Tauberian theorems for statistically \((C, 1, 1)\) summable double sequences

Zerrin Önder\(^1\) · İbrahim Çanak\(^1\)

Received: 19 September 2018 / Accepted: 5 January 2019 / Published online: 11 January 2019
© Springer Nature Switzerland AG 2019

Abstract

In this paper, we obtain some Tauberian conditions in terms of slow oscillation and slow decreasing in certain senses, under which convergence of a double sequence in Pringsheim’s sense follows from its statistical \((C, 1, 1)\) summability.

Keywords

Double sequences · Convergence in Pringsheim’s sense · \((C, 1, 1)\) summability · Statistical convergence · Slowly decreasing sequences · Slowly oscillating sequences · Statistically slowly decreasing sequences · Statistically slowly oscillating sequences · One-sided Tauberian conditions · Two-sided Tauberian conditions · Tauberian theorems

Mathematics Subject Classification 40A05 · 40A35 · 40E05 · 40G05

1 Introduction

In this section, we begin with some remarks about the late history of \((C, 1, 1)\) summability and its Tauberian results, that is about the history from the early part of twentieth century until these days. We shortly mention emergence of the concept of statistical convergence for double sequences and advancement of that in Tauberian theory. In the sequel, bringing together the concepts of \((C, 1, 1)\) summability and statistical convergence for double sequences under the same roof, we refer certain results obtained by several researchers concerning these concepts. After dwelling on studies that encourages us to do this research, we complete this section summarizing theorems and corollaries attained in this article.
Investigated for the first time in detail by Hardy [1] and Bromwich [2], the concept of double sequence has attracted the attention of several researchers dated from its appearance. One of the researchers who performed some studies on summability of double sequences, Agnew [3] obtained the extension of certain theorems on transformations of double sequences. Later on, Knopp [4] revealed some Tauberian results for \((C, 1, 1)\) summable double sequences generalizing conditions which were given for single sequences by Tauber [5]. Móricz [6] presented Tauberian theorems for double sequences which \(P\)-convergence follows from \((C, 1, 1)\) summability under necessary and sufficient conditions and slow decrease conditions in certain senses. Finally, Totur [7] investigated some conditions needed for \((C, 1, 1)\) summable double sequences to be convergent by using different approach.

Contrary to the common belief that the concept of statistical convergence, which is a natural generalization of that of ordinary convergence, was introduced by Fast [8] and Schoenberg [9], this concept was firstly came up with by Zygmund [10] who used the term *almost convergence* in place of statistical convergence and proved some theorems related to it. After the definition of statistical convergence was put into the final form by Fast [8] and Schoenberg [9], it was associated with Tauberian conditions given by several researchers from past to present. Introducing the concepts of statistical convergence and statistical Cauchy for double sequences, Tripathy [11] and Mursaleen and Edely [12] indicated to be a relation between statistical convergence and strong \(p\)-Cesàro summability for double sequences, independent of each other. Later on, Edely and Mursaleen [13] found out that necessary conditions for convergence of sequences which are statistically convergent and statistically \((C, 1, 1)\) summable are 

\[
i \Delta_{01} s_{\nu} = O_L \quad \text{and} \quad j \Delta_{01} s_{\mu, j} = O_L.\]

Móricz [14] presented Tauberian theorems for double sequences which statistical convergence follows from statistical \((C, 1, 1)\) summability under necessary and sufficient conditions and slow decrease conditions in certain senses. Totur and Çanak [15] obtained some Tauberian theorems for statistical convergence and statistical \((C, 1, 1)\) summability method by imposing some conditions on the difference sequence between a double sequence and its different arithmetic means.

Besides the studies mentioned up to now, the study that encourages us to do this research is in fact that including some results obtained by Móricz [16] for Cesàro (or \((C,1)\)) summable sequences of real and complex numbers. Móricz formulated these results as follows, respectively:

**Theorem 1** ([16]) If the sequence \((s_n)\) of real numbers is statistically \((C, 1)\) summable to \(\mu\) and slowly decreasing, then \((s_n)\) is convergent to \(\mu\).

**Theorem 2** ([16]) If the sequence \((s_n)\) of complex numbers is statistically \((C, 1)\) summable to \(\mu\) and slowly oscillating, then \((s_n)\) is convergent to \(\mu\).

Here, our aim is to extend theorems presented by Móricz for Cesàro summable sequences of real and complex numbers to \((C, 1, 1)\) summability method for double sequences of real and complex numbers.

In this paper, we indicate that some conditions under which \(P\)-convergence follows from statistical \((C, 1, 1)\) summability for double sequences of real and complex numbers. In Sect. 2, we recall basic definitions and notations with reference to double sequences. In Sect. 3, we firstly present some lemmas which will be benefited in
the proofs of our main results for double sequences of real numbers. In the sequel, we establish a Tauberian theorem for double sequences of real numbers which $P$-convergence follows from statistically $(C, 1, 1)$ summability under condition of slow decrease in certain senses and we present a corollary related to this theorem. In Sect. 4 in parallel with Sect. 3, we attain some lemmas which will be benefited in the proofs of our main results for double sequences of complex numbers at first. In the sequel, we prove a Tauberian theorem for double sequence of complex numbers which $P$-convergence follows from statistically $(C, 1, 1)$ summability under condition of slow oscillation in certain senses and we present a corollary related to this theorem.

2 Preliminaries

In this section, we begin with basic definitions and notations with reference to double sequences that will be needed throughout this paper. In the sequel, we mention about how connections exist between defined notions and we give some examples with regard to statements which hold for single sequences but not for double sequences. We end this section by introducing concepts of slow decrease and slow oscillation in certain senses and we state how a transition exists between them in pursuit of defining these concepts.

A double sequence $s = (s_{\mu \nu})$ is a function $s$ from $\mathbb{N} \times \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) into the set $\mathbb{K}$ ($\mathbb{R}$ is the set of real numbers or complex numbers). The real or complex number $s_{\mu \nu}$ denotes the value of the function at a point $(\mu, \nu) \in \mathbb{N} \times \mathbb{N}$ and is called the $(\mu, \nu)$-term of the double sequence. We denote the set of all double sequences of real and complex numbers by $w^2(\mathbb{R})$ and $w^2(\mathbb{C})$, respectively.

A double sequence $(s_{\mu \nu})$ is said to be convergent in Pringsheim’s sense (or $P$-convergent) to $\ell$ if for every $\epsilon > 0$ there exists a positive integer $v_0(\epsilon)$ such that $|s_{\mu \nu} - \ell| < \epsilon$ whenever $\mu, \nu \geq v_0$ (see [17]). The number $\ell$ is called the Pringsheim limit of $s$ and we denote by $P - \lim_{\mu, \nu \to \infty} s_{\mu \nu} = \ell$.

We exactly say that a double sequence $(s_{\mu \nu})$ converges to $\ell$ if $(s_{\mu \nu})$ tends to $\ell$ as both $\mu$ and $\nu$ tend to infinity independently of one another. We denote the set of all $P$-convergent double sequences of real and complex numbers by $c^2(\mathbb{R})$ and $c^2(\mathbb{C})$, respectively.

We write down that convergence mentioned throughout this study is convergence in Pringsheim’s sense.

A double sequence $(s_{\mu \nu})$ is bounded if there exists a positive number $M$ such that $|s_{\mu \nu}| < M$ for all $\mu$ and $\nu$.

We denote the set of all bounded double sequences of real and complex numbers by $\ell^2(\mathbb{R})$ and $\ell^2(\mathbb{C})$, respectively.

We write down that every $P$-convergent double sequences may not be bounded contrary to the case in single sequences. For instance, the sequence $(s_{\mu \nu})$ defined by...
\[ s_{\mu \nu} = \begin{cases} 2^\nu & \text{if } \mu = 1; \ \nu = 0, 1, 2, \ldots, \\ 2^{\mu+3} & \text{if } \nu = 4; \ \mu = 0, 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases} \]

is unbounded and \( P \)-convergent.

The \((C, 1, 1)\) mean of double sequence \((s_{\mu \nu})\) is defined by

\[ \sigma_{\mu \nu} := \frac{1}{(\mu + 1)(\nu + 1)} \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} s_{ij} \quad (1) \]

for all nonnegative integers \(\mu\) and \(\nu\).

A double sequence \((s_{\mu \nu})\) is said to be \((C, 1, 1)\) summable to \(\ell\) if \((\sigma_{\mu \nu})\) converges to the same number in Pringsheim’s sense.

We write down that every \(P\)-convergent double sequences need not be \((C, 1, 1)\) summable. For instance, the sequence \((s_{\mu \nu})\) defined by

\[ s_{\mu \nu} = \begin{cases} w_\mu & \text{if } \nu = 0; \ \mu = 0, 1, 2, \ldots, \\ w_\nu & \text{if } \mu = 0; \ \nu = 0, 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases} \]

where \((w_\nu) = \left(\sum_{k=0}^{\nu} (-1)^{k+1} k\right)\) is convergent to 0. On the other hand, one can check that

\[ s_{\mu \nu} = \begin{cases} \frac{-\mu}{2} & \text{if } \nu = 0; \ \mu = 2k, \ k = 0, 1, 2, \ldots, \\ \frac{\mu+1}{2} & \text{if } \nu = 0; \ \mu = 2k + 1, \ k = 0, 1, 2, \ldots, \\ \frac{-\nu}{2} & \text{if } \mu = 0; \ \nu = 2q, \ q = 0, 1, 2, \ldots, \\ \frac{\nu+1}{2} & \text{if } \mu = 0; \ \nu = 2q + 1, \ q = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \]

Then we have from the definition of \((C, 1, 1)\) means that

\[ \sigma_{\mu \nu} = \begin{cases} 0 & \text{if } \mu, \nu \text{ are even,} \\ \frac{1}{2(\mu+1)} & \text{if } \mu \text{ is even, } \nu \text{ is odd,} \\ \frac{1}{2(\nu+1)} & \text{if } \mu \text{ is odd, } \nu \text{ is even,} \\ \frac{1}{4} & \text{if } \mu, \nu \text{ are odd.} \end{cases} \]

Since the limit

\[ \lim_{\mu, \nu \to \infty} \sigma_{\mu \nu} = \lim_{\mu, \nu \to \infty} \frac{1}{(\mu + 1)(\nu + 1)} \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} s_{ij} = \begin{cases} \frac{1}{4} & \text{if } \mu, \nu \text{ are odd} \\ 0 & \text{otherwise,} \end{cases} \]

the sequence \((\sigma_{\mu \nu})\) is not convergent and hence \((s_{\mu \nu})\) is not \((C, 1, 1)\) summable.
In addition to this, every $P$-convergent double sequence is $(C, 1, 1)$ summable to same number under the boundedness condition of double sequence (see [3]). However, the converse of this statement is not always true. In other words, a double sequence which is $(C, 1, 1)$ summable and bounded may not be $P$-convergent. An example indicating this case was constructed by Mursaleen and Edely [12].

We now give the definition of natural density of $K \subset \mathbb{N} \times \mathbb{N}$ and present statistically convergent double sequences by using this concept. Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two dimensional subset of positive integers and $K_{\mu \nu} = \{(i, j) \in K : i \leq \mu, j \leq \nu\}$. Then, the set $K$ has a double natural density if the sequence $\left(\frac{|K_{\mu \nu}|}{(\mu + 1)(\nu + 1)}\right)$ has a limit in Pringsheim’s sense. In this case, we write

$$\delta_2(K) = \lim_{\mu, \nu \to \infty} \frac{|K_{\mu \nu}|}{(\mu + 1)(\nu + 1)},$$

where the vertical bar denotes the cardinality of the enclosed set.

A double sequence $(s_{\mu \nu})$ is said to be statistically convergent to $\ell$ if for every $\epsilon > 0$,.

$$\lim_{\mu, \nu \to \infty} \frac{1}{(\mu + 1)(\nu + 1)} \left|\{(i, j) \in \mathbb{N} \times \mathbb{N} : |s_{ij} - \ell| \geq \epsilon \text{ and } i \leq \mu, j \leq \nu\}\right| = 0.$$

We denote the set of all statistically convergent double sequences of real and complex numbers by $st^2(\mathbb{R})$ and $st^2(\mathbb{C})$, respectively. In this case, we write $st_2 - \lim_{\mu, \nu \to \infty} s_{\mu \nu} = \ell$.

We write down that every $P$-convergent double sequence is statistically convergent to same number but the converse of this statement is not necessarily true (see [13]). Namely, a statistically convergent double sequence may not be $P$-convergent. For instance, the sequence $(s_{\mu \nu})$ defined by

$$s_{\mu \nu} = \begin{cases} 1, & k^2 = \mu, q^2 = \nu, \\ 0, & \text{otherwise.} \end{cases}$$

is statistically convergent to 0, but not $P$-convergent.

We say that the double sequence $(s_{\mu \nu})$ is called statistically $(C, 1, 1)$ summable to $\ell$ if $st_2 - \lim_{\mu, \nu \to \infty} s_{\mu \nu} = \ell$.

We write down that every statistically convergent double sequence is statistically $(C, 1, 1)$ summable to same number under the boundedness condition of double sequence (see [13]).

At present, we define concepts of slow decrease and slow oscillation for the double sequences $(s_{\mu \nu})$ of real and complex numbers in certain senses, respectively. In pursuit of defining these concepts, we mention about how a transition exists between them. We say that a double sequence $(s_{\mu \nu})$ of real numbers is slowly decreasing in sense $(1, 1)$ if

$$\lim_{\mu, \nu \to \infty} \min_{1 \leq i \leq \mu, 1 \leq j \leq \nu} (s_{ij} - s_{\mu \nu}) \geq 0. \quad (2)$$

Using $\epsilon$’s and $\delta$’s, (2) is equivalent to the following statement:
To every $\epsilon > 0$ there exist $\nu_0 = \nu_0(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$s_{ij} - s_{\mu\nu} \geq -\epsilon \quad \text{whenever} \quad \nu_0 < \mu < i \leq \lambda\mu \quad \text{and} \quad \nu_0 < \nu < j \leq \lambda\nu.$$ 

We say that a double sequence $(s_{\mu\nu})$ of complex numbers is slowly oscillating in sense $(1, 1)$ if

$$\lim_{\lambda \downarrow 1} \limsup_{\mu, \nu \to \infty} \max_{\mu + 1 \leq i \leq \lambda\mu, \nu + 1 \leq j \leq \lambda\nu} |s_{ij} - s_{\mu\nu}| = 0 \quad (3)$$

Using $\epsilon$’s and $\delta$’s, $(3)$ is equivalent to the following statement:

To every $\epsilon > 0$ there exist $\nu_0 = \nu_0(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$|s_{ij} - s_{\mu\nu}| \leq \epsilon \quad \text{whenever} \quad \nu_0 < \mu < i \leq \lambda\mu \quad \text{and} \quad \nu_0 < \nu < j \leq \lambda\nu.$$ 

Here, by $\lambda\nu$ we denote the integral part of the product $\lambda\nu$.

It can be easily seen from $(3)$ that every $P$-convergent double sequences of complex numbers is slowly oscillating in sense $(1, 1)$, but converse of that is not always true. An example demonstrating this case was constructed by Çakalli and Patterson [18].

We say that a double sequence $(s_{\mu\nu})$ of real numbers is said to be slowly decreasing in sense $(1, 0)$ if

$$\lim_{\lambda \downarrow 1} \liminf_{\mu, \nu \to \infty} \min_{\mu + 1 \leq i \leq \lambda\mu, \nu + 1 \leq j \leq \lambda\nu} (s_{ij} - s_{\mu\nu}) \geq 0. \quad (4)$$

We say that a double sequence $(s_{\mu\nu})$ of complex numbers is said to be slowly oscillating in sense $(1, 0)$ if

$$\lim_{\lambda \downarrow 1} \limsup_{\mu, \nu \to \infty} \max_{\mu + 1 \leq i \leq \lambda\mu, \nu + 1 \leq j \leq \lambda\nu} |s_{ij} - s_{\mu\nu}| = 0. \quad (5)$$

We say that a double sequence $(s_{\mu\nu})$ of real numbers is said to be slowly decreasing in the strong sense $(1, 0)$ if $(4)$ is satisfied with

$$\min_{\mu + 1 \leq i \leq \lambda\mu, \nu + 1 \leq j \leq \lambda\nu} (s_{ij} - s_{\mu\nu}) \quad \text{instead of} \quad \min_{\mu + 1 \leq i \leq \lambda\mu} (s_{i\nu} - s_{\mu\nu}). \quad (6)$$

We say that a double sequence $(s_{\mu\nu})$ of complex numbers is said to be slowly oscillating in the strong sense $(1, 0)$ if $(5)$ is satisfied with

$$\max_{\mu + 1 \leq i \leq \lambda\mu, \nu + 1 \leq j \leq \lambda\nu} |s_{ij} - s_{\mu\nu}| \quad \text{instead of} \quad \max_{\mu + 1 \leq i \leq \lambda\mu} |s_{i\nu} - s_{\mu\nu}|. \quad (7)$$

Similarly, conditions of slow decrease and slow oscillation of double sequence $(s_{\mu\nu})$ of real and complex numbers in sense $(0, 1)$ and the strong sense $(0, 1)$ can be analogously defined.

We say that a double sequence $(s_{\mu\nu})$ of real numbers satisfies one-sided Tauberian condition of Landau type in sense $(1, 0)$ if there exist positive constants $\nu_0$ and $H$ such that

$$i(s_{i\nu} - s_{i-1,\nu}) \geq -H \quad \text{whenever} \quad i, \nu > \nu_0. \quad (8)$$
It is clear that if condition (8) holds, then \((s_{\mu \nu})\) is slowly decreasing in both sense \((1, 0)\) and the strong sense \((1, 0)\).

Similarly, one-sided Tauberian condition of Landau type in sense \((0, 1)\) can be analogously defined and any double sequence \((s_{\mu \nu})\) satisfying this condition is slowly decreasing in both sense \((0, 1)\) and strong sense \((0, 1)\).

We say that a double sequence \((s_{\mu \nu})\) of complex numbers satisfies two-sided Tauberian condition of Hardy type in sense \((1, 0)\) if there exist positive constants \(\nu_0\) and \(H\) such that

\[
i |s_{i \nu} - s_{i-1, \nu}| \leq H \text{ whenever } i, \nu > \nu_0.
\]

It is clear that if condition (9) holds, then \((s_{\mu \nu})\) is slowly oscillating in both sense \((1, 0)\) and the strong sense \((1, 0)\).

Similarly, two-sided Tauberian condition of Hardy type in sense \((0, 1)\) can be analogously defined and any double sequence \((s_{\mu \nu})\) satisfying this condition is slowly oscillating in both sense \((0, 1)\) and strong sense \((0, 1)\).

We write down that if the double sequence \((s_{\mu \nu})\) of real number is slowly decreasing in senses \((1, 0)\), \((0, 1)\) and slowly decreasing in the strong sense \((1, 0)\) or \((0, 1)\), then \((s_{\mu \nu})\) is slowly decreasing in sense \((1, 1)\).

In fact, suppose that \((s_{\mu \nu})\) is slowly decreasing in senses \((1, 0)\), \((0, 1)\) and in the strong sense \((1, 0)\) without loss of generality. For all large enough \(\mu\) and \(\nu\), that is, \(\mu, \nu \geq \nu_0\) and \(\lambda > 1\), we have

\[
\begin{align*}
\min_{\mu+1 \leq i \leq \lambda \mu, \nu+1 \leq j \leq \lambda \nu} (s_{ij} - s_{\mu \nu}) & = \min_{\mu + 1 \leq i \leq \lambda \mu, \nu + 1 \leq j \leq \lambda \nu} (s_{ij} - s_{\mu j} + s_{\mu j} - s_{\mu \nu}) \\
& \geq \min_{\mu + 1 \leq i \leq \lambda \mu, \nu + 1 \leq j \leq \lambda \nu} (s_{ij} - s_{\mu j}) + \min_{\nu + 1 \leq j \leq \lambda \nu} (s_{\mu j} - s_{\mu \nu}).
\end{align*}
\]

Taking the lim inf and the limit of both sides of (10) as \(\mu, \nu \to \infty\) and \(\lambda \to 1^+\) respectively, we attain that the terms on the right-hand side of (10) are greater than 0. Thus, we reach that \((s_{\mu \nu})\) is slowly decreasing in sense \((1, 1)\).

Similarly, if the double sequence \((s_{\mu \nu})\) of complex number is slowly oscillating in senses \((1, 0)\), \((0, 1)\) and slowly oscillating in the strong sense \((1, 0)\) or \((0, 1)\), then \((u_{mn})\) is slowly oscillating in sense \((1, 1)\).

### 3 Double sequences of real numbers

This section essentially consists of two parts. In the first part, we present some lemmas which will be used in the proofs of our main results for real sequences. In the second part, we obtain some Tauberian conditions under which \(P\)-convergence follows from statistically \((C, 1, 1)\) summability. In the sequel, we end this section by a corollary.
3.1 Lemmas

In this subsection, we express and prove the following assertions which will be benefited in the proofs of our main results for double sequences of real numbers. The following lemma presents two representations of the difference between the general terms of the double sequences \((s_{\mu\nu})\) and \((\sigma_{\mu\nu})\).

**Lemma 1** [6, Lemma 1] (i) If \(\lambda > 1, \lambda\mu > \mu\), and \(\lambda\nu > \nu\), then

\[
s_{\mu\nu} - \sigma_{\mu\nu} = \left(\frac{\lambda\mu + 1}{\lambda\mu - \mu}\right) \left(\frac{\lambda\nu + 1}{\lambda\nu - \nu}\right) \left(\sigma_{\lambda\mu,\lambda\nu} - \sigma_{\lambda\mu,\nu} - \sigma_{\mu,\lambda\nu} + \sigma_{\mu\nu}\right) + \left(\frac{\lambda\mu + 1}{\lambda\mu - \mu}\right) \left(\sigma_{\lambda\mu,\nu} - \sigma_{\mu\nu}\right) + \left(\frac{\lambda\nu + 1}{\lambda\nu - \nu}\right) \left(\sigma_{\mu,\lambda\nu} - \sigma_{\mu\nu}\right) + \frac{1}{(\lambda\mu - \mu)(\lambda\nu - \nu)} \sum_{i=\mu+1}^{\lambda\mu} \sum_{j=\nu+1}^{\lambda\nu} (s_{\mu\nu} - s_{ij}).
\]

(ii) If \(0 < \lambda < 1, \lambda\mu < \mu\), and \(\lambda\nu < \nu\), then

\[
s_{\mu\nu} - \sigma_{\mu\nu} = \left(\frac{\lambda\mu + 1}{\mu - \lambda\mu}\right) \left(\frac{\lambda\nu + 1}{\nu - \lambda\nu}\right) \left(\sigma_{\lambda\mu,\lambda\nu} - \sigma_{\lambda\mu,\nu} - \sigma_{\mu,\lambda\nu} + \sigma_{\mu\nu}\right) + \left(\frac{\lambda\mu + 1}{\mu - \lambda\mu}\right) \left(\sigma_{\mu\nu} - \sigma_{\lambda\mu,\nu}\right) + \left(\frac{\lambda\nu + 1}{\nu - \lambda\nu}\right) \left(\sigma_{\mu\nu} - \sigma_{\mu,\lambda\nu}\right) + \frac{1}{(\mu - \lambda\mu)(\nu - \lambda\nu)} \sum_{i=\lambda\mu+1}^{\mu} \sum_{j=\lambda\nu+1}^{\nu} (s_{\mu\nu} - s_{ij}).
\]

Vijayaraghavan [19] proved that there exists a constant \(K = K(c)\) corresponding to every constant \(c\) such that \(s_{\mu} - s_{\eta} > -(K \log(\mu/\eta) + c)\) for all values of \(\mu, \eta\) providing that the sequence \((s_{\mu})\) is slowly decreasing. In the sequel, Móricz [16] reconstructed this statement for sequences by using a condition less restrictive than the slow decrease condition. On the strength of these statement, we express two lemmas playing a crucial role in the proofs of subsequent lemmas which are necessary to achieve our main results for double sequences.

**Lemma 2** Let \((s_{\mu\nu})\) be a double sequence. If there exist a positive integer \(\nu_0\) and \(\lambda > 1\) such that

\[s_{i\nu} - s_{\mu\nu} \geq -1 \quad \text{whenever} \quad \nu_0 \leq \mu < \lambda\mu \text{ and } \nu_0 \leq \nu,
\]

then there exists a positive constant \(H\) such that

\[s_{i\nu} - s_{\mu\nu} \geq -H \log\left(\frac{i}{\mu}\right) \quad \text{whenever} \quad 1 \leq \mu \leq \frac{i}{\lambda} \text{ and } \nu_0 \leq \nu.
\]
Proof Suppose that \( \nu_0 \) is large enough to satisfy the condition
\[
\frac{2\lambda}{\lambda - 1} \leq \nu_0
\] (12)
without loss of generality. Let \( \nu_0 < i \). We define the subsequence
\[
i_0 := i \quad \text{and} \quad i_r := 1 + \left\lfloor \frac{i_{r-1}}{\lambda} \right\rfloor, \quad r = 1, 2, \ldots, q
\]
where \( q \) is specified by means of the condition \( i_{q+1} \leq \nu_0 < i_q \). We get from the definition of the subsequence \((i_r)\) that
\[
i_r < i_{r-1} < \lambda i_r, \quad r = 1, 2, \ldots, q + 1.
\]
Choose \( \mu \) such that \( 1 \leq \mu \leq \frac{i}{\lambda} \). We investigate chosen \( \mu \) in two cases such that \( \nu_0 \leq \mu \leq \frac{i}{\lambda} \) and \( 1 \leq \mu < \nu_0 \). We firstly take into consideration the case \( \nu_0 \leq \mu \leq \frac{i}{\lambda} \).

Then, we define \( r \) such that
\[
i_{r+1} \leq \mu < i_r \quad \text{for some} \quad 1 \leq r \leq q.
\] (13)
If we regard assumption, we attain for \( \nu_0 \leq v \)
\[
s_i v - s_{\mu v} = (s_i v - s_{i_1} v) + (s_{i_1} v - s_{i_2} v) + \cdots + (s_{i_{r-1}} v - s_{i_r} v) + (s_{i_r} v - s_{\mu v}) \\
\geq -(r + 1).
\] (14)
By the definition of the subsequence \((i_r)\), we obtain
\[
i_1 := 1 + \left\lfloor \frac{i}{\lambda} \right\rfloor \leq 1 + \frac{i}{\lambda}, \quad i_2 \leq 1 + \frac{i_1}{\lambda} \leq 1 + \frac{i}{\lambda^2}, \ldots,
\]
\[
i_r \leq 1 + \frac{i}{\lambda} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^{r-1}} + \frac{i}{\lambda^r} = \sum_{k=0}^{r-1} \left( \frac{1}{\lambda} \right)^k + \frac{i}{\lambda^r}
\]
\[
\leq \sum_{k=0}^{\infty} \left( \frac{1}{\lambda} \right)^k + \frac{i}{\lambda^r} = \frac{\lambda}{\lambda - 1} + \frac{i}{\lambda^r}.
\]
Thus, we reach
\[
\frac{1}{2} \lambda^r \leq \left( 1 - \frac{\lambda}{(\lambda - 1)v_0} \right) \lambda^r \leq \left( 1 - \frac{\lambda}{(\lambda - 1)i_r} \right) \lambda^r < \frac{i}{i_r} < \frac{i}{\mu}
\]
by using (12) and (13), and so we find
\[
r \leq \frac{1}{\log \lambda} \log \frac{2i}{\mu}, \quad i_{r+1} \leq \mu < i_r \quad \text{for some} \quad 1 \leq r \leq q.
\] (15)
If we consider together (14) and (15), then we conclude for $\nu_0 < \nu$

$$s_{i\nu} - s_{\mu\nu} \geq -\left(1 + \frac{1}{\log \lambda} \log \frac{2i}{\mu}\right), \text{ whenever } \nu_0 \leq \mu \leq \frac{i}{\lambda}. \quad (16)$$

On other respects, we take into consideration the case $1 \leq \mu < \nu_0$. Continuing the similar process above and considering the assumption, for $\nu_0 < \nu$ we find

$$s_{i\nu} - s_{\mu\nu} = (s_{i\nu} - s_{i1\nu}) + (s_{i1\nu} - s_{i2\nu}) + \cdots + (s_{iq\nu} - s_{\nu_0\nu}) + (s_{\nu_0\nu} - s_{\mu\nu}) \geq -(q + 1) + c. \quad (17)$$

where $c := \min \left\{0, \min_{1 \leq \mu < \nu_0} (s_{\nu_0\nu} - s_{\mu\nu})\right\}$. Following a similar process to (15), we attain

$$q \leq \frac{1}{\log \lambda} \log \frac{2i}{\mu} \text{ whenever } 1 \leq \mu < \nu_0. \quad (18)$$

If we consider together (17) and (18), then we conclude for $\nu_0 < \nu$

$$s_{i\nu} - s_{\mu\nu} \geq -\left(1 + \frac{1}{\log \lambda} \log \frac{2i}{\mu}\right) + c, \text{ whenever } 1 \leq \mu < \nu_0. \quad (19)$$

Due to $c \leq 0$, it follows from (16) and (19) that we have for $\nu_0 < \nu$

$$s_{i\nu} - s_{\mu\nu} \geq -1 + c - \frac{\log 2}{\log \lambda} - \frac{\log \frac{i}{\mu}}{\log \lambda}$$

$$= \left(H - \frac{1}{\log \lambda}\right) (-\log \lambda) - \frac{\log \frac{i}{\mu}}{\log \lambda}$$

$$\geq \left(H - \frac{1}{\log \lambda}\right) (-\log \frac{i}{\mu}) - \frac{\log \frac{i}{\mu}}{\log \lambda}$$

$$= -H \log \frac{i}{\mu}$$

whenever $1 \leq \mu \leq \frac{i}{\lambda}$ provided that

$$H := \frac{1}{\log \lambda} \left(2 + \frac{\log 2}{\log \lambda} - c\right).$$

\(\square\)

**Lemma 3** Let $(s_{\mu\nu})$ be a double sequence. If there exist a positive integer $\nu_0$ and $\lambda > 1$ such that

$$s_{\mu j} - s_{\mu\nu} \geq -1 \text{ whenever } \nu_0 \leq \nu < j < \lambda \nu \text{ and } \nu_0 \leq \mu,$$

\(\square\) Springer
then there exists a positive constant $H$ such that

$$s_{µ,j} - s_{µ,v} \geq -H \log \left( \frac{j}{v} \right)$$

whenever $1 \leq v \leq \frac{j}{λ}$ and $v_0 \leq µ$.

**Proof** We can prove Lemma 3 following a procedure which is similar to the proof of Lemma 2.

Armitage and Maddox [20] indicated that the sequence $(ω_n) = (\sum_{k=0}^{n}(u_n - u_k))$ is bounded below providing that the sequence $(u_n)$ is slowly decreasing. On the strength of this statement, we prove that double sequences given in Lemmas 4, 5 and 6 are bounded below under slow decrease conditions in certain senses.

**Lemma 4** If the double sequence $(s_{µ,v})$ is slowly decreasing in sense $(1, 0)$, then the sequence

$$\left( \frac{1}{i+1} \sum_{µ=0}^{i}(s_{i,v} - s_{µ,v}) \right)$$

is bounded below.

**Proof** Suppose that $(s_{µ,v})$ is slowly decreasing in sense $(1, 0)$. Then there exist a positive integer $v_0$ and $λ > 1$ such that

$$s_{i,v} - s_{µ,v} \geq -1$$

whenever $v_0 \leq µ < i < λµ$ and $v_0 \leq v$. (20)

It follows from Lemma 2 that there exists a positive constant $H$ such that

$$s_{i,v} - s_{µ,v} \geq -H \log \left( \frac{i}{µ} \right)$$

whenever $1 \leq µ \leq \frac{i}{λ}$ and $v_0 \leq v$. (21)

If we particularly take $µ = 1$, we have

$$s_{i,v} - s_{1,v} \geq -H \log i$$

for $λi ≤ i$ and $v_0 ≤ v$. Accordingly, we attain that the sequence

$$\left( \frac{1}{i+1} \sum_{µ=1}^{i}(s_{i,v} - s_{µ,v}) \right)$$

is bounded below. In order to reach the end of the proof, it is enough to indicate that

$$\sum_{µ=1}^{i}(s_{i,v} - s_{µ,v}) = \left[ \frac{i}{µ} \right] + \sum_{µ=\lceil \frac{i}{λ} \rceil+1}^{i}(s_{i,v} - s_{µ,v})$$

$$\geq -H \sum_{µ=1}^{\left\lfloor \frac{i}{λ} \right\rfloor} \log \left( \frac{i}{µ} \right) - \left( i - \left\lceil \frac{i}{λ} \right\rceil \right)$$

$$\geq -H \sum_{µ=1}^{i} \log \left( \frac{i}{µ} \right) - i$$
which means that the sequence is bounded below. □

Lemma 5 If the double sequence \((s_{\mu v})\) is slowly decreasing in sense \((0, 1)\), then the sequence

\[
\left( \frac{1}{j + 1} \sum_{v=0}^{j} (s_{\mu j} - s_{\mu v}) \right)
\]

is bounded below.

Proof We can prove Lemma 5 following a procedure which is similar to the proof of Lemma 4. □

Lemma 6 If the double sequence \((s_{\mu v})\) is slowly decreasing in senses \((1, 0)\), \((0, 1)\) and slowly decreasing in the strong sense \((1, 0)\) or \((0, 1)\), then the sequence

\[
\left( \frac{1}{(i + 1)(j + 1)} \sum_{\mu=0}^{i} \sum_{v=0}^{j} (s_{ij} - s_{\mu v}) \right)
\]

is bounded below.

Proof Suppose that \((s_{\mu v})\) is slowly decreasing in senses \((1, 0)\), \((0, 1)\) and slowly decreasing in the strong sense \((1, 0)\) without loss of generality. Then there exist positive integers \(\nu_0\), \(\nu_1\) and \(\lambda > 1\) such that

\[
s_{i \nu} - s_{\mu v} \geq -1 \quad \text{whenever} \quad \nu_0 \leq \mu < i < \lambda \mu \quad \text{and} \quad \nu_0 \leq \nu \quad (22)
\]

and

\[
s_{\mu j} - s_{\mu v} \geq -1 \quad \text{whenever} \quad \nu_1 \leq \nu < j < \lambda \nu \quad \text{and} \quad \nu_1 \leq \mu, \quad (23)
\]

respectively. It follows from Lemmas 2 and 3 that there exist positive constants \(H_0\) and \(H_1\) such that

\[
s_{i \nu} - s_{\mu v} \geq -H_0 \log \left( \frac{i}{\mu} \right) \quad \text{whenever} \quad 1 \leq \mu \leq \frac{i}{\lambda} \quad \text{and} \quad \nu_0 \leq \nu \quad (24)
\]

and

\[
s_{\mu j} - s_{\mu v} \geq -H_1 \log \left( \frac{j}{\nu} \right) \quad \text{whenever} \quad 1 \leq \nu \leq \frac{j}{\lambda} \quad \text{and} \quad \nu_1 \leq \mu, \quad (25)
\]
respectively. If we particularly take $\mu, \nu = 1$, we have $s_{ij} - s_{1j} \geq -H_0 \log i$ for $\lambda \leq i$, $v_0 \leq j$ and $s_{1j} - s_{11} \geq -H_1 \log j$ for $v_0 < \lambda \leq j$. Using these inequalities, Lemmas 4 and 5, if we take into consideration that the sequences $\left(\frac{1}{i+1} (s_{i\nu} - s_{0\nu})\right)$ and $\left(\frac{1}{j+1} (s_{\mu j} - s_{\mu 0})\right)$ is bounded below, which were attained in proofs of Lemmas 4 and 5 respectively, then we can find

$$\frac{1}{(i+1)(j+1)} \left( \sum_{\mu=1}^{i} (s_{ij} - s_{\mu 0}) + \sum_{\nu=1}^{j} (s_{ij} - s_{0\nu}) + (s_{ij} - s_{00}) \right)$$

$$= \frac{1}{(i+1)(j+1)} \left( \sum_{\mu=1}^{i} (s_{ij} - s_{\mu j}) + \sum_{\mu=1}^{i} (s_{\mu j} - s_{\mu 0}) \right)$$

$$+ \frac{1}{(i+1)(j+1)} \left( \sum_{\nu=1}^{j} (s_{ij} - s_{iv}) + \sum_{\nu=1}^{j} (s_{iv} - s_{0\nu}) \right)$$

$$+ \frac{1}{(i+1)(j+1)} \left( (s_{ij} - s_{1j}) + (s_{1j} - s_{11}) + (s_{11} - s_{00}) \right)$$

$$\geq - \left( \frac{C_0}{i+1} + \frac{i}{i+1} C_1 \right) - \left( \frac{C_2}{i+1} + \frac{j}{j+1} C_3 \right)$$

$$- \left( \frac{\log i}{i+1} \frac{H_0}{j+1} + \frac{H_1 \log j}{i+1} \frac{H_2}{(i+1)(j+1)} \right)$$

$$\geq - 4 \max\{C_0, C_1, C_2, C_3\} - 3 \max\{H_0, H_1, H_2\}.$$

In conclusion, we reach that the sequence

$$\left( \frac{1}{(i+1)(j+1)} \left( \sum_{\mu=1}^{i} (s_{ij} - s_{\mu 0}) + \sum_{\nu=1}^{j} (s_{ij} - s_{0\nu}) + (s_{ij} - s_{00}) \right) \right)$$

is bounded below. In order to reach the end of the proof, it is enough to indicate that $\left( \frac{1}{(i+1)(j+1)} \sum_{\mu=1}^{i} \sum_{\nu=1}^{j} (s_{ij} - s_{\mu \nu}) \right)$ is bounded below. If we consider $v_2 = \min\{v_0, v_1\}$ and combine (22), (23), (24) and (25), we get for $\lambda, v_2 \leq i$ and $\lambda, v_2 \leq j$

$$\sum_{\mu=1}^{i} \sum_{\nu=1}^{j} (s_{ij} - s_{\mu \nu}) = \sum_{\mu=1}^{i} \sum_{\nu=1}^{j} (s_{ij} - s_{\mu j}) + \sum_{\mu=1}^{i} \sum_{\nu=1}^{j} (s_{\mu j} - s_{\mu \nu})$$

$$= \left\{ \sum_{\mu=1}^{i} \left[ \sum_{\mu=1}^{\left\lfloor \frac{i}{2} \right\rfloor + 1} \sum_{\nu=1}^{j} (s_{ij} - s_{\mu j}) \right] \right\}$$
\begin{align*}
+ \sum_{\mu=1}^{i} \left\{ \sum_{v=1}^{\left\lfloor \frac{i}{\mu} \right\rfloor + 1} + \sum_{v=1}^{\left\lfloor \frac{j}{\mu} \right\rfloor + 1} \right\} (s_{\mu j} - s_{\mu v})
\geq -H_0 \sum_{\mu=1}^{i} \sum_{v=1}^{j} \log \left( \frac{i}{\mu} \right) - \left( i - \left\lfloor \frac{i}{\lambda} \right\rfloor \right) j \\
- H_1 \sum_{\mu=1}^{i} \sum_{v=1}^{j} \log \left( \frac{j}{v} \right) - \left( j - \left\lfloor \frac{j}{\lambda} \right\rfloor \right) i \\
\geq -H_0 \sum_{\mu=1}^{i} \sum_{v=1}^{j} \log \left( \frac{i}{\mu} \right) - H_1 \sum_{\mu=1}^{i} \sum_{v=1}^{j} \log \left( \frac{j}{v} \right) - 2ij \\
\geq -H \sum_{\mu=1}^{i} \sum_{v=1}^{j} \left\{ \log \left( \frac{i}{\mu} \right) - \log \left( \frac{j}{v} \right) \right\} - 2ij \\
= -H \sum_{\mu=1}^{i} \sum_{v=1}^{j} \left\{ \log i - (\log \mu + \log v) \right\} - 2ij \\
= -H \mu j \log i j + Hj \sum_{\mu=2}^{i} \log \mu + Hi \sum_{v=2}^{j} \log v - 2ij \\
\geq -H \mu j \log i j + Hj \int_{1}^{i} \log u du + Hi \int_{1}^{j} \log w dw - 2ij \\
= -2ij(H + 1) + (i + j)H \\
\geq -2ij(H + 1)
\end{align*}

where \( \max\{H_0, H_1\} = H \). In conclusion, we reach that the sequence is bounded below. \( \Box \)

At once, we refer a lemma authenticated by Chen and Chang [21] for statistical convergence studied as a summability method.

**Lemma 7** [21, Theorem 2.3] If the double sequence \((s_{\mu v})\) is statistically convergent to a number \( \ell \) and slowly decreasing in sense \((1, 1)\), then \((s_{\mu v})\) is \(P\)-convergent to \( \ell \).

**Lemma 8** [6, Corollary 1] If the double sequence \((s_{\mu v})\) is \((C, 1, 1)\) summable to a number \( \ell \) and slowly decreasing in sense \((1, 1), (1, 0)\) and \((0, 1)\), then \((s_{\mu v})\) is \(P\)-convergent to \( \ell \).
3.2 Main results

In this subsection, we prove a Tauberian theorem for double sequence of real numbers that $P$-convergence follows from statistically $(\ell, 1, 1)$ summability under condition of slow decrease in certain senses and we present a corollary related to this theorem.

**Theorem 3** Let the double sequence $(s_{\mu \nu})$ be statistically $(\ell, 1, 1)$ summable to a number $\ell$. If $(s_{\mu \nu})$ is slowly decreasing in senses $(1, 0)$ and $(0, 1)$ and slowly decreasing in the strong sense $(1, 0)$ or $(0, 1)$, then $(s_{\mu \nu})$ is $P$-convergent to $\ell$.

**Proof** Suppose that $(s_{\mu \nu})$ being statistically $(\ell, 1, 1)$ summable to $\ell$ is slowly decreasing in senses $(1, 0), (0, 1)$ and slowly decreasing in the strong sense $(1, 0)$ without loss of generality. In order to prove that $(s_{\mu \nu})$ is $P$-convergent to the same number, we firstly indicate that if $(s_{\mu \nu})$ is slowly decreasing in senses $(1, 0), (0, 1)$ and the strong sense $(1, 0)$, then $(s_{\mu \nu})$, which is the arithmetic means of $(s_{\mu \nu})$, is slowly decreasing in sense $(1, 1)$. Given $\epsilon > 0$. From the hypothesis, there exist positive integers $\nu_0 = \nu_0(\epsilon), \nu_1 = \nu_1(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$s_{i\nu} - u_{\mu \nu} \geq -\epsilon \quad \text{whenever} \quad \nu_0 \leq \mu < i \leq \lambda \mu \quad \text{and} \quad \nu_0 \leq \nu$$

and

$$s_{\mu j} - s_{\mu \nu} \geq -\epsilon \quad \text{whenever} \quad \nu_1 \leq \nu < j \leq \lambda \nu \quad \text{and} \quad \nu_1 \leq \mu,$$

respectively. Additionally, it is known that if $(s_{\mu \nu})$ is slowly decreasing in senses $(1, 0), (0, 1)$ and slowly decreasing in the strong sense $(1, 0)$, then it is also slowly decreasing in sense $(1, 1)$. In this case, there exist positive integer $\nu_2 = \min\{\nu_0, \nu_1\}$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$s_{ij} - s_{\mu \nu} \geq -\epsilon \quad \text{whenever} \quad \nu_2 \leq \mu < i \leq \lambda \mu \quad \text{and} \quad \nu_2 \leq \nu < j \leq \lambda \nu.$$
\[
\begin{align*}
&+ \frac{1}{(i + 1)(j + 1)} \sum_{p=\mu+1}^{i} \sum_{q=\nu+1}^{j} s_{pq} \\
&+ \frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu\nu} \\
&+ \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu q} \\
&+ \frac{(j - \nu)}{(i + 1)(j + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{p v} \\
&- \frac{1}{(\mu + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} \\
&- \frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu\nu} \\
&- \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu q} \\
&- \frac{(j - \nu)}{(i + 1)(j + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{p v} \\
&= \frac{1}{(i + 1)(j + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} + \frac{1}{(i + 1)(j + 1)} \sum_{p=0}^{\mu} \sum_{q=\nu+1}^{j} s_{pq} \\
&+ \frac{1}{(i + 1)(j + 1)} \sum_{p=\mu+1}^{i} \sum_{q=0}^{v} s_{pq} \\
&+ \frac{1}{(i + 1)(j + 1)} \sum_{p=\mu+1}^{i} \sum_{q=\nu+1}^{j} s_{pq} \\
&+ \frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu\nu} \\
&+ \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu q} \\
&+ \frac{(j - \nu)}{(i + 1)(j + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{p v}
\end{align*}
\]
\[
\frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} \\
- \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} \\
- \frac{(j - \nu)}{(i + 1)(j + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} \\
- \frac{1}{(i + 1)(j + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} \\
- \frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu q} \\
- \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu q} \\
- \frac{(j - \nu)}{(i + 1)(j + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\nu v} \\
= \frac{1}{(i + 1)(j + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} + \frac{1}{(i + 1)(j + 1)} \sum_{p=\mu+1}^{i} \sum_{q=0}^{\nu+1} s_{pq} \\
+ \frac{1}{(i + 1)(j + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} \\
+ \frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu v} \\
+ \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu q} \\
+ \frac{(j - \nu)}{(i + 1)(j + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\nu v} \\
- \frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} \\
- \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq} \\
- \frac{(j - \nu)}{(i + 1)(j + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{pq}
\[
\begin{align*}
&\quad - \frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu \nu} \\
&\quad - \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{\mu q} \\
&\quad - \frac{(j - \nu)}{(i + 1)(j + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} s_{p \nu} \\
&\quad = \frac{1}{(i + 1)(j + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{i} (s_{pq} - s_{p \nu}) \\
&\quad + \frac{1}{(i + 1)(j + 1)} \sum_{p=0}^{\mu+1} \sum_{q=0}^{i} (s_{pq} - s_{\mu q}) \\
&\quad + \frac{1}{(i + 1)(j + 1)} \sum_{p=0}^{\mu+1} \sum_{q=0}^{j} (s_{pq} - s_{\mu v}) \\
&\quad + \frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} (s_{\mu \nu} - s_{pq}) \\
&\quad + \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} (s_{\mu q} - s_{pq}) \\
&\quad + \frac{(j - \nu)}{(i + 1)(j + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} (s_{p \nu} - s_{pq}).
\end{align*}
\]

From Lemmas 4, 5 and 6, there exist positive constants \( H_0, H_1 \) and \( H_2 \) such that

\[
\frac{1}{\mu + 1} \sum_{p=0}^{\mu} (s_{\mu q} - s_{pq}) \geq -H_0 \quad \text{and} \quad \frac{1}{\nu + 1} \sum_{q=0}^{v} (s_{p \nu} - s_{pq}) \geq -H_1,
\]

\[
\frac{1}{(\mu + 1)(\nu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} (s_{\mu v} - s_{pq}) \geq -H_2
\]

for all nonnegative integers \( \mu \) and \( \nu \). Using slow decrease conditions in certain senses and these inequalities, we attain

\[
\sigma_{ij} - \sigma_{\mu \nu} \geq \frac{(\mu + 1)}{(i + 1)(j + 1)} (j - \nu) (-\epsilon) + \frac{(i - \mu)}{(i + 1)(j + 1)} (\nu + 1) (-\epsilon) + \frac{(i - \mu)}{(i + 1)(j + 1)} (j - \nu) (-\epsilon) \\
+ \frac{(i - \mu)}{(i + 1)(j + 1)} (\nu + 1) (-H_2) + \frac{(i - \mu)}{(i + 1)(j + 1)} (\nu + 1) (-H_0) \\
+ \frac{(i + 1)}{(i + 1)(j + 1)} (j - \nu) (-H_1)
\]
Tauberian theorems for statistically \((C, 1, 1)\) summable… 909

\[
\begin{align*}
&= \frac{(\mu + 1)}{(i + 1)} \frac{(j - \nu)}{(j + 1)} (-\epsilon - H_1) \\
&+ \frac{(i - \mu)}{(i + 1)} \frac{(v + 1)}{(j + 1)} (-\epsilon - H_0) + \frac{(i - \mu)}{(i + 1)} \frac{(j - \nu)}{(j + 1)} (-\epsilon - H_2) \\
&\geq (\lambda - 1)(-\epsilon - H_1) + (\lambda - 1)(-\epsilon - H_0) + (\lambda - 1)^2(-\epsilon - H_2),
\end{align*}
\] (27)

since we have for \(\lambda > 1\)

\[
\frac{\mu + 1}{i + 1} < 1, \quad \frac{i - \mu}{i + 1} = 1 - \frac{\mu + 1}{i + 1} < 1 - \frac{1}{\lambda} < \lambda - 1
\] (28)

and

\[
\frac{\nu + 1}{j + 1} < 1, \quad \frac{j - \nu}{j + 1} = 1 - \frac{\nu + 1}{j + 1} < 1 - \frac{1}{\lambda} < \lambda - 1
\] (29)

whenever \(\mu < i \leq \lambda \mu\) and \(\nu < j \leq \lambda \nu\). If we take \(H = \max\{H_0, H_1, H_2\}\) and \(1 < \lambda \leq \sqrt{\frac{\epsilon}{\epsilon + H} + 1}\), then we find \(\sigma_{ij} - \sigma_{\mu \nu} \geq -\epsilon\) whenever \(v_2 \leq \mu < i \leq \lambda \mu\) and \(v_2 \leq \nu < j \leq \lambda \nu\). Thus, we arrive that \((\sigma_{\mu \nu})\) is slowly decreasing in sense \((1, 1)\). Since \((\sigma_{\mu \nu})\) is slowly decreasing in sense \((1, 1)\) and statistically convergent to \(\ell\), we get that \((\sigma_{\mu \nu})\) is convergent to \(\ell\) by Lemma 7. If we consider that conditions of slow decrease in senses \((1, 0)\), \((0, 1)\) and in the strong sense \((1, 0)\) imply condition of slow decrease in sense \((1, 1)\), then we conclude by virtue of Lemma 8 that \((s_{\mu \nu})\) is \(P\)-convergent to \(\ell\). \(\square\)

Since it is known that if one-sided Tauberian conditions of Landau type in senses \((1, 0)\) and \((0, 1)\) is valid, then the double sequence is slowly decreasing in senses \((1, 0)\) and \((0, 1)\) and also slowly decreasing in the strong senses \((1, 0)\) and \((0, 1)\) respectively, we can present Corollary 1.

**Corollary 1** Let the double sequence \((s_{\mu \nu})\) be statistically \((C, 1, 1)\) summable to a number \(\ell\). If \((s_{\mu \nu})\) satisfies one-sided Tauberian conditions of Landau type in senses \((1, 0)\) and \((0, 1)\), then \((s_{\mu \nu})\) is \(P\)-convergent to \(\ell\).

### 4 Double sequences of complex numbers

This section essentially consists of two parts. In the first part, we present some lemmas which will be used in the proofs of our main results for double sequences of complex numbers. In the second part, we obtain some Tauberian conditions under which \(P\)-convergence follows from statistically \((C, 1, 1)\) summability. In the sequel, we end this section by a corollary.

#### 4.1 Lemmas

In this subsection, we express and prove the following assertions which will be benefited in the proofs of our main results for double sequences of complex numbers.
Vijayaraghavan [19] proved that there exists a constant $K = K(c)$ corresponding to every constant $c$ such that $s_\mu - s_\eta > - (K \log(\mu/\eta) + c)$ for all values of $\mu, \eta$ providing that the sequence $(s_\mu)$ of real numbers is slowly decreasing. In the sequel, Móricz [16] reconstructed this statement for sequences of real numbers by using a condition less restrictive than the slow decrease condition and extended it to sequences of complex numbers with the help of a similar proof process. On the strength of these statement, we express two lemmas playing a crucial role in the proofs of subsequent lemmas which are necessary to achieve our main results for double sequences.

**Lemma 9** Let $(s_{\mu \nu})$ be a double sequence. If there exist a positive integer $\nu_0$ and $\lambda > 1$ such that

$$|s_{i\nu} - s_{\mu \nu}| \leq 1 \quad \text{whenever} \quad \nu_0 \leq \mu < i\lambda \mu \text{ and } \nu_0 \leq \nu,$$

then there exists a constant $H$ such that

$$|s_{i\nu} - s_{\mu \nu}| \leq H \log \left( \frac{i}{\mu} \right) \quad \text{whenever} \quad 1 \leq \mu \leq \frac{i}{\lambda} \text{ and } \nu_0 \leq \nu.$$

**Proof** Suppose that $\nu_0$ is large enough to satisfy the condition

$$\frac{2\lambda}{\lambda - 1} \leq \nu_0$$

without loss of generality. Let $\nu_0 < i$. We define the subsequence

$$i_0 := i \quad \text{and} \quad i_r := 1 + \left[ \frac{i_{r-1}}{\lambda} \right], \quad r = 1, 2, \ldots, q$$

where $q$ is specified by means of the condition $i_{q+1} \leq \nu_0 < i_q$. We get from the definition of the subsequence $(i_r)$ that

$$i_r < i_{r-1} < \lambda i_r, \quad r = 1, 2, \ldots, q + 1.$$

Choose $\mu$ such that $1 \leq \mu \leq \frac{i}{\lambda}$. We investigate chosen $\mu$ in two cases such that $\nu_0 \leq \mu \leq \frac{i}{\lambda}$ and $1 \leq \mu < \nu_0$. We firstly take into consideration the case $\nu_0 \leq \mu \leq \frac{i}{\lambda}$. Then, we define $r$ such that

$$i_{r+1} \leq \mu < i_r \quad \text{for some} \quad 1 \leq r \leq q.$$ (31)

If we regard assumption, we attain for $\nu_0 \leq \nu$

$$|s_{i\nu} - s_{\mu \nu}| \leq |s_{i\nu} - s_{i_1 \nu}| + |s_{i_1 \nu} - s_{\mu \nu}|$$

$$\leq |s_{i\nu} - s_{i_1 \nu}| + |s_{i_1 \nu} - s_{i_2 \nu}| + \cdots + |s_{i_{r-1} \nu} - s_{i_r \nu}| + |s_{i_r \nu} - s_{\mu \nu}|$$

$$\leq r + 1.$$ (32)
By the definition of the subsequence \((i_r)\), we obtain

\[
i_1 := 1 + \left[ \frac{i}{\lambda} \right] \leq 1 + \frac{i}{\lambda}, \quad i_2 \leq 1 + \frac{i_1}{\lambda} \leq 1 + \frac{1}{\lambda} + \frac{i}{\lambda^2}, \ldots,
\]

\[
i_r \leq 1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^{r-1}} + \frac{i}{\lambda^r} = \sum_{k=0}^{r-1} \left( \frac{1}{\lambda} \right)^k + \frac{i}{\lambda^r}
\]

\[
\leq \sum_{k=0}^{\infty} \left( \frac{1}{\lambda} \right)^k + \frac{i}{\lambda^r} = \frac{\lambda}{\lambda - 1} + \frac{i}{\lambda^r}.
\]

Thus, we reach

\[
\frac{1}{2} \lambda^r \leq \left( 1 - \frac{\lambda}{(\lambda - 1)v_0} \right) \lambda^r \leq \left( 1 - \frac{\lambda}{(\lambda - 1)i_r} \right) \lambda^r < \frac{i}{i_r} < \frac{i}{\mu}
\]

by using (30) and (31), and so we find

\[
r \leq \frac{1}{\log \lambda} \log \frac{2i}{\mu}, \quad i_{r+1} \leq \mu < i_r \quad \text{for some} \quad 1 \leq r \leq q.
\] (33)

If we consider together (32) and (33), then we conclude for \(v_0 < v\)

\[
|s_{iv} - s_{\mu v}| \leq 1 + \frac{1}{\log \lambda} \log \frac{2i}{\mu}, \quad \text{whenever} \quad v_0 \leq \mu \leq \frac{i}{\lambda}.
\] (34)

On other respect, we take into consideration the case \(1 \leq \mu < v_0\). Continuing the similar process above and considering the assumption, for \(v_0 < v\) we find

\[
|s_{iv} - s_{\mu v}| \leq |s_{iv} - s_{i_1 v}| + |s_{i_1 v} - s_{\mu v}|
\]

\[
\leq |s_{iv} - s_{i_1 v}| + |s_{i_1 v} - s_{i_2 v}| + \cdots + |s_{i_q v} - s_{v_0 v}| + |s_{v_0 v} - s_{\mu v}|
\]

\[
\leq (q + 1) + c.
\] (35)

where \(c := \max_{1 \leq \mu < v_0} |s_{v_0 v} - s_{\mu v}|\). Following a similar process to (33), we attain

\[
q \leq \frac{1}{\log \lambda} \log \frac{2i}{\mu} \quad \text{whenever} \quad 1 \leq \mu < v_0.
\] (36)

If we consider together (35) and (36), then we conclude for \(v_0 < v\)

\[
|s_{iv} - s_{\mu v}| \leq 1 + c + \frac{1}{\log \lambda} \log \frac{2i}{\mu}, \quad \text{whenever} \quad 1 \leq \mu < v_0.
\] (37)
It follows from (34) and (37) that we have for \( \nu_0 < \nu \)

\[
|s_{i\nu} - s_{i\nu_{\nu}}| \leq 1 + c + \frac{\log 2}{\log \lambda} + \frac{\log \frac{i}{\mu}}{\log \lambda} = \left( H - \frac{1}{\log \lambda} \right) \log \lambda + \frac{\log \frac{i}{\mu}}{\log \lambda} \\
\leq \left( H - \frac{1}{\log \lambda} \right) \log \frac{i}{\mu} + \frac{\log \frac{i}{\mu}}{\log \lambda} \\
= H \log \frac{i}{\mu}
\]

whenever \( 1 \leq \mu \leq \frac{i}{\lambda} \) provided that

\[
H := \frac{1}{\log \lambda} \left( 2 + c + \frac{\log 2}{\log \lambda} \right).
\]

Lemma 10 Let \((s_{\mu\nu})\) be a double sequence. If there exist a positive integer \( \nu_0 \) and \( \lambda > 1 \) such that

\[
|s_{\mu j} - s_{\mu\nu}| \leq 1 \quad \text{whenever} \quad \nu_0 \leq \nu < j < \lambda \nu \quad \text{and} \quad \nu_0 \leq \mu,
\]

then there exists a constant \( H \) such that

\[
|s_{\mu j} - s_{\mu\nu}| \leq H \log \left( \frac{j}{\nu} \right) \quad \text{whenever} \quad 1 \leq \nu \leq \frac{j}{\lambda} \quad \text{and} \quad \nu_0 \leq \mu.
\]

Proof We can prove Lemma 10 following a procedure which is similar to the proof of Lemma 9.

Armitage and Maddox [20] indicated that the sequence \((\omega_n) = (\sum_{k=0}^{n}(u_n - u_k))\) of real numbers is bounded below providing that the sequence \((u_n)\) of real numbers satisfies slow decrease condition which is less restrictive than slow oscillation condition. On the strength of this statement, we prove that the double sequences given in Lemmas 11, 12 and 13 are bounded under slow oscillation conditions in certain senses.

Lemma 11 If the double sequence \((s_{\mu\nu})\) is slowly oscillating in sense \((1, 0)\), then the sequence

\[
\left( \frac{1}{i + 1} \sum_{\mu=0}^{i} |s_{i\nu} - s_{i\nu_{\nu}}| \right)
\]

is bounded.

Proof Suppose that \((s_{\mu\nu})\) is slowly oscillating in sense \((1, 0)\). Then there exist a positive integer \( \nu_0 \) and \( \lambda > 1 \) such that

\[
|s_{i\nu} - s_{i\nu_{\nu}}| \leq 1 \quad \text{whenever} \quad \nu_0 \leq \mu < i < \lambda \mu \quad \text{and} \quad \nu_0 \leq \nu.
\]
It follows from Lemma 9 that there exists a constant $H$ such that

$$|s_{i\nu} - s_{\mu\nu}| \leq H \log \left( \frac{i}{\mu} \right) \quad \text{whenever} \quad 1 \leq \mu \leq \frac{i}{\lambda} \quad \text{and} \quad \nu_0 \leq \nu. \quad (39)$$

If we particularly take $\mu = 1$, we have $|s_{i\nu} - s_{1\nu}| \leq H \log i$ for $\lambda \leq i$ and $\nu_0 \leq \nu$. Accordingly, we attain that the sequence $\left( \frac{1}{i+1} \sum_{\mu=1}^{i} |s_{i\nu} - s_{\mu\nu}| \right)$ is bounded. In order to reach the end of the proof, it is enough to indicate that $\left( \frac{1}{i+1} \sum_{\mu=1}^{i} |s_{i\nu} - s_{\mu\nu}| \right)$ is bounded. Using (38) and (39), we get for $\lambda \nu_0 \leq i$ and $\nu_0 \leq \nu$

$$\sum_{\mu=1}^{i} |s_{i\nu} - s_{\mu\nu}| = \left\{ \sum_{\mu=1}^{\lfloor \frac{i}{\lambda} \rfloor} + \sum_{\mu=\lfloor \frac{i}{\lambda} \rfloor + 1}^{i} \right\} |s_{i\nu} - s_{\mu\nu}|$$

$$\leq H \sum_{\mu=1}^{\lfloor \frac{i}{\lambda} \rfloor} \log \left( \frac{i}{\mu} \right) + \left( i - \left\lfloor \frac{i}{\lambda} \right\rfloor \right)$$

$$\leq H \sum_{\mu=1}^{i} \log \left( \frac{i}{\mu} \right) + i$$

$$= H \left\{ i \log i - \sum_{\mu=2}^{i} \log \mu \right\} + i$$

$$\leq H \left\{ \int_{1}^{i} \log w \, dw \right\} + i$$

$$\leq (H + 1)i,$$

which means that the sequence is bounded. \qed

**Lemma 12** If the double sequence $(s_{\mu\nu})$ is slowly oscillating in sense $(0, 1)$, then the sequence

$$\left( \frac{1}{j+1} \sum_{\nu=0}^{j} |s_{\mu j} - s_{\mu\nu}| \right)$$

is bounded.

**Proof** We can prove Lemma 12 following a procedure which is similar to the proof of Lemma 11. \qed

**Lemma 13** If the double sequence $(s_{\mu\nu})$ is slowly oscillating in senses $(1, 0)$, $(0, 1)$ and slowly oscillating in the strong sense $(1, 0)$ or $(0, 1)$, then the sequence

$$\left( \frac{1}{(i+1)(j+1)} \sum_{\mu=0}^{i+1} \sum_{\nu=0}^{j} |s_{ij} - s_{\mu\nu}| \right)$$

is bounded.
Proof Suppose that \((s_{\mu \nu})\) is slowly oscillating in senses \((1, 0), (0, 1)\) and slowly oscillating in the strong sense \((1, 0)\) without loss of generality. Then there exist positive integers \(\nu_0, \nu_1\) and \(\lambda > 1\) such that

\[
|s_{i\nu} - s_{\mu\nu}| \leq 1 \quad \text{whenever} \quad \nu_0 \leq \nu \leq \lambda \mu \quad \text{and} \quad \nu_0 \leq \nu \leq \lambda 
\]

and

\[
|s_{\mu j} - s_{\mu\nu}| \leq 1 \quad \text{whenever} \quad \nu_1 \leq \nu \leq j \quad \text{and} \quad \nu_1 \leq \nu \leq \mu, 
\]

respectively. It follows from Lemma 9 and Lemma 10 that there exist constants \(H_0\) and \(H_1\) such that

\[
|s_{i\nu} - s_{\mu\nu}| \leq H_0 \log \left( \frac{i}{\mu} \right) \quad \text{whenever} \quad 1 \leq \mu \leq \frac{i}{\lambda} \quad \text{and} \quad \nu_0 \leq \nu 
\]

and

\[
|s_{\mu j} - s_{\mu\nu}| \leq H_1 \log \left( \frac{j}{\nu} \right) \quad \text{whenever} \quad 1 \leq \nu \leq \frac{j}{\lambda} \quad \text{and} \quad \nu_1 \leq \mu, 
\]

respectively. If we particularly take \(\mu, \nu = 1\), we have

\[
|s_{ij} - s_{1j}| \leq H_0 \log i \quad \text{and} \quad |s_{ij} - s_{11}| \leq H_1 \log j \quad \text{for} \quad \lambda \leq i, \nu_0 \leq j \quad \text{and} \quad \nu_1 \leq \mu. 
\]

Using these inequalities, Lemmas 11 and 12, if we take into consideration that the sequences

\[
\left( \sum_{\mu=1}^{i} |s_{ij} - s_{\mu0}| + \sum_{\nu=1}^{j} |s_{ij} - s_{0\nu}| + |s_{ij} - s_{00}| \right) 
\]

and

\[
\left( \sum_{\nu=1}^{i} |s_{\mu j} - s_{\mu\nu}| \right) 
\]

is bounded, which were attained in proofs of Lemma 11 and Lemma 12 respectively, then we can find

\[
\frac{1}{(i + 1)(j + 1)} \left( \sum_{\mu=1}^{i} |s_{ij} - s_{\mu0}| + \sum_{\nu=1}^{j} |s_{ij} - s_{0\nu}| + |s_{ij} - s_{00}| \right) 
\]

\[
\leq \frac{1}{(i + 1)(j + 1)} \left( \sum_{\mu=1}^{i} |s_{ij} - s_{\mu0}| + \sum_{\mu=1}^{i} |s_{\mu j} - s_{\mu0}| \right) 
\]

\[
+ \frac{1}{(i + 1)(j + 1)} \left( \sum_{\nu=1}^{j} |s_{ij} - s_{\nu j}| + \sum_{\nu=1}^{j} |s_{\nu i} - s_{0\nu}| \right) 
\]

\[
+ \frac{1}{(i + 1)(j + 1)} \left( |s_{ij} - s_{1j}| + |s_{1j} - s_{11}| + |s_{11} - s_{00}| \right) 
\]

\[
\leq \left( \frac{C_0}{j + 1} + \frac{i}{i + 1} C_1 \right) + \left( \frac{C_2}{i + 1} + \frac{j}{j + 1} C_3 \right) 
\]

\[
+ \left( \frac{\log i}{i + 1} \frac{H_0}{i + 1} + \frac{H_1}{j + 1} \frac{\log j}{j + 1} + \frac{H_2}{(i + 1)(j + 1)} \right) 
\]

\[
\leq 4 \max\{C_0, C_1, C_2, C_3\} + 3 \max\{H_0, H_1, H_2\}. 
\]

which means that the sequence

\(
\sum_{\mu=1}^{i} |s_{ij} - s_{\mu0}| + \sum_{\nu=1}^{j} |s_{ij} - s_{0\nu}| + |s_{ij} - s_{00}| \)

is bounded.
Tauberian theorems for statistically \((C, 1, 1)\) summable… 915

\[
\left( \frac{1}{(i+1)(j+1)} \sum_{\mu=1}^{i} |s_{ij} - s_{\mu 0}| + \sum_{v=1}^{j} |s_{ij} - s_{0v}| + |s_{ij} - s_{00}| \right)
\]

is bounded. In order to reach the end of the proof, it is enough to indicate that
\[
\frac{1}{(i+1)(j+1)} \sum_{\mu=1}^{i} \sum_{v=1}^{j} |s_{ij} - s_{\mu v}|
\]

is bounded. If we consider \(\nu_2 = \max\{\nu_0, \nu_1\}\) and combine (40), (41), (42) and (43), we get for \(\lambda \nu_2 \leq i\) and \(\lambda \nu_2 \leq j\)

\[
\sum_{\mu=1}^{i} \sum_{v=1}^{j} |s_{ij} - s_{\mu v}| \leq \sum_{\mu=1}^{i} \sum_{v=1}^{j} |s_{ij} - s_{\mu,j}| + \sum_{\mu=1}^{i} \sum_{v=1}^{j} |s_{\mu,j} - s_{\mu,v}|
\]

\[
= \left\{ \sum_{\mu=1}^{\left[\frac{i}{\lambda}\right]} + \sum_{\mu=\left[\frac{i}{\lambda}\right]+1}^{i} \right\} \sum_{v=1}^{j} |s_{ij} - s_{\mu,j}|
\]

\[
+ \sum_{\mu=1}^{i} \left\{ \sum_{v=1}^{\left[\frac{j}{\lambda}\right]} + \sum_{v=\left[\frac{j}{\lambda}\right]+1}^{j} \right\} |s_{\mu,j} - s_{\mu,v}|
\]

\[
\leq H_0 \sum_{\mu=1}^{i} \sum_{v=1}^{j} \log \left( \frac{i}{\mu} \right) + \left( i - \left[\frac{i}{\lambda}\right]\right) j
\]

\[
+ H_1 \sum_{\mu=1}^{i} \sum_{v=1}^{j} \log \left( \frac{j}{v} \right) + \left( j - \left[\frac{j}{\lambda}\right]\right) i
\]

\[
\leq H_0 \sum_{\mu=1}^{i} \sum_{v=1}^{j} \log \left( \frac{i}{\mu} \right) + H_1 \sum_{\mu=1}^{i} \sum_{v=1}^{j} \log \left( \frac{i}{v} \right) + 2ij
\]

\[
\leq H \sum_{\mu=1}^{i} \sum_{v=1}^{j} \left\{ \log \left( \frac{i}{\mu} \right) + \log \left( \frac{j}{v} \right) \right\} + 2ij
\]

\[
= H \sum_{\mu=1}^{i} \sum_{v=1}^{j} \left( \log ij - (\log \mu + \log v) \right) + 2ij
\]

\[
= Hij \log ij - H_j \sum_{\mu=2}^{i} \log \mu - Hi \sum_{v=2}^{j} \log v + 2ij
\]

\[
\leq Hij \log ij - H_j \int_{1}^{i} \log u du - Hi \int_{1}^{j} \log w dw + 2ij
\]

\[
= 2ij(H+1) - (i+j)H
\]

\[
\leq 2ij(H+1)
\]
where \( \max\{H_0, H_1\} = H \). In conclusion, we reach that the sequence is bounded. □

At once, we refer a lemma authenticated by Chen and Chang [21] for statistical convergence studied as a summability method.

**Lemma 14** [21, Theorem 2.3] *If the double sequence \( (s_{\mu\nu}) \) is statistically convergent to a number \( \ell \) and slowly oscillating in sense \((1, 1)\), then \( (s_{\mu\nu}) \) is \( P \)-convergent to \( \ell \).*

As a result of Lemma 8 given by Móricz [6], we can give the following lemma.

**Lemma 15** *If the double sequence \( (s_{\mu\nu}) \) is \((C, 1, 1)\) summable to a number \( \ell \) and slowly oscillating in sense \((1, 1), (1, 0)\) and \((0, 1)\), then \( (s_{\mu\nu}) \) is \( P \)-convergent to \( \ell \).*

### 4.2 Main results

In this subsection, we prove a Tauberian theorem for double sequence of complex numbers that \( P \)-convergence follows from statistically \((C, 1, 1)\) summability under condition of slow oscillation in certain senses and we present a corollary related to this theorem.

**Theorem 4** *Let the double sequence \( (s_{\mu\nu}) \) be statistically \((C, 1, 1)\) summable to a number \( \ell \). If \( (s_{\mu\nu}) \) is slowly oscillating in senses \((1, 0)\) and \((0, 1)\) and slowly oscillating in the strong sense \((1, 0)\) or \((0, 1)\), then \( (s_{\mu\nu}) \) is \( P \)-convergent to \( \ell \).*

**Proof** Suppose that \( (s_{\mu\nu}) \) being statistically \((C, 1, 1)\) summable to \( \ell \) is slowly oscillating in senses \((1, 0)\), \((0, 1)\) and slowly oscillating in the strong sense \((1, 0)\) without loss of generality. In order to prove that \( (s_{\mu\nu}) \) is \( P \)-convergent to the same number, we firstly indicate that if \( (s_{\mu\nu}) \) is slowly oscillating in senses \((1, 0)\), \((0, 1)\) and the strong sense \((1, 0)\), then \( (\sigma_{mn}) \), which is the arithmetic means of \( (s_{\mu\nu}) \), is slowly oscillating in sense \((1, 1)\). Given an \( \epsilon > 0 \). From the hypothesis, there exist positive integers \( \nu_0 = \nu_0(\epsilon), \nu_1 = \nu_1(\epsilon) \) and \( \lambda = \lambda(\epsilon) > 1 \) such that

\[
|s_{i\nu} - u_{\mu\nu}| \leq \epsilon \quad \text{whenever} \quad \nu_0 \leq \mu < i \leq \lambda \mu \quad \text{and} \quad \nu_0 \leq \nu
\]

and

\[
|s_{\mu j} - s_{\mu\nu}| \leq \epsilon \quad \text{whenever} \quad \nu_1 \leq \nu < j \leq \lambda \nu \quad \text{and} \quad \nu_1 \leq \mu,
\]

respectively. Additionally, it is known that if \( (s_{\mu\nu}) \) is slowly oscillating in senses \((1, 0)\), \((0, 1)\) and slowly oscillating in the strong sense \((1, 0)\), then it is also slowly oscillating in sense \((1, 1)\). In this case, there exist positive integer \( \nu_2 = \max\{\nu_0, \nu_1\} \) and \( \lambda = \lambda(\epsilon) > 1 \) such that

\[
|s_{ij} - s_{\mu\nu}| \leq \epsilon \quad \text{whenever} \quad \nu_2 \leq \mu < i \leq \lambda \mu \quad \text{and} \quad \nu_2 \leq \nu < j \leq \lambda \nu.
\]
Let \( \nu_2 \leq \mu < i \leq \lambda_\mu \) and \( \nu_2 \leq \nu < j \leq \lambda_\nu \). It follows from (26) that we obtain

\[
|\sigma_{ij} - \sigma_{\mu \nu}| \leq \frac{1}{(i + 1)(j + 1)} \sum_{p=0}^{\mu} \sum_{q=\nu+1}^{j} |s_{pq} - s_{pq}| \\
+ \frac{1}{(i + 1)(j + 1)} \sum_{p=\mu+1}^{i} \sum_{q=0}^{v} |s_{pq} - s_{pq}| \\
+ \frac{1}{(i + 1)(j + 1)} \sum_{p=\mu+1}^{i} \sum_{q=0}^{j} |s_{\mu \nu} - s_{pq}| \\
+ \frac{(i - \mu)(j - \nu)}{(i + 1)(j + 1)(\mu + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} |s_{\mu \nu} - s_{pq}| \\
+ \frac{(i - \mu)}{(i + 1)(j + 1)(\mu + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} |s_{\mu \nu} - s_{pq}| \\
+ \frac{(j - \nu)}{(i + 1)(j + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} |s_{\mu \nu} - s_{pq}|. \tag{44}
\]

By Lemmas 11, 12 and 13, there exist constants \( H_0, H_1 \) and \( H_2 \) such that

\[
\frac{1}{(\mu + 1)} \sum_{p=0}^{\mu} |s_{pq} - s_{pq}| \leq H_0 \quad \text{and} \quad \frac{1}{\nu + 1} \sum_{q=0}^{v} |s_{pq} - s_{pq}| \leq H_1, \\
\frac{1}{(\mu + 1)(v + 1)} \sum_{p=0}^{\mu} \sum_{q=0}^{v} |s_{\mu \nu} - s_{pq}| \leq H_2
\]

for all nonnegative integers \( \mu \) and \( \nu \). Using slow oscillation conditions in certain senses and these inequalities, we attain

\[
|\sigma_{ij} - \sigma_{\mu \nu}| \leq \frac{(\mu + 1)}{(i + 1)} \frac{(j - \nu)}{(j + 1)} (\epsilon + H_1) + \frac{(i - \mu)}{(i + 1)} \frac{(v + 1)}{(j + 1)} (\epsilon + H_0) \\
+ \frac{(i - \mu)}{(i + 1)} \frac{(j - \nu)}{(j + 1)} (\epsilon + H_2) \\
\leq (\lambda - 1)(\epsilon + H_1) + (\lambda - 1)(\epsilon + H_0) + (\lambda - 1)^2(\epsilon + H_2), \tag{45}
\]
since we have for $\lambda > 1$

$$\frac{\mu + 1}{i + 1} < 1, \quad \frac{i - \mu}{i + 1} = 1 - \frac{\mu + 1}{i + 1} < 1 - \frac{1}{\lambda} < \lambda - 1 \quad (46)$$

and

$$\frac{\nu + 1}{j + 1} < 1, \quad \frac{j - \nu}{j + 1} = 1 - \frac{\nu + 1}{j + 1} < 1 - \frac{1}{\lambda} < \lambda - 1 \quad (47)$$

whenever $\mu < i \leq \lambda \mu$ and $\nu < j \leq \lambda \nu$. If we take $H = \max\{H_0, H_1, H_2\}$ and $1 < \lambda < \sqrt{\frac{\epsilon}{\epsilon + \pi}} + 1$, then we find $|\sigma_{ij} - \sigma_{\mu \nu}| \leq \epsilon$ whenever $v_2 \leq \mu < i \leq \lambda \mu$ and $v_2 \leq \nu < j \leq \lambda \nu$. Thus, we arrive that $(\sigma_{\mu \nu})$ is slowly oscillating in sense $(1, 1)$. Since $(\sigma_{\mu \nu})$ is slowly oscillating in sense $(1, 1)$ and statistically convergent to $\ell$, we get that $(\sigma_{\mu \nu})$ is convergent to $\ell$ by Lemma 14. If we consider that conditions of slow oscillation in senses $(1, 0)$, $(0, 1)$ and in the strong sense $(1, 0)$ imply condition of slow oscillation in sense $(1, 1)$, then we conclude by virtue of Lemma 15 that $(s_{\mu \nu})$ is $P$-convergent to $\ell$. \hfill \Box

Since it is known that if two-sided Tauberian conditions of Hardy type in senses $(1, 0)$ and $(0, 1)$ is valid, then the double sequence is slowly oscillating in senses $(1, 0)$ and $(0, 1)$ and also slowly oscillating in the strong senses $(1, 0)$ and $(0, 1)$ respectively, we can present Corollary 2.

**Corollary 2** Let the double sequence $(s_{\mu \nu})$ be statistically $(C, 1, 1)$ summable to a number $\ell$. If $(s_{\mu \nu})$ satisfies two-sided Tauberian conditions of Hardy type in senses $(1, 0)$ and $(0, 1)$, then $(s_{\mu \nu})$ is $P$-convergent to $\ell$.

**Acknowledgements** This study is supported by Ege University Scientific Research Projects Coordination Unit. Project Number 513.

**References**

1. Hardy, G.H.: On the convergence of certain multiple series. Proc. Lond. Math. Soc. 2(1), 124–128 (1904)
2. Bromwich, T.J.I.: An Introduction to the Theory of Infinite Series. MacMillan, London (1908)
3. Agnew, R.P.: On summability of multiple sequences. Am. J. Math. 1, 62–68 (1934)
4. Knopp, K.: Limitierungs-Umkehrsätze für Doppelfolgen. Math. Z 45, 573–589 (1939)
5. Tauber, A.: Ein Satz aus der Theorie der unendlichen Reihen. Monatsh Math. Phys. 8, 273–277 (1897)
6. Móricz, F.: Tauberian theorems for Cesàro summable double sequences. Stud. Math. 110, 83–96 (1994)
7. Totur, Ü.: Classical Tauberian theorems for the $(C, 1, 1)$ summability method. An. Științ. Univ. Al. I. Cuza. Iași. Mat. (N.S.). 61, 401–414 (2015)
8. Fast, H.: Sur la convergence statistique. Colloq. Math. 2, 241–244 (1952)
9. Schoenberg, I.J.: The integrability of certain functions and related summability methods. Am. Math. Mon. 66, 361–375 (1959)
10. Zygmund, A.: Trigonometrical Series. Dover Publications, New York (1955)
11. Tripathy, B.C.: Statistically convergent double sequences. Tamkang J. Math. 34, 231–237 (2003)
12. Mursaleen, M., Edely, O.H.H.: Statistical convergence of double sequences. J. Math. Anal. Appl. 288, 223–231 (2003)
13. Edely, O.H.H., Mursaleen, M.: Tauberian theorems for statistically convergent double sequences. Inf. Sci. 176, 875–886 (2006)
14. Móricz, F.: Tauberian theorems for double sequences that are statistically summable \((C, 1, 1)\). J. Math. Anal. Appl. 286, 340–350 (2003)
15. Totur, Ü., Çanak, İ.: Tauberian theorems for the statistical convergence and the statistical \((C, 1, 1)\) summability. Filomat 32, 101–116 (2018)
16. Móricz, F.: Ordinary convergence follows from statistical summability \((C, 1)\) in the case of slowly decreasing or oscillating sequences. Colloq. Math. 99, 207–219 (2004)
17. Pringsheim, A.: Zur Theorie der zweifach unendlichen Zahlenfolgen. Math. Ann. 53, 289–321 (1900)
18. Çakalli, H., Patterson, R.F.: Functions preserving slowly oscillating double sequences. An. Științ. Univ. Al. I. Cuza. Iași. Mat. (N.S.) 62, 531–536 (2016)
19. Vijayaraghavan, T.: A Tauberian Theorem. J. Lond. Math. Soc. S1–1, 113–120 (1926)
20. Armitage, D.H., Maddox, I.J.: Discrete Abel means. Analysis 10, 177–186 (1990)
21. Chen, C.-P., Chang, C.-T.: Tauberian conditions under which the original convergence of double sequences follows from the statistical convergence of their weighted means. J. Math. Anal. Appl. 332, 1242–1248 (2007)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.