Periodic Solutions to Painlevé VI and Dynamical System on Cubic Surface

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Dedicated to Professor Masuo Hukuhara on his 100th birthday

Abstract

The number of periodic solutions to Painlevé VI along a Pochhammer loop is counted exactly. It is shown that the number grows exponentially with period, where the growth rate is determined explicitly. Principal ingredients of the computation are a moduli-theoretical formulation of Painlevé VI, a Riemann-Hilbert correspondence, the dynamical system of a birational map on a cubic surface, and the Lefschetz fixed point formula.

1 Introduction

Painlevé equations and dynamical systems on complex surfaces are two subjects of mathematics which have been investigated actively in recent years. In this paper we shall demonstrate a substantial relation between them by presenting a fruitful application to the former subject of the latter. We begin with stating our motivation on the side of Painlevé equations.

The global structure of the sixth Painlevé equation $P_{VI}(\kappa)$, especially the multivalued character of its solutions is an important issue in the study of Painlevé equations. In this respect, several authors [3, 4, 6, 11, 12, 20, 21] have been interested in finding algebraic solutions, because they offer a simplest class of solutions with clear global structure in the sense that they have only finitely many branches under analytic continuations along all loops in the domain

$$X = \mathbb{P}^1 - \{0, 1, \infty\}. \quad (1)$$

In another direction of promising research, we are interested in periodic solutions along a single loop, namely, in those solutions which are finitely many-valued along a single loop chosen particularly. Given such a loop, we shall discuss the following problems:

- How many solutions can be periodic of period $N$ among all solutions to $P_{VI}(\kappa)$?
- How rapidly does that number grow as the period $N$ tends to infinity?

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For such a loop we take a Pochhammer loop $\varphi$ as in Figure 1. If $\ell_0$ and $\ell_1$ are standard generators of $\pi_1(X, x)$ as in Figure 2, then $\varphi$ is a loop homotopic to the commutator

$$[\ell_0, \ell_1^{-1}] = \ell_0 \ell_1^{-1} \ell_0^{-1} \ell_1.$$ 

It is a typical loop which often appears in mathematics due to the property that any abelian representation of $\pi_1(X, x)$ is killed along this loop. For example, it is used as an integration contour of Euler integral representation of hypergeometric functions [18]. In the context of this article the Pochhammer loop will be closely related to a certain birational map of a cubic surface whose dynamics is quite relevant to understanding the global structure of the sixth Painlevé equation (see discussions in Sections 6 and 7).

We remark that the same problem for a simplest loop, namely for a loop $\ell_0$ or $\ell_1$ in Figure 2 or a loop $\ell_\infty = (\ell_0 \ell_1)^{-1}$ around the point at infinity, is not interesting or even meaningless, because for any $N > 1$, there are infinitely many periodic solutions of period $N$ along it. In fact it turns out that they are parametrized by points on certain complex curves and hence their cardinality is that of a continuum [17]. On the contrary, along the Pochhammer loop $\varphi$, the cardinality of the periodic solutions of period $N$ turns out to be finite for every $N \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and hence our problem certainly makes sense.

In this article we shall exactly count the number of periodic solutions to $P_{VI}(\kappa)$ along the Pochhammer loop $\varphi$ under a certain generic assumption on the parameters $\kappa$. In particular we shall show that the number grows exponentially as the period tends to infinity, with the growth rate determined explicitly (see Theorem 2.1). As is already mentioned, this result is the fruits
of a good application to Painlevé equations of a dynamical system theory on complex surfaces as
developed in [5, 7]. Algebraic geometry of Painlevé equations, especially a moduli-theoretical
interpretation of Painlevé dynamics [13, 14] is also an essential ingredient of our work.

After stating the main result of this article in Section 2, we shall develop the story of
this article in the following manner. First, \( P_{VI}(\kappa) \) is formulated as a flow, Painlevé flow, on
a moduli space of stable parabolic connections (Section 3). Secondly, it is conjugated to an
isomonodromic flow on a moduli space of monodromy representations via a Riemann-Hilbert
correspondence (Section 4). Thirdly, with a natural identification of the representation space
with a cubic surface, the Poincaré section of \( P_{VI}(\kappa) \) is conjugated to the dynamical system of a
group action on the cubics (Section 5). Especially, analytic continuation along the Pochhammer
loop is connected with a distinguished transformation, called a ‘Coxeter’ transformation, of the
group action. Fourthly, main properties of our dynamical system on the cubics are established
from the standpoint of birational surface dynamics. Fifthly, the number of periodic points of
the Coxeter transformation is counted by using the Lefschetz fixed point formula (Section 8).
Then, back to the original phase space of \( P_{VI}(\kappa) \), we arrive at our final goal, that is, the exact
number of periodic solutions to \( P_{VI}(\kappa) \) of any period along the Pochhammer loop \( \wp \).

The authors would be happy if this article could give a new insight into the global structure
of the sixth Painlevé equation. They are grateful to Yutaka Ishii for valuable discussions.

2 Main Result

Let us describe our main result in more detail. To this end we recall that the sixth Painlevé
equation \( P_{VI}(\kappa) \) in its Hamiltonian form is a system of nonlinear differential equations

\[
\frac{dq}{dx} = \frac{\partial H(\kappa)}{\partial p}, \quad \frac{dp}{dx} = -\frac{\partial H(\kappa)}{\partial q},
\]

with an independent variable \( x \in X \) and unknown functions \((q(x), p(x))\), depending on complex
parameters \( \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \) in a four-dimensional affine space

\[
\mathcal{K} := \{ \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 : 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \},
\]

where the Hamiltonian \( H(\kappa) = H(q, p, x; \kappa) \) is given by

\[
x(x-1)H(\kappa) = (q_0q_1q_x)p^2 - (\kappa_1q_1q_x + (\kappa_2 - 1)q_0q_1 + \kappa_3q_0q_x)p + \kappa_0(\kappa_0 + \kappa_4)q_x,
\]

with \( q_\nu = q - \nu \) for \( \nu \in \{0, 1, x\} \). It is known that system (2) enjoys the Painlevé property,
that is, any meromorphic solution germ at a base point \( x \in X \) of system (2) admits a global
analytic continuation along any path emanating from \( x \) as a meromorphic function.

Geometrically, the sixth Painlevé equation \( P_{VI}(\kappa) \) is formulated as a holomorphic uniform
foliation on the total space of a fibration of certain smooth, quasi-projective, rational surfaces,

\[
\pi_\kappa : M(\kappa) \to X,
\]

transversal to each fiber of the fibration. We refer to [11, 13, 14, 23, 24, 25] for the detailed
accounts of the space \( M(\kappa) \). Especially the papers [13, 14] give a comprehensive description
of it as a moduli space of stable parabolic connections. The fiber \( M_x(\kappa) \) over \( x \in X \), called
the space of initial conditions at time $x$, parametrizes all the solution germs at $x$ of equation (2) most precisely, completing the naïve and incomplete space $\mathbb{C}^2$ of initial values $(q, p)$ at the points $x$. Given a loop $\gamma \in \pi_1(X, x)$, the horizontal lifts of the loop $\gamma$ along the foliation induces a biholomorphism $\gamma_* : M_x(\kappa) \to M_x(\kappa)$, called the Poincaré return map along $\gamma$, which depends only on the homotopy class of $\gamma$. Then the global structure of the sixth Painlevé equation $P_{VI}(\kappa)$ is described by the group homomorphism

$$\text{PS}_x(\kappa) : \pi_1(X, x) \to \text{Aut} M_x(\kappa), \quad \gamma \mapsto \gamma_*,$$

which is referred to as the Poincaré section of the sixth Painlevé dynamics $P_{VI}(\kappa)$.

In this article we are interested in analytic continuations of solutions to equation (2) along the Pochhammer loop $\wp$, namely, in the iteration of the Poincaré return map $\wp_*$. The Poincaré return map along the Pochhammer loop is referred to as the Pochhammer-Poincaré map. Given any $N \in \mathbb{N}$, let $\text{Per}_N(\kappa)$ be the set of all initial points $Q \in M_x(\kappa)$ that come back to the original positions after the $N$-th iterate of the Pochhammer-Poincaré map $\wp_*$,

$$\text{Per}_N(\kappa) := \{ Q \in M_x(\kappa) : \wp_*^N(Q) = Q \}.$$  

The aim of this article is to count the number of $\text{Per}_N(\kappa)$ and to find out its growth rate as the period $N$ tends to infinity.

To avoid certain technical difficulties (see Remark 2.4), we make a generic assumption on the parameters $\kappa \in \mathcal{K}$. To this end we recall an affine Weyl group structure of the parameter space $[13, 16]$. The affine space $\mathcal{K}$ is identified with the linear space $\mathbb{C}^4$ by the isomorphism

$$\mathcal{K} \to \mathbb{C}^4, \quad \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \mapsto (\kappa_1, \kappa_2, \kappa_3, \kappa_4),$$

where the latter space $\mathbb{C}^4$ is equipped with the standard (complex) Euclidean inner product. For each $i \in \{0, 1, 2, 3, 4\}$, let $w_i : \mathcal{K} \to \mathcal{K}$ be the orthogonal reflection having $\{\kappa_i = 0\}$ as its reflecting hyperplane, with respect to the inner product mentioned above. Then the group generated by $w_0, w_1, w_2, w_3, w_4$ is an affine Weyl group of type $D_4^{(1)}$,

$$W(D_4^{(1)}) = \langle w_0, w_1, w_2, w_3, w_4 \rangle \simeq \mathcal{K}.$$  

corresponding to the Dynkin diagram in Figure 3. The reflecting hyperplanes of all reflections in the group $W(D_4^{(1)})$ are given by affine linear relations

$$\kappa_i = m, \quad \kappa_1 \pm \kappa_2 \pm \kappa_3 \pm \kappa_4 = 2m + 1 \quad (i \in \{1, 2, 3, 4\}, m \in \mathbb{Z}).$$

Figure 3: Dynkin diagram of type $D_4^{(1)}$
with any choice of signs ±. Let \textbf{Wall} be the union of all those hyperplanes. Then the generic condition to be imposed on parameters is that \( \kappa \) should lie outside \textbf{Wall}; this is a necessary and sufficient condition for \( PVI(\kappa) \) to admit no Riccati solutions \cite{[13]}

Now the main theorem of this article is stated as follows.

**Theorem 2.1** For any \( \kappa \in \mathcal{K} – \text{Wall} \) the cardinality of the set \( \text{Per}_N(\kappa) \) is given by

\[
\# \text{Per}_N(\kappa) = (9 + 4\sqrt{5})^N + (9 - 4\sqrt{5})^N + 4 \quad (N \in \mathbb{N}). \tag{6}
\]

**Remark 2.2** It should be noted that formula (6) is rewritten as

\[
\# \text{Per}_N(\kappa) - \{(9 + 4\sqrt{5})^N + 4\} = (9 + 4\sqrt{5})^{-N},
\]

which means that the geometric sequence \((9 + 4\sqrt{5})^N\) shifted by 4 approximates the cardinality of \( \text{Per}_N(\kappa) \) up to an exponentially decaying error term \((9 + 4\sqrt{5})^{-N}\), where the growth rate of cardinality and the decay rate of error term are given by the same number \( 9 + 4\sqrt{5} \). Moreover, since \( 9 \pm 4\sqrt{5} \) are the root of the quadratic equation \( \lambda^2 - 18\lambda + 1 = 0 \), the formula (6) is expressed as \( \# \text{Per}_N(\kappa) = C_N + 4 \), where the sequence \( \{C_N\} \) is defined recursively by

\[
C_0 = 2, \quad C_1 = 18, \quad C_{N+2} - 18C_{N+1} + C_N = 0.
\]

**Remark 2.3** Our main theorem can also be stated in terms of a dynamical zeta function. Indeed, as a generating expression of formula (6) for all \( N \in \mathbb{N} \), we have

\[
Z_{\kappa}(z) := \exp \left( \sum_{N=1}^{\infty} \frac{\# \text{Per}_N(\kappa)}{N} z^N \right) = \frac{1}{(1 - z)^4(1 - 18z + z^2)}.
\]

**Remark 2.4** In this article we restrict our attention to the generic case \( \kappa \in \mathcal{K} – \text{Wall} \) only, leaving the nongeneric case \( \kappa \in \text{Wall} \) untouched. The difference between the generic case and the nongeneric case lies in the fact that the Riemann-Hilbert correspondence to be used in the proof becomes a biholomorphism in the former case, while it gives an analytic minimal resolution of Klein singularities in the latter case (see Remark 4.2). The presence of singularities would make the treatment of the nongeneric case more complicated. However it is expected that the basic strategy developed in this article will also be effective in the nongeneric case. The relevant discussion will be made in another place.

### 3 Moduli Space of Stable Parabolic Connections

In order to describe the fibration (3), we first construct an auxiliary fibration \( \pi_\kappa : \mathcal{M}(\kappa) \rightarrow T \) over the configuration space of mutually distinct, ordered, three points in \( \mathbb{C} \),

\[
T = \{ t = (t_1, t_2, t_3) \in \mathbb{C}^3 : t_i \neq t_j \text{ for } i \neq j \},
\]

and then reduce it to the original fibration (3). We put the fourth point \( t_4 \) at infinity. Given any \( (t, \kappa) \in T \times \mathcal{K} \), a \((t, \kappa)\)-parabolic connection is a quadruple \( Q = (E, \nabla, \psi, l) \) such that

1. \( E \) is a rank 2 vector bundle of degree \(-1\) over \( \mathbb{P}^1 \),
| singularities | $t_1$ | $t_2$ | $t_3$ | $t_4$ |
|---------------|-------|-------|-------|-------|
| first exponent | $-\lambda_1$ | $-\lambda_2$ | $-\lambda_3$ | $-\lambda_4$ |
| second exponent | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4 - 1$ |
| difference     | $\kappa_1$ | $\kappa_2$ | $\kappa_3$ | $\kappa_4$ |

Table 1: Riemann scheme: $\kappa_i$ is the difference of the second exponent from the first.

(2) $\nabla : E \to E \otimes \Omega^1_{P1}(D_t)$ is a Fuchsian connection with pole divisor $D_t = t_1 + t_2 + t_3 + t_4$ and Riemann scheme as in Table 1 where $t_4 = \infty$ as mentioned above,

(3) $\psi : \det E \to \mathcal{O}_{P1}(-t_4)$ is a horizontal isomorphism called a determinantal structure, where $\mathcal{O}_{P1}(-t_4)$ is equipped with the connection induced from $d : \mathcal{O}_{P1} \to \Omega^1_{P1}$,

(4) $l = (l_1, l_2, l_3, l_4)$ is a parabolic structure, namely, $l_i$ is an eigenline of $\text{Res}_{t_i}(\nabla) \in \text{End}(E_{t_i})$ corresponding to eigenvalue $\lambda_i$ (whose minus is the first exponent $-\lambda_i$ in Table 1).

There exists a concept of stability for parabolic connections, with which the geometric invariant theory [22] can be worked out to establish the following theorem [13, 14].

**Theorem 3.1** For any $(t, \kappa) \in T \times K$ there exists a fine moduli scheme $\mathcal{M}_t(\kappa)$ of stable $(t, \kappa)$-parabolic connections. The moduli space $\mathcal{M}_t(\kappa)$ is a smooth, irreducible, quasi-projective surface. As a relative setting over $T$, for any $\kappa \in K$, there exists a family of moduli spaces

$$
\pi_\kappa : \mathcal{M}(\kappa) \to T
$$

such that the projection $\pi_\kappa$ is a smooth morphism with fiber $\mathcal{M}_t(\kappa)$ over $t \in T$.

Now the fibration (3) is defined to be the pull-back of (7) by an injection

$$
t : X \hookrightarrow T, \quad x \mapsto (0, x, 1),
$$

The group $\text{Aff}(\mathbb{C})$ of affine linear transformations on $\mathbb{C}$ acts diagonally on the configuration space $T$ and the quotient space $T/\text{Aff}(\mathbb{C})$ is isomorphic to $X$, with the quotient map given by

$$
r : T \to X, \quad t = (t_1, t_2, t_3) \mapsto x = \frac{t_2 - t_1}{t_3 - t_1}.
$$

The map $r$ yields a trivial $\text{Aff}(\mathbb{C})$-bundle structure of $T$ over $X$ and the fibration (7) is in turn the pull-back of the fibration (3) by the map $r$. Hence we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}(\kappa) & \longrightarrow & M(\kappa) \\
\pi_\kappa \downarrow & & \downarrow \pi_\kappa \\
T & \longrightarrow & X. \\
\end{array}
$$
Then the Poincaré section (10) naturally lifts to a collection of isomorphisms $T/S$. Thus we will be concerned with the Poincaré section (10) along this particular braid.

In [13, 14], the Painlevé dynamics $P_{V1}(\kappa)$ is formulated as a holomorphic uniform foliations on the fibration (7) which is compatible with the diagram (9). Thus the Poincaré section (4) is reformulated as a group homomorphism

$$\text{PS}_t(\kappa) : \pi_1(T, t) \to \text{Aut} \mathcal{M}_t(\kappa).$$

Let us describe the fundamental group $\pi_1(T, t)$ in terms of a braid group [2]. We take a base point $t = (t_1, t_2, t_3) \in T$ in such a manner that the three points lie on the real line in an increasing order $t_1 < t_2 < t_3$. To treat them symmetrically, we denote them by $t_i, t_j, t_k$ for a cyclic permutation $(i, j, k)$ of $(1, 2, 3)$ and think of them as cyclically ordered three points on the equator $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\beta_i$ be a braid on three strings as in Figure 4 (left) along which $t_i$ and $t_j$ make a half-turn, with $t_i$ moving in the southern hemisphere and $t_j$ in the northern hemisphere, while $t_k$ is kept fixed as in Figure 4 (right). Then the braid group on three strings is the group generated by $\beta_i, \beta_j$ and $\beta_k$ and the pure braid group $P_3$ is the normal subgroup of $B_3$ generated by the squares $\beta_i^2, \beta_j^2$ and $\beta_k^2$,

$$P_3 = \langle \beta_i^2, \beta_j^2, \beta_k^2 \rangle \triangleleft B_3 = \langle \beta_i, \beta_j, \beta_k \rangle.$$ 

The generators of $B_3$ satisfy relations $\beta_i\beta_j\beta_i = \beta_j\beta_i\beta_j$ and $\beta_k = \beta_i\beta_j\beta_i^{-1}$, and so $B_3$ is generated by $\beta_i$ and $\beta_j$ only. The fundamental group $\pi_1(T, t)$ is identified with the pure braid group $P_3$.

The reduction map (8) induces a group homomorphism $r_* : \pi_1(T, t) \to \pi_1(X, x)$. It is easy to see that the loops $\ell_0$ and $\ell_1$ in Figure 2 are the $r_*$-images of $\beta_1^2$ and $\beta_2^2$ respectively, so that the Pochhammer loop $\varphi$ in $X$ is the $r_*$-image of the pure braid

$$[\beta_1^2, \beta_2^2] = \beta_1^2\beta_2^{-2}\beta_1^{-2}\beta_2^2.$$ 

Thus we will be concerned with the Poincaré section (10) along this particular braid.

The symmetric group $S_3$ acts on $T$ by permuting the entries of $t = (t_1, t_2, t_3)$ and the quotient space $T/S_3$ is the configuration space of mutually distinct, unordered, three points in $\mathbb{C}$. The fundamental group $\pi(T/S_3, s)$ with base point $s = \{t_1, t_2, t_3\}$ is identified with the ordinary braid group $B_3$ and there exists a short exact sequence of groups

$$1 \longrightarrow \pi_1(T, t) \longrightarrow \pi_1(T/S_3, s) \longrightarrow S_3 \longrightarrow 1 \quad \text{and} \quad 1 \longrightarrow P_3 \longrightarrow B_3 \longrightarrow S_3 \longrightarrow 1.$$

Then the Poincaré section (10) naturally lifts to a collection of isomorphisms

$$\beta_* : \mathcal{M}_t(\kappa) \to \mathcal{M}_{\tau(t)}(\tau(\kappa)), \quad (\beta \in B_3)$$

![Figure 4: Basic braid $\beta_i$ in $T$ and the corresponding movement of $t$ in $\hat{\mathbb{C}}$](image-url)
which may be called the half-Poincaré section of $P_{VI}(\kappa)$, where $\tau \in S_3$ denotes the permutation corresponding to $\beta \in B_3$. Note that $\tau \in S_3$ acts on $\kappa \in K$ too by permuting the entries of $(\kappa_1, \kappa_2, \kappa_3)$ in the same manner as it does on $t = (t_1, t_2, t_3)$, since $\kappa_i$ is loaded on $t_i$. Now the permutation corresponding to the basic braid $\beta_i$ is the substitution $\tau_i = (i, j)$ that exchanges $t_i$ and $t_j$ while keeping $t_k$ fixed. Thus there are three basic half-Poincaré maps:

$$\beta_{i*} : M_t(\kappa) \to M_{\tau_i(t)}(\tau_i(\kappa)), \quad (i = 1, 2, 3).$$

(12)

4 Riemann-Hilbert Correspondence

It is rather hopeless to deal with the Painlevé flow directly, since it is a highly transcendental dynamical system on the moduli space of stable parabolic connections. But it can be recast into a more tractable dynamical system, called an isomonodromic flow, on a moduli space of monodromy representations via a Riemann-Hilbert correspondence. We review the construction of such a Riemann-Hilbert correspondence in the sequel.

Let $A := \mathbb{C}^4$ be the complex 4-space with coordinates $a = (a_1, a_2, a_3, a_4)$, called the space of local monodromy data. Given $(t, a) \in T \times A$, let $R_t(a)$ be the moduli space of Jordan equivalence classes of representations $\rho : \pi_1(\mathbb{P}^1 - D_t, *) \to SL_2(\mathbb{C})$ such that $\text{Tr} \rho(\gamma_i) = a_i$ for $i \in \{1, 2, 3, 4\}$, where the divisor $D_t = t_1 + t_2 + t_3 + t_4$ is identified with the point set $\{t_1, t_2, t_3, t_4\}$ and $\gamma_i$ is a loop surrounding $t_i$ as in Figure 5. Any stable parabolic connection $Q = (E, \nabla, \psi, I) \in M_t(\kappa)$, when restricted to $\mathbb{P}^1 - D_t$, induces a flat connection

$$\nabla|_{\mathbb{P}^1 - D_t} : E|_{\mathbb{P}^1 - D_t} \to (E|_{\mathbb{P}^1 - D_t}) \otimes \Omega^1_{\mathbb{P}^1 - D_t},$$

and one can speak of the Jordan equivalence class $\rho$ of its monodromy representations. Then the Riemann-Hilbert correspondence at $t \in T$ is defined by

$$\text{RH}_{t, \kappa} : M_t(\kappa) \to R_t(a), \quad Q \mapsto \rho,$$

(13)

where in view of the Riemann scheme in Table II the local monodromy data $a \in A$ is given by

$$a_i = \begin{cases} 
2 \cos \pi \kappa_i & (i = 1, 2, 3), \\
-2 \cos \pi \kappa_4 & (i = 4).
\end{cases}$$

(14)
As a relative setting over $T$, let $\pi_a : \mathcal{R}(a) \to T$ be the family of moduli spaces of monodromy representations with fiber $\mathcal{R}_t(a)$ over $t \in T$. Then the relative version of Riemann-Hilbert correspondence is formulated to be the commutative diagram

\[
\begin{array}{ccl}
\mathcal{M}(\kappa) & \xrightarrow{\text{RH}_\kappa} & \mathcal{R}(a) \\
\pi_a \downarrow & & \downarrow \pi_a \\
T & = & T,
\end{array}
\]

whose fiber over $t \in T$ is given by $[13]$. Then we have the following theorem $[13][14]$.

**Theorem 4.1** If $\kappa \in \mathcal{K} - \text{Wall}$, then $\mathcal{R}(a)$ as well as each fiber $\mathcal{R}_t(a)$ is smooth and the Riemann-Hilbert correspondence $\text{RH}_\kappa$ in $[13]$ is a biholomorphism.

**Remark 4.2** If $\kappa \in \text{Wall}$, then $\mathcal{R}_t(a)$ is not a smooth surface but a surface with Klein singularities and $[13]$ yields an analytic minimal resolution of singularities, so that $[13]$ gives a family of resolutions of singularities $[13]$. As is mentioned in Remark 2.4 this fact makes the treatment of the nongeneric case more involved and we leave this case in another occasion.

## 5 Cubic Surface and the 27 Lines

The moduli space $\mathcal{R}_t(a)$ of monodromy representations is isomorphic to an affine cubic surface $\mathcal{S}(\theta)$ and the braid group action on $\mathcal{R}_t(a)$ can be made explicit in terms of $\mathcal{S}(\theta)$. Let us recall this construction $[13]$. Given $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta := \mathbb{C}_\theta^4$, consider an affine cubic surface

\[
\mathcal{S}(\theta) = \{ x = (x_1, x_2, x_3) \in \mathbb{C}_x^3 : f(x, \theta) = 0 \},
\]

where the cubic polynomial $f(x, \theta)$ of $x$ with parameter $\theta$ is given by

\[
f(x, \theta) = x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1x_1 - \theta_2x_2 - \theta_3x_3 + \theta_4.
\]

Then there exists an isomorphism of affine algebraic surfaces, $\mathcal{R}_t(a) \to \mathcal{S}(\theta)$, $\rho \mapsto x$, where

\[
x_i = \text{Tr} \rho(\gamma_j \gamma_k), \quad \text{for} \quad \{i, j, k\} = \{1, 2, 3\},
\]

together with a correspondence of parameters, $A \to \Theta$, $a \mapsto \theta$, given by

\[
\theta_i = \begin{cases} 
    a_ia_4 + a_ja_k & (\{i, j, k\} = \{1, 2, 3\}), \\
    a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 & (i = 4).
\end{cases}
\]

With this identification, the Riemann-Hilbert correspondence $[13]$ is reformulated as a map

\[
\text{RH}_t(\kappa) : \mathcal{M}_t(\kappa) \to \mathcal{S}(\theta), \quad \text{with} \quad \theta = \text{rh}(\kappa),
\]

where $\text{rh} : \mathcal{K} \to \Theta$ is the composition of two maps $\mathcal{K} \to A$ and $A \to \Theta$ defined by $[14]$ and $[16]$, which we call the Riemann-Hilbert correspondence in the parameter level. Through the reformulated Riemann-Hilbert correspondence $[17]$, the $i$-th basic half-Poincaré map $\beta_i$ in $[12]$ is conjugated to a map $g_i : \mathcal{S}(\theta) \to \mathcal{S}(\theta')$, $(x, \theta) \mapsto (x', \theta')$, which is explicitly represented as

\[
g_i : (x'_i, x'_j, x'_k, \theta'_i, \theta'_j, \theta'_k, \theta'_4) = (\theta_j - x_j - x_kx_i, x_i, x_k, \theta_j, \theta_i, \theta_k, \theta_4).
\]

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A derivation of this formula can be found in [16] (see also [4, 6, 9, 10, 15, 19]). By Theorem 1.1, the map (17) is an isomorphism and hence (18) is a strict conjugacy of (12). We can easily check the relations $g_i g_j g_k = g_j g_i g_k$ and $g_k = g_i g_j g_k^{-1}$ which are parallel to those for $\beta_i, \beta_j, \beta_k$.

To utilize standard techniques from algebraic geometry and complex geometry, we need to compactify the affine cubic surface $\mathcal{S}(\theta)$ by a standard embedding

$$\mathcal{S}(\theta) \hookrightarrow \overline{\mathcal{S}(\theta)} \subset \mathbb{P}^3, \quad x = (x_1, x_2, x_3) \mapsto [1 : x_1 : x_2 : x_3],$$

where the compactified surface $\overline{\mathcal{S}(\theta)}$ is defined by $\overline{\mathcal{S}(\theta)} = \{ X \in \mathbb{P}^3 : F(X, \theta) = 0 \}$ with

$$F(X, \theta) = X_1 X_2 X_3 + X_0 (X_1^2 + X_2^2 + X_3^2) - X_0^2 (\theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3) + \theta_4 X_0^3.$$ 

It is obtained from the affine surface $\mathcal{S}(\theta)$ by adding three lines at infinity,

$$L_i = \{ X \in \mathbb{P}^3 : X_0 = X_i = 0 \} \quad (i = 1, 2, 3). \quad (19)$$

Here and hereafter the homogeneous coordinates $X = [X_0 : X_1 : X_2 : X_3]$ of $\mathbb{P}^3$ should not be confused with the domain $X$ in (1). The union $L = L_1 \cup L_2 \cup L_3$ is called the tritangent lines at infinity and the intersection point of $L_j$ and $L_k$ is denoted by $p_i$ (see Figure 6). Note that

$$p_1 = [0 : 1 : 0 : 0], \quad p_2 = [0 : 0 : 1 : 0], \quad p_3 = [0 : 0 : 0 : 1].$$

For $i \in \{1, 2, 3\}$, put $U_i = \{ X \in \mathbb{P}^3 : X_i \neq 0 \}$ and take inhomogeneous coordinates of $\mathbb{P}^3$:

$$u = (u_0, u_j, u_k) = (X_0/X_i, X_j/X_i, X_k/X_i) \quad \text{on} \ U_i,$$

$$v = (v_0, v_j, v_k) = (X_0/X_j, X_i/X_j, X_k/X_j) \quad \text{on} \ U_j,$$

$$w = (w_0, w_j, w_k) = (X_0/X_k, X_i/X_k, X_j/X_k) \quad \text{on} \ U_k,$$

where $\{i, j, k\} = \{1, 2, 3\}$. In terms of these coordinates we shall find local coordinates and local equations of $\overline{\mathcal{S}(\theta)}$ around $L$. Since $L \subset U_1 \cup U_2 \cup U_3$, we can divide $L$ into components $L \cap U_i$, $i = 1, 2, 3$, and make a further decomposition $L \cap U_i = \{p_i\} \cup (L_j - \{p_i, p_k\}) \cup (L_k - \{p_i, p_j\})$ into a total of nine pieces. Then a careful inspection of equation $F(X, \theta) = 0$ implies that around those pieces we can take local coordinates and local equations as in Table 2 where $O_m(u_j, u_k) = O((|u_j| + |u_k|)^m)$ denotes a small term of order $m$ as $(u_j, u_k) \to (0, 0)$. 

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Proof. First we show assertion (1). In terms of inhomogeneous coordinates \( u \) in (20), we have
\[
\mathcal{S}(\theta) \cap U_i \cong \{ u = (u_0, u_j, u_k) \in \mathbb{C}^3 : f_i(u, \theta) = 0 \},
\]
where the defining equation \( f_i(u, \theta) \) is given by
\[
f_i(u, \theta) = u_j u_k + u_0 (1 + u_j^2 + u_k^2) - u_0^2 (\theta_i + \theta_j u_j + \theta_k u_k) + \theta_4 u_0^3.
\]
The partial derivatives of \( f_i = f_i(u, \theta) \) with respect to \( u = (u_0, u_j, u_k) \) are calculated as
\[
\frac{\partial f_i}{\partial u_0} = (1 + u_j^2 + u_k^2) - 2 u_0 (\theta_i + \theta_j u_j + \theta_k u_k) + 3 \theta_4 u_0^2
\]
\[
\frac{\partial f_i}{\partial u_j} = u_k + 2 u_0 u_j - \theta_j u_0^2
\]
\[
\frac{\partial f_i}{\partial u_k} = u_j + 2 u_0 u_k - \theta_k u_0^2.
\]
Restricted to $L \cap U_i = (L_j \cap U_i) \cup (L_k \cap U_i)$, these derivatives become

$$\frac{\partial f_i}{\partial u_0} = 1 + u_k^2, \quad \frac{\partial f_i}{\partial u_j} = u_k, \quad \frac{\partial f_i}{\partial u_k} = 0, \quad \text{on } L_j \cap U_i,$$
$$\frac{\partial f_i}{\partial u_0} = 1 + u_j^2, \quad \frac{\partial f_i}{\partial u_j} = 0, \quad \frac{\partial f_i}{\partial u_k} = u_j, \quad \text{on } L_k \cap U_i.$$

Hence the exterior derivative $d_a f_i$ does not vanish on $L \cap U_i$, and the implicit function theorem implies that $\mathcal{S}(\theta)$ is smooth in a neighborhood of $L$. This proves assertion (1). In order to show assertion (2) we recall that the affine surface $S(\theta)$ is smooth if and only if $\theta = rh(\kappa)$ with $\kappa \in \mathcal{K}$ - Wall (see [13]). Then assertion (2) readily follows from assertion (1).

Now let us review some basic facts about smooth cubic surfaces in $\mathbb{P}^3$ (see e.g. [8]). It is well known that every smooth cubic surface $S$ in $\mathbb{P}^3$ can be obtained by blowing up $\mathbb{P}^2$ at six points $P_1, \ldots, P_6$, no three colinear and not all six on a conic, and embedding the blow-up surface into $\mathbb{P}^3$ by the proper transform of the linear system of cubics passing through the six points $P_1, \ldots, P_6$. It is also well known that there are exactly 27 lines on the smooth cubic surface $S$, each of which has self-intersection number $-1$. Explicitly, they are given by

$$E_a \quad (a = 1, \ldots, 6); \quad F_{ab} \quad (1 \leq a < b \leq 6); \quad G_a \quad (a = 1, \ldots, 6),$$

(1) $E_a$ is the exceptional curve over the point $P_a$,

(2) $F_{ab}$ is the strict transform of the line in $\mathbb{P}^2$ through the two points $P_a$ and $P_b$,

(3) $G_a$ is the strict transform of the conic in $\mathbb{P}^2$ through the five points $P_1, \ldots, P_6$.

Here the index $a$ should not be confused with the local monodromy data $a \in A$. All the intersection relations among the 27 lines with nonzero intersection numbers are listed as

$$(E_a, E_a) = (G_a, G_a) = (F_{ab}, F_{ab}) = -1 \quad (\forall a, b),$$
$$(E_a, F_{bc}) = (G_a, F_{bc}) = 1 \quad (a \in \{b, c\}),$$
$$(E_a, G_b) = 1 \quad (a \neq b),$$
$$(F_{ab}, F_{cd}) = 1 \quad (\{a, b\} \cap \{c, d\} = \emptyset).$$

Moreover there are exactly 45 tritangent planes that cut out a triplet of lines on $S$. In our case $S = \mathcal{S}(\theta)$, the plane at infinity $\{X \in \mathbb{P}^3 : X_0 = 0\}$ is an instance of tritangent plane, which cuts out the lines in $\mathcal{S}(\theta)$. Figure 7 offers an arrangement of the 27 lines viewed from the tritangent plane at infinity, where $\{i, j, k\} = \{1, 2, 3\}$ and $\{l, m, n\} = \{4, 5, 6\}$, and

$$L_i = F_{ij}, \quad L_j = F_{kl}, \quad L_k = F_{mn} \quad (21)$$

are allocated for the lines at infinity. Each line at infinity is intersected by exactly eight lines and this fact enables us to divide the 27 lines into three groups of nine lines labeled by lines at infinity. Caution: only the intersection relations among lines of the same group are indicated in Figure 7, with no other intersection relations being depicted.

If $E_0$ is the strict transform of a plane in $\mathbb{P}^2$ not passing through $P_1, \ldots, P_6$ relative to the 6-point blow-up $S \to \mathbb{P}^2$, then the second cohomology group of $S = \mathcal{S}(\theta)$ is expressed as

$$H^2(\mathcal{S}(\theta), \mathbb{Z}) = \mathbb{Z}E_0 \oplus \mathbb{Z}E_i \oplus \mathbb{Z}E_j \oplus \mathbb{Z}E_k \oplus \mathbb{Z}E_l \oplus \mathbb{Z}E_m \oplus \mathbb{Z}E_n, \quad (22)$$

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where a divisor is identified with the cohomology class it represents. It is a Lorentz lattice of rank 7 with intersection numbers

\[(E_a, E_b) = \begin{cases} 1 & (a = b = 0), \\
-1 & (a = b \neq 0), \\
0 & \text{(otherwise)}. \end{cases} \quad (23)\]

In terms of the basis in (22) the lines \(F_{ab}\) and \(G_a\) are represented as

\[F_{ab} = E_0 - E_a - E_b, \quad G_a = 2E_0 - (E_1 + \cdots + \hat{E}_a + \cdots + E_6). \quad (24)\]

We shall describe the 27 lines on our cubic surface \(\mathcal{S}(\theta)\) under the condition that \(\mathcal{S}(\theta)\) is smooth, namely, \(\theta = \text{rh}(\kappa)\) with \(\kappa \in \mathcal{K} - \text{Wall}\). To this end we introduce new parameters \(b = (b_1, b_2, b_3, b_4) \in B := (\mathbb{C}_b^\times)^4\) in such a manner that \(b\) is expressed as

\[b_i = \begin{cases} \exp(\sqrt{-1}\pi \kappa_i) & (i = 1, 2, 3), \\
-\exp(\sqrt{-1}\pi \kappa_4) & (i = 4), \end{cases} \]

as a function of \(\kappa \in \mathcal{K}\). Then the Riemann scheme in Table 1 implies that \(b_i\) is an eigenvalue of the monodromy matrix \(\rho(\gamma_i)\) around the point \(t_i\) and formula (14) implies that \(a_i = b_i + b_i^{-1}\). Here parameters \(b \in B\) should not be confused with the index \(b\) above. In terms of the parameters \(b \in B\), the discriminant \(\Delta(\theta)\) of the cubic surfaces \(\mathcal{S}(\theta)\) factors as

\[\Delta(\theta) = \prod_{i=1}^{4} \left(b_i - b_i^{-1}\right)^2 \prod_{\epsilon \in \{\pm 1\}^4} (b^\epsilon - 1), \quad (25)\]
The birational map of the quadratic equation in each variable $x$ lines $\sigma_q$ show for which parameters $b \in B$ the cubic surface $\mathcal{S}(\theta)$ is smooth or singular.

Let $L_i(b_i, b_4; b_j, b_k)$ denote the line in $\mathbb{P}^3$ defined by the system of linear equations

$$
X_i = (b_i b_4 + b_i^{-1} b_4^{-1}) X_0, \quad X_j + (b_i b_4) X_k = \{b_i (b_k + b_i^{-1}) + b_4 (b_j + b_j^{-1})\} X_0.
$$

Assume that $\mathcal{S}(\theta)$ is smooth, namely, $\Delta(\theta) \neq 0$. Then, as is mentioned earlier, for each $i \in \{1, 2, 3\}$ there are exactly eight lines on $\mathcal{S}(\theta)$ intersecting $L_i$, but not intersecting the remaining two lines at infinity, $L_j$ and $L_k$. They are just $\{E_i, G_j\}, \{E_j, G_i\}, \{F_{km}, F_{ln}\}, \{F_{km}, F_{im}\}$ as in Figure 7 where two lines from the same pair intersect, but ones from different pairs are disjoint. In terms of parameters $b \in B$ those eight lines are given as in Table 3.

| $i$ | $L_i(b_i, b_4; b_j, b_k)$ | $L_i(1/b_i, 1/b_4; b_j, b_k)$ |
|-----|--------------------------|-----------------------------|
| 1   | $L_i(b_i, b_4; b_j, b_k)$ | $L_i(1/b_i, 1/b_4; b_j, b_k)$ |
| 2   | $L_i(b_j, b_k; b_i, b_4)$ | $L_i(1/b_j, 1/b_k; b_i, b_4)$ |
| 3   | $L_i(1/b_i, b_4; b_j, b_k)$ | $L_i(b_i, 1/b_4; b_j, b_k)$ |
| 4   | $L_i(1/b_j, b_k; b_i, b_4)$ | $L_i(b_j, 1/b_k; b_i, b_4)$ |

Table 3: Eight lines intersecting the line $L_i$ at infinity, divided into four pairs

where $b^e = b_1^e b_2^e b_3^e b_4^e$ for each quadruple sign $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{\pm 1\}^4$. Formula (25) clearly shows for which parameters $b \in B$ the cubic surface $\mathcal{S}(\theta)$ is smooth or singular.

Involutions on Cubic Surface

The affine cubic surface $\mathcal{S}(\theta)$ is a $(2, 2, 2)$-surface, that is, its defining equation $f(x, \theta) = 0$ is a quadratic equation in each variable $x_i, i = 1, 2, 3$. Therefore the line through a point $x \in \mathcal{S}(\theta)$ parallel to the $x_i$-axis passes through a unique second point $x' \in \mathcal{S}(\theta)$ (see Figure 8). This defines an involution $\sigma_i : \mathcal{S}(\theta) \to \mathcal{S}(\theta), x \mapsto x'$, which is explicitly given by

$$
\sigma_i : (x_i', x_j', x_k') = (\theta_i - x_i - x_j x_k, x_j, x_k), \quad (i = 1, 2, 3).
$$

The automorphism $\sigma_i$ of the affine surface $\mathcal{S}(\theta)$ extends to a birational map of the projective surface $\overline{\mathcal{S}}(\theta)$, which will also be denoted by $\sigma_i$. In terms of the homogeneous coordinates $X$ of $\mathbb{P}^3$, the birational map $\sigma_i : X \mapsto X'$ is expressed as

$$
[X_i' : X_j' : X_k'] = [X_i^2 : \theta_i X_i^2 - X_i X_i + X_i X_i : X_i X_j : X_i X_k] = [X_i^2 : \theta_i X_i^2 - X_i X_i + X_i X_j : X_i X_j : X_i X_k]
$$

We shall investigate the behavior of the birational map $\sigma_i$ in a neighborhood of the tritangent lines $L$ at infinity. To this end let us introduce the following three points

$$
q_1 = [0 : 0 : 1 : 1], \quad q_2 = [0 : 1 : 0 : 1], \quad q_3 = [0 : 1 : 1 : 0],
$$

where $q_i$ may be thought of as the “mid-point” of $p_j$ and $p_k$ on the line $L_i$.

**Lemma 6.1** The birational map $\sigma_i$ has the following properties (see Figure 9).

1. $\sigma_i$ blows down the line $L_i$ to the point $p_i$, 
Figure 8: Involutions of a (2, 2, 2)-surface

(2) $\sigma_i$ restricts to the automorphism of $L_j$ that fixes $q_j$ and exchanges $p_i$ and $p_k$, 
(3) $\sigma_i$ restricts to the automorphism of $L_k$ that fixes $q_k$ and exchanges $p_i$ and $p_j$, 
(4) $p_i$ is the unique indeterminacy point of $\sigma_i$,

Proof. In order to investigate $\sigma_i$, we make use of inhomogeneous coordinates of $\mathbb{P}^3$ in (20) and 
local coordinates and local equations of $\mathcal{S}(\theta)$ in Table 2 with target coordinates being dashed.

In terms of inhomogeneous coordinates $v$ and $u'$ of $\mathbb{P}^3$, the map $\sigma_i : v \mapsto u'$ is expressed as 

$$
u'_0 = \frac{v^2_0}{\theta_i v^2_0 - v_0 v_i - v_k}, \quad u'_j = \frac{v_0}{\theta_i v^2_0 - v_0 v_i - v_k}, \quad u'_k = \frac{v_0 v_k}{\theta_i v^2_0 - v_0 v_i - v_k},$$

(28) 

In a neighborhood of $L_i - \{p_j, p_k\}$ in $\mathcal{S}(\theta)$, using $v_i = O(v_0)$, we observe that 

$$\theta_i v^2_0 - v_0 v_i - v_k = -v_k \{1 + O(v^2_0)\},$$

which is substituted into (28) to yield

$$u'_j = \frac{-v_0}{v_k \{1 + O(v^2_0)\}} = \frac{-v_0}{v_k \{1 + O(v^2_0)\}} = -v_0 \{1 + O(v^2_0)\}.$$ 

In particular putting $v_0 = 0$ leads to $u'_j = u'_k = 0$. This means that $\sigma_i$ maps a neighborhood of 
$L_i - \{p_j, p_k\}$ to a neighborhood of $p_i$, collapsing $L_i - \{p_j, p_k\}$ to the single point $p_i$.

In a similar manner, in a neighborhood of $p_j$ in $\mathcal{S}(\theta)$ we observe that

$$v_0 = -(v_i v_k) \{1 + O_2(v_i, v_k)\}, \quad \theta_i v^2_0 - v_0 v_i - v_k = -v_k \{1 + O_2(v_i, v_k)\},$$

which are substituted into (28) to yield

$$u'_j = v_i \{1 + O_2(v_i, v_k)\}, \quad u'_k = (v_i v_k) \{1 + O_2(v_i, v_k)\}.$$ 

In particular putting $v_i = 0$ leads to $u'_j = u'_k = 0$. This means that $\sigma_i$ maps a neighborhood of 
$p_j$ to a neighborhood of $p_i$, collapsing a neighborhood in $L_i$ of $p_j$ to the single point $p_i$. Using $w$ 
and $u'$ in place of $v$ and $u'$, we can make a similar argument in a neighborhood of $p_k$. Therefore 
$\sigma_i$ blows down $L_i$ to the point $p_i$, which proves assertion (1). Moreover it is clear from the 
argument that there is no indeterminacy point on the line $L_i$. 

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Figure 9: The birational map $\sigma_i$ restricted to $L$

In terms of inhomogeneous coordinates $u$ and $u'$ of $\mathbb{P}^3$ the map $\sigma_i : u \mapsto u'$ is expressed as

$$u'_0 = \frac{u_0^2}{\theta_i u_0^2 - u_0 - u_j u_k}, \quad u'_j = \frac{u_0 u_j}{\theta_i u_0^2 - u_0 - u_j u_k}, \quad u'_k = \frac{u_0 u_k}{\theta_i u_0^2 - u_0 - u_j u_k}.$$  \hspace{1cm} (29)

In a neighborhood of $L_j - \{p_i, p_k\}$ in $\mathcal{S}(\theta)$, using $u_j = -(u_k + 1/u_k)u_0 + O(u_0^2)$, we have

$$\theta_i u_0^2 - u_0 - u_j u_k = u_0 \{u_k^2 + O(u_0)\},$$

which is substituted into (29) to yield

$$u'_0 = \frac{u_0}{u_k^2 + O(u_0)} = \frac{u_0}{u_k^2} + O(u_0^2), \quad u'_k = \frac{u_k}{u_k^2 + O(u_0)} = \frac{1}{u_k} + O(u_0).$$

In particular putting $u_0 = 0$ leads to $u'_0 = 0$ and $u'_k = 1/u_k$. This means that $\sigma_i$ restricts to an automorphism of a neighborhood of $L_i - \{p_i, p_j\}$ in $\mathcal{S}(\theta)$ that induces a unique automorphism of $L_i$ fixing $q_j$ and exchanging $p_i$ and $p_k$. This proves assertion (2) and also shows that there is no indeterminacy point on $L_j - \{p_i, p_k\}$. Assertion (3) and the nonexistence of indeterminacy point on $L_k - \{p_i, p_j\}$ are established just in the same manner.

From the above argument we have already known that there is no indeterminacy point other than $p_i$. Then the point $p_i$ is actually an indeterminacy point, because $\sigma_i$ is an involution blowing down $L_i$ to $p_i$ and hence blows up $p_i$ to $L_i$ reciprocally. This proves assertion (4). \hspace{1cm} \Box

Later we will need some information about how the involution $\sigma_i$ transforms a line to another curve, which is stated in the following lemma.

**Lemma 6.2** The involution $\sigma_i$ satisfies the following properties:

1. $\sigma_i(E_i)$ intersects $E_i$ at two points counted with multiplicity,

2. $\sigma_i(E_i)$ intersects $E_j$ at one point counted with multiplicity,

3. $\sigma_i$ exchanges the lines $E_k$ and $G_l$; $E_l$ and $G_k$; $E_m$ and $G_n$; $E_n$ and $G_m$, respectively.
Proof. By Table 3 we may put \( E_i = L_i(b_i, b_4; b_j, b_k) \) and \( E_j = L_j(b_j, b_k; b_i, b_4) \). Assertion (1) of Lemma 6.1 implies that \( \sigma_i(E_i) \) does not intersect \( E_i \) nor \( E_j \) at any point at infinity. So we can work with inhomogeneous coordinates \( x = (x_1, x_2, x_3) \). In view of (26) the line \( E_i \) is given by

\[
x_i = b_i b_4 + (b_i b_4)^{-1}, \quad x_j + (b_i b_4) x_k = a_k b_i + a_j b_4.
\]

(30)

In a similar manner, by exchanging \( (b_i, b_4) \) and \( (b_j, b_k) \) in (26), the line \( E_j \) is given by

\[
x_i = b_j b_k + (b_j b_k)^{-1}, \quad x_j + (b_j b_k) x_k = a_i b_j + a_i b_k.
\]

(31)

Moreover, by applying formula (27) to (30), the curve \( \sigma_i(E_i) \) is expressed as

\[
\theta_i - x_i - x_j x_k = b_i b_4 + (b_i b_4)^{-1}, \quad x_j + (b_i b_4) x_k = a_k b_i + a_j b_4.
\]

(32)

Note that the second equations of (30) and (32) are the same.

In order to find out the intersection of \( \sigma_i(E_i) \) with \( E_i \), let us couple (30) and (32). Eliminating \( x_i \) and \( x_j \) we obtain a quadratic equation for \( x_k \),

\[
(b_i b_4) x_k^2 - (a_k b_i + a_j b_4) x_k + \theta_i - 2\{b_i b_4 + (b_i b_4)^{-1}\} = 0.
\]

For each root of this equation we have an intersection point of \( \sigma_i(E_i) \) with \( E_i \); for a double root we have an intersection point of multiplicity two. This proves assertion (1).

Next, in order to find out the intersection of \( \sigma_i(E_i) \) with \( E_j \), let us couple (31) and (32). From the first equation of (31) the \( x_i \)-coordinate is already fixed. The second equations of (31) and (32) are coupled to yield a linear system for \( x_j \) and \( x_k \), whose determinant

\[
b_j b_k - b_i b_4 = b_i b_4 (b_i^{-1} b_j b_k b_4^{-1} - 1)
\]

is nonzero from the assumption that \( \overline{S}(\theta) \) is smooth, that is, the discriminant \( \Delta(\theta) \) in (25) is nonzero. Then the linear system is uniquely solved to determine \( x_j \) and \( x_k \). Now we can check that the first equation of (32) is redundant, that is, automatically satisfied. Therefore \( \sigma_i(E_i) \) and \( E_j \) has a simple intersection, which implies assertion (2).

Finally we see that \( \sigma_i \) exchanges \( E_k \) and \( G_i \). We may put \( E_k = L_j(b_j, b_4; b_k, b_i) \) and \( G_i = L_j(1/b_j, 1/b_4; b_k, b_i) \). By formula (23) (with indices suitably permuted), these lines are given by

\[
x_j = b_j b_4 + (b_j b_4)^{-1}, \quad x_k + (b_j b_4) x_i = a_i b_j + a_k b_4,
\]

(33)

\[
x_j = b_j b_4 + (b_j b_4)^{-1}, \quad x_k + (b_j b_4)^{-1} x_i = a_i b_j^{-1} + a_k b_4^{-1}.
\]

(34)

Using formula (27) we can check that equations (33) and (34) are transformed to each other by \( \sigma_i \). This together with similar argument for the other lines establishes assertion (3). \( \square \)

7 Dynamical System on Cubic Surface

Let \( G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \) be the group of birational transformations on \( \overline{S}(\theta) \) generated by the involutions \( \sigma_1, \sigma_2, \sigma_3 \). We are interested in the dynamics of the \( G \)-action on \( \overline{S}(\theta) \). Usually the dynamics of a group action is more involved than that of a single transformation; more techniques and tools have been developed for the latter rather than for the former. So it may
be better to pick up a single transformation from the group $G$ and study its dynamics. For such a transformation we take a composition of the three basic involutions,

$$c = \sigma_i \circ \sigma_j \circ \sigma_k : \mathcal{S}(\theta) \, \circlearrowright. \quad (35)$$

If $G$ is regarded as a nonlinear reflection group with basic ‘reflections’ $\sigma_1$, $\sigma_2$, $\sigma_3$, then $c$ may be thought of as a ‘Coxeter’ transformation and it is expected that the dynamics of the transformation $c$ plays a dominant role in understanding the dynamics of the whole $G$-action.

The relevance of the transformation $(35)$ to our main problem is stated as follows.

**Lemma 7.1** Via the Riemann-Hilbert correspondence [17] the Pochhammer-Poincaré map $\varphi_* : \mathcal{M}_2(\kappa) \, \circlearrowright$ is strictly conjugated to the square $c^2 : \mathcal{S}(\theta) \, \circlearrowright$ of the Coxeter transformation $(62)$, restricted to the affine part $\mathcal{S}(\theta)$ of the cubic surface $\mathcal{S}(\theta)$.

**Proof.** Since the transformation $g_i$ in $(18)$ is a strict conjugacy of the half-Poincaré map $\beta_i$, in $(12)$, a glance at $(11)$ and $(12)$ shows that the commutator $[g_i^2 : g_j^2] = g_i^2 g_j^{-2} g_i^{-2} g_j^2$ is a strict conjugacy of the Pochhammer-Poincaré map $\varphi_*$. On the other hand, using $(18)$ and $(27)$, we can directly check that $g_i^2 g_j^{-2} g_i^{-2} g_j^2 = (\sigma_i \sigma_j \sigma_k)^2 = c^2$. Hence $c^2$ is a strict conjugacy of $\varphi_*$. \[\square\]

A general theory of dynamical systems for bimeromorphic maps of surfaces is developed in [5]. We shall apply it to our map $(35)$ upon reviewing some rudiments of the article [5]. Let $S$ be a compact complex surface, $f : S \, \circlearrowright$ a bimeromorphic map. Then $f$ is represented by a compact complex surface $\Gamma$ and proper modifications $\pi_1 : \Gamma \rightarrow S$ and $\pi_2 : \Gamma \rightarrow S$ such that $f = \pi_2 \circ \pi_1^{-1}$ on a dense open subset. For $i = 1, 2$, let $\mathcal{E}(\pi_i) := \{ x \in \Gamma : \# \pi_i^{-1}(\pi_i(x)) = \infty \}$ be the exceptional set for the projection $\pi_i$. The images $I(f) := \pi_1(\mathcal{E}(\pi_1))$ and $\mathcal{E}(f) := \pi_1(\mathcal{E}(\pi_2))$ are called the indeterminacy set and the exceptional set of $f$ respectively. In our case where $S = \mathcal{S}(\theta)$ and $f = \sigma_i$, Lemma 6.1 implies that these sets are described as follows.

**Lemma 7.2** $I(\sigma_i) = \{ p_i \}$ and $\mathcal{E}(\sigma_i) = L_i$ for $i = 1, 2, 3$.

If $S$ is Kähler, two natural actions of $f$, pull-back and push-forward, on the Dolbeault cohomology group $H^{1,1}(S)$ are defined in the following manner: A smooth $(1, 1)$-form $\omega$ on $S$ can be pulled back as a smooth $(1, 1)$-form $\pi_2^* \omega$ on $\Gamma$ and then pushed forward as a $(1, 1)$-current $\pi_1 \circ \pi_2^{-1} \circ \pi_1^* \omega$ on $S$. Hence we define the pull-back $f^* \omega := \pi_1 \circ \pi_2^{-1} \circ \pi_1^* \omega$ and also the push-forward $f_* \omega = (f^{-1})^* \omega := \pi_2 \circ \pi_1^* \omega$. The operators $f^*$ and $f_*$ commute with the exterior differential $d$ and the complex structure of $S$ and so descend to linear actions on $H^{1,1}(S)$. For general bimeromorphic maps $f$ and $g$, the composition rule $(f \circ g)^* = g^* \circ f^*$ is not necessarily true. But a useful criterion under which this rule becomes true is given in [5].

**Lemma 7.3** If $f(\mathcal{E}(f)) \cap I(g) = \emptyset$, then $(f \circ g)^* = g^* \circ f^* : H^{1,1}(S) \, \circlearrowright$.

We apply this lemma to our Coxeter transformation $c = \sigma_i \circ \sigma_j \circ \sigma_k$.

**Lemma 7.4** We have $c^* = \sigma_k^* \circ \sigma_j^* \circ \sigma_i^* : H^{1,1}(\mathcal{S}(\theta)) \, \circlearrowright$.

**Proof.** First we apply Lemma 7.3 to $f = \sigma_i$ and $g = \sigma_j \circ \sigma_k$. By Lemmas 6.1 and 7.2 we have $\mathcal{E}(\sigma_i) = L_i$ and $I(\sigma_j \circ \sigma_k) = \{ p_k \}$ and so $\sigma_i(\mathcal{E}(\sigma_i)) \cap I(\sigma_j \circ \sigma_k) = \{ p_i \} \cap \{ p_k \} = \emptyset$, which means that the condition of Lemma 7.3 is satisfied. Then the lemma yields $(\sigma_i \circ \sigma_j \circ \sigma_k)^* = (\sigma_j \circ \sigma_k)^* \circ \sigma_i^*$. Next we apply Lemma 7.3 to $f = \sigma_j$ and $g = \sigma_k$. Again by Lemmas 6.1 and 7.2 we have $\mathcal{E}(\sigma_j) = L_j$ and $I(\sigma_k) = \{ p_k \}$ and so $\sigma_j(\mathcal{E}(\sigma_j)) \cap I(\sigma_k) = \{ p_j \} \cap \{ p_k \} = \emptyset$, which means that
these two steps together, we obtain 

$$\sigma_i^* \sigma^j = \begin{pmatrix} 6 & 2 & 2 & 2 & 2 & 2 \\ -3 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & -2 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & 1 & 0 \end{pmatrix} \quad \sigma^k = \begin{pmatrix} 6 & 2 & 2 & 3 & 3 & 2 & 2 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \end{pmatrix}$$

Table 4: Matrix representations of $\sigma^*_i, \sigma^*_j, \sigma^*_k, c^* : H^2(\mathcal{S}(\theta), \mathbb{Z}) \otimes$
where in the third equality we have used the fact that \( \sigma_i \) is an involution; \( \sigma_{i\ast} = (\sigma_i^{-1})^\ast \). By assertions (1) and (2) of Lemma 6.2 we have \((\sigma_i^\ast E_i, E_i) = 2 \) and \((\sigma_i^\ast E_i, E_j) = 1 \) and likewise \((\sigma_i^\ast E_j, E_j) = 2 \) and \((\sigma_i^2 E_j, E_i) = 1 \). Then the first formula of (37) yields

\[
\xi_{ii} = \xi_{jj} = -2, \quad \xi_{ij} = \xi_{ji} = -1.
\]

The assertion (3) of Lemma 6.2 together with the second formula of (24) yields

\[
\begin{align*}
\sigma_i^\ast E_k &= 2E_0 - E_i - E_j - E_k - E_m - E_n, \\
\sigma_i^\ast E_l &= 2E_0 - E_i - E_j - E_l - E_m - E_n, \\
\sigma_i^\ast E_m &= 2E_0 - E_i - E_j - E_k - E_l - E_m, \\
\sigma_i^\ast E_n &= 2E_0 - E_i - E_j - E_k - E_l - E_n,
\end{align*}
\]

It follows from (38) and (39) that the matrix representation for \( \sigma_i^\ast \) takes the form

\[
\sigma_i^\ast = \begin{pmatrix}
\ast & \ast & \ast & 2 & 2 & 2 \\
\ast & -2 & -1 & -1 & -1 & -1 \\
\ast & -1 & -2 & -1 & -1 & -1 \\
\ast & \ast & \ast & -1 & 0 & -1 & -1 \\
\ast & \ast & \ast & 0 & -1 & -1 & -1 \\
\ast & \ast & \ast & -1 & -1 & -1 & 0 \\
\ast & \ast & \ast & -1 & -1 & 0 & -1
\end{pmatrix},
\]

where the entries denoted by \( \ast \) are yet to be determined. The \((2,1)\)-block of (40) is easily determined by the second formula in (37). The final ingredient taken into account is the fact that \( \sigma_i \) blows down \( L_i = E_0 - E_i - E_j \) to a point \( p_i \) (see Lemma 6.1), which leads to

\[
\sigma_i^\ast E_0 - \sigma_i^\ast E_i - \sigma_i^\ast E_j = 0.
\]

This means that the first column is the sum of the second and third columns in the matrix (40). Using the second formula in (37) repeatedly, we see that the matrix representation of \( \sigma_i^\ast \) is given as in the first matrix of Table 4. Those of \( \sigma_j^\ast \) and \( \sigma_j^\ast \) are obtained in the same manner. Applying Lemma 7.4 to these results yields the desired representation for \( c^\ast \) as in the last matrix of Table 4. Now it is easy to calculate the characteristic polynomial of \( c^\ast \) as in (30). The assertion for its roots, namely, for the eigenvalues of \( c^\ast \) is straightforward.

We recall some more rudiments from [5]. Given a bimeromorphic map \( f \) of a compact Kähler surface \( S \), there is the concept of first dynamical degree \( \lambda_1(f) \) defined by

\[
\lambda_1(f) := \lim_{n \to \infty} \|(f^n)^\ast\|^{1/n},
\]

where \( \| \cdot \| \) is an operator norm on \( \text{End} H^{1,1}(S) \). The limit certainly exists and one has \( \lambda_1(f) \geq 1 \). It is usually difficult to evaluate this number in a simple mean. But there is a distinguished class of bimeromorphic maps for which the first dynamical degree can be equated to a more tractable quantity, namely, the class of maps which are called analytically stable. Here a bimeromorphic map \( f : S \to S \) is said to be analytically stable (AS for short) if for any \( n \in \mathbb{N} \) there is no curve \( V \subset S \) such that \( f^n(V) \subset I(f) \). From [5] we have the following lemma.
Lemma 7.6 If \( f : S \circlearrowleft \) is an AS bimeromorphic map, then the first dynamical degree \( \lambda_1(f) \) is equal to the spectral radius \( \rho(f^*) \) of the linear map \( f^* : H^{1,1}(S) \circlearrowleft \).

With this lemma in hand we continue to investigate the Coxeter transformation (35).

Proposition 7.7 The birational map \( c = \sigma_i \circ \sigma_j \circ \sigma_k \) enjoys the following properties:

1. its indeterminacy set is given by \( I(c) = \{ p_k \} \),
2. its exceptional set is given by \( \mathcal{E}(c) = L \) with image \( c(\mathcal{E}(c)) = \{ p_i \} \),
3. its tangent map \( (dc)_{p_i} \) at \( p_i \) is zero, that is, \( p_i \) is a superattracting fixed point,
4. it is AS, and
5. its first dynamical degree is given by \( \lambda_1(c) = 2 + \sqrt{5} \).

Proof. Lemma [6.1] implies that \( \sigma_k^{-1}(I(\sigma_j)) = \{ p_k \} \) and \( \sigma_k^{-1} \circ \sigma_j^{-1}(I(\sigma_i)) = \sigma_k^{-1}(\{ p_j \}) = \{ p_k \} \). Thus the indeterminacy set of \( c \) is given by \( I(c) = \{ p_k \} \), which proves assertion (1). In order to see assertion (2), we again apply Lemma 6.1 to obtain

\[
\begin{align*}
c(L_i) &= \sigma_i \circ \sigma_j \circ \sigma_k(L_i) = \sigma_i(L_i) = \{ p_i \}, \\
c(L_j) &= \sigma_i \circ \sigma_j \circ \sigma_k(L_j) = \sigma_i(\{ p_j \}) = \{ p_i \}, \\
c(L_k) &= \sigma_i \circ \sigma_j \circ \sigma_k(L_k) = \sigma_i(\{ p_k \}) = \{ p_i \}.
\end{align*}
\]

This means that \( \mathcal{E}(c) \) is given by the union \( L = L_i \cup L_j \cup L_k \) with \( c(\mathcal{E}(c)) = \{ p_i \} \). Thus assertion (2) follows. From assertion (2) we notice that \( p_i \) is a fixed point of \( c \) and all points on \( L_j - \{ p_k \} \) and on \( L_k - \{ p_j \} \) are taken to the point \( p_i \) by \( c \). So the tangent map \( (dc)_{p_i} \) is zero along the linearly independent directions of the lines \( L_j \) and \( L_k \) with origin at \( p_i \) and hence \( (dc)_{p_i} \) itself is zero, which proves assertion (3). We show assertion (4) by contradiction. Assume that \( V \subset \overline{S}(\theta) \) is an irreducible curve such that \( c^n(V) \subset I(c) = \{ p_k \} \) for some \( n \in \mathbb{N} \). If \( V \) intersects the affine surface \( S(\theta) \), then it cannot happen that \( c^n(V) \subset \{ p_k \} \), because \( c \) is bijective on \( S(\theta) \). Thus \( V \) must lie in \( L \) and hence \( V = L_i, L_j, \) or \( L_k \). But also in this case assertion (2) implies that \( c^n(L_a) = \{ p_k \} \) for \( a = i, j, k \), leading to a contradiction. Hence assertion (4) is proved. Finally, since \( c \) is AS, Proposition [7.5] and Lemma [7.6] immediately imply assertion (5). □

We conjecture that the topological entropy of \( c \) agrees with the logarithm of its first dynamical degree:

\[
h_{\text{top}}(c) = \log \lambda_1(c) = \log(2 + \sqrt{5}).
\]

8 Lefschetz Fixed Point Formula

We are interested in the periodic points of the Coxeter transformation \( c : \overline{S}(\theta) \circlearrowleft \). Given any \( N \in \mathbb{N} \) we can consider the set of periodic points of period \( N \) on the projective surface \( \overline{S}(\theta) \),

\[
\text{Per}_N(c) := \{ X \in \overline{S}(\theta) - I(c^N) : c^N(X) = X \}.
\]

as well as the set of periodic points of period \( N \) on the affine surface \( S(\theta) \),

\[
\text{Per}_N(c) := \{ x \in S(\theta) : c^N(x) = x \}.
\]

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Lemma 8.1

For any \( N \in \mathbb{N} \), the Coxeter transformation \( c : \mathfrak{S}(\theta) \overset{\circ}{\rightarrow} \) admits no curves of periodic points of period \( N \).

Proof. By Proposition 7.8, the Coxeter transformation \( c^* : H^2(\mathfrak{S}(\theta), \mathbb{Z}) \overset{\circ}{\rightarrow} \) has eigenvalues 0, -1, \( 2 \pm \sqrt{5} \), among which 0 and \( 2 \pm \sqrt{5} \) are simple eigenvalues, while -1 is a quadruple eigenvalue whose eigenspace is spanned by four eigenvectors

\[
V_0 = 2E_0 - E_i - E_j - E_k - E_l - E_m - E_n, \quad V_i = E_i - E_j, \quad V_j = E_k - E_l, \quad V_k = E_m - E_n.
\]

In view of (21), (23) and (24), there are orthogonality relations

\[
(V_a, L_b) = 0 \quad (a = 0, i, j, k, \quad b = i, j, k). \tag{42}
\]

We prove the lemma by contradiction. Assume that \( c \) admits a curve (an effective divisor) \( D \subset \mathfrak{S}(\theta) \) of periodic points of some period \( N \). Then we have \((c^*)^N D = (c^N)^* D = D \) in \( H^2(\mathfrak{S}(\theta), \mathbb{Z}) \), where \((c^*)^N = (c^N)^* \) follows from the fact that \( c \) is AS. So \((c^*)^N \) has eigenvalue 1 with eigenvector \( D \). This eigenvalue arises as the \( N \)-th power of eigenvalue \(-1\) of \( c^* \) so that \( N \) must be even and \( D \) must be a linear combination of \( V_0, V_i, V_j, V_k \). Hence (42) implies that

\[
(D, L_a) = 0 \quad (a = i, j, k). \tag{43}
\]

We now write \( D = D' + m_i L_i + m_j L_j + m_k L_k \), where \( D' \) is either empty or an effective divisor not containing \( L_i, L_j, L_k \) as an irreducible component of it and \( m_i, m_j, m_k \) are nonnegative integers. Since \((L_a, L_b) = -1\) for \( a = b \) and \((L_a, L_b) = 1\) for \( a \neq b \), the formula (43) yields

\[
0 = (D, L_i) = (D', L_i) - m_i + m_j + m_k,
\]

\[
0 = (D, L_j) = (D', L_j) + m_i - m_j + m_k,
\]

\[
0 = (D, L_k) = (D', L_k) + m_i + m_j - m_k,
\]

which sum up to

\[
(D', L_i) + (D', L_j) + (D', L_k) + m_i + m_j + m_k = 0. \tag{44}
\]
Since none of $L_i$, $L_j$, $L_k$ is an irreducible component of $D'$, the intersection number $(D', L_a)$ must be nonnegative for any $a = i, j, k$. Since $m_i, m_j, m_k$ are also nonnegative, formula (44) implies that $(D', L_i) = (D', L_j) = (D', L_k) = 0$ and $m_i = m_j = m_k = 0$. Hence $D = D'$ and $(D, L_i) = (D, L_j) = (D, L_k) = 0$. It follows that $D$ is an effective divisor with $(D, L_a) = 0$ not containing $L_a$ as its irreducible component for every $a = i, j, k$. This means that the compact curve $D$ does not intersect $L = L_i \cup L_j \cup L_k$ and hence must lie in the affine cubic surface $S(\theta) = \overline{S}(\theta) - L$. But no compact curve can lie in any affine variety. This contradiction establishes the lemma. \[ \Box \]

For each $N \in \mathbb{Z}$ let $\Gamma_N \subset \overline{S}(\theta) \times \overline{S}(\theta)$ be the graph of the $N$-th iterate $c^N : \overline{S}(\theta) \to \overline{S}(\theta)$, and $\Delta \subset \overline{S}(\theta) \times \overline{S}(\theta)$ be the diagonal. Note that $\Gamma_N = \Gamma_N^\vee$, where $\Gamma_N^\vee$ is the reflection of $\Gamma_N$ with respect to the diagonal $\Delta$. Moreover let $I_N \subset \overline{S}(\theta)$ denote the indeterminacy set of $c^N$. Then the Lefschetz fixed point formula consists of two equations concerning the intersection number $(\Gamma_N, \Delta)$ of $\Gamma_N$ and $\Delta$ in $\overline{S}(\theta) \times \overline{S}(\theta)$,

$$ (\Gamma_N, \Delta) = \sum_{q=0}^4 (-1)^q \text{Tr} \left[ \left( c^N \right)^q : H^q(\overline{S}(\theta), \mathbb{Z}) \right], $$

$$ (\Gamma_N, \Delta) = \# \text{Per}_N(c) + \sum_{p \in I_N} \mu((p, p), \Gamma_N \cap \Delta), $$

where $\mu((p, p), \Gamma_N \cap \Delta)$ denotes the multiplicity of intersection between $\Gamma_N$ and $\Delta$ at $(p, p)$. Lemma 8.1 guarantees that all terms involved in (45) and (46) are well defined and finite.

**Lemma 8.2** Formula (45) becomes $(\Gamma_N, \Delta) = (2 + \sqrt{5})^N + (2 - \sqrt{5})^N + 4(-1)^N + 2$.

**Proof.** We put $T_N^0 = \text{Tr} \left[ \left( c^N \right)^0 : H^0(\overline{S}(\theta), \mathbb{Z}) \right]$. Because $\overline{S}(\theta)$ is a smooth rational surface,

$$ H^q(\overline{S}(\theta), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (q = 0, 4), \\ 0 & (q = 1, 3). \end{cases} $$

Naturally we have $T_N^0 = 1$ and $T_N^1 = T_N^2 = 0$. Since $c$ and so $c^N$ are birational, we have $T_N^3 = 1$. By assertion (4) of Proposition 7.7 the map $c$ is AS, and so Lemma 7.3 implies that $(c^N)^* = (c^*)^N : H^2(\overline{S}(\theta), \mathbb{Z}) \to \overline{S}(\theta)$. Recall that $c^*$ has eigenvalues $0$, $-1$ and $2 \pm \sqrt{5}$, where the eigenvalue $-1$ is quadruple while the remaining ones are simple (see Proposition 7.5). Thus we have $T_N^2 = 0^N + 4(-1)^N + (2 + \sqrt{5})^N + (2 - \sqrt{5})^N$. Substituting these data into (45) yields the assertion of the lemma. \[ \Box \]

**Lemma 8.3** Formula (46) becomes $(\Gamma_N, \Delta) = \# \text{Per}_N(c) + 1$ with $\# \text{Per}_N(c) = \# \text{Per}_N(c) + 1$.

**Proof.** By Proposition 7.7, for any $N \in \mathbb{N}$, the point $p_k$ is the unique indeterminacy point of $c^N$ and the point $p_i$ is the unique fixed point of $c^N$ on $L$. Namely we have $I_N = \{ p_k \}$ and $\text{Per}_N(c) = \text{Per}_N(c) \cup \{ p_i \}$, which implies that formula (46) is rewritten as

$$ (\Gamma_N, \Delta) = \# \text{Per}_N(c) + \mu((p_k, p_k), \Gamma_N \cap \Delta), $$

$$ \# \text{Per}_N(c) = \# \text{Per}_N(c) + \nu(p_i, c^N), $$

where $\nu(p_i, c^N)$ is the local index of the map $c^N$ around the fixed point $p_i$. By assertion (3) of Proposition 7.7, for any $N \in \mathbb{N}$, the point $p_i$ is a superattracting fixed point of $c^N$ and so

$$ 23 $$
Figure 10: The indeterminacy point \( p_k \) of \( c^N \) is a superattracting fixed point of \( c^{-N} \)

\[
\det(I - (dc^N)_{p_i}) = \det(I - O) = 1.
\]

This means that \( \nu(p_i, c^N) = 1 \). Likewise, since \( p_k \) is a superattracting fixed point of \( c^{-N} = (c^{-1})^N \) where \( c^{-1} = \sigma_k \circ \sigma_j \circ \sigma_i \) (see Figure 10), the same reasoning as above with \( c \) replaced by \( c^{-1} \) yields \( \nu(p_k, c^{-N}) = 1 \). Therefore we have

\[
\mu((p_k, p_k), \Gamma_N \cap \Delta) = \mu((p_k, p_k), \Gamma^\vee_N \cap \Delta) = \mu((p_k, p_k), \Gamma^-_N \cap \Delta) = \nu(p_k, c^{-N}) = 1.
\]

These arguments imply that (47) is equivalent to the statement of the lemma. \( \square \)

Putting Lemmas 8.2 and 8.3 together, we have established the following theorem.

**Theorem 8.4** For any \( N \in \mathbb{N} \), the cardinalities of periodic points of period \( N \) are given by

\[
\# \text{Per}_N(c) = (2 + \sqrt{5})^N + (2 - \sqrt{5})^N + 4(-1)^N + 1,
\]

\[
\# \text{Per}_N(c) = (2 + \sqrt{5})^N + (2 - \sqrt{5})^N + 4(-1)^N.
\]

Then our main theorem (Theorem 2.1) is an immediate consequence of (44) and the second formula of (48). Thus the proof of Theorem 2.1 has just been completed.

In this article we have seen that geometry of cubic surfaces and dynamics on them played an important part in understanding an aspect of the global structure of the sixth Painlevé equation. Their relevance to other aspects will be explored elsewhere.

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