Benchmarking Hall-Induced Magnetoacoustic to Alfvén Mode Conversion in the Solar Chromosphere

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ABSTRACT

A 2.5D numerical model of magnetoacoustic-Alfvén linear mode conversions in the partially ionised low solar atmosphere induced by the Hall effect is surveyed, varying magnetic field strength and inclination, and wave frequency and horizontal wave number. It is found that only the magnetic component of wave energy is subject to Hall-mediated conversions to Alfvén wave-energy via a process of polarisation rotation. This strongly boosts direct mode conversion between slow magnetoacoustic and Alfvén waves in the quiet low chromosphere, even at mHz frequencies. However, fast waves there, which are predominantly acoustic in nature, are only subject to Hall-induced conversion via an indirect two-step process: (i) a geometry-induced fast-slow transformation near the Alfvén-acoustic equipartition height $z_{eq}$; and (ii) Hall-rotation of the fast wave in $z > z_{eq}$. Thus, for the two-stage process to yield upgoing Alfvén waves, $z_{eq}$ must lie below or within the Hall-effective window $0 \leq z \leq 700$ km. Magnetic field strengths over 100 G are required to achieve this. Since the potency of this Hall effect varies inversely with the field strength but directly with the wave frequency, only frequencies above about 100 mHz are significantly affected by the two-stage process. Increasing magnetic field inclination $\theta$ generally strengthens the Hall convertibility, but the horizontal wavenumber $k_x$ has little effect. The direct and indirect Hall mechanisms both have implications for the ability of MHD waves excited at the photosphere to reach the upper chromosphere, and by implication the corona.

Key words: (magnetohydrodynamics) MHD, waves, Sun: chromosphere, Sun: photosphere

1 INTRODUCTION

The solar atmosphere is dominated by magnetohydrodynamic oscillations (Banerjee et al. 2007). Many studies have highlighted the potential significance of MHD waves in transport of non-thermal energy, and their contribution to sustaining the temperature structure of the solar atmosphere (Alfvén 1947; Schwarzschild 1948; Osterbrock 1961; De Pontieu et al. 2005; Matsumoto & Shibata 2010; Morton et al. 2012). However, the ultimate source and the exact energy-distribution mechanism of these oscillations are yet to be unveiled.

The three classic MHD wave types – slow, intermediate (or Alfvén) and fast – behave differently as they propagate up from the solar photosphere through the chromosphere and towards the corona. The fast and slow, collectively known as the magnetoacoustic waves, are both compressive to various extents at various levels. In the high-$\beta$ photosphere and lower chromosphere, the fast wave is essentially acoustic in nature, and readily shocks and thereby heats the atmosphere locally. Depending on where the Alfvén-acoustic equipartition level sits, the acoustic ( erstwhile fast) wave may reach it in a shocked or unshocked state. In either case though, it splits into distinct acoustically-dominated (now slow) and magnetically dominated (now fast) components in the overlying low-$\beta$ upper chromosphere. In the case of shocks, the resulting slow wave is smoothed and soon re-shocks, but the fast wave remains a shock (Pennicott & Cally 2021). However, the fast wave in the upper chromosphere is strongly reflected by the steep Alfvén speed gradient, or the transition region if it reaches it, and so contributes little directly to the corona (Srivastava et al. 2021). This leaves the Alfvén wave to do the ‘heavy lifting’ (Mathioudakis et al. 2013).

Since its discovery by Alfvén (1942), many observational and theoretical studies have been launched to corroborate the existence and perhaps preponderance of Alfvén waves in the upper atmosphere and explain the extreme temperature of the corona (Alfvén 1947; Kuperus et al. 1981; Poedts 2002; Cranmer & van Ballegooijen 2005; De Pontieu et al. 2005, 2007; Tomczyk et al. 2007). While some works concentrate on uninterrupted transmission of energy by Alfvén waves from photospheric heights all the way to the corona (Cranmer & van Ballegooijen 2005), others have been investigating the significance of mode converted Alfvén waves elicited from originally magnetoacoustic oscillations (Cally & Goossens 2008; Cally 2017; Cally & Khomenko 2015, 2018; González-Morales et al. 2019; Raboonik & Cally 2019).

In ideal MHD, magnetoacoustic oscillations can undergo mode transformation where the local sound speed matches the radially variant Alfvén speed (Schunker & Cally 2006). This may occur even in 2D. However, ideal MHD Alfvén-magnetoacoustic transformations can only happen in 3D (Cally & Goossens 2008; Cally & Hansen 2011). On the other hand, non-ideal MHD mode conversion mechanisms pervade the partially ionised solar photosphere/lower...
chromosphere. Amongst these processes, the Hall effect is of particular interest in that it exerts a conservative force (for a full discussion on Hall MHD see Pandey & Wardle 2008), leading to lossless conversions of energy. Thus, compressive forms of wave-energy can potentially wind up as Alfvén waves higher up in the atmosphere.

In a magnetically driven fluid, any kind of fluctuation seeds MHD waves. Thus, it is no surprise that the highly energetic small- and large-scale granulation arising from convective motions in the photosphere have been suspected as viable wave drivers with enough kinetic energy to fuel the hot corona. Withbroe & Noyes (1977) estimate the total coronal energy loss to be of order of $10^5$ erg cm$^{-2}$ s$^{-1}$ in the quiet sun and coronal holes, and $10^7$ erg cm$^{-2}$ s$^{-1}$ in active regions.

Premised on the theory of sound generation developed by Lighthill (1952), Musielak et al. (1994) show that subsonic photospheric convective motions in the Sun can trigger the production of acoustic waves of about $5 \times 10^7$ erg cm$^{-2}$ s$^{-1}$ with a frequency peak around 100 mHz. Thus, if transportable, this would be more than enough energy required to heat the solar corona.

In a different study, Cranmer & van Ballegooijen (2005) isolate transverse waves by modelling the solar granular motions as a non-rotational stochastic velocity field and use it as boundary conditions to excite ‘Alfvénic’ waves at the photosphere. Their model estimates a total wave-energy flux density of about $10^8$ erg cm$^{-2}$ s$^{-1}$ at the photosphere (see Figure 12 in that paper). The transverse oscillations are then traced all the way to interplanetary space and dissipated using several nonlinear turbulent damping processes. Their results shows comparable heating rates to empirical data measured at 2R$_\odot$.

However, high-$\beta$ transverse oscillations – which Cranmer & van Balleloogijen (2005) refer to as ‘Alfvénic’ waves – are a mixture of Alfvén and magnetic slow waves, depending on their local polarisation. That is, Alfvén waves are polarised along $k \times B$ where $k$ is the wavevector as defined in the WKB approximation at sufficiently high frequencies, while high-$\beta$ slow waves oscillate in the direction of $k \times (k \times B)$. Part of the present study is motivated by the prospect of obtaining upgoing Alfvén waves out of input slow waves at the base.

The remainder of the paper is laid out as follows. first, an exposition of the model including the choice of the boundary conditions necessary to ensure exclusive excitation of slow waves is presented in Section 2. We then proceed to presenting the results in Section 3. Finally, we conclude the paper with a discussion in Section 4.

2 MATHEMATICAL MODEL

Our setup consists of the plane-stratified quiet Sun atmospheric Model C of Vernazza et al. (1981), permeated by a uniform 2D magnetic field of selectable inclination $\theta$ from the vertical, prescribed by $B_0 = B_0(\sin \theta, 0, \cos \theta)$. The model covers a solar region including the partially ionised photosphere up to mid chromosphere, spanning $\theta - \sin \theta \cos \theta = -0.5$ Mm to $z_{eq} = 1.6$ Mm, to which we will refer as the ‘main box’. The interval over which the Hall effect is of major significance sits inside the partially ionised patch, and in line with Cally & Khomenko (2015), will be called the ‘Hall window’.

The motivation here is to set up numerical MHD-wave dispersion experiments as they pass through a gravitationally stratified semi-realistic solar setting to isolate the role of the Hall effect, and determine its efficacy in mode converting high-$\beta$ slow/fast waves into upgoing Alfvén waves. The phrase ‘semi-realistic’ implies that despite the employed realistic model atmosphere, appropriate artificial isothermal boundaries are imposed to craft pure slow or fast waves at the base, and annihilate any incoming waves at the top. Even though the enforced isothermality condition may be somewhat justifiable at $\beta \gg 1$, it most certainly does not capture the essence of low-$\beta$ regions. Therefore, we entirely ignore the impacts of the transition region and the corona. The dynamic and highly structured nature of the chromosphere is also ignored.

Upon entering the Hall window, an input slow or fast wave encounters two possibilities depending on the location where the sound and Alfvén speeds $c_s$ and $a$ are equal ($\beta = 2/\gamma$). This special location is known as the equipartition level $z_{eq}$. If $z_{eq}$ is situated in the Hall window, then the wave can undergo another conversion induced by geometric effects, which may have vital implications in the production of Alfvén waves. This process is referred to as a two-stage conversion and is addressed in Section 3.2.

To set the scene we first introduce the bifurcation diagram of MHD, dispersion diagrams offer revealing insights into the behaviour of individual modes in $z$-$k_z$ phase space. Figure 1 depicts the ray trajectories of the three MHD modes at 12 mHz, where distinctly clear wave types (based on their loci in the diagram) are labelled by the first letter of their names (–Alfvén, Fast and Slow – see equation (22) of Raboonik & Cally 2019, and the dispersion diagrams therein). The solid line signifies the equipartition level. As can be seen, the curves belonging to the slow and the Alfvén waves are nearly coincident up to $z_{eq}$ whereas the upgoing fast and Alfvén trajectories come very close only past $z_{eq}$. These extended regions of adjacency indicate similar asymptotic behaviours between the participant waves and provide the capacity for mode conversions to take place. However, whether or not any transformation will actually transpire comes down to availability of conversion mechanisms in the system.

In the absence of partial ionisation processes such as the Hall effect, the magnetoacoustic-Alfvén transformation can only occur in full 3D, provided the magnetic field $B_0$, gravity $g$ and the wave vector $k$ are not coplanar. However, upon aligning these vectors into the same plane while allowing for the oscillations to develop in 3D, the system takes on the so-called 2.5 dimensionality. Thence, owing to its unique polarisation, the Alfvén wave polarised in the perpendicular direction would decouple identically from the magnetoacoustic waves that are polarised entirely in the plane. Thus, despite the mode transformation potentiality of the system as suggested by Figure 1, there would be no way for the magnetoacoustic oscillations to communicate and convert to Alfvén waves and vice versa. Nevertheless, the slow and the fast waves can still mode convert to one another at $z_{eq}$ (Schunker & Cally 2006).

On the other hand, once partial ionisation is activated, the Hall effect can furnish a unique inter-modal communication means for magnetoacoustic-Alfvén conversions (Cally & Khomenko 2015). In this paper, $B_0$ and $k$ are confined in the $x-z$ plane, hence any oscillation along the $y$-axis is readily identified as the Alfvén type, whereas the polarisation vectors of the pure fast and slow waves will be confined in the same plane as $B_0$.

There are two significant nuances in the present study in contrast with Raboonik & Cally (2019). First, the incorporated realistic anisothermal atmospheric model, and second, here we dispense with the ‘small Hall parameter $\epsilon_H$’ assumption which was crucial to the semi-analytical solutions in that paper. The latter improvement allows the waves to remain transformable across the conversion zone and continuously exchange energy in a feedback loop.

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It is easy to think that the distinction between slow and Alfvéén waves is moot, given their very similar asymptotic behaviours at high plasma-β. However, Fig. 1 suggests that they are very different globally. In the absence of any mode conversion, the slow and Alfvén loci are very different around and above \( z_{\text{eq}} \), with implications for upper atmospheric heating. Any conversion at low altitudes between slow and Alfvén therefore has global implications.

### 2.1 Governing Equations

Assuming time-homogeneity in the time interval of interest, i.e., \( \partial / \partial t = -i\omega \), and a static equilibrium state, i.e., \( v_0 = 0 \), the set of linearised Lagrangian MHD equations governing the system in the Cartesian coordinates is given by,

\[
-i\omega \rho_0 \xi = -\nabla p_1 + \rho_1 \mathbf{g} + j_1 \times \mathbf{B}_0 \tag{1a}
\]

\[
p_1 = -\nabla \cdot (\rho_0 \xi) \tag{1b}
\]

\[
\mathbf{b} = \nabla \times \left( \xi \times \mathbf{B}_0 - \frac{i\rho_0 \mu_0 H}{\omega \rho_0 B_0} j_1 \times \mathbf{B}_0 \right) \tag{1c}
\]

\[
j_1 = \frac{1}{\mu_0} \nabla \times \mathbf{b} \tag{1d}
\]

\[
p_1 = -\xi \cdot \nabla \rho_0 - \gamma \rho_0 \nabla \cdot \xi. \tag{1e}
\]

where \( \xi \) is the vector plasma displacement, \( \mathbf{b} \) is the perturbed magnetic field, \( \mu_0 \) is the magnetic permeability in vacuum, \( \mu_0 = \rho_0 B_0 / \mu_0 n_e e \) is the Hall parameter (plotted in Fig. 2 for two magnetic field strengths), and the symbols indexed with zero and one indicate the equilibrium and the perturbed quantities, respectively. Now, on account of the 2.5D setup and the symmetry of the system across the \( x = y \) plane, we have

\[
\xi = (\xi(z), \eta(z), \zeta(z)) e^{i(k_x x - \omega t)}. \tag{2}
\]

Combining equations (1) and (2) into a single vectorial differential equation, one arrives at a set of three second order ODEs that fully describe the behaviour of \( \xi, \eta, \) and \( \zeta \) all over the main box (see Appendix B for a derivation).

The significance of the Hall effect at any particular position and wave frequency is best assessed via the dimensionless ‘Hall parameter’ \( \epsilon_H = \omega / f_i \Omega_i = \omega \mu_H / \rho a^2 \), where \( \Omega_i = ZeB_0 / m_i \) is the ion cyclotron frequency, \( f_i \) is the ionisation fraction, \( m_i \) is the mean ion mass, \( Z \) is the ion mean charge state, and \( e \) is the elementary charge. Note in particular that it increases linearly with wave frequency, and varies inversely with magnetic field strength (since \( \mu_H \propto B_0 \) and \( a^2 \propto B_0^2 \)).

### 2.2 Boundary Conditions

To close the system, we need to impose suitable boundary conditions to inject the system with a pure slow, fast or Alfvén wave from below and disallow any other incoming waves at top or bottom. As briefly touched on before, we artificially extend the main box by replicating the end-values of all the atmospheric variables down to \( z = -0.887 \) Mm at the bottom and up to \( z = 7.11 \) Mm at the top\(^1\), and place the numerical boundaries at these new locations. The extension slabs are chosen to be isothermal so that we may use the exact solutions of Hansen & Cally (2009) to set up the boundary conditions.

The exact solutions in the isothermal slabs may be expressed in...
terms of \( 2F_3 \) generalised hypergeometric functions (Luke 1975), using the magnetic field coordinate system \((\hat{e}_x, \hat{e}_y, \hat{e}_z)\), where \( \hat{e}_x \) and \( \hat{e}_y \) indicate perpendicular and parallel to \( B_0 \) in the \( x-z \) plane, respectively, and \( \hat{e}_z \) remains the same as the unit polarisation vector of the Alfvén wave oriented along the \( y \)-axis. These are entirely equivalent to the Meijer-G function solutions of Zhugzhda & Dzialalov (1984), but more convenient to use. The exact isothermal mixed-mode solution (to equation (A1a)) for magnetoacoustic oscillations is given by

\[
\xi_\perp(s) = C_1 s^{-2k} 2F_3 \left[ -\kappa - ik_x + \frac{1}{2}, -\kappa + ik_x + \frac{1}{2}; 1 - 2k, -ik \tan \theta - \kappa - ik_0 + \frac{1}{2}, -s^2 \sec^2 \theta \right] \\
+ C_2 s^{2k} 2F_3 \left[ -\kappa + ik_x + \frac{1}{2}, \kappa + ik_x + \frac{1}{2}; 2k + 1, -ik \tan \theta + \kappa + ik_0 + \frac{1}{2}, -s^2 \sec^2 \theta \right] \\
+ C_3 s^{1+is(2k \tan \theta - 2k_0)} 2F_3 \left[ -ik_0 - ik_x + ik \tan \theta + 1, -ik_0 + ik_x + ik \tan \theta + 1; 1 - 2ik_0, \frac{3}{2}; -s^2 \sec^2 \theta \right] \\
+ C_4 s^{1+is(2k \tan \theta + 2k_0)} 2F_3 \left[ ik_0 - ik_x + ik \tan \theta + 1, ik_0 + ik_x + ik \tan \theta + 1; 2ik_0 + 1, \frac{3}{2}; -s^2 \sec^2 \theta \right] \\
= \sum_{i=1}^{4} C_i \xi_{\perp,i}(s). \tag{3}
\]

Here, \( C_i \) are arbitrary constant coefficients, and we have defined the following dimensionless parameters

\[
y = \frac{\omega H}{c_s} \tag{4}
\]

\[
s = \nu e^{-\gamma/2H} \tag{5}
\]

\[
n = \sqrt{y - 1} \tag{6}
\]

\[
\kappa = k \sqrt{\frac{H}{s}} \tag{7}
\]

\[
k_0 = \sqrt{\frac{\nu^2 \sec^2 \theta - \frac{1}{4}}{\nu^2}} \tag{8}
\]

\[
k_\perp = \sqrt{\frac{(\nu^2 - \nu^2)k^2}{\nu^2} - \frac{1}{4}} \tag{9}
\]

in which \( H \) is the density scale height, \( s \) is the dimensionless measure of height, \( \gamma \) is the heat capacity ratio, and \( n \) is the Brunt-Väisälä frequency. The three \( \kappa \) coefficients are dimensionless wavenumbers that naturally arise in the analysis of non-magnetic acoustic gravity waves. Equation (3) together with equation (A1b) fully describe the behaviour of \( \xi \) and \( \zeta \) in the isothermal boundary slabs, and will be analyzed in section 2.2.2 to find suitable choices for the \( C_i \).

The magnetoacoustic displacement components in the Cartesian and magnetic field coordinates are related through \( \xi_{\parallel} = \xi \cos \theta + \zeta \cos \theta \) and \( \xi_{\perp} = \xi \sin \theta + \zeta \sin \theta \).

On the other hand, the behaviour of the Alfvén wave in the 2.5D isothermal regime outside the Hall-window (governed by equation (A1c)) is conveniently given by types 1 and 2 of the Hankel functions of zeroth order, according to

\[\eta(s) = s^{2is\tan \theta} \left( D_1 H_{0}^{(1)}(2s \sec \theta) + D_2 H_{0}^{(2)}(2s \sec \theta) \right), \tag{10}\]

where \( D_1 \) are arbitrary constants corresponding respectively to down-going and up-going Alfvén waves.

2.2.1 Boundary conditions on \( \eta \)
As shown in Figure 2, the Hall parameter is negligible anywhere outside the interval \(-0.23 \text{Mm} < z < 2.2 \text{Mm}\). Thus, within the boundary slabs, \( \eta \) obeys equation (10). Therefore, to eliminate incoming Alfvén waves at the boundaries, we must choose \( D_1 = 0 \) at the top, and \( D_2 = 0 \) at the bottom.

2.2.2 Boundary conditions on \( \xi \) and \( \zeta \)
Away from \( z_{eq} \), the mixed-mode solution for magnetoacoustic oscillations given by equation (3) can be asymptotically broken down into slow and fast wave contributions.

At the top boundary where \( s \to 0 \) (i.e., \( z \to \infty \)), equation (3) takes on the asymptotic form

\[\xi_{\perp, \text{Top}}(s) \sim C_1 s^{-2k} + C_2 s^{2k} + C_3 s^{1+is(2k \tan \theta - 2k_0)} + C_4 s^{1+is(2k \tan \theta + 2k_0)}, \tag{11}\]

from which the solution to \( \xi_{\parallel} \) follows readily by solving equation (A1b). Assuming \( \kappa > 0 \), the individual terms in equation (11) in order from left to right correspond to, the exponentially growing fast mode (nonphysical), the exponentially decaying fast mode, the upgoing slow mode, and the downgoing slow mode. As stated earlier, for the purposes of this study we immediately set \( C_1 = C_4 = 0 \), to dispense with the nonphysical solution and the incoming slow wave at the top.

At the bottom boundary where \( s \to \infty \) (i.e., \( z \to -\infty \)), each of the four solutions couple to four leading order asymptotic behaviours according to,

\[\xi_{\perp, \text{Bot}}(s) \sim c_1 s^{-1/2(2i2k \tan \theta + 2i2k \tan \theta)} e^{2is \sec \theta} + c_2 s^{-1/2(2i2k \tan \theta + 2i2k \tan \theta)} e^{2is \sec \theta} \]

\[+ c_3 s^{-1/2(2i2k \tan \theta + 2i2k \tan \theta)} + c_4 s^{-1/2(2i2k \tan \theta + 2i2k \tan \theta)}, \tag{12}\]

whose first and second terms are identified as the downgoing and the upgoing slow waves, respectively. However, the remaining two terms may be interpreted depending on the position of the system in the acoustic-gravity diagram as presented in Figure 3. In Region I, where waves are essentially driven by the gas pressure, these terms represent the downgoing and the upgoing fast waves, respectively, whereas the propagation directions are reversed in Region II, where gravity dominates. Ultimately, all waves are evanescent in Regions III and IV and hence carry no energy. Therefore, the general solution at the bottom boundary can be written in terms of these behaviours as

\[\xi_{\perp, \text{Bot}}(s) \sim c_1 s^{-1/2(2i2k \tan \theta + 2i2k \tan \theta)} e^{2is \sec \theta} + c_2 s^{-1/2(2i2k \tan \theta + 2i2k \tan \theta)} e^{2is \sec \theta} \]

\[+ c_3 s^{-1/2(2i2k \tan \theta + 2i2k \tan \theta)} + c_4 s^{-1/2(2i2k \tan \theta + 2i2k \tan \theta)}, \tag{13}\]

where the vector coefficients \( C = (C_1, C_2, C_3, C_4)^T \) and \( e = (c_1, c_2, c_3, c_4)^T \) connect by

\[e = A C, \tag{14}\]

in which \( A = (a_{ij}) \) is a known \( 4 \times 4 \) matrix (Hansen et al. 2016).

Thus, setting \( c_4 = 0 \) in Region I and \( c_3 = 0 \) in Region II, to block out the upgoing fast wave at the bottom boundary, and \( c_1 = 1 \) to inject it with a normalised slow wave, equation (14) can be used to
determine the remaining coefficients \(c_i\). These coefficients determine how much of the incident slow wave will reflect, mode convert to fast, and transmit through the barrier of the equipartition level. Similar considerations can be employed to craft pure normalised fast waves instead.

Having set up the boundary conditions, it remains to solve equations (1) within the main box and find the energy fluxes corresponding to each asymptotically decoupled (or locally weakly coupled) slow, fast, and Alfvén oscillations that passes through the end-planes. For this task, the equations are first combined and simplified to a set of three coupled ODEs in \(\xi\) as listed in equations (B1) (see Appendix B), and then solved numerically using a shooting method.

### 2.2.3 Nature of the asymptotic solutions

It is important to distinguish the essence of MHD waves around \(z_{eq}\) and far away from it. For instance, at and in the vicinity of \(z_{eq}\), 2.5D magnetoacoustic waves have an indivisible mixed-mode characteristic which can only be described by the exact solution given in equation (3). However, as we recede far from \(z_{eq}\) in either direction, they begin to locally exhibit simpler behaviours corresponding to the asymptotic formulae (11) and (13) identifiable as decoupled slow and fast waves. In other words, although the exact \(F\) solutions represent global mixed-modes containing both fast and slow characteristics, these behaviours may be decoupled locally in \(z \ll z_{eq}\) and \(z \gg z_{eq}\). Figure 1 indicates that such a decoupling is well justified in the upper and lower reaches of our computational domain.

It is also of note that the leading order asymptotic behaviour for \(\eta\) of the Alfvén solutions expressed in equation (10) as \(s \to \infty\) are identical to the downgoing and upgoing slow wave asymptotic terms for \(\xi_\perp\) of equation (13). This near-degeneracy of the Alfvén wave \(\eta\) and the slow wave \(\xi_\perp\) where \(a \ll c\) only applies to these components through; the slow wave also exhibits a plasma velocity component parallel to the field, which the Alfvén wave does not. Nevertheless, we might expect a similar rate of generation of slow and Alfvén waves at the photosphere. Once again, we stress the importance of dispersion diagrams in comprehending this picture. As shown in Figure 1, this near degeneracy manifests itself as the near-matching loci of the slow and Alfvén modes below \(z_{eq}\).

This notion of asymptotic analysis of MHD waves is also applicable more broadly. For example, in fully-ionized 3D MHD where the Alfvén mode is no longer decoupled from magnetoacoustic oscillations due to geometric effects, there are no individual Alfvén, slow, or fast modes in the global sense. Nor there are any known functions with which to formulate the exact global solutions of such a system in any generality. Nevertheless, looking at the dispersion curves (e.g., see Figure 1 of Cally & Hansen 2011), we may still learn about the local asymptotic behaviour of the resultant MHD waves following a dispersion experiment far from \(z_{eq}\), where the coupling effects are weak and the waves begin to possess simple-wave characteristics. In particular, these diagrams are very useful in discerning which types and under what circumstances waves reach the upper atmosphere and potentially contribute to upper chromospheric and coronal heating.

### 2.3 Energy Flux

The total vector wave-energy flux is comprised of the pressure work and the Poynting flux vector as follows (Cally 2001; Hansen et al. 2016).

\[
\mathbf{F} = \text{Re} \left[ i \omega \mathbf{p}_1 \mathbf{\xi}^* - \frac{i \omega}{\mu_0} \left( \mathbf{\xi}^* \times \mathbf{B}_0 \right) \times \mathbf{b} \right].
\]

However, we are only interested in the energy transport in the \(z\)-direction. Additionally, it is instructive to break down the total energy flux into the acoustic, magnetic, and Alfvén components (energy flux functions hereafter) denoted respectively by, \(F_{ac}\) and \(F_{mag}\), and \(F_A\), according to

\[
F_{ac} = -\omega \text{Im} \left[ p_1 \mathbf{\xi}^* \cdot \mathbf{\hat{z}} \right] \quad \text{(16a)}
\]

\[
F_{mag} = \omega \text{Im} \left[ \frac{1}{\mu_0} \left( \mathbf{\xi}^*_\perp \mathbf{\hat{e}}_\perp \times \mathbf{B}_0 \right) \times \mathbf{b} \right] \cdot \mathbf{\hat{z}} \quad \text{(16b)}
\]

\[
F_A = \frac{\omega}{\mu_0} \text{Im} \left[ \left( \mathbf{\eta}^*_\perp \mathbf{\hat{e}}_\perp \times \mathbf{B}_0 \right) \times \mathbf{b} \right] \cdot \mathbf{\hat{z}}. \quad \text{(16c)}
\]

However, caution must be taken as these are interconnected through the displacement vector \(\xi\) and hence their identity does not remain the same throughout the domain. Nevertheless, they may locally represent the energy flux associated with certain weakly coupled MHD waves under specific circumstances in the absence of coupling mechanisms or far from \(z_{eq}\) as explained in section 2.2.3. For instance, it is well established that in the \(\beta \gg 1\) regime, the slow magnetoacoustic wave is almost perfectly of pure magnetic attributes, while the fast wave is characterised as acoustic there. This is reflected in Figure 4, and means that the slow and the fast waves tend to decouple to a high degree in sub-photospheric depths. However, as we recede from high-\(\beta\) regions, various coupling mechanisms weigh in, rendering these modes as part acoustic and part magnetic.

Now, given the coefficients \(c\) and \(C\), which are most conveniently applied for the \(\beta > 1\) and \(\beta < 1\) sub-domains respectively, it is possible to directly calculate the energy flux of the individual MHD modes by

\[
F_{FS} = \frac{p_{mag} \omega^2}{c_s} \begin{cases} C^+_\Phi C, & z > z_{eq} \\ C^-\Psi C, & z < z_{eq} \end{cases}, \quad \text{(17a)}
\]

\[
F_A = \frac{p_{mag} \omega^2 \cos^2 \theta}{\pi \nu} D^\top D. \quad \text{(17b)}
\]

Here \(F_{FS}\) is the magnetoacoustic contribution to the energy flux containing the fast and the slow components, \(p_{mag}\) is the equilibrium magnetic pressure, \(D = (D_1, D_2)\) (see equation (10)), \(\Psi = A^\top \Phi A\), and following Hansen et al. (2016)

\[
\Phi = \begin{pmatrix} 0 & \frac{i \omega}{\nu} & 0 & 0 \\ \frac{-i \omega}{\nu} & 0 & 0 & 0 \\ 0 & 0 & \phi_{33} \mathcal{U} \left( \frac{k_0^2}{\nu^2} \right) & \phi_{34} \mathcal{U} \left( \frac{k_0^2}{\nu^2} \right) \\ 0 & 0 & \phi_{34} \mathcal{U} \left( -\frac{k_0^2}{\nu^2} \right) & \phi_{44} \mathcal{U} \left( \frac{k_0^2}{\nu^2} \right) \end{pmatrix}, \quad \text{(18)}
\]

where \(\mathcal{U}\) is the unit step function, and the \(\phi_{ij}\) elements can be found in the Appendix of Hansen et al. (2016). Equation (17a) implicates that one may isolate the energy flux components due to the fast and the slow waves whenever the association between the \(C\) elements and the individual magnetoacoustic modes is known.

Lastly, the outgoing energy fluxes at the boundaries due to each individual asymptotically decoupled wave can be determined by evaluating equations (17) at \(z_{top}\) and \(z_{bot}\). As dictated by the BCs, the energy fluxes at the top can only be due to the outgoing ‘transmitted’ (or ‘converted’) slow and ‘Hall-induced mode converted’ Alfvén waves. These are denoted by \(F_{S}^\top\) and \(F_{A}^\top\), respectively. However, at the bottom, the upgoing slow (or fast) flux is normalised to unity by construction, and hence \(F_{S}^\top = 1\) (or \(F_{F}^\top = 1\)). The remaining
Fluxes are due to the downgoing ‘reflected’ slow $\mathcal{F}_{S1}$ (or ‘converted’ fast $\mathcal{F}_{F1}$), the downgoing ‘mode converted’ fast $\mathcal{F}_{F1}$ (or ‘reflected’ slow $\mathcal{F}_{S1}$), and the Hall-induced mode converted downgoing Alfvén $\mathcal{F}_{A1}$. Note the difference between the superscript and the subscript arrows. In any case, in view of the normalised injected wave and conservation of energy, we require that

$$\left(\mathcal{F}_{S1}^+ + \mathcal{F}_{A1}^+\right) + \left(\mathcal{F}_{S1}^- + \mathcal{F}_{F1}^- + \mathcal{F}_{A1}^\mp\right) = 1. \quad (19)$$

3 RESULTS

We wish to establish a connection between the free parameters $f$, $B_0$, $k_x$, and $\theta$ and the ‘effectiveness’ of the Hall term in generating upgoing Alfvén waves from the input slow waves. Here $f = \omega/2\pi$ is the wave frequency measured in Hertz. Since the total injected energy flux is normalised to unity, we define this effectiveness simply as the magnitude of the outgoing Alfvén energy flux evaluated at the top, i.e., $\mathcal{F}_{A1}^\mp$. The results are presented using (i) contour plots of the local outgoing slow and Alfvén energy fluxes measured at the boundaries of the numerical box, and (ii) plots of the energy flux functions described by equation (16) versus $\zeta$.

3.1 Pure Slow to Alfvén Conversion

As mentioned above, without the Hall effect we would expect a zero Alfvén energy flux because the injected magnetoacoustic waves are in the 2D vertical $x$–$z$ plane and there is no geometric coupling mechanism. Nevertheless, magnetoacoustic waves can still communicate and exchange energy between each other in the vicinity of the equipartition level, where they undergo mode transformation, reflection, and transmission through pure resonant coupling.

When the Hall term is switched on, the energy fuelled by the slow wave may partially or entirely convert to an upgoing Alfvén wave depending on the convertibility strength across the weakly ionised region. Partial transformations occur when the conversion effect is either too weak or too strong. In the latter case, conversion overshoots happen where an input slow wave transforms all the way to Alfvén wave and back to slow wave in a cyclic fashion. In other words, upon entering the Hall window, the slow wave will transit into a superposition (or inter-modal) state of part slow- and part Alfvén-like waves with a rotating polarisation vector replacing its original ($x$–$z$)-plane counterpart. It is the variable circular speed of this vector that determines the local strength of the Hall conversion. Hence stronger Hall conversions lead to more rapidly rotating polarisation vectors, which will manifest themselves as cyclic patterns in the flux diagrams. These cyclic patterns would then eventually cease at the end of the Hall domain, where the convertibility breaks and the hybrid wave splits into its ‘local’ amounts of slow and Alfvén wave-energies. Therefore, assuming a set of parameters $(f, B_0, k_x, \theta)$ that greatly amplifies the strength of the Hall conversion, even slight shifts in any one of these values can drastically impact the effectiveness. As we shall see, this introduces interesting behaviours in the effectiveness function $\mathcal{F}_{A1}^\mp$.

Figure 4 illustrates the behaviour of the energy flux functions $\mathcal{F}_{\alpha c}$ (green), $\mathcal{F}_{\text{mag}}$ (magenta), and $\mathcal{F}_A$ (blue) of an input pure slow wave at $f = 12$ mHz versus height for three different magnetic field inclinations. The colour-shaded background represents the profile of the Hall parameter $\mu_H(z)$, with the weakly ionised Hall-dominated patch of the atmosphere sandwiched between the monotone blue portions. The left and the right columns correspond to the same experiment with and without the Hall effect, respectively. As can be seen, in all the cases, virtually all of the injected slow wave at the base ($\beta \gg 1$) is stored as pure magnetic energy. The downgoing (negative) acoustic energy is supplied by mode conversion to the fast wave near the equipartition level. The Alfvén wave is non-existent below the Hall window, i.e., $z < -0.23$ Mm. The solid horizontal line represents the total upward energy flux, which should remain constant everywhere as required by conservation of energy. Thus, it gives a measure of numerical errors involved in each case. The dashed vertical line flags the location of $z_{eq}$.

In the right column, $\mathcal{F}_A$ is identically zero everywhere, as expected. Notice the heavily reflected slow and converted/reflected fast waves at $\theta = 0$, as implicated by the amplitudes of $\mathcal{F}_{\alpha c}$ and $\mathcal{F}_{\text{mag}}$ near the base. The small magnitude of the net upgoing flux ($\approx 4\%$) suggests that nearly 96% of the original slow wave was reflected/mode converted into downgoing slow and fast waves. On the other hand, the small positive amplitude of $\mathcal{F}_{\text{mag}}$ at the base indicates that about 92% of the input slow wave has been reflected as downgoing slow wave, and hence leaving about 4% for the downgoing mode converted fast wave. Additionally, there is almost no reflection/conversion taking place at $\theta = 60^\circ$, which is suggestive that all of the injected slow wave has been transmitted through the barrier of the $\beta = 1$ layer.

In the left column, the Hall conversion strength at $\theta = 0$ is precisely the right amount for a perfect slow-Alfvén transformation. This is a critical amount beyond which $\mathcal{F}_A^\mp$ begins to oscillate (see the plots for $\theta = 30^\circ$ and $60^\circ$). Remember that the 2.5D configuration gains full 3D generality at $\theta = 0$. That is, despite the 2.5D nature of our equations, a full 3D approach would still yield identical results in the case of the vertical magnetic field. It is noteworthy that even though at $\theta = 0$ the high-$\beta$ slow wave in a vertically stratified medium becomes asymptotically incompressible, it is still intrinsically distinct from the Alfvén wave owing to their unique polarisation vectors.

Furthermore, contrary to the non-Hall case and as a direct byprod-
uct of the efficient slow-Alfvén conversion mediated by the Hall effect, no one of the waves is reflected back regardless of the angle \( \theta \). This is verified by looking at the zero tails of \( F_{\text{ac}} \) and \( F_A \) at the bottom wall. At \( \theta = 30^\circ \), \( F_A \) whittles down as the Hall conversion strength ratchets up in comparison with that of the vertical field. Lastly, the case of \( \theta = 60^\circ \) yields even less effectiveness in view of the much stronger conversion effect that spawns the cyclic pattern that culminates where the local slow energy flux is larger than that of the Alfvén wave. Thus, we observe that

(i) The strength of the slow-Alfvén Hall conversion per unit length is enhanced by widening of the angle \( \theta \).
(ii) The length of the magnetic field lines in the main box is increased with \( \theta \) by a factor of \( \sec \theta \). As a result, field-guided waves such as the Alfvén and the high-\( \beta \) slow waves remain transformable for longer windows.
(iii) The effectiveness can significantly decline or increase with heightening of the conversion effect.

As an important distinction, notice that the second observation above cannot apply to an input high-\( \beta \) fast wave by virtue of its acoustic (or gravity, depending on \( f \) and \( k_z \)) characteristics. We will elaborate on this in Section 3.2.

For the remainder of this subsection, we shall progress by pairing up the parameters into three sub-spaces and survey the energy flux functions at the boundaries of the main box in each case.

### 3.1.1 Survey parameters: \( f = k_x \)

The choice of \( k_x \) and \( f \) reflects the expected photospheric length and time scales of the wave sources. For granules, with typical extent \( 1 \) Mm and timescale of a few minutes, we might focus on \( k_x = O(1) \) Mm\(^{-1} \) and \( f \) of a few mHz. For quiet Sun-Sun magnetic field strengths of \( B_0 \approx 20 \) G say, the Alfvén speed at \( z = -0.2 \) Mm is only around \( 80 \) m s\(^{-1} \) and for \( f = 12 \) mHz, as in Fig. 1, we have \( k_x \cos \theta \approx \omega_f / a \approx 945 \) Mm\(^{-1} \), so excited slow waves with \( k_x = O(1) \) would be essentially vertical \( (k_z \gg k_x) \) at the base. This is consistent with the rapid divergence of the slow and Alfvén loci in Fig. 1 as \( z \) diminishes below \( z_{\text{eq}} \).

However, \( k_x \) becomes important at higher altitudes where \( k_z \) has become comparable to \( k_x \). In the figure, if \( k_x \) were identically zero, the loci would be up-down symmetric, as would also be the case for vertical magnetic field \( \theta = 0^\circ \). The main effect of the non-zero \( k_z \) is on the proximity of the Alfvén and fast loci. Although this has no practical consequences in 2D, because there is no available conversion mechanism, it does lead to powerful Alfvén-slow conversion in 3D, where wave vector, gravity and magnetic field direction are not co-planar (\text{Cally & Hansen} 2011). In that case, a dispersion diagram similar to Fig. 1 (not shown) indicates that an upcoming fast wave would mode convert predominantly to the upcoming Alfvén wave. But if the figure were flipped \( (k_x \approx 0^\circ) \), it would be the descending fast wave, post-reflection, that predominantly transforms to the (now descending) Alfvén wave.

On the other hand, the slow-Alfvén conversion all occurs where \( k_z \) is large, and therefore the effect of \( k_z \) is minor. This will be seen in our computational results, where we observe near-symmetry in the Hall-mediated top escaping Alfvén flux about \( \theta = 0^\circ \).

Figure 5 displays the contour plots of outgoing slow and Hall-induced mode converted Alfvén energy fluxes as functions of \( f \) and \( k_x \) at two nominated magnetic intensities \( 10 \) G (left column) and \( 50 \) G (right column). The fast wave-energy flux is left out as it is zero for \( z > z_{\text{eq}} \) and is deductible by equation (19) for \( z < z_{\text{eq}} \). As seen in the first row, the effectiveness \( (F_A^1) \)

(i) Is very weakly dependent on \( k_x \) for \( |k_x| \leq 1.5 \) Mm\(^{-1} \);
(ii) Oscillates with \( f \);
(iii) Is remarkably scaled down with the larger \( B_0 \), requiring frequencies higher than \( 35 \) mHz to achieve any significant upward Alfvén flux.

The weak sensitivity of the effectiveness to \( k_x \) at these frequencies results from the much larger vertical wavenumbers \( k_z \) of the waves outweighing the effect of \( k_x \) in the slow-Alfvén interaction region (where the loci are close in the dispersion diagram).

Increasing magnetic field strength \( B_0 \) reduces the conversion because of the Hall effect’s inverse dependence on the ion cyclotron frequency \( \Omega_i = ZeB_0/m_i \) through the dimensionless Hall parame-
Figure 5. Outgoing slow and Alfvén wave-energy flux profiles at the boundaries as functions of the horizontal wavevector $k_x$ and the frequency $f$. Note the different colour-scaling of each panel. The plots correspond to $\theta = 0$ and two representative magnetic strengths 10 G (left column) and 50 G (right column), as labelled. The upgoing Alfvén energy flux at the top manifests almost no variation across $1 \text{•} 5 \text{ Mm}$ at either of the field intensities. This is because the vertical wave vector $k_z$ of the mode converted Alfvén wave is much greater than the $k_x$ of the source wave in the slow-Alfvén interaction region.

Figure 6. The energy flux functions associated with representative input slow waves at six frequencies from 15 to 20 mHz, as labelled. The magnetic field parameters are fixed to $B_0 = 10$ G and $\theta = 0$ across the board. The strength of the Hall conversion is consistently augmented by increasing $f$. The effectiveness, however, oscillates with $f$, as the flux curves get modified with the variable conversion strength while the behaviour of the Hall-mediated region remains unchanged.

3.1.2 Survey parameters: $f - \theta$

As discussed earlier, an increase in $\theta$ influences the effectiveness in two ways: by 1) directly bolstering the Hall conversion strength, and 2) lengthening the segment of the field lines lying within the weakly ionised portion of the atmosphere, and hence prolonging the conversion window for field-guided waves. With that reminder, Figure 7 is presented to analyse the energy fluxes across the parameter sub-space $f - \theta$ at two representative magnetic intensities 10 G (left column) and 50 G (right column).

In the left column, once-off slow to Alfvén mode conversions (similar to the top-left panel of Figure 4) of varied effectiveness take place over the region bounded by $3 \text{ mHz}$ and $13 \text{ mHz}$, beyond which the fluxes are impacted by $k_x$. However, the influence of $B_0$ remains similar.

As demonstrated by Figure 6, the periodicity of $F_A^\perp(f)$ originates in the ever-increasing strength of the Hall conversion with frequency via $\epsilon_H$ and size-invariability of the Hall domain. That is, as the transformation intensifies with $f$ (signified by more dense cyclic patterns), the position of the intersection point of $F_A^\perp$ and the upper boundary of the Hall domain traces out a periodic curve.

3.1.3 Survey parameters: $f - B_0$

The last parameter sub-space we wish to look at is $(f, B_0)$. Figure 8 depicts the familiar four flux functions measured at the walls of the...
Hall-Induced Magnetoacoustic to Alfvén Conversion

3.2 Two-stage Fast to Alfvén Hall Conversion

Hall transformation of magnetoacoustic and Alfvén waves is a conservative process of polarisation rotation. As discussed in Sec. 3.1, in a high-\(\beta\) plasma it directly affects the slow MHD wave, which is magnetically dominated and asymptotically transverse to \(k\) and to \(k\times B_0\) as \(\beta \to \infty\).

On the other hand, in a low-\(\beta\) plasma, it is the fast wave that is magnetically dominated and asymptotically transverse to \(B_0\) (as \(\beta \to 0\)). The role and efficacy of the Hall effect in this domain was explored by Cally & Khomenko (2015) and confirmed computationally by González-Morales et al. (2019) at \(B_0 = 500\) G, though frequencies of order 1 Hz were then found necessary to produce a significant effect. This may be understood in terms of the dimensionless Hall parameter \(\epsilon_H = \omega / f \Omega_i\) introduced in Sec. 3.1.1. Only as \(\epsilon_H\) approaches order unity is there substantial rotation.

With this in mind, we now explore the two-stage process by which fast waves launched from the base convert to Alfvén waves via the Hall effect, as distinct from the 3D geometric process of Cally & Hansen (2011). To do this, the predominantly acoustic fast wave in \(a < c_s\) partially transmits as an acoustic wave (now slow) in \(a > c_s\) through the equipartition layer \(z_{eq}\), and partially converts to...
a magnetically dominated (now fast) wave. Provided this fast wave is still in the Hall window, it may be rotated and Alfvén waves produced. But for this to occur, \( z_{eq} \) must be lowered substantially compared to the cases of Figs. 4 and 6 to place it in the Hall region, which requires stronger magnetic fields. But then, higher frequencies will be required.

In this subsection, we first confirm that no significant Alfvén flux arises from fast waves at low field strengths and low frequencies, but then explore field strengths of 100 to 400 G characteristic of weak geometrical coupling effects at these frequencies. Let us now modify the field strength such that \( z_{eq} \) is re-located inside the Hall window, and amplify the frequency to guarantee stronger fast-slow transformations. Figure 10 shows the flux functions of an input fast wave at \( f = 600 \) mHz and three nominated magnetic field strengths, with the Hall term switched on (left columns) and off (right column). As evident in the right column, all fast-slow conversions in the absence of the Hall effect are only weakly sensitive to the location of \( z_{eq} \) at these field strengths. Notice that due to very large values of \( k_z \) at such high frequencies, \( k \sim k_z \). Moreover, the acoustic energy carried by the input fast wave undergoes substantial transformation at \( z_{eq} \) and splits into magnetic and acoustic parts. Without the Hall effect, all the resultant magnetic energy is reflected back as (magnetic) slow waves.

Upon introducing the Hall effect (left column), the resultant magnetic energy derived from fast-slow conversions can now undergo a second transformation into Alfvén waves through the Hall effect. The mildly jagged patterns in the net upward energy flux (solid
horizontal line) indicate minor numerical errors due to highly oscillatory solutions. Notice the larger net upward fluxes compared with those of the right column. Furthermore, notice the relatively larger leaps in $F_{\text{tag}}$ at $z_{eq}$. This is due to the fact that magnetooacoustic and Alfvén waves in 2.5D are geometrically decoupled, and hence $F_A$ is invisible to geometric coupling effects. Thus, as acoustic energy transforms into the Alfvén type via the Hall effect, it depletes the available magnetic energy budget, which is then compensated by further geometry-induced acoustic-magnetic conversion. Finally, looking at the position of $z_{eq}$ set by $B_0$, it is clear that the strength of the Hall conversion depends much more strongly on $B_0$ itself, rather than $z_{eq}$. That is, even though $z_{eq}$ sits much lower in the atmosphere at $B_0 = 400$ G, nevertheless such a large field intensity severely diminishes the strength of the Hall conversion to the point that the net effect is the relatively smaller effectiveness in contrast with the lower $B_0$ cases.

To close this section, let us investigate the effectiveness of two-stage fast-Alfvén conversion in the $\theta - f$ parameter subspace. As a point of caution, substantial fast-slow conversions require high frequencies ($\sim$ Hz) which entail highly oscillatory behaviours and pose numerical problems. Due to these numerical challenges, we were only able to cover frequencies up to 600 mHz and magnetic inclinations no more than 15°. However, revealing aspects are still present at these ranges. Figure 11 depicts the contour maps of the slow and Alfvén energy fluxes at $z_{top}$ and $z_{bot}$ for two representative field intensities $B_0 = 200$ G and $B_0 = 400$ G. The bands of zero flux associated with $\theta \sim 0$ are due to vertical propagation of the input acoustic fast waves handing all the energy directly to acoustic slow waves at the top. Considerable amounts of upgoing Alfvén waves begin to appear for $\theta \geq 2^\circ$ and $f \geq 500$ mHz. The effectiveness maximises for $4^\circ \leq \theta \leq 10^\circ$, and analogous to the slow-Alfvén conversion, favours higher frequencies. However, the crucial difference here is that not only do high frequencies boost the Hall conversion effect, but they are required to ensure significant fast-slow conversions if Alfvén waves are to be generated.

4 CONCLUSION

Dispersion diagrams such as Figure 1 provide useful insights into where MHD wave modes may interact, either because their loci are very close over an extended interval, such as for slow and Alfvén curves in $a < c_s$ and fast and Alfvén in $a > c_s$, or because of avoided crossings, seen for slow and fast curves near $a = c_s$. In the case of the fast-slow avoided crossings, the closeness of the gap indicates the strength of the convertibility, which operates in ideal MHD in both 2D and 3D.

However, the other two opportunities are only realised if a conversion mechanism is present, in addition to phase space proximity. In the case of fast-Alfvén this is afforded by three-dimensionality, where $k$, $g$ and $B_0$ are non-coplanar (Cally & Hansen 2011), but both slow-Alfvén (in $a < c_s$) and fast-Alfvén (in $a > c_s$) are also potentially mediated by the Hall effect, provided the Hall parameter $\epsilon_H$ is of order unity in the relevant regions.

Even at low frequencies ($\sim$ 10 mHz) and low magnetic field strengths ($\sim$ 10 G), the Hall effect is very potent in mode converting slow and Alfvén waves at low altitudes (see Figs. 4 and 6), and may in fact produce several oscillations from fast to Alfvén and back again, with ultimate net effect depending on how many such oscillations fit within the Hall window.

Fast waves in $z < z_{eq}$ are not transformable to Alfvén waves, since their phase space loci are well-separated and because their predominantly acoustic natures are not susceptible to the Hall effect. However, on (ideal) mode conversion to now magnetically dominated and predominantly $B_0$-transverse fast waves in $z > z_{eq}$, they are then prone to Hall conversion to Alfvén slightly below the fast wave reflection height. This only occurs to appreciable extent though if the interaction is in the Hall window, which in turn can only happen if $z_{eq}$ is low enough. This requires stronger magnetic fields (hundreds of gauss), and therefore frequencies of hundreds of mHz, since $\epsilon_H \propto \omega / B_0$.

For the various scenarios considered, dispersion diagrams (Fig. 1), flux vs. height plots (Figs. 4, 6, 9 and 10), and several two-parameter net flux contour plots (Figs. 5 $[f$ and $k_s$], 7 $[f$ and $\theta$], 8 $[f$ and $B_0]$ and 11 $[f$ and $\theta$ for high $B_0$]), reveal a rich picture of Hall conversion to Alfvén waves, with implications for upper atmospheric heating and interpretation of Alfvénic wave observations.

One shortcoming of our analysis is that it is only 2.5D, having assumed that $k$ is in the vertical ($x-z$) plane of the magnetic field lines. It therefore does not address geometrical coupling between slow and Alfvén, which will be considered elsewhere. However,
this is irrelevant in the important case of vertical magnetic field \( \theta = 0^\circ \), since the choice of \( x \)-direction is now arbitrary. In that case, our various diagrams indicate that strong direct slow-to-Alfvén conversion at low altitudes is unavoidable at low field strengths.

Our survey tells us that in regions of weak magnetic field, both

(i) Slow waves generated by convective motions near the solar surface easily convert to Alfvén waves on their way upwards, perhaps many times over, especially at higher frequencies; and

(ii) Alfvén waves generated by convective motions near the solar surface easily convert to slow waves on their way upwards, perhaps many times over, especially at higher frequencies.

This operates even at low frequencies, and is amplified further by magnetic field inclination. Higher magnetic field strengths require proportionately higher frequencies for similar effects. The overall effect is a sort of shuffling process between slow and Alfvén, with only the latter able to contribute to the Alfvénic ‘dark energy’ content of the corona (McIntosh et al. 2011; McIntosh & De Pontieu 2012).

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APPENDIX A: LINEARISED MHD EQUATIONS AT THE BOUNDARIES

Here, we present the ODEs for the plasma displacement vector \( \xi \) drawn from equations (1) at the isothermal boundaries.

As discussed earlier, magnetoacoustic-Alfvén coupling at the boundaries is entirely terminated due to vanishing of the Hall parameter in those heights (Figure 2). Thus, setting \( \mu_H = 0 \) in equation (1c) and substituting equations (1b – 1e) into equation (1a), we obtain a vectorial ODE of 4-th order. Using the magnetic field coordinate system \( (\hat{e}, \hat{e}_\perp, \hat{e}_\|$) introduced in Section 2.2, the individual components of this equation can then be sifted out according to,

\[
\begin{align*}
\xi_{\|} & = \Delta^{-1} \left[ 2 i \gamma y v^2 s^3 \cos^2 \theta \xi_{\|}^{(3)} + 2 i \gamma v^2 \cos \theta \left(3 \gamma - 2 \gamma \cos \theta - 2 \gamma \sin \theta \left( \gamma^2 + v^2 \right) \right) \xi_{\|}^{(4)} \\
+ 4 i \gamma y v^2 s^2 \left( \gamma \cos \theta - 2 \gamma \sin \theta \left( \gamma^2 + v^2 \right) \right) \xi_{\|}^{(3)} + s^2 \xi_{\|}^{(4)} = 0, \tag{A1a} \end{align*}
\]

\[
\begin{align*}
\xi_{\perp} & = \xi_{\|} + \Delta^{-1} \left[ 2 i \gamma y v^2 s^3 \cos^2 \theta \xi_{\perp}^{(3)} + 2 i \gamma v^2 \cos \theta \left(3 \gamma - 2 \gamma \cos \theta - 2 \gamma \sin \theta \left( \gamma^2 + v^2 \right) \right) \xi_{\perp}^{(4)} \\
+ 4 i \gamma y v^2 s^2 \left( \gamma \cos \theta - 2 \gamma \sin \theta \left( \gamma^2 + v^2 \right) \right) \xi_{\perp}^{(3)} + s^2 \xi_{\perp}^{(4)} \right], \tag{A1b} \end{align*}
\]

\[
\begin{align*}
\xi_{\|} & = \Delta^{-1} \left[ 2 i \gamma y v^2 s^3 \cos^2 \theta \xi_{\|}^{(3)} + 2 i \gamma v^2 \cos \theta \left(3 \gamma - 2 \gamma \cos \theta - 2 \gamma \sin \theta \left( \gamma^2 + v^2 \right) \right) \xi_{\|}^{(4)} \\
+ i \gamma y v^2 s \left( -2 \gamma \sin \theta \left( 4 \gamma^2 - 1 \right) + 2 \right) \cos 2\theta \\
+ \cos \left( 4 \gamma \sin \theta - 2 \gamma \sin \theta \left( \gamma^2 + v^2 \right) \right) \right] \xi_{\|}^{(3)} + s^2 \xi_{\|}^{(4)} = 0, \tag{A1c} \end{align*}
\]

where primes denote \( s \)-derivatives and

\[
\Delta = 8 s^2 \left( 2 \gamma \cos \theta \left( \gamma^2 v^2 - \left( \gamma - 1 \right) \right) \right) - \gamma^2 v^4 - i \left( \gamma - 2 \right) \gamma v^2 \sin 2\theta \\
+ \kappa^2 \left( i \gamma \left( 2 \gamma v^2 - 1 \right) \right). \tag{A2}
\]

The various parameters used in the above equations are set out in equations (4) through to (9). Note that although the 2.5D magnetoacoustic oscillations are jointly controlled by \( \xi_{\|} \) and \( \xi_{\perp} \), using the

DATA AVAILABILITY

The data underlying this article will be shared upon reasonable request to the corresponding author.
magnetic field coordinates allows for the independence of $\xi_\parallel$ from $\xi_\perp$ seen in equation (A1a).

**APPENDIX B: HALL COUPLED LINEARISED MHD EQUATIONS IN THE MAIN BOX**

Here, we lay out the set of second order ODEs in terms of $\xi_\perp$, $\xi_\parallel$, and $\eta$ to be solved numerically throughout the domain.

To combine equations (1) into a single vectorial ODE in the magnetic field coordinates, this time we start by eliminating $j_x \times B_0$ by substituting it from the momentum equation (1a) into the induction equation (1b) to obtain $j$ in terms of $\xi$, $\rho_1$, $\rho_1'$, and their derivatives. Finally, the new $j$, along with $\rho_1$, $\rho_1'$, and $\eta$ from equations (1b), (1c), and (2) are substituted back into equation (1a), which yields the $x$-component of the momentum equation

$$\begin{align*}
\xi & \left( -a^2 k_x^2 \rho_0 \cos^2 \theta - c_s^2 k_x^2 \rho_0 + a^2 \rho_2 \right) \\
+ \xi & \left( a^2 k_x^2 \rho_0 \sin \theta \cos \theta - i g k_x \rho_0 \right) \\
- a^2 \rho_0 \xi'' \sin \theta \cos \theta + a^2 \rho_0 \rho''_0 \cos^2 \theta + i c_s^2 k_x \rho_0 \xi' \\
+ \eta & \left( i \omega k_x^2 \mu_H \cos \theta - i \omega \mu'_H \cos \theta - i \omega \mu_H \eta'' \cos \theta = 0 \right), \quad (B1a)
\end{align*}$$

the $y$-component

$$\begin{align*}
\xi & \left( i h k_x \rho_0 \left( (c_s^2)'' + f \right) + a^2 \sin \theta \cos \theta \left( H^2 \rho_0 - c_s^2 k_x \rho_0 \right) \right) \\
+ \xi & \left( H \rho_0 \left( c_s^2 ight)' - c_s^2 \rho_0 \right) + \xi' \left( a^2 \sin^2 \theta H \rho_0 + c_s^2 H \rho_0 \right) \\
+ H & \xi \rho_0 \left( a^2 - a^2 \sin^2 \theta k_x^2 \right) - a^2 \sin \theta \cos \theta H \rho_0 \rho'' \\
+ i c_s^2 H k_x \rho_0 \xi' + \eta & \left( i \omega \sin \theta \mu_H' - i \omega \sin \theta H k_x^2 \mu_H' \right) \\
+ 2 i & \omega \sin \theta H \mu_H' + i \omega \sin \theta H \mu_H \eta'' = 0, \quad (B1b)
\end{align*}$$

and the $z$-component

$$\begin{align*}
\omega H^2 & \left( a^2 - a^2 \sin^2 \theta k_x^2 \right) \eta_0 \rho_0' + i a^2 \omega H^2 \sin 2 \theta k_x \eta_0 \rho_0' \\
+ a^2 & \omega^2 \cos^2 \theta H^2 \eta'' \rho_0 + i \omega^2 \cos \theta H^2 \mu_H \xi'' \rho_0' \\
+ \xi'' & \left( g i \cos \theta \mu_H^2 \rho_0^2 H^2 \right) \\
+ c_s^2 & \cos \theta \mu_H k_x^2 - a^2 \omega^2 \sin \theta \rho_0^2 \mu_H k_x \\
+ 2 i & \omega^2 \cos \theta \mu_H \rho_0^2 \mu_H' \\
+ & \left( g \cos \theta k_x \rho_0^2 \mu_H H^2 + c_s^2 \cos \theta k_x \rho_0 \mu_H \rho_0' H^2 \\
- c_s^2 & \cos \theta k_x \rho_0^2 \mu_H H^2 \right) \xi'' \\
+ & \left( c_s^2 \cos \theta k_x \mu_H \rho_0^2 H^2 + c_s^2 \cos \theta k_x \rho_0 \mu_H \rho_0' H^2 \\
+ a^2 & \omega^2 \cos \theta k_x \rho_0^2 \mu_H H^2 + i c_s^2 \sin \theta k_x \rho_0^2 \mu_H \rho_0' H^2 \\
- i c_s^2 & \sin \theta k_x \rho_0^2 \mu_H H^2 + 2 g \cos \theta k_x \rho_0 \mu_H H^2 \\
- & \cos \theta k_x \rho_0 \mu_H \left( \left( g - c_s^2 \right) \rho_0 - \rho_0' \right) H^2 \\
- c_s^2 & \cos \theta k_x \rho_0^2 \mu_H^2 H^2 - g \cos \theta k_x \rho_0^2 \mu_H H \\
+ & \xi'' \left( g \cos \theta k_x \mu_H \rho_0^2 H^2 + g \cos \theta k_x \rho_0 \mu_H \rho_0' H^2 \\
+ a^2 & \omega^2 \cos \theta k_x \rho_0^2 \mu_H H^2 + i c_s^2 \sin \theta k_x \rho_0^2 \mu_H \rho_0' H^2 \\
- i c_s^2 & \sin \theta k_x \rho_0^2 \mu_H H^2 + 2 g \cos \theta k_x \rho_0 \mu_H H^2 \\
- & \cos \theta k_x \rho_0 \mu_H \left( \left( g - c_s^2 \right) \rho_0 - \rho_0' \right) H^2 \\
- c_s^2 & \cos \theta k_x \rho_0^2 \mu_H^2 H^2 - g \cos \theta k_x \rho_0^2 \mu_H H \right) \right), \quad (B1c)
\end{align*}$$

where primes now denote $z$-derivatives. A non-trivial numerical test of the equations is provided by the observation that the vertical component of the total wave energy flux is constant in $z$, as it should be.

This paper has been typeset from a TeX/TeX file prepared by the author.