Entanglement entropy and flow in two dimensional QCD: parton and string duality

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We discuss quantum entanglement between fast and slow degrees of freedom, in a two dimensional (2D) large $N_c$ gauge theory with Dirac quarks, quantized on the light front. Using the ’t Hooft wave functions, we construct the reduced density matrix for an interval in the momentum fraction $x$-space, and calculate its von Neumann entropy in terms of structure functions, that are measured by DIS on mesons (hadrons in general). We found that the entropy is bounded by an area law with logarithmic divergences, proportional to the rapidity of the meson. The evolution of the entanglement entropy with rapidity, is fixed by the cumulative singlet PDF, and bounded from above by a Kolmogorov-Sinai entropy of 1. At low-$x$, the entanglement exhibits an asymptotic expansion, similar to the forward meson-meson scattering amplitude in the Regge limit. The evolution of the entanglement entropy in parton-$x$ per unit rapidity, measures the meson singlet PDF. The re-summed entanglement entropy along the single meson Regge trajectory, is string-like. We suggest that its extension to multi-meson states, models DIS scattering on a large 2D 'nucleus'. The result, is a large rate of change of the entanglement entropy with rapidity, that matches the current Bekenstein-Bremermann bound for maximum quantum information flow. This mechanism may be at the origin of the large entropy deposition and rapid thermalization, reported in current heavy ion colliders, and may extend to future electron-ion colliders.

I. INTRODUCTION

Quantum entanglement permeates most of our quantum description of physical laws. It follows from the fact that quantum states are mostly superposition states, and two non-causally related measurements can be correlated, as captured by the famed EPR paradox. A quantitative measure of this correlation is given by the quantum entanglement entropy. The entanglement entropy of quantum many body system and quantum field theory has

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been extensively explored in the literature [1–5]. Less known perhaps, is the concept of quantum entanglement flow, and its relation to quantum information flow and storage. A maximum flow is expected in the most ideal quantum systems, following from the bound in energy change imposed by the uncertainty principle [6, 7].

In hadron physics, quantum entanglement is inherent to a hadron undergoing large longitudinal boosts, with its wavefunction described either by *wee partons* [8], or *string bits* [9–11]. Entanglement entropies are currently measured in diffractive *pp* scattering at large $\sqrt{s}$ in current collider facilities [12–14], and will be measured in *ep* scattering at low-x at future eIC facilities [12–15], with better accuracy. Entanglement entropies in relation to hadronic processes, have also been discussed in [16, 17].

In ultra-relativistic heavy-ion collisions, these inherently large entanglement entropies are at the origin of the prompt flow of *wee* entropy, likely at the boundary of our quantum laws. They may also explain, the almost instantaneous thermalization of the current strongly coupled plasma delivered initially at the RHIC facility, and later at the LHC facility [12, 13, 18]. The duality between the low-x partons and the string bits [13, 19, 20], explains why their entanglement provides for the most efficient mechanism for scrambling information, matching only that produced by gravitational black holes [9, 10].

In this work we discuss entanglement in longitudinal partonic momentum or Bjorken-x space, and also in rapidity space or $\ln \frac{1}{x}$ using 2-dimensional QCD. In the large number of colors limit, 2D QCD is solvable with a dual partonic [21] and string-like description [22]. The purpose of this work is to elucidate the concept of entanglement in single hadron states, or along a fixed Regge trajectory, as probed with DIS kinematics. As an example of DIS scattering on a 2D nucleus, we will address the entanglement in a multi-meson state (recall that all hadrons are similar on the light front), and show how its growth rate in rapidity, saturates the current bound on quantum information flow.

The outline of the paper is as follows: In section II we briefly review the light cone formulation of 2D QCD with Dirac quarks. In the large number of color limit, the two-body sector decouples and solves the ’t Hooft equation [21]. In section III, we detail the entangled density matrix in a single meson state, with a single parton-x cut, as probed by DIS scattering. The evolution of the entanglement entropy with rapidity, is fixed by the cumulative PDF, obeys a Kolmogorov-Sinai bound of 1 [23] (and references therein), and reduces to the longitudinal meson structure function at low-x. The entanglement entropy is shown to be universal in the 2D scaling limit. In section IV we recast 2D QCD as a string on the light front in the 2-particle sector. We show that the entanglement of the string bits, follows by resumming over the one meson Regge trajectory, thanks to duality. We suggest that the resummation over multi-meson Regge trajectories may describe DIS scattering on a 2D *nucleus* on the light front. The evolution of the ensuing entanglement entropy with rapidity is extensive in the classical and longitudinal string entropy. The rate of change matches the Bekenstein-Bremermann bound [6, 7] for the maximum flow of quantum information. Our conclusions are in section V. More details are given in the
II. DISCRETE LIGHT-CONE QUANTIZATION OF 2D QCD

To construct the reduced density matrix we first provide a review of the discrete light-cone quantization of the theory \cite{24–26}. The system is put in a finite box in the light-front space $-\frac{L^2}{2} < x^- < \frac{L^2}{2}$. After choosing anti-periodic boundary condition, the momenta are labeled as

$$k_p^+ = \frac{\pi}{L} (2p + 1) .$$  \hspace{1cm} (1)

The good component $\psi_{+i}$ of the fermion field has the mode decomposition as

$$\psi_{+i}(x^-) = \frac{1}{\sqrt{2L}} \sum_{p=0}^{N} \left( a_{i,p} e^{-i \frac{\pi (2p+1)}{L} x^-} + b_{i,p}^\dagger e^{i \frac{\pi (2p+1)}{L} x^-} \right) ,$$  \hspace{1cm} (2)

which satisfies the anti-commutation relation

$$[\psi_{+i}(x^-), \psi_{+j}(x^-)]_+ = \delta(x^-_1 - x^-_2) \delta_{ij} .$$  \hspace{1cm} (3)

Here $i = 1, \ldots, N_c$ is the color indices of the fermion field, which will be omitted below to avoid cluttering. The total number of $N$ for a finite system with lattice cutoff $a$ is given by $N = [\frac{L^2}{2a}] - 1$. Of all the $N$ independent frequencies, half are unfilled ($a_p$) and half are filled ($b_p$). In terms of the above Light Front (LF) free field, the LF momentum $P^+$ and LF Hamiltonian are given by \cite{25, 26}

$$P^+ L^- = 2\pi \sum_{p=0}^{N} \left( p + \frac{1}{2} \right) \left( a_{p}^\dagger a_{p} + b_{p}^\dagger b_{p} \right) ,$$  \hspace{1cm} (4)

and

$$\frac{P^-}{L^-} \equiv H = \frac{M^2}{2\pi} H_0 + \frac{1}{L^-} V ,$$  \hspace{1cm} (5)

with $H_0$ the mass contribution

$$H_0 = \sum_{p=0}^{N} \frac{a_{p}^\dagger a_{p} + b_{p}^\dagger b_{p}}{p + \frac{1}{2}} ,$$  \hspace{1cm} (6)

Here $V$ consists of four-quark contributions which can be computed from the interaction term

$$V = \frac{g_{1+1}^2}{2} \int_{-\frac{L^-}{2}}^{\frac{L^-}{2}} dx^- \psi_+^{\dagger} \psi_+ \frac{1}{(i\partial_-)^2} \psi_+^{\dagger} \psi_+ .$$  \hspace{1cm} (7)
Expressed in terms of $a_p, b_p$, $H$ is independent of $L^-$. Using the explicit formula of $P^+$ above, it is clear that to describe a given hadron state with total momentum $P^+$, not all the modes are required. We only need those $p$ below

$$p \leq \frac{L^- P^+}{2\pi} - \frac{1}{2} \equiv \Lambda^- - \frac{1}{2}.$$  \hfill (8)

Therefore, $\Lambda^-$ provides a natural truncation of the Hilbert space. The momentum fractions are labeled by

$$\frac{1}{2\Lambda^-} \leq x_p = \frac{1}{2\Lambda^-} (2p + 1) \leq 1.$$  \hfill (9)

Below, we use the label $x$ for all momenta. For a generic $\Lambda^-$, the states described above are purely discrete and breaks the Lorentz invariance. We expect that for $\Lambda^- \to \infty$, the spectrum of $H$ goes to zero as $\frac{M^2}{\Lambda}$, and the Lorentz invariant dispersion relation $P^+ P^- = 2M^2$ is restored. In particular, the meson state can be constructed as

$$|n\rangle = \frac{1}{\sqrt{\Lambda^-}} \sum_{0 < p < \Lambda^-} \varphi_p a_p^\dagger b_p^\dagger |0\rangle.$$  \hfill (10)

At large $N_c$, the above two-body state closes under the action of $P^-$. Requiring it to be an eigenstate of $P^-$ leads to the equation

$$\left(\Lambda^-\right)^2 m_R^2 \frac{\varphi_p}{(p + \frac{1}{2})(\Lambda^- - p)} + \Lambda^- \frac{g_{1+1}^2 N_c}{\pi} \sum_{l \neq p} \frac{\varphi_p - \varphi_l}{(p - l)^2} = M^2 \varphi_p.$$  \hfill (11)

In the continuum limit $\Lambda^- \to \infty$, and with the identification $x = \frac{p + \frac{1}{2}}{\Lambda^-}$ and $y = \frac{l + \frac{1}{2}}{\Lambda^-}$, (11) reduces to the ’t Hooft integral equation [21] in the continuum

$$\frac{m_R^2}{x x} \varphi_n(x) + \frac{g_{1+1}^2 N_c}{\pi} \text{PV} \int_0^1 dy \frac{\varphi_n(x) - \varphi_n(y)}{(x - y)^2} = M_n^2 \varphi_n(x).$$  \hfill (12)

The gauge coupling is related to the string tension $g_{1+1}^2 N_c / 2 = \sigma_T$ (see below). The renormalized quark mass is $m_R^2 = m_Q^2 - 2\sigma_T/\pi$. The ensuing spectrum is discrete, with eigenvalues and eigenvectors labeled by $M_n^2$ and $\varphi_n(x)$, respectively. They form a complete set of states in $L^2[0,1]$,

$$\sum_n \varphi^\dagger_n(x) \varphi_n(x') = \delta(x - x').$$  \hfill (13)

Their semi-classical and asymptotic behaviors are briefly reviewed in Appendix A.
III. ENTANGLEMENT ENTROPY IN 2D QCD

We now consider how different parts of a meson light front wave function as a bound quark-anti-quark state, are entangled in the quark longitudinal momentum $k^+ = xP^+ [13–15, 27]$. In particular, we will focus on the entanglement on a single asymmetric cut in longitudinal momentum, by analogy with a DIS experiment where a single parton-x is singled out, say in the segment $x_0 \leq \frac{1}{2}$, including the low-x region. We start by carefully reviewing the structure of the Hilbert space, and then define the pertinent single cut entanglement entropy.

A. Density matrix in longitudinal momentum

Since the color will be always traced out, here we simply omit the color factor. This will not modify our calculation of the entanglement entropy for two body states. With this in mind, for each $x$ we have the quark and anti-quark operators $a_x, b_x$, and their corresponding 2D Fock space. The total un-constrained Hilbert-space is their tensor product

$$\mathcal{H} = \bigotimes_{0<x<1} \mathcal{H}_x \otimes \bar{\mathcal{H}}_x ,$$

where

$$\mathcal{H}_x = \text{Span}(|0\rangle_x, a_x^\dagger |0\rangle_x) \ , \bar{\mathcal{H}}_x = \text{Span}(|\bar{0}\rangle_x, b_x^\dagger |\bar{0}\rangle_x) .$$

The total dimension of the Hilbert space is then $2^{[\Lambda^- - \frac{1}{2}] + 1} \times 2^{[\Lambda^- - \frac{1}{2}] + 1}$, spanned by quark and antiquarks. In a confining theory, however, not all of the states in the above Hilbert space are physical. In the 2D QCD, it can be shown that in the large $N_c$ limit, the physical spectrum consists of bound states formed by quark and anti-quarks, more precisely, the meson wave function reads

$$|n\rangle = \frac{1}{\sqrt{\Lambda^-}} \sum_{0<x<1} \varphi_n(x) |x, \bar{x}\rangle ,$$

where the basis $|x, \bar{x}\rangle$ can be written in full tensorial form as

$$|x, \bar{x}\rangle = a_x^\dagger |0\rangle_x \otimes b_x^\dagger |\bar{0}\rangle_x \otimes y \neq x |0\rangle_y \otimes |\bar{0}\rangle_y .$$

For finite $\Lambda^-$, $\varphi_n(x, \Lambda^-)$ satisfies a discrete version of the ’t Hooft equation, but as $\Lambda^- \to \infty$ $\varphi_n(x, \Lambda^-)$ should converge to its continuum version given above. Below we will always use the continuum version of the wave function. Unlike the free-quark and anti-quark states, the total dimension of the Hilbert space spanned by the ’t Hooft wave functions is not $\mathcal{H}$,
but only the two-quark states spanned by the set of bases $|x, \bar{x}\rangle$ defined above. Indeed, using the completeness equation of the 't Hooft equation one can show that
\[
\sum_n |n\rangle\langle n| = \sum_{0<x<1} |x, \bar{x}\rangle\langle x, \bar{x}| .
\] (18)
which is nothing but the projection operator into these quark-antiquark two body states. The total dimension of these states is only $\Lambda^-$, but not $4\Lambda^-$. Given the above meson state, one can construct its density matrix as
\[
\rho_n = \frac{1}{\Lambda^-} \sum_{x,x'} \varphi_n^\dagger(x') \varphi_n(x)|x, \bar{x}\rangle\langle x', \bar{x}'| .
\] (19)
Below we investigate its entanglement entropy with respect to the tensor product structure in Eq. (14).

**B. Reduced density matrix**

The entanglement in longitudinal space is captured by the reduced matrix
\[
\rho_n(x, x') = \text{tr}_A \rho_n(x, x') .
\] (20)
where $A$ denotes the part of the Hilbert space spanned by $a_x, b_x$, with $x$ lying in one or more sub-intervals of $[0, 1]$. How to choose $A$ depends on the probe experiment of interest. For instance, when probing a hadron in a DIS experiment via hard scattering, the virtual photon selects a quark or antiquark with fixed parton-$x$, say $x_0 < \frac{1}{2}$ in the range $\bar{A} = [0, x_0]$. The DIS event traces the hadron density matrix over the remaining, and unobserved longitudinal momentum range $A = [x_0, 1]$. This is particularly clear, when probing a hadron using semi-inclusive DIS production of heavy mesons. In the large $N_c$ or planar approximation, the process is dominated by Reggeon exchange, with the measured parton-$x$, kinematically limited to small $x_0 \ll 1$. This reduction of the density matrix is asymmetric in parton-$x$. A more symmetric but rather *academic* reduction, is discussed in Appendix B.

With this in mind, we now perform the partial trace in the tensor product in Eq. (14), over all the $\mathcal{H}_x$ and $\mathcal{H}_{\bar{x}}$ where $x > x_0$ with $x_0 < \frac{1}{2}$. To carry the partial trace, it is clear that for $x < x_0$ and $x' < x_0$, we are left with the quark contribution
\[
\frac{1}{\Lambda^-} \sum_{x<x_0} |\varphi_n(x)|^2 a_x^\dagger |0\rangle_x (0)_{x,a_x} \bigotimes_{y<x_0, y'<x_0, y\neq x} |0\rangle_y |0\rangle_{y'} |0\rangle_y .
\] (21)
Similarly, for $x > 1 - x_0$ and $x' > 1 - x_0$, we have the anti-quark contribution
\[
\frac{1}{\Lambda^-} \sum_{x<x_0} |\varphi_n(\bar{x})|^2 b_x^\dagger |0\rangle_x (0)_{x,b_x} \bigotimes_{y<x_0, y'<x_0, y\neq x} |0\rangle_y |0\rangle_{y'} |0\rangle_y .
\] (22)
It is easy to see that in the cases where \( x < x_0 \), \( x' > 1 - x_0 \) or \( x > 1 - x_0 \), \( x' < x_0 \), there are no partial traces that can be formed since in both of these two cases there will be one quark below \( x_0 \) and another quark above \( x_0 \). The case \( x_0 < x < 1 - x_0 \) and \( x_0 < x' < 1 - x_0 \) should be considered, since in this case both the quark and antiquark are above \( x_0 \), and should be traced out. This will leads to the contribution

\[
\frac{1}{\Lambda^{-}} \sum_{x < x < 1 - x_0} |\varphi_n(x)|^2 \otimes \sum_{y < x_0, y' < x_0} |0_y\rangle \langle 0_y'| \langle 0_y'| 0_{y'} \rangle.
\]

(23)

The contribution is proportional to the vacuum contribution \(|0\rangle\langle 0|\) for all the momentum modes below \( x_0 \) since they should not be traced over. Summing over the above, we found that for the two-body LF wave functions of a meson state, the reduced density matrix is diagonal and can be written schematically as

\[
\hat{\rho}_n(x_0) = \frac{1}{\Lambda^{-}} \sum_{x < x_0} \left[ |\varphi_n(x)|^2 |x\rangle_+ \langle x|_+ + |\varphi_n(x)|^2 |x\rangle_+ \langle x|_+ \right] + \frac{1}{\Lambda^{-}} \sum_{x_0 < x < 1 - x_0} |\varphi_n(x)|^2 |0\rangle \langle 0|
\]

(24)

The first contribution in (24) stems from the valence quark-antiquark pair in an \( n \)-meson state, and is expected. The second contribution stems from the vacuum state (zero-modes) assumed normalizable, and is unexpected. The trace of the reduced density matrix is 1, using the normalization condition of the wave function

\[
\sum_{0 < x < 1} \frac{\langle x|x \rangle}{\Lambda^{-}} |\varphi_n(x)|^2 = 1.
\]

(25)

with the light-like cutoff \( \Lambda^{-} \)

\[
\langle x|x \rangle = 2\pi \delta(0_x) = 2\pi P^+ \delta(0_{k^+}) = \frac{P^+}{0_{k^+}} \equiv \Lambda^-. \]

(26)

From the light cone discretization of 2D QCD, we identify \( 0_{k^+} = 1/2L^- \) as the lowest resolved longitudinal momentum, for a meson with total longitudinal momentum \( P^+ \). In the parton model, \( N = P^+/0_{k^+} \) counts the number of wee partons, with the larger the momentum, the larger \( N \) (see also below). We identify \( \chi = \ln \Lambda^- \) with the rapidity, which is fixed by DIS kinematics as \( \chi \sim \ln(Q^2/x) \) at low-\( x \).
C. Von Neumann entropy

Given the reduced density matrix, the corresponding von Neumann entanglement entropy is given by

\[
S_n(x_0) = -\text{tr}\hat{\rho}_n(x_0) \ln \hat{\rho}_n(x_0) = \ln \Lambda^- \int_0^{x_0} dx \left[ |\varphi_n(x)|^2 + |\varphi_n(\bar{x})|^2 \right] \\
- \int_0^{x_0} dx \left[ |\varphi_n(x)|^2 \ln |\varphi_n(x)|^2 + |\varphi_n(\bar{x})|^2 \ln |\varphi_n(\bar{x})|^2 \right] - \int_{x_0}^{1-x_0} dx |\varphi_n(x)|^2 \ln \int_{x_0}^{1-x_0} dx |\varphi_n(x)|^2 .
\]

(27)

Since the n-state quark and antiquark PDF for a meson is given by

\[
q_n(x) = \varphi_n^2(x), \quad \bar{q}_n(x) = \varphi_n^2(\bar{x}),
\]

(28)

the entanglement entropy is specifically

\[
S_n(x_0) = \ln \Lambda^- \int_0^{x_0} dx \left[ q_n(x) + \bar{q}_n(x) \right] - \int_0^{x_0} dx \left[ q_n(x) \ln q_n(x) + \bar{q}_n(x) \ln \bar{q}_n(x) \right] \\
- \int_{x_0}^{1/2} dx \left[ q_n(x) + \bar{q}_n(x) \right] \ln \int_{x_0}^{1/2} dx \left[ q_n(x) + \bar{q}_n(x) \right] .
\]

(29)

which is symmetric under the exchange of a quark to an anti-quark. Note that for \(x_0 = \frac{1}{2}\), the result simplifies

\[
S_n\left(\frac{1}{2}\right) = \ln \Lambda^- - \int_0^{1} dx q_n(x) \ln q_n(x) \rightarrow \ln \Lambda^- - \left(1 - \ln 2\right) .
\]

(30)

with the rightmost result following from the WKB approximation. We have checked that for other hadrons (nucleons, exotics), the extensive part in (29) with the rapidity, is also multiplied by the cumulative probabilities of each parton in that state. (29) is the first major result of this paper.

**Area law and Kolmogorov-Sinai bound [23]:**

Since the entanglement entropy depends on the length of the interval \(x_0\Lambda^-\) only through \(\ln\), it trivially satisfies an area law. Similarly to the spatial entanglement in 2D gapped system [1], the entanglement contains a log-divergent term \(\propto \ln \Lambda^-\) and a finite term. However, unlike the spatial entanglement entropy, the coefficient of the log-term depends also on the length of the interval. The logarithmic dependence leads to an evolution in rapidity, and is bounded from above as

\[
\frac{dS_n(x_0)}{d\chi} = \int_0^{x_0} dx \left[ q_n(x) + \bar{q}_n(x) \right] \equiv C(x_0) \leq 1 ,
\]

(31)
In a way, the analogue of the central charge, is played by the cumulative parton probability $C(x_0)$, with $C(\frac{1}{2}) = 1$ saturating the bound.

If we identify the logarithmic dependence on $P^+$ as an evolution in rapidity, then (31) can be viewed as the Kolmogorov-Sinai bound for the entanglement entropy for an $n$-meson in 2-dimensional QCD, and identify the Kolmogorov-Sinai entropy $S_{KS} = 1$ (sum of the positive Lyapunov exponents).

The bound (31) can be understood in the following way. For the 't Hooft wave functions, the reduced density matrix contains only one-body and zero body (vacuum) terms, therefore its Schmidt decomposition allows at most $2x_0\Lambda^-$ terms, which implies an upper-bound $S_n \leq \ln \Lambda^- + \ln 2x_0$. However, our result shows that this is an over-estimate. For the two-body wave function, it is the finite probability of the zero-mode contribution (vacuum state), that reduces the over-estimation. For three- and higher-body wave functions, we show in Appendix D, that the naive upper bound $\propto (k - 1) \ln \Lambda^-$, for a generic state where $k$ is the maximal number of partons, is also an over-estimate.

**Structure function:**

At low-$x$, (31) is the $n$-meson $F_2^\rho$ structure function

$$\frac{dS_n(x_0 \sim 0)}{d\chi} \sim x_0(q_0(x_0) + \bar{q}_0(x_0)) = F_2^\rho(x_0 \sim 0) ,$$

in agreement with the analysis in higher dimensions [12–15]. In 2D (32) measures the low-$x$ partons in the $n$-meson state

$$\frac{dS_n(x_0 \sim 0)}{d\chi} \sim 2C_n^2 \frac{x_0^{2\beta+1}}{2\beta + 1} ,$$

where we used that at the edges $x = 0$ and $x = 1$. The 't Hooft wave function has an asymptotic expansion, in terms of the dynamically generated coefficient $\beta$ as

$$\varphi_n(x) = C_n x^\beta, \quad \pi \beta \cot \pi \beta = -\frac{\pi m_Q^2}{2\sigma_T} + 1 .$$

A more refined analysis detailed in Appendix E, gives

$$S_n(x_0) = 2C_n^2 \frac{x_0^{2\beta+1}}{2\beta + 1} \left( \ln(e\Lambda^-) + 2\beta \frac{1 + (2\beta + 1) \ln \frac{1}{x_0}}{(2\beta + 1)} + O(x_0^2) \right).$$

for $\beta > 0$. The result is consistent with (32), if we note that the second contribution in (E2) is suppressed in the chiral limit, i.e. $\beta \sim m_Q/\sqrt{\sigma_T}$. In passing, we also note the non-commutativity of the chiral limit with the low-$x$ limit in 2D QCD.

For theories in which there are non-trivial logarithms running in rapidity, for example 4-dimensional QCD, (32) measures the growth of low-$x$ partons carried by the quark sea.
This is consistent with the forward meson-meson (elastic n-n→n-n) scattering amplitude in the Regge limit in 2D [28]

\[
\sigma_n(s) \sim \frac{1}{s} \Im A_n(s,0) \sim s^{-(2\beta+1)} \sim F_n^2(x_0 \sim 0) , \tag{36}
\]

with a negative Reggeon intercept \(\alpha_R = -2\beta\) (In 4D the forward limit is dominated by the Pomeron with positive intercept \(\alpha_P > 0\), with the stringy relation \(\alpha_R + 1 = \alpha_P\)). The forward elastic cross section, is a measure of the n-meson structure function. It is also consistent with the elastic 2D n-meson form factor \(F_n(-q^2) \sim 1/(-q^2)^{\beta+1}\), both of which are dominated by the t-channel single Reggeon exchange, which amounts to a full quantum open string exchange after re-summation, as we show below.

**Valence PDF:**

The longitudinal evolution of the entanglement entropy (29) with parton-\(x\), is highly non-linear

\[
\frac{dS_n(x_0)}{dx_0} = -\left( q_n(x_0) \ln q_n(x_0) + \bar{q}_n(x_0) \ln \bar{q}_n(x_0) \right) \\
+ \left( q_n(x_0) + \bar{q}_n(x_0) \right) \ln \left( \frac{\Lambda^-}{e^{\int_{x_0}^1 dx (q_n(x) + \bar{q}_n(x))}} \right) , \tag{37}
\]

with most of the non-linearity arising from the entanglement with the vacuum contribution in (24). For large rapidities \(\chi\), the longitudinal growth per unit rapidity is linear, and is a direct measure of the n-meson valence PDF

\[
\frac{d^2S_n(x_0)}{d\chi dx_0} = q_n(x_0) + \bar{q}_n(x_0) . \tag{38}
\]

We expect a similar relation to hold for more general wave functions, e.g. baryons and exotics.

**Scaling limit:**

Another interesting limit is the so-called scaling limit, which consists on zooming on the large-n meson states to exhibit the scale invariance of 2D QCD [28, 29]. More specifically, consider the limit \(\mu_n^2 = M_n^2/m_0^2 \to \infty\) with fixed ratio \(\xi = x\mu_n^2\), where \(m_0^2 = 2\sigma_T/\pi\). In this limit, the wave function approaches a universal function \(\phi(\xi)\):

\[
\varphi_n \left( \frac{\xi}{\mu_n^2} \right) \to \phi(\xi) . \tag{39}
\]

In this case, if we set \(x_0 = \frac{\xi_0}{\mu_n^2}\), then the cumulative parton distribution

\[
\int_0^{x_0} dx (q_n + \bar{q}_n) = \frac{2}{\mu_n^2} \int_0^{\xi_0} d\xi \phi^2(\xi) , \tag{40}
\]
and
\[
\int_0^{x_0} dx q_n \ln q_n = \int_0^{x_0} dx \bar{q}_n \ln \bar{q}_n = \frac{1}{\mu_n^2} \int_0^{\xi_0} d\xi \phi^2(\xi) \ln \phi^2(\xi) .
\] (41)

The leading $O(1/\mu_n^2)$ entanglement entropy in this case is therefore purely expressed in terms of the universal function
\[
\mu_n^2 S_{\pi}\left(\frac{\xi_0}{\mu_n^2}\right) \to 2 \ln e \Lambda - \int_0^{\xi_0} d\xi \phi^2(\xi) - 2 \int_0^{\xi_0} d\xi \phi^2(\xi) \ln \phi^2(\xi) .
\] (42)

For large $\xi_0$, the first term diverge linearly in $\xi_0$, while the second term diverges logarithmically.

**Pseudo-Goldstone state:**
In the limit $m_Q^2/2\sigma_T \ll 1$, 2D QCD admits a massless pseudo-Goldstone mode with a light front wavefunction $\varphi_0(x) = \theta(x \bar{x})$ (modulo the end points). This is a Berezinski-Kosterlitz-Thouless state. In 4-dimensional QCD, the pion is a true Goldstone mode, and massless even for a fixed and large constituent mass $m_Q$, yet the pion longitudinal wave-function is also totally delocalized in x-Bjorken with $\varphi_\pi(x) \approx \theta(x \bar{x})$ in the chiral limit, and for point-like interactions [30] (and references therein).

With this in mind, the density matrix for the pseudo-Goldstone state reads
\[
\rho_{\pi} = \frac{1}{\Lambda} \sum_{n=1}^{\infty} \sum_{x,x'} \theta(x \bar{x}) \theta(x' \bar{x'}) |x, \bar{x}\rangle \langle x', \bar{x}'| .
\] (43)

modulo the end-points. When traced over the interval $x_0 = 1 - x_0$, the entanglement entropy is
\[
S_{\pi}(x_0) = 2x_0 \ln \Lambda - (\bar{x}_0 - x_0) \ln(\bar{x}_0 - x_0) ,
\] (44)

which is considerably simpler than (29). The change in rapidity of the pion entanglement entropy
\[
\frac{dS_{\pi}(x_0)}{d\chi} = 2x_0
\]

This result is similar to the one we derive below for the entanglement entropy summed over the full Regge trajectory. This is perhaps the signature of the collective nature of the pseudo-Goldstone mode, on the light front. We note that in the massless Schwinger model, the light front wave function of the “meson” state with mass $m^2 = g^2/\pi$, is
\[
|\gamma\rangle = \frac{1}{\sqrt{\Lambda}} \sum_{0<x<1} |x, \bar{x}\rangle .
\] (45)

with the same entanglement entropy (44) as in the pseudo-Goldstone state.
IV. 2D QCD AS A STRING ON THE LIGHT FRONT

Two-dimensional QCD is non-conformal, but solvable in the large number of colors limit [21], as we discussed using the discretized light front quantization earlier. Remarkably, the solution in this limit is identical to that following from a 2-dimensional relativistic string with massive end-points [22]. To show this, we recall that the 2D light front Hamiltonian (squared mass) for a string with massive ends is [22]

\[ H_{LF} = \frac{m^2_Q}{x\bar{x}} + 2P^+\sigma_T|\vec{r}| \rightarrow \frac{m^2_Q}{x\bar{x}} + 2\sigma_T \left| \frac{id}{dx} \right| , \]  

(46)

with \( 0 \leq x = k^+/P^+ \leq 1 \) the momentum fraction of the quark (\( \bar{x} = 1 - x \) is that of the anti-quark) in a meson with longitudinal momentum \( P^+ \). The relative light-front distance \( \vec{r} \rightarrow id/dk^+ \) is conjugate to \( k^+ \). The string tension is \( \sigma_T \). The eigenstates of (46) solve

\[ H_{LF}\varphi_n(x) = \left( \frac{m^2_Q}{x\bar{x}} + 2\sigma_T \left| \frac{id}{dx} \right| \right) \varphi_n(x) = M_n^2 \varphi_n(x) , \]  

(47)

with squared radial meson masses as eigenvalues. The confining potential in the Bjorken-x representation is given by the Fourier transform

\[ \langle x|P^+|r^-||y \rangle = \int_{-\infty}^{+\infty} \frac{dq}{2\pi} e^{iq(x-y)} |q| \rightarrow \text{PV} \frac{-1}{\pi(x-y)^2} + \frac{-1}{\pi x\bar{x}} , \]  

(48)

with the principal value prescription. Using (48) in (47) yields ’t Hooft equation (12) with the gauge coupling identified through \( \sigma_T = g_{T+1}^2 N_c/2 \). A brief semi-classical analysis of the string states is given in Appendix A. In sum, we can regard the even and odd solutions of the ’t Hooft equation, as the even and odd standing waves of a meson as a string, flying on the light front with either Dirichlet or Neumann boundary conditions modulo the small mass corrections at the edges.

A. Stringy entanglement: Resummed Regge trajectory

In the eikonalized approximation, dipole-dipole (open string) scattering in 2D QCD, sums over all n-meson (Reggeons) exchanges in the t-channel. This re-summed exchange is string-like. To describe it, we need to resum over the full meson Regge trajectory in 2D QCD. However, this is not needed as we now show.
Indeed, the full density matrix of the string $\hat{\rho}_{\text{string}}$, can be reconstructed from the n-meson density matrix $\rho_n$, by noting that each of the meson state on the Regge trajectory, maps onto a stationary state of the open string with massive end-points. The orthonormality and completeness of these states, imply that the full string density matrix is diagonal in-$n$,

$$
\hat{\rho}_{\text{string}} = \frac{1}{\Lambda} \sum_{n=1}^{\infty} \sum_{x,x'} \varphi_n^\dagger(x') \varphi_n(x) |x,\bar{x}\rangle \langle x',\bar{x}'| .
$$

(49)

Using the completeness relation

$$
\sum_n \varphi_n^\dagger(x) \varphi_n(x') = \delta(x - x') ,
$$

(50)

(49) is the projection operator onto the two-body states

$$
\hat{\rho}_{\text{string}} = \frac{1}{\Lambda} \sum_{0 < \bar{x} < 1} |x,\bar{x}\rangle \langle x,\bar{x}| .
$$

(51)

The reduced density matrix, following by tracing over the segment $\bar{x}_0 = 1 - x_0$, yields the entanglement entropy

$$
S(x_0) = 2x_0 \ln \Lambda - (\bar{x}_0 - x_0) \ln(\bar{x}_0 - x_0) .
$$

(52)

which is independent of the mass at the end-points of the string. It is surprisingly similar to (44) for the pseudo-Goldstone mode, even though the string density matrix (49) is diagonal in longitudinal space, while the one associated to the pseudo-Goldstone mode (43) is off-diagonal.

**B. Stringy entanglement: Multi-meson state**

The above density matrix takes into account only single meson states. As we argued earlier, this entangled density matrix captures a DIS measurement of the quark distribution in a meson state, in the interval of length $x_0$ in parton-x. Suppose that we want to use a DIS measurement of the quark distribution for the same $x_0$ interval, in a state composed of many identical hadrons flying on the light front (a 2D nucleus, or a 4D nucleus reduced to its longitudinal components). For that, we extend our analysis to multi-meson states, with the corresponding Fock-space spanned by all the mesons. Using the completeness relation, it is clear that the corresponding density matrix is now given by

$$
\rho = \frac{1}{\text{Dim}} \sum_k \rho_k ,
$$

(53)
where

\[ \rho_k = \frac{1}{\text{Dim}} \sum_{0 < x_1 < x_2 < \ldots < x_k < 1} |x_1, x_2, \ldots x_k \rangle \langle x_1, \ldots x_k | , \]  

(54)

are spanned by all \( k \)-particle tensor product of the fundamental basis, under the constraint that the same \( |x, \bar{x} \rangle \) appears at most \( N_c \) times. For large \( N_c \), each can appear infinitely many times.

**\( N_c = 1 \) case:**

To help understand the book-keeping for general \( N_c \), let us first consider the case with \( N_c = 1 \), with no 2 mesons allowed to occupy the same longitudinal phase space region. In this case \( \text{Dim} = 2^{\Lambda^-} \). After tracing over the segment \( (1 - x_0) \), the reduced density matrix for this case is

\[ \rho(x_0) = \frac{1}{2^{\Lambda^-}} \sum_{0 \leq k \leq x_0 \Lambda^-} C_{N_1}^{N_1 + k} \sum_{k=0}^{N_1} \sum_{0 \leq x_1 < x_2 < \ldots < x_k < x_0} |x_1, \ldots x_k \rangle \langle x_1, \ldots x_k | , \]  

(55)

where \( N_1 = (1 - x_0)\Lambda^- \). After summing over all \( k \) with the help of the binomial theorem, and replacing \( k - i \) by \( \tilde{k} \), the result is

\[ \rho(x_0) = \frac{1}{2^{x_0 \Lambda^-}} \sum_{0 \leq k \leq x_0 \Lambda^-} \sum_{0 \leq x_1 < x_2 < \ldots < x_k < x_0} |x_1, \ldots x_k \rangle \langle x_1, \ldots x_k | . \]  

(56)

This is simply the projection operator onto the subspace with \( x_0 \Lambda^- \) digits, corresponding to the part of the Hilbert space kept. The dimension of the space is \( \text{Dim}(x_0) = 2^{x_0 \Lambda^-} \), and the corresponding entanglement entropy is now

\[ S_E = \ln \text{Dim}(x_0) = \ln 2 \times x_0 \Lambda^- . \]  

(58)

This is the **maximal entropy**, following from the reduction of any density matrix to the small-\( x \) interval.

**General \( N_c \) case:**

For general \( N_c \), and after tracing over the \( (1 - x_0) \), we clearly get again the projection operator onto the subspace spanned by all the \( |x_1, \ldots x_k \rangle \), with the constraint that \( x_k \leq x_0 \) and that each \( x_i \) appears at most \( N_c \) times, due to the fermionic character of the underlying quark constituents in any of the colorless meson. The dimension of this Hilbert space is simply \( (N_c + 1)^{x_0 \Lambda^-} \), hence

\[ S_E = \ln(N_c + 1) \times x_0 \Lambda^- . \]  

(59)
The rate of change with rapidity of the string entanglement entropy $S_E(x_0)$, the sum total of all entanglements along each of the exchanged Regge trajectories for fixed $x_0 \leq \frac{1}{2}$, is extensive in $\Lambda^{-}$

$$\frac{dS_E(x_0)}{d\chi} = \ln(N_c + 1)x_0\Lambda^{-}.$$  \hspace{1cm} (60)

In the low-$x$ regime, dominated by the vacuum zero-modes on the light front, (60) simplifies to

$$\frac{dS_E(x_0 \sim 0)}{d\chi} = \ln(N_c + 1)\frac{1}{2}e^{-\chi x_0} = \frac{1}{2} \ln(N_c + 1),$$  \hspace{1cm} (61)

using the DIS identification $x_0 = \frac{1}{2} e^{-\chi}$.

**Kolmogorov-Sinai bound [23]:**

The rate of increase of $S_E(x_0 \sim 0)$ with the rapidity $\chi$, saturates the Kolmogorov-Sinai bound at low-$x$, with $S_{KS} = \frac{1}{2} \ln(N_c + 1)$. The longitudinal quantum entanglement, for the re-summed mesons (Reggeon) as open strings in 2D, is to be compared to the transverse quantum entanglement of $\frac{D_\perp}{6}$ for the re-summed glueballs (Pomeron) as a closed string exchange in $2 + D_\perp$ dimensions [12–14]. At low-$x$, the entanglement is fixed by the $D_\perp$ transverse quantum vibrations of the string light-like (analogue of Luscher term space-like).

**Classical string entropy:**

Away from low-$x$, the change in $S_E(x_0)$ is extensive in the invariant cut-off $\Lambda^{-}$, e.g.

$$\frac{dS_E(x_0)}{d\chi} = \ln(N_c + 1)x_0\Lambda^{-}$$

This scaling is commensurate with the growth of the string entropy $S_S$ under large boosts. Indeed, a free string as a chain undergoing random walks in 1D, generates $N_S = 2^{L/l_S}$ states (for a free string back-tracking is allowed). The corresponding string entropy $S_S = \ln N_S = \ln 2 L/l_S$. Under large longitudinal boosts $P^+$, the longitudinal length of the string expands (recall that the string bits are considered wee [19, 31], they carry low momentum, and are oblivious to large boosts). As a result, $L/l_S = P^+/0_{k^+} = x_0\Lambda^{-}$ counts the number of string bits or wee partons, and the string entropy is $S_S = \ln 2 x_0\Lambda^{-}$, which is seen to scale similarly to (60), in particular

$$\frac{dS_E(x_0)}{d\chi} = \frac{\ln(N_c + 1)}{\ln 2}S_S.$$  \hspace{1cm} (62)

This large and quantum *wee* entropy stored in the longitudinal evolution in rapidity of open strings (Reggeons), when released in a collision, may contribute to the fast scrambling of information in hadronic collisions at ultra-relativistic energies. Perhaps more so, then the quantum *wee* entropy released from the evolution in rapidity of closed strings...
(Pomerons) [12], provided that $x_0$ is not asymptotically small as in (61). We note that the string bits interactions may hamper the back-tracking, and somehow reduce the entanglement rate in (62).

**Bekenstein-Bremermann bound [6, 7]:**
Quantum information theory sets a bound on the maximum rate of flow of information $I$ in physical systems, as first noted by Bremermann for single channel systems, based on an argument using Shannon entropy and the quantum uncertainty principle [6]. The bound was revisited by Bekenstein on general grounds, using the maximum entropy storage in a black-hole and causality [7]

$$\frac{dS_{\text{max}}}{dt} \leq 2\pi E \to 2\pi TS \ .$$

The rightmost equality follows from the second law. (Here information $I$ is interpreted as entropy in bits units or $I/S = \ln_2 e$). If we recall that the rapidity $\chi$ relates to the Gribov time $t_\chi = \sqrt{\alpha'} \chi$ with $\alpha' = l_S^2$ the open string Regge slope [14, 32], then a comparison of (62) with (63) shows that for $N_c = 1$, the Bekenstein-Bremermann bound is saturated, with $T = T_H = 1/(2\pi l_S)$ the Hagedorn temperature (equivalently, the temperature at the Rindler horizon of a black-hole). Remarkably, for the multi-meson state result with $N_c > 1$ in (62), the bound is still maintained, provided that the temperature exceeds (logarithmically) the Hagedorn temperature.

**V. CONCLUSIONS**

In the large number of colors, the 2-particle sector of 2D QCD on the light front decouples. The eigen-modes in this sector, have a dual description in terms of partons or string modes. We have shown that in the partonic language, the entanglement in longitudinal momentum is captured by an exact reduced density matrix, that is a tensor product of both the valence and vacuum states. The entanglement entropy for a single meson with a single cut in parton-x, as probed by DIS kinematics, is a non-linear function of the meson PDF.

For fixed parton-x, the evolution in rapidity of the single meson entanglement entropy, is the *cumulative* quark single PDF. It is bound by a Kolmogorov-Sinai entropy of 1. At low parton-x, it reduces to the longitudinal structure function, as measured in DIS scattering. It is in agreement with the Regge behavior of the pertinent meson-meson scattering in 2D QCD. Alternatively, for fixed rapidity, the evolution in parton-x is shown to probe directly the meson singlet PDF.

The sum total of the entanglement entropies for a fixed Regge trajectory, is string-like and extensive with the rapidity, as noted in 4D. We have suggested that DIS scattering on
a nucleus in 2D, can be modeled by DIS scattering on a multi-hadron state composed of 2D mesons, modulo Fermi statistics (amusingly shared by mesons through longitudinal space exclusion for $N_c = 1$). The evolution in rapidity of the ensuing entanglement entropy, is found to be extensive in the longitudinal string entropy in 2D. The rate of change of this entropy matches the maximum rate of quantum information flow, as given by the Bekenstein-Bremermann bound.

A highly boosted multi-meson state in 2D (a sort of 2D nucleus as all hadrons are similar on the light front), exhibits a growth rate in its wee parton entanglement entropy, that is only matched by the largest information rate flow allowed by the quantum laws of physics, a fit only exhibited by gravitational black holes. Remarkably, this flow exhibits an energy cost which is fixed by the Hagedorn temperature of the underlying longitudinal string.

The highly entangled wee partons in a boosted string as a mock nucleus, carry an entanglement entropy that is commensurate with the classical string entropy $S_S$. Their prompt release by smashing, in current colliders at large rapidities $\chi = \ln s$, may explain why a large quantum entanglement entropy of about $\chi S_S$ is promptly released, over a short time scale $1/l_S$, and at temperatures in (slight) excess of the Hagedorn temperature $T_H = 1/(2\pi l_S)$.

We will elaborate further on some of these issues and their extension to 4D next.

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**Appendix A: WKB analysis of the string states**

In this Appendix, we qualitatively review the semi-classical solutions to 2D QCD, using the dual string form. In particular, the masses are given by the WKB quantization condition

$$\int_{x_-}^{x_+} dx \left( M_n^2 - \frac{m_Q^2}{x} \right) = M_n^2 - m_Q^2 \ln \left( \frac{x_+ - x_-}{x_+ - x_-} \right) = 2\pi \sigma T n , \quad (A1)$$

with the turning points

$$x_\pm = \frac{1}{2} \left( 1 \pm \left( 1 - \frac{4m_n^2}{M_n^2} \right)^{\frac{1}{2}} \right) , \quad (A2)$$
and with $M_n \geq 2m_Q$. The mass gap vanishes for $m_Q \to 0$ with a radial Regge trajectory $M^2_n = n/\alpha'$, and $\alpha' = 1/2\pi\sigma_T$ the slope of the open bosonic string.

A simple understanding of the light front wavefunctions, can be obtained directly from (47) by noting that for $m_Q^2/2\sigma_T \gg 1$, the mass contribution acts as a confining potential at the end-points $x = 0, 1$, with $\varphi_n(x)$ standing waves solutions to

$$
\left| \frac{id}{dx} \varphi_n(x) \right| \approx \frac{M_n^2}{2\sigma_T} \varphi_n(x),
$$

with Dirichlet boundary conditions. The normalized solutions are $\varphi_n(x) \approx \sqrt{2} \sin((n + 1)\pi x)$. A simple estimate of the mass correction for large $n$ follows from first order perturbation theory $M^2_n \approx n/\alpha' + 2m_Q^2 \ln n$. In the opposite limit of $m_Q^2/2\sigma_T \ll 1$, the confining potential can be ignored to first approximation, in which case the standing waves follow from Neumann boundary conditions, with $\varphi_n(x) \approx \sqrt{2} \cos(n\pi x)$, with an identical reggeized semi-classical spectrum. The effects of the mass is to cause a rapid distortion of the light front wavefunction in a narrow region of $x$ near the end-points (see below).

Appendix B: Symmetric interval

The reduced density matrix in parton-x, was defined by tracing over the length $\bar{x}_0 = 1 - x_0$ for fixed $x_0 \leq 1/2$, as motivated by a DIS measurement. This reduction is asymmetric with respect to the quark-antiquark content of the light front meson wavefunction. A more symmetric but academic reduction, is to trace over the symmetric length $x_0 < x < \bar{x}_0$. The reduced density matrix is then

$$
\hat{\rho}_S(n) = \int_{x_0}^{\bar{x}_0} q_n(x)|0\rangle_S \langle 0|_S + |\bar{\Phi}\rangle \langle \bar{\Phi}|,
$$

where one has

$$
|\bar{\Phi}\rangle = \frac{1}{\sqrt{\Lambda}} \left( \sum_{0<x<x_0} + \sum_{\bar{x}_0<x<1} \right) \varphi_n(x)|x, \bar{x}\rangle.
$$

The above density matrix represents a binomial distribution, with the independent pair of eigenvalues $(p_n(x_0), 1 - p_n(x_0))$ where

$$
p_n(x_0) = \int_0^{x_0} dx \left( q_n(x) + \bar{q}_n(x) \right).
$$

The corresponding entanglement entropy is therefore

$$
S_S(n, x_0) = -p_n(x_0) \ln p_n(x_0) - (1 - p_n(x_0)) \ln(1 - p_n(x_0)).
$$
and is independent of $\Lambda^-$. As $x_0 \to 0$, one has

$$p_n(x_0) \to \frac{2C_n^2 x_0^{2\beta+1}}{2\beta + 1},$$

thus

$$S_S(n, x_0) = 2C_n^2 x_0^{2\beta+1} \ln \frac{1}{x_0} - 2C_n^2 x_0^{2\beta+1} \ln \frac{2C_n^2}{e(2\beta + 1)} + O(x_0^{4\beta+2}).$$

The leading contribution is also proportional to $x_0^{2\beta+1} \ln \frac{1}{x_0}$.

**Appendix C: General interpolating interval**

In this Appendix, we trace over an asymmetric interval centered around $\frac{1}{2}$, that interpolates between the symmetric and asymmetric reduction discussed above. In this case, the reduced density matrix traced over $[x_0, \bar{x}_0 + \delta]$ with $0 < \delta < x_0$, is now

$$\hat{\rho} = \frac{1}{\Lambda^-} \sum_{x_0 < x < 1-x_0} |\varphi_n(x)|^2 \langle 0 |_{[x_0, 1-x_0]} \langle 0 |_{[0, x_0]} + \frac{1}{\Lambda^-} \sum_{x_0 - \delta < x < x_0} |\varphi_n(x)|^2 |x\rangle \langle x|$$

$$+ \frac{1}{\Lambda^-} \sum_{1-x_0 < x < 1-x_0 + \delta} |\varphi_n(x)|^2 \langle \tilde{x} | \langle \tilde{x}|$$

where the state $|\tilde{\Phi}\rangle$ reads

$$|\tilde{\Phi}\rangle = \frac{1}{\Lambda^-} \left( \sum_{0 < x < x_0 - \delta} + \sum_{1-x_0 + \delta < x < 1} \right) \varphi_n(x) |x, \bar{x}\rangle.$$

The entanglement entropy is therefore given by

$$S_n(x_0, \delta) = \ln \Lambda^- \int_{x_0-\delta}^{x_0} dx \left( q_n(x) + \bar{q}_n(x) \right) - \int_{x_0-\delta}^{x_0} dx \left( q_n \ln q_n + \bar{q}_n \ln \bar{q}_n \right)$$

$$- \ln \left( \int_{x_0}^{\frac{1}{2}} dx (q_n + \bar{q}_n) \right) \left( \int_{x_0}^{\frac{1}{2}} dx (q_n + \bar{q}_n) - \ln \left( \int_{0}^{x_0-\delta} dx (q_n + \bar{q}_n) \right) \right).$$

Clearly, it interpolates between the two special cases considered above. When $\delta = x_0$, it reduces to the totally asymmetric case, while for $\delta = 0$, it reduces to the symmetric case. The coefficient of the $\Lambda^-$ measures this asymmetry

$$\frac{dS_n(x_0, \delta)}{d \ln \Lambda^-} = \int_{x_0-\delta}^{x_0} dx \left( q_n(x) + \bar{q}_n(x) \right),$$

which is always less or equal to 1 but non-negative. It vanishes only for $\delta = 0$. 
Appendix D: Naive bound for an $n$-parton state

Consider a generic wave function with maximally $n$-partons
\[ |\Phi\rangle = \sum_{i=1}^{n} \frac{1}{\sqrt{\Lambda}} \sum_{x_1,...x_i} \varphi_i(x_1,...,x_i)|x_1,...,x_i\rangle \]  
(D1)

After tracing over $A = [x_0, 1 - x_0]$, the reduced density matrix has the form
\[ \hat{\rho}_A = \sum_{i,j} \rho_{ij} |i\rangle\langle j| , \]  
(D2)

with $|i\rangle$ a generic state with $i$ particles. It is important to observe, that the diagonal terms follow from tracing $|i\rangle\langle i|$, since partial tracing cannot change the difference in particle numbers. Therefore, the diagonal terms form a reduced density matrix $\hat{\rho}_{Adia}$, which contains more entropy compared to the full reduced density matrix, in general. They could be used to derive a super-bound. Specifically, if we retain only the diagonal terms, the reduced density matrix reads
\[ \hat{\rho}_{Adia} = \sum_{i=0}^{n-1} \frac{1}{(\Lambda^2)^{i-1}} |x_1',x_2',...,x_i'|\langle x_1',x_2',...,x_i'| \]
\[ \times \sum_{j=i+1}^{n} \int_{y \in E^{(j)}_{i}(x,x')} dy \varphi_j\dagger(x_1',...,x_i';y_{i+1},...,y_j)\varphi_j(x_1',...,x_i';y_{i+1},...,y_j) , \]  
(D3)

with $E^{(j)}_{i}(x,x')$ the region of the $j$-particle phase space which should be traced over, for a reduction to the $i$-body phase spaces. Performing another diagonal approximation, the entanglement entropy is bounded by
\[ S_E(x_0) \leq \ln \Lambda^{-n} \sum_{i=0}^{n-1} i p_i(x_0) + C . \]  
(D4)

Here, $p_i(x_0)$ is the sum of the cumulative probabilities
\[ p_i(x_0) = \sum_{j=i+1}^{n} \int_{x \in A^j_{i}(x_0)} dx_1,...,x_j |\varphi_j(x_1,...,x_j)|^2 , \]  
(D5)

where $A^j_{i}(x_0)$ is the part of the $j$-body phase space that after tracing, reduces to the $i$-body state. While the sum over all the probabilities is less then 1, the sum over the $i$-weighted probabilities is not a priori less than 1.

Appendix E: Low-x analysis in 2D QCD
The low-x analysis of the entanglement entropy in the single and asymmetric cut interval $A = [0, x_0 \leq \frac{1}{2}]$, can be carried exactly for $x_0 \to 0$, in 2D QCD. More specifically, using (34) allows to unwind each contributions in (29) as

$$\int_{x_0}^{x_0} dx \left[ q_n(x) + \bar{q}_n(x) \right] = 2C^2_n \frac{x_0^{2\beta+1}}{2\beta+1} + O(x_0^2),$$

$$- \int_{x_0}^{x_0} dx \left[ q_n(x) \ln q_n(x) + \bar{q}_n(x) \ln \bar{q}_n(x) \right] = 4\beta C^2_n 1 + (2\beta + 1) \ln \frac{1}{x_0} x_0^{2\beta+1} + O(x_0^2),$$

$$- \int_{x_0}^{\frac{1}{2}} dx \left[ q_n(x) + \bar{q}_n(x) \right] \ln \int_{x_0}^{\frac{1}{2}} dx \left[ q_n(x) + \bar{q}_n(x) \right] = 2C^2_n \frac{x_0^{2\beta+1}}{2\beta+1} + O(x_0^2),$$

(E1)

and therefore, for $\beta > 0$

$$S_n(x_0) = 2C^2_n \frac{x_0^{2\beta+1}}{2\beta+1} \left( \ln(e\Lambda^*) + 2\beta 1 + (2\beta + 1) \ln \frac{1}{x_0} \right) + O(x_0^2),$$

(E2)

which is the result quoted in the text.

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