Cluster polylogarithms for scattering amplitudes

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Abstract
Motivated by the cluster structure of two-loop scattering amplitudes in $N = 4$ Yang-Mills theory we define cluster polylogarithm functions. We find that all such functions of weight four are made up of a single simple building block associated with the $A_2$ cluster algebra. Adding the requirement of locality on generalized Stasheff polytopes, we find that these $A_2$ building blocks arrange themselves to form a unique function associated with the $A_3$ cluster algebra. This $A_3$ function manifests all of the cluster algebraic structure of the two-loop $n$-particle MHV amplitudes for all $n$, and we use it to provide an explicit representation for the most complicated part of the $n = 7$ amplitude as an example.

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(Some figures may appear in colour only in the online journal)

1. Introduction

There exists vast mathematical literature on cluster algebras (see for example [1]), and also a large amount of literature on the mathematical structure of generalized polylogarithm functions. One of the general themes to emerge from [2] was the observation that the perturbative scattering amplitudes in $N = 4$ supersymmetric Yang–Mills (SYM) theory, which have also been the object of intense study in recent years, tie these two topics intimately together\(^1\).

\(^1\) See also [3] for a different relation between cluster algebras and scattering amplitudes.
In this paper we take a few steps towards a systematic investigation of the intersection between these fields of mathematics. To that end we define and study the simplest examples of cluster polylogarithm functions—pure transcendental functions which ‘depend on’ (in a way to be made precise below) only the cluster coordinates of some cluster algebra. Even the mere existence of non-trivial examples of such functions is not a priori obvious—for example, we will see that the Gr(3,5) cluster algebra admits only a single non-trivial cluster function of weight four. Special functions of this kind are apparently not known in the mathematical literature, but we know they exist and are likely to have remarkable properties since SYM theory evidently provides (in addition to numerous other generous mathematical gifts) at least one infinite class of such functions: the two-loop n-particle MHV amplitudes [4].

In addition to the purely mathematical motivation for exploring this new class of special functions, our work also has a practical application for physicists. The symbol of all two-loop n-particle MHV amplitudes has been known for almost three years from the work of Caron-Huot [5], but it remains an interesting outstanding problem to write explicit analytic formulas for these amplitudes. So far this has been accomplished [20, 21] only for the very special case of n = 6, where the result surprisingly can be written entirely in terms of the classical polylogarithm functions Li_k [22] (a curiosity which we ‘explain’ below). To write analytic results for generic scattering amplitudes, even in SYM theory, requires giving up the crutch of working with only the relatively tame classical functions and entering the much larger and wilder world of generalized polylogarithm functions. Several impressive analytic results of this type have been achieved for higher-loop or non-MHV n = 6 amplitudes in SYM theory by Dixon and collaborators [23, 24]. Applying similar technology at higher n looks challenging because the required computer power grows rapidly with n. Ultimately this is due to the fact that in the absence of other guidance, one would run the risk of being forced to work with a vastly overcomplete basis of functions not specifically tailored to the problem at hand.

When studying n > 6 amplitudes in SYM theory it is desirable, for both practical as well as aesthetic reasons, to seek out functional representations which manifest (to the extent possible) all of the important properties of an amplitude. For example, the GSVV formula [22], unlike the previously known DDS formula, makes three important properties of the two-loop n = 6 MHV amplitude trivially manifest: it is classical, dihedral invariant, and real-valued everywhere inside the Euclidean domain. (There is also one interesting property which the GSVV formula does not make manifest: the fact that it is positive-valued everywhere inside the positive domain.) Working with functional representatives which manifest as many properties as possible has enormous practical advantages over working with a generic basis of functions, and also helps to elucidate the deeper mathematical structure of the amplitudes. Of course as time passes we may discover new, previously unnoticed properties, allowing us the opportunity to further upgrade the class of functions we work with.

Suppose we were to commission some very special custom non-classical polylogarithm function (or collection of functions) specifically suited for the purpose of expressing the two-loop n-point MHV amplitudes. Based on what we know about these amplitudes today, what properties should we demand these special functions manifest? The most basic property we should impose is that the symbol alphabet should consist of the cluster A-coordinates of the Gr(4, n) cluster algebra, a property of the amplitudes which is manifest in the result of [5].

Considerable analytic progress has been made both at two loops and (in some cases) far beyond for special kinematic configurations, including for example multi-Regge kinematics [6–10], the near-collinear limit [11–13], and 2d kinematics [14–17] (see also [18] for comments on cluster structure in 2d kinematics). Fully analytic formulas for the differential of the two-loop MHV amplitude for all n were computed in [19].
We call such functions ‘cluster functions’. For the special case of the \( \text{Gr}(4, 6) \) algebra, relevant to \( n = 6 \) particle amplitudes, functions of this type (and satisfying also various other physical constraints) were extensively studied, and fully classified through weight six in [23–25].

Taking inspiration from this program of classifying allowed functions, we consider here two additional properties of the two-loop MHV amplitudes which were recently observed in [2, 4, 26]: (1) the coproduct of these amplitudes can be expressed entirely in terms of \( X \)-coordinates [27] on the cluster Poisson variety \( \text{Conf}_n(\mathbb{P}^3) \) (or equivalently, the \( \text{Gr}(4, n) \) cluster algebra), and moreover (2) that the \( \Lambda^2 B_2 \) component of the coproduct can be expressed entirely in terms of pairs of variables which Poisson commute.

Remarkably we find that the two simplest non-trivial cluster polylogarithm functions exactly fit the bill. Specifically, we find at transcendentality weight four that the \( \text{Gr}(3, 5) \) cluster algebra (also called \( A_2 \)) admits a unique non-trivial function satisfying property (1), and the \( \text{Gr}(4, 6) \) (or \( A_3 \)) cluster algebra admits a unique non-trivial function (itself a linear combination of \( A_2 \) functions) which in addition satisfies property (2). We have checked explicitly for a small handful of cluster algebras that the functions associated with all \( A_2 \) and \( A_3 \) subalgebras provide a complete basis for all weight-four functions satisfying these properties. It is certainly an interesting mathematical problem to explore the universe of cluster functions for general algebras, but for the more limited purpose of expressing two-loop MHV amplitudes it seems that the six-particle \( A_3 \) function is all we need. Although the structure of the \( n \)-point amplitude stabilizes at relatively small \( n \) (that means that higher \( n \) amplitudes can be written in terms of building blocks involving smaller values of \( n \)), it is rather surprising that the basic two-loop building block seems to involve only six particles.

In section 2 we briefly review some mathematical background necessary for formulating our definition of cluster functions. In sections 3 and 4 respectively we discuss the \( A_2 \) and \( A_3 \) functions, and in section 5 we comment on the problem of expressing two-loop \( n \)-point MHV amplitudes in terms of \( A_3 \) functions, providing an explicit result for \( n = 7 \) as an example.

### 2. Cluster polylogarithm functions

#### 2.1. Polylogarithm functions, symbols, and the coproduct

We begin by recalling some elementary mathematical facts about polylogarithm functions from [28, 29] (see [2, 30–32] for recent reviews written for physicists). To each such transcendental function of weight \( k \) is associated an element of the \( k \)-fold tensor product of the multiplicative group of rational functions modulo constants called its symbol. For example, the classical polylogarithm function \( \text{Li}_k(x) \) has symbol

\[
-(1 - x) \otimes x \otimes \cdots \otimes x.
\]  

A trivial way to make a function of weight \( k \) is to multiply two functions of lower weights \( k_1, k_2 \) with \( k = k_1 + k_2 \). It is often useful to exclude such products from consideration and to focus on the most complicated, intrinsically weight \( k \), part of a function. This may be accomplished via a projection operator \( \rho \) which annihilates all products of functions of lower weight. It is defined recursively by

\[
\rho(a_1 \otimes \cdots \otimes a_k) = \frac{k - 1}{k} \left[ \rho(a_1 \otimes \cdots \otimes a_{k-1}) \otimes a_k - \rho(a_2 \otimes \cdots \otimes a_k) \otimes a_1 \right]
\]  

\( k \)-1 times
beginning with \( \rho(a_1) = a_1 \). Here, in a slight abuse of notation which we will perpetuate throughout this section, we display for simplicity not how \( \rho \) acts on a general weight-\( k \) function but rather how it acts on the symbol of such a function.

We use \( \mathcal{L}_\bullet \) to denote the algebra of polylogarithm functions modulo products of functions of lower weight. It is a commutative graded Hopf algebra with a coproduct \( \delta : \mathcal{L}_\bullet \to \Lambda^2 \mathcal{L}_\bullet \) which satisfies \( \delta^2 = 0 \), giving it the structure of a Lie coalgebra. Explicitly, \( \delta \) may be computed (again, at the level of symbols) by

\[
\delta(a_1 \otimes \cdots \otimes a_k) = \sum_{n=1}^{k-1} (a_1 \otimes \cdots \otimes a_n) \wedge (a_{n+1} \otimes \cdots \otimes a_k). \tag{2.3}
\]

We let \( B_k \) denote the Bloch group \([33, 34]\) defined as the quotient of \( \mathcal{L}_k \) by the subspace of functional equations for the classical logarithm function \( \text{Li}_k \). The case \( k = 1 \) is trivial (any linear combination of logarithm functions can be combined into a single logarithm) so we simply write \( \log x \) to denote the function \( x \). These satisfy

\[
\delta \{ x \} = \begin{cases} (1 + x) \wedge x & k = 2, \\ \{ x \} \otimes x & k > 2. \end{cases} \tag{2.4}
\]

For \( k = 2, 3 \) it is a theorem that \( \mathcal{L}_k \cong B_k \). At weight four, for the first time, the coproduct has two separate components

\[
\delta(a_1 \otimes a_2 \otimes a_3 \otimes a_4) \big|_{\mathcal{L}_4} = \rho(a_1 \otimes a_2) \wedge \rho(a_3 \otimes a_4), \quad \tag{2.5}
\]

\[
\delta(a_1 \otimes a_2 \otimes a_3 \otimes a_4) \big|_{B_4 \otimes \mathbb{C}^*} = \rho(a_1 \otimes a_2 \otimes a_3) \otimes a_4 - \rho(a_2 \otimes a_3 \otimes a_4) \otimes a_1. \tag{2.6}
\]

The classical function \( \text{Li}_4(x) \) has coproduct components

\[
\delta \text{Li}_4(x) \big|_{\mathcal{L}_4} = 0, \tag{2.7}
\]

\[
\delta \text{Li}_4(x) \big|_{B_4 \otimes \mathbb{C}^*} = - \{ -x \} \otimes x. \tag{2.8}
\]

so it is clear that any polylogarithm function of weight four which has a nonzero \( \Lambda^2 B_2 \) content cannot possibly be written in terms of classical functions. It is moreover conjectured that the converse is true \([28]\)\(^5\). In this sense we can say that it is the \( \Lambda^2 B_2 \) coproduct component which measures the ‘non-trivial part’ of a weight-four polylogarithm function.

\(^3\) The top line is an element of \( \Lambda^2 \mathcal{L}_1 \), while the bottom is the element of the summand in \( \Lambda^2 (\mathcal{L}_{k-1} \oplus \mathcal{L}_1) \) given by vectors of the form \( f_{n-1} \otimes f' - f' \otimes f_{n-1} \), and we use the standard notation of denoting such an element simply by \( f_{n-1} \otimes f' \in \mathcal{L}_{k-1} \oplus \mathcal{L}_1 \).

\(^4\) These expressions are simply transcriptions of the components of equation (2.3). Specifically, equation (2.5) is the \( n = 2 \) term in equation (2.3), while equation (2.6) is the sum of the \( n = 1 \) and \( n = 3 \) terms in equation (2.3), with the \( n = 1 \) term receiving a minus sign when put into the order shown in equation (2.6) as implied by the \( \wedge \) in equation (2.3). We have chosen to write some \( \rho \)’s explicitly in equations (2.5) and (2.6) but they are not necessary since \( \rho(x) \) and \( x \) represent the same element in \( \mathcal{L}_4 \). For the same reason, we could have acted on both sides of the \( \wedge \) in equation (2.3) with the \( \rho \) operator without changing the meaning of that definition.

\(^5\) More generally, it is conjectured that a weight-\( k \) function \( f_k \) can be written in terms of the classical polylogarithm \( \text{Li}_k \) if and only if all components of \( \delta f_k \) vanish except possibly \( \mathcal{L}_{k-1} \otimes \mathbb{C}^* \).
2.2. Integrability

Next we discuss the integrability condition which plays the crucial role in the following two sections. A second application of $\delta$ at weight four maps each of the two components to $\Lambda^2 \otimes \Lambda^2 \mathbb{C}^*$, as indicated in the diagram:

\[
\begin{array}{ccc}
\Lambda^2 B_2 & \rightarrow & B_3 \otimes \mathbb{C}^* \\
& \downarrow & \\
B_2 \otimes \Lambda^2 \mathbb{C}^* & \rightarrow & \Lambda^2 B_2 \\
\end{array}
\]

where the bottom two arrows are given explicitly by

\[
\delta(\{x\}_2 \wedge \{y\}_2) = \{y\}_2 \otimes (1 + x) \wedge x - \{x\}_2 \otimes (1 + y) \wedge y, \quad (2.9)
\]

\[
\delta(\{x\}_3 \otimes y) = \{x\}_2 \otimes x \wedge y. \quad (2.10)
\]

Given arbitrary elements $b_{22} \in \Lambda^2 B_2$ and $b_{31} \in B_3 \otimes \mathbb{C}^*$, there does not necessarily exist any function $f_4 \in \mathcal{L}_4$ whose coproduct components are $b_{22}$ and $b_{31}$. A necessary and sufficient condition for such a function to exist is that the integrability condition

\[
0 = \delta^2 f_4 = \delta(b_{22}) + \delta(b_{31}) \quad (2.11)
\]

is satisfied. Equivalently, we can say that a pair $b_{22}$, $b_{31}$ satisfying (2.11) uniquely determines a weight-four polylogarithm function (modulo products of functions of lower weight).

It is important to note that given any element $b_{22} \in \Lambda^2 B_2$ there does exist some function $f_4$ with $b_{22}$ as its coproduct component (indeed Goncharov has written down [35] an explicit map $\kappa: \Lambda^2 B_2 \rightarrow B_3 \otimes \mathbb{C}^*$ such that the pair $b_{22}$, $\kappa(b_{22})$ satisfies (2.11) for any $b_{22} \in \Lambda^2 B_2$), but for generic $b_{22}$ the $B_3 \otimes \mathbb{C}^*$ component $\kappa(b_{22})$ of that function will not have any cluster algebra structure of the type we study below.

2.3. Cluster $\mathcal{A}$- and $\mathcal{X}$-coordinates

Next we provide a lightning review (see [2] for details) of the types of variables which make an appearance in the study of scattering amplitudes in SYM theory: cluster $\mathcal{A}$- and cluster $\mathcal{X}$-coordinates. Much of what we have to say about cluster polylogarithm functions may be interesting to investigate in the context of general algebras, but we restrict our attention here largely to Grassmannian cluster algebras, and in particular to the $\text{Gr}(4, n)$ algebra relevant to the kinematic configuration space $\text{Conf}_n(\mathbb{P}^3)$ of $n$-particle scattering in SYM theory.

Examples of $\mathcal{A}$-coordinates on $\text{Gr}(4, n)$ include the ordinary Plücker coordinates $(ijkl)$ as well as certain particular homogeneous polynomials in them such as

\[
\text{Note that this is not a commutative diagram; indeed according to equation (2.11) it is better thought of as an anticommutative diagram.}
\]
\[ \langle ab(c)(de)(fg) \rangle \equiv \langle abde \rangle \langle acfg \rangle - \langle abfg \rangle \langle acde \rangle, \quad (2.12) \]
\[ \langle ab(cde) \cap (fg)h \rangle \equiv \langle acde \rangle \langle bfgh \rangle - \langle bcde \rangle \langle afgh \rangle, \quad (2.13) \]
while the \( \mathcal{X} \)-coordinates are certain cross-ratios which can be built from \( \mathcal{A} \)-coordinates.

For \( n > 7 \) there exist arbitrarily more complicated \( \mathcal{A} \)-coordinates on \( \text{Gr}(4, n) \). These appear to play no role at two loops (they likely do appear at higher loop order) since the symbol of the \( n \)-point two-loop MHV amplitude was computed in [5] and nothing more exotic than the examples shown in equations (2.12) and (2.13) occurs.

We emphasize that not every homogeneous polynomial of Plücker coordinates is an \( \mathcal{A} \)-coordinate, nor is every cross-ratio one can write down an \( \mathcal{P} \)-coordinate. The only surefire algorithm for determining such coordinates is via the mutation algorithm (see [2]), but we note here an empirical rule for selecting \( \mathcal{X} \)-coordinates for which we know no counter-example: a conformally invariant ratio \( x \) of \( \mathcal{A} \)-coordinates is an \( \mathcal{X} \)-coordinate if \( 1 + x \) also factors into a ratio of products of \( \mathcal{A} \)-coordinates and if \( x \) is positive-valued everywhere inside the positive domain (this is the subset of \( \text{Conf}_n(\mathbb{P}^3) \) for which \( \langle abcd \rangle > 0 \) whenever \( a < b < c < d \)).\footnote{It is a logical possibility that there could exist some \( x \) which satisfies this criterion yet which is not an \( \mathcal{X} \)-coordinate, though we have never encountered such an object in various explorations through \( n = 9 \).}

This algorithm reveals for example that between

\[
\begin{align*}
\langle 1235 \rangle \langle 1278 \rangle \langle 2456 \rangle \langle 5678 \rangle & & \quad \text{and} & & \quad \langle 2 \rangle \langle 13 \rangle \langle 56 \rangle \langle 78 \rangle \langle 5 \rangle \langle 12 \rangle \langle 46 \rangle \langle 78 \rangle \langle 123 \rangle \langle 456 \rangle \\
\langle 1256 \rangle \langle 2578 \rangle \langle 78 \rangle \langle 123 \rangle \langle 456 \rangle & & \quad \text{and} & & \quad \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \langle 4 \rangle \langle 56 \rangle \langle 78 \rangle \langle 456 \rangle
\end{align*}
\]

(whose difference is 1) only the first is an \( \mathcal{X} \)-coordinate.

### 2.4. Cluster polylogarithm functions

Now we turn to the heart of the paper: providing a definition of\textit{ cluster polylogarithm functions}. Good definitions in mathematics must lie in a Goldilocks zone: they must be sufficiently constraining so as to select out only certain objects with interesting behavior, yet they must not be so constraining as to preclude the existence of any examples. In defining cluster polylogarithm functions we are guided by the physics of two-loop MHV amplitudes in SYM theory: these functions certainly exist, yet have properties which render them very special amongst the class of all weight-four polylogarithm functions on \( \text{Conf}_n(\mathbb{P}^3) \).

We first define a cluster \( \mathcal{A} \)-function of weight \( k \) to be a conformally invariant function of transcendental weight \( k \) whose symbol can be written with only the \( \mathcal{A} \)-coordinates of some cluster algebra appearing in its entries. Functions of this type for the \( \text{Gr}(4, 6) \) cluster algebra (and satisfying various other physical constraints) were extensively classified and studied in the papers [23–25].

Our goal here is to impose additional mathematical constraints to focus on a different (and at least for larger \( n \), much smaller) collection of functions: those which ‘depend on’ only the cluster \( \mathcal{X} \)-coordinates of some cluster algebra. At weight \( k < 4 \), where we know that the classical polylogarithm functions generate all of \( L_k \), we can make this statement immediately precise: a cluster \( \mathcal{X} \)-function of weight \( k < 4 \) is a linear combination of the functions \( -\text{Li}_2(-x) \) for \( x \) drawn from the set of \( \mathcal{X} \)-coordinates of some cluster algebra.

At weight one there is no distinction between cluster \( \mathcal{A} \)- and \( \mathcal{X} \)-functions because any conformally invariant cross-ratio can be expressed as a ratio of products of \( \mathcal{X} \)-coordinates. Hence any conformally invariant linear combination of logarithms of \( \mathcal{A} \)-coordinates can be re-expressed as a linear combination of logarithms of \( \mathcal{X} \)-coordinates.

At weight two there is still no distinction; cluster \( \mathcal{A} \)-functions consist of all functions \( -\text{Li}_2(-y) \) for which both \( y \) and \( 1 + y \) factor into ratios of products of \( \mathcal{A} \)-coordinates. But then
either $y$ or $-(1+y)$ (whichever is positive throughout the positive domain) is a cluster $\mathcal{X}$-coordinate by the criterion discussed above. If $y$ is not the $\mathcal{X}$-coordinate then we can represent the function $-\text{Li}_2(-y)$ equivalently by $\text{Li}_2(1+y)$ (modulo $\pi^2$ and products of logs), establishing that it is a cluster $\mathcal{X}$-function.

At weight three there is a third term in the polylogarithm identity

$$-\text{Li}_3(-y) - \text{Li}_3(1+y) - \text{Li}_3(1+1/y) = 0 \tag{2.15}$$

which implies that `only half’ of weight-three cluster $\mathcal{A}$-functions are $\mathcal{X}$-functions. More precisely: if $m$ is the dimension of the space spanned by the functions $-\text{Li}_3(-x)$ for all cluster $\mathcal{X}$-coordinates $x$, then the space of weight-three cluster $\mathcal{A}$-functions is $2m$ dimensional, containing in addition all functions of the form $-\text{Li}_3(1+x)$.

Weight four is the first place where things become nontrivial. We first need a more precise definition of cluster $\mathcal{X}$-functions, since not every weight-four polylogarithm can be expressed in terms of the classical function $\text{Li}_4$ only. Motivated by the results of [2] we define a weight-four cluster $\mathcal{X}$-function (henceforth referred to simply as a cluster polylogarithm function or just cluster function) to be a cluster $\mathcal{A}$-function whose coproduct components can be written as a linear combination of $\{x_i\}_2 \wedge \{x_j\}_2$ or $\{x_i\}_3 \otimes x_j$ for cluster $\mathcal{X}$-coordinates $x_i, x_j$. Of course the classical function $-\text{Li}_4(-x)$ is trivially such a cluster $\mathcal{X}$-function whenever $x$ is an $\mathcal{X}$-coordinate, so we will often use the word `nontrivial’ to denote those cluster $\mathcal{X}$-functions with nonzero $\Lambda^3 B_2$ content.

We do not yet propose a definition of cluster functions for weight greater than four. As mentioned above, an appropriate definition would be as restrictive as possible without ruling out the existence of non-trivial examples, and should include interesting examples of functions from SYM theory. We believe that the identification of a suitable definition requires first a better understanding of the structure of MHV amplitudes at higher loop order, of which the only example currently in the literature is the tour de force calculation of the three-loop MHV amplitude for $n=6$ in [24].

In the next two sections we classify and study the properties of the cluster functions for the simplest nontrivial cluster algebras.
Let us begin with the simplest nontrivial cluster algebra, the \( \text{Gr}(3, 5) \) (or \( A_2 \)) algebra. This algebra has five cluster \( X \)-coordinates which may be generated from an initial pair \( x_1, x_2 \) via the relation

\[
x_{i+1} = \frac{1 + x_j}{x_{i-1}}.
\]

Several relevant pieces of information about this algebra are encoded graphically in the pentagon shown in figure 1. To each oriented edge is associated a cluster \( X \)-variable \( x_i \); in each case \( x_1 \) would be associated to the same edge with opposite orientation. To each vertex is associated the pair of variables (the \( \text{cluster} \)) given by the edge variables emanating away from that vertex—so, for example, the cluster associated with the top vertex in the figure contains the variables \( x_2, 1/x_1 \).

We seek nontrivial cluster polylogarithm functions of weight four—that is, solutions of equation (2.11) for which \( b_{32} \) and \( b_{31} \) can be written simply in terms of the five available cluster \( X \)-coordinates. Since \( A_2 \) is a finite cluster algebra, this is a simple problem in linear algebra. The dimension of \( B_1 \) is 5—spanned by the five multiplicatively independent \( X \)-coordinates, the dimension of \( B_2 \) is 4—spanned by the five \( \{ x_i \}_2 \) subject to the Abel identity

\[
\sum_{i=1}^{5} [x_i]_2 = 0,
\]

and the dimension of \( B_3 \) is again 5—spanned by the five \( \{ x_i \}_3 \), which are independent.

It is simple to check that in the 10-dimensional space \( \Lambda_2 B_2 \), there is a unique element \( b_{22} \) for which there exists a \( b_{31} \) in the 25-dimensional \( \mathfrak{g} \otimes \mathfrak{s}^* \) satisfying equation (2.11). We call this solution the \( A_2 \) function (or the pentagon function). The \( \mathfrak{g} \otimes \mathfrak{s}^* \) component of the \( A_2 \) function is not uniquely fixed by equation (2.11) since one always has the freedom to add any linear combination of the five \( \text{Li}_4 (-x_i) \). We fix this freedom by choosing to define the \( A_2 \) function to have the coproduct components

\[
\delta f_{A_2}(x_1, x_2)|_{B_2} = \sum_{i,j=1}^{5} [x_i]_2 \wedge [x_{i+j}]_2,
\]

\[
\delta f_{A_2}(x_1, x_2)|_{B_3 \otimes \mathbb{C}^*} = 5 \sum_{i=1}^{5} \{ [x_{i+1}]_3 \otimes x_i - [x_i]_3 \otimes x_{i+1} \}.
\]

This is the unique choice which is skew-dihedral invariant—that means it is (1) cyclically invariant under \( x_i \rightarrow x_{i+1} \) and (2) changes sign under \( x_i \rightarrow x_{6-i} \). A very important facet of this definition is the antisymmetry of \( \delta f_{A_2}(x_1, x_2)|_{B_3 \otimes \mathbb{C}^*} \) under \( \{ x \}_3 \otimes y \rightarrow \{ y \}_3 \otimes x \). In some sense we can therefore consider \( f_{A_2} \) to be a ‘purely non-classical’ cluster function (although this notion is not precisely defined), since any linear combination of the classical functions \( -\text{Li}_4 (-x_i) \) functions has a naturally symmetric \( B_3 \otimes \mathbb{C}^* \) component. This antisymmetry property of the \( A_2 \) function makes them useful building blocks for expressing scattering amplitudes, as discussed below in section 5.

It is also interesting to note that the \( B_3 \otimes \mathbb{C}^* \) content of \( f_{A_2} \) can be expressed in an evidently ‘local’ manner—by this we mean that the two \( X \)-coordinates in each term \( \{ x_i \}_3 \otimes x_j \) always have \( j = i \pm 1 \) and therefore appear together inside some cluster and moreover have Poisson bracket \( \{ x_i, x_{i \pm 1} \} = \pm 1 \). In contrast, the \( \Lambda^2 B_2 \) component is
non-local: the two variables appearing in each term \{x_i \} _2 \land \{x_j \} _2 do not in general appear together in a common cluster and do not have any particularly simple Poisson bracket with each other.

Let us pause to clarify one point of notation which will allow us to avoid confusion later. All five \(X\)-coordinates appear on the right-hand sides of \(3.3\), but we appropriately write \(f_{A_2}(x_1, x_2)\) as a function of only two variables since the others may be expressed in terms of these via the relation \((3.1)\). Below we will frequently need to discuss \(A_3\) subalgebras of larger cluster algebras. Any such subalgebra is generated by a pair of \(X\)-coordinates which appear together inside some cluster and which have Poisson bracket \(\{x, y\} = 1\). When this happens the corresponding \(A_2\) function is simply \(f_{A_2}(x, y)\). To summarize using the quiver notation reviewed in \([2]\): \(f_{A_2}(x, y)\) is a function of any two \(X\)-coordinates appearing inside a quiver as \(x \rightarrow y\).

We emphasize that the equations \((3.3)\) completely and unambiguously define the \(A_2\) function as an element of \(\mathcal{L}_4\) — i.e., modulo products of functions of lower weight. Nevertheless, the reader with an appetite for seeing an actual function with these coproduct components may turn to the appendix for satisfaction, and we can write here a relatively simple expression for the symbol of a representative of \(f_{A_2}\):

\[
\text{symbol} \left(f_{A_2}(x_1, x_2)\right) \sim \frac{5}{4} \sum \text{skew-dihedral} x_1 \otimes x_2 \otimes x_1 \otimes \frac{x_2}{x_5} + x_1 \otimes x_2 \otimes x_2 \otimes \frac{x_1}{x_3}. \tag{3.4}
\]

We write \(\sim\) instead of \(=\) because as long as we consider \(f_{A_2}\) only as an element of \(\mathcal{L}_4\) its symbol is not even well-defined — equation \((3.4)\) represents one particular way of fixing the ambiguity associated with products of lower-weight functions (it is the choice which makes the symbol an eigenvector of \(\rho\)), but we are not yet ready to commit to any choice.

Although we believe this function to be new (and hopefully interesting) to the mathematics community, it may seem that this example is too trivial to be relevant to SYM theory, where the relevant algebras are \(\text{Gr}(4, n)\). For sure, \(\text{Gr}(4, n)\) contains many \(A_2\) subalgebras, and we may evaluate \(f_{A_2}\) on each of these, but are there any other solutions of \((2.11)\) for these algebras? Surprisingly, we have checked in addition to \(A_2\) the finite algebras \(A_3, A_4\) and \(D_4\), and in each case we have found that there are no other solutions — for these cluster algebras, \(\text{all non-trivial weight-four cluster functions are linear combinations of } A_2 \text{ functions}\)\(^8\)!

It remains an interesting mathematical problem to determine, for general cluster algebras (even for infinite ones), the set of non-trivial cluster polynlogarithm functions; that is, the subspace of \(A^6 B_2\) on which \((2.11)\) can be solved in terms of an element \(b_{12}\), expressible purely in terms of cluster \(X\)-coordinates. However, even if more exotic solutions exist in general, for the limited purpose of studying two-loop \(n\)-point MHV amplitudes it seems clear that the \(A_2\) functions are completely sufficient, in part because these amplitudes only live in a finite (and small) piece of the relevant cluster algebras, as discussed in section 5.

4. The \(A_3\) function

We now turn our attention to cluster polynlogarithms for the \(A_3\) cluster algebra, beginning with the seed quiver \(x_1 \rightarrow x_2 \rightarrow x_3\)\(^9\). This quiver generates the following 15 cluster \(X\)-coordinates:

\(^8\) We were unable to check this for the finite algebra \(E_6 (= \text{Gr}(4, 7)\) due to a lack of sufficient computer power. In this case the relevant spaces \(A^3 B_1 \otimes \mathbb{C}^n\) and \(B_1 \otimes A^3 \mathbb{C}^n\) have dimension 8646, 15 246 and 113 652 respectively.

\(^9\) Note that this is really shorthand for ‘a triplet of \(X\)-coordinates \(\{x_1, x_2, x_3\}\) that are all in the same cluster (this distinguishes between \(x_i\) and \(1/x_i\) and have the Poisson structure \(\{x_i, x_j\} = \{x_2, x_3\} = 1, \{x_1, x_3\} = 0\)’.
The structure of the algebra is summarized in the Stasheff polytope shown in figure 2. The polytope has 9 faces (comprising six pentagons and three quadrilaterals), 14 vertices, and 21 edges, each of which is labeled by a $X$-coordinate.

We now review a few facts about the natural Poisson structure [27] on $Conf_3(P^3)$ following [2]. A pair of cluster $X$-coordinates has a simple Poisson bracket (‘simple’ means $\pm 1$ or 0) only if they appear together inside some cluster. The coordinates in equation (4.1) have the following Poisson structure:

\[
x_{1,1} = x_1, \quad x_{1,2} = \frac{1}{x_3}, \quad x_{2,1} = \frac{x_1 x_2 + x_2 + 1}{x_1}, \quad x_{2,2} = \frac{x_2 x_3 + x_3 + 1}{x_1 x_2}, \quad x_{3,1} = \frac{x_2 x_3 + x_3 + 1}{x_2}.
\]

\[
e_1 = \frac{x_1 x_2 x_3 + x_2 x_3 + x_3 + 1}{x_1 + 1}, \quad e_2 = \frac{1}{x_2 + 1}, \quad e_3 = \frac{x_1 (x_3 + 1)}{x_1 x_2 x_3 + x_2 x_3 + x_3 + 1},
\]

\[
e_4 = \frac{x_2 + 1}{x_1 x_2}, \quad e_5 = \frac{x_1 x_2 x_3 + x_2 x_3 + x_3 + 1}{x_1 x_2 x_3 + x_2 x_3 + x_3 + 1}, \quad e_6 = x_2.
\]

(4.1)

The polytope has 9 faces (comprising six pentagons and three quadrilaterals), 14 vertices, and 21 edges, each of which is labeled by an $X$-coordinate.

The three quadrilateral faces are shaded blue to distinguish them visually from the six pentagonal faces. The interior of this polytope can be identified with the blow-up of the positive domain in $Conf_3(P^3)$, see for example [36].
where $v$ and $x$ have indices mod 3 and $e$ has indices mod 6. This means that there are 3 pairs of $X$-coordinates that Poisson commute and 30 pairs with Poisson bracket $\pm 1$.

Quadrilateral faces of a Stasheff polytope are in correspondence with pairs of cluster $X$-coordinates which Poisson commute, thereby generating $A_1 \times A_1$ subalgebras. Pentagonal faces of a Stasheff polytope correspond to $A^2$ subalgebras, generated by pairs of cluster $X$-coordinates which have Poisson bracket $\pm 1$. For the $A^3$ algebra there are 30 such pairs—5 (one at each vertex) each for the six pentagonal faces evident in figure 2. The sign of the Poisson bracket is unfortunately not manifest in the figure, so we record here explicitly the five $X$-coordinates $(x_1, \ldots, x_5)$ (following the notation of section 3) for each of the six $A_2$ subalgebras:

\[
\begin{align*}
(e_4, 1/e_6, x_{1,1}, v_3, x_{2,2}), & \quad (e_5, 1/e_1, x_{2,2}, v_1, x_{3,1}), \\
(e_6, 1/e_2, x_{3,1}, v_2, x_{1,2}), & \quad (e_1, 1/e_3, x_{1,2}, v_3, x_{2,1}), \\
(e_2, 1/e_4, x_{2,1}, v_1, x_{3,2}), & \quad (e_3, 1/e_5, x_{3,2}, v_2, x_{1,1}),
\end{align*}
\]

Each cyclically adjacent pair of variables appearing here, for example $\{e_6, 1/e_4\}$ or $\{x_{2,1}, e_1\}$, has Poisson bracket $+1$. The three entries in the left column can be read off from figure 2 by going around the pentagons clockwise (as seen from outside the Stasheff polytope), while the three entries in the right column must be read off counterclockwise.

Finally we come to the question of cluster functions for the $A_3$ algebra. As revealed already at the end of the previous section, it is a simple problem in linear algebra to verify that the equation (2.11) admits solutions only when $b_{22}$ lies in the 6-dimensional subspace of $A^2B_2$ spanned by the six $A_2$ functions associated with (4.3). We may represent these six functions as $f_{A_3}(e_i, 1/e_{i+2})$ for $i = 1, \ldots, 6$ thanks to the cyclic invariance of the $A_3$ function.

It is now time, in our quest to cook up a fine selection of special functions for the two-loop MHV amplitudes, to toss in one more very special ingredient. Beyond the fact that they are cluster polylogarithm functions, an even more amazing property of these amplitudes is that they have $A^2B_2$ content which can be expressed entirely in terms of pairs of cluster $X$-coordinates $\{x_i\}_{2} \wedge \{x_j\}_{2}$ which Poisson commute: $\{x_i, x_j\} = 0$! This was shown to be true for $n = 7$ in [2], and is in fact known to be true for all $n$ [4, 26].

For the $A_3$ algebra it is simple to check that there is a unique linear combination of the six $A_2$ functions with this property, which we naturally call the $A_3$ function:

\[
f_{A_3} = \frac{1}{10} \sum_{i=1}^{6} (-1)^{f_{A_2}(e_i, 1/e_{i+2})}.
\]
The coproduct of the $A_3$ function has the spectacularly simple, ‘local’ $A^2 B_2$ content

$$\delta f_{A_3} \big|_{A^2 B_2} = \sum_{i=1}^{3} \{ x_{i,1} \} \wedge \{ x_{i,2} \}.$$  \hspace{1cm} (4.5)

We do not write the $B_3 \otimes \mathbb{C}^n$ component since it does not simplify beyond the alternating sum of six copies of the corresponding component from the $A_2$ function.

We observed beneath equation (3.3) that the $B_3 \otimes \mathbb{C}^n$ content of the $A_2$ function is ‘local’ (involving only pairs of variables which appear in a common cluster), and the $A_3$ function obviously inherits this property. However the $A_2$ function has a non-local $A^2 B_2$ component, so it is rather amazing that the particular linear combination of $A_2$ appearing inside $A_3$ give rise to the completely local equation (4.5). Moreover, the two coproduct components see distinct aspects of the geometry of the Stasheff polytope—the $A^2 B_2$ component involves the three quadrilateral faces (i.e., the $A_1 \times A_1$ subalgebras) while the $B_3 \otimes \mathbb{C}^n$ component involves the six pentagonal faces (the $A_2$ subalgebras). It is tempting to anticipate the possibility that this notion of locality within the Stasheff polytope might underlie the structure of SYM theory scattering amplitudes in a very deep way. If this proves to be so, we cannot help but wonder (following somewhat the motivation espoused by [3]) whether there exists an alternative formulation of SYM theory scattering amplitudes which makes this ‘locality in the Stasheff polytope’ manifest.

A conjecture central to our approach is that the set of $f_{A_3}$ for all possible $A_3$ subalgebras of $\text{Gr}(4, n)$ spans the space of all weight-four cluster polylogarithm functions whose coproduct components are completely ‘local’ (involving only quadrilaterals in $A^2 B_2$ and only pentagons in $B_3 \otimes \mathbb{C}^n$).

We now display a simple realization of the $A_3$ function in a familiar setting: the $\text{Gr}(4, 6)$ algebra, relevant to 6-particle scattering, which is in fact isomorphic to $A_3$. In order to align with the notation in [2], we consider $(x_1, x_2, x_3) = (x^+_i, e_i, 1/x^+_i)$ and relate $x_{i,1} = x^+_i$ and $x_{i,2} = x^+_i$. The 15 $X$-coordinates can then be written as

$$v_1 = \langle 1246 \rangle \langle 1345 \rangle, \quad v_2 = \langle 1235 \rangle \langle 2456 \rangle, \quad v_3 = \langle 1356 \rangle \langle 2346 \rangle, \quad v_4 = \langle 1256 \rangle \langle 2345 \rangle, \quad v_5 = \langle 1236 \rangle \langle 1245 \rangle, \quad v_6 = \langle 1456 \rangle \langle 2345 \rangle, \quad v_7 = \langle 1235 \rangle \langle 1456 \rangle, \quad v_8 = \langle 1256 \rangle \langle 1345 \rangle, \quad v_9 = \langle 1236 \rangle \langle 1345 \rangle, \quad v_{10} = \langle 1246 \rangle \langle 1345 \rangle, \quad v_{11} = \langle 1456 \rangle \langle 2346 \rangle, \quad v_{12} = \langle 1236 \rangle \langle 1345 \rangle, \quad v_{13} = \langle 1234 \rangle \langle 3456 \rangle, \quad v_{14} = \langle 1234 \rangle \langle 1356 \rangle, \quad v_{15} = \langle 1234 \rangle \langle 2456 \rangle.$$  \hspace{1cm} (4.6)

Notably absent from this list are the three cross-ratios $u_1, u_2, u_3$ often used in the physics literature; these are related to the $v_i$ by $u_i = 1/(1 + v_i)$. Evaluating equation (4.4) on the variables in (4.6) generates what we will call the $\text{Gr}(4, 6)$ function.

It is interesting to note that the transformation of the $\text{Gr}(4, 6)$ function with respect to the dihedral group acting on the 6 particles is opposite to that of the 5-particle dihedral group acting on the $A_2$ function. Specifically, the $\text{Gr}(4, 6)$ function is invariant under flipping particle $i$ to particle $7 - i$, but it is antisymmetric under a cyclic rotation $i \rightarrow i + 1$. This antisymmetry is manifest for example in equation (4.5) upon noting that the $x^+_i$ transform under a cyclic rotation according to
The \( \text{Gr}(4, 6) \) algebra has an additional involution of order 2, called parity in \([2]\) (it corresponds to complex conjugation in Minkowski space kinematics), under which the \( X \)-coordinates transform according to
\[
{x_i}^\pm \mapsto {x_i}^\mp, \quad e_i \mapsto e_{i+3}.
\] (4.7)

The \( \text{Gr}(4, 6) \) function is antisymmetric under this parity operation.

The fact that MHV amplitudes are required to be fully invariant under both parity and cyclic symmetry, yet the unique non-classical weight-four function with the right cluster properties is antisymmetric under these symmetries, ‘explains why’ the two-loop 6-particle MHV amplitude \([22]\) must be expressible in terms of classical polylogarithms\(^{10}\).

### 5. Cluster polylogarithms for \( \text{Gr}(4, 7) \) and the amplitude \( R_7^{(2)} \)

We now demonstrate the utility of the \( A_3 \) function for two-loop MHV scattering amplitudes by providing, as an illustrative example, an explicit representation of the two-loop 7-particle MHV amplitude (modulo products of functions of lower weight, as always). We have carried out this exercise for \( n > 7 \) (where the cluster algebras \( \text{Gr}(4, n) \) are of infinite type) with no difficulty, but we relegate a detailed analysis of these more complicated results to a future publication \([4]\).

First let us take a look at the \( A_2 \) subalgebras. The \( \text{Gr}(4, 7) = E_6 \) cluster algebra has 1071 \( A_2 \) subalgebras (i.e., 1071 pentagonal faces on its generalized Stasheff polytope) on which the \( A_2 \) function can be evaluated, but only 504 of these give distinct results. We can tabulate here the 504 ‘distinct \( A_2 \) subalgebras’ by providing their quivers, in terms of cluster \( X \)-coordinates for \( \text{Gr}(4, 7) \). First we have
\[
\begin{align*}
\langle 1245 \rangle \langle 1567 \rangle & \mapsto \langle 1247 \rangle \langle 1256 \rangle \langle 1345 \rangle, \\
\langle 1257 \rangle \langle 1456 \rangle & \mapsto \langle 1234 \rangle \langle 1257 \rangle \langle 1456 \rangle, \\
\langle 1237 \rangle \langle 1245 \rangle \langle 4567 \rangle \langle 2457 \rangle \langle 1(23)(45)(67) \rangle & \mapsto \langle 1267 \rangle \langle 1457 \rangle \langle 2345 \rangle \langle 2457 \rangle \langle 1(23)(45)(67) \rangle,
\end{align*}
\] (5.1)
and their cyclic images (2 \( \times \) 7 = 14 total quivers). It suffices to take just the cyclic images because both parity and \( i \rightarrow 8 - i \) map this set back to itself. Next we have
\[
\begin{align*}
\langle 1236 \rangle \langle 1245 \rangle & \mapsto \langle 1237 \rangle \langle 1246 \rangle, \\
\langle 1234 \rangle \langle 1256 \rangle & \mapsto \langle 1234 \rangle \langle 1267 \rangle, \\
\langle 1237 \rangle \langle 1246 \rangle & \mapsto \langle 1247 \rangle \langle 1345 \rangle, \\
\langle 1234 \rangle \langle 1267 \rangle & \mapsto \langle 1234 \rangle \langle 1356 \rangle, \\
\langle 1236 \rangle \langle 1567 \rangle & \mapsto \langle 1237 \rangle \langle 1256 \rangle \langle 1346 \rangle, \\
\langle 1267 \rangle \langle 1356 \rangle & \mapsto \langle 1234 \rangle \langle 1267 \rangle \langle 1356 \rangle, \\
\langle 1246 \rangle \langle 1345 \rangle & \mapsto \langle 1247 \rangle \langle 1346 \rangle, \\
\langle 1234 \rangle \langle 1456 \rangle & \mapsto \langle 1234 \rangle \langle 1467 \rangle.
\end{align*}
\] (5.2)

along with their cyclic and parity images (7 \( \times \) 14 = 98 total quivers). In this case it suffices to take only these images since \( i \rightarrow 8 - i \) maps this set back to itself. And finally,

\(^{10}\) An explanation with the same flavor, but based on more physical constraints (rather than our more mathematical constraints) was given in \([23]\).
The $A_2$ function evaluates to 504 distinct results on these 504 algebras, but the 504 resulting quantities are not linearly independent: there are 56 linear relationships amongst these $A_2$ functions. It would be interesting to clarify the geometric origin of these linear relations. We conjecture, but lack the computer power to prove by explicit computation, that these 504 quantities span the space of nontrivial weight-four cluster functions for the $\text{Gr}(4, 7)$ algebra.

The $\text{Gr}(4, 7)$ algebras has 476 $A_3$ subalgebras [2] on which we can evaluate $f_{A_3}$, but only 364 of these give distinct results. We conjecture that these 364 quantities span the space of non-trivial weight-four cluster functions with completely local coproducts having the desired Poisson structure properties ($0$ in $\Lambda^2 B_2$ and $\pm 1$ in $B_3 \otimes C^\ast$).

We can list the 364 distinct $A_3$ evaluations by separating them in to three classes, and providing one (out of a possible six) generating quiver for each. First of all there are $14 \times 2 = 28 A_3$ generated by the quivers

\[
\begin{align*}
\langle 1236 \rangle \langle 1245 \rangle & \rightarrow \langle 1246 \rangle \langle 1345 \rangle \rightarrow \langle 1234 \rangle \langle 1256 \rangle \rightarrow \langle 1235 \rangle \langle 1267 \rangle \langle 1456 \rangle, \\
\langle 1234 \rangle \langle 1256 \rangle & \rightarrow \langle 1234 \rangle \langle 1456 \rangle \rightarrow \langle 1236 \rangle \langle 1245 \rangle \langle 1567 \rangle, \\
\langle 1236 \rangle \langle 1245 \rangle & \rightarrow \langle 1246 \rangle \langle 2345 \rangle \rightarrow \langle 1234 \rangle \langle 1256 \rangle \rightarrow \langle 1235 \rangle \langle 1267 \rangle \langle 2456 \rangle, \\
\langle 1234 \rangle \langle 1256 \rangle & \rightarrow \langle 1234 \rangle \langle 2456 \rangle \rightarrow \langle 1236 \rangle \langle 1245 \rangle \rightarrow \langle 1236 \rangle \langle 1245 \rangle \langle 2567 \rangle, \\
\langle 1234 \rangle \langle 1256 \rangle & \rightarrow \langle 1235 \rangle \langle 1267 \rangle \langle 3456 \rangle \rightarrow \langle 1235 \rangle \langle 4567 \rangle \rightarrow \langle 1236 \rangle \langle 1245 \rangle, \\
\langle 1236 \rangle \langle 1245 \rangle & \rightarrow \langle 1236 \rangle \langle 5(12) \rangle \langle 34 \rangle \langle 67 \rangle \rangle \rightarrow \langle 5(12) \rangle \langle 34 \rangle \langle 67 \rangle \rangle \rightarrow \langle 1234 \rangle \langle 1256 \rangle, \\
\langle 1235 \rangle \langle 1456 \rangle & \rightarrow \langle 1237 \rangle \langle 1245 \rangle \rightarrow \langle 1237 \rangle \langle 1245 \rangle \rightarrow \langle 1247 \rangle \langle 2345 \rangle, \\
\langle 1256 \rangle \langle 1345 \rangle & \rightarrow \langle 1234 \rangle \langle 1257 \rangle \rightarrow \langle 1234 \rangle \langle 1257 \rangle \rightarrow \langle 1234 \rangle \langle 2457 \rangle, \\
\langle 1234 \rangle \langle 1257 \rangle & \rightarrow \langle 1235 \rangle \langle 1267 \rangle \langle 2457 \rangle \rightarrow \langle 1236 \rangle \langle 1456 \rangle \rightarrow \langle 1237 \rangle \langle 1246 \rangle, \\
\langle 1237 \rangle \langle 1245 \rangle & \rightarrow \langle 1237 \rangle \langle 1245 \rangle \langle 2567 \rangle \rightarrow \langle 1256 \rangle \langle 1346 \rangle \rightarrow \langle 1234 \rangle \langle 1267 \rangle, \\
\langle 1234 \rangle \langle 1356 \rangle & \rightarrow \langle 1235 \rangle \langle 1367 \rangle \langle 3456 \rangle \rightarrow \langle 1234 \rangle \langle 1356 \rangle \rightarrow \langle 1235 \rangle \langle 1567 \rangle \langle 3456 \rangle, \\
\langle 1236 \rangle \langle 1345 \rangle & \rightarrow \langle 1236 \rangle \langle 1345 \rangle \langle 3567 \rangle \rightarrow \langle 1236 \rangle \langle 1345 \rangle \rightarrow \langle 1256 \rangle \langle 1345 \rangle \langle 3567 \rangle, \\
\langle 1234 \rangle \langle 1356 \rangle & \rightarrow \langle 1235 \rangle \langle 1367 \rangle \langle 3456 \rangle \rightarrow \langle 1234 \rangle \langle 1356 \rangle \rightarrow \langle 1235 \rangle \langle 1567 \rangle \langle 3456 \rangle, \\
\langle 1235 \rangle \langle 1567 \rangle & \rightarrow \langle 1237 \rangle \langle 1256 \rangle \langle 1345 \rangle, \\
\langle 1234 \rangle \langle 1256 \rangle \langle 1356 \rangle & \rightarrow \langle 1236 \rangle \langle 1567 \rangle \langle 3456 \rangle \rightarrow \langle 1256 \rangle \langle 1346 \rangle \langle 3567 \rangle. \\
\langle 1237 \rangle \langle 1256 \rangle \langle 1345 \rangle & \rightarrow \langle 1236 \rangle \langle 1567 \rangle \langle 3456 \rangle \rightarrow \langle 1256 \rangle \langle 1346 \rangle \langle 3567 \rangle. \\
\langle 1237 \rangle \langle 1256 \rangle \langle 1345 \rangle & \rightarrow \langle 1236 \rangle \langle 1567 \rangle \langle 3456 \rangle \rightarrow \langle 1256 \rangle \langle 1346 \rangle \langle 3567 \rangle. \\
\langle 1237 \rangle \langle 1256 \rangle \langle 1345 \rangle & \rightarrow \langle 1236 \rangle \langle 1567 \rangle \langle 3456 \rangle \rightarrow \langle 1256 \rangle \langle 1346 \rangle \langle 3567 \rangle. \\
\langle 1237 \rangle \langle 1256 \rangle \langle 1345 \rangle & \rightarrow \langle 1236 \rangle \langle 1567 \rangle \langle 3456 \rangle \rightarrow \langle 1256 \rangle \langle 1346 \rangle \langle 3567 \rangle. \\
\langle 1237 \rangle \langle 1256 \rangle \langle 1345 \rangle & \rightarrow \langle 1236 \rangle \langle 1567 \rangle \langle 3456 \rangle \rightarrow \langle 1256 \rangle \langle 1346 \rangle \langle 3567 \rangle.
\end{align*}
\]

along with their dihedral and parity images ($14 \times 28 = 392$ total quivers).

The $A_2$ function evaluates to 504 distinct results on these 504 algebras, but the 504 resulting quantities are not linearly independent: there are 56 linear relationships amongst these $A_2$ functions. It would be interesting to clarify the geometric origin of these linear relations. We conjecture, but lack the computer power to prove by explicit computation, that these 504 quantities span the space of nontrivial weight-four cluster functions for the $\text{Gr}(4, 7)$ algebra.
along with their dihedral images. Next there are \(14 \times 6 = 84\) \(A_3\) generated by the quivers
\[
\begin{align*}
\langle 1245 \rangle (3456) &\rightarrow \langle 1235 \rangle (1456) &\rightarrow \langle 1234 \rangle (1256) \\
\langle 1456 \rangle (2345) &\rightarrow \langle 1256 \rangle (1345) &\rightarrow \langle 1236 \rangle (1245) \\
\langle 1234 \rangle (1257) &\rightarrow \langle 1237 \rangle (1256) &\rightarrow \langle 1245 \rangle (1567) \\
\langle 1237 \rangle (1245) &\rightarrow \langle 1235 \rangle (1267) &\rightarrow \langle 1257 \rangle (1456) \\
\langle 1267 \rangle (1356) &\rightarrow \langle 1346 \rangle (3567) &\rightarrow \langle 1236 \rangle (1345) \\
\langle 1236 \rangle (1567) &\rightarrow \langle 1367 \rangle (3456) &\rightarrow \langle 1234 \rangle (1356) \\
\langle 1237 \rangle (1245) &\rightarrow \langle 1234 \rangle (2457) &\rightarrow \langle 1257 \rangle (2345) (4567) \\
\langle 1234 \rangle (1257) &\rightarrow \langle 1247 \rangle (2345) &\rightarrow \langle 1245 \rangle (2567) (3457) \\
\langle 1237 \rangle (1246) &\rightarrow \langle 1234 \rangle (1267) (1456) &\rightarrow \langle 1256 \rangle (1346) (4567) \\
\langle 1234 \rangle (1267) &\rightarrow \langle 1247 \rangle (1256) (1346) &\rightarrow \langle 1246 \rangle (1567) (3456) \\
\langle 1257 \rangle (2345) (4567) &\rightarrow \langle 1247 \rangle (2567) &\rightarrow \langle 1237 \rangle (1245) \\
\langle 1245 \rangle (2567) (3457) &\rightarrow \langle 1267 \rangle (2457) &\rightarrow \langle 1234 \rangle (1257) \\
\end{align*}
\]
along with their cyclic and parity images. Finally we have the \(9 \times 28 = 252\) \(A_3\) generated by the quivers
\[
\begin{align*}
\langle 1256 \rangle (4567) &\rightarrow \langle 1246 \rangle (1567) &\rightarrow \langle 1236 \rangle (1245) \\
\langle 1567 \rangle (2456) &\rightarrow \langle 1267 \rangle (1456) &\rightarrow \langle 1234 \rangle (1256) \\
\langle 1236 \rangle (1245) &\rightarrow \langle 1234 \rangle (1456) &\rightarrow \langle 1256 \rangle (1345) (4567) \\
\langle 1234 \rangle (1256) &\rightarrow \langle 1246 \rangle (1345) &\rightarrow \langle 1245 \rangle (1567) (3456) \\
\langle 1245 \rangle (1567) &\rightarrow \langle 1235 \rangle (1456) &\rightarrow \langle 1234 \rangle (1257) \\
\langle 1257 \rangle (1456) &\rightarrow \langle 1256 \rangle (1345) &\rightarrow \langle 1237 \rangle (1245) \\
\langle 1234 \rangle (1257) &\rightarrow \langle 1237 \rangle (1256) &\rightarrow \langle 1245 \rangle (2567) \\
\langle 1237 \rangle (1245) &\rightarrow \langle 1235 \rangle (1267) &\rightarrow \langle 1257 \rangle (2456) \\
\langle 1245 \rangle (2567) &\rightarrow \langle 1235 \rangle (2456) &\rightarrow \langle 1234 \rangle (1257) \\
\langle 1257 \rangle (2456) &\rightarrow \langle 1256 \rangle (2345) &\rightarrow \langle 1237 \rangle (1245) \\
\langle 1257 \rangle (4567) &\rightarrow \langle 1247 \rangle (1567) &\rightarrow \langle 1237 \rangle (1245) \\
\langle 1567 \rangle (2457) &\rightarrow \langle 1267 \rangle (1457) &\rightarrow \langle 1234 \rangle (1257) \\
\langle 1246 \rangle (1567) &\rightarrow \langle 1236 \rangle (1456) &\rightarrow \langle 1234 \rangle (1267) \\
\langle 1267 \rangle (1456) &\rightarrow \langle 1256 \rangle (1346) &\rightarrow \langle 1237 \rangle (1246) \\
\langle 1246 \rangle (1567) &\rightarrow \langle 1236 \rangle (2456) &\rightarrow \langle 1234 \rangle (1267) \\
\langle 1267 \rangle (1456) &\rightarrow \langle 1256 \rangle (2346) &\rightarrow \langle 1237 \rangle (1246) \\
\langle 1345 \rangle (1567) &\rightarrow \langle 1235 \rangle (3456) &\rightarrow \langle 1234 \rangle (1357) \\
\langle 1357 \rangle (1456) &\rightarrow \langle 1356 \rangle (2345) &\rightarrow \langle 1237 \rangle (1345) \\
\end{align*}
\]
along with their dihedral and parity images.

While this collection of functions is dramatically more tame than the vastly overcomplete space of completely general non-classical polylogarithms at weight four, there are still 169 functional identities amongst these 364 \(f_{A_3}\). Again, it would be very interesting to understand these relations geometrically.

We now turn our attention to the two-loop 7-point MHV amplitude, whose coproduct was first calculated in [2]. The \(B_3 \otimes C^*\) portion of the coproduct can be separated into symmetric and antisymmetric parts under \([x]_3 \otimes y \rightarrow [y]_3 \otimes x\). The antisymmetric part,
which corresponds to non-classical polylogarithms, can be fit to $A_3$ functions of the $\text{Gr}(4, 7)$ cluster algebra, and the symmetric part can be fit to $\text{Li}_d(-X)$-coordinate.

The functional identities amongst $A_3$ functions prevent us from writing down a unique representation of $R_f^{(2)}$ at this point. We settle here for the shortest possible representation\(^\text{11} \):

$$
R_f^{(2)} \sim \frac{1}{2} A_{1} \left( \frac{\langle 1245 \rangle \langle 1567 \rangle}{\langle 1257 \rangle \langle 1456 \rangle}, \frac{\langle 1235 \rangle \langle 1456 \rangle}{\langle 1256 \rangle \langle 1345 \rangle}, \frac{\langle 1234 \rangle \langle 1257 \rangle}{\langle 1237 \rangle \langle 1245 \rangle} \right) \\
+ \frac{1}{2} A_{1} \left( \frac{\langle 1345 \rangle \langle 1567 \rangle}{\langle 1357 \rangle \langle 1456 \rangle}, \frac{\langle 1235 \rangle \langle 3456 \rangle}{\langle 1356 \rangle \langle 2345 \rangle}, \frac{\langle 1234 \rangle \langle 1357 \rangle}{\langle 1237 \rangle \langle 1245 \rangle} \right) \\
+ \text{Li}_d \left( \frac{\langle 1234 \rangle \langle 1256 \rangle}{\langle 1236 \rangle \langle 1245 \rangle} \right) + \text{Li}_d \left( \frac{\langle 1234 \rangle \langle 1257 \rangle}{\langle 1237 \rangle \langle 1245 \rangle} \right) \\
+ \frac{1}{4} \text{Li}_d \left( \frac{\langle 1234 \rangle \langle 1357 \rangle}{\langle 1237 \rangle \langle 1345 \rangle} \right) + \frac{1}{4} \text{Li}_d \left( \frac{\langle 1234 \rangle \langle 1456 \rangle}{\langle 1246 \rangle \langle 1345 \rangle} \right) \\
- \frac{1}{4} \text{Li}_d \left( \frac{\langle 1234 \rangle \langle 1257 \rangle \langle 1356 \rangle}{\langle 1237 \rangle \langle 1256 \rangle \langle 1345 \rangle} \right) + \frac{1}{4} \text{Li}_d \left( \frac{\langle 1234 \rangle \langle 1267 \rangle \langle 1356 \rangle}{\langle 1237 \rangle \langle 1256 \rangle \langle 1346 \rangle} \right) \\
+ \frac{1}{4} \text{Li}_d \left( \frac{\langle 1235 \rangle \langle 1456 \rangle \langle 1345 \rangle}{\langle 1256 \rangle \langle 1345 \rangle} \right) \\
- \frac{1}{4} \text{Li}_d \left( \frac{\langle 1234 \rangle \langle 1257 \rangle \langle 1456 \rangle}{\langle 1247 \rangle \langle 1256 \rangle \langle 1345 \rangle} \right) - \frac{1}{4} \text{Li}_d \left( \frac{\langle 1234 \rangle \langle 1267 \rangle \langle 1456 \rangle}{\langle 1247 \rangle \langle 1256 \rangle \langle 1346 \rangle} \right) \\
- \frac{1}{4} \text{Li}_d \left( \frac{\langle 1234 \rangle \langle 1457 \rangle \langle 1345 \rangle}{\langle 1247 \rangle \langle 1345 \rangle} \right) + \text{dihedral + parity conjugate}. 
\quad (5.7)
$$

As indicated by the $\sim$, this result expresses the ‘most complicated part’ of $R_f^{(2)}$—the difference between the function presented here and the actual amplitude is some weight-four polynomial in the functions $-\text{Li}_d(-x)$ for $k = 1, 2, 3$ (and $\pi$), with arguments $x$ drawn from the 385 $X$-coordinates of the $\text{Gr}(4, 7)$ cluster algebra.

We end this section by reiterating some important features of the $A_2$ and $A_3$ functions as conveyed in equation (5.7). Given the known symbol of $R_f^{(2)}$ there is no difficulty in principle to find a representation of the non-classical component of this amplitude in terms of (for example) the collection $\text{Li}_{3,2}$ functions (see the appendix) with simple ratios of $X$-coordinates as arguments. The problem with fitting the non-classical portion of the amplitude to some general basis of this type is that these functions in general have non-$\mathcal{A}$ coordinates as entries in their symbols. Therefore, the remaining classical $\text{Li}_d$ needed to express the full amplitude could then have arbitrarily complicated algebraic functions of $X$-coordinates as arguments, which makes constructing an ansatz exceptionally difficult. The $A_2$ function solves this problem because it has only $\mathcal{A}$-coordinates in its symbol, therefore providing a basis which is sufficient to capture the non-classical component while ensuring that the remaining classical $\text{Li}_d$ can be taken to have only (minus) $X$-coordinates as arguments. The packaging of $A_2$ functions into the $A_3$ manifests even more structure of the amplitude $R_f^{(2)}$—namely the complete (i.e., term-by-term) locality and Poisson structure of its coproduct components.

\(^{11}\) Future developments may reveal that a different, longer representation is ‘better’ by manifesting other properties, either a physical property such as smooth behavior under the collinear limit \([4]\) or possibly even an additional, so far unnoticed mathematical property.
6. Conclusion

Motivated by the cluster structure apparently underlying the structure of amplitudes in SYM theory [2], in this paper we defined and studied the simplest few examples of cluster polylogarithm functions at transcendentality weight four. We found that the $A_2$ algebra admits a single non-trivial function $f_{A_2}$ of this type, and for several other cluster algebras which we were able to analyze by explicit computation we found that the space of cluster functions is spanned by $f_{A_2}$ evaluated on all available $A_2$ subalgebras. Interestingly, we found that these functions all have ‘Stasheff polytope local’ $B_3 \otimes \mathbb{C}^*$ content which can be expressed in terms of $[x]_3 \otimes y - [y]_3 \otimes x$ with pairs $x, y$ having Poisson bracket 1 (and therefore associated to pentagonal faces of the appropriate generalized Stasheff polytope).

We then considered an even more special collection of ‘Stasheff polytope local’ functions which have $A^2B_2$ content expressible in terms of $\{x\}_2 \wedge \{y\}_2$ with pairs $x, y$ having Poisson bracket 0 (and therefore associated to quadrilateral faces). For the $A_3$ algebra we found a unique nontrivial function $f_{A_3}$ with this property, and conjectured that the space of such functions for more general algebras is spanned by the function $f_{A_3}$ evaluated on all available $A_3$ subalgebras.

Obviously it would be of mathematical interest to further explore these classes of functions, as well as suitable generalizations of them at higher weight and for more general cluster algebras (especially algebras of infinite type).

We used the $A_3$ function to write an explicit formula for the ‘most complicated part’ of the two-loop 7-particle MHV amplitude in SYM theory. We are confident that the $A_3$ function suffices to similarly express two-loop MHV amplitudes for all $n$, both because we have checked some cases explicitly but more importantly because we know [4, 26] that these amplitudes have completely local coproducts in the sense mentioned a moment ago.

However a number of important questions about the cluster structure of these amplitudes remain. For example, attention was called in [2] to the curious fact that the $A^2B_2$ component of the 7-particle amplitude can be written as a 42-term linear combination of $\{x\}_2 \wedge \{y\}_2$ involving only 42 out of the 1785 distinct pairs of Poisson commuting $X$-coordinates available in the $\text{Gr}(4, 7)$ cluster algebra. It was natural to wonder whether there is any characteristic of these 42 which distinguishes them from the rest, and which might be able to explain ‘why’ the amplitude’s coproduct can be expressed in terms of only these 42. Unfortunately we are no closer to answering this question than [2] was. The first obstacle is that the formula (5.2) in [2] is not manifestly expressible in terms of $A_3$ functions: there does not exist any $A_3$ subalgebra of $\text{Gr}(4, 7)$ which has a quadrilateral face corresponding to the first term in (5.2) (nor to any of its symmetric images). In contrast, if we evaluate the $A^2B_2$ content of the amplitude by starting with equation (5.7) and associating to each $f_{A_3}$ the corresponding coproduct component shown in equation (4.5) we obtain a 56-term linear combination which is nontrivially equal to the 42-term expression presented in (5.2) of [2]12.

Also, the results of this paper unfortunately shed no light on the curiosity noted in [2] that for both $n = 6, 7$, the coproduct of the $n$-particle two-loop MHV amplitude can be expressed in terms of only $3/5$ of the $X$-coordinates available in the $\text{Gr}(4, n)$ cluster algebra.

Our exploration of the appropriate function space for two-loop MHV amplitudes at arbitrary $n$ was strongly motivated by a similar exploration of functions appropriate for non-MHV and higher-loop $n = 6$ amplitudes by Dixon and collaborators [23–25]. It would be very

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12 It turns out that 14 terms in the former correspond exactly to the 14 terms on the second line of (5.2) of the latter; so perhaps it would be better to say that the nontrivial relation is between a 42-term expression and a 28-term expression in $A^2B_2$. 

interesting to explore the (necessarily very close) relationship between their ‘hexagon functions’ and the various cluster functions we have explored, which we leave to future work. Here however we have focused exclusively on purely mathematical constraints: the $A$-coordinate condition on symbols, the $X$-coordinate condition on functions, and the locality and Poisson structure constraints on the coproduct. These are listed in order of increasing mathematical power, but also in order of increasing physical obscurity. We confess to having no physical explanation of why SYM theory should select weight-four polylogarithm functions whose coproducts are local in the generalized Stasheff polytope or have any particular relation to the Poisson structure, except to speculate that it might be related to the integrability of SYM theory. Notice also that clusters represent sets of coordinates that are compatible in some way. For instance, it is known [27] that for $Gr(2, n)$ the cluster structure is isomorphic to that of polygon triangulations, and that in turn to planar tree diagrams. To each tree corresponds a cluster, which can therefore be thought of as a channel for the tree amplitude. Cluster coordinates are then compatible in the sense that they correspond to possible simultaneous poles in planar scattering. Perhaps some more sophisticated version of this argument will hold here. It is natural to wonder if there exists an alternative formulation for SYM theory amplitudes which makes these (and perhaps other, still hidden) cluster algebraic properties manifest.

With our current understanding of how to write down the most complicated part of the two-loop MHV amplitudes it is reasonable to contemplate finding fully analytic expressions for them. To this end the next step is to begin applying various physical constraints to fix ambiguities involving products of functions of lower weight as well as beyond-the-symbol terms. The most obvious such constraints include the first- and last-entry conditions on the symbol, the requirement of smooth behavior under collinear limits, and especially the highly constraining requirement of analyticity inside the Euclidean kinematic region. We believe the last of these, in particular, might be strong enough to fix a unique (or almost unique) ‘analytic tail’ to the $A_3$ function, perhaps similar in form to the analytic tail which appears in the $L(x^+, x^-)$ building block of the GSVV formula [22]. Adding these terms of lower weight will help us resolve the ambiguities present in equation (5.7), where we had to arbitrarily choose one out of many possible representations in terms of $A_3$ functions. Moreover we suspect that ‘the right’ completion of the $A_3$ function (once it is found) will continue to be the unique non-classical building block for all $n$-particle two-loop MHV amplitudes.

Based on the surprising fact that the fundamental building block of the two-loop MHV amplitudes seems to be a function involving only $n = 6$ particles, it is natural to hope that the available results on higher-loop and NMHV functions for $n = 6$, when supplemented by suitable ‘cluster algebraic’ constraints of the type we have discussed in this paper, may serve as a springboard for unlocking the structure of $n$-particle MHV and NMHV amplitudes at higher loop order.

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Appendix. Functional representatives

We present here functional representations for the $A_2$ and $A_3$ functions studied in the paper. These functions are completely defined (as always, modulo products of lower-weight functions) by their coproducts, shown in equations (3.3) and (4.5), but some readers may be comforted by seeing concrete functional representations for them. However, we relegate these formulas to the appendix because they are provided as is, with no express or implied warranty, and certainly not the implied warranty of suitability for numerically evaluating actual SYM theory amplitudes. For such an application one would first need to append to each of the functions shown below a suitable ‘analytic tail’ comprising a carefully chosen product of lower-weight functions, specially crafted to give the functions the right analytic properties. Nevertheless we do believe that these functions capture the ‘most complicated part’ of all two-loop MHV amplitudes in SYM theory.

There are several different types of generalized polylogarithm functions in terms of which non-classical functions can be expressed. At weight four it suffices to use the function $L_{2,2}(x, y)$ (see for example [30] for a discussion), whose symbol is

\[(y - 1) \otimes (x - 1) \otimes x \otimes y + (y - 1) \otimes (x - 1) \otimes y \otimes x + (y - 1) \otimes x \otimes (x - 1) \otimes y \otimes x - (xy - 1) \otimes x \otimes y - (xy - 1) \otimes (x - 1) \otimes y \otimes x - (xy - 1) \otimes (x - 1) \otimes x \otimes y + (xy - 1) \otimes x \otimes x \otimes y + (xy - 1) \otimes y \otimes (y - 1) \otimes y + (xy - 1) \otimes y \otimes y \otimes x + (xy - 1) \otimes y \otimes x \otimes x + (xy - 1) \otimes y \otimes (y - 1) \otimes y.
\]

(A.1)

A.1. The $A_2$ function

The $A_2$ function may be represented as

\[f_{A_2} \sim \sum_{i,j}^5 j L_{2,2}(x_i, x_{i+j})\]

in terms of

\[L_{2,2}(x, y) = \frac{1}{2} \text{Li}_{2,2}\left(\frac{x}{y}, -y\right) + \frac{1}{6} \left(\text{Li}_4\left(\frac{1 + x}{xy}\right) + \text{Li}_4\left(\frac{x(1 + y)}{y(1 + x)}\right)\right) + \frac{1}{5} \left(\text{Li}_4\left(\frac{1 + x}{xy}\right) + \frac{1}{2} \text{Li}_4\left(\frac{1 + x}{1 + y}\right)\right) + \frac{1}{2} \text{Li}_3\left(\frac{x}{y}\right) \log\left(\frac{1 + x}{1 + y}\right) - (x \leftrightarrow y).\]

(A.3)

The factor of $j$ in the summand may seem awkward, but when fully expanded out the sum generates a total of 20 $\text{Li}_{2,2}$ terms, each with coefficient $\pm \frac{3}{2}$ or $\pm \frac{1}{2}$ (each possibility occurs five times). Note that the function $L_{2,2}$ has the simple coproduct $\delta L_{2,2}(x, y)|_{\mathfrak{B}_2} = \{x\} \wedge \{y\}$.
(it is therefore very similar to Goncharov’s $\kappa(x, y)$ function [35]). The rather strange looking $\text{Li}_4$ terms in equation (A.3) of course make no contribution to $\Lambda^2 B_2$; they are carefully tuned to ensure that equation (A.2) has clustery $\mathcal{B}_3$ content. The $\text{Li}_3 \cdot \log$ terms are of course irrelevant inside $\mathcal{L}_4$, but they are required for $f_{A_3}$ to be a cluster $\mathcal{A}$-function of the $A_2$ algebra. The symbol of equation (A.2) is not identical to the one shown in equation (3.4), but the difference between the two is annihilated by $\rho$ (i.e., they differ by products of functions of lower weight).

A.2. The $A_3$ function

The $A_3$ function may of course be written as the sum of equation (A.3)’s for the six pentagons in $A_3$, but the simple form of $\delta f_{A_3}(x_1, x_2, x_3)$ suggests that there is a more concise functional representation. Indeed, we find that a representative of the $A_3$ function can be written as

$$f_{A_3}(x_1, x_2, x_3) \sim \sum_{i=1}^{3} K_{2,2}(x_{i,1}, x_{i,2}) + \frac{1}{2} \sum_{i=1}^{6} (-1)^i \text{Li}_4(-e_i) \quad (A.4)$$

where the $x_{ij}$ and $e_i$ are defined in (4.1) and we use here the new combination

$$K_{2,2}(x, y) = \frac{1}{2} \text{Li}_2(x/y, -y) - \text{Li}_4(x/y) - \frac{2}{3} \text{Li}_3(x/y) \log(y) - (x \leftrightarrow y). \quad (A.5)$$

As was the case for the $A_2$ function, the $\text{Li}_3 \cdot \log$ terms are chosen so that the symbol of (A.4) is expressible entirely in terms of cluster $\mathcal{A}$-coordinates of the $A_3$ algebra.

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