Robust Linear Regression for General Feature Distribution

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Abstract

We investigate robust linear regression where data may be contaminated by an oblivious adversary, i.e., an adversary than may know the data distribution but is otherwise oblivious to the realizations of the data samples. This model has been previously analyzed under strong assumptions. Concretely, (i) all previous works assume that the covariance matrix of the features is positive definite; and (ii) most of them assume that the features are centered (i.e. zero mean). Additionally, all previous works make additional restrictive assumption, e.g., assuming that the features are Gaussian or that the corruptions are symmetrically distributed.

In this work we go beyond these assumptions and investigate robust regression under a more general set of assumptions: (i) we allow the covariance matrix to be either positive definite or positive semi definite, (ii) we do not necessarily assume that the features are centered, (iii) we make no further assumption beyond boundedness (sub-Gaussianity) of features and measurement noise. Under these assumption we analyze a natural SGD variant for this problem and show that it enjoys a fast convergence rate when the covariance matrix is positive definite. In the positive semi definite case we show that there are two regimes: if the features are centered we can obtain a standard convergence rate; otherwise the adversary can cause any learner to fail arbitrarily.

1 Introduction

The remarkable recent success of Machine Learning (ML) models has lead to their wide adoption in numerous fields. However, deploying ML models in real world scenarios brings several challenges. One such challenge of paramount importance is robustness, i.e., the need to design models that are immune to data contamination. The latter might arise due to adversarial corruptions, extreme events, or malfunctioning sensors, amongst other causes. Even beyond ML, designing robust models has proven to be crucial in various applications, including Economics (Zaman et al., 2001), Computer Vision (Gustafsson et al., 2020), Biology (Yeung et al., 2002), and Healthcare (Davies et al., 2004).
In this paper, we investigate robustness in the context linear regression, one of the most fundamental ML tasks. Concretely, we explore robust regression under the assumption that a fraction $\alpha$ of the observations were contaminated by an adversary. In this context, it is well known that standard regression methods are highly sensitive to outliers, and might break down even in the presence of a single contaminated data point.

Past research on robust linear regression roughly falls into one of two categories, depending on the power of the adversary: (a) An adaptive adversary is allowed to contaminate the data after observing the data samples. This setting was explored, e.g., in Candes and Tao (2005); Charikar et al. (2017); Klivans et al. (2018); Liu et al. (2019); Dalalyan and Thompson (2019); Diakonikolas et al. (2019a,b). It is well known that adaptive adversaries may cause any learner to incur a non vanishing error, depending on the fraction of contaminated samples, irrespective of the number of data points. Conversely, (b) an oblivious adversary is not allowed to observe the samples, but may know the true statistical properties of the data. This setting was explored, e.g., in Tsakonas et al. (2014); Suggala et al. (2019); Pesme and Flammarion (2020); Sun et al. (2020); D’Orsi et al. (2021), which have remarkably shown that one can obtain vanishing error by increasing the number of data samples, for any fraction of contamination.

In this paper we focus on robust linear regression with oblivious adversaries. While past works on this topic have highly advanced the understanding of this setting, they were done under limiting assumptions. Concretely: (i) all previous works assume that the covariance matrix of the features $\Sigma$ is positive definite; and (ii) most works assume that the features are centered (i.e., have zero mean). Additionally, all previous works make additional restrictive assumptions, e.g., that the features are Gaussian or that the corruptions are symmetrically distributed.

In this work, we analyze robust regression under a broader set of assumptions. Concretely, our work applies to general feature and noise distributions with bounded (or just sub-Gaussian) distributions. Our contributions are as follows (see Table 1):

- For the case of $\Sigma \succeq 0$ we analyze two regimes: Under the assumption that the features are centered, we provide an SGD (Stochastic Gradient Descent) variant that ensure an error rate of $O\left(\frac{1}{(1-\alpha)\sqrt{T}}\right)$ after observing $T$ samples. This is the first result for this case. Conversely, if the features are non-centered, we show that any algorithm may completely fail.

- For strictly positive definite $\Sigma$, we provide an SGD variant that ensures an estimation error of $\tilde{O}\left(\frac{1}{(1-\alpha)^2T}\right)$. This is done without any assumption on the centering of the features.

We make no further assumptions regarding the adversary/data. Moreover, we provide SGD variants that do not require the knowledge of the contamination fraction $\alpha$. Finally, we allow the adversary to inject unbounded perturbations into the contaminated measurements.

On the technical side, our work builds on utilizing the Huber loss (Huber, 1964), which is a robust loss function, instead of the $\ell_2$ loss. This is done in conjunction to SGD variants that employ feature centering.
Table 1: Assumptions and rates of related work.

| Paper                              | Features          | Noise & Adversary | Rates for $\Sigma > 0$ | Rates for $\Sigma \geq 0$ |
|------------------------------------|-------------------|-------------------|------------------------|---------------------------|
| Tsakonas et al. (2014)             | $x \sim \mathcal{N}(0, I_d)$ | $\epsilon \sim \mathcal{N}(0, \sigma^2)$ | $\mathcal{O}\left(\frac{1}{(1-\alpha)^2 T}\right)$ | N / A                     |
| Suggala et al. (2019)              | $x \sim \mathcal{N}(0, \Sigma)$ | $\epsilon \sim \text{subG}(\sigma^2)$ | $\mathcal{O}\left(\frac{1}{(1-\alpha)^2 T}\right)$ | N / A                     |
| Pesme and Flammarion (2020)        | $x \sim \mathcal{N}(0, \Sigma)$ | $\epsilon \sim \mathcal{N}(0, \sigma^2)$ | $\mathcal{O}\left(\frac{1}{(1-\alpha)^2 T}\right)$ | N / A                     |
| D’Orsi et al. (2021)               | A deterministic design matrix $X$ | $\epsilon + b$ has a symmetric distribution around 0 | $\mathcal{O}\left(\frac{1}{(1-\alpha)^2 T}\right)$ | N / A                     |
| This paper                         | $x \sim \text{subG}(\kappa)$ | $\epsilon \sim \text{subG}(\sigma^2)$ | $\mathcal{O}\left(\frac{1}{(1-\alpha)^2 T}\right)$ | $\mathcal{O}\left(\frac{1}{(1-\alpha)\sqrt{T}}\right)$ |

Related Work

Robust statistics dates back to the works of Tukey and Huber (Tukey, 1960; Huber, 1964), Classical robust statistics has mainly focused on asymptotic performance (Huber, 1973; Bassett Jr and Koenker, 1978; Pollard, 1991; Van der Vaart, 2000; McMahan et al., 2013), and many of the approaches were not computable in polynomial time (Rousseeuw, 1984, 1985).

A popular approach towards robust regression with oblivious adversaries relies on replacing the $\ell_2$ loss with more robust loss, predominantly either the $\ell_1$ loss or the Huber loss (Huber, 1964) (which are convex), as well other non-convex robust losses (Tukey, 1960).

Finite Time Guarantees for Robust regression: Non-asymptotic guarantees for robust regression were recently explored in several works; all of them rely on employing a convex robust loss function (either $\ell_1$ or Huber). Moreover, as we detail in Table 1, all previous works assume strictly positive definite covariance matrix, i.e. $\Sigma \succeq \rho \cdot I$ for $\rho > 0$, and centered features, in addition to other restrictive assumptions detailed below (see also Table 1).

Tsakonas et al. (2014), assume that the features $x$ and measurement noise $\epsilon$ have zero-mean Gaussian distribution. Their algorithmic approach is to apply ERM (Empirical Risk Minimization) while utilizing the Huber loss. Suggala et al. (2019) similarly assume that $x$ is Gaussian yet allow the measurement noise be sub-Gaussian. They suggest an algorithm, AdaCRR, which makes several passes over the dataset while thresholding suspicious points. Pesme and Flammarion (2020) makes the same Gaussianity assumptions as Tsakonas et al. (2014), and are the first to provide guarantees for an efficient online algorithm, namely SGD with $\ell_1$ loss.

The recent work of D’Orsi et al. (2021) has significantly improved the theoretical understanding by relaxing the Gaussian assumptions of previous works. Similarly to Tsakonas et al. (2014) they provide guarantees to ERM over the Huber loss. Nevertheless, they make two limiting assumptions. First, they assume that both the measurement noise and adversarial perturbations are symmetrically distributed around zero, which highly weakens the adversary. They also make an assumption called spreadness regarding the features, which limits the setup (Nonetheless, they allow the features to be non-centered).
2 Problem Formulation

We consider a linear model where the observations may be contaminated by an oblivious adversary,

\textbf{Model 2.1.} \quad y = \langle w^*, x \rangle + \epsilon + b, \quad \text{(1)}

where \( x \in \mathbb{R}^d \) is a feature vector, \( \epsilon \in \mathbb{R} \) is a zero-mean additive noise that is statistically independent of \( x \), and \( \langle \cdot, \cdot \rangle \) denotes the standard inner product. The corruptions \( b \in \mathbb{R} \) is chosen by an adversary that knows \( w^* \) as well as the probability distributions of \( x \) and \( \epsilon \), but is otherwise oblivious to their realizations. We assume that adversary chooses the corruptions \( b \) according to some probability distribution that is constrained to satisfy \( \mathbb{P}(b \neq 0) = \alpha \), where \( \alpha \in [0, 1) \) is the nominal fraction of samples the adversary is allowed to corrupt.

In what follows we denote by \( \mathcal{P} \) the distribution over non-corrupted data samples \((x, y)\) (i.e., \( b = 0 \)), and by \( \mathcal{Q} \) the distribution over corrupted samples (for which \( b \neq 0 \)). Thus, the contaminated samples in the Model 2.1 are coming from the mixture \((x, y) \sim (1 - \alpha)\mathcal{P} + \alpha\mathcal{Q}\). This model is known as the \( \alpha \)-Huber contamination model (Huber, 1964).

\textbf{Robust Linear Regression:} A learner is given \( T \) independent samples \((x_i, y_i)_{i \in \{1, 2, \ldots, T\}}\) from Model 2.1, and is required to learn a parameter vector \( w \in \mathbb{R}^d \) for either the \textit{prediction} problem with respect to the non-corrupted data

\[
\min_w F(w) := \mathbb{E}_{(x,y) \sim \mathcal{P}} \left[ \frac{1}{2} \left( \langle w, x \rangle - y \right)^2 \right] \quad \text{(1)}
\]

or the \textit{estimation} problem,

\[
\arg \min_w \|w - w^*\|^2,
\]

where \( \| \cdot \| \) denotes the \( \ell_2 \) norm.

In the ordinary prediction or estimation problems, when there is no adversary (\( \alpha = 0 \)), a standard solution method is \textit{least squares} (Gauss, 1809), in which the vector \( w \) which minimizes an empirical version of either the prediction error or estimation error. However, this method is known to be fragile for \( \alpha \neq 0 \); As the next simple example demonstrates, for any \( \alpha \in (0, 1) \) the adversary can make the prediction error arbitrarily large, even for infinite number of samples.

\textbf{Example 2.2.} Let \( w^* \in \mathbb{R} \), and assume \( \epsilon = 0 \) with probability 1, and \( x \) is distributed uniformly over \{0, 2\}. Also consider the following adversary,

\[
b_i := \begin{cases} 
\frac{c}{\alpha} & \text{, with probability } \alpha \\
0 & \text{, otherwise.}
\end{cases}, \quad \forall i \in \{1, \ldots, T\}.
\]

\[\text{Note we can equivalently write } \mathbb{E}_{(x,y) \sim \mathcal{P}} \left[ (\langle w, x \rangle - y)^2 \right] = \mathbb{E}_{x,\epsilon} \left[ (\langle w, x \rangle - y)^2 \right].\]
Then \( y_i = x_i \cdot w^* + b_i \) and on the population level (with infinite number of samples) the expected \( L_2 \) loss is
\[
F(w) = (w - w^*)^2 + \mathbb{E}[b] \cdot (w - w^*) + \mathbb{E}[b^2]
\]
whose minimal value is attained for \( w = w^* + \frac{1}{2} \mathbb{E}[b] = w^* + \frac{1}{2} C \), which might be arbitrarily far from \( w^* \).

We make the following assumptions throughout the paper:

**Assumption 2.3** (Bounded Parameter Vector). \( \|w^*\| \leq D \),
\[
w^* \in \mathcal{W} := \{ w \in \mathbb{R}^d : \|w\| \leq D \}.
\]

**Assumption 2.4** (Bounded Zero-Mean Noise). \( |\epsilon| \leq \sigma \) with probability 1 and \( \mathbb{E}[\epsilon] = 0 \).

**Assumption 2.5** (Bounded Feature Vector). \( \|x\| \leq 1 \) with probability 1.

In Section 6.1 we also extend to the case where the features and measurement noise are sub-Gaussian.

We denote by \( \Sigma \) covariance matrix of the feature vector,
\[
\Sigma := \mathbb{E} \left[ (x - \mathbb{E}[x]) (x - \mathbb{E}[x])^T \right].
\]

In later sections, in order to obtain guarantees on the estimation error we would also assume the following,

**Assumption 2.6** (Strictly Positive Definite Covariance Matrix of the Feature Vector). \( \Sigma \succeq \rho \cdot I \), where \( \rho > 0 \) and \( I \) is the identity matrix.

**Additional Definitions & Notations:** We will use Assumption 2.6 to show that the expected loss is a strongly convex function in \( \mathcal{W} \), and this property will be extensively used. So, we remind the reader the following properties:

**Property 2.7.** Let \( f \) be twice continuously differentiable. \( f \) is \( \lambda \)-strongly convex in \( \mathcal{W} \) if and only if,
\[
\forall w \in \mathcal{W} : \nabla^2 f(w) \succeq \lambda I.
\]

**Property 2.8.** Let \( f \) be \( \lambda \)-strongly convex over \( \mathcal{W} \). Denote \( w_{opt} := \arg\min_{w \in \mathcal{W}} f(w) \). Then, \( \forall w \in \mathcal{W} \),
\[
f(w) - f(w_{opt}) \geq \frac{\lambda}{2} \|w - w_{opt}\|^2.
\]

We also denote the orthogonal projection onto \( \mathcal{W} \) by \( \Pi_{\mathcal{W}}(\cdot) \), i.e., \( \Pi_{\mathcal{W}}(u) := \arg\min_{w \in \mathcal{W}} \|w - u\| \).
3 Huber Loss For Robust Regression

As stated in the previous section, the goal of the learner is either to find a good predictor or an accurate estimator under Model 2.1. The straightforward approach to doing so is to minimize the $\ell_2$ loss while using the contaminated data, but as Example (2.2) shows this might completely fail. This is mainly due to the sensitivity of the $\ell_2$ loss to outliers.

Our approach is based on utilizing a robust loss function and minimizing it using the contaminated data. Concretely, we employ the Huber loss (Huber, 1964) which is known to have the two desired properties: it behaves similarly to the $\ell_2$ loss in a region around the origin, yet far away from the origin it behaves similarly to the $\ell_1$ loss and its gradients are bounded over $\mathbb{R}$.

The Huber loss $h_{R}: \mathbb{R} \mapsto \mathbb{R}$ is a convex function, parameterized by a radius parameter $R$. It is defined as follows,

$$h_{R}(s) := \begin{cases} \frac{1}{2}s^2, & \text{if } |s| \leq R \\ R(|s| - \frac{1}{2}R), & \text{otherwise} \end{cases}.$$ 

We will also denote $\phi_{R}(s) := \min \{R, \max \{s, -R\}\}$.

Note that $\nabla_w h_{R}(\langle a, w \rangle) = \phi_{R}(\langle a, w \rangle) \cdot a$. Also note that $\|\phi_{R}(s)\| \leq R$.

Using the Huber loss is a popular technique in robust regression, see e.g. Tsakonas et al. (2014); D’Orsi et al. (2021). Nevertheless, in contrast to previous works, our analysis is done under a broader (less restrictive) set of assumptions, and our algorithm is a simple variant of SGD.

**Choice of $R$:** The radius parameter $R$ can be adjusted by the learner, and involves a trade-off between unnecessarily clipping clean gradients vs. limiting corrupted gradients. Our choice of $R$ is made such that the loss of non-corrupted observations is in the quadratic regime (so its gradient is not affected), while the loss for highly corrupted observations is in the linear regime (so its gradient’s norm will be bounded). Next we detail on how to choose $R$.

Note that for a non-corrupted sample $i$ ($b_i = 0$) and $w \in \mathcal{W}$,

$$|\langle w, x_i \rangle - y_i| \leq |\langle w, x_i \rangle| + |\langle w^*, x_i \rangle| + |\epsilon_i|$$

$$\leq \|x_i\| \cdot (\|w\| + \|w^*\|) + |\epsilon_i|$$

$$\leq \|w\| + \|w^*\| + \sigma,$$

and we limit our search to $\mathcal{W}$ since we know that $w^* \in \mathcal{W}$. Thus, by taking $R := 6D + \sigma$ we obtain that the gradient of non-corrupted samples is unaffected\(^2\). With this choice, for any

\(^2\)It suffices to take $R = 2D + \sigma$, but we slightly increase the radius for consistency with derivations in Section 5.
$w \in W$ and non-corrupted sample $(x_i, y_i)$ we have
\[
h_R(\langle w, x_i \rangle - y_i) = \frac{1}{2} (\langle w, x_i \rangle - y_i)^2,
\] (2)
with probability 1.

**Robust Objective Function.** We can now introduce the definition of expected Huber loss with respect to the contaminated data,
\[
L_R(w) := \mathbb{E}_{x, \epsilon, b} [h_R(\langle w, x \rangle - y)],
\]
and note that this definition implies that $(x, y)$ are sampled from the contaminated Model 2.1. Since $L_R(w)$ is defined with respect to the contaminated data distribution we can efficiently compute unbiased estimates of its gradients and apply SGD. We will use $L_R(w)$ as a proxy to the true expected loss $F(w)$ appearing in Equation (1).

The next Lemma shows that for the right choice of $R$ we can relate $L_R(\cdot)$ to $F(\cdot)$,

**Lemma 3.1.** Let $R = 6D + \sigma$. Then, $\forall w \in W$,
\[
L_R(w) = (1 - \alpha)F(w) + \alpha H(w),
\]
where $H(w) := \mathbb{E}_{x, \epsilon, b} [h_R(\langle w - w^*, x \rangle - \epsilon - b) \mid b \neq 0]$ represents the expected Huber loss of corrupted samples.

**Proof.** Take $w \in W$. We use the law of total expectation with respect to $b$ on $L_R(w)$. We start with conditioning on $b = 0$: For $R$ as stated in the lemma,
\[
\begin{align*}
\mathbb{E}_{x, \epsilon, b} & \left[ h_R(\langle w - w^*, x \rangle - \epsilon) \mid b = 0 \right] \\
& = \mathbb{E}_{x, \epsilon, b} \left[ \frac{1}{2} (\langle w - w^*, x \rangle - \epsilon)^2 \mid b = 0 \right] \\
& = \mathbb{E}_{x, \epsilon} \left[ \frac{1}{2} (\langle w - w^*, x \rangle - \epsilon)^2 \right] \\
& = F(w),
\end{align*}
\]
where the first equality follows from our choice of $R$ and Equation (2). The second equality follows since $b, \epsilon$ and $x$ are statistically independent and last equality follows from the definition of $F(w)$.

Then, by the law of total expectation
\[
L_R(w) = (1 - \alpha) \mathbb{E}_{x, \epsilon, b} \left[ h_R(\langle w - w^*, x \rangle - \epsilon) \mid b = 0 \right] \\
+ \alpha \mathbb{E}_{x, \epsilon, b} \left[ h_R(\langle w - w^*, x \rangle - \epsilon - b) \mid b \neq 0 \right] \\
= (1 - \alpha)F(w) + \alpha H(w).
\]
\[\square\]
4 The General Feature Covariance Matrix Case

In this section, we make no assumptions on $\Sigma$, that is, allow it to have vanishing eigenvalues. We show that there are two regimes, (i) non-centered features, i.e., $\mathbb{E}[x] \neq 0$ and (ii) centered features, i.e., $\mathbb{E}[x] = 0$. For the first regime, we provide an example showing that obtaining a low prediction error is impossible, even for a known $\mathbb{E}[x]$. If $\mathbb{E}[x] = 0$, we show that a good predictor can be learned (Theorem 4.2). Note that our algorithm does not require the knowledge of the corruptions fraction $\alpha$.

4.1 Low Prediction Error is Generally Impossible

Let $\alpha = \frac{1}{2}$. Consider two one-dimensional models with parameter vectors $w_1^*, w_2^*$, where $w_2^* = -w_1^* = 1$. For both models we assume $\epsilon = 0, x = 1$ with probability 1; thus $\mathbb{E}[x]$ is known and equals 1. The adversary chooses his corruptions for every model $m \in [1, 2]$ as follows,

$$b^{(m)} := \begin{cases} -2 \cdot w_m^* & \text{, with probability } \alpha = 1/2 \\ 0 & \text{, otherwise.} \end{cases}$$

Then $y^{(m)} = w_m^* + b^{(m)}$ and both $y^{(1)}$ and $y^{(2)}$ has the same probability distribution of $y^{(m)} = \pm w_m^*$ with probability $\alpha = \frac{1}{2}$. Since the contaminated data has the same distribution in both cases, then a learner cannot distinguish between these models. Thus, this adversary can cause any learner to incur a fixed (non-decreasing) prediction error, irrespective of the number of available samples.

4.2 A Prediction Error Bound for Centered Features

We next show that if $\mathbb{E}[x] = 0$, then a low prediction error can be achieved even if $\Sigma$ has vanishing eigenvalues.

Next we present our key lemma which shows that we can bound the true error by the error of the expected Huber loss.

Lemma 4.1. For any $w \in \mathcal{W}$, the following applies,

$$F(w) - F(w^*) \leq \frac{1}{1 - \alpha} (L_R(w) - L_R(w^*))$$
Algorithm 1 Huber SGD

Input: $D > 0, R > 0, \mathcal{W} \subset \mathbb{R}^d, T \in \mathbb{N}_+$

$w_1 := 0$

$\eta := \frac{D}{R \sqrt{T}}$

for $t = 1$ to $T$

Draw $(x_t, y_t)$ from (the contaminated) Model 2.1

$g_t := \phi_R ((w_t, x_t) - y_t) \cdot x_t$

$w_{t+1} := \Pi_{\mathcal{W}} (w_t - \eta g_t)$

end for

Return: $\bar{w} := \frac{1}{T} \sum_{t=1}^T w_t$

Proof. We first show that $H(w) \geq H(w^*)$ for all $w \in \mathbb{R}^d$. Indeed, for any $w \in \mathbb{R}^d$:

$$H(w) := \mathbb{E}_{x,\epsilon,b} \left[ h_R ((w - w^*, x) - \epsilon - b) \mid b \neq 0 \right]$$

$$(a) \geq \mathbb{E}_{\epsilon,b} \left[ h_R \left( \mathbb{E}_x ((w - w^*, x) - \epsilon - b) \right) \mid b \neq 0 \right]$$

$$(b) \leq \mathbb{E}_{\epsilon,b} \left[ h_R \left( \mathbb{E}_x (w - w^*) - \epsilon - b \right) \mid b \neq 0 \right]$$

$$(c) = \mathbb{E}_{\epsilon,b} \left[ h_R (\epsilon - b) \mid b \neq 0 \right]$$

$$= \mathbb{E}_{x,\epsilon,b} \left[ h_R ((w^* - w^*, x) - \epsilon - b) \mid b \neq 0 \right]$$

$$= H(w^*) ,$$

where $(a)$ follows from Jensen’s inequality applied to the convex function $h_R (\cdot)$, as well as from the independence assumption, $(b)$ follows from the linearity of the inner product, $(c)$ follows from the assumption $\mathbb{E} [x] = 0$. Using $H(w) \geq H(w^*)$ together with Lemma 3.1, gives $\forall w \in \mathcal{W}$,

$$F(w) - F(w^*) = \frac{1}{1 - \alpha} (L_R(w) - L_R(w^*))$$

$$- \frac{\alpha}{1 - \alpha} (H(w) - H(w^*))$$

$$\leq \frac{1}{1 - \alpha} (L_R(w) - L_R(w^*)) .$$

Thus, Lemma 4.1 implies that by applying SGD to the expected Huber loss $L_R(\cdot)$ (while using the contaminated data), we can can obtain guarantees for the true expected prediction error. This is exactly what we do in Algorithm 1. Its guarantees are formalized in the next theorem,
Theorem 4.2. Let \( \bar{w} \) be the output of Algorithm 1 after observing \( T \) samples from Model 2.1. Then,

\[
\mathbb{E}[F(\bar{w})] - F(w^*) \leq \frac{RD}{(1 - \alpha)\sqrt{T}}
\]

Proof. Note that \( L_R(\cdot) \) is convex and \( W \) has a finite diameter \( D \). Also, note that for a given \( w_t \in W \) then \( g_t := \phi_R (\langle w_t, x_t \rangle - y_t) \cdot x_t \) is an unbiased estimate for the gradient of \( L_R(\cdot) \) at \( w_t \). Moreover, since \( \phi_R(\cdot) \) is bounded by \( R \) and the features are bounded, then the norm of \( g_t \) is bounded by \( R \). Thus, applying projected SGD as is done in Algorithm 1 ensures that (see e.g. Shalev-Shwartz (2012, Theorem 14.8)),

\[
\mathbb{E}[L_R(\bar{w})] - L_R(w^*) \leq \frac{RD}{\sqrt{T}}.
\]

Combining this with Lemma 4.1 concludes the proof.

5 The Strictly Positive Definite Feature Covariance Matrix Case

Here we assume that \( \Sigma \succeq \rho \cdot I \) for some known, strictly positive, \( \rho > 0 \). In contrast to the general case discussed in the previous section, here we show that it is possible to obtain guarantees even if \( \mathbb{E}[x] \neq 0 \). We also establish faster convergence rates compared to the general case.

We first consider the case in which the expectation of the features \( \mathbb{E}[x] \) is known to the learner, so it can perfectly center the features. We show that feeding these centered samples to an appropriate variant of SGD leads to an accurate estimation of \( w^* \) with an error rate of \( O \left( \frac{1}{(1 - \alpha)^2 T} \right) \) (Theorem 5.4).

We then show that the same estimation rate can be achieved even if the expectation of the features is unknown to the learner, but rather estimated from data. In Theorem 5.7 we show that this can be done at the same rate as in the known expectation case up to logarithmic factors in \( T \).

5.1 The Known Expectation Case

When the feature vector is not centered, the minimizer of \( L_R(\cdot) \) might be different from \( w^* \). A natural way to avoid it is to center the features, i.e. \( x_i - \mathbb{E}[x] \). In fact, Model 2.1 can be written with centered feature vectors as follows,

Model 5.1.

\[
y = \langle w^*, x - \mathbb{E}[x] \rangle + \langle w^*, \mathbb{E}[x] \rangle + \epsilon + b,\]

which is still a linear model albeit with two differences. First, the norm of the centered features might be larger than 1. To resolve this we recall that the radius parameter of the
Huber loss was set to $R = 6D + \sigma$, and so for any non-corrupted sample in Model 5.1
\[
\left| \langle w, x - \mathbb{E} [x] \rangle - y \right| \leq D \left( \| x \| + \| \mathbb{E} [x] \| \right) + |y| \\
\leq 2D + D + \sigma \leq R,
\]
where here, the first inequality follows from the triangle inequality and the second one follows from Assumption 2.5. Thus, our choice of $R$ ensures that for any $w \in \mathcal{W}$ and non-corrupted sample $(x_i, y_i)$ we have, $h_R \left( \langle w, x_i - \mathbb{E} [x] \rangle - y_i \right) = \frac{1}{2} \left( \langle w, x_i - \mathbb{E} [x] \rangle - y_i \right)^2$ with probability 1.

A second difference in Model 5.1, is that it has an additional unknown quantity, to wit $\langle w^*, \mathbb{E} [x] \rangle$. As we next show, this is inconsequential. With slight abuse of notation define,

\begin{equation}
L_R(w) = \mathbb{E}_{x, \epsilon, b} \left[ h_R \left( \langle w, x - \mathbb{E} [x] \rangle - y \right) \right].
\end{equation}

This expected Huber loss of centered features is also minimized by $w^*$:

**Lemma 5.2.** $L_R(w)$ is $(1 - \alpha)\rho$-strongly convex in $\mathcal{W}$ and $w^* = \arg\min_{w \in \mathcal{W}} L_R(w)$.

**Proof Sketch.** In a high level, we show that $L_R(\cdot)$ can be decomposed to a sum of a $(1 - \alpha)\rho$-strongly convex function and a convex function, and so it is a $(1 - \alpha)\rho$-strongly convex function. This decomposition is similar to the one in Lemma 3.1. The optimality of $w^*$ is also done similarly to Lemma 3.1. The full proof can be found in Section A, which establishes Lemma A.1, a more general version of this lemma.

**Huber Loss SGD with Centered Features (Algorithm 2):** Lemma 5.2 is instrumental in achieving fast convergence of the Huber loss, and in turn for the estimation error $\| \bar{w} - w^* \|^2$. As the lemma implies $L_R(\cdot)$ is strongly convex and maintains the same global optimum $w^*$ as the true expected loss $F(\cdot)$. Thus, it is natural to apply an appropriate version of SGD for strongly-convex functions to $L_R(\cdot)$ while using the contaminated data with centered features, in order to obtain guarantees to $\| \bar{w} - w^* \|^2$. This is exactly what we do in Algorithm 2. Concretely, Algorithm 2 utilizes a SGD with $\frac{1}{2}$-suffix averaging with the following guarantees:

**Lemma 5.3 (Rakhlin et al. (2012, Theorem 5)).** Consider SGD with $\frac{1}{2}$-suffix averaging and with step size $\eta_t := \frac{1}{\lambda t}$. Suppose $f$ is $\lambda$-strongly convex, and that $\mathbb{E} [\| g_t \|^2] \leq G^2$ for all $t$. Then for any $T$, it holds that $\forall w \in \mathcal{W}$

\[
\mathbb{E} [f(\bar{w})] - f(w) \leq \frac{9G^2}{\lambda T}.
\]

Now, by using Lemma 5.2 we can bound the estimation error of Algorithm 2 as follows,
Algorithm 2 Huber SGD for known expectation

Input: $R > 0, \lambda > 0, \mathcal{W} \subset \mathbb{R}^d, T \in \mathbb{N}_+, \mathbb{E}[x]$

$w_1 := 0$

for $t = 1$ to $T$ do
    Draw $(x_t, y_t)$ from (the contaminated) Model 2.1
    $\eta_t := 1/\lambda t$
    $g_t := \phi_R \left( \langle w_t, x_t - \mathbb{E}[x]\rangle - y_t \right) \cdot (x_t - \mathbb{E}[x])$
    $w_{t+1} := \Pi_{\mathcal{W}} (w_t - \eta_t g_t)$
end for

Return: $\bar{w} := \frac{2}{T} \sum_{t=1+T/2}^T w_t$

Theorem 5.4. Let $\bar{w}$ be the output of Algorithm 2 with input $\left( R, (1 - \alpha)\rho, \mathcal{W}, T, \mathbb{E}[x] \right)$, then,

$$\mathbb{E} \left[ \|\bar{w} - w^*\|^2 \right] \leq \frac{72R^2}{((1 - \alpha)\rho)^2 \cdot T}.$$

Proof. Note that the $g_t$ that we employ in Algorithm 2 is an unbiased gradient estimate for the Huber loss $\nabla L_R(w_t)$ (recall the definition in Equation (3)). Also, since $\phi_R(\cdot)$ is bounded by $R$ and the feature norms are bounded by 1, then $\|g_t\| \leq 2R$, $\forall t$. Moreover, Lemma 5.2 implies that $L_R(\cdot)$ is $(1 - \alpha)\rho$-strongly-convex with optimum $w^* \in \mathcal{W}$.

Now, since Algorithm 2 actually applies strongly-convex SGD with $\frac{1}{2}$-suffix averaging, then Lemma 5.3 with $G := 2R$ yields,

$$\mathbb{E} \left[ L_R(\bar{w}) \right] - L_R(w^*) \leq \frac{36R^2}{((1 - \alpha)\rho) \cdot T}.$$

Combining the above with the strong-convexity of $L_R(\cdot)$ while using Property 2.8 establishes the theorem. \qed

5.2 The Unknown Expectation Case

In this section we present Algorithm 3 which generalizes what we did in the previous section for to the case where $\mathbb{E}[x]$ is unknown. We assume that $2T$ samples are provided to the algorithm. Similarly to the previous section, the learning algorithm is based on centering the features before feeding them to an SGD with $\frac{1}{2}$-suffix averaging. The only difference is that in Algorithm 3 we center the features using the empirical mean based on first $T$ samples, $\mu := \frac{1}{T} \sum_{t=1}^T z_t$.

Similarly to the previous section, with this definition, Model 2.1 can be written as follows,

Model 5.5. $y = \langle w^*, x - \mu \rangle + \langle w^*, \mu \rangle + \epsilon + b$.

Our proposed Algorithm 3 is based on sample splitting, and has two phases: In the first phase, $\mu$ is computed using $T$ samples, and in the second phase, $T$ steps of SGD are
Algorithm 3 Huber SGD for unknown expectation

**Input:** $R > 0, \lambda > 0, W \subset \mathbb{R}^d, T \in \mathbb{N}_+$

**Phase 1:** Compute $\mu$

Draw $T$ samples $\{(z_i, y_i)\}_{i=1}^T$ from (the contaminated) Model 2.1, and estimate the mean, $\mu := \frac{1}{T} \sum_{i=1}^T z_i$

**Phase 2:** SGD with $\frac{1}{2}$-suffix averaging given $\mu$

$w_1 := 0$

for $t = 1$ to $T$

Draw $(x_t, y_t)$ from (the contaminated) Model 2.1

$\eta_t := \frac{1}{\lambda t}$

$g_t := \phi_R((w_t, x_t - \mu) - y_t) \cdot (x_t - \mu)$

$w_{t+1} := \Pi_{W}(w_t - \eta_t g_t)$

end for

Return: $\bar{w} := \frac{2}{T} \sum_{t=1+\tau/2}^T w_t$

Algorithm 3 performed on fresh samples that are centered using $\mu$. Since the samples used in the SGD algorithm are independent of the estimator $\mu$ (which is a function of different samples), the analysis of the SGD algorithm can be made conditionally on $\mu$, without affecting the distribution of the samples.

With slight abuse of notation define

$$L_R(w) = \mathbb{E}_{x, \epsilon, b} \left[ h_R((w - w^*, x - \mu) - (w^*, \mu) - \epsilon - b) \right].$$

**Lemma 5.6.** $\forall \mu \in \mathbb{R}^d : L_R(w)$ is $(1 - \alpha)\rho$-strongly convex in $W$.

**Proof.** Because $\mu$ is given, the proof is immediate from the more general Lemma A.1 with $v = q = \mu$. \qed

Similarly to the case of known expectation, Algorithm 3 applies a strongly-convex variant of SGD to the above defined $L_R(\cdot)$. Unfortunately, In contrast to the case of known expectation, the optimum of $L_R(\cdot)$ in $W$ is not necessarily $w^*$. Nevertheless, we are still able to establish a fast convergence rate for Algorithm 3 as can be seen below,

**Theorem 5.7.** Let $\bar{w}$ be the output of Algorithm 3 with input $(R, (1 - \alpha)\rho, W, T)$, then,

$$\mathbb{E} \left[ \|\bar{w} - w^*\|^2 \right] \leq \frac{72R^2 \cdot (2 \log T + 1)}{((1 - \alpha)\rho)^2 \cdot T}.$$

**Analysis.** Prior to proving Theorem 5.7 we need to introduce some definitions and establish some auxiliary lemmas.

**Lemma 5.8.** $h_R(\cdot)$ is $2R$-Lipschitz with probability 1.
Proof. For any \( w \in \mathcal{W} \)
\[
\left\| \nabla h_R \left( \left \langle w - w^*, x - \mathbb{E} [x] \right \rangle - \langle w^*, \mu \rangle - \epsilon - b \right) \right\| \\
= \left\| \phi_R \left( \left \langle w - w^*, x - \mathbb{E} [x] \right \rangle - \langle w^*, \mu \rangle - \epsilon - b \right) \cdot \left( x - \mathbb{E} [x] \right) \right\| \\
\leq 2R ,
\]
where the equality follows from definition and the inequality follows from the definition of \( \phi_R (\cdot) \) and Assumption 2.5 with probability 1. \(\square\)

The following is a version of Hoeffding’s inequality for random vectors,

**Lemma 5.9** (Kakade (2010, Theorem 2.1)). \(\text{Assume that } \{x_i \in \mathbb{R}^d\}_{i=1}^T \text{ are random variables sampled i.i.d and } \|x_i\| \leq K \text{ almost surely. Then with probability greater than } 1 - \delta,\)
\[
\left\| \frac{1}{T} \sum_{i=1}^T x_i - \mathbb{E} [x] \right\| \leq 6K \sqrt{\frac{\log \frac{1}{\delta}}{T}} .
\]

The main challenge is that now \( w^* \) is not necessarily the optimum of \( L_R (\cdot) \) in \( \mathcal{W} \). To circumvent this, we will analyze an auxiliary expected loss function which we define below,
\[
\tilde{L}_R (w) := \mathbb{E}_{x,\tilde{v},\tilde{b}} \left[ h_R \left( \left \langle w - w^*, x - \mathbb{E} [x] \right \rangle - \langle w^*, \mu \rangle - \epsilon - b \right) \right] .
\]
This function is not the one minimized by the algorithm, however, it has the following desirable property:

**Lemma 5.10.** \(\forall \mu \in \mathbb{R}^d : \tilde{L}_R (w) \text{ is } (1-\alpha)\rho \text{-strongly convex in } \mathcal{W} \text{ and } w^* = \arg \min_{w \in \mathcal{W}} \tilde{L}_R (w).\)

**Proof.** Because \( \mu \) is given, the proof is immediate from the more general Lemma A.1 with \( v = \mathbb{E} [x] \) and \( q = \mu). \(\square\)

Now, we can use Property 2.8 to bound the estimation error: for a given \( \mu, \)
\[
\frac{(1-\alpha)\rho}{2} \| \bar{w} - w^* \|^2 \leq \tilde{L}_R (\bar{w}) - \tilde{L}_R (w^*) \\
= \bar{L}_R (\bar{w}) - L_R (\bar{w}) + L_R (\bar{w}) - L_R (w^*) + L_R (w^*) - \tilde{L}_R (w^*) ,
\]
where the last term equals zero by the definitions of \( L_R (w^*) \) and \( \tilde{L}_R (w^*) \). So, By taking expectation we obtain,
\[
C \cdot \mathbb{E} [\| \bar{w} - w^* \|^2] \leq \mathbb{E} \left[ \tilde{L}_R (\bar{w}) - L_R (\bar{w}) \right] + \mathbb{E} \left[ L_R (\bar{w}) - L_R (w^*) \right] , \tag{4}
\]
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where $C = \left(\frac{1-\alpha}{2}\right)\rho$. We now bound each of these terms.

For the second term, since Algorithm 3 applies SGD to the strongly-convex function $L_R(\cdot)$, then the expectation of $L_R(\bar{w}) - L_R(w^*)$ is upper bounded by $O\left(\frac{1}{T}\right)$ due to Lemma 5.3, with the parameter choice $F = L_R, G = R, \lambda = (1 - \alpha) \rho$.

The first term above can be upper bounded as follows,

**Lemma 5.11.** Let $C = \frac{(1-\alpha)\rho}{2}$. Then for a given $\mu$,

$$
\mathbb{E} \left[ \tilde{L}_R(\bar{w}) - L_R(\bar{w}) \right] \leq \frac{C}{2} \cdot \mathbb{E} \left[ \|\bar{w} - w^*\|^2 \right] + \frac{2R^2}{C} \cdot \frac{36 \log T + 4}{T}
$$

**Proof.** Take $C = \frac{(1-\alpha)\rho}{2}$. Then,

$$
\tilde{L}_R(\bar{w}) - L_R(\bar{w}) =
\begin{align*}
& \overset{(a)}{=} \mathbb{E}_{x,\epsilon}[h_R \left( \langle \bar{w} - w^*, x - \mathbb{E}[x] \rangle - \langle w^*, \mu \rangle - \epsilon - b \right) - h_R \left( \langle \bar{w} - w^*, x - \mu \rangle - \langle w^*, \mu \rangle - \epsilon - b \right) ] \\
& \overset{(b)}{\leq} 2R \cdot \| \bar{w} - w^*, \mu - \mathbb{E}[x] \| \\
& \overset{(c)}{\leq} C \cdot \frac{2R}{C} \cdot \| \bar{w} - w^* \| \cdot \| \mu - \mathbb{E}[x] \| \\
& \overset{(d)}{\leq} \frac{C}{2} \| \bar{w} - w^* \|^2 + \frac{2R^2}{C} \| \mu - \mathbb{E}[x] \|^2,
\end{align*}
$$

where (a) follows from the definitions of $\tilde{L}_R(\cdot)$ and $L_R(\cdot)$, (b) follows from Lemma 5.8, (c) follows from the Cauchy-Schwarz’s inequality and (d) follows from Young’s inequality: $a \cdot b \leq \frac{1}{2}(a^2 + b^2)$ where $a = \|\bar{w} - w^*\|$ and $b = \frac{2R}{C} \cdot \| \mu - \mathbb{E}[x] \|$.

Note that $\bar{w}$ and $\mu$ are random as they are that depend on $(z_i, y_i)_{i\in[1,2,\ldots,T]}$ and $(x_t, y_t)_{t\in[1,2,\ldots,T]}$.

By taking an expectation with respect to the $2T$ samples on both sides we have

$$
\mathbb{E} \left[ \tilde{L}_R(\bar{w}) - L_R(\bar{w}) \right] \leq \mathbb{E} \left[ \frac{C}{2} \| \bar{w} - w^* \|^2 + \frac{2R^2}{C} \| \mu - \mathbb{E}[x] \|^2 \right].
$$

We conclude the proof by bounding $\mathbb{E} \left[ \| \mu - \mathbb{E}[x] \|^2 \right]$. We make use of Lemma 5.9 by defining $B$ as the event for which $\| \mu - \mathbb{E}[x] \| \leq 6\sqrt{\frac{\log 1/\delta}{T}}$. Then,

$$
\begin{align*}
\mathbb{E} \left[ \| \mu - \mathbb{E}[x] \|^2 \right] & \overset{(a)}{=} \mathbb{E}_{z_1, z_2, \ldots, z_T} \left[ \| \mu - \mathbb{E}[x] \|^2 \right] \\
& \overset{(b)}{\leq} \mathbb{P}(B) \cdot \mathbb{E}_{z_1, z_2, \ldots, z_T} \left[ \| \mu - \mathbb{E}[x] \|^2 \right] B + \mathbb{P}(B^c) \cdot \mathbb{E}_{z_1, z_2, \ldots, z_T} \left[ \| \mu - \mathbb{E}[x] \|^2 \right] B^c \\
& \overset{(c)}{\leq} (1 - \delta) \cdot \frac{36 \log 1/\delta}{T} + \delta \cdot 4 \\
& \overset{(d)}{\leq} \frac{36 \log T + 4}{T},
\end{align*}
$$

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where (a) follows from the i.i.d assumption on the features, (b) follows from the law of total expectation, (c) follows from the definition of $B$ and Lemma 5.9 (which is true for every $\delta \in (0, 1)$) and a naive upper bound of $\|\mu - \mathbb{E}[x]\| \leq 2$ with probability 1 (which follows from Assumption 2.5) and (d) follows from taking $\delta = \frac{1}{T}$. Plugging the above into Equation (5) concludes the proof.

We are now ready to prove Theorem 5.7.

Proof of Theorem 5.7. By plugging Lemma 5.3 and Lemma 5.11 back into Equation (4), we obtain,

$$C \cdot \mathbb{E}[\|\bar{w} - w^*\|^2] \leq \frac{9R^2}{(1-\alpha)\rho T} + \frac{C}{2} \cdot \mathbb{E}[\|\bar{w} - w^*\|^2] + \frac{2R^2}{C} \cdot \frac{36 \log T + 4}{T}.$$ 

Recalling $C = \frac{(1-\alpha)\rho}{2}$, the above implies,

$$\mathbb{E}[\|\bar{w} - w^*\|^2] \leq \frac{72R^2 \cdot (2 \log T + 1)}{((1-\alpha)\rho)^2 \cdot T},$$

which establishes the theorem.

6 Extensions

Here we present two extensions for the case where $\Sigma \succeq \rho \cdot I$: In Section 6.1 we go beyond the assumptions of bounded features & noise, and extend our results to the sub-Gaussian case. In Section 6.2, we point out to an algorithm that does not require knowledge of the fraction of contaminated samples $\alpha$, but rather implicitly adapts to it. This improves over Algorithms 2 & 3 which require $\alpha$.

6.1 Sub-Gaussian noise and feature vectors

We next relax the assumptions that the noise and the norm of the feature vectors are bounded with probability 1, and replace them with the following sub-Gaussian assumptions.

Assumption 6.1 (Sub-Gaussian Feature Vector). The feature vector $x$ is sub-Gaussian with variance proxy $\kappa^2$, that is $\mathbb{P}(\|x\| > u) \leq 2e^{-\frac{u^2}{2\kappa^2}}$ for all $u \in \mathbb{R}$.

Assumption 6.2 (Sub-Gaussian Noise). The noise $\epsilon$ is sub-Gaussian with variance proxy $\sigma^2$, that is $\mathbb{P}(|\epsilon| > u) \leq 2e^{-\frac{u^2}{2\sigma^2}}$ for all $u \in \mathbb{R}$.

Assumption 6.1 replaces Assumption 2.5 and Assumption 6.2 replaces Assumption 2.4. We show the following,
Theorem 6.3. Under Assumptions 6.1 & 6.2, define, $R = C \cdot (\kappa D + \sigma) \sqrt{\log T}$, where $C > 0$ is an explicit constant, which depends logarithmically on $\kappa, \rho, \sigma$. Denote $\bar{w}$ as the output of Algorithm 3 with input $(R, (1 - \alpha)\rho, \mathcal{W}, T)$, then,

$$
\mathbb{E} \left[ \|\bar{w} - w^*\|^2 \right] \leq \left( \frac{D^2}{14} + \frac{288R^2 \cdot (2\log T + 1)}{((1 - \alpha)\rho)^2} \right) \cdot \frac{1}{T}.
$$

Proof sketch. We denote by $\mathcal{F}^c$ the event in which both the features and the noise values are bounded by some appropriately chosen constants $u_1, u_2$, that is

$$
\mathcal{F}^c := \cap_{i \in [1,2,\ldots,T]} \|x_i\| \leq u_1 \cap |\epsilon_i| \leq u_2.
$$

The constants $u_1, u_2$ are chosen such that $\mathbb{P}(\mathcal{F}) \leq \delta := \min \left\{ \frac{1}{2}, \frac{\rho^2}{105\kappa^2 \cdot \frac{1}{T^2}} \right\}$. Then, by the law of total expectation to decompose the expected estimation error:

$$
\mathbb{E} \left[ \|\bar{w} - w^*\|^2 \right] = \mathbb{P}(\mathcal{F}) \cdot \mathbb{E} \left[ \|\bar{w} - w^*\|^2 \Big | \mathcal{F} \right] + \mathbb{P}(\mathcal{F}^c) \cdot \mathbb{E} \left[ \|\bar{w} - w^*\|^2 \Big | \mathcal{F}^c \right],
$$

The first term (conditioned on $\mathcal{F}$) is bounded by Cauchy-Schwarz’s inequality and a naive bound of $\|\bar{w} - w^*\| \leq 2D$. The second term, which conditions on $\mathcal{F}^c$, and thus implies that the feature vector and the noise are bounded, can in principle be bounded using the analysis in the bounded setting. The main technical part is that it need to be established that the covariance matrix of the feature vector is still strictly convex, even after conditioning on the event $\mathcal{F}^c$. To this end, Lemma B.3 shows that for any event $\mathcal{G}$ such that $\mathbb{P}(\mathcal{G}) \leq \delta$ for some $\delta \in (0, \frac{1}{2})$ it holds that

$$
\tilde{\Sigma} := \mathbb{E} \left[ (x - \mathbb{E}[x|\mathcal{G}^c]) (x - \mathbb{E}[x|\mathcal{G}^c])^T \Big | \mathcal{G}^c \right] \geq \left( \rho - 105\kappa^2 \sqrt{\delta} \right) \cdot I.
$$

By choosing a sufficiently small $\delta$ we assure that $\tilde{\Sigma} \succeq \frac{\rho}{2} \cdot I$. Then, conditioned on $\mathcal{F}^c$, and with a sufficiently large radius $R$ for the Huber loss, the analysis of Section 5.2 is in tact, and the convergence rate that was stated in Theorem 5.7 holds for this conditional expectation. Summing the bounds of both terms then results with the bound on the squared error. The full proof can be found in Theorem B.1.

6.2 Adaptivity to Contamination Fraction $\alpha$

In Section 5 we have assumed that $\alpha$ is known in order to find a good estimation of $w^*$. Our derivation shows that the expected Huber loss $L_R(\cdot)$ is $(1 - \alpha)\rho$-strongly-convex, and we
encode this information into the learning rate of SGD with $\frac{1}{T}$-suffix averaging that we employ as a part of Algorithms 2 & 3, as these algorithms require the strong-convexity parameter in order to ensure fast convergence.

In practice, it is unrealistic to assume that the fraction of contamination $\alpha$ is known. Fortunately, Cutkosky and Orabona (2018) have recently presented a novel and practical first order algorithm for stochastic convex optimization that enables to implicitly adapt to the strong-convexity of the problem at hand. So, if we apply this algorithm instead of SGD with $\frac{1}{T}$-suffix averaging, then we immediately obtain adaptivity to both $\alpha$ and $\rho$. The full description of the algorithm appears in Section 6 of Cutkosky and Orabona (2018).

7 Experiments

We perform 2 experiments under the assumptions of Model 5.5 in $d = 5$ dimensions. The optimum $w^*$ is sampled from the unit ball and is the same for both experiments. We also let $\epsilon \sim \text{Uniform}[-0.1, 0.1]$, and $x \sim \text{Uniform}[-1/\sqrt{d}, 0]^d$ such that $\|x\| \leq 1$ w.p. 1.1 The adversary picks $b = 10^5$ with probability $\alpha$, and 0 with probability $1 - \alpha$. We test two cases $\alpha \in \{0.01, 0.7\}$.

We compare 3 algorithms: (i) Projected SGD over $\ell_2$ loss, (ii) Non-Centered Huber, which is Huber SGD without centering, and (iii) Centered Huber (Algorithm 3). All methods use the same learning rate $\eta_t := \eta_0/t$ where $\eta_0 = 1/(1 - \alpha) \rho$. Also all methods use the same number of samples. In Centered Huber we compute the estimate of feature means on the fly rather than using extra samples (i.e., we use $\mu_t := \frac{1}{t} \sum_{i=1}^{t-1} x_i$ instead of $\mu = \frac{1}{T} \sum_{t=1}^{T} z_t$). We repeat each experiment 15 times and add confidence intervals. The experiments, shown in Figure 1, clearly demonstrate the benefit of our approach compared to the baselines.
8 Conclusion

In this paper, we have analyzed robust linear regression under general assumptions on the feature vectors, noise and the oblivious adversary. We have shown that low prediction error requires either centered features or strict positivity of the features vector covariance matrix, and provided efficient SGD-style algorithms and established error rate convergence bounds. Finally, we have provided SGD variants that do not require prior knowledge on the fraction of contamination. While the linear regression model is both basic and important, an important avenue for future work is to generalize the algorithms and the error bounds to more elaborated ML models.

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A Generalization of Lemma 5.2, Lemma 5.6 and Lemma 5.10

Lemma A.1. Given $v, q \in \mathbb{R}^d$ such that $\|v\|, \|q\| \leq 1$ and $R = 6D + \sigma$, let

$$G_R(w) := \mathbb{E}_{x,\epsilon,b} [h_R (\langle w - w^*, x - v \rangle - \langle w^*, q \rangle - \epsilon, b)].$$

Then, $G_R$ is $(1 - \alpha)\rho$-strongly convex. Furthermore, if $v = \mathbb{E} [x]$ we have $w^* = \arg\min_{w \in \mathcal{W}} G_R(w)$.

Proof. We show that $G_R(w)$ is a sum of a $(1 - \alpha)\rho$-strongly function and a convex function and as such it is a $(1 - \alpha)\rho$-strongly convex.

Define

$$H(w) := \mathbb{E}_{x,\epsilon,b} [h_R (\langle w - w^*, x - v \rangle - \langle w^*, q \rangle - \epsilon - b) \mid b \neq 0],$$

which is convex as an average of convex functions. Also define

$$F(w) := \mathbb{E}_{x,\epsilon,b} [h_R (\langle w - w^*, x - v \rangle - \langle w^*, q \rangle - \epsilon - b) \mid b = 0]
\overset{(a)}{=} \mathbb{E}_{x,\epsilon,b} \left[ \frac{1}{2} (\langle w - w^*, x - v \rangle - \langle w^*, q \rangle - \epsilon)^2 \right] \mid b = 0
\overset{(b)}{=} \mathbb{E}_{x,\epsilon} \left[ \frac{1}{2} (\langle w - w^*, x - v \rangle - \langle w^*, q \rangle - \epsilon)^2 \right]
= \mathbb{E}_{x,\epsilon} \left[ \frac{1}{2} \langle w - w^*, x - v \rangle^2 \right]
+ \mathbb{E}_{x,\epsilon} [\epsilon \cdot \langle w - w^*, x - v \rangle + \langle w^*, q \rangle \cdot \langle w - w^*, x - v \rangle] + S,$$

where $S$ is a constant independent of $w$. $(a)$ follows from the boundness assumptions on $x, v, q, \epsilon$ and $R = 6D + \sigma$ and $(b)$ follows from the assumption that $x, \epsilon, b$ are statistically independent.

$F(w)$ is a polynomial and hence twice continuously differentiable. We take the second derivative and show it is positive definite,

$$\nabla^2 F(w) = \mathbb{E}_{x} [(x - v)(x - v)^T]
= \mathbb{E}_{x} [(x - \mathbb{E} [x] + \mathbb{E} [x] - v)(x - \mathbb{E} [x] + \mathbb{E} [x] - v)^T]
= \mathbb{E}_{x} [(x - \mathbb{E} [x])(x - \mathbb{E} [x])^T + (\mathbb{E} [x] - v)(\mathbb{E} [x] - v)^T]
+ \mathbb{E}_{x} [(x - \mathbb{E} [x])(\mathbb{E} [x] - v)^T + (\mathbb{E} [x] - v)(x - \mathbb{E} [x])^T]
= \Sigma + (\mathbb{E} [x] - v)(\mathbb{E} [x] - v)^T
\geq \Sigma$$
last equality follows from the definition of Σ and the fact that v is some given vector. First inequality follows from the fact that \((\mathbb{E}[x] - \mu)(\mathbb{E}[x] - \mu)^T\) is positive semi-definite by definition.

So, Assumption 2.5 with \(\rho > 0\) and Property 2.7 assure that \(F\) is a \(\rho\)-strongly convex function. Then, by the law of total expectation with respect to \(b\):

\[
G_R(w) = (1 - \alpha)F(w) + \alpha H(w).
\]

Since \((1 - \alpha)F(w)\) is \((1 - \alpha)\rho\)-strongly convex, so is \(G_R(w)\).

If \(v = \mathbb{E}[x]\) we can also show that \(w^* = \arg\min_{w \in W} G_R(w)\). For any \(w \in W\)

\[
G_R(w) := \mathbb{E}_{x,\epsilon,b} \left[ h_R \left( \langle w - w^*, x - \mathbb{E}[x] \rangle - \langle w^*, q \rangle - \epsilon - b \right) \right]
\]

\[
\geq (a) \mathbb{E}_{\epsilon,b} \left[ h_R \left( \mathbb{E}_x \left[ \langle w - w^*, x - \mathbb{E}[x] \rangle - \langle w^*, q \rangle - \epsilon - b \right] \right) \right]
\]

\[
= (b) \mathbb{E}_{\epsilon,b} \left[ h_R \left( \langle w - w^*, \mathbb{E}_x [x - \mathbb{E}[x]] \rangle - \langle w^*, q \rangle - \epsilon - b \right) \right]
\]

\[
= \mathbb{E}_{\epsilon,b} \left[ h_R \left( \langle w^* - w^*, x - \mathbb{E}[x] \rangle - \langle w^*, q \rangle - \epsilon - b \right) \right]
\]

\[
= G_R(w^*),
\]

where \((a)\) follows from convexity of the Huber loss and Jensen’s inequality and \((b)\) follows from the linearity of the inner product. Moreover, \(w^*\) is the unique minimizer of \(G_R(w)\) in \(W\). This is because according to Property 2.8, if \(G_R(w) = G_R(w^*)\) for some \(w \in W\) then

\[
0 = G_R(w^*) - G_R(w) \geq \frac{(1 - \alpha)\rho}{2} \|w - w^*\|^2.
\]

Since \(\frac{(1 - \alpha)\rho}{2} > 0\) is assumed this implies \(w_{opt} = w^*\). ☐

24
B Proofs for Section 6.1

Remark B.1. We have stated the sub-Gaussian assumptions (Assumption 6.1 and Assumption 6.2) in terms of the tails of the probability density functions. We refer the reader to Vershynin (2018, Chapter 2), for equivalent definitions of sub-Gaussian variables in terms of the moment generating function or in terms of integer moments (all these definitions are essentially equivalent).

We will use the following bounds on the moments of sub-Gaussian random variables.

Lemma B.2. Let $z$ be a sub-Gaussian random variable with variance proxy $\lambda^2$, that is $\mathbb{P} (|z| > u) \leq 2e^{-\frac{u^2}{2\lambda^2}}$ for all $u \in \mathbb{R}$. Then, for any $p \geq 1$

$$
\mathbb{E} [|z|^p] \leq \sqrt{2\pi \lambda^2} \cdot p \cdot \lambda^p \frac{2^{p/2} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}}
$$

where $\Gamma(\cdot)$ is the Gamma function. Specifically, $\mathbb{E} [|z|] \leq \sqrt{2\pi \lambda}$, $\mathbb{E} [|z|^2] \leq 4\lambda^2$, and $\mathbb{E} [|z|^4] \leq 24\lambda^4$.

Proof. Let $n \sim \mathcal{N}(0, \lambda^2)$. Then, it holds that

$$
\mathbb{E} [|z|^p] \overset{(a)}{=} \int_0^\infty pu^{p-1} \mathbb{P} (|z| \geq u) \, du \\
\overset{(b)}{\leq} \int_0^\infty |u|^{p-1} \mathbb{P} (|z| > u) \, du \\
= \sqrt{2\pi \lambda^2} \cdot p \cdot \mathbb{E} [|z|^{p-1}] \\
= \sqrt{2\pi \lambda^2} \cdot p \cdot \lambda^p \frac{2^{p/2} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}},
$$

where $(a)$ follows from the tail representation of the absolute moments $p \in (0, \infty)$ of a non-negative random variable $z$ (Vershynin, 2018, Exercise 1.2.3), $(b)$ follows from the sub-Gaussian assumption, $(c)$ follows from the known formula of the central absolute moments of the Gaussian distribution.

Lemma B.3. Let $\mathcal{G}$ be an event such that $\mathbb{P} (\mathcal{G}) \leq \delta$ for some $\delta \in (0, \frac{1}{2})$. Then,

$$
\breve{\Sigma} := \mathbb{E} \left[ \left( x - \mathbb{E} \left[ x \middle| \mathcal{G}^c \right] \right) \left( x - \mathbb{E} \left[ x \middle| \mathcal{G}^c \right] \right)^T \middle| \mathcal{G}^c \right] \succeq \left( \rho - 105\kappa^2 \sqrt{\delta} \right) \cdot I.
$$

Proof. Let $\breve{\Sigma}$ be the conditional covariance matrix of the features. We first relate it to the
unconditional covariance matrix by the decomposition

\[
\Sigma = \mathbb{E} \left[ (x x^T) \bigg| G^c \right] - \mathbb{E} \left[ x \big| G^c \right] \mathbb{E} \left[ x^T \big| G^c \right] \\
= (a) \frac{\mathbb{E} \left[ x x^T \right] - \mathbb{E} \left[ x x^T \mathbb{I}(G) \right]}{\mathbb{P}(G^c)} - \mathbb{E} \left[ x \big| G^c \right] \mathbb{E} \left[ x^T \big| G^c \right] \\
= (b) \Sigma + \left( \frac{1}{\mathbb{P}(G^c)} - 1 \right) \frac{\mathbb{E} \left[ x x^T \right]}{:=G_1} + \mathbb{E} \left[ x \big| G^c \right] \mathbb{E} \left[ x^T \big| G^c \right] - \mathbb{E} \left[ x x^T \cdot \mathbb{I}(G) \right]\frac{\mathbb{P}(G^c)}{:=G_3} - \mathbb{E} \left[ x \big| G^c \right] \mathbb{E} \left[ x^T \big| G^c \right], \tag{6}
\]

where (a) follows from the law of total expectation (b) follows from the definition of the unconditional covariance matrix of the features \( \Sigma := \mathbb{E} \left[ x x^T \right] - \mathbb{E} \left[ x \right] \mathbb{E} \left[ x^T \right] \). For the matrix \( G_1 \) in the last display, the assumptions \( \Sigma \geq \rho \cdot \mathbb{I} \) and \( \delta \leq \frac{1}{2} \) imply that

\[
G_1 \succeq 0. \tag{7}
\]

We next bound the maximal value of \( |v^T G_2 v| \) and \( |v^T G_3 v| \) over all unit vectors \( v \in \mathbb{R}^d \) (with \( ||v|| = 1 \)). For \( G_2 \), we further decompose to

\[
G_2 = \mathbb{E} \left[ x \big| G^c \right] \mathbb{E} \left[ x^T \big| G^c \right] - \mathbb{E} \left[ x \big| G^c \right] \mathbb{E} \left[ x^T \big| G^c \right] \\
= (a) \mathbb{E} \left[ x \right] \left( \mathbb{E} \left[ x^T \right] - \mathbb{E} \left[ x^T \big| G^c \right] \right) + \left( \mathbb{E} \left[ x \right] - \mathbb{E} \left[ x \big| G^c \right] \right) \mathbb{E} \left[ x^T \big| G^c \right],
\]

where (a) follows by adding and subtracting the common term \( \mathbb{E} \left[ x \right] \mathbb{E} \left[ x^T \big| G^c \right] \). Now, for any \( v \in \mathbb{R}^d \) with \( ||v|| = 1 \), it holds that

\[
|v^T G_{2,1} v| = \left| v^T \mathbb{E} \left[ x \right] \left( \mathbb{E} \left[ x^T \right] - \mathbb{E} \left[ x^T \big| G^c \right] \right) v \right| \\
\leq (a) \left\| \mathbb{E} \left[ x \right] \right\| \cdot \left\| \mathbb{E} \left[ x \right] - \mathbb{E} \left[ x \big| G^c \right] \right\| \\
\leq (b) \mathbb{E} \left[ x \right] \cdot \frac{\left\| (\mathbb{P}(G^c) - 1) \mathbb{E} \left[ x \right] - \mathbb{E} \left[ x \cdot \mathbb{I}(G) \right] \right\|}{\mathbb{P}(G^c)} \\
\leq (c) \mathbb{E} \left[ ||x|| \right] \cdot \frac{\left\| (\mathbb{P}(G^c) - 1) \mathbb{E} \left[ ||x|| \right] - \mathbb{E} \left[ ||x|| \cdot \mathbb{I}(G) \right] \right\|}{\mathbb{P}(G^c)} \\
\leq (d) \mathbb{E} \left[ ||x|| \right] \cdot \frac{(1 - \mathbb{P}(G^c)) \mathbb{E} \left[ ||x|| \right] + \mathbb{E} \left[ ||x|| \cdot \mathbb{I}(G) \right]}{\mathbb{P}(G^c)} \\
\leq (e) \mathbb{E} \left[ ||x|| \right] \cdot \frac{(1 - \mathbb{P}(G^c)) \mathbb{E} \left[ ||x|| \right] + \sqrt{\mathbb{E} \left[ ||x||^2 \right]} \cdot \mathbb{P}(G^c)}{\mathbb{P}(G^c)} \\
\leq (f) 18 \kappa^2 \cdot \sqrt{\delta}.
\]
where (a) follows from Cauchy-Schwarz’s inequality, (b) follows from the law of total expectation, (c) follows from Jensen’s inequality, (d) follows from the triangle inequality and Jensen’s inequality, (e) follows from Cauchy-Schwarz’s inequality, (f) follows from the assumptions that $x$ is sub-Gaussian with variance parameter $\kappa^2$ and Lemma B.2, with the assumption $\mathbb{P}(G) = \delta \leq \frac{1}{2}$ (and as $\delta < \sqrt{\delta}$).

For $G_{2,2}$ we use a similar bounding method, except that now the bound on $\left\| \mathbb{E}[x] \right\|$ is replaced by a bound on $\left\| \mathbb{E}[x | G^c] \right\|$ (in Equation (8)). This conditional expectation can be bounded as follows:

$$\left\| \mathbb{E}[x | G^c] \right\| \leq \frac{\left\| \mathbb{E}[x] \right\| + \mathbb{E}[\left\| x \right\| \cdot \mathbb{P}(G)]}{\mathbb{P}(G^c)} \leq \sqrt{8\pi \kappa} + 4\kappa \sqrt{\delta}$$

where (a) follows from the law of total expectation, (b) follows by similar steps in the analysis of $G_{2,1}$, using the triangle, Jensen’s and Cauchy-Schwarz’s inequalities, and (c) follows from Lemma B.2 and the assumptions. With this bound, as in the analysis of $G_{2,1}$, it holds for any $v \in \mathbb{R}^d$ with $\left\| v \right\| = 1$ that

$$\left| v^T G_{2,2} v \right| \leq 80\kappa^2 \sqrt{\delta} ,$$

using $\delta \sqrt{\delta} \leq \delta \leq \sqrt{\delta}$.

From the bounds on $\left| v^T G_{2,1} v \right|$ and $\left| v^T G_{2,2} v \right|$ we deduce that

$$\left| v^T G_2 v \right| = \left| v^T (G_{2,1} + G_{2,2}) v \right| \leq \left| v^T G_{2,1} v \right| + \left| v^T G_{2,2} v \right| \leq 98\kappa^2 \sqrt{\delta} .$$

For $G_3$ it holds for any $v \in \mathbb{R}^d$ with $\left\| v \right\| = 1$ that

$$\left| v^T G_3 v \right| = \frac{\left\| \mathbb{E}[v^T xx^T v \cdot I(G)] \right\|}{\mathbb{P}(G^c)} \leq \frac{\mathbb{E}[\left\| x \right\|^2 \cdot I(G)]}{\mathbb{P}(G^c)} \leq \frac{\sqrt{\mathbb{E}[\left\| x \right\|^4] \cdot \mathbb{P}(G)}}{\mathbb{P}(G^c)} \leq 7\kappa^2 \sqrt{\delta} ,$$

where (a) follows from Cauchy-Schwarz’s inequality in $\mathbb{R}^d$, (b) follows from Cauchy-Schwarz’s inequality in $L_2$, and (c) follows from Lemma B.2 and the assumptions. Using the decomposition of $\tilde{\Sigma}$ in Equation (6) and the bounds on $G_1, G_2, G_3$ in Equation (7), Equation (9)
and Equation (10), respectively, it holds for any $v \in \mathbb{R}^d$ with $\|v\| = 1$ that
\[
v^T \Sigma v \geq \rho + 98 \kappa^2 \sqrt{\delta} - 7 \kappa^2 \sqrt{\delta} ,
\]
which directly implies to the stated claim. \hfill \Box

### B.1 Proof for Theorem 6.3

**Theorem B.4.** Given Assumption 6.1 and Assumption 6.2. Define
\[
R := 2 \sqrt{8 \kappa^2 \log \left( \frac{21 \kappa}{\sqrt{\rho}} + 8 \right) \cdot T} \cdot D + \sqrt{8 \sigma^2 \log \left( \frac{21 \sigma}{\sqrt{\rho}} + 8 \right) \cdot T} .
\]
Then,
\[
\mathbb{E} [\| \bar{w} - w^* \|^2] \leq \left( \frac{D^2}{14} + \frac{288 R^2 \cdot (2 \log T + 1)}{(1 - \alpha) \rho^2} \right) \cdot \frac{1}{T} .
\]

**Proof.** Let a time $T$ be given, and consider the events
\[
\mathcal{F}_x(u) := \bigcup_{i \in [1,2,\ldots,T]} \{ \| x_i \| > u \} ,
\]
and
\[
\mathcal{F}_\epsilon(u) := \bigcup_{i \in [1,2,\ldots,T]} \{ | \epsilon_i | > u \} .
\]
Further let $u_1 := \sqrt{2 \kappa^2 \log(\frac{4T}{\delta})}$ and $u_2 := \sqrt{2 \sigma^2 \log(\frac{4T}{\delta})}$ and set
\[
\mathcal{F} := \mathcal{F}_\epsilon(u_1) \cup \mathcal{F}_x(u_2).
\]
By a union bound over $i \in [1,2,\ldots,T]$ and computing probabilities over $\epsilon$ and $x$, the sub-Gaussian assumptions implies that $\mathbb{P}(\mathcal{F}) \leq \delta$. We choose $\delta = \min \left\{ \frac{1}{2}, \frac{\rho^2}{210 \sqrt{\kappa^2}} \cdot \frac{1}{T^2} \right\}$. Note, that with this choice of $\delta$, and by identifying $\mathcal{G} = \mathcal{F}$, Lemma B.3 implies that $\tilde{\Sigma} \succeq \frac{\rho}{2} \cdot I$ for all $T \geq 1$.

We next evaluate the error of the SGD algorithm by considering two events – the event $\mathcal{F}^c$ in which both $\| x_i \|$ and $\epsilon_i$ are bounded for all $i \in \{1,2,\ldots,T\}$, and the event $\mathcal{F}$, which has a vanishing probability $\delta = O(T^{-2})$. Specifically, by the law of total expectation
\[
\mathbb{E} [\| \bar{w} - w^* \|^2] = \mathbb{P}(\mathcal{F}) \cdot \mathbb{E} [\| \bar{w} - w^* \|^2 \bigg| \mathcal{F}] + \mathbb{P}(\mathcal{F}^c) \cdot \mathbb{E} [\| \bar{w} - w^* \|^2 \bigg| \mathcal{F}^c] , \quad (11)
\]
where $\mathcal{F}^c$ is the complement of the event $\mathcal{F}$. The first term in Equation (11) is upper bounded as follows:

\[
P(\mathcal{F}) \mathbb{E} \left[ \|\bar{w} - w^*\|^2 \mid \mathcal{F} \right] = \mathbb{E} \left[ \|\bar{w} - w^*\|^2 \cdot I(\mathcal{F}) \right]
\]

\[
\begin{align*}
&\leq \sqrt{\mathbb{E} \left[ \|\bar{w} - w^*\|^4 \right] \cdot \mathbb{E} \left[ I(\mathcal{F}) \right]} \\
&\leq 4D^2 \cdot \sqrt{P(\mathcal{F})}) \\
&\leq \frac{4D^2 \rho}{210\kappa^2} \cdot \frac{1}{T}, \quad (b) \\
&\leq \frac{D^2}{14} \cdot \frac{1}{T}, \quad (c)
\end{align*}
\]

where (a) follows from Cauchy-Schwarz’s inequality, (b) follows from Assumption 2.3, and the fact that $\bar{w} \in \mathcal{W}$, (c) follows since $P(\mathcal{F}) \leq \delta$ and the choice of $\delta$, and (d) follows since

\[
\rho \leq \max \mathbb{E} \left[ \left( \frac{1}{2} \cdot \frac{\rho^2}{210\kappa^4} \cdot \frac{1}{T^2} \right) \right] \leq \frac{8\kappa^2 \log \left( \left( \frac{21\kappa}{\sqrt{\rho}} + 8 \right) \cdot T \right)}{\max \left( \frac{21\kappa}{\sqrt{\rho}} + 8 \right) \cdot T},
\]

and similarly

\[
u_{2} \leq \sqrt{\frac{8\sigma^2 \log \left( \left( \frac{21\sigma}{\sqrt{\rho}} + 8 \right) \cdot T \right)}{\max \left( \frac{21\sigma}{\sqrt{\rho}} + 8 \right) \cdot T}}.
\]

Then, by taking

\[
R := 2\sqrt{8\kappa^2 \log \left( \left( \frac{21\kappa}{\sqrt{\rho}} + 8 \right) \cdot T \right)} \cdot D + \sqrt{8\sigma^2 \log \left( \left( \frac{21\sigma}{\sqrt{\rho}} + 8 \right) \cdot T \right)},
\]

For the second term in Equation (11), we note that conditioned on $\mathcal{F}^c$ the noise and the feature vectors are bounded, that is $\|x_i\| \leq u_1$ and $\|\epsilon_i\| \leq u_2$ for all $i \in [1, 2, \ldots, T]$. This model is similar to the one discussed in previous sections, in particular to Model 5.5 in Section 5.2, where the expectation is unknown, with two differences. First, as said, by the choice of $\delta$ the conditional covariance matrix of the features has minimal eigenvalue of $\frac{\rho}{2}$, instead of $\rho$ for the unconditional covariance matrix. The second difference is that $\mathbb{E} \left[ \epsilon \mid \mathcal{F}^c \right]$ may not equal zero. However, it can be easily verified that the result of Section A holds, since the noise related terms in the second derivative of the function $F$ therein vanish.

We will follow the same steps as in Section 5.2: computing $\mu$ with $T$ samples conditioned on $\mathcal{F}^c$ and feed them to Algorithm 3 with different input, that is because the radius parameter $R$, is different and will depend on $u_1$ and $u_2$.

The derivation of the new radius parameter $R$ is similar to derivation made in Section 3, that is bounding the norm of the features & noise by $u_1$ and $u_2$ respectively. By our choice of $\delta = \min \left\{ \frac{1}{2}, \frac{\rho^2}{210\kappa^4} \cdot \frac{1}{T^2} \right\}$ we may further bound

\[
u_{1} = \sqrt{2\kappa^2 \log \left( 4 \cdot \max \left\{ 2T, \frac{210^2\kappa^4 T^3}{\rho^2} \right\} \right)} \leq \sqrt{8\kappa^2 \log \left( \left( \frac{21\kappa}{\sqrt{\rho}} + 8 \right) \cdot T \right)},
\]

and similarly

\[
u_{2} \leq \sqrt{8\sigma^2 \log \left( \left( \frac{21\sigma}{\sqrt{\rho}} + 8 \right) \cdot T \right)}.
\]

Then, by taking

\[
R := 2\sqrt{8\kappa^2 \log \left( \left( \frac{21\kappa}{\sqrt{\rho}} + 8 \right) \cdot T \right)} \cdot D + \sqrt{8\sigma^2 \log \left( \left( \frac{21\sigma}{\sqrt{\rho}} + 8 \right) \cdot T \right)},
\]
it holds by Theorem 5.7 that
\[
P(F^c) \mathbb{E}\left[\|\tilde{w} - w^*\|^2 \bigg| F^c\right] \leq \frac{72R^2 \cdot (2\log T + 1)}{((1 - \alpha)^{\frac{1}{2}})^2 T} = \frac{288R^2 \cdot (2\log T + 1)}{((1 - \alpha)^{\rho})^2 T}, \tag{14}
\]
where we use the fact that $0 \geq P(F^c) \leq 1$.

Note that $w^*$ is deterministic and doesn’t change when conditioned of $F^c$, then by combining the bounds in Equation (12) and Equation (14) into Equation (11) completes the proof.

\[\square\]

C  Discussing Our Results with respect to the Spreadness Assumption

The paper of D’Orsi et al. (2021) assumes that a condition called spreadness applies the to empirical covariance matrix of the features. Moreover, in the context of their work they show a hardness result: i.e., that the spreadness assumption is necessary in order to obtain meaningful guarantees for any algorithm. In contrast, our work shows that one can obtain meaningful guarantees without making this assumption. The goal of the section is to show that the hardness result of D’Orsi et al. (2021) does not apply under our assumptions of bounded optimal solution and bounded features, which explains how these two results can hold simultaneously without contradiction.

We will not describe the spreadness since it is unnecessary for establishing our point. Instead, we will describe the hardness result of D’Orsi et al. (2021) and show that it is not relevant in our case.

**The Example of D’Orsi et al. (2021):** The hardness result of D’Orsi et al. (2021) (Appendix A.2.1 therein) is due to the following example: Assume an oblivious contamination as in Model 2.1 such that $\epsilon = 0$ w.p. 1, the features are 1 dimensional i.e. $x \in \mathbb{R}$, and the contaminations are distributed as follows\(^3\),

\[
\begin{aligned}
    b_i &\sim \mathcal{N}(0, \sigma^2) &\text{, with probability } \alpha \\
    b_i &= 0 &\text{, otherwise}.
\end{aligned}
\]

\[
\forall i \in [1, \ldots, T],
\]

here the fraction of contaminations is $\alpha$, and the adversary injects Gaussian noise with variance $\sigma^2$; note that the adversary may choose $\sigma$ to be arbitrarily large.

It is also assumed that the number of non-zero features in the training set is equal to

\[^3\text{We stick to the notations in our paper where } \alpha \text{ is the fraction of contaminated samples. Conversely, D’Orsi et al. (2021) denote the fraction of contaminated samples by } 1 - \alpha.\]
where \( C > 0 \) is a large enough constant that does not depend on \( \alpha, T \). With these assumptions, their hardness proof is based on the following statement (Lemma A.5 therein):

**Lemma C.1** (D’Orsi et al. (2021, Lemma A.5)). Consider the robust linear regression task with the above assumptions. Then for any estimate \( \hat{\mathbf{w}} \) which is based on the contaminated data, there exists an optimal solution \( \mathbf{w}^* \) (of the problem with non-corrupted data) such that the following holds,

\[
\frac{1}{T} \sum_{i=1}^{T} \| \langle \mathbf{x}_i, \hat{\mathbf{w}} - \mathbf{w}^* \rangle \|^2 = \Omega \left( \frac{\sigma^2}{T} \right).
\]

Now since the adversary can choose \( \sigma \) to be arbitrarily large this means that one cannot obtain meaningful guarantees in this case. Nevertheless, as we show below, the above counterexample is meaningless under our assumptions.

**Lemma C.2.** Consider the robust linear regression task with the above assumptions. Also assume that there exists an optimal solution \( \mathbf{w}^* \) such that \( \| \mathbf{w}^* \| \leq D \) and that the features are bounded, i.e., \( \| \mathbf{x} \| \leq 1 \) w.p. 1. Then there exists a trivial estimator \( \hat{\mathbf{w}} \) such that the following holds,

\[
\frac{1}{T} \sum_{i=1}^{T} \| \langle \mathbf{x}_i, \hat{\mathbf{w}} - \mathbf{w}^* \rangle \|^2 = O \left( \frac{D^2}{(1-\alpha)T} \right). \tag{15}
\]

Importantly, the above lemma implies that for this example there does exist an estimator that achieves a vanishing prediction error that does not depend on \( \sigma \). Next we prove the above lemma,

**Proof of Lemma C.2.** Consider the following trivial predictor \( \hat{\mathbf{w}} = 0 \). In this case, since the features are bounded by 1, and only \( \frac{1}{C(1-\alpha)} \) of them are non-zero we obtain,

\[
\frac{1}{T} \sum_{i=1}^{T} \| \langle \mathbf{x}_i, \hat{\mathbf{w}} - \mathbf{w}^* \rangle \|^2 = \frac{1}{T} \sum_{i=1}^{T} \| \langle \mathbf{x}_i, \mathbf{w}^* \rangle \|^2 \\
\leq \frac{1}{T} \sum_{i=1}^{T} \| \mathbf{x}_i \|^2 \cdot \| \mathbf{w}^* \|^2 \\
\leq \frac{D}{T} \sum_{i=1}^{T} \| \mathbf{x}_i \|^2 \\
\leq \frac{CD}{(1-\alpha)T},
\]

here the first line uses \( \hat{\mathbf{w}} = 0 \), the second line uses Cauchy-Schwarz’s inequality, later we use the fact that \( \| \mathbf{w}^* \| \leq D \), and the last line uses \( \| \mathbf{x}_i \| \leq 1 \), and the fact that only \( 1/C(1-\alpha) \) of them are non zero. This concludes the proof.

4In the example appearing in their appendix A.2.1. there is a typo where they mistakenly write that number of non-zero features is \( \frac{T}{C(1-\alpha)} \), but it should actually be \( \frac{1}{C(1-\alpha)} \). We validated this with the authors of D’Orsi et al. (2021).
To conclude, under our assumptions, the hardness result of D’Orsi et al. (2021) is irrelevant, and so fast rates of convergence are possible.