definition of semantic entailment:

**Definition 35.**

\[ \Gamma : t \models \Delta : t \]

iff

\[ [\Gamma]_t \leq [\Delta]_t \]

for every assignment.

### 5.2 Soundness

The proof of this is very standard.

**Proposition 12.** The rules \( \text{LW}, \text{RW}, \text{LC}, \text{RC}, \lor \text{R}, \land \text{L} \) and \( \top \text{R} \) are sound for \( \models \).

**Proof.** Standard. □

**Proposition 13.** The rules \( \lor \text{L} \) and \( \land \text{R} \) are sound for \( \models \).

**Proof.** This follows from the distributivity of \( \land \) over \( \lor \) and vice versa. □

**Proposition 14.** The rules \( \neg \text{L} \) and \( \neg \text{R} \) are sound for \( \models \).

**Proof.** Standard. □

**Proposition 15.** \( f^* \neg \text{L} \) and \( f^* \neg \text{R} \) are sound for \( \models \).
Proof. Note first that, because $f^*$ is a poset morphism of Boolean algebras preserving $\top$, $\wedge$ and $\lor$, we have $f^*\neg A = \neg f^*A \lor f^*\bot$. The soundness of the rules follows from this.

**Proposition 16.** $\sqsubseteq L$, $\sqsubseteq R$ and $\sqsubseteq LR$ are sound for $\vdash$.

Proof. The first two are immediate: for the third, we use Corollary[7]

**Proposition 17.** $\vdash$ is sound for $\coprod fL$ and $\coprod fR$

Proof. For $\coprod fR$, this follows from the unit of the adjunction together with the semantics of $\sqsubseteq$. For $\coprod fL$, we use the counit of the adjunction together with Beck-Chevalley.

Finally

**Proposition 18.** The cut rule is sound for $\vdash$.

Proof. Standard.

So, putting all these results together, we have

**Theorem 9.** $\vdash$ is sound for our sequent calculus.

### 5.3 Completeness

**Theorem 10.** The semantics is complete: that is, if, for a given base category $\mathcal{C}$, and for an object $t$ of $\mathcal{C}$,

$$[\Gamma]_t \leq [\Delta]_t$$

for two contexts $\Gamma : t$ and $\Delta : t$, then

$$\Gamma \vdash \Delta.$$

This theorem will be proved by constructing a term, or generic, model, which we define as follows.

**Definition 36.** Let $\mathcal{C}$ be a category with fibre products. The term model, $\mathcal{E}_{\mathcal{C}}$, over $\mathcal{C}$ is given by the following data:

- **Objects** these are given by pairs $P : s$, where $s$ is an object of $\mathcal{C}$ and $P$ is a proposition of type $s$

- **Morphisms** a morphism between $P : s$ and $Q : t$ is given by a proof

$$P \vdash f^*Q$$

for some morphism $f : s \to t$ of $\mathcal{C}$. Two such morphisms are equal iff their source and target are the same, and the corresponding morphisms of $\mathcal{C}$ are equal.
Composition suppose we have two morphisms corresponding to proofs

\[ \Pi \text{ and } \Pi' \]

\[ P : s \vdash f^*(Q : t) \quad Q : t \vdash g^*(R : u) \]

Their composition is given by the proof

\[ \begin{array}{c}
\Pi \\
\vdash \\
\Pi'
\end{array}
\]

\[ \begin{array}{c}
P : s \vdash f^*(Q : t) \\
\vdash \\
Q : t \vdash g^*(R : u)
\end{array}
\]

\[ \begin{array}{c}
f^*Q \vdash f^*g^*R \\
\vdash \\
(P \vdash f^*g^*R)
\end{array} \]

Identity morphisms these are given by the proofs

\[ P : t \vdash P : t \\
P : t \vdash \text{id}^*P \]

2-cells Homsets are posets, and there is a 2-cell between \( f, g : P : s \rightarrow Q : t \) iff \( f \sqsubseteq g \).

The display functor this is the map \( p \) which sends a typed proposition \( P : t \) to the object \( t \), and a proof of \( P : s \vdash f^*(Q : t) \) to the morphism \( f : s \rightarrow t \).

Liftings we lift 1-cells as follows. Let \( f : s \rightarrow t \) be a morphism in the base, and let \( P : t \) be an object of the fibre \( E_t \) over \( t \): let the lifting of \( f \) be the following proof:

\[ f^*P \vdash f^*P \]

Adjoint Left adjoints to the substitution functors \( f^* \) are given by \( \Pi f \).

We now prove

Proposition 19. \( E_\mathcal{C} \) is 2-fibred in Boolean algebras over \( \mathcal{C} \): the pullbacks along 1-cells are \( \top, \wedge, \vee \)-preserving poset morphisms. Pullbacks along 1-cells furthermore possess left adjoints satisfying Beck-Chevalley with respect to comma squares.

Proof. It is clear than \( E_\mathcal{C} \) is a locally posetal 2-category (equality of 1-cells is so strong that laws like associativity follow directly from the corresponding laws for \( \mathcal{C} \)). It is likewise clear that our “display functor”, \( p \), is actually a functor. We have to check that the liftings are Cartesian: so, consider composable morphisms \( \alpha : s \rightarrow t \) and \( \beta : t \rightarrow u \) in the base, together with a proof \( \Pi \) of \( P : s \vdash \{ R : t \vdash g^*(R : u) \}

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$u \beta \alpha$ lying over $\alpha \circ \beta$. We need to produce a proof of $P : s \vdash \{R^* : t\}^\alpha$ (commutativity of the resulting diagram is trivial). But this is immediate:

\[
\begin{array}{c}
\vdots \\
P : s \vdash gf^*(R : u) \\
P : s \vdash f^*(g^*R)
\end{array}
\]

This establishes the functoriality of the $\alpha^*$. The universal property for 2-cells follows directly from the corresponding universal property in the base. Consequently, $\mathcal{E}_\mathcal{E}$ is 2-fibred over $\mathcal{C}$: the fibres are Boolean algebras, because the inference rules are a superset of the normal classical inference rules.

We need to show that $\prod f$ is a left adjoint $f^*$. Functoriality is easy. For example, the following construction, which produces a proof of $\prod f P \vdash \prod f Q$ from a proof of $P \vdash Q$, establishes functoriality for $\prod f$:

\[
\begin{array}{c}
\vdots \\
P : s \vdash Q : s \\
\prod P \vdash f^* \prod f P
\end{array}
\]

(where we have used the admissible rules of Lemma 17).

Given functoriality, we only need to establish the unit and counit for the adjunction. The unit for $\prod f \dashv f^*$ is proven using the (admissible) $\prod f R'$, and the counit using $\prod f L'$.

Finally, we must verify the Beck-Chevalley conditions: for $\prod f$, we prove these as follows. Given the usual comma square, we have to prove that $f^* \prod g P$ and $\prod \hat{g} f^* P$ are equivalent. The direction

\[
\prod \hat{g} f^* P \vdash f^* \prod g P
\]

follows from functoriality and the adjunction. We prove the other direction as follows:

\[
\begin{array}{c}
\vdash f^* P \vdash f^* P \\
\vdash \hat{g} f^* P \vdash \prod g P \\
\vdash \hat{g} f^* P \vdash \prod f L
\end{array}
\]

This concludes the proof that $\mathcal{E}_\mathcal{E}$ is a category fibred over $\mathcal{C}$ with the desired properties.

**Definition 37.** Let $\Gamma$ be a left context: let the *propositionalisation* of $\Gamma$, $\mathcal{P} \Gamma$, be defined as the conjunction of its members. The conjunction of a right context is the disjunction of its members.

**Lemma 20.** For any $\Gamma$ and $\Delta$,

$$\Gamma \vdash \Delta$$
Proof. The obvious induction.

**Proof of Theorem 10.** Suppose that \( \Gamma : t \vdash \Delta : t \). Define an interpretation of the language in \( \mathcal{E}_c \) by sending \( P : t \) to \( P : t \) as an object of \( \mathcal{E}_t \). We establish, by induction, that, with respect to this interpretation, \( \llbracket \Gamma \rrbracket_t = \Gamma \), and \( \llbracket \Delta \rrbracket_t = \Delta \). Since \( \Gamma : t \vdash \Delta : t \), we must have \( \llbracket \Gamma \rrbracket_t \leq \llbracket \Delta \rrbracket_t \), and, consequently, \( \Gamma \leq \Delta \): by the definition of \( \mathcal{E}_c \), this means that \( \Gamma \vdash \Delta \). By the lemma, we have \( \Gamma \vdash \Delta \). □

### 5.3.1 Kripke Models

We can make the model theory somewhat more specific in the following way.

**Definition 38.** A Reiter Kripke model over a category \( \mathcal{C} \) is a covariant functor \( \Psi \) from \( \mathcal{C} \) to the category of sets and partial functions. Given a Reiter Kripke model, the associated Reiter category is the category, fibred in Boolean algebras over \( \mathcal{C} \), where the fibre over an object \( t \) of \( \mathcal{C} \) is the powerset of \( \Psi(t) \), and where the pullback morphism over \( \alpha : s \rightarrow t \) is given by the inverse image of subsets of \( \Psi(t) \).

Implicit in the above definition is

**Lemma 21.** The associated Reiter category of a Reiter Kripke model is a Reiter category.

**Proof.** Routine calculation. □

We can, then, define a Kripke semantics for our logic: given a Reiter Kripke model over a category \( \mathcal{C} \), and an object \( t \) of \( \mathcal{C} \), we will call the elements of \( \Psi(t) \) the possible worlds of type \( t \). We will assign sets of possible worlds to atomic propositions, and it is clear how to define, inductively, the semantic value of general propositions.

Since each Reiter Kripke model defines a Reiter category, it is clear that the Kripke semantics is sound. It is also complete:

**Proposition 20.** The Kripke semantics is complete for our logic.

**Proof.** Suppose that we have \( \Gamma : t \not\vdash \Delta : t \). By our soundness theorem, we have a Reiter category model with \( \llbracket \Gamma \rrbracket_t \not\leq \llbracket \Delta \rrbracket_t \). By Stone duality (Johnstone, 1982), we can assume that the Reiter category model is given by a covariant functor \( \Psi \) from \( \mathcal{C} \) to the category of Stone spaces and continuous maps: the fibre over an object \( t \) will be the algebra of clopen sets of \( \Psi(t) \), and the pullbacks along \( \alpha : s \rightarrow t \) will be given by inverse images along the continuous morphism \( \Psi(\alpha) \). If we now compose \( \Psi \) with the underlying set functor, we get a Kripke model: it is trivial to verify that the construction of semantic values commutes with the underlying set functor, and so, as required, we have a Kripke model in which

\[ \llbracket \Gamma \rrbracket_t \not\leq \llbracket \Delta \rrbracket_t. \]

□
6 Internal Languages and Applications

We can now use the machinery that we have used in order to investigate partial Cartesian categories by studying their domain fibrations. We obtain internal languages, both for partial Cartesian categories and also for partial Cartesian categories with comma objects: we show that the internal language of the former coincides with a logic of partial functions independently defined by Palmgren and Vickers \cite{PalmgrenVickers2007}, whereas the internal logic of the latter is an extension of the Palmgren-Vickers logic with a Heyting operation. Using this logic, we show how to present partial Cartesian categories, and partial Cartesian categories with comma objects, by means of signatures consisting of generators and relations. This, in turn, allows us to express our original fibred logic with more explicit expressions for the objects and morphisms of the base (and, in fact, with admissible comprehension rules for internal equalities); so, finally, we can give a formalisation of one of the key philosophical examples which motivated this work, namely Davidson’s argument for the meaningfulness of talk of equality of actions \cite{Davidson1980d, Davidson1980}. So this section will be an explicit construction of the fibrations which we have been considering abstractly in the previous part of this paper.

Firstly, some clarification. We have, in Section 3.5, defined both 2-covariant and 2-contravariant domain fibrations: the 2-covariant domain fibration will turn out to be the one appropriate for the semantics of internal languages as they are usually conceived.

6.1 Bicategories of Partial Maps

6.1.1 Primitives

We will start with the case of partial cartesian categories, i.e. bicategories of partial maps with weak products. The primitives of our internal languages can be motivated as follows. As we describe above (p. 5), we describe actions by means of partially defined functions which take possible worlds, at the state before the action is performed, to possible worlds at the state after the action is performed. We will, loosely following Scott \cite{Scott1979}, use a partially defined equality relation to reason about partially defined functions.

So, if we have two actions, $\alpha(x : s)$ and $\beta(y : t)$, whose values are possible worlds at the same state, then $\alpha(x) \cong \beta(y)$ is a partially defined binary relation between possible worlds at different states, $s$ and $t$. Clearly, if we go on like this, we will need $n$-ary relations between possible worlds: it is easiest to work with $n$-tuples of worlds, of the form

$$\langle x_1 : s_1, \ldots, x_n : s_n \rangle,$$

and we shall write such a tuple, in boldface, as $\mathbf{x} : s$, or, where the states are clear from the context, as $\mathbf{x}$. We will also be interested in $k$-tuples of actions

$$\langle \alpha_1(x_{i_1}), \ldots, \alpha_k(x_{i_k}) \rangle,$$
Our intended notion of partial equality will be as follows: as we shall show in Section 6.3.1 it can also be defined in terms of the abstract structure of partial cartesian categories.

**Definition 39.** Let \( \alpha(x) \) and \( \beta(y) \) be two partial functions whose values are possible worlds at the same tuple of states: then

\[
\alpha(x) = \beta(y)
\]

is true at worlds \( x \) and \( y \) iff

1. \( \alpha \) and \( \beta \) are both defined at those worlds, and
2. the values of \( \alpha \) and \( \beta \) are equal.

This notion of equality can be axiomatised by a system due to Palmgren and Vickers (2007): we give in in Table 4. Notice that the equality \( f(x) ≏ f(x) \), which we shall abbreviate to \( f \downarrow \), is true iff the functions of \( f \) are all defined at \( x \).

### 6.1.2 Formulae, Contexts, and Sequents

We will first use this language to define locally posetal bicategories: we will then show that these bicategories are, firstly, models of the language, and, secondly, free bicategories of partial maps.

**Definition 40.** A signature, \( \Sigma = (I, J) \), will be a pair of sets: \( I \) is a set whose elements will be called situations, and \( J \) is a set whose elements will be called actions: each action will be associated to a pair of situations, its source and its target.

We want to talk about partially defined equalities between tuples of actions, so, on the basis of our signature, we define a language \( \mathcal{L}_\Sigma \) for reasoning about such equalities.

**Definition 41.** Suppose that we have, for each situation \( s \in I \), a supply of variables \( x : s, x' : s, \ldots \) of type \( s \). A variable tuple, written \( x : s \), will be a tuple of variables \( (x_1 : s_1, \ldots, x_k : s_k) \): the zero length tuple will be written ()). An action tuple, written \( \alpha(x : s) \), will be a tuple of actions \( (\alpha_1, \ldots, \alpha_l) \): the source \( s_{i_1} \) of \( \alpha_i \) must be one of the \( s_i \). \( s \) will be called the source type of the action tuple: its target type will be \( (t_1, \ldots, t_l) \), where, for all \( i, t_i \) is the target of \( \alpha_i \).

Given a pair of action tuples \( \alpha(x) \) and \( \beta(x) \) with the same source and target types, the partial equation (or partial equation tuple) \( \alpha(x) ≏ \beta(x) \) will be defined to be the tuple of partial equations

\[
\alpha_1 ≏ \beta_1, \ldots, \alpha_l = \beta_l.
\]

Given a tuple of situations, we define \( T \) at that source typel to be the empty tuple of partial equations: its target type is, of course, the empty tuple of
situations. Given two partial equations \( \zeta(x : s) \) and \( \eta(x : s) \), with the same source type, we define \( \zeta \land \eta \) to be the concatenation of the corresponding tuples. We will abbreviate the equation \( \alpha(x) \equiv \alpha(x) \downarrow \).

We define the entailment relation \( \zeta(x : s) \implies \eta(x : s) \), between partial equations with the same source tuples, by the rules in Table \([\text{4}]\). We should note that, in all of the rules with premises, the source tuples of premises and conclusion must be the same.

A theory, \( T \), will be a set of equation tuples. As usual, we say that two theories are the same if they have the same closure under entailment.

**Example 4.** If we have a variable tuple \( x : s = x_1 : s_1, x_2 : s_2, x_3 : s_3 \), then

\[
\alpha(x : s) = \alpha_1(x_2 : s_2), \alpha_2(x_3 : s_3), \alpha_3(x_1 : s_1), \alpha_4(x_2 : s_2)
\]

is a valid action tuple of that source type. So also is the empty action tuple (): this shows that the source type of an action tuple is a structure assigned to it, rather than a property.

We can now define a two-category of contexts for our logic.

**Definition 42.** Given a signature \( \Sigma \), together with a theory \( T \) in that signature, we define a locally posetal two-category \( C(\Sigma, T) \) as follows.

**objects** An object of \( C(\Sigma, T) \) will be written \( \{ x : s \mid \zeta(x : s) \} \), where \( x : s \) is a variable tuple and where \( \zeta \) is a partial equation tuple with that source type.

**1-cells** A 1-cell will be written

\[
\langle \alpha(x : s) | \zeta(x : s) \rangle : \{ x : s \mid \eta(x : s) \} \to \{ y : t \mid \vartheta(y : t) \}
\]

where we require that

1. \( s \) is the source type of \( \alpha \), of \( \zeta \) and of \( \eta \).
2. \( t \) is the target type of \( \alpha \) and the source type of \( \vartheta \), and
3. the entailment

\[ \langle \zeta(x : s), \eta(x : s) \rangle \Downarrow \vartheta(\alpha) \]

holds (informally, we require that \( \langle \alpha(x : s) | \zeta(x : s) \rangle \) should factor through the subobject of \( \{ y : t \} \) defined by \( \vartheta \)).

Composition of 1-cells is defined as follows. Suppose that

\[
\langle \alpha(x) | \zeta(x) \rangle : \{ x : s \mid \vartheta(x : s) \} \to \{ y : t \mid \vartheta'(y : t) \}
\]
\[
\langle \beta(y) | \eta(y) \rangle : \{ y : t \mid \vartheta'(y : t) \} \to \{ z : u \mid \vartheta''(z : u) \}
\]

then

\[
\langle \beta(y) | \eta(y) \rangle \langle \alpha(x) | \zeta(x) \rangle = \langle \beta(\alpha(x)) | \eta(\alpha(x)), \zeta(x) \rangle
\]

The unit 1-cell, on an object \( \{ x : s \mid \zeta(x : s) \} \), is \( \{ x \mid \zeta(x) \} \).
The 2-cells of our category will be defined as follows. Given $\langle \alpha | \zeta \rangle, \langle \alpha' | \zeta' \rangle : \{x|\eta(x)\} \to \{y|\vartheta(y)\}$, we say that $\langle \alpha | \zeta \rangle \sqsubseteq \langle \alpha' | \zeta' \rangle$ iff $T, \alpha \downarrow, \zeta, \eta \vdash \alpha \asymp \alpha' \land \eta'$; informally, whenever $\langle \alpha | \zeta \rangle$ is defined, then so too is $\langle \alpha' | \zeta' \rangle$ and they are equal.

We also define the following subcategory of $\mathcal{C}_{(\Sigma, T)}$: we will need it for our proof of freeness.

**Definition 43.** Let $\mathcal{C}^0_{(\Sigma, T)}$ be the full subcategory of $\mathcal{C}_{(\Sigma, T)}$ whose objects have no constraints, i.e. are all of the form $\{x : s\}$. After some calculation, we can prove

**Lemma 22.** $\mathcal{C}_{(\Sigma, T)}$ and $\mathcal{C}^0_{(\Sigma, T)}$ are locally posetal bicategories.

**Remark 7.** The system of Table 4 is due to Palmgren & Vickers (2007), and is a sound and complete axiomatisation for equalities between partially-defined functions between sets. We will generally not be pedantic about notation: we will often omit types (or, indeed, variables) where it is obvious from the context. We will also frequently write the object $\{x : s\}$ as $x : s$, and the morphism $\langle \alpha(x : s) | \zeta(x) \rangle$ as $\alpha(x : s)$ (or, for that matter, as $\alpha$ when the source type is obvious). We will also use $\land$ and the comma interchangeably for concatenation of tuples of partial equations.

**Proposition 21.** Let $\Sigma$ be a signature and $T$ be a theory. Then $\mathcal{C}_{(\Sigma, T)}$ is a bicategory of partial maps.

**Proof.** Given objects $\{x|\zeta(x)\}$ and $\{y|\eta(y)\}$, with $x$ and $y$ disjoint tuples of variables (which can always be achieved, up to isomorphism of objects, by renaming), we let

$$\{x|\zeta(x)\} \otimes \{y|\eta(y)\} = \{x, y|\zeta(x), \eta(y)\};$$

for morphisms $\langle \alpha(x) | \zeta(x) \rangle$ and $\langle \beta(y) | \eta(y) \rangle$ (with, again, disjoint tuples of source and target variables), we let

$$\langle \alpha(x) | \zeta(x) \rangle \otimes \langle \beta(y) | \eta(y) \rangle = \langle \alpha(x), \beta(y) | \zeta(x), \eta(y) \rangle.$$

The unit object is given by

$$I = \{()|\top\}.$$
(i.e. the empty tuple with no constraints), whereas the morphism ! is given, for any X, by

$$!_X = \langle ()|T(x) \rangle$$

(i.e. the empty tuple of function symbols (in the appropriate variables) with no constraints. It is straightforward, if tedious, to verify that these make $C_{X,Y}$ into a strict monoidal bicategory. For example, the fact that morphisms are lax !-homomorphisms comes down, in the case of a morphism $(\alpha|\eta)$, to

$$\langle ()|\eta(\alpha(x)) \rangle \sqsubseteq \langle ()|T(\alpha(x)) \rangle.$$  

We now define the comonoid structure: if $X = \{\vartheta|x(\vartheta)\}$, then

$$\Delta_X = \langle x, x|T \rangle : \{x|\vartheta(x)\} \to \{x, y|\vartheta(x), \vartheta(y)\}$$

$$\nabla_X = \langle x|x \equiv y \rangle : \{x, y|\vartheta(x), \vartheta(y)\} \to \{x|\vartheta(x)\}$$

This is the only possible comonoid structure on X: we can write a candidate structure in components as $f, g$, and then it is clear, after some manipulation, that $f = g = \text{Id}_X$.

Verification of the adjunction is straightforward: we need

$$\eta : \text{Id}_X \sqsubseteq \nabla_X \Delta_X \quad \text{i.e.} \quad \text{Id}_{\{x|\vartheta(x)\}} \sqsubseteq \langle x|T \rangle$$

$$\epsilon : \Delta_X \nabla_X \sqsubseteq \text{Id}_X \quad \text{i.e.} \quad \langle x, y|x \equiv y \rangle \sqsubseteq \text{Id}_{\{x|\vartheta(x)\} \times \{y|\vartheta(y)\}}$$
and these clearly hold. The identities (4) and (5) amount to
\[
\langle x, y \mid x \equiv y \rangle = \langle x, y \mid x \equiv y \rangle
\]
\[
\langle \alpha|\zeta, \alpha \equiv \beta, \eta \rangle \subseteq \langle \alpha|\zeta \rangle
\]
when \(X = \{ x|\vartheta(x) \}\), \(f = \langle \alpha|\zeta \rangle \), and \(g = \langle \beta|\eta \rangle \).

\textbf{Lemma 23.} \( \mathcal{C}_{(\Sigma, T)} \) is a split bicategory of partial maps.

\textit{Proof.} We first show that any coreflexive \( f : \{ x : s|\zeta(x : s) \} \rightarrow \{ x : s|\zeta(x : s) \} \) is of the form \( \langle x : s|\zeta(x : s) \wedge \eta(x : s) \rangle \). Suppose, then, that \( \langle \alpha(x : s)|\eta(x : s) \rangle \subseteq \text{Id}_{\{ x : s\mid\zeta(x : s) \}} \): by definition of \( \sqsubseteq \), we have
\[
T, \alpha(x) \downarrow, \zeta(x), \eta(x) \vdash \alpha(x) \equiv x
\]
and from this follows that
\[
\langle \alpha|\eta \rangle \subseteq \langle \text{Id}|\eta \rangle \quad \text{and} \quad \langle \text{Id}|\eta \rangle \subseteq \langle \alpha|\eta \rangle, \quad \text{i.e.} \\
\langle \alpha|\eta \rangle \equiv \langle \text{Id}|\eta \rangle
\]
but now we can split this coreflexive by the pair of morphisms
\[
i = \langle x \mid \rangle \quad \{ x|\zeta(x), \eta(x) \} \rightarrow \{ x|\eta(x) \} \\
j = \langle x|\eta(x) \rangle \quad \{ x|\zeta(x) \} \rightarrow \{ x|\zeta(x), \eta(x) \}.
\]

\textbf{Corollary 9.} \( \mathcal{C}_{(\Sigma, T)} \), the category of objects and total morphisms of \( \mathcal{C}_{(\Sigma, T)} \), is cartesian (that is, it has all finite limits).

\textit{Proof.} According to Carboni \cite{Carboni1987}, Theorem 2.3), this is the case iff \( \mathcal{C}_{(\Sigma, T)} \) is functionally complete, and we have just shown that it is.

\textbf{6.1.3 Examples}

We now need to do some work on rephrasing the concepts of partial cartesian categories in the more concrete terms of \( \mathcal{C}_{(\Sigma, T)} \).

\textbf{Definition 44.} We say that a morphism
\[f : A \rightarrow B\]
in a partial cartesian category is \textit{total} if
\[!_A f = !_B .\]
Lemma 24. In $C_{\Sigma,T}$, a morphism

$$\langle \alpha|\eta \rangle : \{x|\vartheta(x)\} \rightarrow \{y|\vartheta'(y)\}$$

is total iff

$$\vartheta \models \alpha \downarrow \land \eta$$

Proof. Routine.

Definition 45. In a bicategory of partial maps, we define $\land : \text{hom}(X,Y) \times \text{hom}(X,Y) \rightarrow \text{hom}(X,Y)$ to be the map which takes $(f,g)$ to $\nabla_Y f \otimes g \Delta_X$.

Lemma 25. $\land$ is a least upper bound on the poset $\text{hom}(X,Y)$.

Proof. See Carboni (1987, Lemma 2.1).

Example 5. In $C_{\Sigma,T}$, maps $X \rightarrow I$ are of the form $\langle ()|\zeta(x) \rangle$. We find, by easy calculation, that $\langle ()|\zeta(x) \rangle \land \langle ()|\eta(x) \rangle = \langle ()|\zeta(x) \land \eta(x) \rangle$: so the $\land$ that we have just defined agrees with the $\land$ in Palmgren-Vickers logic.

Example 6. If $\alpha(x), \beta(x) \in \text{hom}_{C_{\Sigma,T}}(\{x:s: \top\}, \{y:t|\top\})$, then

$$\alpha(x) \land \beta(x) = \langle \alpha|\alpha \equiv \beta \rangle$$

and

$$\langle ()|\alpha \equiv \beta \rangle = \iota_!(\alpha \land \beta)$$

$$= \langle \iota_!\alpha \rangle \land \langle \iota_!\beta \rangle;$$

if $\zeta(x), \eta(x) \in \text{hom}_{C_{\Sigma,T}}(\{x:s: \top\}, I)$, then

$$\zeta \land \eta = \langle ()|\zeta(x) \equiv \eta(x) \rangle$$

$$= \langle ()|\zeta(x) \land \eta(x) \rangle$$

$$= \langle ()|\zeta(x), \eta(x) \rangle$$

Proof. Routine computation.

Remark 8. This result is perhaps not surprising: in the paradigm model of these things – that is, sets and partial functions – $I$ is a one-element set, and all that $\equiv$ then worries about is definedness: so the coincidence of $\land$ and $\equiv$ is only to be expected.
6.1.4 The Internal Model in $C^0_{(\Sigma, T)}$

We will first show how $C^0_{(\Sigma, T)}$ is related to the Palmgren-Vickers logic. So, let $T$ be a theory, with signature $\Sigma$, in this logic. We will now show how to associate, to each formula in the signature $\Sigma$, a semantic value in a homset of $C^0_{(\Sigma, T)}$, and we will show that this gives a sound and complete model of $T$.

**Definition 46.** Given a formula $\zeta(x)$, with free variables $x : s$, of $L(\Sigma)$, associate to it the semantic value

$$\llbracket \zeta \rrbracket_s = \langle () | \zeta(x) \rangle : s \to I.$$

(where, as usual, we abbreviate $\{ x : s | \top \}$ to $s$).

Given a pair of formulae $\zeta(x : s)$ and $\eta(x : s)$, with the same variables, we say that

$$\zeta \vdash \eta$$

(in words: $\zeta$ semantically entails $\eta$) if, in $\text{hom}_{C^0_{(\Sigma, T)}}(s, I)$,

$$\llbracket \zeta \rrbracket_s \leq \llbracket \eta \rrbracket_s.$$

Before we prove soundness and completeness, we need a lemma.

**Lemma 26.** (Cf. [Palmgren & Vickers, 2007, Lemma 3.3]) Given $\langle \alpha | \rangle : s \to t$, and a formula of the Palmgren-Vickers logic $\zeta(y : t)$, then

$$\llbracket \zeta(\alpha(x)) \rrbracket_s \land \llbracket \alpha(x) \downarrow \rrbracket_s = \llbracket \zeta \rrbracket_t \langle \alpha \rangle.$$

**Proof.** We argue by cases. If $\zeta = \top$, then the left hand side is (by computation, and using Example 6) $\langle () | \top \rangle \land \langle () | \alpha \downarrow \rangle$, whereas the right hand side is $\langle () \rangle \langle \alpha \rangle$. An easy computation shows that the two are equal.

If $\zeta = (\beta(y) \equiv \gamma(y))$, with $\beta, \gamma : t \to u$, then (again using Example 6) we find that the left hand side is $\langle !_u \beta(\alpha) \rangle \land \langle !_u \gamma(\alpha) \rangle \land \langle !_u \alpha \rangle$: the right hand side is $\langle !_u (\beta \land \gamma) \alpha \rangle$. The two can easily be seen to be equal using the naturality of $\land$ and the fact that it is a supremum, together with the lax naturality of $!$.

If $\zeta = \eta \land \vartheta$, then the left hand side is $\llbracket \eta(\alpha) \rrbracket \land (\vartheta(\alpha)) \land (\land_! \alpha)$, whereas the right hand side is $(\eta \land \vartheta)(\alpha)$: equality follows, as before, by the properties of $\land$ and $!$.

**Theorem 11.** The internal model is sound and complete for $T$.

**Proof.** We first show that the model is sound: we start with the rules in Table 4. Axiom and cut are inherited from the partial order structure on the homsets of $C_{(\Sigma, T)}$: reflexivity follows from a routine computation. For the equality axiom,
we argue as follows. Given \( \zeta : s \to I \) and tuples of variables \( x, y : s \), then

\[
\begin{align*}
[\zeta(x) \land x \equiv y]_s &= (\zeta \land \nabla_s)(\Delta_s \otimes \text{id}_s) \\
&= (\zeta \land \text{id}_s)(\text{id}_s \otimes \Delta_s)(\Delta_s \otimes \text{id}_s) \\
&= (\zeta \land \text{id}_s)\Delta_s \nabla_s \quad \text{by (4)} \\
&= (\text{id}_s \land \zeta)\Delta_s \nabla_s \quad \text{by symmetry of } \land \\
&= ([x \equiv y \land \zeta(y)]_s \\
&\leq [\zeta(y)]_s \quad \text{by properties of } \land
\end{align*}
\]

The strictness properties follow from Lemma 26. We also need to show that the elements of \( T \) are all interpreted as tautologies, i.e. that, for \( \tau(x : x) \in T \), we have \([\tau]_s = T\). But this trivially follows from the definition of \( \leq \) in homsets.

Completeness is likewise trivial. Suppose that we have \( \zeta, \eta \), with

\[
[\zeta]_s \leq [\eta]_s.
\]

Then, by definition of semantic values,

\[
\langle \zeta \rangle_s \leq \langle \eta \rangle_s;
\]

and so, by definition of \( \leq \) in homsets,

\[
\mathcal{T}, \zeta \vdash \eta
\]

which was to be proved.

\( \square \)

### 6.1.5 Models and Free Categories

In this section we shall relate \( \mathcal{C}_0^0(\Sigma, T) \) to the theory of bicategories by showing that it is the free bicategory of partial maps on the signature \( (\Sigma, T) \), and that \( \mathcal{C}(\Sigma, T) \) is the free functionally complete bicategory of partial maps on that signature.

Now in general free things are produced by a left adjoint to some forgetful functor: in this case the functor will produce, from a bicategory of partial maps, a theory in a signature. So we need first to define the corresponding category structure on theories in signatures.

**Definition 47.** We define a locally posetal bicategory theories, whose objects are theories in signatures, as follows. Given theories \( (\Sigma, T) \) and \( (\Sigma', T') \), a morphism of signatures is a map \( \Phi \) from the situations of \( \Sigma \) to the situations of \( \Sigma' \), together with a map \( \Psi \) from the actions of \( \Sigma \) to those of \( \Sigma' \), compatible with the typing of actions and such that the induced map on formulae sends \( T \) to a subset of \( T' \). \( \langle \Phi, \Psi \rangle \subseteq \langle \Phi', \Psi' \rangle \) iff \( \Phi = \Phi' \) and if, for all actions \( f \), \( T' \vdash \Psi(f) \subseteq \Psi'(f) \).
We define the forgetful functor \( U \) as follows: if \( C \) is a bicategory of partial maps, then \( U(C) \) will have, for situations, the objects of \( C \) and for actions the 1-cells of \( C \). An entailment will be in the corresponding theory if it holds in \( C \).

Now in order to show the required adjunction, we must show that the posets \( \text{Hom}_{BPM}(\mathfrak{C}_0^{C^0}(\Sigma, T), C) \) and \( \text{Hom}_{\text{theories}}((\Sigma, T), U(C)) \) are naturally isomorphic. We do this by showing that they are both isomorphic to the poset of models of \((\Sigma, T)\) in \( C \); we define models as follows.

**Definition 48.** Given a signature \( \Sigma \), a theory \( T \), and a category of partial maps \( C \), then a \( \Sigma \)-structure in \( C \) is given by the following data:

1. For each type \( s \) of \( \Sigma \), an object \( [s] \) of \( C \)
2. for each action symbol \( \alpha : s \to t \) of \( \Sigma \), a 1-cell \( [\alpha]_s : [s] \to [t] \) of \( C \)

We can now interpret formulae in \( C \): to be precise, we will associate, to each variable tuple \( x : s \) of our logic an object \( [x] \) of \( C \), to each formula \( \eta(x : s) \) in that context a 1-cell \( [\eta]_s : [s] \to I \), and, to each 1-cell \( \langle \alpha|\eta \rangle \) a 1-cell (with appropriate source and target) of \( C \).

**Definition 49.** We define the following semantic values by mutual recursion:

1. \([x] = [s_1] \otimes \cdots \otimes [s_k]\]
2. \([\eta(x : s)]_s \) is defined by induction on the structure of \( \eta \):
   
   (a) \( [\top]_s \) is
   
   \[
   [s] \xrightarrow{1[s]} I
   \]

   (b) \( [\alpha(x : s) \equiv \beta(x : s)]_s \) is defined as follows (suppose that \( \alpha, \beta : s \to t \))
   
   \[
   [s] \xrightarrow{\Delta[s]} [s] \otimes [s] \xrightarrow{[\alpha]_s \otimes [\beta]_t} [t] \otimes [t] \xrightarrow{\nabla[t]} [t] \xrightarrow{1[t]} I
   \]

   (c) \( [\zeta \land \eta]_s = [\zeta]_s \land [\eta]_s \)

   (d) If the logic has Heyting operations, then \( [\eta \zeta]_s = [\eta]_s \cdot [\zeta]_s \)

3. We define the semantic values of 1-cells as follows:

   (a) An action symbol \( \alpha : s \to t \) of \( \Sigma \) has semantic value
   
   \([\alpha]_s : [s] \to [t]\)

   (b) A 1-cell of the form
   
   \( \langle \alpha_i(x_i : s_i) \rangle : s \to t \)
has semantic value

\[
[s_1] \otimes \cdots \otimes [s_k] \xrightarrow{!_{s_1} \otimes \cdots \otimes [s_k]_s \otimes \cdots \otimes I} I \otimes \cdots \otimes [t] \otimes \cdots \otimes I \\
\equiv \Downarrow \Downarrow [t]
\]

(c) A 1-cell of the form

\[\langle \alpha, \alpha' \rangle : s \to t \otimes t'\]

has semantic value

\[
[s] \xrightarrow{\Delta_{[s]}} [s] \otimes [s] \xrightarrow{[\alpha]_s \otimes [\alpha']_s} [t] \otimes [t'] \equiv [t \otimes t']
\]

We now have the following notion of a model of a theory.

**Definition 50.** Given a theory \( T \) in a signature \( \Sigma \), we say that a structure \( [\cdot] \), with values in \( C \), is a model if, for all

\[
\zeta(x : s) \vdash \eta(x : s) \quad \text{in} \ T, \\
[\zeta][s] \sqsubseteq [\eta][s] \quad \text{in} \ \text{hom}_C([s], I)
\]

This notion of model, which is defined by recursion on the structure of the language, can, in fact, be greatly simplified: models, as we have defined them, are the same as 2-functors from \( C_0(\Sigma, T) \).

**Theorem 12.** Every model of \( T \) in a structure \( \Sigma \) corresponds to a 2-functor, preserving the structure of a bicategory of partial maps:

\[
C_0^{(\Sigma, T)} \to C
\]

**Proof.** Suppose that we are given a model; it will assigns semantic values in \( C \) to the objects and 1-cells of \( C_0(\Sigma, T) \), and we prove first that these semantic values make up a functor. Note first that, because \( C \) is a model under the given assignment of semantic values, it preserves the \( \sqsubseteq \) relation on homsets, and thus, in particular, preserves equality of morphisms. Consider the semantic values assigned to \( \Delta_s, !_s \), for situation tuples \( s \): computation shows that \( [\cdot]_s \) is a comonoid homomorphism, so that, since \( \Delta_{[s]} \) is the unique comonoid structure on \( [s] \), we must have \( [\Delta_s]_s = \Delta_{[s]} \) and \( [!_s]_s = !_{[s]} \). Further computation also shows that \( [\alpha \otimes \beta]_s \otimes t = [\alpha]_s \otimes [\beta]_t \). Furthermore, the adjunction between \( \Delta \) and \( \nabla \) is given by equations and inequalities, and these are, by hypothesis, preserved by \( [\cdot]_s \); we also know that, since our categories are locally posetal, adjunctions are unique, and so \( [\cdot] \) preserves \( \nabla \). The semantic values of formulae \( \zeta(z) \) are all defined in terms of \( \Delta, \nabla, \otimes \) and \(!\); consequently, \( [\cdot] \) preserves the semantic values of formulae, in the sense that

\[
[\eta(x : s)]^C_\alpha = [[\eta(x : s)]^C_\alpha]_s
\]
where \([\|\cdot\|_C]\) is the semantic value in our given model and \([\|\cdot\|_I]\) is the semantic value in the internal model.

Conversely, suppose that we have a functor \(\phi : \mathcal{C}_{(\Sigma, T)} \to C\). Applying \(\phi\) to the semantic values of the internal model, we get semantic values in \(C\), and they are easily verified to give a model. Furthermore, this gives a one-to-one correspondence between models of \(T\) with signature \(\Sigma\): the argument of the previous section shows that such functors are given by their values on \(\mathcal{C}_0^{(\Sigma, T)}\).

**Theorem 13.** The above semantics is sound and complete for \(\cdot \vdash \equiv \cdot\).

**Proof.** Consider (12). Suppose that \(\zeta \vdash \equiv \eta\) is valid in the Vickers-Palmgren logic: then \([\|\zeta\|_I]\subseteq [\|\eta\|_I]\), since the internal model is sound. But we get the model in \(C\) by applying a suitable 2-functor to the internal model, and so we have \([\|\zeta\|_C]\subseteq [\|\eta\|_C]\). So we have soundness. Completeness is straightforward: we already have completeness for the internal model, and the internal model is a model in the above sense, obtained from the trivial interpretation in \(\mathcal{C}_0^{(\Sigma, T)}\).

We can rephrase Theorem 12 as

**Corollary 10.** \(\mathcal{C}_0^{(\Sigma, T)}\) is the free bicategory of partial maps on the signature \((\Sigma, T)\).

**Proof.** As explained above, we must show that the posets \(\text{Hom}_{BPM}(\mathcal{C}_0^{(\Sigma, T)}, C)\) and \(\text{Hom}_{\text{theories}}((\Sigma, T), U(C))\) are naturally isomorphic. Theorem 12 shows that the former poset is naturally isomorphic to the poset of models of \(T\); the latter poset, on the other hand, is trivially isomorphic to the poset of models.

### 6.2 Weak Comma Objects

The above correspondences can be extended to the case of bicategories of partial maps with weak comma objects. As we have remarked above, we will continue using the 2-covariant domain fibration for the semantics of the internal language of these, even though, for bicategories with weak comma objects, we can also define a 2-contravariant domain fibration.

The existence of weak comma objects, as shown above, is equivalent to the Heyting operation on domain posets; it is thus natural to augment the internal language with a corresponding primitive. The Palmgren-Vickers language consisted of conjunctions of equations between partial functions: the language for weak comma objects will be generated by the Heyting operation and conjunction from equations between partial functions. The language and the calculus are defined in Table 5; we call it the Heyting-Palmgren-Vickers logic, which we abbreviate to HPV.

### 6.2.1 Free Categories with Comma Objects

We can now define free categories with comma objects: the definition exactly replicates the definition for partial cartesian categories, except that the logic is now the HPV logic.
\[ \vartheta(x) := f(x) \cong g(x) \mid \vartheta(x) \land \vartheta \mid \vartheta(x) \\vartheta(x) \]

### Structural Rules

| Rule | Description |
|------|-------------|
| \( \vartheta(x) \vdash \vartheta(x) \) | Ax | \( \vartheta(x) \vdash \vartheta'(x) \) \( \vartheta'(x) \vdash \vartheta''(x) \) | Cut |
| \( \vartheta(y) \vdash \vartheta'(y) \) | | \( \alpha(x) \downarrow \land \vartheta[\alpha/y] \vdash \vartheta'[\alpha/y] \) | Substitution |

### Equality

| Rule | Description |
|------|-------------|
| \( \top \vdash \varphi \) | Reflexivity | \( x \cong y \land \varphi \vdash \varphi[y/x] \) | Equality |
| \( \alpha(x) \cong \beta(x) \vdash \alpha(x) \downarrow \land \beta(x) \downarrow \) | Strictness 1 |
| \( \alpha(\beta(x)) \downarrow \vdash \beta(x) \downarrow \) | Strictness 2 |

### Conjunctions

| Rule | Description |
|------|-------------|
| \( \vartheta(x) \vdash \top(x) \) | | \( \vartheta(x) \vdash \vartheta'(x) \) \( \vartheta(x) \vdash \vartheta''(x) \) \( \vartheta(x) \vdash \vartheta'(x) \land \vartheta''(x) \) | \( \land \) |
| \( \vartheta(x) \land \vartheta'(x) \vdash \vartheta(x) \) | \( \land \) \( \pi_1 \) | \( \vartheta(x) \land \vartheta'(x) \vdash \vartheta(x) \) \( \land \) \( \pi_2 \) |

### Heyting

| Rule | Description |
|------|-------------|
| \( \vartheta(x), \vartheta'(x) \vdash \vartheta''(x) \) | HeytingI |
| \( \vartheta(x) \vdash \vartheta'(x) \land \vartheta''(x) \) | HeytingE |
| \( \vartheta(x) \vdash \vartheta'(x) \land \vartheta''(x) \) | HeytingE |

Table 5: The Heyting-Palmgren-Vickers Logic
Definition 51. Given a signature $\Sigma$, together with a theory $T$ of the HPV logic in that signature, we define a locally posetal two-category $\mathcal{C}^h_{(\Sigma, T)}$ as follows.

**objects** An object of $\mathcal{C}^h_{(\Sigma, T)}$ will be written $\{ x : s | \zeta(x : s) \}$, where $x : s$ is a variable tuple and where $\zeta$ is a term of the HPV logic with that source type.

1-cells A 1-cell will be written $\langle \alpha(x : s) | \zeta(x : s) \rangle : \{ x : s | \eta(x : s) \} \to \{ y : t | \vartheta(y : t) \}$

where we require that

1. $s$ is the source type of $\alpha$, of $\zeta$ and of $\eta$.
2. $t$ is the target type of $\alpha$ and the source type of $\vartheta$, and
3. the entailment $T, \zeta, \eta \vdash h \equiv \vartheta(\alpha)$

holds (informally, we require that $\langle \alpha(x : s) | \zeta(x : s) \rangle$ should factor through the subobject of $\{ y : t \}$ defined by $\vartheta$).

Composition of 1-cells is defined as follows. Suppose that

$\langle \alpha(x : s) | \zeta(x : s) \rangle : \{ x : s | \vartheta(x : s) \} \to \{ y : t | \vartheta'(y : t) \}$

$\langle \beta(y) | \eta(y) \rangle : \{ y : t | \vartheta'(y : t) \} \to \{ z : u | \vartheta''(z : u) \}$

then

$\langle \beta(y) | \eta(y) \rangle \langle \alpha(x) | \zeta(x) \rangle = \langle \beta(\alpha(x)) | \eta(\alpha(x)) \rangle$,

$\zeta(x)$

The unit 1-cell, on an object $\{ x : s | \zeta(x : s) \}$, is $\langle x | \zeta(x) \rangle$.

2-cells The 2-cells of our category will be defined as follows. Given

$\langle \alpha | \zeta \rangle, \langle \alpha' | \zeta' \rangle : \{ x | \eta(x) \} \to \{ y | \vartheta(y) \}$

we say that $\langle \alpha | \zeta \rangle \sqsubseteq \langle \alpha' | \zeta' \rangle$ iff

$T, \alpha \downarrow, \zeta, \eta \vdash h \quad \alpha = \alpha' \wedge \eta'$

informally, whenever $\langle \alpha | \zeta \rangle$ is defined, then so too is $\langle \alpha' | \zeta' \rangle$ and they are equal.

We also define the following subcategory of $\mathcal{C}^h_{(\Sigma, T)}$: we will need it for our proof of freeness.

Definition 52. Let $\mathcal{C}^h_0_{(\Sigma, T)}$ be the full subcategory of $\mathcal{C}^h_{(\Sigma, T)}$ whose objects have no constraints, i.e. are all of the form $\{ x : s \}$.

We can, as before, prove, by routine calculation

Lemma 27. $\mathcal{C}^h_{(\Sigma, T)}$ and $\mathcal{C}^h_0_{(\Sigma, T)}$ are locally posetal bicategories.
We can also prove

**Lemma 28.** \( C_{(Σ, T)}^{k} \) and \( C_{(Σ, T)}^{h0} \) have terminal objects and weak comma objects.

**Proof.** Terminal objects are straightforward. We define weak comma objects as follows: suppose that we have morphisms

\[
\begin{align*}
\{ x : A | ϑ(x) \} & \quad \{ x'' : C | ϑ(x'') \} \\
(f|ζ(x)) & \quad (g|ζ'(x''))
\end{align*}
\]

Then we define the comma object

\[
\langle f|ζ(x) \rangle \odot (g|ζ'(x')) = \{ x : A, y : B | ϑ(x), ϑ'(y), (f ≏ g ∧ ϑ') (f ↓ ∧ ζ) \}
\]

and, after some calculation, we can show that it has the required universal property. \(\square\)

### 6.3 Comprehensions

We now revert to the fibrational setting.

#### 6.3.1 Equality in the Fibres

Because of our results on presentations of partial cartesian categories, we can use a term-based notation for contexts in our calculus: thus, over an object such as \( A \otimes A \otimes B \), we can write sequents in this form:

\[
x : A, y : A, z : B | Γ, x \vdash Δ(x, y, z)
\]

Now we define equalities in more abstract terms.

**Definition 53.** Given a context \( x : A, y : A, z : B \), define the proposition \( x = y \) as

\[
¬(∇_{A} \otimes \text{Id}_{B}) \ast ⊥
\]

**Proposition 22.** The following rules for \( = \) are admissible:

\[
\frac{x : A, y : A, z : B | Γ, x \vdash Δ(x, y, z)}{x : A, z : B | Γ[y/x] \vdash Δ'[y/x]}
\]

**Proof.** Expanding the definition of \( = \), this is equivalent to the admissibility of

\[
\frac{x : A, y : A, z : B | Γ \vdash Δ^∗ \perp, Δ'}{x : A, z : B | Δ^∗Γ \vdash Δ', Δ^∗Γ'}
\]

and this follows from the adjunction \( Δ^∗ \dashv Σ^∗ \), together with Lemma [10] to show that \( Δ^∗Γ \) and \( Σ^∗Γ' \) can be identified with \( Γ[y/x] \) and \( Γ'[y/x] \). \(\square\)
By [Jacobs, 1999, Prop. 3.2.3], this is enough to show that $\equiv$ is a genuine equality.

We can also relate equality in the fibres to equality in the base. Suppose we have an object $A$ of $\mathcal{C}$, and let $A_0 = \{A|\alpha \equiv \beta\}$ with inclusion $i : A_0 \rightarrow A$.

**Lemma 29.** $\prod_i T$ is $\alpha = \beta$.

**Proof.** Because $A_0$ arises from the splitting of a coreflexive, we have

$$i : A_0 \rightarrow A, \quad j : A \rightarrow A_0,$$

with $i \dashv j$ and $ji = \text{Id}_{A_0}$.

Now, by Lemma 13 we have

$$\prod_i T = \neg j^* \bot$$

$$= \neg j^* i^* \bot$$

since $j^*$ is a boolean algebra homomorphism

$$= \neg (ij)^* \bot$$

$$= (\alpha = \beta)$$

by definition of $\equiv$.

$\square$

**Corollary 11.** The following rule is admissible:

$$A, \alpha \equiv \beta | i^* \Gamma \vdash i^* Q$$

$$A | \alpha = \beta, \Gamma \vdash Q$$

**Proof.** This is simply the adjunction $\prod_i i^*$, together with Lemma 29 $\square$.

We have, by induction,

**Corollary 12.** The following rule is admissible, where $\zeta$ is in the fragment generated by $\land$ from equalities, and where we write, by abuse of notation, $\zeta$ both in the base and in the fibres:

$$A, \zeta | \Gamma \vdash \Delta$$

$$A | \zeta, \Gamma \vdash \Delta$$
7 Davidson’s Example

We conclude with an extended example: this is of the philosopher Davidson’s argument about the equality of actions. It is important because, in the philosophical community, the notion of equality of action seems to have significant consequences: roughly speaking, first-class objects are those which have meaningful equalities. However, we have a system in which we can define equalities on actions on fairly weak premises: they come from a well-established treatment of partiality, together with quite weak assumptions about the existence of limits. Consequently, we can show that Davidson’s argument probably establishes less than he takes it to. But in order to do that, we have to show that our equalities are capable of playing the same argumentative role as Davidson’s equalities: and this is what we do in this section.

7.1 Davidson

The philosopher Donald Davidson (following Austin (1956–7)) considers the following pattern of reasoning.

‘I didn’t know that it was loaded’ belongs to one standard pattern of excuse. I do not deny that I pointed the gun and pulled the trigger, nor that I shot the victim. My ignorance explains how it happens that I pointed the gun and pulled the trigger intentionally, but did not shoot the victim intentionally. . . . The logic of this sort of excuse includes, it seems, at least this much structure: I am accused of doing \( b \), which is deplorable. I admit I did \( a \), which is excusable. My excuse for doing \( b \) rests upon my claim that I did not know that \( a = b \). (Davidson, 1980d, p. 109)

Davidson, then, is arguing for two things:

1. equalities between actions are meaningful, and
2. we use these equalities in common-sense reasoning about action.

These claims of Davidson’s have given rise to a great deal of argument, of which the main protagonists are Davidson (1980a) and Kim (1993). Although this debate has generated a lot of high-quality philosophy, it has been strangely inconclusive, and, perturbingly, strangely orthogonal to other issues in the semantics of natural language.

Example 7 (Davidson). The facts in Davidson’s example can now be expressed as follows. Suppose that we have actions: \( pt \) stands for “pull trigger”, \( sh \) stands for “shoot”, and \( kw \) stands for “kill”. Suppose, also, that we have propositions \( \text{loaded} \) and \( \text{aimed} \). The semantics of these propositions will be as follows. \( \text{loaded}(x) \) will be true in a world \( x \) iff the gun is loaded in that world: similarly, \( pt(x) = sh(x) \) is true in a world \( x \) iff the result of pulling the trigger is the same as the result of shooting. Thus, (13) says that the gun is loaded iff pulling the trigger

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Table 6: The Davidsonian Scenario

is the same as shooting it: similarly, \((14)\) says that the gun is aimed iff shooting it is the same as killing the victim.

We formulate the effects of these actions in context in the following axioms:
we use partial equality and are careful to stipulate that actions are performable.
Let \(sh\) stand for ‘shoot’ and \(pt\) stand for ‘pull trigger’.

\[
\begin{align*}
& \vdash_{\vartheta_2} \text{loaded}(x) \rightarrow \text{pt}(x) \equiv \text{sh}(x) \tag{13} \\
& \vdash_{\vartheta_2} \text{sh}(x) \equiv \text{sh}(x), \text{pt}(x) \equiv \text{pt}(x) \text{ loaded}(x) \vdash \text{pt}(x) = \text{sh}(x) \tag{14}
\end{align*}
\]

We also describe the effects of shooting as follows:

\[
\begin{align*}
& \vdash_{\vartheta_2} \text{sh}(x) \equiv \text{sh}(x), \text{pt}(x) \equiv \text{pt}(x) \text{ loaded}(x) \vdash \text{pt}(x) = \text{sh}(x)
\end{align*}
\]

Here we assume that \(\text{loaded}, \text{alive}\) and \(\text{dead}\) are predicates which are defined for all values of \(x\). \(sh\) and \(pt\), on the other hand, are actions which may not be performable in all circumstances (there may not be a gun to hand, for example), and thus these entailment have non-trivial contexts.

We prove the unfortunate consequence in Table 6. We use two contexts here: \(\vartheta_2\) is the base context consisting of the facts holding. \(\vartheta_1\) is \(\{\vartheta_2 | \text{pt}(x) = \text{sh}(x)\}\); let \(i\) be the inclusion of \(\vartheta_1\) in \(\vartheta_2\).

The inferences are annotated as follows: \(\text{subs}\) is the substitution rule from the type theory (i.e. \(i^*\) of Table 3). \(\text{rep}\) is the equality rule of Proposition 22 and \(\text{comp}\) is the rule of Corollary 11. \(\text{subs}\) and \(\text{rep}\) are used as axioms. \(\dagger\) is the unfortunate fact that the gun is loaded, likewise used as an axiom.

This is a deduction of the eventual death, using equational reasoning, and starting from the axioms describing the initial situation. The reasoning in Davidson’s example is probably best regarded as abductive, and this can be handled in this system in terms of proof search, starting from the observed death and assuming \(\dagger\).

7.1.1 Evaluation

Davidson uses this example to argue for the first-class status of actions, on the basis that equalities between them are meaningful and, in fact, used in reasoning about action. However, although our formalisation accounts for the inferences in Davidson’s story, and deals with them equationally, it hardly supports
Davidson’s reading: the partial function semantics of our logic treats equality as equality between function values. But Davidson’s argument, based on this reasoning, would need equalities between the functions themselves rather than, as we do, between their values. However, our semantics of equality seems to be difficult to avoid: given a plausible treatment of partiality – that given by partial cartesian categories, and independently discovered by Palmgren and Vickers – we get equality for free, and it is definitely an equality between function values.

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