Multiplicities of Singular Points in Schubert Varieties of Grassmannians

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Abstract. We give a closed-form formula for the Hilbert function of the tangent cone at the identity of a Schubert variety $X$ in the Grassmannian in both group theoretic and combinatorial terms. We also give a formula for the multiplicity of $X$ at the identity, and a Gröbner basis for the ideal defining $X \cap O^-$ as a closed subvariety of $O^-$, where $O^-$ is the opposite cell in the Grassmannian. We give conjectures for the Hilbert function and multiplicity at points other than the identity.

1 Introduction

The first formulas for the multiplicities of singular points on Schubert varieties in Grassmannians appeared in Abhyankar’s results [3] on the Hilbert series of determinantal varieties (recall that a determinantal variety gets identified with the opposite cell in a suitable Schubert variety in a suitable Grassmannian). Herzog-Trung [6] generalized these formulas to give determinantal formulas for the multiplicities at the identity of all Schubert varieties in Grassmannians. Using standard monomial theory, Lakshmibai-Weyman [7] obtained a recursive formula for the multiplicities of all points in Schubert varieties in a minuscule $G/P$; Rosenthal-Zelevinsky [9] used this result to obtain a closed-form determinantal formula for multiplicities of all points in Grassmannians.

2 Summary of Results

Let $K$ be the base field, which we assume to be algebraically closed, of arbitrary characteristic. Let $G$ be $SL_n(K)$, $T$ the subgroup of diagonal matrices in $G$, and $B$ the subgroup of upper diagonal matrices in $G$. Let $R$ be the root system of $G$ relative to $T$, and $R^+$ the set of positive roots relative to $B$. Let $W$ be the Weyl group of $G$. Note that $W = S_n$, the group of permutations of the set of $n$ elements. Let $P_d$ be the maximal parabolic subgroup

$$P_d = \left\{ A \in G \left| A = \begin{pmatrix} * & \cdots & * \\ 0_{(n-d) \times d} & * \end{pmatrix} \right. \right\}.$$  

Let $R_{P_d}$, $R_{P_d}^+$, and $W_{P_d}$ denote respectively the root system, set of positive roots, and Weyl group of $P_d$. The quotient $W/W_{P_d}$, with the Bruhat order, is a distributive lattice. The map $\alpha \mapsto s_\alpha W_{P_d}$ taking a positive root to its
corresponding reflection, embeds \( R^+ \setminus R^+_P \) in \( W/W_P \). We shall also denote the image by \( R^+ \setminus R^+_P \). It is a sublattice of \( W/W_P \).

A multiset is similar to a set, but with repetitions of entries allowed. Define the cardinality of a multiset \( S \), denoted by \(|S|\), to be the number of elements in \( S \), including repetitions. Define a uniset to be a multiset which has no repetitions. If \( S \) is a set, define \( S^* \) to be the collection of all multisets which are made up of elements of \( S \).

A chain of commuting reflections in \( W/W_P \) is a nonempty set of pairwise-commuting reflections \( \{s_{\alpha_1}, \ldots, s_{\alpha_t}\} \), \( \alpha_i \in R^+ \setminus R^+_P \), such that \( s_{\alpha_t} > \cdots > s_{\alpha_1} \); we refer to \( t \) as the length of the chain. For a multiset \( S \in (R^+ \setminus R^+_P)^* \), define the chainlength of \( S \) to be the maximum length of a chain of commuting reflections in \( S \).

Fix \( w \in W/W_P \). Define \( S_w \) to be the multisets \( S \) of \( (R^+ \setminus R^+_P)^* \), such that the product of every chain of commuting reflections in \( S \) is less than or equal to \( w \); similarly, define \( S'_w \) to be the unisets of \( (R^+ \setminus R^+_P)^* \) having the same property. For \( m \) a positive integer, define

\[
S_w(m) = \{ S \in S_w : |S| = m \}
\]

\[
S'_w(m) = \{ S \in S'_w : |S| = m \}.
\]

We can now state our two main results. First, letting \( X(w) \) denote the Schubert variety of \( G/P_d \) corresponding to \( w \in W/W_P \), the Hilbert function of the tangent cone to \( X(w) \) at the identity is given by

**Theorem 1** \( h_{TC,dX(w)}(m) = |S_w(m)|, m \in \mathbb{N}. \)

Second, letting \( M \) denote the maximum cardinality of any element of \( S'_w \), the multiplicity at the identity is given by

**Theorem 2** \( \text{mult}_{id} X(w) = |\{ S \in S'_w : |S| = M \}|. \)

3 Preliminaries

3.1 Multiplicity of an Algebraic Variety at a Point

Let \( B \) be a graded, affine \( K \)-algebra such that \( B_1 \) generates \( B \) (as a \( K \)-algebra). Let \( X = \text{Proj}(B) \). The function \( h_B(m) \) (or \( h_X(m) \)) = \( \dim_K B_m/\mathfrak{m}^m \), \( m \in \mathbb{Z} \) is called the *Hilbert function* of \( B \) (or \( X \)). There exists a polynomial \( P_B(x) \) (or \( P_X(x) \)) \( \in \mathbb{Q}[x] \), called the *Hilbert polynomial* of \( B \) (or \( X \)), such that \( f_B(m) = P_B(m) \) for \( m \gg 0 \). Let \( r \) denote the degree of \( P_B(x) \). Then \( r = \dim(X) \), and the leading coefficient of \( P_B(x) \) is of the form \( c_B / r! \), where \( c_B \in \mathbb{N} \). The integer \( c_B \) is called the *degree* of \( X \), and denoted \( \deg(X) \). In the sequel we shall also denote \( \deg(X) \) by \( \deg(B) \).

Let \( X \) be an algebraic variety, and let \( P \in X \). Let \( A = \mathcal{O}_{X,P} \) be the stalk at \( P \) and \( \mathfrak{m} \) the unique maximal ideal of the local ring \( A \). Then the *tangent
cone to $X$ at $P$, denoted $TC_P(X)$, is defined to be $\text{Spec}(\text{gr}(A, m))$, where $\text{gr}(A, m) = \bigoplus_{j=0}^{\infty} m^j/m^{j+1}$. The multiplicity of $X$ at $P$, denoted $\text{mult}_P(X)$, is defined to be $\deg(\text{Proj}(\text{gr}(A, m)))$. If $X \subset K^n$ is an affine closed subvariety, and $m_P \subset K[X]$ is the maximal ideal corresponding to $P \in X$, then $\text{gr}(K[X], m_P) = \text{gr}(A, m)$.

### 3.2 Monomial Orders, Gröbner Bases, and Flat Deformations

Let $A$ be the polynomial ring $K[x_1, \cdots, x_n]$. A monomial order $\succ$ on the set of monomials in $A$ is a total order such that given monomials $m, m_1, m_2, m \neq 1, m_1 \succ m_2$, we have $mm_1 \succ m_1$ and $mm_1 \succ mm_2$. The largest monomial (with respect to $\succ$) present in a polynomial $f \in A$ is called the initial term of $f$, and is denoted by $\text{in}(f)$.

The lexicographic order is a total order defined in the following manner. Assume the variables $x_1, \ldots, x_n$ are ordered by $x_n > \cdots > x_1$. A monomial $m$ of degree $r$ in the polynomial ring $A$ will be written in the form $m = x_{i_1} \cdots x_{i_r}$, with $n \geq i_1 \geq \cdots \geq i_r \geq 1$. Then $x_{i_1} \cdots x_{i_r} \succ x_{j_1} \cdots x_{j_s}$ in the lexicographic order if and only if either $r > s$, or $r = s$ and there exists an $l < r$ such that $i_1 = j_1, \ldots, i_l = j_l, i_{l+1} > j_{l+1}$. It is easy to check that the lexicographic order is a monomial order.

Given an ideal $I \subset A$, denote by $\text{in}(I)$ the ideal generated by the initial terms of the elements in $I$. A finite set $G \subset I$ is called a Gröbner basis of $I$ (with respect to the monomial order $\succ$), if $\text{in}(I)$ is generated by the initial terms of the elements of $G$.

**Flat Deformations:** Given a monomial order and an ideal $I \subset A$, there exists a flat family over $\text{Spec}(K[t])$ whose special fiber ($t = 0$) is $\text{Spec}(A/\text{in}(I))$ and whose generic fiber ($t$ invertible) is $\text{Spec}(A/I \otimes K[t, t^{-1}])$. Further, if $I$ is homogeneous, then the special fiber and generic fiber have the same Hilbert function (see [4] for details).

### 3.3 Grassmannian and Schubert Varieties

**The Plücker Embedding:** Let $d$ be such that $1 \leq d < n$. The Grassmannian $G_{d,n}$ is the set of all $d$-dimensional subspaces $U \subset K^n$. Let $U$ be an element of $G_{d,n}$ and $\{a_1, \ldots, a_d\}$ a basis of $U$, where each $a_j$ is a vector of the form

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}, \text{ with } a_{ij} \in K.$$

Thus, the basis $\{a_1, \cdots, a_d\}$ gives rise to an $n \times d$ matrix $A = (a_{ij})$ of rank $d$, whose columns are the vectors $a_1, \cdots, a_d$. 
We have a canonical embedding

\[ p : G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d K^n),\quad U \mapsto [a_1 \wedge \cdots \wedge a_d] \]
called the Plücker embedding. Let

\[ I_{d,n} = \{ \hat{i} = (i_1, \ldots, i_d) \in \mathbb{N}^d : 1 \leq i_1 < \cdots < i_d \leq n \} \, . \]

Then the projective coordinates (Plücker coordinates) of points in \( \mathbb{P}(\wedge^d K^n) \) may be indexed by \( I_{d,n} \); for \( \hat{i} \in I_{d,n} \), we shall denote the \( \hat{i} \)-th component of \( p \) by \( p_{\hat{i}} \), or \( p_{i_1, \ldots, i_d} \). If a point \( U \) in \( G_{d,n} \) is represented by the \( n \times d \) matrix \( A \) as above, then \( p_{i_1, \ldots, i_d}(U) = \det(A_{i_1,\ldots,i_d}) \), where \( A_{i_1,\ldots,i_d} \) denotes the \( d \times d \) submatrix whose rows are the rows of \( A \) with indices \( i_1, \ldots, i_d \), in this order.

**Identification of \( G/P_d \) with \( G_{d,n} \):** Let \( G, T, B, \) and \( P_d \) be as in Section 2. Let \( \{ e_1, \ldots, e_n \} \) be the standard basis for \( K^n \). For the natural action of \( G \) on \( \mathbb{P}(\wedge^d K^n) \), the isotropy group at \( [e_1 \wedge \cdots \wedge e_d] \) is \( P_d \), while the orbit through \( [e_1 \wedge \cdots \wedge e_d] \) is \( G_{d,n} \). Thus we obtain an identification of \( G/P_d \) with \( G_{d,n} \). We also note that \( W/W_{P_d} = S_n/(S_d \times S_{n-d}) \) may be identified with \( I_{d,n} \).

**Schubert Varieties:** For the action of \( G \) on \( G_{d,n} \), the \( T \)-fixed points are precisely \( \{ [e_\hat{i}] : \hat{i} \in I_{d,n} \} \), where \( e_\hat{i} = e_{i_1} \wedge \cdots \wedge e_{i_d} \). The Schubert variety \( X_\hat{i} \) associated to \( \hat{i} \) is the Zariski closure of the \( B \)-orbit \( B[e_\hat{i}] \) with the canonical reduced scheme structure.

We have a bijection between \{Schubert varieties in \( G_{d,n} \)\} and \( I_{d,n} \). The partial order on Schubert varieties given by inclusion induces a partial order (called the Bruhat order) on \( I_{d,n} \) (\( = W/W_{P_d} \)); namely, given \( \hat{i} = (i_1, \ldots, i_d) \), \( \hat{j} = (j_1, \ldots, j_d) \in I_{d,n} \),

\[ \hat{i} \geq \hat{j} \iff i_t \geq j_t, \text{ for all } 1 \leq t \leq d. \]

We note the following facts for Schubert varieties in the Grassmannian (see [3] or [8] for example):

- **Bruhat Decomposition**: \( X_{\hat{i}} = \bigcup_{\hat{j} \leq \hat{i}} B[e_\hat{j}] \).
- **Dimension**: \( \dim X_{\hat{i}} = \sum_{1 \leq t \leq d} i_t - t \).
- **Vanishing Property of a Plücker Coordinate**: \( p_{\hat{i}}|_{X_{\hat{i}}} \neq 0 \iff \hat{i} \geq \hat{j} \).
Standard Monomials: A monomial $f = p_{\theta_1} \cdots p_{\theta_t}$, $\theta_i \in W/W_{P_d}$ is said to be standard if

$$\theta_1 \geq \cdots \geq \theta_t.$$ (1)

Such a monomial is said to be standard on the Schubert variety $X(\theta)$, if in addition to (1), we have $\theta \geq \theta_1$.

Let $w \in W/W_{P_d}$. Let $R(w) = K[X(w)]$, the homogeneous coordinate ring for $X(w)$, for the Plücker embedding. Recall the following two results from standard monomial theory (cf. [5]).

**Theorem 3** The set of standard monomials on $X(w)$ of degree $m$ is a basis for $R(w)_m$.

**Theorem 4** For $w \in W/W_{P_d}$, let $I_w$ be the ideal in $K[G_{d,n}]$ generated by \{ $p_\theta, \theta \not\in w$ \}. Then $R(w) = K[G_{d,n}]/I_w$.

The Opposite Big Cell $O^-$: Let $U^-$ denote the unipotent lower triangular matrices of $G = SL_n(K)$. Under the canonical projection $G \to G/P_d$, $g \mapsto gP_d (= g[e_{id}])$, $U^-$ maps isomorphically onto its image $U^-[e_{id}]$. The set $U^-[e_{id}]$ is called the opposite big cell in $G_{d,n}$, and is denoted by $O^-$. Thus, $O^-$ may be identified with

$$\begin{pmatrix}
\text{Id}_{d \times d} \\
x_{d+11} & \cdots & x_{d+1d} \\
\vdots \\
x_{n1} & \cdots & x_{nd}
\end{pmatrix}, \quad x_{ij} \in K, \quad d+1 \leq i \leq n, 1 \leq j \leq d.$$ (2)

Thus we see that $O^-$ is an affine space of dimension $(n - d) \times d$, with $id$ as the origin; further $K[O^-]$ can be identified with the polynomial algebra $K[x_{-\beta}, \beta \in R^+/R_{P_d}^+]$. To be very precise, denoting the elements of $R$ as in [3], we have $R^+/R_{P_d}^+ = \{ e_{ij} - e_{ii}, d+1 \leq i \leq n, 1 \leq j \leq d \}$; given $\beta \in R^+/R_{P_d}^+$, say $\beta = e_{ij} - e_{ii}$, we identify $x_{-\beta}$ with $x_{ij}$. We denote by $s_{(i,j)}$ (or $s_{(j,i)}$) the reflection corresponding to $\beta$, namely, the transposition switching $i$ and $j$.

Evaluation of Plücker Coordinates on $O^-$: Let $j \in I_{d,n}$. We shall denote the Plücker coordinate $p_j|_{O^-}$ by $f_j$. Let us denote a typical element $A \in O^-$ by $\begin{pmatrix} \text{Id}_{d \times d} \\ X \end{pmatrix}$. Then $f_j$ is simply a minor of $X$ as follows. Let $j = (j_1, \ldots, j_d)$, and let $j_r$ be the largest entry $\leq d$. Let $\{k_1, \ldots, k_{d-r}\}$ be the complement of $\{j_1, \ldots, j_r\}$ in $\{1, \ldots, d\}$. Then this minor of $X$ is given by column indices $k_1, \ldots, k_{d-r}$ and row indices $j_{r+1}, \ldots, j_d$ (here the rows of $X$ are indexed as $d+1, \ldots, n$).

Conversely, given a minor of $X$, say, with column indices $b_1, \ldots, b_s$, and row indices $i_{d-s+1}, \ldots, i_d$, then that minor is the evaluation of $f_j$ at $X$, where
\( \mathbf{j} = (j_1, \ldots, j_d) \) may be described as follows: \( \{j_1, \ldots, j_{d-s}\} \) is the complement of \( \{b_1, \ldots, b_s\} \) in \( \{1, \ldots, d\} \), and \( j_{d-s+1}, \ldots, j_d \) are simply the row indices (again, the rows of \( X \) are indexed as \( d+1, \ldots, n \)).

Note that if \( j = (1, \ldots, d) \), then \( p_j \) evaluated at \( X \) is 1. In the above discussion, therefore, we must consider the element 1 (in \( K[O^-] \)) as the minor of \( X \) with row indices (and column indices) given by the empty set.

Example 1 Consider \( G_{2,4} \). Then

\[
O^- = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix}, \ x_{ij} \in K \right\}.
\]

On \( O^- \), we have \( p_{12} = 1 \), \( p_{13} = x_{32}, p_{14} = x_{42}, p_{23} = x_{31}, \ p_{24} = x_{41}, \ p_{34} = x_{31}x_{42} - x_{41}x_{32} \).

Note that each of the Plücker coordinates is homogeneous in the local coordinates \( x_{ij} \).

4 The Hilbert Function of \( TC_{id}X(w) \)

In view of the Bruhat decomposition, in order to determine the multiplicity at a singular point \( x \), it is enough to determine the multiplicity of the \( T \)-fixed point in the \( B \) orbit \( Bx \). In this section, we shall discuss the behavior at a particular \( T \)-fixed point, namely the identity.

4.1 The Variety \( Y(w) \)

We define \( Y(w) \subset G_{d,n} \) to be \( X(w) \cap O^- \). Since \( Y(w) \subset X(w) \) is open dense, and \( id \in Y(w) \), we have that \( TC_{id}Y(w) = TC_{id}X(w) \). As a consequence of Theorem 4, \( Y(w) \subset O^- \) is defined as an algebraic subvariety by the homogeneous polynomials \( f_\theta, \theta \not\in w \); further, \( id \in O^- \) corresponds to the origin. Thus we have that \( \text{gr}(K[Y(w)], m_{id}) = K[Y(w)] \). Hence,

\[
TC_{id}X(w) = TC_{id}Y(w) = \text{Spec}(\text{gr}(K[Y(w)], m_{id}))
\]

\[= \text{Spec}(K[Y(w)]) = Y(w). \quad (3)\]

4.2 Monomials and Multisets

For a monomial \( p = x_{\alpha_{i_1}} \cdots x_{\alpha_{i_m}} \in K[O^-] \), define \( \text{Multisupp}(p) \) to be the multiset \( \{\alpha_{i_1}, \ldots, \alpha_{i_m}\} \). It follows immediately from the definition that \( \text{Multisupp} \) gives a bijection between the monomials of \( K[O^-] \) and the multisets of \( (R^+ \setminus R_{P_d}^+) \), pairing the square-free monomials with the unisets. Let \( w \in W/W_{P_d} \). We call a monomial \( w \)-good if it maps under \( \text{Multisupp} \) to an
element of $S_w$. Note that the w-good square-free monomials are precisely those which map to $S'_w$.

Define a monomial order $\succ$ on $K[O^-]$ in the following manner. We say $x_{i,j} \succ x_{i',j'}$ if $i > i'$, or if $i = i'$ and $j < j'$. Note that this extends the partial order $x_\alpha > x_\beta \iff s_\alpha > s_\beta$ (in the Bruhat order). The monomials are then ordered using the lexicographic order.

Define the monomial ideal $J_w \subset K[O^-]$ to be the ideal generated by $\{\inf_{\theta, \theta \not\in w}, \theta \not\in w\}$, and let $A_w = K[O^-]/J_w$. With our ordering, $\text{MultSupp}(\inf_{\theta})$ is a commuting chain of reflections whose product is $\theta$. Thus the non w-good monomials form a vector space basis for $J_w$, and therefore the w-good monomials form a basis for $A_w$.

4.3 Sketch of Proof of Theorems 1 and 2

In view of (3) and the above discussion, Theorem 1 follows immediately from Lemma 1

$\text{h}_{K[Y(w)]}(m) = h_{A_w}(m), m \in \mathbb{N}$.

Theorem 2 is also a consequence. Indeed,

$\text{mult}_{\text{id}}X(w) = \deg(K[\text{TC}_{\text{id}}X(w)]) = \deg(K[Y(w)]) = \deg(A_w)$.

Since $A_w$ is an affine quotient of an ideal generated by square-free monomials, letting $M$ be the maximum degree of a square-free monomial in $A_w$, we have (cf. (3))

$\deg(A_w) = |\{ p \in A_w : p \text{ is a square-free monomial and } \deg(p) = M \}|$

$= |\{ p \in K[O^-] : p \text{ is a square-free w-good monomial and } \deg(p) = M \}|$

$= |\{ S \in S'_w : |S| = M \}|$

yielding Theorem 2.

The proof of Lemma 2 relies on an inductive argument which shows directly that both functions agree for all positive integers $m$. Note that $K[Y(w)] = K[X(w)]_{(p_w)}$. Thus, as a consequence of Theorem 2, $K[Y(w)]$ has a basis consisting of monomials of the form $f_{\theta_1} \cdots f_{\theta_l}, w \geq \theta_1 \geq \cdots \geq \theta_l.$ If $SM_w(m)$ denotes the basis elements of degree $m$, then $h_{Y(w)}(m) = |SM_w(m)|$. Letting $d = d_w$ be the degree of $w$ (see section 4.4 below for definition), as a consequence of standard monomial theory we have

$SM_w(m + d) = SM_w(m) \cup SM_H(m + d)$ (4)

where $SM_H(m + d) = \bigcup_{w_i} SM_{w_i}(m + d)$, the union being taken over the divisors $X(w_i)$ of $X(w)$ (cf. (3)).

We have that $|SM_H(m + d)| = |\bigcup_{w_i} SM_{w_i}(m + d)|$ can be set-theoretically written as the integral linear combination of terms of the form $|SM_{w_i}(m + d)|$
and terms of the form \( |SM_{w_j}(m + d) \cap \cdots \cap SM_{w_k}(m + d)| \). Further, it can be shown that

\[
SM_{w_j}(m + d) \cap \cdots \cap SM_{w_k}(m + d) = SM_\theta(m + d),
\]

where \( \theta \) is given by \( X(\theta) = X(w_j) \cap \cdots \cap X(w_k) \). (Note that \( I_{d,n} \) being a distributive lattice implies that for \( \tau, \phi \in I_{d,n} \), \( X(\tau) \cap X(\phi) \) is irreducible.) Thus,

\[
|SM_H(m + d)| = \sum_{w' < w} a_{w'}|SM_{w'}(m + d)|, \quad \text{for some } a_{w'} \in \mathbb{Z}. \tag{5}
\]

Taking cardinalities of both sides of (4), we obtain

\[
h_K[Y(w)](m + d) = h_K[Y(w)](m) + \sum_{w' < w} a_{w'}h_K[Y(w')](m + d).
\]

Equivalently, \( h_K[Y(w)] \) satisfies the difference equation

\[
\phi(w, m + d) = \phi(w, m) + \sum_{w' < w} a_{w'}\phi(w', m + d). \tag{6}
\]

To prove Lemma 1, it suffices to show that \( h_{A_w}(m) \) satisfies (5) for all \( m \in \mathbb{Z}_{\geq 0} \), since it is a straightforward verification that \( h_K[Y(w)](m) \) and \( h_{A_w}(m) \) have the same initial conditions.

As stated earlier, \( K[A_w] \) has as basis the \( w \)-good monomials of \( K[O^-] \), which are in bijection with the elements of \( S_w \). Thus \( h_K[A_w](m) = |S_w(m)| \), and it suffices to show that \( |S_w(m)| \) satisfies (5). We can write

\[
S_w(m + d) = (S_w(m + d) \setminus S_H(m + d)) \cup S_H(m + d), \tag{7}
\]

where \( S_H(m + d) = \bigcup_{w_i} S_{w_i}(m + d) \), the union being over the divisors \( X(w_i) \) of \( X(w) \). Following the identical arguments used to deduce (5) (replacing “\( SM \)” by “\( S \)” everywhere), one obtains

\[
|S_H(m + d)| = \sum_{w' < w} a_{w'}|S_{w'}(m + d)|, \tag{8}
\]

for the same integers \( a_{w'} \) as in (5).

Establishing an explicit bijection between \( S_w(m + d) \setminus S_H(m + d) \) and \( S_w(m) \) completes the proof, for then (taking cardinalities of both sides of (7)), one sees that \( h_{A_w}(m) \) satisfies (5) for all \( m \in \mathbb{Z}_{\geq 0} \).

In view of the discussion of flat deformations in Section 3.2, Lemma 1 also implies

**Corollary 1** The set \( \{ f_\theta, \theta \not\in w \} \subset K[O^-] \) forms a Gröbner basis for the ideal it generates.
4.4 Combinatorial Interpretation

We call a multiset \( S \) of \((R^+ \backslash R^+_P)^*\) a t-multipath, if the chainlength of \( S \) is \( t \). If \( S \) has no repeated elements (i.e. it is a uniset), then we call it a t-unipath. Define \( s \in S \) to be a chain-maximal element of \( S \) if there is no element in \( S \) strictly greater than \( s \) which commutes with \( s \). Any t-multipath \( S \) can be written in the following manner as the union of \( t \) nonintersecting 1-multipaths: if \( S_i \) is the \( i \)th 1-multipath, then \( S_{i+1} \) is the multiset of chain-maximal elements (including repetitions) of \( S \cup \bigcup_{k=1}^{i} S_k \) (for \( i = 0, \cdots, t-1 \), where \( S_0 \) is defined to be the empty set). If the t-multipath \( S \) is a t-unipath, then each \( S_i \) will be a 1-unipath.

Fix \( w \in W/W_P \). There is a unique expression \( w = s_{\alpha_1} \cdots s_{\alpha_i} d_w \) such that \( s_{\alpha_k} > s_{\alpha_{k+1}} \) for all \( k \), and all the reflections pairwise commute; \( d_w \) is called the degree of \( w \).

**Example 2** Let \( w = (3,5,7,8) \in I_{4,8} \). Then \( w = s_{(8,1)} s_{(7,2)} s_{(5,4)} \), where \( s_{(8,1)} > s_{(7,2)} > s_{(5,4)} \) is a chain of commuting reflections. Thus \( d_w = 3 \).

Let \( H_j = \{ \alpha \in R^+ \backslash R^+_P | s_\alpha \leq s_{\alpha_{ij}} \} \). We say that a t-multipath \( S \) is \( w \)-good if, when written as the union of weighted 1-multipaths \( \bigcup_{k=1}^{t} S_k \) as above, we have that the elements of \( S_j \) are in \( H_j \), \( j = 1, \cdots, t \). Any multiset in \((R^+ \backslash R^+_P)^*\) is a t-multipath for some \( t \); it is said to be \( w \)-good if the corresponding t-multipath is \( w \)-good.

It can be seen that the combinatorial property that a multiset (resp. uniset) \( S \) of \((R^+ \backslash R^+_P)^*\) is \( w \)-good is equivalent to the group-theoretic property that \( S \in S_w \) (resp. \( S \in S'_w \)). Thus Theorem 2 is equivalent to the assertion that \( h_{TC,S}(w)(m) \) is the number of \( w \)-good multisets of \((R^+ \backslash R^+_P)^*\) of degree \( m \). Letting \( M \) be the maximum cardinality of a \( w \)-good uniset, Theorem 2 is equivalent to the assertion that \( \text{mult}_{id} X(w) \) is the number of \( w \)-good unisets of cardinality \( M \).

**Example 3** Let \( w = s_{(15,2)} s_{(13,4)} s_{(10,5)} \in I_{7,16} \). We have that \( s_{(15,2)} > s_{(13,4)} > s_{(10,5)} \) is a chain of commuting reflections, and thus \( d_w = 3 \).

The diagram below shows the lattice \( R^+ \backslash R^+_P \), where the reflection \( s_{(i,j)} \) is denoted by \( i,j \). The set \( S \) of reflections which lie along the three broken-line paths is an example of a \( w \)-good uniset of maximum cardinality. In fact, any \( w \)-good uniset of maximum cardinality can be seen as the set of reflections lying on three paths in the lattice, satisfying the following properties:

- One path starts and ends at “X”, the second at “Y”, and the third at “Z”.
- Each path can move only down or to the right.
- The paths do not intersect.

Thus the number of ways of drawing three such paths is \( \text{mult}_{id} X(w) \).
5 Conjectures on the Behavior at Other Points

Let \( w, \tau \in W/W_p \). Define \( S_{w,\tau} \) to be the multisets \( (R^+ \setminus R^+_p)^* \), such that for every chain of commuting reflections \( s_{\alpha_1} \circ \cdots \circ s_{\alpha_t} \), \( s_{\alpha_i} \in S \), we have that \( w \geq s_{\alpha_1} \circ \cdots \circ s_{\alpha_t} \); define \( S'_{w,\tau} \) to be the unisets of \( (R^+ \setminus R^+_p)^* \) having the same property. For \( m \) a positive integer, define

\[
S_{w,\tau}(m) = \{ S \in S_{w,\tau} : |S| = m \}
\]

\[
S'_{w,\tau}(m) = \{ S \in S'_{w,\tau} : |S| = m \}.
\]

We state two conjectures. First, the Hilbert function \( h_{TC, X(w)}(m) \) of the tangent cone to \( X(w) \) at \( \tau \) is given by

**Conjecture 1** \( h_{TC, X(w)}(m) = |S_{w,\tau}(m)|, m \in \mathbb{N}. \)

Second, letting \( M \) denote the maximum cardinality of an element of \( S'_{w,\tau} \), the multiplicity \( \text{mult}_\tau X(w) \) of \( X(w) \) at \( \tau \) is given by

**Conjecture 2** \( \text{mult}_\tau X(w) = |\{ S \in S'_{w,\tau} : |S| = M \}|. \)

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