Constellation ensembles and interpolation in ensemble averages

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Abstract We introduce constellation ensembles, in which charged particles on a line (or circle) are linked with charged particles on parallel lines (or concentric circles). We present formulas for the partition functions of these ensembles in terms of either the Hyperpfaffian or the Berezin integral of an appropriate alternating tensor. Adjusting the distances between these lines (or circles) gives an interpolation between a pair of limiting ensembles, such as one-dimensional \(\beta\)-ensembles with \(\beta = K\) and \(\beta = K^2\).

1 Introduction

Consider the one-dimensional log-gas particle model (whose relationship with the classical \(\beta\)-ensembles of random matrix theory was first observed by Dyson in [5] and further developed by Forrester in [6]). A finite number of charged particles are placed on an infinite wire represented by the real line. The charges of the particles are assumed to be integers with the same sign, and the particles are assumed to repel each other with logarithmic interactions. We assume any two particles of the same charge are indistinguishable. The wire is imbued with a potential which discourages the particles from escaping to infinity in either direction, and heat is applied to the system according to a parameter we call inverse temperature \(\beta\).

Herein, we modify this setup by copying the system onto a parallel line (translated vertically in the complex plane). Note, the particle locations on the new line are not independent of the original particle locations. In particular, each particle in the combined system is bound to an image at the same horizontal position (same real part). Despite these dependencies, we allow the new particles to interact (by repulsion, contributing additional potential energy to the system) as independent particles would. In addition to the internal interactions between particles on the same line, particles from different lines are able to interact with each other, with the strength of this interaction depending on the distance between the lines. The resulting model is an example of what we will call a linear constellation ensemble, with constellation referring to a pair of linked particles.

Recalling the dependence between linked particles (whose locations share a real part), we should think of one matching pair as representative of a single object whose location is described by a real variable. Then, there is an induced interaction between one pair and another (equivalently, one real variable and another), different from, but still related to, the normal logarithmic interaction between individual log-gas particles. These induced interactions are what is primarily of interest to us, and we will consider several variations on this setup:

1. The \((K\text{-fold})\) First Constellation Ensemble, in which charge \(L = 1\) particles are copied onto \(K\) many parallel lines, subject to inverse temperature \(\beta = 1\).
2. The \((K\text{-fold})\) Monocharge Constellation Ensemble, in which particles of the same integer charge \(L\) are copied onto \(K\) many lines.
3. The \((K\text{-fold})\) Homogeneous Constellation Ensemble, in which particles on the same line have the same integer charge \(L_k\), but particles on different lines may have different charges.
4. The \((K\text{-fold})\) Multicomponent Constellation Ensemble, in which the original line may have particles of different charges, but all the parallel lines are copies, featuring the same charges in the same horizontal positions.

The first is a special case of the second, which is a special case of either the third or the fourth. In Sect. 2.5, we will also consider circular constellation ensembles of concentric circles in the complex plane.

These ensembles of constellations should be thought of as a natural generalization of the method of image charges, wherein a system of imaginary particles are contrived for the purpose of gaining insight into a real system (see section 15.9 of [6]). In fact, allowing the images to be oppositely charged (as is traditionally the case with image charges) falls within the scope of our algebraic machinery, but that model (and its analogues) will not be the focus of this document. Moreover, constellations of linked particles behave somewhat like the complex conjugate pairs of eigenvalues observed in the real Ginibre ensemble (random matrices with all real entries), discussed by Forrester and Nagao in [10], and further explored by Borodin and Sinclair in [3].

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Next, by varying the distances between the parallel lines (or concentric circles), we obtain an interpolation between limiting ensembles of classical importance. For example, taking the limit of the first constellation ensemble as the distances between the lines (or circles) goes to zero produces a one-dimensional $\beta = K^2$ ensemble. On the other end, taking the limit as the distances between the lines (or circles) increases without bound produces a one-dimensional $\beta = K$ ensemble.

Historically, a crucial step toward solvability for several random matrix models, as well as some more general log-gas models, has been expressing the partition functions in particularly exploitable algebraic forms (in terms of determinants and Pfaffians). The relevant correlation functions are then expressed in a related and similarly accessible algebraic form. In this document, we focus on that first step, deriving analogous algebraic expressions for the partition functions of our constellation ensembles. Note, the partition functions of more complicated constellation ensembles are expressed in terms of Hyperpfaffians and Berezin integrals (see Sect. 4) instead of the better understood determinant or Pfaffian. As such, although the Hyperpfaffian expressions are honest generalizations of the Pfaffian expressions, it is not yet known what the next (algebraic) steps toward solvability should be in the Hyperpfaffian case. However, our constellation setup should be thought of as a general strategy still able to produce new models with Pfaffian partition functions, some of which are legitimately solvable by known methods.

For example, in the realm of solvable log-gas models, we are highly limited in which $\beta$ the known methods can handle. However, in [17], Shum demonstrated the 2-fold first circular constellation ensembles (of charge $L = 1$ particles on only $K = 2$ concentric circles) to be completely solvable (with Pfaffian correlation functions), even with the distance between the concentric circles left as an unknown parameter. This gave a continuum of solvable models between the previously solvable $\beta = 2$ and $\beta = 4$ ensembles.

Consider instead charge $L = 1$ particles on a curve in the upper half plane. This model is desirable because behavior near intersections and cusps will be entirely new, but this model is not readily solvable by known methods. However, if we reflect the system across the real axis, pairing each particle with an image on the conjugate curve in the lower half plane, the new model will have a Pfaffian partition function, inviting solvability (in theory, depending on how elaborate the curve is). For example, Shum also explored paired particles on the unit circle (obtained by reflecting the upper half circle) in addition to paired particles on a vertical line (obtained by reflecting the positive imaginary axis).

On one hand, insight gained from the constellation model (such as behavior near intersections and cusps) can be applied to the original (one curve) model from which it was generated. On the other hand, the presence of the image particles produces new behavior as well. As previously mentioned, the induced repulsion between a linked pair of particles and another linked pair is stronger when the curve is closer to the real line (and therefore its complex conjugate reflection), corresponding to a higher effective $\beta$ (hot spot). Conversely, the induced repulsion is weaker when the curve is further from the real line, corresponding to a lower effective $\beta$ (cold spot). As such, we can think of this (nonlinear) constellation model as equivalent to a one-dimensional log-gas subject to a temperature gradient, transitioning between regions of higher and lower temperature (lower and higher effective $\beta$, respectively).

### 1.1 The monocharge setup

Let $\tilde{x} \in \mathbb{R}^M$, and let $\tilde{y} \in \mathbb{R}^K$ such that $0 \leq y_1 < \cdots < y_K$. We call $\tilde{y}$ the translation vector of the system, giving the locations of the $K$ many lines $\mathbb{R} + i y_k$ in the complex plane. Consider $M$ many charge $L \in \mathbb{Z}_{\geq 0}$ particles on each line $\mathbb{R} + iy_k$ having the same real parts, meaning for each location $x_m \in \mathbb{R}$, and $1 \leq k \leq K$, there is a charge $L$ particle at location $x_m + iy_k$. Denote the (total $KM$) particle locations by

$$x = (x_1^1, x_2^1, \ldots, x_M^1) \in \mathbb{C}^{KM},$$

where $x_m^k = x_m + iy_k = (x_m + iy_1, x_m + iy_2, \ldots, x_m + iy_K) \in \mathbb{C}^K$. We call $x$ the location vector of the system, in which each $x_m^k \in \mathbb{C}$ gives the location of a particle. We call $x_m^k$ the location vector of the constellation of $K$ many particles which all share the same real part $x_m$. We call $\tilde{x} = (x_1, \ldots, x_M)$ the location vector of the real parts which generate each constellation.

The particles are assumed to interact logarithmically so that the contribution of energy to the system by the two (charge $L$) particles at locations $x_m + iy_k$ and $x_n + iy_j$ is given by $-L^2 \log |(x_m + iy_k) - (x_n + iy_j)|$. Let $U : \mathbb{R} \to \mathbb{R}$ be a potential on the real axis. Let $\hat{U} : \mathbb{C} \to \mathbb{R}$ be an extension of this potential to the entire complex plane such that $\hat{U}(z) = U(\text{Re}(z))$. Without loss of generality, we can assume $x_1 < \cdots < x_M$. Then at inverse temperature $\beta$, the total potential energy of the system is given by

$$E(\tilde{x}, \tilde{y}) = \beta L \sum_{k=1}^{K} \sum_{m=1}^{M} \hat{U}(x_m + iy_k) - \beta L^2 \sum_{k=1}^{K} \sum_{n<m} \log |(x_m + iy_k) - (x_n + iy_j)|$$

$$- \beta L^2 \sum_{j \neq k} \sum_{m=1}^{M} \log |(x_m + iy_k) - (x_m + iy_j)|$$

$$- \beta L^2 \sum_{j \neq k} \sum_{n<m} \log |(x_m + iy_k) - (x_n + iy_j)| + \log |(x_n + iy_j) - (x_n + iy_k)|.$$

The first iterated sum in the first line accounts for the potential $\hat{U}$. We can substitute $\hat{U}(x_m + iy_k) = U(x_m)$ of which there are $K$ many for each $m$. The second iterated sum in the first line accounts for interactions between particles which share a line. Note, the
In the constellation model, the induced interaction between real parts \( x_m \) and \( y_n \) is given by

\[
E(\vec{x}, \vec{y}) = \beta L K \sum_{m=1}^{M} U(x_m) - \beta L^2 K \sum_{n<m}^{M} \log(x_m - x_n) - \beta L^2 M \sum_{j<k}^{K} \log|i(y_k - y_j)|
\]

\[- \beta L^2 \sum_{j<k}^{K} \sum_{n<m}^{M} \log((x_m - x_n)^2 + (y_k - y_j)^2).
\]

With this setup, the relative density of states (corresponding to varying location vectors \( \vec{x} \) and translation vectors \( \vec{y} \)) is given by the Boltzmann factor

\[
\Omega(\vec{x}, \vec{y}) = \exp(-E(\vec{x}, \vec{y}))
\]

\[
= |\Delta(x)|^{\beta L^2} \prod_{m=1}^{M} e^{-\beta L K U(x_m)}
\]

\[
= \Delta(\vec{x})^{\beta L^2} \prod_{m=1}^{M} \left( (-i)^{L(K-1)/2} e^{-U(x_m)} \right)^{\beta L K},
\]

where \( \Delta(x) \) denotes the Vandermonde determinant, evaluated at the variables \( x \). Note, the last equality comes from \(|i| = (i)(-i)\), of which there are \( \beta L^2 M(K/2) \) many instances. Observe, in a one-dimensional log-gas, the interaction between one particle at location \( x_m \) and another particle at location \( x_n \) is given by

\[
d(x_m, x_n) = |x_m - x_n|^{\beta L^2}.
\]

In the constellation model, the induced interaction between real parts \( x_m \) and \( y_n \) is given by

\[
d(\vec{x}, \vec{y}) = \left( |x_m - x_n|^{K} \prod_{j<k} \left| (x_m - x_n)^2 + (y_k - y_j)^2 \right| \right)^{\beta L^2},
\]

so that

\[
d(\vec{x}, \vec{y}) = \lim_{y_k \to y_j \to \infty} \frac{d(\vec{x}, \vec{y})}{\prod_{j<k} \left( 1 + (y_k - y_j)^2 \right)} = d(x_m, x_n)^{K} = |x_m - x_n|^{\beta L^2 K},
\]

while

\[
d(\vec{x}, \vec{y}) = \lim_{y_k \to y_j \to 0} \frac{d(\vec{x}, \vec{y})}{\prod_{j<k} \left( 1 + (y_k - y_j)^2 \right)} = d(x_m, x_n)^{K^2} = |x_m - x_n|^{\beta L^2 K^2}.
\]

Thus, we can interpolate between an induced \( \tilde{\beta} = \beta K^2 \) as the lines collapse onto the real line and an induced \( \tilde{\beta} = \beta K \) as the lines escape from each other.

Returning to translation vector \( \vec{y} \) fixed, the probability of finding the system in a state corresponding to a location vector \( \vec{x} \) is given by the joint probability density function

\[
\rho(\vec{x}, \vec{y}) = \frac{\Omega(\vec{x}, \vec{y})}{Z_M(\vec{y})},
\]

where the partition function (of the \( K \)-fold monocharge constellation ensemble) \( Z_M(\vec{y}) \) is the normalization constant given by

\[
Z_M(\vec{y}) = \int_{-\infty < x_1 < \cdots < x_M < \infty} \Omega(\vec{x}, \vec{y}) \, dx_1 \cdots dx_M
\]

\[
= \int_{-\infty < x_1 < \cdots < x_M < \infty} \Delta(\vec{x})^{\beta L^2} \, d\mu(x_1) \cdots d\mu(x_M),
\]

in which \( d\mu(x) = (\frac{-i}{L(K-1)/2}) e^{-U(x)} \) as the potential \( U \) is one for which \( Z_M(\vec{y}) \) is finite.
Note, unit charges (meaning L = 1) at inverse temperature β = b² have the same Boltzmann factor (and resulting density function) as charge L = b particles at inverse temperature β = 1 (subject to different but related potentials U(x)). In general, replacing β with β = β/b² and replacing L with L = bL leaves Δ(x)βL² unchanged. Then replacing U with U = bU leaves Ω(\vec{x}, \vec{y}) unchanged. Thus, for computational purposes, we can change to β = 1 (provided √βL ∈ Z for the original β).

The partition function Z_M(\vec{y}) and its analogues are the primary objects of interest to us. Though we assume β = 1 for computational purposes, Z_M(\vec{y}) is inherently a function of β, among other parameters. The potential U dictates the external forces experienced by each particle individually, affecting the (complex) measures μ against which we are integrating. The charge L and the inverse temperature β influence the strength of the interactions between the particles, affecting the exponents on the interaction terms in the Boltzmann factor. Unique to the constellation setup, as y_k - y_j → 0, the corresponding interaction factors shrink, and the potential energy grows. Conversely, as y_k - y_j → ∞, the corresponding interaction factors grow, and the potential energy shrinks.

Recall, this Z_M(\vec{y}) is an iterated integral in M many variables. In the style of Sinclair in [18] and [19], and using the methods established by Wells and Wolff in [21], our goal herein is not to compute these integrals for any particular choice of several parameters. Instead, we demonstrate, in general, how to write Z_M(\vec{y}) as a Hyperpfaffian (or Berezin integral in the multicomponent case) of an alternating tensor whose coefficients are only single or double integrals of (potentially orthogonal or skew orthogonal) polynomials.

1.2 Example formulae

Let \vec{p} = \{p_j\}_{j=1}^{2M} be a complete 2M-family of monic polynomials, meaning each p_j is a monic polynomial of degree j - 1. Let A be the matrix whose entries are defined by a skew inner product

\[ A_{j,k} = \langle p_j, p_k \rangle \]

\[ = \int_{-\infty}^{\infty} \det \left[ p_j(x) p'_k(x) \right] d\mu(x) \]

\[ = \int_{-\infty}^{\infty} \left( p_j(x)p_k'(x) - p_j'(x)p_k(x) \right) d\mu(x). \]

When K = 1, β = 1, and L = 2 (equivalently, β = 4 and L = 1), such as in the case of charge L = 2 particles on the real line, the partition function Z_M (of the classical β = 4 ensemble and the corresponding log-gas model) is given by

\[ Z_M = \int_{-\infty x_1 < \cdots < x_M < \infty} \prod_{j<k} |x_k - x_j|^4 \prod_{j=1}^{M} d\mu(x_j) = \text{Pf} A, \]

where Pf A denotes the Pfaffian of the skew symmetric matrix A. This well-known expression above is known as one of the de Bruijn integral identities [4]. Note, the Pfaffian of a skew symmetric 2M × 2M matrix A can be defined as

\[ \text{Pf} A = \frac{1}{2^M M!} \sum_{\sigma \in \Sigma_M} \text{sgn}(\sigma) \prod_{j=1}^{M} A_{\sigma(2j-1), \sigma(2j)}. \]

which is just a sum of signed products of matrix entries like a determinant (except taking into account the redundancy in the entries of a skew symmetric matrix). In fact, \text{Pf} A = \sqrt{\det A}. Equivalently, the Pfaffian can be defined (using the Berezin integral) for an associated alternating tensor of rank 2.

More generally, fix a positive even integer L. Let V be an ML × ML matrix whose entries in the first L columns are real-valued functions of the variable x_1. Further suppose the entries in the next L columns are the same functions evaluated at the variable x_2, and so on up through x_M in the last L columns. Explicitly, the entries are

\[ V_{j,(m-1)L+k} = f_{j,k}(x_m) \]

for some family of real-valued functions \{f_{j,k}\}_{j,k=1}^{ML}. Let A be the L-dimensional array whose (m₁, ..., m_L)-entry A_{m₁,...,m_L} is given by

\[ A_{m₁,...,m_L} = \int_{-\infty}^{\infty} \det \left[ f_{m_j,k}(x) \right]_{j,k=1}^{L} dx. \]

Note, these integrands are simply determinants of the L × L univariate submatrices of V, obtained by taking L many rows from the L many columns which share a variable. The following theorem is a special case of Theorem 3.4 from [21] (referred to therein as a generalized de Bruijn identity):

**Theorem 1.1** Let V and A be defined as above. Then,

\[ \int_{-\infty x_1 < \cdots < x_M < \infty} \text{det} V d\mu(x_1) \cdots d\mu(x_M) = \text{Pf} A, \]

where Pf A denotes the Hyperpfaffian of A.

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As with the Pfaffian, it is correct to think of the Hyperpfaffian as a sum of signed products of array entries, related to a hyperdeterminant. Explicitly,

\[
\text{PF}(A) = \frac{1}{(L!)^M M!} \sum_{\sigma \in S_{ML}} \text{sgn}(\sigma) \prod_{j=1}^{M} A_{\sigma((j-1)L+1), \ldots, \sigma(jL)}.
\]

Equivalently, the Hyperpfaffian can be defined (using the Berezin integral) for an associated alternating tensor of rank \( L \) (see Sect. 4.1). Now divorced from the context of computing partition functions, the above theorem is a general statement about integrating multivariate determinants. The main results of this document are obtained by substituting in the particulars of different ensembles. As such, the Hyperpfaffian expressions presented herein should not be surprising. The main takeaway should be general methods for determining the matrix \( V \) which gives the desired Boltzmann factor (joint density function) as its determinant (free of absolute value).

2 Statement of results

For any positive integer \( N \), let \( N \) denote the set \( \{1, \ldots, N\} \). Assuming positive integers \( L \leq N \), let \( t : \{1, \ldots, L\} \not\rightarrow \{1, \ldots, N\} \) denote a strictly increasing function from \( L \) to \( N \), meaning

\[1 \leq t(1) < t(2) < \cdots < t(L) \leq N.\]

It will be convenient to use these increasing functions to track indices used in denoting submatrices and alternating tensors, among other things (often in place of, but sometimes in conjunction with, permutations). For example, given an \( N \times N \) matrix \( V \), \( V_t \) might denote the \( L \times L \) submatrix composed of the rows \( t(1), \ldots, t(L) \), taken from the first \( L \) columns of \( V \). However, for an \( L \)-dimensional array \( A \), \( A_t \) might denote the \((t(1), \ldots, t(L))\)-entry of the array.

Recall (from Sect. 1.1), we took \( L \) to be the charge of each particle, \( K \) the number of the parallel lines, \( M \) the number of independent particles on each line, and \( d\mu(x) = \left((-i)^{L(K-1)/2} e^{-U(x)}\right)^L K \ dx \) (also, \( \beta = 1 \) without loss of generality). Let \( \tilde{p} \) be a complete \( N \)-family of monic polynomials, where \( N = LKM \). Define the alternating tensor

\[
\gamma_L(\vec{y}) = \sum_{t: L \not\rightarrow N} \int \text{Wr} \otimes \text{Pr}_\gamma(\tilde{p}_t, x) d\mu(x) \varepsilon_t,
\]

where \( \varepsilon_t \) denotes the rank \( LK \) pure tensor \( \varepsilon_t = \varepsilon_{t(1)} \wedge \cdots \wedge \varepsilon_{t(LK)} \), \( \tilde{p}_t = \{p_{(m)}\}_{m=1}^{LK} \) denotes a subfamily of \( LK \) many polynomials, and \( \text{Wr} \otimes \text{Pr}_\gamma \) is a determinant defined by

\[
\text{Wr} \otimes \text{Pr}_\gamma(\tilde{p}_t, x) = \det \left[ \frac{1}{(L-1)!} \frac{d^{L-1}}{dx^{L-1}} p_{t((n-1)L+j)}(x + iy_k) \right]_{j=1}^{L},
\]

The first column of the above matrix is \( LK \) many monic polynomials evaluated at \( x + iy_1 \). The second column is the first derivatives of those polynomials evaluated at the same \( x + iy_1 \), and so on until the first \( L \) many columns have been exhausted. The next \( L \) many columns are the same functions and derivatives evaluated at \( x + iy_2 \), and so on until all \( y_k \) have been exhausted. The resulting \( LK \times LK \) matrix will have \( L \times L \) Wronskian blocks evaluated at each of the \( K \) many \( x + iy_k \).

Provided we can write the Boltzmann factor integrand \( \Omega(\vec{x}, \vec{y}) \) as a determinant of an \( N \times N \) matrix (see Sect. 3) with univariate minors of the form \( \text{Wr} \otimes \text{Pr}_\gamma(\tilde{p}_t, x) \), Theorem 1.1 immediately gives us the desired Hyperpfaffian expression

\[
Z_M(\vec{y}) = \text{PF}(\gamma_L(\vec{y}))
\]

when \( LK \) is even. When \( LK \) is odd, we additionally need to define

\[
\eta_L(\vec{y}) = \sum_{t: L \not\rightarrow N} \sum_{s: L \not\rightarrow N} \int \int_{-\infty < x_1 < x_2 < \infty} \text{Wr} \otimes \text{Pr}_\gamma(\tilde{p}_t, x_1) \text{Wr} \otimes \text{Pr}_\gamma(\tilde{p}_s, x_2) d\mu(x_1) d\mu(x_2) \varepsilon_t \wedge \varepsilon_s,
\]

taking two univariate minors of odd dimensions at a time to produce an alternating tensor of even rank \( 2LK \). Finally, we can state our main result.

Theorem 2.1 (\( K \)-fold monocharge partition function)

\[
Z_M(\vec{y}) = \text{PF}(\omega(\vec{y})),
\]

where \( \omega(\vec{y}) \) is defined by:

1. If \( LK \) is even, then \( \omega(\vec{y}) = \gamma_L(\vec{y}) \).
2. If \( LK \) is odd, but \( M \) is even, then \( \omega(\vec{y}) = \eta_L(\vec{y}) \).
3. If $LKM$ is odd, then $\omega(\vec{y}) = \eta_k(\vec{y}) + \gamma_k(\vec{y}) \wedge \xi_{LKM}$.

Note, $\xi_{LKM} = \epsilon_{N+1} \times \cdots \times \epsilon_{N+LKM}$ is a rank $LKM$ pure tensor which upgrades $\gamma_k(\vec{y})$ from an odd rank $LKM$ alternating tensor and makes $\omega(\vec{y})$ homogeneous so that the Hyperpfaffian $PF(\omega(\vec{y}))$ is well-defined. Recall, the first constellation ensemble is the special case in which $L = 1$. In that case, we write

$$\text{Wr} \otimes \text{Pr}_L(\vec{p}_1, x) = \text{Pr}_L(\vec{p}_1, x) = \det[p_{(\alpha)}(x + iy_k)]_{n,k=1}^K,$$

featuring translations but not derivative columns. Alternatively, any one-dimensional log-gas with a single species is a special case of a constellation ensemble in which $K = 1$ (meaning only one line). The Theorem 2.1 agrees with known Hyperpfaffian expressions proven for $L$ even by Luque and Thibon in [11] and expanded to $L$ odd by Sinclair in [18]. Explicitly, we get Wronskian minors

$$\text{Wr} \otimes \text{Pr}_L(\vec{p}_1, x) = \text{Wr}(\vec{p}_1, x) = \det \left[ \frac{1}{(L-1)!} \frac{d^{L-1}}{dx^{L-1}} p_{(\alpha)}(x) \right]_{n,l=1}^L$$

when $K = 1$.

2.1 Homogeneous constellation ensembles

Let $\vec{L} \in (\mathbb{Z},0)^K$ be a vector of positive integers which we will call the charge vector of the system. Modify the setup of Sect. 1.1 by changing the charge of each $x_m + iy_k$ particle from $L$ to $L_k$. Note, the $M$ many particles on each line $\mathbb{R} + iy_k$ will all have the same charge $L_k$. However, each $x_m$ will generate a constellation of $K$ many particles with charges $L_1,\ldots,L_K$. In this homogeneous constellation model, it is straightforward to verify the induced interaction between real parts $x_m$ and $x_n$ is given by

$$d_{L_k}^L(x_m, x_n) = \prod_{k=1}^K [x_m - x_n]^{L_k^2} \prod_{j<k}^K (x_m - x_n)^2 + (y_k - y_j)^2 \frac{1}{L_k^L}.$$

Let $R_1 = \sum_{k=1}^K L_k$, let $R_2 = \sum_{j<k}^K L_j L_k$, and define $d\mu(x) = (-i)^{R_2} e^{-R_1 U(x)} dx$. Define

$$\text{Wr}^{\vec{L}} \otimes \text{Pr}_L(\vec{p}_1, x) = \det \left[ \frac{1}{(L-1)!} \frac{d^{L-1}}{dx^{L-1}} p_{(\alpha)}(x + iy_k) \right]_{n,l=1}^L.$$

The first column is $R_1$ many functions evaluated at $x + iy_1$. The second column is the first derivatives of those functions evaluated at the same $x + iy_1$, and so on until the first $L_1$ many columns have been exhausted. The next $L_2$ many columns are $L_2$ many derivatives of the same functions evaluated at $x + iy_2$, and so on until all $y_k$ have been exhausted. The resulting $R_1 \times R_1$ matrix will have $L_k \times L_k$ Wronskian blocks evaluated at one of the $K$ many $x + iy_k$.

Define $\gamma_k(\vec{y})$ from $\gamma_k(\vec{y})$ and $\eta_k(\vec{y})$ from $\eta_k(\vec{y})$ by replacing all instances of $\text{Wr} \otimes \text{Pr}_L$ with $\text{Wr}^{\vec{L}} \otimes \text{Pr}_L$ and replacing all instances of $L$ with $R_1$. In particular, $N = R_1 M$. Substituting these changes into Theorem 2.1, we generalize the Hyperpfaffian expressions to the partition functions of homogeneous constellation ensembles. In fact, homogeneous constellation ensembles are the most general classification (in this document) for which the partition functions are Hyperpfaffians (because of homogeneous $\omega(\vec{y})$).

2.2 Nonlinear constellation ensembles

Let $f_1,\ldots,f_K : \mathbb{R} \to \mathbb{R}$. Modify the setup of Sect. 1.1 by setting $y_k = f_k(x)$ and $y_{K+k} = -f_k(x)$ for $1 \leq k \leq K$. This is equivalent to replacing each line $\mathbb{R} + iy_k$ with the curve $z_k(x) = x + if_k(x)$ and its conjugate curve $\overline{z}_k(x) = x - if_k(x)$. Using the established algebraic machinery, this nonlinear constellation ensemble will also have a Hyperpfaffian partition function, provided particles on $z_k(x)$ and $\overline{z}_k(x)$ have the same charge $L_k$ (or opposite charge, technically). Explicitly, we need only substitute $y_k = f_k(x)$ and $y_{K+k} = -f_k(x)$ into $\text{Wr}^{\vec{L}} \otimes \text{Pr}_L(\vec{p}_1, x)$ so that $x + iy_k \mapsto z_k(x)$. Moreover, more general $z_k(x) = g_k(x) + if_k(x)$ is also permissible. Note (as seen with the parallel lines of Sect. 1.1), the addition of the complex conjugates (image charges) is exactly what is needed to drop the absolute value from the Boltzmann factor (density function).

Finally, the functions $f_k$ and $g_k$ might feature some additional parameter like the original $y_k$, fixed but allowed to vary continuously (such as the slope of a line, or the amplitude or period of a wave function). When $L = 1$ and $K = 1$, so that the system contains only charge 1 particles on the two curves, $z(x) = g(x) + if(x)$ and its conjugate curve $\overline{z}(x) = g(x) - if(x)$, the partition function is an honest Pfaffian of the $2M \times 2M$ skew symmetric matrix $A$ whose entries are given by a skew inner product

$$A_{j,k} = (p_j, p_k) = \int_{-\infty}^{\infty} \det \left[ \frac{p_j(z(x))}{p_k(z(x))} \frac{p_j(\overline{z}(x))}{p_k(\overline{z}(x))} \right] d\mu(x).$$

As a simple example (actually still linear), consider the ray $z(x) = x(1 + i\tan \theta)$ extending out from the origin at an angle of $0 < \theta < \pi/2$. As mentioned in Sect. 1, the induced interaction will effectively be that of a low temperature region near the origin, with the temperature rising away from the origin. As $\theta \to 0$, the cold region near the origin expands out over the positive real axis. Alternatively, we could consider $z(x) = x + ia|\chi|^n$ for $a > 0$ and $n \in \mathbb{R}$ or $z(x) = x + ia(1 - \cos bx)$ for $a, b > 0$.  

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As mentioned at the end of Sect. 1, these Pfaffian models are likely solvable even with known methods, established by Tracy and Widom in [20]. In particular, the next step is essentially inverting the matrix $A$. Note, we are free to choose any complete family of monic polynomials $\tilde{p}$. Historically, skew orthogonal polynomials (with respect to $(\cdot, \cdot)$) have been used to diagonalize the matrix $A$. For the classical $\beta = 1$ and $\beta = 4$ ensembles, Pfaffian correlation functions were obtained first for the circular ensembles by Dyson in [5], then for Gaussian ensembles by Mehta in [13], and finally for general weights $(\mu)$ by Mehta and Mahoux in [12], except in the case $\beta = 1$ and $M$ odd. The last remaining case was given by Adler, Forrester, and Nagao in [1]. Shum uses skew orthogonal polynomials throughout [16] as well as well. Solvability of new models will obviously depend on the complexity of $z(x)$ and will likely required a clever choice of weight function $\mu$, at least at first, as evident from the historical development of the $\beta$-ensembles.

2.3 Multicomponent constellation ensembles

In Sect. 1.1, we demonstrated the Boltzmann factor of the monocharge constellation ensemble is the same as the Boltzmann factor of the single-species $\beta$-ensemble with $\beta = L^2$ but with the $KM$ many translated variables $x$ substituted in. In both cases, the Boltzmann factors are demonstrably determinantal. The $\text{Wr} \otimes \text{Pr}_x$ minors of the former resemble the normal Wronskian minors of the latter except with the addition of extra columns generated by the translation vector $\tilde{y}$. Likewise, the alternating tensors which give the partition functions for multicomponent constellation ensembles are simply the expanded versions of the alternating tensors which give the partition functions for multicomponent log-gases in one dimension.

In a multicomponent ensemble, we allow the particles to have possibly different charges, provided the charges are all integers of the same sign. We assume any two particles of the same charge, which we will call same species, are indistinguishable. As in [19] and [21], we consider two ensembles:

1. The Canonical Ensemble, in which the number of particles of each species is fixed; in this case, we say fixed population.

2. The Isocharge Grand Canonical Ensemble, in which the sum of the charges of the particles is fixed, but the number of particles of each species is allowed to vary; in this case, we say the total charge of the system is fixed.

In contrast, the grand canonical ensemble traditionally refers to the ensemble in which the total number of particles is not fixed. For computational purposes, it is beneficial to group configurations which share the same total charge. The grand canonical ensemble is then a disjoint union (over all possible sums of charges) of our isocharge ensembles.

For our constellation ensembles. For reference, we will start by giving abbreviated versions of the setup and main results from the (roughly, the probability of the occurrence of any one particle of a given charge). Likewise, we will consider analogous versions for our constellation ensembles. For reference, we will start by giving abbreviated versions of the setup and main results from the one-dimensional case.

Let $J \in \mathbb{Z}_{>0}$ be a positive integer, the maximum number of distinct charges in the system. Let $\tilde{L} = (L_1, \ldots, L_J) \in (\mathbb{Z}_{>0})^J$ be a vector of distinct positive integers which we will call the charge vector of the system. We will assume the first $r$ many $L_j$ are even, and the remaining $J - r$ many are odd. Let $\tilde{M} \in (\mathbb{Z}_{>0})^J$ be a vector of non-negative integers which we will call the population vector of the system. Each $M_j$ gives the number (possibly zero) of indistinguishable particles of charge $L_j$ on the real axis. Let $N = L \cdot \tilde{M}$ be the total charge of the system, and let $U : \mathbb{R} \to \mathbb{R}$ be a potential on the real axis.

For any complete $N$-family of monic polynomials $\tilde{p}$, define

$$y_j = \sum_{t|L_j} \int_{\mathbb{R}} \text{Wr}(\tilde{p}_t, x) d\mu_j(x) \, \varepsilon_t,$$

and define

$$\eta_{j,k} = \sum_{t|L_j} \sum_{s|L_k} \int_{x<y} \text{Wr}(\tilde{p}_t, x) \text{Wr}(\tilde{p}_s, y) d\mu_j(x) d\mu_k(y) \, \varepsilon_t \wedge \varepsilon_s,$$

where $d\mu_j(x) = \exp(-L_j U(x)) \, dx$. Note, we get Wronskians, and therefore alternating tensors, of potentially different sizes. Under this setup, we inherit the following lemma from [21]:

**Lemma 2.1** (Canonical partition function) When $N$ is even,

$$Z_{\tilde{M}} = \int \frac{y_1^{\wedge^M_1}}{M_1!} \wedge \cdots \wedge \frac{y_r^{\wedge^M_r}}{M_r!} \wedge \sum_{\sigma \in \text{Sh}(M_{r+1}, \ldots, M_J)} \frac{1}{J!} \wedge \sum_{j=1}^{J} \eta_j^{\wedge_{j,k}} \, d\varepsilon_{\text{vol}},$$

and when $N$ is odd,

$$Z_{\tilde{M}} = \int \frac{y_1^{\wedge^M_1}}{M_1!} \wedge \cdots \wedge \frac{y_r^{\wedge^M_r}}{M_r!} \wedge \sum_{\sigma \in \text{Sh}(M_{r+1}, \ldots, M_J)} \frac{1}{K!} \wedge \sum_{j=1}^{J} \eta_j^{\wedge_{j,k}} \wedge \gamma_{\sigma^{-1}(K_a)} \, d\varepsilon_{\text{vol}}.$$
where $\int \cdot d\varepsilon_{\text{vol}}$ denotes the Berezin integral (see Sect. 4) of an alternating tensor (with respect to the volume form $\varepsilon_{\text{vol}} = \varepsilon_1 \wedge \cdots \wedge \varepsilon_N$). Here, $K_\sigma = \sum_{j=r+1}^J M_j$, and $\text{Sh}(M_{r+1}, \ldots, M_J) \subset S_{K_\sigma}$ denotes the subset of shuffle permutations. These are permutations which satisfy $\sigma(n) < \sigma(m)$ whenever

$$M_{r+1} + M_{r+2} + \cdots + M_J < n < m \leq M_{r+1} + M_{r+2} + \cdots + M_{J+1},$$

meaning each block of $M_j$ many things retains the same relative order in the image of $\sigma$. If we define

$$\tilde{\lambda} = (L_{r+1}, \ldots, L_{r+1}, L_{r+2}, \ldots, L_J, \ldots, L_J),$$

which has $M_j$ copies of each $L_j$ for $r + 1 \leq j \leq J$, then each $M_{\sigma}^{j,k}$ is the number of times $\lambda_{\sigma^{-1}(2m - 1)} = L_j$ and $\lambda_{\sigma^{-1}(2m)} = L_k$. When $N$ is even, $K = \sum_{j,k} M_{\sigma}^{j,k}$. When $N$ is odd, $K = 1 + \sum_{j,k} M_{\sigma}^{j,k}$.

Heuristically, the sum is taken over all possible orderings of the particles which carry an odd charge $L_j$. Since particles of the same charge are indistinguishable, we restrict the sum to shuffle permutations which preserve the relative order of blocks of $M_j$ many particles. $K_\sigma$ is the total number of these particles with odd charges. $\lambda'$ is the vector of charges, each of which is repeated $M_j$ many times. Pairs $(L_j, L_k)$ of adjacent charges produce the double-Wronskian $\eta_{j,k}$ factor by combining an $L_j \times L_j$ Wronskian in one variable with an $L_k \times L_k$ Wronskian in another variable. Because the charges are repeated, a permutation $\sigma$ may produce multiple copies of the same pair $(L_j, L_k)$ and multiple copies of the same $\eta_{j,k}$. $K$ is the number of these pairs, also the number of $\eta_{j,k}$. $K$ factors in the wedge product. When $N$ is odd, the last particle lacks a partner and produces only a single-Wronskian $\gamma_j$ factor.

Allowing the number of particles of each species to vary, let $P(\hat{M})$ be the probability of finding the system with population vector $\hat{M}$. Let $z = (z_1, \ldots, z_J) \in (\mathbb{R}_{>0})^J$ be a vector of positive real numbers called the fugacity vector. Classically, the probability $P(\hat{M})$ is given by

$$P(\hat{M}) = \frac{z_1^{M_1} z_2^{M_2} \cdots z_J^{M_J} Z_{\hat{M}}}{Z_N},$$

where $Z_N$ is the partition function of the isocharge grand canonical ensemble (corresponding to fixed total charge $N$) given by

$$Z_N = \sum_{\hat{L}, M = \hat{N}} z_1^{M_1} z_2^{M_2} \cdots z_J^{M_J} Z_{\hat{M}}.$$

In the above expression, the vector $\hat{L}$ of allowed charges is fixed, so we’re summing over allowed population vectors $\hat{M}$. A population vector is valid only when the sum of the charges $\sum_{j=1}^J L_j M_j$ is equal to the prescribed total charge $N$. Taking the fugacity vector $z$ to be a vector of indeterminates, $Z_N$ is a polynomial in these indeterminates which generates the partition functions of the canonical ensembles. Combining partition functions of the canonical type, we obtained the following theorem (as given in [21]):

**Theorem 2.2** (Isocharge grand canonical partition function) When $N$ is even,

$$Z_N = \text{BE} \text{vol} \left( \sum_{j=1}^r z_j y_j + \sum_{j=r+1}^J \sum_{k=r+1}^J z_j z_k \eta_{j,k} \right),$$

and when $N$ is odd,

$$Z_N = \text{BE} \text{vol} \left( \sum_{j=1}^r z_j y_j + \sum_{j=r+1}^J \sum_{k=r+1}^J z_j z_k \eta_{j,k} + \sum_{j=r+1}^J z_j y_j \wedge \varepsilon_{N+1} \right),$$

where $\text{BE} \text{vol}(\cdot) = \int \exp(\cdot) d\varepsilon_{\text{vol}}$.

Note, this is the first case (in this document) in which the partition function is not Hyperpfaffian (because the tensor inside the parentheses is not homogeneous).

Recall, the constellation version is created by copying the system onto parallel lines. Then with population vector $\hat{M}$ fixed, the total charge of the expanded system is $N = K \left( \hat{L} \cdot \hat{M} \right)$. Without writing out the full Boltzmann factor for the interactions between these particles, it is straightforward to verify the correct (complex) measure $\mu_j$ for each species should be

$$d\mu_j(x) = \left( (-i)^{L_j(K-1)/2} e^{-U(x)} \right)^{L_j(K-1)} dx.$$

Then we obtain $\gamma_j(\tilde{y})$ from $\gamma_j$ and $\eta_{j,k}(\tilde{y})$ from $\eta_{j,k}$ by replacing each instance of $L_j$ with $L_j K$ and each instance of $\varepsilon_{r+1}$ with $\varepsilon_{r+1} \otimes \varepsilon_{N+1}$. When $K$ is odd, we’re done (because the parity of $L_j K$ is determined by the parity of $L_j$). However, when $K$ is even, all of our minors have even dimensions $L_j K$, and the total charge $N = K \left( \hat{L} \cdot \hat{M} \right)$ is even as well. Thus, for $K$ even (notably no additional restrictions on $L_j$), we get the “all even” version of our Berezin integral expressions:
Theorem 2.3 When $K$ is even,

$$Z_M(\tilde{y}) = \int \frac{\gamma_1(\tilde{y})^{M_1}}{M_1!} \wedge \cdots \wedge \frac{\gamma_J(\tilde{y})^{M_J}}{M_J!} \, d\text{e}_{\text{vol}},$$

and

$$Z_N(\tilde{y}) = \text{BE}_{\text{vol}} \left( \sum_{j=1}^{J} z_j \gamma_j(\tilde{y}) \right).$$

Note, the fugacity parameters $z_j$ are no longer the probabilities of individual charge $L_j$ particles appearing. Instead, $z_j$ is the probability of a constellation of $K$ many points all having the same charge $L_j$.

A more general perspective on multicomponent ensembles is yet viable. For each homogenous species, whether it be a species of constellations or a species of individual particles, there exists an associated alternating tensor $\gamma_j$, whose entries are some Wronskian variant, and whose Hyperpfaffian generally gives the partition function. In the case $\gamma_j$ is odd rank, the double-Wronskian $\eta_{j,k}$ is used instead. A version of Theorem 2.2 holds even when there is little resemblance between species. Finally, if the relevant $\gamma_j$ of each even rank species or $\eta_{j,k}$ of each pair of odd rank species have the same rank, the Berzien integral (of the exponential of the sum) is actually a Hyperpfaffian. Suppose two solvable Pfaffian species have associated inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ (as in Sect. 2.2). Then the mixed-species model will also be a Pfaffian model with an associated inner product of the form $\langle \cdot, \cdot \rangle = z_1 \langle \cdot, \cdot \rangle_1 + z_2 \langle \cdot, \cdot \rangle_2$.

An ensemble of charge 1 and charge 2 particles on the unit circle was first investigated by Forrester in [8] and [7], then considered on the real line by Rider, Sinclair, and Xu in [15]. The limiting behavior for the circular case was later studied by Shum and Sinclair in [17]. A recent paper by Forrester and Li [9] extends these results by expressing the skew orthogonal polynomials for classical weight functions in terms of hypergeometric polynomials. As mentioned in Sect. 1, the eigenvalues of the real Ginibre ensemble come in two species, individual real eigenvalues or complex conjugate pairs of eigenvalues. In [16], Shum solved the case of one species on the real line and a second species on the imaginary axis, in which unique behavior is observed at the origin, where the lines intersect.

2.4 Limits of constellations ensembles

In the monocharge setup of Sect. 1.1, it was noted how the induced interaction between real parts $x_m$ and $x_n$ interpolates between $d_0(x_m, x_n) = \|x_m - x_n\|^{\beta L^2 K}$ (as $\tilde{y} \to 0$) and $d_\infty(x_m, x_n) = \|x_m - x_n\|^{\beta L^2 K}$ (as each $y_k - y_j \to \infty$). More generally, in a homogeneous constellation ensemble with charge vector $\tilde{L}$, the induced interaction interpolates between $d_0(x_m, x_n) = \|x_m - x_n\|^{\beta L_1 + \cdots + L_K^2}$ (as $\tilde{y} \to 0$) and $d_\infty(x_m, x_n) = \|x_m - x_n\|^{\beta L_1 + \cdots + L_K^2}$ (as each $y_k - y_j \to \infty$).

In terms of (monocharge) partition functions,

$$\lim_{\tilde{y} \to 0} \frac{Z_M(\tilde{y})}{\Delta(i\tilde{y})^{2L^2 M}} = \lim_{\tilde{y} \to 0} \frac{\text{PF}(\gamma_1(\tilde{y}))}{\Delta(i\tilde{y})^{2L^2 M}} \int_{\mathbb{R}} \text{Wr} \otimes \text{Pr}_{\tilde{\gamma}}(\tilde{\mu}, x) \, d\mu(x) \, \epsilon_1,$$

when $L K$ is even, with the limit turning translation columns into further derivative columns, a total of $LK$ many. An analogous result holds for $LK$ odd and for the more general homogeneous constellation ensembles (for which $\text{Wr}^L \otimes \text{Pr}_{\tilde{\gamma}}$ becomes an $R_1 \times R_1$ Wronskian). However, the limit as $y_k - y_j \to \infty$ which works for the Boltzmann factor is not yet compatible with the Hyperpfaffian partition function.

Still, for the 2-fold first circular constellation ensembles in [16], Shum was able to compute the limits at the kernel and correlation function level, producing the expected kernels of the limiting ensembles in both directions (classical $\beta = 4$ as $y \to 0$ and classical $\beta = 2$ as $y \to \infty$). As such, the limiting ensembles were demonstrated to be solvable Pfaffian models without needing to express the limiting partition functions as Pfaffians themselves.

Analogously, once it known generally how to obtain Hyperpfaffian correlation functions from Hyperpfaffian partition functions, $\beta$ ensembles for $\beta = L^2$ become solvable via Luque, Thibon, and Sinclair’s Hyperpfaffian formulae. Moreover, this limit perspective promises to solve the square-free $\beta = L$ cases even though an explicit Hyperpfaffian expression does not exist for the partition functions in those cases.
2.5 Circular constellation ensembles

We will begin with homogeneous circular constellation ensembles of which monocharge circular constellation ensembles are a special case. Consider $K$ concentric circles in the complex plane with radii $r_j \in (\mathbb{R}_{>0})^K$. Modify the setup of Sect. 2.1 by replacing $\mathbb{R}$ with $[0, 2\pi]$ in the definition of $x$. For each angle $x_m \in [0, 2\pi)$, and $1 \leq k \leq K$, place a charge $L_k$ particle at location $y_k e^{ix_m}$. Denote the (total $KM$) particle locations by

$$z = (z_1, z_2, \ldots, z^M) \in \mathbb{C}^{KM},$$

where $z^m = \tilde{y} e^{ix_m} = (y_1 e^{ix_m}, y_2 e^{ix_m}, \ldots, y_K e^{ix_m}) \in \mathbb{C}^K$. Assuming logarithmic interaction between the particles, it is straightforward to verify the induced interaction between a real angle $x_m$ and a real angle $x_n$ is given by

$$d\gamma(x_m, x_n) = \frac{1}{(1-x)_!} \prod_{j<k} [y_j e^{ix_m} - y_k e^{ix_n}]^{L_j L_k} \cdot \prod_{j,k} [y_j e^{ix_m} - y_k e^{ix_n}]^{L_j L_k}.$$}

Recall (from Sect. 2.1), $R_1 = \sum_{k=1}^K L_k$ and $R_2 = \sum_{j<k} L_j L_k$. Also, $W_{\gamma}^L \otimes \mathcal{P} \gamma$ is created by evaluating polynomials and derivatives at the (vertically) translated variables $x_m + iy_j$. For circular ensembles, instead define analogous

$$\mathcal{C}_{\gamma}^L(f, x) = \det[f_n(y_k e^{ix})]_{n,k=1}^{K},$$

and

$$W_{\gamma}^L \otimes \mathcal{C}_{\gamma}^L(f, x) = \det \left[ \frac{1}{(l-1)!} \frac{d^{l-1}}{dt^{l-1}} f_n(t) \bigg|_{t=e^{ix}} \right]_{n,k=1}^{R_1,K}.$$}

Proceeding as in Sect. 2.1, define new (circular) $\gamma^L_\gamma$ and $\eta^L_\gamma$ from the linear versions by replacing all instances of $W_{\gamma}^L \otimes \mathcal{P} \gamma$ with $W_{\gamma}^L \otimes \mathcal{C}_{\gamma}^L$ and setting $d\mu(x) = (-i e^{-ix})^{R_1(M-1)/2} (e^{-ix})^{R_2} \, dx$, with $R_3 = \sum_{j=1}^K L_j L_k$ (see Sect. 3.5 for why this $d\mu(x)$).

With these modifications, the (Hyperpfaffian) partition functions of the homogeneous circular constellation ensembles are structurally the same as those for the linear constellation ensembles. The same is true of multicomponent circular constellation ensembles, provided we use

$$d\mu_j(x) = (-i e^{-ix})^{K^2 L_j T/2} (e^{-ix})^{L_j^2 K/2} \, dx,$$

where $T = -L_j + \sum_{k=1}^K L_k M_k$ for charge vector $\tilde{L} \in (\mathbb{Z}_{>0})^J$ and population vector $\tilde{M} \in (\mathbb{Z}_{\geq 0})^J$.

As in Sect. 2.4, we consider limits as the distances between the circles shrink to zero or increase without bound, with slight modification in the former case as we use $\tilde{y} \to \tilde{1}$ so that the radii $y_1 = \cdots y_K = 1$ in the limit. As expected, the limiting ensembles are the circular versions of the (one-dimensional) limiting ensembles of the linear case.

At this point, it is worth noting, the Wronskian of a collection of monomials will again be a monomial. Explicitly, consider $\tilde{g} = \{x^{n-1}\}_{n=1}^{N}$. Then, for any $t : K' \not\to \mathbb{N}$, we have

$$W_{\tilde{g}}(\tilde{g}_t, x) = \chi^{\sum_n t(k) - k} \frac{\Delta(t(K))}{\Delta(K)}.$$}

Thus, for $R \in \mathbb{Z}$,

$$\int_0^{2\pi} W_{\tilde{g}}(\tilde{g}_t, e^{ix}) (e^{-ix})^R \, dx = \int_0^{2\pi} r^{\sum_n t(k) - k} \frac{\Delta(t(K))}{\Delta(K)} (e^{ix})^{-R + \sum_n t(k) - k} \, dx = 0,$$

unless $-R + \sum n t(k) - k = 0$. This gives a sum condition which all $t$ of the same size must satisfy. Likewise,

$$\mathcal{C}_{\gamma}^L(\tilde{g}_t, x) = \det[y_k^{(j)-1}]_{j,k=1}^{K^{(L-1)}} (e^{ix})^{-K + \sum t(k)}.$$}

Thus,

$$\int_0^{2\pi} \mathcal{C}_{\gamma}^L(\tilde{g}_t, x) (e^{-ix})^R \, dx = \det[y_k^{(j)-1}]_{j,k=1}^{K^{(L-1)}} \int_0^{2\pi} (e^{ix})^{-R - K + \sum t(k)} \, dx = 0,$$

unless $-R - K + \sum t(k) = 0$. This condition is actually quite strong and makes our $\gamma(\tilde{y})$ tensors quite sparse. For example, when $K = 2$, knowing $t(1) \in \mathbb{N}$ determines $t(2) = t(1) + R + 2$, no matter how big $N$ is.

As mentioned in Sect. 2.2, being able to “diagonalize” the form $\gamma$ by a clever choice of (potentially orthogonal or skew orthogonal) polynomials is incredibly useful in obtaining Pfaffian correlation functions from the Pfaffian partition functions. We expect this to be the case with Hyperpfaffian partition functions and correlation functions as well, though this is admittedly still speculation.
3 Proofs

As mentioned in Sect. 2, the remaining task is to write the Boltzmann factor (density function) of various models as a determinant free of absolute value (to be compatible with Theorem 1.1). Fix \( L = (L_1, \ldots, L_M) \in (\mathbb{Z}_{>0})^M \), and let \( N = \sum_{m=1}^{M} L_m \). Let \( \tilde{f} = \{ f_n \}_{n=1}^{N} \) be a family (not necessarily complete) of \( \max(L_1, \ldots, L_M) - 1 \) times differentiable functions. Define the confluent alternant (with respect to shape \( \tilde{L} \)) to be the \( N \times N \) matrix

\[
V_{f \tilde{L}}(\vec{x}) = \begin{bmatrix} V_{f,1}(x_1) & V_{f,2}(x_2) & \cdots & V_{f,L_M}(x_L) \end{bmatrix},
\]

where each \( V_{f,m}(x_m) \) is an \( N \times L_m \) matrix defined by

\[
V_{f,m}(x_m) = \left[ \frac{1}{(l-1)!} \int_{x_m}^{x_{m+1}} f_m(x_m) \right]_{n,l=1}^{N,L_m}.
\]

Then each variable \( x_m \) appears in \( L_m \) many consecutive columns, generated from \( \tilde{f} \) by taking derivatives. Note, any increasing function \( t: L_m \not\rightarrow N \) defines an \( L_m \times L_m \) submatrix with Wronskian determinant corresponding to the polynomials \( \tilde{f}_t = \{ f_{n(t)} \}_{n=1}^{L_m} \). Explicitly,

\[
\det V_{f,\tilde{L}}(x_m) = \mathrm{Wrt}(\tilde{f}_t, x_m).
\]

Let \( \tilde{g} = \{ x^{n-1} \}_{n=1}^{N} \). If \( \tilde{p} \) is any complete \( N \)-family of monic polynomials, then

\[
\det V_{f,\tilde{L}}(\vec{x}) = \det V_{\tilde{p}}(\vec{x})
\]

because \( V_{\tilde{p}}(\vec{x}) \) can be obtained from \( V_{\tilde{g}}(\vec{x}) \) by performing elementary column operations. This is only because the \( p_j \) are assumed to be monic, and \( \tilde{p} \) is complete, containing a \( p_j \) of each degree. We call \( V_{\tilde{p}}(\vec{x}) \) the confluent Vandermonde matrix (with respect to shape \( \tilde{L} \), in variables \( \vec{x} \)). We omit the \( \tilde{L} \) subscript when it is clear from context which family of functions is being used.

For all \( L_m \) are the same \( L \), we write \( V_{L}^{L} \) confluent Vandermonde matrix (in variables \( \vec{x} \)). Observe, the \( 1^{st} \) confluent Vandermonde matrix is the ordinary Vandermonde matrix (in \( M \) many variables) whose determinant is

\[
\Delta(\vec{x}) = \det V_{\tilde{g}}^{L}(\vec{x}) = \prod_{1 \leq n < m \leq M} (x_m - x_n).
\]

More generally, it is known [14]

\[
\det V_{\tilde{p}}^{L}(\vec{x}) = \prod_{1 \leq n < m \leq M} (x_m - x_n)^{L_m L_n}
\]

for any complete \( N \)-family of monic polynomials \( \tilde{p} \). In particular,

\[
\det V_{\tilde{p}}^{L}(\vec{x}) = \prod_{1 \leq n < m \leq M} (x_m - x_n)^{L^2} = \Delta(\vec{x})^L^2.
\]

In the previous, more general case, we will write \( \Delta^{L^2} = \det V_{\tilde{p}}^{L}(\vec{x}) \) to denote the confluent Vandermonde determinant with different exponents \( L_m L_n \) on each difference \( x_m - x_n \).

3.1 Monocharge constellation ensembles

Recall (from Sect. 1.1),

\[
x = (x_1, \ldots, x_M) \in \mathbb{C}^M,
\]

where \( x^m = x_n + i \vec{y} = (x_n + iy_{1}, \ldots, x_n + iy_{K}) \in \mathbb{C}^K \) gives all the vertical translations of the real variable \( x_n \). Then, the partition function of the \( (K\text{-fold}) \) monocharge constellation ensemble \( Z_M(\vec{y}) \) (after setting \( \beta = 1 \) without loss of generality) is given by

\[
Z_M(\vec{y}) = \int_{-\infty < x_1 < \cdots < x_M < \infty} \Delta(x)^L_{\text{det}} \, d\mu(x_1) \cdots d\mu(x_M).
\]

Notably, the integrand is just the \( L \)th confluent Vandermonde determinant \( \Delta(x)^L_{\text{det}} = \det V_{L}(x) \), evaluated at the translated variables \( x \). Previously, we noted a confluent alternant has Wronskian minors. Similarly, \( V_{L}(x) \) will have \( LK \times LK \) univariate minors of the form

\[
\det V_{L}(x_m) = \mathrm{Wrt} \otimes \mathrm{Pr}^L_{\tilde{f}_t}(x_m)
\]

for each \( t: LK \not\rightarrow N \). Thus, when \( LK \) is even, applying Theorem 1.1 completes the proof of the first case of Theorem 2.1. When \( LK \) is odd, the “odd” version of Theorem 1.1 is used (part of the same Theorem 3.4 from [21]).
3.2 Homogeneous constellation ensembles

In the monocharged case, all of the particles have the same charge $L$. As such, there are $L$ many columns of derivatives in $V^L(x)$ for each particle location $x_m+i y_k$. In the homogeneous constellation ensemble, a particle at location $x_m+i y_k$ has charge $L_k$ instead. Let $L = (L, \ldots, L) \in (\mathbb{Z}_{>0})^{KM}$. It is straightforward to check

$$Z_M(\bar{y}) = \int_{-\infty<x_1<\cdots<x_M<\infty} \det V^L(x_1) \cdots \det V^L(x_M).$$

Finally, the confluent Vandermonde matrix $V^L(x)$ will have $R_1 \times R_1$ univariate minors of the form

$$\det V^L_{\mu}(x_m) = \text{Wr}^L \otimes \text{Pr}_{\mu}(\tilde{f}_1, x_m)$$

for each $\mu : R_1 \not\rightarrow N$, featuring $L_k$ many derivative columns for each particle location $x_m+i y_k$. Thus, Theorem 2.1 can be extended to homogeneous constellation ensembles in the way outlined in Sect. 2.1.

3.3 Nonlinear constellation ensembles

By now, it is clear whatever expressions which describe the particle locations are the expressions which get substituted into the confluent Vandermonde determinant and its Wronskian minors. Recall, dropping the absolute value from the interaction terms from the log-gas setup is necessary to use Theorem 1.1. Consider three log-gas particles at locations $z_1 = x_1 + y_1 i$, $z_2 = x_1 - y_1 i$, and $z_3 = x_2$ in the complex plane with $y_1 > 0$. Observe:

$$\prod_{j<k} |z_k - z_j| = |-2y_1 i||x_2 - x_1| + y_1 i||x_2 - x_1| - y_1 i|$$

$$= |-2y_1 i|(x_2 - x_1)^2 + (y_1)^2||x_2 - x_1| + y_1 i|$$

$$= i(2y_1 i)((x_2 - x_1)^2 + (y_1)^2)$$

$$= i \prod_{j<k}(z_k - z_j).$$

Similarly, with four log-gas particles at locations $z_1 = x_1 + y_1 i$, $z_2 = x_1 - y_1 i$, $z_3 = x_2 + y_2 i$, and $z_4 = x_2 - y_2 i$, we get

$$\prod_{j<k} |z_k - z_j| = |-2y_1 i||-2y_2 i||(x_2 - x_1)^2 + (y_2 - y_1)^2||x_2 - x_1| + y_2 + y_1^2|$$

$$= i^2(-2y_1 i)(-2y_2 i)((x_2 - x_1)^2 + (y_2 - y_1)^2)(x_2 - x_1)^2 + (y_2 + y_1)^2$$

$$= - \prod_{j<k}(z_k - z_j).$$

Thus, for any model in which the locations $z_j$ come in complex conjugate pairs, the absolute value can be resolved by introducing sufficiently many factors of $i$. Moreover, because these distances are translation invariant, the same method can be applied to linear constellation ensembles.

3.4 Multicomponent constellation ensembles

Let $x = (x^1, \ldots, x^J) \in \mathbb{R}^{M_1} \times \cdots \times \mathbb{R}^{M_J}$ so that $x^j = (x^j_1, \ldots, x^j_{M_j}) \in \mathbb{R}^{M_j}$ gives the real parts of all particles of charge $L_j$. Define $x^j = (x^j_1 + i y_1, \ldots, x^j_{M_j} + i y_j) \in \mathbb{C}^{K_{M_j}}$ so that

$$x^j = (x^j_1 + i y_1, \ldots, x^j_{M_j} + i y_j) \in \mathbb{C}^{K_{M_j}},$$

with $x^j_m + i y_j = (x^j_m + i y_1, \ldots, x^j_m + i y_k) \in \mathbb{C}^{K}$. As a list, $x^j$ is generated from $x$ by replacing each real location $x^j_m$ with $x^j_m + i y_j$, the list of its $K$ many translations.

With any homogeneous constellation ensemble, we could assume the real parts were ordered $x_1 < \cdots < x_M$ because all particles on the same line were indistinguishable (same charge). This was necessary to drop the absolute value from the Boltzmann factor. In particular, whenever the real parts are labeled with the same order as the domain of integration, all differences in the confluent Vandermonde determinant are positive. In the case of differently-charged particles, the order in which they occur is relevant, and some additional tools are needed.

For $\sigma \in S_n$, let $Q_n(\sigma) \subset \mathbb{R}^n$ denote the region with $x_{\sigma^{-1}(1)} < \cdots < x_{\sigma^{-1}(n)}$. The sign of the confluent Vandermonde determinant is constant on this region of $\mathbb{R}^n$. Taking the real absolute value of the confluent Vandermonde determinant equates to replacing the matrix with one in which the columns have been reordered to match $\sigma$ (so that all differences in the determinant are positive on $Q_n(\sigma)$).
Given confluent type matrix $V \tilde{Z}(\vec{x})$, let $V \tilde{Z}(\vec{y})$ denote the matrix obtained from $V \tilde{Z}(\vec{x})$ by permuting the columns so that the columns with $x_{\sigma^{-1}(1)}$ come first (of which there are $L_{\sigma^{-1}(1)}$ many), then the columns with $x_{\sigma^{-1}(2)}$ come next (of which there are $L_{\sigma^{-1}(2)}$ many), and so on until the $x_j$ are exhausted. Then for $\vec{x} \in Q_n(\sigma)$, we have $\det V \tilde{Z}(\vec{x}) = \det V \tilde{Z}(\vec{y})$.

Let $\Lambda = (L_1, \ldots, L_1, L_2, \ldots, L_2, \ldots, L_j, \ldots, L_j)$ with each $L_j$ appearing $K M_j$ consecutive times. Then, the partition function for the canonical multicomponent constellation ensemble (meaning fixed population $M$) is given by

$$Z_M(\vec{y}) = \frac{1}{M_1! \cdots M_J!} \sum_{\sigma \in S_T} \int_{Q_T(\sigma)} \det V_{\Lambda}(\vec{x}_j) d\mu(\vec{x}_j),$$

where $T = \sum_{j=1}^J K M_j$. Note, the factorial denominators appear because there are $M_j$ indistinguishable real parts which share the same charge $L_j$. This same technique (of decomposing the domain of integration over the possible orderings on the variables to account for the absolute value) is also necessary in the one-dimensional case, so nothing unique to constellation ensembles is happening here. Notably, this is where the sum in Lemma 2.1 comes from.

### 3.5 Circular constellation ensembles

Recall (from Sect. 2.5), the induced interaction between a real angle $x_m$ and another real angle $x_n$ is given by

$$d_\gamma(x_m, x_n) = \prod_{k=1}^K |y_k e^{ix_m} - y_k e^{ix_n}| \prod_{j<k} |y_k e^{ix_m} - y_j e^{ix_n}| |y_j e^{ix_m} - y_k e^{ix_n}|.$$ 

As observed in [21], we can express these interactions without absolute value using the following identities:

$$|y_k e^{ix_m} - y_k e^{ix_n}| = -i e^{-i(x_m+x_n)/2} (y_k e^{ix_m} - y_k e^{ix_n}) \sgn(x_m - x_n).$$

In the homogeneous cases, we can also assume without loss of generality $\vec{x} \in Q_M(id)$, meaning $x_1 < \cdots < x_M$. Then $\sgn(x_m - x_n) = 1$ for all $n < m$. Finally, the weight function $d\mu(x) = (-i e^{-ix_1})^{R_1(M-1)/2}(-i e^{ix_m})^{R_2} dx$ comes from the factors of $-i$ and $e^{-ix_m}$ which appear here, with $R_2$ and $R_3$ coming from the exponents in $d_\gamma(x_m, x_n)$.

In the multicomponent case, we follow the setup of the linear case but replace each instance of $x_m^j + i y_k$ with $y_k e^{ix_m^j}$. For particles $y_k e^{ix_m^j}$ and $y_k e^{ix_n^j}$ on the same circle,

$$|y_k e^{ix_m^j} - y_k e^{ix_n^j}| = -i e^{-i(x_m^j+x_n^j)/2} (y_k e^{ix_m^j} - y_k e^{ix_n^j}) \sgn(x_m^j - x_n^j).$$

Using what we know from the one-dimensional case in [21], the sign correction factors in $x_m^j$ are

$$(-i e^{-ix_m^j})^{KL_j T/2},$$

where

$$T = -L_j + \sum_{k=1}^J L_k M_k.$$ 

Next, for particles $y_k e^{ix_m^j}$ and $y_l e^{ix_m^j}$ which share an angle $x_m^j$,

$$|y_k e^{ix_m^j} - y_l e^{ix_m^j}| = e^{-ix_m^j} (y_k e^{ix_m^j} - y_l e^{ix_m^j}),$$

giving us the sign correction factor

$$e^{-ix_m^j} = \frac{L_j^2(k)}{2}.$$ 

Finally, for particles $y_k e^{ix_m^j}$ and $y_h e^{ix_m^j}$, which share neither an angle nor a radius,

$$|y_k e^{ix_m^j} - y_h e^{ix_m^j}| = -i e^{-i(x_m^j+x_n^j)} (y_k e^{ix_m^j} - y_h e^{ix_m^j}) (y_h e^{ix_m^j} - y_k e^{ix_m^j}),$$

giving us the last sign correction factor

$$(-i e^{-ix_m^j})^{(k)} L_j T.$$
Thus, \[ d\mu_j(x) = (-ie^{-ix})K^2L_jT/2\left(e^{-ix}\right)L_j^2(\xi_x)\,dx. \]

4 The Berezin integral

Let \( \varepsilon_1, \ldots, \varepsilon_N \) be a basis for \( \mathbb{R}^N \). For any injection \( t : \mathbb{K} \to \mathbb{N} \), let \( \varepsilon_t \in \bigwedge^K(\mathbb{R}^N) \) denote \( \varepsilon_t = \varepsilon_{t(1)} \wedge \varepsilon_{t(2)} \wedge \cdots \wedge \varepsilon_{t(K)}. \)

Then \( \{ \varepsilon_t \mid t : \mathbb{K} \not\to \mathbb{N} \} \) is a basis for \( \bigwedge^K(\mathbb{R}^N) \). In particular, \( \bigwedge^N(\mathbb{R}^N) \) is a one-dimensional subspace we call the determinantal line, spanned by \( \varepsilon_{\text{vol}} = \varepsilon_{\text{id}} = \varepsilon_1 \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_N, \)

which we call the volume form (in \( \mathbb{R}^N \)). For each \( 0 < n \leq N, \) define \( \frac{\partial}{\partial \varepsilon_n} : \bigwedge^K(\mathbb{R}^N) \to \bigwedge^{K-1}(\mathbb{R}^N) \) on basis elements by

\[
\frac{\partial}{\partial \varepsilon_n} \varepsilon_t = \begin{cases} (-1)^k \varepsilon_{t(1)} \wedge \cdots \wedge \varepsilon_{t(k-1)} \wedge \varepsilon_{t(k+1)} \wedge \cdots \wedge \varepsilon_{t(K)} & \text{if } k = t^{-1}(n) \\ 0 & \text{otherwise} \end{cases},
\]

and then extend linearly. If \( n \in t(\mathbb{K}) \), meaning \( \varepsilon_n \) appears as a factor in \( \varepsilon_t \), then \( \frac{\partial}{\partial \varepsilon_n} \) is the result of permuting \( \varepsilon_n \) to the front and then removing it, picking up a sign associated with changing the order in which the basis elements occur. If \( \varepsilon_t \) does not have \( \varepsilon_n \) as a factor, then \( \frac{\partial}{\partial \varepsilon_n} = 0 \). Given an injection \( s : L \to \mathbb{N} \), we define the Berezin integral [2] (with respect to \( \varepsilon_s \)) as a linear operator \( \bigwedge^s(\mathbb{R}^N) \to \bigwedge^N(\mathbb{R}^N) \) given by

\[
\int \varepsilon_t \, d\varepsilon_s = \int \varepsilon_t \, d\varepsilon_{s(1)} \, d\varepsilon_{s(2)} \cdots d\varepsilon_{s(L)} = \frac{\partial}{\partial \varepsilon_{s(1)}} \cdots \frac{\partial}{\partial \varepsilon_{s(2)}} \frac{\partial}{\partial \varepsilon_{s(1)}}^{-\varepsilon_t}.
\]

Note, if \( \varepsilon_t \in \bigwedge^K(\mathbb{R}^N) \) for any \( K < N \), then

\[
\int \varepsilon_t \, d\varepsilon_{\text{vol}} = 0
\]

because \( \varepsilon_t \) is missing some \( \varepsilon_k \) as a factor. Thus, the Berezin integral with respect to \( \varepsilon_{\text{vol}} \) is a projection operator \( \bigwedge(\mathbb{R}^N) \to \bigwedge^N(\mathbb{R}^N) \cong \mathbb{R} \). In particular, if \( \sigma \in S_N \), then

\[
\int \varepsilon_{\sigma} \, d\varepsilon_{\text{vol}} = \text{sgn}(\sigma).
\]

4.1 Exponentials and hyperpfaffians

For \( \omega \in \bigwedge(\mathbb{R}^N) \) and positive integer \( m \), we write \( \omega^\wedge m = \omega \wedge \cdots \wedge \omega \),

with \( \omega \) appearing as a factor \( m \) times. By convention, \( \omega^0 = 1 \). We then define the exponential

\[
\exp(\omega) = \sum_{m=0}^{\infty} \frac{\omega^\wedge m}{m!}.
\]

Moreover, suppose \( \omega = \omega_1 + \omega_2 + \cdots + \omega_J \) where each \( \omega_j \in \bigwedge^{L_j}(\mathbb{R}^N) \) and each \( L_j \) even, then (we say each \( \omega_j \) is a homogeneous even form of length \( L_j \) and it is easily verified

\[
\exp(\omega) = \exp(\omega_1 + \cdots + \omega_J) = \exp(\omega_1) \wedge \cdots \wedge \exp(\omega_J).
\]

We get a homogeneous form in all cases but the multicomponent constellation ensemble. In the homogeneous cases, exactly one summand in the exponential will live at the determinantal line. Assuming \( \omega \in \bigwedge^K(\mathbb{R}^N) \) with \( K \leq N \), we get

\[
\int \exp(\omega) \, d\varepsilon_{\text{vol}} = \int \sum_{m=0}^{\infty} \frac{\omega^\wedge m}{m!} \, d\varepsilon_{\text{vol}} = \int \frac{\omega^\wedge M}{M!} \, d\varepsilon_{\text{vol}} = \text{PF}(\omega),
\]

where \( \text{PF}(\omega) \) is the Hyperpfaffian of \( \omega \). We can take this to be a definition; \( \text{PF}(\omega) \) is the real number coefficient on \( \varepsilon_{\text{vol}} \) in \( \frac{\omega^\wedge M}{M!} \). To avoid confusing this Berezin integral with other integrals which appear in our computations, we write

\[
\text{BE}_{\text{vol}}(\omega) = \int \exp(\omega) \, d\varepsilon_{\text{vol}}.
\]
where the subscript on the left hand side indicates which form we are integrating with respect to.

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