Asphericity of groups defined by graphs

VADIM BEREZNYUK

A graph $\Gamma$ labelled by a set $S$ defines a group $G(\Gamma)$ whose generators are the set of labels $S$ and whose relations are all words which can be read on closed paths of this graph. We introduce the notion of aspherical graph and prove that such a graph defines an aspherical group presentation. This result generalizes a theorem of Dominik Gruber on graphs satisfying graphical $C(6)$-condition and also allows to get new graphical conditions of asphericity analogous to some classical conditions.

1 Introduction

Every group can be defined by a set of generators and a set of relations among these generators. The latter set can be excessive: there can be some non-trivial identities among its elements. For example, in a group $\langle a, b, c \mid ab^{-1}, bc^{-1}, ac^{-1} \rangle$ a relation $ac^{-1}$ can be written as $ab^{-1}bc^{-1}$. Roughly speaking, a presentation is called aspherical if all identities among its relations are trivial. It can be formalized in various ways, so there are quite a few different definitions of asphericity (see, for example, [1]).

It is well known that asphericity follows from the classical small cancellation conditions. In 2003 Mikhail Gromov briefly introduced a graphical analogue of small cancellation theory in his paper [2]. After that Yann Ollivier gave a combinatorial proof of a theorem of Gromov, which in particular states asphericity of groups defined by graphs satisfying graphical $C'(1/6)$-condition [3]. In 2015 Dominik Gruber proved asphericity of groups defined by graphs satisfying graphical $C(6)$-condition [4].

We introduce a notion of aspherical graph and suggest to consider it as graphical analogue of diagrammatic asphericity. That notion allows to transfer known classical conditions which imply diagrammatic asphericity to graphical case. We show that not only graphical analogue of condition $C(6)$ implies asphericity of a group but also graphical analogues of conditions $C(4)\&T(4)$ and $C(3)\&T(6)$. Moreover we show how a car-crash lemma from [5] can be applied to prove asphericity in graphical case.

Classical small cancellation theory operates with presentations where every two distinct relations have quite short common parts. In graphical small cancellation theory, a group is defined by a labelled graph. The set of generators is the set of labels and the set of relations is the set of all words which can be read on closed paths of the graph. Thus every relation corresponds to a closed path where this relation can be read. Unlike classical case, two distinct relations can have a long common part, but only if this common part originates from the graph. It means that paths of the graph corresponding to these relations have the same common part as relations themselves.

Recall that a reduction pair in a diagram is a pair of distinct faces of the diagram such that their boundary cycles share a common edge and such that their boundary cycles, read starting from that edge, clockwise for one of the faces and counter-clockwise for the other, are equal as words. A spherical diagram is reduced if there are no reduction pairs. If there exists no reduced spherical diagram over a presentation, then the presentation is called diagrammatically aspherical. It is well known that presentations satisfying classical small cancellation conditions are diagrammatically aspherical.

In graphical case, we call a pair of faces a graphical reduction pair if these faces share an edge originating from the graph. A spherical diagram is graphically reduced if there exists no graphical reduction pair in this diagram. If there exists no graphically reduced diagram over a presentation whose set of relations is the set of labels of all simple closed paths
of the graph, then we call this graph aspherical. It is easy to show that asphericity of a graph follows from the graphical small cancellation $C(6)$-condition.

It turns out that asphericity of a graph implies topological asphericity of the corresponding group. Thus asphericity of a graph can be considered as a graphical analogue of diagrammatic asphericity that allows to transfer different conditions which imply diagrammatic asphericity to graphical case.

Note that, when we define a group by a graph, we can restrict the set of relations to the set of labels of all simple closed paths. It does not change the group. This set of relations can be reduced further. We can choose an arbitrary basis of the fundamental group of the graph. Then the set of relations will be the set of cyclically reduced paths of that basis. Again it does not change the group. The obtained presentation is topologically aspherical if the graph is aspherical.

The paper begins with a brief introduction to theory of groups defined by graphs. In Section 1.2 the main result is formulated and in Section 1.3 the proof is outlined. In Section 2 we give exact definitions of main notions. Section 3 is devoted to the link between identities among relations of a presentation and spherical diagrams over this presentation. The full proof of the main theorem can be found in Section 4. At the end we show how to transfer classical conditions of asphericity to graphical case.

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1.1 Groups defined by graphs and graphical small cancellation conditions

Let $\Gamma$ be an oriented graph every edge of which is labelled by an element of a finite set $S$. Then each path $p$ in that graph can be mapped to a word $\ell(p)$ in the alphabet $S \sqcup S^{-1}$, which is called the label of the path $p$. This word is equal to a product (without reductions) of the labels of edges of this path, considering that if orientation of the edge in the path doesn’t match orientation of the edge in the graph then the label belongs to the product with exponent $-1$.

Let $R_c$ be a set of labels of all closed paths in $\Gamma$, $R_s$ be a set of labels of all simple closed paths in $\Gamma$ and $R_f$ be a set of cyclically reduced labels of paths which generate a basis of a fundamental group of each connected component of the graph $\Gamma$ (note that $R_c$ and $R_s$ are determined by the graph $\Gamma$ itself while $R_f$ depends on the chosen basis of the fundamental group of $\Gamma$). Then a group $G(\Gamma)$ are defined by one of the three following presentations: $\langle S \mid R_c \rangle$, $\langle S \mid R_s \rangle$ or $\langle S \mid R_f \rangle$. Clearly, all these presentations define the same group.

![Fig. 1: $G(\Gamma) \cong \langle a, b, c \mid bbc, c^{-1}bc^{-1}, b^{-1}a^{-1}b^{-1} \rangle$](image)

Definition 1. A lift of a word $w$ in the graph $\Gamma$ is such a path $\bar{p}$ in the graph that $\ell(\bar{p}) \equiv w$ (i.e., the label of the path $\bar{p}$ coincides with the word $w$ character by character).

Definition 2. A word $w$ is a piece (with respect to $\Gamma$) if it has two (or more) distinct lifts in the graph $\Gamma$. 

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Definition 3. Let p be a path in a graph labelled by a set S. A lift of the path p in the graph Γ is such a path ¯p in the graph that ℓ(¯p) ≡ ℓ(p) (i.e., the label of the path p coincides with the label of the path ¯p character by character).

Definition 4. Let p be a path in a graph labelled by a set S. The path p is a piece (with respect to Γ) if it has two (or more) distinct lifts in the graph Γ.

Recall that a cycle in a graph is a set of all cyclic shifts of some closed path.

Definition 5. Let γ be a cycle in a graph labeled by a set S. A lift of the cycle γ in the graph Γ is such a cycle ¯γ in the graph together with a map f : γ → ¯γ, that f commutes with cyclic shifts and f(¯p) is a lift of p for all p ∈ γ.

Everywhere further it will be clear about which graph Γ we talk, so we will call words and paths just “pieces”, not “pieces with respect to Γ”.

Consider an example. Let Γ be a graph as in Figure 1. Then the words b, b⁻¹, c and e⁻¹ are all pieces of lengths 1. The words a and a⁻¹ are not pieces. The words bb and (bb)⁻¹ are all pieces among reduced words of length 2.

A labelling of a graph Γ is reduced if any two distinct edges starting at the same vertex have distinct labels and any two distinct edges ending at the same vertex have distinct labels.

Definition 6. Let Γ be a labelled graph and let k ∈ N. We say Γ satisfies graphical C(k)-condition (or Γ is a C(k)-graph) if:

- the labelling of Γ is reduced and
- no simple closed path is a concatenation of strictly fewer than k pieces.

Note that if a graph Γ satisfies graphical condition C(2) then that graph has a reduced labelling and any word from R_2 has a unique lift in the graph.

A graph as in Figure 1 satisfies graphical C(2)-condition, but does not satisfy graphical C(3)-condition because the simple closed path with the label bbc is a concatenation of the pieces bb and c.

Let Γ be a C(2)-graph and let D be a diagram over the presentation ⟨S | R_s⟩ (see the next section for definitions). Let p be a path lying in intersection of some positively oriented boundary path of a face Π_1 and some negatively oriented boundary path of a face Π_2. A word from R_s are written on the boundary of any face of D. Thus the boundary of any face has a lift in the graph Γ. That lift is unique since Γ is a C(2)-graph.

Lift the boundary of the face Π_1 in the graph. After that the path p, as subpath of the boundary, maps to some path p_1 in the graph. Similarly lift the boundary of the face Π_2 and determine a path p_2. We say that the path p originates from the graph Γ if p_1 = p_2. Roughly speaking, a path p originates from the graph Γ if faces Π_1 and Π_2 share the same path in the diagram and in the graph Γ itself.

Note that if a path p, lying between faces Π_1 and Π_2, does not originate from the graph Γ then it is a piece. Indeed, if p does not originate from the graph then its lifts via Π_1 and via Π_2 are distinct. Therefore this path have two distinct lifts in the graph Γ, i.e., this path is a piece.

1.2 Main result

Recall that a presentation complex K(S; R) of a presentation ⟨S | R⟩ is a 2-complex which has a 1-skeleton which consists of a single vertex and a loop labelled by s for every element s from S, and which have a face with the boundary label r attached to the 1-skeleton for every element r from R.

Definition 7. A presentation ⟨S | R⟩ is called aspherical if its presentation complex K(S; R) is aspherical, i.e., π_q(K(S; R)) = 0 for all q ≥ 2.
Definition 8. We say that a diagram $D$ over the presentation $\langle S \mid R_s \rangle$ is graphically reduced if it does not have edges originating from the graph $\Gamma$.

Definition 9. We say that a graph $\Gamma$ is aspherical if it satisfies graphical $C(2)$-condition and there exists no graphically reduced spherical diagram over the presentation $\langle S \mid R_s \rangle$.

The following theorem is the main result of this paper.

Theorem. If a graph $\Gamma$ is aspherical, then the presentation $\langle S \mid R_f \rangle$ is aspherical.

1.3 Idea of proof

Propositions 1.3 and 1.5 of the paper [1] implies that a presentation $\langle S \mid R \rangle$, where all relations from $R$ are not empty and freely reduced, is aspherical if and only if the presentation is concise, no relation is a proper power and any identity among relations of this presentation is trivial.

Recall that a presentation $\langle S \mid R \rangle$ is concise if for any two distinct relations $r$ and $r'$ from $R$ nor $r$, neither $r^{-1}$ is conjugate to $r'$.

Also recall a notion of identity among relations of a presentation $\langle S \mid R \rangle$. Let $\pi = (p_1, \ldots, p_n)$ be a sequence such that $p_i = u_i r_i^{e_i} u_i^{-1}$, where $r_i \in R$, $u_i \in F(S)$ and $e_i \in \{+1, -1\}$. It is called an identity if a product of its elements is equal to the identity element of the free group, i.e., $p_1 \cdots p_n = 1$ in $F(s)$. There are identities which we should consider as trivial. For this reason Peiffer transformations are introduced:

1. Replace any pair of consecutive elements $(p_i, p_{i+1})$, either by the pair $(p_i p_{i+1} p_i^{-1}, p_i)$ or by the pair $(p_{i+1} p_{i+1}^{-1} p_i, p_{i+1})$.

2. Delete the pair of consecutive elements $(p_i, p_{i+1})$ if $p_ip_{i+1} = 1$ in $F(S)$.

3. Insert at any place a pair of inverse elements $(p, p^{-1})$.

An identity is called trivial if it can be transformed to an empty identity by the finite number of Peiffer transformation.

Lemma 2.22 of the paper [4] guarantees the first two conditions: conciseness of a presentation and absence of proper powers. Thus we only should show that any identity among relations is trivial.

To prove this fact we use a link between identities among relations of a presentation and spherical diagrams over this presentation which was obtained in [6]. A plan of the proof is the following. Assume the contrary, that there exists non-trivial identities over the presentation $\langle S \mid R_f \rangle$ or, equivalently, that there exists non-trivial spherical diagrams over the presentation $\langle S \mid R_f \rangle$. Consider a part of an 1-skeleton of a spherical diagram which consists of all edges not originating from a graph $\Gamma$. We call such a part a not originating skeleton. Consider a non-trivial spherical diagram with the smallest not originating skeleton. Delete all edges originating from the graph from this diagram. It turns out that the obtained diagram is
a diagram over the presentation \( \langle S | R \rangle \). Moreover, it is graphically reduced because we deleted all originating edges. A contradiction with asphericity of the graph \( \Gamma \).

## 2 Main notions

Definitions of graphs and diagrams are given in this section.

### 2.1 Graphs

We use a definition of graph according to [7], i.e., a graph is a union of two sets \( V \) and \( E \) with three maps \( \alpha : E \to V \), \( \omega : E \to V \), \( \cdot^{-1} : E \to E \). The elements of \( V \) are called vertices and the elements of \( E \) are called edges. If \( e \in E \) then \( \alpha(e) \) is called the initial point and \( \omega(e) \) the terminal point. The map \( \cdot^{-1} \) assigns to every edge its inverse. For convenience we write \( \cdot^{-1}(e) \) as \( e^{-1} \). The map \( \cdot^{-1} \) should be an involution without fixed elements such as \( \alpha(e^{-1}) = \omega(e) \) and \( \omega(e^{-1}) = \alpha(e) \). In fact, it is a definition of undirected graph, because for every edge \( e \) graph also contains \( e^{-1} \).

A path in a graph is a finite sequence of edges \( p = (e_1, \ldots, e_n) \), such that \( \alpha(e_{i+1}) = \omega(e_i) \) for \( 1 \leq i < n \). A path begins at a point \( \alpha(p) = \alpha(e_1) \) and ends at a point \( \omega(p) = \omega(e_n) \).

A labelling of a graph \( \Gamma \) by a set \( S \) is such a map \( \ell : E \to S^{-1} \) that \( \ell(e^{-1}) = \ell(e)^{-1} \). A labelled graph is a graph with its labelling. One can think of a labelled graph as an oriented graph every edge of which labelled by an element of \( S \). A labelling of a graph \( \Gamma \) is reduced if any two distinct edges starting at the same vertex have distinct labels and any two distinct edges ending at the same vertex have distinct labels.

Continue the map \( \ell \) to a set of all paths in the graph \( \Gamma \). Let \( p = (e_1, \ldots, e_n) \) be a path in the graph, then put \( \ell(p) = \ell(e_1) \cdots \ell(e_n) \), where \( w_1 w_2 \) is concatenation. Thus \( \ell(p) \) is a word over an alphabet \( S \sqcup S^{-1} \) (not necessarily reduced). We call \( \ell(p) \) as the label of the path \( p \).

Let \( p = (e_1, \ldots, e_n) \) be a path in a graph. The path \( p \) is reduced if it contains no subpaths \( (e, e^{-1}) \). The path is trivial if it becomes empty after consecutive deletion of all subpaths \( (e, e^{-1}) \). The path is closed if its initial point coincides with its terminal point or if it is empty. The path is simple if it is not empty and it does not contain non-empty closed subpaths. The path \( p \) is simple closed if it is not trivial, is closed and no proper subpaths of \( p \) is closed. A set of cyclic shifts of a closed path is called a cycle. A path \( p \) is called an arc if all its vertices besides endpoints have degree 2. An arc is called a spur if at least one its endpoint has degree 1.

### 2.2 Diagrams over graphs

Let \( R \) be a set of words, then we define \( R_{sym} \) as the set obtained from the sets \( R \) and \( R^{-1} \) by considering all its elements up to cyclic shifts.

A singular disk diagram in the alphabet \( S \) is a finite and simply connected 2-dimensional CW-complex embedded into \( \mathbb{R}^2 \) such that its 1-skeleton is a graph labelled by the set \( S \). The closures of its 1-cells and 2-cells are called edges and faces, respectively. The label of a face is a cyclic word obtained by reading its boundary path in a counterclockwise direction.

A singular disk diagram over a presentation \( \langle S | R \rangle \) is a singular disk diagram in the alphabet \( S \) every face of which has a label from \( R_{sym} \). A simple disk diagram is a singular disk diagram homeomorphic to a disk.

We introduce spherical diagrams following [9]. First we inductively define a spherical complex. A 2-sphere is a spherical complex. If \( D \) is a spherical complex, then \( D' \) obtained by attaching a simple curve or a 2-sphere to a point of \( D \) is also a spherical complex.

A spherical complex is a complex which can be obtained from a 2-sphere by the finite number of such attachments. In other words, a spherical complex is a tree embedded into \( \mathbb{R}^3 \) some vertices of which (at least one) are replaced by spheres or by some number of spheres attached to each other. A spherical diagram over a presentation \( \langle S | R \rangle \) is a 2-dimensional
CW-complex homeomorphic to some spherical complex, such that its 1-skeleton is a graph labelled by the set $S$ and such that the labels of all its faces lie in $R_{sym}$. A simple spherical diagram is a spherical diagram homeomorphic to a sphere.

Let denote a cycle consisting of positively oriented (i.e., obtained by reading the boundary in a counterclockwise direction) boundary paths of a face $\Pi$ as $\partial\Pi^+$, and a cycle consisting of negatively oriented as $\partial\Pi^-$. It is clear that $\partial\Pi^- = \{\gamma^{-1}: \gamma \in \partial\Pi^+\}$. If $P$ and $Q$ are two sets of paths then $P \sqcup Q = \{r: \exists p \in P, \exists q \in Q \text{ such as } p = rp', q = rq'\}$.

**Definition 10.** Let $D$ be a diagram over the presentation $\langle S \mid R_\gamma \rangle$. Let $\Pi_1$ and $\Pi_2$ be two faces of $D$ (not necessary distinct) and let $p \in \partial\Pi_1^+ \sqcup \partial\Pi_2^-$. We say that the path $p$ originates from the graph $\Gamma$ if there exists such lifts of $\partial\Pi_1^+$ and $\partial\Pi_2^-$ in $\Gamma$, that a lift of $p$ in $\Gamma$ via $\partial\Pi_1^+$ and via $\partial\Pi_2^-$ are equal.

Note that according to this definition edges on the boundary of a diagram are not originating. And note again that if the graph $\Gamma$ satisfies graphical $C(2)$-condition then lifts of $\partial\Pi_1^+$ and $\partial\Pi_2^-$ are unique.

### 3 Identities and spherical diagrams

For any identity $\pi = (p_1, \ldots, p_n)$ over a presentation $\langle S \mid R \rangle$ we can construct a spherical diagram over the same presentation by the so-called van Kampen construction ([7], [6]). Recall it. First, note that for any word $w$ in the alphabet $S \sqcup S^{-1}$ a linear graph $p$ labelled by the set $S$ such that $\ell(p) \equiv w$ can be constructed.

Now, fix a point $v$ on the plane. After that for each $p_i = u_i r_i^+ u_i^{-1}$ draw a face with a spur on the plane such that boundary of the face is a closed arc with a label $r_i^+$ and the spur has a label $u_i$. Then the boundary label of the obtained diagram is equal to the word $p_1 \cdots p_n$. So this boundary label is trivial because $\pi$ is an identity. It means that there exists 2 consecutive edges with opposite labels on the boundary. Glue these edges together. We obtain again a diagram with a trivial boundary label but with the smaller boundary size. Consecutively gluing pairs of edges with opposite labels we finally obtain a spherical diagram $D$ which is called a diagram for $\pi$ (see [6] Section 1.5) for details.

Note that obtained diagram is spherical by our definition because if a sphere arises after gluing then this sphere touches the boundary of a diagram only by one vertex.

This procedure can be reversed. By “ungluing” faces of a spherical diagram $D$ over a presentation $\langle S \mid R \rangle$ along edges we can obtain a bouquet of faces with spurs that gives us a sequence of elements $\pi$ over the same presentation. Moreover, $\pi$ is an identity because the diagram was spherical. Note that $D$ is a diagram for $\pi$ because we can reverse the described procedure.

These procedures are not unique. We may obtain different diagrams and identities changing the order of gluings and un gluings. But at the same time Proposition 8 from [8] implies that if $D$ is a diagram for $\pi_1$ and a diagram for $\pi_2$ then $\pi_1$ is trivial if and only if $\pi_2$ is. It allows us to define trivial spherical diagrams with correct correspondence to trivial identities. We will call $D$ a trivial spherical diagram if a trivial identity can be obtained from $D$. Due to the fact noted above, every identity obtained from a trivial spherical diagram $D$ will be trivial. Thus only trivial identities can be obtained from a trivial diagram, and only trivial diagrams can be obtained from a trivial identity.

Note also that actually a diagram $D$ which is a diagram for some identity $\pi$ has the marked point (the initial vertex $v$ from the construction). Intuitively it is clear that replacing of the marked point does not change triviality of a diagram. To strictly prove that we should use original definition of triviality of diagram from [6], which states that a trivial diagram is a diagram which can be transformed to the trivial diagram consisting of only one vertex by finite number of certain transformations. Note that these transformations do not depend on the marked point that means that if a diagram can be transformed to trivial then after
replacing of the marked point it still can be transformed to trivial. Thus replacing of the marked point does not change triviality of a diagram so further we will not specify which point we consider as marked.

And finally note that a spherical diagram can have spurs. But Proposition 8 from [6] implies that inserting and deleting of spurs does not change triviality of the diagram.

4 Proof of the theorem

In this section \(\Gamma\) is a labelled by the set \(S\) graph and \(R_c, R_s\) and \(R_f\) are the sets defined at the beginning of the section 1.1.

Definition 11. Let \(D\) be a diagram over the presentation \(\langle S \mid R_s \rangle\). A not originating skeleton of \(D\) is a graph that consists of all edges of \(D\) which do not originate from the graph \(\Gamma\).

Lemma 1. Suppose \(\Gamma\) satisfies graphical \(C(2)\)-condition and let \(D\) be a simple spherical diagram over the presentation \(\langle S \mid R_s \rangle\). Then a not originating skeleton of \(D\) has no spurs.

Proof. Assume the contrary. Let \(D\) be a simple spherical diagram whose not originating skeleton contains a spur \(e\) with endpoint \(v\). It means that all other edges incidental to \(v\) originate from the graph \(\Gamma\) because otherwise the edge \(e\) would not be a spur in the originating skeleton.

We may assume that a part of the diagram in a neighborhood of the point \(v\) is embedded into the plane. Draw a circle with a center at \(v\) with quite a small radius such that the circle intersects all incidental to \(v\) edges and only them. Once if an edge is not a loop and twice if an edge is a loop. This circle intersects some number of consecutive faces \(\Pi_1, \ldots, \Pi_n\). Also it intersects the edges \(e_1, \ldots, e_{n-1}, e_n = e\), where \(e_i\) lies in the intersection of faces \(\Pi_i\) and \(\Pi_{i+1}\).

The boundary of each face has a unique lift to the graph because \(\Gamma\) satisfies graphical \(C(2)\)-condition. Let \(v_i\) be a vertex of the graph obtained by lifting \(v\) via the boundary of the face \(\Pi_i\). The edge \(e_i\) originates from the graph for \(i = 1, \ldots, n - 1\), therefore \(v_i = v_{i+1}\) for \(i = 1, \ldots, n - 1\). Thus \(v_1 = v_n\). It means that lifts of the edge \(e\) via \(\partial\Pi_i^+\) and via \(\partial\Pi_i^-\) have a common vertex in the graph and, moreover, have the same label. It implies that these lifts coincide because a labelling of the graph is reduced. Thus \(e\) originates from the graph. A contradiction.

Definition 12. We will call \(D\) a minimal non-trivial spherical diagram over the presentation \(\langle S \mid R_f \rangle\) if it is a non-trivial spherical diagram such that

1. A not originating skeleton of \(D\) has the smallest number of edges among all non-trivial diagrams and among all such diagrams a not originating skeleton of \(D\) has the smallest number of vertices.
2. $D$ has the smallest number of edges among all diagrams satisfying the first condition.

Note that a minimal non-trivial spherical diagram $D$ is always a simple spherical diagram. Indeed, assume the contrary, let $D$ not be a simple spherical diagram. We noted earlier that insertion and deletion of spurs does not affect its triviality. Therefore $D$ does not have spurs. It means that there exists such a point $v$ in $D$ that the diagram splits into two spherical diagrams $D_1$ and $D_2$ after cutting $D$ at $v$. Let $\pi_1$ be an identity obtained from the diagram $D_1$ and $\pi_2$ be an identity obtained from the diagram $D_2$. These identities are trivial because $D$ is a minimal non-trivial spherical diagram. Then an identity $\pi = (\pi_1, \pi_2)$ is trivial as well. But $\pi$ can be obtained from the diagram $D$. Thus $D$ is a trivial spherical diagram that contradicts its definition.

Let $D$ be a simple spherical diagram. Denote by $\overline{D}$ a simple spherical diagram obtained from $D$ by erasing all edges originating from the graph $\Gamma$ (if isolated vertices are left we delete them). Clearly, a 1-skeleton of $\overline{D}$ coincides with a not originating skeleton of $D$. Note that actually $\overline{D}$ may have some not simply connected “faces”, but we still consider them as faces and call them not simply connected faces. Note that the boundary of a not simply connected face consists of some connected components.

Lemma 2. Let $\Gamma$ be a $C(2)$-graph. Let $D$ be a minimal non-trivial spherical diagram over the presentation $\langle S \mid R_f \rangle$. Then the boundary label of any face of the diagram $\overline{D}$ is reduced (the boundary label of any connected component for not simply connected faces).

Proof. Assume the contrary. Let $\overline{\Pi}$ be a face with a not reduced boundary label. Let $e$ and $f$ be the edges where reduction is occurred. We will use diamond moves introduced in [6]. For definiteness assume that $\omega(e) = \omega(f)$ and $\ell(e) = \ell(f) = a$. Consider in detail only the case when all 3 points $\omega(e), \alpha(e), \alpha(f)$ are distinct. Transform the diagram $D$ as shown in Fig. 4. Note that the not originating skeleton was reduced at least by one edge because one of the two new edges appears to be inside the face $\overline{\Pi}$. But the not originating skeleton does not have spurs so this edge originates from the graph.

![Fig. 4: The first case of the Lemma 2.](image)

Diamond moves for other cases can be found in [6] Section 1.4]. We do not consider these cases in detail because they do not differ much from the first case: after applying a diamond move two old edges are replaced by two new ones and one of the two new edges appears to be inside the face $\overline{\Pi}$ and therefore originates from the graph. Thus the not originating skeleton is reduced anyway.

Lemma 3. Let $p$ be a simple closed path in the graph $\Gamma$. Then there exists a diagram $D$ over the presentation $\langle S \mid R_f \rangle$ such that $\partial D$ lifts to the path $p$ and all internal edges of $D$ originates from the graph $\Gamma$.

Proof. Forget for a while that $\Gamma$ is labelled by the set $S$. Label every edge $e$ of the graph $\Gamma$ by the new unique label $t_e$. Assume the path $p$ lies in the connected component $\Gamma'$ of the graph $\Gamma$ and let $B'$ be the basis of $\Pi(\Gamma', v)$ from which some relations of $R_f$ are obtained. Let $q$ be
a path which runs from the vertex $\alpha(p)$ to the vertex $v$ and let $B = \{qb'^{-1} | b' \in B'\}$. Then the path $p$ are generated by the paths from $B$, i.e., $p = b_1^\ell \cdots b_n^\ell$, $b_i \in B$. Each $b_i = u_i r_i u_i^{-1}$ where $r_i$ is a cyclic reduction of the path $b_i$. Due to the definition of the set $R_f$, the label of each path $r_i$ lie in $R_f$.

Fix a point $s$ on the plane. For each $b_i^\ell$ draw a face with a spur such that the spur starts at $s$ and has a label $\ell(u_i)$ and such that the boundary label of the face is $\ell(r_i^\ell)$. Thus we obtain a diagram $E$ such that its boundary label is freely equal to the label of the path $p$. If there exists a pair of consecutive edges with the same labels and opposite directions on the boundary of $E$ then glue them. Doing this several times we obtain a reduced boundary label. The labelling of $\Gamma$ is reduced and $p$ is a simple closed path so its label is reduced. Thus the boundary label of $E$ and the label of $p$ is equal as words.

Note that all internal edges of $E$ originate from the graph because they have unique labels and therefore have unique lifts to the graph. Now recall the original labelling by the set $S$. After replacing unique labels by original ones all internal edges still originate from the graph (because boundaries of the faces still have unique lifts to the graph and these lifts coincide with lifts to the graph with unique labels) and the boundary label of the diagram still is equal to the label of $p$ as words. Thus the desired diagram for the path $p$ over the presentation $\langle S | R_f \rangle$ was obtained.

**Lemma 4.** Let $\Gamma$ be a C(2)-graph. Let $D$ be a minimal non-trivial spherical diagram over the presentation $\langle S | R_f \rangle$ and let $\Pi$ be a face of $D$. Then if $\Pi$ is simply connected and its boundary is a simple closed path then this boundary lifts to a simple closed path in the graph $\Gamma$.

**Proof.** Assume the contrary, that there exists a face $\Pi$ in $\tilde{\Gamma}$ such that it have a simple boundary which lifts to a not simply connected path in the graph. By Lemma 2, the boundary label of $\Pi$ is reduced so there exists a subpath $p$ of the path $\partial \Pi$ which lifts to a simple closed path $\tilde{p}$ in the graph. Let $s$ and $t$ be respectively the start and the end of the path $p$. Transform the diagram as shown in Fig. 5: cut the diagram along the path $p$ and glue together $s$ and $t$. Both $\partial \Pi_1$ and $\partial \Pi_2$ lift to the simple closed path $\tilde{p}$ in the graph $\Gamma$. By the previous lemma, there exists a diagram $E$ over the presentation $\langle S | R_f \rangle$ such that its boundary coincides with $\tilde{p}$ and all internal edges originate from the graph. Glue $E$ at the place of $\Pi_1$ and a diagram symmetric to $E$ at the place of $\Pi_2$. Thus we again obtain a spherical diagram over the presentation $\langle S | R_f \rangle$. Denote it by $D'$.

Note that $\Pi \cap \Pi_1 = \partial \Pi_1$ originates from the graph because it lifts to the simple closed path in the graph and due to the graphical C(2)-condition such a lift is unique. Therefore $D'$ has a smaller not originating skeleton than $D$ because we glued together the vertices $s$ and $t$ which lay in a not originating skeleton of $D$ and so we reduced the number of vertices in the not originating skeleton while the number of not originating edges are still the same, because all internal edges of $E$ originate from the graph.

Moreover, $D'$ is not trivial because we glued into $D$ the set of opposite faces and we can think of it as adding to the not trivial identity $\pi$ some sequence $(p_1, \ldots, p_n, p_n^{-1}, \ldots, p_1^{-1})$ that corresponds to the $n$-fold application of the insert Peiffer transformation. Thus we obtained a not trivial spherical diagram with a smaller not originating skeleton than $D$ has. A contradiction.

**Lemma 5.** Let $\Gamma$ be an aspherical graph and let $D$ be a minimal non-trivial spherical diagram over the presentation $\langle S | R_f \rangle$. Then every face of $\tilde{D}$ is simply connected and the boundary of every face of $\tilde{D}$ lifts to a simple closed path in the graph $\Gamma$.

**Proof.** Assume the contrary. Then there are two types of bad faces: not simply connected faces and simply connected faces with a not simple boundary path which does not lift to a simple closed path in the graph. Note that since a not originating skeleton does not have spurs every face with a not simple boundary encloses some subdiagram which has at least one face. Similarly every not simply connected face does.
Consider a face $\Pi$ which is the innermost bad face. It means that a subdiagram $\Delta$ enclosed by this face contains only simply connected faces which boundaries lift to simple closed paths.

First examine the case when $\Pi$ is a simple connected face. As before let $p$ be a subpath of $\partial \Pi$ which lifts to a simple closed path and let $s$ and $t$ be respectively the start and the end of the path $p$. If $s \neq t$ then acting as in the proof of Lemma 4 we obtain a contradiction with minimality of $D$. Thus $s = t$.

Let $q$ be such a subpath of $\partial \Pi$ that $pq \in \partial \Pi^+$. Note that either $p$ or $q$ encloses some subdiagram $\Delta'$ of the diagram $\Delta$. If $q$ does then we can again assume that there is a subpath $p'$ in $q$ with the end points $s'$ and $t'$ which lifts to a simple closed path. Arguing as before we obtain $s' = t'$. And now $p'$ encloses some subdiagram $\Delta'$ of the diagram $\Delta$. Note that $\Delta'$ as a subdiagram of $\Delta$ does not contains bad faces. So in the both cases there is a sub path which encloses some subdiagram without bad faces and which lifts to a simple closed path. We denote this path by $p$ and this subdiagram by $\Delta$.

Since $\Delta$ does not contain bad faces boundaries of all faces of $\Delta$ lift to a simple closed paths. But the boundary of $\Delta$ itself, which equals to a path $p$, lifts to a simple closed path. Thus we can think of $\Delta$ as a spherical diagram over the presentation $\langle S | R_s \rangle$ no edge of which originates from the graph. But it contradicts asphericity of the graph $\Gamma$.

In the case of a not simple connected face we act similarly. Let $\Delta$ be a subdiagram enclosed by $\Pi$. Due to the choice of $\Pi$ all faces of $\Delta$ lift to simple closed paths. If $\partial \Delta$ lifts to a simple closed path in the graph then we again contradict asphericity of the graph. Otherwise we take a subpath $p$ of $\partial \Delta$ with the end points $s$ and $t$ which lifts to a simple closed path in the graph. If $s \neq t$ then we contradict minimality of $D$. If $s = t$ then a subdiagram enclosed by $p$ gives a spherical diagram over the presentation $\langle S | R_s \rangle$ where no edge originates from the graph that contradicts asphericity of the graph.

\[\square\]
The following lemma can be found in [4].

Lemma 6 (Gruber). Let $\Gamma$ be a connected $C(2)$-labelled graph. Let $R$ be the set of cyclic reductions of words read a set of free generators of $\pi_1(\Gamma, v)$ for some $v \in \Gamma$. Let $(D, v)$ be a simple disk diagram over $R$ with freely trivial boundary word such that every interior edge originates from $\Gamma$. Then any sequence obtained from $(D, v)$ is a trivial identity sequence.

Now we can prove the theorem.

Proof of the theorem. Assume the contrary, that there exists non-trivial identities. Let $D$ be a minimal non-trivial spherical diagram over the presentation $\langle S \mid R \rangle$. Consider a not originating skeleton of $D$. If the skeleton is empty then all edges of the diagram originate from the graph. In this case we can unglue $D$ along some edge connecting distinct vertices and we obtain a simple disk diagram with a trivial boundary label all edges of which originate from the graph $\Gamma$. Clearly, all these edges lift to the same connected component of the graph. Then, by the previous lemma, $D$ is a diagram for a trivial identity that contradicts its definition. Thus the not originating skeleton is not empty. By the all previous lemmas every face of $\bar{D}$ is simply connected and lifts to a simple closed path of the graph. Therefore $\bar{D}$ is a a diagram over the presentation $\langle S \mid R_s \rangle$ where no edge originates from the graph that contradicts asphericity of the graph.

\[\square\]

5 Corollary of the main result

5.1 Small cancellation conditions

Recall that classical small cancellations conditions $C(q)&T(p)$ (see, for example, [7]) mean that every face in any reduced spherical diagram consists of at least $q$ arcs and every vertex has degree at least $p$ or equal to 2. Graphical analogue of condition $C(q)$ was already introduced in the article, so it only remains to formulate an analogue of condition $T(p)$.

Let $\Gamma$ be a labelled by the set $S$ $C(2)$-graph. Let $r_1$ and $r_2$ be two elements of $R_s$. We say that $r_1$ and $r_2$ mutually originate from the graph if $r_1 = r'_1 c$, $r_2 = c^{-1} r'_2$ and lifts of $c$ in the graph via $r_1$ and via $r_2$ coincide (recall that $r_1$ and $r_2$ have unique lifts since they are elements of $R_s$ and $\Gamma$ is a $C(2)$-graph). Now we can modify a classical definition from [7] to obtain a definition of graphical condition $T(q)$.

Definition 13 (Graphical condition $T(p)$). Let $\Gamma$ be a labelled by the set $S$ $C(2)$-graph and let $3 \leq h < p$. Assume $r_1, \ldots, r_h$ to be elements of $R_s$ such that consecutive elements $r_i, r_{i+1}$ are not mutually originating from the graph. Then at least one product $r_1 r_2, \ldots, r_{h-1} r_h, r_h r_1$ is reduced.

This definition preserves geometric meaning of condition $T(p)$: every inner vertex of any graphically reduced diagram has degree at least $p$ or equal to 2.

Now we show that each graphical condition $C(6)[&T(3)], C(4)&T(4)$ or $C(3)&T(6)$ implies asphericity of a graph. To do this we will need a notion of $[p, q]$-diagrams introduced in [7].

Let $p$ and $q$ be positive integers such that $1/p + 1/q = 1/2$. Degree of a face is a number of edges in its boundary path. A face of a diagram is called interior if it has no common edges with the boundary of the diagram. If $D$ is a non-empty diagram such that each interior vertex of $D$ has degree at least $p$ and all faces of $D$ have degree at least $q$, then $D$ is called $[p, q]$-diagram. If $D$ is a non-empty diagram such that each interior vertex of $D$ has degree at least $p$ and each interior face of $D$ have degree at least $q$, then $D$ is called $(p, q)$-diagram.

Let us show that there exists no simple spherical $(p, q)$-diagram. For the number $c$ of faces and the number $d$ of edges in such a diagram we have an inequality $c \leq 2d/q$. On the other hand, for the number $v$ of vertices and the number $d$ of edges in such a diagram we have an inequality $v \leq 2d/p$. Summing these inequalities and recalling that $1/p + 1/q = 1/2$, we obtain a contradiction with Euler’s formula.
We also need an operation of “forgetting” vertices of degree 2. Let \( D \) be a diagram without spurs. We call an arc in the diagram \( D \) full if its endpoints have degree more than 2. Then if we replace every full arc in \( D \) by an edge, we obtain a diagram where no vertex has degree equal to 2. Therefore after “forgetting” vertices of degree 2 the diagram \( D \) becomes a simple spherical \((p,q)\)-diagram which does not exist. A contradiction.

**Corollary 1.** If \( \Gamma \) satisfies any of the conditions \( C(6), C(4)&T(4) \) or \( C(3)&T(6) \), then the presentation \( \langle S \mid R_4 \rangle \) is aspherical.

For condition \( C(6) \) this result was obtained for the first time by Dominic Gruber in [4]. Consider an example. Let \( \Gamma \) be a graph as in figure 7, which defines a group \( \langle a, b, c, d \mid (abcd)^2ad^{-1}b = 1, (abcd)^2a = c^2b \rangle \). It is easy to check that this presentation doesn’t satisfy classical condition \( C(6) \). Neither it satisfies classical condition \( T(4) \), because we can consider relations \( c^2ba^{-1}(abcd)^{-2} \), \( (abcd)^2ad^{-1}b \) and \( b^{-1}a^{-1}c^2ba^{-1}(abcd)^{-1}d^{-1}c^{-1} \). The graph \( \Gamma \) doesn’t satisfy graphical condition \( C(6) \) because, for example, a simple cycle \((abcd)^2ad^{-1}b\) can be written as \( abcd \cdot beda \cdot d \cdot d \cdot b \).

Nevertheless, we can check that \( \Gamma \) satisfies graphical condition \( C(4)&T(4) \) and therefore this presentation is aspherical due to previous corollary. It is easy to check graphical condition \( C(4) \).

Let us check that \( \Gamma \) satisfies graphical condition \( T(4) \). We should show that for any triple of relations \( r_1, r_2, r_3 \in R_4 \), such that each pair of consecutive elements have a reduction, a mutually originating pair exists. To check that each pair in triple \( r_1, r_2, r_3 \) has a reduction it is sufficient to consider only first and last letters of each relation. And since all cyclic shifts lie in \( R_4 \) it is sufficient to consider only two-letter subwords of relations.

Note that any two-letter subword of a relation from \( R_4 \) looks like \( xy \), where either \( x, y \in \{a, b, c, d\} \) or \( x, y \in \{a^{-1}, b^{-1}, c^{-1}, d^{-1}\} \), only except subwords \( ab^{-1}, ba^{-1}, a^{-1}c, c^{-1}a \). It is clear that if in a triple of subwords each pair have a reduction then at least one of them should be an exceptional subword. Considering possible cases we obtain only 4 appropriate triples: \((ab^{-1}, bd, d^{-1}a^{-1}),(a^{-1}c, c^{-1}b^{-1}, ba)\) and inverse to them. Finally note that a letter \( a \) originates in all this triples.

### 5.2 Car-crash lemma

Shortly consider one more method for proving diagrammatic asphericity, which is based on a topological lemma from [5]. Let \( S \) be a simple spherical diagram and let there be a moving point (a car) on the boundary of some face of the diagram. We say that a car moves properly
if it moves along the boundary in the positive direction continuously, perpetually, with no stops, no reverses and visiting every point of the boundary infinitely many times.

**Lemma 7** (Klyachko). *Let $S$ be a simple spherical diagram. Let there be a car on the boundary of each face and let the cars move properly. Then there exists at least 2 points of the sphere in which complete collision happens. A collision is called complete, if in a point of multiplicity $k$ collide $k$ cars simultaneously.*

This lemma can be used to prove diagrammatic asphericity in the following way. Assume that a motion is defined for each relation such that for any diagram over that presentation collisions occurs only on common boundaries of reducible pairs of faces. Then there exist no reduced spherical diagrams because otherwise there exists a motion of cars on a sphere without collisions that contradicts the lemma.

Consider, for example, a presentation $\langle a, b \mid aba^{-1}b^{-1} \rangle$. Let us prove that it is aspherical. Define a motion of cars. Let each car evenly moves along its face with period 1. If a car moves along a face with relation $aba^{-1}b^{-1}$ then first it traverses an edge with a letter $a$, then it traverses an edge with a letter $b$ and so on. If a car moves along a face with relation $bab^{-1}a^{-1}$ then first it traverses an edge with a letter $a$, then it traverses an edge with a letter $b^{-1}$ and so on.

In the moments of time from $(k, k+1/4)$ all cars move along edges with a letter $a$. In the moments of time from $(k+2/4, k+3/4)$ all cars move along edges with a letter $a^{-1}$. Hence there can be no collisions in these moments of time. In the moments of time from $(k+1/4, k+2/4) \cup (k+3/4, k+1)$ all cars move along edges with letters $b^{\pm 1}$. Hence there can be no complete collision in a vertex because if it occurs in the moment $t$ then in the moment $t+\epsilon$ one part of the cars should move along edges with letters $a^{\pm 1}$ and another part should move along edges with letters $b^{\pm 1}$ that is impossible. In the moments of time $k+3/8$ and $k+7/8$ a collision on an edge with a letter $b$ can occur but in this case a collision occurs between a reducible pair of faces. Thus collisions occurs only on common boundaries of reducible pairs of faces that imply asphericity of the presentation $\langle a, b \mid aba^{-1}b^{-1} \rangle$.

This method can be transferred to the graphical case in the following way: let us allow collisions to occur on edges originating from the graph.

**Corollary 2.** Let $\Gamma$ be a $C(2)$-graph and let a proper motion be given for each (up to conjugation) element of $R_s$ such that complete collisions in diagrams over $\langle S \mid R_s \rangle$ occur only on edges originating from the graph and on vertices incident to originating edges. Then the presentation $\langle S \mid R_f \rangle$ is aspherical.

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