UNIFORMLY CONVEX METRIC SPACES

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Abstract. In this paper the theory of uniformly convex metric spaces is
developed. These spaces exhibit a generalized convexity of the metric from
a fixed point. Using a (nearly) uniform convexity property a simple proof of
reflexivity is presented and a weak topology of such spaces is analyzed. This
topology called co-convex topology agrees with the usual weak topology in
Banach spaces. An example of a $CAT(0)$-spaces with weak topology which is
not Hausdorff is given. This answers questions raised by Monod 2006, Kirk
and Panyanak 2008 and Espínola and Fernández-León 2009.

In the end existence and uniqueness of generalized barycenters is shown
and a Banach-Saks property is proved.

In this paper we summarize and extend some facts about convexities of the metric
from a fixed point and give simpler proofs which also work for general metric spaces.
In its simplest form this convexity of the metric just requires balls to be convex
or that $x \mapsto d(x,y)$ is convex for every fixed $y \in X$. It is easy to see that both
conditions are equivalent on normed spaces with strictly convex norm. However,
in [BP79] (see also [Foe04, Example 1]) Busemann and Phadke constructed spaces
whose balls are convex but its metric is not. Nevertheless, a geometric condition
called non-positive curvature in the sense of Busemann (see [BH99, Bač14]) implies
that both concepts are equivalent, see [Foe04, Proposition 1].

The study of stronger convexities for Banach spaces [Cla36] has a long tradition.
In the non-linear setting so called $CAT(0)$-spaces are by now well-understood, see
[BH99, Bač14]. Only recently Kuwae [Kuw13] based on [NS11] studied spaces with
a uniformly $p$-convexity assumption similar to that of Banach spaces. Related to
this are Ohta’s convexities definitions [Oht07] which, however, seem more restrictive
than the ones defined in this paper.

In the first section of this article an overview of convexities of the metric and
some easy implications are given. Then existence of the projection map onto convex
subsets and existence and uniqueness of barycenters of measures is shown. For this
we give simple proofs using an old concept introduced by Huff in [Huf80].

In the third section we introduce weak topologies. The lack of a naturally defined
dual spaces similar to Banach space theory requires a more direct definition either
via convex sets, i.e. the co-convex topology (first appeared in [Mon06]), or via
asymptotic centers (see historical remark at the end of [Bač14, Chapter 3]). Both
topologies might not be equivalent and/or comparable. For $CAT(0)$-spaces it is

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easy to show that the convergence via asymptotic centers is stronger than the co-
convex topology. However, the topologies do not agree in general, see Example 19.
With this example we answer questions raised in [KP08] and [EFL09].
In the last sections, we use the results show existence of generalized barycenters
and prove the Banach-Saks property. The proof extends a proof recently found by
Yokota [Yok13] in the setting of CAT(1)-spaces with small diameter. In the end a
discussion about further extending convexities is given.

Convexity of the metric

Let \((X, d)\) be a complete metric space. We say that \((X, d)\) admits midpoints if
for every \(x, y \in X\) there is an \(m(x, y)\) such that \(d(x, m(x, y)) = d(y, m(x, y)) = \frac{1}{2}d(x, y)\). One easily sees that each such space is a geodesic space.

Now for \(p \in [1, \infty)\) and all non-negative real numbers \(a, b\) we define the \(p\)-mean

\[
\mathcal{M}^p(a, b) := \left( \frac{1}{2} a^p + \frac{1}{2} b^p \right)^{\frac{1}{p}}.
\]

Furthermore, the case \(p = \infty\) can be defined as a limit, i.e. \(\mathcal{M}^\infty(a, b) = \max\{a, b\}\).

**Definition 1** (\(p\)-convexity). Suppose the metric space \((X, d)\) admits midpoints.
Then it is called \(p\)-convex for some \(p \in [1, \infty]\) if for each triple \(x, y, z \in X\) and each
midpoint \(m(x, y)\) of \(x\) and \(y\)

\[
d(m(x, y), z) \leq \mathcal{M}^p(d(x, z), d(y, z)).
\]

It is called strictly \(p\)-convex for \(p \in (1, \infty]\) if the inequality is strict whenever \(x \neq y\)
and strictly 1-convex if the inequality is strict whenever \(d(x, y) > d(x, z) - d(y, z)\).

**Remark.** (1) In [Foe04] Foertsch defined 1-convexity and \(\infty\)-convexity under the
name distance convexity and ball convexity.

A \(p\)-convex space is \(p'\)-convex for all \(p' \geq p\) and it is easy to see that balls are
convex iff the space is \(\infty\)-convex. Furthermore, one sees that any strictly \(\infty\)-convex
space is uniquely geodesic.

Instead of just requiring convexity from a fixed point one can assume that assume
a convexity of \(t \mapsto d(x_t, y_t)\) there \(x_t\) and \(y_t\) are constant speed geodesics. This gives the following condition.

**Definition 2** (\(p\)-Busemann curvature). A metric space \((X, d)\) admitting midpoints
is said to satisfy the \(p\)-Busemann curvature condition for some \(p \in [1, \infty]\) if for all
triples quadruples \(x_0, x_1, y_0, y_1 \in X\) with midpoints \(x'_2 = m(x_0, x_1)\) and \(y'_2 = m(y_0, y_1)\) it holds

\[
d(x'_2, y'_2) \leq \mathcal{M}^p(d(x_0, y_0), d(x_1, y_1)).
\]

In such a case we will say that \((X, d)\) is \(p\)-Busemann.

It is not difficult to show (see e.g. [Bac14, Proposition 1.1.5]) that in case \(p \in [1, \infty]\) this is equivalent to the more traditional form: for each triples \(x, y, z \in X\) with midpoints \(m_1 = m(x, z)\) and \(m_2 = m(y, z)\) it holds

\[
d(m_1, m_2)^p \leq \frac{1}{2} d(x, y)^p.
\]

In particular for \(p = 1\), this is Busemann’s original non-positive curvature assumption. In this case we will just say that \((X, d)\) is Busemann. Busemann’s condition
can be used to show equivalence of all (strict/uniform) $p$-convexity, see Corollary 5. Currently we cannot prove that on $p$-Busemann spaces (strict/uniform) $p$-convexity is equivalent to (strict/uniform) $p'$-convexity for all $p' \geq p$. However, it can be used to get a $p$-Wasserstein contraction of 2-barycenters if Jensen’s inequality holds on the space, see Proposition 40 below.

In [Foe04] Foertsch also defines uniform distance/ball convexity. We adapt his definition as follows:

**Definition 3** (uniform $p$-convexity). Suppose $(X,d)$ admits midpoints and let $p \in [1,\infty]$. Then we say it is uniformly $p$-convex if for all $\epsilon > 0$ there is a $\rho_p(\epsilon) \in (0,1]$ such that for all triples $x,y,z \in X$ satisfying $d(x,y) > \epsilon \mathcal{M}^p(d(x,z),d(y,z))$ for $p > 1$ and $d(x,y) > |d(x,z) - d(y,z)| + \epsilon \mathcal{M}^1(d(x,z),d(y,z))$ for $p = 1$ it holds

$$d(m(x,y),z) \leq (1 - \rho_p(\epsilon))\mathcal{M}^p(d(x,z),d(y,z)).$$

**Remark.** (1) W.l.o.g. we assume that $\rho_p$ is monotone in $\epsilon$ so that $\rho_p(\epsilon) \to 0$ requires $\epsilon \to 0$.

(2) Uniform $p$-convexity for $p \in (1,\infty)$ is equivalent to the existence of a $\tilde{\rho}_p(\epsilon) > 0$ such that

$$d(m(x,y),z)^p \leq (1 - \tilde{\rho}_p(\epsilon))\mathcal{M}^p(d(x,z),d(y,z))^p,$$

just let $\tilde{\rho}_p(\epsilon) = 1 - (1 - \rho_p(\epsilon))^p$.

(3) The usual definition of uniform convexity for functions is as follows: A function $f$ is uniformly convex if for $x,y \in X$ with midpoint $m$:

$$f(m) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) - \omega(d(x,y)),$$

where $\omega$ is the modulus of convexity and $\omega(r) > 0$ if $r > 0$. For $p \geq 2$ and $\omega(r) = Cr^p$ and $f = d(\cdot,z)^p$ one recovers Kuwae’s $p$-uniform convexity [Kuw13]. However, in this form one does not see whether $p$-convexity implies $p'$-convexity. Furthermore, one gets a restriction that $\omega(r) \geq Cr^2$, i.e. the cases $p \in (1,2)$ are essentially excluded. And whereas our definition is multiplicative, matching the fact that $Cd(\cdot,\cdot)$ is also a metric, the usual uniform convexity is only multiplicative by adjusting the modulus of convexity.

**Example.** (1) Every $CAT(0)$-space is uniformly 2-convex with $\rho(\epsilon) = \left(\frac{\epsilon}{2}\right)^2$. More generally any $R_\kappa$-domain of a $CAT(\kappa)$-space is uniformly 2-convex with $\rho(\epsilon) = c_\kappa \epsilon^2$.

(2) Every $p$-uniformly convex space as defined in [NS11, Kuw13] is uniformly $p$-convex with $\rho(\epsilon) = c_k \epsilon^p$.

**Lemma 4.** A uniformly $p$-convex metric space $(X,d)$ is uniformly $p'$-convex for all $p' \geq p$.

**Proof.** First note that $\mathcal{M}^p(a,b) \leq \mathcal{M}^{p'}(a,b)$.

Assume first that $1 < p < p' \leq \infty$. If $x,y,z \in X$ is a triple satisfying the condition for $p'$ then it also satisfies the condition for $p$ and thus for $m = m(x,y)$

$$d(m,z) \leq (1 - \rho_p(\epsilon))\mathcal{M}^p(d(x,z),d(y,z)) \leq (1 - \rho_p(\epsilon))\mathcal{M}^{p'}(d(x,z),d(y,z)).$$

Hence setting $\rho_p(\epsilon) := \rho_p(\epsilon)$ gives the result.

For $p = 1$ we skip the case $p' = \infty$ as this was proven in [Foe04, Proposition 1]: Let $x,y,z \in X$ be some triple with $d(x,z) \geq d(y,z)$. If

$$d(x,y) > |d(x,z) - d(y,z)| + \frac{\epsilon}{2}\mathcal{M}^1(d(x,z),d(y,z))$$

then
then
\[ d(m(x, y), z) \leq \left( 1 - \rho_1 \left( \frac{c}{2} \right) \right) M^1(d(x, z), d(y, z)) \]
\[ \leq \left( 1 - \rho_1 \left( \frac{c}{2} \right) \right) M^{p'}(d(x, z), d(y, z)). \]

So assume
\[ d(x, y) \leq d(x, z) - d(y, z) + \frac{c}{2} M^{p'}(d(x, z), d(y, z)). \]

If \( d(x, z) - d(y, z) \leq \frac{c}{2} M^{p'}(d(x, z), d(y, z)) \) then
\[ d(x, y) \leq c M^{p'}(d(x, z), d(y, z)). \]

Hence we can assume \( d(x, z) - d(y, z) > \frac{c}{2} M^{p'}(d(x, z), d(y, z)) \). Now for \( p' \geq 2 \)
Clarkson’s inequality
\[ \left( \frac{1}{2} a + \frac{1}{2} b \right)^{p'} + c_{p'}(a - b)^{p'} \leq \frac{1}{2} a^{p'} + \frac{1}{2} b^{p'} \]
holds. Thus using 1-convexity and our assumption we get
\[ d(m, z)^{p'} \leq \left( \frac{1}{2} d(x, z) + \frac{1}{2} d(y, z) \right)^{p'} \]
\[ \leq \frac{1}{2} d(x, z)^{p'} + \frac{1}{2} d(y, z)^{p'} - c_{p'}(d(x, z) - d(y, z))^{p'} \]
\[ \leq \left( 1 - c_p \left( \frac{c}{2} \right)^{p'} \right) M^{p'}(d(x, z), d(y, z))^{p'}. \]

Choosing \( \rho_{p'}(e) = \min \{ \rho_1 \left( \frac{c}{2} \right), 1 - (1 - c_p \left( \frac{c}{2} \right)^{p'})^{\frac{1}{p'}} \} \) gives the result.

For \( 1 < p' < 2 \) we use the other Clarkson inequality
\[ \left( \frac{1}{2} a + \frac{1}{2} b \right)^{q} + c_{p'}(a - b)^{q} \leq \left( \frac{1}{2} a^{q} + \frac{1}{2} b^{q} \right)^{q} \]
where \( \frac{1}{p} + \frac{1}{q} = 1 \). By similar arguments we get
\[ d(m, z)^{q} + c_{p'} \left( \frac{c}{2} \right)^{\frac{q}{p'}} \left( M^{p'}(d(x, z), d(y, z))^{q} \leq (M^{p'}(d(x, z), d(y, z))^{q}. \]

Choosing in this case
\[ \rho_{p'}(e) = \min \left\{ \rho_1 \left( \frac{c}{2} \right), 1 - \left( 1 - c_{p'} \left( \frac{c}{2} \right)^{\frac{q}{p'}} \right)^{\frac{1}{q}} \right\} \]
finishes the proof. \( \square \)

**Corollary 5.** Assume \((X, d)\) is Busemann. Then \((X, d)\) is (strictly/uniformly) \(p\)-convex for some \( p \in [1, \infty) \) if and only if it is (strictly/uniformly) \( p\)-convex for all \( p \in [1, \infty) \).

**Proof.** This is just using [Foe04, Proposition 1] who proved that (strict/uniform) \( \infty \)-convexity implies (strict/uniform) \( 1 \)-convexity. \( \square \)

Any \( CAT(0) \)-space is both Busemann and uniformly \( 2 \)-convex, hence uniformly \( p \)-convex for every \( p \in [1, \infty] \).
Convex subsets and reflexivity

In a geodesic metric space, we say that subset \( C \subset X \) is convex if for each \( x, y \in C \) and each geodesic \( \gamma \) connecting \( x \) and \( y \) also \( \gamma \subset C \). Given any subset \( A \subset X \) we define the convex hull of \( A \) as follows:

\[
G_0 = A
\]

\[
G_n = \bigcup_{x,y \in G_{n-1}} \{ \gamma_t \mid \gamma \text{ is a geodesic connecting } x \text{ and } y \text{ and } t \in [0,1] \}
\]

\[
\text{conv } A = \bigcup_{n \in \mathbb{N}} G_n.
\]

The closed convex hull is just the closure \( \text{conv } A \) of \( \text{conv } A \).

The projection map onto (convex) sets can be defined as follows: Given a non-empty subset \( C \) of \( X \) define \( r_C : X \to [0, \infty) \) by

\[
r_C(x) = \inf_{c \in C} d(x, c)
\]

and \( P_C : X \to 2^C \) by

\[
P_C(x) = \{ c \in C \mid r_C(x) = d(x, c) \}.
\]

In case \( |P_C(x)| = 1 \) for all \( x \in X \) we say that the set \( C \) is Chebyshev. In that case, just assume \( P_C \) is a map from \( X \) to \( C \).

It is well-known that a Banach space is reflexive iff any decreasing family of closed bounded convex subsets has non-empty intersection. Thus it makes sense for general metric spaces to define reflexivity as follows.

**Definition 6** (Reflexivity). A metric space \((X, d)\) is said to be reflexive if for every decreasing family \((C_i)_{i \in I}\) of non-empty bounded closed convex subsets, i.e. \( C_i \subset C_j \) whenever \( i > j \) where \( I \) is a directed set then it holds

\[
\bigcap_{i \in I} C_i \neq \emptyset.
\]

It is obvious that any proper metric space is reflexive. The following was defined in [Huf80]. We will simplify Huff’s proof of [Huf80, Theorem 1] to show that nearly uniform convexity implies reflexivity using a proof via the projection map, see e.g. [Bač14, Proofs of 2.1.12(i) and 2.1.16]. However, since the weak topology (see below) is not necessarily Hausdorff, we cannot show that nearly uniform convexity also implies the uniform Kadec-Klee property.

We say that a family of points \((x_i)_{i \in I}\) is \( \epsilon \)-separated if \( d(x_i, x_j) \geq \epsilon \) for \( i \neq j \), i.e.

\[
\text{sep}((x_i)_{i \in I}) = \inf d(x_i, x_j) \geq \epsilon.
\]

**Definition 7** (Nearly uniformly convex). A \( \infty \)-convex metric space \((X, d)\) is said to be nearly uniformly convex, if for any \( R > 0 \) for any \( \epsilon \)-separated infinite family \((x_i)_{i \in I}\) with \( d(x_i, y) \leq R \leq R \) there is a \( \rho = \rho(\epsilon, R) > 0 \) such that

\[
B_{(1-\rho)r}(y) \cap \text{conv}(x_i)_{i \in I} \neq \emptyset.
\]

Note that uniform \( \infty \)-convexity implies nearly uniform convexity, an even stronger statement is formulated in Theorem 25. However, not every nearly uniformly convex space is uniformly convex, see [Huf80].

**Theorem 8.** For every closed convex subset \( C \) of a nearly uniformly convex metric space the projection \( P_C \) has non-empty compact images, i.e. \( P_C(x) \) is non-empty and compact for every \( x \in X \).
Corollary 9. If \((X, d)\) is nearly uniformly convex and strictly \(\infty\)-convex then every closed convex set is Chebyshev.

Proof of the Theorem. Let \(C\) be a closed convex subset, \(x \in X\) be arbitrary and set \(r = r_C(x)\). For each \(n \in \mathbb{N}\) there is an \(x_n \in C\) such that \(r \leq d(x, x_n) \leq r + \frac{1}{n}\). In particular, \(d(x, x_n) \to r\) as \(n \to \infty\). If \(r = 0\) or every subsequence of \((x_n)\) admits a convergent subsequence we are done.

So assume \((x_n)\) w.l.o.g. that \((x_n)\) is \(\epsilon\)-separated for some \(\epsilon > 0\). By nearly uniform convexity there is a \(\rho = \rho(\epsilon) > 0\) such that
\[
A_n = B_{(1-\rho)(r+\frac{\epsilon}{n})}(x) \cap \overline{\text{conv}}(x_{m \geq n}) \neq \emptyset.
\]
For sufficiently large \(n\) and some \(0 < \rho' < \rho\) we also have \(B_{(1-\rho)(r+\frac{\epsilon}{n})}(x) \subset B_{(1-\rho')r}(x)\), i.e. \(d(x, y) < r\) for some \(y \in A_n\). But this contradicts the fact that \(\overline{\text{conv}}(x_{m \geq n}) \subset C\), i.e. \(d(x, y) \geq r\) for all \(y \in A_n\).

\[
\square
\]

Theorem 10. A nearly uniformly convex metric space is reflexive.

Proof. Let \((C_i)_{i \in I}\) be a non-increasing family of bounded closed convex subsets of \(X\) and let \(x \in X\) be some arbitrary point. For each \(i \in I\) define \(r_i = \inf_{y \in C_i} d(x, y)\). Since \((C_i)_{i \in I}\) is non-increasing so the net \((r_i)_{i \in I}\) is non-decreasing and bounded, hence convergent to some \(r\). By the previous theorem there are \(x_i \in C_i\) such that \(d(x, x_i) = r_i\). If \(r = 0\) or \((x_i)_{i \in I}\) admits a convergent subnet we are done.

So assume there is an \(\epsilon\)-separated subnet \((x_i')_{i' \in I'}\) for some \(\epsilon > 0\). Now nearly uniform convexity implies that for some \(\rho = \rho(\epsilon) > 0\)
\[
\emptyset \neq A_i = B_{(1-\rho)r}(x) \cap \overline{\text{conv}}(x_{j \geq i}) \subset C_i.
\]
Since the subnet \((x_i')\) is also convergent to \(r\) there is some \(i\) and \(0 < \rho' < \rho\) such that
\[
B_{(1-\rho)r}(x) \subset B_{(1-\rho')r_i}(x).
\]
However, this implies that \(d(x, y_i) < r_i\) for all \(y_i \in A_i\) contradicting the definition of \(r_i\).

\[
\square
\]

In order to use reflexivity to characterize the weak topology defined below better we need the following equivalent description. We say that a collection of sets \((C_i)_{i \in I}\) has the finite intersection property if any finite subcollection has non-empty intersection, i.e. for every finite \(I' \subset I\), \(\bigcap_{i \in I'} C_i \neq \emptyset\).

Lemma 11. The space \((X, d)\) is reflexive iff every collection \((C_i)_{i \in I}\) of closed bounded convex subsets with finite intersection property satisfies
\[
\bigcap_{i \in I} C_i \neq \emptyset.
\]

Proof. The if-direction is obvious. So assume \((X, d)\) is reflexive and \((C_i)_{i \in I}\) be a collection of closed bounded convex subsets with finite intersection property.

Let \(\mathcal{I}\) be the set of finite subsets of \(I\). This set directed by inclusion and the sets
\[
\tilde{C}_i = \bigcap_{i \in I} C_i
\]
are non-empty closed and convex. Furthermore, the family \((\tilde{C}_i)_{i \in \mathcal{I}}\) is decreasing. By reflexivity
\[
\bigcap_{i \in I} C_i = \bigcap_{i \in \mathcal{I}} \tilde{C}_i \neq \emptyset.
\]
Theorem 12. Assume \((X, d)\) is strictly \(\infty\)-convex and nearly uniformly convex. Then the midpoint map \(m\) is continuous.

\textit{Proof.} Since \((X, d)\) is strictly \(\infty\)-convex we see that geodesics are unique. Thus the midpoint map by \(m : (x, y) \mapsto m(x, y)\) is well-defined. Now if \((x_n, y_n) \to (x, y)\) then for all \(\epsilon > 0\) the sequence \(m_n = m(x_n, y_n)\) eventually enters the closed convex and bounded set

\[ A_\epsilon = B_{\frac{1}{2}d(x,y)+\epsilon}(x) \cap B_{\frac{1}{2}d(x,y)+\epsilon}(y). \]

By uniform \(\infty\)-convexity \(\bigcap_{\epsilon>0} A_\epsilon\) is non-empty and contains only the point \(m(x, y)\).

We only need to show that \(\text{diam} A_\epsilon \to 0\) as \(\epsilon \to 0\). Now assume there is a sequence \(x_n \in A_{\frac{1}{2}}\) that is not Cauchy, so assume it is \(\delta\)-separated for some \(\delta > 0\). Then by nearly uniform convexity there is a \(\rho(\delta) > 0\) \(B_{(1-\rho(\delta))\frac{1}{2}d(x,y)}(x) \cap \text{conv}(x_n) \neq \emptyset\).

And thus

\[ \bigcap_{m} \text{conv}(x_n) \cap B_{(1-\rho(\delta))\frac{1}{2}d(x,y)}(x) \neq \emptyset. \]

But this contradicts the fact that

\[ \bigcap_{m} \text{conv}(x_n) \subset \bigcap_{m} A_{\frac{1}{2}} \subset B_{\frac{1}{2}d(x,y)}(y) \]

is disjoint from \(B_{(1-\rho(\delta))\frac{1}{2}d(x,y)+\epsilon}(x)\). \(\square\)

Corollary 13. A strictly \(\infty\)-convex, nearly uniformly convex metric space is contractible.

\textit{Proof.} Take a fixed point \(x_0 \in X\) and define the map

\[ \Phi_t(x) = \gamma_{xx_0}(t) \]

where \(\gamma_{xx_0}\) is the geodesic connecting \(x\) and \(x_0\). Now proof of previous theorem also shows that \(t\)-midpoints are continuous, in particular \(\Phi_t\) is continuous. \(\square\)

Weak topologies

In Hilbert and Banach spaces the concept of weak topologies can be introduced with the help of dual spaces. Since for general metric spaces there is (by now) no concept of dual spaces, a direct definition needs to be introduced. As it turns out the first topology agrees with the usual weak topology, see Corollary 17.

\textbf{Co-convex topology.} The first weak topology on metric spaces is the following. It already appeared in [Mon06]. As it turns out, this topology is agrees with the weak topology on any Banach space, see Corollary 17 below.

\textbf{Definition 14 (Co-convex topology).} Let \((X, d)\) be a metric space. Then the co-convex topology \(\tau_{co}\) is the weakest topology containing all complements of closed convex sets.

Obviously this topology is weaker than the topology induced by the metric and since point sets are convex the topology satisfies the \(T_1\)-separation axiom, i.e. for each two points \(x, y \in X\) there is an open neighborhood \(U_x\) containing \(x\) but not \(y\). Furthermore, the set of weak limit points of a sequence \((x_n)_{n \in \mathbb{N}}\) is convex if the space is \(\infty\)-convex. A useful characterization of the limit points is the following:
Lemma 15. A sequence of points \( x_n \) converges weakly to \( x \) iff for all subsequences \( (x_{n'}) \) it holds
\[
x \in \text{conv}(x_{n'}).
\]
The set of limit point \( \text{Lim}(x_n) \) is the non-empty subset
\[
\bigcap_{(i_n) \subset I_{\text{inf}}} \text{conv}(x_{i_n})
\]
where \( I_{\text{inf}} \) is the set of sequences of increasing natural numbers.

Remark. The same statement holds for also for nets. Below we will make most statements only for sequences if in fact they also hold for nets.

Proof. This follows immediately from the fact that
\[
A(x_{n'}) = \text{conv}(x_{n'})
\]
is closed, bounded and convex and thus weakly closed.

First suppose \( x \notin A(x_{n'}) \) for some subsequence \( (x_{n'}) \). By definition \( x_n \xrightarrow{\tau_{co}} y \) implies that \( x_n \) eventually leaves every closed bounded convex sets not containing \( y \). Since \( (x_{n'}) \subset A(x_{n'}) \) for \( m' \geq n' \), we conclude \( (x_{n'}) \) cannot converge weakly to \( x \).

Conversely, if \( (x_n) \) does not converge to \( x \) then there is a weakly open set \( U \in \tau_{co} \) such that \( (x_n) \not\subset U \) and \( x \in U \). In particular, for some subsequence \( (x_{n'}) \) it holds \( (x_{n'}) \subset X \setminus U \). Since \( \tau_{co} \) is generated by complements of closed convex sets we can assume \( U = X \setminus C \) for some closed convex subset \( C \). Therefore, \( (x_{n'}) \subset C \) and thus \( A(x_{n'}) \subset C \), i.e. \( x \notin A(x_{n'}) \). □

Corollary 16. For any weakly convergent sequence \( (x_n) \) and countable subset \( A \) disjoint from \( \text{Lim}(x_n) \) there is a subsequence \( (x_{n'}) \) such that
\[
A \cap \bigcap_{m \in \mathbb{N}} \text{conv}(x_{n'})_{n' \geq m} = \emptyset.
\]

Proof. First note, by the lemma above there is a subsequence \( (x_{m_n(0)}) \) of \( (x_n) \) such that \( y_0 \in A \) is not contained in \( \text{conv}(x_{m_n(0)}) \). Now inductively constructing \( (x_{m_n(k)}) \) avoiding \( y_k \) using the sequence \( (x_{m_n(k-1)}) \) we can choose the diagonal sequence \( m_n = m_n^{(n)} \) such that
\[
y \notin \bigcap_{m \in \mathbb{N}} \text{conv}(x_{m_n})_{m_n \geq m}
\]
for all \( y \in A \). □

Corollary 17. On any Banach space \( X \) the co-convex topology \( \tau_{co} \) agrees the weak topology \( \tau_w \). In particular, \( \tau_{co} \) is Hausdorff.

Proof. By Corollary 22 below any linear functional \( \ell \in X^* \) is \( \tau_{co} \)-continuous. Hence \( x_n \xrightarrow{\tau_{co}} x \) implies \( x_n \xrightarrow{\tau_w} x \). The converse follows from that fact that for any subsequence \( (x_{n'}) \) the set \( \text{conv}(x_{n'}) \) is \( \tau_w \)-closed and \( x_{n'} \xrightarrow{\tau_w} x \). Therefore, \( x \in \text{conv}(x_{n'}) \) which implies \( x_n \xrightarrow{\tau_w} x \) by Lemma 15 above. □

Now similar to Banach spaces, one can easy show that reflexivity implies weak compactness of bounded closed convex subsets.
Theorem 18. Bounded closed convex subsets are weakly compact iff the space is reflexive.

Proof. By Alexander sub-base theorem it suffices to show that each open cover \((U_i)_{i \in I}\) of \(B\), where \(U_i\) is a complement of a closed convex set, has a finite subcover. For this, note that \(U_i = X \setminus C_i\) and the cover property of \(U_i\) is equivalent to

\[
\bigcap_{i \in I} B \cap C_i = \emptyset.
\]

If we assume that there is no finite subcover then the collection \((B \cap C_i)_{i \in I}\) has finite intersection property. But then Corollary 11 yields \(\bigcap_{i \in I} B \cap C_i \neq \emptyset\), which is a contradiction.

Conversely, assume \((X, d)\) is not reflexive but any bounded closed convex subset is weakly compact. Then there \((C_i)_{i \in I}\) is a decreasing family of non-empty bounded closed convex subsets such that \(\bigcap_{i \in I} C_i = \emptyset\). Assume w.l.o.g. that \(I\) has a minimal element \(i_0\). Then \(U_i = X \setminus C_i\) is an open cover of \(C_{i_0}\), i.e.

\[
C_{i_0} \subset \bigcup_{i \in I} U_i.
\]

Since \((C_i)_{i \in I}\) is decreasing, \((U_i)_{i \in I}\) is increasing. By weak compactness, finitely many of there are sufficient to cover \(C_{i_0}\). Since \((U_i)_{i \in I}\) is increasing, there exists exactly one \(i_1 \in I\) such that \(C_{i_0} \subset U_{i_1} = X \setminus C_{i_1}\). But then \(C_{i_1} = \emptyset\) contradicting our assumption. \(\square\)

Note that on general spaces the co-convex topology is not necessary Hausdorff. Even in case of \(CAT(0)\)-spaces one can construct an easy counterexample.

Example 19 (Euclidean Cone of a Hilbert space). For the construction of Euclidean cones see [BH99, Chapter I.5]. Let \((H, d_H)\) be an infinite-dimensional Hilbert space and \(d_H\) be the induced metric. The Euclidean cone over \((H, d_h)\) is defined as the set \(C(H) = H \times [0, \infty)\) with the metric

\[
d((x, t), (x', t')) = t^2 + t'^2 - 2tt' \cos(d_{\pi}(x, x'))
\]

where \(d_{\pi}(x, x') = \min\{\pi, d_H(x, x')\}\). By [BH99, Theorem II-3.14] \((C(H), d)\) is a \(CAT(0)\)-space and thus uniformly p-convex for any \(p \in [1, \infty)\). In particular, bounded closed convex subsets are compact w.r.t. the co-convex topology. Note that in \((H, d_H)\) the co-convex topology agrees with the usual weak topology. Now let \((\pi_n)_{n \in \mathbb{N}}\) be a sequence in \(C(H)\). We claim that for any subsequence \((\pi_{n'}, 1))\) we have

\[
\bigcap_{m \in \mathbb{N}} \text{conv}((\pi_{n'}, 1))_{n' \geq m} = \{(0, r) \mid r \in [a, b]\}
\]

with \(a < b\) where it is easy to see that \(a\) and \(b\) do not depend on the subsequence. Any point in that intersection is a limit point of \((\pi_n, 1))\) which implies that \(\tau_{\text{co}}(C(H))\) is not Hausdorff. To see this, note that the projection \(p\) onto the line \(\{(0, r) \mid r \geq 0\}\) has the following form

\[
p((x, r)) = (0, r \cos(d_H(x, 0)))
\]

for \(d(x, 0) \leq \frac{\pi}{2}\). In particular, \(d((\pi_n, 1)) = (0, \cos(1))\). Using the weak sequential convergence defined below, this means that \((\pi_n, 1) \rightarrow (0, \cos(1))\), in particular.
Now we will show that the sequence of midpoints \( l_{mn} \) of \((e_n, 1)\) and \((e_m, 1)\) with \(m \neq n\) converges weakly sequentially to some point \((0, r)\) with \(r > \cos(1)\). This immediately implies that \(\bigcap_{m \in \mathbb{N}} \text{conv}(e_n, 1)_{n' \geq m}\) contains more than one point and each is a limit point of \((e_n, 1)\) w.r.t. the co-convex topology.

To show that \(l_{mn}\) does not weakly sequentially converge to \((0, \cos(1))\) we just need to show that \(p(l_{mn}) \neq (0, \cos(1))\). By the calculus of Euclidian cones the points \(l_{mn}\) have the following form

\[
l_{mn} = \left( \frac{e_m - e_n}{2}, r_{\frac{1}{2}} \right)
\]

where \(r_{\frac{1}{2}}\) is the (positive) solution of the equation

\[
r^2 + 1 - 2r \cos \left( \frac{\sqrt{2}}{2} \right) = \frac{1}{4} \left( 2 - 2 \cos \left( \frac{\sqrt{2}}{2} \right) \right),
\]

i.e. 

\[
r_{\frac{1}{2}} = \cos \left( \frac{\sqrt{2}}{2} \right).
\]

Then the projection has the form

\[
p(l_{mn}) = \left( 0, r_{\frac{1}{2}} \cos \left( \frac{\|e_n - e_m\|}{2} \right) \right)
\]

\[
= \left( 0, \cos \left( \frac{\sqrt{2}}{2} \right)^2 \right).
\]

Since \(\cos(1) < \cos(\frac{\sqrt{2}}{2})^2\) we see that \(l_{mn} \not\rightarrow (0, \cos(1))\) w.r.t. weak sequential convergence and

\[
(0, \cos(\frac{\sqrt{2}}{2})^2) \in \bigcap_{m \in \mathbb{N}} \text{conv}(e_n, 1).
\]

And this obviously does not depend on the subsequence.

Note that this space also violates the property \((N)\) defined in [EFL09], more generally any cone over a (even proper) \(CAT(1)\)-space which is not the sphere gives a counterexample. The example also gives a negative answer to Question 3 of [KP08]. This topology is also a counterexample to topologies similar to Monod’s \(T_w\) topology: Let \(\tau_w^p\) be the weakest topology making all maps \(x \mapsto d(x, y)^p - d(x, z)^p\) for \(y, z \in X\) continuous. For Hilbert spaces and \(p = 2\) this is the weak topology, (compare to [Mon06, 18. Example]) which should be \(p = 2\). For the space \((C(H), d)\) one can show that each \(\tau_w^p\) is strictly stronger that the weak sequential convergence.

**Definition 20** (weak lower semicontinuity). A function \(f : X \rightarrow (-\infty, \infty]\) is said to be weakly l.s.c. at a given point \(x \in \text{dom} f\) if

\[
\liminf_{i} f(x_i) \geq f(x)
\]

whenever \((x_i)\) is a net converging to \(x\) w.r.t. \(\tau_{co}\). We say \(f\) is weakly l.s.c. if it is weakly l.s.c. at every \(x \in \text{dom} f\).
Remark. A priori it is not clear if $\tau_{co}$ is first-countable and thus the continuity needs to be stated in terms of nets. In that case it boils down to $\liminf_{n \to \infty} f(x_n) \geq f(x)$.

**Proposition 21.** Assume $(X, d)$ is $\infty$-convex. Then every lower semicontinuous quasi-convex function is weakly lower semicontinuous. In particular, the metric is lower semicontinuous.

**Remark.** A function is quasi-convex iff its sublevels are convex, i.e., whenever $z$ is on a geodesic connecting $x$ and $y$ then $f(z) \leq \max \{f(x), f(y)\}$.

**Proof.** By definition of the co-convex topology, if $x_i \tau_{co} \to x$ and $x_i \in C$ for some closed convex subset $C$ then $x \in C$. Now assume $f$ is not weakly lower semicontinuous at $x$, i.e.

$$\liminf f(x_i) < f(x).$$

Then there is a $\delta > 0$ such that

$$x_i \in A_\delta = \{y \in X \mid f(y) \leq f(x) - \delta\}$$

for all $i \geq i_0$. By quasi-convexity and lower semicontinuity the set $A_\delta$ is closed convex and thus $x \in A_\delta$ which is a contradiction. Hence $f$ is weakly lower semicontinuous. $\square$

A function $\ell : X \to \mathbb{R}$ is called quasi-monotone iff it is both quasi-convex and quasi-concave. Similarly $\ell$ is called linear iff it is both convex and concave. A linear function is obviously quasi-monotone. The converse is not true in general: Every $CAT(0)$-spaces with property $(N)$ (see [EFL09]) admits such functionals; for $x, y \in X$ just set $\ell(x') = d(P_{[x,y]}x', x)$ where $P_{[x,y]}$ is the projection onto the geodesic connecting $x$ and $y$.

**Corollary 22.** Assume $(X, d)$ is $\infty$-convex. Then every continuous quasi-monotone function is weakly continuous.

**Proof.** Just note that the previous theorem implies that a quasi-monotone function is both weakly lower and upper semicontinuous. $\square$

In order to get the Kadec-Klee property one needs to find limit points which are easily representable.

**Definition 23 (countable reflexive).** A reflexive metric space $(X, d)$ is called countable reflexive if for each weakly convergent sequence $(x_n)$ there is a subsequence $(x_{n'})$ such that

$$\text{Lim}(x_n) = \bigcap_{m \in \mathbb{N}} \text{conv}(x_{n'})_{n' \geq m}.$$ 

By diagonal procedure it is easy to see that one only needs to show that for each $\epsilon > 0$ there is a subsequence $(x_{n'})$ such that

$$B_\epsilon(\text{Lim}(x_n)) \supset \bigcap_{m \in \mathbb{N}} \text{conv}(x_{n'})_{n' \geq m}.$$ 

**Lemma 24.** Any reflexive Banach space is countable reflexive. More generally any reflexive metric space admitting quasi-monotone functions separating points is countable reflexive. In this case the co-convex topology is Hausdorff.
Proof. If \( x_n \xrightarrow{\tau_{co}} x \) and \( x \neq y \in \bigcap_m \text{conv}(x_n)_{n \geq m} \) then there is a quasi-monotone functional \( \ell \) such that \( \ell(y) > \ell(x) \). Since \( \ell \) is weakly continuous we have \( \ell(x_n) \to x \) and thus by quasi-convexity of \( \ell \) also \( \ell(y) > \ell(x') \) for all \( x' \in \text{conv}(x_n)_{n \geq m} \) with \( m \in \mathbb{N} \) sufficiently large \( m \in \mathbb{N} \). However, this contradicts \( y \in \bigcap_m \text{conv}(x_n)_{n \geq m} \) and also shows that \( \tau_{co} \) is Hausdorff. \( \square \)

Theorem 25 (Nearly uniform convexity). Let \((X, d)\) be nearly uniformly convex and countable reflexive. Then for any \( \epsilon \)-separated sequence \((x_n)\) in \( B_R(y) \) there is a weak limit point of \((x_n)\) contained in the ball \( B_{(1-\rho)}R(y) \).

Proof. If \((x_n)\) is \( \epsilon \)-separated with \( d(x_n, y) \leq R \) and assume w.l.o.g. that \((x_n)\) is chosen such that

\[
\text{Lim}(x_n) = \bigcap_{m \in \mathbb{N}} C_m
\]

where \( C_m = \text{conv}(x_n)_{n \geq m} \). We know by nearly uniform convexity there is a \( \rho > 0 \) such that

\[
\tilde{C}_m = B_{(1-\rho)}R(y) \cap C_m \neq \emptyset.
\]

Since \( \tilde{C}_m \) is non-decreasing closed convex and non-empty, we see by reflexivity that \( \cap_m \tilde{C}_m \neq \emptyset \) and hence \( B_{(1-\rho)}R(y) \cap \text{Lim}(x_n) \neq \emptyset \). \( \square \)

Theorem 26 (Kadec-Klee property). Let \((X, d)\) be strictly \( \infty \)-convex, nearly uniformly convex and countable reflexive. Suppose some fixed \( y \in X \) and for each weak limit point \( x \) of \((x_n)\) one has \( d(x_n, y) \to d(x, y) \) then \((x_n)\) has exactly one limit point and \((x_n)\) converges strongly, i.e. norm plus weak convergence implies strong convergence.

Corollary 27. If, in addition, \( \tau_{co} \) is Hausdorff then \( x_n \xrightarrow{\tau_{co}} x \) and \( d(x_n, y) \to d(x, y) \) implies \( x_n \to x \).

Proof of the Theorem. Since strong convergence implies weak and norm convergence, we only need to show the converse. For this let \( x_n \) be some weakly convergent sequence. Note that \( d(x, y) = \lim d(x_n, y) = \text{const} \) for all limit points \( x \) of \((x_n)\). Since \( x \mapsto d(\cdot, y) \) is strictly quasi-convex and the set of limit points is convex, there can be at most one limit point, i.e. \( x_n \xrightarrow{\tau_{co}} x \) for a unique \( x \in X \). If \( d(x, y) = 0 \) then \( x = y \) and \( x_n \to x \) strongly.

Now assume \( d(x, y) = R > 0 \). If \((x_n)\) is not Cauchy then there is a subsequence \((x_n')\) still weakly converging to \( x \) which is \( \epsilon \)-separated for some \( \epsilon > 0 \). By the Theorem 25 there is a limit point \( x^* \) of \((x_n')\) such that \( d(x^*, y) < R \). But this contradicts the fact that \( x^* = x \) and \( d(x, y) = R \). Therefore, \( x = y \). \( \square \)

A “topology” via asymptotic centers. A more popular notion of convergence is the weak sequential convergence. Note, however, it is an open problem whether this “topology” is actually generated by a topology, see [Bac14, Question 3.1.8].

Given a sequence \((x_n)\) in \( X \) define the following function

\[
\omega(x, (x_n)) = \limsup_{n \to \infty} d(x, x_n).
\]

Lemma 28. Assume \((X, d)\) is uniformly \( \infty \)-convex. Then function \( \omega(\cdot, (x_n)) \) has a unique minimizer.
Proof. It is not difficult to see that the sublevels of \( \omega(\cdot, (x_n)) \) are closed bounded and convex. This reflexivity implies existence of minimizers. Assume \( x, x' \) are minimizers and \( x_\frac{1}{2} \) their midpoint. If \( \omega(x, (x_n)) = 0 \) then obviously \( x_n \to x = x' \).

So assume \( \omega(x, (x_n)n) = c > 0 \). Then we can choose a subsequence \((x_{n'})_{n'}\) such that \( \lim_{n' \to \infty} d(x, x_{n'}) \) and \( \lim_{n' \to \infty} d(x', x_{n'}) \) exists and are equal. If \( x \neq x' \) then there is an \( \epsilon > 0 \) such that \( d(x, x') \geq 2\epsilon c \). This yields

\[
\limsup_{n' \to \infty} d(x_{\frac{1}{2}}, x_{n'}) \leq (1 - \rho(\epsilon)) \lim_{n' \to \infty} \max\{d(x, x_{n'}), d(x', x_{n'})\} < c.
\]

But this contradicts \( x \) and \( x' \) be minimizers. Hence \( x = x' \).

The minimizer of \( \omega(\cdot, (x_n)) \) is called the asymptotic center. With the help of this we can define the weak sequential convergence as follows.

Definition 29 (Weak sequential convergence). We say that a sequence \((x_n)_{n \in \mathbb{N}}\) converges weakly sequentially to a point \( x \) if \( x \) is the asymptotic center for each subsequence of \((x_n)\). We denote this by \( x_n \overset{w}{\to} x \).

For CAT(0)-spaces it is easy to see that \( x_n \overset{w}{\to} x \) implies \( x_n \overset{\tau_\gamma}{\to} x \), i.e. the weak topology is weaker than the weak sequential convergence (see [Bač14, Lemma 3.2.1]). Later we will show that the weak sequential limits can be strongly approximated by barycenters, which can be seen as a generalization of the Banach-Saks property (see below). If, in addition, the barycenter of finitely many points is in the convex hull of those points, one immediate gets that \( x_n \overset{w}{\to} x \) implies \( x_n \overset{\tau_\gamma}{\to} x \).

Proposition 30. Each bounded sequence \((x_n)\) has a subsequence \((x_{n'})\) such that \( x_{n'} \overset{w}{\to} x \).

Proof. The proof can be found in [Bač14, Proposition 3.2.1]. Since it is rather technical we leave it out.

A different characterization of this convergence can be given as follows (see [Bač14, Proposition 3.2.2]).

Proposition 31. Assume \((X, d)\) is uniformly \( \infty \)-convex. Let \((x_n)\) be a bounded sequence and \( x \in X \). The the following are equivalent:

1. The sequence \((x_n)\) converges weakly sequentially to \( x \).
2. For every geodesic \( \gamma : [0, 1] \to X \) with \( x \in \gamma([0, 1]) \), we have \( P_\gamma x_n \to x \) as \( n \to \infty \).
3. For every \( y \in X \), we have \( P_{[x,y]} x_n \to x \) as \( n \to \infty \).

Proof. (i)\(\Rightarrow\)(ii): Let \( \gamma \) be some geodesic containing \( x \). If

\[
\lim d(P_\gamma x_n, x) \geq 0
\]

then there is a subsequence \((x_{n'})\) such that

\[
P_\gamma y_n \to y \in \gamma([0,1]) \setminus \{x\}.
\]

But then \( d(P_\gamma x_{n'}, x_{n'}) < d(x, x_{n'}) \) which implies

\[
\limsup_{n \to \infty} d(y, x_{n'}) = \limsup_{n \to \infty} d(P_\gamma x_{n'}, x_{n'}) \leq \limsup_{n \to \infty} d(x, x_{n'})
\]

and contradicts uniqueness of the asymptotic center of \((x_{n'})\).

(ii)\(\Rightarrow\)(iii): Trivial.
(iii)⇒(i): Assume \((x_n)\) does not converge weakly sequentially to \(x\). Then for some subsequence \(x_{n'} \to y \in X\setminus\{x\}\). Then by the part above \(P_{[x,y]}x_{n'} \to y\). But this contradicts the assumption \(P_{[x,y]}x_{n'} \to x\). Hence \(x_n \to x\). □

**Corollary 32** (Opial property). Assume \((X,d)\) is uniformly \(\infty\)-convex and \((x_n)\) some bounded sequence with \(x_n \to x\). Then

\[
\liminf d(x, x_n) < \liminf d(y, x_n)
\]

for all \(y \in X\setminus\{x\}\).

**Barycenters in convex metric spaces**

**Wasserstein space.** For \(p \in [1, \infty)\) the \(p\)-Wasserstein space of a metric space \((X,d)\) is defined as the set \(\mathcal{P}_p(X)\) of all probability measures \(\mu \in \mathcal{P}(X)\) such that

\[
\int d^p(x, x_0)d\mu(x)
\]

for some fixed \(x_0 \in X\). Note that by triangle inequality this definition is independent of \(x_0\). We equip this set with the following metric

\[
w_p(\mu, \nu) = \left( \inf_{\pi \in \Pi(\mu, \nu)} \int d^p(x, y)d\pi(x, y) \right)^{\frac{1}{p}}
\]

where \(\Pi(\mu, \nu)\) is the set of all coupling measures \(\pi \in \mathcal{P}(X \times X)\) such that \(\pi(A \times X) = \mu(A)\) and \(\pi(X \times B) = \nu(B)\). It is well-known [Vil09] that \((\mathcal{P}_p(X), w_p)\) is a complete metric space if \((X, d)\) is complete and that it is a geodesic space if \((X, d)\) is geodesic. Furthermore, by Hölder inequality one easily sees that \(w_p \leq w_{p'}\) whenever \(p \leq p'\) so that the limit

\[
w_\infty(\mu, \nu) = \lim_{p \to \infty} w_p(\mu, \nu)
\]

is well-defined and defines a metric on the space \(\mathcal{P}_\infty(X)\) of probability measures with bounded support. An equivalent description of \(w_\infty\) can be given as follows (see [CDJ08]): For a measure \(\pi \in \Pi(\mu, \nu)\) let \(C(\pi)\) be the \(\pi\)-essential support of \(d(\cdot, \cdot)\), i.e.

\[
C(\pi) = \pi - \text{ess sup}_{(x,y) \in X \times X} d(x, y).
\]

Then

\[
w_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} C(\pi).
\]

For a fixed point \(y \in X\) the distance of \(\mu\) to the delta measure \(\delta_y\) has the following form

\[
w_p^p(\mu, \delta_y) = \int d^p(x, y)d\mu(x)
\]

and

\[
w_\infty(\mu, \delta_y) = \sup_{x \in \text{supp} \mu} d(x, y)
\]

where \(\text{supp} \mu\) is the support of \(\mu\).
Existence and uniqueness of barycenters.

**Lemma 33.** Assume $(X,d)$ is $p$-convex then $y \mapsto \int d^p(x,y)d\mu(x)$ is convex for $p \in [1,\infty)$ whenever $\mu \in \mathcal{P}_p(X)$. In case $p > 1$ strict $p$-convexity even implies strict convexity. Furthermore, if $\mu$ is not supported on a single geodesic then $y \mapsto \int d(x,y)d\mu(x)$ is strictly convex if $(X,d)$ is strictly 1-convex.

**Remark.** (1) It is easy to see that for a measure supported on a geodesic the functional $y \mapsto \int d(x,y)d\mu(x)$ cannot be strictly convex on that geodesic.

(2) The same holds for the functional $F_w(y) := \int d^p(x,y) - d^p(x,w)d\mu(x)$ as defined in [Kuw13]

**Proof.** Let $y_0, y_1 \in X$ be two point in $X$ and $y$ be any geodesic connecting $y_0$ and $y_1$. Then by $p$-convexity

$$d^p(x,y) \leq (1-t)d^p(x,y_0) + td^p(x,y_1)$$

which implies convexity of the functional and similarly strict convexity if $p > 1$.

If $\mu$ is not supported on a single geodesic then there is a subset of positive $\mu$-measure disjoint from $\{y_i | t \in [0,1]\}$ such that $d(x,y_i) < (1-t)d(x,y_0) + td(x,y_1)$.

In particular, $y \mapsto \int d(x,y)d\mu(x)$ is strictly convex. $\square$

**Lemma 34.** Assume $(X,d)$ is uniformly $\infty$-convex with modulus $\rho$. Let $\mu \in \mathcal{P}_\infty(X)$ then the function $F : y \mapsto \omega_\infty(\mu, \delta_y)$ is uniformly quasi-convex, i.e. whenever $d(y_0,y_1) < \epsilon \max\{F(y_0),F(y_1)\}$ for some $\epsilon > 0$ then

$$F(y_{\frac{1}{2}}) \leq (1 - \rho(\epsilon)) \max\{F(y_0),F(y_1)\}.$$

**Remark.** In contrast to the cases $1 < p < \infty$ strict $\infty$-convexity is not enough.

**Proof.** Note that $F$ has the following equivalent form

$$F(y) = \sup_{x \in \text{supp } \mu} d(x,y).$$

Take any $y_0, y_1 \in X$ with $d(y_0,y_1) > \epsilon \max\{F(y_0),F(y_1)\}$. Let $x_n$ be a sequence such that $F(y_{\frac{1}{2}}) = \lim_{n \to \infty} d(x_n,y_{\frac{1}{2}})$. By uniform $\infty$-convexity we have

$$\lim_{n \to \infty} d(x_n,y_{\frac{1}{2}}) \leq (1 - \rho(\epsilon)) \max\{d(x_n,y_0),d(x_n,y_1)\} \leq (1 - \rho(\epsilon)) \max\{F(y_0),F(y_1)\}. \square$$

The following was defined in [Kuw13].

**Definition 35** ($p$-barycenter). For $p \in [1,\infty]$ the $p$-barycenter of a measure $\mu \in \mathcal{P}_p(X)$ is defined as the point $y \in X$ such that $w_p(\mu,\delta_y)$ is minimal. If $p < \infty$ and $\mu$ has only $(p-1)$-moments, i.e. $\int d^{p-1}(x,y)d\mu(x) < \infty$, then the $p$-barycenter can be defined as the minimizer of the functional $F_w(y)$ above. If the $p$-barycenter is unique we denote it by $b_p(\mu)$.

**Remark.** (1) This functional $F_w(y)$ is well-defined since

$$|F_w(y)| \leq pd(y,w) \int (d(x,y) + d(x,w))^{p-1} d\mu(x).$$

Furthermore, $F_w(y) - F_w(y)$ is constant and thus the minimizer(s) are independent of $w \in X$. 


(2) The \( \infty \)-barycenters are also called circumcenter. In case \( \mu \) consists of three points it was recently used in [BHJ+14] to define a new curvature condition. From the section above, the \( \infty \)-barycenter only depends on the support of the measure \( \mu \). Hence the \( \infty \)-barycenter of any bounded set \( A \) can be defined as

\[
b_\infty(A) = \arg \min_{y \in X} \sup_{x \in A} d(x, y).
\]

The proofs below work without any change.

**Theorem 36.** On any \( p \)-convex, reflexive metric space \((X, d)\) every measure \( p \)-moment has \( p \)-barycenter.

**Proof.** Define

\[
A^p_r = \{ y \in X \mid w_p(\mu, \delta_y) \leq r \},
\]

which is a closed convex subset of \( X \) which is non-empty for \( r > m^p_\mu = \inf_{y \in X} w_p(\mu, \delta_y) \).

If it is bounded then by reflexivity

\[
A_{m_\mu} = \bigcap_{r > m_\mu} A_r \neq \emptyset.
\]

In this case minimality implies \( w_\infty(\mu, \delta_y) = m_\mu \) for all \( y \in A_{m_\mu} \).

In case \( p = \infty \) note that \( y \mapsto w_\infty(\mu, \delta_y) = \sup_{x \in \supp \mu} d(x, y) \) is finite iff \( \mu \) has bounded support in which case \( A_r \) is bounded as well.

The cases \( p \in (1, \infty) \) where proven in [Kuw13, Proposition 3.1], the assumption on properness can be dropped using reflexivity. For convenience we include the short proof: If \( \mu \in \mathcal{P}_p(X) \) then \( w_p(\mu, \delta_{y_0}) \leq R \). Now take any \( y \in X \) and assume \( w_p(\mu, \delta_y) \leq r \). Since \((X, d)\) is isometrically embedded into \( (\mathcal{P}_p(X), w_p) \) by the map \( y \mapsto \delta_y \) we have

\[
d(y_0, y) = w_p(\delta_{y_0}, \delta_y) \leq w_p(\mu, \delta_{y_0}) + w_p(\mu, \delta_y) \leq R + r,
\]

i.e. \( y \in B_{R+r}(y_0) \) which implies \( A_r \) is bounded. Using a similar argument one can also show that \( A_r \) is bounded if \( \mu \) is only in \( \mathcal{P}_{p-1}(X) \). \( \square \)

**Corollary 37.** Let \( p \in [1, \infty] \) and \((X, d)\) be a strictly \( p \)-convex if \( p \in [1, \infty) \) and uniformly \( \infty \)-convex if \( p = \infty \). Then \( p \)-barycenters are unique for \( p > 1 \). In case \( p = 1 \), all measure admitting 1-barycenters which are not supported on a single geodesic have a unique 1-barycenter.

The \( p \)-product of finitely many metric spaces \( \{(X_i, d_i)\}_{i=1}^n \) is defined the metric space \((X, d)\) with \( X = \times_{i=1}^n X_i \) and

\[
d(x, y) = \left( \sum_{i=1}^n d^p_i(x_i, y_i) \right)^{\frac{1}{p}}.
\]

A minor extension of [Foe04, Theorem 1] shows that for \( p \in (1, \infty) \) the space \((X, d)\) is strictly \( p \)-convex if all \((X_i, d_i)\) are if \( 1 < p < \infty \) and projections onto the factors of geodesic in \((X, d)\) are geodesics in \((X_i, d_i)\).

**Theorem 38.** Let \( \{(X_i, d_i)\}_{i=1}^n \) be finitely many strictly \( p \)-convex reflexive metric spaces and \((X, d)\) be the \( p \)-product of \( \{(X_i, d_i)\}_{i=1}^n \). If \( \mu \in \mathcal{P}_p(X) \) then \( b_p(\mu) = (b_p(\mu_i)) \) where \( \mu_i \) are the marginals of \( \mu \).
With the help of barycenters Jensen’s inequality can be stated as follows.

\[(\text{Jensen’s inequality})\]

Definition 39

The classical Jensen’s inequality states that on a Hilbert space \(H\) for every lower semicontinuous function \(\varphi \in L^1(H, \mu)\) it holds

\[
\varphi \left( \int x d\mu(x) \right) \leq \int \varphi(x) d\mu(x).
\]

With the help of barycenters Jensen’s inequality can be stated as follows.

**Jensen’s inequality.** The classical Jensen’s inequality states that on a Hilbert space \(H\) for any measure \(\mu \in \mathcal{P}_1(H)\) and any convex lower semicontinuous function \(\varphi \in L^1(H, \mu)\) it holds

\[
\varphi \left( \int x d\mu(x) \right) \leq \int \varphi(x) d\mu(x).
\]

Thus by existence for the factors we know

\[
\inf_{y \in X} w_p(\mu, \delta_y) \leq \sum_{i=1}^n w_p(\mu_i, \delta_{b_p(\mu_i)}) = w_p(\mu, (b_p(\mu_i))).
\]

Conversely, suppose there is a \(y\) such that \(w_p(\mu, \delta_y) \leq w_p(\mu, b_p(\mu))\). Since it holds \(w_p(\mu, \delta_y) \geq w_p(\mu_i, b_p(\mu_i))\) we see that \(y\) is a minimizer of \(y \mapsto w_p(\mu, \delta_y)\). Since \((X, d)\) is strictly \(p\)-convex, \(y = b_p(\mu)\).

**Proof.** If is not difficult to see that

\[
w_p^\mu(\mu, \delta_y) = \sum_{i=1}^n w_p^\mu(\mu_i, \delta_{y_i}).
\]

Thus by existence for the factors we know

\[
\inf_{y \in X} w_p(\mu, \delta_y) \leq \sum_{i=1}^n w_p(\mu_i, \delta_{b_p(\mu_i)}) = w_p(\mu, (b_p(\mu_i))).
\]

Conversely, suppose there is a \(y\) such that \(w_p(\mu, \delta_y) \leq w_p(\mu, b_p(\mu))\). Since it holds \(w_p(\mu, \delta_y) \geq w_p(\mu_i, b_p(\mu_i))\) we see that \(y\) is a minimizer of \(y \mapsto w_p(\mu, \delta_y)\). Since \((X, d)\) is strictly \(p\)-convex, \(y = b_p(\mu)\).

**Definition 39** (Jensen’s inequality). A metric space \((X, d)\) is said to admit the \(p\)-Jensen’s inequality if for all measure \(\mu\) admitting a (unique) barycenter \(b_p(\mu)\) and for every lower semicontinuous function \(\varphi \in L^{p-1}(X, \mu)\) it holds

\[
\varphi(b_p(\mu)) \leq b_p(\varphi_\ast \mu)
\]

where \(\varphi_\ast \mu \in \mathcal{P}(\mathbb{R})\) is the push-forward of \(\mu\) via \(\varphi\).

For \(p = 2\) this boils down to

\[
b_2(\varphi_\ast \mu) = \int \varphi d\mu.
\]

Using the existence proofs above one can adapt Kuwae’s proof of [Kuw13, Theorem 4.1] to show that Jensen’s inequality holds spaces satisfying the condition \((B)\), i.e. for any two geodesics \(\gamma, \eta\) with \(\{p_0\} = \gamma \cap \eta\) and \(\pi_\gamma(y) = p_0\) for \(y \in \eta \setminus \{p_0\}\) it holds \(\pi_\eta(x) = p_0\) for all \(x \in \eta\). Kuwae states this as \(\eta \perp_{p_0} \gamma\) implies \(\gamma \perp_{p_0} \eta\).

Using Busemann’s non-positive curvature condition one can then show that if each measure in \(\mathcal{P}_1(X)\) admits unique barycenters and Jensen’s inequality holds on the product space then a Wasserstein contraction holds, i.e.

\[
d(b_2(\mu), b_2(\nu)) \leq w_1(\mu, \nu).
\]

If instead the \(p\)-Busemann holds one still gets the following:

**Proposition 40.** Let \((X, d)\) be \(p\)-Busemann for some \(p \in [1, \infty)\). If the \(2\)-Jensen’s inequality for holds on the \(2\)-product \(X \times X\) then

\[
d(b_2(\mu), b_2(\nu)) \leq w_p(\mu, \nu).
\]

**Proof.** Since \((x, y) \mapsto d(x, y)^p\) is convex on \(X \times X\) we have by Jensen’s inequality

\[
d(b_2(\mu), b_2(\nu))^p \leq \int d^p(x, y) d\pi(x, y)
\]

for any \(\pi \in \Pi(\mu, \nu)\). Hence \(d(b_2(\mu), b_2(\nu)) \leq w_p(\mu, \nu)\). \(\square\)
**Banach-Saks**

The classical Banach-Saks property for Banach spaces is stated as follows: Any bounded sequence has a subsequence \((x_{m_n})\) such that sequence of Cesàro means

\[
\frac{1}{N} \sum_{n=1}^{N} x_{m_n}
\]
converges strongly. In a general metric space there is no addition of two elements defined. Furthermore, convex combinations do not commute (are not associative), i.e. if \((1 - \lambda)x \oplus \lambda y\) denotes the point \(x_\lambda\) on the geodesic connecting \(x\) and \(y\) then in general

\[
\frac{2}{3} \left( \frac{1}{2} x \oplus \frac{1}{2} y \right) \oplus \frac{1}{3} z \neq \frac{1}{3} x \oplus \frac{2}{3} \left( \frac{1}{2} y \oplus \frac{1}{2} z \right),
\]
so that \(\frac{1}{N} \bigoplus_{n=1}^{N} x_n\) does not make sense.

For Hilbert spaces the point \(\frac{1}{N} \bigoplus_{n=1}^{N} x_n\) agrees with the 2-barycenter of the measure

\[
\mu_N = \frac{1}{N} \sum_{n=1}^{N} \delta_{x_n}.
\]

Since this is well-defined on general metric spaces the Banach-Saks property can be formulated as follows.

**Definition 41** \((p\text{-Banach-Saks})\). Let \(p \in [1, \infty]\) and suppose for any sequence \((x_n)\) in a metric space \((X, d)\) the measures \(\mu_n\) admit a unique \(p\)-barycenter. Then \((X, d)\) is said to satisfy the \(p\)-Banach-Saks property if every sequence \((x_n)\) there is a subsequence \((x_{m_n})\) such that the sequence of \(p\)-barycenters of the measures \(\tilde{\mu}_N = \frac{1}{N} \sum_{n=1}^{N} \delta_{x_{m_n}}\) converges strongly.

Since Hilbert spaces satisfy the (traditional) Banach-Saks property they also satisfy the \(p\)-Banach-Saks property. Yokota managed in [Yok13, Theorem C] (see also [Bac14, Theorem 3.1.5]) to show that any \(CAT(1)\)-domain with small radius, in particular any \(CAT(0)\)-space, satisfy the 2-Banach-Saks property and if \(x_n \overset{w}{\rightarrow} x\) then \(b_p(\mu_N) \rightarrow x\) where \(\tilde{\mu}_N\) is defined above. We will adjust his proof to show that for \(p \in (1, \infty)\) any uniformly \(p\)-convex space satisfies the \(p\)-Banach-Saks property and the limit of the chosen subsequence agrees with the weak sequential limit. Since a \(CAT(1)\)-domain with small radius is uniformly \(p\)-convex, our result generalizes Yokota’s when restricted to that convex subset.

We leave the proof of the following statement to the reader.

**Lemma 42.** If \((X, d)\) is uniformly \(p\)-convex and \(x_n \rightarrow x\) then \(b_p(\mu_N) \rightarrow x\) where \(\mu_N = \frac{1}{N} \sum_{n=1}^{N} \delta_{x_n}\).

**Lemma 43.** Assume \((X, d)\) be a metric space admitting midpoints and let \(f : X \rightarrow \mathbb{R}\) be a uniformly convex function with modulus \(\omega\). If \(f\) attains its minimum at \(x_m \in X\) then

\[
f(x) \geq f(x_{\min}) + \frac{1}{2} \omega(d(x, x_{\min})).
\]

**Proof.** Let \(x \in X\) be arbitrary and \(m\) be the midpoint of \(x\) and \(x_m\). Then

\[
f(x_{\min}) \leq f(m) \leq \frac{1}{2} f(x_{\min}) + \frac{1}{2} f(x) - \frac{1}{4} \omega(d(x, x_{\min})).
\]

\[\square\]
**Theorem 44.** Let \((X, d)\) be a uniformly \(p\)-convex metric space. If \(x_n \xrightarrow{w} x\) then there is a subsequence \((x_{m_n})\) such that \(b_p(\mu_N) \rightarrow x\) where \(\mu_N = \frac{1}{N} \sum_{n=1}^{N} \delta_{x_{m_n}}\). In particular, \((X, d)\) satisfies the \(p\)-Banach-Saks property.

**Proof.** By the previous lemma we can assume that \((x_n)\) is \(2\epsilon\)-separated for some \(\epsilon > 0\), since otherwise there is a strongly convergent subsequence fulfilling the statement of the theorem. Furthermore, assume w.l.o.g. that \(d(x_n, x) \rightarrow r\). Then

\[
\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} d(x, x_n)^p \rightarrow r^p.
\]

For any measure \(\mu \in \mathcal{P}_p(X)\) define

\[
\Var_{\mu, p}(y) = \int d(x, y)^p d\mu(x)
\]

and

\[
V(\mu) = \inf_{y \in X} \Var_{\mu, p}(y)
\]

Furthermore, for any finite subset \(I \subset \mathbb{N}\) define

\[
\mu_I = \frac{1}{|I|} \sum_{i \in I} \delta_{x_i}.
\]

Let \(R > 0\) be such that \(d(x_n, x) \leq R\). Note that \(b_p(\mu_I) \in B_{2R}(x)\) for any finite \(I \subset \mathbb{N}\). Furthermore, if \((y_i)_{i \in [0, 1]}\) is a geodesic in \(B_{2R}(x)\) with \(d(y_0, y_1) \geq 3\delta R\) then \(d(y_0, y_1) \geq \delta M^p(d(x_n, y_0), d(x_n, y_1))\) and by uniform \(p\)-convexity

\[
d(x_n, y_{\frac{1}{2}})^p \leq (1 - \hat{\rho}_p(\delta)) \left( \frac{1}{2} d(x_n, y_0) + \frac{1}{2} d(x_n, y_1) \right).
\]

Thus there is a monotone function \(\tilde{\omega} : (0, \infty) \rightarrow (0, \infty)\) such that

\[
d(x_n, y_{\frac{1}{2}})^p \leq (1 - \tilde{\omega}(d(y_0, y_1))) \left( \frac{1}{2} d(x_n, y_0)^p + \frac{1}{2} d(x_n, y_1)^p \right).
\]

This implies

\[
\Var_{\mu_I, p}(y_{\frac{1}{2}}) \leq (1 - \tilde{\omega}(d(y_0, y_1))) \left( \frac{1}{2} \Var_{\mu_I, p}(y_0) + \frac{1}{2} \Var_{\mu_I, p}(y_1) \right)
\]

By \(2\epsilon\)-separation of \((x_n)\) for any finite \(I \subset \mathbb{N}\) there is at most one \(i \in I\) such that \(d(x_i, y) \leq \epsilon\). Hence if \(|I| \geq 2\)

\[
V(\mu_I) \geq \frac{1}{2} r^p.
\]

Combining this with the above inequality we see that for \(\omega(r) = 2e^p \tilde{\omega}(r)\)

\[
\Var_{\mu_I, p}(y_{\frac{1}{2}}) \leq \frac{1}{2} \Var_{\mu_I, p}(y_0) + \frac{1}{2} \Var_{\mu_I, p}(y_1) - \frac{1}{4} \omega(d(y_0, y_1)).
\]

This implies

\[
\Var_{\mu_I, p}(y_i) \leq (1 - t) \Var_{\mu_I, p}(y_0) + t \Var_{\mu_I, p}(y_1) - t(1 - t) \omega(d(y_0, y_1)).
\]

i.e. the functions \(\Var_{\mu_I, p} : B_{2R}(X) \rightarrow \mathbb{R}\) are uniformly convex with modulus \(\omega\).

The next steps follow directly from the proofs of [Yok13, Theorem C] and [Bac14, Theorem 3.1.5] we include the whole proof for convenience of the reader.

**Step 1:** set \(I_k^N = \{(k-1)2^N, \ldots, k2^N\} \subset \mathbb{N}\) for any \(k, N \in \mathbb{N}\). We claim that if

\[
\sup_{N \in \mathbb{N}} \liminf_{k \rightarrow \infty} V(\mu_{I_k^N}) = r^p
\]
then $b_n = b_p(\mu_{IJ})$ converges strongly to $x$. To see this note that for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
\[
\liminf_{k \to \infty} V(I_N^k) \geq r^p - \epsilon.
\]
Then
\[
\liminf_{n \to \infty} \text{Var}_{\mu_{IJ},p}(b_n) \geq \liminf_{k \to \infty} V(\mu_{I_N^k}) \geq r^p - \epsilon.
\]
Since $\epsilon > 0$ is arbitrary we see that $\liminf_{n \to \infty} \text{Var}_{\mu_{IJ},p}(b_n) \geq r^p$.

By uniform convexity of $\text{Var}_{\mu_{IJ},p}$ and Lemma 43 we also have
\[
\text{Var}_{\mu_{IJ},p}(x) \geq \text{Var}_{\mu_{IJ},p}(b_n) + \frac{1}{2} \omega(d(x, b_n)).
\]
Since the left hand side converges to $r^p$ and $\limsup_{n \to \infty} \text{Var}_{\mu_{IJ},p}(b_n) \geq r^p$. This implies that $\limsup_{n \to \infty} \omega(d(x, b_n)) \to 0$, i.e. $d(x, b_n) \to 0$.

Step 2: We will select a subsequence of $(x_n)$ such the assumption of the claim in Step 1 are satisfied. Set $J_k^N = \{k\}$ for $k \in \mathbb{N}$. We construct a sequence of set $J_k^N$ for $N \in \mathbb{N}$ of cardinality $2^N$ such that $J_k^N = J_{k-1}^{N-1} \cup J_m^{m-1}$ for some $m, l \in \mathbb{N}$.

Furthermore, max $J_k^N < J_{k+1}^N$ and
\[
\lim_{k \to \infty} V(\mu_{J_k^N}) = V_N := \limsup_{l, m \to \infty} V(\mu_{J_{k-1}^{N-1} \cup J_{m-1}^{m-1}}).
\]
It is not difficult to see that $V_N \leq V_{N+1} \leq r^p$ for every $N \in \mathbb{N}$. We will show that $V_N \to r^p$ as $N \to \infty$. It is not difficult to see that this follows from the claim below.

Claim. For every $\epsilon' > 0$ there exists a $\delta > 0$ such that whenever $V_N < (r - \epsilon)^p$ then $V_{N+1} > V_N + \delta$.

Proof of claim. Fix $N \in \mathbb{N}$ and for $l \in \mathbb{N}$ let $b_l^N$ be the $p$-barycenter of $\mu_{J_N^N}$. By assumption there is an $l \in \mathbb{N}$ such that $V(\mu_{J_N^N}) < (r - \epsilon'p$ for all $k \geq l$. By the Opial property, Corollary 32, there is a large $m > l$ such that $d(b_l^N, x_i) > r$ for $i \in J_m^N$.

This implies
\[
\frac{1}{2^N} \sum_{i \in J_m^N} d(b_l^N, x_i)^p > r^p > (r - \epsilon)^p > V(\mu_{J_m^N}) = \frac{1}{2^N} \sum_{i \in J_m^N} d(b_l^N, x_i)^p
\]
and hence
\[
2 \max \{d(b_m^N, b_m^N), d(b_l^N, b_l^N)\} \geq d(b_m^N, b_l^N) > \epsilon'
\]
where $b_m^N$ is the $p$-barycenter of $\mu_{J_m^N \cup J_N^N}$. By uniform convexity of $\text{Var}_{\mu_{IJ},p}$ and Lemma 43 we get
\[
V(\mu_{J_m^N \cup J_N^N}) = \frac{1}{2^{N+1}} \left( \sum_{i \in J_m^N} d(b_m^N, x_i)^p + \sum_{i \in J_N^N} d(b_l^N, x_i)^p \right)
\geq \frac{1}{2} \left[ V(\mu_{J_m^N}) + V(\mu_{J_N^N}) + \omega(d(b_m^N, b_m^N)) + \omega(d(b_l^N, b_l^N)) \right]
\geq \frac{1}{2} \left[ V(\mu_{J_m^N}) + V(\mu_{J_N^N}) + 2\omega(\epsilon') \right].
\]
To finish the proof of the theorem, note that \( \bigcap_{N} \bigcup_{k} J_{k}^{N} \subset \mathbb{N} \) is infinite. Denoting its elements in increasing order \((n_{1}, n_{2}, \ldots)\) we see that the sequence \((x_{n_{k}})\), after naming, satisfies the assumption needed in Step 1. \(\square\)

**Corollary 45.** Assume \((X, d)\) is uniformly \(p\)-convex and that for any sequence \((x_{n})\) the \(p\)-barycenter of \(\mu_{I}\) for any finite \(I \subset \mathbb{N}\) is in the convex hull of the point \(\{x_{i}\}_{i \in I}\). Then whenever \(x_{n} \xrightarrow{w} x\) implies \(x \in \text{conv}(x_{n})\). In particular, it holds \(x_{n} \xrightarrow{\tau} x\), i.e. the co-convex topology is weaker than weak sequential topology.

**Remark.** The assumption of the corollary are satisfied on any CAT(0)-space, see [Bač14, Lemma 2.3.3] for the \(2\)-barycenter. More generally Kuwae’s condition \((A)\) is enough as well, see [Kuw13, Remark 3.7 (2)]. In particular, it holds on all spaces that satisfy Jensen’s inequality.

**Generalized Convexities**

Let \(L: (0, \infty) \to (0, \infty)\) be a strictly increasing convex function such that \(L(1) = 1\) and \(L(r) \to 0\) as \(r \to 0\). Then \(L\) has the following form

\[
L(r) = \int_{0}^{r} \ell(s)ds
\]

where \(\ell\) is a positive monotone function. As an abbreviation we also set \(L_{\lambda}(r) = L(r/\lambda)\) for \(\lambda > 0\).

Given \(L\) we define the \(L\)-mean of two non-negative numbers \(a, b \in [0, \infty)\) as follows

\[
\mathcal{M}^{L}(a, b) = L^{-1} \left( \frac{1}{2} L(a) + \frac{1}{2} L(b) \right) \left\{ t > 0 \mid \frac{1}{2} L \left( \frac{a}{t} \right) + \frac{1}{2} L \left( \frac{b}{t} \right) \leq 1 \right\}.
\]

**Definition 46 (L-convexity).** A metric space admitting midpoints is said to be \(L\)-convex if for any triple \(x, y, z \in X\) it holds

\[
d(m(x, y), z) \leq \mathcal{M}^{L}(d(x, z), d(y, z)).
\]

If the inequality is strict whenever \(x \neq y\) then the space is said to be strictly \(L\)-convex.

In a similar way one can use a more elaborate definition of mean: For \(L\) as above define the Orlicz mean

\[
\mathcal{M}^{L}(a, b) = \inf \left\{ t > 0 \mid \frac{1}{2} L \left( \frac{a}{t} \right) + \frac{1}{2} L \left( \frac{b}{t} \right) \leq 1 \right\}.
\]

Now Orlicz \(L\)-convexity can be defined by using \(\mathcal{M}^{L}\) instead of \(\mathcal{M}^{L}\). It is not clear if this definition is meaningful. The existence theorem for Orlicz-Wasserstein barycenters below only uses \(L\)-convexity.

It is easy to see that for \(p \in (1, \infty)\) (strict) \(p\)-convexity is the same as (Orlicz) \(L\)-convexity for \(L(r) = r^{p}\). However, because \(L\) needs to be strictly convex, the cases 1-convexity and \(\infty\)-convexity are not covered, but can be obtained as limits. Strict convexity of \(L\) also implies that the inequality above is strict whenever \(d(x, y) = |d(x, z) - d(y, z)|\), i.e. the condition \(d(x, y) > |d(x, z) - d(y, z)|\) is not needed for strict \(L\)-convexity.
Lemma 47. Suppose $\Phi$ is a convex function with $\Phi(1) = 1$ and $\Phi(r) \to 0$ as $r \to 0$. Then any (strictly) (Orlicz) $L$-convex metric space is (strictly) (Orlicz) $\Phi \circ L$-convex and (strictly) $\infty$-convex. Also, any (strictly) 1-convex space is strictly (Orlicz) $L$-convex.

The proof of this follows directly from convexity of $\Phi$. Similarly one can define uniform convexity.

Definition 48 (uniform $p$-convexity). A strictly $L$-convex metric space is said to be uniformly $L$-convex if for all $\epsilon > 0$ there is a $\rho_L(\epsilon) \in (0,1)$ such that for all triples $x,y,z \in X$ satisfying $d(x,y) > \epsilon M^L(d(x,z), d(y,z))$ it holds
\[
d(m(x,y), z) \leq (1 - \rho_L(\epsilon))M^L(d(x,z), d(y,z)).
\]

Using $L$ one can also define an Orlicz-Wasserstein space $(P_L(X), w_L)$, see [Stu11] and [Kel13, Appendix] for precise definition and further properties. Since $L(1) = 1$ the natural embedding $x \to \delta_x$ is an isomorphism. For $\mu \in P_L(X)$ and $y \in X$ the metric $w_L$ has the following form
\[
w_L(\mu, y) = \inf \{ t > 0 | \int L\left(\frac{d(x,y)}{t}\right) d\mu(x) \leq 1\}.
\]

Note that by [Stu11] the infimum is attained if $w_L(\mu, y) > 0$.

Now the $L$-barycenter $b_L(\mu)$ of a measure $\mu \in P_L(X)$ can be defined as
\[
b_L(\mu) = \arg \min_{y \in X} w_L(\mu, \delta_y).
\]

Theorem 49. Assume $(X,d)$ is reflexive and strictly $L_\lambda$-convex for any $\lambda > 0$. Then any measure $\mu \in P_L(X)$ admits a unique barycenter.

Remark. Since $L(r) = r^p$ is homogeneous, one sees that $(X,d)$ is strictly $L$-convex if it is strictly $L_\lambda$-convex for some $\lambda > 0$.

Proof. By our assumption we see that for any $\lambda > 0$
\[
\int L\left(\frac{d(x,y_0)}{\lambda}\right) d\mu(x) \leq \frac{1}{2} \int L\left(\frac{d(x,y_0)}{\lambda}\right) d\mu(x) + \frac{1}{2} \int L\left(\frac{d(x,y_1)}{\lambda}\right) d\mu(x).
\]

with strict inequality whenever $y_0 \neq y_1$. Hence, if $F_\mu(y_0), F_\mu(y_1) \leq \Lambda$ then
\[
\int L\left(\frac{d(x,y_0)}{\Lambda}\right) d\mu(x) \leq 1,
\]
i.e. $F_\mu(y_0) \leq \Lambda$. Furthermore, the strict inequality is strict if $y_0 \neq y_1$, i.e. $F_\mu$ is strictly quasi-convex and can have at most one minimizer.

This now implies that the sublevels of $F_\mu$ are convex. Closedness follows from continuity of $F_\mu$. In order to see that they are also bounded, just note that $w_L$ is a metric, i.e. implies that $|w_L(\mu, \delta_{y_0}) - w_L(\delta_{y_0}, \delta_y)| \leq w_L(\mu, \delta_y)$. Thus $F_\mu(y_0) \leq R$ for all $y \in X \setminus B_{2R}(y)$ it holds
\[
F_\mu(y) = w_L(\mu, \delta_y) > R.
\]
reflexivity implies now existence of $L$-barycenters.

In a similar way one can obtain a Banach-Saks theorem for spaces which are uniformly $L_\lambda$-convex for each $\lambda > 0$ such that the moduli $(\rho_L)_\lambda \in (0,\infty)$ are equicomparable for compact subsets of $(0,\infty)$. The proof then follows along the line of Theorem 44.
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