Poincaré invariance in low-energy effective field theories for QCD

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Abstract. We present the calculations on deriving relations between the Wilson coefficients in non-relativistic quantum chromodynamics and potential non-relativistic quantum chromodynamics by exploiting the Poincaré invariance. Implications of the constraints are briefly discussed in the context of the effective string theory.

1. Introduction
Quantum Chromodynamics (QCD) has theoretically and experimentally been verified as the solid theory for explaining colour interactions at subatomic scales, during the last decades. However, it contains both perturbative and non-perturbative aspects, manifested by the asymptotic freedom [1, 2, 3] and the confinement [4], respectively. While perturbative QCD (pQCD) [5] is the proper framework for describing interactions at high energies, it is not able to give any feasible description of its dynamics around the confinement phase. Thus, it is helpful to separate the perturbative and non-perturbative regimes.

Separately treating those two different aspects of physics calls upon a framework of the effective field theories (EFTs), which requires to observe a hierarchy of scales in the physical systems of one’s interest. In case of heavy mesons, such as the bottomonium or charmonium states, hierarchy is given such that the rest mass of single quark/antiquark $M$ is much greater than the momentum transfer $Mv$ between them ($v$ being the relative velocity) and the confinement scale $\Lambda_{QCD}$: $M \gg Mv, \Lambda_{QCD}$. As the heavy scale $M$ is integrated out, dynamics of the heavy quark fields enter into the non-relativistic regime, so that the resulting effective theories are either non-relativistic QCD (NRQCD) [6, 7] for heavy quark-antiquark systems or the heavy quark effective theory (HQET) [8, 9] for heavy quark and light quark states, whose Lagrangians are organised with respect to $1/M$ expansion, $L = \sum_n c_n \mathcal{O}_n / M^{d_n-4}$, where operators $\mathcal{O}_n$ consist of effective degrees of freedom (DOFs) with its mass dimension $d_n$, and $c_n$ are undetermined scalar functions, namely the Wilson (or matching) coefficients. These coefficients contain the information of its underlying (or UV) theory, whose exact values are to be determined via matching to the UV theory at the corresponding energy scale.

NRQCD is the useful EFT for analysing productions (or annihilations) of the heavy quark-antiquark states, but it shows some difficulties in explaining the mass splitting of the bound states. In order to resolve these issues, one can construct a valid EFT for such cases, which is the potential NRQCD (pNRQCD)[10]. When the hierarchy of scale is established as such, $M \gg Mv \gg Mv^2$, ($Mv^2$ being the relative kinetic energy between quark and antiquark),
pNRQCD is constructed by integrating out the relative momentum $Mv$ from NRQCD, and its Lagrangian is organised by the multipole expansion in the relative distance $r$ upon the $1/M$ expansion from NRQCD. Just like in NRQCD/HQET, the pNRQCD Lagrangian contains a series of the effective operators along with the Wilson coefficients, and these coefficients are to be determined via matching to NRQCD, the underlying theory of pNRQCD. It is apparent that calculations can easily become more complicated and lengthier if one tries to perform the matching directly from QCD to NRQCD and then to pNRQCD, due to the proliferation of the effective operators; it is, therefore, worthwhile to look for some ways to reduce the amount of the matching calculations, such as by deriving constraints between the coefficients.

There have been two separate ways of finding constraints between Wilson coefficients, both of which exploit symmetry of the EFTs. The first symmetry is the reparametrisation invariance \[11\], and the second is the Poincare invariance \[12, 13\], which we explore in detail here. Since QCD is a Poincare invariant theory, it is natural to observe that its low-energy counterpart, NRQCD in particular, is also to observe Poincare invariance \[1\]. When the Poincare symmetry is realised in NRQCD, pNRQCD is also to preserve the same symmetry, thereby constraints between the Wilson coefficients are established.

Concerning the derivations of the constraints, this report is organised as follows: in Sec. 2, general principles of Poincare transformation of non-relativistic fields are discussed, which we apply to NRQCD and derive the constraints between the coefficients. In Sec. 3, we present similar arguments on the pNRQCD, but certain complications arise due not only to the proliferations of the effective operators, but also to the transformation behaviour of the fields; these issues are addressed in the same section. Implications of the constraints in pNRQCD are briefly discussed in Sec. 4, with the framework of the effective string theory \[14\], which is the EFT for the flux tube model in the heavy quark-antiquark bound system.

2. Poincare invariance in NRQCD

In relativistic field theories, a transformation of a field $\phi$ under the Poincare group is given: $\phi_a(x) \rightarrow M_{ab}(\Lambda)\phi_b(\Lambda^{-1}x)$, with $M_{ab}(\Lambda)$ being the finite dimensional representation of the Poincare group and $\Lambda$ being the transformation matrix which acts on the coordinate $x'^\mu = \Lambda^\mu_\nu x^\nu$.

There are ten independent bases for the transformation matrix, three of which are responsible for the spatial rotations, three are for the spatial boosts, and four of them are the spacetime translations. Since the representation of the Poincare group in the case of the spatial boosts is no longer linear at the non-relativistic regime, let us pay our attention to the boost transformation of a free relativistic field, which can be expressed with $\phi_a(x) \rightarrow (e^{\pm \eta \cdot \Sigma})_{ab}\phi_b(B^{-1}x)$, for $\eta$ is the parameter for the boost, $\Sigma$ is the spin representation, and $B$ is the boost transformation matrix that acts on spacetime coordinates. In case of a massive field with mass $M$, one can rewrite the boost transformation in a slightly different manner when a rest frame of the massive particle, $v = (1, 0, 0, 0)$, is chosen as its reference \[13\]

$$\phi_a(x) \rightarrow \exp \left[ \pm \eta \cdot \frac{\Sigma \times \partial}{M + \sqrt{M^2 - \partial^2}} \right]_{ab} \phi_b(B^{-1}x). \tag{1}$$

The transformation has now changed from global to local (appearance partial derivatives), and one can take a non-relativistic expansion of this expression. For an interacting case, the boost transformation of the field with regard to symmetries under charge, parity and time reversal,
\((C, P, T \text{ in short), as well as spatial rotations, are given by (up to a cubic order in } 1/M)\) [15]

\[
\phi_a(x) \rightarrow \left\{ 1 + iM \eta \cdot x - \frac{ik_1 \eta \cdot \mathbf{D}}{2M} + \frac{k_2 (\Sigma \times \eta) \cdot \mathbf{D}}{2M} - \frac{ik_3 \eta \cdot D^2 D^2}{4M^3} + \frac{k_4 (\Sigma \times \eta) \cdot D \cdot \mathbf{D^2}}{4M^2} + \mathcal{O}(g, 1/M^4) \right\} \phi_a(B^{-1}x).
\]

for \(k\)'s are undetermined coefficients; it is clear that \(\mathbf{D}\) is the gauge covariant derivatives from the interacting theory. Let us now apply this expression to NRQCD whose bilinear parts of the Lagrangian and additional terms up to total derivatives (also up to the order of \(1/M\))

\[
\mathcal{L}_{\text{NRQCD}} \ni \psi^\dagger \left\{ iD \phi_0 + \xi \frac{D^2}{2M} - \xi_2 \frac{D^4}{8M^3} + \xi g \frac{g \cdot \mathbf{B}}{2M} + \xi g \frac{[\mathbf{D}, \mathbf{E}]}{8M^2} + \xi g \frac{\mathbf{E} \times \mathbf{E}}{8M^2} \right\} \psi \\
+ \chi^\dagger \left\{ iD - \xi \frac{D^2}{2M} - \xi_2 \frac{D^4}{8M^3} - \xi g \frac{g \cdot \mathbf{B}}{2M} + \xi g \frac{[\mathbf{D}, \mathbf{E}]}{8M^2} + \xi g \frac{\mathbf{E} \times \mathbf{E}}{8M^2} \right\} \chi.
\]

where \(c\)'s are the Wilson coefficients, \(\psi\) annihilates the heavy quark, and \(\chi\) creates the heavy antiquark. From the Eq. (2), the quark and antiquark fields transform under the boost as (up to \(1/M\) order)

\[
\psi(x) \rightarrow \left\{ 1 + iM \eta \cdot x - \frac{ik_1 \eta \cdot \mathbf{D}}{2M} + \frac{k_2 (\Sigma \times \eta) \cdot \mathbf{D}}{2M} + \mathcal{O}(g, M^{-2}) \right\} \psi(B^{-1}x)
\]

\[
\chi(x) \rightarrow \left\{ 1 - iM \eta \cdot x + \frac{ik_1 \eta \cdot \mathbf{D}}{2M} - \frac{k_2 (\Sigma \times \eta) \cdot \mathbf{D}}{2M} + \mathcal{O}(g, M^{-2}) \right\} \chi(B^{-1}x)
\]

and applying Eq. (4) and Eq. (5) to Eq. (3), we obtain the original form of the bilinear sectors of the Lagrangian and additional terms up to total derivatives (also up to the order of \(1/M\))

\[
\delta \mathcal{L}_{2\psi} = \psi^\dagger \left\{ i(1 - c_1) \eta \cdot \mathbf{D} - \frac{1}{2M} (k_1 - c_1) [D_0 \eta \cdot \mathbf{D}] + \frac{1}{4M^2} (1 - 2c_F + c_s) \eta \cdot (g \mathbf{E} \times \mathbf{E}) \right\} \psi.
\]

This expression is to vanish according to the Poincaré invariance, which implies \(k_1 = c_1 = 1, c_s = 2c_F - 1\). Note that the coefficient \(k_2\) from the boost transformations, Eq. (4) and Eq. (5), is still to be determined. It can be fixed by using the commutation relations between boost generators which is to be equivalent to the rotation generator up to \(1/M\) order; \(k_2\) equals to the unity. These results confirm the values from the literature [12], but the amount of calculations involved here is smaller and simpler than the hitherto known methods, which means that one can apply it for the higher loop calculations with much ease. Similar procedures are applied to the four-quark sector of NRQCD, and the results are in accord with known values [16]; several new features are also to be shown. Full results on the four-quark sector will be discussed in the upcoming paper.

3. Poincaré invariance in pNRQCD

pNRQCD is obtained by integrating out the momentum transfer \(Mv \sim 1/r\) in NRQCD [10], from which arises a slight change in its coordinate system: spatial dependence of the colour singlet and octet fields is on the centre-of-mass \(\mathbf{R} = (\mathbf{x}_1 + \mathbf{x}_2)/2\) as well as on the relative coordinates \(\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2\), where \(\mathbf{x}_1\) and \(\mathbf{x}_2\) are the spatial coordinates of the heavy quark and heavy antiquark, respectively. Also the ultrasoft gluons have only spatial dependence on the relative coordinates; in short, the non-relativistic DOFs in pNRQCD are the singlet \(S(t, \mathbf{R}, \mathbf{r})\) and the octet fields \(O^a(t, \mathbf{R}, \mathbf{r})\), with \(a\) being the colour index, and we denote the gluons with \(A^a_{\mu}(t, \mathbf{R})\). Power counting schemes are followed \(\nabla_{\mathbf{r}}, 1/r \sim Mv, \partial_0, \partial_{\mathbf{R}}, A_{\mu} \sim Mv^2, E, B \sim M^2v^4\), so the bilinear
parts of the singlet field (up to $1/M^2$) are organised by \([12]\)

\[
S'k_S^S S = S' \left( i\partial_0 + \frac{1}{2M} \left\{ c_s^{(1,-2)}, \nabla_r^2 \right\} + \frac{c_s^{(1,0)}}{4M} \nabla_R^2 - V_S^{(0)} - \frac{V_S^{(1)}}{M} + \frac{V_{\rho S a}^2}{8M^2} \nabla_R^2 + \frac{1}{2M^2} \left\{ \nabla_r^2, V_{\rho S b} \right\} \right)
\]

\[
+ \frac{V_{L^2 S a}}{4M^2} (r \times \nabla_r)^2 + \frac{V_{L^2 S b}}{4M^2} (r \times \nabla_r)^2 - \frac{V_{S S}^{(1)}}{M^2} (3(r \cdot \sigma^{(1)})(r \cdot \sigma^{(2)}) - r^2 \sigma^{(1)} \cdot \sigma^{(2)})
\]

\[
- \frac{V_{S S}^{(1)}}{4M^2} \sigma^{(1)} \cdot \sigma^{(2)} + \frac{iV_{L S S a}}{4M^2} (r \times \nabla_r) \cdot (\sigma^{(1)} - \sigma^{(2)}) + \frac{V_{S S}^{(1)}}{4M^2} (r \times \nabla_r) \cdot (\sigma^{(1)} + \sigma^{(2)}) S.
\]

(7)

Note that $c$’s and $V$’s are the undetermined Wilson coefficients; superscripts on $c$’s denote the orders of $1/M$ and $r$, respectively, and $V$’s are the potential terms. Also, the superscript to the Pauli matrices $\sigma^{(1/2)}$ denotes the spin for the heavy quark/antiquark, respectively. Our task is to construct the most general form of the spatial boost which acts on the singlet field with the symmetries under $C, P, T$ transformations in mind, up to the order of $1/M$ and $r$, and it is shown as \([15]\)

\[
S'(t, R, r) = \left( 1 - 2iM \eta \cdot R + \frac{i k_{D}^{(1,0)}}{4M} \eta \cdot \nabla_R + \frac{i}{4M} \left\{ k_{a'}^{(1,0)} \eta \cdot r, \nabla_r \nabla_r \right\} \right)
\]

\[
+ \frac{i}{4M} \left\{ k_{a'}^{(1,0)} \eta \cdot r, \nabla_r (\eta \cdot \nabla_R) \right\} + \frac{i}{4M} \left\{ k_{b}^{(1,0)} \eta \cdot r, (\sigma^{(1)} + \sigma^{(2)}) \right\}
\]

\[
- \frac{k_{c}^{(1,0)}}{8M} \eta \cdot \nabla_R \times \left( \sigma^{(1)} + \sigma^{(2)} \right) + \frac{i k_{D}^{(1,0)}}{8M^2} (\eta \cdot r \times \nabla_R) \left( \eta \cdot r, (\sigma^{(1)} + \sigma^{(2)}) \right)
\]

\[
- \frac{k_{c}^{(1,0)}}{8M^2} (\eta \cdot r \times \left( \sigma^{(1)} + \sigma^{(2)} \right)) \left( \eta \cdot r, (\sigma^{(1)} - \sigma^{(2)}) \right)
\]

\[
+ \frac{1}{8M} \left\{ k_{b}^{(1,1)} (\eta \cdot r, (\sigma^{(1)} - \sigma^{(2)}) \right\} \left( \eta \cdot r, (\sigma^{(1)} - \sigma^{(2)}) \right)
\]

\[
+ \frac{1}{8M} \left\{ k_{b}^{(1,1)} (\eta \cdot r, (\sigma^{(1)} - \sigma^{(2)}) \right\} \left( \eta \cdot r, (\sigma^{(1)} - \sigma^{(2)}) \right)
\]

\[
S(t', R', r').
\]

(8)

At first glance, more problems have arisen due to the emergence of the unknown coefficients $k$ in the boost transformation. Eq. (8) can, however, be simplified by the means of the field redefinitions, unitary transformations in particular: $S = U_S \tilde{S}$ where $\tilde{S}$ is the newly defined singlet field and $U_S$ is the unitary operator for it. A natural choice for $U_S$ is an exponential function with its exponents being anti-hermitian operators. Order of $1/M$ expansion of the exponents is quadratic from the boost transformation of the new singlet field: \[U_S = \exp \left\{ -\frac{1}{4M^2} \left\{ q^{(1,0)}_a \eta \cdot \nabla_R, \nabla_r \nabla_r \right\} - \frac{1}{4M^2} \left\{ q^{(1,0)}_a \eta \cdot \nabla_R, \nabla_r \nabla_r \right\} + \cdots \right\},\]

where the $q$s are parameters we can freely choose. When we apply the unitary transformation to the singlet field and choose $q$s to eliminate as many terms in the boost as possible, there only remain four boost coefficients, $k_{D}^{(1,0)}, k_{a'}^{(1,0)}, k_{b}^{(1,0)}, k_{c}^{(1,0)}$. As shown in the case of NRQCD, there exist additional constraints one can impose on the boost generator. First of which is the commutation relations between the boost generators acting onto the singlet field: \[\left[ 1 - i\xi \cdot k, 1 - i\eta \cdot k \right] S = i(\xi \times \eta) \cdot j S,\]

and it fixes three coefficients: $k_{a'}^{(1,0)} = k_{c}^{(1,0)} = 1, k_{b}^{(1,0)} = 0$. The last remaining one $k_{D}^{(1,0)}$ is fixed in the end along with deriving the constraints between the Wilson coefficients. The boost transformation of the singlet bilinear part of the pNRQCD Lagrangian is obtained by the simplified expression of the boost generator, thereby the original structure of the Lagrangian is kept intact up to total derivatives (organised up to the order of $1/M$)

\[
\delta \mathcal{L}_{2S} = S' \left( i \left( 1 - c_S^{(1,0)} \right) \eta \cdot \nabla_R - \frac{1}{2M} \left( k_D^{(1,0)} - c_S^{(1,0)} \right) \eta \cdot \nabla_R \partial_0 - \frac{i}{M} \left( V_{\rho S a}^2 + V_{L^2 S a} + \frac{1}{2} V_S^{(0)} \right) \eta \cdot \nabla_R 
\]

\[
+ \frac{i}{M^2} \left( V_{L^2 S a} + \frac{r}{2} \partial_r V_S^{(0)} \right) (\eta \cdot r)(\eta \cdot \nabla_R) + \frac{1}{2M} \left( V_{L^2 S a} + \frac{1}{2} V_S^{(0)} \right) \eta \cdot (\sigma^{(1)} - \sigma^{(2)}) \times r \right) S.
\]

(9)
which implies that all of the expressions inside the bracket in Eq. (9) are to vanish. Similar procedures are applied to the octet bilinear as well as to the singlet-octet sector, and they also yield relations between the coefficients. The significance of this procedure is that the form of the boost generator we started with is greatly simplified and eventually coincides with the one inspired by Wigner’s little group formalism [13, 17], due to a particular choice of the unitary transformation.

4. Outlook: effective string theory

Poincaré invariance in low-energy EFTs for QCD sheds light on another effective description of the heavy quark-antiquark bound system, namely the effective string theory (EST) [14]. The EST is valid at the long distance separation, $r_{\Lambda_{QCD}} \gg 1$, and the transversal excitations of the long strings with its two ends fixed are the DOFs. There has been a hypothesis that Wilson loops in NRQCD are related to the EST at long distance as well as at large time scale [18]. On the other hand, Wilson loops in NRQCD at large time are related to the static potential in pNRQCD [19], which implies that one can relate the heavy quark-antiquark potentials to the string field variables. There arise dimensionful parameters in the string description [20], and Poincaré invariance in pNRQCD induces constraints between the parameters in the EST [21]. In [21], constraints are derived, but string variables are only considered up to the leading order, which are subject to be modified when subleading terms are included; it is clear that $1/r^5$ potentials would change its coefficients, which is a part of our work in progress. Full EFT systematics to the EST concerning the inclusion of the subleading terms is to be discussed in the follow-up paper.

5. Acknowledgments

I thank Matthias Berwein, Nora Brambilla, and Antonio Vairo for the collaboration on this work, and International Max Planck Research School (IMPRS) program from Max-Planck-Institut für Physik, München and Excellence Cluster “Universe” for the financial support.

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