Fundamental group in the projective knot theory

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To Volodia Turaev with gratitude
for lifelong friendship and inspiration

Abstract

In this paper, properties of a link \( L \) in the projective space \( \mathbb{RP}^3 \) are related to properties of its group \( \pi_1(\mathbb{RP}^3 \setminus L) \):

- \( L \) is isotopic to a projective line if and only if \( \pi_1(\mathbb{RP}^3 \setminus L) = \mathbb{Z} \).
- \( L \) is isotopic to an affine circle if and only if \( \pi_1(\mathbb{RP}^3 \setminus L) = \mathbb{Z} \ast \mathbb{Z}/2 \).
- \( L \) is isotopic to a link disjoint from a projective plane if and only if \( \pi_1(\mathbb{RP}^3 \setminus L) \) contains a non-trivial element of order two.

A simple algorithm which finds a system of generators and relations for \( \pi_1(\mathbb{RP}^3 \setminus L) \) in terms of a link diagram of \( L \) is provided.

1 Introduction

Projective knot theory

A classical link is a smooth closed 1-submanifold of \( \mathbb{R}^3 \). If a link is connected, then it is called a knot. The classical knot theory studies classical knots and links. Many notions and results of the classical knot theory extend to knots and links in other 3-manifolds. Any closed connected 3-manifold can be presented as a compactification of \( \mathbb{R}^3 \). The 3-sphere \( S^3 \) is the result of adding a single point to \( \mathbb{R}^3 \). This addition is inessential for topology of knots and links. Knots and links in \( S^3 \) are objects of the same classical knot theory.

A substantial enlargement for the set of knots and links happens when we pass to the next simplest ambient 3-manifold, which is the real projective
space $\mathbb{R}P^3$. It can be obtained from $\mathbb{R}^3$ by adding “points at infinity” as any other closed connected 3-manifold, but $\mathbb{R}P^3$ is the only closed 3-manifold for which the added points are all alike and constitute a surface without singularities. The projective space has many other special features, which provide opportunities which are not available for studying links in other 3-manifolds.

On the other hand, the projective space appears naturally outside of topology, and this gives extra reasons for study of links in the projective space.

In the real algebraic geometry the projective space is even more profound than the Euclidean 3-space or 3-sphere. Recently there was a significant progress in real algebraic knot theory, which studies knots that are real algebraic curves. See [11] and [3]. In that area, projective space is the most elementary and natural ambient space.

Any straight line in $\mathbb{R}^3$ gives rise to a projective line in $\mathbb{R}P^3$. Topologically, a projective line is a circle. A configuration of lines in $\mathbb{R}^3$ skew to each other gives rise to a link in $\mathbb{R}P^3$. See [7].

A \textit{projective link} $L$ is a smooth closed 1-submanifold of $\mathbb{R}P^3$. If $L$ is connected, then it is called a \textit{projective knot}. Projective knots and links are studied in the projective knot theory. It occupies a special place between the classical knot theory and theory of links in arbitrary 3-manifolds.

This paper appears as a fragment of a book by both authors on the projective link theory. It is an expanded version of a preprint [15] of the second author.

Some parts of the classical knot theory have no clear generalizations to knots in arbitrary ambient 3-manifold, but have well-developed counterparts in the projective knot theory.

For example, a large part of the classical knot theory deals with \textit{link diagrams}, decorated generic projections of a link to a plane that are used to describe classical links up to a smooth isotopy.

Projective links also can be described up to smooth isotopy by pictures similar to link diagrams that are used to describe classical links. These pictures are called \textit{projective link diagrams}. Projective link diagrams differ from classical link diagrams in the way that a projective link diagram is placed not in the plane, but in a disk; arcs of the diagram meet the boundary of the disk in pairs of antipodal points. Examples of projective link diagrams are shown in Figure 1. For more details see [5].

Most invariants of classical links can be easily calculated in terms of link diagrams. For some invariants, like the Jones polynomial and Kauffman bracket polynomial, the very definitions rely on diagrams. The first author
used projective link diagrams for generalizing these classical links invariants to projective links, see [5].

**Fundamental groups in the knot theory**

The fundamental group plays a central role in topology of 3-manifolds. One of the most powerful topological invariants of a classical link $L$ is the fundamental group $\pi_1(\mathbb{R}^3 \setminus L)$ of the complement. If $L$ is a knot, then $\pi_1(\mathbb{R}^3 \setminus L)$ is even called just the *knot group*. (The term *link group* is used for a certain quotient group of $\pi_1(\mathbb{R}^3 \setminus L)$, see [12].)

Many geometric problems of knot theory have been solved in terms of knot groups. For example, in 1910 Dehn [8] proved that a knot is isotopic to the unknot if and only if its group is isomorphic to $\mathbb{Z}$ (there was a gap in the proof, which was repaired by Papakyriacopoulos in 1956).

The power of the knot group is compromised by algorithmic difficulties in operating with non-commutative groups. To the best of our knowledge, there is no algorithmic problem in the classical knot theory, which have been solved via reducing to a problem about the knot group (with possibly extra structures like group system) followed by an algorithmic solution of the correspondent group-theoretic problem.

Probably, due to preferences to algorithmic solutions, it happens that in the projective knot theory the fundamental group has not been used so far. Of course, some general results are applicable. For example, Stallings Theorem [14] about necessary and sufficient conditions for fibering of a compact 3-manifold over a circle gives a criterion for a projective knot to have complement fibered over circle in terms of the knot group. But there was no results of this direction belonging and specifically to the projective link theory. The conceptual beauty of fundamental group makes them inevitable.

In this paper we prove three theorems about relation between geometric properties of a projective link $L$ and properties of $\pi_1(\mathbb{R}P^3 \setminus L)$. 

![Figure 1: Projective link diagrams](image)
Homological classes of the link components

The homology group \( H_1(\mathbb{R}^3) \) of \( \mathbb{R}^3 \) is trivial. Therefore each connected component of a classical link is zero-homologous.

The homology group \( H_1(\mathbb{R}P^3) \) consists of two elements. Therefore we can distinguish projective knots homologous to zero and non-homologous to zero. In a projective link, each connected component realizes one of these two classes. The distribution of components into two classes has profound effect on other invariants and properties of a projective link. See Lemma 4 below.

Projective unknots

Recall that a classical knot is called an unknot if it is isotopic to a circle in a plane.

![Figure 2: Diagrams of projective unknots](image)

In the projective space, knots of two isotopy types clearly deserve the name of unknot. One of the classes consists of projective knots isotopic to unknots in \( \mathbb{R}^2 \subset \mathbb{R}P^3 \). We call knots of this kind affine unknots. The other isotopy type contains real projective lines. Projective diagrams of the unknots are shown in Figure 2. Notice that there is one unknot in each of the two homology classes: affine unknot is zero-homologous, a projective line is not.

Dehn’s characterization of unknots in \( \mathbb{R}^3 \) (as classical knots with knot group isomorphic to \( \mathbb{Z} \)) has the following natural counterparts for both types of projective unknots.

**Theorem 1.** A knot \( K \subset \mathbb{R}P^3 \) is isotopic to the affine unknot if and only if \( \pi_1(\mathbb{R}P^3 \setminus K) \) is isomorphic to the free product of \( \mathbb{Z} \) and \( \mathbb{Z}/2 \).

**Theorem 2.** A knot \( K \subset \mathbb{R}P^3 \) is isotopic to a projective line if and only if \( \pi_1(\mathbb{R}P^3 \setminus K) \) is isomorphic to \( \mathbb{Z} \).

These theorems are proved in Section 3.
Contractibility

A projective link \( L \) is said to be \textit{contractible} if it is isotopic to a link, which does not intersect the projective plane \( \mathbb{R}P^2 \subset \mathbb{R}P^3 \). The complement \( \mathbb{R}P^3 \setminus \mathbb{R}P^2 \) can be identified with the affine space \( \mathbb{R}^3 \). Therefore a contractible projective link \( L \) can be contracted by a continuous deformation \( L_t, t \in [0, 1] \), which starts with \( L_0 = L \) and is an isotopy except for the last moment \( t = 1 \), when \( L_1 \) is a point.

In the papers of the first author [5], [6] and the second author [15] contractible projective links were called \textit{affine}.

The following theorem provides necessary and sufficient condition for a projective link to be contractible:

\textbf{Theorem 3.} A link \( L \subset \mathbb{R}P^3 \) is contractible if and only if \( \pi_1(\mathbb{R}P^3 \setminus L) \) contains a non-trivial element of order 2.

The problem of determining whether a link is contractible was considered in literature and there are results in this direction, mostly about necessary conditions:

\textbf{Homology condition.} Each connected component of a link \( L \subset \mathbb{R}P^3 \) realizes a homology class, an element of \( H_1(\mathbb{R}P^3; \mathbb{Z}/2) = \mathbb{Z}/2 \). All components of an contractible link \( K \subset \mathbb{R}P^3 \) realize \( 0 \in H_1(\mathbb{R}P^3; \mathbb{Z}/2) \). The converse is not true: there exist knots homological to zero in \( \mathbb{R}P^3 \), which are not contractible.

The homology class of a closed curve in \( \mathbb{R}P^3 \) equals its intersection number with a projective plane modulo 2. On a projective link diagram, the number of its points on the boundary circle is twice the intersection number of the link with a projective plane. Therefore the homology class realized by a projective link is non-zero iff the number of the boundary points of its diagram is not divisible by four.

All three projective knots shown in Figure 3 are not contractible. The first two of them, knots 21 and 52 are zero-homologous, because the numbers of their boundary points equal 4, the third one, 59, is not zero-homologous, as the number of its boundary points is 6.

\textbf{Self-linking number.} If a knot \( K \subset \mathbb{R}P^3 \) realizes \( 0 \in H_1(\mathbb{R}P^3; \mathbb{Z}/2) \), then a \textit{self-linking number} \( \text{sl}(K) \in \mathbb{Z} \) is defined as the linking number modulo 2 of the connected components of the preimage \( \widetilde{K} \subset S^3 \) of \( K \) under the covering \( S^3 \to \mathbb{R}P^3 \), see [5], §7.

If \( K \) is contractible, then \( \text{sl}(K) = 0 \), see [5], §7. For the knot 21 shown in Figure 3, this invariant equals 2, and this is why 21 is not contractible.
Exponents of monomials in the bracket polynomial. If a projective $k$ component link $L$ is contractible, then the exponents of all monomials of its bracket polynomial $V_K$ defined by the first author in [5] are congruent to $2k - 2$ modulo 4, see [5], Theorem 7.

The self-linking and the exponent conditions are independent. For the knot $5_2$ the self-linking condition does not work, as $\text{sl} \ 5_2 = 0$, while the exponent condition works: $V_{5_2} = A^4 + A^2 - 1 - 2A^{-2} + A^{-4} + 2A^{-6} - 2A^{-10} + A^{-14}$. On the other hand, for the knot $5_9$ the self-linking condition works, as $\text{sl} \ 5_9 = 3$, while the exponent condition does not: $V_{5_9} = A^{-8} + A^{-12} - A^{-20}$. See Figure 3 and the projective link table [6].

In [5] the first author introduced a notion of alternating projective link diagram and found the following criterion for contractibility of projective links which admit alternating diagram: a projective link represented by an alternating diagram is contractible if and only if the difference between the greatest and lowest exponents of its bracket polynomial is divisible by 4.

Theorem 3. provides necessary and sufficient condition for contractibility of a link in $\mathbb{R}P^3$. However the criteria formulated in terms of fundamental groups are not easy to verify. Therefore the known necessary conditions of contractibility mentioned above may happen to be more useful for checking if a specific knot is contractible.

Contractibility in general 3-manifold

The property of a projective link of being contractible admits the following obvious reformulation:

A link $L \subset \mathbb{R}P^3$ is contractible if and only if there exists an embedding $i : D^3 \rightarrow \mathbb{R}P^3$ such that $L \subset i(D^3)$.

This property of links in $\mathbb{R}P^3$ admits a generalization to links in an arbitrary 3-manifold. A link $L$ in a 3-manifold $M$ is called contractible if there exists an embedding $i : D^3 \rightarrow M$ of the ball $D^3$ such that $L \subset i(D^3)$. The following conjecture would provide a generalization of Theorem 3 to
closed 3-manifolds with trivial $\pi_2$.

**Conjecture.** Let $M$ be a closed 3-manifold which satisfies the condition $\pi_2(M) = 0$. Then a link $L \subset M$ is contractible if and only if $\pi_1(M \setminus L)$ contains a subgroup $G$ which is mapped isomorphically onto $\pi_1(M)$ by the inclusion homomorphism $\pi_1(M \setminus L) \to \pi_1(M)$.

**Organization of the paper**

In Section 2 we explain how to write down a presentation of the group $\pi_1(\mathbb{R}P^3 \setminus L)$ by generators and relations for a projective link $L$ given by its projective link diagram. Our presentation is similar to the presentation for the classical knot group introduced by Dehn in [8]. Moreover, our presentation can be considered a generalization of slightly enhanced version of the Dehn presentation. We found beneficial preliminary checkerboard coloring of a link diagram both in classical and projective environments.

In the classical knot theory Wirtinger presentations are used more often than Dehn presentations. It also generalizes to the projective environment, but becomes more cumbersome, and we do not consider it.

Section 3 is devoted to proof of Theorems 1 and 2. Section 4 contains a proof of Theorem 3.

## 2 Generators and relations for $\pi_1(\mathbb{R}P^3 \setminus L)$

The material of this section is not necessary for understanding of subsequent sections.

**Geometric origin of generators and relations**

The generators of our presentation correspond to the connected components of the complement of the link projection. Relations are of two types. Relations of the first type correspond to crossings. Relations of the second type correspond to pairs of antipodal arcs of the boundary circle.

For example, the presentation of group for the projective knot $2_1$ in Figure 3 has 5 generators, 2 relations of the first type and 2 relations of the second type.

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1As follows from the Papakyriacopoulos sphere theorem, this condition can be reformulated as non-existence of a 2-sphere $\Sigma$ embedded into the orientation covering space of $M$ in such a way that $\Sigma$ does not bound a 3-ball there.
Projective diagrams

In order to explain how our presentation appears, let us recall that a projective link diagram of $L \subset \mathbb{R}P^3$ comes from the ball model of $\mathbb{R}P^3$, that is a presentation of $\mathbb{R}P^3$ as a ball $D^3$ with each point on the boundary sphere identified with its antipode:

$$\mathbb{R}P^3 = D^3/_{x \sim -x} \text{ for any } x \in \partial D^3.$$ 

Choose the corresponding mapping $D^3 \to \mathbb{R}P^3$ so that the image of the northern and southern poles $N$ and $S$ of the ball does not belong to $L$. Denote by $L'$ the preimage of $L$ in $D^3$. Let $p : L' \to D^2$ be the projection onto the equatorial disk $D^2 \subset D^3$ given by the formula $p : x \mapsto c(x) \cap D^2$, where $c(x)$ is the (metric) circle in $\mathbb{R}^3 \supset D^3$ passing through $x \in L'$ and the poles $N$ and $S$ of $D^3$.

![Figure 4: Projection to $D^2$ in a ball model of $\mathbb{R}P^3$](image)

Then we convert $p(L')$ in a diagram by decorating it as usual near each double point.

Choice of a base point

In order to speak about the fundamental group of $\mathbb{R}P^3 \setminus L$ in detail, we have to choose a base point in $\mathbb{R}P^3 \setminus L$. Denote by $P$ the point in $\mathbb{R}P^3$ obtained
from the poles $N$ and $S$ of $D^3$. In other words, let $P = p(N) = p(S) \in \mathbb{R}P^3$.

The group $\pi_1(\mathbb{R}P^3 \setminus L, P)$ consists of homotopy classes of loops starting in the base point. Observe, that any path in $D^3$ connecting $N$ and $S$ gives rise to a loop like this and hence to an element of $\pi_1(\mathbb{R}P^3 \setminus L, P)$.

**Checkerboard coloring and generators**

By *domains* of the diagram, we mean connected components of $D^2 \setminus p(L')$. Choose a checkerboard coloring of domains.

In each domain choose a point and draw a circular arc connecting the poles $N$ and $S$ through this point. For dark domains orient the arc from $N$ to $S$, for light domains orient the arc from $S$ to $N$. This gives for each domain a path. For a dark domain the path in $D^3$ starts at $N$ and finishes at $S$, for a light domain the path starts at $S$ and finishes at $N$. In $\mathbb{R}P^3$ the paths gives rise to loops at the base point $P$ and, further, to elements of $\pi_1(\mathbb{R}P^3 \setminus L, P)$.

Assign to each domain on the diagram a symbol, and denote the corresponding element of $\pi_1(\mathbb{R}P^3 \setminus L, P)$ by this symbol.

**Lemma 1.** The elements of $\pi_1(\mathbb{R}P^3 \setminus L, P)$ corresponding to domains generate $\pi_1(\mathbb{R}P^3 \setminus L, P)$.

We leave the proof of Lemma 1 to the reader. It is a standard exercise, which can be done in many ways using your favorite technique for calculation of a fundamental group. For example, you can use Seifert - van Kampen Theorem, or transversality theorem.

**Relations corresponding to crossings**

A relation corresponding to a crossing starts with the symbol of an adjacent dark domain, then proceeds with the symbol of the domain adjacent to this first domain and separated from it by an overcrossing arc. Then proceeds with the symbols of the remaining dark domain followed by remaining light domain. All four domains are adjacent to the crossing and are going all the way around the crossing. See Figure 5.

On Figure 6 relations at crossings of a diagram for $2_1$ are shown. We assign relation $adbe = 1$ to the lower crossing if we start with the domain $a$, or $bead = 1$, if we start at the domain $b$. It does not matter where to start, as the relations are equivalent. Similarly, at the upper crossing we get relation $bdce = 1$. 
Relations corresponding to pairs of antipodal boundary arcs

If the domains adjacent to a pair of antipodal boundary arcs have the same color, then relation correspondent to the pair of arcs says that the generators of the adjacent domains are inverse to each other.

On Figure 5 all the pairs of antipodal boundary arcs are adjacent to domains of the same color. Thus, we have relations $c = a^{-1}$ and $e = d^{-1}$. The whole presentation of the group looks as follows:

$$\{a, b, c, d, e \mid adbe = 1, bdce = 1, c = a^{-1}, e = d^{-1}\} = \{a, b, d \mid adbd^{-1} = 1, bda^{-1}d^{-1} = 1\} = \{a, b, d \mid a = db^{-1}d^{-1} = d^{-1}bd\} = \{b, d \mid d^2 = bd^2b\}$$

If the domains adjacent to the pair of antipodal boundary arcs have different colors, then relation correspondent to the pair of arcs says that the generators of the adjacent domains are equal to each other. In this case, it’s convenient to take into account the relations of this kind on the preceding
step, i.e., while assigning symbols of generators to the domains. Of course, this can be done in the other case, too. Given a projective link diagram equipped with checkerboard coloring and symbols, one can write down the correspondent presentation of the group by listing the symbols and writing down the 4-term relations at each crossing.

![Figure 7: The graphic stage of writing down the group representation](image)

The simplest complete examples of writing down the group presentations are shown in Figure 7. The group for the projective link $1_1$, which is a pair of two skew lines, is $\{a, c \mid ac^{-1}a^{-1}c = 1\}$, that is a free abelian group of rank two.

The group for $0_1$, a projective line, has generator $a$ and no relations, so this is an infinite cyclic group.

The group of the last knot is $\{a, b \mid b = b^{-1}\} = \{a, b \mid b^2 = 1\}$, that is the free product $\mathbb{Z} \ast \mathbb{Z}/2$ of an infinite cyclic group by a group of order two.

### A version of Dehn presentation for classical link

If a projective link $L$ is presented by a diagram which does not meet the boundary circle of its disk, the $L$ does not intersect the projective plane which is the quotient space of the boundary sphere $\partial D^3$. Then removing the boundary circle turns the projective diagram into a diagram of the corresponding classical link.

A presentation of the group $\pi_1(\mathbb{R}^3 \setminus L)$ can be obtained from the presentation described above by removing the generator which corresponds to the external domain of the initial projective diagram. This generator had order two in the group of projective link. Cf. Theorem 3.

### 3 Projective unknots

In this section we prove Theorems 1 and 2. Recall their statements.

**Theorem 1.** A knot $K \subset \mathbb{R}P^3$ is isotopic to the affine unknot if and only if $\pi_1(\mathbb{R}P^3 \setminus K)$ is isomorphic to the free product of $\mathbb{Z}$ and $\mathbb{Z}/2$. 

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Theorem 2. A knot $K \subset \mathbb{RP}^3$ is isotopic to a projective line if and only if $\pi_1(\mathbb{RP}^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

Both theorems admit a proof similar to the classical proof of Dehn’s Theorem.

The ”only if” parts are proved on Figure 7, where the groups are found for the standard diagrams of the unknots. The rest of this section is devoted to proof of the ”if” part.

Plan of the proof

Let $N$ be a tubular neighborhood of $K$, and $T$ be its boundary $\partial N$. Since $\mathbb{RP}^3$ is orientable, $N$ is homeomorphic to $S^1 \times D^2$, and $T$ is torus, $T$ is homeomorphic to $S^1 \times S^1$.

First, we prove that the inclusion homomorphism $\pi_1(T) \to \pi_1(\mathbb{RP}^3 \setminus K)$ cannot be injective in either cases.

Second, by a version of Papakyriacopoulous’ Loop Theorem, we conclude that there exists an embedded circle in $T$ which does not bound in $T$, but bounds a smoothly embedded disk $\Delta$ in $\mathbb{RP}^3 \setminus \text{Int} \ N$.

Third, we find the homology class realized by the boundary of $\Delta$ in $H_1(T)$. The inclusion homomorphism $H_1(T) \to H_1(N) = H_1(K)$ maps this class to a generator in the case of Theorem 1 and to the double of a generator in the case of Theorem 2.

Fourth, the disk $\Delta$ is completed by an annulus in $N$ into a disk bounded by $K$, in Theorem 1 and into a projective plane in which $K$ is contained and does not bound, in Theorem 2.

Non-contractible loop on $T$ contractible in $\mathbb{RP}^3 \setminus K$

The assumptions of each of the theorems imply that the inclusion homomorphism $\pi_1(T) \to \pi_1(\mathbb{RP}^3 \setminus K)$ is not injective.

For Theorem 2 it is obvious: there is no monomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^* \mathbb{Z} / 2$.

For Theorem 1 it is also true, but requires more arguments.

Lemma 2. There is no monomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \ast \mathbb{Z} / 2$.

Proof. Due to Kurosh Subgroup Theorem 10, a subgroup of $\mathbb{Z} \ast \mathbb{Z} / 2$ should be a free product of cyclic groups. A group isomorphic to $\mathbb{Z} \times \mathbb{Z}$ cannot be a free product of cyclic groups by Baer-Levi Theorem 2. □
Disk provided by the Loop Theorem

Recall that according to the Loop Theorem (see, e.g., [9]), if \( M \) is a 3-manifold, and the inclusion homomorphism \( \pi_1(\partial M) \to \pi_1(M) \) is not injective, then there exists an embedding \( f : (D^2, \partial D^2) \to (M, \partial M) \) such that it gives a non-contractible loop \( \partial D^2 \to \partial M \).

Lemma 3. Under assumptions of Theorem 1 or of Theorem 2, let \( N \) be a tubular neighborhood of \( K \) and let \( E \) be the exterior \( \mathbb{R}P^3 \setminus \text{Int} \; N \) of \( K \). Then there exists a disk \( \Delta \) properly embedded in \( \mathbb{R}P^3 \setminus \text{Int} \; N \) such that its boundary \( \Sigma = \partial \Delta \) is not contractible in \( \partial N \).

Proof. Apply the Loop Theorem to \( M = E \). It gives an embedding \( f : (D^2, \partial D^2) \to (E, \partial N) \) such that its restriction \( \partial D^2 \to \partial N \) is a non-contractible loop. Let us denote the disk \( f(D^2) \) by \( \Delta \) and the simple closed curve \( f(\partial D^2) \) by \( \Sigma \).

A simple closed curve on a torus non-homological to zero realizes an indivisible homology class. Thus, \( \Sigma \) realizes an indivisible class in \( H_1(T) \).

Homological calculation

There are exactly two elements in \( H_1(\mathbb{R}P^3) = \mathbb{Z}/2 \): the zero and non-zero.

Lemma 4. Let \( K \) be a projective knot. Then

\[
H_1(\mathbb{R}P^3 \setminus K) = \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } K \text{ is zero-homologous in } \mathbb{R}P^3 \\
\mathbb{Z} & \text{if } K \text{ is homologous to a projective line}
\end{cases}
\]

\[
H_1(\mathbb{R}P^3, K) = \begin{cases} 
\mathbb{Z}/2 & \text{if } K \text{ is zero-homologous in } \mathbb{R}P^3 \\
0 & \text{if } K \text{ is homologous to a projective line}
\end{cases}
\]

The group \( H_2(\mathbb{R}P^3, K) = \mathbb{Z} \) independently on the homology class realized by \( K \), but the boundary homomorphism \( H_2(\mathbb{R}P^3, K) \to H_1(K) \) depends: if the homology class of \( K \) is zero, then this is an isomorphism, if not, then this is a monomorphism with the image of index two.

Proof. Consider the homology sequence of pair \((\mathbb{R}P^3, K)\):

\[
\begin{array}{cccccccccccccccc}
H_2(K) & \to & H_2(\mathbb{R}P^3, K) & \to & H_1(K) & \to & H_1(\mathbb{R}P^3) & \to & H_1(\mathbb{R}P^3, K) & \to & H_0(K) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & ? & \to & \mathbb{Z} & \to & \mathbb{Z}/2 & \to & ? & \to & 0 \\
\end{array}
\]
If the middle inclusion homomorphism $H_1(K) \to H_1(\mathbb{R}P^3)$ is zero, then by exactness of the sequence we get $H_1(\mathbb{R}P^3, K) = \mathbb{Z}/2$ and $H_2(\mathbb{R}P^3, K) = \mathbb{Z}$. If the homomorphism is not zero, then $H_1(\mathbb{R}P^3, K) = 0$ and still $H_2(\mathbb{R}P^3, K) = \mathbb{Z}$, but the boundary homomorphism $H_2(\mathbb{R}P^3, K) \to H_1(K)$ in the first case is an isomorphism and in the second case it is a monomorphism with the image $2\mathbb{Z} \subset \mathbb{Z}$.

Further, by duality and the universal coefficient formula

$$H_1(\mathbb{R}P^3 \setminus K) = H^2(\mathbb{R}P^3, K; \mathbb{Z}) = \text{Hom}(H_2(\mathbb{R}P^3, K), \mathbb{Z}) \oplus \text{Ext}(H_1(\mathbb{R}P^3, K), \mathbb{Z})$$

Hence,

$$H_1(\mathbb{R}P^3 \setminus K) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } K \text{ is zero-homologous in } \mathbb{R}P^3 \\ \mathbb{Z} & \text{if } K \text{ is homologous to a projective line} \end{cases}$$

Since $K$ is either zero-homologous or homologous to a projective line, we see that the group $H_1(\mathbb{R}P^3 \setminus K)$ determines which homology class is realized by $K$. Hence, under assumptions of Theorem 1, $K$ is zero homologous, while, under assumptions of Theorem 2, $K$ is homologous to a projective line.

**Mutual position of $K$ and $\Sigma$ in $N$**

**Lemma 5.** Under the assumptions of Theorem 1, the curves $K$ and $\Sigma$ bound an annulus embedded in $N$. Under assumptions of Theorem 2, the curve $\Sigma$ is a boundary of a Möbius band embedded in $N$.

**Proof.** Recall that by Lemma 3 under assumptions of Theorem 1 or Theorem 2 there is a simple closed curve $\Sigma$ on the boundary $T$ of a tubular neighborhood $N$ of $K$.

Denote by $E$ the exterior $\mathbb{R}P^3 \setminus \text{Int } N$ of the knot $K$. The curve $\Sigma$ bounds a disk $\Delta$ in $E$ and is not zero-homologous on $T$. Denote by $\sigma \in H_1(T)$ the homology class realized by $\Sigma$ and by $\delta \in H_2(E, T)$ the homology class realized by the disk $\Delta$.

The knot $K$ is a deformation retract of its tubular neighborhood $N$. Hence the pair $(\mathbb{R}P^3, K)$ is homotopy equivalent to $(\mathbb{R}P^3, N)$, the inclusion homomorphism $H_2(\mathbb{R}P^3, K) \to H_2(\mathbb{R}P^3, N)$ is an isomorphism and, by Lemma 4, $H_2(\mathbb{R}P^3, N) = H_2(\mathbb{R}P^3, K) = \mathbb{Z}$. By excision, $H_2(E, T) = \mathbb{Z}$.

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\[ H_2(\mathbb{R}P^3, N) = \mathbb{Z}. \] So, we have the following commutative diagram:

\[
\begin{array}{cccccc}
\mathbb{Z} & \longrightarrow & H_2(\mathbb{R}P^3, K) & \longrightarrow & H_1(K) & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \text{iso} & & \downarrow \text{in. iso} \\
\mathbb{Z} & \longrightarrow & H_2(\mathbb{R}P^3, N) & \longrightarrow & H_1(N) & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \text{iso} & & \uparrow \text{in.} \\
\mathbb{Z} & \longrightarrow & H_2(E, T) & \longrightarrow & H_1(T) & \longrightarrow & \mathbb{Z} \times \mathbb{Z}
\end{array}
\]

Since \( \sigma \neq 0 \) and \( GS \) is a simple closed curve, \( \sigma \) is indivisible in \( H_1(T) = \mathbb{Z} \times \mathbb{Z} \). Hence \( \delta \) is indivisible in \( H_2(E, T) = \mathbb{Z} \), so \( \delta \) generates \( H_2(E, T) \).

The vertical homomorphism on the left hand side of the diagram maps \( \sigma \) to a generator of \( H_2(\mathbb{R}P^3, K) \). By Lemma 4, this generator is mapped by the top homomorphism \( H_2(\mathbb{R}P^3, K) \to H_1(K) \) under assumptions of Theorem 1 to a generator of \( H_1(K) \), or, under assumptions of Theorem 2 to a doubled generator of \( H_1(K) \).

Completion of the proofs

By Lemma 5, under assumptions of Theorem 1, there exists an annulus \( A \subset N \) bounded by \( \Sigma \cup K \). The union \( A \cup \Delta \) is homeomorphic to disk. It is bounded by \( K \). By an isotopy fixed on \( K \) it can be made smoothly embedded in \( \mathbb{R}P^3 \). One can contract \( K \) along this disk to a small planar circle. This completes our proof of Theorem 1.

By Lemma 5, under assumptions of Theorem 2, \( \Sigma \) is a boundary of a Mobius band \( M \) embedded in \( N \) and \( K \) lies on \( M \) as the midline. The union \( \Delta \cup M \) is homeomorphic to \( \mathbb{R}P^2 \). The original knot \( K \) lies on it as a projective line. By an isotopy the union can be made smooth in \( \mathbb{R}P^2 \). Any smoothly embedded \( \mathbb{R}P^2 \) in \( \mathbb{R}P^3 \) is smoothly isotopic to a projective embedding. Then \( K \) can be deformed by a smooth isotopy to a projective line. This completes our proof of Theorem 2.

4 Contractibility

Proof of Theorem 3. Necessity.

Assume that \( L \) is an contractible link. Since the group \( \pi_1(\mathbb{R}P^3 \setminus L) \) is invariant under isotopy, it suffices to prove that \( \pi_1(\mathbb{R}P^3 \setminus L) \) contains an element of order 2 if \( L \) does not intersect a plane \( \mathbb{R}P^2 \subset \mathbb{R}P^3 \).
If \( L \cap \mathbb{R}P^2 = \emptyset \), then \( L \) is contained in the contractible part \( \mathbb{R}P^3 \setminus \mathbb{R}P^2 = \mathbb{R}^3 \) of \( \mathbb{R}P^3 \), and \( \mathbb{R}P^3 \setminus L \) is a union of \( \mathbb{R}P^3 \setminus (L \cup \mathbb{R}P^2) \) and a regular neighborhood of \( \mathbb{R}P^2 \). The van Kampen Theorem applied to this presentation of \( \mathbb{R}P^3 \setminus L \) implies that \( \pi_1(\mathbb{R}P^3 \setminus L) \) is the free product \( \mathbb{Z}/2 \ast \pi_1(\mathbb{R}^3 \setminus L) \). Hence it contains a non-trivial element of order 2.

A lemma about a map of the projective plane

The proof of sufficiency is prefaced with the following simple homotopy-theoretic lemma:

**Lemma 6.** Let \( f : \mathbb{R}P^2 \to \mathbb{R}P^2 \) be a map inducing isomorphism on fundamental group. Then the covering map \( \tilde{f} : S^2 \to S^2 \) is not null homotopic.

**Proof.** Consider the diagram

\[
\begin{array}{cccccc}
0 & \to & H_2(\mathbb{R}P^2; \mathbb{Z}/2) & \to & H_2(S^2; \mathbb{Z}/2) & \to & H_2(\mathbb{R}P^2; \mathbb{Z}/2) & \to & 0 \\
\| & f_* & \| & f_* & \| & f_* & \|
\end{array}
\]

in which the rows are segments of the Smith sequence (see, e.g., [4]) for the antipodal involution on \( s : S^2 \to S^2 : x \mapsto -x \). The diagram is commutative, because \( \tilde{f} \) commutes with \( s \). Notice that all the groups in this diagram are isomorphic to \( \mathbb{Z}/2 \). Exactness of the Smith sequences implies that the middle horizontal arrows in both rows are trivial, and the horizontal arrows next to them are isomorphism. By assumption, the rightmost vertical arrow is an isomorphism. Therefore the next vertical arrow is an isomorphism. This isomorphism coincides with the homomorphism represented by leftmost vertical arrow. Hence, the next arrow \( \tilde{f}_* : H_2(S^2; \mathbb{Z}/2) \to H_2(S^2; \mathbb{Z}/2) \) is an isomorphism. Thus \( \tilde{f} \) is not null homotopic.

**Proof of Theorem 3. Sufficiency**

Assume that \( \pi_1(\mathbb{R}P^3 \setminus L) \) contains a non-trivial element \( \lambda \) of order two. Realize \( \lambda \) by a smoothly embedded loop \( l : S^1 \to \mathbb{R}P^3 \setminus L \).

**From a loop of order 2 to a singular projective plane in \( \mathbb{R}P^3 \setminus L \)**

Since \( \lambda^2 = 1 \), there exists a continuous map \( D^2 \to \mathbb{R}P^3 \setminus L \) such that its restriction to the boundary circle \( \partial D^2 \) is the square of \( l \). Together with \( l \), this
map gives a continuous map of the projective plane \( P = D^2 / x \sim -x \), for \( x \in \partial D^2 \) to \( \mathbb{R}P^3 \setminus L \). Denote this map by \( g \). So, \( g : P \to \mathbb{R}P^3 \setminus L \) is a generic differentiable map which induces a monomorphism \( g_* \) of \( \pi_1(P) = \mathbb{Z}/2 \) to \( \pi_1(\mathbb{R}P^3 \setminus L) \).

**Singular sphere over the singular projective plane**

Let \( p : S^3 \to \mathbb{R}P^3 \) be the canonical two-fold covering. Consider its restriction \( S^3 \setminus p^{-1}(L) \to \mathbb{R}P^3 \setminus L \). Observe that \( \lambda \) does not belong to the group of this covering, because otherwise \( \pi_1(S^3 \setminus p^{-1}(L)) \) would contain a non-trivial element of order two, which is impossible - a link group does not have any non-trivial element of finite order.

Therefore the covering of the projective plane induced from \( p \) via \( g \) is a non-trivial two-fold covering. Its total space is a 2-sphere. Denote it by \( S \).

The map \( \tilde{g} \) which covers \( g \) maps \( S \) to \( S^3 \setminus p^{-1}(L) \).

**Non-contractibility of the singular sphere in \( S^3 \setminus p^{-1}(L) \)**

Let us choose a point \( x \in L \). Its complement \( \mathbb{R}P^3 \setminus \{x\} \) is homotopy equivalent to \( \mathbb{R}P^2 \). Indeed, the projection from \( x \) to any projective plane, which does not contain \( x \), is a deformation retraction \( \mathbb{R}P^3 \setminus \{x\} \to \mathbb{R}P^2 \).

The composition of \( g : P \to \mathbb{R}P^3 \setminus L \) with the inclusion \( \mathbb{R}P^3 \setminus L \to \mathbb{R}P^3 \setminus \{x\} \) induces an isomorphism \( \pi_1(P) \to \pi_1(\mathbb{R}P^3 \setminus \{x\}) \). Both spaces, \( P \) and \( \mathbb{R}P^3 \setminus \{x\} \), have homotopy type of \( \mathbb{R}P^2 \). Lemma 6 implies that \( \tilde{g} : S \to S^3 \setminus p^{-1}(x) \) is not null-homotopic.

**Existence of a non-singular sphere \( \Sigma_0 \subset S^3 \) that splits \( p^{-1}(L) \)**

Denote by \( \sigma \) the antipodal involution \( S^3 \to S^3 : x \mapsto -x \).

**Lemma 7.** There exists a non-singular polyhedral submanifold \( \Sigma_0 \) of \( S^3 \) homeomorphic to \( S^2 \) such that \( \Sigma_0 \cap p^{-1}(L) = \emptyset \) and the two points of \( p^{-1}(x) \) belong to different connected components of \( S^3 \setminus \Sigma_0 \).

**Proof.** First, let us apply Whitehead’s modification [16] of the Papakyriakopoulos Sphere Theorem [13].

Recall the statement of this theorem (Theorem (1.1) of [16]): For any connected, orientable triangulated 3-manifold \( M \) and subgroup \( \Lambda \subset \pi_2(M) \) which is invariant under the action of \( \pi_1(M) \), if \( \Lambda \neq \pi_2(M) \), then \( M \) contains a non-singular polyhedral 2-sphere which is essential \( \mod \Lambda \).
We will apply this theorem to
\[ M = S^3 \setminus p^{-1}(L), \quad \Lambda = \text{Ker}\left( \text{in}_*: \pi_2(S^3 \setminus p^{-1}(L)) \to \pi_2(S^3 \setminus p^{-1}(x)) \right). \]

We know that \( \pi_2(S^3 \setminus \{x\}) = \pi_2(S^2) = \mathbb{Z} \) and that the homotopy class of \( \tilde{g} \) is non-trivial in \( \pi_2(S^3 \setminus p^{-1}(x)) \). Therefore the homotopy class of \( \tilde{g} \) does not belong to \( \Lambda \), and hence \( \Lambda \neq \pi_2(S^3 \setminus p^{-1}(L)) \). Thus, all the assumptions of the Whitehead theorem are fulfilled.

Let us denote by \( \Sigma_0 \) a non-singular polyhedral 2-sphere whose existence is stated by the Whitehead theorem. By the Alexander Theorem \([1]\), \( \Sigma_0 \) bounds in \( S^3 \) two domains homeomorphic to ball. Since \( \Sigma_0 \) is not null homotopic in \( S^3 \setminus \{x\} \), each of these domains contains a point of \( p^{-1}(x) \).

Improving the splitting sphere

**Lemma 8.** There exists a smooth submanifold \( \Sigma \) of \( S^3 \) homeomorphic to \( S^2 \) such that \( \Sigma \cap p^{-1}(L) = \emptyset \), the two points of \( p^{-1}(x) \) belong to different connected components of \( S^3 \setminus \Sigma \), and either \( \Sigma = \sigma(\Sigma) \) or \( \Sigma \cap \sigma(\Sigma) = \emptyset \).

**Proof.** Any polyhedral compact surface can be approximated by a smooth 2-submanifold. Let \( \Sigma_1 \) be a smooth submanifold of \( S^3 \) approximating \( \Sigma_0 \) in \( S^3 \setminus p^{-1}(L) \).

The antipodal involution \( \sigma: S^3 \to S^3 \) is an automorphism of the covering \( p: S^3 \to \mathbb{RP}^3 \), therefore \( p^{-1}(L) \) is invariant under \( \sigma \) and \( \sigma(\Sigma_1) \subset S^3 \setminus p^{-1}(L) \).

Let us assume that \( \Sigma_1 \) and \( \sigma(\Sigma_1) \) are transversal – this can be achieved by an arbitrarily small isotopy of \( \Sigma_1 \). Then the intersection \( \Sigma_1 \cap \sigma(\Sigma_1) \) consists of disjoint circles.

Take a connected component \( C \) of \( \Sigma_1 \cap \sigma(\Sigma_1) \) which is innermost in \( \sigma(\Sigma_1) \) (i.e., which bounds in \( \sigma(\Sigma_1) \) a disc \( D \) containing no other components of \( \Sigma_1 \cap \sigma(\Sigma_1) \)).

First, assume that \( C \neq \sigma(C) \). In this case, make surgery on \( \Sigma_1 \) along \( D \): remove a regular neighborhood \( N \) of \( C \) from \( \Sigma_1 \) and attach to \( \partial N \) two discs parallel to \( D \). This surgery does not change the homology class with coefficients in \( \mathbb{Z}/2 \) realized by \( \Sigma_1 \) in \( S^3 \setminus p^{-1}(x) \).

Denote by \( \Sigma_2 \) the result of this surgery on \( \Sigma_1 \). This is a disjoint union of two spheres. The sum of the homology classes realized by them is the same non-trivial element of \( H_2(S^2 \setminus p^{-1}(x); \mathbb{Z}/2) = \mathbb{Z}/2 \) which was realized by \( \Sigma_1 \). Therefore, one of the summands is non-trivial. The corresponding component of \( \Sigma_2 \) separates the two points of \( p^{-1}(x) \). Denote this component
by $\Sigma_3$. Since $C \neq \sigma(C)$, the number of connected components of $\Sigma_3 \cap \sigma(\Sigma_3)$ is less than the number of connected components of $\Sigma_1 \cap \sigma(\Sigma_1)$, all other properties of $\Sigma_1$ are inherited by $\Sigma_3$, and we are ready to continue with the next connected component of $\Sigma_3 \cap \sigma(\Sigma_3)$ which bounds in $\sigma(\Sigma_3)$ a disc containing no other components of $\Sigma_3 \cap \sigma(\Sigma_3)$.

Second, consider the case $C = \sigma(C)$. Then the disc $D \subset \sigma(\Sigma_1)$ together with its image $\sigma(D) \subset \sigma(\Sigma_1)$ form an embedded sphere, which is invariant under $\sigma$ and does not meet the rest of $\Sigma \cup \sigma(\Sigma_1)$ besides along $C$, that is $(D \cup \sigma(D)) \cap ((\Sigma_1 \setminus \sigma(D) \cup (\sigma(\Sigma_1) \setminus D)) = \emptyset$.

If $D \cup \sigma(D)$ separates points of $p^{-1}(x)$, we are done: we can smoothen the corner of $D \cup \sigma(D)$ along $C$ keeping it invariant under $\sigma$ and take the result for $\Sigma$.

If $D \cup \sigma(D)$ does not separate points of $p^{-1}(x)$, then $D \cup (\Sigma_1 \setminus \sigma(D))$ separates points of $p^{-1}(x)$ (as well as its image under $\sigma$, that is $\sigma(D) \cup (\sigma(\Sigma_1) \setminus D)$). Indeed, the homology classes realized by $D \cup \sigma(D)$ and $D \cup (\Sigma_1 \setminus \sigma(D))$ in $S^3 \setminus p^{-1}(x)$ differ from each other by the homology class of $\sigma(D) \cup (\Sigma_1 \setminus \sigma(D)) = \Sigma_1$ which is known to be nontrivial. So, if the class of $D \cup \sigma(D)$ is trivial, then the class of $\sigma(D) \cup (\sigma(\Sigma_1) \setminus D)$ is not. Then smoothing of a corner along $C$ turns $D \cup (\Sigma_1 \setminus \sigma(D))$ into a new sphere $\Sigma_2$ such that $\Sigma_2 \cap \sigma(\Sigma_2)$ has less connected components than $\Sigma_1 \cap \sigma(\Sigma_1)$.

By repeating this construction, we will eventually build up a sphere $\Sigma \subset S^3 \setminus p^{-1}(L)$ with the required properties.

Completion of the proof

Let us return to the proof of Main Theorem. If the sphere $\Sigma$ provided by Lemma 8 is invariant under $\sigma$, then $\Sigma$ divides $S^3$ into two balls which are mapped by $\sigma$ homeomorphically to each other. Let $B$ be one of them. The part of $p^{-1}(L)$ contained in $B$ can be moved by an isotopy fixed on a neighborhood of the boundary of $B$ inside an arbitrarily small metric ball in $S^3$. Using $\sigma$, extend this isotopy to a $\sigma$-equivariant isotopy of the whole $S^3$. The equivariant isotopy defines an isotopy of $\mathbb{R}P^3$ which moves $L$ to a link contained in a small metric ball. This proves that $L$ is an contractible link.

Consider now the case in which the sphere $\Sigma$ provided by Lemma 8 is not $\sigma$-invariant, but rather is disjoint from its image $\sigma(\Sigma)$. Then spheres $\Sigma$ and $\sigma(\Sigma)$ divide $S^3$ into three domains: two of them are balls bounded by $\Sigma$ and $\sigma(\Sigma)$, respectively. Let us denote by $B$ the ball bounded by $\Sigma$, then its image $\sigma(B)$ is bounded by $\sigma(\Sigma)$. Denote the third domain by $E$. It is invariant under $\sigma$. 

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If one of the points from $p^{-1}(x)$ belonged to $E$, then the other one also would belong to $E$, and then the sphere $\Sigma$ would be contractible in $S^3 \setminus p^{-1}(x)$. Therefore $B \cap p^{-1}(L) \neq \emptyset$. Denote $B \cap p^{-1}(L)$ by $K$. This is a sublink of $p^{-1}(L)$. It can be moved by an isotopy fixed on a neighborhood of the boundary of $B$ inside an arbitrarily small metric ball in $S^3$. Then this isotopy can be extended to $\sigma$-equivariant isotopy of $S^3$ fixed on $E$. This equivariant isotopy defines an isotopy of $\mathbb{RP}^3$ which moves $p(K)$ to a link contained in a small metric ball.

Thus our link $L$ is presented as a disjoint sum of an contractible link $p(K)$ and the rest $L \setminus p(K)$ of $L$. If $L \setminus p(K) = \emptyset$, then we are done. If not, then $\pi_1(S^3 \setminus L)$ is presented as a free product of $\pi_1(B \setminus K)$ and $\pi_1(\mathbb{RP}^3 \setminus (L \setminus p(K)))$. By the assumption, the group $\pi_1(S^3 \setminus L)$ has a non-trivial element of order 2. The first factor, $\pi_1(B \setminus K)$ cannot contain such an element, because this is a group of a classical link. Hence, the second factor, $\pi_1(\mathbb{RP}^3 \setminus (L \setminus p(K)))$, contains it, and we can apply the constructions and arguments above to the link $L \setminus p(K)$. This link contains less components than the original one, therefore, after several iterations, we will come to the situation in which $p^{-1}(L) \cap B = \emptyset$. \hfill \qed

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